\textbf{Abstract.} With the help of recent adjacent dyadic constructions by Hytönen and the author, we give a simple alternative proof of results of Lechner and Passenbrunner about the $L^p$-boundedness of shift operators acting on functions $f \in L^p(X; E)$ where $1 < p < \infty$, $X$ is a metric space and $E$ is a UMD space.

\section{Introduction}

During the last two decades, the highly influential $T(1)$ theorem of G. David and J.-L. Journé \cite{DavidJourné1989} has been generalized to various settings by different authors (e.g. \cite{Figiel1988, Figiel1987}). One of these generalizations was due to T. Figiel (\cite{Figiel1988, Figiel1987}, different proof by T. Hytönen and L. Weis \cite{HytönenWeis2005}) who proved the theorem for UMD-valued functions $f \in L^p(\mathbb{R}^d; E)$ and scalar-valued kernels using a clever observation that any Caldéron-Zygmund operator on $\mathbb{R}^d$ can be decomposed into sums and products of Haar shifts (or rearrangements), Haar multipliers and paraproducts. Not long ago, this technique was borrowed by P.F.X Müller and M. Passenbrunner \cite{MüllerPassenbrunner2014} to prove the $T(1)$ theorem for UMD-valued functions $f \in L^p(X; E)$, where $X$ is a normal space of homogeneous type (i.e. a space of homogeneous type that satisfies an Ahlfors 1-regularity type condition). One of the key elements of their (and Figiel’s) proof - the $L^p$-boundedness of the shift operators - was simplified and sharpened by R. Lechner and Passenbrunner in their recent paper \cite{LechnerPassenbrunner2015} by proving the claim in a slightly more general form with different techniques.

Roughly speaking, a shift operator permutes the generating Haar functions in such a way that if $h_Q \mapsto h_P$, then the dyadic cubes $P$ and $Q$ are not too far away from each other and they belong to the same generation of the given dyadic system. On the real line, this can be expressed in a very simple form: for every $m \in \mathbb{Z}$, the shift operator $T_m$ is the linear extension of the map $h_I \mapsto h_{I + m |I|}$. In \cite{Figiel1988} Theorem 1, Figiel showed that for UMD-valued functions $f: [0, 1] \to E$ and for every $p \in (1, \infty)$ we have the norm estimate

$$\|T_m f\|_p \leq C \log (2 + |m|)^\alpha \|f\|_p$$  \hspace{1cm} (1.1)

where $\alpha < 1$ depends only on $E$ and $p$, and the constant $C$ depends on $E$, $p$ and $\alpha$ (the same result was formulated for functions $f: \mathbb{R}^d \to E$ in \cite{Figiel1988} Lemma 1). In \cite{MüllerPassenbrunner2014} Proposition 4.5, Müller and Passenbrunner showed that for shift operators in more general settings satisfy a norm estimate that has stronger dependency of the number $m$ than in (1.1), but this estimate was sharpened to match (1.1) by Lechner and Passenbrunner in \cite{LechnerPassenbrunner2015} Theorem 4.3.

In this paper we give an alternative short proof for the estimate (1.1) for functions $f: X \to E$ where $X$ is a metric space equipped with a doubling Borel measure and $E$ is an UMD space. Our proof uses efficiently the auxiliary adjacent dyadic systems that were constructed recently by Hytönen and the author in \cite{Täppola2015}; by splitting a given dyadic system $\mathcal{D}$ into suitable subcollections $\mathcal{D}_A$, we can embed cubes and the shifted cubes inside suitable nested structures that give us a convenient way to approximate certain indicator functions by their conditional expectations. Our main point is that the estimate for shift operators can be achieved by a black-box application of existing results on parallel dyadic structures, in contrast to the ad-hoc construction of similar structures by Lechner and Passenbrunner.
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2. Dyadic cubes, conditional expectations and UMD spaces

2.1. Geometrically doubling metric spaces. Let \( (X, d) \) be a geometrically doubling metric space. That is, there exists a constant \( M \) such that every ball \( B(x, r) := \{ y \in X : d(x, y) < r \} \) can be covered by at most \( M \) balls of radius \( r/2 \). In this subsection we do not assume any measurability of \( (X, d) \) but we note that if \( (Y, d', \mu) \) is a doubling metric measure space, then \( (Y, d') \) is a geometrically doubling metric space.

We use the following two standard lemmas repeatedly in different proofs without referring to them every time we use them.

**Lemma 2.1** ([12] Lemma 2.3]). The following properties hold for \( (X, \mu) \):

1. Any ball \( B(x, r) \) can be covered by at most \( \lceil M \delta^{- \log_2 M} \rceil \) balls \( B(x_i, \delta r) \) for every \( \delta \in (0, 1] \).
2. Any ball \( B(x, r) \) contains at most \( \lceil M \delta^{- \log_2 M} \rceil \) centres \( x_i \) of pairwise disjoint balls \( B(x_i, \delta r) \) for every \( \delta \in (0, 1] \).

**Lemma 2.2** ([17] Lemma 2.2]). For any \( \delta > 0 \) there exists a countable maximal \( \delta \)-separated set \( \mathcal{A}_\delta \subseteq X \):

- \( d(x, y) \geq \delta \) for every \( x, y \in \mathcal{A}_\delta \), \( x \neq y \)
- \( \min_{x \in \mathcal{A}_\delta} d(x, z) < \delta \) for every \( z \in X \).

Since the center points of dyadic cybes (see Theorem 2.5 below) form \( \delta^k \)-separated sets, the following simple lemma is a convenient tool for splitting dyadic systems into smaller sparse systems. We will use the lemma later in Section 5.

**Lemma 2.3.** Let \( D_2 > D_1 > 0 \) and let \( Z \) be a \( D_1 \)-separated set of points in the space \( X \). Then \( Z \) is a disjoint union of at most \( N D_2 \)-separated sets where \( N \) depends only on \( M \) and \( D_1/D_2 \).

