The quantic monoid and degenerate quantized enveloping algebras

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Abstract

We study a monoid associated to complex semisimple Lie algebras, called the quantic monoid. Its monoid ring is shown to be isomorphic to a degenerate quantized enveloping algebra. Moreover, we provide normal forms and a straightening algorithm for this monoid. All these results are proved by a realization in terms of representations of quivers, namely as the monoid of generic extensions of a quiver with automorphism.

1 Introduction

In this paper, we introduce and study the quantic monoid $U$, an object associated to any complex semisimple Lie algebra $g$. Its definition is given in terms of generators and relations (Definition 2.1), which can be read off from the Cartan matrix of the Lie algebra $g$.

We show that the monoid ring $QU$ of the quantic monoid can be viewed as a degenerate quantized enveloping algebra in a natural way. More precisely, we consider a twisted form $U^+_q(g)$ of the positive part of the quantized enveloping algebra of $g$, which can be specialized at $q = 0$. This specialization is isomorphic to $QU$ (the Degeneration Theorem 2.4). Such twisted forms of quantized enveloping algebras already appear in [KT] in the type $A$ case, and in the Hall algebra approach to quantum groups [Ri1].

We give several natural normal forms for the elements of $U$ in root-theoretic terms (the Parametrization Theorem 2.10), using the concept of directed partitions of root systems introduced in [Re2]. Moreover, we provide a straightening rule for $U$ (Proposition 2.7), yielding a simple algorithm for multiplication of elements in normal form.

All these structural results are achieved by realizing the quantic monoid in terms of quiver representations, namely as the monoid of generic extensions of a quiver with automorphism (the Realization Theorem 2.5). This generalizes constructions in [Re1], which deals with the simply laced cases (see also [Re3] for a generalization to the Kac-Moody case).
The step from simply laced to arbitrary root systems shows some surprising features:

The degenerate quantized enveloping algebra is no longer defined by just specializing suitably twisted quantum Serre relations to \( q = 0 \). Instead, one has to impose additional relations (see Definition 2.1), whose nature is quite mysterious from the algebraic point of view. However, they become completely natural from the point of view of quiver representations (see Lemma 5.2).

It is well known that, in contrast to the case of enveloping algebras, there is no embedding of arbitrary quantized enveloping algebras \( \mathcal{U}^+(g) \) into simply laced ones. Our approach shows that this becomes true again in the degenerate case. Indeed, Definition 1.4, in combination with Theorem 2.5, shows that an arbitrary quantic monoid \( \mathcal{U} \) always embeds into a simply laced one. Thus, the same is true for degenerate quantized enveloping algebras by Theorem 2.4.

Whereas [Re1] is mostly formulated from the point of view of quiver representations, the results of the present paper are formulated in a purely root-theoretic language to make them easily accessible. However, all of them depend entirely on techniques from quiver representation theory. In particular, section 5 which constitutes the technical heart of the paper, makes free use of such techniques, for example, the structure of Auslander-Reiten quivers [ARS].

The paper is organized as follows:

In section 2, the quantic monoid is defined, and the structural results mentioned above are stated. They are illustrated with a detailed discussion of a quantic monoid associated to the root system of type \( B_3 \) (Example 2).

Section 3 first recollects several facts on quiver representations which are used in this paper. Several of them are generalized to the case of a quiver together with an automorphism (Lemmas 3.3 to 3.7), which is the right framework to formulate the Realization Theorem 2.5 (for similar material, see also [Hu]). Note that the alternative approach to non-simply laced root systems via species [Ri2] does not apply to the present setup, since the geometry of quiver representations is used in an essential way, requiring an algebraically closed base field.

These geometric methods are taken from [Re1], and are used in section 4 to define the monoid of generic extensions associated to a quiver with an automorphism (Definition 4.4). Several methods of [Re1] are generalized to the present setup (Lemmas 4.5 to 4.9).

In section 5, the Realization Theorem is proved in the form of Theorem 5.1. As noted above, this section makes extensive use of (quiver-) representation theoretic techniques. The theorem is first reduced to a “straightening rule” (Proposition 5.4), which is proved by a reduction to the rank 2 case via several intermediate steps (Lemmas 5.5 to 5.9).

The efforts of section 5 are finally rewarded in section 6, where all statements of section 2 are easily proved using the Realization Theorem and properties of the monoid of generic extensions from section 4.
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2 Statement of the results

Let $C = (a_{ij})_{i,j \in I}$ be a symmetrizable Cartan matrix of finite type (see [Lu]), i.e.

- $a_{ii} = 2$ for all $i \in I$ and $a_{ij} \in \{0, -1, -2, \ldots \}$ for all $i \neq j$ in $I$,
- there exist $d_i \in \mathbb{Z}$ for $i \in I$ such that the matrix $(d_i a_{ij})_{i,j}$ is symmetric,
- $C$ is positive definite.

We assume that the $d_i$ are positive and minimal. Let $(I, \leq)$ be a total ordering of $I$. We will now associate a monoid to the pair $(C, \leq)$, called the quantic monoid. Its definition might look quite arbitrary at first sight. But Theorem 2.4 below shows that this monoid is naturally related to (the quantized enveloping algebra associated to) the Cartan matrix $C$.

Definition 2.1 (Quantic monoid) Define the quantic monoid $U = U(C, \leq)$ as the monoid with set of generators $I$ and relations

$$i^p j^q i^r j^s = i^{p+r} j^{q+s}$$

if $i < j$, and $(p, q), (r, s)$ are two consecutive entries in the following list $L_{ij}$:

- $L_{ij} = ((0, 1), (1, 0))$ if $a_{ij} = 0$,
- $L_{ij} = ((0, 1), (1, 1), (1, 0))$ if $a_{ij} = -1 = a_{ji}$,
- $L_{ij} = ((0, 1), (1, 2), (1, 1), (1, 0))$ if $a_{ij} = -1, a_{ji} = -2$,
- $L_{ij} = ((0, 1), (1, 1), (2, 1), (1, 0))$ if $a_{ij} = -2, a_{ji} = -1$,
- $L_{ij} = ((0, 1), (1, 3), (1, 2), (2, 3), (1, 1), (1, 0))$ if $a_{ij} = -1, a_{ji} = -3$,
- $L_{ij} = ((0, 1), (1, 1), (3, 2), (2, 1), (2, 1), (3, 1), (1, 0))$ if $a_{ij} = -3, a_{ji} = -1$.

Remarks:

a) In the case of a simply laced Cartan matrix $C$, i.e. if $a_{ij} \in \{0, -1\}$ for all $i \neq j$ in $I$, we thus have the following relations for $i < j$ (see [Re1]):

$$ij = ji \text{ if } a_{ij} = 0, \quad \begin{align*}
&ij = ij, \quad ji = ji \text{ if } a_{ij} = -1.
\end{align*}$$

b) The pairs $(p, q)$ in $L_{ij}$ correspond - via $(p, q) \leftrightarrow p\alpha_i + q\alpha_j$ - precisely to the positive roots of the rank 2 root system spanned by the simple roots $\alpha_i, \alpha_j$; the only exception being one root in type $G_2$ which is doubled in $L_{ij}$.
c) The defining relations of $U$ can be rewritten as “framed commutation relations” for all $i < j$ in $I$:

- $[i, j] = 0$ if $a_{ij} = 0$,
- $i[i, j] = 0$, $[i, j] = 0$ if $a_{ij} = -1$,
- $i[i, j] = 0$, $ij[i, j][j] = 0$ if $a_{ij} = -1$, $a_{ji} = -2$,
- $i[i, j] = 0$, $ij[i, j]j^2 = 0$, $ij[i, j]j^2 = 0$, $ij[i, j]j^2 = 0$, $ij[i, j]j^2 = 0$, $ij[i, j]j^3 = 0$.

(The last two cases have obvious dual analogues). The equivalence of these sets of defining relations can be verified by an elementary calculation.

d) The defining relations of $U$ can also be rewritten as commutation relations $[[i, j]^q, i^r j^s] = 0$

if $i < j$, and $((p, q), (r, s))$ are two consecutive entries in $L_{ij}$, together with the relation $ijijj^3 = ij^2jj^2$ (resp. $ij^2ji^2j = ij^3j^3$) in the $G_2$ cases. Again, the equivalence of the defining relations can be verified by an elementary calculation. This reformulation is related to a straightening rule discussed at the end of this section.

We introduce some basic notation related to quantized enveloping algebras.

Let $R = R(C)$ be the root system corresponding to $C$, and let $Q = Q(R)$ be the root lattice, which we identify with $Z$ via $\alpha_i \leftrightarrow i$. Similarly, we denote by $R^+$ the set of positive roots, and by $Q^+ \simeq NI$ the positive span of $R^+$ in $Q$.

Let $U_v^+$ be the positive part of the quantized enveloping algebra (over $Q(v)$) associated to $C$ (see [Lu]); it is given by generators $E_i$ for $i \in I$ and the quantized Serre relations

$$\sum_{p+p'\sim -a_{ij}} (-1)^{p'} \left[ \begin{array}{c} p+p' \\ p \end{array} \right] d_i E_i p E_i p' = 0$$

for all $i \neq j$ in $I$,

where $\left[ \begin{array}{c} p+p' \\ p \end{array} \right]_i$ denotes the usual quantum binomial coefficients defined via the quantum numbers $[n]_i := (v_i^n - v_i^{-n})/(v_i - v_i^{-1})$, $v_i := v^{a_i}$ (see [Lu]).

The algebra $U_v^+$ is $Q^+$-graded by defining the degree of $E_i$ as $a_i \in Q^+$. The degree of a homogeneous element $x \in U_v^+$ is denoted by $|x|$.

