The block-coherence measures and the coherence measures based on positive-operator-valued measures

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We mainly study the block-coherence measures based on resource theory of block-coherence and the coherence measures based on positive-operator-valued measures (POVM). Several block-coherence measures including a block-coherence measure based on maximum relative entropy, the one-shot block coherence cost under the maximally block-incoherent operations, and a coherence measure based on coherent rank have been introduced and the relationships between these block-coherence measures have been obtained. We also give the definition of the maximally block-coherent state and describe the deterministic coherence dilution process by constructing block-incoherent operations. Based on the POVM coherence resource theory, we propose a POVM-based coherence measure by using the known scheme of building POVM-based coherence measures from block-coherence measures, and the one-shot block coherence cost under the maximally POVM-incoherent operations. The relationship between the POVM-based coherence measure and the one-shot block coherence cost under the maximally POVM-incoherent operations is analysed.

I. INTRODUCTION

Quantum coherence is an important ingredient in quantum information processing [1]. Baumgratz et al. proposed the theoretical framework of the resource theory of quantum coherence in 2014 (BCP framework) [2]. The theoretical framework comprises three basic elements: a set of free states which do not contain resource, a corresponding set of free operations that map an arbitrary free state to a free state (generating no resource), and a metric functional [2].

In the resource theory of quantum coherence, the free states are the incoherent states, which can be diagonalized under a fixed reference basis [2]. The free operations (incoherent operations) are some specified classes of physically realizable operations [2]. According to different operational capabilities and physical relevance, the sets of free operations may be: the maximally incoherent operation [3, 4], the dephasing-covariant incoherent operation [3, 5, 6], the incoherent operation [2], the strictly incoherent operation [7, 8], and the physically implementable incoherent operation [9]. In order to quantify coherence, different coherence measures are defined in the resource theory of coherence, such as L1-norm coherence measure [2], relative entropy coherence measure [2] and coherence of formation [7, 10], etc. Coherence measures of different meanings help us better quantify and understand coherence [2, 13].

An interesting problem in the resource theory of quantum coherence is the transformation of states via free operations, specially the transformation between an arbitrary state \( \rho \) and a maximally coherent state [14]. In particular, the process of converting a given state \( \rho \) to the maximally coherent state by incoherent operation is referred to as coherence distillation [14–17]. In contrast to distillation, the dilution process converts the maximally coherent state into the desired target state [14–18]. The processes of asymptotic dilution and distillation are performed under the independent and identically distributed assumption [14–16], which ignores the possible correlation between different state preparations. Therefore, in order to relax the assumption, it is necessary to consider the one-shot scenario, where only one copy of the state is supplied [14–16].

The resource theory of block-coherence was introduced in Ref. [4]. Here we adopt the framework proposed in Ref. [20]. In the resource theory of block-coherence, the block-incoherent states can be considered to be generated by a von Neumann measurement \( \mathbf{P} = \{ P_i \} \) \( i = 1, 2, \cdots, d \), i.e., the block-incoherent state \( \sigma = \sum_{i=1}^{d} P_i \rho P_i \) for state \( \rho \in \mathcal{S} \), where \( \mathcal{S} \) denotes the set of quantum states on the Hilbert space \( \mathcal{H} \), the rank of the orthogonal projector \( P_i \) is arbitrary and the orthogonal projectors form a complete set, i.e., \( \sum_{i=1}^{d} P_i = 1 \).

In 2019, Bischof et al. [20] established the resource theory of coherence based on the positive-operator-valued measures (POVM). The theory is called POVM coherence resource theory. The approach of this theory is to employ the Naimark extension to define the POVM coherence via the block-coherence in a larger Hilbert space, where the quantum states act through an embedded channel in the \( d' \)-dimensional \( (d' > d) \) Hilbert space \( \mathcal{H}' \) (Naimark space), and a POVM \( \mathbf{E} \) is extended to the projection measurement \( \mathbf{P} \) of the Naimark space \( \mathcal{H}' \) \( [20, 21, 22] \). We will give detailed description of the resource theory of block-coherence and the POVM coherence resource theory in the second section of the article.

In this paper, we mainly study the block-coherence measures based on resource theory of block-coherence.
and the coherence measures based on POVM coherence resource theory and analyse the relationship between these block-coherence measures.

The paper is divided into five sections. In Section II, we give main concepts, review the resource theory of block-coherence and POVM coherence resource theory. In Section III, we define two block-coherence measures and the one-shot block coherence cost in the framework of resource theory of block-coherence, and analyse the relationship between these coherence measures. We illustrate the problem of deterministic coherence dilution by constructing block-incoherent operation. In Section IV, a POVM-based coherence measure and the one-shot block coherence cost under the maximally POVM-incoherent operations are defined and analysed.

II. BACKGROUND

A. Block-coherence theoretical framework

In 2006, Åberg introduced the general measurement method of superposition degree of mixed quantum states and applied it to the orthogonal decomposition of Hilbert space, created the resource theory of block-coherence. Similar to the theoretical framework of BCP, it also consists of three elements: the set of block-incoherent states, the set of block-incoherent operations, and the block-coherence measures [1, 20].

The Hilbert space $\mathcal{H}$ is divided into $d$ orthogonal subspaces, the projective measurement $\mathbf{P} = \{P_i\}$ is performed on a set $S$ of quantum states, where $P_i$ is the projector of the $i$th subspace. Block-incoherent states [1, 20, 22] are defined as

$$
\rho_{\text{BI}} = \sum_i P_i \rho P_i = \Delta[\sigma], \ \sigma \in S,
$$

where $\Delta$ denotes the block-dephasing operation. The set of block-incoherent states is denoted as $\mathcal{I}_B(\mathcal{H})$.

We refer to the largest class of (free) operations that cannot produce block-coherence as maximally block-incoherent (MBI) operations. A channel $\Lambda_{\text{MBI}}$ on $S$ is an element of this operation class if and only if it maps any block-incoherent state to a block-incoherent state [1, 20–22], namely

$$
\mathcal{I}_B(\mathcal{H}) \subseteq \mathcal{I}_B(\mathcal{H}),
$$

or equivalently

$$
\Lambda_{\text{MBI}} \circ \Delta = \Delta \circ \Lambda_{\text{MBI}} \circ \Delta.
$$

A quantum channel $\Lambda$ is often expressed by the Kraus operators. Let $\{K_n\}$ be a set of Kraus operators on $\mathcal{H}$, and the operators satisfy the normalization condition $\sum_n K_n K_n^\dagger = \mathbb{I}$. Some Kraus operators have the form

$$
K_n = \sum_i P_i(i)c_nP_i,
$$

where $f$ is the index function, $c_n$ is the complex matrix. $K_n$ is a block-incoherent Kraus operator, if $f$ is an index permutation.

