REMARKS ON $K3$ SURFACES WITH NON-SYMPLECTIC AUTOMORPHISMS OF ORDER 7

SHINGO TAKI

ABSTRACT. In this note, we treat a pair of a $K3$ surface and a non-symplectic automorphism of order $7m$ ($m = 1, 3$ and $6$) on it. We show that if the fixed locus of a non-symplectic automorphism order $7$ is "special" then the pair is unique up to isomorphism. And we describe fixed loci of non-symplectic automorphisms of order $21$ and $42$.

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1. Introduction

Let $X$ be an algebraic $K3$ surface. In the following, we denote by $S_X$, $T_X$ and $\omega_X$ the Néron-Severi lattice, the transcendental lattice and a nowhere vanishing holomorphic 2-form on $X$, respectively. Let $\sigma_I$ be an automorphism on $X$ of finite order $I$. It is called non-symplectic if and only if it satisfies $\sigma_I^* \omega_X = \zeta_I \omega_X$ where $\zeta_I$ is a primitive $I$-th root of unity. Non-symplectic automorphisms have been studied by Nikulin who is a pioneer and several mathematicians.

It is known that the dimension of a moduli space of $K3$ surfaces with a non-symplectic automorphism of order $I$ is $\text{rk}T_X/\Phi(I) - 1$ if $I \neq 2$ or $\text{rk}T_X - 2$ if $I = 2$ [5 Section 11], where $\Phi$ is the Euler function. Then there exists some cases such that the dimension of a moduli space of $K3$ surfaces with a non-symplectic automorphism is zero.

Problem 1.1. Let $X_I$ be a $K3$ surface and $\sigma_I$ a non-symplectic automorphism of order $I$ on $X_I$. When is a pair $(X_I, \langle \sigma_I \rangle)$ unique up to isomorphism?

Vorontsov [10] announced some answers (without proofs) for the problem. Finally these were proved by Kondo, Oguiso and Zhang.
Theorem 1.2. [7 Theorem] Assume that $T_{X_I}$ is unimodular and $\sigma_I$ acts trivially on $S_{X_I}$. If $I = 66, 44, 42, 36, 28$ or $12$ and $\Phi(I) = \text{rk} T_{X_I}$ then there exists a unique (up to isomorphism) $K3$ surface with $\sigma_I$.

Here a lattice $L$ is called unimodular if and only if $L = \text{Hom}(L, \mathbb{Z})$, i.e. $L$ is isomorphic to its dual lattice. If the transcendental lattice is not unimodular then the following theorem is important.

Theorem 1.3. [12 §2, §4] Assume that $T_{X_I}$ is not unimodular and $\sigma_I$ acts trivially on $S_{X_I}$ and $\Phi(I) = \text{rk} T_{X_I}$. If $I = 3, 5, 7, 11, 13, 19, 5^2, 3^2, 3^3$ then there exists a (unique) algebraic $K3$ surface $X_I$ with $\text{rk} T_X = \Phi(I)$.

In some of the above cases, it seems that an assumption about the action of $\sigma_I$ on $S_{X_I}$ is important. We can see some uniqueness theorems by changing assumptions on $\sigma_I$. An important assumption of Theorem 1.4 and Theorem 1.5 is the order of $\sigma_I$. We show uniqueness of $K3$ surfaces with $\sigma_I$ from only $I$.

Theorem 1.4. [8 Main Theorem 1 and 2] Pairs $(X_{66}, \langle \sigma_{66} \rangle), (X_{33}, \langle \sigma_{33} \rangle), (X_{44}, \langle \sigma_{44} \rangle), (X_{50}, \langle \sigma_{50} \rangle), (X_{25}, \langle \sigma_{25} \rangle)$ and $(X_{40}, \langle \sigma_{40} \rangle)$ are unique up to isomorphism, respectively.

Recently the following is proved.

Theorem 1.5. [6] Pairs $(X_{21}, \langle \sigma_{21} \rangle)$ and $(X_{42}, \langle \sigma_{42} \rangle)$ are unique up to isomorphism, respectively.

We remark that these theorems do not assume that non-symplectic automorphisms act trivially on the Néron-Severi lattice. Indeed if $I = 66, 44, 21$ and $42$ then $\sigma_I$ acts trivially on $S_{X_I}$.

If $\Phi(I) < 12$ then the uniqueness of $(X_I, \langle \sigma_I \rangle)$ is not induced by only $I$. An important assumption is the fixed locus of $\sigma_I$, hence forms of fixed loci induce uniqueness.

