Ideal Membership Problem for Boolean Minority

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Abstract

The Ideal Membership Problem (IMP) tests if an input polynomial \(f \in \mathbb{F}[x_1, \ldots, x_n]\) with coefficients from a field \(\mathbb{F}\) belongs to a given ideal \(I \subseteq \mathbb{F}[x_1, \ldots, x_n]\). It is a well-known fundamental problem with many important applications, though notoriously intractable in the general case. In this paper we consider the IMP for polynomial ideals encoding combinatorial problems and where the input polynomial \(f\) has degree at most \(d = O(1)\) (we call this problem \(\text{IMP}^d\)).

A dichotomy result between “hard” (NP-hard) and “easy” (polynomial time) IMPs was recently achieved for Constraint Satisfaction Problems over finite domains\(^2\) and \(\text{IMP}^d\) for the Boolean domain\(^3\), both based on the classification of the IMP through functions called polymorphisms. For the latter result, there are only six polymorphisms to be studied in order to achieve a full dichotomy result for the \(\text{IMP}^d\). The complexity of the \(\text{IMP}^d\) for five of these polymorphisms has been solved in \(^3\) whereas for the ternary minority polymorphism it was incorrectly declared in \(^3\) to have been resolved by a previous result. As a matter of fact the complexity of the \(\text{IMP}^d\) for the ternary minority polymorphism is open.

In this paper we provide the missing link by proving that the \(\text{IMP}^d\) for Boolean combinatorial ideals whose constraints are closed under the minority polymorphism can be solved in polynomial time.

This is achieved by first showing that a Gröbner basis can be efficiently computed in the lexicographic order for these ideals. Since this is insufficient for the efficient solvability of the \(\text{IMP}^d\), we show how this Gröbner basis can be converted to a \(d\)-truncated Gröbner basis in graded lexicographic order in polynomial time which ensures the achievement of the result. This result, along with the results in \(^3\), completes the identification of the precise borderline of tractability for the \(\text{IMP}^d\) for constrained problems over the Boolean domain.

This paper is motivated by the pursuit of understanding the recently raised issue of bit complexity of Sum-of-Squares proofs raised by O’Donnell\(^4\). Raghavendra and Weitz\(^5\) show how the \(\text{IMP}^d\) tractability for combinatorial ideals implies bounded coefficients in Sum-of-Squares proofs.
1 Introduction

A polynomial ideal is a subset of the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ with two properties: for any two polynomials $f, g$ in the ideal, $f + g$ also belongs to the ideal and so does $hf$ for any polynomial $h$. The Hilbert Basis Theorem [8] states that every ideal $I$ is finitely generated by a set $F = \{f_1, \ldots, f_m\} \subset I$, i.e., any polynomial in $I$ is a polynomial combination of elements from $F$. The polynomial Ideal Membership Problem (IMP) is to find out if a polynomial $f$ belongs to an ideal $I$ or not, given a set of generators of the ideal. This fundamental algebraic complexity problem was first pioneered by David Hilbert [9] and has important applications in solving polynomial systems and polynomial identity testing [5, 19]. The IMP is, in general, EXPSPACE-complete and Mayr and Meyer show that the problem for multivariate polynomials over the rationals is solvable in exponential space [14, 15]. The IMP is intractable (can be decided in single exponential time [6]) even when the ideal in question is zero-dimensional (number of common zeros of generators is finite).

The vanishing ideal of a set $S \subseteq \mathbb{F}^n$ is the set of all polynomials in $\mathbb{F}[x_1, \ldots, x_n]$ that vanish at every point of $S$. This set of polynomials forms an ideal. In this paper we consider vanishing ideals of the sets $S$ of feasible solutions that arise from Boolean combinatorial optimization problems. The vanishing ideal of the solution space $S$ is defined as its combinatorial ideal. We consider the IMP for polynomial ideals encoding combinatorial problems. We call such problems where the input polynomial $f$ has degree at most $d = O(1)$ as IMP$_d$. The polynomial ideals that arise from combinatorial optimization problems frequently have special properties: these ideals are finite domain and therefore zero-dimensional and radical. The question of identifying problem restrictions which are sufficient to ensure the IMP$_d$ tractability is important from both a practical and a theoretical viewpoint, and has an immediate application to Sum-of-Squares (SoS) proof systems (or Lasserre relaxations) as explained in the following.

The SoS proof system is an increasingly popular tool to solve combinatorial optimization problems. Especially over the last few decades, SoS has had several applications in continuous and discrete optimization (see, e.g., [12]). It was generally believed that a degree $d$ SoS proof could be computed (if one existed) via the Ellipsoid algorithm in $n^{O(d)}$ time. O’Donnell [16], who initially also believed this, gave a counterexample: a polynomial system and a polynomial which had degree two proofs of non-negativity with coefficients of exponential bit-complexity that forced the Ellipsoid algorithm to take exponential time. O’Donnell [16] raised the open problem to establish useful conditions under which “small” SoS proof can be guaranteed automatically. A first elegant approach to this question is due to Raghavendra and Weitz [17] by providing a sufficient condition on a polynomial system that implies bounded coefficients in SoS proofs. In particular, the work of Raghavendra and Weitz [17] shows that the IMP$_d$ tractability for combinatorial ideals implies polynomially bounded coefficients in SoS proofs. Therefore, the IMP$_d$ tractability yields to degree $d$ SoS proof (if one exists) computation via the Ellipsoid algorithm in $n^{O(d)}$ time. Hence the following question poses itself: Which restrictions on combinatorial problems can guarantee an efficient computation of the IMP$_d$?

In this paper we consider restrictions on the so-called constraint language, namely a set of relations that is used to form the constraints of the considered combinatorial optimization problem. Each constraint language $\Gamma$ gives rise to a particular polynomial ideal membership
problem, denoted $\text{IMP}_d(\Gamma)$, and the goal is to describe the complexity of the $\text{IMP}_d(\Gamma)$ for all constraint languages $\Gamma$. This kind of restrictions on the constraint languages have been successfully applied to study the computational complexity classification (and other algorithmic properties) of the decision version of Constraint Satisfaction Problems (CSP) over a fixed constraint language $\Gamma$ on a finite domain, denoted $\text{CSP}(\Gamma)$ (see Section 1.1). This classification started with the classic dichotomy result of Schaefer [18] for 0/1 CSPs, and culminated with the recent papers by Bulatov [2] and Zhuk [21], settling the long-standing Feder-Vardi dichotomy conjecture for finite domain CSPs. We refer to [3] for an excellent survey. Note that $\text{CSP}(\Gamma)$ corresponds to the very special case of the $\text{IMP}_d(\Gamma)$ with $d = 0$, i.e. where we are only interested in testing if the constant polynomial “1” belongs to the combinatorial ideal (see Appendix B.1 for more details on Ideal-CSP correspondence). In this paper we are interested in the problem with $d \geq 1$.

Mastrolilli [13] recently claimed a dichotomy result for the $\text{IMP}_d(\Gamma)$ that fully answers the above question for 0/1 combinatorial problems: for any constant $d \geq 1$, the $\text{IMP}_d(\Gamma)$ of Boolean combinatorial ideals is either decidable in polynomial time or it is NP-complete. Note that the solvability of $\text{CSP}(\Gamma)$ (and therefore of the $\text{IMP}_0(\Gamma)$) in the Boolean domain is known to admit a nice dichotomy result [18]: it is solvable in polynomial time if all constraints are closed under one of six polymorphisms (majority, minority, MIN, MAX, constant 0 and constant 1), else it is NP-complete. In [13] it is claimed that the $\text{IMP}_d(\Gamma)$ for the Boolean domain also has a nice dichotomy result: it is solvable in polynomial time if all constraints are closed under one of four polymorphisms (majority, minority, MIN, MAX), else it is NP-complete. The complexity of the $\text{IMP}_d(\Gamma)$ for five of these polymorphisms has been solved in [13] whereas for the ternary minority polymorphism it was incorrectly declared in [13] to have been resolved by a previous result. As a matter of fact the complexity of the $\text{IMP}_d(\Gamma)$ for the ternary minority polymorphism is open.

In this paper we solve this issue by providing the missing link and therefore establishing the full dichotomy result claimed in [13]. To ensure efficiency of the $\text{IMP}_d$, it is sufficient to compute a $d$-truncated Gröbner basis in the graded lexicographic order (see Definition 1.5, Section 1.1, and Appendix B for definitions and more details). This is achieved by first showing that a Gröbner basis can be efficiently computed in the lexicographic order for the minority polymorphism. Since this is insufficient for the efficient solvability of the $\text{IMP}_d$, we show how this Gröbner basis can be converted to a $d$-truncated Gröbner basis in the graded lexicographic order in polynomial time. This efficiently solves the $\text{IMP}_d$ for combinatorial ideals whose constraints are over a language closed under the minority polymorphism. Together with the results in [13], our result allows to complete the answer of the aforementioned question by allowing to identify the precise borderline of tractability of the $\text{IMP}_d(\Gamma)$.

