A Method of Descriptions for Graph Labelings

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Abstract. Labeling the graph is the most essential method in graph theory, we develop a rule for some graph labeling instead of labeling functions of a graph elements which arrives the same result as in the graph labeling. Labeling is an allotment of digits to the vertex (dots) or edge (lines) or the two under certain conditions, most popular labeling method trace their origin called functions (labeling). Labeled graphs provide mathematical models called mathematical functions \( f : G \rightarrow X \). The graph labeling of a graph variables are used in various area such as resolutions in the subfield of matrix theory and cryptography to mention a few. Here we develop a rule graph labeling and it is very useful in the field of coding theory.

1. Introduction

The awesome mind learns, receives and observes. As a result, the journey in the pursuit of knowledge turns out to fruitful. A gear in a vehicle man made have provided coding techniques to the research. A countless directions are available to the mind which moves in the pursuit of knowledge. Any concept selected for the search comforts ones mind when one is able to find a result of utility, however insignificant it may be. Graph Theory is too generous and prolific offering innumerable concepts with applications galore. Graph theory is one of the mathematic which growing rapidly and can be used to simplify the
solution of a problem in day today life. Graph theory can be used to modeling a problem that can be easier to see and find the solution for the problem. one of the Graph subject is graph labeling topic. For more results on graph labeling can be found in [1],[2]. Let $G = (V(G), E(G))$ be a finite simple connected graph with vertex set $V(G)$ and edge set $E(G)$ where $e = uv$ if and only if edge $e$ connects vertex $u$ to vertex $v$.

Graph indexing method of the graph is an allotment of digits with dots or lines or both under some conditions and it is introduced in the mid of 1960's. Most graph index methods trace their origin to one introduced by Rosa [7] in 1967, of one given by Graham and Sloane in 1980. Giving numbers to the dots and lines gives important role in the field of coding theory and cryptography etc. The interested reader can consult this survey for more informations about the subject by [2]. Jayenthi and Ramya were proved $S(p_{n} \circ K_{1})$, $S(p_{2} \times p_{4})$, $S(B_{n,n})$, $(B_{n,n} : p_{m})$, $C_{n} \circ K_{2n} \geq 3$ generalized antiprism $A^{m}_{n}$ and the double triangular snake $D(T_{n})$ are super mean graph [3],[4][8]. Vasuki et al.,[15],[16],[17] have studied the super meanness property of the subdivision of the $H$-graph $H_{n}$, $H_{n} \circ K_{1}$, $H_{n} \circ S_{2}$, slanting ladder, $T_{n} \circ K_{1}$, $C_{n} \circ K_{1}$ and $C_{n} \circ C_{m}$. Motivated by these work we applied super mean labeling on star graphs and provided a rule for super mean labeling on two and three star graphs, the discription is very direct than the graph labeling method. By referring few definitions and graph labelings, we provide brief summary of graph labelings and it is very useful for present investigators have since been applied in areas such as coding Theory, radar, radio astronomy, and circuit design, fascinated by a variety of graph labelings we develop rule for graph labelings.

2. Difference Cordial Labeling

Ponraj et al., [6] were introduced a new notion called Difference Cordial Labeling. The wheel Web graph with $n$ vertices($n > four$) is the structure getting by introducing a new dots and lines connecting to every dots of cycle with $n$ vertices. That dot is said to be the apex dot and the dots connecting to cycle with $n$ vertices is said to be rim dots of web graph with $n$ vertices. The lines connecting rim dots is known as rim lines, the next lines is said to be the lines of the graph. The Helm $H_{n}$ is the graph obtained from a wheel $W_{n}$ by attaching a pendant edge to each of the rim vertices. Koh et al., [2] define a web graph is obtained by joining the pendent points of the helm to form a cycle and then adding a single pendent edge to each vertex of the outer cycle. Yang [2] has extended the notion of a web by iterating the process of adding pendent vertices and joining them to form a cycle and then adding pendent point to the new cycle. $W(t, n)$ is the generalised web with $t$ cycles $C_{n}$. Note that $W(1, n)$ is the helm and $W(2, n)$ is the web.
2.1. A Description for DCL

(i) Rule for DCL of Web Fibonacci series \(FBW(s, m)\) are listed. These are the steps should follow the label Web Fibonacci series \(FBW(s, m)\) by DCL.

