Simons-Type Equation for $f$-Minimal Hypersurfaces
and Applications

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Abstract We derive the Simons-type equation for $f$-minimal hypersurfaces in weighted Riemannian manifolds and apply it to obtain a pinching theorem for closed $f$-minimal hypersurfaces immersed in the product manifold $S^n(\sqrt{2(n-1)}) \times \mathbb{R}$ with $f = \frac{r^2}{4}$. Also, we classify closed $f$-minimal hypersurfaces with $L_f$-index one immersed in $S^n(\sqrt{2(n-1)}) \times \mathbb{R}$ with the same $f$ as above.

Keywords Minimal hypersurface · Bakry–Émery curvature · Riemannian manifold

Mathematics Subject Classification Primary 58J50 · Secondary 58E30

1 Introduction

In [18], Simons proved an identity, called Simons’s equation, for the Laplacian of $|A|^2$, the square of the norm of the second fundamental form of minimal hypersurfaces in Riemannian manifolds. Simons’s equation plays an important role in the study of minimal hypersurfaces. For self-shrinkers for the mean curvature flow in the Euclidean space $\mathbb{R}^{n+1}$, the Simons-type equation also holds ([10], [7] Sect. 10.2). By applying it, Huisken [10] proved that an embedded closed self-shrinker of nonnegative mean...
curvature must be a sphere of radius $\sqrt{2n}$, and recently Colding and Minicozzi [7] classified complete embedded self-shrinkers of nonnegative mean curvature with polynomial volume growth. Later, Le and Sesum [12] and Cao and Li [2] used it to obtain gap theorems for self-shrinkers.

Both minimal hypersurfaces and self-shrinkers are special cases of $f$-minimal hypersurfaces in the weighted Riemannian manifolds. See the definition of $f$-minimal hypersurfaces in Sect. 2 and more examples in [4]. An $f$-minimal hypersurface $\Sigma$ is not only a critical point of the weighted volume functional $\int_{\Sigma} e^{-f} d\sigma$ of $\Sigma$, where $d\sigma$ denotes the volume element of $(\Sigma, g)$, but also a minimal hypersurface in $(\tilde{M}, \tilde{g})$, where the new metric $\tilde{g} = e^{-\frac{n}{2}f} g$ of $M$ is conformal to $g$. Recently, Lott [15] and Magni et al. [16] showed that $f$-minimal hypersurfaces arise in the study of the mean curvature flow of a hypersurface in an ambient manifold evolving by Ricci flow. Especially the mean curvature soliton (for the mean curvature flow of a hypersurface in a gradient Ricci soliton solution) introduced by Lott [15] is just an $f$-minimal hypersurface, where $f$ is the potential function of the ambient gradient Ricci soliton.

Recently, Liu [14] studied stable $f$-minimal hypersurfaces in manifolds with nonnegative Bakry–Émery Ricci curvature and gave a partial classification of the ambient space when the dimension is 3 and $f$ is bounded. The present authors [4] studied the stability condition and compactness of $f$-minimal surfaces. Li and Wei [13] gave eigenvalue estimates for closed $f$-minimal hypersurfaces in a compact manifold with positive $m$-Bakry–Émery curvature. We [3] also obtained similar eigenvalue estimates for some closed $f$-minimal hypersurfaces in a complete manifold with positive Bakry–Émery curvature. These estimates have been used to prove compactness theorems for closed $f$-minimal surfaces.

In this paper, we will prove a Simons-type equation for $f$-minimal hypersurfaces in a smooth metric measure space $(M, \bar{g}, e^{-f} d\mu)$, that is, an identity for the weighted Laplacian $\Delta_f$ of $|A|^2$ of $f$-minimal hypersurfaces, involving the Bakry–Émery Ricci curvature $\bar{\text{Ric}}_f$ (see Theorem 3). Also, we derive the equations for the weighted Laplacian $\Delta_f$ of some other geometric quantities on $f$-minimal hypersurfaces, like the mean curvature $H$, etc.; see, for instance, Propositions 1 and 2. Since these equations involve $\bar{\text{Ric}}_f$, we naturally would like to consider the cases in which the ambient manifolds are gradient Ricci solitons, that is, $M$ satisfies $\bar{\text{Ric}}_f = C \bar{g}$; see Corollaries 2, 3 and 4. Further, we apply the equations mentioned above to the special case of $f$-minimal hypersurfaces in the cylinder shrinking soliton $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$ with $f = \frac{t^2}{4}$, where $t$ is the coordinate of the second factor $\mathbb{R}$. Namely, we obtain the following pinching theorem.

**Theorem 1** Let $\Sigma^n$ be a closed immersed $f$-minimal hypersurface in the product manifold $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$, $n \geq 3$, with $f = \frac{t^2}{4}$, $t \in \mathbb{R}$. If the square of the norm of the second fundamental form of $\Sigma$ satisfies

$$\frac{1}{4} \left( 1 - \sqrt{1 - \frac{8}{n-1} \alpha^2 (1-\alpha^2)} \right) \leq |A|^2 \leq \frac{1}{4} \left( 1 + \sqrt{1 - \frac{8}{n-1} \alpha^2 (1-\alpha^2)} \right),$$

\[ \square \]
then $\Sigma$ is $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$, where $\alpha = \langle v, \frac{\partial}{\partial t} \rangle$, $v$ is the outward unit normal to $\Sigma$ and $t$ denotes the coordinate of the factor $\mathbb{R}$ of $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$.

Observe that $n \geq 3$ implies that $\frac{8}{n-2}(1-\alpha^2) \leq 1$ and hence the inequalities in Theorem 1 make sense. Theorem 1 implies that

**Corollary 1** There is no closed immersed $f$-minimal hypersurface in the product manifold $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$, $n \geq 3$, with $f = \frac{t^2}{4}$, $t \in \mathbb{R}$ so that the square of its norm of the second fundamental form satisfies

$$\frac{1}{4} \left( 1 - \sqrt{1 - \frac{2}{n-1}} \right) \leq |A|^2 \leq \frac{1}{4} \left( 1 + \sqrt{1 - \frac{2}{n-1}} \right).$$

Next, we discuss, as another application, the eigenvalues and the index of the operator $L_f$ on $f$-minimal hypersurfaces. The eigenvalues of the $L$-operator for self-shrinkers were discussed in [7] and recently, Hussey [11] studied the index of the operator for self-shrinkers (see Example 1). In this paper, we classify closed $f$-minimal hypersurfaces in the cylinder shrinking soliton $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$ whose $L_f$ operators have index one and prove that

**Theorem 2** Let $\Sigma^n$ be a closed immersed $f$-minimal hypersurface in the product manifold $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$ with $f = \frac{t^2}{4}$. Then $L_f$-ind($\Sigma$) $\geq 1$. Moreover, the equality holds if and only if $\Sigma = \mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$.

For complete noncompact $f$-minimal hypersurfaces in the cylinder shrinking soliton $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$ with $f = \frac{t^2}{4}$, the first and third authors [6] proved the results corresponding to Theorems 1 and 2. In this case complete noncompact $f$-minimal hypersurfaces are assumed to have polynomial volume growth, which is equivalent to properness of immersion, or finiteness of weighted volume (see [5] and [4]).

