Addendum: A separation in modulus property of the zeros of a partial theta function

Vladimir Petrov Kostov
Université Côte d’Azur, CNRS, LJAD, France,
e-mail: kostov@math.unice.fr

Abstract

We consider the partial theta function $\theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$, where $z \in \mathbb{C}$ is a variable and $q \in \mathbb{C}$, $0 < |q| < 1$, is a parameter. Set $D(a) := \{ q \in \mathbb{C}, 0 < |q| \leq a, \arg(q) \in [\pi/2, 3\pi/2] \}$. We show that for $k \in \mathbb{N}$ and $q \in D(0.55)$, there exists exactly one zero of $\theta(q, \cdot)$ (which is a simple one) in the open annulus $|q|^{-k+1/2} < z < |q|^{-k-1/2}$ (if $k \geq 2$) or in the punctured disk $0 < z < |q|^{-3/2}$ (if $k = 1$). For $k = 1, 4, 5, 6, \ldots$, this holds true for $q \in D(0.6)$ as well.

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1 The new results

In this addendum to [3] we consider the partial theta function $\theta(q, z) := \sum_{j=0}^{\infty} q^{j(j+1)/2} z^j$. The series converges for $q \in \mathbb{D}_1$, $z \in \mathbb{C}$, where $\mathbb{D}_a$ denotes the open disk centered at 0 and of radius $a$. We regard $q$ as a parameter and $z$ as a variable.

Definition 1. For $0 < a < b$ and for $q \in \mathbb{D}_1 \setminus 0$ fixed, denote by $U_{a,b} \subset \mathbb{C}$ the open annulus $\{|q|^{-a} < |z| < |q|^{-b}\}$. For $q$ fixed, we say that strong separation (in modulus) of the zeros of $\theta$ takes place for $k \geq k_0$, if for any $k \geq k_0$, there exists a unique zero $\xi_k$ of $\theta(q, \cdot)$ which is a simple one and

$$|\xi_k| \in U_{k-1/2,k+1/2} , \quad \text{if} \quad k \geq 2 , \quad \text{and} \quad \xi_k \in (\mathbb{D}_{|q|^{-3/2}} \setminus 0) , \quad \text{if} \quad k = 1 , \quad (1)$$

while the remaining $k_0 - 1$ zeros (counted with multiplicity) are in $\mathbb{D}_{|q|^{-k_0+1/2}} \setminus 0$. If $k_0 = 1$, then we say that the zeros of $\theta$ are strongly separated. (One can notice that the cases $k_0 = 1$ and $k_0 = 2$ are identical.)

It is shown in [3] that the zeros of $\theta$ are strongly separated for $0 < |q| \leq c_0 := 0.2078750206 \ldots$, see Lemma 1 and its proof therein. Set $D(a) := \{ q \in \mathbb{C}, 0 < |q| \leq a, \arg(q) \in [\pi/2, 3\pi/2] \}$.

Theorem 2. (1) For $q \in D(0.55)$, the zeros of $\theta$ are strongly separated.

(2) For $q \in D(0.6)$, conditions (1) hold true for $k = 1, 4, 5, 6, \ldots$, and there are exactly two zeros of $\theta$ (counted with multiplicity) in $U_{3/2,7/2}$.

Remarks 3. (1) The domain $D(0.55) \cup (\mathbb{D}_{c_0} \setminus 0)$ is much larger than the domain $\mathbb{D}_{c_0} \setminus 0$ in which strong separation of the zeros of $\theta$ is guaranteed by Lemma 1 in [3]. It is impossible to extend
Theorem 2 to the whole of $\mathbb{D}_{0.55} \setminus 0$, because for certain values of $q$ in the right half of $\mathbb{D}_{0.55} \setminus 0$, the function $\theta(q, \cdot)$ has double zeros, see [3] and [4].

(2) Numerical computations suggest that in part (1) of Theorem 2 the number 0.55 can be replaced by 0.6, see Remark 14.

2 Comments

Theorem 2 can be compared with the results of [3]. Set $\alpha_0 := \sqrt{3}/2\pi = 0.2756644777\ldots$. Parts (1) and (4) of Theorem 5 in [3] read:

**Theorem 4.** (1) For $n \geq 5$ and for $|q| \leq 1 - 1/(\alpha_0 n)$, strong separation of the zeros $\xi_k$ of $\theta$ occurs for $k \geq n$.

(2) For $0 < |q| \leq 1/2$, strong separation of the zeros of $\theta$ occurs for $k \geq 4$.

