The Throughput-Reliability Tradeoff in MIMO Channels

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February 1, 2008

Abstract
In this paper, an outage limited MIMO channel is considered. We build on Zheng and Tse's elegant formulation of the diversity-multiplexing tradeoff to develop a better understanding of the asymptotic relationship between the probability of error, transmission rate, and signal-to-noise ratio. In particular, we identify the limitation imposed by the multiplexing gain notion and develop a new formulation for the throughput-reliability tradeoff that avoids this limitation. The new characterization is then used to elucidate the asymptotic trends exhibited by the outage probability curves of MIMO channels.

1 Problem Formulation
This paper revolves around the following question: What does a 3 dB increase in the signal-to-noise ratio (SNR) buy in an outage limited Multi-Input Multi-Output (MIMO) channel? In an Additive White Gaussian Noise (AWGN) setting, it is well known that a 3 dB increase in SNR translates into an extra bit in channel capacity in the high SNR regime. The scenario considered in this paper, however, is more involved. We study an outage limited channel where the randomness of the instantaneous mutual information results in a non-zero lower bound on the probability of error, for non-zero constant transmission rates. Hence, a fundamental tradeoff between the throughput, as quantified by the transmission rate, and reliability, as quantified by the so-called outage probability, arises. Our work explores this tradeoff in the high SNR regime.

To be more specific, we consider a MIMO wireless communication system with $m$ transmit and $n$ receive antennas. We adopt a quasi-static flat-fading setup where the path gains remain constant over $l$ consecutive symbol-intervals (i.e., a block), but change independently from one block to another. We further assume a coherent communication model implying the availability of channel state information (CSI) only at the receiver. Under these assumptions, the channel input-output relation is given by:

$$y = \sqrt{\frac{2}{m}} Hx + w. \quad (1)$$
In \( \mathbf{y} \in \mathbb{C}^n \) has entries \( y_i \) representing the signal received at antenna \( i \in \{1, \cdots, n\} \), \( \mathbf{x} \in \mathbb{C}^m \) has entries \( x_j \) denoting the signal transmitted by antenna \( j \in \{1, \cdots, m\} \), \( \mathbf{H} \in \mathbb{C}^{n \times m} \) has entries \( h_{ij} \) which represents the path gain between receive antenna \( i \in \{1, \cdots, n\} \) and transmit antenna \( j \in \{1, \cdots, m\} \), and \( \mathbf{w} \in \mathbb{C}^n \) represents the unit-variance additive white Gaussian noise. We model \( \{h_{ij}\} \) as i.i.d zero-mean and unit-variance complex Gaussian random variables. Finally, \( \rho \) corresponds to the SNR at each receive antenna.

Our work builds on Zheng and Tse’s formulation of the diversity-multiplexing tradeoff \([1]\). This formulation assumes a family of space-time codes \( \{\mathcal{C}_\rho\} \) indexed by their operating SNR \( \rho \), such that the code \( \mathcal{C}_\rho \) has rate \( R(\rho) \), in bits per channel use (bpcu), and error probability \( P_e(\rho) \). For this family, the multiplexing gain \( r \) and the diversity gain \( d \) are defined by
\[
 r \triangleq \lim_{\rho \to \infty} \frac{R(\rho)}{\log \rho} \quad \text{and} \quad d \triangleq -\lim_{\rho \to \infty} \frac{\log P_e(\rho)}{\log \rho}.
\]

The optimal diversity-multiplexing tradeoff yields the maximum possible diversity gain for every value of \( r \). The main result of \([1]\) is summarized in the following theorem:

**Theorem 1** The optimal diversity gain for the coherent quasi-static MIMO channel with \( m \) transmit and \( n \) receive antennas, at multiplexing gain \( r \), is given by \( d(r) = f(r) \), where \( f(\cdot) \) is the piecewise linear function joining the points \( (k, (m-k)(n-k)) \) for \( k = 0, \ldots, \min\{m, n\} \). Moreover, there exists a code that achieves \( d(r) \) for all block lengths \( l \geq m+n-1 \).

In the sequel, we will use the notation \( d_{\max} = mn \) and \( r_{\max} = \min\{m, n\} \). To motivate our work, we use the diversity-multiplexing tradeoff to make a first attempt towards answering our central question on the utility of a 3 dB SNR gain in quasi-static MIMO channels. Using the extreme points of the tradeoff curve, i.e., \((0, d_{\max})\) and \((r_{\max}, 0)\), a reasonable conjecture is

1. At high enough SNRs, one can fix the transmission rate and obtain \( d_{\max} \) orders of decay in the outage probability (on a log-log scale) for every 10 dB gain in SNR.

2. At high enough SNRs, one can fix the outage probability and obtain a rate increase of \( r_{\max} \) bpcu for every 3 dB gain in SNR.

Fig. 1 and Fig. 2 examine the validity of this conjecture in a \( 2 \times 2 \) MIMO channel. In these figures, the transmission rates and SNR ranges are carefully chosen to illustrate the following points.

1. The slope of the outage probability curves in Fig. 1 is shown to approach the asymptotic value of \( d_{\max} = 4 \), on the log-log scale, as predicted by the first part of our conjecture. The surprising observation, however, is that for a constant outage probability a 4.5 dB gain in SNR is needed to obtain an \( r_{\max} = 2 \) bpcu increase in the throughput (To avoid fractions of a dB, the figure shows a 9 dB spacing for a 4 bpcu throughput increase). This contrasts the second part of our conjecture which predicts the need for only 3 dB for every 2 bpcu. More interestingly, this 4.5 dB horizontal spacing seems to persist as the SNR increases.

\(^1\)Unless otherwise stated, in this paper all logarithms are assumed to be in base 2.
2. Fig. 2, on the other hand, comes in close agreement with the second part of our conjecture. Here, the horizontal spacing, for a 2 bpcu increase in throughput, is seen to be 3 dB. The disagreement in this case, however, is exhibited in the fact that the slope of the outage probability curves, corresponding to fixed rates, seems to stabilize for a wide range of SNRs at a value of 2 (instead of 4).

3. Repeating the experiment for different values of $m$ and $n$ reveals the same trends, i.e., 1) Our conjecture seems to offer partially accurate predictions in certain operating regions and 2) Except for the $1 \times 1$ channel, the predictions for the outage probability rate of decay and horizontal spacing are never simultaneously accurate.

4. Overall, these disagreements clearly disprove our naive conjecture. However, it seems that the conjecture is not completely false as it offers some accurate predictions, at least in certain operating regions.

Inspired by these observations, this paper aims at developing a better understanding of the fundamental throughput-reliability tradeoff in outage limited MIMO channels. It turns out that such an understanding requires a more general formulation which is not limited by the multiplexing gain notion as defined in (2). In particular, the multiplexing gain notion limits the scenarios of interest to asymptotic lines on the $R - \log \rho$ plane as shown in Fig. 3. Our formulation, on the other hand, allows for investigating more general scenarios by relaxing this constraint. Specifically, we shed more light on the relationship between the three quantities $(R, \log \rho, P_e(R, \rho))$, in the asymptotic limit of large $\rho$, when

$$\limsup_{\rho \to \infty} \frac{R}{\log \rho} \neq \liminf_{\rho \to \infty} \frac{R}{\log \rho}.$$  

(3)

It is clear that (3) allows for investigating scenarios defined by arbitrary asymptotic trajectories in the $R - \log \rho$ plane where the multiplexing gain is not defined (Fig. ?? depicts such a trajectory). As argued in the sequel, this freedom of walking along arbitrary trajectories, on the $R - \log \rho$ plane, is the key to obtaining accurate predictions for the outage probability slopes and horizontal spacings in different operating regions. While our characterization is rigorous only in the asymptotic scenario where SNR grows to infinity (i.e., $\rho \to \infty$), we will demonstrate, via numerical results, that it yields very accurate predictions for practically relevant values of SNR.

The rest of the paper is organized as follows. In Section 2 we state our main result formulating the throughput-reliability tradeoff (TRT) for the point-to-point coherent MIMO channel, along with a sketch of the main ideas in the proof. In this section, we also present numerical results and intuitive arguments that demonstrate the utility of our results in predicting the behavior of outage probability curves in the high SNR regime. Section 3 utilizes the TRT to shed more light on the performance of various space-time architectures and further extends our investigation to Automatic Repeat reQuest (ARQ) channels. We offer few concluding remarks in Section 4. In order to enhance the flow of the paper, the proofs are collected in the Appendix.