A real-valued function $C(\rho, \mathbf{P})$ is called block-coherence monotone of quantum state $\rho$ with respect to the projective measurement $\mathbf{P}$, if it satisfies [4, 20, 22]:

1. (B1) Faithfulness: $C(\rho, \mathbf{P}) \geq 0$ with equality if $\rho \in \mathcal{I}_B(\mathcal{H})$.
2. (B2) Monotonicity: $C(\Lambda_{\text{BI}}(\rho), \mathbf{P}) \leq C(\rho, \mathbf{P})$ for any block-incoherent operation $\Lambda_{\text{BI}}$.
3. (B3) Strong monotonicity: $\sum_n p_n C(\rho_n, \mathbf{P}) \leq C(\rho, \mathbf{P})$ for any block-incoherent operation $\Lambda_{\text{BI}}$.
4. (B4) Convexity: $C(\sum_n p_n \rho_n, \mathbf{P}) \leq \sum_n p_n C(\rho_n, \mathbf{P})$ for all states $\rho_n$, and the probability $\{p_n\}$ which satisfies $\sum_n p_n = 1$.

Note that the rank of the above projector $P_i$ is arbitrary, and when the rank of $P_i$ is 1, it is consistent with the standard resource theory of coherence.

B. POVM coherence theoretical framework

The most general quantum measurement refers to the positive-operator-valued measures (POVM) [20]. Let a set $\mathbf{E} = \{E_i\}_{i=1}^n$ of positive-semidefinite operators be a POVM on a $d$-dimensional Hilbert space $\mathcal{H}$, and $\sum_i E_i = \mathbb{I}$, where $E_i$ is called POVM element. Suppose $E_i = A_i^\dagger A_i$ for any $i$, where $\{A_i\}$ is a set of measurement operators for $\mathbf{E}$, and $A_i$ can be written as $A_i = U_i \sqrt{E_i}$ with any unitary operator $U_i$. The $i$th post-measurement state for a given $A_i$ is $\rho_i = \frac{A_i \rho A_i^\dagger}{\text{Tr}(A_i \rho A_i^\dagger)}$ [20, 21, 23].

The POVM coherence resource theory is established via the Naimark extension [20, 24]. Every POVM $\mathbf{E} = \{E_i\}_{i=1}^n$ on a $d$-dimensional Hilbert space $\mathcal{H}$, can be extended to a projective measurement $\mathbf{P} = \{P_i\}_{i=1}^n$ on the Hilbert space $\mathcal{H}'$, if one can embed the $d$-dimensional Hilbert space $\mathcal{H}$ into a larger $d'$-dimensional Hilbert space $\mathcal{H}'$ called the Naimark space, where $d' \geq d$. The general way to embed the original space $\mathcal{H}$ into a larger space $\mathcal{H}'$ is via a direct sum, namely, in the Naimark space $\mathcal{H}'$, the corresponding state $\varepsilon(\rho)$ of quantum state $\rho$ in the $d$-dimensional Hilbert space $\mathcal{H}$ is

$$
\varepsilon(\rho) = \rho \oplus 0,
$$

requiring

$$\text{Tr}[E_i \rho] = \text{Tr}[P_i \varepsilon(\rho)] = \text{Tr}[P_i (\rho \oplus 0)]$$

to hold for all states $\rho$ in a set $S$ of quantum states. Here $\oplus$ denotes the orthogonal direct sum, and $0$ is zero matrix.
The block-coherence measure of dimension $d' - d$. Any projective measurement $P$ that satisfies Eq. (6) is called a Naimark extension of $E$.

The embedding into a larger-dimensional Hilbert space can also be realized via the canonical Naimark extension \cite{20,24}; one attaches a probe, initially in the state $|1\rangle\langle 1|$, via the tensor product $\varepsilon(\rho) = \rho \otimes |1\rangle\langle 1|$. \cite{20}. A canonical Naimark extension projective measurement $P = \{P_i\}_{i=1}^n$ of the POVM $E = \{E_i\}_{i=1}^n$ is described by a unitary matrix $V$ which makes \cite{20,21}

$$P_i := V^\dagger (|i\rangle\langle i|) V, $$

and

$$\text{Tr}[E_i\rho] = \text{Tr}[P_i(\rho \otimes |1\rangle\langle 1|)]$$

(8)

to hold for all states $\rho$ in the quantum state set $\mathcal{S}$. A state $\rho$ is called a POVM-incoherent state \cite{20,21,23}, if

$$E_i\rho E_j = 0, \text{ for all } i \neq j,$$

or equivalently

$$A_i\rho A_j^\dagger = 0, \text{ for all } i \neq j.$$ (10)

The set of POVM-incoherent states is denoted as $\mathcal{I}_P$.

A channel $\Lambda$ is called a POVM-incoherent operation $\mathcal{P}_I$ with respect to the POVM $E = \{E_i\}_{i=1}^n$ if it admits a Kraus decomposition $\Lambda(\rho) = \sum_i K_i\rho K_i^\dagger$ such that all operators $K_i$ with respect to a canonical Naimark extension projective measurement $P = \{P_i\}_{i=1}^n$ of the POVM $E = \{E_i\}_{i=1}^n$ satisfies

$$K_i\rho K_i^\dagger \otimes |1\rangle\langle 1| = K_i'(\rho \otimes |1\rangle\langle 1|)(K_i')^\dagger,$$ (11)

for all $i \in \{1,2,\ldots,n\}$, where $\{K_i'\}$ is a set of the block-incoherent operators on the extended Hilbert space $\mathcal{H}'$ \cite{21}.

The POVM-based coherence measure $C(\rho, E)$ of a state $\rho$ with respect to a POVM $E = \{E_i\}_{i=1}^n$ is defined as the block-coherence measure $C(\varepsilon(\rho), P)$ of $\varepsilon(\rho)$ with respect to the Naimark extension POVM $P$ of $E$ \cite{20,21,23}, namely

$$C(\rho, E) := C(\varepsilon(\rho), P),$$ (12)

where the function $C$ on the right side denotes any unitary-covariant block-coherence measure.

The POVM-based coherence measure $C(\rho, E)$ with respect to a general quantum measurement $E = \{E_i\}_{i=1}^n$ should satisfy:

(P1) Faithfulness: $C(\rho, E) \geq 0$ with equality if $\rho \in \mathcal{I}_P$.

(P2) Monotonicity: $C(\Lambda_P(\rho), E) \leq C(\rho, E)$ for any POVM-incoherent operation $\Lambda_P$.

(P3) Strong monotonicity: $\sum_i p_i C(\rho_i, E) \leq C(\rho, E)$ for all POVM-incoherent operation $\Lambda_P = \{K_i\}$, where

$$p_i = \text{Tr}(K_i\rho K_i^\dagger), \quad K_i = \frac{K_i' p_i}{\sum_i p_i}.$$ (13)

(P4) Convexity: $C(\sum_i p_i \rho_i, E) = \sum_i p_i C(\rho_i, E)$ for all states $\rho_i$, and the probability $\{p_i\}$ satisfying $p_i \geq 0$.

