Group classification of variable coefficient generalized Kawahara equations

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Abstract

An exhaustive group classification of variable coefficient generalized Kawahara equations is carried out. As a result, we derive new variable coefficient nonlinear models admitting Lie symmetry extensions. All inequivalent Lie reductions of these equations to ordinary differential equations are performed. We also present some examples on the construction of exact and numerical solutions.

Keywords: group classification, Lie symmetry, generalized Kawahara equations, equivalence group, admissible transformations

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(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we study generalized Kawahara equations with time-dependent coefficients

\[ u_t + \alpha(t) u^n u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxx} = 0 \]  

from the Lie symmetry point of view. Here \( n \) is an arbitrary nonzero integer, \( \alpha, \beta \) and \( \sigma \) are smooth nonvanishing functions of the variable \( t \).

It is widely known that Lie (point) symmetries of differential equations (DEs) give a powerful tool for finding exact solutions and this is one of the most successful applications
of geometrical studies of DEs [27, 37, 38]. At the same time it is much less known that Lie symmetries can be served also as a selection principle for equations which are important for applications among wide set of possible models. All fundamental equations of mathematical physics, e.g., the Maxwell, Schrödinger, Newton, Laplace, Euler–Lagrange, d’Alembert, Lamé, Hamilton–Jacobi equations, etc, have nontrivial symmetry properties, i.e., they admit multi-dimensional Lie invariance algebras. Moreover, many equations of mathematical physics can be derived just from requirement of invariance with respect to a transformation group. For example, there is only one system of Poincaré-invariant first-order partial DEs for two real vectors \( \mathbf{E} \) and \( \mathbf{H} \), and this is the system of Maxwell equations [15]. The important problem of classification of all possible Lie symmetry extensions for equations from a given class with respect to the equivalence group of the class is called the group classification problem [21, 38, 42].

If \( \alpha, \beta \) and \( \sigma \) are not functions but constants, equations (1) become classical models appearing in the solitary waves theory. Here we present a brief overview on applications of Kawahara equations and the related results. In the usual sense, solitary waves are nonlinear waves of constant form which decay rapidly in their tail regions. The rate of this decay is usually exponential. However, under critical conditions in dispersive systems (e.g., the magneto-acoustic waves in plasmas, the waves with surface tension, etc), unexpected rise of weakly nonlocal solitary waves occurs. These waves consist of a central core which is similar to that of classical solitary waves, but they are accompanied by copropagating oscillatory tails which extend indefinitely far from the core with a nonzero constant amplitude. In order to describe and clarify the properties of these waves Kawahara introduced generalized nonlinear dispersive equations which have a form of the KdV equation with an additional fifth order derivative term, namely,

\[
\frac{du}{dt} + \alpha u u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0,
\]

where \( \alpha, \beta \) and \( \sigma \) are nonzero constants [19, 31]. This equation was heavily studied from different points of view. The exact solitary wave solution was presented in [55]. In [20] the existence of traveling wave solutions of the Kawahara equation being considered as a formal asymptotic approximation for water waves with surface tension was shown. In [8] various numerical computations of both infinite interval and spatially periodic solutions to a one-dimensional wave equation which models capillary-gravity waves were done. Using techniques of exponential asymptotics it was shown in [17] that solitary wave solutions of the Kawahara equation form a one-parameter family characterized by the phase shift of the trailing oscillations. An explicit asymptotic formula relating the oscillation amplitude to the phase shift was obtained therein. Solvability of the Cauchy problem (local and global existence) of the Kawahara equation was studied in [11, 22]. Various studies on behavior of solutions of the Kawahara equations were presented, e.g., in [2, 3, 13, 14, 18, 56].

Generalized constant coefficient models related to the Kawahara equation appeared later. For example, long waves in a shallow liquid under ice cover in the presence of tension or compression were described by the equation \( u_t + u_x + \alpha uu_x + \beta u_{xxx} + \sigma u_{xxxx} = 0 \) [34, 47]. This equation is similar to the classical Kawahara equation with respect to the simple changes of variables: \( \tilde{x} = x - t \), where \( t \) and \( u \) are not transformed, or \( \tilde{u} = 1 + \alpha u \), where \( t \) and \( x \) are not transformed. An analytical theory of radiating and stationary solitons, satisfying the modified Kawahara equations

\[
\frac{du}{dt} + \alpha \tilde{u}^n u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0,
\]

where \( \alpha, \beta \) and \( \sigma \) are nonzero constants, and \( n \in \mathbb{N} \), was given in [28]. The stability in the sense of Lyapunov for solitons described by these equations was studied in [29].
We note that neither the classical Kawahara equation nor its generalization adduced above are integrable by the inverse scattering transform method [36, 48].

Previously, much attention has been paid to variable coefficient models, like variable coefficient KdV, Burgers, and Schrödinger equations [45]. This is due to the fact that variable coefficient equations can model certain real-world phenomena with more accuracy than their constant coefficient counterparts. In the recent paper [30] Lie symmetries were applied for finding exact solutions of variable coefficient Kawahara and modified Kawahara equations, which are of the form (1) with \( n = 1 \) and \( n = 2 \), respectively. The presence of three arbitrary coefficients depending on \( t \) makes the task of finding Lie symmetries too difficult to get complete results without reducing the number of variable coefficients by equivalence transformations. That is why only few results on Lie symmetries were derived in [30]. In the present paper we show that the use of such transformations is a cornerstone in the complete solution of the problem.

The structure of this paper is as follows. All point transformations between equations from class (1) (so-called admissible transformations) are exhaustively described in section 2. Possibilities of reducing equations from this class to a simpler form are discussed therein. We choose the gauge \( \alpha = 1 \) and justify that it is optimal. The classical algorithm based on applying the Lie invariance criterion to (1) and subsequent study of compatibility and direct integration of the derived determining equations is utilized in section 3 to get the complete group classification. As a result, new variable coefficient models with nontrivial Lie symmetry properties are singled out from (1). Section 4 is devoted to the classification of Lie reductions and finding exact and numerical solutions for the variable coefficient generalized Kawahara equations. We give some final remarks and discuss problems for further investigation in section 5.

2. Admissible transformations

If two DEs are connected by a change of variables (a point transformation), they are called similar equations [38]. Then related objects like, e.g., exact solutions, conservation laws, different kinds of symmetries of such equations, are also similar. If they are known for one of these equations, then their counterparts for the other equation can be derived using the aforementioned transformation. This is why when one deals with a class of DEs parameterized by arbitrary elements (constants or functions), it is highly important to study relations between fixed equations from this class that are induced by point transformations. Such similarity relations are called in the literature allowed [53], form-preserving [32], and admissible [43] transformations. An admissible transformation can be interpreted as a triple consisting of two fixed equations from a class and a point transformation that links these equations. The set of admissible transformations considered with the standard operation of composition of transformations is also called the equivalence groupoid [41].

Equivalence transformations generate a subset in a set of admissible transformations. It is important that admissible transformations are not necessarily related to a group structure, but equivalence transformations always form a group. An equivalence transformation applied to any equation from the class always maps it to another equation from the same class. In other words, equivalence transformations preserve differential structure of the class. At the same time, an admissible transformation may exist only for a specific pair of equations from the class under consideration. For example, the point transformation \( t' = e^{bt}/b, x' = x, u' = u - bt \) links equations \( u_t = (e^u)_{xx} + a e^u + b \) and \( u'_t = (e^{u'})_{xx} + a e^{u'} \), where \( a \) and \( b \) are arbitrary constants with \( b \neq 0 \) [21]. Both these equations are members of the class.
\[ \mathcal{L} : u_t = (e^u)_x + Q(u), \text{ where } Q \text{ is a smooth function of } u. \] Acting on another equation from this class, e.g., on \( u_t = (e^a)_x + e^{2u} + b \), this transformation maps it to the equation \( u'_t = (e^b')_x + bt e^{2u} \), that is not constant coefficient one and does not belong to the class \( \mathcal{L} \).