\footnote{A more formal definition of an operating region is presented in the sequel.}
2 The Throughput-Reliability Tradeoff (TRT)

An outage is defined as the event that the instantaneous mutual information does not support the intended rate, i.e.,

$$O_{p(x)} \triangleq \{ H \in \mathbb{C}^{n \times m} | I(x; y| H = H) < R \}.$$  

Notice that the mutual information depends on both the channel realization $H$ and the input distribution $p(x)$. The outage probability $P_o(R, \rho)$ is then defined as

$$P_o(R, \rho) = \inf_{p(x)} \Pr \{ O_{p(x)} \}.$$ 

The following theorem characterizes the relationship between $R$, $\rho$, and $P_o(R, \rho)$.

**Theorem 2** For the $m \times n$ MIMO channel described by (1),

$$\lim_{\rho \to \infty} \frac{\log P_o(R, \rho) - c(k)R}{\log \rho} = -g(k), \quad (4)$$

where $P_o(R, \rho)$ denotes the outage probability at rate $R$ and SNR $\rho$. $\mathcal{R}(k)$ is defined by

$$\mathcal{R}(k) \triangleq \{ R| k + 1 > \frac{R}{\log \rho} > k \} \text{ for } k \in \mathbb{Z}, \min\{m, n\} > k \geq 0. \quad (5)$$

In (4), $c(k)$ and $g(k)$ are given by

$$c(k) \triangleq m + n - (2k + 1), \quad (6)$$

and

$$g(k) \triangleq mn - k(k + 1). \quad (7)$$

Moreover, in the degenerate case $R > \min\{m, n\} \log \rho$, $\lim_{\rho \to \infty} \log P_o(R, \rho)/\log \rho = 0$.

We refer to $g(k)$ as the reliability gain coefficient and $t(k) \triangleq g(k)/c(k)$ as the throughput gain coefficient.

**Proof:** (Sketch) Our proof follows the same lines as the proof of Theorem 1 in [1] except for the fundamental challenge that the multiplexing gain is not defined here. To handle this challenge, we judiciously choose the region $\mathcal{R}(k)$ and find a lower bound on

$$\liminf_{\rho \to \infty} \frac{\log P_o(R, \rho) - c(k)R}{\log \rho},$$

and an upper bound on

$$\limsup_{\rho \to \infty} \frac{\log P_o(R, \rho) - c(k)R}{\log \rho}.$$
and show that the two bounds coincide for the choice of \(c(k)\) and \(g(k)\) given by (6) and (7), respectively. The detailed proof is reported in the Appendix.

It is immediate to check that if the multiplexing gain is well defined, i.e., if

\[
\lim_{\rho \to \infty} \frac{R(\rho)}{\log \rho}
\]

exists, then Theorem 2 reduces to Theorem 1. In the more general case, however, Theorem 2 replaces the restrictive multiplexing gain notion with the new concept of operating regions \(R(k)\). It is worth noting that every operating region corresponds to a line segment in the diversity-multiplexing tradeoff. In fact, this correspondence inspires the following observation

\[
g(k) = d(k) - kd'(k^+), \tag{8}
\]

\[
c(k) = -d'(k^+), \tag{9}
\]

where \(d'(k^+)\) is the slope of the line segment connecting \(d(k)\) and \(d(k + 1)\).

We are now ready to investigate the asymptotic trends of the throughput-reliability tradeoff using Theorem 2. The following discussion hinges on the intuitive interpretation of equation (4).

\[
\log P_o(R, \rho) \approx c(k)R - g(k)\log \rho, \tag{10}
\]

where the approximation in (10) becomes progressively more accurate as \(\rho\) increases. Equation (10) implies that the slope of the outage curve, for a constant rate, is given by \(g(k)\), while the horizontal spacing in dB between two outage curves with a \(\Delta R\) rate difference is given by \(3\Delta R/t(k)\). The key observation here is that in order to fix the transmission rate (or outage probability), while staying in the same operating region, one must deviate from the linear trajectory imposed by the multiplexing gain notion. The following heuristic derivation of (8) and (9) further illustrates this point. To derive (8), we start from two approximate relationships obtained from the diversity-multiplexing tradeoff

\[
\log P_o(R, 2^{\log \rho}) \approx -d\left(\frac{R}{\log \rho}\right) \log \rho, \tag{11}
\]

\[
\log P_o(R, 2^{\log \rho + \Delta \log \rho}) \approx -d\left(\frac{R}{\log \rho + \Delta \log \rho}\right)(\log \rho + \Delta \log \rho). \tag{12}
\]

We further approximate \(d\left(\frac{R}{\log \rho + \Delta \log \rho}\right)\) with the first two terms of its Taylor series expansion, i.e.,

\[
d\left(\frac{R}{\log \rho + \Delta \log \rho}\right) \approx d\left(\frac{R}{\log \rho}\right) - \frac{R \times \Delta \log \rho}{\log \rho(\log \rho + \Delta \log \rho)} d'\left(\frac{R}{\log \rho}\right), \tag{13}
\]

Now (13), together with (11) and (12), gives

\[
\log P_o(R, 2^{\log \rho}) - \log P_o(R, 2^{\log \rho + \Delta \log \rho}) \approx d\left(\frac{R}{\log \rho}\right) - \frac{R}{\log \rho} d'\left(\frac{R}{\log \rho}\right). \tag{14}
\]
Realizing that the left-hand side of (14) gives the slope of \( P_o(R, \rho) \) with respect to \( \rho \), i.e. \( g(k) \), and that \( d(r) - r d'(r) \) remains constant over the line segments of \( d(r) \), we get (8). Deriving (9) follows the same lines. In particular, we first compute the horizontal spacing, \( \Delta \log \rho \), between the outage curves corresponding to rates \( R \) and \( R + \Delta R \). For this purpose, we use (11) to write

\[
\log P_o(R + \Delta R, 2^{\log \rho + \Delta \log \rho}) \approx -d \left( \frac{R + \Delta R}{\log \rho + \Delta \log \rho} \right) (\log \rho + \Delta \log \rho) \tag{15}
\]

Then we expand \( d(\frac{R+\Delta R}{\log \rho+\Delta \log \rho}) \) in a way similar to (13) and equate (11) with (15) to get

\[
\Delta \log \rho \approx -d' \left( \frac{R}{\log \rho} \right) \Delta R,
\]

\[
\Delta \log \rho \approx -\frac{-d' \left( \frac{R}{\log \rho} \right)}{g(k)} \Delta R. \tag{16}
\]

Realizing that \( \Delta \log \rho = \frac{c(k)}{g(k)} \Delta R \), and that \( d'(r) \) remains constant over the line segments of \( d(r) \), we get (9).

Revisiting our naive conjecture, we can now see that \( g(0) = mn = d_{max} \) which agrees with the first part, while \( t(\min\{n, m\} - 1) = \min\{m, n\} = r_{max} \) agrees with the second part. This explains the partial correctness of the conjecture and the fact that, except for the 1 x 1 MIMO channel, the two parts are never simultaneously accurate, since they correspond to different operating regions. We also observe that both the reliability and throughput gain coefficients exhibit a staircase behavior. Moreover, it is easy to see that \( g(k) \) is a decreasing function of \( k \), while \( t(k) \) is an increasing function of \( k \). This implies that, at a fixed rate \( R \) and for sufficiently large SNRs, as \( R/\log \rho \) increases (i.e., \( \rho \) decreases), the decay rate of \( P_o(R, \rho) \) decreases and the horizontal spacing between the outage curves corresponding to a fixed rate difference shrinks. In the following, we present numerical results that validate this observation.

