BOGOMOLOV CONJECTURE FOR CURVES OF GENUS 2
OVER FUNCTION FIELDS

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Abstract. In this note, we will show that Bogomolov conjecture holds for a non-isotrivial curve of genus 2 over a function field.

1. Introduction

Let $k$ be an algebraically closed field, $X$ a smooth projective surface over $k$, $Y$ a smooth projective curve over $k$, and $f : X \rightarrow Y$ a generically smooth semistable curve of genus $g \geq 2$ over $Y$. Let $K$ be the function field of $Y$, $\overline{K}$ the algebraic closure of $K$, and $C$ the generic fiber of $f$. Let $j : C(\overline{K}) \rightarrow \text{Jac}(C)(\overline{K})$ be a morphism given by $j(x) = (2g - 2)x - \omega_C$ and $\| \cdot \|_{NT}$ the semi-norm arising from the Neron-Tate height pairing on $\text{Jac}(C)(\overline{K})$. We set

$$B_C(P; r) = \{ x \in C(\overline{K}) \mid \| j(x) - P \|_{NT} \leq r \}$$

for $P \in \text{Jac}(C)(\overline{K})$ and $r \geq 0$, and

$$r_C(P) = \begin{cases} -\infty & \text{if } \#(B_C(P; 0)) = \infty, \\ \sup \{ r \geq 0 \mid \#(B_C(P; r)) < \infty \} & \text{otherwise.} \end{cases}$$

Bogomolov conjecture claims that, if $f$ is non-isotrivial, then $r_C(P)$ is positive for all $P \in \text{Jac}(C)(\overline{K})$. Even to say that $r_C(P) \geq 0$ for all $P \in \text{Jac}(C)(\overline{K})$ is non-trivial because it contains Manin-Mumford conjecture, which was proved by Raynaud. Further, it is well known that the above conjecture is equivalent to say the following.

**Conjecture 1.1** (Bogomolov conjecture). If $f$ is non-isotrivial, then

$$\inf_{P \in \text{Jac}(C)(\overline{K})} r_C(P) > 0.$$  

Moreover, we can think the following effective version of Conjecture 1.1.

**Conjecture 1.2** (Effective Bogomolov conjecture). In Conjecture 1.1, there is an effectively calculated positive number $r_0$ with

$$\inf_{P \in \text{Jac}(C)(\overline{K})} r_C(P) \geq r_0.$$
In [4], we proved that, if \( f \) is non-isotrivial and the stable model of \( f : X \to Y \) has only irreducible fibers, then Conjecture 1.2 holds. More precisely, 

\[
\inf_{P \in \text{Jac}(C)(\mathbb{K})} r_C(P) \geq \begin{cases} 
\sqrt{12(g-1)} & \text{if } f \text{ is smooth,} \\
\frac{(g-1)^3}{3g(2g+1)}\delta & \text{otherwise,}
\end{cases}
\]

where \( \delta \) is the number of singularities in singular fibers of \( f : X \to Y \). In this note, we would like to show the following result.

**Theorem 1.3.** If \( f \) is non-isotrivial and \( g = 2 \), then \( f \) is not smooth and 

\[
\inf_{P \in \text{Jac}(C)(\mathbb{K})} r_C(P) \geq \frac{2}{135}\delta.
\]

2. Notations and ideas

In this section, we use the same notations as in §1. Let \( \omega^a_{X/Y} \) be the dualizing sheaf in the sense of admissible pairing. (For details concerning admissible pairing, see [5] or [2].) First we note the following theorem. (cf. [5, Theorem 5.6] or [2, Corollary 2.3])

**Theorem 2.1.** If \( (\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a > 0 \), then 

\[
\inf_{P \in \text{Jac}(C)(\mathbb{K})} r_C(P) \geq \sqrt{(g-1)(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a},
\]

where \( (\cdot)_a \) is the admissible pairing.

From now on, we assume \( g = 2 \). By the above theorem, in order to get Theorem 1.3, we need to estimate \( (\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a \). First of all, we can set 

\[
(\omega^0_{X/Y} \cdot \omega^0_{X/Y})_a = (\omega_{X/Y} \cdot \omega_{X/Y}) - \sum_{y \in Y} e_y,
\]

where \( e_y \) is the number coming from the Green function of \( f^{-1}(y) \). This number depends on the configuration of \( f^{-1}(y) \). So, let us consider the classification of semistable curves of genus 2. Let \( E \) be a semistable curve of genus 2 over \( k \) and \( E' \) the stable model of \( E \), that is, \( E' \) is a curve obtained by contracting all \((-2)\)-curves in \( E \). It is well known that there are 7-types of stable curves of genus 2. Thus, we have the classification of semistable curves of genus 2 according to type of \( E' \) as in Table 1. (In Table 1, the symbol \( A_n \) for a node means that the dual graph of \((-2)\)-curves over the node is same as \( A_n \) type graph.) The exact value of \( e_y \) can be found in Table 2 and will be calculated in §3.

