ON THE DEFORMATIONS OF DORFMAN’S AND
SOKOLOV’S OPERATORS

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Abstract. We deform the Dorfman’s and Sokolov’s Hamiltonian operators by the quasi-Miura transformation coming from the topological field theory and investigate the deformed operators.

1. Introduction

The Dorfman’s and Sokolov’s Hamiltonian operators are defined respectively as [2, 11] (D = ∂x)

\begin{align}
J &= D\frac{1}{v_x}D\frac{1}{v_x}D \\
S &= v_xD^{-1}v_x,
\end{align}

which are Hamiltonian operators ( or \( J^{-1} = D^{-1}\frac{1}{v_x}D\frac{1}{v_x}D^{-1} \) and \( S^{-1} = \frac{1}{v_x}D\frac{1}{v_x} \) are symplectic operators). The Dorfman’s operator \( J \) (or \( J^{-1} \)) and the Sokolov’s operator \( S \) are related to integrable equations as follows.

• The Riemann hierarchy

\[
v_{t_n} = v^n v_x = S\delta H_n = \frac{1}{(n+1)(2n+1)}K\delta H_{n+1} = \frac{1}{(n+1)(n+2)}D\delta H_{n+2} = \frac{1}{(n+1)(n+2)(n+3)(n+4)}J\delta H_{n+4},
\]

where \( K = Dv + vD \), \( H_n = \int v^n dx, n = 1, 2, 3 \cdots \),

and \( \delta \) is the variational derivative. When \( n = 1 \), it is called the Riemann equation or dispersion less KdV equation. We
notice that it seems that the Riemann hierarchy is a quarternion-Hamiltonian system. But one can show that $S$ and $J$ is not compatible, i.e., $S + \lambda J$ is not Hamiltonian operator for any $\lambda \neq 0$(see below).

- The Schwarzian KdV equation [10, 13]

\[ v_t = v_{xxx} - \frac{3v_x^2}{2} = v_x \{v, x\} = S\delta H_1 = J^{-1}\delta H_2, \]

where $\{v, x\}$ is the Schwartz derivative and

\[
H_1 = \frac{1}{2} \int \left( v_x^{-2} v_{xx}^2 \right) dx, \quad H_2 = \frac{1}{2} \int \left( -v_x^{-2} v_{xxx}^2 + \frac{3}{4} v_x^{-4} v_{xx}^4 \right) dx.
\]

Remark: It is not difficult to verify that $J^{-1}$ is also a Hamiltonian operator and, then, $J$ is also a symplectic operator; however, $S^{-1} = \frac{1}{v_x} D \frac{1}{v_x}$ is not a Hamiltonian operator and, then, $S$ is not a symplectic operator.

Next, to deform the operators $J$ and $S$, we use the free energy in topological field theory of the famous KdV equation

\[ u_t = uu_x + \frac{\epsilon^2}{12}u_{xxx} \]

to construct the quasi-Miura transformation as follows. The free energy $F$ of KdV equation [14] in TFT has the form $F_0 = \frac{1}{6}v^3$.

\[
F = \frac{1}{6}v^3 + \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g(v; v_x, v_{xx}, v_{xxx}, \cdots, v^{(3g-2)}).
\]

Let

\[
\Delta F = \sum_{g=1}^{\infty} \epsilon^{2g-2} F_g(v; v_x, v_{xx}, v_{xxx}, \cdots, v^{(3g-2)})
\]

\[= F_1(v; v_x) + \epsilon^2 F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}) + \epsilon^4 F_3(v; v_x, v_{xx}, v_{xxx}, v_{xxxx}, \cdots, v^{(7)}) + \cdots. \]
The $\Delta F$ will satisfy the loop equation (p.151 in [4])

$$
\begin{align*}
\sum_{r \geq 0} \frac{\partial \Delta F}{\partial v^{(r)}} \frac{1}{v - \lambda} + \sum_{r \geq 1} \frac{\partial \Delta F}{\partial v^{(r)}} \sum_{k=1}^{r} \binom{r}{k} \\
\partial_x^{k-1} \frac{1}{\sqrt{v - \lambda}} \partial_x^{r-k+1} \frac{1}{\sqrt{v - \lambda}} \\
= \frac{1}{16\lambda^2} - \frac{1}{16(v - \lambda)^2} - \frac{\kappa_0}{\lambda^2} \\
+ \frac{\epsilon^2}{2} \sum_{k,l \geq 0} \left[ \frac{\partial^2 \Delta F}{\partial v^{(k)} \partial v^{(l)}} + \frac{\partial \Delta F}{\partial v^{(k)}} \frac{\partial \Delta F}{\partial v^{(l)}} \right] \partial_x^{k+1} \frac{1}{\sqrt{v - \lambda}} \partial_x^{l+1} \frac{1}{\sqrt{v - \lambda}} \\
- \frac{\epsilon^2}{16} \sum_{k \geq 0} \frac{\partial \Delta F}{\partial v^{(k)}} \partial_x^{k+2} \frac{1}{(v - \lambda)^2}.
\end{align*}
$$

