SINAI-RUELLE-BOWEN MEASURES FOR PIECEWISE HYPERBOLIC MAPS WITH TWO DIRECTIONS OF INSTABILITY IN THREE-DIMENSIONAL SPACES

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ABSTRACT. A class of piecewise twice-differentiable Lozi-like maps in three-dimensional Euclidean spaces is introduced, and the existence of Sinai-Ruelle-Bowen measures is studied, where the dimension of the instability is equal to two. Further, an example with computer simulations is provided to illustrate the theoretical results.

1. Introduction. An important characterization of a chaotic system is the sensitive dependence on initial conditions in which a small change in the initial state of a deterministic nonlinear system can generate large differences in a later state. The instability of a system is a main reason to generate chaotic dynamics for a deterministic system. The instability can be observed in some interesting systems, for example, the Logistic map, Hénon map, Lorenz attractor, and so on [4, 5, 20, 23].

The study of the existence of an invariant measure of a map is an interesting problem in dynamical systems. In smooth ergodic theory, based on a series of Sinai’s work on Anosov diffeomorphisms [26], Bowen and Ruelle’s results on Axiom-A attractors [7], an invariant Borel measure, which has absolutely continuous conditional measure on unstable manifolds with respect to the Lebesgue measure, is called a Sinai-Ruelle-Bowen (SRB) measure. Later, some interesting and deep results have been obtained for non-uniformly hyperbolic systems by Pesin [2, 3], singular systems by Katok et al. [16], lots of work on the billiard systems [9], piecewise twice-differentiable diffeomorphisms with unbounded derivatives by Jakobson and Newhouse [14], transformations with infinitely many hyperbolic branches by Sánchez-Salas [24], and so on. For more information on SRB measures, please refer to Young’s work [31].

A powerful tool to study the existence of SRB measures and some statistical properties of the systems is the transfer operator approach combined with some function spaces [25, 29, 30]. Some discussions on the statistical properties of two-dimensional piecewise smooth dynamical systems could be found in [11]. The well-known Lasota-Yorke inequality and some generalization contributed to the development of the chaos theory greatly [8, 18]. A useful construction is the “Young
The piecewise maps can be used to represent some interesting systems in physics [6], and have practical applications [12, 15, 17]. The Lozi map is a very important model in the development of the theory of dynamical systems [10, 27]. In [27], Young studied the existence of SRB measures for a type of piecewise $C^2$ Lozi-like maps in two dimensions. In [12], Elhadj summarized both the theory and applications of two-dimensional Lozi map since its discovery in 1978. In our present work, we provide some rigorous parameter conditions for maps with the existence of SRB measures and chaotic dynamics, which can be thought of as the generalization of Lozi maps in three-dimensional spaces. In [21], Pesin studied a type of systems on a Riemannian manifold with singularities and strong hyperbolic dynamical behavior, including the generalized Lozi maps, and proved the existence of Gibbs $u$-measures, which are analogous to SRB measures for classical hyperbolic attractors, and the method is a generalization of that used in [22]. In [1], Baladi and Gouëzel studied the piecewise cone-hyperbolic systems satisfying a bunching condition and consider the transfer operators acting on anisotropic Sobolev spaces. A complete description of the SRB measures for two-dimensional piecewise hyperbolic maps could also be found in [11]. Some results for our present work can be obtained by the methods used in [1, 21], we provide the checkable conditions and different approaches to show the existence of SRB measures. In our present work, we apply the bounded variation function method [19] to study the existence of SRB measures. This method can be used to construct chaotic piecewise maps with several directions of instability in higher-dimensional spaces.

The rest is organized as follows. In Section 2, the main result is introduced. In Section 3, the proof of the main result is provided. In Section 4, an example with computer simulations is given to illustrate the theoretical results.

2. Main results. In this section, the main results are introduced and some basic concepts and lemmas are given.

Given any $0 = a_0 < a_1 < \cdots < a_p < a_{p+1} = 1$ and $0 = b_0 < b_1 < \cdots < b_q < b_{q+1} = 1$, denote

$$S_1 := \{a_1, ..., a_p\} \times [0, 1] \times [0, 1], \quad S_2 := [0, 1] \times \{b_1, ..., b_q\} \times [0, 1],$$

and

$$S := S_1 \cup S_2, \quad R := [0, 1]^3. \quad \quad \quad (1)$$

There is a natural partition of the set $[0, 1]^2 \setminus (\{a_0, a_1, ..., a_p, a_{p+1}\} \times [0, 1]) \cup ([0, 1] \times \{b_0, b_1, ..., b_q, b_{q+1}\}))$. Without loss of generality, suppose this partition is

$$\bigcup_{k=1}^{(p+1)(q+1)} \Omega_k = [0, 1]^2 \setminus (\{a_0, a_1, ..., a_p, a_{p+1}\} \times [0, 1] \cup ([0, 1] \times \{b_0, b_1, ..., b_q, b_{q+1}\})),$$

where $\Omega_{k_1} \cap \Omega_{k_2} = \emptyset$, if $k_1 \neq k_2$, and $\bigcup \Omega_k = [0, 1]^2$. Assume that

$$\Omega_k \cap \{(x, u)\; u \in [0, 1] = \{x\} \times (a_k(x), b_k(x)), \forall x \in [0, 1], \; 1 \leq k \leq (p+1)(q+1);$$

and

$$\Omega_k \cap \{(u, y)\; u \in [0, 1] = (c_k(y), d_k(y)) \times \{y\}, \forall y \in [0, 1], \; 1 \leq k \leq (p+1)(q+1).$$
So, for any $1 \leq k \leq (p + 1)(q + 1)$, one has
\[ \prod_k \cap \{ (x, u) \}_{u \in [0, 1]} = x \times [a_k(x), b_k(x)] \text{ and } \prod_k \cap \{ (u, y) \}_{u \in [0, 1]} = [c_k(y), d_k(y)] \times \{ y \} . \]

Consider a map $f$ on $R$ satisfying that $f(R) \subset R$ and $| \det(D(f)) | < 1$, and

(A0). $f \mid (R-S)$ is a twice-differentiable map, $f \mid (R-S)$ and $f^{-1} \mid (R-S)$ have bounded first and second derivatives, respectively.