(ii) if \(m = 1, 3, 5, \ldots\):

Start with any line of web of dots and assign the digit one to its single dot. That line of web is the first line of web with the dot written as \(SD_1\). Remaining lines of web are denoted as \(SD_2, SD_3\) and continuing this process, start with the line of web in moving to the right direction. The pendant dot numbered one get into and give the digit to the neighborhood dot as two. Circulate along the outer circle to the next neighborhood dot assign the number three and go out to the single dot, assign the number four. The single dot in the next line of web (jump from single dot to single dot) is given the number 5. Get into and number the adjacent dot and go along the outer circle. Continuing the same process to get the dot \(K_1\).

if \(m = 2, 4, 6, \ldots\):

Starting with first out circle, give the digit to any dot as one and this is the first string of dots \(SD_1\). Circulating along the clockwise direction the next strings are denoted by \(SD_2, SD_3\) etc. The single dot \(SD_1\) which is assigned by going out and it is taken as digit two. The single dot in the next line of web \(SD_2\) has taken the digit three then going into and circulating along the outer circle and continuing this procedure so that all the dots are labeled.

3. Labeling for Product Cordial:

The PCL the roles of dots and lines are interchanged. For a graph contains no far away dot from G, and define a \(\lambda : L(G) \rightarrow (0, 1)\) is said to be an line (edge) PCL of G, for the induced dot numbering function defined by the multiple of numbers of incident lines to every dot is the way that all the lines with number 0 and all the lines with label one which is differed by almost one and all the dots with label 0 and all the dots with label one also differed by almost one.

Generating of a dot \(d\) of a graph \(G\) creates a new graph \(G'\) by inserting a new dot \(d'\) such that \(N(d') = N(d)\). Otherwise, \(d'\) is said to be a generation of \(d\) if all the dots which are join with \(d\) in \(G\) are also joining with \(d'\) in \(G'\). Otherwise, \(d''\) is said to be a double generation of \(d\) if all the dots which is joined with \(d\) in \(G\) are also joining with \(d''\) in \(G''\) such that \(N(d) = N(d') = N(d'')\). The graph got from generation of each of the dots \(w_i\) for \(i=1,2,3,\ldots,n\) by a new lines \(f_i\) in \(SF_n\) is an line (edge) product cordial graph, one such
theorem made by udayan et al.\cite{13} also they proved two different cases if \( n = 1, 3, 5, \ldots \) and if \( n = 2, 4, 6, \ldots \). The flower graph \( Fl_n \) is the graph obtained from a helm \( H_n \) by joining each pendant vertex to the apex of the helm. Single duplication of a flower graph becomes sunflower graph \( SF_n \). But here we introduce double duplication of a flower graph and it is denoted by \( SF_nD_2 \), since the sunflower having 8 petals with double duplication \( D_2 \). The rule for edge product cordial labeling is given instead of graph labelings and it is very easy to label the graph for non-mathematicians also in the field of Coding theory, Cryptography and to mention a few. The Sunflower graph \( SF_nD_2 \) is Sunflower \( SF_n \) doubly duplicating. By duplicating \( w_i \) once the dots \( u_i \) are obtained and generating \( u_i \) again dots \( y_i \) are obtained. So, the dots of \( SF_nD_2 \) are \( v_0, v_1, v_2, \cdots, v_8, w_1, w_2, \cdots, w_8, u_1, u_2, \cdots, u_8, y_1, y_2, \cdots, y_8 \).

3.1. Product Cordial Labeling (PCL)

**Definition 3.1.** A binary dot labeling \( \lambda : L(G) \rightarrow \{0, 1\} \) of a graph \( G \) with induced line numbering \( \lambda^* : L(G) \rightarrow \{\text{zero, one}\} \) given by \( \lambda^*(xy) = \lambda(x)\lambda(y) \) is called PCL if
\[
|d_\lambda(0) - d_\lambda(1)| \leq 1 \quad \text{and} \quad |l_\lambda(0) - l_\lambda(1)| \leq 1,
\]
where \( d_\lambda(0), d_\lambda(1) \) denote the number of dots of \( G \) having labels 0 and 1 under \( \lambda \) and \( l_\lambda(0), l_\lambda(1) \) denote the number of lines of \( G \) having labels zero, one respectively under \( \lambda \). Thus \( G \) is PCG if it permits PCL.

