Research Article

Analysis of Stochastic Nicholson-Type Delay System under Markovian Switching on Patches

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1. Introduction

In 1980, Gurney et al. [1] established Nicholson’s blowflies equation according to experimental data of Nicholson [2]. In recent decades, there have been a large amount of results related to the dynamical behaviors for this model and its modification, see [3–13].

In ecosystems, the pattern of complex population dynamics is inevitably subject to some kind of environmental noises. As a matter of fact, the phenomenon of stochasticity plays a critical role in understanding the evolutionary dynamics and ecological characteristics of species. Particularly, May [14] has revealed that due to environmental fluctuations, the parameters in a system should be stochastic. Environmental noises are classified into two categories: the first is white noise, and the second one is coloured noise. Stochastic population models [15–20] are more realistic compared to deterministic population models. Wang et al. [21] first studied a scalar stochastic Nicholson’s blowflies delayed equation

\[ dx(t) = \left( -ax(t) + px(t-\tau)e^{-\gamma(t-\tau)} \right) dt + \sigma x(t)dB(t). \] (1)

Notice, however, that white noise is unable to depict the phenomena that the species may be invaded by the alien population [22] or suffer sudden catastrophic shocks [23]. And in recent years, some significant progress has been made in the theory of the stochastic population models with regime switching, see [24–27] and the references therein. In [28], Zhu et al. considered a stochastic equation with Markovian switching:

\[ dx(t) \left( -a_{r}x(t) + p_{r}x(t-\tau_{r})e^{-\gamma_{r}(t-\tau_{r})} \right) dt + \sigma_{r}x(t)dB(t), \] (2)

where continuous-time Markov chain \( \{r_{t}\}_{t\geq0} \) is defined on a state space \( S = \{1, 2, \ldots, m\} \).

On the contrary, migration is a ubiquitous phenomenon in the nature. Both continuous reaction-diffusion models and discrete patchy systems could incorporate and explain the phenomenology of spatial dispersion [29] in the literature of mathematical ecology. Objectively speaking, patch-structured models illustrate the spatial heterogeneity of species, depending on a lot of factors, such as ecological systems in different geographic types (e.g., nature reserves and other regions), various food-rich patches of habitats, and many other circumstances. Besides, models in the patchy environment include disease systems as well, such as the two-compartment model of the cancer cell population. In order to take the dispersal phenomenon into
consideration, Berezansky et al. [30] introduced the Nicholson-type delay system on patches as follows:

\[
\begin{align*}
  x_1'(t) &= -a_1 x_1(t) - b_2 x_1(t) + b_1 x_2(t) + \phi_1 x_1(t - \tau) e^{-\gamma_1 x_1(t - \tau)}, \\
  x_2'(t) &= -a_2 x_2(t) - b_1 x_2(t) + b_2 x_1(t) + \phi_2 x_2(t - \tau) e^{-\gamma_2 x_2(t - \tau)},
\end{align*}
\]

(3)

which includes the novel two-compartment models of leukemia dynamics and the systems of marine protected areas.

In particular, considering that the parameters \(a_i\) of system (3) are affected by the white noise, Yi and Liu [31] formulated the stochastic diffusion system which consists of two patches:

\[
\begin{align*}
  dx_1(t) &= \left[ -a_1 (\ell(t)) x_1(t) - b_2 (\ell(t)) x_1(t) + b_1 (\ell(t)) x_2(t) + \phi_1 (\ell(t)) x_1(t - \tau(\ell(t))) e^{-\gamma_1 (\ell(t)) x_1(t - \tau(\ell(t)))} \right] dt + \sigma_1 x_1(t) dB_1(t), \\
  dx_2(t) &= \left[ -a_2 (\ell(t)) x_2(t) - b_1 (\ell(t)) x_2(t) + b_2 (\ell(t)) x_1(t) + \phi_2 (\ell(t)) x_2(t - \tau(\ell(t))) e^{-\gamma_2 (\ell(t)) x_2(t - \tau(\ell(t)))} \right] dt + \sigma_2 x_2(t) dB_2(t).
\end{align*}
\]

(4)

where \(\delta > 0\), \(\rho_{ij} \geq 0\) for \(i \neq j\), and \(\sum_{j \in \mathcal{M}} \rho_{ij} = 0\), \(i, j \in \mathcal{M}\). Suppose that \(\ell(t)\) is irreducible and has the unique stationary distribution \(\pi = (\pi_1, \pi_2, \ldots, \pi_N)\). Hence, we obtain the stochastic Nicholson-type system under Markovian switching on the patch structure as follows:

\[
\begin{align*}
  dx_1(t) &= \left[ -a_1 (\ell(t)) x_1(t) - b_2 (\ell(t)) x_1(t) + b_1 (\ell(t)) x_2(t) + \phi_1 (\ell(t)) x_1(t - \tau(\ell(t))) e^{-\gamma_1 (\ell(t)) x_1(t - \tau(\ell(t)))} \right] dt + \sigma_1 x_1(t) dB_1(t), \\
  dx_2(t) &= \left[ -a_2 (\ell(t)) x_2(t) - b_1 (\ell(t)) x_2(t) + b_2 (\ell(t)) x_1(t) + \phi_2 (\ell(t)) x_2(t - \tau(\ell(t))) e^{-\gamma_2 (\ell(t)) x_2(t - \tau(\ell(t)))} \right] dt + \sigma_2 x_2(t) dB_2(t).
\end{align*}
\]

(6)

We can further model random shift in different regimes by a continuous-time Markov chain \(\{\ell(t)\}_{t \geq 0}\) defined on a state space \(\mathcal{M} = \{1, 2, \ldots, N\}\). Let \(\{\ell(t)\}_{t \geq 0}\) be right-continuous and \(\Gamma = (\rho_{ij})_{N \times N}\) be its generator of \(\{\ell(t)\}_{t \geq 0}\), i.e.,

\[
\mathbb{P}\{\ell(t + \delta) = j \mid \ell(t) = i\} = \begin{cases} 
\rho_{ij}\delta + o(\delta), & \text{if } j \neq i, \\
1 + \rho_{ii}\delta + o(\delta), & \text{if } j = i,
\end{cases}
\]

(5)

with initial conditions

\[
x(t) = \varphi(t) = (\varphi_1(t), \varphi_2(t))^T, \quad t \in [-\tau, 0], \ell(0) = \ell_0 \in \mathcal{M},
\]

(7)

where \(\varphi_h \in C([-\tau, 0]; [0, +\infty))\) and \(\varphi_h(0) > 0\) for \(h = 1, 2\) and \(\tau = \max_{i \in \mathcal{M}} \{\tau(i)\}\).