Moreover, we believe the techniques described in this paper can be generalized for a finite domain with prime $p$ elements. The basis of this claim comes from the fact that constraints that are linear equations (mod $p$) are associated with an affine polymorphism [11]. We claim that the $\text{IMP}_d$ is tractable for problems that are constrained as linear equations (mod $p$). The details are currently being worked out and will soon be updated in the full version of this paper. This is a first step towards the long term and challenging goal of generalizing the dichotomy results of solvability of the $\text{IMP}_d$ for finite domains.

**Structure of the paper:** Section 1.1 contains the basic definitions required for this paper, although a reader unfamiliar with CSPs over a constraint language or algebraic...
geometry and Gröbner bases is strongly recommended to read the standard literature \cite{4, 5} or Appendix B.

We concretely state our results in Section 1.2. In Section 2 we show that the reduced Gröbner basis in lexicographic order can be efficiently computed for combinatorial problems constrained under the minority polymorphism. This is achieved in Section 2 by first computing a Gröbner basis in modular arithmetic and then transforming it into a Gröbner basis $G_1$ in regular arithmetic. However, this Gröbner basis is in the lexicographic monomial ordering, and does not guarantee the efficient solvability of the $\text{IMP}_d$. In Section 3 we show how to convert $G_1$ to a $d$-truncated Gröbner basis $G_2$ in graded lexicographic monomial ordering. We prove that this conversion can be obtained in polynomial time for any fixed $d = O(1)$.

A simple example is provided in Section 4.

1.1 Preliminaries

Let $D$ denote a finite set (domain). By a $k$-ary relation $R$ on a domain $D$ we mean a subset of the $k$-th cartesian power $D^k$; $k$ is said to be the arity of the relation. We often use relations and (affine) varieties interchangeably since both essentially represent a set of solutions. A constraint language $\Gamma$ over $D$ is a set of relations over $D$. A constraint language is finite if it contains finitely many relations, and is Boolean if it is over the two-element domain $\{0, 1\}$. In this paper, $D$ is the Boolean domain.

A constraint over a constraint language $\Gamma$ is an expression of the form $R(x_1, \ldots, x_k)$ where $R$ is a relation of arity $k$ contained in $\Gamma$, and the $x_i$ are variables. A constraint is satisfied by a mapping $\phi$ defined on the $x_i$ if $(\phi(x_1), \ldots, \phi(x_k)) \in R$.

Definition 1.1. The (nonuniform) Constraint Satisfaction Problem (CSP) associated with language $\Gamma$ over $D$ is the problem $\text{CSP}(\Gamma)$ in which: an instance is a triple $\mathcal{C} = (X, D, C)$ where $X = \{x_1, \ldots, x_n\}$ is a set of $n$ variables and $C$ is a set of constraints over $\Gamma$ with variables from $X$. The goal is to decide whether or not there exists a solution, i.e. a mapping $\phi : X \rightarrow D$ satisfying all of the constraints. We will use $\text{Sol}(\mathcal{C})$ to denote the set of solutions of $\mathcal{C}$.

Moreover, we follow the algebraic approach to Schaefer’s dichotomy result \cite{18} formulated by Jeavons \cite{10} where each class of CSPs that are polynomial time solvable is associated with a polymorphism.

Definition 1.2. An operation $f : D^m \rightarrow D$ is a polymorphism of a relation $R \subseteq D^k$ if for any choice of $m$ tuples from $R$ (allowing repetitions), it holds that the tuple obtained from these $m$ tuples by applying $f$ coordinate-wise is in $R$. If this is the case we also say that $f$ preserves $R$, or that $R$ is invariant or closed with respect to $f$. A polymorphism of a constraint language $\Gamma$ is an operation that is a polymorphism of every $R \in \Gamma$.

In this paper we deal with the minority polymorphism:

Definition 1.3. For a finite domain $D$, a ternary operation $f$ is called a minority polymorphism (denoted as $\text{Minority}$) if $f(a, a, b) = f(a, b, a) = f(b, a, a) = b$ for all $a, b \in D$.

Note that there is only one minority polymorphism ($\text{Minority}$ in short) for the Boolean domain.
Example 1.1. Consider relations \( R_1 = \{(0,0,1),(1,0,0),(0,1,1),(1,1,0)\} \) and \( R_2 = \{(1,1),(0,1)\} \) associated with language \( \Gamma \) over \( D = \{0,1\} \). Observe that both \( R_1 \) and \( R_2 \) are closed under \textit{Minority}. Consider the instance \((X = \{x,y,z\}, D, C = \{C_1, C_2\})\) where constraint \( C_1 = R_1(x,y,z) \) and \( C_2 = R_2(x,z) \). The assignment \( \phi \) where \( \phi(x) = 0, \phi(y) = 0, \phi(z) = 1 \) is a solution to this instance of \( \text{CSP}(\Gamma) \).

For a given instance \( C \) of \( \text{CSP}(\Gamma) \), the \textit{combinatorial ideal} \( I(Sol(C)) \) is defined as the vanishing ideal of set \( Sol(C) \), (see Definition 1.1 in Appendix B). We call polynomials of the form \( x_i(x_i - 1) \) \textit{domain polynomials}, denoted by \( \text{dom}(x_i) \), and it is easy to see that they belong to \( I(Sol(C)) \) for every \( i \in [n] \) as they describe the fact that \( Sol(C) \subseteq D^n \). For a more detailed Ideal-CSP correspondence we refer to Appendix B.1.

Definition 1.4. The \textbf{Ideal Membership Problem} associated with language \( \Gamma \) is the problem \( \text{IMP}(\Gamma) \) in which the input consists of a polynomial \( f \in \mathbb{F}[X] \) and a \( \text{CSP}(\Gamma) \) instance \( C = (X, D, C) \). The goal is to decide whether \( f \) lies in the combinatorial ideal \( I(Sol(C)) \). We use \( \text{IMP}_d(\Gamma) \) to denote \( \text{IMP}(\Gamma) \) when the input polynomial \( f \) has degree at most \( d \).

The Gröbner basis \( G \) of an ideal is a set of generators such that \( f \in \langle G \rangle \iff f|_G = 0 \), where \( f|_G \) denotes the remainder of \( f \) divided by \( G \) (see [5] or Appendix B.2 for more details and notations).

Definition 1.5. If \( G \) is a Gröbner basis of an ideal, the \textbf{\( d \)-truncated Gröbner basis} \( G' \) of \( G \) is defined as
\[
G' = G \cap \mathbb{F}[x_1, x_1, \ldots, x_n]_d,
\]
where \( \mathbb{F}[x_1, x_1, \ldots, x_n]_d \) is the set of polynomials of degree less than or equal to \( d \).

It is not necessary to compute a Gröbner basis of \( I(Sol(C)) \) in its entirety to solve the \( \text{IMP}_d \). Since the input polynomial \( f \) has degree \( d = O(1) \), the only polynomials from \( G \) that can possibly divide \( f \), in the graded lexicographic order (see Definition B.5 in Appendix B.2), are those that are in \( G' \). The remainders of such divisions are also in \( \mathbb{F}[x_1, x_1, \ldots, x_n]_d \). Therefore, by Proposition B.3 and Corollary B.4, the membership test can be computed by using only polynomials from \( G' \) and therefore we have
\[
f \in I(Sol(C)) \cap \mathbb{F}[x_1, x_1, \ldots, x_n]_d \iff f|_{G'} = 0.
\]

From the previous observations it follows that if we can compute \( G' \) in \( n^{O(d)} \) time then this yields an algorithm that runs in \( n^{O(d)} \) time for the \( \text{IMP}_d \) (note that the size of the input polynomial \( f \) is bounded by \( n^{O(d)} \)).

1.2 Our contributions

In this paper we focus on instances \( C = (X = \{x_1, \ldots, x_n\}, D = \{0,1\}, C) \) of \( \text{CSP}(\Gamma) \) (see Definition 1.1) where \( \Gamma \) is a language that is closed under \textit{Minority} (see Definition 1.3). We first produce the reduced Gröbner basis \( G_1 \) of \( I(Sol(C)) \) according to the lexicographic order. Note that this Gröbner basis does not guarantee finding a solution to the \( \text{IMP}_d(\Gamma) \) in polynomial time. In Section B we show how to convert \( G_1 \) to a \( d \)-truncated Gröbner basis \( G_2 \) for a graded lexicographic monomial ordering. We prove that this computation can be
obtained in polynomial time for any fixed \( d = O(1) \). As pointed out at the end of Section 1.1, an efficient computation of \( G_2 \) yields an efficient algorithm for the \( \text{IMP}_d \). A simple example is provided in Section 4. Thus we have the following main results:

**Theorem 1.1.** The \( d \)-truncated reduced Gröbner basis of a Boolean combinatorial ideal whose constraints are closed under the minority polymorphism can be computed in \( n^{O(d)} \) time, assuming the graded lexicographic ordering of monomials.