Then we can determine $F_1, F_2, F_3, \ldots$ recursively by substituting $\Delta F$ into equation (5). For $F_1$, one obtains

$$
\frac{1}{v - \lambda} \frac{\partial F_1}{\partial v} - \frac{3}{2} \frac{v_x}{(v - \lambda)^2} \frac{\partial F_1}{\partial v_x} = \frac{1}{16\lambda^2} - \frac{1}{16(v - \lambda)^2} - \frac{\kappa_0}{\lambda^2}.
$$

From this, we have

$$
\kappa_0 = \frac{1}{16}, \quad \lambda = \frac{1}{24} \log v_x.
$$

For the next terms $F_2(v; v_x, v_{xx}, v_{xxx}, v_{xxxx})$, it can be similarly computed and the result is

$$
F_2 = \frac{v_{xxxx}}{1152v_x^2} - \frac{7v_{xx}v_{xxx}}{1920v_x^3} + \frac{v_{xx}^3}{360v_x^4}.
$$

Now, one can define the quasi-Miura transformation as

$$
(6) \quad u = v + \epsilon^2 (\Delta F)_{xx} = v + \epsilon^2 (F_1)_{xx} + \epsilon^4 (F_2)_{xx} + \cdots
$$

$$
= v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + \epsilon^4 \left( \frac{v_{xxxx}}{1152v_x^2} - \frac{7v_{xx}v_{xxx}}{1920v_x^3} + \frac{v_{xx}^3}{360v_x^4} \right)_{xx} + \cdots.
$$

One remarks that Miura-type transformation means the coefficients of $\epsilon$ are homogeneous polynomials in the derivatives $v_x, v_{xx}, \ldots, v^{(m)}$ (p.37 in [4], [5]) and "quasi" means the ones of $\epsilon$ are quasi-homogeneous rational functions in the derivatives, too (p.109 in [4] and see also [12]).
The truncated quasi-Miura transformation

\[ u = v + \sum_{n=1}^{g} \epsilon^{2n} \left[ F_n(v; v_x, v_{xx}, \cdots, v^{(3g-2)}) \right]_{xx} \]

has the basic property (p.117 in [4]) that it reduces the Magri Poisson pencil [6] of KdV equation [4]

\[ \{u(x), u(y)\}_\lambda = [u(x) - \lambda] D \delta(x-y) + \frac{1}{2} u_x(x) \delta(x-y) + \frac{\epsilon^2}{8} D^3 \delta(x-y) \]

to the Poisson pencil of the Riemann hierarchy (??):

\[ \{v(x), v(y)\}_\lambda = [v(x) - \lambda] D \delta(x-y) + \frac{1}{2} v_x(x) \delta(x-y) + O(\epsilon^{2g+2}). \]

One can also say that the truncated quasi-Miura transformation (7) deforms the KdV equation (4) to the Riemann equation \( v_t = vv_x \) up to \( O(\epsilon^{2g+2}) \).

**Remark:** A simple calculation shows that, under the transformation \( u = \frac{\epsilon^2}{24}(m, x) \), the KdV equation (4) is transformed into the Schwarzian KdV equation

\[ m_t = \frac{\epsilon^2}{12} m_x \{m, x\} = \frac{\epsilon^2}{12} (m_{xxx} - \frac{3}{2} m_x^2). \]

Furthermore, after a direct calculation, one can see that the Magri Poisson bracket

\[ K(\epsilon) = \{u(x), u(y)\} = u(x) D \delta(x-y) + \frac{1}{2} u_x(x) \delta(x-y) + \frac{\epsilon^2}{8} D^3 \delta(x-y) \]

is transformed into the Dorfman’s symplectic operator \( J^{-1} \) (\( m = v \))

\[ \{m(x), m(y)\} = -\frac{\epsilon^2}{8} D^{-1} m_x D^{-1} m_x D^{-1} \delta(x-y). \]

