Effective-Range Expansion of Neutron-Deuteron Scattering Studied by a Quark-Model Nonlocal Gaussian Potential

Kenji Fukukawa and Yoshikazu Fujiwara

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

(Received October 11, 2010; Revised February 18, 2011)

The S-wave effective-range parameters of the neutron-deuteron (nd) scattering are calculated in the Faddeev formalism using a nonlocal Gaussian potential based on the quark-model baryon-baryon interaction fss2. The spin-doublet low-energy eigenphase shift is sufficiently attractive to reproduce predictions by the AV18 plus Urbana three-body force, yielding almost correct values of the scattering length and the triton binding energy without the three-nucleon force. This feature is due to the strong distortion effect of the deuteron in this spin channel, which is very sensitive to the nonlocal description of the short-range repulsion in the quark-model nucleon-nucleon interaction. We incorporate the Coulomb force by extending the framework of the Coulomb externally corrected approximation and calculate the differential cross sections of the pd scattering.

Subject Index: 200, 205

§1. Introduction

Few-nucleon systems are best suited to study the underlying nucleon-nucleon (NN) interaction and its extension to few-nucleon forces, since many sophisticated techniques to solve the systems yield equivalent results that can be compared with ample experimental data.1,2) In fact, the three-nucleon (3N) system has already been solved for many realistic meson-exchange potentials, yielding insufficient binding energies of the triton missing 0.5 to 1 MeV without the 3N force.3) An attempt to reproduce the NN and 3N data consistently has been made using the chiral effective field theory, but a complete reproduction of all the 3N data by this approach is still not possible.4,5) We have applied the quark-model (QM) baryon-baryon (BB) interaction6,7) to the triton and hypertriton using the Faddeev formalism and obtained many interesting results.8)–11) The most recent model, fss2, gives a nearly correct binding energy of the triton with the proper root-mean-square radius, preserving the sufficient strength of the tensor force for the deuteron and the exact 1S0 NN scattering length.8) This result indicates that fss2 is sufficiently attractive in the 2S1/2 channel of the 3N system without the 3N force. In this channel, the deuteron distortion effect related to the strong short-range repulsion of the NN interaction is very important. In the QM BB interaction, this repulsion is mainly described by the quark-exchange nonlocal kernel of the one-gluon exchange Fermi-Breit interaction, which has quite different off-shell properties from the phenomenological repulsive core described by local potentials in the standard meson-exchange models. The nonlocal effect resulting from the exact antisymmetrization of six quarks is very important in reproducing a nearly correct triton binding energy.

The QM BB interaction is constructed for two three-quark clusters in the frame-
work of the resonating-group method (RGM). It is characterized not only by the dominant nonlocality from the interaction kernel but also by the energy dependence originating from the normalization kernel. In the first evaluation of the triton and hypertriton binding energies, this energy dependence is determined self-consistently by calculating the expectation value of the two-cluster Hamiltonian with the square integrable three-cluster wave function.\(^8\)–\(^10\) This method is, however, not applicable to the three-body scattering problem, since the scattering wave function is not square-integrable. In the final form of the triton and hypertriton Faddeev calculations,\(^11\) the energy dependence of the RGM kernel is eliminated by a standard off-shell transformation using the square root of the normalization kernel.\(^12\) An extra nonlocality emerges from this off-shell transformation as a result of eliminating the energy dependence of the RGM kernel. It is shown in Ref. 11) that this renormalized RGM gives a slightly less attractive effect of the triton and hypertriton binding energies in comparison with the previous self-consistent treatment. The 350 keV deficiency of the triton binding energy predicted by fss2 after the charge dependence correction of the \(NN\) interaction is still much smaller than the 0.5 to 1 MeV given by the standard meson-exchange potentials. It is therefore interesting to examine the nonlocal effect of the QM \(NN\) interaction on the \(3N\) scattering observables, such as the scattering lengths, the differential cross sections and the spin polarization, in this energy-independent QM \(NN\) interaction.

The numerical calculation of three-body scattering using the QM \(BB\) interaction is very time-consuming owing to the complex structure of the interaction. We therefore construct a nonlocal Gaussian potential in the isospin basis by applying the Gauss-Legendre integration formula to special functions appearing in the exchange RGM kernels.\(^13\) The nonlocality and energy dependence of the QM \(BB\) interaction are strictly preserved in this nonlocal Gaussian potential. We find that the 15-point Gauss-Legendre integration formula is satisfactory for carrying out few-body calculations accurately. The \(NN\) phase shifts predicted by this potential are essentially the same as those predicted by fss2 with an accuracy of better than 0.1°. We will show that the difference in the triton binding energy between fss2 and this nonlocal Gaussian potential is only 15 keV.

Here, we study the low-energy neutron-deuteron \((nd)\) elastic scattering below the deuteron breakup threshold on the basis of the formulation developed in Ref. 14). For this purpose, the \(S\)-wave effective-range theory in the channel-spin formalism is very useful. The channel spin \(S_c\) is composed of the \(NN\) total angular momentum \(I\) and the spin 1/2 of the third nucleon; i.e., \((I^{1/2})S_c\). Since the deuteron channel with \(I = 1\) only survives in the asymptotic region, the scattering amplitudes for the \(nd\) elastic scattering are specified by the channel spin \(S_c = 1/2\) (the spin-doublet channel) and \(S_c = 3/2\) (the spin-quartet channel). These two channels have considerably different characteristics with respect to the deuteron distortion effect. Namely, in the spin-quartet channel the incident neutron cannot penetrate deep inside the deuteron owing to the effect of the Pauli principle, resulting in the weak distortion effect of the deuteron. On the other hand, the neutron can freely approach the deuteron in the spin-doublet channel, causing strong distortion effects reflecting the strong sensitivity to details of the \(NN\) interaction. In this sense, the spin-doublet
scattering length $a_{nd}$ is an important observable to determine whether the $NN$ interaction is appropriate or not. It has been known for a long time that a larger triton binding energy corresponds to a smaller $a_{nd}$. This linear correlation is known as the Phillips line, and has been confirmed by many theoretical calculations. Since the binding energy of the triton is not reproduced in the $NN$ meson-exchange potentials, the experimental value, $a_{nd} = 0.65 \pm 0.04$ fm, is not reproduced either. The calculated $a_{nd}$ is more than 0.9 fm if only an $NN$ force is used. It is therefore a common practice to add the $3N$ force to reproduce the triton binding energy as well as the correct $a_{nd}$. A thorough investigation of the triton binding energy and the scattering lengths using a number of meson-exchange potentials and various $3N$ forces is given in Ref. 22). We expect that the nonlocal effect of $fss2$ will lead to a good reproduction of the doublet scattering length since $fss2$ gives a large triton binding energy close to the experimental value. In Ref. 23), the nonlocal interaction was applied to the $nd$ scattering lengths using the $NN$ interaction based on the chiral constituent quark model. Since the Faddeev calculation in these studies did not properly treat the energy dependence of the QM $NN$ interaction, an insufficient triton binding energy and a large doublet scattering length, almost comparable to those of the meson-exchange predictions, were obtained.

The most accurate method of determining the scattering lengths is to calculate the zero-energy scattering amplitude directly, as carried out in Refs. 16), 17), 22), 23), etc. In this approach, however, only the zero-energy $nd$ scattering is examined. A more extensive study of the low-energy $nd$ elastic scattering can be achieved by employing the effective-range theory, in which the pole structure existing in the effective-range function for the doublet-$S$ channel should be properly taken into account. We can discuss the energy dependence of the low-energy $S$-wave phase shifts using this approach. On the other hand, this method has a problem of numerical inaccuracy at extremely low energies below 100 keV in the center-of-mass (cm) system, since the solution of the basic equation becomes singular in the momentum representation.

In this paper, we start with the nonlocal Gaussian QM $NN$ interaction and eliminate the energy dependence numerically using the above-mentioned renormalized RGM formalism. We then apply this interaction to the $nd$ scattering and solve the Alt-Grassberger-Sandhas (AGS) equation to obtain the scattering amplitudes. The elastic scattering amplitudes are conveniently parameterized by the standard eigenphase shifts and mixing parameters defined in Ref. 33). The spin-doublet and -quartet $S$-wave effective-range parameters, i.e., $a_{nd}$, $(\tilde{r}_e)_{nd}$, $a_{nd}$ and $(\tilde{r}_e)_{nd}$, together with the pole parameter $qQ$ in the doublet case, are calculated by employing the $S$-wave single-channel effective-range formula. We find reasonable agreement of the low-energy differential cross sections with the $nd$ experimental data. Since the $pd$ data are more precise than the $nd$ data, we also evaluate the differential cross sections of the $pd$ elastic scattering in the framework of the so-called Coulomb externally corrected approximation with some modifications to the nuclear phase shifts. We find that the $2S_{1/2}$ eigenphase shift predicted by our model is sufficiently attractive to reproduce the doublet scattering length $a_{nd}$ and the triton binding energy without the $3N$ force. It will be shown that the $pd$ differential cross sections
below the deuteron breakup threshold are well reproduced by the large Coulomb modification to the \( ^4S_{3/2} \) nuclear phase shift.

