Kepler’s laws

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Kepler’s laws
with introduction to differential calculus
(book for scholars, who are interested in physics and mathematics)
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Abstract

We explain the solution of the following two problems: obtaining of Kepler’s laws from Newton’s laws (so called two bodies problem) and obtaining the fourth Newton’s law (the formula for gravitation) as a corollary of Kepler’s laws.

This small book is devoted to the scholars, who are interested in physics and mathematics. We also make a series of digressions, where explain some technique of the higher mathematics, which are used in the proofs.
1 Introduction

Both Newton’s laws and Kepler’s laws are classical facts, known from the school course of physics. One of the beautiful facts in physics is that Kepler’s laws are equivalent to the Newton’s law of the gravitation.

Remind the formulation of Kepler’s laws.

**Kepler’s first law**: The orbit of a planet is an ellipse with the Sun at one of the two foci.

**Kepler’s second law**: A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.

**Kepler’s third law**: The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Remind the formulation of Newton’s laws:

**Newton’s first law**: if the vector sum of all forces acting on an object is zero, then the velocity of the object is constant.

**Newton’s second law**: the constant force $\vec{F}$, which acts on the body of the mass $m$, produces the constant acceleration $\vec{a} = \frac{\vec{F}}{m}$ in the direction of $\vec{F}$.

**Newton’s third law**: all forces between two objects exist in equal magnitude and opposite direction: if one object $A$ exerts a force $F_A$ on a second object $B$, then $B$ simultaneously exerts a force $F_B$ on $A$, and the two forces are equal in magnitude and opposite in direction: $F_A = -F_B$.

**Newton’s fourth law**: every point mass attracts every single other other point mass by a force pointing along the line intersecting both points. The force is proportional of the two masses and inversely proportional to the square of the distance between them.

Notice, that 1-th and 3-rd Newton’s law are technical. They are formulated in the style of Euclidean axioms and thus, it is impossible to try follow them from some complicated facts. We consider the second Newton’s law as the definition of the force, considering the acceleration as the mathematical notion of the second derivative.

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Thus, the equivalence of the fourth Newton’s law and Kepler’s laws is the unique non-trivial fact, which may exist as mathematical theorem and it is.

The proof of the mentioned equivalence is not elementary. It uses derivatives, integration, some basis of vector analysis, polar coordinates. In this book we explain these notions as detailed, as it is possible in the small book for scholars and our the main deal is to show young people the powerfulness of mathematics and inspirit them to study mathematics in more detailed, using other books, dedicated not especially to Kepler’s laws, or the connection of mathematics and physics.

Moreover, we give in Section 2.3 some non-standard proves of two classical facts. In Section 2.3.1 we proof “from physical reasonings” the formulas for the derivative of functions $\sin x$ and $\cos x$. In Section 2.3.2 we prove “by trivial reasonings” the formula for the radius of the curvature of the plain curve.
2 Derivatives and integrals

2.1 Physical approach

2.1.1 Derivatives

The use of derivatives (precisely the invention of derivatives) in the 17-th century completely transformed physics, because opened the possibility for the complicated mathematically strict calculations as the solutions of the physical (mechanical) problems.

Kepler’s laws were one of the first illustration of the powerfulness of the new mathematical (and physical) notion of derivative. From another hand, the derivative is not more then the mathematical formalization of notion of velocity.

Suppose that a point $A$ moves on the line (say $X$-axis) and denote $x = s(t)$ the position function of $A$, i.e. $s(t)$ is the coordinate of $A$ at the time $t$. Then the classical notion of the instantaneous velocity of $A$ will be some function $v(t)$, dependent on $t$. The mathematical formalization of the instantaneous velocity is exactly the derivative of the function $s$ and it is traditionally denoted $v = s'$. $\text{Fix moment of time $t_0$. The velocity of $A$ at this moment (the instantaneous velocity) can be found as} \quad s'(t_0) \approx \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad (2.1)\text{where $\Delta t$ is “small enough”, i.e. $\Delta t \approx 0$, but $\Delta t \neq 0$. Moreover, the symbol $\approx$ is used in (2.1) only to notice that $\Delta t$ is not an exact number, but is “very small, but non-zero”. In fact, the formula (2.1) defines some certain function, which is as certain, as clear as the physical notion of instantaneous velocity. Suppose that $dt$ is so small value of $\Delta t$, that (2.1) can be considered as equality} \quad s'(t_0) = \frac{s(t_0 + dt) - s(t_0)}{dt}, \quad (2.2)\text{i.e. we can ignore the absolute value $|s'(t_0) - \frac{s(t_0 + dt) - s(t_0)}{dt}|$ assuming that it is zero. Now denote} \quad ds(t_0) = s(t_0 + dt) - s(t_0)$
and rewrite (2.2) as

\[ s'(t_0) = \frac{ds(t_0)}{dt}. \tag{2.3} \]

Traditionally, people do not write \( t_0 \) in (2.3), i.e. left just

\[ s' = \frac{ds}{dt}, \tag{2.4} \]

but understand (2.4) with all the remarks before it. Notice ones more, that \( s \) and \( s' \) in (2.4) are some functions, and \( t \) is their argument. Moreover, mathematics has a powerful techniques how to find derivative \( s' \) by given \( s \).

**Example 1.** Find the derivative of the function \( s(t) = t^2 \).

Plug \( s(t) = t^2 \) into (2.1) and get

\[
\frac{s(t+h) - s(t)}{h} = \frac{(t+h)^2 - t^2}{h} = \frac{(t^2 + 2th + h^2) - t^2}{h} = 2t + h.
\]

Evidently, if \( h \approx 0 \), then \( 2t + h \approx 2t \), whence

\[ s'(t) = 2t. \]

We will need the following properties of the derivative for our further reasonings.

1. For arbitrary function \( f(x) \) and arbitrary constant \( a \) the formula \( (af(x))' = a \cdot f'(x) \) holds. This formula follows from (2.2) as the definition of the derivative.

2. Let \( f(x) \) and \( g(x) \) be arbitrary functions. Denote \( h(x) = f(g(x)) \). Then the derivative \( h'(x) \) equals

\[
h'(x) = [f(g(x))]' = f'(g(x)) \cdot g'(x). \tag{2.5}
\]

We will give at first an example of the use of (2.5) and follow it by the proof.

**Example 2.** Find the derivative of \( p(t) = t^4 \).

Denote \( s(t) = t^2 \) and \( p(t) = t^4 \). We have obtained in Example 1 that \( s'(t) = 2t \). Clearly, \( p(t) = s(s(t)) \). For make our reasonings more clear, denote \( f(t) = g(t) = t^2 \) whence \( p(t) = f(g(t)) \). The expression \( f'(g(x)) \) means that function \( g \) should be plugged into \( f' \), whence \( f'(g(t)) = 2 \cdot g(t) = 2t^2 \). Thus, by (2.5), \( p'(t) = 2t^2 \cdot 2t = 4t^3 \).
2.1 Physical approach

Proof of (2.5). The derivative of \( h \), which is written the by definition

\[
h'(x) = \frac{h(x + t) - h(x)}{t}, \quad t \approx 0,
\]

can be rewritten as follows.

\[
\frac{h(x + t) - h(x)}{t} = \frac{f(g(x + t)) - f(g(x))}{g(x + t) - g(x)} \cdot \frac{g((x + t) - g(x))}{t}, \quad t \approx 0.
\]

The first fraction here is \( f' \), where \( g \) is plugged. The second multiplier is \( g'(x) \), whence (2.5) is proved.

Also derivative \( s'(t) \) is the tangent of the angle between \( x \)-axis and the tangent-line for the graph \( s(t) \) at the point, corresponding to \( t \). Indeed, denote \( A(t, s(t)) \) and \( B(t + \Delta t, s(t + \Delta t)) \). Then the tangent of the line \( AB \) (which equals to the tangent of the angle \( x \)-axis and \( AB \)) equals to the expression from formula (2.1). If \( \Delta t \approx 0 \), then \( AB \) transforms to the tangent and (2.1) becomes the derivative.

There are known a lot of methods of finding the derivatives in mathematics. Precisely, derivatives of all the functions, which are studied at school (i.e. \( y = x^n \), \( y = a^x \), \( y = \sin x \), \( y = \cos x \), \( y = \tan x \)) can be found explicitly.

2.1.2 Conservation laws

Consider the following classical physical problem. Suppose that a point of the mass \( m \) moves with the velocity \( v \) and constant force \( F \) acts on the point in the direction of the movement.

The following physical quantities are know in this case:

1. The kinetic energy of the point is \( \frac{mv^2}{2} \).
2. The impulse of the point is \( mv \).
3. If the force acts during the distance \( l \), then the work of the force is \( Fl \).
4. If the force acts during the time \( t \), then the impulse of the force is \( Ft \).

Moreover, is we ignore other factors, which cause the movement of the body, then impulse of the force transforms to the impulse of the body and also the work of the force transforms to

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the kinetic energy of the body. In other words, the equality

\[ m \Delta v = F \Delta t \]  \hspace{1cm} (2.6)

and

\[ \frac{m(\Delta v)^2}{2} = F \Delta l \]  \hspace{1cm} (2.7)

hold. The equality (2.6) is called the conservation of the impulse law and (2.7) is conservation of the energy law.

School course of physics does not contain the clear explanation of the notions of energy (kinetic energy) and impulse. In fact, laws (2.6) and (2.7) are considered like definition of these two notions and it is frequently said that (2.6) and (2.7) are obtained “experimentally”.

The same situation is about the notion of the “force”. By second Newton’s law, the constant force \( F \) causes the acceleration \( a \) with the same direction as \( F \) and such that

\[ F = am. \]  \hspace{1cm} (2.8)

The precisely, formula (2.8) is not more than definition of the force. If there is an acceleration, then (2.8) defines something and this “something” is called force. Moreover, the existence of the force should cause the acceleration, again by (2.8).

The acceleration is the derivative of the velocity, or, which is the same, the “second derivative” of the position of the point (i.e. the derivative of the derivative). It is better here to understand the derivative just as “new function, which is constructed by some certain rulers from the former function”. In any case, say that

\[ \begin{cases} a = v'; \\ a = s'' \end{cases} \]  \hspace{1cm} (2.9)

**Example 3.** Second Newton’s law implies the law of conversation of the impulse.

Rewrite the 2-nd Newton’s law (2.8) as \( m \frac{dv}{dt} = F \), which is equivalent to

\[ m \, dv = F \, dt. \]

and is exactly the same as (2.6).
2.2 Mathematical approach

**Remark 1.** Notice, that reasonings from the Example 3 can be inverted and, thus, the following proposition can be obtained: Suppose that material point moves on a line an the formula $mv^2 = Ft$ holds, where $F$ is some constant. Then this point moves with constant acceleration $a = \frac{F}{m}$.

**Example 4.** Suppose that material point moves on a line an the formula (2.7) holds, where $F$ is some constant. Then this point moves with constant acceleration, which can be found by (2.8).

**Proof.** Suppose that at some (former) moment of time the velocity and the length equal zero, i.e. $v(0) = l(0) = 0$. Then we can rewrite (2.7) as

$$\frac{mv^2}{2} = Fl.$$

Since left hand side and right hand side of the obtained equality are equal, then their derivatives are equal too, i.e.

$$\frac{m}{2} \cdot 2v v' = Fl'.$$

But $v' = a$ and $l' = v$, whence $mva = Fv$, and, after cancellation, we get (2.8). 

**Remark 2.** Notice, that reasonings from the Example 3 can be inverted and, thus, the following proposition can be obtained: Suppose that material point moves on a line an the formula $\frac{mv^2}{2} = Ft$ holds, where $F$ is some constant. Then this point moves with constant acceleration $a = \frac{F}{m}$.

2.2 Mathematical approach

2.2.1 Derivatives

If a function $s = s(t)$ is given by quite simple formula, then there are mathematical methods, which give us quite simple formulas for the derivative $v = s'(t)$.

