Hecke Eigensystems for \((\text{mod } p)\) Modular Forms of PEL-type and Algebraic Modular Forms

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1 Introduction

In a letter to J. Tate written in 1987 ([Ser96]), J-P. Serre gives an adelic interpretation of the systems of Hecke eigenvalues occurring in the space of $\pmod{p}$ elliptic modular forms: Serre proves that these eigensystems coincide with the Hecke eigensystems obtained from locally constant endomorphisms of a fixed superspecial elliptic curve over $\mathbb{Q}$ associated to an inner form $I$. The data define the connected reductive $\mathbb{Q}$-group $G$ such that the systems of (mod $p$) algebraic modular forms, as it was done by B. Gross in [Gro99], naturally appears when evaluating modular forms at supersingular elliptic curves over $\mathbb{F}_p$.

This result fits into the more general settings of a (mod $p$) Langlands philosophy, and it was originally closely related to the study of Serre conjecture; it can also be regarded as the starting point for defining the notion of algebraic modular forms, as it was done by B. Gross in [Gro99].

In [Ghi04a], A. Ghita generalizes Serre’s result to (mod $p$) Siegel modular forms of genus $g > 1$; in this context the algebraic group $\mathfrak{B}^\times$ is replaced by $GU_g(\mathfrak{B})$, an inner form of $GSp_{2g}$ over $\mathbb{Q}$. It is interesting to notice that to obtain the Hecke correspondence for $g > 1$, it is not enough to restrict modular forms to the supersingular locus of the Siegel moduli variety over $\mathbb{F}_p$, since this locus has positive dimension: one needs to restrict modular forms to the superspecial locus of the Siegel variety.

It was suggested to us by B. Gross that one could expect results similar to the ones of Serre and Ghita in more general settings, the basic idea being that the systems of (mod $p$) Hecke eigenvalues for modular forms associated to some reductive group $G$ over $\mathbb{Q}$ should coincide with the systems arising from (mod $p$) algebraic modular forms associated to an inner form $I$ of $G$ over $\mathbb{Q}$ satisfying some properties: for example $I$ should be compact modulo center at infinity.

In this paper we investigate the existence of the above correspondence in the context of modular forms arising from Shimura varieties of PEL-type; while the procedure we follow generalizes to any PEL-situation of type A or C, provided the supersingular locus of the corresponding Shimura variety is non-empty, we present full details only for the case of modular forms arising from Picard modular varieties (type A).

To state the main result we fix some notation: let $p$ be an odd prime and $k = \mathbb{Q}(\sqrt{\alpha})$ be a quadratic imaginary field in which $p$ is inert; let $g$ be a positive even integer and let $r, s$ be non-negative integers such that $r + s = g$. These data define the connected reductive $\mathbb{Q}$-group $G = GU_g(k; r, s)$ of type A, which is the unitary group associated to the extension $k/\mathbb{Q}$ and of signature $(r, s)$: fixing an embedding of $k$ inside the quaternion $\mathbb{Q}$-algebra $\mathfrak{B}$ of endomorphisms of a fixed superspecial elliptic curve over $\mathbb{F}_p$, we can define an inner form $I$ of $G$ over $\mathbb{Q}$ so that:

$$I(\mathbb{Q}) = \{ X \in GU_g(\mathfrak{B}) : X \cdot \Phi = \Phi \cdot X \},$$

where $\Phi$ is the matrix $\left( \begin{array}{cc} -\sqrt{\alpha} \cdot I_r & 0_{r,s} \\ 0_{s,r} & \sqrt{\alpha} \cdot I_s \end{array} \right)$. Let $N \geq 3$ be an integer not divisible by $p$ and set:

$$U(N) : = \ker \left( GU_g \left( \mathcal{O}_k \otimes_\mathbb{Z} \mathbb{Z}_p^g ; r, s \right) \to GU_g \left( \mathcal{O}_k \otimes_\mathbb{Z} \hat{\mathbb{Z}}_p^g ; r, s \right) \right),$$

$$U_p : = \ker \left( I(\mathbb{F}_p) \to G(U_r \times U_s)(\mathbb{F}_p^*) \right),$$

where $\pi$ denotes the reduction map modulo a uniformizer of $\mathfrak{B}$. The main result of the paper, contained in Theorem 5.18, is:

**Theorem 1.1** The systems of Hecke eigenvalues arising from $(r, s)$-unitary PEL-modular forms (mod $p$) for the quadratic imaginary field $k$, having genus $g$, fixed level $N$ and any possible weight $\rho : GL_g \to GL_{m(\rho)}$, are the same as the systems of Hecke eigenvalues arising from (mod $p$) algebraic modular forms for the group $I$ having level $U_p \times U(N) \subset I(k,f)$ and any possible weight $\rho' : G(U_r \times U_s) \to GL_{m(\rho')}$. 

An important tool to prove the above result will be for us the use of a uniformization result for some isogeny classes on PEL-moduli varieties, due to M. Rapoport and Th. Zink (Theorem 13). We fix a $p$-divisible group with
additional structure \((X, \overline{\lambda}_X, i_X)\), where \(X = G_{1/2}^p\) is superspecial, \(\overline{\lambda}_X\) is the \(\mathbb{Z}_p\)-class of a fixed principal polarization of \(X\), and \(i_X\) is an action of the order \(O_k\) on \(X\). We can adapt the uniformization result of Rapoport and Zink in order to uniformize only the finitely many points in the supersingular locus of the Picard moduli variety having \(p\)-divisible group isomorphic to \((X, \overline{\lambda}_X, i_X)\) (Corollary 4.13). This finite set will be our superspecial locus.

Since the cotangent space of the fixed \(p\)-divisible group \((X, \overline{\lambda}_X, i_X)\) has automorphism group naturally isomorphic to \(G(U_r \times U_s)(\mathbb{F}_{p^2})\), we are able to put differentials into the picture, and realize all the \((\text{mod } p)\) Hecke eigenvalues of algebraic modular forms for the group \(I\) in the space of superspecial modular forms, and vice versa (Proposition 5.16). Finally, we use an argument that is a modification of the one appearing in [Ghi04a] Th. 28, to conclude that the set of superspecial Hecke eigenvalues coincide with the set of all the \((\text{mod } p)\) Hecke eigenvalues.

In the more general settings of PEL-varieties over \(\overline{\mathbb{F}}_p\) of type A and C, whose superspecial locus is non-empty, the group \(I\) should be taken to be the \(\mathbb{Q}\)-group of automorphisms of a fixed triple \((A_0, i_0, \overline{\lambda}_0)\) defining a point in the superspecial locus of the PEL-variety; the uniformization result of Rapoport and Zink can be applied in this general context, since the \(p\)-divisible group of a superspecial abelian variety is basic (cf. 2.1.3).

As an application of the main theorem of the paper, we obtain (cf. Theorem 5.19) that the number \(N\) of \((\text{mod } p)\) Hecke eigensystems arising from unitary modular forms of signature \((r, s)\), fixed level \(N\) as above and any weight is finite and satisfies the following asymptotic with respect to \(p\):

\[ N \in O(p^{g^2 + g + 1 - rs}). \]

The paper is organized as follows: Section 2 recalls the basic definitions and results that one needs in order to state a representability theorem for some moduli functor of \(p\)-divisible groups, as considered by Rapoport and Zink in [MR96b]; in Section 3 we recall the basic facts on PEL-moduli varieties and PEL-modular forms; in Section 4 we present the construction of the uniformization morphism of Rapoport and Zink for basic isogeny classes in PEL-Shimura varieties, and the modification we need to parametrize our superspecial locus. The final Section 5 presents the settings and the computations that allow us to compare \((\text{mod } p)\) Hecke eigensystems for PEL-modular forms of unitary type, and \((\text{mod } p)\) Hecke eigensystems for algebraic forms for the group \(I\); the main theorem is here stated and proven, and the number of \((\text{mod } p)\) Hecke eigenvalues is estimated.

The reader might prefer to start reading the paper from section 4.2.2 and might refer to paragraph 2.2 for the definition of the moduli problem of \(p\)-divisible groups that appears in the formulation of Theorem 4.3, and to paragraph 4.1 for the description of the Rapoport-Zink uniformization morphism.

I would like to address my gratitude to C. Khare for introducing me to the topics studied in this paper, and to B.H. Gross for suggesting to me the problem treated here. I would also like to thank Khare for his valuable supervision, and for sharing with me a note of N. Fakhruddin solving a problem with the Hecke equivariance of a map considered in [Ghi04a].

I am grateful to N. Fakhruddin for signaling a problem contained in a previous version of the paper relatively to the Kodaira-Spencer isomorphism, and for other valuable remarks. I thank H. Hida and M-H. Nicole for explaining to me that, in suitable circumstances, the natural map from the Picard moduli scheme to the Siegel moduli scheme is a closed immersion. I would like to further thank J. Tilouine for an interesting conversation we had on the subject of this paper.
2 Moduli of $p$-divisible groups of PEL-type

2.1 $p$-divisible groups

We recall basic facts about $p$-divisible group; the main references are [Dem72], [Fon77], [Mes72] and [MR96b], 2.1-2.8.

2.1.1 Basic definitions

Fix a scheme $S$. By an $S$-group we mean a f.p.p.f. sheaf of commutative groups on the site $(\text{Sch}/S)_{f.p.p.f.}$, whose underlying category is the category of $S$-schemes, endowed with the Grothendieck f.p.p.f. topology. An $S$-group that is representable is called an $S$-group-scheme. For any positive integer $n$ and any $S$-group $G$ we denote by $G[n]$ the kernel of the $S$-morphism $n \cdot 1_G : G \to G$ given my multiplication by $n$; $G[n]$ is an $S$-group.

Fix a positive prime integer $p$. An $S$-group $G$ is a Barsotti-Tate group (or a $p$-divisible group) if the following three conditions are satisfied: (1) $G = \lim\limits_{\longrightarrow} G[p^n]$; (2) the morphism $[p] : G \to G$ is an epimorphism; (3) $G[p]$ is a finite locally free group-scheme over $S$.

By the theory of finite group-schemes over a field, the rank of the fiber of $G[p]$ at a point $s \in S$ is of the form $p^h(s)$, where $h : S \to \mathbb{Z}$ is a locally constant function on $S$; the rank of the fiber of $G[p^n]$ at $s$ is $p^{nh(s)}$ for any $n \geq 1$. If $h$ is a constant function (e.g., when $S = \text{Spec}(k)$ for a field $k$), its only value is called the height $ht(G)$ of $G$.

A morphism $f : G \to H$ of $p$-divisible groups over $S$ is said to be an isogeny if it is an epimorphism of f.p.p.f. sheaves whose kernel is representable by a finite locally free $S$-group-scheme. If $S$ is a scheme over $\text{Spec}(\mathbb{Z}_p)$ in which $p$ is locally nilpotent, then the kernel of an isogeny $f : G \to H$ is finite of rank $p^h$ where $h' : S \to \mathbb{Z}$ is locally constant; if $h'$ is constant, its only value is the height of $f$.

The $\mathbb{Z}$-module $\text{Hom}_S(G, H)$ of homomorphisms from $G$ to $H$ is a torsion-free $\mathbb{Z}_p$-module. A quasi-isogeny $f$ from $G$ to $H$ is global section of the sheaf $\text{Hom}_S(G, H) \otimes \mathbb{Z}_p$ such that any point $s$ of $S$ has a Zariski open neighborhood on which $p^n f : X \to Y$ is an isogeny for some positive integer $n = n(s)$. We denote by $\text{Qis}_S(G, H)$ the group of quasi-isogenies from $G$ to $H$.

We have the following rigidity property (cf. [MR96b], 2.8):

**Proposition 2.1** Let $G$ and $H$ be $p$-divisible groups over a scheme $S$ in which $p$ is locally nilpotent; let $S' \subset S$ be a closed subscheme whose defining sheaf of ideals is locally nilpotent. Then the canonical homomorphism $\text{Qis}_S(G, H) \to \text{Qis}_{S'}(G_{S'}, H_{S'})$ is bijective.

Recall that for any finite flat group scheme $X$ over $S$, the Cartier dual $D(X)$ (or $X^D$) of $X$ is the finite locally free $S$-group-scheme defined by $D(X)(T) := \text{Hom}_T(G_T, G_{m_T} \times S_T)\ (T\text{ any }S\text{-scheme})$. The assignment $D$ induces an additive anti-duality on the category of finite and locally free $S$-group schemes. Let $G$ be a $p$-divisible group over $S$. The **Serre dual** of $G$, denoted by $\hat{G}$ (or $G^D$, or $D(G)$), is the $p$-divisible group defined as $\hat{G} := \lim\limits_{\longrightarrow} D(G[p^n])$, where the map $D(G[p^n]) \to D(G[p^{n+1}])$ in the direct system is given by $D([p])$ for each $n$.

If $G$ is a $p$-divisible group over $S$, then $\hat{G}[p^n] = D(G[p^n])$ for any $n$. The assignment $G \to \hat{G}$ extends to morphisms in an obvious way and gives rise to an anti-duality in the category of $p$-divisible groups over $S$ (notice this category is not abelian) which is compatible with base changes. There is a canonical isomorphism of $p$-divisible groups $G \cong \hat{\hat{G}}$.

An S-**polarization** of a $p$-divisible group $G$ over $S$ is an anti-symmetric $S$-quasi-isogeny $\lambda : G \to \hat{G}$. A $Q_p$-**homogeneous S-polarization** of $G$ is the set $\lambda = Q_p^\times \lambda$ of $p$-adic non-zero multiples of an $S$-polarization of $G$. A **principal S-polarization** is an $S$-polarization that is also an isomorphism of $p$-divisible groups. (A polarization $\lambda$ of $G$ is anti-symmetric in the sense that $\hat{\lambda} = -\lambda$, where $\hat{\lambda}$ denotes the Serre dual of $\lambda$, viewed in the canonical way as a map from $G$ to $\hat{G}$; the reason one requires the polarization to be anti-symmetric can be found in [Oda69], Prop. 1.12: a polarization of an abelian variety $A$ over $\mathbb{F}_p$ induces a polarization of the associated $p$-divisible group in the sense of the above definition).

Let $O$ be a $\mathbb{Z}_p$-algebra with involution $\ast$; an action of $(O; \ast)$ on $G$ is a homomorphism of $\mathbb{Z}_p$-algebras $i : O \to \text{End}_S(G)$. If $G$ is endowed with such an action $i$, then $\hat{G}$ is endowed with the dual action $\hat{i}$ given by setting $\hat{i}(a) := i(a^\ast)$ for any $a \in O$, and we say that an $S$-polarization $\lambda : G \to \hat{G}$ respects the $O$-action if $\lambda \circ i = \hat{i} \circ \lambda$. 


2.1.2 Dieudonné modules

Fix a perfect field of characteristic \( p > 0 \) and denote by \( \sigma \) its absolute Frobenius morphism. Denote by \( \mathcal{W} \) the ring scheme over \( k \) of Witt vectors, and let \( V : \mathcal{W} \to \mathcal{W} \) be the Verschiebung morphism. The absolute Frobenius morphism \( F \) on \( \mathcal{W} \) is a ring-scheme homomorphism and one has \( FV = VF = p \cdot id_{\mathcal{W}} \). If \( R \) is a \( k \)-algebra, we denote by \( W_R \) (or by \( W(R) \), or simply by \( W \) if no confusion arises) the ring \( \mathcal{W}(R) \).

For the proof of most of the results that follow, cf. [Dem72] and [Fon77]; cf. also [KZL98], 1-5 and [OB09].

Let \( W \) be the ring \( W_k \) and denote by \( K_0 \) its quotient field. Also, denote by \( \sigma \) the absolute Frobenius morphism induced by \( \sigma : k \to k \) on \( W \) and on \( K_0 \).

An isocrystal over \( K_0 \) is a finite dimensional \( K_0 \)-vector space \( D_0 \) endowed with a Frobenius \( K_0 \)-semilinear automorphism \( F : D_0 \to D_0 \); we denote isocrystals as pairs \((D_0, F)\) and we will call the \( K_0 \)-automorphism \( pF^{-1} \) the Verschiebung of the isocrystal, and denote it by \( V \). The dimension of the vector space \( D_0 \) over \( K_0 \) is called the height of the isocrystal; the \( p \)-adic valuation of \( \det F \) is called the dimension of the isocrystal.

Let \( K \) be a finite field extension of \( K_0 \). An isocrystal over \( K \) is an isocrystal \((D_0, F)\) over \( K_0 \) with a decreasing filtration \( F^{il} \) of \( D_0 \) stable under the Verschiebung map. If \( K \) is the sum of the subspaces of \( N \) appearing according to its multiplicity. We say that \( D \) is isoclinic if it is a perfect pairing.

Theorem 2.2 If \( k = \overline{\mathbb{F}} \), the abelian category of \( K_0 \)-isocrystals is semi-simple, and the simple objects of this category are given, up to isomorphism and without repetition of isomorphism classes, by the \( D_{\lambda} \) \((\lambda \in \mathbb{Q})\).

Let \( k \) be algebraically closed; if \( N \) is an isocrystal over \( K_0 = W(k)[\frac{1}{p}] \), the component of the slope \( \lambda \in \mathbb{Q} \) in \( N \) is the sum of the subspaces of \( N \) isomorphic to \( D_{\lambda} \). The multiplicity of \( \lambda \) is the \( K_0 \)-dimension of this component. The slope sequence of \( N \) is the non-decreasing sequence \( \lambda_1 \leq \ldots \leq \lambda_h \) \((h = \dim_{K_0} N)\) of all slopes of \( N \), each appearing according to its multiplicity. We say that \( N \) is isoclinic if \( \lambda_1 = \lambda_h \).

Let \( W[F, V] \) be the quotient of the associative free \( W \)-algebra generated by the indeterminates \( F, V \) with respect to the relations: \( Fa = a^eF, Va^e = aV, VF = VF = p \) \((\text{any } a \in W)\). A Dieudonné \( W \)-module is a finitely generated \( W \)-module \( F \times W \)-algebra. An \( F \)-lattice over \( W \) is a \( W \)-free module of finite rank endowed with a Frobenius semilinear injective endomorphism \( F \). An \( F \)-lattice embedded as a subobject in an isocrystal \((D_0, F)\) is called a crystal if it is stable under the Verschiebung map.

If \( D \) is an \( F \)-lattice over \( W \) with Frobenius \( F \) such that \( pD \subseteq FD \), there is a unique operator \( V : D \to D \) that can be defined on \( D \) in such a way that \( D \) becomes a module over \( W[F, V] \) \((\text{i.e. a Dieudonné module that is finite and free as a } W \text{-module})\): if \( x \in D \), we set \( Vx := y \) if and only if \( px = Fy \). If \( D \) is an \( F \)-lattice over \( W \), then \( D \otimes^L W \mathbb{Q}_p \) is an isocrystal over \( K_0 \); on the other side, every \( F \)-lattice can be realized as an \( F \)-stable \( W \)-sublattice of some isocrystal. In particular, \( W \)-free finite Dieudonné modules are just \( W \)-sublattices of isocrystals that are stable under \( F \) and \( V := pF^{-1} \).

Let \( D \) be a Dieudonné module over \( W \) that is a finite free \( W \)-module; a polarization of \( D \) is a \( W \)-bilinear non-degenerate alternating form \( \langle , \rangle : D \times D \to W \) such that \( (Fx, y) = \langle x, Vy \rangle \) for all \( x, y \in D \); a \( \mathbb{Z}_p \)-homogeneous polarization of \( D \) is the equivalence class of \( \mathbb{Z}_p^e \)-multiples of a given polarization. A polarization is called a principal polarization if it is a perfect pairing.

If \( D_0 \) is an isocrystal over \( K_0 \), a polarization of \( D_0 \) is a \( K_0 \)-bilinear non-degenerate alternating pairing of isocrystals \( \langle , \rangle : D_0 \times D_0 \to K_0(1) \), where \( K_0(1) = (K_0, p) \); this means that \( \langle , \rangle \) is a \( K_0 \)-bilinear non-degenerate alternating pairing such that \( \langle Fx, Fy \rangle = p \langle x, y \rangle \) for all \( x, y \in D_0 \). One defines \( \mathbb{Q}_p \)-homogeneous polarizations in a similar way as above.

If \( D \) is a Dieudonné module over \( W \) that is finite and free as \( W \)-module, and it is endowed with a polarization \( \langle , \rangle : D \times D \to W \), then \( D_0 := D[\frac{1}{p}] \) is canonically endowed with a polarization of isocrystals \( \langle , \rangle : D_0 \times D_0 \to K_0(1) \).
Let except at most a finite number of them. This decomposition (up to isogeny) is uniquely determined by $\lambda$ of $G$ is a bijection between the set of quasi-isogenies $G$ where $M$ is the covariant Dieudonné functor, by using duality of commutative formal groups. In the sequel, we will sometimes use $M$, instead of $M$, as it is easier to work with covariant functors than with contravariant.

If $D = M(G)$ for some $p$-divisible group $G$ over $k$, then $D[\frac{1}{p}] := D \otimes_W \mathbb{Q}_p$ is an isocrystals over $K_0$ and $D$ is an $F$-lattice over $W$ (and also a crystal). The functor $M$ of Theorem 2.3 gives rise to an additive anti-equivalence between the category of $p$-divisible groups over $k$ and the category of $F$-lattices $D$ over $W$ such that $pD \subseteq FD$. If $G_1$ and $G_2$ are two $p$-divisible groups over $k$, a morphism $f : G_1 \to G_2$ is an isogeny if and only if $M(f) : M(G_2) \to M(G_1)$ is injective, if and only if $\text{coker}(M(f))$ is finite, and if and only if $M(f)$ induces an isomorphism $M(G_2)[\frac{1}{p}] \cong M(G_1)[\frac{1}{p}]$ of isocrystals over $K_0$. The classification of $p$-divisible groups over $k$ up to isogenies is equivalent to the classification of isocrystal over $K_0$ that contain a lattice stable under $F$ and $pF^{-1}$. Also, there is a bijection between the set of quasi-isogenies $G_1 \to G_2$ and the set of isomorphisms $M(G_2)[\frac{1}{p}] \cong M(G_1)[\frac{1}{p}]$ of isocrystals.

**Theorem 2.4 (Dieudonné - Manin)** Let $k$ be algebraically closed; then every $p$-divisible group $G$ over $k$ can be written, up to isogeny, as:

$$G \sim \bigoplus_{0 \leq \lambda \leq 1} G^g(\lambda),$$

where $\lambda$ is a rational number between $0$ and $1$ (included), and $g(\lambda)$ is a non-negative integer, equal to zero for all $\lambda$ except at most a finite number of them. This decomposition (up to isogeny) is uniquely determined by $G$.

If $G$ is a $p$-divisible group over $k$ and we write $G \sim \bigoplus_{0 \leq \lambda \leq 1} G^g(\lambda)$ as in the above theorem, the slope sequence of $G$ is the slope sequence of the associated isocrystal, hence it is given by $0 \leq \lambda_1 \leq \ldots \leq \lambda_h \leq 1$ (where $h = ht(G)$) and $G$ is isoclinic if and only if $G \sim G^g_\lambda$ for some slope $\lambda$ and some $g \geq 0$. We have that $h = ht(G)$ and $ht(G) = \dim G + \dim \hat{G}$: furthermore $\hat{G}$ has slope sequence $0 \leq 1 - \lambda_h \leq \ldots \leq 1 - \lambda_1 \leq 1$.

The following result will be crucial later (cf. [Dem72], page 92):

**Proposition 2.5** Let $k$ be algebraically closed. If the $p$-divisible group $G$ over $k$ is isogenous to $G_{1/r}$ (resp. to $G_{(r-1)/r}$), then $G$ is isomorphic to it.

Let $G$ be a $p$-divisible group over $k$ endowed with a left action $i$ of a $\mathbb{Z}_p$-algebra with involution $(O, *)$: then $M(G)$ is endowed, by functoriality, with a left action of $O^{opp}$. On the other side, $M_*(G)$ is endowed with an action of $O$. The anti-equivalence of Theorem 2.3 gives the following:

**Proposition 2.6** Let $k$ be any perfect field of characteristic $p$. 


1. A polarization $\lambda : G \to \hat{G}$ of a $p$-divisible group over $k$ that is also an isogeny determines a polarization of $M(\hat{G}) = M(G)^\hat{\lambda}$ which is principal if and only if $\lambda$ is principal (in this case, it can then be identified with a principal polarization of $M(G)$). Viceversa, a polarization (resp. principal polarization) of $M(\hat{G})$ determines a polarization (resp. principal polarization) of $G$. If $\lambda : G \to \hat{G}$ is a polarization (and not necessarily an isogeny), it induces a polarization of $M(\hat{G})[\frac{1}{p}]$, and viceversa.

2. Let $G$ be a $p$-divisible group endowed with the action $i$ of a $\mathbb{Z}_p$-algebra with involution $(O, \ast)$, and let $\lambda$ be a polarization of $G$ respecting such an action; then the polarization $\langle \cdot, \cdot \rangle$ induced on the left $O^{opp}$-module $M(\hat{G})[\frac{1}{p}]$ is skew-hermitian with respect to the involution acting on $M(\hat{G})$, i.e. $\langle bf, g \rangle = \langle f, b^\ast g \rangle$ for all $b \in O^{opp}, f, g \in M(\hat{G})$. The viceversa is also true.

**Proof.** Let $\lambda : G \to \hat{G}$ be a polarization that is also an isogeny of $p$-divisible groups. Then $M(\lambda) : M(\hat{G}) \to M(G)$ is a monomorphism of Dieudonné modules. Using the canonical identification $M(\hat{G}) = \text{Hom}_{W}(M(G), W)$ coming from Theorem 2.3, we obtain a non-degenerate $W$-linear pairing $\langle \cdot, \cdot \rangle : M(\hat{G}) \times M(\hat{G}) \to W$ defined by $\langle f, g \rangle := f(M(\lambda)(g))$ for any $f, g \in \text{Hom}_{W}(M(G), W)$. We have $\langle F f, g \rangle = (F f)(M(\lambda)(g)) = f(V M(\lambda)(g))^{\ast}$; since $M(\lambda)$ is a map of Dieudonné modules, it commutes with $V$, so that $\langle F f, g \rangle = f(M(\lambda)(V g))^{\ast} = (f, V g)^{\ast}$. Furthermore, $\langle f, g \rangle = g(M(\lambda)(f))$, so that $\langle \cdot, \cdot \rangle$ is alternating since $\lambda$ is anti-symmetric. Notice that $\lambda$ is principal if and only if $M(\lambda)$ is an isomorphism, i.e. if and only if $\langle \cdot, \cdot \rangle$ is a perfect W-pairing. Using the fact that $M$ is an anti-equivalence of categories, we obtain that viceversa we can pass from polarizations of Dieudonné modules that are finite and free as $W$-modules to polarizations of $p$-divisible groups. The statement for isocrystals follows after inverting $p$.

For the second statement, let $b \in O, f, g \in M(\hat{G})$ and assume that $\lambda : G \to \hat{G}$ is an isogeny. Notice that the compatibility of $\lambda$ with $i$ implies that $M(i(b)) \circ M(\lambda) = M(\lambda) \circ M(i(b))$ and similarly for $b^{\ast}$. Then we have:

$$\langle bf, g \rangle = \langle M(i(b))(f), g \rangle = (M(i(b)^{\ast}))^{-1}(f)(M(\lambda)g) = (f \circ M(i(b^{\ast})) \circ M(\lambda))(g) = \langle f, b^{\ast}g \rangle.$$

In a similar way, one can prove that a skew-hermitian polarization on $M(\hat{G})[\frac{1}{p}]$ induces a $*$-compatible polarization of $G$. ■

**2.1.3 Isocrystals associated to a $K_0$-rational element of a reductive group**

We still assume that $k$ is a perfect field of positive characteristic $p$, with $W = \mathcal{W}(k)$ and $K_0 = W[\frac{1}{p}]$. We also fix a connected algebraic group $G$ over $\mathbb{Q}_p$ and an element $b \in G(K_0)$. If $\rho : G \to GL(V)$ is a rational algebraic $\mathbb{Q}_p$-representation of $G$ of finite dimension we define the isocrystal associated to $(V; b)$ by the pair:

$$(V \otimes_{\mathbb{Q}_p} K_0, \rho(b)(id_V \otimes \sigma)).$$

(We denote $\rho(b)(id_V \otimes \sigma)$ also by $b(id_V \otimes \sigma)$ or $b\sigma$ is no confusion arises). An exact functor remains therefore defined, from the category of rational $\mathbb{Q}_p$-representation of $G$ of finite dimension, to the category of $K_0$-isocrystal. Notice that if $g \in G(K_0)$, and we set $b' := gb(g)^{-1}$, then the functors associated to $b$ and $b'$ are isomorphic ($b$ and $b'$ are in this case said to be $\sigma$-conjugate in $G(K_0)$).

**The slope morphism** Assume that $k$ is algebraically closed. Let $\mathbb{D}$ denote the diagonalizable pro-algebraic group over $\mathbb{Q}_p$, whose character group $X^*(\mathbb{D})$ (in the sense of pro-algebraic groups) is isomorphic to $\mathbb{Q}$: we can think of $\mathbb{D}$ as the universal cover of $G_m$ in the sense of quasi-algebraic groups (cf. [Ser70], 7.3). Notice that the canonical inclusion $\mathbb{Z} \subset \mathbb{Q}$ induces a morphism $\mathbb{D} \to G_m$ and hence an inclusion $\text{Hom}_{\mathbb{Q}_p}(G_m, G) \to \text{Hom}_{\mathbb{Q}_p}(\mathbb{D}, G)$. Furthermore, for any morphism $f \in \text{Hom}_{\mathbb{Q}_p}(\mathbb{D}, G)$ there is a positive integer $n$ such that $nf \in \text{Hom}_{\mathbb{Q}_p}(G_m, G)$.

In [Kot85] §4, Kottwitz defines a morphism of over $K_0$:

$$\nu : \mathbb{D} \to G_{K_0}.$$
characterized by the following property: for any $\mathbb{Q}_p$-rational finite-dimensional representation $\rho : G \to GL(V)$ of $G$, let $\nu_\rho \in \text{Hom}_{K_0}(\mathbb{D}_{K_0}, GL(V_K))$ be the only morphism for which the action of $\mathbb{D}$ on the isotypical component of the isocrystal $(V \otimes_{\mathbb{Q}_p} K_0, \rho(b)(id_V \otimes \sigma))$ of slope $\lambda \in \mathbb{Q}$ is given by the character $\lambda \in X^*(\mathbb{D})$. Then $\nu$ is the only morphism over $K_0$ such that $\rho \circ \nu = \nu_\rho$ for any representation $\rho$ as above. In other words, $\nu$ is characterized by the fact that for any representation $V$ of $G$, the grading induced by $\nu$ on $V \otimes K_0$ coincides with the slope decomposition of $(V \otimes K_0, b(id_V \otimes \sigma))$. We call such $\nu$ the slope morphism associated to $b \in G(K_0)$; that this definition does not depend upon the choice of $\sigma$-conjugacy class of $b$.

**Basic elements** Let us keep assuming that $k$ is algebraically closed and furthermore that $G$ is a connected and reductive algebraic group over $K_0$. Following [Ko85], 5.1 we say that the element $b \in G(K_0)$ is basic if the corresponding slope morphism $\nu : \mathbb{D}_{K_0} \to G_{K_0}$ factors through the center of $G_{K_0}$.

Let $J$ be the functor on $\mathbb{Q}_p$-algebras defined by:

$$J(R) := \{ g \in G(R \otimes_{\mathbb{Q}_p} K_0) : g(b \sigma) = (b \sigma)g \},$$

for any $\mathbb{Q}_p$-algebra $R$ (where we see $b \sigma \in G(K_0) \rtimes \langle \sigma \rangle$ and the action by conjugation of $\sigma$ on elements of $G(K_0)$ is the one obtained by viewing $\sigma$ as an automorphism of $K_0$). Then $J$ defines a smooth affine group scheme over $\mathbb{Q}_p$ ([MR96b], 1.12) and Kottwitz proves in [Ko85] that $b$ is basic if and only if $J$ is an inner form of $G$. For example, if $V$ is a finite dimensional $\mathbb{Q}_p$-vector space and $G = GL(V)$, an element $b \in G(K_0)$ is basic if and only if the corresponding isocrystal $(V \otimes_{\mathbb{Q}_p} K_0, b \sigma)$ is isoclinic.

**Admissibility** We assume here that $k$ is a perfect field of characteristic $p$, with $K_0 = W(k)[\frac{1}{p}]$. Let $K$ be a finite field extension of $K_0$. We say that a $K$-filtered isocrystal $D$ over $K_0$ is admissible if the following condition is satisfied: for any $K$-filtered sub-isocrystal $D'$ of $D$, the rightmost endpoint of the Newton polygon of $D'$ lies on or above the rightmost endpoint of the Hodge polygon of $D'$, with equality if $D = D'$ (for the definition of Newton and Hodge polygons see for example [OB09], 8). The category of admissible $K$-filtered isocrystal over $K_0$ is abelian.

Let $B_{cris}$ denote Fontaine’s crystalline ring; it is a $K_0$-algebra domain, endowed with a continuous action of the absolute Galois group $G_K$ of $K$, such that $B'_{cris} = K_0$; furthermore $B_{cris}$ is endowed with a $\sigma$-linear Frobenius endomorphism, and with a (non Frobenius-stable) $K$-filtration (i.e. a filtration of $K \otimes_{K_0} B_{cris}$) induced by the canonical inclusion:

$$K \otimes_{K_0} B_{cris} \hookrightarrow B_{dR},$$

where $B_{dR}$ is the de Rham period ring, with its canonical filtration (cf. [Pon94]).

Let $\text{Rep}_{\mathbb{Q}_p}(G_K)$ be the category of $\mathbb{Q}_p$-linear continuous representations of $G_K$ of finite dimension and, for any $W \in \text{Rep}_{\mathbb{Q}_p}(G_K)$ define $D_{cris}(W) := (W \otimes_{\mathbb{Q}_p} B_{cris})^{G_K}$ and say that $W$ is crystalline if $\text{dim}_{\mathbb{Q}_p} W = \text{dim}_{K_0} D_{cris}(W)$. Let $\text{Rep}_{\mathbb{Q}_p}^{cris}(G_K)$ be the subcategory of $\text{Rep}_{\mathbb{Q}_p}(G_K)$ consisting of crystalline representations; then $D_{cris}$ defines a fully faithful exact tensor functor from $\text{Rep}_{\mathbb{Q}_p}^{cris}(G_K)$ into the category of $K$-filtered isocrystal over $K_0$. It was proven by Colmez and Fontaine ([PC99]) that this functor gives rise to an equivalence between the category $\text{Rep}_{\mathbb{Q}_p}^{cris}(G_K)$ and the category of $K$-filtered isocrystal over $K_0$ that are admissible. Therefore in the sequel an admissible filtered isocrystal will be the same as an isocrystal coming from a crystalline representation via the functor $D_{cris}$.

