Differences between mean-field dynamics and \(N\)-particle quantum dynamics as a signature of entanglement

Christoph Weiss\(^1\) and Niklas Teichmann\(^2\)

\(^1\)Laboratoire Kastler Brossel, École Normale Supérieure, Université Pierre et Marie-Curie-Paris 6, 24 rue Lhomond, CNRS, F-75231 Paris Cedex 05, France
\(^2\)Institut Henri Poincaré, Centre Emile Borel, 11 rue P. et M. Curie, F-75231 Paris Cedex 05, France

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A Bose-Einstein condensate in a tilted double-well potential under the influence of time-periodic potential differences is investigated in the regime where the mean-field (Gross-Pitaevskii) dynamics become chaotic. For some parameters near stable regions, even averaging over several condensate oscillations does not remove the differences between mean-field and \(N\)-particle results. While introducing decoherence via piecewise deterministic processes reduces those differences, they are due to the emergence of mesoscopic entangled states in the chaotic regime.

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Experimentally it is possible to generate precisely controllable double-well potentials for Bose-Einstein condensates (BECs) (Ref. \(^1\) and references therein). A future goal for this system is the realization of mesoscopic entangled states in the chaotic regime.

\[ H = -\frac{\hbar \Omega}{2} (\hat{a}_1 \hat{a}_2 + \hat{a}_1^\dagger \hat{a}_2^\dagger) + \hbar \kappa (\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1^\dagger \hat{a}_1) + \hbar (\mu_0 + \mu_1 \sin(\omega t)) \left( \hat{a}_2^\dagger \hat{a}_1^\dagger - \hat{a}_1 \hat{a}_2 \right), \] (1)

where \(\hat{a}_j\) creates (annihilates) a boson in well \(j\); \(\mu_0\) models the tilt and \(\mu_1\) the driving amplitude. Such Hamiltonians have been used for schemes of entanglement generation \(^{17,18}\); without the periodic driving, entanglement has been investigated in BECs \(^{19,20}\). Other applications include high precision measurements, many-body quantum coherence \(^{21,22}\) and spin systems \(^{23}\).

On the level of the Gross-Pitaevskii equation for the above model, a wave function is characterized by the variables \(\theta\) and \(\phi\), where \(\cos^2(\theta/2) \sin^2(\theta/2)\) is the probability of finding the condensate in well 1 (well 2) and \(\exp(i\phi)\) is the phase between the two wells. The corresponding \(N\)-particle wave-function ("atomic coherent states") \(^{24}\) with all particles in this state reads (in an expansion in the Fock-basis \(|n, N-n\rangle\) with \(n\) atoms in well 1):

\[ |\theta, \phi\rangle = \sum_{n=0}^{N} \binom{N}{n}^{1/2} \cos^n(\theta/2) \sin^{N-n}(\theta/2) \times e^{i(N-n)\phi} |n, N-n\rangle. \] (2)

The mean-field dynamics can be mapped to that of a nonrigid pendulum \(^{15,22}\); including periodic driving the Hamilton function reads (\(z = \cos^2(\theta/2) - \sin^2(\theta/2)\)):

\[ H_{mf} = \frac{N \kappa}{\Omega} \frac{z^2}{2} - \sqrt{1 - z^2} \cos(\phi) \]
\[ - 2z \left( \frac{\mu_0}{\Omega} + \frac{\mu_1}{\Omega} \sin(\frac{\pi}{2} \tau) \right), \quad \tau = t \Omega. \] (3)

The experimentally measurable \(^{11}\) population imbalance \(z/2\) can be used to characterize the mean-field dynamics. Fig. \(^{11}\) shows typical Poincaré surfaces of section. The initial parameters were chosen such that tunneling
in the driven, tilted double-well potential is enhanced by “photon”-assisted tunneling [26] (cf. Ref. [27]). If the interaction is not too low ($N\kappa/\Omega \gtrsim 0.4 \ldots 0.6$), regular and chaotic dynamics coexist (Fig. 1a cf. 28), for low interaction the dynamics are regular (Figs. 1 b and 1c).

