Circular Orbits in the Taub-NUT and mass-less Taub-NUT Space-time

Parthapratim Pradhan

Department of Physics, Hiralal Mazumdar Memorial College For Women, Dakshineswar, Kolkata-700035, India.

Received Day Month Year
Revised Day Month Year
Communicated by Managing Editor

In this work we study the equatorial causal geodesics of the Taub-NUT (TN) space-time in comparison with mass-less TN space-time. We emphasized both on the null circular geodesics and time-like circular geodesics. From the effective potential diagram of null and time-like geodesics, we differentiate the geodesics structure between TN spacetime and mass-less TN space-time. It has been shown that there is a key role of the NUT parameter to changes the shape of pattern of the potential well in the NUT spacetime in comparison with mass-less NUT space-time. We compared the ISCO (innermost stable circular orbit), MBCO (marginally bound circular orbit) and CPO (circular photon orbit) of the said space-time with graphically in comparison with mass-less cases. Moreover, we compute the radius of ISCO, MBCO and CPO for extreme TN black hole (BH). Interestingly, we show that these three radii coincides with the Killing horizon i.e. the null geodesic generators of the horizon. Finally in Appendix-A, we compute the center-of-mass (CM) energy for TN BH and mass-less TN BH. We show that in both cases the CM energy is finite. For extreme NUT BH, we have found that the diverging nature of CM energy. First, we have observed that a non-asymptotic flat, spherically symmetric and stationary extreme BH is showing such feature.

Keywords: ISCO, MBCO, CPO, Taub-NUT spacetime, CM energy

1. Introduction

The Taub-NUT (Newman, Unti and Tamburino) space-time \cite{12} is a stationary, spherically symmetric and non-asymptotically flat solution of the vacuum Einstein equation in general theory of relativity. The space-time has topology $\mathbb{R} \times S^3$ with Lorentzian signature $\mathbb{M}$. The NUT space-time is described by two parameters: one is the mass parameter $M$ and another one is the NUT parameter $n$. There is no modification required in the Einstein-Hilbert action to accommodate the NUT charge \cite{6} or "dual mass" \cite{78} or "gravito-magnetic mass" or "gravito-magnetic monopole" \cite{9}. This dual mass is an intrinsic feature of general theory of relativity. The space-time contains closed time-like curve and null lines. It is a geodetically

*pppradhan77@gmail.com.
incomplete space-time \[3\]. Bonor \[4\] has given a new interpretation of the NUT spacetime and it describes ‘the field of a spherically symmetric mass together with a semi-infinite massless source of angular momentum along the axis of symmetry’. On the other hand, Manko and Ruiz \[5\] analyzed the mass and angular momentum distributions in case of generalized NUT spacetime using Komar integral approach.

't Hooft and Polykov \[10,11\] have demonstrated that the magnetic monopole present in certain non-Abelian gauge theories. Zee \[6\] observed that there is an existence of a gravitational analog of Dirac’s magnetic monopole \[12\]. The author is also discussed regarding the mass quantization following the idea of Dirac quantization rule. He also claimed that there is certainly no experimental evidence of mass quantization. Moreover, he proposed that if mass is quantized there may have profound consequences in physics. For example, if a magnetic monopole moving around a nucleus then the mass quantization rule suggests that the binding energy of every level in the nucleus is also quantized. Friedman and Sorkin \[13\] observed that the gravito-pole may exist in topological solution. Dowker \[14\] proposed that the NUT spacetime as a ‘gravitational dyon’.

The Euclidean version of the space-time is closely related to the dynamics of BPS (Bogomol’nyi-Prasad-Sommerfield) monopoles \[15\]. The experimental evidence of this dual mass has not been verified till now. There may be a possibilities of experimental evidences in near future and it was first proposed by Lynden-Bell and Nouri-Zonoz \[16\] in 1998. Letelier and Vieira \[17\] have observed that the manifestation of chaos for test particles moving in a TN space-time perturbed by dipolar halo using Poincare sections. The geodesics structure in Euclidean TN space-time has been studied in the Ref. \[18\].

The gravito-magnetic lensing effect in NUT space-time was first studied by Nouri-Zonoz et al. \[19\] in 1997. They proved that all the geodesics in NUT spacetime confined to a cone with the opening angle \(\delta\) defined by

\[
\sin \delta = \frac{2n}{D \sqrt{1 + \frac{4n^2}{D^2}}} 
\tag{1}
\]

where \(D = \frac{L}{E}\) is the impact factor. For small \(\alpha\) and in the limit \(\frac{2n}{D} \ll 1\), it should be

\[
\alpha \approx \frac{2n}{D} 
\tag{2}
\]

It should also be noted that the opening angle is proportional to the NUT parameter \(n\).

Furthermore, they also examined the lensing of light rays passing through the NUT deflector. This properties modified the observed shape, size and orientation of a source. It has been also studied there that there is an extra shear due to the presence of the gravito-magnetic monopole, which changes the shape of the source.

\[\text{This parameter is defined in Eq. (18) for null circular geodesics.}\]
The same author also studied the electromagnetic waves in NUT space through the solutions of the Maxwell equations via Newman-Penrose null tetrad formalism to further deeper insight of the physical aspects of the dual mass. Since the TN space-time has gravito-magnetic monopole that changes the structure of the accretion disk and might offer novel observational prospects [20,21].

