On a method for constructing the Lax pairs for nonlinear integrable equations

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Abstract
We suggest a direct algorithm for searching the Lax pairs for nonlinear integrable equations. It is effective for both continuous and discrete models. The first operator of the Lax pair corresponding to a given nonlinear equation is found immediately, coinciding with the linearization of the considered nonlinear equation. The second one is obtained as an invariant manifold to the linearized equation. A surprisingly simple relation between the second operator of the Lax pair and the recursion operator is discussed: the recursion operator can immediately be found from the Lax pair. Examples considered in the article are convincing evidence that the found Lax pairs differ from the classical ones. The examples also show that the suggested objects are true Lax pairs which allow the construction of infinite series of conservation laws and hierarchies of higher symmetries. In the case of the hyperbolic type partial differential equation our algorithm is slightly modified; in order to construct the Lax pairs from the invariant manifolds we use the cutting off conditions for the corresponding infinite Laplace sequence. The efficiency of the method is illustrated by application to some equations given in the Svinolupov–Sokolov classification list for which the Lax pairs and the recursion operators have not been found earlier.

Keywords: Lax pair, symmetry, invariant manifold, Laplace sequence, discrete equations, sine-Gordon type equations, recursion operator

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1. Introduction

In the present article we suggest an algorithm for constructing the Lax pairs for nonlinear integrable models. Our scheme is based on the symmetry approach. Let us explain the algorithm with the evolutionary type partial differential equation (PDE) of the form
\[ u_t = f(x, t, u, u_1, u_2, \ldots, u_k), \quad u_j = \frac{\partial^j u}{\partial x^j}. \tag{1.1} \]

Recall that equation
\[ u_r = g(x, t, u, u_1, u_2, \ldots, u_m) \tag{1.2} \]
is called a symmetry for equation (1.1) if function \( g \) satisfies the following differential equation,
\[ \left( D_t - \frac{\partial f}{\partial u} - \frac{\partial f}{\partial u_1} D_t - \cdots - \frac{\partial f}{\partial u_k} D_t^k \right) g = 0, \tag{1.3} \]
where \( D_t \) and \( D_x \) are the operators of the total derivatives with respect to \( t \) and \( x \) correspondingly. Note that equation (1.3) is overdetermined and for any fixed value of \( m \) one can effectively find all of its solutions of the form \( g = g(x, t, u, u_1, u_2, \ldots, u_m) \) (see [1]).

An ordinary differential equation
\[ u_m = G(x, t, u, u_1, u_2, \ldots, u_{m-1}) \tag{1.4} \]
defines an invariant manifold for (1.1) if the following condition is satisfied,
\[ D_t G = D_x^m f \bigg|_{(1.1),(1.4)} = 0. \tag{1.5} \]

Here \( D_t G \) is evaluated by means of equation (1.1) and all the \( x \)-derivatives of the order greater than \( m - 1 \) are expressed due to equation (1.4) and its differential consequences. Equation (1.5) generates a PDE with unknown \( G \) admitting a large class of solutions. However it is a hard problem to find these solutions since the equation is not overdetermined. Some of the invariant manifolds to (1.1) can be found from the stationary part of the symmetry (1.2) by taking \( g(x, t, u, u_1, u_2, \ldots, u_m) = 0 \).

We now concentrate on the linearization of equation (1.1) around its arbitrary solution \( u = u(x, t) \),
\[ v_t - \frac{\partial f}{\partial u} v - \frac{\partial f}{\partial u_1} v_1 - \cdots - \frac{\partial f}{\partial u_k} v_k = 0. \tag{1.6} \]

Actually (1.6) defines a family of differential equations, depending on \( u \). The important fact that \( u \) ranges across the whole set of solutions to (1.1) is formalized in the following way. We assume that in addition to the natural independent dynamical variables \( v, v_1, v_2, \ldots \) the variables \( u, u_1, u_2, \ldots \) are also considered as independent ones.

Let us find the invariant manifold of the form
\[ v_m = \sum_{j=0}^{m-1} a(j) v_j, \quad v = v_0, \tag{1.7} \]
to equation (1.6). This means that the condition
\[ D_t \left( \sum_{j=0}^{m-1} a(j) v_j \right) - D_x^m \left( \sum_{j=0}^{k} \frac{\partial f}{\partial u} v_j \right) \bigg|_{(1.1),(1.6),(1.7)} = 0 \tag{1.8} \]
holds identically for all values of \( u, u_1, u_2, \ldots \). Here we assume that \( \forall j \) function \( a(j) \) depends on \( x, t \) and a finite number of the dynamical variables \( u, u_1, u_2, \ldots \). In (1.8) the variables \( u, u_1, u_2, \ldots \) are expressed by means of equation (1.1) and the variables \( v, v_1, \ldots, v_{n-1,2} \) are expressed due to (1.6). Equation (1.8) splits down into a system of \( m \) PDEs with the unknown functions \( a(0), a(1), \ldots, a(m-1) \), coefficients of the decomposition (1.7). Due to the presence of the additional dynamical variables \( u, u_1, u_2, \ldots \) the system is overdetermined and hence can effectively be investigated. Actually, equation (1.7) defines a bundle of manifolds depending on the infinite set of variables \( u, u_1, u_2, \ldots \). Suppose that such an invariant manifold is found. Then we can interpret a pair of equations (1.6) and (1.7) as the Lax pair to equation (1.1). In the examples the order \( k \) of equation (1.1) and the order \( m \) of equation (1.7) coincide.

As an illustrative example we consider the well known Korteweg–de Vries (KdV) equation

\[
u_t = u_3 + uu_1. 
\]

(1.9)

The linearized equation

\[
v_t = v_3 + uv_1 + u_1v \]

(1.10)

obtained by virtue of the rule (1.6) admits the third-order invariant manifold defined by the equation

\[
v_3 = \frac{u_2}{u_1}v_2 - \left( \frac{2}{3}u + \lambda \right) v_1 + \left( \left( \frac{2}{3}u + \lambda \right) \frac{u_2}{u_1} - u_1 \right) v.
\]

(1.11)

It is easily checked that equations (1.10) and (1.11) are consistent if and only if the function \( u = u(x, t) \) satisfies equation (1.9). Therefore these equations constitute the Lax pair to the KdV equation. It differs from the usual one found earlier in [2]. It is worth stressing that there is not any second-order invariant manifold of the form \( v_2 = a(u, u_1, u_2)v_1 + b(u, u_1, u_2)v \) for equation (1.10). But there is a first order one \( v_1 = \frac{u_2}{u_1}v \) which however does not contain \( \lambda \) and therefore does not generate any true Lax pair.

The recursion operator is an important attribute of the integrability theory. It gives a compact description for the hierarchies of both symmetries and conservation laws. Various methods for studying the recursion operators can be found in the literature (see, for instance [3–11] and the references therein).

We observe that the invariant manifold for the linearized equation is closely connected with the recursion operator for the original equation. Indeed examples in section 6 show that the equation defining the invariant manifold, which provides the second operator of the Lax pair, can be rewritten as a formal eigenvalue problem of the form

\[ Rv = \lambda v \]

for the recursion operator \( R \). For instance, equation (1.11) is easily rewritten as (see section 6)

\[
\left( D_t^2 + \frac{2}{3}u + \frac{1}{3}u_1D_t^{-1} \right)v = \lambda v,
\]

where the operator on the lth is nothing else but the recursion operator for the KdV hierarchy.

Therefore our scheme of constructing the Lax pairs provides an alternative tool for searching the recursion operator. On the other hand when the recursion operator is known the invariant manifold and hence the Lax pair can be found by simple manipulations (see section 6). At this point we have an intersection with the pioneering articles [1], where nonstandard Lax pairs are given for some of the KdV type equations in terms of the recursion
operators. However these Lax pairs differ from those found within our scheme since they are nonlocal and do not contain any spectral parameter.

There is a great variety of approaches for searching the Lax pairs from the Zakharov–Shabat dressing [12, 13] and prolongation structures by Wahlquist and Estabrook [14] to the three-dimensional (3D) consistency approach developed in [15–17]. We also mention approaches proposed in [1, 18, 19]. An advantage of our scheme is that it can be applied to any integrable model (at least in the 1 + 1-dimensional case), the first of the operators is easily found and the second is effectively computed. They allow finding the conservation laws, higher symmetries and invariant surfaces for the corresponding nonlinear equation. The found Lax pairs are more complicated since they are based on differential operators of orders greater than usual. This is their disadvantage. The question remains open whether the Lax pairs of this kind allow any new solution to be found for the well studied models.

In the case of hyperbolic type integrable equations the algorithm should be slightly modified. Let us explain it with the example of the sine-Gordon equation

\[ u_{xy} = \sin u. \] (1.12)

Let us find the simplest but nontrivial, i.e. depending on a parameter, invariant manifold for the linearized equation

\[ v_{xy} = (\cos u)v. \] (1.13)

It is the 3D surface in the space of the dynamical variables \( v, v_x, v_y, v_{xy}, \ldots \) defined by the following two linear equations, with the coefficients, depending on the field variable \( u(x, t) \) and its derivatives,

\[ v_{xy} = u_y(\cot u)v_x + \lambda \frac{u_x}{\sin u} v_x - \lambda v = 0, \] (1.14)

\[ v_{xx} = u_x(\cot u)v_x + \lambda^{-1} \frac{u_x}{\sin u} v_y - \lambda^{-1} v = 0. \] (1.15)

Here \( \lambda \) is a complex parameter. Note that equations (1.14) and (1.15) are not independent. One of them is immediately found from the other by differentiation by means of equations (1.12) and (1.13). Equations (1.13)–(1.15) can be rewritten (see the end of section 3) as a pair of systems of ordinary differential equations providing the Lax pair realized in \( 3 \times 3 \) matrices. The method for deriving the Lax pair from the triple (1.13)–(1.15) is based on constructing the infinite Laplace cascade for the linearized equation (1.13) and obtaining the finite reduction of the cascade.

Let us give a brief comment on the structure of the article. In sections 2 and 3 we discuss the well-known Laplace cascade for linear and nonlinear hyperbolic type equations. In section 3 the problem of finding finite reductions of the infinite Laplace sequence is studied. The Lax pair to the sine-Gordon equation is derived from the Laplace cascade. In section 4 the definition of the invariant manifold for hyperbolic type equations is recalled. The Lax pair is constructed via invariant manifolds for hyperbolic equation (4.12) found in [20]. In section 5 the Lax pairs are constructed by evaluating invariant manifolds for the evolutionary type integrable equations. Beside the explanatory examples here we consider two equations (5.19) and (5.20) found in [21] as equations possessing infinite hierarchies of conserved quantities. To the best of our knowledge the Lax pairs for equations (4.12), (5.19) and (5.20) have never been found before. In section 6 we illustrate applications of the newly found Lax pairs. The Lax pair obtained in the previous sections is used to construct conservation laws for a Volterra type chain. We also show that the second operators of our Lax pairs are closely connected with the recursion operators for the associated nonlinear
equations. In the appendix we give all of the computational details used when we evaluated the invariant manifold for the linearization of the sine-Gordon equation.

2. Laplace cascade for the linear hyperbolic type equations

Let us recall the main steps of the Laplace cascade method (see [22, 23]). Consider a linear second order hyperbolic type PDE of the form

\[ v_{xy} + a(x, y)v_x + b(x, y)v_y + c(x, y)v = 0. \]  

(2.1)

It can easily be checked that functions

\[ h_{[0]} = a_x + ab - c, \quad k_{[0]} = b_y + ab - c \]

(2.2)
do not change under the linear transformation \( v \to \lambda(x, y)v \) with arbitrary smooth factors \( \lambda(x, y) \) applied to equation (2.1). They are called the Laplace invariants for (2.1).

