Almost all elliptic curves are Serre curves.

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Abstract

Using a multidimensional large sieve inequality, we obtain a bound for the mean square error in the Chebotarev theorem for division fields of elliptic curves that is as strong as what is implied by the Generalized Riemann Hypothesis. As an application we prove a theorem to the effect that, according to height, almost all elliptic curves are Serre curves, where a Serre curve is an elliptic curve whose torsion subgroup, roughly speaking, has as much Galois symmetry as possible.

1 Introduction.

Let $E$ be an elliptic curve defined over $\mathbb{Q}$ and denote by

$$\phi_{N,E} : G_{\mathbb{Q}} \to \text{Aut}(E[N])$$

the representation of $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the $N$-torsion of $E$. Fixing a $\mathbb{Z}/N\mathbb{Z}$-basis of $E[N]$, we identify $\text{Aut}(E[N])$ with $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and write

$$\phi_{N,E} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

The image $\phi_{N,E}(G_{\mathbb{Q}})$ is exactly the Galois group of the $N$th division field of $E$ over $\mathbb{Q}$, i.e. the field obtained by adjoining to $\mathbb{Q}$ the $x$ and $y$ coordinates of the $N$-torsion of a given Weierstrass model of $E$. Taking the inverse limit over all $N \geq 1$ with the bases chosen compatibly, we obtain the full torsion representation

$$\phi_E : G_{\mathbb{Q}} \to \text{GL}_2(\hat{\mathbb{Z}}) := \lim_{\leftarrow} \text{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

It is natural to wonder how large the image of $\phi_E$ in $\text{GL}_2(\hat{\mathbb{Z}})$ is.

Definition 1. The integer $N$ is said to be exceptional for $E$ if $\phi_{N,E}$ is not surjective.

To wonder about the size of the image of $\phi_E$ in $\text{GL}_2(\hat{\mathbb{Z}})$ is simply to wonder about which numbers $N$ are exceptional for $E$, and about “how exceptional each $N$ is,” i.e. about the index $[\text{GL}_2(\mathbb{Z}/N\mathbb{Z}) : \phi_{N,E}(G_{\mathbb{Q}})]$.

When $E$ has complex multiplication, every $N$ except possibly $N = 2$ is exceptional, and so the image $\phi_E(G_{\mathbb{Q}})$ has infinite index in $\text{GL}_2(\hat{\mathbb{Z}})$. On the
other hand, when $E$ does not have CM, Serre [15] has shown that the index $[GL_2(\hat{\mathbb{Z}}) : \phi(G_Q)]$ is finite. Equivalently, there exists an integer $n_E$ so that

$$\phi_E(G_Q) = \pi^{-1}(\phi_{n_E,E}(G_Q)),$$

(1)

where $\pi : GL_2(\hat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/n_E\mathbb{Z})$ is the natural projection. In particular, this implies that any fixed elliptic curve $E$ has only finitely many exceptional primes, since any such exceptional prime must divide $n_E$. One might wonder how the integer $n_E$ (chosen minimally so that (1) still holds) depends on the curve $E$.

Various results exist which bound the largest possible exceptional prime for $E$. For example, Mazur [13] proves that if $E$ is semistable then no prime $N \geq 11$ can be exceptional for $E$.

Define the height $H(E)$ of the elliptic curve by

$$H(E) = \max(|r|^3, |s|^2),$$

where $r$ and $s$ are the unique integers so that $E$ has a model of the form $y^2 = x^3 + rx + s$ and $\gcd(r^3, s^2)$ is twelfth-power free. Duke [7] shows that, when counted according to height, almost all elliptic curves have no exceptional primes. Stated precisely, he shows that if

$$C(X) := \{ \text{isomorphism classes elliptic curves } E \text{ over } \mathbb{Q} : H(E) \leq X^6 \}$$

and $\varepsilon(X)$ is the set of $E \in C(X)$ which have at least one exceptional prime, then

$$\lim_{X \to \infty} \frac{|\varepsilon(X)|}{|C(X)|} = 0.$$

(2)

He does this by using a two-dimensional large sieve inequality to prove a result which bounds the mean-square error in the Chebotarev density theorem for the $N$th division fields of $E$ over curves of bounded height. Using this, he shows

$$|\varepsilon(X)| \ll X^4 \log^B X$$

with an absolute (but ineffective) constant. Since

$$C(X) = \frac{4}{\zeta(10)}X^5 + O(X^3)$$

(3)

(c.f. [1]), this implies (2).

In [11], Grant obtains an asymptotic formula for $\varepsilon(X)$. He shows that the curves which are exceptional at the primes 2 and 3 contribute the main term of $|\varepsilon(X)|$, and that, for an explicit constant $C$,

$$|\varepsilon(X)| = CX^3 + O(X^{2+\epsilon}).$$

for all $\epsilon > 0$.

This paper gives a different generalization. The statement that an elliptic curve $E$ has no exceptional primes may be viewed as saying that the Galois representation $\phi_E$ has “large image.” In this paper we extend (2) to a result that almost all elliptic curves have $\phi_E(G_Q)$ “as large as possible.”
2 Acknowledgment.

This paper contains results of my Ph. D. dissertation. I am grateful to my advisor William Duke for his guidance.

3 Statement of Results.

Our main result is a theorem bounding the mean-square error in the Chebotarev theorem for division fields of elliptic curves. Fix a positive integer level $N$ and a conjugacy class

$$\mathcal{C} \subset GL_2(\mathbb{Z}/N\mathbb{Z}).$$

We denote by

$$\pi_E(X; N, C) := |\{p \leq X : \phi_{N,E}(\text{Frob}_p) \subseteq C\}|$$

the function which counts the number of primes up to $X$ which are unramified in $\mathbb{Q}(E[N])$ and whose Frobenius class is contained in $C$, and as usual

$$\pi(X; N, d) = |\{p \leq X : p \equiv d \mod N\}|.$$

**Theorem 2.** For $X \geq 1$, one has

$$\frac{1}{|C(X)|} \sum_{E \in C(X)} (\pi_E(X; N, C) - \frac{|C|\varphi(N)}{|GL_2(\mathbb{Z}/N\mathbb{Z})|}\pi(X; N, d))^2 \ll N^8 X,$$

where $\varphi(N)$ denotes the Euler-phi function, and the implied constant is absolute.

In [7], Duke proves this (without the $N^8$ factor) for prime level $N$ and the where the conjugacy class $C$ is replaced by a set of the form

$$G_{t,d} := \{A \in GL_2(\mathbb{Z}/N\mathbb{Z}) : \text{tr} A = t, \text{det} A = d\}$$

Such sets are unions of conjugacy classes. For example, even when $N$ is prime, the set $G_{2\lambda,\lambda^2}$ contains two conjugacy classes, represented by the matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

respectively. Theorem 2 distinguishes between these two cases.

Our second result is an application of this theorem to the problem of counting elliptic curves $E$ for which $\phi_E(G_Q)$ is as large as possible. First of all, how large can this image be? Does there exist an elliptic curve $E$ with $\phi_E$ surjective? In other words, is there a curve $E$ with $n_E = 1$? Serre [15] answers no. For each elliptic curve $E$, there is an index two subgroup $H_E \subseteq GL_2(\mathbb{Z})$ (for a precise definition, see section 5) so that

$$\phi_E(G_Q) \subseteq H_E.$$  \hspace{1cm} (4)

**Definition 3.** We call an elliptic curve $E$ a Serre curve when equality holds in (4).

Our second theorem is

**Theorem 4.** Let $C_{\text{Serre}}(X)$ denote the set

$$\{E \in C(X) : E \text{ is a Serre curve} \}.$$ 

Then,

$$\lim_{X \to \infty} \frac{|C_{\text{Serre}}(X)|}{|C(X)|} = 1.$$ 

In order to obtain this result that “almost all elliptic curves are Serre curves”, we prove an algebraic lemma which gives a sufficient condition on an elliptic curve $E$ to be a Serre curve.

**Lemma 5.** Suppose $E$ over $\mathbb{Q}$ is an elliptic curve such that

1. $E$ has no exceptional primes.
2. $E$ is not exceptional at 4 or 9.
3. The index $[GL_2(\mathbb{Z}/8\mathbb{Z}) : \phi_8(E(G_\mathbb{Q}))] \neq 2$.
4. There is a prime number $p > 3$ which divides the Serre number $M_{\Delta, f(E)}$ (For the definition of the Serre number $M_{\Delta, f(E)}$, see section 5).

Then, $E$ is a Serre curve.

This lemma is used together with Theorem 2 to give Theorem 4. In a subsequent paper we plan to use Theorem 4 to compute the average value over elliptic curves of the Lang-Trotter constants, answering a question of David and Pappalardi [5].

The paper is organized as follows: in section 4 we prove Theorem 2. Section 5 gives the complete definition of a Serre curve, and section 6 devotes itself to a proof of Lemma 5. Finally in section 7 we prove Theorem 4 and in section 8 we produce an example of a one-parameter family of elliptic curves which are exceptional at $N = 4$ but not at $N = 2$.