C. The max-relative entropy and the coherent rank

In the theoretical framework of BCP, the max-relative entropy between quantum state $\rho \geq 0$ and quantum state $\sigma \geq 0$ is defined as \cite{25,26}

$$D_{\max}(\rho||\sigma) = \log_2 \min\{\lambda | \rho \leq \lambda \sigma \}. $$ (13)

One equivalent definition of $D_{\max}(\rho||\sigma)$ \cite{26} is

$$D_{\max}(\rho||\sigma) := \log_2 \min\{\lambda | \text{Tr}[P^\dagger_+ (\rho - \lambda \sigma)] = 0\}, $$ (14)

where $P^\dagger_+$ is the projector of $\rho - \lambda \sigma$ with positive eigenvalues.

The coherent rank $C_r$ of a pure state $|\varphi\rangle = \sum_{i=1}^R |\varphi_i\rangle |i\rangle$ (not necessarily normalized) with $\varphi_i \neq 0$ is defined as the number of terms with $\varphi_i \neq 0$ \cite{7,27}, i.e.,

$$C_r(\varphi) = R.$$ (15)

III. THE BLOCK-COHERENCE MEASURES

Based on the max-relative entropy, we first define a block-coherence measure, which is a generalization of the coherence measure in Ref. \cite{18}.

Definition 1. The block-coherence measure $C_{\max}(\rho, P)$ of a quantum state $\rho$ with respect to the projective measurement $P$ is defined as

$$C_{\max}(\rho, P) = \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\rho||\sigma). $$ (16)

Then, we have the following result.

Proposition 1. The block-coherence measure $C_{\max}(\rho, P)$ is a block-coherence monotone under MBI operations and it is quasi-convex.

Proof. First, we show that $C_{\max}(\rho, P) \geq 0$ with the equality if and only if $\rho \in \mathcal{I}_B(\mathcal{H})$.

By the definition, we known \cite{25,26}

$$C_{\max}(\rho, P) = \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\rho||\sigma) = \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} \log_2 \min\{\lambda | \rho \leq \lambda \sigma \}. $$ (17)

Since $\rho \leq \lambda \sigma$, we have $\text{Tr}(\lambda \sigma - \rho) \geq 0$. So $\lambda \geq 1$ holds. Hence

$$C_{\max}(\rho, P) \geq 0. $$ (18)

From Ref. \cite{25,26}, we know that $D_{\max}(\rho||\sigma) = 0$ if and only if $\rho = \sigma$. Then, when

$$C_{\max}(\rho, P) = \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\rho||\sigma) = 0, $$ (19)
$\rho$ must be a block-incoherent state. This implies that $C_{\text{max}}(\rho, P)$ satisfies (B1).

Second, we can prove that for any MBI operation with $\{K_n\}$, $C_{\text{max}}(\rho, P)$ satisfies (B2). According to Ref. [25], we know that the max-relative entropy $D_{\text{max}}(\rho\|\sigma)$ are monotonic under completely positive trace-preserving map (CPTP) $\Lambda$. Hence

$$D_{\text{max}}(\Lambda(\rho)\|\Lambda(\sigma)) \leq D_{\text{max}}(\rho\|\sigma).$$

(20)

As any MBI operation $\Lambda_{\text{MBI}}$ with $\{K_n\}$ is a CPTP map, we have

$$\min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\text{max}}(\Lambda_{\text{MBI}}(\rho)\|\Lambda_{\text{MBI}}(\sigma)) \leq \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\text{max}}(\rho\|\sigma).$$

(21)

Therefore,

$$C_{\text{max}}\left(\sum_n K_n \rho K_n^\dagger, P\right) \leq C_{\text{max}}(\rho, P).$$

(22)

It means that $C_{\text{max}}(\rho, P)$ satisfies (B2).

Next we will show that $C_{\text{max}}(\rho, P)$ is quasi-convex, i.e.,

$$C_{\text{max}}(\sum_n p_n \rho_n, P) \leq \max_n C_{\text{max}}(\rho_n, P),$$

(23)

where $p_n = \text{Tr}(K_n \rho K_n^\dagger)$, $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$.

In order to prove the above conclusion we need the following result: For self-adjoint operators $A, B$ and any positive operator $0 \leq P \leq I$, we have [26, 28, 29]

$$\text{Tr}[P(A - B)] \leq \text{Tr}[[A \geq B](A - B)],$$

$$\text{Tr}[P(A - B)] \geq \text{Tr}[[A \leq B](A - B)].$$

(24)

Now let’s prove that $C_{\text{max}}(\rho, P)$ is quasi-convex. For any mixture of states, $\rho = \sum_n p_n \rho_n$, we can construct a block-incoherent state $\sigma = \sum_n p_n \sigma_n$, where every $\sigma_n$ is a block-incoherent state. Another equivalent definition of the max-relative entropy $D_{\text{max}}(\rho\|\sigma)$ [25] is

$$D_{\text{max}}(\rho\|\sigma) := \log_2 \min[\lambda |\text{Tr}[P_+^{\lambda}(\rho - \lambda \sigma)] = 0],$$

(25)

where $P_+^{\lambda}$ is the projector of $\rho - \lambda \sigma$ with positive eigenvalues. By (24), we have [26]

$$0 \leq \text{Tr}[P_+^{\lambda}(\rho - \lambda \sigma)] = \sum_n p_n \text{Tr}[P_+^{\lambda}(\rho_n - \lambda \sigma_n)] \leq \sum_n p_n \text{Tr}[P_+^{\lambda,n}(\rho_n - \lambda \sigma_n)],$$

(26)

where $P_+^{\lambda,n}$ is the projector of $\rho_n - \lambda \sigma_n$ with positive eigenvalues. Set $\lambda = \max \lambda_n$, where for each $n$, $\lambda_n$ is defined by $\log_2 \lambda_n = C_{\text{max}}(\rho_n, P)$.

For this choice of $\lambda$, there is $\text{Tr}[P_+^{\lambda}(\rho - \lambda \sigma)] = 0$, and hence $\log_2 \lambda \geq C_{\text{max}}(\rho, P)$, i.e.,

$$C_{\text{max}}(\sum_n p_n \rho_n, P) \leq \max_n C_{\text{max}}(\rho_n, P).$$

(27)

So $C_{\text{max}}(\rho, P)$ is the quasi-convex. ■

**Definition 2.** A maximally block-coherent state is defined by

$$|\psi_N\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} P_k |\psi_d\rangle,$$  

(28)

where $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$ is maximally coherent state in the $d$-dimensional Hilbert space, and the rank of projective measurement $P_k$ is arbitrary and the number of $P_k$ in the projective measurement $P = \{P_k\}$ is $N$ ($N \leq d$).

Obviously, for a maximally block-coherent state $|\psi_N\rangle$, we have

$$C_{\text{max}}(\psi_N, P) = \log_2 N,$$  

(29)

namely the value of $C_{\text{max}}(\psi_N, P)$ depends on the number $N$ of projectors in the space.

One-shot scenario is the most general conversion case, where the conversion is from an initial state to a final state. One-shot block coherence dilution process is to convert the maximally block-coherent state $|\psi_N\rangle$ into the desired state $\rho$ via the maximally block-incoherent operation [14, 18, 19].

