Model of hydrogen atom for twisted acceleration-enlarged Newton-Hooke space-times

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Abstract

We define the model of hydrogen atom for twist-deformed acceleration-enlarged Newton-Hooke space-time. Further, using time-dependent perturbation theory, we find in first step of iteration procedure the solution of corresponding Schrödinger equation as well as the probability of transition between two different energy-eigenstates.

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1 Introduction

Recently, there appeared a lot of papers dealing with noncommutative classical and quantum mechanics (see e.g. [1]-[3]) as well as with field theoretical models (see e.g. [4], [5]), in which the quantum space-time is employed. The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [6]. Recently, there were also found formal arguments based mainly on Quantum Gravity [7] and String Theory models [8], indicating that space-time at Planck scale should be noncommutative, i.e. it should have a quantum nature. On the other side, the main reason for such considerations follows from the suggestion that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincare invariance still remains valid at larger distances [9], [10].

Presently, it is well known that in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries, one can distinguish three basic types of quantum spaces [11], [12]:

1) Canonical ($\theta^{\mu\nu}$-deformed) space-time

\[ [x_\mu, x_\nu] = i\theta^{\mu\nu} ; \quad \theta^{\mu\nu} = \text{const}, \quad (1) \]

introduced in [13], [14] in the case of Poincare quantum group and in [15] for its Galilean counterpart.

2) Lie-algebraic modification of classical space

\[ [x_\mu, x_\nu] = i\theta^{\rho}_{\mu\nu} x_\rho, \quad (2) \]

with particularly chosen coefficients $\theta^{\rho}_{\mu\nu}$ being constants. This type of noncommutativity has been obtained as the representations of the $\kappa$-Poincare [16] and $\kappa$-Galilei [17] as well as the twisted relativistic [18] and nonrelativistic [15] symmetries, respectively.

3) Quadratic deformation of Minkowski and Galilei space

\[ [x_\mu, x_\nu] = i\theta^{\rho\tau}_{\mu\nu} x_\rho x_\tau, \quad (3) \]

with coefficients $\theta^{\rho\tau}_{\mu\nu}$ being constants. This kind of deformation has been proposed in [19], [20], [18] at relativistic and in [15] at nonrelativistic level.

Besides, it has been demonstrated in [21] that in the case of so-called acceleration-enlarged Newton-Hooke Hopf algebras $\mathcal{U}_0(\hat{N}H_{\pm})$ the (twist) deformation provides the new space-time noncommutativity, which is expanding ($\mathcal{U}_0(\hat{N}H_{+})$) or periodic ($\mathcal{U}_0(\hat{N}H_{-})$) in time, i.e. it takes the form

\[ [t, x_i] = 0 \quad , \quad [x_i, x_j] = if_{\pm} \left( \frac{t}{\tau} \right) \theta_{ij}(x), \quad (4) \]

$^1x_0 = ct$. 

with time-dependent functions

\[ f_+ \left( \frac{t}{\mathcal{T}} \right) = f \left( \sinh \left( \frac{t}{\mathcal{T}} \right), \cosh \left( \frac{t}{\mathcal{T}} \right) \right), \quad f_- \left( \frac{t}{\mathcal{T}} \right) = f \left( \sin \left( \frac{t}{\mathcal{T}} \right), \cos \left( \frac{t}{\mathcal{T}} \right) \right), \]

and \( \theta_{ij}(x) \sim \theta_{ij} = \text{const} \) or \( \theta_{ij}(x) \sim \theta_{ij} x_k \). Such a kind of noncommutativity follows from the presence in acceleration-enlarged Newton-Hooke symmetries \( \mathcal{U}_0(\mathcal{NH}_\pm) \) of the time scale parameter (cosmological constant) \( \mathcal{T} \). As it was demonstrated in [21] that just this parameter is responsible for oscillation or expansion of space-time noncommutativity.

