A LOWER BOUND ON THE WIDTH OF SATELLITE KNOTS

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ABSTRACT. Thin position for knots in $S^3$ was introduced by Gabai in [2] and has been used in a variety of contexts. We conjecture an analogue to a theorem of Schubert and Schultens concerning the bridge number of satellite knots. For a satellite knot $K$, we use the companion torus $T$ to provide a lower bound for $w(K)$, proving the conjecture for $K$ with a 2-bridge companion. As a corollary, we find thin position for any satellite knot with a braid pattern and 2-bridge companion.

1. INTRODUCTION

Thin position for knots in $S^3$ was introduced by Gabai in [2] and has since been studied extensively. Although thin position has been used in a variety of different proofs, there are relatively few methods for putting specific knots into thin position. Thin position of a knot always provides a useful surface; either a level sphere is a bridge sphere for the knot or the thinnest thin sphere is incompressible in the complement of the knot, as shown by Wu [8].

In some sense, width can be considered to be a refinement of bridge number, although recently it has been shown in [11] that one cannot recover the bridge number of a knot $K$ from the thin position of $K$. On the other hand, if $K$ is small, then $w(K) = 2 \cdot b(K)^2$ and any thin position of $K$ is a bridge position. In his classic paper on the subject [6], Schubert proved that for any two knots $K_1$ and $K_2$, $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$. This was later reproved by Schultens in [7].

Unfortunately, we cannot hope for a similar statement to hold for width. In [5], Scharlemann and Schultens establish $\max\{w(K_1), w(K_2)\}$ as a lower bound for $w(K_1 \# K_2)$, and Blair and Tomova prove that this bound is tight in some cases [11], while Rieck and Sedwick [4] demonstrate that the bound is not tight for small knots. Both Schubert and Schultens also prove the following:
**Theorem 1.1.** Let \( K \) be a satellite knot with pattern \( \hat{K} \) and companion \( J \), where \( n \) is the winding number of \( \hat{K} \). Then
\[
b(K) \geq n \cdot b(J).
\]

We make an analogous conjecture:

**Conjecture 1.** Let \( K \) be a satellite knot with pattern \( \hat{K} \) and companion \( J \), where \( n \) is the winding number of \( \hat{K} \). Then
\[
w(K) \geq n^2 \cdot w(J).
\]

In this paper, we provide a weaker lower bound for \( w(K) \). Our main theorem is as follows:

**Theorem 1.2.** Let \( K \) be a satellite knot with pattern \( \hat{K} \), where \( n \) is the winding number of \( \hat{K} \). Then
\[
w(K) \geq 8n^2.
\]

This proves the conjecture in the case that the companion \( J \) is a 2-bridge knot, since the width of such \( J \) is 8. As a corollary, if \( K \) is a satellite with a 2-bridge companion and its pattern \( \hat{K} \) is a braid with index \( n \), then any thin position is a bridge position for \( K \).

## 2. Preliminaries

Let \( K \) be a knot in \( S^3 \), and let \( \mathcal{M}(K) \) denote the collection of Morse functions \( h : S^3 \to \mathbb{R} \) with exactly two critical points on \( S^3 \), denoted \( \pm \infty \), and such that \( h \mid_K \) is also Morse. (Equivalently, we could fix some Morse function \( h \) and look instead at the collection of embeddings of \( K \) into \( S^3 \).) For every \( h \in \mathcal{M}(K) \), let \( c_0 < c_1 < \cdots < c_n \) denote the critical values of \( h \mid_K \). Choose regular levels \( c_0 < r_1 < c_1 < \cdots < r_n < c_n \), and define
\[
w(h) = \sum_{i=1}^{n} |K \cap h^{-1}(r_i)|,
\]
\[
b(h) = \frac{n + 1}{2},
\]
\[
\text{trunk}(h) = \max |K \cap h^{-1}(r_i)|.
\]