By Ovsiannikov, the equivalence group consists of the nondegenerate point transformations of the independent and dependent variables and of the arbitrary elements of the class, where transformations for independent and dependent variables are projectible on the space of these variables [38]. After appearance of other kinds of equivalence group the one used by Ovsiannikov is called now usual equivalence group. If the transformations for independent and dependent variables involve arbitrary elements, then the corresponding equivalence group is called the generalized equivalence group [35]. If new arbitrary elements appear to depend on old ones in a nonlocal way (e.g., new arbitrary elements are expressed via integrals of old ones), then the corresponding equivalence group is called extended [24]. Generalized extended equivalence group possesses both the aforementioned properties. A number of examples of usage of different kinds of equivalence groups are presented, e.g., in [25, 51].

If any admissible transformation in a given class is induced by a transformation from its equivalence group (usual/generalized/extended/generalized extended), then this class is called normalized in the corresponding sense.

We search for admissible transformations in class (1) using the direct method [32]. Suppose that equation (1) is similar to an equation from the same class,

\[ \tilde{u}_t + \tilde{\alpha}(i) \tilde{u}_{ix} + \tilde{\beta}(i) \tilde{u}_{xxx} + \tilde{\sigma}(i) \tilde{u}_{xxxxx} = 0, \quad (2) \]

with respect to a nondegenerate point transformation in the space of variables \((t, x, u)\). We can restrict ourselves by consideration of point transformations of the form

\[ \tilde{t} = T(t), \quad \tilde{x} = X(t)x + X(0), \quad \tilde{u} = U(t, x)u + U(0), \quad (3) \]

where \(T, X, X^0, U^1\) and \(U^0\) are arbitrary smooth functions of their variables with \(T_0X^1U \neq 0\). This restriction does not lead to any loss of generality for a subclass of the normalized class of evolution equations,

\[ u_t = F(t)u_n + G(t, x, u, u_1, \ldots, u_{n-1}), \quad F \neq 0, \quad G_{u_iu_{i-1}} = 0, \quad i = 1, \ldots, n - 1, \]

where \(n \geq 2, u_0 = \frac{\partial F}{\partial x}, F \) and \(G\) are arbitrary smooth functions of their variables [50]. Under transformations (3) partial derivatives involved in (1) are transformed as follows

\[ \tilde{u}_t = \frac{1}{T_t} (U^1_1 u + U^1 u + U^0_0) - \frac{X^1_1 x + X^0_0}{T_x X^1} (U^1_1 u + U^1 u + U^0_0), \]

\[ \tilde{u}_x = \frac{U^1_1 u + U^1 u + U^0_0}{X^1_x} = U^1_{xxx} u + 3U^1_{xss} u_s + 3U^1_{xxs} u_s + U^1_{xxx} u_{xx} + U^0_{xxxx}, \]

\[ \tilde{u}_{xxxxx} = \frac{U^1_{xxxxx} u + 5U^1_{xxxxx} u_x + 10U^1_{xxxxx} u_{xx} + 10U^1_{xxxxx} u_{xxx} + 5U^1_{xxxxx} u_{xxxx} + U^1_{xxxxx} u_{xxxxx} + U^0_{xxxxx}}{(X^1)^5}. \]

Rewriting (2) in terms of the untilded variables, we further substitute \( u_t = -\alpha(t)u^n u_s - \beta(t)u_{xxx} - \sigma(t)u_{xxxxx} \) to the obtained equation in order to confine it to the manifold defined by (1) in the fifth-order jet space with the independent variables \((t, x)\) and the dependent variable \(u\). Splitting the obtained identity with respect to the derivatives of \(u\) leads to the determining equations on the functions \(T, X^1, X^0, U^1\) and \(U^0\). In particular, we get the following conditions

\[ \tilde{n} = n, \quad U^1_1 = 0, \quad \tilde{\beta}T_t - \beta(X^1)^3 = 0, \quad \tilde{\sigma}T_t - \sigma(X^1)^5 = 0. \]

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The further splitting depends on whether \( n \neq 1 \) or \( n = 1 \). The rest of the determining equations in these cases are the following

\[
\begin{align*}
  n \neq 1 : & \quad U^0 = U_t^1 = X^1_t = X^0_t = 0, \quad \tilde{a}(U^1)t - \alpha X^1_t = 0; \\
  n = 1 : & \quad U_t^1X^1 + \tilde{a}U^0T - \alpha X^1 = 0, \quad \tilde{a}U^1T - \alpha X^1_t = 0, \quad U^0(X^1)^5 - (X^1_t x + X^0_t)U_x^0(X^1)^4 + \tilde{a}T U^0U_x^0(X^1)^4 + \tilde{b}T U^0_{xxxx}X^1 + \tilde{a}U^0_{xxxx} = 0.
\end{align*}
\]

Solving these equations we get exactly the statements presented in theorems 1 and 2.

**Theorem 1.** The usual equivalence group \( G^- \) of class (1) consists of the transformations

\[
\begin{align*}
  \tilde{i} &= T(t), \quad \tilde{x} = \delta_1 x + \delta_2, \quad \tilde{u} = \delta_3 u, \\
  \tilde{a}(\tilde{i}) &= \frac{\delta_1}{\delta_2} \alpha, \quad \tilde{b}(\tilde{i}) = \frac{\delta_3}{\delta_2} \beta, \quad \tilde{\sigma}(\tilde{i}) = \frac{\delta_3}{\delta_2} \sigma, \quad \tilde{n} = n,
\end{align*}
\]

where \( \delta_j, \ j = 1, 2, 3 \), are arbitrary constants with \( \delta_1 \delta_3 \neq 0 \), \( T \) is an arbitrary smooth function with \( T \neq 0 \).

**Theorem 2.** The generalized extended equivalence group \( G^-_{new} \) of the class

\[
\begin{align*}
  \omega + \alpha(t) \omega + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0
\end{align*}
\]

is formed by the transformations

\[
\begin{align*}
  \tilde{i} &= T(t), \quad \tilde{x} = (x + \delta_1)X^1 + \delta_0, \quad \tilde{u} = \frac{\delta_0}{X^1} u - \delta_2 \delta_3 (x + \delta_1), \\
  \tilde{a}(\tilde{i}) &= \frac{(X^1)^2}{\delta_2 T} \alpha, \quad \tilde{b}(\tilde{i}) = \frac{(X^1)^3}{\delta_2 T} \beta, \quad \tilde{\sigma}(\tilde{i}) = \frac{(X^1)^5}{\delta_2 T} \sigma, \quad \tilde{n} = n,
\end{align*}
\]

where \( X^1 = (\delta_3 \int \alpha(t) dt + \delta_4)^{-1}, \delta_j, \ j = 0, \ldots, 4 \), are arbitrary constants with \( \delta_2 \delta_3 \neq 0 \); \( T = T(t) \) is a smooth function with \( T \neq 0 \). The usual equivalence group \( G^-_{new} \) of class (4) comprises the above transformations with \( \delta_1 = \delta_3 = 0 \).