We rewrite equation (2.1) as a system of two equations

\[ \left( \frac{\partial}{\partial y} + a \right)v = v_{[1]}, \quad \left( \frac{\partial}{\partial x} + b \right)v_{[1]} = h_{[0]}v. \]  

(2.3)

When the invariant \( h_{[0]} \) does not vanish then one can exclude \( v \) from (2.3) and obtain a linear PDE for \( v_{[1]} \)

\[ v_{[1]xy} + a_{[1]}v_{[1]x} + b_{[1]}v_{[1]y} + c_{[1]}v_{[1]} = 0, \]  

(2.4)

where the coefficients are evaluated as follows,

\[ a_{[1]} = a - \frac{\partial}{\partial y} \log(h_{[0]}), \quad b_{[1]} = b, \quad c_{[1]} = a_{[1]}b_{[1]} + b_{[1]y} - h_{[0]}. \]  

(2.5)

Thus we define a transformation of equation (2.1) into equation (2.4). This transformation is called the Laplace \( y \)-transformation. Iterations of the transformation generate a sequence of equations

\[ v_{[i]xy} + a_{[i]}v_{[i]x} + b_{[i]}v_{[i]y} + c_{[i]}v_{[i]} = 0 \]  

(2.6)

for \( i \geq 1 \) where the coefficients are given by

\[ a_{[i]} = a_{[i-1]} - \frac{\partial}{\partial y} \log(h_{[i-1]}), \quad b_{[i]} = b_{[i-1]}, \quad c_{[i]} = a_{[i]}b_{[i]} + b_{[i]y} - h_{[i-1]}. \]  

(2.7)

Here we assume that \( a_{[0]} = a, \quad b_{[0]} = b, \quad c_{[0]} = c \). Eigenfunctions \( v_{[i]} \) are related by the equations

\[ \left( \frac{\partial}{\partial y} + a_{[i]} \right)v_{[i]} = v_{[i+1]}, \quad \left( \frac{\partial}{\partial x} + b_{[i]} \right)v_{[i+1]} = h_{[i]}v_{[i]}. \]  

(2.8)

Due to the relation \( c_{[i]} = \frac{\partial}{\partial x}a_{[i]} + a_{[0]}b_{[i]} - h_{[i]} \) system (2.7) is rewritten as

\[ a_{[i]} = a_{[i-1]} - \frac{\partial}{\partial y} \log(h_{[i-1]}), \quad h_{[i]} = h_{[i-1]} + a_{[i]y} - b_{[i]y}, \quad b_{[i]} = b. \]  

(2.9)

The reasonings above define the functions \( a_{[i]}, b_{[i]}, h_{[i]} \) only for \( i \geq 1 \). However they can be prolonged for \( i \leq 0 \) by virtue of the same formulas rewritten as follows,
Summarizing the computations above we obtain a dynamical system of the form
\[
\frac{\partial}{\partial y} \log(h_{[i]}) = a_{[i]} - a_{[i+1]}, \quad a_{[i]x} = h_{[i]} - h_{[i-1]} + b_{\gamma}, \quad b_{[i]} = b,
\]
which is reduced to the well-known Toda lattice
\[
\frac{\partial}{\partial y} \log(h_{[i]}) = p_{[i]} - p_{[i+1]}, \quad p_{[i]x} = h_{[i]} - h_{[i-1]},
\]
where \( p_{[i]} = a_{[i]} - \bar{b} \) and \( \bar{b} = b_{\gamma} \).

We define two linear operators
\[
L_i = D_y + a_{[i]} - D_{[i]}, \quad M_i = D_x + b_{[i-1]} - h_{[i-1]}D_i^{-1},
\]
where \( D_x, D_y \) are the operators of differentiation with respect to \( x, y \) correspondingly and \( D_i \) is the shift operator acting as follows: \( D_ia_{[i]} = a_{[i+1]}, D_ih_{[i]} = h_{[i+1]} \), etc. We summarize all the reasonings above as a statement.

**Proposition 1.** The operators \( L_i, M_i \) commute for all \( i \) iff their coefficients satisfy the system (2.9).

**Corollary.** Equations (2.8) constitute the Lax pair for the system (2.11).

The Laplace \( x \)-transformation can be interpreted in a similar way.

### 3. Laplace cascade for the nonlinear hyperbolic type equations. Formal Lax pairs

Let us explain how the Laplace cascade is adapted to the nonlinear case [23] (see also [24]). Consider a second order nonlinear hyperbolic type PDE
\[
u_{xy} = F(x, y, u, u_x, u_y).
\]
Its linearization around a solution \( u(x, y) \) derived by substituting \( u = u(x, y, \varepsilon) = u(x, y, 0) + \varepsilon v(x, y) + \cdots \) with \( v(x, y) = \frac{\partial u(x, y, \varepsilon)}{\partial \varepsilon} \big|_{\varepsilon=0} \) into (3.1) is an equation of the form
\[
v_{xy} + av_x + bv_y + cv = 0,
\]
where the coefficients \( a = -\partial F/\partial u_x, b = -\partial F/\partial u_y, c = -\partial F/\partial u \) depend explicitly on the independent variables \( x, y \) and the dynamical variables \( u, u_x, u_y \). Let us assign the Laplace sequence (2.8)–(2.11) to the linearized equation (3.2). However now instead of the operators \( \partial/\partial x, \partial/\partial y \) in (2.8)–(2.11) we use the operators \( D_x, D_y \) of the total differentiation with respect to \( x \) and \( y \). Denote through \( u_i, \bar{u}_i, i = 0, 1, \ldots \) the \( i \)th order derivatives of the variable \( u \) with respect to \( x \) and \( y \) correspondingly,
\[
D^i_xu = u_i, \quad D^i_yu = \bar{u}_i.
\]
Evidently we have explicit expressions for the operators \( D_x, D_y \) acting on the class of smooth functions of \( x, y \) and a finite number of the dynamical variables \( u_i, \bar{u}_i \), i.e.
The Laplace invariants corresponding to equation (3.2) are evaluated as

\[ h_{[0]} = D_x(a) + ab - c, \quad k_{[0]} = D_y(b) + ab - c. \]  

The linear system (2.8) in this case converts into

\[ (D_x + a_{[i]})(v_{[i]} - v_{[i+1]}) = (D_y + b_{[i]})(v_{[i]}). \]  

The coefficients \( a_{[i]}, b_{[i]}, h_{[i]} \) are evaluated via the equations

\[ D_x(\log(h_{[i]})) = a_{[i]} - a_{[i+1]}, \quad D_y(a_{[i]}) = h_{[i]} - h_{[i-1]} + D_y(b), \]
\[ b_{[i]} = b, \quad a_{[0]} = a. \]

All of the mixed derivatives \( u_{xy}, u_{x,y}, \ldots \) are replaced by means of equation (3.1) and its differential consequences.

Infinite-dimensional system (3.7) defines a sequence of linear operators

\[ L_i = D_y + a_{[i]} - D_x, \quad M_i = D_x + b_{[i-1]} - h_{[i-1]}D_{[i]}^{-1}, \]  

satisfying the commutativity conditions

\[ \forall i \quad [L_i, M_i] = 0. \]  

Thus one can define a pair of commuting operators \( L_i, M_i \) depending on an integer parameter \( i \) for an arbitrarily chosen equation (3.1). Roughly speaking the sequence of commuting operators (3.9) and (3.10) recovers equation (3.1). Hence system (3.7) defines a (formal) Lax pair for the arbitrary (generally non-integrable) equation (3.1). It is not very surprising since for the non-integrable case system (3.7) is of infinite dimension. However as it is confirmed below by several examples for the integrable case the system is either finite (Liouville type equations) or admits a finite-dimensional reduction (sine-Gordon type equations).

Example 1. As an illustrative example of the non-integrable equation with infinite-dimensional Lax pair consider the equation

\[ u_{xy} = u^2. \]  

For its linearization

\[ v_{xy} = 2uv \]  

we have \( a_{[0]} = h_{[0]} = 0, h_{[1]} = -2u, a_{[1]} = -\frac{a}{2}, \) \( h_{[1]} = u + \frac{a}{2}u. \) One can find all of the coefficients \( a_{[i]}, h_{[i]} \) due to equation (3.8) as functions of the dynamical variables and therefore completely define the system (3.7). We now go back and suppose that the system evaluated above is consistent and show that its consistency uniquely defines equation (3.11). Indeed the consistency of (3.7) implies \( D_x(a_{[1]}) = h_{[1]} - h_{[0]} \) equivalent to

\[ \left( \frac{-a_{[1]}}{u} \right)_x = \frac{a_{[1]}}{u} - u \]  

which gives (3.11).

Example 2. As an example with the finite system (3.7) we take the Liouville equation

\[ u_{xy} = e^u \]  

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for which \( h_0 = h_{-1} = e^u \) and \( h_1 = h_{-2} = 0 \). For \( i > 1 \) and \( i < -2 \) the Laplace invariants \( h_i \) are not defined. The coefficient \( a_{ii} \) is defined only for the following three values of \( i \): \( a_{11} = -u_y \), \( a_{00} = 0 \), \( a_{-1-1} = u_y \). Evidently \( h_{00} = 0 \). System (3.7) for equation (3.13) contains only seven equations

\[
(D_y - u_y)v_{[1]} = v_{[2]}, \quad D_yv_{[2]} = h_{[1]}v_{[1]},
\]

\[
D_yv_{[0]} = v_{[1]}, \quad D_yv_{[1]} = h_{[0]}v_{[0]},
\]

\[
(D_y + u_y)v_{[-1]} = v_{[0]}, \quad D_yv_{[0]} = h_{[-1]}v_{[-1]},
\]

\[
D_yv_{[-1]} = h_{[-2]}v_{[-2]}.
\]

There is a freedom in choosing \( v_{[2]}, v_{[-2]} \). If we make them equal to zero, then the obtained system gives the Lax pair for (3.13)

\[
\Psi_t = A\Psi, \quad \Psi_y = B\Psi,
\]

where \( \Psi = (v_{[1]}, v_{[0]}, v_{[-1]})^T \) and

\[
A = \begin{pmatrix}
0 & e^u & 0 \\
0 & 0 & e^u \\
0 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
u_y & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & -u_y
\end{pmatrix}.
\]

Note that there are some degenerate cases where the commutativity condition of the operators (3.9) does not exactly define the initial equation (3.1) but some other equation connected with (3.1) by a Miura type transformation. We illustrate it with the following example.

**Example 3.** Consider the equation

\[
u_{xy} = e^{u+u}.
\]

(3.14)

The coefficients of its linearization

\[
v_{xy} = e^r(v + v_y)
\]

depend on \( r = u + u_y \). Thus the commutativity condition of the operators (3.9) assigned to (3.14) implies the equation \( r_{xy} = e^r + r_{xx} \), connected with (3.14) by a very simple Miura type transformation \( u_x + u = r \).

**3.1. A more symmetrical form of the Laplace sequence**

Let us change the dependent variables in the system (3.7) to make formulas more symmetrical. We introduce new dependent variables \( w_{[i]}, i \in (-\infty, \infty) \) in such a way that \( w_{[i]} = v_{[i]} \) for \( i \geq 0 \) and \( w_{[i]} = h_{[-i-1]}h_{[-i+2]}...h_{[-1]}v_{[i]} \) for \( i \leq -1 \).

Then the set of equations (3.7) and (3.8) is changed to the form below, where \( i \geq 0 \),

\[
(D_x + a_{[i]})w_{[i]} = w_{[i+1]}, \quad (D_x + b_{[0]})w_{[i+1]} = h_{[i]}w_{[i]},
\]

\[
D_x(\log(h_{[i]})) = (a_{[i]} - a_{[i+1]}), \quad D_x(a_{[i]}) = h_{[i]} - h_{[i-1]} + D_x(b_{[0]}),
\]

\[
(D_x + \hat{b}_{[-i-1]}^i)w_{[-i]} = w_{[-i+1]}, \quad (D_x + a_{[0]})w_{[-i-1]} = h_{[-i-1]}w_{[-i]},
\]

\[
\hat{b}_{[-i-1]} = \hat{b}_{[-i]} - D_x(\log(h_{[-i-1]})), \quad \hat{b}_{[0]} = b_{[0]}.
\]

(3.16)
3.2. Sine-Gordon equation

More than two decades ago an important property of the Laplace invariants of the Liouville type integrable equations was observed [24, 25]. It was proved that the hyperbolic equation (3.1) is an integrable equation of the Liouville type if and only if the set of its Laplace invariants is terminated on both sides. It is also worth mentioning recent results obtained in [26]. The problem of describing the properties of the Laplace invariants characterizing the sine-Gordon type integrable PDE is discussed in [27]. Our investigation reveals that a connection between the sine-Gordon type equations and the Laplace cascade is clearly formulated in terms of the cascade eigenfunctions. Let us explain our observation with an example.

We consider the sine-Gordon equation

\[ u_{xy} = \sin u. \]  

(3.17)

It can be shown that the Laplace invariants \( h_{(i)} \) for the linearized equation

\[ v_{xy} = (\cos u)v \]  

(3.18)

do not vanish identically for any integer \( i \). Thus system (3.7) provides an infinite-dimensional Lax pair for (3.17). Below we show that for this case (3.7) admits a finite-dimensional reduction, bringing (3.7) to the symmetric form (3.16).

**Proposition 2.** The system (3.16) corresponding to the sine-Gordon equation (3.17) with \( w_{[0]} = v, \ a_{[0]} = 0, \ b_{[0]} = 0, \ c_{[0]} = -\cos u \) is consistent with the following cutting off boundary conditions

\[
\begin{align*}
\hat{w}_{[1]} & = \alpha (-1)w_{[-1]} + \alpha (0)w_{[0]} + \alpha (1)w_{[1]}, \\
\hat{w}_{[-1]} & = \beta (-1)w_{[-1]} + \beta (0)w_{[0]} + \beta (1)w_{[1]},
\end{align*}
\]

(3.19)

where

\[
\begin{align*}
\alpha (-1) & = -\lambda \frac{u_y}{\sin u}, & \alpha (0) & = \lambda, & \alpha (1) & = \frac{u_y}{\cos u \sin u}, \\
\beta (-1) & = \frac{u_x}{\cos u \sin u}, & \beta (0) & = \lambda^{-1}, & \beta (1) & = -\lambda^{-1} \frac{u_x}{\sin u},
\end{align*}
\]

(3.20)

\( \lambda \) is a complex parameter.