Using the ordering on $I$, we can consider the (non-symmetric) inner product $\langle , \rangle$ on $Q$ given by

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i a_{ij}, & i < j \\ d_i, & i = j \\ 0, & i > j \end{cases}$$

(I)

Note that the symmetrization of this form is the symmetric form $(d_i a_{ij})_{ij}$, which is a Cartan datum in the sense of ([Lu], 1.1.1.). The bilinear form $\langle , \rangle$ allows us to define a variant of $U_v^+$ as follows:
**Definition 2.2** Define a new multiplication $\ast$ on $U^+_v$ by

$$x \ast y = v^{-\langle |x|,|y| \rangle} x \cdot y$$

on homogeneous elements $x, y \in U^+_v$. Set $q = v^2 \in \mathbb{Q}(v)$ and define $U^+_q$ as the $\mathbb{Q}[[q]]$-subalgebra generated by the $E_i$ for $i \in I$.

**Remarks:**

a) In other words, we produce a version of $U^+_v$ which can be specialized at $q = 0$ by “breaking the symmetry” of $U^+_v$ and considering a natural $\mathbb{Q}[[q]]$-subalgebra.

b) A short calculation using the definition of the twisted multiplication $\ast$ shows that the algebra $U^+_q$ fulfills the “desymmetrized Serre relations” (see [Ri1]) given in the lemma below. But in general, these relations are no longer defining. Examples for this can be seen in the proof of Theorem 2.4 in section 6.

**Lemma 2.3** The following relations hold in $U^+_q$:

$$\sum_{p+p'=1-a_{ij}} (-1)^{p'} \left\{ \begin{array}{c} p + p' \\ p \end{array} \right\}_i q_i^{p'(p'-1)/2} E_i E_{j'} = 0,$$

$$\sum_{p+p'=1-a_{ji}} (-1)^p \left\{ \begin{array}{c} p + p' \\ p \end{array} \right\}_j q_j^{p(p-1)/2} E_j E_{i'} = 0$$

for all $i < j$ in $I$, where $q_i = q^{d_i}$, and the quantum binomial coefficient is defined via the quantum numbers \{n\}_i = (q_i^n - 1)/(q_i - 1).

We can now formulate the first main result of this paper:

**Theorem 2.4 (Degeneration)** The specialization $\mathbb{Q} \otimes \mathbb{Q}[[q]] U^+_q$ of $U^+_q$ at $q = 0$ is isomorphic to the monoid ring $\mathbb{Q}U$.

In other words, the monoid ring $\mathbb{Q}U$ can be viewed as a degenerate quantized enveloping algebra. In particular, the theorem justifies the – at first sight – complicated defining relations of $U$.

The second result concerns an explicit realization of $U$ in terms of quiver representations. In fact, this realization is the basis for the proof of all other statements in this section. The precise formulation of the theorem will be postponed to section 5 (see Theorem 5.1).

**Theorem 2.5 (Realization)** The quantic monoid $U(C, \leq)$ is isomorphic to the monoid of generic extensions of a quiver with automorphism associated to $(C, \leq)$. 

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Choosing an enumeration \( i_1 < i_2 < \ldots < i_n \) of \( I \), we can view any element \( d = \sum_{i \in I} d_i \alpha_i \) in \( Q^+ \) as an element of \( U \) via

\[
(d) = i_1^{d_{i_1}} \cdots i_n^{d_{i_n}} \in U.
\]

To analyse the behaviour of these natural elements of \( U \) under multiplication, we need some additional notation.

Let \( NR^+ \) be the set of functions on \( R^+ \) with values in \( N \). Define the weight \( |a| \) of a function \( a \in NR^+ \) as \( |a| = \sum_{\alpha \in R^+} a(\alpha) \alpha \in Q^+ \). Using the decomposition \( d = \sum_{\alpha \in R^+} a_\alpha \alpha \) provided by the following lemma, we associate to any \( d \in Q^+ \) a function \( a_d \in NR^+ \) by \( a_d(\alpha) = a_\alpha \). We have \( |a_d| = d \).

**Lemma 2.6** Given \( d \in Q^+ \), there exists a unique decomposition

\[
d = \sum_{\alpha \in R^+} a_\alpha \alpha
\]

such that \( \langle \alpha, \beta \rangle \geq 0 \) if \( a_\alpha \neq 0 \neq a_\beta \).

Using this notation, we can formulate:

**Proposition 2.7 (Straightening rule)** Given \( d, e \in Q^+ \), assume that

\[
\langle \beta, \alpha \rangle \geq 0 \text{ for all } \alpha, \beta \in R^+ \text{ such that } a_d(\alpha) \neq 0 \neq a_e(\beta).
\]

Then

\[
(d) \cdot (e) = (d + e) \in U.
\]

Finally, we will construct several parametrizations of the elements of \( U \) by using the concept of a directed partition (see [Re2], [Re4]).

**Definition 2.8** Define a directed partition \( I_* \) of \( R^+ \) to be a partition into disjoint subsets \( R^+ = I_1 \cup \ldots \cup I_k \) such that

1) \( \langle \alpha, \beta \rangle \geq 0 \) for \( \alpha, \beta \in I_s \), \( 1 \leq s \leq k \),

2) \( \langle \alpha, \beta \rangle \geq 0 \geq \langle \beta, \alpha \rangle \) for all \( \alpha \in I_s \), \( \beta \in I_t \), \( 1 \leq s < t \leq k \).

**Lemma 2.9** There exists an enumeration \( \alpha^1, \ldots, \alpha^\nu \) of \( R^+ \) such that \( \langle \alpha^k, \alpha^l \rangle \geq 0 \) if \( k \leq l \) and \( \langle \alpha^l, \alpha^k \rangle \leq 0 \) if \( k > l \).

**Remark:** This lemma shows that directed partitions do exist: given an enumeration as above, \( R^+ = \{ \alpha^1 \} \cup \ldots \cup \{ \alpha^\nu \} \) is obviously a directed partition.

Fix a directed partition \( I_* \). We associate an element of \( U \) to any function \( a \in NR^+ \) by

\[
(a) = \left( \sum_{\alpha \in I_1} a(\alpha) \alpha \right) \cdot \ldots \cdot \left( \sum_{\alpha \in I_k} a(\alpha) \alpha \right) \in U.
\]
Theorem 2.10 (Parametrization) The map

\[ p_\alpha : \mathbb{N}R^+ \to U, \quad a \mapsto (a) \]

is a bijection.

Remark: In other words, relative to a directed partition, we get a parametrization of the elements of \( U \), as well as a normal form for them. In fact, we will see in section 4 that Proposition 2.7 (or, more precisely, a special case of it) can be viewed as a straightening rule, which allows us to straighten an arbitrary word in the alphabet \( I \) to the form provided by the theorem.

Example: We illustrate the results of this section in a particular example. Let \( C \) be the Cartan matrix of type \( B_3 \) over the index set \( I = \{1 < 2 < 3\} \). The matrix \( C \) and the matrix representing the non-symmetric form \((I)\) are thus given, respectively, by

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -1 & 2
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & -1 & 0 \\
0 & 1 & -2 \\
0 & 0 & 2
\end{pmatrix}.
\]

The following diagram gives the positive roots, where \((l_1, l_2, l_3)\) denotes the root \( l_1\alpha_1 + l_2\alpha_2 + l_3\alpha_3 \in R^+ \). They are presented in the form of a graph (in fact, a “symmetrized” Auslander-Reiten quiver), such that there exists a path from \( \alpha \) to \( \beta \) if \( \langle \alpha, \beta \rangle > 0 \) or \( \langle \beta, \alpha \rangle < 0 \). Thus, reading the diagram from the left to the right gives an ordering as in Lemma 2.9:

\[
(111) \rightarrow (010) \rightarrow (100) \quad (011) \rightarrow (121) \rightarrow (110) \quad (001) \rightarrow (021) \rightarrow (221) \rightarrow (212) \rightarrow (112) \rightarrow (122) \rightarrow (222)
\]

The quantic monoid \( U \) has generators 1, 2 and 3, subjected to the defining relations

\[
121 = 112, \quad 212 = 122, \quad 13 = 31, \quad 323 = 233, \quad 2323 = 22233, \quad 2232 = 2223.
\]

Applying Theorem 2.10 to the trivial directed partition provided by the above enumeration of \( R^+ \), we see that each element of \( U \) can be written as

\[
3^a \cdot (23)^b \cdot (123)^c \cdot (223)^d \cdot (1223)^e \cdot 2^f \cdot (11223)^g \cdot (12)^h \cdot 1^i \quad \text{(II)}
\]

for \( a, \ldots, i \in \mathbb{N} \). The following table gives straightening rules for all the root elements \((\alpha)\) for \( \alpha \in R^+ \). The entry at position \((\alpha), (\beta)\) in the table gives a rewriting of the product \((\alpha) \cdot (\beta)\) in \( U \) in the form \((II)\). All relations are easily
verified using Proposition 2.7.

|   | 3 | 23 | 123 | 223 | 1223 | 2 | 11223 | 12 |
|---|---|----|-----|-----|------|---|-------|----|
| 3 |   |    |     |     |      |   |       |    |
| 23|   |    |     |     |      |   |       |    |
| 123|   |    |     |     |      |   |       |    |
| 223|   |    |     |     |      |   |       |    |
| 1223|   |    |     |     |      |   |       |    |
| 2 |   |    |     |     |      |   |       |    |
| 11223|   |    |     |     |      |   |       |    |
| 12 |   |    |     |     |      |   |       |    |
| 1 |   |    |     |     |      |   |       |    |

The subsets

\[ I_1 = \{(001), (011), (111)\}, \]

\[ I_2 = \{(021), (121), (010)\}, \]

\[ I_3 = \{(221), (110), (100)\}\]

form a directed partition of \( R^+ \). Theorem 2.10 implies that any element of \( U \) can be written in the form

\[ 1^c_2b+c_3a+b+c_42d+e_3f_2g+e_412g+h+i2g+h_j3g \]

for \( a, \ldots, i \in \mathbb{N} \). In other words, the set of monomials

\[ 1^{x_1}2^{x_2}3^{x_3}1^{x_4}2^{x_5}3^{x_6}1^{x_7}2^{x_8}3^{x_9} \]

such that

\[ 0 \leq x_1 \leq x_2 \leq x_3, \ 0 \leq x_4 \leq x_6, \ 2x_6 \leq x_4, \ 0 \leq x_9, \ 2x_9 \leq x_8 \leq x_7 \]

gives a parametrization of the elements of \( U \).