For the parameters corresponding to the Poincaré surface of section in Fig. 1a, Fig. 2a displays the differences between N-particle and mean-field dynamics by numerically calculating (using the Shampine-Gordon-routine [29]) the time-average of the (experimentally measurable [1]) population imbalance $\langle J_z \rangle / N$ which corresponds to the mean-field $\langle z/2 \rangle$:

$$\frac{\langle J_z \rangle}{N} = \frac{1}{NT} \int_0^T dt \frac{1}{2} \langle \psi | \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2 | \psi \rangle ,$$

(4)

where for $\langle J_z \rangle / N = \pm 0.5$ the entire condensate is in the left, respectively, right well. Each point represents an initial condition [2]. The differences are small if the mean-field dynamics are regular (cf. Fig. 1a) while they can be rather large in the chaotic regime (up to half the theoretical limit, $\max\{|z/2 - \langle J_z \rangle / N|\} = 1$). Most of the deviations between N-particle dynamics and mean-field dynamics in Fig. 2a lie within twice the root-mean-square (r.m.s.)-fluctuations of the N-particle dynamics. However, contrary to the preliminary results of Ref. [13], for many initial conditions in the (classically) chaotic regime the differences can be very small; they are large near the boundaries of stable regions.

In Fig. 2b, the time-averaged r.m.s.-fluctuations of $\langle J_z \rangle / N$ reproduce many features displayed in the Poincaré section in Fig. 1a. Note that the values for the r.m.s.-fluctuations are well above those expected for $N = 100$ particles in an atomic coherent state, $\sin(\theta)/(2\sqrt{N}) \leq 0.05$, thus clearly indicating that more than one atomic coherent state is involved. Bose-Einstein condensates of $N \approx 100$ have been realized experimentally [30], both the validity of the two-mode approximation will be better and life-times of mesoscopic entangled states will be longer than in larger condensates. However, even when the calculation is repeated for $N = 1000$ particles, the differences in the chaotic regime remain. As the (non-linear) Gross-Pitaevskii equation does not allow any superpositions, decoherence should reduce the differences between mean-field and quantum dynamics.

In this Letter, we use a piecewise deterministic process (PDP) (Ref. [31], cf. [32]) to model decoherence. To avoid to have to introduce decoherence also on the mean-field level (the atomic coherent states [2]) become orthogonal in the limit $N \to \infty$, we use the projection on the atomic coherent states [24]:

$$1 = \frac{N + 1}{4\pi} \int d\theta \sin(\theta) \int d\phi \langle \theta, \phi \rangle \langle \theta, \phi \rangle .$$

(5)

Now, the PDP simplifies to having jumps on one of the atomic coherent states [2] after time $t$ with probability

$$p_{\text{jump}} = 1 - \exp(-\alpha t) , \quad \alpha = \text{const.} > 0 ,$$

(6)

and Hamiltonian dynamics [1] between jumps. The state on which the wave-function is projected is determined by the probability distribution

$$p_{\theta, \phi} d\Omega = \frac{N + 1}{4\pi} |\langle \psi | \theta, \phi \rangle|^2 \sin(\theta) d\theta d\phi .$$

(7)
by the curves (to the mean-field dynamics. Many dots lie in the area defined
regime: Fig. 4 shows that, at least for 
\( \Delta J_z \) reaches 5). A perfect agreement cannot be expected as the
peaks of the differences between mean-field and quantum dynamics by a factor of
therein). This decreases the peaks of the differences between mean-field and quantum
distribution (7) (see Fig. 4, cf. Refs. [2, 35] and references
To numerically identify if a given wave-function \( \psi \) is in a mesoscopic superposition, we start by searching the
atomic coherent states (2) are
states (see the explanation before Eq. (2)): \[ |\psi_{sp}\rangle = \eta \left( |\theta_1, \phi_1 \rangle + e^{i\gamma}|\theta_2, \phi_2 \rangle \right) , \ 0 \leq \gamma \leq 2\pi \]
If both parts hardly overlap, \(|\langle \theta_1, \phi_1 |\theta_2, \phi_2 \rangle | \ll 1 , \) the normalization \( \eta \approx 1/\sqrt{2} \) and \(|\psi_{sp}\rangle \) is a highly entangled
expected as the averaged probability distribution on the mean-field level is always added whereas in quantum mechanics also
destructive interference can occur.