Suppose now, that we have a material point, which moves not just on a line, but in a 3-dimensional space. It means that the function $s = s(t)$ should be consisted of three functions, write

$$s(t) = (x(t), y(t), z(t)).$$

(2.10)
The derivative $s'$ of $s$ of the form (2.10) also is defined by formulas (2.1), (2.2) and (2.3) with correspond remarks, but now $s$ is a vector-function there, whence the differences $s(t_0 + \Delta t) - s(t_0)$, $s(t_0 + dt) - s(t_0)$ and $ds(t_0)$ are differences of vectors and, thus, vectors.

**Fact 1.** If the function is given by (2.10), then its derivative equals

$$s'(t) = (x'(t), y'(t), z'(t)).$$

**Fact 2.** If (2.10) defines the position function of a point, then the derivative $v(t) = s'(t)$ defines the velocity (which is, clearly, a vector) and the derivative $a(t) = v'(t) = (s'(t))'$ defines the acceleration (which is a vector too). Notice, that in this case $(s'(t))'$ is called second derivative and also can be denoted as $s''(t)$ which means the same as $(s'(t))'$, being the derivative of the derivative.

**Fact 3.** If $g, h$ are functions $\mathbb{R} \to \mathbb{R}$, then the derivative of the function $f(x) = g(h(x))$ can be found as

$$f'(x) = g'(t) \cdot h'(x),$$

plugging $t = h(x)$. This fact is called the chain rule of the differentiation.

**Fact 4.** If $g, h$ are functions $\mathbb{R} \to \mathbb{R}$, then the derivative of the function $f(x) = g(x) \cdot h(x)$ can be found as

$$f'(x) = g'(x) \cdot h(x) + g(x) \cdot h'(x).$$

This fact is called the product rule of the differentiation.

**Fact 5.** **Table of derivatives:**

$\sin' x = \cos x$ and $\cos' x = -\sin x$.

**Fact 6.** The derivative of the function $s(t) = \frac{1}{t}$ is

$$s'(t) = \frac{-1}{t^2}.$$
2.2 Mathematical approach

2.2.2 Integrals

The derivative, which we have mentioned above, can be considered now just as some rule of finding the new function \( g = f' \) by the given function \( f \).

The converse problem also can be considered: by the given function \( g \) find the function \( f \) such that \( g = f' \). In this case the function \( g \) is called the indefinite integral of the function \( f \). The nature of this name will become evident after we will understand the following fact.

Let a function \( y = g(x) : \mathbb{R} \rightarrow \mathbb{R} \) such that \( g(x) > 0 \) for all \( x \) be given and let the graph of \( g \) be constructed. Fix any point \( a \in \mathbb{R} \) and denote by \( A(t) \), \( t \geq a \) the area between the graph of \( y = g(x) \) and \( X \)-axis for \( x \in [a, t] \) (see Fig. 1).

\[ f(x) = A(x) + c \] (2.11)

for all \( x \in \mathbb{R} \), whence (2.11) is the complete description of all the functions \( f \) such that \( f' = g \). This fact is called the Newton-Leibnitz Theorem.

From another hand, since the function \( A(x) \) is defined as some area, we can write it in “a bit strange manner”, using the function \( g \), which in used for this area. Let \( a \in \mathbb{R} \) as above and \( x > a \) be fixed. Then geometrical figure (whose are we are calculating to obtain \( A(x) \)) may be considered as consisted of many-many rectangles of the small with \( \Delta x \) (which will become \( dx \) later) and of the height \( g(s) \) (for each \( s \in [a, x] \)). Denote \( n = \frac{x-a}{\Delta x} \). Then the area \( A(x) \) can be

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calculated

\[ A(x) \approx \sum_{k=0}^{n} g(a + k\Delta x)\Delta x. \]  

\( \text{(2.12)} \)

Figure 2:

When we decrease \( \Delta x \) (in fact, change \( \Delta x \) to the differential \( dx \)), then the limit value of the expression \((2.12)\) is denoted by

\[ A(x) = \int_{a}^{x} g(s) \, ds. \]  

\( \text{(2.13)} \)

The expression \((2.13)\) is called the definite integral (of the function \( g \) with boundaries \( a \) and \( x \)). The word “integral” here means the collection of “infinitely many rectangles of infinitely small width” such that the sum of the areal of these rectangles equals to the area \( A(x) \).

Figure 3:

For example, if we need to find the area “between real points” \( b \) and \( c \) (i.e. the areas, which is denoted at Figure 3), then we calculate it as

\[ A = A(c) - A(b) \]  

\( \text{(2.14)} \)
and write $A = \int_b^c g(s) \, ds$. Notice, that it would not be good enough if we will use $x$ in the notation

$$\int_a^x g(x) \, dx$$

in (2.13), because $x$ in the upper bound of the integral and $x$ in the expression $g(x)$ have the completely different meaning. In the same time, we are free to write both $A = \int_b^c g(s) \, ds$ or $A = \int_b^c g(x) \, dx$ for the area (2.14).

Thus, we can formulate the general rule for the calculating the area $\int_{x_1}^{x_2} g(t) \, dt$:

1. Find the function $f$ such that $f' = g$.
2. The necessary area will be $f(x_2) - f(x_1)$.

We need to notice, that we meant “not exactly the area” under the word “area” in this section. We have calculated the area by the formula (2.12), assuming that $\Delta x \approx 0$ but with the additional assumption that $g(s) > 0$ for $s \in [a, x]$. If the function $g$ is negative at some (or all) points of $[a, x]$, then (2.12) will, clearly, have negative summands and the entire sum can also be negative. Such method of the calculation of the area (precisely, such “area”) is called the the **signed area**. If we need “real area”, we, evidently should to calculate it as $\int_a^x |g(s)| \, ds$.

Notice, that the finding of the function $f$ (“finding” here means “finding the explicit formula for $f'$”) such that $f' = g$ (for the given $g$ needs some special techniques, which, clearly, depends of the function $g$).

Suppose that we have an equation

$$\frac{f(x)}{g(y)} = \frac{dy}{dx}, \quad (2.15)$$

where $y$ is considered as unknown function on $x$, which has to be found, whenever functions $f$ and $g$ are given. Notice, that (2.15) is also called the differential equation of the separated variables, because it can be rewritten as

$$f(x) \, dx = g(y) \, dy.$$  

\[1\]This rule is not “completely correct”. It should be specified for some “complicated situation”, i.e. if the function $f$ has vertical asymptotes, or is braking. But we write this rule just for “the first acquaintance”.

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2.3 Interesting theorems about derivatives and integrals

i.e. in the form where all the “letters” \( x \) are left at one hand of the equation and all the letter \( y \) are put to another.

**Fact 7.** The solution of (2.15) can be written in the form

\[
\int f(x) \, dx - \int g(y) \, dy = C(x),
\]

where \( C'(x) = 0 \).

2.3 Interesting theorems about derivatives and integrals

2.3.1 Calculating of derivatives of \( \sin x \) and \( \cos x \) as simple corollary of the physical interpretation of uniform movement by a circle

Suppose that a point moves anti clock wise on a circle with radius 1 and center at the Origin. Clearly, the equation of the movement will be

\[
\begin{aligned}
    x &= \cos(\omega t + \omega_0) \\
    y &= \sin(\omega t + \omega_0),
\end{aligned}
\]

where \( \omega \) is the radial speed and \( \omega_0 \) is the angle at time 0 between the radius vector of our point and the \( X \)-axis. For the simplicity of the further reasonings assume that \( \omega_0 = 0 \).

Since the period of functions \( \sin \) and \( \cos \) is \( 2\pi \), then the period of the rotation of our point can be calculated from the equation \( \omega T = 2\pi \), whence

\[
T = \frac{2\pi}{\omega}.
\]

Since the length of the circle is \( 2\pi \), then the constant speed is

\[
|\vec{v}| = \frac{2\pi}{T} = \omega.
\]

Evidently the vector of the velocity is the tangent vector to the circle, i.e. \( \vec{v} \) is perpendicular to \( s = (\cos \omega t, \sin \omega t) \), whence

\[
v = k(- \sin \omega t, \cos \omega t)
\]

for some \( k \in \mathbb{R} \).

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From another hand, $|k| = \omega$, since $|\vec{v}| = \omega$. Moreover, $k > 0$, because the movement is anti clock wise. Thus, $\vec{v} = \omega(-\sin \omega t, \cos \omega t)$ and this means that
\[
\begin{cases}
\cos' \omega t = -\omega \sin \omega t, \\
\sin' \omega t = \omega \cos \omega t.
\end{cases}
\]
plug $\omega = 1$ and obtain formulas
\[
\begin{cases}
\cos' t = -\sin t, \\
\sin' t = \cos t.
\end{cases}
\]

2.3.2 Simple proof of the formula for the radius of curvature for the plain curve

When we introduced the derivative, we looked at any curve as the collection of line segments, whose direction changes from point to point. Thus, each point, “together with its neighbor point” determine some line (or the direction of line). In the same manner we can consider each point of the curve at “two its neighbor points”. These three points, being the vertices of the triangle (if our curve does not contain line segments) will determine a circle, which circumscribes this triangle.

We will need some mathematical facts.

**Fact 8.** Let $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$ be two non-collinear vectors on the plane $XOY$. Denote

$$P_3[\vec{u}, \vec{v}] = x_1y_2 - x_2y_1.$$  

Then the area of the parallelogram, whose sides are $\vec{u}$ and $\vec{v}$ equals

$$A = |P_3[\vec{u}, \vec{v}]|.$$  

**Proof.** Denote $\theta$ the angle between $\vec{a}$ and $\vec{b}$. By the formula for a dot-product we have

$$x_ax_b + y_ay_b = ||a|| \ ||b|| \cos \theta. \quad (2.16)$$

By the formula for the area of a parallelogram we have

$$A = ||a|| \ ||b|| \sin \theta. \quad (2.17)$$

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If follows from (2.16) and (2.17) that
\[
A^2 = ||a||^2 ||b||^2 \left( 1 - \frac{(x_a x_b + y_a y_b)^2}{||a||^2 ||b||^2} \right) = (x_a^2 + y_a^2) (x_b^2 + y_b^2) - (x_a x_b + y_a y_b)^2 =
\]
\[
= (\frac{x_a^2 x_b^2}{x_a x_b} + x_a^2 y_b + x_b^2 y_a + y_a^2 y_b) - (\frac{x_a^2 x_b^2}{x_a x_b} + 2x_a x_b y_a + y_a^2 y_b) =
\]
\[
= x_a^2 y_b + x_b^2 y_a - 2x_a x_b y_a =
\]
\[
= (x_a y_b - x_b y_a)^2.
\]

Let us have vectors \( \overrightarrow{a} = (x_a, y_a) \) and \( \overrightarrow{b} = (x_b, y_b) \) and find the area of the parallelogram, whose sides are \( \overrightarrow{a} \) and \( \overrightarrow{b} \).

Notice, that \([\overrightarrow{a}, \overrightarrow{b}]\) is called vector product of \( \overrightarrow{u} \) and \( \overrightarrow{v} \) and is a vector
\[
[\overrightarrow{u}, \overrightarrow{v}] = (0, 0, x_1 y_2 - x_2 y_1)
\]
and \( P_3 \) is the projection of a vector to its third coordinate, i.e.
\[
P_3(x, y, z) = z.
\]

We will explain some basic things about vector product in Section 7.8.1 Nevertheless, we do not need the notion of vector product now and we will be satisfied with Fact 8.

The following properties of \( P_3[\overrightarrow{u}, \overrightarrow{v}] \) follow from the definition:
\[
P_3[\overrightarrow{u}_1 + \overrightarrow{u}_2, \overrightarrow{v}] = P_3[\overrightarrow{u}_1, \overrightarrow{v}] + P_3[\overrightarrow{u}_2, \overrightarrow{v}],
\]
\[
P_3[\overrightarrow{u}, \overrightarrow{v}_1 + \overrightarrow{v}_2] = P_3[\overrightarrow{u}, \overrightarrow{v}_1] + P_3[\overrightarrow{u}, \overrightarrow{v}_2],
\]
\[
P_3[\alpha \overrightarrow{u}_1, \beta \overrightarrow{v}_1] = \alpha \beta P_3[\overrightarrow{u}_1, \overrightarrow{v}_1],
\]
\[
P_3[\overrightarrow{u}_1, \overrightarrow{v}_1] = 0,
\]
for all vectors \( \overrightarrow{u}_1, \overrightarrow{u}_2, \overrightarrow{v}_1 \) and \( \overrightarrow{v}_2 \) and numbers \( \alpha, \beta \in \mathbb{R} \).