The organization of this paper is as follows. In §2.1, a brief description of the Faddeev formalism is given for the bound-state and \( nd \) scattering problems. The spin-isospin factors and rearrangement factors for the permutation operator are explicitly given in Appendix A. In §2.2, we recapitulate the procedure to obtain the eigenphase shifts and their \( J \)-averaged central phase shifts from the solutions of the AGS equation. The single-channel effective-range expansion is explained in §2.3. In §3.1, the \( nd \) eigenphase shifts of our model are compared with those of Argonne V18 (AV18) plus Urbana (UR) 3\( N \) potentials obtained by the \( K \)-harmonics technique. The \( nd \) and \( pd \) differential cross sections below the deuteron breakup threshold are also discussed. The effective-range parameters are given in §3.2, together with the analysis of the \( S \)-wave contributions to the \( nd \) total cross sections. The last section is devoted to a summary.

§2. Formulation

2.1. Faddeev approach to the triton and the \( nd \) scattering

We start with the three-body Schrödinger equation

\[
[E - H_0 - V_\alpha^{\text{RGM}} - V_\beta^{\text{RGM}} - V_\gamma^{\text{RGM}}] \Psi = 0 ,
\]

where \( V_\alpha^{\text{RGM}} \) denotes the energy-independent renormalized RGM kernel for which the detailed derivation is given in Ref. 14). The subscripts \( \alpha, \beta \) and \( \gamma \) in Eq. (2.1), with \( (\alpha, \beta, \gamma) \) being the cyclic permutation of \( (1, 2, 3) \), specify the three types of Jacobi coordinates related to the particle and the residual pair in the usual way. For systems of three identical particles, the Faddeev equation for the bound state is

\[
\psi = G_0 t P \psi ,
\]

where \( P = P_{12}P_{13} + P_{13}P_{12} \) is a sum of the permutation operators for the nucleon rearrangement. The \( NN \) \( t \)-matrix in the three-body model space is derived from the standard Lippmann-Schwinger equation \( t = v + vG_0 t \) with \( v = V^{\text{RGM}} \), where \( G_0 = (E - H_0)^{-1} \) is the three-body Green function for the free motion. The Green function is composed of the negative total energy \( E \) in the cm system and the three-body kinetic-energy operator \( H_0 = h_0 + \bar{h}_0 \). The operators \( h_0 \) and \( \bar{h}_0 \) correspond to the kinetic energies for \( p_3 = \frac{1}{2}(k_1 - k_2) \) and \( q_3 = \frac{1}{3}(2k_3 - k_1 - k_2) \), respectively, when the Jacobi coordinates with \( \gamma = 3 \) are chosen. The vector \( k_i \) \( (i = 1 - 3) \) is the individual momentum of particle \( i \). The Faddeev component \( \psi \) is defined through \( \psi = G_0 v \Psi \) using the total wave function \( \Psi = \sum_\alpha \psi_\alpha \) in Eq. (2.1). In the partial wave expansion, we use the channel-spin formalism specified by \( (I^1_2)S_c \). The channel spin \( S_c \) is coupled with the relative angular momentum \( \ell \) between the spectator nucleon and the \( NN \) subsystem and makes the total angular momentum \( (\ell S_c)J \). The \( NN \) channel is specified by \( (\lambda s)I; t \), where \( \lambda, s \) and \( t \) are the orbital angular momentum, the spin and the isospin of the \( NN \) system, respectively. The \( NN \) isospin \( t \) is uniquely specified by \( \lambda \) and \( s \) from the Pauli principle \((-)^{\lambda+s+t} = -1 \). We further
set the parity restriction $\pi = (-)^{J + \ell}$, which is conserved for each $J$. The angular-spin-isospin wave functions are thus defined by

$$|p, q; 123\rangle = \sum_\gamma |p, q, \gamma\rangle \langle \gamma | \hat{p}, \hat{q}; 123\rangle,$$

$$\langle \hat{p}, \hat{q}; 123|\rangle = \left[ Y_0(\hat{q}) \left( [\chi(\hat{p}) \chi_{st}(1, 2)] \chi_{1, 1/2}(3) \right) \right]_{J_z; \frac{1}{2}T_z},$$

(2.3)

with $\gamma = [\ell([\lambda s]I_{\frac{1}{2}})S_c]JJ_z; (t_\frac{1}{2} \frac{1}{2} T_z)$ in the channel-spin representation. The partial wave expansion of the Faddeev equation in Eq. (2.2) is given by

$$\psi_{\gamma}(p, q) = -\frac{M}{\hbar^2 \kappa^2_t + p^2 + (3/4)q^2} \sum_{\gamma', \gamma''} \int_0^\infty dq' q'^2 \int_0^\infty dp' p'^2 \int_0^\infty dp'' q''^2$$

$$\times \int_0^\infty dp''' p'''^2 \langle p, q, \gamma|t|p', q', \gamma'\rangle \langle p', q', \gamma'|P|p'', q'', \gamma''\rangle \psi_{\gamma''}(p'', q''),$$

(2.4)

where $M = (M_n + M_p)/2$ is the averaged nucleon mass in the isospin formalism. The binding energy of the triton is deduced from $E_B(-E) = (\hbar^2/M)\kappa^2_t$. The coupled integral equation for $p$ and $q$ in Eq. (2.4) is solved by the Lanczos-Arnoldi method after the necessary process of discretization. The $NN\ t$-matrix is factorized as

$$\langle p, q, \gamma|t|p', q', \gamma'\rangle = \frac{4\pi}{(2\pi)^3} \frac{\delta(q - q')}{qq'} t_{\gamma\gamma'}(p, p'; E - \frac{3\hbar^2}{4Mq^2}).$$

(2.5)

The matrix elements of the permutation operator are evaluated as

$$\langle p, q, \gamma|P|p', q', \gamma'\rangle = \frac{1}{2} \int_{-1}^1 dx \frac{\delta(p - p_1)}{p^{\lambda+2}} g_{\gamma, \gamma'}(q, q', x) \frac{\delta(p' - p_2)}{p'^{\lambda+2}},$$

(2.6)

with $p_1 = p(q', q/2; x), p_2 = p(q, q'/2; x)$ and $p(a, b; x) = \sqrt{a^2 + b^2 + 2abx}$. The basic rearrangement coefficients $g_{\gamma, \gamma'}(q, q', x)$ contain the spin-isospin factors, and the explicit expression depends on a specific type of the channel-coupling scheme, as given in Appendix A.

For the $nd$ scattering, the three-body scattering amplitudes are obtained by solving the AGS equation

$$U|\phi\rangle = G_0^{-1}P|\phi\rangle + PtG_0U|\phi\rangle,$$

(2.7)

where $|\phi\rangle = |q_0, \psi_d\rangle$ is the plane-wave channel wave function with $|\psi_d\rangle$ being the deuteron wave function. The total cm energy $E$ in $G_0 = (E + i0 - H_0)^{-1}$ is expressed as $E = E_{cm} + \varepsilon_d$, where $E_{cm} = (3\hbar^2/4M)q_0^2$ is the cm incident energy of the neutron and $\varepsilon_d = -\varepsilon_d$ is the deuteron binding energy. In the AGS equation, we have in general two types of singularities, but the notorious moving singularity does not appear for energies below the deuteron breakup threshold. The other singularity related to the deuteron pole of the $NN\ t$-matrix is directly incorporated into the AGS equation by the Noyes-Kowalski method. For the detailed procedure to overcome difficulties caused by these singularities, Ref. 14) should be referred to. Here, we
recapitulate the procedure to derive the elastic scattering amplitudes $\langle \phi | U | \phi \rangle$. We solve Eq. (2.7) in the form

$$\tilde{Q} | \phi \rangle = \tilde{P} | \phi \rangle + W \tilde{Q} | \phi \rangle ,$$

where $\tilde{t}$ is the singularity-free $NN$ $t$-matrix and $W$ is given by

$$W = G_0 P - P | \phi \rangle Z^{-1} | \phi P \rangle \quad \text{with} \quad Z = \langle \phi | G_0^{-1} P | \phi \rangle .$$

Furthermore, $\tilde{Q} | \phi \rangle$ and $\tilde{P} | \phi \rangle$ are related to $G_0 U | \phi \rangle$ and $P | \phi \rangle$, respectively, through

$$G_0 U | \phi \rangle = \tilde{Q} | \phi \rangle \langle \phi | U | \phi \rangle , \quad \tilde{P} | \phi \rangle = P | \phi \rangle Z^{-1} .$$

The elastic scattering amplitude $\langle \phi | U | \phi \rangle$ is obtained by solving a simple algebraic equation including $\langle \phi | X | \phi \rangle = \langle \phi | \tilde{P} t \tilde{Q} | \phi \rangle$. The partial wave components of the scattering amplitude, $U_{(l'S_c'),(l'S_c)}^{J} = \langle \phi_{l'S_c'} | U | \phi_{l'S_c} \rangle$, defined through

$$\langle \phi_qf; S_c'S_{cz} | U | \phi_qi; S_cS_{cz} \rangle = \sum_{l'l''J_z} U_{l'l''}^{J} |_{(l'S_c'),(l'S_c)}
\times \sum_{m'} \langle l'm'S_c'S_{cz} | J J_z \rangle Y_{l'm'}(\tilde{q}_f) \sum_{m} \langle l'mS_cS_{cz} | J J_z \rangle Y^{*}_{l'm}(\tilde{q}_i) ,$$

with $|q_f| = |q_i| = q_0$, are converted to the coupled-channel $S$-matrix by $S_{(l'S_c'),(l'S_c)}^{J} = \delta_{l'l'} \delta_{S_cS_{c'}} - 4\pi i (q_0 M / 3h^2) U_{(l'S_c'),(l'S_c)}^{J}$.  