Part (1) of the theorem implies that for $k \geq n \geq 5$, one has

$$(1 - 1/(\alpha_0 k))^{-k+1/2} =: m_k \leq |\xi_k| \leq M_k := (1 - 1/(\alpha_0 k))^{-k-1/2}.$$  

The following table gives an idea how the numbers $m_n$ and $M_n$ decrease as $n$ increases (both of them tend to $e^{1/\alpha_0} = 37.6223665\ldots$ as $n \to \infty$). We list the truncations up to the second decimal for the numbers $\tau_n := 1 - 1/(\alpha_0 n)$ and up to the first decimal for $m_n$ and $M_n$. The numbers $m_n$ and $M_n$ are the internal and external radius of the open annulus to which the zero $\xi_n$ belongs for $|q| = \tau_n$.

| $n$  | 5   | 6   | 7   | 8   | 9   | 10  | 15  | 20  | 25  | 30  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\tau_n$ | 0.27 | 0.39 | 0.48 | 0.54 | 0.59 | 0.63 | 0.75 | 0.81 | 0.85 | 0.87 |
| $m_n$   | 336.2 | 164.5 | 115.2 | 92.8 | 80.2 | 72.2 | 55.3 | 49.5 | 46.5 | 44.7 |
| $M_n$   | 1225.1 | 416.1 | 239.1 | 169.8 | 134.4 | 113.4 | 73.0 | 60.5 | 54.4 | 50.9 |

3 Proof of Theorem 2

We use the same method of proof as the one of the proof of Theorem 5 in [3], but with more accurate estimations. We remind that (see [3]) $\theta = \Theta^* - G$, where $\Theta^*(q, z) := \sum_{j=-\infty}^{\infty} q^{(j+1)/2} z^j$ and $G(q, z) := \sum_{j=-\infty}^{1} q^{(j+1)/2} z^j = \sum_{j=1}^{\infty} q^{(j-1)/2} z^{-j}$. Set

$$Q := \prod_{j=1}^{\infty} (1 - q^j), \quad U := \prod_{j=1}^{\infty} (1 + z q^j) \quad \text{and} \quad R := \prod_{j=1}^{\infty} (1 + q^{j-1}/z).$$

We recall that by formula (4) in [3] (resulting from the Jacobi triple product), one has $\Theta^* = QUR$.

**Lemma 5.** For $q \in D(0.6)$, one has $|Q| \geq 1.2$.

**Proof.** We set $Q := Q_0 Q_1$, where $Q_0 := \prod_{j=1}^{11} (1 - q^j)$ and $Q_1 := \prod_{j=12}^{\infty} (1 - q^j)$. For the quantity $|Q_1|$ one obtains the minoration
Lemma 9. For \( \theta \geq 0 \), the inequality

\[
|Q_1| \geq \prod_{j=12}^{\infty} (1 - |q|^j) \geq \prod_{j=12}^{\infty} (1 - 0.6^j) = \gamma := 0.9945691384\ldots.
\]

Hence to prove the lemma it suffices to show that for \( q \in D(0.6) \), the inequality

\[
|Q_0| \geq 1.2/\gamma = 1.206552620\ldots
\]

holds true. This can be proved using the maximum principle. The polynomial \( Q_0 \) has no zeros in \( D(0.6) \) hence \( \min_{\partial D(0.6)} |Q_0| = 1/\max_{\partial D(0.6)} |1/Q_0| \) is attained on the border \( \partial D(0.6) \) of the domain \( D(0.6) \). The restrictions of \( |Q_0|^2 \) to the segment (resp. to the arc) of \( \partial D(0.6) \) are a polynomial and a trigonometric polynomial respectively and the latter displayed inequality is readily checked numerically. The check can be limited to the upper half-plane, because all coefficients of \( Q \) are real hence \( Q(\bar{q}) = \overline{Q(q)} \).

\[\square\]

Lemma 6. For \( q \in D(0.6), k \in \mathbb{N}, k \geq 5 \) and \( |z| = |q|^{-k+1/2}, \) one has \( \theta(q, z) \neq 0 \).

Proof. One can minorize \( |U| \) and \( |R| \) as follows:

\[
|U| \geq \prod_{j=1}^{\infty} |zq^j| - 1 | = \prod_{j=1}^{\infty} |q|^{-k+j+1/2} - 1 | \geq \prod_{j=1}^{\infty} |0.6^{-k+j+1/2} - 1 |
\]

\[
\geq \prod_{j=1}^{\infty} |0.6^{-9/2+j} - 1 | = \eta := 0.2411047426\ldots
\]

and

\[
|R| \geq \prod_{j=0}^{\infty} |1 - |q^j/z|| | = \prod_{j=0}^{\infty} |1 - |q|^{k+j-1/2}| \geq \prod_{j=0}^{\infty} |1 - |q|^{9/2+j}|
\]

\[
\geq \prod_{j=0}^{\infty} |1 - 0.6^{9/2+j} | = \xi := 0.7715882456\ldots.
\]

Thus using Lemma 5 one obtains the inequality

\[
|QRU| \geq 1.2 \cdot \eta \cdot \xi = 0.2232403024\ldots. \tag{2}
\]