Sketch of proof. We look for the functions (3.20) providing consistency of the following overdetermined system of equations obtained from (3.16) by imposing (3.19)

\[
\begin{align*}
w_{[1]} & = (\alpha (1) - a_{[1]})w_{[1]} + \alpha (0)w_{[0]} + \alpha (-1)w_{[-1]}, \\
w_{[0]} & = w_{[1]}, \\
w_{[-1]} & = h_{[-1]}w_{[0]},
\end{align*}
\]

(3.21)

and

\[
\begin{align*}
w_{[1]} & = h_{[0]}w_{[0]}, \\
w_{[0]} & = w_{[-1]}, \\
w_{[-1]} & = \beta (1)w_{[1]} + \beta (0)w_{[0]} + \left( \beta (-1) - \hat{b}_{[-1]} \right)w_{[-1]},
\end{align*}
\]

(3.22)

The compatibility conditions \( (w_{[1]}), (w_{[0]}), (w_{[-1]}) \) generate a system of nonlinear equations for the functions \( \alpha (j), \beta (j) \).
Here all the given coefficients $a_{[1]}$, $\hat{b}_{[-1]}$, $h_{[-2]}$, $h_{[-1]}$, $h_{[0]}$, $h_{[1]}$ are linear functions of the derivatives $u_x, u_y$

\begin{align*}
  h_{[0]} &= h_{[-1]} = \cos u, \\
  h_{[1]} &= h_{[-2]} = \frac{1}{\cos u} + \frac{u_x u_y}{\cos^2 u}, \\
  a_{[1]} &= u_x \tan u, \\
  \hat{b}_{[-1]} &= u_x \tan u,
\end{align*}

therefore we can assume that $\alpha(j), \beta(j)$ also linearly depend on the first derivatives of $u$

\begin{align*}
  \alpha(j) &= \alpha(j, u, u_x, u_y) = p(j, u)u_x + q(j, u), \\
  \beta(j) &= \beta(j, u, u_x) = r(j, u)u_x + s(j, u), \quad j = 1, 0, -1. \tag{3.25}
\end{align*}

We substitute expressions (3.24) and (3.25) into equation (3.23) and then compare the coefficients before the independent combinations of the dynamical variables $u_x, u_y$. As a result one obtains 24 equations for 12 functions $p(j, u), q(j, u), r(j, u), s(j, u), j = 1, 0, -1$, depending on $u$ only. We write down explicitly only a part of the equations since the others are obtained from these by applying the replacement $p(j) \leftrightarrow r(-j), q(j) \leftrightarrow s(-j)$

\begin{align*}
  \frac{dr(-1)}{du} + r(1)p(-1) &= \frac{1}{\cos^2 u}, \\
  \frac{ds(-1)}{du} + s(1)p(-1) &= 0, \\
  r(1)q(-1) &= 0, \\
  r(-1)\sin u + s(1)q(-1) &= \frac{1}{\cos u}, \\
  \frac{dr(0)}{du} + r(1)p(0) &= 0, \\
  \frac{ds(0)}{du} + s(1)p(0) &= 0, \\
  r(-1)\cos u + r(1)q(0) &= 0, \\
  r(0)\sin u + s(-1)\cos u + s(1)q(0) &= 0, \\
  \frac{dr(1)}{du} - r(1)\tan u + r(1)p(1) &= 0, \\
  \frac{ds(1)}{du} - s(1)\tan u + s(1)p(1) &= 0, \\
  r(0) + r(1)q(1) &= 0, \\
  r(1)\sin u + s(0) + s(1)q(1) &= 0. \tag{3.26}
\end{align*}

By solving the overdetermined system of equations (3.26) we find explicit expressions (3.20) for the coefficients of the constraint (3.19). Now the systems (3.21) and (3.22) can be rewritten as

\begin{align*}
  \Psi_t = A\Psi, & \quad \Psi_j = B\Psi, \tag{3.27}
\end{align*}

where $\Psi = (w_{[1]}, w_{[0]}, w_{[-1]})^T$ and

\begin{align*}
  A &= \begin{pmatrix} 0 & \cos u & 0 \\ 0 & 0 & 1 \\ -u_x & \frac{1}{\lambda} & u_x \cot u \end{pmatrix}, &
  B &= \begin{pmatrix} u_x \cot u & \lambda & -\lambda u_x \\ 1 & 0 & 0 \\ 0 & \cos u & 0 \end{pmatrix}. \tag{3.28}
\end{align*}

It is easily checked that (3.27) and (3.28) define the Lax pair for the sine-Gordon equation (3.17). We failed to reduce it to the well-known usual one found in [28].
4. Invariant manifolds of the hyperbolic type PDE

Let us recall the definition of the invariant manifold of the hyperbolic type equation (3.1). Consider an equation of the form

\[ G(x, y, u_k, u_{k-1}, \ldots, \bar{u}_1, \bar{u}_2, \ldots \bar{u}_m) = 0. \]  

(4.1)

Note that \( G \) depends on \( x, y \) and a set of dynamical variables \( u, u_1, \ldots, u_j = \frac{\partial u}{\partial y}, \ldots \)

We take the differential consequences of (4.1)

\[ G_1(x, y, u_{k+1}, \ldots, \bar{u}_1, \bar{u}_2, \ldots \bar{u}_m) = 0, \]  

(4.2)

\[ G_2(x, y, u_k, \ldots, u_j, \bar{u}_2, \ldots \bar{u}_{m+1}) = 0, \]  

(4.3)

where \( G_1, G_2 \) are evaluated by applying the operators \( D_x, D_y \): \( G_1 = D_x G, G_2 = D_y G \) and subsequent replacement of the mixed derivatives by means of equation (3.1) and its differential consequences. Equation (4.1) defines an invariant manifold for (3.1) if the following equation is satisfied

\[ D_x D_y G \mid_{(3.1)} = 0. \]  

(4.4)

Example 4. We show that equation

\[ u_{xx} + \frac{1}{2} u_x^2 \tan u = 0 \]  

(4.5)

defines an invariant manifold for the sine-Gordon equation (3.17). Here \( G = u_{xx} + \frac{1}{2} u_x^2 \tan u, G_1 = D_x G = u_{xxx} + \frac{1}{2} u_x^3 \) and \( G_2 = D_y G = \frac{u_x}{\cos u} + \frac{u_x^2 u_y}{2(\cos u)^2} \). It is easily verified that

\[ D_x D_y G = D_x \left( \frac{u_x}{\cos u} + \frac{u_x^2 u_y}{2(\cos u)^2} \right) = 0 \quad \text{mod} \ (4.5), \ G = 0, \ G_1 = 0, \ G_2 = 0). \]

Therefore equation (4.4) holds and thus (4.5) defines an invariant manifold for (3.17).

4.1. From the Laplace cascade to invariant manifolds

Here we show that the reduced system (3.16) and (3.19) is closely connected with the invariant manifolds of the linearized equation (3.18). Indeed, equations (3.16) imply that \( w_{[2]} = (D_x + a_{[1]})(D_x + a_{[0]})w_{[0]}, w_{[1]} = (D_x + a_{[0]})w_{[0]}, w_{[-1]} = (D_x + b_{[0]})w_{[0]}, w_{[-2]} = (D_x + b_{[-1]})(D_x + b_{[0]})w_{[0]} \). Therefore since \( a_{[0]} = b_{[0]} = 0 \), the boundary conditions (3.19) turn into the equations

\[ (D_x + a_{[1]})D_x w_{[0]} = \alpha (-1)D_x w_{[0]} + \alpha (0)w_{[0]} + \alpha (1)D_x w_{[0]}; \]  

\[ (D_x + b_{[-1]})D_x w_{[0]} = \beta (-1)D_x w_{[0]} + \beta (0)w_{[0]} + \beta (1)D_x w_{[0]}. \]

We simplify the equations obtained by using explicit expressions (3.20) and (3.24) and find equations

\[ L_x w_{[0]} = \left( D_x^2 - u_x \cot u D_x + \lambda \frac{u_y}{\sin u} D_x - \lambda \right)w_{[0]} \mid_{3.17} = 0, \]  

(4.6)
\[ \mathcal{L}_{\mathcal{A}} w_0 = \left( D_x^2 - u_x \cot u \frac{u_x}{\sin u} - \lambda^{-1} \frac{u_x}{\sin u} - \lambda^{-1} \right) w_{0|x} + (3.17), w_{0|xy} = (\cos u) w_{0|y} = 0, \]  
\[ (4.7) \]

which define the invariant manifold for (3.18) discussed in the introduction (see (1.14) and (1.15) above).

This observation leads to an alternative algorithm to look for the Lax pair. Instead of the cutting off boundary conditions to the lattice (3.16) one searches an invariant manifold for the linearized equation (3.18).

By construction we have
\[ M w_0 = (D_x D_y - \cos u) w_{0|x}. \]  
\[ (4.8) \]

Commutators of the operators \( L_x, L_y, M \) satisfy the relations
\[ [L_x, L_y] = 2A_y \left( \lambda^{-1} L_y - \lambda L_x \right), \]
\[ [M, L_x] = B_x L_y - B_y L_x + \left( A_{xx} - \lambda^{-1} A_{xy} \right) M, \]
\[ [M, L_y] = B_y L_x - B_x L_y + \left( A_{yy} - \lambda A_{xy} \right) M, \]  
\[ (4.9) \]

where \( A = \log \cot \frac{u}{2}, \ B = \log \sin u, \ A_x = D_x(A), \ A_y = D_y(A), \ A_{xx} = D_x^2(A) \) and so on. Consequently any element of the Lie ring generated by the operators \( L_x, L_y, M \) is represented as a linear combination of the same three operators.

Linear equations (4.6) and (4.7) define a manifold parametrized by \( w_0, w_{0|x}, w_{0|y} \) and the dynamical variables \( u, u_x, u_y, \ldots \). By applying \( D_x \) to equations (4.6) and (4.7) and then simplifying due to equations (3.17) and (3.18) one gets another parametrization of the manifold
\[ w_{0|x|x} = \frac{u_{xx}}{u_x} w_{0|xx} + \left( \lambda^{-1} - \frac{u_x^2}{u_x} \right) w_{0|y} - \lambda^{-1} \frac{u_{xx}}{u_x} w_{0|y}, \]  
\[ (4.10) \]
\[ w_{0|y} = -\lambda \frac{\sin u}{u_x} w_{0|xx} + \lambda (\cos u) w_{0|y} + \frac{\sin u}{u_x} w_{0|y}, \]  
\[ (4.11) \]

where the parameters \( w_0, w_{0|x}, w_{0|xx} \) are taken as independent ones. It is shown below that this parametrization is closely connected with the Lax pair for the potential KdV equation being a symmetry of the sine-Gordon equation.

4.2. Evaluation of the invariant manifolds and the Lax pair for the equation
\[ u_{xy} = f(u) \sqrt{1 + u_x^2}, \quad f' = \gamma f \]

In this section we construct a Lax pair to the equation
\[ u_{xy} = f(u) \sqrt{1 + u_x^2}, \quad f'' = \gamma f \]  
\[ (4.12) \]

found in [20]. It is known that the \( S \)-integrable equation of the form (4.12) by an appropriate point transformation can be reduced either to the case \( f(u) = u \) or \( f(u) = \sin u \) (see [20]). By analogy with the sine-Gordon equation considered in the previous section we look for the invariant manifold of the form
\[ v_{yy} + av_y + bv_x + cv = 0 \]  
\[ (4.13) \]
for the linearized equation

\[ v_{xy} = f'(u) \sqrt{1 + u_x^2} v + \frac{f(u) u_x}{\sqrt{1 + u_x^2}} v_x. \]  

(4.14)

We apply the operator \( D_y \) to (4.13) and rewrite the result as

\[
v_{xy} = -\frac{1}{b} \left( 2vu_x f(u)f'(u) + v_x f^2(u) + D_y(b) v_x + cv_y + D_x(c)v + D_x(a)v_y \right)
- \frac{u_x u_x v_x + (v_y + av) \left( 1 + u_x^2 \right) f'(u) + \left( pu_x v \left( 1 + u_x^2 \right) + au_x v_x \right) f(u)}{b \sqrt{1 + u_x^2}}.
\]

(4.15)

Now we apply \( D_y \) to (4.15), simplify the result due to the equations above and get an equation of the form

\[ v_{xy} + \tilde{a} v_y + \tilde{b} v_x + \tilde{c} v = 0, \]

with the coefficients \( \tilde{a} \), \( \tilde{b} \), \( \tilde{c} \) depending on a finite number of dynamical variables. According to the definition of the invariant manifold equations (4.13) and (4.16) should coincide. This fact implies a system of three equations on the sought functions \( a, b, c \), namely

\[
\begin{align*}
2f(u) u_x b f'(u) + D_x(c) b - D_x(b) D_x(a) + D_x(D_x(a)b - abD_x(a)) \sqrt{1 + u_x^2} \\
+ \left( 2pu_x b u_x^2 - D_x(a) u_x b + 2pu_x b \right) f(u) - \left( u_x^2 + 1 \right) D_y(b) f'(u) = 0,
\end{align*}
\]