Assume now that $k$ is algebraically closed. Let $G$ be a reductive connected group defined over $\mathbb{Q}_p$ and fix a cocharacter $\mu : G_m \to G$ defined over $K$ (we dropped the subscript $K$). If $V$ is a finite dimensional $\mathbb{Q}_p$-rational representation of $G$ and $(V \otimes_{\mathbb{Q}_p} K_0, b \sigma)$ is the associated isocrystal over $K_0$, we can construct a $K$-filtered $K_0$-isocrystal $\mathcal{I} = \mathcal{I}(V; b; \mu)$ associated to the triple $(V; b; \mu)$ by letting:

$$\mathcal{I} := (V \otimes_{\mathbb{Q}_p} K_0, b \sigma, V_K^\bullet)$$

where, for any $i \in \mathbb{Z}$, we set:

$$V_K^i = \bigoplus_{j \geq i} (V \otimes_{\mathbb{Q}_p} K)_j,$$

with the subscript $j$ denoting the $j$ weight space for the action of $G_m$ on $V_K := V \otimes_{\mathbb{Q}_p} K$ induced by $\mu$.

Keeping the above notation, we say that the pair $(b; \mu)$ is admissible if for any finite dimensional $\mathbb{Q}_p$-rational representation $V$ of $G$, the filtered isocrystal $\mathcal{I}(V; b; \mu)$ is admissible in the sense explained above. Notice that this is equivalent to say that $\mathcal{I}(V; b; \mu)$ is admissible for some faithful finite dimensional $\mathbb{Q}_p$-rational representation $V$ of $G$ (cf. [MR96b], 1.18).
2.2 Moduli of \( p \)-divisible groups of PEL-type

We refer for this section to [MR96b], Chapter 3; cf. also [Bou97].

2.2.1 Definition of local PEL-data

We define local PEL-data; we will define later on PEL-data of global type, that will be the starting point for the construction of moduli spaces of abelian schemes of PEL-type.

Let \( B \) be a finite dimensional semi-simple \( \mathbb{Q}_p \)-algebra endowed with an involution \( * \). Let \( V \neq \{ 0 \} \) be a finitely generated left \( B \)-module and \( \langle , \rangle : V \times V \to \mathbb{Q}_p \) a non-degenerate, alternating \( \mathbb{Q}_p \)-bilinear form which is skew-Hermitian with respect to \( * \), i.e. \( \langle tv, w \rangle = \langle v, t^*w \rangle \) for all \( v, w \) in \( V \) and all \( t \) in \( B \). These objects define a reductive algebraic group scheme \( G := G(B,^*, V, \langle \cdot, \cdot \rangle) \) over \( \mathbb{Q}_p \) whose \( R \)-points, for a fixed \( \mathbb{Q}_p \)-algebra \( R \), are given by:

\[
G(R) = \left\{ (g, s) \in GL_B \otimes_{\mathbb{Q}_p} (V \otimes_{\mathbb{Q}_p} R) \times G_m(R) : \langle gv, gw \rangle = s \langle v, w \rangle \ \forall v, w \in V \right\}.
\]

The map \( G(R) \to R^\times \) given by \( (g, s) \mapsto s \) defines a homomorphism of \( \mathbb{Q}_p \)-algebraic groups \( c : G \to \mathbb{G}_m \) that is called the similitude character of \( G \); the kernel of \( c \) is a reductive \( \mathbb{Q}_p \)-subgroup of \( G \) denoted by \( G_1 \). By abuse of notation, an element \((g, s) \in G(R)\) will be often denoted by \( g \).

Notice that \( * \) defines on the \( \mathbb{Q}_p \)-algebra \( C := \text{End}_B V \) an involution \( x \mapsto x^* \) via the identity \( \langle xv, w \rangle = \langle v, x^*w \rangle \) \( \forall v, w \in V \).

We therefore have functorial isomorphisms:

\[
G(R) \simeq \left\{ x \in C \otimes_{\mathbb{Q}_p} R : xx^* \in R^\times \right\},
\]

where \( R \) is any \( \mathbb{Q}_p \)-algebra.

**Definition 2.7** Fix a tuple \((B,^*, V, \langle \cdot, \cdot \rangle)\) satisfying the above properties. Let \( k \) be an algebraically closed field of characteristic \( p > 0 \) and set \( W = W(k) \), \( K_0 = W[\frac{1}{p}] \); denote by \( \sigma \) the Frobenius automorphism of \( W \) and \( K_0 \).

1. The datum \( \mathcal{D} := (B,^*, V, \langle \cdot, \cdot \rangle) \) is called a \( \mathbb{Q}_p \)-PEL-datum; the group \( G = G(B,^*, V, \langle \cdot, \cdot \rangle) \) is the algebraic group associated to \( \mathcal{D} \). A \( \mathbb{Q}_p \)-PEL-datum with integral structure (or with a \( \mathbb{P} \)-lattice) is the datum \( \mathcal{D}_\mathcal{O} := (B,^*, V, \langle \cdot, \cdot \rangle, \mathcal{O}_B, \Lambda) \), where \((B,^*, V, \langle \cdot, \cdot \rangle)\) is a \( \mathbb{Q}_p \)-PEL-datum, \( \mathcal{O}_B \) is a maximal \( \mathbb{Z}_p \)-order in \( B \) stable under the involution \( * \), and \( \Lambda \subset V \) is a \( \mathbb{Z}_p \)-lattice in \( V \) which is also an \( \mathcal{O}_B \)-submodule and which is self-dual with respect to the pairing \( \langle \cdot, \cdot \rangle \) (i.e. the restriction of \( \langle \cdot, \cdot \rangle \) to \( \Lambda \times \Lambda \) defines a perfect pairing of \( \mathbb{Z}_p \)-modules).

2. A \( \mathbb{Q}_p \)-PEL-datum for modules of \( p \)-divisible groups over \( k \) is the datum \( \mathcal{D}_\text{mod} := (B,^*, V, \langle \cdot, \cdot \rangle, \mathcal{O}_B, \Lambda, b, \mu) \), where \((B,^*, V, \langle \cdot, \cdot \rangle, \mathcal{O}_B, \Lambda)\) is a \( \mathbb{Q}_p \)-PEL-datum with integral structure and associated group \( G \), \( b \) is a fixed element of \( G(K_0) \), and \( \mu : \mathbb{G}_m / K \to G_K \) is a co-character of \( G \) defined over some finite field extension \( K \) of \( K_0 \). We furthermore require that the following conditions are satisfied:

   (a) \((b, \mu)\) is an admissible pair in the sense of [2, L.3]
   (b) the isocrystal \((N, F) := (V \otimes_{\mathbb{Q}_p} K_0, b\sigma)\) associated to \( \mathcal{D}_\text{mod} \) has slopes in the interval \([0,1]\);
   (c) the weight decomposition of \( V \otimes_{\mathbb{Q}_p} K \) with respect to \( \mu \) contains only the weights 0 and 1: \( V \otimes_{\mathbb{Q}_p} K = V_0 \oplus V_1 \);
   (d) if \( v : \mathbb{D}_K_0 \to G_K \) denotes the slope morphism associated to \( b \) and \( c : G \to \mathbb{G}_m \) is the similitude character of \( G \) over \( \mathbb{Q}_p \), then \( cv : \mathbb{D}_K_0 \to \mathbb{G}_m / K_0 \) is the character of \( \mathbb{D}_K_0 \) corresponding to the rational number 1.

The reflex field (or local Shimura field) of \( \mathcal{D}_\text{mod} \) is the field of definition of the conjugacy class of the co-character \( \mu \).

Each of the above \( \mathbb{Q}_p \)-PEL-data will be called simple if the algebra \( B \) is simple. The \( \mathbb{Q}_p \)-PEL-data \( \mathcal{D}_\mathcal{O} \) or \( \mathcal{D}_\text{mod} \) have good reduction if \( B \) is an unramified \( \mathbb{Q}_p \)-algebra (i.e. its center is the product of matrix algebras over unramified field extensions of \( \mathbb{Q}_p \)). If the PEL-data \( \mathcal{D}_\mathcal{O} \) or \( \mathcal{D}_\text{mod} \) has good reduction at \( p \), the associated group \( G_{Q_p} \) is unramified, i.e. it is quasi-split over \( \mathbb{Q}_p \) and split over an unramified extension of \( \mathbb{Q}_p \) or, equivalently, \( G(Q_p) \) has an hyperspecial subgroup. In particular \( G_{Q_p} \) has a reductive model \( \mathcal{G} \) over \( \mathbb{Z}_p \) such that for any \( \mathbb{Z}_p \)-algebra \( R \) we have:

\[
\mathcal{G}(R) = \left\{ g \in GL_{R \otimes_{\mathbb{Q}_p} \mathbb{Q}_p} (\Lambda \otimes_{\mathbb{Z}_p} R) : \langle g\lambda_1, g\lambda_2 \rangle = c(g) \langle \lambda_1, \lambda_2 \rangle, c(g) \in R^\times \right\}.
\]
Let us make some comments on the definition of local PEL-datum for moduli of $p$-divisible groups (cf. [MR96b, 3.19]). We keep the notation from the above definition, and fix a local PEL-datum for moduli of $p$-divisible groups $\mathcal{D}_{\text{mod}}$. We also denote by $\mathcal{D}_s(\cdot)$ the covariant functor that assigns to a $p$-divisible group over a $\mathbb{Z}_p$-scheme of characteristic $p$ the associated Dieudonné crystal (cf. [PB82, Mes72]).

- The condition $b$ on $\mathcal{D}_{\text{mod}}$ is equivalent to requiring that the $K_0$-isocrystal $(N,F)$ comes from a $p$-divisible group $X$ (defined over $k$) via the covariant Dieudonné functor $M_s$ (cf. Th. 2.21); such an $X$ is uniquely determined up to $k$-quasi-isogeny and we fix for the sequel such a choice of $X$.
- Assume that there are a $p$-divisible group $X$ over the ring of integers $\mathcal{O}_K$ of $K$ and a $k$-quasi-isogeny $\varphi : X \to X_\kappa$; we can lift $\varphi$ to a quasi-isogeny of $p$-divisible groups over $\text{Spec}(\mathcal{O}_K/(p))$:

$$\varphi : X \times_{\text{Spec} k} \text{Spec}(\mathcal{O}_K/(p)) \xrightarrow{\sim} X \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_K/(p)).$$

Applying the covariant functor $\mathcal{D}_s(\cdot)$ to the above quasi-isogeny, and then evaluating the corresponding Dieudonné crystals of $\mathcal{O}_{\text{Spec}(\mathcal{O}_K/(p))}$-modules at the PD-thickening $\text{Spec}(\mathcal{O}_K/(p)) \to \text{Spec}(\mathcal{O}_K)$ belonging to the small crystalline site of $\text{Spec}(\mathcal{O}_K/(p))$, we obtain an isomorphism of $K$-isocrystals:

$$\varphi : (N,F)^{(p)} \otimes_{K_0} K \xrightarrow{\sim} \mathcal{D}_s(X \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_K/(p))))_{\text{Spec}(\mathcal{O}_K)} \otimes \mathbb{Q},$$

where we used the fact that, by assumption, $M_s(X)[p^{-1}] = (N,F)$. If we assume that, under this identification, the Hodge filtration of the crystal $\mathcal{D}_s(X \times_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_K/(p))))_{\text{Spec}(\mathcal{O}_K)} \otimes \mathbb{Q}$ corresponds to the filtration induced by $\mu$ on $V \otimes_{\mathcal{O}_K} K$: $0 \to V_1 \to V \otimes_{\mathcal{O}_K} K \to V_0 \to 0$, then the conditions $a), b), c)$ of the above definition of $\mathcal{D}_{\text{mod}}$ are automatically satisfied: for the pair $(b,\mu)$ is, in this case of "good reduction", admissible in the sense of section 2.13. In the later application of these constructions, we will be in the "good reduction" case, so we will not need to check conditions $a), b), c)$ on the PEL-data we will have in hands.

- Let $\chi : G \to G_m$ be a $\mathbb{Q}_p$-rational character of $G$. Then $(K_0(\chi),b\sigma)$ is a one dimensional $K_0$-isocrystal whose only slope is $\text{ord}_p \chi(b)$, since $(b\sigma)(W) = \chi(b)W = p^\text{ord}_p \chi(b)W$. If the pair $(b,\mu)$ is admissible, the $K$-filtered $K_0$-isocrystal associated to $(K_0(\chi),b\sigma)$ and $\mu$ need to satisfy the admissibility equation $\sum_{i \in \mathbb{Z}} i \cdot \dim_{K_0} K(\chi)^i = \text{ord}_p \chi(b)$, where $K(\chi)^i$ is the $i^{th}$ term in the filtration that is induced by $\mu$ on $K(\chi)$. We conclude that $(\mu,\chi) = \text{ord}_p \chi(b)$.

In particular:

- taking $\chi = \text{det}_{\mathcal{O}_K}$, defined by viewing $G \subset GL(V)$, we obtain $\dim_{K_0} V_1 = \text{ord}_p(\text{det}_{\mathcal{O}_K}(b; V \otimes_{\mathcal{O}_K} K_0))$;
- taking $\chi = c$ to be the similitude factor of $G$, condition $d)$ from the definition of $\mathcal{D}_{\text{mod}}$ implies that $\langle \mu, c \rangle = \text{ord}_p c(b) = 1$. Hence the subspaces $V_0$ and $V_1$ of $V \otimes_{\mathcal{O}_K} K$ are isotropic for the pairing induced by $\langle , \rangle$ on $V \otimes_{\mathcal{O}_K} K$. (In fact if $v \in V_0$, we have $\langle v,v \rangle = \langle \mu(x)v, \mu(x)v \rangle = x\langle v,v \rangle$ for all $x \in K^\times$; similarly for $V_1$).

Fix a local PEL-datum for moduli of $p$-divisible groups $\mathcal{D}_{\text{mod}}$; denote by $(N,F)$ the associated isocrystal. The action of $B$ on $V$ by left multiplication induces an action of $B$ on $N$. Furthermore, since $k$ is algebraically closed and $\text{ord}_p c(b) = 1$, we can write $c(b) = p \cdot u^\sigma(u)^{-1}$ for some $u \in W^\times$. Define a map $\Psi : N \times N \to K_0$ by setting $\Psi(v,w) := u^{-1} \cdot \langle v,w \rangle$ for all $v,w \in N$; $\Psi$ is a non-degenerate $K_0$-bilinear pairing, which is alternating and skew-Hermitian with respect to $\ast$. Furthermore, $\Psi(Fv,Fw) = p\Psi(v,w)^\ast$, so that $\Psi$ defines a polarization of isocrystals $N \times N \to 1(1)$.

Notice that any other choice of $u$ gives rise to a multiple of $\Psi$ by an element of $\mathbb{Q}_p^\times$, so that our local PEL-datum gives rise to a well defined triple $(N,F,\mathbb{Q}_p^\times \Psi)$ that is to a $\mathbb{Q}_p$-homogeneously polarized $K_0$-isocrystal endowed with an action of $B$ (the polarization form is skew-Hermitian with respect to the involution of $B$).

### 2.2.2 The moduli functor for $p$-divisible groups

Fix a $\mathbb{Q}_p$-PEL-datum for moduli of $p$-divisible groups $\mathcal{D}_{\text{mod}} := (B,\ast, V,\langle , \rangle, \mathcal{O}_B, \Lambda, b, \mu)$ over the algebraically closed field $k$ of characteristic $p$. Denote by $G$ the associated algebraic $\mathbb{Q}_p$-group, and let $(N,F,\mathbb{Q}_p^\times \Psi)$ be the $\mathbb{Q}_p$-homogeneously polarized $K_0$-isocrystal with $B$-action associated to $\mathcal{D}_{\text{mod}}$, as in the previous section.

Pick a $p$-divisible group $X$ over $k$ whose $K_0$-isocrystal constructed via $M_s$ is isomorphic to $(N,F)$; by Prop. 2.10 $B$ acts on the group of quasi-isogenies of $X$, so that we have a $\mathbb{Q}_p$-algebra homomorphism:

$$i_X : B \to \text{End}(X) \otimes_{\mathbb{Z}_p} \mathbb{Q};$$
furthermore $\mathbb{Q}_p^x \Psi$ induces a $\mathbb{Q}_p$-homogeneous polarization $\lambda_X : (X, i_X) \rightarrow (\overset{\circ}{X}, i_{\overset{\circ}{X}})$ of $X$ that respect the action of $B$ (this polarization needs not to be principal). We have therefore associated to $\mathcal{D}_{\text{mod}}$ the triple $(X, i_X, \overset{\circ}{X})$ consisting on a $\mathbb{Q}_p$-homogeneously polarized $p$-divisible group over $k$ endowed with an action of $B$. Such a triple is unique only up to quasi-isogenies, and it is assumed fixed in the rest of this paragraph.

Denote by $\overset{\circ}{E}$ the complete unramified extension of the local Shimura field $E$ of $\mathcal{D}_{\text{mod}}$, which has residue field $k$ and is contained inside $K$, the field of definition of the quasi-character $\mu$; we have $\overset{\circ}{E} = EK_0$. Let $\mathcal{O}_{\overset{\circ}{E}}$ be the ring of integers of $\overset{\circ}{E}$ and denote by $\text{NILP}_{\mathcal{O}_{\overset{\circ}{E}}}$ the category of locally noetherian $\text{Spec} \mathcal{O}_{\overset{\circ}{E}}$-schemes $(S, \mathcal{O}_S)$ such that the ideal sheaf $p\mathcal{O}_S$ is locally nilpotent; for such an $S$, we denote by $\overline{S}$ the closed subscheme of $S$ defined by $p\mathcal{O}_S$: it is a scheme over $\text{Spec} k$. For example, we could take $S = \text{Spec} k = \overline{S}$.

The moduli problem for $p$-divisible groups associated to the above data, as defined in [MR96b]. 3.21, is the following:

**Definition 2.8** Fix a local PEL-datum $\mathcal{D}_{\text{mod}}$ for moduli of $p$-divisible groups over $k$. Denote by $E$ its local Shimura field and let $(X, i_X, \overset{\circ}{X})$ be a choice of $\mathbb{Q}_p$-homogeneously polarized $p$-divisible group over $k$ with $B$-action associated via $M_*$ to the triple $(N, F, \mathbb{Q}_p^x \Psi)$ defined by $\mathcal{D}_{\text{mod}}$.

We let $\mathcal{M}$ be the contravariant functor from $\text{NILP}_{\mathcal{O}_{\overset{\circ}{E}}}$ to $\text{SETS}$ defined as follows: if $S$ is a scheme in $\text{NILP}_{\mathcal{O}_{\overset{\circ}{E}}}$, $\mathcal{M}(S)$ consists of the equivalence classes of tuples $(X, i, \overset{\circ}{X}; \rho)$ where:

1. $X$ is a $p$-divisible group over $S$;
2. $i : \mathcal{O}_B \rightarrow \text{End} X$ is a $\mathbb{Z}_p$-algebra homomorphism satisfying the determinant condition, i.e. we require an equality of polynomial functions: $\det_{\mathcal{O}_S}(a, \text{Lie} X_S) = \det_K(a, V_0)$ for all $a \in \mathcal{O}_B$;
3. $\lambda : (X, i) \rightarrow (\overset{\circ}{X}, \overset{\circ}{i})$ is a principal polarization of $(X, i)$, and $\overset{\circ}{X}$ is the corresponding $\mathbb{Q}_p$-homogeneous (principal) polarization;
4. $\rho : (X, i_X) \rightarrow (X, i_X)$ is a quasi isogeny of $p$-divisible groups over $\overline{S}$ that respects the $\mathcal{O}_B$-structure and such that $\overset{\circ}{\rho} \circ \lambda_{\overset{\circ}{X}} \circ \rho \in \mathbb{Q}_p^x(\lambda_X)$.

Two tuples $(X, i, \overset{\circ}{X}; \rho), (X', i', \overset{\circ}{X}'; \rho') \in \mathcal{M}(S)$ are said to be equivalent if the $\overline{S}$-quasi-isogeny $\rho' \circ \rho^{-1}$ lifts to an isomorphism $f : (X, i) \rightarrow (X', i')$ of $p$-divisible groups over $S$ with $\mathcal{O}_B$-action, such that $f \circ \lambda' \circ f^{-1} \in \mathbb{Z}_p^x \lambda$.

Notice that the definition of the functor $\mathcal{M}$ depends - up to isomorphism - only upon the isocrystal $(N, F, \mathbb{Q}_p^x \Psi)$, and not upon the choice of $p$-divisible group $(X, i_X, \overset{\circ}{X})$ that is associated to $(N, F, \mathbb{Q}_p^x \Psi)$ via covariant Dieudonné theory.

It was proven by Rapoport ant Zink that the above functor is representable by a formal scheme: this is the content of Theorem 3.25 of [MR96b].

**Theorem 2.9** The functor $\mathcal{M}$ defined above is representable by a formal scheme (still denoted by $\mathcal{M}$) which is formally locally finite type over $\text{Spf} \mathcal{O}_{\overset{\circ}{E}}$. Furthermore, if the original local PEL-datum $\mathcal{D}_{\text{mod}}$ has good reduction, we have $\overset{\circ}{E} = K_0$ and the formal scheme representing $\mathcal{M}$ is formally smooth over $\text{Spf} \mathcal{W}$.

**Determinant condition** The determinant condition required in the above definition is due to Kottwitz. A precise formulation, a bit hidden behind the one we gave, is contained in [Kot92]. 5, and also [MR96b]. 3.23: let $\mathcal{V}$ be the $\mathbb{Z}_p$-scheme whose values in any $\mathbb{Z}_p$-algebra $R$ are given by $\mathcal{O}_B \otimes_{\mathbb{Z}_p} R$: fix an $\mathcal{O}_B$-invariant $\mathcal{O}_K$-lattice $\Gamma \subset V_0$ and define a map of $\mathcal{O}_K$-schemes $\det_K(-, V_0) : \mathcal{V}_{\mathcal{O}_K} \rightarrow \mathbb{A}^1_{\mathcal{O}_K}$ by setting, for any $\mathcal{O}_K$-algebra $R$ and any $a \in \mathcal{O}_B \otimes_{\mathbb{Z}_p} R$, $\det_K(a, V_0) := \det(a; \Gamma \otimes_{\mathcal{O}_K} R)$. It is clear that this map does not depend upon the choice of lattice $\Gamma$ and is defined over $\mathcal{O}_E$ ($E$ is the local Shimura field of the datum), so that it defines a morphism of $\mathcal{S}$-schemes $\det_K(\cdot, V_0) : \mathcal{V}_S \rightarrow \mathbb{A}^1_S$. In a similar way, $\det_{\mathcal{O}_S}(\cdot, \text{Lie} X)$ can be seen as a morphism of $\mathcal{S}$-schemes:

$$
det_{\mathcal{O}_S}(\cdot, \text{Lie} X) : \mathcal{V}_S \rightarrow \mathbb{A}^1_S.
$$

The determinant condition is the requirement that these two morphisms of schemes over $S$ coincide. By the definition of $\mathcal{V}$, we could rephrase this condition by requiring that for any $\mathcal{S}$-scheme $S'$ and any $a \in \mathcal{O}_B \otimes_{\mathcal{O}_S} \mathcal{S}'$, we have an identity of polynomial functions $\det_{\mathcal{O}_{S'}}(a, \text{Lie} X_{S'}) = \det_K(a, V_0)$; notice that by fixing a basis for the $\mathbb{Z}_p$-free module $\mathcal{O}_B$ - say, of rank $t$ - then $\det_K(a, V_0)$ for a variable $a \in \mathcal{O}_B$ can be written as a polynomial of the algebra $\mathcal{O}_E[X_1, ..., X_t]$. 

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3 Moduli of abelian schemes of PEL-type

3.1 Global PEL-data

We recall the definition of a PEL-data over $\mathbb{Q}$, following [Kot92].

3.1.1 Definition of a PEL-data over $\mathbb{Q}$

Let $B$ be a finite dimensional semi-simple $\mathbb{Q}$-algebra endowed with a positive involution $^*$ (positivity means that $Tr_{B/Q}(xx^*) > 0$ for any $x$ in $B - \{0\}$). Let $V \neq \{0\}$ be a finitely generated left $B$-module and $(,): V \times V \to \mathbb{Q}$ a non-degenerate, alternating $\mathbb{Q}$-bilinear form which is skew-hermitian with respect to $^*$, i.e. $\langle bv, w \rangle = \langle v, b^*w \rangle$ for all $v, w$ in $V$ and all $b$ in $B$. These objects define a reductive algebraic group $G$ over $\mathbb{Q}$ whose $\mathbb{R}$-points, for a fixed $\mathbb{Q}$-algebra $R$, are given by:

$$G(R) = \{ g \in GL_{B \otimes \mathbb{Q}}(V \otimes \mathbb{Q} R): (gv, gw) = c(g) \langle v, w \rangle \ \forall v, w \in V; c(g) \in R^\times \}.$$ 

The map $G(R) \to \mathbb{R}^\times$ given by $g \mapsto c(g)$ defines a homomorphism of $\mathbb{Q}$-algebra groups $c: G \to \mathbb{G}_m$ (the similitude character of $G$); the kernel of $c$ is a reductive $\mathbb{Q}$-subgroup of $G$ denoted by $G_1$.

Notice that $^*$ defines on the $\mathbb{Q}$-algebra $C := \text{End}_B V$ an involution $x \mapsto x^*$ via the identity $\langle xv, w \rangle = \langle v, x^*w \rangle \ (v, w \in V)$.

We therefore have functorial isomorphisms:

$$G(R) \simeq \{ x \in C \otimes \mathbb{Q} R : xx^* \in R^\times \},$$

where $R$ is any $\mathbb{Q}$-algebra.

**Definition 3.1** Fix a tuple $(B^*, V, \langle, \rangle)$ satisfying the above properties, and let $p$ be a fixed prime number.

1. The datum $\mathcal{D} = (B^*, V, \langle, \rangle)$ is called a $\mathbb{Q}$-PEL datum; the group $G$ is the algebraic group associated to $\mathcal{D}$. A $\mathbb{Q}$-PEL datum with integral structure at $p$ (or with a $p$-adic PEL-lattice) is the datum $\mathcal{D}_O := (B^*, V, \langle, \rangle, O_B, \Lambda)$ where $(B^*, V, \langle, \rangle)$ is a $\mathbb{Q}$-PEL datum; $O_B$ is a $\mathbb{Z}_p$-order of $B$ stable under the involution $^*$ and such that $O_B \otimes \mathbb{Z}_p$ is a maximal order in $B \otimes \mathbb{Q}_p$; $\Lambda \subset V \otimes \mathbb{Q}_p$ is a $\mathbb{Z}_p$-lattice and an $O_B$-submodule such that the restriction of $\langle, \rangle \otimes \mathbb{Q}_p$ to $\Lambda \times \Lambda$ gives a perfect pairing of $\mathbb{Z}_p$-modules.

2. A $\mathbb{Q}$-PEL datum for moduli of abelian schemes (at $p$) is the datum $\mathcal{D}_{mod} := (B^*, V, \langle, \rangle, O_B, \Lambda, h, K^p, \nu)$ where $(B^*, V, \langle, \rangle, O_B, \Lambda)$ is a PEL datum with integral structure at $p$; $K^p \subset G(A^p_f)$ is an open compact subgroup of $G(A^p_f)$; $\nu: \mathbb{Q} \to \mathbb{Q}_p$ is an embedding of fields; $h: \mathbb{C} \to \text{End}_B V \otimes \mathbb{Q} \mathbb{R}$ is an $\mathbb{R}$-algebra homomorphism such that:

(a) $h(z) = h(z)^*$ for all $z$ in $\mathbb{C}$ (i.e. $h$ is a $^*$-homomorphism);

(b) the symmetric $\mathbb{R}$-bilinear form $(,): V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{R}$ defined by $\langle v, w \rangle := \langle v, h(\sqrt{-1})w \rangle$ is positive definite.

Such a map $h$ is called a polarization for the PEL-datum $(B^*, V, \langle, \rangle)$.

Each of the above PEL-data over $\mathbb{Q}$ will be called simple if $B$ is a simple $\mathbb{Q}$-algebra. The PEL-data $\mathcal{D}_O$ or $\mathcal{D}_{mod}$ will be said to have good reduction at $p$ if the algebra $B \otimes \mathbb{Q}_p$ is unramified and, in case $\text{End}_B V \otimes \mathbb{Q} \mathbb{R}$ has a factor isomorphic to $M_n(\mathbb{H})$ for some $n > 0$, then $p$ is odd (here $\mathbb{H}$ denotes the division algebra of real quaternions). (Cf. [Wed99], 1.4).

Let $\mathcal{D} = (B^*, V, \langle, \rangle)$ be a PEL-datum endowed with a polarization $h: \mathbb{C} \to \text{End}_B V \otimes \mathbb{Q} \mathbb{R}$; denote by $G$ the associated algebraic group. Denote tensoring (or extension of scalars) with a subscript. Since $h$ is a $^*$-homomorphism, $h(z) \in G(\mathbb{R})$ for any $z \in \mathbb{C}^\times$. Define the map:

$$h_C : \mathbb{C} \times \mathbb{C} \to \text{End}_{B_\mathbb{R}} V_\mathbb{R} \otimes \mathbb{R} \mathbb{C}$$

$$(z_1, z_2) \mapsto 1 \otimes \frac{z_1 + z_2}{2} + h(\sqrt{-1}) \otimes \frac{z_1 - z_2}{2\sqrt{-1}}.$$ 

If we view $\mathbb{C} \times \mathbb{C}$ as a $\mathbb{C}$-algebra via the diagonal embedding, $h_C$ is a $\mathbb{C}$-algebra map such that $h_C(z_1, z_2) \in G(\mathbb{C})$ if $z_1, z_2 \in \mathbb{C}^\times$. Let $S := \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m/\mathbb{C})$ with the usual identifications $S(\mathbb{R}) = \mathbb{C}^\times$ and $S(\mathbb{C}) = (\mathbb{C} \otimes \mathbb{R} \mathbb{C})^\times \simeq \mathbb{C}^\times \times \mathbb{C}^\times$,
where $C$ is embedded into $C \otimes \mathbb{R}$ by $z \mapsto z \otimes 1$. By the above considerations, a morphism of real algebraic groups $h : S \to G_\mathbb{R}$ remains defined by setting:

$$h(C) := h |_{C \times \mathbb{R}}; \quad h(\mathbb{R}) := h |_{\mathbb{R}}.$$ 

for if $z \in C^\times$ we have $h(C)(z, \overline{z}) = (\text{Re } z \cdot 1 + \text{Im } z \cdot h(\sqrt{-1})) \otimes 1$ that we identify with $h(\mathbb{R})(z)$. We denote the above morphism of algebraic groups by $h$ and no confusion should arise.

By definition, our original map $h : C \to \text{End}_F V \otimes \mathbb{R}$ endows $V_F$ with a complex structure, hence it defined a Hodge structure of type $(0, -1), (-1, 0)$ on the vector space $V' = V^{(0, -1)} \oplus V^{(-1, 0)}$, where:

$$V^{(0, -1)} = \{ v \in V_C : h(C)(z_1, z_2)v = z_2v, z_1, z_2 \in C^\times \}, \quad V^{(-1, 0)} = \overline{V^{(0, -1)}}.$$ 

Observe that $C \subset C \otimes \mathbb{R}$ acts on $V^{(0, -1)}$ via the character $z \mapsto \overline{z}$, while it acts on $V^{(-1, 0)}$ via the character $z \mapsto z$.

Now let $\mu : \mathbb{G}_{m/C} \to G_C$ be the map induced by the assignment $z \mapsto h(C)(z, 1)$ for $z \in C^\times$; $V^{(0, -1)}$ is the subspace of $V_C$ on which $\mathbb{G}_{m/C}$ acts through the identity character (weight zero), and $V^{(-1, 0)}$ is the subspace of $V_C$ on which $\mathbb{G}_{m/C}$ acts through the trivial character (weight one), so that we will write the decomposition $V_C = V^{(0, -1)} \oplus V^{(-1, 0)}$ as $V_C = V_{C,0} \oplus V_{C,1}$.

Both $V_{C,0}$ and $V_{C,1}$ have an action of $B_C$, so that we obtain a semisimple complex representation of the $\mathbb{Q}$-algebras $B$:

$$\rho : B \to \text{End}_C V_{C,0}.$$ 

The reflex field (or global Shimura field) of the PEL-datum $D = (B, V, \langle , \rangle)$ endowed with a polarization $h$ is the field of definition of $E = E(D, h)$ of the isomorphism class of the complex representation $\rho : B \to \text{End}_C V_{C,0}$. If $\nu : \mathbb{C} \to \mathbb{C}_p$ is an embedding of fields, the $\nu$-adic completion of $E$ will be denote by $E_\nu$ and called the $\nu$-adic Shimura field associated to our set of data.

Fix embeddings $\mathbb{C} \to \mathbb{C}$ and $\nu : \mathbb{C} \to \mathbb{C}_p$; keeping the above assumptions, we have that $E = \mathbb{Q}(\text{Tr}(\rho(b)) : b \in B)$ is a number field; we can characterize $E$ as the field of definition of the $G^0(\mathbb{C})$-conjugacy class of the co-character $\mu : \mathbb{G}_{m/C} \to G_C$. On the other side, the field of definition of the $G^0(\mathbb{Q}_p)$-conjugacy class of $\mu$ coincide with $E_\nu$ and is a finite extension of $\mathbb{Q}_p$.

Notice that the decomposition $V_C = V_{C,0} \oplus V_{C,1}$ induced by the polarization $h$ is defined over a finite extension $K'$ of $\mathbb{Q}_p$, so that we can write $V_{K'} = V_{K',0} \oplus V_{K',1}$, where $V_{K',0}$ is the $K'$-subspace of $V_{K'}$ on which $\mathbb{G}_{m/K'}$ acts trivially, and similarly for $V_{K',1}$.