On the other hand the quantity \( |G| \) can be majorized as follows:

\[
|G| \leq \sum_{j=1}^{\infty} |q|^{j(j-1)/2} |z|^j = \sum_{j=1}^{\infty} |q|^{j(j-1)/2+(k-1/2)j}
\]

\[
\leq \sum_{j=1}^{\infty} |q|^{j(j-1)/2+9j/2} \leq \sum_{j=1}^{\infty} 0.6^{j(j-1)/2+9j/2} = 0.1066576686\ldots. \tag{3}
\]

The lemma follows from inequalities (2) and (3) – for \( |z| = |q|^{-k+1/2}, \) one has at the same time \( \theta = \Theta^* - G, |\Theta^*| > 0.22 \) and \( |G| < 0.11, \) so \( \theta = 0 \) is impossible.

\[\square\]

Lemma 7. For \( q \in D(0.6) \) and \( |z| = |q|^{-7/2}, \) one has \( \theta(q, z) \neq 0. \)

Lemma 8. For \( q \in D(0.6) \) and \( |z| = |q|^{-3/2}, \) one has \( \theta(q, z) \neq 0. \)

Lemma 9. For \( q \in D(0.55) \) and \( |z| = |q|^{-5/2}, \) one has \( \theta(q, z) \neq 0. \)
The last three lemmas are proved in Sections 4, 5 and 6 respectively. We explain how Theorem 2 results from them and from Lemma 6. We recall that for \(|q| \leq 0.108\), all zeros of \(\theta(q,.)\) are simple (see [2]). For any \(k \in \mathbb{N}\) fixed and for \(|q|\) sufficiently small, there exists a single zero of \(\theta(q,.)\) which is \(\sim -q^{-k}\) as \(q \to 0\), see Proposition 10 in [1]. Hence for \(|q|\) sufficiently small, this zero satisfies the conditions (1). These conditions hold true as \(q\) varies along any segment \(S\) belonging to a half-line passing through the origin and such that \(S \subset (D(0.55) \cup \mathbb{D}_{\epsilon_0})\). Hence these conditions hold true for \(q \in D(0.55)\). In the same way one sees that, with the possible exception of \(\xi_2\) and \(\xi_3\), the zeros of \(\theta\) satisfy conditions (1) for \(q \in D(0.6)\).

4 Proof of Lemma 7

For \(|G|\) we obtain the majorization

\[
|G| \leq \sum_{j=1}^{\infty} |q|^{(j-1)/2+7j/2} \leq \sum_{j=1}^{\infty} 0.6^{(j-1)/2+7j/2} = 0.1851580824 \ldots .
\]

We minorize the quantity \(|Q|\) using Lemma 5. There remains to minorize \(|U|\) and \(|R|\).

We observe that \(\theta(\bar{q},\bar{z}) = \theta(q,z)\), therefore when \(|\theta|\) is majorized, we can assume that \(\arg(q) \in [0,\pi]\). Suppose that \(q \in D(0.6), \arg(q) \in [3\pi/4,\pi]\). We define the sectors \(S_j\) in \(\mathbb{C}\) by the formula

\[
S_j := \{ \arg(z) \in [(j-1)\pi/4, j\pi/4) \}, \quad j = 1, \quad 2, \quad 3, \quad S_4 := \{ \arg(z) \in [3\pi/4, \pi] \},
\]

and \(S_{-j}\) as the symmetric of \(S_j\) w.r.t. the real axis. Suppose that \(\zeta \in S_{\pm j}\). Set \(\psi := \arg(\zeta)\). By the cosine theorem

\[
|1 - \zeta|^2 = 1 + |\zeta|^2 - 2|\zeta| \cos(\psi).
\]

The function \(\cos(t)\) being even and decreasing on \([0,\pi]\) one obtains the minoration

\[
|1 - \zeta| \geq \mu_j(|\zeta|) := (1 + |\zeta|^2 - 2|\zeta| \cos((j - 1)\pi/4))^{1/2}.
\]

Remarks 10. (1) For \(\zeta \neq 0\), one has \(\mu_4(|\zeta|) > \mu_3(|\zeta|) > \mu_2(|\zeta|) > \mu_1(|\zeta|)\).

(2) For \(j\) fixed and for \(|\zeta| \geq 1\), the right-hand side of (1) is an increasing function in \(|\zeta|\).

(3) One can prove by straightforward computation that for \(1 \geq |\zeta_1| > |\zeta_2|\) and for \(3 \geq \ell > m \geq 1\), one has \(\mu_\ell(|\zeta_1|) \cdot \mu_m(|\zeta_2|) \geq \mu_\ell(|\zeta_2|) \cdot \mu_m(|\zeta_1|)\).

