ALGEBRAIC CONNECTIONS ON PROJECTIVE MODULES
WITH PRESCRIBED CURVATURE

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Abstract. We construct algebraic connections on a class of finitely generated projective modules using universal enveloping algebras of Lie-Rinehart algebras. We also calculate the curvature of the connections. The main aim of the paper is to construct for any projective Lie-Rinehart algebra $L$ a subring $\text{Char}(L)$ of $H^*(L, B)$ - the characteristic ring of $L$. This ring is defined purely in terms of the Lie-Rinehart cohomology $H^*(L, B)$ and has the property that it equals the image of the Chern character $\text{Ch} : K(L) \rightarrow H^*(L, B)$.

Contents

1. Introduction 1
2. Lie-Rinehart cohomology and extensions 2
3. Families of universal enveloping algebras of Lie-Rinehart algebras 6
4. Application I: Deformations of filtered algebras 11
5. Applications II: Connections on families of projective modules 15
References 19

1. Introduction

In the following paper we generalize classical notions on Lie algebras and universal enveloping algebras of Lie algebras (see [5] and [7]) to Lie-Rinehart algebras and universal enveloping algebras of Lie-Rinehart algebras. As a consequence we get new examples of finitely generated projective modules with no flat algebraic connections. We also construct families of (mutually non-isomorphic) finitely generated projective modules of arbitrary high rank using families of universal enveloping algebras of Lie-Rinehart algebras (see Example 5.6). The main theorem (see Theorem 5.3) is that for any Lie-Rinehart algebra $\{L, \alpha\}$ which is projective as $B$-module and any cohomology class $c \in H^2(L, B)$ there is a finitely generated projective $B$-module $E$ with $c_1(E) = c$. One application of this result is the following construction: For any Lie-Rinehart algebra $L$ which is projective as left $B$-module, there is a subring $\text{Char}(L) \subseteq H^*(L, B)$ which is defined purely in terms of the cohomology ring $H^*(L, B)$. The subring $\text{Char}(L)$ equals the image $\text{Im}(\text{Ch})$ of the Chern character

$$\text{Ch} : K(L) \rightarrow H^*(L, B).$$
It is an unsolved problem to calculate the generators of $K(L)$ in general and this problem is eliminated in the study of $\text{Im}(Ch)$ since the definition of $\text{Char}(L)$ only involves the cohomology group $H^2(L, B)$.

We also relate the cohomology group $H^2(L, B)$ where $\{L, \alpha\}$ is a Lie-Rinehart algebra which is projective as left $B$-module to deformations of filtered associative algebras. Let $A(\text{Sym}^*_B(L))$ be the deformation groupoid of the Lie-Rinehart algebra $\{L, \alpha\}$ parametrizing filtered associative algebras $\{U, U_i\}$ whose associated graded algebra $\text{Gr}(U)$ is isomorphic to $\text{Sym}^*_B(L)$ as graded $B$-algebra. There is a one-to-one correspondence between $H^2(L, B)$ and the set of isomorphism classes of objects in $A(\text{Sym}^*_B(L))$ (see Theorem 4.9).

2. Lie-Rinehart cohomology and extensions

In this section we extend well known results on Lie algebras, cohomology of Lie algebras and extensions to cohomology of Lie-Rinehart algebras and extensions of Lie-Rinehart algebras. We give an interpretation of the cohomology groups $H^i(L, W)$ for $i = 1, 2$ in terms of derivations of Lie-Rinehart algebras and equivalence classes of extensions of Lie-Rinehart algebras. The results are straight forward generalizations of existing results for Lie algebras and are included because of lack of a good reference.

Let in the following $h : A \to B$ be a map of commutative rings with unit. Let $L$ be a left $B$-module and an $A$-Lie algebra and let $\alpha : L \to \text{Der}_A(B)$ be a map of left $B$-modules and $A$-Lie algebras.

Recall the following definition:

**Definition 2.1.** The pair $\{L, \alpha\}$ is a Lie-Rinehart algebra if the following equation holds for all $x, y \in L$ and $a \in B$:

$$[x, ay] = a[x, y] + \alpha(x)(a)y.$$ 

The map $\alpha$ is usually called the anchor map.

Let $W$ be a left $B$-module and let $\nabla : L \to \text{End}_A(W)$ be a $B$-linear map.

**Definition 2.2.** The map $\nabla$ is an $L$-connection if the following equation holds for all $x \in L, a \in B$ and $w \in W$:

$$\nabla(x)(aw) = a\nabla(x)(w) + \alpha(x)(a)w.$$ 

Let $\{W, \nabla\}$ be a connection. Recall the definition of the Lie-Rinehart complex of the connection $\nabla$: Let

$$C^p(L, W) = \text{Hom}_B(\wedge^p L, W)$$

with differentials

$$d^p : C^p(L, W) \to C^{p+1}(L, W)$$

defined by

$$d^p(\phi)(x_1 \wedge \cdots \wedge x_p) = \sum_k (-1)^{k+1} \nabla(x_k)(\phi(x_1 \wedge \cdots \wedge \overline{x_k} \wedge \cdots \wedge x_p)) +$$

$$\sum_{i,j} (-1)^{i+j} \phi([x_i, x_j] \wedge x_1 \wedge \cdots \wedge \overline{x_i} \wedge x_{i+1} \wedge \cdots \wedge \overline{x_j} \wedge \cdots \wedge x_p).$$

One checks the following:
where
\[d^0(w)(x) = \nabla(x)(w),\]
\[d^1(\phi)(x \wedge y) = \nabla(x)(\phi(y)) - \nabla(y)(\phi(x)) - \phi([x, y]),\]
and
\[d^1(d^0(w))(x \wedge y) = R_\nabla(x \wedge y)(w),\]
where
\[R_\nabla(x \wedge y) = [\nabla(x), \nabla(y)] - \nabla([x, y]).\]

We let \(R_\nabla\) be the curvature of the connection \(\nabla\). One checks that the sequence of groups and maps given by \(\{C^p(L, W), d^p\}\) is a complex of \(A\)-modules if and only if the curvature \(R_\nabla\) is zero.

**Definition 2.3.** Let \(\{W, \nabla\}\) be a flat connection. Let \(Z^i(L, W) = \ker(d^i)\) and \(B^i(L, W) = \text{im}(d^{i-1})\). Let for all \(i \geq 0\) \(H^i(L, W) = Z^i(L, W)/B^i(L, W)\) be the \(i\)th Lie-Rinehart cohomology group of \(L\) with values in \(\{W, \nabla\}\).

It follows the abelian group \(H^i(L, W)\) is a left \(A\)-module.

In this section we are interested in the group \(H^i(L, W)\) for \(i = 1, 2\) where \(\{W, \nabla\}\) is a flat connection.

We get a map
\[d^2 : C^2(L, W) \rightarrow C^3(L, W)\]
where for any element
\[f \in C^2(L, A) = \text{Hom}_B(\wedge^2 L, W)\]
it follows
\[d^2(f)(x_1 \wedge x_2 \wedge x_3) = \nabla(x_1)(f(x_2 \wedge x_3)) - \nabla(x_2)(f(x_1 \wedge x_3)) + \nabla(x_3)(f(x_1 \wedge x_2)) - f([x_1, x_2] \wedge x_3) + f([x_1, x_3] \wedge x_2) - f([x_2, x_3] \wedge x_1).\]

It follows \(Z^2(L, W)\) is the set of \(B\)-bilinear maps
\[f : L \times L \rightarrow W\]
satisfying \(f(x, x) = 0\) for all \(x \in L\) and such that \(d^2(f) = 0\).