This proves the following:

**Corollary 1.2.** The \( \text{IMP}_d(\Gamma) \), over the Boolean domain, can be solved in polynomial time for \( d = O(1) \) if the solution space of every constraint in \( \Gamma \) is closed under the minority polymorphism.

**Structure of the proof:** A high level description of the proof structure is as follows. Each constraint that is closed under the minority polymorphism can be written in terms of linear equations (mod 2) (see e.g. [4]). In Section 2, we first express these equations in their reduced row echelon form: that is to say the ‘leading variable’ (the variable that comes first in the lexicographic order or \( \text{lex} \) in short, see Definition B.5) in each equation does not appear in any other (mod 2) equation. We then show how each polynomial in (mod 2) translates to a polynomial in regular arithmetic with exactly the same 0/1 solutions. The use of elementary symmetric polynomials allows for an efficient computation of the polynomials in regular arithmetic. Using these, we produce a set of polynomials \( G_1 \) and prove that \( G_1 \) is the reduced Gröbner basis of \( I(\text{Sol}(\mathcal{C})) \) in the \( \text{lex} \) order. As already mentioned, a Gröbner basis in the \( \text{lex} \) order does not guarantee the efficient solvability of the \( \text{IMP}_d \). We provide a conversion algorithm in Section 3 which converts \( G_1 \) to the \( d \)-truncated reduced Gröbner basis \( G_2 \) of \( I(\text{Sol}(\mathcal{C})) \) in the graded lexicographic ordering (\( \text{grlex} \) for short, see Definition B.5). In Section 3.1 we show how polynomials in \( G_1 \) from Section 2 are handled so our conversion algorithm in Section 3.2 works in polynomial time. Theorem 3.3 proves the correctness and polynomial running time of the conversion algorithm. This gives the proof of the main results of the paper stated in Theorem 1.1 and Corollary 1.2.

## 2 Gröbner bases in \( \text{lex} \) order

Consider an instance \( \mathcal{C} = (X = \{x_1, \ldots, x_n\}, D = \{0, 1\}, C) \) of CSP(\( \Gamma \)) where \( \Gamma \) is a language that is closed under \( \text{Minority} \). Any constraint of \( \mathcal{C} \) can be written as a system of linear equations over GF(2) (see e.g. [4]). These linear systems with variables \( x_1, \ldots, x_n \) can be solved by Gaussian elimination. If there is no solution, then we have from Hilbert’s Weak Nullstellensatz (Theorem B.2) that \( 1 \in I(\text{Sol}(\mathcal{C})) \) \( \iff \) \( \text{Sol}(\mathcal{C}) = \emptyset \) \( \iff \) \( I(\text{Sol}(\mathcal{C})) = \mathbb{R}[x] \). If \( 1 \in I(\text{Sol}(\mathcal{C})) \) the reduced Gröbner basis is \( \{1\} \). We proceed only if \( \text{Sol}(\mathcal{C}) \neq \emptyset \). In this section, we assume the \( \text{lex} \) order \( >_{\text{lex}} \) with \( x_1 >_{\text{lex}} x_2 >_{\text{lex}} \cdots >_{\text{lex}} x_n \). We also assume that the linear system has \( r \leq n \) equations and is already in its reduced row echelon form with \( x_i \) as the leading monomial of the \( i \)-th equation. Let \( \text{Supp}_i \subset [n] \) such that \( \{x_j : j \in \text{Supp}_i\} \) is the set of variables appearing in the \( i \)-th equation of the linear system except for \( x_i \). Let the \( i \)-th equation be \( R_i = 0 \) (mod 2) where

\[
R_i := x_i \oplus f_i,
\]
with \( i \in [r] \) and \( f_i \) is the Boolean function \((\bigoplus_{j \in \text{Supp}_i} x_j) \oplus \alpha_i \) and \( \alpha_i = 0/1 \).

### 2.1 From \((\text{mod } 2)\) to regular arithmetic Gröbner basis

In this section, we show how to transform \( R_i \)'s into polynomials in regular arithmetic. The idea is to map \( R_i \) to a polynomial \( R'_i \) over \( \mathbb{R}[x_1, \ldots, x_n] \) such that \( a \in \{0, 1\}^n \) satisfies \( R_i = 0 \) if and only if \( a \) satisfies \( R'_i = 0 \). Moreover, \( R_i \) is such that it has the same leading term as \( R'_i \). We produce a set of polynomials \( G_1 \) and prove that \( G_1 \) is the reduced Gröbner basis of \( I(Sol(C)) \) over \( \mathbb{R}[x_1, \ldots, x_n] \) in the lex ordering. We define \( R'_i \) as

\[
R'_i := x_i - M(f_i) \tag{2}
\]

where

\[
M(f_i) = \begin{cases} 
\sum_{k=1}^{\left|\text{Supp}_i\right|} (-1)^{k-1} \cdot 2^{k-1} \sum_{\{x_{j_1}, \ldots, x_{j_k}\} \subseteq \text{Supp}_i} x_{j_1}x_{j_2} \cdots x_{j_k} & \text{when } \alpha_i = 0 \\
1 + \sum_{k=1}^{\left|\text{Supp}_i\right|} (-1)^{k} \cdot 2^{k-1} \sum_{\{x_{j_1}, \ldots, x_{j_k}\} \subseteq \text{Supp}_i} x_{j_1}x_{j_2} \cdots x_{j_k} & \text{when } \alpha_i = 1
\end{cases} \tag{3}
\]

**Lemma 2.1.** Consider the following set of polynomials:

\[
G_1 = \{R'_1, \ldots, R'_r, x_{r+1}^2 - x_{r+1}, \ldots, x_n^2 - x_n\}, \tag{4}
\]

where \( R'_i \) is from Eq. (2). \( G_1 \) is the reduced Gröbner basis of \( I(Sol(C)) \) in the lexicographic order \( x_1 >_{\text{lex}} x_2 >_{\text{lex}} \ldots >_{\text{lex}} x_n \).

**Proof.** For any two Boolean variables \( x \) and \( y \),

\[
x \oplus y = x + y - 2xy \tag{5}
\]

By repeatedly using Eq. (5) to obtain the equivalent expression for \( f_i \), we see that \( R_i = 0 \ (\text{mod } 2) \) and \( R'_i = 0 \) have the same set of 0/1 solutions. Therefore \( V(\langle G_1 \rangle) \) is equal to \( Sol(C) \). This implies that \( \langle G_1 \rangle \subseteq I(Sol(C)) \). Moreover, \( \text{LM}(R_i) = \text{LM}(R'_i) = x_i \), by construction. For every pair of polynomials in \( G_1 \) the reduced \( S \)-polynomial is zero as the leading monomials of any two polynomials in \( G_1 \) are relatively prime. By Buchberger's Criterion (see Theorem B.5) it follows that \( G_1 \) is a Gröbner basis of \( \langle G_1 \rangle \) over \( \mathbb{R}[x_1, \ldots, x_n] \) (according to the lex order). In fact, it can be seen by inspection that \( G_1 \) is the reduced Gröbner basis of \( \langle G_1 \rangle \). To prove that \( I(Sol(C)) = \langle G_1 \rangle \), we need to prove that any \( p \in I(Sol(C)) \implies p \in \langle G_1 \rangle \). It is enough to prove that \( p|_{G_1} = 0 \) as this implies \( p \in \langle G_1 \rangle \). We have that \( p|_{G_1} \) cannot contain variable \( x_i \) for all \( 1 \leq i \leq r \). Hence \( p|_{G_1} \) is multilinear in \( x_{r+1}, x_{r+2}, \ldots, x_n \). Each tuple of \( D^{n-r} \) extends to exactly that \( n \)-tuple in \( Sol(C) \) whose coordinate associated with \( x_i \ (1 \leq i \leq r) \) is the unique value \( x_i \) takes to satisfy \( x_i \oplus f_i = 0 \) (see Eq. (1) and Eq. (2)). As \( p|_{G_1} \) is multilinear in \( x_{r+1}, x_{r+2}, \ldots, x_n \), there are at most \( 2^{n-r} \) coefficients. Since every point of \( D^{n-r} \) is a solution of \( p|_{G_1} \), we see that every coefficient of \( p|_{G_1} \) is zero and hence \( p|_{G_1} \) is the zero polynomial. Hence \( G_1 \) is the reduced Gröbner basis of \( I(Sol(C)) \).
Example 2.1. Consider a system with just one equation with $R_1 := x_1 \oplus x_2 \oplus x_3 = 0$ where $x_1 >_{\text{lex}} x_2 >_{\text{lex}} x_3$. Then $f_1 := x_2 \oplus x_3$ and $M(f_1) := x_2 + x_3 - 2x_2x_3$. The polynomial corresponding to Eq. (2) is

$$R'_1 := x_1 - x_2 - x_3 + 2x_2x_3.$$ 

The equations $R_1 = 0$ and $R'_1 = 0$ have the same set of 0/1 solutions and $\text{LM}(R_1) = \text{LM}(R'_1) = x_1$. For every pair of polynomials in $G = \{R'_1, x_2^2 - x_2, x_3^2 - x_3\}$ the reduced $S$-polynomial is zero. By Buchberger’s Criterion (see e.g. [4] or Theorem B.5 in the appendix) it follows that $G$ is a Gröbner basis over $\mathbb{R}[x_1, x_2, x_3]$ (according to the specified lex order).