**Fact 9.** Let \( s(t) \) be a vector equation of the curve. Then this curve can be locally considered as a circle with radius
\[
R = \frac{|s'(t_0)|^3}{|P_3[s'(t_0), s''(t_0)]|}.
\]

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Proof. Let \( s(t) \) be an equation of our curve. Consider an arbitrary moment \( t_0 \) and \( \Delta t \approx 0 \). The circle, which we are talking about passes through points \( s(t_0), s(t_0 + \Delta t) \) and \( s(t_0 + 2\Delta t) \). Notice that these three are vectors, whence can be considered as points on a plain, is we assume that the vectors have start point at the Origin.

We will find now the radius of the circle, which is described about the triangle with vertices \( s(t_0), s(t_0 + d\ t) \) and \( s(t_0 + 2d\ t) \).

We can use differentials to express \( s(t_0 + d\ t) \) and \( s(t_0 + 2d\ t) \) in terms of \( s(t_0), s'(t_0) \) and \( s''(t_0) \). Since

\[
s'(t_0) = \frac{s(t_0 + d\ t) - s(t_0)}{d\ t},
\]

then

\[
s(t_0 + d\ t) = s(t_0) + s'(t_0) d\ t.
\]

Also by

\[
s''(t_0) = \frac{s'(t_0 + d\ t) - s'(t_0)}{d\ t} = \frac{s(t_0 + 2d\ t) - s(t_0 + d\ t) - s'(t_0)}{d\ t}
\]

obtain

\[
s(t_0 + 2d\ t) = (s''(t_0) d\ t + s'(t_0)) d\ t + s(t_0 + d\ t),
\]

whence

\[
s(t_0 + 2d\ t) = s(t_0) + 2s'(t_0) d\ t + s''(t_0)(d\ t)^2.
\]

Thus, we want to find the radius of the circle, which is described about the triangle with sides \( \vec{a} = s(t_0 + d\ t) - s(t_0), \vec{b} = s(t_0 + 2d\ t) - s(t_0 + d\ t) \) and \( \vec{c} = s(t_0 + 2d\ t) - s(t_0) \). We can express \( \vec{a}, \vec{b} \) and \( \vec{c} \) as

\[
\vec{a} = s'(t_0) d\ t;
\]

\[
\vec{b} = s''(t_0)(d\ t)^2 + s'(t_0) d\ t;
\]

\[
\vec{c} = 2s'(t_0) d\ t + s''(t_0)(d\ t)^2.
\]

We know, that the radius of the circle, which is described over the triangle, can be found as

\[
R = \frac{|\vec{a}| |\vec{b}| |\vec{c}|}{4A},
\]

where \( |\vec{a}|, |\vec{b}|, |\vec{c}| \) are sides of the triangle and \( A \) is its area.

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By Fact 8 the area of the triangle, formed by these vectors is

\[ A = P_3[\vec{a}, \vec{b}] = P_3[s'(t_0) \, dt, \, s''(t_0)(dt)^2 + s'(t_0) \, dt] \]

and can be simplified as

\[ A = P_3[s'(t_0) \, dt, \, s''(t_0)(dt)^2 + s'(t_0) \, dt] = P_3[s'(t_0) \, dt, \, s''(t_0)(dt)^2] = P_3[s'(t_0), \, s''(t_0)](dt)^3 \]

Since \( dt \approx 0 \), then \((dt)^2\) is much smaller than \( dt\). Whence, we can suppose that

\[ |\vec{a}| \approx |s'(t_0)| \, dt, \]
\[ |\vec{b}| \approx |s'(t_0)| \, dt \]

and

\[ |\vec{c}| \approx 2|s'(t_0)| \, dt. \]

Thus,

\[ R = \frac{|\vec{a}| \, |\vec{b}| \, |\vec{c}|}{2P_3[\vec{a}, \vec{b}]} \approx \frac{|s'(t_0)|^3}{|P_3[s'(t_0), s''(t_0)]|.} \]

Notice, that \((dt)^2\) is as close to 0, comparing with \( dt \), as \( dt \) itself is close to 0. But \( dt \) is “as close to 0, as possible”, whence

\[ R = \frac{|s'(t_0)|^3}{|P_3[s'(t_0), s''(t_0)]|} \]

and the equality is exact, but not just “approximate”.

\[ \square \]

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3 Additional mathematical notions and facts

In this section we will formulate mathematical facts, which we will use them in our explanations about Kepler’s laws. We will try to explain these facts as clear as possible (without proves, just explanations of facts themselves), but we will not stay on technical calculations.

Reader, for whom these facts are new, has to believe them for the first reading, but to try to understand, what these facts claim, i.e. to understand, what is stated in the facts, because they will be used in further computations. The experienced mathematician can understand the level of math, which will be used later. More or less detailed explanations of the formulated facts are given in section 7.

3.1 Polar coordinates

Any coordinates of a point in a plane are exactly the way of the coding the its position by numbers. Precisely, the known cartesian coordinates of a point $A$ are projections $(x_A, y_A)$ of the vector $OA$ to the fixed “coordinate lines” $Ox$ and $Oy$, passing through the point $O$.

Polar coordinates are constructed as follows. Suppose that we already have cartesian coordinates (the the origin $O$ and axes $Ox$ and $Oy$) on the plane. Call the ray $Ox$ the axis (it will be unique, so we do not need to mention that it is $x$-axis) and call $\theta$ the angle between $OA$ and $Ox$, calculating from $Ox$ to $OA$ (for example, points on the positive $y$-axis have $\theta = 90^0$; points on the negative $y$-axis have $\theta = 270^0$ and points on the line $y = x$ have $\theta = 45^0$ and $\theta = 135^0$, dependently on the quadrant). Denote $r$ the length of the segment $OA$, whence the pair

$$(r, \theta), \ r \geq 0, \ \theta \in [0, 360^0)$$

(3.1)

defines all the points on the plain. Moreover, if $r > 0$, then the correspondence is one-to-one and $O$ corresponds to $r = 0$ with any $\theta$.

Fact 10. If $(r, \theta)$ are polar coordinates (3.1) of a point $A$ on a plane, then its cartesian coordinates are

$$\begin{align*}
    x &= r \cos \theta, \\
    y &= r \sin \theta.
\end{align*}$$

(3.2)
Fact 11. Suppose that the position-vector of a point is given by polar coordinates \((3.1)\). Then speed of this point can be found by

\[
v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2
\]

(3.3)

3.2 Equation of the plane

Fact 12. The set of points on the space, which satisfy the equation

\[Ax + By + Cz = D\]

is a plain. Moreover, the equation of each plain is of this form.

3.3 Ellipse

An ellipse is a plane geometrical figure, which is defined as follows. Fix two points, say \(F_1, F_2\) on the plain, which will contain the ellipse. The ellipse is consisted of all the points \(M\) of the plain such that \(MF_1 + MF_2 = 2a\), where \(a\) is fixed at the very beginning. Points \(F_1\) and \(F_2\) are called focuses of the ellipse.

Figure 4 contains an ellipse, whose so called "focal line" (i.e. line, which contains focuses) is the \(x\)-axis and origin \(O\) is the midpoint of \(F_1F_2\). Points of the intersection of the ellipse and the \(x\)-axis are denoted \(A\) and \(C\). It is clear that in this case \(a = AO = OC\). Denote \(B, D\) the intersections of the ellipse with the \(y\)-axis and \(b = BO = OD\). Notice, that \(F_1O = \sqrt{a^2 - b^2}\), whence \(a \geq b\).

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Fact 13. If the Cartesian coordinates of the focuses of the ellipse are $F_1(-f, 0)$ and $F_2(f, 0)$, then

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

is an equation of the ellipse.

The number

$$e = \frac{\sqrt{a^2 - b^2}}{a}$$

is called eccentricity of the ellipse. It is clear that $e \in [0, 1)$ and $e = 0$ if and only if the ellipse is a circle.

Fact 14. If $F_1$ is in the polar origin and $F_2$ is on the polar axis, then the equation of the ellipse is

$$r = \frac{p}{1 - e \cos \theta}$$

where

$$p = \frac{b^2}{a}$$

and is called semi-latus rectum of the ellipse.

Fact 15. The area of the ellipse can be calculated as $A = \pi ab$.

3.4 Mathematical expression of the Second Kepler’s law

Fact 16. Let the trajectory of the planet is given in Cartesian coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a > b > 0$. Then the Second Kepler’s law can be rewritten as

$$x \frac{dy}{dt} - y \frac{dx}{dt} = C,$$

where $C$ is a constant, dependent on the planet and independent on time. Moreover,

$$C = \frac{A}{T},$$

where $A$ is the area of the ellipse, which is the trajectory of the planet and $T$ is the period of the rotation.

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Fact 17. If the Sun is in the Origin, then the Second Kepler’s law in polar coordinates is expressed as

\[ r^2 \frac{d\theta}{dt} = C, \]

where \( C \) is the constant from Fact 16.

Fact 18. Denote \( u = \frac{1}{r} \). Then in polar coordinates the Second Kepler’s Law can be written as

\[ v^2 = C^2 \left[ \left( \frac{du}{dt} \right)^2 + u^2 \right]. \]
4 Results about Kepler’s laws

The main deal of this section is to show the equivalence of the Newton’s Laws and the Kepler’s Laws. This demonstration will have the pure nature os some pure mathematical computations. The ideas of these computations is taken from [1].

Nevertheless, since the Newton’s Laws contain the notion of the force, we need some remark, how we will understand the Force from the mathematical point of view, since there is not this notion in mathematics.

We will consider the second Newton’s Law as the definition of the force. In other words, if the mass of some body is fixed (all masses will be considered as fixed) and force \( F \) acts on his body, then acceleration 
\[
a = \frac{F}{m}
\]
will appear. We will understand any acceleration (calculated mathematically) as the result of some force and controversially, is the mathematical calculation will show us an acceleration, the we will interpret it as the result of some force.

4.1 Two-bodies problem

Let two bodies with masses \( m \) and \( M \) (say Sun and a planet) move such that the 4-th Newton’s Law holds. In other words, denote \( r(t) \) the distance between these bodies and assume that there is a force
\[
F(t) = k \cdot \frac{mM}{r^2(t)},
\]
which acts to each of these body in the direction of another. Clearly, this force (these forces) cause the acceleration \( \vec{a}_S \) of the Sun with absolute value
\[
|\vec{a}_S| = k \cdot \frac{m}{r^2(t)}
\]
and the acceleration of the planet \( \vec{a}_P \) with absolute value
\[
|\vec{a}_P| = k \cdot \frac{M}{r^2(t)}.
\]
4.1 Two-bodies problem

We will fix the Sun at the origin. This will mean that the speed and the acceleration of the Sun will be 0. It will give all other points of the coordinate system the additional acceleration $a_s$, directed to the Origin. Thus, the acceleration $\vec{a}$ of the planet in “the fixed” coordinate system will be

$$|\vec{a}| = |\vec{a}_S - \vec{a}_P|.$$ 

Since the directions of $\vec{a}_S$ and $\vec{a}_P$ are opposite, then

$$|a(t)| = k \cdot \frac{m}{r^2(t)} + k \cdot \frac{M}{r^2(t)} = \frac{k(m + M)}{r^2(t)}.$$ 

Denote $\mu = k(m + M)$ and we are ready now to formulate the mathematical problem, which we will solve.

