PARAFERMIONS, BRANE DISTRIBUTIONS AND FRW HIERARCHIES *

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Abstract

I review a class of exact string backgrounds, which appear in hierarchies, where the boundary of the target space of an exact sigma model is itself the target space of another exact model. From the worldsheet viewpoint this is due to the existence of (1,1) operators based on parafermions. From the target space side, it is reminiscent of the structure of maximally symmetric Friedmann–Robertson–Walker cosmological solutions, with broken homogeneity though. Cosmological evolution in this framework raises again the question of the nature of time in string theory.

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1 Parafermions and brane deformations

The investigation of string moduli space has been an important issue for a variety of reasons. Exact worldsheet theories are good starting points for further deformations, using integrable marginal operators, whenever available. Conformal models based on group manifolds $G$ and stabilized with the help of Kalb–Ramond fields (Wess–Zumino–Witten models) offer the ideal arena for investigating deformations. They possess a high degree of symmetry $(G \times G)$ realized in terms of holomorphic and antiholomorphic currents. Bilinears of the latter are marginal operators which have been discussed prolifically in the literature [1, 2, 3, 4, 5, 6, 7, 8].

Gauged WZW models, hereafter called non-Abelian conformal cosets $G/H$, are also exact conformal theories, with poor symmetry though. The subgroup $H$ acts vectorially (or axially) and spoils completely the symmetry, contrary to what happens in ordinary geometric cosets, where the action is one-sided. The $(1, 0)$ and $(0, 1)$ currents of the original WZW model do not survive but new holomorphic and antiholomorphic operators appear, with remarkable conformal and braiding properties: the parafermions [9]. For compact groups, the anomalous dimensions of these operators obey $h, \bar{h} < 1$.

It has been recently realized that the parafermions can be used as building blocks for new marginal operators, in a framework which does also enable to clarify the brane interpretation of the deformed background [10]. The starting point in this analysis is the gauged WZW model $SL(2, \mathbb{R})/U(1) \times SU(2)/U(1)$, which appears as the non-trivial part of the target space of a continuous distribution of NS5-branes on a circle, in the near-horizon limit [11]. Deforming the circle into an ellipsis, preserves the original supersymmetry and breaks half of the isometry $(SO(2) \times SO(2) \rightarrow SO(2))$. This deformation keeps the exact nature of the solution and is driven by an identifiable marginal operator which is a bilinear in the holomorphic and antiholomorphic parafermions of $SU(2)/U(1)$ dressed with a non-holomorphic conformal operator of the non-compact $SL(2, \mathbb{R})/U(1)$.

This result opened the way to a new class of integrable marginal deformations triggered by non-left-right-factorizable operators, available in sigma models without symmetries, namely gauged WZW models. The class of models based on (pseudo)orthogonal groups was studied in [12]. Although it remains still unclear whether a brane distribution exists, which could reproduce these backgrounds as near-horizon geometries, these models are of special interest because of their potential cosmological applications.

2 FRW hierarchies in string theory

In cosmology, isotropy and homogeneity of space imply the existence of a co-moving frame where
\[
\text{d}s^2 = -\text{d}t^2 + a^2(t) \gamma_{ij}(x) \text{d}x^i \text{d}x^j.
\] (2.1)
The Euclidean metric $\gamma_{ij}$ is homogeneous and maximally symmetric and describes therefore a geometric coset of a (pseudo)orthogonal group. It solves three-dimensional Einstein’s equations with pure cosmological constant $\Lambda = R^{(3)}/6$, where $R^{(3)}$ is the constant three-
dimensional curvature scalar. Three situations appear:

\[
\begin{align*}
S^3 &: \quad SO(4)/SO(3) \quad \Lambda > 0, \\
H_3 &: \quad SO(3,1)/SO(3) \quad \Lambda < 0, \\
E_3 &: \quad \text{flat space} \quad \Lambda = 0.
\end{align*}
\]

