On the restricted partition function via determinants with Bernoulli polynomials. II
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Abstract
Let \( r \geq 1 \) be an integer, \( \mathbf{a} = (a_1, \ldots, a_r) \) a vector of positive integers and let \( D \geq 1 \) be a common multiple of \( a_1, \ldots, a_r \). We prove that, if \( D = 1 \) or \( D \) is a prime number then the restricted partition function \( p_\mathbf{a}(n) := \) the number of integer solutions \( (x_1, \ldots, x_r) \) to \( \sum_{j=1}^{r} a_j x_j = n \) with \( x_1 \geq 0, \ldots, x_r \geq 0 \) can be computed by solving a system of linear equations with coefficients which are values of Bernoulli polynomials and Bernoulli Barnes numbers.

Keywords: restricted partition function, Bernoulli polynomial, Bernoulli Barnes numbers.

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1 Introduction

Let \( \mathbf{a} := (a_1, a_2, \ldots, a_r) \) be a sequence of positive integers, \( r \geq 1 \). The restricted partition function associated to \( \mathbf{a} \) is \( p_\mathbf{a} : \mathbb{N} \to \mathbb{N}, p_\mathbf{a}(n) := \) the number of integer solutions \( (x_1, \ldots, x_r) \) of \( \sum_{i=1}^{r} a_i x_i = n \) with \( x_i \geq 0 \). Let \( D \) be a common multiple of \( a_1, \ldots, a_r \). According to [5], \( p_\mathbf{a}(n) \) is a quasi-polynomial of degree \( r - 1 \), with the period \( D \), i.e.

\[
p_\mathbf{a}(n) = d_{\mathbf{a}_r-1}(n)n^{r-1} + \cdots + d_{\mathbf{a}_1}(n)n + d_{\mathbf{a}_0}(n), \quad (\forall)n \geq 0,
\]

(1.1)

where \( d_{\mathbf{a}_m}(n+D) = d_{\mathbf{a}_m}(n), (\forall)0 \leq m \leq r-1, n \geq 0 \), and \( d_{\mathbf{a}_{r-1}}(n) \) is not identically zero.

The restricted partition function \( p_\mathbf{a}(n) \) was studied extensively in the literature, starting with the works of Sylvester [14] and Bell [5]. Popoviciu [11] gave a precise formula for \( r = 2 \). Recently, Bayad and Beck [4, Theorem 3.1] proved an explicit expression of \( p_\mathbf{a}(n) \) in terms of Bernoulli-Barnes polynomials and the Fourier Dedekind sums, in the case that \( a_1, \ldots, a_r \) are are pairwise coprime. In [7] we proved that the computation of \( p_\mathbf{a}(n) \) can be reduced to solving the linear congruence \( a_1 j_1 + \cdots + a_r j_r \equiv n \pmod{D} \) in the range \( 0 \leq j_1 \leq \frac{D}{a_1}, \ldots, 0 \leq j_r \leq \frac{D}{a_r} \). In [9] we proved that if a determinant \( \Delta_{r,D} \), which depends only on \( r \) and \( D \), with entries consisting in values of Bernoulli polynomials is nonzero, then \( p_\mathbf{a}(n) \) can be computed in terms of values of Bernoulli polynomials and Bernoulli Barnes numbers. The aim of our paper is to tackle the same problem, from another perspective which relies on the arithmetic properties of Bernoulli polynomials.

First we recall some definitions. The Barnes zeta function associated to \( \mathbf{a} \) and \( w > 0 \) is

\[
\zeta_\mathbf{a}(s,w) := \sum_{n=0}^{\infty} \frac{p_\mathbf{a}(n)}{(n+w)^s}, \quad \text{Re} \, s > r;
\]

see [3] and [13] for further details. It is well known that \( \zeta_\mathbf{a}(s,w) \) is meromorphic on \( \mathbb{C} \) with poles at most in the set \( \{1, \ldots, r\} \). We consider the function

\[
\zeta_\mathbf{a}(s) := \lim_{w \to 0} (\zeta_\mathbf{a}(s,w) - w^{-s}).
\]