The maximal analytic extension or Kruskal like extension of the TN space-time shows that it has some unusual properties [22]. Maximal analytic extension is needed in order to understand the global properties of the space-time. Misner and Taub have shown that TN space is maximally analytic i.e. it has no Hausdorff extension [3]. Whereas Hajicek [23] showed that the non-Hausdorff property occurs only on the Killing horizons and causes no geodesics to bifurcate.

Chakraborty and Majumdar [24] have derived the exact Lense-Thirrring precession (inertial frame dragging effect) in case of the TN space-time in comparison with the mass-less TN space-time. The mass-less dual mass (i.e. TN space-time with $M = 0$) concept was first introduced by Ramaswamy and Sen [7]. They also proved that ‘in the absence of gravitational radiation magnetic mass requires either that the metric be singular on a two dimensional world sheet or the space-time contain closed time-like lines, violating causality’. After that Ashtekar and Sen [25] demonstrated that the consequences of magnetic mass in quantum gravity. They also proved that the dual mass implies the existence of ‘wire singularities’ in certain potentials for Weyl curvature. Finally, Mueller and Perry [26] showed that the ‘mass quantization’ rule regarding the NUT space-time.

In [27], the author has been studied SU(2) time-dependent tensorial perturbations of Lorentzian TN space-time and proved that Lorentzian TN space-time is unstable. Geodesics of accelerated circular orbits on the equatorial plane has been studied in detail of the NUT space using Frenet-Serret procedure [28].

However, in the present work we wish to investigate the complete geodesic structure of the TN space-time in the equatorial plane. We compare the circular geodesics in the TN space-time with mass-less TN space-time and zero NUT parameter by analyzing the effective potential graphically for both null cases and time-like cases. The presence of the dual mass can changes the geodesic structure in comparison with the mass-less dual mass and zero dual mass. This is clearly manifested in the effective potential diagram. We also differentiate graphically the ISCO, MBCO and CPO of the said space-time in comparison with the mass-less cases. Moreover, we examine the circular geodesics in the $L - r$ plane for different values of energy i.e. $E^2 > 1$, $E^2 < 1$ and $E^2 = 1$ in case of TN and mass-less TN spacetime, and plotted graphically. Furthermore, we have studied more exotic cases i.e. extreme cases, where we find surprising results that the radii of three important class of orbits namely, the radius of ISCO ($r_{isco}$), the radius of MBCO ($r_{mbco}$) and the the radius

---

It should be noted that due to many pathological properties of the TN spacetime such as closed time-like curves, one may think that it is not a physical solution to the Einstein field equations therefore in this sense the TN spacetime is unlikely to be physical.
of CPO \((r_{\text{cpo}})\) are coincident with the horizon. From the best of my knowledge, this is the first we have observed in this work for any spherically symmetric, stationary and non-asymptotically flat spacetime showing such feature. Finally, we compute the CM energy for these spacetimes and we have found that for non-extreme TN BH, the CM energy is finite whereas for extreme TN BH the CM energy is diverging in nature.

The circular geodesics studied earlier by several authors [21,9] but they have not been studied in more graphically. We here show the differences between the two spacetime with the mass parameter and the mass-less parameter in visually. It may be noted that circular orbits of arbitrary radii are not possible, there exists a minimum radius below which no circular orbits are possible. The geodesic structure has been studied earlier for Schwarzschild BH [29], Reissner Nordstrøm (RN) BH [29], Kerr-Taub-NUT (KTN) BH [21] and Kerr-Newman-Taub-NUT (KNTN) more recently [30]. In this work, we have specialized on the cases when the parameter \(a = Q = 0\). By studying the geodesic structure, we can extract more information about the back ground space-time. Different observables like Lens-Thirrring effect, gravitational time delay, gravitational bending of light etc. all are the phenomenon related to the geodesic structure of the space-time. This is one of the major motivation behind to study them.

The structure of the manuscript is as follows. In section 2, we discuss the basics of TN BH. In section 3, we study the equatorial geodesic properties of the said BH. Section 4 devoted to study the extreme TN BH. The conclusions are given in the section 5.

2. The TN Space-time:

The metric is given by [1,22,31,32]

\[
\begin{align*}
\text{ds}^2 &= -\mathcal{H}(r) \left( dt + 2n \cos \theta d\phi \right)^2 + \frac{dr^2}{\mathcal{H}(r)} + \left( r^2 + n^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \quad (3) \\
\mathcal{H}(r) &= 1 - \frac{2(Mr + n^2)}{r^2 + n^2}. \quad (4)
\end{align*}
\]

where, \(M\) denotes the gravito-electric mass or ADM mass and \(n\) denotes the gravito-magnetic mass or dual mass or magnetic mass of the space-time. It is clearly evident that there are two types of singularities are present in the metric (3). One is at \(\mathcal{H}(r) = 0\) which give us the Killing horizons or BH horizons:

\[
r_{\pm} = M \pm \sqrt{M^2 + n^2} \quad \text{and} \quad r_+ > r_-, \quad (5)
\]

\(r_+\) is called event horizon and \(r_-\) is called Cauchy horizon.