The rest of this paper is organized as follows: In Sect. 2 some definitions and notation are given. In Sect. 3 we prove the Simons-type equation and the equations for $\Delta_f$ of other geometric quantities. In Sect. 4, we calculate the index of the $L_f$ operator on closed $f$-minimal hypersurfaces in the cylinder shrinking soliton $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$. In Sect. 5, we prove Theorem 1 for closed $f$-minimal hypersurfaces in the cylinder shrinking soliton $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$.

2 Definitions and Notation

Let $(M^{n+1}, g, e^{-f} d\mu)$ be a smooth metric measure space, which is an $(n+1)$-dimensional Riemannian manifold $(M^{n+1}, g)$ together with a weighted volume form $e^{-f} d\mu$ on $M$, where $f$ is a smooth function on $M$ and $d\mu$ the volume element induced by the metric $g$. In this paper, unless otherwise specified, we denote by a bar all quantities on $(M, g)$, for instance by $\bar{\nabla}$ and $\bar{\nabla}$, the Levi-Civita connection and the Ricci curvature tensor of $(M, g)$ respectively. For $(M, g, e^{-f} d\mu)$, the $\infty$-Bakry–Émery Ricci curvature tensor $\hat{\text{Ric}}_f$ (for simplicity, Bakry–Émery Ricci curvature), which is defined by
\[ \overline{\text{Ric}_f} := \overline{\text{Ric}} + \nabla^2 f, \]

where \( \nabla^2 f \) is the Hessian of \( f \) on \( M \). If \( f \) is constant, \( \overline{\text{Ric}_f} \) is the Ricci curvature \( \overline{\text{Ric}} \).

Now let \( i : \Sigma^n \to M^{n+1} \) be an \( n \)-dimensional smooth immersion. Then \( i \) induces a metric \( g = i^* \overline{g} \) on \( \Sigma \) so that \( i : (\Sigma^n, g) \to (M, \overline{g}) \) is an isometric immersion. We will denote for instance by \( \nabla \), \( \text{Ric} \), \( \Delta \) and \( d\sigma \), the Levi-Civita connection, the Ricci curvature tensor, the Laplacian, and the volume element of \( (\Sigma, g) \) respectively.

The restriction of \( f \) on \( \Sigma \), still denoted by \( f \), yields a weighted measure \( e^{-f} d\sigma \) on \( \Sigma \) and hence an induced smooth metric measure space \( (\Sigma^n, g, e^{-f} d\sigma) \). The associated weighted Laplacian \( \Delta_f \) on \( \Sigma \) is defined by

\[ \Delta_f u := \Delta u - \langle \nabla f, \nabla u \rangle. \]

The second-order operator \( \Delta_f \) is a self-adjoint operator on \( L^2(e^{-f} d\sigma) \), the space of square integrable functions on \( \Sigma \) with respect to the measure \( e^{-f} d\sigma \).

We define the second fundamental form \( A \) of \( \Sigma \) by

\[ A : T_p \Sigma \times T_p \Sigma \to \mathbb{R}, \quad A(X, Y) = -\langle \nabla_X Y, \nu \rangle, \]

where \( p \in \Sigma \), \( X, Y \in T_p \Sigma \), \( \nu \) is a unit normal vector at \( p \).

In a local orthonormal system \( \{e_i\}, i = 1, \ldots, n \) of \( \Sigma \), the components of \( A \) are denoted by \( a_{ij} = A(e_i, e_j) = \langle \nabla_{e_i} \nu, e_j \rangle \). The shape operator \( A \) and the mean curvature \( H \) of \( \Sigma \) are defined by

\[ A : T_p \Sigma \to T_p \Sigma, \quad AX = \nabla_X \nu, \quad X \in T_p \Sigma; \quad H = \text{tr} A = \sum_{i=1}^n a_{ii}. \]

With the above notation, we have the following

**Definition 1** The weighted mean curvature \( H_f \) of the hypersurface \( \Sigma \) is defined by

\[ H_f = H - \langle \nabla f, \nu \rangle. \]

\( \Sigma \) is called an \( f \)-minimal hypersurface if it satisfies

\[ H = \langle \nabla f, \nu \rangle. \]  \hspace{1cm} (1)

**Definition 2** The weighted volume of \( \Sigma \) is defined by

\[ V_f(\Sigma) := \int_{\Sigma} e^{-f} d\sigma. \]  \hspace{1cm} (2)

It is known that \( \Sigma \) is \( f \)-minimal if and only if it is a critical point of the weighted volume functional. On the other hand, we can view it in another manner: \( \Sigma \) being \( f \)-minimal in \( (M, \overline{g}) \) is equivalent to \( (\Sigma, i^* \overline{g}) \) being minimal in \( (M, \overline{g}) \), where the conformal metric \( \overline{g} = e^{-\frac{2f}{n}} \overline{g} \) (cf. [4]).
Now we assume that $\Sigma$ is a two-sided hypersurface, that is, there is a globally defined unit normal $\nu$ on $\Sigma$.

**Definition 3** For a two-sided hypersurface $\Sigma$, the $L_f$ operator on $\Sigma$ is given by

$$L_f := \Delta_f + |A|^2 + \operatorname{Ric}_f(\nu, \nu),$$

where $|A|^2$ denotes the square of the norm of the second fundamental form $A$ of $\Sigma$.

The operator $L_f = \Delta_f + |A|^2 + \operatorname{Ric}_f(\nu, \nu)$ is called the $L_f$-stability operator of $\Sigma$.

**Example 1** For self-shrinkers in $\mathbb{R}^{n+1}$, the operator $L_f$, where $f = \frac{|x|^2}{4}$, is just the $L$ operator in [7]:

$$L = \Delta - \frac{1}{2} \langle x, \nabla \cdot \rangle + |A|^2 + \frac{1}{2}.$$

**Definition 4** A two-sided $f$-minimal hypersurface $\Sigma$ is said to be $L_f$-stable if for any compactly supported smooth function $\varphi \in C_\infty^\infty(\Sigma)$, it holds that

$$-\int_\Sigma \varphi L_f \varphi e^{-f} d\sigma \geq 0,$$

or equivalently,

$$\int_\Sigma \left[ |\nabla \varphi|^2 - (|A|^2 + \operatorname{Ric}_f(\nu, \nu))\varphi^2 \right] e^{-f} d\sigma \geq 0. \quad (4)$$

$L_f$-stability means that the second variation of the weighted volume of $f$-minimal hypersurface $\Sigma$ is nonnegative. Further, one has the definition of the $L_f$-index of $f$-minimal hypersurfaces. Since $\Delta_f$ is self-adjoint in the weighted space $L^2(e^{-f} d\sigma)$, we may define a symmetric bilinear form on the space $C_\infty^\infty(\Sigma)$ of compactly supported smooth functions on $\Sigma$ by, for $\varphi, \psi \in C_\infty^\infty(\Sigma)$,

$$B_f(\varphi, \psi) := -\int_\Sigma \varphi L_f \psi e^{-f} d\sigma = \int_\Sigma \left[ \langle \nabla \varphi, \nabla \psi \rangle - (|A|^2 + \operatorname{Ric}_f(\nu, \nu))\varphi \psi \right] e^{-f} d\sigma. \quad (5)$$

**Definition 5** The $L_f$-index of $\Sigma$, denoted by $L_f\text{-ind}(\Sigma)$, is defined to be the maximum of the dimensions of negative definite subspaces for $B_f$.