**Theorem 3.** A variable coefficient equation from class (1) is reducible to constant coefficient equation from the same class if and only if the coefficients \( \alpha, \beta \) and \( \sigma \) satisfy the conditions

\[
\begin{align*}
  \left( \frac{\beta}{\alpha} \right)_t = \left( \frac{\sigma}{\alpha} \right)_t = 0, \quad \text{for} \quad n \neq 1, \quad (5) \\
  \left( \frac{1}{\alpha} \left( \frac{\beta}{\alpha} \right)_t \right)_t = 0, \quad \left( \frac{\sigma}{\beta^3} \right)_t = 0, \quad \text{for} \quad n = 1. \quad (6)
\end{align*}
\]

The presence of the arbitrary function \( T(t) \) in the equivalence transformations adjoined in theorems 1 and 2 allows one to gauge one of the arbitrary functions \( \alpha, \beta \) and \( \sigma \) to a simple constant value, e.g., to 1. An interesting question is which one of the three possible gauges is preferable for further consideration. Class (4) with \( \beta = 1 \) or \( \sigma = 1 \) is still normalized only in the generalized extended sense, since transformations of independent and dependent variables still involve \( \int \alpha(t) dt \). At the same time class (4) with \( \alpha = 1 \) is normalized with respect to its usual equivalence group, as \( X^1 \) appearing in theorem 2 in this case takes the form \( X^1 = (\delta_3 t + \delta_4)^{-1} \). This is why we can expect that in the case \( n = 1 \) it is easier to carry out the group classification under the gauge \( \alpha = 1 \) rather than under other possible gauges. If \( n \neq 1 \) all the three suggested gauges look equally convenient, and we choose the gauge \( \alpha = 1 \) just to present the group classification in the uniform way.

The gauge \( \alpha = 1 \) is realized by the point transformation

\[
\hat{i} = \int \alpha(t) dt, \quad \hat{x} = x, \quad \hat{u} = u.
\]
Then class (1) is mapped to its subclass with \( \alpha = 1, \beta = \beta / \alpha \) and \( \sigma = \sigma / \alpha \). Therefore, without loss of generality we can restrict ourselves to the study of the class

\[ u_t + u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0, \tag{8} \]

since all results on symmetries, conservation laws, classical solutions and other related objects for equations (1) can be found using the similar results derived for equations from class (8).

To derive the equivalence group for subclass of class (1) with \( \alpha = 1 \) we set \( \tilde{\alpha} = \alpha = 1 \) in the transformations presented in theorems 1 and 2.

**Corollary 1.** The generalized equivalence group \( \hat{G}_{\mu=1} \) of class (8) comprises the transformations

\[
\begin{align*}
\tilde{t} &= \delta_1 \delta_3^{-n} t + \delta_0, & \tilde{x} &= \delta_1 x + \delta_2, & \tilde{u} &= \delta_3 u, \\
\tilde{\beta}(\tilde{t}) &= \delta_1^2 \delta_3^{n} \beta(t), & \tilde{\sigma}(\tilde{t}) &= \delta_1^4 \delta_3^n \sigma(t), & \tilde{n} &= n,
\end{align*}
\tag{9}
\]

where \( \delta_j, j = 0, 1, 2, 3 \), are arbitrary constants with \( \delta_1 \delta_3 \neq 0 \).

**Remark 1.** If we assume that the constant \( n \) varies in class (8), then the equivalence group \( \hat{G}_{\mu=1} \) is generalized since \( n \) is involved explicitly in the transformation of the variable \( t \). From the other hand, \( n \) is invariant under the action of transformations from the equivalence group, so class (8) can be considered as the union of all its subclasses with fixed \( n \). For each such subclass the group \( \hat{G}_{\mu=1} \) is usual equivalence group.

In the case \( n = 1 \) we put \( \alpha = \tilde{\alpha} = 1 \) in transformation from theorem 2 and redenote the constants \( \delta_j, j = 0, \ldots, 4 \), to write the transformations in a more compact form.

**Corollary 2.** The usual equivalence group \( G_{\mu=n=1} \) of the class

\[ u_t + u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} = 0 \tag{10} \]

consists of the transformations

\[
\begin{align*}
\tilde{t} &= \frac{at + b}{ct + d}, & \tilde{x} &= \frac{e_2 x + e_1 t + e_0}{ct + d}, & \tilde{u} &= \frac{e_2 (ct + d) u - e_2 cx - e_0 c + e_1 d}{\Delta}, \\
\tilde{\beta} &= \frac{e_2^3}{ct + d}\beta, & \tilde{\sigma} &= \frac{e_2^3}{(ct + d)^3} \sigma, \tag{11}
\end{align*}
\]

where \( a, b, c, d, e_0, e_1, e_2 \) are arbitrary constants with \( \Delta = ad - bc \neq 0 \) and \( e_2 \neq 0 \), the tuple \((a, b, c, d, e_0, e_1, e_2)\) is defined up to a nonzero multiplier and hence without loss of generality we can assume that \( \Delta = \pm 1 \).

### 3. Lie symmetries

The group classification of equations of the form (8) with \( n \neq 1 \) up to \( \hat{G}_{\mu=n=1} \)-equivalence (resp. up to \( G_{\mu=n=1}^- \)-equivalence if \( n = 1 \)) coincides with the group classification of equations of the form (1) with \( n \neq 1 \) up to \( G^- \)-equivalence (resp. up to \( G_{\mu=1}^- \)-equivalence if \( n = 1 \)). In order to carry out the group classification of class (8) we use the classical algorithm based on direct integration of determining equations implied by the infinitesimal invariance criterion [37, 38] (see modern discussion on algebraic method of group classification, e.g., in [4]). We search for symmetry generators of the form \( Q = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u \) and require that

\[ Q^{(5)} \{ u_t + u_x + \beta(t) u_{xxx} + \sigma(t) u_{xxxxx} \} = 0 \tag{12} \]
identically, modulo equation (8). Here $Q^{(5)}$ is the fifth prolongation of the operator $Q$ [37, 38], i.e., in our case $Q^{(5)} = Q + \eta \partial_u + \eta^0 \partial_u + \eta^{xxx} \partial_{uxx} + \eta^{xxxx} \partial_{uxxx}$, where

\[ \eta^i = D_\tau (\eta) - u D_\tau (\tau) - u D_\tau (\xi), \quad \eta^{xxx} = D_\tau (\eta^{xx}) - u D_\tau (\tau) - u D_\tau (\xi), \]

\[ \eta^{xxxx} = D_\tau (\eta^{xxxx}) - u D_\tau (\tau) - u D_\tau (\xi), \quad \eta^{i} = D_\tau (\eta) - u D_\tau (\tau) - u D_\tau (\xi), \]

\[ \tau = \tau (t), \quad \xi = \xi (t, x), \quad \eta = \eta (t, x) u + \eta^0 (t, x), \]

where $\tau$, $\xi$, $\eta^i$ and $\eta^0$ are arbitrary smooth functions of their variables. The rest of the determining equations have the form

\[ \eta^i = 2\xi_{xx} - 3(\eta^1 - \xi_{xx}) \beta + 5(2\eta^2_{xx} - \xi_{xxxx}) \sigma = 0, \]

\[ \tau \sigma_i = (5\xi_{xx} - \tau_i) \sigma, \quad \tau \beta_i = (3\xi_{xx} - \tau_i) \beta + 10(\xi_{xx} - \eta^1) \sigma, \]

\[ \eta^i u^{i+1} + \eta^i u^0 + (\eta^1 + \eta^1 - \beta + \eta^{xxxx} \sigma) u + \eta^0 + \eta^0 \beta + \eta^0 \sigma = 0, \]

\[ (\tau - \xi_{xx} + 3\eta^1) u^0 + \eta^0 u^0 + \eta^0 u^0 + 2(3\eta^1_{xx} - \xi_{xx}) \beta + (5\eta^3_{xxx} - \xi_{xxxx}) \sigma - \xi_{xx} = 0. \]

The determining equations were verified using the GeM software package [10]. As $\tau$ and $\sigma$ are functions of $t$ only, the equation $\tau \sigma_i = (5\xi_{xx} - \tau_i) \sigma$ implies $\xi_{xx} = 0$. Then the first determining equation gives $\eta^1 = 0$. The latter two equations can be split with respect to different powers of $u$. Special cases of the splitting arise if $n = 0, 1$. If $n = 0$ equations (8) are linear ones and, therefore, excluded from consideration. The cases $n \neq 1$ and $n = 1$ will be investigated separately.