(A1). $\inf \left\{ \left| \frac{\partial f_1}{\partial x} - \frac{\partial f_1}{\partial y} \right|, \left| \frac{\partial f_2}{\partial x} - \frac{\partial f_2}{\partial y} \right| \right\} = \lambda > 1 . \]

(A2). $\sup \left\{ \left| \frac{\partial f_1}{\partial z} \right|, \left| \frac{\partial f_2}{\partial z} \right|, \left| \frac{\partial f_3}{\partial x} \right|, \left| \frac{\partial f_3}{\partial y} \right| \right\} \leq \frac{1}{\lambda} . \]

(A3). There is $N \in \mathbb{N}$ such that $\lambda^N > 2$ and $f_1(f^{k-1}(S_1)) \cap \{ a_1, ..., a_p \} = \emptyset$, $f_2(f^{k-1}(S_2)) \cap \{ b_1, ..., b_q \} = \emptyset$, $1 \leq k \leq N . \]

Remark 1. The assumption (A1) means that the action of $Df$ when projected onto the $x$-axis and $y$-axis is uniformly expanding, respectively. An example satisfying all the assumptions above is given in the last section.

Let $(X, B)$ be a compact metric space with Borel $\sigma$-algebra, $M(X)$ be the set of Borel probability measures on $X$. Assume that $C(X)$ is the set of continuous real-valued functions on $X$ with the sup-norm, which is a Banach space. Let $C(X)^*$ be the dual space, that is, the space of continuous linear functionals on $C(X)$. The weak star topology on $M$ is defined as $\lim_{n \to \infty} \mu_n = \mu$ if and only if $\int \phi dm_n = \int \phi dm$ for any $\phi \in C(X)$. With the weak star topology, one has that $M(X)$ is compact and the set of invariant Borel probability measures with respect to a continuous map is non-empty [28].

Since the map has singular points, the global unstable manifolds might not exist, it is still possible to define the local unstable manifolds [28]. So, we adopt the following definition for the SRB measures [27]:

Definition 2.1. An $f$-invariant Borel probability measure $\mu$ on $R$ is said to be an SRB measure if

(i) $(f, \mu)$ has positive Lyapunov exponents $\mu$-almost everywhere;
(ii) there exist a sequence of measurable partitions $\{ P_n \}_{n \in \mathbb{N}}$ of $R$ with $P_n \subset P_{n+1}$, $n \geq 1$, and a sequence of measurable sets $\{ V_n \}$ with $V_n \subset V_{n+1}$, $n \geq 1$, such that

(a) $\mu(V_n) \to 1$ as $n \to \infty$;
(b) each element of $P_n \cap V_n$ is an open subset of some unstable manifold;
(c) for $\gamma \in P_n \cap V_n$, let $\mu_\gamma$ denote the conditional measure of $\mu$ on $\gamma$, and $m_\gamma$ be the Riemannian measure on $\gamma$, then $\mu_\gamma$ is absolutely continuous with respect to $m_\gamma$, denoted by $\mu_\gamma \ll m_\gamma$.

Now, we introduce the bounded variation functions [13, 19]. Let $m$ be the Lebesgue measure on $\mathbb{R}^2$ and $h = h(x, y) \in L^1(m)$. Set

$||h||_{BV} := \sup_{\psi \in C^r_0(\mathbb{R}^2, \mathbb{R}^2), \| \psi \|_{\infty} \leq 1} \int_{\mathbb{R}^2} h \cdot \text{div} \psi \, dm,$

where $\psi = (\psi_1(x,y), \psi_2(x,y))$, $\text{div} \psi = \partial_x \psi_1 + \partial_y \psi_2$, and $C^r_0$ represents the vector space of $r$-times differentiable functions with compact support. The bounded variation functions are a subset of $L^1$ with $|| \cdot ||_{BV}$ finite.
Lemma 2.2. [19, Lemma A.1]

(i) There exists a constant $C_0 > 0$ such that for any $h \in BV(\mathbb{R}^2)$ and $\Omega \subset \mathbb{R}^2$,
$$\|\mathbb{1}_\Omega h\|_{L^1} \leq m(\Omega)^{1/2}\|h\|_{L^2} \leq C_0 m(\Omega)^{1/2}\|h\|_{BV};$$

(ii) for any $h \in BV(\mathbb{R}^2)$, for almost $x$ and $y$,
$$h(x, \cdot), h(\cdot, y) \in BV(\mathbb{R}) \subset L^\infty(\mathbb{R}, m);$$

(iii) for each $h \in BV(\mathbb{R}^2)$ and all $\psi = (\psi_1, \psi_2) \in L^\infty(\mathbb{R}^2, m)$ of compact support satisfying that almost surely, $\psi_1(\cdot, y)$ and $\psi_2(x, \cdot) \in C^0(\mathbb{R}, \mathbb{R})$, and $\partial_x \psi_1(\cdot, y)$ and $\partial_y \psi_2(x, \cdot) \in L^1(\mathbb{R}, m)$, one has
$$\left| \int_{\mathbb{R}^2} h \cdot \text{div} \psi \, dm \right| \leq \|h\|_{BV} \|\psi\|_{L^\infty}.$$

The main result is stated as follows:

Theorem 2.3. For the map $f$ satisfying (A0)–(A3), there exists an invariant measure, which is an SRB measure.

3. The existence of SRB measures. In this section, it is to show Theorem 2.3.

It follows from the definition of the map $f$ that the unstable manifolds are piecewise smooth surfaces zigzag across $R$, which are turning around at unknown places. To avoid the singular set $S$ specified in (1), the strategy is to construct an invariant measure $\mu$ with good dynamical behavior on a neighborhood of the singular set.

For any subset $A \subset [0, 1]^2$ and twice-differentiable function $\alpha : A \to [0, 1]$, the graph of $\alpha$ is denoted by
$$\text{Graph} (\alpha) := \{ (x, y, \alpha(x, y)) : (x, y) \in A \}.$$ 

Lemma 3.1. Given any surface represented by the graph of a twice-differentiable function $\alpha : [0, 1]^2 \to [0, 1]$, suppose that the map $f$ satisfies (A1) and (A2). If the angle between the normal vector of the surface and the $z$-axis (including both the positive and negative axes) is less than 45 degrees, then the angle between the normal vector of $f(\text{Graph}(\alpha))$ and the $z$-axis is also less than 45 degrees, except points contained in the image of the singular set $S$.