Let \(\alpha : L \rightarrow \text{Der}_A(B)\) and \(\tilde{\alpha} : \tilde{L} \rightarrow \text{Der}_A(B)\) be Lie-Rinehart algebras. Let
\[p : \tilde{L} \rightarrow L\]
be a map of left \(B\)-modules and \(A\)-Lie algebras.

**Definition 2.4.** We say \(p\) is a map of Lie-Rinehart algebras if \(\alpha \circ p = \tilde{\alpha}\).

Let \(p : \tilde{L} \rightarrow L\) be a surjective map of Lie-Rinehart algebras and let \(W = \ker(p)\).

It follows \(W\) is a sub-\(B\)-module and a sub-\(A\)-Lie algebra of \(\tilde{L}\). We get an exact sequence
\[0 \rightarrow W \rightarrow \tilde{L} \rightarrow L \rightarrow 0.\]
of left \(B\)-modules and \(A\)-Lie algebras. Define the following action:
\[\tilde{\nabla} : \tilde{L} \rightarrow \text{End}(W)\]
by
\[\tilde{\nabla}(z)(w) = [z, w]\]
where \([,]\) is the Lie-product on \(\tilde{L}\) and \(z \in \tilde{L}, w \in W\).

**Lemma 2.5.** The map \(\tilde{\nabla}\) is a flat \(L\)-connection on \(W\).
Proof. The proof is left to the reader as an exercise. □

Assume now \( W = \ker(p) \subseteq \tilde{L} \) is an abelian sub-algebra of \( \tilde{L} \). Assume \( z \in \tilde{L} \) is an element with \( p(z) = x \in L \). Let \( w \in W \). Define the following map:

\[
\rho : L \to \text{End}(W)
\]

by

\[
\rho(x)(w) = [z, w].
\]

Assume \( p(z') = x \). It follows \( z' = z + v \) where \( v \in W \). We get \( [z + v, w] = [z, w] + [v, w] = [z, w] \). Hence the element \( \rho(x) \in \text{End}(W) \) does not depend on choice of the element \( z \) mapping to \( x \). It follows \( \rho \) is a well defined map. One checks using the proof of Lemma 2.5 that \( \rho \) is a \( B \)-linear map

\[
\rho : L \to \text{End}_A(W).
\]

Lemma 2.6. The map \( \rho \) is a flat \( L \)-connection \( W \).

Proof. The proof is an exercise. □

Fix a flat connection

\[
\nabla : L \to \text{End}_A(W)
\]

on the Lie-Rinehart algebra \( L \) and assume \( p : \tilde{L} \to L \) is a surjective map of Lie-Rinehart algebras. Assume \( W = \ker(p) \) is an abelian sub-algebra of \( \tilde{L} \). Assume the induced connection

\[
\rho : L \to \text{End}_A(W)
\]

from Lemma 2.6 equals \( \nabla \).

Definition 2.7. The extension

\[
0 \to W \to \tilde{L} \to L \to 0
\]

is an extension of \( L \) by the flat connection \( \{W, \nabla\} \).

Two extensions \( L_1, L_2 \) of \( L \) by \( \{W, \nabla\} \) are equivalent if there is an isomorphism \( \phi : L_1 \to L_2 \) of Lie-Rinehart algebras making the two obvious diagrams commute.

Definition 2.8. Let \( \text{Ext}^1(L, W, \nabla) \) be the set of equivalence classes of extensions of \( L \) by the flat connection \( \{W, \nabla\} \).

Let \( f \in Z^2(L, W) \) be an element. It follows \( f : L \times L \to W \) is \( B \)-linear in both variables with \( f(x, x) = 0 \) for all \( x \in L \) and \( d^2(f) = 0 \). Define the following product on \( W \oplus L \):

\[
[(w, x), (v, y)] = (\nabla(x)(v) - \nabla(y)(w) + f(x, y), [x, y]).
\]

Let \( L(f) \) be the left \( B \)-module \( W \oplus L \) equipped with the product \([,]\). Define a map \( \alpha_f : L(f) \to \text{Der}_A(B) \) by \( \alpha_f(w, x) = \alpha(x) \).

Lemma 2.9. The left \( B \)-module \( L(f) \) is a Lie-Rinehart algebra. The sequence

\[
0 \to W \to L(f) \to L \to 0
\]

is an extension of \( L \) by the flat connection \( \{W, \nabla\} \).

Proof. The proof is an exercise. □

Lemma 2.10. Let \( \alpha : L \to \text{Der}_A(B) \) be a Lie-Rinehart algebra and let \( f, g \in Z^2(L, W) \) be two cocycles. There is an isomorphism \( \phi : L(f) \to L(g) \) of extensions if and only if there is a \( \rho \in C^1(L, W) \) with \( d^1 \rho = f - g \).
Proof. The proof is an exercise. □

It follows we get a well defined map of sets
\[ \beta : \mathbb{Z}^2(L, W) \to \text{Ext}^1(L, W, \nabla), \]
defined by sending \( f \) to the equivalence class in \( \text{Ext}^1(L, W, \nabla) \) determined by \( L(f) \).

Let \( f + d^1 \rho \) be an element in \( \mathbb{Z}^2(L, W) \) with \( \rho \in C^1(L, W) \). It follows from Lemma 2.10 that \( \beta(f) = \beta(f + d^1 \rho) \). We get a well defined map
\[ \overline{\beta} : H^2(L, W) \to \text{Ext}^1(L, W, \nabla) \]
defined by
\[ \overline{\beta}(f) = L(f). \]

**Theorem 2.11.** If \( \{L, \alpha\} \) is an arbitrary Lie-Rinehart algebra the map \( \overline{\beta} \) is an injection of sets. If \( L \) is a projective \( B \)-module it follows the map \( \overline{\beta} \) is an isomorphism of sets.

Proof. The proof is an exercise. □

Note: One may construct an \( A \)-module structure on \( \text{Ext}^1(L, W, \nabla) \) and one checks that the map \( \overline{\beta} \) is an \( A \)-linear map.

One checks that
\[ H^1(L, W) = \text{Der}(L, W)/\text{Der}^{\text{inn}}(L, W). \]

**Example 2.12.** Cohomology of Lie algebras.

The following result is well known from the cohomology theory of Lie algebras:

**Corollary 2.13.** Let \( L \) be a Lie algebra over a field \( k \) and let \( W \) be a left \( L \)-module. There is a bijection between \( H^2(L, W) \) and the set of equivalence classes of extensions of \( L \) by \( W \).