We have in conclusion:

**Lemma 3.2** Let $D = (B, V, \langle , \rangle)$ be a $\mathbb{Q}$-PEL datum with associated group $G$. Define $B_p := B \otimes \mathbb{Q}_p$, $V_p := V \otimes \mathbb{Q}_p$, $\langle , \rangle_p := \langle , \rangle \otimes \mathbb{Q}_p$, $G_p = G_{\mathbb{Q}_p}$.

1. The tuple $\mathcal{D}_p = (B_p, V_p, \langle , \rangle_p)$ is a $\mathbb{Q}_p$-PEL datum with associated group $G_p$. If $\mathcal{D}_\mathbb{O} := (B, V, \langle , \rangle, O_B, \Lambda)$ is a $\mathbb{Q}$-PEL datum with integral structure at $p$, and $O_{B_p} := O_B \otimes \mathbb{Z}_p$, then $\mathcal{D}_p := (B_p, V_p, \langle , \rangle_p, O_{B_p}, \Lambda)$ is a $\mathbb{Q}_p$-PEL datum with integral structure.

2. If $\mathcal{D}_\text{mod} := (B, V, \langle , \rangle, O_B, \Lambda, h, K^p, \nu)$ is a $\mathbb{Q}$-PEL datum for moduli of abelian schemes at $p$, then the $p$-adic field of definition of the $G^0(\mathbb{Q}_p)$-conjugacy class of the associated cocharacter $\mu : \mathbb{G}_{m/C} \to G_C$ coincide with the $\nu$-adic completion $E_\nu$ of the Shimura field $E$ of $\mathcal{D}_\text{mod}$; furthermore there is a finite field extension $K'$ of $\mathbb{Q}_p$ such that the weight decomposition $V_{C} = V_{C,0} \oplus V_{C,1}$ under the action of $\mu$ is defined over $K'$.

If any of the above global PEL-data is simple, then the corresponding local PEL-datum is simple; if $\mathcal{D}_\mathbb{O}$ has good reduction at $p$, then $\mathcal{D}_p$ has good reduction.

Let $D = (B, V, \langle , \rangle)$ be a global PEL-datum with polarization $h$ cocharacter $\mu : \mathbb{G}_{m/C} \to G_C$ as before. If $K'$ is a finite field extension of $\mathbb{Q}_p$ such that $V_{K'} = V_{K',0} \oplus V_{K',1}$ is the weight decomposition of $V_{K'}$ under $\mu$, then we can define a morphism of schemes over $O_{K'}$ as in [2.2.2]

$$\det_{K'}(\langle , \rangle, V_{K',0}) : O_{K'} \to \mathbb{A}^1_{O_{K'}}.$$ 

This morphism is actually defined over the ring $O_E \otimes \mathbb{Z}_p$, embedded in $O_{E_\nu}$ via $\nu$. 

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3.1.2 Data of type A and C

Fix a \( \mathbb{Q} \)-PEL datum \( D_{\text{mod}} = (B^*, V, \langle, \rangle, \mathcal{O}_B, A, h, K^p, \nu) \) for moduli of abelian schemes (at \( p \)); let \( G \) be the corresponding algebraic group and \( E \) the Shimura field. Assume that \( D_{\text{mod}} \) is simple, so that \( G \simeq M_{N}(D) \) for some skew field \( D \) of characteristic zero. Let \( F \) be the center of \( B \), and \( F_0 \) be the subfield of \( F \) fixed by the involution \( * \); \( F \) is a number field endowed with a positive involution, so that \( F_0 \) is totally real and either \( F = F_0 \) (and \( * \) is called an involution of the first kind), or \( F/F_0 \) is a quadratic totally complex extension (and \( * \) is said to be of the second kind). Then there is a reductive algebraic group \( G_0 \) over \( F_0 \) such that \( G_1 = \text{Res}_{F_0/\mathbb{Q}} G_0 \), namely

\[
G_0(R) = \{ x \in \text{End}_B V \otimes_{F_0} R : xx^* = 1 \} \quad \text{for any } F_0\text{-algebra } R. \]

Set \( m := \lfloor F : F_0 \rfloor \cdot (\dim_{F}(\text{End}_B V))^1/2 \).

Since \( \text{End}_B V \otimes_{\mathbb{Q}} \mathbb{R} \) is endowed with a complex structure given by \( h \), we have \( m = 2n \) for some integer \( n \). We have three possible situations:

(A) if \( * \) is of the second kind, then \( G_0 \) is an inner form of the quasi-split unitary group over \( F_0 \) associated to the quadratic imaginary extension \( F/F_0 \). Over an algebraically closure of \( F_0 \), the group \( G_0 \) is of type \( A_{n-1} \); in this case \( \text{End}_B V \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_n(\mathbb{C})^{[F_0: \mathbb{Q}]} \);

(C) if \( * \) is of the first kind and \( \text{End}_B V \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_{2n}(\mathbb{R})^{[F_0: \mathbb{Q}]} \), then over an algebraically closure of \( F_0 \), the group \( G_0 \) is a symplectic group in \( 2n \) variables: it is of type \( C_n \);

(D) if \( * \) is of the first kind and \( \text{End}_B V \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_{n}(\mathbb{H})^{[F_0: \mathbb{Q}]} \), then over an algebraically closure of \( F_0 \), the group \( G_0 \) is an orthogonal group in \( 2n \) variables: it is of type \( D_n \).

As remarked in [Kot92], 7, if the PEL-datum falls into cases A or C, the associated reductive group \( G \) is connected, while in case D it has \( 2^{[F_0: \mathbb{Q}]} > 1 \) connected components. Furthermore, in case A and C, the derived subgroup \( G' \) is simply-connected. If \( G \) is of type A with \( n \) even (notation as above) or of type C, then \( G \) satisfies the Hasse principle, while in case D, the group \( G \) does not satisfy the Hasse principle. Notice finally that, in case D, in order to guarantee good reduction of our PEL-datum, we need to exclude the prime \( p = 2 \). For these reasons we won’t consider PEL-data of type D.

It is enough for us to consider simple PEL-data as above, in which \( B \) is a division \( \mathbb{Q} \)-algebra, endowed with the positive involution \( * \). Define \( d^2 := [B : F] \), \( e := [F : \mathbb{Q}] \), \( e_0 := [F_0 : \mathbb{Q}] \), where as above \( F = \mathbb{Z}(B) \) and \( F_0 = F^{* = \text{id}} \).

By Albert’s classification of division algebras with positive involutions, we only have four possibilities for \( (B, *) \) (cf. [Mum74], 21, Th. 2), that reduce, if we exclude the case of PEL-data of type D, to the following three:

(C-I) \( B = F = F_0 \) is a totally real number field, with the trivial involution \( * = \text{id}_B \);

(C-II) \( F = F_0 \) is a totally real number field, and \( B \) is a quaternion division algebra over \( F \) such that \( B \otimes_{F, e} \mathbb{R} \simeq M_2(\mathbb{R}) \) for any real embedding \( e : F \hookrightarrow \mathbb{R} \); the involution \( * \) on such a \( B \) is given by conjugating the natural involution \( x \mapsto Tr_B/F(x) - x \) of \( B \) by some element \( a \in B^* \) such that \( a^2 \in F \) is totally negative in \( F \); any such map is a positive involution on \( B \). In this case we can choose an isomorphism \( B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_2(\mathbb{R})^{e_0} \) carrying the involution \( * \) into the involution \( (X_1, ..., X_{e_0}) \mapsto (X_1^t, ..., X_{e_0}^t) \);

(A) \( F \) is a totally imaginary quadratic extension of the totally real field \( F_0 \), with complex conjugation \( c \) and, for any finite place \( v \) of \( F \) we have \( inv_v B = 0 \) if \( v = cv \), and \( inv_v B + inv_{cv} B = 0 \) otherwise; there is also a positive involution \( t \) on \( B \) and an isomorphism \( B \otimes_{\mathbb{Q}} \mathbb{R} \simeq M_d(\mathbb{C})^{e_0} \) which carries \( t \) into the involution \( (X_1, ..., X_{e_0}) \mapsto (\overline{X}_1, ..., \overline{X}_{e_0}) \). Given such an involution \( t \), any other positive involution is obtained by conjugating it through an element \( a \in D \) such that \( a = a^t \) and such that the image of \( a \) via the above isomorphism is of the form \( (A_1, ..., A_{e_0}) \) where each matrix \( A_i \) is Hermitian and positive definite.

Let \( 2g = \dim_{\mathbb{Q}} V \) and define, for each of the above cases, the integer \( r \) as follows (cf. [Yu02], 2):

\[
r := \begin{cases} 
\frac{d}{e_0} & \text{(Type C-I)} \\
\frac{d}{e_0} & \text{(Type C-II)} \\
\frac{d}{e_0} & \text{(Type A)}
\end{cases}
\]

Being \( B \) a simple algebra over \( \mathbb{Q} \), \( B_{\mathbb{C}} := B \otimes_{\mathbb{Q}} \mathbb{C} \) is a semisimple algebra over \( \mathbb{C} \) and \( V_{\mathbb{C}, 0} \) becomes a finite dimensional \( B_{\mathbb{C}} \)-module, so that it decomposes as the direct sum of irreducible \( B_{\mathbb{C}} \)-modules (the irreducible \( B_{\mathbb{C}} \)-modules are in one to one correspondence with the irreducible modules of each simple constituent of \( B_{\mathbb{C}} \)).
Using this fact, one sees (cf. [Shi66], 5) that in cases C-I and C-II (and also in case D), the representation \( \rho : B \to \text{End}_\mathbb{C} V_{\mathbb{C},0} \) is uniquely determined and has to be a multiple of a reduced \( \mathbb{C} \)-representation of \( B \) over \( \mathbb{Q} \), so that the reflex field is \( E = \mathbb{Q} \). In case A, the situation is less rigid: we know that \( F \) is a quadratic imaginary extension of the totally real field \( F_0 \); let \( \{\tau_i\}_{i=1}^n \) be a CM-type of \( F \), and let \( n_i \) (resp. \( \pi_i \)) be the multiplicity of the standard representation of \( B \otimes F, \mathbb{C} \) (resp. \( B \otimes F, \mathbb{C} \)) in \( V_{\mathbb{C},0} \), for \( i = 1, \ldots, e_0 \) (notice that the standard representations of \( B \otimes F, \mathbb{C} \) and of \( B \otimes F, \mathbb{C} \) have dimension \( q \) over \( \mathbb{C} \)). Any fixed collection of pairs \( (n_i, \pi_i) \) such that \( n_i + \pi_i = dr \) \((1 \leq i \leq e_0)\) gives rise in the obvious way to a unique finite dimensional complex representation of \( B ; \rho \) has to be isomorphic to a representation of this form.

We now present two cases that will be interesting for us: the first is of type C-I, the third is of type A. We will always assume fixed a choice of square root of \(-1\) in \( \mathbb{C} \), denoted by \( \sqrt{-1} \).

**PEL-datum of type C-I**  Let \( B = F \) be a totally real, finite, Galois extension of \( \mathbb{Q} \) of degree \( f \). Denote by \( \{\tau_1, \ldots, \tau_f\} \) the distinct embeddings of \( F \) into \( \mathbb{R} \) and endow \( F \) with the identity involution. Assume that the fixed prime \( p \) is inert in \( F/\mathbb{Q} \), so that \( F \otimes \mathbb{Q}_p \) is the unramified extension of \( \mathbb{Q}_p \) of degree \( f \) in a fixed algebraic closure of \( \mathbb{Q}_p \). Let \( V = F^{2g} \) for a fixed integer \( g > 0 \) and denote by \( \langle ., . \rangle \) the map \( V \times V \to \mathbb{Q} \) defined by setting \( \langle v, w \rangle := Tr_{F/\mathbb{Q}} (v^t J_{2g} w) \) for all \( v, w \in V \).

Identifying \( C = \text{End}_F V \) with the matrix algebra \( M_{2g}(F) \), the induced positive involution on \( C \) is given by \( A \mapsto A^* := J_{2g}^{-1} A^t J_{2g} \). We identify the \( \mathbb{R} \)-algebras \( \mathbb{R} \otimes \mathbb{Q} \mathbb{R} \) and \( \mathbb{R}^f \) by the map \( x \otimes r \mapsto (\tau_1(x)r, \ldots, \tau_f(x)r) \); an identification \( C_\mathbb{R} = M_{2g}(F) \otimes \mathbb{Q} \mathbb{R} \cong M_{2g}(\mathbb{R})^\otimes f \) remains therefore defined via the embeddings \( \tau_i \). Notice that the involution * acts componentwise on \( M_{2g}(\mathbb{R})^\otimes f \), i.e. \( (X_1, \ldots, X_f)^* = (J_{2g}^{-1} X_1 J_{2g}, \ldots, J_{2g}^{-1} X_f J_{2g}) \).

Let \( h : \mathbb{C} \to M_{2g}(\mathbb{R})^\otimes f \) be the \( \mathbb{R} \)-algebra map defined by the assignment \( a + b\sqrt{-1} \mapsto (a + J_{2g}^{-1} b)^\otimes f \). The algebraic \( \mathbb{Q} \)-group \( G \) associated to the above data is isomorphic to the reductive connected group \( GSp_{2g}(F)^{\otimes f} \) of symplectic similitudes of \( F \); if \( R \) is a \( \mathbb{Q} \)-algebra

\[ GSp_{2g}(F)(R) = \{ A \in GL_{2g}(F \otimes \mathbb{Q} R) : A^t J_{2g} A = c(A) J_{2g}, c(A) \in R^\times \}. \]

Furthermore, \( G_1 = Sp_{2g}(F) \) and \( G_0 = Sp_{2g}(F)/F \) is a group of type \( C_g \) when viewed over \( F \); \( G \) satisfies the Hasse principle.

By making the identification \( V_{\mathbb{C}} \cong (\mathbb{R}^{2g})^{\otimes f} \otimes \mathbb{C} \), we write, for any \( z_1, z_2 \in \mathbb{C}^f \):

\[ h(\mathbb{C})(z_1, z_2) = (1, \ldots, 1) \otimes \frac{z_1 + z_2}{2} + (J_{2g}^t, \ldots, J_{2g}^t) \otimes \frac{z_1 - z_2}{2\sqrt{-1}} \in M_{2g}(\mathbb{R})^{\otimes f} \otimes \mathbb{C}. \]

If \( \{e_1, \ldots, e_g, f_1, \ldots, f_g\} \) is the standard ordered basis of \( \mathbb{R}^{2g} \), denote by \( \{e_i B_h, f_j B_h : 1 \leq i, j \leq g, 1 \leq h \leq f \} \) the corresponding canonical ordered basis of \( (\mathbb{R}^{2g})^{\otimes f} \), so that for example \( e_i B_h \) is the vector \( (0, \ldots, e_i, \ldots, 0) \in (\mathbb{R}^{2g})^{\otimes f} \) where \( e_i \) appears in position \( h \). We have:

\[ V_{\mathbb{C},0} = \langle e_i B_h \otimes 1 + f_i B_h \otimes \sqrt{-1} : 1 \leq i \leq g, 1 \leq h \leq f \rangle, \]

\[ V_{\mathbb{C},1} = \langle e_i B_h \otimes 1 - f_i B_h \otimes \sqrt{-1} : 1 \leq i \leq g, 1 \leq h \leq f \rangle. \]

If \( x \in F \), then \( \rho(x) \) acts on the vector \( e_i B_h \otimes 1 + f_i B_h \otimes \sqrt{-1} \in V_{\mathbb{C},0} \) as multiplication by \( (\tau_1(x), \ldots, \tau_f(x)) \otimes 1 \in F_\mathbb{R} \otimes \mathbb{R} \mathbb{C} = \mathbb{R}^f \otimes \mathbb{R} \mathbb{C} \), so that:

\[ \rho(x) (e_i B_h \otimes 1 + f_i B_h \otimes \sqrt{-1}) = \tau_h(x) (e_i B_h \otimes 1 + f_i B_h \otimes \sqrt{-1}), \]

for all \( 1 \leq i \leq g \) and \( 1 \leq h \leq f \). Referring this linear transformation the ordered basis of \( V_{\mathbb{C},0} \) given by:

\[ e_1 B_1 \otimes 1 + f_1 B_1 \otimes \sqrt{-1}, \ldots, e_1 B_f \otimes 1 + f_1 B_f \otimes \sqrt{-1}, \]

\[ e_2 B_1 \otimes 1 + f_2 B_1 \otimes \sqrt{-1}, \ldots, e_2 B_f \otimes 1 + f_2 B_f \otimes \sqrt{-1}, \ldots, \]

we obtain the matrix form of the representation \( \rho : F \to M_{gf}(\mathbb{C}) \):\n
\[ \rho : x \mapsto \text{diag}(\tau_1(x), \tau_2(x), \ldots, \tau_f(x))^{\otimes g} \quad (x \in F); \]
ρ is the g-fold multiple of the reduced representation of \( F \) over \( Q \), and the reflex field for our PEL-datum is \( E = Q \). Observe that \( \det(\rho)(x) = (Nm_{F/Q}(x))^g \). If \( \{b_1, ..., b_f\} \) is a \( \mathbb{Z}(p) \)-basis of the free \( \mathbb{Z}(p) \)-module \( \mathcal{O}_B := \mathcal{O}_F \otimes \mathbb{Z}(p) \simeq \mathcal{O}_{F,(p)} \), then the determinant polynomial function is:

\[
 f(X_1, ..., X_f) := \prod_{i=1}^{f} (\tau_i(b_i)X_1 + ... + \tau_f(b_f)X_f)^g = (Nm_{F/Q}(b_1X_1 + ... + b_fX_f))^g. 
\]

Notice that this polynomial belongs to \( \mathbb{Z}(p)[X_1, ..., X_f] \). If we set \( \Lambda := (\mathcal{O}_B \otimes \mathbb{Z}(p))^{2g} \) and pick any compact open subgroup \( K^p \) of \( G(\mathbb{Z}^p) \) and any choice of embedding \( \nu : \mathbb{Q} \hookrightarrow \mathbb{Q}_p \), we have all the information necessary to define a simple \( Q \)-PEL datum with good reduction at \( p \). We denote this datum by \( \mathcal{D}^{Sp(F)} \) and notice that \( G \) is defined over \( Z \), by setting for any \( Z \)-algebra \( R \): \( G(R) = \{A \in GL_{2g}(\mathcal{O}_F \otimes Z) : A^tJ_{2g}A = c(A)J_{2g}, c(A) \in R^\times \} \).

**PEL-datum of type \( A \)**

Let \( B = k \) be a quadratic imaginary field, say \( k = Q(\sqrt{\alpha}) \), where \( \alpha \) is a negative square-free integer; fix an embedding \( \tau : k \hookrightarrow C \) such that \( \tau(\sqrt{\alpha}) = \sqrt{-1}\sqrt{-\alpha} \); via this embedding we make the identification \( k \otimes_Q \mathbb{R} \simeq C \). Assume that the fixed prime \( p \) is inert in the extension \( k/Q \), so that \( k \otimes_Q \mathbb{Q}_p = \mathbb{Q}_p(\sqrt{\alpha}) \) is the quadratic unramified extension of \( \mathbb{Q}_p \) in a fixed algebraic closure of \( \mathbb{Q}_p \).

Let \( x \mapsto j \ (x \in k) \) denote the non-trivial field automorphism of \( k \): it is a positive involution on \( k \). Set \( V = k^g \) for a positive even integer \( g = 2n \); fix two non-negative integers \( r \) and \( s \) whose sum is \( g \) and let:

\[
 H := H_{r,s} := \begin{pmatrix} \sqrt{\alpha}l_r & 0_{r,s} \\ 0_{s,r} & \sqrt{-\alpha}l_s \end{pmatrix}. 
\]

Let us denote by \( \langle, \rangle : V \times V \rightarrow Q \) the map defined by setting \( \langle v, w \rangle := Tr_{k/Q}(\overline{\tau}Hw) \) for every \( v, w \in V \). Notice that \( \langle, \rangle \) is a \( \mathbb{Q} \)-bilinear non-degenerate skew-Hermitian pairing. Letting \( C = \text{End}_k V = M_g(k) \), we find that the involution induced by \( * \) on \( C \) is \( A \mapsto A^* := H^{-1}AH \), where \( A \) is the matrix obtained from \( A \) by applying the field automorphism \( * \) to the entries.

Let \( h : C \rightarrow C \otimes Q R = M_g(k) \otimes Q R \) be the \( R \)-algebra homomorphism defined by:

\[
 a + b\sqrt{-1} \mapsto 1 \otimes a - H \otimes \frac{b}{\sqrt{-\alpha}}. 
\]

The algebraic group \( G \) associated to the above data is identified with the \( Q \)-group \( GU_g(k; r, s) \): for any \( Q \)-algebra \( R \) we have

\[
 GU_g(k \otimes Q R; r, s) = \left\{ A \in GL_g(k \otimes Q R) : \overline{A}^tHA = c(A)H, c(A) \in R^\times \right\}. 
\]

Furthermore, \( G_1 = U_g(k; H) \) is an inner form of the quasi-split unitary group over \( Q \) associated to the extension \( k/Q \), hence it is a group of type \( A_{r-1} \) when viewed over \( \mathbb{Q} \): being \( g \) even, \( G \) satisfies the weak Hasse principle. We also notice \( G \) is connected.

The \( R \)-algebra \( C_R \) is isomorphic to \( M_g(C) \) via the fixed embedding \( k \hookrightarrow C \), and by making the identification \( V_C = C^g \otimes Q C \) we write for any \( z_1, z_2 \in C^\times \):

\[
 h(C)(z_1, z_2) = 1 \otimes \frac{z_1 + z_2}{2} - \frac{H}{\sqrt{-\alpha}} \otimes \frac{z_1 - z_2}{2\sqrt{-1}} \in G(C). 
\]

If \( \{e_1, ..., e_g, f_1, ..., f_g\} \) denotes the standard ordered basis of \( C^{2g} \), it is easy to see that:

\[
 V_{C,0} = \left\{ \sqrt{-1}e_i \otimes 1 - e_i \otimes \sqrt{-1}, \sqrt{-1}f_j \otimes 1 + f_j \otimes \sqrt{-1} : 1 \leq i \leq r, 1 \leq j \leq s \right\}, \\
 V_{C,1} = \left\{ \sqrt{-1}e_i \otimes 1 + e_i \otimes \sqrt{-1}, \sqrt{-1}f_j \otimes 1 - f_j \otimes \sqrt{-1} : 1 \leq i \leq r, 1 \leq j \leq s \right\},
\]

and the representation \( \rho : B \rightarrow \text{End}_C(V_{C,0}) \simeq M_g(C) \) is obviously induced by the assignment:

\[
 \sqrt{\alpha} \mapsto \begin{pmatrix} \sqrt{-1}l_r & 0_{r,s} \\ 0_{s,r} & \sqrt{-\alpha}l_s \end{pmatrix}. 
\]

The reflex field for our PEL-datum is \( E = Q \) if \( r = s (= n) \), and it is \( E = k \) otherwise. The multiplicity of \( \psi \otimes_k \mathbb{C} \simeq \mathbb{C} \) in \( \rho \) is equal to \( s \) and the multiplicity of \( \psi \otimes_k \mathbb{C} \) is \( r \). Moreover:

\[
 f(X_1, X_2) := \det(X_1 + \sqrt{\alpha}X_2; V_{C,0}) = (X_1 - \sqrt{\alpha}X_2)^t(X_1 + \sqrt{\alpha}X_2)^s \in O_E[X_1, X_2].
\]
Finally, set $\mathcal{O}_B := \mathcal{O}_k \otimes \mathbb{Z}(p) \simeq \mathcal{O}_{k,(p)}$, $\Lambda := (\mathcal{O}_B \otimes \mathbb{Z}(p), \mathbb{Z}_p)^{2g}$ (notice that $p$ does not divide $\alpha$ in $\mathbb{Z}$, so that our pairing $(,)\text{ restricts to a perfect pairing on } \Lambda^2$). Then for any compact open subgroup $K^p$ of $G(\mathbb{Z}_p)$ and for any choice of embedding $\nu : \mathbb{Q} \rightarrow \mathbb{Q}_p$ we have all the information necessary to define a simple $\mathbb{Q}$-PEL datum with good reduction at $p$. Denote this datum by $D_{(r,s),p}$ and note that $G$ is defined over $\mathbb{Z}$, by setting for any $\mathbb{Z}$-algebra $R$:

$$G(R) = \left\{ A \in GL_g(\mathcal{O}_k \otimes \mathbb{Z}_p) : \bar{A}HA = c(A)H, c(A) \in R^\times \right\}.$$

### 3.2 The moduli functor for abelian schemes

#### 3.2.1 Abelian schemes up to prime-to-$p$ isogenies

We assume fixed in this paragraph a global PEL-datum for moduli of abelian schemes (at $p$) $D_{\text{mod}} = (B^+, \tau, \langle , \rangle, \mathcal{O}_B, \Lambda, h, K^p, \nu)$; we write $G$ for the associated algebraic group, and $E$ for the reflex field; we furthermore assume that our datum has good reduction at $p$. Let us fix a locally noetherian base scheme $S$; we recall some definitions following [MR96], 6.3 and [Lan08], 1.3.1 and 1.3.2.

The category of abelian $O_B$-schemes over $S$ up to isogeny of order prime to $p$, denoted by $AV_{O_B/S}$ or simply by $AV$, is defined as follows: its objects are pairs $(A, i)$ where $A$ is an abelian scheme over $S$, and $i$ is a homomorphism of $\mathbb{Z}(p)$-algebras $i : \mathcal{O}_B \rightarrow \text{End} A \otimes \mathbb{Z}(p)$. A morphism $f : (A_1, i_1) \rightarrow (A_2, i_2)$ in $AV$ is an element of the group $\text{Hom}_{\mathcal{O}_B}(A_1, A_2) \otimes \mathbb{Z}(p)$, where $\text{Hom}_{\mathcal{O}_B}(A_1, A_2)$ is the module of morphisms of abelian $S$-schemes that respect the action of $\mathcal{O}_B$.

An isogeny $\phi : (A_1, i_1) \rightarrow (A_2, i_2)$ in $AV$ is a quasi-isogeny of abelian $S$-schemes $A_1 \rightarrow A_2$ which is also a morphism of $AV$; its kernel is the kernel of the corresponding isogeny of $p$-divisible groups $A_1(p) \rightarrow A_2(p)$, so that ker $\phi$ is a finite locally free group scheme whose order is locally a power of $p$, and all isogenies have degree a power of $p$ (locally). A quasi-isogeny in $AV$ is a quasi-isogeny of abelian schemes that respects the action of $\mathcal{O}_B$.

If $(A, i)$ is an object of $AV$, then we define an object of $AV$ by $(A, i)^\circ := (\hat{A}, \hat{i})$, where $\hat{A}$ is the dual abelian scheme of $A$, and $\hat{i} : \mathcal{O}_B \rightarrow \text{End} \hat{A} \otimes \mathbb{Z}(p)$ is given by $\hat{i}(b) = i(b^*)^\circ$. If $\phi : (A_1, i_1) \rightarrow (A_2, i_2)$ is an isogeny in $AV$, then the dual quasi-isogeny $\phi^\circ : A_2 \rightarrow A_1$ is an isogeny in $AV$, called the dual isogeny of $\phi$ in $AV$.

A polarization of $(A, i)$ in $AV_{O_B/S}$ is a quasi-isogeny $\lambda : (A, i) \rightarrow (\hat{A}, \hat{i})$ in $AV$ such that there exists a positive integer $n$ for which $n\lambda$ is induced by an ample line bundle on $A$. Such a $\lambda$ is called a principal polarization if furthermore $\lambda$ is an isomorphism in $AV$. A $\mathbb{Q}$-homogeneous (resp. $\mathbb{Z}(p)$-homogeneous) polarization $\lambda : (A, i) \rightarrow (\hat{A}, \hat{i})$ is the set of (locally on $S$) $Q^\times$-multiples (resp. $\mathbb{Z}(p)^\times$-multiples) of a polarization $\lambda$ of $(A, i)$ in $AV$; such a set is called a principal $\mathbb{Q}$-homogeneous (resp. $\mathbb{Z}(p)$-homogeneous) polarization if there is an element $\lambda \in \lambda$ that is a principal polarization in $AV$.

An isogeny of polarized (resp. $\mathbb{Q}$-homogeneously polarized; $\mathbb{Z}(p)$-homogeneously polarized) abelian varieties $\phi : (A_1, i_1; \lambda_1) \rightarrow (A_2, i_2; \lambda_2)$ in $AV$ is an isogeny $\phi : (A_1, i_1) \rightarrow (A_2, i_2)$ in $AV$ such that $\phi \circ \lambda_2 \circ \phi = \lambda_1$ (resp. $\phi \circ \lambda_2 \circ \phi \in \mathbb{Z}(p)^\times$; resp. $\phi \circ \lambda_2 \circ \phi \in \mathbb{Z}(p)^\times$).

Notice that if $\lambda : (A, i) \rightarrow (\hat{A}, \hat{i})$ is a polarization in $AV$, then $i(b^*) = i(b^\dagger)$ for any $b \in \mathcal{O}_B$, where $^\dagger$ denotes the Rosati involution induced by $\lambda$.

An isomorphism $\phi : (A_1, i_1) \rightarrow (A_2, i_2)$ in $AV$ is of the form $f \circ r$ where $f$ is an isogeny of abelian schemes endowed with $\mathcal{O}_B$-action whose degree is prime to $p$, and $r \in \mathbb{Z}(p)^\times$. We can also say that $\phi$ is an isogeny of $AV$ of prime-to-$p$ degree. Furthermore, if $\phi : (A_1, i_1; \lambda_1) \rightarrow (A_2, i_2; \lambda_2)$ is an isomorphism of polarized abelian varieties in $AV$, then $i_2$ and $\lambda_2$ are determined uniquely by $\phi$, $i_1$ and $\lambda_1$. If $\phi : (A_1, i_1) \rightarrow (A_2, i_2)$ is an isomorphism in $AV$ and $\lambda_2 : (A_2, i_2) \rightarrow (\hat{A}_2, \hat{i}_2)$ is a principal polarization in $AV$, then $\phi \circ \lambda_2 \circ \phi$ is a principal polarization of $(A_1, i_1)$ and $\lambda_2^{-1}$ is a principal polarization of $(\hat{A}_2, \hat{i}_2)$.

#### Level structure

Recall that our PEL-datum comes with an open compact subgroup $K^p$ of $G(\mathbb{A}_f^p)$ ($\mathbb{A}_f^p$ denotes the ring of finite adèles over $\mathbb{Q}$ with trivial $p$-component). Let $(A, i; \lambda)$ be a principally polarized abelian scheme in $AV_{O_B/S}$, where the base scheme $S$ is assumed to be a connected locally noetherian scheme over $O_E \otimes \mathbb{Z}(p)$. Let $s$ be a geometric point of $S$ and consider:

$$H_1(A_s, \mathbb{A}_f^p) = \left( \prod_{l \neq p} T_l(A_s) \right) \otimes \mathbb{Q},$$
the Tate $\mathbb{A}_p^g$-module of the abelian variety $A_\lambda$. It is endowed with a continuous action of $\pi_1(S, s)$. The action of $O_B$ on $A$ endows $H_1(A, \mathbb{A}_p^g)$ with a structure of $B$-module, and the principal polarization $\lambda$ of $(A, i)$ induces a canonical skew-symmetric $\mathbb{A}_p^g$-pairing (the Weil pairing):

$$H_1(A_\lambda, \mathbb{A}_p^g) \times H_1(A_\lambda, \mathbb{A}_p^g) \to \mathbb{A}_p^g(1)$$

which is non-degenerate and skew-Hermitian with respect to $\ast$. On the other side, by definition of PEL-datum, $V_{\mathbb{A}_p^g} := V \otimes_\mathbb{Q} \mathbb{A}_p^g$ is endowed with an action of $B$ and a skew-Hermitian (with respect to $\ast$) non-degenerate $\mathbb{A}_p^g$-pairing with values in $\mathbb{A}_p^g$.

A level structure of type $K^p$ on $(A, i; \lambda)$ is the left $K^p$-orbit $\mathfrak{p}$ of an isomorphism $\alpha : H_1(A_\lambda, \mathbb{A}_p^g) \to V_{\mathbb{A}_p^g}$ of skew-Hermitian $B$-modules such that $\mathfrak{p}$ is fixed by $\pi_1(S, s)$. Here by isomorphisms of skew-Hermitian $B$-modules we mean an isomorphism of $B$-modules carrying one alternating form into a $(\mathbb{A}_p^g)^\ast$-multiple of the other.

Assume now that the group $G/\mathbb{Q}$ associated to the PEL-datum $D_{mod}$ has a model $G/\mathbb{Z}$ over $\mathbb{Z}$ (but it does not need to be smooth over $\mathbb{Z}$). Let $N \geq 1$ be an integer not divisible by the prime $p$ and let $(A, i; \lambda)$ be a principally polarized abelian scheme in $AV$. A principal level-$N$ structure on $(A, i; \lambda)$ is a level structure of type:

$$U(N) = \text{Ker} \left( (G/\mathbb{Z}^p) \to G(\mathbb{Z}^p/N\mathbb{Z}^p) = G(\mathbb{Z}/N\mathbb{Z}) \right).$$

Notice that $U(N) = \prod_{p \nmid N} U_1(N)$, where $U_i(N) = G(\mathbb{Z}_i)$ if $i \neq p$ and $i \mid N$, and $U_i(N) = \text{Ker} (G(\mathbb{Z}_i) \to G(\mathbb{Z}_i/\mathbb{Z}_i^\ast \mathbb{Z}_i))$, where $n_i = \text{ord}_i N$. If $K^p$ is a compact open subgroup of $(G/\mathbb{A}_p^g)^\ast$ contained inside $U(N)$ for some $N \geq 3$ not divisible by $p$, then “Serre’s lemma” implies that $K^p$ is neat in the sense of Pink (cf. [Lan08] 1.4.1.9-10).

The determinant condition Let $V_C = V_{C,0} \otimes V_{C,1}$ be the Hodge decomposition of $V_C$ as in $\ref{def:1.1.3}$ where $\mathbb{C}^\times$ act on $V_{C,0}$ via $\mu$ through the trivial character. We have recalled above the morphism of schemes $\text{det}_{K^p}(\cdot, V_{C,0}^{\mathbb{Q}}) : V_{O_K^\mathbb{Q}} \otimes \mathbb{A}_p^g \to \mathbb{A}_p^g$, that is defined over $O_E \otimes \mathbb{Z}_p(\mathfrak{p}) \to O_E$; let us denote this morphism by $\text{det}_{E}(\cdot, V_0)$. If $S$ is a locally noetherian scheme over $O_E \otimes \mathbb{Z}_p$, we can therefore define a morphism of $S$-schemes $\text{det}_{E}(\cdot, V_0) : V_{S} \to \mathbb{A}_p^g(1)$. Similarly (cf. Remark $\ref{rem:2.2.2}$), we have, for any object $(A, i)$ in $AV_{O_B/S}$, a well defined morphism of $S$-schemes $\text{det}_{O_B}(\cdot, \text{Lie} A) : V_{S} \to \mathbb{A}_p^g(1)$.