Now, let
\[
w(K) = \min_{h \in \mathcal{M}(K)} w(h),
\]
\[
b(K) = \min_{h \in \mathcal{M}(K)} b(h),
\]
\[
\text{trunk}(K) = \min_{h \in \mathcal{M}(K)} \text{trunk}(h).
\]
These three knot invariants are called the width, bridge number, and the trunk of $K$, respectively. Width was defined by Gabai in [2], and trunk was defined by Ozawa in [3]. Observe that $b(K)$ is the least number of maxima of any embedding of $K$. If $h \in \mathcal{M}(K)$ satisfies $w(K) = w(h)$, we say that $h$ is a thin position for $K$. If $h \in \mathcal{M}(K)$ satisfies $b(K) = b(h)$ and all maxima of $h|_K$ occur above all minima, then we say that $h$ is a bridge position for $K$.

In [5], the authors give an alternative formula for computing width, which involves thin and thick levels. Let $h \in \mathcal{M}(K)$ with critical and regular values as defined above. Then we say $h^{-1}(r_i)$ is a thick level if $|K \cap h^{-1}(r_i)| > |K \cap h^{-1}(r_{i-1})|, |K \cap h^{-1}(r_{i+1})|$ and $h^{-1}(r_i)$ is a thin level if $|K \cap h^{-1}(r_i)| < |K \cap h^{-1}(r_{i-1})|, |K \cap h^{-1}(r_{i+1})|$, where $1 < i < n$. Note that if $h$ is a bridge position for $K$, then $h$ has exactly one thick level and no thin levels. Letting $a_1, \ldots, a_m$ denote the number of intersections of the thick levels with $K$ and $b_1, \ldots, b_{m-1}$ denote the number of intersection of the thin levels with $K$, the width of $h$ is given by

$$w(h) = \frac{1}{2} \left( \sum_{i=1}^{m} a_i^2 - \sum_{i=1}^{m-1} b_i^2 \right).$$

In particular, we see that for every $h \in \mathcal{M}(K)$, there exists $a_i \geq \text{trunk}(K)$, which implies that

$$w(K) \geq \frac{\text{trunk}(K)^2}{2}.$$

The knots we will be concerned with are satellite knots, defined below:

**Definition 2.1.** Let $\hat{K} \subset V$ be a knot contained in a solid torus $V$ with core $C$ and such that every meridian of $V$ intersects $\hat{K}$, and let $J$ be any nontrivial knot. Suppose that $\varphi : V \to S^3$ is an embedding such that $\varphi(C)$ is isotopic to $J$ in $S^3$. Then $K = \varphi(\hat{K})$ is called a satellite knot with companion $J$ and pattern $\hat{K}$.

Essentially, to construct a satellite knot $K$, we start with a pattern in a solid torus and then tie the solid torus in the shape of the companion $J$. We will need several more definitions to state the main result:

**Definition 2.2.** Let $\hat{K}$ be a pattern contained in a solid torus $V$. The winding number of $\hat{K}$, $\#(\hat{K})$, is the absolute value of the algebraic intersection number of any meridian disk of $V$ with $\hat{K}$.

Equivalently, if $\alpha : S^1 \to V$ is an embedding such that $\alpha(S^1) = \hat{K}$ and $r : V \to S^1$ is a strong deformation retract of $V$ onto its core, then $\#(\hat{K})$ agrees with the degree of the map $r \circ \alpha$. 

**Definition 2.3.** Let $\hat{K}$ be a pattern contained in a solid torus $V$. We say that $\hat{K}$ is a braid of index $n$ if there is a foliation of $V$ such that every leaf is a meridian disk intersecting $\hat{K}$ exactly $n$ times.

In the case that $\hat{K}$ is a braid of index $n$, it is clear that $\#(\hat{K}) = n$. For an example, consider Figure 1. On the left, we see a braid pattern of index 3, $\hat{K}$, contained in a solid torus $V$. On the right, $V$ is embedded in such a way that its core is a trefoil. Thus, the knot $K$ on the right is a satellite knot with trefoil companion and pattern $\hat{K}$.