Proof. The proof follows from Theorem 2.11. Let \( A = B = k \). □

**Example 2.14.** Singular cohomology of complex algebraic manifolds.

Assume \( A \) is a finitely generated regular algebra over the complex numbers and let \( X = \text{Spec}(A) \) be the associated affine scheme. Let \( X(C) \) be the complex manifold associated to \( X \) and let \( L = \text{Der}_C(A) \) be the Lie-Rinehart algebra of derivations of \( A \). It follows there is an isomorphism
\[ H^i(L, A) \cong H^i_{\text{sing}}(X(C), C) \]
of cohomology groups where \( H^i_{\text{sing}}(X(C), C) \) is singular cohomology of \( X(C) \) with complex coefficients. It follows we get an isomorphism
\[ \text{Ext}^1(L, A, \alpha) \cong H^2_{\text{sing}}(X(C), C) \]
of complex vector spaces. Hence to each cohomology class \( \gamma \in H^2_{\text{sing}}(X(C), C) \) we get an extension
\[ 0 \to A \to L(\gamma) \to L \to 0 \]
of Lie-Rinehart algebras. The class \( \gamma \) is a purely topological object and the extension \( L(\gamma) \) is a purely algebraic object: \( L(\gamma) \) is an infinite dimensional extension of the complex Lie algebra \( L = \text{Der}_C(A) \) of \( C \)-derivations of \( A \).
3. Families of universal enveloping algebras of Lie-Rinehart algebras

In this section we generalize some constructions for Lie algebras and enveloping algebras of Lie algebras from [5] and [7] to the case of Lie-Rinehart algebras and universal enveloping algebras of Lie-Rinehart algebras. For an arbitrary Lie-Rinehart algebra \( \{L, \alpha\} \) and an arbitrary cocycle \( f \in Z^2(L, B) \) we define the universal enveloping algebra of type \( f \) denoted \( U(B, L, f) \) and prove some basic properties of this algebra. We prove a Poincare-Birkhoff-Witt Theorem for \( U(B, L, f) \) when \( L \) is a projective \( B \)-module generalizing the Poincare-Birkhoff-Witt Theorem proved by Rinehart in [5].

Let \( \alpha : L \to \text{Der}_A(B) \) be a Lie-Rinehart algebra and let \( f \in Z^2(L, B) \) be a cocycle. Let \( z \) be a generator for the free \( B \)-module \( F \) and let

\[
0 \to F \to L(f) \to L \to 0
\]

be the extension corresponding to \( f \). Let \( \nabla : L \to \text{End}_A(W) \) be an \( L \)-connection.

**Definition 3.1.** We say \( \nabla \) is an \( L \)-connection of curvature type \( f \) if the following is satisfied: For all \( x, y \in L \) and \( v \in W \) the following formula holds:

\[
R_\nabla(x \wedge y)(v) = f(x, y)v.
\]

Here \( R_\nabla \) is the curvature of \( \nabla \).

**Lemma 3.2.** Let \( W \) be a left \( B \)-module. There is a one-to-one correspondence between the set of \( L \)-connections of curvature type \( f \) on \( W \) and the set of flat \( L \)-connections on \( W \) with \( \nabla(z) = Id_W \).

**Proof.** The proof is an exercise. \( \square \)

For any elements \( u = az + x, v = bz + y \in L(f) \) the following holds:

\[
[u, v] = [az + x, bz + y] = (\alpha(x)(b) - \alpha(y)(a) + f(x, y), [x, y]).
\]

Write \( x(b) = \alpha(x)(b) \). The pair \( \{L(f), \alpha_f\} \) where \( \alpha_f(az + x) = \alpha(x) \in \text{Der}_A(B) \) is by the results in the previous section a Lie-Rinehart algebra. Hence \( L(f) \) is a left \( B \)-module and an \( A \)-Lie algebra.

Let \( T(L(f)) = \oplus_{k \geq 0} L(f)^{\otimes A^k} \) be the tensor algebra (over \( A \)) of the \( A \)-Lie algebra \( L(f) \). Let \( T^0(L(f)) = \oplus_{k \geq 0} L(f)^{\otimes A_k} \) and let \( T_p(L(f)) = \oplus_{k=0}^p L(f)^{\otimes A_k} \). Let \( U_f \) be the two sided ideal in \( T(L(f)) \) generated by the set of elements

\[
u \otimes v - v \otimes u - [u, v]
\]

with \( u, v \in L(f) \). Let \( U(L(f)) = T(L(f))/U_f \) be the universal enveloping algebra of the \( A \)-Lie algebra \( L(f) \).

Let \( p : T(L(f)) \to U(L(f)) \) be the canonical map and let \( U^+ = p(T^1(L(f))) \).

Let

\[
p_B : B \to U^+
\]

be defined by

\[
p_B(b) = p(bz)
\]

for all \( b \in B \). Let

\[
P_L : L \to U^+
\]

be defined by

\[
P_L(x) = p(x)
\]
for $x \in L$ Let finally

$$p_{L(f)} : L(f) \to U^+$$

be defined by

$$p_{L(f)}(w) = p(w)$$

for $w \in L(f)$. Let $J_f$ be the two sided ideal in $U^+$ generated by the following set:

$$\{p_{L(f)}(bw) - p_B(b)p_{L(f)}(w) : \text{where } b \in B \text{ and } w \in L(f)\}.$$

Let $U(B, L, f) = U^+/J_f$. By definition $U(B, L, f)$ is an associative $A$-algebra.

Let $p_1 : T^1(L(f)) \to U(B, L, f)$ be the canonical map. Let $U^p(B, L, f) = p(T^p(L(f)))$ and $U_p(B, L, f) = p(T_p(L(f)))$. We get a filtration

$$\cdots \subseteq U_k(B, L, f) \subseteq U_{k-1}(B, L, f) \subseteq \cdots \subseteq U^1(B, L, f) = U(B, L, f)$$
called the descending filtration of $U(B, L, f)$. We moreover get a filtration

$$U_1(B, L, f) \subseteq U_2(B, L, f) \subseteq \cdots \subseteq U_k(B, L, f) \subseteq \cdots \subseteq U(B, L, f)$$
called the ascending filtration of $U(B, L, f)$.

Note: If $\rho \in C^1(L, B)$ is a cocycle it follows there is an isomorphism $L(f) \cong L(f + d^1 \rho)$ of extensions. It follows there is an isomorphism

$$U(B, L, f) \cong U(B, L, f + d^1 \rho)$$
of associative $A$-algebras. We get for any cohomology class $c \in H^2(L, B)$ a universal enveloping algebra $U(B, L, c) = U(B, L, f)$ where $f$ is some element in $Z^2(L, B)$ representing the cohomology class $c$. The $A$-algebra $U(B, L, c)$ is by the above discussion well defined up to isomorphism of $A$-algebras.

**Definition 3.3.** Let $f \in Z^2(L, B)$. Let $U(B, L, f)$ be the universal enveloping algebra of $\{L, \alpha\}$ of type $f$.

The algebra $U(B, L, f)$ is a simultaneous generalization of the universal enveloping algebra $U(B, L)$ of a Lie-Rinehart algebra $L$ introduced by Rinehart in [5] and the twisted universal enveloping algebra $g_f$ of a Lie algebra $g$ introduced by Sridharan in [7]. If $f = 0$ it follows $U(B, L, 0) = U(B, L)$ and if $B = A$ it follows $U(A, L, f) = g_f$ where $g = L$.

**Proposition 3.4.** There is a one-to-one correspondence between the set of left $U(B, L, f)$-modules and the set of $L$-connections of curvature type $f$.