From Fig. 1 we can see the horizon structure of TN and mass-less TN BH. There is a qualitative difference between two horizon structure with NUT parameter and without NUT parameter. In the limit \(n = 0\), one obtains the Schwarzschild BH. Interestingly, when \(M = 0\), we get mass-less TN space-time and the horizons at

\[
r_{\pm} = \pm n \quad \text{and} \quad r_+ > r_- \quad. \quad (6)
\]
The other type of singularity occurs at $\theta = 0$ and $\theta = \pi$, where the determinant of the metric component vanishes. Misner first demonstrated that in order to remove the apparent singularities at $\theta = 0$ and $\theta = \pi$, $t$ must be identified modulo $8\pi n$. Provided that $r^2 + n^2 \neq 2(Mr + n^2)$. It should be noticed here that the NUT parameter actually measures deviation from the asymptotic flatness at infinity which may be manifested in the off-diagonal components of the metric and this is happening due to presence of the Dirac-Misner type of singularity.

When,

$$M^2 + n^2 \geq 0,$$

the TN metric describes a BH, otherwise it has a naked singularity. When $M^2 + n^2 = 0$, we find extreme TN BH.

The characteristics of the variation of $-g_{tt}$ with radial coordinate is shown in Fig. 2. The “red-shift factor” $R$ for TN BH is given by

$$R = \frac{d\tau}{dt} = \frac{1}{u^t} = \sqrt{\frac{r^2 - 2Mn - n^2}{r^2 + n^2}}.$$  

(8)

and for mass-less TN BH, it is

$$R_{ml} = \sqrt{\frac{r^2 - n^2}{r^2 + n^2}}.$$  

(9)

The variation of redshift factor with radial coordinate is shown in Fig. 3.

The “red-shift” $z$ is given by

$$z = \sqrt{\frac{r^2 + n^2}{r^2 - n^2} - 1}.$$  

(10)

For mass-less TN BH, it is

$$z_{ml} = \sqrt{\frac{r^2 + n^2}{r^2 - n^2} - 1}.$$  

(11)
Fig. 2. The figure shows the variation of \( g_{tt} \) with \( r \) for TN and massless TN BH.

Fig. 3. The figure shows the variation of red-shift factor with \( r \) for TN and massless TN BH.

The variation of redshift with radial coordinate is depicted in Fig. 4.

The thermodynamic properties of TN BH could be found in detail in [34] and the BH temperature of \( \mathcal{H}^+ \) was calculated there

\[
T_+ = \frac{r_+ - r_-}{8\pi (Mr_+ + n^2)}.
\]  

In the limit \( r_+ = r_- \), \( T_+ = 0 \), this indicates that there must exists extreme TN BH. The geodesic properties have been studied in Sec. 4.

\(^c \mathcal{H}^+ \) denotes event horizon of the BH.
3. Equatorial circular geodesics of the TN BH:

Since the equatorial plane is the good location where we can see the causal characteristics of the geodesics so in this section we would like to study them. It should be noted that $r = r_0$, a constant is called circular geodesics. Also, the study of geodesics in the TN space-time is governed by the laws of conservation of energy and angular momentum because the space-time is independent of the coordinates $t$ and $\phi$. The TN space-time possesses time-like isometry generated by the time-like Killing vector $\xi \equiv \partial_t$ whose projection along the four velocity $u$ of geodesics: $\xi \cdot u = -E$, is conserved along such geodesics and the another conserved quantity is angular momentum followed by the relation $L \equiv \zeta \cdot u$ (where $\zeta \equiv \partial_\phi$). Where $\zeta$ is the space-like Killing vector field due to the rotational isometry.

Using these conditions together with the normalization of the four velocity, one can easily derived the radial equation for TN BH on the $\theta = \frac{\pi}{2}$ plane:

$$\dot{r}^2 = E^2 - V_{eff} = E^2 - \mathcal{H}(r) \left( \frac{L^2}{r^2 + n^2} - \epsilon \right).$$

(13)

where the effective potential [3] is

$$V_{eff} = \mathcal{H}(r) \left( \frac{L^2}{r^2 + n^2} - \epsilon \right).$$

(14)

Here, $\epsilon = -1$ for time-like geodesics, $\epsilon = 0$ for light-like geodesics and $\epsilon = +1$ for space-like geodesics.

3.1. Lightrays orbit:

For light rays orbit, the effective potential becomes

$$U_{eff} = \frac{L^2}{r^2 + n^2} \left( \frac{r^2 - 2Mr - n^2}{r^2 + n^2} \right).$$

(15)
Fig. 5. The figure shows the variation of $U_{eff}$ with $r$ for TN BH and mass-less TN BH.

Fig. 6. The figure shows the variation of $U_{eff}$ with $r$ for TN BH and mass-less TN BH for different values of $n$.

First we see the behaviour of the test particle in the potential well diagram. In Fig. 5, Fig. 6, Fig. 7, Fig. 8, Fig. 9, Fig. 10, Fig. 11 and Fig. 12, we show how the effective potential for photon changes for different values of angular momentum parameter and NUT parameter. From the effective potential diagram, it has been observed that the presence of dual mass parameter effectively changes the shape of the potential well in comparison with absence of the dual mass parameter. The structure of the potential well also changes in the presence of ADM mass parameter and in the absence of ADM mass parameter. Now by introducing the impact parameter $D = \frac{L}{E}$, one can reparametrization of any null geodesics described by
Fig. 7. The figure shows the variation of $U_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

Fig. 8. The figure shows the variation of $U_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

d the parameter $L$ and $E$. $D$ be the angular momentum of null geodesics when it is reparametrized to have unit energy. Now the plot of $D$ with $r$ gives the information of radial motion for null geodesics.