Consider for \(|z| = |q|^{-7/2}\) (i.e. for \(z = |q|^{-7/2} \omega, \ |\omega| = 1\)), the moduli of three consecutive factors \(u_k := 1 + zq^k\) of \(U\), for \(k = k^*, k^* + 1\) and \(k^* + 2\). (The role of \(\zeta\) will be played by the numbers \(-zq^k\)). Notice that the numbers \(u_k\) are of the form

\[
1 + |q|^{-7/2+k} \omega_k \quad , \quad |\omega_k| = 1.
\]

Remarks 11. (1) At least one of the three numbers \(-zq^k\) belongs to the left half-plane, because \(\arg(q) \in [\pi/2, \pi]\). Hence to the corresponding modulus \(|u_k|\) minoration \(\mu_3\) is applicable. If at most one of the other two numbers \(-zq^k\) belongs to \(S_1 \cup S_{-1}\), then to the corresponding modulus \(|u_k|\) minoration \(\mu_1\), and to the third modulus minoration \(\mu_2\) are applicable respectively.

(2) Suppose that at least two of the three numbers \(-zq^k\) belong to \(S_1 \cup S_{-1}\). Then these correspond to \(k = k^*\) and \(k = k^* + 2\), because \(\arg(q) \in [\pi/2, \pi]\). Moreover, \(-zq^{k^*+1} \in S_4 \cup S_{-4}\), so minoration \(\mu_4\) is applicable to \(|u_{k^*+1}|\) and minoration \(\mu_1\) to \(|u_{k^*+1}|\) and \(|u_{k^*+2}|\).
Consider the three numbers \( u_1, u_2, u_3 \). When represented in the form \( [5] \), their exponents 
\(-7/2 + k\) are negative, so \(|-zq^k| > 1\). For fixed \( \omega_k \), the modulus \(|-zq^k|\) decreases as \( |q| \) increases. 
Hence one can apply part (2) of Remarks [10] and minorize 
use of Remarks [11] one finds that the product 
\[ \mu_3(0.6^{-5/2}) \cdot \mu_2(0.6^{-3/2}) \cdot \mu_1(0.6^{-1/2}) \] 
where \((i_1, i_2, i_3)\) is a permutation of \((1,2,3)\). This is the number 
\[ \mu_3(0.6^{-5/2}) \cdot \mu_2(0.6^{-3/2}) \cdot \mu_1(0.6^{-1/2}) = 1.742379963 \ldots =: \chi_0. \]
Now consider for \( j = 1, 2, \ldots, \) a triple \(|u_{3j+1}|, |u_{3j+2}|, |u_{3j+3}|, j \in \mathbb{N}\). Hence \(|-zq^k| < 1\) and by Remarks [10] one has 
\[ \tilde{u}_j := |u_{3j+1}| \cdot |u_{3j+2}| \cdot |u_{3j+3}| \geq \min(A_j, B_j), \] 
where 
\[ A_j := \mu_3(|q|^{-5/2+3j}) \cdot \mu_2(|q|^{-3/2+3j}) \cdot \mu_1(|q|^{-1/2+3j}), \] 
and 
\[ B_j := \mu_1(|q|^{-5/2+3j}) \cdot \mu_4(|q|^{-3/2+3j}) \cdot \mu_1(|q|^{-1/2+3j}). \]
Set \( \rho := |q| \). We prove Lemma [7] with the help of the following result (the proof is given at the end of this section): 

**Lemma 12.** The quantities \( A_j(\rho) \) and \( B_j(\rho) \) are decreasing in \( \rho \) for \( \rho \in (0, 0.6) \).

This means that one can minorize the product \( \tilde{u}_j \) by \( \chi_j := \min(A_j(0.6), B_j(0.6)) \). For \( j = 1, 2, 3 \) and 4, the values of \( \chi_j \) are respectively 
\[ 0.1749135662 \ldots, 0.7772399345 \ldots, 0.9492771959 \ldots, 0.9889171980 \ldots \]
(in all cases they equal \( A_j(0.6) \)). For \( k \geq 16 \), the factors \(|u_k|\) can be minorized by \(|1 - |q|^{-7/2+k}|\), and then by \(|1 - 0.6^{-7/2+k}|\). We set \( \chi_5 := \prod_{k=16}^{\infty} |1 - 0.6^{-7/2+k}| = 0.9957913379 \ldots \). Thus we minorize \(|U|\) by \( \chi_0 \cdot \chi_1 \cdot \chi_2 \cdot \chi_3 \cdot \chi_4 \cdot \chi_5 \).

To minorize the product \( R \) one can observe that for \(|z| = |q|^{-7/2}\), each number \( 1 + q^{j-1}/z \) is of the form \([5]\) with \( k \geq 7 \). Therefore when minorizing \(|R|\) one can use the same reasoning as for \(|U|\) and obtain a minoration by \( \chi_2 \cdot \chi_3 \cdot \chi_4 \cdot \chi_5 \) (the first value of the index \( k \) being 7, not 1, one has to skip the analogs of the minorations of \(|u_k|\) for \( k = 1, \ldots, 6 \), i.e. to skip \( \chi_0 \) and \( \chi_1 \)).