**Proof.** First, notice that any ball of radius \( D_2 \) can contain at most boundedly many, say \( M_1 \), points of \( Z \) by the first part of Lemma 2.1. By Lemma 2.2 we can choose a maximal \( D_2 \)-separated subset \( Z_1 \) from \( Z \). By applying the same lemma \( M_1 \) times, we can choose maximal \( D_2 \)-separated subsets \( Z_k \subseteq Z \setminus \bigcup_{i=1}^{k-1} Z_i \) for every \( k = 1, 2, \ldots, M_1 \). We claim that now \( Z \setminus \bigcup_{k=1}^{M_1} Z_k = \emptyset \).

For contradiction, suppose that there exists a point \( x \in Z \setminus \bigcup_{k=1}^{M_1} Z_k \). By maximality, \( B(x, D_2) \cap Z_k \neq \emptyset \) for every \( k = 1, 2, \ldots, M_1 \) since otherwise the point \( x \) would belong to one of the collections \( Z_k \). Thus, the ball \( B(x, D_2) \) contains \( M_1 + 1 \) points of \( Z \), which is a contradiction.

In the construction of metric dyadic cubes we need maximal \( \delta^k \)-separated sets for every \( k \in \mathbb{Z} \). For this we can use Lemma 2.2 or the following stronger result:

**Theorem 2.4** ([17] Theorem 2.4]). For every \( \delta \in (0, 1/2) \) there exist maximal nested \( \delta^k \)-separated sets \( \mathcal{A}_k := \{ z_{\alpha k}^k : \alpha \in \mathcal{N}_k \}, k \in \mathbb{Z} \):

- \( \mathcal{A}_k \subseteq \mathcal{A}_{k+1} \) for every \( k \in \mathbb{Z} \);
- \( d(z_{\alpha k}^k, z_{\beta k}^{k+1}) \geq \delta^k \) for \( \alpha \neq \beta \);
- \( \min_{x \in \mathcal{A}_k} d(x, z_{\alpha k}^k) < \delta^k \) for every \( x \in X \) and every \( k \in \mathbb{Z} \), where \( \mathcal{N}_k = \{ 0, 1, \ldots, n_k \} \) if the space \( (X, d) \) is bounded, and \( \mathcal{N}_k = \mathbb{N} \) otherwise.

2.2. Adjacent dyadic systems in metric spaces. The following theorem is an improved version of the famous constructions of (quasi)metric dyadic cubes by M. Christ [5] and E. Sawyer and R. L. Wheeden [20]. This version was proved by Hytönen and A. Kairema [19] Theorem 2.2 and it has been adapted for different dyadic constructions in [17] (see [17] Theorem 2.9) including Theorem 2.6 below.
Theorem 2.5. Let $(X, d)$ be a doubling metric space and $\delta \in (0, 1)$ be small enough. Then for given nested maximal sets of $\delta^k$-separated points $\{z^k_\alpha; \alpha \in \mathcal{A}_k\}, k \in \mathbb{Z}$, there exist a countable collection of dyadic cubes $\mathcal{D} := \{Q^k_\alpha; k \in \mathbb{Z}, \alpha \in \mathcal{A}_k\}$ such that

i) $X = \bigcup_{\alpha} Q^k_\alpha$ for every $k \in \mathbb{Z}$;
ii) $P, Q \in \mathcal{D} \Rightarrow P \cap Q \in \{\emptyset, P, Q\}$;
iii) $B(z^k_\alpha, 1/2\delta^k) \subseteq Q^k_\alpha \subseteq B(z^k_\alpha, 3\delta^k)$;
iv) $Q^k_\alpha = \bigcup_{B, \mathcal{Q}^k_{\alpha} \subseteq \mathcal{Q}^k_\alpha} \mathcal{Q}^k_{\alpha+m}$ for every $m \in \mathbb{N}$.

For every dyadic system $\mathcal{D}$ and cube $Q := Q^k_\alpha \in \mathcal{D}$ we use the following notation:

- $\ell(Q) := j$ (level/generation of the cube $Q$)
- $\mathcal{D}^k := \{Q^k_\alpha \in \mathcal{D}; \alpha \in \mathcal{A}_k\}$ (cubes of level $k$)
- $B_Q := B(z^k_\alpha, 3\delta^k)$ (ball containing cube $Q$)
- $x_Q := z^k_\alpha$ (the center point of the cube $Q$).

Like we mentioned earlier, the key idea of our techniques in Section 4 is to construct additional nested structures that help us approximate certain given indicators by their conditional expectations. For this we use adjacent dyadic systems which have turned out to be a convenient tool for approximating arbitrary balls and other objects by cubes both in $\mathbb{R}^n$ and more abstract settings (see e.g. [20, 23]). In quasimetric spaces they were first constructed by Hytönen and Kairema [15, Theorem 2.6] (based on the ideas of Hytönen and H. Martikainen [16]) but by restricting ourselves to a strictly metric setting we can use systems with more powerful properties. The following theorem was proved recently by Hytönen and the author for $n = 1$:

Theorem 2.6. Let $(X, d)$ be a doubling metric space with a doubling constant $M$ and let $n \in \mathbb{N}$ be fixed. Then for $\delta < 1/(n \cdot 168 M^8)$ there exist a bounded number of adjacent dyadic systems $\mathcal{D}(\omega), \omega = 1, 2, \ldots, K = K(\delta)$, such that

I) each $\mathcal{D}(\omega)$ is a dyadic system in the sense of Theorem 2.5.
II) for a fixed $p \in \mathbb{N}$ and fixed balls $B_1, B_2, \ldots, B_n$ there exist $\omega \in \{1, 2, \ldots, K\}$ and cubes $Q_{B_1}, Q_{B_2}, \ldots, Q_{B_n} \in \mathcal{D}(\omega)$ such that for every $i \in \{1, 2, \ldots, n\}$ we have

i) $B_i \subseteq Q_{B_i}$;
ii) $\ell(Q_{B_i}) \leq \delta^{-2}\ell(B_i)$;
iii) $\delta^{-p} B_i \subseteq Q_{B_i}^{(p)}$,

where $\ell(Q) = \delta^k$ if $Q = Q^k_\alpha$, $r(B)$ is the radius of the ball $B$ and $Q_{B_i}^{(p)}$ is the unique dyadic ancestor of $Q_{B_i}$ of generation $\ell(Q_{B_i}) - p$.