**Proof.** Let $L(f) = Bz \oplus L$ and let $\alpha_f(az + x) = \alpha(x)$. Let

$$\sigma : L(f) \to U(B, L, f)$$

be the canonical map and let $\sigma_L : L \to U(B, L, f)$ be defined by

$$\sigma_L(x) = \sigma(x).$$

Let $\sigma_B : B \to U(B, L, f)$ be defined by $\sigma_B(b) = \sigma(bz)$. Let $W$ be a left $U(B, L, f)$-module. Define for any $x \in L$ and $w \in W$ the following map: $\nabla(x)(w) = \sigma_L(x)w$. One checks that $\nabla$ is an $L$-connection on $W$. Assume $x, y \in L$ and $w \in W$. It follows that

$$\sigma_L(x)\sigma_L(y) - \sigma_L(y)\sigma_L(x) = \sigma_L([x, y]) + \sigma_B(f(x, y))$$
in $U(B, L, f)$ hence

$$[\nabla(x), \nabla(y)](w) = \nabla([x, y])(w) + f(x, y)w.$$
It follows that
\[ R_\nabla(x, y)w = f(x, y)w \]
hence \( \nabla \) is an \( L \)-connection of curvature type \( f \). Conversely let \( \nabla : L \to \text{End}_A(W) \) be an \( L \)-connection of curvature type \( f \). Define the following action
\[ \phi : T^1(L(f)) \to \text{End}_A(W) \]
by
\[ \phi(\otimes_i (b_i z + x_i)) = \prod_i (b_i \text{Id}_W + \nabla(x_i)). \]

One checks the action \( \phi \) gives a map
\[ U(B, L, f) \to \text{End}_A(W). \]
One checks this construction sets up the desired correspondence and the Proposition is proved.

**Corollary 3.5.** Let \( 0 \in Z^2(L, B) \) be the zero cocycle. There is a one-to-one correspondence between the set of left \( U(B, L, 0) \)-modules and the set of flat \( L \)-connections.

**Proof.** The Corollary follows from Proposition 3.4.

Let \( U(B, L) = U(B, L, 0) \).

**Definition 3.6.** Let \( U(B, L) \) be the universal enveloping algebra of \( L \).

The algebra \( U(B, L) \) defined in Definition 3.6 was first introduced by Rinehart in [5]. It follows \( U(B, L) \) has a descending filtration \( U_k(B, L) \) and an ascending filtration \( U_k(B, L) \).

Let \( Bw \) be the free rank one \( B \)-module on the element \( w \) and let \( \tilde{L} = Bw \oplus L(f) \) with the following Lie-product:
\[ [aw + u, bv + v] = (u(b) - v(a))w + [u, v]. \]
Here \( u(b) = \alpha_f(u)(b) \) where \( \alpha_f : L(f) \to \text{Der}_A(B) \) is the anchor map of \( L(f) \). As left \( B \)-module it follows \( \tilde{L} = Bw \oplus Bz \oplus L \). There is a canonical map
\[ \tilde{\alpha} : \tilde{L} \to \text{Der}_A(B) \]
defined by
\[ \tilde{\alpha}(aw + bz + x) = \alpha(x) \]
and the pair \( \{\tilde{L}, \tilde{\alpha}\} \) is a Lie-Rinehart algebra. Let \( U(B, L(f)) \) be the universal enveloping algebra of the pair \( \{L(f), \alpha_f\} \). Let
\[ q_1 : T^1(\tilde{L}) \to U(B, L(f)) \]
be the canonical map. We get a map
\[ q : \tilde{L} \to U(B, L(f)) \]
defined by
\[ q(w) = q_1(w) \]
for \( w \in \tilde{L} \). Let \( z' = q(z) \) and \( w' = q(w) \). Let \( U(B, L(f), z') = U(B, L(f))(z' - 1) \). It follows \( U(B, L(f), z') \) has a descending filtration \( U_k(B, L(f), z') \) and an ascending filtration \( U_k(B, L(f), z') \).
Theorem 3.7. There is a canonical isomorphism of filtered $A$-algebras and left $B$-modules

$$\phi : U(B, L(f), z') \cong U(B, L, f).$$

Proof. Define the map $\phi'$ as follows:

$$\phi' : T^1(L) \to U(B, L, f)$$

by

$$\phi'(aw + bz + x) = (a + b)z + x.$$ One checks $\phi'$ gives a well defined map

$$\phi : U(B, L(f), z') \to U(B, L, f)$$
of $A$-algebras. One shows $\phi$ has an inverse hence the first claim follows. The map $\phi$ maps the descending (resp. ascending) filtration of $U(B, L(f), z')$ to the descending (resp. ascending) filtration of $U(B, L, f)$. The theorem follows.

Let $q_f : L(f) \to U(B, L(f))$ be the canonical map of left $B$-modules.

Lemma 3.8. The module $U_k(B, L(f))$ is generated as left $B$-module by the set

$$\{q_f(x_{i_1})q_f(x_{i_2})\cdots q_f(x_{i_l}) : \text{ with } x_{i_j} \in L(f) \text{ and } l \leq k.\}$$

Proof. We prove the result by induction in $k$. For $k = 1$ it is obvious. Assume the result is true for the case $p = k - 1$. Assume $p = k$. Let $q = q_{f}$ and let $w = q(z_1)\cdots q(z_k) \in U_k(B, L(f))$ with $z_i \in L(f)$. We get by the induction hypothesis the following equality:

$$q(z_2)\cdots q(z_k) = \sum_I a_I q(x_{i_1})\cdots q(x_{i_l})$$

with $a_I \in B$ and $x_{i_j} \in L(f)$ for all $I, i_j$. We may write $z_1 = az + x \in L(f)$. We get

$$q(z_1)q(z_2)\cdots q(z_k) = \sum_I (az + x)a_I q(x_{i_1})\cdots q(x_{i_l}) =$$

$$\sum_I a_I a_I q(x_{i_1})\cdots q(x_{i_l}) + a_I q(x)q(x_{i_1})\cdots q(x_{i_l}) + \alpha(x)(a_I q(x_{i_1})\cdots q(x_{i_l})$$

hence the claim holds for $p = k$. The Lemma follows.

Corollary 3.9. There is a canonical surjective map of left $B$-modules

$$\phi : \text{Sym}^k_B(L(f)) \to U_k(B, L(f))/U_{k-1}(B, L(f)).$$

Proof. Assume $x_1, \ldots, x_k \in L(f)$. By induction one proves the following result: Assume $\sigma$ is a permutation of the set $\{1, 2, \ldots, k\}$. The following formula holds:

$$q(x_1)\cdots q(x_k) = q(x_{\sigma(1)})\cdots q(x_{\sigma(k)}) + w$$

with $w \in U_{k-1}(B, L(f))$. Define the following map:

$$\phi : \text{Sym}^k_B(L(f)) \to U_k(B, L(f))/U_{k-1}(B, L(f))$$

by

$$\phi(x_1 \cdots x_k) = \overline{q(x_1)\cdots q(x_k)}.$$ It follows

$$\phi(x_1 \cdots x_k) = \phi(x_{\sigma(1)} \cdots x_{\sigma(k)})$$
hence $\phi$ is well defined. By Lemma 3.8 it follows the map $\phi$ is a surjective map of left $B$-modules and the Corollary is proved.
Lemma 3.10. Assume \( L(f) \) is a projective \( B \)-module. For all \( k \geq 1 \) there is a canonical isomorphism of left \( B \)-modules

\[
U_k(B, L(f), z')/U_{k-1}(B, L(f), z') \cong \text{Sym}_B^k(L).
\]

Proof. Let \( q_f : L(f) \to U(B, L(f)) \) be the canonical map and let \( z' = q_f(z) \).