### 3 Quivers with automorphisms and their representations

Let \( \Gamma \) be a quiver, i.e. a finite oriented graph with set of vertices \( \Gamma_0 \). Let \( \gamma \) be an automorphism of \( \Gamma \), i.e. a bijection \( \gamma : \Gamma_0 \rightarrow \Gamma_0 \) such that for all \( i, j \in \Gamma_0 \),
There is an arrow from $i$ to $j$ if and only if there is an arrow from $\gamma_i$ to $\gamma_j$.

We always assume $\Gamma$ to be of Dynkin type, i.e., the unoriented graph underlying $\Gamma$ is assumed to be a disjoint union of Dynkin diagrams of type $A$, $D$ and $E$.

A case by case analysis shows that, if $\Gamma$ is connected and $\gamma$ is not the identity, then there are precisely the following possibilities for the pair $(\Gamma, \gamma)$ (see [Lu]):

- $\Gamma$ of type $A_{2n-1}$, $\gamma$ of order 2 (type $C_n$),
- $\Gamma$ of type $D_{n+1}$, $\gamma$ of order 2 (type $B_n$),
- $\Gamma$ of type $D_4$, $\gamma$ of order 3 (type $G_2$),
- $\Gamma$ of type $E_6$, $\gamma$ of order 2 (type $F_4$).

We associate to the quiver $\Gamma$ a Cartan matrix $\tilde{C} = (\tilde{a}_{ij})_{i,j \in \Gamma_0}$ by defining $-\tilde{a}_{ij}$ as the number of arrows between $i$ and $j$ (in either direction) for $i \neq j$. As in the previous section, we denote by $\tilde{R}^+$ and $\tilde{Q}^+$ the corresponding set of positive roots and the positive part of the root lattice, respectively.

Moreover, we associate to the pair $(\Gamma, \gamma)$ a Cartan matrix over a totally ordered index set as follows (see also ([Lu], 14.1.1):

Let $I$ be the set of $\gamma$-orbits $i$ in $\Gamma_0$, and choose a total ordering on $I$ such that $i < j$ if there exists an arrow from a vertex $i \in i$ to a vertex $j \in j$; such an ordering exists, since $\Gamma$, being a Dynkin quiver, has no oriented cycles.

**Definition 3.1** For $i \neq j$, define $-a_{ij}$ as the number of arrows between some vertex in $i$ and some vertex in $j$ (in either direction), divided by the cardinality of $i$.

It is then easy to see that $C = (a_{ij})_{i,j \in I}$ is a symmetrizable Cartan matrix, with the cardinality $d_i$ of the orbit $i$ as symmetrization index.

Using this notation, we can identify $Q^+$ with $(\tilde{Q}^+)^\gamma$, the $\gamma$-fixed elements in $\tilde{Q}^+$, via $\alpha_i \leftrightarrow \sum_{i \in I} \alpha_i$. This induces an identification of $R^+$ with the $\gamma$-symmetrizations of elements of $\tilde{R}^+$. In the following, we will freely use these identifications; in particular, we will not distinguish between $\gamma$-fixed elements $d \in (\tilde{Q}^+)^\gamma$ and their induced elements in $Q^+$.

Let $k$ be an algebraically closed field, and denote by $\text{mod}_k \Gamma$ the category of finite dimensional $k$-representations of $\Gamma$. This is an abelian $k$-linear category, since it is equivalent to the category of finite dimensional representations of the path algebra $k\Gamma$ of $\Gamma$ over $k$ (see [ARS]). The Auslander-Reiten quiver of $\Gamma$ encodes the structure of this category: its vertices correspond to the isomorphism classes of indecomposable representations in $\text{mod}_k \Gamma$, its arrows are given by irreducible maps, and there is an additional graph endomorphism $\tau$ corresponding to the Auslander-Reiten translation (see [ARS] for details).

For representations $M, N \in \text{mod}_k \Gamma$, we denote by $[M, N]$ (resp. $[M, N]^1$) the $k$-dimension of $\text{Hom}_{k\Gamma}(M, N)$ (resp. of $\text{Ext}_{k\Gamma}^1(M, N)$). We have $[\gamma M, \gamma N] = \cdots$
We define a non-symmetric bilinear form $\langle \cdot, \cdot \rangle$ (the Euler form) on $N\Gamma_0$ by $\langle i, i \rangle = 1$, and $\langle i, j \rangle$ equals the number of arrows from $i$ to $j$ in $\Gamma$ for $i \neq j$. Denoting by $\dim M = \sum_{i \in \Gamma_0} \dim_k M_i \in N\Gamma_0$ the dimension vector of a representation $M$, we have

$$\langle \dim M, \dim N \rangle = [M, N] - [M, N]^1.$$  

Via the identification of $\tilde{Q}$ and $N\Gamma_0$, the restriction of $\langle \cdot, \cdot \rangle$ to $\tilde{\gamma} \simeq Q$ identifies with the non-symmetric bilinear form (I) on $Q$ introduced in the previous section.

The graph automorphism $\gamma$ of $\Gamma$ induces an algebra automorphism $\gamma$ of $k\Gamma$, which is necessarily of finite order (since $\gamma$ is so). Given a representation $M \in \text{mod} k\Gamma$, we define a new representation $\gamma M \in \text{mod} k\Gamma$ with the same underlying $k$-vector space and the twisted multiplication $a \cdot m = \gamma(a)m$ for $a \in k\Gamma$, $m \in M$. We define $M$ to be $\gamma$-symmetric if $\gamma M \simeq M$. In this case, $M$ is called $\gamma$-indecomposable if $M$ has no $\gamma$-symmetric direct summands except 0 and $M$ itself.

For a vertex $i \in \Gamma_0$, denote by $E_i$, $P_i$, $I_i$ the simple, resp. indecomposable projective, resp. indecomposable injective representation associated to $i$. For an orbit $i \in I$, denote by $E_i$ the $\gamma$-symmetric representation $\oplus_{i \in I} E_i$; define $P_i$ and $I_i$ similarly. The support $\text{supp} M$ of a representation $M \in \text{mod} k\Gamma$ is the full subquiver of $\Gamma$ on all vertices $i \in \Gamma_0$ such that $E_i$ appears as a composition factor in $M$, or, equivalently, such that $P_i$ maps to $M$. If $M$ is $\gamma$-symmetric, we define its symmetrized support $\text{supp} M$ as the set of all $i \in I$ such that $i \in \text{supp} M$ for some $i \in i$.

**Definition 3.2** Let $U \in \text{mod} k\Gamma$ be an indecomposable representation. Let $n \geq 1$ be minimal such that $\gamma^n U \simeq U$; this number is called the symmetrization index of $U$. Then we define the $\gamma$-symmetrization $\tilde{U}$ of $U$ by $\tilde{U} = \bigoplus_{k=0}^{n-1} \gamma^k U$. Note that the representation $\tilde{U}$ is obviously $\gamma$-symmetric.

**Lemma 3.3**

a) Any $\gamma$-symmetric representation is a direct sum of $\gamma$-indecomposables, and this decomposition is unique up to isomorphisms and permutations.

b) The $\gamma$-indecomposables are of the form $\tilde{U}$ for indecomposables $U \in \text{mod} k\Gamma$.

**Proof:** Part a) follows immediately from the definitions and the Krull-Schmidt theorem for $\text{mod} k\Gamma$. To prove part b), let $M$ be $\gamma$-indecomposable, and let $U$ be an indecomposable direct summand of $M$. Since $\gamma M \simeq M$, all $\gamma^k U$ are again direct summands of $M$. Thus, $\tilde{U}$ is a direct summand of $M$ which is symmetric, and we conclude that $M \simeq \tilde{U}$ by $\gamma$-indecomposability of $M$. □
In particular, for each \( \gamma \)-orbit \( i \), we have \( E_i = \hat{E}_i \) for each vertex \( i \in i \), and similarly for \( P_i \) and \( I_i \).

By Gabriel’s theorem (see e.g. [ARS], VIII), the isomorphism classes of indecomposables in \( \text{mod} k \Gamma \) correspond bijectively to the positive roots \( \tilde{R}^+ \). Moreover, there exists a partial ordering \( \preceq \) on the isomorphism classes of indecomposables in \( \text{mod} k \Gamma \) such that \([U, V] \neq 0\) or \([V, U]^1 \neq 0\) implies \( U \preceq V \). This ordering can be defined by setting \( U \preceq V \) if there exists a chain of non-zero maps \( U = U_0 \rightarrow U_1 \rightarrow \ldots \rightarrow U_n = V \), where all \( U_k \) are indecomposable.