We focus on the meaning of parameters with respect to fish population in marine protected area \(A_1\) and fishing area \(A_2\). \(x_1(t)\) and \(x_2(t)\) are the number of fish populations in \(A_1\) and \(A_2\), respectively; for \(h = 1, 2\) and \(i \in \mathcal{M}\), \(a_i(\ell)\) and \(a_2(\ell)\) are the mortality rate in \(A_1\) and \(A_2\), respectively; let \(G(x_1(t - \tau)) = p_{h_1}(x_1(t - \tau)) e^{-\gamma_1(x_1(t - \tau))}\) be the fish growth rates; \(p_1(\ell)\) and \(p_2(\ell)\) represent the maximum per adult yearly birth rate in \(A_1\) and \(A_2\), respectively; \(y_1(\ell) > 0\); \(1/\gamma_1(\ell)\) and \(1/\gamma_2(\ell)\) are the number at which the reproduction at their maximum birth rate in \(A_1\) and \(A_2\), respectively; \(\tau(\ell)\) is the maturation time; \(B_{h}(t)\) is the standard Brownian motion defined on the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\); and \(\sigma_h(\ell) \geq 0\), for any \(i \in \mathcal{M}\) and \(h = 1, 2\). We assume \(\ell(t)\) is \(\mathcal{F}_t\)-adapted. Nevertheless, suppose \(\ell(t)\) and \(B_{h}(t)\) are independent of each other, \(h = 1, 2\).

Especially, system (6) can reduce to the model in [32] if \(\tau(\ell) \equiv \tau, \ell \in \mathcal{M}\). By contrast, our work differs from and improves [32], which will be depicted further in detail.

In the field of ecology, it is important to use mathematics to study extinction of species, see [33, 34] and the references therein. However, no work has yet been done on the problem of extinction for scalar equation (1), not to mention the scalar equation with Markovian switching (2) and system (4). In order to prove the extinction of species, the conventional method is to construct a proper Lyapunov function or functional and then estimate the upper bound of the drift term of its Itô differential. Taking system (6) for example, \(x_1(t)\) and \(x_2(t)\) are likely to appear in the denominator of the expression of \(LV\), and coefficients in front of them are positive, for a general Lyapunov function \(V(x_1, x_2)\). Unfortunately, this leads to some difficulties in finding the upper bound of \(LV\). So, based on this, we give a new method for investigating extinction of species.
Complexity

Especially, system (6) reduces to (1), (2), (4), or the system in [32] when parameters of system (6) assume some special values. That is to say, we have derived extinction of the above systems at the same time.

In this paper, system (6) is more general than the model of [21, 28, 30–32]. In addition, our results improve and generalize the corresponding results in these literature studies.

The remainder of this paper is built up as follows. In Section 2, we show the global existence of almost surely positive solution. The asymptotic estimates for the solution, stochastically ultimate boundedness, and boundedness for the average in time of the $\theta$th moment of the solution are then constructed in Section 3. In Section 4, we discuss the pathwises properties of the solution. Sufficient conditions for extinction of species are obtained in Section 5. Numerical investigations are then given in Section 6. The last part is a conclusion.

2. Preliminary Results

To simplify, denote the solution of (6) with initial values (7):

$$x(t) = x(t; \varphi, \epsilon_0),$$

where $x(t) = (x_1(t), x_2(t))^T$. Let

$$\beta_1(i) = a_1(i) + b_2(i),$$

$$\beta_2(i) = a_2(i) + b_1(i), \quad i \in \mathcal{M}. $$

We denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}_+^2 = \{(x_1, x_2)^T \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$, and $\mathbb{R}^{2^{\mathcal{M}}} = \{(w_{uv})_{u,v} \in \mathbb{R}^{2^{\mathcal{M}}} : w_{uv} > 0, u, v = 1, 2\}$. For any $\Phi: \mathcal{M} \rightarrow \mathbb{R}$, let $\Phi = \min_{\pi \in \mathcal{P}} \Phi(\pi)$ and $\Phi = \max_{\pi \in \mathcal{P}} \Phi(\pi)$. Let $\|\cdot\|$ denote Euclidean norm in $\mathbb{R}^2$. Denote the trace norm $|A| = \sqrt{\text{trace}(A^TA)}$ for matrix $A$.

**Lemma 1.** Given any initial values (7), system (6) has a unique solution $x(t) \in \mathbb{R}_+^2$ for all $t \in [-r, \infty)$ almost surely.

**Proof.** We omit the proof since it is analogous to that of [31] by making use of the generalized Itô formula (see, e.g., Theorem 1.45 in [35]) to $\sum_{k=1}^{\infty} (x_{k+1} - x_k + 1 - \log x_k)$. \square

**Remark 1.** The delay stochastic Nicholson-type model under regime switching on patches (6) is a direct extension of the models in [21, 28, 30–32]. From Lemma 1, it is worthy to point out that priori conditions $\alpha > \sigma/2$ in [21] are unnecessary. Therefore, Lemma 1 improves and generalizes Lemma 2.2 in [21]. In addition, this lemma shows that both white noise and telegraph noise will not destroy a great property that the solution of (3) does not explode.

3. Boundedness

Because of resource constraints, asymptotic boundedness is the core of the research in ecosystems. And it is the main purpose of the present section. For simplicity, we use the following notations. For any $i \in \mathcal{M}$, denote

$$K_1(i) = a_1(i) - \frac{1}{\theta} (b_1(i) - b_2(i)) - \frac{1}{2} (\theta - 1)a_1^2(i) - (\theta - 1),$$

$$K_2(i) = a_2(i) - \frac{1}{\theta} (b_2(i) - b_1(i)) - \frac{1}{2} (\theta - 1)a_2^2(i) - (\theta - 1),$$

$$H_\pi(i) = \theta \cdot \sup_{s \in \mathbb{R}} \left\{ -K_h(i)x_h + \frac{p_h(i)x_h}{y_h(i)e^{-\gamma s}} \right\},$$

$$\hat{H}_h(i) = \max_{i \in \mathcal{M}} H_\pi(i),$$

$$A_1(\theta) = \frac{2^{\theta/2}}{\theta(\theta - 1)} \sum_{i \in \mathcal{M}} \pi_i [H_1(i) + H_2(i)],$$

$$A_2(\theta) = \frac{2^{\theta/2}}{\theta(\theta - 1)} \sum_{i \in \mathcal{M}} \hat{H}(i),$$

$$A_3(\theta) = \frac{2^{\theta/2}}{\theta(\theta - 1)} \sum_{i \in \mathcal{M}} \hat{H}(i), \quad h = 1, 2.$$  (10)

Firstly, inspired by the work of Wang and Chen [32], we give this theorem.

**Theorem 1.** Let $\theta > 1$ such that $K_h(i) > 0, h = 1, 2, i \in \mathcal{M}$. Given any initial values (7), solution $(x_1(t), x_2(t))$ of (6) satisfies

$$\limsup_{t \to -\infty} \int_{-\infty}^{t} \frac{1}{t} E[x_h(s)^\theta] ds \leq A_1(\theta), \quad h = 1, 2,$$  (11)

and

$$\limsup_{t \to -\infty} E(x_1^\theta(t) + x_2^\theta(t)) \leq A_2(\theta).$$  (12)

In particular,

$$\limsup_{t \to -\infty} E|x(t)|^\theta \leq A_3(\theta).$$  (13)

That is, system (6) is ultimately bounded.