(1.2)
In [7, Lemma 2.6] we proved that

\[ \zeta_a(s) = \frac{1}{D^s} \sum_{m=0}^{r-1} \sum_{v=1}^{D} d_{a,m}(v) D^m \zeta(s - m, \frac{v}{D}), \]  

where

\[ \zeta(s, w) := \sum_{n=0}^{\infty} \frac{1}{(n + w)^s}, \text{ Re } s > 1, \]

is the Hurwitz zeta function. See also [8]. The Bernoulli numbers \( B_j \) are defined by

\[ \frac{e^z}{e^z - 1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!}, \]

\( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30} \) and \( B_n = 0 \) if \( n \) is odd and greater than 1. The Bernoulli polynomials are defined by

\[ \frac{z e^{xz}}{(e^z - 1)} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}. \]

They are related with the Bernoulli numbers by

\[ B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} x^k. \]  

(1.4)

It is well known, see for instance [2, Theorem 12.13], that

\[ \zeta(-n, w) = \frac{B_{n+1}(w)}{n+1}, \forall n \in \mathbb{N}, w > 0. \]  

(1.5)

The Bernoulli-Barnes polynomials are defined by

\[ \frac{z^r e^{xz}}{(e^{a_1 z} - 1) \cdots (e^{a_r z} - 1)} = \sum_{j=0}^{\infty} B_j(x; a) \frac{z^j}{j!}. \]

The Bernoulli-Barnes numbers are defined by

\[ B_j(a) := B_j(0; a) = \sum_{i_1 + \cdots + i_r = j} \binom{j}{i_1, \ldots, i_r} B_{i_1} \cdots B_{i_r} a_1^{i_1-1} \cdots a_r^{i_r-1}. \]

According to [12, Formula (3.10)], it holds that

\[ \zeta_a(-n, w) = \frac{(-1)^r n!}{(n+r)!} B_{r+n}(w; a), \forall n \in \mathbb{N}. \]  

(1.6)
From (1.2) and (1.6) it follows that

$$\zeta_\alpha(-n) = \frac{(-1)^r n!}{(n+r)!} B_{r+n}(\alpha), \ (\forall)n \geq 1.$$  
(1.7)

From (1.3), (1.5) and (1.7) it follows that

$$\sum_{m=0}^{r-1} \sum_{v=1}^{D} d_{a,m}(v) D^{n+m} B_{n+m+1}(\frac{v}{D}) = \frac{(-1)^{r-1} n!}{(n+r)!} B_{r+n}(\alpha), \ (\forall)n \geq 1,$$
(1.8)

Let \(\alpha_1 < \alpha_2 < \cdots < \alpha_r \) be a sequence of integers with \(\alpha_1 \geq 2\). Substituting \(n\) with \(\alpha_j - 1, 1 \leq j \leq rD\), in (1.8) and multiplying with \(D\), we obtain the system of linear equations

$$\sum_{m=0}^{r-1} \sum_{v=1}^{D} d_{a,m}(v) D^{\alpha_j+m} B_{\alpha_j+m}(\frac{v}{D}) = \frac{(-1)^{r-1}(\alpha_j-1)!D}{(\alpha_j+r-1)!} B_{\alpha_j+r-1}(\alpha), \ 1 \leq j \leq rD,$$
(1.9)

which has the determinant

$$\Delta_{r,D}(\alpha) := \begin{vmatrix} D^0 B_{\alpha_1}(\frac{1}{D}) & \cdots & D^0 B_{\alpha_1}(1) & \cdots & D^0 B_{\alpha_1+r-1}(\frac{1}{D}) & \cdots & D^0 B_{\alpha_1+r-1}(1) \\ D^0 B_{\alpha_2}(\frac{1}{D}) & \cdots & D^0 B_{\alpha_2}(1) & \cdots & D^0 B_{\alpha_2+r-1}(\frac{1}{D}) & \cdots & D^0 B_{\alpha_2+r-1}(1) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ D^0 B_{\alpha_r}(\frac{1}{D}) & \cdots & D^0 B_{\alpha_r}(1) & \cdots & D^0 B_{\alpha_r+r-1}(\frac{1}{D}) & \cdots & D^0 B_{\alpha_r+r-1}(1) \\ \end{vmatrix}.$$  
(1.10)

Note that, with the notation given in (2.10), we have that \(\Delta_{r,D} = \Delta_{r,D}(0,1, \ldots, rD-1)\), here omitting the condition \(\alpha_1 \geq 2\).

