PURE STATE TRANSFORMATIONS INDUCED BY LINEAR OPERATORS

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Abstract. We generalise Wigner's theorem to its most general form possible for \( B(\mathfrak{h}) \) in the sense of completely characterising those vector state transformations of \( B(\mathfrak{h}) \) that appear as restrictions of duals of linear operators on \( B(\mathfrak{h}) \). We then use this result to similarly characterise all pure state transformations of general \( C^* \)-algebras that appear as restrictions of duals of linear operators on the underlying algebras. This result may variously be interpreted as either a non-commutative Banach-Stone theorem, or (in the bijective case) a pure state based description of Wigner symmetries. These results extend the work of Shultz [Sh] (who considered only the case of bijections), and also complements and completes the investigation of linear maps with pure state preserving adjoints begun in [LMa].

1. Preliminaries

Notation in this paper will be based on that of [BRo] and [KRi]. We briefly review the essentials. In general \( A, B \) will denote \( C^* \)-algebras with \( A_{sa} \) denoting the self-adjoint portion of a given \( C^* \)-algebra \( A \). Unless otherwise stated we will generally assume our algebras to be unital with \( I \) being used to denote the unit. In this context a linear map \( \varphi : A \to B \) is called a Jordan \( \ast \)-homomorphism if it preserves both the Jordan product and adjoints, i.e. if \( \varphi(AB + BA) = \varphi(A)\varphi(B) + \varphi(B)\varphi(A) \) and \( \varphi(A^\ast) = \varphi(A)^\ast \) for all \( A, B \in A \).

Given a \( C^* \)-algebra \( A \), the state space will be denoted by \( S_A \) with \( \omega, \rho \) denoting typical states on \( A \). At a slight variance with the more common practice we will use \( \mathcal{P}_A \) to denote merely the set of pure states of \( A \), rather than the weak\(^\ast\)-closure of this set. If \( B(\mathfrak{h}) \) is the space of bounded operators on some Hilbert space \( \mathfrak{h} \), a representation \( \pi : A \to B(\mathfrak{h}) \) of \( A \) as bounded operators on \( \mathfrak{h} \) will be denoted by \((\pi, \mathfrak{h})\). If indeed this is the representation canonically engendered by some state \( \omega \) via the Gelfand-Neumark-Segal construction, this will be indicated by a suitable subscript. If now \( A \) is a concrete \( C^* \)-algebra on some Hilbert space \( \mathfrak{h} \), a vector state \( A \to \mathbb{C} : A \mapsto (Ax, x) \) yielded by some norm-one vector \( x \in \mathfrak{h} \) will be denoted by \( \omega_x \).

When focussing specifically on von Neumann algebras, we will employ the notation \( M_1, M_2, \ldots \) for these algebras. Given a von Neumann algebra \( M \), its projection lattice will be denoted by \( \mathbb{P}_M \). In this context an orthoisomorphism between two
von Neumann algebras is understood to be a bijection between their respective projection lattices which preserves orthogonality of projections. More specifically given two von Neumann algebras \( M_1 \) and \( M_2 \), \( \psi : \mathcal{P}_{M_1} \to \mathcal{P}_{M_2} \) is an orthoisomorphism if it injectively maps \( \mathcal{P}_{M_1} \) onto \( \mathcal{P}_{M_2} \) and also satisfies the condition that
\[
EF = 0 \quad \text{if and only if} \quad \psi(E)\psi(F) = 0
\]
for all \( E, F \in \mathcal{P}_{M_1} \).

Given two \( C^* \)-algebras \( A \) and \( B \) each corresponding to some quantum mechanical system, it is natural to consider two such algebras to be physically equivalent if in some phenomenological sense they are the same. Based on the fact that the essential physical information of a quantum mechanical system is encoded in its set of possible states, Emch defined two representations \((\pi_1, \mathfrak{h}_1)\) and \((\pi_2, \mathfrak{h}_2)\) of a \( C^* \)-algebra \( A \) to be physically equivalent if indeed
\[
\{\omega \circ \pi_1 | \omega \in \mathcal{S}_{\pi_1(A)}\} = \{\omega \circ \pi_2 | \omega \in \mathcal{S}_{\pi_2(A)}\}
\]
(see [E], p 107). Taking this idea a step further, it seems natural to consider \( A \) and \( B \) to be physically equivalent if their respective state spaces are structurally identical. More precisely \( A \) and \( B \) are called physically equivalent if we can find an affine weak*-continuous bijection \( \varphi^\sharp \) from the state space of \( B \) onto that of \( A \). Such a map \( \varphi^\sharp \) will then be considered to be a map which preserves all relevant physical information in a change of contexts from \( A \) to \( B \). At the level of von Neumann algebras physical equivalence of two von Neumann algebras \( M_1 \) and \( M_2 \) may similarly be defined in terms of the structural similarity of their respective normal state spaces. In this setting the mappings which preserve all relevant physical information in a change of contexts from \( M_1 \) to \( M_2 \) (thereby ensuring the physical equivalence of these algebras) may be identified with affine bijections from the normal state space of \( M_2 \) onto that of \( M_1 \).

Our first task will be to show that various alternative ways of formalising this notion are in fact equivalent and invariably tend to lead one to a Jordan \(*\)-isomorphism between the algebras. To afford such results we need the following lemmas. The second lemma is of course well-known for \(*\)-isomorphisms but for our purposes the Jordan case needs to be included. Since both these lemmas will be needed again later on, we show how Lemma 1.1 may be used to reduce Lemma 1.2 to the \(*\)-isomorphism case.

**Lemma 1.1.** Let \( M_1 \) and \( M_2 \) be von Neumann algebras and let \( \varphi \) be a Jordan \(*\)-isomorphism from \( M_1 \) onto \( M_2 \). Then there exists a central projection \( E \in M_1 \cap M_1' \) such that \( \varphi(E) = F \) is a central projection in \( M_2 \cap M_2' \) for which \( F \varphi(\cdot)F = \varphi F \) defines a \(*\)-isomorphism from \( (M_1)_E \) onto \( (M_2)_F \) and \((\mathbb{1} - F)\varphi(\cdot)(\mathbb{1} - F) = \varphi(\mathbb{1} - F)\) a \(*\)-antiisomorphism from \( (M_1)_{(1 - E)} \) onto \( (M_2)_{(1 - F)} \).

**Proof.** The proof follows from an application of Kadison’s result [BRo, 3.2.2] regarding the existence of a projection \( F \in M_2 \cap (M_2)' \) such that \( \varphi_F \) is a \(*\)-homomorphism and \( \varphi(\mathbb{1} - F) \) a \(*\)- antimorphism from \( M_1 \) onto \( (M_2)_F \) and \((M_2)_{(1 - F)} \) respectively.

**Lemma 1.2.** Let \( M_1 \) and \( M_2 \) be von Neumann algebras and let \( \varphi \) be Jordan \(*\)-isomorphism from \( M_1 \) onto \( M_2 \). Then \( \varphi \) is \( \sigma \)-weakly and \( \sigma \)-strongly continuous.

**Proof.** Let \( E \) be as in the preceding lemma and suppose that \( \{A_\lambda\} \) converges \( \sigma \)-weakly (alternatively \( \sigma \)-strongly) to \( A_0 \) in \( M_1 \). But then \( \{A_\lambda E\} \) and \( \{A_\lambda(\mathbb{1} - E)\} \)
converge $\sigma$-weakly (alt. $\sigma$-strongly) to $A_0E$ and $A_0(1-E)$ respectively [BRo, 2.4.1 & 2.4.2]. By for example [BRo, 2.4.23] applied to the previous lemma it follows that each of $\varphi_F$ and $\varphi_I-F$ is $\sigma$-weakly (alt. $\sigma$-strongly) continuous on $(\mathcal{M}_1)_E$ and $(\mathcal{M}_1)_{1-E}$ respectively. Consequently $\{\varphi(A_\lambda)F\}$ and $\{\varphi(A_\lambda)(1-F)\}$ converge $\sigma$-weakly (alt. $\sigma$-strongly) to $\varphi(A_0)F$ and $\varphi(A_0)(1-F)$ respectively. Clearly $\{\varphi(A_\lambda)\}$ will then converge $\sigma$-weakly (alt. $\sigma$-strongly) to $\varphi(A_0)$. \hfill $\Box$

Besides establishing the framework for the results that follow, the following collection of known results has a lot of philosophical significance. From the point of view of the preservation of relevant physical information in a change of contexts from one algebra to another, this array of results suggests that in the category of von Neumann algebras not just the structural similarity of the state spaces (respectively normal state spaces) will guarantee the physical equivalence of two von Neumann algebras, but also the similarity of any one of their metric structures, their order structures, and (provided we exclude the case of the $2 \times 2$ matrices) even the similarity of their quantum propositional calculi. This result may also be interpreted in terms of Wigner-symmetries. In elementary quantum mechanics the states of a system may be taken to correspond to the one-dimensional subspaces of some Hilbert space $\mathfrak{h}$. Given two unit vectors $x, y \in \mathfrak{h}$, the transition probability between the corresponding states $\text{span}\{x\}$ and $\text{span}\{y\}$ is defined to be $|(x,y)|^2$. In this context Wigner [W] defined a symmetry to be a bijection on these states which preserves the transition probabilities between states, before going on to show that the class of all such symmetries corresponds canonically to the class consisting of both unitarily implemented $\ast$-automorphisms and anti-unitarily implemented $\ast$-anti-automorphisms on $B(\mathfrak{h})$ (or equivalently the class of all Jordan $\ast$-automorphisms on $B(\mathfrak{h})$ - see for example p 210 and Example 3.2.14 of [BRo]).

For a fuller account of the connection between Wigner symmetries and Jordan $\ast$-automorphisms the reader is referred to either p 210 of [BRo] or p 150 of [E]. In the light of the above discussion it therefore makes sense to consider Jordan $\ast$-isomorphisms from one $C^\ast$-algebra onto another as some sort of Wigner symmetry. With this in mind, the following result may be interpreted as saying that not just its action on the relevant state spaces serves to identify a transformation as a Wigner symmetry, but its action on various other structures as well.

**Theorem 1.3.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras. The set of all Jordan $\ast$-isomorphisms from $\mathcal{M}_1$ onto $\mathcal{M}_2$ is in a one to one correspondence with each of the following in the sense of each appearing as either a suitable restriction of a surjective Jordan $\ast$-isomorphism, or of the adjoint of such:

1. Affine bijections from the normal states of $\mathcal{M}_2$ onto the normal states of $\mathcal{M}_1$.
2. Weak$^\ast$ continuous affine bijections from the state space of $\mathcal{M}_2$ onto that of $\mathcal{M}_1$.
3. Linear identity-preserving isometries from $\mathcal{M}_1$ onto $\mathcal{M}_2$.
4. Linear identity-preserving order-isomorphisms from $\mathcal{M}_1$ onto $\mathcal{M}_2$.

If indeed the von Neumann algebra $\mathcal{M}_1$ contains no direct summand of type $I_2$, then the set of all orthoisomorphisms from $\mathcal{P}_{\mathcal{M}_1}$ onto $\mathcal{P}_{\mathcal{M}_2}$ is also in one to one correspondence with the set of surjective Jordan $\ast$-isomorphisms.

**Proof.** The one direction of statement (1) follows from the lemma. The other from a result of Kadison cf. [BRo, 3.2.8]. Statement (2) may be found on p 264 of [Stø2].
(see the discussion following Corollary 5.8). Statements (3) and (4) are contained in [BRo, 3.2.3]. The one direction of the final statement may be deduced from [L, 3.7 & 4.2] and the other from [D] (see p 83).

Yet another possible description can be had from investigating those pure state bijections which correspond to Jordan ∗-isomorphisms between the relevant algebras. From a physical point of view the importance of such a description is due to the fact that it demonstrates that there is enough information internally encoded in the pure states of a C∗-algebra (respectively von Neumann algebra) to either guarantee its physical equivalence to another algebra, or alternatively to be able to identify a pure state transformation as some sort of Wigner symmetry. The most impressive result in this regard seems to be a result of Shultz [Sh] which serves to characterise those pure state bijections which correspond to surjective ∗-isomorphisms. We review the relevant results before indicating possible extensions and improvements.

Following Shultz [Sh] we define the transition probability for pure states as follows:

**Definition 1.4.** Let $A$ be a C∗-algebra and let $\omega_0$ and $\omega_1$ be pure states. We call $\omega_0$ and $\omega_1$ unitarily equivalent if we can find a unitary element $U \in A$ such that $\omega_1(U^*(\cdot)U) = \omega_0(\cdot)$, and unitarily inequivalent if this is not possible. If $\omega_0$ and $\omega_1$ are unitarily inequivalent we define the transition probability from $\omega_0$ to $\omega_1$ to be zero. If $\omega_1$ and $\omega_2$ are unitarily equivalent, there exists an irreducible representation $(\pi, \mathcal{h})$ and unit vectors $x_0, x_1 \in \mathcal{h}$ such that $\omega_0 = \omega_{x_0} \circ \pi$ and $\omega_1 = \omega_{x_1} \circ \pi$ [KRi, 10.2.3 & 10.2.6]. In this case the transition probability from $\omega_0$ to $\omega_1$ is defined to be $|\langle x_0, x_1 \rangle|^2$. By for example [KRi, 10.3.7] this covers all possibilities.