## 4 Bounding Mean-Square Chebotarev error.

In this chapter we prove Theorem 2. We first remark that although it gives a bound as strong as the appropriate Generalized Riemann Hypothesis, the proof is unconditional. It employs the following large sieve inequality of Gallagher (see Lemma A of [10]) and follows along the same lines as the proof of Theorem 2 of [7].
Lemma 6. For each prime number $p$ let $\Omega(p) \subseteq (\mathbb{Z}/p\mathbb{Z})^k$ be any subset. For each fixed $m \in \mathbb{Z}^k$ we define

$$P(X; m) = \{p \leq X : m \equiv \Omega(p) \mod p\}$$

and

$$P(X) = \sum_{p \leq X} |\Omega(p)| p^{-k}.$$ 

Let $B$ be a box in $\mathbb{R}^k$ whose sides are parallel to the coordinate planes which has minimum width $W(B)$ and volume $V(B)$. If $W(B) \geq X^2$, then

$$\sum_{m \in B \cap \mathbb{Z}^n} (P(X; m) - P(X))^2 \ll_k V(B) P(X).$$

We will take $k = 2$ and define the set $\Omega_C(p) \subseteq (\mathbb{Z}/p\mathbb{Z})^2$. In any case (including the supersingular case), the ring $\text{End}_{\mathbb{F}_p}(E_p)$ is isomorphic to an imaginary quadratic order (see Theorem 4.2 of [19]), whose discriminant we denote by $\Delta = \Delta(E_p)$. The comparison of discriminants yields

$$\Delta b^2 = a^2 - 4p.$$ (7)

We associate to $E_p$ the following matrix of trace $a$ and determinant $p$:

$$\sigma(E_p) = \begin{pmatrix} (a + b\delta)/2 & b \\ b(\Delta - \delta)/4 & (a - b\delta)/2 \end{pmatrix}$$ (8)

where for a discriminant $\Delta$ we have $\delta = 0, 1$ according to whether $\Delta \equiv 0, 1 \mod 4$. Because of (7), $\sigma$ has integer entries.
Theorem 7. Suppose the elliptic curve \( E_p \) over \( \mathbb{Z}/p\mathbb{Z} \) is the reduction modulo \( p \) of an elliptic curve \( E \) over \( \mathbb{Q} \) and \( p \) is prime to \( N \). Then \( p \) is unramified in \( \mathbb{Q}(E[N]) \) and the integral matrix \( \sigma \) defined in (8), when reduced modulo \( N \), represents the class of the Frobenius of \( p \) in \( \text{Gal}(\mathbb{Q}(E[N])/\mathbb{Q}) \).

Now suppose \( p > 3 \) is a prime number and for \((r, s) \in \mathbb{F}_p^2\), let \( E_{r,s} \) denote the curve given by the equation
\[
y^2 = x^3 + rx + s,
\]
and \( \Delta_{r,s} = -16(4r^3 + 27s^2) \) the associated discriminant. For any conjugacy class \( C \subset \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \) we define \( \Omega_C(2) = \Omega_C(3) = \emptyset \) and for \( p > 3 \),
\[
\Omega_C(p) := \{(r, s) \in \mathbb{F}_p^2 : \Delta_{r,s} \neq 0 \text{ and } \sigma(E_{r,s}) \mod N \in C\}.
\]
Since the discriminant \( \Delta_{r,s} \) of the curve \( E_{r,s} \) is related to its minimal discriminant \( \Delta \) by
\[
\Delta_{r,s} = e^{12} \Delta
\]
for some \( e \) dividing 6, we see from Theorem 7 that (5) holds. We now turn to verifying (6).

4.2 The asymptotic in \( p \) of \( |\Omega_C(p)| \).

The goal of this section is to give the asymptotic of \( |\Omega_C(p)| \) as \( p \) ranges through the set of prime numbers for which \( \Omega_C(p) \neq \emptyset \). Our proof will show that in fact,
\[
\Omega_C(p) \neq \emptyset \iff p \equiv \det C \mod N.
\]

Theorem 8. For \( p \) prime congruent to \( \det C \) modulo \( N \) we have
\[
|\Omega_C(p)| = \frac{|C| \varphi(N)}{|\text{GL}_2(\mathbb{Z}/N\mathbb{Z})|} p^2 + O(N^5 p^{3/2})
\]
where the implied constant is absolute.

We observe that (8) follows upon partial summation. Thus, Theorem 2 will follow from Theorem 8.

To prove Theorem 8 we first express \( |\Omega_C(p)| \) in terms of a weighted class number. Define the set
\[
\mathcal{T}_C(p) := \{A \in M_{2 \times 2}(\mathbb{Z}) : \det A = p, A \mod N \in C\},
\]
and the subset of elliptic matrices
\[
\mathcal{E}_C(p) := \{A \in \mathcal{T}_C(p) : (\text{tr } A)^2 - 4 \det A < 0\}.
\]
Since \( C \) is stable by \( \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \)-conjugation, both of the above sets are stable by \( \text{SL}_2(\mathbb{Z}) \)-conjugation.

Note: Throughout the rest of this paper we will use the standard notation
\[
\Gamma(1) := \text{SL}_2(\mathbb{Z}).
\]
Proposition 9.
\[ |\Omega_C(p)| = \frac{p-1}{2} \sum_{\alpha \in \mathcal{T}_C(p) // \Gamma(1)} \frac{1}{|\Gamma(1)\alpha|}, \]
where \( \mathcal{T}_C(p) // \Gamma(1) \) is the set of \( \Gamma(1) \)-conjugation orbits in \( \mathcal{T}_C(p) \) and
\[ \Gamma(1)\alpha := \{ \gamma \in \Gamma(1) : \gamma \alpha = \alpha \gamma \}. \]

This proposition, together with the following lemma, imply Theorem 8.

Lemma 10. If \( p \equiv \det C \mod N \) then
\[ \sum_{\alpha \in \mathcal{T}_C(p) // \Gamma(1)} \frac{1}{|\Gamma(1)\alpha|} = \frac{|C|}{|\text{SL}_2(\mathbb{Z}/N\mathbb{Z})|} p + O(N^5 p^{1/2}), \]
with an absolute constant.

Proof. This is Corollary 8 of [12].

The remainder of this section is devoted to proving Proposition 9. We note that
\[ \Omega_C(p) = \{(r, s) \in (\mathbb{Z}/p\mathbb{Z})^2 : \Delta_{r,s} \neq 0 \text{ and } \sigma(E_{r,s}) \in \mathcal{T}_C(p)\}. \]

At this point we must give a finer description of \( C \). For any divisor \( M \) of \( N \) and integers \( T, D \mod \frac{N}{M} \), define
\[ \mathcal{T}_{N/M}(T, D) = \{ A \in M_{2 \times 2}(\mathbb{Z}/(N/M)\mathbb{Z}) : (\text{tr} A, \text{det} A) \equiv (T, D) \mod N/M \} \]
and
\[ \mathcal{T}^*_{N/M}(T, D) = \{ A \in \mathcal{T}_{N/M}(T, D) : A \text{ is non-scalar mod each prime } l \mid N/M \}. \]

The following lemma is a corollary of Proposition 7 of [12] describing the structure of conjugacy classes in the group \( \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \).

Lemma 11. Any conjugacy class
\[ \mathcal{C} \subset \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \]
has the form
\[ \mathcal{C} = \lambda I + M \mathcal{T}^*_{N/M}(T, D), \]
where \( \lambda \) is an integer satisfying \( 0 \leq \lambda < M \).

We would like to partition \( \mathcal{T}_C(p) \) into subsets which are stable by \( \Gamma(1) \)-conjugation. Let \( \mathcal{T}^*(T, D, f) \) denote
\[ \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{Z}) : \text{tr } A = T, \text{det } A = D, \gcd(b, d-a, c) = f \}. \]
We note then that the trace $t$ and determinant $d$ of any matrix in the set $\lambda I + M T^*(T, D, f)$ satisfy

$$t = 2\lambda + MT, \quad d = \lambda^2 + M\lambda T + M^2 D, \quad \text{and} \quad t^2 - 4d = M^2 (T^2 - 4D). \quad (10)$$

Thus, from Lemma 11 we see that

$$\mathcal{C}_\mathcal{C}(p) = \bigsqcup_{(T, D)} \bigsqcup_{f \geq 1 \text{ gcd}(f, N/M) = 1} (\lambda I + M T^*(T, D, f)), \quad (11)$$

where $(T, D)$ runs over integer pairs satisfying $(T, D) \equiv (T, D) \mod N/M, p = \lambda^2 + M\lambda T + M^2 D$, and $(2\lambda + MT)^2 < 4p$.