Set \( \chi_* := \chi_2 \cdot \chi_3 \cdot \chi_4 \cdot \chi_5 \). Thus one can minorize the product \(|Q| \cdot |U| \cdot |R|\) by 
\[ 1.2 \cdot \chi_0 \cdot \chi_1 \cdot \chi_*^2 = 0.1930636291 \ldots > 0.1851580824 \ldots \geq |G| \]
from which the lemma follows.

**Proof of Lemma 12.** One has \( A_j^2 = CEF \), where 
\[ C := 1 + \rho^{6j+6}, \quad E := 1 + \rho^{6j+4} - \sqrt{2} \rho^{3j+2}, \quad F := 1 + \rho^{6j+2} - 2 \rho^{3j+1} \].

For \( \rho \in (0, 0.6) \), each factor \( C, E \) and \( F \) is positive-valued. Clearly \( E' = (3j+2)\rho^{3j+1}((2\rho^{3j+2} - \sqrt{2}) < 0 \), because \( 2\rho \leq 1.2 < \sqrt{2} \). We show that \((CF)' < 0 \) from which and from \( A_j > 0 \) follows that \( A_j' < 0 \). A direct computation shows that 
\[ (CF)' = \rho^{3j}(6j+6)\rho^{3j+5} + (6j+2)\rho^{3j+1} + (12j+8)\rho^{3j+7} - (6j+2) - (18j+14)\rho^{6j+6}. \]
For \( j \geq 1 \) and \( \rho \in (0, 0.6] \), the first three summands inside the brackets are majorized respectively by \((6j + 6) \cdot 0.6^8\), \((6j + 2) \cdot 0.6^4\) and \((12j + 8) \cdot 0.6^{16}\). Their sum is less than \(6j + 2\), so \((CF)' < 0\).

We set \( B_j^2 = MNW \), where

\[
M := (1 - \rho^{3j+1})^2 \quad , \quad N := 1 + \rho^{6j+4} + \sqrt{2} \rho^{3j+2} \quad , \quad W := (1 - \rho^{3j+3})^2 .
\]

Obviously \( W' < 0 \). We show that \((MN)' < 0\) which together with \( M > 0 \), \( N > 0 \), \( W > 0 \) and \( B_j > 0 \) proves that \( B_j' < 0 \). One has \((MN)' = (K_1 + K_2 - L_1) + (K_3 + K_4 - L_2) + (K_5 - L_3)\), where

\[
K_1 := (6j + 2)\rho^{6j+1} \quad , \quad K_2 := \sqrt{2}(3j + 2)\rho^{3j+1} \quad , \quad L_1 := (6j + 2)\rho^{3j} \quad ,
\]

\[
K_3 := (6j + 4)\rho^{6j+3} \quad , \quad K_4 := \sqrt{2}(9j + 4)\rho^{9j+3} \quad , \quad L_2 := 2\sqrt{2}(6j + 3)\rho^{6j+2} \quad ,
\]

\[
K_5 := (12j + 6)\rho^{12j+5} \quad \text{and} \quad L_3 := (18j + 10)\rho^{9j+4} .
\]

The inequality \( K_5 < L_3 \) is evident. One has \( K_4/L_2 \leq (3/4)\rho^{3j+1} \leq (3/4) \cdot 0.6^4 = 0.0972 \) and \( K_3/L_2 \leq (10/9) \cdot (0.6/2\sqrt{2}) = 0.2357022603 \ldots \), so \( K_3 + K_4 < L_2 \). Finally, \( K_1/L_1 \leq 0.6^{3j+1} < 0.6^4 = 0.1296 \) and \( K_2/L_1 \leq \sqrt{2} \cdot 0.6 = 0.8485281372 \ldots \) hence \( K_1 + K_2 < L_1 \) and \((MN)' < 0\). \( \square \)

## 5 Proof of Lemma 8

We set \( \theta^\dagger(q, z) := \theta(q, z/q) = \sum_{j=0}^{\infty} q^{j(j-1)/2}z^j = 1 + z + qz^2 + q^3z^3 + \cdots \). We show that \( \theta^\dagger(q, z) \neq 0 \) for \( |z| = |q|^{-1/2} \) from which the lemma follows. As \( \theta^\dagger(q, \bar{z}) = \theta^\dagger(q, z) \), we consider only the case \( \arg(q) \in [\pi/2, \pi] \). We assume that \(|q| \in [c_0, 0.6]\), because for \(|q| \leq c_0\), the zeros of \( \theta \) are strongly separated in modulus, see \([3]\). Set

\[
b_0 := 1 \quad , \quad b_1 := z \quad , \quad b_2 := qz^2 \quad \text{and} \quad r_k := \sum_{j=k}^{\infty} q^{j(j-1)/2}z^j .
\]

