Magnonic qudit and algebraic Bethe Ansatz

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Abstract. A magnonic qudit is proposed as the memory unit of a register of a quantum computer. It is the $N$-dimensional space, extracted from the $2^N$-dimensional space of all quantum states of the magnetic Heisenberg ring of $N$ spins $1/2$, as the space of all states of a single magnon. Three bases: positional, momentum, and that of Weyl duality are described, together with appropriate Fourier and Kostka transforms. It is demonstrated how exact Bethe Ansatz (BA) eigenfunctions, classified in terms of rigged string configurations, can be coded using a collection of magnonic qudits. To this aim, the algebraic BA is invoked, such that a single magnonic qudit is prepared in a state corresponding to a magnon in one of the states provided by spectral parameters emerging from the corresponding BA equations.

1. Introduction

It is known the Bethe Ansatz (BA) [1, 2, 3], which provides the exact solution to the eigenproblem of the Heisenberg Hamiltonian for a magnetic ring of $N$ nodes with the spin $1/2$ and isotropic nearest neighbour interactions, has an adequate formulation in terms of quantum computations [4, 5]. We recall here briefly that the set $\mathbb{Z} = \{+, -\} = \{0, 1\}$ of single-node spin projections can be recognized as a bit, i.e. the memory unit of a (classical) computer, the set $\tilde{N} = \{1, 2, \ldots, N\}$ of magnetic nodes describes the size of computer memory, and the set $2^\tilde{N} = \{f : \tilde{N} \rightarrow \mathbb{Z}\}$ of all magnetic configurations is the register of a classical computer. Moreover, the linear span of the bit $2$ over the field $\mathbb{C}$ of complex numbers becomes a qubit, and the space $\mathcal{H}$ of all quantum states of the magnetic ring, i.e. the span of the set $2^\tilde{N}$, can serve as a register of a quantum computer. In this way, the results of BA can be adapted within the midst of quantum informatics.

The adaptation mentioned above, with the spin $1/2$ in the role of a qubit, is quite natural but not the only way of use of BA. Here we propose another comparison of BA and quantum informatics, based on the notion of a magnonic qudit [5]. Our approach also uses the magnetic ring of $N$ spins $1/2$ interacting via the Heisenberg Hamiltonian as a prototypic model, but proposes another extraction of the single-particle space, namely the subspace consisting of all states with a single spin deviation from the ferromagnetic saturation state. Clearly, this is the space isomorphic to $\mathbb{C}^N$, with the states corresponding to localization of this deviation on a single node as an orthonormal basis. We refer to this space as to the magnonic qudit $h = \mathbb{C}^N$, and to this basis, labeled by nodes, as to the computational basis. Our aim is to demonstrate how an arbitrary exact Bethe Ansatz eigenfunction for the sector of $r$ overturned spins, $r \leq N/2$, can be encoded within at most $r$ magnonic qudits. Our demonstration uses the formalism of algebraic Bethe Ansatz [3].
2. The magnonic qudit

Let \( |\tilde{N}j\rangle = |+\ldots+\rangle \), \( j \in \tilde{N}\), be the magnetic configuration with the single overturned spin at the \( j\)-th node. We treat the set \( \{ |\tilde{N}j\rangle \mid j \in \tilde{N} \} \) as the computational orthonormal basis, and the linear closure of this set over the field \( \mathbb{C} \) of complex numbers,

\[
h = \text{lc}_\mathbb{C} \{ |\tilde{N}j\rangle \mid j \in \tilde{N} \}
\]

is a linear unitary space referred to as the magnonic qudit. Within BA, \( h \) is the space of all quantum states of the system, consisting of a single spin deviation, or a single Bethe pseudoparticle on the magnetic ring \( \tilde{N} \). We assume it to be a single memory unit of a (hypothetical) quantum computer, so that the algebra

\[
\text{End } h = \text{lc}_\mathbb{C} U(N)
\]

of all operators in \( h \), which can be seen as the linear closure of the unitary group \( U(N) \), encompasses all single-qudit quantum gates.