First, we define a block-coherence measure, the one-shot block coherence cost of quantifying block coherence dilution.

**Definition 3.** Let MBI denote the set of the maximally block-incoherent operations. For a given state $\rho$ and $\epsilon \geq 0$, the one-shot block coherence cost under the maximally block-incoherent operations is defined as

$$C_{\text{MBI}}(\rho, p) = \min_{\Lambda \in \text{MBI}} \{\log_2 N |\Lambda(|\psi_N\rangle, \rho) > 1 - \epsilon\},$$

(30)

while $F(\rho, \sigma) = (\text{Tr}[\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}])^2$ is the fidelity between two quantum states $\rho$ and $\sigma$, where $|\psi_N\rangle$ is the maximally block-coherent state.

Since the one-shot scenario allows errors to exist, in the presence of the error $\epsilon$, we use

$$C^\epsilon(\rho) := \min_{\rho^\prime : F(\rho, \rho^\prime) \geq 1 - \epsilon} C(\rho^\prime)$$

(31)

to characterize the coherence measure of state $\rho$ [18]. That is in order to define the coherence cost with a certain error $\epsilon$, one can use a smoothing to the measure $C(\rho)$ by minimizing over states $\rho^\prime$ satisfying $F(\rho, \rho^\prime) \geq 1 - \epsilon$ to smooth the measure $C(\rho)$.

Next we discuss the relationship between the coherence measure $C_{\text{max}}^\epsilon(\rho, P)$ and the one-shot block coherence cost $C_{\text{MBI}}^\epsilon(\rho, P)$. We have

**Theorem 1.** For $\epsilon > 0$, the coherence measures satisfy

$$C_{\text{max}}(\rho, P) \leq C_{\text{MBI}}^\epsilon(\rho, P) \leq C_{\text{max}}^\epsilon(\rho, P) + 1.$$  

(32)
We calculate the critical value of $\lambda$. Here

$$C_{\text{max}}(\rho, P) = \min_{\delta \in I(\mathcal{P})} D_{\text{max}}(\Lambda_{\text{MBI}}(\psi_{N'}), \|\delta\|)$$

(33)

Then

$$C_{\text{max}}(\rho, P) \leq D_{\text{max}}(\psi_{N'} \| \sigma) = \log_2 N' = C_{\text{MBI}}(\rho, P).$$

(34)

So

$$C_{\text{max}}(\rho, P) \leq C_{\text{MBI}}(\rho, P).$$

(35)

Next we prove the right side of Eq. (32). Assume that the state $\rho'$ reaches minimum, so

$$C_{\text{max}}(\rho, P) = C_{\text{max}}(\rho', P) = D_{\text{max}}(\rho' \| \delta)$$

(36)

Set $N'' = [\lambda]$, then $\rho' \leq N'' \delta$. Consider the following mapping

$$\Lambda(\omega) = \frac{1}{N'' - 1} (N''(T_{\psi_{N''} \circ \omega}) - 1) \rho' + \frac{N''}{N'' - 1} (1 - T_{\psi_{N''} \circ \omega}) \delta,$$

(37)

where $\psi_{N''} \circ \omega = \langle \psi_{N''} | \psi_{N''} \rangle \omega$, $T_{\psi_{N''} \circ \omega} = \langle \psi_{N''} | \psi_{N''} \rangle \omega$. For all $\delta = \sum_{i=1}^{N'} P_i \rho_i \in I(\mathcal{H})$, we have $T_{\psi_{N''} \circ \delta} = \frac{1}{N''} \rho'$ and $\Lambda(\delta) = \delta \in I(\mathcal{H})$. So $\Lambda \in \text{MBI}$. On the other hand, it is easy to obtain $\Lambda(\psi_{N''}) = \rho'$. One can also write the mapping as

$$\Lambda(\omega) = \frac{1}{N'' - 1} (N''(T_{\psi_{N''} \circ \omega}) - 1) \rho' + \frac{N''}{N'' - 1} (1 - T_{\psi_{N''} \circ \omega}) \delta,$$

(38)

$$= \frac{1}{N'' - 1} (\text{Tr}[\psi_{N''} \circ \omega] \rho' - \frac{1}{N'' - 1} \rho') + \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \omega]) \delta.$$
(maximally block-coherent state) and the final state $|\phi\rangle$ satisfy the majorization relation 30, i.e.,

$$\mu(\psi_d) = (||\psi_d||^2, ||\psi_d||^2, \ldots, ||\psi_d||^2)^T$$

$$\prec \mu(\phi) = (\phi_1^2, \phi_2^2, \ldots, \phi_d^2)^T,$$

where $||\psi_d||^2 = \frac{1}{d}$.

According to the protocol 31 for the deterministic transformations of the coherent states for which the majorization relation can be satisfied, for the case $\frac{1}{\sqrt{d}} \leq \phi_i, i = 1, 2, \ldots, d - 1$, the set of $d$ permutations in that case turns out to be

$$\{U^i|i = 1, 2, \ldots, d\}$$

$$= \{I_d, |1\rangle \leftrightarrow |d\rangle, |2\rangle \leftrightarrow |d\rangle, \ldots, |d-1\rangle \leftrightarrow |d\rangle\};$$

the probabilities in that case turn out to be 31

$$p^1 = 1 - \sum_{i=2}^{d} p^i, \quad p^i = \frac{\phi_{i-1}^2 - \psi_{i-1}^2}{\phi_{i-1}^2 - \phi_d^2};$$

the set of Kraus operators of the incoherent operation 31 is

$$\{K^i = U^i \sqrt{p^i} \sum_{j=1}^{d} c_{ij} \sqrt{d}|j\rangle \langle j|, i = 1, 2, \ldots, d\},$$

where $c_{ij}$ is the $(ij)$th element of the $d \times d$ matrix $c$ satisfying

$$U^1(c_{11}, c_{12}, \cdots, c_{1d})^T = (\phi_1, \phi_2, \cdots, \phi_d)^T.$$ (47)

For example, when $d = 4, N = 4$, we have $P_1 = |1\rangle \langle 1|$, $P_2 = |2\rangle \langle 2|$, $P_3 = |3\rangle \langle 3|$ and $P_4 = |4\rangle \langle 4|$. The maximally block-coherent state $|\psi_4\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$. We choose the dilution state $|\phi\rangle = \sum_i \phi_i |i\rangle = \sqrt{0.4}|1\rangle + \sqrt{0.3}|2\rangle + \sqrt{0.28}|3\rangle + \sqrt{0.02}|4\rangle$. So $\mu(\phi) = (0.4, 0.3, 0.28, 0.02)$ and $\mu(\psi_4) = (0.25, 0.25, 0.25, 0.25)$. It is easy to see

$$\mu(\psi_4) = (0.25, 0.25, 0.25, 0.25)$$

$$\prec \mu(\phi) = (0.4, 0.3, 0.28, 0.02).$$

Obviously $|\phi\rangle$ and $|\psi_4\rangle$ satisfy $\frac{1}{\sqrt{d}} \geq \phi_d, \frac{1}{\sqrt{d}} \leq \phi_i, i = 1, 2, \ldots, d - 1$, when $d = 4$. The set of permutations for this case should be

$$\{I_4, |1\rangle \leftrightarrow |4\rangle, |2\rangle \leftrightarrow |4\rangle, |3\rangle \leftrightarrow |4\rangle\}. $$

