A new topology on the space of Lorentzian metrics on a fixed manifold

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Abstract

We give a covariant definition of closeness between (time oriented) Lorentzian metrics on a manifold $M$, using a family of functions which measure the difference in volume form on one hand and the difference in causal structure relative to a volume scale on the other hand. These functions will distinguish two geometric properties of the Alexandrov sets $A(p,q)$, $\tilde{A}(p,q)$ relative to two space time points $q$ and $p$ and metrics $g$ and $\tilde{g}$. It will be shown that this family generates uniformities and consequently a topology on the space of Lorentzian metrics which is Hausdorff when restricted to strongly causal metrics. This family of functions will depend on parameters for a volume scale, a length scale (relative to the volume scale) and an index which labels a submanifold with compact closure of the given manifold $M$.

1 Introduction

In this paper we follow the convention that $M$ is a Hausdorff, paracompact, oriented $C^\infty$ $d+1$-dimensional manifold. When $M$ is compact, we shall assume it has a boundary otherwise the chronology violating set wouldn’t be empty, i.e. there exist timelike closed curves [8]. In the other case $M$ is assumed to be a manifold without boundary. We will first consider compact space-times for technical reasons, later on we will propose a generalization.
to the non compact case.
There exist already a great deal of topologies, most of them however do not use the specific properties which emerge from the symmetry and signature of a Lorentzian metric. An example is the $W^k$ compact - open topology which is defined by the open sets :

$$B_{X\epsilon}(g) = \{ \tilde{g} \mid \| g - \tilde{g}, X \|_{W^k} < \epsilon \}$$

and

$$\| g - \tilde{g}, X \|_{W^k} = \sqrt{\sum_{p=0}^{k} \int_{X} \| D^p(g - \tilde{g}) \|^2 d\sigma}$$

$d\sigma$ is the volume form induced by a reference Riemannian metric $h$, $X$ is an open submanifold with compact closure in $M$, $D$ is a reference derivative operator and

$$\| D^p(g - \tilde{g}) \| = \sqrt{(g - \tilde{g})_{ab;c_1...c_p} h^{ad} h^{be} h^{c_1 f_1} ... h^{c_p f_p} (g - \tilde{g})_{de;f_1...f_p}}$$

It has been proven that this topology is independent of the chosen Riemannian metric and derivative operator. This topology is more local than the $C^k$ open topology which demands that the difference of the metrics and their derivatives becomes uniformly small. The $W^k$ topology is therefore coarser than the uniform convergence topology.

A more "physical" approach was taken by L. Bombelli and R. D. Sorkin, using the fact that causal structure and conformal structure are the same when the Lorentzian metrics are future and past distinguishing. In that philosophy they defined a set of functions which compare the volume elements and causal structures of two metrics $g$, $\tilde{g}$ separately. They prove that this topology is Hausdorff when restricted to $C^2$ future and past distinguishing metrics. By definition this means that for every point $p$ there exists a unique maximal geodesic with starting point $p$ and initial direction $X_p \in T_p M$. Moreover the geodesic depends continuously on $p$ and $X_p$ in the sense that $\forall (p, X_p) \in TM \quad \forall V \subset M, V$ an open neighborhood of $\exp(p, X_p) \quad \exists U$ open in $TM$ such that $\exp(q, X_q) \in V \forall (q, X_q) \in U$ whenever $\exp(q, X_q)$ is defined.

Geometrically it seems reasonable that we do not interpret perturbations of the form $g + \delta g$, where $\delta g$ results from an infinitesimal diffeomorphism, as genuine perturbations. Therefore all "distance" functions should be fully
diffeomorphism invariant in the sense that

\[ d(g, \tilde{g}) = d(g, \psi^* \tilde{g}) \quad \forall \psi \in Diff(M) \]

In [11, 12] one starts from functions which are diagonally diffeomorphism invariant in the following sense:

\[ d(g, \tilde{g}) = d(\psi^* g, \psi^* \tilde{g}) \quad \forall \psi \in Diff(M) \] as will be the case for my construction. Later on it is argued in [11, 12] that one can take the quotient \( \mathbb{L}(M)/Diff(M) \) in a well defined way, but one cannot make any prediction anymore about the Hausdorff character of the resulting topology.

In this article we do not try to solve the problem of constructing a fully diffeomorphism invariant metric topology. We will try to give a method which will solve the problem as good as possible (within our knowledge) relying only upon contemporary functional analytical methods. There are two key concepts: on one hand there is the choice of the topology of the diffeomorphism group (the Schwartz topology) and on the other hand there is the concept of amenability. A topological group \( G \) is amenable if there exists a positive translation invariant functional \( A \) on the Banach space of the bounded Borel measurable functions on \( G \) such that \( A(1_G) = 1 \). The difficulty is captured in the concept of measurability. The famous Banach Hausdorff Tarski paradox proves the existence of a finitely additive measure on all subsets of \( \mathbb{R}^d \) when \( d \leq 2 \) which is invariant under all euclidian isometries. It says moreover that such a measure does not exist in higher dimensions. Von Neumann remarked that this curiosity is due to the structure of the isometry group and not to the space it acts upon. He proved that \( SO(3) \) endowed with the discrete topology contains a free group of two generators, and therefore fails to be amenable, ie there does not exist a translation invariant mean on the space of all bounded functions of \( SO(3) \) [4, 5]. However it is clear that \( SO(3) \) as a compact Lie group is amenable; the measurable sets here are just the Borel sets.

Suppose for the moment that the space-time manifold \( M \) is compact and define:

\[ d_{cau}(g, \tilde{g}) = \int_{M \times M} \alpha_{g, \tilde{g}}(p, q) dV(p) dV(q) \]

where

\[ \alpha_{g, \tilde{g}}(p, q) = \begin{cases} \frac{V(A(p, q) \triangle \tilde{A}(p, q))}{V(A(p, q) \cup \tilde{A}(p, q))} & \text{if } V(A(p, q) \cup \tilde{A}(p, q)) > 0 \\ 0 & \text{otherwise} \end{cases} \]
A(p, q) and \( A(p, q) \) denote the Alexandrov sets with respect to the metrics \( g \) and \( \tilde{g} \) respectively. One can define now a group action of \( Diff(M) \) on \( \mathbb{L}(M) \) by \( \psi \rightarrow \psi^* \) where \( \psi^* g \) is the pull back of \( g \) with respect to \( \psi \). It is clear that \( d_{\text{cau}}(g, \tilde{g}) \) is diagonally diffeomorphism invariant but in general

\[
d_{\text{cau}}(g, \tilde{g}) \neq d_{\text{cau}}(g, \psi^* \tilde{g})
\]

In the next chapter we will define functions \( d_{\text{vol}} \) and \( d_{\text{geo}} \) which are also diagonally but not fully diffeomorphism invariant. We will prove that these functions generate a uniformity of countable basis; hence the corresponding uniform topology is generated by a single pseudodistance \( \tilde{d} \) which is proven to be a distance on the space of the strongly causal metrics. This pseudodistance is evidently also diagonally diffeomorphism invariant.

Define now \( \forall g, \tilde{g} \) on \( Diff(M) \) the following map:

\[
f_{g, \tilde{g}}: Diff(M) \rightarrow \mathbb{R}: \psi \rightarrow d_{\text{cau}}(g, \psi^* \tilde{g})
\]

The logical question now is if for the Schwartz topology on \( Diff(M) \) and possible causal restrictions on \( g \) and \( \tilde{g} \) this map is Borel measurable. Moreover one has to raise the question if \( Diff(M) \) is amenable and, if not, whether one can find a "large enough" subgroup which is? Suppose for now that \( G \subset Diff(M) \) is a maximal amenable subgroup and \( A \) is an invariant mean.

Our strategy will be the following: first we prove that \( d_{\text{cau}}(g, \tilde{g}) \) is nonzero on the space of strongly causal \( C^2 \) metrics, then we prove that when \( g, \tilde{g} \) are globally hyperbolic the functions \( \alpha_{g, \tilde{g}} \) and \( f_{g, \tilde{g}} \) are continuous and hence measurable. It will be obvious then that the functions \( f_{g, \tilde{g}}: \phi \rightarrow d_{\text{vol}}(g, \phi^* \tilde{g}) \) and \( f_{g, \tilde{g}}: \phi \rightarrow d_{\text{geo}}(g, \phi^* \tilde{g}) \) are also continuous in the Schwartz topology. This implies that the function \( f_{g, \tilde{g}}: \psi \rightarrow d(g, \psi^* \tilde{g}) \) is continuous in the Schwartz topology on \( Diff(M) \). If we then define the function :

\[
\tilde{d}: (g, \tilde{g}) \rightarrow \begin{cases} 
A(f_{g, \tilde{g}} + f_{\tilde{g}, g}) & \text{when } \tilde{g} \neq \psi^* g \ \forall \psi \in Diff(M) \\
0 & \text{otherwise}
\end{cases}
\]

then \( \tilde{d} \) is a \( G \) invariant pseudodistance on the space of globally hyperbolic metrics. \( \tilde{d} \) is by definition symmetric, so we have to prove the triangle inequality. We have that \( \forall \phi, \psi \in G: \)

\[
d(g, \phi^* \tilde{g}) \leq d(g, \psi^* \tilde{g}) + d(\psi^* \tilde{g}, \phi^* \tilde{g})
\]

\[
\leq d(g, \psi^* \tilde{g}) + d(\tilde{g}, (\phi \psi^{-1})^* \tilde{g})
\]
because $d$ is diagonally diffeomorphism invariant. Application of $A$ to this inequality (interpreted as an inequality in functions of $\phi$), using the right invariance of $A$ and $A(1) = 1$, we get:

$$A(f_{g,\tilde{g}}) \leq d(g, \psi^*\tilde{g}) + A(f_{\tilde{g},\tilde{g}})$$

Interpreting this inequality as an inequality in functions of $\psi$ we get analogously:

$$A(f_{g,\tilde{g}}) \leq A(f_{g,\tilde{g}}) + A(f_{\tilde{g},\tilde{g}})$$

Symmetrization over $g$ and $\tilde{g}$ gives the desired result. We have made the assumption that $g$, $\tilde{g}$ and $\tilde{g}$ were not diffeomorphism equivalent; the case where one pair is equivalent is trivial. The $G$ invariance of $\tilde{d}$ is a consequence of the left invariance of $A$.

We have no argument to say that $\tilde{d}$ is a distance, but this issue is discussed in greater detail later on. So when we restrict the problem to globally hyperbolic metrics on a compact space-time, the only open question which remains is the one concerning the "size" of a maximal amenable subgroup. When $M$ is not compact the situation becomes even more complex as one will see later on. This ends the mathematical presentation of the problem.

Although the problem is purely mathematical - and it will also be treated in this way - defining a fully diffeomorphism invariant topology on $L(M)$ is an important issue in physics, especially in general relativity and quantum gravity. In general relativity and more accurately in cosmology, one would like to make a gauge invariant perturbation of a given space-time (for example a Robertson-Walker space-time). The constructions in [11, 12, 13] propose an answer to this problem. The solution in [11, 12] however has a few problems. The defining functions of the uniform topology generated by the pseudodistance $\tilde{d}$ on $L(M)$ are too sensitive to small perturbations of future and past distinguishing metrics. As an illustration of this unfortunate property, the authors show that a (non conformal) perturbation of compact support $\delta g$ of the Minkowski metric $\eta$ on $\mathbb{R}^{d+1}$ is at maximal distance 1 from the latter. This means that the mappings $\psi \rightarrow \tilde{d}(g, \psi^*\tilde{g})$ ($g \neq \tilde{g}$ future and past distinguishing) are not continuous at the identity in $Diff(M)$ equipped with the Schwartz topology. This is a property which I think is in contrast with our physical intuition. In this article we modify the definitions of [11, 12] in such a way that - as mentioned before - the actions $f_{g,\tilde{g}}$ are continuous - at least when $g$ and $\tilde{g}$ are globally hyperbolic. Since we do not give up diagonal diffeomorphism invariance (at least in the compact case), we
could at this point do exactly the same as in \cite{11,12} by taking the quotient $L(M)/\text{Diff}(M)$ and hence obtaining a fully diffeomorphism invariant topology on $L(M)$. Taking the quotient however might lead us to considerable problems concerning the Hausdorff property of the quotient topology. We will touch upon this very delicate issue in the epilogue. The problem is that $\text{Diff}(M)$ is just a too big object to handle! Our proposal - as mentioned before - is to give up the idea of a fully diffeomorphism invariant topology, and reduce the invariance to a maximal amenable subgroup. In this way we hope to recover a finer topology. We remark that in our construction as well as in \cite{11,12} the manifold $M$ is kept fixed. On the other hand is a fully diffeomorphism invariant metric topology on $L(M)$ is a vital ingredient in some formulations of quantum gravity where one wants to take a "sum" over isometrically or topologically inequivalent (compact) space times. In fact such a program has only been rigorously performed in two space time dimensions, where the genus of a compact orientable surface provides the parameter which distinguishes between homeomorphism (= diffeomorphism) classes of surfaces. In dimensions 3 and 4 one does not have a classification of homeomorphism equivalent manifolds yet. The problem becomes even more difficult in dimension 4 in the sense that topological structure is not the same as differential structure; Donaldson gave examples of four dimensional topological spaces which do not have a smooth structure, he also constructed four dimensional topological spaces which have an infinite number of inequivalent smooth structures \cite{15}. In Euclidian quantum gravity, the most serious attempt until now is the dynamical triangulations approach which is inspired on results of Gromov, who proved that the space of isometry classes of compact metric spaces is complete in the Gromov-Hausdorff topology. This approach is investigated by J. Ambjørn, R. Loll, M. Carfora and others \cite{14}. They have proven that every Riemannian structure $(M, h)$ of bounded geometry is the Gromov Hausdorff limit of a sequence of dynamically triangulated manifolds (this is a piecewise linear manifold with fixed edge length) - so in this formalism one takes the sum over inequivalent isometry structures. This approach however, makes heavy use of the properties of a Riemannian metric. Bombelli defined in \cite{13} a distance on isometry classes of Lorentzian structures by introducing the original idea of statistical Lorentzian geometry and as such tried to do the same for Lorentzian structures as Gromov did for the Riemannian counterpart. This construction however is restricted to Lorentzian structures $(M, g)$ such that $M$ has finite volume with respect to the invariant volume form defined by $g$; we believe that a generalization of
the results in [13] to structures of infinite volume is very unlikely without braking isometry invariance. We think however that further investigation of [13] is needed. In the epilogue of this article, we will propose a generalization a la Gromov, which defines a pseudodistance on the space of isometry classes of compact, future and past distinguishing structures.