Within the Schwinger description [6] of finite-dimensional Hilbert spaces, the computational basis vectors \( |\tilde{N}j\rangle \) acquires the meaning of the position of a Bethe pseudoparticle [7, 8, 9], whereas the (finite) Fourier transform

\[
|Bk\rangle = \frac{1}{\sqrt{N}} \sum_{j \in \tilde{N}} e^{-2\pi i kj/N} |\tilde{N}j\rangle
\]

yields the basis of quasimomenta in the magnonic qudit \( h \). Here \( k \in B \) is the quasimomentum of a pseudoparticle, and

\[
B = \left\{ k = 0, \pm 1, \pm 2, \ldots, \pm (N/2 - 1), N/2 \quad \text{for } N \text{ even} \right\}
\]

is the Brillouin zone of the chain \( \tilde{N} \), that is, the dual group of the cyclic group \( C_N \) on the ring \( \tilde{N} \) - the translational symmetry group of the magnetic ring. The two bases in \( h \), positions \( |\tilde{N}j\rangle \) and quasimomenta \( |Bk\rangle \), are related mutually by the Fourier operator \( \hat{F} \in \text{End } h \) as

\[
\hat{F} |\tilde{N}j\rangle = |B\beta(j)\rangle, \quad j \in \tilde{N},
\]

where \( \beta : \tilde{N} \to B \) is a bijection between the positions and quasimomenta of the system, given by

\[
\beta(j) = j \mod B, \quad j \in \tilde{N},
\]

(\( j \mod B \) denotes reduction of an integer \( j \mod N \) to the range provided by Eq. (4) \( (\text{quasimomenta } j > N/2 \text{ become negative}) \)). Within the basis of positions, matrix elements of the Fourier operator are

\[
\langle \tilde{N}j_1 | \hat{F} |\tilde{N}j_2\rangle = \frac{1}{\sqrt{N}} e^{-2\pi i j_1 j_2/N}, \quad j_1, j_2 \in \tilde{N},
\]

in accordance with Eq. (3). It is worth to observe that the operator \( \hat{P} = \hat{F}^2 \) satisfies

\[
\hat{P} |\tilde{N}j\rangle = |\tilde{N}, -j \mod N\rangle, \quad j \in \tilde{N},
\]

\[
\hat{P} |Bk\rangle = |B, -k\rangle, \quad k \in B,
\]

and thus can identified as the parity on the ring \( \tilde{N} \).
Schwinger operators $\hat{U} \in \text{End} \, \mathcal{H}$ and $\hat{V} \in \text{End} \, \mathcal{H}$, defined as shifts in the position and momentum bases, that is by

$$\hat{U}|\tilde{N}j\rangle = |\tilde{N}, (j + 1) \bmod N\rangle, \quad j \in \tilde{N},$$

and

$$\hat{V}|Bk\rangle = |B, (k + 1) \bmod B\rangle, \quad k \in B,$$

respectively, provide a natural basis in the algebra $\text{End} \, \mathcal{H}$ of observables for this system, namely

$$\text{End} \, \mathcal{H} = \text{lcC}\{U^jV^j \mid j \in \tilde{N}, k \in B\} \cong \text{lcC}(\tilde{N} \times B).$$

In this way, the set $\tilde{N} \times B$ acquires the meaning of the "classical" phase space for the magnonic qudit.

The magnonic qudit $\mathcal{H}$ can be also seen as the carrier space of the transitive representation $R^{\{N-1,1\}}$ of the symmetric group $\Sigma_N$ acting on the set $\tilde{N}$, with Young subgroup $\Sigma^{\{N-1,1\}} = \Sigma_{N-1} \times \Sigma_1 \subset \Sigma_N$ as the stabilizer. The decomposition

$$R^{\{N-1,1\}} = \Delta^{\{N\}} + \Delta^{\{N-1,1\}}$$

into irreps $\Delta^\lambda$ of $\Sigma_N$ provides a third basis in $\mathcal{H}$, in a form

$$|\mu\lambda y\rangle = \sum_{j \in \tilde{N}} K_{j,\mu y}|\tilde{N}j\rangle,$$

with $K_{j,\mu y}$ being Kostka matrices at the level of bases [10] for the transitive representation $R^\mu$ with $\mu = \{N-1,1\}$ denoting the weight, $\lambda$ being the shape of the Young diagram corresponding to rhs of Eq. (13), and $y \in \text{SYT}(\lambda)$ denoting the standard Young tableau of the shape $\lambda$. We refer to the basis (14) of magnonic qudit to as the irreducible basis of the Schur-Weyl duality, since it is related to mutually dual actions $A : \Sigma_N \times \mathcal{H} \to \mathcal{H}$ and $B : U(2) \times \mathcal{H} \to \mathcal{H}$ of the symmetric group $\Sigma_N$ and the unitary group $U(2)$ (the latter realized by its action on $\text{lcC}(2)$) on the space $\mathcal{H}$ of all quantum states of the magnetic ring.