The equations evaluating the radius $r_c$ of the unstable circular photon orbit at $E = E_c$ and $L = L_c$ by introducing the impact parameter $D_c = \frac{D}{E_c}$ are

\begin{align*}
  r_c^2 + n^2 + \left( \frac{2Mr_c + n^2 - r_c^2}{(r_c^2 + n^2)^2} \right) D_c^2 &= 0 , \\
  r_c - \left[ \frac{Mr_c^2 - Mn^2 + 2n^2r_c}{(r_c^2 + n^2)^2} \right] D_c^2 &= 0 .
\end{align*}
From Eq. (17), we find

\[ D_c = \pm \sqrt{r_c(r_c^2 + n^2)^2} \]  

\[ = \pm \sqrt{M r_c^2 + 2n^2 r_c - Mn^2} . \]  

(18)

The behaviour of the impact parameter can be shown from the Fig. 13. Putting the Eq. (18) in Eq. (16), we obtain the equation of CPO [21,30]:

\[ r_c^3 - 3Mr_c^2 - 3n^2 r_c + Mn^2 = 0 . \]  

(19)

For mass-less TN BH [21], the root of the Eq. becomes

\[ r_c = \pm \sqrt{3} n . \]  

(20)
Fig. 11. The figure shows the variation of $U_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

Fig. 12. The figure shows the variation of $U_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

The variation of CPO with $r_c$ for TN BH and mass-less TN BH could be found in the Fig. 13.

The gravitational bending of light analyzed in [19] and the bending angle on the cone is computed in [19] $\alpha = \frac{4M_D}{r}$. In terms of opening angle it should be $\alpha = \frac{2M_D}{r}$. For massless TN spacetime, this angle reduces to zero value.
Parthapratim Pradhan

3.2. Circular Time-like Geodesics:

For time-like geodesic we have to set $\epsilon = -1$ then the effective potential becomes

$$V_{eff} = \left( \frac{r^2 - 2Mr - n^2}{r^2 + n^2} \right) \left( 1 + \frac{L^2}{r^2 + n^2} \right). \quad (21)$$

The qualitative behaviour of the test particle can be obtained by studying this potential. First we consider the zero angular momentum geodesics for this the effective
Fig. 15. The figure shows the variation of $V_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

potential reduces to

$$V_{\text{eff}} = \left( \frac{r^2 - 2Mr - n^2}{r^2 + n^2} \right).$$

(22)

One can observe the qualitative behaviour of the geodesics in the presence of dual mass and without dual mass and also the mass-less parameter. This can be seen from the Fig. 15. In this plot, we can see that the presence of the dual mass parameter deforms the shape of the radial effective potential in comparison with zero dual mass parameter. This behaviour can be seen from the mass-less case also.

For $L \neq 0$, the behaviour of the test particle in the potential well could be seen from the following Fig. 10 Fig. 11 Fig. 12 Fig. 13 Fig. 14 Fig. 15 Fig. 16 Fig. 17 and Fig. 23. In the above figures, we have seen that the radial dependency of the effective potential with dual mass parameter, without dual mass parameter and $M = 0$. When $n = 0$, the effective potential at large radial distance does not change much more with the increasing of $L$. When we introduced the NUT parameter, the shape of the effective potential deforms in comparison with NUT less case and it also changes for different values of angular momentum parameter. Finally, when we increase the value of $n$ the height of the potential well decreases. This work has been done earlier for the parameter $a$, $Q$ and $n$ [30]. This work is specially for the parameter values $a = Q = 0$ and for mass-less cases which has been not studied previously.

Now re-write the Eq. 13 for time-like geodesics as

$$i^2 \left( r^2 + n^2 \right)^2 = \left( E^2 - 1 \right) \left( r^2 + n^2 \right)^2 + 2Mr \left( r^2 + n^2 \right) - L^2 \left( r^2 - 2Mr - n^2 \right) + 2n^2 \left( r^2 + n^2 \right).$$

(23)
For circular geodesics \( r = r_0 \), we have the following condition:

\[
\dot{r}^2|_{r=r_0} = \frac{d\dot{r}^2}{dr}|_{r=r_0} = 0 .
\]  

(24)

From this condition, we find the energy and angular momentum for circular orbit:

\[
E_{0}^2 = \frac{r_0 (r_0^3 - 2M r_0 - n^2)^2}{(r_0^2 + n^2) (r_0^3 - 3M r_0^2 - 3n^2 r_0 + M n^2)} .
\]  

(25)
Fig. 18. The figure shows the variation of $V_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

Fig. 19. The figure shows the variation of $V_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

and

$$L_0^2 = \frac{(r_0^2 + n^2) \left( M r_0^2 - 2 n^2 r_0 - M n^2 \right)}{(r_0^2 - 3 M r_0^2 - 3 n^2 r_0 + M n^2)}.$$  \hspace{1cm} (26)

These equations require for energy square and angular momentum square positive definite i.e. $r_0^2 - 3 M r_0^5 - 3 n^2 r_0 + M n^2 > 0$. Compared with the Eq. (19), it implies that the minimum radius for time-like circular orbit is the radius of of the unstable CPO. Interestingly for mass-less case, these values are

$$E_0^2 = \frac{(r_0^2 - n^2)^2}{(r_0^2 + n^2) (r_0^3 - 3 n^2 r_0)}.$$  \hspace{1cm} (27)
Fig. 20. The figure shows the variation of $V_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

Fig. 21. The figure shows the variation of $V_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

We have plotted their behaviour in the Fig. 24 and Fig. 25.