These results extend to the \( \gamma \)-symmetrized case. The isomorphism classes of \( \gamma \)-indecomposables thus correspond bijectively to \( \tilde{R}^+ \), and there is a partial ordering, again denoted by \( \preceq \), on the set of \( \gamma \)-indecomposables defined by \( V_1 \preceq V_2 \) if there exist indecomposable direct summands \( U_1 \) of \( V_1 \) and \( U_2 \) of \( V_2 \) such that \( U_1 \preceq U_2 \). That this indeed defines a partial ordering (more precisely, that \( \preceq \) is anti-symmetric), follows easily from the next lemma.

**Lemma 3.4** If \( V \) is \( \gamma \)-indecomposable, then \( \text{End}_{k \Gamma}(V) \) is isomorphic to \( n \) copies of \( k \), where \( n \) is the symmetrization index of \( V \), and \( \text{Ext}^1_{k \Gamma}(V, V) = 0 \).

**Proof:** By Lemma 3.3, \( V \) is of the form \( \hat{U} \) for an indecomposable \( U \in \text{mod} k \Gamma \). Suppose there exists a \( k \in \{0, \ldots, n-1\} \) such that \([U, \gamma^k U] \neq 0\) or \([\gamma^k U, U]^1 \neq 0\). Then \( U \preceq \gamma^k U \) by the properties of the partial ordering \( \preceq \) mentioned above. This leads to a chain \( U \preceq \gamma^k U \preceq \gamma^{2k} U \preceq \ldots \preceq \hat{U} \). But this implies \( U \preceq \gamma^k U \), contradicting the choice of \( n \). Since \( U \) fulfills \([U, U] = 1\) and \([U, U]^1 = 0\), this proves the lemma. \(\square\)

**Lemma 3.5** If \( V \) is \( \gamma \)-indecomposable and \( M \) is \( \gamma \)-symmetric, then \([V, V] \) divides \([M, V], [V, M], [M, V]^1 \) and \([V, M]^1 \). More precisely, if \( V = \hat{U} \), then

\[
\frac{[M, \hat{U}]}{[U, \hat{U}]} = [M, U].
\]

**Proof:** By Lemma 3.3, we have \( V \simeq \hat{U} = \oplus_{k=0}^{n-1} \gamma^k U \) for some indecomposable \( U \in \text{mod} k \Gamma \) and \( n \) as in Definition 3.2. Thus, we find

\[
[M, V] = \sum_{k=0}^{n-1} [M, \gamma^k U] = n \cdot [M, U],
\]

since \( M \) is \( \gamma \)-symmetric. By Lemma 3.4, we have \([V, V] = n \). The other parts of the lemma are proved in the same way. \(\square\)

A morphism \( f \in \text{Hom}_{k \Gamma}(M, N) = \text{Hom}_{k \Gamma}(\gamma M, \gamma N) \) between \( \gamma \)-symmetric representations \( M, N \in \text{mod} k \Gamma \) is called \( \gamma \)-symmetric if there exist isomorphisms \( \phi : \gamma M \cong M \) and \( \psi : \gamma N \cong N \) such that \( f \phi = \psi f \). It is then easy to see that kernels, images and cokernels of \( \gamma \)-symmetric morphisms are \( \gamma \)-symmetric.
Lemma 3.6 If \( M, N \in \text{mod}k\Gamma \) are \( \gamma \)-symmetric such that \([M, N] \neq 0\), then there exists a non-zero \( \gamma \)-symmetric morphism from \( M \) to \( N \).

**Proof:** By Lemma 3.3, we can assume without loss of generality that \( N = \hat{U} \) for an indecomposable \( U \) with symmetrization index \( n \). Since \([M, N] \neq 0\), we also have \([M, U] \neq 0\) by Lemma 3.5; let \( f : M \to U \) be non-zero. Under the isomorphism \( \text{Hom}_k(M, U) = \text{Hom}_k(\gamma^k M, \gamma^k U) \simeq \text{Hom}_k(M, \gamma^k U) \), the morphism \( f \) corresponds to a morphism \( \gamma^k f : M \to \gamma^k U \) for each \( k = 0, \ldots, n-1 \).

It is then easy to see that the morphism

\[
\bigoplus_{k=0}^{n-1} \gamma^k f : M \to \bigoplus_{k=0}^{n-1} \gamma^k U = N
\]

is \( \gamma \)-symmetric. \(\square\)

Given a dimension vector \( d = \sum_{i \in \Gamma_0} d_i i \in \mathbb{N}_{\Gamma_0} \), denote by \( R_d \) the affine space \( R_d = \bigoplus_{\alpha: i \to j} \text{Hom}_k(k^{d_i}, k^{d_j}) \), and by \( G_d \) the group \( G_d = \prod_{i \in \Gamma_0} \text{GL}(k^{d_i}) \). The linear reductive algebraic group \( G_d \) acts on the affine variety \( R_d \) by conjugation, i.e. by

\[
(g_i)_i \cdot (M_\alpha)_\alpha = (g_j M_\alpha g_i^{-1})_{\alpha: i \to j}.
\]

The orbits of \( G_d \) in \( R_d \) correspond bijectively to the isoclasses of representations of \( \Gamma \) of dimension vector \( d \). We denote by \( O_M \) the orbit corresponding to the isoclass \([M]\). We say that a representation \( M \) degenerates to \( N \) and write \( M \leq N \) if the orbit closure (in the Zariski topology) of the orbit \( O_M \) contains the orbit \( O_N \). This defines a partial ordering on the isoclasses. By [Be], a degeneration \( M \leq N \) implies \([U, M] \leq [U, N]\) and \([U, M]^1 \leq [U, N]^1\) for all representations \( U \).

Since there are only finitely many orbits of \( G_d \) in the affine space \( R_d \), there has to exist a dense one, whose corresponding representation is denoted by \( E_d \). Thus, we have \( E_d \leq M \) for all representations \( M \) of dimension vector \( d \). The representation \( E_d \) is characterized by the property \([E_d, E_d]^1 = 0\). The next property follows immediately.

**Lemma 3.7** If \( d \) is a \( \gamma \)-symmetric dimension vector, then the representation \( E_d \) is \( \gamma \)-symmetric.

### 4 The monoid of generic extensions

We continue to use the notation of the previous section. In particular, let \( \Gamma \) be a quiver of Dynkin type, and let \( \gamma \) be an automorphism of \( \Gamma \). The following lemmas are proved in [Re1] using the geometry of the representation varieties \( R_d \).

**Lemma 4.1** Given representations \( M, N \in \text{mod}k\Gamma \), there exists a unique (up to isomorphism) representation \( X \in \text{mod}k\Gamma \) such that
a) $X$ is an extension of $M$ by $N$, i.e. there exists an exact sequence

$$0 \to N \to X \to M \to 0,$$



b) $\dim_k \text{End}(X)$ is minimal among all extensions of $M$ by $N$.

We denote the representation $X$ provided by this lemma by $M \ast N$, and call it the generic extension of $M$ by $N$.

**Lemma 4.2** For all representations $L, M, N \in \text{mod}_k \Gamma$, we have

$$(L \ast M) \ast N \simeq L \ast (M \ast N).$$

Thus, the set of isoclasses $[M]$ of representations $M \in \text{mod}_k \Gamma$, together with the operation $[M] \ast [N] = [M \ast N]$, defines a monoid $\tilde{M}$ with unit element $[0]$, the isoclass of the zero representation.

**Lemma 4.3** IF $M, N \in \text{mod}_k \Gamma$ are $\gamma$-symmetric, then $M \ast N$ is so.

**Proof:** Applying $\gamma$ to the exact sequence defining $M \ast N$, we get an exact sequence

$$0 \to \gamma N \to \gamma(M \ast N) \to \gamma M \to 0.$$

Since $\dim_k \text{End}(\gamma(M \ast N)) = \dim_k \text{End}(M \ast N)$, both conditions of Lemma 4.1 defining the generic extension of $M$ by $N$ are fulfilled, thus $\gamma(M \ast N) \simeq M \ast N$ by uniqueness. $\square$

The $\gamma$-symmetric representations of $\text{mod}_k \Gamma$ thus form a submonoid $M$ of $\tilde{M}$.

**Definition 4.4** The submonoid $M(\Gamma, \gamma) = M \subset \tilde{M}$ is called the monoid of generic extensions of the pair $(\Gamma, \gamma)$.

**Remark:** The statements of section 2 are proved by showing that $M(\Gamma, \gamma) \simeq U(C, \leq)$, where $(C, \leq)$ is the Cartan matrix over a totally ordered index set constructed from $(\Gamma, \gamma)$ in Definition 3.1.

In the remaining part of this section, we generalize some results of (Re1, 3.) to the monoid $M$.

We enumerate the $\gamma$-indecomposables in $\text{mod}_k \Gamma$ as $V_1, \ldots, V_\nu$, in such a way that $V_k \preceq V_l$ implies $k \leq l$.

**Lemma 4.5** Any element $[M]$ of $M$ can be written as

$$[M] = [V_1]^{m_1} \ast \ldots \ast [V_\nu]^{m_\nu}$$

for certain $m_k \in \mathbb{N}$ in a unique way.
Proof: By Lemma 3.3, $M$ can be decomposed uniquely as $M \cong \bigoplus_{k=1}^{r} V_{k}^{m_{k}}$. If $k \leq l$, then $[V_{k}, V_{l}]^{1} = 0$, thus $[V_{k}] \ast [V_{l}] = [V_{k} \oplus V_{l}]$. Iterating this, we can write $[M]$ in $M$ in the desired form. \hfill \Box

Lemma 4.6 The monoid $M$ is generated by the $\gamma$-symmetrizations of the simples in $\text{mod} \Gamma$, i.e. by the $[E_{i}]$ for $i \in I$.

