Nonlinear magnetic response of the magnetized vacuum to applied electric field

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We find first nonlinear correction to the field, produced by a static charge at rest in a background constant magnetic field. It is quadratic in the charge and purely magnetic. The third-rank polarization tensor - the nonlinear response function - is written within the local approximation of the effective action in an otherwise model- and approximation-independent way within any P-invariant nonlinear electrodynamics, QED included.

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I. INTRODUCTION

In Maxwell electrodynamics the superposition principle is true, which reads that electromagnetic fields do not directly interact between themselves and may be linearly combined independently. This is not the case in nonlinear electrodynamics, wherein only small electromagnetic fields are mutually independent.

A popular example of a nonlinear electrodynamics in the vacuum is provided by the Born-Infeld model [1], also by a noncommutative \( U(1) \) gauge theory, in this respect considered, e.g., in [2, 3]. Many issues of nonlinear electrodynamics are thoroughly elaborated in [4]. Another, practically most important example is quantum electrodynamics (QED). The reason why it is nonlinear is that an electromagnetic field, say a photon, may create virtual electron-positron pairs that interact with this field itself and/or with any other, "external" field. This makes a mechanism that lets electromagnetic fields sense each other.

The well-known nonlinear effect of QED, present already in the vacuum without any external field, is light-by-light scattering. When taken off the photon mass shell the corresponding probability amplitude becomes as a matter of fact responsible for the leading nonlinear (cubic) correction to the electric Coulomb field [5] that can be conveniently written as

\[
E_{nl} = E \left( 1 - \frac{2\alpha}{45\pi} \frac{e^2}{m^2} \right)^2. \tag{1}
\]

Here \( E = (q/4\pi r^2) \) is the standard Coulomb field in Heaviside units produced by the point charge \( q \) at the distance \( r \), while \( e \) and \( m \) are the electron charge and mass, \( \alpha = (e^2/4\pi) = 1/137 \) is the fine-structure constant. It is generally known, and also seen in this equation, that in QED the nonlinearity is determined by the ratio of the electromagnetic field to Schwinger's characteristic value \( (m^2/e) = 4.4 \cdot 10^{13} \) cgse units that makes \( 1.3 \cdot 10^{16} \) V/cm when one measures an electric field, and \( 4.4 \cdot 10^{13} \) G if a magnetic field is concerned. Electromagnetic fields should be comparable in strength to these values in order that the interaction between them might become essential. The nonlinear correction in (1) becomes valuable when one is interested to approach a sufficiently small-sized charge sufficiently close. Say, to approach the nucleus of a not too heavy atom as close as a few Fm. On the other hand, electric fields, large in the Schwinger scale, up to \( 10^{18} - 10^{19} \) V/cm, occur [6] at the surface of strange quark stars [7], depending on whether the matter is in the superconducting state [8]. For such fields the vacuum is unstable, and the Schwinger effect of...
spontaneous electron-positron pairs by the vacuum becomes already efficient, which requires a special treatment, see the book [9]. We do not consider the corresponding complications in the present paper, however.

In this paper we are dealing with another nonlinear phenomenon, also associated with strong electric field, namely the production by it of a magnetic field: this magneto-electric effect becomes possible if an external magnetic field is present.

The linear correction to the Coulomb field of a charge due to the vacuum polarization in a magnetic field was studied earlier [10,12] with the result that the hydrogen ground energy level saturates [10?], as the magnetic field grows, and that a string is formed [11]. Some hints were thereby produced for considering [13] interquark potential in QCD. The nonlinear (purely magnetic) correction to the field of a charge in a magnetic field to be considered now for the first time is based on the known fact that in this case not only the photon-by-photon scattering exists, but also the photon splitting into two (also two-photon merging into one). The splitting is enhanced by the strength of the external magnetic field as compared to the vacuum case above. It was elaborated in theory [14] and is thought of as being efficient in a pulsar magnetosphere with the magnetic fields above $10^{12}$ G [15], essentially contributing to the electron-positron plasma production and to the radiation pattern of pulsars. Again, the same as above, when taken outside the photon mass shell, the corresponding probability amplitudes become responsible for a nonlinear induction of time-independent current (and, hence, of the stationary magnetic field) by static charges or, equivalently, by static electric fields created by them. The magnetic field produced by a static charge in an external magnetic field is even (quadratic in the lowest order of nonlinearity) with respect to its magnitude and linearly disappears with the external field – in agreement with the generalized Furry theorem of Ref. [3] that states that the numbers of electric and magnetic legs in every diagram should be each even. It also agrees with this theorem in that there are no corrections to the static electric field in the lowest (second-power) nonlinear order. Previously, magneto-electric effect was considered in [2,8] for classical noncommutative electrodynamics, and within QED as a linear response to a static charge by the vacuum filled with external electric and magnetic fields [10].

In the next Section II, for the most general case of a constant and homogeneous external electromagnetic field, we outline the derivation of nonlinear Maxwell equations keeping only the first and the second powers of the electromagnetic field living above that external field background, and define a notion of a current, nonlinearly induced by a static electric field (or by a static charge). The nonlinear field equations are served by the second- and third-rank polarization tensors. In Section III we restrict the external background to the magnetic-like field, i.e. the one that is purely magnetic in a class of special Lorentz frames. Then the involved polarization tensors are given in small-4-momentum limit, called also infrared or local approximation, in terms of the derivatives of the effective Lagrange density over the background field invariants, bearing in mind that in the local approximation this density does not depend upon space-time derivatives of the background field strength. In Section IV we are working in a special frame, where the background field is purely magnetic and the static charge is at rest. We calculate the nonlinearly-induced current and its magnetic field as expressed through the static electric field produced by the charge. The limiting cases of very large and very small background magnetic field are discussed within QED referring to the one-loop Euler-Heisenberg effective Lagrangian. In Conclusions the results are resumed, and numerical estimates of the domains of their applicability are given. Detailed calculations of the second- and third-rank derivatives of the effective action used in the work are presented in Appendix within the necessary local approximation.