3.2. Labeling and Verification on a Doubly generated Sunflower Graph \( SF_nD_2 \)

**Description of \( SF_nD_2 \):**
The sunflower graph \( SF_8 \) with 8 petals is doubly duplicated in order to get 3 whorls in every petal. With a pair of dots \( v_i, v_{i+1} (i = 1, 2, \cdots, 8) \) and with \( v_1, v_8 \), 8 petals with 3 whorls are available \( w_i, u_i \) and \( y_i \) are the top vertices of the inner, middle and outer whorls. Each whorl contains two lines one on the left and the other on the right. There are 6 such lines for every petal and so 48 such edges totally, half of them \( LWE \) (Left whorl edges) and the other half of them \( RWE \) (Right whorl lines). There are 8 rim lines and 8 more lines called the spoke lines adding the apex dot to all the 8 rim dots \( v_i \). \( SF_nD_2 \) is an line (edge) product cordial graph for \( m = 2, 4, 6, \ldots \) only.

**Verification:**

(i) The spoke e 8, the rim lines 8 and the LWE and RWE between \( v_5 \) to \( v_8 \) and to \( v_1 \) of the inner and middle petals 16 are assigned the number 0 (32 edges).
(ii) The 24 LWE and RWE of all the three whorls (Inner, middle and outer) from $v_1$ to $v_4$ and the LWE and RWE of the outer whorls between $v_5$ to $v_8$ and $v_8$ to $v_1$ are assigned the number 1 (32 edges).

From (1) and (2), $|e_f(0) - e_f(1)| = 0 \leq 1$ is satisfied.

(iii) The rim vertices $v_i$ is 8, the apex $v_0$ is 1 and $w_i$ and $u_i$ ($i = 5$ to $8$) is 8 are assigned the numbers 0, a total of 17.

(iv) The vertices $u_i$ and $w_i$ ($i = 1, 2, 3, 4$) is 8 and the vertices $y_i$ ($i = 1$ to $8$) is 8 are assigned the number 1, a total of 16.

From (4) and (5) $|v_f(0) - v_f(1)| \leq 1$.

Therefore (3) and (6) shows that the labeling done on $SF_8D_2$ is the edge product cordial labeling.

Vaidya and Barasara [14] were proved the theorem gear graph $G_n$ is an edge product cordial graph for $n$ odd and even. Motivated by the theorem we develop rule for vertex product cordial labeling for the Gear graph. If the domain of the mapping is the set of dots then the labeling is called a dot product cordial labeling. Let $l = xy$ be an line of G and $z$ is not a dot of G. The line $l$ is subdivided when it is replaced by the lines $l' = xz$ and $l'' = zy$. The gear graph $G_n$ is getting from the cycle with the spoke lines $W_n$ by subdividing each of its rim line. we apply dot PCL for gear graph.

3.3. A Description and Verification on a Gear Graph $G_9$

**Description of $G_n$ if $n$ is odd:**
A gear graph $G_n$ when $n$ odd is an vertex product cordial graph. Here $G_9$, the gear graph with 9 rim vertices $(v_i)$ and 9 vertices due to the subdivision $(u_i)$ is taken.

**Verification:**
(i) The vertices \( v_1 \) to \( v_5 \) and \( u_1 \) to \( u_4 \) and the apex vertex are allotted number 1. A total of 10 vertices take the value 1.

(ii) The rim vertices \( v_6 \) to \( v_9 \) and \( u_5 \) to \( u_9 \) are allotted number 0. A total of 9 vertices take the value 0.

From (7) and (8), \( |v_f(0) - v_f(1)| = 1 \leq 1 \). The definition of Vertex product cordial labeling is satisfied here.