Theorem 1.6. The followings hold by [10, Theorem 3, Theorem 4] [11, Main Theorem 4] [13, Theorem 1.5 (3)]:

1. If $X_3^{73}$ consists of only (smooth) rational curves and possibly some isolated points and contains at least 6 rational curves then a pair $(X_3, \langle \sigma_3 \rangle)$ is unique up to isomorphism.
2. If $X_2^{72}$ consists of only (smooth) rational curves and contains at least 10 rational curves then a pair $(X_2, \langle \sigma_2 \rangle)$ is unique up to isomorphism.
3. If $X_5^{75}$ contains no curves of genus $\geq 2$, but contains at least 3 rational curves then a pair $(X_5, \langle \sigma_5 \rangle)$ is unique up to isomorphism.
4. Put $M := \{ x \in H^2(X_{11}, \mathbb{Z}) | \sigma_{11}^*(x) = x \}$. A pair $(X_{11}, \langle \sigma_{11} \rangle)$ is unique up to isomorphism if and only if $M = U \oplus A_{10}$.

It is well known that if $I$ is prime then $I \leq 19$. But these theorems miss the case of $I = 7$. Moreover Jang [6] does not determine fixed loci of automorphisms. The main purpose of this paper is to prove the following theorem:

Main Theorem. (1) If $X_7^7$ consists of only (smooth) rational curves and some isolated points and contains at least 2 rational curves then a pair $(X_7, \langle \sigma_7 \rangle)$ is unique up to isomorphism.
(2) The fixed locus of $\sigma_{21}$ consists of exactly 11 isolated points and one $\mathbb{P}^1$.
(3) The fixed locus of $\sigma_{42}$ consists of exactly 9 isolated points and one $\mathbb{P}^1$. 
Remark 1.7. It is easy to see that if $\sigma_I$ holds the theorem (2) or (3) then pairs $(X_{21}, (\sigma_{21}))$ and $(X_{42}, (\sigma_{42}))$ are unique up to isomorphism by Theorem 1.5.

We know further results for uniqueness. See also [15].

Throughout this article we shall denote by $A_m$, $D_n$, $E_l$ the negative-definite root lattice of type $A_m$, $D_n$, $E_l$ respectively. We denote by $U$ the even indefinite unimodular lattice of rank 2 and $U(m)$ the lattice whose bilinear form is the one on $U$ multiplied by $m$.

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2. Preliminaries

In this section, we collect some basic results for non-symplectic automorphisms on a K3 surface. For the details, see [9] and [2], and so on.

Lemma 2.1. Let $\sigma_I$ be a non-symplectic automorphism of order $I$ on $X_I$. Then

1. The eigen values of $\sigma_I^* | T_{X_I}$ are the primitive $I$-th roots of unity, hence $\sigma_I^* | T_{X_I} \otimes \mathbb{C}$ can be diagonalized as:

$$
\begin{pmatrix}
\zeta_1 E_q & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \zeta^p E_q & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \cdots & \zeta_I^{-1} & E_q
\end{pmatrix}
$$

where $E_q$ is the identity matrix of size $q$ and $1 \leq n \leq I - 1$ is co-prime with $I$.

2. Let $P$ be an isolated fixed point of $\sigma_I$ on $X_I$. Then $\sigma_I^*$ can be written as

$$
\begin{pmatrix}
\zeta_i & 0 \\
0 & \zeta_i
\end{pmatrix}
$$

under some appropriate local coordinates around $P$.

3. Let $C$ be an irreducible curve in $X_I$ and $Q$ a point on $C$. Then $\sigma_I^*$ can be written as

$$
\begin{pmatrix}
1 & 0 \\
0 & \zeta_i
\end{pmatrix}
$$

under some appropriate local coordinates around $Q$. In particular, fixed curves are non-singular.

Lemma 2.1 (1) implies that $\Phi(I)$ divides $\text{rk} T_X$ and Lemma 2.1 (2) and (3) imply that the fixed locus of $\sigma_I$ is either empty or the disjoint union of non-singular curves and isolated points:

$$
X_I = \{p_1, \ldots, p_M\} \amalg C_1 \amalg \cdots \amalg C_N,
$$

where $p_i$ is an isolated fixed point and $C_j$ is a non-singular curve.

The global Torelli Theorem gives the following.

Remark 2.2. [8, Lemma (1.6)] Let $X$ be a K3 surface and $g_i$ ($i = 1, 2$) automorphisms of $X$ such that $g_i^*|S_X = g_j^*|S_X$ and that $g_i^* \omega_X = g_j^* \omega_X$. Then $g_1 = g_2$ in $\text{Aut} (X)$. 