II. NONLINEAR ELECTROMAGNETIC FIELD EQUATIONS OVER A CONSTANT FIELD BACKGROUND

In QED and in any other U(1)-gauge-invariant nonlinear electrodynamics, the field equations, when written up to terms, quadratic in the small electromagnetic field potential $a^\nu(x)$, have the form

$$\left[\eta_{\rho\nu} \square - \partial^\rho \partial^\nu \right] a^\nu(x) + \int d^4x' \Pi_{\rho\nu}(x, x') a^\nu(x') + \frac{1}{2} \int d^4x' d^4x'' \Pi_{\rho\sigma}(x, x' x'') a^\nu(x') a^\sigma(x'') = j_\rho(x),$$

(2)

where $j_\rho(x)$ is a (small) source of the field, Greek indices span the 4-dimensional Minkowski space taking the values 1,2,3,0, the metric tensor is $\eta_{\rho\nu} = (1, 1, 1, -1)$, and $\square = \nabla^2 - \partial^2_0$. The second- and the third-rank polarization tensors, $\Pi_{\rho\nu}$ and $\Pi_{\rho\nu\sigma}$, here are, in the presence of an external field potential $A^\rho(x) = A_{\text{ext}}^\rho(x)$, defined as

$$\Pi_{\rho\tau}(x, x') = \left. \frac{\delta^2 \Gamma}{\delta A^\rho(x) \delta A^\tau(x')} \right|_{A=A_{\text{ext}}}$$

(3)
\begin{equation}
\Pi_{\mu\tau\sigma}(x, x', x'') = \frac{\delta^3 \Gamma}{\delta A^\mu(x) \delta A^\tau(x') \delta A^\sigma(x'')} \bigg|_{A=A_{\text{ext}}} \tag{4}
\end{equation}
in terms of the effective action

\begin{equation}
\Gamma = \int \mathcal{L}(z) d^4 z, \tag{5}
\end{equation}
the generating functional of all-rank polarization tensors \( \Gamma \) of vertex functions known in QED as the Legendre transform of the generating functional of the Green functions \[17\]. The parameter of the power expansion, to which Eq.\((2)\) provides two lowest terms, depends on a field scale of a definite dynamical theory. We shall discuss this issue in Section IV below for QED.

We did not write the zero-power term \((a^\nu(x))^0\), an external macroscopic current, in equation \((2)\), because we assumed that the external field had been subjected to the sourceless field equation

\begin{equation}
\frac{\delta S}{\delta A^\beta(y)} \bigg|_{A=A_{\text{ext}}} = 0, \tag{6}
\end{equation}

where

\begin{equation}
S = \int L(z) d^4 z, \quad L(z) = -\mathfrak{F}(z) + \mathcal{L}(z) \tag{7}
\end{equation}
are the total action and the total Lagrangian, respectively. Here \(\mathfrak{F}(z) = (1/4) F_{\mu\nu} F^{\mu\nu} \) is the (free) Maxwell Lagrangian, \(F_{\alpha\beta}(z) = \partial^\alpha A^\beta(z) - \partial^\beta A^\alpha(z) \) is the field-strength tensor. In what follows we shall only deal with external fields \( F_{\alpha\beta} = \partial^\alpha A^\beta_{\text{ext}} - \partial^\beta A^\alpha_{\text{ext}} \), which are independent of the 4-coordinate \(z\), and with the case where the effective Lagrangian \(\mathcal{L}(z)\) may depend on \(z\) only through the field tensor \(F_{\alpha\beta}(z)\) and its space-time derivatives, and not explicitly. The latter property is fulfilled in QED and will be also assumed for other theories subject to our consideration. Under this assumption the constant field does satisfy the exact sourceless nonlinear field equation \((6)\). To see this, we fulfill the variational derivative in it

\begin{equation}
\frac{\delta S}{\delta A^\beta(x)} \bigg|_{A=A_{\text{ext}}} = 2 \sum_n \int \frac{\delta S}{\delta F^{(n)}_{\alpha\beta}(z)} \bigg|_{F=F} \frac{\partial}{\partial z^\alpha} \delta^{(n)}(x-z) d^4 z, \tag{9}
\end{equation}

where \((n)\) marks the derivative with respect to any space-time component. Once the variational derivative \(\frac{\delta S}{\delta F^{(n)}_{\alpha\beta}(z)}\), when restricted onto the coordinate-independent fields \(F_{\mu\nu}(z) = F_{\mu\nu}\), cannot depend on \(z\), the integration by parts turns this integral to zero.

The above presentation explains why Eq. \((2)\) is the field equation for small electromagnetic perturbations \(a^\beta(x) = A^\beta(x) - A^\beta_{\text{ext}(x)}\) over the external field of a constant field strength, caused by a small external current \(j_\rho(x)\) and taken to the lowest-power nonlinearity.

Polarization tensors of every rank \(\Pi_{\mu\tau...\sigma}(x, x', ...x'')\) satisfy the continuity relations with respect to every argument and every index (the transversality property)

\begin{equation}
\frac{\partial}{\partial x^\tau} \Pi_{\mu...\tau...\sigma}(x,...x',...x'') = 0, \tag{8}
\end{equation}
necessary to provide invariance of every term in the expansion of \(\Gamma\) in powers of the field \(a^\nu\) under the gauge transformation of it. Note that this is the primary property of \(\Gamma\) as a functional given on field strengths and their space-time derivatives only.

In our case of the external field with space- and time-independent strength the translational invariance holds true, which fact makes the all-rank polarization tensors depending on their coordinate differences.

With the definition of the photon propagator \(D_{\mu\nu}(x,x')\)

\begin{equation}
D_{\mu\nu}^{-1}(x-x') = \left[ \eta_{\mu\nu} \Box - \partial^\mu \partial^\nu \right] \delta^{(4)}(x'-x) + \Pi_{\mu\nu}(x-x') \tag{9}
\end{equation}
the nonlinear field equations \((2)\) take the form of (the set of) integral equations

\begin{equation}
a^\lambda(x) = \int d^4 y D^{\lambda\rho}(x-y) j_\rho(y) + \int d^4 y D^{\lambda\rho}(x-y) j_\rho^{\text{nl}}(y), \tag{10}
\end{equation}
\[ j_{\mu}^{nl}(x) = -\frac{1}{2} \int d^4y d^4u \Pi_{\mu\rho\sigma}(x - u, y - u) a^\rho(y) a^\sigma(u), \] (11)

where we have introduced the notation \( j_{\mu}^{nl}(x) \) for what we shall be calling "nonlinearly induced current".