Proof. The graph of $\alpha$ is $(x, y, \alpha(x, y))$. The normal vector is the cross product of the vectors $< 1, 0, \alpha_x >$ and $< 0, 1, \alpha_y >$, that is, $< -\alpha_x, -\alpha_y, 1 >$. The cosine of the angle between the normal vector and the $z$-axis is $\frac{1}{\sqrt{1 + \alpha_x^2 + \alpha_y^2}}$ or $\frac{-1}{\sqrt{1 + \alpha_x^2 + \alpha_y^2}}$. The assumption that the angle between the normal vector and the $z$-axis is less than 45 degrees is equivalent to
$$\left| \frac{1}{\sqrt{1 + \alpha_x^2 + \alpha_y^2}} \right| \geq \frac{\sqrt{2}}{2}. \quad (2)$$

Since $f(\text{Graph}(\alpha)) = (f_1(x, y, \alpha), f_2(x, y, \alpha), f_3(x, y, \alpha))$, one has that the tangent vectors are
$$\left( \frac{\partial f_1}{\partial x} + \frac{\partial f_1}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial f_2}{\partial \alpha} \frac{\partial \alpha}{\partial x} + \frac{\partial f_3}{\partial \alpha} \frac{\partial \alpha}{\partial x} \right)$$
and
$$\left( \frac{\partial f_1}{\partial y} + \frac{\partial f_1}{\partial \alpha} \frac{\partial \alpha}{\partial y} + \frac{\partial f_2}{\partial \alpha} \frac{\partial \alpha}{\partial y} + \frac{\partial f_3}{\partial \alpha} \frac{\partial \alpha}{\partial y} \right).$$
Phic maps:

with respect to the map \( f \) that is, a sequence of smooth surfaces \( \{ f_i \} \). It follows from this definition that it is reasonable to define the following set:

\[
\int \left[ \frac{x}{y} \right] \text{d}x \text{d}y
\]

So, the cross product is

\[
\left[ \left( \frac{\partial f_2}{\partial x} \frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial x} \frac{\partial f_1}{\partial y} \right) + \left( \frac{\partial f_2}{\partial x} \frac{\partial f_3}{\partial z} - \frac{\partial f_3}{\partial x} \frac{\partial f_1}{\partial z} \right) \frac{\partial \alpha}{\partial y} + \left( \frac{\partial f_2}{\partial y} \frac{\partial f_3}{\partial z} - \frac{\partial f_3}{\partial y} \frac{\partial f_1}{\partial z} \right) \frac{\partial \alpha}{\partial x} \right]^j
\]

\[
- \left[ \left( \frac{\partial f_1}{\partial x} \frac{\partial f_3}{\partial y} - \frac{\partial f_3}{\partial x} \frac{\partial f_1}{\partial y} \right) + \left( \frac{\partial f_1}{\partial x} \frac{\partial f_3}{\partial z} - \frac{\partial f_3}{\partial x} \frac{\partial f_1}{\partial z} \right) \frac{\partial \alpha}{\partial y} + \left( \frac{\partial f_1}{\partial y} \frac{\partial f_3}{\partial z} - \frac{\partial f_3}{\partial y} \frac{\partial f_1}{\partial z} \right) \frac{\partial \alpha}{\partial x} \right]^j
\]

\[
+ \left[ \left( \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial y} \right) + \left( \frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial z} - \frac{\partial f_2}{\partial x} \frac{\partial f_1}{\partial z} \right) \frac{\partial \alpha}{\partial y} + \left( \frac{\partial f_1}{\partial y} \frac{\partial f_2}{\partial z} - \frac{\partial f_2}{\partial y} \frac{\partial f_1}{\partial z} \right) \frac{\partial \alpha}{\partial x} \right]^k
\]

:= A\vec{i} + B\vec{j} + C\vec{k}.

The absolute value of the cosine of the angle between the normal vector and the z-axis is \( \frac{|C|}{\sqrt{A^2 + B^2 + C^2}} \). By (A1), (A2), and (2), one has that \( |C| \geq |A| + |B| \), which implies that \( \frac{|C|}{\sqrt{A^2 + B^2 + C^2}} \geq \frac{\sqrt{2}}{2} \). Hence, the angle between the normal vector and the z-axis is less than 45 degrees.

This completes the proof. \( \square \)

In the following discussions, assume that \( f \) satisfies (A0)-(A3).

Let \( p_x : R \rightarrow [0,1] \) and \( p_y : R \rightarrow [0,1] \) be the projection onto the x-axis and y-axis, respectively. The Lebesgue measure on \([0,1]^2\) is denoted by \( m \). Let \( J = [a, b] \times [c, d] \subset [0,1]^2 \) be a closed rectangle and \( \alpha : J \rightarrow [0,1] \) be a twice-differentiable function with the angle between the normal vector and the z-axis less than 45 degrees. Set \( J_k := J \cap \mathbb{R}^2_k \) and \( L_k \) as a part of Graph(\( \alpha \)) with \( (x, y) \in J_k \). Without loss of generality, we only consider the sets \( J_k \) satisfying \( m(J_k) > 0 \). By (A1), one has that \( f(L_k) \) is homeomorphic to \( (p_x \times p_y)(f(L_k)) \). By Lemma 3.1, the angle between the normal vector of \( f(L_k) \) and the z-axis is less than 45 degrees. So, it is reasonable to define the following set:

\[
L_{k1} := \left( (p_x \times p_y)^{-1} \left( \left( |(p_x \times p_y)(f(L_k))| \cap \mathbb{R}^2_x \right) \right) \right) \cap f(L_k).
\]

It follows from this definition that \( f(L_k) = \bigcup L_{k1} \). By (A0) and (A1), \( f \) is differentiable on \( L_{k1} \) and \( L_{k1} \) is homeomorphic to \( f(L_{k1}) \). Inductively, one could find a sequence of smooth surfaces \( \{ L_{k1} \} \) such that \( f(L_{k1}) = \bigcup_{j \geq 1} L_{k1} \), and the angle between the normal vector of \( L_{k1} \) and the z-axis is less than 45 degrees.