Recall that \( L(f) = Bz \oplus L \) where \( z \) is a generator for the free rank one submodule \( Bz \) of \( L(f) \). The element \( z' \) is a central element in \( U(B, L(f)) \): For all elements \( w \in U(B, L(f)) \) it follows that \( z'w = wz' \). It follows \( (z' - 1)w = w(z' - 1) \) for all \( w \in U(B, L(f)) \). It follows the two sided ideal in \( U(B, L(f)) \) generated by \( z' - 1 \) is the following set:

\[
\{ w(z' - 1) : \text{ where } w \in U(B, L(f)) \}.
\]

We get a commutative diagram of exact sequences of left \( B \)-modules

\[
\begin{array}{ccccccccc}
0 & \rightarrow & U_k(B, L(f))(z' - 1) & \rightarrow & U_k(B, L(f)) & \rightarrow & U_k(B, L(f), z') & \rightarrow & 0 \\
& v \downarrow & & \downarrow v & & \downarrow w & & \downarrow . & \\
0 & \rightarrow & U_{k-1}(B, L(f))(z' - 1) & \rightarrow & U_{k-1}(B, L(f)) & \rightarrow & U_{k-1}(B, L(f), z') & \rightarrow & 0
\end{array}
\]

Since \( \ker(v) = \ker(w) = \ker(u) = 0 \) we get by the snake lemma a short exact sequence of left \( B \)-modules

\[
0 \rightarrow \text{coker}(u) \rightarrow {}^i \text{coker}(v) \rightarrow {}^j \text{coker}(w) \rightarrow 0
\]

and there is by definition an isomorphism of left \( B \)-modules

\[
\text{coker}(w) \cong U_k(B, L(f), z')/U_{k-1}(B, L(f), z').
\]

By assumption there is a canonical isomorphism of left \( B \)-modules

\[
\text{Sym}_B^k(L(f)) \cong U_k(B, L(f))/U_{k-1}(B, L(f)).
\]

There is also an isomorphism

\[
\text{Sym}_B^k(L(f)) \cong \text{Sym}_B^{k-1}(L(f))z \oplus \text{Sym}_B^k(L).
\]

One checks that \( \text{im}(i) = \text{Sym}_B^{k-1}(L(f))z \) hence we get an isomorphism

\[
\text{Sym}_B^k(L) \cong \text{coker}(w) \cong U_k(B, L(f), z')/U_{k-1}(B, L(f), z')
\]

and the Lemma is proved. \( \square \)

Corollary 3.11. Assume \( L \) is a projective \( B \)-module. There is a canonical isomorphism of graded \( B \)-algebras

\[
\text{Sym}_B^*(L) \cong \text{Gr}(U(B, L, f)).
\]

Proof. The Corollary follows from Theorem 3.7 and Lemma 3.10 \( \square \)

Note: When \( f = 0 \) is the zero cocycle we get the following result: There is a canonical isomorphism of graded \( B \)-algebras

\[
\text{Sym}_B^*(L) \cong \text{Gr}(U(B, L)).
\]

This result gives a simultaneous generalization of the PBW-Theorem proved by Rinehart in [5] in the case when \( f = 0 \) and by Sridharan in [7] when \( B = A \).
4. Application I: Deformations of filtered algebras

In this section we give an interpretation of $H^2(L, B)$ in terms of isomorphism classes of filtered algebras.

Let $U$ be a filtered associative algebra with filtration

$$U_0 \subseteq U_1 \subseteq \cdots \subseteq U_k \subseteq \cdots$$

where $U_0 = B$ and $h : A \to B$ an arbitrary map of commutative rings with unit. Assume $A \subseteq \text{Center}(U)$ and let $L$ be a fixed left $B$-module. We say that $U$ has graded commutative multiplication if the following holds: Assume $x_1, \ldots, x_k \in U_1$ and assume $\sigma$ is a permutation of $k$ elements. Then there is an equality

$$x_1 \cdots x_k = x_{\sigma(1)} \cdots x_{\sigma(k)} + y_{k-1}$$

where $y_{k-1} \in U_{k-1}$.

**Example 4.1.** Rings of differential operators.

The ring of differential operators $D_A(B) \subseteq \text{End}_A(B)$ has a filtration

$$B = D_A^0(B) \subseteq D_A^1(B) \subseteq \cdots \subseteq D_A(B).$$

The ring $D_A(B)$ has graded commutative multiplication.

**Lemma 4.2.** The algebra $U$ has graded commutative multiplication if and only if the associated graded algebra $\text{Gr}(U)$ is commutative.

**Proof.** The proof is an exercise. □

Assume $U$ has graded commutative multiplication and that there is an isomorphism $L \cong U_1/U_0$ of left $B$-modules. Let

$$\gamma_U : \text{Sym}_B^*(L) \to \text{Gr}(U)$$

be the canonical map of graded $B$-algebras.

**Definition 4.3.** We say $U$ has $L$-graded commutative multiplication if $U$ has graded commutative multiplication and the canonical map $\gamma_U$ is an isomorphism of graded $B$-algebras.

Assume $L = D_A^1(B)/D_A^0(B)$ and consider the canonical map

$$\gamma : \text{Sym}_B^*(L) \to \text{Gr}(D_A(B)).$$

The map $\gamma$ is neither surjective nor injective in general.

Assume in the following that $U$ has $L$-graded commutative multiplication.

We get an exact sequence of left $B$-modules

$$0 \to U_0 \to U_1 \to L \to 0.$$

Consider the following map

$$\psi : U_0 \times U_1 \to L$$

where

$$\psi(b, z) = b\overline{z}$$

where $\overline{z} \in L = U_1/U_0$ is the equivalence class of $z$. Since $U$ is an associative algebra it follows $U_1$ is a left and right $B$-module and since $\text{Sym}_B^*(L)$ is a commutative $B$-algebra it follows the element $b\overline{z} - \overline{z}b$ is zero in $L$. It follows the commutator $[z, b] = zb - bz$ is an element in $U_0 \subseteq U_1$. We get a map

$$\tilde{\gamma} : U_1 \to \text{End}(B)$$
defined by
\[ \tilde{\gamma}(z)(b) = [z, b]. \]
It follows immediately that \( \tilde{\gamma}(z) \in \text{End}_A(B) \) for any element \( z \in U_1 \). We moreover get the following equation:
\[ \tilde{\gamma}(z)(ab) = [z, ab] = zab - azb + azb - abz = [z, a]b - a[z, b] = \gamma(z)(a)b + a\gamma(z)(b) \]
hence
\[ \tilde{\gamma}(z) \in \text{Der}_A(B). \]
It follows we get a map
\[ \tilde{\gamma} : U_1 \to \text{Der}_A(B). \]

**Lemma 4.4.** The pair \( \{U_1, \tilde{\gamma}\} \) is a Lie-Rinehart algebra.

**Proof.** The proof is an exercise. \( \square \)

Since \( U_0 \subseteq U_1 \) is an ideal we get an induced structure of \( A \)-Lie algebra on \( L = U_1/U_0 \). By definition \( B = U_0 \subseteq U_1 \) is an abelian sub-algebra. It follows the exact sequence
\[ 0 \to B \to U_1 \to L \to 0 \]
is an exact sequence of Lie-Rinehart algebras. We get an induced Lie-Rinehart structure
\[ \gamma : L \to \text{Der}_A(B). \]

**Definition 4.5.** Assume \( \{U, U_i\} \) has \( L \)-graded commutative multiplication. We say \( \{U, U_i\} \) is a filtered algebra of type \( \alpha \) if there is an equality \( \gamma = \alpha \) of Lie-Rinehart algebras.