**Proof.** Define

$$V_1(x_1, x_2) = x_1^\theta + x_2^\theta.$$  (14)

The generalized Itô formula, together with the fact $p_h(i)y_h e^{-\gamma s} \leq (p_h(i)y_h(i)e^{-\gamma e})$ and the elementary inequality $A^T B^{1-\epsilon} \leq A e + B (1 - \epsilon)$ for any $A, B \geq 0$ and $\epsilon \in [0, 1]$, yields

\[\text{...}\]
According to (12), (18), and the fact that

\[ \theta > 1, \]

\[ |x|^{\theta} \leq 2^\theta \max \{x_1^{\theta}, x_2^{\theta}\} \leq 2^\theta V_1(x), \]

it follows that (11) and (13) hold. The proof is therefore complete. \( \square \)

**Remark 2.** In Theorem 1, the parameter \( \theta \) is greater than 1 in the result. Although ultimate boundedness in the \( \theta \)th moment was derived for \( \theta \) restricted to the precondition \( \theta > 1 \), \( \theta \)th moment of system (6) can be obtained when \( \theta \leq 1 \) by Hölder’s equality.

**Remark 3.** Without regime switching or without migration and regime switching, Theorem 1 improves the corresponding results in [21, 31]. If \( r(i) \equiv r \), system (6) is a direct extension of the model in [32]. Besides, no proof of ultimate boundedness in the \( \rho \)th moment is given in [32], which is
shown in Theorem 1. Therefore, this theorem extends and improves Theorem 3.1 in [21], Theorem 2.2 in [28], Theorem 3.3 in [31], and Theorem 3.2 in [32].

**Theorem 2.** Given any initial values (7), solution \((x_1(t), x_2(t))\) of (6) satisfies

\[
\limsup_{t \to \infty} E[x(t)] \leq \limsup_{t \to \infty} E[x_1(t) + x_2(t)] \leq \frac{\hat{p}_1}{\gamma_1 e^\lambda} + \frac{\hat{p}_2}{\gamma_2 e^\lambda},
\]

where \(\lambda = \min[\bar{a}_1, \bar{a}_2]\). That is, (6) is ultimately bounded in mean.

**Proof.** Let \(\nabla_1(t, x_1, x_2) = e^{\theta t}(x_1 + x_2)\). Then,

\[
E(x_1(t) + x_2(t)) \leq e^{-\theta t} E_1(0, \phi_1(0), \phi_2(0)) + \left(\frac{\hat{p}_1}{\gamma_1 e^\lambda} + \frac{\hat{p}_2}{\gamma_2 e^\lambda}\right) \int_0^t e^{(s-t)\lambda} ds.
\]

Finally, (22) follows by letting \(t \to \infty\). The proof is therefore complete.

**Remark 4.** Compared with Theorem 1, this theorem describes the case that \(\theta = 1\), which does not require any conditions. If \(\varphi(t) = \varphi\), we get \((\hat{p}_1/\gamma_1 e^\lambda) + (\hat{p}_2/\gamma_2 e^\lambda) \leq (c/a)\), where \((c/a)\) is defined in [32]. So, this theorem improves and extends Theorem 1.4 in [21] and Theorem 3.1 in [32].

**Theorem 3.** System (6) is stochastically ultimately bounded.

**Proof.** By (22), we derive

\[
\limsup_{t \to \infty} E|x_1(t)| \leq \frac{\hat{p}_1}{\gamma_1 e^\lambda}, \quad h = 1, 2.
\]

By the Chebyshev inequality, it yields, for any \(\varepsilon \in (0, 1)\),

\[
\limsup_{t \to \infty} P[x_h(t) \geq H] \leq H^{-1}\left(\frac{\hat{p}_1}{\gamma_1 e^\lambda} + \frac{\hat{p}_2}{\gamma_2 e^\lambda}\right) = \varepsilon,
\]

where \(H = (1/\varepsilon)(((\hat{p}_1/\gamma_1 e^\lambda) + (\hat{p}_2/\gamma_2 e^\lambda))\). The proof is therefore complete.

**Remark 5.** Theorem 3 can be seen as the extension and improvement of [31, 32].

4. **Asymptotic Pathwise Estimation**

We shall estimate a sample Lyapunov exponent in what follows.

**Lemma 2.** If \(a \in \mathbb{R}\) and \(b \in \mathbb{R}_+\), then \((ax^2 + bx/1 + x^2) \leq K(a)\) for \(x \in \mathbb{R}\), where \(K(a) = (a + \sqrt{a^2 + b^2}/2)\).

The proof of this lemma is easy and so is omitted. In the process of finding \(K(a)\), we know that the precondition is \(a - K(a) < 0\). In this case, we can choose \(K(a)\) which satisfies \(K(a) = (a + \sqrt{a^2 + b^2}/2)\). We have to mention that it has no relation with the sign of parameter \(a\). If \(a < 0\), we get \((a + \sqrt{a^2 + b^2}/2) < - (b^2/4a)\) by simple computation. So, this lemma is an improvement of Lemma 1.2 in [28] and Lemma 2.1 in [32].

**Theorem 4.** Given any initial values (7), solution \(x(t)\) of (6) satisfies

\[
\limsup_{t \to \infty} \frac{1}{t} \log x_h(t) \leq \limsup_{t \to \infty} \frac{1}{t} \log |x(t)| \leq \frac{Q}{2}, \quad \text{a.s.}
\]

where \(h = 1, 2, \quad Q = \max_{i, \epsilon} \epsilon \min_{i, \epsilon} \left[Q_1(i, \epsilon) + Q_2(i, \epsilon)\right]\) with

\[
Q_1(i, \epsilon) = \frac{\sqrt{[2\beta_1(i) - \sigma_1^2(i) - (b_1(i) + b_2(i))\epsilon]^2 + 4(\beta_1(i)/\gamma_1(i)\epsilon)^2} - [2\beta_1(i) - \sigma_1^2(i) - (b_1(i) + b_2(i))\epsilon]^2}{2},
\]

\[
Q_2(i, \epsilon) = \frac{\sqrt{[2\beta_2(i) - \sigma_2^2(i) - b_1(i) + b_2(i))\epsilon]^2 + 4(\beta_2(i)/\gamma_2(i)\epsilon)^2} - [2\beta_2(i) - \sigma_2^2(i) - (b_1(i) + b_2(i))\epsilon]^2}{2},
\]
for any positive constant $\varepsilon$. Proof. The generalized Itô formula, together with Lemma 2 and the Cauchy–Schwarz inequality, yields

\[
\log(1 + x_i^2(t) + x_j^2(t)) \leq \log(1 + x_i^2(0) + x_j^2(0)) + \int_0^t \left( -2 \beta_i(\ell(s)) - \sigma_i^2(\ell(s)) - (b_1(\ell(s)) + b_2(\ell(s))) \right) x_i(s) ds + \frac{1}{t} \log(1 + x_i^2(t) + x_j^2(t)) + Q t + 4 \log m,
\]

where for any $h \in \{1, 2\}$,

\[
M_{h}(t) = 2 \int_0^t \sigma_h(\ell(s)) x_h(s) dB_h(s),
\]

with the quadratic variation

\[
\langle M_{h}(t), M_{h}(t) \rangle = 4 \int_0^t \sigma_h^2(\ell(s)) x_h^2(s) ds.
\]