(1) If $n \neq 1$ then the splitting results in $\eta^0 = \eta^1 = \xi_{xx} = 0$ and $\tau_i - \xi_{xx} + 3\eta^1 = 0$. We solve this system together with the derived earlier conditions $\xi_{xx} = \eta^1 = 0$ and get the solution $\tau = c_1 t + c_2$, $\xi = (c_1 + nc_0) x + c_3$, $\eta^1 = c_0$, $\eta^0 = 0$, where $c_i$, $i = 0, \ldots, 3$, are arbitrary constants. Thus, the general form of the infinitesimal generator is

\[ Q = (c_1 t + c_2) \partial_t + ((c_1 + nc_0) x + c_3) \partial_x + c_0 u \partial_u. \]

The classifying equations on $\beta$ and $\sigma$ are

\[ (c_1 t + c_2) \beta_i = (2c_1 + 3nc_0) \beta, \quad (c_1 t + c_2) \sigma_i = (4c_1 + 5nc_0) \sigma. \]

To derive the kernel $A^{(5)}$ of maximal Lie invariance algebras $A^{(5)}$ of equations from class (8) (i.e., the Lie invariance algebra admitted by (8) for arbitrary $\beta$ and $\sigma$) we split (13) with respect to $\beta$, $\sigma$ and their derivatives. Then $c_0 = c_1 = c_2 = 0$ and, therefore, $Q = c_3 \partial_x$. Thus, the kernel algebra is the one-dimensional algebra $(\partial_x)$. To get possible extensions of $A^{(5)}$ we consider (13) not as two identities but as a system of equations on $\beta$ and $\sigma$ of the form

\[ (pt + q) \beta_i = r \beta, \quad (pt + q) \sigma_i = \frac{1}{2} (5r + 2p) \sigma, \]

where $p$, $q$, and $r$ are arbitrary constants, $p^2 + q^2 \neq 0$. The equivalence transformations (9) act on the coefficients $p$, $q$, and $r$ of system (14) as follows

\[ \tilde{p} = \kappa p, \quad \tilde{q} = \kappa (q \delta \tau \delta \gamma - r \delta \beta_0), \quad \tilde{r} = \kappa r, \]

where $\kappa$ is a nonzero constant. Therefore, there are three inequivalent triples $(p, q, r)$: $(1, 0, \rho)$, $(0, 1, 1)$ and $(1, 0, 0)$. We integrate (14) for these three equivalence classes of $(p, q, r)$. Up to $G_{w1}$-equivalence
$\beta(t) \quad \sigma(t) \quad \text{Basis of } A^{\text{max}}$

| $n \neq 1$ | This case is classified up to $G^-$-equivalence. | | | |
|---|---|---|---|---|
| 0 | $\forall$ | $\forall$ | $\partial_t$ |
| 1 | $\lambda t^\rho$ | $\delta t^\lambda e^\rho$ | $\partial_t, 3n\partial_t + (\rho + 1)n\partial_x + (\rho - 2)u\partial_u$ |
| 2 | $\lambda e^\rho$ | $\delta e^\lambda$ | $\partial_t, 3n\partial_t + n\partial_x + u\partial_u$ |
| 3 | $\lambda$ | $\delta$ | $\partial_t, \partial_x$ |

| $n = 1$ | This case is classified up to $G_{\text{max}}^1$-equivalence. | | | |
|---|---|---|---|---|
| 0' | $\forall$ | $\forall$ | $\partial_t, \partial_x + \partial_u$ |
| 1' | $\lambda t^\rho$ | $\delta t^\lambda e^\rho$ | $\partial_t, \partial_x + \partial_u, 3\partial_x + (\rho + 1)x\partial_x + (\rho - 2)u\partial_u$ |
| 2' | $\lambda e^\rho$ | $\delta e^\lambda$ | $\partial_t, \partial_x + \partial_u, 3\partial_x + x\partial_x + u\partial_u$ |
| 3' | $\lambda$ | $\delta$ | $\partial_t, \partial_x + \partial_u, \partial_u$ |
| 4' | $\lambda(t^2 + 1)^2 e^{\lambda\text{arctan} t}$ | $\delta(t^2 + 1)^2 e^{\lambda\text{arctan} t}$ | $\partial_t, \partial_x + \partial_u, (t^2 + 1)\partial_x + (t + v)\partial_x + ((v - t)u + x)\partial_u$ |

Here $\alpha = 1 \text{ mod } G^-$, $\rho$ and $v$ are arbitrary constants, $\rho \geq 1/2$, $v \geq 0$; and $\lambda$ are nonzero constants, $\delta = \pm 1 \text{ mod } G^-$. 

$(\beta, \sigma)$ take the values from the set $\{(\lambda t^\rho, \delta t^\lambda e^\rho), (\lambda e^\rho, \delta e^\lambda), (\lambda, \lambda), (\lambda, \lambda)\}$. Here $\rho$, $\lambda$, and $\delta$ are arbitrary constants with $\lambda \delta \neq 0$, $\delta = \pm 1 \text{ mod } G_{\text{max}}^1$. The respective forms of the infinitesimal generators are $Q = c_1\partial_t + (\sum c_i x + c_3)\partial_x + \sum c_i \partial_u$, $Q = 3nc_0\partial_x + (nc_0 x + c_3)\partial_x + c_0\partial_u$, and $Q = c_2\partial_x + c_3\partial_u$, where $c_0$, $c_1$, $c_2$ and $c_3$ are arbitrary constants.

We have proven the following assertion.

**Theorem 4.** The kernel of the maximal Lie invariance algebras of equations from class (8) (resp. (1)) with $n \neq 1$ coincides with the one-dimensional algebra $(\partial_t)$. All possible $G_{\text{max}}^1$- inequivalent (resp. $G^-$-inequivalent) cases of extension of the maximal Lie invariance algebras are exhausted by the cases 1–3 of Table 1.

(II) If $n = 1$ then the determining equations lead to the system $\eta_1 = \xi_{\xi t} = \xi_{\xi x} = 0$, $\eta_0 = \xi_t$, $\tau_t - \xi_t + \eta_1 = 0$, $\eta_0 + \eta_1 = 0$, $\eta_0 = 0 = \eta_{1t} = 0$, $\tau\beta_t = (3\xi_t - \xi)\beta$, and $\tau\sigma_t = (5\xi_t - \tau)\sigma$.

We solve firstly the equations that do not contain arbitrary elements and get the form of the infinitesimal generator

$Q = (c_2t^2 + 2c_1t + c_0)\partial_t + ((c_2 + c_1 + c_3)x + c_4t + c_5)\partial_x$

$+((c_3 - c_1 - c_2)u + c_2x + c_4)\partial_u$,

where $c_i$, $i = 0, \ldots, 5$, are arbitrary constants. The system of classifying equations takes the form

$(c_2t^2 + 2c_1t + c_0)\beta_t = (c_2t^2 + c_1 + 3c_3)\beta$,

$(c_2t^2 + 2c_1t + c_0)\sigma_t = (3c_2t^2 + 3c_1 + 5c_3)\sigma$.

If $\beta$ and $\sigma$ are arbitrary we can split the latter equations with respect to them and their derivatives. As a result we obtain that $c_1 = c_2 = c_3 = c_0 = 0$. Therefore, $Q = (c_2t + c_3)\partial_x + c_4\partial_u$ and the kernel $A^{\text{ker}}$ of the maximal Lie invariance algebras of equations from class (10) coincides with the two-dimensional algebra $(\partial_x, \partial_u)$. 