Defining $\Omega^*(\lambda, M, T, D, f)$ by

$$\{(r, s) \in \mathbb{Z}/p\mathbb{Z} : \Delta_{r,s} \neq 0 \text{ and } \sigma(E_{r,s}) \in \lambda I + M T^*(T, D, f)\},$$

Proposition 9 is reduced to showing that

$$|\Omega^*(\lambda, M, T, D, f)| = \frac{p-1}{2} \sum_{a \in (\lambda I + M T^*(T, D, f))//\Gamma(1) \alpha} \frac{1}{|\Gamma(1)\alpha|}. \quad (11)$$

**Lemma 12.** $\Omega^*(\lambda, M, T, D, f)$ is equal to

$$\{(r, s) \in (\mathbb{Z}/p\mathbb{Z})^2 : \Delta_{r,s} \neq 0, b(E_{r,s}) = Mf \text{ and } a(E_{r,s}) = 2\lambda + MT\}.$$

**Proof.** The containment “$\Omega^*(\lambda, M, T, D, f) \subseteq \ldots$” is immediate from (8) and (10). The reverse containment comes from the fact that, for fixed $t$ and $p$, the two equations

$$t = 2\lambda + MT \quad \text{and} \quad p = \lambda^2 + M\lambda T + M^2 D$$

have a unique solution $(\lambda, T, D) \in \{0, 1, \ldots, M - 1\} \times \mathbb{Z}^2$, if they have one at all. This fact is immediate when $M$ is odd. If $M$ is even, we see from the first equation that the only way two distinct solutions can exist is if one solution looks like $(\lambda, T, D)$ with $\lambda \in \{0, 1, \ldots, M/2 - 1\}$ and the other solution has the form $(\lambda + M/2, T - 1, D')$ for some integer $D'$. But then the second equation gives us the contradiction that

$$\lambda^2 + M\lambda T - p \equiv 0 \mod M^2 \quad \text{and} \quad \lambda^2 + M\lambda T - p \equiv \frac{M^2}{4} (1 - 2T) \mod M^2.$$
\( K \) which contains a basis of \( K \) over \( \mathbb{Q} \) and has rank 2 as a free abelian group. For each negative number \( \Delta \) satisfying
\[
\Delta \equiv 0 \text{ or } 1 \quad \text{mod } 4,
\]
there is a unique imaginary quadratic order of discriminant \( \Delta \), which we will denote by \( \mathcal{O}(\Delta) \). Orders \( \mathcal{O}(\Delta') \) which contain a given order \( \mathcal{O}(\Delta) \) are exactly those orders whose discriminant \( \Delta' \) satisfies
\[
f^2 \Delta' = \Delta, \quad f = [\mathcal{O}(\Delta') : \mathcal{O}(\Delta)].
\]
Every imaginary quadratic order \( \mathcal{O} \) is contained in a unique maximal imaginary quadratic order
\[
\mathcal{O} \subseteq \mathcal{O}_{\text{max}} = \mathcal{O}_K \subset K,
\]
which is the ring of integers of \( K \). The ideal class group \( \mathcal{C}(\mathcal{O}) \) is the group of invertible fractional ideals of \( \mathcal{O} \) modulo the subgroup of principal fractional ideals. This is a finite group whose size we denote by \( h(\mathcal{O}) \).

**Lemma 13.** Suppose \( p \geq 5 \) is prime and \( t \) is any integer satisfying \( t^2 < 4p \). Let \( \mathcal{O} \) be any imaginary quadratic order containing the order of discriminant \( t^2 - 4p \). The number of elliptic curves \( E_{r,s} \) over \( \mathbb{F}_p \) of the form \( (9) \) which satisfy
\[
\text{tr}(\phi_p) = t \quad \text{and} \quad \text{End}_{\mathbb{F}_p}(E_{r,s}) = \mathcal{O}
\]
is given by
\[
\frac{p - 1}{|\mathcal{O}^*|} h(\mathcal{O}),
\]
where \( \mathcal{O}^* \) is the group of units of \( \mathcal{O} \).

**Proof.** The following theorem restates Theorems 4.2 and 4.5 of [19], specialized to our situation. See also [14], which corrects a small error in the original proof. The original work is due to Deuring [6].

**Theorem 14.** Let \( t \) be any integer satisfying \( t^2 < 4p \). Then the following are precisely the rings which occur as rings of \( \mathbb{F}_p \)-endomorphisms of some elliptic curve \( E_p \) over \( \mathbb{F}_p \) satisfying \( a(E_p) = t \):

- \( t \neq 0 \): all complex quadratic orders containing \( \mathcal{O}(t^2 - 4p) \);
- \( t = 0 \): all complex quadratic orders \( \mathcal{O} \) satisfying
\[
\mathcal{O}(-4p) \subset \mathcal{O} \quad \text{and} \quad p \nmid [\mathcal{O}_{\text{max}} : \mathcal{O}].
\]

Furthermore, given such an order \( \mathcal{O} \), the number of \( \mathbb{F}_p \)-isomorphism classes of elliptic curves \( E_p \) over \( \mathbb{F}_p \) satisfying
\[
a(E_p) = t \quad \text{and} \quad \text{End}_{\mathbb{F}_p}(E_p) = \mathcal{O}
\]
is equal to \( h(\mathcal{O}) \).
Note that, since \( p \geq 5 \), every \( \mathbb{F}_p \)-isomorphism class contains an elliptic curve of the form (9). By the theorem, the proof of Lemma 13 is reduced to showing that whenever \( E_{r,s} \) is the form (9) with \( \text{tr} \phi_p = t \) and \( \text{End}_{\mathbb{F}_p}(E_{r,s}) = \mathcal{O} \), the number of elliptic curves of the same form which are isomorphic over \( \mathbb{F}_p \) to \( E_{r,s} \) is \( (p - 1)/|\mathcal{O}^*| \). Such elliptic curves are exactly those given by the equations

\[
E_{ru^4, su^6} : y^2 = x^3 + ru^4 x + su^6, \quad u \in (\mathbb{Z}/p\mathbb{Z})^*.
\]

In case \( |\mathcal{O}^*| = 2 \), the \( j \)-invariant \( j(E_{r,s}) \) cannot be equal to 0 or 1728, i.e. neither \( r \) nor \( s \) can be equal to zero. In this case, \( E_{ru^4, su^6} = E_{r(u')^4, s(u')^6} \) if and only if \( u = \pm u' \) and we count exactly \( (p - 1)/2 \) distinct \( E_{ru^4, su^6} \)'s. The case \( |\mathcal{O}^*| = 4 \) occurs exactly when \( \mathcal{O} = \mathcal{O}(-4) = \mathbb{Z}[i] \) is the ring of Gaussian integers, and this happens only if \( j(E_{r,s}) = 1728 \) and \( s = 0 \). Since \( \mathcal{O}(t^2 - 4p) \subset \mathcal{O}(-4) \), we see by relating the discriminants that \( t \) must be even and that \( p \equiv 1 \mod 4 \).

Choosing \( i_p \in (\mathbb{Z}/p\mathbb{Z})^* \) satisfying \( i_p^2 = -1 \), we note that in this case \( E_{ru^4, su^6} = E_{r(u')^4, s(u')^6} \) if and only if \( u/u' \in \{ \pm ip, \pm 1 \} \), and so there are again exactly \( (p - 1)/|\mathcal{O}^*| \) elliptic curves of the form (9) isomorphic over \( \mathbb{F}_p \) to \( E_{r,s} \). The case \( j(E_{r,s}) = 0 \) case is quite similar, so we omit it. This finishes the proof of Lemma 13.

Returning to the verification of (11), we see by the two lemmas and (10) that

\[
|\Omega^*(\lambda, M, T, D, f)| = \frac{p - 1}{|\mathcal{O}\left(\frac{T^2 - 4D}{f^2}\right)|} h\left(\mathcal{O}\left(\frac{T^2 - 4D}{f^2}\right)\right).
\]

Now we use a theorem which equates the counting of weighted \( \Gamma(1) \)-orbits of matrices of a fixed trace and determinant (of negative discriminant) with the counting of weighted ideal classes in the imaginary quadratic order of the same discriminant. We denote by \( Q^*(\Delta) \) the set of primitive integral binary quadratic forms of discriminant \( \Delta \) and \( Q^+\Delta) \) the subset of positive definite forms, both acted on by the classical \( \Gamma(1) \)-action

\[
f : \left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right)(x, y) = f(ax + by, cx + dy).
\]

By \( Q^*(\Delta) // \Gamma(1) \) and \( Q^+\Delta) // \Gamma(1) \) we denote the corresponding orbit spaces under this action.

