LONG-TERM REGULARITY OF TWO DIMENSIONAL NAVIER-STOKES-POISSON EQUATIONS

CHANGZHEN SUN

ABSTRACT. This article is devoted to the long-term regularity of the 2-d Navier-Stokes-Poisson system. We allow the initial density to be close to a constant and the potential part of the initial velocity to be small independently of the rescaled viscosity parameter $\varepsilon$ while the rotational part of the initial velocity is assumed to be small compared to $\varepsilon$. We then show that the lifespan of the system $T_\varepsilon$ satisfies $T_\varepsilon > \varepsilon^{-1-\vartheta}$, where the small constant $\vartheta$ is the size of the initial perturbation in some suitable space. The normal form transformation and the classical parabolic energy estimates are the main ingredients of the proof.

1. Introduction

In the present paper, we are concerned with the large time regularity of the scaled two dimensional Navier-Stokes-Poisson system, which is a hydrodynamical model of plasma describing the dynamics of electrons and ions that interact with their self-consistent electric field. If the motion of ions is negligible, the dynamics of electrons can be described as the following Navier-Stokes-Poisson (NSP) system:

$$
\begin{align*}
\partial_t \rho^\varepsilon + \text{div} (\rho^\varepsilon u^\varepsilon) &= 0, \\
\partial_t (\rho^\varepsilon u^\varepsilon) + \text{div} (\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \varepsilon \mathcal{L} u^\varepsilon + \nabla P(\rho^\varepsilon) - \rho^\varepsilon \nabla \varphi^\varepsilon &= 0, \\
\Delta \varphi^\varepsilon &= \rho^\varepsilon - 1, \\
u|_{t=0} = u^\varepsilon_0, \rho|_{t=0} = \rho^\varepsilon_0.
\end{align*}
$$

(1.1)

Here $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, the unknowns $\rho^\varepsilon(t, x) \in \mathbb{R}_+, u^\varepsilon \in \mathbb{R}^2, \nabla \varphi^\varepsilon \in \mathbb{R}^2$ are the electron density, the electron velocity and the self-consistent electric field respectively. The thermal pressure $P(\rho^\varepsilon)$ is assumed to follow a polytropic $\gamma$-law: $P(\rho^\varepsilon) = \rho^\varepsilon^\gamma$, $\gamma > 1$, while the viscous term is under the form $\mathcal{L} u^\varepsilon = \mu \Delta u^\varepsilon + (\mu + \lambda) \nabla \text{div} u^\varepsilon$, where the Lamé coefficients $\mu, \lambda$ are supposed to be constants which satisfy the condition: $\mu > 0, 2\mu + \lambda > 0$. For the conciseness of the presentation, we shall assume $\mu = 1, \lambda = 0$ and $\gamma = 2$, since there are no specific cancellations arising from this choice. Note also that the scaled parameter $\varepsilon$ in front of the diffusion term is the inverse of the Reynolds number and is assumed to be small in this paper, that is $\varepsilon \in (0, 1]$.

There has been extensive studies concerning the global well-posedness of (NSP) under small and smooth perturbations of the constant equilibrium $((\rho^0, u^0) = (1, 0))$ when the scaled parameter $\varepsilon = 1$ and the spatial dimension $d = 3$. We refer for example to [19] [29], where the global existence in $H^N$ for $N \geq 4$ is proved under the assumption that the initial perturbation is small in $H^3$. Besides, the smoothing effect of the diffusion term allows them to prove some time decay rate for the perturbation. In [19], the global existence of (1.1) is obtained in hybrid Besov spaces when the initial perturbation is close to equilibrium in critical $L^2$ norm by considering low and high frequencies differently, inspired by the former work on compressible Navier-Stokes equations [4]. This result was then generalized to the critical $L^p$ spaces [3] [25] [31]. Although these works are done in spatial dimension $d = 3$, global existence in $H^N$ ($N \geq 3$) for $d = 2$ could still be proved by the same arguments as in [29]. That is, by using the dissipation for $u$ provided by the
diffusion term $\varepsilon \Delta u^\varepsilon$ and the damping for $\rho^\varepsilon - 1$ resulting from the coupling structure, one could control the nonlinear term as long as some lower order Sobolev norm of $(\rho^\varepsilon - 1, u^\varepsilon, \nabla \varphi^\varepsilon)$ is small. Nevertheless, if we consider the scaled equations, that is $\varepsilon \in (0, 1]$, the above strategy requires the initial perturbation $(\rho_0^\varepsilon - 1, u_0^\varepsilon, \nabla \varphi_0^\varepsilon)$ to be small proportional to $\varepsilon$ in some suitable Sobolev spaces.

On the other hand, when $\varepsilon = 0$, the system (1.1) reduces to the so-called Euler-Poisson (EP) equation. Regarding (EP), Guo [9] constructs in dimension $d=3$ the global smooth solutions close to the reference equilibrium $(1, 0)$ under neutral ($\int_{\mathbb{R}^d} (\rho_0^\varepsilon - 1) dx = 0$), irrotational, small perturbation to the equilibrium. The good dispersive properties due to the presence of the electric field and the normal form transformation technique developed by Shatah [24] are the main two ingredients of his proof. More recently, similar results was obtained in dimension $d = 2$ by Ionescu-Pausader [10] and Li-Wu [18] independently and $d = 1$ by Guo-Han-Zhang [10]. See also the result about the large time regularity of 2-d (EP) on the torus [30].

Nevertheless, since in practical physics, the Reynolds number is usually very high (that is $\varepsilon$ very small), it is natural to ask global existence results that hold uniformly in $\varepsilon$. In [24], we successfully combine the parabolic energy estimate which works for (NSP) and normal form transformation used in the works for (EP) to prove a uniform stability result for 3-d (NSP) system. That is, we construct the global smooth solutions around the constant equilibria $(1, 0)$ with a smallness assumption on the perturbation which is independent of $\varepsilon$ except for the curl part of the velocity (recall that for $\varepsilon = 0$ we have global smooth solutions only for irrotational data). In this paper, we aim to prove the analogous results in 2-d. However, due to the weaker dispersion in 2-d, the rotational part of the velocity is driven by a source term whose $L^2_{x,t}$ norm enjoys only at best the critical time decay $(1 + t)^{-1}$. Consequently, the rotational part of the velocity has a logarithmic growth which prevents one from establishing the global existence. Therefore, in this paper, we only devote ourselves to proving a large time existence result of 2-d (EP) system.

We shall denote by $\mathcal{P} = \text{Id} - \nabla \Delta^{-1} \text{div}$ the Leray projector which projects a vector to its divergence-free (or rotational) part. Denote also $\mathcal{P}^\perp = \text{Id} - \mathcal{P}$, which projects a vector field to its curl-free (or potential) part.

**Theorem 1.1.** There exist two constants $\vartheta, C$, such that for any $\varepsilon \in (0, 1], \vartheta \in (0, \vartheta_0]$, if the following assumption holds:

$$
\| (\rho_0^\varepsilon - 1, \mathcal{P}^\perp u_0^\varepsilon, \nabla \varphi_0^\varepsilon) \|_{Y^4} \leq \frac{1}{C} \vartheta, \quad \| \mathcal{P} u_0^\varepsilon \|_{H^3} \leq \vartheta \varepsilon,
$$

where the $Y^4$ norm is defined in (1.4), then the system (1.1) admits a solution in $C([0, T), H^3)$ with $T > \varepsilon^{-(1-\vartheta)}$.

**Remark 1.2.** We remark that the assumption we add on the rotational part of the initial velocity, that is, $\mathcal{P} u_0^\varepsilon$ small proportional to $\varepsilon$ is rather natural. Indeed, as the rotational part $\mathcal{P} u^\varepsilon$ is driven by a source term of size $\varepsilon$, even if we assume it to vanish initially, a rotational part of size $\varepsilon$ is instantaneously created.

**Remark 1.3.** As explained before, the rotational part of the velocity satisfies at the leading order a heat equation with a source term of size $\varepsilon$ and critical time decay in $L^2_{x,t}$, so that we can not even extend its existence time to $\varepsilon^{-1}$.

A natural attempt to prove Theorem 1.1 is to consider the highly coupled equations for the potential part and rotational part of the velocity respectively. One expects to use the dispersive property from the coupled equations (see (3.3) below) for density and potential part of velocity to prove the time decay of $\rho^\varepsilon - 1$ and $\mathcal{P}^\perp u^\varepsilon$, and to use the smoothing effect to prove the large time existence of of the rotational part $\mathcal{P} u$. Nevertheless, there will be some difficulties stemming
from the interactions between the rotational part and the potential part. To be more precise, on one hand, by the dispersive estimate, the $L^\infty_x$ norm of the solution to (3.3) decays at best at rate $(1 + t)^{-1}$. On the other hand, since $Pu^\varepsilon$ is governed by the heat equation with a source term $\varepsilon P[(\frac{1}{\rho^0} - 1)\Delta P \cdot u^\varepsilon]$, whose $L^2_x$ norm has at best critical time decay $(1 + t)^{-1}$, it is far from being bounded in $L^2_x$, which in turn prevents us from proving the time decay of $(\rho^\varepsilon - 1, P \cdot u^\varepsilon)$ in $L^\infty_x$. Moreover, owing to the presence of the diffusion term, the eigenvalues $\lambda_{\pm} = \varepsilon \Delta \pm i \sqrt{\langle \nabla \rangle^2 - (\varepsilon \Delta)^2}$ of the linearized matrix for the dispersive part of the system are far from $\pm i \langle \nabla \rangle$ — the eigenvalues for (EP). It seems necessary to cut the frequency to isolate the dispersive effects and dissipation effects, which forces us to control the interactions between different frequencies. We prefer not to take this way since it is more sophisticated to treat the potential-rotational interactions and low-high frequencies interactions in the same time.

Another attempt is to write the solution of NSP $(\rho^0 - 1, \nabla \varphi^0, u^0)$ by that of EP $(\rho^0, \nabla \varphi^0, u^0)$ plus a remainder, and try to control the remainder by the dissipation term. However in that case the equation satisfied by the remainder has source term $\varepsilon \Delta u^0$ which has size $\varepsilon$ but without any decay, this forces the remainder to grow linearly (in $L^2_x$) and leads to the time existence only at order $O(1)$.

We shall thus adopt the same strategy developed in [23] where the uniform stability problem for 3d NSP is investigated. More precisely, we split the (NSP) into two viscous systems, with initial data $(\rho^0_0 - 1, \nabla \varphi^0_0, P_0 \cdot u^0_0)$ and $(0, 0, P_0 u^0_0)$ respectively. The first one will have global solutions under $\varepsilon$-independent assumptions on the initial data $(\rho^0_0 - 1, \nabla \varphi^0_0, P_0 \cdot u^0_0)$ and the solutions will enjoy the same time decay as the 2-d (EP) system. The other is the perturbation of the original system (1.1) by the former one. The source term $\varepsilon (\rho - 1) \Delta u$ in this system is small compared to $\varepsilon$ and has critical decay in $L^2_x$. We can thus get the desired lifespan by merely energy estimates. More precisely, we write the solution $(\rho^\varepsilon, u^\varepsilon, \nabla \varphi^\varepsilon)$ of (NSP) as

$$
(\rho^\varepsilon, u^\varepsilon, \nabla \varphi^\varepsilon) = (\rho, u, \nabla \varphi) + (n, v, \nabla \psi),
$$

where $(\rho, \nabla \phi, u)$ and $(n, \nabla \psi, v)$ are the solutions of the following systems

$$
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t u + u \cdot \nabla u - \varepsilon \Delta u + \nabla \rho - \nabla \varphi &= 0, \\
\Delta \varphi &= \rho - 1, \\
u|_{t=0} &= u_0 = P_0 \cdot u^\varepsilon_0, \rho|_{t=0} = \rho_0 = \rho^\varepsilon_0.
\end{align*}
$$

(1.2)

$$
\begin{align*}
\partial_t n + \text{div} (\rho v + nu + nv) &= 0, \\
\partial_t v + u \cdot \nabla v + v \cdot (\nabla u + \nabla v) - \varepsilon \Delta v + \nabla n - \nabla \psi &= \varepsilon (\frac{1}{\rho + n} - 1)(\Delta v + \nabla u), \\
\Delta \psi &= n, \\
v|_{t=0} &= P_0 \cdot u^\varepsilon_0, n|_{t=0} = 0.
\end{align*}
$$

(1.3)

Note that we skip the $\varepsilon$ dependence of the solutions in our notation for the last two systems. We also point out that we choose this kind of splitting mainly to ensure that the smooth solutions of the first system remain irrotational which is crucial to establish the global existence. As we shall see below, system (1.2) is a good viscous approximation of the Euler-Poisson system, in the sense that the linear part of this system has the same decay properties for low frequencies as the (EP) system, that is for localized initial data, the $L^p_x$ norm of $\nabla (\rho - 1, \nabla \phi, u)$ decay like $(1 + t)^{-1 - \frac{n}{p}}$ uniformly for $\varepsilon \in (0, 1]$.

To prove the global existence of (1.2), we shall use the following norm for the initial data:

$$
\| (\rho_0 - 1, u_0, \nabla \varphi_0) \|_{Y^\varepsilon} \triangleq \| (\rho_0 - 1, u_0, \nabla \varphi_0)^L \|_{W^{4+1.1}} + \| x (\rho_0 - 1, u_0, \nabla \varphi_0)^L \|_{H^{4+4+\delta}} + \| (\rho_0 - 1, u_0, \nabla \varphi_0)^h \|_{L^2} + \| (\rho_0 - 1, u_0, \nabla \varphi_0) \|_{H^{11+2\varepsilon}}
$$

(1.4)
where $\sigma \geq 0$ is a positive parameter and $\delta = \frac{1}{1000}$.

We now state our results for system (1.2) and (1.3):

**Theorem 1.4.** Let $\sigma \geq 0$. There exist two constants $C_1 > 0$, $\vartheta_1 > 0$ such that for any $\varepsilon \in (0, 1]$, any $\vartheta \in (0, \vartheta_1]$ if

$$
\|(u_0, \rho_0 - 1, \nabla \varphi_0)\|_{Y^\sigma} \leq \vartheta,
$$

then the system (1.2) admits a global solution $(u, \rho, \nabla \varphi)$ in $C([0, \infty), H^{\sigma+7})$, which enjoys the uniform (in $\varepsilon$) time decay: for any $t > 0$,

$$
(1 + t)\|(\rho - 1, \nabla u, \nabla \varphi)(t)\|_{W^{\sigma, \infty}} + \|(\rho - 1, \nabla u, \nabla \varphi)(t)\|_{H^{\sigma+7}} \leq C_1 \vartheta.
$$

Once the above theorem proved, Theorem 1.1 is an easy consequence of the following one:

**Theorem 1.5.** Let $(\rho, u, \nabla \phi)$ be the solutions constructed in the Theorem 1.4 with $\sigma = 4$. There exists $C > 1$, $\vartheta_0 \in (0, \vartheta_1]$, such that for any $\varepsilon \in (0, 1]$, $\vartheta \in (0, \vartheta_0]$, if the following assumption holds:

$$
\|(\rho_0^\vartheta - 1, P^\perp u_0^\vartheta, \nabla \varphi_0^\vartheta)\|_{Y^4} \leq \frac{\vartheta}{C}, \quad \|P u_0^\vartheta\|_{H^3} \leq \vartheta \varepsilon,
$$

then the system (1.3) admits a solution in $C([0, T), H^3)$ with $T > \varepsilon^{-(1-\vartheta)}$.

Since Theorem 1.5 is quite easy to obtain by merely energy estimates, we shall only explain the difficulties and strategies for proving Theorem 1.4

We introduce the new unknown $U = (\frac{\nabla}{|\nabla|}(\rho - 1), \frac{\div}{|\nabla|} u)$. Using the curl-free condition, it suffices for us to consider the system:

$$
\partial_t U + AU = F(U, U), \quad A = \begin{pmatrix} 0 & \langle \nabla \rangle \\ -\langle \nabla \rangle & -2 \varepsilon \Delta \end{pmatrix}, \quad (1.5)
$$

where $F$ is a quadratic form defined in (3.3). Simple computations shows that the eigenvalues of $A(\xi)$ are $\lambda_{\pm} = -\varepsilon|\xi|^2 \pm \sqrt{1 + |\xi|^2 - \varepsilon^2 |\xi|^4} \triangleq -\varepsilon|\xi|^2 \pm b(\xi)$. To present our ideas about decay estimate, we thus consider the toy model (we change the nonlinear term for clarity since there will be no loss of derivative in energy estimate):

$$
\begin{cases}
\partial_t \beta - \varepsilon \Delta \beta + i b(D) \beta = \beta^2 \\
\beta|_{t=0} = \beta_0
\end{cases}
$$

where $\beta \in \mathbb{C}^2$. As indicated in the 3d case [23], we need to consider different frequencies to isolate the dispersive effects and dissipation effects. On one hand, when we focus on low frequency (say $\varepsilon |\xi|^2 \lesssim \kappa_0$ where $\kappa_0$ is very small but independent of $\varepsilon$), $e^{t \Delta}$ is useless since we want to get estimate uniformly for $\varepsilon$, but we can expect that $e^{t b(D)}$ behave very like $e^{t |\nabla|}$, which shall provide us the dispersive estimate uniformly in $\varepsilon \in (0, 1]$. On the other hand, when we focus on the high frequency (say $\varepsilon |\xi|^2 \gtrsim \kappa_0$), we have that $\Re \lambda_{\pm} \leq -c(\kappa_0)$ where $c(\kappa_0)$ is independent of $\varepsilon \in (0, 1]$, the operator $e^{t \lambda_{\pm}(D)}$ can provide exponential decay, we thus expect the solution has good decay even in $L^2_x$ norm.

In practice, we first try to introduce some norms which indicate the decay properties for both low and high frequency. In order to do this, let us choose a compactly supported function $\chi$, which equals to 1 on the unit ball $B_1(\mathbb{R}^2)$ and vanishes outside $B_2(\mathbb{R}^3)$. Denote then $\chi^L(\xi) = \chi(\sqrt{\kappa_0} |\xi|)$, $\chi^H = 1 - \chi^L$ where $\kappa_0$ is a threshold to be chosen later. We define the norm (the reason for evolving the weighted norm will be explained later):

$$
\|\beta\|_{X_T} = \sup_{t \in [0, T)} \langle t \rangle \|\beta^L(t)\|_{W^{1, \infty}} + \|xe^{itb(D)} \beta^L\|_{H^4} + (1 + t)\|\beta^H(t)\|_{H^9} + \langle t \rangle^{-\delta}\|\beta(t)\|_{H^{10}} + \|\beta(t)\|_{H^8}.
$$
where $\beta^L = \chi^L(D)\beta$, $\beta^H = \chi^H(D)\beta$. Note that in the definition of the norm, we have time decay of rate $(1 + t)^{-1}$ rather than $e^{-ct}$ for high frequency due to the weak decay property for low frequency. To get the a priori estimate, we need to consider several interactions between different frequencies. However, due to the slow decay provided by dispersive estimate for low frequency, the low frequency output of the interactions between the low frequency and the high frequency is difficult to close.

More precisely, in order to estimate the low frequency, by rewriting $\beta^2 = (\beta^L)^2 + 2\beta^L\beta^H + (\beta^H)^2$, we need to estimate the term $\int_0^t e^{-i(D)(t-s)}\chi^L(D)(\beta^H\beta^L)(s)ds$ which can be estimated as, by dispersive estimate for $e^{ib(D)}\chi^L$ (see Lemma 3.1)

$$\| \int_0^t e^{-i(D)(t-s)}\chi^L(D)(\beta^H\beta^L)(s)ds\|_{L^\infty} \lesssim \int_0^t (1 + t - s)^{-1}\|\beta^H\beta^L\|_{W^{2,1}}ds$$

$$\lesssim \int_0^t (1 + t - s)^{-1}(1 + s)^{-1}ds\|\beta^L\|_X^2 \lesssim (1 + t)^{-\epsilon}\|U\|_X^2 \quad (1.6)$$

where $0 < \epsilon < 1$. Unfortunately, the desired case $\epsilon = 1$ is not true. To overcome this difficulty, we need more accurate splitting of frequencies. We observe that one can indeed split the frequency into three parts, namely, lowest frequency: $\{\epsilon|\xi|^2 \leq \rho_0\}$, intermediate frequency $\{\frac{\rho_0}{\epsilon} \leq \epsilon|\xi|^2 \leq 3\rho_0\}$ and highest frequency $\{\epsilon|\xi|^2 \geq \frac{3}{2}\rho_0\}$. In this way, due to the lack of interaction lowest $\times$ lowest $\rightarrow$ highest, we could expect the highest frequency enjoys faster decay. What is more, the intermediate frequency part has also good decay since on this region, we have $e^{\lambda_+\epsilon(\xi)} \leq e^{-ct}$ for some $c > 0$ independent of $\epsilon$. The lowest frequency is now manageable since for the lowest $\times$ intermediate $\rightarrow$ lowest interaction we could use normal form transformation (see details below) by noticing that the intermediate frequency still lies in the region that dispersive property holds. To summarize, after some crude analysis, we expect the lowest frequency part enjoys the $L^\infty_x$ decay of $(1 + t)^{-1}$, the intermediate part enjoy the $L^p_x$ $(2 \leq p \leq 4)$ decay like $(1 + t)^{-\left(2 - \frac{2}{p}\right)}$, and the high frequency parts have $L^2_x$ decay like $(1 + t)^{-2}$. We explain for instance the high frequency case. By choosing three smooth radial function $\chi^l, \chi^m, \chi^h$ which satisfies $\chi^l(\xi - \eta)\chi^m(\eta)\chi^h(\xi) = 0$ (see the definition in Section 2) and defining $\chi^L = \chi^l + \chi^m, \beta^L = \beta^l + \beta^m$, one can write $(\beta^2)^h = (2\beta^L\beta^h + (\beta^h)^2 + (\beta^m)^2 + 2\beta^L\beta^m)^h$. We could expect the worst part $\beta^L\beta^m$ enjoys $L^2$ decay of $(1 + t)^{-2}$.

We thus need to modify our norm to be (with $N > 10$ to be chosen) :

$$\|\beta\|_{X_T} \overset{\Delta}{=} \sup_{t \in [0,T)} \langle t \rangle \|\beta^L\|_{L^\infty} + \langle t \rangle \|\beta^m\|_{H^{N-1}} + \langle t \rangle^2 \|\beta^h\|_{H^{N-1}}$$

$$+ \|xe^{ib(D)}\beta^L\|_{H^1} + \langle t \rangle^{-\delta}\|\beta\|_{H^N} + \|\beta\|_{H^{N-2}} \quad (1.7)$$

We now explain Low $\times$ Low $\rightarrow$ Low estimate where only dispersive estimate is available. To overcome the difficulty of quadratic nonlinearity, the normal form transformation (or more generally ’space-time resonance’ philosophy [5]) should be enforced. To be more precise, we set $\alpha = e^{ib(D)}\beta^L$ and write

$$\int_0^t e^{-i(t-s)b(D)}e^{c(t-s)}\Delta \chi^L(D)(\beta^L)^2ds = \mathcal{F}^{-1}(e^{-i\phi_1(\xi,\eta)}e^{\epsilon t(t-s)|\xi|^2}\chi^L(\xi)\hat{\alpha}(\xi - \eta)\hat{\alpha}(\xi)d\xi d\eta)$$

$$(1.8)$$

where $\phi_1 = b(\xi) - b(\xi - \eta) - b(\eta) < 0$ on the support of $\chi^L(\xi)\chi^L(\xi - \eta)\chi^L(\eta)$. Following the ‘space-time resonance’ philosophy, by identity $e^{i\alpha} = \frac{1}{\alpha}\partial_\alpha e^{i\alpha}$, one integrate by parts in time so that $(1.8)$ becomes

$$-\int_0^t e^{-i(t-s)b(D)}e^{\epsilon t(t-s)}\Delta \chi^L(D)(\epsilon \Delta T\frac{1}{\epsilon\phi_1}(\beta^L,\beta^L) + T\frac{1}{\epsilon\phi_1}(\epsilon \Delta \beta^L + (\beta^2)^L, \beta^L))ds$$

$$(1.9)$$
plus boundary terms and symmetric terms which can be handled similarly. Here, $T_{\frac{1}{\alpha t^\beta}}$ is the bilinear operator defined by (2.1). Note that we have also used the equation satisfied by $\alpha$:

$$\partial_t \alpha = \varepsilon \Delta \alpha + e^{itb(D)}(\beta^2)^L.$$  

In view of (1.9), besides the viscous terms, we need to estimate the typical term:

$$\int_0^t e^{\lambda-(D)(t-s)}\xi^L(D)(\beta^L\beta^L_L)^L\beta^L(s)ds.$$  

Nevertheless, the same problem like (1.6) emerges, since we could estimate $\|\beta^L\beta^L_L\|_{W^{2,1}}$ by $\|\beta^L\|_{L^2}^2\|\beta^L\|_{L^\infty}$ which has only the decay $(1+s)^{-1}$. Following [16, 18], the 'vector field-like' norm $e^{-itb(D)}_\varepsilon e^{itb(D)}\beta^L$ needs to be involved to detect some space resonance information of the phase function.

We now explain the extra difficulty due to the dissipation term $\varepsilon \Delta \beta$. By noticing that $e^{\varepsilon(t-s)\Delta} \xi^L e^{\Delta} \xi^L$ is a multiplier in $L^2$ with norm $(1 + t - s)^{-1}$, we expect that $\varepsilon \Delta \beta^L$ has $L^2$ decay like $(1 + t)^{-1}$. However, we shall still encounter the difficulty that $\|\beta^L\|^2_{L^2}$ have decay like $(1 + t)^{-1}$, which forces us to use normal form transformation (or integrate by parts in time) again. This will increase the complexity of computations. The trick to simplify the arguments is that in the process of performing normal form transformations, we could introduce $\hat{\alpha} = e^{-\varepsilon t\Delta} e^{itb(D)} \beta^L$ as the intermediate profile. By defining the complex phase function $\phi = i \phi_1 + \varepsilon (|\xi|^2 - |\xi - \eta|^2 - |\eta|^2)$ which does not vanish on the support of $\chi^L(\xi)\chi^L(\xi - \eta)\chi^L(\eta)$, we could integrate by parts in time as before to get:

$$\int_0^t e^{-it(t-s)\beta(D)} e^{i(t-s)\Delta} \xi^L(D)(\beta^L)^2 ds = F^{-1}(e^{-itb(D)}(\xi)) \chi^L(D)(\xi) \hat{\alpha}(\xi - \eta) \hat{\alpha}(\eta) d\eta ds$$  

which allows us not to care about $\varepsilon \Delta \beta^L$. Note that there is no singularity on $\frac{1}{\beta}$ since $i \phi_1$ never vanishes. Moreover, computations (see Section 3) show that the bilinear operator $T_{\frac{1}{\beta}}$ enjoys the same good quasi-product estimates as $T_{\frac{1}{\alpha t^\beta}}$. The strategy for dealing with this term shall then have similarities with [16, 18] where the global existence for 2-d (EP) is proved.