Let \( c(U) \in \text{Ext}^1(L, B, \alpha) \) be the characteristic class defined by the extension
\[ 0 \to B \to U_1 \to L \to 0. \]
We say \( U \) is the trivial deformation if \( c(U) = 0 \).

Assume now \( L \) is a projective \( B \)-module and consider the exact sequence
\[ 0 \to U_0 \to U_1 \to^p U_1/U_0 \to 0. \]

Assume \( t \) is a right splitting hence \( t : U_1/U_0 \to U_1 \) is left \( B \)-linear and \( p \circ t = \text{id} \).
Let
\[ \phi_{U,1} : L \to U_1/U_0 \]
be the first component of the graded isomorphism \( \phi_U : \text{Sym}^*_B(L) \cong Gr(U) \). Let \( \phi_{U,1}^{-1} \) be the inverse and let \( T = t \circ \phi_{U,1} \) and \( P = p \circ \phi_{U,1}^{-1} \). We get an exact sequence
\[ 0 \to U_0 \to U_1 \to^p L \to 0 \]
which is right split by \( T \).

Assume \( p(z) = x \) and let \( \gamma : L \to \text{Der}_A(B) \) be defined by
\[ \gamma(x)(b) = [T(x) - b] = T(x)b - bT(x). \]

Assume \( \{U, U_i\} \) is a filtered algebra of type \( \alpha \). This means that
\[ \gamma(x)(b) = [T(x), b] = T(x)b - bT(x) = \alpha(x)(b). \]
Assume moreover that
\[ [T(x), T(y)] - T([x, y]) = f(x, y) \in B \subseteq U_1 \]
Lemma 4.6. Define \( \sigma_1 : T^1(L(f)) \to U(B, L, f) \).

Define

\[ T' : T^1(L(f)) \to U \]

by

\[ T'(a_1 z + x_1) \otimes \cdots \otimes (a_k z + x_k) = \prod_i (a_i + T(x_i)). \]

It follows

\[ T'(a_1 z + x_1) \otimes (b_2 + y) - (b_2 + y) \otimes (a_1 z + x) - [a_1 z + b_2 + y] = \]

\[ (a + T(x))(b + T(y)) - (a + T(x))(b + T(y)) - \alpha(x)(b) - \alpha(y)(a) + f(x, y)] z - T([x, y]) = \]

\[ ab + aT(y) + T(x) + (x) - ba - bT(x) - T(y)a - T(y)f(x) - \alpha(x)(b) + \alpha(y)(b) = f(x, y) - T([x, y]) = 0 \]

since \( T(x)b - bT(x) = \alpha(x)(b) \). Moreover for any \( b \in B \) and \( w = az + x \in L(f) \) it follows

\[ T'(\alpha_1(bw) - \alpha_1(b)\alpha_1(w)) = T'(b_1 z + x_1 + b_2 a_2 z - b_2 x) = 0 \]

hence \( T' \) induce a map

\[ \overline{T} : U(B, L, f) \to U \]

of filtered algebras:

\[ \overline{T}(x_1 \cdots x_k) = T(x_1) \cdots T(x_k) = t(\phi_U^{i_1}(x_1)) \cdots t(\phi_U^{i_k}(x_k)) \]

for \( x_i \in L \). Since \( p \circ t \circ \phi_U^{i_1} = \phi_U^{i_1} = t \circ \phi_U^{i_1} \) it follows

\[ \overline{T}(x_1 \cdots x_k) = \phi_U^{i_1}(x_1) \cdots \phi_U^{i_k}(x_k). \]

Lemma 4.6. There is a commutative diagram

\[
\begin{array}{ccc}
Gr(U(B, L, f)) & \xrightarrow{Gr(T)} & Gr(U) \\
\phi_f \downarrow & & \downarrow \phi_U \\
\mathrm{Sym}_{U}^{L}(L) & \xrightarrow{\phi_U} & \\
\end{array}
\]

Proof. The proof follows from the discussion above. \( \square \)

Hence there is an equality \( Gr(T) \circ \phi_f = \phi_U \) hence \( Gr(T) = \phi_U \circ \phi_f^{-1} \). It follows the map

\[ Gr(T) : Gr(U(B, L, f)) \to Gr(U) \]

is an isomorphism of filtered algebras.

Lemma 4.7. The map \( \overline{T} : U(B, L, f) \to U \) is an isomorphism of associative rings.

Proof. Since \( Gr(T) \) is an isomorphism it follows the induced map

\[ \overline{T} : U_0(B, L, f) \to U_0 \]

is an isomorphism. Assume the induced map

\[ \overline{T} : U_{k-1}(B, L, f) \to U_{k-1} \]
is an isomorphism. We get a commutative diagram of exact sequences

\[
\begin{array}{cccccc}
0 & U_{k-1}(B, L, f) & U_k(B, L, f) & U_k(B, L, f)/U_{k-1}(B, L, f) & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
0 & U_{k-1} & U_k & U_k/U_{k-1} & 0
\end{array}
\]

It follows from the snake Lemma that the induced morphism

\[\tilde{T} : U_k(B, L, f) \rightarrow U_k\]

is an isomorphism. The Lemma follows by induction. \hfill \Box

Let \(A(\text{Sym}^*_B(L))\) be the following category: Let the objects of \(A(\text{Sym}^*_B(L))\) be the set of pairs \(\{U, \psi_U\}\) where \(U\) is a filtered algebra of type \(\alpha\) and where

\[\psi_U : \text{Sym}^*_B(L) \rightarrow \text{Gr}(U)\]

is a fixed isomorphism of graded \(B\)-algebras. A morphism \(\theta : \{U, \psi_U\} \rightarrow \{V, \psi_V\}\) in \(A(\text{Sym}^*_B(L))\) is a map of filtered algebras

\[\theta : U \rightarrow V\]

such that the induced map on associated graded rings

\[\text{Gr}(\theta) : \text{Gr}(U) \rightarrow \text{Gr}(V)\]

satisfies \(\text{Gr}(\theta) \circ \psi_U = \psi_V\). Since \(\psi_U\) and \(\psi_V\) are isomorphisms it follows that

\[\text{Gr}(\theta) = \psi_V \circ \psi_U^{-1}\]

hence the map \(\text{Gr}(\theta)\) is an isomorphism of graded \(B\)-algebras. It follows the map \(\theta\) is an isomorphism of filtered algebras. The inverse \(\theta^{-1}\) is a map in \(A(\text{Sym}^*_B(L))\) hence the category \(A(\text{Sym}^*_B(L))\) is a groupoid.