2 Definitions and an example

From now on the reader may assume that all Lorentz metrics are globally hyperbolic, however this severe restriction is absolutely not necessary and the interested reader may find in appendix A the definition of ”Class $\mathcal{A}$” space times which is more adapted to our needs. Before we state our main definitions we explain some notations:

- The symmetric difference $A \triangle B$ of two sets $A$ and $B$ is defined as:
  \[ A \triangle B = (A \setminus B) \cup (B \setminus A) \]

- The Alexandrov sets $A(p, q)$ are defined as the set of all points $s$ which are in the causal future of $p$ and the causal past of $q$.

Since all metrics are assumed to be globally hyperbolic the Alexandrov sets are compact; however the Alexandrov sets for ”Class $\mathcal{A}$” space times are not necessarily closed. We will modify now the definitions in [11, 12] in an appropriate way:

**definition 1** With $g$ and $\tilde{g} \in \mathbb{L}(M)$ and $\forall p, q \in M$ $A(p, q)$ and $\tilde{A}(p, q)$ the Alexandrov sets for $g$ respectively $\tilde{g}$, define:

1. \[
\alpha_{g,\tilde{g}}(p, q) = \begin{cases} 
\frac{V(A(p, q) \triangle \tilde{A}(p, q))}{V(A(p, q) \cup \tilde{A}(p, q))} & \text{if } 0 < V(A(p, q) \cup \tilde{A}(p, q)) < \infty \\
0 & \text{otherwise}
\end{cases}
\]

2. Put $W_i \subset W_{i+1} \subset M$, $\cup_{i \in \mathbb{N}} W_i = M$ and the closure of $W_i$ is compact. Put $f : \mathbb{R}^+ \to \mathbb{R}^+$ an increasing function such that $\exists \sigma \geq 1 : f(ax) \leq a^\sigma f(x)$ $\forall a \geq 1$ and $x \in \mathbb{R}^+$. Define then the following set of functions:

\[
d_{\text{cau}}(g, \tilde{g}) = \frac{1}{f(V(W_i))} \int_{W_i \times W_i} \alpha_{g,\tilde{g}}(p, q) dV(p) dV(q)
\]
3. 

\[ d_{vol}^i(g, \tilde{g}) = \sup_{p \in W_i} \left| \ln \left( \frac{\sqrt{-g(p)}}{\sqrt{-\tilde{g}(p)}} \right) \right| \]

so this is a pseudodistance which measures the difference in volume elements.

4. 

\[ d_{geo}^i(g, \tilde{g}) = \sup_{p,q \in W_i} |\lambda(p,q) - \tilde{\lambda}(p,q)| \]

where \( \lambda(p,q) \) is zero when \( q \notin J^+(p) \) and otherwise it equals \( \sup_{\gamma \in C(p,q)} L[\gamma] \). We refer the reader for the definition of \( L \) to [8]. When \( g \) is globally hyperbolic it has been proven [8] that \( \lambda \) is continuous in \( p \) and \( q \), moreover one has that \( \lambda(p,q) \) equals the length of a non-spacelike \( g \)-geodesic curve from \( p \) to \( q \). \( d_{geo}^i(g, \tilde{g}) \) is clearly a pseudodistance.

In the introduction we already stated the main advantage of \( d_{cau}^i \) over the supremum definition:

\[ \sup_{p,q \in M:V(A(p,q) \cup \tilde{A}(p,q)) \geq \zeta} \alpha_{gg}(p,q) \]

where \( \zeta \) is a minimal volume scale that has been introduced to avoid that points which are ”too close” and are causally related in \( g \) (\( \tilde{g} \)) but not in \( \tilde{g} \) (\( g \)) put \( \alpha_{gg} \) (\( \alpha_{g\tilde{g}} \)) and hence the supremum equal to 1 [11]. But this is not the only problem of the definition, there exist also cases where the points \( p \) and \( q \) are ”far away” and define for one metric a very tiny Alexandrov set of considerable volume and of zero volume for the other one. The function \( d_{cau}^i \) distinguishes between conformally inequivalent strongly causal metrics (see proposition 3) as does the supremum definition for future and past distinguishing metrics. The pseudodistance \( d_{vol}^i \) compares the volume forms defined by both Lorentz metrics. We will use it to ”symmetrize” the distance functions \( d_{cau}^i \). The last metric compares ”the geodesic length between the points \( p \) and \( q \)” , this tells us something about the shape of the Alexandrov sets and thus delivers extra information. The author wishes to stress that for future and past distinguishing Lorentz metrics \( g \) and \( \tilde{g} \), \( d_{geo}^i(g, \tilde{g}) > 0 \) if and only if \( g \) and \( \tilde{g} \) are conformally inequivalent. This makes our topology more restrictive, which is necessary since the definition of \( d_{cau}^i \) is a considerable relaxation of the supremum definition. One could give an other definition of
by introducing another parameter $\epsilon$ as follows:

$$d^i_{\text{geo}}(g, \tilde{g}) = \sup_{p,q \in W_i : V(A(p,q)) \geq \epsilon} \left| \ln \left( \frac{\lambda(p,q)}{\tilde{\lambda}(p,q)} \right) \right|$$

We do not choose this distance function because the function $\epsilon \rightarrow d^i_{\text{vol}}(g, \tilde{g})$ is not upper continuous when $g, \tilde{g}$ are globally hyperbolic and $M$ is compact. We will come back to this later on. We will discuss the use of the function $f$ after proposition 2.

The reader might ask if the resulting topology does depend on the choice of the sequence $W_i$: we will prove this is not the case. In the sequel we will follow quite accurately the structure of [11], but we make some crucial modifications where necessary. We call two metrics $g$ and $\tilde{g}$ $(i, \zeta, \epsilon, \alpha)$ close if and only if

$$\begin{cases}
  d^i_{\text{vol}}(g, \tilde{g}) \leq \zeta \\
  d^i_{\text{cau}}(g, \tilde{g}) \leq \epsilon \\
  d^i_{\text{geo}}(g, \tilde{g}) \leq \alpha
\end{cases}$$

It is straightforward that if $g$ and $\tilde{g}$ are $(i, \zeta, \epsilon, \alpha)$ close that they are $(j, \zeta, f(V(W_j)))\epsilon, \alpha)$ close $\forall j \leq i$. In the two following propositions we prove a generalised symmetry and transitivity property.

**Proposition 1** If $(g, \tilde{g})$ are $(i, \zeta, \epsilon, \alpha)$ close then $(\tilde{g}, g)$ are $(i, \zeta, e^{(4+\sigma)}\epsilon, \alpha)$ close.

**Proof**

$$d^i_{\text{vol}}(\tilde{g}, g) \leq \zeta$$ is obvious because $d^i_{\text{vol}}$ is a pseudodistance. The inequality for $d^i_{\text{cau}}$ follows from the fact that for all Lebesgue measurable regions $O$:

$$e^{-\zeta}V(O) \leq V(O) \leq e^{\zeta}V(O)$$

The Alexandrov sets are Lebesgue measurable [23] and consequently:

$$\frac{\tilde{V}(A(p,q) \triangle \tilde{A}(p,q))}{V(A(p,q) \cup \tilde{A}(p,q))} \leq e^{2\zeta} \frac{\tilde{V}(A(p,q) \triangle \tilde{A}(p,q))}{V(A(p,q) \cup \tilde{A}(p,q))}$$

for all $p$ and $q$ such that $V(A(p,q) \cup \tilde{A}(p,q)) > 0$. Using this property we get that:

$$\frac{1}{f(\tilde{V}(W_i))} \int_{W_i \times W_i} \alpha_{\tilde{g},g}(p,q)d\tilde{V}(p)d\tilde{V}(q) \leq \frac{f(V(W_i))}{f(\tilde{V}(W_i))}e^{4\zeta}d^i_{\text{vol}}(g, \tilde{g})$$
Because $V(W_i) \leq e^\xi \tilde{V}(W_i)$, the properties of $f$ imply that:

$$\frac{f(V(W_i))}{f(V(W_i))} \leq e^{\sigma \xi}$$

which gives the result.

That $d^\text{geo}_{\text{geo}}(\tilde{g}, g) \leq \alpha$ is obvious. □

The proof clearly shows why the properties of $f$ were necessary if we later want to define a uniformity. One also immediately remarks that $\forall j, \gamma, \delta, \nu$ there exist $i, \xi, \epsilon, \alpha$ such that if $(g, \tilde{g})$ are $(i, \xi, \epsilon, \alpha)$ close then $(\tilde{g}, g)$ are $(j, \gamma, \delta, \nu)$ close.

**Proposition 2** If $(g, \tilde{g})$ and $(\tilde{g}, \bar{g})$ are $(i, \xi, \epsilon, \alpha)$ close then $(g, \bar{g})$ are $(i, 2\xi, 2(1+e^{(4+\sigma)\xi})\epsilon, 2\alpha)$ close.

**Proof** That $d^\text{vol}_{\text{vol}}(g, \bar{g}) \leq 2\xi$ is obvious because $d^\text{vol}_{\text{vol}}$ is a pseudodistance. To prove the second assertion assume $p$ and $q$ are points such that

$$\alpha_{g\tilde{g}}(p, q), \alpha_{\tilde{g}g}(p, q) < 1$$

We start by observing the following inequality:

$$V(A(p, q) \triangle \bar{A}(p, q)) \leq V(A(p, q) \triangle \bar{A}(p, q)) + V(\tilde{A}(p, q) \triangle \bar{A}(p, q))$$

so that:

$$\frac{V(A(p, q) \triangle \bar{A}(p, q))}{V(A(p, q) \cup \bar{A}(p, q))} \leq \frac{V(A(p, q) \triangle \bar{A}(p, q))}{V(A(p, q) \cup \bar{A}(p, q))} + e^\xi \tilde{V}(\tilde{A}(p, q) \triangle \bar{A}(p, q))$$

We can estimate that:

$$V(A(p, q) \cup \bar{A}(p, q)) \leq V(A(p, q) \cup \bar{A}(p, q)) + V(A(p, q) \triangle \bar{A}(p, q))$$

so we have that:

$$V(A(p, q) \cup \bar{A}(p, q)) \geq (1 - \alpha_{g\tilde{g}}) V(A(p, q) \cup \bar{A}(p, q))$$
A similar calculation shows that:

$$\alpha_{g\bar{g}}(p, q) \leq \frac{\alpha_{\tilde{g}\tilde{g}}(p, q)}{1 - \alpha_{g\bar{g}}(p, q)} + e^{2\zeta} \frac{\alpha_{\tilde{g}\tilde{g}}(p, q)}{1 - \alpha_{\tilde{g}\tilde{g}}(p, q)}$$

Now the function $x \to \frac{x}{1-x}$ is monotonically increasing and exceeds 1 at $x = \frac{1}{2}$. Because the function $\alpha_{g\bar{g}}(p, q)$ can at most obtain the value 1, we can conclude that:

$$\alpha_{g\bar{g}}(p, q) \leq 2\alpha_{g\bar{g}}(p, q) + 2e^{2\zeta} \alpha_{\tilde{g}\tilde{g}}(p, q)$$

for all $p$ and $q$. Using this result, we obtain that:

$$d_{\text{cau}}^i(g, \tilde{g}) \leq 2d_{\text{cau}}^i(g, \tilde{g}) + 2e^{(4+\sigma)\zeta} d_{\text{cau}}^i(\tilde{g}, \tilde{g})$$

which proves the claim. $\square$

Again it is easy to see that $\forall j, \gamma, \delta, \nu$ there exist $i, \zeta, \epsilon, \alpha$ such that if $(g, \tilde{g})$ and $(\tilde{g}, \tilde{g})$ are $(i, \zeta, \epsilon, \alpha)$ close then $(g, \tilde{g})$ are $(j, \gamma, \delta, \nu)$ close.

**Remark 1** All the previous properties remain unchanged when we replace $\alpha_{g\bar{g}}(p, q)$ by $z^\beta(\alpha_{g\bar{g}}(p, q))$ where $\beta \in \mathbb{R}^+$, $z$ is a $C^2$ convex function with $z(0) = 0$ and $\frac{dz}{dx}(0) \geq 0$ and $z^\beta(x) = z(\beta x)$. This is the consequence of the next inequality:

$$z^\beta(\alpha_{g\bar{g}}(p, q)) \leq \frac{1}{e^{2\zeta} + 1} z^{2\beta(1+e^{2\zeta})}(\alpha_{g\bar{g}}(p, q)) + \frac{e^{2\zeta}}{e^{2\zeta} + 1} z^{2\beta(1+e^{2\zeta})}(\alpha_{\tilde{g}\tilde{g}}(p, q))$$

One also immediately notices that the resulting topology will remain unchanged.

Now we will say more about the function $f$ introduced in definition 1. Let us introduce the following notations:

- $d'_{\text{cau}}(g, \tilde{g}) = \limsup_{i \to \infty} d_{\text{cau}}^i$
- $d_{\text{vol}}(g, \tilde{g}) = \sup_{p,q \in M} | \ln \left( \frac{\sqrt{-g(p)}}{\sqrt{-\tilde{g}(p)}} \right) |$
- $d_{\text{geo}}(g, \tilde{g}) = \sup_{p,q \in M} | \lambda(p, q) - \tilde{\lambda}(p, q) |$
and as before we define \( g \) and \( \tilde{g} \) to be \((\zeta,\epsilon,\alpha)\) close if:

\[
\begin{cases}
    d_{\text{vol}}(g, \tilde{g}) \leq \zeta \\
    d'_{\text{cau}}(g, \tilde{g}) \leq \epsilon \\
    d_{\text{geo}}(g, \tilde{g}) \leq \alpha
\end{cases}
\]

The reader can easily check then that the following modified versions of proposition 1 and 2 are valid:

- if \((g, \tilde{g})\) are \((\zeta,\epsilon,\alpha)\) close then \((\tilde{g}, g)\) are \((\zeta, e^{(4+\sigma)\zeta} \epsilon, \alpha)\) close

- if \((g, \tilde{g})\) and \((\tilde{g}, \bar{g})\) are \((\zeta,\epsilon,\alpha)\) close then \((g, \bar{g})\) are \(2\zeta, 2(1+e^{(4+\sigma)\zeta})\epsilon, 2\alpha\) close

One can also give appropriate versions of the remarks concerning propositions 1 and 2. This shows - as the reader will understand later - that the functions \(d_{\text{vol}}, d'_{\text{cau}}, d_{\text{geo}}\) define a uniform topology. There are however two objections against this construction:

- The function \(d'_{\text{cau}}\) depends on the sequence \(W_i\) and is therefore not canonically determined by \(M\).