### 3. Reducing the Saddle Points on the Companion Torus

From this point on, we set the convention that $K$ is a satellite knot with companion $J$ and pattern $\hat{K}$ contained in a solid torus $\hat{V}$, $\varphi$ is an embedding of $\hat{V}$ into $S^3$ that takes a core of $\hat{V}$ to $K'$, $V = \varphi(\hat{V})$, and $T = \partial V$. Further, we will let $h \in \mathcal{M}(K)$ and perturb $V$ slightly so that $h |_T$ is Morse. We wish to restrict our investigation to tori $T$ with only certain types of saddle points. In this vein, we follow [7], from which the next definition is taken.

**Definition 3.1.** Consider the singular foliation, $F_T$, of $T$ induced by $h |_T$. Let $\sigma$ be a leaf corresponding to a saddle point. Then one component of $\sigma$ is the wedge of two circles $s_1$ and $s_2$. If either is inessential in $T$, we say that $\sigma$ is an inessential saddle. Otherwise, $\sigma$ is an essential saddle.

The next lemma is the Pop Over Lemma from [7]:

**Lemma 3.2.** If $F_T$ contains inessential saddles, then after a small isotopy of $T$, there is an inessential saddle $\sigma$ in $T$ such that...
Figure 2. First, we cancel the inessential saddle, shown center. Then we isotope any part of $K$ or $T$ contained in $B$ along an increasing arc $\alpha$, increasing maxima of $T$ is necessary, so that $h \mid_{K}$ is unchanged with respect to the end product of our isotopy, shown at right.

(1) $s_1$ bounds a disk $D_1 \subset T$ such that $F_T$ restricted to $D_1$ contains only one maximum or minimum,

(2) for $L$ the level surface of $h$ containing $\sigma$, $D_1$ co-bounds a 3-ball $B$ with a disk $\tilde{D}_1 \subset L$ such that $B$ does not contain $\pm \infty$ and such that $s_2$ lies outside of $\tilde{D}_1$.

In the following lemma, we mimic Lemma 2 of [7] with a slight modification to preserve the height function $h$ on $K$:

**Lemma 3.3.** There exists an isotopy $f_t : S^3 \to S^3$ such that $f_0 = id$, $h = h \circ f_1$ on $K$, and the foliation of $T$ induced by $h \circ f_1$ contains no inessential saddles.

**Proof.** Suppose that $T$ has an inessential saddle, $\sigma$, lying in the level 2-sphere $L$. By the previous lemma, we may suppose that $\sigma$ is as described above, and suppose without loss of generality that $D_1$ contains only one maximum. By slightly pushing $D_1$ into $\text{int}(B)$, we can create a new closed ball $B'$ such that $B' \cap D_1 = \emptyset$ and $(K \cup T) \cap \text{int}(B) \subset B'$. First, we isotope $B'$ vertically until it lies below $L$, and then isotope $D_1$ down until the maximum of $D_1$ cancels out the saddle point $\sigma$. Now, there exists a monotone increasing arc beginning at the highest point of $B'$, passing through the disk $\tilde{D}_2$ bounded by $s_2$, intersecting only maxima of $T$, and disjoint from $K$. Thus, we may isotope $B'$ vertically through a regular neighborhood of $\alpha$, increasing the heights of maxima of $T$ if necessary, until the heights of maxima and minima of $K \cap \text{int}(B')$ are the same as before any of the above isotopies. We see that after isotopy $T$ has one fewer inessential saddle and no new critical points have been created. See Figure 2. Repeating this process, we eliminate all inessential saddles via isotopy. \[\square\]
Thus, from this point forward, we may replace any \( h \in \mathcal{M}(K) \) with \( h \circ f_1 \) from the lemma without changing the information carried by \( h|_K \); thus we may suppose that the torus \( T = \partial V \) contains no inessential saddles. It follows that if \( \gamma \) is a loop contained in a level 2-sphere that bounds a disk \( D \subset T \), then \( D \) contains exactly one critical point, a minimum or a maximum. If not, \( D \) must contain a saddle point, which is necessarily inessential.