Proof. In [17, Theorem 5.9] the case $n = 1$ was proved by showing that if $B(x, r)$ is a ball such that $\delta^{k+2} < r \leq \delta^{k+1}$, then

$$P_\omega \left( \left\{ \omega \in \Omega: x \in \left( \bigcup_\alpha \partial_{\delta^k} Q^k_{\alpha} - P(\omega) \cup \bigcup_\alpha \partial_{\delta^{k+1}} Q^k_{\alpha}(\omega) \right) \right\} \right) \leq 168 M^8 \delta < 1 \tag{2.7}$$

where $P_\omega$ is the natural probability measure of the finite set $\Omega := \{0, 1, \ldots, \lfloor 1/\delta \rfloor\}$, $Q(\omega)$ is a cube of the dyadic system $\mathcal{D}(\omega)$ and

$$\partial_{\varepsilon} A := \{x \in A: d(x, A^c) < \varepsilon\} \cup \{x \in A^c: d(x, A) < \varepsilon\}.$$

Given (2.7), the proof for general $n \in \mathbb{N}$ is simple. Let $B_1, B_2, \ldots, B_n$ be balls and denote $B_i := B(x_i, r_i), \delta^{k_i+2} < r_i \leq \delta^{k_i+1}$. Then

$$P_\omega \left( \left\{ \omega \in \Omega: x_i \in \left( \bigcup_\alpha \partial_{\delta^{k_i}} Q^{k_i}_{\alpha} - P(\omega) \cup \bigcup_\alpha \partial_{\delta^{k_i+1}} Q^{k_i}_{\alpha}(\omega) \right) \text{ for some } i \right\} \right) \leq n \cdot 168 M^8 \delta < 1.$$ 

Thus, there exists $\omega \in \Omega$ such that $x_i \notin \left( \bigcup_\alpha \partial_{\delta^{k_i}} Q^{k_i}_{\alpha} - P(\omega) \cup \bigcup_\alpha \partial_{\delta^{k_i+1}} Q^{k_i}_{\alpha}(\omega) \right)$ for every $i = 1, 2, \ldots, n$, which is enough to prove the claim. \qed
Lemma 2.10. Let \( \omega \) denote the function where \( \exists m \in \mathbb{N} \) such that \( 2mB_{\omega} \subseteq B_{\omega}^m \) for \( p_m \in \mathbb{N} \) such that \( 2mB_{\omega}^m \leq 1 \).

2.3. Conditional expectations. Conditional expectations are mostly used in the field of probability theory but have turned out to be extremely useful also with many questions related to more classical analysis (see e.g. [13]). It is well known among specialists that most of the results related to conditional expectations remain true in more general measure spaces but, unfortunately, it is difficult to find a comprehensive presentation of this extended theory in the literature. We refer to [24] for some basic properties of conditional expectations in \( \sigma \)-finite measure spaces and Chapter 9 for a presentation of the classical probabilistic theory of conditional expectations.

Let \( (X, \mathcal{F}, \mu, d) \) be a metric measure space such that \( \mu \) is a doubling Borel measure, i.e. there exists a constant \( D := D_{\mu} \) such that
\[
\mu(2B) \leq D\mu(B) < \infty
\]
for every ball \( B \). By construction we know that if \( \mathcal{G} \) is a dyadic system given by Theorem 2.5 then \( \mathcal{G} \subseteq \text{Bor} \, X \). In particular, the \( \sigma \)-algebra generated by any subcollection of \( \mathcal{G} \) is a subset of \( \mathcal{F} \).

Let us denote \( \mathcal{G}_0 := \{ G \in \mathcal{G} : \mu(G) < \infty \} \) for every \( \sigma \)-algebra \( \mathcal{G} \subseteq \mathcal{F} \), and let \( L^1_{\sigma}(\mathcal{G}) \) be the space of functions that are integrable over all \( G \in \mathcal{G}_0 \).

Definition 2.9. Let \( \mathcal{G} \) be \( \sigma \)-finite sub-\( \sigma \)-algebra of \( \mathcal{F} \) and let \( f : X \rightarrow E \) be a \( \mathcal{F} \)-measurable function where \( E \) is a Banach space. Then a \( \mathcal{G} \)-measurable function \( g \) is a conditional expectation of \( f \) with respect to \( \mathcal{G} \) if
\[
\int_G f \, d\mu = \int_G g \, d\mu
\]
for every \( G \in \mathcal{G} \), \( \mu(G) < \infty \).

It is not difficult to prove that if the conditional expectation exists, it is unique a.e. Thus, we denote \( \mathbb{E}[f|\mathcal{G}] := g \) if \( g \) is a conditional expectation of \( f \) with respect to \( \mathcal{G} \). Concerning existence, we only need the following elementary case in this paper.

Lemma 2.10. Let \( A := \{ A_i : i \in \mathbb{N} \} \subseteq \mathcal{F} \) be a countable partition of the space \( X \) such that \( \mu(A_i) < \infty \) for every \( i \in \mathbb{N} \) and let \( \mathcal{A} \) be the \( \sigma \)-algebra generated by \( A \). Then for every \( f \in L^1_{\sigma}(\mathcal{F}) \) we have
\[
E[f|\mathcal{A}] = \sum_{A \in A} 1_A(f)A.
\]

Proof. Let \( G \in \mathcal{A}_0 \). Then there exist pairwise disjoint sets \( A_1^G, A_2^G, \ldots \in A \) such that \( G = \bigcup_i A_i^G \).