In the following discussions, assume that \( N = 1 \), that is, \( \lambda > 2 \). If \( N > 1 \), we could define the sets \( L_{k1} \) and \( L_{k1} \) as above with respect to the map \( f \).

By (A1), (A2), and the discussions above, there exist two homeomorphism maps:

\[
H_{k1} : L_{k1} \rightarrow (p_x \times p_y)L_{k1} \quad \text{and} \quad H_{k2} : f(L_{k1}) \rightarrow (p_x \times p_y)f(L_{k1}).
\]

So, one could introduce a homeomorphic map \( T_{k1} \) as follows:

\[
T_{k}(x, y) := H_{k1} \circ f \circ H_{k2}^{-1}(x, y).
\]

By (A0) and (A1), one has that \( T_{k1} \) is twice-differentiable in the interior of \( (p_x \times p_y)L_{k1} \), and

\[
\frac{\partial T_{k1}}{\partial x} \geq \lambda \quad \text{and} \quad \frac{\partial T_{k1}}{\partial y} \geq \lambda,
\]

where

\[
T_{k} = (T_{k1}, T_{k2}).
\]
Since $T_k$ is homeomorphic, denote the inverse map of $T_k$ as

$$T_k^{-1} := (T_k^{-1}, T_k^{-1}).$$

**Lemma 3.2.** For any given $L_{i_1, i_2, \ldots, i_k}$ and any $w_1$ and $w_2$ with $T_l \circ T_{l-1} \circ \cdots \circ T_1(w_1), T_l \circ T_{l-1} \circ \cdots \circ T_1(w_2) \in L_{i_1, i_2, \ldots, i_k}$, $1 \leq l \leq k-1$, one has that

$$\frac{\det D(T_k^{-1} \circ T_{k-2} \circ \cdots \circ T_1)(w_1)}{\det D(T_k^{-1} \circ T_{k-2} \circ \cdots \circ T_1)(w_2)} \leq \exp^{C_l T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_1) - T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_2)},$$

where $C_l$ is a positive constant independent on $k$.

**Proof.** For any $w_1, w_2 \in L_{i_1, i_2, \ldots, i_k}$, it follows from the assumption (A0) that

$$|\det DT_i(w_1) - \det DT_i(w_2)|$$

$$= \left| \det \begin{pmatrix} \partial_x T_{i,1}(w_1) & \partial_y T_{i,1}(w_1) \\ \partial_x T_{i,2}(w_1) & \partial_y T_{i,2}(w_1) \end{pmatrix} \right| - \left| \det \begin{pmatrix} \partial_x T_{i,1}(w_2) & \partial_y T_{i,1}(w_2) \\ \partial_x T_{i,2}(w_2) & \partial_y T_{i,2}(w_2) \end{pmatrix} \right|$

$$= |(\partial_x T_{i,1}(w_1) \partial_y T_{i,2}(w_1) - \partial_y T_{i,1}(w_1) \partial_x T_{i,2}(w_1)) - (\partial_x T_{i,1}(w_2) \partial_y T_{i,2}(w_2) - \partial_y T_{i,1}(w_2) \partial_x T_{i,2}(w_2))|$$

$$\leq |\partial_x T_{i,1}(w_1)(\partial_y T_{i,2}(w_1) - \partial_y T_{i,2}(w_2)) + |\partial_y T_{i,2}(w_2)(\partial_x T_{i,1}(w_1) - \partial_x T_{i,1}(w_2))|$$

$$+ |\partial_y T_{i,1}(w_1)(\partial_x T_{i,2}(w_1) - \partial_x T_{i,2}(w_2)) + |\partial_x T_{i,2}(w_2)(\partial_y T_{i,1}(w_1) - \partial_y T_{i,1}(w_2))|$$

$$\leq C_1|w_1 - w_2|,$$

where $C_1$ is a positive constant only dependent on $f$. By (4), there is a positive constant $C_2$ such that

$$|\det D(T_k)| \geq C_2,$$

where $C_2$ is independent on $k$.

So, by (4), (6), and (7), one has

$$\log \frac{\det D(T_k^{-1} \circ T_{k-2} \circ \cdots \circ T_1)(w_1)}{\det D(T_k^{-1} \circ T_{k-2} \circ \cdots \circ T_1)(w_2)}$$

$$\leq \sum_{i=1}^{k-1} |\log \det DT_i(T_{i-1} \circ \cdots \circ T_1)(w_1) - \log \det DT_i(T_{i-1} \circ \cdots \circ T_1)(w_1)|$$

$$\leq \sum_{i=1}^{k-1} C_3|T_{i-1} \circ \cdots \circ T_1(w_1) - T_{i-1} \circ \cdots \circ T_1(w_2)|$$

$$\leq \sum_{i=1}^{k-1} \lambda^{l-k+1} C_3 |T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_1) - T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_2)|$$

$$\leq C_0 |T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_1) - T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_2)|.$$

This completes the proof. \qed

Next, it is to study the density of the invariant measure on the unstable manifolds. And, it is to show that the density has bounded variation.

**Lemma 3.3.** For the given surface $\alpha$ as above, there exist an invariant Borel probability measure $\mu$ on $\alpha$ and a function $\rho : [0, 1]^2 \to \mathbb{R}$ of bounded variation such that $d((p_x \times p_y)_* \mu) = \rho \mu.$
Proof. If $\mu$ is any Borel measure on $R$, then $f_*\mu$ is defined by $f_*\mu(E) = \mu(f^{-1}(E))$, where $E$ is a Borel set. Let $\mu_0$ be a Borel measure on the surface $\alpha$ such that $(p_x \times p_y)_*\mu_0$ is the normalized Lebesgue measure on $J$. Set $\mu_k := (f^k)_*\mu_0$. The density of $(p_x \times p_y)_*\mu_k$ on $(p_x \times p_y)L_{i_1i_2\cdots i_k}$ and $(p_x \times p_y)_*\mu_k$ on $J$ are denoted by $\rho_{i_1i_2\cdots i_k}$ and $\hat{\rho}_k$, respectively. Since $f^k(\text{Graph}(\alpha)) = \bigcup_{i_1, \ldots, i_k} L_{i_1i_2\cdots i_k}$, one has that $
abla_{i_1i_2\cdots i_k} \rho_{i_1i_2\cdots i_k} = \hat{\rho}_k$.