**Mathematical problem.** Denote

$$\vec{s}(t) = (x(t), y(t), z(t))$$

the position vector of some point and denote

$$r(t) = \sqrt{x^2(t) + y^2(t) + z^2(t)}.$$ 

Suppose that

$$\frac{d^2 \vec{s}}{dt^2} = -\frac{\mu}{r^2(t)} \cdot \frac{\vec{s}(t)}{|\vec{s}(t)|}. \quad (4.1)$$

We need to find (to describe) the function $\vec{s}(t)$. Precisely, we need to prove that $\vec{s}(t)$ satisfies the Kepler’s Laws.

Notice, that the equation (4.1) means nothing more than the acceleration $\vec{a} = \frac{d^2 \vec{s}}{dt^2}$ has the absolute value $\frac{\mu}{r^2(t)}$ (we suppose $\mu$ to be positive) and the multiplier $\frac{\vec{s}(t)}{|\vec{s}(t)|}$ together with minus-sign before the entire expression mans that the direction of $\vec{a}$ is opposite to $\vec{s}$, i.e. the acceleration is directed from the position of the Planet to the origin.

We can rewrite (4.1) as

$$\frac{d^2 x}{dt^2} = -\mu \frac{x}{r^3}, \quad \frac{d^2 y}{dt^2} = -\mu \frac{y}{r^3}, \quad \frac{d^2 z}{dt^2} = -\mu \frac{z}{r^3}. \quad (4.2)$$

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The obtained (4.2) is called the “second order system of differential equations”. The name “differential equation” is because the differentials (derivatives) are parts of equations of this system. The order of a system of differential equations is the maximal derivative, which appears in the equations (clearly, this maximum here is 2, being, for instance, the order of the derivative \( \frac{d^2x}{dt^2} \), but the third or more order derivative does not appear in the system of equations).

Notice, that tree equations of (4.2) are not independent, because \( r \), being the distance from the planet to the origin, is dependent on \( x, y, z \), precisely, for example the first equation of (4.2) is, in fact,

\[
\frac{d^2x}{dt^2} = -\frac{\mu x}{(\sqrt{x^2 + y^2 + z^2})^3}
\]

and in the same way other two equations can be rewritten.

There are no general rules of solving the systems of differential equations. Students of mathematical faculties of universities study the classical course, which is called differential equations theory. They study there the methods of solving different types of differential equations, or system of differential equations. Clearly, specialists on differential equations consider the equations, which are studied in the university course to by “standard” and “the simplest”. Knowing of methods of solving of “standard differential equations” is considered like a “time table”, which should be clear for everybody, who studies differential equations. If a differential equation has no standard method to solve, the it is necessary to combine the knowing standard methods, or in any way use the talent or intuition.

The useful notion, which can help to solve a differential equation is so called the first integral. Let one have a system of differential equations with respect to unknown functions \( x(t), y(t) \) and \( z(t) \) (as in our case). The first integral if some functional expression, which contains the symbols for \( t, x, y, z \) and all the derivatives, such that the substitution of the solution to this expression transforms it to zero. The usefulness of the first integral is in that, it may help to make some conclusions about the properties of the solution, without finding the solutions explicitly. Clearly, more than one the first integral can be necessary to find the final (explicit) solution of the differential equation.

Let us start to find on of the first integral of (4.2). Multiply the second and third equation...
4.1 Two-bodies problem

of (4.2) by \( z \) and \( y \) respectively and obtain

\[
\begin{align*}
  z \frac{d^2 y}{dt^2} &= -\mu \frac{yz}{r^3}, \\
  y \frac{d^2 z}{dt^2} &= -\mu \frac{yz}{r^3}.
\end{align*}
\]

The substraction of these equations leads to

\[
y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2} = 0.
\] (4.3)

Notice that

\[
\frac{d}{dt} \left( y \frac{dz}{dt} - z \frac{dy}{dt} \right) = y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2},
\] (4.4)

because

\[
\frac{d}{dt} \left( y \frac{dz}{dt} - z \frac{dy}{dt} \right) = \left( \frac{dy}{dt} \frac{dz}{dt} + y \frac{d^2 z}{dt^2} \right) - \left( \frac{dz}{dt} \frac{dy}{dt} + z \frac{d^2 y}{dt^2} \right) = y \frac{d^2 z}{dt^2} - z \frac{d^2 y}{dt^2}.
\]

Denote \( q(t) = y \frac{dz}{dt} - z \frac{dy}{dt} \), whence rewrite equalities (4.3) and (4.4) as \( \frac{dq}{dt} = 0 \). This means that \( q \) is a constant function, i.e. there exists \( A = \text{const} \) such that

\[
y \frac{dz}{dt} - z \frac{dy}{dt} = A.
\]

Do the analogical transformations with another pairs of equations of (4.2). Multiply the first and the third equation by \( z \) and \( x \), then subtract them and obtain \( \frac{d}{dt} \left( z \frac{dx}{dt} - x \frac{dz}{dt} \right) = 0 \). Similarly obtain \( \frac{d}{dt} \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) = 0 \) from the first two equations of (4.2). Thus,

\[
\begin{cases}
  y \frac{dz}{dt} - z \frac{dy}{dt} = A, \\
  z \frac{dx}{dt} - x \frac{dz}{dt} = B, \\
  x \frac{dy}{dt} - y \frac{dx}{dt} = C,
\end{cases}
\] (4.5)

where \( A, B, C \) are some constants.

Multiply equations of (4.5) by \( x, y \) and \( z \) respectively, then add them and obtain

\[
Ax + By + Cz = 0.
\] (4.6)

By Fact 12. This equation describes a plain, i.e. the orbit of the planet is a plan curve and the Sun (the Origin) belongs to the plain of this curve.

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Notice ones more, that (4.6) is one more example of the first integral of (4.2).

Since the movement of the planet is plain, then we can simplify the former system of equations (4.2). Suppose that the orbit belongs to the XOY-plane, whence \( z(t) \) will become a zero-function, and (4.5) will be simplified as

\[
\frac{x}{d} \frac{dy}{dt} - y \frac{dx}{dt} = C_1,
\]

where \( C_1 \) is some new constant, which is not necessary to be equal to \( C \) from (4.5).

Notice, that the obtained equality is exactly the **Second Kepler’s Law** from Fact 16.

We will find now one more first integral of (4.2). Multiply the first equality by \( 2 \frac{dx}{dt} \) and the second by \( 2 \frac{dy}{dt} \). After the addition obtain

\[
2 \left[ \frac{dx}{dt} \frac{d^2 x}{dt^2} + \frac{dy}{dt} \frac{dy}{dt} \right] = - \frac{2\mu}{r^3} \left[ x \frac{dx}{dt} + y \frac{dy}{dt} \right].
\]

Remind, that we have already obtained got that \( z = 0 \), thus (4.2) contains only two equations.

For the simplification of (4.7), observe that

\[
\frac{dv^2}{dt} = 2 \frac{dx}{dt} \frac{d^2 x}{dt^2} + 2 \frac{dy}{dt} \frac{d^2 y}{dt^2}
\]

and

\[
\frac{d}{dt} \left( r^2 \right) = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.
\]

Indeed,

\[
v^2 = \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2
\]

implies

\[
\frac{dv^2}{dt} = \frac{d}{dt} \left( \left( \frac{dx}{dt} \right)^2 \right) + \frac{d}{dt} \left( \left( \frac{dy}{dt} \right)^2 \right) = 2 \frac{dx}{dt} \frac{d^2 x}{dt^2} + 2 \frac{dy}{dt} \frac{d^2 y}{dt^2}.
\]

From another hand, since

\[
r^2 = x^2 + y^2,
\]

then

\[
\frac{d}{dt} \left( r^2 \right) = \frac{d}{dt} \left( x^2 + y^2 \right) = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.
\]
Thus, the equation \((4.7)\) is equivalent to
\[
\frac{dv^2}{dt} = -\frac{2\mu}{r^3} \frac{dr^2}{dt}.
\] (4.8)

Moreover, notice that
\[
-\frac{\mu}{r^3} \frac{d}{dt}(r^2) = -\frac{\mu}{r^3} \cdot r \frac{dr}{dt} = -\frac{\mu}{r^2} \frac{dr}{dt}
\]
and
\[
\frac{d}{dt} \left( \frac{2\mu}{r} \right) = -\frac{\mu}{r^2} \frac{dr}{dt}.
\]

Thus, we can rewrite \((4.8)\) as
\[
\frac{dv^2}{dt} = \frac{d}{dt} \left( \frac{2\mu}{r} \right),
\]
or
\[
\frac{d}{dt} \left( v^2 - \frac{2\mu}{r} \right) = 0.
\]

Thus, there is a constant \(h\) such that
\[
v^2 - \frac{2\mu}{r} = h. \tag{4.9}\]

We are ready now to obtain the equation of the movement of the planet. We will make the further calculations in polar coordinates.

By Fact 18, we can rewrite the Second Kepler’s Law in polar coordinates as
\[
v^2 = c^2 \left[ \left( \frac{du}{dt} \right)^2 + u^2 \right],
\]
where \(c\) is some constant and \(u = \frac{1}{r}\). This lets us to rewrite \((4.9)\) as
\[
c^2 \left( \frac{du}{d\theta} \right)^2 = h + 2\mu u - c^2 u^2.
\]

This differential equation can be reduced to one of separate variables as
\[
d\theta = \pm \frac{d(cu)}{\sqrt{h + 2\mu u - c^2 u^2}}. \tag{4.10}
\]

Keeping in mind the fact that
\[
\int \frac{-dx}{\sqrt{1-x^2}} = \arccos x,
\]

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transform the expression under the square root in the denominator of (4.10) as

\[ h + 2\mu u - c^2 u^2 = \left( \frac{\mu^2}{c^2} + h \right) - \left( cu - \frac{\mu}{c} \right)^2. \]

Denote \( q = \frac{\mu^2}{c^2} + h \) and \( \xi = cu - \frac{\mu}{c} \), notice that \( d \left( cu - \frac{\mu u}{c} \right) = d \xi \), whence rewrite (4.10) as

\[
d\theta = \pm \frac{d\xi}{\sqrt{q^2 - \xi^2}} = \pm \frac{d\left(\frac{\xi}{q}\right)}{\sqrt{1 - \left(\frac{\xi}{q}\right)^2}}.\]

Then

\[ \pm \theta = \arccos \left( \frac{\xi}{q} \right) + k. \]

Whence

\[ \frac{\xi}{q} = \pm \cos(\theta - k), \]

or

\[ \xi = \pm q \cos(\theta - k). \]

Plug the previous expressions for \( q \), \( \xi \) and \( u \) and obtain

\[ \frac{c}{r} - \frac{\mu}{c} = \pm \sqrt{\frac{\mu^2}{c^2} + h \cos(\theta - k)}, \]

or

\[ \frac{c}{r} = \frac{\mu}{c} \left[ 1 \pm \sqrt{1 + \frac{c^2 h}{\mu^2} \cos(\theta - k)} \right]. \]

Resolve the obtained expression with respect to \( r \) and obtain

\[
r = \frac{\frac{c^2}{\mu}}{1 \pm \sqrt{1 + \frac{c^2 h}{\mu^2} \cos(\theta - k)}}, \tag{4.11}
\]

Notice, that choose \( \theta \) for \( \theta + 180^\circ \) means the choose of the direction of the polar axis, whence without losss of generality assume that we have “−” instead of ±. Now denote \( p = \frac{c^2}{\mu} \), \( e = \sqrt{1 + \frac{c^2 h}{\mu^2}} \) and \( \nu = \theta - k \) for obtain

\[ r = \frac{p}{1 - e \cos \nu}, \]

which is the known equation of the curve of the second order.

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Since all the closed curves of the second order are ellipse, hyperbola and parabola, claim that for the planets from the Solar system trajectory is an ellipse. Clearly, that for other pair of bodies (for example, the Sun and some comets) this trajectory can be as hyperbola, as parabola too.