In particular, $\Sigma$ is $L_f$-stable if and only if $L_f - \text{ind}(\Sigma) = 0$. The $L_f$-index of $\Sigma$ has the following equivalent definition: Consider the Dirichlet eigenvalue problems of $L_f$ on a compact domain $\Omega \subset \Sigma$:

$$L_f u = -\lambda u, \quad u \in \Omega; \quad u|_{\partial \Omega} = 0.$$

$L_f\text{-ind}(\Sigma)$ is defined to be the supremum over compact domains of $\Sigma$ of the number of negative (Dirichlet) eigenvalues of $L_f$ (cf. [9]).
It is known that an $f$-minimal hypersurface $(\Sigma, g)$ is $L_f$-stable if and only if $(\Sigma, i^* \tilde{g})$ is stable as a minimal surface in $(M, \tilde{g})$. Further, the Morse index of $L_f$ on $(\Sigma, g)$ is equal to the Morse index of the Jacobi operator on minimal hypersurface $(\Sigma, i^* \tilde{g})$ (see [4]).

We will take the following convention for tensors. For instance, under a local frame field on $M$, suppose that $T = (T_{j_1, \ldots, j_r})$ is an $(r, 0)$-tensor on $M$. The components of the covariant derivative $\nabla T$ are denoted by $T_{j_1, \ldots, j_r; i}$, that is,

$$T_{j_1, \ldots, j_r; i} = (\nabla_{e_i} T)(e_{j_1}, \ldots, e_{j_r}).$$

Meanwhile, under a local frame field on $\Sigma$, suppose that $S = (S_{k_1, \ldots, k_s})$ is an $(s, 0)$-tensor on $\Sigma$. The components of the covariant derivative $\nabla S$ are denoted by $S_{k_1, \ldots, k_s, l}$, that is,

$$S_{k_1, \ldots, k_s, l} = (\nabla_{e_l} S)(e_{k_1}, \ldots, e_{k_s}).$$

Throughout this paper, we assume that the $f$-minimal hypersurfaces are orientable and without boundary. For a closed hypersurface, we choose $\nu$ to be the outer unit normal. Finally, we refer the interested reader to [1], [3], [4], [8] and the references therein for more details about $f$-minimal hypersurfaces.

3 Simons-Type Equation for $f$-Minimal Hypersurfaces

First, we calculate the weighted Laplacian $\Delta_f$ for mean curvature $H$ of $f$-minimal hypersurfaces.

**Proposition 1** Let $(\Sigma^n, g)$ be an $f$-minimal hypersurface isometrically immersed in a smooth metric measure space $(M, \tilde{g}, e^{-f} d\mu)$. Then the mean curvature $H$ of $\Sigma$ satisfies

$$\Delta_f H = 2 \text{tr}_g (\nabla^3 f (\cdot, \nu, \cdot)|_\Sigma) - \text{tr}_g (\nabla^3 f (\nu, \cdot, \cdot)|_\Sigma) + 2 \langle A, \nabla^2 f \rangle_g - \text{Ric}_f (\nu, \nu) H - |A|^2 H,$$

or equivalently,

$$\Delta_f H = 2 \sum_{i=1}^n (\nabla^3 f)_{i\nu i} - \sum_{i=1}^n (\nabla^3 f)_{\nu i i} + 2 \sum_{i,j=1}^n a_{ij} (\nabla^2 f)_{ij} - \text{Ric}_f (\nu, \nu) H - |A|^2 H,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame field on $\Sigma$, $\nu$ denotes the unit normal to $\Sigma$, and $|\cdot|_\Sigma$ denotes the restriction to $\Sigma$.

**Proof** We choose a local orthonormal frame field $\{e_i\}_{i=1}^{n+1}$ for $M$ so that, restricted to $\Sigma$, $\{e_i\}_{i=1}^n$ are tangent to $\Sigma$, and $e_{n+1} = \nu$ is the unit normal to $\Sigma$. Throughout this paper, for simplicity of notation, we substitute $\nu$ for the subscript $n + 1$ in the components of
Simons-Type Equation for $f$-Minimal Hypersurfaces

The tensors on $M$, for instance, $R^\nu_{ijk} = R^\nu_m(e_i, e_k, e_j)$, $(\nabla^2 f)_vi = (\nabla^2 g)(v, e_i)$. Differentiating the mean curvature $H = (\nabla f, v)$, we have, for $1 \leq i \leq n$,

$$e_i H = e_i (\nabla f, v) = (\nabla e_i (\nabla f), v) + (\nabla f, \nabla e_i v) = \nabla^2 f(v, e_i) + \sum_{k=1}^{n} a_{ik}(\nabla f, e_k).$$  \hspace{1cm} (8)

Then for $1 \leq i, j \leq n$,

$$e_j e_i (H) = e_j (\nabla^2 f(v, e_i)) + \sum_{k=1}^{n} e_j (a_{ik}) f_k + \sum_{k=1}^{n} a_{ik}(e_j (\nabla f, e_k)).$$  \hspace{1cm} (9)

For a fixed point $p \in \Sigma$, we may further choose the local orthonormal frame \{$e_1, \ldots, e_n$\} so that $\nabla e_i e_j(p) = (\nabla e_i e_j)^\top(p) = 0$, $1 \leq i, j \leq n$. Then at $p$, for $1 \leq i, j \leq n$,

$$e_j (\nabla^2 f(v, e_i)) = \nabla e_j (\nabla^2 f)(v, e_i) = \nabla^3 f(e_j, v, e_i) + \sum_{k=1}^{n} a_{jk} \nabla^2 f(e_k, e_i) + \nabla^2 f(v, (\nabla e_j e_i, v)v) = \nabla^3 f_{jiv} + \sum_{k=1}^{n} a_{jk}(\nabla^2 f)_{ki} - a_{ji}(\nabla^2 f)_{vv}. \hspace{1cm} (10)$$

In the third equality of (10), we used the assumption: $\nabla e_j e_i(p) = 0$, $1 \leq i, j \leq n$. Also by this assumption and the Codazzi equation, we have at $p$, for $1 \leq i, j \leq n$,

$$e_j (a_{ik}) = a_{ijk} = a_{ij,k} + R^\nu_{vikj}$$

$$e_j (\nabla f, e_k) = (\nabla e_j (\nabla f), e_k) + (\nabla f, \nabla e_j e_k) = (\nabla^2 f)_{jk} - a_{jk} f_v$$

$$(\nabla^2 H)(e_j, e_i) = e_j e_i H.$$  

Substituting these equalities and (10) into (9), we have at $p$, for $1 \leq i, j \leq n$,

$$\nabla^2 H(e_j, e_i) = \nabla^3 f_{jvi} + \sum_{k=1}^{n} a_{jk}(\nabla^2 f)_{ki} - a_{ji}(\nabla^2 f)_{vv}$$

$$+ \sum_{k=1}^{n} a_{ij,k} f_k + \sum_{k=1}^{n} R^\nu_{vikj} f_k$$

$$+ \sum_{k=1}^{n} a_{ik}(\nabla^2 f)_{jk} - \sum_{k=1}^{n} a_{ik} a_{jk} f_v. \hspace{1cm} (11)$$
On the other hand, it holds that on $\Sigma$,

\[
(\nabla^3 f)_{ivj} = (\nabla^2 f)_{vji} = (\nabla^2 f)_{jvi} = (\nabla^2 f)_{ji;v} + \sum_{k=1}^{n+1} f_k R_{kji}v
\]

\[
= (\nabla^3 f)_{vji} + f_i R_{vivj} + \sum_{k=1}^{n} f_k R_{vikj}.
\]