Now, it is to show that there are two positive constants $C_4$ and $C_5$ such that

$$C_4 \leq \hat{\rho}_k \leq C_5.$$  \hspace{1cm} (8)

By Lemma 3.2, one has

$$\frac{\rho_{i_1i_2\cdots i_k}(w_1)}{\rho_{i_1i_2\cdots i_k}(w_2)} \leq \exp C_0|T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_1) - T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_2)|,$$

where $T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1 \in L_{i_1 \cdots i_k}$. By the fact that for any two finite positive sequences $\{a_i\}_{i \in I}$ and $\{b_i\}_{i \in I}$, $\sum a_i \leq \max b_i$, one has that

$$\hat{\rho}_k(w_1) \leq \exp C_0|T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_1) - T_{k-1} \circ T_{k-2} \circ \cdots \circ T_1(w_2)|.$$

This, together with $\int \hat{\rho}_k \mu_0 = 1$, yields that (8) holds.

Now, it is to show that there is a positive constant $C_d$ such that $\|\hat{\rho}_k+1\|_{BV([0,1]^2)} \leq C_d$ for any $k \geq 0$.

First, it is to show an inequality:

$$\|\hat{\rho}_k+1\|_{BV([0,1]^2)} \leq C_d + \frac{1}{\tau} \|\hat{\rho}_k\|_{BV([0,1]^2)}, \quad \forall k \geq 1,$$  \hspace{1cm} (9)

where $C_d > 0$ and $\tau > 1$ are two constants determined later.

Since $\sum_{i_1, i_2, \ldots, i_k} \rho_{i_1i_2\cdots i_k} = \hat{\rho}_k$, one has

$$\|\hat{\rho}_k+1\|_{BV([0,1]^2)} = \sum_{i_1, i_2, \ldots, i_k} \sum_j \|\rho_{i_1i_2\cdots i_k j} \|_{BV([0,1]^2)}.$$

Fix any $L_{i_1i_2\cdots i_k}$ and suppose that $m((p_x \times p_y)L_{i_1i_2\cdots i_k}) > 0$. For convenience, for $T_k$ in (3), suppose that $T = T_k$ and $(T_1, T_2) = (T_{k-1}, T_{k-2})$, and $T^{-1} = T_{k-1}^{-1}$ and $(T_{k-1}^{-1}, T_{k-2}^{-1}) = (T_{k-1}, T_{k-2}).$

It follows from direct calculation and the fact that the interior of $(p_x \times p_y)L_{i_1i_2\cdots i_k}$ is disjoint with respect to the index $j$ that

$$\sum_j \|\rho_{i_1i_2\cdots i_k j} \|_{BV([0,1]^2)} = \sum_j \int_0^1 \int_0^1 \rho_{i_1i_2\cdots i_k j}(x,y) \left( \frac{\partial \phi_1(x,y)}{\partial x} + \frac{\partial \phi_2(x,y)}{\partial y} \right) dx dy$$

$$= \int_0^1 \int_0^1 \sum_j \frac{\rho_{i_1i_2\cdots i_k j}(T_{k-1}^{-1}(x,y), T_{k-2}^{-1}(x,y))}{\det(DT(T_{k-1}^{-1}(x,y), T_{k-2}^{-1}(x,y)))} \left( \frac{\partial \phi_1(x,y)}{\partial x} + \frac{\partial \phi_2(x,y)}{\partial y} \right) dx dy$$

$$= \int_0^1 \int_0^1 \rho_{i_1i_2\cdots i_k j}(x,y) \left( \frac{\partial \phi_1}{\partial x} \circ T + \frac{\partial \phi_2}{\partial y} \circ T \right) dx dy,$$

where $\phi = (\phi_1, \phi_2)$ with $\phi \in C_1^0(\mathbb{R}^2, \mathbb{R}^2)$ and $\|\phi\|_{\infty} \leq 1$.

By the chain rules, one has

$$\left( \frac{\partial (\phi_1 \circ T)}{\partial \phi_1 \circ T} \right) = \left( \frac{\partial T_1}{\partial x} \frac{\partial T_2}{\partial y} \right) \left( \frac{\partial \phi_1}{\partial x} \circ T \right), \quad i = 1, 2.$$  \hspace{1cm} (10)
Denote
\[ DT := \left( \begin{array}{cc} \frac{\partial T_1}{\partial x} & \frac{\partial T_1}{\partial y} \\ \frac{\partial T_2}{\partial x} & \frac{\partial T_2}{\partial y} \end{array} \right) \] (11)
and
\[ DT^{-1} = \frac{1}{\partial_x T_1 \partial_y T_2 - \partial_x T_2 \partial_y T_1} \left( \frac{\partial T_2}{\partial x} \frac{\partial T_1}{\partial y} - \frac{\partial T_2}{\partial y} \frac{\partial T_1}{\partial x} \right) := \left( \begin{array}{cc} \[(DT)^{-1}\]_{11} & \[(DT)^{-1}\]_{12} \\ \[(DT)^{-1}\]_{21} & \[(DT)^{-1}\]_{22} \end{array} \right). \] (12)

By (10) and (12), one has
\[ \frac{\partial \phi_1}{\partial x} \circ T + \frac{\partial \phi_2}{\partial y} \circ T \]
\[ = \left( \[(DT)^{-1}\]_{11} \frac{\partial (\phi_1 \circ T)}{\partial x} + \[(DT)^{-1}\]_{21} \frac{\partial (\phi_2 \circ T)}{\partial x} \right) \]
\[ + \left( \[(DT)^{-1}\]_{12} \frac{\partial (\phi_1 \circ T)}{\partial y} + \[(DT)^{-1}\]_{22} \frac{\partial (\phi_2 \circ T)}{\partial y} \right) \]
\[ = \frac{\partial}{\partial x} \left( \[(DT)^{-1}\]_{11} \phi_1 \circ T + \[(DT)^{-1}\]_{21} \phi_2 \circ T \right) \]
\[ + \frac{\partial}{\partial y} \left( \[(DT)^{-1}\]_{12} \phi_1 \circ T + \[(DT)^{-1}\]_{22} \phi_2 \circ T \right) \]
\[ - \left( \phi_1 \circ T \frac{\partial}{\partial x} \[(DT)^{-1}\]_{11} \right)_{11} + \phi_2 \circ T \frac{\partial}{\partial x} \[(DT)^{-1}\]_{21} \]
\[ + \phi_1 \circ T \frac{\partial}{\partial y} \[(DT)^{-1}\]_{12} + \phi_2 \circ T \frac{\partial}{\partial y} \[(DT)^{-1}\]_{22}. \]