Next we need to think an estimation of \( (\omega_{X/Y} \cdot \omega_{X/Y}) \) in terms of type of \( f^{-1}(y) \). According to Ueno [4], there is the canonical section \( s \) of 

\[
H^0(Y, \det(f_*(\omega_{X/Y}))^{10})
\]

such that \( d_y = \text{ord}_y(s) \) for \( y \in Y \) can be exactly calculated under the assumption that \( \text{char}(k) \neq 2,3,5 \). The result can be found in Table 2. Prof. Liu points out that by works of T. Saito [3] and Q. Liu [1], the value \( d_y \) in Table 2 still holds even if \( \text{char}(k) \leq 5 \).
Let $\delta_y$ be the number of singularities in $f^{-1}(y)$. Then, by Noether formula,

$$\deg(\det(f_*(\omega_{X/Y}))) = \left(\frac{\omega_{X/Y} \cdot \omega_{X/Y}}{12} + \sum_{y \in Y} \delta_y\right).$$

On the other hand, by the definition of $d_y$,

$$\sum_{y \in Y} d_y = 10 \deg(\det(f_*(\omega_{X/Y}))).$$

Thus, we have

$$(2.3) \quad (\omega_{X/Y} \cdot \omega_{X/Y}) = \sum_{y \in Y} \left(\frac{6}{5} d_y - \delta_y\right).$$

Hence, by (2.2) and (2.3),

$$(2.4) \quad (\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a = \sum_{y \in Y} \left(\frac{6}{5} d_y - \delta_y - e_y\right).$$

Therefore, using (2.4) and Table 2 we have the following theorem. (Note that an inequality

$$\frac{abc}{ab + bc + ca} \leq \frac{a + b + c}{9}$$

holds for all positive numbers $a, b, c$.)

**Theorem 2.5.** If $f$ is non-isotrivial, then $f$ is not smooth and

$$(\omega^a_{X/Y} \cdot \omega^a_{X/Y})_a \geq \frac{2}{135} \delta,$$

where $\delta = \sum_{y \in Y} \delta_y$.

Note that non-smoothness of $f$ can be easily derived from the fact that the moduli space $M_2$ of curves of genus 2 is an affine variety.

3. Calculation of $e_y$

Let us start calculations of $e_y$. If the stable model of a fiber is irreducible, $e_y$ is calculated in [2]. Thus it is sufficient to calculate $e_y$ for II(a), IV(a,b), VI(a,b,c) and VII(a,b,c). In these cases, the stable model has two irreducible components. Let $f^{-1}(y) = C_1 + \cdots + C_n$ be the irreducible decomposition of $f^{-1}(y)$. We set

$$D_y = \sum_{i=1}^{n} (\omega_{X/Y} \cdot C_i)v_i,$$

where $v_i$ is the vertex in $G_y$ corresponding to $C_i$. Especially, we denote by $P$ and $Q$ corresponding vertexes to stable components. Then, $D_y = P + Q$. Let $\mu$ and $g_\mu$ be the measure and the Green function defined by $D_y$. In the same way as in the Proof of Theorem 5.1 in [2],

$$e_y = -g_\mu(D_y, D_y) + 4c(G_y, D_y),$$

where $c(G_y, D_y)$ is the constant coming from $g_\mu$. By the definition of $c(G_y, D_y)$,

$$c(G_y, D_y) = g_\mu(P, P) + g_\mu(P, D_y).$$
Therefore, we have
\[ e_y = 7g_\mu(P, P) - g_\mu(Q, Q) + 2g_\mu(P, Q). \]

Here claim:

**Lemma 3.1.** \( g_\mu(P, P) = g_\mu(Q, Q). \) In particular,
\[ e_y = 6g_\mu(P, P) + 2g_\mu(P, Q). \]

**Proof.** By the definition of \( c(G_y, D_y), \)
\[ c(G_y, D_y) = g_\mu(P, P) + g_\mu(P, P + Q) = g_\mu(Q, Q) + g_\mu(Q, P + Q). \]
Thus, we can see \( g_\mu(P, P) = g_\mu(Q, Q). \) \( \square \)

In the following, we will calculate \( e_y \) for each type II(a), IV(a,b), VI(a,b,c) and VII(a,b,c). First we present the dual graph of each type and then show its calculation.