It should be noted that both Hopf structures \( \mathcal{U}_0(\mathcal{NH}_\pm) \) contain, apart from rotation \( (M_{ij}) \), boost \( (K_i) \) and space-time translation \( (P_i, H) \) generators, the additional ones denoted by \( F_i \), responsible for constant acceleration. Consequently, if all generators \( F_i \) are equal zero we obtain the twisted Newton-Hooke quantum space-times [22], while for time parameter \( \mathcal{T} \) running to infinity we get the acceleration-enlarged twisted Galilei Hopf structures proposed in [21]. In particular, due to the presence of generators \( F_i \), for \( \mathcal{T} \to \infty \) we get the new cubic and quartic type of space-time noncommutativity

\[ [x_\mu, x_\nu] = i \alpha_{\mu\nu}^{\rho_1\ldots\rho_n} x_{\rho_1} \ldots x_{\rho_n}, \quad (5) \]

with \( n = 3 \) and \( 4 \) respectively, whereas for \( F_i \to 0 \) and \( \mathcal{T} \to \infty \) we reproduce the canonical [11], Lie-algebraic [2] and quadratic [3] (twisted) Galilei spaces provided in [15].

Finally, it should be noted that all mentioned above noncommutative space-times have been defined as the quantum representation spaces, so-called Hopf modules (see [23], [24], [13], [14]), for quantum acceleration-enlarged Newton-Hooke Hopf algebras, respectively.

Recently, in the series of papers [25]-[32] there has been discussed the impact of different kinds of quantum spaces on the dynamical structure of physical systems. Particulary, it has been demonstrated that in the case of classical oscillator model [28] as well as in the case of nonrelativistic particle moving in constant external field force \( \mathcal{F} \) [29], there are generated by space-time noncommutativity additional force terms. Such a type of investigation has been performed for quantum oscillator model as well [28], i.e., it was demonstrated that the quantum space in nontrivial way affects on the spectrum of energy operator. Besides, in article [30] there has been considered model of particle moving on the \( \kappa \)-Galilei space-time in the presence of gravitational field force. It has been demonstrated that in such a case there is produced force term, which can be identified with so-called Pioneer anomaly [34], and the value of deformation parameter \( \kappa \) can be fixed by comparison of obtained result with observational data.

From the abovementioned point of view, the case of two spatial directions commuting to the function of classical time seems to be most interesting. As it was demonstrated in articles [28] and [29], just this type of space-time noncommutativity produces additional time-dependent force terms, which appear in Hamiltonian function of the models. Usually, such a situation is interpreted as the interaction (by radiation) of considered system with (some) external source [35], [36].

In this article we consider the hydrogen atom model defined on the twist-deformed
acceleration-enlarged Newton-Hooke space-times. As one can see, it introduces due to the presence of function \( f(t) \) the time-dependent interaction term \( V_{\text{int}}(t, \bar{x}) \). Such a system can be analyzed with use of so-called time-dependent perturbation theory, i.e. one can find by iteration procedure solution of the corresponding Schrödinger equation as well as the probability of transition between two different energy-eigenstates. Surprisingly, for particular choice of function \( f(t) \), we get the interaction term similar to the potential function describing hydrogen atom present in time-periodic electric field. Such a result indicates that the choice of different functions \( f(t) \) may correspond to the choice of different kinds of external radiation sources.

The paper is organized as follows. In first section we recall commonly-known facts which concern hydrogen atom model defined on classical space-time. In Section 2 we introduce its noncommutative counterpart given on twist-deformed acceleration-enlarged Newton-Hooke space-time. Particularly, we find the solution of corresponding Schrödinger equation at first order of perturbation series as well as we calculate the probability of transition between two different energy-eigenstates. The final remarks are discussed in the last section.