So

\[
\sum_{k=1}^{n} f_k R_{vikj} = (\nabla^3 f)_{ivj} - (\nabla^3 f)_{vji} - f_v R_{vivj}. \tag{12}
\]

Substituting (12) into (11) and noting that $f_v = H$, we have at $p$, for $1 \leq i, j \leq n$,

\[
(\nabla^2 H)(e_j, e_i) = (\nabla^3 f)_{jvi} + (\nabla^3 f)_{ivj} - (\nabla^3 f)_{vji} + \sum_{k=1}^{n} a_{jk}(\nabla^2 f)_{ki} + \sum_{k=1}^{n} a_{ik}(\nabla^2 f)_{jk} + \sum_{k=1}^{n} a_{ij,k} f_k
\]

\[- HR_{ivjv} - a_{ji}(\nabla^2 f)_{vv} - \sum_{k=1}^{n} a_{ik} a_{kj} H. \tag{13}
\]

Taking the trace, we have that at $p$,

\[
\Delta H = 2 \sum_{i=1}^{n} (\nabla^3 f)_{ivi} - \sum_{i=1}^{n} (\nabla^3 f)_{vii} + 2 \sum_{i,k=1}^{n} a_{ik}(\nabla^2 f)_{ki}
\]

\[+ \langle \nabla f, \nabla H \rangle - \text{Ric}_f(v, v) H - |A|^2 H. \tag{14}
\]

Since $p \in \Sigma$ is arbitrary and (14) is independent of the choice of frame, (14) holds on $\Sigma$. By (14) and $\Delta_f = \Delta - \langle \nabla f, \nabla H \rangle$, we obtain (7) and also the equivalent identity (6).

Proposition 1 yields the following

**Corollary 2** With the same assumption and notation as in Proposition 1,

\[
L_f H = 2 \text{tr}_g(\nabla^3 f(\cdot, v, \cdot)|_\Sigma) - \text{tr}_g(\nabla^3 f(v, \cdot, \cdot)|_\Sigma) + 2\langle A, \nabla^2 f|_\Sigma \rangle_g, \tag{15}
\]

or equivalently,

\[
L_f H = 2 \sum_{i=1}^{n} (\nabla^3 f)_{ivi} - \sum_{i=1}^{n} (\nabla^3 f)_{vii} + 2 \sum_{i,j=1}^{n} a_{ij}(\nabla^2 f)_{ij}. \tag{16}
\]

Next we will derive the Simons-type equation for $f$-minimal hypersurfaces.
Theorem 3 Let \((\Sigma^n, g)\) be an \(f\)-minimal hypersurface isometrically immersed in \((M, \overline{g}, e^{-f} \, d\mu)\). Then the square of the norm of the second fundamental form of \(\Sigma\) satisfies

\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + 2 \sum_{i,j,k=1}^n a_{ijk}(\overline{\text{Ric}}_f)_{jk} - (\overline{\text{Ric}}_f)_{iv} |A|^2 - |A|^4
\]

\[
+ 2 \sum_{i,j=1}^n a_{ij}(\overline{\text{Ric}}_f)_{ivj} - \sum_{i,j=1}^n a_{ij}(\overline{\text{Ric}}_f)_{ijv} + \sum_{i,j=1}^n a_{ij} \overline{R}_{ijv} - \sum_{i,j,k=1}^n a_{ijk} \overline{R}_{ijk},
\]

where the notation is the same as in Proposition 1.

Proof Simons [18] proved the following identity (see, e.g., [17] (1.20)) under a local orthonormal frame \(e_1, \ldots, e_n\) of \(\Sigma\):

\[
\Delta a_{ij} = \nabla^2 H(e_j, e_i) + \sum_{k=1}^n \overline{R}_{vik};j + \sum_{k=1}^n \overline{R}_{vij};k
\]

\[
- H \overline{R}_{vijv} - \overline{\text{Ric}}_{viv} a_{ij} + H \sum_{k=1}^n a_{ik} a_{kj} - |A|^2 a_{ij}
\]

\[
+ \sum_{k,l=1}^n (a_{ik} \overline{R}_{kijl} + a_{jk} \overline{R}_{kijl} + 2a_{kl} \overline{R}_{iljk}).
\]

Observe that

\[
\sum_{k=1}^n \overline{R}_{vik};j = (\overline{\text{Ric}})_{vij} = (\overline{\text{Ric}})_{ivj},
\]

\[
\sum_{k=1}^n \overline{R}_{vij};k = \sum_{k=1}^n \overline{R}_{jki};v = - \sum_{k=1}^n \overline{R}_{jki};v - \sum_{k=1}^n \overline{R}_{jkk};i
\]

\[
= -(\overline{\text{Ric}})_{ijv} + \overline{R}_{ijv};v + (\overline{\text{Ric}})_{ivj};i,
\]

\[
\sum_{k,l=1}^n a_{ik} \overline{R}_{kijl} = \sum_{k=1}^n a_{ik} (\overline{\text{Ric}})_{kj} - \sum_{k=1}^n a_{ik} \overline{R}_{kvj},
\]

\[
\sum_{k,l=1}^n a_{jk} \overline{R}_{kijl} = \sum_{k=1}^n a_{jk} (\overline{\text{Ric}})_{ki} - \sum_{k=1}^n a_{jk} \overline{R}_{kiv}.
\]
Using the same local frame as in the proof of Proposition 1 and substituting (13), (19), (20), (21) and (22) into (18), we have that at \( p \), for \( 1 \leq i, j \leq n \),

\[
\Delta a_{ij} = (\overline{\text{Ric}}_f)_{ij;v} + (\overline{\text{Ric}}_f)_{jv; i} - (\overline{\text{Ric}}_f)_{ij;v} + \overline{R}_{ijv;v} \\
+ \sum_{k=1}^{n} a_{ik}(\overline{\text{Ric}}_f)_{kj} + \sum_{k=1}^{n} a_{jk}(\overline{\text{Ric}}_f)_{ki} \\
- \sum_{k=1}^{n} a_{ik}\overline{R}_{kv;v} - \sum_{k=1}^{n} a_{jk}\overline{R}_{kvi} - 2 \sum_{l,k=1}^{n} a_{lk}\overline{R}_{likj} \\
+ \sum_{k=1}^{n} a_{ij,k}\overline{f}_k - (\overline{\text{Ric}}_f)_{\nu\nu} a_{ij} - |A|^2 a_{ij}.
\]

(23)