The Remark says that for study of non-symplectic automorphisms, the action on $S_X$ is important. Hence the invariant lattice $S_{X^7}^7 := \{ x \in S_X | \sigma_7^* (x) = x \}$ plays an essential role for the classification of non-symplectic automorphisms.

**Proposition 2.3.** [2, Theorem 6.3] The fixed locus $X_7^7$ is of the form

$$X_7^7 = \begin{cases} \{ p_1, p_2, p_3 \} \amalg E & \text{if } S_{X^7}^7 = U \oplus K_7, \\ \{ p_1, p_2, \ldots \} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } S_{X^7}^7 = U(7) \oplus K_7, \\ \{ p_1, p_2, \ldots, p_8 \} \amalg \mathbb{P}^1 & \text{if } S_{X^7}^7 = U(7) \oplus E_8, \\ \{ p_1, p_2, \ldots, p_13 \} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1 & \text{if } S_{X^7}^7 = U \oplus E_8 \oplus A_6. \end{cases}$$

Here $E$ is a non-singular curve of genus 1 and $K_7$ is the even negative definite lattice given by Gram matrix $\begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$.

3. **Uniqueness of K3 surfaces with a certain fixed locus**

In this section, we treat a pair $(X_7, \langle \sigma_7 \rangle)$ whose the fixed locus $X_7^7$ consists of (smooth) rational curves and isolated points and contains at least 2 rational curves. We show that the pair $(X_7, \langle \sigma_7 \rangle)$ is unique up to isomorphism.

**Proposition 3.1.** The automorphism $\sigma_7$ acts trivially on $S_{X^7}$.

**Proof.** Since $X_7^7$ has at least 2 rational curves, $X_7^7 = \{ p_1, p_2, \ldots, p_13 \} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1$ and $S_{X^7}^7 = U \oplus E_8 \oplus A_6$ by Proposition 2.3 We know that $\text{rk} T_{X^7} \leq 6$ by Lemma 2.1 (1) and $\text{rk} S_{X^7} \geq 16$ since it contains the invariant lattice $S_{X^7}^7$ which is of rank 16. This gives $\text{rk} T_{X^7} \leq 6$ so that $\text{rk} T_{X^7} = 6$ and $\text{rk} S_{X^7} = 6$, hence $S_{X^7}$ coincides with $S_{X^7}^7$. This implies that the action of $\sigma_7$ is trivial on the $S_{X^7}$.

The following Corollary follows from Proposition 3.1 and Proposition 2.3.

**Corollary 3.2.** $S_{X^7} = U \oplus E_8 \oplus A_6$, $T_{X^7} = U \oplus U \oplus K_7$ and the fixed locus $\sigma_7$ has 2 non-singular rational curves and 13 isolated points: $X_7^7 = \{ p_1, p_2, \ldots, p_13 \} \amalg \mathbb{P}^1 \amalg \mathbb{P}^1$.

We recall that the dimension of a moduli space of K3 surfaces with a non-symplectic automorphism of order 7 is $\text{rk} T_{X^7}/\Phi(7) - 1$. In our case, its dimension is 0. Indeed we have the following.

**Theorem 3.3.** A pair $(X_7, \langle \sigma_7 \rangle)$ is unique up to isomorphism.

**Proof.** It follows from Proposition 3.1 and Theorem 1.3.

**Example 3.4.** [7, (7.5)] Put

$$X_{Ko} : y^2 = x^3 + 5^3 x + 7^3,$$ $\sigma_{Ko}(x, y, t) = \langle \zeta^2 t x, \zeta y, \zeta^2 t \rangle$.

Then $X_{Ko}$ is a K3 surface with $S_{X_{Ko}} = U \oplus E_8 \oplus A_6$ and $\sigma_{Ko}$ is a non-symplectic automorphism of order 7 acting trivially on $S_{X_{Ko}}$.

**Example 3.5.** [12, §4] Put $X_{OZ} : y^2 = x^3 + 5^3 x + 7^4$. Then $X_{OZ}$ is a K3 surface with $S_{X_{OZ}} = U \oplus E_8 \oplus A_6$ and a non-symplectic automorphism of order 7.

In [12], a non-symplectic automorphism of order 7 is not constructed. But $\phi(x, y, t) = \langle \zeta^2 x, \zeta y, \zeta^2 t \rangle$ is a non-symplectic automorphism of order 7 on $X_{OZ}$.

Of course, it is easy to see that these examples are the same, by analysing the elliptic fibration.
4. THE FIXED LOCUS OF A NON-SYMPLECTIC AUTOMORPHISM OF ORDER 21

We describe the fixed locus of a non-symplectic automorphism of order 21. First we recall the following.

**Proposition 4.1.** [6 Theorem 2.1] A non-symplectic automorphism of order 21 \( \sigma_{21} \) acts trivially on \( S_{X_{21}} \).