2 Model of hydrogen atom in commutative space-time

In this section we recall basic facts associated with hydrogen atom model defined on commutative space-time. Firstly, it should be noted that the dynamics of considered system is described by the following Schrödinger equation

\[
i\hbar \frac{\partial \psi(t, \bar{x})}{\partial t} = H(\bar{p}, \bar{x})\psi(t, \bar{x}),
\]

where

\[
H(\bar{p}, \bar{x}) = H_0(\bar{p}) + V(\bar{x}) ; \quad H_0(\bar{p}) = \frac{\bar{p}^2}{2m} , \quad V(\bar{x}) = -\frac{Ze^2}{r},
\]

and

\[
r = \sqrt{x_1^2 + x_2^2 + x_3^2} , \quad \bar{a} = (a_1, a_2, a_3) , \quad Z = 1.
\]

It is well-known that the energy spectrum of such defined model is degenerate, i.e. the eigenvalues of the energy operator take the form

\[
E_n = -\frac{me^4Z^2}{2\hbar n^2} ; \quad n = 1, 2, 3, \ldots,
\]

---

\(^2\)The choice of acceleration-enlarged Newton-Hooke quantum spaces is dictated by the fact that they constitute the most general known deformation of classical nonrelativistic space-time.

\(^3\)Particularly, for function \( f(t) = f_{\kappa_1}(t) = \kappa_1 \) we recover the well-known noncommutative hydrogen atom model proposed in paper [3].
while the corresponding eigenfunctions look as follows

$$\psi_{nlm}(r, \theta, \phi) = N_{nlm} e^{-\frac{r}{r_0}} L_{n+l}^{(2l+1)} \left(\frac{2r}{r_0}\right) P_{lm}^m(\cos \theta) e^{im\phi}, \quad (10)$$

with \(l \leq n - 1, -l < m < l\),

$$N_{nlm} = N_{nl}N_{lm}N_m, \quad N_m = \frac{1}{2\pi}, \quad a = \frac{\hbar^2}{me^2}, \quad r_0 = \sqrt{\frac{-\hbar^2}{2mE}}, \quad (11)$$

$$N_{lm} = \sqrt{\frac{(2l+1)(l-m)!}{2(l+m)!}}, \quad N_{nl} = \left(\frac{2Z}{a}\right)^{3/2} \frac{1}{n^2} \sqrt{\frac{(n-l-1)!}{2[(n+l)]^{3/2}}}, \quad (12)$$

and \(L_n(x), P^m_n(x)\) being the Laguere and Legendre polynomials respectively. Besides, one can check that the above eigenvectors constitute the orthonormal base in Hilbert space of square integrated functions \(L^2(\mathbb{R}^3; d^3x)\)

$$\langle \psi_{nlm}(\vec{x}), \psi_{n'l'm'}(\vec{x}) \rangle = \delta_{nn'}\delta_{mm'}\delta_{ll'} \quad (13)$$

Finally, it should be noted that due to the stationary character of potential function \(V\), the solution of Schrödinger equation \((6)\) is given by

$$\psi(t, \vec{x}) = \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} c_{nlm} e^{iE_n(t-t_0)} \psi_{nlm}(\vec{x}), \quad (14)$$

with symbol \(t_0\) denoting initial time of evolution.

3 Model of hydrogen atom for twisted acceleration-enlarged Newton-Hooke space-times

3.1 Twisted acceleration-enlarged Newton-Hooke space-times

In this subsection we turn to the twisted acceleration-enlarged Newton-Hooke space-times equipped with two spatial directions commuting to classical time, i.e. we consider spaces of the form \(21\)

$$[t, \dot{x}_i] = [\dot{x}_1, \dot{x}_3] = [\dot{x}_2, \dot{x}_3] = 0, \quad [\dot{x}_1, \dot{x}_2] = if(t) \quad i = 1, 2, 3, \quad (15)$$
with function \( f(t) \) given by\(^4\)

\[
f(t) = f_{\kappa_1}(t) = f_{\pm,\kappa_1} \left( \frac{t}{\tau} \right) = \kappa_1 C_\pm \left( \frac{t}{\tau} \right), \tag{16}
\]