**Type II(a).**

\[
\begin{array}{c}
| & G & |
\end{array}
\]
\[ P \rightarrow G \rightarrow Q \quad \text{length}(G) = a \]

In this case, \( \mu = \frac{\delta_P}{2} + \frac{\delta_Q}{2} \) by [3, Lemma 3.7]. We fix a coordinate \( s : G \rightarrow [0, a] \) with \( s(P) = 0 \) and \( s(Q) = a \). If we set
\[ g(x) = -\frac{s(x)}{2} + \frac{a}{4}, \]
then, \( \Delta(g) = \delta_P - \mu \) and \( \int_G g_\mu = 0 \). Thus, \( g(x) = g_\mu(P, x) \). Hence
\[ g_\mu(P, P) = \frac{a}{4} \quad \text{and} \quad g_\mu(P, Q) = -\frac{a}{4}. \]
Thus
\[ e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = a. \]

**Type IV(a,b).**

\[
\begin{array}{c}
G_1 \quad G_2
\end{array}
\]
\[ P \rightarrow G_1 \rightarrow Q \rightarrow G_2 \quad \text{length}(G_1) = a \quad \text{length}(G_2) = b \]
We fix coordinates \( s : G_1 \rightarrow [0, a] \) and \( t : G_2 \rightarrow [0, b] \) with \( s(P) = 0 \), \( s(Q) = a \) and \( t(Q) = 0 \). In this case, \( \mu = \frac{\delta_P}{2} + \frac{dt}{2b} \) by [5, Lemma 3.7]. We set
\[
g(x) = \begin{cases} 
-\frac{s(x)}{2} + \frac{b + 12a}{48} & \text{if } x \in G_1, \\
\frac{1}{2} \left( \frac{(t(x))^2}{2b} - \frac{t(x)}{2} \right) + \frac{b - 12a}{48} & \text{if } x \in G_2.
\end{cases}
\]

Then, \( g \) is continuous, \( \Delta(g|_{G_1}) = \frac{\delta_P}{2} - \frac{\delta_Q}{2} \), and \( \Delta(g|_{G_2}) = \frac{\delta_P}{2} - \frac{dt}{2b} \). Thus, \( \Delta(g) = \delta_P - \mu \).

Moreover, \( \int_{G} g \mu = 0 \). Therefore, \( g(x) = g_\mu(P, x) \). Hence
\[
g_\mu(P, P) = \frac{b + 12a}{48} \quad \text{and} \quad g_\mu(P, Q) = \frac{b - 12a}{48}.
\]

Thus
\[
e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = a + \frac{b}{6}.
\]

**Type VI(a,b,c).**

\[
\begin{array}{c}
\hline
G_2 \\
\hline
\bullet & G_1 & G_3 \\
\hline
P & Q \\
\hline
\end{array}
\]

\[
\begin{array}{l}
\text{length}(G_1) = a \\
\text{length}(G_2) = b \\
\text{length}(G_3) = c
\end{array}
\]

We fix coordinates \( s : G_1 \rightarrow [0, a] \), \( t : G_2 \rightarrow [0, b] \) and \( u : G_3 \rightarrow [0, c] \) with \( s(P) = 0 \), \( s(Q) = a \), \( t(P) = 0 \) and \( u(Q) = 0 \). In this case, \( \mu = \frac{dt}{2b} + \frac{du}{2c} \) by [5, Lemma 3.7]. We set
\[
g(x) = \begin{cases} 
\frac{1}{2} \left( \frac{(t(x))^2}{2b} - \frac{t(x)}{2} \right) + \frac{b + c + 12a}{48} & \text{if } x \in G_2, \\
-\frac{s(x)}{2} + \frac{b + c + 12a}{48} & \text{if } x \in G_1, \\
\frac{1}{2} \left( \frac{(u(x))^2}{2c} - \frac{u(x)}{2} \right) + \frac{b + c - 12a}{48} & \text{if } x \in G_3.
\end{cases}
\]

Then, \( g \) is continuous, \( \Delta(g|_{G_1}) = \frac{\delta_P}{2} - \frac{\delta_Q}{2} \), \( \Delta(g|_{G_2}) = \frac{\delta_P}{2} - \frac{dt}{2b} \), and \( \Delta(g|_{G_3}) = \frac{\delta_Q}{2} - \frac{du}{2c} \). Thus, \( \Delta(g) = \delta_P - \mu \). Moreover, \( \int_{G} g \mu = 0 \). Therefore, \( g(x) = g_\mu(P, x) \). Hence
\[
g_\mu(P, P) = \frac{b + c + 12a}{48} \quad \text{and} \quad g_\mu(P, Q) = \frac{b + c - 12a}{48}.
\]

Thus
\[
e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = a + \frac{b + c}{6}.
\]
Type VII(a,b,c).