Before proceeding to the numerical results, we need the following rule of thumb for determining the operating regions, for large but finite values of \( \rho \) and \( R \), such that the approximation in (10) is accurate. The operating point \((R, \rho)\) is in operating region \( \mathcal{R}(k) \) if and only if

\[
\frac{\rho^k}{2^R} \leq \delta, \quad \text{and} \quad \frac{2^R}{\rho^{k+1}} \leq \delta,
\]

where \( \delta \) is a small value which determines the accuracy of the approximation. It is now straightforward to see that the high SNR segment of Fig. 11 falls in the region \( \mathcal{R}(0) \) and, indeed, the 4 levels of diversity and 4.5 dB spacing (for every 2 bpcu throughput increase) in this figure correspond precisely to \( g(0) = 4 \) and \( t(0) = 4/3 \). Similarly, the high SNR segment of Fig. 2 falls in \( \mathcal{R}(1) \) and, again, the 2 levels of diversity and 3 dB spacing, for every 2 bpcu throughput increase, agree with \( g(1) = 2 \) and \( t(1) = 2 \). Fig. 3 and Fig. 4 are different from the previous two figures in that the high SNR segments of the outage curves fall within both of the two regions. As a result, the slope of the curves and the spacing between them change as the operating point leaves one operating region and enters another. Again, the values of the slopes, spacings, and operating points at which the change occurs (which can be read from Fig. 5) match nicely with the predictions of the TRT as formulated by Theorem 2. Fig. 8 through Fig. 11 correspond to a 3 x 3
MIMO system. As can be seen from Fig. 8, the solid segment of the curve corresponding to $R = 10$ bpcu, falls in $\mathcal{R}(1)$ and, as predicted, we observe a slope of $g(1) = 7$. It should be noted, however, that the tail of the curve corresponding to $R = 4$ bpcu is leaving $\mathcal{R}(1)$ and entering $\mathcal{R}(0)$ and thus the slope of this curve is larger than 7 (about 7.7). For the same reason, the horizontal spacing between the two curves (almost 10 dB) is larger than the value predicted for $k = 1$ (i.e., 7.7 dB). The solid segments of the outage curves corresponding to $R = 58$ and 64 bpcu in Fig. 9 fall in $\mathcal{R}(2)$, and therefore, we observe 3 levels of diversity and 3 dB of spacing, for every 3 bpcu throughput increase, which correspond precisely to $g(2) = 3$ and $t(2) = 3$. Fig. 10 and Fig. 11 depict the case where the high SNR segments fall within two different regions ($k = 2$ and $k = 1$). Again the slopes and spacings are in agreement with the predictions of the TRT.

3 Applications

The following result establishes the operational significance of Theorem 2 by showing that the optimal space-time code probability of error exhibits the same asymptotic behavior as the outage probability.

**Theorem 3** The probability of error for the optimal coding/decoding scheme used in conjunction with channel (1) satisfies

$$\lim_{\rho \to \infty} \frac{\log P_e(R, \rho) - c(k)R}{\log \rho} = -g(k),$$

(17)

where $\mathcal{R}(k)$, $c(k)$, and $g(k)$ are given by (5), (6), and (7), respectively. Moreover, there exists a coding scheme that achieves (17) for $l \geq m + n - 1$.

**Proof:** (Sketch) The proof follows the same lines as [1]. In particular, the converse is obtained via a careful use of Fano’s inequality. The achievability is established using an ensemble of Gaussian codebooks along with the appropriate use of the union bound.

One can also derive the TRT achievable by certain suboptimal space-time architectures. In this paper, we restrict our study to square V-BLAST protocols and orthogonal space-time constellations. In the V-BLAST architecture, the input stream is split into $m$ sub-streams. These sub-streams are then encoded independently and transmitted over the $m$ transmit antennas [2]. The following theorem characterizes the throughput-reliability tradeoff for this protocol when a maximum likelihood decoder is employed.

**Theorem 4** The ML error probability for a V-BLAST communication system with $m$ transmit and $m$ receive antennas satisfies

$$\lim_{\rho \to \infty} \frac{\log P_e(R, \rho) - R}{\log \rho} = -m,$$

(18)

where $\mathcal{R}_{eb}$ is given by

$$\mathcal{R}_{eb} \triangleq \{R | m > \frac{R}{\log \rho} > 0\}.$$

Moreover, there exists a coding scheme that achieves (18) for $l \geq 2m - 1$.

3The subscript “vb” stands for V-BLAST.
Proof: Please refer to the appendix for a detailed proof.

Fig. 12 depicts the outage curves corresponding to \( R = 8 \) and 12 bpcu for a \( 2 \times 2 \) V-BLAST scheme with ML decoding. As can be seen from this figure, the high SNR segments of the outage curves achieve 2 levels of diversity with a horizontal spacing of 6 dB. These values agree with \( g_{\text{vb}}(0) = 2 \) and \( t_{\text{vb}}(0) = 2 \). Fig. 13, on the other hand, compares the outage behavior of \( 2 \times 2 \) MIMO and ML V-BLAST schemes for \( R = 4, 16 \) and 32 bpcu. As can be seen from this figure, the outage curves for the two schemes coincide while \( R/\log \rho > 1 \). In particular, the curve corresponding to \( R = 32 \) bpcu is almost identical. However, for \( R/\log \rho < 1 \), the sub-optimality of the V-BLAST becomes evident. In fact, the curve corresponding to the \( 2 \times 2 \) MIMO with \( R = 4 \) bpcu approaches 4 levels of diversity very rapidly, while the curve corresponding to the V-BLAST only attains 2 levels.

Similarly, orthogonal space-time constellations allow for a simple TRT characterization. An orthogonal constellation of size \( m \), length \( l \), and rate \( k/l \) (in symbols per channel use (spcu)) is a space-time code \( X \in \mathbb{C}^{m \times l} \) such that

\[
XX^H = \left( \sum_{i=1}^{k} |x_i|^2 \right) \times I_m, \tag{19}
\]

where \( \{x_i\}_{i=1}^{k} \) denote the symbols to be sent, \( I_m \) is the \( m \times m \) identity matrix and \( X^H \) denotes the hermitian of matrix \( X \). As an example, consider the orthogonal constellation with \( m = 2 \), \( l = 2 \), and rate one, which is known as the Alamouti code. In this case

\[
X = \begin{bmatrix} x_1 & -x_2^* \\ x_2^* & x_1 \end{bmatrix},
\]

where \( x^* \) denotes the complex conjugate of \( x \). Notice that

\[
XX^H = (|x_1|^2 + |x_2|^2) \times I_2,
\]

as required by (19). The received signal matrix \( Y \in \mathbb{C}^{n \times l} \) at the destination can be written as

\[
Y = \sqrt{\frac{\rho}{m}}HX + W. \tag{20}
\]

The ML receiver performs linear processing on \( Y \) to yield the following equivalent parallel channel model

\[
\hat{y}_i = \sqrt{\frac{\rho}{m}}\|H\|^2 x_i + \tilde{w}_i, \text{ for } i = 1, \cdots, k. \tag{21}
\]

In (21), \( \|H\|^2 \) denotes the Frobenius norm of \( H \) (i.e., \( \|H\|^2 \triangleq \sum |h_{ij}|^2 \), where \( \{h_{ij}\} \) are the entries of \( H \)) and \( \{\tilde{w}_i\}_{i=1}^{k} \) are i.i.d complex Gaussian random variables of zero mean and unit variance. The following theorem gives the throughput-reliability tradeoff for orthogonal constellations.

**Theorem 5** The optimal throughput-reliability tradeoff for an orthogonal constellation of size \( m \), length \( l \), rate \( k/l \) spcu (R bpcu) and \( n \) receive antennas satisfies

\[
\lim_{\rho \to \infty} \frac{\log P_e(R, \rho) - \frac{\log(\rho)}{2}mnR}{\log \rho} = -mn, \tag{22}
\]

\( R \in \mathcal{R}_{\text{oc}} \)

\(^{4}\)The subscript “oc” stands for an orthogonal constellation.
where $\mathcal{R}_{oc}$ is given by

$$\mathcal{R}_{oc} \triangleq \{ R | \frac{k}{l} > \frac{R}{\log \rho} > 0 \},$$

Moreover, there exists an outer coding scheme (one that maps the information bits into symbols $\{x_i\}_{i=1}^k$) that achieves (22) for $k \geq mn$.

**Proof:** The proof follows immediately from Theorem 3 and the fact that the orthogonal constellation of interest effectively converts the underlying $m \times n$ MIMO channel of rate $R$ (as given by (20)) into a $1 \times mn$ channel of rate $\frac{k}{l}R$ (as given by (21)).

Fig. 14 depicts the outage curves corresponding to $R = 4$ and 8 bpcu for a $2 \times 2$ Alamouti scheme. As can be seen from this figure, the high SNR segments of the outage curves achieve 4 levels of diversity with a horizontal spacing of 12 dB. These values agree with $g_{oc}(0) = 4$ and $t_{oc}(0) = 1$. Fig. 15, on the other hand, compares the outage behavior of the $2 \times 2$ MIMO channel and the Alamouti constellation for $R = 4$, 16 and 32 bpcu. As can be seen from this figure, while the outage curves for the two schemes coincide for small rates, the sub-optimality of the Alamouti scheme becomes evident at higher values of $R$. In particular, the curves corresponding to $R = 4$ bpcu are almost identical and for $R = 32$ bpcu the curve corresponding to the Alamouti scheme lags that of MIMO by more than 40 dB.

