Semiclassical theory of weak values

Atushi Tanaka
Department of Physics, Tokyo Metropolitan University,
Hachioji, Tokyo 192-0397, Japan

23 March 2002

Abstract
Aharonov-Albert-Vaidman’s weak values are investigated by a semiclassical method. Examples of the semiclassical calculation that reproduces “anomalous” weak values are shown. Furthermore, a complex extension of Ehrenfest’s quantum-classical correspondence between quantum expectation values of the states with small quantum fluctuation, and classical dynamics, is shown.

1 Introduction
A standard approach of quantum theoretical description of the interaction between a measurement apparatus and the system that is subject to the measurement was introduced by von Neumann. [1]. Recently, it is revealed that the notion of weak (expectation) values [2, 3] emerges from the von Neumann theory, in the limit that the influence of the apparatus on the system is weak so as to avoid the collapse of the state of the system. The weak values and the conventional expectation values of quantum theory coincide for the quantum ensembles that are specified by only the preselections (preparations) of initial states [3]. However, once we consider the quantum ensembles that are specified not only by the preselections of initial states, but also by the postselections of final states, the weak values depart from the conventional expectation values [4, 5, 6]. In particular, the weak values can take “anomalous” values that lie outside of the range of the eigenvalues of the corresponding operators. More precisely, there are two kinds of anomalous weak values, surprisingly large weak values of bounded operators, and complex-valued weak values of Hermite operators. On one hand, Aharonov, Albert and Vaidman showed an experimental setup, in which weak measurements of a component of the spin of the spin-$\frac{1}{2}$ particle can turn out to be 100 [2] (see also, [3]). Experimental studies confirms the surprisingly large weak values [3, 4]: On the other hand, Aharonov, Albert and Vaidman [5] mentioned that the weak values can be a complex number. This is also shown to be experimentally accessible [6], and moreover, to provide useful theoretical notions [2, 3, 7].
This letter reports a study on weak values by a semiclassical method. The usefulness of the semiclassical approach for the weak values was anticipated by Berry’s asymptotic (i.e. semiclassical) analysis of superosillations. Although the anomalous weak values suggest us that the notion of weak values is completely foreign to classical concepts, the semiclassical theory is shown to reproduce examples of anomalous weak values. This is achieved by extending the classical trajectories into complex-valued phase space. It is well-known that the semiclassical method employs complex-valued classical trajectories to describe classically forbidden phenomena, e.g. tunneling and nonadiabatic transitions. Furthermore, the recent investigations of the classically forbidden processes in “quantum chaos” revealed that the complex extension of classical trajectory is indispensable.

In the following, I remind the basic point of the weak measurements. Consider the quantum ensemble that is specified not only by a preselection (preparation) of an initial state but also a postselection of a final state: let the state vectors of the initial and the final states be \( |\psi'\rangle \) (at time \( t' \)) and \( \langle \psi''| \) (at time \( t''(>t') \)), respectively. The corresponding “expectation value” of an observable \( \hat{A} \) at an intermediate time \( t \) \( (t' \leq t \leq t'') \) is called a weak value \( W(\hat{A}, t) \), whose definition is

\[
W(\hat{A}, t) \equiv \frac{\langle \psi''| \hat{U}(t'', t) \hat{A} \hat{U}(t, t') |\psi'\rangle}{\langle \psi''| \hat{U}(t'', t') |\psi'\rangle}(1)
\]

where \( \hat{U}(t_1, t_0) \) is the time evolution operator during the time interval \([t_0, t_1]\).