Now
\[
\int_G f \, d\mu = \sum_i \int_{A_i^G} \left( \int_{A_i^G} f \, d\mu \right) d\mu = \int_G \sum_i 1_{A_i^G} \left( \int_{A_i^G} f \, d\mu \right) d\mu = \int_G \left( \sum_{A \in A} 1_A \int_{A_i} f \, d\mu \right) d\mu
\]
which proves the claim.

2.4. UMD spaces; type and cotype of Banach spaces. Let \( (X, d, \mathcal{F}, \mu) \) be a metric measure space and let \( (\mathcal{F}_k), k = 0, 1, \ldots, N \), be a sequence of sub-\( \sigma \)-algebras of \( \mathcal{F} \) such that \( \mathcal{F}_k \subseteq \mathcal{F}_{k+1} \) for all \( k \). For simplicity, let us denote
\[
\| \cdot \|_p := \| \cdot \|_{L^p(X, E)}
\]
where \( \| \cdot \|_{L^p(X, E)} \) is the \( L^p \)-Bochner norm.
Definition 2.11. A sequence of functions \( (d_k)_{k=1}^N \) is a martingale difference sequence if \( d_k \) is \( \mathcal{F}_k \)-measurable and \( \mathbb{E}[d_k | \mathcal{F}_{k-1}] = 0 \) for every \( k \).

Definition 2.12. A Banach space \( (E, \| \cdot \|) \) is a UMD (unconditional martingale difference) space if for every \( p \in (1, \infty) \) there exists a constant \( \beta_p \) such that

\[
\left\| \sum_{i=1}^N \varepsilon_i d_i \right\|_p \leq \beta_p \left\| \sum_{i=1}^N d_i \right\|_p
\]

for all \( E \)-valued \( L^p \)-martingale difference sequences \( (d_i)_{i=1}^N \) (i.e. \( (d_i) \) is a martingale difference sequence such that \( d_i \in L^p(X, \mathcal{F}; E) \) for every \( i \)) and for all choices of signs \( (\varepsilon_i)_{i=1}^N \in \{-1, +1\}^N \).

UMD spaces are the key spaces in Banach space valued harmonic analysis due to their many good properties; for example, a Banach space \( E \) is a UMD space if and only if the Hilbert transform is bounded on \( L^p(\mathbb{R}; E) \). They give us a natural setting for analysis that is based on techniques used in probability spaces, as we will next see.

Let \( (d_i) \) be a martingale difference sequence and let \( (\varepsilon_i) \) be a sequence of random signs, i.e. independent random variables on some probability space \( (\Omega, \mathcal{F}, P) \), with distribution \( P(\varepsilon_i = -1) = P(\varepsilon_i = +1) = 1/2 \). Then for every \( \eta \in \Omega \) the sequence \( (\varepsilon_i(\eta)d_i) \) is a martingale difference sequence. In particular, the UMD property gives us

\[
\left\| \sum_{i=1}^N \varepsilon_i d_i \right\|_{\Omega, p} \leq \beta_{p, \sigma} \left\| \sum_{i=1}^N \varepsilon_i d_i \right\|_{\Omega, p},
\]

for every \( p \in (1, \infty) \).

The following inequality by J. Bourgain is a standard tool in UMD valued analysis. Its original scalar-valued version was due to E. Stein.

Theorem 2.14 (See e.g. [19, Proposition 3.8]). Let \( (f_k) \) be a sequence of functions in \( L^p(X, \mathcal{F}; E) \) and \( \mathcal{F}_k \) a sequence of \( \sigma \)-finite \( \sigma \)-algebras such that \( \mathcal{F}_k \subseteq \mathcal{F}_{k+1} \subseteq \mathcal{F} \) for every \( k \in \mathbb{N} \). Then for any sequence of random signs \( (\varepsilon_k) \) we have

\[
\left\| \sum_k \varepsilon_k \mathbb{E}[f_k | \mathcal{F}_k] \right\|_{\Omega, p} \leq \beta_{p, \sigma} \left\| \sum_k \varepsilon_k f_k \right\|_{\Omega, p}.
\]

In our proofs we also need the following version of the well-known principle of contraction by J.-P. Kahane. It holds in all Banach spaces.

Theorem 2.15 ([19, Theorem 5 (Section 2.6)]). Suppose that \( (\varepsilon_i) \) is a sequence of random signs and the series \( \sum_i \varepsilon_i x_i \) converges in \( E \) almost surely. Then for any bounded sequence of scalars \( (c_i) \) the series \( \sum_i \varepsilon_i c_i x_i \) converges in \( E \) almost surely and

\[
\int_{\Omega} \left\| \sum_i \varepsilon_i c_i x_i \right\|^p_E d\mathbb{P} \leq \left( \sup_i |c_i| \right)^p \int_{\Omega} \left\| \sum_i \varepsilon_i x_i \right\|^p_E d\mathbb{P}.
\]

2.4.1. Type and cotype of Banach spaces.

Definition 2.16. Let \( (E, \| \cdot \|) \) be a Banach space. We say that \( E \) has type \( t \in [1, 2] \) if there exists a constant \( C_t > 0 \) such that for every finite sequence \( (x_i) \) in \( E \) and finite sequence \( (\varepsilon_i) \) of random signs we have

\[
\int_{\Omega} \left\| \sum_i \varepsilon_i x_i \right\|^t_E d\mathbb{P} \leq C_t \left( \sum_i \|x_i\|^t \right)^{1/t}.
\]

In a similar fashion, we say that \( E \) has cotype \( q \in [2, \infty] \) if there exists a constant \( C_q > 0 \) such that

\[
\left( \sum_i \|x_i\|^q \right)^{1/q} \leq C_q \int_{\Omega} \left\| \sum \varepsilon_i x_i \right\|^q_E d\mathbb{P}.
\]
The notion of type and cotype of Banach spaces was introduced by B. Maurey and G. Pisier in the 1970's and it has become an important part of analysis on Banach spaces. Out of this rich theory, we need the following results:

i) If \( Y \) is a \( \sigma \)-finite measure space and \( E \) is a Banach space of type \( r \) and cotype \( s \), then \( L^p(X; E) \) has type \( \min\{p, r\} \) and cotype \( \max\{p, s\} \).

ii) If \( E \) is a UMD space, then \( E \) has a non-trivial type \( s > 1 \) and non-trivial cotype \( t < \infty \).