Note that the reduced Gröbner basis in Eq. (4) can be “efficiently” computed by exploiting the high degree of symmetry in each $M(f_i)$ and using a version of the elementary symmetric polynomials.

3 Conversion of basis

Now that we have the reduced Gröbner basis in lex order, we show how to obtain the $d$-truncated reduced Gröbner basis in grlex order in polynomial time for any fixed $d = O(1)$. Before we describe our conversion algorithm, we show how to expand a product of Boolean functions. This expansion will play a crucial step in our algorithm.

3.1 Expansion of a product of Boolean functions

In this section, we show a relation between a product of Boolean functions and (mod 2) sums of the Boolean functions, which is heavily used in our conversion algorithm in Section 3.2. We have already seen from Eq. (5) that if $f, g$ are two Boolean functions, then

$$2 \cdot f \cdot g = f + g - (f \oplus g).$$

Hence it can be proved by repeated use of the above equation that the following holds for Boolean functions $f_1, f_2, \ldots, f_m$:

$$f_1 \cdot f_2 \cdots f_m = \frac{1}{2^{m-1}} \left[ \sum_{i \in [m]} f_i - \sum_{\{i,j\} \subset [m]} (f_i \oplus f_j) + \sum_{\{i,j,k\} \subset [m]} (f_i \oplus f_j \oplus f_k) + \cdots + (-1)^{m-1} (f_1 \oplus f_2 \oplus \cdots \oplus f_m) \right].$$ (6)

We call each Boolean function of the form $(f_{i_1} \oplus \cdots \oplus f_{i_k})$ in Eq. (6) a Boolean term. We call the Boolean term $(f_1 \oplus f_2 \oplus \cdots \oplus f_m)$ as the longest Boolean term of the expansion. Thus, a product of Boolean functions can be expressed as a linear combination of Boolean terms. Note that Eq. (6) is symmetric with respect to $f_1, f_2, \ldots, f_m$ as any $f_i$ interchanged with $f_j$ produces the same expression. It is no coincidence that we chose the letter $f$ in the

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1 We earlier considered Boolean variables, but the same holds for Boolean functions.
above equation: we later apply this identity using \( f_j \) from \( R_i := x_j \oplus f_j \) (see Section 2). When we use Eq. (3) in the conversion algorithm, we will have to evaluate a product of at most \( d \) functions, i.e. \( m \leq d = O(1) \). We now see in the right hand side of Eq. (6) that the coefficient \( 1/2^{m-1} \) is of constant size and there are \( O(1) \) many Boolean terms.

### 3.2 Our conversion algorithm

The FGLM \([7]\) conversion algorithm is well known in computer algebra for converting a given reduced Gröbner basis of a zero dimensional ideal in some ordering to the reduced Gröbner basis in any other ordering. However, it does so with \( O(nD((G_1))^3) \) many arithmetic operations, where \( D((G_1)) \) is the dimension of the \( \mathbb{R} \)-vector space \( \mathbb{R}[x_1, \ldots, x_n] / \langle G_1 \rangle \) (see Proposition 4.1 in \([7]\)). \( D((G_1)) \) is also equal to the number of common zeros (with multiplicity) of the polynomials from \( \langle G_1 \rangle \), which would imply that for the combinatorial ideals considered in this paper, \( D((G_1)) = O(2^{n-r}) \). This exponential running time is avoided in our conversion algorithm by exploiting the symmetries in Eq. (4) and by truncating the computation up to degree \( d \).

Some notations necessary for the algorithm are as follows: \( G_1 \) and \( G_2 \) are the reduced Gröbner basis of \( \langle G_1 \rangle \) in lex and grlex ordering respectively. \( \text{LM}(G_i) \) is the set of leading monomials of polynomials in \( G_i \) for \( i \in \{1,2\} \). Since we know \( G_1 \), we know \( \text{LM}(G_1) \), whereas \( G_2 \) and \( \text{LM}(G_2) \) are constructed by the algorithm. \( B(G_1) \) is the set of monomials that cannot be divided (considering the lex order) by any monomial of \( \text{LM}(G_1) \). Therefore, \( B(G_1) \) is the set of all multilinear monomials in variables \( x_{r+1}, \ldots, x_n \). Similarly, \( B(G_2) \) is the set of monomials that cannot by divided (considering the grlex order) by any monomial of \( \text{LM}(G_2) \).

Recall the definition of \( f_i \) for \( i \leq r \) from Section 2. For \( i > r \), for notational purposes, we define the Boolean function \( f_i := x_i \).

**Lemma 3.1.** Consider a monomial \( q \) such that \( \text{deg}(q) \leq d \). Then \( q|_{G_1} \) can be expressed as a linear combination of Boolean terms.

**Proof.** Consider \( q = x_{i_1}x_{i_2} \cdots x_{i_k} \) where \( k \leq d \). Then from Eqs. (1) and (2), \( q|_{G_1} = f_{i_1}f_{i_2} \cdots f_{i_k} \) and the lemma holds using Eq. (4). \( \square \)

Let elements \( b_i \) of \( B(G_2) \) be arranged in increasing grlex order. We construct a set \( C \) in our algorithm such that its elements \( c_i \) are defined as \( c_i = b_i|_{G_1} \) written as linear combinations of Boolean terms using Lemma 3.1. We say that a Boolean term \( f \) of \( c_i \) “appears in \( c_j \)” for some \( j < i \) if the longest Boolean term of \( c_j \) is \( f \oplus \alpha \) where \( \alpha = 0/1 \).

Let \( Q \) be the set of all monomials \( m \) such that \( 1 <_{\text{grlex}} \text{deg}(m) \leq_{\text{grlex}} d \). We recommend the reader to refer to the example in Section 4 and Appendix A for an intuitive working of the algorithm. We now describe the algorithm in full (we assume \( 1 \notin I(Sol(C)) \), else \( G_1 = \{1\} = G_2 \) and we are done):

**Inputs:** Degree \( d \), \( G_1 \), \( Q \)

**Initial states:** \( G_2 = \emptyset \), \( B(G_2) = \{1(= b_1)\} \), \( C = \{1(= c_1)\} \), \( q = x_n \).

**Outputs:** \( d \)-Truncated versions of \( G_2 \), \( B(G_2) \).
• **Main loop:** Find \( q_{|G_1} \), by which we simply replace any occurrence of \( x_i \) by the Boolean functions \( f_i \). Expand \( q_{|G_1} \) by using Eq. (6).

  - Suppose the longest Boolean term of \( q_{|G_1} \) does not appear in any \( c \in C \). Then \( q_{|G_1} \) is written as a linear combination of \( b_{i|G_1} \) and its longest Boolean term (see Lemma 3.2). This polynomial is added to \( C \) and \( q \) is added to \( B(G_2) \). Go to **Termination check**.

  - If the longest Boolean term of \( q_{|G_1} \) appears in some \( c \in C \), then every Boolean term of \( q_{|G_1} \) can be written as linear combinations of \( b_{j|G_1} \)'s. Note that if the longest Boolean term \( f \) appears in \( c \) as \( f \oplus 1 \), then we use \( f \oplus 1 = 1 - (f) \) (see Eq. (4)). Thus we have \( q_{|G_1} = \sum_j k_j b_{j|G_1} \implies q - \sum_j k_j b_j \in \langle G_1 \rangle \). The polynomial \( q - \sum_j k_j b_j \) is added to \( G_2 \) and \( q \) to \( LM(G_2) \). Go to **Termination check**.

• **Termination check:** We delete the occurrence of \( q \) from \( Q \). If \( q \) was added to \( LM(G_2) \) then we delete any monomial in \( Q \) that \( q \) can divide. The algorithm terminates if \( Q \) is empty, else go to **Next monomial**.

• **Next monomial:** Choose the smallest (according to **grlex** order) monomial in \( Q \) as \( q \). Go to Main loop.