In his analysis of pure state transformations Shultz used the notions of transition probabilities and orientation to characterise those pure state bijections which correspond to ∗-isomorphisms. The concept of orientation of the set of pure states is based on the fact that if two distinct pure states are unitarily equivalent the face of the state space they generate is a 3-ball, with the face being a line segment if they are inequivalent. Given this fact the notion of orientation is nothing but an extension of the fact each 3-ball can effectively be oriented in one of two ways using either a right-hand or left-hand set of axes. For further details the reader is referred to the paper of Shultz [Sh].

With regard to the following result, recall that the bidual $A^{**}$ of a C∗-algebra $A$ may be identified with the double commutant of its universal representation [KRi, 10.1.1]. With this in mind the atomic part of $A^{**}$ is then the direct sum of all the direct summands of $A^{**}$ which are type I factors. However unitarily equivalent pure states of $A$ generate equivalent irreducible representations [KRi, 10.2.3 & 10.2.6] and hence with regard to the GNS construction correspond to ∗-isomorphic copies of type I factors in the direct sum representation of $A^{**}$. If among these type I factors one “removes all duplicates”, the result is what is called the reduced atomic representation. This will be discussed in further detail later on.

**Theorem 1.5** ([Sh]). Let $A$ and $B$ be (not necessarily unital) C∗-algebras. A bijection $\varphi^\sharp$ from the set $P_B$ of pure states of $B$ onto $P_A$ is induced by a ∗-isomorphism $\varphi$ from the atomic part of $A^{**}$ onto the atomic part of $B^{**}$ in the sense that $\varphi^\sharp$ appears as a restriction of the adjoint of $\varphi$ if and only if $\varphi^\sharp$ preserves both transition probabilities and orientation. Moreover a bijection $\varphi^\sharp$ from $P_B \cup \{0\}$ onto $P_A \cup \{0\}$ with
\( \varphi^\sharp(0) = (0) \) is induced by a \(*\)-isomorphism \( \varphi \) from \( \mathcal{A} \) onto \( \mathcal{B} \) if and only if \( \varphi^\sharp \) is a uniform \( \sigma(\mathcal{B}^*, \mathcal{B})\)-\( \sigma(\mathcal{A}^*, \mathcal{A}) \) homeomorphism which preserves transition probabilities and orientation.

We will see that if in the above result one deletes the condition regarding the preservation of orientation the result is a corresponding relationship between pure state bijections and Jordan \(*\)-isomorphisms between the relevant algebras. More importantly we will indicate how one may use the result of Dye \([D]\) listed in Theorem 1.3 to show that if no irreducible representation of \( \mathcal{A} \) is of the form \( M_2(\mathbb{C}) \), then significantly less information is needed to get the same correspondences. In particular in this case we don’t need to know that all transition probabilities are preserved, but rather only the orthogonal ones. This complements and extends similar results by Cassinelli, et al \([CdVLL]\) who considered only the case \( \mathcal{A} = B(\mathfrak{h}) \).

More generally we will precisely describe those transformations between the pure state spaces of two \( C^* \)-algebras which in a canonical way correspond to linear mappings on the underlying algebras (with no assumptions of bijectivity). To achieve such a description we also investigate the relationship between transformations which behave well with regard to unitary equivalence and orthogonality of pure states. These results of course extend the work of Shultz \([Sh]\) who considered only the case of bijections, and focused on the preservation of transition probabilities rather than orthogonality. We point out that the results presented in this paper complement the characterisations of linear maps on \( C^* \)-algebras with pure state preserving adjoints given in \[\text{Stol}\] and \[\text{LMa}\] (\[\text{Stol}\] Theorem 5.6] a local version of this result and \[\text{LMa}\] Theorem 5] a global version). In the case of \( C(K) \) spaces (\( K \) compact Hausdorff) a linear operator \( \varphi : C(K) \to C(S) \) is called a composition operator if it is induced by some transformation \( T : S \to K \) in the sense that \( \varphi(f) = f \circ T \) for each \( f \in C(K) \). Keeping in mind that up to a homeomorphism \( K \) and \( S \) are effectively just the pure state spaces of \( C(K) \) and \( C(S) \), these cycles of results therefore effectively lay the groundwork for a noncommutative theory of composition operators on \( C^* \)-algebras.

2. Orthogonality of pure states

We analyse the concept of orthogonality of pure states, before proceeding to investigate transformations between pure state spaces that behave well with regard to orthogonality. Ultimately we will see that for such transformations it is precisely their behaviour with regard to orthogonality together with some continuity restrictions that determine whether they are induced by adjoints of linear maps on the underlying algebras or not.

**Definition 2.1.** Let \( \mathcal{A} \) be a \( C^* \)-algebra. Two states \( \omega_0 \) and \( \omega_1 \) are called *orthogonal* if \( \| \omega_0 - \omega_1 \| = \| \omega_0 \| + \| \omega_1 \| \), ie. \( \| \omega_0 - \omega_1 \| = 2 \).

**Remark 2.2.** We note that the concept of orthogonality used here is that presented in the book of Conway \([Co]\), and is slightly different to the concept of orthogonality presented in the book of Bratteli and Robinson \([BRG]\) being in some sense more geometrical in flavour. To illustrate the point note that for any two states \( \omega_0 \) and \( \omega_1 \) the fact that all positive functionals assume their norm at \( I \in \mathcal{A} \) ensures that \( \| \omega_0 + \omega_1 \| = \omega_0(I) + \omega_1(I) = \| \omega_0 \| + \| \omega_1 \| \). Hence for states the above definition boils down to a concept of orthogonality of \( \omega_0 \) and \( \omega_1 \) based on the fact that the vectors \( \omega_0 + \omega_1 \) and \( \omega_0 - \omega_1 \) have the same length.
The so-called reduced atomic representation of a C*-algebra will prove to be an important tool in establishing the main theorem of this and the next section. For the sake of clarity of exposition we therefore proceed to review the most important structural information regarding this representation. For further details the reader is referred to for example (Kri).

Remark 2.3. If (π, ℏ) is the reduced atomic representation of the C*-algebra A, then π is faithful and for some maximal set \{ (π_α, ℏ_α) \} of pairwise inequivalent irreducible representations, the representation π(A) ⊂ B(ℏ) is of the form π = ⊕_{α} π_α with ℏ = ⊕_{α} ℏ_α and π(A)'' = ⊕_{α} B(ℏ_α). (See [Kri 10.3.10] and the discussion preceding it.) From for example [Kri, 10.2.3 & 10.2.5] it is clear that the pure states of A correspond exactly to the norm-one vectors of ⊕_{α} ℏ_α of the form x = ⊕_{α} x_α where x_α = 0 for all but one α ∈ A. Now given a vector x_0 in ℏ_0 we will write \( x_b \) for the vector in ℏ = ⊕_{α} ℏ_α with the \( b^{th} \) coordinate precisely \( x_b \) and all other coordinates zero. Given pure states \( \omega_0 \) and \( \omega_1 \) corresponding to say \( x_0 \) and \( x_1 \), it is clear from [Kri 10.3.7] that \( \omega_0 \) and \( \omega_1 \) are disjoint if and only if \( x_0 \neq x_1 \) (ie. if and only if \( \omega_0 \) and \( \omega_1 \) are inequivalent). Finally considering Definition 2.4 in the context of the above construction, it is clear that the transition probability of two pure states \( \omega_0 \) and \( \omega_1 \) of A corresponding to say \( x_0 \) and \( x_1 \) is precisely \( |\langle x_0, x_1 \rangle|^2 \). We need the following facts regarding orthogonal pure states. Although probably known, the author is not aware of a concise and explicit statement of these facts, and therefore for the sake of completeness has elected to reproduce the proof locally.

**Proposition 2.4.** Let A be a C*-algebra and \( \omega_0 \) and \( \omega_1 \) pure states of A. Then the following are equivalent:

1. \( \omega_0 \) and \( \omega_1 \) are orthogonal;
2. \( \omega_0 \) and \( \omega_1 \) are either disjoint or there exists an irreducible representation \( (\pi_0, ℏ_0) \) of A and orthonormal vectors \( x_0, x_1 \in ℏ_0 \) such that
   \[\omega_0(\cdot) = \omega_{x_0} \circ \pi_0(\cdot) \quad \omega_1(\cdot) = \omega_{x_1} \circ \pi_0(\cdot)\];
3. if \( (\pi, ℏ) \) is the reduced atomic representation of A, we may find orthonormal vectors \( x_0, x_1 \in ℏ \) such that
   \[\omega_0(\cdot) = \omega_{x_0} \circ \pi(\cdot) \quad \omega_1(\cdot) = \omega_{x_1} \circ \pi(\cdot)\];
4. if \( (\pi, ℏ) \) is the reduced atomic representation of A and if by \( \tilde{\omega}_0 \) and \( \tilde{\omega}_1 \) we denote the unique normal extensions of \( \omega_0 \circ \pi^{-1} \) and \( \omega_1 \circ \pi^{-1} \) to all of \( \pi(A)'' \left[ \text{LMa Lemma 11} \right] \), there exists an orthogonal projection \( E \in \pi(A)'' \) such that
   \[\tilde{\omega}_0(E\pi(\cdot)E) = \omega_0, \quad \tilde{\omega}_0((I - E)\pi(\cdot)(I - E)) = 0\]
   and
   \[\tilde{\omega}_1(E\pi(\cdot)E) = 0, \quad \tilde{\omega}_1((I - E)\pi(\cdot)(I - E)) = \omega_1\]

**Proof.** Without loss of generality we may identify A with its reduced atomic representation.

(2) ⇔ (3): In the light of the information garnered from Remark 2.3 this is a straightforward exercise.

(3) ⇔ (4): Let \( \omega_0 \) and \( \omega_1 \) be pure states respectively corresponding to norm-one vectors \( x_{a_0} \) and \( x_{a_1} \) in the sense described in Remark 2.3.
Similarly and to \( \| \omega \| \) of \( \| \omega \| \) by condition (4). Since in general \( \| \omega \| \) and \( \| \omega \| \) are orthogonal projection \( E \in A'' \). Now since \( 2E - I \) is self-adjoint with \( (2E - I)^2 = I \), we clearly have \( \|2E - I\| = 1 \). But then
\[
\| \omega_0 - \omega_1 \| \geq (\omega_0 - \omega_1)(2E - I) = (\omega_0 - \omega_1)(E - (I - E)) = \omega_0(E) + \omega_1(I - E) = \omega_0(I) + \omega_1(I) = \|\omega_0\| + \|\omega_1\|
\]
by condition (4). Since in general \( \| \omega_0 - \omega_1 \| \leq \| \omega_0 \| + \| \omega_1 \| \), we conclude that \( \| \omega_0 - \omega_1 \| = \| \omega_0 \| + \| \omega_1 \| \).

(4) \Rightarrow (1): Let \( \omega_0, \omega_1 \) be pure states which satisfy condition (4) for some orthogonal projection \( E \in A'' \). Since by assumption \( \omega_0 \) and \( \omega_1 \) are normal, each of \( \omega_0, \omega_1 \) and \( \omega_0 - \omega_1 \) extends uniquely and without change of norm to \( A'' \) [KR10.1.11]. Now since \( 2E - I \) is self-adjoint with \( (2E - I)^2 = I \), we clearly have \( \|2E - I\| = 1 \). But then
\[
\| \omega_0 - \omega_1 \| \geq (\omega_0 - \omega_1)(2E - I) = (\omega_0 - \omega_1)(E - (I - E)) = \omega_0(E) + \omega_1(I - E) = \omega_0(I) + \omega_1(I) = \|\omega_0\| + \|\omega_1\|
\]
by condition (4). Since in general \( \| \omega_0 - \omega_1 \| \leq \| \omega_0 \| + \| \omega_1 \| \), we conclude that \( \| \omega_0 - \omega_1 \| = \| \omega_0 \| + \| \omega_1 \| \).