**Organization of the paper:** We first introduce some notations in Section 2. To prove Theorem 1.4 some reformulations and useful lemmas (linear estimates, bilinear estimates) are presented in Section 3. The local existence in weighted space for system (1.2) shall be shown in Section 4. Section 5 to Section 8 are dedicated to establish several a priori estimates. The conclusion for Theorem 1.4 are then made in Section 9. Theorem 1.5 shall be proved in Section 10. Finally, in appendix, we sketch the proofs for part of low frequency estimates.

### 2. Notations

- We denote $a_+ (\text{resp.} a_-)$ for a constant larger (resp. smaller) but arbitrarily closed to $a$.
- We choose three radial smooth functions $\chi_1, \chi_2, \chi_3 : \mathbb{R}^2 \to \mathbb{R}$ s.t. $\chi_1 + \chi_2 + \chi_3 = 1$ and Supp $\chi_1(\xi) \subset \{ |\xi| \leq 1 \}$, Supp $\chi_2(\xi) \subset \{ \frac{1}{2} \leq |\xi| \leq 3 \}$, Supp $\chi_3(\xi) \subset \{ |\xi| \geq \frac{3}{2} \}$. Denote $\chi^l = \chi\left(\sqrt{\frac{\xi}{\kappa_0}}\right)$, $\chi^m = \frac{\xi}{\kappa_0}$, $\chi^H = \frac{\xi}{\kappa_0}$, $\chi^L = \chi^l + \chi^m$, $\chi^H = \chi^m + \chi^H$. We also write: $f^L = F^{-1}(\chi^L(D)f(D))$, $f^l = F^{-1}(\chi^l(D)f(D))$, $f^m = F^{-1}(\chi^m(D)f(D))$, $f^H = F^{-1}(\chi^H(D)f(D))$.  

We first remark that since \( \text{curl}(P_1) \) and \( \text{curl} \), to symmetrize the system, we first introduce the new unknowns:

\[
\begin{align*}
T_m(f, g) & \triangleq \mathcal{F}^{-1}\left( \int m(\xi, \eta) \hat{f}(\xi - \eta) \hat{g}(\eta) d\eta \right) \\
T_m(f, g, h) & \triangleq \mathcal{F}^{-1}\left( \int m(\xi, \eta, \sigma) \hat{f}(\xi - \eta) \hat{g}(\eta - \sigma) \hat{h}(\eta) d\eta \right)
\end{align*}
\]

We recall the classical Littlewood-Paley decomposition: choose a cut-off function \( \Psi, 0 \leq \Psi \leq 1, \Psi \equiv 1 \) on \( B_{3/2} \) and vanish on \( B_{5/3}^c \). We set

\[
\Phi_j(x) = \Phi\left( \frac{x}{2^j} \right), \quad \text{where} \quad \Phi(x) = \Psi(x) - \Psi(2x).
\]

Note that \( \Phi(x) \) supported on the annulus \( \{ \frac{3}{4} \leq |x| \leq \frac{5}{4} \} \) and \( 1 = \Psi(x) + \sum_{j \in \mathbb{N}} \Phi_j(x) \).

Recall the homogeneous dyadic block: \( \Delta_k f \triangleq \mathcal{F}^{-1}(\Phi_k(\xi)\hat{f}(\xi)) \) (\( k \in \mathbb{Z} \)), inhomogeneous dyadic block: \( \Delta_{-1} f \triangleq \mathcal{F}^{-1}(\Phi(\xi)\hat{f}(\xi))\), \( \Delta_1 f \triangleq \mathcal{F}^{-1}(\Phi_k(\xi)\hat{f}(\xi)) \), \( l \in \mathbb{N} \), and \( S_k = \sum_{-1 \leq j \leq k-1} \Delta_j \).

3. Preliminaries

Set \( \varrho = \rho - 1 \), system (1.1) is equivalent to the following system:

\[
\begin{align*}
\partial_t \varrho + \text{div} u + \text{div} (\rho u) &= 0, \\
\partial_t u + u \cdot \nabla u - \varepsilon \mathcal{L} u + \nabla \varrho - \nabla \varphi &= 0, \\
\Delta \varphi &= \varrho \\
|u|_{t=0} &= \mathcal{P}^\perp u_0, \quad |\varrho|_{t=0} = \rho_0 - 1
\end{align*}
\]

We first remark that since \( \text{curl}(\mathcal{P}^\perp u_0) = 0 \), standard energy estimates indicate that this curl-free property will propagate as long as smooth solution exists. Note also that by identity \( \Delta u = -\text{curl} \text{curl} u + \nabla \text{div} u \), we have \( \mathcal{L} u = \Delta u + \nabla \text{div} u = 2\Delta u \).

To symmetrize the system, we first introduce the new unknowns:

\[
a = \frac{\nabla}{|\nabla|} \varrho, \quad c = \text{div} \left( \frac{\nabla}{|\nabla|} u \right), \quad U = (a, c)^	op
\]

It is direct to see that \((a, c)\) satisfies the system:

\[
\begin{align*}
\partial_t a + \langle \nabla \rangle c &= \langle \nabla \rangle \text{div} \left( \frac{|\nabla|}{|\nabla|} a \right) \mathcal{R} c = \langle \nabla \rangle \mathcal{R} \cdot \left( \frac{|\nabla|}{|\nabla|} a \right) \mathcal{R} c \\
\partial_t c - \langle \nabla \rangle a - 2\varepsilon \Delta c &= \frac{1}{2} \text{div} \left( \frac{|\nabla|}{|\nabla|} \nabla |\mathcal{R} c|^2 \right) = -\frac{1}{2} |\nabla| |\mathcal{R} c|^2 \\
a|_{t=0} = \langle \nabla \rangle |\nabla| \varrho_0, \quad c|_{t=0} = \text{div} \left( \frac{|\nabla|}{|\nabla|} u_0 \right)
\end{align*}
\]

ie.

\[
\partial_t U + \begin{pmatrix}
0 & \langle \nabla \rangle \\
-\langle \nabla \rangle & -2\varepsilon \Delta
\end{pmatrix} U = \begin{pmatrix}
\langle \nabla \rangle \mathcal{R} \cdot \left( \frac{|\nabla|}{|\nabla|} a \right) \mathcal{R} c \\
-\frac{1}{2} |\nabla| |\mathcal{R} c|^2
\end{pmatrix} \triangleq \begin{pmatrix}
F_1(a, c) \\
F_2(a, c)
\end{pmatrix} = F(a, c)
\]

where we denote \( \mathcal{R} = \sum_j \frac{\Phi_j}{|\nabla|} \) the Riesz potential. Note also that we have used the fact that \( u = \mathcal{R} c \) which is a consequence of \( \text{curl} u = 0 \).

Define

\[
A = \begin{pmatrix}
0 & \langle \nabla \rangle \\
-\langle \nabla \rangle & -2\varepsilon \Delta
\end{pmatrix}
\]

and
By elementary computation, we get that the eigenvalues of $-\hat{A}(\xi)$ are:
\[ \lambda_{\pm} = -\varepsilon|\xi|^2 \pm i\sqrt{1+|\xi|^2 - \varepsilon^2}|\xi|^4 \triangleq -\varepsilon|\xi|^2 \pm ib(\xi) \]  
(3.5)
where we cut the lower half imaginary axis in the definition of square root of a complex function. What is more, one can easily check that the Green matrix is
\[ e^{-t\hat{A}(\xi)} = \frac{1}{\lambda_+ - \lambda_-} \left( \lambda_+ e^{\lambda_+t} - \lambda_- e^{\lambda_-t} \right) \left( e^{\lambda_+t} - e^{\lambda_-t} \right) (\xi) \triangleq \left( \mathcal{G}_1 \mathcal{G}_2 \mathcal{G}_3 \right) \]  
(3.6)
Note $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$ are well defined everywhere since there is no singularity when $\lambda_+ = \lambda_-$. When we focus on the Low frequency, i.e., when $\varepsilon|\xi|^2 \leq 3\kappa_0 << 1$, we can diagonalize $A$ as:
\[ A(D) = \left( \begin{array}{cc} -\frac{\lambda_+}{(\xi)} & 0 \\ 0 & -\frac{\lambda_-}{(\xi)} \end{array} \right) \left( \begin{array}{cc} -\lambda_+ & 0 \\ 0 & -\lambda_- \end{array} \right) \left( \begin{array}{cc} \langle \nabla \rangle & \langle \nabla \rangle \\ \langle \nabla \rangle & -\langle \nabla \rangle \end{array} \right) \frac{1}{2ib} \]
\[ \triangleq \mathcal{Q} \left( \begin{array}{cc} -\lambda_+ & 0 \\ 0 & -\lambda_- \end{array} \right) \mathcal{Q}^{-1}, \quad \mathcal{Q}^{-1} = \left( \begin{array}{cc} \frac{\lambda_+}{\langle \nabla \rangle} & \langle \nabla \rangle \\ \langle \nabla \rangle & -\langle \nabla \rangle \end{array} \right) \frac{1}{2ib} \]  
(3.7)
We denote then $W = Q^{-1}\chi^L U \triangleq (w, \bar{w})$ for which the first component satisfies the equation:
\[ \partial_t w - \lambda_-(D)w = \frac{\lambda_+}{2ib}\chi^L(D)F_1(a,c) + \langle \nabla \rangle \frac{\lambda_+}{2ib}\chi^L(D)F_2(a,c) \]
\[ = \frac{\lambda_+}{2ib}\chi^L(D)F_1(a^L,c^L) + \langle \nabla \rangle \frac{\lambda_+}{2ib}\chi^L(D)F_2(a^L,c^L) \]
\[ + [Q^{-1}\chi^L(D)(F(a^h,c^L) + F(a^L,c^h) + F(a^h,c^h))]_1 \]
\[ \triangleq \mathcal{R}(B(w,w) + \langle \nabla \rangle \chi^L H) \]  
(3.8)
where, $H = \mathcal{R}a^L \mathcal{R}c^h + \mathcal{R}c^L \mathcal{R}a^h + \mathcal{R}a^h + \mathcal{R}c^h \approx \mathcal{R}U^L \mathcal{R}U^h + \mathcal{R}U^h \mathcal{R}U^h$ and by relation $a^L = w + \bar{w}, c^L = -(\frac{\lambda_+}{\langle \nabla \rangle} w + \frac{\lambda_+}{\langle \nabla \rangle} \bar{w})$, $B(w, w)$ is defined by
\[ \mathcal{F}B(w,w) = \sum_{\mu, \nu \in \{+, -\}} \int m_{\mu, \nu}(\xi, \eta) \mathcal{R}w^\mu(\xi - \eta) \mathcal{R}w^{\nu}(\eta) d\eta \]  
(3.9)
with $m_{\mu, \nu}(\xi, \eta) = \langle \xi \rangle n_\mu(\xi - \eta)n_\nu(\eta)\chi^L(\xi)\chi^L(\xi - \eta)\chi^L(\eta)$, $n_+ \in \{-\frac{\lambda_+}{\langle \nabla \rangle}, 1\}, n_- \in \{-\frac{\lambda_-}{\langle \nabla \rangle}, 1\}$ and exponent $\{\pm\} = \{Id, conjugate\}$.

Note that in the above (and hereafter), for notational convenience, we do not make difference between ‘real’ Riesz potential and general zero order Fourier multipliers whose symbol satisfies zero homogeneous condition and is smooth away from the origin, since they have similar properties. For example, they are both bounded operators in $L^p(1 < p < \infty)$. Moreover, we do not distinguish the scalar Riesz potential $\sum_{j}(\xi, \eta)$ and vector one $\sum_{j}(\xi, \eta)$. One easily checks that in the above, $\mathcal{R}$ can represent anyone of the set $\{\sum_{j}(\xi, \eta) \chi^L(D), \sum_{j}(\xi, \eta) \chi^L(D), \sum_{j}(\xi, \eta) \sum_{j}(\xi, \eta)\}$.

After recalling the definition: $U^L = \chi^L(D)U = \chi^L(D)U + \chi^m(D)U = U^i + U^m, U^m = \chi^m(D)U, U^h = \chi^h(D)U$, we define the following norm:
\[ \|U\|_X \triangleq \sup_{t \in [0,T]} \langle t \rangle \|\nabla \|_{\langle \nabla \rangle} Q^{-1}\|U^L(t)\|_{W^{4,4}} + \|xe^{itb(D)}w(t)\|_{W^{4,4} + \frac{\|\nabla \|_{\langle \nabla \rangle}}{2}} + \|U^L(t)\|_{H^{2s+N'}} \]
\[ + \langle t \rangle^{1 - 3\delta} \|U^m(t)\|_{H^{2s+N-1}} + \langle t \rangle^{\frac{\alpha}{2}} \|U^m(t)\|_{W^{1,4}} + \langle t \rangle^\alpha \|U^h(t)\|_{H^{2s+N-2}} + \langle t \rangle^{-\delta} \|U(t)\|_{H^{2s+N}}, \]  
(3.10)
potential in the nonlinear term (see (3.8)). We shall prove the global existence of system (3.3) in the Banach space $X_T^p$ defined by the norm $\|\cdot\|_{X_T^p}$. In the sequel, for notational clarity, we shall assume that $\sigma = 0$ (and denote $X_T^0 = X_T$), since the case $\sigma > 0$ can be easily generalized. We first remark that by dispersive estimate (3.11) and Hölder’s inequality, for any $0 \leq t < T$

\[
\|U^L(t)\|_{W^2,1} \lesssim \|w(t)\|_{W^2,1} \lesssim \langle t \rangle^{-(1-2\delta)} \|e^{it(D)}w(t)\|_{W^{4(1-\delta)}}, \quad \frac{1}{2} \lesssim \langle t \rangle^{-(1-2\delta)} \|U\|_{X_T}.
\]

Moreover, we have:

\[
\|\nabla u(t)\|_{L^\infty} = \|\nabla RU(t)\|_{L^\infty} \leq \|\nabla RU^L(t)\|_{L^\infty} + \|\nabla RU^h(t)\|_{L^\infty} \leq \sum_k 2^{\frac{k}{2}} (2^k)^{-1} \|\Delta_k |\nabla|^\frac{1}{2} (\nabla) w(t)\|_{L^\infty} + \langle t \rangle^{-\alpha} \|U\|_{X_T} \lesssim \|\nabla |\nabla|^\frac{1}{2} (\nabla) w(t)\|_{L^\infty} + \langle t \rangle^{-\alpha} \|U\|_{X_T} \lesssim \langle t \rangle^{-1} \|U\|_{X_T}.
\]

In the following of this section, we will give some preliminary lemmas which will be used later.

3.1. **Linear estimates.** We present in this subsection the linear estimates for Low (lowest and intermediate) and highest frequency.

3.1.1. **Linear estimates for Low frequency.**

**Lemma 3.1.** Dispersive estimate for $e^{it(D)}\chi^L$.

For every $\kappa_0$ is small enough (say $\kappa_0 \leq \frac{1}{200}$) and for any $2 \leq p \leq \infty$, we have the following dispersive estimate:

\[
\|e^{it(D)}\chi^L(D) f\|_{L^p} \lesssim_{\kappa_0} (1 + |t|)^{-(\frac{1}{2})} \|f\|_{W^{2(1-\frac{1}{p}),\frac{1}{p}'}} \quad \forall t \in \mathbb{R}.
\] (3.11)

**Proof.** Indeed, (3.11) holds when $e^{it(D)}\chi^L(D)$ is replaced by $e^{it(\nabla)}$, which follows from the classical stationary phase arguments. Nevertheless, when $\kappa_0$ is chosen small enough, $e^{it(D)}\chi^L(D)$ enjoys the similar algebraic properties as $e^{it(\nabla)}$. One can refer to Corollary 3.7 of [23] for the detailed proof.

**Lemma 3.2.** $L^p \to L^p$ boundedness for $e^{it(D)}\chi^L$.

Suppose $\kappa_0 \leq \frac{1}{200}$. For any $1 < p < \infty$, we have the following estimate:

\[
\|\Delta_k e^{it(D)}\chi^L(D) u\|_{L^p} \lesssim_{\kappa_0} \langle t \rangle^{\frac{1}{2}} (2^k)^{\frac{1}{2}} \|\Delta_k u\|_{L^p}, \quad (k \geq -1)
\] (3.12)

\[
\|e^{it(D)}\chi^L(D) u\|_{L^p} \lesssim_{\kappa_0} \langle t \rangle^{\frac{1}{2}} \|u\|_{W^{s,p}}.
\] (3.13)

where $s = \lfloor 1 - \frac{2}{p} \rfloor$.

**Proof.** This lemma has essentially been proved in Lemma 2.2 of [18] where $e^{it(\nabla)}$ rather than $e^{it(D)}\chi^L(D)$ is considered. We will sketch the proof of (3.12) for $p = 1, 2, \infty$, the other case for (3.12) and (3.13) follows from interpolation and summation respectively. By Young’s inequality, it suffices for us to show:

\[
\|\mathcal{F}^{-1}(\Phi_k(\xi)e^{it(D)}\chi^L(\xi))\|_{L^1} \lesssim_{\kappa_0} \langle t \rangle \langle 2^k \rangle, \quad k \geq 0; \quad \|\mathcal{F}^{-1}(\Psi(\xi)e^{it(D)}\chi^L(\xi))\|_{L^1} \lesssim_{\kappa_0} \langle t \rangle
\] (3.14)

where $\Phi_k, \Psi_k$ is defined in (2.3). To prove (3.14), one uses the inequality: $\|f\|_{L^1} \lesssim \|f\|_{L^2}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}}$ and elementary estimate:

\[
\|\Phi_k(\xi)e^{it(D)}\chi^L(\xi)\|_{L^2} \lesssim 2^k, \quad \|\partial_x^2(\Phi_k(\xi)e^{it(D)}\chi^L(\xi))\|_{L^2} \lesssim_{\kappa_0} 2^{-k}(2^k t)^2, \quad \|\Psi(\xi)e^{it(D)}\chi^L(\xi)\|_{L^2} \lesssim 1, \quad \|\partial_x^2(\Phi_k(\xi)e^{it(D)}\chi^L(\xi))\|_{L^2} \lesssim_{\kappa_0} \langle t \rangle^2.
\]
3.1.2. Linear estimate for high frequency.

**Lemma 3.3.** Linear estimate for \(e^{-tA} \chi^h\).

There exists a constant \(c = c(\kappa_0)\), such that, for any real number \(s\), we have:

\[
\|e^{-tA} \chi^h u\|_{H^s} \lesssim_{\kappa_0} e^{-cs}\|u\|_{H^s}.
\]

**Proof.** One needs to study carefully the Green matrix (3.6) localized on high frequency, since the algebraic computations do not depend on the dimension, one can refer to Lemma 3.5 of [23] where the similar property is shown in dimension 3.

\[\Box\]

3.1.3. Additional estimate for intermediate frequency. For the intermediate frequency, we could use the spectral localization to get the boundedness of \(e^{-tA} \chi^m\) from \(W^{1-\frac{2}{p},p}\) to \(L^p(1 < p < \infty)\).

**Lemma 3.4.** Recall \(\chi^m(\xi) = \chi_2\left(\sqrt{\frac{\epsilon}{\kappa_0}}\xi\right)\) where \(\chi_2\) is smooth function supported on \(\{\xi|\frac{1}{2} \leq |\xi| \leq 3\}\).

We have for any \(1 < p < \infty\),

\[
\|e^{-tA} \chi^m(D) u\|_{L^p} \lesssim_{\kappa_0} e^{-\frac{\kappa_0}{p}t}\|u\|_{W^{1-\frac{2}{p},p}}.
\]

**Proof.** We first prove

\[
\|e^{\pm t\Delta} \chi^m(D)f\|_{L^p} \lesssim_{\kappa_0} e^{-\frac{1}{2}\kappa_0 t}\|f\|_{L^p}.
\]

which follows from the Young’s inequality and the fact: \(\|f\|_{L^1} \lesssim \|f\|_{L^p}^\frac{1}{p}\|f\|_{L^q}^\frac{1}{q}\). Indeed, one has:

\[
\|F^{-1}(e^{-\epsilon|\xi|^2t} \chi^m(\xi))\|_{L^1} \lesssim \|F^{-1}(e^{-\kappa_0|\xi|^2t} \chi_2(\xi))\|_{L^1} \lesssim e^{-\frac{1}{2}\kappa_0 t}\|\chi_2\|_{L^2} \lesssim e^{-\frac{1}{2}\kappa_0 t}.
\]

By the definition of the Green matrix,

\[
e^{-tA(\xi)} = \frac{1}{\lambda_+ - \lambda_-} \left( \begin{array}{cc} \lambda_+ e^{-\lambda_+ t} - \lambda_- e^{-\lambda_- t} & (e^{\lambda_+ t} - e^{\lambda_- t})(\xi) \\ (e^{\lambda_+ t} - e^{\lambda_- t})(\xi) & \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t} \end{array} \right).
\]

and eigenvalue \(\lambda_\pm = \epsilon \Delta \pm ib(D)\), we see that \(e^{-tA}\) is indeed the combination of terms like \(e^{\lambda_\pm(D)} q(D)\) where \(q(D) \in \{\frac{\Delta}{b(D)}, \frac{\nabla}{b(D)}, Id\}\). Therefore, by Lemma 3.2 and the definition of \(\chi^m\),

\[
\|e^{ib(D)} e^{\pm t\Delta} \chi^m q(D)(u)\|_{L^p} \lesssim_{\kappa_0} \|e^{ib(D)} \chi^m e^{\pm t\Delta} \chi^m q(D)\|_{L^p} \lesssim_{\kappa_0} \langle t \rangle^{1-\frac{2}{p}} \|e^{\pm t\Delta} \chi^m q(D) u\|_{W^{1-\frac{2}{p},p}} \lesssim_{\kappa_0} \langle t \rangle^{1-\frac{2}{p}} e^{-\frac{1}{2}\kappa_0 t}\|\chi^m q(D) u\|_{W^{1-\frac{2}{p},p}} \lesssim_{\kappa_0} e^{-\frac{1}{2}\kappa_0 t}\|u\|_{W^{1-\frac{2}{p},p}}.
\]

\[\Box\]
3.2. Bilinear estimates. As we shall use the normal form transformation, it is necessary for us to get some continuous properties for bilinear operators defined by \( m \). To start, we present some elementary properties of bilinear multipliers which is useful to derive the bilinear estimates.