\[
f(t) = f_{\kappa_2}(t) = f_{\pm,\kappa_2} \left( \frac{t}{\tau} \right) = \kappa_2 \frac{t}{\tau} C_\pm \left( \frac{t}{\tau} \right) S_\pm \left( \frac{t}{\tau} \right), \tag{17}
\]

\[
f(t) = f_{\kappa_3}(t) = f_{\pm,\kappa_3} \left( \frac{t}{\tau} \right) = \kappa_3 \frac{t^2}{\tau^2} S_\pm^2 \left( \frac{t}{\tau} \right), \tag{18}
\]

\[
f(t) = f_{\kappa_4}(t) = f_{\pm,\kappa_4} \left( \frac{t}{\tau} \right) = 4\kappa_4 \frac{t}{\tau} \left( C_\pm \left( \frac{t}{\tau} \right) - 1 \right)^2, \tag{19}
\]

\[
f(t) = f_{\kappa_5}(t) = f_{\pm,\kappa_5} \left( \frac{t}{\tau} \right) = \pm \kappa_5 \frac{t^2}{\tau^2} \left( C_\pm \left( \frac{t}{\tau} \right) - 1 \right) C_\pm \left( \frac{t}{\tau} \right), \tag{20}
\]

\[
f(t) = f_{\kappa_6}(t) = f_{\pm,\kappa_6} \left( \frac{t}{\tau} \right) = \pm \kappa_6 \frac{t^3}{\tau^3} \left( C_\pm \left( \frac{t}{\tau} \right) - 1 \right) S_\pm \left( \frac{t}{\tau} \right); \tag{21}
\]

\[
C_{+/-} \left( \frac{t}{\tau} \right) = \cosh / \cos \left( \frac{t}{\tau} \right) \quad \text{and} \quad S_{+/-} \left( \frac{t}{\tau} \right) = \sinh / \sin \left( \frac{t}{\tau} \right).
\]

As it was already mentioned in Introduction, in \( \tau \to \infty \) limit the above quantum spaces reproduce the canonical \(^1\), Lie-algebraic \(^2\), quadratic \(^3\) as well as cubic and quartic \(^4\) type of space-time noncommutativity, with\(^5\)

\[
f_{\kappa_1}(t) = \kappa_1, \tag{22}
\]

\[
f_{\kappa_2}(t) = \kappa_2 t, \tag{23}
\]

\[
f_{\kappa_3}(t) = \kappa_2 t^3, \tag{24}
\]

\[
f_{\kappa_4}(t) = \kappa_4 t^4, \tag{25}
\]

\[
f_{\kappa_5}(t) = \frac{1}{2} \kappa_5 t^2, \tag{26}
\]

\[
f_{\kappa_6}(t) = \frac{1}{2} \kappa_6 t^3. \tag{27}
\]

Of course, for all parameters \( \kappa_a \ (a = 1, \ldots, 6) \) running to zero the above deformations disappear.

### 3.2 Model of hydrogen atom

Let us now turn to the main aim of our investigations - to the model of hydrogen atom defined on quantum space-times \(^{16}\)-\(^{21}\). In first step of our construction we extend the

\(^4\)The indexes + and − in formulas \(^{16}\)-\(^{21}\) correspond to the acceleration-enlarged Newton-Hooke quantum spaces associated with Hopf structures, which can be get partially by the contraction of De-Sitter and Anti-De-Sitter Hopf algebras respectively.

\(^5\)Space-times \(^{22}\), \(^{23}\) correspond to the twisted Galilei Hopf algebras provided in \(^{15}\), while the quantum space \(^{27}\) is associated with acceleration-enlarged Galilei Hopf structure \(^{21}\).