According to the exponential martingale inequality (see, e.g., [36]), for any integer $m > 0$, we have

\[
P \left( \sup_{0 \leq t \leq m} M_{h}(t) > 2 \log m \right) \leq \frac{1}{m^{h/2}}, \quad h = 1, 2.
\]

Since $\sum_{i=1}^{\infty} 1/m^2 < \infty$ and Borel–Cantelli’s lemma (see, e.g., [36]), there exist $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ and an integer $m_0 = m_0(\omega)$ such that

\[
M_{h}(t) \leq 2 \int_0^t \sigma_h^2(\ell(s)) x_h^2(s) ds + 2 \log m,
\]

for all $\omega \in \Omega_0$, $0 \leq t \leq m$. Substituting the above inequality into (28), for any $\omega \in \Omega_0$, $m \geq m_0$, $0 \leq t \leq m$, we have

\[
\log(1 + x_i^2(t) + x_j^2(t)) \leq \log(1 + x_i^2(0) + x_j^2(0)) + Q t + 4 \log m,
\]

which yields

\[
\frac{1}{t} \log(1 + x_i^2(t) + x_j^2(t)) \leq \frac{1}{m - 1} \left[ \log(1 + x_i^2(0) + x_j^2(0)) + Q m + 4 \log m \right],
\]

for all $\omega \in \Omega_0$, $0 \leq m - 1 \leq t \leq m$, $m \geq m_0$. Letting $m \to \infty$ and using the inequality $y \leq (1/2)(1 + y^2)$ for any $y \in (-\infty, +\infty)$, we obtain

\[
\limsup_{t \to \infty} \frac{1}{t} \log x_i(t) \leq \limsup_{t \to \infty} \frac{1}{2(m - 1)} \log(1 + x^2(0)) + Q m + 4 \log m = \frac{Q}{2}, \quad a.s.
\]

The proof is therefore complete.

\[
\square
\]

Remark. Without migrations, we get $Q_1(i) = (\sqrt{[2\beta_i(i) - \sigma_i^2(i)]^2 + 4(p_i(i)/\gamma_i(i))\varepsilon} - [2\beta_i(i) - \sigma_i^2(i)])/2$. By comparison, we find that $Q_1(i) \leq C_i$, where $C_i$ is defined in [28]. In addition, without migration and regime switching, we can get $Q$ in Theorem 4 is less than $K$, where $K$ is defined in [21]. Furthermore, the condition $2\alpha_1 - \sigma_1^2 - (b_1 + b_2)e > 0$, $2\alpha_1 - \sigma_1^2 - (b_1 + b_2)e > 0$, $2\alpha_1 - \sigma_1^2 - (b_1 + b_2)e > 0$ in [31] means that the parameter $\varepsilon$ needs to be satisfied: $(b_1 + b_2)/2\alpha_1 - \sigma_1^2 < \varepsilon < 2\alpha_1 - \sigma_1^2/(b_1 + b_2)$. However, we know that this condition is unnecessary from the above theorem. Despite all this, if we let the parameter $\varepsilon$ satisfy $\varepsilon \in ((b_1 + b_2)/2\alpha_2 - \sigma_2^2), (2\alpha_2 - \sigma_2^2/(b_1 + b_2))$, we compute that $Q$ in Theorem 4 is less than $Q$ in [31]. Therefore, the above work is a promotion of Theorem 4.1 in [21], Theorem 2.2 in [28], and Theorem 4.1 in [31].
5. Extinction

Sufficient conditions for extinction are the subject of this section. Unless otherwise stated, we hypothesize $\tau(i) \equiv \tau, \quad i \in \mathcal{M}$ in this section. We first rewrite (6) as follows:

$$dx(t) = f_1(x(t), x(t-\tau), \ell(t))dt + f_2(x(t), \ell(t))dB(t),$$

(36)

where the operator $f_1: \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathcal{M} \rightarrow \mathbb{R}_+^2$ is defined as

$$f_1(x, y, i) = \left( -\beta_1(i)x_1 + b_1(i)x_2 + p_1(i)e^{-\gamma_1(i)y}, \
-\beta_2(i)x_1 + b_2(i)x_2 + p_2(i)y_2e^{-\gamma_1(i)y} \right),$$

(37)

the operator $f_2: \mathbb{R}_+^2 \times \mathcal{M} \rightarrow \mathbb{R}_+^{2 \times 2}$ is defined as

$$f_2(x, i) = \left( \sigma_1(i)x_1 0 \quad 0 \sigma_2(i)x_2 \right),$$

and $dB(t) = \left( dB_1(t) \quad dB_2(t) \right)$.

We first note that

$$f_1(0, 0, i) \equiv 0, \quad f_2(0, i) \equiv 0,$$

(38)

for $i \in \mathcal{M}$, whence (6) admits a trivial solution corresponding to $\phi(0) = 0$.

Before our result, we give a lemma.

**Lemma 3.** For system (36), the terms $f_1(x, y, i)$ and $f_2(x, i)$ are locally bounded in $(x, y)$ while uniformly bounded in $i$. That is, for any $m > 0$, there is $K_m > 0$ satisfying

$$|f_1(x, y, i)| \vee |f_2(x, i)| \leq K_m,$$

(39)

for all $i \in \mathcal{M}, \ x, y \in \mathbb{R}_+^2$ with $|x| \vee |y| \leq m$.

The proof is not particularly difficult, so we omit the proof.

**Theorem 5.** Assume that

$$2\beta_1 > \sigma_1^2 + b_1 + (1 + \sqrt{2})b_2 + \left( 2 + \frac{1}{\sqrt{2}} \right)p_1 + (1 + \sqrt{2})p_2,$$

$$2\beta_2 > \sigma_2^2 + b_1 + b_2 + (1 + \sqrt{2})p_1 + \left( 2 + \frac{1}{\sqrt{2}} \right)p_2,$$

(40)

Then, the solution of (36) satisfies $\lim_{t \to \infty} x(t) = 0$, a.s., for any initial values (7). That is, all populations in system (36) go to extinction with probability one.

