HOW LIKELY ARE TWO INDEPENDENT RECURRENT EVENTS TO OCCUR SIMULTANEOUSLY DURING A GIVEN TIME?

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Abstract. We determine the probability $P$ of two independent events $A$ and $B$, which occur randomly $n_A$ and $n_B$ times during a total time $T$ and last for $t_A$ and $t_B$, to occur simultaneously at some point during $T$. Therefore we first prove the precise equation

$$P^* = \frac{t_A + t_B}{T} - \frac{t_A^2 + t_B^2}{2T^2}$$

for the case $n_A = n_B = 1$ and continue to establish a simple approximation equation

$$P \approx 1 - \left(1 - n_A \frac{t_A + t_B}{T}\right)^{n_B}$$

for any given value of $n_A$ and $n_B$. Finally we prove the more complex universal equation

$$P = 1 - \frac{(T^+ - t_A n_A - t_B n_B)^{n_A + n_B}}{(T^+ - t_A n_A)^{n_A} (T^+ - t_B n_B)^{n_B}} \pm E^\pm,$$

which yields the probability for $A$ and $B$ to overlap at some point for any given parameter, with $T^+ := T + \frac{t_A + t_B}{2}$ and a small error term $E^\pm$.

1. Introduction

Let us consider two independent and recurring events $A$ and $B$, which take place during a total time $T$. The events occur exactly $n_A$ and $n_B$ times randomly during this total time and last for $t_A$ and $t_B$ until the next event occurrence may happen. Since both events occur independently, they may eventually take place simultaneously at some point during $T$ – i.e. they overlap. This may happen in case an occurrence

1. of $A$ and $B$ take place at the same moment,
2. of $A$ takes places while an occurrence of $B$ still takes place or
3. of $B$ takes place while an occurrence of $A$ still takes place.

We divide all time parameters in discrete units $\Delta \rightarrow 0$ and let $P$ denote the probability for at least one overlap – i.e. it exists at least one time unit, during which both events take place. We will derive expressions to determine $P$ under given parameters $T$, $t_A$, $t_B$, $n_A$ and $n_B$. The following examples and notes clarify the setting and will be revisited later:

Example I — John works inside his office for 2 hours. A blue car will occur 10 times on the nearby street and remains visible for 1 minute each time. John looks outside 5 times for 3 minutes each. How likely is John going to see a blue car? $T = 120$ min, $t_A = 3$ min, $t_B = 1$ sec, $n_A = 5$, $n_B = 10$ and we are about to determine $P(120, 3, 1, 5, 10)$.

Example II — We choose a total time of 3 seconds. Both events happen only once for 1 second during this time: How likely are both events to overlap at some point?: $T = 3$ sec, $t_A = 1$ sec, $t_B = 1$ sec, $n_A = 1$, $n_B = 1$ and we are about to determine $P(3, 1, 1, 1, 1)$.

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Note: Since all time values are given with same unit, we will stop to note the unit in further calculations.
Definitions and Notes:
(a) If the duration time of $A$ and $B$ differs, let always $A$ denote the event with the longer duration time: $t_A \geq t_B$.
(b) The duration of the event-overlap is not relevant – i.e. an overlap of a few minutes or a millisecond are both counted as valid overlap.
(c) We consider a possible overlap with the end of the total time as more natural and thus valid - e.g. John may start to look outside 4 seconds before the 120 minutes total time are over.
(d) Both events occur precisely $n_A$ and $n_B$ times - we do not perceive these parameters as random in the proofs. When the events occur is the only random aspect first. We remark in Section 6 how the equation changes with $n_A$ and $n_B$ as random.
(e) Occurrences of the same event must not overlap with themselves – e.g. John must not start to look outside while he is already looking. Thus $T \leq t_A \cdot n_A$ implies $P = 1$.