### 2.2. Eigenphase shifts and differential cross sections

In the channel-spin representation, the asymptotic channel wave function $|\phi; (l'S_c)JJ_z \rangle$ in the partial wave expansion is specified by $(l'S_c)J = (J \pm 3/2, 3/2)J$, $(J \pm 1/2, 1/2)J$ and $(J \mp 1/2, 3/2)J$ for the parity $\pi = (-)^J + 1/2$. The $S$-matrix $S_{(l'S_c'),(l'S_c)}^{J}$ is a two-dimensional matrix for $J = 1/2$ and a three-dimensional matrix for $J \neq 1/2$. The $S$-matrix can be diagonalized as

$$S = U^{\dagger} e^{2i \Delta} U ,$$

where $\Delta$ is the diagonal matrix of the eigenphase shifts $\delta_{l'S_c}^{J}$. (We follow the notation in Ref. 33), but we should note that $\ell$ and $S_c$ are not good quantum numbers.) The unitary matrix $U$ can be parameterized in terms of the mixing parameters $\varepsilon, \xi$ and $\eta$,

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varepsilon & \sin \varepsilon & 0 \\ 0 & -\sin \varepsilon & \cos \varepsilon & 0 \\ \cos \xi & \cos \eta & \sin \xi & 0 \\ \cos \xi \sin \eta & \sin \xi \sin \eta & \sin \xi \cos \eta & 0 \\ -\sin \xi \sin \eta & \sin \xi \cos \eta & -\cos \xi \sin \eta & \cos \xi \cos \eta \end{pmatrix} ,$$

(2.13)
Effective-Range Expansion of \(^{2}P_{J} - ^{4}P_{J}\) couplings).

Since the difference in the eigenphase shifts with the same \(\ell S_c\) but different \(J\) is very small, the \(J\)-averaged central phase shifts \(\delta_{\ell S_c}\) are convenient for discussing the scattering cross sections approximately. These are defined by taking an average over all possible \(J\) states of \(^{2S_c+1} \ell J\) with the weight factor \((2J+1)\),

\[
\delta_{\ell S_c} = \frac{1}{(2\ell + 1)(2S_c + 1)} \sum_J (2J+1) \delta_{\ell S_c}^J .
\]

The differential cross sections for the \(nd\) elastic scattering are calculated by summing over the final spin states and by taking an average over the initial spin states. This is given by

\[
\frac{d\sigma}{d\Omega} = \frac{1}{3} \left( \frac{d\sigma}{d\Omega} \right)_{S_c=1/2} + \frac{2}{3} \left( \frac{d\sigma}{d\Omega} \right)_{S_c=3/2} ,
\]

where the differential cross sections for each channel spin \(S_c\) are calculated from the scattering amplitudes \(f_{\ell S_c} = (1/q_0)e^{i\delta_{\ell S_c}} (\sin \delta_{\ell S_c})\) through

\[
\left( \frac{d\sigma}{d\Omega} \right)_{S_c} = \sum_\ell (2\ell + 1) f_{\ell S_c} P_\ell (\cos \theta) \right) \right|^2 .
\]

In order to compare our results with the \(pd\) experimental data, we have to take the Coulomb force into account. A comparison with the \(pd\) data is desirable since they are abundant and more accurate than those of the \(nd\) scattering. For example, in Refs. 38)–43) many phase shift analyses have been carried out for the \(pd\) data with high accuracy. The exact treatment of the Coulomb force in the three-body problem is still a challenging task\(^{34),35),44)–46}\) and preliminary results by a new method are given in Ref. 47) for \(s2s2\). Here, we incorporate the Coulomb force by extending the
framework of the Coulomb externally corrected approximation\cite{34,35} using the two-potential formula for three-body scattering.\cite{46} In this framework, the single-channel differential cross sections are given by

$$
\left( \frac{d\sigma}{d\Omega} \right)_{Sc} = f^C(\theta) + \sum_{\ell} (2\ell + 1)e^{2i\sigma_{\ell}}e^{i\delta_{\ell,Sc}^{N}}P_{\ell}(\cos\theta) \right|^{2},
$$

(2.18)

where \( f^C(\theta) \) is the standard Coulomb scattering amplitude and \( \sigma_{\ell} \) is the partial wave Coulomb phase shift. The scattering amplitude from the strong interaction, \( f^{N}_{\ell,Sc} = (1/q_0)e^{i\delta_{\ell,Sc}^{N}}(\sin\delta_{\ell,Sc}^{N}) \), is evaluated from the nuclear phase shifts \( \delta_{\ell,Sc}^{N} \) of fss2 with/without the Coulomb force.

2.3. Effective-range expansion for the nd scattering

If we use the effective-range theory, we can study the energy dependence of the phase shifts reflected in the effective range \( r_e \) and discuss the contributions to the total cross sections from the S-wave components between the neutron and the deuteron. We first calculate the eigenphase shifts for the \( J^\pi = 1/2^+ \) and \( 3/2^+ \) states. For the \( J^\pi = 3/2^+ \) state, the \( 4S_{3/2}, 2D_{3/2} \) and \( 4D_{3/2} \) channels are coupled and the dominant eigenphase shift is that for the spin-quartet S-state. For neutron energies below the deuteron breakup threshold, the phase shifts and mixing parameters are in general very small except for the dominant S-wave eigenphase shift, so that the effective-range expansion formula for a single-channel problem can be applied to this component to obtain the effective-range parameters. In the quartet S-channel, we expand the effective-range function \( K(q_0) = q_0 \cot \delta \), which is the real part of the inverse scattering amplitude, in a power series of \( q_0^2 \):

$$
K(q_0) = -\frac{1}{a} + \frac{1}{2}r_e q_0^2 + \mathcal{O}(q_0^4).
$$

(2.19)

Here, \( a \) is the scattering length, \( r_e \) is the effective range and \( \delta = \delta_{0}^{3/2} \) in \( K(q_0) = q_0 \cot \delta \) is the S-wave eigenphase shift for the quartet channel with \( q_0 \) being the relative wave number between the neutron and the deuteron.

For the \( J^\pi = 1/2^+ \) state, the \( 2S_{1/2} \) and \( 4D_{1/2} \) channels are coupled and the doublet-S channel is dominant. In this case, the effective-range function \( K(q_0) \) has a pole slightly below the elastic threshold.\cite{28,29,30,31} We parameterize the effective-range function in the doublet channel as

$$
K(q_0) = -\frac{1}{a} + \frac{1}{2}r_e q_0^2 + \mathcal{O}(q_0^4) \left( 1 + \frac{q_0}{q_0/q_Q} \right)^2,
$$

(2.20)

where the pole parameter \( q_Q \) specifies the pole position and \( \delta = \delta_{0}^{1/2} \) is the S-wave eigenphase shift for the doublet channel. The origin of this pole structure is studied by means of \( N/D \) equations.\cite{29,30,31} In the \( N/D \) formalism, the partial wave components of the scattering amplitude are given by \( N(z)/D(z) \), where \( N(z) \) and \( D(z) \) are analytic functions of the complex and dimensionless energy variable \( z \) defined by...
Effective-Range Expansion of \( nd \) Scattering

\[ z = \frac{E_{\text{cm}}}{|\varepsilon_d|}. \]

From the solution of the \( N/D \) equation, it was found that this singularity is brought about by both the dominant single-nucleon exchange and other effects such as the two-nucleon exchange, which slowly vary for small \( z \). In the doublet channel, the single-nucleon exchange, which is by far the longest-range force, is attractive. Thus, nothing prevents the other effects from affecting the low-energy scattering. On the other hand, the single-nucleon exchange is strongly repulsive in the quartet channel. Therefore, the nucleons in this channel cannot penetrate to the region where the other attractive forces can act. This is why the pole structure is found only in the doublet channel.