**Lemma 4.2.** The Euler characteristic of \( X_{21}^{\sigma_{21}} \) is 3 + tr(\( \sigma_{21}^*|S_{X_{21}} \)) = 13.

**Proof.** We apply the topological Lefschetz formula to the fixed locus \( X_{21}^{\sigma_{21}} \): \( \chi(X_{21}^{\sigma_{21}}) = 2 + \text{tr}(\sigma_{21}^*|S_{X_{21}}) + \text{tr}(\sigma_{21}^*|T_{X_{21}}) \). By [6 Theorem 3.1], \( \text{tr}(\sigma_{21}^*|T_{X_{21}}) = \zeta_{21} + \zeta_{21}^* + \zeta_{21}^2 + \zeta_{21}^3 + \zeta_{21}^4 + \zeta_{21}^5 + \zeta_{21}^6 + \zeta_{21}^7 + \zeta_{21}^8 = -(1 + \zeta_{21} + \zeta_{21}^2 + \zeta_{21}^3 + \zeta_{21}^4 + \zeta_{21}^5 + \zeta_{21}^6 + \zeta_{21}^7 + \zeta_{21}^8) = -(0 + (\zeta_3 + \zeta_3^2) = -(0 - 1) = 1 \). Since \( \Phi(21) = 12 \), \( \text{rk} \ S_{X_{21}} = 10 \).

**Proof.** Since \( \sigma_{21}^3(P_{21}^{i,j}) \) is a fixed point of \( \sigma_7 \), \( P_{21}^{i,j} \) is mapped to \( P_7^{i,j'} \) (mod 7). Thus

\[
\begin{align*}
& P_{21}^{2.20}, P_{21}^{2.20}, P_{21}^{9.13} \xrightarrow{\sigma_{21}^3} P_{7}^{2.6}, \\
& P_{21}^{6.16}, P_{21}^{5.17}, P_{21}^{10.12} \xrightarrow{\sigma_{21}^3} P_{7}^{3.5}, \\
& P_{21}^{4.18}, P_{21}^{11.11} \xrightarrow{\sigma_{21}^3} P_{7}^{4.4}, \\
& P_{21}^{7.15}, P_{21}^{8.14} \xrightarrow{\sigma_{21}^3} Q_7,
\end{align*}
\]

where \( Q_7 \) is a point on fixed curves of \( \sigma_7 = \sigma_{21}^3 \). Since \( \text{rk} \ S_{X_{21}} = 10 \) and Proposition [41] we have

\[
\begin{align*}
&m_{21}^{2.20} + m_{21}^{6.16} + m_{21}^{9.13} \leq 4, \\
&m_{21}^{3.19} + m_{21}^{5.17} + m_{21}^{10.12} \leq 3, \\
&m_{21}^{4.18} + m_{21}^{11.11} \leq 1.
\end{align*}
\] (4.1)

by [2 Theorem 2.4].

Moreover \( \sigma_{21}^2(P_{21}^{i,j}) \) is a fixed point of \( \sigma_7 \). If \( i \) or \( j \equiv 0 \) mod 3 then \( P_{21}^{i,j} \) is mapped to a point on a fixed curve of \( \sigma_3 = \sigma_{21}^7 \) and if \( i \) and \( j \not\equiv 0 \) mod 3 then \( P_{21}^{i,j} \) is mapped to \( P_3^{2.2} \). Since \( \text{rk} \ S_{X_{21}} = 10 \) and Proposition [41] we have

\[
\begin{align*}
&m_{21}^{2.20} + m_{21}^{5.17} + m_{21}^{8.14} + m_{21}^{11.11} \leq 4.
\end{align*}
\] (4.2)

by [1 Theorem 2.2] and [14 Proposition 3.2]
We apply the holomorphic Lefschetz formula ([3] page 542 and [4] page 567) to \(X_{21}^{\sigma_2}:
\[2 \sum_{k=0}^{\infty} \text{tr}(\sigma_2^{*k}|H^k(X_{21}, \mathcal{O}_{X_{21}})) = \sum_{i+j=22}^M a(P_{21}^{i,j}) + \sum_{i=1}^N b(C_i),\]
where \(a(P_{21}^{i,j}) = 1/((1 - \zeta_2^i)(1 - \zeta_2^j))\) and \(b(C_i) = (1 - g(C_i))/(1 - \zeta_2) - \zeta_2 C_i^2/(1 - \zeta_2^2).\) Hence

\[1 + \zeta_2^{20} = \sum_{i+j=22, 2 \leq i \leq j} \frac{m_{i,j}^{20}}{(1 - \zeta_2^i)(1 - \zeta_2^j)} + \sum_{i=1}^N \frac{(1 + \zeta_2^i)(1 - g(C_i))}{(1 - \zeta_2)^2}.
\]