**Lemma 3.2.** The set \( C \) is such that every \( c_i \) is a linear combination of existing \( b_{j|G_1} \)'s \((j < i)\) and the longest Boolean term of \( b_{i|G_1} \).

**Proof.** By definition, element \( c_i \) is added to \( C \) when a monomial \( q \) is added to \( B(G_2) \) where \( b_i = q \) and \( c_i = b_{i|G_1} \) expressed in Boolean terms (see Main loop). This means that \( q \) is not divisible by any monomial in \( LM(G_2) \). We prove the lemma by induction on the degree of \( q \). Note that \( b_1 = 1 \) and hence \( c_1 = b_{1|G_1} = 1 \).

If \( \deg(q) = 1 \), then \( q \) is some \( x_i \) and \( x_i_{|G_1} \) is one of 0, 1 or \( f_i \). If \( x_i_{|G_1} \) is either 0 or 1, then it then appears in \( c_1 \). We are now in the second case of the Main loop, so \( q \) should be added to \( LM(G_2) \) and not \( B(G_2) \). Hence \( x_i_{|G_1} \) can be neither 0 nor 1 and the lemma holds for \( \deg(q) = 1 \) as \( f_i \) is the longest Boolean term.

Let us assume the statement holds true for all monomials with degree less than \( m \). Consider \( q \) such that \( \deg(q) = m \) and \( q = x_{i_1} x_{i_2} \ldots x_{i_m} \), where \( i_j \)'s need not be distinct, and the lemma holds for every monomial \( <_{\text{grlex}} q \). Then \( q_{|G_1} = f_{i_1} \cdot f_{i_2} \cdots f_{i_m} \). Let \( (f_{j_1} \oplus \cdots \oplus f_{j_k}) \) be a Boolean term in the expansion of \( q_{|G_1} \) (by using Eq. (3)), that is not the longest Boolean term, so \( \{j_1, \ldots, j_k\} \subset \{i_1, \ldots, i_m\} \) and \( k < m \). Consider the monomial \( x_{j_1} x_{j_2} \ldots x_{j_k} \). We will now prove that \( x_{j_1} x_{j_2} \ldots x_{j_k} \) is in fact some \( b_l \in B(G_2) \) and there exists \( c_l \in C \) which is a linear combination of \( b_{l_{|G_1}} \)'s and \( (f_{j_1} \oplus \cdots \oplus f_{j_k}) \). The monomial \( x_{j_1} x_{j_2} \ldots x_{j_k} \) either belongs to \( LM(G_2) \) or \( B(G_2) \). If \( x_{j_1} x_{j_2} \ldots x_{j_k} \in LM(G_2) \) then it divides \( q \), a contradiction to our choice of \( q \). Therefore, \( x_{j_1} x_{j_2} \ldots x_{j_k} = b_l \in B(G_2) \). Clearly \( b_l <_{\text{grlex}} q \) and the induction hypothesis applies, so there exists \( c_l \in C \) such that

\[
b_l_{|G_1} = c_l = \sum_{i<l} a_i b_{i_{|G_1}} + a_0 (f_{j_1} \oplus \cdots \oplus f_{j_k})
\]
where $a_i$’s are constants. Then we simply use the above equation to substitute for the Boolean term $f_{j_1} \oplus \cdots \oplus f_{j_k}$ in $q|_{G_1}$ as a linear combination of $b_i|_{G_1}$ where $i \leq l$. We can do this for every Boolean term of $q|_{G_1}$ except the longest one. Hence the lemma holds.

**Theorem 3.3.** The conversion algorithm terminates for every input $G_1$ and correctly computes a $d$-truncated reduced Gröbner basis, with the grlex ordering, of the ideal $\langle G_1 \rangle$ in polynomial time.

**Proof.** The Main loop runs at most $|Q| = O(n^d)$ times. Evaluation of any $q|_{G_1}$ can be done in $O(n)$ steps (see Eq. (6)), checking if previous $c_i$’s appear (and replacing every Boolean term appropriately if it does) takes at most $O(n^d)$ steps since there are at most $|Q|$ many elements in $C$. Hence the running time of the algorithm is $O(n^{2d})$.

Suppose the set of polynomials \{$g_1, g_2, \ldots, g_k$\} is the output of the algorithm for some input $G_1$. Clearly, $\deg(g_i) \leq d$ for all $i \in [k]$. We now prove by contradiction that the output is the $d$-truncated Gröbner basis of the ideal $\langle G_1 \rangle$ with the grlex ordering. Suppose $g$ is a polynomial of the ideal with $\deg(g) \leq d$, but no $\text{LM}(g_i)$ can divide $\text{LM}(g)$. In fact, since every $g_i \in \langle G_1 \rangle$ we can replace $g$ by $g|_{\{g_1, g_2, \ldots, g_k\}}$ ($g$ generalises the reduced $S$-polynomial). The fact that $g \in \langle G_1 \rangle$ and $g|_{G_1} = 0$ implies that $\text{LM}(g)$ is a linear combination of monomials that are less than $\text{LM}(g)$ (in the grlex order) and hence must be in $B(G_2)$, i.e

$$g|_{G_1} = 0 \implies \text{LM}(g)|_{G_1} = \sum_i k_ib_i|_{G_1}$$

where every $b_i \in B(G_2)$ and $b_i <_{\text{grlex}} \text{LM}(g)$. When the algorithm runs for $q = \text{LM}(g)$, since $q$ was not added to $\text{LM}(G_2)$,

$$\text{LM}(g)|_{G_1} = \sum_j k_j b_j|_{G_1} + f$$

where $f$ is the longest Boolean term of $\text{LM}(g)|_{G_1}$ which does not appear in any previous element of $C$. But the two equations above imply that $\sum_i k_ib_i|_{G_1} = \sum_j k_j b_j|_{G_1} + f$, which proves that there exists some $b_i \in B(G_2)$ such that $c_i$ has $f$ as its longest Boolean term, so $f$ should have appeared in $c_i$, a contradiction. Therefore the output is a $d$-truncated Gröbner basis. Although unnecessary for the IMP$_d$, we also prove that the output is reduced: every non leading monomial of every polynomial in the output comes from $B(G_2)$ and no leading monomial is a multiple of another by construction (see Termination check).

Thus we have proof of the main theorem and corollary (see Theorem 1.1 and Corollary 1.2).

### 4 An example

We provide a simple example in Table 1 where we convert the reduced Gröbner basis in lex order of a combinatorial ideal to one in grlex order. Consider the problem formulated by the following (mod 2) equations: $x_1 \oplus x_3 \oplus x_4 = 0$ and $x_2 \oplus x_3 \oplus x_5 \oplus 1 = 0$. The example is explained in more detail in Appendix A.
| #  | \( q \) | \( B(G_2) \) | \( C \) | \( G_2 \) |
|---|---|---|---|---|
| 0 | - | 1 | 1 | \( \emptyset \) |
| 1 | \( x_5 \) | \( x_5 \) | \( x_5 \) | - |
| 2 | \( x_4 \) | \( x_4 \) | \( x_4 \) | - |
| 3 | \( x_3 \) | \( x_3 \) | \( x_3 \) | - |
| 4 | \( x_2 \) | \( x_2 \) | \( x_3 \oplus x_5 \oplus 1 \) | - |
| 5 | \( x_1 \) | \( x_1 \) | \( x_3 \oplus x_4 \) | - |
| 6 | \( x_5^2 \) | - | - | \( x_5^2 - x_5 \) |
| 7 | \( x_4x_5 \) | \( x_4x_5 \) | \( \frac{1}{2}[x_4|G_1 + x_5|G_1 \) \( -(x_4 \oplus x_5)] \) | - |
| 8 | \( x_4^2 \) | - | - | \( x_4^2 - x_4 \) |
| 9 | \( x_3x_5 \) | - | - | \( x_3x_5 - \frac{1}{2}[x_2 + x_3 + x_5 - 1] \) |
| 10 | \( x_3x_4 \) | - | - | \( x_3x_4 - \frac{1}{2}[-x_1 + x_3 + x_4] \) |
| 11 | \( x_3^2 \) | - | - | \( x_3^2 - x_3 \) |
| 12 | \( x_2x_5 \) | - | - | \( x_2x_5 - \frac{1}{2}[x_2 + x_3 + x_5 - 1] \) |
| 13 | \( x_2x_4 \) | \( x_2x_4 \) | \( \frac{1}{2}[x_2|G_1 + x_4|G_1 \) \( -(x_3 \oplus x_4 \oplus x_5 + 1)] \) | - |
| 14 | \( x_2x_3 \) | - | - | \( x_2x_3 - \frac{1}{2}[x_2 + x_3 + x_5 - 1] \) |
| 15 | \( x_2^2 \) | - | - | \( x_2^2 - x_2 \) |
| 16 | \( x_1x_5 \) | - | - | \( x_1x_5 + x_2x_4 - \frac{1}{2}[x_1 + x_2 + x_4 + x_5 - 1] \) |
| 17 | \( x_1x_4 \) | - | - | \( x_1x_4 - \frac{1}{2}[x_1 - x_3 + x_4] \) |
| 18 | \( x_1x_3 \) | - | - | \( x_1x_3 - \frac{1}{2}[x_1 + x_3 - x_4] \) |
| 19 | \( x_1x_2 \) | - | - | - |
| 20 | \( x_1^2 \) | - | - | \( x_1^2 - x_1 \) |