**Theorem 15.** Let \( T \) and \( D \) be integers and \( f \) a positive integer satisfying

\[
T^2 - 4D < 0 \quad \text{and} \quad \frac{T^2 - 4D}{f^2} \in \mathbb{Z}, \quad \frac{T^2 - 4D}{f^2} \equiv 0, 1 \mod 4.
\]

Then there are set bijections

\[
T^*(T, D, f) // \Gamma(1) \longleftrightarrow Q^*\left(\frac{T^2 - 4D}{f^2}\right) // \Gamma(1)
\]
and

\[ Q_+^* \left( \frac{T^2 - 4D}{f^2} \right) \big/ \Gamma(1) \hookrightarrow \mathcal{C} \left( \mathcal{O} \left( \frac{T^2 - 4D}{f^2} \right) \right), \]

Proof. We first observe that, whenever \( T^*(T, D, f) \neq \emptyset \) (which is equivalent to the three given conditions), there are unique integers \( T', D' \) and \( \lambda \in \{0, 1, \ldots, f-1\} \) so that

\[ T^*(T, D, f) = \lambda I + fT^*(T', D', 1). \]

Since \( T^2 - 4D = f^2((T')^2 - 4D') \), the first bijection in the theorem is induced by the bijection

\[ T^*(T', D', 1) \leftrightarrow Q^* \left( (T')^2 - 4D' \right) \]

given by sending the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) to the form \( cx^2 + (d-a)xy - by^2 \) and the form \( \alpha x^2 + \beta xy + \gamma y^2 \) to the matrix \( \begin{pmatrix} (t-\beta)/2 & -\gamma \\ \alpha & (t-\beta)/2 \end{pmatrix} \). The second bijection is Theorem 7.7 in [3].

We observe that for any matrix \( \alpha \in T^*(T, D, f) \), we have

\[ |\Gamma(1)_{\alpha}| = |\mathcal{O} \left( \frac{T^2 - 4D}{f^2} \right)^*|, \]

and the common value can be greater than 2 only when \( \frac{T^2 - 4D}{f^2} \in \{-3, -4\} \), in which case \( h(\mathcal{O} \left( \frac{T^2 - 4D}{f^2} \right)) = 1 \). We conclude:

Corollary 16.

\[ \frac{2}{|\mathcal{O} \left( \frac{T^2 - 4D}{f^2} \right)^*|} h \left( \mathcal{O} \left( \frac{T^2 - 4D}{f^2} \right) \right) = \sum_{\alpha \in \lambda I + MT^*(T, D, f)/\Gamma(1)} \frac{1}{|\Gamma(1)_{\alpha}|}. \]

By the corollary, (11) follows and we have proved Proposition 9.

5 The definition of a Serre curve.

We now describe the subgroup \( H_E \) mentioned in Definition 3 following the discussion proceeding Proposition 22 of [15]. Suppose that \( E \) is given by the equation

\[ y^2 = x^3 + rx + s = (x - e_1)(x - e_2)(x - e_3). \]

Then \( \{e_1, e_2, e_3\} \) is the set of \( x \)-coordinates of the non-trivial 2-torsion of \( E \). The discriminant \( \Delta \) of this model of \( E \) is given by

\[ \Delta = ((e_1 - e_2)(e_1 - e_3)(e_2 - e_3))^2. \]

Thus, one has

\[ \mathbb{Q}(\sqrt{\Delta}) \subset \mathbb{Q}(E[2]). \]
Because of the action of $\text{Aut } E[2] \simeq GL_2(\mathbb{Z}/2\mathbb{Z})$ on the $e_i$'s we have a group isomorphism between $GL_2(\mathbb{Z}/2\mathbb{Z})$ and the symmetric group on three letters:

$$GL_2(\mathbb{Z}/2\mathbb{Z}) \simeq S_3.$$  

By (12) we see that for any Galois automorphism $\sigma \in \text{Gal } (\mathbb{Q}(E[2]) / \mathbb{Q}) \subset S_3$,  

$$\sigma : \sqrt{\Delta} \mapsto \varepsilon(\sigma)\sqrt{\Delta},$$  

where $\varepsilon$ denotes the signature character on $S_3$. In particular we note that if $\sqrt{\Delta} \in \mathbb{Q}$ then  

$$\text{Gal } (\mathbb{Q}(E[2]) / \mathbb{Q}) \subset A_3 = \text{ the alternating group on 3 letters.}$$  

In this case we define the Serre number $M_1$ to be 2 and the Serre subgroup $H_2$ by  

$$H_2 := A_3.$$  

Otherwise, $\mathbb{Q}(\sqrt{\Delta})$ is a quadratic extension, which in particular is abelian. Since each abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension, one may choose a positive integer $D$ so that  

$$\mathbb{Q}(\sqrt{\Delta}) \subset \mathbb{Q}(\zeta_D) \subset \mathbb{Q}(E[D]),$$  

where as usual $\zeta_D$ denotes a primitive $D$-th root of unity and the second containment comes from the Weil pairing (c.f. [18], for example).

**Lemma 17.** Let $W$ be any square-free integer and define the positive integer $D_W$ by  

$$D_W = \begin{cases}  |W| & \text{if } W \equiv 1 \mod 4 \\ 4|W| & \text{otherwise}. \end{cases}$$  

Then we have  

$$\mathbb{Q}(\sqrt{W}) \subset \mathbb{Q}(\zeta_D) \Leftrightarrow D_W \text{ divides } D.$$  

Furthermore, for such a $D$ and $\sigma \in \text{Gal } (\mathbb{Q}(E[D]) / \mathbb{Q}) \subseteq GL_2(\mathbb{Z}/2\mathbb{Z})$, we have  

$$\sigma : \sqrt{W} \mapsto \left( \frac{W}{\det \sigma} \right) \sqrt{W}.$$  

(14)  

Here we use the Kronecker symbol $\left( \frac{W}{.} \right) := \left( \frac{W/|W|}{.} \right) \cdot \prod_{p|W} \left( \frac{p}{.} \right)$, where  

$$\left( \frac{2}{.} \right) := (-1)^{((.)^2-1)/8}, \quad \text{and} \quad \left( \frac{\pm 1}{.} \right) = (\pm 1)^{((.)-1)/2}.$$  

Proof. These are standard results from algebraic number theory, together with Theorem 6.6 of [17].
By the lemma we see that
\[ \mathbb{Q}(\sqrt{\Delta}) \subset \mathbb{Q}(\zeta_D) \iff D_{\Delta, sf} \text{ divides } D, \]  
where \( \Delta_{sf} = \Delta_{sf}(E) \) is the square-free part of the discriminant \( \Delta \) of \( E \). For any square-free number \( W \) we define the “Serre number”
\[ M_W = \begin{cases} 2|W| & \text{if } W \equiv 1 \mod 4 \\ 4|W| & \text{otherwise} \end{cases}, \]
to be the least common multiple of 2 and \( D_W \). Thus in particular, \( \mathbb{Q}(E[M_{\Delta, sf}]) \) is the compositum of \( \mathbb{Q}(E[2]) \) and \( \mathbb{Q}(E[D_{\Delta, sf}]) \). We furthermore define the subgroup \( H_{MW} \) by
\[ H_{MW} = \ker \left( \left( \frac{W}{\det(\cdot)} \right) \varepsilon(\cdot) \right) \subset GL_2(\mathbb{Z}/M_W\mathbb{Z}), \]
where here we have extended the definition of the signature character \( \varepsilon \) in the natural way to any even level:
\[ \varepsilon : GL_2(\mathbb{Z}/2m\mathbb{Z}) \to GL_2(\mathbb{Z}/2\mathbb{Z}) \to \{\pm 1\}. \]
Later in the paper we will casually refer to “\( \ker \varepsilon \)”, hoping that in each instance its domain will be clear from context.

By virtue of (13) and (14), we see that
\[ \text{Gal} \left( \mathbb{Q}(E[M_{\Delta, sf}])/\mathbb{Q} \right) \subseteq H_{M_{\Delta, sf}}. \]

The subgroup \( H_E \) of \( GL_2(\hat{\mathbb{Z}}) \) referred to in (4) is simply
\[ H_E = \pi_{M_{\Delta, sf}}^{-1}(H_{M_{\Delta, sf}}), \]
where \( \pi_{M_{\Delta, sf}} : GL_2(\hat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/M_{\Delta, sf}\mathbb{Z}) \) is the natural projection. \( H_E \) is evidently an index 2 subgroup of \( GL_2(\hat{\mathbb{Z}}) \) and
\[ \phi_E(G_\mathbb{Q}) \subseteq H_E. \]

An elliptic curve \( E \) is a Serre curve if \( \phi_E(G_\mathbb{Q}) = H_E \). In other words, an elliptic curve is a Serre curve exactly when, for every integer \( m \), we have
\[ [GL_2(\mathbb{Z}/m\mathbb{Z}) : \phi_{m,E}(G_\mathbb{Q})] = \begin{cases} 2 & \text{if } M_{\Delta, sf}(E) \mid m \\ 1 & \text{otherwise.} \end{cases} \]

We will refer to \( H_{M_{\Delta, sf}(E)} \subset GL_2(\mathbb{Z}/M_{\Delta, sf}(E)\mathbb{Z}) \) (and by abuse of notation, also to \( H_E \subset GL_2(\hat{\mathbb{Z}}) \)) as the “Serre subgroup associated to \( E \).”
6 Which elliptic curves are Serre curves?

If \( N \) is exceptional for \( E \) (see Definition 1) then so is any multiple of \( N \).