To show that \( \theta^\dagger \neq 0 \) we prove that the modulus of the sum of some or all of the terms \( b_0, b_1 \) and \( b_2 \) (we denote the set of the chosen terms by \( S \)) is larger than the modulus of the sum of the remaining terms of the series of \( \theta^\dagger \). We distinguish the following cases according to the intervals to which \( \arg(z) \) and \( \arg(q) \) belong:

**Case 1** \( \text{Re } z \geq 0 \), i.e. \( \arg(z) \in [-\pi/2, \pi/2] \), and \( \arg(q) \in [\pi/2, \pi] \). We set \( S := \{b_0, b_1\} \).

One has

\[
|r_2| \leq \sum_{j=2}^{\infty} q^{j(j-1)/2-j/2} =: \Phi_\theta(|q|) \quad , \quad |r_3| \leq \sum_{j=3}^{\infty} q^{j(j-1)/2-j/2} = \Phi_\theta(|q|) - 1 \quad (6)
\]

and

\[
|1 + z| \geq (1 + |z|^2)^{1/2} = (1 + |q|^{-1})^{1/2} =: \Phi_\theta(|q|) > \Phi_\theta(|q|) .
\]

The last inequality follows from \( \Phi_\theta(|q|) \) and \( \Phi_\theta(|q|) \) being respectively decreasing and increasing on \([c_0, 0.6] \) and

\[
\Phi_\theta(0.6) = 1.632993162 \ldots > 1.618354488 \ldots = \Phi_\theta(0.6) .
\]
Case 2) \( \arg(z) \in [\pi, 3\pi/2] \) and \( \arg(q) \in [\pi/2, \pi] \). Then \( \arg(qz) \in [3\pi/2, 5\pi/2] \), i.e. \( \Re(qz) \geq 0 \). We set \( S := \{b_1, b_2\} \). Hence

\[
|z + qz^2| = |z| \cdot |1 + qz| \geq |q|^{-1/2} \cdot (1 + |q|)^{1/2} = \Phi_\ast(|q|) .
\]

The other terms of \( \theta^3 \) are 1, \( q^2 z^3, q^6 z^4, \ldots \). For \( |z| = |q|^{-1/2} \), their moduli equal respectively 1, \( |q|^3/2, |q|^4, \ldots \), which are precisely the terms of the series \( \Phi_\ast \), so as in Case 1) one concludes that \( |z + qz^2| \geq \Phi_\ast(|q|) > \Phi_\ast(|q|) \geq 1 + |r_3| \).

Case 3) \( \arg(z) \in [\pi/2, 3\pi/4] \) and \( \arg(q) \in [\pi/2, \pi] \). The case is subdivided into five subcases in all of which we set \( S := \{b_0, b_1, b_2\} \).

Case 3A) \( \arg(z) \in [\pi/2, 3\pi/4] \) and \( \arg(q) \in [3\pi/4, \pi] \). Then \( \arg(qz^2) \in [7\pi/4, 5\pi/2] \). If \( \arg(qz^2) \in [2\pi, 5\pi/2] \), then

\[
|1 + z + qz^2| \geq \Im(1 + z + qz^2) \geq \Im(z) \geq \sin(3\pi/4) \cdot |q|^{-1/2} \geq (2 \cdot 0.6)^{-1/2} = 0.9128709292 \ldots > 0.618354488 \ldots = \Phi_0(0.6) - 1 \geq |r_3| ,
\]

see [7]. Set \( \tau(|q|) := (2 \cdot |q|)^{-1/2} - 2^{-1/2} = \sin(3\pi/4) \cdot |q|^{-1/2} + \sin(7\pi/4) = -\cos(3\pi/4) \cdot |q|^{-1/2} - \cos(7\pi/4) \). If \( \arg(qz^2) \in [7\pi/4, 2\pi] \), then

\[
\Im(1 + z + qz^2) \geq \tau(|q|) \quad \text{and} \quad \Re(1 + z + qz^2) \geq 1 - \tau(|q|)
\]

(one can observe that \( 1 - \tau(|q|) > 0 \) for \( q \geq c_0 \)). Thus

\[
|1 + z + qz^2| \geq (\tau^2 + (1 - \tau)^2)^{1/2} = ((1 - 2\tau)^2/2 + 1/2)^{1/2} .
\]

The latter quantity is minimal for \( \tau = 1/2 \), i.e. for \( |q| = 0.3431457506 \ldots \), when it equals \( 1/\sqrt{2} = 0.7071067814 \ldots > \Phi_0(0.6) - 1 \geq |r_3| \).