The matrix $c$ corresponding to this set of permutations

is

$$c = \left(\begin{array}{cccc}
\sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{3}{5}} \\
\sqrt{\frac{1}{5}} & \sqrt{\frac{2}{5}} & \sqrt{\frac{3}{5}} & \sqrt{\frac{1}{5}} \\
\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{3}{5}} & \sqrt{\frac{1}{5}} \\
\sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{3}{5}}
\end{array}\right).$$

Thus, the set of probabilities is found to be

$$p^2 = \frac{\phi_2^2 - \psi_2^2}{\phi_2^2 - \phi_4^2} = \frac{15}{38},$$

$$p^3 = \frac{\phi_3^2 - \psi_3^2}{\phi_3^2 - \phi_4^2} = \frac{5}{28},$$

$$p^4 = \frac{\phi_4^2 - \psi_4^2}{\phi_4^2 - \phi_4^2} = \frac{3}{26},$$

$$p^1 = 1 - p^2 - p^3 - p^4 = \frac{2153}{6916}.$$ (51)

Then the Kraus operators are

$$K^1 = U^1 \sqrt{p^1} (2c_{11}|1\rangle \langle 1| + 2c_{12}|2\rangle \langle 2| + 2c_{13}|3\rangle \langle 3| + 2c_{14}|4\rangle \langle 4|),$$

$$K^2 = U^2 \sqrt{p^2} (2c_{21}|1\rangle \langle 1| + 2c_{22}|2\rangle \langle 2| + 2c_{23}|3\rangle \langle 3| + 2c_{24}|4\rangle \langle 4|),$$

$$K^3 = U^3 \sqrt{p^3} (2c_{31}|1\rangle \langle 1| + 2c_{32}|2\rangle \langle 2| + 2c_{33}|3\rangle \langle 3| + 2c_{34}|4\rangle \langle 4|),$$

$$K^4 = U^4 \sqrt{p^4} (2c_{41}|1\rangle \langle 1| + 2c_{42}|2\rangle \langle 2| + 2c_{43}|3\rangle \langle 3| + 2c_{44}|4\rangle \langle 4|),$$

where $U^1 = I_4$ is the identity transformation, $U^2 = |1\rangle \leftrightarrow |4\rangle$, $U^3 = |2\rangle \leftrightarrow |4\rangle$, $U^4 = |3\rangle \leftrightarrow |4\rangle$. The Kraus operators can be expressed in the following form

$$K^1 = U^1 \left(\begin{array}{cccc}
\sqrt{\frac{4396}{8045}} & 0 & 0 & 0 \\
0 & \sqrt{\frac{6459}{17290}} & 0 & 0 \\
0 & 0 & \sqrt{\frac{2153}{53458}} & 0 \\
0 & 0 & 0 & \sqrt{\frac{2153}{53458}}
\end{array}\right),$$

$$K^2 = U^2 \left(\begin{array}{cccc}
\sqrt{\frac{2}{\sqrt{19}}} & 0 & 0 & 0 \\
0 & \sqrt{\frac{3}{\sqrt{19}}} & 0 & 0 \\
0 & 0 & \sqrt{\frac{45}{85}} & 0 \\
0 & 0 & 0 & \frac{2\sqrt{19}}{\sqrt{19}}
\end{array}\right).$$

(53)
A block-coherence measure based on the quantity $\rho$ is taken over all possible pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $\rho$, $|\psi_i\rangle\langle\psi_i|$, is the number of $P_j$ satisfying $\langle\psi_i|P_j|\psi_i\rangle \neq 0$.

Next we define another coherence measure based on coherent rank.

**Definition 4.** A block-coherence measure based on coherent rank is defined as

$$C_0(\rho, \mathbf{P}) = \min_{\{p_i, |\psi_i\rangle\}} \max_i \log_2 M(|\psi_i\rangle),$$

where $\mathbf{P}$ is the projective measurement, the minimum is taken over all possible pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ with $p_i \geq 0$ and $\sum_i p_i = 1$, $M(|\psi_i\rangle)$ is the number of $P_j$ satisfying $\langle\psi_i|P_j|\psi_i\rangle \neq 0$.

We have the following result.

**Proposition 2.** The quantity $C_0(\rho, \mathbf{P})$ is a coherence monotone under the block-incoherent operation $\Lambda_{BI}$.

**Proof.** Apparently $C_0(\rho, \mathbf{P}) \geq 0$. Next we prove that the quantity $C_0(\rho, \mathbf{P})$ satisfies $C_0(\rho, \mathbf{P}) = 0 \Leftrightarrow \rho \in \mathcal{I}_B(H)$.

Suppose $C_0(\rho, \mathbf{P}) = 0$ and the corresponding ensemble of $\rho$ is $\{p_i, |\psi_i\rangle\}$, we can deduce for all $i$, $|\psi_i\rangle\langle\psi_i| = P_i |\psi_i\rangle\langle\psi_i| P_i$, which means that $\rho \in \mathcal{I}_B(H)$. Conversely, suppose $\rho = \sum_i P_i |\psi_i\rangle\langle\psi_i| P_i$, we can choose $\{\delta_j, P_j|\psi_j\rangle\}$ as a decomposition, which leads to $C_0(\rho, \mathbf{P}) = 0$. Hence $C_0(\rho, \mathbf{P})$ satisfies (B1).

Then, we prove that for any block-incoherent operation with $\{K_n\}$, there is

$$C_0(\sum_n K_n \rho K_n^\dagger, \mathbf{P}) \leq C_0(\rho, \mathbf{P}).$$

Before we prove above conclusion, let’s introduce the following lemma proved in Ref. [7].

**Lemma 1.** If $|\psi\rangle = \frac{|K_1\psi\rangle}{\sqrt{\text{Tr}[|K_1\psi\rangle\langle\psi|K_1^\dagger]]}}$, where $\{K_1\}$ is a set of incoherent-preserving Kraus operators, then $C_0(|\psi\rangle\langle\psi|) \leq C_0(|\phi\rangle\langle\phi|)$.

It is easy to see that the Lemma 1 also holds when $\{K_1\}$ is a block-incoherent operation.

Let $\{p_i, |\psi_i\rangle\}$ be the decomposition such that $C_0(\rho, \mathbf{P}) = \max_i \log_2 M(|\psi_i\rangle)$.