The group classification of class (10) is equivalent to the integration of the classifying equations up to the $G_{\text{max}}^1$-equivalence. Combined with multiplication by a nonzero constant, each transformation from the equivalence group $G_{\text{max}}^1$ can be extended to the coefficient quadruple $(p, q, r, s)$ of the system

$(pt^2 + qt + r)\beta_t = (pt + s)\beta$, \quad $(pt^2 + qt + r)\sigma_t = (3pt + (5s + 2q)/3)\sigma$. \quad \quad (8)
where \( p, q, r \) and \( s \) are arbitrary constants, \( p^2 + q^2 + r^2 \neq 0 \), in the following way
\[
\tilde{p} = \kappa (pd^2 - qcd + rc^2), \quad \tilde{q} = \kappa (-2pbd + q(ad + bc) - 2rac), \\
\tilde{r} = \kappa (pb^2 - qab + ra^2), \\
\tilde{s} = \kappa (rac + qbc - pbd + s\Delta),
\]
where \( \Delta = ad - bc \) and \( \kappa \) is an arbitrary nonzero constant.

It can be proved that there are only three \( G_{a_{n=1}} \)-inequivalent values of the triple \((p, q, r)\) depending upon the sign of \( D = a^2 - 4pr \).

- \((0, 1, 0)\) if \( D > 0 \), \((0, 0, 1)\) if \( D = 0 \), and \((1, 0, 1)\) if \( D < 0 \).

The technique of the proof can be found in \[51\]. The remaining task is to consider whether there is possibility to scale the constant \( s \) in each of the three distinct cases for \((p, q, r)\). As a result we get the following statement.

**Proposition 1.** Up to \( G_{a_{n=1}} \)-equivalence the parameter quadruple \((p, q, r, s)\) can be assumed to belong to the set
\[
\{(0, 1, 0, \rho), \ (0, 0, 1, \rho), \ (0, 0, 1, 0), \ (1, 0, 1, \bar{s})\},
\]
where \( \bar{s} \) is an arbitrary constant, \( \rho \geq \frac{1}{2}, \bar{s} \geq 0 \).

We integrate (15) for the values of \((p, q, r, s)\) presented in proposition 1, then substitute the derived forms of \( \beta \) and \( \sigma \) to the classifying equations in order to get \( c_i, i = 0, \ldots, 3 \), (the constants \( c_4 \) and \( c_5 \) are arbitrary). We get that all \( G_{a_{n=1}} \)-inequivalent cases of Lie symmetry extension are exhausted by the following:

1. \((\beta, \sigma) = (\lambda \rho^2, \delta \tan\beta), \quad \rho \geq \frac{1}{2} : \)
   \[
   Q = 2c_1 t \delta_x + \left( \frac{4}{5} (\rho + 1) c_1 x + c_1 t + c_5 \right) \delta_t + \left( \frac{4}{5} (\rho - 2) c_2 u + c_3 \right) \delta_u;
   \]

2. \((\beta, \sigma) = (\lambda \rho^2, \delta e^{\beta t}), \quad \rho \geq \frac{1}{2} : \)
   \[
   Q = 3c_4 \delta_x + \left( c_3 x + c_1 t + c_5 \right) \delta_t + \left( c_3 u + c_4 \right) \delta_u;
   \]

3. \((\beta, \sigma) = (\lambda \rho^2, \delta \tan\beta), \quad \rho \geq \frac{1}{2} : \)
   \[
   Q = c_3 \delta_x + \left( c_2 t + c_5 \right) \delta_t + \left( c_2 u + c_4 \right) \delta_u.
   \]

In all four adduced cases the maximal Lie invariance algebras are three-dimensional. The following assertion is true.

**Theorem 5.** The kernel of the maximal Lie invariance algebras of equations from class (10) (resp. (4)) coincides with the two-dimensional algebra \([\delta_t, t\delta_x + \delta_u] \). All possible \( G_{a_{n=1}} \)-inequivalent (resp. \( \hat{G}_{a_{n=1}} \)-inequivalent) cases of extension of the maximal Lie invariance algebras are exhausted by the cases \( 1-4' \) of table 1.

To derive the complete list of Lie symmetry extensions for the entire class (1), where arbitrary elements are not simplified by point transformations, we use the equivalence-based approach \[49\]. The results are collected in table 2.

The presented group classification reveals equations of the form (1) that may be of interest for applications and for which the classical Lie reduction method can be used.

### 4. Symmetry reductions and exact solutions

The Lie symmetry operators derived as a result of solving the group classification problem can be applied to construction of exact solutions of the corresponding equations. The reduction method with respect to subalgebras of Lie invariance algebras is algorithmic and well-known; we refer to the classical textbooks on the subject \[37, 38\]. In order to get an optimal system of group-invariant solutions reductions should be performed with respect to subalgebras from the optimal system \[37, section 3.3\].
Table 2. The group classification of the class $u_t + \alpha u_x u_x + \beta u_{xxx} + \sigma u_{xxxxx} = 0$, $n \neq 0$, $\alpha \beta \sigma \neq 0$, using no equivalence.

| $\beta(t)$ | $\sigma(t)$ | Basis of $A^{\text{max}}$ |
|-----------|------------|--------------------------|
| $0$       | $\forall$  | $\partial_t$             |
| $1$       | $\lambda_1(T + l)^\rho$ | $\lambda_2(T + l)^{\frac{\nu+1}{\nu}}$ |
| $2$       | $\lambda_3 e^{\xi t}$     | $\lambda_2 e^{\frac{\xi t}{n}}$ |
| $3$       | $\lambda_3 \alpha$        | $\lambda_2 \alpha$ |

Here $\lambda_1, \lambda_2, \alpha, b, c, d, l, m, \rho$ and $\nu$ are arbitrary constants, $\lambda_1, \lambda_2 (c^2 + d^2) \neq 0$, $\Delta = ad - bc \neq 0$, $\alpha$ is an arbitrary nonvanishing smooth function of $t$, $T = \int a(t) \, dt$. 
Table 3. Optimal systems of one-dimensional subalgebras of $A^{\text{max}}$ presented in table 1.

| Case       | Optimal system |
|------------|----------------|
| $1_{\rho \neq -1}$ | $g_0 = \langle \partial_t \rangle$, $g_{1,1} = \langle 3(\rho + 1)x\partial_x + (\rho - 2)u\partial_u \rangle$ |
| $1_{\rho = -1}$ | $g_0 = \langle \partial_\rho \rangle$, $g_{1,1} = \langle u\partial_u - a\partial_x \rangle$ |
| 2          | $g_0 = \langle \partial_\rho \rangle$, $g_2 = \langle 3\partial_u + nx\partial_x + u\partial_u \rangle$ |
| 3          | $g_0 = \langle \partial_\rho \rangle$, $g_3 = \langle \partial_x + a\partial_u \rangle$ |
| 0          | $g_0 = \langle \partial_\rho \rangle$, $g_0 = \langle (t + a)\partial_t + \partial_u \rangle$ |
| $1'_{\rho \neq -1, 2}$ | $g_0 = \langle \partial_\rho \rangle$, $g_{1,1} = \langle (t + \sigma)\partial_t + \partial_u \rangle$, $g_{1,2} = \langle (t\partial_t + a\partial_\rho - u\partial_u) \rangle$ |
| $1'_{\rho = -1}$ | $g_0 = \langle \partial_\rho \rangle$, $g_0 = \langle (t + \sigma)\partial_t + \partial_u \rangle$, $g_{1,1} = \langle (t\partial_t + a\partial_\rho - u\partial_u) \rangle$, $g_{1,2} = \langle (t\partial_\rho + (x + at)\partial_x + u\partial_u) \rangle$ |
| 2          | $g_0 = \langle \partial_\rho \rangle$, $g_2 = \langle (t + \sigma)\partial_t + \partial_u \rangle$, $g_2 = \langle (3\partial_x + x\partial_x + u\partial_u) \rangle$ |
| 3          | $g_0 = \langle \partial_\rho \rangle$, $g_3 = \langle \partial_x + a\partial_u \rangle$, $g_{0,1} = \langle \partial_x \rangle$, $g_{1,2} = \langle a\partial_x + 2u\partial_u + a\partial_u \rangle$ |
| 4          | $g_0 = \langle \partial_\rho \rangle$, $g_4 = \langle (t^2 + 1)x\partial_x + (x + (v - t)a)^2u\partial_u \rangle$ |

In all cases $a \in \mathbb{R}, n \neq 0, \sigma \in \{-1, 0, 1\}$.