3. The algebraic form of Bethe Ansatz

The algebraic BA, proposed by Faddeev and Takhtajan [3] (cf. also [11]), expresses the famous exact eigenstates of Bethe [1] in terms of some operators which can be readily interpreted in the language of quantum gates [12]. It is rooted in the identity of Dirac [13], which involves a pair of single-node Pauli matrices. For the case of two qubits, that is for the space $\mathbb{C}^2 \otimes \mathbb{C}^2$, Dirac identity reads

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 + I \otimes I = u_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\sigma^x, \sigma^y, \sigma^z$ are standard Pauli matrices, which form, together with the two-dim unit matrix $I$, a basis of the algebra $\text{End} \, \mathbb{C}^2$ - the algebra which encompasses all single-qubit gates, $\vec{\sigma}_j$ is the vector matrix for the qubit labeled by $j$ (at this moment, $j \in \{1, 2\}$), and thus scalar product $\vec{\sigma}_1 \cdot \vec{\sigma}_2 \in \text{End} \, (\mathbb{C}^2 \otimes \mathbb{C}^2)$ reads

$$\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z,$$

$u_{12} \in \text{End} \, (\mathbb{C}^2 \otimes \mathbb{C}^2)$ is the transposition operator for the two qubits, presented in the computational basis by the rhs matrix of Eq. (15). We recall here that the formal identification
of \( \vec{s}_j = \frac{1}{2} \vec{\sigma}_j \) as the vector operator of the electron spin localized at the \( j \)-th node was the key ingredient of the Heisenberg theory of magnetism at the early stages of development of quantum mechanics.

The space \( \mathcal{H} \) of all quantum states of the magnet is the \( N \)-th tensor power \( (\mathbb{C}^2)^{\otimes N} \) of a single qubit \( \mathbb{C}^2 \), so that its computational basis is determined by the set

\[
\mathcal{N} = \{ f : \mathcal{N} \to \hat{2} \}
\]

(17)
of all magnetic configurations, i.e. mappings \( f : \mathcal{N} \to \hat{2} \). Each magnetic configuration \( f \in \mathcal{N} \) corresponds to a separable state

\[
|f\rangle = |f(1)\rangle \otimes \ldots \otimes |f(N)\rangle \in \mathcal{H}.
\]

(18)

Accordingly, the algebra \( \text{End} \mathcal{H} \) is generated from single-node operators by

\[
s^a_j = I \otimes \ldots \otimes \hat{s}^a \otimes \ldots \otimes I \in \text{End} \mathcal{H}, \quad j \in \mathcal{N}, \quad a \in \{x, y, z\},
\]

(19)

with \( s^a \) corresponding to the \( j \)-th factor of the tensor power \( (\mathbb{C}^2)^{\otimes N} \) and unit \( I \) to all the other ones.

The algebraic BA is formulated within an extended system consisting of \( N + 1 \) qubits \( \mathbb{C}^2 \), \( N \) of which correspond to nodes of the magnetic ring, and the last qubit, denoted by \( V = \mathbb{C}^2 \), is referred to as an auxiliary one. The space of the extended system is \( \mathcal{H} \otimes V \), so that the auxiliary qubit can be seen as an “environment” of the magnetic ring. Now, one introduces the Lax operator, or, more specifically, a family of operators in \( \mathcal{H} \otimes \mathcal{V} \) referred to as an auxiliary one. The space of the extended system is

\[
\mathcal{H} \otimes \mathcal{V},
\]

where the transposition operator \( u_{ja} \) for qubits \( (\mathbb{C}^2)_j \) and \( V \) is embedded into \( \text{End} (\mathcal{H} \otimes \mathcal{V}) \), i.e. \( u_{ja} \in \text{End} (\mathcal{H} \otimes \mathcal{V}) \), in such a way that it acts trivially on all other \( (N - 1) \) qubits of the space \( \mathcal{H} \otimes V \cong (\mathbb{C}^2)^{\otimes N+1} \). The Lax operator in the computational basis of the auxiliary qubit \( V \), that is, in the basis of the environment of the magnetic ring, has the form

\[
L_{ja}(\lambda) = \left( \lambda - \frac{i}{2} \right) \text{id}_\mathcal{H} \otimes \text{id}_\mathcal{V} + i u_{ja},
\]