Multiply (23) by \( a_{ij} \) and take the trace. Then it holds that at \( p \),

\[
\frac{1}{2} \Delta_f |A|^2 = \frac{1}{2} \Delta |A|^2 - \frac{1}{2} \langle \nabla f, \nabla |A|^2 \rangle \\
= \sum_{i,j=1}^{n} a_{ij} \Delta a_{ij} + \sum_{i,j,k=1}^{n} a_{ij,k}^2 - \sum_{i,j,k=1}^{n} a_{ij}\overline{f}_k a_{ij,k} \\
= |\nabla A|^2 + 2 \sum_{i,j,k=1}^{n} a_{ij} a_{ik}(\overline{\text{Ric}}_f)_{kj} - (\overline{\text{Ric}}_f)_{\nu\nu} |A|^2 - |A|^4 \\
+ 2 \sum_{i,j=1}^{n} a_{ij}(\overline{\text{Ric}}_f)_{ij;v} - \sum_{i,j=1}^{n} a_{ij}(\overline{\text{Ric}}_f)_{ij;v} \\
+ \sum_{i,j=1}^{n} a_{ij}\overline{R}_{ijv;v} \\
- 2 \sum_{i,j,k=1}^{n} a_{ij} a_{ik}\overline{R}_{kvj;v} - 2 \sum_{i,j,k,l=1}^{n} a_{ij} a_{lk}\overline{R}_{likj}.
\]

(24)

Thus

\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + 2\langle A^2, \overline{\text{Ric}}_f |\Sigma \rangle - (\overline{\text{Ric}}_f)_{\nu\nu} |A|^2 - |A|^4 \\
+ 2\langle A, \nabla(\overline{\text{Ric}}_f)(\cdot, \nu, \cdot) |\Sigma \rangle - \langle A, \nabla(\overline{\text{Ric}}_f)(\nu, \cdot, \cdot) |\Sigma \rangle \\
+ \langle A, \nabla(\overline{Rm})(\nu, \cdot, \cdot, \nu) |\Sigma \rangle \\
- 2\langle A^2, \overline{Rm}(\cdot, \nu, \cdot, \nu) |\Sigma \rangle - 2 \sum_{i,j,k,l=1}^{n} a_{ij} a_{lk}\overline{R}_{likj}.
\]

(25)

Since (25) is independent of the choice of the coordinates, (17) holds on \( \Sigma \). \( \square \)
When the ambient space $M$ has the property $\text{Ric}_f = C \bar{g}$, i.e., $M$ is a gradient Ricci soliton, the Simons-type equation for $f$-minimal hypersurfaces is the following (26).

**Corollary 3** Let $(M^{n+1}, \bar{g}, e^{-f} d\mu)$ be a smooth metric measure space satisfying $\text{Ric}_f = C \bar{g}$, where $C$ is a constant. If $(\Sigma, g)$ is an $f$-minimal hypersurface isometrically immersed in $M$, then it holds that on $\Sigma$

$$\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + C |A|^2 - |A|^4 + \sum_{i, j=1}^{n} a_{ij} \bar{R}_{ij}^\nu \nu$$

$$- 2 \sum_{i, j, k=1}^{n} a_{ij} a_{ik} \bar{R}_{jvk}^\nu - 2 \sum_{i, j, k, l=1}^{n} a_{ij} a_{lk} \bar{R}_{ilj}^\nu,$$  \hspace{1cm} (26)

where the notation is the same as in Theorem 3.

Finally, in this section, we prove the following identity for $L_f$ operator, which is useful for the study of the eigenvalues of $L_f$ and $L_f$-index of $f$-minimal hypersurfaces.

**Proposition 2** Let $(M^{n+1}, \bar{g}, e^{-f} d\mu)$ be a smooth metric measure space and $X$ a parallel vector field on $M$. If $(\Sigma^n, g)$ is an $f$-minimal hypersurface isometrically immersed in $M$, then the function $\alpha : \Sigma \to \mathbb{R}$ defined by $\alpha = \langle X, \nu \rangle$ satisfies

$$\Delta_f \alpha = \text{Ric}_f(X, \nu) - |A|^2 \alpha - (\text{Ric}_f)_{\nu\nu} \alpha,$$  \hspace{1cm} (27)

$$L_f \alpha = \text{Ric}_f(X, \nu),$$  \hspace{1cm} (28)

where the notation is the same as in Theorem 3.

**Proof** Choose a local field of orthonormal frame $\{e_1, \ldots, e_n, e_{n+1}\}$ on $M$ as in the proof of Proposition 1. Then for $1 \leq i \leq n$,

$$e_i \alpha = \langle X, \nabla e_i \nu \rangle$$

$$= \sum_{j=1}^{n} a_{ij} \langle X, e_j \rangle.$$  \hspace{1cm} (29)

Note that $\nabla e_i e_j(p) = 0$, $1 \leq i, j \leq n$. The Hessian of $\alpha$ at $p$ is given by

$$\nabla^2 \alpha (e_k, e_i) = e_k e_i \alpha = \sum_{j=1}^{n} a_{ij, k} \langle X, e_j \rangle + \sum_{j=1}^{n} a_{ij} \langle X, \nabla e_k e_j \rangle$$

$$= \sum_{j=1}^{n} a_{ij, k} \langle X, e_j \rangle - \sum_{j=1}^{n} a_{ij} a_{kj} \langle X, \nu \rangle$$

$$= \sum_{j=1}^{n} a_{ik, j} \langle X, e_j \rangle + \sum_{j=1}^{n} \bar{R}_{ij} \langle X, e_j \rangle - \sum_{j=1}^{n} a_{ij} a_{jk} \alpha.$$  \hspace{1cm} (30)
Take the trace in \((30)\). Then
\[
\Delta \alpha = \langle \nabla H, X \rangle + \sum_{j=1}^{n} (\overline{\text{Ric}})_{v_j} \langle X, e_j \rangle - |A|^2 \alpha.
\] (31)

Also, from \((8)\) and \((30)\) we have at \(p\),
\[
\langle \nabla H, X \rangle = \sum_{j=1}^{n} (\nabla^2 f)_{v_j} \langle X, e_j \rangle + \sum_{j,k=1}^{n} a_{jk} f_k \langle X, e_j \rangle
\]
\[
= \sum_{j=1}^{n} (\nabla^2 f)_{v_j} \langle X, e_j \rangle + \sum_{k=1}^{n} f_k \alpha_k
\]
\[
= \sum_{j=1}^{n} (\nabla^2 f)_{v_j} \langle X, e_j \rangle + \langle \nabla f, \nabla \alpha \rangle.
\]

Substituting the above identity into \((31)\), we have
\[
\Delta \alpha = \sum_{j=1}^{n} (\overline{\text{Ric}}_f)_{v_j} \langle X, e_j \rangle + \langle \nabla f, \nabla \alpha \rangle - |A|^2 \alpha
\]
\[
= \overline{\text{Ric}}_f (X, v) + \langle \nabla f, \nabla \alpha \rangle - |A|^2 \alpha - (\overline{\text{Ric}}_f)_{vv} \alpha,
\] (32)

that is,
\[
\Delta_f \alpha = \overline{\text{Ric}}_f (X, v) - |A|^2 \alpha - (\overline{\text{Ric}}_f)_{vv} \alpha.
\] (33)

Since \(p\) is arbitrary and \((33)\) is independent of the frame, we have proved \((27)\) and then \((28)\).
\[\square\]