(4.17)

\[
\begin{align*}
\left( -D_y(b) u_x^2 + ab - D_x(b) \right) f(u) + u_x b \left( 3 + 2u_x^2 \right) f'(u) f(u) \sqrt{1 + u_x^2} \\
- b^2D_x(a) - D_x(c) b - D_x(D_x(b) + D_x(b)c + D_x(b)D_x(b)) \left( 1 + u_x^2 \right)^{3/2} \\
+ \left( u_x b^2 + \left( pu_x^2 u_x b + D_x(a) u_x b - D_x(b)au_x \right) \left( 1 + u_x^2 \right) \right) f(u) \\
+ u_x \left( 1 + u_x^2 \right) \left( b^2 u_x + u_y b + au_x b - D_x(b) u_x \right) f'(u) = 0,
\end{align*}
\]

(4.18)

\[
\begin{align*}
2pf(u) u_x u_x b + u_x \left( -ab - 2D_x(b) \right) f'(u) f(u) + 3f'(u) u_x^2 u_x b \\
- D_x(b) D_x(c) - cbD_x(a) + D_x(D_x(c)b) \sqrt{1 + u_x^2} + b \left( u_x^2 + 3 \right) f'(u) f(u) \\
+ \left( -D_y(b) a + D_x(a) b + D_x(b) b + pu_x^2 b \right) \left( 1 + u_x^2 \right) + u_x u_x b^2 \right) f'(u) \\
+ \left( pu_x b + au_x b + u_x b^2 - D_x(b) u_x \right) \left( 1 + u_x^2 \right) - D_x(c) u_x b f'(u) = 0.
\end{align*}
\]

(4.19)

Assuming that the sought functions depend only on \( u, u_x, u_y \), i.e. \( a = a(u, u_x, u_y) \), \( b = b(u, u_x, u_y) \) and \( c = c(u, u_x, u_y) \), we substitute these functions into (4.17)–(4.19) and eliminate mixed derivatives of \( u \) using (4.12) from the resulting equations. Thus we obtain three equations of the form

\[ \alpha_i(u, u_x, u_y) u_x u_y + \beta_i(u, u_x, u_y) u_{xx} + \gamma_i(u, u_x, u_y) u_{yy} + \delta_i(u, u_x, u_y) = 0, \]

(4.20)
\( i = 1, 2, 3 \). These relations are satisfied only if the conditions

\[
\alpha_i(u, u_x, u_y) = 0, \quad \beta_i(u, u_x, u_y) = 0, \quad \gamma_i(u, u_x, u_y) = 0, \quad \delta_i(u, u_x, u_y) = 0
\]

(4.20)

hold identically for all values \( u, u_x \) and \( u_y, i = 1, 2, 3 \). Here

\[
\alpha_1 = \left( ba_{u,u_x} - a_u b_u \right) \sqrt{1 + u_x^2},
\]

\[
\alpha_2 = \left( bb_{u,u_x} - b_u b_{u_x} \right) \left( 1 + u_x^2 \right)^{3/2},
\]

\[
\alpha_3 = \left( bc_{u,u_x} - c_u b_{u_x} \right) \sqrt{1 + u_x^2},
\]

\[
\beta_1 = \left( bc_{u,u_x} + bu_x a_{u_{xx}} - b_{u_x} a_{u_x} - ab a_{u_{xx}} \right) \sqrt{1 + u_x^2} + f(u) \left( 1 + u_x^2 \right) \left( ba_{u,u_x} - b_u a_{u_x} \right),
\]

\[
\beta_2 = - \left( 1 + u_x^2 \right) \left( b_{u_x} b_u a_{u_{xx}} - bu_x b_{u_{xx}} + b^2 a_{u_{xx}} \right) \sqrt{1 + u_x^2} + f(u) \left( 1 + u_x^2 \right) \left( ba_{u,u_x} - b_u a_{u_x} \right) \left( 1 + u_x^2 \right) + b^2,
\]

\[
\beta_3 = \left( bu_x c_{u_{xx}} - cb_{u_x} - b_{u_x} a_{u_x} \right) \sqrt{1 + u_x^2} + \left( 1 + u_x^2 \right) \left( ba_{u,u_x} - b_u a_{u_x} \right) \left( 1 + u_x^2 \right) + b^2 f(u),
\]

\[
\gamma_1 = u_x \left( -a_u b_{u_x} + ba_{u_{xx}} \right) \sqrt{1 + u_x^2} + \left( 1 + u_x^2 \right) \left( ba_{u,u_x} - b_u a_{u_x} \right) f(u) - f'(u) b_u,
\]

\[
\gamma_2 = \left( 1 + u_x^2 \right) \left( -b_{u_x} b_u u_x + bb_{u_{xx}} u_x - f(u) b_{u_{xx}} - cb_{u_x} + bc_{u_x} \right) \sqrt{1 + u_x^2} + \left( ba_{u,u_x} - b_u a_{u_x} \right) f(u) + u_x \left( b - b_{u_x} u_x \right) f'(u),
\]

\[
\gamma_3 = u_x \left( -c_u b_{u_x} - 2 b_{u_x} f(u) f'(u) \right) \left( 1 + u_x^2 \right) + \left( 1 + u_x^2 \right) \left( pb - b_{u_x} c_{u_x} + ba_{u,u_x} - pu b_{u_x} \right) f(u) + \left( -ab a_{u_x} + ba_{u_{xx}} \right) f'(u),
\]

\[
\delta_1 = \left[ \left( 1 + u_x^2 \right) \left( -b_{u_x} a_{u_x} + ba_{u,u_x} \right) f''(u) + \left( -u_x^2 b_{u_x} + 2u_x b - b_{u_x} \right) f'(u) \right] f(u)
\]

\[
- u_x \left( b_{u_x} a_{u_x} + ba_{u,u_x} - bc_{u_x} + b_{u_x} a_{u_{xx}} \right) \sqrt{1 + u_x^2} + b \left( u_x a_{u_{xx}} + c_{u_x} + 2 pu_y - aa_{u_x} + a_{u_{xx}} u_x \right) \left( 1 + u_x^2 \right) f(u)
\]

\[
\times \left( ba_{u_x} - \left( b_{u_x} a_{u_x} + b_{u_x} u_x - a_{u_x} u_{xx} \right) \left( 1 + u_x^2 \right) \right) f(u) + \left( 1 + u_x^2 \right) \left( ba_{u,u_x} - b_{u_x} u_x + ba_{u,u_x} \right) f'(u),
\]
Thus the problem is reduced to a system of equations (4.20). We look for the functions \( a, b \) and \( c \) depending on the variable \( u \), linearly,

\[ a = a_1(u, u_x) u_y + a_2(u, u_x), \]
\[ b = b_1(u, u_x) u_y + b_2(u, u_x), \]
\[ c = c_1(u, u_x) u_y + c_2(u, u_x). \]

Then equation \( \alpha_2 = 0 \) is essentially simplified to \( b_2(b_1)_{u_x} - b_1(b_2)_{u_x} = 0 \). Assume that \( b_2 \equiv 0 \), then

\[ b = b_1(u, u_x) u_y. \quad (4.21) \]

From equations \( \alpha_1 = 0 \) and \( \alpha_3 = 0 \) we obtain

\[ u_x a_{u_x, u} - a_u = 0, \quad u_x c_{u_x, u} - c_u = 0. \]

Assume that \( a_u = c_u = 0 \) then

\[ a = a_1(u) u_x + a_2(u), \quad c = c_1(u) u_x + c_2(u). \quad (4.22) \]

Substituting the functions (4.21) and (4.22) into \( \gamma_1 = 0 \) (see (4.20) above) we obtain the equation

\[ b_1(u, u_x) \left[ (f'(u) + a_1(u)f(u))(1 + u_x^2) + \sqrt{1 + u_x^2} u_x a_2(u) \right] = 0. \]
Since functions $f, a_1$ and $a_2$ depend only on $u$, we obtain
\[ f'(u) + a_1(u)f(u) = 0, \quad a_2'(u) = 0. \]
Consequently
\[ a_1(u) = -\frac{f'(u)}{f(u)}, \quad a_2(u) = a_3, \]
where $a_3$ is an arbitrary constant. Then from the equality $\gamma_3 = 0$ we get
\[ c_1(u) = -a_3 f'(u)/f(u), \quad c_2(u) = -f^2(u) + c_3, \]
where $c_3$ is an arbitrary constant. Analyzing the equation $\delta_1 = 0$ we define
\[ b_1 = \frac{b_3}{f(u)\sqrt{1 + u_x^2}}, \]
where $b_3 = 0$ is an arbitrary constant. From $\delta_2 = 0$ we find that $a_1 = 0$ and $c_1 = -b_1$. It is easily checked that equalities $b_i = 0, i = 1, 2, 3, \gamma_2 = 0, \delta_3 = 0$ are automatically satisfied.

Thus summarizing the reasonings above we can claim that equations
\[ v_{yy} = \frac{f'(u)}{f(u)} u_x v_x + \frac{\lambda u_x}{f(u)\sqrt{1 + u_x^2}} v_y - \left(\frac{f^2(u) + \lambda}{\lambda} \right) v = 0, \quad (4.23) \]
\[ v_{xx} = \left(\frac{f'(u)}{f(u)} + \frac{u_{xx}}{u_x^2 + 1}\right) u_x v_x + \frac{u_x \sqrt{u_x^2 + 1}}{\lambda f(u)} v_y - \left(\frac{u_x^2 + 1}{\lambda}\right) v = 0 \quad (4.24) \]
define an invariant manifold for the linearized equation $(4.14)$. Here $\lambda$ is the spectral parameter.

We now derive the Lax pair for equation $(4.12)$ from the invariant manifold $(4.23)$ and $(4.24)$. To this end we evaluate the Laplace sequence of the form $(3.16)$ for the linearized equation $(4.14)$. In what follows we will need explicit expressions for several first coefficients of the system $(3.16)$
\[ a_{[0]} = a_{[-1]} = -\frac{u_x f(u)}{\sqrt{u_x^2 + 1}}, \quad b_{[0]} = b_{[1]} = b_{[-1]} = 0, \]
\[ k_{[0]} = f'(u)\sqrt{u_x^2 + 1}, \quad h_{[0]} = -\frac{u_x f(u)}{u_x^2 + 1} + \frac{f'(u)}{\sqrt{u_x^2 + 1}}, \]
\[ a_{[1]} = \frac{(u_x^2 + 1)f(u)(\gamma u_x \sqrt{u_x^2 + 1} - f'(u)) + u_x u_{xx} f^2(u) - f'(u) u_x u_{xx} \sqrt{u_x^2 + 1}}{u_x^2 + 1 \left(f'(u)(u_x^2 + 1) - f(u) a_\alpha \right)} \quad (4.25) \]
The constraints $(4.23)$ and $(4.24)$ generate the cutting off boundary conditions of the form
\[ \begin{cases} v_{[-2]} = \alpha(1)v_{[1]} + \alpha(0)v_{[0]} + \alpha(-1)v_{[-1]}, \\ v_{[-3]} = \beta(1)v_{[1]} + \beta(0)v_{[0]} + \beta(-1)v_{[-1]} \end{cases} \quad (4.26) \]
imposed on the infinite system $(3.16)$. We find the coefficients $\alpha(i)$ and $\beta(i)$ in the relation $(4.26)$. Evidently equations $(3.16)$ imply
Set \( v = v_{[0]} \) and simplify (4.27) by virtue of (4.26) and the relations \( v_{[1]} = (D_x + a_{[0]}) v \) and \( v_{[-1]} = (D_x + b_{[0]}) v \). As a result we obtain

\[
\begin{align*}
    v_{yy} + \left( a_{[0]} + a_{[1]} - \alpha(1) \right) v_y - \alpha(-1) v_x \\
    + \left( a_{[0]} v_{[1]} + a_{[0]} a_{[1]} - a_{[0]} \alpha(1) - \alpha(0) - \alpha(-1) b_{[0]} \right) v &= 0, \\
    v_{xx} + \left( b_{[-1]} - \beta(-1) \right) v_x - \beta(1) v_y \\
    + \left( b_{[-1]} + b_{[0]} b_{[-1]} - a_{[0]} \beta(1) - \beta(0) - \beta(-1) b_{[0]} \right) v &= 0. 
\end{align*}
\] (4.28)

The last equations should coincide with (4.23) and (4.24). Comparison of the corresponding coefficients allows one to derive explicit formulas for the sought functions \( \alpha(i) \) and \( \beta(i) \)

\[
\begin{align*}
    \alpha(1) &= a_{[0]} + a_{[1]} + \frac{u_x f'(u)}{u}, \\
    \alpha(-1) &= -\frac{\lambda u_y}{f(u) \sqrt{u_x^2 + 1}}, \\
    \alpha(0) &= f(u)^2 + \lambda + a_x - a_{[0]}^2 - a_{[0]} \frac{u_x f'(u)}{f(u)} + \frac{\lambda u_x b_{[0]}}{f(u) \sqrt{u_x^2 + 1}}. \\
    \beta(1) &= -\frac{u_x \sqrt{u_x^2 + 1}}{\lambda f(u)}, \\
    \beta(-1) &= b_{[-1]} + \frac{u_x f'(u)}{f(u)} + \frac{u_x b_{[-1]}}{u_x^2 + 1}, \\
    \beta(0) &= b_{[0]} + a_{[0]} \left( \frac{u_x \sqrt{u_x^2 + 1}}{\lambda f(u)} \right) - \left( \frac{u_x f'(u)}{f(u)} + \frac{u_x b_{[0]}}{u_x^2 + 1} \right) b_{[0]} + \frac{u_x^2 + 1}{\lambda}.
\end{align*}
\]