**Proof**

Step 1: let

$$V_2(x_1, x_2) = x^T \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} x = x_1^2 + \sqrt{2}x_1x_2 + x_2^2.$$  

(41)

Obviously, $V_2$ is positive-definite and radially unbounded. That is,

$$\lim_{|x| \to \infty} V_2(x_1, x_2) = \infty.$$  

(42)

The generalized Itô formula yields

$$LV_2(x_1(t), x_2(t)) = V_2(\varphi_1(0), \varphi_2(0)) + \int_0^t \left[ \frac{\partial}{\partial x} V_2(x_1(s), x_2(s)) \right] f_2(x(s), \ell) \cdot (s-\tau(\ell(s)), \ell(s))ds$$

$$+ \int_0^t \left[ \frac{\partial}{\partial y} V_2(x_1(s), x_2(s)) \right] f_1(x(s), \ell) \cdot (s)dB(s).$$  

(43)

By computation, we know

$$LV_2(x_1, x_2) \leq -2\beta_1(i)x_1^2 - b_1(i)x_1 - (1 + \sqrt{2})b_2(i)x_2 - p_1(i)\frac{1}{\sqrt{2}}p_2(i)x_1^2$$

$$- 2\beta_2(i)x_2^2 - b_2(i)x_2 - (1 + \sqrt{2})b_1(i)x_1 - p_2(i)\frac{1}{\sqrt{2}}p_1(i)x_2^2$$

$$+ \left( 1 + \frac{1}{\sqrt{2}} \right)p_1(i)y_1^2 + \left( 1 + \frac{1}{\sqrt{2}} \right)p_2(i)y_2^2$$

$$\leq -\lambda_1x_1^2 - \lambda_2x_2^2 + \left( 1 + \frac{1}{\sqrt{2}} \right)p_1(i)y_1^2 + \left( 1 + \frac{1}{\sqrt{2}} \right)p_2(i)y_2^2,$$

(44)
where \( \lambda_1 = 2\beta_1 - \sigma_1^2 - b_1 - (1 + \sqrt{2})b_2 - \beta_1 - (1/\sqrt{2}) \hat{p}_1 - \hat{p}_2 \) and \( \lambda_2 = 2\beta_2 - \sigma_2^2 - (1 + \sqrt{2})b_1 - b_2 - (1/\sqrt{2}) \hat{p}_1 - \hat{p}_2 \). It is straightforward to see from (40) that \( \lambda_k > 0, \) \( h = 1,2. \) For simplicity, we let

\[
F_1(x) = \min\{\lambda_1, \lambda_2\} |x|^2, \\
F_2(x) = \left(1 + \frac{1}{\sqrt{2}}\right) \max\{\hat{p}_1, \hat{p}_2\} |x|^2.
\]

By condition (40) again, we obtain that

\[
F(x) = F_1(x) - F_2(x) = \left[ \min\{\lambda_1, \lambda_2\} - \left(1 + \frac{1}{\sqrt{2}}\right) \max\{\hat{p}_1, \hat{p}_2\} \right] |x|^2 > 0, \quad x \neq 0.
\]

Applying (44) and (46), we derive

\[
\int_0^t LV_2(x(s), x(s - \tau(\ell(s))), \ell(s))ds \\
\leq \int_0^t F_2(x(s))ds - \int_0^t F(x(s))ds.
\]

Substituting the preceding equality into (43), it yields

\[
V_2(x_1(t), x_2(t)) \leq V_2(\varphi_1(0), \varphi_2(0)) + \int_0^t F_2(x(s))ds \\
- \int_0^t F(x(s))ds \\
+ \int_0^t \left[ \frac{\partial}{\partial x} V_2(x_1(s), x_2(s)) \right] f_2(x(s), \ell(s))ds.
\]

Then, the nonnegative semimartingale convergence theorem (see, e.g., [37]) implies

\[
\limsup_{t \to \infty} V_2(x_1(t), x_2(t)) < \infty \quad \text{a.s.} \tag{49}
\]

Moreover, we obtain from (48) that

\[
E \int_0^t F(x(s))ds \leq V_2(\varphi_1(0), \varphi_2(0)) + \int_0^t F_2(x(s))ds.
\]

Then, letting \( t \to \infty \), together with the Fubini theorem, we have

\[
E \int_0^\infty F(x(t))dt < \infty. \tag{51}
\]

Let \( A_k = \{\omega|Y(\omega) = \int_0^\infty F(x(s, \omega))ds > 2^k\} \), where \( k = 1,2, \ldots \) Obviously, \( \{A_k\}^{\infty}_{k=1} \) Combining Chebyshev’s inequality and (51), we see that \( \sum_{k=1}^{\infty} P(A_k) < \infty \). By Borel–Cantelli’s lemma, one can show that \( P(\lim_{k \to \infty} A_k) = P(\omega|Y(\omega) = \infty) = 0 \), that is,

\[
\int_0^\infty F(x(t))dt < \infty \quad \text{a.s.} \tag{52}
\]

Step 2: from (52), we observe

\[
\lim_{t \to \infty} \inf F(x(t)) = 0, \quad \text{a.s.} \tag{53}
\]

One now needs to consider

\[
\lim_{t \to \infty} F(x(t)) = 0 \quad \text{a.s.} \tag{54}
\]

If the above conclusion would not hold, then \( P(\limsup_{t \to \infty} F(x(t)) > 0) > 0 \). So, there is \( \epsilon \in (0, (1/3)) \) satisfying

\[
P(\Omega_1) \geq 3\epsilon, \tag{55}
\]

where

\[
\Omega_1 = \left\{ \limsup_{t \to \infty} F(x(t)) > 2\epsilon \right\}. \tag{56}
\]

Noting that Lyapunov function \( V_2(x(t)) \) and the solution \( x(t) \) of (6) are all continuous, together with (49), it yields

\[
\sup_{t \leq \tau} V_2(x(t)) < \infty, \quad \text{a.s.} \tag{57}
\]

Define

\[
v(r) = \inf_{|x| \geq k} V_2(x), \quad \text{for } k > 0. \tag{58}
\]
Clearly,
\[ \sup_{-T \leq t \leq 0} \nu(|x(t)|) \leq \sup_{-T \leq t \leq 0} V_2(x(t)) < \infty, \quad \text{a.s.} \quad (59) \]

In addition, by (42), we get
\[ \lim_{k \to \infty} \nu(k) = \infty. \quad (60) \]

So,
\[ \sup_{-T \leq t \leq 0} |x(t)| < \infty, \quad \text{a.s.} \quad (61) \]