From Eq. (23), and for circular orbit one obtains

$$L_{\pm} = \pm \sqrt{(E^2 - 1) \left( \frac{n^2}{r_0^2} + \frac{n^2}{n^2} \right)^2 + 2Mr_0 \left( \frac{n^2}{r_0^2} + \frac{n^2}{n^2} \right) + 2n^2 \left( \frac{n^2}{r_0^2} + \frac{n^2}{n^2} \right)} \ .$$

(29)

It should be noted that $L_+ = -L_-$.

From this expression, it follows that the angular momentum parameter explicitly depends upon the energy value. Therefore there

and

$$L_0^2 = \frac{2n^2 \left( \frac{n^2}{r_0^2} + \frac{n^2}{n^2} \right)}{\left( \frac{n^2}{r_0^2} - 3n^2 \right)} \ .$$

(28)
(Circular Orbits in the Taub-NUT and mass-less Taub-NUT Space-time) 17

Fig. 22. The figure shows the variation of $V_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

Fig. 23. The figure shows the variation of $V_{\text{eff}}$ with $r$ for TN BH and mass-less TN BH.

must be difference in the circular orbits in the $L - r$ plane for $E^2 > 1$, $E^2 < 1$ and $E^2 = 1$. We are interested here to look the behaviour of the circular geodesics in this plane by incorporating these energy conditions.

When $E^2 < 1$, we obtain the bound orbits, this can displayed from the $L_+ - r_0$ diagram. It has been shown in the Fig. 26. For $E^2 = 1$, we find the marginally escape orbits. This has been seen from the Fig. 27. Finally for $E^2 > 1$, we find the unbound orbits. This could be seen from the Fig. 28, Fig. 29 and Fig. 30. From Fig. 31, Fig. 32, Fig. 33, Fig. 34 and Fig. 35 one can observe that the variation of $L_+$ with respect to $r_0$ for various values of NUT parameter in the presence of mass parameter and without mass parameter.
Fig. 24. The figure depicts the variation of $E_{0}^{2}$ with $r_{0}$ for TN BH and mass-less TN BH.

Fig. 25. The figure depicts the variation of $E_{0}^{2}$ with $r_{0}$ for TN BH and mass-less TN BH.

Now we see the behaviour of the $L$ with $r$ for different values of energy for a fixed value of the dual mass parameter. Now the ISCO radius can be obtain by using the condition Eq. (24) with additional condition:

$$\left. \frac{d^{2}r^{2}}{dt^{2}} \right|_{r=r_{0}} = 0.$$  \hspace{1cm} (30)

Although it has been derived earlier in [30], We are interested here to see the behaviour of ISCO in the presence of NUT parameter and with out NUT parameter in comparison with mass-less cases in more graphically which has been not shown there. Therefore the ISCO equation for TN BH could be obtain by applying the condition given in Eq. (24) and Eq. (30) (or by putting $a = Q = 0$ in Eq. (81) in
Fig. 26. The figure shows the variation of $L_+$ with $r_0$ for TN BH and mass-less TN BH.

Fig. 27. The figure shows the variation of $L_+$ with $r_0$ for TN BH and mass-less TN BH.

\[ Mr_0^6 - 6M^2r_0^5 - 15Mn^2r_0^4 + (4M^2n^2 - 16n^4)r_0^3 + 15Mn^4r_0^2 - 6M^2n^4r_0 - Mn^6 = 0. \]  \hspace{1cm} (31)

and for the mass-less TN BH, it is

\[ r_0 = 0. \]  \hspace{1cm} (32)

and it also indicates there is no ISCO for massless TN BH. It is a curious result although there exists mass-less CPO. The variation of ISCO in the presence of the NUT parameter and without NUT parameter could be found in the Fig. 36.
3.3. **MBCO:**

Another interesting orbit that has not been considered previously in [21] but considered in [30] for KNTN BH. So we have just set $E_0^2 = 1$ in Eq. (25) (or putting the parameters $a = Q = 0$ in Eq. (98) in [30]), we find the MBCO for TN BH:

$$Mr_0^7 - 4M^2r_0^6 - 7Mn^2r_0^5 + (2M^2n^2 - 4n^4)r_0^4 + (M^2n^4 - 4n^6)r_0^2 + 4Mn^6r_0 - M^2n^6 = 0.$$  \hspace{1cm} (33)

and for the mass-less TN BH, it is

$$r_0^2 + n^2 = 0.$$  \hspace{1cm} (34)
It implies that there is no existence of MBCO for mass-less case. From Eq. (33), in the limit $n = 0$, we get the MBCO for Schwarzschild BH [29]. In Fig. 37, we have plotted the MBCO with $r_0$ in comparison with Schwarzschild BH.