We say that $(A, i)$ satisfies the Kottwitz determinant condition if for any locally noetherian $S$-scheme $S'$ we have:

$$\text{det}_{O_B}(a, \text{Lie} A_{S'}) = \text{det}_{E}(a, V_0) \quad \text{for all } a \in O_B \otimes O_{S'}.$$

Let us fix a basis $\{b_1, \ldots, b_i\}$ of the $\mathbb{Z}_p(\mathfrak{p})$-free module $O_B$ and let $\{X_1, \ldots, X_i\}$ be indeterminates. Set $f(X_1, \ldots, X_i) := \text{det}(b_1X_1 + \cdots + b_iX_i)$. This is a homogeneous polynomial of degree $\dim C_{\mathbb{Q}}$ in the indeterminates $X_1, \ldots, X_i$ with coefficients in $O_E \otimes \mathbb{Z}_p(\mathfrak{p})$. On the other side, $\text{Lie} A$ is as a locally free $O_S$-module with an action of $O_B$; hence it makes sense to consider the polynomial $g(X_1, \ldots, X_i) := \text{det}(b_1X_1 + \cdots + b_iX_i; \text{Lie} A)$, which is homogeneous of degree $\dim C$ with coefficients in the ring of global sections of $O_S$. Since we are assuming that $S$ is a scheme over $O_E \otimes \mathbb{Z}_p$, the condition $f = g$ makes sense: this is equivalent to the above defined determinant condition.

### 3.2.2 The moduli problem

Following [Kot92], one defines the following moduli problem:

**Definition 3.3** Let $D = (B^\ast, V, (\cdot), O_B, \Lambda, h, K^p, \nu)$ be a $\mathbb{Q}$-PEL-datum with good reduction at $p$, having Shimura field $E$ and associated group $G$. The moduli problem $M := M(D)$ associated to the above data is the contravariant functor from the category $\text{SCH}_{O_E \otimes \mathbb{Z}_p(\mathfrak{p})}$ of locally noetherian schemes over $O_E \otimes \mathbb{Z}_p$ to the category of sets defined as follows: if $S$ is an object of $\text{SCH}_{O_E \otimes \mathbb{Z}_p}$, then $M(S)$ is the set of isomorphism classes of tuples $(A, i, \lambda, \overline{\lambda})$ where:

1. $(A, i)$ is an object in $AV$ satisfying the determinant condition;
2. $\overline{A} : (A, i) \to (\hat{A}, \hat{i})$ is a $\mathbb{Q}$-homogeneous principal polarization in $AV$;
3. $\overline{\lambda}$ is a level structure of type $K^p$ on $(A, i, \overline{\lambda})$.

Here we consider two tuples $(A_1, i_1, \lambda_1, \overline{\lambda}_1)$ and $(A_2, i_2, \lambda_2, \overline{\lambda}_2)$ as above isomorphic if there is an isomorphism $f : (A_1, i_1; \lambda_1) \to (A_2, i_2; \lambda_2)$ of $\mathbb{Q}$-homogeneously principally polarized abelian schemes in $AV$ carrying $\overline{\lambda}_1$ into $\overline{\lambda}_2$ (in the sense that $\alpha_2 \circ H_1(f, K^p) \circ \alpha_1^{-1} \in K^p$ and $c(\alpha_2) \cdot c(\alpha_1)^{-1} \in r \cdot c(K^p)$, where $c$ denotes the similitude factor homomorphism, and $r \in \mathbb{Z}_p^\times$ is such that $r \cdot f \circ \lambda_2 \circ f = \lambda_1$).
Notice that any abelian scheme over some scheme $S$ that satisfies the conditions given above has relative dimension over $S$ equal to $\dim_\mathbb{C} V_{\mathbb{C},0} = \frac{1}{2} \dim_\mathbb{Q} V$, in virtue of the determinant condition. Furthermore, the possible determinant conditions that we can impose are subject to rigid constraints; in particular they are only finitely many.

We have the following result (cf. [Kot92], [Lan08], Ch. 2, 1.4.1.14, 7.2.3.10):

**Theorem 3.4** Let $\mathcal{D}$ be a global PEL-datum with Shimura field $E$; assume $\mathcal{D}$ had good reduction at $p$ and assume $K^p$ is neat. Then the associated functor $\mathcal{M} := \mathcal{M}(\mathcal{D})$ is represented by a quasi-projective smooth separated scheme $\mathcal{S}_{\mathcal{D},K^p}$ over $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}(p)$ which is of finite type.

We will later need the following:

**Theorem 3.5** Let $\mathcal{D}$ be a global PEL-datum with Shimura field $E$ and let $k$ be the algebraic closure of the residue field of $E$ at $p$; assume $\mathcal{D}$ had good reduction at $p$ and $K^p$ is neat; let us denote by $\mathcal{S}_{\mathcal{D},K^p}$ the scheme representing $\mathcal{M}(\mathcal{D})$. The canonical map $\mathcal{S}_{\mathcal{D},K^p}(W(k)) \to \mathcal{S}_{\mathcal{D},K^p}(k)$ is surjective.

The proof of the last result can be found in [Lan08], 2.2.4.16, 2.3.2.1, where it is shown that the deformation functor $\text{ART}(W(k)) \to \text{SETS}$ associated to a fixed point $[\xi] \in \mathcal{S}_{\mathcal{D},K^p}(k)$ is pro-representable and formally smooth, and that one can apply Grothendieck’s Formal Existence Theorem to guarantee that we can algebraize the formal scheme defined on $W(k)$ by a projective system of deformations of $[\xi]$ over the rings $W_n(k)$’s. The main hypothesis necessary to prove the above theorem is that the polarization of $\xi$ is separable.

**Hecke action** Let us only consider open compact subgroups $K^p$ of $G(\mathbb{A}^p)$ that are small enough, so that each element in $\mathcal{M}(S)$ has no non-trivial automorphisms, for any $S$ in $\text{SCH}_{\mathbb{Z} \otimes \mathbb{Z}}$: for example, if $\mathcal{D}$ has a $\mathbb{Z}$-model, we consider only open compact subgroups contained inside $U(N)$ for some integer $N \geq 3$. If $K^p \subseteq K'^p$ are two such open compact subgroups of $G(\mathbb{A}^p)$, then the transition map $\mathcal{S}_{\mathcal{D},K'^p} \to \mathcal{S}_{\mathcal{D},K^p}$ induced by $(A,i,\lambda,K'^p_1) \mapsto (A,i,\lambda,K^p_1)$ is a finite étale covering which is Galois, with Galois group $K'^p_2/K^p_1$, if $K^p_1$ is normal in $K^p_2$. Denote by $\mathcal{D}$ the projective systems of the family of schemes $\{\mathcal{S}_{\mathcal{D},K^p}\}_{K^p}$ where the $K^p$’s are small enough; we define the following natural Hecke action of $G(\mathbb{A}^p)$ on $\mathcal{S}_{\mathcal{D}}$: if $g \in G(\mathbb{A}^p)$, then $g$ acts on the right on $\mathcal{S}_{\mathcal{D}}$ via the isomorphism:

$$g : \mathcal{S}_{\mathcal{D},K^p} \to \mathcal{S}_{\mathcal{D},g^{-1}K^pg}$$

defined by $[(A,i,\lambda,\pi)] \cdot g := [(A,i,\lambda,g^{-1} \circ \alpha)]$.

**3.2.3 Modular forms of PEL-type**

We define modular forms of PEL-type, as a generalization of Siegel modular forms (cf. [Gor02], 5.1). We will keep the notation of the previous sections.

Let $\mathcal{D} = (B,^*,V,\langle \cdot,\cdot \rangle,\mathcal{O}_B,\Lambda,h,K^p,\nu)$ be a simple PEL-datum for moduli of abelian schemes, having good reduction at the fixed prime $p$; let $G$ be the associated algebraic group and assume it has a model over $\mathbb{Z}$. Let $E$ be the reflex field of $\mathcal{D}$ and let $g = \dim_{\mathbb{C}} V_{\mathbb{C},0}$; fix an integer $N \geq 3$ not divisible by $p$ and assume $K^p = U(N)$. Let us denote by $\mathcal{S}_{\mathcal{D},N} := \mathcal{S}_{\mathcal{D},U(N)}$ the quasi-projective smooth scheme over $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}(p)$ representing the functor $\mathcal{M} = \mathcal{M}(\mathcal{D})$. Let $\pi : \mathcal{X}_{\mathcal{D},N} \to \mathcal{S}_{\mathcal{D},N}$ be the corresponding universal abelian scheme over $\mathcal{S}_{\mathcal{D},N}$, and let $0 : \mathcal{S}_{\mathcal{D},N} \to \mathcal{X}_{\mathcal{D},N}$ be its zero section. Denote by $\Omega^1_{\mathcal{X}_{\mathcal{D},N}/\mathcal{S}_{\mathcal{D},N}}$ the sheaf of relative invariant differentials of $\mathcal{X}_{\mathcal{D},N}$ over $\mathcal{S}_{\mathcal{D},N}$: it is a locally free sheaf of $\mathcal{O}_{\mathcal{X}_{\mathcal{D},N}}$-modules over $\mathcal{X}_{\mathcal{D},N}$, having rank $g$. Its pull-back via the zero section 0, i.e. the sheaf of relative cotangent vectors at the origin of $\mathcal{X}_{\mathcal{D},N}$:

$$\mathcal{E} := 0^* \left( \Omega^1_{\mathcal{X}_{\mathcal{D},N}/\mathcal{S}_{\mathcal{D},N}} \right)$$

is the Hodge bundle of the PEL-scheme $\mathcal{S}_{\mathcal{D},N}$; it is a locally free sheaf of $\mathcal{O}_{\mathcal{S}_{\mathcal{D},N}}$-modules over $\mathcal{S}_{\mathcal{D},N}$ and its rank equals $g$. If $\rho : GL_g \to GL_m$ is an $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}(p)$-representation of the algebraic group $GL_g$, we denote by $\mathcal{E}_{\rho}$ the locally free sheaf of rank $m$ on $\mathcal{S}_{\mathcal{D},N}$ obtained by twisting $\mathcal{E}$ via $\rho$ (cf. [Ghi01a], 2.2.1).

**Definition 3.6** Let $\mathcal{S}_{\mathcal{D},N}$ and $\rho$ be as above. For any $\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}(p)$-algebra $\mathfrak{R}$ we define the $\mathfrak{R}$-module:

$$M^\rho_{\mathfrak{R}}(\mathcal{D};\mathfrak{R}) := H^0(\mathcal{S}_{\mathcal{D},N} \otimes_{\mathcal{O}_E \otimes_\mathbb{Z} \mathbb{Z}(p)} \mathfrak{R}; \mathcal{E}_{\rho} \otimes \mathfrak{R}),$$

and we call it the space of PEL-modular forms over $\mathfrak{R}$ of weight $\rho$ relative to the moduli problem $\mathcal{M}(\mathcal{D})$. If no confusion arises we also denote this space as $M^\rho_{\mathfrak{R}}(N;\mathfrak{R})$ and we say that the modular forms in $M^\rho_{\mathfrak{R}}(N;\mathfrak{R})$ have genus $g$ and (full) level $N$. 

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Assume there is an isomorphism above, a PEL-modular forms over $\mathfrak{R}$ be a tuple such that $[\xi]$ is an element of $(S_{D,N} \otimes_{O_{E \otimes_{\mathbb{Z}[\eta]} \mathbb{Z}[p]}} \mathfrak{R})(S)$, for some $\mathfrak{R}$-scheme $S$; let $\phi : S \to S_{D,N} \otimes_{O_{E \otimes_{\mathbb{Z}[\eta]} \mathbb{Z}[p]}} \mathfrak{R}$ be the morphism of $\mathfrak{R}$-schemes parametrizing the element $[\xi]$. We have:

$$f([\xi]) \in \phi^*(\mathbb{E} \otimes \mathfrak{R})_\rho(S) = \phi^*(0^\rho \Omega^1_{X_{D,N} \otimes \mathfrak{R}(S_{D,N} \otimes \mathfrak{R})})_\rho(S) \simeq (\tau_{A/S})_\rho(S),$$

where the last isomorphism depends upon the choice of representative for $[\xi]$. We conclude that for $D, p, N, \rho, \mathfrak{R}$ as above, a PEL-modular forms over $\mathfrak{R}$ of weight $\rho$ relative to the moduli problem $M(D)$ is a rule $f$ that assigns, to any tuple $\xi := (A, i, \lambda, \overline{\lambda})/\mathfrak{R}$ such that $[\xi]$ is an element of $(S_{D,N} \otimes_{O_{E \otimes_{\mathbb{Z}[\eta]} \mathbb{Z}[p]}} \mathfrak{R})(S)$ an element $f(\xi)$ of $(\tau_{A/S})_\rho(S)$ in such a way that the rule $f$ is compatible isomorphisms and commute with base change.

Pick a modular form $f$ described as in the above proposition and let $\xi := (A, i, \lambda, \overline{\lambda})/\mathfrak{R}$ and $\xi' := (A', i', \lambda', \overline{\lambda'})/\mathfrak{R}$ be two isomorphic tuples, say $\varphi : \xi \to \xi'$ is an $S$-isomorphism; let $\varphi^* : (\tau_{A/S})_\rho(S) \to (\tau_{A'/S})_\rho(S)$ be the induced isomorphism on the cotangent spaces. The compatibility of $f$ with respect to isomorphism means that $\varphi^* f(\xi') = f(\xi).$ Let $\xi$ be as above and fix an $S$-scheme $S'$; denote by $\xi : \xi \otimes_S S' \to S'$ the canonical map. The commutativity of $f$ with base extension means that $\varphi^* f(\xi) = f(\xi \otimes S')$ of elements of $(\tau_{A \otimes_S S'/S'})_\rho(S').$

**Modular forms mod $p$**

Let $\mathbb{F}_p$ be a fixed algebraic closure of the field with $p$ elements. If $A/\mathbb{F}_p$ is an abelian variety of dimension $g$, we denote for brevity $\tau_{A/\mathbb{F}_p}$ : it is a vector space over $\mathbb{F}_p$ of dimension $g$.

Let $A$ and $A'$ be two abelian varieties over $\mathbb{F}_p$ of dimension $g$, and let $\rho : GL_g \to GL_m$ as above; let us fix isomorphisms of vector spaces $\gamma : \tau_{A/\mathbb{F}_p} \to \mathbb{F}_p^m$ and $\gamma' : \tau_{A'/\mathbb{F}_p} \to \mathbb{F}_p^m$. If $\rho : A \to A'$ is an isomorphism of abelian varieties over $\mathbb{F}_p$, then $\varphi^* : \tau_{A'/\mathbb{F}_p} \to \tau_{A/\mathbb{F}_p}$ is defined. By definition of the functor $(\cdot)_\rho$, we can make the identifications $$(\tau_{A'/\mathbb{F}_p})_\rho = (\tau_{A/\mathbb{F}_p})_\rho \mathbb{F}_p^m, \text{ so that } \varphi^* = \rho(\gamma' \circ \varphi^* \circ \gamma')^{-1}$$
is a morphism $(\tau_{A'/\mathbb{F}_p})_\rho \to (\tau_{A/\mathbb{F}_p})_\rho$.

Let us compute $H^0((S_{D,N} \otimes \mathbb{F}_p)(\mathbb{F}_p); \mathbb{E}_\rho \otimes \mathbb{F}_p)$. Fix a modular form $f$ and a tuple $\xi := (A, i, \lambda, \overline{\lambda})/\mathbb{F}_p$ such that $[\xi]$ is an element of $(S_{D,N} \otimes \mathbb{F}_p)(\mathbb{F}_p)$; fix also an isomorphism of vector spaces $\gamma : \tau_{A/\mathbb{F}_p} \to \mathbb{F}_p^m$ that will allow us to identify these two spaces in the sequel (the choice of $\gamma$ will not be influential). If $\eta := (\eta_1, \ldots, \eta_g)$ is an ordered basis for $\tau_{A/\mathbb{F}_p}$ over $\mathbb{F}_p$, denote by the same symbol the matrix $\eta = [\eta_1 | \cdots | \eta_g] \in GL_g(\mathbb{F}_p)$ obtained by placing the vectors $\eta_j \in \mathbb{F}_p^m$ as columns. Write $\mu = [\nu_1 | \cdots | \nu_m] := \rho(\eta)$, so that we can find a unique column vector $x \in \mathbb{F}_p^m$ such that $f(\xi) = \sum_{j=1}^m x_j \nu_j = \rho(\eta) \cdot x$.

We can define an assignment $\tilde{f}$ on tuples $(A, i, \lambda, \overline{\lambda})$, where $(A, i, \lambda, \overline{\lambda})/\mathbb{F}_p$ is as above and $\eta$ is an ordered basis for $\tau_{A/\mathbb{F}_p}$ over $\mathbb{F}_p$, by setting:

$$\tilde{f} : (A, i, \lambda, \overline{\lambda}) \to x \iff f(A, i, \lambda, \overline{\lambda}) = \rho(\eta) \cdot x.$$

Notice that if $M \in GL_g(\mathbb{F}_p)$, then:

$$\tilde{f}(A, i, \lambda, \overline{\lambda}, \eta M) = \rho(M)^{-1} \cdot \tilde{f}(A, i, \lambda, \overline{\lambda}).$$

Assume that we are given another tuple $\xi' = (A', i', \lambda', \overline{\lambda'})/\mathbb{F}_p$ such that $[\xi']$ is an element of $(S_{D,N} \otimes \mathbb{F}_p)(\mathbb{F}_p)$; fix an isomorphism of vector spaces $\gamma' : \tau_{A'/\mathbb{F}_p} \to \mathbb{F}_p^m$ and pick an ordered basis $\eta' \in \mathbb{F}_p$ of invariant differentials for $A'$. Assume there is an isomorphism $\varphi : (\xi, \eta) \to (\xi', \eta')$, i.e. an isomorphism of abelian varieties (up to primo-to-$p$ isogeny) with additional structure $\varphi : \xi \to \xi'$ such that $\varphi^* \eta' = \eta$. Notice that by functoriality of the $\rho$-twisting we have $\rho(\eta) = \varphi^* \rho(\eta')$, since $\varphi^* \eta' = \eta$. We compute, using the fact that $f$ is compatible with isomorphisms:

$$\tilde{f}(\xi, \eta) = \rho(\eta')^{-1} \cdot f(\xi') = (\varphi^* \rho(\eta')^{-1} \cdot \varphi^* f(\xi') = \rho(\eta')^{-1} \cdot f(\xi') = f(\xi', \eta').$$
We have shown:

**Proposition 3.7** Let $D, p, N, \rho$ be as above; then a PEL-modular forms over $\mathbb{F}_p$ of weight $\rho$ relative to the moduli problem $M(D)$ is a rule $f$ that assigns, to any tuple $(A, i, \lambda, \alpha, \eta)/F_p$ such that $[(A, i, \lambda, \alpha, \eta)]$ is an element of $(S_{D,N} \otimes \mathbb{F}_p)(F_p)$ and $\eta$ is an ordered basis for $t^*A/F_p$ over $\mathbb{F}_p$, an element $f(A, i, \lambda, \alpha, \eta) \in \mathbb{F}_m$ of in such a way that:

(a) $f(A, i, \lambda, \alpha, \eta)M = \rho(M)^{-1} \cdot f(A, i, \lambda, \alpha, \eta)$ for all $M \in GL_g(F_p)$;

(b) if $(A, i, \lambda, \alpha, \eta) \cong (A', i', \lambda', \alpha', \eta')$ then $f(A, i, \lambda, \alpha, \eta) = f(A', i', \lambda', \alpha', \eta')$.

**Hilbert-Siegel modular forms** Let $N \geq 3$ be an integer prime to $p$, $F$ a totally real Galois extension of $\mathbb{Q}$ having degree $f$ in which $p$ is unramified. We consider the functor $M(D_{Sp}(F))_{2g,p}$ associated to the PEL-datum $D_{Sp}(F)$ with good reduction at $p$ (cf. 3.1.2), with $K_p = U(N)$; in this case the Shimura field is $E = \mathbb{Q}$. Fix a rational $\mathbb{Z}(p)$-representation $\rho : GL_{fg} \to GL_m$ and let $\mathfrak{R}$ be any $\mathbb{Z}(p)$-algebra. The $\mathfrak{R}$-module:

$$M^{fg}_{\rho}(F; N; \mathfrak{R})_{HS} := M_{\rho}(D_{2g,p}^{Sp(F)}; \mathfrak{R})$$

is the space of Hilbert-Siegel $\mathfrak{R}$-modular forms of genus $fg$, level $N$, weight $\rho$ and relative to the field $F$. If $\mathfrak{R} = \mathbb{C}$, the Hermitian symmetric domain associated to the corresponding Shimura variety $S_{D_{2g,p}^{Sp(F)},N}$ is the product $h^f$ of $f$ copies of the genus-$g$ Siegel upper half plane $h_g$. If furthermore $f = 1$, we obtain the classical space of Siegel modular forms of genus $g$. (Cf. [GF90], Ch.V and [Lan08] 1.4.1-3 for the comparison between $M(D_{Sp}(\mathbb{Q}))$ and the classical Mumford’s functor $A_{g,1,N}$).

**Unitary modular forms** We consider the functor $M(D_{U(r,s),p}^U)$ associated to the PEL-datum $D_{U(r,s),p}^U$ with good reduction at $p$ as defined in 3.1.2. Let $N \geq 3$ denotes an integer prime to $p$. Recall that the PEL-datum comes with a quadratic imaginary field $k$ in which $p$ is inert. For any $\mathbb{Z}(p)$-algebra $\mathfrak{R}$, we call

$$M^{(r,s)}_{\rho}(k; N; \mathfrak{R})_{U} := M_{\rho}(D_{U(r,s),p}^U; \mathfrak{R})$$

the space of unitary $\mathfrak{R}$-modular forms of signature $(r, s)$ for the field $k$, having genus $g := r + s$, level $N$ and weight $\rho$. Over the complex numbers, these forms can be constructed analytically starting from the Hermitian domain $h_{r,s} := \{z \in M_{r,s}(\mathbb{C}) : 1 - z^*z \text{ positive Hermitian}\}$ (the Picard space).
4 Uniformization results for the supersingular and the superspecial loci

We recall a uniformization result for isogeny classes in a PEL-moduli space due to Rapoport and Zink ([MR96], Ch. 6); we then present a modification of this result that allows us to parametrize the superspecial locus.

4.1 The result of Rapoport and Zink

We fix some notation. Let $D = (B_\nu^*, V, \langle, \rangle, O_B, \Lambda, h, K^p, \nu)$ be a simple $\mathbb{Q}$-PEL-datum for moduli of abelian schemes with good reduction at $p$, and neat level $K^p$. Let $G$ the associated reductive group over $\mathbb{Q}$, and $E$ the Shimura field. The completion $E_\nu$ of $E$ at $\nu$ coincide with the field of definition of the $G^0(\mathbb{Q}_p)$-conjugacy class of $\mu$. Let $k = \mathbb{F}_p$ be a fixed algebraic closure of the residue field of $E_\nu$, and let $W = W(\mathbb{F}_p)$, $K_0 = W[\frac{1}{p}]$ and $\sigma$ the Frobenius morphism of $W$; fix a finite extension $K$ of $K_0$ such that $\mu$ is defined over $K$, so that the corresponding weight decomposition $V_K = V_{K,0} \oplus V_{K,1}$ is also defined over $K$. Set $\hat{E} = E_\nu K_0$ as in [2222] since we are in the good reduction case, we have $\hat{E} = K_0$. Set $B_p := B \otimes \mathbb{Q}_p$, $V_p := V \otimes \mathbb{Q}_p$, $\langle, \rangle_p := \langle, \rangle \otimes \mathbb{Q}_p$, $G_p = G_{\mathbb{Q}_p}$, $O_{B_p} := O_B \otimes \mathbb{Z}_p$.

4.1.1 From global PEL-data to local PEL-data

Let $S_{D, K^p}$ be the quasi-projective smooth scheme over $O_E \otimes \mathbb{Z}(p)$ representing the functor $M = M(D)$ of Theorem 3.3 and let us fix a point $[(A_0, i_0, \lambda_0, \sigma_0)] \in S_{D, K^p}(\mathbb{F}_p)$, where $\lambda_0$ denotes a principal polarization of $(A_0, i_0)$. Correspondingly we have a $p$-divisible group $X := A_0(p)$ over $\mathbb{F}_p$, endowed with the action $i_X : O_{B_p} \rightarrow \text{End} X$ of $O_{B_p}$ induced by $i_0$; the principal polarization $\lambda_0$ induces a principal polarization $\lambda_X : A_0(p) \rightarrow \tilde{A}_0(p) = A_0(p)$ of $p$-divisible groups (cf. [Oda69], 1.8, 1.12) respecting the $O_{B_p}$-action; $\lambda_X$ is well defined up to a constant in $\mathbb{Q}_p^\times$. The triple $(X, i_X, \tilde{\lambda_X})$ is well defined modulo isomorphisms by the given point $[(A_0, i_0, \lambda_0, \sigma_0)] \in S_{D, K^p}(\mathbb{F}_p)$.

By covariant Dieudonné theory, we can associate to $(X, i_X, \tilde{\lambda_X})$ the isocrystal $(N := M_*(X)[\frac{1}{p}], F)$ over $K_0$ endowed with an action of $B_p$ and with a non-degenerate bilinear form of isocrystals $\Psi : N \times N \rightarrow 1(1)$ (well defined up to an element of $\mathbb{Q}_p^\times$) that is skew-Hermitian with respect to $\ast$. The quasi-isogeny class of the principally polarized $B_p$-isocrystal $(N, F, \mathbb{Q}_p^\times \Psi)$ depends upon the isomorphism class of $(X, i_X, \tilde{\lambda_X})$.

We fix an isomorphism of $B \otimes \mathbb{Q}_p$-modules $N \simeq V \otimes \mathbb{Q}_p$ that respects the skew-symmetric forms on both sides. We then write the action of Frobenius on the right hand side as $F = b \otimes \sigma$ for a unique element $b \in G_p(K_0)$. By construction we have, for any $x, y \in V \otimes \mathbb{Q}_p$:

$$c(b) \langle x, y \rangle = \langle b \otimes \sigma \cdot x, b \otimes \sigma \cdot y \rangle^\sigma = p \langle x, y \rangle,$$

so that $c(b) = p$. The isocrystal $V \otimes \mathbb{Q}_p$ has slopes in the interval $[0, 1]$, and in the decomposition of the $K$-vector space $V \otimes K$ under the co-character $\mu$ only the weights 0 and 1 appear. Finally, the pair $(b, \mu)$ is admissible in the sense of Definition 2.1.3. For, since $\lambda_0$ is a separable character, the point $[(A_0, i_0, \lambda_0, \sigma_0)] \in S_{D, K^p}(\mathbb{F}_p)$ is liftable to a point $[(\tilde{A}_0, \tilde{i}_0, \tilde{\lambda}_0, \tilde{\sigma}_0)] \in S_{D, K^p}(W)$, by Theorem 3.3 this implies that the $p$-divisible group $(X, i_X, \tilde{\lambda_X})$ over $\mathbb{F}_p$ can be correspondingly lifted to a $p$-divisible group $(\tilde{X}, i_{\tilde{X}}, \tilde{\lambda}_{\tilde{X}})$ over $W$, so that the $K$-filtered isocrystal over $K_0$ given by $(V \otimes K_0, b \otimes \sigma, \{V_{K,1} \subset V_K\})$ is associated to a $p$-divisible group over $W \subset O_K$. By the considerations we made in 2.2.1 this implies that $(b, \mu)$ is admissible, since we are in the good reduction case.

We conclude that the choice of a global PEL-datum $D$ plus a fixed point $[(A_0, i_0, \lambda_0, \sigma_0)] \in S_{D, K^p}(\mathbb{F}_p)$ determines a simple $\mathbb{Q}_p$-PEL datum for moduli of $p$-divisible groups:

$$D_p := (B_p^*, , V_p, \langle, \rangle_p, O_{B_p}, \Lambda, b, \mu),$$

in the sense of Definition 2.7, having good reduction at $p$ and Shimura field equal to $E_\nu$. Denote by $\hat{M}$ the contravariant functor from $NILP_{\mathbb{Q}_p}$ to $SETS$ defined in 2.8 starting from the $\mathbb{Q}_p$-homogeneously principally polarized $p$-divisible group $(X, i_X, \tilde{\lambda_X})$ endowed with the action of $O_{B_p}$: $\hat{M}$ is representable by a formal scheme $\tilde{M}$ which is formally locally of finite type over $\text{Spf} O_{\mathbb{Q}_p}$. Furthermore $E_\nu = K_0$ and the $\tilde{M}$ is formally smooth over $\text{Spf} W$. 

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The groups $I$ and $J$ Denote by $J(Q_p)$ the group of automorphisms of the polarized $B \otimes Q K_0$-isocrystal $(V \otimes Q K_0; b \otimes \sigma)$: this means that $J(Q_p)$ is the group of $K_0$-automorphisms of the isocrystal $(V \otimes Q K_0; b \otimes \sigma)$ equivariant for the action of $B_p$ and preserving the polarization form induced by $\langle , \rangle$ up to a non-zero scalar in $Q_p$. $J(Q_p)$ is the group of $Q_p$-rational points of an algebraic group $J$ defined over $Q_p$. By covariant Dieudonné theory, $J(Q_p)$ is isomorphic to the group $\hat{J}(Q_p)$ of quasi-isogenies $f : (X, i_X) \rightarrow (X, i_X)$ of $p$-divisible groups over $\mathbb{F}_p$ such that $\hat{f} \circ \lambda_X \circ f \in Q_p^* \lambda_X$. The isomorphism $J(Q_p) \rightarrow \hat{J}(Q_p)$ depends upon the choice of isomorphism of $B \otimes Q K_0$-modules $N \simeq V \otimes Q K_0$ that respects the skew-symmetric forms on both sides (recall that $N$ is the isocrystal associated to $X = A_0(p)$)), hence it depends upon the choice of $b \in G_p(K_0)$. We have fixed such an isomorphism, so that we think $J(Q_p) = \hat{J}(Q_p)$. The isomorphism class of $J(Q_p)$ only depends upon the quasi-isogeny class of $(X, i_X, \lambda_X)$.

The group $J(Q_p)$ acts on $\mathcal{M}$ from the left by the rule:

$$g \cdot [(X, i, \lambda; \rho)] := [(X, i, \lambda; \rho \circ g^{-1})],$$

where $[(X, i, \lambda; \rho)] \in \mathcal{M}(S)$ for some scheme $S$ in $\text{NILP}_{\mathbb{F}_p}$. Denote by $I(Q)$ the group of quasi-isogenies of $(A_0, i_0, \lambda_0)$, i.e. the group of quasi-isogenies of $\mathbb{F}_p$-abelian variety $(A_0, i_0) \rightarrow (A_0, i_0)$ that send $\lambda_0$ into itself: $I(Q)$ is the group of rational points of an algebraic group $I$ defined over $Q$ and the isomorphism class of $I$ only depends upon the quasi-isogeny class of $(A_0, i_0, \lambda_0)$.

The group $I(Q)$ acts by quasi-isogenies on the tuple $(X, i_X, \lambda_X)$, hence on $(\mathbb{N}, f)$, $\mathbb{Q}_p^\times \Psi)$, defining a morphism $I(Q) \rightarrow J(Q_p)$ factoring through $I(Q) \rightarrow I(Q_p)$. Such a monomorphism depends upon the choice of an isomorphism $J'(Q_p) \simeq J(Q_p)$. Since we fixed an isomorphism of isocrystal (with additional structure) $N \simeq V \otimes Q K_0$ at the beginning of this section, we denote by $\alpha_{0,p} : I(Q) \rightarrow J(Q_p)$ the corresponding map.

Notice that $I(Q)$ acts by skew-Hermitian symplectic $B$-equivariant similitudes on the space $H_1(A_0, \Lambda_f^p)$, that we identified with $V \otimes Q \Lambda_f^p$ through the isomorphism $\alpha_0$, and hence we also have a homomorphism $\alpha_0^p : I(Q) \rightarrow G(\Lambda_f^p)$ depending of the choice of a representative $\alpha_0$ for the class $\overline{\alpha_0}$ (hence well defined modulo $K^p$).

In conclusion, once $b$ and $\alpha_0$ are fixed, we have well defined group homomorphisms $\alpha_{0,p} : I(Q) \rightarrow J(Q_p)$ and $\alpha_0^p : I(Q) \rightarrow G(\Lambda_f^p)$. We can make the identification:

$$J(Q_p) = \{ g \in G(K_0) : g \cdot b \sigma = b \sigma \cdot g, c(g) \in Q_p^* \}. $$

4.1.2 The uniformization morphism

The formal scheme $\mathcal{M}$ is not defined, in general, over $\mathcal{O}_{E_\nu}$, i.e. over the (ring of integer of the) local Shimura field. There is a suitable completion of $\mathcal{M}$ that can be written as $\mathcal{M} \otimes \text{Spf}(\mathcal{O}_{E_\nu}) \text{Spf}(\mathcal{O}_{E_\nu}^\circ)$ for a pro-formal scheme $\mathcal{M}$ over $S_{\text{Spf}}(\mathcal{O}_{E_\nu})$. The uniformization result of Rapoport and Zink that we recall here can be stated in terms of $\mathcal{M}$.