Proof: Let $M$ be $\gamma$-symmetric. We want to show that $[M]$ can be written as a product of the $[E_{i}]$ in $M$. By the previous lemma, we can assume without loss of generality that $M$ is $\gamma$-indecomposable. In particular, we have $[M, M]^{1} = 0$ by Lemma 3.4. Let $i$ be a sink in $\text{supp} M$. Then there exists an embedding $E_{i} \to M$, thus a $\gamma$-symmetric embedding $E_{i} \to M$ by Lemma 3.6, since $M$ is $\gamma$-symmetric. Denoting by $N$ the ($\gamma$-symmetric) cokernel of this embedding, we have $[M] = [N] \ast [E_{i}]$, since $M$ has no self-extensions. Proceeding by induction on the dimension of $M$ we are done. \hfill \Box

Proposition 4.7 Let $M, N \in \text{mod} \Gamma$ be representations without self-extensions, and assume that $[N, M]^{1} = 0$. Then $M \ast N$ has no self-extensions.

Proof: This is a slight generalization of ([Be], Theorem 4.5.). Let $X$ be the representation $E_{\dim M + \dim N}$ without self-extensions of dimension vector $\dim M + \dim N$. Then $X$ degenerates to $M \oplus N$, and we have $[N, X]^{1} \leq [N, M \oplus N]^{1} = 0$ by both ([Be], 2.1.) and the assumptions. Thus, we have

$$[N, X] - [N, M \oplus N] = [N, X]^{1} - [N, M \oplus N]^{1} = 0,$$

using the properties of the Euler form. Now Theorem 2.4. of [Be] shows that we also have an embedding $N \to X$, since $N$ embeds into $M \oplus N$, the representation $X$ degenerates to $M \oplus N$, and $[N, X] = [N, M \oplus N]$. Denoting by $L$ the cokernel of this embedding, we arrive at the situation

\[
\begin{array}{cccccc}
0 & \to & N & \to & M \ast N & \to & M & \to & 0 \\
\| & & \| & & \| & & \| & & \|
\end{array}
\]

\[
\begin{array}{cccccc}
0 & \to & N & \to & X & \to & L & \to & 0
\end{array}
\]

By Proposition 2.4. of [Re1], this yields a degeneration of $M \ast N$ to $X$. Thus $M \ast N \simeq X$ since $X$ has no self-extensions. \hfill \Box

Corollary 4.8 Let $M$ and $N$ be $\gamma$-indecomposables such that $M \preceq N$. Then $M \ast N$ has no self-extensions.

Proof: By Lemma 3.4, $\gamma$-indecomposables have no self-extensions. By the properties of the partial ordering $\preceq$, we have $[N, M]^{1} = 0$. Thus, Proposition 4.7 applies. \hfill \Box
Lemma 4.9 Let \( i_1, \ldots, i_n \) be an enumeration of \( I \) such that \( k < l \) if there exists an arrow from some vertex in \( i_k \) to some vertex in \( i_l \). Then for all \( d = \sum_{k=1}^{n} d_k \alpha_{i_k} \in Q^+ \), we have

\[
[E_d] = [E_{i_1}]^{*d_1} \ast \ldots \ast [E_{i_n}]^{*d_n} \text{ in } M.
\]

Proof: We just have to note that each representation of dimension vector \( d \) has a composition series

\[
M = M_n \supset \ldots \supset M_1 \supset M_0 = 0
\]

which successive subquotients \( M_k/M_{k-1} \simeq E_{d_k}^{d_k} \) for \( k = 1 \ldots n \), and that \( E_d \) has no self-extensions by definition. □

5 Isomorphism of the monoids \( M \) and \( U \)

In this section, we prove the Realization Theorem 2.5 in the following form:

Theorem 5.1 Let \( \Gamma \) be a quiver of Dynkin type, and let \( \gamma \) be an automorphism of \( \Gamma \). Let \( (C, \leq) \) be the pair constructed from \( (\Gamma, \gamma) \) in Definition 3.1. Then

\[
U(C, \leq) \simeq M(\Gamma, \gamma).
\]

Remark: This theorem generalizes Theorem 4.2. of [Re1], whose proof contains a gap. Namely, it is not clear whether \( U_q^+ \) is isomorphic to the generic Hall algebra over \( \mathbb{Q}[[q]] \), as used there.

We start the proof by constructing a monoid morphism from \( U = U(C, \leq) \) to \( M = M(\Gamma, \gamma) \).

Lemma 5.2 The defining relations of \( U \) hold in \( M \) if \( i \) is replaced by \([E_i]\), i.e.

\[
[E_i]^{*p} \ast [E_j]^{*q} \ast [E_i]^{*r} \ast [E_j]^{*s} = [E_i]^{*(p+r)} \ast [E_j]^{*(q+s)}
\]

for \( i < j \) in \( I \), and \( ((p, q), (r, s)) \) being two consecutive entries of the list \( L_{ij} \) as in Definition 2.4.

Proof: Without loss of generality, we can work in a rank 2 situation, i.e. we can assume \( I = \{i, j\} \). In case \( a_{ij} = 0 = a_{ji} \), we obviously have \([E_i, E_j]^1 = 0 = [E_j, E_i]^1\), and thus \([E_i] \ast [E_j] = [E_i \oplus E_j] = [E_j \ast [E_i] \ast [E_i] \ast [E_j] \ast [E_j] \ast [E_i]\). So assume that \( a_{ij} \neq 0 \). We only treat the cases where \( a_{ij} = -1 \); the other cases can be proved dually and are left to the reader. The pair \( (C, \leq) \) is then associated to the quiver

\[
i_1 \quad \quad i_2 \quad \quad i_3
\]

\[
i_1 \rightarrow j_1 \text{ if } a_{ji} = -1, 
\]

\[
i_1 \quad \quad i_2 \quad \quad i_3
\]

\[
j_1 \rightarrow j_1 \text{ if } a_{ji} = -2, 
\]

\[
i_1 \quad \quad j_1 \quad \quad i_3
\]

\[
i_1 \rightarrow j_1 \text{ if } a_{ji} = -3,
\]

\[
i_2 \rightarrow j_1
\]

\[
i_3
\]
respectively, where \( \{i_1, \ldots, i_{-a_i}\} \) forms the \( \gamma \)-orbit \( i \), and the \( \gamma \)-orbit \( j \) consists of the single element \( j_1 \).

Calculating the Auslander-Reiten quiver (see [ARS]), we get the following directed enumerations of the \( \gamma \)-indecomposables:

- \((j_1, i_1 j_1, i_1)\) if \( a_{ji} = -1 \),
- \((j_1, i_1 j_1 \oplus i_2 j_1, i_1 \oplus i_2)\) if \( a_{ji} = -2 \),
- \((j_1, i_1 j_1 \oplus i_2 j_1, i_1 i_2 i_3 j_1, i_1 i_2 i_3 j_1, i_1 \oplus i_2 \oplus i_3)\) if \( a_{ji} = -3 \),

respectively, where \( i_1^{d_1} \ldots i_k^{d_k} j_1^{e_1} \) stands for the indecomposable representation in \( \text{mod} k \Gamma \) of dimension vector \( \sum_{i=1}^k d_i i + e_j \).

Using the procedure of the proof of Lemma 4.6 or Proposition 4.7, we see that the elements of \( M \) corresponding to the above \( \gamma \)-indecomposables can be written as

- \(([E_j], [E_i] \ast [E_j], [E_i])\) if \( a_{ji} = -1 \),
- \(([E_j], [E_i] \ast [E_j]^{\ast 2}, [E_i] \ast [E_j], [E_i])\) if \( a_{ji} = -2 \),
- \(([E_j], [E_i] \ast [E_j]^{\ast 3}, [E_i] \ast [E_j]^{\ast 2}, [E_i]^{\ast 2}, [E_i]^{\ast 2}, [E_i] * [E_j], [E_i], [E_i])\) if \( a_{ji} = -3 \),

respectively. From this, we see that all pairs \((i p j q, i r j s)\) which enter in the defining relations of \( U \) correspond in \( M \) to pairs of \( \gamma \)-indecomposables satisfying \( U \preceq V \). Thus, Corollary 4.8 applies and the relations are proved by Lemma 4.9.

As a consequence, we get:

**Corollary 5.3** The map \( i \mapsto [E_i] \) for \( i \in I \) extends to a surjective monoid homomorphism \( \eta: U \rightarrow M \).

**Proof:** The map \( i \mapsto [E_i] \) extends to a monoid homomorphism since the defining relations of \( U \) hold in \( M \). By Lemma 4.6, the elements \([E_i]\) generate \( M \). \( \square \)

The main difficulty in the proof of Theorem 5.1 is to show the injectivity of the comparison map \( \eta \). We first reduce the problem to the following “straightening rule”, which is the analogue of Proposition 4.7 in \( U \):

**Proposition 5.4** Let \( M, N \in \text{mod} k \Gamma \) be \( \gamma \)-symmetric representations without self-extensions, and assume that \([N, M]^1 = 0\). Then \((\dim M) \cdot (\dim N) = (\dim M + \dim N)\) in \( U \).

Using the above comparison map \( \eta \), we can now reduce Theorem 5.1 to Proposition 5.4.