Proposition 3.5. Define the phase function
\[
\phi_{\mu,\nu}(\xi + \eta, \eta) = i(b(\xi + \eta) - \mu b(\xi) - \nu b(\eta)) + Z(\xi, \eta),
\]
and multiplier function
\[
m_{\mu,\nu}(\xi + \eta, \eta) = (\xi + \eta)\chi^L(\xi)\chi^L(\eta)\chi^L(\xi + \eta)n_\mu(\xi)n_\nu(\eta)
\]
where \( b(\xi) = \sqrt{1 + |\xi|^2 - \varepsilon^2|\xi|^4}, Z(\xi, \eta) = \varepsilon(|\xi| + |\eta|^2 - |\xi|^2 - |\eta|^2) \) and \( \mu, \nu \in \{+, -\} \). Suppose \( \kappa_0 \leq \frac{1}{200} \), then for any multi-index \( \alpha, \beta \in \mathbb{N}^2 \), the following estimate holds uniformly in \( \varepsilon \in (0, 1) \):
\[
|\partial_{\xi}^\alpha \partial_{\eta}^\beta m_{\mu,\nu}(\xi + \eta, \eta)| \lesssim_{\alpha,\beta,\kappa_0} \langle \xi + \eta \rangle \min\{b(\xi), b(\eta), b(\xi + \eta)\}
\]

Proof. We only present the proof for \( \mu = \nu = ' \), since the others are easier or can be obtained by symmetry. At first, we have
\[
\frac{1}{\phi_{++}} = \frac{i(b(\xi) + b(\eta) + b(\xi + \eta) + iZ(\xi, \eta))}{(b(\xi) + b(\eta) + iZ(\xi, \eta))^2 - b^2(\xi + \eta)} \triangleq \frac{i(b(\xi) + b(\eta) + b(\xi + \eta) + iZ(\xi, \eta))}{B},
\]
where
\[
B = \left(b(\xi) + b(\eta)\right)^2 - b^2(\xi + \eta) - Z^2(\xi, \eta) + 2iZ(\xi, \eta)(b(\xi) + b(\eta))
\]
\[
\triangleq A = A - Z^2(\xi, \eta) + 2iZ(\xi, \eta)(b(\xi) + b(\eta))
\]
Note that \( A \) has the lower bound:
\[
A = 1 + 2b(\xi)b(\eta) - 2\xi \cdot \eta + \varepsilon^2(|\xi|^4 + |\eta|^4 - |\xi + \eta|^4)
\]
\[
\geq 1 - 27\kappa_0^2 + 2b(\xi)b(\eta) - 2\xi \cdot \eta
\]
\[
= \frac{(1 - 27\kappa_0^2 + 2b(\xi)b(\eta))^2 - 4|\xi \cdot \eta|^2}{1 - 27\kappa_0^2 + 2b(\xi)b(\eta) + 2\xi \cdot \eta} \gtrsim \frac{(b(\xi) + b(\eta))^2}{b(\xi)b(\eta)} \gtrsim 1.
\]
(3.15)

We will prove that on the support of \( \chi^L(\xi)\chi^L(\eta)\chi^L(\xi + \eta), \) for any multi-index \( \alpha, \beta \in \mathbb{N}^2, \) the following property holds:
\[
|\partial_{\xi}^\alpha \partial_{\eta}^\beta \frac{1}{B}| \lesssim_{\alpha,\beta,\kappa_0} \frac{1}{|B|}
\]
which is an easy consequence of Leibniz’s rule and
\[
|\partial_{\xi}^\alpha \partial_{\eta}^\beta B| \lesssim_{\alpha,\beta,\kappa_0} |B|, \quad \forall \alpha, \beta \in \mathbb{N}^2.
\]
(3.17)
However, \( (3.17) \) can be derived once we have the estimate for \( A \):
\[
|\partial_{\xi}^\alpha \partial_{\eta}^\beta A| \lesssim_{\alpha,\beta,\kappa_0} A, \quad \forall \alpha, \beta \in \mathbb{N}^2.
\]
(3.18)
Indeed, since
\[
|\partial_{\xi}^\alpha \partial_{\eta}^\beta Z(\xi, \eta)| + |\partial_{\xi}^\alpha \partial_{\eta}^\beta (\xi, \eta)(b(\xi) + b(\eta))| \leq P(\alpha, \beta, \kappa_0), \quad \forall (\xi, \eta) \in \text{Supp} m(\xi, \eta)
\]
where \( P(\alpha, \beta, \kappa_0) \) is a polynomial with respect to \( \kappa_0 \) which can be bounded by a constant \( C(\alpha, \beta) \) if we choose \( \kappa_0 \) small (say \( \kappa_0 \leq \frac{1}{200} \)), we can use \( (3.18), (3.15) \) to get that:
\[
|\partial_{\xi}^\alpha \partial_{\eta}^\beta B| \leq |\partial_{\xi}^\alpha \partial_{\eta}^\beta A| + C(\alpha, \beta) \lesssim_{\alpha,\beta,\kappa_0} A + C(\alpha, \beta) \lesssim_{\alpha,\beta,\kappa_0} A \lesssim_{\alpha,\beta,\kappa_0} |B|.
\]
Nevertheless, we note that the estimate of (3.18) has been proved in the appendix of [23]. Therefore, inequality (3.16) holds, which leads to the following computation:

\[
\left| \partial_\xi \partial_\eta^3 m_{++}(\xi, \eta) \right| = \left| \sum c_{\alpha_1 \alpha_2 \beta_1 \beta_2} \partial_\xi^{\alpha_1} \partial_\eta^{\alpha_2} (m_{++}(\xi, \eta)) \partial_\xi^{\beta_1} \partial_\eta^{\beta_2} b(\xi) + b(\eta) + b(\xi + \eta) + iZ(\xi, \eta) \right| B
\]
\[
\leq \kappa_0 \langle \xi + \eta \rangle (b(\xi) + b(\eta) + b(\xi + \eta)) \frac{1}{A}
\]
\[
\leq \kappa_0 \langle \xi + \eta \rangle \min\{b(\xi), b(\eta), b(\xi + \eta)\}.
\]

□

This proposition in hand, we then show the following bilinear estimate:

**Lemma 3.6.** Let \( m_{\mu \nu}, \phi_{\mu \nu} \) be defined as the last proposition, one has bilinear estimate:

\[
\| T_{\mu \nu} f,g \|_{L^p} \lesssim_\kappa_0 \| f \|_{W^{2+q_1} r_1} \| g \|_{W^{2+q_2} r_1} + \| f \|_{W^{2+q_2} r_1} \| g \|_{W^{2+q_2} r_2} \quad (3.19)
\]

where \( \frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2} \), \( 1 < r_1, r_2 \leq +\infty \), \( 1 \leq q_1, q_2 < +\infty \), \( T_{\mu \nu} \) is the bilinear operator defined in (2.1) and \( k_+ \) is a real number slightly larger than \( k \).

**Proof.** As before, we only treat the case \( T_{m_{++}} \). Let \( \psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^4) \) which satisfy the following conditions:

\[
\begin{cases}
\psi_1 + \psi_2 = 1 \\
\text{Supp } \psi_1 \subset \{(\xi, \eta) \mid \langle \xi - \eta \rangle \geq \frac{\langle \eta \rangle}{2}\}, \\
\text{Supp } \psi_2 \subset \{(\xi, \eta) \mid \langle \eta \rangle > \langle \xi - \eta \rangle\}.
\end{cases}
\]

We write

\[
m_{++}(\xi, \eta) = m_{+++}(\xi, \eta) - m_{++-}(\xi, \eta) = m_{+++}(\xi, \eta) + m_{++-}(\xi, \eta)
\]
\[
\Phi_{++}(\xi, \eta) \ni m_{+++}(\xi, \eta) = M_1(\xi, \eta) + M_2(\xi, \eta),
\]

By Proposition 3.5 we have for any \( \alpha, \beta \) with \( |\alpha| + |\beta| \leq 3 \),

\[
|\partial_\xi^{\alpha} \partial_\eta^{\beta} M_1| \leq I_{\langle \xi \rangle \geq \frac{\langle \eta \rangle}{2}} \langle \xi - \eta \rangle^{-1+\langle \eta \rangle}.
\]

Therefore, \( M_1, \partial_\xi^2 M_1, \partial_\eta^2 M_1 \in L^2(\mathbb{R}^4) \), which leads to the fact: \( \mathcal{F}^{-1}(M_1)(x, y) \in L^1_{x,y} \). Indeed,

\[
\| \mathcal{F}^{-1}(M_1)(x, y) \|_{L^1_{x,y}} \lesssim \| (1 + |x|^3 + |y|^3)^{-1} \|_{L^2_{x,y}} \| M_1 \|_{L^2} + \| \partial_\xi^2 M_1 \|_{L^2} + \| \partial_\eta^2 M_1 \|_{L^2}.
\]

By the definition of bilinear operator \( T_{m_{++}} \) and Fourier transform:

\[
T_{M_1}(\xi)^{2+\langle \eta \rangle^2} f,g = \int (\mathcal{F}^{-1} M_1)(x', y' - x') \langle D_x^2 \rangle^{2+} f(x' - x') \langle D_x^2 \rangle g(x', y') dx' dy'.
\]

Therefore, By Minkowski's inequality,

\[
\| T_{M_1} f,g \|_{L^p} \leq \int \| D_x^2 g \|_{L^{r_1}} \| (\mathcal{F}^{-1} M_1)(x', y' - x') \langle D_x^2 \rangle^{2+} f(x' - x') dx' \|_{L^{r_1}} dy'
\]
\[
\leq \| \mathcal{F}^{-1} M_1 \|_{L^1_{x,y}} \| f \|_{W^{2+q_1} r_1} \| g \|_{W^{2+q_2} r_1}.
\]

The similar result for \( M_2 \) can be derived in the same fashion.

□

**Remark 3.7.** It is easy to adapt the proof of the above lemma to get that:

\[
\| T_{\mu \nu} f,g \|_{L^p} \lesssim_\kappa_0 \| f \|_{W^{3+q_1} r_1} \| g \|_{W^{1+r_1}} + \| f \|_{W^{1+r_2} r_2} \| g \|_{W^{3+q_2} r_2}.
\]

(3.20)
Remark 3.8. Of course the norm of $T_{wuw}$ in (3.19) and (3.20) is dependent on $2_{+} - 2$ or $3_{+} - 3$, but when we use this lemma, we fixed $2_{+}$ and $3_{+}$.

Remark 3.9. From now on, we will fix $\kappa_{0} = \frac{1}{200}$.

Corollary 3.10. Recall $B(w,w) \approx R(\nabla)(R w)^{2}$ is defined in (3.9), the following trilinear estimate holds

$$\|T_{w}(R B(w,w), R w)\|_{W^{s,p}} \lesssim \|w\|_{W^{2,p_{1}}} \|w\|_{W^{2,p_{2}}} \|w\|_{W^{s+3_{+},p_{3}}}$$

where $1 < p_{1}, p_{2}, p_{3} < \infty$ and $\frac{1}{p} = \frac{1}{p_{1}} + \frac{1}{p_{2}} + \frac{1}{p_{3}}$.

3.3. Useful Lemmas for local existence. In this subsection, we give some preliminary lemmas which will be used in the proof of local existence.

Lemma 3.11. Let $\zeta$ be a compactly support smooth function and $\theta \in \dot{W}^{1,\infty}$, and denote $\zeta_{R} = \zeta(\frac{\cdot}{R})$. then

$$\|[\zeta_{R}(\cdot), \Theta(D)]f\|_{L^{2}} \lesssim \frac{1}{R} \|f\|_{L^{2}}$$

Proof.

$$\mathcal{F}(\zeta_{R}, \Theta(D))f) = \int \hat{\zeta}_{R}(\eta) \hat{f}(\xi - \eta) \Theta(\xi - \eta - \Theta(\xi)) d\eta$$

$$= - \int \hat{\zeta}_{R}(\eta) \hat{f}(\xi - \eta) \int_{0}^{1} \eta \cdot \nabla \Theta(\xi - \tau \eta) d\tau d\eta$$

$$= i \int \hat{f}(\xi - \eta) \nabla \zeta_{R}(\eta) \cdot \int_{0}^{1} \nabla \Theta(\xi - \tau \eta)) d\tau d\eta. \quad (3.21)$$

Therefore, by Parseval’s inequality and Young’s inequality,

$$\|[\zeta_{R}, \Theta(D)]f\|_{L^{2}} \lesssim \|\mathcal{F}(\zeta_{R}, \Theta(D))f\|_{L^{2}}$$

$$\lesssim \|\nabla \zeta_{R}\|_{L^{1}} \|\hat{f}\|_{L^{2}} \|\nabla \Theta\|_{L^{\infty}} \lesssim \frac{1}{R} \|f\|_{L^{2}}.$$  

Note that in the above, we have used the fact: $\|\nabla \zeta_{R}\|_{L^{1}} \lesssim \frac{1}{R}$. \hfill \Box

Corollary 3.12. Denote $\Phi_{j}$ the $j$-th dyadic function coming from Littlewood-Paley theory, and $\chi^{L}(\xi) = (\chi_{1} + \chi_{2})(\sqrt{\frac{\xi}{\kappa_{0}}} \xi)$ (See the definitions in Section 2). Then, for any $t \in (0,1]$, $2 \leq p < \infty$,

$$\|\Phi_{j}, e^{i t \lambda_{-}(D)} \langle \nabla \rangle^{2} \chi^{L}(D)\|_{L^{p}} \lesssim_{\kappa_{0}, \varepsilon, p} 2^{-j} \|f\|_{L^{2}}. \quad (3.22)$$

Proof. By virtue of Hausdorff’s inequality, identity (3.21) and Young’s inequality, we have:

$$\|\Phi_{j}, e^{i t \lambda_{-}(D)} \langle \nabla \rangle^{2} \chi^{L}(D)\|_{L^{p}} \lesssim \|\mathcal{F}(\Phi_{j}, e^{i t \lambda_{-}(D)} \langle \nabla \rangle^{2} \chi^{L}(D)\|_{L^{p}}$$

$$\lesssim \|\nabla \Phi_{j}\|_{L^{1}} \|\hat{f}\|_{L^{2}} \|\chi^{L}(e^{i t \lambda_{-}(\xi)}(\xi)^{2} \chi^{L})\|_{L^{\frac{2p}{p-2}}}$$

$$\lesssim 2^{-j} \left(\frac{\varepsilon}{\kappa_{0}}\right)^{-\frac{1}{2} - \frac{1}{p}} \left(1 + \frac{\kappa_{0}}{\varepsilon}\right) \|f\|_{L^{2}}.$$  

We will need to estimate ’weighted product term’ like $xfg$, the following lemma allows us not to lose derivative on weighted term.
Lemma 3.13. For $s \geq 0$, $\nu > 0$, the following weighted product estimate holds:

$$\|xfg\|_{H^s} \lesssim \min \left\{ \|xf\|_{L^2} \|g\|_{H^{s+1}}, \|xf\|_{L^\infty} \|g\|_{H^{s+1}} \right\} + \|xg\|_{L^2} \|f\|_{H^{s+1}} + \|f\|_{H^{s+1}} \|g\|_{H^{s+1}}.$$  

Proof. Write $fg = \sum_{j \geq 1} S_j f \Delta_j g + \Delta_j f S_j g$. Thanks to the Bernstein inequality and Young’s inequality the first term can be estimated as:

$$\|x \sum_{j \geq 1} S_j f \Delta_j g\|_{H^s} = \sum_{j \geq 1} 2^{js} \|x, S_{j-1} f \Delta_j g\|_{L^2} + \|S_{j-1}(xf) \Delta_j g\|_{L^2}$$

$$\leq \sum_{j \geq 1} 2^{js} \|f\|_{L^2} \|\Delta_j g\|_{L^2} + \min \left\{ \|xf\|_{L^2}, \|xf\|_{L^\infty} \right\} \|g\|_{L^2}$$

$$\leq \min \left\{ \|xf\|_{L^2} \|g\|_{H^{s+1}}, \|xf\|_{L^\infty} \|g\|_{H^{s+1}} \right\} + \|xf\|_{L^2} \|g\|_{H^{s+1}}.$$  

The second term can be controlled similarly, we omit the detail. \qed

4. Local existence and time continuity of weighted norm

By classical iteration technique, one could construct solution for system (3.1) in $C([-T_0, T_0]; H^N)$, ($N \geq 3$) for some $T_0 > 0$ (please refer to [20] for example), which leads to the local existence for system (3.3). We thus focus on the local boundedness of weighted norm $\|xe^{i\theta(D)}w(t)\|_{W^4, \frac{1}{2}}$ and its continuity in time. We start with the weighted $L^2$ estimate for high frequency which shall be useful later.

Lemma 4.1. There exists a constant $M_0 > 0$, such that for small but fixed time $T_0 < 1$, the following a-priori estimate holds

$$\sup_{t \in [0, T_0]} \|x(u^h, \varphi^h)\|_{L^2} \lesssim M_0 e^{M_0 T_0} T_0 (1 + \|x(u^L, \varphi^L)\|_{L^\infty([0, T_0], L^\infty)}).  \quad (4.1)$$

Proof. We consider the system satisfied by the high frequency:

$$\begin{cases}
\partial_t \varphi^h + \text{div} u^h + \text{div} (gu)^h = 0, \\
\partial_t u^h + (u \cdot \nabla u)^h - 2\varepsilon \Delta u^h + \nabla \varphi^h - \nabla \varphi^h = 0, \\
\Delta \varphi^h = \varphi^h, \\
u_{\mid t=0} = P^1 u_0^\varepsilon, \varphi_{\mid t=0} = \varphi_0^\varepsilon.
\end{cases}  \quad (4.2)$$

Set $\psi_R(x) = x \psi(\frac{\varepsilon}{R})$, where the compactly supported function $\Psi \equiv 1$ on $B_{\frac{\varepsilon}{4}}$ and vanish on $B_{\frac{\varepsilon}{4}}$. Multiplying the system (4.2) by $\psi_R(x)$, and test $\psi_R(\varphi^h, u^h)$, one gets the energy equality:

$$\frac{1}{2} \frac{d}{dt} \|\psi_R(\varphi^h, u^h)\|^2_{L^2}$$

$$= - \int \psi_R^2 \text{div} u^h - \nabla \varphi^h \cdot u^h dx + \int \psi_R \Delta \varphi^h \psi_R u^h dx + 2\varepsilon \int \psi_R \Delta u^h \psi_R u^h dx$$

$$\leq T_1 + T_2 + \cdots T_5.$$

We now estimate $T_1, \cdots T_5$. For $T_1$, integration by parts and Hölder inequality yield:

$$T_1 = 2 \int \psi_R \nabla \varphi^h \nabla \psi_R u^h dx \leq \|\psi_R \varphi^h\|_{L^2} \|\nabla \psi_R u^h\|_{L^2} \lesssim \|\psi_R \varphi^h\|_{L^2} \|\nabla u^h\|_{L^2}.$$  

Note that $\nabla \psi_R$ is pointwise bounded uniformly in $R$. For $T_2$, by Hölder inequality,

$$T_2 \lesssim \|\psi_R u^h\|_{L^2} \|\psi_R \varphi^h\|_{L^2} \lesssim \|\psi_R u^h\|_{L^2} \|\psi_R \varphi^h\|_{L^2} \|u^h\|_{L^2} \|\varphi^h\|_{L^2}.$$
where we have used
\[ \psi_R \nabla \varphi^h = \psi_R \nabla (\Delta)^{-1} \chi^h \vartheta^h = [\psi_R, \nabla (\Delta)^{-1} \chi^h] \vartheta^h + \nabla (\Delta)^{-1} \chi^h (D) (\psi_R \vartheta^h). \]

Notice that by the Lemma 3.11 and the spectral localization of \( \chi^h \), one has that:
\[ \| [\psi_R, \nabla (\Delta)^{-1} \chi^h] \vartheta^h \|_{L^2} \lesssim \| \mathcal{F}^{-1} (\nabla \psi_R) \|_{L^1} \| \partial_\xi (\xi^{-2} \chi^h) \|_{L^\infty} \| \vartheta^h \|_{L^2} \lesssim \frac{\varepsilon}{\kappa_0} \| \vartheta^h \|_{L^2}, \]
\[ \| \nabla (\Delta)^{-1} \chi^h (D) (\psi_R \vartheta^h) \|_{L^2} \lesssim \sqrt{\frac{\varepsilon}{\kappa_0}} \| \psi_R \vartheta^h \|_{L^2}. \]

For \( K_3 \), it is direct to see
\[ T_3 + 2\varepsilon \int |\psi_R \nabla u|^2 dx = -4\varepsilon \int \psi_R u^h \nabla u \psi_R dx \lesssim \| \psi_R u^h \|_{L^2} \| \nabla u \|_{L^2}. \]

For \( T_4 \), using again \( [\psi_R, \chi^h (D)] \) belongs to \( \mathcal{L}(L^2(\mathbb{R}^2)) \) whose norm is independent of \( R \), we get:
\[ T_4 = -\int [\psi_R, \chi^h (D)] (u \cdot \nabla u) \psi_R u^h + \chi^h (D) (\psi_R u \cdot \nabla u) \psi_R u^h dx \lesssim \| \psi_R u^h \|_{L^2} (\| u \cdot \nabla u \|_{L^2} + \| \psi_R u^h \|_{L^2} \| \nabla u \|_{L^\infty} + \| \psi_R u^L \|_{L^\infty} \| \nabla u \|_{L^2}). \]

Similarly, for \( T_5 \), we have:
\[ T_5 = -\int [\psi_R, \chi^h (D)] \text{div} (\varphi u) \psi_R \vartheta^h + \chi^h (\nabla \vartheta \cdot \psi_R u + \psi_R \vartheta \text{div} u) \psi_R \vartheta^h dx \lesssim \| \psi_R \vartheta^h \|_{L^2} (\| \text{div} (\varphi u) \|_{L^2} + \| \psi_R (u^h, \vartheta^h) \|_{L^2} \| (u, \vartheta) \|_{W^{1,\infty} + \| \psi_R (u^L, \vartheta^L) \|_{L^\infty} \| (\vartheta, u) \|_{H^1}). \]

Summing up the above estimates, we finally get:
\[ \partial_t \| \psi_R (u^h, \vartheta^h) \|_{L^2} \lesssim \| \psi_R (u^h, \vartheta^h) \|_{L^2} (1 + \| (u, \vartheta) \|_{W^{1,\infty}}) + (1 + \| (u, \vartheta) \|_{L^\infty}) \| (u, \vartheta) \|_{H^1} + \| \psi_R (u^L, \vartheta^L) \|_{L^\infty} \| (\vartheta, u) \|_{H^1}. \]

Grönwall’s inequality then gives us for any \( t < T_0 \), there exists constant \( M_0 \) which depends on \( \| (u, \vartheta) \|_{L^\infty([0, T_0], H^1 \cap W^{1,\infty})} \),
\[ \| \psi_R (u^h, \vartheta^h) (t) \|_{L^2} \lesssim M_0 e^{M_0 T_0} T_0 (1 + \| \psi_R (u^L, \vartheta^L) \|_{L^\infty([0, T_0], L^\infty)}). \]

Letting \( R \) tends to \( +\infty \), we obtain 111.

One can easily adapt the above proof to derive the following Corollary:

**Corollary 4.2.** There exists another constant \( M'_0 \) which is dependent on \( \| (u, \vartheta) \|_{L^\infty([0, T_0], H^1 \cap W^{1,\infty} \cap W^{1,4})} \), such that for any \( j \geq 0 \),
\[ \sup_{t \in [0, T_0]} \| \Phi_j (\cdot) (u^h, \vartheta^h) (\cdot) \|_{L^2} \lesssim M'_0 e^{M'_0 T_0} T_0 (2^{-j} + \| \Phi_j (u^L, \vartheta^L) \|_{L^\infty([0, T_0], L^\infty)}). \]

where \( \Phi_j \) is the dyadic function defined in 2.3.
4.1. Weighted norm for low frequency: \(x(u^L, g^L)\). By the Duhamel formula,
\[
w = e^{i\lambda_-(D)}w_0 + \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D) \frac{\lambda_+(D)}{2ib} \langle \nabla \rangle (gu) + \frac{\langle \nabla \rangle}{2ib} |\nabla| |u|^2 \, ds. \tag{4.3}
\]
Since \(g^L = \frac{\langle \nabla \rangle}{\langle \nabla \rangle} (w + \bar{w})\), \(u^L = -\mathcal{R} \left( \frac{\lambda_-(D)}{\langle \nabla \rangle} w + \frac{\lambda_-(D)}{\langle \nabla \rangle} w \right)\), we define the linear and nonlinear flow for \(g^L, u^L\):
\[
\mathcal{L}_{u^L} = \mathcal{R} \frac{\lambda_-(D)}{\langle \nabla \rangle} e^{i\lambda_-(D)} w_0 \quad \mathcal{L}_{g^L} = \frac{\langle \nabla \rangle}{\langle \nabla \rangle} e^{i\lambda_-(D)} w_0 \tag{4.4}
\]

\[
\mathcal{N}_{u^L} = \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D) \left[ \mathcal{R} \frac{\lambda_-(D)}{2ib} \div (gu) + \frac{\lambda_-(D)}{2ib} \langle \nabla \rangle |u|^2 \right] \, ds \]
\[
= \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D) \left[ \mathcal{R} \frac{\langle \nabla \rangle}{2ib} \langle \nabla \rangle (gu) + \frac{\lambda_-(D)}{2ib} \langle \nabla \rangle |u|^2 \right] \, ds
\]
\[
\mathcal{N}_{g^L} = \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D) \left[ \frac{\lambda_+(D)}{2ib} \div (gu) - \frac{\div}{2ib} \langle \nabla \rangle |u|^2 \right] \, ds
\]

Inspired by [18], we need to prove the following claim:

**Claim:** For \( j \geq 0 \), one has:
\[
\sup_{t \in [0,T_0]} \| \Phi_j(\cdot) (\langle \nabla \rangle u^L, \langle \nabla \rangle g^L)(\cdot) \|_{L^4_x} \lesssim 2^{-j} \tag{4.5}
\]

We will postpone the proof of (4.5) and first show the local boundedness of \( \| xe^{ib(D)} w \|_{W^{4, \frac{1}{2}}_{\text{loc}}} \) and its continuity in time. By (4.5), we have for any \( j \in \mathbb{N} \),
\[
\sup_{t \in [0,T_0]} \| \Phi_j (u^L, g^L) \|_{W^{1, 4}} \lesssim 2^{-j}.
\]

which leads to, by Sobolev embedding,
\[
\sup_{t \in [0,T_0]} \| x(u^L, g^L) \|_{L^\infty_x} \lesssim 1.
\]

Note that by (1.1), we have also \( \sup_{t \in [0,T_0]} \| x(u^h, g^h)(t) \|_{L^2_x} < +\infty \).

- Local boundedness of \( \| xe^{ib(D)} w \|_{W^{4, \frac{1}{2}}_{\text{loc}}} \). The boundedness of the linear term \( xw_0 \) stems from the assumption imposed upon the initial data. We thus focus on the boundedness of the nonlinear term. In light of (4.3), it suffices for us to consider the typical term :
\[
x \int_0^t e^{isb(D)} e^{i(t-s)\Delta} \langle \nabla \rangle \mathcal{R} \chi^L (gu) \, ds.
\]

Nevertheless, since for arbitrary function \( g \),
\[
\| x\mathcal{R} g \|_{W^{4, \frac{1}{2}}_{\text{loc}}} \lesssim \| \langle \nabla \rangle^{-1} g \|_{W^{4, \frac{1}{2}}_{\text{loc}}} + \| xg \|_{W^{4, \frac{1}{2}}_{\text{loc}}} \lesssim \| g \|_{H^{4+\delta}} + \| xg \|_{H^{4+\delta}}, \tag{4.6}
\]

it remains for us to control \( \| x \int_0^t e^{isb(D)} e^{i(t-s)\Delta} \langle \nabla \rangle \chi^L (gu) \, ds \|_{H^{4+\delta}} \).
To begin with, we write:

\[ x \int_0^t e^{isb(D)} e^{t-s} \Delta \langle \nabla \rangle \chi^L (g u) ds \]

\[ = \int_0^t e^{isb(D)} e^{t-s} \Delta (sb(D) + 2e(t-s)\nabla) \chi^L (g u) ds \]

\[ + \int_0^t e^{isb(D)} e^{t-s} \Delta (\chi^L (\cdot))' (D) (g u) ds + \int_0^t e^{isb(D)} e^{t-s} \Delta \chi^L (\langle x \rangle u) ds \]

\[ \triangleq I + II + III. \]

On one hand, the first two terms can be controlled directly by:

\[ ||I + II||_{H^{4+\delta}} \lesssim \int_0^t \|qu\|_{H^{5+\delta}} ds \lesssim T_0 \sup_{t \in [0,T_0]} ||g||_{H^{5+\delta}} ||u||_{H^{5+\delta}}. \] (4.7)

On the other hand, by Lemma 3.13

\[ ||III||_{H^{4+\delta}} \lesssim \int_0^t \|xgu\|_{H^{5+\delta}} ds \]

\[ \lesssim T_0 \sup_{t \in [0,T_0]} \left( \|x(u^L,g^L)\|_{L^\infty} \|u,\varrho\|_{H^{5+2\delta}} \right. \]

\[ + \|x(u^h,g^h)\|_{L^2} \|u,\varrho\|_{H^{6+2\delta} + \|(u,\varrho)\|_{H^{5+2\delta}}^2}. \] (4.8)

- Continuity in time of the weighted norm. By (4.6) again, it suffices for us to prove that:

\[ \|xe^{s\Delta} \int_0^t e^{isb(D)} e^{-s\Delta} \chi^L (\langle x \rangle u) ds \|_{H^{4+\delta}} \]

is continuous in time.