4. THE CONNECTIVITY GRAPH

For each regular value \( r \) of \( h|_{T,K} \), we have that \( h^{-1}(r) \) is a level 2-sphere \( S^2 \) and \( h^{-1}(r) \cap T \) is a collection of simple closed curves. Let \( \gamma_1, \ldots, \gamma_n \) denote these curves.

A bipartite graph is a graph together with a partition of its vertices into two sets \( \mathcal{A} \) and \( \mathcal{B} \) such that no two vertices from the same set share an edge. We will create a bipartite graph \( \Gamma_r \) from \( h^{-1}(r) \) as follows: Cut the 2-sphere \( h^{-1}(r) \) along \( \gamma_1, \ldots, \gamma_n \), splitting \( h^{-1}(r) \) into a collection of planar regions \( R_1, \ldots, R_m \). The vertex set \( \{v_1, \ldots, v_m\} \) of \( \Gamma_r \) corresponds to the regions \( R_1, \ldots, R_m \), and the edges correspond to the curves \( \gamma_1, \ldots, \gamma_n \) that do not bound disks in \( T \). For each such \( \gamma_i \), make an edge between \( v_j \) and \( v_k \) if \( \gamma_i = R_j \cap R_k \) in \( h^{-1}(r) \). To see that \( \Gamma_r \) is bipartite, we create two vertex sets \( \mathcal{A}_r \) and \( \mathcal{B}_r \), letting \( v_i \in \mathcal{A}_r \) if \( R_i \subset V \), and \( v_i \in \mathcal{B}_r \) otherwise. We call \( \Gamma_r \) the \textbf{essential connectivity graph} with respect to the regular value \( r \) of \( h \), where the word “essential” emphasizes the fact that edges correspond to only those \( \gamma_i \) that are essential in \( T \). Note that since each \( \gamma_i \) separates \( h^{-1}(r) \), the graph \( \Gamma_r \) must be a tree. An endpoint of \( \Gamma_r \) is a vertex that is incident to exactly one edge.
For instance, in Figure 3 we see a possible level 2-sphere and corresponding essential connectivity graph. Observe that since \( V \) is a knotted solid torus, \( T \) is only compressible on one side, and every compression disk for \( T \) is a meridian of \( V \). This leads to the third lemma:

**Lemma 4.1.** If \( v_i \in \Gamma_r \) is an endpoint, then \( v_i \in \mathcal{A}_r \).

*Proof.* Suppose \( R_i \) is the region in \( h^{-1}(r) \) corresponding to \( v_i \). Then \( \partial R_i \) contains exactly one essential curve in \( T \), call it \( \gamma_i \) and some (possibly empty) set of curves that bound disks in \( T \). Since each of these disks contains only one maximum or minimum by the discussion above, any two must be pairwise disjoint. Thus, we can glue each disk to \( R_i \) to create an embedded disk \( D \) such that \( \partial D = \gamma_i \). Now, push each glued disk into a collar of \( T \) in \( V \), so that \( T \cap \text{int}(D) = \emptyset \), and thus \( D \) is a compression disk for \( T \). We conclude \( D \subset V \) and \( R_i \cap D \neq \emptyset \), implying \( R_i \subset V \) and \( v_i \in \mathcal{A}_r \). \( \square \)

Using similar arguments, we prove the next lemma:

**Lemma 4.2.** Suppose that \( v_1, \ldots, v_n \subset \Gamma_r \) are endpoints corresponding to regions \( R_1, \ldots, R_n \subset h^{-1}(r) \), where each \( R_i \) contains exactly one curve \( \gamma_i \) that is essential in \( T \). Then \( \gamma_1, \ldots, \gamma_n \) bound meridian disks \( D_1, \ldots, D_n \subset V \) such that \( K \cap D_i \subset R_i \) for all \( i \).