(iii) The rim edges \( v_i u_i \) and \( u_i v_{i+1} \) \((i = 1 \text{ to } 4)\), and the spoke edges \( v_0 v_i \) \((i = 1 \text{ to } 5)\) are assigned the number 1. A total of 13 edges take the value 1.

(iv) The rim edges \( v_i u_i \) \((i = 5 \text{ to } 9)\) and \( u_i v_{i+1} \) \((i = 5 \text{ to } 8)\), the edge \( u_9 v_1 \) and the spoke edges \( v_0 v_i \) \((i = 6 \text{ to } 9)\), are assigned the value 0. A total of 14 edges take the value 0.

From (10) and (11), \( |e_f(0) - e_f(1)| = 0 \leq 1 \) is satisfied here.

From (9) and (12), the label is found to be (vertex) product cordial labeling.

4. Labeling for Super Mean:
A graph is called a mean graph if there is an injective function from the vertices of G to such that when each edge is labeled with if is even and if it is odd, the resultant line numbers are distinct. Further more, the concept of super mean labeling was introduced by Ponraj and Ramya [5][12]. Let be an injection on. It is possible to label all the dots and lines with SML for every values of \( p, q \) and \( r \) for two star graph \( K_{1,q} \cup K_{1,r} \) and 3 star graph \( K_{1,t} \cup K_{1,q} \cup K_{1,r} \) not avoiding any digits from 1 to \( a + b \). Rule for assigning the SML on 2 and 3 star graphs is described below.
4.1. Super mean labeling (SML):

Definition 4.1. The SML of a graph is a graph $G$ be a $(a, b)$, dots and lines and $\lambda : V(G) \rightarrow \{1, 2, 3, \ldots, a+b\}$ be a $1-1$ and on to mapping. All the line $l=xy$, let $\lambda^*(l)=\frac{\lambda(x)+\lambda(y)}{2}$ if $\lambda(x) + \lambda(y)$ is even and $\lambda^*(l)=\frac{\lambda(x)+\lambda(y)+1}{2}$ if $\lambda(x) + \lambda(y)$ is odd. Thus $\lambda$ is called SML if $\lambda(A) \cup \{\lambda^*(l) : l \in L(G)\} = \{1, 2, 3, \ldots, a+b\}$. A graph which permits SML is said to be a SMG.

4.2. Descriptions for SML on Two Star:

(i) For giving names to the graph using SML of $K_{1, q} \cup K_{1, r}$, $q \leq r$ are shown. Let $a$ and $b$ be define the total of dots also lines, $a = 2 + q + r$, $b = q + r$, $a + b = 2 + 2q + 2r$. The numbers from 1 to $2 + 2q + 2r$ must be given to the non pendant dots and the single dots, line numbers gets credit. A number present twice is not allowed. Here $\lambda(x)$, $\lambda(y)$, $\lambda(x_i)$ and $\lambda(y_i)$ are the digits given to the top dots and the pendant dots of the 1st and the 2nd star. Method for allotting the line values is $\frac{\lambda(x) + \lambda(x_i)}{2}$ or $\frac{\lambda(x) + \lambda(x_i) + 1}{2}$ where the line join $x$ and $x_i$. See the line number can take real mean or adjusting real mean.

The mean of the biggest and the before number is $\frac{(2 + 2q + 2r) + (2 + 2q + 2r - 1)}{2} = 2 + 2q + 2r$.

This is not possible since repeating labels are not taken.

So, neither the line value nor the single dots can exceed $2 + 2q + 2r$. Therefore it is possible to label a two star $K_{1, p} \cup K_{1, r}$ with super mean labeling for any number without deleting any digits from 1 to $(a + b)$.

(ii) If $(\lambda(x) \lambda(x_i))$ or $(\lambda(y) \lambda(y_i))$ if the two terms are odd terms or two terms are even terms the the line value is the real mean, if it is not so the line value is the adjustable mean.

(iii) Similarly $\lambda(x)$ is odd, as $\lambda(x_i) = 2s$, $\lambda(x_{i+1}) = 2s + 1$, it gives the same line value and hence to be deleted, implies $\frac{1+6}{2}$ and $\frac{1+7}{2}$ must be the same line value 4 when $\lambda(x) = 1$.