Denote \( z(t) = \int_0^t e^{isb(D)} e^{-s\Delta} \chi^L (\langle x \rangle u) ds \). Notice that \( xe^{s\Delta} z(t) = e^{s\Delta} x(t) e^{s\Delta} (x(t)) \).

Since \( e^{s\Delta} \) is a continuous operator in \( H^{4+\delta} \), we reduce the problem to the continuity of \( \|xz(t)\|_{H^{4+\delta}} \), which is the consequence of the following:

\[ \sup_{t \in [0,T_0]} \|xe^{it\Delta} e^{-s\Delta} \chi^L (\langle x \rangle u) (t) \|_{H^{2+\delta}} < +\infty. \]

However, it has essentially been included in the proof of (4.7), (4.8). One notes here that \( e^{-s\Delta} \chi^L \) is a \( L^2 \) multiplier whose norm is less than \( e^{t\Delta} \leq e^{c_0 t} \).

4.2. Proof of the claim. We are now in position to prove (4.5).

- Linear flow estimate. In light of (4.4) and the crude approximation:

\[ w_0 = \frac{\lambda_+}{2ib(D)} \frac{\langle \nabla \rangle g^L_0}{|\nabla|} + \frac{\langle \nabla \rangle \varrho L}{2ib} R \cdot u_0^L = R\langle \frac{\langle \nabla \rangle g^L_0}{|\nabla|} + u_0^L \rangle = \langle \langle \nabla \rangle R \cdot \nabla \varphi^L_0 + u_0^L \rangle, \]

it suffices for us to show that for \( \forall j \geq 0 \),

\[ \|\Phi_j R n_1(D) e^{it\lambda_-(D)} g\|_{L^1} \lesssim 2^{-j} \|\langle x \rangle g\|_{H^{\frac{1}{2}}} \] (4.9)

where we denote \( n_1(D) = \lambda_-(D) \) or \( |\nabla| \), \( g = u_0^L \) or \( \langle \nabla \rangle \varphi^L \). Nevertheless, we have by Sobolev embedding, Hausdorff-Young inequality that,

\[ \|xR n_1(D) e^{it\lambda_-(D)} \chi^L(D) g\|_{L^1} \]

\[ \lesssim \|n_1(D) e^{it\lambda_-(D)} \chi^L(D) g\|_{L^2} + \|x n_1(D) e^{it\lambda_-(D)} \chi^L(D) g\|_{H^{\frac{1}{2}}} \]

\[ \lesssim \|g\|_{H^1} + \|n_1(D) e^{it\lambda_-(D)} \chi^L(D) x g\|_{H^{\frac{1}{2}}} + \|[x,n_1(D) e^{it\lambda_-(D)} \chi^L(D)] g\|_{H^{\frac{1}{2}}} \]

\[ \lesssim \|g\|_{H^1} + \|xg\|_{H^{\frac{1}{3}}} + (1 + t) \|g\|_{H^{\frac{1}{2}} \frac{17}{14}} \lesssim \langle t \rangle \|\langle x \rangle g\|_{H^{\frac{1}{2}}} \].
• Nonlinear flow estimate.

For notational brevity, we replace again $\chi^L \frac{\lambda_0}{2\theta}$, $\chi^L \frac{\lambda}{2\theta}$, $\chi^L \frac{\lambda}{2\theta}$, $\chi^L \frac{\lambda}{\theta}$ by $\mathcal{R}$ since they have the similar properties. Therefore, it remains for us to show: for any $j \in \mathbb{Z}$

$$
\|\Phi_j(x) \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D) \mathcal{R}(\nabla)^2 (gu + |u|^2) ds \|_{L^4} \lesssim_{\varepsilon,\kappa_0, T_0} 2^{-j} 
$$

(4.10)

By Corollary 3.12

$$
\|\Phi_j(x) \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D) \mathcal{R}(\nabla)^2 (gu + |u|^2) ds \|_{L^4} \\
\lesssim_{\varepsilon,\kappa_0, T_0} 2^{-j} + \| \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D)(\Phi_j(x) \mathcal{R}(gu + |u|^2)) ds \|_{L^4} \\
\lesssim_{\varepsilon,\kappa_0, T_0} 2^{-j} + \| \int_0^t e^{i(t-s)\lambda_-(D)} \chi^L(D)(\Phi_j(x) \mathcal{R}(gu + |u|^2)) ds \|_{L^2} \\
\lesssim_{\varepsilon,\kappa_0, T_0} 2^{-j} + T_0 (1 + \frac{K_0 \varepsilon}{\kappa}) \sup_{t \in [0, T_0]} \|\Phi_j(x) \mathcal{R}(gu + |u|^2)(t) \|_{L^2} 
$$

(4.11)

To prove (4.10), we are left to establish that:

$$
\sup_{t \in [0, T_0]} \|\Phi_j(x) \mathcal{R}(\Phi_{\geq j+2} + \Phi_{\leq j-2})(gu + |u|^2) \|_{L^2} \lesssim 2^{-j} 
$$

(4.12)

We show for example the case of $\Phi_{\geq j+2}$ as the other is similar. Denote $\tilde{\Delta}_k \mathcal{R} \chi^L g = G_k * g$, where

$$
G_k(x) = \int \frac{e^{ix\xi} \Phi_k(\xi)}{\xi} d\xi = 2^k \int \frac{e^{i2^k x \cdot \xi} \Phi(\xi)}{\xi} d\xi
$$

By virtue of the identity $e^{i2^k x \cdot \xi} = \frac{1 - i2^k x \cdot \xi}{2^k} e^{i2^k x \cdot \xi}$, one can integrate by parts to get that, for any non-negative integer $l$,

$$
|G_k(x)| \lesssim 2^k \langle 2^k |x| \rangle^{-l}.
$$

Note that there is no singularity when the derivative hits on $\frac{\xi}{|\xi|}$ since $\Phi(\xi)$ is supported on annulus. Therefore, we have that

$$
\|\Phi_j(x) \mathcal{R} \Phi_{\geq j+2}(gu + |u|^2) \|_{L^2} \lesssim \sum_{k \in \mathbb{Z}} \|\Phi_j(x) \tilde{\Delta}_k \mathcal{R} \Phi_{\geq j+2}(gu + |u|^2) \|_{L^2} \\
\lesssim \sum_{k \in \mathbb{Z}} \|G_k I_{|x| \geq 2^j} \Phi_{\geq j+2}(gu + |u|^2) \|_{L^2} \lesssim \sum_{k \in \mathbb{Z}} \|G_k I_{|x| \geq 2^j} \|_{L^2} \|gu + |u|^2\|_{L^1} \\
\lesssim \sum_{k \in \mathbb{Z}} \langle 2^k 2^j \rangle^{-3/2} \|g(u, u)\|^2_{L^2} \lesssim 2^{-j} \|g(u, u)\|^2_{L^2}. 
$$

where the following two inequalities has been used:

$$
\|G_k I_{|x| \geq 2^j} \|_{L^2} \lesssim 2^k \left( \int_{|x| \geq 2^j} (2^k |x|)^{-8} dx \right)^{1/2} \lesssim \langle 2^k 2^j \rangle^{-3/2} k, \\
\sum_{k \in \mathbb{Z}} \langle 2^k 2^j \rangle^{-3/2} k \lesssim \left( \sum_{k \leq -j} + \sum_{k \geq -j} \right) \langle 2^k 2^j \rangle^{-3/2} k \lesssim 2^{-j}. 
$$

Finally, denote $\tilde{\Phi}_j = \Phi_{j-1} + \tilde{\Phi}_j + \Phi_{j+1}$, Corollary 1.2 implies:

$$
\|\tilde{\Phi}_j \mathcal{R} \tilde{\Phi}_j (gu + |u|^2) \|_{L^2} \lesssim \|\tilde{\Phi}_j (gu + |u|^2) \|_{L^2} \\
\lesssim \|\tilde{\Phi}_j (u^L, \varrho^L) \|_{L^4} \|((u, \varrho)) \|_{L^1} + \|\tilde{\Phi}_j (u^L, \varrho^L) \|_{L^2} \|u, \varrho\|_{L^\infty} \\
\lesssim_{\varepsilon, \kappa_0} (1 + T_0) (2^{-j} + \|\tilde{\Phi}_j (u^L, \varrho^L) \|_{L^4}). 
$$

(4.13)
Define \( y_j = \| \Phi_j(x)(\langle \nabla \rangle u^f, \langle \nabla \rangle g^f) \|_{L^4} \), by (4.9), (4.11)-(4.13), one gets that, as long as \( T_0 \) is chosen small enough,
\[
y_j \lesssim 2^{-j} + \frac{1}{16} (y_{j-1} + y_j + y_{j+1}),
\]
which, combined with the fact \( y_j \lesssim 1 \) \((j \geq 0)\) and iteration arguments, yields that \( y_j \lesssim 2^{-j} \). We thus finish the proof of (4.5).

5. Weighted \( L^2 \) norm for high frequency: a priori estimate

The goal of this section is to get the weighted \( L^2 \) estimate for \((\varrho, u, \nabla \varphi)^h\) which shall be used in Section 7.

**Lemma 5.1.** There exists a constant \( \vartheta_2 > 0 \), for any \( \varepsilon \in (0, 1) \), if \( \| U \|_{X_T} \leq \vartheta_2 \), we have the a priori estimate:
\[
\| \langle x \rangle (\varrho, u, \nabla \varphi)^h(t) \|_{L^2} \lesssim \langle t \rangle^{2\delta} (\| \langle x \rangle (\varrho_0, u_0, \nabla \varphi_0)^h \| + \| U \|_{X_T}), \quad \forall t \in [0, T)
\]  
(5.1)

**Proof.** We first claim that the following estimate holds,
\[
\| x(\varrho, u, \nabla \varphi)^L(t) \|_{W^{1,4}} \lesssim \langle t \rangle^{2+\delta} \| U \|_{X_T}.
\]  
(5.2)

Indeed, by virtue of the Hardy-Littlewood-Sobolev inequality, the \( L^p \) boundedness of \( e^{itb(D)} \tilde{x}^L \) (Lemma 3.2),
\[
\| x(\varrho, u, \nabla \varphi)^L \|_{W^{1,4}} = \| xR\tilde{x}^Lw \|_{W^{1,4}} \approx \| \langle x \rangle w \|_{W^{1,4}} \lesssim \langle t \rangle^{2+\delta} \| U \|_{X_T},
\]  
(5.3)

with \( \langle x \rangle \) defined in (4.13), \( b(D) = \nabla \cdot \nabla \varphi^h = 0 \), \( \varphi^h = \nabla \cdot u^h + \nabla \varphi^h \), \( \varphi^h = \varphi^0 \).

Note that \( f = e^{-itb(D)}w \). Hereafter, unless specifically emphasized, the spatial Sobolev norm is estimated in each temporary time \( t \in [0, T) \).

The system projected onto the high frequency reads:
\[
\begin{aligned}
\partial_t g^h + \text{div } u^h + \text{div } (g^h) &= 0, \\
\partial_t u^h + (u \cdot \nabla u)^h - 2\varepsilon \Delta u^h + \nabla g^h - \nabla \varphi^h &= 0, \\
\Delta \varphi^h &= \varphi^h, \\
u|_{t=0} &= P^+ u_0, \varphi|_{t=0} = \varphi_0.
\end{aligned}
\]  
(5.4)

Multiplying the system (5.4) by \( x \), and testing it by \( x(g^h, u^h) \), we obtain the energy equality:
\[
\frac{1}{2} \frac{d}{dt} \| x(g^h, u^h, \nabla \varphi^h) \|_{L^2}^2 + 2\varepsilon \int |x\nabla u^h|^2 dx
\]  
\[= - \int x^2(\text{div } u^h g^h - \nabla g^h u^h) dx + \int x^2((\nabla \varphi)^h u^h - \nabla \varphi^h \nabla (\Delta)^{-1} \text{div } u^h) dx - 4\varepsilon \int \nabla u^h x u^h dx
\]  
\[= - \int x \text{div } (u g^h) x g^h dx - \int x(u \cdot \nabla u)^h x u^h dx - \int x \nabla \varphi^h x \nabla (\Delta)^{-1} \text{div } (g u)^h dx
\]  
\[\triangleq G_1 + G_2 + \cdots G_6.
\]

The following task is to estimate \( G_1, \cdots, G_6 \). At first, by integration by parts and Hölder inequality, \( G_1 \) can be controlled as:
\[
G_1 = -2 \int x g^h u^h dx \lesssim \| x g^h \|_{L^2} \| u^h \|_{L^2}.
\]
Similarly, by using the fact \( \|\varphi^h\|_{L^2} \lesssim \sqrt{\frac{\varepsilon}{\kappa_0}} \|\nabla \varphi^h\|_{L^2} \),

\[
G_2 = 2 \int \varphi^h x (\nabla (\Delta)^{-1} \text{div} - 1) u^h \, dx 
\lesssim \|\varphi^h\|_{L^2} (\|x u^h\|_{L^2} + \|x, \nabla (\Delta)^{-1} \text{div} \chi^h\|_{L^2}) 
\lesssim \|\varphi^h\|_{L^2} (\|x u^h\|_{L^2} + \|\partial_k \left( \frac{\xi \xi_j}{|\xi|^2} \chi^h \right)\|_{L^2} \|u^h\|_{L^2}) 
\lesssim \left( \sqrt{\frac{\varepsilon}{\kappa_0}} + \frac{\varepsilon}{\kappa_0} \right) \|\nabla \varphi^h\|_{L^2} (\|u^h\|_{L^2} + \|x u^h\|_{L^2}).
\]

For \( G_3 \), we just use Hölder inequality: \( G_3 \leq 2\varepsilon \|\nabla u^h\|_{L^2} \|x u^h\|_{L^2} \).

By setting \( P(f, g) = f^h g + f^L g^h + f^L g^m + f^m g^l \), we estimate \( G_4 \) as follows:

\[
G_4 = - \int \text{div} (q u^h) x g^h \, dx = - \int |\langle x, \chi^h | \text{div} (q u^h) + \chi^h x (P(\nabla \varphi, u) + P(\varphi, \text{div} u)) \rangle | x g^h \, dx 
\lesssim \|x g^h\|_{L^2} (\sqrt{\frac{\varepsilon}{\kappa_0}} \|\text{div} (q u^h)\|_{L^2} + \|x (P(\nabla \varphi, u) + P(\varphi, \text{div} u))\|_{L^2}) 
\lesssim \|x g^h\|_{L^2} \sqrt{\frac{\varepsilon}{\kappa_0}} (\|\varphi, u\|_{L^2} \|\nabla (\varphi, u)\|_{L^\infty} + \|x g^h\|_{L^2} (\langle t \rangle - (1-\delta) \|U\|_{X_T}^2 + \|x (g^h, u^h)\|_{L^2} (\langle t \rangle -1) \|U\|_{X_T}^2) 
\lesssim \langle t \rangle^{-1} \|U\|_{X_T} (\|x g^h\|_{L^2}^2 + \|x u^h\|_{L^2}^2) + \langle t \rangle^{-1} (\delta) \|U\|_{X_T}^2 \|x g^h\|_{L^2}^2.
\]

where we have used that:

\[
\|x P(\nabla \varphi, u)\|_{L^2} + \|x P(\varphi, \text{div} u)\|_{L^2} \lesssim (1 + t)^{-1} \|x (\varphi, u)\|_{L^2}^2 + (1 + t)^{-1} \|U\|_{X_T}^2.
\]

For example, since \((\|u^m, \varphi^m\|_{W^{1,4}}) \lesssim \langle t \rangle^{-\frac{7}{4}} \|U\|_{X_T}^2, \|u^h\|_{H^1} \lesssim \langle t \rangle^{-\alpha} \|U\|_{X_T}, (\alpha = 2 - 5\delta)\), one has that:

\[
\|x P(\nabla \varphi, u)\|_{L^2} \lesssim \|x u^h\|_{L^2} \|\nabla \varphi^h\|_{L^\infty} + \|x f^L\|_{L^4} \|\nabla \varphi^h\|_{L^4} + \|x \nabla \varphi^h\|_{L^4} \|u^h + u^m\|_{L^4} + \|x u^h\|_{L^4} \|\nabla \varphi^m\|_{L^4} 
\lesssim (1 + t)^{-1} \|x (\varphi, u)\|_{L^2}^2 + (1 + t)^{-1} \|U\|_{X_T}^2.
\]

The estimate of \( G_5, G_6 \) is similar, we thus omit the details. To summarize, we have obtained:

\[
\frac{d}{dt} (\|x \varphi, u, \nabla \varphi\|_{L^2}^2) \lesssim (1 + t)^{-1} \|U\|_{X_T} \|x (\varphi, u, \nabla \varphi)\|_{L^2}^2 
+ (1 + t)^{-\alpha} \|U\|_{X_T} \|x (\varphi, u, \nabla \varphi)\|_{L^2} (5.4)
\]

In the same fashion, one can show that:

\[
\frac{d}{dt} (\|\varphi, u, \nabla \varphi\|_{L^2}^2) \lesssim (1 + t)^{-1} \|U\|_{X_T} \|\varphi, u, \nabla \varphi\|_{L^2}^2
\]

Finally, we set \( g(t) = \|x (\varphi, u, \nabla \varphi)\|_{L^2} \). Summing up the above two estimates (5.4)-(5.5), we see that there exists three constants \( c_1, c_2, c_3 \) which are independent of \( \varepsilon \), such that:

\[
\frac{d}{dt} g(t) \leq c_1 (1 + t)^{-1} \|U\|_{X_T} g(t) + c_2 (1 + t)^{-1-\delta} \|U\|_{X_T}^2 + c_3 (1 + t)^{-\alpha} \|U\|_{X_T}.
\]

Suppose that \( \|U\|_{X_T} \leq \vartheta_2 \leq \frac{\delta}{c_1} \), then for any \( 0 \leq t < T \), the Grönwall inequality leads to:

\[
g(t) \lesssim \langle t \rangle^\delta g(0) + \langle t \rangle^{2\delta} \|U\|_{X_T}^2 + \|U\|_{X_T} \lesssim \langle t \rangle^{2\delta} (\|x (\varphi_0, u_0, \nabla \varphi_0)\|_{L^2} + \|U\|_{X_T}).
\]

\[\square\]
6. Estimate of Sobolev norm

In this section, we aim to get the highest Sobolev estimate for $U$: $\|U\|_{H^N}$ and Sobolev estimate for high and intermediate frequencies: $\|U^h\|_{H^{N-2}}$, $\|U^m\|_{H^{N-1}}$ and $\|U^m\|_{W^{1,4}}$.

6.1. Control of highest Sobolev norms. Define the energy norm

$$E_N(t) = \sum_{|\alpha| \leq N} \frac{1}{2} \int \rho |\partial^\alpha u(t)|^2 + |\partial^\alpha \rho(t)|^2 + |\partial^\alpha \nabla \phi(t)|^2 dx.$$ 

In our former paper [23] where the 3d NSP is considered, it has been showed that if $\|\varrho\|_{L^\infty} \lesssim \frac{1}{6}$, then the following energy inequality holds:

$$\frac{d}{dt} E_N(t) \lesssim (\|u(t)\|_{W^{1,\infty}} + \|\varrho(t)\|_{L^\infty}) E_N(t).$$

However, such an inequality is not enough to close the energy estimate in 2d case. Indeed, due to the presence of Riesz potential in the quadratic nonlinearity (see (3.1)), one could only expect that $\|\nabla u(t)\|_{L^\infty}$ rather than $\|u(t)\|_{W^{1,\infty}}$ has the critical decay $(1 + t)^{-\frac{1}{2}}$. Nevertheless, it is not hard to modify the proof in [23] to get that:

$$\frac{d}{dt} E_N(t) \lesssim (\|\nabla u(t)\|_{L^\infty} + \|\varrho(t)\|_{L^\infty}) E_N(t).$$

(6.1)

Indeed, denote $E_\alpha = \frac{1}{2} \int \rho |\partial^\alpha u|^2 + |\partial^\alpha \rho|^2 + |\partial^\alpha \nabla \phi|^2 dx$, we then have by using the equations (3.1) that:

$$\frac{d}{dt} E_\alpha = \int \rho \partial^\alpha u \cdot [\partial^\alpha \varrho \nabla u + \int \partial^\alpha \rho \nabla \varrho \div (\partial^\alpha \varrho u)] + 2 \varepsilon \int \rho \partial^\alpha u \cdot \partial^\alpha \Delta u$$

$$= L_1 + L_2 + L_3 + L_4$$

One can estimate all the terms in the same way as that in [23] except the term $L_3$. However, for any $|\alpha| \geq 1$, it can be rewritten as

$$L_3 = \int \rho \partial^\alpha u \partial^\alpha \nabla \phi + \partial^\alpha \varphi \partial^\alpha \div \varrho u dx$$

$$= \int \rho \partial^\alpha u \partial^\alpha \nabla \phi + \rho \partial^\alpha (\varrho \nabla u) \partial^\alpha \varphi + [\partial^\alpha \varrho \nabla \varphi + \partial^\alpha \div \varrho \partial^\alpha \varphi] + \partial^\alpha \varphi \partial^\alpha \nabla \varphi \cdot u \partial^\alpha \nabla \phi dx$$

$$= \int \partial^\alpha \varphi \div [\partial^\alpha \varrho \nabla u + \nabla \varrho \cdot \partial^\alpha u] + [\partial^\alpha \varrho \nabla \varphi \partial^\alpha \varphi - \partial^\alpha \div \varrho \partial^\alpha \varphi - \partial^\alpha \div \varrho \partial^\alpha \varphi]$$

$$+ \partial^\alpha \varphi \cdot \nabla u \cdot \partial^\alpha \nabla \varphi dx$$

Notice that in the above expressions, there is at least one spatial derivative in front of $u$, we thus conclude by standard commutator estimate that:

$$|L_3| \leq (\|\nabla u\|_{L^\infty} + \|\varrho\|_{W^{1,\infty}})(\|u\|_{H^{1,\infty}_H}^2 + \|\varrho\|_{H^{1,\infty}_H}^2 + \|\nabla \varphi\|_{H^{1,\infty}_H}^2 + ||\varrho\||_{H^{1,\infty}_H}^2 + ||\nabla \varphi\||_{H^{1,\infty}_H}^2 + ||\varrho\||_{H^{1,\infty}_H}^2),$$

which ends the proof (6.1).
We only detail the estimation of $\|U\|_{H^N}^2 \approx E_N$ by noting the relation $u = R_c, \varrho = \frac{|\nabla|}{(\nabla)}a$. This, combined with (6.1) and the definition of $X_T$ norm (3.10), yields:

$$\|U\|_{H^N}^2 \lesssim E_N(t) \lesssim E_N(0) + \int_0^t (\|\nabla u\|_{L^\infty} + \|\varrho\|_{L^\infty})\|U(s)\|_{H^N}^2 ds \lesssim E_N(0) + (1 + t)^{2\delta}\|U\|_{X_T}^2.$$

### 6.2. High and intermediate frequency estimate

We have firstly the following estimate for nonlinear term $F(a, c) = F(U, U)$.