Recalling (7), we know that the initial values satisfy \( \phi_h \in C([-\tau, 0]; [0, +\infty)) \) for \( h = 1, 2 \). We therefore could find an integer \( m > 0 \), depending on \( \epsilon \), sufficiently large for \( |\phi(s)| < m \) for \( s \in [-\tau, 0] \) almost surely, while
\[ \mathbb{P}(\Omega_2) \geq 1 - \epsilon, \quad (62) \]

where \( \Omega_2 = \{ \sup_{-t \leq t < 0} |x(t)| < m \} \). By (55) and (62), one implies
\[ \mathbb{P}(\Omega_1 \cap \Omega_2) \geq \mathbb{P}(\Omega_1) - \mathbb{P}(\Omega^c_1) \geq 2\epsilon, \quad (63) \]

where \( \Omega^c_1 \) is the complement of \( \Omega_1 \). Let
\[ \rho_1 = \inf\{ t \geq 0: F(x(t)) \geq 2\epsilon \}, \]
\[ \rho_{2j} = \inf\{ t \geq 0: f(x(t)) \leq \epsilon \}, \quad j = 1, 2, \ldots, \]
\[ \rho_{2j+1} = \inf\{ t \geq \rho_{2j}: F(x(t)) \leq 2\epsilon \}, \quad j = 1, 2, \ldots, \]
\[ \sigma_m = \inf\{ t \geq 0: |x(t)| \geq m \}. \quad (64) \]

From (53) and the definitions of \( \Omega_1 \) and \( \Omega_2 \), we have
\[ [\Omega_1 \cap \Omega_2] \subset \left\{ \sigma_m = \infty \cap \bigcap_{j=1}^{\infty} [\rho_j < \infty] \right\}. \quad (65) \]

Hence, we define \( [\zeta(t)] = [x(t \wedge \sigma_m)] \) for \( t > -\tau \), and its differential is
\[ d\zeta(t) = \tilde{f}_1(t)dt + \tilde{f}_2(t)dB(t), \quad (66) \]

where
\[ \tilde{f}_1(t) = f_1(x(t), x(t-\tau), \ell(t))I_{[0, \sigma_m]}(t), \]
\[ \tilde{f}_2(t) = f_2(x(t), \ell(t))I_{[0, \sigma_m]}(t). \quad (67) \]

Here, \( I_A \) is the indicator function of \( A \). Recalling Lemma 3, we know
\[ |\tilde{f}_1(t)| \vee |\tilde{f}_2(t)| \leq K_m, \quad \text{a.s}., \quad (68) \]

for any \( t \geq -\tau \), \( \ell(t) \in \mathcal{M} \), and \( |x(t)| \vee |x(t-\tau)| \leq m \). Moreover, the definition of \( [\zeta(t)] \leq m \), \( t \geq 0 \). We also note that, for all \( \omega \in \Omega_1 \cap \Omega_2 \) and \( j \geq 1 \),
\[ F(\zeta(\rho_{2j-1})) - F(\zeta(\rho_{2j})) = \epsilon, \quad F(\zeta(t)) \geq \epsilon, \quad t \in [\rho_{2j-1}, \rho_{2j}] \]. \quad (69) \]

In the close ball \( \mathbb{S}_m = \{ x \in \mathbb{R}^2: |x| \leq m \} \), \( F(\cdot) \) is uniformly continuous. Therefore, there exists \( \xi = \xi(\epsilon) > 0 \) small sufficiently such that
\[ |F(\zeta) - F(\tilde{\zeta})| < \epsilon, \quad \xi, \tilde{\xi} \in \mathbb{S}_m \text{ with } |\zeta - \tilde{\zeta}| < \xi. \quad (70) \]

For \( \omega \in \Omega_1 \cap \Omega_2 \), we emphasize that if \( |\zeta(\rho_{2j-1} + t) - \zeta(\rho_{2j-1})| < \xi \) for some \( T > 0 \) and \( t \in [0, T] \), then \( \rho_{2j} - \rho_{2j-1} \geq T \). Furthermore, let the number \( T = T(\epsilon, \xi, m) > 0 \) be small enough such that
\[ 2K_m^2(T + 4) \leq k\xi^2. \quad (71) \]

By (63) and (65), we can obtain that
\[ \mathbb{P}(\rho_{2j} < \infty) \geq 2\epsilon. \quad (72) \]

In particular, if \( \rho_{2j} < \infty \), then \( |\zeta(\rho_{2j})| < m \). Hence, the definition of \( \zeta(t) \) implies \( \rho_{2j} < \sigma_m \). So,
\[ \zeta(t, \omega) = x(t, \omega), \quad (73) \]

for all \( 0 \leq t \leq \rho_{2j} \), \( \omega \in [\rho_{2j} < \infty] \). Then, from the Hölder inequality and the Burkholder–Davis–Gundy inequality, it follows that
which together with (71) and the Chebyshev inequality, imply easily that
\[
P \left( |\rho_{j-1} - \rho_j (t) - \frac{\zeta(t)}{\omega(j-1)}| \geq \xi \right) \leq \varepsilon.
\]  
(75)

Obviously, we observe that \( \rho_{j-1} < \infty \) if \( \rho_j < \infty \). Hence, by (72) and (75), we can derive that

\[
P \left( \rho_{j-1} < \infty \cap \left| \sup_{t \leq T} \zeta (\rho_{j-1} + t) - \zeta (\rho_j - t) \right| \geq \xi \right) = P \left( \rho_{j-1} < \infty \cap \sup_{t \leq T} \zeta (\rho_{j-1} + t) - \zeta (\rho_j - t) \geq \xi \right)
\]  
\[\geq \varepsilon.
\]  
(76)

This, together with (70), we can get that

\[
P \left( \rho_{j-1} < \infty \cap (\rho_j - \rho_{j-1}) \geq T \right) \geq \varepsilon.
\]  
(77)

Recalling (51), (69), and (77), for all \( \omega \in \Omega_1 \cap \Omega_2 \) and \( j \geq 1 \), we compute

\[
\lim_{t \to \infty} x(t) = 0, \text{ a.s.}
\]  
(80)

for any initial value \( x(t) = \phi(t), t \in [-\tau, 0], \phi(0) > 0, \phi \in C([-\tau, 0], [0, +\infty)) \). That is, all populations in equation (2) go to extinction with probability one.

**Corollary 2.** Assume that \( a_{h}, b_{h}, p_{h}, \gamma_{h}, \sigma_{h}, \tau \) are nonnegative constants, \( h = 1, 2 \), and

\[
2a_1 > a_1^2 + b_1 + (\sqrt{2} - 1)b_2 + \left( 2 + \frac{1}{\sqrt{2}} \right)p_1 + \left( 1 + \sqrt{2} \right)p_2,
\]  
\[2a_2 > a_2^2 + (\sqrt{2} - 1)b_1 + b_2 + (1 + \sqrt{2})p_1 + \left( 2 + \frac{1}{\sqrt{2}} \right)p_2.
\]  
(81)

Then, solution \( x(t) \) of (4) obeys

\[
\lim_{t \to \infty} x(t) = 0, \text{ a.s.}
\]  
(82)

for any initial value \( x_h(t) = \phi_h(t), t \in [-\tau, 0], \phi_h(0) > 0, \phi_h \in C([-\tau, 0]; [0, +\infty)) \). That is, all populations in model (4) go to extinction with probability one.