2. Notation
Overview of main notation:

- $A, B$ – labels of the events
- $T$ – total time
- $t_A, t_B$ – duration of event $A, B$
- $n_A, n_B$ – number of occurrences of event $A, B$ during total time
- $P, \bar{P}$ – probability for an (no) overlap
- $\Delta$ – infinitesimal unit, divides time in discrete pieces

Overview of specific notations for local purpose:

- $S_1, S_2 = \sum_{i=0}^{\frac{T}{\Delta} - 2} \left( \frac{t_A}{\Delta} + i \right), \sum_{j=1}^{\frac{T}{\Delta} - 1} \left( \frac{t_A}{\Delta} + t_B \Delta - 1 - j \right)$
- $T' = \frac{T}{\Delta}$
- $N_A, N_B = \left( \frac{t_A}{\Delta} - 1 \right) \cdot n_A, \left( \frac{t_B}{\Delta} - 1 \right) \cdot n_B$
- $M_A, M_B, M = T' - N_A - n_A, T' - N_B - n_B, T' - N_A - n_A - N_B - n_B$
- $T'' = T + \frac{t_A + t_B}{2}\tau$
- $\alpha_A, \alpha_B = T + t_A - t_A n_A - t_B n_B, \alpha_A = \alpha_A - \frac{t_A - t_B}{2}$
- $\tau = \max\{t_A n_A, t_B n_B\}$

3. Precise Equation for $n_A = n_B = 1$
We start with deriving an equation for the case $n_A = n_B = 1$, since

(1) this case has shown to occur unexpectedly often in applications
(2) prepares main techniques for the proof of Theorem 3 and
(3) the equation yields precise results without error term.

Let $P^*(T, t_A, t_B)$ denote the probability function for this case.

Theorem 1. The probability$^2$ of $A$ and $B$ to overlap at some point during the total time $T$ in case of $n_A = n_B = 1$ is given by

$$P^*(T, t_A, t_B) = \frac{t_A + t_B}{T} - \frac{t_A^2 + t_B^2}{2T^2}.$$  

$^2$although $|t_A - t_B| > 0$ (see Section 3)

$^3$Note: Due to reading purpose we will use just 'A' instead of 'the A occurrence(s)' – same for $B$.  


Lemma 1.1. We remember the well known arithmetic summation formula
\[ 1 + 2 + \ldots + n = \sum_{k=1}^{n} k = \frac{1}{2} n \cdot (n + 1). \]

Proof. See [2]. □

Proof of Theorem. First, we divide the time in pieces of size \( \Delta \). Thus, we get \( T/\Delta \) as limited number of positions at which the events may start, so there are \( (T/\Delta)^2 \) possible arrangements. By determining the number of positions \( x_B(x_A) \) of \( B \) to overlap with \( A \) for every possible position \( x_A \) of \( A \), we are able to obtain the probability \( P^* \) with
\[ P^* = \lim_{\Delta \to 0} \frac{\sum_{x_A} x_B(x_A)}{(T/\Delta)^2}. \] (3.1)

In general, if \( A \) is located somewhere in the center of \( T \) – i.e. in a sufficient distance to beginning and end of the total time – there are
\[ \frac{t_A}{\Delta} + \frac{t_B}{\Delta} - 1 \] (3.2)
possible positions \( x_B \) for \( B \) to overlap with \( A \). This holds true for
\[ \frac{T}{\Delta} - \frac{t_A}{\Delta} - \frac{t_B}{\Delta} + 1 \] positions \( x_A \) of \( A \), since there are at \( \frac{t_B}{\Delta} - 1 \) positions (beginning of \( T \)) and at \( \frac{t_A}{\Delta} - 1 \) positions (end of \( T \)) less possible positions for \( B \) to overlap with \( A \). Therefore (3.1) becomes
\[ P^* = \lim_{\Delta \to 0} \frac{\Delta^2 S_1 + (t_A + t_B - \Delta)(T - t_A - t_B + \Delta) + \Delta^2 S_2}{(T/\Delta)^2}. \] (3.3)

By applying Lemma [1] the first sum simplifies to
\[ \Delta^2 S_1 = \Delta^2 \sum_{i=0}^{\Delta-2} \left( \frac{t_A}{\Delta} + i \right) = (t_B - \Delta) t_A + \Delta^2 \sum_{i=1}^{\Delta-2} i = (t_B - \Delta) t_A + \frac{1}{2} (t_B - \Delta) (t_B - 2\Delta), \]
and the second sum to
\[ \Delta^2 S_2 = \Delta^2 \sum_{j=1}^{\Delta-1} \left( \frac{t_A}{\Delta} + \frac{t_B}{\Delta} - 1 - j \right) = (t_A - \Delta) (t_A + t_B - \Delta) - \Delta^2 \sum_{j=1}^{\Delta-1} j = (t_A - \Delta) (t_A + t_B - \Delta) - \frac{1}{2} t_A (t_A - \Delta). \]