**Lemma 6.1.** For every $t \in [0, T)$, the following estimate holds:

$$\|\chi^h F(U, U)(t)\|_{H^{N-2}} \lesssim (1 + t)^{-(2-5\delta)}\|U\|_X^2,$$  \hspace{1cm} (6.2)

$$\|F(U, U)(t)\|_{H^{N-1}} \lesssim (1 + t)^{-(1-3\delta)}\|U\|_{X_T}^2. \hspace{1cm} (6.3)$$

**Proof.** We begin with the proof of (6.2). By the definition of truncation functions, one has $\chi^l(\xi - \eta)\chi^h(\eta) = 0$, which leads to the decomposition:

$$\chi^h(D)F(U, U) = \chi^h(D)(F(U^L, U^h) + F(U^h, U) + F(U^l, U^m) + F(U^m, U^l) + F(U^m, U^m)).$$

We only detail the estimation of $F(U^l, U^m)$, the other terms are much easier. By the definition, one has that $F(U, U) \approx \langle \nabla \rangle R(\mathcal{R}U \cdot \mathcal{R}U)$. Therefore, owing to the tame estimate, Sobolev embedding and the definition of $X_T$ norm,

$$\|F(U^l, U^m)\|_{H^{N-2}} \lesssim \|\mathcal{R}U^l \cdot \mathcal{R}U^m\|_{H^{N-1}} \lesssim \|\mathcal{R}U^l\|_{L^\infty} \|\mathcal{R}U^m\|_{H^{N-1}} \lesssim \|\mathcal{R}U^l\|_{W^{1, \frac{d}{d+1}}} \|U^m\|_{H^{N-1}} \lesssim (1 + t)^{-(2-5\delta)}\|U\|_{X_T}^2. \hspace{1cm} (6.4)$$

We next show (6.3), by splitting $F(U, U)$ into:

$$F(U, U) = F(U^L, U^L) + F(U^L, U^h) + F(U^h, U).$$

Similar to (6.4), $F(U^L, U^L)$ can be controlled as:

$$\|F(U^L, U^L)\|_{H^{N-1}} \lesssim (\|\mathcal{R}U^L\|_{H^N})^2 \lesssim \|\mathcal{R}U^L\|_{W^{1, \frac{d}{d+1}}} \|\mathcal{R}U^L\|_{H^N} \lesssim (1 + t)^{-(1-3\delta)}\|U\|_{X_T}^2.$$ 

The other two terms are easier, we omit the detail. \hfill \Box

#### 6.2.1. High frequency estimate: control of $\|U^h\|_{H^{N-2}}$

By Duhamel’s formula:

$$U^h(t) = e^{-tA}U^h_0 + \int_0^t e^{-\tau A}\chi^h F(U, U)(s)ds,$$

Lemma 3.3 and Lemma 6.1 then imply:

$$\|U^h\|_{H^{N-2}} \lesssim e^{-ct}\|U_0\|_{H^{N-2}} + \int_0^t e^{-c(t-s)}\|\chi^h F(U, U)\|_{H^{N-2}} ds \lesssim e^{-ct}\|U_0\|_{H^{N-2}} + \int_0^t e^{-c(t-s)}(1 + s)^{-\alpha}\|U\|_{X_T}^2 ds \lesssim e^{-ct}\|U_0\|_{H^{N-2}} + (1 + t)^{-\alpha}\|U\|_{X_T}^2.$$
6.2.2. Intermediate frequency estimate: control of \(\|U^m\|_{H^{N-1}}\) and \(\|U^m\|_{W^{1,4}}\). By Duhamel’s formula, Lemma 5.1 and Lemma 3.3 one can control the Sobolev norm of intermediate frequency as follows

\[
\|U^m\|_{H^{N-1}} \lesssim e^{-ct}\|U^m_0\|_{H^{N-1}} + \int_0^t e^{-c(t-s)}\|F(U, U)(s)\|_{H^{N-1}} ds
\]

\[
\lesssim e^{-ct}\|U_0\|_{H^{N-1}} + \int_0^t e^{-c(t-s)(1 + s)^{-3\delta}}\|U\|_{X_T}^2 ds
\]

\[
\lesssim e^{-ct}\|U_0\|_{H^{N-1}} + (1 + t)^{-1 - 3\delta}\|U\|_{X_T}^2.
\]

We can estimate \(\|U^m\|_{W^{1,4}}\) in the same fashion. In fact, by Corollary 3.4 (we will use \((\frac{1}{2})_+ = \frac{3}{4}\)), Lemma 7.1 (we use relation \(\frac{11}{4} \leq \frac{2}{5} \cdot 7\)), the definition of \(X_T\) norm, we get:

\[
\|U^m\|_{W^{1,4}} \lesssim e^{-ct}\|U^m_0\|_{W^{1,4}} + \int_0^t e^{-c(t-s)}\|F(U, U)\|_{W^{1,4}} ds
\]

\[
\lesssim e^{-ct}\|U_0\|_{W^{1,4}} + \int_0^t e^{-c(t-s)}\|U\|_{W^{1,4}} ds
\]

\[
\lesssim e^{-ct}\|U_0\|_{H^4} + \int_0^t e^{-c(t-s)(1 + s)^{-\frac{2}{3}}\|U\|_{X_T}^2 ds
\]

\[
\lesssim e^{-ct}\|U_0\|_{H^4} + (1 + t)^{-\frac{2}{3}}\|U\|_{X_T}^2.
\]

7. Low frequency estimate

In this section, we focus on the a priori estimate of Low frequency: \(\|\nabla|^{\frac{1}{2}}(\nabla)Q^{-1}U^L\|_{L^\infty},\|xe^{itb(D)w}\|_{W^{4,\frac{4}{1-\frac{7}{2}}}},\|U^L\|_{H^{N'}}\). In practice, we shall perform the decay estimate and weighted estimate in the same time.

By equation (3.8) and Duhamel principle:

\[
w = e^{\lambda_-(D)}w_0 + R\int_0^t e^{(t-s)\lambda_-(D)}(B(w, w) + n(D)\chi^LH)(s)ds
\]

\[
= K_1 + R(K_2 + K_3)
\]

(7.1)

To close the decay estimate for RK2, the ‘space-time resonance’ philosophy that change the quadratic nonlinearity to the cubic one needs to be enforced. More specifically, we rewrite (2) in the following fashion. Recall the definition of the phase function \(\phi_{\mu,\nu}(\xi, \eta) = i(b(\xi) - \mu b(\xi - \eta) - \nu b(\eta)) + \varepsilon(|\xi|^2 - |\xi - \eta|^2 - |\eta|^2)\) (\(\mu, \nu \in \{+, -\}\)) and the bilinear operator \(T_m\) in (2.1). Denote \(\tilde{f} = e^{-t\lambda_-(D)}w\), then \(f\) is governed by

\[
e^{t\lambda_-(D)}\partial_f\tilde{f} = R^2(B(w, w) + (\nabla)\chi^LH).
\]

One thus has by identity \(e^{s\phi_{\mu,\nu}} = \frac{1}{\phi_{\mu,\nu}}\partial_s e^{s\phi_{\mu,\nu}}\) and integration by parts in time that:

\[
\int_0^t e^{(t-s)\lambda_-(D)}B(w, w)ds
\]

\[
= e^{t\lambda_-(D)}\sum_{\mu, \nu \in \{+, -\}} F^{-1}\left(\int_0^t e^{-s\lambda_-(D)}m_{\mu\nu}(\xi, \eta)\hat{R}f\hat{w}^{\mu}(s, \xi - \eta)\hat{R}f\hat{w}^{\nu}(s, \eta)ds\right)
\]

\[
= e^{t\lambda_-(D)}\sum_{\mu, \nu \in \{+, -\}} F^{-1}\left(\int_0^t e^{s\phi_{\mu,\nu}}m_{\mu\nu}(\xi, \eta)\hat{R}\tilde{f}\hat{w}^{\mu}(s, \xi - \eta)\hat{R}f\hat{w}^{\nu}(s, \eta)ds\right)
\]

23
Similarly, if

\[ \sum_{\mu, \nu \in \{+,-\}} \left[ T_{\mu \nu} \left( R w^\mu(t), R w^\nu(t) \right) - e^{it - \phi \Delta} e^{it \Delta} T_{\mu \nu} \left( R w_0^\mu, R w_0^\nu \right) \right] - \int e^{-i(t-s)B(D)} e^{i(t-s)\Delta} T_{\mu \nu} \left( R (B(w, w) + \langle \nabla \rangle \chi^L H)^\mu, R w^\nu \right) ds + \text{symmetric terms} \]

\[ \triangleq I_1 + \cdots + I_4 + \text{symmetric terms}. \quad (7.2) \]

Note that we denote \( R^2 = R \) as they have the same property (they are both \( L^p (1 < p < \infty) \) multiplier). It is also worthy to remark that the operator \( e^{itb(D)} \tilde{\omega} \) is well defined as \( \tilde{\omega} \) is supported on the low frequency region. For notational brevity, we shall not distinguish \( m_{\mu, \nu} \) (just write them as \( m \)) and ignore the summation on \( \mu, \nu \). Therefore, in the following, we will write \( I_1 - I_4 \) as follows:

\[ I_1 = T_{\mu}(R w(t), R w(t)), \quad I_2 = -T_{\mu}(R w_0, R w_0), \]
\[ I_3 = -\int e^{-i(t-s)B(D)} e^{i(t-s)\Delta} T_{\mu} (R B(w, w), R w) ds, \]
\[ I_4 = -\int e^{-i(t-s)B(D)} e^{i(t-s)\Delta} T_{\mu} (R \langle \nabla \rangle \chi^L H, R w) ds. \]

7.1. Decay estimate and weighted estimate.

7.1.1. Estimate of \( K_1 \) and boundary terms \( I_1, I_2 \). We begin with the decay estimate of \( K_1 \). Since \( w_0 = (Q^{-1} \chi^L U_0) \approx R \frac{\nabla \varphi}{|\nabla \varphi|} g_0 + iR^* u_0 \), we have by the dispersive estimate (3.11)

\[ ||| \nabla \frac{1}{2} e^{-itb(D)} e^{it \Delta} w_0 |||_{W^{1, \infty}} \lesssim ||| \nabla \frac{1}{2} w_0 |||_{W^{3,1}} \lesssim \| (g_0, u_0, \nabla \varphi_0) \|_{H^{\frac{1}{2} + \delta, 1}}. \]

Note that the last inequality in the above arises from the fact that \( R \Delta_j \) is \( L^1 \) multiplier for any \( j \in \mathbb{Z} \). Next, for the decay estimate for \( RI_2 \), we take benefits of the Sobolev embedding, dispersive estimate (3.11), bilinear estimate (3.19) (use \( 2_+ = \frac{9}{4} - \delta \)) to get:

\[ ||| \nabla \frac{1}{2} R e^{itb(D)} e^{it \Delta} T_{\mu} (R w(0), R w(0)) |||_{W^{2,1}} \lesssim \langle t \rangle^{1 - \frac{1}{p}} \sum_{j \in \mathbb{Z}} 2^{j \frac{1}{4}} (2_j)^3 ||| T_{\mu} (R w(0), R w(0)) |||_{L^1} \]
\[ \lesssim \langle t \rangle^{1 - \frac{1}{p}} \sum_{j \in \mathbb{Z}} 2^{j \frac{1}{4}} (2_j)^3 \underbrace{||| T_{\mu} (R w(0), R w(0)) |||_H}_{H^{\frac{1}{2} + \delta, 1}} \lesssim \langle t \rangle^{-1} \| w(0) \|_{H^{\frac{3}{2}}} \| w(0) \|_{H^2}. \]

As for the decay estimate of \( RI_1 \), it is helpful to establish the following lemma:

**Lemma 7.1.** For any \( 2 \leq p < \infty \), and \( k < \frac{2}{p} N' + \left( 1 - \frac{2}{p} \right) \frac{3}{2} \), we have for every \( t \in [0, T) \),

\[ ||| U^L(t) |||_{W^{k, p}} \leq (1 + t)^{-(1 - \frac{2}{p})} \| U \|_{X_T}. \quad (7.3) \]

Similarly, if \( k < \frac{2}{p} N + \left( 1 - \frac{2}{p} \right) \frac{3}{2} \), then

\[ ||| U^L(t) |||_{W^{k, p}} \leq (1 + t)^{-(1 - \frac{2}{p}) + \frac{3}{p} \delta} \| U \|_{X_T}. \quad (7.4) \]

**Proof.** We only detail the proof of (7.3) since (7.4) can be treated in the same manner. We shall use decomposition \( U^L = \Delta_{-1} U^L + \sum_{j \geq 0} \Delta_j U^L \). On one hand, the low frequency can be dealt with as follows:

\[ \| \Delta_{-1} U^L \|_{W^{k, p}} \lesssim \| U^L \|_{L^p} \lesssim (1 + t)^{-(1 - \frac{2}{p})} \| U \|_{X_T}. \]

\[ \int_{2^{-4}}^{2^{-3}} \{ A \} dt < \infty. \]

\[ \| U^L \|_{W^{k, p}} \lesssim \| U^L \|_{L^p} \lesssim (1 + t)^{-(1 - \frac{2}{p})} \| U \|_{X_T}. \]

\[ \| \Delta_{-1} U^L \|_{W^{k, p}} \lesssim \| U^L \|_{L^p} \lesssim (1 + t)^{-(1 - \frac{2}{p})} \| U \|_{X_T}. \]
On the other hand, the high frequency term can be controlled by interpolation and the definition of $X_T$ norm:

$$
\sum_{j \geq 0} \| \Delta_j U^L \|_{W^{k,p}} \lesssim \sum_{j \geq 0} 2^{kj} \| \Delta_j U^L \|_{L^2}^2 \| \Delta_j U^L \|_{L^\infty}^{-\frac{p}{2}} \\
\lesssim \sum_{j \geq 0} 2^{j(k - N') \frac{p}{2} (1 - \frac{k}{p})} (2N')^j \| \Delta_j U^L \|_{L^2}^2 \| \nabla \Delta_j U^L \|_{L^\infty}^{1 - \frac{p}{2}} \\
\lesssim (1 + t)^{-1} \| U \|_{X_T}.
$$

□

In light of (3.19), (7.3) and condition $\frac{19}{4} < \frac{2}{3} \cdot 7 + \frac{1}{2}$, one has that

$$
\| \Re |\nabla|^\frac{1}{2} T_{\phi^w} (\Re w(t), \Re w(t)) \|_{W^{1,\infty}} \lesssim \| T_{\phi^w} (\Re w(t), \Re w(t)) \|_{H^{\frac{2}{3} + \delta}} \\
\lesssim \| w(t) \|_{W^{1,\frac{2}{3}}} \| \Re w \|_{W^{2,\infty}} \lesssim (1 + t)^{-1} \| U \|_{X_T}^2.
$$

(7.5)

We are now committed to the weighted estimate. Let us first detail the estimate of boundary terms: $x \Re e^{itb(D)} (I_1, I_2)$. Using (4.6) again, it suffices to estimate $\| xe^{itb(D)} (I_1, I_2) \|_{H^{4+\delta}}$. Denote $f = e^{itb(D)} w$ the profile of $w$, one then writes

$$
x e^{itb(D)} T_{\phi^w} (\Re w(t), \Re w(t)) = x e^{it\Im \phi} T_{\phi^w} (\Re f(t), \Re f(t)) \\
= -t e^{itb(D)} T_{\phi^w} \partial_x (\Im \phi) (\Re w(t), \Re w(t)) + i e^{itb(D)} T_{\phi^w} (\Re f(t), \Re w(t)) \\
+ e^{itb(D)} T_{\phi^w} (e^{itb(D)} x \Re f(t), \Re w(t)).
$$

where $\Im \phi = b(\xi) \pm b(\xi - \eta) \pm b(\eta)$. Thanks to (3.19), (7.3) and relation $\frac{25}{4} \leq \frac{16}{7} \cdot 7$, the first term can be controlled as:

$$
\| te^{itb(D)} T_{\phi^w} (\Re w(t), \Re w(t)) \|_{H^{4+\delta}} \lesssim t \| \Re w \|_{W^{2,\frac{7}{3}}} \| \Re w \|_{W^{2,\frac{2}{3}}} \lesssim \| U \|_{X_T}^2.
$$

The second term is easier, since it does not contain prefactor $t$. Moreover, the quadratic form $T_{\phi^w}$ admits the similar bilinear estimates as $T_{\phi^w}$. The third term is much involved since one could not put the loss of derivative on the weighted term. We thus write:

$$
F (e^{itb(D)} T_{\phi^w} (e^{itb(D)} x \Re f(t), \Re w(t))) \\
= \int e^{it\Im \phi} \frac{m}{\phi} (\xi, \eta) e^{-ib(\xi - \eta)} \Re f(\xi - \eta) \Re w(\eta) \chi_{\{\xi - \eta \leq \langle \eta \rangle \}} d\eta \\
+ \int e^{it\Im \phi} \frac{m}{\phi} (\xi, \eta) \partial_x \Re f(\xi - \eta) \Re f(\eta) \chi_{\{\xi - \eta > \langle \eta \rangle \}} d\eta \triangleq I_{131} + I_{132},
$$

By virtue of (3.19), (7.3) and condition $7.5 \leq \frac{2}{3} \cdot 11 + \frac{1}{2}$, one gets that:

$$
\| I_{131} \|_{H^{4+\delta}} \lesssim \| T_{\phi^w} \chi_{\{\xi - \eta \leq \langle \eta \rangle \}} (e^{itb(D)} x \Re f(t), \Re w(t)) \|_{H^{4+\delta}} \\
\lesssim \| e^{itb(D)} x \Re f(t) \|_{W^{2,\frac{2}{3}}} \| \Re w \|_{W^{2,\frac{2}{3}}} \\
\lesssim \langle t \rangle^{2\delta} \| \langle x \rangle f \|_{W^{2,\frac{2}{3}}} \| w \|_{W^{7.5,3}} \lesssim \langle t \rangle^{-1} \| U \|_{X_T}^2.
$$
For the term $I_{132}$, thanks to identity $\partial_t \widehat{R}f(\xi - \eta) = -\partial_\eta \widehat{R}f(\xi - \eta)$, one could integrate by parts in \( \eta \) to rewrite it as:

\[
I_{132} = T\frac{w}{\xi} \chi_{\{\langle \xi - \eta \rangle > \langle \eta \rangle \}} (\mathcal{R}w(t), e^{itb(D)} x\mathcal{R}f(t)) + T\frac{b}{\xi} \chi_{\{\langle \xi - \eta \rangle > \langle \eta \rangle \}} (\mathcal{R}w(t), \mathcal{R}w(t))
\]

+ \( itT\frac{w}{\xi} \chi_{\{\langle \xi - \eta \rangle > \langle \eta \rangle \}} \partial_\eta (Im\phi) (\mathcal{R}w(t), \mathcal{R}w(t)) \).

Nevertheless, the first term in the above can be estimated exactly as $I_{311}$, the last two terms can be treated in the same manner as that of \( \|I_1\|_{H^{2+\delta}} \), see (7.5).

We are now in position to show the estimate of $xe^{itb(D)}I_2$. By definition,

\[
xe^{it\Delta}T\frac{w}{\xi} (\mathcal{R}w_0, \mathcal{R}w_0) = 2e^{it\Delta} \varepsilon \nabla T\frac{w}{\xi} (\mathcal{R}w_0, \mathcal{R}w_0) + e^{it\Delta} T\frac{w}{\xi} (x\mathcal{R}w_0, \mathcal{R}w_0) + iT\frac{\xi}{\xi} (\mathcal{R}w_0, \mathcal{R}w_0).
\]

Let us focus on the estimate of the first two terms, since the last one is easier. Owing to the bilinear estimate (8.19), one has

\[
\|e^{it\Delta} \varepsilon \nabla T\frac{w}{\xi} (\mathcal{R}w_0, \mathcal{R}w_0)\|_{H^{2+\delta}} + \|e^{it\Delta} T\frac{w}{\xi} (\chi_{\{\langle \xi - \eta \rangle \leq \langle \eta \rangle \}}) (x\mathcal{R}w_0, \mathcal{R}w_0)\|_{H^{2+\delta}}
\]

\[
\lessapprox \epsilon^{\frac{1}{2}} \|\mathcal{F}^{-1} (e^{-\varepsilon t|\xi|^2} |\xi|^2 \xi)\|_{L^2} \|T\frac{w}{\xi} (\mathcal{R}w_0, \mathcal{R}w_0)\|_{W^{2+\delta,1}} + \|T\frac{w}{\xi} (\chi_{\{\langle \xi - \eta \rangle \leq \langle \eta \rangle \}}) (x\mathcal{R}w_0, \mathcal{R}w_0)\|_{H^{2+\delta}}
\]

\[
\lessapprox \|T\frac{w}{\xi} (\mathcal{R}w_0, \mathcal{R}w_0)\|_{W^{2+\delta,1}} + \|x\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{W^{2,3}} \|\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{W^{2+\delta,6}}
\]

\[
\lessapprox \|\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{H^2} \|\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{H^{2+\delta}} + \|\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{H^2} \|\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{H^7}
\]

\[
\lessapprox \|\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{H^2} \|\mathcal{R}(\phi_0, \varphi_0, u_0)\|_{H^7}.
\]

Note that in the above, the following fact has been used:

\[
\|\mathcal{F}^{-1} (e^{-\varepsilon t|\xi|^2} |\xi|^2 \xi)\|_{L^2} \lessapprox e^{-\frac{\epsilon}{4} t^{-1}}, \quad (7.6)
\]

\[
\|x\mathcal{R}f\|_{L^3} \lessapprox \|xf\|_{L^3} + \|\nabla^{-1} f\|_{L^3} \lessapprox \|xf\|_{H^{\frac{1}{2}}} + \|f\|_{L^4} \lessapprox \|\langle x \rangle f\|_{H^{\frac{1}{2}}}.
\]

The estimate of $T\frac{w}{\xi} (\chi_{\{\langle \xi - \eta \rangle > \langle \eta \rangle \}}) (x\mathcal{R}w_0, w_0)$ can be obtained by integrating by parts in \( \eta \) as before.

7.1.2. Estimate of $K_3$ and $R\mathcal{I}_4$. We begin with the decay estimate of $K_3$ (which is defined in (7.1)). By dispersive estimate, $\|\mathcal{R}K_3\|_{L^\infty}$ can be estimated as follows:

\[
\|\mathcal{R} \int_0^t e^{i(t-s)\lambda_- (D)} |\nabla|^{\frac{\lambda_-}{2}} \langle \nabla \rangle \chi^L H(s) ds\|_{L^\infty}
\]

\[
\lessapprox \int_0^t (1 + t - s)^{-1} \sum_{j \in \mathbb{Z}} 2^j \|\hat{\Delta}_j H(s)\|_{L^1} ds \lesssim \int_0^t (1 + t - s)^{-1} \|H(s)\|_{B_{5,2}} ds
\]

\[
\lesssim \int_0^t (1 + t - s)^{-1} \|U^{\lambda_3} H\|_{H^2} ds \lesssim \int_0^t (1 + t - s)^{-1} (1 + s)^{-\alpha} \|U\|^2_{X_T} ds \lesssim (1 + t)^{-1} \|U\|^3_{X_T}
\]

26
We now prove the weighted estimate of $K_3$. According to (4.6), it suffices to estimate $\|x(3)\|_{H^{4+\delta}}$. Rewrite $H = RU^hRU^L + RU^hRU^h \triangleq H_1 + H_2$, one has that by the definition of $W : W = Q^{-1}U^L$,

$$
x \int_0^t e^{isb(D)}e^{\varepsilon(t-s)}\Delta(\nabla)\chi^L H_1 ds
$$

$$
= \mathcal{F}^{-1}(\int_0^t \int_0^t s(b'(\xi) \pm b'(\xi - \eta)) + 2\varepsilon(t - s)i\xi e^{isb(\xi)}e^{-\varepsilon(t-s)|\xi|^2} \overline{RU^h(\eta)} \overline{RU^L(\xi - \eta)} d\eta ds
$$

$$
+ \int_0^t e^{isb(\xi)}e^{-\varepsilon(t-s)|\xi|^2} \partial_\xi \langle \xi \rangle \chi^L(\xi) H_1(\xi) ds
$$

$$
+ \int_0^t e^{isb(\eta)}e^{-\varepsilon(t-s)|\xi|^2} \langle \xi \rangle \chi^L(\xi) \overline{RU^h(\eta)} \partial_\xi \overline{Q^L} \overline{Rf}(\xi - \eta) d\eta ds
$$

$$
\triangleq (3)_{11} + (3)_{12} + (3)_{13}
$$

(7.7)

For $(3)_{11}$, by virtue of (7.6), the fact $b'\chi^L(D)$ is $L^p(1 < p < \infty)$ multiplier as well as Young's inequality:

$$
\| (3)_{11} \|_{H^{4+\delta}} \lesssim \int_0^t s\|RU^hRU^L\|_{H^{5+\delta}} ds + \int_0^t \|RU^hRU^L\|_{W^{5+\delta,1}} ds
$$

$$
\lesssim \int_0^t s\|RU^h\|_{H^{5+\delta}} \|RU^L\|_{W^{1,4}} ds + \int_0^t \|RU^h\|_{H^{5}} \|RU^L\|_{L^2} ds
$$

$$
\lesssim \int_0^t s\|RU^h\|_{H^{5+\delta}} \|RU^L\|_{W^{1,4}} ds + \int_0^t \|RU^h\|_{H^{5}} \|RU^L\|_{L^2} ds
$$

The estimate of $(3)_{12}$ is similar, we thus skip it. For $(3)_{13}$, by (3.19), (3.13),

$$
\| (3)_{13} \|_{H^{4+\delta}} \lesssim \int_0^t \|RU^h\|_{W^{5+\delta,1}} e^{isb(D)} xQ\overline{\chi}^L Rf \|_{L^2} ds
$$

$$
\lesssim \int_0^t \|RU^h\|_{H^{6}} \|xQ\overline{\chi}^L Rf\|_{W^{5+\delta,1}} ds \lesssim \int_0^t \|RU^h\|_{H^{6}} \|xQ\overline{\chi}^L Rf\|_{W^{5+\delta,1}} ds \lesssim \|U\|_{L^2}.\]

Note that in the above, we have also used the fact $xQ\overline{\chi}^L Rf \approx |\nabla|^{-1}Rf + R(xf)$ which gives:

$$
\|xQ\overline{\chi}^L Rf\|_{W^{5+\delta,1}} \lesssim \|Rf\|_{W^{5+\delta,1}} + \|xf\|_{W^{5+\delta,1}} \lesssim \|f\|_{H^{6}} + \|xf\|_{W^{5+\delta,1}}.\]

For the case of $H_2$, since in the original definition $H_2 = \theta^h u^h$ or $(u^h)^2$, one can split it into three terms:

$$
x \int_0^t e^{isb(D)}e^{\varepsilon(t-s)}\Delta(\nabla)\chi^L H_2 ds
$$

$$
= \int_0^t \int_0^t sb'(D) + 2(t - s)i\xi e^{isb(D)}e^{\varepsilon(t-s)}\Delta(\xi) H_2 ds
$$

$$
+ \int_0^t e^{isb(D)}e^{\varepsilon(t-s)}\Delta \mathcal{F}^{-1}(\partial_\xi \langle \xi \rangle \chi^L(\xi) H_1(\xi)) ds + \int_0^t e^{isb(D)}e^{\varepsilon(t-s)}\Delta(\nabla)\chi^L(x\theta^h u^h) ds
$$

$$
\triangleq (3)_{21} + (3)_{22} + (3)_{23}
$$

(7.8)

The estimates of $(3)_{21}, (3)_{22}$ are similar to that of $(3)_{11}, (3)_{12}$ and thus can be omitted. For $(3)_{23}$, one uses Lemma (3.1), the estimate (4.1) to get:

$$
\| \langle \nabla \rangle \chi^L (x\theta^h u^h) \|_{H^{4+\delta}} \lesssim \|xu^h\|_{L^2} \|\theta^h\|_{H^{4+\delta,\infty}} + \|x\theta^h\|_{L^2} \|u^h\|_{H^{4+2\delta}} + \|\theta^h\|_{H^{5+2\delta}} \|u^h\|_{H^{5+2\delta}}
$$

$$
\lesssim (s)^{-2\delta} (\|U\|_{L^2} + \|x(\varrho^0, u_0, \nabla \varphi_0)\|_{L^2}^2).
$$
7.1.3. Estimate of $I_4$. One first observes that

\[
\|\nabla|^j (\nabla) \mathcal{R} I_4\|_{L^\infty} \lesssim \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} (2^j)^{\frac{1}{2}} \|\hat{\Delta}_j e^{itb(D)} I_4\|_{L^1} \lesssim \langle t \rangle^{-1} \|e^{itb(D)} I_4\|_{W^{\frac{3}{2} + \delta, 1}}.
\]