Table 1: Example

5 Conclusion

The \( \text{IMP}_d \) tractability for combinatorial ideals has useful practical applications as it implies bounded coefficients in Sum-of-Squares proofs. A dichotomy result between “hard” (NP-hard) and “easy” (polynomial time) IMPs was recently achieved for the \( \text{IMP}_0 \) over the finite domain nearly thirty years after that over the Boolean domain [18]. The \( \text{IMP}_d \) for \( d = O(1) \) over the Boolean domain was tackled by Mastrolilli [13] based on the classification of the IMP through polymorphisms, where the complexity of the \( \text{IMP}_d \) for five of six polymorphisms was solved. We solve the remaining problem, i.e. the complexity of the \( \text{IMP}_d(\Gamma) \) when \( \Gamma \) is closed under the ternary minority polymorphism. This is achieved by showing that the \( d \)-truncated reduced Gröbner basis can be computed in polynomial time, thus completing the missing link in the dichotomy result of [13].

Moreover, we believe the techniques described in this paper can be generalized for a finite domain with prime \( p \) elements, as constraints that are linear equations (mod \( p \)) are associated with an affine polymorphism [11]. We claim that the \( \text{IMP}_d \) is tractable for problems that are constrained as linear equations (mod \( p \)). This is a step in identifying the borderline of tractability, if it exists, for the general \( \text{IMP}_d \). We believe that generalizing the dichotomy results of solvability of the \( \text{IMP}_d \) for a finite domain is an interesting and challenging goal.
that we leave as an open problem.

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A Example (in detail)

Note that \( f_1 = x_3 \oplus x_4 \), \( f_2 = x_3 \oplus x_5 \oplus 1 \), \( f_3 = x_3 \), \( f_4 = x_4 \) and \( f_5 = x_5 \). The reduced Gröbner basis in the lex order is \( G = G_1 = \{ x_1 - M(f_1), x_2 - M(f_2), \text{dom}(x_3), \text{dom}(x_4), \text{dom}(x_5) \} \).

We start with \( G_2 = \text{LM}(G_2) = \emptyset \), \( B(G_2) = C = \{ 1 \} \) (so \( b_1 = c_1 = 1 \)) and \( q = x_5 \). For the problem of \( d = 2 \), we have

\[
Q = \{ x_5, x_4, x_3, x_2, x_1, x_5^2, x_5 x_4, x_4^2, x_5 x_3, x_4 x_3, x_3^2, x_5 x_2, x_4 x_2, x_3 x_2, x_2^2, x_5 x_1, x_4 x_1, \]
\[
x_3 x_1, x_2 x_1, x_1^2 \}.
\]

We start with \( q = x_5 \) and since \( q|_{G_1} = x_5 \) and does not appear as the longest Boolean term of any element of \( C \), we have that \( x_5 \) is added to \( C \) (so \( c_2 = x_5 \)) and \( x_5 \) is added to \( B(G_2) \) (so \( b_2 = x_5 \)). The Termination check of the algorithm deletes \( x_5 \) from \( Q \) and Next monomial chooses \( q = x_4 \). The iterations are similar for \( q = x_4 \) and \( q = x_3 \), so we have \( b_3 = c_3 = x_4 \) and \( b_4 = c_4 = x_3 \) and \( x_4, x_3 \) are deleted from \( Q \). When Next monomial chooses \( q = x_2 \), we have \( q|_{G_1} = f_2 = (x_3 \oplus x_4 + 1) \), and since the Boolean term does not appear in any \( c \in C \), we add \( (x_3 \oplus x_5 + 1) \) to \( C \) (so \( c_5 = (x_3 \oplus x_5 + 1) \)) and \( x_2 \) to \( B(G_2) \) (so \( b_5 = x_2 \)).

For similar reasons, when \( q = x_1 \), we add \( c_6 = (x_3 \oplus x_4) \) to \( C \) and \( b_6 = x_1 \) to \( B(G_2) \).

After the 5-th iteration (see Table \ref{table3}) is complete, we only have degree-two monomials in \( Q \). Next monomial chooses \( q = x_5^2 \) and \( q|_{G_1} = x_5 \). Since \( c_1 = x_5 \), \( x_5 \) appears as a Boolean term in \( c_1 \). Since the longest Boolean term appears already in \( C \), \( q|_{G_1} \) must be a linear combination of existing \( b_i|_{G_1} \)’s. That is to say, \( x_5^2|_{G_1} = c_1 = b_1|_{G_1} = x_5|_{G_1} \implies x_5^2|_{G_1} = x_5|_{G_1} \), so the polynomial \( x_5^2 - x_5 \) is added to \( G_2 \). Termination check adds \( x_5^2 \) to \( \text{LM}(G_2) \) and deletes \( x_5^2 \) from \( Q \).

Next monomial chooses \( q = x_5 x_4 \), so

\[
x_5 x_4|_{G_1} = f_4 \cdot f_5 = \frac{1}{2} [x_4 + x_5 - (x_4 \oplus x_5)] = \frac{1}{2} [x_4|_{G_1} + x_5|_{G_1} - (x_4 \oplus x_5)].
\]

The longest Boolean term of \( q|_{G_1} \) is \( (x_4 \oplus x_5) \) which does not appear in any \( c \in C \), so \( c_7 = 1/2[x_4|_{G_1} + x_5|_{G_1} - (x_4 \oplus x_5)] \) is added to \( C \) and \( b_7 = x_5 x_4 \) is added to \( B(G_2) \). Next monomial chooses \( q = x_4^2 \), this is similar to the case when \( q = x_5^2 \), we see that when \( q = x_4^2 \), and \( x_4^2 - x_4 \) is added to \( G_2 \) and \( x_4^2 \) to \( \text{LM}(G_2) \). When Next monomial chooses \( q = x_3 x_4 \), we have

\[
x_3 x_4|_{G_1} = f_3 \cdot f_5 = \frac{1}{2} [x_3 + x_5 - (x_3 \oplus x_5)].
\]

Note that \( (x_3 \oplus x_5 + 1) \) appears in \( c_5 \in C \). We use the fact that \( (f \oplus 1) = 1 - f \) (see Main loop), and we have

\[
x_3 x_5|_{G_1} = \frac{1}{2} [x_3 + x_5 - (x_3 \oplus x_5)] = \frac{1}{2} [x_3|_{G_1} + x_5|_{G_1} - (1 - (x_3 \oplus x_5 + 1))]
\]
\[
= \frac{1}{2} [x_2|_{G_1} + x_3|_{G_1} + x_5|_{G_1} - 1|_{G_1}]
\]

and thus \( x_3 x_5 - 1/2[x_2 + x_3 + x_5 - 1] \) is added to \( G_2 \) and \( x_3 x_5 \) to \( \text{LM}(G_2) \). The rest of the polynomials in \( B(G_2), G_2, C \) are as shown in Table \ref{table3}. It can be seen that after the 20-th iteration, \( Q \) becomes empty and Termination check halts the algorithm. This gives the 2-truncated reduced Gröbner basis \( G_2 \) of the combinatorial ideal. Note that this is in fact the reduced Gröbner basis in its entirety for this example (see Termination check).
B Ideals, Varieties and Constraints

Let $\mathbb{F}$ denote an arbitrary field (for the applications of this paper $\mathbb{F} = \mathbb{R}$). Let $\mathbb{F}[x_1, \ldots, x_n]$ be the ring of polynomials over a field $\mathbb{F}$ and indeterminates $x_1, \ldots, x_n$. Let $\mathbb{F}[x_1, \ldots, x_n]_d$ denote the subspace of polynomials of degree at most $d$.

Definition B.1. The ideal (of $\mathbb{F}[x_1, \ldots, x_n]$) generated by a finite set of polynomials $\{f_1, \ldots, f_m\}$ in $\mathbb{F}[x_1, \ldots, x_n]$ is defined as

$$I(f_1, \ldots, f_m) \overset{\text{def}}{=} \left\{ \sum_{i=1}^m t_i f_i \mid t_1, \ldots, t_m \in \mathbb{F}[x_1, \ldots, x_n] \right\}.$$  

The set of polynomials that vanish in a given set $S \subset \mathbb{F}^n$ is called the vanishing ideal of $S$ and denoted: $I(S) \overset{\text{def}}{=} \left\{ f \in \mathbb{F}[x_1, \ldots, x_n] : f(a_1, \ldots, a_n) = 0 \ \forall (a_1, \ldots, a_n) \in S \right\}$.