The weak values are experimentally accessible by a kind of von Neumann type apparatus whose “pointer” position has large quantum fluctuation. After the apparatus weakly contact with the system and the system is successively postselected, we can gain the information of the corresponding weak value by examining the statistics of the pointer. At each readout, the pointer value, which is obtained by the standard quantum measurement that invokes wavepacket reduction, is very noisy due to the initial fluctuation of the pointer and the weakness of the interaction with the system. Accordingly we need to accumulate the readouts of the pointer to make sense. The peak of the pointer position distribution determines the real part of the weak value. In addition, the imaginary part of the weak value is determined by the peak of the momentum (i.e. the conjugate quantity of the pointer position) distribution of the pointer. Thus both the real and the imaginary parts of weak values are experimentally accessible.

The plan of this letter is as follows. In Sec. 2, I develop a theory to evaluate weak values by a semiclassical method. Examples are shown in Sec. 3. We encounter two kinds of “anomalous” weak values, complex-valued weak values (Sec. 3.1), and surprisingly large weak values (Sec. 3.2). It is shown that they provide an “anomalously-extended” classical-quantum correspondence. I discuss the limitation of the present semiclassical argument in Sec. 4. Finally, Sec. 5 summarizes this letter.
2 Semiclassical evaluation of weak values

For brevity, I employ a one degree-of-freedom system, whose position and momentum operators are \( \hat{q} \) and \( \hat{p} \), respectively. Let us consider the quantum ensemble specified by an initial state \(|\psi'\rangle \equiv |q'\rangle\) at \( t = t' \) and a final state \( \langle \psi'' | \equiv \langle q'' | \) at \( t = t'' (> t') \), where \( |q\rangle \) is the \( \hat{q} \)'s eigenvector whose eigenvalue is \( q \). In the evaluation of weak value \( W(\hat{A}, t) \), the following generating functional is useful:

\[
Z(\zeta(\cdot), \hat{A}) \equiv \langle \psi'' | \exp\left\{ -\frac{i}{\hbar} \int_{t'}^{t''} \left( \hat{H} - \hat{A} \zeta(t) \right) dt \right\} |\psi'\rangle
\]

where \( \exp(\cdot) \) is the time-ordered exponential and \( \hat{H} \) is the Hamiltonian of the system. It is straightforward to show that

\[
W(\hat{A}, t) = -i\hbar \frac{\delta \ln Z(\zeta(\cdot), \hat{A})}{\delta \zeta(t)} \bigg|_{\zeta(\cdot) = 0}
\]

holds. This is evaluated by a semiclassical method in the following.

Let \( A(q, p) \) and \( H(q, p) \) be the classical counterparts of the operator \( \hat{A} \) and the Hamiltonian \( \hat{H} \) of the system, respectively. I ignore the operator ordering problem, since it changes the result only \( \mathcal{O}(\hbar) \), i.e., within the accuracy of the following semiclassical argument.

In the evaluation of \( Z(\zeta(\cdot), \hat{A}) \) \(^{2}\), I employ the semiclassical approximation that evaluates the Feynman path integral representation of \( Z \) by the stationary phase method \(^{3}\). In order to carry out the semiclassical evaluations, I introduce an important assumption: for infinitesimally small values of \( \zeta(\cdot) \), quantum interference between multiple classical trajectories do not present in the semiclassical evaluation. Namely, the semiclassical generating functional have a contribution only from one classical trajectory \( (q(t), p(t)) \), which satisfies the Hamilton equation with the classical Hamiltonian \( H(q, p) - A(q, p) \zeta(t) \) and the boundary condition \( q(t') = q' \) and \( q(t'') = q'' \), which are specified by the initial and the final states \( |q'\rangle \) and \( \langle q'' | \), respectively\(^{4}\). In other words, \( Z \) is assumed to be in a single-term form \(^{2}\)

\[
Z \simeq E \exp(iS/\hbar)
\]

where \( E \) and \( S \) are the amplitude factor and classical action, respectively,

\[
E \equiv \frac{1}{\sqrt{2\pi\hbar \partial q''/\partial p'}}
\]

\[
S \equiv \int_{t'}^{t''} \{ p(t)\dot{q}(t) - H(q(t), p(t)) \}
+ A(q(t), p(t))\zeta(t) \} dt
\]

\(^{1}\)The Hamilton equation with the boundary condition \( q(t') = q' \) and \( q(t'') = q'' \) can have multiple solutions. If \( t'' - t' \) is small enough, it is proved that the boundary-value problem has a unique solution. See, e.g., Ref. \(^{14}\), Chap. 12.