\(^6\)Technically, we expand the r.h.s of formulas \(^{16}\)-\(^{21}\) in Taylor series with respect variable \( \frac{t}{\tau} \) and take the limit \( \tau \to \infty \). In such a way there survive only in the commutation relation \(^{22}\)-\(^{27}\) \( t^n \) terms with \( n = 0, 1, 2, 3 \) and 4 respectively.
described in previous subsection spaces to the whole algebra of momentum and position operators as follows

\[ [\hat{x}_1, \hat{x}_2] = i f_{\kappa_a}(t) , \quad [\hat{x}_1, \hat{x}_3] = [\hat{x}_2, \hat{x}_3] = [\hat{p}_i, \hat{p}_j] = 0 , \quad (28) \]
\[ [\hat{x}_i, \hat{p}_j] = i \hbar \delta_{ij} ; \quad i, j = 1, 2, 3 . \quad (29) \]

One can check that relations (28), (29) satisfy the Jacobi identity and for deformation parameters \( \kappa_a \) approaching zero become classical.

Next, by analogy to the commutative case we define the Hamiltonian operator

\[ H(\hat{p}, \hat{x}) = \frac{\hat{p}^2}{2m} - \frac{Ze^2}{\hat{r}} , \quad (30) \]

with \( \hat{r} = \sqrt{\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2} \).

In order to analyze the above system we represent the noncommutative operators \((\hat{x}_i, \hat{p}_i)\) by classical ones \((x_i, p_i)\) as (see e.g. [31]-[33])

\[ \hat{x}_1 = x_1 - \frac{f_{\kappa_a}(t)}{2\hbar} p_2 , \quad \hat{x}_2 = x_2 + \frac{f_{\kappa_a}(t)}{2\hbar} p_1 , \quad \hat{x}_3 = x_3 , \quad \hat{p}_i = p_i , \quad (31) \]

where

\[ [x_i, x_j] = 0 = [p_i, p_j] , \quad [x_i, p_j] = i \hbar \delta_{ij} . \quad (32) \]

Then, the Hamiltonian (30) takes the form

\[ H(\hat{p}, \hat{x}, t) = \frac{\hat{p}^2}{2m} - \frac{Ze^2}{r} - \frac{Ze^2 L_3 f_{\kappa_a}(t)}{2r \hbar^3} + \mathcal{O}(\kappa_a) = \]
\[ = H_0(\hat{p}, \hat{x}) + V_{\text{int}}(t, \hat{x}) + \mathcal{O}(\kappa_a) , \quad (33) \]

with \( \mathcal{O}(\kappa_a) \) denoting terms quadratic in deformation parameter \( \kappa_a \) and \( L_3 = x_1 p_2 - x_2 p_1 \).

Besides, present in the above formula additional potential function

\[ V_{\text{int}}(t, \bar{x}) = -\frac{Ze^2 L_3 f_{\kappa_a}(t)}{2r \hbar^3} , \quad (34) \]

describes interaction (by radiation) of quantum particle with (some) external source.

It is well-known that such a system can be analyzed with use of time-dependent perturbation theory [35], [36]. Then, the solution of corresponding Schrödinger equation

\[ i\hbar \frac{\partial \psi(t, \bar{x})}{\partial t} = [H_0(\hat{p}, \hat{x}) + V_{\text{int}}(t, \hat{x})] \psi(t, \bar{x}) , \quad (35) \]

is given by

\[ \psi(t, \bar{x}) = \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} c_{n,m}(t) e^{i E_n(t-t_0)} \psi_{n,m}(\bar{x}) , \quad (36) \]
where symbols $E_n$ and $\psi_{nlm}(\bar{x})$ denote eigenvalues $[9]$ and eigenfunctions $[10]$ for classical hydrogen atom respectively. The coefficients $c_{nlm}(t)$ are defined as the solutions of the following differential equations

$$\frac{dc_{nlm}(t)}{dt} = \frac{1}{i\hbar} \sum_{n'=0}^{\infty} \sum_{l'=-l}^{l} \sum_{m=-l}^{l} (\psi_{nlm}(\bar{x}), V_{\text{int}}(t, \bar{x})) \psi_{n'l'm'}(\bar{x}) c_{n'l'm'}(t_0) \cdot e^{i\omega_{n'l'm'}(t-t_0)} ; \quad \omega_{n'l'm'} = \frac{1}{\hbar} (E_n - E_{n'}) ,$$