**Definition 18.** We call an integer \( N \) minimal exceptional for \( E \) if it is exceptional for \( E \) and none of its proper nontrivial divisors are exceptional for \( E \).

For example, if \( E \) is a Serre curve, then the Serre number \( M_{\Delta,E}(E) \) (see section 5) is a minimal exceptional number for \( E \). Also, any exceptional prime \( p \) of \( E \) is minimal exceptional.

The proof of Lemma 5 uses only the theory of the groups \( GL_2(\mathbb{Z}/N\mathbb{Z}) \) (especially for \( N \) divisible by 2 and 3, complimenting [16]) as well as a few facts about cyclotomic fields. The arguments are similar to those given in Kani’s appendix to [2]. Two separate issues arise: (1) which numbers \( N \) can actually occur as minimal exceptional numbers for an elliptic curve and (2) the stability of the Serre number \( M_{\Delta,E}(E) \). We treat them in that order.

We will make repeated use of

**Lemma 19.** The commutator subgroup \((GL_2(\mathbb{Z}/p^n\mathbb{Z}))'\) of \( GL_2(\mathbb{Z}/p^n\mathbb{Z}) \) is given by

\[
(GL_2(\mathbb{Z}/p^n\mathbb{Z}))' = \begin{cases} 
SL_2(\mathbb{Z}/p^n\mathbb{Z}) & \text{if } p \neq 2 \\
\ker(\varepsilon) \cap SL_2(\mathbb{Z}/2^n\mathbb{Z}) & \text{if } p = 2
\end{cases}
\]

(see [16].) For \( p \geq 5 \), the group \( SL_2(\mathbb{Z}/p^n\mathbb{Z}) \) is equal to its own commutator:

\[
(SL_2(\mathbb{Z}/p^n\mathbb{Z}))' = SL_2(\mathbb{Z}/p^n\mathbb{Z}) \quad (p \geq 5).
\]

6.1 Minimal exceptional numbers of elliptic curves.

The following lemma gives us a restriction on which positive integers \( N \) can occur as a minimal exceptional number of an elliptic curve. Throughout the remainder of the paper we will sometimes use the abbreviation

\[ G_N := Gal(Q(E[N])/Q), \]

suppressing the dependence on the elliptic curve \( E \).

**Lemma 20.** Let \( E \) be an elliptic curve over \( Q \). Suppose that \( N \in \mathbb{N} \) is minimal exceptional for \( E \). Then,

\[ N \in \{ \text{prime numbers} \} \cup \{ M_{\Delta,E}(E) \} \cup \{ 4, 8, 9 \}. \]

If 8 is a minimal exceptional number for \( E \), then

\[ [GL_2(\mathbb{Z}/8\mathbb{Z}) : \phi_{E,8}(G_\mathbb{Q})] = 2. \]
Proof. Let us assume that \( N \) is not prime. If \( N \) is exceptional for \( E \) then we have
\[
G_N \subseteq GL_2(\mathbb{Z}/N\mathbb{Z}).
\]
If \( N \) is minimal exceptional, we have \( G_d = GL_2(\mathbb{Z}/d\mathbb{Z}) \) for each proper divisor \( d \) of \( N \). Therefore the canonical map
\[
G_N \twoheadrightarrow GL_2(\mathbb{Z}/d\mathbb{Z}), \quad \forall d \mid N
\]
is a surjection. By the surjectivity of the Weil pairing, we also see that the determinant map
\[
det : G_N \twoheadrightarrow (\mathbb{Z}/N\mathbb{Z})^*
\]
is surjective. We consider the question: for which composite numbers \( N \) does there exist a subgroup \( G_N \) of \( GL_2(\mathbb{Z}/N\mathbb{Z}) \) satisfying conditions (17) and (18)?

We divide the investigation into cases according to whether \( N \) is a prime power or not. We tackle the latter case first:

**Case 1**: \( N \) is not a prime power. Let \( p \) be the smallest prime divisor of \( N \). Suppose \( p^n \mid|N \) and write \( M := N/p^n \neq 1 \). By Galois theory we must have
\[
Q \subseteq Q(E[p^n]) \cap Q(E[M]) =: F
\]
Let \( H = \text{Gal}(F/\mathbb{Q}) \). If \( H \) is not simple, replace it by any nontrivial simple quotient, and replace \( F \) by the corresponding field. The next lemma is a corollary of the discussion on page IV-25 of [16].

**Lemma 21.** If \( N_1 \) and \( N_2 \) are relatively prime positive integers than the groups \( GL_2(\mathbb{Z}/N_1\mathbb{Z}) \) and \( GL_2(\mathbb{Z}/N_2\mathbb{Z}) \) have no common simple nonabelian quotient.

Since \( H \) is a common quotient of the groups
\[
\text{Gal}(Q(E[M])/\mathbb{Q}) = GL_2(\mathbb{Z}/M\mathbb{Z}) \quad \text{and} \quad \text{Gal}(Q(E[p^n])/\mathbb{Q}) = GL_2(\mathbb{Z}/p^n\mathbb{Z})
\]
we conclude that \( H \) is abelian. From this and Lemma [19] it follows that
\[
F \subset Q(\zeta_M).
\]
If \( p > 2 \) then we must similarly have \( F \subset Q(\zeta_{p^n}) \). Since
\[
Q(\zeta_M) \cap Q(\zeta_{p^n}) = Q,
\]
we conclude that \( F = Q \), contradicting that \( H \) is nontrivial. Therefore we must have \( p = 2 \). But then using Lemma [19] we similarly conclude that
\[
Q \neq F \subset Q(\sqrt{\Delta_E}, \zeta_{2^n}) \cap Q(\zeta_M).
\]
If \( n \leq 1 \) then we must have \( F = Q(\sqrt{\Delta_E}) \), and we see that \( N \) is a multiple of the Serre number \( M_{\Delta_{E_f}(E)} \). If \( n \geq 2 \) then we reason as follows: since the Galois group \( \text{Gal}(Q(\sqrt{\Delta_E}, \zeta_{2^n})/\mathbb{Q}) \) has order a power of two, \( F \) must contain a
quadratic subfield $K$. By (19), we conclude that if $n = 2$, $K$ must be one of the fields

$$\mathbb{Q}(\sqrt{\Delta_E}), \mathbb{Q}(\sqrt{-\Delta_E})$$

and if $n \geq 3$ that $K$ must be one of the fields

$$\mathbb{Q}(\sqrt{\Delta_E}), \mathbb{Q}(\sqrt{-\Delta_E}), \mathbb{Q}(\sqrt{2\Delta_E}), \mathbb{Q}(\sqrt{-2\Delta_E}).$$

Thus in any case by (15), $N$ is a multiple of the Serre number of $E$, which implies that $N$ is the Serre number of $E$, since $N$ is assumed to be minimal exceptional. We have shown that the Serre number of $E$ is the only minimal exceptional number which is not a prime power.

**Case 2:** $N = p^n$ is a prime power with $n \geq 2$. If $p \geq 5$, we reason as follows:

Taking commutators of (17) we have a surjection

$$(G_{p^n}(E))' \to SL_2(\mathbb{Z}/p^{n-1}\mathbb{Z}) = (GL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})).$$

By Lemma 3 on page IV-23 of [16], this implies that $(G_{p^n}(E))' = SL_2(\mathbb{Z}/p^n\mathbb{Z})$. But now since

$$SL_2(\mathbb{Z}/p^n\mathbb{Z}) \subset G_{p^n}(E)$$

we conclude by (18) that $G_{p^n}(E) = GL_2(\mathbb{Z}/p^n\mathbb{Z})$, contradicting the fact that $p^n$ is exceptional. We conclude that $p \in \{2, 3\}$.