Before proceeding, the following explanation seems to be in order. Within the present approach the electromagnetic field \( a^\lambda(x) \) is not quantized, this is not needed unless we leave the electromagnetic sector. The nonlinear equations written in this section are classical and will be treated classically below in understanding that the effective action is known. In QED the latter is the final product of quantum theory, obtained by continual integration over fermions \[17\].

The effective Lagrangian and all-rank polarization tensors involved are subject to approximate quantum calculations and, hence, are functions containing the Plank constant, electron mass and charge. Available is the effective action in the local limit referred to in the next section, which is known as the Euler-Heisenberg action when it is calculated within the approximation of one electron-positron loop, see \[3\], and as Ritus action when it is calculated with two-loop accuracy \[18\]. The second-rank polarization tensor \( \tilde{\mathcal{F}} \) was calculated in the one-loop approximation when the external background is formed by a constant and homogeneous electromagnetic field of the most general form (when the both its invariants \( \tilde{\mathcal{F}} \) and \( \mathcal{G} \) are nonvanishing) in \[19\]. One-loop diagrams with three photon legs corresponding to the third-rank tensor \[11\] were calculated both on and off the photon mass shell for QED with external magnetic-like \( (\tilde{\mathcal{F}} > 0, \mathcal{G} = 0) \) and crossed \( (\tilde{\mathcal{F}} = \mathcal{G} = 0) \) fields in \[13\], and for charge-asymmetric electron-positron plasma without external field using the temperature Green function techniques, in \[20\]. The calculations of Stoneham in \[14\] might become a basis for extending the results of the next sections beyond the local approximation used in getting them, but they are overcomplicated and not well-structured, so we leave this extension for future.

In the next sections we stick to the general form of the effective Lagrangian and refer to its specific Euler-Heisenberg form only at the very last steps for getting numerical estimates.

### III. LOCAL LIMIT

From now on we shall restrict ourselves only to slowly varying fields \( a^\lambda(x) \) and, correspondingly, to consideration of the sources \( j_{\mu}(y) \) that give rise to such fields via equations \[10, 11\]. To this end we may take the effective action in the local limit, where the field derivatives are disregarded from this functional. This is equivalent to going to the infrared asymptotic limit in the second- and third-rank polarization operators, i.e. to keeping, respectively, only the second and the third powers of four-momentum \( k_\mu \) in their Fourier transforms. Aiming at the local limit, we may admit that the effective Lagrangian \( \mathcal{L} \) depends only on (the relativistic and gauge invariant combinations of) the fields \( F_{\rho\sigma} \). Moreover, as long as constant fields are concerned all such combinations may be expressed as functions of the two field invariants \( \mathbf{S} = \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \) and \( \mathbf{G} = \frac{1}{2} F_{\rho\sigma} F_{\eta\sigma} \), where the dual field tensor is defined as \( \tilde{F}_{\rho\sigma} = \frac{1}{2} \epsilon_{\rho\sigma\alpha\beta} F^{\alpha\beta} \), with the completely antisymmetric unit tensor defined in such a way that \( \epsilon_{1230} = 1 \). Then the variational derivatives in \[3\] and \[11\] can be calculated in terms of derivatives of \( \mathcal{L}(\mathbf{S}, \mathbf{G}) \) with respect to the field invariants reduced to the space-and time-independent external field. Henceforth we shall be interested in the special case, where the external field is a constant purely magnetic field in a certain class of reference frames, called special below. Since in other Lorentz frames the electric field is also present we refer to this case as magnet-like. The invariant conditions that specialize the magnetic-like case are \( \mathbf{S} > 0, \mathbf{G} = 0 \). Once the invariant \( \mathbf{G} \) is a pseudoscalar, the Lagrangian of a P-invariant theory, QED included, may contain it only in an even power. Hence all the odd derivatives of \( \mathcal{L}(\mathbf{S}, \mathbf{G}) \) with respect to it disappear after being reduced to the external magnetic-like field:

\[
\begin{align*}
\frac{\partial \mathcal{L}(\mathbf{S}, \mathbf{G})}{\partial \mathbf{G}} & \bigg|_{F = F, \mathbf{G} = 0} = \frac{\partial^2 \mathcal{L}(\mathbf{S}, \mathbf{G})}{\partial \mathbf{G}^2} \bigg|_{F = F, \mathbf{G} = 0} = 0 \\
\frac{\partial^3 \mathcal{L}(\mathbf{S}, \mathbf{G})}{\partial \mathbf{G}^3} & \bigg|_{F = F, \mathbf{G} = 0} = 0.
\end{align*}
\] (12)

We calculate \[3\] and \[11\] in Appendix using the rule

\[
\frac{\delta F_{\alpha\beta}(z)}{\delta A^\mu(x)} = \left( \eta_{\mu\beta} \frac{\partial}{\partial z^\alpha} - \eta_{\mu\alpha} \frac{\partial}{\partial z^\beta} \right) \delta^4(x - z),
\] (13)

(understood as integrated over \( z \) with any function of \( z \)) by repeatedly applying the relation

\[
\frac{\delta \Gamma}{\delta A^\mu(x)} = \int \left[ \frac{\partial \mathcal{L}(\mathbf{S}(z), \mathbf{G}(z))}{\partial \mathbf{G}(z)} F_{\alpha\mu}(z) + \frac{\partial \mathcal{L}(\mathbf{S}(z), \mathbf{G}(z))}{\partial \mathbf{S}(z)} \tilde{F}_{\alpha\mu}(z) \right] \frac{\partial}{\partial z^\alpha} \delta^4(x - z) d^4z
\] (14)
and reducing the final results onto the external field. Then, taking eqs. (12) into account and using the notations

\[
\mathcal{L}_{\bar{\sigma}} = \left. \frac{d\mathcal{L}(\bar{\sigma}, 0)}{d\bar{\sigma}} \right|_{F=F_{\sigma}}, \quad \mathcal{L}_{\bar{\sigma}\bar{\sigma}} = \left. \frac{d^2\mathcal{L}(\bar{\sigma}, 0)}{d\bar{\sigma}^2} \right|_{F=F_{\sigma}}, \quad \mathcal{L}_{\bar{\sigma}\bar{\sigma}\bar{\sigma}} = \left. \frac{d^3\mathcal{L}(\bar{\sigma}, 0)}{d\bar{\sigma}^3} \right|_{F=F_{\sigma}}, \quad \mathcal{L}_{\bar{\sigma}\bar{\sigma}\bar{\sigma}\bar{\sigma}} = \left. \frac{\partial^2\mathcal{L}(\bar{\sigma}, \bar{\sigma})}{\partial\bar{\sigma}^2} \right|_{F=F_{\sigma}, \bar{\sigma}=0},
\]