Inserting these results in (3.3) and considering \( \Delta \to 0 \) yields
\[ P^* = \frac{t_A t_B + \frac{1}{2} t_A^2 + (t_A + t_B)(T - t_A - t_B) + t_A(t_A + t_B) - \frac{1}{2} t_A^2}{T^2} \]
\[ = \frac{t_A t_B + \frac{1}{2} t_A^2 + (t_A + t_B)(T - t_B) + t_A(t_A + t_B) - \frac{1}{2} t_A^2}{T^2} \]
\[ = \frac{t_A t_B + t_B T - \frac{1}{2} t_A^2 - \frac{1}{2} t_B^2}{T^2} = \frac{t_A + t_B}{T} - \frac{t_A^2 + t_B^2}{2T^2}, \]
which concludes the proof. □
Example I (modified) — John looks outside once for 5 minutes and a blue car occurs once for 2 minutes during 1 hour. He will see a blue car with a probability of about 11.26%:

\[
P^*(60, 5, 2) = \frac{5 + \frac{2}{60} \cdot \frac{5^2 + 2^2}{2}}{60 - \frac{5}{2}} \approx 0.1126.
\]

Example II — Two events with \(t_A = t_B = 1\) sec will overlap during a total time \(T = 3\) sec with a probability of about 55.56%:

\[
P^*(3, 1, 1) = \frac{1 + \frac{1}{3} \cdot \frac{1^2 + 1^2}{2} \cdot 3}{2} \approx 0.5556.
\]

4. Approximation of Universal Equation

Before we derive a more complex universal equation (see Section 5), we will find an approximation, which shall apply for any given \(n_A\) and \(n_B\).

**Theorem 2.** The probability of \(A\) and \(B\) to overlap at some point during the total time \(T\), with \(A\) and \(B\) occurring \(n_A\) and \(n_B\) times for \(t_A\) and \(t_B\), can be approximated by

\[
P(T, t_A, t_B, n_A, n_B) \approx 1 - \left(1 - n_A \frac{t_A + t_B}{T} \right)^{n_B}.
\]

**Proof of Theorem.** Since \(P = 1 - \bar{P}\), we will establish an equation for the probability \(\bar{P}\) for the events not to occur at the same time at any point. Let \(k \in \{0, 1, ..., t_B - 1\}\) number the \(n_B\) occurrences of \(B\). There are

\[
\frac{T}{\Delta} - k \cdot \frac{t_B}{\Delta}
\]

possible positions for the \(k\)-th \(B\) occurrence to start, but we consider \(T \gg t_B\) and approximate (4.1) with \(T/\Delta\). Every \(B\) occurrence may overlap at about

\[
\frac{t_A}{\Delta} + \frac{t_B}{\Delta}
\]

positions (see (3.2)) with any occurrence of \(A\). Thus we have for the probability \(\bar{p}\) of a \(B\) occurrence not to overlap at some point with any \(A\) occurrence

\[
\bar{p} \approx 1 - n_A \cdot \left(\frac{t_A + t_B}{T/\Delta}\right) = 1 - n_A \frac{t_A + t_B}{T}
\]

To find \(\bar{P}\) this has to be true for every occurrence of \(B\) and we get

\[
P = 1 - \bar{P} = 1 - (\bar{p})^{n_B} \approx 1 - \left(1 - n_A \frac{t_A + t_B}{T}\right)^{n_B},
\]

which confirms the approximation. \(\square\)

Example I — John will see a blue car with a probability of about 83.85%:

\[
P(120, 3, 1, 5, 10) \approx 1 - \left(1 - 5 \cdot \frac{3 + 1}{120}\right)^{10} \approx 0.8385.
\]