**Lemma 5.5** Theorem 5.1 holds provided that Proposition 5.4 holds.
Proof: Recall the enumeration \( V_1, \ldots, V_\nu \) of (the isoclasses of) the \( \gamma \)-indecomposables from the previous section. Assume that Proposition \( \ref{prop:main} \) holds. Let

\[
w = (\dim V_i) \cdot \ldots \cdot (\dim V_\nu)
\]

be a word in \( U \). Note that any word in \( U \) can be written in this form, since

\[
i_1 \ldots i_m = (\dim E_{i_1}) \cdot \ldots \cdot (\dim E_{i_m})
\]

using an enumeration \( i_1, \ldots, i_m \) of \( I \) as in Lemma \( \ref{lem:enumeration} \). We prove that \( w \) can be rewritten in the form

\[
w = (\dim V_1)^{n_1} \ldots (\dim V_\nu)^{n_\nu}.
\]

Assume there exists an index \( k \) such that \( i_k > i_{k+1} \) (otherwise we are done). Then \( [V_{i_{k+1}}, V_{i_k}] = 0 \) by the properties of the partial ordering \( \preceq \). Thus, we can apply Proposition \( \ref{prop:main} \) to get

\[
(\dim V_{i_k}) \cdot (\dim V_{i_{k+1}}) = (\dim V_k + \dim V_{i_{k+1}})
\]

in \( U \). Set \( X = E_d \) for \( d = \dim V_{i_k} + \dim V_{i_{k+1}} \). Writing \( X = \oplus_l V_l^{m_l} \) as in the proof of Lemma \( \ref{lem:induction} \), we have \( [V_l, V_{i_k}] = 0 \) whenever \( m_l \neq 0 \neq m_{i_k} \). Moreover, we have \( i_{k+1} \leq l \leq i_k \) whenever \( m_l \neq 0 \) by \( \gamma \)-indecomposability of \( V_{i_k}, V_{i_{k+1}} \).

Applying Proposition \( \ref{prop:main} \) again several times, we get

\[
(\dim V_{i_k} + \dim V_{i_{k+1}}) = (\dim V_1)^{m_1} \ldots (\dim V_{i_k})^{m_{i_k}} \ldots (\dim V_{i_{k+1}})^{m_{i_{k+1}}}
\]

in \( U \). Putting these two equations together, we arrive at the following rewriting of \( w \):

\[
(\dim V_i) \cdot \ldots \cdot (\dim V_{i_{k-1}}) \cdot (\dim V_{i_k})^{m_1} \cdot \ldots \cdot (\dim V_{i_{k+1}})^{m_{i_{k+1}}} \cdot \ldots \cdot (\dim V_{i_{m}}).
\]

After a finite number of such rewritings, we obviously arrive at the desired form

\[
w = (\dim V_1)^{n_1} \ldots (\dim V_\nu)^{n_\nu}.
\]

But the image of \( w \) under the map \( \eta \) is, by definition, the product

\[
\eta(w) = [V_1]^{*n_1} \ast \ldots \ast [V_\nu]^{*n_\nu},
\]

proving the injectivity of \( \eta \) by Lemma \( \ref{lem:injectivity} \). We conclude using Corollary \( \ref{cor:injectivity} \) that the map \( \eta \) is a bijection.

To prove Proposition \( \ref{prop:main} \), we perform a sequence of reductions, until finally arriving at a rank 2 situation.

Lemma 5.6 Proposition \( \ref{prop:main} \) holds provided it holds for all pairs \( M, N \in \text{mod } k\Gamma \) of \( \gamma \)-symmetric representations without self-extensions such that \( [N, M]^1 = 0 \), but
\( [N, V]^1 \neq 0 \) for all \( \gamma \)-symmetric non-zero proper subrepresentations \( V \) of \( M \),

\( [W, M]^1 \neq 0 \) for all \( \gamma \)-symmetric non-zero proper factor representations \( W \) of \( N \).

**Proof:** Let \( M, N \) be representations as in Proposition 5.4. We only prove the first condition, the second one can be treated dually. We proceed by induction on the dimension of \( M \). Assume that there exists a \( \gamma \)-symmetric non-zero proper subrepresentation \( V \) of \( M \) such that \( [N, V]^1 = 0 \), and consider the exact sequence \( 0 \to V \to M \to X \to 0 \). We apply the functors \( \text{Hom}_{k{\Gamma}}(M, \_ ) \) and \( \text{Hom}_{k{\Gamma}}(\_ , X) \) and get surjections

\[
\text{Ext}^1_{k{\Gamma}}(M, M) \to \text{Ext}^1_{k{\Gamma}}(M, X) \to \text{Ext}^1_{k{\Gamma}}(V, X),
\]

hence \( [V, X]^1 = 0 \) since \( [M, M]^1 = 0 \). Set \( M_1 = E_{\dim X} \) and \( M_2 = E_{\dim V} \).

Since \( V \) is \( \gamma \)-symmetric, its dimension vector \( \dim V \) is \( \gamma \)-symmetric, and so is \( \dim X = \dim M - \dim V \). Hence \( M_1 \) and \( M_2 \) are \( \gamma \)-symmetric by Lemma 3.7. We have degenerations \( M_1 \leq X \) and \( M_2 \leq V \), thus \( [M_2, M_1]^1 \leq [V, X]^1 = 0 \). Thus, we can apply Proposition 4.7 and get \( M_1 \star M_2 \simeq M \).

Denote by \( W \) the representation \( W = M_2 \star N \). From the long exact sequence induced by \( \text{Hom}_{k{\Gamma}}(\_ , M_1) \) on the defining exact sequence

\[
0 \to N \to W \to M_2 \to 0,
\]

we get \([W, M_1]^1 = 0\). By induction, we can apply Proposition 5.4 to the pairs \((M_1, M_2), (M_2, N)\) (since \([N, M_2]^1 \leq [N, V]^1 = 0\) by assumption) and \((M_1, W)\) (since \([W, W]^1 = 0\) by Proposition 4.7), respectively, and get:

\[
\dim M \cdot \dim N = (\dim M_1) \cdot (\dim M_2) \cdot (\dim N) = (\dim M_1) \cdot (\dim M_2 + \dim N)
\]

\[
= (\dim M_1) \cdot (\dim W) = (\dim M_1 + \dim W) = (\dim M + \dim N).
\]

Note that, in particular, representations \( M \) and \( N \) as in the lemma have to be \( \gamma \)-indecomposable.

**Lemma 5.7** Let \( M, N \in \text{mod} k{\Gamma} \) be \( \gamma \)-symmetric representations without self-extensions such that \( [N, M]^1 = 0 \), and

\( [N, V]^1 \neq 0 \) for all \( \gamma \)-symmetric non-zero proper subrepresentations \( V \) of \( M \),

\( [W, M]^1 \neq 0 \) for all \( \gamma \)-symmetric non-zero proper factor representations \( W \) of \( N \).

Then at least one of the two following statements holds:

\( [X, M]^1 \neq 0 \) for all \( \gamma \)-symmetric non-zero proper factor representations \( X \) of \( M \),

\( [X, N]^1 \neq 0 \) for all \( \gamma \)-symmetric non-zero proper factor representations \( X \) of \( N \),
• \([N, Y]^1 \neq 0\) for all \(\gamma\)-symmetric non-zero proper subrepresentations \(Y\) of \(N\).

**Proof:** Assume that there exists a \(\gamma\)-symmetric proper factor representation \(X \neq 0\) of \(M\) such that \([X, M]^1 = 0\). This defines an exact sequence

\[
0 \to V \to M \to X \to 0,
\]

and we have \([N, V]^1 \neq 0\) by assumption. Considering the long exact sequence induced by \(\text{Hom}_{k, \Gamma}(N, M)\), we get \([N, X]^1 = 0\) since \([N, M]^1 = 0\). Let \(f : N \to X\) be a \(\gamma\)-symmetric non-zero morphism provided by Lemma 3.6. From the embedding \(\text{Im} f \subset X\) and the assumption \([X, M]^1 = 0\), we deduce \([\text{Im} f, M]^1 = 0\). But \(\text{Im} f \neq 0\) is a \(\gamma\)-symmetric subrepresentation of \(N\), so we have \(\text{Im} f \cong N\), i.e. \(f\) is injective.

If there exists a \(\gamma\)-symmetric proper subrepresentation \(Y \neq 0\) of \(N\) such that \([N, Y]^1 = 0\), we can dualize the above argument to construct a \(\gamma\)-symmetric non-zero morphism \(g : Y \to M\), which has to be surjective since \([N, \text{Im} g]^1 = 0\). This would yield a chain of inequalities

\[
\dim Y \leq \dim N \leq \dim X \leq \dim M \leq \dim Y,
\]

a contradiction. \(\square\)

Assume from now on - without loss of generality - that the first case of Lemma 5.7 holds (the other case can be treated dually).

**Lemma 5.8** Assume that \(M \in \text{mod}\Gamma\) is a \(\gamma\)-symmetric representation without self-extensions such that \([X, M]^1 \neq 0\) for all \(\gamma\)-symmetric non-zero proper factor representations \(X\) of \(M\). Then \(M\) is simple, or \(\text{supp} M\) is of type \(G_2\), or \(M = P_i\) for the quiver

\[
\bullet \to \ldots \to j_1
\]

\[
i_1
\]

\[
\bullet \to \ldots \to j_2,
\]

and \(\gamma\) is of order 2.

**Proof:** Assume that \(M\) has the above properties; in particular, \(M\) is already \(\gamma\)-indecomposable. Let \(i\) be an orbit in the support \(\text{supp} M \subset \Gamma_0\) of \(M\). Then we can choose a \(\gamma\)-symmetric non-zero homomorphism \(f : P_i \to M\). If \(f\) is surjective, then \(i\) has to be the unique source in \(\text{supp} M \subset I\) by definition. Otherwise, we get a non-split exact sequence \(0 \to \text{Im} f \to M \to X \to 0\), such that \(X \neq 0\) is \(\gamma\)-symmetric. Thus \([X, M]^1 \neq 0\) by assumption. Consider the induced exact sequence

\[
0 \to \text{Hom}_{k, \Gamma}(X, M) \to \text{End}_{k, \Gamma}(M) \to \text{Hom}_{k, \Gamma}(\text{Im} f, M) \to
\]

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→ \text{Ext}^1_{k\Gamma}(X, M) \to \text{Ext}^1_{k\Gamma}(M, M) = 0.