When the ambient manifold is a gradient Ricci soliton, we obtain

**Corollary 4** Let \((M^{n+1}, \overline{g}, e^{-f} d\mu)\) be a smooth metric measure space satisfying \(\overline{\text{Ric}}_f = C \overline{g}\), where \(C\) is a constant. Suppose that \(X\) is a parallel vector field on \(\Sigma\). If \((\Sigma, g)\) is an \(f\)-minimal hypersurface isometrically immersed in \(M\), then \(\alpha = \langle X, v \rangle\) satisfies that on \(\Sigma\),
\[
L_f \alpha = C \alpha.
\] (34)

**Example 2** [7] Let \(M = \mathbb{R}^{n+1}\) and \(f = \frac{|x|^2}{4}\). The \(f\)-minimal hypersurfaces are self-shrinkers. Suppose that \(\Sigma\) is a self-shrinker. Then
\[
\frac{1}{2} \Delta_f H^2 = |\nabla H|^2 + \left(\frac{1}{2} - |A|^2\right) H^2,
\]
\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + \left(\frac{1}{2} - |A|^2\right) |A|^2,
\]
\[
L_f H = H.
\]
If $V$ is a constant vector in $\mathbb{R}^{n+1}$ and $v$ is the unit normal to $\Sigma$, then

$$L_f \langle V, v \rangle = \frac{1}{2} \langle V, v \rangle.$$

4 \( L_f \)-Index of \( f \)-Minimal Hypersurfaces

In this section, we study the \( L_f \)-index of closed \( f \)-minimal hypersurfaces immersed in the product manifold $S^n(a) \times \mathbb{R}$, $n \geq 2$, with $f(x, t) = \frac{(n-1)t^2}{4a^2}$, where $(x, t) \in S^n(a) \times \mathbb{R}$ and $S^n(a)$ denotes the round sphere in $\mathbb{R}^{n+1}$ of radius $a$. For simplicity of notation, we only consider $a = \sqrt{2(n-1)}$ and hence $f = \frac{t^2}{4}$. The cases of other $a$ are analogous. $S^n(\sqrt{2(n-1)}) \times \mathbb{R}$ has the metric

$$g_{\bar{g}} = g_{S^n(\sqrt{2(n-1)})} + dt^2,$$

where $g_{S^n((\sqrt{2(n-1)}))}$ denotes the canonical metric of $S^n(\sqrt{2(n-1)})$.

Let $\{\bar{e}_1, \ldots, \bar{e}_{n+1}\}$ be a local orthonormal frame on $S^n(\sqrt{2(n-1)}) \times \mathbb{R}$. By a straightforward computation, one has the components of the curvature tensor and Ricci curvature tensor of $S^n(\sqrt{2(n-1)}) \times \mathbb{R}$ given by, for $1 \leq i, j, k, l \leq n+1$,

$$R_{ijkl} = \frac{1}{2(n-1)} \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \langle \bar{e}_j, \frac{\partial}{\partial t} \rangle \langle \bar{e}_l, \frac{\partial}{\partial t} \rangle \delta_{ik} - \langle \bar{e}_i, \frac{\partial}{\partial t} \rangle \langle \bar{e}_k, \frac{\partial}{\partial t} \rangle \delta_{jl} + \langle \bar{e}_j, \frac{\partial}{\partial t} \rangle \langle \bar{e}_k, \frac{\partial}{\partial t} \rangle \delta_{il} + \langle \bar{e}_i, \frac{\partial}{\partial t} \rangle \langle \bar{e}_l, \frac{\partial}{\partial t} \rangle \delta_{jk} \right),$$

and

$$\text{Ric}_{ik} = \frac{1}{2} \left( \delta_{ik} - \langle \bar{e}_i, \frac{\partial}{\partial t} \rangle \langle \bar{e}_k, \frac{\partial}{\partial t} \rangle \right).$$

On the other hand,

$$\nabla f = \frac{t}{2} \frac{\partial}{\partial t},$$

$$(\nabla^2 f)_{ik} = \frac{1}{2} \langle \bar{e}_i, \frac{\partial}{\partial t} \rangle \langle \bar{e}_k, \frac{\partial}{\partial t} \rangle.$$

By (36) and (37), we have

$$(\text{Ric})_{ik} + (\nabla^2 f)_{ik} = \frac{1}{2} \bar{g}_{ik}.$$

Hence $(S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, e^{-f} d\mu)$ is a smooth metric measure space with $\text{Ric}_f = \frac{1}{2} \bar{g}$. In addition, in the theory of Ricci flow, $(S^n(\sqrt{2(n-1)}) \times \mathbb{R}, \bar{g}, f)$ is a shrinking gradient soliton.

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For an $f$-minimal hypersurface $\Sigma$ immersed in $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$,

$$0 = H_f = H - \frac{t}{2} \langle \frac{\partial}{\partial t}, \nu \rangle = H - \frac{t}{2} \alpha,$$

where $\alpha = \langle \frac{\partial}{\partial t}, \nu \rangle$. So $\Sigma$ satisfies

$$H = \frac{t \alpha}{2}.$$

The operator $L_f$ on $\Sigma$ is

$$L_f = \Delta - \frac{t}{2} \left( \frac{\partial}{\partial t} \right)^T \cdot \nabla \cdot + |A|^2 + \frac{1}{2}. \quad (39)$$

**Lemma 1** The slice $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$ is an $f$-minimal hypersurface in $\mathbb{S}^n(\sqrt{2(n-1)}) \times \mathbb{R}$. Moreover, a complete $f$-minimal hypersurface $\Sigma$ is immersed in a horizontal slice $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{t\}$, where $t \in \mathbb{R}$ is fixed, if and only if $\Sigma$ is $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$.

**Proof** The unit normal $\nu$ to $\Sigma$ satisfies $\nu = \frac{\partial}{\partial t}$ and hence $AX = \nabla_X \nu = 0$, $X \in T\Sigma$. Thus $\Sigma$ is totally geodesic. Meanwhile,

$$\langle \nabla f, \nu \rangle = \frac{t}{2},$$

$$H_f = H - \langle \nabla f, \nu \rangle = -\frac{t}{2}. \quad (40)$$

It follows that $\Sigma$ is $f$-minimal if and only if $t = 0$. Further, by Gauss’s equation we know $\Sigma$ has constant positive section curvature and hence is closed. Since the closed $\Sigma$ has dimension $n$ and $\mathbb{S}^n$ is simply connected, $\Sigma$ must be $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$. \(\square\)

We will prove that

**Lemma 2** $L_f$-ind($\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$) = 1.

**Proof** On $\mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}$, we have $\nabla f = (\nabla f)^T = 0$, $|A|^2 = 0$. Hence,

$$L_f = \Delta_{\mathbb{S}^n(\sqrt{2(n-1)})} + \frac{1}{2}. \quad (41)$$

Thus the eigenvalues of $L_f$ are

$$\mu_k = \lambda_k - \frac{1}{2},$$

where $\lambda_k = \frac{k(k+n-1)}{2(n-1)}$, $k = 0, 1, \ldots$, are the eigenvalues of the Laplacian $\Delta_{\mathbb{S}^n(\sqrt{2(n-1)})}$. Observe that

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\[
\begin{align*}
\mu_0 &= -\frac{1}{2}, \\
\mu_k &= 0, \quad \text{for all } k \geq 1,
\end{align*}
\]
that is, \(L_f\) has a unique negative eigenvalue with multiplicity one. Therefore, the \(L_f\)-index of \(S^n(\sqrt{2(n-1)}) \times \{0\}\) is 1.