To this purpose, let $s \in \mathbb{Z}$ such that $D_{K_0} \overset{\nu}{\rightarrow} D_{K_0} \overset{\nu}{\rightarrow} G_{K_0}$ factors, via the canonical projection $D_{K_0} \rightarrow \mathbb{G}_{m/K_0}$ induced by $\mathbb{Z} \subset Q$, through a map $s v : \mathbb{G}_{m/K_0} \rightarrow G_{K_0}$. Since $\mathbb{F}_p$ is algebraically closed, and if we assume $G$ to be connected, we can assume that $b$ satisfies the decency (equiv) $G(\mathbb{G}_{m/K_0}) = (\langle s v \rangle(p)) \sigma^s$ in $G(K_0) \times \{ \sigma \}$, and that $b \in G(Q_{p,s})$ (MR965, 1.8.9). Let $\gamma_s := p^s \cdot (sv(p))^{-1} \in G(K_0)$; since $\gamma_s \cdot b \sigma = b \sigma \cdot \gamma_s$ and $c(\gamma_s) \in Q_p^*$, we have $\gamma_s \in J(Q_p)$. By what we saw above, $\gamma_s$ acts upon the functor $\mathcal{M}$, and we define $\mathcal{M}_s$ to be the sheaf associated to the functor:

$$S \longmapsto \mathcal{M}(S)/\gamma_s^p,$$

($S \in \text{NilP}_{\mathcal{O}_{E_\nu}}$). By MR965, 3.42, $\mathcal{M}_s$ is represented by a formal scheme locally of finite type over $\text{Spf}(\mathcal{O}_{E_\nu})$; notice that $J(Q_p)$ continues to act on $\mathcal{M}_s$.

The formal schemes $\mathcal{M}$ and $\mathcal{M}_s$ have a natural Weyl descent datum relative to the extension $\mathcal{O}_{E_\nu}/\mathcal{O}_{E_\nu}$ (cf. MR965, 3.45–46); this descent datum is effective on each $\mathcal{M}_s$ for $s$ as above. Then the projective limit of the $\mathcal{M}_s$ is a completion of $\mathcal{M}$ that comes as base-change from a pro-formal scheme $\mathcal{M}$ defined over $\text{Spf}(\mathcal{O}_{E_\nu})$:

$$\mathcal{M} = \lim_{\leftarrow} \mathcal{M}_s / \text{Spf}(\mathcal{O}_{E_\nu}).$$

The action of $J(Q_p)$ on $\mathcal{M}$ commutes with the descent datum, and then gives an action of $J(Q_p)$ on $\mathcal{M}$.

From now on, we shall assume that the group $G$ is connected and satisfies the Hasse principle, unless otherwise stated; this is because under this assumption, the uniformization result has an easier formulation. Recall that we fixed $[(A_0, i_0, \lambda_0, \overline{\rho})] \in \mathcal{S}_{D,K^r}(\mathbb{F}_p)$; we also keep the notation introduced at the beginning of the section.
Definition 4.1 We let \( Z([[(A_0, i_0, \lambda_0, \pi_0)]) \subseteq S_{D, K^p}(\mathbb{F}_p) \) to be the set of points \( [(A, i, \lambda, \pi)] \in S_{D, K^p}(\mathbb{F}_p) \) such that there exists an isogeny \( (A_0, i_0) \to (A, i) \) in \( AV \) sending the \( \mathbb{Q} \)-homogeneous polarization \( \lambda_0 \) into \( \lambda \) (no condition on the level structure).

Notice that \( Z([[(A_0, i_0, \lambda_0, \pi_0)]) \) is also the set of points \( [(A, i, \lambda, \pi)] \in S_{D, K^p}(\mathbb{F}_p) \) whose underlying abelian scheme \( (A, i, \lambda) \) is quasi-isogenous to \( (A_0, i_0, \lambda_0) \). By a result of Rapoport and Richartz (cf. [MR96a], [MR96b] 6.26-27), one knows:

**Theorem 4.2 (Rapoport, Richartz)** If the \( p \)-divisible group \( X \) of \( (A_0, i_0, \lambda_0) \) is basic, then \( Z([[(A_0, i_0, \lambda_0, \pi_0)]) \) is the set of \( \mathbb{F}_p \)-valued points of a closed subset \( Z \subseteq S_{D, K^p} \otimes \mathbb{F}_p \).

We can now state:

**Theorem 4.3 (Rapoport, Zink)** Let us fix \( [\lambda(A_0, K)] \subseteq S_{D, K^p}(\mathbb{F}_p) \) such that the \( p \)-divisible group \( X \) of \( (A_0, i_0, \lambda_0) \) is basic; denote by \( Z \subseteq S_{D, K^p} \) the closed subspace whose \( \mathbb{F}_p \)-points are the points of \( Z([[(A_0, i_0, \lambda_0, \pi_0)]) \subseteq S_{D, K^p}(\mathbb{F}_p) \). Let \( \bar{S}_{D, C^p, Z} \) be the formal completion of the scheme \( S_{D, K^p} \) along \( Z \). If the algebraic group \( G \) associated to the PEL-datum \( D \) is connected and satisfies the Hasse principle, then there is a canonical isomorphism of formal schemes over \( \text{Spf} \mathcal{O}_{E^p} : \)

\[ \partial_{K^p} : I(\mathbb{Q}) \backslash M \times G(A^p) / K^p \to \bar{S}_{D, C^p, Z} , \]

where \( I(\mathbb{Q}) \) acts on \( M \) via \( \alpha_{o,p} \), and on \( G(A^p) \) via \( \alpha_{G} \). The system of morphisms \( \{ \partial_{K^p} \}_{K^p} \) is equivariant with respect to the right Hecke \( G(A^p) \)-action on the projective systems of both sides above. Furthermore, \( I \) is an inner form of \( G \), and we have canonical identifications \( I(A^p) = I(K^p) / I(\mathbb{Q}_p) = I(\mathbb{Q}_p) / I(\mathbb{R}) \) is compact modulo its center.

Recall that the right action of \( G(A^p) \) on the projective system \( \{ \bar{S}_{D, K^p / Z} \}_{K^p} \) is defined at 3.22. In a similar way, notice that if \( K_p^q \subseteq K_p^r \) are open compact subgroups of \( G(A^p) \), we have a transition map \( I(\mathbb{Q}) \backslash M \times G(A^p) / K^p \to I(\mathbb{Q}) \backslash M \times G(A^p) / K^r \); we obtain correspondingly a projective system of formal schemes. If \( g \in G(A^p) \), we define a morphism:

\[ g : I(\mathbb{Q}) \backslash M \times G(A^p) / K^p \to I(\mathbb{Q}) \backslash M \times G(A^p) / g^{-1} K^p g \]

induced by the map \( x \cdot K^p \mapsto g^{-1} x g \cdot g^{-1} K^p g \) (\( x \in G(A^p) \)). The equivariance of the isomorphisms \( \partial_{K^p} \)'s stated in the above theorem is with respect to these two right actions of \( G(A^p) \) (that we will call the action of the Hecke operators of \( G(A^p) \)).

The statement about \( I(A^p) \) and \( I(\mathbb{Q}_p) \) in the above theorem is a consequence of the basicity of \( X \) (cf. [MR96a], 6.29).

To prove Theorem 4.3 Rapoport and Zink first define a morphism of functors over \( \text{NILP}_{\mathcal{O}_{E^p}} : \)

\[ \Theta_{K^p} : I(\mathbb{Q}) \backslash M \times G(A^p) / K^p \to \bar{S}_{D, K^p} \otimes \mathcal{O}_{E^p} \mathcal{O}_{\mathbb{F}_p} \]

and then they show the crucial:

**Proposition 4.4** Assume that \( G \) is connected and satisfies the Hasse principle; the morphism of functors \( \Theta_{K^p} \) defines a canonical isomorphism:

\[ \Theta_{K^p}(\mathbb{F}_p) : I(\mathbb{Q}) \backslash M(\mathbb{F}_p) \times G(A^p) / K^p \to Z(\mathbb{F}_p) \]

Both the domain and the codomain of \( \Theta_{K^p} \) have a Weyl descent datum to \( \mathcal{O}_{E^p} \); one can prove that \( \Theta_{K^p} \) is compatible with such a descent, and then deduce Theorem 4.3 from the above Proposition. The morphism \( \Theta_{K^p} \) is called the uniformization morphism in [MR96a], 6.15.

(In some special situations, \( Z \) coincides with the whole special fiber of the scheme \( S_{D, K^p} \) over \( \mathbb{Z}_p \); in this cases, the result of Rapoport and Zink gives a \( p \)-adic uniformization of the whole Shimura variety at \( p \). This happens, for example, in the case considered by Cherednik in [Che76], where \( B \) is a rational quaternion algebra unramified at infinity, \( G = B^x \) and \( p \) a prime at which \( B \) is ramified).
4.1.3 Description of the uniformization morphism over $\overline{p}$

We keep the assumptions of the previous sections. In particular, we have fixed an element $[(A_0/i_0, \overline{X}, \overline{x_0})] \in \mathcal{S}_{\mathcal{D}, K^p}(\overline{p})$ that gives rise to the $p$-divisible group $(X, i_X, \overline{X})$ over $\overline{p}$, and then to the isocrystalline $((N, F), i, Q_p^{\times} \Psi)$ over $K_0$; we also have fixed an isomorphism $N \simeq V \otimes K_0$ of $B \otimes K_0$-modules that respects the skew-symmetric forms on both sides, and such that $F = b \otimes \sigma$. We still assume that $X$ is basic and that $G$ is connected and satisfies the Hasse principle.

We recall the definition of the uniformization morphism of functors over $NILP_{\mathcal{O}_{E_v}}$:

$$\Theta_{K^p} : I(\mathbb{Q}) \backslash \mathcal{M}(S) \times G(\mathbb{A}_f)/K^p \rightarrow \mathcal{S}_{\mathcal{D}, K^p} \otimes \mathcal{O}_{E_v} \mathcal{O}_{E_v}.$$  

that appears in Proposition 4.4. We need:

**Lemma 4.5** Let $S \in NILP_{\mathcal{O}_{E_v}}$, and let $A'$ be an object in $AV_{\mathcal{O}_B/S}$; denote by $X'$ the $p$-divisible group of $A'$. For any quasi-isogeny $\xi : X' \rightarrow X''$ of $p$-divisible groups over $S$ that respects the $\mathcal{O}_{B_v}$-action, there exists an element $\xi''$ of $AV_{\mathcal{O}_B/S}$ whose $p$-divisible group is $X''$ and a quasi-isogeny $\xi : A' \rightarrow A''$ of $AV_{\mathcal{O}_B/S}$ inducing $\xi : X' \rightarrow X''$. Furthermore the arrow $\xi : A' \rightarrow A''$ in $AV_{\mathcal{O}_B/S}$ is uniquely determined; we denote $A''$ by $\xi(A')$. This construction is functorial, i.e. $(\xi_2 \xi_1)_* : A'' \rightarrow A''$.

Under the hypothesis of the above Lemma, if $A'$ comes with a polarization $\lambda : A' \rightarrow \hat{A}'$, then $\xi$ defines a polarization $\xi \ast \lambda := (\xi^{-1})^* \lambda \xi^{-1}$ on $A''$. If furthermore $A'$ comes with a rigidification $\alpha : H_1(A', \mathbb{A}^\flat_f) \rightarrow V \otimes K^p$ (i.e. a symplectic $\mathcal{O}$-equivariant isomorphism), then $A''$ comes with the rigidification $\xi \ast \lambda$.

Let $S$ be a fixed scheme in $NILP_{\mathcal{O}_{E_v}}$; we denote by $(\overline{X}, \overline{i}_X, \overline{\lambda}_X)$ a fixed lifting of $(X, i_X, \overline{X})$ to $\mathcal{O}_{E_v}$, and we let $(\overline{A}_0, \overline{i}_0, \overline{\lambda}_0, \overline{\lambda}_0)$ be the corresponding lifting of $(A_0, i_0, \overline{X}, \overline{\lambda}_0)$ over $\mathcal{O}_{E_v}$ (cf. Prop. 3.5). We can consider base changes of these objects to $S$ and to $\overline{\mathcal{O}}$ (which is the closed subscheme of $\mathcal{S}$ defined by the ideal sheaf $p\mathcal{O}_S$); we denote it by the subscripts $\overline{-}$ and $\overline{-}$ respectively.

Consider a $p$-divisible group with additional structure $[(X, i, \overline{\lambda})] \in \mathcal{M}(S)$; $\rho : (X, i_X) \rightarrow (X, i)$ is an $\overline{\mathcal{O}}$-quasi-isogeny of $p$-divisible groups with $\mathcal{O}_{B_v}$-action such that $\overline{\rho} \circ \lambda_X \circ \rho \in Q_p^{\times} \overline{\lambda}_X$; by Prop. 2.1 $\rho$ lifts uniquely to a quasi-isogeny $\overline{\rho} : (\overline{X}, \overline{i}_X) \rightarrow (X, i)$ of $p$-divisible groups over $S$ with an action of $\mathcal{O}_{B_v}$. By the above lemma, we obtain therefore an abelian scheme $\overline{\rho}_*: (\overline{A}_0/S)$ over $S$ endowed with an action $\overline{\rho}_*(\overline{\lambda}_0)$ of $\mathcal{O}_B$, a polarization $\overline{\rho}_*(\overline{\lambda}_0)$ and a level structure $\overline{\rho}_*(\overline{\lambda}_0)$, such that:

$$[(\overline{\rho}_*(\overline{A}_0/S), \overline{\rho}_*(\overline{i}_0), \overline{\rho}_*(\overline{\lambda}_0)) \in \mathcal{S}_{\mathcal{D}, K^p} \otimes \mathcal{O}_{E_v} \mathcal{O}_{E_v}(S).$$

We define a morphism of functors $\Theta_{K^p}$ over $NILP_{\mathcal{O}_{E_v}}$ by letting, for any $S \in \text{Obj}(NILP_{\mathcal{O}_{E_v}})$:

$$\Theta_{K^p}(S) : \mathcal{M}(S) \times G(\mathbb{A}^\flat_f)/K^p \rightarrow \mathcal{S}_{\mathcal{D}, K^p}(S),$$

$$[(X, i, \overline{\lambda})] \times gK^p \mapsto [(\overline{\rho}_*(\overline{A}_0/S), \overline{\rho}_*(\overline{i}_0), \overline{\rho}_*(\overline{\lambda}_0), g^{-1} \cdot \overline{\rho}_*(\overline{\lambda}_0))].$$

We need to explain better the meaning of the notation $g^{-1} \cdot \overline{\rho}_*(\overline{\lambda}_0)$. More properly, we assumed fixed a representative $\alpha_0$ for $\overline{\lambda}_0$, and $g^{-1} \cdot \overline{\rho}_*(\overline{\lambda}_0)$ denotes the $K^p$-class of the isomorphism $g^{-1} \circ \alpha_0 \circ H_1(\rho^{-1})$; this class does not depend on the choice of representative for the coset $gK^p$. In the sequel, we will use without any further explanation the notation just introduced.

We shall use the shorter notation:

$$\Theta_{K^p}(S) : [(X, i, \overline{\lambda})] \times gK^p \mapsto [(\overline{\rho}_*(\overline{A}_0/S, \overline{i}_0), g^{-1} \cdot \overline{\rho}_*(\overline{\lambda}_0))]$$

Notice that $\Theta_{K^p}(S)$ is well defined (all the choices we made above give the same tuples, up to isomorphism). Furthermore, $\{\Theta_{K^p}\}_{K^p}$ is equivariant with respect to the right $G(\mathbb{A}^\flat_f)$-action on the projective systems of both sides above.
We now define a left action of $I(\mathbb{Q})$ on the left hand side. Fix $S \in \text{Obj}(\text{NILP}_{\mathbb{Q}_\ell})$, $[(X, i, \lambda; \rho)] \in \hat{M}(S)$, $g \in G(\mathbb{A}_p)$ and $\lambda \in I(\mathbb{Q})$; we set:

$$\xi : \left( [(X, i, \lambda; \rho)] \times gK^p \right) := [(X, i, \lambda; \rho \circ \alpha_{0,p}(\xi^{-1}))] \times \alpha_{0,p}(\xi)gK^p,$$

where $\alpha_{0,p} : I(\mathbb{Q}) \to J(\mathbb{Q}_p)$ and $\alpha_{0}^\prime : I(\mathbb{Q}) \to G(\mathbb{A}_p^c)$ are the homomorphisms defined in [4.1]. It is easy to check that $\Theta_{K^p}$ is invariant under the left action of $I(\mathbb{Q})$ just defined: in fact since $\alpha_{0}^\prime(\xi^{-1}) : (X, i, \lambda, \lambda X) \to (X, i, \lambda, \lambda X)$ is a quasi-isogeny coming from $\xi^{-1} : (A_0, i_0, \lambda_0) \to (A_0, i_0, \lambda_0)$, we have canonical identifications

$$\alpha_{0}^\prime(\xi^{-1}) : (A_0, i_0, \lambda_0) = (A_0, i_0, \lambda_0),$$

$$(\rho \circ \alpha_{0,p}(\xi^{-1}))^\prime : (A_0, i_0, \lambda_0) = \tilde{\rho}_s(A_0, \tilde{i}_0, \tilde{\lambda}_0).$$

Under these identifications, the level structures associated to $\xi : \left( [(X, i, \lambda; \rho)] \times gK^p \right)$ and $[(X, i, \lambda; \rho)] \times gK^p$ coincide.

We call the uniformization morphism

$$\Theta_{K^p} : I(\mathbb{Q})\backslash \hat{M} \times G(\mathbb{A}_p^c)/C^p \to \mathcal{S}_{D, K^p} \otimes \mathcal{O}_{E_\iota} \mathcal{O}_{E_\iota}$$

the map induced by $\Theta_{K^p}$ defined above modulo the left action of $I(\mathbb{Q})$ on $\hat{M} \times G(\mathbb{A}_p^c)/C^p$.

Let us now consider the map on geometric points $\Theta_{K^p}(\mathbb{F}_p)$: it is clear that its image coincide with $Z(\mathbb{F}_p)$. Furthermore $\Theta_{K^p}(\mathbb{F}_p)$ is injective: assume that we start with two points $[(X_j, i_j, \lambda_j; \rho_j)] \in \hat{M}(\mathbb{F}_p)$ such that $f \circ \rho_1 = f \circ \rho_2$. Then, $X_j \subseteq i_j \lambda_j / p \lambda_j$. We say that $A$ is a bijection in the first case, otherwise $A$ is said to be supersingular. In higher dimension the situation is richer.

An abelian variety $A$ over $\mathbb{F}_p$ is said to be supersingular if $A(p)$ is isogenous to $G^{\phi}_{1/2}$ (over $\mathbb{F}_p$); it is said to be superspecial if $A(p)$ is isomorphic to $G^{\phi}_{1/2}$ (over $\mathbb{F}_p$). An abelian variety $A'$ over a finite extension of $\mathbb{F}_p$ is said to be supersingular (resp. superspecial) if its base change to $\mathbb{F}_p$ is supersingular (resp. superspecial).

If $g = 1$, there is no difference between supersingular and superspecial abelian varieties. We recall some properties of supersingular elliptic curves (cf.: [Sil86], Ch. V; [Tat66]; [Ghi03], 2.2.).

An elliptic curve $E$ defined over $\mathbb{F}_p$ is supersingular if and only if its endomorphism ring $\text{End} E := \text{End}_{\mathbb{Q}_p} E$ is isomorphic to a maximal order in the quaternion $\mathbb{Q}$-algebra ramified exactly at the places $\{p, \infty\}$; if $E/\mathbb{F}_p$ is supersingular, then there exists a unique (up to $\mathbb{F}_p^2$-isomorphism) elliptic curve $E'$ defined over $\mathbb{F}_p$, such that $E = E' \otimes_{\mathbb{F}_p^2} \mathbb{F}_p$ and such that the geometric Frobenius $E' \to E'(p) = E'$ equals $[p]$. Furthermore, $\text{End}_{\mathbb{Q}_p} E' = \text{End}_{\mathbb{Q}_p} E$ and the association $E \mapsto E'$ is functorial. The cotangent space $\omega(E)$ of $E$ has a canonical $\mathbb{Q}_p^2$-structure.

If $E$ is a supersingular elliptic curve over $\mathbb{F}_p$ and $E'$ is its canonical model over $\mathbb{F}_p^2$, then $E'(p)$ is a canonical model of $E(p)$ whose covariant Diedonné module $A'_{1/2} := M_*(E'(p))$ (over $W(\mathbb{F}_p^2)$) is isomorphic to:

$$\left( W(\mathbb{F}_p^2), F = \begin{pmatrix} 0 & 1 \\ -p & 0 \end{pmatrix}, \sigma, V = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix} \sigma^{-1} \right).$$
where $\sigma$ denotes the Frobenius morphism of $W(\mathbb{F}_p)$. Furthermore, $A_{1/2} := M_*(E(p)) \simeq \frac{W(\mathbb{F}_p)[E,V]}{W(\mathbb{F}_p)[F,V][F-V]}$ (cf. Th. 2.3), and the ring of Dieudonné module endomorphisms $\text{End} A_{1/2} = \text{End} A_{1/2}'$ is isomorphic to the maximal order in the quaternion division algebra over $\mathbb{Q}_p$ (cf. [Ghi04a], Corollary 7).

There are finitely many isomorphism classes of supersingular elliptic curves over $\mathbb{F}_p$, and they all are isogenous. It is a result of Deligne and Ogus that if $g \geq 2$ and $E_1, \ldots, E_{2g}$ are supersingular elliptic curves over $\mathbb{F}_p$, then we have an isomorphism over $\mathbb{F}_p$: $E \times \cdots \times E_g \simeq E_{g+1} \times \cdots \times E_{2g}$. For references on the proof of this and the following results, cf. [KZL98], 1.6.:

Proposition 4.6 Let $A$ be an abelian variety over $\mathbb{F}_p$ of dimension $g \geq 2$. Then the following are equivalent:

1. $A$ is supersingular;
2. all $2g$ slopes of the Newton polygon of $A$ are equal to $1/2$;
3. $A$ is isogenous over $\mathbb{F}_p$ to $E^g$ for some (any) supersingular elliptic curve $E$ over $\mathbb{F}_p$.

Furthermore, the following are equivalent:

1. $A$ is superspecial;
2. $M_{*}(A(p)) \simeq A_{1/2}^g$ as Dieduonné $W(\mathbb{F}_p)$-modules;
3. $A$ is isomorphic over $\mathbb{F}_p$ to $E^g$ for some (any) supersingular elliptic curve $E$ over $\mathbb{F}_p$.

If $A$ is a superspecial abelian variety over $\mathbb{F}_p$ of dimension $g \geq 2$, then we see that $A$ has a canonical model $A'$ over $\mathbb{F}_{p^*}$, in which the geometric Frobenius equals $[-p]$. Furthermore, the association $A \mapsto A'$ is functorial. If $A = E^g$ is a superspecial abelian variety over $\mathbb{F}_p$ of dimension $g \geq 1$, then $A$ comes with a canonical principal polarization $A \to A'$ induced from the canonical polarization of the elliptic curve $E$. For this reason, in this case we will identify $A$ and $A'$. As a consequence, $M(A(p))$ and $M_{*}(A(p))$ are canonically isomorphic as Dieduonné modules.

4.2.2 A variant of the moduli problem for $p$-divisible groups

Let us fix integers $g \geq 1$, $N \geq 3$ and a prime number $p$ not dividing $N$; let us denote by $A_{g,1,N}$ the Siegel moduli scheme associated to the PEL-datum $\mathcal{D}_{2g,p}$ having good reduction at $p$, and to the choice of principal level $U(N)$ (cf. [KZL98]. If $g = 1$, $A_{1,1,N}(\mathbb{F}_p)$ contains a finite number of supersingular elliptic curves, which form an isogeny class; if $g > 1$, the supersingular abelian varieties living in $A_{g,1,N}(\mathbb{F}_p)$ define a closed subset of positive dimension (cf. [KZL98], 4.9), hence "too big" for our purposes. For this reason, in [Ghi04a], the author needs to consider superspecial abelian varieties, instead of supersingular varieties, in order to construct a map from geometric to algebraic eigenforms of Siegel type. The above situation also occurs in other cases of PEL-type.

Fix a prime number $p$ and denote by $W = W(\mathbb{F}_p)$ the ring of Witt vectors of $\mathbb{F}_p$ and by $K_0$ its fraction field, endowed with the Frobenius automorphism $\sigma$. We assume fixed a simple $\mathbb{Q}_p$-PEL-datum with good reduction at $p$ for moduli of $p$-divisible groups over $\mathbb{F}_p$: $\mathcal{D}_{\text{mod}} = (B_p, V_p, \zeta), \mathcal{O}_{B_p}, A, b, \mu)$. We denote by $K$ the finite extension of $K_0$ over which the co-character $\mu$ is defined, by $E_p$ the Shimura field of the datum; we let $\tilde{E}_p := E_pK_0$.

The above data define an isocrystal $(N, F) := (V_p \otimes \mathbb{Q}_p K_0, \sigma)$ endowed with an action $i$ of $B_p$ and a skew-hermitian non-degenerate form of isocrystals $\Psi : N \times N \to \mathfrak{1}(1)$. By our assumptions, this isocrystal comes - via the covariant Dieudonné functor - from some $p$-divisible group over $\mathbb{F}_p$ - endowed with action of $B_p$ and polarization - that is uniquely determined only up to isogeny. We fix a choice of isomorphism class $(X, i_X, \overline{\lambda}_X)$ of polarized $p$-divisible group over $\mathbb{F}_p$ (with Shimura field $E_p$) associated to $(N, i, \mathbb{Q}_p^e \Psi)$. We furthermore assume that $\lambda_X$ is a principal polarization and that $i_X$ comes from an action of $\mathcal{O}_{B_p}$ on $X$ (this will always be automatically true in our applications).

Definition 4.7 Let us fix $(X, i_X, \overline{\lambda}_X)$ as above. The set:

$$\mathcal{M}'(\mathbb{F}_p) := \mathcal{M}'(X, i_X, \overline{\lambda}_X)(\mathbb{F}_p)$$
is the collection of equivalence classes of quasi-isogenies $\rho : (X, i_X, \overline{\lambda}_X) \to (X, i_X, \overline{\lambda}_X)$ of the $p$-divisible group $X$ over $\overline{\mathbb{F}}_p$ that respect the $\mathcal{O}_{B_p}$-structure and such that $\overline{\rho} \circ \lambda_X \circ \rho \in \mathbb{Q}_p^\times \lambda_X$. Two quasi-isogenies $\rho$ and $\rho'$ are said to be equivalent if the $\overline{\mathbb{F}}_p$-quasi-isogeny $f := \rho' \circ \rho^{-1}$ is an isomorphism $(X, i_X, \overline{\lambda}_X) \to (X, i_X, \overline{\lambda}_X)$ of $p$-divisible groups over $\overline{\mathbb{F}}_p$ with $\mathcal{O}_{B_p}$-action, such that $f \circ \lambda_X \circ f \in \mathbb{Z}_p^\times \lambda_X$.

The set $\mathcal{M}'(\overline{\mathbb{F}}_p)$ is non-empty since $[(X, i_X, \overline{\lambda}_X)] \in \mathcal{M}'(\overline{\mathbb{F}}_p)$. Furthermore $\mathcal{M}'(\overline{\mathbb{F}}_p) \subseteq \hat{\mathcal{M}}(\overline{\mathbb{F}}_p)$ is closed. Notice also that in the definition of $\mathcal{M}'(\overline{\mathbb{F}}_p)$ we can forget about the determinant condition that appears in the definition of $\mathcal{M}$, since it is automatically satisfied.

Let $J(\mathbb{Q}_p)$ denote the group of quasi-isogenies $\rho : (X, i_X) \to (X, i_X)$ over $\overline{\mathbb{F}}_p$ such that $\overline{\rho} \circ \lambda_X \circ \rho \in \mathbb{Q}_p^\times \lambda_X$, as in 4.1.1 and let $J(\mathbb{Z}_p)$ be the subgroup of isomorphisms $(X, i_X) \to (X, i_X)$ preserving the polarization form up to a factor in $\mathbb{Z}_p^\times$. Although the space $\tilde{\mathcal{M}}(\overline{\mathbb{F}}_p)$ is somehow mysterious, we have a better understanding of $\mathcal{M}'(\overline{\mathbb{F}}_p)$:

**Proposition 4.8** There is a natural bijection $\mathcal{M}'(\overline{\mathbb{F}}_p) \simeq J(\mathbb{Q}_p)/J(\mathbb{Z}_p)$.

**Proof.** Clear from the definition of $\mathcal{M}'(\overline{\mathbb{F}}_p)$: the bijection associates to $[\rho] \in \mathcal{M}'(\overline{\mathbb{F}}_p)$ the left coset $\rho^{-1}J(\mathbb{Z}_p)$, where we view $\rho^{-1}$ as an element of $J(\mathbb{Q}_p)$. ■

### 4.2.3 Uniformization of the superspecial locus

We now apply the above results to the situation we are interested in. For convenience we recall some notation already introduced. Fix $D = (\mathbb{B},^*, \mathfrak{V}, \langle , \rangle, \mathcal{O}_B, \Lambda, h, K^\times, \nu)$ a simple $\mathbb{Q}$-PEL-datum for moduli of abelian schemes with good reduction at $p$, and next level $K^\times$. Denote by $G$ the associated algebraic group, and assume it is connected and satisfies the Hasse principle. Let $E$ be the Shimura field of $D$ and $E_{\nu}$ the completion of $E$ at $\nu$. Let $\overline{\mathbb{F}}_p$ be a fixed algebraic closure of the residue field of $E_{\nu}$, and $\nu = \nu(\overline{\mathbb{F}}_p)$, $K_0 = W[\frac{1}{p}]$ and $\sigma$ the Frobenius morphism of $W$; fix a finite extension $K$ of $K_0$ such that $\sigma$ is induced over $K$; set $E_{\nu} = E_{\nu}K_0$. The objects $B_p$, $V_p$, $\langle , \rangle_p$, $G_p$, $\mathcal{O}_{B_p}$ are defined as before.

Let $\mathcal{S}_{D, K^\times}$ be the quasi-projective smooth scheme over $\mathcal{O}_E \otimes \mathbb{Z}_p(\mathfrak{m})$ representing $\mathbf{M}(D)$; we see $\mathcal{S}_{D, K^\times}$ as a scheme over $\mathcal{O}_{E_{\nu}}$. Suppose that the common dimension of the abelian schemes parametrized by $\mathcal{S}_{D, K^\times}$ is $g := \dim_{\mathbb{C}} \mathfrak{V}_{0,0} \geq 2$; fix a supersingular elliptic curve $E_0$ over $\overline{\mathbb{F}}_p$, and denote its canonical model over $\overline{\mathbb{F}}_p$ by $E_0$. Let $A_0 = \mathbb{E}_0^\ell$ be the corresponding superspecial abelian variety over $\overline{\mathbb{F}}_p$, endowed with the identity principal polarization $\lambda_0 := \text{id}^\ell_{E_0}$ (we identify canonically $E_0$ and $\mathbb{E}_0^\ell$).

Assume that the moduli scheme $\mathcal{S}_{D, K^\times}$ contains a point of the form $[(A_0, i_0, \overline{\lambda}_0, \overline{\sigma}_0)] \in \mathcal{S}_{D, K^\times}(\overline{\mathbb{F}}_p)$ that we fix; the $p$-divisible group $X = A_0(p)$ over $\overline{\mathbb{F}}_p$ is isomorphic to $G_{1/2}^{\text{sg}}$ and is endowed with the action $i_X$ of $\mathcal{O}_{B_0}$ induced by $i_0$ and the principal polarization $\lambda_X : A_0(p) \to \mathcal{A}_0(p) \simeq G_{1/2}^{\text{sg}}$. The triple $(X, i_X, \overline{\lambda}_X)$ is well defined modulo isomorphisms by the given point $[(A_0, i_0, \overline{\lambda}_0, \overline{\sigma}_0)] \in \mathcal{S}_{D, K^\times}(\overline{\mathbb{F}}_p)$. As usual, we associate to $(X, i_X, \overline{\lambda}_X)$ the isocrystal $(N := M(X)[1], \overline{\lambda}_X)$ over $K_0$ endowed with an action of $B_0$ and with a non-degenerate bilinear form of isocrystals $\Psi : N \times N \to \mathbb{1}_{(1)}$. We fix an isomorphism of $B \otimes_{\mathbb{Z}_p} K_0$-modules $N \cong \nu \otimes_{\mathbb{Q}} K_0$ that respects the skew-symmetric forms on both sides and we then write the action of Frobenius on the right hand side as $\mathbf{F} = b \otimes \sigma$ for some $b \in G_p(K_0)$. Since $N$ is isoclinic, the slope morphism associated to $G$ and over $K_0$ has image contained inside the center of $G$, so that $b$ is basic in the sense of 2.1.5.

We have in hands also a simple $\mathbb{Q}_p$-PEL datum for moduli of $p$-divisible groups $\mathcal{D}_p := (\mathbb{B}_p,^*, \mathfrak{V}_p, \langle , \rangle_p, \mathcal{O}_B, \Lambda, b, \mu)$, having good reduction at $p$ and Shimura field equal to $E_{\nu}$. The closed subscheme $\mathcal{M}'(\overline{\mathbb{F}}_p) := \mathcal{M}'(X, i_X, \overline{\lambda}_X)(\overline{\mathbb{F}}_p)$ of $\hat{\mathcal{M}}(\overline{\mathbb{F}}_p)$ is then defined and identified with $J(\mathbb{Q}_p)/J(\mathbb{Z}_p)$.

**Definition 4.9** We let $\mathcal{Z}(\overline{\mathbb{F}}_p) := \mathcal{Z}'(\overline{\mathbb{F}}_p) = \mathcal{Z}'([(A_0, i_0, \overline{\lambda}_0, \overline{\sigma}_0)])(\overline{\mathbb{F}}_p) \subseteq \mathcal{S}_{D, K^\times}(\overline{\mathbb{F}}_p)$ be the set of points $[(A, i, \overline{\lambda}, \overline{\sigma})] \in \mathcal{Z}(\overline{\mathbb{F}}_p)$ such that the principally polarized $p$-divisible group $(A(p), i, \overline{\lambda})$ of $(A, i, \overline{\lambda})$ is isomorphic to $(X, i_X, \overline{\lambda}_X)$. We call $\mathcal{Z}(\overline{\mathbb{F}}_p)$ the **superspecial locus** associated to $(X, i_X, \overline{\lambda}_X)$.

The set $\mathcal{Z}(\overline{\mathbb{F}}_p)$ is a closed subset of $\mathcal{Z}(\overline{\mathbb{F}}_p)$; furthermore if the class $[(A, i, \overline{\lambda}, \overline{\sigma})]$ belongs to $\mathcal{Z}(\overline{\mathbb{F}}_p)$, the $p$-divisible group of $A$ is isomorphic to $G_{1/2}^{\text{sg}}$, so that $A \simeq A_0$ is superspecial.