The scale factor \(a(t)\) satisfies Friedmann–Lemaître equations. In the absence of matter (pure cosmological constant) these equations are again exactly solvable and the four-dimensional space–time is also maximally symmetric with constant scalar curvature \(R^{(4)} = 4\Lambda = 2R^{(3)}/3:\)

\[
\begin{align*}
dS_4 &: \quad SO(4,1)/SO(3,1) \quad \Lambda > 0 \quad \text{with spatial sections } S^3, \\
\text{Einstein } - dS_4 &: \quad \Lambda > 0 \quad \text{with spatial sections } E_3, \\
\text{AdS}_4 &: \quad SO(3,2)/SO(3,1) \quad \Lambda < 0 \quad \text{with spatial sections } H_3.
\end{align*}
\]

In all previous expressions, \(S, H, dS\) and \(\text{AdS}\) stand for spheres, hyperbolic planes, de-Sitter and anti-de Sitter spaces.

This hierarchical structure in which four-dimensional maximally symmetric space–times are foliated with three-dimensional maximally symmetric spaces is intimately related to the underlying orthogonal-group geometric-coset structure or equivalently to the maximal symmetry.

Geometric cosets are not exact string\(^1\) backgrounds. Supplemented with antisymmetric tensors, they solve the supergravity equations – with vanishing dilaton since they have constant curvature – and appear as factors of near-horizon geometries of branes. As I mentioned in Sec. II however, gauged WZW models are exact theories, and it is remarkable that \textit{despite the absence of isometries, a similar though weaker hierarchy holds} in that case too.

Consider the following Euclidean-signature conformal cosets

\[
\begin{align*}
\text{CH}_{d,k} &= \frac{SO(d,1)_{-k}}{SO(d)_{-k}}, \quad d = 2, 3, \ldots \\
\text{CS}_{d,k} &= \frac{SO(d+1)_k}{SO(d)_k}, \quad d = 2, 3, \ldots
\end{align*}
\]  

(2.2)

(2.3)

where \(k\) and \(-k\) indicate the level of the corresponding current algebras and their signs ensure that the target space has Euclidean signature. This notation reminds of the geometric cosets but it should be kept in mind that the latter are different from the conformal ones (gauged WZW) discussed here. Explicit forms of the corresponding backgrounds have been worked out in the literature for various values of \(d\), both to lowest order in \(\alpha' \sim 1/k\)\(^{13, 14, 15, 16, 17, 18, 19}\) and to all orders \(20, 21\).

Our observation can be formulated as follows: the radial infinity of the space of the non-compact coset \(\text{CH}_{d,k+2d-4}\) is the full space of the compact coset \(\text{CS}_{d-1,k}\) times a decoupled

\(^{1}\)The three-dimensional anti de Sitter and the three-sphere are exceptions because they are also group manifolds.
scalar (the radial coordinate) with linear dilaton $R_{Q,k,d}$, where $Q_{k,d} = \frac{1 - d}{2\sqrt{k + d - 3}}$ is an appropriate background charge. As a consequence, $CS_{d-1,k}$ does not appear as a leaf of $CH_{d,k+2d-4}$ at finite radial coordinate but only when this coordinate becomes infinite. A similar property holds for Minkowskian-signature conformal cosets $CAdS_{d,k}$.

3 An example in three dimensions

The above property of conformal cosets can be proven exactly, to all orders in $1/k$ [12]. Here, I would like to illustrate how the argument goes by analyzing the three-dimensional example in the large-$k$ regime. Using global variables [19], the metric and dilaton for $CH_{3,k+2}$ read

$$ds^2_{(3)} = 2k \left( \frac{\hat{b}^2}{\hat{b}^2 - 1} + \frac{\hat{b} - 1}{\hat{b} + 1} \frac{du^2}{4\hat{b}^2} - \frac{\hat{b} + 1}{\hat{b} - 1} \frac{d\theta^2}{4\hat{b}^2} \right),$$