all with \( F_{\mu\nu} = F_{\mu\nu} \) substituted (hence, from now on, \( \bar{\sigma} = \frac{1}{2}F_{\rho\sigma}F_{\rho\sigma} > 0 \) and \( \bar{\sigma} = \frac{1}{2}F^{\rho\sigma}F_{\rho\sigma} = 0 \), we get for the second- and for the third-rank tensor in the infrared limit

\[
\Pi^{R}_{\mu\nu}(x-y) = \mathcal{L}_{\bar{\sigma}} \left( \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}} - \eta_{\mu\nu}\square \right) \delta^4(x-y) - (\mathcal{L}_{\bar{\sigma}\bar{\sigma}} F_{\alpha\mu} F_{\beta\tau} + \mathcal{L}_{\bar{\sigma}\bar{\sigma}\bar{\sigma}} F_{\alpha\mu} F_{\beta\tau}) \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial x_{\beta}} \delta^4(x-y).
\]

(15)

and for the third-rank tensor in the infrared limit

\[
\Pi^{R}_{\mu\nu\rho}(x-y, x-u) = -O_{\mu\nu\rho\sigma\beta\gamma} \frac{\partial}{\partial x_{\alpha}} \left( \left( \frac{\partial}{\partial x_{\beta}} \delta^4(y-x) \right) \left( \frac{\partial}{\partial x_{\gamma}} \delta^4(x-u) \right) \right),
\]

(16)

where

\[
O_{\mu\nu\rho\sigma\beta\gamma} = \mathcal{L}_{\bar{\sigma}\bar{\sigma}} \left[ \bar{F}_{\gamma\sigma} \epsilon_{\alpha\mu\beta\tau} + \bar{F}_{\alpha\mu} \epsilon_{\beta\gamma\sigma} + \bar{F}_{\beta\tau} \epsilon_{\alpha\mu\gamma\sigma} \right] +
+ \mathcal{L}_{\bar{\sigma}\bar{\sigma}} \left[ (\eta_{\mu\tau}\eta_{\alpha\beta} - \eta_{\mu\beta}\eta_{\alpha\tau}) F_{\gamma\sigma} + F_{\alpha\mu} (\eta_{\tau\sigma}\eta_{\beta\gamma} - \eta_{\beta\gamma}\eta_{\tau\sigma}) + F_{\beta\tau} (\eta_{\mu\sigma}\eta_{\alpha\gamma} - \eta_{\alpha\gamma}\eta_{\mu\sigma}) \right]
+ \mathcal{L}_{\bar{\sigma}\bar{\sigma}\bar{\sigma}} \left[ F_{\alpha\mu} \bar{F}_{\beta\nu} F_{\gamma\sigma} + F_{\alpha\mu} F_{\beta\nu} \bar{F}_{\gamma\sigma} + F_{\alpha\mu} F_{\beta\nu} F_{\gamma\sigma} \right] + \mathcal{L}_{\bar{\sigma}\bar{\sigma}\bar{\sigma}\bar{\sigma}} F_{\alpha\mu} F_{\beta\nu} F_{\gamma\sigma},
\]

(17)

(The reader may consult Appendix for detailed calculations.) This tensor turns to zero, when there is no external field, \( F = 0 \), in agreement with the Furry theorem. The two transversality conditions (8) for (15) are provided by that the matrix in the brackets is antisymmetric under each permutation \( \mu \leftrightarrow \alpha \) and \( \tau \leftrightarrow \beta \), while the first term in (15) is transverse explicitly. The three transversality conditions (8) for (16) are provided by that matrix (17) is antisymmetric under each permutation \( \mu \leftrightarrow \alpha, \tau \leftrightarrow \beta \) and \( \sigma \leftrightarrow \gamma \). Thanks to the two latter antisymmetries, by using (16), (17) in (11) we obtain for the nonlinearly induced current the expression

\[
j^{nl}_{\mu}(x) = \frac{1}{8} O_{\mu\nu\rho\sigma\beta\gamma} \frac{\partial}{\partial x_{\alpha}} (f^{\beta\tau} f^{\gamma\sigma}),
\]

(18)

that includes only the field intensity tensors \( f^{\beta\tau} = \frac{\partial}{\partial x_{\beta}} a^{\tau}(x) - \frac{\partial}{\partial x_{\tau}} a^{\beta}(x) \). Therefore, the nonlinearly induced current is gauge-invariant: it depends only on field intensities and, besides, it is conserved, \( \frac{\partial}{\partial x_{\mu}} j^{nl}_{\mu}(x) = 0 \) due to the first antisymmetry \( \mu \leftrightarrow \alpha \).

We have to approach the nonlinear set (10), (11) by looking for its solution in a power series in the field \( a^{\lambda}(x) \). Within the first iteration, to which we shall as a matter of fact confine ourselves, we substitute the linear approximation to the solution of equation (10)

\[
a^{lin}_{\nu}(x) = \int d^4x' D_{\nu\rho}(x-x') j^{\rho}(x')
\]

(19)

for \( a(x) \) into (11). In other words, we should use the electromagnetic field \( f^{\beta\tau} = f^{lin}_{\beta\tau} = \frac{\partial}{\partial x_{\tau}} a^{lin}_{\beta}(x) - \frac{\partial}{\partial x_{\beta}} a^{lin}_{\tau}(x) \) linearly produced by the source \( j_{\mu}(x) \) in the expression for the nonlinearly induced current (18).
IV. MAGNETIC FIELD OF A STATIC CHARGE AT REST IN EXTERNAL MAGNETIC FIELD

We are in a position to start studying the nonlinear effect of production of a magnetic field by a static charge at rest in a constant and homogeneous external magnetic field in a special frame. The linear effect of the external magnetic field on the electrostatic field of a charge was studied earlier (beyond the infrared approximation) in [10–12].