The image \( J \) of \( \text{Hom}_{k\Gamma}(X, M) \) in \( \text{End}_{k\Gamma}(M) \) consists entirely of non-invertible \( \gamma \)-symmetric endomorphisms, since the above short exact sequence is non-split. But since the endomorphism ring of each indecomposable in \( \text{mod}_{k\Gamma} \) is trivial, this means that \( J = 0 \), hence \([X, M] = 0\). Thus, we get \([\text{Im}f, M] \geq [M, M] + [X, M]^1\). Applying Lemma 3.5, this yields an estimate

\[
\frac{[P_i, M]}{[M, M]} \geq \frac{[\text{Im}f, M]}{[M, M]} \geq \frac{[M, M]}{[M, M]} + [X, M]^1 \geq 2.
\]

Since \( M \) is \( \gamma \)-indecomposable, we have \( M = \hat{U} \) for some indecomposable \( U \in \text{mod}_{k\Gamma} \) by Lemma 3.3. Again by Lemma 3.3, we thus have \( \sum_{i \in j} \dim_i U = [P_i, U] \geq 2 \).

Applying the above argument to all \( i \) in the support \( \text{supp} M \), we arrive at one of the following two situations:

a) \( \sum_{i \in j} \dim_i U \geq 2 \) for all \( i \in \text{supp} U \) or

b) There exists a unique source \( i \in \text{supp} U \subset I \) such that \( \hat{U} \) is a factor of \( P_i \), and \( \sum_{i \in j} \dim_i U \geq 2 \) for all \( i \neq j \in \text{supp} U \).

We start by analyzing situation a). By a direct inspection of the root systems of type \( A, D \) and \( E \) (using the classification of possible automorphisms \( \gamma \) of section 3), we conclude that \( \dim U \) has to be the maximal positive root for the root system of type \( \text{supp} U \), and the pair \( (\text{supp} U, \gamma) \) has to be one of the following:

- \( \text{supp} U \) of type \( D_4 \), \( \gamma \) of order 3,
- \( \text{supp} U \) of type \( E_8 \), \( \gamma \) trivial,
- \( \text{supp} U \) of type \( E_6 \), \( \gamma \) of order 2.

In particular, \( M = \hat{U} = U \), since the maximal root is always \( \gamma \)-symmetric. In the first case, we are done. In the second case, we choose an immediate successor \( X \) of \( U \) with respect to the ordering \( \preceq \) on indecomposables in \( \text{mod}_{\Gamma} \). Since \( \dim U \) is the maximal root, \( X \) is a proper factor of \( U \), and \([X, U]^1 = 0\) by the properties of \( \preceq \), a contradiction. In the third case, we consider again the immediate successors of \( U \). Since \( U \) is \( \gamma \)-symmetric, it belongs to the \( \tau \)-orbit of a projective indecomposable \( P_i \), where \( i \) is one of the two \( \gamma \)-fixed vertices of \( \text{supp} U \). In any case, \( U \) has an odd number of immediate successors, so among them, there is a \( \gamma \)-symmetric one \( X \). Arguing as in the second case, we obtain a contradiction.

So assume we are in situation b). Note that \( \dim_i P_i \) equals 1 if there exists a path from \( i \) to \( j \) in \( \Gamma \), and 0 otherwise. Using this, we can again proceed by a direct inspection of the root systems, and we arrive at one of the following situations:
• $U$ is simple,
• $\text{supp } U$ of type $D_4$, $\gamma$ of order 3,
• $\text{supp } U$ is the quiver

\[
\begin{array}{c}
\bullet \\
\downarrow \gamma \\
\bullet \rightarrow \cdots \rightarrow \bullet
\end{array}
\]

$i_1$

\[
\begin{array}{c}
\bullet \\
\downarrow \gamma \\
\bullet \rightarrow \cdots \rightarrow \bullet
\end{array}
\]

$\gamma$ is of order 2, and $U = P_{i_1}$.

In each of these cases, we are done.  \(\square\)

Using this lemma, we can now perform the final reduction.

**Lemma 5.9** Assume that $M$ and $N$ are as in Lemma 5.4. Then $\text{supp } M \cup \text{supp } N \subset I$ is at most of rank 2.

**Proof:** In the second case of Lemma 5.8, i.e. $\text{supp } M$ being of type $G_2$, there is nothing to prove, since then $I$ has to be of type $G_2$. So assume we are in the first or the third case. Let $i$ be in $\text{supp } N$. Then there exists a $\gamma$-symmetric non-zero morphism $P_i \rightarrow N$. If $N$ is not a factor of $P_i$, then an argument similar to the one in the proof of Lemma 5.8 shows that $[P_i, M] \neq 0$, which means $i \in \text{supp } M$. Again as in the proof of Lemma 5.8, we arrive at one of the following situations:

• $\text{supp } N \subset \text{supp } M$ or
• there exists a unique source $i \in \text{supp } N$ such that $N$ is a factor of $P_i$, and $\text{supp } N \setminus \{i\} \subset \text{supp } M$.

Dually, we see that

• $\text{supp } M \subset \text{supp } N$ or
• there exists a unique sink $i \in \text{supp } M$ such that $M$ is a subrepresentation of $I_i$, and $\text{supp } M \setminus \{i\} \subset \text{supp } N$.

This analysis gives us enough information to prove the lemma. In case $M$ is simple, this is obvious. So assume that $M$ is as in the third case of Lemma 5.8.

In particular, $\text{supp } M$ is of type $C_n$. If $\text{supp } N$ is not contained in $\text{supp } M$, then the second of the above situations applies. Then $\Gamma$ has to be one of the following quivers:

\[
\begin{array}{c}
1 \rightarrow 2 \\
\downarrow \\
4 \rightarrow 6
\end{array}
\]

\[
\begin{array}{c}
3 \\
\uparrow \\
1 \rightarrow 2 \\
\downarrow \\
4 \rightarrow 6
\end{array}
\]

or

\[
\begin{array}{c}
3 \rightarrow 5 \\
\uparrow \\
1 \rightarrow 2 \\
\downarrow \\
4 \rightarrow 6
\end{array}
\]

or

\[
\begin{array}{c}
3 \\
\uparrow \\
1 \rightarrow 2 \\
\downarrow \\
4 \rightarrow 6
\end{array}
\]

or

\[
\begin{array}{c}
3 \rightarrow 5 \\
\uparrow \\
1 \rightarrow 2 \\
\downarrow \\
4 \rightarrow 6
\end{array}
\]
M = P_2, and \(N\) a factor of \(P_1\). In both cases, we easily get a contradiction by direct inspection of the Auslander-Reiten quivers. If \(\text{supp} \, M\) is contained in \(\text{supp} \, N\), then we use the possible situations for \(\text{supp} \, M\) to conclude that \(\text{supp} \, M \setminus \{j_1, j_2\} \subset \text{supp} \, N \subset \text{supp} \, M\). Again, this makes the possible cases for \(M\) and \(N\) explicit, and we can conclude that \(\text{supp} \, M \cup \text{supp} \, N\) already is of type \(B_2\) by inspection of the Auslander-Reiten quiver of \(\text{supp} \, M\). The details are left to the reader. \(\square\)

Thus, we only have to study the rank 2 cases to prove Proposition 5.4.

**Proof of Proposition 5.4**: The possible rank 2 situations are listed at the beginning of this section. Using the Auslander-Reiten quivers, one easily enumerates all possible pairs \((M, N)\) which are as in Lemma 5.7. Apart from trivial relations as \((i) \cdot (j) = (i + j)\), the relations claimed in Proposition 5.4 are precisely the defining relations of \(\mathcal{U}\).

We conclude that Theorem 5.1 is proved.

6 Proofs of the statements of section 2

Using the Realization Theorem 5.1, we can now easily prove all the statements of section 3. We start with the Degeneration Theorem 2.4.

**Proof of Theorem 2.4**: Denote by \(\mathcal{U}_q^+\) the specialization \(\mathbb{Q} \otimes_{\mathbb{Z}[q]} \mathcal{U}_q^+\) of \(\mathcal{U}_q^+\) at \(q = 0\). First we show that the map \(i \mapsto E_i\) extends to an algebra homomorphism \(\theta : \mathcal{U} \rightarrow \mathcal{U}_0^+\). Thus, we have to verify that the defining relations of \(\mathcal{U}\) hold in \(\mathcal{U}_0^+\).

Without loss of generality, we can assume to be in the rank 2 case. In case \(a_{ij} = 0 = a_{ji}\), there is nothing to prove; so assume by symmetry that \(a_{ij} = -1\).

In case \(a_{ij} = -1\), the \(q\)-Serre relations of Lemma 2.3 directly specialize at \(q = 0\) to the defining relations \(iji = i^2j, jij = ij^2\) of \(\mathcal{U}\).