4.2 Equivalence of the Third Kerpler’s Law and the fourth Newton’s law

**Fact 19.** Suppose that the Sun is fixed in the Origin and a planet moves by the elliptic orbit with polar equation

\[ r = \frac{p}{1 - e \cos \theta} \]

and the equation

\[ r^2 \frac{d\theta}{dt} = C \] \hspace{1cm} (4.12)

holds. Then the acceleration of the planet is

\[ \overrightarrow{a} = -\frac{\overrightarrow{r}}{|\overrightarrow{r}|} \cdot C^2 \frac{\overrightarrow{r}}{pr^2}. \] \hspace{1cm} (4.13)

The proof of this fact is simple, but technical. The idea of the proof is the following. Since we have polar coordinates, we have equalities

\[ \begin{align*} x &= r \cos \theta; \\
y &= r \sin \theta, \end{align*} \] \hspace{1cm} (4.14)

where both \( \theta \) and \( r \) are unknown functions, dependent on \( t \). Formula (4.12) lets us to express \( x \) and \( y \) in terms of \( \theta \) as

\[ x(t) = r(t) \cos \theta(t) = \frac{p \cos \theta(t)}{1 - e \sin \theta(t)} \]

and analogously for \( y \).

By Fact 2 we need to find the second derivative of the position function \( s(t) = (x(t), y(t)) \). We will use Fact 1 which means that we will find separately the derivatives \( x''(t) \) and \( y''(t) \), saying that \( a(t) = (x''(t), y''(t)) \).

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We can find the derivative $\theta'(t)$ from (4.12), because, in fact, this equation is nothing, but this derivative. We will use this derivative together with the Chain rule (Fact 3), Product rule (Fact 4) and some table derivatives (Fact 5).

The proof of Fact 19 in details is given at Section 7.6.

Fact 19 has an important corollary. The proof of this corollary uses only Fact 19 and some technical calculations with parameters of an ellipse, which were introduced in Section 3.3.

Fact 20. Suppose that the planets moves such that Newton’s the First and the Second Kepler’s laws hold. Then the Third Kepler’s law is equivalent to the existence of the constant $k$ (which is the same for all planets) such that

$$|\vec{a}| = \frac{k}{r^2},$$

(4.15)

where $a$ is the acceleration of a planet and $r$ is its distance to the Sun.

Proof. Suppose that the Third Kepler’s law holds, i.e. there exists a constant $d$ (which is the same for all planets) such that

$$d = \frac{a^3}{T^2},$$

(4.16)

where $a$ is the half of the bigger diameter of the ellipse, which is the trajectory of the planet (we will use notations from Section 3.3). By Facts 14 and 15, i.e. using formulas

$$A = \pi ab$$

and

$$p = \frac{b^2}{a},$$

we can rewrite (4.16) as

$$\pi^2 d = \frac{\pi^2 a^3}{T^2} = \frac{\pi^2 a^2 b^2}{T^2} \cdot \frac{a}{b^2} = \frac{A^2}{p T^2}.$$  

Next, by Fact 16 we have $A = CT$ (where $C$ is taken from the Second Kepler’s Law, i.e. it is the same $C$, which is in the formulation of Fact 19). Thus, using Fact 19 calculate the acceleration (4.13) as

$$|\vec{a}| = \frac{C^2}{p r^2} = \frac{A^2}{p r^2 T^2} = \frac{\pi^2 d}{r^2}.$$  

Now denote $k = \pi^2 d$ and (4.15) follows.

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Conversely, suppose (4.15). By Fact 19 write

$$|\vec{a}| = \frac{k}{r^2} = \frac{C^2}{p r^2}.$$ 

Using Fact 15 rewrite the obtained equality as

$$k = \frac{A^2}{p C^2}.$$ 

Express, as in the fist part of the proof, $A$ and $p$ by Facts 14 and 15 obtain

$$k = \frac{\pi^2 a^2 b^2}{C^2} \cdot \frac{a}{b^2} = \frac{\pi^2 a^3}{T^2}.$$ 

Denote

$$d = \frac{k}{\pi^2}$$

and we are done. \qed
5 Analytic representation of the movement

Suppose that we know, that a planet moves according to Kepler’s Laws. In this section we will use mathematical tools to specify some properties of the movement of this planet.

Since we know that the trajectory of the planet is an ellipse, we can write its polar equation as

\[ r(t) = \frac{p}{1 - \varepsilon \cos \theta(t)} . \]  

Thus the function \( \theta = \theta(t) \) define the law of the movement of our planet by its orbit. This function can be found from the Second Kepler’s Law, written as in Fact 17, i.e.

\[ r^2 \frac{d\theta}{dt} = C . \]

We can use this and the above equation of the ellipse to express the derivative of \( \theta \) as

\[ \theta' = \frac{C}{p^2} (1 - \varepsilon \cos \theta)^2 , \]

whence rewrite the Second Kepler’s Law as

\[ \frac{d\theta}{(1 - \varepsilon \cos \theta)^2} = \frac{C}{p^2} \, dt . \]

This is the differential equation of one of the simplest form, which is called differential equation with separated variables. Using quite standard techniques from the mathematical analysis, we can solve this it and obtain the evident formulas, which will express \( t \) in terms of \( \theta \).

Also we can use these formulas (after we will obtain them) for plotting (and understanding) some properties of the movement of planets by their orbits.

5.1 The parametric equations of the motion of a planet around the sun and their corollaries

We will find the expression for \( \theta(t) \) from the equation (5.3) and this will give the law of the motion of the planet on its orbit.

Denote

\[ \Theta(\theta) = \frac{1}{(1 - \varepsilon \cos \theta)^2} . \]
According to Fact 7, we can write the solution of (5.3) as

\[ \int \Theta(\theta) \, d\theta - \int \frac{C}{p^2} \, dt = \mathcal{T}(t), \]

where \( \mathcal{T}'(t) = 0 \) for all \( t \).

Denote \( I(\theta) \) an infinite integral of \( \Theta(\theta) \), i.e. \( I(\theta) \) is a function such that \( I'(\theta) = \Theta(\theta) \). By Newton-Leibnitz Theorem we can determine \( I(\theta) \) as

\[ I(\theta) = \int_0^\theta \frac{d x}{(1 - \varepsilon \cos x)^2}. \]

This implies that the solution of (5.3) is

\[ \int_0^\theta \frac{d x}{(1 - \varepsilon \cos x)^2} = \frac{C}{p^2} \cdot t + \mathcal{T}(t), \]

where \( \mathcal{T}'(t) = 0 \) for all \( t \).

Equation (5.6) can be considered as law, which lets to find an polar angle \( \theta \) of the planet at the time \( t \). The value \( \frac{C}{p^2} \) depends on “how quickly the time passes”, i.e. whether we measure time in seconds, hours, years etc.

The graphs of the function \( \Theta(\theta) \) for \( \theta \in [0, 3\pi] \) are given at Figures 5, 6 and 7.

The explicit formula for the \( I(\theta) \) of the form (5.5) can be found with the use of the following fact.

**Fact 21.** Suppose that \( \varepsilon \in (0, 1) \). Then

\[ \int \frac{d x}{(1 - \varepsilon \cos x)^2} = \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{2}{(1 - \varepsilon)(1 - \varepsilon^2)} \left( \arctan t + \frac{t \varepsilon}{t^2 + 1} \right), \]

where \( t = \tan \left( \frac{\varepsilon}{2} \cdot \frac{\sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}} \right) \).

We prove Fact 21 in Section 7.7.

Using Fact 21 we can write the solution (5.6) of (5.3) as

\[ \frac{C}{p^2} \cdot t = \mathcal{I}(\theta) + \mathcal{T}(t), \]

(5.7)

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5.1 The parametric equations of the motion of a planet around the sun and their corollaries

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5.1 The parametric equations of the motion of a planet around the sun and their corollaries

where

\[ I(\theta) = \frac{\sqrt{1-\varepsilon}}{\sqrt{1+\varepsilon}} \cdot \frac{2}{(1-\varepsilon^2)(1-\varepsilon)} \times \left( \arctan \left( \tan \left( \frac{2}{\sqrt{1+\varepsilon}} \right) \right) + \frac{\varepsilon \cdot \tan \left( \frac{2}{\sqrt{1+\varepsilon}} \right)}{\tan^2 \left( \frac{2}{\sqrt{1+\varepsilon}} \right) + 1} \right) \]  

and

\[ T'(t) = 0 \]

for all \( t \).

This formula lets to plot (clearly, by some computer techniques, not manually) the graph of the function \( t = t(\theta) \), i.e. plot the dependence of time and the the polar angle of the planet. Remind ones more, that the multiplier \( \frac{C}{p^2} \) determines “how quick the time passes” and an be ignored (can be supposed to be equal 1) in our computations.

The graph of the equation (5.7) is given at pictures 8b and 9b (for \( \varepsilon = 0,1 \) and \( \varepsilon = 0,3 \) respectively).

![Graph of \( \Theta \) of the form (5.4)](image1)

![Graph of \( I(\theta) \) of the form (5.5)](image2)

Figure 8: \( \varepsilon = 0,1 \)

![Graph of \( \Theta \) of the form (5.4)](image3)

![Graph of \( I(\theta) \) of the form (5.5)](image4)

Figure 9: \( \varepsilon = 0,3 \)

It is clear from the graphs, that the functions, which are plotted at Figures 8b and 9b are discontinuous, whence they can not be areas under the graphs of functions from Figures 8a.

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5.2 The speed of the movement

We have seen in Section 5.1 that the law of the dependence of $\theta$ on $t$ (especially in the case when $\varepsilon \approx 1$) can be complicated. In the same way we can establish the dependence of the speed of the planet on time. Using Fact 11 write

$$v^2 = \left( \frac{d}{dt} \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d}{dt} \frac{d\theta}{dt} \right)^2.$$

If follows from (5.1) by chain rule that

$$r'(t) = \frac{-p\varepsilon \sin \theta}{(1 - \varepsilon \cos \theta)^2} \theta'(t).$$

Thus, using (5.2), obtain

$$v^2 = \left( \frac{-p\varepsilon \sin \theta}{(1 - \varepsilon \cos \theta)^2} \right)^2 + r^2 \theta'^2 =$$

$$= \left( \frac{-p\varepsilon \sin \theta}{(1 - \varepsilon \cos \theta)^2} \right)^2 + \left( \frac{p}{1 - \varepsilon \cos \theta(t)} \right)^2 \cdot \frac{C^2}{p^4} (1 - \varepsilon \cos \theta)^4.$$

M. Plakhotnyk: “Kepler’s laws with introduction to differential calculus”
5.2 The speed of the movement

Figure 10: Correct graph of $I(\theta)$ of the form (5.5) for $\varepsilon = 0.3$

a. Graph of $\Theta$ of the form (5.4)  

b. Graph of $I(\theta)$ of the form (5.5)

Figure 11: $\varepsilon = 0.5$

a. Graph of $\Theta$ of the form (5.4)  

b. Graph of $I(\theta)$ of the form (5.5)

Figure 12: $\varepsilon = 0.7$

a. Graph of $\Theta$ of the form (5.4)  

b. Graph of $I(\theta)$ of the form (5.5)

Figure 13: $\varepsilon = 0.9$

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5.2 The speed of the movement

\[
= \left( \left( \frac{\varepsilon \sin \theta}{1 - \varepsilon \cos \theta} \right)^2 + 1 \right) \cdot \frac{C^2}{p^2} \cdot (1 - \varepsilon \cos \theta)^2.
\]

Graphs of the speed in the assumption \(\frac{C}{p} = 1\), dependent on angle, i.e. graphs of the equations

\[
|\vec{v}| = \sqrt{\left( \left( \frac{\varepsilon \sin \theta}{1 - \varepsilon \cos \theta} \right)^2 + 1 \right) \cdot (1 - \varepsilon \cos \theta)^2}
\]

are given on Figures 14, 15 and 16 for \(\theta \in [0, 2\pi]\). These graphs show, how many times the biggest speed is greater the smallest, dependently on \(\varepsilon\).