Now the infinite system of equations (3.16) is reduced to a pair of third order systems of ordinary differential equations

\[
\begin{align*}
    v_{[0]} &= v_{[1]} - a_{[0]} v_{[0]}, \\
    v_{[1]} &= (\alpha(1) - a_{[1]}) v_{[1]} + \alpha(0) v_{[0]} + \alpha(-1) v_{[-1]}, \\
    v_{[-1]} &= k_{[0]} v_{[0]} - a_{[0]} v_{[-1]}, \\
    v_{[0]} &= v_{[-1]} - b_{[1]} v_{[1]}, \\
    v_{[1]} &= b_{[0]} v_{[0]} - b_{[0]} v_{[1]}, \\
    v_{[-1]} &= \beta(1) v_{[1]} + \beta(0) v_{[0]} + \left( \beta(-1) - b_{[-1]} \right) v_{[-1]}. 
\end{align*}
\]
which can be specified as

$$
\begin{pmatrix}
  v_0 \\
  v_1 \\
  v_{-1}
\end{pmatrix}_{x} =
\begin{pmatrix}
  \frac{u_y f(u)}{u_x^2 + 1} & 0 & 0 \\
  \lambda & -\frac{u_y f(u)}{u_x^2 + 1} - \frac{u_y \lambda}{f(u)\sqrt{u_x^2 + 1}} & 0 \\
  f'(u)\sqrt{u_x^2 + 1} & 0 & \frac{u_y f(u)}{u_x^2 + 1}
\end{pmatrix}
\begin{pmatrix}
  v_0 \\
  v_1 \\
  v_{-1}
\end{pmatrix},
$$

(4.29)

$$
\begin{pmatrix}
  v_0 \\
  v_1 \\
  v_{-1}
\end{pmatrix}_{x} =
\begin{pmatrix}
  0 & 0 & 0 \\
  \frac{u_y f(u)}{(u_x^2 + 1)^2} & 0 & 0 \\
  \frac{1}{\lambda} & -\frac{u_y u_x^2 + 1}{\lambda f(u)} & \frac{u_y f(u)}{u_x^2 + 1}
\end{pmatrix}
\begin{pmatrix}
  v_0 \\
  v_1 \\
  v_{-1}
\end{pmatrix}.
$$

(4.30)

Systems (4.29) and (4.30) define the Lax pair for equation (4.12).

5. Searching the Lax pairs for the evolutionary type integrable equations

Let us consider the evolutionary type integrable equations, for which the Laplace cascade is not defined. Here we use the scheme set out in the introduction.

5.1. Korteweg–de Vries equation

As an illustrative example we consider the KdV equation

$$u_t = u_{xxx} + uu_x.
$$

(5.1)

Its linearization evidently has the form

$$v_t = v_{xxx} + vv_x.
$$

(5.2)

Direct computations show that equation (5.2) does not admit any invariant manifold of the form $v_{xxx} = a(u, u_x, u_{xx})v_t + b(u, u_x, u_{xx})v$ for arbitrary solution $u(x, t)$ of (5.1).

Let us look for the invariant manifold of order three

$$v_{xxx} = av_x + bv_t + cv.
$$

(5.3)

where the coefficients $a$, $b$, $c$ depend on a finite number of dynamical variables $u$, $u_1$, $u_2$,... .

According to the definition the condition

$$\left( v_{xxx} \right)_t = \left( v_t \right)_{xxx}
$$

(5.4)

should be valid.

Replacing in (5.4) $v_{xxx}$ and $v_t$ due to (5.3) and (5.2) respectively and then comparing the coefficients before the independent variables $v_{xxx}$, $v_x$ and $v$ we obtain the equations

$$3aD_x(b) + 6u_{xxx} + D_x^2(a) + 3D_x(a)b + u_xa + uD_x(a) + 3D_x(a)^2
$$

$$+ 3aD_x^2(a) + 3a^2D_x(a) + 3D_x(c) - D_t(a) + 3D_x^2(b) = 0,
$$

(5.5)

$$D_x^2(b) + 3D_x^2(c) + 3bD_x(a) + 3D_x(a)D_x(b) + uD_x(b) + 4u_{xxx} + 3abD_x(a)
$$

$$+ 3D_x^2(a)b - 3au_{xx} - D_t(b) + 3cD_x(a) + 2u_xb = 0,
$$

(5.6)
\[ 3acD_t(a) + uu_{xxx} + uD_x(c) + D_t^3(c) + 3D_x^2(a)c - au_{xxx} + 3D_x(b)c - D_t(c) + 3u_t c + 3D_x(a)D_t(c) - bu_{xx} = 0. \] (5.7)

It is reasonable to assume that \( a = a(u, u_x, u_{xx}), \quad b = b(u, u_x, u_{xx}) \) and \( c = c(u, u_x, u_{xx}) \). We substitute these expressions into (5.5)–(5.7) and then exclude all the mixed derivatives of \( u \) due to equation (5.1). As a result we obtain three equations of the form

\[ \alpha_i(u, u_x, u_{xx})u_{xxx} - \beta_i(u, u_x, u_{xx})u_{xxx}^3 - \gamma_i(u, u_x, u_{xx})u_{xxx}^2 \]
\[ - \delta_i(u, u_x, u_{xx})u_{xxx} - \epsilon_i(u, u_x, u_{xx})u_{xx} = 0, \quad i = 1, 2, 3. \]

Since \( u_{xxx}, u_{xxx}^3, u_{xxx}^2, u_{xxx} \) are independent variables then these equations split down into 15 equations as

\[ \alpha_i(u, u_x, u_{xx}) = 0, \quad \beta_i(u, u_x, u_{xx}) = 0, \quad \gamma_i(u, u_x, u_{xx}) = 0, \]
\[ \delta_i(u, u_x, u_{xx}) = 0, \quad \epsilon_i(u, u_x, u_{xx}) = 0 \] (5.8)

hold for all values of \( u, u_x \) and \( u_{xx}, \ i = 1, 2, 3 \). Thus the sought coefficients \( a, b, c \) satisfy a highly overdetermined system of differential equation (5.8).

We now specify and analyze the system. It can be verified that the three equations \( \beta_i = 0, i = 1, 2, 3 \) immediately imply \( a_{u_{xxx}u_{xx}} = 0, \beta_{u_{xxx}u_{xx}} = 0 \) and \( c_{u_{xxx}u_{xx}} = 0 \). Therefore

\[ a = a_1(u, u_x)u_{xx} + a_2(u, u_x)u_{xx} + a_3(u, u_x), \]
\[ b = b_1(u, u_x)u_{xx} + b_2(u, u_x)u_{xx} + b_3(u, u_x), \]
\[ c = c_1(u, u_x)u_{xx} + c_2(u, u_x)u_{xx} + c_3(u, u_x). \]

Equations \( \alpha_i = 0, i = 1, 2, 3 \) are of the form

\[ a_{u_x}u_xu_{xxx} + a_{u_{xx}}u_{xx} + a_{u_{xxx}}u_{xx} = 0, \] (5.9)
\[ b_{u_x}u_xu_{xxx} + b_{u_{xx}}u_{xx} + b_{u_{xxx}}u_{xx} + ba = 0, \] (5.10)
\[ 3c_{u_x}u_{xxx} + 3c_{u_{xx}}u_x + 3c_{u_{xxx}}u_{xx} + 3ca = 1. \] (5.11)

Then since functions \( a, b \) and \( c \) depend only on the variables \( u, u_x \) and \( u_{xx} \), the coefficients at \( u_{xxx} \) vanish, i.e. we get

\[ a = a_1(u, u_x)u_{xx} + a_2(u, u_x), \]
\[ b = b_1(u, u_x)u_{xx} + b_2(u, u_x), \]
\[ c = c_1(u, u_x)u_{xx} + c_2(u, u_x). \]

Substituting \( a, b \) and \( c \) into equations (5.9)–(5.11) we obtain

\[ (a_1^2 + (a_1)u_x)u_{xx} + a_1a_2 + u_x(a_1) + b_1 = 0, \]
\[ (b_1)u_x + a_1b_1+u_x(b_1) + a_1b_2 = 0, \]
\[ 3(c_1)u_x + a_1c_1)u_{xx} + 3a_1c_2 + 1 = 0. \]

Since functions \( a_i, b_i, c_i, i = 1, 2 \) depend only on \( u \) and \( u_x \) the coefficients at \( u_{xx} \) vanish and we get the system of equations

\[ \]
If we concentrate on the last system, it is easy to check that \( a_1 \neq 0 \) and we can assume that \( b_1 \neq 0 \) and \( c_1 \neq 0 \). From system (5.12) then we find

\[
\begin{align*}
 a_1 &= \frac{1}{a_4 + a_6(u)}, \\
 b_1 &= \frac{b_4(u)}{a_4 + a_6(u)}, \\
 c_1 &= \frac{c_4(u)}{a_4 + a_6(u)}, \\
 a_2 &= -\frac{u_s(a_4)_u + b_1}{a_1}, \\
 b_2 &= -\frac{c_1 + u_s(b_1)_u}{a_1}, \\
 c_2 &= -\frac{1 + 3u_s(c_1)_u}{3a_1}.
\end{align*}
\]  

(5.13)

Now equations \( \gamma_i = 0, i = 1, 2, 3 \) are satisfied automatically. From equation \( \delta_1 = 0 \) we find that \( b_4 = b_5 \) and \( a_4 = a_6 u + a_6 \), where \( b_5, a_5 \) and \( a_6 \) are arbitrary constants. From equation \( \delta_2 = 0 \) we get \( c_4 = \frac{2}{3} u + c_5 \), where \( c_5 \) is an arbitrary constant. Equation \( \delta_3 = 0 \) implies \( a_5 = b_5 \). Then from equation \( \epsilon_1 = 0 \) we obtain \( a_5 = a_6 = b_5 = 0 \). It is easy to check that equations \( \epsilon_2 = 0 \) and \( \epsilon_3 = 0 \) are identically satisfied.

Thus equation (5.3) is of the form

\[
v_{xxx} = \frac{u_{xx} v_{xx}}{u_x} - \left( \frac{2}{3} u + \lambda \right) v_x + \left( \frac{2}{3} u + \lambda \right) \frac{u_{xx}}{u_x} - u_x \right) v.
\]  

(5.14)

Here \( \lambda = c_5 \) is an arbitrary parameter.

**Proposition 3.** *The pair of equations (5.2) and (5.14) defines the Lax pair for the KdV equation.*

5.2. Potential and modified KdV equations

Let us concentrate on the potential KdV equation

\[
u_s = u_{xxxx} + \frac{1}{2} u_x^3.
\]  

(5.15)

Its linearization

\[
v_r = u_{xxx} + \frac{1}{2} w^2 v_s, \text{ where } w = u_s,
\]  

(5.16)

does not admit any second order invariant manifold of the necessary form \( v_{xx} = a(u, u_s, u_{xx}, \ldots) v_x + b(u, u_s, u_{xx}, \ldots) v \). However it admits a third order invariant manifold given by

\[
v_{xxx} = \frac{w}{w_x} v_{xx} - (w^2 + \lambda) v_x + \lambda \frac{w_x}{w} v.
\]  

(5.17)

Here \( \lambda \) is an arbitrary parameter. The consistency condition of equations (5.16) and (5.17) is equivalent to the modified KdV equation
\[ w_t = w_{xxx} + \frac{1}{2} w^2 w_x \]  

(5.18)

connected with (5.15) by a very simple substitution \( w = u_x \). In other words (5.16) and (5.17) define the Lax pair for equation (5.18) as well.

Now let us return to the sine-Gordon equation (3.17). Recall that equation (5.15) is a symmetry of the sine-Gordon equation. It is easily seen that equation (5.17) coincides with (4.10) up to notation differences.