4. Geodesics in Extreme TN Spacetime:

In the previous section, we have discussed the complete geodesic structure for non-extreme TN BH. Moreover, we have clearly explained the difference of geodesic structure between TN spacetime and mass-less TN spacetime in graphically. In the present section we shall discuss more interesting case i.e. extreme TN spacetime.
What happens the geodesic structure in the extreme limit i.e. $r_+ = r_-$. This is the main aim in this section. Proceeding similarly, we can write the effective potential for massive particles in the extreme limit [using Eq. (21)] given by

$$V_{eff} = \left( \frac{r - M}{r + M} \right) \left( 1 + \frac{L^2}{r^2 - M^2} \right).$$

(35)

The qualitative behaviour of the test particle may be seen from the effective potential diagram (See Fig. 38).

Proceeding analogously, we apply for circular geodesics $r = r_0$, one obtains the
energy and angular momentum for extreme TN BH [using Eq. (25) and Eq. (26)]:

\[ E_0 = \sqrt{\frac{r_0}{r_0 + M}} . \]  \hspace{1cm} (36)

and

\[ L_0 = \sqrt{M(r_0 + M)} . \]  \hspace{1cm} (37)

How \( E_0 \) and \( L_0 \) are varied with \( r_0 \), it can be seen from the Fig. 39. Now we have derived the ISCO equation for non-extreme TN BH but here we will see what would be the ISCO radius in the extremal limit which is most important class of circular
Fig. 36. The figure depicts the variation of ISCO with $r_0$ in the presence of NUT parameter and without NUT parameter. Where $y = M r_0^6 - 6 M^2 r_0^5 - 15 M n^2 r_0^4 + (4 M^2 n^2 - 16 n^4) r_0^3 + 15 M n^4 r_0^2 - 6 M^2 n^3 r_0 - M n^6$.

Fig. 37. Here $y = M r_0^6 - 4 M^2 r_0^5 - 7 M n^2 r_0^4 + (2 M^2 n^2 - 4 n^4) r_0^3 + (M^2 n^4 - 4 n^6) r_0^2 + 4 M n^6 r_0 - M^2 n^6$.

orbit in astrophysics. We find for extreme TN BH the ISCO radius [using Eq. (31)] should be

\[ r_0 = r_{isco} = M. \]  

(38)

The corresponding ISCO energy and ISCO angular momentum are

\[ E_{isco} = \frac{1}{\sqrt{2}}. \]  

(39)

\[ L_{isco} = \sqrt{2} M. \]  

(40)
Fig. 38. The variation of $V_{\text{eff}}$ with $r$ for extreme TN BH, Here $M = 1$.

Fig. 39. The variation of energy and angular momentum with $r_0$ for extreme TN BH, Here $M = 1$.

For photons, the effective potential in the extreme limit [using Eq. (15)] reduces to

$$U_{\text{eff}} = \frac{L^2}{(r + M)^2}. \quad (41)$$

Its behaviour for different values of angular momentum parameter can be seen from the Fig. 10. Therefore, one obtains the CPO [using Eq. (19)] for extreme TN BH:

$$r_c = r_{\text{cpo}} = M. \quad (42)$$
Similarly, we can find the radius of MBCO [using Eq. (33)] for extreme TN BH:

$$r_0 = r_{mbco} = M .$$

Interestingly, we observed that for extreme TN BH the three radii namely ISCO, MBCO and CPO coincides with the horizon i.e.

$$r_{isco} = r_{mbco} = r_{cpo} = M .$$

This has not been previously examined in the literature that for extreme TN BH three orbits are coincident with the Killing horizons i.e. null geodesic generators of the horizon. For spherically symmetric extreme string BH, this result has been observed in [36]. Probably, this is a first time we have seen that such type of features for any spherically symmetric, stationary and non-asymptotic flat extreme BH.

5. Discussion:

In this paper we examined the geodesic motion of test particles in the background geometry of TN spacetime in comparison with mass-less (zero mass) TN spacetime. We considered the both massive and mass-less cases. We differentiated the ISCO, MBCO and CPO in graphically between TN spacetime and mass-less TN spacetime. From effective potential diagram, we showed the presence of the NUT parameter changes the shape of the potential well in comparison with zero NUT paprameter and mass-less spacetime. We studied the circular orbits in the $L − r$ plane for different values of energy i.e. $E^2 > 1$, $E^2 < 1$ and $E^2 = 1$ in case of TN and mass-less TN spacetime.

We also derived the geodesic motion of test particles (both massive and mass-less) in case of extreme TN BH. Interestingly, we showed that the three radii i.e. the radius of ISCO, the radius of MBCO and the radius of CPO are coincident with
the Killing horizon radius. This is a very surprising result because probably we first obtained this result for any spherically symmetric, stationary and non-asymptotic flat spacetime. In Appendix-A, we derived the CM energy for extreme TN BH and we found that the diverging value of CM energy.

Appendix A. CM Energy of Particle Collision near the horizon of TN BH:

In this section, we should study what is the role of NUT parameter in the Bañados, Silk and West (BSW) effect which was predicted by Bañados, Silk and West \[35\] several years ago. Does TN BH could be act as a particle accelerator with arbitrarily high energy when the BH is extremal. This is the main aim in this section and motivated by the previous section from the analysis of geodesic structure. We have considered the particle acceleration and collision in the CM frame. To determine the CM energy, we consider first two particles coming from infinity with \( E_{\text{1}}/m_0 = E_{\text{2}}/m_0 = 1 \) approaching the TN BH with different angular momenta \( L_{\text{1}} \) and \( L_{\text{2}} \) and colliding at some radius \( r \). Later, we choose the collision point is at \( r \) to approach the horizon \( r = r_+ \). Also we have assumed that the particles are initially to be at rest at infinity.