Finally, we extend our analysis to MIMO-ARQ channels. In this setup, the transmitter starts by picking up a message from the transmission buffer. It then uses a space-time encoder to map the message to a sequence of blocks $X_p \in \mathbb{C}^{m \times l}, L \geq p \geq 1$. During transmission-round $p$, the transmitter sends $X_p$ one column at a time over its $m$ antennas. The receiver then tries to decode the message. If successful, it sends back a positive acknowledgement signal (ACK), which causes the transmitter to start sending the next message. However, if the receiver detects an error, it requests another round of transmission by feeding back a negative acknowledgement signal (NACK). The only exception to this rule is when $L$ rounds of transmission have already been sent, in which case the transmitter abandons sending the current message and goes to the next one. In this paper, we address the long-term static channel scenario, where all of the transmission-rounds corresponding to a message take place over the same channel realization. We also impose a short term power constraint on the transmitter, such that power-control is not possible. At this point, we need to distinguish between two closely related parameters, namely, the first-round transmission rate and the long-term average throughput. Assume that each message consists of $b$ information bits which means that the first-round transmission rate is $R_1 = b/l$ bpcu. Since some of the messages take more than one transmission-round to be sent, the long-term average throughput $\eta$ is strictly less than $R_1$. The gap between the two quantities, however, diminishes as the SNR grows. This is due to the fact that at high SNRs, most of the messages are decoded error-free after the first round of transmission and the ARQ retransmission-rounds are used only for those rare events in which the message does not get through with only one round of transmission. Recognizing the operational significance of $\eta$, in the following we state the TRT for ARQ channels in terms of $\eta$, rather than $R_1$.

**Theorem 6** The optimal throughput-reliability tradeoff for the coherent block-fading MIMO ARQ channel with $m$ transmit antennas, $n$ receive antennas, $L$ maximum number of transmission-rounds, under the long-term static channel and short-term power constraint
assumptions is given by\(^5\)

\[
\lim_{\rho \to \infty} \frac{\log P_e(\eta, \rho) - c_{ls}(k)\eta}{\log \rho} = -g_{ls}(k),
\]

(23)

where \(\eta\) denotes the long-term average data rate. In (23), \(R_{ls}(k)\), \(c_{ls}(k)\) and \(g_{ls}(k)\) are defined by

\[
R_{ls}(k) \triangleq \left\{ \begin{array}{ll}
\{ \eta \mid (k+1)L > \eta / \log \rho > kL \} & \text{for } k \in \mathbb{Z}, \left\lfloor \frac{\min\{m,n\}}{L} \right\rfloor > k \geq 0 \\
\{ \eta \mid \min\{m,n\} > \frac{\eta}{\log \rho} > \left\lfloor \frac{\min\{m,n\}}{L} \right\rfloor L \} & \text{for } k = \left\lfloor \frac{\min\{m,n\}}{L} \right\rfloor ,
\end{array} \right.
\]

\[
c_{ls}(k) \triangleq \frac{c(k)}{L} \quad \text{and} \quad g_{ls}(k) \triangleq g(k),
\]

respectively. \(c(k)\) and \(g(k)\) are given by (6) and (7).

**Proof**: (Sketch) The proof follows the same lines as that of Theorem 5 in [6]. In particular, the converse is obtained by lower-bounding the error probability of the ARQ protocol with that of a ML decoder that operates on the whole codeword \(\{X_p\}_{p=1}^L\). The achievability, on the other hand, is established through the use of an ensemble of Gaussian code-books, along with a bounded-distance decoder. The main idea here is to differentiate between the undetected-errors (i.e., the ones for which the receiver sends back an ACK signal) and those errors that the decoder makes after requesting \(L\) rounds of transmission. It can then be shown that, through judicious choice of decoder threshold-distance, the latter error type becomes dominant, and hence, the lower and upper bounds become tight. It is then straightforward to argue the existence of codes in the ensemble that perform at least as well as the ensemble average.

### 4 Conclusions

We have developed a new asymptotic relationship between \(P_e, \rho,\) and \(R\) in outage limited MIMO channels. By relaxing the restriction imposed by the multiplexing gain notion, our characterization sheds more light on the throughput-reliability tradeoff in the high SNR regime. We presented numerical results which validate our claim that, the throughput-reliability tradeoff offers accurate predictions on the *worth* of a 3 dB SNR gain in a MIMO wireless system operating in the high SNR regime. For our results to be valid, the only requirement is that the operating point be within certain well defined regions in the \(R - \log \rho\) plane. Characterizing the performance in the transitional regions remains an open problem.

### 5 Acknowledgment

The authors would like to thank Profs. G. Caire and M. O. Damen for inspiring discussions.

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\(^5\)The subscript “ls” stands for long-term static.
6 Appendix

6.1 Proof of Theorem 2

For the channel described by (1), an outage is defined as the event that the realized mutual information does not support the intended rate, i.e.

\[ O_p(x) \triangleq \{ H \in \mathbb{C}^{n \times m} | I(x; y| H = H) < R \}. \]  

(24)

Notice that the mutual information depends on both, the realized channel \( H \) and the input distribution \( p(x) \). For this channel, the outage probability \( P_o(R, \rho) \) is defined as

\[
P_o(R, \rho) = \inf_{p(x)} \Pr\{O_p(x)\}.
\]

It is shown in [1] that

\[
P_o(R, \rho) \leq \Pr\{\log \det(I_n + \frac{\rho}{m} \mathbf{H} \mathbf{H}^H) < R\} \quad \text{and} \quad P_o(R, \rho) \geq \Pr\{\log \det(I_n + \rho \mathbf{H} \mathbf{H}^H) < R\}.
\]

These equations can be re-written as

\[
P_o(R, \rho) \leq \Pr\{\log(\prod_{i=1}^{\min\{m,n\}} (1 + \frac{\rho}{m} \mu_i)) < R\} \quad \text{and}
\]

\[
P_o(R, \rho) \geq \Pr\{\log(\prod_{i=1}^{\min\{m,n\}} (1 + \rho \mu_i)) < R\},
\]

(25)

(26)

where \( \mu_{\min\{m,n\}} \geq \cdots \geq \mu_1 \geq 0 \) represent the ordered eigenvalues of \( \mathbf{H} \mathbf{H}^H \). The joint Probability Density Function (PDF) of \( (\mu_1, \cdots, \mu_{\min\{m,n\}}) \) is given by the Wishart distribution, i.e.

\[
p(\mu) = K_{m,n}^{-1} \prod_{i=1}^{\min\{m,n\}} \mu_i^{\frac{m-n}{2}} \prod_{i<j} (\mu_i - \mu_j)^2 e^{-\sum_i \mu_i},
\]

(27)

where \( \mu \triangleq (\mu_1, \cdots, \mu_{\min\{m,n\}}) \) and \( K_{m,n} \) is a normalizing factor. Now, let us focus on (25) and introduce the change of variables

\[
\alpha_i \triangleq \log(1 + \frac{\rho}{m} \mu_i) \frac{R}{R}.
\]

(28)

This implies that \( \alpha_{\min\{m,n\}} \geq \cdots \geq \alpha_1 \geq 0 \). In terms of the new variables, (25) can be written as

\[
P_o(R, \rho) \leq \Pr\{\mathcal{A}\},
\]

(29)

where

\[
\mathcal{A} \triangleq \{ \alpha | \alpha_{\min\{m,n\}} \geq \cdots \geq \alpha_1 \geq 0, 1 - \sum_i \alpha_i > 0 \}.
\]

(30)
In (30), \( \alpha \equiv (\alpha_1, \ldots, \alpha_{\min\{m,n\}}) \). On the other hand, (27) becomes

\[
p(\alpha) = KR_{\min\{m,n\}} \rho^{-mn} 2^R \sum_i \alpha_i \times \prod_{i=1}^{\min\{m,n\}} (2^{\alpha_i R} - 1)^{m-n} \prod_{i<j} (2^{\alpha_i R} - 2^{\alpha_j R})^2 e^{-\sum_i \frac{m(2^{\alpha_i R} - 1)}{\rho}}, \tag{31}
\]

where \( K \equiv K_{m,n}^{-1} (\ln 2)^{\min\{m,n\}} m^{mn} \). Next, we define \( \mathcal{R}_\delta(k) \), for integer \( k \)'s, as

\[
\mathcal{R}_\delta(k) \equiv \begin{cases} 
\{ R \frac{1}{2} > \log \frac{\rho}{R} > 1 + \delta \} & \text{if } k = 0 \\
\{ R \frac{1}{k+1} - \delta > \log \frac{\rho}{R} > \frac{1}{k+1} + \delta \} & \text{if } k \in \mathbb{Z}, \min\{m,n\} > k > 0
\end{cases}, \tag{32}
\]

where \( \delta \) denotes a small positive value. Notice that \( \delta = 0 \) reduces \( \mathcal{R}_\delta(k) \) to \( \mathcal{R}(k) \), as given by (3). Now, it follows from (29) that

\[
P_o(R, \rho) 2^{-c(k)R} \leq 2^{-c(k)R} \int_A p(\alpha) d\alpha, \quad R \in \mathcal{R}_\delta(k)
\]