**Definition 4.8.** The category \(A(\text{Sym}^*_B(L))\) is the deformation groupoid of \(\{L, \alpha\}\).

Let \(\text{Iso}(A(\text{Sym}^*_B(L)))\) be the set of isomorphism classes of objects in \(A(\text{Sym}^*_B(L))\) and define the following map:

\[h : H^2(L, B) \rightarrow \text{Iso}(A(\text{Sym}^*_B(L)))\]

by

\[h(\tilde{f}) = \{U(B, L, f), \phi_f\}\]

where

\[\phi_f : \text{Sym}^*_B(L) \rightarrow U(B, L, f)\]

is the canonical isomorphism of graded \(B\)-algebras. The map is well defined since for two elements \(f, f + d^1 \rho\) representing the cohomology class \(\tilde{f}\) in \(H^2(L, B)\) it follows there is an isomorphism

\[U(B, L, f) \cong U(B, L, f + d^1 \rho)\]

of filtered algebras.

**Theorem 4.9.** The map \(h\) is a one to one correspondence.
Proof. By Lemma 4.7 it follows $h$ is a surjective map. Assume $h(f) = h(g)$ for two elements $f, g \in \mathbb{Z}^2(L, B)$. It follows we get an isomorphism
$$U(B, L, f) \cong U(B, L, g)$$
of filtered algebras.

It follows we get isomorphic extensions of Lie-Rinehart algebras $L(f) \cong L(g)$ hence there is an element $\rho \in C^1(L, B)$ with $d^1 \rho = f - g$ hence $\tilde{f} = \tilde{g}$ in $H^2(L, B)$. The Theorem is proved.

Theorem 4.9 was first proved in [7] for Lie algebras over an arbitrary base ring $K$.

Let $\text{Sch}(H^2(L, B)) = \text{Spec}(\text{Sym}^*A((H^2(L, B))^*))$. It follows
$$\pi : \text{Sch}(H^2(L, B)) \to \text{Spec}(A)$$
is a scheme over $A$. If $H^2(L, B)$ is a locally free $A$-module it follows $\text{Sch}(H^2(L, B))$ is a vector bundle over $\text{Spec}(A)$. Theorem 4.9 shows that isoclasses in $A(\text{Sym}^*B((L, B)))$ are parametrized by the points of the scheme $\text{Sch}(H^2(L, B))$. This result may be phrased in the language of representable functors.

Assume now that $\alpha : L \to \text{Der}_A(B)$ is a Lie-Rinehart algebra which is projective as left $B$-module. Assume $f \in \mathbb{Z}^2(L, B)$ is a 2-cocycle of $L$. Let Mod$(L, f)$ be the category of $L$-connections of curvature type $f$.

Corollary 4.10. Let $\{U, U_i\}$ be a filtered algebra of type $\alpha$ and let Mod$(U)$ be the category of left $U$-modules. It follows there is an element $f \in \mathbb{Z}^2(L, B)$ and an equivalence of categories
$$\text{Mod}(U) \cong \text{Mod}(L, f).$$

Proof. The Corollary follows from Theorem 4.9 since $U \cong U(B, L, f)$ for some $f \in \mathbb{Z}^2(L, B)$.

5. Applications II: Connections on families of projective modules

In the paper [2] a formula for the curvature of an algebraic connection
$$\nabla : L \to \text{End}_A(E)$$
where $E$ is a finitely generated projective $B$-module was established. Hence if one is interested in explicit calculations of the image of the Chern character
$$ch : K(L) \to H^*(L, B)$$
one picks a set of generators $\{E_i\}_{i \in I}$ of $K(L)$ and calculates connections
$$\nabla_i : L \to \text{End}_A(E_i)$$
and the curvature $R_{\nabla_i}$ for all $i \in I$. The problem about this approach is that it is difficult to calculate the grothendieck group $K(L)$ for general Lie-Rinehart algebras $L$. The aim of this section is to use the constructions given in the previous sections to construct a subring $\text{Char}(L) \subseteq H^*(L, B)$ which is defined purely in terms of the cohomology ring $H^*(L, B)$. There is an equality $\text{Char}(L) = \text{Im}(Ch)$ hence if we are interested in the study of the image $\text{Im}(Ch)$ we do not need to calculate generators of $K(L)$. We simply study the ring $\text{Char}(L)$.

We use the constructions in the previous sections to study algebraic connections of curvature type $f$ on finitely generated projective $B$-modules. We prove that any cohomology class in $H^2(B, L)$ is the first Chern class of a finitely generated
projective $B$-module. We also construct families of mutually non-isomorphic $B$-modules of arbitrary high rank. As a consequence we prove that for any affine algebraic manifold $X$ over the complex numbers and any topological class $c \in H^2_{\text{sing}}(X, \mathbb{C})$ there is a finite rank algebraic vector bundle $E$ on $X$ with $c_1(E) = c$. Hence the first Chern class map
\[ c_1 : K(X) \to H^2_{\text{sing}}(X, \mathbb{C}) \]
where $K(X)$ is the grothendieck group of finite rank algebraic vector bundles on $X$, is surjective.

Assume $A \to B$ is a map of commutative rings where $A$ contains a field $k$ of characteristic zero. Let $\alpha : L \to \text{Der}_A(B)$ be a Lie-Rinehart algebras which is a finitely generated projective $B$-module. Let $f \in Z^2(L, B)$ be a cocycle and let $U = U(B, L, f)$ be the universal enveloping algebra of $L$ of type $f$. Let $U^k = U^k(B, L, f)$ be the descending filtration of $U$. It follows $U^k$ is a filtration of two sided ideals in $U$.

**Definition 5.1.** Let for any $k \geq 1$ and $i \geq 1$ $V^{k,i}(B, L, f) = U^k/U^{k+i}$.

By definition it follows $V^{k,i}(U, L, f)$ is a left and right $U(B, L, f)$ module for all $k, i \geq 1$. Assume $rk(L) = l$ as projective $B$-module. It follows by the results in the previous section that $U^k(B, L, f)$ and $V^{k,i}(B, L, f)$ are projective $B$-modules for all $k, i \geq 1$. Let $r(k, i, f) = rk(V^{k,i}(B, L, f))$.

**Lemma 5.2.** For all $k, i \geq 1$ the following formula holds:
\[ r(k, i, f) = \binom{l + k + i - 1}{l} - \binom{l + k - 1}{l}. \]

**Proof.** The proof is left to the reader as an exercise. \hfill \square

Since $V^{k,i}(B, L, f)$ is a left $U(B, L, f)$-module we get for all $k, i \geq 1$ algebraic connections $\nabla : L \to \text{End}_A(V^{k,i}(B, L, f))$ of curvature type $f$. Recall from Proposition 3.4 that this means that for any $x, y \in L$ and $w \in V^{k,i}(B, L, f)$ it follows $R\nabla(x, y)(w) = f(x, y)w$.

Let $F = \frac{1}{r(k, i, f)} f \in Z^2(L, B)$. We get by Proposition 3.4 a connection $\tilde{\nabla} : L \to \text{End}_A(V^{k,i}(B, L, F))$ of curvature type $F$. Let $c = \overline{f} \in H^2(L, B)$.