For proofs, see e.g. [22, Chapter 9] for i) and [2, Theorem 11.1.14], [25, Proposition 3] for ii).

2.5. Structural constants. We say that \( c \) is a structural constant if it depends only on the doubling constant \( D \), the UMD constant \( \beta_p \) for a fixed \( p \in (1, \infty) \) and the type and cotype constants \( C_t \) and \( C_q \). We do not track the dependencies of our bounds on the structural constants and thus, we use the notation \( a \lesssim b \) if \( a \leq cb \) for some structural constant \( c \) and \( a \approx b \) if \( a \lesssim b \lesssim a \).

3. Embedding cubes into larger cubes

In this section we prove a decomposition result for dyadic systems using Theorem 2.6. We formulate the result in such a way that it is easy to apply it in Section 4 but we note that it is simple to modify the proof for other similar decompositions.

Let \( \mathcal{D} \) be a dyadic system with \( \delta < 1/(2 \cdot 168 M^3) \) and \( \{\mathcal{D}_\omega\}_\omega \) be adjacent dyadic systems for the same \( \delta \) given by Theorem 2.6. Let us fix a number \( m \geq 1 \) and an injective function \( \tau: \mathcal{D} \rightarrow \mathcal{D} \) such that \( \tau(Q) \subseteq mB_Q \) for every \( Q \in \mathcal{D} \) and \( \tau\mathcal{D}_\omega \subseteq \mathcal{D}_\omega \) for every \( k \in \mathbb{Z} \).

**Proposition 3.1.** The system \( \mathcal{D} \) is a disjoint union of boundedly many subcollections \( \mathcal{D}_\lambda \subseteq \mathcal{D} \), \( \lambda = (i, j, \omega) \), with the following property: for every \( Q \in \mathcal{D}_\lambda \) there exist cubes \( P_Q, P_{\tau(Q)}, P^*_Q \in \mathcal{D}(\omega) \) such that

\[
Q \subseteq P_Q, \quad \tau(Q) \subseteq P_{\tau(Q)}, \quad P_Q \cup P_{\tau(Q)} \cup 2mB_Q \subseteq P^*_Q;
\]

if \( Q_1, Q_2 \in \mathcal{D}_\lambda \cap \mathcal{D}_\omega, Q_1 \neq Q_2 \), then \( (P_Q, P_{\tau(Q)}, P^*_Q) \cap (P_{Q_2}, P_{\tau(Q_2)}^*) = \emptyset \);

In other words, we split the collection \( \mathcal{D} \) into sparse subcollections \( \mathcal{D}_\lambda \) such that we can embed every cube \( Q \in \mathcal{D}_\lambda \) and its image \( \tau(Q) \) into some larger cubes \( P_Q \) and \( P_{\tau(Q)} \) such that \( P_Q \) and \( P_{\tau(Q)} \) belong to the same dyadic system and they have a mutual dyadic ancestor \( P^*_Q \).

We form the sets \( \mathcal{D}_\lambda \) with the help of next technical lemma.

**Lemma 3.5.** The collection \( \mathcal{D} \) is a disjoint union of \( L = L(X) \) subcollections \( \mathcal{D}_i \) such that for every \( k \in \mathbb{Z} \) and \( Q_1, Q_2 \in \mathcal{D}_i \cap \mathcal{D}_k \) we have

\[
3\delta^{-3}B_{R_1} \cap 3\delta^{-3}B_{R_2} = \emptyset
\]

where \( R_1 \in \{Q_1, \tau(Q_1)\} \) and \( R_2 \in \{Q_2, \tau(Q_2)\} \), \( R_1 \neq R_2 \), and the number \( L \) is independent of \( m \).

**Proof.** Basically, we only need to use basic properties of geometrically doubling metric spaces with the help of the observation that if \( Q, P \in \mathcal{D}_k \) and \( d(x(Q), x(P)) \geq 12\delta^{-k-m} \), then \( 3\delta^{-3}B_Q \cap 3\delta^{-3}B_P = \emptyset \).

Let \( k \in \mathbb{Z} \) be fixed. For any subcollection \( \mathcal{D} \subseteq \mathcal{D}_k \) and any set \( A \) of center points of cubes, let us denote

\[
Y_{\mathcal{D}} := \{x(Q): Q \in \mathcal{D}\}, \quad \mathcal{D}_A := \{Q \in \mathcal{D}: x(Q) \in A\}.
\]

We split the set \( Y_{\mathcal{D}} \) into smaller sets in three steps. To keep our notation simple, \( i \) is an index whose role may change from one occurrence to the next.

1) By Lemma 2.3, we can split the \( \delta^k \)-separated set \( Y_{\mathcal{D}} \) into boundedly many \( 12\delta^{-k-3} \)-separated subsets \( Y_{i,k}^1 \).

2) For every \( Q \in \mathcal{D}_{Y_{i,k}^1} \), the ball \( 3\delta^{-3}B_Q \) intersects at most boundedly many balls \( 3\delta^{-3}B_{\tau(P)} \) where \( P \in \mathcal{D}_{Y_{i,k}^1} \). Thus, we can split the set \( Y_{i,k}^1 \) into boundedly many subsets \( Y_{i,k}^2 \) such that \( 3\delta^{-3}B_Q \cap 3\delta^{-3}B_{\tau(P)} = \emptyset \) for every \( Q, P \in \mathcal{D}_{Y_{i,k}^2}, Q \neq \tau(P) \).
3) For every \( Q \in \mathcal{Q}_{Y^3_{i,k}} \), the ball \( 3\delta^{-3}B_{r(Q)} \) intersects at most boundedly many balls \( 3\delta^{-3}B_{r(P)} \), \( P \in \mathcal{Q}_{Y^3_{i,k}} \). Thus, we can split the set \( Y^3_{i,k} \) into boundedly many subsets \( Y^3_{i,k} \) such that \( 3\delta^{-3}B_{r(Q)} \cap 3\delta^{-3}B_{r(P)} = \emptyset \) for every \( Q, P \in \mathcal{Q}_{Y^3_{i,k}} \), \( Q \neq P \).