Then we have

\[
\begin{align*}
m_{21}^{6,16} &= -\frac{m_{21}^{3,19} + m_{21}^{5,17}}{2} + 3 \sum_{i=1}^N (1 - g(C_i)), \\
m_{21}^{7,15} &= 1 - 3m_{21}^{3,19} + 8 \sum_{i=1}^N (1 - g(C_i)), \\
m_{21}^{8,14} &= 1 - \frac{9m_{21}^{3,19} + 3m_{21}^{5,17}}{2} + 17 \sum_{i=1}^N (1 - g(C_i)), \\
m_{21}^{9,13} &= 1 - 5m_{21}^{2,20} - m_{21}^{3,19} - 2m_{21}^{5,17} + 18 \sum_{i=1}^N (1 - g(C_i)), \\
m_{21}^{10,12} &= 3 + \frac{15m_{21}^{2,20} + m_{21}^{3,19} + m_{21}^{5,17}}{6} - 3m_{21}^{4,18} + 21 \sum_{i=1}^N (1 - g(C_i)), \\
m_{21}^{11,11} &= 1 - 3m_{21}^{2,20} - m_{21}^{4,18} + 9 \sum_{i=1}^N (1 - g(C_i)).
\end{align*}
\]

(4.3)

**Proposition 4.4.** The fixed locus of \(\sigma_{21}\) consists of exactly 11 isolated points and one \(\mathbb{P}^1:\)

\[X_{21}^{\sigma_{21}} = \{P_{21}^{2,20}, P_{21}^{3,20}, P_{21}^{4,20}, P_{21}^{5,19}, P_{21}^{7,15}, P_{21}^{4,18}, P_{21}^{5,17}, P_{21}^{6,16}, P_{21}^{7,15}, P_{21}^{5,15}\} \cup \mathbb{P}^1.
\]

**Proof.** We remark inequalities in Lemma [4.3] equations (4.3) and \(m_{i,j}^{20}\) is a non-negative integer.

If \(m_{21}^{4,18} + m_{21}^{11,11} < 1\) then \(m_{21}^{4,18} = m_{21}^{11,11} = 0\) and \(m_{21}^{2,20} = 1/3 + 3 \sum (1 - g(C_i)).\)

This is a contradiction.

If \(m_{21}^{2,20} + m_{21}^{6,16} + m_{21}^{9,13} = 3\) (resp. 2, 0) then \(m_{21}^{3,19} + m_{21}^{5,17} - 5 \sum (1 - g(C_i)) = -4/3\) (resp. \(-2/3, 2/3\)). These are not integer. If \(m_{21}^{2,20} + m_{21}^{6,16} + m_{21}^{9,13} = 1\) then \(m_{21}^{5,17} = -m_{21}^{3,19} + 5 \sum (1 - g(C_i))\) and \(m_{21}^{8,19} = 1 - 4 \sum (1 - g(C_i)).\) Hence \(m_{21}^{5,17}\) or \(m_{21}^{8,19}\) is negative. Thus \(m_{21}^{2,20} + m_{21}^{6,16} + m_{21}^{9,13} = 4.

If \(m_{21}^{3,19} + m_{21}^{5,17} + m_{21}^{10,12} = 2\) (resp. 1) then \(m_{21}^{4,18} - 2 \sum (1 - g(C_i)) = -2/3\) (resp. \(-1/3\)). These are not integer. Assume \(m_{21}^{3,19} + m_{21}^{5,17} + m_{21}^{10,12} = 0.\) Since \(m_{21}^{10,12} = 0,\) we have \(m_{21}^{5,17} = -2 - m_{21}^{3,19} + 5 \sum (1 - g(C_i)).\) This contradicts \(m_{21}^{3,19} = m_{21}^{5,17} = 0.\) Hence we have \(m_{21}^{3,19} + m_{21}^{5,17} + m_{21}^{10,12} = 3.\)

If \(m_{21}^{2,20} + m_{21}^{5,17} + m_{21}^{8,14} + m_{21}^{11,11} = 2\) (resp. 1, 0) then \(m_{21}^{3,19} = 5 - 5 \sum (1 - g(C_i))\) or \(m_{21}^{7,15} = -5 + 2 \sum (1 - g(C_i))\) (resp. \(-8 + 2 \sum (1 - g(C_i)), -11 + 2 \sum (1 - g(C_i))\).