5.3. Lax pairs for the KdV type equations from the Svinolupov–Sokolov list

Consider the following two third order differential equations

\[
\begin{align*}
\psi_t &= \psi_{yyy} - \frac{\gamma}{2} \psi^3_y - \frac{3}{2} f^2(u) \psi_y, \\
\psi_x &= \psi_{xxx} - \frac{3 \psi_x \psi_x^2}{2 (1 + \psi_x^2)} - \frac{\gamma}{2} \psi_x^3,
\end{align*}
\]

(5.19)

(5.20)

possessing infinite hierarchies of conservation laws [21]. As established in [20] these equations are symmetries of equation (4.12). We have proved above in section 4 that equations

\[
\begin{align*}
\varphi_{yy} &= \frac{f'(u)}{f(u)} \varphi_y \varphi_y + \frac{\lambda \varphi_x}{f(u) \sqrt{1 + \varphi_x^2}} \varphi_x - \left( f^2(u) + \lambda \right) \varphi = 0, \\
\varphi_{xx} &= \left( \frac{f'(u)}{f(u)} + \frac{\varphi_x x}{u_x^2 + 1} \right) \varphi_x \varphi_x + \frac{u_x \varphi_x \sqrt{u_x^2 + 1}}{\lambda f(u)} \varphi_y - \left( \frac{u_x^2 + 1}{\lambda} \right) \varphi = 0
\end{align*}
\]

(5.21)

(5.22)

define an invariant manifold for the linearized equation (4.14). It is reasonable to expect that invariant manifolds for the linearizations

\[
\varphi_t = \varphi_{yyy} - \frac{3}{2} \left( \gamma \varphi_x^2 + f^2(u) \right) \varphi_y - 3 f(u) f'(u) \varphi_x \varphi_y
\]

(5.23)

and

\[
\varphi_x = \varphi_{xxx} - \frac{3 \varphi_x \varphi_x^2}{1 + \varphi_x^2} \varphi_x - \frac{3}{2} \left( \frac{1 - \varphi_x^2}{1 + \varphi_x^2} \right) \varphi_x^2 + \gamma \varphi_x^2 \varphi_x
\]

(5.24)

of the symmetries (5.19) and (5.20) are closely connected with the same manifold. Indeed by applying the operators \( D_y \) and \( D_x \) to equation (5.21) and respectively to (5.22) one can deduce the two third order ordinary differential equations

\[
\begin{align*}
\varphi_{yyy} &= \frac{\varphi_{yy}}{\varphi_y} \varphi_y - \left( \gamma \varphi_x^2 + f^2(u) + \lambda \right) \varphi_y \\
&\quad + \left( \frac{f^2(u) + \lambda}{\varphi_y} \right) \varphi_{yy} - 3 f(u) f'(u) \varphi_x \varphi_y \varphi_y = 0,
\end{align*}
\]

(5.25)
\[ v_{xxx} = \frac{(1 + 3u_x^2)u_{xx}}{1 + u_x^2}v_{xx} \]

\[-\left(\frac{\lambda^{-1} + \gamma u_x^2 + u_xu_{xxx}}{1 + u_x^2} - \frac{3u_x^2u_{xx}^2}{(1 + u_x^2)^2}\right)v_x + \frac{u_{xx}u_x}{u_x} \lambda^{-1}v = 0. \quad (5.26)\]

It is easily checked by a direct computation that equations (5.25) and (5.26) define invariant manifolds for (5.23) and (5.24) respectively.

**Proposition 4.**

(1) Linear equations (5.23) and (5.25) define the Lax pair for equation (5.19)

(2) Linear equations (5.24) and (5.26) define the Lax pair for equation (5.20).

### 5.4. Volterra type integrable chains

In this section we discuss the semi-discrete equations of the form

\[ u_{n+1} - u_n = \partial_t \left( \frac{\partial}{\partial p} \right)_n \]

with the sought function \( u = u_n(t) \), depending on discrete \( n \) and continuous \( t \). The direct method for constructing the Lax pairs through linearization can be applied to the discrete models as well. As illustrative examples we consider the modified Volterra chain

\[ \frac{dp_n}{dt} = -p_n^2(p_{n+1} - p_{n-1}) \quad (5.27) \]

and the equation

\[ \frac{du_n}{dt} = \frac{1}{u_{n+1} - u_{n-1}} \quad (5.28) \]

found in [29]. These two equations are related to each other by a very simple Miura type transformation

\[ \frac{dv_n}{dt} = -p_n^2(v_{n+1} - v_{n-1}) \quad (5.30) \]

of equation (5.28) depend on the variable \( p_n \). This explains why we study these two equations together. We look for the invariant manifold of the third order

\[ v_{n+2} = av_{n+1} + bv_n + cv_{n-1} \quad (5.31) \]

to equation (5.30) with the coefficients \( a, b, c \) depending on a finite set of dynamical variables \( p_n, p_{n+1}, \ldots \). Actually we suppose that (5.31) defines an invariant manifold for any choice of the solution \( p = p_n(t) \) to the equation (5.27).

The coefficients \( a, b, c \) are found from the equation

\[ \frac{d}{dt}(av_{n+1} + bv_n + cv_{n-1}) = D_n^2\left(-p_n^2(v_{n+1} - v_{n-1})\right). \quad (5.32)\]

Studying equation (5.32) we assume that the variables \( \{p_k\}_{k=-\infty}^{\infty}, \ v_n, v_{n+1}, v_{n-1} \) are independent dynamical variables. Omitting the simple but tediously long computations we
Therefore the invariant manifold sought is of the form

$$v_{n+2} = \left( -\frac{p_n}{p_{n+1}} + \frac{\lambda}{p_{n+1}^2} \right) v_{n+1}$$

$$+ \left( 1 - \frac{\lambda}{p_n p_{n+1}} \right) v_n + \frac{p_n}{p_{n+1}} v_{n-1}. \quad (5.33)$$

**Proposition 5.** The consistency condition of equations (5.30) and (5.33) coincides with equation (5.27).

### 6. Construction of the recursion operators and conservation laws via newly found Lax pairs

It was observed that the Lax pairs for the integrable equations found above essentially differ from their classical counterparts. In this section we discuss some useful properties of the newly found Lax pairs. We show, for instance, that they provide a very convenient tool for searching the recursion operators and conservation laws for integrable models. As illustrative examples we take the KdV and potential KdV equations, the Volterra type chain (5.28), etc. We show that the equation of the invariant manifold to the linearized equation is easily transformed into the recursion operator.

#### 6.1. Evaluation of the recursion operators for KdV type equations

Let us start with the KdV equation (1.9). We can rewrite equation (1.11) of the invariant manifold in the form

$$\left( D_x^3 - \frac{u_{xx}}{u_x} D_x^2 + \frac{2u}{3} D_x + u_x - \frac{2u u_{xx}}{3 u_x} \right) v = \lambda u_x D_x \frac{1}{u_x} v. \quad (6.1)$$

We now multiply (6.1) from the left by the operator $u_x D_x^{-1} \frac{1}{u_x}$ and obtain a formal eigenvalue problem of the form

$$R v = \lambda v \quad (6.2)$$

for the operator

$$R = u_x D_x^{-1} \frac{1}{u_x} \left( D_x^3 - \frac{u_{xx}}{u_x} D_x^2 + \frac{2u}{3} D_x + u_x - \frac{2u u_{xx}}{3 u_x} \right). \quad (6.3)$$

An amazing fact is that $R$ coincides with the recursion operator for the KdV equation. Indeed, we have
It can be simplified due to the relation
\[
\frac{uv_x}{u_x} - \frac{uu_{xx} v}{u_x^2} = \left( \frac{uv}{u_x} \right)_x - v
\]
and reduced to the form
\[
Rv = \left( D_x^2 + \frac{2}{3} u + \frac{1}{3} u_x D_x^{-1} \right)v.
\]
In a similar way we derive from (5.17) the recursion operator to the potential KdV equation (5.15). Indeed, we rewrite (5.17) as
\[
\left( D_x^3 - \frac{u_{xx}}{u_x} D_x^2 + u_x^2 D_x \right)v = -\lambda u_x D_x^{-1} v.
\]
Equation (6.4) implies
\[
Rv = -\lambda v,
\]
where
\[
R = u_x D_x^{-1} \left( D_x^3 - \frac{u_{xx}}{u_x} D_x^2 + u_x^2 D_x \right).
\]
We simplify the expression for \( Rv \) so that
\[
Rv = u_x D_x^{-1} \left( \frac{uv_{xx} u_x - uu_{xx} v}{u_x^2} + u_x v \right) = v_{xx} + u_x D_x^{-1} u_x D_x v.
\]
Clearly operator \( R = D_x^2 + u_x D_x^{-1} u_x D_x \) coincides with the recursion operator for (5.15).
Invariant manifolds (5.25) and (5.26) for the linearized equations (5.23) and (5.24) allow the construction of the recursion operators
\[
R = D_y^2 - \gamma u_y^2 - \frac{f^2(u)}{u_y} + u_x D_y^{-1} \left( \gamma u_{yy} - ff' \right)
\]
and
\[
R = D_x^2 - \frac{2u_x u_{xx}}{1 + u_x^2} + u_x D_x^{-1} \left( \frac{u_{xxx}}{1 + u_x^2} + \frac{u_x u_{xx}^2}{(1 + u_x^2)^2} - \gamma u_x \right) D_x
\]
for the Svinolupov–Sokolov equations (5.19) and (5.20), respectively.

6.2. Recursion operators via the Lax pair for a Volterra type chain

We proceed with an example of the discrete model (5.28). Let us write equation (5.33) defining the invariant manifold for (5.30) as
where $D_n$ is the shift operator acting due to the rule $D_n a(n) = a(n + 1)$. We multiply (6.5) from the left by the factor $p_n(D_n - 1)^{-1} p_{n+1}$ and then get

$$R V = \lambda V,$$

(6.6)

where $R = p_n(D_n - 1)^{-1} p_{n+1} \left( D_n^2 + \frac{p_n}{p_{n+1}} D_n - 1 - \frac{p_n}{p_{n+1}} D_n^{-1} \right)$. After some elementary transformations the operator $R$ is reduced to the form

$$R = p_n^2 \left( D_n + D_n^{-1} \right) + 2 p_n p_{n-1} + 2 p_n (D_n - 1)^{-1} (p_{n-1} - p_{n+1}).$$

(6.7)

Operator $R$ given by (6.7) defines the recursion operator for the chain (5.28). For instance, by applying the operator $R$ to

$$u_{n+1} \left( \begin{array}{l} \frac{p_n}{p_{n+1}} \end{array} \right),$$

(6.9)

we obtain the rhs of the equation constructed in [30]

$$u_{n+1} = \left( u_{n+1} - u_{n-1} \right) \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 1 \end{array} \right),$$

(6.10)

as a fifth-point symmetry of equation (5.28). Note that the whole hierarchy of symmetries for the chain (5.28) is described in [31].

### 6.3. Conservation laws via the Lax pair for a Volterra type chain

Let us consider an equation of the form

$$\frac{d u_n}{d t} = \frac{1}{u_{n+1} - u_{n-1}}.$$  

Due to the relations (5.29) between $u$ and $p$ a pair of equations (5.30) and (5.33) define a Lax pair to the chain (6.9) as well.

We rewrite the scalar Lax pair (5.30) and (5.33) in the matrix form

$$y_{n+1} = f y_n, \quad \frac{d y_n}{d t} = g y_n,$$

(6.10)

where

$$f = \left( \begin{array}{c} - \frac{p_n}{p_{n+1}} + \frac{\lambda}{p_{n+1}} 1 - \frac{\lambda}{p_{n+1}} \frac{p_n}{p_{n+1}} \\ 1 0 0 \\ 0 1 0 \end{array} \right),$$

(6.11)

and

$$g = \left( \begin{array}{c} \frac{p_n p_{n+1}}{p_n} - \lambda \frac{p_{n+1}}{p_n} - p_n p_{n+1} \\ - p_n^2 0 p_n^2 \\ p_n - 1 - \frac{p_{n-1}}{p_n} + \lambda \end{array} \right).$$

(6.12)

To construct the conservation laws, we apply the method of formal diagonalization suggested in [32, 33] and developed in [34].