In this frame the external magnetic field is defined as \( B_i = (1/2)\epsilon_{ijk}F_{jk} = \mathcal{F}_{i0}, B = |\mathbf{B}| \), while the external electric field disappears \( E_i = \mathcal{F}_{0i} = 0 \). The roman indices span the 3D subspace in this reference frame, \( \epsilon_{ijk} \) is the fully antisymmetric tensor, \( \epsilon_{123} = 1 \).

Consider now a static charge given in that frame by the 4-current \( j_\mu(x) = j_\mu(x)\delta_{\mu0} \). In the linear approximation [10], naturally, only an electrostatic field is generated in that frame. Hence, the components with \( \alpha = \beta = \gamma = 0, \tau, \sigma \neq 0 \) do not contribute to (18), so we need only the components

\[
\mathcal{O}_{\alpha0\beta\gamma\delta} = -\nabla_\delta j_{\alpha0} + \nabla_\delta j_{\alpha0} - \nabla_\delta j_{\alpha0} - \nabla_\delta j_{\alpha0}
\]

and the ones obtained from (20) by permutations between the second and the fifth, and between the third and the sixth indices, while \( \mathcal{O}_{\alpha0\beta\gamma\delta} = 0 \) according to (17). Therefore \( j_0^{nl}(x) = 0 \), i.e. there is no nonlinear (quadratic) correction to the static charge within the current quadratic approximation: the induced current (18) is purely spatial:

\[
j_i^{nl}(x) = \frac{1}{2}\mathcal{O}_{\alpha0\beta\gamma\delta} \frac{\partial}{\partial x_j} (j_0^{lin} j_0^{lin}) = \frac{1}{2} (\nabla \times B)_i \left[ \mathcal{L}_{\delta\delta} E^2 - \mathcal{L}_{\delta\delta} \mathcal{E}_i (B\mathcal{E})^2 \right] - \mathcal{L}_{\delta\delta} \left( \nabla \times \mathcal{E}_i \right) (B\mathcal{E}), \tag{21}\]

where \( \mathcal{E}_n = \mathcal{E}_n(x) = j_0^{lin} = \frac{-\delta_{\alpha0}}{4\pi} a_0^{lin}(x) \) is the time-independent electric field, linearly produced following Eq. (19), and the differential operator \( \nabla \) acts on everything to the right of it. The magnetic field strength \( h(x) \) generated by this current according to the Maxwell equation \( \nabla \times h(x) = j^{nl}(x) \) is

\[
h_i(x) = h_i(x) + \nabla_i \Omega, \tag{22}\]

where

\[
h_i(x) = \frac{B}{2} \left[ \mathcal{L}_{\delta\delta} (\mathcal{E}(x))^2 - \mathcal{L}_{\delta\delta} \mathcal{E}_i (B\mathcal{E}(x))^2 \right] - \mathcal{E}_i(x) \mathcal{L}_{\delta\delta} \mathcal{E}_i (B\mathcal{E}(x)), \tag{23}\]

because \( \nabla \times \nabla \Omega = 0 \), and the scalar function \( \Omega \) should be subjected to the Poisson equation

\[
\nabla^2 \Omega = -\nabla_j h_j(x) \tag{24}\]

to make the magnetic field \( h(x) \) obey the other Maxwell equation \( \nabla h(x) = 0 \). Hence, the magnetic field is the transverse part of (23):

\[
h_i(x) = \left( \delta_{ij} - \frac{\nabla_i \nabla_j}{\nabla^2} \right) h_j(x) = h_i(x) + \frac{\nabla_i \nabla_j}{4\pi} \int \frac{h_j(y)}{|x-y|} \, d^3y, \tag{24}\]

Note that the substitution of the field of a point-like charge into (24) through (23) would cause the divergency of the integral in (24) near \( y = 0 \): the present approach fails near the point charge, since it is not applicable to its strongly inhomogeneous field. Dealing with the point charge would require going beyond the infrared approximation followed to in the present work. Nevertheless, Eq. (24) is sound as applied to extended charges.

Eq. (24) would coincide with the magnetic induction \( b(x) = \nabla \times a^{nl}(x) \) if the linear vacuum magnetization effect might be neglected, i.e. if the nonlinear correction to the field in (10)

\[
a_{nl}(x) = \int d^4y D^{\lambda\nu}(x-y) j_\nu^{nl}(y) \tag{25}\]

might be taken without the contribution of the linear response function (10) \( \Pi_{\mu\nu}(x-x') \) in the photon propagator (9). Taking this contribution into account results in more complicated integrals. The situation remains simple, however, when we may disregard the anisotropy of the linear magnetic response. The inverse magnetic permeability tensor inherent in the second-rank polarization tensor (19) is, in the special frame, the constant tensor (21).
\[ \mu_{ij} = (1 - \mathcal{L}_\delta) \delta_{ij} - \mathcal{L}_\delta B_i B_j, \]

whose two\(^3\) eigenvalues \(\mu_{\perp}^{-1} = 1 - \mathcal{L}_\delta\), and \(\mu_{\parallel}^{-1} = 1 - \mathcal{L}_\delta - 2 \mathcal{S}_\delta \mathcal{S}_\delta\) are responsible for magnetizations linearly caused by certain conserved constant straight-linear currents flowing along the external magnetic field, and across it, respectively (see Appendix in [22]). In QED, the values \(\mathcal{L}_\delta\) and \(2 \mathcal{S}_\delta \mathcal{S}_\delta\) are of the order of the fine structure constant \(\alpha = 1/137\), but depend on the field \(B\). When \(B\) is very large, \(B > m^2/e\), these quantities, as found from the Euler-Heisenberg one-loop effective Lagrangian, behave as, see, e.g., [23]

\[ \mathcal{L}_\delta \approx \frac{\alpha}{3\pi} \ln \frac{eB}{m^2}, \quad 2 \mathcal{S}_\delta \mathcal{S}_\delta \approx \frac{\alpha}{3\pi}. \]

So, when \(\frac{eB}{m^2} > 2.7\), the contribution of \(2 \mathcal{S}_\delta \mathcal{S}_\delta\) may be neglected as compared to \(\mathcal{L}_\delta\), and the linear magnetization becomes isotropic, \(\mu_{\perp}^{-1} = \mu_{\parallel}^{-1}\). Therefore, in this limit, we finally have for the nonlinear magnetic induction

\[ b(x) = (1 - \mathcal{L}_\delta)^{-1} h(x). \quad (26) \]