We will prove Theorem 2, which says that \(S^n(\sqrt{2(n-1)}) \times \{0\}\) is the unique closed \(f\)-minimal hypersurface of \(L_f\)-index one.

**Proof of Theorem 2** On \(\Sigma\),
\[
\nabla t = (\nabla t)^\top = \frac{\partial}{\partial t} - \langle \frac{\partial}{\partial t}, v \rangle v.
\]
So
\[
|\nabla t|^2 = 1 - (\frac{\partial}{\partial t}, v)^2 = 1 - \alpha^2.
\]
(42)
Since \(\Sigma\) is closed, there is a point \(p \in \Sigma\) such that \(t(p) = \max_\Sigma t\) and \(|\nabla t|(p) = 0\).

By equation (42), we have
\[
0 = |\nabla t|^2(p) = 1 - \alpha^2(p).
\]
(43)
Hence \(\alpha(p) = \pm 1\) and so \(\alpha \neq 0\). Since \(\frac{\partial}{\partial t}\) is a parallel vector field on \(S^n(\sqrt{2(n-1)}) \times \mathbb{R}\) and \(\text{Ric}_f = \frac{1}{2}g\), Proposition 2 implies that
\[
L_f \alpha = \text{Ric}_f \left( \frac{\partial}{\partial t}, v \right) = \frac{1}{2} \alpha.
\]
Thus \(\alpha\) is an eigenfunction of \(L_f\) with eigenvalue \(-\frac{1}{2}\) and this implies that \(L_f\)-ind(\(\Sigma\)) \(\geq 1\).

Now we consider the equality case. Lemma 2 says that \(S^n(\sqrt{2(n-1)}) \times \{0\}\) has \(L_f\)-index one. Conversely, if \(L_f\)-ind(\(\Sigma\)) = 1, then \(-\frac{1}{2}\) is the first eigenvalue. Then the corresponding eigenfunction \(\alpha\) cannot change sign. We may assume that \(\alpha > 0\). On the other hand, \(L_f \alpha = \frac{1}{2} \alpha\) and \(\text{Ric}_f = \frac{1}{2}g\) imply
\[
\Delta_f \alpha + |A|^2 \alpha = 0.
\]
Hence
\[
\Delta_f \alpha \leq 0.
\]
(44)
By the maximum principle, \(\alpha\) is constant on \(\Sigma\). On the other hand, by (43), there is a point \(p \in \Sigma\) such that \(\alpha(p) = \pm 1\). Since \(\alpha\) is positive, \(\alpha \equiv 1\). Hence \(\nabla t = 0\) and thus \(\Sigma\) is in a horizontal slice \(S^n(\sqrt{2(n-1)}) \times \{t\}\). By Lemma 1, \(\Sigma\) must be \(S^n(\sqrt{2(n-1)}) \times \{0\}\). \(\square\)
5 Pinching Theorem

First, we will derive various identities, including a Simons-type equation (see (47)) for \( f \)-minimal hypersurfaces immersed in \( S^n(\sqrt{2(n-1)}) \times \mathbb{R} \). Next, we apply them to obtain a pinching result for \( f \)-minimal hypersurfaces. We use the same notation as in Sect. 4.

**Proposition 3** Let \( \Sigma \) be an \( f \)-minimal hypersurface immersed in the product manifold \( S^n(\sqrt{2(n-1)}) \times \mathbb{R} \) with \( f = \frac{t^2}{4} \). Then

\[
\frac{1}{2} \Delta_f \alpha^2 = |\nabla \alpha|^2 - |A|^2 \alpha^2, \tag{45}
\]

\[
\frac{1}{2} \Delta_f H^2 = |\nabla H|^2 - (|A|^2 + \frac{1}{2})H^2 + \frac{1}{2} (\nabla \alpha^2, \nabla f), \tag{46}
\]

\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + |A|^2 (\frac{1}{2} - |A|^2) - \frac{1}{n-1} (|\nabla \alpha|^2 - \alpha^2 |A|^2)
- \frac{1}{n-1} (\alpha^2 f - (\nabla \alpha^2, \nabla f)). \tag{47}
\]

**Proof** Choose a local orthonormal frame field \( \{e_i\}_{i=1}^{n+1} \) for \( M \) so that, restricted to \( \Sigma \), \( \{e_i\}_{i=1}^{n} \) are tangent to \( \Sigma \), and \( e_{n+1} = \nu \) is the unit normal to \( \Sigma \). Recall that Proposition 2 states that

\[
\Delta_f \alpha = \mathbf{Ric}_f(v, \frac{\partial}{\partial t}) - |A|^2 \alpha - \mathbf{Ric}_f(v, v) \alpha. \tag{48}
\]

Substituting (38) in (48), we have

\[
\Delta_f \alpha = -|A|^2 \alpha. \tag{49}
\]

(49) implies (45):

\[
\frac{1}{2} \Delta_f \alpha^2 = |\nabla \alpha|^2 + \alpha \Delta_f \alpha
= |\nabla \alpha|^2 - |A|^2 \alpha^2.
\]

Now we prove (46). Note \( f = \frac{1}{4} t^2 \). \( \nabla^3 f = 0 \). This and Proposition 1 yield

\[
\Delta_f H = 2 \sum_{i,j=1}^{n} a_{ij}(\nabla^2 f)_{ij} - \mathbf{Ric}_f(v, v) H - |A|^2 H. \tag{50}
\]

Substituting (37) and (38) into (50), we have

\[
\Delta_f H = \sum_{i,j=1}^{n} a_{ij} \langle e_i, \frac{\partial}{\partial t} \rangle \langle e_j, \frac{\partial}{\partial t} \rangle - \frac{1}{2} H - |A|^2 H
= \langle \nabla \alpha, \frac{\partial}{\partial t} \rangle - \frac{1}{2} H - |A|^2 H.
\]
Then,

\[
\frac{1}{2} \Delta_f H^2 = |\nabla H|^2 + H \Delta_f H
\]

\[
= |\nabla H|^2 + H \langle \nabla \alpha, \frac{\partial}{\partial t} \rangle - (|A|^2 + \frac{1}{2})H^2
\]

\[
= |\nabla H|^2 + \frac{t \alpha}{2} \langle \nabla \alpha, \frac{\partial}{\partial t} \rangle - (|A|^2 + \frac{1}{2})H^2
\]

\[
= |\nabla H|^2 + \frac{1}{2} \langle \nabla \alpha^2, \nabla f \rangle - (|A|^2 + \frac{1}{2})H^2.
\]

In the above, we used \( H = \frac{t \alpha}{2} \) and \( \nabla f = \frac{t}{2} \frac{\partial}{\partial t} \). Thus (46) holds. Finally we prove (47). Since \( S^n(\sqrt{2}(n-1)) \times \mathbb{R} \) is a symmetric space, \( \nabla R = 0 \). By the Simons-type equation (Corollary 3), it holds that

\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + |A|^2 \left( \frac{1}{2} - |A|^2 \right)
\]

\[
- 2 \sum_{i,j,k=1}^{n} a_{ij} a_{ik} \bar{R}_{kivj} - 2 \sum_{i,j,k,l=1}^{n} a_{ij} a_{lk} \bar{R}_{iklj}.
\]