(20)

where the transposition operator \( u_{ja} \) for qubits \( (\mathbb{C}^2)_j \) and \( V \) is embedded into \( \text{End} (\mathcal{H} \otimes \mathcal{V}) \), i.e. \( u_{ja} \in \text{End} (\mathcal{H} \otimes \mathcal{V}) \), in such a way that it acts trivially on all other \( (N - 1) \) qubits of the space \( \mathcal{H} \otimes V \cong (\mathbb{C}^2)^{\otimes N+1} \). The Lax operator in the computational basis of the auxiliary qubit \( V \), that is, in the basis of the environment of the magnetic ring, has the form

\[
L_{ja}(\lambda) = \left( \lambda + \frac{i}{2} \right) \text{id}_\mathcal{H} \otimes \text{id}_\mathcal{V} + i \hat{s}_j \cdot \hat{\sigma}_a,
\]

(21)

where

\[
\hat{s}_j^\pm = \hat{s}_j^x \pm i \hat{s}_j^y \in \text{End} \mathcal{H}.
\]

It is thus a \( 2 \times 2 \) matrix, with entries being operators in \( \mathcal{H} \).
The Lax operator provides a kind of connection between qubits of the extended system. The product of Lax operators over consecutive nodes of the ring $\tilde{N}$ defines the monodromy matrix

$$\mathcal{M}(\lambda) = L_{1a}(\lambda)L_{2a}(\lambda)\ldots L_{Na}(\lambda) \in \text{End} (\mathcal{H} \otimes V),$$

presented in the computational basis of the auxiliary qubit $V$ as

$$\mathcal{M}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},$$

with $A(\lambda), B(\lambda), C(\lambda), D(\lambda) \in \text{End} \mathcal{H}$ being some operators in $\mathcal{H}$. We sketch here some important features of these operators.

All these operators, $A, B, C$ and $D$, are polynomials on the spectral parameter $\lambda$, of degree not larger than $N$. It justifies the adjective "algebraic" BA. The trace

$$F(\lambda) = A(\lambda) + B(\lambda),$$

i.e. the transfer matrix of Baxter [14], yields the family of mutually commuting operators,

$$[F(\lambda), F(\lambda')] = 0, \quad \lambda, \lambda' \in \mathbb{C},$$

for arbitrary values of the spectral parameter $\lambda$, and thus yields the subalgebra of integrals of motion in $\text{End} \mathcal{H}$ – the key for integrability of the XXX model.

We explore in this note the fact, that the operators $B(\lambda)$, when acting on the ferromagnetic vacuum, $|\text{vac}\rangle = |++\ldots+\rangle$, create the magnon with the spectral parameter $\lambda$. In our notation it can be written as

$$B(\lambda)|\text{vac}\rangle = |\lambda\rangle \in \mathcal{H},$$

with $|\lambda\rangle$ being a state of the magnonic qudit $h$, introduced in Section 2. The state $|\lambda\rangle$ is fully determined by the creation operator $B(\lambda)$, for a given value of the spectral parameter $\lambda$. We show in the next section how to encode an arbitrary exact BA eigenstate as a sequence of magnonic qudits, each in a state represented by an appropriate spectral parameter, emerging from BA equations.

4. Encoding of rigged string configurations in the magnonic qudit

In order to encode exact BA eigenstates using the magnonic qudit, we first recall that (i) the space $\mathcal{H}$ of all quantum states of the magnetic ring decomposes as

$$\mathcal{H} = \sum_{r=0}^N \mathcal{H}(r)$$

into subspaces $\mathcal{H}(r)$ with $r$ overturned spins, or, equivalently, with the $z$-projection $M = N/2 - r$ of the total spin of the magnet; (ii) each subspace $\mathcal{H}(r)$ decomposes further as

$$\mathcal{H}(r) = \sum_{r'=0}^{\min(r,N-r)} \mathcal{H}(r,r')$$

into subspaces $\mathcal{H}(r,r')$ with the total spin $S = N/2 - r'$; (iii) each subspace $\mathcal{H}(r,r')$ is spanned on the orthonormal basis $|\nu\rangle$ of exact BA eigenstates, called rigged string configurations. Here $\nu \vdash r'$ is a partition of the integer $r'$ called a configuration of strings, each each part of $\nu$, or equivalently, each row of the Young diagram of $\nu$, is a string with the length $t$ equal to the
number of boxes in the row, and $L$ is the collection of riggings, i.e. integers (quantum numbers), one for each string, which characterize its state of motion, and vary within a prescribed range, dependent on the length $l$. More specifically, let $m_l$ be the number of $l$-strings in the string configuration $\nu \vdash r'$, so that
\[ \sum_l lm_l = r'. \tag{30} \]
Then
\[ L = \{ L_{lv} | l = 1, 2, \ldots, v = 1, 2, \ldots, m_l \}, \tag{31} \]
and the riggings $L_{lv}$ satisfy selection rules
\[ 0 \leq L_{lv} \leq L_{lv'} \leq P_l \quad \text{for } 1 \leq v \leq v' \leq m_l, \tag{32} \]
where
\[ P_l = N - 2Q_l, \tag{33} \]
and $Q_l$ is the number of boxes in the first $l$ columns of the Young diagram of $\nu$ (c.f., e.g. [4] for more details).