Note that this expression is true, regardless of the choice for \( c(k) \), i.e., at this point we regard \( c(k) \) as an arbitrary function of \( k \). This inequality can be written as

\[
P_o(R, \rho) 2^{-c(k)R} \leq A_1(R, \rho, \epsilon) + A_2(R, \rho, \epsilon), \quad R \in \mathcal{R}_\delta(k)
\]

where

\[
A_1(R, \rho, \epsilon) \equiv 2^{-c(k)R} \int_{A_1} p(\alpha) d\alpha, \quad A_2(R, \rho, \epsilon) \equiv 2^{-c(k)R} \int_{A_2} p(\alpha) d\alpha
\]

and

\[
A_1 \equiv \{ \alpha \in \mathcal{A} | \alpha_{\min\{m,n\}} > \frac{\log \rho}{R} + \epsilon \}, \quad A_2 \equiv \{ \alpha \in \mathcal{A} | \frac{\log \rho}{R} + \epsilon > \alpha_{\min\{m,n\}} \}. \tag{33}
\]

This means that

\[
\limsup_{\rho \to \infty} \frac{\log P_o(R, \rho) - c(k)R}{\log \rho} \leq \limsup_{\rho \to \infty} \frac{\log (1 + A_1(R, \rho, \epsilon)/A_2(R, \rho, \epsilon))}{\log \rho} + \frac{\log A_2(R, \rho, \epsilon)}{\log \rho}, \tag{34}
\]

To characterize the first term in the right-hand side of (33), we notice that

\[
A_1(R, \rho, \epsilon) \leq KR_{\min\{m,n\}} \rho^{-mn} 2^{-c(k)R} \times \int_{A_1} 2^R \sum_i \alpha_i \prod_{i=1}^{\min\{m,n\}} 2^{\alpha_i m-n|R} \prod_{i<j} 2^{2\alpha_i R} e^{-m(2^{\alpha_i R} - 1)} d\alpha \leq KR_{\min\{m,n\}} \rho^{-mn} 2^{-c(k)R} \int_{A_1} 2^{f(\alpha)R} e^{-m(2^{\alpha R} - 1)} d\alpha,
\]
where
\[ f(\alpha) \triangleq \sum_{i=1}^{\min\{m,n\}} \left( |m - n| + 2i - 1 \right) \alpha_i. \] (35)

Realizing that \( \text{Vol}\{A_1\} \leq 1 \), we conclude
\[ A_1(R, \rho, \epsilon) \leq KR^{\min\{m,n\}} \left( 2eR \right)^{f_1 - c(k)} e^{-m2R} e^\epsilon \rho^{-mn}, \] (36)
where
\[ f_1 \triangleq \sup_{A_1} f(\alpha). \]

On the other hand
\[
A_2(R, \rho, \epsilon) \geq KR^{\min\{m,n\}} e^{-\min\{m,n\} \rho} R^{\epsilon - c(k)R} \int_{A_2} e^{-\sum_i m2^{-\left(\log R - \alpha_i\right)R} 2R \sum_i \alpha_i} \times \prod_{i=1}^{\min\{m,n\}} (1 - 2^{-\alpha_i R}) |m-n|2^{m-n}|\alpha_i R} \prod_{i<j} (1 - 2^{-(\alpha_j - \alpha_i) R)} 2^{2\alpha_j R} d\alpha, \]
\[
\geq KR^{\min\{m,n\}} e^{-\min\{m,n\} \rho} R^{\epsilon - c(k)R} \int_{A_{e_1}} e^{-\min\{m,n\} 2^{-\epsilon R} 2R \sum_i \alpha_i} \times \prod_{i=1}^{\min\{m,n\}} (1 - 2^{-\epsilon R}) |m-n|2^{m-n}|\alpha_i R} \prod_{i<j} (1 - 2^{-\epsilon R}) 2^{2\alpha_j R} d\alpha, \] (37)
where
\[ A_{e_1} \triangleq \{ \alpha \in A_2 | \frac{\log \rho}{R} - \alpha_{\min\{m,n\}} > \epsilon_1, \alpha_1 > \epsilon_1, |\alpha_j - \alpha_i| > \epsilon_1 \forall i \neq j \}. \]

Realizing that \( e^{-2^{-\epsilon R}} \geq (1 - 2^{-\epsilon R}) \), (37) yields
\[ A_2(R, \rho, \epsilon) \geq KR^{\min\{m,n\}} e^{-\min\{m,n\} \rho} (1 - 2^{-\epsilon R})^{m(n+\min\{m,n\})} \rho^{-mn} e^{-c(k)R} \int_{A_{e_1}} 2^{f(\alpha) R} d\alpha, \]
where, as before, \( f(.) \) is given by (35). Let us define \( \alpha^* \) as
\[ \alpha^* \triangleq \arg \sup_{\alpha \in A_{e_1}} f(\alpha). \]

Then it follows from the continuity of \( f(.) \) that, for any \( \epsilon_2 > 0 \), there exists a neighborhood \( I_{\epsilon_2} \) of \( \alpha^* \), within which
\[ f(\alpha) \geq f(\alpha^*) - \epsilon_2. \]