(1) \Rightarrow (2): We verify this implication for vector states of \( B(\mathfrak{h}) \) (\( \mathfrak{h} \) an arbitrary Hilbert space). By Remark 23 it is clear that this will suffice to establish the implication for the case of equivalent pure states of \( A \). Thus let \( x, y \in \mathfrak{h} \) be unit vectors, let \( \omega_0 = \omega_x \) and \( \omega_1 = \omega_y \), and assume that \( \|\omega_0 - \omega_1\| = 2 \). Now by replacing \( y \) with \( \tilde{y} = e^{i\theta}y \) if necessary where \( (x, y) = |\langle x, y \rangle|e^{i\theta} \), we may assume without loss of generality that \( \langle x, y \rangle = |\langle x, y \rangle| \). Since vector states are clearly \( \sigma \)-weakly continuous, \( \omega_0 \) and \( \omega_1 \) belong to the predual of \( B(\mathfrak{h}) \), and hence the Hahn-Banach theorem assures us that there exists \( A \in (B(\mathfrak{h}))^* = B(\mathfrak{h}) \) with
\[
\| A \| = 1 \quad \text{and} \quad \omega_0(A) - \omega_1(A) = \| \omega_0 - \omega_1 \| = 2.
\]
Now since \( |\omega_0(A)| \leq \| \omega_0 \| \| A \| = 1 \) and \( |\omega_1(A)| \leq 1 \), this can only be the case if
\[
\omega_0(A) = 1 \quad \text{and} \quad \omega_1(A) = -1.
\]
On bringing the Cauchy-Schwarz inequality into play we may now conclude that
\[
1 = \omega_0(A) = \langle Ax, x \rangle \leq \|Ax\| \| x \| \leq \| A \| \| x \|^2 = 1
\]
and hence that \( \langle Ax, x \rangle = \|Ax\| = \| x \| = 1 \). Consequently
\[
\| Ax - x \|^2 = \| Ax \|^2 - 2\Re(Ax, x) + \| x \|^2 = 0,
\]
ie. \( Ax = x \). Similarly \( Ay = -y \). But then
\[
\| x - y \| = \| A(x + y) \| \leq \| x + y \| = \| A(x - y) \| \leq \| x - y \|,
\]

that is \( \|x - y\| = \|x + y\| \). Squaring both sides and cancelling like terms reveals that \(-2\Re(x, y) = 2\Re(x, y)\). Since by assumption \((x, y) = |(x, y)|\), we conclude that \((x, y) = \Re(x, y) = 0\) as required. \(\square\)

**Definition 2.5.** Let \(A, B\) be \(C^*\)-algebras. A transformation \(\varphi\) from a subset of the pure states of \(B\) into the set of pure states of \(A\), is called **orthogonal** if for any two pure states \(\omega_0, \omega_1 \in \text{dom}(\varphi)\), we have that \(\varphi(\omega_0)\) and \(\varphi(\omega_1)\) are orthogonal whenever \(\omega_0\) and \(\omega_1\) are orthogonal.

If on the other hand \(\omega_0\) and \(\omega_1\) are orthogonal whenever \(\varphi(\omega_0)\) and \(\varphi(\omega_1)\) are orthogonal, we call \(\varphi\) co-orthogonal.

If \(\varphi\) is both orthogonal and co-orthogonal, it will be deemed to be bi-orthogonal.

**Definition 2.6.** Let \(A, B\) be \(C^*\)-algebras and let \(\varphi\) be a transformation from the set of pure states of \(B\) into the set of pure states of \(A\). For a pure state \(\omega\) the set of all other pure states unitarily equivalent to \(\omega\) will be denoted by \([\omega]\). We call \(\varphi\) **fibre-preserving** if \(\varphi([\omega]) \subseteq [\varphi(\omega)]\) for each pure state \(\omega\) of \(B\).

Given a map \(\varphi\) from \(P_B\) into \(P_A\), we say that a property holds **locally** for \(\varphi\) if it holds for the restriction of \(\varphi\) to each equivalence class \([\omega]\) of \(P_B\). For example we call \(\varphi\) locally injective if for each \(\omega \in P_B\) \(\varphi\) acts injectively on \([\omega]\), locally orthogonal if for each \(\omega \in P_B\), \(\varphi([\omega])\) is orthogonal, etc.

**Definition 2.7.** Let \(A, B\) be \(C^*\)-algebras and let \(\varphi\) be a fibre-preserving transformation from \(P_B\) into \(P_A\). For a subset \(V\) of say \(P_A\), we write \(V^\perp\) for the set

\[
V^\perp = \{ \omega \in P_A | \omega \perp \rho, \rho \in V \}.
\]

We call the range of \(\varphi\) **locally solid** if \(\varphi([\omega]) = \varphi([\omega])^{\perp \perp}\) for each \(\omega \in P_B\).

**Remark 2.8.** Let \(A\) be a commutative \(C^*\)-algebra. In this setting it follows from for example [KR] 4.4.1 that pure states \(\omega_0\) and \(\omega_1\) of \(A\) are unitarily equivalent if and only if \(\omega_0 = \omega_1\). Thus “equivalence classes” of pure states are just singletons. Since in general two pure states are inequivalent if and only if they are disjoint [KR] 10.2.3 & 10.3.7, we may apply Proposition 2.4(2) to the above fact to conclude that in the commutative setting pure states \(\omega_0\) and \(\omega_1\) are orthogonal if and only if \(\omega_0 \neq \omega_1\). Consequently any transformation between the pure state spaces of commutative \(C^*\)-algebras is automatically both co-orthogonal and fibre-preserving, with a transformation being orthogonal precisely when it is injective. The orthogonality of distinct pure states (point evaluations) in the case \(A = C(K)\), may also be deduced from Urysohn’s lemma. So in a very real sense the requirement of orthogonality in the non-commutative setting compensates for the lack of a suitable “non-commutative” Urysohn’s lemma.

Our primary task is of course to identify those transformations \(\varphi : P_B \to P_A\) that are induced by linear maps on the underlying algebras. Taking our cue from the commutative case elucidated above, a good place to start seems to be among the co-orthogonal fibre-preserving transformations. We therefore proceed to investigate the relationship between these two properties.

**Lemma 2.9.** Let \(A, B\) be \(C^*\)-algebras and \(\varphi\) a transformation from \(P_B\) into \(P_A\). Consider the following statements:

1. \(\varphi\) is co-orthogonal.
2. \(\varphi\) is locally co-orthogonal.
3. \(\varphi\) is fibre-preserving.
The implications (1) $\iff$ (2) $\implies$ (3) hold in general.

Proof. The implication (1) $\implies$ (2) is of course trivial. We therefore proceed to verify that (2) $\implies$ (3). Firstly assume that both $A, B$ are reduced atomically represented and that $\varphi^\sharp$ is locally co-orthogonal. Now let $\omega_0, \omega_1 \in \mathcal{P}_B$ be unitarily equivalent pure states. We show that $\varphi^\flat(\omega_0)$ and $\varphi^\flat(\omega_1)$ are then necessarily also unitarily equivalent. In the notation of Remark 2.3 $\omega$ Lemma 2.10.

2.4 is orthogonal to neither $\omega^\sharp$ must also be orthogonal. If $\omega_0$ and $\omega_1$ are orthogonal, then by the local co-orthogonality of $\varphi^\sharp$, neither are $\varphi(\omega_0)$ and $\varphi^\flat(\omega_1)$. Hence in this case $\varphi^\flat(\omega_0)$ and $\varphi^\flat(\omega_1)$ are equivalent. If however $\omega_0$ and $\omega_1$ are indeed orthogonal but unitarily equivalent pure states (ie. $x_a \perp y_a$), then $\omega_2 = \omega_{z_a}$ where $z_a = \frac{1}{\sqrt{2}}(x_a + y_a) \in \mathfrak{f}_a$ is a pure state of $B$ which by Proposition 2.3 is orthogonal to neither $\omega_0$, nor $\omega_1$. Again the local co-orthogonality of $\varphi^\sharp$ now ensures that $\varphi^\flat(\omega_2)$ is orthogonal to neither $\varphi^\flat(\omega_0)$ nor $\varphi^\flat(\omega_1)$. Thus both $\varphi^\flat(\omega_0)$ and $\varphi^\flat(\omega_1)$ are unitarily equivalent to $\varphi^\flat(\omega_2)$, and hence equivalent to each other.

It remains to show that (2) $\implies$ (1). To this end let $\varphi^\sharp$ be locally co-orthogonal and let $\omega_0$ and $\omega_1$ be pure states of $B$ with $\varphi^\flat(\omega_0)$ and $\varphi^\flat(\omega_1)$ orthogonal. If $\omega_0$ is unitarily equivalent to $\omega_1$, then by the local co-orthogonality of $\varphi^\sharp$, $\omega_0$ and $\omega_1$ must also be orthogonal. If $\omega_0$ is not unitarily equivalent to $\omega_1$, then we necessarily already have that $\omega_0$ and $\omega_1$ are orthogonal. Thus $\varphi^\sharp$ is co-orthogonal.

Again as in the commutative case there is a clear link between orthogonality and injectivity.

Lemma 2.10. Let $A, B$ and $\varphi^\sharp$ be as in the previous lemma. If $\varphi^\sharp$ is bi-orthogonal (alternatively locally bi-orthogonal), then it is injective (locally injective).

Proof. Due to the similarity of the proofs we prove only the first claim. Hence assume that both $A, B$ are reduced atomically represented and that $\varphi^\sharp$ is bi-orthogonal. Let $\omega_0, \omega_1 \in \mathcal{P}_B$ be two distinct pure states. If $\omega_0$ and $\omega_1$ are orthogonal, then by the bi-orthogonality of $\varphi^\sharp$, so are $\varphi^\flat(\omega_0)$ and $\varphi^\flat(\omega_1)$. Thus in this case $\varphi^\flat(\omega_0)$ and $\varphi^\flat(\omega_1)$ are clearly distinct. If now $\omega_0$ and $\omega_1$ are not orthogonal, they must of course necessarily be unitarily equivalent. In the notation of Remark 2.3, $\omega_0$ and $\omega_1$ are of the form $\omega_0 = \omega_{z_a}$ and $\omega_1 = \omega_{y_a}$ where for some fixed $a \in \mathbb{B}$ both $x_a$ and $y_a$ are unit vectors in $\mathfrak{f}_a$. Now by Proposition 2.3 the two vectors $x_a$ and $y_a$ are distinct but not orthogonal. Thus in the two-dimensional subspace spanned by these vectors we can find a third unit vector $z_a$ which is orthogonal to $x_a$ but not to $y_a$. By Proposition 2.3 this unit vector corresponds to pure state $\omega_2 \in \mathcal{P}_B$ which is unitarily equivalent to both $\omega_0$ and $\omega_1$, orthogonal to $\omega_0$, but not to $\omega_1$. Since $\varphi^\sharp$ is (locally) bi-orthogonal, it follows that $\varphi^\sharp(\omega_2)$ is orthogonal to $\varphi^\flat(\omega_0)$, but not to $\varphi^\flat(\omega_1)$. This can clearly only be the case if $\varphi^\flat(\omega_0)$ and $\varphi^\flat(\omega_1)$ are distinct. Thus $\varphi^\flat$ is injective.

We have already proposed co-orthogonality and the concomitant preservation of fibres as properties that may help to identify those pure state transformations that come from linear mappings on the underlying algebras. In addition to these some local properties will no doubt also be needed. However since in the commutative case (unitary) equivalence classes of pure states are just singletons, it is not so easy to use this case to formulate a conjecture regarding the necessary local properties. We therefore proceed to investigate the case of $B(\mathfrak{h})$ in order to get some idea of
what may be needed. Classically Wigner used preservation of transition probabilities to identify those vector state bijections which come from linear maps on $B(\mathfrak{h})$ ([W; cf. SH Theorem 1]). The explicit use of orthogonality to achieve the same end, seems to be a result due to Cassinelli, de Vito, et al [CdVLL]. We sketch a proof of their result before extending both these results to the most general (non-bijective) case possible. To prove the bijective case we need the following lemma. Although well-known, the author has not been able to find an explicit statement of the anti-isomorphic case, and hence has once again elected to reproduce the proof locally.

**Lemma 2.11.** Let $\varphi$ be a $*$-isomorphism or $*$-antiisomorphism from $B(\mathfrak{h})$ onto $B(\mathfrak{k})$. Then

$$Tr_\mathfrak{h} = Tr_\mathfrak{k} \circ \varphi.$$  

**Proof.** Without loss of generality assume that $\dim(\mathfrak{h}) = \infty$. First note that with $\varphi$ as above we may find a unitary operator $U : \mathfrak{k} \to \mathfrak{h}$ such that either

$$\varphi(A) = U^*AU \quad \text{for all } A \in B(\mathfrak{h})$$  

or

$$\varphi(A) = U^*c^*A^*cU \quad \text{for all } A \in B(\mathfrak{h})$$

where $c$ is the anti-unitary operator on $\mathfrak{h}$ induced by complex conjugation. To see this one may for example suitably adapt the proof of [BRo, Example 3.2.14]. (In this regard note that the map $\sigma_0$ in [BRo, Example 3.2.14] corresponds to transposition with respect to some orthonormal base and hence to all intents of purposes is of the form $\sigma_0(A) = c^*A^*c$.