**Proposition 1.1.** With the above notations, if \(\Delta_{r,D}(\alpha) \neq 0\), then

$$d_{a,m}(v) = \frac{\Delta_{r,D}(\alpha)}{\Delta_{r,D}(\alpha) v^m}, \ (\forall)1 \leq v \leq D, 0 \leq m \leq r-1,$$

where \(\Delta_{r,D}^{m,v}(\alpha)\) is the determinant obtained from \(\Delta_{r,D}(\alpha)\), as defined in (1.10), by replacing the \((mD + v)\)-th column with the column \((\frac{(-1)^{r-1}(\alpha_j-1)!D}{(\alpha_j+r-1)!} B_{\alpha_j+r-1}(\alpha))\) \(1 \leq j \leq rD-1\). Moreover,

$$p_{\alpha}(n) = \frac{1}{\Delta_{r,D}(\alpha)} \sum_{m=0}^{r-1} \Delta_{r,D}^{m,v}(\alpha)n^m, \ (\forall)n \in \mathbb{N}.$$

**Proof.** It follows from (1.8) and (1.10) by Cramer’s rule. The last assertion follows from (1.1).

Our main theorem is the following:

**Theorem 1.2.** Let \(r \geq 1\) and let \(D = 1\) or \(D \geq 2\) is a prime number. There exists a sequence of integers \(\alpha : \alpha_1 < \alpha_2 < \cdots < \alpha_r \), \(\alpha_1 \geq 2\), such that \(\Delta_{r,D}(\alpha) \neq 0\). In particular, we can compute \(p_{\alpha}(n)\) in terms of values of Bernoulli polynomials and Bernoulli-Barnes numbers.

We believe that the result holds for any integer \(D \geq 1\). Unfortunately, our method based on p-adic value and congruences for Bernoulli numbers and for values of Bernoulli polynomials, is not refined enough to prove it.
We recall several properties of the Bernoulli polynomials. We have that:
\[ B_n(1 - x) = (-1)^n B_n(x), \quad (\forall) x \in \mathbb{R}, \ n \in \mathbb{N}. \] (2.1)

For any integers \( n \geq 1 \) and \( 1 \leq v \leq D \), using (1.4), we let
\[ \tilde{B}_n(x) := D^n(B_n(x) - B_n) = \sum_{j=1}^{n-1} \binom{n}{j} D^j(xD)^{n-j} \] (2.2)

According to [1, Theorem 1], it holds that
\[ \tilde{B}_n(vD) \in \mathbb{Z}, \quad (\forall) 1 \leq v \leq D. \] (2.3)

According to the a result of T. Clausen and C. von Staudt (see [6],[15]), we have that
\[ B_{2n} = A_{2n} - \sum_{p-1|2n} \frac{1}{p}, \quad (\forall) n \geq 1, \] (2.4)

where \( A_{2n} \in \mathbb{Z} \) and the sum is over the all primes \( p \) such that \( p - 1 \mid 2n \).

Let \( p \) be a prime. For any integer \( a \), the \( p \)-adic order of \( a \) is \( v_p(a) := \max\{k : p^k \mid a\} \), if \( a \neq 0 \), and \( v_p(0) = \infty \). For \( q = \frac{a}{b} \in \mathbb{Q} \), the \( p \)-adic order of \( q \) is \( v_p(q) := v_p(a) - v_p(b) \). Note that (2.4) implies
\[ v_p(B_{2n}) = \begin{cases} -1, & p - 1 \mid 2n \\ 0, & p - 1 \nmid 2n \end{cases}. \] (2.5)

Lemma 2.1. For any integer \( n \geq 1 \), it holds that:
1. \( \tilde{B}_n(\frac{1}{2}) = 0 \), if \( n \) is odd, and \( \tilde{B}_n(\frac{1}{2}) \equiv 1 \pmod{2} \), if \( n \) is even.
2. If \( p \) is a prime, then \( \tilde{B}_n(\frac{v}{p}) \equiv v^n \pmod{p} \), \((\forall) 1 \leq v \leq p - 1 \).

Proof. (1) From (2.1) it follows that \( B_n(\frac{1}{2}) = 0 \) if \( n \) is odd, hence, as \( B_n = 0 \), we get
\[ \tilde{B}_n(\frac{1}{2}) = D^n(B_n(\frac{1}{2}) - B_n) = 0. \]

Assume \( n \) is even. According to (2.2), we have that
\[ \tilde{B}_n(\frac{1}{2}) = \sum_{j=0}^{n} \binom{n}{j} B_j 2^j. \]

Since \( 2|2nB_1 = -n \) and \( v_2(2^j B_j) \geq 1 \) for any \( j \geq 2 \), the conclusion follows immediately.