In case \(a_{ji} = -2\), denote by

\[
S^+ = E_i^2E_j - (q^2 + 1)E_iE_jE_i + q^2E_jE_i^2,
\]

\[
S^- = E_j^2E_i - (q^2 + 1)E_jE_iE_i + q^2E_iE_j - q^3E_iE_iE_iE_i
\]

the elements defining the \(q\)-Serre relations, i.e. \(S^+ = S^- \in \mathcal{U}_0^+\). Specializing these elements to \(q = 0\) gives the relations \(iji = i^2j\) and \(jij = ij^2\). To get the third relation \(ij^2ij = i^2j^3\), consider the element

\[
q^{-1}(S^+E_j^2 - E_iS^-) = E_iE_jE_iE_j^2 + qE_jE_iE_jE_j - (q^2 + q + 1)E_iE_jE_iE_j + q^2E_iE_jE_jE_i
\]

This is a well-defined element of \(\mathcal{U}_0^+\), which by definition equals zero in this algebra. It specialises to \(ij^2ij = ij^2ij\) at \(q = 0\), so we derive

\[
ij^2ij = (ijj)^2 = (i^2j)^2 = i^2j^3,
\]
as desired. Note however that this relation is not a consequence of the first two relations in U.

We proceed similarly in case \( a_{ji} = -3 \). The calculations get quite involved, so we only sketch them here. Again, define elements

\[
S^+ = E_i^3 E_j - (q^3 + 1)E_i E_j E_i + q^3 E_j E_i^2,
\]
\[
S^- = E_i E_j^4 - (q^3 + q^2 + q + 1)E_i E_j E_i^3 + q(q^4 + q^3 + 2q^2 + q + 1)E_j^2 E_i E_j^2 -
\]
\[-q^3(q^3 + q^2 + q + 1)E_i^3 E_j E_i + q^6 E_i^4 E_i.
\]

Define the following elements inductively; direct calculations show that each of them in fact belongs to \( U_+^q \):

\[
X = q^{-1}(S^+ E_j^3 - E_i S^-),
\]
\[
U = q^{-1}(XE_j - E_i E_j S^-),
\]
\[
V = q^{-2}E_i X + (q^{-1} + q^{-2})(S^+ E_i E_j - S^+ E_i E_j - E_i E_j S^+)E_j^2,
\]
\[
Y = q^{-1}(VE_j + E_i U - 2E_i^2 E_j S^- + S^+ E_i E_j^4).
\]

Specializing the elements \( S^-, U, X, Y, V, S^+ \) at \( q = 0 \) gives (after some tedious calculations) the six defining relations of U, in their order of appearance in the list \( L_{ij} \). For example, let us verify the relation \( E_i E_j^2 E_i E_j^3 = E_i^3 E_j^3 \), assuming that the other five defining relations of U hold in \( U_+^0 \). The element \( Y \) evaluates to

\[
\begin{align*}
-2q^4 - 2q^3 - 3q^2 - 2q - 1 & \cdot E_i^3 E_j^3 E_i E_j^2 + (q^2 + q) \cdot E_i^2 E_j^2 E_i E_j^3 + \\
-2q^4 + q^2 + 2 & \cdot E_i E_j E_i E_j^3 + (2q^3 + 3q^2 + 2q^2 + q) \cdot E_i^2 E_j^2 E_i E_j + \\
-2q^4 - q^4 & \cdot E_i^3 E_j^3 E_i + (q^2 - q - 1) \cdot E_j E_i^3 E_j^4 + \\
(q + 1) & \cdot E_j E_i^3 E_j E_i E_j^3 + (-q - 1) \cdot E_i E_j^2 E_i E_j^3,
\end{align*}
\]

which proves that \( Y \) belongs to \( U_+^q \). Specializing to \( q = 0 \) yields the relation

\[-E_i^2 E_j^3 E_i E_j^2 + 2E_i E_j E_i E_j^3 E_j^4 - E_i E_j^3 E_i^3 E_j^4 + E_j E_i^2 E_j E_i E_j E_j^3 - E_i E_j^2 E_i E_j^2 E_j^3 = 0. \] (III)

Using the five relations already known, we have

\[
E_i E_j E_i E_j^3 = E_j E_i^3 E_j^3 = E_j E_i E_j^3,
\]
\[
E_i E_j E_i E_j^4 = E_i E_j E_i E_j^4 = E_i E_j E_i E_j^3 E_j =
\]
\[
E_i E_j E_i E_j^3 = E_j E_i^3 E_i^3
\]
and

\[
E_i^2 E_j^3 E_i E_j = E_i^2 E_j E_i E_j E_j = E_i^2 E_j^3 E_j = E_i^2 E_j E_i E_j^3.
\]
Substituting these relations in (III) gives the desired relation
\[ E_i E_j E_k = E_k E_j E_i. \]
The other relations are treated similarly.

So we see that \( \theta \) extends to an algebra homomorphism. It is obviously surjective, since \( U_0^+ \) is generated by the elements \( E_i \) for \( i \in I \).

To prove injectivity, we consider the natural \( Q^+ \)-gradings on \( QU \), \( U^+ \) and \( U_i^+ \), respectively, which are given by setting the degree of the generating element \( i \) (resp. \( E_i \)) to \( \alpha_i \in Q^+ \). By definition of \( U_i^+ \), we have the following chain of inequalities for each \( d \in Q^+ \):
\[
\dim_{Q(d)} (U_i^+) d \leq \dim_{Q(q)} (U_i^+) d \leq \dim_{Q(q)} (U_i^+) d.
\]
Since the quantized enveloping algebra \( U_i^+ \) has a PBW type basis ([14], Corollary 40.2.2.), the leftmost term of the above chain equals the value \( P(d) \) of Kostant’s partition function at \( d \). On the other hand, the rightmost term equals the number of isoclasses of \( \gamma \)-symmetric representations of dimension vector \( d \). By Lemma 3.3 and the fact that the roots for \( C \) correspond to the \( \gamma \)-symmetrizations of roots for \( \tilde{C} \), the rightmost term also equals \( P(d) \). We conclude that equality holds in each step of the above chain, and that the map \( \theta : QU \to U_i^+ \) is already an isomorphism.

\[ \square \]

Proof of Lemma 2.6: The element \( d \in Q^+ \) corresponds to an element \( d \in (Q^+)^\gamma \). By Lemma 3.7, the unique representation without self-extensions \( M \) of dimension vector \( d \) is \( \gamma \)-symmetric, hence has a unique decomposition \( M \cong \bigoplus_{a \in R^+} V_{\alpha}^{a} \) into \( \gamma \)-indecomposables, which correspond to roots in \( R^+ \); this yields a decomposition \( d = \sum_{a \in R^+} a \alpha \). If \( a \alpha \neq 0 \neq a \beta \), then
\[
(\alpha, \beta) = (\dim_{V_{\alpha}}, \dim_{V_{\beta}}) = [V_{\alpha}, V_{\beta}] - [V_{\alpha}, V_{\beta}]_{=0} \geq 0.
\]
This proves the existence of the claimed decomposition.

To prove uniqueness, start with a decomposition \( d = \sum_{a \in R^+} a \alpha \) as in the lemma, and define a \( \gamma \)-symmetric representation \( M = \bigoplus_{a \in R^+} V_{\alpha}^{a} \). If \( [V_{\alpha}, V_{\beta}] \neq 0 \) for some \( \alpha, \beta \in R^+ \), then \( [V_{\alpha}, V_{\beta}] = 0 \), thus \( \langle \alpha, \beta \rangle < 0 \) by the properties of the partial ordering \( \leq \). This yields \([M, M] = 0 \). But since a representation without self-extensions is uniquely determined by its dimension vector, this proves uniqueness.

\[ \square \]

Proof of Proposition 2.7: Given \( d, e \in Q^+ \) as in the proposition, consider the representations without self-extensions \( M \) (resp. \( N \)) of dimension vector \( d \) (resp. \( e \)). Similar to the previous proof, the assumption that \( \langle \beta, \alpha \rangle \geq 0 \) whenever \( a_{\beta}(\alpha) \neq 0 \neq a_{\alpha}(\beta) \) translates into the property \( [N, M]^{1} = 0 \). Now Proposition 5.4 yields the desired result \( (d) \cdot (e) = (d + e) \) in \( U \).
Finally, we prove the Parametrization Theorem 2.10. It was already noted in [Re4] that the root-theoretic definition 2.8 of a directed partition is equivalent to the following representation-theoretic one:

A partition $\mathcal{I}_1 \cup \ldots \cup \mathcal{I}_k$ of the set of isoclasses of $\gamma$-indecomposables is directed if and only if

a) $[U, V] = 0$ for all $U, V \in \mathcal{I}_s$, $1 \leq s \leq k$,

b) $[U, V] = 0 = [V, U]$ for all $U \in \mathcal{I}_s$, $V \in \mathcal{I}_t$, $1 \leq s < t \leq k$.

Now Lemma 2.9 follows directly from the existence of the partial ordering $\preceq$ on $\gamma$-indecomposables.

**Proof of Theorem 2.10:** Given a $\gamma$-symmetric representation $M$ of $\Gamma$, we decompose it as $M = M_1 \oplus \ldots \oplus M_k$, where $M_s$ denotes the direct sum of all $\gamma$-indecomposable direct summands $U \in \mathcal{I}_s$ of $M$. By the definition of a directed partition, we have $[M_s, M_t] = 0$ for all $1 \leq s \leq t \leq k$. This yields the identity $[M] = [M_1] \ast \ldots \ast [M_k]$ in $M$ by definition. Using Lemma 4.9, the right hand side of this identity translates under the isomorphism $U \cong M$ into the element $(\dim M_1) \cdot \ldots \cdot (\dim M_k)$. Now these elements constitute precisely the image of the map $p_{\mathcal{I}_s} : \mathbb{N}R^+ \to U$ of the theorem, whereas the elements $[M]$ constitute precisely the elements of $M \cong U$. We see that the theorem is proved. □

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