- This topology depends on the function \(f\) for which there doesn’t seem to be a good proposal. The following example will force us to make such a proposal for \(f\), and there is no guarantee that this is a good choice under all circumstances.

**example 1** We consider flat two dimensional Minkowski space - time with the usual metric \(\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\). Now we make a perturbation \(\delta g\) of this metric of compact support \(S\). For computational simplicity we will assume that \(S\) is the rectangle of length \(m\) and height \(l\) centered around the origin. The perturbed metric looks like:

\[
\eta + \delta g = \eta + \epsilon \chi_S(1,-1) \otimes (1,-1)
\]

where \(\epsilon > 0\) is small and \(\chi_S\) is the characteristic function of support \(S\). Furthermore we define sets \(W_s = B(0,s)\) \(s \in \mathbb{N}_0\) and assume that \(s \gg l,m\).
Figure 1: shows the difference in conformal structure in the perturbed area

Using the convention that $\alpha \equiv \alpha(\epsilon)$ is the difference in opening angle of the light cones, we get the following picture:

We calculate now $d^s_{\text{cau}}(g, \tilde{g})$ for $s \to \infty$, to do this we split the double integral in a few parts. In the sequel the shaded area indicates the range of the point $q$ we integrate over. The terminology "maximal order" indicates the order of the leading term in the radius $s$ of the domain. The next picture describes a part of the integral which contributes a term of maximal order $\mathcal{O}(s^3)$ to the integral.

All other possibilities except the one in figure 6 contribute at most a factor of $\mathcal{O}(s^2)$. So if we want to know the leading order, we should first calculate the contribution described in figure 2. Keep $p$ fixed and put $x, y$ the lightcone coordinates, we obtain that:

$$V(A(p, q) \triangle \tilde{A}(p, q)) = l^2 \beta + l^2 \beta^2 + (x - \delta(p)) \sqrt{2l^2}$$

with

$$\beta = \frac{\tan(\alpha)}{1 + \tan(\alpha)}$$

and $\delta(p)$ is the riemannian length of the section of the null curve for $g$ with initial point $p$ and end point the intersection of this curve with the top
Figure 2: gives the picture where q is any point in the shaded area and p lies under the rectangle between the parallel lines.

One calculates that:

\[
\int_{W_i} \alpha_{\bar{g}, g}(p, q) dV(q) = \int_{\delta(p)}^{s} dx \int_{\sqrt{2}l\beta}^{s} dy \frac{l^2 \beta + l^2 \beta^2 + (x - \delta(p)) \sqrt{2}l\beta}{xy} + O(s^2)
\]

Integration yields:

\[
\delta(p)\sqrt{2}l\beta(l^2 \beta + l^2 \beta^2)ln\left(\frac{s}{\delta(p)}\right)ln\left(\frac{s}{\sqrt{2}l\beta}\right) + \sqrt{2}l\beta(s-\delta(p))ln\left(\frac{s}{\sqrt{2}l\beta}\right) + O(s^2)
\]

Integration over \(p\), multiplying by a factor 2 and dividing by \(f\) gives the following result:

\[
d_{\text{cau}}(g, \bar{g}) = \frac{1}{f(V(B(0, s)))} \left( \frac{l\beta m}{\sqrt{2}} s^2 ln(s) + O(s^2) \right)
\]

\[
= \frac{1}{f(V(B(0, s)))} \left( \frac{\beta V(S)}{\sqrt{2}} s^2 ln(s) + O(s^2) \right)
\]
This shows that in 1 + 1 dimensions and for $V(B(0, s)) = x \gg V(S)$ large enough $d^s_{\text{cau}} (\eta, \eta + \delta g)$ looks like:

$$d^s_{\text{cau}} (\eta, \eta + \delta g) \sim \frac{\beta V(S)}{2\sqrt{2\pi f(x)}} x \ln(x)$$

In order to make the limit $s \to \infty$ nonzero and finite, $f$ has to be proportional to $x \ln(x)$ for $x$ big enough. It is easy to see that $\forall x \geq e$ and $a \geq 1$:

$$(ax)\ln(ax) \leq a^2 x \ln(x)$$

Therefore it is convenient to define:

$$f(x) = \begin{cases} 
    x & \text{if } 0 \leq x \leq e \\
    x \log(x) & \text{otherwise}
\end{cases}$$

This shows -as anticipated before- that the asymptotics of $f$ is completely determined by this example modulo a proportionality constant. The author tried to check if one gets the same result in higher dimensions, but this seemed to be an impossible task (for example the calculation of the intersection of
Figure 4: is of maximal order $O(s^2)$

regular volumes is not a triviality). Probably the only approach here is the numerical one. This example also shows that the result depends on the volume of the distorted area and a "deflection parameter" $\beta$ which is obviously a much better result than the one obtained in [11, 12].

All the previous remarks lead us to the conclusion that in the non compact case we have to make another approach than just taking the limsup and therefore we have to dispose of $d'_{\text{cau}}$. In section 6 the reader will see that our final proposal for a topology does not depend of the function $f$ and therefore the reader can assume that $f = 1$ unless stated otherwhise.

3 Properties of the distance functions when certain causality requirements are satisfied

We begin this section by a review some terminology [8].

terminology

- $p \prec q$ if and only if $\exists$ a future oriented timelike curve from $p$ to $q$
- $p \preceq q$ if and only if $\exists$ causal curve from $p$ to $q$
- $p \rightarrow q$ if and only if $q \in J^+(p) \setminus I^+(p)$
Figure 5: is of maximal order $O(s^2)$

- If $O$ is open in $M$, $\prec_O$, $\preceq_O$ and $\rightarrow_O$ are defined analogously with the extra requirement that all connecting causal (timelike) curves must lie in $O$. These are the partial order relations defined by the sets $I^\pm(p, O), J^\pm(p, O)$

- The restriction of $\prec$ to $O \prec_{|O}$ is defined in the usual way.

- Define by $C(p, q)$ the set of all $C^0$ causal curves with initial end point $p$ and final endpoint $q$.

- A map $f$ from a topological space $O$ to the set of all subsets of $O$ is called outer continuous in a point $p \in O$ if and only if $\forall K \subset O \setminus f(p)$ compact, there exists a neighborhood $V$ of $p$ such that for all $r \in V$ it follows that $f(r) \cap K = \emptyset$.

- A space-time $(M, g)$ is causally continuous if and only if the map

  $$I^+: p \rightarrow I^+(p)$$

  is outer continuous.
remark 2 In the definition of outer continuity for the map $f$ it is usually assumed that $f(p)$ is open $\forall p \in O$. We do not follow this convention here because it is not essential.

remark 3 On $C(p,q)$ we define a topology with the following basis:

$$O_W(\gamma) = \{ \lambda \in C(p,q) | \lambda \subset W \}$$

$W$ is an open neighborhood of $\gamma$ in $M$. Now $g$ is globally hyperbolic on $M$ if and only if $g$ is strongly causal and $C(p,q)$ is compact $\forall p, q \in M$. This implies that if $K_1, K_2$ are compact then $C(K_1, K_2)$ is compact. Another (equivalent) criterion is that $g$ is strongly causal and $A(p,q)$ is compact $\forall p, q \in M$.

With these concepts, we can now prove the following theorem.

**Proposition 3** If $g$ is future and past distinguishing and $\tilde{g}$ is a strongly causal metric which are not conformally related in $p \in M$ then $\exists i_0 \in \mathbb{N}$ such that $\forall j \geq i_0 : d_{\text{can}}^j (g, \tilde{g}) > 0$

**Proof**
Choose \( i_0 \) large enough such that \( p \in W_{i_0} \). All the following sets are assumed to be subsets of \( W_{i_0} \). We have to prove that \( \exists X \) of compact closure such that \( \bar{X} \subset I^+(p) \setminus \bar{J}^+(p) \) and an open subset \( Y \) such that \( \forall r \in Y : X \subset I^+(r) \setminus \bar{J}^+(r) \). Then \( \alpha_{\bar{g}\bar{g}}(r, s) = 1 \) on the productset \( X \times Y \).

Choose \( \bar{U} \) to be a simple neighborhood of \( p \) for \( \bar{g} \) such that \( \bar{g}_{|\bar{U}} \) is causally continuous. Because \( \bar{g} \) is strongly causal there exists \( \bar{V} \subset U \) such that no \( \bar{g} \) - causal curve intersects \( \bar{V} \) more than once. This implies that on \( \bar{V} \), \( \bar{\lambda} \) is equivalent to \( \bar{\lambda}\bar{V} \) and the same for \( \bar{\lambda}\). Now it is obvious that on \( \bar{V} \) \( \exists X \) of compact closure such that \( \bar{X} \subset I^+(p) \setminus \bar{J}^+(p) \)

(otherwise \( g \) and \( \bar{g} \) would be conformally related in \( p \) because both of them are future and past distinguishing in \( p \)). Because of the causal continuity of \( \bar{g}_{|\bar{U}} \) there exists a \( \bar{Z} \) such that \( p \in \bar{Z} \) and \( \forall r \in Z : \bar{X} \cap \bar{J}^+(r, \bar{V}) = \emptyset \)

but this implies that \( \bar{X} \cap \bar{J}^+(p) = \emptyset \) because of the properties of \( \bar{V} \). Put \( Y = Z \cap I^-(p) \) , this set clearly satisfies all the above properties. \( \square \)

This theorem indicates that our construction is not trivial. We know that the function \( \alpha_{\bar{g}\bar{g}} \) is measurable, we now prove that it is continuous almost everywhere if \( g \) and \( \bar{g} \) are both globally hyperbolic.

**Proposition 4** If \( g \) and \( \bar{g} \) are globally hyperbolic then \( \alpha_{\bar{g}\bar{g}} \) is almost everywhere continuous.

**Proof**

We split the proof in a few parts.

- Suppose \( p, q \in M \) we prove that the function \( (r, s) \rightarrow A(r, s) \) is outer continuous in \( (p, q) \) (and of course the same works for \( \bar{g} \)). Assume not, then one has a compact set \( K \subset M \setminus A(p, q) \), sequences \( r_n \rightarrow p \) and \( s_n \rightarrow q \) such that \( r_n \prec r_m \prec p \) and \( q \prec s_m \prec s_n \) \( \forall n < m \). Moreover one has causal curves \( \lambda_n \) with initial endpoint \( r_1 \) and final endpoint \( s_1 \), passing trough \( r_n \) and \( s_n \), which intersect \( K \). Because \( C(1, s_1) \) and \( K \) are compact we can find a subsequence \( \lambda_{n_k} \rightarrow \lambda \in C(1, s_1) \) where \( \lambda \) is a causal curve passing trough \( p \) and \( q \) which intersects \( K \). This is a contradiction.

- It is now obvious that the function \( (r, s) \rightarrow A(r, s) \cup \bar{A}(r, s) \) is outer continuous in \( (p, q) \). Choose \( K \subset M \setminus (A(p, q) \cup \bar{A}(p, q)) \) then \( K \subset M \setminus A(p, q) \), \( M \setminus \bar{A}(p, q) \) so there exist \( R, \bar{R} \) neighborhoods of \( p \) and \( S, \bar{S} \) neighborhoods of \( q \) such that \( \forall r \in R \cap \bar{R}, s \in S \cap \bar{S} : K \cap A(r, s) = \emptyset = K \cap \bar{A}(r, s) \) so \( K \cap (A(p, s) \cup \bar{A}(r, s)) = \emptyset \).
We can now say that \((r, s) \to V(A(r, s) \cup \bar{A}(r, s))\) is continuous in \((p, q)\). It is obvious that \(\forall K \subset A(p, q) \cup \bar{A}(p, q)\) compact \(\exists R, S\) neighborhoods of \(p\) and \(q\) respectively such that \(\forall r \in R, s \in S : K \subset A(r, s) \cup \bar{A}(r, s)\). Now taken in account that \(A(p, q) \cup \bar{A}(p, q)\) is compact and \(M\) is Hausdorff, \(\forall \epsilon > 0\) sufficiently small \(\exists K_\epsilon\) compact and \(U_\epsilon\) open such that \(K_\epsilon \subset A(p, q) \cup \bar{A}(p, q) \subset U_\epsilon\) and \(V(U_\epsilon) < V(A(p, q) \cup \bar{A}(p, q)) + \epsilon\) and \(V(K_\epsilon) > V(A(p, q) \cup \bar{A}(p, q)) - \epsilon\). Because of the previous remark and the outer continuity of \((r, s) \to A(r, s) \cup \bar{A}(r, s)\) we can find neighborhoods \(R_\epsilon\) of \(p\) and \(S_\epsilon\) of \(q\) such that \(K_\epsilon \subset A(r, s) \cup \bar{A}(r, s) \subset U_\epsilon\ \forall \epsilon \in R_\epsilon\) and \(s \in S_\epsilon\). The result is now obvious.

With the same techniques it is easy to prove that \((p, q) \to V(A(p, q) \triangle \bar{A}(p, q))\) is continuous everywhere.

The only problem which might arise in the definition of \(\alpha_{\bar{g}}\) is when \(V(A(p, q) \cup \bar{A}(p, q)) = 0\). Suppose now that \(A(p, q) = \emptyset = \bar{A}(p, q)\) then it is easy to see that exist open neighborhoods \(X\) of \(p\) and \(Y\) of \(q\) of compact closure such that \(\forall r \in X, s \in Y : A(r, s) = \emptyset = \bar{A}(r, s)\) so the continuity is proven in this case.

There is only left the case where \(A(p, q)\) or \(\bar{A}(p, q)\) is nonempty but has zero volume. This implies that \(q \in \partial J^+(p)\) or \(q \in \partial \bar{J}^+(p)\), but this set has zero volume; the result is now obvious. □

In the proof we used the simple fact that \(V(A(p, q)) > 0\) if and only if \(p \in I^-(q)\) which was proven in [11]. Proposition 3 will give the key ingredient for the fact that the resulting topology on \(\mathbb{L}(M)\) will be Hausdorff on the subset of strongly causal \(C^2\) metrics. It is now interesting to remark that if \(g\) is strongly causal the manifold topology is the same as the corresponding Alexandrov topology [8]. Moreover the group of conformal \(C^\infty\) diffeomorphisms is in this case equal to the group of the \(\rho_g\) - homeomorphisms, where the \(\rho_g\) - topology is defined as the finest topology for which \(E \subset M\) belongs to \(\rho_g\) if and only if \(\forall \gamma (\gamma\ a\ g\ timelike\ curve) \exists O\ open\ in\ M\ such\ that\ \O \cap \gamma = E \cap \gamma\). This fact is surprising because at first sight the \(\rho_g\) - topology is not that much finer than the manifold topology. This is a very useful characterization of the conformal group, so in this case the set
of conformal structures agrees with the quotient of \( \mathbb{L}(M) \) with the following equivalence relation:

\[
g \sim \tilde{g} \iff \exists \psi \in Diff_{\rho_g}(M)
\]
such that

\[
\psi^* g = \tilde{g}
\]

\( Diff_{\rho_g}(M) \) is the subgroup of the \( \rho_g \) continuous homeomorphisms \[10].