The electric field \(\mathbf{E} = -\nabla a_0^{\text{lin}}(x)\) to be substituted in (21) and (23) is the one that is linearly produced via Eq. (19) by a static charge distribution within the same infrared approximation. To determine it, note that in (19) only the propagator component \(D_{00}\) participates, that, in the Fourier representation, is \(D_{00} = (k^2 - \alpha x^2)^{-1}\), with \(\alpha x^2\) being one (out of three) eigenvalues of the second-rank polarization tensor \(\mathbf{\epsilon}\) taken in the static limit \(k_0 = 0\) in the special reference frame. Once the polarization tensor is considered in its infrared limit (16), this quantity is \(\mathcal{S}_\delta \mathcal{S}_\delta\), which is one (out of three) eigenvalues of the dielectric tensor [21] \(\epsilon_{ij} = (1 - \mathcal{L}_\delta) \delta_{ij} + \mathcal{L}_\delta \mathbf{B}_i \mathbf{B}_j\), responsible for polarizations, linearly caused by homogeneously charged planes parallel and orthogonal to \(\mathbf{B}\), respectively, \(x_\perp\) and \(x_\parallel\) are the coordinate components across and along \(\mathbf{B}\). For large magnetic field one gets the linearly growing asymptote from the Euler-Heisenberg Lagrangian \(2 \mathcal{S}_\delta \mathcal{S}_\delta\) \(\approx \frac{\alpha}{3\pi} \frac{eB}{m^2}\). This means that if \(\frac{eB}{m^2} > 2.7\), the dielectric component \(\epsilon_{\parallel}\) dominates over \(\epsilon_{\perp}\), i.e., the electrization becomes highly anisotropic, in contrast to the magnetization. In this asymptotic region Eq. (27) becomes (if we disregard the polarization in \(\epsilon_{\perp}\) by setting \(\epsilon_{\perp} = 1\)) the large-distance behavior of the potential of a point charge in a strong magnetic field calculated in the linear approximation in [10, 11] beyond the infrared approximation of the polarization tensor. Note that Eq. (27), as well as its high-field limit, is only valid far from the charge. In that domain, however, it also fits any charge, with the total value \(q\), distributed over a finite region.

\[ a_0^{\text{lin}}(x) = \frac{q}{4\pi} \frac{1}{\sqrt{\epsilon\epsilon_1 x_1^2 + \epsilon\epsilon_2 x_2^2}}, \quad (27) \]

where \(\epsilon_1 = 1 - \mathcal{L}_\delta\) and \(\epsilon_\parallel = 1 - \mathcal{L}_\delta + 2 \mathcal{S}_\delta \mathcal{S}_\delta\) are eigenvalues of the dielectric tensor [21] \(\epsilon_{ij} = (1 - \mathcal{L}_\delta) \delta_{ij} + \mathcal{L}_\delta \mathbf{B}_i \mathbf{B}_j\), responsible for polarizations, linearly caused by homogeneously charged planes parallel and orthogonal to \(\mathbf{B}\), respectively, \(x_\perp\) and \(x_\parallel\) are the coordinate components across and along \(\mathbf{B}\). For large magnetic field one gets the linearly growing asymptote from the Euler-Heisenberg Lagrangian \(2 \mathcal{S}_\delta \mathcal{S}_\delta\) \(\approx \frac{\alpha}{3\pi} \frac{eB}{m^2}\). This means that if \(\frac{eB}{m^2} > 2.7\), the dielectric component \(\epsilon_{\parallel}\) dominates over \(\epsilon_{\perp}\), i.e., the electrization becomes highly anisotropic, in contrast to the magnetization. In this asymptotic region Eq. (27) becomes (if we disregard the polarization in \(\epsilon_{\perp}\) by setting \(\epsilon_{\perp} = 1\)) the large-distance behavior of the potential of a point charge in a strong magnetic field calculated in the linear approximation in [10, 11] beyond the infrared approximation of the polarization tensor. Note that Eq. (27), as well as its high-field limit, is only valid far from the charge. In that domain, however, it also fits any charge, with the total value \(q\), distributed over a finite region.

V. SOME NUMERICAL ESTIMATES

To analyze the large magnetic field limits of the induced current (21), of the resulting magnetic field (24) and of its induction (20) one should also bear in mind the asymptotic behavior \(\mathcal{L}_\delta \mathcal{S}_\delta\) \(= -\frac{\alpha}{3\pi} m^2 B\). Then it follows from the large external magnetic field asymptotic behavior, \(\frac{eB}{m^2} > \frac{3\pi}{\alpha}\), of the other derivatives of the Euler-Heisenberg Lagrangian involved in (23) that were listed above that in this limit

\[ \frac{eB}{m^2} \sim \frac{\alpha}{6\pi} \left[ \left( \frac{e\mathcal{E}_\parallel}{m^2} \right)^2 + \left( \frac{e\mathcal{E}_\perp}{m^2} \right)^2 \right], \]

\[ \frac{eB}{m^2} > \frac{\alpha}{\epsilon \mathcal{E}_\parallel \mathcal{E}_\perp} > \frac{\alpha}{3\pi} m^2 > \frac{3\pi}{\alpha}. \]

\(^3\) The constant background magnetic-like field makes an uniaxial medium in any of the special frames [21].
The minus sign in the first line indicates that the induced magnetic field diminishes the external field in the large external field regime.