In Example I in Section 5 the error of this result will be shown to be insignificant.
5. Universal Equation

Theorem 3. The probability of A and B to overlap at some point during the total time T, with A and B occurring \(n_A\) and \(n_B\) times for \(t_A\) and \(t_B\), is given by

\[
P(T, t_A, t_B, n_A, n_B) = 1 - \frac{(T^+ - t_{An_A} - t_{Bn_B})^{n_A+n_B}}{(T^+ - t_{An_A})^{n_A} (T^+ - t_{Bn_B})^{n_B}} \pm E^\pm
\]

with \(T^+ := T + \frac{t_A + t_B}{2}\) and a small error term

\[
0 \leq E^\pm \leq \frac{(T^+ - t_{An_A} - t_{Bn_B})^{n_A+n_B}}{(T^+ - t_{An_A})^{n_A} (T^+ - t_{Bn_B})^{n_B}} \cdot \left( \frac{\alpha_A(\alpha_B + \tau)}{\alpha_B(\alpha_A + \tau)} \right)^{n_A+n_B} - 1
\]

with \(\alpha_A := t - t_{A} - t_{An_A} - t_{Bn_B}, \alpha_B := \alpha_A - \frac{t}{2} t_B\) and \(\tau := \max\{t_{An_A}, t_{Bn_B}\} \).

Lemma 3.1. For constant \(y \in \mathbb{N}\) we have

\[
\lim_{X \to \infty} \frac{(X + y)!}{X!X^y} = 1.
\]

Proof. The above fraction can be rearranged to

\[
\lim_{X \to \infty} \frac{(X + 1) \cdot \ldots \cdot (X + y)}{X^y} = \lim_{X \to \infty} \prod_{i=1}^{y} \left(1 + \frac{l}{X}\right) = 1.
\]

\[\square\]

Lemma 3.2. For constant \(u, v \in \mathbb{R}\) with \(u \leq v\) following inequality holds true:

\[
\frac{\alpha_B + u}{\alpha_A + u} \leq \frac{\alpha_B + v}{\alpha_A + v}.
\]

Proof. Rearranging the inequality yields

\[
(\alpha_A + v)(\alpha_B + u) \leq (\alpha_A + u)(\alpha_B + v)
\]

\[
\alpha_A u + \alpha_B v \leq \alpha_A v + \alpha_B u
\]

\[
\alpha_B(v - u) \leq \alpha_A(v - u).
\]

Inequality \((5.1)\) holds true since \(v - u \geq 0\) and \(\alpha_B := \alpha_A - \frac{1}{2} t_A + \frac{1}{2} t_B \leq \alpha_A\) due to Definition (a) in Section 11 that \(t_A \geq t_B\). \[\square\]

Proof of Theorem. Similar to the approximation we determine the probability \(\tilde{P}\) for no overlap to occur. Therefore we divide the time in parts of size \(\Delta\) again and count the number of possible arrangements of all A and B occurrences. There are

\[
\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1\right) \cdot n_A
\]

positions for the \(n_A\) occurrences of A left, so that they do not overlap with themselves. Thus the number of possible arrangements of the A occurrences is given by

\[
\left(\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1\right) \cdot n_A\right)
\]

and for the B occurrences by

\[
\left(\frac{T}{\Delta} - \left(\frac{t_B}{\Delta} - 1\right) \cdot n_B\right).
\]

The number of ways to order the \(n_A\) occurrences of A and the \(n_B\) occurrences of B is

\[
\left(\frac{n_A + n_B}{n_A}\right).
\]

Similar to \((5.2)\) and \((5.3)\) we can express the number of ways to arrange the A and B occurrences per order of \((5.1)\) as

\[
\left(\frac{T}{\Delta} - \left(\frac{t_A}{\Delta} - 1\right) \cdot n_A - \left(\frac{t_B}{\Delta} - 1\right) \cdot n_B\right).
\]

\[(5.5)\]
Since the probability $\tilde{P}$ describes the ratio of arrangements without overlap to the total number of possible arrangements of the $A$ and $B$ occurrences, we have