We fix coordinates $s : G_1 \to [0, a]$, $t : G_2 \to [0, b]$ and $u : G_3 \to [0, c]$ with $s(P) = 0$, $s(Q) = a$, $t(P) = 0$, $t(Q) = b$, $u(P) = 0$ and $u(Q) = c$. In this case, $\mu = \frac{ds}{3a} + \frac{dt}{3b} + \frac{du}{3c}$ by Lemma 3.7. We set

$$g(x) = \begin{cases} \frac{s(x)^2}{6a} - \left( \frac{1}{6} + \frac{1}{2} \frac{bc}{ab + bc + ca} \right) s(x) + \frac{a + b + c}{108} + \frac{abc}{4(ab + bc + ca)} & \text{if } x \in G_1, \\ \frac{t(x)^2}{6b} - \left( \frac{1}{6} + \frac{1}{2} \frac{ac}{ab + bc + ca} \right) t(x) + \frac{a + b + c}{108} + \frac{abc}{4(ab + bc + ca)} & \text{if } x \in G_2, \\ \frac{u(x)^2}{6c} - \left( \frac{1}{6} + \frac{1}{2} \frac{ab}{ab + bc + ca} \right) u(x) + \frac{a + b + c}{108} + \frac{abc}{4(ab + bc + ca)} & \text{if } x \in G_3. \end{cases}$$

Then, $g$ is continuous and $\Delta(g) = \Delta(g|_{G_1}) + \Delta(g|_{G_2}) + \Delta(g|_{G_3}) = \delta_P - \mu$. Moreover, $\int_G g\mu = 0$. Therefore, $g(x) = g_\mu(P, x)$. Hence

$$e_y = 6g_\mu(P, P) + 2g_\mu(P, Q) = \frac{2}{27}(a + b + c) + \frac{abc}{ab + bc + ca}.$$
| Type of $E$ | Description of the stable model $E'$ of $E$ | Figure of $E'$ and types of singularities by contracting $(−2)$-curves in $E$ |
|---|---|---|
| I | a smooth curve of genus 2 | 

\[
g = 2
\]

| II(a) | two elliptic curves meeting at one point transversally | 

\[
\begin{tikzpicture}
    \draw (0,0) node {$A_a$};
    \draw (-2,0) -- (2,0);
    \draw (-1,0) -- (1,0);
    \draw (-1,1) -- (1,-1);
    \draw (-1,-1) -- (1,1);
    \node at (-1,1) {$g = 1$};
    \node at (1,1) {$g = 1$};
\end{tikzpicture}
\]

| III(a) | an elliptic curve with one node | 

\[
\begin{tikzpicture}
    \draw (0,0) node {$A_a$};
    \draw (0,-2) .. controls (1,-1) and (-1,-1) .. (0,2);
    \node at (0,-2) {$g = 1$};
\end{tikzpicture}
\]

| IV(a,b) | a smooth elliptic curve and a rational curve with one node, which meet at one point transversally | 

\[
\begin{tikzpicture}
    \draw (-2,0) .. controls (-1,-1) and (1,-1) .. (2,0);
    \draw (-2,0) -- (-2,-2);
    \draw (2,0) -- (2,-2);
    \node at (-2,0) {$A_b$};
    \node at (2,0) {$A_a$};
    \node at (-2,-2) {$g = 0$};
    \node at (2,-2) {$g = 1$};
\end{tikzpicture}
\]

| V(a,b) | a rational curve with two nodes | 

\[
\begin{tikzpicture}
    \draw (-2,0) .. controls (-1,-1) and (1,-1) .. (2,0);
    \draw (-2,0) -- (-2,-2);
    \draw (2,0) -- (2,-2);
    \node at (-2,0) {$A_a$};
    \node at (2,0) {$A_b$};
    \node at (-2,-2) {$g = 0$};
\end{tikzpicture}
\]

---

Table 1. Classification of semistable curve $E$ of genus 2
| Type      | \(\delta_y\) | \(d_y\) | \(e_y\) |
|----------|--------------|---------|---------|
| I        | 0            | 0       | 0       |
| II(a)    | \(a\)        | \(2a\)  | \(a\)   |
| III(a)   | \(a\)        | \(a\)   | \(\frac{a}{6}\) |
| IV(a,b)  | \(a + b\)    | \(2a + b\) | \(a + \frac{b+6}{6}\) |
| V(a, b)  | \(a + b\)    | \(a + b\) | \(\frac{a+b}{6}\) |
| VI(a,b,c)| \(a + b + c\) | \(2a + b + c\) | \(a + \frac{b+c}{6}\) |
| VII(a,b,c)| \(a + b + c\) | \(a + b + c\) | \(\frac{2}{27}(a + b + c) + \frac{abc}{ab + bc + ca}\) |

Table 2. \(\delta_y\), \(d_y\) and \(e_y\)