Table 4. Similarity reductions of the equations $u_t + \alpha u_x + \beta(t)u_{xxx} + \sigma(t)u_{xxxx} = 0$.

| Case  | $\omega$ | Ansatz, $u =$ | Reduced ODE |
|-------|----------|---------------|-------------|
| $1_{\rho \neq -1}$ | $x = \frac{\rho + 1}{t^2 \omega} \psi(\omega)$ | $\delta\psi'''' + \lambda \psi'' + (\psi'''' - \frac{\rho + 1}{\rho + 2} \psi) = 0$ |
| $1_{\rho = -1}$ | $x = - \frac{1}{\ln t} \psi(\omega)$ | $\delta\psi'''' + \lambda \psi'' + (\psi'''' - \frac{\rho + 1}{\rho + 2} \psi) = 0$ |
| 2     | $x = \frac{1}{\sqrt{2}} \psi(\omega)$ | $\delta\psi'''' + \lambda \psi'' + (\psi'''' - \frac{\rho + 1}{\rho + 2} \psi) = 0$ |
| 3     | $x = \frac{1}{\sqrt{2} \omega} \psi(\omega)$ | $\delta\psi'''' + \lambda \psi'' + (\psi'''' - \frac{\rho + 1}{\rho + 2} \psi) = 0$ |
| 0     | $x = \frac{1}{\sqrt{2} \omega} \psi(\omega)$ | $\delta\psi'''' + \lambda \psi'' + (\psi'''' - \frac{\rho + 1}{\rho + 2} \psi) = 0$ |

Specific reductions for $n = 1$:

| Case  | Ansatz, $u =$ | Reduced ODE |
|-------|---------------|-------------|
| 3     | $x = \frac{1}{\sqrt{2} \omega} \psi(\omega)$ | $\delta\psi'''' + \lambda \psi'' + (\psi'''' - \frac{\rho + 1}{\rho + 2} \psi) = 0$ |
| 4     | $x = \frac{1}{\sqrt{2} \omega} \psi(\omega)$ | $\delta\psi'''' + \lambda \psi'' + (\psi'''' - \frac{\rho + 1}{\rho + 2} \psi) = 0$ |

Here $a$ is an arbitrary constant.

Consider firstly, the structure of the two and three-dimensional Lie algebras spanned by the generators presented in table 1, using notations of [40]. In Cases 1–3 and Case 0 the maximal Lie-invariance algebras are two-dimensional. In Case 0, Case 1 with $\rho = -1$, and Case 3 they are Abelian (A1). The algebras adduced in Case 1 with $\rho \neq -1$ and Case 2 are non-Abelian (A2). The algebras with basis operators presented in Cases 1’–4’ are three-dimensional. In Case 1’ with $\rho \neq -1, 2$ the maximal Lie invariance algebra is of the type $A_{3,3}$ if $\rho = 1/2$, $A_{2,2}$ with $a = \frac{\rho - 1}{\rho + 1} \omega$ or $a = \frac{\rho + 1}{\rho - 1} \omega$ if $\rho > 1/2$ or $\rho < 1/2$, respectively. If $\rho = -1$ or $\rho = 2$, then $A^{\text{max}}$ from Case 1’ is $A_1 \oplus A_2$. In other cases the maximal Lie invariance algebras are of the following types: Case 2’—$A_{3,3}$, Case 3’—the Weyl algebra $A_3$, Case 4’—$A_{3,2}^\#$, with $a = |\nu|$. If a one-dimensional invariance algebra is spanned by an operator $Q = \xi \partial_x + \eta \partial_u$, then the associated ansatz reducing the corresponding PDE with two independent variables to an ODE is found as a solution of the invariant surface condition $Q[u] := \tau u_t + \xi u_x - \eta = 0$. In practice the related characteristic system $\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta}$ has to be solved. Ansatzes and reduced equations obtained for equations from class (8) using one-dimensional subalgebras from table 3 are collected in table 4. Reductions associated with the subalgebra $g_0$ are not considered since they lead to constant solutions only. We do not present reductions with respect to the subalgebras $g_{1,1}, g_{1,2}$ and $g_2$ since these subalgebras are specifications of the subalgebras $g_{1,1}, g_{1,2}$ and $g_2$ for the case $n = 1$. The reduction for the case $V_{\mu = 2}$ is not performed because this case is equivalent to $V_{\mu = 1}$. Indeed, the equations $u_t + uu_x + \lambda \eta^4 u_{xxx} + \delta \eta^4 u_{xxxx} = 0$ and
that such variable coefficient equations from class (1) are reducible to constant coefficient both papers exact solutions were derived for equations whose coefficients obey additional to construct exact solutions of Kawahara equations with time-dependent coefficients. In Kawahara equations.

\[ \frac{u'}{u'} + u' + \lambda/\tau' \frac{u''}{u''' + \delta/\tau' \frac{u'''}{u''}} = 0 \] are linked by the transformation \( \tau' = 1/t', x' = -x/t, u' = tu - x \).

The first-order reduced equation from table 4, \( (\omega + a)\phi' + \phi = 0 \), gives the ‘degenerate’ solution of (8) for arbitrary values of \( \beta(t) \) and \( \sigma(t) \), \( u = (x + c)/(t + a) \), where \( c \) and \( a \) are arbitrary constants. Using transformation (7) we get the ‘degenerate’ exact solution of equation (4) in the form

\[ u = \frac{x + c}{\int \alpha(t) \, dt + a}. \] (16)

Consider fifth-order reduced ODEs from table 4. Cases 3 and 3’ correspond to the constant coefficient generalized Kawahara equations. The corresponding ODEs were heavily studied in the literature, see, e.g., [1, 12, 33, 39] and references therein. We concentrate our attention on variable coefficient cases.

### 4.1. Exact solutions for equations reducible to their constant coefficients counterparts

In recent papers [30, 52] different techniques for finding exact solutions were applied to construct exact solutions of Kawahara equations with time-dependent coefficients. In both papers exact solutions were derived for equations whose coefficients obey additional constraints, namely, when all the coefficients are proportional to each other. Theorem 3 implies that such variable coefficient equations from class (1) are reducible to constant coefficient Kawahara equations.

In our opinion, the optimal way to get exact solutions for equations from (1) that are reducible to the constant coefficient equations from this class is to take known solutions for constant coefficient equations and then to make a corresponding change of variables. In such a way it is possible to construct exact solution not only for the case when the coefficients in (1) are proportional but also (if \( n = 1 \)) for equations of the form (1) whose coefficients satisfy conditions (6).

