Designing, implementing and analysing optimal controllers on a non-linear reaction wheel pendulum

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Abstract. This paper discusses a reaction wheel control system. A reaction wheel pendulum is a good example of a non-linear and underactuated system, which attracts control system researchers to develop many control algorithms. The reaction wheel plant is usually used for studying advanced control system courses. In this paper, a mathematical model of the state space is discussing. A proposed LQR control algorithm is explained. Simulation and real time experiments have been carried out to verify the performance of the proposed closed loop system. The LQR optimisation algorithm was able to find the optimum feedback gains. The simulation and real time experiments show that the reaction wheel pendulum could stabilize the pendulum at upright position.

1. Introduction
Underactuated mechanical systems are well known of systems that present interesting characteristics for test linear, nonlinear and intelligent control algorithms, since this kind of systems represent an interesting challenge from the viewpoint of control theory and engineering [1]. Pendulums are present everywhere around us, translated in different forms. To give a definition, a pendulum is simply an object swinging freely relative to its equilibrium position [2]. A Pendulum system has been a popular demonstration platform for nonlinear and underactuated control at least fifty years [3]. There are some control experiments performed in different pendulum systems or balancing system. It is typical to use one control law to swing up the pendulum and another to balance it around the upright position [4]. In recent time optimal control provides the best possible solution to process control has been presented in [5].

One of pendulums is reaction wheel pendulum. The reaction wheel is a spinning mass which provides a reaction torque coming from the rotational acceleration [6]. The Reaction Wheel Pendulum is perhaps the simplest of the various pendulum systems in terms of its dynamic properties, consequently, its controllability properties. The Reaction Wheel Pendulum is one of the newest modification form of pendulum. The Reaction wheel pendulum is a system contains from pendulum and rotating disk. The Reaction Wheel Pendulum is a simple pendulum with a rotating wheel, or bob, at the end. The level of background knowledge assumed is that of a first course in control, together with some rudimentary knowledge of dynamics of physical systems [7]. The Reaction Wheel Pendulum exhibits several
properties, such as underactuation and nonlinear, that make it an attractive and useful system for research and advanced education [8]. The Reaction Wheel Pendulum is suited for education for university students.

2. System modelling

2.1. Lagrange’s equation

Represented the motion equation of a complex system dynamic is using Lagrange’s equation.

\[
\begin{align*}
K &= \frac{1}{2} (m_1 L_1^2 + m_2 L_2^2 + I_1 + I_2) \dot{\theta}^2 + I_2 \dot{\Theta} \dot{\Phi} + \frac{1}{2} I_2 \dot{\Phi}^2. \\
\end{align*}
\]  \hspace{1cm} (2.1)

The potential energy of the reaction wheel pendulum is obtained as follows

\[
P = (m_1 L_1 + m_2 L_2) g \cos \theta \hspace{1cm} (2.2)
\]

The Lagrange function \( L \) is expressed by the kinetic energy \( K \) and the potential energy \( P \). Where the kinetic energy function in terms of the generalised coordinate \( q \) and its derivative \( \dot{q} \). The Lagrange function is expressed as

\[
L = K - P \hspace{1cm} (2.3)
\]

\[
L = \frac{1}{2} (m_1 L_1^2 + m_2 L_2^2 + I_1 + I_2) \dot{\theta}^2 + I_2 \dot{\Theta} \dot{\Phi} + \frac{1}{2} I_2 \dot{\Phi}^2 - (m_1 L_1 + m_2 L_2) g \cos \theta \hspace{1cm} (2.4)
\]

From (2.4) we can calculate the Euler – Lagrange equations of motion, based on formula

\[
Q_1 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \hspace{1cm} (2.5)
\]

Where

\[
q = [q_1, q_2]^T = [\theta_1, \theta_2]^T \hspace{1cm} (2.6)
\]

It is noted that the produced motor torque \( T_r \) is spinning the wheel which results in motor torque acting on the pendulum rod in the opposite direction. Thus, the Euler – Lagrange equations as below

\[
(m_1 L_1^2 + m_2 L_2^2 + I_1 + I_2) \ddot{\theta} + I_2 \ddot{\Theta} - (m_1 L_1 + m_2 L_2) g \cos \theta = 0 \hspace{1cm} (2.7)
\]

\[
l_2(\dot{\theta}_1 + \dot{\Theta}) = T_r \hspace{1cm} (2.8)
\]
2.2. State space representation

The reaction wheel pendulum can be represented in state – space form as

\[ \dot{x} = Ax + Bu \] (2.9)
\[ y = Cx + Du \] (2.10)

From Euler – Lagrangian’s equation we can get the controllable linear system with matrices as

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi} \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{b}{a} & 0 & 0 & 0 \\
-b/a & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta} \\
\phi \\
\dot{\phi}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-\frac{1}{a} \\
0
\end{bmatrix}
Tr
\frac{(a + l_2)/(a - l_2)}{Tr}
\] (2.11)
\[
\begin{bmatrix}
\theta \\
\dot{\theta} \\
\phi \\
\dot{\phi}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta} \\
\phi \\
\dot{\phi}
\end{bmatrix}
\] (2.12)

Where, \( a = m_1 L_1^2 + m_2 L_2^2 + I_1 \) and \( = (m_1 L_1 + m_2 L_2)g \).