Now, a natural question arises: under the truncated quasi-Miura transformation (7), are the deformed Dorfman’s operator \( J(\epsilon) \) and Sokolov’s operator \( S(\epsilon) \) still Hamiltonian operators up to \( O(\epsilon^{2g+2}) \)? For simplicity, we consider only the case \( g = 1 \), i.e.,

\[ u = v + \frac{\epsilon^2}{24} (\log v_x)_{xx} + O(\epsilon^4) \]
or

\[ v = u - \frac{\epsilon^2}{24} (\log u_x)_{xx} + O(\epsilon^4). \]

The answer is true for the Dorfman’s operator \( J(\epsilon) \) but it’s false for the Sokolov’s operator \( S(\epsilon) \). It’s the purpose of this article.
2. Deformations under quasi-Miura Transformation

In the new "u-coordinate", $J$ and $S$ will be given by the operators

\begin{align*}
J(\epsilon) &= M^* D \left( \frac{1}{u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}} D \frac{1}{u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}} D M \right) + O(\epsilon^4); \\
S(\epsilon) &= M^* (u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}) D^{-1} (u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx}) M + O(\epsilon^4),
\end{align*}

where

\begin{align*}
M &= 1 - \frac{\epsilon^2}{24} D \frac{1}{u_x} D^2 \\
M^* &= 1 + \frac{\epsilon^2}{24} D^2 \frac{1}{u_x} D,
\end{align*}

$M^*$ being the adjoint operator of $M$. Then we have the following

**Theorem 1.** (1) $J(\epsilon)$ is a Hamiltonian operator up to $O(\epsilon^4)$. (2) $S(\epsilon)$ is not a Hamiltonian operator up to $O(\epsilon^4)$.

**Proof.** (1) The fact that $J(\epsilon)$ is a skew-adjoint (or $J^*(\epsilon) = -J(\epsilon)$) differential operator (up to $O(\epsilon^4)$) follows immediately from (13). Rather than prove the Poisson form \[7\] of the Jacobi identity for $J(\epsilon)$, it is simpler to prove that the symplectic two form

\[ \Omega_J(\epsilon) = \int \{ du \wedge J(\epsilon)^{-1} du \} dx + O(\epsilon^4) \]
is closed \[ d\Omega_J(\epsilon) = O(\epsilon^4). \]

A simple calculation can yield

\[
J(\epsilon)^{-1} = (1 + \frac{\epsilon^2}{24} D^1 \frac{1}{u_x} D^2) D^{-1}(u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx}) D^{-1}(u_x - \frac{\epsilon^2}{24} (\log u_x)_{xxx}) D^{-1}
\]

\[
= (D^{-1} u_x - \frac{\epsilon^2}{24} D^{-1}(\log u_x)_{xxx} + \frac{\epsilon^2}{24} D^1 \frac{1}{u_x} Du_x^0 D^{-1}(u_x D^{-1} - \frac{\epsilon^2}{24}(\log u_x)_{xxx} D^{-1} - \frac{\epsilon^2}{24} u_x D^{-1} D^{-1} (\log u_x))_{xxx} D^{-1} - \frac{\epsilon^2}{24} u_x D^{-1} D^{-1}(\log u_x)_{xxx} D^{-1} + O(\epsilon^4)
\]

\[
= D^{-1} u_x D^{-1} u_x D^{-1} + \frac{\epsilon^2}{24} [D^1 \frac{1}{u_x} Du_x^0 D^{-1} - D^{-1}(\log u_x)_{xxx} D^{-1} u_x D^{-1} - D^{-1} u_x D^{-1} u_x D^{-1} (\log u_x)_{xxx} D^{-1}] + O(\epsilon^4)
\]

\[
= D^{-1} u_x D^{-1} u_x D^{-1} + \frac{\epsilon^2}{24} [Du_x^0 D^{-1} - D^{-1} u_x D + (\log u_x)_{xxx} u_x D^{-1} + D^{-1} (\log u_x)_{xxx} u_x] + O(\epsilon^4).
\]

Let \( \psi \) denote the potential function for \( u \), i.e., \( u = \psi_x \). Thus, formally,

\[
D_x^{-1}(du) = d\psi
\]

and hence, after a series of integration by parts, one has

\[
\Omega_J(\epsilon) = \int \left\{ \left[ (D^{-1} d(\frac{\psi_x^2}{2}) \wedge d(\frac{\psi_x^2}{2}) - \psi_x d\psi \wedge d(\frac{\psi_x^2}{2}) \right] + \frac{\epsilon^2}{24} [2\psi_{xx} d\psi \wedge d\psi_{xx} + 2\psi_{xxx} d\psi_x \wedge d\psi_x] \right\} dx + O(\epsilon^4).
\]

So

\[
d\Omega_J(\epsilon) = \int \left\{ 0 + \frac{\epsilon^2}{12} [d\psi_{xxx} \wedge d\psi_x \wedge d\psi_x] \right\} dx + O(\epsilon^4)
\]

\[
= \frac{\epsilon^2}{12} \int \left\{ (d\psi_{xx} \wedge d\psi_x \wedge d\psi_x) \right\} dx + O(\epsilon^4) = O(\epsilon^4).
\]

This completes the proof of (1).