Notice, that we have obtained in Section 4.1 the formula (4.9), i.e.

\[
v^2 - \frac{2\mu}{r} = h,
\]

where \(\mu\) and \(h\) are constants. This formula lets us to make the conclusion, that the speed of the planet strongly depends on \(r\) and is as huge, as small is \(r\). The constructed graphs confirm this claim. Indeed, the smallest speed appear to be at \(\theta = 0\) and the biggest at \(\theta = \pi\).
5.2 The speed of the movement

Figure 14: Graphs of the speed of the planet

Figure 15: Graphs of the speed of the planet

Figure 16: Graphs of the speed of the planet

M. Plakhotnyk: “Kepler’s laws with introduction to differential calculus”
6 Physical remarks about the Kepler’s laws

The notion of centripetal acceleration is well known from the school course of physics. The fact is following: if a material point of the mass $m$ moves on the circle of radius $R$ with constant speed $v$, then the force $F = \frac{mv^2}{R}$ appears.

6.1 Centripetal acceleration

6.1.1 Physical proof from the majority of textbooks

Just for the completeness, we will present the proof of the formula for the centripetal acceleration, which appear at the majority of books.

Figure 17:

Write a circle (see Fig. 17) and let $AC$ be a vector of the velocity at point $A$. Let $A_1$ be any another point of the circle and $A_1C_1$ be the vector of velocity at this point. Thus, $\overrightarrow{AC} = \overrightarrow{v}(t)$ and $\overrightarrow{A_1C_1} = \overrightarrow{v}(t + \Delta t)$.

Take point $B$ such that $AC = BC_1$. Continue $AC$ and $BC_1$ to obtain points $C_0$ and $B_0$ at the intersection with $AC$ and $OA_1$ respectively. Clearly, $\angle C_0 = \angle B_0$, because $AC$ is parallel to $BC_1$ by the construction. From triangles $OAC_0$ and $B_0A_1C_1$ obtain that

$$\angle AOA_1 = 90^0 - \angle C_0 = 90^0 - \angle B_0 = \angle A_1C_1B_0 = \angle A_1C_1B,$$
whence
\[ \angle AOA_1 = \angle A_1C_1B. \]

Thus, \( \triangle AOA_1 \sim \triangle BC_1A_1 \), because they are isosceles with equal angles between equal sides. From their similarity obtain
\[ \frac{A_1C_1}{A_1O} = \frac{A_1B}{A_1A}. \]
Plug physical values and obtain
\[ \frac{v}{R} = \frac{\Delta v}{\Delta r} \Rightarrow \Delta v = \frac{v\Delta r}{R}. \]

Now,
\[ a = \frac{\Delta v}{\Delta t} = \frac{v}{R} \cdot \frac{\Delta v}{\Delta t} = \frac{v^2}{R} \]
and we are done.

### 6.1.2 Physical proof without triangles

When a point moves by circle of radius \( R \), the vector of its velocity makes the entire rotation (rotation by \( 360^\circ \)) around the origin (we we consider the origin as the start points of the vector.

Thus, the change of the vector of the velocity equals to the length of the circle of the radius \( R \), i.e. \( 2\pi|v| \). The time, which is necessary for this rotation of the time of the rotation of a point, which moves with the velocity \( v \) by circle of radius \( R \), i.e. \( t = \frac{2\pi R}{|v|} \). Thus, the acceleration is
\[ \frac{v^2}{R}. \]

### 6.1.3 Pure mathematical proof

Suppose that a point moves with a constant speed over a circle of radius \( R \) and center at origin. Then Cartesian coordinates of this movement will be
\[
\left\{
\begin{align*}
x &= R \cos \omega t, \\
y &= R \sin \omega t.
\end{align*}
\right. \tag{6.1}
\]

The constant \( \omega \) here is called radial speed and determines, how quickly the point moves on the circle.

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Since the period of functions sin and cos is $2\pi$, then the period of the rotation of our point can be calculated from the equation $\omega T = 2\pi$, whence

$$T = \frac{2\pi}{\omega}.$$ 

Since the length of the circle is $2\pi R$, then the constant speed is

$$|\vec{v}| = \frac{2\pi R}{T} = \omega R.$$ 

Find the velocity vector as the derivative of (6.1).

$$\begin{cases} 
x' = -\omega R \sin \omega t, \\
y' = \omega R \cos \omega t. 
\end{cases}$$

and acceleration will be

$$\begin{cases} 
x'' = -\omega^2 R \cos \omega t, \\
y'' = -\omega^2 R \sin \omega t. 
\end{cases}$$

Thus,

$$|\vec{a}| = \omega^2 R = \frac{v^2}{R}$$

and this result is the same as the results, which were obtained in previous sections.

### 6.2 Forces, which act on a planet during its movement

Thus, we know, that the formula for the centripetal acceleration of the moving on the circle with constant speed is nothing more that the law of the change of the vector of the velocity.

Clearly, the came, quite simple answer can be found by reasonings, which look in such way, like mathematical is not used there. From another hand it is clear, that “mathematics is used” in “physical reasonings” too, this mathematics is almost the same and, in fact, is nothing more that the second derivative.

We will understand in this section the meaning and role of the centripetal acceleration in the moving of a planet by its elliptical orbit, assuming that this orbit is not a circle.

We will try to understand: is there any “centripetal acceleration”? What is its direction and what is its value?

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When we worked with equation of the movement of a planet over its orbit, we have the acceleration of the planet. We supposed that this acceleration is caused by a force of Gravity, which is directed to the Sun and, in fact, is directed to one of the focuses of the ellipse. Clearly, there is no other acceleration of the planet, i.e. the equation of the movement of the planet is an exact function and this function has some exact second derivative, which is the acceleration and determines the force, which acts on the planet.

Nevertheless, we will find now some “another” acceleration.

Our idea will be the following. We will consider an ellipse to be “locally a circle”. This will mean, that is we look “at some point” and an ellipse and look at its “neighbor” points, then they look like a segment of a circle. This circle (the radius and the center) changes from point to point, but each point “has its” circle. We will explain this idea a bit later, but suppose that this is true and such circles exist. Is any movement on the ellipse is considered as a movement by a circle, and speed of the point (planet) and each position is found, we can find the centripetal acceleration and can try to understand, which it will be.

Let’s come back the the understanding of the ellipse as “local circles”.

**Fact 22.** The projection of the acceleration of the point, whose movement is described by vector-equation $s(t)$, to the normal to the velocity at a point $t_0$ equals

$$|a_\nu| = \frac{v^2}{R},$$

where $v = s'(t_0)$ and $R$ is from Fact 9.
7 Detailed mathematical computations

7.1 Derivatives and differentials

7.2 Polar coordinates

Fact (Fact 11). Suppose that the position-vector of a point is given by polar coordinates (3.1). Then speed of this point can be found by

\[ v^2 = \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \]

Proof. Let \((r(t), \theta(t))\) be the equation of the movement in polar coordinates. This means that \(x(t) = r(t)\cos\theta(t)\) and \(y(t) = r(t)\sin\theta(t)\) is the law of the movement in Cartesian coordinates. Thus, we can calculate \(v^2 = (x')^2 + (y')^2\) as follows.

\[
x'(t) = r'(t)\cos\theta(t) - r(t)\theta'(t)\sin\theta(t);
\]
\[
y'(t) = r'(t)\sin\theta(t) + r(t)\theta'(t)\cos\theta(t)).
\]

Thus

\[
v^2(t) = (r'(t)\cos\theta(t) - r(t)\theta'(t)\sin\theta(t))^2 + \]
\[+ (r'(t)\sin\theta(t) + r(t)\theta'(t)\cos\theta(t))^2 =
\]
\[= (r(t))^2 \cos^2\theta(t) + (r(t))^2 (\theta'(t))^2 \sin^2\theta(t) -
\]
\[ - 2r(t)r'(t)\theta'(t)\sin\theta(t)\cos\theta(t) + (r(t))^2 \sin^2\theta(t) +
\]
\[+ (r(t))^2 (\theta'(t))^2 \cos^2\theta(t) + 2r(t)r'(t)\theta'(t)\sin\theta(t)\cos\theta(t) =
\]
\[= (r'(t))^2 + (r(t))^2 (\theta'(t))^2 + 0 =
\]
\[= (r'(t))^2 + (r(t))^2 (\theta'(t))^2.
\]

This finishes the proof. \(\square\)

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7.3 Integration

Fact (Fact 7). The solution of the differential equation

$$\frac{f(x)}{g(y)} = \frac{dy}{dx}$$

can be written in the form

$$\int f(x) \, dx - \int g(y) \, dy = C(x),$$

where $C'(x) = 0$.

Suppose that we have an equation

$$\frac{f(x)}{g(y)} = \frac{dy}{dx}, \quad (7.1)$$

where $y$ is considered as unknown function on $x$, which has to be found, whenever functions $f$ and $g$ are given.

Suppose that we know functions $F$ and $G$ such that $F' = f$ and $G' = g$. Then

$$\frac{d(F(x) - G(y))}{dx} = F'(x) - G'(y) \cdot \frac{dy}{dx} = f(x) - g(y) \cdot \frac{dy}{dx}.$$  

Now (7.1) implies

$$f(x) - g(y) \cdot \frac{dy}{dx} = 0,$$

whence

$$\frac{d(F(x) - G(y))}{dx} = 0. \quad (7.2)$$

The equation (7.2) means that $F(x) - G(y(x))$ is such function, whose derivative as the function of $x$ is zero. The most simple example of such function is zero function (i.e. which is zero everywhere), or constant function, or piecewise-constant function.

More formally, we can say that the solution of the equation (7.1) is

$$\int f(x) \, dx - \int g(y) \, dy = C(x),$$

where $C'(x) = 0$.

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7.4 Ellipse

An ellipse is a plane geometrical figure, which is defined as follows. Fix two points, say \( F_1, F_2 \) on the plain, which will contain the ellipse. The ellipse is consisted of all the points \( M \) of the plain such that

\[
MF_1 + MF_2 = 2a,
\]

(7.3)

where \( a \) is fixed at the very beginning. Points \( F_1 \) and \( F_2 \) are called focuses of the ellipse.

**Fact** (Fact 13). If the Cartesian coordinates of the focuses of the ellipse are \( F_1(-f, 0) \) and \( F_2(f, 0) \), then

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

is an equation of the ellipse.

**Proof.** Denote the cartesian coordinates of focuses of an ellipse by \( F_1(-f, 0) \) and \( F_2(f, 0) \), whence obtain the equation of this ellipse in cartesian plain. Rewrite (7.3) as

\[
\sqrt{(x+f)^2 + y^2} + \sqrt{(x-f)^2 + y^2} = 2a;
\]

\[
(x+f)^2 + y^2 = (2a - \sqrt{(x-f)^2 + y^2})^2;
\]

\[
x^2 + 2xf + f^2 + y^2 = 4a^2 - 4a\sqrt{(x-f)^2 + y^2} + x^2 - 2xf + f^2 + y^2;
\]

\[
2xf = 4a^2 - 4a\sqrt{(x-f)^2 + y^2} - 2xf;
\]

\[
a\sqrt{(x-f)^2 + y^2} = a^2 - xf;
\]

\[
a^2x^2 - 2ax^2f + a^2f^2 + a^2y^2 = a^4 - 2a^2xf + x^2f^2;
\]

\[
x^2(a^2 - f^2) + a^2y^2 = a^2(a^2 - f^2);
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{a^2 - f^2} = 1.
\]

(7.4)

If follows from the equality of the triangle \( F_1F_2 \leq MF_1 + MF_2 \) that \( 2f \leq 2a \). Thus, denote

\[
b^2 = a^2 - f^2,
\]

(7.5)

and rewrite (7.4) as

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.
\]

(7.6)

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Equation (7.6) is called the **canonical equation of an ellipse** in Cartesian form. Due to (7.6) we can imagine an ellipse as an oval figure, which is bounded by lines $x = \pm a$ and $y = \pm b$. If follows from (7.5) that $a \geq b$ and

$$f = \sqrt{a^2 - b^2}. \quad (7.7)$$

The expression

$$\varepsilon = \frac{f}{a} \quad (7.8)$$

is called the **eccentricity** and characterizes the differ of the ellipse and a circle. Notice, that $\varepsilon \in [0, 1)$. Moreover, $\varepsilon = 0$ if and only if an ellipse is the circle. Controversially, if $\varepsilon \approx 1$, then an ellipse transforms to a figure, which is proximate to an line segment between points $(-a, 0)$ and $(a, 0)$.