with initial condition of the form $c_{n'l'm'}(t_0) = (\psi_{n'l'm'}(\bar{x}), \psi_0(\bar{x}))$ for some wave function $\psi_0(\bar{x})$. Usually, the above system can be solve by iteration procedure with zero-approximated term given by

$$c_{nlm}^{(0)}(t) = (\psi_{nlm}(\bar{x}), \psi_0(\bar{x})) ,$$

and with two neighboring higher steps of approximation combined into the following equation

$$\frac{dc_{nlm}^{(j)}(t)}{dt} = \frac{1}{i\hbar} \sum_{n'=0}^{\infty} \sum_{l'=0}^{l} \sum_{m'=-l}^{l} (\psi_{nlm}(\bar{x}), V_{\text{int}}(t, \bar{x})) \psi_{n'l'm'}(\bar{x}) c_{n'l'm'}^{(j-1)}(t) \cdot e^{i\omega_{n'l'm'}(t-t_0)} .$$

Particularly, in accordance with the above formula, for $j = 1$ and for $\psi_0(\bar{x}) = \psi_{n''l''m''}(\bar{x})$, we get ($t_0 = 0$)

$$c_{nlm}^{(1)}(t) = \delta_{n'n''} \delta_{l'l''} \delta_{m'm''} + \frac{1}{i\hbar} \int_0^t dt_1 e^{i\omega_{n''l''m''} t_1} (\psi_{nlm}(\bar{x}), V_{\text{int}}(t_1, \bar{x})) \psi_{n''l''m''}(\bar{x}) ,$$

and, by direct calculation, we obtain

$$c_{nlm}^{(1)}(t) = \delta_{n'n''} \delta_{lm'm''} - \frac{m''^2 e^2}{2 \hbar^2} g(t) \left( \psi_{nlm}(\bar{x}), \frac{1}{r^3} \psi_{n''l''m''}(\bar{x}) \right) ;$$

$$g(t) = g_{\kappa_1}(t) = g_{\pm, \kappa_1} \left( \frac{t}{\tau} \right) = \frac{\kappa_1}{4} \left( \tau S_\pm \left( \frac{2t}{\tau} \right) + 2t \right) ,$$

$$g(t) = g_{\kappa_2}(t) = g_{\pm, \kappa_2} \left( \frac{t}{\tau} \right) = \pm \frac{\kappa_2}{4} \tau^2 \left( C_\pm \left( \frac{2t}{\tau} \right) - 1 \right) ,$$

$$g(t) = g_{\kappa_3}(t) = g_{\pm, \kappa_3} \left( \frac{t}{\tau} \right) = \pm \frac{\kappa_3}{4} \tau^2 \left( \tau S_\pm \left( \frac{2t}{\tau} \right) + 2t \right) ,$$

$$g(t) = g_{\kappa_4}(t) = g_{\pm, \kappa_4} \left( \frac{t}{\tau} \right) = \frac{\kappa_4}{4} \tau^4 \left( \tau S_\pm \left( \frac{2t}{\tau} \right) - 8 \tau S_\pm \left( \frac{t}{\tau} \right) + 6t \right) ,$$

$$g(t) = g_{\kappa_5}(t) = g_{\pm, \kappa_5} \left( \frac{t}{\tau} \right) = \pm \frac{\kappa_5}{4} \tau^2 \left( \tau S_\pm \left( \frac{2t}{\tau} \right) + 4 \tau S_\pm \left( \frac{t}{\tau} \right) + 2t \right) ,$$

$$g(t) = g_{\kappa_6}(t) = g_{\pm, \kappa_6} \left( \frac{t}{\tau} \right) = \frac{\kappa_6}{2} \tau^4 \left( C_\pm \left( \frac{t}{\tau} \right) - 2 \right) C_\pm \left( \frac{t}{\tau} \right) + 1 \right) ,$$