\(^{2}\)I omit Maslov’s index \(^{20}\), since this is irrelevant to the present argument.
and \( p' \equiv p(t') \) \[21\]. The single-term condition \((4)\) holds when \( \hbar \) is small or the time scale in question is short. The details are discussed in Sec. 4. The single-term condition \((4)\) implies

\[
W(\hat{A}, t) = \left. \frac{\delta S}{\delta \zeta(t)} \right|_{\zeta(t) \equiv 0} + O(\hbar). \tag{7}
\]

Applying \((3)\) to this, the main result is obtained:

\[
W(\hat{A}, t) = A(q(t), p(t)) + O(\hbar) \tag{8}
\]

where \( \zeta(\cdot) \equiv 0 \) is imposed on \((q(t), p(t))\). The first term of eq. \((8)\) is a “classical” quantity: It persists in the classical limit \( \hbar \to 0 \) and is almost independent of \( \hbar \), in general. The weak values are accordingly determined, with an error of \( O(\hbar) \), by the classical trajectory \((q(t), p(t))\) that composes the semiclassical evaluation of the Feynman kernel \( \langle \psi'\prime | \hat{U}(t', t) | \psi' \rangle = Z(\zeta(\cdot) \equiv 0) \).

I emphasize the significance of the single-term assumption \((4)\), which optimize the shape of semiclassical kernel, by using a simple wave \((1)\) that is composed by a single classical trajectory. The resultant estimation \((3)\) accordingly exclude the effect of the quantum interference phenomena among multiple semiclassical amplitudes. Namely, eq. \((3)\) establishes a correspondence between a weak value and a single classical trajectory.

The generalization of the result \((3)\) to various initial and final states (e.g. the eigenstates of the momentum operator, and coherent states) can be obtained straightforwardly with the help of the semiclassical algebra \((22, 23)\), as long as the single-term approximation (cf. \((4)\)) holds for the corresponding semiclassical generating function \( Z \).

The “variance” of the weak value of \( \hat{A} \) is \( W(\{ \hat{A} - W(\hat{A}) \}^2) \) \[3\]. According to the semiclassical evaluation of weak values \((3)\), the “weak variance” is \( O(\hbar) \), when the single-term approximation \((4)\) for the Feynman kernel holds.

3 Examples: “anomalous” weak values

3.1 Coherent state path integral: An “anomalous” extension of quantum-classical correspondence into complex-valued phase-space

Firstly, I show an example that the semiclassical theory above reproduces complex-valued weak values. The semiclassical evaluation \((3)\) suggests that we encounter complex-valued weak values for classically forbidden phenomena (e.g. tunneling phenomena and nonadiabatic transitions). One of the simplest ways to investigate the classically forbidden phenomena is to study semiclassical coherent state path integrals, which are generically composed of complex-valued classical trajectory \[24\]. The present argument accordingly provides an extension of quantum-classical correspondence into complex-valued phase-space.
I use the coherent states \( |q'p'\rangle \) that are characterized with the help of a complex symplectic transformation \( [Q, P] = \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \) \( (9) \)

where \( q \) and \( p \) are the position and its canonical conjugate momentum of the system, respectively. The quantized operators \( \hat{Q} \) and \( \hat{P} \) are creation and annihilation operators, respectively, of a harmonic oscillator. The coherent state \( |q'p'\rangle \) that is employed here is \( \hat{P} \)'s eigenstate whose eigenvalue is \( (p' - iq')/\sqrt{2} \).