The formula that we have used here suggested first by BSW \[35\] should read

\[
\left( \frac{E_{\text{cm}}}{\sqrt{2m_0}} \right)^2 = 1 - g_{\mu\nu}u_1^\mu u_2^\nu. \tag{A.1}
\]

Since in the previous section we have calculated total geodesic structure confined in the equatorial plane for TN BH, so we should not repeat it again. Due to the time-like isometry and space-like isometry the space-time possesses two conserved quantities one is the energy and other is the angular momentum. Thus for massive particles, the components of the four velocity are

\[
u^t = \dot{t} = \frac{E}{\mathcal{H}(r)} \tag{A.2}
\]

\[
u^r = \dot{r} = \pm \sqrt{E^2 - \mathcal{H}(r) \left( 1 + \frac{L^2}{r^2 + n^2} \right)} \tag{A.3}
\]

\[
u^{\theta} = \dot{\theta} = 0 \tag{A.4}
\]

\[
u^{\phi} = \dot{\phi} = \frac{L}{r^2 + n^2} \tag{A.5}
\]

and

\[
u_1^\mu = \left( \frac{E_1}{\mathcal{H}(r)}, -X_1, 0, \frac{L_1}{r^2 + n^2} \right) \tag{A.6}
\]

\[
u_2^\mu = \left( \frac{E_2}{\mathcal{H}(r)}, -X_2, 0, \frac{L_2}{r^2 + n^2} \right) \tag{A.7}
\]
Putting these values in (A.1), we find the expression for CM energy:

$$
\left( \frac{E_{cm}}{\sqrt{2m_0}} \right)^2 = 1 + \frac{E_1 E_2}{H(r)} - \frac{X_1 X_2}{H(r)} \sqrt{1 + L_1^2 + L_2^2 + r^2 + n^2}. \tag{A.8}
$$

where,

$$
X_1 = \sqrt{\frac{E_1^2}{H(r)} - \frac{L_1^2}{r^2 + n^2}}, \quad X_2 = \sqrt{\frac{E_2^2}{H(r)} - \frac{L_2^2}{r^2 + n^2}}.
$$

For simplicity, $E_1 = E_2 = 1$ and reverting back the value of $H(r)$, we obtain the CM energy near the event horizon ($r_+$) of TN space-time:

$$
E_{cm} \big|_{r \to r_+} = \sqrt{2m_0} \sqrt{\frac{4(r_+^2 + n^2) + (L_1 - L_2)^2}{2(r_+^2 + n^2)}}. \tag{A.9}
$$

and for Cauchy horizon ($r_-$) the CM energy is

$$
E_{cm} \big|_{r \to r_-} = \sqrt{2m_0} \sqrt{\frac{4(r_-^2 + n^2) + (L_1 - L_2)^2}{2(r_-^2 + n^2)}}. \tag{A.10}
$$

where $r_\pm$ is defined previously. It implies that the CM energy is finite and depends upon the values of angular momentum parameter. It also suggests that the CM energy depends upon the NUT parameter. It may also playing a key role in the BSW effect as we have seen from the above expression.

In the limit $n = 0$, the above expression reduces to the following form

$$
E_{cm} = \sqrt{2m_0} \sqrt{\frac{16M^2 + (L_1 - L_2)^2}{8M^2}}. \tag{A.11}
$$

which is exactly CM energy of the Schwarzschild BH found by BSW in [35].

Now see what happens in case of mass-less case and extreme case? First we consider the mass-less case, in this case the CM energy is given by

$$
E_{cm} \big|_{r \to r_\pm} = \sqrt{2m_0} \sqrt{\frac{8n^2 + (L_1 - L_2)^2}{4n^2}}. \tag{A.12}
$$

It indicates that the CM energy is finite depends on the NUT parameter. More interesting case i.e. the extreme case where the CM energy is given by

$$
E_{cm} \big|_{r \to M} = \sqrt{2m_0} \sqrt{\frac{4(M^2 + n^2) + (L_1 - L_2)^2}{2(M^2 + n^2)}}. \tag{A.13}
$$

Since we know for extreme case, $r_+ = r_- = M$ and $M^2 + n^2 = 0$. Therefore we find for extreme TN BH:

$$
E_{cm} \big|_{r \to M \to \infty}. \tag{A.14}
$$

i.e the CM energy is divergent. This is an amusing result. Because we first reported a non-asymptotic flat, spherically symmetric and stationary extreme BH spacetime possesses such properties.
(Circular Orbits in the Taub-NUT and mass-less Taub-NUT Space-time) 29