§3. Results and discussion

3.1. Eigenphase shifts and differential cross sections

Since our Coulomb treatment for fss2 in Ref. 47) is still at a preliminary stage, we show in Table I our \( nd \) phase shifts for the energies of \( E_n =1, 2 \) and 3 MeV, together with other theoretical predictions by the Pisa group,\(^{42}\) which are calculated using AV18 and AV18+UR3N potentials.\(^{48),49}\) Here, \( E_n = (3/2)E_{\text{cm}} \) is the neutron incident energy measured in a laboratory system. We have included the NN interaction up to the total angular momentum \( I_{\text{max}} = 3 \) or 4 and the momentum mesh points \( n \equiv n_1-n_2-n_3 = 6-6-5 \), using the notation defined in §3.1 of Ref. 14). The UR3N potential has a sizable effect of about 4°–5° only in the \( J^{\pi} = 1/2^+ \) channel. We immediately find an noteworthy feature in the \( J^{\pi} = 1/2^+ \) channel. Namely, our results obtained by fss2 are very similar not to the AV18 results but to the AV18+UR3N results shown in the parentheses. This is not surprising since fss2 gives a nearly correct binding energy without the \( 3N \) force. On the other hand, the phase shifts of the \( J^{\pi} = 3/2^+ \) state are very similar between fss2 and AV18 (or AV18+UR3N), and the difference between fss2 and AV18+UR3N is less than 0.1°. In this state, the effect of the UR3N force is very small owing to the Pauli principle. For the \( P \) states, some of the eigenphase shifts show a somewhat larger difference from the AV18 or AV18+UR3N results, especially at \( E_n = 3 \) MeV, but the difference is still less than 1°. After all, we have found good correspondence of the nuclear phase shifts between our fss2 and the AV18+UR3N potentials.

The resemblance seen in Table I becomes clearer in Table II if we compare the \( J \)-averaged central phase shifts defined by Eq. (2.15). These are used to evaluate the \( nd \) differential cross sections through Eqs. (2.16) and (2.17). We have illustrated the results in Figs. 1 – 3 with dot-dashed curves, but they almost overlap with the solid curves representing the exact calculations. We find that the \( D \)-wave components give an appreciable contribution to the differential cross sections, even at such a low energy as \( E_n = 1 \) MeV. This is of course because of the \( D \)-wave component of the deuteron wave function. This analysis encourages us to study the \( pd \) differential cross sections by using a simple approximation for the Coulomb effect, discussed in §2.2. We first assume, for the nuclear scattering amplitude \( f_N^{\ell S_c} \) in Eq. (2.18), the \( nd \) scattering amplitude \( f_{\ell S_c} = (1/q_0)e^{i\delta_{\ell S_c}}(\sin \delta_{\ell S_c}). \)

---

Downloaded from https://academic.oup.com/ptp/article-abstract/125/5/957/2938817
by guest on 30 July 2018
Table I. The \( nd \) eigenphase shifts and mixing parameters (in degrees) obtained from the model fss2. The maximum angular momentum for the \( NN \) system, \( I_{\text{max}} = 3 \) (1, 2 MeV) or 4 (3 MeV), and the momentum mesh points \( n = 6-6-5 \) are used. The corresponding parameters calculated by the Pisa group from the AV18 potential models are also listed for comparison.\(^{42} \)

The parameters in the parentheses are predictions by the AV18+UR3N potentials.

| Model | \( E_n (\text{MeV}) \) | fss2 | AV18 | fss2 | AV18 | fss2 | AV18 |
|-------|------------------|------|------|------|------|------|------|
| \( ^4D_{1/2} \) | 1.0 | -0.979 | -0.980 | -2.52 | -2.53 | -3.82 | -3.85 |
| | | — | (-0.976) | — | (-2.52) | — | (-3.84) |
| \( ^2S_{1/2} \) | 1.0 | -14.9 | -18.1 | -24.3 | -28.3 | -30.8 | -35.3 |
| | | — | (-14.3) | — | (-24.0) | — | (-30.8) |
| \( \eta_{1/2}^+ \) | 1.0 | 1.29 | 0.928 | 1.48 | 1.08 | 1.55 | 1.12 |
| | | — | (1.39) | — | (1.47) | — | (1.45) |
| \( ^2P_{1/2} \) | 1.0 | -4.13 | -4.13 | -6.56 | -6.57 | -7.44 | -7.49 |
| | | — | (-4.13) | — | (-6.58) | — | (-7.50) |
| \( ^4P_{1/2} \) | 1.0 | 11.8 | 12.0 | 19.4 | 19.9 | 23.6 | 24.2 |
| | | — | (12.1) | — | (20.1) | — | (24.5) |
| \( \varepsilon_{1/2}^- \) | 1.0 | 3.49 | 3.47 | 5.00 | 4.98 | 6.76 | 6.68 |
| | | — | (3.53) | — | (5.07) | — | (6.82) |
| \( ^4S_{3/2} \) | 1.0 | -46.7 | -46.7 | -60.6 | -60.8 | -69.6 | -69.9 |
| | | — | (-46.6) | — | (-60.7) | — | (-69.7) |
| \( ^2D_{3/2} \) | 1.0 | 0.563 | 0.564 | 1.50 | 1.51 | 2.34 | 2.36 |
| | | — | (0.564) | — | (1.51) | — | (2.36) |
| \( ^4D_{3/2} \) | 1.0 | -1.05 | -1.05 | -2.71 | -2.72 | -4.12 | -4.14 |
| | | — | (-1.05) | — | (-2.71) | — | (-4.14) |
| \( \varepsilon_{3/2}^+ \) | 1.0 | 0.605 | 0.621 | 0.690 | 0.686 | 0.762 | 0.747 |
| | | — | (0.623) | — | (0.688) | — | (0.754) |
| \( \xi_{3/2}^+ \) | 1.0 | 0.515 | 0.511 | 0.955 | 0.948 | 1.36 | 1.35 |
| | | — | (0.514) | — | (0.948) | — | (1.35) |
| \( \eta_{3/2}^+ \) | 1.0 | -0.105 | -0.107 | -0.231 | -0.231 | -0.363 | -0.363 |
| | | — | (-0.105) | — | (-0.228) | — | (-0.356) |
| \( ^4F_{3/2} \) | 1.0 | 0.122 | 0.121 | 0.490 | 0.488 | 0.919 | 0.920 |
| | | — | (0.121) | — | (0.489) | — | (0.921) |
| \( ^2P_{3/2} \) | 1.0 | -4.07 | -4.08 | -6.40 | -6.41 | -7.10 | -7.18 |
| | | — | (-4.08) | — | (-6.43) | — | (-7.20) |
| \( ^4P_{3/2} \) | 1.0 | 13.7 | 13.9 | 21.9 | 22.3 | 25.5 | 26.0 |
| | | — | (14.0) | — | (22.3) | — | (26.0) |
| \( \varepsilon_{3/2}^- \) | 1.0 | -1.28 | -1.24 | -1.92 | -1.86 | -2.71 | -2.62 |
| | | — | (-1.27) | — | (-1.89) | — | (-2.66) |
| \( \xi_{3/2}^- \) | 1.0 | -0.197 | -0.177 | -0.338 | -0.262 | -0.427 | -0.265 |
| | | — | (-0.177) | — | (-0.259) | — | (-0.256) |
| \( \eta_{3/2}^- \) | 1.0 | -1.00 | -1.00 | -2.15 | -2.17 | -3.57 | -3.52 |
| | | — | (-1.04) | — | (-2.16) | — | (-3.53) |

(continued)
This prescription yields a rather large overestimation of the differential cross sections, as plotted in Figs. 1 – 3 with dotted curves. This implies that a large effect of Coulomb modification on \( f^N_{\ell S_c} \) is necessary for the present low-energy \( pd \) scattering, which is a well-known assertion claimed by many authors.\(^{34,35,44,45} \) Here, we use an extended version of the Coulomb externally corrected approximation,\(^{35} \) in which nuclear eigenphase shifts for the \( pd \) scattering, \( \delta^N_{\ell S_c}(pd) \), are calculated from our \( \delta_{\ell S_c}(nd) \) by adding the difference between those evaluated by fss2.\(^{47} \) The Coulomb modification, \( \Delta_{\ell S_c} = \delta^N_{\ell S_c}(pd) - \delta_{\ell S_c}(nd) \), is significant (more than 1°) only for the \( ^2S, \, ^4S \) and \( ^4P \) channels, and the modified \( J \)-averaged central phase shifts in these channels are shown in Table III. The \( pd \) differential cross sections calculated using