The first equation in (6.10) has singular point $\lambda = \infty$ ($f$ has a pole at $\lambda = \infty$). It can be checked that the potential $f$ is represented by

$$f = \alpha \Omega \beta,$$
where

\[
\alpha = \begin{pmatrix}
\frac{1}{p_{n+1}} & -\frac{p_{n+1}}{p_n} & \lambda^{-1} & 0 & 0 \\
\lambda^{-1} & \frac{p_{n+1}}{p_n} & 0 & 0 & 1 \\
0 & 1 & 1 & & 
\end{pmatrix}
\]  \hspace{1cm} (6.13)

\[
\beta = \begin{pmatrix}
1 & -\frac{p_{n+1}}{p_n} & \frac{p_{n+1}}{p_n} & \lambda & \frac{p_{n+1}}{p_n} \\
0 & 1 & 0 & \lambda - \frac{p_{n+1}}{p_n} & \frac{p_{n+1}}{p_n} \\
0 & 0 & 0 & \lambda^{-1} & \frac{p_{n+1}}{p_n} \\
\end{pmatrix}
\]  \hspace{1cm} (6.14)

are analytic and non-degenerate around \( \lambda = \infty \), and \( Z \) is a diagonal matrix of the form

\[
Z = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \lambda^{-1} \\
\end{pmatrix}
\]  \hspace{1cm} (6.15)

The change of variables \( \psi = \beta y \) reduces the first system in (6.10) to the special form

\[
\psi_{n+1} = PZ\psi_n,
\]  \hspace{1cm} (6.16)

where

\[
P = D_{\beta}(\beta) \alpha
\]

\[
= \begin{pmatrix}
\frac{1}{p_{n+1}} & -\frac{p_{n+1}}{p_n} & \lambda^{-1} & 1 - \frac{p_{n+1}}{p_n} & \lambda^{-1} \\
\lambda^{-1} & \frac{p_{n+1}}{p_n} & 0 & 1 - \frac{p_{n+1}}{p_n} & 0 \\
0 & 1 & 1 & \lambda & \frac{p_{n+1}}{p_n} \\
& & & & \\
\end{pmatrix}
\]

The function \( P(\lambda) \) together with \( P^{-1}(\lambda) \) are analytic around \( \lambda = \infty \), so

\[
P(\lambda) = \sum_{i=0}^{\infty} p^{(i)}(\lambda^{-1}),
\]  \hspace{1cm} (6.17)

where

\[
p^{(0)} = \begin{pmatrix}
\frac{1}{p_{n+1}} & -\frac{p_{n+2}}{p_n} & 0 \\
0 & \frac{p_{n+1}}{p_n} & 0 \\
0 & 0 & \frac{p_{n+1}^2}{p_n^2} \\
\end{pmatrix},
\]

\[
p^{(1)} = \begin{pmatrix}
-\frac{p_{n+1}(p_n + p_{n+2})}{p_{n+1}^2} & p_{n+1} & 0 \\
\frac{p_{n+2}}{p_{n+1}} & 1 & -\frac{p_{n+2}}{p_{n+1}} \\
0 & \frac{p_{n+1}^3}{p_{n+2}^2} & \frac{p_{n+1}^3}{p_{n+2}^2} \\
\end{pmatrix},
\]

\[
p^{(2)} = \begin{pmatrix}
0 & \frac{p_{n+1}^2}{p_{n+2}^2} & \frac{p_{n+2}}{p_{n+1}} \\
-\frac{p_{n+1}^2}{p_{n+2}^2} & 0 & \frac{p_{n+1}^2}{p_{n+2}^2} \\
0 & \frac{p_{n+1}^2}{p_{n+2}^2} & \frac{p_{n+1}^2}{p_{n+2}^2} \\
\end{pmatrix},
\]

\[
p^{(3)} = \begin{pmatrix}
0 & \frac{p_{n+1}^3}{p_{n+2}^3} & \frac{p_{n+2}^3}{p_{n+1}^3} \\
-\frac{p_{n+1}^3}{p_{n+2}^3} & 0 & \frac{p_{n+1}^3}{p_{n+2}^3} \\
0 & \frac{p_{n+1}^3}{p_{n+2}^3} & \frac{p_{n+1}^3}{p_{n+2}^3} \\
\end{pmatrix},
\]

e etc.
The leading principal minors of $P(\lambda)$

$$\det_1 P(\lambda = \infty) = \frac{1}{p_{n+1}}, \quad \det_2 P(\lambda = \infty) = \frac{1}{p_{n}p_{n+1}},$$

$$\det_3 P(\lambda = \infty) = \det P(\lambda = \infty) = \frac{p_{n+1}}{p_{n}},$$

do not vanish if the variable $p_{n}$ satisfies the inequality $p_{n} \neq 0$ for all $n$. According to proposition 1 in [33] the first system in (6.10) can be diagonalized, i.e. there exist formal series

$$T = T^{(0)} + T^{(1)}\lambda^{-1} + T^{(2)}\lambda^{-2} + \cdots, \quad (6.18)$$

$$h = h^{(0)} + h^{(1)}\lambda^{-1} + h^{(2)}\lambda^{-2} + \cdots \quad (6.19)$$

such that the formal change of variables $\psi = T\varphi$ converts the system (6.16) to the system of diagonal form

$$\varphi_{n+1} = hZ\varphi_{n}. \quad (6.20)$$

Thus we see that the formal change of variables $y = R\varphi = \beta^{-1}T\varphi$, reduces the first system in investigated Lax pair (6.10) to the form (6.20). By construction $R = \beta^{-1}T$ is a formal series of the form

$$R = R^{(0)} + R^{(1)}\lambda^{-1} + R^{(2)}\lambda^{-2} + \cdots. \quad (6.21)$$

It follows from (6.19)–(6.21) that diagonalizable system (6.20) admits an asymptotic representation of the solution to the direct scattering problem

$$y_{n}(\lambda) = R(n, \lambda) e^{\sum_{s=n}^{n-1} \log h(s, \lambda)} \lambda^{n} \quad (6.22)$$

with ‘amplitude’ $A = R(n, \lambda)$ and ‘phase’ $\phi = n \log Z + \sum_{s=n}^{n-1} \log h(s, \lambda)$.

Let us turn back to the problem of diagonalization. By solving the equation

$$D_{\lambda}(T)h = P(\lambda)T, \quad \hat{T} = ZT^{-1} \quad (6.23)$$

we find the formal series $T$ and $h$.

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p_{n+1}p_{n} & 1 \\ 0 & p_{n-1}p_{n} & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{p_{n}p_{n+2}}{p_{n}} & 0 & 0 \\ \frac{p_{n}^{2}p_{n+2}}{p_{n}} & 0 & 0 & 0 \\ 0 & p_{n}p_{n+1}(p_{n} - p_{n-2}) & 0 & 0 \\ p_{n}^{2}(2p_{n-1} + p_{n+1}) & 0 & p_{n}p_{n+1} & \frac{p_{n}^{2}p_{n+2}(p_{n+1} + p_{n})}{p_{n}} \end{pmatrix} \lambda^{-1}$$

$$\quad + \begin{pmatrix} 0 & \frac{p_{n}^{3}(p_{n+1} + p_{n})}{p_{n}} & 0 & 0 \\ \frac{p_{n}^{4}p_{n+2}}{p_{n}} & \frac{p_{n}^{2}(p_{n+1} + p_{n})}{p_{n}} & 0 & 0 \\ p_{n}^{2}(2p_{n-1} + p_{n+1}) & 0 & p_{n}p_{n+1} & \frac{p_{n}^{2}p_{n+2}(p_{n+1} + p_{n})}{p_{n}} \end{pmatrix} \lambda^{-2} + \cdots.$$
\[
\begin{align*}
&h = \begin{pmatrix}
\frac{1}{p_{n+1}} & 0 & 0 \\
0 & \frac{p_{n+1}}{p_n} & 0 \\
0 & 0 & p_{n+1}^2
\end{pmatrix} + \begin{pmatrix}
-\frac{p_{n+1}(p_n + p_{n+2})}{p_{n+1}} & 0 & 0 \\
0 & \frac{p_n^2(p_{n+1}^2 - p_n)}{p_n} & 0 \\
0 & 0 & p_{n+1}^3(p_n + p_{n+2})
\end{pmatrix} \lambda^{-1} \\
&+ \begin{pmatrix}
-P_p p_{n+2} & 0 & 0 \\
0 & h_{22}^{(2)} & 0 \\
0 & 0 & p_n^3(p_{n+1} + p_{n+2})
\end{pmatrix} \lambda^{-2} + \cdots,
\end{align*}
\]

where
\[
T_{32}^{(2)} = p_n p_{n-1} \left( p_{n-2}^2 p_{n-1} + p_{n-2}^2 p_{n-3} p_{n-1} - p_{n-2}^2 p_{n-1} p_{n-3} + p_n^2 p_{n+1} - 2p_n^2 p_{n-1} - p_n^2 p_{n+2} - p_{n-1}^2 p_{n-2} + p_n^2 p_{n+1} p_{n-1} - p_n^2 p_{n+1} p_{n+2} \right),
\]
\[
h_{22}^{(2)} = -\frac{p_{n+1}^3 - p_{n+2}^2 p_{n+1} - p_{n+1}^2 p_{n+2} - p_{n+1}^3 p_{n+1} + p_n^2 p_{n+1} + p_n^3 p_{n+1} + p_n^3 p_{n+1} p_{n+2}}{p_n}.
\]

According to the general scheme the second system of the Lax pair (6.10) is diagonalized by the same linear change of variables \( y = R \varphi \), where
\[
R = \beta^{-1} T = \begin{pmatrix}
1 & -\frac{p_{n+1}}{p_n} & 0 \\
0 & 1 & 0 \\
0 & -p_{n-1} p_n & p_n^2
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
0 & -\frac{p_{n+1}^2 p_{n+2}}{p_n} & p_n p_{n+1} \\
0 & 0 & -p_n^2 \\
p_{n-1} p_n^3 & p_n^2 p_{n-1} p_n & p_n^2 \left( p_{n+1} + p_{n-1} \right)
\end{pmatrix} \lambda^{-1}
\]

\[
+ \begin{pmatrix}
R_{11}^{(2)} & R_{12}^{(2)} & R_{13}^{(2)} \\
R_{21}^{(2)} & R_{22}^{(2)} & R_{23}^{(2)} \\
R_{31}^{(2)} & R_{32}^{(2)} & R_{33}^{(2)}
\end{pmatrix} \lambda^2 + \cdots,
\]

\[
R_{11}^{(2)} = p_n p_{n+1}^2 p_{n+2},
\]
\[
R_{12}^{(2)} = -p_n^2 \left( 2p_{n-1} + p_{n+1} \right),
\]
\[
R_{13}^{(2)} = p_n^2 \left( p_n + p_{n+2} \right),
\]
\[
R_{21}^{(2)} = -p_n^3, \quad R_{22}^{(2)} = p_n^2 p_{n+1}^2 + 3p_n^2 p_{n+1}^2 + p_n^2 p_{n+1}^2,
\]
\[
R_{23}^{(2)} = -3p_n^3 p_{n+1},
\]

\[
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\]
This change of variables reduces the second system in (6.10) to the form
\[ \frac{dy}{dt} = S y \] with
\[ S = -R^{-1}R_t + R^{-1}gR = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \lambda 
+ \begin{pmatrix} 2p_n p_{n+1} & 0 & 0 \\ 0 & p_n (p_{n-1} - p_{n+1}) & 0 \\ 0 & 0 & -2p_n p_{n+1} \end{pmatrix} + \begin{pmatrix} S_{11}^{(2)} & 0 & 0 \\ 0 & S_{22}^{(2)} & 0 \\ 0 & 0 & S_{33}^{(2)} \end{pmatrix} \lambda^{-1} + \cdots, \]
where
\[ S_{11}^{(2)} = p_n p_{n+1} (p_{n+1} p_{n+2} + p_{n-1} p_n), \]
\[ S_{22}^{(2)} = p_n p_{n+1} (p_{n-1} p_n - p_{n+1} p_{n+2}), \]
\[ S_{33}^{(2)} = -p_n p_{n+1} (p_{n+1} p_{n+2} + p_{n-1} p_n). \]

According to the paper [33] the equation
\[ D_t \ln b = (D_n - 1)S \]
generates an infinite series of conservation laws for equation (5.27). We write down in an explicit form three conservation laws from the infinite sequence obtained by the diagonalization procedure
\[ D_t \left( \ln \frac{1}{p_{n+1}} \right) = (D_n - 1)p_n p_{n+1}, \]
\[ D_t \left( -p_{n+1} (p_n + p_{n+2}) \right) = (D_n - 1)p_n p_{n+1} (p_{n-1} p_n + p_{n+1} p_{n+2}), \]
\[ D_t \left( -2p_n^2 p_{n+1} p_{n+2} - \frac{1}{2} p_n^2 p_{n+1}^2 - \frac{1}{2} p_n^2 p_{n+2}^2 \right) = (D_n - 1)p_n^2 p_{n+2} (2p_{n-1} p_{n+2} + p_{n+1} p_{n+2} + p_{n-1} p_n). \]

Being rewritten in terms of \( u_n \) these relations give the conservation laws for equation (6.9)
\[ D_t (u_{n+2} - u_n) = (D_n - 1) \frac{1}{(u_{n+1} - u_{n-1})(u_{n+2} - u_n)}, \]
\[ D_t \left( \frac{u_n - u_{n+1}}{(u_{n+1} - u_{n-1})(u_{n+2} - u_n)} \right) = (D_n - 1) \left( \frac{1}{(u_n - u_{n-2})(u_{n+1} - u_{n-1})^2(u_{n+2} - u_n)} \right). \]
These conservation laws coincide with those found earlier (see, for instance, [18, 29]).

**Conclusions**

There is a large set of classification methods allowing classes of integrable nonlinear PDEs and their discrete analogues to be described. For studying the analytical properties of these equations one needs the Lax pairs. Therefore the problem of creating convenient algorithms for constructing the Lax pairs is relevant. In the present article such a method is suggested. For the evolution type integrable equation the Lax pair consists of the linearized equation and the equation of its invariant manifold. In the case of the hyperbolic equations to obtain the Lax pair we use the Laplace cascade in addition to the invariant manifold. The method is applied to equations (4.12), (5.19) and (5.20) known to be integrable for which the Lax pairs have not been constructed before.

An interesting observation is connected with the Laplace cascade of the sine-Gordon equation. It is proved that in this case the cascade admits a finite-dimensional reduction which generates the Lax pair to the sine-Gordon model. We conjecture that the Laplace cascade corresponding to any hyperbolic type integrable equation admits a finite-dimensional reduction.