This means that
\[
A_2(R, \rho, \epsilon) \geq KR^{\min\{m,n\}} e^{-\min\{m,n\} \rho} (1 - 2^{-\epsilon R})^{m(n+\min\{m,n\})} \rho^{-mn} e^{-c(k)R} \int_{A_{e_1} \cap I_{\epsilon_2}} 2^{f(\alpha) R} d\alpha, \]
\[
A_2(R, \rho, \epsilon) \geq KR^{\min\{m,n\}} e^{-\min\{m,n\} \rho} (1 - 2^{-\epsilon R})^{m(n+\min\{m,n\})} \rho^{-mn} 2^{f(\alpha^*) - c(k) \epsilon_2 R} \times \text{Vol}\{A_{e_1} \cap I_{\epsilon_2}\}. \] (38)
Now, from (36), (38) and the fact that 
\[
e^{-m \min\{m,n\} - (2^\epsilon \frac{f_2 - f(\alpha^*) + \epsilon}{\epsilon}) e^{-m(2^R) \epsilon}} \leq 1,
\]
we conclude that
\[
\frac{A_1(R, \rho, \epsilon)}{A_2(R, \rho, \epsilon)} \leq (1 - (2^R)^{-\frac{1}{\epsilon}})^{-m \min\{m,n\}} (2^R)^{f_2 - f(\alpha^*) + \epsilon} e^{-m(2^R) \epsilon} \Vol^{-1} \{ A_1 \cap I_{\epsilon_2} \}. \tag{39}
\]
This means that
\[
\limsup_{\rho \to \infty} \frac{\log (1 + A_1(R, \rho, \epsilon)/A_2(R, \rho, \epsilon))}{\log \rho} = 0 \tag{40}
\]
Note that (40) holds, whether \( \rho \) growing to infinity and \( R \in \mathcal{R}_\delta(k) \) result in \( R \) growing to infinity or not. This is because the right hand side of (39) decays exponentially with \( 2^R \), while it only grows polynomially with the same variable. To characterize the second term on the right-hand side of (34), we note that
\[
A_2(R, \rho, \epsilon) \leq KR^{\min\{m,n\}} \rho^{-m} 2^{-c(k)R} \int_{A_2} 2^R \Sigma_i \alpha_i \prod_{i=1}^{\min\{m,n\}} 2^{\alpha_i} \prod_{i<j} 2^{2\alpha_i} R \, d\alpha,
\]
\[
\leq KR^{\min\{m,n\}} 2^{(f_2-c(k))R} \rho^{-mn}.
\]
This means that
\[
\frac{\log A_2(R, \rho, \epsilon)}{\log \rho} \leq \log(K R^{\min\{m,n\}}) + (f_2 - c(k)) \frac{R}{\log \rho} - mn, \tag{41}
\]
where
\[
f_2 \triangleq \sup_{A_2} f(\alpha).
\]
To derive \( f_2 \), one needs to consider two different cases. The first case is when \( R \in \mathcal{R}_\delta(0) \), in which case
\[
f_2 = m + n - 1, \quad R \in \mathcal{R}_\delta(0). \tag{42}
\]
In this case, the supremum is achieved at \( \alpha^* = (0, \cdots, 0, 1). \) The second case is when \( R \in \mathcal{R}_\delta(k), \) for \( k \in \mathbb{Z} \) and \( \min\{m,n\} > k > 0 \), where
\[
f_2 = [m + n - (2k + 1)] + k(k + 1)(\log \rho \frac{R}{R} + \epsilon), \quad R \in \mathcal{R}_\delta(k), \min\{m,n\} > k > 0. \tag{43}
\]
The supremum happens at
\[
\alpha^* = (0, \cdots, 1 - k(\log R \frac{R}{R} + \epsilon), \underbrace{\log R \frac{R}{R} + \epsilon, \cdots, \log R \frac{R}{R} + \epsilon}_{k \text{ times}}).
\]
Notice that, assuming \( \epsilon \leq \delta_3 \), (32) guarantees that \( 1 - k(\log R \frac{R}{R} + \epsilon) > 0 \). Plugging in for \( f_2 \) in (41), we conclude
\[
\frac{\log A_2(R, \rho, \epsilon)}{\log \rho} \leq \log(K R^{\min\{m,n\}}) + g(k) + (mn - g(k)) \frac{R}{\log \rho} \epsilon + (\tilde{c}(k) - c(k)) \frac{R}{\log \rho}, \quad \text{for} \quad R \in \mathcal{R}_\delta(k). \tag{44}
\]
In this expression, \( g(k) \) is given by (7) and \( \bar{c}(k) \) is defined as
\[
\bar{c}(k) \triangleq m + n - (2k + 1). \tag{45}
\]
Now, from (34), (40) and (44), together with the fact that \( \epsilon \) can be made arbitrarily small, one concludes
\[
\limsup_{\rho \to \infty} \frac{\log P_o(R, \rho) - c(k)R}{\log \rho} \leq -g(k) +
\]
\[
(\bar{c}(k) - c(k)) \times \limsup_{\rho \to \infty} \frac{R}{\log \rho}. \tag{46}
\]
Next we turn our attention to (26) and introduce the following change of variables.
\[
\beta_i \triangleq \log(1 + \rho \mu_i) R. \tag{47}
\]
This implies that \( \beta_{\min{m,n}} \geq \cdots \geq \beta_1 \geq 0 \). In terms of the new variables, (26) can be written as
\[
P_o(R, \rho) \geq \Pr\{\mathcal{B}\}, \tag{48}
\]
where
\[
\mathcal{B} \triangleq \{\beta|\beta_{\min{m,n}} \geq \cdots \geq \beta_1 \geq 0, 1 - \sum_i \beta_i > 0\}. \tag{49}
\]
In (49), \( \beta \triangleq (\beta_1, \cdots, \beta_{\min{m,n}}) \). On the other hand, (27) becomes
\[
p(\beta) = KR_{\min{m,n}} \rho^{-mn} 2^{R \sum_i \beta_i} \times
\prod_{i=1}^{\min{m,n}} 2^{\beta_i R - 1} \prod_{i<j} (2^{\beta_i R - 2\beta_j R})^2 e^{-\sum_i \frac{\beta_i R - 1}{\rho}}, \tag{50}
\]
where \( K \triangleq K_{\min{m,n}}^{-1}(\ln 2)_{\min{m,n}} \). This means that
\[
P_o(R, \rho) 2^{-c(k)R} \geq 2^{-c(k)R} \int_{\mathcal{B}} p(\beta) d\beta, \quad R \in \mathcal{R}_\delta(k).
\]
Thus
\[
P_o(R, \rho) 2^{-c(k)R} \geq KR_{\min{m,n}} e^{-\min{m,n} \frac{c(k)}{\rho}} \rho^{-mn} 2^{-c(k)R} \int_{\mathcal{B}} e^{-\sum_i 2^{-\beta_i} \cdot \frac{R - \beta_i R}{\rho}} 2^{R \sum_i \beta_i} \times
\prod_{i=1}^{\min{m,n}} (1 - 2^{-\beta_i R})^{m-n} |2^{m-n}| \beta_i R \prod_{i<j} (1 - 2^{-(\beta_j - \beta_i) R})^{2^{2\beta_j R}} d\beta,
\]
\[
\geq KR_{\min{m,n}} e^{-\min{m,n} \frac{c(k)}{\rho}} \rho^{-mn} 2^{-c(k)R} \int_{\mathcal{B}_{i=1}} e^{-\min{m,n} 2^{-c(k) R} 2^{R \sum_i \beta_i} \times
\prod_{i=1}^{\min{m,n}} (1 - 2^{-c_1 R})^{m-n} |2^{m-n}| \beta_i R \prod_{i<j} (1 - 2^{-(c_1 R)})^{2^{2\beta_j R}} d\beta, \tag{51}
\]
where
\[ \mathcal{B}_{\epsilon_1} \triangleq \{ \beta \in \mathcal{B} | \log \frac{\rho}{R} > \epsilon_1 > \beta_{\min(m,n)}, \beta_1 > \epsilon_1, |\beta_j - \beta_i| > \epsilon_1 \forall i \neq j \}. \]

Realizing that \( e^{-2\epsilon_1 R} \geq (1 - 2^{-\epsilon_1 R}) \), (51) yields
\[
P_o(R,\rho)2^{-c(k)R} \geq KR^{\min(m,n)}e^{\frac{\min(m,n)}{R}}(1 - 2^{-\epsilon_1 R})^{mn+\min(m,n)}\rho^{mn}2^{-c(k)R} \int_{\mathcal{B}_{\epsilon_1}} 2f(\beta)R d\beta,
\]
where, as before, \( f(.) \) is given by (35). Let us define \( \beta^* \) as
\[ \beta^* \triangleq \arg \sup_{\beta \in \mathcal{B}_{\epsilon_1}} f(\beta). \]

Again, it follows from the continuity of \( f(.) \) that, for any \( \epsilon_2 > 0 \), there exists a neighborhood \( I_{\epsilon_2} \) of \( \beta^* \), within which
\[ f(\beta) \geq f(\beta^*) - \epsilon_2. \]

This means that
\[
P_o(R,\rho)2^{-c(k)R} \geq KR^{\min(m,n)}e^{\frac{\min(m,n)}{R}}(1 - 2^{-\epsilon_1 R})^{mn+\min(m,n)}\rho^{mn}2^{-c(k)R} \int_{\mathcal{B}_{\epsilon_1} \cap I_{\epsilon_2}} 2f(\beta)R d\beta,
\]
Thus
\[
\liminf_{\rho \to \infty} \frac{\log P_o(R,\rho) - c(k)R}{\log \rho} \geq -mn + \int_{\mathcal{B}_{\epsilon_1} \cap I_{\epsilon_2}} f(\beta^*) - c(k) - \epsilon_2 \rho \log \rho.
\]

Since (52) is valid for arbitrarily small values of \( \epsilon_1 \) and \( \epsilon_2 \), we conclude
\[
\liminf_{\rho \to \infty} \frac{\log P_o(R,\rho) - c(k)R}{\log \rho} \geq -g(k) + (\tilde{c}(k) - c(k)) \times \liminf_{\rho \to \infty} \frac{R \log \rho}{\log \rho},
\]
where we have used the fact that \( \mathcal{B}_0 = \mathcal{A}_{2|\epsilon=0} \), which means that \( f(\beta^*) \) can be easily derived from (42) and (43) by simply plugging in \( \epsilon = 0 \). Examining (46) and (53) reveals that the choice
\[ c(k) = \tilde{c}(k), \forall k \]
guarantees the existence of
\[
\lim_{\rho \to \infty} \frac{\log P_\sigma(R, \rho) - c(k)R}{\log \rho} = -g(k),
\] (55)

regardless of whether \(\lim_{\rho \to \infty} \frac{R}{\log \rho}\) exits or not. Now, since (55) holds for arbitrarily small
values of \(\delta\), we get (4). Note that (54), together with (45), result in (6) and thus complete
the proof.