Now we can set \( \mathcal{D}_i := \bigcup_{k \in \mathbb{Z}} \mathcal{Q}_{Y^3_{i,k}} \) for every \( i \).

Let \( \{ \mathcal{D}_i \} \) be the partition of \( \mathcal{D} \) given by the previous lemma and let \( T \in \mathbb{N}, T \geq 1 \), be the smallest number such that

\[ 2m\delta^T \leq 1. \]

Recall Theorem \( \ref{thm:finite-doubling} \) and denote

\[ \gamma(R) := \min \left\{ \omega: Q_{B_R}, Q_{B_{(r)}}, \in \mathcal{D}(\omega), \delta^{-T}B_R \subseteq Q_{B_R} \right\} \]

for every cube \( R \in \mathcal{D} \) and

\[ \mathcal{D}_{i,\omega} := \{ R \in \mathcal{D}_i: \gamma(R) = \omega \} \]

for every \( i = 1, 2, \ldots, L \) and \( \omega = 1, 2, \ldots, K \). Then the collections \( \mathcal{D}_{i,\omega} \) satisfy properties \( \ref{prop:finite-doubling-1} \) and \( \ref{prop:finite-doubling-2} \) but they are still not suitable for property \( \ref{prop:finite-doubling-3} \). Thus, we split collections \( \mathcal{D}_{i,\omega} \) into smaller collections whose cubes have large enough generation gaps: we set

\[ \mathcal{D}_{i,j,\omega} := \bigcup_{k \in \mathbb{Z}} (\mathcal{D}_{i,\omega} \cap \mathcal{D}_{j+4kT}) \]

for every \( j = 0, 1, \ldots, 4T - 1 \). Notice that the indices \( i, j \) and \( \omega \) are independent of each other.

**Proof of Proposition \( \ref{prop:finite-doubling} \)** Clearly we only need to show the claim for the collections \( \mathcal{D}_{i,0,\omega} := \mathcal{D}_i \).

Recall

Notice first that

\[ 2m \cdot r(B_Q) = 6m \delta^{4kT} \leq \delta^{-T} 3\delta^{4kT} = \delta^{-T} \cdot r(B_Q) \]

for every \( Q := Q^{4kT}_{\alpha} \in \mathcal{D}_i \). Thus, by Remark \( \ref{rem:finite-doubling} \) and the definition of \( \mathcal{D}_i \), for every cube \( Q \in \mathcal{D}_i \) there exist cubes \( P_Q, P_{\tau(Q)} \in \mathcal{D}(\omega) \) such that

\[ B_Q \subseteq P_Q, \quad B_{\tau(Q)} \subseteq P_{\tau(Q)}, \quad 2mB_Q \subseteq P_{\tau(Q)} \subseteq P_Q^* \]

Let us then show that the cubes \( P_Q, P_{\tau(Q)} \) and \( P_Q^* \) satisfy properties \( \ref{prop:finite-doubling-1} \) - \( \ref{prop:finite-doubling-3} \).

\( \ref{prop:finite-doubling-1} \) Since \( Q, \tau(Q) \subseteq 2mB_Q \), we know that \( P_Q \cap P_Q^* \neq \emptyset \) and \( P_{\tau(Q)} \cap P_Q^* \neq \emptyset \). Thus, since \( \mathcal{D}(\omega) \) is a dyadic system and \( \lev(P_Q^*) < \lev(P_Q) = \lev(P_{\tau(Q)}) \), we have \( P_Q \cup P_{\tau(Q)} \subseteq P_Q^* \).

\( \ref{prop:finite-doubling-3} \) Since \( x(Q) \in P_Q \) for every cube \( Q \in \mathcal{D}_i \), we have

\[ P_Q \subseteq B(x(P_Q), 3\delta^{4kT-3}) \subseteq B(x(Q), 6\delta^{4kT-3}) = 2\delta^{-3}B_Q \]

for every cube \( Q \in \mathcal{D}_i \). Thus, the property \( \ref{prop:finite-doubling-3} \) follows directly from Lemma \( \ref{lem:finite-doubling} \).

\( \ref{prop:finite-doubling-4} \) Suppose that \( R \subseteq Q := Q^{4kT}_{\alpha} \). Then \( \lev(R) \geq (4k + 4)T \) and thus, \( \lev(P_R) \geq (4k + 4)T - 3 \) and

\[ \lev(P_R^*) \geq (4k + 4)T - 3 - T \geq 4kT = \lev(Q) \geq \lev(P_Q) \]

since \( T \geq 1 \). In particular, \( P_R \subseteq P_Q \) since \( P_R^*, P_Q \in \mathcal{D}(\omega) \) and \( \mathcal{D}(\omega) \) is a dyadic system.

\( \square \)

4. \( L^p \)-boundedness of shift operators

In this section, we show that with the help of Proposition \( \ref{prop:finite-doubling} \) we can give a simple proof for the \( L^p \)-boundedness of the shift operators in doubling metric measure spaces. We follow some basic ideas of \( \ref{Advances} \) and \( \ref{Advances-2} \) but mostly we rely on our own dyadic constructions.