Applying Lemma 3.2 for $p = 1$, we get:

\[
\|e^{itb(D)} I_4\|_{W^{\frac{3}{2}, \frac{3}{2}}} \lesssim \int_0^t s \|T^\omega (\langle \nabla \rangle \mathcal{R} \chi^L H, \mathcal{R} w)\|_{W^{\frac{3}{2} + \delta, 1}} ds \\
\lesssim \int_0^t s \|\nabla \chi^L H\|_{H^{\frac{3}{2}}} \|w\|_{H^{\frac{3}{2}}} ds \lesssim \int_0^t s (s)^{3 - 7\delta} ds \|U\|_X^3 \lesssim \|U\|_{X_T}^3.
\]

where the following crude estimate has been used

\[
\|\nabla\chi^L \mathcal{R} H\|_{H^{N - 3}} \lesssim \|H\|_{H^{N - 2}} \lesssim \|U\|_{L^\infty} \|U^h\|_{H^{N - 2}} \lesssim \langle s \rangle^{-(3 - 7\delta)} \|U\|_{X_T}^2.
\quad (7.9)
\]

We are now devoted to proving the estimate of $\|\chi e^{itb(D)} I_4\|_{W^{3, \frac{3}{2}}}$. As before, let us write

\[
\chi e^{itb(D)} I_4 = \mathcal{F}^{-1} \left( \int_0^t \mathfrak{m}\langle \nabla \rangle \chi^L \mathcal{R} H(s) \hat{\mathcal{R}} f(s) ds \right) \approx J_{41} + J_{42} + J_{43}.
\]

The first term $J_{41}$ can be dealt with similarly as the term $(3)_{11}$. Indeed, by using Lemma 3.2, Lemma 3.6 and (7.9), one obtains

\[
\|J_{41}\|_{W^{\frac{3}{2}, \frac{3}{2}}} \lesssim \int_0^t \langle s \rangle^{1 + \delta} \|T^\omega (\langle \nabla \rangle \chi^L \mathcal{R} H, \mathcal{R} w)(s)\|_{W^{\frac{3}{2} + 2\delta, \frac{3}{2}}} ds \\
\quad + \int_0^t \langle s \rangle^{\delta} \|T^\omega (\langle \nabla \rangle \chi^L \mathcal{R} H, \mathcal{R} w)(s)\|_{W^{\frac{3}{2} + 2\delta, \frac{3}{2}}} ds \\
\lesssim \int_0^t \langle s \rangle^{1 + \delta} \|\mathcal{R} w\|_{W^{\frac{3}{2}}} \|\nabla \chi^L \mathcal{R} H\|_{W^{\frac{3}{2}}} ds + \int_0^t \langle s \rangle^{\delta} \|\mathcal{R} w\|_{W^{\frac{3}{2}}} \|\nabla \chi^L \mathcal{R} H\|_{H^{\frac{3}{2}}} ds \\
\lesssim \int_0^t \langle s \rangle^{1 + \delta} \|\mathcal{R} w\|_{W^{\frac{3}{2}}} \|\chi\|_{X_T}^3 \lesssim \|U\|_{X_T}^3.
\]

$J_{42}$ can be estimated in the same manner, we thus do not detail it. For $J_{43}$, one splits it into two terms:

\[
J_{43} = \mathcal{F}^{-1} \left( \int (\chi_{\langle \xi - \eta \rangle \leq \langle \eta \rangle} + \chi_{\langle \xi - \eta \rangle \geq \langle \eta \rangle}) \cdots ds \right) \approx J_{431} + J_{432}.
\]
The estimate of \( J_{431} \) is easy since we can put all the derivatives onto \( H \). Indeed, by (7.9), Lemma 3.2 and the Sobolev embedding, one obtains that

\[
\|J_{431}\|_{W^{4, \frac{2}{\epsilon}}} \lesssim \int_0^t \|(\nabla) \chi^L H\|_{W^{4, \frac{2}{1+3s}}} e^{i\delta h(D)} x RF \|_{W^{2,3}} ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{-(3-7\delta)} \langle s \rangle^\frac{5}{2} \|\langle x \rangle f\|_{W^{3, \frac{2}{1+3s}}} ds \lesssim \|U\|_{X_T}^3.
\]

For \( J_{432} \), we use the identity \( \partial_\eta \hat{R} f(\eta - \eta) = -\partial_\eta \hat{R} f(\eta - \eta) \) to integrate by parts in \( \eta \). Eventually, we get the terms like \( J_{41}, J_{42}, J_{431} \) as well as the term:

\[
\int_0^t e^{i\delta h(D)} e^{(t-s)\Delta} T_{m(\xi - \eta) \geq \langle \eta \rangle_0} (x(\nabla) \chi^L(D) RH, Rw) ds.
\] (7.10)

Besides,

\[
x(\nabla) \chi^L(D) RH = \frac{\nabla}{\langle \nabla \rangle} \chi^L RH + \langle \nabla \rangle ((i\chi^L)'(D) RH + \chi^L(D)x RH)
\]

\[
\approx \frac{\nabla}{\langle \nabla \rangle} \chi^L RH + \langle \nabla \rangle ((i\chi^L)'(D) RH + \chi^L(D)|\nabla|^{-1} H + \chi^L(D)x RH).
\]

To continue, the following estimate for \( H \) shall be useful:

**Proposition 7.2.**

\[
\|(\chi^L)'(D) RH + \chi^L(D)|\nabla|^{-1} H\|_{W^{4, \frac{2}{\epsilon}}} \lesssim \langle t \rangle^{-(2-5\delta)} \|U\|_{X_T}^3,
\]

\[
\|\chi^L RH\|_{L^2} \lesssim \langle t \rangle^{-1-4\delta} (\|U\|_{X_T}^3 + \|(\xi, u_0)\|_{L^2}^2).
\]

We postpone the proof of this proposition and finish firstly the estimate of term (7.10). Indeed, Lemma 7.1, Proposition 7.2 combined with the Sobolev embedding \( W^{4, \frac{2}{2}} \hookrightarrow W^{4, \infty}, H^8 \hookrightarrow W^{4, \infty} \) yield that:

\[
\int_0^t \|\mathcal{R} w\|_{W^{4, \infty}} \|\chi^L RH + \langle \nabla \rangle ((\chi^L)'(D) RH + \chi^L(D)|\nabla|^{-1} H)\|_{W^{2, \frac{2}{\epsilon}}} ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{-(3-7\delta)} \langle s \rangle^\frac{5}{2} + \langle s \rangle^{-(\frac{5}{2}-\delta)} \langle s \rangle^{-(1-4\delta)} (\|U\|_{X_T}^3 + \|(\xi, u_0)\|_{L^2}^2) ds
\]

\[
\lesssim \|\langle \xi, u_0, \nabla \varphi_0\|_{L^2}^2 + \|U\|_{X_T}^3.
\]

We are now left to prove Proposition 7.2.

**Proof of Proposition 7.2.** First, since \( (\chi^L)'(D) \mathcal{R} \) is \( L^2 \) multiplier with norm \( \lesssim \sqrt{\frac{2}{\kappa_0}} \), one has by Sobolev embedding, the definition \( H \approx \mathcal{R} U^h \mathcal{R} U \) that:

\[
\|(\chi^L)'(D) RH\|_{W^{4, \frac{2}{\epsilon}}} \lesssim \sqrt{\frac{\kappa}{\kappa_0}} \|H\|_{W^{4, \frac{2}{\epsilon}}} \lesssim \|U^h\|_{H^4} \|U\|_{H^4} \lesssim \langle t \rangle^{-(2-5\delta)} \|U\|_{X_T}^3.
\]

Similarly, by Hardy-Littlewood-Sobolev inequality,

\[
\|\|\nabla|^{-1} H\|_{W^{4, \frac{2}{\epsilon}}} \lesssim \|H\|_{W^{4, \frac{2}{\epsilon}}} \lesssim \|\mathcal{R} U^h\|_{H^4} \|\mathcal{R} U\|_{H^4} \lesssim \langle t \rangle^{-(2-5\delta)} \|U\|_{X_T}^3.
\]
We are now ready to estimate $R x H$. Notice that in the original definition $H = \rho^L u^h + \rho^h u$ or $u^L \cdot u^h + u^h \cdot u$. Therefore, due to the Sobolev embedding $W^{1, \frac{3}{2}} \hookrightarrow L^\infty$ and weighted Sobolev estimate for high frequency [5.1]:

$$\| \chi^L R x H \|_{L^2} \lesssim \| x (\varphi, u)^h \|_{L^2} \| (\varphi, u) \|_{L^\infty}$$

$$\lesssim \langle t \rangle^{3\delta} \| (\varphi, u) \|_{X_T} + \| (x) (\varphi, u_0, \nabla \varphi_0)^h \|_{L^2} \langle t \rangle^{-(1-2\delta)} \| U \|_{X_T}$$

$$\lesssim \langle t \rangle^{-(1-4\delta)} \left( \| U \|_{X_T}^2 + \| (x) (\varphi, u_0, \nabla \varphi_0)^h \|_{L^2}^2 \right).$$

7.1.4. **Estimate of $RI_3$.** In view of the definition of $B(w, w)$ (see [3.9]), we could indeed consider $B(w, w)$ as $\langle \nabla \rangle R (R w)^2$ for simplicity. Therefore, by recalling the definition of profile $f(s) = e^{isb(D)} w$, we see that $I_3 = e^{-ibt(D)} I_3'$ with $I_3'$ under the form

$$I_3'(t) = \int_0^t \int e^{is\tilde{\phi}(\xi, \eta, \sigma)} e^{-\epsilon(t-s)} \langle t \rangle^{2m} \phi(\xi, \eta) R(\eta) R f(\xi - \eta) R f(\eta - \sigma) R f(\sigma) d\sigma d\eta ds$$

(7.11)

where $\tilde{\phi} = b(\xi) \pm b(\xi - \eta) \pm b(\eta - \sigma) \pm b(\sigma)$. For the weighted estimate, thanks to Bernstein’s inequality and Young’s inequality, one has that

$$\| x R I_3' \|_{W^{1, \frac{1}{2}}} \lesssim \sum_{k \in \mathbb{Z}} 2^{4k^+} \| \Delta_k \chi x R I_3' \|_{L^\infty_x} \lesssim \sum_{k \in \mathbb{Z}} 2^{4k^+} \left( \| [\nabla, \Delta_k] I_3' \|_{L^\infty_x} + \| \Delta_k x I_3' \|_{L^\infty_x} \right)$$

$$\lesssim \sum_{k \in \mathbb{Z}} 2^{4k^+} \left( 2^{6k} \| \Delta_k I_3' \|_{L^1_x} + 2^{\frac{2}{3} \delta k} \| \Delta_k x I_3' \|_{L^2_x} \right)$$

$$\lesssim \| I_3' \|_{H^5} + \sup_{k \in \mathbb{Z}} 2^{-\frac{1}{3} \delta k} \| x [\Delta_k, I_3'] \|_{L^2_x}$$

where we denote $2_\delta = \frac{2}{1 + \frac{3}{4} \delta}$ and $k^- = \max(-k, 0), k^+ = \max(k, 0)$. Similarly, by dispersive estimate (3.11), Hölder’s inequality,

$$\| \nabla \|^{\frac{1}{2}} I_3' \|_{W^{1, \infty}} \lesssim \langle t \rangle^{-1} \sum_{k \in \mathbb{Z}} 2^{k^+} 2^{3k^+} \| \Delta_k I_3' \|_{L^1} \lesssim \langle t \rangle^{-1} \left( \| I_3' \|_{H^4} + \sup_{k \in \mathbb{Z}} 2^{\delta k} \| x [\Delta_k, I_3'] \|_{L^2} \right)$$

By Lemma [5.2] one has that: $\| e^{isb(D)} \chi^L(D) g \|_{L^2} \lesssim \langle s \rangle^{\frac{1}{2}} \| g \|_{W^{5, 2}}$. Therefore, the commutator term can be bounded by applying Corollary [3.10] and Lemma [7.1]

$$\| [x, \Delta_k] I_3' \|_{W^{4+25.25}} = \left\| \int_0^t e^{is \Phi(D)} \langle \nabla \rangle \left( (\Phi(D)) R^2 (\nabla \langle \Phi(D) \rangle (\nabla w)^2, R w) \right) \right\|_{W^{4+25.25}} ds$$

$$\lesssim \int_0^t \langle s \rangle^{\frac{1}{4}} \| T_{w} (\nabla \langle \Phi(D) \rangle (\nabla w)^2, R w) \|_{W^{4+25.25}} \| F^{-1} (2^{-k} \Phi'(2^{-k} \cdot)) \| \| x \|_{L^{10}_{5+45}} ds$$

$$\lesssim 2^{-\frac{1}{3} \delta k} \int_0^t \langle s \rangle^{\frac{1}{4}} \| w \|_{X_T} \| w \|_{W^{2, 4}} \| w \|_{W^{2, \frac{8}{3}}} ds \lesssim 2^{-\frac{1}{3} \delta k} \int_0^t \langle s \rangle^{\frac{1}{4}} \langle s \rangle^{-\frac{1}{6} \delta} \| U \|_{X_T}^3 \lesssim 2^{-\frac{1}{3} \delta k} \| U \|_{X_T}^3$$
It now remains for us to estimate \( \sup_{k \in \mathbb{Z}} 2^{\frac{3}{2}k - 2(4 + \delta)k} \| \Delta_k x I_3' \|_{L^2} \). By the expression of \( I_3' \), we have that \( x I_3' = \sum_{j=1}^{4} Z_j \) where

\[
Z_1 = \int e^{i \text{sgn}(D)} e^{\varepsilon(t-s)} \Delta \frac{\partial \phi}{\partial t} \phi \| \mathcal{R} \langle \nabla \rangle (\mathcal{R} w)^2 \|_{L^2} \, ds,
\]

\[
Z_2 = \int e^{i \text{sgn}(D)} e^{\varepsilon(t-s)} \Delta \frac{\partial \phi}{\partial t} \phi \| \mathcal{R} \langle \nabla \rangle (\mathcal{R} w)^2 \|_{L^2} \, ds,
\]

\[
Z_3 = \int e^{i \text{sgn}(D)} e^{\varepsilon(t-s)} \Delta \frac{\partial \phi}{\partial t} \phi \| \mathcal{R} \langle \nabla \rangle (\mathcal{R} w)^2 \|_{L^2} \, ds,
\]

\[
Z_4 = \int e(t-s) \Delta \frac{\partial \phi}{\partial t} \phi \| \mathcal{R} \langle \nabla \rangle (\mathcal{R} w)^2 \|_{L^2} \, ds.
\]

We first remark that \( \| \Delta_k x Z_4 \|_{W^{4+2\delta, 2}} \) can be estimated in the same manner as that of \( \| [x, \Delta_k] I_3' \| \), the only difference is that at this stage we use the fact the \( L^{5+\delta} \) norm of the kernel of \( \Delta_k \varepsilon(t-s) \nabla e^{\varepsilon(t-s) \Delta} \) is less than \( 2^{-\frac{3}{2}k} \) uniformly in \( \varepsilon \in (0, 1) \). Indeed, one can think of \( \Delta_k \varepsilon(t-s) \nabla e^{\varepsilon(t-s) \Delta} \) as \( \tilde{\Delta_k} \varepsilon |\nabla|^{-1} \), since \( \mathcal{F}^{-1} (\varepsilon(t-s) \xi^2 e^{-\varepsilon(t-s) \xi^2}) \|_{L^1} \) is uniformly bounded. We point out here that we choose \( 2\delta = \frac{\sigma}{1+\sigma} \) mainly to manage to control the commutator term \( [x, \Delta_k] I_3' \) and \( Z_4 \), since the situation is better if \( 2\delta \) is closer to 2. We also emphasize that the presence of the half derivative when we control the \( L^\infty \) norm of \( I_3 \) is necessary in here. Indeed, as we explained in the introduction, due to the weak dispersive estimate, we need to control it by the weighted norm: \( \| x I_3 \|_{L^2} \). Nevertheless, when we deal with the \( Z_4 \|_{L^2} \) which corresponds to the frequency derivative hits on \( e^{\varepsilon(t-s) \Delta} \), to compensate the growth of \( (t-s) \), the best one can use is: \( \| \nabla e^{\varepsilon(t-s) \Delta} \|_{L^2} \lesssim (\varepsilon(t-s))^{-1} \), which is obviously not enough. Based on this, the extra derivative can help in the sense that we could find some \( 1^+ \), such that \( \| \nabla^{1^+} e^{\varepsilon(t-s) \Delta} \|_{L^2} \lesssim (\varepsilon(t-s))^{-1} \).

The estimations for \( Z_1 - Z_3 \) are more involved since in these cases we cannot use any information of heat kernel. However, since it has been showed that \( e^{it \text{sgn}(D)} \chi^L(D) \), Im \( \phi(\xi, \eta, \sigma) = b(\xi) \pm b(\xi - \eta) \pm b(\eta) \) and \( \tilde{\phi}(\xi, \eta, \sigma) = b(\xi) \pm b(\xi - \eta) \pm b(\eta - \sigma) \pm b(\sigma) \) has the same properties as \( e^{it \langle \nabla \rangle} \), \( \langle \xi \rangle \pm \langle \xi - \eta \rangle \pm \langle \eta \rangle \) and \( \langle \xi \rangle \pm \langle \xi - \eta \rangle \pm \langle \eta - \sigma \rangle \pm \langle \sigma \rangle \) respectively, they can be achieved by the similar arguments as in \[16\] where the global existence of 2-d Euler-Poisson is proved. We thus just sketch them in the appendix for completeness and reader’s convenience.

### 7.2. Estimate of \( H^{N'} \)

In this short subsection, we deal with the estimate of \( \| U^L \|_{H^{N'}} \). By virtue of the definition \( U^L = Q^{-1} \chi^L W \) and the fact that \( Q^{-1} \chi^L \) is a \( L^2 \) multiplier, one easily sees that \( \| U^L \|_{H^{N'}} \lesssim \| W \|_{H^{N'}} \). By \[1] \( \alpha \geq 1 \), \[7.2\],

\[
w = (1) + (3) + I_1 + I_2 + I_3 + I_4
\]

where \( (1), (3) \) are defined in \[7.1\] and \( I_1 - I_4 \) is defined in \[7.2\].

We shall only show the estimation of \( I_1 - I_4 \). For \( I_1, I_2 \), by bilinear estimate \[3.19\] (we use \( 2^+ = \frac{9}{4} - 2\delta \), Sobolev embedding and the definition of \( X_T \) norm,

\[
\| I_1 \|_{H^{N'}} \lesssim \| T_w \phi(\mathcal{R} w(t)) \|_{H^{N'}} \lesssim \| \mathcal{R} w \|_{H^{N'+\frac{3}{2}}} \| \mathcal{R} w \|_{W^{2, \frac{3}{2}}} \lesssim \| w_0 \|_{H^{N'+\frac{3}{2}}} \| w_0 \|_{W^{2, \infty}} \lesssim \| w_0 \|^2_{H^{N'+\frac{3}{2}}},
\]

\[
\| I_2 \|_{H^{N'}} \lesssim \| T_w \phi(\mathcal{R} w(t)) \|_{H^{N'}} \lesssim \| w_0 \|^2_{H^{N'+\frac{3}{2}}} \| w_0 \|_{W^{2, \infty}} \lesssim \| w_0 \|^2_{H^{N'+\frac{3}{2}}}.
\]
For $I_3$, we use Corollary \ref{corollary-3.10} with $3_+ = \frac{13}{4} - 2\delta$ and assumption $N' = N - 4$ to get:

$$
\|I_3\|_{H^{N'}} \lesssim \int_0^t \|T_{\frac{s}{\bar{s}}} (\mathcal{R}B(\mathcal{R}w, \mathcal{R}w), \mathcal{R}w(s))\|_{H^{N'}} ds \\
\lesssim \int_0^t \|w\|_{H^{N' + \frac{1}{2}}} \|w\|^2_{W^{2, \frac{1}{2}}} ds \lesssim \int_0^t \langle s \rangle^{-2(1 - 3\delta)} \|U\|_{X_T}^3 ds \lesssim \|U\|_{X_T}^3.
$$

$I_4$ can be estimated in the similar fashion, by bilinear estimate and the definition of $H$, where

$$
\|I_4\|_{H^{N'}} \lesssim \int_0^t \|T_{\frac{s}{\bar{s}}} (\mathcal{R}(\nabla)\chi^L H, \mathcal{R}w(s))\|_{H^{N'}} ds \\
\lesssim \int_0^t \|\langle \nabla \rangle H\|_{W^{N', 4 \delta + 3, d}} \quad \|w\|_{W^{2, \frac{1}{2}}} + \|\mathcal{R}(\nabla)H\|_{W^{2, 4}} \|w\|_{W^{N', 4 \delta + 3, d}} ds \\
\lesssim \int_0^t \|U\|_{L_2} \|U\|_{H^{N', 4 \delta + 3, d}} \|w\|_{W^{2, \frac{1}{2}}} + \|U\|_{W^{3, 4}} \|U\|_{L_2} \|w\|_{H^{N', 4 \delta + 3, d}} ds \\
\lesssim \int_0^t \langle s \rangle^{-2(1 - 3\delta)} \|U\|_{X_T}^3 + \langle s \rangle^{-2(1 - 3\delta)} \|U\|_{X_T}^3 ds \lesssim \|U\|_{X_T}^3.
$$

8. Conclusion of Theorem \ref{theorem-1.4}

By collecting the estimates in Section 6 and Section 7, we find that if $\|U\|_{X_T} \leq \vartheta_2$, there exists three constants $d_1, d_2, d_3$ such that for any $\varepsilon \in (0, 1)$, any $T > 0$,

$$
\|U\|_{X_T} \leq d_1 \|(u_0, \varphi_0, \nabla \varphi_0)\|_Y + d_2 \|U\|_{X_T}^3 + d_3 \|U\|_{X_T}^3.
$$

where

$$
\|(u_0, \varphi_0, \nabla \varphi_0)\|_Y \triangleq \|(u_0, \varphi_0, \nabla \varphi_0)^L\|_{W^{4, 1}} + \|\nabla u_0, \varphi_0, \nabla \varphi_0\|_{H^{4 + \delta}} \\
+ \|\nabla (u_0, \varphi_0, \nabla \varphi_0)^h\|_{L^2} + \|(u_0, \varphi_0, \nabla \varphi_0)\|_{H^N}.
$$

Combining with the local existence shown in Section 4, the global existence stems from the standard bootstrap arguments. Indeed, assume $\|(u_0, \varphi_0, \nabla \varphi_0)\|_Y \leq \vartheta_1$, and set

$$
T^* = \sup \{T | U \in C([0, T), H^N), \|U\|_{X_T} \leq 2d_1 \bar{\vartheta} \}
$$

Suppose that $T^* < +\infty$, then by (8.1), for any $t < T^*$,

$$
\|U\|_{X_T} \leq d_1 \bar{\vartheta} + d_2 (2d_1 \bar{\vartheta})^2 + d_3 (2d_1 \bar{\vartheta})^3 \leq \frac{3}{2} d_1 \bar{\vartheta}
$$

if $\bar{\vartheta}$ is chosen small enough, say $\bar{\vartheta} \leq \vartheta_3$. By the time continuity of $X_t$ norm, one gets that: $\|U\|_{X_{T^*}} \leq \frac{3}{2} d_1 \bar{\vartheta}$, which contradicts with the local existence and the definition of $T^*$. We thus finish the proof of Theorem \ref{theorem-1.4} by setting $\bar{\vartheta}_1 = \min \{\vartheta_3, \frac{\vartheta_1}{2d_1} \}$.