Case 3B) \( \arg(z) \in [5\pi/8, 3\pi/4] \subset [\pi/2, 3\pi/4] \) and \( \arg(q) \in [\pi/2, 3\pi/4] \). Then \( \arg(qz^2) \in [7\pi/4, 9\pi/4] \subset [7\pi/4, 5\pi/2] \) and one proves as in Case 3A) that \( |1 + z + qz^2| > |r_3| \).

Case 3C) \( \arg(z) \in [\pi/2, 5\pi/8] \) and \( \arg(q) \in [5\pi/8, 3\pi/4] \). Then \( \arg(qz^2) \in [13\pi/8, 2\pi] \). If \( |q| \in [0.3, 0.6] \), then

\[
|1 + z + qz^2| \geq \Re(1 + z + qz^2) \geq 1 + |q|^{-1/2} \cdot \cos(5\pi/8) + \cos(13\pi/8) \geq 1 + 0.3^{-1/2} \cdot \cos(5\pi/8) + \cos(13\pi/8) > 0.68 > \Phi_0(0.6) - 1 \geq |r_3| .
\]

If \( |q| \in [c_0, 0.3] \), then

\[
1 + |q|^{-1/2} \cdot \cos(5\pi/8) + \cos(13\pi/8) \geq 1 + c_0^{-1/2} \cdot \cos(5\pi/8) + \cos(13\pi/8) \geq 0.5433422972 \ldots > \Phi_0(0.3) - 1 = 0.1725370862 \ldots \geq |r_3| .
\]

Case 3D) \( \arg(z) \in [9\pi/16, 5\pi/8] \) and \( \arg(q) \in [\pi/2, 5\pi/8] \). Then \( \arg(qz^2) \in [13\pi/8, 15\pi/8] \). Just as in Case 3C) one obtains \( |1 + z + qz^2| > |r_3| \).

Case 3E) \( \arg(z) \in [\pi/2, 9\pi/16] \) and \( \arg(q) \in [\pi/2, 5\pi/8] \). Then \( \arg(qz^2) \in [3\pi/2, 7\pi/4] \) and
We set $z := q\xi$ hence $|\xi| = |q|^{-3/2}$. Thus $\theta = \sum_{j=0}^{\infty} q^{j(j-1)/2} \xi^j$. Next we set $A := 1 + \sum_{j=4}^{\infty} q^{j(j-1)/2} \xi^j$ and $B := 1 + q\xi + q^2 \xi^2$, so $\theta = A + \xi B$. Finally we set $\xi := q\xi$, thus $B = 1 + \xi + q\xi^2$ and $|\xi| = |q|^{-1/2}$.

6 Proof of Lemma 9

We consider four subcases in all of which we set $S$ := \{b_0, b_1, b_2\}:

Case 4A) $\arg(z) \in [3\pi/4, \pi]$ and $\arg(q) \in [\pi/2, \pi]$. Then $\arg(qz^2) \in [2\pi, 3\pi]$, i.e. $\Im(qz^2) > 0$. We consider four subcases in all of which we set $S$ := \{b_0, b_1, b_2\}:

Case 4B) $\arg(z) \in [5\pi/6, 11\pi/12]$ and $\arg(q) \in [\pi/2, \pi]$. Hence $\arg(qz^2) \in [13\pi/6, 17\pi/6]$ (with $\sin(13\pi/6) = \sin(17\pi/6) = 1/2$) and

$|1 + z + qz^2| \geq \Im(1 + z + qz^2) \geq 0.6^{-1/2} \cdot \sin(5\pi/6) = 0.64 \ldots > \Phi_5(0.6) - 1 \geq |r_3|$.

Case 4C) $\arg(z) \in [11\pi/12, \pi]$ and $\arg(q) \in [\pi/2, 5\pi/6]$ hence $\arg(qz^2) \in [7\pi/3, 17\pi/6]$. If $\arg(qz^2) \in [7\pi/3, 5\pi/2]$, then

$|1 + z + qz^2| \geq \Im(1 + z + qz^2) \geq \Im(qz^2) \geq \sin(7\pi/3) = \sqrt{3}/2 = 0.86 \ldots > \Phi_5(0.6) - 1 \geq |r_3|$.

If $\phi := \arg(qz^2) \in [5\pi/2, 17\pi/6]$, then

$\Im(1 + z + qz^2) \geq \Im(qz^2) = \sin(\phi)$,\\
$|\Re(1 + z + qz^2)| \geq |0.6^{-1/2} \cdot \cos(11\pi/12) + 1 + \cos(\phi)|$ and\\
$|1 + z + qz^2| \geq T(\phi) := (\sin^2(\phi) + (0.6^{-1/2} \cdot \cos(11\pi/12) + 1 + \cos(\phi))^2)^{1/2}$.

The function $T(\phi)$ takes only values larger than 1 (hence larger than $|r_3|$) for $\phi \in [5\pi/2, 17\pi/6]$.