$$\tilde{P} = \lim_{\Delta \to 0} \frac{\binom{n_A+n_B}{\Delta - (\frac{n_A}{n} - 1)n_B - (\frac{n_B}{n} - 1)n_A} \binom{n_A+n_B}{\Delta - (\frac{n_A}{n} - 1)n_B - (\frac{n_B}{n} - 1)n_A}}{\binom{n_A}{\Delta - (\frac{n_A}{n} - 1)n_B} \binom{n_B}{\Delta - (\frac{n_B}{n} - 1)n_A} \binom{T - N_A - N_B}{\Delta - (\frac{n_A}{n} - 1)n_B \Delta - (\frac{n_B}{n} - 1)n_A}} = \lim_{\Delta \to 0} \frac{\binom{n_A+n_B}{\Delta - (\frac{n_A}{n} - 1)n_B - (\frac{n_B}{n} - 1)n_A} \binom{T - N_A - N_B}{\Delta - (\frac{n_A}{n} - 1)n_B \Delta - (\frac{n_B}{n} - 1)n_A}}{\binom{n_A}{\Delta - (\frac{n_A}{n} - 1)n_B} \binom{n_B}{\Delta - (\frac{n_B}{n} - 1)n_A} \binom{T - N_A - N_B}{\Delta - (\frac{n_A}{n} - 1)n_B \Delta - (\frac{n_B}{n} - 1)n_A}}.$$  

Rewriting the binomial coefficients and rearrange the fraction yields

$$\tilde{P} = \lim_{\Delta \to 0} \frac{(n_A + n_B)![(T' - N_A - N_B)!n_A][(T' - N_A - n_A)N_B][(T' - N_B - n_B)!]}{(T' - N_A - N_B)!(T' - N_A - n_A)!n_B![n_A + n_B]![(T' - N_B - n_B)!]} = \lim_{\Delta \to 0} \frac{(T' - N_A - N_B)!(T' - N_A - n_A)!}{(T' - N_A - N_B)!(T' - N_A - n_A)!}.$$  

$$= \lim_{\Delta \to 0} \frac{(T' - N_A - N_B)!(T' - N_A - n_A)!}{(T' - N_A - N_B)!(T' - N_A - n_A)!}.$$  

$$= \lim_{\Delta \to 0} \frac{(M + M_A + n_B)M_A M_B!}{(M + M_A + n_B)!(M_B + n_B)M!}.$$  

Due to the limit we have $\Delta \to 0$ and thus $M_A, M_B, M \to \infty$. Therefore we are allowed to apply Lemma 3.1 on (5.6):

$$\tilde{P} = \lim_{\Delta \to 0} \frac{M!M_A M_B!}{M_A M_B M_A M_B! M!} = \lim_{\Delta \to 0} \frac{M!M_A M_B!}{M_A M_B M_A M_B! M!}.$$  

$$= \lim_{\Delta \to 0} \frac{(T - N_A - N_B + n_A + n_B)M_A M_B!}{(T - N_A - N_B) M_A M_B!}.$$  

$$= \lim_{\Delta \to 0} \frac{(T - N_A - N_B + n_A + n_B)M_A M_B!}{(T - N_A - N_B) M_A M_B!}.$$  

This equation yields the probability in case that the occurrences may not overlap the total time. Since we defined in (c) in Section 1 the occurrences may overlap with the end of the total time, we have to extend the total time $T$ by some $\delta > 0$:

$$\tilde{P}_\delta := \frac{(T + \delta - t_{A+B} - t_{B+B})^{n_A+n_B}}{(T + \delta - t_{A+B})^{n_A+n_B} (T + \delta - t_{B+B})^{n_B}}.$$  

Obviously, $t_B \leq \delta \leq t_A$ and we choose $\delta^* := \frac{t_A + t_B}{2}$. Therefore let us redefine

$$\tilde{P} := \tilde{P}_{\delta} \pm E^\pm = \frac{(T + \delta - t_{A+B} - t_{B+B})^{n_A+n_B}}{(T + \delta - t_{A+B})^{n_A+n_B} (T + \delta - t_{B+B})^{n_B}} \pm E^\pm$$  

with $0 \leq E^\pm$ as small error term. $E^\pm$ approaches 0 if $|t_A - t_B|$ approaches 0 or if the ratio between $T$ and the event time $t_{A+B} + t_{B+B}$ increases.