On the level of quantum dynamics, the differences could be due to either a distribution of many atomic
cohert states - or maybe even mesoscopic superpositions.
For our model all mesoscopic quantum superpositions of all \( N \) particles being either in one quantum state or in
another can be expressed as a sum of two atomic coherent

\[ \langle \theta_1, \phi_1 \rangle \text{ and } |\theta_2, \phi_2 \rangle \text{ should hardly overlap, the second maximum } m_2 = (|\psi|\theta_2, \phi_2 \rangle |^2 \text{ is searched out}
outside the set R1. The set R2 is defined analogously to R1 by \(|\langle \theta, \phi |\theta_1, \phi_1 \rangle |^2 > 10^{-3} \text{ (cf. Fig 5c)}\). As both parts of the
mesoscopic superposition |\psi\rangle = |0, \phi_1 \rangle \text{ and } |\pi, \phi_2 \rangle \rangle \text{ (cf. Fig 5c)}\) such a state is a
bimodal distribution (for \( N \to \infty \): two delta-peaks).

To numerically identify if a given wave-function \( \psi \) is in a mesoscopic superposition, we start by searching the
atomic coherent state \(|\theta_1, \phi_1 \rangle \) for which \(|\langle \psi |\theta, \phi \rangle |^2 \) reaches its maximum, \( m_1 \). Around \(|\theta_1, \phi_1 \rangle \rangle \text{ we define the set R1}
by \(|\langle \theta, \phi |\theta_1, \phi_1 \rangle |^2 > 10^{-3} \text{ (cf. Fig 5c)}\). As both parts of the
mesoscopic superposition \(|\psi\rangle \rangle \text{ should hardly overlap, the second maximum } m_2 = (|\psi|\theta_2, \phi_2 \rangle |^2 \text{ is searched out}
outside the set R1. The set R2 is defined analogously to R1 by \(|\langle \theta, \phi |\theta_2, \phi_2 \rangle |^2 > 10^{-3} \). The fidelity \(|\langle \psi |\psi_{sp} \rangle |^2 \) still is a function of \( \gamma \), taking its maximum and excluding large
overlaps (\( R1 \cap R2 \neq \emptyset \)) yields:
\[ p_{\text{fid}} = \begin{cases} 0 & : R1 \text{ and R2 overlap} \\ \frac{1}{2} (\sqrt{m_1} + \sqrt{m_2})^2 & : \text{else} \end{cases} \]

Yet, this only indicates entanglement if \( p_{\text{fid}} > 0.5 \). With
\[ \sigma_{\text{ent}} = \frac{m_2}{m_1} p_{\text{fid}}, \quad \sigma_{\text{ent}} \leq p_{\text{fid}} \]
even values of \( \sigma_{\text{ent}} \leq 0.5 \) can identify mesoscopic superpositions (Fig. 5c). In Fig. 5a, the maximum value of
entanglement (evaluated at \( \tau = 5 \) and 10) is plotted as a function of the initial condition \((\theta_0, \phi_0)\) within the
chaotic regime (left), entanglement generation happens on faster time-scales than in the regular regime (right); for longer time-scales (Fig. 5b) the entanglement in the
entire chaotic regime is more pronounced. It reaches particularly high values near initial conditions with large
differences in the time-averaged population imbalances.
at times \(\tau\) in Fig. 1.a (left column) and as in Fig. 1.c (right column).

**FIG. 5:** (color online) Entanglement (10) for parameters as in Fig. 2.a. We obtained qualitatively similar results also for larger BECs.

To conclude, generation of mesoscopic entangled states can be a signature of quantum chaos for a BEC in a periodically driven double well potential. We investigated the driving near multi-“photon” tunneling resonances [29] which were recently observed experimentally for a BEC in an optical lattice [30]. While decoherence can lead to a “chaotic” behavior similar to the predictions of the Gross-Pitaevskii equation, the differences between quantum dynamics and mean-field dynamics are due to the emergence of mesoscopic superpositions. If the mean-field dynamics are chaotic, the entanglement generation is accelerated and its values are enhanced.

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* Electronic address: weiss@theorie.physik.uni-oldenburg.de

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