In the next section we will give a very quick introduction to uniformities and topologies. The reader who wants a more thorough treatment is invited to read \[22\].

4 A quick review of uniformities and topologies

Let \((X,d)\) be a topological space where \(d\) is a (pseudo) distance and denote by \(\tau\) the corresponding locally compact topology. It is an elementary fact that the open balls \(B_{1/n}(p)\) with radius \(1/n : n \in \mathbb{N}_0\) around \(p\) define a countable basis for \(\tau\) in \(p\). In this chapter \(I,J\) will denote index sets. A \((X, \tau)\) cover \(C\) is defined as follows:

\[
C = \{A_i | A_i \in \tau, i \in I\}
\]
such that

\[
\bigcup_{i \in I} A_i = X
\]

If \(C = \{A_i | A_i \in \tau, i \in I\}, D = \{B_j | B_j \in \tau, j \in J\}\) are \((X, \tau)\) covers then we say that \(C\) is finer than or is a refinement of \(D\), \(C < D\) if and only if

\[
\forall i \in I \quad \exists j \in J : A_i \subset B_j
\]

Next we define a few operations on the set of covers \(C(X, \tau)\):

**Operations on covers**

- Let \(C, D\) be as before,

\[
C \land D = \{A_i \cap B_j | A_i, B_j \in \tau, i \in I, j \in J\}
\]

\(C \land D\) is obviously a cover, moreover the doublet \(C(X, \tau), \land\) is a commutative semigroup.
• For $A \subset X$ the star of $A$ with respect to $C$ is defined as follows:

$$St(A, C) = \bigcup_{A_i \in C : A \cap A_i \neq \emptyset} A_i$$

• The star of $C, C^*$ is then defined as:

$$C^* = \{St(A_i, C) | A_i \in C\}$$

Remark that $C < C^* < C^{**} \ldots$ and that if $I$ is finite then there exists a $n \in \mathbb{N}$ such that after $n$ star operations $C$ has become the trivial cover.

Using the topological basis of open balls, we can define elementary covers $C_n$, $n \in \mathbb{N}_0$ as follows:

$$C_n = \{B_{1/n}(p) | p \in X\}$$

These elementary covers now define a subset $U$ of $C(X, \tau)$:

$$U = \{C \in C(X, \tau) | \exists C_n : C_n < C\}$$

The set $U$ satisfies the following obvious properties:

1. If $C \in U$ and $C < D$ then $D \in U$
2. If $C, D \in U$ then $C \wedge D \in U$
3. If $C \in U$ then $\exists D \in U : D^* < C$

From now on we take the above properties as a definition for a uniformity:

**definition 2** Let $X$ be a set, a cover $C$ is defined as:

$$C = \{A_i | A_i \subset X, i \in I\}$$

such that

$$\bigcup_{i \in I} A_i = X$$

A collection of covers $U$ is called a uniformity for $X$ if and only if

1. If $C \in U$ and $C < D$ then $D \in U$
2. If $C, D \in U$ then $C \wedge D \in U$

3. If $C \in U$ then $\exists D \in U : D^* < C$

where all definitions of $<$, $\wedge$ and $*$ are independent of $\tau$.

It has been proven that any uniformity can be generated by a family of pseudodistances [24]. This indicates a uniformity defines a topology. For our applications we need a different ingredient.

**definition 3** Let $I$ be a directed net, and suppose $B_i(x) \subset X$ satisfy the following properties:

1. $x \in B_i(x)$ $\forall x \in X, i \in I$

2. If $i \leq j$ then $B_i(x) \subset B_j(x)$ $\forall x \in X$

3. $\forall i \in I, \exists j \in I$ such that $\forall y \in B_j(x) : x \in B_i(y)$

4. $\forall i \in I, \exists j \in I$ such that if $z \in B_j(y), y \in B_j(x)$ then $z \in B_i(x)$.

then we call the family of all $B_i(x)$ a **uniform neighborhood system**.

Now it has been proven that if $\{B_i(x) \mid x \in X, i \in I\}$ is a uniform neighborhood system then the family of covers:

\[
C_i = \{B_i(x) \mid x \in X\}
\]

$i \in I$ is a basis for a uniformity on $X$ [22]. On the other hand every uniformity can be constructed from a uniform neighborhood system.

The topology $\tau_U$ defined by a uniformity $U$, the **uniform topology**, is constructed as follows:

\[
O(x) \in \tau_U \iff \exists C \in U : St(x,C) \subset O(x)
\]

so $\{St(x,C) \mid x \in X, C \in U\}$ defines a basis for the topology. The topology is Hausdorff if and only if $\bigcap_{O(x) \in \tau_U} O(x) = \{x\}$ but it is not difficult to see that this is equivalent with:

\[
\bigcap_{i \in I} B_i(x) = \{x\}
\]

where $\{B_i(x) \mid i \in I, x \in X\}$ is the uniform neighborhood system which generates $U$. To indicate the reasoning followed in [11] we will just state a few facts about **quotient uniformities**. **Terminology**
• Let \((X, U)\) and \((Y, V)\) be uniform spaces, a map \(f : X \to Y\) is uniformly continuous if and only if

\[\forall C \in V : f^{-1}(C) \in U\]

where for \(C = \{A_i | i \in I\}\), \(f^{-1}(C) = \{f^{-1}(A_i) | i \in I\}\).

• A uniformity \(\tilde{U}\) on \(X\) is finer than \(U\) if and only if every cover in \(U\) belongs to \(\tilde{U}\).

• Let \(\pi : X \to \tilde{X}\) be a surjective map and \((X, U)\) a uniform space, the quotient uniformity \(\tilde{U}\) on \(\tilde{X}\) is the finest uniformity which makes \(\pi\) uniformly continuous.

Notice that the existence of a quotient uniformity is guaranteed by the lemma of Zorn, the uniqueness is immediate. The obvious question now is if \(\tau_{\tilde{U}}\) is equal to the quotient topology of \(\tau_{U}\). The answer is in general no, but under some special circumstances it works.

definition 4 A uniform neighborhood system \(\{B_i(x) | x \in X, i \in I\}\) is compatible with an equivalence relation on \(X\) if and only if

\[\forall i \in I, x' \sim x \text{ and } y \in B_i(x) \exists y' \sim y : y' \in B_i(x')\]

As envisaged, compatibility implies that \(\tau_{\tilde{U}}\) is equal to the quotient topology of \(\tau_{U}\).

Proposition 5 If \(U\) is generated by \(\{B_i(x) | i \in I, x \in X\}\) which is compatible with \(\sim\) which is for example defined by a surjective map, then the quotient uniformity \(\tilde{U}\) on \(\tilde{X} = X/\sim\) is generated by the uniform neighborhood system defined by:

\[\tilde{B}_i(\tilde{x}) = \{\tilde{y} | \exists x \in \tilde{x} \text{ and } y \in \tilde{y} : y \in B_i(x)\}\]

\[\forall \tilde{x} \in \tilde{X}, i \in I. \text{ Moreover } \tau_{\tilde{U}} \text{ is equal to the quotient topology of } \tau_{U} \text{ and a basis of neighborhoods of } \tilde{x} \in \tilde{X} \text{ is } \{\tilde{B}_i(\tilde{x}) | i \in I\}\]

As mentioned, every uniformity can be generated by a family of pseudodistances. In the case that the uniformity is generated by a countable uniform neighborhood system, the topology is defined by one pseudodistance, which is a distance when the uniformity is Hausdorff. Suppose \(C_n = \{B_n(x) | x \in X\}\),
$n \in \mathbb{N}$, is a countable basis for a uniformity $U$, then we can find a subsequence $(n_k)_k$ such that:

$$\forall k, \ w \in B_{n_k}(z), z \in B_{n_k}(y), y \in B_{n_k}(x) \Rightarrow w \in B_{n_{k-1}}(x)$$

Assume $C_n$ is such a basis.

**Proposition 6** Let $C_n$ be a countable basis of $U$, then with

$$\rho(x, y) = \inf_{\{n \geq 0, y \in B_n(x)\}} 2^{-n}$$

the function

$$d(x, y) = \inf_{K \in \mathbb{N}, x_k} \sum_{k=1}^{K} \frac{1}{2} (\rho(x_{k-1}, x_k) + \rho(x_k, x_{k-1}))$$

is a pseudodistance which generates $U$. $\{x_0, \ldots, x_K\}$ with $x_0 = x, x_K = y$ is a path in $X$. If $U$ is Hausdorff then $d$ is a distance.

Note that the function $d$ depends on the choice of basis $C_n$ and is therefore not canonical. In the next chapter we apply this machinery to the distance functions $d^c_{\text{cau}}, d^i_{\text{vol}}$ and $d^i_{\text{geo}}$.

5 A topology on $\mathbb{L}(M)$ in the case $M$ is compact

In this section we first introduce the Schwartz topology on $\text{Diff}(M)$, next we prove that $f^c_{g,\tilde{g}}$ is continuous in this topology when $g, \tilde{g}$ are globally hyperbolic. Then we give some characterizations of amenability and give some hints why we think that large amenable subgroups $G$ can be found. With a left invariant mean on $L^\infty(G)$ we can symmetrize the distance functions $d^c_{\text{cau}}, d^i_{\text{vol}}, d^i_{\text{geo}}$, these functions will define a uniformity and thus a $G$ invariant pseudodistance on the space of globally hyperbolic metrics.

5.1 A topology on $\text{Diff}(M)$

We first state the result (in a slightly more general way as needed) and then show the major lines of the construction [21] [20].
Proposition 7  Let $M$ and $N$ be ordinary manifolds with $M$ compact. Then $C^\infty(M,N)$ has the structure of a $C^c$ manifold. The local model at $f \in C^\infty(M,N)$ is given by the nuclear Fréchet space $C^\infty_f(M,TN)$.

One proves that $\text{Diff}(M)$ is a $C^\infty$ open submanifold of $C^\infty(M,M)$, moreover $\text{Diff}(M)$ is a $C^\infty$ Lie group. It is also proven that the composition and inversion are continuous.

First we recall when a map $f$ is $C^c$. Let $E,F$ be locally convex Hausdorff linear spaces and $W \subset E$ be open, then $f : W \to F$ is $C^1$ if and only if there exists a linear mapping $Df : E \to L(E,F)$ such that

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} = Df(x)v \quad \forall v \in E, \quad x \in W, \quad t \in \mathbb{R}$$

such that the mapping $W \times E \to F : (x,y) \to Df(x)y$ is continuous. The set of all $C^1$ mappings is a linear space. The space $C^k_c$ is defined by recursivity: a map $f$ is $C^k_c$ if $D^{k-1}f : W \times E^{k-1} \to F$ is $C^1_c$. Finally

$$C^\infty_c = \bigcap_{k \geq 1} C^k_c$$

Now for Fréchet spaces this concept of smoothness is equivalent to the following: a map $f : W \subset E \to F$ is smooth if and only if every $C^\infty_c$ curve on $W$ is mapped to a $C^\infty_c$ curve on $F$. Many results, which we will state now, can be generalized to a larger class of locally convex Hausdorff linear spaces [18]. Let $W \subset \mathbb{R}^n$, then the topology on $C^\infty(W,F)$ where $F$ is an euclidian space, is defined by the family $\rho_{m,K}$ of seminorms. Let $m \in \mathbb{N}_0$, $p = (p_1,p_2,\ldots,p_n) \in \mathbb{N}^n$ and $|p| = \sum_{i=1}^n p_i$, then for any compact $K \subset W$

$$\rho_{m,K}(f) = \sup_{|p| \leq m} \sup_{x \in K} \| D^p f(x) \|$$

where

$$D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_n^{p_n}}$$

It is well known that this family of seminorms makes $C^\infty_c(W,F)$ into a nuclear Fréchet space [3,4]. Now let $M$ be any ordinary manifold and $(U,\phi)$ be a chart, the map

$$\phi^* : C^\infty(U,F) \to C^\infty(\phi(U),F) : f \to f \circ \phi^{-1}$$
is a linear isomorphism and induces on $C^\infty(U, F)$ the structure of a nuclear Fréchet space. Because $M$ is second countable there exists a countable covering of charts $(U_k, \phi_k)$ such that one can construct the following restriction maps

$$C^\infty(M, F) \to C^\infty(U_k, F)$$

which implies that

$$C^\infty(M, F) = \lim_k C^\infty(U_k, F)$$

where this limit is a projective limit of nuclear Fréchet spaces and is hence a nuclear Fréchet space \cite{17, 1, 3}. This topology is known as the Schwartz topology. One can introduce jet bundles in order to characterize the Schwartz topology, the reader is referred to \cite{20}.

Let $\exp : U \subset TN \to N$ be the exponential map associated to a Riemannian metric $h$ on $N$, $U$ an open neighborhood of the zero section on which the exponential map is defined and $\pi_N : TN \to N$ the canonical projection. One can choose $U$ such that

$$\nu \equiv (\pi_N, \exp) : U \to N \times N : v \mapsto (\pi_N(v), e^\pi_N(v)(v))$$

is a diffeomorphism from $U$ to a neighborhood $V$ of the diagonal in $N \times N$. Now we can define the local model $C^\infty_f(M, TN)$ of $f \in C^\infty(M, N)$:

$$C^\infty_f(M, TN) = \{ g \in C^\infty(M, TN) \mid \pi_N \circ g = f \}$$

$C^\infty_f(M, TN)$ is a linear space which can be identified with $Sec(E)$ in the bundle $(E, \pi_E, M)$ where $E$ is defined as follows:

$$E = \{(x, v) \mid v \in TN_{f(x)} \}$$

The topology on $E$ is defined by the open sets $O_{W, U} = \{(x, v_{f(x)}) \mid x \in W \text{ and } v_{f(x)} \in U \}$ where $W$ is open in $M$ and $U$ is open in $TN$ with $f(W) \subset \pi_N(U)$. The differential structure is defined by the charts $(O_{W, U}, (\chi, pr_2 \circ \psi))$ where $(W, \chi), (U, \psi)$ are charts in $M$ respectively $TN$, with $\psi(U) \subset V \times F$ and $pr_2$ is the projection on the second factor. We clearly can endow the $C^\infty$ sections on $E$ with the Schwartz topology, so this induces on $C^\infty_f(M, TN)$ the structure of a nuclear Fréchet space.