The semiclassical evaluation of \( K(q''p''; q'p') = \langle q''p''|U(t'', t')|q'p'\rangle \), which is the Feynman kernel in the coherent state representation, is obtained by the stationary phase evaluation of the coherent state path integral representation of \( K \) \( (27) \). The boundary condition (at \( t = t', t'' \)) of the classical trajectories for semiclassical coherent state path integral is obtained by Klauder \( (24) \)

\[ P(t') = P' \left( \equiv (p' - iq')/\sqrt{2} \right) \]
\[ Q(t'') = Q'' \left( \equiv (q'' - ip'')/\sqrt{2} \right) \] \( (10) \)

One way to explain the Klauder’s boundary condition \( (10) \) is to remember the fact that \( |q'p'\rangle \) is a right-eigenvector of \( \hat{P} \) and \( \langle q''p''| \) is a left-eigenvector of \( \hat{Q} \): The corresponding eigenvalues determine the values of classical variables \( P(t') \) and \( Q(t'') \). The classical trajectory \( (q(t), p(t)) \) during the time interval \( t' < t < t'' \) is complex-valued in general, except the case that the real-valued classical time evolution carries the point in the phase space \( (q', p') \) at time \( t' \) to \( (q'', p'') \) at time \( t'' \).

The single-term condition (cf. \( (4) \)) holds for semiclassical Feynman kernel in the coherent state representation, when the time interval \( t'' - t' \) is small enough. Accordingly the semiclassical estimation \( (8) \) implies

\[ (W(\hat{q}, t), W(\hat{p}, t)) = (q(t), p(t)) + \mathcal{O}(\hbar). \] \( (11) \)

Namely, the weak values \( W(\hat{q}, t) \) and \( W(\hat{p}, t) \) approximately obey the classical equation of motion. Furthermore, since \( (q(t), p(t)) \) are generally complex-valued as is explained above, so are the weak values \( W(\hat{q}, t) \) and \( W(\hat{p}, t) \).

The estimation \( (11) \) provides an extension of a correspondence between real-valued classical trajectories and the expectation values of quantum systems. The Ehrenfest theorem implies that the expectation values of quantum system obeys the corresponding classical theory, as long as the quantum fluctuation in a phase space representation is small \( (28) \). Although this concerns only for the real-valued, “ordinary” correspondence, the semiclassical theory of weak values developed in this letter extends the argument to the “anomalous”-valued (i. e. complex-valued) trajectories. Note that the complex extension \( (11) \) is carried out without any discrimination between real- and complex-valued trajectories. This suggests that the distinction between real (“normal”) and complex (“anomalous”) trajectories is only superficial in the framework of the weak measurements.
Let us consider an example, the vanishing Hamiltonian $H = 0$, which is the simplest, yet nontrivial example in the studies of semiclassical coherent state path integral \[24\]. I remind that the semiclassical evaluation for this system is exact. Let the initial ($t = t'$) and final ($t = t''$) states be $|q'p'angle$ and $\langle q''p''|$, respectively. During the time interval $t' < t < t''$, the position and the momentum of the classical trajectory in the semiclassical Feynman kernel are complex-valued in general \[24\]:

\[
q = \frac{1}{2}(q'' + q') - \frac{i}{2}(p'' - p') \quad (12)
\]

\[
p = \frac{1}{2}(p'' + p') + \frac{i}{2}(q'' - q') \quad (13)
\]

These are nothing but complex-valued weak values $W(\hat{q},t)$ and $W(\hat{p},t)$ during the time interval $t' < t < t''$.