**Theorem 5.3.** The following holds:
\[ c_1(V^{k,i}(B, L, F)) = c \in H^2(L, B). \]

**Proof.** By the results in [3] we may construct the first Chern class of $V^{k,i}(B, L, F)$ in $H^2(L, B)$ by taking the trace of the curvature $R_{\tilde{\nabla}}$. It follows
\[ tr(R_{\tilde{\nabla}}) = tr(FId) = \frac{1}{r(k, i, f)} tr(Id) = f. \]

Hence
\[ c_1(V^{k,i}(B, L, F)) = \overline{f} = c \in H^2(L, B). \]

The Theorem is proved. \hfill \square
Corollary 5.4. Any cohomology class in $H^2(L, B)$ is the first Chern class of a finitely generated projective $B$-module.

Proof. The Corollary follows from Theorem 5.3 since $f$ is an arbitrary element in $Z^2(L, B)$.

Corollary 5.5. Assume $A$ is a ring of characteristic zero. Fix $k, i \geq 1$ and let $f_1, f_2 \in Z^2(L, B)$. Assume $\tilde{f}_1 \neq \tilde{f}_2$ in $H^2(L, B)$. It follows $V^{k,i}(B, L, f_1)$ and $V^{k,i}(B, L, f_2)$ are non-isomorphic as left $B$-modules.

Proof. Assume $V^{k,i}(B, L, f_1) \cong V^{k,i}(B, L, f_2)$ as left $B$-modules. Since $A$ has characteristic zero, it follows $c_1(V^{k,i}(B, L, f_1)) = \hat{d}f_1 = \hat{d}f_2 = c_1(V^{k,i}(B, L, f_2))$ in $H^2(L, B)$ where $d = rk(V^{k,i}(B, L, f_1))$. This leads to a contradiction and the Corollary follows.

Example 5.6. Families of finitely generated projective modules.

Assume in this example that $A$ is a ring of characteristic zero. Assume $\tilde{f} = \tilde{g} \in H^2(L, B)$. It follows there is an isomorphism $U(B, L, f) \cong U(B, L, g)$ of filtered algebras. It follows for all $k \geq 1$ there is an isomorphism

$$U^k(B, L, f) \cong U^k(B, L, g)$$

of left and right $B$-modules hence $V^{k,i}(B, L, f) \cong V^{k,i}(B, L, g)$ as left and right $B$-modules for all $k, i \geq 1$. We may define for any cohomology class $c \in H^2(L, B)$

$$V^{k,i}(B, L, c) = V^{k,i}(B, L, f)$$

where $f \in Z^2(L, B)$ is a representative for the class $c$. Hence when we consider the left and right $B$-module $V^{k,i}(B, L, c)$ for varying $c \in H^2(L, B)$ we get a family of finitely generated projective $B$-modules of constant rank parametrized by $H^2(L, B)$. From Lemma 5.5 it follows that different classes in $H^2(B, L)$ gives non-isomorphic modules.

Example 5.7. Singular cohomology of a complex algebraic manifold.

Let $A$ be a finitely generated regular algebra over the complex numbers $\mathbb{C}$ and let $X = \text{Spec}(A)$. Let $X(\mathbb{C})$ be the underlying complex manifold of $X$ in the strong topology. Let $L = \text{Der}_{\mathbb{C}}(A)$ be the Lie-Rinehart algebra of derivations of $A$. It follows there is an isomorphism

$$H^2(L, A) \cong H^2_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$$

where $H^2_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$ is singular cohomology of $X(\mathbb{C})$ with complex coefficients. It follows any topological class $c \in H^2_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$ is the first Chern class of an algebraic vector bundle on $X(\mathbb{C})$. Hence if $K(A)$ is the grothendieck group of finitely generated projective $A$-modules, it follows the Chern class map

$$c_1 : K(A) \to H^2_{\text{sing}}(X(\mathbb{C}), \mathbb{C})$$

is surjective.

Example 5.8. The image of the Chern character for Lie-Rinehart algebras.
Let $A$ contain a field $k$ of characteristic zero and consider the map 
\[ \exp : \mathbb{H}^2(L, B) \to \oplus_{k \geq 0} \mathbb{H}^{2k}(L, B) \]
defined by 
\[ \exp(x) = \sum_{k \geq 0} \frac{1}{k!} x^k. \]

**Lemma 5.9.** The map $\exp$ is a map of abelian groups.

**Proof.** We view the element $\exp(x)$ as an element in the multiplicative subgroup of $\mathbb{H}^{2*}(L, B)$ with “constant term” equal to one. Let $x, y \in \mathbb{H}^2(L, B)$ be two cohomology classes. We get
\[ \exp(x + y) = \sum_{k \geq 0} \frac{1}{k!} (x + y)^k = \]
\[ \sum_{k \geq 0} \frac{1}{k!} \sum_{i + j = k} \binom{k}{i} x^i y^j = \]
\[ \sum_{k \geq 0} \sum_{i + j = k} \frac{1}{i!} x^i y^j = (\sum_{i \geq 0} \frac{1}{i!} x^i)(\sum_{j \geq 0} \frac{1}{j!} y^j) = \exp(x) \exp(y). \]

Let $\text{Char}(L) = \mathbb{Z}\{\exp(x) : x \in \mathbb{H}^2(L, B)\}$ be the $\mathbb{Z}$-lattice spanned by the image of $\exp$.

**Definition 5.10.** Let $\text{Char}(L)$ be the characteristic ring of $L$.

**Lemma 5.11.** Let $\text{Ch} : K(A) \to \mathbb{H}^*(L, B)$ be the Chern character. There is an equality $\text{Im}(\text{Ch}) = \text{Char}(L)$ as subrings of $\mathbb{H}^{2*}(L, B)$.

**Proof.** By definition it follows $\text{Im}(\text{Ch}) \subseteq \text{Char}(L)$. By Corollary 5.10 it follows $\text{Char}(L) \subseteq \text{Im}(\text{Ch})$ and the Lemma is proved.

**Example 5.12.** The classical Chern character.

Assume $A$ is a regular algebra of finite type over the complex numbers and let $X = \text{Spec}(A)$. Let $X(C)$ be the underlying complex manifold of $X$ in the strong topology and let 
\[ \text{Ch} : K(A) \to \mathbb{H}^*(\text{Der}_C(A), A) \]
be the classical Chern character. There is a filtration - the gamma filtration - on $K(A)$ and the associated graded group $\text{Gr}(K(A))$ is isomorphic to the Chow group $A(X)$ of $X$. There is an isomorphism
\[ \mathbb{H}^*(\text{Der}_C(A), A) \cong \mathbb{H}^*_{\text{sing}}(X(C), C) \]
and a cycle map
\[ \gamma : A(X) \to \mathbb{H}^*_{\text{sing}}(X(C), C). \]
If there is an equality $\text{Im}(Ch) = \text{Im}(\gamma)$ in $H^*_\text{sing}(X(C), C)$ it follows we have given a definition of $\text{Im}(\gamma)$ independent with respect to choice of generators for $A(X)$. It is well known that the problem of calculating generators for the group $A(X)$ is unsolved.

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