(51)

Substituting the curvature tensors (35) into (51) and computing directly, we obtain

\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + |A|^2 \left( \frac{1}{2} - |A|^2 \right) - \frac{1}{n-1} \left( H^2 - \alpha^2 |A|^2 \right)
\]

\[
- 2H \sum_{i,j=1}^{n} a_{ij} \langle e_i, \frac{\partial}{\partial t} \rangle \langle e_j, \frac{\partial}{\partial t} \rangle + \sum_{i,j,k=1}^{n} a_{ij} a_{ik} \langle e_j, \frac{\partial}{\partial t} \rangle \langle e_k, \frac{\partial}{\partial t} \rangle.
\]

Note that the function \( \alpha \) satisfies \( \alpha_i = \sum_{j=1}^{n} a_{ij} \langle e_j, \frac{\partial}{\partial t} \rangle \). Hence,

\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + |A|^2 \left( \frac{1}{2} - |A|^2 \right)
\]

\[
- \frac{1}{n-1} \left( H^2 - \alpha^2 |A|^2 - 2H \langle \nabla \alpha, \frac{\partial}{\partial t} \rangle + |\nabla \alpha|^2 \right)
\]

\[
= |\nabla A|^2 + |A|^2 \left( \frac{1}{2} - |A|^2 \right) - \frac{1}{n-1} \left( |\nabla \alpha|^2 - \alpha^2 |A|^2 \right)
\]

\[
- \frac{1}{n-1} (H^2 - 2H \langle \nabla \alpha, \frac{\partial}{\partial t} \rangle).
\]

Using \( \nabla f = \frac{t}{2} \frac{\partial}{\partial t} \) and \( H = \frac{t \alpha}{2} \), we obtain (47):
\[
\frac{1}{2} \Delta_f |A|^2 = |\nabla A|^2 + |A|^2 \left( \frac{1}{2} - |A|^2 \right) - \frac{1}{n-1} \left( |\nabla \alpha|^2 - \alpha^2 |A|^2 \right) \\
- \frac{1}{n-1} (\alpha^2 f - \langle \nabla \alpha^2, \nabla f \rangle).
\]

Proposition 3 implies the following equations:

**Lemma 3** If \( \Sigma \) is a closed orientable \( f \)-minimal hypersurface immersed in \( M = \mathbb{S}^n / (\sqrt{2}(n-1)) \times \mathbb{R} \), then

\[
\int_{\Sigma} |\nabla \alpha|^2 e^{-f} - \int_{\Sigma} \alpha^2 |A|^2 e^{-f} = 0, 
\]

\[
-\int_{\Sigma} |\nabla H|^2 e^{-f} + \int_{\Sigma} H^2 |A|^2 e^{-f} + \frac{1}{4} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0, 
\]

\[
\int_{\Sigma} |\nabla A|^2 e^{-f} + \int_{\Sigma} |A|^2 \left( \frac{1}{2} - |A|^2 \right) e^{-f} - \frac{1}{2(n-1)} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0. 
\]

**Proof** (52) can be obtained by integrating (45) directly. Now we prove (53). Since

\[
\Delta f = \text{tr} \nabla^2 f = \sum_{i=1}^{n} \left[ (\nabla^2 f)_{ii} - a_{ii} f_v \right] \\
= \frac{1}{2} \sum_{i=1}^{n} (e_i, \frac{\partial}{\partial t})^2 - H f_v \\
= \frac{1}{2} (1 - \alpha^2) - \langle \nabla f, v \rangle^2,
\]

\[
\Delta_f f = \frac{1}{2} (1 - \alpha^2) - \langle \nabla f, \nabla f \rangle - \langle \nabla f, v \rangle^2 \\
= \frac{1}{2} (1 - \alpha^2) - |\nabla f|^2 \\
= \frac{1}{2} (1 - \alpha^2) - f.
\]

Integrating (46) and using (55), we obtain

\[
-\int_{\Sigma} |\nabla H|^2 e^{-f} + \int_{\Sigma} (|A|^2 + \frac{1}{2}) H^2 e^{-f} \\
= \frac{1}{2} \int_{\Sigma} \langle \nabla \alpha^2, \nabla f \rangle e^{-f} \\
= -\frac{1}{2} \int_{\Sigma} \alpha^2 (\Delta_f f) e^{-f} \\
= -\frac{1}{4} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} + \frac{1}{2} \int_{\Sigma} H^2 e^{-f}.
\]
In the above we have used \( \int_{\Sigma} \Delta f H^2 e^{-f} = 0 \) and \( H^2 = \alpha^2 f \). Thus, we get (53). Finally we prove (54). Integrating (47) and using (52) and (55), we have

\[
\int_{\Sigma} |\nabla A|^2 e^{-f} + \int_{\Sigma} |A|^2 (\frac{1}{2} - |A|^2) e^{-f} = \frac{1}{n-1} \int_{\Sigma} (\alpha^2 f - \langle \nabla \alpha^2, \nabla f \rangle) e^{-f} = \frac{1}{n-1} \int_{\Sigma} (\alpha^2 f + \alpha^2 \Delta f) e^{-f} = \frac{1}{2(n-1)} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f}.
\]

Using Lemma 3, we may prove Theorem 1.

**Proof of Theorem 1** Observe that, for \( n \geq 3 \),

\[
|A|^2 (\frac{1}{2} - |A|^2) - \frac{1}{2(n-1)} \alpha^2 (1 - \alpha^2) = -(|A|^2 - \frac{1}{4})^2 + \frac{1}{4} [1 - \frac{8}{n-1} \alpha^2 (1 - \alpha^2)] \geq 0
\]

if and only if

\[
\frac{1}{4} \left(1 - \sqrt{1 - \frac{8}{n-1} \alpha^2 (1 - \alpha^2)}\right) \leq |A|^2 \leq \frac{1}{4} \left(1 + \sqrt{1 - \frac{8}{n-1} \alpha^2 (1 - \alpha^2)}\right).
\]

So (54) implies that on \( \Sigma \), for \( n \geq 3 \),

\[\nabla A \equiv 0,\]

and

\[|A|^2 \left(\frac{1}{2} - |A|^2\right) - \frac{1}{2(n-1)} \alpha^2 (1 - \alpha^2) = 0.\]

Hence, \( |A|^2 \) and \( H \) are constants. Substituting in (53), we obtain

\[
\int_{\Sigma} |A|^2 H^2 e^{-f} + \frac{1}{4} \int_{\Sigma} \alpha^2 (1 - \alpha^2) e^{-f} = 0.
\]

So

\[\alpha^2 (1 - \alpha^2) = 0.\]

This implies that on \( \Sigma \),

\[\alpha \equiv 0 \quad \text{or} \quad \alpha^2 \equiv 1.\]

Since \( \Sigma \) is closed, \( \alpha^2 \equiv 1 \). Without loss of generality, we choose \( \alpha \equiv 1 \). So \( \Sigma \) is in a horizontal slice \( \mathbb{S}^n(\sqrt{2(n-1)}) \times \{t\} \). By Lemma 1, we conclude that \( \Sigma \) is \( \mathbb{S}^n(\sqrt{2(n-1)}) \times \{0\}. \)

\( \square \)
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