### 3.2 Spin coherent state path integral: an example of anomalously large weak values

The weak values investigated in ref. \[2\] for a spin-$\frac{1}{2}$ system can be reproduced by the semiclassical theory of spin coherent state path integral \[24\]. Let $(\theta, \phi)$ be the orientation of the spin in the polar coordinate, which parameterizes spin coherent states $|\theta, \phi\rangle$ \[25\]. The classical trajectory $(\theta(t), \phi(t))$ that composes the semiclassical evaluation of a Feynman kernel $\langle \theta'', \phi''|\hat{U}(t'', t')|\theta', \phi'\rangle$ satisfies Klauder’s boundary condition \[24\]

\[
e^{i\phi'}\tan(\theta'/2) = e^{i\phi(t')}\tan(\theta(t')/2)
\]

\[
e^{-i\phi''}\tan(\theta''/2) = e^{-i\phi(t'')}\tan(\theta(t'')/2)
\]

as well as the classical equation of motion. Note that the classical variables $\theta(t)$ and $\phi(t)$ are complex-valued in general. The semiclassical weak values of the ensemble that is specified by an initial state $|\theta', \phi'\rangle$ and a final state $\langle \theta'', \phi''|$ are obtained with the help of the semiclassical formula \[3\], if we do not encounter multiple classical trajectories.

For example, when the Hamiltonian of the system vanishes (i.e. $H \equiv 0$), the weak values of the all components of the spin of the spin-$\frac{1}{2}$ particle exactly agree with the values obtained by the semiclassical evaluation \[3\]. I demonstrate an example $(\theta', \phi') = (2\alpha, 0)$, and $(\theta'', \phi'') = (\frac{\pi}{2}, \pi)$, where $0 < \alpha < \pi/2$. The weak values, during the time interval $t' < t < t''$, are

\[
W(\hat{\sigma}_x) = -1 \quad (15)
\]

\[
W(\hat{\sigma}_y) = -i \frac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} \quad (16)
\]

\[
W(\hat{\sigma}_z) = \frac{\cos \alpha + \sin \alpha}{\cos \alpha - \sin \alpha} \quad (17)
\]

The value of $W(\hat{\sigma}_x)$ is determined by the postselection; $W(\hat{\sigma}_y)$ is an example of complex-valued weak value; $W(\hat{\sigma}_z)$ can take surprisingly large values.

\[3\] For brevity, I simplified the example studied in ref. \[2\].
4 Limitation of the semiclassical argument

In order to obtain the semiclassical formula (8), the single-term assumption (cf. (4)) is essential. In this section, I discuss about the breakdown of this assumption. I focus on the coherent state representation discussed in Sec 3.1. Concerning to other pairs of initial and final states, the similar phenomena occur (more precisely, see ref. [29]). When $t'' - t'$ is small, the semiclassical Feynman kernel have a significant contribution only from the single classical trajectory that are placed around the real-valued classical trajectory. Since (phase space) caustics [17, 30], which are bifurcation points of classical trajectories, exist far from the real-valued trajectory, the influence from the caustics is small. The single-term assumption accordingly holds. As $t'' - t'$ become larger, several caustics approaches to the real-valued trajectory [31]. Consequently the influence from the caustics to the Feynman kernel become significant to produce the quantum interference phenomena between multiple classical trajectories. Hence the single-term assumption breaks down [17]. Furthermore, the caustics induce the divergence of the semiclassical amplitude factor (cf. eq. (3)). Such divergence induces the divergence of $O(h)$ contribution to $W(A)$ [8] as well as the divergence of the semiclassical evaluation of weak variance $W(A - W(A))^2)$. Although the semiclassical method itself does not breakdown even in the presence of the quantum interference, the semiclassical expressions of the weak values become complicated in general.

In summary, at the breakdown of the single-term assumption, the semiclassical evaluation encounters large fluctuations due to the caustics. After the large fluctuations, the interference between the multiple classical trajectories emerge.

5 Summary

The present argument establishes an intimate relationship between weak values and classical trajectories that appear in the semiclassical evaluations of Feynman kernels, when the quantum interference between multiple classical trajectories are negligible. In particular, it is shown that the semiclassical theory has an ability to reproduce complex-valued or surprisingly large, “anomalous”, weak values, with the help of the complex-valued classical trajectories.