Let $\Lambda_{BI}$ be any block-incoherent operation with $\Lambda_{BI}(\rho) = \sum_n K_n \rho K_n^\dagger$. For a given state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, then the post-measurement state of $n$th outcome is

$$|\psi_i^n\rangle = \frac{K_n |\psi_i\rangle}{\sqrt{\text{Tr}[K_n |\psi_i\rangle\langle\psi_i| K_n^\dagger]]}}.$$

Then we can get an ensemble $\{p(i|n), |\psi_i^n\rangle\}$, where the probability is

$$p(i|n) = \frac{\text{Tr}[K_n |\psi_i\rangle\langle\psi_i| K_n^\dagger]}{\text{Tr}[K_n \rho K_n^\dagger]}.$$ Then the corresponding density operator $\rho_n$ of $n$th outcome is

$$\rho_n = \sum_i p_i K_n |\psi_i\rangle\langle\psi_i| K_n^\dagger \frac{\text{Tr}[K_n \rho K_n^\dagger]}{\text{Tr}[K_n \rho K_n^\dagger]}.$$ According to the Lemma 1, we can know that for the minimum ensemble decomposition, there is $C_0(\rho_n, \mathbf{P}) \leq C_0(\rho, \mathbf{P})$, and then

$$C_0(\sum_n K_n \rho K_n^\dagger, \mathbf{P}) \leq C_0(\rho, \mathbf{P}).$$

This implies $C_0(\rho, \mathbf{P})$ satisfies (B2).

Now we discuss the relationship between $C_{\text{MBI}}(\rho, \mathbf{P})$ and $C_0(\rho, \mathbf{P})$. We have the following conclusion.

**Theorem 2.** For $\epsilon > 0$, the value of the one-shot block coherence cost under MBI is equal to $C_0^\epsilon(\rho, \mathbf{P})$, namely

$$C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) = C_0^\epsilon(\rho, \mathbf{P}).$$

**Proof.** First we study the lower bound on $C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$.

Let $\log_2 N = C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$, then there exists an operation $\Lambda_{MBI}$ such that $F[\Lambda_{MBI}(\psi_N), \rho] \geq 1 - \epsilon$. Then we have

$$C_{\rho}(\rho, \mathbf{P}) \leq C_0(\Lambda_{MBI}(\psi_N), \mathbf{P}) \leq C_0(\rho, \mathbf{P}) \leq C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) = \log_2 N = C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$$

For the upper bound on $C_{\text{MBI}}(\rho, \mathbf{P})$, we select the state $\rho', \rho$ reaching minimum such that $C_0(\rho', \mathbf{P}) = C_0(\rho', \mathbf{P})$. Let $C_0(\rho', \mathbf{P}) = \log_2 N'$, as the MBI operation is constructed in the deterministic coherence dilution process discussed above, there is a $\Lambda_{MBI}$ such that $F[\Lambda_{MBI}(\psi_{N'}), \rho] = \bar{F}[\rho', \rho] \geq 1 - \epsilon$, thus

$$C_{\text{MBI}}(\rho, \mathbf{P}) \leq C_{\text{MBI}}(\rho', \mathbf{P}) = \log_2 N' = C_0(\rho', \mathbf{P}) = C_0(\rho, \mathbf{P}).$$

Therefore, we obtain

$$C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) = C_0^\epsilon(\rho, \mathbf{P}).$$
IV. THE POVM-BASED COHERENCE MEASURES

For a POVM $E = \{ E_i = A_i^d A_i \}_{i=1}^n$ on a $d$-dimensional Hilbert space $\mathcal{H}$, a canonical Naimark extension projective measurement $P = \{ P_i \}_{i=1}^n$ of $E$ is described by a unitary matrix $V$ on Naimark space $\mathcal{H}'$ as

$$ P_i = V^\dagger \overline{P}_i V, \quad (67) $$

where

$$ V = \sum_{i,j=1}^n A_{ij} \otimes |i\rangle \langle j|, \quad (68) $$

with $\{ A_{ij} \}_{i,j=1}^n$ satisfying the conditions.

$$ \sum_{i=1}^n A_{ij}^\dagger A_{i'j} = \delta_{jj} I_d, \quad \sum_{j=1}^n A_{ij}^\dagger A_{j'i} = \delta_{ii} I_d, \quad A_{i1} = A_i, \quad (69) $$

and

$$ \overline{P} = ( \overline{P}_i = I_d \otimes |i\rangle \langle i| )_{i=1}^n. \quad (70) $$

Let $C(\rho', \overline{P})$ be a unitary invariant block-coherence measure, that is,

$$ C(\rho', \overline{P}) = C(U \rho'^\dagger U^\dagger, U \overline{P} U^\dagger) \quad (71) $$

for any unitary transformation $U$ on the Hilbert space. The POVM-based coherence measure $C(\rho, E)$ of $\rho$ under POVM $E$ is defined [20]

$$ C(\rho, E) = C(\varepsilon(\rho), \overline{P}) = C(\rho \otimes |1\rangle \langle 1|, \overline{P}), \quad (72) $$

where

$$ \varepsilon(\rho) = \sum_{i,j=1}^n A_i \rho A_j^\dagger \otimes |i\rangle \langle j| \quad (73) $$

is a state on the embedded state Hilbert space $\mathcal{H}'$.

From the conclusions in references [20, 21, 22], we know that the quantity $C(\rho, E)$ is a POVM-based coherence measure satisfying the conditions (P1), ..., (P4).

Next, we discuss a concrete POVM-based coherence measure.

**Proposition 3.** Let $E = \{ E_i = A_i^d A_i \}_{i=1}^n$ be a POVM on the Hilbert space $\mathcal{H}$, the quantity based on the maximal relative entropy

$$ C_{\max}(\rho, E) = C_{\max}(\varepsilon(\rho), \overline{P}) = \min_{\sigma \in \mathcal{I}_{\text{BI}}(\mathcal{H}_z)} \log_2 \min \{ \lambda | \varepsilon(\rho) - \lambda \sigma \} \quad (74) $$

is a block-coherence monotone and it is quasi-convex. Here $\mathcal{I}_{\text{BI}}(\mathcal{H}_z)$ is the set of block-incoherent states in the Hilbert space $\mathcal{H}_z$.

**Proof.** We first prove that $C_{\max}(\varepsilon(\rho), \overline{P})$ is invariant under unitary transformation. The quantity

$$ C_{\max}(\varepsilon(\rho), \overline{P}) = \min_{\sigma \in \mathcal{S}(\mathcal{H}_z)} \log_2 \min \{ \lambda | \varepsilon(\rho) - \lambda \sigma \} \quad (75) $$

where $\sigma$ is an arbitrary density operator on the state set $\mathcal{S}(\mathcal{H}_z)$.

For any unitary transformation $U$ on $\mathcal{H}_z$, we have

$$ C_{\max}(\varepsilon(\rho) U^\dagger, U \overline{P} U^\dagger) = \min_{\sigma \in \mathcal{S}(\mathcal{H}_z)} \log_2 \min \{ \lambda | \varepsilon(\rho) U^\dagger - \lambda \sigma \} \quad (76) $$

Then, we show that $C_{\max}(\varepsilon(\rho), \overline{P})$ is a block-coherence monotone.