Let \( (X, d) \) be a metric space, \( \mu \) a doubling Borel measure on \( X \) and \( (E, || \cdot ||) \) an UMD space. Since the doubling property of \( \mu \) implies the geometrical doubling property of \( d \), there exists a finite geometrical doubling constant \( M \). Thus, we may fix a dyadic system \( \mathcal{D} \) for \( \delta < 1/(2 \cdot 168M^8) \) and adjacent dyadic systems \( \{ \mathcal{D}(\omega) \}_\omega \) given by Theorem \( \ref{thm:finite-doubling} \) for the same \( \delta \).
4.1. Haar functions. There are various different ways to construct Haar functions in metric spaces (see e.g. [11] Section 5)) and thus, we do not want to fix any particular construction. We do, however, refer to the construction in [14] Section 4]) with the choice $b \equiv 1$) for a system of Haar functions that satisfy the properties in the following definition. In [14] the construction is done in $\mathbb{R}^n$ for a non-doubling measure but it is simple to generalize the result for our setting.

**Definition 4.1.** A collection of functions $h^\theta_Q : X \to \mathbb{R}$, $Q := Q^\alpha_0 \in \mathcal{D}$, $\theta = 1, \ldots, n(Q) \leq \Theta$, is a system of Haar functions if it satisfies the following properties: for every $Q$ and $\theta$ we have

- $\text{supp} \ h^\theta_Q \subseteq Q$;
- $h^\theta_Q$ is constant on every child cube $Q^\alpha_{\theta+1} \subseteq Q$;
- $\int h^\theta_Q = 0$ if $\theta \neq \theta'$;
- $\|h^\theta_Q\|_2 = 1$;

and the space of finite linear combinations of the functions $h^\theta_Q$ is dense in $L^2(X; \mu)$.

The number $\Theta$ in the previous definition depends only on $M$ or, more precisely, the maximum number of child cubes $Q^\alpha_{\theta+1}$ a cube $Q^\alpha_0$ can have. Henceforth, we fix some $\theta = \theta(Q)$ for each $Q \in \mathcal{D}$ and drop the dependency on $\theta$ in the notation.

Let $h_Q = \sum_k v_k 1_{Q_k}$ be a Haar function, where $Q_k$ are the child cubes of $Q$. The following properties are straightforward consequences of the previous definition:

$$\|h_Q\|_{\infty} = \max |v_k| \approx \frac{1}{\mu(Q)^{1/2}}; \quad \|h_Q\|_1 \approx \mu(Q)^{1/2}. \tag{4.2}$$

In particular,

$$\frac{1}{\mu(Q)^{1/2}} \lesssim |h_Q(x)| \lesssim \frac{1}{\mu(Q)^{1/2}} \quad \text{for every } x \in Q \text{ and some } Q_k. \tag{4.4}$$

The previous properties give us the following lemma:

**Lemma 4.5.** For every $p \in (1, \infty)$ and finite collection of cubes $Q$ we have

$$\left\| \sum_Q x_Q h_Q \right\|_p \approx \left\| \sum_{Q, k} \varepsilon_k \sigma_k^\theta x_Q h_{Q_\alpha^k} \right\|_p \tag{4.4}$$

**Proof.** Let us denote $\sum_Q x_Q h_Q = \sum_k \sum_{\alpha} x_{Q_\alpha^k} h_{Q_\alpha^k}$ and let $(\varepsilon_Q)$ be a sequence of random signs. Then for every $y \in X$ and $k \in \mathbb{Z}$ there exists at most one $Q_\alpha^k$ such that $h_{Q_\alpha^k}(y) \neq 0$. Let $\sigma_k^\theta \in \{-1, +1\}$ be such that $\sigma_k^\theta h_{Q_\alpha^k} (y) = |h_{Q_\alpha^k}(y)|$ for every $x \in X$ and $k \in \mathbb{Z}$. Then, for a fixed $y \in X$, $(\sigma_k^\theta \varepsilon_k x_{Q_\alpha^k})_k$ is a sequence of random signs. Since the functions $h_Q$ form a martingale difference sequence and by [4, 4] we know that $|h_Q\|\mu(Q)^{1/2} \lesssim 1$ for every $Q$, we have

$$\left\| \sum_{Q} x_Q h_Q \right\|_p \approx \int \left( \sum_k \left\| \sum_{\alpha} \sigma_k^\theta \varepsilon_k x_{Q_\alpha^k} h_{Q_\alpha^k} \right\|_p \right) d\mu(y) \tag{4.4}$$

by the UMD property of $E$, Fubini’s theorem and Kahane’s contraction principle. Let us then denote $\sum_{Q} x_Q h_Q = \sum_{i=1}^N x_i h_{Q_i}$, where $\text{lev}(Q_1) \leq \text{lev}(Q_2) \leq \ldots \leq \text{lev}(Q_N)$. Then by Lemma 2.10 we have $E[|h_Q|, \|\mathcal{F}_i\| = 1_Q (|h_Q|)_Q$ where $\mathcal{F}_i$ be the $\sigma$-algebra generated by $\mathcal{G}_{\text{lev}(Q)}$. Thus,
since $1/\mu(Q)^{1/2}\langle |h_Q| \rangle_Q \approx 1$, the previous estimates, Stein’s inequality and Kahane’s contraction principle (in this order) give us

$$\left\| \sum_{i=1}^{N} x_i h_{Q_i} \right\|_p^p \approx \left\| \sum_{i=1}^{N} \epsilon_i x_i |h_{Q_i}| \right\|_{\Omega,p}^p \geq \left\| \sum_{i=1}^{N} \epsilon_i x_i \mathbb{E}[|h_{Q_i}| |\mathcal{F}_i]| \right\|_{\Omega,p}^p = \left\| \sum_{i=1}^{N} \epsilon_i x_i \frac{1}{\mu(Q_i)^{1/2}} \langle |h_{Q_i}| \rangle_{Q_i} \right\|_{\Omega,p}^p \geq \left\| \sum_{i=1}^{N} \epsilon_i x_i \frac{1}{\mu(Q_i)^{1/2}} \right\|_{\Omega,p}^p,$$

which proves the claim. \qed

4.2. **Shift operators.** Let us fix the number $m \geq 1$ and let $\tau : \mathcal{D} \rightarrow \mathcal{D}$ be an injective function such that