If $\lambda(x_i) = 2s + 1 \lambda(x_{i+1}) = 2s + 2$, implies different line values and hence it is allotted.

Implies, $\frac{1+7}{2} = four$ and $\frac{1+8}{2} = five$ give different line values.

When $\lambda(x)$ is even then reverse the process.

while giving label to the pendant dots the following things to be noted.

(i) Take one and $a + b$ as $\lambda(x)$ and $\lambda(y)$, $\lambda(x_i) \neq two$, for the line value is two, but
\( \lambda(y_i) = \text{two} \) is allowed.

(ii) \( \lambda(x_1) = \text{three} \), then \( \lambda(y_1) = \text{four} \) and if \( \lambda(x_1) = \text{five} \), then \( \lambda(y_1) = \text{two} \).

Giving a number to \( x_1 \) and give the next least digit to \( y_1 \) of the 2\textsuperscript{nd} star, the \( x_2 \) and \( y_2 \) are named in the same process. Twice or once it may have to continue with giving to \( x_i \)'s successively, for removing the repeating numbers. This process makes giving label to a two star graph using SML. An example for 2 star with SML is given below.

4.3. Description for SML on Three Star Graph

(i) Descriptions on SML of \( K_{1,p} \cup K_{1,q} \cup K_{1,r} \), \( p \leq q \leq r \) are presented. Here \( a \) and \( b \) denote the number of dots and lines, \( a = 3 + p + q + r \), \( b = p + q + r \), \( a + b = 3 + 2p + 2q + 2r \).

The digits from 1 to \( 3 + 2p + 2q + 2r \) be allotted for the non pendant dots then single dots, then line digits are numbered. Digits repetition is not allowed. Here \( \lambda(x) \), \( \lambda(y) \), \( \lambda(z) \), \( \lambda(x_1) \), \( \lambda(y_1) \) and \( \lambda(z_1) \) are the digits given to the top dots and the pendant dots and \( \lambda^*(xx) \), \( \lambda^*(yy) \) and \( \lambda^*(zz) \) are the digits given to the lines for all the three components of \( G \).

(i) Take 1, \( a + b + \text{one} \) and \( a + b \) as \( \lambda(x) \), \( \lambda(y) \) and \( \lambda(z) \) respectively \( \lambda(x_1) \neq 2 \), corresponding line value is 2 when \( \lambda(x_1) = \text{two} \), but \( \lambda(y_1) = \text{two} \) is permitted as \( \lambda(y) \neq 1 \).

(ii) If \( \lambda(x_1) = \text{three} \), then \( \lambda(y_1) = \text{four} \) and if \( \lambda(x_1) = \text{five} \), then \( \lambda(y_1) = \text{two} \).

Giving digits to the 1\textsuperscript{st} component of \( G \) is finish, then \( 2\textsuperscript{nd} \) and \( 3\textsuperscript{rd} \) objects are label one by one. An illustration for SML is given below:

Give the allotment for the smallest digit avoided in 1\textsuperscript{st} component for the first single dot of the 2\textsuperscript{nd} component. Suddenly, give the next least digit to the 1\textsuperscript{st} pendant dot of the 3\textsuperscript{rd} component. Implies \( \lambda(y_1) \) and \( \lambda(z_1) \) should given one by one. same procedure is followed for giving integers to the single dots of the remaining components.

5. Labeling for Sub Super Mean:

One such theorem were proved, that the star graph \( K_{1,n} \), \( n > 3 \) is not a super mean labeling. Motivated by these work sub super mean labeling was found by [9], [10].

The word ‘sub’ is used as \( \lambda(A) \cup \{\lambda^*(l) : l \in L(G)\} \subset \{1, 2, 3, \cdots, a + b\} \) thereby allowing omissions which lead to repetitions and a special a case of SSML(\( V_4, E_2 \)). A specific SSML is defined on two star and three star graphs where the digits given to the pendant dots differ by four then corresponding line values differ by two, almost everywhere and so it is
named as SSML($V_4, E_2$) aptly. If the number of repetitions = number of omissions = $t$ or $(t - 1)$ with respect to SSML($V_4, E_2$), then the two star graph is called $t$ RO or $(t - 1)$ RO sub super mean graph where $n = m + 2t$ or $n = m + (2t + 1)$ with $|m - n| > 3$, for two star graph.