Alternatively one may apply [LMa, Theorem 16.B(1)] to conclude that there exist injective partial isometries $U : \mathfrak{t} \to \mathfrak{h}$ and $V : \mathfrak{t} \to \mathfrak{h}$ such that either

$$\varphi(A) = V^*AU \quad \text{for all } A \in B(\mathfrak{h})$$  

or

$$\varphi(A) = V^*c^*A^*cU \quad \text{for all } A \in B(\mathfrak{h})$$

where $c$ is as before. (The surjectivity of $\varphi$ excludes part $B(2)$ of [LMa, Theorem 16.B] as a possibility.) To see that [LMa, Theorem 16.B(1)] is indeed applicable we may combine Lemma 1.2, Theorem 1.5 and [LMa, Corollary 20] to conclude that $\varphi^*$ maps the $\sigma$-weakly continuous extreme points of the dual ball of $B(\mathfrak{t})$ onto extreme points of the dual ball of $B(\mathfrak{h})$. Having obtained the above formulae one may then note that $\varphi(1) = \varphi(E)$ where $E$ is the range projection of $U$, and then conclude from the injectivity of $\varphi$ that $I = E$. Thus $U$ must in fact be surjective, and hence a unitary. It is then a simple matter to see that since $I = \varphi(1)$, we must then have that $V^* = V^*UU^* = \varphi(1)U^* = IU^* = U^*$.

Now for any set of orthonormal bases $\{x_\lambda\}_\Lambda$ and $\{y_\lambda\}_\Lambda$ of $\mathfrak{t}$, $\{\bar{x}_\lambda\}_\Lambda = \{cU(x_\lambda)\}_\Lambda$ and $\{\bar{y}_\lambda\}_\Lambda = \{cU(y_\lambda)\}_\Lambda$ are orthonormal bases of $\mathfrak{h}$. If therefore $\varphi$ is of the form
then
\[
\sum_{\lambda \in \Lambda} (A\bar{x}_\lambda, \bar{y}_\lambda) = \sum_{\lambda \in \Lambda} (AcU x_\lambda, cU y_\lambda) = U y_\lambda, c^* (AcU x_\lambda) \\
= \sum_{\lambda \in \Lambda} (y_\lambda, U^* c^* AcU x_\lambda) \\
= \sum_{\lambda \in \Lambda} (y_\lambda, \varphi(A^*) x_\lambda) \\
= \sum_{\lambda \in \Lambda} (\varphi(A) y_\lambda, x_\lambda)
\]

for each \(A \in B(h)\). In particular it then follows that \(\varphi(A)\) is trace-class whenever \(A\) is trace class, and that
\[
\operatorname{Tr}_h(A) = \sum_{\lambda \in \Lambda} (A\bar{x}_\lambda, \bar{x}_\lambda) = \sum_{\lambda \in \Lambda} (\varphi(A) x_\lambda, x_\lambda) = \operatorname{Tr}_k(\varphi(A))
\]
for each trace-class element of \(B(h)\). A similar conclusion holds if \(\varphi\) is of the form (2.1).

The one-dimensional subspaces of \(h\) are clearly in a one-one correspondence with the vector states of \(B(h)\). More precisely if \(x \in h\) is a unit vector and \(E_x\) the minimal projection onto the ray spanned by \(x\), then \(\omega_x = \operatorname{Tr}_h(E_x \cdot E_x)\). Thus with reference to Definition 1.4 and the discussion preceding Theorem 1.3, the following result is the first step towards showing that on condition we exclude the case of \(M_2(C)\), we don’t need to know that all transition probabilities are preserved in order to identify a pure state transformation as a Wigner symmetry. As noted earlier this result is of course known (see for example Cassinelli, de Vito, Lahti and Levrero [CdVLL]). We therefore content ourselves with merely sketching how this result may be deduced from the previous lemma by means of Dye’s result [D, p. 83]. We hasten to add that the dimensional restriction can not be removed as the implication may in fact fail in the two-dimensional case ([U]; cf [CdVLL, Example 4.1]).

**Theorem 2.12** ([CdVLL]). Let \(\varphi^\#\) be a bijection from the set of vector states of \(B(k)\) onto the set of vector states of \(B(h)\). If either \(\dim(h) \neq 2\) or \(\dim(k) \neq 2\), then \(\varphi^\#\) is bi-orthogonal in the sense that for any norm-one vectors \(x, y \in \ell\) and \(\vec{x}, \vec{y} \in h\) with
\[
\varphi^\#(\omega_x) = \omega_{\vec{x}} \quad \text{and} \quad \varphi^\#(\omega_y) = \omega_{\vec{y}},
\]
we always have that
\[
x \perp y \quad \text{if and only if} \quad \vec{x} \perp \vec{y}
\]
if and only if there exists either a \(*\)-isomorphism or a \(*\)-antiisomorphism \(\varphi\) from \(B(h)\) onto \(B(k)\) such that
\[
\varphi^\#(\omega_x) = \omega_x \circ \varphi \quad \text{for each norm-one} \quad x \in \ell.
\]

**Proof.** Let \(\varphi^\#\) be a bijection between the respective sets of vector states satisfying the stated hypothesis and suppose that \(B(h) \neq M_2(C)\). We show that \(\varphi^\#\) canonically induces a bijection \(\varphi\) from the rank-one orthogonal projections of \(B(h)\) onto those of \(B(\ell)\) in a way that “preserves orthogonality”, before proceeding to show
that in fact \( \varphi \) extends to an orthoisomorphism from \( \mathbb{P}_B(h) \) onto \( \mathbb{P}_B(t) \). Now for any norm-one vector \( x \in \mathfrak{t} \) we will write \( \tilde{x} \) for the corresponding norm-one vector in \( h \) such that \( \varphi^2(\omega_x) = \omega_{\tilde{x}} \). Given a norm-one vector \( x \in \mathfrak{t} \), we set \( \varphi(E_{\tilde{x}}) = E_x \).

(Here \( E_x \) and \( E_{\tilde{x}} \) are the orthogonal projections onto \( \text{span}\{x\} \) and \( \text{span}\{\tilde{x}\} \) respectively.) It is now an exercise to show that the fact that \( E_{\tilde{x}}E_{\tilde{y}} = 0 \) if and only if \( \varphi(E_{\tilde{x}})\varphi(E_{\tilde{y}}) = 0 \) is inherited from the so-called bi-orthogonality of \( \varphi^4 \). To see that \( \varphi \) extends to an orthoisomorphism from \( \mathbb{P}_B(h) \) onto \( \mathbb{P}_B(t) \) we need only show that by means of \( \varphi^2 \) we may identify the closed linear subspaces of \( h \) with those of \( t \) in a one-to-one way that preserves mutual orthogonality of subspaces. To this end let \( h_0 \) be a closed linear subspace of \( h \) and let \( \{\tilde{x}_\lambda\}_A \subset h_0 \) be a set of unit vectors such that \( h_0 = \overline{\text{span}}\{\tilde{x}_\lambda\}_A \). Now select \( \{x_\lambda\}_A \subset t \) so that \( \varphi^2(\omega_{x_\lambda}) = \omega_{\tilde{x}_\lambda} \) for each \( \lambda \in A \). Now with \( \mathfrak{t}_0 = \overline{\text{span}}\{x_\lambda\}_A \), let \( \varphi(E_{h_0}) = E_{\mathfrak{t}_0} \). (Here \( E_{h_0} \) and \( E_{\mathfrak{t}_0} \) respectively denote the orthogonal projections onto \( h_0 \) and \( \mathfrak{t}_0 \).) We show that \( E_{\mathfrak{t}_0} \) is uniquely defined and that \( \varphi(E_{h_0^\perp}) = E_{\mathfrak{t}_0^\perp} \).

Let \( \{z_\mu\} \) be an orthonormal base for \( h_0^\perp \) with as before \( \{z_\mu\} \) selected so that \( \varphi^2(\omega_{z_\mu}) = \omega_{\tilde{z}_\mu} \) for each \( \mu \). By the hypothesis \( \{z_\mu\} \) is again an orthonormal system with \( \{x_\lambda|\lambda \in A\} \subset \{z_\mu\} \) and hence \( \mathfrak{t}_0 \subset \{z_\mu\}^\perp \). To achieve our stated objective we only need to show that in fact \( \mathfrak{t}_0 = \{\{z_\mu\}\}^\perp \). (The uniqueness of \( \mathfrak{t}_0 \) then follows from the fact that \( \varphi^2 \) identifies any set of norm-one vectors generating \( \{\{z_\mu\}\}^\perp \) with a set of norm-one vectors generating \( \{\{z_\mu\}\} \). By interchanging the roles of \( h_0 \) and \( h_0^\perp \) it then also follows from this that \( \varphi^2 \) identifies \( h_0^\perp \) with \( \mathfrak{t}_0^\perp \).)

Since \( \mathfrak{t}_0 \subset \{\{z_\mu\}\}^\perp \), it suffices to show that \( \mathfrak{t}_0^\perp \cap \{\{z_\mu\}\} = \{0\} \) in order to see that \( \mathfrak{t}_0 = \{\{z_\mu\}\}^\perp \). Now if it were possible to find a norm-one vector

\[
y \in \mathfrak{t}_0^\perp \cap \{\{z_\mu\}\} = \{x_\lambda|\lambda \in A\} \cap \{z_\mu\} \quad \text{then for } \tilde{y} \in h \text{ with } \varphi^2(\omega_{\tilde{y}}) = \omega_{\tilde{y}} \text{ we would by the hypothesis have to have}
\]

\[
\tilde{y} \in \{\tilde{x}_\lambda|\lambda \in A\} \cap \{\tilde{z}_\mu\} = h_0^\perp \cap h_0^\perp \quad \text{a situation which is clearly impossible. Thus as claimed } \varphi \text{ canonically extends to an orthoisomorphism.}
\]

By a result of Dye [10, p. 83] \( \varphi \) then extends further to either a *-isomorphism or *-antisomorphism from \( B(h) \) onto \( B(t) \). This extension will also be denoted by \( \varphi \). It remains to show that \( \varphi^2 \) appears as a restriction of the adjoint of \( \varphi \). To this end let \( x \in \mathfrak{t} \) be a norm-one vector, and \( \tilde{x} \) the related norm-one vector in \( h \). By construction we have that \( \varphi(E_{\tilde{x}}) = E_x \). It is an exercise to conclude that \( \omega_x(\cdot) = Tr_\mathfrak{t}(E_x \cdot E_x) \) and \( \omega_{\tilde{x}}(\cdot) = Tr_h(E_{\tilde{x}} \cdot E_{\tilde{x}}) \). By making use of the previous lemma we may now conclude that

\[
\varphi^2(\omega_x)(A) = \omega_{\tilde{x}}(A) = Tr_h(E_{\tilde{x}}AE_{\tilde{x}}) = Tr_h \circ \varphi^{-1}(\varphi(E_{\tilde{x}})\varphi(A)\varphi(E_{\tilde{x}})) = Tr_\mathfrak{t}(E_x\varphi(A)E_x) = \omega_x(\varphi(A)) \quad \text{for each } A \in B(h).
\]

For the converse note that as in the previous lemma, we may show that \( \varphi \) is either of the form

\[
\varphi(A) = U^*AU \quad \text{for all } A \in B(h)
\]
or
\[ \varphi(A) = U^* c^* A^* c U \quad \text{for all} \quad A \in B(\mathfrak{h}), \]
where \( c \) is as before and \( U : \mathfrak{k} \to \mathfrak{h} \) is a unitary. Using this description it is a simple matter to verify the claim regarding the bi-orthogonality of \( \varphi^\dagger \). The result follows. \( \square \)

Let \( \varphi^\dagger \) be a transformation from the set of vector states of \( B(\mathfrak{k}) \) into the set of vector states of \( B(\mathfrak{h}) \). We recall from the discussion preceding Theorem 2.12 that the vector states are all states of the form \( \omega = Tr(\rho_0 \cdot E_0) \) for some minimal projection. By analogy with the earlier definitions for pure states we say that

\[ \varphi(\mathfrak{h}) = \{ E_0 | \rho = \varphi^\dagger(\omega), \omega \text{ a vector state of } B(\mathfrak{k}) \} \]

there exists a vector state \( \omega_0 \) of \( B(\mathfrak{k}) \) such that \( \varphi^\dagger(\omega_0) = Tr_{\mathfrak{h}}(E_0 \cdot E_0) \).

At the Hilbert space level the set \( \mathcal{E}_0 = \{ E_0 | \rho = \varphi^\dagger(\omega), \omega \text{ a vector state of } B(\mathfrak{k}) \} \) is of course nothing but the closure of \( \mathfrak{h}_0 = \text{span}\{ x \in \mathfrak{h} | \varphi^\dagger(x) = \varphi^\dagger(\omega), \omega \text{ a vector state of } B(\mathfrak{k}) \} \). In this context local solidness is then nothing more than the claim that any unit vector \( x \in \mathfrak{h}_0 \) must in fact belong to \( \mathfrak{h}_0 \), that is that \( \mathfrak{h}_0 \perp \perp = \mathfrak{h}_0 \).