**Remark 4.10** Assume that $D$ is the PEL-datum of type $C$ defined in 2.1.2 with $c = 1$ (so that $G(\mathbb{Q}) = GSp_{2g}(\mathbb{Q})$); let $A_0$ be as above a fixed superspecial abelian variety of dimension $g > 1$ over $\overline{\mathbb{F}}_p$. In [Ek87], it is shown that
the isomorphism classes of principal polarizations on $A_0$ form a single genus class: this means in particular that if $\lambda$ and $\lambda'$ are two principal polarizations on $A_0$, then the $p$-adic polarizations associated to $\lambda$ and $\lambda'$ respectively on the Dieudonné module of $A_0$ are isomorphic. Hence $(A_0(p), \lambda) \simeq (A_0(p), \lambda')$ as principally polarized $p$-divisible groups over $\mathbb{F}_p$.

As a consequence, in the case $D$ is of type $C$ with $e = 1$, we have:

$$Z'(\mathbb{F}_p) = \{(A_0, \lambda, \alpha) : \lambda \text{ a principal polarization on } A_0, \alpha \text{ a } K^p\text{-level structure on } A_0\}$$

**Proposition 4.11** The uniformization morphism $\Theta_{K^p}(\mathbb{F}_p)$ of Proposition 4.4 induces a canonical Hecke-equivariant isomorphism:

$$\Theta'_{K^p}(\mathbb{F}_p) : I(\mathbb{Q}) \setminus \mathcal{M}'(\mathbb{F}_p) \times G(A^p_\mathbb{F})/K^p \rightarrow Z'(\mathbb{F}_p).$$

We call $\Theta'_{K^p}(\mathbb{F}_p)$ the uniformization morphism for the superspecial locus.

**Proof.** First recall that under our assumptions on $G$, and by the basicity of $b$, the map $\Theta_{K^p}(\mathbb{F}_p)$ is a well defined Hecke-equivariant isomorphism. The action of $I(\mathbb{Q})$ on $\mathcal{M}(\mathbb{F}_p)$ determines an action on $\mathcal{M}'(\mathbb{F}_p) \subseteq \mathcal{M}(\mathbb{F}_p)$, so that we obtain a natural injective Hecke-equivariant map:

$$I(\mathbb{Q}) \setminus \mathcal{M}'(\mathbb{F}_p) \times G(A^p_\mathbb{F})/K^p \hookrightarrow I(\mathbb{Q}) \setminus \mathcal{M}(\mathbb{F}_p) \times G(A^p_\mathbb{F})/K^p.$$ 

Define $\Theta'_{K^p}(\mathbb{F}_p)$ by precomposing this last map with $\Theta_{K^p}(\mathbb{F}_p)$. In order to determine the image of $\Theta'_{K^p}(\mathbb{F}_p)$, we follow the construction of the uniformization morphism over the field $\mathbb{F}_p$ (cf. [4.1.3] and [MR96b, 6.13-14]).

Pick an element $[\rho] \in \mathcal{M}'(\mathbb{F}_p)$; the quasi-isogeny $\rho : (X, i_X, \lambda_X) \rightarrow (X, i_X, \lambda_X)$ determines, by Lemma 4.5, a principally polarized abelian variety:

$$(\rho_* A_0, \rho_* i_0, \rho_* \lambda_0)$$

in $AV_{\mathbb{Q}_p}$, whose $p$-divisible group is isomorphic to $(X, i_X, \lambda_X)$, so that the image of $\Theta'_{K^p}(\mathbb{F}_p)$ is contained inside the superspecial locus.

Viceversa, let $[(A, i, \lambda, \alpha)] \in Z'(\mathbb{F}_p)$ and choose a quasi-isogeny of principally polarized abelian varieties $\rho : (A_0, i_0, \lambda_0) \rightarrow (A, i, \lambda)$. Then $\rho$ defines a quasi-isogeny of the corresponding $p$-divisible groups $\rho : (X, i_X, \lambda_X) \rightarrow (A(p), i, \lambda)$. Precomposing $\rho$ with an isomorphism $\mu : (A(p), i, \lambda) \rightarrow (X, i_X, \lambda_X)$ we obtain an element $[\mu \circ \rho] \in \mathcal{M}'(\mathbb{F}_p)$ such that $(\mu \circ \rho)_*(A_0, i_0, \lambda_0) = \mu_*(A, i, \lambda) \simeq (A, i, \lambda)$. Let now $g \in G(A^p_\mathbb{F})$ defined by $g := (\mu \circ \rho)_* \alpha_0 \circ \alpha^{-1}$; the pre-image of $[(A, i, \lambda, \alpha)] \in Z'(\mathbb{F}_p)$ under $\Theta'_{K^p}(\mathbb{F}_p)$ is the $I(\mathbb{Q})$-class represented by $[\mu \circ \rho] \times gK^p$. □

**Remark 4.12** We have seen that the group $I(\mathbb{Q}) := (\text{End}_{\mathbb{Q}_p}(A_0, \lambda_0) \otimes \mathbb{Q})^\times$ acts on the left upon $\mathcal{M}'(\mathbb{F}_p)$ through the map $\alpha_{0,p} : I(\mathbb{Q}) \rightarrow J(\mathbb{Q}_p)$. We have therefore the canonical identification:

$$I(\mathbb{Q}) \setminus \mathcal{M}'(\mathbb{F}_p) \cong I(\mathbb{Q}) \setminus J(\mathbb{Q}_p)/J(\mathbb{Z}_p),$$

where the action of $I(\mathbb{Q})$ on $\mathcal{M}'(\mathbb{F}_p)$ is the one described in [1.1.1], so that (cf. proof of Proposition 4.8) the action of $I(\mathbb{Q})$ on the coset space $J(\mathbb{Q}_p)/J(\mathbb{Z}_p)$ is given by $x \cdot gJ(\mathbb{Z}_p) = (M_x(x) \cdot g)J(\mathbb{Z}_p)$, for all $x \in I(\mathbb{Q})$ and all $g \in J(\mathbb{Q}_p)$. More easily, we will write $x \cdot gJ(\mathbb{Z}_p) = xgJ(\mathbb{Z}_p)$.

**Corollary 4.13** There is a canonical Hecke equivalent isomorphism:

$$\Theta'_{K^p}(\mathbb{F}_p) : I(\mathbb{Q}) \setminus (J(\mathbb{Q}_p)/J(\mathbb{Z}_p) \times G(A^p_\mathbb{F})/K^p) \rightarrow Z'(\mathbb{F}_p),$$

where the action of $I(\mathbb{Q})$ on $J(\mathbb{Q}_p)/J(\mathbb{Z}_p)$ is the natural one, described in Remark 4.12. Furthermore, $Z'(\mathbb{F}_p)$ is a finite set.

**Proof.** We just need to show the finiteness of $Z'(\mathbb{F}_p)$. By Theorem 4.3 we have canonical identifications $I(A^p_\mathbb{F}) = G(A^p_\mathbb{F})$ and $J(\mathbb{Q}_p) = I(\mathbb{Q}_p)$, so that we can rewrite the domain of the morphism $\Theta'_{K^p}(\mathbb{F}_p)$ as:

$$I(\mathbb{Q}) \setminus \left( I(\mathbb{Q}_p)/I(\mathbb{Z}_p) \times I(A^p_\mathbb{F})/C^p \right) = I(\mathbb{Q}) \setminus I(A^p_\mathbb{F})/C,$$

where $C^p$ is the image of $K^p$ in $I(A^p_\mathbb{F})$ and $C = I(\mathbb{Z}_p) \times C^p$. By the proof of 6.23 in [MR96b], $I(\mathbb{Q})$ is a discrete subgroup of $I(\mathbb{Q}_p) \times I(A^p_\mathbb{F}) = I(A^p_\mathbb{F})$; by Proposition 1.4 of [Gro99], the quotient space $I(\mathbb{Q}) \setminus I(A^p_\mathbb{F})$ is therefore compact, so that $I(\mathbb{Q}) \setminus I(A^p_\mathbb{F})/C$ is finite. □
Comparing Hecke eigensystems

We apply the results of the last section to study systems of Hecke eigenvalues coming from unitary modular forms. We begin by recalling some notation: although this notation will be assumed fixed in the rest of the chapter, we will specify in the following paragraphs additional requirements on the objects considered below.

Fix an algebraic closure $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}}_p$) of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) and an embedding $\overline{\mathbb{Q}} \subset \mathbb{C}$. Fix a simple $\mathbb{Q}$-PEL-datum $D = (B, \sigma, V, \langle \cdot, \cdot \rangle, \mathcal{O}_B, \Lambda, h, K^p, \nu)$ for moduli of abelian schemes with good reduction at $p$, and neat level $K^p$; denote by $G$ the associated algebraic group, and assume it is connected and satisfies the Hasse principle. Let $\mu$ be the co-character associated to $h$, and let $E \subset \overline{\mathbb{Q}}$ be the Shimura field of $D$. Let $\mathbb{F}_p$ be a fixed algebraic closure of the residue field of $E_0 \subset \overline{\mathbb{Q}}_p$; set $W = W(\mathbb{F}_p)$, $K_0 = W(\overline{\mathbb{Q}}_p)$ and denote by $\sigma$ the Frobenius morphism of $W$; recall that $E_0 \subset K_0$. Fix a finite extension $K \subset \overline{\mathbb{Q}}_p$ of $K_0$ such that $\mu$ is defined over $K$; set $\tilde{E}_0 = E_0 K_0 = K_0$. Define $B_p, V_p, \langle \cdot, \cdot \rangle_p, G_p, \mathcal{O}_{B_p}$ as usual.

Let $\mathcal{S}_{D,K^p}$ be the quasi-projective smooth scheme over $\mathcal{O}_{E_0}$ defined in Th. 3.4: suppose that the common relative dimension of the abelian schemes parametrized by $\mathcal{S}_{D,K^p}$ is $g := \dim_{\mathbb{C}} \mathcal{V}_{\mathcal{O}_0} \geq 2$; fix a supersingular elliptic curve $E_0$ over $\mathbb{F}_p$, and denote its canonical model over $\mathbb{F}_{p^2}$ by $E'_0$. Let $A_0 := E'_0 \otimes_{\mathbb{F}_p} \mathbb{F}_p$ be the corresponding superspecial abelian variety over $\mathbb{F}_p$, endowed with the identity principal polarization $\lambda_0 := id_{E'_0 / \mathbb{F}_p}$ (we will always identify canonically $E_0$ and $\tilde{E}_0$). Denote by $\mathfrak{R}$ the $\mathbb{Z}$-algebra $End \ E_0 = End_{\mathbb{F}_p} E'_0$ of endomorphisms of the supersingular elliptic curve $E_0$ over $\mathbb{F}_p \otimes \mathcal{R}$; $\mathfrak{R}$ is a maximal order in $\mathfrak{B} := End_{\mathbb{F}_p} E'_0 = End_{\mathbb{F}_p} \otimes \mathbb{Q}$, where $\mathfrak{B}$ is a quaternion algebra over $\mathbb{Q}$ whose ramification set is $\{p, \infty\}$. If $v$ is a place of $\mathbb{Q}$, we denote by $\mathfrak{B}_v$ the $\mathfrak{Q}_v$-algebra $\mathfrak{B} \otimes_{\mathbb{Q}} \mathfrak{Q}_v$; we also denote by $\overline{\mathfrak{B}}$ the canonical involution of $\mathfrak{B}$ (cf. [Vig80]).

Assume that $\mathcal{S}_{D,K^p}$ contains a point of the form $[(A_0, i_0, \lambda_0, \pi_0)] = \mathcal{S}_{D,K^p} (\mathbb{F}_p)$ for some $i_0, \lambda_0$ and $\pi_0$. Let us fix such a point; the $p$-divisible group $\mathfrak{X} := A_0(p)$ over $\mathbb{F}_p$ is isomorphic to $\mathbb{D}^{1/2}_{1/2}$ and is endowed with the action $i_{\mathfrak{X}}$ of $\mathcal{O}_{B_p}$ induced by $i_0$ and the principal polarization $\lambda_X : A_0(p) \to \tilde{A}_0(p) \simeq G_{1/2}$. By covariant Dieudonné theory, we associate to $(\mathfrak{X}, i_X, \tilde{\lambda}_X)$ a Dieudonné module $\mathfrak{M} := M_\lambda(\mathfrak{X})$ over $\mathcal{O}_{B_p}$ and a principal polarization $e_{\mathfrak{M}} : \mathfrak{M} \times \mathbb{M} \to W$ of Dieudonné modules, which is skew-Hermitian with respect to $*,$ and well defined only up to a scalar factor in $\mathbb{Z}^\times_\mathfrak{B}$. By inverting $p$, we obtain an isocrystal $(N := [M_{\overline{\mathbb{Q}}_p}^g], \mathcal{F})$ over $K_0$ endowed with an action of $B_p$ and with a non-degenerate bilinear form of isocrystals $\Psi : N \times N \to 1(1)$. We fix an isomorphism of $B \otimes_{\mathcal{O}} K_0$-modules $\mathcal{N} \simeq V \otimes_{\mathcal{O}} K_0$ that respects the skew-symmetric forms on both sides and we then write the action of Frobenius on the right hand side as $\mathcal{F} = b \circ \sigma$ for some $b \in G_p(K_0)$. Since $N$ is isoclinic, $b$ is basic in the sense of 2.1.3.

We have a simple $\mathfrak{Q}_p$-PEL datum for moduli of $p$-divisible groups $\mathfrak{D}_p := (B_p, \sigma, V_p, \langle \cdot, \cdot \rangle_p, \mathcal{O}_{B_p}, \Lambda, h, \mu)_p$ having good reduction at $p$ and Shimura field equal to $E_0$. Associated to $\mathfrak{D}_p$ we have the moduli functor $\mathcal{M}$. The fixed choice of $(\mathfrak{X}, i_X, \tilde{\lambda}_X)$ gives rise to a closed subset $\mathcal{M}(\mathbb{F}_p) \subseteq \mathcal{M}(\mathcal{O}_{E_0})$ (cf. Def. 4.7), and a uniformization Hecke-equvariant isomorphism of finite sets:

$$\Theta'_K(\mathbb{F}_p) : I(\mathbb{Q}) \setminus \mathcal{M}(\mathbb{F}_p) \times G(\mathfrak{A}_p^g) / K^p \to Z'(\mathbb{F}_p),$$

or:

$$\Theta'_K(\mathbb{F}_p) : I(\mathbb{Q}) \setminus (J(\mathbb{Q}) / J(\mathbb{Z}) \times G(\mathfrak{A}_p^g) / K^p) \to Z'(\mathbb{F}_p).$$

Since $A_0 = E'_0$, we obtain canonical isomorphisms $End A_0 = M_g(\mathfrak{R})$ and $End^0 A_0 = M_g(\mathfrak{B})$. Under these identifications, and the canonical isomorphism $A_0 \to \tilde{A}_0$, the principal polarization $\lambda_0 = id_{E'_0}$ coincides with the identity matrix $I_g \in M_g(\mathfrak{R})$, so that the auto-quasi-isogenies of the principally (homogeneously) polarized abelian variety $(A_0, \lambda_0)$ are identified with the elements of the unitary quaternion similitude group:

$$GU_g(\mathfrak{B}; I_g) := \{ X \in GL_g(\mathfrak{B}) : X^* X = c(X) \cdot I_g, \ c(X) \in \mathbb{Q}^\times \},$$

where $X^* := \overline{X}$. Similarly, the automorphisms of the pair $(A_0, \lambda_0)$ are given by $GU_g(\mathfrak{R}; I_g)$, and the automorphisms of $(A_0, \lambda_0)$ viewed as a polarized abelian variety up to prime-to-$p$ isogeny are given by $GU_g(\mathfrak{R} \otimes_{\mathbb{Z}} \mathbb{Z}(p); I_g)$.

Notice that $GU_g(\mathfrak{B}; I_g)$ defines an algebraic group over $\mathbb{Q}$, which is reductive, since it is a form of the reductive group $GSp_{2g}/\mathfrak{Q}$, and such that $U_g(\mathfrak{B}; I_g)$ is compact at infinity (since $U_g(\mathfrak{B}_{\infty}; I_g) \subset O(4g)$).

5.1 The settings

Let us assume that $\mathfrak{D} = D^{(r,s)}(\mathfrak{R}, p)$ is the PEL-datum of type $A$ constructed in 3.1.2; in particular, $\mathfrak{D}$ is associated to a quadratic imaginary field $k = \mathbb{Q}(\sqrt{a}) \subset \mathbb{Q}$ ($a \in \mathbb{Z}_{<0}$ square free) in which $p$ is inert. By our previous assumptions,
we are given an embedding \( \tau : k \to \mathbb{C} \) of fields such that \( \tau(\sqrt{\alpha}) = \sqrt{-1}\sqrt{-\alpha} \) for our fixed choice of square root of \(-1\); we denote by \( \tau \) the non-trivial field automorphism of \( k \). Recall that we set \( V = k^g, g = 2n; r, s \) are two positive integers such that \( r + s = g \). The algebraic \( \mathbb{Q} \)-group associated to \( \mathcal{D}_{(r, s), p}^U \) is the unitary group relative to \( k \) and to the form:

\[
H = \begin{pmatrix} -\sqrt{\alpha}I_r & 0_{r, s} \\ 0_{s, r} & \sqrt{\alpha}I_s \end{pmatrix},
\]

i.e. it is \( G = GU_g(k; r, s) \); \( G \) is connected, satisfies the Hasse principle and can be defined over \( \mathbb{Z} \). We have shown in [3.1.2] that the reflex field of our datum is \( \mathbb{E} = \mathbb{Q} \) if \( r = s = n \), and \( \mathbb{E} = k \) otherwise. The determinant polynomial is:

\[
f(X_1, X_2) = (X_1 - \sqrt{\alpha}X_2)^s (X_1 + \sqrt{\alpha}X_2)^s \in \mathcal{O}_E[X_1, X_2].
\]

We take \( K^p := U(N) \) for a fixed integer \( N \geq 3 \) not divisible by \( p \). Assume for simplicity that \( p \neq 2 \).

The embedding \( \nu : \mathbb{F}_p \to \mathbb{F}_p \), that we fixed identifies \( k_\nu \) with the only degree two unramified extension of \( \mathbb{Q}_p \) inside \( K_0 \); if \( \mathbb{F}_p^2 \) denotes the residue field of \( k_\nu \), we obtain an embedding \( \mathbb{F}_p^2 \subset \mathbb{F}_p^2 \).

**Proposition 5.1** There is a supersingular elliptic curve \( E_0 \) over \( \mathbb{F}_p^2 \) whose endomorphism ring \( \text{End}E_0 \) contains an element \( \varphi_\alpha \) such that:

1. \( \varphi_\alpha^2 = \alpha \);
2. the tangent map \( \text{Lie}\varphi_\alpha : \text{Lie}E_0 \to \text{Lie}E_0 \) is multiplication by the scalar \( \sqrt{\alpha}(\mod p) \in \mathbb{F}_p^2 \subset \mathbb{F}_p^2 \), where here \( \sqrt{\alpha} \in k \) is viewed as an element of \( k_\nu \) via \( \nu \).

**Proof.** Let \( E \) be any fixed supersingular elliptic curve over \( \mathbb{F}_p \). By [Vig80], Théorème 3.8, page 78, there is an embedding of \( \mathbb{Q} \)-algebras \( j : k \hookrightarrow \text{End}^0 E \); there is a maximal order \( R \) of \( \text{End}^0 E \) containing \( j(\sqrt{\alpha}) \): in fact we can write \( \text{End}^0 E = j(k) \oplus j(k)u \) for some \( u \in \text{End}^0 E \) (cf. [Vig80], Corollaire 2.2, page 6), and the left order of the ideal \( \mathcal{O} + \mathcal{Z}j(\sqrt{\alpha}) + \mathcal{Z}u + \mathcal{Z}j(\sqrt{\alpha})u \) of \( \text{End}E \) clearly contains \( j(\sqrt{\alpha}) \). By work of Deuring, it is known that there is an elliptic curve \( E_0 \) over \( \mathbb{F}_p \) and a quasi-isogeny \( f : E_0 \to E \) such that \( R = f \circ \text{End}E_0 \circ f^{-1} \), so that \( \text{End}E_0 \) contains an element \( \varphi'_\alpha \) whose square equals \( \alpha \) (cf. [Wed07], 2.15).

The tangent morphism \( \text{Lie}\varphi'_\alpha \) can be canonically identified with an element of \( \mathbb{F}_p^2 \), since \((\text{Lie}\varphi'_\alpha)^2 = \alpha \mod p \), we have \( \text{Lie}\varphi'_\alpha = \pm\sqrt{\alpha}(\mod p) \in \mathbb{F}_p^2 \) (recall we are given a fixed embedding \( \mathbb{F}_p^2 \subset \mathbb{F}_p^2 \)). We define \( \varphi_\alpha := \pm\varphi'_\alpha \) depending on \( \text{Lie}\varphi'_\alpha \) being equal to \( \pm\sqrt{\alpha}(\mod p) \) respectively. The pair \((E_0, \varphi_\alpha)\) we just constructed satisfies the requirement of the proposition. \( \blacksquare \)

Fix a pair \((E_0, \varphi_\alpha)\) over \( \mathbb{F}_p \) as in the above proposition (recall that the choice of isomorphism class of \( E_0 \) will not be relevant later on, since \( g \geq 2 \)) and set \( A_0 = E_0^g, \lambda_0 = id_{E_0}^g; \) recall that \( \mathfrak{R} \) is the \( \mathbb{Z} \)-algebra \( \text{End}E_0 \) (containing \( \varphi_\alpha \)), and \( \mathfrak{B} := \text{End}^0 E_0 \). Let \( \iota : k \hookrightarrow \mathfrak{B} \) be the \( \mathbb{Q} \)-algebra homomorphism such that \( \iota(\sqrt{\alpha}) = \varphi_\alpha \). Since \( p \) is odd, we have \( \mathcal{O}_B := \mathcal{O}_{k,(p)} = \mathcal{Z}_{(p)}[\sqrt{\alpha}] \) (cf. [Vig80], Corollaire 2.2, page 6); therefore we have:

\[
\iota_0(\sqrt{\alpha}) = -\iota_0(\sqrt{\alpha}) = \iota_0(\sqrt{\alpha})^*,
\]

so that \( \lambda_0 \circ \iota_0(b) = \iota_0(b) \circ \lambda_0 \) for any \( b \in \mathcal{O}_{k,(p)} \), and \( \lambda_0 \) is a principal polarization for \((A_0, \iota_0)\) (equivalently, \( \iota_0(b^*) = \iota_0(b) \lambda_0 \) for any \( b \in \mathcal{O}_{k,(p)} \)), and \( \lambda_0 \) is the Rosati involution associated to \( \lambda_0 \).

Fix an ordered basis \( \{t_1, ..., t_g\} \) for the \( \mathbb{F}_p \)-vector space \( A_0 = (\text{Lie}E_0)^g \) such that \( \{t_i\} \) is (the natural image of) a basis for the Lie algebra of the simple \( i^{th} \) factor of \( A_0 = E_0^g \) (1 \( \leq i \leq g \)). With respect to \( \{t_1, ..., t_g\} \), by construction Lie \( \iota_0(\sqrt{\alpha}) \) acts on \( \text{Lie}A_0 \) via the matrix:

\[
\begin{pmatrix} -\sqrt{\alpha}(\mod p) \cdot I_r \\ 0_{s, r} \\ \sqrt{\alpha}(\mod p) \cdot I_s \end{pmatrix} \in GL_g(\mathbb{F}_p^2) \subset GL_g(\mathbb{F}_p).
\]
We conclude that the fixed pair \((A_0, i_0)\) satisfies Kottwitz’ determinant condition, since we have the following equalities in \(\mathbb{F}_p^2\):
\[
\det(X_1 + \sqrt{\alpha}X_2; \text{Lie } A_0) = \det \begin{pmatrix} (X_1 - \sqrt{\alpha}X_2)_I & 0_{r,s} \\ 0_{s,r} & (X_1 + \sqrt{\alpha}X_2)_I \end{pmatrix} (\text{mod } p) \\
= (X_1 - \sqrt{\alpha}X_2)_f (X_1 + \sqrt{\alpha}X_2)_s (\text{mod } p) \\
= f(X_1, X_2) (\text{mod } p).
\]

We fix a \(U(N)\)-orbit of an isomorphism \(\alpha_0 : H_1(A_0, \mathbb{A}_f^\mathbb{Q}) \to V \otimes_{\mathbb{Q}} \mathbb{A}_f^\mathbb{Q}\) of skew-hermitian modules with \(k\)-action. By definition of our moduli variety, we have determined a point:
\[
[(A_0, i_0, \overline{\alpha}_0, \overline{\alpha}_0)] \in \mathcal{S}_{\mathcal{D}, U(N)}(\mathbb{F}_p)
\]
that we consider fixed for the remaining of this section. In particular, we have a canonical Hecke-equivariant isomorphism associated to \((X, i_X, \overline{\alpha}_X)\) from Corollary 4.13:
\[
\Theta_{U(N)}(\mathbb{F}_p) : I(\mathbb{Q}) \backslash \left( J(\mathbb{Q}_p)/J(\mathbb{Z}_p) \times G(\mathbb{A}_f^\mathbb{Q})/U(N) \right) \to Z'(\mathbb{F}_p).
\]
Recall that both sides above are finite sets.

### 5.1.1 The groups \(I, J\) and \(G\)

We recall the nature of the algebraic groups \(I, J, G\) appearing in the domain of the uniformization morphism. Recall that \(G = GU_g(k; r, s)\) is the reductive group over \(\mathbb{Q}\) associated to the quadratic imaginary field extension \(k/\mathbb{Q}\) and the Hermitian form determine by the matrix \(H\); furthermore:
\[
U(N) := \ker \left( GU_g \left( \mathcal{O}_k \otimes_{\mathbb{Z}} \mathcal{H}^p; r, s \right) \to GU_g \left( \mathcal{O}_k \otimes_{\mathbb{Z}} \mathcal{H}^p; r, s \right) \right)
\]
is a compact open subgroup of \(G(\mathbb{A}_f^\mathbb{Q})\).

By definition, \(I(\mathbb{Q})\) is the group of auto-\(\mathcal{O}_{k,(p)}\)-quasi-isogenies of the homogeneously principally polarized abelian variety \((A_0, \overline{\alpha}_0)\); let us denote by \(\Phi\) the matrix:
\[
\Phi := \Phi_\alpha = \begin{pmatrix} -\varphi_\alpha I_r & 0_{r,s} \\ 0_{s,r} & \varphi_\alpha I_s \end{pmatrix} \in M_g(\mathfrak{R}).
\]

Then we have \(I(\mathbb{Q}) = \{ X \in GU_g(\mathfrak{B}; I_g) : X\Phi = \Phi X \}\), and for any \(\mathfrak{Q}\)-algebra \(S\) we have \(I(S) = \{ X \in GU_g(\mathfrak{B} \otimes_{\mathfrak{Q}} S; I_g) : X\Phi = \Phi X \}\).

On the other side, \(J(\mathbb{Q}_p)\) is the group of \(K_0\)-automorphisms of the homogeneously principally polarized isocrystal with \(k\)-action \((N, \mathfrak{P}; \mathbb{Q}_p^S \Psi)\): it has a compact subgroup \(J(\mathbb{Z}_p)\) that is given by the \(W\)-automorphisms of the homogeneously principally polarized Dieudonné module \((\mathcal{M}, \delta_{\mathcal{M}}, \tau_{\mathcal{M}})\) endowed with \(\mathcal{O}_{k,(p)}\)-action. Since \(\mathfrak{M} = M_s(A_0(p)) \simeq A_{1/2}^{\mathfrak{g}}\) as principally polarized Dieudonné modules, where \(A_{1/2}^{\mathfrak{g}}\) is endowed with the product polarization coming from the polarization \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) on \(A_{1/2}\), one deduces \([\text{Ghi04a}, \text{Cor. 10}]\), keeping track of the action of \(\mathcal{O}_{k,(p)}\):
\[
J(\mathbb{Q}_p) \simeq \{ X \in GU_g(\mathfrak{B}_p; I_g) : X\Phi = \Phi X \} = I(\mathbb{Q}_p),
\]
\[
J(\mathbb{Z}_p) \simeq \{ X \in GU_g(\mathfrak{R}_p; I_g) : X\Phi = \Phi X \} =: I(\mathbb{Z}_p),
\]
where \(\mathfrak{R}_p\) denotes the unique maximal order of the skew-field \(\mathfrak{B}_p\). This result can also be deduced from \([\text{MR96b}6.29]\); the above isomorphisms are canonical.

We deduce from Corollary 4.13

### Proposition 5.2
There is a canonical Hecke-equivariant isomorphism of finite sets:
\[
\Theta_{U(N)}(\mathbb{F}_p) : I(\mathbb{Q})\backslash I(\mathbb{A}_f)/U_N \to Z'(\mathbb{F}_p),
\]
where \(U_N := J(\mathbb{Z}_p) \times U(N)\) is viewed as an open compact subgroup of \(I(\mathbb{A}_f) = J(\mathbb{Q}_p) \times G(\mathbb{A}_f^\mathbb{Q})\).
If \( \mathbb{L}_q \) is a finite field of cardinality \( q \), there is, up to isomorphism, a unique Hermitian space of dimension \( m \) associated to the quadratic extension \( \mathbb{L}_{q^2}/\mathbb{L}_q \) (cf. [Lew82]). We denote the associated unitary group - defined over \( \mathbb{L}_q \) - by \( U_m \); in particular:

\[
GU_m(\mathbb{L}_{q^2}) = \{ X \in GL_m(\mathbb{L}_{q^2}) : X^*X = c(X) \cdot I_m, c(X) \in \mathbb{L}_q^\times \}.
\]

We will also need to consider the algebraic group \( G(U_{m_1} \times U_{m_2}) \subset GU_{m_1} \times GU_{m_2} \) defined over \( \mathbb{L}_q \), whose \( \mathbb{L}_q \)-points are:

\[
G(U_{m_1} \times U_{m_2})(\mathbb{L}_{q^2}) = \left\{ g = \begin{pmatrix} X & 0_{m_1,m_2} \\ 0_{m_2,m_1} & Y \end{pmatrix} : g \in GU_{m_1+m_2}(\mathbb{L}_{q^2}) \right\} = \left\{ \begin{pmatrix} X & 0_{m_1,m_2} \\ 0_{m_2,m_1} & Y \end{pmatrix} : X^*X = cI_r, Y^*Y = cI_s, c \in \mathbb{L}_q^\times \right\}.
\]

(Notice the slight abuse of notation). Via the embedding \( \iota : k \hookrightarrow \mathfrak{B} \) that we fixed, we obtain a natural epimorphism \( \mathfrak{R}_p \to \mathbb{F}_p^2 \), indeed we can write \( \mathfrak{R}_p = \mathbb{Z}_p[\varphi_\alpha] \oplus \mathbb{Z}_p[\varphi_\alpha]\Pi \) for a choice of uniformizer \( \Pi \) such that \( \Pi^2 = p \), so that:

\[
\frac{\mathbb{Z}_p[\varphi_\alpha]}{(p)} \cong \frac{\mathbb{Z}_p[\varphi_\alpha]}{(\sqrt{p})} \cong \mathbb{F}_p^2.
\]

**Lemma 5.3** Let \( G(p) := G(U_r \times U_s)(\mathbb{F}_p^2) \); there is a short exact sequence of groups (defining \( U_p \)):

\[
1 \to U_p \to J(\mathbb{Z}_p) \xrightarrow{\pi} G(p) \to 1,
\]

where the map \( \pi : J(\mathbb{Z}_p) \to G(p) \) is induced by the canonical epimorphism \( \mathfrak{R}_p \to \mathbb{F}_p^2 \) arising from our fixed embedding \( \iota : k \hookrightarrow \mathfrak{B} \).

**Proof.** By previous considerations, we have a natural identification:

\[
J(\mathbb{Z}_p) = \{ X \in GU_2(\mathfrak{R}_p, I_g) : X\Phi = \Phi X \}.
\]

Via the embedding \( \iota \), we identify \( \varphi_\alpha(\text{mod } \Pi) \) with \( \sqrt{\alpha}(\text{mod } p) \in \mathbb{F}_p^2 \); if \( X \in J(\mathbb{Z}_p) \), then \( \pi(X) \in GU_2(\mathbb{F}_p^2) \) and the equation \( X\Phi = \Phi X \) for an \((r,s)\)-block matrix \( X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GU_2(\mathfrak{R}_p, I_g) \) reduces to the equation in \( M_2(\mathbb{F}_p^2) \):

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} (\text{mod } \Pi) \cdot \begin{pmatrix} \sqrt{\alpha}I_r & 0_{r,s} \\ 0_{s,r} & \sqrt{\alpha}I_s \end{pmatrix} (\text{mod } p) = \begin{pmatrix} -\sqrt{\alpha}I_r & 0_{r,s} \\ 0_{s,r} & \sqrt{\alpha}I_s \end{pmatrix} (\text{mod } p) \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} (\text{mod } \Pi)
\]

We deduce that \( B(\text{mod } \Pi) = 0_{r,s} \) and \( C(\text{mod } \Pi) = 0_{s,r} \), since \( p \neq 2 \), so that \( \pi(X) \in G(p) \).

On the other side, if \( Y = \begin{pmatrix} T & 0_{r,s} \\ 0_{s,r} & S \end{pmatrix} \in G(p) \), then we can lift \( Y \) to a matrix in \( G(U_r \times U_s)(\mathcal{O}_{k_\alpha}) \), and we can see \( G(U_r \times U_s)(\mathcal{O}_{k_\alpha}) \subset J(\mathbb{Z}_p) \) via \( \iota \), so that \( \pi \) is onto. ■

### 5.2 Unitary Dieudonné modules and invariant differentials

We will need the following general fact:

**Lemma 5.4** Let \( L \) be a subfield of \( \mathbb{F}_p \) and let \( \sigma \) denote the restriction of the Frobenius of \( W \) to \( W(L) \). Let \( M \) be a Dieudonné module over \( W(L) \) endowed with the \( \mathbb{Z}(\sigma) \)-linear action of a \( \mathbb{Z}(\sigma) \)-algebra with involution \( O \); assume \( M \) comes with a principal polarization \( \epsilon : M \times M \to W(L) \) that is skew-Hermitian with respect to the \( O \)-action. Then the assignment:

\[
\langle , \rangle : \frac{M}{FM} \times \frac{M}{VM} \to L, \quad (\tau, \psi) \mapsto \epsilon(x, Fy)(\text{mod } p)
\]

is a well-defined, perfect pairing which is \( L \)-linear in the first variable and \( L \)-semilinear (with respect to \( \sigma \)) in the second variable. Furthermore, \( \langle , \rangle \) is skew-Hermitian with respect to the action of \( O \).