**Remark 7.** This theorem reveals that the solutions of (6) will all tend to the origin asymptotically with probability one when the intensities of noises and the parameters satisfy condition (40). However, [21, 28, 31, 32] do not study extinction of populations. Besides, this method can be extended to research extinction in the above literature studies.
Corollaries 1 and 2 give the conditions of extinction of (2) and (4), respectively. Therefore, our work is the extension of [21, 28, 31, 32].

6. Numerical Simulations

Based on [38], we show numerical simulations in the present section.

Here, we consider model (6) with the same initial data $\varphi_1(0) = 1$, $\varphi_2(0) = 0.5$ and the same $\{\ell(t)\}_{t \geq 0}$ on $\mathcal{M} = \{1, 2, 3\}$ with

$$\Gamma = \begin{pmatrix}
-10 & 4 & 6 \\
2 & -3 & 1 \\
3 & 5 & -8
\end{pmatrix}. \quad (83)$$

Then, we know the Markov chain $\ell(t)_{t \geq 0}$ is irreducible and has a unique stationary distribution $\pi = (0.1845, 0.6019, 0.2136)$.

In Figure 1, we give a simulation of the sample path of $\ell(t)_{t \geq 0}$ with $\ell(0) = 3$.

In Figure 2, we can choose $a_1 = [1, 2, 3]$, $b_1 = [0.12, 0.23, 0.22]$; $p_1 = [4, 5, 6]$; $\gamma_1 = [0.4, 0.5, 0.6]$; $\sigma_1 = [1.2, 1.1, 2]$; $a_2 = [3, 1, 2]$; $b_2 = [0.23, 0.16, 0.12]$; $p_2 = [7, 4, 3]$; $\gamma_2 = [0.3, 0.4, 0.5]$; $\sigma_2 = [0.45, 1.5, 0.25]$; $\tau = [1, 2, 3]$. It is easy to see that $1 = a_{22} \leq (\sigma^2_{22}/2) = 1.125$ when $\xi(t) = 2$. There is a good agreement between Lemma 1 and Figure 2. By Theorem 2, we know $\lim\sup_{t \to \infty} (\bar{P}_1/\bar{F}_1) + (\bar{P}_2/\bar{F}_2) = 14.1020$. Furthermore, we get $Q = \max \{9.5553, 6.3244, 3.8081\}$ by calculation. Therefore, conditions of Theorem 4 have been checked. So, $\lim\sup_{t \to \infty} (1/t) \ln x_{1}(t) \leq 4.7777$, a.s.

In Figure 3, we can choose $a_1 = [2, 1, 2, 2]; b_1 = [0.11, 0.13, 0.12]$; $p_1 = [3, 2, 3, 3]$; $\gamma_1 = [0.83, 0.85, 0.86]$; $\sigma_1 = [0.62, 0.61, 0.63]$; $a_2 = [2, 2, 1, 2]; b_2 = [0.11, 0.13, 0.12]$; $p_2 = [3, 1, 3, 3, 3]$; $\gamma_2 = [0.81, 0.8, 0.83]$; $\sigma_2 = [0.71, 0.75, 0.71]$. Let $\theta = 2$. Then, conditions of Theorem 1 could be checked. By calculation, we get $\lim\sup_{t \to \infty} (1/t) \int_0^t E(x_{1}(s))^3 ds \leq 31.9797$, $\lim\sup_{t \to \infty} E(x_{2}(t) + x_{3}(t)) \leq 19.3208$, $\lim\sup_{t \to \infty} E(x(t)^3) \leq 38.6416$. Figure 3 clearly supports this result.

In Figure 4, we can choose $a_1 = [17, 18, 19]; b_1 = [0.14, 0.35, 0.15]$; $p_1 = [4, 5, 6, 5.43]$; $\gamma_1 = [0.8, 0.5, 0.6]$; $\sigma_1 = [0.5, 0.45, 0.12]$; $a_2 = [16.5, 16.9, 17.5]$; $b_2 = [0.5, 0.13, 0.22]$; $p_2 = [3.23, 4.67, 6]$; $\gamma_2 = [0.3, 0.4, 0.8]$; $\sigma_2 = [0.49, 1, 0.65]$; $\tau = [3, 3, 3]$. By calculation, we get $\tau(1) = \tau(2) = \tau(3) = 3, 35 = 2p_1 > \sigma^2_1 + b_1 + (1 + \sqrt{2}) b_2 + (2 + (1/\sqrt{2}) = \bar{p}_1 + (1 + \sqrt{2})); \bar{p}_2 = 32.5350$, and $33.28 = 2p_2 > \sigma^2_2 + (1+ \sqrt{2}) b_1 + b_2 + (1 + \sqrt{2}) \bar{p}_1 + (2 + (1/\sqrt{2}) \bar{p}_2 = 33.0729$.

Therefore, conditions of Theorem 5 have been checked. Thus, from Theorem 5, all species become extinct. Figure 4 clearly supports this result.
By the conclusion of Lemma 1, it is worthy to point out that the Brownian noise and colored noise will not destroy a great property that the solution of (6) may not explode. Especially, system (6) reduces to (1)–(4) or the model in [32] when parameters of system (6) take some special values. From Lemma 1, the condition $\alpha > (\sigma^2/2)$ in [21] is too strict and unnecessary. In Theorem 1, we comprehensively analyze ultimate boundedness in the $\theta$th moment and boundedness for the average in time of the $\theta$th moment of solution, which is the improvement of Theorem 3.1 in [21], Theorem 2.2 in [28], Theorem 3.3 in [31], and Theorem 3.2 in [32]. In Theorem 4, we find an upper bound $Q/2$ of the sample Lyapunov exponent. When parameters of system (6) take some special values, we compute that the upper bound $Q/2$ is less than the corresponding upper bound in [21, 28]. Furthermore, we find that the condition $2\alpha_1 - \sigma_1^2 - (b_1 + b_2)\epsilon > 0$, $2\alpha_2 - \sigma_2^2 - b_1 + b_2\epsilon > 0$ in [31] is not necessary. Despite all this, if we let parameter $\epsilon$ satisfy the above conditions, we compute that $Q/2$ is less than the upper bound in [31]. One point should be stressed is that the method for extinction in Theorem 5 can be used successfully for the models in [21, 28, 31, 32]. And then, Corollaries 1 and 2 give the conditions of extinction of (2) and (4), respectively. From Remarks 1–7, our work is a generalization and promotion of the corresponding work in [21, 28, 30–32]. To some extent, our proposed approaches are both more robust and more efficient than the existing methods.

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### Authors’ Contributions

All the authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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### References

[1] W. S. C. Gurney, S. P. Blythe, and R. M. Nisbet, “Nicholson's blowflies revisited,” *Nature*, vol. 287, no. 5777, pp. 17–21, 1980.

[2] A. Nicholson, “An outline of the dynamics of animal populations,” *Australian Journal of Zoology*, vol. 2, no. 1, pp. 9–65, 1954.

[3] W. Chen and B. Liu, “Positive almost periodic solution for a class of Nicholson's blowflies model with multiple time-varying delays,” *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 2090–2097, 2011.