We considered examples showing that our method leads to true Lax pairs having useful applications. For the Lax pair of the Volterra type chain we found an asymptotic eigenfunction which allowed an infinite set of conservation laws to be constructed. It is also shown that these Lax pairs allow the recursion operators to be constructed describing infinite hierarchies of the higher symmetries and invariant manifolds for the given nonlinear equation.

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Appendix. How to look for the invariant manifolds for the linearization of a nonlinear hyperbolic type equation

It is well known that the linear hyperbolic type equation \( \nu_{xy} = p(x, y)\nu_x + q(x, y)\nu_y + r(x, y) \) might admit linear invariant manifolds \((35)\). Here we consider equation (3.2) obtained by linearizing an essentially nonlinear equation (3.1). We request that at least one of the coefficients \(a, b, c\) in (3.2) depends on at least one of the dynamical variables \(u, u_1, \tilde{u}_1\). We look for a linear invariant manifold of the form
\[
\left( p(j)D_j^j + p(j - 1)D_j^{j-1} + \ldots + p(1)D_x + p(0) + q(1)D_x + \ldots + q(k)D_x^k \right)\nu = 0.
\]

(7.1)

It is assumed that all the coefficients \(p(i)\) and \(q(m)\) can depend on \(x, y\) and on a finite set of the dynamical variables \(u, u_1, \tilde{u}_1, u_2, \tilde{u}_2, \ldots\). More precisely we study a family of equations (3.2) and (7.1) depending on a functional parameter \(u(x, y)\) such that when \(u = u(x, y)\) ranges across a set of all solutions to equation (3.1) then (7.1) ranges across a set of invariant manifolds for the corresponding equation (3.2). In this case equation (7.1) generates an overdetermined system of differential equations for defining the sought coefficients \(p(i)\) and \(q(m)\).

As an illustrative example we consider equation (3.18) obtained by linearizing the sine-Gordon equation. We search an invariant manifold for (3.18) in the form
\[
\nu_{xy} + a\nu_y + b\nu_x + c\nu = 0.
\]

(7.2)

Here \(a, b\) and \(c\) are functions depending on a finite number of dynamical variables \(u, u_1, \tilde{u}_1, u_2, \tilde{u}_2, \ldots\). We apply the operator \(D_x\) to (7.2) and rewrite the obtained result as
\[
v_{xx} = \frac{1}{b}\left( \sin(u)\nu_y - \cos(u)(\nu_y + av) - D_x(a)\nu_y - D_x(b)\nu_x - D_x(c)\nu - cv_x \right).
\]

(7.3)

We now apply the operator \(D_y\) to this equation and simplify the result by means of equation (3.18). We arrive at the equation
\[
v_{yy} + \tilde{a}\nu_y + \tilde{b}\nu_x + \tilde{c}v = 0,
\]

which should coincide according to the definition above with equation (7.2), i.e.
\[
\tilde{a} = a, \quad \tilde{b} = b, \quad \tilde{c} = c.
\]

These conditions give rise to the equalities
\[
D_xD_x(a)b - D_x(b)\cos(u) - 2\sin(u)a_{xy}b + D_x(c)b - D_x(a)(ab + D_x(b)) = 0,
\]

(7.4)

\[
D_yD_y(a)b - D_y(b)D_x(a)b + D_y(c)b - D_y(a)eb - b^2D_x(a) = 0,
\]

(7.5)

\[
D_xD_y(c)b - D_y(b)D_x(c) - c\theta D_x(a) + \cos(u)\left( -u_{yy}b + D_x(a)b + D_x(b)b - D_x(b)a \right) + \sin(u)\left( -u_{xy}b - b^2u_e - au_xb + D_x(b)u_x \right) = 0.
\]

(7.6)

Assume that \(a = a(u, u_1, u_2), \quad b = b(u, u_1, u_2)\) and \(c = c(u, u_1, u_2)\) and substitute these functions into (7.4)–(7.6). Eliminating the mixed derivatives of \(u\) due to equation (3.17) we obtain three equations of the form
\[
\alpha_i(u, u_1, u_2)u_{xx}u_{xy} + \beta_i(u, u_1, u_2)u_{xx} + \gamma_i(u, u_1, u_2)u_{xy} + \delta_i(u, u_1, u_2) = 0,
\]

\(i = 1, 2, 3\). Since the functions \(\alpha_i, \beta_i, \gamma_i, \delta_i\) depend only on \(u, u_1,\) and \(u_2\), we should have the coefficients at \(u_{xx}, u_{xy}, u_{x}, u_{y}\) and remaining terms equal to zero, i.e.
\[ \alpha_i(u, u_x, u_y) = 0, \quad \beta_i(u, u_x, u_y) = 0, \quad \gamma_i(u, u_x, u_y) = 0, \quad \delta_i(u, u_x, u_y) = 0 \quad (7.7) \]

for all \( u, u_x \) and \( u_y \), \( i = 1, 2, 3 \). Here

\[ \alpha_1 = ba_{u,u} - b_{u}a_{u}, \quad (7.8) \]
\[ \alpha_2 = bb_{u,u} - b_{u}b_{u}, \quad (7.9) \]
\[ \alpha_3 = bc_{u,u} - c_{u}b_{u}, \quad (7.10) \]
\[ \beta_1 = bc_{u,u} - ab_{u,u} - b_{u}u_xa_{u} - b_{u} \sin(u)a_{u}, \]
\[ + ba_{u,u} \sin(u) + ba_{u}u_y, \quad (7.11) \]
\[ \beta_2 = bb_{u,u}u_y + bb_{u}u_y \sin(u) - b_{u}u_xb_{u} - b^2a_{u} - b_{u}^2 \sin(u), \quad (7.12) \]
\[ \beta_3 = bc_{u,u,u} \sin(u) + bc_{u}u_yu_x + bb_{u,u} \cos(u) - cba_{u}, \]
\[ - b_{u} \sin(u)c_{u} - b_{u}u_xc_{u}, \quad (7.13) \]
\[ \gamma_1 = - \cos(u)h_{u} + b \sin(u)d_{u,u} + bu_xa_{u} + b_{u}a_xu_x - b_{u}a_u \sin(u), \]
\[ \gamma_2 = bu_{x}b_{u,u} - b_{u}b_{u}u_x + b \sin(u)b_{u}u_x - b_{u}^2 \sin(u) + bc_{u}u_x - cba_{u}, \]
\[ \gamma_3 = b_{u}c_{u}u_x - b \sin(u) - b_{u}c_{u} \sin(u) + ba_{u} \cos(u) + \sin(u)u_xb_{u}, \]
\[ + bu_{x}c_{u}u_x - a \cos(u)b_{u}u_x + b \sin(u)c_{u}u_x, \]
\[ \delta_1 = \left( b_{u}a_{u} - ba_{u,u} \right) \cos^2(u) - b_{u} \cos(u) \sin(u) \]
\[ + \left( ba_{u}u_y + ba_{u}u_x - b_{u}u_x \right) \cos(u) \]
\[ + \left( ba_{u} - 2a_{u}b + ba_{u,u}u_x - b_{u}u_xa_{u} \right) \sin(u) \]
\[ + \left( -ab_{u}a_{u,u} + ba_{u}u_x - ba_{u}u_x + bc_{u} \right) \sin(u) \]
\[ + bu_{x}a_{u,u}u_x - b_{u}u_xa_{u,u} - b_{u}a_{u,u} - b_{u}a_u + bc_{u}u_x, \]
\[ \delta_2 = \left( b_{u}b_{u,u} - bb_{u,u} \right) \cos^2(u) + b \left( b_{u}u_x + bu_{x}u_x \right) \cos(u) \]
\[ + \left( -b_{u}u_xb_{u,u} - b_{u}u_xb_{u} + bu_{x}b_{u,u} \right) \sin(u) \]
\[ + \left( bb_{u,u}u_y - b^2a_{u} - cb_{u} + bb_{u} + bc_{u} \right) \sin(u) \]
\[ + bb_{u,u}u_x + bc_{u}u_x - b_{u}^2u_xu_x - bb_{u}u_y - b^2a_{u}u_x + bu_{x}b_{u}u_x - b_{u}b_{u,u}, \]
\[ \delta_3 = \left( b_{u}c_{u,u} - bc_{u,u,u} - u_{y}b_{u} \right) \cos^2(u) + \left( ba_{u} - ab_{u} + bb_{u} \right) \cos(u) \sin(u) \]
\[ + \left( bb_{u}u_x - ab_{u}u_x + ba_{u}u_x + bc_{u}u_x + bc_{u}u_y - u_{y}^2b \right) \cos(u) \]
\[ + \left( bu_{x}c_{u,u} - au_{y}b + bc_{u,u}u_x - b_{u}c_{u,u}u_x \right) \sin(u) \]
\[ + \left( -cb_{u}u_x - bu_{x}c_{u,u} - b_{u}^2u_x + bc_{u} + u_{y}^2b \right) \sin(u) \]
\[ + bu_{x}c_{u,u}u_x - b_{u}c_{u,u}u_x - cba_{u}u_x + u_{y}b_{u} - b_{u}c_{u}u_x + bc_{u,u,u}. \]

Hence the problem of searching equation (7.2) is reduced to the system of equations (7.7).

We now look for functions \( a, b \) and \( c \) depending linearly on the variable \( u_y \)

\[ a = a_1(u, u_x)u_y + a_2(u, u_x), \quad (7.14a) \]
Then equation $\alpha_2 = 0$ (see (7.9)) takes the form $b_2(b_1)_{u_x} - b_1(b_2)_{u_x} = 0$. This equation is satisfied only if at least one of the following conditions holds,

$$b_1 = 0, \quad b_2 = 0, \quad \frac{(b_1)_{u_x}}{b_1} = \frac{(b_2)_{u_x}}{b_2}.$$ 

Let us consider the case $b_2 = 0$. Then function $b$ defined by formula (7.14b) takes the form

$$b = b_1(u, u_x)u_y.$$

Furthermore equations $\alpha_1 = 0$ and $\alpha_3 = 0$ become

$$u_y a_{u_x} - a_{u_x} = 0, \quad u_x c_{u_x} - c_{u_x} = 0.$$

These equations hold if at least one of the following four conditions is satisfied,

1. $a_{u_x} = 0, \quad c_{u_x} = 0$;
2. $a_{u_x} = 0, \quad c_{u_x} \neq 0, \quad \frac{c_{u_x}}{a_{u_x}} = \frac{1}{u_y}$;
3. $c_{u_x} = 0, \quad a_{u_x} \neq 0, \quad \frac{a_{u_x}}{c_{u_x}} = \frac{1}{u_y}$;
4. $a_{u_x}, c_{u_x} \neq 0, \quad \frac{a_{u_x}}{a_{u_x}} = \frac{c_{u_x}}{c_{u_x}}$.

It can be proved that cases (2)–(4) lead to contradiction, so we concentrate on case (1) which implies

$$a = a_1(u)u_x + a_2(u), \quad c = c_1(u)u_y + c_2(u).$$

By substituting functions (7.15) and (7.16) into equalities $\beta_i = 0$, $i = 1, 2, 3$ (see (7.11), (7.12) and (7.13)) one can verify that equation $\beta_1 = 0$ is satisfied. Equations $\beta_2 = 0$ and $\beta_3 = 0$ lead to

$$b_1(b_1)_{u_x} - (b_1)_{u_x} = 0 \quad \text{and} \quad (\cos u)u_x^2 b_1(b_1)_{u_x} = 0$$

correspondingly. Thus $b_1(u, u_x) = F_1(u)$ for some function $F_1(u)$. The equality $\gamma_1 = 0$ becomes

$$F_1(u)(\cos u + u_x a_2^2(u) + a_1(u)\sin u) = 0.$$

Clearly, the functions $a_1$ and $a_2$ are defined by the formulas

$$a_1(u) = -\cot u, \quad a_2(u) = C_1,$$

where $C_1$ is an arbitrary constant. According to that we rewrite the equality $\gamma_3 = 0$ as

$$F_1(u)(u_x c_2^2(u) + c_1(u)\sin u + \cos u C_2) = 0.$$

Since $u$ and $u_x$ are regarded as independent variables the last equation leads to

$$c_1(u) = -C_1 \cot u, \quad c_2(u) = C_2.$$
where $C_2$ is an arbitrary constant. Let us turn to the equation $\delta_1 = 0$ which gives
\[
u u^3 u_x \left( F(u) \cos u + F'(u) \sin u \right) = 0.
\]
Integration of the equation leads to
\[F_1 = \frac{C_3}{\sin u}.
\]
Now the equation $\delta_2 = 0$ takes the form
\[C_3 u_x^2 \left( C_1 u_y \sin u + (C_3 + C_2) \cos u \right) = 0.
\]
We get $C_2 + C_3 = 0$, $C_1 = 0$. It is easy to check that equalities $\delta_3 = 0$ and $\gamma_2 = 0$ are identically satisfied. Thus we have
\[a = -u_y \cot u, \quad b = \frac{\lambda u_y}{\sin u}, \quad c = -\lambda.
\]
Therefore the invariant manifold (7.2) is of the form
\[v_{yy} - u_x \cot u v_y + \frac{\lambda u_y}{\sin u} v_x - \lambda v = 0
\]
and coincides with (4.6), while (7.3) gives (4.7).

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