This is negative. Assume \(m_{21}^{2,20} + m_{21}^{5,17} + m_{21}^{8,14} + m_{21}^{11,11} = 3.\) Then it is easy to see \(M = \sum m_{21}^{1,j} = 10 - 2 \sum (1 - g(C_i)).\) In particular \(m_{21}^{2,20} = 3 \sum (1 - g(C_i)), m_{21}^{5,17} = -3 + 3 \sum (1 - g(C_i)), m_{21}^{8,14} = -4 - 4 \sum (1 - g(C_i))\) and \(m_{21}^{11,11} = 2 - 2 \sum (1 - g(C_i))\).

Since \(m_{21}^{2,20}, m_{21}^{5,17}, m_{21}^{8,14}, m_{21}^{11,11} = 0,\) we have \(\sum (1 - g(C_i)) = 1\) and \(M = 8.\) It follows from \(\chi(X_{21}^{\sigma_{21}}) = M + \sum (2 - 2g(C_i))\) and Lemma [4.2] that \(\text{tr}(\sigma_{21}^2|S_{X_{21}}) = 7.\)

This is a contradiction for Proposition [4.1] hence \(m_{21}^{2,20} + m_{21}^{5,17} + m_{21}^{8,14} + m_{21}^{11,11} = 4.\)
In conclusion inequalities in Lemma 5.3 are equations. Moreover by Lemma 5.2, we have

\[
\begin{align*}
m_{21}^{220} & = 3 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{3.19} & = 2 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{4.18} & = -1 + 2 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{5.17} & = -2 + 3 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{6.16} & = -1 + 2 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{7.15} & = 1 + 2 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{8.14} & = 4 - 4 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{9.13} & = 5 - 5 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{10.12} & = 5 - 5 \sum_{i=1}^{N} (1 - g(C_i)), \\
m_{21}^{11.11} & = 2 - 2 \sum_{i=1}^{N} (1 - g(C_i))
\end{align*}
\]

and \( M = \sum m_{21}^{i,j} = 13 - 2 \sum_{i=1}^{N} (1 - g(C_i)). \)

If \( \sum_{i=1}^{N} (1 - g(C_i)) \neq 1 \) then \( m_{21}^{4.18} \) or \( m_{21}^{8.14} \) are negative. Thus we have \( \sum_{i=1}^{N} (1 - g(C_i)) = 1, \ M = 11 \) and \( \chi(X_{21}^{\sigma_4}) = M + \sum_{i=1}^{N} (2 - 2g(C_i)) = 11 + 2 = 13. \)

5. The fixed locus of a non-symplectic automorphism of order 42

We describe the fixed locus of a non-symplectic automorphism of order 42. The following is a key in this section.

**Proposition 5.1.** [6 Corollary 2.6] A non-symplectic automorphism of order 42 \( \sigma_{42} \) acts trivially on \( S_{X_{42}} \).

**Lemma 5.2.** The Euler characteristic of \( X_{42}^{\sigma_{42}} \) is \( 1 + \text{tr}(\sigma_{42}^{2}|S_{X_{42}}) = 11 \).

**Proof.** We apply the topological Lefschetz formula to the fixed locus \( X_{42}^{\sigma_{42}}: \chi(X_{42}^{\sigma_{42}}) = \sum_{i=0}^{1} (-1)^{i}\text{tr}(\sigma_{42}^{i}|H^i(X_{42}, \mathbb{R})) = 1 - 0 + \text{tr}(\sigma_{42}^{2}|S_{X_{42}}) + \text{tr}(\sigma_{42}|T_{X_{42}}) - 0 + 1. \) By [9] Theorem 3.1, \( \text{tr}(\sigma_{42}^{2}|T_{X_{42}}) = \zeta_{42} + \zeta_{14}^{11} + \zeta_{14}^{13} + \zeta_{14}^{17} + \zeta_{14}^{19} + \zeta_{14}^{21} + \zeta_{14}^{23} + \zeta_{14}^{29} + \zeta_{14}^{37} + \zeta_{14}^{39} - (1 + \zeta_{42} + \zeta_{14}^{2} + \cdots + \zeta_{14}^{19}) + (\zeta_{3}^{3} + \zeta_{14}^{7} + \zeta_{14}^{9} + \zeta_{14}^{11} + \zeta_{14}^{13} + \zeta_{14}^{19} + \zeta_{14}^{21} + \zeta_{14}^{23} + \zeta_{14}^{29} + \zeta_{14}^{37} + \zeta_{14}^{39}) = -(0 + \zeta_{14}^{3} + \zeta_{14}^{7} + \zeta_{14}^{9} + \zeta_{14}^{11} + \zeta_{14}^{13} + \zeta_{14}^{19} + \zeta_{14}^{21} + \zeta_{14}^{23} + \zeta_{14}^{29} + \zeta_{14}^{37} + \zeta_{14}^{39}) = -0 + 0 + 1 = 0. \) Since \( \Phi(21) = 12, \ \text{rk} S_{X_{42}} = 10. \)