We derive the corresponding changes of variables using theorem 1 for the case \( n \neq 1 \) and theorem 2 for the case \( n = 1 \). The following statement is true. The equations from class (1)

\[ u_t + \alpha(t)u u_x + \beta \alpha(t)u_{xxx} + \sigma \alpha(t)u_{xxxxx} = 0, \quad \text{and,} \]

\[ u_t + \alpha(t)u u_x + \beta \alpha(t) \left( \delta_3 \int \alpha(t) \, dt + \delta_4 \right) u_{xxx} + \sigma \alpha(t) \left( \delta_3 \int \alpha(t) \, dt + \delta_4 \right)^3 u_{xxxxx} = 0, \] (17)

where \( \alpha(t) \) is a smooth nonvanishing function, reduce to the constant coefficient Kawahara equations

\[ \ddot{u}_t + \tilde{\alpha} \ddot{u} \ddot{u}_x + \tilde{\beta} \ddot{u}_{xxx} + \tilde{\sigma} \ddot{u}_{xxxxx} = 0, \quad \text{and} \]

\[ \ddot{u}_t + \tilde{\alpha} \ddot{u} \ddot{u}_x + \tilde{\beta} \ddot{u}_{xxx} + \tilde{\sigma} \ddot{u}_{xxxxx} = 0 \] (19)

via the transformations

\[ \ddot{t} = \int \alpha(t) \, dt, \quad \ddot{x} = x, \quad \ddot{u} = \tilde{\alpha}^{-\frac{1}{2}} u, \quad \text{and} \]

\[ \ddot{t} = \int \alpha(t) \left( \delta_3 \int \alpha(t) \, dt + \delta_4 \right)^{-2} \, dt, \quad \ddot{x} = (x + \delta_1) \left( \delta_3 \int \alpha(t) \, dt + \delta_4 \right)^{-1}, \]

\[ \ddot{u} = \frac{1}{\tilde{\alpha}} \left( \left( \delta_3 \int \alpha(t) \, dt + \delta_4 \right) u - (x + \delta_1) \delta_3 \right). \] (20)

respectively. Here \( \delta_i, i = 1, 3, 4, \tilde{\alpha}, \tilde{\beta}, \text{ and } \tilde{\sigma} \) are arbitrary constants with \( \tilde{\alpha} \tilde{\beta} \tilde{\sigma} (\delta_3^2 + \delta_4^2) \neq 0 \).
We take a family of solitary wave solutions of the Kawahara equation (20) of the form

\[
\tilde{u} = \frac{-264992\tilde{\sigma}^2\tilde{k}^5 - 7280\tilde{\sigma}\tilde{k}^3 - 31\tilde{\beta}^2\tilde{k} + 507\tilde{\sigma} \mu}{507\tilde{\sigma} \tilde{k}}
\]

with \(\tilde{k}\) given by

\[
\tilde{k} = \frac{-\tilde{\beta}}{13\tilde{\alpha}}, \quad \tilde{k} = \frac{280\tilde{\sigma}^2(\tilde{\beta} - 104\tilde{\sigma}\tilde{k}^2)}{13\tilde{\alpha}} \tanh^2(\tilde{k}\tilde{\chi} + \mu\tilde{t} + \chi) - \frac{1680\tilde{\sigma}\tilde{k}^4\tanh^4(\tilde{k}\tilde{\chi} + \mu\tilde{t} + \chi)}{\tilde{\alpha}}
\]

and \(\tilde{\alpha} = \frac{\sqrt{-13\tilde{\beta}\tilde{\sigma}}}{26\tilde{\sigma}}, \quad \tilde{k} = \frac{\sqrt{65\tilde{\beta}\tilde{\alpha}(31 - 3i\sqrt{31})}}{260\tilde{\sigma}}, \quad \tilde{\kappa}_5 = \frac{\pm \sqrt{65\tilde{\beta}\tilde{\sigma}(31 + 3i\sqrt{31})}}{260\tilde{\sigma}}\), \(\mu\) and \(\chi\) being arbitrary constants \([33]\). The corresponding exact solution of (18), derived with the usage of (21), is

\[
u = \frac{1}{\delta_3 \int \alpha(t) dt + \delta_1} \left( \delta_3(x + \delta_1) - \frac{264992\tilde{\sigma}^2\tilde{k}^5 - 7280\tilde{\sigma}\tilde{k}^3 - 31\tilde{\beta}^2\tilde{k} + 507\tilde{\sigma} \mu}{507\tilde{\sigma} \tilde{k}} \right) - \frac{280}{13} \tilde{k}^2(\tilde{\beta} - 104\tilde{\sigma}\tilde{k}^2) \tanh^2(\tilde{k}\tilde{\chi} + \mu\tilde{t} + \chi) - 1680\tilde{\sigma}\tilde{k}^4\tanh^4(\tilde{k}\tilde{\chi} + \mu\tilde{t} + \chi),
\]

where \(\tilde{\beta} = \int \alpha(t) dt + \delta_2 \tilde{\chi}, \tilde{\chi} = (x + \delta_1) + \delta_3 \int \alpha(t) dt + \delta_4 \tilde{\chi}, \delta_1, \mu\) and \(\chi\) are arbitrary constants, \(\kappa\) takes the six values adduced above.

A family of solutions for equation (17) with \(n = 2\) has the form

\[
u = \frac{40\tilde{\beta}^2 - \tilde{\beta}^4 + 6\tilde{k}^2\sqrt{-10\tilde{\sigma}} \tanh^2 \left( \frac{\tilde{k}x - \frac{k}{10\tilde{\sigma}}(240\tilde{\sigma}^2 + \tilde{\beta}^2\tilde{\chi})}{\int \alpha(t) dt + \chi} \right)}{\sqrt{-10\tilde{\sigma}}},
\]

where \(k\) and \(\chi\) are arbitrary constants with \(k \neq 0\). On figures 1–3 we present the graphs of solution (22) for certain values of parameters and different time inhomogeneities.

### 4.2. Numerical solutions using Lie symmetries

Exact solutions of the fifth-order ODEs presented in Cases 1, 2 and 4’ of table 4 are not known. At the same time behavior of solutions for variable coefficients models is what we are most interested in. The Lie reductions obtained can be useful in seeking solutions of equations (1) accompanied with boundary conditions that are invariant with respect to the corresponding Lie symmetry algebras \([5]\).

Consider a class of boundary value problems (BVPs) for variable coefficient generalized Kawahara equations,

\[
u_t + \nu^6 \nu_x + \lambda t^n \nu_{xxx} + \delta t^{-\nu_{xxxxx}} = 0, \quad t > t_0, \quad x > 0, \quad n \in \mathbb{N},
\]

where \(\gamma, i = 0, \ldots, 4, \lambda\) and \(\delta\) are arbitrary constants with \(\gamma, \lambda \neq 0\). Both equation and boundary conditions are invariant with respect to the scaling symmetry operator \(Q = 3nt \lambda \partial_t + (\lambda + 1)n x \partial_x + (\lambda - 2)\mu \partial_x\) (Case 1 of table 1). Using the corresponding ansatz (Case 1, \(\rho = 1\) of table 4) this problem reduces to the initial value problem (IVP) for a fifth-order ODE,

\[
\delta \varphi''' + \lambda \varphi'' + \left( \varphi^n - \frac{\rho + 1}{3} \omega \right) \varphi' + \frac{\rho - 2}{3\rho} \varphi = 0,
\]

\[
\varphi(0) = \gamma_i, \quad i = 1, \ldots, 4.
\]
Figure 1. Solution (22) for $\alpha(t) = 1/t$, $\sigma = -0.1$, $\beta = -1$, $k = 1$, $\chi = 0$.

Figure 2. Solution (22) for $\alpha(t) = 1/t^2$, $\sigma = -0.1$, $\beta = -1$, $k = 1$, $\chi = -17$.

Figure 3. Solution (22) for $\alpha(t) = \sqrt{t}$, $\sigma = -0.1$, $\beta = -1$, $k = 1$, $\chi = 15$. 