**Fact (Fact 14).** If $F_1$ is in the polar origin and $F_2$ is on the polar axis, then the equation of the ellipse is

$$r = \frac{p}{1 - \varepsilon \cos \theta},$$

where

$$p = \frac{b^2}{a}$$

and is called semi-latus rectum of the ellipse.

**Proof.** Put the origin to the left focus $F_1$ and the $X$-axis (and polar axis) out through the second focus $F_2$. Then cartesian coordinates of $F_2$ would be $(2c, 0)$.

Let $M$ be a “general point” of the ellipse ad let $(r, \theta)$ be its polar coordinates. Then cartesian coordinates of $M$ are $M(r \cos \theta, r \sin \theta)$. Then

$$MF_2 = \sqrt{(2f)^2 + r^2 \cos^2 \theta - 4rf \cos \theta + r^2 \sin^2 \theta} =$$

$$= \sqrt{r^2 - 4rf \cos \theta + 4f^2}$$

and we can rewrite (7.3) as

$$r + \sqrt{r^2 - 4rf \cos \theta + 4f^2} = 2a.$$
Subtract \( r \) from both sides, square this equality and obtain
\[
r^2 - 4rf \cos \theta + 4f^2 = 4a^2 + r^2 - 4ar.
\]
After the evident simplification get
\[
-rf \cos \theta + f^2 = a^2 - ar.
\]
Now express \( r \)
\[
r = \frac{a^2 - f^2}{a - f \cos \theta} = \frac{a^2 - f^2}{a \left(1 - \frac{f}{a} \cos \theta\right)},
\]
and we are done.

If follows from (7.5) and (7.8) that
\[
r = \frac{p}{1 - \varepsilon \cos \theta},
\]
where \( p = \frac{b^2}{a} \). Notice, that \( p \) is called semi-latus rectum of the ellipse.

**Fact (Fact 15).** The area of the ellipse can be calculated as \( A = \pi ab \).

**Proof.** Before the beginning of the proof of this fact we should give the explanation of the notion of the area. It is a well known fact that area of the rectangle is the product of its sides. The same is true for the rectangle with sides \( \Delta x \) and \( \Delta y \), which are “very small” and are parallel to \( X \)-axis and \( Y \)-axis respectively. The area of any geometrical is defines as the sum of all squares \( \Delta x \times \Delta y \), which belong to our figure and such that \( \Delta x \approx 0 \) and \( \Delta y \approx 0 \) (in fact \( \Delta x = d x, \Delta y = d y \) and the area is some integral, but we will not come into these details).

Suppose that \( a = b \) and our ellipse is a circle. Clearly, in this case its area is \( \pi b^2 \). Now we sketch in the \( X \)-axis direction the big square, where the circle is inscribed in such way that the width of the appeared rectangle will become \( a \), i.e. we sketch \( \frac{a}{b} \) times. The equation of the new figure will be
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]
but it is exactly the equation (7.6) of the ellipse.

The area of the new geometrical figure can be calculates follows. Since each rectangle \( \Delta x \times \Delta y \) is transformed to \( \frac{a}{b} \cdot \Delta x \times \Delta y \), then the former area \( \pi b^2 \) is multiplied by \( \frac{a}{b} \) and becomes equal to \( \pi ab \).
7.5 Mathematical expression of the Second Kepler’s law

Fact (Fact 16). Let the trajectory of the planet is given in Cartesian coordinates as
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \]
where \( a > b > 0 \). Then the Second Kepler’s law can be rewritten as
\[ \frac{dy}{dt} = \frac{dx}{dt} = C, \]
where \( C \) is a constant, dependent on the planet and independent on time. Moreover,
\[ C = \frac{A}{T}, \]
where \( A \) is the area of the ellipse, which is the trajectory of the planet and \( T \) is the period of the rotation.

Proof. Denote \( r(t) = (x(t), y(t)) \) the coordinates of the planet of the moment \( t \). By Fact 8 (above in this section), the area between vectors \( \vec{r}(t) \) and \( \vec{r}(t + \Delta t) \) equals
\[ A(t, t + \Delta t) = \text{abs} \left( x(t)y(t + \Delta t) - y(t)x(t + \Delta t) \right). \] \hfill (7.9)

Second Kepler’s Law claims that there is a constant \( C \) such that
\[ A(t, t + \Delta t) = C \Delta t. \] \hfill (7.10)

For the entire ellipse (trajectory of a planet) denote \( A \) its area and \( T \) the period of the movement we, evidently, have
\[ A = CT, \]
whence
\[ C = \frac{A}{T}. \] \hfill (7.11)

Since the direction (clockwise or, anti clockwise) of the movement of a planet is constant (it trivially follows from the Second Kepler’s law), then we can remove the absolute value sign in (7.9) and obtain from (7.10) and (7.11) that
\[ (x(t)y(t + \Delta t) - y(t)x(t + \Delta t)) = \frac{A}{T} \Delta t. \] \hfill (7.12)
Divide both sides by $\Delta t$ and consider $\Delta t \approx 0$, i.e. take the difference $d\ t$ instead of $\Delta t$, and obtain
\[
\frac{dx}{dt} \frac{dy}{dt} - \frac{y}{x} \frac{dx}{dt} = \frac{A}{T}.
\] (7.13)

**Fact (Fact 17).** If the Sun is in the Origin, then the Second Kepler’s law in polar coordinates is expressed as
\[
r^2 \frac{d\theta}{dt} = C,
\]
where $C$ is the constant from Fact 16.

**Proof.** We will rewrite (7.13) (the Second Kepler’s law from Fact 16) in Polar coordinates. If
\[
\begin{align*}
x(t) &= r(t) \cos(\theta(t)) \\
y(t) &= r(t) \sin(\theta(t))
\end{align*}
\]
then
\[
\begin{align*}
x'(t) &= r'(t) \cos(\theta(t)) - r(t) \theta'(t) \sin \theta(t) \\
y'(t) &= r'(t) \sin(\theta(t)) + r(t) \theta'(t) \cos \theta(t)
\end{align*}
\]
and we can rewrite (7.13) as
\[
C = r \cos \theta \cdot (r' \sin \theta + r \theta' \cos \theta) - r \sin \theta \cdot (r' \cos \theta - r \theta' \sin \theta) =
\]
\[
r^2 \theta' \cdot (\cos^2 \theta + \sin^2 \theta) = r^2 \theta'
\]
and we are done. \qed

**Fact (Fact 18).** Denote $u = \frac{1}{r}$. Then in polar coordinates the Second Kepler’s Law can be written as
\[
v^2 = C^2 \left[ \left( \frac{du}{dt} \right)^2 + u^2 \right].
\]

**Proof.** Notice, that, by Fact 11 we have
\[
v^2 = \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2.
\] (7.14)
7.6 Equivalence of the Third Kerpler’s Law and the Fourth Newton’s Law

If follows from Fact 17 that
\[ \frac{d\theta}{dt} = \frac{C}{r^2}, \]
whence
\[ \frac{dr}{dt} = \frac{d}{d\theta} \cdot \frac{dr}{dt} = \frac{C}{r^2} \cdot \frac{dr}{d\theta}. \]
Thus, we can rewrite (7.14) as
\[ v^2 = C^2 \left[ \left( \frac{1}{r^2} \frac{dr}{d\theta} \right)^2 + \frac{1}{r^2} \right]. \]
Denote
\[ u = \frac{1}{r}, \]
whence
\[ \frac{du}{dt} = -\frac{1}{r^2} \cdot \frac{dr}{dt} \]
and we are done.

\[ \Box \]

7.6 Equivalence of the Third Kerpler’s Law and the Fourth Newton’s Law

We will present in this Section the proof of Fact 19.

Fact (Fact 19). Suppose that the Sun is fixed in the Origin and a planet moves by the elliptic orbit with polar equation
\[ r = \frac{p}{1 - e \cos \theta} \quad (7.15) \]
and the equation
\[ r^2 \frac{d\theta}{dt} = C \]
holds. Then the acceleration of the planet is
\[ \vec{a} = -\frac{\vec{v}^2}{|\vec{v}|} \cdot \frac{C^2}{pr^2}. \]
Proof. We will find the vector of the acceleration by differentiating
\[
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta.
\end{align*}
\]
two times as follows:
\[
\begin{align*}
    x' &= r' \cos \theta - r \theta' \sin \theta \\
    y' &= r' \sin \theta + r \theta' \cos \theta. \\
    x'' &= r'' \cos \theta - 2r' \theta' \sin \theta - r \theta'' \sin \theta - r \theta'^2 \cos \theta \\
    y'' &= r'' \sin \theta + 2r' \theta' \cos \theta + r \theta'' \cos \theta - r \theta'^2 \sin \theta.
\end{align*}
\] (7.16)

The derivative of \( r \) can be found from (7.15) as
\[
r' = \frac{-p \varepsilon \sin \theta}{(1 - \varepsilon \cos \theta)^2} \theta'.
\]

Notice, that if follows from (7.15) that
\[
\theta' = \frac{C}{r^2}.
\] (7.17)

Thus, we can simplify the derivative \( r' \) as
\[
r' = \frac{-C \varepsilon \sin \theta}{p}.
\]

Next,
\[
r'' = \frac{-C \varepsilon \cos \theta}{p} \theta'
\]
and (7.17) implies
\[
r'' = \frac{-C^2 \varepsilon \cos \theta (1 - \varepsilon \cos \theta)^2}{p^3}.
\]

Again by (7.17) find \( \theta'' \) as
\[
\theta'' = \frac{2 C \varepsilon \sin \theta}{p^2} (1 - \varepsilon \cos \theta) \theta',
\]
whence
\[
\theta'' = \frac{2 C^2 \varepsilon \sin \theta}{p^4} (1 - \varepsilon \cos \theta)^3.
\]

Using expressions for \( r', r'', \theta' \) and \( \theta'' \), we can rewrite the first line of (7.16) as
\[
x'' = r'' \cos \theta - 2r' \theta' \sin \theta - r \theta'' \sin \theta - r \theta'^2 \cos \theta =
\]

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7.7 Analytical representation of the movement

We will prove

Fact (Fact 21). Suppose that $\varepsilon \in (0, 1)$. Then

$$\int \frac{dx}{(1 - \varepsilon \cos x)^2} = \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{2}{(1 - \varepsilon^2)(1 - \varepsilon)} \left( \arctan t + \frac{t \varepsilon}{t^2 + 1} \right),$$

where $t = \tan \left( \frac{\pi}{2} \cdot \frac{\sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}} \right)$. 

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We will need two additional facts for the proof of Fact 21.

**Fact 23.**
\[ \int \frac{dx}{(x^2 + 1)^2} = \frac{\arctan x}{2} + \frac{x}{2x^2 + 2} + C \]

**Proof.** Differentiate the expression \( \frac{x}{x^2 + 1} \) and obtain
\[
\frac{d}{dx} \left( \frac{x}{x^2 + 1} \right) = \frac{1}{x^2 + 1} - \frac{2x^2}{(x^2 + 1)^2} = \frac{1}{x^2 + 1} - 2 \left( \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} \right) = \\
= \frac{-1}{x^2 + 1} + \frac{2}{(x^2 + 1)^2}.
\]

Thus, taking integrals from the former and the last part of this equality obtain
\[
\frac{x}{x^2 + 1} = -\int \frac{1}{x^2 + 1} \, dx + 2 \int \frac{1}{x^2 + 1} \, dx,
\]
whence
\[
\int \frac{1}{x^2 + 1} \, dx = \frac{1}{2} \left( \arctan x + \frac{x}{x^2 + 1} \right).
\]

**Fact 24.**
\[ \int \frac{x^2 \, dx}{(x^2 + 1)^2} = \frac{\arctan x}{2} - \frac{x}{2x^2 + 2} + C \]

**Proof.**
\[
\int \frac{x^2 \, dx}{(x^2 + 1)^2} = \int \left( \frac{1}{x^2 + 1} - \frac{1}{(x^2 + 1)^2} \right) \, dx.
\]

Applying Fact 23 obtain
\[
\int \frac{x^2 \, dx}{(x^2 + 1)^2} = \arctan x - \frac{1}{2} \left( \arctan x + \frac{x}{x^2 + 1} \right)
\]
obtain the necessary.