*Proof.* The existence of the disks \( D_1, \ldots, D_n \) is given in the proof of Lemma 3. Thus, suppose that \( \Delta \) is a disk glued to \( R_i \) to construct \( D_i \). When we push \( \Delta \) into a collar of \( T \), we can choose this collar to be small enough so that it does not intersect \( K \). Thus, we may suppose that \( \Delta \cap K = \emptyset \) for every such \( \Delta \), which implies that all intersections of \( K \) with \( D_i \) must be contained in \( R_i \). \( \square \)

We note that the Lemmas 3 and 4 are inspired by the proof of Theorem 1.9 of [3]. Essentially, Lemma 4 demonstrates that even though the set of meridian disks \( D_1, \ldots, D_n \) may not be level, we may assume they are level for the purpose of counting intersections of \( K \) with \( h^{-1}(r) \), since any intersection of \( K \) with one of these disks occurs in one of the level regions \( R_i \). Hence, we define the trunk of a level 2-sphere.

**Definition 4.3.** Let \( r \) be a regular value of \( h \mid_{T,K} \). We define the trunk of the level 2-sphere \( h^{-1}(r) \), denoted \( \text{trunk}(r) \), to be the number of endpoints of \( \Gamma_r \).

For example, if \( r \) is the regular value whose essential connectivity graph is pictured in Figure 3, then \( \text{trunk}(r) = 6 \). We are now in a position to use the winding number of the pattern \( \hat{K} \).

**Lemma 4.4.** Let \( r \) be a regular value of \( h \mid_{T,K} \).
• If \( \text{trunk}(r) \) is even, then \( |K \cap h^{-1}(r)| \geq \#(\hat{K}) \cdot \text{trunk}(r) \);
• if \( \text{trunk}(r) \) is odd, then \( |K \cap h^{-1}(r)| \geq \#(\hat{K}) \cdot [\text{trunk}(r) + 1] \).

Proof. First, suppose that \( m = \text{trunk}(r) \) is even and let \( n = \#(\hat{K}) \). Since each meridian of \( V \) has algebraic intersection \( \pm n \) with \( K \), we know that each meridian must intersect \( V \) in at least \( n \) points. Let \( v_1, \ldots, v_m \) be endpoints of \( \Gamma_r \) corresponding to regions \( R_1, \ldots, R_m \). By Lemma 4, \( |K \cap R_i| = |K \cap D_i| \geq n \) for each \( i \). Further, since these regions are pairwise disjoint, it follows that \( |K \cap h^{-1}(r)| \geq n \cdot m \), completing the first part of the proof.

Now, suppose that \( m \) is odd. If \( N_1 \) is the algebraic intersection number of \( K \) with \( R = \cup R_i \), we have that
\[
N_1 = \sum_{i=1}^{m} \pm n.
\]
In particular, as \( m \) is odd it follows that \( |N_1| \geq n \). Let \( R' = \overline{h^{-1}(r) - R} \). Then \( R' \cap R \subset T \), so \( K \) does not intersect \( R' \cap R \). Let \( N_2 \) denote the algebraic intersection number of \( K \) with \( R' \). Since \( h^{-1}(r) \) is a 2-sphere which bounds a ball in \( S^3 \), \( h^{-1}(r) \) is homologically trivial, implying that the algebraic intersection of \( K \) with \( h^{-1}(r) \) is zero. This means \( N_1 + N_2 = 0 \), so \( |N_2| \geq n \) and thus \( |K \cap R'| \geq n \). Finally, putting everything together, we have
\[
|K \cap h^{-1}(r)| = |K \cap R| + |K \cap R'| = \sum_{i=1}^{m} |K \cap R_i| + |K \cap R'| \geq n \cdot (m + 1).
\]

\[\square\]