| \( E_n (\text{MeV}) \) | \( ^4G_{5/2} \) | \( ^2D_{5/2} \) | \( ^4D_{5/2} \) | \( ^\varepsilon_{5/2^+} \) | \( ^\xi_{5/2^+} \) | \( ^\eta_{5/2^+} \) | \( ^4P_{5/2} \) | \( ^2F_{5/2} \) | \( ^4F_{5/2} \) | \( ^\varepsilon_{7/2^+} \) | \( ^\xi_{7/2^+} \) | \( ^\eta_{7/2^+} \) | \( ^2G_{9/2} \) | \( ^4G_{9/2} \) | \( ^\varepsilon_{9/2^+} \) | \( ^4F_{9/2} \) | \( ^4G_{11/2} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|--|
| 1.0             | -0.015          | 0.558           | -1.11           | -0.262          | -0.231          | -0.664          | 13.0            | -0.063          | 0.127           | 0.400           | 0.386           | -0.118          | 13.0            | -0.62            | 0.127           | -0.211          | 0.075           | -0.015          | -0.124          | -0.015          |
| 1.0             | -0.015          | 0.559           | -1.11           | -0.277          | -0.272          | -0.821          | 13.2            | -0.063          | 0.127           | 0.447           | 0.390           | -0.123          | 13.2            | (13.2)          | 0.127           | -0.211          | 0.075           | -0.016          | 0.124           | -0.015          |
| 2.0             | -0.091          | 1.48            | -2.90           | -0.289          | -0.485          | -1.44           | 21.4            | -0.252          | 0.510           | 0.481           | 0.696           | -0.243          | 21.7            | (13.2)          | 0.510           | -0.251          | 0.472           | 0.684           | -0.239          |
| 2.0             | -0.090          | 1.49            | -2.90           | -0.297          | -0.494          | -1.49           | 21.7            | -0.251          | 0.510           | 0.472           | 0.684           | -0.239          | 25.8            | (21.8)          | 0.947           | -0.465          | 0.515           | 0.938           | -0.343          |
| 2.0             | -0.206          | 2.31            | -4.44           | -0.313          | -0.716          | -2.02           | 26.0            | -0.466          | 0.951           | 0.538           | 0.926           | -0.334          | -0.466          | (26.3)          | 0.951           | -0.466          | 0.538           | 0.926           | -0.334          |
| 3.0             | -0.206          | 2.33            | -4.46           | -0.315          | -0.701          | -2.04           | 3.0             | -0.466          | 0.951           | 0.538           | 0.926           | -0.334          | -0.466          | (26.3)          | 0.951           | -0.466          | 0.538           | 0.926           | -0.334          |
| 3.0             | -0.206          | 2.33            | -4.46           | -0.315          | -0.701          | -2.04           | 3.0             | -0.466          | 0.951           | 0.538           | 0.926           | -0.334          | -0.466          | (26.3)          | 0.951           | -0.466          | 0.538           | 0.926           | -0.334          |
...and introducing the 3-$\ell_S$ spin-doublet and -quartet channels should also be reproduced by fss2 without in-
es...eigenphase shifts, we can expect that the...

The results of the phase shift analysis (PSA) at $E_N = 3$ MeV are also shown for comparison.

These Coulomb modified nuclear scattering amplitudes $f_{\ell S_c}^N$ are plotted in Figs. 1–3.

3.2. The nd effective-range parameters and the total cross sections

Since a good correspondence between fss2 and AV18+UR3N is found in the eigenphase shifts, we can expect that the S-wave effective-range parameters for the spin-doublet and -quartet channels should also be reproduced by fss2 without introducing the 3N force. The S-wave effective-range parameters in Table IV are...
Effective-Range Expansion of nd Scattering

Fig. 1. fss2 predictions of the \( nd \) and \( pd \) differential cross sections obtained from various formulae for the phase shifts at \( E_N=1 \) MeV (The subscript \( N \) denotes the nucleon): the exact \( nd \) calculation (solid curve), the \( J \)-averaged central phase shifts (dot-dashed curve), the Coulomb externally corrected approximation using \( \delta_{\ell S_c}(nd) \) (dotted curve) and the Coulomb modified nuclear phase shifts in Table III (dashed curve). For details of the Coulomb modification, see the text. The experimental data are cited from Ref. 50) (\( nd \) with error bars) and Ref. 51) (\( pd \) with filled circles).

Fig. 2. Same as Fig. 1, but for \( E_N=2 \) MeV.

Fig. 3. Same as Fig. 1, but for \( E_N=3 \) MeV. The experimental data are cited from Ref. 52) (\( nd \) with error bars) and Ref. 53) (\( pd \) with empty circles).
Table IV. The triton binding energy $E_B(^3\text{H})$, S-wave scattering lengths $^2a_{nd}$ and $^4a_{nd}$, and effective ranges ($^2r_{e,nd}$ and $^4r_{e,nd}$) predicted by fss2 for various model spaces with the maximum angular momentum of the $NN$ interaction ($I_{\text{max}}$) included. The nonlocal Gaussian potential with 15-point quadrature is used for fss2. In “$S + D$”, only the $^3S_1 + ^3D_1$ and $^1S_0$ $NN$ channels are included. The pole energy for the doublet channel ($E_Q$) is also shown. The calculated deuteron binding energy is 2.2206 MeV. The momentum mesh points with $n=6-10$-5 are used.

| $I_{\text{max}}$ | $E_B(^3\text{H})$ (MeV) | $^2a_{nd}$ (fm) | $^2r_{e,nd}$ (fm) | $E_Q$ (keV) | $^4a_{nd}$ (fm) | $^4r_{e,nd}$ (fm) |
|------------------|--------------------------|-----------------|------------------|------------|----------------|-----------------|
| $S + D$          | 8.247                    | 0.65            | −149             | −147       | 6.30           | 1.84            |
| 1                | 7.948                    | 0.94            | −102             | −207       | 6.31           | 1.85            |
| 2                | 8.213                    | 0.72            | −133             | −163       | 6.30           | 1.84            |
| 3                | 8.298                    | 0.67            | −146             | −151       | 6.30           | 1.84            |
| 4                | 8.307                    | 0.66            | −148             | −148       | 6.30           | 1.84            |

obtained by applying Schlessinger’s point method\(^{54}\) to the effective-range function $K(q_0) = q_0 \cot \delta$. This method is convenient for approximating a function with a pole such as the doublet effective-range function and for taking into account the contributions of the higher-order terms in Eqs. (2.19) and (2.20) in a natural way. It is very difficult in the momentum representation to maintain sufficient accuracy of the eigenphase shift for extremely low energies of less than 100 keV if we use the realistic deuteron wave function including the $D$-wave component. We therefore use the sample points of energies between $E_{\text{cm}} = 200$ keV and 2 MeV shown in Figs. 4 (the doublet channel) and 5 (the quartet channel) unless the denominator of the fraction appearing in Schlessinger’s point method becomes zero. In these calculations, we finely discretize points of $p$ for the $NN$ relative motion in the low-momentum region to achieve eigenphase shifts with accuracy better than 0.01° and scattering lengths with accuracy of 0.01 fm. We use a set of mesh points $n=6$-10-5 and take the $NN$ partial waves up to the $G$-wave ($I_{\text{max}} = 4$). The effective-range functions obtained by using the effective-range parameters in Table IV with $I_{\text{max}}=4$ are shown in Figs. 4 and 5 as solid curves. We reconfirm that a pole structure exists in the doublet channel at $E_Q = -(3\hbar^2 q_0^2/4M) \simeq -150$ keV.

Table IV lists the model-space dependence of the triton binding energy and effective-range parameters. As to the converged triton binding energy $E_B(^3\text{H}) = 8.311$ MeV with $I_{\text{max}} = 6$, the small difference from the original result of fss2 (8.326 MeV with $I_{\text{max}} = 6$)\(^{11}\) originates mainly from the nonlocal Gaussian approximation to the interaction kernel adopted in this study. Since we have three $NN$ pairs in the triton, this 15 keV difference in the triton binding energy is consistent with the 4 keV difference in the deuteron binding energy, which is the difference between 2.2250 MeV (the original result of fss2)\(^6\) and 2.2206 MeV (the result of the nonlocal Gaussian potential based on fss2). We find that the spin-quartet scattering length $^4a_{nd}$ is reasonably insensitive to the model space adopted. This insensitivity is related to the small distortion effect of the deuteron due to the Pauli repulsion of the $nd$ interaction in this channel, which is a kinematical constraint imposed by the spin-isospin quantum numbers. On the other hand, the spin-doublet scattering length $^2a_{nd}$ is subject to a strong channel coupling effect as seen in Table IV. We
Effective-Range Expansion of \( nd \) Scattering

Fig. 4. The effective-range function \( K(q_0) = q_0 \cot \delta \) for the doublet \( S \)-state as a function of \( E_{cm} \). The filled circles are the fss2 results and the solid curve shows the approximation by the effective-range parameters in Table IV with \( I_{max} = 4 \).

Fig. 5. Same as Fig. 4, but for \( K(q_0) = q_0 \cot \delta \) in the quartet \( S \)-state.

find that the simplest five-channel (\( S + D \)) calculation, incorporating the \( ^3S_1 + ^3D_1 \) and \( ^1S_0 \) \( NN \) channels only, yields a value very close to the converged one, namely, 0.66 fm. However, the \( P \)-wave \( NN \) interaction has a repulsive effect and a well-converged value is achieved after many partial waves, up to at least the \( G \)-wave of the \( NN \) interaction, are included. The values of \( |E_Q|/(2a_{nd}) \sim 220 \text{ keV/fm} \) are almost independent of the model space. This linear correlation has already been suggested in Ref. 31) for separable potentials. A strong correlation between \( E_B(\text{\textsuperscript{3}H}) \) and \( 2a_{nd} \) is also apparent in Table IV.