Now we are ready to prove Fact 21.

**Proof of Fact 21.** Denote \( \Theta = \frac{1}{(1-\varepsilon \cos \theta)^2} \), \( J = \frac{d\theta}{(1-\varepsilon \cos \theta)^2} \), and \( I = \int J \).

Denote
\[
\psi = \tan \frac{\theta}{2}
\]

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Then

\[ \cos \theta = \frac{1 - \psi^2}{1 + \psi^2} \]

and

\[ d\theta = \frac{2d\psi}{1 + \psi^2} \]

Thus,

\[
\frac{d\theta}{(1 - \varepsilon \cos \theta)^2} = \frac{1}{(1 - \varepsilon \left(\frac{1 - \psi^2}{1 + \psi^2}\right))^2} \cdot \frac{2d\psi}{1 + \psi^2} = \frac{2(1 + \psi^2)d\psi}{(1 + \psi^2 - \varepsilon(1 - \psi^2))^2} = \frac{2(1 + \psi^2)d\psi}{(\psi^2(1 + \varepsilon) + (1 - \varepsilon))^2}.
\]

We should notice, that the obtained expression is “similar” to examples, which are given in Facts 23 and 24, but we need some additional transformations for the direct use of these facts.

\[
\frac{2(1 + \psi^2)d\psi}{(\psi^2(1 + \varepsilon) + (1 - \varepsilon))^2} = \frac{1}{(1 - \varepsilon)^2} \cdot \frac{2(1 + \psi^2)d\psi}{\left(\frac{\psi\sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}}\right)^2 + 1}
\]

\[
= \frac{1}{(1 - \varepsilon)^2} \cdot 2 \left(\frac{1 + \psi^2}{1 - \varepsilon} + \left(\frac{\psi\sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}}\right)^2\right) d\psi \cdot \frac{1 - \varepsilon}{1 + \varepsilon} = \frac{1}{1 - \varepsilon^2} \cdot 2 \left(\frac{1 + \psi^2}{1 - \varepsilon} + \left(\frac{\psi\sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}}\right)^2\right) d\psi
\]

Denote \( \zeta = \frac{\psi\sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}} \). Then \( d\psi = \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} d\zeta \), whence

\[
I = \frac{1}{1 - \varepsilon^2} \cdot 2 \left(\frac{1 + \zeta^2}{\zeta^2 + 1} + \zeta^2\right) \cdot \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} d\zeta = \frac{2\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{1}{1 - \varepsilon^2} \left(\frac{1 + \varepsilon}{1 - \varepsilon} \cdot \frac{d\zeta}{\zeta^2 + 1} + \frac{\zeta}{2\zeta^2 + 2}\right).
\]

Using Facts 23 and 24 simplify

\[
\int I = \frac{2\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{1}{1 - \varepsilon^2} \left(\frac{1 + \varepsilon}{1 - \varepsilon} \cdot \frac{\text{arctan} \zeta}{2} + \frac{\zeta}{2\zeta^2 + 2} \right) + \left(\frac{\text{arctan} \zeta}{2} - \frac{\zeta}{2\zeta^2 + 2}\right).
\]

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\[
= \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{1}{1 - \varepsilon^2} \left( \frac{1 + \varepsilon}{1 - \varepsilon} \left( \arctan \zeta + \frac{\zeta}{\zeta^2 + 1} \right) + \left( \arctan \zeta - \frac{\zeta}{\zeta^2 + 1} \right) \right) = \\
= \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{1}{1 - \varepsilon^2} \left( \arctan \zeta \left( \frac{1 + \varepsilon}{1 - \varepsilon} + 1 \right) + \frac{\zeta}{\zeta^2 + 1} \left( \frac{1 + \varepsilon}{1 - \varepsilon} - 1 \right) \right) = \\
= \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{2}{1 - \varepsilon^2} \left( \arctan \zeta \cdot \frac{2}{1 - \varepsilon} + \frac{\zeta}{\zeta^2 + 1} \cdot \frac{2\varepsilon}{1 - \varepsilon} \right) = \\
= \frac{\sqrt{1 - \varepsilon}}{\sqrt{1 + \varepsilon}} \cdot \frac{2}{(1 - \varepsilon^2)(1 - \varepsilon)} \left( \arctan \zeta + \frac{\zeta \varepsilon}{\zeta^2 + 1} \right).
\]

Notice, that since \( \psi = \tan \frac{\theta}{2} \) and \( \zeta = \frac{\psi \sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}} \), then \( \zeta = \tan \left( \frac{\theta}{2} \cdot \frac{\sqrt{1 + \varepsilon}}{\sqrt{1 - \varepsilon}} \right) \).

### 7.8 Physical remarks about Kepler’s laws

**Fact** (Fact 22). The projection of the acceleration of the point, whose movement is described by vector-equation \( s(t) \), to the normal to the velocity at a point \( t_0 \) equals

\[
|a_\nu| = \frac{v^2}{R},
\]

where \( v = s'(t_0) \) and \( R \) is from Fact 9.

**Proof.** Remind that \( R \) here is the radius of a circle, such that we can “locally” consider our curve as a circle with this radius. Thus, “from physics” obtain that \( \vec{a}_c \) with

\[
a_c = \frac{(v(t_0))^2}{R}
\]

is centripetal acceleration, which should appear, where \( v(t_0) = s'(t_0) \) is the velocity of a point, which can be found as the derivative.

From another hand, it is clear that the tangent to our curve (to the trajectory of the planet), which is parallel to the velocity \( s'(t_0) \) should also be the tangent to the circle, which “locally” is equaled to the curve. This means that the center of this circle should belong to the perpendicular to the tangent at \( t_0 \). Denote \( \nu \) the direction vector of this perpendicular, i.e. \( \nu \cdot s'(t_0) = 0 \).

Find the projection \( a_\nu \) of the acceleration \( \vec{a} \) of our point on \( \nu \). If \( \alpha \) is the angle between \( a \) and \( \nu \), then it is clear that

\[
a_\nu = a \cos \alpha.
\]

M. Plakhotnyk: “Kepler’s laws with introduction to differential calculus”
Since \( \nu \) is perpendicular to \( v \), then \( \beta = 90^\circ - \alpha \) is the angle between \( a \) and \( v \). Find the area of the parallelogram, whose sides are \( a \) and \( v \). From one hand it is \( |a| \cdot |v| \sin \beta \). From another hand, it follows from Fact 8 that this area is \( P_3[\vec{a}, \vec{v}] \). Thus,

\[
|a| \cdot |v| \sin \beta = |P_3[\vec{a}, \vec{v}]|,
\]

whence

\[
\sin \beta = \cos \alpha = \frac{|P_3[\vec{a}, \vec{v}]|}{|s''(t_0)| \cdot |s'(t_0)|}
\]

and

\[
|a_v| = |a| \cos \alpha = \frac{|P_3[\vec{a}, \vec{v}]|}{|s'(t_0)|}
\]

Notice, that this expression of exactly the same as

\[
\frac{v^2}{R},
\]

where \( v = s'(t_0) \) and \( R \) is found above.

\[\square\]

### 7.8.1 About vector product

We have already mentioned in Fact 8 that if we have two vectors \( a \) and \( b \) of the plane XOY, then

\[
A = |P_3[a, b]|
\]

is the area of the parallelogram, whose sides are vectors \( a \) and \( b \). In fact, the projection \( P_3 \) and the fact that \( a \), and \( b \) belong to the plane XOY is not necessary in Fact 8. The more general fact is true.

**Fact 25.** For any 3-dimensional vectors \( a \) and \( b \) the area of the parallelogram with sides \( a \) and \( b \) equals

\[
A = |[a, b]|,
\]

where \([a, b]\) is the vector product of \( a \) and \( b \).

Clearly, we will explain now the definition of vector product and will prove Fact 25.

M. Plakhotnyk: “Kepler’s laws with introduction to differential calculus”
Consider the 3-dimensional space, and denote
\[ i = (1, 0, 0), \]
\[ j = (0, 1, 0) \]
and
\[ k = (0, 0, 1). \]

Define the vector product as follows:
1. \([i, j] = k, \) \([j, k] = i \) and \([k, i] = j. \]
2. \([a, b] = -[b, a] \) for all \(a, b. \)
3. \([\alpha a + \beta b, c] = \alpha [a, c] + \beta [b, c]. \)

Now we are ready to prove Fact 25.

**Proof of Fact 25.** Suppose that \(a = (x_1, y_1, z_1) \) and \(b = (x_2, y_2, z_2). \) In other words \(a = x_1i + y_1j + z_1k \) and \(b = x_2i + y_2j + z_2k. \) Then
\[
[a, b] = [x_1i + y_1j + z_1k, x_2i + y_2j + z_2k] = \\
= x_1y_2k + x_1z_2(-j) + y_1x_2(-k) + y_1z_2i + z_1x_2j + z_1y_2(-i) = \\
= (y_1z_2 - z_1y_2)i + (z_1x_2 - x_1z_2)j + (x_1y_2 - y_1x_2)k.
\]

Clearly,
\[
a^2 = x_1^2 + y_1^2 + z_1^2, \\
b^2 = x_2^2 + y_2^2 + z_2^2
\]
and
\[
a \cdot b = x_1x_2 + y_1y_2 + z_1z_2.
\]

Denote \(\alpha\) the angle between \(a\) and \(b.\) Then \((a \cdot b)^2 = a^2 b^2 \cos^2 \alpha, \) whence
\[
a^2 b^2 \sin^2 \alpha = a^2 b^2 - (a \cdot b)^2.
\]

By direct calculation check that
\[
a^2 b^2 - (a \cdot b)^2 = [a, b]^2,
\]

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7.8 Physical remarks about Kepler’s laws

\[ a \cdot [a, b] = 0 \]

and

\[ b \cdot [a, b] = 0 \]

and we are done.
8 Some information about Solar System

Eccentricities of planets of Solar systems are given in table below (see [2]).

| Planet | Eccentricity |
|--------|--------------|
| Mercury| 0.20563069   |
| Venus  | 0.00677323   |
| Earth  | 0.01671022   |
| Mars   | 0.09341233   |
| Jupiter| 0.04839266   |
| Saturn | 0.05415060   |
| Uranus | 0.04716771   |
| Neptune| 0.00858587   |

We can see, that all of them, except Mercury, are less than 0.1, whence all the orbits are very close to circles.
References

[1] Bojarchenko I.H., Gulak Yu.K., Radzimaha G.S. and Sandakova E.V. *Astronomy (in Ukrainian)*. Kyiv, 1971.

[2] Wikipedia. List of gravitationally rounded objects of the solar system. https://en.wikipedia.org/wiki/List_of_gravitationally_rounded_objects_of_the_Solar_System.
6 Physical remarks about the Kepler’s laws

6.1 Centripetal acceleration

6.1.1 Physical proof from the majority of textbooks

6.1.2 Physical proof without triangles

6.1.3 Pure mathematical proof

6.2 Forces, which act on a planet during its movement

7 Detailed mathematical computations

7.1 Derivatives and differentials

7.2 Polar coordinates

7.3 Integration

7.4 Ellipse

7.5 Mathematical expression of the Second Kepler’s law

7.6 Equivalence of the Third Kepler’s Law and the Fourth Newton’s law

7.7 Analytical representation of the movement

7.8 Physical remarks about Kepler’s laws

7.8.1 About vector product

8 Some information about Solar System