(2) The skew-adjoint property of the deformed Sokolov’s operator \( S(\epsilon) \) (14) is obvious. To see whether \( S(\epsilon) \) is Hamiltonian operator or not, we must check \( S(\epsilon) \) whether satisfy the Jacobi identity up to \( O(\epsilon^4) \). Following [7, 8], we introduce the arbitrary basis of tangent vector \( \Theta \), which is then conveniently manipulated according to the rules of exterior calculus. The Jacobi identity is given by the compact expression
where $P(\epsilon) = S(\epsilon)\Theta$, $I = \frac{1}{2}\Theta \wedge P(\epsilon)$ and $\delta$ denotes the variational derivative. The vanishing of the tri-vector \((15)\) modulo a divergence is equivalent to the satisfaction of the Jacobi identity.

After a tedious calculation, one can obtain

\[
S(\epsilon) = M^*(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx})D^{-1}(u_x - \frac{\epsilon^2}{24}(\log u_x)_{xxx})M + O(\epsilon^4)
\]

\[
= [u_x + \frac{\epsilon^2}{24}(D^3 + D^2(\log u_x)_x - (\log u_x)_{xxx})]D^{-1}[u_x - \frac{\epsilon^2}{24}(D^3 - (\log u_x)_x D^2 \\
+ (\log u_x)_{xxx})] + O(\epsilon^4)
\]

\[
= u_x D^{-1}u_x + \frac{\epsilon^2}{24}[D^2 u_x + D^2(\log u_x)_x D^{-1}u_x - (\log u_x)_{xxx} D^{-1}u_x - u_x D^2 \\
+ u_x D^{-1}(\log u_x)_{xx} D^2 - u_x D^{-1}(\log u_x)_{xxx}] + O(\epsilon^4)
\]

\[
= u_x D^{-1}u_x + \frac{\epsilon^2}{24}[D^2 u_x - u_x D^2 + (\log u_x)_x D u_x + u_x D(\log u_x)_x] + O(\epsilon^4)
\]

\[
= u_x D^{-1}u_x + \frac{\epsilon^2}{12}[D u_{xx} + u_{xx} D] + O(\epsilon^4).
\]

So

\[
P(\epsilon) = S(\epsilon)\Theta = u_x D^{-1}(u_x \Theta) + \frac{\epsilon^2}{12}[2u_{xx} \Theta_x + u_{xxx} \Theta] + O(\epsilon^4).
\]

Hence

\[
I = \frac{1}{2}\Theta \wedge P(\epsilon) = \frac{1}{2} u_x \Theta \wedge D^{-1}(u_x \Theta) + \frac{\epsilon^2}{12} u_{xx} \Theta \wedge \Theta_x + O(\epsilon^4)
\]

and then

\[
\delta I = -\frac{1}{2} [\Theta \wedge D^{-1}(u_x \Theta)]_x - \frac{1}{2} u_x \Theta \wedge D^{-1}(\Theta_x) + \frac{\epsilon^2}{12} [\Theta \wedge \Theta_x]_{xx} + O(\epsilon^4)
\]

\[
= -\frac{1}{2} \Theta_x \wedge D^{-1}(u_x \Theta) + \frac{\epsilon^2}{12} [\Theta \wedge \Theta_x]_{xx} + O(\epsilon^4).
\]
Finally,

\[
P(\epsilon) \land \delta I = \{u_x D^{-1}(u_x \Theta) + \frac{\epsilon^2}{12} [2u_{xx} \Theta_x + u_{xxx} \Theta] \} \land \{-\frac{1}{2} \Theta_x \land D^{-1}(u_x \Theta) \\
+ \frac{\epsilon^2}{12} [\Theta \land \Theta_x]_{xx} \} + O(\epsilon^4)
\]

\[
= 0 + \frac{\epsilon^2}{12} \{-\frac{1}{2} u_{xxx} \Theta_x \land \Theta_x \land D^{-1}(u_x \Theta) + u_{xxx} D^{-1}(u_x \Theta) \land \Theta \land \Theta_x \\
+ 3u_{xx} u_x \Theta \land \Theta \land \Theta_x + u^2 \Theta_x \land \Theta \land \Theta_x \} + O(\epsilon^4)
\]

\[
= 0 + \frac{\epsilon^2}{24} u_{xxx} \Theta \land \Theta_x \land D^{-1}(u_x \Theta),
\]

which can be easily checked that it can’t be expressed as a total divergence. So \(S(\epsilon)\) can’t satisfy the Jacobi identity and therefore \(S(\epsilon)\) is not a Hamiltonian operator. This completes the proof of (2). \(\square\)