In Table V, we compare \( E_B(\text{\textsuperscript{3}H}) \) and \( 2a_{nd} \) obtained by fss2 with some results of other calculations using meson-exchange potentials and the \( 3N \) force. Here, fss2 does not incorporate the charge dependence of the \( NN \) interaction, but the other calculations in Table V include this effect. As in \( E_B(\text{\textsuperscript{3}H}) \), the charge dependence should have an appreciable effect on \( 2a_{nd} \). We will estimate the maximum shift of \( 2a_{nd} \) by simply assuming the relationship given by the Phillips line for the triton binding energy. For fss2, the slope of the Phillips line, \(-0.686 \text{ fm/MeV}\), yields 0.13 fm for the charge dependence effect on \( 2a_{nd} \). After the charge dependence correction, \( 2a_{nd} \) for fss2 turns out to be \( 2a_{nd} \sim 0.76 - 0.80 \text{ fm} \). Table V shows that the effect of the \( 3N \) force is more important than the charge dependence of the \( NN \) force. When only \( NN \) meson-exchange potentials are used, \( 2a_{nd} \) is more than 0.9 fm. The experimental values for \( 2a_{nd} \) and \( E_B(\text{\textsuperscript{3}H}) \) are reproduced only when the \( 3N \) force is included. The model fss2 almost reproduces \( E_B(\text{\textsuperscript{3}H}) \) and \( 2a_{nd} \) simultaneously without introducing the \( 3N \) force.
Table V. Comparison of the \( nd \) scattering lengths predicted by using fss2 \( (I_{\text{max}} = 4) \) with other models. For the fss2 results, the charge dependence of the \( NN \) force is neglected. The heading \( NN \) denotes the calculation using only the \( NN \) force, and \( NN+\text{TM99} \) denotes the calculation including the Tucson-Melbourne 99 (TM99) \( 2\pi \)-exchange \( 3N \) force.\(^{55,56} \) The results obtained by CD-Bonn 2000, AV18 and Nijm I are cited from Ref. 22) \( (I_{\text{max}} = 5) \). The experimental values are cited from Ref. 24). The values of \( ^4a_{nd} \) are insensitive to the \( 3N \) force.

| Model       | \( NN \) | \( NN+\text{TM99} \) | \( NN \) | \( NN+\text{TM99} \) | \( NN(\text{+TM99}) \) |
|-------------|---------|-----------------|---------|-----------------|-----------------|
| fss2        | 8.307   | —               | 0.66    | —               | 6.30            |
| CD-Bonn 2000 | 8.005  | 8.482           | 0.925   | 0.569           | 6.347           |
| AV18        | 7.628   | 8.482           | 1.248   | 0.587           | 6.346           |
| Nijm I      | 7.742   | 8.485           | 1.158   | 0.594           | 6.342           |

Table VI. The quartet \( (^4S) \) and doublet \( (^2S) \) \( S \)-wave contributions to the total cross sections \( (\sigma_{\text{tot}}) \) calculated from the effective-range parameters for fss2 with \( I_{\text{max}} = 4 \) in Table IV. The total cross sections \( \sigma_{\text{tot}} \) for fss2 are cited from Ref. 14).

| \( E_n \) (MeV) | \( ^4S \) (mb) | \( ^2S \) (mb) | \( S \)-wave (mb) | \( \sigma_{\text{tot}} \) (mb) |
|-----------------|--------------|--------------|-----------------|-----------------|
| 0 MeV          | 3325 (99.5%) | 18.2 (0.5%)  | 3343            | —               |
| 1 MeV          | 2055 (73%)   | 122.4 (4.3%) | 2177 (77%)      | 2832            |
| 2 MeV          | 1467 (59%)   | 158.4 (6.3%) | 1625 (66%)      | 2480            |
| 3 MeV          | 1128 (54%)   | 165.5 (7.9%) | 1294 (62%)      | 2104            |

We should keep in mind that the mechanism involved to reproduce \( ^2a_{nd} \) is considerably different from that in the spin-quartet case. The large positive value for \( ^4a_{nd} \) is related to the Pauli repulsion for the loosely bound deuteron cluster. On the other hand, the value of \( ^2a_{nd} \) entirely originates from the dynamical effect related to the reasonably large triton binding energy \( E_B^{\text{exp}}(^3\text{H}) = 8.482 \text{ MeV} \) and the existence of the pole structure in the effective-range function slightly below the elastic threshold. In this special situation, it is natural that the more attractive \( nd \) interaction, afforded by the \( 3N \) force in the meson-exchange potentials or by fss2 without the \( 3N \) force, can reproduce a smaller value for \( ^2a_{nd} \), which is consistent with the Phillips line.

In Table VI, we show the \( S \)-wave contributions from the quartet \((^4S)\) and doublet \((^2S)\) channels to the total cross sections, calculated from the effective-range parameters in Table IV for fss2 with \( I_{\text{max}} = 4 \). We find that the total cross sections are dominated by the \( S \)-wave contribution, which is more than 60% even for \( E_n = 3 \text{ MeV} \). Furthermore, the quartet state is far more important than the doublet state owing to the small values of \( |q_0\cot\delta| \), even considering the statistical weight factor \((2S_c + 1)\). As the energy increases, the contribution from the doublet state becomes appreciable owing to the avoidance of the pole structure slightly below the elastic...
Effective-Range Expansion of nd Scattering 973

threshold but is still less than 10% at $E_n=3$ MeV. This implies a very special situation in which the extra attraction to the $^2S_{1/2}$ state by the $3N$ force is unimportant in reproducing the differential cross sections of the low-energy $nd$ scattering. The low-energy $nd$ cross sections are mainly determined by the magnitude of the repulsive $^4S_{3/2}$ eigenphase shift.

§4. Summary

Motivated by the success of the QM baryon-baryon interaction fss2, which reproduces the triton binding energy almost correctly without the $3N$ force,$^{11}$ we have extended the Faddeev calculation to low-energy $nd$ scattering by employing a new algorithm$^{14}$ to solve the AGS equation.$^{32}$ The QM NN interaction, formulated by the two-cluster RGM, has various types of nonlocality that the standard meson-exchange potentials do not possess.$^6$ In addition to the dominant nonlocality from the RGM interaction kernel, an extra nonlocality emerges from the off-shell transformation to eliminate the energy dependence connected to the normalization kernel,$^{12}$ which has sometimes been neglected in similar works.$^{23},^{27}$ To reduce the computational time for three-body calculations, we have developed and used a non-local Gaussian potential constructed from the model fss2.$^{13}$ The nonlocality and energy dependence of fss2 are, however, strictly preserved in this potential model, resulting in almost the same amount of triton binding energy, i.e., only 15 keV less bound.

In this paper, we have examined the effective-range parameters of the $nd$ scattering and the low-energy differential cross sections below the deuteron breakup threshold. The elastic scattering amplitudes in the channel-spin representation are parameterized by the eigenphase shifts and mixing parameters,$^{33}$ from which the $S$-wave effective-range parameters are derived by employing the single-channel effective-range formula. We have reconfirmed that the improved single-channel effective-range formula should be used for the channel-spin doublet ($S_c = 1/2$) state to incorporate the pole structure of the effective-range function, which exists slightly below the elastic threshold.$^{28}–^{31}$ The effective-range parameters predicted by fss2 are $^2a_{nd} = 0.66$ fm, $(^2r_e)_{nd} \sim -150$ fm, $E_Q = -(3\hbar^2q^2/4M) \sim -150$ keV, $^4a_{nd} = 6.30$ fm and $(^4r_e)_{nd} = 1.84$ fm without the charge dependence of the $NN$ interaction. After the charge dependence correction, $^2a_{nd}$ for fss2 turns out to be $^2a_{nd} \sim 0.76 - 0.80$ fm. It is found that almost the same $^2a_{nd}$ value is obtained in the restricted model space involving only the $^3S_1+^3D_1$ and $^1S_0$ NN interactions. However, the deuteron distortion effect in the spin-doublet channel is so strong that a sufficient number of partial waves, up to at least the $G$-wave, are necessary to obtain a converged result. On the other hand, the positive value $^4a_{nd} = 6.30$ fm, implying the repulsive nature of the $nd$ interaction in the spin-quartet ($S_c = 3/2$) channel, is rather insensitive to the expansion of the model space owing to the kinematical constraint imposed by the effect of the Pauli principle.