9. Proof of Theorem \ref{theorem-1.3}

This section is devoted to the proof of Theorem \ref{theorem-1.3} concerning the life span of system (1.3):

$$
\begin{cases}
\partial_t n + \text{div} (\rho v + nu + nv) = 0, \\
\partial_t v + u \cdot \nabla v + v \cdot (\nabla u + \nabla v) - \varepsilon \frac{1}{\rho + n} \Delta v + \nabla n - \nabla \psi = \varepsilon (\frac{1}{\rho + n} - 1) \Delta u, \\
\Delta \psi = n, \\
v|_{t=0} = P u_0^\varepsilon, n|_{t=0} = 0, \nabla \psi|_{t=0} = 0.
\end{cases}
$$
Proof of Theorem 1.3. The local existence of the above system in $C([0,T_\varepsilon,H^3])$ results from the local existence of system (1.1) and (1.2), it thus suffices for us to extend the existence time to $\varepsilon^{-(1-\theta)}$ which follows from the energy estimates. We define the energy functional:
\[
E_3 = \sum_{|\alpha| \leq 3} E_{\alpha} = \sum_{|\alpha| \leq 3} \frac{1}{2} \int (1 + \rho + n)|\partial^\alpha v|^2 + |\partial^\alpha n|^2 + |\partial^\alpha \nabla \psi|^2 \, dx.
\]
Taking the time derivative of the above energy functional and using the equations (1.3), we get:
\[
\partial_t E_\alpha + \varepsilon \int |\partial^\alpha \nabla v|^2 + |\partial^\alpha \text{div} \, v|^2 \, dx = \sum_{j=1}^{9} F_j, \quad (9.1)
\]
where:
\[
F_1 = - \int \rho \partial^\alpha v [\partial^\alpha, u + v] \nabla v \, dx, \quad F_2 = - \int \partial^\alpha n [\partial^\alpha, \rho \varepsilon] \text{div} \, v \, dx,
\]
\[
F_3 = - \int \partial^\alpha \nabla \psi [\partial^\alpha, \rho \varepsilon] v \, dx, \quad F_4 = - \int \partial^\alpha n \partial^\alpha \text{div} \, (nu) \, dx,
\]
\[
F_5 = = \int \partial^\alpha \nabla \psi \partial^\alpha (nu) \, dx, \quad F_6 = - \int \rho \varepsilon \partial^\alpha v \partial^\alpha (v \cdot \nabla u) \, dx,
\]
\[
F_7 = \int \partial^\alpha n (\nabla \rho \varepsilon \partial^\alpha v - \partial^\alpha (\nabla \rho \varepsilon v)) \, dx, \quad F_8 = \varepsilon \int \rho \varepsilon \partial^\alpha v \partial^\alpha \left[ \left( \frac{1}{\rho \varepsilon} - 1 \right) \Delta u \right],
\]
\[
F_9 = \varepsilon \int \rho \varepsilon \partial^\alpha v \left[ \partial^\alpha, \rho \varepsilon \right] (\Delta v + \nabla \text{div} \, v) \, dx.
\]

We recall that $\rho \varepsilon = \rho + n = 1 + \rho + n$. It is easy to see that: $F_1 = F_2 = F_3 = F_9 = 0$ if $|\alpha| = 0$. By standard commutator estimates: (we assume $|\alpha| \geq 1$ in the estimates of $F_1, F_2, F_3, F_9$)
\[
|F_1| \lesssim \|v\|_{H^1}^2 (\|\nabla v\|_{L^\infty} + \|\nabla u\|_{W^{1,\infty}}), \quad |F_2 + F_3| \lesssim \|n, v, \nabla \psi\|_{H^1}^2 (\|(v, n)\|_{W^{1,\infty}} + \|\rho\|_{W^{1,\infty}}),
\]
\[
|F_4| \lesssim \|n\|_{H^1}^2 \|\nabla u\|_{W^{1,\infty}}, \quad |F_5| \lesssim \|n, \nabla \psi\|_{H^1} \|\nabla u\|_{W^{1,\infty}},
\]
\[
|F_6| \lesssim \|v\|_{H^1}^2 \|\nabla u\|_{W^{1,\infty}}, \quad |F_7| \lesssim \|n, v\|_{H^1}^2 (\|(\nabla n, \nabla v)\|_{L^\infty} + \|\nabla \rho\|_{W^{1,\infty}}),
\]
\[
|F_8| \lesssim \varepsilon \|v\|_{H^1}^2 \|\Delta u\|_{W^{1,\infty}} (\|(n, \nabla \psi)\|_{H^1} + \|\nabla \rho\|_{W^{1,\infty}}),
\]
\[
|F_9| \lesssim \varepsilon \|v\|_{H^1}^2 (\|\Delta u\|_{W^{1,\infty}} + \varepsilon \|\Delta u\|_{W^{1,\infty}} + \|\rho\|_{H^1}^2)
\]
\[
|F_5| \lesssim \int \partial^\alpha \nabla \psi \cdot u \partial^\alpha u \, dx + \int \partial^\alpha \nabla \psi [\partial^\alpha, u] \, dx
\]
\[
= - \int \partial^\alpha \nabla \psi \cdot \nabla (\partial^\alpha \nabla \psi \cdot u) \, dx + \int \partial^\alpha \nabla \psi [\partial^\alpha, u] \, dx
\]
\[
= \frac{1}{2} \int |\partial^\alpha \nabla \psi|^2 \text{div} \, u \, dx - \int \partial^\alpha \nabla \psi \cdot (\partial^\alpha \nabla \psi \cdot u) \, dx + \int \partial^\alpha \nabla \psi [\partial^\alpha, u] \, dx
\]
\[
\lesssim \|n, \nabla \psi\|_{H^1}^2 \|\nabla u\|_{W^{1,\infty}}.
\]

We only detail the estimate of $F_5$, which seems not direct. Indeed, by the Poisson equation $\Delta \psi = n$, we have:
\[
|F_5| \lesssim \int \partial^\alpha \nabla \psi \cdot u \partial^\alpha u \, dx + \int \partial^\alpha \nabla \psi [\partial^\alpha, u] \, dx
\]
\[
= - \int \partial^\alpha \nabla \psi \cdot \nabla (\partial^\alpha \nabla \psi \cdot u) \, dx + \int \partial^\alpha \nabla \psi [\partial^\alpha, u] \, dx
\]
\[
= \frac{1}{2} \int |\partial^\alpha \nabla \psi|^2 \text{div} \, u \, dx - \int \partial^\alpha \nabla \psi \cdot (\partial^\alpha \nabla \psi \cdot u) \, dx + \int \partial^\alpha \nabla \psi [\partial^\alpha, u] \, dx
\]
\[
\lesssim \|n, \nabla \psi\|_{H^1}^2 \|\nabla u\|_{W^{1,\infty}}.
\]

Define
\[
T_* = \sup_T \left\{ T \mid \sup_{0 \leq t \leq T} E_3(t) \leq 4 \partial^2 \varepsilon^{2-\theta} \right\},
\]
where \(0 \leq \theta \leq \theta_0 \leq \theta_1, \theta_1\) is defined in Theorem 1.4 and \(\theta_0\) is to be chosen.

Summing up the above estimates for any \(|\alpha| \leq 3\), we obtain that, there exist three constants \(C_2 > 0, C_3 > 0, C_4 > 0\), for any \(0 < \theta \leq \theta_1\), if \(\|\rho_0^\theta - 1, P^\perp u_0^\theta, \nabla c_0^\theta\|_{W^4} \leq \frac{\theta}{C_3}\) (which yields \((1 + t)\|\nabla u, \theta\|_{W^4, \infty} + (\nabla u, \theta)\|_{H^\infty} \leq \frac{\theta}{C_3}\) by Theorem 1.4), such that the following energy inequality holds:

\[
\partial_t \mathcal{E}_3 + \varepsilon \|\nabla v\|^2_{H^3} \leq C_2 \mathcal{E}_3^\frac{3}{2} + (1 + t)^{-1} \varepsilon \mathcal{E}_3 + C_4 \mathcal{E}_3 \|\nabla v\|_{H^3}^2 (\frac{\theta}{C_3} + \mathcal{E}_3). \tag{9.2}
\]

Now one can choose \(\theta_0\) small enough, such that for any \(0 \leq t < T_\ast\),

\[
\|\mathcal{E}_3\|_{L^\infty} \leq \frac{1}{4}, \quad C_4 (\frac{\theta_0}{C_3} + \mathcal{E}_3) \leq \frac{1}{2}
\]

which leads to:

\[
\partial_t \mathcal{E}_3 \leq C_2 \mathcal{E}_3^\frac{3}{2} + (1 + t)^{-1} \varepsilon \mathcal{E}_3 + (1 + t)\mathcal{E}_3 \quad \forall t \in [0, T_\ast). \tag{9.3}
\]

We are now ready to show \(T_\ast \geq \varepsilon^{-1(1-\theta)}\). Indeed, for any \(t \leq t_0 \triangleq \min\{\varepsilon^{-1(1-\theta)}, T_\ast\}\), one has by (9.3), Gronwall’s inequality and assumptions: \(\theta \leq \frac{1}{2}, 16C_2^2 \theta \leq 1, \mathcal{E}_3(0) \leq \varepsilon\mathcal{E}^2\) :

\[
\mathcal{E}_3(t) \leq e^{\int_0^t \theta(1+\tau)^{-1}d\tau} \mathcal{E}_3(0) + \int_0^t e^{\int_\tau^t \theta(1+\tau)^{-1}d\tau} (C_2 \mathcal{E}_3^\frac{3}{2} + \varepsilon \mathcal{E}_3^2) \leq (1 + t)^\theta \mathcal{E}_3(0) + (1 + t)^\theta \int_0^t (1 + s^{-\theta}) (C_2 \varepsilon \mathcal{E}_3^2 + (1 + s)^{-1} \varepsilon^2) ds \leq 2^\theta e^{-\theta(1-\theta)} \varepsilon^2 + \frac{8}{1 - \theta} e^{-\theta(1-\theta)} C_2^2 \varepsilon^3 \mathcal{E}_3^{-2\theta} + \varepsilon^3 e^{-\theta(1-\theta)} \leq \frac{7}{2} \varepsilon^2 \mathcal{E}^{2-\theta}, \tag{9.4}
\]

which ensures \(t_0 = \varepsilon^{-1-\theta} < T_\ast\). Note that since \(\frac{1}{2} \leq \rho_0^\theta \leq \frac{3}{2}\), the assumption \(\|\mathcal{(n, v, \nabla \psi)}\|_{H^3} \leq \varepsilon\mathcal{E}\) leads to \(\mathcal{E}_3(0) \leq \varepsilon\mathcal{E}^2\). We thus finish the proof of Theorem 1.5 by choosing \(C = C_1 C_3\).

\[\square\]

10. Appendix

We sketch in this appendix the proof of the decay and weighted norm of \(\mathcal{R}I_3 \triangleq \mathcal{R}e^{-ib(D)}I_3\): \(W^{4+26, 25}\) (recall \(2\delta = \frac{2}{1 + \delta + 5}\) norm of \(Z_1, Z_2, Z_3\). Let us begin with the estimate \(Z_2\) which is the easiest. By Lemma 3.2 Corollary 3.10 and Sobolev embedding,

\[
\|Z_2\|_{W^{4+26, 25}} \lesssim \int_0^t \langle s \rangle^{\frac{5}{4}} \|T_{\delta}(\mathcal{R}(\nabla)(\mathcal{R}w)^2, \mathcal{R}w)\|_{W^{4+26, 25}} ds \lesssim \int_0^t \langle s \rangle^{\frac{5}{4}} \|\mathcal{E}_3\|_{H^8} \|w\|_{W^{2, 12}}^2 ds \lesssim \int_0^t \langle s \rangle^{\frac{5}{4}} \|\mathcal{E}_3\|_{X^T} \lesssim \|U\|_{X^T}^3.
\]

For \(Z_3\), we split it into two terms:

\[
Z_3 = \int_0^t e^{ib(D)} e^{-(t-s)\Delta} T_m \chi_{\{x_\| x \in (\xi - \eta) \leq (0)\}} (\mathcal{R}(\nabla)(\mathcal{R}w)^2, e^{-ib(D)}x \mathcal{R}f) ds \triangleq Z_{31} + Z_{32}
\]

The estimate of \(Z_{31}\) is similar to that of \(Z_2\), as we can put all the loss of derivative on the term \(B(\mathcal{R}w, \mathcal{R}w)\). Applying lemma 3.2 with \(p = 2\delta = \frac{2}{1 + \delta + 5}\), bilinear estimate 3.19, Lemma 7.3 with
\[7.5 \leq \frac{2}{3} N + \frac{1}{2},\] we control \(Z_{31}\) as:

\[
\|Z_{31}\|_{W^{4+2\delta,2\delta}} \lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \|T \mathcal{W} \chi_{\{(\xi - \eta , \xi - \eta ) \geq (0,j)\}} (\mathcal{R}(\nabla) (\mathcal{R}w)^2, e^{-i s b(D)} x \mathcal{R} f)\|_{W^{1+\frac{3\delta}{8}}} \, ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \|\mathcal{R} w\|_{L^\infty} \|\mathcal{R} w\|_{W^{7+4\delta,1+\frac{3\delta}{8}}} \|e^{-i s b(D)} x \mathcal{R} f\|_{W^{1+\frac{3\delta}{8}}} \, ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \|\mathcal{R} w\|_{W^{1+\frac{1}{2}}} \|\mathcal{R} w\|_{W^{1.5,3}} \|\langle x \rangle f\|_{W^{3,3}} \langle s \rangle^{\frac{\delta}{2}} \, ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{\frac{1+\delta}{2}} \langle s \rangle^{-(1-2\delta)} \langle s \rangle^{-\frac{\delta}{2}} \, ds \|U\|_{X_T}^3 \lesssim \|U\|^3_{X_T}.
\]

For \(Z_{32}\), one splits it again into two terms:

\[
Z_{32} = \mathcal{F}^{-1} \left( \int_0^t \left( \Psi \left( \frac{\xi - \eta}{\langle s \rangle^\alpha} \right) + 1 - \Psi \left( \frac{\xi - \eta}{\langle s \rangle^\alpha} \right) \right) \cdots \, ds \right) \triangleq Z_{321} + Z_{322}
\]

For \(Z_{321}\), Corollary 3.10 and Sobolev embedding lead to:

\[
\|Z_{321}\|_{W^{4+2\delta,2\delta}} \lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \|T \mathcal{W} \chi_{\{(\xi - \eta) \geq (0,j)\}} (\mathcal{R}(\nabla) (\mathcal{R}w)^2, P_{\leq (s)} e^{-i s b(D)} x \mathcal{R} f)\|_{W^{1+\frac{3\delta}{8}}} \, ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \|w\|_{W^{2,12}} \|P_{\leq (s)} e^{-i s b(D)} x \mathcal{R} f\|_{W^{8,3}} \, ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \|w\|_{W^{2,12}} \langle s \rangle^{5\delta_0} \langle s \rangle^{\frac{\delta}{2}} \|x \mathcal{R} f\|_{W^{3,3}} \, ds
\]

\[
\lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \langle s \rangle^{\frac{\delta}{2}} \|U\|_{X_T}^3 \langle s \rangle^{\frac{\delta}{2}} + 5\delta_0 \|U\|_{X_T} \lesssim \|U\|^3_{X_T}.
\]

Note we can choose \(\delta < \delta_0 \leq \frac{1}{50}\).

For \(Z_{322}\), we define \(M_1(\xi, \eta) = \mathcal{W} \chi_{\{(\xi - \eta) \geq (0,j)\}} \chi_{\langle \xi - \eta \rangle \langle \eta \rangle} \) and recall \(\hat{\phi} = b(\xi) \pm b(\xi - \eta) \pm b(\eta - \sigma) \pm b(\sigma)\). Using identity \(\partial_\eta \tilde{\mathcal{R}} f(\xi - \eta) = -\partial_\eta \tilde{\mathcal{R}} f(\eta)\) to integrate by parts in \(\eta\), one gets:

\[
Z_{322} = \mathcal{F}^{-1} \left( \int_0^t \int \int is \partial_\eta \hat{\phi} e^{is \hat{\phi} e^{-\varepsilon(t-s)}} |\xi|^2 M_1(\xi, \eta) \mathcal{R}(\eta) \tilde{\mathcal{R}} f(\eta - \sigma) \tilde{\mathcal{R}} f(\sigma) \tilde{\mathcal{R}} f(\xi - \eta) \eta \, d\eta \, d\sigma \, ds \right.
\]

\[+ \int_0^t \int \int e^{is \hat{\phi} e^{-\varepsilon(t-s)}} |\xi|^2 \partial_\eta M_1(\xi, \eta) \mathcal{R}(\eta) \tilde{\mathcal{R}} f(\eta - \sigma) \tilde{\mathcal{R}} f(\sigma) \tilde{\mathcal{R}} f(\xi - \eta) \eta \, d\eta \, d\sigma \, ds \]

\[+ \int_0^t \int \int e^{is \hat{\phi} e^{-\varepsilon(t-s)}} |\xi|^2 M_1(\xi, \eta) |\eta|^{-1} \tilde{\mathcal{R}} f(\eta - \sigma) \tilde{\mathcal{R}} f(\sigma) \tilde{\mathcal{R}} f(\xi - \eta) \eta \, d\eta \, d\sigma \, ds \]

\[+ \int_0^t \int \int e^{is \hat{\phi} e^{-\varepsilon(t-s)}} |\xi|^2 M_1(\xi, \eta) \mathcal{R}(\eta) \tilde{\mathcal{R}} f(\eta - \sigma) \tilde{\mathcal{R}} f(\sigma) \tilde{\mathcal{R}} f(\xi - \eta) \eta \, d\eta \, d\sigma \, ds \]

\[\lesssim Z_{3221} + \cdots Z_{3224}.\]
The estimation of $Z_{3222} = \int_0^t e^{isb(D)} e^{i(t-s)\Delta} T_{\partial bM_1}(\mathcal{R}(\mathcal{R}w)^2, \mathcal{R}w) \, ds$ is similar to that of $Z_2$, we thus do not detail it. For $G_{3223}$, thanks to Corollary 3.19 and Hardy-Littlewood-Sobolev inequality,

$$
\|Z_{3223}\|_{W^{4+25,2s}} \lesssim \int_0^t \langle s \rangle^{\frac{6}{5}} \| T_{M_1} (|\nabla|^{-1}(\mathcal{R}w)^2, P_{\geq (\cdot)^0}(\mathcal{R}w)) \|_{W^{4+35,2s}} \, ds
$$

$$
\lesssim \int_0^t \langle s \rangle^{\frac{6}{5}} \| P_{\geq (\cdot)^0}(\mathcal{R}w) \|_{H^s} \| (\mathcal{R}w)^2 \|_{W^{2,2s}} \, ds
$$

$$
\lesssim \int_0^t \langle s \rangle^{\frac{6}{5}} \| (\mathcal{R}w)^2 \|_{H^s} \| (\mathcal{R}w)^2 \|_{W^{2,2s}} \, ds
$$

Note $N = 11$, the last inequality holds if we choose $\delta \leq \delta_0$.

The term $Z_{3223}$ can be estimated in the similar manner as that of $Z_{31}$, we omit the detail. The estimate of $Z_{3221}$ is similar to that of $Z_1$, we thus only focus on the estimate of $Z_1$ in the following.

As before, we split it as:

$$
Z_1 = \mathcal{F}^{-1} ( \int_0^t (\Psi(\hat{\xi} - \hat{\eta} + \hat{\eta}) + 1 - \Psi(\hat{\xi} - \hat{\eta})) \cdots ds ) \triangleq Z_{11} + Z_{12}.
$$

Split $Z_{12}$ further as:

$$
Z_{12} = \int_0^t s e^{isb(D)} e^{i(t-s)\Delta} T_{M_2}((P_{\geq (\cdot)^0} + P_{\leq (\cdot)^0})(\mathcal{R}(\mathcal{R}w)^2, P_{\geq (\cdot)^0} \mathcal{R}w) \, ds
$$

where $M_2 = \frac{m}{\phi} \langle \eta \rangle (i\partial \xi \hat{\phi})$. Therefore, by bilinear estimate lemma 3.6 and the Sobolev embedding,

$$
\|Z_{12}\|_{W^{4+25,2s}} \lesssim \int_0^t s \langle s \rangle^{\frac{6}{5}} \| (\mathcal{R}(\mathcal{R}w)^2 \|_{W^{2,2s}} \| P_{\geq (\cdot)^0} \mathcal{R}w \|_{H^s} + \| P_{\geq (\cdot)^0} \mathcal{R}(\mathcal{R}w)^2 \|_{H^s} \| \mathcal{R}w \|_{W^{2,2s}} \) \, ds
$$

$$
\lesssim \int_0^t s \langle s \rangle^{\frac{6}{5}} \| \mathcal{R}w \|_{W^{2,2s}} \| (\mathcal{R}w)^2 \|_{H^s} \| (\mathcal{R}w)^2 \|_{W^{2,2s}} \| s \rangle^{\frac{6}{5}} \| (\mathcal{R}w)^2 \|_{H^s} \| (\mathcal{R}w)^2 \|_{W^{2,2s}} \, ds
$$

if we assume $N = 11$, $\delta_0 \geq 3\delta$.

For $Z_{11}$, one uses decomposition $q^2 = gP_{\geq (\cdot)^0}g + P_{\leq (\cdot)^0}gP_{\geq (\cdot)^0}g + P_{\leq (\cdot)^0}gP_{\leq (\cdot)^0}g$ to split it into three terms and denote as $Z_{111} + Z_{112} + Z_{113}$. For the term $Z_{111}$, one can use bilinear estimate 3.19 Sobolev embedding and the spectral localization of each term to get:

$$
\|Z_{111}\|_{W^{4,2s}} \lesssim \int_0^t \langle s \rangle^{1+\frac{6}{5}} \| P_{\leq (\cdot)^0} \mathcal{R}w \|_{W^{7,2s}} \| P_{\geq (\cdot)^0} \mathcal{R}w \|_{W^{7,2s}} \| P_{\geq (\cdot)^0} \mathcal{R}w \cdot \mathcal{R}w \|_{W^{2,2s}} \, ds
$$

$$
+ \| P_{\leq (\cdot)^0} \mathcal{R}w \|_{W^{7,2s}} \| P_{\geq (\cdot)^0} \mathcal{R}w \cdot \mathcal{R}w \|_{W^{7,2s}} \| P_{\geq (\cdot)^0} \mathcal{R}w \cdot \mathcal{R}w \|_{W^{2,2s}} \, ds
$$

$$
\lesssim \int_0^t \langle s \rangle^{1+\frac{6}{5}} \| P_{\leq (\cdot)^0} \mathcal{R}w \|_{W^{7,2s}} \| P_{\geq (\cdot)^0} \mathcal{R}w \|_{W^{7,2s}} \langle s \rangle^{1+\frac{6}{5}} \| (\mathcal{R}w)^5 \|_{H^3} \| P_{\geq (\cdot)^0} \mathcal{R}w \|_{H^3} \| P_{\geq (\cdot)^0} \mathcal{R}w \|_{H^3} \, ds
$$

$$
\lesssim \int_0^t \langle s \rangle^{1+\frac{6}{5}} \| (\mathcal{R}w)^2 \|_{H^3} \| (\mathcal{R}w)^2 \|_{H^3} \| U \|_{X_T}^3 \lesssim \| U \|_{X_T}^3
$$

as $N = 11$, and $\delta_0 \geq 5\delta$. The estimate of $Z_{112}$ is similar to that of $Z_{111}$, we thus skip it. Now, we focus on the estimation of

$$
Z_{113} = \int_0^t s e^{isb(D)} e^{i(t-s)\Delta} T_{M_2}(\mathcal{R}(P_{\leq (\cdot)^0} \mathcal{R}w) \mathcal{R}w, P_{\leq (\cdot)^0} \mathcal{R}w) \, ds
$$

$$
= \mathcal{F}^{-1} \left( \int_0^t \mathcal{F} e^{is\hat{b}} e^{i(t-s)\Delta} m \mathcal{F}(\xi, \eta) \mathcal{F}(\eta) \mathcal{F}(\xi) \mathcal{F}(\xi - \eta) \mathcal{F}(\xi - \eta)(\xi) \mathcal{F}(\xi - \eta) ds \right)
$$

$$
\Phi_{\leq (\cdot)^0} \mathcal{F}(\xi - \eta) \mathcal{F}(\xi - \eta) \mathcal{F}(\xi - \eta) \mathcal{F}(\xi - \eta) \mathcal{F}(\xi - \eta) (\xi - \eta) \mathcal{F}(\xi - \eta) ds
$$
Recall that $\tilde{\phi} = b(\xi) \pm b(\xi - \eta) \pm b(\eta - \sigma) \pm b(\sigma)$. For the case $'+++', '++-', '+-+, '++-', '-++', '-+-', \ldots$, one could easily prove that: $\frac{1}{\nu} > \nu_0 \min\{\langle \xi \rangle, \langle \xi - \eta \rangle, \langle \eta - \sigma \rangle, \langle \sigma \rangle\}$. Therefore, we shall use the identity $\frac{1}{i \nu} \partial_\nu e^{i s \tilde{\phi}} = e^{i s \tilde{\phi}}$ to integrate by parts in time, which gives us essentially the quartic terms that are easy to handle, we do not detail them. The remaining three terms $'---, '-+-, '+--$ are more involved, we will detail the $'---'$ case for instance. The other two cases are a little bit easier. Firstly, for $\tilde{\phi} = b(\xi) + b(\xi - \eta) - b(\eta - \sigma) - b(\sigma)$, we have $\partial_\nu \tilde{\phi} = \frac{(1 - 2 \xi^2)|\xi|^2 \xi}{b(\xi)} + \frac{(1 - 2 \eta^2)|\eta|^2 \eta}{b(\xi - \eta)}$. In this case, by Lemma 10.2 below, one can find two matrices $Q_1, Q_2, st., \partial_\nu \tilde{\phi}(\xi, \eta, \sigma) = -2Q_1(\xi, \eta)\partial_\eta \tilde{\phi} - Q_2(\xi, \eta, \sigma)\partial_\sigma \tilde{\phi},$ and $Q_j(j = 1, 2)$ satisfy the condition:

$$\|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma Q_j(\xi, \eta, \sigma)\| \lesssim_{\alpha, \beta, \gamma, \nu} (|\xi| + |\eta| + |\sigma|)^3.$$

We now split $Z_{113}$ again:

$$Z_{113} = \mathcal{F}^{-1}\left(\int_0^t \left(\Psi_{\geq s} - \Theta_0(\eta) + \Psi_{\leq s} - \Theta_0(\eta)\right) \cdots d\eta ds\right) = Y_1 + Y_2.$$

Let us see $Y_1$. In this case, there is no singularity for $R(\eta)$ as $\eta$ does not vanish. Therefore, we could use the identity:

$$is \partial_\nu \tilde{\phi} e^{i s \tilde{\phi}} = -2Q_1(\xi, \eta, \sigma)\partial_\eta (e^{i s \tilde{\phi}}) - Q_2(\xi, \eta, \sigma)\partial_\sigma (e^{i s \tilde{\phi}})$$

and integrate by parts in $\eta$ and $\sigma$ respectively. We only detail the situation of integration by parts in $\eta$ as the other case is similar. To continue, we denote $m_j(\xi, \eta, \sigma) = Q_j m(\xi, \eta)R(\eta)(\eta)\Psi_{\leq s} - \Theta_0(\eta)^2$, we have

$$Y_1 = \int_0^t e^{i s b(D)} e^{i s (t-s)} \Delta T_{\partial_\nu m_1} (Rw, Rw, Rw) ds$$

However, these two terms can be easily treated once we have the following lemma.

**Lemma 10.1.** $m_j(\xi, \eta, \sigma), j = 1, 2$ is defined as follows, the following trilinear estimates hold:

$$\|T_{\partial_\nu m_j}(f, g, h)\|_{L^p} \lesssim \langle s \rangle^{2d_0} \|f\|_{L^p} \|g\|_{L^p} \|h\|_{L^p},$$

$$\|T_{\partial_\nu m_j}(f, g, h)\|_{L^p} \lesssim \langle s \rangle^{3d_0} \|f\|_{L^p} \|g\|_{L^p} \|h\|_{L^p}$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$.