**Remark:** Using the technics of the last proof, one can show that \(J\) and \(S\) is not compatible. Since \(J\) and \(S\) are Hamiltonian operators, what we are going to do is show that \([7, 8]\)

\[
\tilde{Q}(\Theta) \land \delta R + Q(\Theta) \land \tilde{\delta} R \neq 0, \quad (\text{mod. div.})
\]

where

\[
Q(\Theta) = v_x D^{-1}(v_x \Theta), \quad R = \frac{1}{2} \Theta \land Q(\Theta)
\]

\[
\tilde{Q}(\Theta) = \left(\frac{1}{v_x} \left(\frac{\Theta_x}{v_x}\right)_x\right)_x, \quad \tilde{R} = \frac{1}{2} \Theta \land \tilde{Q}(\Theta) = -\frac{1}{2v_x^2} \Theta_x \land \Theta_{xx}.
\]

Then

\[
\delta R = -\frac{1}{2} [\Theta \land D^{-1}(v_x \Theta)]_x - \frac{1}{2} v_x \Theta \land D^{-1}(\Theta_x) = -\frac{1}{2} \Theta_x \land D^{-1}(v_x \Theta)
\]

and

\[
\delta \tilde{R} = -\frac{1}{v_x^3} \Theta_x \land \Theta_{xx}.\]
Hence

\[
\tilde{Q}(\Theta) \wedge \delta R + Q(\Theta) \wedge \delta \tilde{R} = (\frac{1}{v_x} (\frac{\Theta_x}{v_x})_x \wedge \frac{-1}{2} \Theta_x \wedge D^{-1}(v_x \Theta)) - v_x D^{-1}(v_x \Theta) \wedge (\frac{1}{v_x^3} \Theta_x \wedge \Theta_{xx})_x \\
= \frac{1}{2} \frac{1}{v_x} (\frac{\Theta_x}{v_x})_x \wedge [\Theta_{xx} \wedge D^{-1}(v_x \Theta) + v_x \Theta_x \wedge \Theta] + [v_{xx} D^{-1}(v_x \Theta) + v_x^2 \Theta] \\
\wedge (\frac{1}{v_x^3} \Theta_x \wedge \Theta_{xx}) \\
= \frac{1}{2} \frac{1}{v_x} \Theta_{xx} \wedge \Theta_x \wedge \Theta - \frac{v_{xx}}{2 v_x^3} \Theta_x \wedge \Theta_{xx} \wedge D^{-1}(v_x \Theta) + \frac{v_{xx}}{v_x^3} D^{-1}(v_x \Theta) \wedge \Theta_x \wedge \Theta_{xx} \\
+ \frac{1}{v_x} \Theta \wedge \Theta_x \wedge \Theta_{xx} \\
= \frac{1}{2} \frac{1}{v_x} \Theta \wedge \Theta_x \wedge \Theta_{xx} + \frac{v_{xx}}{2 v_x^3} \Theta_x \wedge \Theta_{xx} \wedge D^{-1}(v_x \Theta) \\
\neq 0 \quad \text{(mod. div.)},
\]

as required.

3. Concluding Remarks

- That \( J(\epsilon) \) is a Hamiltonian operator (up to \( O(\epsilon^4) \)) is proved in [1]. One gives another proof here, which remarkably simplifies the proof given in [1].
- We notice that all the deformed operators \( J(\epsilon)[13] \), \( D(\epsilon)(= D + O(\epsilon^4)) \), \( K(\epsilon)[10] \) under the quasi-Miura transformation [3] are Hamiltonian operators (up to \( O(\epsilon^4) \)). That the deformed Sokolov’s operator \( S(\epsilon) \) is not Hamiltonian is a little surprised, which means that the Poisson bracket of the Hamiltonians \( H_m(u; \epsilon), H_n(u; \epsilon) \) for \( S(\epsilon) \)

\[
\{H_m(u; \epsilon), H_n(u; \epsilon)\}_{S(\epsilon)}
\]

won’t be \( O(\epsilon^4) \) but \( O(\epsilon^2) \), i.e., it can’t be a conserved quantity of the Riemann hierarchy.

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