**Theorem 2.13 (Generalised Wigner Theorem).** Let \( \varphi^\dagger \) be a transformation from the set of vector states of \( B(\mathfrak{k}) \) into the set of vector states of \( B(\mathfrak{h}) \). Consider the following statements:

1. There exists a linear map \( \varphi : B(\mathfrak{h}) \to B(\mathfrak{k}) \) such that \( \varphi^\dagger(\omega_x) = \omega_x \circ \varphi \) for every vector state of \( B(\mathfrak{k}) \).
2. \( \varphi^\dagger \) preserves transition probabilities and has a locally solid range.
3. \( \varphi^\dagger \) is bi-orthogonal (in the sense defined in the previous theorem) and has a locally solid range.

The implications (1) \( \iff \) (2) \( \implies \) (3) hold in general with all three statements being equivalent if \( \dim(\mathfrak{k}) \neq 2 \).

Moreover any linear map \( \varphi : B(\mathfrak{h}) \to B(\mathfrak{k}) \) which induces a vector state transformation \( \varphi^\dagger \) in the manner described above is either of the form

\[ \varphi(A) = U^* A U \quad \text{for all} \quad A \in B(\mathfrak{h}) \]

or

\[ \varphi(A) = U^* c^* A^* c U \quad \text{for all} \quad A \in B(\mathfrak{h}) \]

where \( c \) is the anti-unitary operator on \( \mathfrak{h} \) induced by complex conjugation and \( U \) a linear isometry from \( \mathfrak{k} \) into \( \mathfrak{h} \).

**Proof.** (1) \( \implies \) (2): Suppose we can find such a linear map \( \varphi : B(\mathfrak{h}) \to B(\mathfrak{k}) \) with

\[ \varphi^\dagger(\omega_x) = \omega_x \circ \varphi \]

for every vector state of \( B(\mathfrak{k}) \). Then \( \varphi \) is clearly positivity preserving (since \( A \in B(\mathfrak{h})^+ \) then implies that \( \langle \varphi(A)x, x \rangle \geq 0 \) for each \( x \in \mathfrak{k} \) and hence bounded. It now easily follows from the hypothesis that the dual \( \varphi^* \) will map the normal states of \( B(\mathfrak{k}) \) (the norm-closed convex hull of the vector states \([\text{K&R} \ 7.1.12 \ & 7.1.13]\)) into the normal states of \( B(\mathfrak{h}) \). On regarding say \( B(\mathfrak{k}) \), as a subspace of \( B(\mathfrak{h}) \), this means that \( \varphi^*(B(\mathfrak{k})^+) \subset B(\mathfrak{h})^+ \), and hence that \( \varphi^* : B(\mathfrak{k})^+ \to B(\mathfrak{h})^+ \) restricts to a map from \( B(\mathfrak{k})_+ \) into \( B(\mathfrak{h})_+ \). It is now an exercise to show that the dual of this induced map is exactly \( \varphi \). Therefore \( \varphi \) is in fact a dual operator and hence
necessarily weak* to weak* continuous. We may therefore apply [St67 Lemma 5.4] to see that there exists a linear isometry $U$ from $\mathfrak{t}$ into $\mathfrak{h}$ such that either
\begin{equation}
\varphi(A) = U^*AU \quad \text{for all} \quad A \in B(\mathfrak{h})
\end{equation}

or
\begin{equation}
\varphi(A) = U^*c^*A^*cU \quad \text{for all} \quad A \in B(\mathfrak{h})
\end{equation}

where $c$ is the anti-unitary operator on $\mathfrak{h}$ induced by complex conjugation. (As was shown in [LAMa Theorem 5] the case where $\mathfrak{t} = \mathbb{C}$ and $\varphi$ is a vector (pure) state also reduces to the form (2.5) above. Just select $x_\varphi \in \mathfrak{h}$ so that $\varphi = (\cdot x_\varphi, x_\varphi)$ and define $U$ by $1 \rightarrow x_\varphi$.)

If $\varphi$ is of the form (2.6) above it may be regarded as a map generated by the $*$-anti-automorphism $A \mapsto c^*A^*c$ on $B(\mathfrak{h})$ composed with the map $B(\mathfrak{h}) \rightarrow B(\mathfrak{t}): B \mapsto U^*BU$. Now by ([W]; cf. [SH Theorem 1]) the dual of the $*$-anti-automorphism $A \mapsto c^*A^*c$ yields a bijection on the vector states of $B(\mathfrak{h})$ which preserves transition probabilities. Thus all that remains is to show that a map of the form (2.5) above preserves transition probabilities and has locally solid range. Therefore assume that $\varphi$ is of this form. But then $\varphi^*$ will map a vector state
\[ \omega(\cdot) = (\cdot x_\omega, x_\omega) \quad x_\omega \in \mathfrak{t} \]
on to
\[ \varphi^*(\omega)(\cdot) = (U^* \cdot x_\omega, x_\omega) = (Ux_\omega, Ux_\omega). \]

The injectivity of $U$ ensures that $U^*U = I$ and hence for any two unit vectors $x, y \in \mathfrak{t}$ we have that
\[ |(x, y)|^2 = ||(U^*Ux, y)||^2 = ||(Ux, y)||^2. \]

It therefore clearly follows that $\varphi^*$ preserves transition probabilities.

Next let $E_\varphi = UU^*$ (the projection onto $U(\mathfrak{t})$). Since $U^*$ restricts to a linear isometry from $E_\varphi(\mathfrak{h}) = U(\mathfrak{t})$ onto $\mathfrak{t}$, it is an exercise to show that $A \mapsto U^*AU$ and $A \mapsto U^*c^*A^*cU$ respectively induce an onto $*$-isomorphism and an onto $*$-anti-isomorphism from $B(E_\varphi(\mathfrak{h}))$ onto $B(\mathfrak{t})$. Thus any map $\varphi$ of the form (2.6) above may be written as
\begin{equation}
\varphi = \psi(E_\varphi \cdot E_\varphi)
\end{equation}
where $\psi$ is a $*$-isomorphism from $B(E_\varphi(\mathfrak{h}))$ onto $B(\mathfrak{t})$. Now as we noted in the discussion preceding this theorem, at the Hilbert space level the set $\{E_\rho| \rho = \varphi^*(\omega), \omega \text{ a vector state of } B(\mathfrak{t})\}$ corresponds to nothing more than the closure of $\mathfrak{h}_0 = \text{span}\{\bar{x} \in \mathfrak{h}|\omega_{\bar{x}} = \varphi^*(\omega), \omega \text{ a vector state of } B(\mathfrak{t})\}$. However $\text{span}\{\bar{x} \in \mathfrak{h}|\omega_{\bar{x}} = \varphi^*(\omega), \omega \text{ a vector state of } B(\mathfrak{t})\}$ is of course precisely $E_{\varphi}(\mathfrak{h}) = U(\mathfrak{t})$, and hence we may conclude that here
\[ E_\varphi = \{E_\rho| \rho = \varphi^*(\omega), \omega \text{ a vector state of } B(\mathfrak{t})\}. \]

Now since $\psi$ is a bijective $*$-isomorphism we may apply ([W]; cf. [SH Theorem 1]) to conclude that the dual $\psi^*$ maps the set of vector states of $B(\mathfrak{t})$ onto that of $B(E_\varphi\mathfrak{h})$. Now any minimal projection $E_0$ of $B(\mathfrak{h})$ with $E_0 \leq E_\varphi$ is of course also a minimal projection of $B(E_\varphi\mathfrak{h})$. Hence from what we have just shown, for such an $E_0$ we can always find a minimal projection $F_0 \in B(\mathfrak{t})$ with
\[ \psi^*(\text{Tr}(F_0 \cdot F_0)) = \text{Tr}_{E_\varphi\mathfrak{h}}(E_0 \cdot E_0), \]
and hence by (2) above
\[ \varphi^* (Tr_b(F_0 \cdot F_0)) = Tr_b(E_0 E_\varphi \cdot E_\varphi E_0) = Tr_b(E_0 \cdot E_0). \]
Thus \( \varphi^* \) has a locally solid range.

(2) ⇒ (1): Let \( \varphi^* \) be a transformation from the vector states of \( B(\mathfrak{h}) \) into the vector states of \( B(\mathfrak{t}) \) which preserves transition probabilities and has a locally solid range. By Lemma 2.10, \( \varphi^* \) is injective. Now let
\[ E_{\varphi} = \{ E_\rho | \rho = \varphi^*(\omega), \omega \text{ a vector state of } B(\mathfrak{t}) \}. \]
We show that the dual of the map \( W_{E_{\varphi}} : B(\mathfrak{h}) \to B(E_{\varphi} \mathfrak{h}) : A \mapsto E_{\varphi}AE_{\varphi} \) bijectively maps the vector state space of \( B(E_{\varphi} \mathfrak{h}) \) onto the range of \( \varphi^* \) in a way that preserves transition probabilities. From this it then follows that \( \varphi^* \) may be written as a bijection, say \( \psi^* \), from the vector state space of \( B(\mathfrak{t}) \) onto that of \( B(E_{\varphi} \mathfrak{h}) \), composed with a restriction of the dual of \( W_{E_{\varphi}} \). The claim will then follow from an application of \( (\mathbb{W} ; \text{ cf. Sh} \) Theorem 1) to \( \psi^* \).

For any minimal projection \( E_0 \) of \( B(\mathfrak{h}) \) majorised by \( E_{\varphi} \), we have by hypothesis that \( Tr_{\mathfrak{h}}(E_0 \cdot E_0) \) corresponds to an element of the range of \( \varphi^* \). Moreover all elements of the range of \( \varphi^* \) are of this form. These projections of course correspond exactly to the minimal projections of \( B(E_{\varphi} \mathfrak{h}) \). So for each such a minimal projection \( E_0 \in B(E_{\varphi} \mathfrak{h}) \), the dual of \( W_{E_{\varphi}} \) will map the vector state \( Tr_{E_{\varphi} \mathfrak{h}}(E_0 \cdot E_0) \) of \( B(E_{\varphi} \mathfrak{h}) \) onto the corresponding element \( Tr_{\mathfrak{h}}(E_0 \cdot E_0) \in B(\mathfrak{h}) \) of the range of \( \varphi^* \). Finally note that for any two minimal projections \( E, F \in B(\mathfrak{h}) \) with \( E, F \leq E_{\varphi} \), we clearly have
\[ Tr_{E_{\varphi} \mathfrak{h}}(EF) = Tr_{\mathfrak{h}}(E_{\varphi} EF E_{\varphi}) = Tr_{\mathfrak{h}}(EF). \]
This fact can be shown to be equivalent to the statement that \( W_{E_{\varphi}} \) preserves transition probabilities. The claim regarding \( W_{E_{\varphi}} \) therefore follows.

(3) ⇒ (1): The proof of this implication is very similar to the proof of (2) ⇒ (1) with the main difference being that here we use the previous theorem instead of \( (\mathbb{W} ; \text{ cf. Sh} \) Theorem 1). We therefore leave this as an exercise. \( \square \)

Based on the information garnered from the above result, a good place to start searching for those transformations \( \varphi^* : \mathcal{P}_B \to \mathcal{P}_A \) that are induced by linear maps on the underlying algebras, would be among the co-orthogonal transformations that are locally bi-orthogonal with a locally solid range. However for now we defer such matters to the next section and content ourselves with the following observation:

**Corollary 2.14.** Let \( A, B \) be \( C^* \)-algebras and \( \varphi^* \) a transformation from \( \mathcal{P}_B \) into \( \mathcal{P}_A \). If no irreducible representation of \( B \) is of the form \( M_2(\mathbb{C}) \) and if \( \varphi^* \) is fibre-preserving with locally solid range, then \( \varphi^* \) is locally bi-orthogonal if and only if \( \varphi^* \) locally preserves transition probabilities

**Proof.** The “if” part is of course trivial and we therefore indicate how the “only if” part may be verified. So suppose that \( \varphi^* \) is fibre-preserving, locally bi-orthogonal, with locally solid range. Let \( \omega \in \mathcal{P}_B \) be given. By our supposition \( \varphi^* \) then restricts to a bi-orthogonal transformation from \( [\omega] \) into \( [\varphi^* (\omega)] \). By Remark 2.8 the equivalence classes \( [\omega] \) and \( [\varphi^* (\omega)] \) respectively correspond to the set of vector states of some \( B(\mathfrak{t}) \) and \( B(\mathfrak{h}) \), with \( B \) and \( A \) irreducibly represented on these two algebras. Call the induced vector state map \( \psi^* \). Now by hypothesis \( B(\mathfrak{t}) \neq M_2(\mathbb{C}) \). So to be able to deduce the corollary from the preceding theorem, all we need to do is to show that the local solidness of the range of \( \varphi^* \) is enough to ensure the local
solidness of the range of $\psi^\dagger$ as defined in the discussion preceding this theorem. This in turn is an exercise depending on Remark 2.3 and Proposition 2.4. □

3. Pure state transformations induced by linear mappings

We remind the reader that our primary challenge in this section is to use the preceding results to identify those transformations $\varphi^\dagger : \mathcal{P}_B \to \mathcal{P}_A$ that are induced by linear maps on the underlying algebras. We will do this in three phases. Suppose that both $\mathcal{A}$ and $\mathcal{B}$ are reduced atomically represented. Our first cycle of results will focus on identifying those pure state transformations $\varphi^\dagger$ that correspond to linear maps from $\mathcal{A}''$ into $\mathcal{B}''$. In the second cycle we will show that under the assumption that $\mathcal{A}$ and $\mathcal{B}$ are reduced atomically represented, linear maps from $\mathcal{A}$ into $\mathcal{B}$ that have pure state preserving duals, live naturally in the class of linear maps from $\mathcal{A}'''$ into $\mathcal{B}'''$. Hence the correspondence established in the first cycle is therefore a good place to start in searching for a solution to our primary challenge. In the third and final cycle we will strengthen the correspondence obtained in the first cycle by means of the introduction of some additional continuity restrictions on $\varphi^\dagger$ to finally obtain the solution to our problem.