(2) According to (2.2), we have that
\[ \tilde{B}_n(\frac{a}{p}) = \sum_{j=0}^{n} \binom{n}{j} B_j v^n -jp^j. \]

From (2.5), we have that \( v_p(p^j B_j) \geq 1 \) for \( j \geq 1 \), hence the conclusion follows immediately. \( \square \)
Lemma 2.2. If \( p \) is a prime such that \( p \nmid D \) then \( \tilde{B}_p(\frac{v}{D}) \equiv 0(\text{mod } p) \), \((\forall)1 \leq v \leq D - 1.

Proof. We have that

\[
\tilde{B}_p(\frac{v}{D}) = \sum_{j=0}^{p} \binom{p}{j} B_j v^{p-j} D^j.
\]

Since \( v_p(B_j) \geq 0 \) for \( j \leq p - 2 \), it follows that

\[
v_p(\binom{p}{j} B_j) \geq 1, \quad (\forall)1 \leq j \leq p - 2. \tag{2.6}
\]

On the other hand, from (2.4), it follows that

\[
v^p + \binom{p}{p-1} B_{p-1} v D^{p-1} \equiv v^p - vD^{p-1} \equiv v^p - v \equiv 0(\text{mod } p), \tag{2.7}
\]

hence we get the required result.

\[\Box\]

3 Preliminary results

Proposition 3.1. (Case \( D = 1 \)) Let \( p_1 < p_2 < \cdots < p_r \) be some primes such that \( p_1 > 2 \) and \( p_{j+1} - p_j > r \), \((\forall)1 \leq j \leq r \). Let \( \alpha_j := p_j - j, 1 \leq j \leq r \). We have that \( \Delta_{r,1}(\alpha) \neq 0 \).

Proof. Note that (2.4) implies \( B_n(1) = B_n \) for any \( n \geq 2 \). It follows that

\[
\Delta_{r,1}(\alpha) = \begin{vmatrix}
\frac{B_{\alpha_1}}{\alpha_1} & \frac{B_{\alpha_1+1}}{\alpha_1+1} & \cdots & \frac{B_{\alpha_1+r-1}}{\alpha_1+r-1} \\
\frac{B_{\alpha_2}}{\alpha_2} & \frac{B_{\alpha_3}}{\alpha_3} & \cdots & \frac{B_{\alpha_2+r-1}}{\alpha_2+r-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{B_{\alpha_r}}{\alpha_r} & \frac{B_{\alpha_1+1}}{\alpha_1+1} & \cdots & \frac{B_{\alpha_r+r-1}}{\alpha_r+r-1}
\end{vmatrix}. \tag{3.1}
\]

From (2.4) it follows that \( v_{p_j}(B_{\alpha_j+1}) = -1 \) and \( v_{p_j}(B_{\alpha_j+k-1}) \geq 0 \) for and \( 1 \leq k \leq r \) with \( k \neq j \). Moreover, if \( 1 \leq \ell < j \leq r \), then, by hypothesis, \( v_{p_j}(B_{\alpha_j+1}) \geq 0 \) for any \( 1 \leq k \leq r \) (We implicitly used the fact that \( B_n = 0 \) if \( n \geq 3 \) is odd). It follows that, in the expansion of \( \Delta_{r,1}(\alpha) \) written in (3.1), the term

\[
\prod_{j=1}^{r} \frac{D_{\alpha_j+1} B_{\alpha_j+1-j}}{\alpha_j+1-j}
\]

cannot be simplified, hence \( \Delta_{r,1}(\alpha) \neq 0. \) \[\Box\]