6.2 Proof of Theorem 3

The proof follows that of Theorem 2 in [1]. In particular, we prove (17) in two steps. The first step is to show that
\[
\liminf_{\rho \to \infty} \frac{\log P_e(R, \rho) - c(k)R}{\log \rho} \geq -g(k).
\] (56)

Towards this end, let us denote the ML error probability, conditioned on a certain channel
realization \(H\), by \(P_{E|H}\). It then follows that
\[
P_e(R, \rho) = E_H \{P_{E|H}(R, \rho)\},
\]
\[
= \int P_{E|H}(R, \rho)p(H)dH.
\]
Thus
\[
P_e(R, \rho) \geq \int_C P_{E|H}(R, \rho)p(H)dH,
\] (57)
where \(C\) denotes any subset of the set of all channel realizations. Let us define \(C_\epsilon\) as
the set of channel realizations for which the conditional ML error probability cannot be
made smaller than \(\epsilon\), i.e.
\[
C_\epsilon \triangleq \{H|P_{E|H}(R, \rho) \geq \epsilon\}. \quad (58)
\]

From (57) and (58) one concludes that
\[
P_e(R, \rho) \geq \epsilon P_{C_\epsilon}(R, \rho),
\]
where
\[
P_{C_\epsilon}(R, \rho) = \Pr\{C_\epsilon\}. \quad (59)
\]
This means that
\[
\frac{\log P_e(R, \rho) - c(k)R}{\log \rho} \geq \frac{\log P_{C_\epsilon}(R, \rho) - c(k)R}{\log \rho} + \frac{\log \epsilon}{\log \rho},
\]
where \(c(k)\) is given by (6), or
\[
\liminf_{\rho \to \infty} \frac{\log P_e(R, \rho) - c(k)R}{\log \rho} \geq \liminf_{\rho \to \infty} \frac{\log P_{C_\epsilon}(R, \rho) - c(k)R}{\log \rho}, \quad (60)
\]
\[
R \in \mathcal{R}_\delta(k)
\]
where $\mathcal{R}_\delta(k)$ is given by (32). Now, application of Fano’s inequality reveals that

$$P_{E|H}(R, \rho) \geq 1 - \frac{I(x; y|H = H)}{R} - \frac{1}{Rl},$$

where $l$ denotes the codeword length [1]. This, together with (58), means that

$$\{H|1 - \frac{I(x; y|H = H)}{R} - \frac{1}{Rl} \geq \epsilon\} \subseteq \mathcal{C}_\epsilon,$$

which using (24) results in

$$P_{C_\epsilon}(R, \rho) \geq P_o((1 - \epsilon - \frac{1}{Rl})R, \rho) \text{ or }$$

$$\lim_{\rho \to \infty} \frac{\log P_{C_\epsilon}(R, \rho) - c(k)R}{\log \rho} \geq \lim_{\rho \to \infty} \frac{\log P_o((1 - \epsilon)R, \rho) - c(k)R}{\log \rho}. \quad (61)$$

Now, from (60) and (61), together with the fact that both, $\delta$ and $\epsilon$ can be made arbitrarily small, we conclude that

$$\lim_{\rho \to \infty} \frac{\log P_e(R, \rho) - c(k)R}{\log \rho} \geq \lim_{\rho \to \infty} \frac{\log P_o(R, \rho) - c(k)R}{\log \rho}. \quad (62)$$

But, from Theorem 2 (refer to (11)), we know that the right-hand side equals $-g(k)$ (given by (7)). This proves (56) and thus completes the first step.

The second step in proving (17) is to show that

$$\limsup_{\rho \to \infty} \frac{\log P_e(R, \rho) - c(k)R}{\log \rho} \leq -g(k), \quad (62)$$

provided that the codeword length, $l$, satisfies $l \geq m + n - 1$. To prove this, consider a Gaussian code-book with $2^{Rl}$ codewords of length $l$. It is straightforward to verify that the ML error probability, conditioned on a certain channel realization, is upper-bounded by

$$P_{E|H}(R, \rho) \leq 2^{Rl} \det(I_n + \frac{\rho}{2m}HH^H)^{-l},$$

$$= 2^{Rl} \prod_{i=1}^{\min\{m, n\}} (1 + \frac{\rho}{2m} \mu_i)^{-l}, \quad (63)$$

where $\mu_{\min\{m, n\}} \geq \cdots \geq \mu_1 \geq 0$ represent the ordered eigenvalues of $HH^H$. The joint PDF of $(\mu_1, \cdots, \mu_{\min\{m, n\}})$ is given by (27). The change of variables

$$\gamma_i \triangleq \frac{\log(1 + \frac{\rho}{2m} \mu_i)}{R},$$

changes (63) and (27), into

$$P_{E|\gamma}(R, \rho) \leq 2^{(1 - \sum_{i=1}^{\min\{m, n\}} \gamma_i)Rl} \quad (64)$$
and
\[ p(\gamma) = KR^{\min\{m,n\}} \rho^{-mn} 2^R \prod_{i=1}^{\min\{m,n\}} (2^{\gamma_i R} - 1)^{|m-n|} \prod_{i<j} (2^{\gamma_i R} - 2^{\gamma_j R})^2 e^{-\sum_{i=1}^{2m(\gamma_i R - 1)}} , \]
where \( \gamma \triangleq (\gamma_1, \ldots, \gamma_{\min\{m,n\}}) \) and \( K \triangleq K^{-1}_{m,n}(\ln 2)^{\min\{m,n\}}(2m)^{mn} \).

Next we define \( D \) as
\[ D \triangleq \{ \gamma \gamma_{\min\{m,n\}} \geq \ldots \geq \gamma_i \geq 0, 1 - \sum_{i=1}^{\min\{m,n\}} \gamma_i \geq 0 \} . \]

Referring to (64) reveals that \( D \) consists of those channel realizations for which the upper-bound on the ML error probability cannot be made arbitrarily small, even through the use of infinitely long codewords. For these channel realizations, we upper-bound \( P_{E|\gamma}(R, \rho) \) by 1, i.e.
\[ P_e(R, \rho) = P_{E,D}(R, \rho) + P_{E,D}(R, \rho), \]
\[ P_e(R, \rho) \leq P_{E,D}(R, \rho) + P_{D}(R, \rho), \quad (65) \]
where \( D^c \) denotes the complement of \( D \).

Let us first focus on \( P_{D}(R, \rho) \). Realizing that \( D \) is precisely the same set as \( A \) (refer to (30)) and that, up to a scaling factor, \( p(\gamma) \) is identical to \( p(\alpha) \) (refer to (31)), it follows immediately that
\[ \limsup_{\rho \to \infty} \frac{\log P_{D}(R, \rho) - c(k)R}{\log \rho} \leq -g(k). \quad (66) \]

Now, turning our attention back to \( P_{E,D}(R, \rho) \), we realize that
\[ \lim_{\rho \to \infty} P_{E,D}(R, \rho) 2^{-c(k)R} = 0, \quad (67) \]
which means that
\[ \limsup_{\rho \to \infty} \frac{\log P_{E,D}(R, \rho) - c(k)R}{\log \rho} = \limsup_{\rho \to \infty} \frac{\log P_{E,D}(R, \rho) - c(k)R}{\log \rho}, \quad (68) \]
where
\[ D^c_1 \triangleq \{ \gamma \notin D \mid \gamma_{\min\{m,n\}} > \frac{\log \rho}{R} \} \quad \text{and} \quad D^c_2 \triangleq \{ \gamma \notin D \mid \gamma_{\min\{m,n\}} \leq \frac{\log \rho}{R} \} . \]
Notice that (67) holds for exactly the same reason as (40) does (refer to the comment after (40)). Now, using (64), we have
\[ P_{E,D_c}(R, \rho) 2^{-c(k)R} = 2^{-c(k)R} \int_{D^c_2} P_{E|\gamma}(R, \rho)p(\gamma)d\gamma, \]
\[ \leq KR^{\min\{m,n\}} \rho^{-mn} 2^{-c(k)R} \int_{D^c_2} 2^{f(\gamma) + l(1 - \sum_{i=1}^{\min\{m,n\}} \gamma_i)} R \ d\gamma, \]
\[ \leq \frac{\rho^{(k)m\min\{m,n\}}}{2^{(k)2n^2}}. \]
where
\[ f(\gamma) \triangleq 2^{\sum_i (|m-n|+2i-1)\gamma_i R}. \]

Thus
\[ P_{E,D_2^c}(R,\rho)2^{-c(k)R} \leq KR^{\min\{m,n\}}\rho^{-mn}2^{(f_2-c(k))R}\Vol\{D_2^c\}, \tag{69} \]
where
\[ f_2 \triangleq \sup_{D_2^c} f(\gamma) + l(1 - \sum_{i=1}^{\min\{m,n\}} \gamma_i). \]

Realizing that for \( l \geq m+n-1 \), the supremum occurs at \( \gamma = \gamma^* \), such that \( 1 - \sum \gamma^*_i = 0 \), \( f_2 \) can be easily derived from (42) and (43) by simply plugging in \( \epsilon = 0 \). Therefore, (69) gives
\[ \limsup_{\rho \to \infty} \frac{\log P_{E,D_2^c}(R,\rho) - c(k)R}{\log \rho} \leq -g(k). \tag{70} \]

Now, from (70), (68) and (66), we conclude (62), which together with (56), proves (17).

