Application of two-dimensional truncated singular value decomposition in image restoration

Hongyu Zhou¹*, Ting Kou² and Xu Song¹

¹ School of Computer and Information Engineering, Anyang Normal University, Anyang, Henan Province, China
² Qingdao Hisense Group, Qingdao, Shandong Province, China

*Corresponding author. Email: zhouhongyupear@163.com

Abstract. Large solution size and ill-posed problem often exists in image restoration. In order to overcome these problems, the application of two-dimensional truncated singular value decomposition (2-D TSVD) in large-scale image restoration is studied in this paper. The 2-D TSVD overcomes the inherent ill-posed problem and also solves the problem caused by limited storage in image restoration, which plays a regularization role. Experimental results show that this method is very effective in image restoration.

1. Introduction

The research about the field of image restoration begins in the fifties and sixties of the 20th century, image restoration has been accompanied by the continuous development of digital image technology and development. In the fifties and sixties of the last century, people has spent a lot of manpower and financial resources to get a lot of photos in practice, but these photos are very fuzzy which can not be seen clearly at that time and have no value at all . Fortunately scientists believe that these images contain a wealth of information, and finally through the unremitting efforts to restore the original information of the images, today these technology is relatively mature and widely used in image restoration. After 60 years of research, technology has been widely used in image restoration, such as space exploration, astronomical observation, material research, remote sensing, remote sensing, military science, medical imaging, traffic monitoring, criminal detection, and many other areas[1-3].

Although the sources of the observed images in these areas are different, the observation system has different nature, but research shows that the observation system model can usually be simplified to linear shift invariant convolution blur combined with related gaussian white noise to the image signal. The Linear shift invariant convolution blur can be generated by using the observation system Point Spread Function (PSF, Point Spread Function). In this sense, the image is restored according to the observation, the observation system of point spread function (PSF) to obtain the estimate of a real image.

2. Degradation model of image

Many image restoration problems encountered in practice, when the degradation is not too serious, the model of linear shift invariant system is commonly used to restore images. A linear spatial displacement constant imaging system is considered in this paper, for the nonlinear displacement variable system, modern calculation method can be used to solve it. After some transformation of the image degradation models, such problems can be transformed into the first kind of convolution...
integral equations. In the process of imaging, a point on the landscape is not only reflected in a corresponding point on the image, but is diffused into an area on the image plane. Therefore each point on the image is a reflection of many points in the scenery by mixed stack\[4\]. With hybrid stack imaging process, this can be described by a superposition integral in mathematics:

\[ g(x,y) = \iint H(x,\xi; y,\eta)f(\xi,\eta)d\xi d\eta + R(x,y) \]  

(1)

The function \( H(x,\xi; y,\eta) \) is called point spread function, which represents that each pixel of the discrete image diffus influenced by the function \( H(x,\xi; y,\eta) \). Image degradation is exactly caused by the function \( H(x,\xi; y,\eta) \), the system is unchanged in most cases, reflected in the image is displacement unchanged, so the function \( H(x,\xi; y,\eta) \) can be denoted by the function \( H(x-\xi, y-\eta) \). Therefore the continuous degradation model can be further denoted by the convolution integral equations below:

\[ \iint H(x-\xi, y-\eta)f(\xi,\eta)d\xi d\eta + R(x,y) \]  

(2)

Due to the problem is the first kind Fredholm integral equation, which belongs to the inverse problem of mathematical physics equation, therefore itself is ill-posed, and using the general method to solve this kinds of problem won't get effective solution. The 2-D TSVD method which will be discussed below is a kind of regularization method that can effectively solve this kinds of problems\[5-6\].

3. Discretization of two-dimensional integral equation

Now we discuss two-dimensional deconvolution problem whose kernel can separate. That is to say the two-dimensional first kind Fredholm integral equation should be processed, the general form as follows

\[ \int_0^1 \int_0^1 \kappa(x-x')\omega(y-y')f(x',y')dx'dy' = g(x,y) \]  

(3)

Here \( \kappa \) and \( \omega \) are functions. When the image is blurred by the Gauss point spread function there

\[ \kappa(t) = \omega(t) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{1}{2\sigma^2}t^2\right) \]

is

The discrete result can be described by the following symmetric banded Toeplitz matrix\[7\].

\[ (K_\sigma)_{ij} = \begin{cases} e^{-\frac{(i-j)^2}{2\sigma^2}} & \text{i-j-band} \\ 0 & \text{other one} \end{cases} \]

(4)

We use the rectangular quadrature to discrete the integral equations. Thus, the integral of \( y' \) can be approximated as the summation of n items.