$$e^{-2\Phi_{(3)}} = e^{-2\Phi_{(2)}} \left( \hat{b}^2 - 1 \right) (\hat{v} - \hat{u} - 2),$$

where $|\hat{b}| > 1, 0 < \hat{v} < \hat{u} + 2 < 2$. At large $\hat{b}$ (radial coordinate), the above are better expressed in coordinates

$$\hat{b} = \exp 2x, \quad \hat{u} = -2\sin^2 \theta \cos^2 \phi, \quad \hat{v} = 2\sin^2 \theta \sin^2 \phi. \quad (3.3)$$

Keeping also the subleading term (in $1/\hat{b} = \exp -2x$) in the metric, one finds:

$$ds^2_{(3)} = 2kdx^2 + ds^2_{(2)} + 4ke^{-2x} \left[ 2\tan \theta \sin 2\phi d\theta d\phi - \cos 2\phi (d\theta^2 + \tan^2 \theta d\phi^2) \right]$$

and

$$e^{-2\Phi_{(3)}} = 2e^{4x} e^{-2\Phi_{(2)}} \quad \text{with} \quad e^{-2\Phi_{(2)}} = \cos^2 \theta. \quad (3.5)$$

The leading contributions

$$ds^2_{(\text{3-lead})} = 2kdx^2 + ds^2_{(2)}$$

$$= 2kdx^2 + 2k(d\theta^2 + \tan^2 \theta d\phi^2) \quad (3.6)$$

and $\Phi_{(2)}$ are the background fields of $R_{Q,k} \times CS_{2,k}$, as advertised previously. The radial coordinate $x$ supports a linear dilaton with background charge $Q_{3,k}$, which is found to be $-1/\sqrt{k}$ by comparing the normalization of the field $x$ with the slope of the linear dilaton (Eq. [3.5]). With this information, one easily checks that the central charges indeed match, by using the general following formulas (exact in $k$):

$$c_{CH_{3,k+2}} = \frac{6(k + 2)}{k} - \frac{3(k + 2)}{k + 1}, \quad (3.7)$$

$$c_{CS_{2,k}} = \frac{3k}{k + 1} - 1, \quad (3.8)$$

$$c_{R_{Q,k}} = 1 + 12Q^2_{3,k} = 1 + \frac{12}{k}. \quad (3.9)$$

Details can be found in [12], where as already stressed, the above arguments are shown to hold beyond the large-$k$ approximation and for all $d$. 

3
4 The role of parafermions

At large spatial infinity, the three-dimensional exact coset $CH_{3,k+2}$ is factorized in an exact two-dimensional $CS_{2,k}$ times a free scalar with background charge $\mathbb{R}_{Q_{3,k}}$. Hence, it must be possible to dynamically generate the full three-dimensional theory, i.e. beyond its asymptotic region, by using an integrable marginal perturbation driven by a $(1,1)$ operator of $\mathbb{R}_{Q_{3,k}} \times CS_{2,k}$. This operator is read off in the subleading correction of the metric (3.4):

$$\delta L = 4ke^{-2x} \left[ \tan \theta \sin 2\phi (\partial_+ \partial \phi + \partial_+ \partial_\theta - \tan^2 \theta \partial_+ \phi \partial_\phi) - \cos 2\phi (\partial_+ \partial \theta - \tan^2 \theta \partial_+ \phi \partial_\phi) \right].$$

(4.1)

The latter can indeed be reexpressed in terms of natural objects in the $CS_{2,k}$ conformal field theory. For the $CS_{2,k}$ factor the natural objects are the parafermions [13].