**Proof.** It is not hard to check that: $\partial_\xi^2 \partial_\eta^\beta \partial_\sigma^\gamma m_1 \lesssim \langle s \rangle^{d_0 + |\beta| + 5d_0}$, which yields

$$\|\mathcal{F}^{-1}(m_j(\xi, \eta, \sigma))\|_{L^1} \lesssim \|(1 + \partial_\xi^2 + \partial_\eta^4 + \partial_\sigma^4) m_j\|_{L^2} \lesssim \langle s \rangle^{d_0 + |\beta| + 5d_0}$$

Similarly, $\|\mathcal{F}^{-1}(\partial_\nu m_j(\xi, \eta, \sigma))\|_{L^1} \lesssim \langle s \rangle^{3d_0}$. We thus finish the proof by noticing the explicit formulae of $T(f, g, h)$

$$T_{m_j}(f, g, h) = \int \int \mathcal{F}^{-1}(m_j)(y, z - y, w - z)f(x - y)g(x - z)h(x - w)d\eta dz dw$$
We now estimate $Y_{12}$ for example, by the spectral localization and Lemma 3.2:

$$
\|Y_{12}\|_{W^{4+2\delta,2\delta}} \lesssim \int_0^t \langle s \rangle^{\frac{\delta}{2}} \langle s \rangle^{(4+3\delta)\delta_0} \|T_{m_3(\xi)^{\delta}}(e^{isb(D)} x^R f, R w, R w)\|_{L^2_{\delta}} ds
$$

$$
\lesssim \int_0^t \langle s \rangle^{17\delta_0} \|e^{isb(D)} x^R f\|_{L^2_{\frac{1}{2}\xi - 2\delta_{-}}(\xi_2, \xi_1)} \|R w\|_{L^2_{\frac{1}{\delta}}} ds
$$

$$
\lesssim \int_0^t \langle s \rangle^{17\delta_0} \langle s \rangle^{-2(1-2\delta)} \langle s \rangle^{4\delta} ds \|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3.
$$

where the following fact has been used:

$$
\|e^{isb(D)} x^R f\|_{L^2_{\frac{1}{2}\xi - 2\delta_{-}}(\xi_2, \xi_1)} \lesssim \langle s \rangle^{4\delta} \|x^R f\|_{W^{2,\frac{1}{2}}}.
$$

We now go back to estimate $Y_2$, as before, we split it into two terms:

$$
Y_2 = \mathcal{F}^{-1} \left( \int_0^t \int (\Psi_{\leq 3(s)} - \delta_0/5)(\xi - \eta) + \Psi_{\geq 3(s)} - \delta_0/5)(\xi - \eta) \right) \cdot ds dy ds \triangleq Y_{21} + Y_{22}
$$

For $Y_{21}$, one can use the specific form of $\partial_\xi \hat{\varphi} = \frac{1-2\varepsilon|\xi|^2}{b(\xi)} \xi + \frac{1-2\varepsilon|\xi - \eta|^2}{b(\xi - \eta)}(\xi - \eta)$. The observation is that, having projected to the low frequency for $\xi - \eta$, we could use of $\xi - \eta$ and $\eta$ appearing in $\partial_\xi \varphi$. We also recall that $(1-2\varepsilon|\xi|^2)$ here is bounded on the support of $\chi^c(\xi)$. Formally, we could write $Y_2$ as

$$
\int_0^t i s e^{isb(D)} e^{(t-s)} \nabla T_{\frac{1-2s|\xi|^2}{b(\xi)}}(\mathcal{P}_{\leq 3(s)} - \delta_0 (P_{\leq 3(s)} - \delta_0 R w)^2, P_{\geq 3(s)} - \delta_0 R w) ds
$$

and similar term. One can estimate (10.1) as follows:

$$
\|Y_2\|_{W^{4+2\delta,2\delta}} \lesssim \int_0^t \langle s \rangle^{1+\delta} \langle s \rangle^{-\delta_0} \|R w\|_{W^{2,\frac{1}{\delta}}} \|R w\|_{W^{s,(\frac{1}{2}\xi - 2\delta_{-})}} ds
$$

$$
\lesssim \int_0^t \langle s \rangle^{1+\delta - \frac{1}{\delta} \delta_0} \langle s \rangle^{-2(1-2\delta)} ds \|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3
$$

if we choose $\delta_0 \geq 50\delta$

For $Y_{22}$, we need to split again into two terms.

$$
Y_{22} = \mathcal{F}^{-1} \left( \int_0^t \int (\Psi_{\geq 2(s)} - \delta_0 (\sigma) + \Psi_{\leq 2(s)} - \delta_0 (\sigma)) \cdot ds dy ds \right) \triangleq Y_{221} + Y_{222}
$$

Let us see $Y_{221}$, in this case we have $|\sigma - \eta| > |\sigma| - |\eta| > \frac{1}{3}|\sigma|$. Besides, one can find a matrix $Q_3$, such that:

$$
\partial_\sigma \hat{\varphi} = \left( \frac{1-2\varepsilon|\sigma|^2}{b(\sigma)} \sigma + \frac{1-2\varepsilon|\sigma - \eta|^2}{b(\sigma - \eta)}(\sigma - \eta) \right) = Q_3(2\sigma - \eta)
$$

so we have: $|\partial_\sigma \hat{\varphi}| \gtrsim \|Q_3^{-1}\|^{-1} |2\sigma - \eta| \gtrsim \left( \frac{|\sigma|}{(\sigma)(\sigma - \eta)} \right) \gtrsim \langle s \rangle^{-3\delta_0}$. We thus could use identity $e^{isb(D)} = \frac{\partial_\sigma \hat{\varphi} \partial_\sigma \hat{\varphi}}{is|\sigma|^2} \partial_\sigma e^{isb(D)}$ and integrate by parts in $\sigma$, this leads to two terms:

$$
Y_{2211} = \int_0^t e^{isb(D)} e^{(t-s)} \nabla T_{m_3(\mathcal{R} T_{\partial_\sigma \hat{\varphi}}(R w)^2, R w)} ds
$$

$$
Y_{2212} = \int_0^t e^{isb(D)} e^{(t-s)} \nabla T_{m_3(R T_{\hat{\varphi}}(e^{isb(D)} x^R f, R w), R w)} ds
$$

38
where we denote
\[ m_3(\xi, \eta) = \frac{m}{\phi} \langle \eta \rangle \Psi_{\leq (s-s_0)} \langle \eta \rangle \Psi_{\leq (s-s_0)} \langle \xi - \eta \rangle \Psi_{\leq (s-s_0)} \langle \xi - \eta \rangle, \]
\[ \tilde{m}(\eta, \sigma) = \frac{\partial \phi}{\partial \sigma} \langle \eta \rangle \Psi_{\geq 2(s-s_0)} \langle \sigma \rangle \Psi_{\leq (s-s_0)} \langle \eta - \sigma \rangle \tilde{\chi}^L(\sigma - \eta) \tilde{\chi}^L(\sigma). \]

Similar to Lemma 10.1, one has the following inequality:
\[ \| T_{m_1}(f, g) \|_{L^p} \lesssim \langle s \rangle^{6s_0} \| f \|_{L^1} \| g \|_{L^2} \]
\[ \| T_{\tilde{m}_1}(u, v) \|_{L^p} \lesssim \langle s \rangle^{14s_0} \| u \|_{L^p} \| v \|_{L^2} \]
for any \( 1 \leq p \leq \infty \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \). These inequalities in hand, the estimates of \( Y_{2211}, Y_{2212} \) are direct. For example:
\[ \| Y_{2212} \|_{W^{4s,2s}} \lesssim \int_0^t \langle s \rangle^{2s} \langle s \rangle^{14s_0} \| e^{is\hat{\phi}(D)} \| \| \mathcal{R} f \|_{L^{(\frac{3}{2} - s_0)}^{-1}} \| \mathcal{R} w \|_{L^{\frac{3}{2}}} \]
\[ \lesssim \int_0^t \langle s \rangle^{14s_0} \| U \|_{X_T}^3 \lesssim \| U \|_{X_T}^3 \]

Note we could choose \( \delta < \delta_0 \leq \frac{1}{10} \).

Finally, it remains for us to estimate \( Y_{222} \). In this case, there is no structure for \( \partial \phi \) can be used. Nevertheless, as noted in [8], one can employ kind of ‘partial normal form’. We notice that:
\[ b(\eta) + b(\sigma) = 2 \frac{1}{b(\eta - \sigma) + 1} \| | \eta - \sigma \|^2 + 2 \frac{1}{b(\eta - \sigma) + 1} \| \sigma \|^2, \]
the observation is that we could use \( | \eta - \sigma \|^2 \) and \( | \sigma \|^2 \) appearing in this quantity. On the other hand, \( b(\xi) + b(\xi - \eta) - 2 = \frac{1}{b(\eta)} \| | \xi \|^2 + 1 - \frac{2}{b(\eta)} \| \xi - \eta \|^2 \geq \frac{1}{b(\eta)} \| \xi - \eta \|^2 \geq \langle s \rangle^{s_0/5} \). We thus use identity:
\[ e^{is\hat{\phi}} = e^{is(\hat{\phi} + \xi(\eta - \xi))} e^{-is(b(\eta) + b(\eta - \xi))} \]
\[ = \frac{-i}{b(\xi) + b(\xi - \eta)} \partial_s (e^{is(\hat{\phi} + \xi(\eta - \xi))}) e^{-is(b(\eta) + b(\eta - \xi))} \]
to integrate by parts in \( s \):
\[ Y_{222} = -te^{itb(D)} T_{m_4}(RT_{\tilde{m}_1}(Rw, Rw), Rw) + \int_0^t se^{itb(D)} e^{s(t-s)} \Delta \| \Delta T_{m_4}(RT_{\tilde{m}_1}(Rw, Rw), Rw)ds \]
\[ + \int_0^t se^{itb(D)} e^{s(t-s)} \Delta T_{m_4}(RT_{\tilde{m}_1}(b(\eta) + b(\eta - \xi), b(\eta - \xi) - 2)) (Rw, Rw, Rw)ds \]
\[ - \int_0^t se^{itb(D)} e^{s(t-s)} \Delta T_{m_4}(RT_{\tilde{m}_1}(e^{it\delta(D)} \partial_s \mathcal{R} f, Rw), Rw)ds + \text{ similar term} \]
\[ + \int_0^t se^{itb(D)} e^{s(t-s)} \| \mathcal{T}_{\tilde{m}_1}(RT_{\tilde{m}_1}(Rw, Rw), Rw) + T_{m_4}(RT_{\tilde{m}_1}(Rw, Rw, Rw))ds \]
\[ \triangleq Y_{2221} + \cdots Y_{2225} \]
where the following notations has been used:
\[ m_4(\xi, \eta, s) = \frac{\partial \phi}{\partial \sigma} \langle \eta \rangle \Psi_{\leq (s-s_0)} \langle \eta \rangle \Psi_{\leq (s-s_0)} \langle \xi - \eta \rangle \Psi_{\leq (s-s_0)} \langle \xi - \eta \rangle \]
\[ \tilde{m}_1(\eta, \sigma, s) = \Psi_{\leq (s-s_0)} \langle \eta \rangle \Psi_{\leq (s-s_0)} \langle \eta - \sigma \rangle \Psi_{\leq (s-s_0)} \langle \eta - \sigma \rangle \]

We have again, as in Lemma 10.1 for any \( 1 \leq p \leq \infty \), \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \),
\[ \| T_{m_4}(f, g) \|_{L^p} \lesssim \langle s \rangle^{s_0} \| f \|_{L^p} \| g \|_{L^p}, \]
\[ \| T_{\tilde{m}_1}(u, v) \|_{L^p} \lesssim \| u \|_{L^p} \| v \|_{L^p}. \]
For $Y_{2221}$, 
\[
\|Y_{2221}\|_{W^{4+25,2}} \lesssim t \langle t \rangle^{14b_0} \|\mathcal{R}T_{\tilde{m}_1}(\mathcal{R}w,\mathcal{R}w)\|_{L^2_y} \|\mathcal{R}w\|_{L^\infty_y}\langle t \rangle^{-25}\langle t \rangle^{-1}
\]
\[
\lesssim \langle t \rangle^{1+14b_0-2(1-\delta_0)}\|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3
\]
if $\delta \leq \delta_0 \leq \frac{1}{50}$.

For $Y_{2222}$, owing to the above estimate and the fact $e^{(t-s)\Delta} \mathcal{R} \mathcal{R}_x^{2-\delta}$ is $L^{2-\delta}$ multiplier with norm less than $(t-s)^{-1}$, we get that:
\[
\|Y_{2222}\|_{W^{4+25,2}} \lesssim \int_0^t \langle t-s \rangle^{-1}\langle t \rangle^{-18b_0}\|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3
\]

For $Y_{2223}$, since on the support of $m_4(t,\xi,\eta,\epsilon)$, $|\xi - \eta| \geq |\eta|$, and $|\xi - \eta| \gtrsim \langle s \rangle^{-\delta_0}$, it is not hard to see that
\[
\|\mathcal{F}^{-1}(m_4(t,\xi)^{1+3\delta})\|_{L^1} \lesssim \|(1 + \partial_\xi^3 + \partial_\eta^3)(m_4(t,\xi)^{1+3\delta})\|_{L^2} \lesssim \langle s \rangle^{-\delta_0}.
\]

Write also $b(\eta) + b(\sigma) - 2 = \frac{1 - \epsilon^2}{(\eta - \sigma)^2 + 1/2} - \epsilon^2 - \epsilon^2 \frac{\sigma^2}{(\eta - \sigma)^2}$, we thus could estimate $Y_{2223}$ by
\[
\|Y_{2223}\|_{W^{4+25,2}} \lesssim \int_0^t \langle t \rangle^{1+\delta}\langle s \rangle^{-\delta_0} \|\nabla^2 P_{\langle s \rangle^{-\delta_0}} \mathcal{R}w\|_{L_x^4} \|\mathcal{R}w\|_{L_x^\infty} \|w\|_{W^{x,(\frac{1}{5},-25)} - 1} ds
\]
\[
\lesssim \int_0^t \langle t \rangle^{1+2\delta}\langle s \rangle^{-2\delta_0} \langle s \rangle^{-(2(1-\delta))} \|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3
\]
if $10\delta < \delta_0$. For $Y_{2224}$ it is not tough because it is essentially quartic. Similar to that of $Y_{2221}$, we have
\[
\|Y_{2224}\|_{W^{4+25,2}} \lesssim \int_0^t \langle t \rangle^{1+\delta}\langle s \rangle^{14b_0} \|\mathcal{R}w\|_{L_x^4} \|e^{i\sigma b(D)} x f\|_{L^\infty_y}\langle t \rangle^{-25} - 1 ds
\]
\[
\lesssim \int_0^t \langle t \rangle^{1+\delta}\langle s \rangle^{14b_0} \langle s \rangle^{-(2(1-\delta))} \|U\|_{X_T}^3 \lesssim \|U\|_{X_T}^3
\]
where in the above, the following identity has been used
\[
e^{i\sigma b(D)} \partial_x \mathcal{R}f = \mathcal{R}\varepsilon \Delta w + \mathcal{R}\langle \nabla \rangle (\mathcal{R}w)^2 + \mathcal{R}f
\]
from which one easily gets: $\|e^{i\sigma b(D)} \partial_x \mathcal{R}f\|_{H^1} \lesssim \langle s \rangle^{-1} \|U\|_{X_T}^3$.

For the last term $Y_{2225}$, one notices that when we take the time derivative on $m_4(t,\xi,\eta,\epsilon)$ or $\tilde{m}_1(t,\sigma,\eta,\epsilon)$, there will emerge a power $(s)^{-1}$ which is enough for us to close the estimate. For instance,
\[
\partial_s \Psi((s)^{\delta_0} \eta) = \eta \cdot \nabla \Psi((s)^{\delta_0} \eta) (s)^{\delta_0} \langle s \rangle^{-1} \sqrt{\Psi((s)^{\delta_0} \eta)} (s)^{\delta_0} \langle s \rangle^{-1}
\]
where $\tilde{\Psi}$ has the same properties as $\psi$ that we need: compactly supported, smooth.

**Lemma 10.2.** Recall $b(x) = \sqrt{1 + |x|^2 - \varepsilon^2 |x|^4}$, with $x \in \mathbb{R}^2$, $\varepsilon \in (0,1]$. There exists a $2 \times 2$ matrix $S$ such that the following identity holds:
\[
\frac{1 - \varepsilon^2 |x|^2}{b(x)} - \frac{1 - \varepsilon^2 |y|^2}{b(y)} = S(x,y)(x-y)
\]
Besides, if \( \varepsilon|x|^2 \leq 3\kappa_0, \varepsilon|y|^2 \leq 3\kappa_0 \) with \( \kappa_0 \leq \frac{1}{200} \), then \( S \) is invertible. Moreover, for any \( \varepsilon \in (0,1) \) and \( \alpha, \beta \in \mathbb{N}^2 \), the following uniform (in \( \varepsilon \)) estimate holds:

\[
|\partial_x^\alpha \partial_y^\beta S(x,y)| \lesssim_{\alpha,\beta,\kappa_0} \frac{1}{\langle y \rangle},
\]

\[
|\partial_x^\alpha \partial_y^\beta S^{-1}(x,y)| \lesssim_{\alpha,\beta,\kappa_0} (\langle x \rangle + \langle y \rangle)^3.
\]

Proof.

\[
\frac{(1 - 2\varepsilon^2|x|^2)x}{b(x)} - \frac{(1 - 2\varepsilon^2|y|^2)y}{b(y)} = \frac{(1 - 2\varepsilon^2|x|^2)x(1 - b(y)) + (1 - 2\varepsilon^2|y|^2)y(1 - b(x))}{b(y)}
\]

\[
= -\frac{(1 - 2\varepsilon^2|x|^2)(|x|^2 - |y|^2)}{b(y)} x - \frac{1 - \varepsilon^2(|x|^2 + |y|^2)}{b(x)(x) + b(y))} + \frac{(1 - 2\varepsilon^2|x|^2)(x - y) - 2\varepsilon^2(|x|^2 - |y|^2)y}{b(y)}
\]

\[
= \frac{1 - 2\varepsilon^2|x|^2}{b(y)}[\text{Id}_{2\times 2} - \frac{\varepsilon^2(|x|^2 + |y|^2)}{b(x)(x) + b(y))} x \otimes (x + y) - 2\frac{\varepsilon^2}{1 - 2\varepsilon^2|x|^2} y \otimes (x + y)](x - y)
\]

\[\triangleq S(x - y).
\]

We now compute \( \det S \).

\[
\det S = \left(\frac{1 - 2\varepsilon^2|x|^2}{b(y)}\right)^2 \left[1 - \frac{1 - \varepsilon^2(|x|^2 + |y|^2)}{b(x)(x) + b(y))} x + \frac{2\varepsilon^2}{1 - 2\varepsilon^2|x|^2} x \cdot (x + y)
\]

\[
= \frac{(1 - 2\varepsilon^2|x|^2)^2}{b^2(y)b(x)(x) + b(y))} \left(1 + \varepsilon^2(|x|^2 + |y|^2) + b(x)b(y) - (1 - \varepsilon^2(|x|^2 + |y|^2))x \cdot y - \frac{2\varepsilon^2(x + y) \cdot y b(x)(b(x) + b(y))}{1 - 2\varepsilon^2|x|^2}\right).
\]

We note that if \( \varepsilon \leq 1, \kappa_0 \leq \frac{1}{100}, 1 - 2\varepsilon^2|x|^2 \geq \frac{1}{2} \) and

\[
4\varepsilon^2 b(x)(x + y) \cdot y \leq 9\varepsilon^2(|x|^2 + |y|^2)(b^2(x) + b^2(y)) \leq 108\kappa_0(\varepsilon + 3\kappa_0) \leq \frac{2}{3}.
\]

We thus have:

\[
\det S \geq \frac{1}{4b^2(y)} \left[ \frac{1}{3} + b(x)b(y) - x \cdot y \right].
\]

It is thus easy to see that:

\[
|\partial_x^\alpha \partial_y^\beta \left(\frac{1}{\det Q}\right)| \lesssim_{\alpha,\beta,\kappa_0} \langle x \rangle^2 \langle y \rangle / (\langle x \rangle + \langle y \rangle)^2.
\]

Besides, direct computations shows that:

\[
|\partial_x^\alpha \partial_y^\beta S(x,y)| \lesssim_{\alpha,\beta,\kappa_0} \langle y \rangle^{-1},
\]

which combined with (10.2), yields:

\[
|\partial_x^\alpha \partial_y^\beta S^{-1}(x,y)| \lesssim_{\alpha,\beta,\kappa_0} (\langle x \rangle + \langle y \rangle)^3.
\]
ACKNOWLEDGEMENT

The author is indebted to his supervisor Professor Frédéric Rousset for his kind guidance, careful reading of the paper and constructive suggestions which improve the presentation. The author also benefits from the conversation with professor Corentin Audiard on the uniform stability of the Navier-Stokes-Korteweg system.

REFERENCES

[1] H. Bahouri, J.-Y. Chemin, and R. Danchin. Fourier analysis and nonlinear partial differential equations, volume 343. Springer Science & Business Media, 2011.
[2] Y. Cai, Z. Lei, F. Lin, and N. Masmoudi. Vanishing viscosity limit for incompressible viscoelasticity in two dimensions. Comm. Pure Appl. Math., 72(10):2063–2120, 2019.
[3] N. Chikami and R. Danchin. On the global existence and time decay estimates in critical spaces for the Navier-Stokes-Poisson system. Math. Nachr., 290(13):1939–1970, 2017.
[4] R. Danchin. Global existence in critical spaces for compressible Navier-Stokes equations. Invent. Math., 141(3):579–614, 2000.
[5] P. Germain. Space-time resonances. Journées Équations aux dérivées partielles, pages 1–10, 2010.
[6] P. Germain, N. Masmoudi, and B. Pausader. Nonneutral global solutions for the electron Euler-Poisson system in three dimensions. SIAM J. Math. Anal., 45(1):267–278, 2013.
[7] L. Grafakos. Classical Fourier analysis, volume 249 of Graduate Texts in Mathematics. Springer, New York, second edition, 2008.
[8] L. Grafakos. Modern Fourier analysis, volume 250 of Graduate Texts in Mathematics. Springer, New York, second edition, 2009.
[9] Y. Guo. Smooth irrotational flows in the large to the Euler-Poisson system in $\mathbb{R}^{3+1}$. Comm. Math. Phys., 195(2):249–265, 1998.
[10] Y. Guo, L. Han, and J. Zhang. Absence of shocks for one dimensional Euler-Poisson system. Arch. Ration. Mech. Anal., 223(3):1057–1121, 2017.
[11] Y. Guo and B. Pausader. Global smooth ion dynamics in the Euler-Poisson system. Comm. Math. Phys., 303(1):89–125, 2011.
[12] Y. Guo and Y. Wang. Decay of dissipative equations and negative Sobolev spaces. Comm. Partial Differential Equations, 37(12):2165–2208, 2012.
[13] C. Hao and H.-L. Li. Global existence for compressible Navier-Stokes-Poisson equations in three and higher dimensions. J. Differential Equations, 246(12):4791–4812, 2009.
[14] D. Hoff and K. Zumbrun. Multi-dimensional diffusion waves for the Navier-Stokes equations of compressible flow. Indiana Univ. Math. J., 44(2):603–676, 1995.
[15] A. D. Ionescu and V. Lie. Long term regularity of the one-fluid Euler-Maxwell system in 3D with vorticity. Adv. Math., 325:719–769, 2018.
[16] A. D. Ionescu and B. Pausader. The Euler-Poisson system in 2D: global stability of the constant equilibrium solution. Int. Math. Res. Not. IMRN, (4):761–826, 2013.
[17] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. Comm. Pure Appl. Math., 41(7):891–907, 1988.
[18] D. Li and Y. Wu. The Cauchy problem for the two dimensional Euler-Poisson system. J. Eur. Math. Soc. (JEMS), 16(10):2211–2266, 2014.
[19] H.-L. Li, A. Matsumura, and G. Zhang. Optimal decay rate of the compressible Navier-Stokes-Poisson system in $\mathbb{R}^3$. Arch. Ration. Mech. Anal., 196(2):681–713, 2010.
[20] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ., 20(1):67–104, 1980.
[21] K. Nakanishi and W. Schlag. Invariant manifolds and dispersive Hamiltonian evolution equations. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2011.
[22] Š. Nečasová, X. Blanc, R. Danchin, B. Ducomet, et al. The global existence issue for the compressible euler system with poisson or helmholtz couplings. arXiv:1906.08075, 2019.
[23] F. Rouset and C. Sun. Stability of equilibria uniformly in the inviscid limit for the navier-stokes-poisson system, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2021, to appear.
[24] J. Shatah. Normal forms and quadratic nonlinear Klein-Gordon equations. Comm. Pure Appl. Math., 38(5):685–696, 1985.
[25] W. Shi and J. Xu. A sharp time-weighted inequality for the compressible Navier-Stokes-Poisson system in the critical $L^p$ framework. *J. Differential Equations*, 266(10):6426–6458, 2019.

[26] T. C. Sideris. Formation of singularities in three-dimensional compressible fluids. *Communications in mathematical physics*, 101(4):475–485, 1985.

[27] E. M. Stein. *Harmonic Analysis (PMS-43), Volume 43: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. (PMS-43)*, volume 43. Princeton University Press, 2016.

[28] C. Sun. Large time existence of euler-korteweg equations and two-fluid euler-maxwell equations with vorticity. *arXiv:2008.08354*, 2020.

[29] Y. Wang. Decay of the Navier-Stokes-Poisson equations. *J. Differential Equations*, 253(1):273–297, 2012.

[30] F. Zheng. Long-term regularity of the periodic Euler-Poisson system for electrons in 2D. *Comm. Math. Phys.*, 366(3):1135–1172, 2019.

[31] X. Zheng. Global well-posedness for the compressible Navier-Stokes-Poisson system in the $L^p$ framework. *Nonlinear Anal.*, 75(10):4156–4175, 2012.

[32] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

Laboratoire de Mathématiques d’Orsay (UMR 8628), Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay Cedex, France

Email address: changzhen.sun@universite-paris-saclay.fr