Following a necessary technical lemma, we proceed to describe those pure state transformations $\varphi^\dagger$ that correspond to linear maps from $\mathcal{A}''$ into $\mathcal{B}''$.

**Lemma 3.1.** Suppose that $\mathcal{A}$ is a $C^*$-algebra which is reduced atomically represented. Then the $\sigma$-weakly continuous pure states of $\mathcal{A}''$ canonically correspond to the pure states of $\mathcal{A}$.

**Proof.** The one direction of this correspondence follows from [LMa, Lemma 4] and the necessary normality ($\sigma$-weak continuity) of the pure states of $\mathcal{A}$ in this representation (see Remark 2.3). For the converse recall that any normal ($\sigma$-weakly continuous) state $\omega$ of $\mathcal{A}''$ is of the form $\omega = \sum_{k=1}^\infty \lambda_k x_k$ where $\lambda_k \geq 0$ for each $k$, $\sum_{k=1}^\infty \lambda_k = 1$, and $\{x_k\} \in \mathcal{B}^\prime = \oplus_{a \in \mathcal{A}} a^\prime a$ is an orthonormal sequence [KR] 7.1.12. If therefore $\omega$ is to be an extreme point of the state space of $\mathcal{A}''$, it may not be a convex combination of distinct vector states and hence must itself be a vector state, i.e. $\lambda_k = 0$ for all but one $k$. Thus suppose that $\omega = \omega_x$ for some norm-one $x \in \oplus_{a \in \mathcal{A}} a^\prime a$. Such an $x$ is of course of the form $x = \oplus_{a \in \mathcal{A}} \mu_a x_a = \oplus_{a \in \mathcal{A}} \mu_a \hat{x}_a$ with $||\hat{x}_a|| = 1$ for each $a \in \mathcal{A}$ and $\sum_{a \in \mathcal{A}} |\mu_a|^2 = 1$. Now since each $\hat{x}_a$ is orthogonal to all the elements of $\mathcal{B}$ for which the $a^{th}$ coordinate is zero, a careful consideration of the action of $\omega = \omega_x$ on $\mathcal{A}'' = \oplus_{a \in \mathcal{A}} B(\mathcal{h}_a)$ reveals that it may be written in the form

$$\omega_x = \oplus_{a \in \mathcal{A}} |\mu_a|^2 \omega_{x_a} = \sum_{a \in \mathcal{A}} |\mu_a|^2 \omega_{\hat{x}_a}.$$  

As before it is now clear that if $\omega = \omega_x$ is to be an extreme point of the state space of $\mathcal{A}''$, we must have that $\mu_a = 0$ for all but one $a \in \mathcal{A}$. From the discussion in Remark 2.3 it is now clear that such an $\omega$ canonically corresponds to a pure state of $\mathcal{A}$. □

**Theorem 3.2.** Let $\mathcal{A}, \mathcal{B}$ be $C^*$-algebras with $(\pi_\mathcal{A}, \mathcal{h}_\mathcal{A})$ and $(\pi_\mathcal{B}, \mathcal{h}_\mathcal{B})$ their respective reduced atomic representations, and let $\varphi^\dagger$ be a transformation from $\mathcal{P}_B$ into $\mathcal{P}_A$. Consider the following statements:

1. There exists a linear map $\bar{\varphi} : \pi_\mathcal{A}(\mathcal{A}''') \to \pi_\mathcal{B}(\mathcal{B})''$ such that $\varphi^\dagger(\omega) = \omega \circ \bar{\varphi}$ for every pure state $\omega \in \mathcal{P}_B$. 

(1)
(2) \( \varphi^\sharp \) is fibre-preserving, locally preserves transition probabilities, and has a locally solid range.

(3) \( \varphi^\sharp \) is locally bi-orthogonal (and hence fibre-preserving by Lemma 2.13) and has a locally solid range.

The implications (1) \( \Leftrightarrow \) (2) \( \Rightarrow \) (3) hold in general with all three statements being equivalent if no irreducible representation of \( B \) is of the form \( M_2(\mathbb{C}) \).

**Proof.** The implication (2) \( \Rightarrow \) (3) is obvious in the light of the results in section two. In addition the fact that (3) \( \Rightarrow \) (2) under the assumption that no irreducible representation of \( B \) is of the form \( M_2(\mathbb{C}) \), is a direct consequence of Lemma 2.9 and Corollary 2.13. Thus we need only prove that (1) \( \Leftrightarrow \) (2).

(1) \( \Rightarrow \) (2): Suppose that (1) holds. For the sake of simplicity we may of course assume that both \( A \) and \( B \) are reduced atomically represented. Then \( A'' \) and \( B'' \) are of the form \( \oplus_{a \in A} B(h_a) \) and \( \oplus_{b \in B} B(t_b) \) respectively.

We first show that \( \hat{\varphi} \) is necessarily a contractive adjoint preserving map. To this end let \( A = A^* \in \oplus_{a \in A} B(h_a) \) be given. A typical \( \sigma \)-weakly continuous pure state \( \omega \) of \( B'' \) is of course of the form \( (\hat{z}_b, \hat{z}_b) \) for some unit vector \( \hat{z}_b \in \oplus_{b \in B} t_b \) (corresponding to some \( z_b \in t_b \) – see Remark 2.3). The transformation \( \varphi^\sharp \) will then map this pure state onto a similar looking \( \sigma \)-weakly continuous pure state \( \rho \) of \( A'' \); say \( (\hat{x}_{a''}, \hat{x}_{a''}) \) where \( x_{a''} \in h_{a''} \). Then by the assumption on \( \varphi \) we have that

\[
(\hat{\varphi}(A)\hat{z}_b, \hat{z}_b) = (A\hat{x}_{a''}, \hat{x}_{a''}) \in \mathbb{R}
\]

with

\[
\| (\hat{\varphi}(A)\hat{z}_b, \hat{z}_b) \| \leq \| A \|.
\]

Since \( b_0 \in B \) and \( z_{b_0} \in t_{b_0} \) was arbitrary, this suffices to prove that \( \hat{\varphi} \) preserves adjoints with \( \| \hat{\varphi} \| \leq 1 \). In particular since \( \varphi^\sharp \) is induced by the dual of \( \hat{\varphi} \), we then have that

\[
\| \varphi^\sharp(\omega_0) - \varphi^\sharp(\omega_1) \| \leq \| \omega_0 - \omega_1 \|.
\]

Thus \( \varphi^\sharp \) must then be co-orthogonal (since \( \| \omega_0 - \omega_1 \| < 2 \Rightarrow \| \varphi^\sharp(\omega_0) - \varphi^\sharp(\omega_1) \| < 2 \)) and hence fibre-preserving by Lemma 2.14.

It remains to show that \( \varphi^\sharp \) locally preserves transition probabilities and that it has a locally solid range. To this end let \( \omega_0 = (\hat{z}_b, \hat{z}_b) \) be as before and consider the action of \( \hat{\varphi}^* \) on [\( \omega_0 \)]. Recall that [\( \omega_0 \)] corresponds to the vector states of \( B(t_{b_0}) \) in that the members of [\( \omega_0 \)] are precisely the vector states of the form \( (\hat{y}_b, \hat{y}_b) \) for some unit vector \( \hat{y}_b \in t_{b_0} \) – see Remark 2.3. Now let \( B \to E_0BE_0 \) be the canonical compression from \( B'' = \oplus_{b \in B} B(t_b) \) onto \( B(t_{b_0}) \). It is then an exercise to show that the action of \( \hat{\varphi}^* \) from [\( \omega_0 \)] into \( P_A \) corresponds canonically to the action induced by \( E_0 \hat{\varphi} E_0 \) on the vector states of \( B(t_{b_0}) \). If necessary we may therefore replace \( \hat{\varphi} \) by \( E_0 \hat{\varphi} E_0 \), and assume that \( B'' = B(t) \) where now the vector states of \( B(t) \) plays the role of [\( \omega_0 \)].

Now since \( \hat{\varphi}^* \) is fibre-preserving, all the vector states of \( B(t) \) get mapped onto a single equivalence class [\( \varphi^\sharp(\omega_0) \)] of \( A'' = \oplus_{a \in A} B(h_a) \). This single equivalence class corresponds to the vector states of precisely one of the \( B(h_a) \)'s; say \( B(h_{a''}) \). So in the notation of Remark 2.3 this means that for every vector state \( \omega_z = (\hat{z}, \hat{z}) \) of \( B(t) \) we have

\[
\varphi^\sharp(\omega_z) = (\hat{x}_{a''}, \hat{x}_{a''})
\]
for some unit vector \( x_{a_0} \in h_{a_0} \). In particular this means that for any unit vector \( z \in B(\mathfrak{t}) \) and any \( A = \bigoplus_{a \in \mathcal{A}} A_a \in A'' = \bigoplus_{a \in \mathcal{A}} B(h_a) \) we have
\[
(\tilde{\varphi}(A)z, z) = (A\hat{x}_{a_0}, \hat{x}_{a_0}) = (A_{a_0}x_{a_0}, x_{a_0}).
\]

Clearly \( \tilde{\varphi} \) annihilates all elements \( A = \bigoplus_{a \in \mathcal{A}} A_a \) of \( A'' \) with 0 in the \( a_0 \)th coordinate. More to the point if by \( \hat{A}_{a_0} \) we denote the element of \( A'' \) with \( A_{a_0} \) in the \( a_0 \)th coordinate and zeros elsewhere, this shows that \( \tilde{\varphi} \) factors through \( B(h_{a_0}) \) in that we may write it as a composition of the maps
\[
A'' = \bigoplus_{a \in \mathcal{A}} B(h_a) \to B(h_{a_0}) : \bigoplus_{a \in \mathcal{A}} A_a \mapsto A_{a_0}
\]

and
\[
\varphi_0 : B(h_{a_0}) \to B(\mathfrak{t}) : A_{a_0} \to \tilde{\varphi}(\hat{A}_{a_0}).
\]

It is now clear from equation (3.1) that \( \varphi_0^* \) maps the vector states of \( B(\mathfrak{t}) \) into the vector states of \( B(h_{a_0}) \) in a way that canonically agrees with the action of \( \varphi^* \) from \([\omega_0]\) to \([\varphi^*(\omega_0)]\). We may therefore apply Theorem 2.13 to see that \( \varphi_0^* \) preserves transition probabilities and that (by abuse of notation for the sake of clarity)
\[
\varphi_0^*([\omega_0])^{\perp\perp} = \varphi_0^*([\omega_0])
\]
(complements taken with respect to \( B(h_{a_0}) \)). It is then not difficult to see that this is the same as saying that \( \varphi^* \) preserves transition probabilities on \([\omega_0]\) and that
\[
\varphi^*([\omega_0])^{\perp\perp} = \varphi^*([\omega_0])
\]
(complements taken with respect to \( A'' = \bigoplus_{a \in \mathcal{A}} B(h_a) \)). In the light of Proposition 2.11 and the discussion preceding Theorem 2.13 this last statement is just another way of saying that for a linear subspace \( S \) of \( h_{a_0} \) the claim \( S^{\perp\perp} = S \) (complements taken in \( h_{a_0} \)) is equivalent to the claim \( \{\hat{x}_{a_0}|x_{a_0} \in S\}^{\perp\perp} = \{\hat{x}_{a_0}|x_{a_0} \in S\} \) (complements taken in \( \bigoplus_{a \in \mathcal{A}} h_a \)). Since our original choice of \( \omega_0 \) was arbitrary, this proves the required implication.