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with \( n = n'' \) and

\[
c_{nlm}^{(1)}(t) = -\frac{m'' Ze^2}{i2\hbar^2} h(t) \left( \psi_{nlm}(\bar{x}), \frac{1}{r^3} \psi_{n''l''m''}(\bar{x}) \right) ;
\]

\[
h(t) = h_{k_1}(t) = h_{\pm,k_1} \left( \frac{t}{\tau} \right) = -\frac{ik_1}{2\omega_{nn''}(\tau^2\omega_{nn''}^2 \pm 4)} \left[ e^{it\omega_{nn''}} \left( \tau^2\omega_{nn''} C_\pm \left( \frac{2t}{\tau} \right) + \tau^2\omega_{nn''}^2 \pm 2i\tau\omega_{nn''} S_\pm \left( \frac{2t}{\tau} \right) \mp 4 \right) \right],
\]

\[
h(t) = h_{k_2}(t) = h_{\pm,k_2} \left( \frac{t}{\tau} \right) = \frac{k_2\tau^2}{2(\tau^2\omega_{nn''}^2 + 4)} \left[ e^{it\omega_{nn''}} \left( 2C_\pm \left( \frac{2t}{\tau} \right) - i\tau\omega_{nn''} \cdot \right) \right]
• \ S_\pm \left( \frac{2t}{\tau} \right) - 2 \right],
\]

\[
h(t) = h_{k_3}(t) = h_{\pm,k_3} \left( \frac{t}{\tau} \right) = \frac{\kappa_3\tau^2}{2(\tau^2\omega_{nn''}^2 + 4)} \left[ e^{it\omega_{nn''}} \left( \tau^2\omega_{nn''} C_\pm \left( \frac{2t}{\tau} \right) + \tau^2\omega_{nn''}^2 \pm 2i\tau\omega_{nn''} S_\pm \left( \frac{2t}{\tau} \right) \mp 4 \right) \right],
\]

\[
h(t) = h_{k_4}(t) = h_{\pm,k_4} \left( \frac{t}{\tau} \right) = -\frac{2i\kappa_4\tau^4}{\omega_{nn''}(\tau^4\omega_{nn''}^4 + 5\tau^2\omega_{nn''}^4 + 4)} \left[ e^{it\omega_{nn''}} \left( 3\tau^4\omega_{nn''}^4 + 4\tau^2\omega_{nn''}^4 \right) \right]
• \ 4i\tau^3\omega_{nn''} S_\pm \left( \frac{2t}{\tau} \right) + 4\tau^3\omega_{nn''}^3 S_\pm \left( \frac{2t}{\tau} \right) - 4\tau^2\omega_{nn''}^2 \left( \tau^2\omega_{nn''}^2 + 4 \right) C_\pm \left( \frac{2t}{\tau} \right) + \tau^2\omega_{nn''}^2 \left( \tau^2\omega_{nn''}^2 + 1 \right) C_\pm \left( \frac{2t}{\tau} \right) + 15\tau^2\omega_{nn''}^2 - 16i\tau\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) + 2i\tau\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) + 12 \right),
\]

\[
h(t) = h_{k_5}(t) = h_{\pm,k_5} \left( \frac{t}{\tau} \right) = -\frac{i\kappa_5\tau^2}{2\omega_{nn''}(\tau^4\omega_{nn''}^4 + 5\tau^2\omega_{nn''}^4 + 4)} \left[ e^{it\omega_{nn''}} \left( \tau^4\omega_{nn''}^4 + 4\tau^2\omega_{nn''}^4 \right) \right]
• \ 2i\tau^3\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) + 2i\tau^3\omega_{nn''} \left( \tau^2\omega_{nn''}^4 + 4 \right) C_\pm \left( \frac{t}{\tau} \right) + \tau^2\omega_{nn''} \left( \tau^2\omega_{nn''}^4 + 1 \right) \cdot \ C_\pm \left( \frac{t}{\tau} \right) + 5\tau^2\omega_{nn''}^2 + 8i\tau\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) + 2i\tau\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) - 2i\tau^3\omega_{nn''} \cdot \ S_\pm \left( \frac{t}{\tau} \right) + 12 \right),
\]