For any \( l, 1 \leq l \leq (p + 1)(q + 1) \), and any \((x, y) \in [0, 1]^2 \setminus \{(a_1, \ldots, a_p) \times [0, 1] \cup (0, 1) \times \{b_1, \ldots, b_q\}\}\), set
\[ \Phi_{1,l}(x, y) := \left[ (DT)^{-1} \right]_{11} \phi_1 \circ T(x, y) \mathbb{I}_{[c_l(y), d_l(y)]}(x) + \left[ (DT)^{-1} \right]_{21} \phi_2 \circ T(x, y) \mathbb{I}_{[c_l(y), d_l(y)]}(y), \]
\[ \Phi_{2,l}(x, y) := \left[ (DT)^{-1} \right]_{12} \phi_1 \circ T(x, y) \mathbb{I}_{[a_l(x), b_l(x)]}(y) + \left[ (DT)^{-1} \right]_{22} \phi_2 \circ T(x, y) \mathbb{I}_{[a_l(x), b_l(x)]}(y), \]
\[ \Phi_{1,l}(y) := \Phi_{1,l}(c_l(y), y), \quad \Phi_{1,l}^{+}(y) := \Phi_{1,l}(d_l(y), y), \]
\[ \Phi_{2,l}(x) := \Phi_{2,l}(x, a_l(x)), \quad \Phi_{2,l}^{+}(x) := \Phi_{2,l}(x, b_l(x)). \]

It follows from (12), (A1), (A2), and (A3) that there exist \( \delta > 0 \) and \( \tau > 1 \) such that
\[ \sup_{(x, y) \in [0, 1]^2} \sup_{l: [x - \delta, x + \delta] \cap [c_l(y), d_l(y)] \neq \emptyset} \left\{ \sum_{l: [x - \delta, x + \delta] \cap [c_l(y), d_l(y)] \neq \emptyset} \right\} \leq \tau^{-1}. \] (13)

Denote
\[ \eta_{1,l,g}(v) := \begin{cases} 0 & \text{if } v \in (-\infty, c_l(y) - \delta) \\ \Phi_{1,l}(v - c_l(y) + \delta) & \text{if } v \in [c_l(y) - \delta, c_l(y)) \\ 0 & \text{if } v \in [c_l(y), d_l(y)] \\ \Phi_{1,l}^{+}(d_l(y) + \delta - v) & \text{if } v \in (d_l(y), d_l(y) + \delta) \\ 0 & \text{if } v \in [d_l(y) + \delta, +\infty) \end{cases}, \]
Thus, by Lemma 2.2, one has that
\[
\eta_{2,l,x}(v) := \begin{cases} 
0 & \text{if } v \in (-\infty, a_l(x) - \delta) \\
\Phi^-_{2,l}(x)(v - a_l(x) + \delta)\delta^{-1} & \text{if } v \in [a_l(x) - \delta, a_l(x)] \\
0 & \text{if } v \in [a_l(x), b_l(x)] \\
\Phi^+_{2,l}(x)(b_l(x) + \delta - v)\delta^{-1} & \text{if } v \in (b_l(x), b_l(x) + \delta) \\
0 & \text{if } v \in [b_l(x) + \delta, +\infty) 
\end{cases} 
\]

Set
\[
\Phi_{1,l}(x, y) := \Phi_{1,l}(x, y) + \eta_{1,l,y}(x), \quad \Phi_{2,l}(x, y) := \Phi_{2,l}(x, y) + \eta_{2,l,x}(y), 
\]
\[
\Theta_1(x, y) := \sum_{l=1}^{(p+1)(q+1)} \Phi_{1,l}(x, y), \quad \Theta_2(x, y) := \sum_{l=1}^{(p+1)(q+1)} \Phi_{2,l}(x, y). 
\]

By the construction above, $\Phi_{1,l}$ and $\Phi_{2,l}$ are continuous functions, and for $(x, y) \in [0, 1]^2 \setminus (\{ a_1, ..., a_p \} \times [0, 1]) \cup ([0, 1] \times \{ b_1, ..., b_q \})$, by (13) and $\| \phi \|_{\infty} \leq 1$, one has
\[
\max\{|\Theta_1(x, y)|, |\Theta_2(x, y)|\} \leq \sup \left\{ \sum_l \left( \sup_{x \in [c_l(y), d_l(y)]} \Phi_{1,l}(x, y) \right) \mathbb{I}_{[c_l(y), d_l(y)]}(x) \right\} \leq \tau^{-1}. 
\]

Further, set
\[
\Theta_1(x, y) := \sum_l \int_0^x \partial_v \Phi_{1,l}(v, y) dv = \int_0^x \sum_l \partial_v \Phi_{1,l}(v, y) dv, 
\]
\[
\Theta_2(x, y) := \sum_l \int_0^y \partial_v \Phi_{2,l}(x, v) dv = \int_0^y \sum_l \partial_v \Phi_{2,l}(x, v) dv, 
\]
and
\[
\Theta(x, y) := (\Theta_1(x, y), \Theta_2(x, y)). 
\]

Hence, one has that
\[
\text{div} \Theta(x, y) = \partial_x \Theta_1 + \partial_y \Theta_2 
\]
\[
= \sum_l \left( \partial_x \left( [(DT)^{-1}]_{11} \phi_1 \circ T + [(DT)^{-1}]_{21} \phi_2 \circ T \right) \mathbb{I}_{[c_l(y), d_l(y)]}(x) 
\]
\[
+ \partial_y \left( [(DT)^{-1}]_{12} \phi_1 \circ T + [(DT)^{-1}]_{22} \phi_2 \circ T \right) \mathbb{I}_{[a_l(x), b_l(x)]}(y) \right) 
\]
\[
+ \delta^{-1} \sum_l \left( \Phi_{1,l}^+(y) \mathbb{I}_{[c_l(y) - \delta, c_l(y)]}(x) - \Phi_{1,l}^-(y) \mathbb{I}_{[d_l(y), d_l(y) + \delta]}(x) 
\]
\[
+ \Phi_{2,l}^+(x) \mathbb{I}_{[a_l(x) - \delta, a_l(x)]}(y) - \Phi_{2,l}^-(x) \mathbb{I}_{[b_l(x), b_l(x) + \delta]}(y) \right). 
\]