\[ \int_0^1 \omega(y-y')f(x',y')dy' \approx n^{-1}\sum_{i=1}^{n} \omega(y-y_i)\tilde{f}(x',y_i) = \phi(x',y) \]
Here $y'_i$ is integral nod, $\tilde{f}$ is the approximate solution. Then use another summation to approximate $x'$.

$$\int_0^1 \kappa(y - y')\phi(x', y)dx' \approx m^{-1}\sum_{k=1}^m \kappa(x - x'_k)\phi(x'_k, y) = \psi(x', y)$$

Here $x'_k$ is integral nod too, and take on $mn$ points $(x_i, y_j)$

$$\psi(x_i, y_j) = g(x_i, y_j), \quad i = 1, \cdots, m, j = 1, \cdots, n$$

Finally a $mn$ order linear equations can be constructed which contains $mn$ unknown quantities $\tilde{f}(x_i, y_j)$.

Four matrix $A, \tilde{A}, F, G$ are introduced below \[8\]:

$$A_j = m^{-1}\kappa(x_i - x'_j), \quad \tilde{A}_{ij} = n^{-1}\omega(y_j - y'_i), \quad F_{ij} = \tilde{f}(x_i, y_j), \quad G_{ij} = g(x_i, y_j)$$

Now, define a $m \times n$ order matrix $\Phi$, Make $\Phi$ integral to $y'$, then

$$\Phi_{kj} = \phi(x'_k, y'_j) = n^{-1}\sum_{j=1}^n \omega(y_j - y'_i)\tilde{f}(x'_k, y'_j), \quad j = 1, \cdots, n, k = 1, \cdots, m$$

So you can get $\Phi = F\tilde{A}^T$.

Similarly, define a $m \times n$ order matrix $\Psi$, Make $\Psi$ integral to $x'$,then

$$\Psi_{ij} = \psi(x_i, y_j) = m^{-1}\sum_{k=1}^m \kappa(x_i - x'_k)\phi(x'_k, y_j), \quad i = 1, \cdots, m, j = 1, \cdots, n$$

Then $\Psi = A\Phi = AF\tilde{A}^T$, and $\Psi = G$. So the discretization form $AF\tilde{A}^T = G$ of the two-dimensional deconvolution problem can be obtained.

Finally, in order to convert it into standard form of linear equations, we should calculate Kronecker inner product of matrices $A, \tilde{A}$, the result is a $mn \times mn$ matrix \[9\]:

$$\tilde{A} \otimes A = \begin{pmatrix} \pi_{11}A & \pi_{12}A & \cdots & \pi_{1n}A \\ \pi_{21}A & \pi_{22}A & \cdots & \pi_{2n}A \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{m1}A & \pi_{m2}A & \cdots & \pi_{mn}A \end{pmatrix}$$

At the same time we need to explain the concept of ‘vec’, if $X$ is a $m \times n$ matrix, writes it in columns $X = (x_1, \cdots, x_n)$, and defines a vector whose length is $vec(X) = (x_1 \cdots x_n)^T$.

The connection between the 'Kronecker' inner product and the definition of 'vec' by the following important relation \[10\]

$$(\tilde{A} \otimes A)vec(X) = vec(AX\tilde{A}^T) \quad (5)$$

So there is

$$AF\tilde{A}^T = G \iff (\tilde{A} \otimes A)vec(F) = vec(G) \quad (6)$$

On the right hand side of the formula (6) is a linear equations whose coefficient matrix has $n^2 \times n^2$ elements, There are $n^2$ unknown quantities in the linear equations, and the corresponding
linear system is one-dimensional which is suitable for theoretical research. On the left hand side of the formula (6) is the matrix expressions which is suitable for numerical calculation. It's worth mentioning that if $A$ and $\bar{A}$ are Toeplitz matrix, then $\bar{A} \otimes A$ is block-Toeplitz matrix. This paper mainly discusses the situation with this special structure.

### 4. 2-D TSVD method

From the discussion above we can see that singular value decomposition of the fuzzy matrix directly has a large amount of calculation, and often appears incapable of action when faced to practical problems. Previously we have introduced the representation of the 2-D discretization problem which can be separated by tensor product of two Toeplitz matrixes. Combined with the special nature we are now searching for a more practical, efficient algorithm which can solve image inverse fuzzy problems. From this point of view, we will study 2-D TSVD solution of the problem, All the computation about the matrix $\bar{A} \otimes A$ whose size is $mn \times mn$ can be avoided, then the corresponding calculation efficiency will be greatly improved.

Now, the discrete forms of the two-dimensional convolution integral equation as shown in a formula (5) will be processed.