Definition B.2. An ideal $I$ is radical if $f^m \in I$ for some integer $m \geq 1$ implies that $f \in I$.

Another common way to denote $I(f_1, \ldots, f_m)$ is by $\langle f_1, \ldots, f_m \rangle$ and we will use both notations interchangeably.

Definition B.3. Let $\{f_1, \ldots, f_m\}$ be a finite set of polynomials in $\mathbb{F}[x_1, \ldots, x_n]$. We call $V(f_1, \ldots, f_m) \overset{\text{def}}{=} \{(a_1, \ldots, a_n) \in \mathbb{F}^n \mid f_i(a_1, \ldots, a_n) = 0 \ \forall 1 \leq i \leq m\}$ the affine variety defined by $f_1, \ldots, f_m$.

Definition B.4. Let $I \subseteq \mathbb{F}[x_1, \ldots, x_n]$ be an ideal. We will denote by $V(I)$ the set $V(I) = \{(a_1, \ldots, a_n) \in \mathbb{F}^n \mid f(a_1, \ldots, a_n) = 0 \ \forall f \in I\}$.

Theorem B.1 ([5], Th.15, p.196). If $I$ and $J$ are ideals in $\mathbb{F}[x_1, \ldots, x_n]$, then $V(I \cap J) = V(I) \cup V(J)$.

B.1 The Ideal-CSP Correspondence

Indeed, let $C = (X, D, C)$ be an instance of the CSP($\Gamma$) (see Definition I.1). Without loss of generality, we shall assume that $D \subset \mathbb{N}$ and $D \subseteq \mathbb{F}$.

Let $Sol(C)$ be the (possibly empty) set of all feasible solutions of $C$. In the following, we map $Sol(C)$ to an ideal $I_C \subseteq \mathbb{F}[X]$ such that $Sol(C) = V(I_C)$.

Let $Y = (x_i, \ldots, x_k)$ be a $k$-tuple of variables from $X$ and let $R(Y)$ be a non-empty constraint from $C$. In the following, we map $R(Y)$ to a generating system of an ideal such that the projection of the variety of this ideal onto $Y$ is equal to $R(Y)$ (see [20] for more details).

Every $v = (v_1, \ldots, v_k) \in R(Y)$ corresponds to some point $v \in \mathbb{F}^k$. It is easy to check [5] that $I(\{v\}) = \langle x_i - v_1, \ldots, x_k - v_k \rangle$, where $\langle x_i - v_1, \ldots, x_k - v_k \rangle \subseteq \mathbb{F}[Y]$ is radical. By Theorem B.1, we have

$$R(Y) = \bigcup_{v \in R(Y)} V(I(\{v\})) = V(I_{R(Y)}) \quad \text{where} \quad I_{R(Y)} = \bigcap_{v \in R(Y)} I(\{v\}),$$ (7)
where $I_{R(Y)} \subseteq \mathbb{F}[Y]$ is zero-dimensional and radical ideal since it is the intersection of radical ideals (see [5], Proposition 16, p.197). Equation (7) states that constraint $R(Y)$ is a variety of $\mathbb{F}^k$. It is easy to find a generating system for $I_{R(Y)}$:

$$I_{R(Y)} = \langle \prod_{v \in R} (1 - \prod_{j=1}^k \delta_{v_j}(x_{i_j})), \prod_{j \in D} (x_{i_1} - j), \ldots, \prod_{j \in D} (x_{i_k} - j) \rangle,$$

where $\delta_{v_j}(x_{i_j})$ are indicator polynomials, i.e. equal to one when $x_{i_j} = v_j$ and zero when $x_{i_j} \in D \setminus \{v_j\}$; polynomials $\prod_{j \in D} (x_{i_k} - j)$ force variables to take values in $D$ and will be denoted as domain polynomials.

The smallest ideal (with respect to inclusion) of $\mathbb{F}[X]$ containing $I_{R(Y)} \subseteq \mathbb{F}[x]$ will be denoted $I_{F[X]}^{R(Y)}$ and it is called the $\mathbb{F}[X]$-module of $I$. The set $Sol(C) \subseteq \mathbb{F}^n$ of solutions of $C = (X, D, C)$ is the intersection of the varieties of the constraints:

$$Sol(C) = \bigcap_{R(Y) \in C} V(I_{R(Y)}) = V(I_C),$$

$$I_c = \sum_{R(Y) \in C} I_{F[X]}^{R(Y)}. \quad (10)$$

The following properties follow from Hilbert’s Nullstellensatz.

**Theorem B.2.** Let $C$ be an instance of the CSP($\Gamma$) and $I_C$ defined as in (10). Then

1. (Weak Nullstellensatz)
   $$V(I_C) = \emptyset \iff 1 \in I(I_C) \iff I_C = \mathbb{F}[X],$$
2. (Strong Nullstellensatz)
   $$I(V(I_C)) = \sqrt{I_C},$$
3. (Radical Ideal)
   $$\sqrt{I_C} = I_C. \quad (13)$$

Theorem B.2 follows from a simple application of the celebrated and basic result in algebraic geometry known as Hilbert’s Nullstellensatz. In the general version of Nullstellensatz it is necessary to work in an algebraically closed field and take a radical of the ideal of polynomials. In our special case it is not needed due to the presence of domain polynomials. Indeed, the latter implies that we know a priori that the solutions must be in $\mathbb{F}$ (note that we are assuming $D \subseteq \mathbb{F}$).

### B.2 Gröbner bases.

In this section we suppose a fixed monomial ordering $>$ on $\mathbb{F}[x_1, \ldots, x_n]$ (see [5], Definition 1, p.55), which will not be defined explicitly. We can reconstruct the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ from the $n$-tuple of exponents $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. This establishes a one-

to-one correspondence between the monomials in $\mathbb{F}[x_1, \ldots, x_n]$ and $\mathbb{Z}^n_{\geq 0}$. Any ordering $>$ we establish on the space $\mathbb{Z}^n_{\geq 0}$ will give us an ordering on monomials: if $\alpha > \beta$ according to this ordering, we will also say that $x^\alpha > x^\beta$. The two monomial orderings that we use in this paper are the lexicographic order $>_{\text{lex}}$ and the graded lexicographic ordering $>_{\text{grlex}}$. 17
Definition B.5. Let $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_{\geq 0}^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i, |\beta| = \sum_{i=1}^n \beta_i$.

(i) We say $\alpha >_{\text{lex}} \beta$ if, in the vector difference $\alpha - \beta \in \mathbb{Z}^n$, the left most nonzero entry is positive. We will write $x^\alpha >_{\text{lex}} x^\beta$ if $\alpha >_{\text{lex}} \beta$.

(ii) We say $\alpha >_{\text{grlex}} \beta$ if $|\alpha| > |\beta|$, or $|\alpha| = |\beta|$ and $\alpha >_{\text{lex}} \beta$.

Definition B.6. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ let $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. Let $f = \sum_{\alpha} a_\alpha x^\alpha$ be a nonzero polynomial in $\mathbb{F}[x_1, \ldots, x_n]$ and let $>$ be a monomial order.

(i) The multidegree of $f$ is $\text{multideg}(f) \overset{\text{def}}{=} \max(\alpha \in \mathbb{Z}_{\geq 0}^n : a_\alpha \neq 0)$.

(ii) The degree of $f$ is $\text{deg}(f) = |\text{multideg}(f)|$. In this paper, this is always according to grlex order.

(iii) The leading coefficient of $f$ is $\text{LC}(f) \overset{\text{def}}{=} a_{\text{multideg}(f)} \in \mathbb{F}$.

(iv) The leading monomial of $f$ is $\text{LM}(f) \overset{\text{def}}{=} x^{\text{multideg}(f)}$ (with coefficient 1).

(v) The leading term of $f$ is $\text{LT}(f) \overset{\text{def}}{=} \text{LC}(f) \cdot \text{LM}(f)$.

The concept of reduction, also called multivariate division or normal form computation, is central to Gröbner basis theory. It is a multivariate generalization of the Euclidean division of univariate polynomials.

Definition B.7. Fix a monomial order and let $G = \{g_1, \ldots, g_t\} \subset \mathbb{F}[x_1, \ldots, x_n]$. Given $f \in \mathbb{F}[x_1, \ldots, x_n]$, we say that $f$ reduces to $r$ modulo $G$, written $f \rightarrow_G r$, if $f$ can be written in the form $f = A_1 g_1 + \cdots + A_t g_t + r$ for some $A_1, \ldots, A_t, r \in \mathbb{F}[x_1, \ldots, x_n]$, such that:

(i) No term of $r$ is divisible by any of $\text{LT}(g_1), \ldots, \text{LT}(g_t)$.