We can now define the charts on $C^\infty(M, N)$. Define

$$U_f = \{ g \in C^\infty(M, N) \mid (f(x), g(x)) \in V \ \forall x \in M \}$$
This induces the map:

\[ u_f : U_f \rightarrow C_f^\infty(M, TN) \]

with \( u_f(g)(x) = \exp_{f(x)}^{-1}(g(x)) = (\nu^{-1} \circ (f \times g))(x) \).

\( u_f \) maps \( U_f \) bijectively to \( \{ s \in C_f^\infty(M, TN) | s(x) \in U \} \) which is by definition open in the Schwartz topology because \( M \) is compact. Now the inverse of \( u_f \) is given by:

\[ u_f^{-1}(s)(x) = \exp_{f(x)}(s(x)) \]

so \( u_f^{-1}(s) = (\text{pr}_2 \circ \nu) \circ s \). The atlas on \( C^\infty(M, N) \) is hence defined as

\[ \{(U_f, u_f) | f \in C^\infty(M, N)\} \]

One calculates the transition maps and proves that they are \( C^\infty_c \). It is clear that if \( U_f \cap U_g \neq \emptyset \) that \( u_f \circ u_g^{-1} : u_g(U_f \cap U_g) \rightarrow u_f(U_f \cap U_g) \) is given by:

\[ ((u_f \circ u_g^{-1})(s))(x) = u_f(x)(\exp_g(x)(s(x))) = \exp_{f(x)}^{-1}(\exp_g(x)(s(x))) \]

or

\[ u_f \circ u_g^{-1} = \exp_{f(x)}^{-1} \circ \exp_g \]

This transition map is \( C^\infty_c \) if and only if they map smooth curves to smooth curves. But the smooth curves of \( C_f^\infty(M, TN) \) correspond with the smooth sections of the bundle

\[ \mathbb{R} \times E \rightarrow \mathbb{R} \times M \]

which are preserved by transition maps. It is easy to see that the differential structure is independent of the chosen Riemannian metric so we arrive at our result. It is worthwhile noticing that \( \exp \) is defined everywhere on \( TM \) and that in \( M \) every two points can be connected by a unique \( h \)-geodesic (this is a special case of the Hopf - Rinow theorem) [9].

**Proposition 8** Let \( g, \tilde{g} \) be globally hyperbolic, the map \( f_{g, \tilde{g}}^{cu} \) is continuous in the Schwartz topology on \( \text{Diff}(M) \)

**Proof**

In this proof \( \tilde{d} \) denotes the distance on \( M \) corresponding with the Riemannian metric \( h \); we shall also use the shorthand notation \( V^2 \equiv V \times V \). We will prove
the upper continuity, the lower continuity is identical. It is clearly sufficient to prove the continuity of \( f^\text{cau}_{\tilde{g}} \) in the identity diffeomorphism, for every other diffeomorphism the proof repeats almost ad verbatim. The Alexandrov sets of \( \phi^*\tilde{g} \) satisfy the following property:

\[
A_{\phi^*\tilde{g}}(p, q) = \phi^{-1}(\tilde{A}(\phi(p), \phi(q)))
\]

This implies that:

\[
\alpha_{g\phi^*\tilde{g}}(p, q) = \frac{V(A(p, q) \triangle \phi^{-1}(\tilde{A}(\phi(p), \phi(q))))}{V(A(p, q) \cup \phi^{-1}(\tilde{A}(\phi(p), \phi(q))))}
\]

We introduce the following notations:

- \( \mathcal{N} = \{(p, q) | q \in J^+(p) \setminus \tilde{J}^+(p)\} \)
- \( \tilde{J}^+ = \{(p, q) | q \in \tilde{J}^+(p)\} \)
- \( \mathcal{O}_\phi = \{(p, q) | q \notin \tilde{J}^+(p) \text{ but } \phi(q) \in \tilde{J}^+(\phi(p))\} \)

It is clear that \( \alpha_{g\phi^*\tilde{g}}(p, q) > 0 \) implies that \((p, q) \in \mathcal{N} \cup \mathcal{O}_\phi \cup \tilde{J}^+ \). This implies that

\[
d_{\text{cau}}(g, \phi^*\tilde{g}) \leq V^2(\mathcal{N}) + V^2(\mathcal{O}_\phi) + \int_{\tilde{J}^+} \alpha_{g\phi^*\tilde{g}}(p, q)dV(p)dV(q)
\]

Choose \( \epsilon > 0 \), we show first that the second term on the right hand side can be made smaller than \( \frac{\epsilon}{4} \). For this purpose we define for every \( X \subset M \) the set \( X_\zeta = \{y \in M \setminus X | \exists x \in X : \bar{d}(x, y) < \zeta\} \). The reader can check that the mapping \((\zeta, p) \rightarrow V((J^+(p))_\zeta)\) is continuous. Hence there exists a \( \delta_0 \) such that

\[
V^2(\bigcup_{p \in M} \{p\} \times (\tilde{J}^+(p))_\delta) < \frac{\epsilon}{4}
\]

for all \( \delta < \delta_0 \). If \( \phi \in C^\infty_{id}(M, TM) \) satisfies the property that \( \forall p \in M : \bar{d}(p, \phi(p)) < \delta_0 \) then one has that:

\[
\mathcal{O}_\phi \subset \bigcup_{p \in M} \{p\} \times (\tilde{J}^+(p))_{\delta_0}
\]

and hence

\[
V^2(\mathcal{O}_\phi) < \frac{\epsilon}{4}
\]

29
The condition on $\phi$ determines an open subset $U_{\delta_0} \subset C^\infty_{id}(M, TM)$. In order to make further estimations we have to split $\tilde{J}^+$ in two parts.

Let $\beta_p : \tilde{J}^+(p) \to \mathbb{R} : q \to V(\tilde{A}(p, q))$ and $S^\delta_p = \{ p \times \{ q \in \tilde{J}^+(p) | \beta_p(q) \leq \delta \} \}$. It is proven in appendix B that there exists a $\delta_1 > 0$ such that $V^2(\cup_{p \in M} S^\delta_p) \leq \frac{\epsilon}{2}$. If we use the notation that $M_{\delta_1} = \{ (p, q) | V(\tilde{A}(p, q)) \geq \delta_1 \} \subset M \times M$ then (2) becomes:

$$d_{cau}(g, \phi^* \tilde{g}) \leq \frac{3\epsilon}{4} + V^2(N) + \int_{M_{\delta_1}} \alpha_{g\phi^* \tilde{g}}(p, q) dV(p) dV(q)$$

(3)

If we show that there exists an open subset $U_{\delta_1} \subset C^\infty_{id}(M, TM)$ such that for all $X \in U_{\delta_1}$ and $\phi = \exp_{id}(X)$:

$$\int_{M_{\delta_1}} \alpha_{g\phi^* \tilde{g}}(p, q) dV(p) dV(q) < \int_{M_{\delta_1}} \alpha_{g\tilde{g}}(p, q) dV(p) dV(q) + \frac{\epsilon}{4}$$

(4)

then

$$d_{cau}(g, \phi^* \tilde{g}) < \epsilon + V^2(N) + \int_{M_{\delta_1}} \alpha_{g\tilde{g}}(p, q) dV(p) dV(q) < \epsilon + d_{cau}(g, \tilde{g})$$

On $M_{\delta_1}$ one can easily bound $\alpha_{g\phi^* \tilde{g}}(p, q)$ by:

$$\frac{V(\tilde{A}(p, q) \Delta \tilde{A}(p, q)) + V(\tilde{A}(\phi(p), \phi(q)) \Delta \tilde{A}(p, q)) + V(\phi^{-1}(\tilde{A}(\phi(p), \phi(q)) \Delta \tilde{A}(\phi(p), \phi(q))))}{V(\tilde{A}(p, q) \cup \tilde{A}(p, q)) - V(\phi^{-1}(\tilde{A}(\phi(p), \phi(q)) \Delta \tilde{A}(p, q)))}$$

(5)

In order to make the final estimates, we need two lemma’s.

**Lemma 1** $\forall \kappa > 0 \ \exists U_\kappa \subset C^\infty_{id}(M, TM)$ such that $X \in U_\kappa$ implies that for $\phi = \exp_{id}(X)$:

$$V(\tilde{A}(p, q) \Delta \tilde{A}(\phi(p), \phi(q))) < \kappa V(\tilde{A}(p, q))$$

$\forall (p, q) \in M_{\delta_1}$.

**Proof**

Define the function

$$F : M_{\delta_1}^2 \to \mathbb{R} : ((p, q), (r, s)) \to \frac{V(\tilde{A}(p, q) \Delta \tilde{A}(r, s))}{V(\tilde{A}(p, q))}$$

30
In appendix C it is proven that $F$ is continuous. This proves that $\exists \delta < \delta_0$ such that $\forall (p, q) \in M_{\delta_1}$ and $p', q'$ such that $\bar{d}(p, p'), \bar{d}(q, q') < \delta$:

$$V(\bar{A}(p, q) \triangle \bar{A}(p', q')) < \kappa V(\bar{A}(p, q))$$

$\bigcup_{p \in M} \{p\} \times B(p, \delta)$ clearly determines an open neighborhood of the diagonal in $M^2$ and this yields the open neighborhood $U_\kappa \subset U_{\delta_0}$. □

**lemma 2** \(\forall \kappa > 0,\) then there exists a neighborhood $W_\kappa$ of the zero section such that $\forall (p, q) \in M_{\delta_1}$ we have that :

$$V(\bar{A}(\phi(p), \phi(q)) \triangle \phi^{-1}\bar{A}(\phi(p), \phi(q))) < \kappa V(\bar{A}(p, q))$$

$\phi \in W_\kappa$ implies that $d_{vol}(\phi_*g, g) < ln(3)$.

**remark 4** The condition that $d_{vol}(\phi_*g, g) < ln(3)$ makes restrictions on $\bar{d}(r, \phi(r))$ $\forall r \in M$ and on the determinant of the Jacobian of $\phi$ in any chart of the cover $U_k$. Choose $p \in M$ and charts $(U_p, \chi), (U_{\phi(p)}, \nu)$ belonging to the cover, we denote the coordinates in $U_p$ with $x^\alpha$ and the coordinates in $U_{\phi(p)}$ with $y^\beta$. We note $\phi^\beta(x^\alpha) = y^\beta(\nu \circ \phi \circ \chi^{-1}(x^\alpha))$, so we can estimate :

$$| ln \left( \frac{\sqrt{-|\phi_*g(\phi^\beta(x^\alpha))|}}{\sqrt{-|g(\phi^\beta(x^\alpha))|}} \right) |$$

by

$$| ln \left( \frac{\partial x^\alpha}{\partial \phi^\beta(x^\nu)} \right) | + | ln \left( \frac{\sqrt{-|g(x^\alpha)|}}{\sqrt{-|g(\phi^\beta(x^\alpha))|}} \right) |$$

These restrictions determine open sets in $C_{id}^\infty(M, TM)$ so the condition is justified.

The proof of lemma 2 is given in appendix D. The result of proposition 8 follows now immediately. Lemma 1 and 2 determine an open neighborhood $W_\kappa \cap U_\kappa$ of the identity such that

$$V(\bar{A}(\phi(p), \phi(q)) \triangle \phi^{-1}\bar{A}(\phi(p), \phi(q))) < \kappa V(\bar{A}(p, q))$$

and

$$V(\bar{A}(\phi(p), \phi(q)) \triangle \bar{A}(p, q)) < \kappa V(\bar{A}(p, q))$$
Substitution of these inequalities in (5) implies that $\alpha_{g^{\star \tilde{g}}}(p, q)$ can be bounded on $M_{\delta_1}$ by:

$$\frac{1}{1 - 2\kappa} \alpha_{g^{\tilde{g}}}(p, q) + \frac{2\kappa}{1 - 2\kappa}$$

This can be bounded again by

$$\alpha_{g^{\tilde{g}}}(p, q) + \frac{4\kappa}{1 - 2\kappa}$$

Putting this in inequality (4) we obtain that $\kappa < \frac{\epsilon}{16M^2 + 2\epsilon}$ yields the result.

The proof is lengthy because of the construction of $M_{\delta_1}$, which is the key idea of the proof. One has to avoid to come arbitrarily close to the boundary of the lightcone because long skinny Alexandrov sets will force the diffeomorphism to become very small. This situation can be restored in the compact case, this is obviously impossible when $M$ is not compact! A much more interesting - and difficult - question is under which causal restrictions on $g, \tilde{g}$ the mapping $f_{g^{\tilde{g}}}$ is measurable in the Schwartz topology. We didn’t find an answer to this question yet. It is obvious that the function $f_{g^{\tilde{g}}}$ is continuous in the Schwartz topology. The continuity of $f_{g^{\star \tilde{g}}}$ needs a bit of explanation.

Choose $\epsilon > 0$, we have for all $\psi \in Diff(M)$ that:

$$|d_{geo}(g, \tilde{g}) - d_{geo}(g, \psi^{\star \tilde{g}})| \leq d_{geo}(\tilde{g}, \psi^{\star \tilde{g}})$$

Using that $\lambda_{\psi^{\star \tilde{g}}}(p, q) = \tilde{\lambda}(\psi(p), \psi(q))$ we only have to prove that there exists an open neighborhood $V$ of the identity diffeomorphism in $Diff(M)$ such that $\psi \in V$ implies that

$$|\tilde{\lambda}(\psi(p), \psi(q)) - \tilde{\lambda}(p, q)| < \epsilon$$

One can easily proof this using the continuity of $(p, q) \rightarrow \tilde{\lambda}(p, q)$ and applying a doubling trick such as in the proof of lemma 2.

## 5.2 Amenability

The aim of this paragraph is to introduce the reader to the concept of amenability. It contains no new results. The intention is to give the reader an idea of the technical difficulties which show up. We will first state some
of the main results for general topological groups; later on we give a list of equivalent characterizations in case of locally compact topological groups or Lie groups. For more concise treatments, the reader is referred to [4, 6, 2].