1) $\tau \mathcal{D}^k \subseteq \mathcal{D}^k$ for every $k \in \mathbb{Z}$;
2) for every $Q \in \mathcal{D}$ we have $\tau(Q) \subseteq mB_Q$;
3) the measures of cubes $Q$ and $\tau(Q)$ are approximately the same:

$$\mu(Q) \approx \mu(\tau(Q)). \quad (4.6)$$

Let $\{h_Q\}_{Q \in \mathcal{D}}$ be a system of Haar functions. Then we can define the shift operator $T := T_{\tau}$ as the linear extension of the operator $T$,

$$\hat{T} h_Q = h_{\tau(Q)}.$$

It is easy to see that without condition [4.6] an estimate of the type $[1.1]$ is out of reach for all $p \in (1, \infty)$. More precisely: by property [4.4] we have $\|h_Q\|_p \approx \mu(Q)^{1/p} \mu(Q)^{1/2}$ for every cube $Q$ and thus, without condition [4.6] the estimate cannot hold simultaneously for all $p \in (1, 2]$ and for all $q \in (2, \infty)$. We note that the condition [4.6] is automatically valid in normal spaces of homogeneous type just by property 1) [23] Definition 3.2].

4.3. **$L^p$-boundedness of shift operators.** Using Proposition 3.1 and Lemma 4.5 we can now prove the following theorem easily.

**Theorem 4.7.** Let $p \in (1, \infty)$ and $f \in L^p(X; E)$. Then

$$\|T f\|_p \leq C \left( \log(2m) + 1 \right)^n \|f\|_p$$

where $C = C(p, X, E, \alpha), \alpha = 1/\min\{t_E, p\} - 1/\max\{q_E, p\} < 1$ and $t_E$ and $q_E$ are the type and cotype of the space $E$.

**Proof.** Suppose that $f \in L^p(X; E)$. Then, by the properties of the Haar functions and Proposition 3.1 we may assume that the function $f$ is of the form

$$f = \sum_{i=1}^{L_1} \sum_{j=0}^{4T-1} \sum_{\omega=1}^{K} \sum_{Q \in \mathcal{D}_{j, \omega}} x_Q h_Q$$

where $x_Q \neq 0$ only for finitely many $Q$. Thus, we can denote $f = \sum_{i,j,\omega} \sum_{k=1}^{n} x_k h_{Q_k}$ where $\text{lev}(Q_1) \leq \text{lev}(Q_2) \leq \ldots \leq \text{lev}(Q_n)$.

For every $k = 1, 2, \ldots, n$, let $\mathcal{F}_k$ be the $\sigma$-algebra generated by

$$F_k := \left( \mathcal{D}(\omega)^{|\text{lev}(Q_k)|-3} \setminus \bigcup_{l=1,\ldots,n} \left\{ P_{Q_l}, P_{\tau(Q_l)} \right\} \right) \cup \bigcup_{l=1,\ldots,n} \left\{ P_{Q_l} \cup P_{\tau(Q_l)} \right\}.$$
Notice that if \( \text{lev}(Q_{k_1}) = \text{lev}(Q_{k_2}) \), then \( F_{k_1} = F_{k_2} \). By property (3.3) we know that \( F_k \) is a partition of the space \( X \) and by property (3.4) we know that the sequence \( (\mathcal{F}_k) \) is nested. Thus, for every \( k = 1, 2, \ldots, n \) we have

\[
\mathbb{E}[1_{Q_k}|\mathcal{F}_k] \lesssim 1_{P_{Q_k} \cup P_{T(Q_k)}} 1_{Q_k} \lesssim 1_{P_{Q_k} \cup P_{T(Q_k)}} 1_{P_{Q_k}} \approx 1_{P_{Q_k} \cup P_{T(Q_k)}}. \tag{4.8}
\]

In particular,

\[
\left\| \sum_k x_k h_{\tau}(Q_k) \right\|_p \lesssim \left\| \sum_k \frac{x_k}{\mu(\tau(Q_k))^{1/2}} 1_{\tau(Q_k)} \right\|_{\Omega,p} \lesssim \left\| \sum_k \frac{x_k}{\mu(Q_k)^{1/2}} 1_{P_{Q_k} \cup P_{T(Q_k)}} \right\|_{\Omega,p} \lesssim \left\| \sum_k \frac{x_k}{\mu(Q_k)^{1/2}} \mathbb{E}[1_{Q_k}|\mathcal{F}_k] \right\|_{\Omega,p} \lesssim \left\| \sum_k \frac{x_k}{\mu(Q_k)^{1/2}} 1_{Q_k} \right\|_{\Omega,p} .
\]

Finally, since by Section 2.4.1 the space \( L^p(X; E) \) has a non-trivial type \( t > 1 \) and a non-trivial cotype \( q < \infty \), we have

\[
\left\| \sum_{i,j,\omega} \sum_k x_k h_{\tau}(Q_k) \right\|_p \lesssim \left( \sum_{i,j,\omega} \left\| \sum_k \frac{x_{Q_k}}{\mu(Q_k)^{1/2}} 1_{Q_k} \right\|_{\Omega,p}^t \right)^{1/t} \lesssim (4TKL)^{1/t-1/q} \left( \sum_{\omega,i,j} \left\| \sum_k \frac{x_{Q_k}}{\mu(Q_k)^{1/2}} 1_{Q_k} \right\|_{\Omega,p}^q \right)^{1/q} \lesssim T^{1/t-1/q} \left\| \sum_{\omega,i,j} \sum_k \frac{x_{Q_k}}{\mu(Q_k)^{1/2}} 1_{Q_k} \right\|_{\Omega,p} \lesssim (\log(2m) + 1)^{1/t-1/q} \left\| f \right\|_p
\]

by Hölder’s inequality. \( \square \)

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