In order to determine the maximum error we consider, that the probability for an overlap decreases with longer $T$. Since we defined $t_A \geq t_B$ (see Definition (a) in Section 1), we have

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4 number of arrangements without overlap, 5 see notation in Section 2
5 see notation in Section 2
6 Notation: ‘∼’ denotes ‘proportional to’
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\[ P_{A^c} - P_{t_A} \geq P_{t_B} - P_{B^c} \]  
and the maximum error is given by  
\[ E^\pm \leq P_{B^c} - P_A = (1 - P_{B^c}) - (1 - P_A) = P_{B^c} \cdot \left( \frac{P_A}{P_{B^c}} - 1 \right) \]

\[ = P_{B^c} \cdot \left( \frac{\alpha_A}{\alpha_B} \right)^{n_A+n_B} \cdot \left( \frac{\alpha_B + t_B n_B}{\alpha_A + t_B n_A} \right)^{n_A} \cdot \left( \frac{\alpha_B + t_B n_A}{\alpha_A + t_B n_B} \right)^{n_B} - 1 \]

\[ \leq P_{B^c} \cdot \left( \frac{\alpha_A}{\alpha_B} \right)^{n_A+n_B} \cdot \left( \frac{\alpha_B + \tau}{\alpha_A + \tau} \right)^{n_A+n_B} - 1 \]  
(see Lemma 5.2)

\[ = P_{B^c} \cdot \left( \frac{\alpha_B(\alpha_B + \tau)}{\alpha_B(\alpha_A + \tau)} \right)^{n_A+n_B} - 1 . \]

Finally the probability for an overlap is given by \( P = 1 - \bar{P} \), which concludes the proof. □

--- Example —

**Example I** — John will see an about a blue car with a probability of about 85.46 ± 1.77 %. Since \( T^+ = 120 + \frac{3\cdot5}{2} = 122 \), \( \alpha_A = 120 + 3 \cdot 5 - 1 \cdot 10 = 98 \) and \( \alpha_B = 98 - \frac{3\cdot5}{2} = 97 \), we have

\[ P(120,3,1,5,10) = 1 - \left( \frac{122 - 3 \cdot 5 - 1 \cdot 10}{122 - 3 \cdot 5} \right)^{5+10} \approx 0.8546 ± E^\pm \]

and, due to \( \max\{3 \cdot 5, 1 \cdot 10\} = 3 \cdot 5 \),

\[ E^\pm \leq 85.46% \cdot \left( \frac{98 \cdot (97 + 3 \cdot 5)}{97 \cdot (98 + 3 \cdot 5)} - 1 \right) \approx 1.77 \% . \]

This confirms the precision of the approximation in Section 4.

**Example II** — Two events with \( t_A = t_B = 1 \) sec will overlap during a total time \( T = 3 \) sec at some point with a probability of about 55.56 %. Since \( T^+ = 3 + \frac{1+1}{2} = 4 \), we have

\[ P(3,1,1,1,1) = 1 - \left( \frac{4 - 1 \cdot 1 - 1 \cdot 1}{4 - 1 \cdot 1} \right)^{1+1} \approx 0.5555 \pm 0 \]

without any error due to \( |t_A - t_B| = |1 - 1| = 0 \). This result is consistent with Section 3.

6. Remark: \( n_A \) and \( n_B \) as random parameters

Instead of determining \( n_A \) and \( n_B \) as precise numbers of occurrences, it is more natural to define \( \rho_A \) and \( \rho_B \), which represent the probability \( p \) for the event to occur in a given time \( s \). \( p := p/s \). In order to calculate the probability for a possible overlap, the number of expected occurrences \( n = p \cdot T \) has to be calculated first. The universal equation would simply change to

\[ P(T, t_A, t_B, \rho_A, \rho_B) = 1 - \left( \frac{T^+ - t_A T \rho_A - t_B T \rho_B}{T^+ - t_A T \rho_A} \right)^{\rho_A + \rho_B} \cdot \left( T^+ - t_B T \rho_B \right)^{\rho_B} \pm E^\pm . \]

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