Thus, by Lemma 2.2, one has that
\[
\sum_j \int_0^1 \int_0^1 \rho_{i_1 \cdots i_k} \circ \text{div} \phi dy dx \leq \int_0^1 \int_0^1 \rho_{i_1 \cdots i_k} \cdot \text{div} \Theta dy dx + \frac{2\|\rho_{i_1 \cdots i_k}\|_{L^1}}{\tau \delta} 
\]
\[
+ \int_0^1 \int_0^1 \rho_{i_1 \cdots i_k} \sum_l \left( \phi_1 \circ T \frac{\partial}{\partial x} [(DT)^{-1}]_{11} + \phi_2 \circ T \frac{\partial}{\partial x} [(DT)^{-1}]_{21} \right) 
\]
Hence, for

Thus, one has that

where \( C_6 = \frac{2}{\tau} + C_7 \).

Hence, one has that \((9)\) holds. Hence, for any \( k \),

\[
\| \hat{\rho}_k \|_{BV([0,1]^2)} \leq C_6 \sum_{i=0}^{\infty} \left( \frac{1}{r} \right)^i := C_d < \infty.
\]

Thus, one has that

\[
\left\| \frac{1}{n} \sum_{k=1}^{n} \hat{\rho}_k \right\|_{BV} \leq C_d, \quad \forall n \geq 1.
\]

This, together with Lemma 2.2, implies that there is a convergent subsequence of \( \{n^{-1} \sum_{k=1}^{n} \hat{\rho}_k \}_{n \in \mathbb{N}} \) in \( L^1([0,1]^2, m) \), denoted by \( \rho \). By \((8)\),

\[
C_4 \leq \rho \leq C_5.
\]

So, the limit measure denoted by \( \mu \) is absolutely continuous with respect to the Lebesgue measure, and is a Borel measure.

This completes the proof. \( \square \)

For any invariant measure \( \nu \), suppose that \( \nu = \rho, m \) and there is a positive constant \( D_\nu \) such that \( |\rho_\nu| \leq D_\nu \), where \( m \) is the Lebesgue measure of \( R \). One has

\[
\sum_{k=0}^{\infty} \nu(f^k(D(S, \delta \lambda^{-k}))) \leq E_0 \sum_{k=0}^{\infty} \nu(D(S, \delta \lambda^{-k})) \leq 4(p+1)(q+1)\delta D_\nu E_0 \sum_{k=0}^{\infty} \lambda^{-k} < \infty,
\]

where \( \delta > 0 \) and \( D(S, \delta) \) is the \( \delta \)-neighborhood of the singular set \( S \), and \( E_0 > 0 \) is a constant dependent on the degree of \( f \). It follows from the Borel-Cantelli Lemma that \( w \) is in \( f^k(D(S, \delta \lambda^{-k})) \) for at most finitely many \( k \), \( \nu \)-almost everywhere. Hence, for \( \nu \)-almost everywhere, there are \( w \) and \( \delta(w) > 0 \) such that \( f^{-k}(w) \notin D(S, \delta(w) \lambda^{-k}) \) for all \( k > 0 \), implying the existence of local unstable manifold \( W^u_{\delta(w)}(w) \) \( \nu \)-almost everywhere by [16], where the existence of negative Lyapunov exponents is derived from the assumption that \( |\det(Df)| < 1 \). This, together with \((4)\) and the method in the classical Pesin theory \([3]\), yields the existence of positive Lyapunov exponents almost everywhere with respect to the measure \( \nu \).

Pick some \( w \) such that the local unstable manifold exists, denoted by \( W^u_{\delta(w)}(w) \). Let \( W^u_{\delta(w)}(w) \) be the smooth surface \( \alpha \) introduced as above, the angle between the normal vector of this surface and the \( z \)-axis is less than 45 degrees can be derived from \((A1)\) and \((A2)\). Let \( \mu \) be the invariant measure on \( W^u_{\delta(w)}(w) \) given by Lemma 3.3.

Finally, it is to show the following result.

**Lemma 3.4.** There exist a sequence of measurable partitions \( \{P_n\}_{n \in \mathbb{N}} \) of \( R \) with \( P_n \subset P_{n+1}, \ n \geq 1, \) and a sequence of measurable sets \( \{V_n\} \) with \( V_n \subset V_{n+1}, \ n \geq 1, \) such that

(a) \( \lim_{n \to \infty} \mu(V_n) = 1; \)

(b) each element of \( P_n[V_n] \) is an open subset of some unstable manifolds;
(c) for $\gamma \in \mathcal{P}_n | V_n$, one has $\mu_\gamma \ll m_\gamma$.

Proof. For any $\delta > 0$, set
\[ \Lambda_\delta := \{ w \in R : d(f^{-k}w, S) \geq \delta \lambda^{-k}, \forall k \geq 0 \} \] and $\Lambda_0 := \lim_{\delta \to 0} \Lambda_\delta$.

Next, it is to define the measurable partition $\mathcal{P}_n$ for any $n \in \mathbb{N}$. Let
\[ U_{j,l}^n := \left\{ (x,y,z) : \frac{j-1}{2^n} \leq x \leq \frac{j}{2^n}, \frac{l-1}{2^n} \leq y \leq \frac{l}{2^n}, 0 \leq z \leq 1 \right\}, 1 \leq j,l \leq 2^n. \]

For $w \in U_{j,l}^n \cap \Lambda_{1/2^n}$, set $\gamma(w) := W_{1/2^n}(w) \cap U_{j,l}^n$,
\[ V_n := \bigcup_{j,l \in \Lambda_{1/2^n} \cap U_{j,l}^n} \gamma(w), \text{ and } \mathcal{P}_n := \{ \gamma(w) : w \in V_n \cup \{ R \setminus V_n \} \}. \]

So, one has that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$, $n \geq 1$.