The semiclassical expressions for the chiral parafermions (holomorphic) in terms of space variables are (a factor involving $k$ is ignored)

$$\Psi_\pm = (\partial_+ \theta \mp i \tan \theta \partial_+ \phi) e^{\mp i(\phi + \phi_1)},$$

(4.2)

where the phase is

$$\phi_1 = -\frac{1}{2} \int^+ J^1_+ d\sigma^+ + \frac{1}{2} \int^- J^1_- d\sigma^-, \quad J^1_\pm = \tan^2 \theta \partial_\pm \phi.$$

(4.3)

The phase obeys on-shell the condition $\partial_+ \partial_\phi \phi_1 = \partial_- \partial_\phi \phi_1$ and is well defined, due to the classical equations of motion. Similarly, the expressions for the antichiral parafermions (antiholomorphic) are

$$\bar{\Psi}_\pm = (\partial_- \theta \mp i \tan \theta \partial_- \phi) e^{\pm i(\phi - \phi_1)}.$$  

(4.4)

The exact conformal weights of the parafermions are $(1 - 1/2k, 0)$ for the chiral and $(0, 1 - 1/2k)$ for the antichiral ones.

The correction (4.1) is reproduced as

$$\delta L = -2kV_{3,k} (\Psi_+ \bar{\Psi}_- + \Psi_- \bar{\Psi}_+),$$

(4.5)

with $V_{3,k} = \exp -2x$ a vertex operator of weights $(1/2k, 1/2k)$. Therefore $\delta L$ has indeed conformal weights $(1,1)$ as it should. Its exactness is inherited from the relation established between the two theories $CH_{3,k+2}$ and $CS_{2,k} \times \mathbb{R}_{Q_{3,k}}$, which, being exact, are necessarily connected by an integrable marginal perturbation. Notice that Eq. (4.5) is valid at any finite $k$ whereas (4.1)–(4.4) are semiclassical expressions, as the whole analysis of Sec. 3.

Similar considerations hold for higher dimensions and I refer to the already cited paper for further reading.

5 Comments

Ordinary FRW universes, i.e. space–times based on homogeneous spaces, are obtained by solving string equations to some order in $\alpha'$. Time evolution usually emerges through a time-dependent dilaton as well as in a warping factor of the spatial metric. This genuine
time evolution is often identified at late times, with RG evolution where the two-dimensional scale plays the role of time [22] [23] [24].

In the present set-up, exact \( d \)-dimensional backgrounds \( \mathcal{B} \) are constructed, whose \((d-1)\)-dimensional “boundaries” \( \partial \mathcal{B} \) are also exact conformal field theories. Supplemented with an extra free field with background charge, the theory on \( \partial \mathcal{B} \) admits a truly marginal deformation that allows to reconstruct the theory on \( \mathcal{B} \).

If \( \mathcal{B} \) is Minkowskian and if \( \partial \mathcal{B} \) is space-like, we are in a situation similar to that of the ordinary – inhomogeneous though – FRW universes in the following sense: the universe is generated by letting evolve in time some initial space, which is per se a solution of the equations of motion in one dimension less. This evolution is however more involved than a simple warping (see Eq. (3.1), which is similar to its Minkowskian analogue – CAdS\(_{d,k}\) or CdS\(_{d,k}\)) and can never be identified with an RG evolution since it corresponds to a marginal deformation. This latter property persists of course in the case of Euclidean \( \mathcal{B} \) making it hard to speculate about holography.

Whether the above ideas could be of any practical use in string cosmology is questionable. There is no doubt, however, that they open yet another window to the deep problem of understanding the very concept of time in string theory and its relation with the Liouville field.

The emergence of parafermions as building blocks of exactly marginal operators, when appropriately dressed, is in the heart of the present analysis. These appear either for the deformation of a brane distribution or for establishing the advertised FRW-like hierarchy in string theory. Our proof that these operators are indeed integrable is indirect and relies on the independent observation that they generate a line of continuous deformation with vanishing beta-functions to all orders. A proof based on genuine conformal-field-theory techniques would require mastering of (non-)Abelian quantum parafermions, which is a notoriously difficult subject.

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