In the following, we assume \( D \geq 2 \) and we consider the determinant

\[
\tilde{\Delta}_{r,D}(\alpha) := \begin{vmatrix}
\frac{B_{\alpha_1}}{\alpha_1} & \frac{B_{\alpha_1+1}}{\alpha_1+1} & \cdots & \frac{B_{\alpha_1+r-1}}{\alpha_1+r-1} \\
\frac{B_{\alpha_2}}{\alpha_2} & \frac{B_{\alpha_3}}{\alpha_3} & \cdots & \frac{B_{\alpha_2+r-1}}{\alpha_2+r-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{B_{\alpha_r}}{\alpha_r} & \frac{B_{\alpha_1+1}}{\alpha_1+1} & \cdots & \frac{B_{\alpha_r+r-1}}{\alpha_r+r-1}
\end{vmatrix}. \tag{3.2}
\]
Let \( p_1 < p_2 < \ldots < p_r \) be some primes such that
\[ p_1 \geq \alpha_{r(D-1)} + r \text{ and } p_j + 1 - p_j > r, (\forall) 1 \leq j \leq r - 1. \]

We let
\[ \alpha_{r-D+j} := p_j - j, (\forall) 1 \leq j \leq r. \tag{3.3} \]

According to Lemma 2.2 and (3.3), we have that
\[ v_{p_j} \left( \frac{D^a_{r+j}B_{a}^j}{\alpha + j} \right) \geq 0, (\forall) 1 \leq j, \ell \leq r, 1 \leq v \leq D - 1. \tag{3.4} \]

On the other hand, since \( p_j \geq \alpha_{r(D-1)} + r \), from Lemma 2.2 it follows that
\[ v_{p_j} \left( \frac{D^a_{r+j}B_{a}^j}{\alpha + j} \right) \geq 0, (\forall) 1 \leq j, \ell \leq r, 1 \leq t \leq r(D-1), 1 \leq v \leq D - 1. \tag{3.5} \]

Also, from (2.5) and (3.3), it follows that
\[ v_{p_j} \left( \frac{D^a_{r+j}B_{a}^j}{\alpha + j} \right) \geq 0, \quad v_{p_j} \left( \frac{D^a_{r+j}B_{a}^j}{\alpha + j} \right) = -1, 1 \leq j, \ell \leq r, j \neq \ell, 1 \leq v \leq D - 1. \tag{3.6} \]

From (1.40), using the basic properties of determinants and (2.2), it follows that
\[ \Delta_{r,D}(\alpha) := \begin{vmatrix} \hat{B}_{a_1}(\frac{1}{\alpha_1}) & \hat{B}_{a_1+1}(\frac{1}{\alpha_1}) & \hat{D}^{a_1}B_{a_1} & \hat{B}_{a_1+r-1}(\frac{1}{\alpha_1}) & \hat{D}^{a_1}B_{a_1+r-1} \\ \hat{B}_{a_2}(\frac{1}{\alpha_2}) & \hat{B}_{a_2+1}(\frac{1}{\alpha_2}) & \hat{D}^{a_2}B_{a_2} & \hat{B}_{a_2+r-1}(\frac{1}{\alpha_2}) & \hat{D}^{a_2}B_{a_2+r-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \hat{B}_{a_r}(\frac{1}{\alpha_r}) & \hat{B}_{a_r+1}(\frac{1}{\alpha_r}) & \hat{D}^{a_r}B_{a_r} & \hat{B}_{a_r+r-1}(\frac{1}{\alpha_r}) & \hat{D}^{a_r}B_{a_r+r-1} \end{vmatrix} \tag{3.7} \]

**Proposition 3.2.** With the above assumptions, \( \Delta_{r,D}(\alpha) \neq 0 \) if and only if \( \tilde{\Delta}_{r,D}(\alpha) \neq 0 \).

**Proof.** The conclusion follows from (3.2), (3.1), (3.5), (3.6) and (3.7), using a similar argument as in the proof of Proposition 3.1. \( \square \)

**Proposition 3.3.** (Case \( D = 2 \)) With the above assumptions, \( \Delta_{r,2}(\alpha) \neq 0 \).

**Proof.** By Proposition 3.2, it is enough to prove that \( \tilde{\Delta}_{r,2}(\alpha) \neq 0 \). We have that
\[ \tilde{\Delta}_{r,2}(\alpha) = \begin{vmatrix} \hat{B}_{a_1}(\frac{1}{\alpha_1}) & \hat{B}_{a_1+1}(\frac{1}{\alpha_1}) & \hat{B}_{a_1+r-1}(\frac{1}{\alpha_1}) \\ \hat{B}_{a_2}(\frac{1}{\alpha_2}) & \hat{B}_{a_2+1}(\frac{1}{\alpha_2}) & \hat{B}_{a_2+r-1}(\frac{1}{\alpha_2}) \\ \vdots & \vdots & \vdots \\ \hat{B}_{a_r}(\frac{1}{\alpha_r}) & \hat{B}_{a_r+1}(\frac{1}{\alpha_r}) & \hat{B}_{a_r+r-1}(\frac{1}{\alpha_r}) \end{vmatrix}. \tag{3.8} \]