After the problem for the latter IVP is solved numerically, then the corresponding solution of BVP (23)–(24) can be recovered using the similarity transformation \( u = \frac{t}{\omega^2} \psi(\omega) \) with \( \omega = x t^{-1/4} \).

We illustrate the usage of Lie symmetries for the construction of numerical solutions for the Kawahara equations with time-dependent coefficients by the following example.

**Example 1.** Consider the equation

\[
v_t + v_x + \frac{3}{2} \epsilon v v_x + \frac{1}{2} \kappa v_{xxx} + \frac{1}{4} \gamma v_{xxxx} = 0
\]

that arises as a model describing the propagation of long nonlinear waves in the water covered by ice [16, 23, 34, 47, 54]. Here

\[
\epsilon = \frac{a}{H}, \quad \kappa = \frac{h}{\rho_0 g \lambda^2 (\sigma_0 - \sigma_{\alpha s})}, \quad \gamma = \frac{EH^3}{12(1 - \nu^2) \rho_0 g \lambda^4},
\]

where \( \nu \) is the dimensionless amplitude of the oscillations of the under-ice surface of the fluid about the horizontal equilibrium position, \( a \) is the characteristic wave amplitude, \( H \) is the depth of the fluid, \( 2 \pi \lambda \) is the characteristic wavelength, \( \rho_0 \) and \( \rho_i \) are the densities of the fluid and ice, respectively; \( h, E, \) and \( v \) are the thickness, Young’s modulus and Poisson’s ratio of the ice, and \( \sigma_{\alpha s} \) is a component of the ice sheet stress tensor, \( \sigma_0 = gH(\rho_0 H/(3h) + \rho_i) \). It is assumed that \( \sigma_{\alpha s} \approx 10^8 \text{ N m}^{-2} \) is the result of external forces [23].

We suppose that the growth of ice thickness is described by the law \( h = 0.04 \sqrt{t} \), which for certain weather conditions is in well agreement with the data obtained for the sea of Azov for 10 days (240 h) of observations of ice growth starting from \( h = 0.1 \text{ m} \) [9]. Then for the values \( \lambda \approx 100 \text{ m}, H \approx 10 \text{ m}, E \approx 3 \times 10^9 \text{ N m}^{-2}, a \approx 0.1 \text{ m}, \rho_0 \approx 1030 \text{ kg m}^{-3}, \rho_i \approx 916 \text{ kg m}^{-3} \) and \( \sigma_0 \approx 1.2 \times 10^6 \text{ N m}^{-2} \) that is calculated for average ice thickness \( h_0 \approx 0.3 \text{ m} \) we will have a model equation of the form

\[
v_t + v_x + \alpha vv_x + \lambda \beta v_{xxx} + \delta t^2 v_{xxxx} = 0,
\]

where \( \alpha = 1.5 \times 10^{-2}, \beta \approx 2.20215 \times 10^{-5} \) and \( \delta \approx 1.00566 \times 10^{-8} \) (after converting time in \( E, \sigma_0 \) and \( \sigma_{\alpha s} \) in hours). To reduce this equation to the form (23) we make the change of the dependent variable \( u = 1 + \alpha v \) and get the equation

\[
u_t + u u_x + \lambda \beta u_{xxx} + \delta t^2 u_{xxxx} = 0 \tag{26}
\]

where \( \lambda \) and \( \delta \) remain the same. We consider the boundary conditions

\[
u(t, 0) = \gamma t^{-1/4}, \quad u_x(t, 0) = 0, \quad u_{xx}(t, 0) = 0, \quad u_{xxx}(t, 0, 0) = 0, \quad u_{xxxx}(t, 0, 0) = 0, \tag{27}
\]

that are invariant with respect to the operator of scaling symmetry \( 2i \partial_t + x \partial_x - u \partial_x \) of the latter equation. Such a BVP reduces to the following IVP

\[
\delta \psi'''' + \lambda \psi''' + (\psi - \frac{1}{2} \nu) \psi' - \frac{1}{2} \psi = 0,
\]

\[
\psi(0) = \gamma_0, \quad \psi'(0) = \psi'(0) = \psi''(0) = \psi'''(0) = 0. \tag{28}
\]

The numerical solution for this IVP is presented on figure 4. The corresponding numerical solution of equation (26) with the associated boundary conditions (27) is presented on figure 5.

### 5. Conclusion and discussion

In the present paper the group classification problem for class (1) of variable coefficient generalized Kawahara equations was solved exhaustively. As a result, new variable coefficient nonlinear models admitting Lie symmetry extensions were derived. This became possible due to an appropriate gauge of arbitrary elements of the class. Namely, the gauge \( \alpha = 1 \) was
The use of different equivalence groups for the cases $n \neq 1$ and $n = 1$, which were found in the course of the study of admissible transformations in class (1), allowed us to write down the classification list in a simple and concise form (see table 1). For convenience of further applications, in table 2 we also presented the classification list extended by the equivalence transformations. Then one-dimensional subalgebras of Lie symmetry algebras admitted by equations from class (1) were classified and all inequivalent reductions with respect to such subalgebras were performed. It is obvious that the extension of known solutions of constant coefficient equations from class (1) by equivalence transformations of this class is a preferable way for the construction of exact solutions to the equations (1) that are reducible to constant coefficient equations, i.e., to equations of the form (17) and (18). Some exact solutions for...
the classes of equations (18) and equations (17) with \( n = 2 \) were constructed as illustrative examples. We also presented a class of boundary value problems for variable coefficient Kawahara equations possessing scaling symmetry and constructed a numerical solution for the specific equation that could be of interest for applications.

In the framework of modern group analysis of differential equations the following problems for equations from class (1) can also be studied.

- **The study of conservation laws.** Of course, using modern computer algebra packages it is easy to compute low-order conservation laws, not to mention zero-order conservation laws. The main problem is to prove that the set of orders of conservation laws is bounded and then to describe exhaustively the entire space of conservation laws. The obvious zero-order conservation laws of equations of the form (8) are given by conserved vectors with characteristics 1 and \( u \)

\[
\left( u, \frac{1}{n+1} \alpha(t) u^{n+1} + \beta(t) u_{xx} + \sigma(t) u_{xxxx} \right),
\]

\[
\left( \frac{1}{2} u^2, \frac{1}{n+2} \alpha(t) u^{n+2} + \beta(t) \left( u u_{xx} - \frac{1}{2} u_x^2 \right) + \sigma(t) \left( u u_{xxxx} - u_x u_{xxx} + \frac{1}{2} u_{xx}^2 \right) \right).
\]

These are conservation laws of momentum and energy, respectively.

- **Potential symmetries** [7, 46]. For each equation of the form (1) we can construct the potential systems,

\[
v_t = u, \quad v_x = -\frac{1}{n+1} \alpha(t) u^{n+1} - \beta(t) u_{xx} - \sigma(t) u_{xxxx},
\]

\[
v_t = \frac{1}{2} u_x^2,
\]

\[
v_x = -\frac{1}{n+2} \alpha(t) u^{n+2} - \beta(t) \left( u u_{xx} - \frac{1}{2} u_x^2 \right) - \sigma(t) \left( u u_{xxxx} - u_x u_{xxx} + \frac{1}{2} u_{xx}^2 \right),
\]

associated with the above conservation laws. The complete description of potential systems needs the exhaustive classification of local conservation laws of such equations [6, 26, 44]. Then Lie symmetries of the constructed potential systems should be found, which may result in nontrivial potential symmetries for the generalized Kawahara equations.

In these problems the investigation can be restricted to subclass (8) of class (1) without loss of generality, i.e., it can be assumed that \( \alpha = 1 \). The justification is presented in section 2.

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