Firstly we prove that $C_{\max}(\varepsilon(\rho), \overline{P}) \geq 0$, with equality if and only if $\varepsilon(\rho) = \sum_{i=1}^n \overline{P}_i \sigma \overline{P}_i$, i.e., $\varepsilon(\rho)$ is the block-incoherent state on $\mathcal{H}_z$.

By the definition, we known

$$ C_{\max}(\varepsilon(\rho), \overline{P}) = \min_{\sigma \in \mathcal{S}(\mathcal{H})} \log_2 \min \{ \lambda | \varepsilon(\rho) - \lambda \sigma \} \quad (77) $$

since $\varepsilon(\rho) \leq \lambda \sum_{i=1}^n \overline{P}_i \sigma \overline{P}_i$, we have $\text{Tr}(\lambda \sum_{i=1}^n \overline{P}_i \sigma \overline{P}_i - \varepsilon(\rho)) \geq 0$. So $\lambda \geq 1$ holds. Hence,

$$ C_{\max}(\varepsilon(\rho), \overline{P}) \geq 0. \quad (78) $$

According to the properties of maximum relative entropy, the equality holds if and only if $\varepsilon(\rho) = \sum_{i=1}^n \overline{P}_i \sigma \overline{P}_i$, thus $\varepsilon(\rho)$ is a block-incoherent state on $\mathcal{H}_z$. This implies that $C_{\max}(\varepsilon(\rho), \overline{P})$ satisfies (B1).

The monotonicity of $C_{\max}(\varepsilon(\rho), \overline{P})$ can be easily derived from the properties of the max-relative entropy. Hence $C_{\max}(\varepsilon(\rho), \overline{P})$ satisfies (B2).

It is easy to show that $C_{\max}(\varepsilon(\rho), \overline{P})$ is also quasi-convex, i.e.,

$$ C_{\max}(\sum_{i=1}^n p_i \varepsilon_i(\rho), \overline{P}) \leq \max_{i} C_{\max}(\varepsilon_i(\rho), \overline{P}), \quad (79) $$

where $p_i = \text{Tr}(K_i^\dagger \varepsilon(\rho) (K_i^\dagger)^\dagger)$, $\varepsilon_i(\rho) = \frac{K_i^\dagger \varepsilon(\rho) K_i^\dagger}{p_i}$, $\{ K_i^\dagger \}$ is the set of the Kraus operations.

Combining the results above, we know that the quantity $C_{\max}(\rho, E)$ is a block-coherence monotone and quasi-convex.

Now, we define one-shot block coherence cost under the maximally POVM-incoherent operations.
Definition 5. Let $E = \{E_i = A_i^\dagger A_i\}_{i=1}^n$ be a POVM on the $d$-dimensional Hilbert space $\mathcal{H}$, and $P = \{P_i = V_i | i\rangle \langle i| V_i^\dagger\}_{i=1}^n$ be a canonical Naimark extension of $E = \{E_i\}_{i=1}^n$. We use $O$ to denote the set of the maximally POVM-incoherent operations. For a state $\rho$ and $\epsilon \geq 0$, the one-shot block coherence cost under $O$ is defined as

$$C_O(\rho, E) = \min_{\Lambda \in O} \{\log_2 N' | F[\Lambda_O(\psi_{N'}), \rho \otimes |1\rangle \langle 1|] \geq 1 - \epsilon\},$$

where $F(\rho, \sigma) = (\text{Tr}[\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}])^2$ is the fidelity between two quantum states $\rho$ and $\sigma$, and $d_{\mathcal{H}}$ is the dimension of the Hilbert space $\mathcal{H}$.

For this one-shot block coherence cost under the maximally POVM-incoherent operations the following conclusion holds.

**Theorem 3.** For quantum state $\rho$ and $\epsilon > 0$, we have

$$C_{\text{max}}(\rho, E) \leq C_O(\rho, E) \leq C_{\text{max}}(\rho, E) + 1.$$  

**Proof.** Let $\Delta(\cdot) = \sum_{i,j} P_i \cdot P_j$ be the block-dephasing operator of the canonical Naimark extension $P = \{P_i = V_i | i\rangle \langle i| V_i^\dagger\}_{i=1}^n$, with $V(\rho \otimes |1\rangle \langle 1|) V^\dagger = \sum_{i,j} A_i \rho A_j^\dagger \otimes |i\rangle \langle j|$. Then

$$V(\rho \otimes |1\rangle \langle 1|) V^\dagger = \sum_{i,j} A_i \rho A_j^\dagger \otimes |i\rangle \langle j|.$$  

We first prove the left side of Eq. (82). Let $\log_2 N' = C_O(\rho, E)$, and $\sigma = \sum_{i=1}^n P_i \sigma \otimes |1\rangle \langle 1| P_i$. The definition of $C_O(\rho, E)$ means that there is a maximally POVM-incoherent operation $\Lambda_O$ such that $F[\Lambda_O(\psi_{N'}), \rho \otimes |1\rangle \langle 1|] \geq 1 - \epsilon$. Then

$$C_{\text{max}}(\rho, E) = C_{\text{max}}(\rho \otimes |1\rangle \langle 1|, P) \leq C_{\text{max}}(\Lambda_O(\psi_{N'}), P) = \min_{\delta \in S(\mathcal{H}'), \Delta(\delta) \in \mathcal{I}_B(\mathcal{H}')} D_{\text{max}}(\Lambda_O(\psi_{N'}), \Delta(\delta)) \leq \min_{\sigma \in \mathcal{I}_B(\mathcal{H}')} D_{\text{max}}(\Lambda_O(\psi_{N'}), \Delta(\delta)) \leq \min_{\sigma \in \mathcal{I}_B(\mathcal{H}')} D_{\text{max}}(\Lambda_O(\psi_{N'}), \sigma) = \log_2 N' = C_O(\rho, E).$$

Here $\mathcal{H}'$ is the Naimark space.

Next we prove the right side of Eq. (82). Suppose that the state $\rho'$ satisfies

$$C_O(\rho, E) = C_{\text{max}}(\rho \otimes |1\rangle \langle 1|, P) = C_{\text{max}}(\rho', P) = \min_{\tau \in \mathcal{I}_B(\mathcal{H}')} D_{\text{max}}(\rho' \| \tau) \leq \min_{\tau \in \mathcal{I}_B(\mathcal{H}')} \log_2 \min \{\lambda | \lambda' \leq \lambda'\tau\} = \log_2 \lambda.$$

Set $N'' = [\lambda]$, then $\rho' \leq N'' \tau$. Consider the following mapping in the $d$-dimensional Hilbert space $\mathcal{H}$,

$$\Lambda(\omega) = \frac{1}{N'' - 1} (N'' \text{Tr}[\psi_{N''} \circ \epsilon(\omega)] - 1) \epsilon^{-1}(\rho') + \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \epsilon(\omega)]) \epsilon^{-1}(\tau),$$

where $\epsilon(\cdot)$ is the (positive) one-shot block coherence cost under the maximally POVM-incoherent operation in the Hilbert space $\mathcal{H}$. The mapping also can be written as

$$\Lambda(\omega) = \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \epsilon(\omega)]) \epsilon^{-1}(\tau) + \text{Tr}[\psi_{N''} \circ \epsilon(\omega)] \epsilon^{-1}(\rho').$$