Case 4D) $\arg(z) \in [11\pi/12, \pi]$ and $\arg(q) \in [5\pi/6, \pi]$, so $\arg(qz^2) \in [8\pi/3, 3\pi]$ and

$\Re(1 + z + qz^2) \leq 0.6^{-1/2} \cdot \cos(11\pi/12) + 1 + \cos(8\pi/3) = -0.7470048804 \ldots$.

Hence $|1 + z + qz^2| \geq |\Re(1 + z + qz^2)| > \Phi_5(0.6) - 1 \geq |r_3|$.
Lemma 13. (1) For $|q| \leq 0.6$ and $|\xi| \leq |q|^{-3/2}$, one has $|A| \leq a_0 := 1 + \sum_{j=4}^{\infty} 0.6^{j-1/2-3j/2} = 2.330487021\ldots$

(2) For $|q| \leq 0.6$, $\text{arg}(q) \in [2\pi/3, \pi]$ and $|\xi| \leq |q|^{-3/2}$, one has $|\xi B| > a_0 \geq |A|$ hence $\theta \neq 0$.

(3) For $|q| \leq 0.55$ and $|\xi| \leq |q|^{-3/2}$, one has $|\xi B| > a_0 \geq |A|$ hence $\theta \neq 0$.

Remark 14. For $\text{arg}(q) \in [\pi/2, 2\pi/3]$, it is possible to show numerically that $|\xi B| > a_0 \geq |A|$ also for $|q| \in [0.55, 0.6]$. To this end one can compute the quantities $|\xi B|$ and $|A|$ with sufficiently small steps in $\text{arg}(q)$ and $|q|$. To obtain a majoration of $|A|$ one can set $A := A^* + A^{**}$ with $A^* := 1 + \sum_{j=4}^{7} q^{j(j-1)/2} / \xi j$ and $A^{**} := \sum_{j=8}^{\infty} q^{j(j-1)/2} / \xi j$. For $|\xi| = |q|^{-3/2}$ and $|q| \leq 0.6$, one has $|A^{**}| \leq \sum_{j=8}^{\infty} 0.6^{j(j-1)/2-3j/2} = 0.0002925303367\ldots$. Thus there remains to estimate $|\xi B|$ and $|A^*|$ which contain finitely-many terms.

Proof of Lemma 13. Part (1) follows from $|A| \leq 1 + \sum_{j=4}^{\infty} |q|^{j(j-1)/2-3j/2} \leq a_0$. To prove parts (2) and (3) we set $q := re^{i\omega}$, $\rho \geq 0$, $\omega \in [\pi/2, \pi]$, and $\zeta := e^{i\psi/2}$, $\psi \in [0, 2\pi]$. Observe that $|q\zeta^2| = 1$. Thus

$$B = 1 + \rho^{-1/2}e^{i\psi} + e^{i(2\psi+\omega)} \quad \text{and}$$

$$|B|^2 = (1 + \cos(2\psi + \omega) + \rho^{-1/2} \cos \psi)^2 + (\sin(2\psi + \omega) + \rho^{-1/2} \sin \psi)^2$$

$$= 2 + \rho^{-1} + 2\rho^{-1/2} \cos(\psi + \omega) + 2 \cos(2\psi + \omega) + 2\rho^{-1/2} \cos \psi$$

$$= 2 + \rho^{-1} + 4\rho^{-1/2} \cos(\psi + \omega/2) \cos(\omega/2) + 2 \cos(2\psi + \omega)$$

$$= \rho^{-1} + 4\rho^{-1/2} \cos(\psi + \omega/2) \cos(\omega/2) + 4 \cos^2(\psi + \omega/2).$$

Set $a := \cos(\psi + \omega/2) \in [-1, 1]$ and $b := \cos(\omega/2) \in [0, \sqrt{2}/2]$ (because $\omega/2 \in [\pi/4, \pi/2]$). The polynomial $4a^2 + 4\rho^{-1/2}ab + \rho^{-1}$ (considered as a polynomial in $a$ with $b$ as a parameter) takes its minimal value for $a = -\rho^{-1/2}b/2$. This value is $(1 - b^2)\rho^{-1}$. Therefore for $b \leq 1/2$ (hence for $\omega \in [2\pi/3, \pi]$), one has $|B|^2 \geq 3 \cdot \rho^{-1/4} \geq 3 \cdot 0.6^{-1/4}$. So for $|\xi| = |q|^{-3/2} \geq 0.6^{-3/2}$, one has

$$|\xi B| = |q|^{-3/2} \cdot |B| \geq 0.6^{-3/2} \cdot (3 \cdot 0.6^{-1/4})^{1/2} = 2.405626123\ldots > a_0 \geq |A|.$$

For $\rho = |q| \leq 0.55$, one has $|B|^2 \geq \rho^{-1/2}$ and $|\xi B| \geq 0.55^{-2}/\sqrt{2} = 2.337543079\ldots > a_0 \geq |A|$. □

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