If the number of repetitions = number of omissions = $t$ or $(t - 1)$ with respect to SSML($V_4, E_2$), then the two star graph is called $t$ RO or $(t - 1)$ RO sub super mean graph where $n = m + 2t$ or $n = m + (2t + 1)$ with $|m - n| > 3$, for two star graph.

5.1. Sub Super mean labeling (SSML):

**Definition 5.1.** The super mean graph is a graph $G$ with $(a, b)$ vertices and edges and $\lambda : D(G) \rightarrow \{1, 2, 3, \ldots, a+b\}$ be a one to one and on to mapping. All the line $l = xy$, such that,

$$\lambda^*(l) = \begin{cases} 
\frac{\lambda(x) + f(y)}{2} & \text{if } \lambda(x) + \lambda(y) \text{ is even;} \\
\frac{\lambda(x) + \lambda(y) + 1}{2} & \text{if } \lambda(x) + \lambda(y) \text{ is odd}
\end{cases}$$

then $\lambda$ is called SSML if $\lambda(A) \cup \{\lambda^*(l) : l \in L(G)\} \subset \{1, 2, 3, \ldots, a+b\}$. A graph that admits a SSML is called a Sub Super Mean Graph.

While assigning numbers to the pendant dots using SSML($V_4, E_2$) an even integer gets omitted moving from $K_1, q$ to $K_1, r$. This even integer is the first even integer omitted and is denoted by FEIO. while assigning numbers to the pendant dots using SSML($V_4, E_2$), an even integer gets omitted in moving from $K_1, q$ to $K_1, r$. This even integer is the second even integer omitted and is denoted by $SEIO = 2p + 2q + 4$. while assigning numbers to the pendant dots using SSML($V_4, E_2$), an odd integer gets omitted in moving from $K_1, q$ to $K_1, r$. This odd integer is the first odd integer omitted and is denoted by $FOIO = 4p + 7$.

Theorems were proved [9] the two star $G = K_1, q \cup K_1, r$ on which the sub super mean labeling SSML($V_4, E_2$) is defined turns out to be a 1 RO sub super mean graph if $|q - r| \leq 3$. The rule for SSML for two star if $|q - r| > 3$ and three star for all the values is given below:

The discussion of SSML ($V_4, E_2$) for $G = K_1, q \cup K_1, r$ every values of $q$ and $r$, for $q \leq r$ is provided below, which is required for the following theorem.

Let $G = K_1, q \cup K_1, r$

$a = 1 + q + 1 + r = 2 + q + r$

$b = q + r$

$a + b = 2q + 2r + 2$
In first copy of \( G = K_{1,q} \cup K_{1,r} \) the labeling is done by:
\[
\begin{align*}
\lambda(x) &= 1 \\
\lambda(x_1) &= 3 \\
\lambda(x_i) &= 3 + 4(i-1), \quad 2 \leq q \\
\lambda(x_m) &= 3 + 4(q-1) = 4q - 1 \\
\lambda^*(xx_1) &= 2 \\
\lambda^*(xx_i) &= 2 + 2(i-1), \quad 2 \leq q \\
\lambda^*(xx_q) &= 2q.
\end{align*}
\]

For figure is that referred by \[9\.

In the second copy of \( G = K_{1,q} \cup K_{1,r} \), the labeling is done as follows:
Define \( \lambda(y) = \lambda(x_p) + 4 = 4q + 3 \)
\[
\begin{align*}
\lambda(y_1) &= 5 \\
\lambda(y_j) &= 5 + 4(j-1), \quad 2 \leq j \leq J, \\
\text{where } J &= \left[ \frac{2q + 2r - 3}{4} \right] \\
\lambda(y_{J+1}) &= 2q + 2r + 2 \\
\lambda(y_{J+2}), \lambda(y_{J+3}) \text{ etc are assigned the remaining odd integers.}
\end{align*}
\]

For figure is that referred by \[9\.