A detailed comparison of the $nd$ eigenphase shifts predicted by fss2 and by AV18 plus the UR3N potential$^{42}$ shows a strong resemblance, especially for the $^2S_{1/2}$
state. The UR3N potential has a sizable effect of about $4^\circ - 5^\circ$ only in this state and very small effects on other partial waves. We find reasonable agreement with the $nd$ experimental data for the low-energy differential cross sections. Since the $pd$ data are more precise, we have also examined the $pd$ differential cross sections. We first examined a simple formula involving the Coulomb amplitude added to the $nd$ scattering amplitudes, called the Coulomb externally corrected approximation,\textsuperscript{34,35} which yields too large differential cross sections for $E_p = 1 - 3$ MeV. This implies that Coulomb modification of the nuclear phase shifts is very important in this low-energy region. The Coulomb modified nuclear phase shifts are introduced by adding the major difference (more than $1^\circ$) between the $nd$ and $pd$ eigenphase shifts predicted by fss2.\textsuperscript{47} The modification is carried out with respect to the $J$-averaged central phase shifts only for the $^2S$, $^4S$ and $^4P$ channels. The experimental $pd$ differential cross sections for $E_p = 1, 2$ and $3$ MeV are well reproduced. The differential cross sections below the deuteron breakup threshold are dominated by the $^4S_{3/2}$ contribution, in which the Coulomb modification of the eigenphase shift is especially large. The $pd$ phase shift is more attractive than the $nd$ phase shift by $10^\circ - 7^\circ$ in the energy range $E_p = 1 - 3$ MeV.

On the basis of these analyses, we can conclude that the nonlocal effects of the QM $NN$ interaction fss2 are very important in making the easily distortable spin-doublet $nd$ channel sufficiently attractive to reproduce almost correct values of the doublet scattering length and the triton binding energy without introducing the $3N$ force. The $nd$ and $pd$ differential cross sections are consistently reproduced by fss2, in which the spin-doublet contribution is rather minor. In a separate paper,\textsuperscript{47} we have shown that the spin polarization observables are also correctly reproduced by the direct incorporation of the Coulomb force in our model fss2. The breakup differential cross sections of the $nd$ scattering are discussed in Ref. 63).

Acknowledgements

The authors would like to thank Professor B. F. Gibson and Professor S. Ishikawa for informing them of the pole structure of the spin-doublet effective-range function. They also thank Professor K. Sagara for providing them with the $pd$ experimental data obtained by the Kyushu University group. This work was supported by a Grant-in-Aid for Scientific Research on Priority Areas (Grant No. 2002803) and by a Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan. It was also supported by core-stage backup subsidies from Kyoto University. The numerical calculations were carried out on Altix3700 BX2 at YITP in Kyoto University.

Appendix A

Spin-Isospin Factors and the Rearrangement Coefficients in Eq. (2.6)

In this appendix, we give the rearrangement coefficients $g_{\gamma,\gamma'}(q, q', x)$ in Eq. (2.6) for the permutation operator $P = P_{12}P_{13} + P_{13}P_{12}$, together with the spin-isospin
factors for the $3N$ system. The two contributions from $P_{12}P_{13}$ and $P_{13}P_{12}$ become equal owing to the Pauli principle for particle 1 and particle 2. We therefore only need to calculate the rearrangement coefficients for $(-2)P_{13}$. These are given by

$$g_{\gamma,\gamma'}(q, q', x) = \sum_{\lambda_1+\lambda_2=\lambda} \sum_{\lambda_1' + \lambda_2' = \lambda'} q^{\lambda_1+\lambda_2} q'^{\lambda_1+\lambda_2'} \left(\frac{1}{2}\right)^{\lambda_2+\lambda_2'} \sum_k (2k+1)g_{\gamma,\gamma'}^{\lambda_1\lambda_1'k} P_k(x),$$

with

$$g_{\gamma,\gamma'}^{\lambda_1\lambda_1'k} = \sum_{LS} (X_N)^{LSJ}_{\gamma,\gamma'} G^{\lambda_1\lambda_1'kL}_{(\lambda\ell), (\lambda'\ell')},$$

where $P_k(x)$ are Legendre polynomials. The spatial rearrangement factors $G^{\lambda_1\lambda_1'kL}_{(\lambda\ell), (\lambda'\ell')}$ are given by

$$G^{\lambda_1\lambda_1'kL}_{(\lambda\ell), (\lambda'\ell')} = G^{\lambda_1\lambda_1'kL}_{(\lambda'\ell'), (\lambda\ell)} = 4\pi \left[ \frac{(2\lambda+1)(2\lambda'+1)!}{(2\lambda_1+1)(2\lambda_2+1)(2\lambda_1'+1)(2\lambda_2'+1)!} \right]^{\frac{1}{2}} \times \int d\hat{q} d\hat{q}' \left[ Y_{(\lambda\ell)}(\hat{q}, \hat{q}')Y_{(\lambda'\ell')}(\hat{q}, \hat{q}') \right]^*_{LM} P_k(\hat{q} \cdot \hat{q}') \left[ Y_{(\lambda_1\lambda_1')}^{\lambda_1'}(\hat{q}, \hat{q}')Y_{(\lambda_1'\lambda_1)}^{\lambda_1}(\hat{q}, \hat{q}') \right]_{LM}$$

$$= \left[ \frac{(2\lambda+1)(2\lambda'+1)!}{(2\lambda_1+1)(2\lambda_2+1)(2\lambda_1'+1)(2\lambda_2'+1)!} \right]^{\frac{1}{2}} \tilde{\lambda} \tilde{\lambda}' \ell \ell' \sum_{ff'} (\lambda_2\ell_20\ell_0|f0)(\lambda_2'\ell_2'0\ell_0'|f'0) \langle k0\lambda_10|f0 \rangle \times \langle k0\lambda_1'0|f0 \rangle \left\{ \begin{array}{ccc} f & L & \lambda_1 \\ \ell & \lambda_2 & \lambda_1' \end{array} \right\} \left\{ \begin{array}{ccc} f' & L & \lambda_1' \\ \ell' & \lambda_2' & \lambda_1 \end{array} \right\} \left\{ \begin{array}{ccc} \lambda & f & k \\ 1 & f & \lambda \end{array} \right\}.$$

Here, $\lambda_2 = \lambda - \lambda_1$ and $\lambda_2' = \lambda' - \lambda_1'$ with $\lambda_1 = 0 - \lambda$ and $\lambda_1' = 0 - \lambda'$, and the usual notation $\tilde{\lambda} = \sqrt{2\lambda+1}$ is used. The explicit expression for the spin-isospin factors $(X_N)^{LSJ}_{\gamma,\gamma'}$ depends on the channel specification scheme. For the $LS$ coupling scheme

$$\langle \hat{p}, \hat{q}; 123|\gamma \rangle = \left[ Y_{(\lambda\ell)L}(\hat{p}, \hat{q}) \left[ \chi_{st}(1, 2)\chi_{lT}^{\frac{1}{2}}(3) \right]_{SSz; \frac{1}{2}Tz} \right]_{JJ_z},$$

with $\gamma = [(\lambda\ell)L(s\frac{1}{2})SJJ_z; (t\frac{1}{2})\frac{1}{2}Tz]$, we do not need the $LS$ sum in Eq. (A-2) since the orbital angular momentum $L$ and the total spin $S$ are separately conserved. We modify Eq. (A-2) to

$$(X_N)^{LSJ}_{\gamma,\gamma'} \equiv -2X^S_{s,s',t'} X^L_{t,t'},$$

where a common definition of the spin and isospin factors,

$$X^S_{s,s'} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad X^L_{s,s'} = \begin{pmatrix} \frac{1}{2} & -\sqrt{3} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

is used. In Eq. (A-6), the upper row (or the leftmost column) corresponds to $s = 0$ ($s' = 0$) and the second row (or the rightmost column) corresponds to $s = 1$ ($s' = 1$). For the $jj$-coupling scheme

$$\langle \hat{p}, \hat{q}; 123|\gamma \rangle = \left[ Y_{\lambda}(\hat{p})\chi_{st}(1, 2)\chi_{lT}^{\frac{1}{2}}(3) \right]_{JJ_z; \frac{1}{2}Tz},$$

we use a common definition of the spin and isospin factors,
with $\gamma = [(\lambda s)I(\ell_{1/2}^s)jJJ_z; (t_{1/2}^s)T_z$, we use the superposition of Eq. (A.2) with

$$(X_N)_{\gamma, \gamma'}^{LSJ} = \left[ \begin{array}{ccc} \lambda & s & I \\ \ell & \frac{1}{2} & j \\ L & S & J \end{array} \right] \left[ \begin{array}{ccc} \lambda' & s' & I' \\ \ell' & \frac{1}{2} & j' \\ L' & S' & J' \end{array} \right] \left( X_N^{S_{1/2}^{LSJ}} \right)_{st, st'}^{LSJ}. \quad (A.8)$$

The coefficients in the channel-spin formalism using Eq. (2.3) are similarly obtained as

$$(X_N)_{\gamma, \gamma'}^{LSJ} = \sum_{jj'} \left[ \begin{array}{ccc} 0 & \ell & \ell \\ I & \frac{1}{2} & S_c \\ I & j & J \end{array} \right] \left[ \begin{array}{ccc} \lambda & s & I \\ \ell & \frac{1}{2} & j \\ L & S & J \end{array} \right] \left[ \begin{array}{ccc} \lambda' & s' & I' \\ \ell' & \frac{1}{2} & j' \\ L' & S' & J' \end{array} \right] \left( X_N^{S_{1/2}^{LSJ}} \right)_{st, st'}^{LSJ}. \quad (A.9)$$