Let $G$ be a topological group and $X$ a closed subspace of $L^\infty(G)$. $X$ is assumed to be one of the following subspaces:

- $X = L^\infty(G)$
- $X = CB(G)$ (continuous bounded functions)

A mean $m$ on $X$ is a positive element of the dual $X^*$ such that

$$m(1) = 1 = \| m \|$$

This condition is equivalent with the requirement that

$$\inf_{x \in G} f(x) \leq m(f) \leq \sup_{x \in G} f(x) \quad \forall f \in X$$

Now we are especially interested in invariant means (IM). To define invariance we first define the right and left translations $f_g$ respectively $f^g$ of a function $f \in X$ by an element $g \in G$:

$$f_g(x) = f(xg)$$

and

$$f^g(x) = f(g^{-1}x)$$

It is clear that

$$f_{gh}(x) = f(xgh) = (f_h)_g(x)$$

so right translation defines a representation of $G$ on $X$ and the same is true for the left translation. A RIM $m$ is a mean which satisfies :

$$m(f_x) = m(f) \quad \forall x \in G, f \in X$$

and left invariant means (LIM) are defined in the same way. An IM is a LIM and RIM. It is obvious that an IM on $X = L^\infty(G)$ is also a IM on $X = CB(G)$, the converse however is not true in general. However when $G$ is locally compact the two notions are the same [4, 5]. Because $Diff(M)$ is not locally compact and considering proposition 8 it is obvious that our
purpose is to consider an IM on $X = CB(G)$. Because the inversion and multiplication are continuous in the Schwartz topology on $\text{Diff}(M)$, the existence of a LIM (RIM) on $CB(G)$ where $G$ a l.c. subgroup of $\text{Diff}(M)$ implies the existence of an IM ([4] page 2, [5] p. 36). We will call groups which have a IM on the space of continuous bounded functions amenable (this is just our convention, the word amenable is also used in other meanings).

$CB(G)$ endowed with the sup norm is a Banach space. It is well known that the unit ball in $X^*$ is weak* compact, hence it is obvious that the convex set of IM is weak* compact. This implies that if $G = \cup_{\alpha \in I} H_{\alpha}$, $I$ a directed set and $H_{\alpha}$ a net of closed amenable subgroups, we have that $G$ is amenable ([4] p. 30). The lemma of Zorn implies then that we can find maximal amenable subgroups. Now the most interesting properties and characterizations arise when $G$ is locally compact. Intuitively most locally compact groups are amenable, it is a disadvantage however that a RIM is not unique. It is also important to point out that the right invariant Haar measure is not a RIM unless $G$ is compact, this is logical since a mean of a function of compact support is zero when $G$ is not compact. For more information concerning amenable l.c. groups we refer the reader to appendix E. There it is shown that a great deal of l.c. groups are amenable, so one could raise the question if in $\text{Diff}(M)$ one can find ”large” maximal amenable subgroups. ”Large” means for example that the subgroup has infinite dimension. This kind of questions are not dealt with yet.

In the following paragraphs we always assume that $G$ is a maximal closed amenable subgroup of the diffeomorphism group.

5.3 A $G$ - invariant uniformity and pseudodistance on the space of globally hyperbolic metrics

Now we return to the results starting at section 3. We call two globally hyperbolic metrics $g, \tilde{g}$ on a compact space time $(\zeta, \epsilon, \alpha)$ close if and only if :

$$
\begin{align*}
  d_{\text{vol}}(g, \tilde{g}) & \leq \zeta \\
  d_{\text{cau}}(g, \tilde{g}) & \leq \epsilon \\
  d_{\text{geo}}(g, \tilde{g}) & \leq \alpha
\end{align*}
$$

We say that $(\zeta', \epsilon', \alpha') < (\zeta, \epsilon, \alpha)$ if and only if $\zeta' < \zeta$, $\epsilon' < \epsilon$, $\alpha' < \alpha$. This turns $\mathbb{R}^3$ into an oriented net. Define now:

$$
B_{\zeta,\epsilon,\alpha}(g) = \{\tilde{g}|(g, \tilde{g}) \text{ close with } (\zeta', \epsilon', \alpha') < (\zeta, \epsilon, \alpha)\}
$$

34
Proposition 1 and 2 and the following remarks prove that the $B_{\zeta,\epsilon,\alpha}(g)$ form a uniform neighborhood system. So we know that the covers

$$C_{\zeta,\epsilon,\alpha} = \{ B_{\zeta,\epsilon,\alpha}(g) | g \in \mathbb{L}(M) \}$$

form a basis of a uniform neighborhood system $U$. Clearly we have a countable subbasis so according to proposition 6 there exists a diagonally invariant pseudometric $d$. However there is no guarantee that $d$ is bounded, therefore we introduce a cut off scale $\eta$. A modified version of proposition 6 gives the following result.

**Proposition 9** Let $C_n = \{ B_n(g) | g \in \mathbb{L}(M) \}$ be a countable basis of $U$ which satisfies the property that :

$$\forall n, \ g \in B_n(g_1), g_1 \in B_n(g_2), g_2 \in B_n(\tilde{g}) \Rightarrow g \in B_{n-1}(\tilde{g})$$

and let $\eta > 0$ be a cut off scale. Put

$$\rho(g, \tilde{g}) = \inf_{\{n, n, \tilde{g} \in B_n(g) \}} 2^{-n(g, \tilde{g})}$$

then we have that the function

$$d(g, \tilde{g}) = \min \{ \inf_{K \in \mathbb{N}, x_k} \sum_{k=1}^{K} \frac{1}{2}(\rho(x_{k-1}, x_k) + \rho(x_k, x_{k-1})), \eta \}$$

is a diagonally diffeomorphism invariant pseudodistance which generates $U - \{ x_0, \ldots, x_K \}$ with $x_0 = g, x_K = \tilde{g}$ is a path in $\mathbb{L}(M)$. Moreover $d$ is a distance on the space of strongly causal metrics.

*Proof*

The fact that $d$ is a distance on the space of strongly causal metrics follows from proposition 3. $\Box$

One could have esthetical objections against this cut off scale from the mathematical point of view. For a physicist however, this cut off scale is quite natural, one is a priori only interested in these space times $\tilde{g}$ which are a "good approximation" of $g$. Moreover the cut off doesn’t change the topology generated by $d$. 

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35
Let \( g, \tilde{g} \) be globally hyperbolic, we show now that \( f_{g, \tilde{g}} \) is continuous in the Schwartz topology. Choose \( \epsilon > 0 \), from the triangle inequality we get that:

\[
|d(g, \psi^* \tilde{g}) - d(g, \tilde{g})| \leq d(\tilde{g}, \psi^* \tilde{g})
\]

Proposition 9 implies that there exists a \( n_0 \) such that \( \forall n \geq n_0 : B_n(\tilde{g}) \subset B^d(\tilde{g}) \) where \( 2^{-n_0} < \epsilon \). The continuity of \( f_{g, \tilde{g}}^{cau}, f_{g, \tilde{g}}^{vol} \) and \( f_{g, \tilde{g}}^{geo} \) implies that there exists an open neighborhood \( V \) of the identity in \( Diff(M) \) such \( \psi \in V \) implies that \( \psi^* \tilde{g} \in B(n_0(\tilde{g})) \). This proves the continuity.

As stated in the introduction:

\[
\tilde{d} : (g, \tilde{g}) \rightarrow \begin{cases} A(f_{g, \tilde{g}} + f_{\tilde{g}, g}) & \text{when } \tilde{g} \neq \psi^* g \text{ } \forall \psi \in Diff(M) \\ 0 & \text{otherwise} \end{cases}
\]

(with \( A \) the \( G \) invariant mean) is a \( G \) invariant pseudodistance on the space of globally hyperbolic metrics. Now we make some remarks concerning the Hausdorff character of \( \tilde{d} \). Obviously one has:

\[
\tilde{d}(g, \tilde{g}) = 0 \Rightarrow \inf_{\phi \in Diff(M)} d(g, \phi^* \tilde{g}) = 0
\]

but the implication doesn’t necessary go the other way. This is clearly an advantage in comparison with the construction given in [11, 12]. On the other hand, in the same paper one obtained a fully diffeomorphism invariant pseudodistance, whereas ours is only fully \( G \) invariant. There remains the question what size this \( G \) has, when for example under some topological restrictions on \( M \) the Lie algebra of \( G \) has infinite cardinality. In example 3 one has a group \( G \subset Diff(M) \) such that \( \inf_{\phi \in G} d(g, \phi^* \tilde{g}) = 0 \) but \( \tilde{d}(g, \tilde{g}) > 0 \).

Let \( g, \tilde{g} \) be globally hyperbolic, as anticipated before on page 7, we investigate the continuity of \( \epsilon \rightarrow d_{\epsilon}(g, \tilde{g}) \) (we have dropped the index “\( i \)” in the notation). This function is clearly left continuous. This can be seen as follows; choose \( \epsilon > 0 \) and \( \kappa > 0 \), and suppose there exists a sequence \( p_n, q_n \) such that \( V(A(p_n, q_n)) \geq \epsilon - \frac{1}{n}, \ V(A(p_n, q_n)) \geq \epsilon - \frac{1}{n} \) and

\[
\frac{\lambda(p_n, q_n)}{\lambda(p_n, q_n)} \leq e^{-(d_{\epsilon}(g, \tilde{g}) + \kappa)}
\]

or

\[
\frac{\lambda(p_n, q_n)}{\lambda(p_n, q_n)} \geq e^{d_{\epsilon}(g, \tilde{g}) + \kappa}
\]


Without loss of generality we can assume the latter and because $M$ is compact we can find accumulation points $p$ and $q$ such that:

$$\frac{\lambda(p, q)}{\lambda(p, q)} \geq e^{d_i(g, \tilde{g}) + \kappa}$$

and $V(A(p, q)) \geq \epsilon$, $V(\tilde{A}(p, q)) \geq \epsilon$ which is a contradiction. This proves our claim. In general one does not have right continuity and it is not so difficult to find counterexamples. One even can construct counterexamples where $M$ is compact and $\epsilon \rightarrow d_i(g, \tilde{g})$ is not continuous in a countably infinite number of points. This is due to the non locality of the definition of $d_i$.

## 6 The noncompact case

We will now make a similar construction as before, however some more care is needed. There are 3 obvious difficulties:

- It is clear that the functions $d^i_{\text{vol}}, d^i_{\text{cau}}$ and $d^i_{\text{geo}}$ are not diagonally diffeomorphism invariant when one considers diffeomorphisms on $M$. They are however invariant with respect to the subgroup of all diffeomorphisms $\phi$ such that $\phi|_{W_i} \in Diff(W_i)$.

- If $g, \tilde{g}$ are globally hyperbolic on $M$, one could be tempted to consider the restrictions of these metrics to $W_i$. It is however meaningless to look at $W_i$ as a space - time by itself because the restrictions are not necessarily globally hyperbolic. So the causal relations we consider on $W_i \times W_i$ are of the type $\prec_{|W_i}$ and not $\prec_{W_i}$. In that way we preserve the qualities of global hyperbolicity we needed to prove theorem 8.

- When $M$ is not compact, the Schwartz topology on $Diff(M)$ gets more complicated. One has that $Diff(M)$ is an open $C^\infty_c$ submanifold of $C^\infty_{FD}(M, M)$, composition and inversion are smooth. The Lie algebra of the smooth infinite dimensional Lie group $Diff(M)$ is the convenient vector space $\mathfrak{x}_{\text{cpt}}(M)$ of all smooth vector fields on $M$ with compact support, equipped with the negative of the usual Lie bracket and with the $D$ topology. The exponential mapping $\exp : \mathfrak{x}_{\text{cpt}}(M) \rightarrow Diff(M)$ is the flow mapping to time 1, and it is smooth. A definition of the $FD$ topology can be found in [20].
As in the previous chapter we make of $\mathbb{N}_0 \times \mathbb{R}^3$ a directed net as follows:

$$(i', \zeta', \epsilon', \alpha') < (i, \zeta, \epsilon, \alpha)$$

if and only if

$$i' > i, \zeta' < \zeta, \epsilon' < \epsilon, \alpha' < \alpha$$

In the same way one can define a countable basis of neighborhoods $B_n(g) = B_{\frac{1}{n}, \frac{1}{n}, \frac{1}{n}}(g)$ as follows:

$$B_n(g) = \{ \tilde{g} | (g, \tilde{g}) \text{ are } (i', \zeta', \epsilon', \alpha') \text{ close with } (i', \zeta', \epsilon', \alpha') < (n, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}) \}$$

It is clear that the uniform topology constructed here doesn’t depend on the choice of the sequence $W_i$. Suppose one has two sequences $W_i$ and $V_j$, then for every $i$ there exists a $j$ such that $W_i \subset V_j$. It is obvious then that for all $\zeta, \epsilon, \alpha$ one has that $B_{\frac{1}{n}, \frac{1}{n}, \frac{1}{n}}(g) \subset B_{\frac{1}{n}, \frac{1}{n}, \frac{1}{n}}(g)$ $\forall g \in \mathbb{L}(M)$ and one can make the same reasoning for the $V_j$.

We show now that the topology defined by the $B_n(g)$ does not depend on the function $f$ we defined in the beginning. Our restrictions on $f$ made sure that 3 out of 4 defining properties for a uniform neighborhood system are satisfied. However the second defining property can be broken. This can be seen as follows; if $g$ and $\tilde{g}$ are $(i, \zeta, \epsilon, \alpha)$ close then they are $(j, \zeta, \frac{f(V(W_i))}{f(V(W_j))} \epsilon, \alpha)$ close $\forall j \leq i$. As one notices there is a volume dependent factor $\frac{f(V(W_i))}{f(V(W_j))}$ in the formula, which makes the partial order $<$ dependent on the volume scale of $g$, and hence $<$ cannot define a uniform neighborhood system. The most obvious thing to do is to redefine $<$ as $\tilde{<}$ with:

$$(i', \zeta', \epsilon', \alpha') \tilde{<} (i, \zeta, \epsilon, \alpha)$$

if and only if

$$i' = i, \zeta' < \zeta, \epsilon' < \epsilon, \alpha' < \alpha$$

This defines the neighborhoods $\tilde{B}_n(g)$ as:

$$\tilde{B}_n(g) = \{ \tilde{g} | (g, \tilde{g}) \text{ are } (i', \zeta', \epsilon', \alpha') \text{ close with } (i', \zeta', \epsilon', \alpha') \tilde{<} (n, \frac{1}{n}, \frac{1}{n}, \frac{1}{n}) \}$$

It is easy to see that the topology defined by the sets $\tilde{B}_n(g)$ is the same as the one defined by the $B_n(g)$.