The discussion of SSML\((V_4, E_2)\) on \( G = K_{1,p} \cup K_{1,q} \cup K_{1,r} \), \( p \leq q \leq r \) any number of \( p,q \) and \( r \) is provided below, as it is required for the theorem proved. Let \( G = K_{1,p} \cup K_{1,q} \cup K_{1,r} \), \( a = 3 + p + q + r \), \( b = p + q + r \), \( a + b = 3 + 2p + 2q + 2r \).

First copy of \( G = K_{1,p} \cup K_{1,q} \cup K_{1,r} \) the labeling is done:

Top dot value is 1 in first star,
the pendant dots= three \(+4(i-1), 1 \leq i \leq p\).

The corresponding line labeling is given by \( \lambda^*(xx_i) = (2i), \quad on \leq i \leq \ell \). For figure is that referred by \[10, 11\.

In the second copy of \( G = K_{1,p} \cup K_{1,q} \cup K_{1,r} \) the labeling is done by:
Define: \( \lambda(y) = \lambda(x_\ell) + 4 = (4p - 1) + 4 = 4m + 3 \);
\[
\begin{align*}
\lambda(y_j) &= 5 + 4(j-1), \quad 1 \leq j \leq q \\
\text{Then the line label is given by} \\
\lambda^*(yy_j) &= \frac{(4p + 3) + 5 + 4(j-1)}{2} = 2p + 2j + 2, \quad 1 \leq j \leq q; \\
FEIO &= 2\ell + 2.
\end{align*}
\]

The pendant dots of the first and second copies differ by two correspondingly. As the \( p^{th} \) pendant dot of the first copy = \( 4p - 1 \), the \( p^{th} \) pendant dot of the second copy is \( 4p + 1 \), the \( p^{th}, (p + 1)^{st}, (p + 2)^{nd}, \ldots \) pendant dots of the second copy are \( 4p + 1, 4\ell + 5, 4\ell + 9, \ldots \),
up to the $\ell$th place, no odd integers are omitted and $4 \ell + 3$ is allotted to $\lambda(y)$.

So, FOIO is $4p + 7$, this should be the first pendant dot of the third copy of $G = K_1, p \cup K_1, q \cup K_1, r$. For figure is that referred by [10][11]

In the third copy of $G = K_1, \ell \cup K_1, m \cup K_1, n$ the labeling as follows:

Define $\lambda(z) = \lambda(y_m) + 4 = (4q + 1) + 4 = 4q + 5$ ;

$\lambda(z_k) = (4p + 7) + 4(k - 1), \quad 1 \leq k \leq K.$

To find $K$:

As $a + b = 2p + 2q + 2r + 3$, $(4p + 7) + 4(k - 1) \leq 2p + 2q + 2r + 3$, $4k \leq 2q + 2r - 2\ell$,

$k \leq \left\lfloor \frac{r + q - p}{2} \right\rfloor$, $K = \left\lfloor \frac{r + q - p}{2} \right\rfloor$, $\lambda(z_{K+1}) = 2p + 2q + 2 + 3 + 4\ell + 9$, $\lambda(z_{K+2})$ to $\lambda(z_n)$ are allotted all the remaining odd integers, FEIO and SEIO if required, the corresponding line labeling is given by

$$\lambda^*(zz_k) = \frac{(4q + 5) + (4p + 7) + 4(k - 1)}{2}$$

$$= 2p + 2q + 2r + 4, \quad 1 \leq k \leq K.$$  

The edge values $\lambda^*(zz_{K+1})$ to $\lambda^*(zz_n)$ assume only odd integral values whether the end dots are odd or FEIO or SEIO. For figures is that referred by [10][11].

6. Conclusion and Suggestion for Future Research:

The researcher have been developed a rule for few graph labeling in a simple method of descriptions, but in general, graph labeling are defined by equations. This method can easily understood by mathematicians also non-mathematicians and it is very useful in the field of coding theory, cryptography to mention a few to share the secret messages using graph labeling. Here we discussed, a rule for graph labeling for few graphs and we can able to develop the rule for other graph families also.

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