\[
h(t) = h_{k_6}(t) = h_{\pm,k_6} \left( \frac{t}{\tau} \right) = -\frac{i\kappa_6\tau^2}{2\omega_{nn''}(\tau^4\omega_{nn''}^4 + 5\tau^2\omega_{nn''}^4 + 4)} \left[ e^{it\omega_{nn''}} \left( \tau^4\omega_{nn''}^4 + 4\tau^2\omega_{nn''}^4 \right) \right]
• \ 2i\tau^3\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) + 2i\tau^3\omega_{nn''} \left( \tau^2\omega_{nn''}^4 + 4 \right) C_\pm \left( \frac{t}{\tau} \right) + \tau^2\omega_{nn''} \left( \tau^2\omega_{nn''}^4 + 1 \right) \cdot \ C_\pm \left( \frac{t}{\tau} \right) + 5\tau^2\omega_{nn''}^2 + 8i\tau\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) + 2i\tau\omega_{nn''} S_\pm \left( \frac{t}{\tau} \right) - 2i\tau^3\omega_{nn''} \cdot \ S_\pm \left( \frac{t}{\tau} \right) + 12 \right),
\]
\[ h(t) = h_{\kappa_0}(t) = h_{+\kappa_0} \left( \frac{t}{\tau} \right) = \pm \frac{\kappa_0 \tau^4}{2(\tau^4 \omega_{mn}^2 + 5 \tau^2 \omega_{mn}^2 + 4)} \left[ e^{i\omega_{mn}t} (e^{-2(\tau^2 \omega_{mn}^2 t + 4)} \cdot C_\pm \left( \frac{2t}{\tau} \right) - 2i\tau \omega_{mn} S_\pm \left( \frac{t}{\tau} \right) (\tau^2 \omega_{mn}^2 + 1) \right] + 6 \right], \tag{55} \]

for \( n \neq n'' \).

Consequently, the solution of Schrödinger equation at first level of iteration procedure takes the form

\[ \psi(t, \bar{x}) = \sum_{n=0}^{\infty} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} c_{nlm}^{(1)}(t) e^{iE_n t} \psi_{nlm}(\bar{x}) \tag{56} \]

Finally, it should be noted that coefficients \( c_{nlm}(t) \) have quite simple physical interpretation, i.e. their module is proportional to the probability of transition from initial state \( \psi_0(\bar{x}) \) to the final wave function \( \psi_{nlm}(\bar{x}) \). Particularly, in accordance with formula \( \text{(55)} \) for \( \psi_0(\bar{x}) = \psi_{n''l''m''}(\bar{x}) \) and \( j = 1 \), one gets

\[ P^{(1)}(t) = |c_{nlm}(t)|^2 = \left( \frac{m'' Ze^2}{2\hbar^2} \right)^2 \hbar^2(t) \left| \psi_{nlm}(\bar{x}), \frac{1}{\sqrt{3}} \psi_{n''l''m''}(\bar{x}) \right|^2. \tag{57} \]

Of course, for cosmological constant \( \tau \) approaching infinity we obtain the proper formula associated with quantum spaces \( \text{(22)-(27)} \).

4 Final remarks

In this article we investigate the hydrogen atom model defined on twisted acceleration-enlarged Newton-Hooke space-times \( \text{(21)} \). We find the solution of corresponding Schrödinger equation at first order of perturbation series as well as we calculate the probability of transition between two different energy-eigenstates. It should be noted that the presented algorithm can be extended to much more complicated atomic systems such as, for example, helium, orto- and para-hydrogen atoms \( \text{(35), (36)} \). The works in this direction already started and are in progress.

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\footnote{We consider transition associated with two different energy levels.}
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