In this way, each BA eigenstate $|\nu L\rangle$ is determined by a rigging $L_{lv}$, associated with each row $(l, v)$ of the Young diagram $\nu$. The algebraic BA provides an equivalent determination of this eigenstate, in terms of a collection of $r'$ spectral parameters $\lambda_{lv}^m$, such that each box $(l, v, m)$ of the Young diagram $\nu$ corresponds to a distinct spectral parameter. These parameters are labeled, for each string $(l, v)$, by the integer or half-integer $m$, varying by step 1 within the range
\[ -\frac{l - 1}{2} \leq m \leq \frac{l - 1}{2}, \tag{34} \]
so that the quantity $(l - 1)/2$ resembles "the spin of an $l$-string". It is worth to add that the string hypothesis of Bethe implies that for $N \to \infty$ the spectral parameters tend to the form
\[ \lambda_{lv}^m = \lambda_{lv} + im \tag{35} \]
with real $\lambda_{lv}$ and $m$, and that in the most of the known cases for a finite $N$ the actual results for spectral parameters are quite close to the asymptotic form (35) (but there are severe exceptions in several, but asymptotically remote cases - c.f. e.g. [15]).

We arrive at a conclusion that each exact eigenstate $|\nu L\rangle$ cane be written within algebraic BA in a form
\[ |\nu L\rangle = \prod_{(l,v,m)\in\nu} \hat{B} (\lambda_{lv}^m) |\text{vac}\rangle, \tag{36} \]
where the collection $\{\lambda_{lv}^m\}$ of spectral parameters satisfies the algebraic system of $r'$ equations (Bethe equations), of the standard form
\[ \left( \frac{\lambda_{lv}^m - i \frac{l}{2}}{\lambda_{lv}^m + i \frac{l}{2}} \right)^N = \prod_{(l',v',m')\neq(lv)} \frac{\lambda_{lv}^m - \lambda_{l'v'}^{m'} - i}{\lambda_{lv}^m + \lambda_{l'v'}^{m'} + i} \tag{37} \]
and the product in the rhs of Eq. (36) runs over all boxes of the Young diagram $\nu$ of the configuration of strings. Thus $r'$ copies of the magnonic qudit $h$ described in Section 2, one copy for each state $\hat{B}(\lambda_{lv}^m)|\text{vac}\rangle \in h$, are sufficient for construction of the state $|\nu L\rangle$. In other words, the collection of $r'$ qudits forms a register in a quantum computer composed of qudits, for the rigged string configuration $|\nu L\rangle$. 

5. Discussion

We have described a hypothetical unit of memory of a quantum computer, extracted from the space of quantum states of the one-dimensional Heisenberg ring, as the space $h = \mathbb{C}^N$ of all states with a single overturned spin, referred to as the magnonic qudit. To keep a close connection with the prototypic Heisenberg model of magnetism, we pointed out three natural orthonormal bases in the qudit space $h$: (i) the computational basis, referred to as the positional one, (ii) the basis of momenta, introduced in terms of the finite Fourier transform along the Schwinger unitary geometry, and (iii) the irreducible basis of the Schur-Weyl duality, realized by Kostka transformation at the level of bases.

We pointed out the way of encoding exact BA eigenfunctions, classified by rigged string configurations $|\nu\L⟩$, in a register of a quantum of magnonic qudits. To this aim, we have reviewed in short the algebraic form of BA, based on the monodromy matrices and the associated operators $A, B, C, D$, dependent on the spectral parameter $\lambda$. We have pointed out the fact that a single magnonic qudit can be prepared in a state $B(\lambda)|\text{vac}\rangle \in h$, which represents a single Bethe pseudoparticle with the spectral parameter $\lambda$. It follows that a collection of $r' \leq r$ magnonic qudits carries the full information on the rigged string configuration $|\nu\L⟩$ when each qudit is in the state $\hat{B}(\lambda_{\nu}^{|m|})|\text{vac}\rangle$, corresponding to on of the collection of spectral parameters corresponding to this state.

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