(2) \( \Rightarrow \) (1): Suppose that (2) holds. By Remark 2.11 we have that
\[
A'' = \bigoplus_{a \in \mathcal{A}} B(h_a) \quad \text{and} \quad B'' = \bigoplus_{b \in \mathcal{B}} B(\mathfrak{t}_b)
\]
with the classes of vector states of each distinct \( B(h_a) \) and \( B(\mathfrak{t}_b) \) corresponding in a unique way to distinct equivalence classes of pure states of \( \mathcal{A} \) and \( \mathcal{B} \) respectively. Now since \( \varphi^* \) is fibre-preserving, we may re-index the \( B(\mathfrak{t}_b) \)'s with a double index where for any \( a \in \mathcal{A} \), the collection \( B(\mathfrak{t}_b^{(\lambda)}) \) \((\lambda \in \Lambda_a)\) denotes all the \( B(\mathfrak{t}_b) \)'s that correspond to equivalence classes of pure states of \( \mathcal{B} \) that map into the single equivalence class of \( \mathcal{P}_A \) corresponding to \( B(h_a) \). (Note that some of the \( \Lambda_a \)'s may be empty.) It then surely follows that
\[
B'' = \bigoplus_{a \in \mathcal{A}} (\bigoplus_{\lambda \in \Lambda_a} B(\mathfrak{t}_b^{(\lambda)})).
\]

Now fix a non-empty \( \Lambda_a \) and let \( \lambda \in \Lambda_a \) be given. Since the action of \( \varphi^* \) from the equivalence class corresponding to \( B(\mathfrak{t}_b^{(\lambda)}) \) into the one corresponding to \( B(h_a) \) preserves transition probabilities and has a locally solid range, we may apply Theorem 2.13 to obtain a contractive linear map \( \tilde{\varphi}^{(\lambda)} : B(h_a) \to B(\mathfrak{t}_b^{(\lambda)}) \) whose dual in a canonical way induces the action of \( \varphi^* \) on the corresponding equivalence classes of pure states. Summing over \( \lambda \) we get a contractive map
\[
\tilde{\varphi}_a : B(h_a) \to \bigoplus_{\lambda \in \Lambda_a} B(\mathfrak{t}_b^{(\lambda)}) : A_a \mapsto \bigoplus_{\lambda \in \Lambda_a} \tilde{\varphi}^{(\lambda)}(A_a)
\]
whose dual again in a canonical way induces the action of \( \varphi^\dagger \) from all the equivalence classes corresponding to the \( B(\mathfrak{t}_\lambda^{(a)}) \)'s \( (\lambda \in \Lambda_a) \) to the one corresponding to \( B(h_a) \). Summing over \( A_0 = \{ a \in \mathfrak{A} | \Lambda_a \neq \emptyset \} \), we get a contractive map

\[
\tilde{\varphi} = \oplus_{a \in A_0} \varphi_a : \oplus_{a \in A_0} B(h_a) \rightarrow \oplus_{a \in A_0} (\oplus_{\lambda \in \Lambda_a} B(\mathfrak{t}_\lambda^{(a)})) = \mathcal{B}''
\]

which may be extended to all of \( \mathcal{A}'' = \oplus_{a \in A} B(h_a) \) by defining it in such a way that it annihilates all the \( B(h_a) \)'s for which the corresponding \( \Lambda_a \) is empty. By construction the dual of this extended map induces the action of \( \varphi^\dagger \). Thus (1) holds.

We list the details of the bijective case separately because of its more elegant behaviour. For this case we note that while the statement regarding bi-orthogonal bijections is new, the case pertaining to bijections which preserve transition probabilities is basically just a version of Shultz’s result (see Theorem 1.5) with the condition regarding orientation removed.

**Proposition 3.3.** Let \( \mathcal{A}, \mathcal{B} \) be \( C^* \)-algebras with \( (\pi_\mathcal{A}, h_\mathcal{A}) \) and \( (\pi_\mathcal{B}, h_\mathcal{B}) \) their respective reduced atomic representations.

For any Jordan \(*\)-isomorphism \( \tilde{\varphi} \) from \( \pi_\mathcal{A}(\mathcal{A})'' \) onto \( \pi_\mathcal{B}(\mathcal{B})'' \) the adjoint of \( \tilde{\varphi} \) restricts to bijection \( \varphi^\dagger \) from the set of pure states of \( \mathcal{B} \) onto the set of pure states of \( \mathcal{A} \) which preserves all the transition probabilities. In particular \( \varphi^\dagger \) is bi-orthogonal.

Conversely if either \( \varphi^\dagger \) is a bijection from the pure states of \( \mathcal{B} \) onto the pure states of \( \mathcal{A} \) which preserves transition probabilities or if \( \varphi^\dagger \) is a bi-orthogonal bijection and either \( \mathcal{A} \) or \( \mathcal{B} \) has the property that no irreducible representation is of the form \( M_2(\mathbb{C}) \), then we can find a Jordan \(*\)-isomorphism \( \tilde{\varphi} \) from \( \pi_\mathcal{A}(\mathcal{A})'' \) onto \( \pi_\mathcal{B}(\mathcal{B})'' \) such that

\[
\varphi^\dagger(\omega)(\cdot) = \omega(\pi_\mathcal{B}^{-1} \tilde{\varphi}(\cdot) \pi_\mathcal{A})
\]

for each pure state \( \omega \) of \( \mathcal{B} \).

(In the above statements we have again identified the pure states of \( \mathcal{A} \) and \( \mathcal{B} \) with their unique normal extensions to \( \pi_\mathcal{A}(\mathcal{A})'' \) and \( \pi_\mathcal{B}(\mathcal{B})'' \) [LMa, Lemma 11].)

**Proof.** In the proof we concentrate on the case pertaining to bi-orthogonality. The proofs of the two cases are similar with Wigner’s result ([W], cf. [Sh, Theorem 1]) being used in the former case instead of Theorem [LMa, Theorem 1].

Without any loss of generality we may identify both \( \mathcal{A} \) and \( \mathcal{B} \) with their respective reduced atomic representations.

First let \( \varphi^\dagger \) be an orthogonal bijection from the set of pure states of \( \mathcal{B} \) onto the set of pure states of \( \mathcal{A} \) and suppose that no irreducible representation of \( \mathcal{A} \) is of the form \( M_2(\mathbb{C}) \). Now by assumption \( \mathcal{A}'' \) and \( \mathcal{B}'' \) are of the form \( \oplus_{a \in A} B(h_a) \) and \( \oplus_{b \in B} B(t_b) \) respectively, with each equivalence class of pure states of \( \mathcal{A} \) and \( \mathcal{B} \) corresponding to the set of vector states of one of the \( B(h_a) \)'s and \( B(t_b) \)'s respectively in the sense described in Remark 2.9. Now by Lemma 2.9 both \( \varphi^\dagger \) and its inverse is fibre-preserving. It therefore induces a bijection from the set of equivalence classes of pure states of \( \mathcal{B} \) onto the set of equivalence classes of pure states of \( \mathcal{A} \). We may therefore re-index the \( B(t_b) \)'s \( (b \in B) \) with the index set \( \mathfrak{A} \) in such a way that \( \mathcal{B}'' = \oplus_{a \in A} B(t_a) \), with \( \varphi^\dagger \) for each \( a \in \mathfrak{A} \) mapping the equivalence class corresponding to the vector states of \( B(t_a) \) onto the equivalence class corresponding to the vector states of \( B(h_a) \).

Now by the assumption on \( \mathcal{A} \) we have that \( B(h_a) \neq M_2(\mathbb{C}) \) for each \( a \in \mathfrak{A} \). Since \( \varphi^\dagger \) is also bi-orthogonal, it now follows from Proposition 2.9 and Theorem...
that for each \( a \in \mathbb{A} \), \( \varphi^\dagger \) induces either a \(*\)-isomorphism or \(*\)-antisomorphism \( \varphi_a \) from \( B(\mathcal{H}_a) \) onto \( B(\mathfrak{t}_a) \). Moreover for each \( a \in \mathbb{A} \) the transformation that \( \varphi^\dagger \) induces from the vector states of \( B(\mathfrak{t}_a) \) onto the vector states of \( B(\mathcal{H}_a) \) appears as a restriction of the adjoint of \( \varphi_a \). Recall that in the notation of Remark 2.3, pure states of \( B \) correspond to vector states of the form \( \omega_{\hat{x}_a} \) for some \( a \in \mathbb{A} \) and some norm one vector \( x_a \in \mathfrak{t}_a \). With this in mind it is now a simple matter to verify that \( \hat{\varphi} = \oplus_{a \in \mathbb{A}} \varphi_a \) is a Jordan \(*\)-isomorphism from \( \mathcal{A}'' = \oplus_{a \in \mathbb{A}} B(\mathcal{H}_a) \) onto \( B'' = \oplus_{a \in \mathbb{A}} B(\mathfrak{t}_a) \) with the property that
\[
\varphi^\dagger(\omega_{\hat{x}_a}) = \omega_{\hat{x}_a} \circ \hat{\varphi}
\]
for each \( a \in \mathbb{A} \) and each norm one vector \( x_a \in \mathfrak{t}_a \). This clearly suffices to establish the claim.

Conversely let \( \hat{\varphi} \) be a Jordan \(*\)-isomorphism from \( \mathcal{A}'' \) onto \( B'' \). By Lemma 2.2 \( \hat{\varphi} \) is then necessarily a \( \sigma \)-weak homeomorphism. Moreover by [LMa] Theorem 5] the adjoint of \( \hat{\varphi} \) restricts to a bijection from the pure states of \( B'' \) onto the pure states of \( \mathcal{A}'' \). Thus \( \hat{\varphi}^* \) trivially maps the \( \sigma \)-weakly continuous pure states of \( B'' \) onto the \( \sigma \)-weakly continuous pure states of \( \mathcal{A}'' \).

It remains to show that the restriction of \( \hat{\varphi} \) to the pure states of \( B \) preserves transition probabilities. We once again identify the pure states of \( \mathbb{A} \) and \( B \) with their unique \( \sigma \)-weakly continuous extensions to \( \mathcal{A}'' \) and \( B'' \). The fact that \( \varphi^\dagger \) is bi-orthogonal is a straightforward consequence of the fact that \( \hat{\varphi} \), and hence also its dual, is a surjective linear isometry [BRG 3.2.3]. Thus given any two pure states \( \omega_0 \) and \( \omega_1 \) of \( B \) it easily follows that \( \| \omega_0 - \omega_1 \| = 2 \) if and only if \( \| \varphi^\dagger(\omega_0) - \varphi^\dagger(\omega_1) \| = \| \omega_0 \circ \hat{\varphi} - \omega_1 \circ \hat{\varphi} \| = 2 \).

Establishing that \( \varphi^\dagger \) preserves transition probabilities requires a bit more work and may in fact be deduced from Wigner’s classical result. We show how to do this using the available structure. As before bi-orthogonality ensures that both \( \varphi^\dagger \) and its inverse are fibre-preserving. Hence it is enough to prove the preservation of transition probabilities for sets of unitarily equivalent pure states only. Recall that \( \mathcal{A}'' \) and \( B'' \) are of the form \( \mathcal{A}'' = \oplus_{a \in \mathbb{A}} B(\mathcal{H}_a) \) and \( B'' = \oplus_{b \in \mathbb{B}} B(\mathfrak{t}_b) \). The fact that both \( \varphi^\dagger \) and its inverse are fibre-preserving ensures that we may re-index the expression for \( B'' \) with the index set \( \mathbb{A} \) in such a way that for any given \( a \in \mathbb{A} \), \( \varphi^\dagger \) identifies the pure states corresponding to the vector states of \( B(\mathfrak{t}_a) \) with the pure states corresponding to the vector states of \( B(\mathcal{H}_a) \) (see Remark 2.3). For any \( B = \oplus_{a \in \mathbb{A}} B_a \in \oplus_{a \in \mathbb{A}} B(\mathcal{H}_a) \) and any given \( d \in \mathbb{A} \), we then have that \( \omega_{\hat{x}_a}(B) = 0 \) for all unit vectors \( x_a \in \mathfrak{t}_a \) if and only if \( \omega_{\hat{x}_a}(\varphi(\mathfrak{t})) = 0 \) for all unit vectors \( y_d \in \mathfrak{t}_d \) (again notation is as in Remark 2.3). It is now an exercise to see that this ensures that for each \( a \in \mathbb{A} \), \( \hat{\varphi} \) maps the copy of \( B(\mathcal{H}_a) \) in \( \mathcal{A}'' \) onto the copy of \( B(\mathfrak{t}_a) \) in \( B'' \). All that remains to be done is to check that the map that \( \varphi^\dagger \) induces from the vector states of \( B(\mathfrak{t}_a) \) onto those of \( B(\mathcal{H}_a) \), is a restriction of the dual of the map \( \hat{\varphi} \) induces from \( B(\mathcal{H}_a) \) to \( B(\mathfrak{t}_a) \). Since the map induced by \( \hat{\varphi} \) is necessarily either a \(*\)-isomorphism or a \(*\)-anti-isomorphism [BRG 3.2.2], we may then directly apply Wigner’s theorem [W; cf. SH Theorem 1] to get the required conclusion.

The above results provide the context for a non-commutative version of the Banach-Stone theorem [KR 3.4.3]. To see this recall that for any compact Hausdorff set \( K \) the pure state space of \( C(K) \) corresponds exactly to the set of point evaluations engendered by elements of \( K \). Endowed with the weak* topology, this set of point evaluations is of course homeomorphic to \( K \) itself. The link between
Thus their respective reduced atomic representations. For any linear map \(\varphi\) from \(A\) into \(B\) whose duals preserve pure states, as a subclass of the \(\sigma\)-weakly continuous linear maps from \(A''\) into \(B''\) whose duals preserve \(\sigma\)-weakly continuous pure states.