We choose \( \alpha_j := 2^{j+1} \), where \( 2^t \geq r \). From (2.3) and Lemma 2.1(1) it follows that
\[ v_2(\tilde{B}_{a_j}(\frac{1}{\alpha_j})) = 0, v_2(\tilde{B}_{a_{j+1}}(\frac{1}{\alpha_j})) \geq 0, (\forall) 1 \leq j, \ell \leq r, j \neq \ell. \tag{3.9} \]
hence $\tilde{\Delta}_{r,2}(\alpha) \neq 0$, as required.

In the following, we assume $D \geq 3$. Let $N := \lfloor \frac{(D-1)r}{2} \rfloor$. We also assume that $\alpha_t$ is odd for all $1 \leq t \leq N$, and $\alpha_t$ is even for all $N + 1 \leq t \leq r(D - 1)$. Let $k := \lfloor \frac{D-1}{2} \rfloor$ and $\tilde{k} := \lceil \frac{D-1}{2} \rceil$. From (2.11) and (2.2) it follows that

$$\tilde{B}_{\alpha_t+1-j}(\frac{D-v}{D}) + \tilde{B}_{\alpha_t+1-j}(\frac{v}{D}) = \begin{cases} 0, & \alpha_t + j - 1 \text{ is odd} \\ 2\tilde{B}_{\alpha_t+1-j}(\frac{v}{D}), & \alpha_t + j - 1 \text{ is even,} \end{cases} \quad (3.11)$$

for all $1 \leq t \leq r(D - 1)$, $1 \leq v \leq \tilde{k}$ and $1 \leq j \leq r$. We consider the determinants:

$$\tilde{\Delta}'_{r,D}(\alpha) := \begin{vmatrix} \tilde{B}_1(\frac{1}{\alpha_1}) & \cdots & \tilde{B}_{N+1}(\frac{1}{\alpha_{N+1}}) \\ \frac{\alpha_1}{\alpha_2} & \cdots & \frac{\alpha_{N+1}}{\alpha_{N+2}} \\ \vdots & \ddots & \vdots \\ \tilde{B}_N(\frac{1}{\alpha_N}) & \cdots & \tilde{B}_{N+1}(\frac{1}{\alpha_{N+2}}) \end{vmatrix} \quad (3.12)$$

$$\tilde{\Delta}''_{r,D}(\alpha) := \begin{vmatrix} \tilde{B}_{\alpha_1+1}(\frac{1}{\alpha_{\alpha_1+1}}) & \cdots & \tilde{B}_{\alpha_{N+1}+1}(\frac{1}{\alpha_{\alpha_{N+1}+1}}) \\ \frac{\alpha_{\alpha_1+1}}{\alpha_{\alpha_2}} & \cdots & \frac{\alpha_{\alpha_{N+1}+1}}{\alpha_{\alpha_{N+2}}} \\ \vdots & \ddots & \vdots \\ \tilde{B}_{\alpha_{rD-R}}(\frac{1}{\alpha_{\alpha_{rD-R}}}) & \cdots & \tilde{B}_{\alpha_{rD-R}+1}(\frac{1}{\alpha_{\alpha_{rD-R}+1}}) \end{vmatrix} \quad (3.13)$$

**Proposition 3.4.** With the above assumptions, it holds that

$$\tilde{\Delta}_{r,D}(\alpha) = C\tilde{\Delta}'_{r,D}(\alpha)\tilde{\Delta}''_{r,D}(\alpha),$$

where $C \neq 0$. In particular, if $\tilde{\Delta}'_{r,D}(\alpha) \neq 0$ and $\tilde{\Delta}''_{r,D}(\alpha) \neq 0$ then $\tilde{\Delta}_{r,D}(\alpha) \neq 0$.