Now consider the exact sequence

$$1 \to K \to GL_2(\mathbb{Z}/p^n\mathbb{Z}) \to GL_2(\mathbb{Z}/p^{n-1}\mathbb{Z}) \to 1.$$ 

Here $K = I + p^{n-1}M_2(\mathbb{Z}/p\mathbb{Z})$. Since $G_{p^n}$ surjects onto $GL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$, we have the exact sequence

$$1 \to K \cap G_{p^n} \to G_{p^n} \to GL_2(\mathbb{Z}/p^{n-1}\mathbb{Z}) \to 1.$$ 

First we show that if $n \geq 3$ then

$$I + p^{n-1}\{ \text{traceless matrices} \} \subseteq K \cap G_{p^n}. \quad (20)$$

This is seen by choosing any preimage

$$\begin{pmatrix} 1 & p^{n-2} \\ 0 & 1 \end{pmatrix} + p^{n-1} A \in G_{p^n}$$

of the matrix $\begin{pmatrix} 1 & p^{n-2} \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}/p^{n-1}\mathbb{Z})$ and observing that, if $n \geq 3$,

$$\left(\begin{pmatrix} 1 & p^{n-2} \\ 0 & 1 \end{pmatrix} + p^{n-1} A\right)^p \equiv \begin{pmatrix} 1 & p^{n-1} \\ 0 & 1 \end{pmatrix} \mod p^n, \quad (21)$$

which shows that the matrix $I + p^{n-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in K \cap G_{p^n}$. Now let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be any matrix in $GL_2(\mathbb{Z}/p\mathbb{Z})$ and choose a matrix $A \in G_{p^n}(E)$ with

$$A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod p.$$
We then have
\[
A(I + p^{n-1}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})A^{-1} = I + \frac{1}{ad - bc}p^{n-1}\begin{pmatrix} -ac & a^2 \\ -c^2 & ac \end{pmatrix} \in K \cap G_{p^n}(E).
\]

Letting the matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) vary modulo \( p \) we see that (20) holds. From this we see that \( G_{p^n} \) must be an index 2 subgroup of \( GL_2(\mathbb{Z}/p^n\mathbb{Z}) \). Thus, there is a character
\[
\chi : GL_2(\mathbb{Z}/p^n\mathbb{Z}) \to \{\pm 1\} \quad \text{with} \quad G_{p^n}(E) = \ker \chi.
\]
(22)

By Lemma 19, we see that if \( p = 3 \) then \( \chi \) factors through the determinant, i.e. that there is a character
\[
\delta : (\mathbb{Z}/3^n\mathbb{Z})^* \to \{\pm 1\}
\]
with \( \chi = \delta \circ \det \). This implies that \( SL_2(\mathbb{Z}/3^n\mathbb{Z}) \subset G_{3^n}(E) \), which says that \( G_{3^n}(E) = GL_2(\mathbb{Z}/3^n\mathbb{Z}) \) by (18), a contradiction. Thus, the only (non-prime) power of 3 which can be minimal exceptional for \( E \) is 9.

Now let us return to (22) with \( p = 2 \). In this case Lemma 19 says that either \( \chi \) or \( \chi \in \rangle \) factors through the determinant, according to whether
\[
\chi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = 1 \text{ or } -1.
\]
Now if \( \chi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = 1 \) then \( SL_2(\mathbb{Z}/2^n\mathbb{Z}) \subseteq G_{2^n} \), a contradiction. Thus we must have \( \chi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = -1 \). Therefore since \( \chi \in \rangle \) factors through the determinant, we have
\[
\chi = \varepsilon \cdot (\delta \circ \det),
\]
where \( \delta : (\mathbb{Z}/2^n\mathbb{Z})^* \to \{\pm 1\} \) is a character. Now pick \( X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + 2^{n-1}A \in G_{2^n} \). We have det \( X = 1 \) or \( 1 + 2^{n-1} \). One verifies by induction that for \( n \geq 3 \),
\[
1 + 2^{n-1} \equiv 5^{2^{n-3}} \mod 2^n,
\]
so for \( n > 3 \) we must have \( \delta(\det X) = 1 \), contradicting (22). We have shown that for any elliptic curve \( E \), if \( N \neq M_{\Delta_f}(E) \) is a composite minimal exceptional number for \( E \), then \( N = 4, 8 \) or 9, where if \( N = 8 \) there is a real character
\[
\chi : GL_2(\mathbb{Z}/8\mathbb{Z}) \to \{\pm 1\}
\]
with \( G_8(E) = \ker \chi \). This concludes the proof of Lemma 20. \( \square \)
6.2 Stability of the Serre number $M_{\Delta_{sf}}(E)$.  

Continuing the proof of Lemma 5, we will now show that under the assumptions stated therein, we have

$$G_{M_{\Delta_{sf}}}(E) = H_{M_{\Delta_{sf}}} = \text{the Serre subgroup}$$

and also that for each integer $N$ we have

$$G_N(E) = \begin{cases} \pi_{N,M_{\Delta_{sf}}}^{-1}(H_{M_{\Delta_{sf}}}) & \text{if } M_{\Delta_{sf}} \mid N \\ GL_2(\mathbb{Z}/N\mathbb{Z}) & \text{otherwise,} \end{cases}$$

where $\pi_{N,M_{\Delta_{sf}}}$ denotes the natural projection

$$GL_2(\mathbb{Z}/N\mathbb{Z}) \longrightarrow GL_2(\mathbb{Z}/M_{\Delta_{sf}}\mathbb{Z}).$$

We make use of the following technical lemma.

**Lemma 22.** Let $N > 1$ be any integer which is divisible by some prime $p \geq 5$ and by some prime $p < 5$. Write

$$N = N_1 \cdot N_2$$

where $N_1$ is not divisible by any prime $p \geq 5$ and $N_2$ is not divisible by any prime $p < 5$. Suppose that $G_a \subset GL_2(\mathbb{Z}/N\mathbb{Z})$ is a subgroup such that

$$G_a \cap SL_2(\mathbb{Z}/N\mathbb{Z}) = (GL_2(\mathbb{Z}/N\mathbb{Z}))'.$$

Finally, assume $G_b \subset G_a$ is a subgroup for which the canonical maps

$$G_b \rightarrow GL_2(\mathbb{Z}/N_1\mathbb{Z}) \quad \text{and} \quad G_b \rightarrow GL_2(\mathbb{Z}/N_2\mathbb{Z})$$

as well as the determinant map

$$\det : G_b \rightarrow (\mathbb{Z}/N\mathbb{Z})^*$$

are surjections. Then $G_b = G_a$.

**Proof.** Write $N_1 = 2^r3^s$. By (25), we find by taking commutators that

$$G_b' \rightarrow (GL_2(\mathbb{Z}/N_1\mathbb{Z}))(\ker \epsilon \cap SL_2(\mathbb{Z}/2^r\mathbb{Z})) \times SL_2(\mathbb{Z}/3^s\mathbb{Z})$$

and

$$G_b' \rightarrow (GL_2(\mathbb{Z}/N_2\mathbb{Z}))(\ker \epsilon \cap SL_2(\mathbb{Z}/2^r\mathbb{Z})) \times SL_2(\mathbb{Z}/3^s\mathbb{Z}).$$

are also surjections. We are now in a position to apply the Goursat lemma:

**Lemma 23.** Let $G_1$ and $G_2$ be groups. Denote by $\pi_i : G_1 \times G_2 \rightarrow G_i$ $(i = 1, 2)$ the projection map. Suppose that $G \subseteq G_1 \times G_2$ is a subgroup such that $\pi_i(G) = G_i$ for $i = 1, 2$ and define

$$H_1 = \pi_1(G \cap (G_1 \times \{e_2\})) \quad \text{and} \quad H_2 = \pi_2(G \cap (\{e_1\} \times G_2)).$$

Then,

$$G_1/H_1 \simeq G_2/H_2$$

and the graph of this isomorphism is induced by $G$. 

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We argue that the containment \( \subseteq \) and conclude that \( (\ker \varepsilon \cap SL_2(\mathbb{Z}/2^r\mathbb{Z})) \times SL_2(\mathbb{Z}/3^s\mathbb{Z}) \), \( G_2 = SL_2(\mathbb{Z}/N_2\mathbb{Z}) \), and \( G = G'_2 \) and conclude that \( (\ker \varepsilon \cap SL_2(\mathbb{Z}/2^r\mathbb{Z})) \times SL_2(\mathbb{Z}/3^s\mathbb{Z}) \) and \( SL_2(\mathbb{Z}/N_2\mathbb{Z}) \) have a common quotient group \( Q \). If \( Q \) is nontrivial then it has a nontrivial simple quotient \( Q_s \). Since Lemma \( 24 \) continues to hold with \( GL_2 \) replaced by \( SL_2 \), we see that \( Q_s \) must be abelian. Since \( (SL_2(\mathbb{Z}/N_2\mathbb{Z}))' = SL_2(\mathbb{Z}/N_2\mathbb{Z}) \), we conclude that \( Q_s = 1 \). This shows that \( Q \) was trivial to begin with. We conclude that \( G_1 = H_1 \) and \( G_2 = H_2 \), i.e. that

\[
(GL_2(\mathbb{Z}/N_1\mathbb{Z}))' \times \{1\} \subset G'_b \quad \text{and} \quad \{1\} \times (GL_2(\mathbb{Z}/N_2\mathbb{Z}))' \subset G'_b,
\]

which implies that

\[
G'_b = (GL_2(\mathbb{Z}/NZ))' .
\]

But now from the exact sequence

\[
1 \to (GL_2(\mathbb{Z}/NZ))' \to G_a \to (\mathbb{Z}/NZ)^* \to 1
\]

and

\[
\text{det} : G_b \to (\mathbb{Z}/NZ)^*
\]

we conclude that \( (GL_2(\mathbb{Z}/NZ))'/G_b = G_a \). So since \( (GL_2(\mathbb{Z}/NZ))' \subset G_b \), we have \( G_b = G_a \).