**Proposition 5.5** If \( F + V = 0 \) on \( M \), then \( \langle , \rangle \) defines a \( \sigma^{-1} \)-alternating pairing \( \frac{M}{VM} \times \frac{M}{VM} \to L \).
Proof. For $x, y, m, m' \in M$ we have:

$$e(x + Fm, F(y + Vm')) = e(x, Fy) + e(Fm, Fy) + e(x, Fm, Vm') =$$

$$= e(x, Fy) + e(m, VFy)^\sigma + e(x + Fm, pm') \equiv$$

$$\equiv e(x, Fy) (\text{mod } p),$$

so that $\langle , \rangle$ is well defined; it is clearly $L$-linear in the first variable, and $\sigma$-semilinear in the second since $F$ is $\sigma$-semilinear. If $b \in O$ and $x, y \in M$ we have:

$$\langle b, x, y \rangle = \langle bx, Fy \rangle (\text{mod } p) =$$

$$= e(x, b^*Fy) (\text{mod } p) = e(x, Fb^*y) (\text{mod } p) =$$

$$\langle x, b^*y \rangle .$$

To show that $\langle , \rangle$ is non-degenerate we need to show that the $L$-linear map:

$$\varepsilon : \frac{M}{FM} \to \text{Hom}_{\sigma, \text{lin}}(\frac{M}{VM}, L)$$

induced by $\langle , \rangle$ is an isomorphism of $L$-vector spaces. Let $x \in M$ such that $\langle x, y \rangle = 0$ for all $y \in M$. By assumption we have $e(y, Vx) \in pW(L)$ for all $y \in M$; if we denote by $\mu : M \to \text{Hom}_W(M, W)$ the isomorphism defined by $\mu(m) := e(\cdot, m)$, we have that $\mu(Vx)$ has image contained inside $pW(L)$, so that there is $z \in M$ such that $\frac{1}{p^g} \mu(Vx) = \mu(z)$; we obtain $Vx = pz$, so that $px = FVx = pFz$, hence $x = Fz \in FM$. We deduce that $\varepsilon$ is injective.

Similarly one can show that $\langle , \rangle$ induces an injective $\sigma$-semilinear map of $L$-space:

$$\frac{M}{VM} \hookrightarrow \text{Hom}_L(\frac{M}{FM}, L).$$

We conclude that $\dim_L \frac{M}{VM} = \dim_L \frac{M}{FM} < \infty$ and $\varepsilon$ is forced to be an isomorphism.

Finally, if $F + V = 0$, we have $FM = VM$ and if $x, y \in M$ one computes:

$$\langle x, y \rangle = e(x, Fy) (\text{mod } p) = -e(x, Vy) (\text{mod } p) =$$

$$= -e(Fx, y)^\sigma (\text{mod } p) = e(y, Fx)^\sigma (\text{mod } p) =$$

$$= \langle y, x \rangle^{\sigma^{-1}} .$$

This says that $\langle , \rangle$ is $\sigma^{-1}$-alternating. $\blacksquare$

Fix a triple $(A_0, i, \lambda)$ such that $[(A_0, i, \lambda, \overline{\sigma})] \in Z'(\mathbb{F}_p)$ for some level structure $\overline{\sigma}$; we know that $(A_0, i, \lambda)$ has a canonical $\mathbb{F}_p$-structure, that we denote by $(A_0', i, \lambda)$; by functoriality, the various object that we associated to $(A_0, i, \lambda)$ (as $p$-divisible groups, Dieudonné modules, polarizations, actions of algebras) are obtained as base changes to $\mathbb{F}_p$ (or to $W$) of analogous objects defined over $\mathbb{F}_p^\varphi$ (or over $W(\mathbb{F}_p^\varphi)$). It is therefore equivalent to work with $(A_0', i, \lambda)$ or with $(A_0, i, \lambda)$.

Let $X': = A_0'(p)$ be the $p$-divisible group of $A_0'$; it is defined over $\mathbb{F}_p^\varphi$; its covariant Dieudonné module $M' = M(X')$ is a $W(\mathbb{F}_p^\varphi)$-module, and $M' = (A'_1)^{\varphi^g}$, (cf. [2.1]); furthermore $M = M' \otimes_{W(\mathbb{F}_p^\varphi)} W$. The symbols $i_{M'}$ and $e_{M'}$ have the obvious meanings, relatively to the given action $i$ and polarization $\lambda'$.

By [PFS2] 3.3.1, there is a positive integer $m$ such that the canonical map of cotangent spaces (at the origin)

$$t^*_{A_0'(p)} = t^*_{X'} \rightarrow t^*_{A_0'[p^m]}$$

is an isomorphism. Furthermore, the closed immersion $A_0'[p^m] \to A_0'$ of $\mathbb{F}_p^\varphi$-group schemes induces an epimorphism of $\mathbb{F}_p^\varphi$-vector spaces $t^*_{A_0'} \twoheadrightarrow t^*_{A_0'[p^m]} = t^*_{X'}$. Since $A_0'$ is superspecial, we have $\dim X' = g = \dim A_0'$, so that we obtain canonical identifications $t^*_{A_0'} = t^*_{X'}$ and $\text{Lie} A_0' = \text{Lie} X'$.

By covariant Dieudonné theory, we have a canonical isomorphism of $\mathbb{F}_p^\varphi$-vector spaces:

$$\text{Lie}(X') = \frac{M'}{VM'}.$$
All the above isomorphisms and identifications respect the $O_k(p)$-structures of the modules considered, and also the polarizations induced by $\lambda$.

By our assumptions, $F + V = 0$ on $M'$, so that by Proposition 4 the principal polarization $e_{M'} : M' \times M' \to W(F_{p^2})$ induces a non-degenerate pairing of $F_{p^2}$-spaces:

$$\langle \cdot, \cdot \rangle : \frac{M'}{VM'} \times \frac{M'}{VM'} \to F_{p^2}$$

which is linear in the first argument, $\sigma$-linear in the second argument, and $\sigma$-alternating (i.e. $\langle x, y \rangle = \langle y, x \rangle^\sigma$; recall that $\sigma^2 = 1$ here); hence $(\frac{M'}{VM'}; i_{M'}, \langle \cdot, \cdot \rangle)$ is a Hermitian space over $F_{p^2}$ of dimension $g = 2n$, endowed with an action of $O_k(p)$ with respect to which the pairing $\langle \cdot, \cdot \rangle$ is skew-symmetric. Since $e_{M'}$ is determined only up to a constant in $Z_p^\times$, the pairing $\langle \cdot, \cdot \rangle$ is determined up to a constant in $F_{p^2}^\times$.

We need to study a bit more the Dieudonné module $M'$ (cf. for example [Yu02] §3, or [OB06] 2.1). Via the fixed embedding $\nu$ we obtain the decomposition:

$$M' = M'_- \oplus M'_+,$$

where:

$$M'_\pm := \{ m \in M' : \nu m = \pm \sqrt{\alpha} m \}, \quad \nu k_W(F_{p^2}) M'_\pm = g.$$

One easily sees that $VM'_\pm \subseteq M'_\pm$, $FM'_\pm \subseteq M'_\pm$ and that $e_{M'}(M'_+, M'_-) = 0$, $e_{M'}(M'_-, M'_-) = 0$ (i.e. $M'_-$ and $M'_+$ are totally isotropic with respect to the principal polarization). We have:

$$\frac{M'}{VM'} = \frac{M'_-}{VM'_-} \oplus \frac{M'_+}{VM'_+}$$

Notice that this is a decomposition as $F_{p^2}$-vector spaces with action of $O_k(p)$, where $\sqrt{\alpha}$ acts as $-\sqrt{\alpha}$ (mod $p$) in $F_{p^2}$ on the first summand, and as $\sqrt{\alpha}$ (mod $p$) on the second. Furthermore:

$$\dim_{F_{p^2}} \frac{M'_-}{VM'_-} = r, \quad \dim_{F_{p^2}} \frac{M'_+}{VM'_+} = s.$$

**Proposition 5.6** Let $(A_0, i, \overline{\alpha})$ be a triple over $\overline{F}_p$ such that for some level structure $\overline{\alpha}$ we have $[(A_0, i, \overline{\alpha}, \overline{\alpha})] \in Z'(\overline{F}_p)$; let $(A'_0, i, \overline{\alpha})$ be the canonical $\overline{F}_p$-structure of $(A_0, i, \overline{\alpha})$. The automorphism group of the Hermitian space with $O_k(p)$-action $(\frac{M'}{VM'}; \nu i_{M'}, \langle \cdot, \cdot \rangle)$ is isomorphic to the finite group $G(p) = G(U_r \times U_s)(F_{p^2})$.

**Proof.** Let $\Sigma_{\pm} := \frac{M'_\pm}{VM'_\pm}$. Let $B^+$ (resp. $B^-$) be a fixed ordered basis of $\Sigma_+$ (resp. $\Sigma_-$). If $\overline{\alpha}$ is an automorphism of $\frac{M'}{VM'}$ which commutes with the action of $O_k(p)$, we have $\Sigma_{\pm} \subseteq \Sigma_{\pm}$, so that the matrix representing $\overline{\alpha}$ with respect to $B := B^- \cup B^+$ is of the form:

$$X = \begin{pmatrix} X_- & 0_{r,s} \\ 0_{s,r} & X_+ \end{pmatrix} \in GL_d(F_{p^2}).$$

Vice versa, any such matrix represents - with respect to $B$ - an automorphism of $\frac{M'}{VM'}$ commuting with the action of $O_k(p)$.

Since $FM'_\pm \subseteq M'_\pm$ and $e_{M'}(M'_-, M'_-) = 0$, we deduce that $\langle \Sigma_-, \Sigma_+ \rangle = 0$, by definition of the pairing $\langle \cdot, \cdot \rangle$. This implies that $\langle \cdot, \cdot \rangle$ is represented, with respect to $B$, by a Hermitian diagonal matrix $\begin{pmatrix} U_- & 0_{r,s} \\ 0_{s,r} & U_+ \end{pmatrix} \in GL_d(F_{p^2})$, so that if $X$ is as above, we have that $X$ represents an automorphism of $(\frac{M'}{VM'}; \nu i_{M'}, \langle \cdot, \cdot \rangle)$ with respect to $B$ if and only if:

$$X_+ \cdot U_- \cdot X_- = c U_+,$$

where $c \in F_{p^2}^\times$ is a scalar depending only on $X$.

We conclude that the automorphism group of $(\frac{M'}{VM'}; \nu i_{M'}, \langle \cdot, \cdot \rangle)$ is isomorphic to the group:

$$G = \{(X_-, X_+) \in GU_r(F_{p^2}; U_-) \times GU_s(F_{p^2}; U_+) : c_-(X_-) = c_+(X_+)\},$$

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where \(c_\pm\) is the similitude factor homomorphism in \(GU(F_\rho; U_\pm)\). The unitary spaces \((F_\rho^+; U_-)\) and \((F_\rho^-; I_r)\) are isomorphic, hence we can find an isomorphism \(GU_r(F_\rho^2; U_-) \simeq GU_r(F_\rho^2)\) preserving the similitude factor of corresponding matrices in each group; similarly we can do for \(GU_s(F_\rho^2; U_+)\). Putting things together we obtain an isomorphism \(G \simeq G(p)\). 

We now need to switch to cotangent spaces in our considerations. As usual, \(t^*_\mathbb{A}_0\) denotes the cotangent space (at the origin) of \(\mathbb{A}_0\). As vector spaces over \(F_\rho^2\), we have:

\[
t^*_\mathbb{A}_0 = \text{Hom}_{F_\rho^2} \left( \frac{M'}{\sqrt{M^*}}, F_\rho^2 \right).
\]

For simplicity, let \(\mathcal{L} := \frac{M'}{\sqrt{M^*}} = \mathcal{L}_- \oplus \mathcal{L}_+\), where \(\mathcal{L}_\pm := \frac{M'}{\sqrt{M^*}}\), so that \(t^*_\mathbb{A}_0 = \mathcal{L}^*\). The action of \(O_k(p)\) on \(\mathcal{L}\) induces by functoriality an algebra homomorphism:

\[
i^\vee : O_{k(p)} \to \text{End}_{F_\rho^2}(\mathcal{L}^*)
\]

defined by \(i^\vee(b)(\eta) := \eta \circ i(b)\) for all \(b \in O_{k(p)}\) and \(\eta \in t^*_\mathbb{A}_0\). Notice that \(\sqrt{\alpha} \in O_{k(p)}\) acts on \(\mathcal{L}_\pm^*\) as \(\pm \sqrt{\alpha}\) \((\text{mod } p)\), via \(i^\vee\).

The non-degenerate Hermitian pairing \(\langle , \rangle\) on \(\mathcal{L}\) induces a \(\sigma\)-semilinear isomorphism of \(F_\rho^2\)-spaces \(\varepsilon : \mathcal{L} \to \mathcal{L}^*\) by setting \(\varepsilon_v : w \mapsto \langle w, v \rangle\) for all \(v, w \in \mathcal{L}\). This allows us to define a pairing \(\langle , \rangle\) on \(\mathcal{L}^*\) by setting \(\varepsilon_{v_1, v_2} := \langle v_1, v_2 \rangle\) for all \(v_1, v_2 \in \mathcal{L}\). One can check that we have obtained a non-degenerate pairing \(\langle , \rangle : \mathcal{L}^* \times \mathcal{L}^* \to F_\rho^2\), which is \(\sigma\)-semilinear in the first variable, linear in the second, and such that \(\langle \eta_1, \eta_2 \rangle = (\eta_2, \eta_1)^\sigma\) for all \(\eta_1, \eta_2 \in \mathcal{L}^*\).

Furthermore \(i^\vee(b)\eta_1, \eta_2) = (\eta_1, i^\vee(b)\eta_2)\) for all \(b \in O_{k(p)}\), and \(\langle \mathcal{L}^*_-, \mathcal{L}^*_+ \rangle = 0\). We have therefore obtained a \(F_\rho^2\)-Hermitian space:

\[
(t^*_\mathbb{A}_0 = \mathcal{L}^* : i^\vee, (,))
\]
of dimension \(g\), endowed with an action \(i^\vee\) of \(O_{k(p)}\) with respect to which the pairing \((, )\) is skew-Hermitian.

**Lemma 5.7** There is an isomorphism of groups:

\[
\text{Aut}_{F_\rho^2}(t^*_\mathbb{A}_0 ; i^\vee, (,)) \simeq G(p).
\]

**Proof.** The result follows from Proposition 5.6 since the map \(\mathfrak{X} \mapsto (\mathfrak{X}^*)^{-1}\) defines a canonical isomorphism of groups \(\text{Aut}_{F_\rho^2}(t^*_\mathbb{A}_0 ; i^\vee, (,)) \to \text{Aut}_{F_\rho^2}(\text{Lie } \mathbb{A}_0; i, (,)).\)

We can now give the following:

**Definition 5.8** Let \((A_0, i, \overline{\lambda})\) be a triple over \(F_\rho\) such that for some level structure \(\overline{\pi}\) we have \(\{A_0, i, \overline{\lambda}, \overline{\pi}\} \in Z'(F_\rho)\); let \((A_0', i, \overline{\lambda})\) be the canonical \(F_\rho^2\)-structure of \((A_0, i, \overline{\lambda})\). A basis of invariant differentials of \((A_0', i, \overline{\lambda})\) (over \(F_\rho^2\)) is a choice of an ordered (similitude) Hermitian basis \(\eta = (\eta_-, \eta_+)\) of the Hermitian module \((t^*_\mathbb{A}_0 ; i^\vee, (,))\) such that \(\eta_\pm\) is a basis for \(t^*_\mathbb{A}_0 ; \pm \) := \(\left( \frac{M'}{\sqrt{M^*}} \right)^*\).

We have:

**Lemma 5.9** Let \((A_0', i, \overline{\lambda})\) be as above:

(a) there is a basis of \(t^*_\mathbb{A}_0\) with respect to which the automorphisms of \((t^*_\mathbb{A}_0 ; i^\vee, (,))\) are represented by the matrices of \(G(p) = G(U_r \times U_s)(F_\rho^2)\);

(b) there are basis of invariant differentials for \((A_0', i, \overline{\lambda})\);

(c) let \(\mathcal{B}\) be a basis of \(t^*_\mathbb{A}_0\) as in (a) and let \(\eta \in F_\rho^2\) be the coordinate column vector of a basis of invariant differentials for \((A_0', i, \overline{\lambda})\). Then any other coordinate vector (with respect to \(\mathcal{B}\)) of a basis of invariant differentials for \((A_0', i, \overline{\lambda})\) is of the form \(M\eta\) for some \(M \in G(p)\).
5.3 Superspecial modular forms

We assume fixed from now on on a basis $\eta_0$ of invariant differentials for $(A'_0, i_0, \overline{\lambda}_0)$.

**Definition 5.10** The symbol $Z'_{d_{\text{iff}}}^\text{pr}(\mathbb{F}_p)$ denotes the set of equivalence classes of tuples $(A, i, \overline{\lambda}, \eta)$, where $(A, i, \overline{\lambda})$ is a representative for an equivalence class $[(A, i, \overline{\lambda})] \in Z'(\mathbb{F}_p)$, and $\eta$ is a choice of basis of invariant differentials for the triple $(A', i', \overline{\lambda}')$ defined over $\mathbb{F}_{p^2}$. Two tuples $(A, i, \overline{\lambda}, \eta)$ and $(A_1, i_1, \overline{\lambda}_1, \eta_1)$ are equivalent if there is an isomorphism $f : (A, i, \overline{\lambda}, \eta) \to (A_1, i_1, \overline{\lambda}_1, \eta_1)$ such that $f^*(\eta_1) = \eta$, where $f^* : \mathfrak{t}_A^0 \to \mathfrak{t}_A^0$ is the cotangent map induced by $f$.

**Remark 5.11** Let $g \in J(\mathbb{Z}_p) \subset GL_g(\mathfrak{R}_p)$ and let $v \in \text{Lie } A'_0 \simeq \mathbb{F}_{p^2}^g, \omega \in \mathfrak{t}_A^0 = (\text{Lie } A'_0)^*$. Then $g$ acts on $v$ and on $\omega$ as follows:

\[
g \cdot v := g (\text{mod } \Pi)v \quad \quad g \cdot \omega := \omega \circ g (\text{mod } \Pi),
\]

where $\Pi$ is a uniformizer for $\mathfrak{R}_p$.

**Proposition 5.12** For any fixed choice of left transversal $\mathcal{Y}$ (resp. $\mathcal{G}$) for $J(\mathbb{Z}_p)$ in $J(\mathbb{Q}_p)$ (resp. $U_p$ in $J(\mathbb{Z}_p)$), the uniformization morphism for the superspecial locus $\Theta'(\mathbb{F}_p)$ induces a Hecke-equivariant isomorphism $\Theta'_{d_{\text{iff}}}^\text{pr}(\mathbb{F}_p) = \Theta'_{d_{\text{iff}}}^\text{pr}(\mathbb{F}_p) U(N) : I(\mathbb{Q}_p) \setminus \left( J(\mathbb{Q}_p)/U_p \times G(\mathbb{A}_f)/U(N) \right) \to Z'_{d_{\text{iff}}}^\text{pr}(\mathbb{F}_p)$.

**Proof.** (In this proof, for $\xi \in I(\mathbb{Q})$ we will sometimes write $\xi$ to denote $\alpha_{p, \xi}(\eta) \in J(\mathbb{Q}_p)$, or vice versa, if no ambiguity arises, cf. 4.1.11.) Let $I(\mathbb{Q}) \setminus \left( \rho^{-1}U_p \times U(N) \right)$ be a fixed element in the above left hand side; there are uniquely determined $y^{-1} \in \mathcal{Y}$ and $g^{-1} \in \mathcal{G}$ such that $\rho^{-1}U_p = y^{-1}g^{-1}U_p$. By the definition of $\Theta'(\mathbb{F}_p)$, we obtain a well defined tuple $(y, A_0, y, i_0, y, \overline{\lambda}_0) \in \mathcal{G}$ representing a class in $Z'(\mathbb{F}_p)$. Since the $p$-divisible group of $(y, A_0, y, i_0, y, \overline{\lambda}_0)$ coincides with $(\mathcal{X}, i, X, \overline{\lambda})$, $\eta_0$ is a basis of invariant differentials for the $\mathbb{F}_{p^2}$-model of $(y, A_0, y, i_0, y, \overline{\lambda}_0)$, via the canonical identification:

\[
(\text{Lie } y, A'_0)^* = (\text{Lie } X')^*;
\]

therefore $\eta_0 g$ is also a basis of invariant differentials for the $\mathbb{F}_{p^2}$-model of $(y, A_0, y, i_0, y, \overline{\lambda}_0)$. We set:

\[
\Theta'_{d_{\text{iff}}}^\text{pr}(\mathbb{F}_p) : I(\mathbb{Q}) \setminus \left( \rho^{-1}g^{-1}yU_p \times U(N) \right) \mapsto [(y, A_0, y, i_0, y, \overline{\lambda}_0, x^{-1} \cdot y, \overline{\lambda}_0, \eta_0 g)].
\]

Notice the above assignment is well defined, once we fixed the transversals $\mathcal{Y}$ and $\mathcal{G}$: we only need to check that the map constructed factors through $I(\mathbb{Q})$. Let $\xi \in I(\mathbb{Q})$, so that $\xi = \alpha_{p, \xi}(\eta) \in J(\mathbb{Q}_p)$. Then $\xi y^{-1}g^{-1}U_p = y^{-1}g^{-1}U_p$ for uniquely determined $y^{-1} \in \mathcal{Y}$ and $g^{-1} \in \mathcal{G}$ (notice that here $\xi y^{-1}g^{-1}$ should be more properly be written as $\alpha_{p, \xi}(\eta) y^{-1}g^{-1}$). Write $y_1 = f \cdot y \xi^{-1}$ with $f = g^{-1}u \in J(\mathbb{Z}_p)$, for some $u \in U_p$. The isomorphism $f$ induces an isomorphism:

\[
f_{ab} : (y, A_0, y, i_0, y, \overline{\lambda}_0) \sim (y, A_0, y, i_0, y, \overline{\lambda}_0).
\]

By definition of the action of $I(\mathbb{Q})$ on $G(\mathbb{A}_f)/U(N)$ we have:

\[
\xi : xU(N) = \alpha_{p, \xi}(\eta) xU(N) = \alpha_0 \circ H_1(\xi) \circ \alpha_0^{-1} \circ xU(N).
\]

The level structure on $(y_1, A_0, y_1, i_0, y_1, \overline{\lambda}_0)$ associated to the pair $y^{-1}g^{-1}U_p \times \alpha_{p, \xi}(\eta) xU(N)$ is therefore induced by:

\[
x^{-1} \alpha_{p, \xi}(\eta) y_1, (\alpha_0) = x^{-1} \alpha_0 H_1(\xi) \alpha_0^{-1} \alpha_0 H_1(y_1^{-1}) = x^{-1} \alpha_0 H_1(y^{-1} f^{-1}) = x^{-1} y_1, (\alpha_0) \circ H_1(f^{-1}),
\]

so that $f_{ab}$ is an isomorphism:

\[
f_{ab} : (y, A_0, y, i_0, y, \overline{\lambda}_0, x^{-1} \cdot y, \overline{\lambda}_0) \sim (y_1, A_0, y_1, i_0, y_1, \overline{\lambda}_0, x^{-1} \alpha_0(\xi) y_1, (\alpha_0))
\]

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The cotangent map induced by \( f_{ab} \) gives: \( f_{ab}^*(\eta_0g_1) = \eta_0g_1f = \eta_0ug = \eta_0g \), so that \( f_{ab} \) preserves the choices of invariant differentials.

The map \( \Theta'_{diff}(\mathbb{F}_p) \) is surjective: let \([A, i, \overline{x}, \eta] \in Z'_{diff}(\mathbb{F}_p)\); we can find \( y^{-1} \in \mathcal{Y}, \overline{g}^{-1} \in J(\mathbb{Z}_p) \) and \( x \in G(\mathbb{A}_f^\infty) \) such that \([A, i, \overline{x}, \eta] \) is isomorphic to a tuple of the form:

\[
(y_*, A_0, y_*i_0, y_*, \overline{\alpha}_0, x^{-1} \cdot y_* \overline{\alpha}_0, \eta_0g).
\]

Let \( g^{-1} \in \mathcal{G} \) such that \( \overline{g}^{-1}U_p = g^{-1}U_p \); then \([A, i, \overline{x}, \eta] = \Theta'_{diff}(\mathbb{F}_p)(I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times xU(N))) \).

The map \( \Theta'_{diff}(\mathbb{F}_p) \) is injective: let us fix \( y^{-1}, g^{-1}, g_1^{-1} \in \mathcal{G} \) and \( x, x_1 \in G(\mathbb{A}_f^\infty) \) such that there is an isomorphism:

\[
f_{ab} : (y_*, A_0, y_*i_0, y_*, \overline{\alpha}_0, x^{-1} \cdot y_* \overline{\alpha}_0, \eta_0g) \xrightarrow{\sim} (y_1, A_0, y_1i_0, y_1, \overline{\alpha}_0, x_1^{-1} \cdot y_1 \overline{\alpha}_0, \eta_0g_1).
\]

Denote by \( f \in J(\mathbb{Z}_p) \) the automorphism induced by \( f_{ab} \) on \((X, \xi, \overline{X})\) and let \( \xi = (y^{-1}f^{-1}y_1)_{ab} \in I(\mathbb{Q}) \) be the auto-quasi-isogeny of \((A_0, i_0, \overline{\alpha}_0) \) inducing \( y^{-1}f^{-1}y_1 \) on \((X, \xi, \overline{X})\). Since \( f \) is an isomorphism, we have:

(i) \( x_1^{-1}y_1, (\overline{\alpha}_0) \cdot H_1(f) = x^{-1}y_1, (\overline{\alpha}_0) \), so that \( x \equiv a_0H_1(y^{-1}f^{-1}y_1a_0^{-1}x_1 \mod U(N)) \);

(ii) \( \eta_0g_1f = \eta_0g \), hence \( g_1f \equiv g(\mod U_p) \).

Then we have:

\[
I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times x_1U(N)) = I(\mathbb{Q}) \cdot (\xi y^{-1}g^{-1}U_p \times \xi a_0^{-1} \cdot x_1U(N)) = I(\mathbb{Q}) \cdot ((y^{-1}g_1f)^{-1}U_p \times a_0H_1(y^{-1}f^{-1}y_1a_0^{-1} \cdot x_1U(N)) \ (ii) = I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times a_0H_1(y^{-1}f^{-1}y_1a_0^{-1} \cdot x_1U(N)) = I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times xU(N)).
\]

The Hecke-equivariance of \( \{\Theta'_{diff}(\mathbb{F}_p)_{K^r} \} \) (with respect to the projective systems of domain and codomain obtained by varying the level structures) is an easy consequence of the definition of the Hecke operators in this context: for \( \gamma \in G(\mathbb{A}_f^\infty) \) let us denote by \( H_\gamma \) the corresponding Hecke operator acting on the domain or codomain of \( \Theta'_{diff}(\mathbb{F}_p) \), for a suitable level subgroup. For \( y^{-1} \in \mathcal{Y}, g^{-1} \in \mathcal{G} \) and \( x \in G(\mathbb{A}_f^\infty) \) we have:

\[
I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times xU(N)) \xrightarrow{H_\gamma} I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times \gamma^{-1}x \cdot \gamma^{-1}U(N) \gamma) \xrightarrow{\Theta'_{diff}(\mathbb{F}_p)} [(y_*, A_0, y_*i_0, y_*, \overline{\alpha}_0, \gamma^{-1}x \cdot y_* (\gamma^{-1} \overline{\alpha}_0), \eta_0g)];
\]

\[
I(\mathbb{Q}) \cdot (y^{-1}g^{-1}U_p \times xU(N)) \xrightarrow{H_\gamma} [(y_*, A_0, y_*i_0, y_*, \overline{\alpha}_0, x^{-1} \cdot y_* \overline{\alpha}_0, \eta_0g)];
\]

and \( \gamma^{-1}x^{-1} \cdot \gamma \cdot y_* (\gamma^{-1} \overline{\alpha}_0) = \gamma^{-1} x^{-1} \cdot y_* \overline{\alpha}_0 \).

The isomorphism \( \Theta'_{diff}(\mathbb{F}_p) \) described above depends upon the choices of transversal \( \mathcal{G} \) and \( \mathcal{Y} \). We assume from now on that such transversal have been fixed.

For an algebraic \( \mathbb{F}_p \)-representation \( \rho : GL_d \to GL_d \) of the group \( GL_d \), we denote by \( M_\rho(N; \mathbb{F}_p) := M_\rho(D^U_{(r,s), p}; \mathbb{F}_p) \) the \( \mathbb{F}_p \)-vector space of unitary modular forms \( (mod p) \) of signature \((r, s)\) for the field \( k \), having weight \( \rho \) and level \( N \). Denote by \( \iota : Z' \hookrightarrow S_{D(U,N)} \otimes \mathbb{F}_p \) the closed immersion of \( \mathbb{F}_p \)-schemes associated to the inclusion of sets \( Z'(\mathbb{F}_p) \subset S_{D(U,N)}(\mathbb{F}_p) \). We want to consider "restrictions" of modular forms to the superspecial locus, as in [Cher04]. At this purpose, let \( \tau : G(U_r \times U_s) \to GL_m \) be an algebraic \( \mathbb{F}_p \)-representation of the \( \mathbb{F}_p \)-group \( G(U_r \times U_s) \).

**Definition 5.13** The space \( M_\tau^\dagger(N; \mathbb{F}_p) \) of \((r, s)\)-unitary superspecial modular forms \( (mod p) \) for the field \( k \), having weight \( \tau \) and level \( N \) is the finite dimensional \( \mathbb{F}_p \)-vector space whose elements \( f \) are rules that assign, to any tuple \((A, i, \overline{X}, \eta) / \mathbb{F}_p \) such that \([A, i, \overline{X}, \eta] \) is an element of \( Z'(\mathbb{F}_p) \) and \( \eta \) is an ordered basis of invariant differentials for \((A', i, \overline{X}')\), an element \( f(A, i, \overline{X}, \eta) \in \mathbb{F}_p^\dagger \) in such a way that:
(a) \( f(A, i, \lambda, \pi, \eta M) = \tau(M)^{-1} f(A, i, \lambda, \pi, \eta) \) for all \( M \in G(p) \simeq \text{Aut}_{F_p^2}(t^*_A; i^*, (., .)) \);

(b) if \( (A, i, \lambda, \pi, \eta) \simeq (A_1, i_1, \lambda_1, \pi_1, \eta_1) \) then \( f(A, i, \lambda, \pi, \eta) = f(A_1, i_1, \lambda_1, \pi_1, \eta_1) \).

We have another description of the above space:

**Proposition 5.14** For any algebraic \( F_p \)-representation \( \rho : GL_d \rightarrow GL_d \), denote by \( \text{Res} \rho \) its restriction to \( G(U_r \times U_s) \). Then:

\[
M_{\text{Res} \rho}^*(N; F_p) = H^0(Z'(F_p), \iota^* E_p).
\]

**Proof.** By Proposition 3.7, if \( f \in H^0(Z'(F_p), \iota^* E_p) \), then \( f \) satisfies (a) and (b) of Def. 5.13 Viceversa, let \( f \in M_{\text{Res} \rho}^*(N; F_p) \) so that \( f \) is an assignment on tuples \( (A, i, \lambda, \pi, \eta) \in F_p \), as in Def. 5.13 in particular \( \eta \) here is a basis of invariant differentials for \( A', i, \lambda' \). If \( \omega \) is any ordered basis of \( t_A \), there is a unique \( X_{\omega, \eta} \in GL_2(F_p) \) such that \( \omega = \eta X_{\omega, \eta} \) and we define \( f(A, i, \lambda, \pi, \omega) := \rho(X_{\omega, \eta})^{-1} : f(A, i, \lambda, \pi, \eta) \). This assignment is well defined as \( \rho \) is a representation of \( GL_2 \), and allows us to view \( f \) as an element of \( H^0(Z'(F_p), \iota^* E_p) \).

The definition of \( M_{\text{Res} \rho}^*(N; F_p) \) depends upon \( Z'(F_p) \), hence upon the choice of \( (A_0, i_0, \lambda_0, \pi_0) \) that we have fixed at the beginning. It is clear that the Hecke operators act upon \( M_{\text{Res} \rho}^*(N; F_p) \).

5.3.1 Algebraic modular forms

We briefly recall the definition of algebraic modular forms mod \( p \) in our context (cf. [Gro99]). By Proposition 4.3 we have identifications \( I(\mathbb{Q}_p) = J(\mathbb{Q}_p) \) and \( I(A'_p) = G(A'_p) \); we set \( U := U_p \times U(N) \) and view it as an open compact subgroup of \( I(A_f) \). The group \( I(\mathbb{Q}) \) is discrete inside \( I(A_f) \), by [MR96b], 6.23, so that the double coset space \( I(\mathbb{Q}) \backslash I(A_f) / U \) is finite, because \( I(\mathbb{Q}) \backslash I(A_f) \) is compact (cf. [Gro99], Prop. 1.4). Assume fixed an algebraic \( F_p \)-representation \( \tau : G(U_r \times U_s) \rightarrow GL_m \).

**Definition 5.15** The space of algebraic modular forms (mod \( p \)) for the group \( I \), having level \( U \) and weight \( \tau \) is the \( F_p \)-vector space (of finite dimension):

\[
M_{\tau}^{\text{alg}}(N; F_p) := \{ f : I(\mathbb{Q}) \backslash I(A_f) / U \rightarrow F_p^m : f(M) = \tau(M)^{-1} f(M), M \in G(p), \mathfrak{g} \in I(\mathbb{Q}) \backslash I(A_f) / U \},
\]

where the right action of \( G(p) \) on \( I(\mathbb{Q}) \backslash I(A_f) / U \) is induced by the identification \( G(p) = J(A_p) / U_p \).