5. Bounding the width of satellite knots

We will use the trunk of the level surfaces to impose a lower bound on the trunk of a \( K \), which in turn forces a lower bound on the width of the \( K \). We need the following lemma, which is Claim 2.4 in [3]:

Lemma 5.1. Let \( S \) be a torus embedded in \( S^3 \), and let \( h : S^3 \to \mathbb{R} \) be a Morse function with two critical points on \( S^3 \) such that \( h \mid_S \) is also Morse. Suppose that for every regular value \( r \) of \( h \mid_S \), all curves in \( h^{-1}(r) \cap S \) that are essential in \( S \) are mutually parallel in \( h^{-1}(r) \). Then \( S \) bounds solid tori \( V_1 \) and \( V_2 \) in \( S^3 \) such that \( V_1 \cap V_2 = T \).

As a result of this lemma, we have

Corollary 5.2. There exists a regular value \( r \) of \( h \mid_{T,K} \) such that \( \text{trunk}(r) \geq 3 \).
Proof. Suppose not, and let \( r \) be any regular value of \( h |_{T,K} \) such that \( h^{-1}(r) \) contain essential curves in \( T \). Such a regular value must exist; otherwise \( T \) could not contain a saddle point. By assumption, \( \text{trunk}(r) \leq 2 \), so \( \Gamma_r \) has exactly two endpoints, \( v_1 \) and \( v_2 \). But this implies that \( \Gamma_r \) is a path, and thus all essential curves in \( h^{-1}(r) \) are mutually parallel. As this is true for every such regular value \( r \), we conclude by Lemma 6 that \( V \) is an unknotted solid torus, contradicting the fact that \( K \) is a satellite knot with nontrivial companion \( J \).

This brings us to our main theorem.

**Theorem 5.3.** Suppose \( K \) is a satellite knot with pattern \( \hat{K} \), where \( n = \#(\hat{K}) \). Then

\[
w(K) \geq 8n^2.
\]

Proof. Choose a height function \( h \in \mathcal{M}(K) \) such that \( \text{trunk}(h) = \text{trunk}(K) \). Since \( K \) is a satellite knot, \( K \) is contained in a knotted solid torus \( V \). Let \( T = \partial V \), and if necessary perturb \( T \) slightly so that \( h |_{T} \) is also Morse. By Corollary 1 above, there exists a regular value \( r \) of \( h \) such that \( \text{trunk}(r) \geq 3 \). From Lemma 5, it follows that \( \|K \cap h^{-1}(r)\| \geq 4n \). Since \( \text{trunk}(K) = \text{trunk}(h) \), and \( \text{trunk}(h) \) corresponds to the level of \( h \) with the greatest number of intersections with \( K \), we have \( \text{trunk}(h) \geq 4n \). Finally, using the lower bound for width based on trunk,

\[
w(K) \geq \frac{\text{trunk}(K)^2}{2} \geq 8n^2,
\]

as desired.

**Corollary 5.4.** Suppose \( K \) is a satellite knot, with pattern \( \hat{K} \) and companion \( J \). If \( \hat{K} \) is a braid of index \( n \) and \( J \) is a 2-bridge knot, then \( w(K) = 8n^2 \) and any thin position for \( K \) is a bridge position.

Proof. For such \( K \) we can exhibit a Morse function \( h \in \mathcal{M}(K) \) such that \( w(h) = 8n^2 \), \( b(h) = 2n \), and \( \text{trunk}(h) = 4n \). By [7], \( b(K) = b(h) \), so \( h \) is both a bridge and thin position for \( h \), and further every bridge position \( h' \) for \( K \) satisfies \( w(h') = 8n^2 \) and is also thin. It follows from the proof of the above theorem that \( \text{trunk}(K) = 4n \), so any \( h \in \mathcal{M}(K) \) that is not a bridge position satisfies \( w(h) > 8n^2 \).

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\]

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