**Lemma 5.3.** The following inequalities and equations hold:

\[
\begin{align*}
m_{42}^{2.41} + m_{42}^{20.23} & \leq 3, \\
m_{42}^{3.40} + m_{42}^{19.24} & \leq 2, \\
m_{42}^{4.39} + m_{42}^{18.25} & \leq 1, \\
m_{42}^{5.38} + m_{42}^{17.26} & \leq 1, \\
m_{42}^{6.37} + m_{42}^{16.27} & \leq 1, \\
m_{42}^{7.36} + m_{42}^{15.28} & \leq 3, \\
m_{42}^{8.35} & = m_{42}^{9.34} = m_{42}^{10.33} = m_{42}^{11.32} = m_{42}^{12.31} = m_{42}^{13.30} = m_{42}^{14.29} = 0.
\end{align*}
\]

**Proof.** Since \( \sigma_{21}^{2}(P_{i,j}) \) is a fixed point of \( \sigma_{21} \), \( P_{i,j} \) is mapped to \( P_{i',j'} \) \( (i \equiv i', j \equiv j' \mod 21) \). It is easy to see these inequalities and equations by Theorem 4.4.
Proposition 5.4. The fixed locus of \( \sigma_{42} \) consists of exactly 9 isolated points and one \( \mathbb{P}^1 \):

\[
X_{42}^* = \{ P_{42}^{2,41}, P_{42}^{2,41}, P_{42}^{2,41}, P_{42}^{3,40}, P_{42}^{3,40}, P_{42}^{3,39}, P_{42}^{5,38}, P_{42}^{6,37}, P_{42}^{7,36} \} \cup \mathbb{P}^1.
\]

Proof. We apply the holomorphic Lefschetz formula ([3] page 542 and [4] page 567) to \( X_{42}^* \):

\[
1 + \zeta_{42}^{41} = \sum_{i+j=43, \ 2 \leq i \leq j} \frac{m_{42}^{i,j}}{(1 - \zeta_{42}^i)(1 - \zeta_{42}^j)} + \sum_{l=1}^N \frac{(1 + \zeta_{42})(1 - g(C_l))}{(1 - \zeta_{42})^2}.
\]

Then we have

\[
m_{42}^{15,28} = 0,
\]
\[
m_{42}^{16,27} = 4m_{42}^{2,41} + 2m_{42}^{3,40} + 4m_{42}^{5,38} - 3m_{42}^{6,37} - m_{42}^{7,36} - 16 \sum_{l=1}^N (1 - g(C_l)),
\]
\[
m_{42}^{17,26} = -1 + 12m_{42}^{2,41} + 6m_{42}^{3,40} + 7m_{42}^{5,38} - 4m_{42}^{6,37} - 2m_{42}^{7,36} - 48 \sum_{l=1}^N (1 - g(C_l)),
\]
\[
m_{42}^{18,25} = -2 + 26m_{42}^{2,41} + 12m_{42}^{3,40} + m_{42}^{4,39} + 12m_{42}^{5,38} - 6m_{42}^{6,37} - 3m_{42}^{7,36} - 104 \sum_{l=1}^N (1 - g(C_l)),
\]
\[
m_{42}^{19,24} = 5 - 58m_{42}^{2,41} - 23m_{42}^{3,40} - 6m_{42}^{4,39} - 16m_{42}^{5,38} + 14m_{42}^{6,37} + m_{42}^{7,36} + 23 \sum_{l=1}^N (1 - g(C_l)),
\]
\[
m_{42}^{20,23} = 4 - 51m_{42}^{2,41} - 20m_{42}^{3,40} - 6m_{42}^{4,39} - 14m_{42}^{5,38} + 4m_{42}^{6,37} + m_{42}^{7,36} + 204 \sum_{l=1}^N (1 - g(C_l)),
\]
\[
m_{42}^{21,22} = 2 - 24m_{42}^{2,41} - 8m_{42}^{3,40} - 4m_{42}^{4,39} - 4m_{42}^{5,38} + 94 \sum_{l=1}^N (1 - g(C_l)).
\]