**Proof.** In (3.2), we add the $(j + tr)$-th column over the $(D - j + tr)$-th column, where $1 \leq j \leq k$ and $0 \leq t \leq r - 1$. The conclusion follows from (3.11), (3.12) and (3.13) using the basic properties of determinants. 

$$j + t = v_2(\alpha_j + j - 1) > v_2(\alpha_j + \ell - 1), \quad (\forall) 1 \leq j, \ell \leq r, \quad j \neq \ell. \quad (3.10)$$

From (3.9), (3.10) and (3.11), it follows that

$$v_2(\tilde{\Delta}_{r,2}(\alpha)) = v_2 \left( \prod_{j=1}^{r} \tilde{B}_{\alpha_j + j - 1}(\frac{1}{\alpha_j + j - 1}) \right) = -rt - \left( \frac{r}{2} \right) < \infty,$$

On the other hand,

$$\tilde{\Delta}_{r,2}(\alpha) \neq 0,$$
4 Proof of Theorem 1.2

The case \( D = 1 \) was proved in Proposition 3.1. Also, the case \( D = 2 \) was proved in Proposition 3.3. Assume that \( D := p > 2 \) is a prime number. Let \( k := \lfloor \frac{D-1}{2} \rfloor \). According to Proposition 3.4, it is enough to prove that \( \tilde{\Delta}'_{r,p}(\alpha) \neq 0 \) and \( \tilde{\Delta}''_{r,p}(\alpha) \neq 0 \). Let

\[
\log_p(r-1) < t_1 < t_2 < \cdots < t_r,
\]

be a sequence of positive integers. We define

\[
\alpha_{j+(s-1)k} := \begin{cases} 2jp^s - s + 1, & \text{if } s \text{ is even}, \\ (2j-1)p^s - s + 1, & \text{if } s \text{ is odd}, \end{cases} \quad (\forall) 1 \leq s \leq r, 1 \leq j \leq k.
\]

From (4.1) and (4.2) it follows that

\[
v_p(\alpha_{j+(s-1)k} + s - 1) = t_s, \quad (\forall) 1 \leq s \leq r, 1 \leq j \leq k.
\]

(4.3)

On the other hand, using the Vandermonde formula, we have

\[
v_p(\alpha_{j+(s-1)k} + \ell) < t_1, \quad (\forall) 1 \leq s \leq r, 1 \leq j \leq k \text{ and } 0 \leq \ell \leq r - 1 \text{ with } \ell \neq s - 1.
\]

(4.4)

We consider the determinants

\[
M_s := \det \left( B_{\alpha_{j+(s-1)k}+s-1}(\frac{\alpha_j}{p}) \right)_{1 \leq j,v \leq k}, 1 \leq s \leq r.
\]

(4.8)

From (4.3) it follows that

\[
M_s \equiv \det \left( v^{2jp^s} \right)_{1 \leq j,v \leq k} \equiv \det \left( v^{2j} \right)_{1 \leq j,v \leq k} \mod p \quad \text{for } s \text{ even},
\]

(4.9)

\[
M_s \equiv \det \left( v^{2(j-1)p^s} \right)_{1 \leq j,v \leq k} \equiv \det \left( v^{2j-1} \right)_{1 \leq j,v \leq k} \mod p \quad \text{for } s \text{ odd}.
\]

(4.10)

On the other hand, using the Vandermonde formula, we have

\[
\det \left( v^{2j} \right)_{1 \leq j,v \leq k} = v^2 \prod_{1 \leq i < j \leq k} (j-i)(j+i) \neq 0 \mod p,
\]

(4.11)

\[
\det \left( v^{2j-1} \right)_{1 \leq j,v \leq k} = v \prod_{1 \leq i < j \leq k} (j-i)(j+i) \neq 0 \mod p.
\]

(4.12)

From (4.9), (4.10), (4.11) and (4.12) it follows that

\[
v_p(M_s) = 0, \quad (\forall) 1 \leq s \leq r,
\]

(4.13)

hence, in particular \( M_s \neq 0 \). From (3.12), (4.3), (4.7), (4.8) and (4.13) it follows that

\[
v_p(\tilde{\Delta}'_{r,p}(\alpha)) = -(t_1 + t_2 + \cdots + t_r)k,
\]

hence, in particular, \( \tilde{\Delta}'_{r,p}(\alpha) \neq 0 \). Similarly, one can prove that \( \tilde{\Delta}''_{r,p}(\alpha) \neq 0 \).

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