Now suppose that \( E \) is an elliptic curve over \( \mathbb{Q} \) which satisfies the hypotheses of Lemma \( 5 \). First we use Lemma \( 22 \) to show (23). This is done by applying the lemma with \( N = M_{\Delta,s} \). Set \( G_a = H_{M_{\Delta,s}} \) and \( G_b = G_{M_{\Delta,s}}(E) \). Write

\[
M_{\Delta,s} = 2^r 3^s M'
\]

where \( M' > 1 \) and is co-prime to 6. To see that

\[
H_{M_{\Delta,s}} \cap SL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z}) = (\ker \varepsilon \cap SL_2(\mathbb{Z}/2^r 3^s\mathbb{Z})) \times SL_2(\mathbb{Z}/M'\mathbb{Z}),
\]

we argue that the containment \( \subset \) follows from the definition of \( H_{M_{\Delta,s}} \). To see the reverse containment, we use isomorphism theorems from group theory and count:

\[
\frac{|H_{M_{\Delta,s}}|}{|H_{M_{\Delta,s}} \cap SL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z})|} = \frac{|GL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z})|}{|SL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z})|}.
\]

Now since the index \( |GL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z}) : H_{M_{\Delta,s}}| = 2 \) we see that

\[
|SL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z}) : H_{M_{\Delta,s}} \cap SL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z})| = 2.
\]

Since the index \( |SL_2(\mathbb{Z}/M_{\Delta,s}\mathbb{Z}) : (\ker \varepsilon \cap SL_2(\mathbb{Z}/2^r 3^s\mathbb{Z})) \times SL_2(\mathbb{Z}/M'\mathbb{Z})| \) is obviously equal to 2, we conclude that (26) holds.

We next verify the surjectivity conditions

\[
G_{M_{\Delta,s}}(E) \to GL_2(\mathbb{Z}/2^r 3^s\mathbb{Z}) \text{ and } G_{M_{\Delta,s}}(E) \to GL_2(\mathbb{Z}/M'\mathbb{Z}).
\]

(27)
Thus, if \( N \) dividing \( E \) and the assumptions on \( M \) and \( G \) preclude these possibilities. Therefore \( G_{\Delta f} E \to GL_2(\mathbb{Z}/2^3\mathbb{Z}) \) is surjective. Similarly, if \( G_{\Delta f} E \to GL_2(\mathbb{Z}/M'\mathbb{Z}) \) is not surjective, then \( E \) has some minimal exceptional \( d \) dividing \( M' \). By Lemma [20] we see that \( d \) cannot be even and so \( d \) cannot be equal to \( M_{\Delta f} E \). The assumption that \( E \) has no exceptional primes precludes the first possibility, and since \( 2 \nmid M' \), \( d \) cannot be even and so \( d \) cannot be equal to \( M_{\Delta f} E \), which is always even. We have verified the conditions (27). Finally, the surjectivity of

\[
\text{det} : G_{\Delta f} E \to (\mathbb{Z}/M_{\Delta f} \mathbb{Z})^*
\]

is the surjectivity of the Weil pairing. By Lemma [22] we conclude that

\[
G_{\Delta f} E = H_{M_{\Delta f}}.
\]

Now we verify (24). First let \( N \) be any positive integer and suppose \( \Delta f N \). If \( G_N(E) \nsubseteq GL_2(\mathbb{Z}/N\mathbb{Z}) \) then \( E \) has some minimal exceptional number \( d \) dividing \( N \). Clearly \( d \) cannot be equal to \( N \) or \( d \) cannot be equal to \( M_{\Delta f} E \), so again by Lemma [20] and the assumptions on \( E \) from Lemma [5] we arrive at a contradiction. Thus, if \( \Delta f N \) we have

\[
G_N(E) = GL_2(\mathbb{Z}/N\mathbb{Z}).
\]

Now suppose \( \Delta f N \). We apply Lemma [22] with \( G_a = \pi^{-1}_{N,M_{\Delta f}} H_{M_{\Delta f}} \), and \( G_b = G_N(E) \). The verification of the conditions on \( G_a \) and \( G_b \) are almost identical to those done in the previous paragraph, so we omit them. We conclude in this case that \( G_N(E) = \pi^{-1}_{N,M_{\Delta f}} H_{M_{\Delta f}} \).

We have shown that for any elliptic curve satisfying the hypotheses of Lemma [5] that (23) and (24) hold, and so our proof of Lemma [5] is now complete.

7 **Almost all elliptic curves are Serre curves.**

We now show how Lemma [5] and Theorem [2] together imply Theorem [4].

For \( N \in \{4, 6, 8, 9, 12, 24\} \) define

\[
\varepsilon_N(X) := \begin{cases} 
{E \in C(X) : E \text{ is minimal exceptional at } N} & \text{if } N \in \{4, 9\} \\
{E \in C(X) : [GL_2(\mathbb{Z}/8\mathbb{Z}) : G_8(E)] = 2} & \text{if } N = 8 \\
{E \in C(X) : G_N(E) \subseteq H_N} & \text{if } N \in \{6, 12, 24\}.
\end{cases}
\]

By Lemma [5] we have that the set of non-Serre curves \( C_{ns} X \subset C(X) \) satisfies

\[
C_{ns} X \subseteq \varepsilon(X) \cup \bigcup_{N \in \{4, 6, 8, 9, 12, 24\}} \varepsilon_N(X).
\]

By (2), to prove Theorem [4] it suffices to estimate the sets \( \varepsilon_N(X) \).
**Definition 24.** Let $W$ be any integer and let $(t, d) \in (\mathbb{Z}/W\mathbb{Z})^2$ be any pair of integers modulo $W$ with $d \in (\mathbb{Z}/W\mathbb{Z})^*$. Suppose that $G \subseteq GL_2(\mathbb{Z}/W\mathbb{Z})$ is any subgroup. We say that $G$ represents the pair $(t, d)$ if there is a matrix $g \in G$ satisfying

$$\text{tr}(g) = t, \quad \det(g) = d.$$  

The next two lemmas guarantee that when an elliptic curve fails to be a Serre curve by being exceptional at $N$, there must be some pair $(t, d)$ not represented by $G_N(E)$.

**Lemma 25.** Let $W > 2$ be any positive integer and let

$$\chi : GL_2(\mathbb{Z}/W\mathbb{Z}) \to \{\pm 1\}$$

be any nontrivial real character. Suppose that $G \subseteq \ker \chi$ is any subgroup. Then there exist integers $t$ and $d$ modulo $W$ so that the pair $(t, d)$ is not represented by $G$.

**Proof.** Lemma 19 implies that $\chi$ is either of the form $\delta \circ \det$ or $\varepsilon \cdot \delta \circ \det$, where $\delta$ is nontrivial and the second possibility may only occur if $W$ is even. Choose any $f \in (\mathbb{Z}/W\mathbb{Z})^*$ with $\delta(f) = -1$ and set $(t, d) = (1, f)$.

**Lemma 26.** Let $p = 2$ or $3$ and suppose $G \subseteq GL_2(\mathbb{Z}/p^2\mathbb{Z})$ is a subgroup which represents every trace-determinant pair $(t, d) \in (\mathbb{Z}/p^2\mathbb{Z}) \times (\mathbb{Z}/p^2\mathbb{Z})^*$ and which surjects onto $GL_2(\mathbb{Z}/p\mathbb{Z})$. Then, $G = GL_2(\mathbb{Z}/p^2\mathbb{Z})$.

**Proof.** We consider the intersection

$$G \cap K$$

of $G$ with $K = \ker$ of the projection

$$GL_2(\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow GL_2(\mathbb{Z}/p\mathbb{Z}).$$

Our goal is to show that $G$ actually contains $K$. From here we divide the argument into cases, according to whether $p$ is 2 or 3.

**Case:** $p = 3$. Under the given hypothesis, we may find a matrix $g \in G$ with $\text{tr} \, g = 3$ and $\det \, g = 1$. Such a matrix must have the form

$$X + 3Y, \quad X \in \left\{ \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \right\},$$
and the (mod 3) coefficients of the matrix \( Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) satisfy the conditions

- \( a + d = 1, b - c = 1 \) if \( X = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \)
- \( a + d = 1, b - c = 2 \) if \( X = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \)
- \( a + d = 0, a + b + c - d = 0 \) if \( X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \)
- \( a + d = 0, a - b - c - d = 2 \) if \( X = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \)
- \( a + d = 0, a - b - c - d = 0 \) if \( X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \)
- \( a + d = 0, a + b + c - d = 1 \) if \( X = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} \).

In each case, the first equation comes from the trace condition on \( g \) and the second one comes from the determinant condition. One computes:

\[(X + 3Y)^4 \equiv I + 3X \mod 9.\]

Since in this case the discriminant \( t^2 - 4d = 5 \) is nonzero modulo 3 we see by Lemma [11] that all six of the matrices \( X \), when reduced modulo 3, are \( GL_2(\mathbb{Z}/3\mathbb{Z}) \)-conjugate to one another. From this and the fact that the various \( X \) span the \( \mathbb{Z}/3\mathbb{Z} \)-vector space \( M_2(\mathbb{Z}/3\mathbb{Z}) \) we conclude that

\[G \cap K = I + 3M_2(\mathbb{Z}/3\mathbb{Z}),\]

i.e. that \( K \subseteq G \), and so \( G = GL_2(\mathbb{Z}/9\mathbb{Z}) \) in this case.