We may apply the results (23), (24) to small external magnetic field \((eB/m^2) << 1\), as well. With the Euler-Heisenberg Lagrangian density, one has in this regime:

\[
\mathcal{L}_{\delta \delta} = \frac{4\alpha}{45\pi} \left( \frac{e}{m^2} \right)^2, \quad \mathcal{L}_{\delta \Phi} = \frac{7\alpha}{45\pi} \left( \frac{e}{m^2} \right)^2, \quad \mathcal{L}_{\delta \Phi \Phi} = \frac{\alpha}{315\pi} \left( \frac{e}{m^2} \right)^4.
\]

The third coefficient \(\mathcal{L}_{\delta \Phi \Phi}\) does not contribute in the leading order in \((eB/m^2) << 1\) to the estimates

\[
h_\parallel \sim B \frac{\alpha}{45\pi} \left[ \frac{2}{3} \left( \frac{e\mathcal{E}}{m^2} \right)^2 - 7 \left( \frac{e\mathcal{E}_\parallel}{m^2} \right)^2 \right],
\]

\[
h_\perp \sim -B \frac{7\alpha}{45\pi} \left( \frac{e\mathcal{E}_\perp}{m^2} \right) \left( \frac{e\mathcal{E}_\parallel}{m^2} \right), \quad \frac{eB}{m^2} << 1.
\]

In this approximation we may set \(\epsilon_\perp = \epsilon_\parallel = \mu_\perp = \mu_\parallel = 1\). Therefore, \(h = b\), and for the electric field of a charge outside of it, one may use here the standard Coulomb law \(\mathcal{E} = (q/4\pi) x/|x|^3\) instead of (24).

Note that \(\alpha/45\pi = 5 \cdot 10^{-5}\). So, for the electric field value close to Schwinger’s \(1.3 \cdot 10^{16}\) V/cm, the nonlinearly produced magnetic field makes up to \(3 \cdot 10^{-4}\) of the external magnetic field, which must be kept below Schwinger’s \(4.4 \cdot 10^{13}\) G in this case.

VI. CONCLUSION

In this paper we have found an expression for the magnetic field \(h(x)\) produced by a static charge \(q\) placed into an external magnetic field \(B\), Eqs. (23), (24). It is shown that in QED this nonlinear magneto-electric effect, not considered before, occurs already in the simplest approximation, where the effective Lagrangian \(\mathcal{L}\) is taken in its local limit, and only second power of the charge \(q\) and/or of its electric field \(\mathcal{E}(x)\) are kept. As for the background magnetic field \(B\), to reveal the effect suffice it to take it into account in the linear approximation, \(\sim B\), although magnetic field \(B\) of arbitrary magnitude is included in our result, as well. The final formulas depend on the first three derivatives of the effective action \(\mathcal{L}\) with respect to the external field invariants, which complies with the fact that, minimally, diagrams with three photon legs are responsible for the effect in the given approximation.

The results are model-independent and relate not only to QED, but also to any nonlinear electrodynamics provided the standard postulates of U(1)-gauge-, Lorentz-, translation-, C-, P-, T- invariances are respected. When applying them to QED we take the Euler-Heisenberg Lagrangian for \(\mathcal{L}\) to estimate the regimes of weak and strong \(B\). In QED all electromagnetic fields appear in ratios to the Schwinger characteristic value \(m^2/e\) of \(4.4 \cdot 10^{13}\) cgse units. The nonlinear magneto-electric effect we are reporting on is efficient, if the electric field of a charge is comparable, but still smaller than \(m^2/e\). Such fields take place near atomic nuclei and at the surface of a strange quark star. Strange quark stars can be strongly magnetized, besides \(24\). When the Schwinger value is exceeded by the electric field, the nonlinearity can no longer be treated via the power expansion \(2\), and also electron-positron pair creation from the vacuum must be taken into account.

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The second variational derivative of the local effective action is

\[
\frac{\delta^2 \Gamma}{\delta A^\mu(x) \delta A^\nu(y)} = \int d^4 z \left\{ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} \left( \eta_{\mu\rho} \eta_{\alpha\beta} - \eta_{\mu\beta} \eta_{\alpha\tau} \right) + \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}(z)} \epsilon_{\alpha\beta\tau} + \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} F_{\alpha\beta}(z) F_{\beta\gamma}(z) + \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}(z)} \tilde{F}_{\alpha\beta}(z) + \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} \left[ F_{\alpha\mu}(z) \tilde{F}_{\beta\tau}(z) + F_{\alpha\mu}(z) \tilde{F}_{\beta\tau}(z) \right] \right\} \left( \frac{\partial}{\partial z_\alpha} \delta^4(x - z) \right) \left( \frac{\partial}{\partial z_\beta} \delta^4(y - z) \right).
\] (28)

\(\delta \mathcal{F}_{\alpha\beta}(z) = \epsilon_{\alpha\beta\gamma} \frac{\partial}{\partial x_\gamma} \delta^4(x - z)\)

following from (13).

After reduced to the constant and homogeneous magnetic-like external field this gives the second-rank polarization tensor (15). The latter can be also used when the external field is "crossed", i.e. when \(\mathcal{G} = 0, \mathcal{F} = 0\), but \(\mathcal{F}^{\rho\sigma} \neq 0\) (electric and magnetic field vectors are mutually orthogonal and equal in length in every Lorentz frame).

The next derivative of (28) is

\[
\frac{\delta^3 \Gamma}{\delta A^\mu(x) \delta A^\nu(y) \delta A^\rho(u)} = \int d^4 z \left\{ \left( \eta_{\mu\rho} \eta_{\alpha\beta} - \eta_{\mu\beta} \eta_{\alpha\tau} \right) \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} \left( \eta_{\alpha\beta} \frac{\partial}{\partial z_\beta} \eta_{\mu\tau} \frac{\partial}{\partial z_\tau} \right) - \eta_{\alpha\beta} \frac{\partial}{\partial z_\tau} \eta_{\mu\tau} \frac{\partial}{\partial z_\tau} \right] + \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}(z)} \epsilon_{\alpha\beta\tau} \frac{\partial}{\partial z_\beta} \delta^4(x - z) + \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{F}(z)} \left( \eta_{\alpha\beta} \frac{\partial}{\partial z_\beta} \eta_{\mu\tau} \frac{\partial}{\partial z_\tau} \right) + \eta_{\alpha\beta} \frac{\partial}{\partial z_\tau} \eta_{\mu\tau} \frac{\partial}{\partial z_\tau} \right] \right\} \left( \frac{\partial}{\partial z_\alpha} \delta^4(x - z) \right) \left( \frac{\partial}{\partial z_\beta} \delta^4(y - z) \right).
\] (29)

Each \(z\)-derivative here and in the next relation is meant to apply only to a single \(\delta\)-function, the closest on the right of it.