Now proposition 6 again defines a pseudodistance on $\mathbb{L}(M)$ as before, but this
pseudodistance is only diagonally invariant with respect to diffeomorphisms $\phi$ such that $\phi_{|W_i} \in Diff(W_i)$. So it seems we have to symmetrize twice, thus we have to prove that $f_{g,\tilde{g}}$ and $\psi \to A(f_{\psi^* g,\tilde{g}})$ are continuous in the $FD$ topology. Choose $\epsilon > 0$, then one can bound

$$|A(f_{\psi^* g,\tilde{g}}) - A(f_{g,\tilde{g}})|$$

by

$$\|f_{\psi^* g,\tilde{g}} - f_{g,\tilde{g}}\|_\infty$$

This can again be bounded by $d(g, \psi^* g)$ because of the triangle inequality. Now there exists an $n_0$ such that $\psi^* g \in B_{n_0}(g)$ implies that $\psi^* g \in B_\epsilon^d(g)$. It is obvious now that because $\bar{W}_{n_0}$ is compact, a similar result of proposition 8 for $\bar{W}_{n_0}$ implies our result. So we are only left with proving that:

**Proposition 10** For any $i \in \mathbb{N}_0$ and $g, \tilde{g}$ globally hyperbolic on $M$, we have that

- $\phi \to d_{i,cau}^i(g, \phi^* \tilde{g})$
- $\phi \to d_{i,vol}^i(g, \phi^* \tilde{g})$
- $\phi \to d_{i,geo}^i(g, \phi^* \tilde{g})$

are continuous in the $FD$ topology.

**Proof**

One has to keep in mind that $\bar{W}_i$ is not necessarily a compact submanifold of $M$. So one cannot assume a differential structure on $\bar{W}_i$. Let the $U_i$ be a countable cover of charts of $M$ as in section 5.1 and assume $U_k$ is a finite subcover of $\bar{W}_i$ then the conditions on the determinant of $\phi$ obtained in proposition 11 are meant with respect to the finite subcover $U_k$. This determines again open neighborhoods of the identity in $Diff(M)$ and we are done. $\square$

Now it is possible as in the compact case to introduce the following pseudodistance:

$$\tilde{d} : (g, \tilde{g}) \to \begin{cases} A(\psi \to A(f_{\psi^* g,\tilde{g}} + f_{\psi^* g,\tilde{g}})) & \text{if and only if } \phi^* g \neq \tilde{g} \ \forall \phi \in Diff(M) \\ 0 & \text{otherwise} \end{cases}$$

The symmetrization would be unnecessary if an analogue to the Fubini theorem would be valid for $A$. We don’t know if this is the case so we symmetrize
anyway. The mean $A$ in the above construction is not necessarily an IM, it is easy to check that if $A$ is a LIM then $\tilde{d}$ satisfies all properties of a pseudometric. There is one disappointing feature to this symmetrization, namely that we have been unable to prove that the resulting topology does not depend on the choice of the pseudometric which generates the uniformity. We give an example which illustrates this.

**Example 2** We consider $1 + 1$ dimensional space time $\mathbb{R}^2$ and the subspace $V \subset L(M)$ defined as follows:

$$V = \{ \begin{pmatrix} -\alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta > 0 \}$$

Define the action of $G = \mathbb{R}_0^+ \subset Diff(\mathbb{R}^2)$ on $\mathbb{R}^2$ as follows:

$$\tilde{\lambda}(x, y) = (x, \sqrt{\lambda}y) \quad \forall \lambda \in G$$

The $FD$ topology on $Diff(M)$ induces the discrete topology on $G$. This is obvious because if $\tilde{\lambda}, \tilde{\gamma}$ are two diffeomorphisms in $G$ which differ only on a set of compact closure then they must be equal. This topological group however is not amenable. We don’t need such a strong topology to make the functions $\phi \to d_{\text{can}}^i (g, \phi^* \tilde{g})$, $\phi \to d_{\text{vol}}^i (g, \phi^* \tilde{g})$ and $\phi \to d_{\text{geo}}^i (g, \phi^* \tilde{g})$ continuous (the $W_i$ are circles of radius $i$ in the standard Riemannian metric on $\mathbb{R}^2$). They will also be continuous if we relax the discrete topology to the standard euclidian one. The action of $G$ on $V$ is as follows:

$$\tilde{\lambda}^* \begin{pmatrix} -\alpha & 0 \\ 0 & \beta \end{pmatrix} = \begin{pmatrix} -\alpha & 0 \\ 0 & \lambda \beta \end{pmatrix}$$

It is clear that the topology on $V \sim \mathbb{R}_0^+ \times \mathbb{R}_0^+$ defined by the uniform neighborhood system $B_{i, \zeta, \epsilon, \alpha}(g)$ is equivalent to the standard euclidian topology on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$. Define on $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ the following two metrics (with cut off scale $\eta$):

$$d_1(\bar{a}, \bar{b}) = \min \{ \| \bar{a} - \bar{b} \|, \eta \}$$

and

$$d_2(\bar{a}, \bar{b}) = \min \{ |a_1 - b_1| + \frac{1}{|a_2|} - \frac{1}{|b_2|}, \eta \}$$

It is obvious that $d_1$ and $d_2$ generate the euclidian topology, they are however not equivalent. If we calculate:

$$\tilde{d}_1(\bar{a}, \bar{b}) = m(x \to m(y \to d_1((a_1, xa_2), (b_1, yb_2))))$$
then one gets that

\[
\tilde{d}_1(\bar{a}, \bar{b}) = \begin{cases} 
\eta & \text{if } \bar{a} \neq \check{\lambda}^\ast \bar{b} \quad \forall \lambda \in G \\
0 & \text{otherwise}
\end{cases}
\]

The same calculation for \(d_2\) gives however a totally different result:

\[
\tilde{d}_2(\bar{a}, \bar{b}) = \begin{cases} 
\min\{|a_1 - b_1|, \eta\} & \text{if } \bar{a} \neq \check{\lambda}^\ast \bar{b} \quad \forall \lambda \in G \\
0 & \text{otherwise}
\end{cases}
\]

So in the first case the quotient topology on \(V/G\) is the discrete topology, in the second case it is the usual euclidian one. This shows how sensitive the quotient topology is to the choice of the generating distance for the uniformity.

This problem becomes even much more complicated when we consider diagonally invariant (with respect to a group action \(G\)) metrics. One can however make the following remark:

**Example 3**  Consider the action of \(\mathbb{Q}_0^+\) by multiplication on \(\mathbb{R}_0^+\) endowed with the metric

\[
d(a, b) = |\ln \left( \frac{a}{b} \right)|
\]

It is clear that \(\sqrt{2}\) is not on the orbit of \(\sqrt{3}\) but this orbit is dense in \(\mathbb{R}_0^+\).

This example is quite pathologic in the sense that the group \(\mathbb{Q}_0^+\) is totally disconnected. This is however not the case for \(\text{Diff}(M)\).

7 Epilogue

We will first compare our results with the results obtained in [11, 12].

- The topology on \(\mathbb{L}(M)/\text{Diff}(M)\) constructed in [11] is unique, however there exist many pseudodistances which might generate this uniform topology. The topology on \(\mathbb{L}(M)/G\) in our case is probably not uniquely determined and depends on the generating pseudodistance of the uniform topology on \(\mathbb{L}(M)\). However as mentioned in the introduction when \(M\) is compact we can also take the quotient \(\mathbb{L}(M)/\text{Diff}(M)\).
• Obviously we don’t know the "size" yet of the maximal amenable subgroup $G$, this is something which has to be investigated.

• Our topology has much better continuity properties with respect to group actions.

• One can also raise the question whether the topology on $\mathbb{L}(M)$ is locally arcwise connected. $Diff(M)$ is locally arcwise connected in the $\mathcal{FD}$ topology so we have that for $\phi$ sufficiently small and $g \in \mathbb{L}(M)$ that $\phi^*g$ is arcwise connected to $g$ by a path in $\mathbb{L}(M)$ which corresponds to a path in $Diff(M)$ from the identity to $\phi$.

Let $d$ be the distance generating the uniform topology (on the space of globally hyperbolic metrics $GLH(M)$). If we would take the quotient $GLH(M)/Diff(M)$, the Hausdorff property would fail if and only if:

$$\inf_{\psi \in Diff(M)} d(g, \psi^* \tilde{g}) = 0$$

for $\tilde{g}$ not diffeomorphism equivalent to $g$. This implies that there exists a sequence of diffeomorphisms $\psi_n$ such that $\psi_n^* \tilde{g} \in B_n(g)$. This means that for $n$ big enough the conformal and volume structures of $\psi_n^* \tilde{g}$ and $g$ are almost the same. There is no argument to exclude that this can happen. By symmetrizing $d$ with an invariant mean we hope that this will happen less, however there is no way we can prove this.

We will show now that our work has hope to further generalization. As mentioned before, one is interested in measuring the distance between isometry classes of Lorentzian structures $(M, g)$ and $(M', g')$. $(M, g)$ and $(M', g')$ are isometrically equivalent if and only if there exists a diffeomorphism $\phi : M \to M'$ such that $\phi_* g = g'$. Suppose that all manifolds considered are compact, and let $d_M$ be the (diagonally invariant) distance defined in proposition 9. We define the pseudodistance $\hat{d}$ between $(M, g)$ and $(M', g')$ as follows:

$$\hat{d}((M, g), (M', g')) = \begin{cases} 
\inf_{\psi \in Diff(M, M')} d_M(g, \psi^* g') & \text{if } Diff(M, M') \neq \emptyset \\
1 & \text{otherwise}
\end{cases}$$
The symmetry of \( \hat{d} \) follows from \( d_M(g, \psi^*g') = d_{M'}(\psi_*g, g') = d_{M'}(g', \psi_*g) \).

The triangle inequality follows from:
\[
\begin{align*}
  d_M(g, \psi^*g'') &\leq d_M(g, \phi^*g') + d_M(\phi^*g', \psi^*g'') \\
  &\leq d_M(g, \phi^*g') + d_{M'}(g', (\psi\phi^{-1})^*g'')
\end{align*}
\]

\( \hat{d} \) is clearly a pseudodistance on isometry classes of metrics. Unfortunately \( \hat{d} \) makes a too rough distinction between two inequivalent differential structures. We propose now a la Gromov a pseudodistance on the space of all compact, future and past distinguishing structures \((M, g)\), which might improve drastically this defect. We denote by \( \mathcal{C}(M, M') \) the space of all continuous functions from \( M \) to \( M' \). We define two structures \((M, g)\) and \((M', g')\) to be \((\zeta, \epsilon, \mu)\) close if and only if there exist \( f \in \mathcal{C}(M, M') \) and \( h \in \mathcal{C}(M', M) \) such that:

1. \[ \sup_{O \in B_M} | \ln \left( \frac{V_{g'}(f(O))}{V_g(O)} \right) | < \epsilon \] and \( \sup_{O \in B_{M'}} | \ln \left( \frac{V_g(h(O))}{V_{g'}(O)} \right) | < \epsilon \) where \( B_M \) and \( B_{M'} \) are the sets of all Borel measurable subsets of nonzero measure of \( M \) and \( M' \) respectively.

2. \[ \sup_{p,q \in M: V_g(f(A_g(p, q)) \cup A_g'(f(p), f(q))) \geq \zeta} \frac{V_{g'}(f(A_g(p, q)) \triangle A_g'(f(p), f(q)))}{V_{g'}(f(A_g(p, q)) \cup A_g'(f(p), f(q)))} \leq \epsilon \] and a similar constraint with \( f \) replaced by \( h \) and \((M, g)\) switched with \((M', g')\).

3. We are also obliged to make a topological constraint in order to construct a uniformity. Intuitively this constraint means that \( h \) is approximately the inverse of \( f \) and vice versa:
\[
\forall A \in \tau_M \text{ such that } V_g(A) \geq \mu : V_g(A \triangle \overline{h(f(A))}) \leq \epsilon V_g(A)
\]

and
\[
\forall A \in \tau_{M'} \text{ such that } V_{g'}(A) \geq \mu : V_{g'}(A \triangle \overline{f(h(A))}) \leq \epsilon V_{g'}(A)
\]

where \( \tau_M \) (\( \tau_{M'} \)) is the set of all closed subsets of \( M \) (\( M' \)).
This notion of closeness defines again a uniformity, and hence a pseudodistance $d$ on the space of all isometry classes of compact, future and past distinguishing structures. The reader might ask why we go back to the supremum function while the natural thing to do would be to introduce the following type of constraint on the space of compact, class $\mathcal{A}$ structures:

$$\int_{M \times M} \frac{V_{g'}(f(A_g(p,q)) \triangle A_{g'}(f(p), f(q)))}{V_{g'}(f(A_g(p,q)) \cup A_{g'}(f(p), f(q)))} dV_g(p) dV_g(q) < \epsilon$$

This seems also possible to us, but one has to demand then that the functions $f$ and $h$ are injective.

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9 Appendix A

In this appendix we introduce the class $A$. In proposition 4 it will be proven that $A$ contains all globally hyperbolic metrics. In [23] one defined with respect to a Lorentz metric $g$ a semiring $H_g$ with identity element as follows:

$$H_g = \{ F \cap P \mid F \text{ is a future set and } P \text{ is a past set} \}$$

It is also proven in [23] that all sets in $H_g$ are Lebesgue measurable. Moreover we have that for any element $H \in H_g$:

$$\mu(H) = \mu(\overline{H}) = \mu(\overline{\overline{H}})$$

where $\mu$ is the four-dimensional Lebesgue measure. $H_g$ is not closed with respect to the union and has the property that the complement of an element $(F \cap P)^c = (F^c \cup P^c)$ equals the union of two elements of $H_g$. Denote by $\mathcal{L}$ the $\sigma$ algebra of all Lebesgue measurable sets, we define two topologies on $\mathcal{L}$ by giving a base. Let $A$ be an open subset and $C$ a closed subset of $M$ satisfying the property that $C = C^c$, the open set $B(A; C)$ is defined as follows:

$$S \in B(A; C) \iff C \subset S \subset A$$

It is easy to check that these sets constitute a basis for the topology $\tau_1$ on $\mathcal{L}$. The topology $\tau_2$ is defined in a slightly different way: the open sets $B(A, C)$ constitute of the following elements:

$$S \in B(A, C) \iff C \subset S \subset O$$

This leads us to the following definition:

**Definition 5** A Lorentz metric $g$ is of class $A$ if and only if the map:

$$A : M \times M \to H_g : (p, q) \to A(p, q)$$

is measurable with respect to one of the Borel $\sigma$ - algebras defined by the topological spaces $(H_g, \tau_1), (H_g, \tau_2)$. In a next paper we will examine class $A$ space times more thoroughly. From now on it is always assumed that the space time metrics are of class $A$. It is not difficult to prove that the map $\alpha_{g\tilde{g}}$ is measurable with respect to class $A$ space times. One can also show that on a compact space time a causally continuous metric is of class $A$, one even has the stronger result that the corresponding map $A$ is continuous with respect to the topology $\tau_2$. 