The solution of 2-D TSVD is defined by the SVD of $\bar{A} \otimes A$. The singular value decomposition of $\bar{A} \otimes A$ can be represented by singular value decomposition of two matrix. Assume that the singular value decomposition of $A$ and $\bar{A}$ can be given by the following equation:

$$A = U \Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T, \quad \bar{A} = \bar{U} \bar{\Sigma} \bar{V}^T = \sum_{i=1}^{n} \bar{u}_i \bar{\sigma}_i \bar{v}_i^T$$

Then, by the property of the Kronecker inner product we can get

$$\bar{A} \otimes A = (√{\bar{U} \otimes U})(Σ ⊗ Σ)(V ⊗ V) = \sum_{i=1}^{m} \sum_{i=1}^{n} (\bar{u}_i \otimes u_i)(\bar{\sigma}_i \sigma_i)(\bar{v}_i \otimes v_i)^T$$

(7)

As you can see, the singular value decomposition of $\bar{A} \otimes A$ is composed by the combination of singular values and vectors, the product of the singular values of $A$ and $\bar{A}$ formed the $mn$ singular values of $\bar{A} \otimes A$, the tensor product of left and right singular value vector of $A$ and $\bar{A}$ formed the left and right singular value vector of $\bar{A} \otimes A$. So the singular value decomposition of $\bar{A} \otimes A$ can be obtained by calculating the singular value decomposition of $A$ and $\bar{A}$.

Let $\text{vec}(F_k)$ represent the solution of 2-D TSVD, its can be represented by linding up the rows or columns of $F_k$. Here, $k$ is the truncation parameter of the 2-D TSVD truncation, It is the number of the singular value that included in the regularization. Then $\text{vec}(F_k)$ can be given by the following formula.

$$\text{vec}(F_k) = \sum_{i,j} \left( \frac{\text{vec}(G)}{\sigma_\bar{\sigma}} \right) \left( \bar{\sigma}_j v_j \otimes v_j \right)$$

The summation of the formula above is to taken former $k$ maximum values of $\bar{\sigma}/\sigma$. Use the relationship $(\bar{A} \otimes A) \text{vec}(X) = \text{vec}(AX\bar{A}^T)$, we can get

$$F_k = \sum_{i,j} \frac{u_i^T G u_j}{\sigma_i \bar{\sigma}_j} v_j v_j^T$$

(8)

The following is a summary of the 2-D TSVD calculation steps:
(1) Decompose the integral kernel which can be divided into tensor products of two smaller matrixes;
(2) Denote the solution of the 2-D TSVD according to the form of 1-D TSVD solution;
(3) According to the properties of tensor product, transforming the vector form solution to the matrix form;
(4) Find the former $k$ maximum values of $mn$ singular value products of two matrixes, and record the corresponding left and right singular value vectors;
(5) Solve it according to formula (8).

5. Numerical Simulation

Now we are going to show the algorithm of 2-D TSVD algorithm by one image inverse fuzzy problem. Here take the original image for J.H. Wilkinson who is the founder of rounding errors.

Fuzzy matrixes $A$ and $\overline{A}$ are obtained by formula (4), let $\sigma = 2.5, \text{band} = 10$, in addition adds a Gauss noise whose mean is 0, and variance is 0.001. The following diagrams represent the original image $F$, fuzzy image plus noise $G = AF\overline{A}^T + E$, 10 reconstruction images $F_k$. In each image above marks the value of $k$. It can be observed that when the value of $k$ increases at first, the image quality is getting better, then the influence of noise emerged. Throughout the process, except for the peak signal-to-noise ratio of the first reconstruction image is 33.01, the peak signal-to-noise ratio of the rest images are in the vicinity of 33.03. Improving the quality of image is not visible.

![Figure 1](image_url)

**Figure 1.** Top row: original and noisy blurred images of dimension $100 \times 75$. Middle and bottom rows: 2-D TSVD solutions $F_k$; the number above each image is $k$.

Now consider to get the results of applying the algorithm only under the fuzzy operator, bedim image through Gauss pulse spectrum function, let $\sigma = 2.5, \text{band} = 10$, then the condition number of matrix $A$ is $1.4368 \times 10^{13}$, this belongs to a highly ill-posed problem. The following diagrams represent the original image $F$, fuzzy image $G = AF\overline{A}^T$, 10 reconstruction images $F_k$. In each image above marks the value of $k$ and the corresponding peak signal to noise ratio.
Figure 2. Top row: original and blurred images of dimension $100 \times 75$. Middle and bottom rows: 2-D TSVD solutions $F_i$; the numbers above each image are $k$ and ISNR.

6. Conclusion

The image restoration discussed in this paper is a kind of two-dimensional deconvolution problem. Ill-posed is the biggest difficulty in handling this kinds of problem. 2-D TSVD algorithm provides a new way to solving image restoration problems, the effect by using this method to restore the blurred image is quite well; As for how to effectively use the method to deal with blurred image with noise, and how to effectively select optimal truncation parameter, these are all behind to further research to solve the problem. These are the problems to discuss and research later.

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