Acknowledgments

A part of this work is carried out at Institute of Physics, University of Tsukuba, where this work is supported by University of Tsukuba Research Projects.

References

[1] J. von Neumann, Mathematische grundlagen der quantenmechanik (Springer Verlag, Berlin, 1932), § VI.3.
[2] Y. Aharonov, D. Z. Albert and L. Vaidman, Phys. Rev. Lett. 60 (1988) 1351.

[3] Y. Aharonov and L. Vaidman, Phys. Rev. A 41 (1990) 11.

[4] Y. Aharonov, D. Z. Albert, A. Casher and L. Vaidman, Phys. Lett. A 124 (1987) 199.

[5] N. W. M. Ritchie, J. G. Story and R. G. Hulet, Phys. Rev. Lett. 66 (1991) 1107.

[6] A. D. Parks, D. W. Cullin, and D. C. Stoudt, Proc. R. Soc. Lond. A 454 (1998) 2997.

[7] A. D. Parks, J. Phys. A. 33 (2000) 2555.

[8] M. S. Wang, Phys. Rev. Lett. 79 (1997) 3319; Phys. Rev. A 57 (1998) 1565.

[9] A. M. Steinberg, Phys. Rev. Lett. 74 (1995) 2405.

[10] M V Berry and K E Mount, Rep. Prog. Phys. 35 (1972) 315.

[11] M Berry, in Quantum Coherence and Reality, eds. J S Anandan and J L Safko (World Scientific, Singapore, 1994) p. 55.

[12] See, e.g., ref. [13], Chap. 12.

[13] D. Bohm, Quantum theory (Dover, New York, 1989).

[14] L. Landau, Physik. Z. Sowjet. 2 (1932) 46; E. C. G. Stueckelberg, Helv. Phys. Acta 5 (1932) 369; J-T. Hwang and P. Pechukas, J. Chem. Phys. 67 (1977) 4640.

[15] W. H. Miller and T. F. George, J. Chem. Phys. 56 (1972) 5668.

[16] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics (Springer-Verlag, New York, 1990).

[17] S. Adachi, Ann. Phys. 195 (1989) 45.

[18] A. Shudo and K. S. Ikeda, Phys. Rev. Lett. 74 (1995) 682; K. Takahashi and K. S. Ikeda, Ann. Phys. (N. Y.) 283 (2000) 94; T. Onishi et al., Phys. Rev. E 64 (2001) 025201.

[19] L. S. Schulman, Techniques and applications of path integration, (Wiley, New York, 1981).

[20] M. C. Gutzwiller, J. Math. Phys. 8 (1967) 1979.

[21] J. H. van Vleck, Proc. Natl. Acad. Sci. USA 14 (1928) 178.

[22] W. H. Miller, Adv. Chem. Phys. 25 (1974) 69.
[23] Y. Weissman, J. Chem. Phys. 76 (1982) 4067.

[24] J. R. Klauder, in Path Integrals, edited by G. J. Papadopoulos and J. T. Devreese, NATO Advanced Summer Institute (Plenum, New York, 1978), p. 5; in Random Media, edited by G. Papanicolaou, (Springer-Verlag, New York, 1987), p. 163.

[25] J. R. Klauder and B.-S. Skagerstam, Coherent states (World Scientific, Singapore, 1985).

[26] P. Kramer, M. Moshinsky and T. H. Seligman, in Group Theory and Its Applications, edited by E. M. Loebl (Academic Press, New York, 1975), Vol. III, p. 249.

[27] I. Daubechies and J. R. Klauder, J. Math. Phys. 26 (1985) 2239.

[28] See, e.g., ref. [13], § 9.26.

[29] R. G. Littlejohn, J. Stat. Phys. 68 (1992) 7.

[30] Ref. [14], Chap. 15.

[31] Atushi Tanaka, (unpublished).