Since we proved (17) using an ensemble of Gaussian codes, it follows that, for any code-length \( l \geq m+n-1 \), there exists at least a code, for which (17) holds. This completes the proof.

### 6.3 Proof of Theorem 4

Realizing that the V-BLAST protocol essentially transforms the \( m \times m \) MIMO channel into a multiple-access channel with \( m \) single-antenna users and a destination with \( m \) receive antennas, we prove (18) by following the same lines as that of Theorem 2 in [3].

In particular, let \( E_S \) denote the event that a certain decoder makes errors in decoding the codewords transmitted by a subset \( S \) of the antennas. It then follows that
\[ \sum_{S \neq \emptyset} \Pr\{E_S\} \geq P_e(R,\rho). \tag{71} \]

It is also clear that
\[ P_e(R,\rho) \geq \Pr\{E_{S^*}\}, \tag{72} \]
where \( S^* \) denotes any non-empty subset of \( \{1, \cdots, m\} \). Let us define \( S^* \) as the non-empty subset of \( \{1, \cdots, m\} \), such that for all other non-empty subsets \( S \), we have
\[ \liminf_{\rho \to \infty} \left( g_S(k) - g_{S^*}(k^*) \right) = \left( \frac{|S|}{m} c_S(k) - \frac{|S^*|}{m} c_{S^*}(k^*) \right) \frac{R}{\log \rho} \geq 0. \tag{73} \]

In (73), \( |S| \) denotes the cardinality of set \( S \). Also, \( \mathcal{R}_S(k) \), \( c_S(k) \) and \( g_S(k) \) denote \( \mathcal{R}(k) \), \( c(k) \) and \( g(k) \), as defined by (5), (6) and (7), for a MIMO channel with \( |S| \) transmit and
receive antennas. The proof of (18) then follows in three steps. First, we prove that for any decoder
\[
\lim_{\rho \to \infty} \frac{P_e(R, \rho) - \frac{|S^*|}{m} c_{S^*}(k) R}{\log \rho} \geq -g_{S^*}(k^*). \tag{74}
\]
This follows immediately from (72) and the fact that \( \Pr\{E_{S^*}\} \) upper-bounds the ML error probability of a MIMO channel with \(|S^*|\) transmit antennas, \(m\) receive antennas and rate \(\frac{|S^*|}{m} R\). The second step in proving (18) is to show that there exists a code, along with a decoder, for which
\[
\lim_{\rho \to \infty} \frac{P_e(R, \rho) - \frac{|S^*|}{m} c_{S^*}(k^*) R}{\log \rho} \leq -g_{S^*}(k^*). \tag{75}
\]
We prove this by showing that the error probability of the joint ML decoder, averaged over the ensemble of Gaussian codes, satisfies (75). The existence of the desired code then follows from the fact that there exist codes in the ensemble that perform at least as well as the average. For this purpose, assume that each of the antennas uses a Gaussian code-book of code-word length \(l = 2m + 1\) and size \(2^{2l} m\) codewords. It then follows from Theorem 3 (refer to (17)), that
\[
\lim_{\rho \to \infty} \frac{\log \Pr\{E_S\} - \frac{|S|}{m} c_S(k) R}{\log \rho} = -g_S(k), \ \forall S \neq \emptyset, \tag{76}
\]
which means
\[
\lim_{\rho \to \infty} \left[ \frac{\log \Pr\{E_S\} / \Pr\{E_{S^*}\}}{\log \rho} + \left( g_S(k) - g_{S^*}(k^*) \right) - \left( \frac{|S|}{m} c_S(k) - \frac{|S^*|}{m} c_{S^*}(k^*) \right) \frac{R}{\log \rho} \right] = 0. \tag{77}
\]
Now, (77), together with (73), results in
\[
\lim_{\rho \to \infty} \frac{\log \Pr\{E_S\} / \Pr\{E_{S^*}\}}{\log \rho} \leq 0, \ \forall S \neq \emptyset. \tag{78}
\]
Returning to (71), we have
\[
\frac{\log(1 + \sum_{S \neq S^*, \emptyset} \frac{\Pr\{E_S\}}{\Pr\{E_{S^*}\}})}{\log \rho} + \frac{\log \Pr\{E_{S^*}\} - \frac{|S^*|}{m} c_{S^*}(k^*) R}{\log \rho} \geq \frac{\log P_e(R, \rho) - \frac{|S^*|}{m} c_{S^*}(k^*) R}{\log \rho}.
\]
Taking the lim sup of both sides, together with (78) and (76) results in (75). Notice that (74) and (75) mean that
\[
\lim_{\rho \to \infty} \frac{P_e(R, \rho) - \frac{|S^*|}{m} c_{S^*}(k)R}{\log \rho} = -g_{S^*}(k^*). \tag{79}
\]

The third and last step in proving (18) is to show that \(|S^*| = 1\). One can prove this directly using the definitions of \(R(k), c(k)\) and \(g(k)\) (equations (5), (6) and (7)). However, we choose to do this using observations (8) and (9). In particular, notice that based on these observations, (73) reduces to finding the subset \(S^*\), for which
\[
d_S\left(\frac{|S|}{m} r\right) - d_{S^*}\left(\frac{|S^*|}{m} r\right) \geq 0, \quad \forall S \neq \emptyset,
\]
where \(d_S(r)\) represents the diversity-multiplexing tradeoff for a MIMO system with \(|S|\) transmit and \(m\) receive antennas. From the proof of Theorem 3 in [3], we know that \(|S^*| = 1\). Now, this together with (79) results in (18) and thus completes the proof.

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Figure 1: Outage curves corresponding to $R = 4, 8$ bpcu, for a $2 \times 2$ MIMO channel.

Figure 2: Outage curves corresponding to $R = 28, 32$ bpcu, for a $2 \times 2$ MIMO channel.
Figure 3: The notion of multiplexing gain restricts the scenarios of interest to those in which $R$ asymptotically scales linearly with $\log \rho$, i.e. $R \sim r \log \rho$.

Figure 4: Relaxing the constraint imposed by the multiplexing gain notion; A multiplexing gain cannot be defined for the depicted trajectory, however, since it remains well within an operating region (i.e. $\mathcal{R}(1)$) TRT analysis can be applied.
Figure 5: The constant rate trajectory with $R = 20$ bpcu passes through different operating regions in a $2 \times 2$ MIMO system.

Figure 6: Outage curves corresponding to $R = 20$ bpcu for a $2 \times 2$ MIMO channel. The solid segment corresponds to the $\mathcal{R}(1)$ operating region.
Figure 7: Outage curves corresponding to $R = 20, 24$ bpcu for a $2 \times 2$ MIMO channel. The solid segments correspond to the $\mathcal{R}(1)$ operating region.

Figure 8: Outage curves corresponding to $R = 4, 10$ bpcu for a $3 \times 3$ MIMO channel. The solid segments correspond to the $\mathcal{R}(1)$ operating region.
Figure 9: Outage curves corresponding to $R = 58, 64$ bpcu for a $3 \times 3$ MIMO channel. The solid segments correspond to the $R(2)$ operating region.

Figure 10: Outage curves corresponding to $R = 40$ bpcu for a $3 \times 3$ MIMO channel. The solid segment corresponds to the $R(2)$ operating region.
Figure 11: Outage curves corresponding to $R = 34, 40$ bpcu for a $3 \times 3$ MIMO channel. The solid segments correspond to the $\mathcal{R}(2)$ operating region.

Figure 12: Outage curves corresponding to $R = 8, 12$ bpcu for a $2 \times 2$ V-BLAST scheme.
Figure 13: Comparison of outage curves corresponding to $R = 4, 16, 32$ bpcu for the $2 \times 2$ MIMO channel and the V-BLAST scheme.

Figure 14: Outage curves corresponding to $R = 4, 8$ bpcu for the $2 \times 2$ Alamouti scheme.
Figure 15: Comparison of outage curves corresponding to $R = 4, 16, 32$ bpcu for the $2 \times 2$ MIMO channel and the Alamouti scheme.