Due to the $\tau \geq \frac{1}{N''} \rho'$, then $\epsilon^{-1}(\tau) - \frac{1}{N''} \epsilon^{-1}(\rho') = \epsilon^{-1}(\tau - \frac{1}{N''} \rho') \geq 0$, so $\Lambda$ is entirely positive. Therefore, there is a positive operator $\Lambda(\omega)$ which is a block-incoherent operation in the Hilbert space $\mathcal{H}$. Then we have

$$C_{\text{max}}(\rho, E) = \log_2 N'' \leq \log_2 (1 + \lambda) \leq \log_2 \lambda + 1 = C_O(\rho, E) + 1.$$  

**V. CONCLUSION**

In the resource theory of block-coherence, we define a block-coherence measure $C_{\text{max}}(\rho, P)$ based on maximum relative entropy, and show that it is a coherence monotone and quasi-convex under the maximally block-incoherent operations. The maximally block-coherent state is introduced, and we obtain that the value of $C_{\text{max}}(|\psi_{N'}\rangle, P)$ only depends on the number $N$ of projectors in the Hilbert space. Furthermore we give the definition of the one-shot block coherence cost under the maximally block-incoherent operations and find the relationship between the coherence measure $C_{\text{max}}(\rho, P)$ and the one-shot block coherence cost. We describe the de-
terministic coherence dilution process by constructing block-incoherent operations based on the resource theory of block-coherence. We also introduce the coherence measure $C_0(\rho, P)$ based on coherent rank, and obtain the relationship with the one-shot block coherence cost. Based on the POVM coherence resource theory, we propose a POVM-based coherence measure by using the known scheme of building POVM-based coherence measures from block-coherence measures, and the one-shot block coherence cost under the maximally POVM-incoherent operations. The relationship between the POVM-based coherence measure and the one-shot block coherence cost under the maximally POVM-incoherent operations is analysed.

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[1] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England, 2000).
[2] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. 113, 140401 (2014).
[3] E. Chitambar and G. Gour, Comparison of incoherent operations and measures of coherence, Phys. Rev. A 94, 052336 (2016).
[4] J. Aberg, Quantifying superposition, arXiv: quant-ph/0612146.
[5] I. Marvian and R. W. Spekkens, How to quantify coherence: Distinguishing speakable and unspeakable notions, Phys. Rev. A 94, 052324 (2016).
[6] E. Chitambar and G. Gour, Critical examination of incoherent operations and a physically consistent resource theory of quantum coherence, Phys. Rev. Lett. 117, 030401 (2016).
[7] A. Winter and D. Yang, Operational resource theory of coherence, Phys. Rev. Lett. 116, 120404 (2016).
[8] B. Yadin, J. Ma, D. Girolami, M. Gu, and V. Vedral, Quantum processes which do not use coherence, Phys. Rev. X 6, 041028 (2016).
[9] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Intrinsic randomness as a measure of quantum coherence, Phys. Rev. A 92, 022124 (2015).
[10] C. Liu, Q. Ding, and D. M. Tong, Superadditivity of convex roof coherence measures, J. Phys. A: Math. Theor. 51, 414012 (2018).
[11] J. I. de Vicente and A. Streltsov, Genuine quantum coherence, J. Phys. A: Math. Theor. 50, 045301 (2017).
[12] X. F. Qi, T. Gao, and F. L. Yan, Measuring coherence with entanglement concurrence, J. Phys. A: Math. Theor. 50, 285301 (2017).
[13] L. M. Zhang, T. Gao, and F. L. Yan, Transformations of multilevel coherent states under coherence-preserving operations, Sci. China-Phys. Mech. Astron. 64, 260313 (2021).
[14] Q. Zhao, Y. Liu, X. Yuan, E. Chitambar, and A. Winter, One-shot coherence distillation: Towards completing the picture, IEEE Trans. Inf. Theory 65, 6441 (2019).
[15] C. L. Liu and D. L. Zhou, Deterministic coherence distillation, Phys. Rev. Lett. 123, 070402 (2019).
[16] B. Regula, K. Fang, X. Wang, and G. Adesso, One-shot coherence distillation, Phys. Rev. Lett. 121, 010401 (2018).
[17] S. Chen, X. Zhang, Y. Zhou, and Q. Zhao, One-shot coherence distillation with catalysts, Phys. Rev. A 100, 042323 (2019).
[18] Q. Zhao, Y. Liu, X. Yuan, E. Chitambar, and X. Ma, One-shot coherence dilution, Phys. Rev. Lett. 120, 070403 (2018).
[19] Y. F. Lian, Y. Luo, and Y. M. Li, Protocol of deterministic coherence distillation and dilution of pure states, Laser Phys. Lett. 17, 055201 (2020).
[20] F. Bischof, H. Kampermann, and D. Bruß, Resource theory of coherence based on positive-operator-valued measures, Phys. Rev. Lett. 123, 110402 (2019).
[21] J. W. Xu, L. H. Shao, and S. M. Fei, Coherence measures with respect to general quantum measurements, Phys. Rev. A 102, 012411 (2020).
[22] T. Theurer, N. Killoran, D. Eglolf, and M. B. Plenio, A resource theory of superposition, arXiv: quant-ph/1703.10943.
[23] F. Bischof, H. Kampermann, and D. Bruß, Quantifying coherence with respect to general quantum measurements, arXiv: quant-ph/1907.08574.
[24] T. Decker, D. Janzing, and M. Rotteler, Implementation of group-covariant positive operator valued measures by orthogonal measurements, J. Math. Phys. 46, 012104 (2005).
[25] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, On quantum Rényi entropies: A new generalization and some properties, J. Math. Phys. 54, 122203 (2013).
[26] N. Datta, Min- and max-relative entropies and a new entanglement monotone, IEEE Trans. Inf. Theory 55, 2816 (2009).
[27] N. Killoran, F. E. S. Steinhoff, and M. B. Plenio, Converting nonclassicality into entanglement, Phys. Rev. Lett. 116, 080402 (2016).
[28] G. Bowen and N. Datta, Beyond i.i.d. in quantum information theory, arXiv: quant-ph/0604013.
[29] H. Nagaoka and M. Hayashi, An information-spectrum approach to classical and quantum hypothesis testing for simple hypotheses, arXiv: quant-ph/0206185.
[30] S. Du, Z. Bai, and Y. Guo, Conditions for coherence transformations under incoherent operations, Phys. Rev. A 91, 052120 (2015).
[31] G. Torun and A. Yildiz, Deterministic transformations of coherent states under incoherent operations, Phys. Rev.
A 97, 052331 (2018).

[32] R. Bhatia, *Matrix Analysis* (Springer-Verlag, New York, 1997).

[33] H. Zhu, Z. Ma, Z. Cao, S. M. Fei, and V. Vedral, Operational one-to-one mapping between coherence and entanglement measures, Phys. Rev. A 96, 032316 (2017).