45
10 Appendix B

We have to prove that for any $\epsilon > 0$, \( \exists \delta_1 > 0 \) such that
\[
V^2(\bigcup_{p \in M} S_p^{\delta_1}) \leq \frac{\epsilon}{2}
\]

Choose $\epsilon > 0$, because $\delta \to V(\{p \mid V(J^+(p)) \leq \delta\})$ is continuous, monotonously increasing and $V(\{p \mid V(J^+(p)) = 0\}) = 0$, we can find a $\delta_2 < 1$ such that $V(\{p \mid V(J^+(p)) \leq \delta_2\}) < \frac{\epsilon}{4}$. We denote in the sequel $\gamma = \frac{\epsilon}{4M}$, $V(M) = \tilde{M}$, $M^{\delta_2} = \{p \mid V(J^+(p)) \geq \delta_2\}$ and define $\forall p \in M^{\delta_2}$ the mapping
\[
\eta_p : [0, \tilde{M}] \to \mathbb{R} : x \to V(S_p^x)
\]

It is not difficult to see that the mappings $\beta_p, \eta_p$ are continuous on $M$ and nonzero on $M^{\delta_2}$, moreover $\forall p \in M^{\delta_2}$ there exists a unique $\delta_p > 0$ such that the mapping $\eta_p$ is strictly increasing for $x < \delta_p$ and constant for $x \geq \delta_p$. It is obvious that $\forall p \in M^{\delta_2}$ there exists a unique $\tilde{\gamma}_p > 0$ such that that $x > \tilde{\gamma}_p$ implies that $\eta_p(x) > \gamma$ or $x > \delta_p$. If we prove now that the mapping $p \to \tilde{\gamma}_p$ is continuous then we are done because $\forall 0 < \nu \leq \min_{p \in M^{\delta_2}} \tilde{\gamma}_p = \delta_1$ we have that $\eta_p(\nu) \leq \gamma \ \forall p \in M^{\delta_2}$ and so
\[
V^2(\bigcup_{p \in M} S_p^{\nu}) < \gamma \tilde{M} + \frac{\epsilon}{4} < \frac{\epsilon}{2}
\]

To prove the continuity of $p \to \tilde{\gamma}_p$, we first show that the mapping
\[
M \times [0, \tilde{M}] \to \mathbb{R} : (p, \nu) \to \eta_p(\nu)
\]
is continuous. Choose $\tilde{\epsilon} > 0$ and $(p, \nu) \in M \times [0, \tilde{M}]$. Because $\eta_p$ is uniformly continuous there exists a $\tilde{\delta}_1$ such that $|x - y| < \tilde{\delta}_1$ implies that $|\eta_p(x) - \eta_p(y)| < \frac{\tilde{\epsilon}}{4}$. Then there exists a $\delta_2 > 0$ such that $\forall q \in B(p, \delta_2) : V(J^+(p) \triangle J^+(q)) < \frac{\tilde{\epsilon}}{2}$ and $|\beta_p(r) - \beta_q(r)| < \frac{\tilde{\delta}_1}{2} \ \forall r \in J^+(p)$ because $J^+(p)$ is compact. This implies that
\[
|\eta_p(\nu) - \eta_q(\chi)| < V(\{r \in J^+(p) \cap J^+(q) \mid \beta_p(r) > \nu \text{ or } \beta_q(r) > \chi \text{ but not both}\}) + \frac{\tilde{\epsilon}}{2}
\]

This is smaller than
\[
V(\{r \in J^+(p) \cap J^+(q) \mid \nu < \beta_p(r) \leq \chi + \frac{\tilde{\delta}_1}{2} \text{ and } \beta_q(r) \leq \chi\}) + \frac{\tilde{\epsilon}}{2}
\]

46
\[
V(\{ r \in \bar{J}^+(p) \cap \bar{J}^+(q) | \chi - \frac{\delta_1}{2} < \beta_p(r) \leq \nu \text{ and } \beta_q(r) > \chi \}) + \frac{\tilde{\epsilon}}{2}
\]

and this can be further bounded by

\[
|\eta_p(\chi + \frac{\delta_1}{2}) - \eta_p(\nu)| + |\eta_p(\nu) - \eta_p(\chi - \frac{\delta_1}{2})| + \frac{\tilde{\epsilon}}{2}
\]

which is smaller than \( \tilde{\epsilon} \) for all \( \chi, \nu \) such that \( |\chi - \nu| < \frac{\delta_1}{2} \).

Now to prove the continuity of \( p \to \tilde{\delta}_p^\gamma \) we have to consider two cases:

- \( \tilde{\delta}_p^\gamma < \tilde{\delta}_p \). Because \( \eta_p \) is strictly increasing in \( \tilde{\delta}_p^\gamma \) there exists for \( \kappa > 0 \) small enough a unique \( \zeta > 0 \) such that \( |\eta_p(\nu) - \eta_p(\tilde{\delta}_p^\gamma)| < \zeta \) implies that \( |\nu - \tilde{\delta}_p^\gamma| < \kappa \). Choose now \( \delta > 0 \) small enough such that \( q \in B(p, \delta) \) implies that
  \[
  |\eta_p(\nu) - \eta_q(\nu)| < \zeta \quad \forall \nu \in [0, \tilde{M}]
  \]
  (this is possible because of the continuity of \( (p, \nu) \to \eta_p(\nu) \)). We obviously have then that \( |\tilde{\delta}_p^\gamma - \tilde{\delta}_q^\gamma| < \kappa \) \( \forall q \in B(p, \delta) \).

- We are left with the case \( \tilde{\delta}_p^\gamma = \tilde{\delta}_p \). Here the result will immediately follow from the previous result and the fact that \( \forall \kappa > 0 \quad \exists \delta > 0 \) such that \( \forall q \in B(p, \delta) : \beta_q(r) < \tilde{\delta}_p + \kappa \quad \forall r \in \bar{J}^+(q) \). Suppose this is false then \( \exists q_n < q_m < p \quad \forall n < m \) and a sequence \( r_n \in \bar{J}^+(q_n) : \beta_{q_n}(r_n) \geq \tilde{\delta}_p + \kappa \). Because \( M \) is compact the sequence \( r_n \) has an accumulation point \( r \). Clearly \( r \in \bar{J}^+(p) \) and \( \beta_p(r) \geq \tilde{\delta}_p + \kappa \) which is impossible.

\( \square \)

11 Appendix C

Here we prove lemma 1. One has to show that the function

\[
F : M^2_{\delta_1} \to \mathbb{R} : ((p, q), (r, s)) \to \frac{V(\bar{A}(p, q) \triangle \bar{A}(r, s))}{V(\bar{A}(p, q))}
\]

\( F \) is continuous.

Proof
Choose $1 > \kappa > 0$, then one easily sees that $V(\tilde{A}(p', q') \triangle \tilde{A}(r', s'))$ can be bounded by:

$$V(\tilde{A}(p, q) \triangle \tilde{A}(p', q')) + V(\tilde{A}(p, q) \triangle \tilde{A}(r, s)) + V(\tilde{A}(r', s') \triangle \tilde{A}(r, s))$$

Now obviously there exists a $\delta > 0$ such that $p' \in B(p, \delta), q' \in B(q, \delta), r' \in B(r, \delta), s' \in B(s, \delta)$ implies that with $\nu = \kappa \delta_1 \frac{1}{3 \delta_1 + \tilde{M}}$ we have that

$$V(\tilde{A}(p, q) \triangle \tilde{A}(p', q')) < \nu V(\tilde{A}(p, q))$$

and

$$V(\tilde{A}(r, s) \triangle \tilde{A}(r', s')) < \nu V(\tilde{A}(p, q))$$

This implies that

$$F(p', q', r', s') < \left(\frac{1}{1 - \nu}\right)(F(p, q, r, s) + \nu \delta_1 + \tilde{M}) < F(p, q, r, s) + \kappa$$

The lower continuity is proven in a similar way.

\[\square\]

12 Appendix D

We prove the following result:

Let $\tilde{\epsilon} > 0$, then there exists a neighborhood $U_{\tilde{\epsilon}}$ of the zero section such that $\forall (p, q) \in M_{\delta_1}$ we have that :

$$V(\tilde{A}(\phi(p), \phi(q)) \triangle \phi^{-1} \tilde{A}(\phi(p), \phi(q))) < \tilde{\epsilon} V(\tilde{A}(p, q))$$

$\phi \in U_{\tilde{\epsilon}}$ implies that $d_{vol}(\phi_* g, g) < \ln(3)$.

Proof

From the previous lemma we can assume that for every $\phi \in U_{\tilde{\epsilon}}$ we have that $V(\tilde{A}(\phi(p), \phi(q)) \triangle \tilde{A}(p, q)) < \frac{\tilde{\epsilon}}{8} V(\tilde{A}(p, q))$. It is easy to see that $V(\tilde{A}(\phi(p), \phi(q)) \triangle \phi^{-1} \tilde{A}(\phi(p), \phi(q)))$ can be bounded by

$$V(\tilde{A}(\phi(p), \phi(q)) \triangle \tilde{A}(p, q)) + V(\tilde{A}(p, q) \triangle \tilde{A}(p, q)) + V(\phi^{-1} \tilde{A}(\phi(p), \phi(q)) \triangle \phi^{-1} \tilde{A}(p, q))$$

The last term is equal to

$$\phi_* V(\tilde{A}(\phi(p), \phi(q)) \triangle \tilde{A}(p, q)) < \frac{3\tilde{\epsilon}}{8} V(\tilde{A}(p, q))$$

48
Thus everything can be bounded by:

\[ \frac{\varepsilon}{2} V(\tilde{A}(p, q)) + V(\tilde{A}(p, q) \triangle \phi^{-1}(\tilde{A}(p, q))) \]

and we are left to prove that there exists an open neighborhood of the identity such that:

\[ \phi_* V(\tilde{A}(p, q) \triangle \phi(\tilde{A}(p, q))) = V(\tilde{A}(p, q) \triangle \phi^{-1}(\tilde{A}(p, q))) < \frac{\varepsilon}{2} V(\tilde{A}(p, q)) \]

\( \forall (p, q) \in M_{\delta_1} \). It is clearly possible to find an open neighborhood \( V \) of the identity such that in one point \((p, q)\) one has:

\[ V(\tilde{A}(p, q) \triangle \phi(\tilde{A}(p, q))) < \frac{\varepsilon}{12} V(\tilde{A}(p, q)) \]

Now it is also obvious that the mapping

\[ (r, s) \rightarrow \frac{V(\tilde{A}(r, s) \triangle \phi(\tilde{A}(r, s)))}{V(\tilde{A}(r, s))} \]

is continuous in \((p, q)\) using similar techniques as above. So we can find a \( \delta > 0 \) such that \( \forall r \in B(p, \delta), s \in B(q, \delta) \) we have that

\[ V(\tilde{A}(r, s) \triangle \phi(\tilde{A}(r, s))) < \frac{\varepsilon}{6} V(\tilde{A}(r, s)) \]

The compactness of \( M_{\delta_1} \) together with \( \phi_* V \leq 3V \) yield the result. \( \square \)

### 13 Appendix E

Let \( G \) be a locally compact group, \( \lambda \) the right invariant Haar measure and \( \Delta \) the (right) modular function i.e. \( \lambda(f^g) = \Delta(g)\lambda(f) \) where \( f^g(x) = f(g^{-1}x) \). A l.c. group is unimodular if and only if the modular function equals 1. We denote with \( K(G) \) the functions of compact support, \( P^p(G) \) are the positive elements of \( L^p(G) \), \( L^\infty(G) \) has here the meaning of the essentially bounded measurable functions (the locally null sets are defined with respect to \( \lambda \)) and \( M^1(G) \) denotes the convex set of the probability measures on \( G \). We denote with \( f * g \) the usual convolution product. The following few properties characterize amenability:
• There exists a net \((\phi_i)\) in \(P^p(G) \cap K(G)\) such that
\[
\lim_i \| \phi_i^a - \phi_i \|_p = 0
\]
whenever \(a \in G\)

• There exists a net \((\phi_i)\) in \(P^1(G)\) such that
\[
\lim_i \| \chi \ast \phi_i - \phi_i \|_1 = 0
\]
whenever \(\chi \in M^1(G)\).

• If \(K\) is a compact subset (resp. \(F\) is a finite subset) and \(\epsilon > 0\) then there exists a measurable set \(U\) in \(G\) of positive finite measure such that
\[
\frac{\lambda(aU \Delta U)}{\lambda(U)} \leq \epsilon \quad \forall a \in K \quad (a \in F)
\]

• If \(K\) is a compact subset (resp. \(F\) is a finite subset) and \(\epsilon > 0\) then there exists a measurable set \(U\) of positive finite measure such that:
\[
\frac{\lambda(KU)}{\lambda(U)} < 1 + \epsilon
\]
and the same for the finite set.

One has of course much more characterizations of amenability; one of them links amenability to properties of the continuous unitary representations of \(G\) [5]. There are special classes of locally compact groups which are known to be (or not to be) amenable. We give a grasp out of a wealth on results:

• Every abelian l.c. group is amenable

• If for every compact neighborhood of the identity in \(G\):
\[
\liminf_{n \to \infty} \frac{\lambda(V_n^{n+1})}{\lambda(V_n)} = 1
\]
then \(G\) is amenable. Notice also that the above condition implies that \(G\) is unimodular. This result implies also that every compact group is amenable. Another consequence is that if the conjugacy classes are relatively compact and \(G\) is totally disconnected then \(G\) is amenable.
• If $G$ is almost connected and semisimple, then $G$ is amenable if and only if $G$ is compact

• if $G$ is almost solvable then it is amenable.

• If $G$ is a discrete free group that is freely generated by elements $a_1 \ldots a_n$ of orders $p_i$ then $G$ is nonamenable unless $n = 2$ and $p_1 = p_2 = 2$.

One proves that any closed subgroup of an amenable l.c. group is amenable, that the finite cartesian product of l.c amenable groups is amenable. So one easily concludes for example that a l.c. group is not amenable if there exists a closed free subgroup of $n$ generators where one generator has order different form 2.
14 References

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