(ii) Whenever $A_i g_i \neq 0$, we have $\text{multideg}(f) \geq \text{multideg}(A_i g_i)$.

The polynomial remainder $r$ is called a normal form of $f$ by $G$ and will be denoted by $f|_G$.

A normal form of $f$ by $G$, i.e. $f|_G$, can be obtained by repeatedly performing the following until it cannot be further applied: choose any $g \in G$ such that $\text{LT}(g)$ divides some term $t$ of $f$ and replace $f$ with $f - \frac{t}{\text{LT}(g)} g$. Note that the order we choose the polynomials $g$ in the division process is not specified.

In general a normal form $f|_G$ is not uniquely defined. Even when $f$ belongs to the ideal generated by $G$, i.e. $f \in I(G)$, it is not always true that $f|_G = 0$.

Example B.1. Let $f = xy^2 - y^3$ and $G = \{g_1, g_2\}$, where $g_1 = xy - 1$ and $g_2 = y^2 - 1$. Consider the graded lexicographic order (with $x > y$) and note that $f = y \cdot g_1 - y \cdot g_2 + 0$ and $f = 0 \cdot g_1 + (x - y) \cdot g_2 + x - y$.

This non-uniqueness is the starting point of Gröbner basis theory.
Definition B.8. Fix a monomial order on the polynomial ring \( \mathbb{F}[x_1, \ldots, x_n] \). A finite subset \( G = \{g_1, \ldots, g_t\} \) of an ideal \( I \subseteq \mathbb{F}[x_1, \ldots, x_n] \) different from \( \{0\} \) is said to be a **Gröbner basis** (or **standard basis**) if \( \langle \text{LT}(g_1), \ldots, \text{LT}(g_t) \rangle = \langle \text{LT}(I) \rangle \), where we denote by \( \langle \text{LT}(I) \rangle \) the ideal generated by the elements of the set \( \text{LT}(I) \) of leading terms of nonzero elements of \( I \).

Definition B.9. A **reduced Gröbner basis** for a polynomial ideal \( I \) is a Gröbner basis \( G \) for \( I \) such that:

(i) \( \text{LC}(g) = 1 \) for all \( g \in G \).

(ii) For all \( g \in G \), \( g \) cannot reduce any other polynomial from \( G \), i.e \( f|_g = f \) for every \( f \in G \setminus \{g\} \).

It is known (see [5], Theorem 5, p.93) that for a given monomial ordering, a polynomial ideal \( I \neq \{0\} \) has a reduced Gröbner basis (see Definition B.9), and the reduced Gröbner basis is unique.

Proposition B.3 ([5], Proposition 1, p.83). Let \( I \subseteq \mathbb{F}[x_1, \ldots, x_n] \) be an ideal and let \( G = \{g_1, \ldots, g_t\} \) be a Gröbner basis for \( I \). Then given \( f \in \mathbb{F}[x_1, \ldots, x_n] \), \( f \) can be written in the form \( f = A_1g_1 + \cdots + A tg_t + r \) for some \( A_1, \ldots, A_t, r \in \mathbb{F}[x_1, \ldots, x_n] \), such that:

(i) No term of \( r \) is divisible by any of \( \text{LT}(g_1), \ldots, \text{LT}(g_t) \).

(ii) Whenever \( A_i g_i \neq 0 \), we have \( \text{multideg}(f) \geq \text{multideg}(A_i g_i) \).

(iii) There is a unique \( r \in \mathbb{F}[x_1, \ldots, x_n] \).

In particular, \( r \) is the remainder on division of \( f \) by \( G \) no matter how the elements of \( G \) are listed when using the division algorithm.

Corollary B.4 ([5], Corollary 2, p.84). Let \( G = \{g_1, \ldots, g_t\} \) be a Gröbner basis for \( I \subseteq \mathbb{F}[x_1, \ldots, x_n] \) and let \( f \in \mathbb{F}[x_1, \ldots, x_n] \). Then \( f \in I \) if and only if the remainder on division of \( f \) by \( G \) is zero.

Definition B.10. We will write \( f|_F \) for the remainder of \( f \) by the ordered \( s \)-tuple \( F = (f_1, \ldots, f_s) \). If \( F \) is a Gröbner basis for \( \langle f_1, \ldots, f_s \rangle \), then we can regard \( F \) as a set (without any particular order) by Proposition B.3.

The “obstruction” to \( \{g_1, \ldots, g_t\} \) being a Gröbner basis is the possible occurrence of polynomial combinations of the \( g_i \) whose leading terms are not in the ideal generated by the \( \text{LT}(g_i) \). One way (actually the only way) this can occur is if the leading terms in a suitable combination cancel, leaving only smaller terms. The latter is fully captured by the so called \( S \)-polynomials that play a fundamental role in Gröbner basis theory.

Definition B.11. Let \( f, g \in \mathbb{F}[x_1, \ldots, x_n] \) be nonzero polynomials. If \( \text{multideg}(f) = \alpha \) and \( \text{multideg}(g) = \beta \), then let \( \gamma = (\gamma_1, \ldots, \gamma_n) \), where \( \gamma_i = \max(\alpha_i, \beta_i) \) for each \( i \). We call \( x^\gamma \) the **least common multiple** of \( \text{LM}(f) \) and \( \text{LM}(g) \), written \( x^\gamma = \text{lcm}(\text{LM}(f), \text{LM}(g)) \). The **\( S \)-polynomial** of \( f \) and \( g \) is the combination \( S(f, g) = \frac{x^\gamma}{\text{LT}(f)} \cdot f - \frac{x^\gamma}{\text{LT}(g)} \cdot g \).
The use of $S$-polynomials to eliminate leading terms of multivariate polynomials generalizes the row reduction algorithm for systems of linear equations. If we take a system of homogeneous linear equations (i.e., the constant coefficient equals zero), then it is not hard to see that bringing the system in triangular form yields a Gröbner basis for the system.

**Theorem B.5 (Buchberger’s Criterion).** (See e.g. [3], Theorem 3, p. 105) A basis $G = \{g_1, \ldots, g_t\}$ for an ideal $I$ is a Gröbner basis if and only if $S(g_i, g_j) \not\to G 0$ for all $i \neq j$.

By Theorem B.5 it is easy to show whether a given basis is a Gröbner basis. Indeed, if $G$ is a Gröbner basis then given $f \in \mathbb{F}[x_1, \ldots, x_n]$, $f|_G$ is unique and it is the remainder on division of $f$ by $G$, no matter how the elements of $G$ are listed when using the division algorithm.

Furthermore, Theorem B.5 leads naturally to an algorithm for computing Gröbner bases for a given ideal $I = \langle f_1, \ldots, f_s \rangle$: start with a basis $G = \{f_1, \ldots, f_s\}$ and for any pair $f, g \in G$ with $S(f, g)|_G \not= 0$ add $S(f, g)|_G$ to $G$. This is known as Buchberger’s algorithm [1] (for more details see Algorithm 1 in Section B.2.1).

Note that Algorithm 1 is non-deterministic and the resulting Gröbner basis is not uniquely determined by the input. This is because the normal form $S(f, g)|_G$ (see Algorithm 1, line 8) is not unique as already remarked. We observe that one simple way to obtain a deterministic algorithm (see [3], Theorem 2, p. 91) is to replace $h := S(f, g)|_G$ in line 8 with $h := S(f, g)|_G$ (see Definition B.10), where in the latter $G$ is an ordered tuple. However, this is potentially dangerous and inefficient. Indeed, there are simple cases where the combinatorial growth of set $G$ in Algorithm 1 is out of control very soon.

**B.2.1 Construction of Gröbner Bases.**

Buchberger’s algorithm [1] can be formulated as in Algorithm 1. The pairs that get placed

1: **Input**: A finite set $F = \{f_1, \ldots, f_s\}$ of polynomials
2: **Output**: A finite Gröbner basis $G$ for $\langle f_1, \ldots, f_s \rangle$
3: $G := F$
4: $C := G \times G$
5: while $C \neq \emptyset$ do
6: choose a pair $(f, g) \in C$
7: $C := C \setminus \{(f, g)\}$
8: $h := S(f, g)|_G$
9: if $h \neq 0$ then
10: $C := C \cup (G \times \{h\})$
11: $G := G \cup \{h\}$
12: end if
13: end while
14: Return $G$

**Algorithm 1**: Buchberger’s Algorithm

in the set $C$ are often referred to as critical pairs. Every newly added reduced $S$-polynomial enlarges the set $C$. If we use $h := S(f, g)|_G$ in line 8 then there are simple cases where the
situation is out of control. This combinatorial growth can be controlled to some extent by eliminating unnecessary critical pairs.