**Case:** \( p = 2 \). The proof in this case is similar. Pick \( g \in G \) with \( \text{tr} \, g = 2 \) and \( \det g = -1 \). Then \( g \) must have the form

\[g = X + 2Y, \quad X \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}, \quad Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\]

where the (mod 2) coefficients of the matrix \( Y \) satisfy the conditions

- \( a + d = 1, b + c = 0 \) if \( X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \)
- \( a + d = 0, a + c + d = 1 \) if \( X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)
- \( a + d = 0, a + b + d = 1 \) if \( X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \).

(The possibility \( X = I + 2Y \) is eliminated since the conditions on the coefficients of \( Y \) in that case read \( a + d = 0, a + d = 1 \).) One computes:

\[(X + 2Y)^2 \equiv I + 2X \mod 4.\]
After conjugating by preimages of elements of $GL_2(\mathbb{Z}/2\mathbb{Z})$, one concludes that

$$G \cap K \supseteq \{I + 2 \cdot \text{span} \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} \}.$$ 

Playing the same game with $t = 0$ and $d = 1$, one sees that in fact

$$G \cap K \supseteq \{ \text{traceless matrices} \}.$$ 

Now $G$ can be at worst an index 2 subgroup of $GL_2(\mathbb{Z}/4\mathbb{Z})$. However, if $G$ is indeed a proper subgroup of index 2, we may apply Lemma 25 and arrive at a contradiction. This concludes the proof in this case.

Lemmas 25 and 26 imply the following corollary.

**Corollary 27.** For $N \in \{4, 6, 8, 9, 12, 24\}$, we have

$$\varepsilon_N(X) = \bigcup_{(t,d) \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon_{N,(t,d)}(X),$$

where

$$\varepsilon_{N,(t,d)} := \{E \in \varepsilon_N(X) : (t, d) \text{ is not represented by } G_N(E) \}.$$ 

**Lemma 28.** For each $N \in \{4, 6, 8, 9, 12, 24\}$, we have

$$|\varepsilon_N(X)| \ll N^{16} X^6 \max_d \pi(X; N, d)^{-2},$$

with an absolute implied constant.

This lemma and its proof are analogous to Lemma 5 of [7], whose statement contains a small error: the “$\ll X^6 \pi(X; N, d)^{-2}$” should be replaced by “$\ll N^{4} X^6 \max_d \pi(X; N, d)^{-2}$.”

Using the Siegel-Walfisz theorem (c.f. [4], p. 133), which gives $\pi(X; N, d) \gg (\varphi(N))^{-1} \pi(X)$, Theorem 4 follows.

8 $N = 4$ occurs as a minimal exceptional number.

If $N = 4$ or 9, the argument given in section 18 is invalid since we may not conclude that (20) holds. In fact, there is a subgroup $H \subset GL_2(\mathbb{Z}/4\mathbb{Z})$ of index four which satisfies conditions (17) and (18). We now describe $H$ and demonstrate an infinite family of non-isomorphic elliptic curves $E$ for which $G_4(E) = H$. Elkies [9] has recently exhibited similar examples for $N = 9$.

First, a geometric description of $H$: Let $L$ be a complex lattice and let $L[4]$ denote the 4-torsion of $C/L$. By choosing a basis, we may identify $L[4]$ with $(\mathbb{Z}/4\mathbb{Z})^2$. Let $l_1, l_2, \ldots, l_6$ denote the lines through the origin in $L[4]$. More precisely, define the equivalence relation on $L[4]$ by declaring $u \sim u'$ exactly if
$u' = \lambda u$ for some $\lambda \in (\mathbb{Z}/4\mathbb{Z})^* = \{\pm 1\}$ and denote the resulting equivalence classes by $l_1, l_2, \ldots, l_6$. Since the Weierstrass $\wp$-function is even, the association $l_i = [u] \mapsto \wp(u)$ identifies $\mathbb{P}((\mathbb{Z}/4\mathbb{Z})^2) := \{l_1, l_2, \ldots, l_6\}$ with the set of $x$-coordinates of the 4-torsion of $E = E_L$, the elliptic curve associated to the lattice $L$. This correspondence identifies the Galois group of $\mathbb{Q}(E[4], x)$ over $\mathbb{Q}$ with a subgroup of $\text{PGL}_2(\mathbb{Z}/4\mathbb{Z})$.

We may extend the natural action of $\text{PGL}_2(\mathbb{Z}/4\mathbb{Z})$ on $\mathbb{P}((\mathbb{Z}/4\mathbb{Z})^2)$ to obtain a $\text{PGL}_2(\mathbb{Z}/4\mathbb{Z})$ action on the set

$$S := \{\{l_{i_1}, l_{i_2}, l_{i_3}\}, \{l_{i_4}, l_{i_5}, l_{i_6}\}\} : \text{ all } i_j \in \{1, 2, \ldots, 6\} \text{ are distinct}\}.$$ 

This action is not transitive. The size 10 set $S$ decomposes into two orbits $S_1$ and $S_2$ of sizes 4 and 6, respectively. To describe these sets, one needs to define an “addition relation” on $\mathbb{P}((\mathbb{Z}/4\mathbb{Z})^2)$. If $l_1, l_2,$ and $l_3$ are lines in $\mathbb{P}((\mathbb{Z}/4\mathbb{Z})^2)$, we say that $l_1 + l_2 = l_3$ exactly when, for some choice of representatives $u_i \in l_i$ we have $u_1 + u_2 = u_3$.

(Note: This is a relation, not a well-defined operation. For example, $[(1, 0)] + [(0, 1)] = [(1, 1)]$ and $[(1, -1)]$.) Then the two orbits are defined by

$$S_1 := \{\{l_{i_1}, l_{i_2}, l_{i_3}\}, \{l_{i_4}, l_{i_5}, l_{i_6}\}\} \in S : l_{i_1} + l_{i_2} = l_{i_3}\}$$

and $S_2 = S - S_1$. Fixing any element $r \in S_1$ we define $H_x = H_x(r) \subset \text{PGL}_2(\mathbb{Z}/4\mathbb{Z})$ to be the stabilizer of $r$. Finally, we define $H = H(r)$ to be the preimage of $H_x$ under the natural projection $\text{GL}_2(\mathbb{Z}/4\mathbb{Z}) \rightarrow \text{PGL}_2(\mathbb{Z}/4\mathbb{Z})$.

To find elliptic curves $E$ with $G_4(E) = H$, we reason as follows: let $x_1, \ldots, x_6$ be the elements of $E[4]$. If $E$ is given in the form

$$E : \quad y^2 = 4x^3 - g_2x - g_3$$

then the minimal polynomial for $x_1, x_2, \ldots, x_6$ is given by

$$f_E(t) = t^6 - \frac{5g_2}{4}t^4 - \frac{5g_3}{16}t^2 - \frac{g_2g_3}{4}t + \frac{g_3^2 - 32g_2^3}{64}.$$ 

The set $S_1$ defined above corresponds to the set of numbers

$$X_1 := \{(x_{i_1} + x_{i_2} + x_{i_3})(x_{i_4} + x_{i_5} + x_{i_6}) : (x_{i_1}, y_{i_1}) \oplus (x_{i_2}, y_{i_2}) = (x_{i_3}, y_{i_3})\},$$

where $\oplus$ refers to the addition law on $E$. $X_1$ is a set of four complex numbers which satisfy the (generically irreducible) polynomial

$$f_{1,E}(t) = t^4 + 3g_2t^3 + \frac{27g_2^2}{8}t^2 + \left(-\frac{37g_3^2}{16} + 108g_2^3\right)t + \frac{81g_2^4}{256}.$$
We note that $G_4(E) \subseteq \text{some } H(r)$ whenever $f_{1,E}(t)$ has a linear factor over $\mathbb{Q}$. Let $s \in \mathbb{Q}$ and denote by $E_s$ the elliptic curve given by the equation

$$y^2 := 4x^3 + \frac{16s^2 + 56s + 81}{3s}x + \frac{(16s^2 + 56s + 81)^2(-1 + 4s)}{864s^2}.$$ 

It may be checked that for each $s$, $f_{1,E_s}(t)$ is divisible by $t + 27 + \frac{96}{3} s + \frac{16}{3} s^2$ and that

$$\text{Gal}(\mathbb{Q}(E_s[4])/\mathbb{Q}(s)) \simeq H.$$ 

The discriminant is computed to be

$$\Delta(E_s) = -\frac{(16s^2 + 56s + 81)^3(4s + 3)^4}{27648s^4},$$ 

and the $j$-invariant is

$$j(E_s) = \frac{1769472s}{(4s + 3)^4}.$$ 

In particular, if we apply the Hilbert irreducibility criterion, we see that there are infinitely many non-isomorphic curves $E_s$, each with Galois group $G_4 \simeq H$.

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