The space \( M_{\tau}^{\text{alg}}(N; F_p) \) is endowed with a natural action of the Hecke algebra and our previous computations give:

**Proposition 5.16** There is an isomorphism of finite dimensional \( F_p \)-vector spaces endowed with Hecke action:

\[
M_{\tau}^{\text{alg}}(N; F_p) \simeq M_{\tau}^{\text{alg}}(N; F_p).
\]

**Proof.** By Proposition 5.12 we have an isomorphism of \( M_{\tau}^{\text{alg}}(N; F_p) \) with the space of functions \( Z_{\text{diff}}(F_p) \rightarrow F_p^m \) satisfying condition (a) of Def. 5.13 (Note that if \( g^{-1} \in G \), \( mU_p \in J(A_p) / U_p \) then by definition \( \eta_0 g \cdot mU_p = \eta_0 m^{-1} g \)).

5.4 The Hecke eigensystems correspondence

To establish the correspondence between Hecke eigensystems coming from algebraic modular forms and PEL-modular forms in the unitary settings, we begins by following [Ghi04a], 4.

Let \( \mathcal{I} \) be the ideal sheaf associated to the closed immersion of \( F_p \)-schemes \( \iota : Z' \rightarrow S_{\mathbb{D}, U(N)} \otimes F_p \), so that the following is an exact sequence of sheaves over \( S := S_{\mathbb{D}, U(N)} \otimes F_p \):

\[
0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_S \rightarrow \iota_* \mathcal{O}_{Z'} \rightarrow 0.
\]
Since $\mathcal{S}$ is of finite type over $\mathbb{P}_p$, it is noetherian, so that $\mathcal{I}$ is a coherent ideal sheaf on $\mathcal{S}$. After tensoring the above sequence with the locally free sheaf of $\mathcal{O}_{\mathcal{S}}$-modules $\mathbb{E}_p$ and then taking cohomology, we obtain the exact sequence:

$$0 \to H^0(\mathcal{S}, \mathcal{I} \otimes \mathbb{E}_p) \to H^0(\mathcal{S}, \mathbb{E}_p) \to H^0(\mathcal{S}, \iota_* \mathcal{O}_{\mathcal{Z}} \otimes \mathbb{E}_p).$$

By [Har77] III, 2.10 we have $H^0(\mathcal{S}, \iota_* \mathcal{O}_{\mathcal{Z}} \otimes \mathbb{E}_p) = H^0(\mathcal{Z}, \iota^* \mathbb{E}_p)$, since the projection formula gives $\iota_* \iota^* \mathbb{E}_p = \iota_* \mathcal{O}_{\mathcal{Z}} \otimes \mathbb{E}_p$. The above exact sequence of vector spaces can therefore be written, by Prop. 5.14 as:

$$0 \to H^0(\mathcal{S}, \mathcal{I} \otimes \mathbb{E}_p) \to M_\rho(N; \mathcal{F}_p) \overset{\iota}{\to} M^{ss}_{\text{Res}}(N; \mathcal{F}_p),$$

where the last map $\iota$ need not to be surjective. Recall that $\text{Res} \rho$ is the restriction of $\rho$ to the algebraic group $G(U_r \times U_s)$. Furthermore, observe that $\iota$ is Hecke-equivariant.

**Proposition 5.17** There exists a positive integer $m_0$ such that the above map $\iota$ is a surjection $M_{\rho \otimes \det^m}(N; \mathcal{F}_p) \overset{\iota}{\to} M^{ss}_{\text{Res}(\rho \otimes \det^m)}(N; \mathcal{F}_p)$ for any $m > m_0$.

**Proof.** The invertible sheaf of $\mathcal{O}_{\mathcal{S}}$-modules $\bigwedge^g \mathbb{E} = \mathbb{E}_{\text{det}}$ is ample over $\mathcal{S}$ (cf. for example [Lan08], Th. 7.24.1). The proposition now follows in the same way as [Ghi03], Prop. 24. ■

With the notation introduced in this section, we have:

**Theorem 5.18** Let $p$ be an odd prime and $k/\mathbb{Q}$ be a quadratic imaginary field extension in which $p$ is inert. Let $n$ be a positive integer and let $r, s$ be non-negative integers such that $r + s = g := 2n$. Let furthermore $N \geq 3$ be an integer not divisible by $p$. Denote by $I$ the reductive $\mathbb{Q}$-group whose $\mathbb{Q}$-rational points are given by $I(\mathbb{Q}) = \{X \in GU_g(\mathfrak{B}; I_B) : X\Phi = \Phi X\}$, where $\mathfrak{B}$ is the quaternion algebra over $\mathbb{Q}$ ramified at $p$ and $\infty$, and $\Phi$ is as in [7.1.1.]

The systems of Hecke eigenvalues coming from $(r, s)$-unitary PEL-modular forms $\mod p$ for the quadratic imaginary field $k$, having genus $g$, fixed level $N$ and any possible weight $\rho : GL_g \to GL_{m(\rho)}$, are the same as the systems of Hecke eigenvalues coming from (mod $p$) algebraic modular forms for the group $I$ having level $U = U_p \times U(N) \subset I(\mathbb{A}_f)$ and any possible weight $\rho' : G(U_r \times U_s) \to GL_{m(\rho')}$.

**Proof.** To prove the theorem we first show that any system of Hecke eigenvalues occurring in the spaces $\{M_\rho(N; \mathcal{F}_p)\}_\tau$ for variable $\rho : GL_g \to GL_{m(\rho)}$ also occurs in the spaces $\{M^{ss}_\tau(N; \mathcal{F}_p)\}_\tau$ for variable $\tau : G(U_r \times U_s) \to GL_{m(\tau)}$. Then we follow the proof of Th. 28 in [Ghi03], to show that the converse is also true, and finally we apply Proposition 5.10. Notice that the first part of this proof is different from the one given in [Ghi03].

For any integer $i \geq 0$ we have an exact sequence of $\mathcal{O}_{\mathcal{S}}$-modules:

$$0 \to \mathcal{I}^{i+1} \to \mathcal{I}^i \to \mathcal{I}^{i}/\mathcal{I}^{i+1} \to 0$$

giving rise to the exact sequence in cohomology:

$$0 \to H^0(\mathcal{S}, \mathcal{I}^{i+1} \otimes \mathbb{E}_p) \to H^0(\mathcal{S}, \mathcal{I}^i \otimes \mathbb{E}_p) \overset{\iota}{\to} H^0(\mathcal{S}, \mathcal{I}^{i}/\mathcal{I}^{i+1} \otimes \mathbb{E}_p),$$

which defines the homomorphisms $\iota_i$ for any $i \geq 0$ (notice that $\iota_0 = \iota$).

For any $j \geq 1$ we also have the exact sequence of sheaves of $\mathcal{O}_{\mathcal{S}}$-modules:

$$\mathcal{I} \otimes \mathcal{I}^j/\mathcal{I}^{j+1} \to \mathcal{O}_{\mathcal{S}} \otimes \mathcal{I}^j/\mathcal{I}^{j+1} \to \iota_* \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{I}^j/\mathcal{I}^{j+1} \to 0.$$

Notice that the image of the first map is zero, so that we obtain isomorphisms of $\mathcal{O}_{\mathcal{S}}$-modules $\mathcal{I}^j/\mathcal{I}^{j+1} \simeq \iota_* \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{I}^j/\mathcal{I}^{j+1}$ for any $j \geq 1$. In cohomology we have therefore canonical isomorphisms $\xi_j$ for any $j \geq 1$:

$$\xi_j : H^0(\mathcal{S}, \mathcal{I}^j/\mathcal{I}^{j+1} \otimes \mathbb{E}_p) \overset{\sim}{\to} H^0(\mathcal{Z}', \iota^*(\mathcal{I}^j/\mathcal{I}^{j+1} \otimes \mathbb{E}_p)).$$

Let $0 \neq f \in M_\rho(N; \mathcal{F}_p) = H^0(\mathcal{S}, \mathbb{E}_p)$ be a Hecke eigenform for some fixed weight $\rho : GL_g \to GL_m$. If $r(f) \neq 0$ then the system of Hecke eigenvalues associated to $f$ occurs in $M^{ss}_{\text{Res} \rho}(N; \mathcal{F}_p)$, since $\iota$ is Hecke-equivariant. If $r(f) = 0$, then $f \in H^0(\mathcal{S}, \mathcal{I}^h \otimes \mathbb{E}_p)$. We claim that there is a positive integer $h$ such that $f \in H^0(\mathcal{S}, \mathcal{I}^h \otimes \mathbb{E}_p)$ and $r_1(f) \neq 0$. Assume not, then $r_1(f) = 0$ and $f \in H^0(\mathcal{S}, \mathcal{I}^2 \otimes \mathbb{E}_p)$, so that we can compute $r_2(f)$ and we need to have $r_2(f) = 0$; therefore $f \in H^0(\mathcal{S}, \mathcal{I}^3 \otimes \mathbb{E}_p)$, etc. Reiterating this procedure we deduce that $f \in H^0(\mathcal{S}, \mathcal{I}^h \otimes \mathbb{E}_p)$.
for all integers $i > 0$ (recall that $\ldots \subseteq H^0(S, T^{i+1} \otimes \mathcal{E}_p) \subseteq H^0(S, T^i \otimes \mathcal{E}_p) \subseteq \ldots \subseteq H^0(S, T \otimes \mathcal{E}_p))$, so that $f = 0$, contradicting our assumption $f \neq 0$. Then there exists a positive integer $h$ such that $f \in H^0(S, T^h \otimes \mathcal{E}_p)$ and $r_h(f) \neq 0$. Let:

$$f^{ss} := \xi_h(r_h(f)) \in H^0(\mathcal{Z}', \iota^*(T^h/\mathcal{T}^{h+1} \otimes \mathcal{E}_p)).$$

Since $\xi_h$ is injective, $f^{ss}$ is non-zero. Observe that $T^h/\mathcal{T}^{h+1} = \text{Sym}^h(\mathcal{T}/\mathcal{T}^2)$ and that $\iota^*(\mathcal{T}/\mathcal{T}^2) = \iota^*(\mathcal{O}_k^1)$ (cf. [Ham77 II, 8.17]).

We now need to use the Kodaira-Spencer isomorphism for our PEL variety: let $\mathcal{X}/\mathfrak{M}_p$ denotes the universal abelian scheme over $\mathcal{S}$, endowed with the polarization $\lambda_{\text{univ}}$ and action $i_{\text{univ}}$ of the ring $\mathcal{O}_{k(p)}$; let $\mathcal{X}/\mathfrak{M}_p$ denotes the dual abelian scheme. By Prop. 2.3.4.2. in [Lan08], the Kodaira-Spencer map induces an isomorphism of $\mathcal{O}_S$-sheaves:

$$K_S : \frac{\mathcal{E}_X \otimes \mathcal{O}_S \mathcal{E}_X^*}{J'} \rightarrow \Omega^1_S,$$

where:

$$J' = \left( \begin{array}{c} \lambda^*_{\text{univ}}(y) \otimes z - \lambda^*_{\text{univ}}(z) \otimes y \\ i_{\text{univ}}(b)^*(x) \otimes y - x \otimes i_{\text{univ}}(b)^*(y) \end{array} \right) : x \in \mathcal{E}_X; y, z \in \mathcal{E}_X^*; b \in \mathcal{O}_{k(p)};$$

(the map $\lambda^*_{\text{univ}}$ denotes the pull-back isomorphism $\lambda^*_{\text{univ}} : \mathcal{E}_X \rightarrow \mathcal{E}_X$ and $\hat{i}_{\text{univ}}(b)^*$ denotes the endomorphism of $\mathcal{E}_X^*$ induced by $i_{\text{univ}}(b)$). Precomposing $K_S$ with $id \otimes \lambda^*$ we get the isomorphism of sheaves:

$$\frac{\text{Sym}^2 \mathcal{E}}{J} \rightarrow \Omega^1_S,$$

where $J = (i_{\text{univ}}(b)^*(x) \otimes y - x \otimes i_{\text{univ}}(b)^*(y)) : x, y \in \mathcal{E}$, $b \in \mathcal{O}_{k(p)}$ and $\mathcal{E} : = \mathcal{E}_X$ as usual. Write $\text{Sym}^2 \mathcal{E} := (\text{Sym}^2 \mathcal{E})/J$ and notice that $J$ is not preserved by the group $GL_q$ (if $rs > 0$), but it has an action of $GL_r \times GL_s$, because of the condition defined in the definition of the moduli problem.

We have:

$$H^0(\mathcal{Z}', \iota^*(T^h/\mathcal{T}^{h+1} \otimes \mathcal{E}_p)) = H^0(\mathcal{Z}', \iota^*(\text{Sym}^h(\mathcal{E}_X^2) \otimes \mathcal{E}_p)).$$

The group $GL_r \times GL_s$ acts on the sheaf $\iota^*(\text{Sym}^h(\mathcal{E}_X^2) \otimes \mathcal{E}_p)$, and this is enough for our purposes, as the space of superspecial modular forms is defined for representations $\tau$ of $G(U_r \times U_s) \subseteq GL_r \times GL_s$. We conclude that:

$$f^{ss} \in H^0(\mathcal{Z}', \iota^*(\text{Sym}^h(\mathcal{E}_X^2) \otimes \mathcal{E}_p)) = M^{ss}_{\text{Sym}^h(\text{Sym}^2 \mathcal{E}) \otimes \mathcal{E}_p},$$

where we are viewing $\text{Sym}^h(\text{Sym}^2(\text{std}) \otimes \text{Res} \rho$ as a representation of $G(U_r \times U_s)$ by restriction (std : $GL_g \rightarrow GL_g$ is the standard representation of $GL_g$). The maps $r_h$ and $\tilde{\xi}_h$ are Hecke equivariant; as N. Fakhruddin showed in [Fak09], the Kodaira-Spencer map is also Hecke-equivariant, modulo a scaling on the Hecke operators acting on $\text{Sym}^2 \mathcal{E}$. We deduce that, after performing the mentioned rescaling, the non-zero form $f^{ss}$ is an Hecke-eigenform with the same eigenvalues as our original $f$.

On the other side, let assume we are given a non-zero eigenform $f^{ss} \in M^{ss}_{\rho'}(N; \mathbb{F}_p)$ for some weight $\rho' : G(U_r \times U_s) \rightarrow GL_{m'}$. There is a rational $\mathbb{F}_p$-representation $\tilde{\rho} : GL_g \rightarrow GL_m$ whose restriction to $G(U_r \times U_s)$ contains $\rho'$. Indeed the algebraically induced representation $\rho'' := \text{Ind}_{G(U_r \times U_s)}^{GL_g} \rho'$ contains (non-canonically) a finite dimensional $G(U_r \times U_s)$-invariant subspace $\tau$ that is $G(U_r \times U_s)$-isomorphic to $\rho'$; by local finiteness there is a finite dimensional $GL_{m'}$-submodule $\tilde{\rho}$ of $\rho''$ containing $\tau$ as a $G(U_r \times U_s)$-submodule.

By Proposition 5.17 there is an integer $c > 0$ divisible by $p^2 - 1$ such that the map:

$$r : M^{ss}_{\tilde{\rho} \otimes \det}(N; \mathbb{F}_p) \rightarrow M^{ss}_{\text{Res}(\tilde{\rho} \otimes \det)}(N; \mathbb{F}_p) = M^{ss}_{\text{Res}(\tilde{\rho})(N; \mathbb{F}_p)}$$

is surjective; since $M^{ss}_{\rho'}(N; \mathbb{F}_p) \subseteq M^{ss}_{\text{Res}(\tilde{\rho})(N; \mathbb{F}_p)}$ and since $r$ is Hecke-equivariant, we see that a system of Hecke eigenvalues occurring in $M^{ss}_{\rho'}(N; \mathbb{F}_p)$ also occurs in $M^{ss}_{\text{Res}(\tilde{\rho})(N; \mathbb{F}_p)}$.

We conclude that the system of Hecke eigenvalues arising from our spaces of modular forms $M_{\rho}(N; \mathbb{F}_p)$ for varying $\rho : GL_g \rightarrow GL_m$, coincide with the systems of Hecke eigenvalues arising from the spaces $M^{ss}_{\rho'}(N; \mathbb{F}_p)$ for varying $\rho' : G(U_r \times U_s) \rightarrow GL_{m'}$. The theorem now follows from Proposition 5.16. ■

We presented here the construction of the Hecke correspondence in the PEL unitary case. One obtains the result of Ghitza (for $g > 1$) by forgetting about the action of the algebra with involution that appears in our
computations; observe that for Siegel modular forms, the superspecial locus has an easier shape, as explained by Remark 5.11.

Under suitable assumptions that guarantee that the superspecial locus is non-empty, one obtains a result similar to the above Theorem 5.18 in the context of Hilbert modular forms, i.e. for PEL-data of type C-I with $e \geq 1$ (cf. 5.1.2). One condition that needs to be satisfied in this case is the existence of an embedding of the fixed totally real field into the subset of symmetric matrices of $M_2(\mathfrak{B})$, for $\mathfrak{B}$ the quaternion algebra over $\mathbb{Q}$ ramified at $p$ and $\infty$ (here a matrix $X \in M_2(\mathfrak{B})$ is said to be symmetric if $X = \overline{X}$).

### 5.5 Computing the number of Hecke eigensystems

We continue to assume fixed the notation introduced in the previous section, in particular we work with unitary (mod $p$) PEL-modular forms of signature $(r,s)$, with $p > 2$. We want to give an estimate of the number of distinct (mod $p$) Hecke eigensystems occurring in the spaces $M_p(N;\overline{\mathbb{F}}_p)$ for $N$ fixed, and varying $\rho$; we follow the lines of [Ghi04b].

Denote by $\mathcal{N} := N(p;k,r,s;N)$ the number of distinct Hecke eigensystems occurring in the totality of spaces $M_p(N;\overline{\mathbb{F}}_p)$'s for $\rho$ varying over the set of representations of $GL_2$ over $\mathbb{F}_p$; by Th. 5.18 and Prop. 5.16, $\mathcal{N}$ is the number of distinct Hecke eigensystems occurring in the totality of spaces $M^\alpha_p(N;\overline{\mathbb{F}}_p)$ where $\rho$ now runs over the finite set $\text{Irr}(G(p))$ of isomorphism classes of irreducible finite-dimensional representations of $G(p) := G(U_r \times U_s)(\mathbb{F}_p)$ over $\mathbb{F}_p$. If $\rho : G(p) \to GL(W_\rho)$ is any fixed element representing a class in $\text{Irr}(G(p))$, we have:

$$M^\alpha_p(N;\overline{\mathbb{F}}_p) = \{ f : Z_{diff}(\overline{\mathbb{F}}_p) \to W_\rho : f([A,i,\overline{\alpha},\overline{\eta},\overline{M}]) = \rho(M)^{-1}f([A,i,\overline{\alpha},\overline{\eta}]) \}, \quad \text{all } M \in G(p), \quad [A,i,\overline{\alpha},\overline{\eta}] \in Z_{diff}(\overline{\mathbb{F}}_p),$$

so that, by definition of $Z'(\overline{\mathbb{F}}_p)$, we have $\dim_{\mathbb{F}_p} M^\alpha_p(N;\overline{\mathbb{F}}_p) = \# Z'(\overline{\mathbb{F}}_p) \cdot \dim_{\mathbb{F}_p} W_\rho$, and:

$$\mathcal{N} \leq \# Z'(\overline{\mathbb{F}}_p) \cdot \sum_{[\rho] \in \text{Irr}(G(p))} \dim_{\mathbb{F}_p} W_\rho. \quad (1)$$

We now give estimates of the two factors appearing in the right hand side of the last inequality.

#### 5.5.1 Estimate of the cardinality of the superspecial locus

In order to compute $\# Z'(\overline{\mathbb{F}}_p)$, one would like to have an explicit mass formula for superspecial varieties of the PEL-type considered here; lacking such an explicit formula, we can instead using what is known for Siegel varieties. Let us denote by $A := A_{g,1,N}$ the Siegel moduli scheme over $O_{k,(p)}$ classifying prime-to-$p$ isogeny classes of tuples $(A,\overline{\alpha},\overline{\eta})$, where $A$ is an abelian projective scheme over some $S \in SCH_{O_{k,(p)}}$ of relative dimension $g$, $\overline{\alpha}$ is a principal homogeneous polarization of $A$, and $\overline{\eta}$ is a full level $N$-structure on $(A,\overline{\alpha})$; there is a natural mapping $j$ from the moduli $O_{k,(p)}$-scheme $S := S_D$ associated to the PEL-datum $D$ of type $A$ that we fixed here, to $A$. More precisely, by fixing an isomorphism of $\mathbb{Q}$-vector spaces $V := k^g \simeq \mathbb{Q}^{2g}$ we obtain a monomorphism of $\mathbb{Q}$-groups $G^\alpha U_g(k,r,s) \to GSp_{2g}(J) \simeq GSp_{2g}$, where $J$ is some symmetric form on $\mathbb{Q}^{2g}$; then by definition, if $S$ is some locally noetherian $O_{k,(p)}$-scheme, $j$ sends the equivalence class $[(A,i,\overline{\alpha},\overline{\eta})] \in S(S)$ to the equivalence class $[(A,\overline{\alpha},\overline{\eta})] \in A(S)$, where $\overline{\eta}$ is the $U'(N)$-orbit of the symplectic isomorphism $\alpha : H_1(A,\mathbb{A}_K^p) \to V \otimes \mathbb{Q} K_r^p$, with $U'(N) := \text{Ker}(GSp_{2g}(\mathbb{Z}^p;J) \to GSp_{2g}(\mathbb{Z}^p/N\mathbb{Z}^p;J)) \) (notice that $U(N) = U'(N) \cap GU_g(O_k \otimes \mathbb{Z}^p; r,s))$.

By works of Kisin and Vasiu $\overline{\eta}$ is a closed immersion. For our purposes, we content ourselves with the fact that $j$ induces an injection on closed $\mathbb{F}_p$-points, and sends injectively the set of closed points $Z'(\overline{\mathbb{F}}_p)$ of the superspecial locus of the unitary PEL-variety $S \otimes \overline{\mathbb{F}}_p$ - relative to our choice of $(\mathcal{A}_0, i_0, \overline{\alpha}_0, \overline{\eta}_0)$ - into the superspecial locus of $A(\overline{\mathbb{F}}_p)$. We can use the estimate given in [Ghi04b], based on the explicit mass formula for superspecial principally polarized abelian $\overline{\mathbb{F}}_p$-varieties due to Ekedahl ([Ek07]) and based on work of Hashimoto-Ibukiyama ([KHS8]):

$$\# Z'(\overline{\mathbb{F}}_p) \leq C_g \cdot \# GSp_{2g}(\mathbb{Z}/N\mathbb{Z}) \cdot \prod_{i=1}^g \left( p^i + (-1)^i \right), \quad (2)$$

where the constant $C_g$ is defined to be:

$$C_g = \frac{-1}{2g} \cdot \left( \zeta(1 - 2i) \right)^{g(3g+1)/2} \cdot \prod_{i=1}^g \zeta(1 - 2i) = \frac{1}{2^g g!} \cdot \prod_{i=1}^g B_{2i},$$

(here $\zeta$ is the Riemann zeta function, and $B_{2i}$ denotes the $2i$-th Bernoulli number).
5.5.2 Estimates on the size of irreducible representations of $G(p)$

All the representations we consider in this section are finite dimensional over the appropriate field. The number of pairwise non-isomorphic irreducible representations of the finite group $G(p)$ over $\mathbb{F}_p$ coincides with the number $k^\pi(G(p))$ of $p$-regular conjugacy classes of $G(p)$; a matrix element $X$ of $G(p)$ is $p$-regular if and only if its minimal polynomial has only simple roots over $\mathbb{F}_p$, that is to say if and only if $X$ is semi-simple (over $\mathbb{F}_p$).

The group $G(p)$ is the set of $\mathbb{F}_p$-points of the connected reductive algebraic group $\mathbb{G} := G(U_r \times U_s)$ defined over $\mathbb{F}_p$; one can compute the center $Z$ and the derived subgroup $G'$ of $\mathbb{G}$ and find:

$$Z = Z^0 \cong \begin{cases} G(U_1 \times U_1) & \text{if } rs \neq 0, \\ GU_1 & \text{if } rs = 0, \end{cases}$$

$$G' = SU_r \times SU_s.$$ 

Since $G'$ is connected, simply-connected and semi-simple with rank:

$$rk(G') = \left\{ \begin{array}{ll} g-2 & \text{if } rs \neq 0, \\ g-1 & \text{if } rs = 0, \end{array} \right.$$ 

by applying Theorem 3.7.6 of [Car85], we have:

$$k^\pi(G(p)) = \#Z^0(\mathbb{F}_p) \cdot p^{rk(G')} = \begin{cases} p^{g-2} \cdot (p-1)(p+1)^2 & \text{if } rs \neq 0, \\ p^{g-1} \cdot (p-1)(p+1) & \text{if } rs = 0. \end{cases} \quad (3)$$

If $t \geq 2$, then $SU_t(\mathbb{F}_p)$ is the set of $\mathbb{F}_p$-points of a group of a simply connected group of type $2 A_{t-1}(p)$, and its order is (cf. [Car85] 2.9) $#SU_t(\mathbb{F}_p^2) = p^{t(r-1)} \cdot \prod_{i=2}^{t-1} (p^i-(1)^i)$. (We set $SU_0(\mathbb{F}_p^2) = SU_1(\mathbb{F}_p^2) := \{1\}$.) Using the exactness of the sequence $1 \to SU_t(\mathbb{F}_p^2) \to U_t(\mathbb{F}_p^2) \to det U_1(\mathbb{F}_p^2) \to 1$ for $t > 0$, one deduces that $#U_t(\mathbb{F}_p^2) = #SU_t(\mathbb{F}_p^2) \cdot (p+1)$. We conclude that for any choice of non-negative integers $r$ and $s$ such that $r+s = g$ we have:

$$#G(U_r \times U_s)(\mathbb{F}_p^2) = \#U_r(\mathbb{F}_p^2) \cdot #U_s(\mathbb{F}_p^2) \cdot (p-1) = p^{r(r-1) + s(s-1)} \cdot \prod_{i=1}^{r} (p^i-(1)^i) \cdot \prod_{i=1}^{s} (p^i-(1)^i) \cdot (p-1).$$

In particular, a $p$-Sylow subgroup of $G(U_r \times U_s)(\mathbb{F}_p^2)$ has order $p^{r(r-1) + s(s-1)}$. Since $G(p)$ is a group with a split $(B,N)$-pair (cf. [Car85] 1.18), we deduce that if $\rho : G(p) \to GL(W_\rho)$ is an irreducible representation of $G(p)$ over $\mathbb{F}_p$, then:

$$\dim_{\mathbb{F}_p} W_\rho \leq p^{\frac{r(r-1) + s(s-1)}{2}}. \quad (4)$$

(The proof of this fact is contained in [Cur70]; cf. esp. corollaries 3.5 and 5.11.) Putting together formulae (3) and (4) we obtain:

$$\sum_{[\rho] \in \text{Irr}(G(p))} \dim_{\mathbb{F}_p} W_\rho \leq \left\{ \begin{array}{ll} p^{g(r-1) + s(s-1)} \cdot p^{g-2} \cdot (p-1)^2 & \text{if } rs \neq 0, \\ p^{g(r-1) + s(s-1)} \cdot p^{g-1} \cdot (p-1)(p+1) & \text{if } rs = 0. \end{array} \right. \quad (5)$$

5.5.3 Upper bound for the number of Hecke eigensystems

Putting formulae (2) and (5) together into formula (1), we obtain:

**Theorem 5.19** Let $p > 2, k, r, s, N$ be fixed as above (in particular $r, s \geq 0$ and $r + s = g$) and set $C_g := 2^{-2g} (g!)^{-1} \prod_{i=2}^{g} B_{2i}$. The number $N := N(p, k, r, s; N)$ of distinct (mod $p$) Hecke eigensystems occurring in the spaces $M_\rho(N; \mathbb{F}_p)$ for varying $\rho$ satisfies the following inequality:

$$N \leq C_g \cdot #GS p_{2g}(\mathbb{Z}/NZ) \cdot \prod_{i=1}^{g} (p^i + (-1)^i) \cdot \left\{ \begin{array}{ll} p^{r(r-1) + s(s-1)} \cdot p^{g-2} \cdot (p-1)^2 & \text{if } rs \neq 0, \\ p^{r(r-1) + s(s-1)} \cdot p^{g-1} \cdot (p-1)(p+1) & \text{if } rs = 0. \end{array} \right.$$ 

In particular, if we keep $k, r, s, N$ fixed and let $p > 2$ vary:

$$N \in O(p^{g^2 + g + 1 - rs}), \quad \text{for } p \to \infty.$$
For an estimate of $N$ in the case of Siegel modular forms, cf. \cite{Ghi04b}; for elliptic modular forms, one can find a conjectural mass formula for the asymptotic with respect to $p$ of two-dimensional odd and irreducible Galois representations of $\mathbb{Q}$ in \cite{Cen09}.
References

[Bou97] J-F. Boutot, *Uniformisation p-adique des varit de shimura*, Sminaire Bourbaki, expos n 831 (1997).

[Car85] R.W. Carter, *Finite groups of lie type. conjugacy classes and complex characters*, John Wiley and Sons, 1985.

[Cen09] T. G. Centeleghe, *A conjectural mass formula for (mod p) galois representations*, Ph.D. thesis, University of Utah, 2009.

[Che76] I. V. Cherednik, *Uniformization of algebraic curves by discrete subgroups of pgl(2,kw) with compact quotient*, Math USSR Sbornik 29 (1976), 55–78.

[Cur70] C.W. Curtis, *Modular representations of finite groups with split (b,n)-pairs, in "seminar on algebraic groups and related finite groups"*, Lecture Notes in Mathematics, vol. 131, Springer, Berlin, 1970.

[Dem72] M. Demazure, *Lectures on p-divisible groups*, Lectures Notes in Mathematics, vol. 302, Springer, Berlin, 1972.

[Eke87] T. Ekedahl, *On supersingular curves and abelian varieties*, Math. Scand. v.60 (1987), 151–178.

[Fak09] N. Fakhruddin, Private correspondence with C. Khare (2009).

[Fon77] J-M. Fontaine, *Groupes p-divisibles sur les corps locaux*, Astrisque 47 - Socit Mathmatique de France, 1977.

[Fon94] ——, *Les corps des priodes p-adiques*, Astrisque 223 (1994), 59–102.

[GF90] C-L. Chai G. Faltings, *Degeneration of abelian varieties*, Springer, New York, 1990.

[Ghi03] A. Ghitza, *Siegel modular forms (mod p) and algebraic modular forms*, Available on ArXiv (2003).

[Ghi04a] ——, *Hecke eigenvalues of siegel modular forms (mod p) and of algebraic modular forms*, J. Number Th. 106 (2004), 345–384.

[Ghi04b] ——, *Upper bound on the number of systems of hecke eigenvalues for siegel modular forms (mod p)*, Int. Math. Research Notices 55 (2004).

[Gor02] E. Z. Goren, *Lectures on hilbert modular varieties and modular forms*, CRM Monograph Series, AMS, vol. 14, 2002.

[Gro99] B. H. Gross, *Algebraic modular forms*, Israel J. Math. 113 (1999), 61–93.

[Har77] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.

[KH80] T. Ibukiyama K. Hashimoto, *On class number of positive definite binary quaternion hermitian forms*, J. Fac. Sci. Univ. Tokyo 27 (1980), 549–601.

[Kot85] R. E. Kottwitz, *Isocrystals with additional structure*, Comp. Math. 2 (1985), 201–220.

[Kot92] ——, *Points on some shimura varieties over finite fields*, J. Amer. Math. Soc. 5 (1992), no. 2, 373–444.

[KZL98] F. Oort K-Z. Li, *Moduli of supersingular abelian varieties*, Lectures Notes in Mathematics, vol. 1680, Springer, Berlin, 1998.

[Lan08] K-W. Lan, *Arithmetic compactification of pel type shimura varieties*, Ph.D. thesis, Harvard University, 2008.

[Lew82] D. W. Lewis, *The isometry classification of hermitian forms over division algebras*, Linear Algebra Appl. 43 (1982), 245–272.

[Mes72] W. Messing, *The crystals associated to barsotti-tate groups: with applications to abelian schemes*, Lectures Notes in Mathematics, vol. 264, Springer, Berlin, 1972.
[MR96a] M. Richartz M. Rapoport, On the classification and specialization of f-isocrystals with additional structure, Comp. Math. 103 (1996), 153–181.

[MR96b] Th. Zink M. Rapoport, Period spaces for p-divisible groups, Annals of Mathematics Studies, Princeton University Press, 1996.

[Mum74] D. Mumford, Abelian varieties, Hindustan Book Agency, India, 1974.

[OB06] T. Wedhorn O. Bltel, Congruence relations for shimura varieties associated to some unitary groups, Jornal of the Inst. of Math. Jussieu 5 (2006), no. 2, 229–261.

[OB09] B. Conrad O. Brinon, Notes on p-adic hodge theory, CMI Summer School Notes (June-2009).

[Oda69] T. Oda, The first de rham cohomology group and dieudonn modules, Ann. Scient. Ec. Norm. Sup. 2 (1969), 63–135.

[PB82] W. Messing P. Berthelot, L. Breen, Thorie de dieudonn cristalline ii, Lecture Notes in Mathematics, vol. 930, Springer, Berlin, 1982.

[PC00] J-M. Fontaine P. Colmez, Construction des representations p-adiques semistables, Inv. Math 140 (2000), 1–43.

[Ser60] J-P. Serre, Groupes proalgebriques, Publications Mathematiques de l'I.H.E.S. 7 (1960), 5–67.

[Ser96] J-P. Serre, Two letters on quaternions and modular forms (mod p), Israel J. Math 95 (1996), 281–299.

[Shi66] G. Shimura, Moduli of abelian varieties and number theory, Algebraic Groups and Discontinuous Subgroups, Proc. Sym. Pure Math. 9 (1966), 306–332.

[Sil86] J. H. Silverman, The arithmetic of elliptic curves, Graduate Texts in Mathematics, vol. 106, Springer, New York, 1986.

[Tat66] J. Tate, Endomorphisms of abelian varieties over finite field, Invent. Math. (1966), 134–144.

[Vig80] M-F. Vignras, Arithmtique des algbrres de quaternions, Lecture Notes in Mathematics, vol. 800, Springer, Berlin, 1980.

[Wed99] T. Wedhorn, Ordinariness in good reduction of shimura varieties of pel-type, Ann. Scient. Ec. Norm. Sup. 32 (1999), 575–618.

[Wed07] T. Wedhorn, The genus of the endomorphisms of a supersingular elliptic curve, Astrisque 312 (2007), 25–47.

[Yu02] C-F. Yu, Lifting abelian varieties with additional structures, Math. Z. 242 (2002), 427–441.