References

[1] A. H. Taub, “Empty Space-Times Admitting a Three Parameter Group of Motions”, Ann. of Maths. 53, 3 (1951).
[2] E. Newman, L. Tamburino, T. Unti, “EmptySpace Generalization of the Schwarzschild Metric”, J. Math. Phys. 4, 915 (1963).
[3] C. W. Misner and A. H. Taub, “A Singularity-free Empty Universe”, Soviet Phys. JETP 28, 122 (1969).
[4] W. B. Bonor, “A new interpretation of the NUT metric in general relativity”, Math. Proc. of the Cambridge Physical Society 66, 145 (1969).
[5] V. S. Manko and E. Ruiz, “Physical interpretation of the NUT family of solutions”, Class. Quant. Grav. 22, 3555 (2005).
[6] A. Zee, “Gravitomagnetic Pole And Mass Quantization”, Phys. Rev. Lett. 55, 2379 (1985), and 56, 1101(E) (1986).
[7] S. Ramaswamy and A. Sen, “Dualmass in general relativity”, J. Math. Phys. 22, 11 (1981).
[8] S. Ramaswamy and A. Sen, “Comment On gravitomagnetic Pole And Mass Quantization”, Phys. Rev. Lett. 57, 1088 (1986).
[9] J. S. Dowker and J. A. Roche, “The gravitational analogues of magnetic monopoles”, Proc. of the Physical Society 92 575 (1967).
[10] G. ’t. Hooft, “Magnetic monopoles in unified gauge theories”, Nucl. Phys. B79, 276 (1974).
[11] A. M. Polyakov, “Particle Spectrum in the Quantum Field Theory”, JETP Lett. 20, 194 (1974).
[12] P. A. M. Dirac, “Quantised Singularities in the Electromagnetic Field”, Proc. Roy. Soc. London A 133, 60 (1931).
[13] J. L. Friedman and R. D. Sorkin, “Spin 1/2 From Gravity”, Phys. Rev. Lett. 44, 1100 (1980), and 45, 148(E) (1980).
[14] J. S. Dowker, “The nut solution as a gravitational dyon”, Gen. Relativ. Grav. 5, 603 (1974).
[15] G. W. Gibbons and N. S. Manton, “Classical and quantum dynamics of BPS monopoles”, Nucl. Phys. B274, 183 (1986).
[16] D. Lynden-Bell and M. Nouri-Zonoz, “Classical monopoles: Newton, NUT space, gravimagnetic lensing and atomic spectra”, Rev. Mod. Phys., 70, 427-445 (1998).
[17] P. S. Letelier and W. M. Vieira, “Chaos and Taub–NUT related space-times”, Phys. Lett., A 244, 324 1998.
[18] D. Vaman, M. Visinescu, “Generalized Killing equations and Taub - NUT spinning space”, Phys. Rev. D 54, 1398 (1996).
[19] D. Lynden-Bell, M. Nouri-Zonoz., “Gravimagnetic lensing by NUT space”, Mon. Not. R. Astron. Soc. 292, 714 (1997).
[20] C. Liu, S. Chen, C. Ding and J. Jing, “Particle Acceleration on the Background of the Kerr-Taub-NUT Spacetime”, Phys. Lett. B 701, 285 (2011).
[21] C. Chakraborty, “Inner-most stable circular orbits in extremal and non-extremal Kerr-Taub-NUT spacetimes”, Eur. Phys. J. C 74, 2759 (2014).
[22] J. G. Miller, M. D. Kruskal, B. B. Godfrey, “Taub-NUT (Newman, Unti, Tamburino) Metric and Incompatible Extensions”, Phys. Rev. D4, 2945 (1971).
[23] P. Hajicek, “Bifurcate SpaceTimes”, J. Math. Phys. 12, 157 (1971).
[24] C. Chakraborty, P. Majumdar, “Strong gravity Lense-Thirring precession in Kerr and Kerr-Taub-NUT spacetimes”, Class. Quant. Grav. 31, 075006 (2014).
[25] A. Ashtekar and A. Sen, “NUT 4momenta are forever”, J. Math. Phys. 23, 2168 (1982).
[26] M. Mueller and M. J. Perry, “Constraints On Magnetic Mass”, *Class. Quant. Grav.* 3, 65 (1986).
[27] G. Holzegel, “A Note on the instability of Lorentzian Taub-NUT-space”, *Class. Quant. Grav.* 23, 3951 (2006).
[28] D. Bini, C. Cherubini, M. Mattia and R. T. Jantzen, “Equatorial Plane Circular Orbits in the Taub-NUT Spacetime”, *Gen. Relativ. Grav.* 35, 2249 (2003).
[29] S. Chandrashekar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford (1983).
[30] P. Pradhan, “Circular geodesics in the Kerr-Newman-Taub-NUT spacetime”, *Class. Quant. Grav.* 32, 165001 (2015).
[31] C. W. Misner, “The Flatter regions of Newman, Unti and Tamburino's generalized Schwarzschild space”, *J. Math. Phys.* 4, 924 (1963).
[32] J. G. Miller, “Global analysis of the Kerr-Taub-NUT metric”, *J. Math. Phys.* 14, 486 (1973).
[33] C. W. Misner, K. S. Thorn, J. A. Wheeler, *Gavitation*, W. H. Freeman (1973).
[34] P. Pradhan, “Thermodynamic product formula for a Taub-NUT black hole”, *JETP* 122, 113 (2016).
[35] M. Bañados, J. Silk, and S. M. West, “Kerr Black Holes as Particle Accelerators to Arbitrarily High Energy”, *Phys. Rev. Lett.* 103, 111102 (2009).
[36] P. Pradhan, “Horizon Straddling ISCOs in Spherically Symmetric String Black Holes”, *Inter. J. Mod. Phys.* D 24, 155086 (2015).