Fix a partition $\mathcal{P}_n$, it is to define a sequence of measures $\{ \tilde{\mu}_k \}_{k \in \mathbb{N}}$ as follows: since $\mu_k = f^k \mu_0$ is defined on $f^k(\text{Graph}(\alpha))$ and $f^k(\text{Graph}(\alpha))$ is a finite union of smooth surfaces, let $U = \bigcup_{i_1,...,i_k} U_{i_1,...,i_k}$ be the Lebesgue measure of $\{(p_x \times p_y) U_{i_1,...,i_k}^n \} \setminus \{(p_x \times p_y) f(L_{i_1,...,i_k}) \}$ is zero. Set $\tilde{\mu}_k(V) := \mu_k(V \cap U)$ for any measurable set $V$. In other words, the support of $\tilde{\mu}_k$ is a part of the support of $\mu_k$ consisting of all the subsets that cross some $U_{i_1,...,i_k}$ fully, when projected onto the $xy$-plane.

Next, it is to show that given any $\epsilon > 0$, there are $N_0$ and $K_0$ such that for any fixed partition $\mathcal{P}_n$ with $n > N_0$ and any $k > K_0$, one has that $\tilde{\mu}_k(R) > 1 - \epsilon$.

For $w \in \text{support}(\mu_k) \setminus \text{support}(\tilde{\mu}_k)$, $w$ is either in a small piece, which crosses some $U_{j,l}^n$ partially, when projected onto the $xy$-plane, or the distance between $w$ and a cusp in $f^k(\text{Graph}(\alpha))$ is less than $1/2^{n-1}$. Hence, one has
\[ 1 - \tilde{\mu}_k(R) = \mu_k(\text{support}(\mu_k) - \text{support}(\tilde{\mu}_k)) \]
\[ = \sum_{i=1}^k \mu_i \{ w : d(f^{-i}w, S) \leq 2^{-n+1} \lambda^{-i} \} + \mu_k(\{ \text{boundary pieces of } f^k(\text{Graph}(\alpha)) \}) \]
\[ = \sum_{i=1}^k \mu_i \{ w : d(w, S) \leq 2^{-n+1} \lambda^{-i} \} + \mu_k(\{ \text{boundary pieces of } f^k(\text{Graph}(\alpha)) \}) \]
\[ \leq 4C_5(p + 1)(q + 1)2^{-n+1} \sum_{i=1}^k \lambda^{-i} + \mu_k(\{ \text{boundary pieces of } f^k(\text{Graph}(\alpha)) \}), \]
where $C_5$ is specified in (8). The first term is very close to zero as $n$ goes to positive infinite, the second term goes to zero as $k$ goes to positive infinite. This verifies the existence of $K_0$ and $N_0$.

Since $\mu$ is a weak star limit of $\frac{1}{n} \sum_{k=1}^n \mu_k$ by Lemma 3.3, there exists a subsequence of $\frac{1}{n} \sum_{k=1}^n \tilde{\mu}_k$, which is convergent to $\tilde{\mu}$ in the weak star topology. It follows from the definitions of $\tilde{\mu}_k$ and $\mu_k$ that $\tilde{\mu} \ll \mu$ and $0 \leq d\tilde{\mu}/d\mu \leq 1$. So, by taking $n$ large enough, one has that $\tilde{\mu}$ is equivalent to $\mu$ except on a set with the $\mu$-measure less than $\epsilon$. Let the support of $\tilde{\mu}$ be the set $V_n$ for sufficiently large $n$. Let $\epsilon$ go to zero, we know that $\lim_{n \to \infty} \mu(V_n) = 1$. This proves (a) and (b). By the discussions above and Lemma 3.3, one has that (c) holds.

This completes the whole proof. \qed

Remark 2. It is easy to obtain similar results for maps with several directions of instability in higher-dimensional spaces.
4. **Example.** In this section, an example is provided to illustrate the theoretical results of Theorem 2.3 by computer simulations. The simulation diagram is obtained by using the software Mathematica.

Consider the map \( f = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) : [0, 1]^3 \to [0, 1]^3 \), where

\[
  f_1(x, y, z) = \begin{cases} 
    k_1 x + k_2 (y + z) & \text{if } x \in [0, 1/3] \\
    -k_1 (x - 1/3) + k_1/3 + k_2(y + z) & \text{if } x \in [1/3, 2/3] \\
    k_1 (x - 2/3) + k_2(y + z) & \text{if } x \in [2/3, 1]
  \end{cases},
\]

\[
  f_2(x, y, z) = \begin{cases} 
    k_1 y + k_2 (x + z) & \text{if } y \in [0, 1/3] \\
    -k_1 (y - 1/3) + k_1/3 + k_2(x + z) & \text{if } y \in [1/3, 2/3] \\
    k_1 (y - 2/3) + k_2(x + z) & \text{if } y \in [2/3, 1]
  \end{cases},
\]

\[
  f_3(x, y, z) = k_3 x,
\]

and satisfies the following assumptions: \( k_1 > 2 + 2k_2, \ 2k_2 + k_1/3 \leq 1, \ k_1 \geq 10k_2 > 0, \ 0 < k_3 < (k_1 - 2k_2)/8, \) and \( k_2k_3(k_1 + k_2) < 1 \). It is easy to verify that this map satisfies all the assumptions in Theorem 2.3 with \( N = 1 \).

Take \( k_1 = 2.4, \ k_2 = 0.08, \) and \( k_3 = 0.25 \). Figures 1 and 2 are the simulation diagrams with different initial values. In Figure 1, the initial value is taken as \((0.2, 0.1, 0.5)\). In Figure 2, the initial value is taken as \((0.5, 0.5, 0.5)\).

![Figure 1](image1.png)

**Figure 1.** The chaotic attractor with \( k_1 = 2.4, \ k_2 = 0.08, \) and \( k_3 = 0.25, \) where the initial value is taken as \((0.2, 0.1, 0.5)\).

![Figure 2](image2.png)

**Figure 2.** The chaotic attractor with \( k_1 = 2.4, \ k_2 = 0.08, \) and \( k_3 = 0.25, \) where the initial value is taken as \((0.5, 0.5, 0.5)\).

In Figure 1, the chaotic dynamical behavior is observed. In Figure 2, a “regular” orbit is observed. We guess that there is only one SRB measure for this map.
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I devote this work to my grandmother, she moved to heaven in 2014. She will always be in my heart.

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