Eq. (29) can be used for calculating higher variational derivatives, when needed.
After we set $\mathcal{G} = 0$ and use (12) we get for the P-even theory

$$
\frac{\delta^3 \Gamma}{\delta A^\mu(x) \delta A^\gamma(y) \delta A^\sigma(u)} \bigg|_{\mathcal{G} = 0} = 
\int d^4 z \left\{ \eta_{\mu\tau} \eta_{\alpha\beta} - \eta_{\mu\beta} \eta_{\alpha\tau} \right\} \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \frac{\partial F_{\gamma\sigma}(z)}{\partial \mathcal{G}(z)} \right] \delta^4(u - z) + 
+ \epsilon_{\alpha\mu\beta} \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \frac{\partial F_{\beta\gamma}(z)}{\partial \mathcal{G}(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ \frac{\partial \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}(z)} \right] F_{\alpha\mu}(z) F_{\beta\gamma}(z) \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ F_{\alpha\mu}(z) F_{\beta\gamma}(z) + F_{\beta\mu}(z) F_{\alpha\gamma}(z) \right] \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ F_{\alpha\mu}(z) F_{\beta\gamma}(z) + F_{\beta\mu}(z) F_{\alpha\gamma}(z) \right] \left[ \frac{\partial \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ F_{\alpha\mu}(z) F_{\beta\gamma}(z) + F_{\beta\mu}(z) F_{\alpha\gamma}(z) \right] \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z). 
$$

It is understood that $\mathcal{G}$ is set equal to zero also in the right-hand sides of this and the next equations. We integrate by parts with the delta-function $\delta^4(\mathcal{x} - \mathcal{y})$ to get

$$
\frac{\delta^3 \Gamma}{\delta A^\mu(x) \delta A^\gamma(y) \delta A^\sigma(u)} \bigg|_{\mathcal{G} = 0} = 
- \int d^4 z \delta^4(\mathcal{x} - \mathcal{y}) \frac{\partial}{\partial z_\alpha} \left\{ \eta_{\mu\tau} \eta_{\alpha\beta} - \eta_{\mu\beta} \eta_{\alpha\tau} \right\} \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \frac{\partial F_{\gamma\sigma}(z)}{\partial \mathcal{G}(z)} \right] \delta^4(u - z) + 
+ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} F_{\alpha\mu}(z) F_{\beta\gamma}(z) \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ F_{\alpha\mu}(z) F_{\beta\gamma}(z) + F_{\beta\mu}(z) F_{\alpha\gamma}(z) \right] \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ F_{\alpha\mu}(z) F_{\beta\gamma}(z) + F_{\beta\mu}(z) F_{\alpha\gamma}(z) \right] \left[ \frac{\partial \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z) + 
+ \left[ F_{\alpha\mu}(z) F_{\beta\gamma}(z) + F_{\beta\mu}(z) F_{\alpha\gamma}(z) \right] \left[ \frac{\partial^2 \mathcal{L}(\mathcal{F}(z), \mathcal{G}(z))}{\partial \mathcal{G}^2(z)} \right] \frac{\partial}{\partial z_\gamma} \delta^4(u - z). 
$$

Here and in the next relation the left-most derivative $\partial/\partial z_\alpha$ acts on all functions of $z$ to the right of it, whereas each other $z$-derivative acts only on the first $\delta$-function placed to the right of it.

The third-rank polarization tensor (16) is obtained from this expression by reducing onto $z$-independent fields $F(z) = \mathcal{F}$.
$$\Pi_{\mu\nu}^{R}(x \rightarrow y, x \rightarrow u) = -\frac{\partial}{\partial x_{\alpha}} \left( \eta_{\mu\nu} \eta_{\alpha\beta} - \eta_{\mu\alpha} \eta_{\nu\beta} \right) \left[ \frac{\partial^{2} \mathcal{L}(\mathcal{F}, \mathcal{G})}{\partial \delta^{2}} \mathcal{F}_{\gamma\sigma} \right] \frac{\partial}{\partial x_{\gamma}} \delta^{4}(u - x) +$$

$$+ \epsilon_{\alpha\mu\beta\gamma} \left[ \frac{\partial^{2} \mathcal{L}(\mathcal{F}, \mathcal{G})}{\partial \delta^{2}} \mathcal{F}_{\alpha\beta\gamma} \right] \frac{\partial}{\partial x_{\gamma}} \delta^{4}(u - x) +$$

$$+ \left[ \frac{\partial^{2} \mathcal{L}(\mathcal{F}, \mathcal{G})}{\partial \delta^{2}} \mathcal{F}_{\alpha\mu} \mathcal{F}_{\beta\gamma} \mathcal{F}_{\gamma\sigma} \right] \frac{\partial}{\partial x_{\gamma}} \delta^{4}(u - x) +$$

$$+ \mathcal{F}_{\alpha\mu} \mathcal{F}_{\beta\gamma} \left[ \frac{\partial^{2} \mathcal{L}(\mathcal{F}, \mathcal{G})}{\partial \delta^{2}} \mathcal{F}_{\gamma\sigma} \right] \frac{\partial}{\partial x_{\gamma}} \delta^{4}(u - x) +$$

$$+ \mathcal{F}_{\alpha\mu} \mathcal{F}_{\beta\gamma} \left[ \frac{\partial^{2} \mathcal{L}(\mathcal{F}, \mathcal{G})}{\partial \delta^{2}} \mathcal{F}_{\gamma\sigma} \right] \frac{\partial}{\partial x_{\gamma}} \delta^{4}(u - x) +$$

$$+ \mathcal{F}_{\alpha\mu} \mathcal{F}_{\beta\gamma} \left[ \frac{\partial^{2} \mathcal{L}(\mathcal{F}, \mathcal{G})}{\partial \delta^{2}} \mathcal{F}_{\gamma\sigma} \right] \frac{\partial}{\partial x_{\gamma}} \delta^{4}(u - x) +$$

It is meant that $\mathcal{G} = \frac{1}{4} \mathcal{F}^{\rho\sigma} \mathcal{F}_{\rho\sigma} = 0$, and $\mathcal{F} = \frac{1}{4} \mathcal{F}_{\rho\sigma} \mathcal{F}^{\rho\sigma}$ here. This formula is also applicable to the crossed field $\mathcal{G} = 0$, $\mathcal{F} = 0$, $\mathcal{F}^{\rho\sigma} \neq 0$.
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