Table 1. System parameters.

| Parameters                      | Values       |
|--------------------------------|--------------|
| Mass of pendulum \((m_1)\)     | 0.212 Kg     |
| Mass of wheel \((m_2)\)        | 0.114 Kg     |
| Reaction wheel upright height\((L_1)\) | 0.50 m       |
| Centre of mass height\((L_2)\) | 0.35 m       |
| Gravitational acceleration\((g)\) | 9.81 m/s²   |
| Reaction wheel moment of inertia\((I_2)\) | 0.0024 Kg.m² |
| Motor torque constant\((Kt)\)  | 487.10³Vs/rad|
| Motor back EMF constant\((Ke)\) | 487.10³Vs/rad|
| Motor gear ratio\((N_g)\)      | 25 : 1       |

By substituting the parameters values in Table 1, the state – space expression of the system is changed into:

\[
\begin{bmatrix}
\dot{\theta} \\
\dot{\phi} \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & 0 & 0 \\
23.6290 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\theta \\
\dot{\theta} \\
\phi \\
\dot{\phi}
\end{bmatrix}
+ \begin{bmatrix}
0 \\
-18.3587 \\
0
\end{bmatrix}
Tr
\frac{435.0254}{Tr}
\] (2.13)

3. LQR method

Linear Quadratic Regulator Controller is based on full state – space feedback control principle (Figure 2) [9].
The optimal linear Quadratic control method aims to obtain a controlling signal \( u(t) \) that will move a linear system state from the initial condition \( x(t_o) \) to the final condition \( x(t) \) that will minimize the cost function, expressed as:

\[
\mathcal{J} = \int_0^\infty (x^TQx + u^2)R dt
\]  

(3.1)

Matrix selection \( Q \) and \( R \) doing by trial and error. Where matrix \( Q \) is symmetrical matrix, positive semidefinite \( (Q \geq 0) \), while matrix \( R \) is positive and real \( (R > 0) \). The relative importance of the error and the energy cost are determined by \( Q \) and \( R \). Therefore, The matrix \( R \) and \( Q \) signify the trade-off between performance and the control effort respectively.

From figure 3, the denominator can be found using formula as

\[
D(s) = sI - (A - BK)
\]

(3.2)

Where \( I \) is an identity matrix. So, the stability and transient response characteristics of the closed loop system are determined by all eigenvalues of \( (A - BK) \). The design is to select the feedback gain \( K \) such that eigenvalues of \( (A - BK) \) have negative real parts [10].

4. Result and analysis

Simulated testing has been done using MATLAB. Figure 3 and Figure 4 show the system response.

Table 2. Parameter of LQR.

| Parameter | Pendulum |
|-----------|----------|
| \( Q \)   | \[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\] |
| \( R \)   | 1        |
| \( K \)   | [-65.2304, -13.6986, -0.4883] |

Using the value of parameter system from table 2, obtained
Figure 3. Graph response with control with gain feedback \((K)\).

From figure 3 the curve in red represents the wheel’s angle in radians, and the curve in blue represents pendulum angle in radian. As you can see, this plot is not satisfactory. Wheel pendulum and pendulum angle’s overshoot appear fine, but their settling times need improvement. So, Need to do changes to the value \(Q\).

Table 3. Parameter of LQR simulation 2.

| Parameter | Pendulum |
|-----------|----------|
| \(Q\)     | \[
         \begin{bmatrix}
         5000 & 0 & 0 & 0 \\
         0 & 0 & 0 & 0 \\
         0 & 0 & 100 & 0 \\
         0 & 0 & 0 & 0 \\
         \end{bmatrix}
     \] |
| \(R\)     | 1        |
| \(K\)     | [-534.5587, -111.3672, -10.000, -4.4596] |

Using the value of parameter system from table 3, obtained

Figure 4. Graph response with control with gain feedback \((K)\).
Figure 4 show the graph with higher $Q$ value than before. With changing the value of $Q$ can make the system becomes more stable.

5. Conclusion

Reaction wheel Pendulum is a non-linear system and underactuation system. This system is unstable and can be stable when modelled by using mathematical equations. From this experiments we get the value of gain feedback $[-534.5587 \ -111.3672 \ -10.000 \ -4.4596]$ to stabilize the reaction wheel pendulum in simulation. From the picture of graphic is known that system can stabilize and can be at the upright position.

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References

[1] Moreno-Valenzuela J, Aguilar-Avelar C and Puga-Guzm’an S 2014 On trajectory tracking control of the inertia wheel pendulum 2014 International Conference on Control, Decision and Information Technologies (CoDIT) 572-577

[2] Elena S L 2015 Cubic Structure Capable of Balancing (Faculty of Electronics, Telecommunications and Information Technology University POLITEHNICA of Bucharest)

[3] Boubaker O 2013 The inverted pendulum benchmark in nonlinear control theory: A survey International Journal of Advanced Robotic System 10 2013

[4] Klaus W W L and Willi H 2009 Handbook of Space Technology

[5] Tabak D 1970 Applications of mathematical programming techniques in optimal control : A survey IEEE Transactions on Automatic Control 15(6) 688-690

[6] Daniel J B, Karl J A and Mark W S 2007 The Reaction Wheel Pendulum in The Reaction Wheel Pendulum

[7] Astrom K J and Furuta K 2000 Swinging up a pendulum by energy control Automatica 36 278-285

[8] Muehlebach M and D’Andrea R 2017 Nonlinear Analysis and Control of a Reaction-Wheel-Based 3-D Inverted Pendulum in IEEE Transactions on Control Systems Technology 25(1) 235-246

[9] Indrazno S 2017 State space control using LQR method for a cart-inverted pendulum linearised model in International Journal of Mechanical and Mechatronics Engineering IJMME-IJENS 17(01)

[10] Ignaciuk P 2012 Linear – quadratic optimal control of periodic – review perishable inventory systems IEEE Transactions on Control Systems Technology 20(5) 1400 – 1407