**Proposition 3.4.** As before let \(A, B\) be \(C^*\)-algebras with \((\pi_A, \mathfrak{h}_A)\) and \((\pi_B, \mathfrak{h}_B)\) their respective reduced atomic representations. For any linear map \(\varphi\) from \(A\) into \(B\) with a pure state preserving dual, \(\pi_B \circ \varphi \circ \pi_A^{-1}\) is \(\sigma\)-weakly continuous and admits of a unique extension to a \(\sigma\)-weakly continuous linear map \(\tilde{\varphi}\) from \(\pi_A(A)''\) into \(\pi_B(B)''\) whose dual preserves \(\sigma\)-weakly continuous pure states. In particular if \(\varphi\) is a Jordan *-isomorphism from \(A\) onto \(B\), then its dual restricts to a bijection between the respective sets of pure states, and the canonical extension \(\bar{\varphi}\) described above is a Jordan *-isomorphism from \(\pi_A(A)''\) onto \(\pi_B(B)''\).

**Proof.** Without loss of generality let \(A, B\) be identified with their reduced atomic representations and let \(\varphi : A \to B\) be a linear map with a pure state preserving dual. It follows from [LMar] Theorem 5 that \(\varphi^*\) is necessarily contractive. The map \(\varphi^*\) is then bounded and so restricts to an affine map from the norm-closed convex hull of the pure states of \(B\) into the norm-closed convex hull of the pure states of \(A\). We show that these norm closed convex hulls are precisely the normal state spaces of \(B\) and \(A\) respectively. By Remark 2.3 all the pure states of say \(A\) are necessarily normal and hence the norm-closed convex hull of the pure states is at least contained in the normal state space. (The fact that the normal state space of a concrete \(C^*\)-algebra is a norm-closed convex set follows from for example [KR1] 10.1.15.) Since by [KR1] 7.1.12 & 10.1.11(i)] the normal state space of say \(A\) is the norm-closed convex hull of the vector states, we need only show that the vector states are contained in the norm-closed convex hull of the pure states to conclude that the two sets are equal. This in turn can be seen as follows: Given a norm-one element \(x \in \mathfrak{h}_A\), we saw in the proof of Lemma 3.1 that in this case \(\omega_x\) can be written in the form

\[
\omega_x = \sum_{a \in h} |\mu_a|^2 \omega_{\hat{a}_a}
\]

where each \(\omega_{\hat{a}_a}\) is a pure state and \(\sum_{a \in h} |\mu_a|^2 = 1\). It follows that \(\varphi^*\) affinely maps the normal state space of \(B\) into the normal state space of \(A\).

Now since the sets of \(\sigma\)-weakly continuous states of \(B\) and \(A\) correspond canonically to the sets of \(\sigma\)-weakly continuous states (i.e. the normal states) of \(B''\) and \(A''\) respectively (see [KR1] 10.1.11(i))), it follows that the linear span of the sets of \(\sigma\)-weakly continuous states of \(A\) and \(B\) correspond canonically to \((A'')_\ast\) and \((B'')_\ast\) respectively. Let \(\tilde{\varphi}\) be the \((\sigma\)-weakly continuous) dual of the map induced by \(\varphi^*\) from \((B'')_\ast\) to \((A'')_\ast\). Then for any normal state \(\omega\) of \(B''\) and any \(A \in A\) we have by construction that

\[
\omega(\tilde{\varphi}(A)) = \tilde{\varphi}_\ast \circ \omega(A) = \varphi^* \circ \omega(A) = \omega(\varphi(A)).
\]

Thus \(\tilde{\varphi}\) is \(\sigma\)-weakly continuous, and since we must have that \(\tilde{\varphi}(A) = \varphi(A)\) for each \(A\), it is of course also an extension of \(\varphi\). The claim regarding the uniqueness of
the extension follows from for example [KRi 10.1.10]. The claim regarding the σ-
weakly continuous pure states is then a trivial consequence of what we just proved,
and Lemma 3.1.

Now if ϕ is a Jordan ∗-isomorphism from \( A \) onto \( B \), then by [LMa Theorem 5] its dual defines a bijection between the respective sets of pure states. We may then mimic the above argument to in this case construct an affine bijection between the normal state spaces of \( B' \) and \( A' \). By Kadison’s result (cf. Theorem 1.3(1)) this affine bijection yields a Jordan ∗-isomorphism \( \tilde{\varphi} \) from \( A' \) onto \( B' \) which by the same argument as before can be shown to be the required extension.

We are now finally ready to present our main theorem. In mathematical terms this amounts to a very general non-commutative Banach-Stone type theorem. At a slightly different level one may interpret the bijective case of this as a result stating that there is indeed enough information internally encoded in the pure state spaces of two \( C^* \)-algebras \( A \) and \( B \) to ensure their physical equivalence via a suitable pure state bijection and also enough to be able to identify those pure state bijections which actually correspond to some type of Wigner symmetry. We also take this opportunity to invite the reader to compare the result below with Theorem 5.7 of [Stø2]. In this result Størmer uses conditions analogous to fibre-preserving and locally solid to describe a class of Jordan ∗-homomorphisms in terms of their action on pure states.

We point out that we have assumed our algebras to be unital. For the result to hold in the non-unital case \( \varphi^\sharp \) is required to admit of a homeomorphic action from \( \mathcal{P}_B \cup \{0\} \) to \( \mathcal{P}_A \cup \{0\} \) which fixes 0 (see [Sh] and [Br]). We take this opportunity to re-emphasise the fact noted prior to Theorem 2.12 that the restriction regarding \( M_2(\mathbb{C}) \) can not be removed.

**Theorem 3.5** (Non-commutative Banach-Stone theorem). Let \( A, B \) be \( C^* \)-algebras and let \( \varphi^\sharp \) be a transformation from \( \mathcal{P}_B \) into \( \mathcal{P}_A \). Consider the following statements:

1. There exists a linear map \( \varphi : A \to B \) such that \( \varphi^\sharp(\omega) = \omega \circ \varphi \) for every pure state \( \omega \in \mathcal{P}_B \).
2. \( \varphi^\sharp \) is uniformly \( \sigma(B^*, B) - \sigma(A^*, A) \) continuous, fibre-preserving, locally preserves transition probabilities, and has a locally solid range.
3. \( \varphi^\sharp \) is uniformly \( \sigma(B^*, B) - \sigma(A^*, A) \) continuous, locally bi-orthogonal (and hence fibre-preserving by Lemma 2.4) and has a locally solid range.

The implications (1) \( \Leftrightarrow \) (2) \( \Rightarrow \) (3) hold in general with all three statements being equivalent if no irreducible representation of \( B \) is of the form \( M_2(\mathbb{C}) \).

**Proof.** The implication (2) \( \Rightarrow \) (3) is fairly clear, whereas the fact that (3) \( \Rightarrow \) (2) whenever no irreducible representation of \( B \) is of the form \( M_2(\mathbb{C}) \), follows from Corollary 2.13. Hence we need only show that (1) \( \Leftrightarrow \) (2).

Firstly let \( \varphi \) be a linear map from \( A \) into \( B \) with a pure state preserving dual. Then all statements in (2) except the claim about uniform weak∗ continuity follows from a combination of Theorem 3.5 and Proposition 3.1. To see the last claim note that since \( \varphi \) is necessarily continuous (see [LMa Theorem 5]), its dual is of course \( \sigma(B^*, B) - \sigma(A^*, A) \) continuous on the \( \sigma(B^*, B) \)-compact unit ball of \( B^* \). Continuity of the dual on a compact superset then ensures that the restriction to the subset \( \mathcal{P}_B \) must necessarily be uniformly \( \sigma(B^*, B) - \sigma(A^*, A) \) continuous.

For the converse we may assume without loss of generality that both \( A \) and \( B \) are reduced atomically represented. It is then clear from Theorem 6.2 that the
hypotheses are sufficient to guarantee that \( \varphi^p \) is induced by a linear map \( \tilde{\varphi} \) from \( A'' \) into \( B'' \). We therefore need only show that requiring \( \varphi^p \) to in addition be uniformly weak\(^*\) continuous is sufficient to guarantee that \( \tilde{\varphi}(A) \subset B \).

Hence suppose that \( \tilde{\varphi}^* \) is uniformly \( \sigma(B^*, B) - \sigma(A^*, A) \) continuous from \( \mathcal{P}_B \) into \( \mathcal{P}_A \). Recall the the biduals of \( A \) and \( B \) may be identified with the double commutants of their respective universal representations \([KR] 10.1.1\]. Since in the universal representation of a \( C^* \)-algebra all states are normal, it follows from \([LMa, Proposition 6]\) that each of \( A^{**} \) and \( B^{**} \) admit of central projections \( E_A \) and \( E_B \) respectively such that \( A_{E_A} \) and \( B_{E_B} \) are \( \sigma \)-weakly \( * \)-isomorphic to the reduced atomic representations of \( A \) and \( B \) respectively. In fact these projections correspond to nothing more than the canonical central projections \( z_A \) and \( z_B \) of \( A^{**} \) and \( B^{**} \) onto their respective atomic parts (as defined and used by both Shultz \([Sh]\) and Brown \([Br]\)). Again in the notation of Brown, \( \tilde{\varphi} \) must then correspond to a linear map from \( z_A A^{**} \) to \( z_B B^{**} \). Therefore given any \( A \in A \), \( \tilde{\varphi}(A) \) corresponds to an element \( B \) of \( z_B B^{**} \). Now since \( A \) defines a linear \( \sigma(A^*, A) \) continuous functional on \( A^* \) and \( \tilde{\varphi}^* \) is uniformly \( \sigma(B^*, B) - \sigma(A^*, A) \) continuous from \( \mathcal{P}_B \) into \( \mathcal{P}_A \), it follows that \( \tilde{\varphi}(A) \) (that is \( B \)) is uniformly \( \sigma(B^*, B) \) continuous on \( \mathcal{P}_B \). Thus by \([Br, Corollary 8]\) we must have that \( B \in zB \). Given that \( zB \) (that is \( B_{E_B} \)) corresponds to the reduced atomic representation of \( B \) and that \( B \) is identified with \( \tilde{\varphi}(A) \) under this correspondence, this shows that \( \tilde{\varphi}(A) \in B \) as required.

**Corollary 3.6.** Let \( A, B \) be \( C^* \)-algebras and let \( \varphi^p \) be a transformation from \( \mathcal{P}_B \) into \( \mathcal{P}_A \). Consider the following statements:

1. There exists a Jordan \( * \)-isomorphism \( \varphi \) from \( A \) onto \( B \) such that \( \varphi^p(\omega) = \omega \circ \varphi \) for every pure state \( \omega \in \mathcal{P}_B \).
2. \( \varphi^p \) is a bijective uniform \( \sigma(B^*, B) - \sigma(A^*, A) \) homeomorphism which preserves transition probabilities.
3. \( \varphi^p \) is bijective and is a biorthogonal uniform \( \sigma(B^*, B) - \sigma(A^*, A) \) homeomorphism.

The implications (1) \( \iff \) (2) \( \Rightarrow \) (3) hold in general with all three statements being equivalent if either \( A \) or \( B \) has the property that no irreducible representation is of the form \( M_2(C) \).

**Proof.** First recall that any Jordan \( * \)-isomorphism \( \varphi \) from \( A \) onto \( B \) is a linear isometry \([Br\alpha, 3.2.3]\). Hence the dual of such an object yields a linear \( \sigma(B^*, B) - \sigma(A^*, A) \) homeomorphism between the respective dual spaces. With this observation in place, the proof of the previous theorem now modifies readily with Proposition \( 5.3 \) being used instead of Theorem \( 5.2 \). Note also that once the existence of \( \tilde{\varphi} \) has been established in the proof of (2) \( \Rightarrow \) (1), it is enough to verify that \( \tilde{\varphi}(A) \subset B \), since by symmetry we will then also have \( \tilde{\varphi}^{-1}(B) \subset A \) (implying that in this case \( \tilde{\varphi} \) restricts to the required Jordan \( * \)-isomorphism \( \varphi \) from \( A \) onto \( B \)).

**Corollary 3.7** (Banach-Stone). A mapping \( \varphi \) of \( C(K) \) into \( C(S) \), with \( K \) and \( S \) compact Hausdorff spaces, is a \( * \)-homomorphism if and only if there exists a continuous transformation \( \nu \) of \( S \) into \( K \) such that

\[
\varphi(f) = f \circ \nu \quad \text{for each } \ f \in C(K).
\]

In particular \( \varphi \) is a \( * \)-isomorphism from \( C(K) \) onto \( C(S) \) if and only if \( \nu \) is a homeomorphism from \( S \) onto \( K \).
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