Moreover by Proposition 5.5 we have

\[
\begin{align*}
\text{Proposition 5.5} \quad & m_{42}^{2,41} = 1 + 2 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{3,40} = -2 + 4 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{4,39} = -1 + 2 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{5,38} = -1 + 2 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{6,37} = -3 + 4 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{7,36} = 1, \\
& m_{42}^{16,27} = 4 - 4 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{17,26} = 2 - 2 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{18,25} = 2 - 2 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{19,24} = 4 - 4 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{20,23} = 2 - 2 \sum_{l=1}^N (1 - g(C_l)), \\
& m_{42}^{21,22} = 2 - 2 \sum_{l=1}^N (1 - g(C_l)).
\end{align*}
\]

and

\[
M = \sum m_{42}^{i,j} = 11 - 2 \sum_{l=1}^N (1 - g(C_l)).
\]
If \( \sum_{i=1}^{N} (1 - g(C_i)) \neq 1 \) then \( m_{42}^{19,40} \) or \( m_{42}^{16,27} \) are negative. Thus \( \sum_{i=1}^{N} (1 - g(C_i)) = 1 \) and \( M = 9 \). If the fixed locus \( X_{21}^{\sigma_2} \) contains a non-singular curve then \( X_{21}^{\sigma_2} \) also contain it. Thus \( X_{42}^{\sigma_2} \) has at most one \( \mathbb{P}^1 \) by Proposition \[5.3\].

**Proposition 5.5.** The following equations hold:

\[
\begin{align*}
m_{42}^{2,41} + m_{42}^{20,23} &= 3, \\
m_{42}^{3,40} + m_{42}^{19,24} &= 2, \\
m_{42}^{4,39} + m_{42}^{18,25} &= 1, \\
m_{42}^{5,38} + m_{42}^{17,26} &= 1, \\
m_{42}^{6,37} + m_{42}^{16,27} &= 1, \\
m_{42}^{7,36} + m_{42}^{15,28} &= 1.
\end{align*}
\]

**Proof.** We remark inequalities in Lemma \[5.3\] and \( m_{i,j}^{42} \) is a non-negative integer.

If \( m_{42}^{4,39} + m_{42}^{18,25} = 0 \) (resp. \( m_{42}^{5,38} + m_{42}^{17,26} = 0 \)) then \( m_{42}^{16,27} = 2/3 - 14m_{42}^{2,41} / 3 - 2m_{42}^{3,40} - 2m_{42}^{4,39} / 3 - m_{42}^{6,37} + 56 \sum (1 - g(C_i)) / 3 \) (resp. \( m_{42}^{16,27} = 1/2 - 2m_{42}^{4,39} - m_{42}^{3,40} - m_{42}^{6,37} + 8 \sum (1 - g(C_i)) \)). These are not integers, respectively.

If \( m_{42}^{6,37} + m_{42}^{16,27} = 0 \) then \( m_{42}^{4,39} = -3/2 - m_{42}^{2,41} + 4 \sum (1 - g(C_i)) \). This is not a integer.

If \( m_{42}^{4,39} + m_{42}^{19,24} = 0 \) (resp. \( m_{42}^{5,38} + m_{42}^{17,26} = 0 \)) then we have \( m_{42}^{6,37} = 2 + 6m_{42}^{3,40} \) and \( m_{42}^{7,36} = 3 - 8m_{42}^{6,37} \). \( m_{42}^{4,39} \) or \( m_{42}^{5,38} \) is negative. If \( m_{42}^{4,39} + m_{42}^{19,24} = 1 \) then \( m_{42}^{6,37} = -3/2 + 4m_{42}^{2,41} + 6m_{42}^{5,38} - 16 \sum (1 - g(C_i)) \). This is not a integer.

If \( m_{42}^{7,36} + m_{42}^{15,28} = 3 \) (resp. \( m_{42}^{5,38} = -1/4 - m_{42}^{2,41} + 4 \sum (1 - g(C_i)) \) (resp. \( = -1/8 - m_{42}^{2,41} + 4 \sum (1 - g(C_i)) \)) = 1/8 - m_{42}^{2,41} + 4 \sum (1 - g(C_i)) \)). These are not integer.

If \( m_{42}^{2,41} + m_{42}^{20,23} = 2 \) (resp. \( m_{42}^{5,38} = -1/2 + 2 \sum (1 - g(C_i)) \) (resp. \( = 1/2 + 2 \sum (1 - g(C_i)) \)) and \( m_{42}^{18,25} = 1 - 2 \sum (1 - g(C_i)) \). \( m_{42}^{6,37} \) or \( m_{42}^{15,28} \) is negative.

\[\Box\]

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Department of Mathematics, Tokai University, 4-1-1, Kitakaname, Hiratsuka, Kanagawa, 259-1292, JAPAN

E-mail address: taki@tsc.u-tokai.ac.jp

URL: http://sm.u-tokai.ac.jp/~taki/