For practical calculations, it is convenient to first calculate the product sum of the 6-j and 9-j coefficients, defined through

$$A^{LSJ}_{(\ell S_c)(\lambda s)} I = \sum_j (-1)^{I+\ell+S_c+j} \tilde{S}_{c J} \gamma \left\{ \frac{1}{2} \right\}_{\ell} \left[ \begin{array}{ccc} \lambda & s & I \\ \ell & \frac{1}{2} & j \\ L & S & J \end{array} \right]. \quad (A.10)$$

Furthermore, the factor of two in Eq. (A.5) cancels with the factor of 1/2 for the $x$ integral. Thus, for Eq. (2.6) we use

$$\frac{1}{2} g_{\lambda \lambda'}^{\chi \chi'}_{t, t'} = -X_{t, t'}^{S_{1/2}^{LSJ}} A^{LSJ}_{(\ell S_c)(\lambda s)} I A^{LSJ}_{(\ell' S_c')(\lambda' \chi') I'} X_{s, s'}^{S_{1/2}^{LSJ}} G_{\lambda \lambda'}^{\chi \chi'}_{t, t'}. \quad (A.11)$$

References

1) D. Hübner, W. Glöckle, J. Golak, H. Witala, H. Kamada, A. Kievsky, S. Rosati and M. Viviani, Phys. Rev. C 51 (1995), 1100.
2) See, for example, H. Kamada, A. Nogga, W. Glöckle, E. Hiyama, M. Kamimura, K. Varga, Y. Suzuki, M. Viviani, A. Kievsky, S. Rosati, J. Carlson, S. C. Pieper, R. B. Wiringa, P. Navrátil, B. R. Barrett, N. Barnea, W. Leidemann and G. Orlandini, Phys. Rev. C 64 (2001), 044001, and references therein.
3) A. Nogga, H. Kamada and W. Glöckle, Phys. Rev. Lett. 85 (2000), 944.
4) E. Epelbaum, H. Kamada, A. Nogga, H. Witala, W. Glöckle and Ulf-G. Meißner, Phys. Rev. Lett. 86 (2001), 4787.
5) E. Epelbaum, A. Nogga, W. Glöckle, H. Kamada, Ulf-G. Meißner and H. Witala, Phys. Rev. C 66 (2002), 064001.
6) Y. Fujiwara, Y. Suzuki and C. Nakamoto, Prog. Part. Nucl. Phys. 58 (2007), 439.
7) Y. Fujiwara, T. Fujita, M. Kohno, C. Nakamoto and Y. Suzuki, Phys. Rev. C 65 (2002), 014002.
8) Y. Fujiwara, K. Miyagawa, M. Kohno, Y. Suzuki and H. Nemura, Phys. Rev. C 66 (2002), 021001(R).
9) Y. Fujiwara, K. Miyagawa, Y. Suzuki, M. Kohno and H. Nemura, Nucl. Phys. A 721 (2003), C983.
10) Y. Fujiwara, K. Miyagawa, M. Kohno and Y. Suzuki, Phys. Rev. C 70 (2004), 024001.
11) Y. Fujiwara, Y. Suzuki, M. Kohno and K. Miyagawa, Phys. Rev. C 77 (2008), 027001.
12) Y. Suzuki, H. Matsumura, M. Orabi, Y. Fujiwara, P. Descouvemont, M. Theeten and D. Baye, Phys. Lett. B 659 (2008), 160.
13) K. Fukukawa, Y. Fujiwara and Y. Suzuki, Mod. Phys. Lett. A 24 (2009), 1035.
14) Y. Fujiwara and K. Fukukawa, Prog. Theor. Phys. 124 (2010), 433.
15) A. C. Phillips, Rep. Prog. Phys. 40 (1977), 905; Nucl. Phys. A 107 (1968), 209.
16) J. L. Friar, B. F. Gibson, G. L. Payne and C. R. Chen, Phys. Rev. C 30 (1984), 1121.
17) C. R. Chen, G. L. Payne, J. L. Friar and B. F. Gibson, Phys. Rev. C 33 (1986), 401.
18) C. R. Chen, G. L. Payne, J. L. Friar and B. F. Gibson, Phys. Rev. C 39 (1989), 1261.
19) C. R. Chen, G. L. Payne, J. L. Friar and B. F. Gibson, Phys. Rev. C 44 (1991), 50.
20) A. Kievsky, M. Viviani and S. Rosati, Nucl. Phys. A 577 (1994), 511.
21) A. Kievsky, Nucl. Phys. A 624 (1997), 125.
22) H. Witala, A. Nogga, H. Kamada, W. Glöckle, J. Golak and R. Skibiński, Phys. Rev. C 68 (2003), 034002.
23) H. García-Lázaro and A. Valcarce, Phys. Rev. C 76 (2007), 057002.
24) W. Dilg, L. Koester and W. Nistler, Phys. Lett. B 36 (1971), 208.
25) D. R. Entem, F. Fernández and A. Valcarce, Phys. Rev. C 62 (2000), 034002.
26) A. Valcarce, H. García-Lázaro, F. Fernández and P. González, Rep. Prog. Phys. 68 (2005), 965.
27) B. Juliá-Díaz, J. Haidenbauer, A. Valcarce and F. Fernández, Phys. Rev. C 65 (2002), 034001.
28) L. M. Delves, Phys. Rev. 118 (1960), 1318.
29) G. Barton and A. C. Phillips, Nucl. Phys. A 132 (1969), 97.
30) A. S. Reiner, Phys. Lett. B 28 (1969), 18.
31) J. S. Whiting and M. G. Fuda, Phys. Rev. C 14 (1976), 167.
32) R. G. Seyler, Nucl. Phys. A 124 (1969), 253.
33) P. Doleschall, W. Grüebler, V. König, P. A. Schmelzbach, F. Sperisen and B. Jenny, Nucl. Phys. A 380 (1982), 72.
34) A. Deltuva, A. C. Fonseca and P. U. Sauer, Phys. Rev. C 71 (2005), 054005.
35) W. Glöckle, The Quantum Mechanical Few-Body Problem, Texts and Monographs in Physics (Springer, Berlin, 1983).
36) W. Glöckle, H. Witala, D. Hüber, H. Kamada and J. Golak, Phys. Rep. 274 (1996), 107.
37) P. A. Schmelzbach, W. Glüeblner, R. E. White, V. König, R. Risler and P. Marmier, Nucl. Phys. A 197 (1972), 273.
38) J. Arvieux, Nucl. Phys. A 221 (1974), 253.
39) J. Chauvin and J. Arvieux, Nucl. Phys. A 247 (1975), 347.
40) L. D. Kнутson, L. O. Lamm and J. E. McAninch, Phys. Rev. Lett. 71 (1993), 3762.
41) A. Kievsky, S. Rosati, W. Tornow and M. Viviani, Nucl. Phys. A 607 (1996), 402.
42) W. Tornow, A. Kievsky and H. Witala, Few-Body Systems 32 (2002), 53.
43) E. O. Alt, W. Sandhas and H. Ziegelmann, Phys. Rev. C 17 (1978), 1981.
44) G. H. Berthold, A. Stadler and H. Zankel, Phys. Rev. C 41 (1990), 1365.
45) E. O. Alt and W. Sandhas, “Collision Theory for Two- and Three-Particle Systems Interacting via Short-Range and Coulomb Forces” in Coulomb Interactions in Nuclear and Atomic Few-Body Collisions, ed. F. S. Levin and D. A. Micha (Plenum Press, New York and London, 1996).
46) K. Fukukawa and Y. Fujiwara, Prog. Theor. Phys. 125 (2011), 729.
47) B. S. Pudliner, V. R. Pandharipande, J. Carlson and R. B. Wiringa, Phys. Rev. Lett. 74 (1995), 4396.
48) A. J. Elwyn, R. O. Lane and A. Langsdorf, Jr., Phys. Rev. 128 (1962), 779.
49) D. C. Kocher and T. B. Clegg, Nucl. Phys. A 132 (1969), 455.
50) P. Schwarz, H. O. Klages, P. Doll, B. Haesner, J. Wilczynski, B. Zeitnitz and J. Kecskemeti, Nucl. Phys. A 398 (1983), 1.
51) R. S. Coon and H. K. Han, Few-Body Systems 30 (2001), 131.
52) R. Machleidt, Adv. Nucl. Phys. 19 (1989), 189.
53) J. M. Clement, P. Stoler, C. A. Goulding and R. W. Fairchild, Nucl. Phys. A 183 (1972), 51.
59) J. D. Seagrave and R. L. Hankel, Phys. Rev. 98 (1955), 666.
60) T. W. Phillips, B. L. Berman and J. D. Seagrave, Phys. Rev. C 22 (1980), 384.
61) R. G. Nuckolls, C. L. Bailey, W. E. Bennett, T. Bergstralh, H. T. Richards and J. H. Williams, Phys. Rev. 70 (1946), 805.
62) H. B. Willard, J. K. Bair and C. M. Jones, Phys. Lett. 9 (1964), 339.
63) Y. Fujiwara and K. Fukukawa, Prog. Theor. Phys. 125 (2011), 979.