[4] M. Yang, “Exponential convergence for a class of Nicholson's blowflies model with multiple time-varying delays,” *Nonlinear Analysis: Real World Applications*, vol. 12, no. 4, pp. 2245–2251, 2011.

[5] L. Bereczky, E. Braverman, and L. Idels, “Nicholson's blowflies differential equations revisited: main results and open problems,” *Applied Mathematical Modelling*, vol. 34, no. 6, pp. 1405–1417, 2010.
[6] T. Faria and G. Röst, “Persistence, permanence and global stability for an n-dimensional Nicholson system,” Journal of Dynamics and Differential Equations, vol. 26, no. 3, pp. 723–744, 2014.

[7] C. Huang, X. Yang, and J. Cao, “Stability analysis of Nicholson’s blowflies equation with two different delays,” Mathematics and Computers in Simulation, vol. 171, pp. 201–206, 2020.

[8] X. Long and S. Gong, “New results on stability of Nicholson’s blowflies equation with multiple pairs of time-varying delays,” Applied Mathematics Letters, vol. 100, Article ID 106027, 2020.

[9] J. Zhang and C. Huang, “Dynamics analysis on a class of delayed neural networks involving inertial terms,” Advances in Difference Equations, vol. 2020, p. 120, 2020.

[10] C. Qian and Y. Hu, “Novel stability criteria on nonlinear density-dependent mortality Nicholson’s blowflies systems in asymptotically almost periodic environments,” Journal of Inequalities and Applications, vol. 2020, p. 13, 2020.

[11] W. Wang, F. Liu, and W. Chen, “Exponential stability of pseudo almost periodic delayed Nicholson-type system with patch structure,” Mathematical Methods in the Applied Sciences, vol. 42, no. 2, pp. 592–604, 2019.

[12] C. Huang, X. Long, L. Huang, and S. Fu, “Stability of almost periodic Nicholson’s blowflies model with involving patch structure and mortality terms,” Canadian Mathematical Bulletin, vol. 63, no. 2, pp. 405–422, 2019.

[13] C. Huang, H. Zhang, and L. Huang, “Almost periodicity analysis for a delayed Nicholson’s blowflies model with nonlinear density-dependent mortality term,” Communications on Pure & Applied Analysis, vol. 18, no. 6, pp. 3337–3349, 2019.

[14] R. M. May, Stability and Complexity in Model Ecosystems, Princeton University Press, Princeton, NJ, USA, 1973.

[15] R. Liu and G. Liu, “Asymptotic behavior of a stochastic two-species competition model under the effect of disease,” Complexity, vol. 2018, Article ID 3127404, 15 pages, 2018.

[16] W. Wang, C. Shi, and W. Chen, “Stochastic Nicholson-type delay differential system,” International Journal of Control, 2019, In press.

[17] W. Wang and W. Chen, “Stochastic delay differential classical growth model,” Advances in Difference Equations, vol. 2019, p. 355, 2019.

[18] X. Mao, “Stationary distribution of stochastic population systems,” Systems & Control Letters, vol. 60, no. 6, pp. 398–405, 2011.

[19] G. Liu, X. Wang, X. Meng, and S. Gao, “Extinction and persistence in mean of a novel delay impulsive stochastic infected predator-prey system with jumps,” Complexity, vol. 2017, Article ID 1950907, 15 pages, 2017.

[20] F. Bian, W. Zhao, Y. Song, and R. Yue, “Dynamical analysis of a class of prey-predator model with Beddington-DeAngelis functional response, stochastic perturbation, and impulsive toxicant input,” Complexity, vol. 2017, Article ID 3742197, 18 pages, 2017.

[21] W. Wang, L. Wang, and W. Chen, “Stochastic Nicholson’s blowflies delayed differential equations,” Applied Mathematics Letters, vol. 87, pp. 20–26, 2019.

[22] J. M. Drake, “Allee effects and the risk of biological invasion,” Risk Analysis, vol. 24, no. 4, pp. 795–802, 2004.

[23] M. Assaf, A. Kamenetz, and B. Meerson, “Population extinction risk in the aftermath of a catastrophic event,” Physical Review E, vol. 79, no. 1, Article ID 011127, 2009.

[24] X. Mao, “Stability of stochastic differential equations with Markovian switching,” Stochastic Processes and Their Applications, vol. 79, no. 1, pp. 45–67, 1999.

[25] Q. Luo and X. Mao, “Stochastic population dynamics under regime switching II,” Mathematical Analysis and Applications, vol. 355, no. 2, pp. 577–593, 2007.

[26] T. K. S. Liao, “Bond pricing under a Markovian regime-switching jump-augmented Vasicek model via stochastic flows,” Applied Mathematics and Computation, vol. 216, no. 11, pp. 3184–3190, 2010.

[27] A. Settati and A. Lahrouz, “Stationary distribution of stochastic population systems under regime switching,” Applied Mathematics and Computation, vol. 244, pp. 235–243, 2014.

[28] Y. Zhu, K. Wang, Y. Ren, and Y. Zhuang, “Stochastic Nicholson’s blowflies delay differential equation with regime switching,” Applied Mathematics Letters, vol. 94, pp. 187–195, 2019.

[29] L. J. S. Allen, “Persistence, extinction, and critical patch number for island populations,” Journal of Mathematical Biology, vol. 24, no. 6, pp. 617–625, 1987.

[30] L. Bereczsány, L. Ídés, and L. Troib, “Global dynamics of Nicholson-type delay systems with applications,” Nonlinear Analysis: Real World Applications, vol. 12, no. 1, pp. 436–445, 2011.

[31] X. Yi and G. Liu, “Analysis of stochastic Nicholson-type delay system with patch structure,” Applied Mathematics Letters, vol. 96, pp. 223–229, 2019.

[32] W. Wang and W. Chen, “Stochastic Nicholson-type delay system with regime switching,” Systems & Control Letters, vol. 136, p. 104603, 2020.

[33] A. Lahrouz and A. Settati, “Necessary and sufficient condition for extinction and persistence of SIRS system with random perturbation,” Applied Mathematics and Computation, vol. 233, pp. 10–19, 2014.

[34] A. Settati, S. Hamdoune, A. Imlahi, and A. Akharif, “Extinction and persistence of a stochastic Gilpin-Ayala model under regime switching on patches,” Applied Mathematics Letters, vol. 90, pp. 110–117, 2019.

[35] X. Mao, Ren, and C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, London, UK, 2006.

[36] X. Mao, Stochastic Differential Equations and Applications, Horwood, Chichester, UK, 2007.

[37] R. S. Liptser and A. N. Shiryayev, Theory of Martingales, Kluwer Academic Publishers, Dordrecht, Netherlands, 1989.

[38] D. J. Higham, “An algorithmic introduction to numerical simulation of stochastic differential equations,” SIAM Review, vol. 43, no. 3, pp. 525–546, 2001.