Six-dimensional regularization of chiral gauge theories

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We propose a regularization of four-dimensional chiral gauge theories using six-dimensional Dirac fermions. In our formulation, we consider two different mass terms having domain-wall profiles in the fifth and the sixth directions, respectively. A Weyl fermion appears as a localized mode at the junction of two different domain walls. One domain wall naturally exhibits the Stora–Zumino chain of the anomaly descent equations, starting from the axial $U(1)$ anomaly in six dimensions to the gauge anomaly in four dimensions. Another domain wall implies a similar inflow of the global anomalies. The anomaly-free condition is equivalent to requiring that the axial $U(1)$ anomaly and the parity anomaly are canceled among the six-dimensional Dirac fermions. Since our formulation is based on a massive vector-like fermion determinant, a nonperturbative regularization will be possible on a lattice. Putting the gauge field at the four-dimensional junction and extending it to the bulk using the Yang–Mills gradient flow, as recently proposed by Grabowska and Kaplan, we define the four-dimensional path integral of the target chiral gauge theory.

Subject Index B00, B01

1. Introduction

Defining chiral fermions on a lattice has been a big challenge since Nielsen and Ninomiya (Refs. [1,2]) proved the no-go theorem about chiral symmetry without unphysical doublers. The problem was partly solved in the formulation of vector-like gauge theories on a lattice (Refs. [3–6]). It has been, however, still difficult to nonperturbatively realize chiral gauge symmetry. To construct chiral gauge theories, one must separate the positive and negative chiral modes. This process usually violates gauge symmetry, and one has to find gauge noninvariant counterterms\textsuperscript{1} to recover it even when the target theory is anomaly-free (Refs. [9–15]).

Recently, an interesting approach was proposed by Grabowska and Kaplan (Ref. [16]), in the 5-dimensional domain-wall fermion formulation. Keeping the 4-dimensional gauge invariance, they succeeded in coupling the gauge fields to only one chiral mode on the domain wall. This was made possible by turning off the gauge fields near the anti-domain wall using the Yang–Mills gradient flow (Ref. [17]). In their new approach, one can distinguish anomalous and nonanomalous theories by the presence and absence of the Chern–Simons (CS) term in the bulk of the 5-dimensional space. If the CS term exists, then 4-dimensional gauge invariance is not closed on the domain wall alone and the gauge current flows in the extra dimension. Thus, no consistent 4-dimensional local effective

\textsuperscript{1} The counterterms are nonperturbatively given for $U(1)$ (Ref. [7]) and $SU(2) \times U(1)$ (Ref. [8]) gauge groups.
theory exists in the low-energy limit. The extra dimension plays a more important role than that for vector-like formulations, as it converts the problem of gauge anomaly into the problem of parity invariance (broken by the CS term) in 5-dimensions (Refs. [18–20]).

The importance of an extra dimension was also discussed in studies on global anomalies (Refs. [21, 22]). It was shown that the global anomaly can be formulated as the complex phase of the bulk 5-dimensional theory, which has 4-dimensional target (massless) fermions on its boundary. They then claimed a more strict definition of the global anomalies: not only on the mapping torus (on which the SU(2) global anomaly was shown (Ref. [23])), but also any fermion determinant on a 5-dimensional compact manifold, must be real and positive, otherwise the theory becomes anomalous when it has a 4-dimensional boundary. The extra dimension is essential since this new notion of anomaly can never be understood within 4-dimensional space alone.

It would therefore be interesting to consider both the perturbative gauge anomaly and the global anomaly at the same time in the new formulation in higher-dimensional space-time, which was not discussed in Ref. [16]. However, we notice that the extra dimension for the perturbative anomaly in Ref. [16] and that for the global anomaly in Refs. [21,22] are quite different. For the former domain-wall fermion formulation, the 5th direction is introduced to separate the left- and right-handed modes. On the other hand, the extra dimension for the global anomaly is introduced as a one-parameter family of the fermion determinant phase where the chiral fermions are already put on a 4-dimensional space. It is then natural to speculate that chiral fermions may need two extra directions, or in total 6 dimensions, to be formulated.

The relation of 4-dimensional Weyl fermions to 6-dimensional space-time is not a new idea but can be found in the literature. In Refs. [24,25], it was shown that the phase of the Weyl fermion determinant can be given by the $\eta$-invariant of a Dirac operator extended in 5-dimensions. However, this $\eta$-invariant needs a variation with respect to another one-parameter family (originally denoted by $u$; see also Ref. [26]) and we integrate it over a finite range from 0 to 1. This fact implies that the phase of the Weyl fermion determinant needs two parameters to be well defined.

A more direct hint at the 6th dimension was well known as the anomaly descent equations by Stora (Refs. [27,28]) and Zumino (Refs. [29–31]). They showed that the 4-dimensional (consistent) gauge anomaly is obtained uniquely from the 6-dimensional axial U(1)$_A$ anomaly up to an overall constant. Soon after, Alvarez-Gaumé and Ginsparg (Ref. [32]) and Sumitani (Ref. [33]) proved that this over-all constant must be unity, and the 4-dimensional gauge anomaly originates from the index theorem in 6 dimensions (Ref. [34]). There has, however, been no theory that reproduces these anomaly descent equations proposed in the literature.

In this work, we formulate a vector-like 6-dimensional Dirac fermion system in which Weyl fermions are localized at the junction of two different kinds of domain walls. One domain wall is made in a conventional way, giving a kink mass (let us take this term in the 6th direction) to the fermions. Another domain wall is made by giving a kink structure in the 5th direction to a background operator which is invariant under U(1)$_A$ rotation. In a sense, our system is a doubly gapped 6-dimensional topological insulator. Apart from the domain walls, the Dirac fermions are gapped by two types of masses having different quantum numbers. Each of the domain walls eliminates one mass term from the boundary modes, and a gapless mode or our target Weyl fermion appears only at the domain-wall junction.

As will be shown in this paper, these two domain walls play different roles in the anomaly cancelations. The conventional mass domain wall (we call it the $M$ domain wall) converts the 6-dimensional U(1)$_A$ anomaly into the CS term, or the 5-dimensional parity anomaly on it. The CS term breaks the
gauge symmetry at the domain-wall junction, which is absorbed by the gauge anomaly of the Weyl fermion. Namely, the $M$ domain wall mediates the perturbative anomaly inflow

\[
\begin{align*}
\text{6D } & \text{U}(1)_A \text{ anomaly} \\
\uparrow & \\
\text{5D parity anomaly (CS term)} \\
\uparrow & \\
\text{4D (perturbative) gauge anomaly.}
\end{align*}
\] (1.1)

On the other hand, another domain wall ($\mu$ domain wall) is only sensitive to the zero modes which cannot appear in the index theorem of the U(1)$_A$ symmetry. In fact, the fermion modes localized on this domain wall produce an almost real determinant (except those from the domain-wall junction) and sensitive to the flips of sign, or mod-two types of the anomaly. This is true even when the perturbative anomaly is absent. Therefore, we conjecture that the anomaly mediated by this $\mu$ domain wall corresponds to a kind of global anomaly. For the fundamental representation of the SU(2) group, e.g., we will show that this anomaly inflow is consistent with a ladder of the mod-two indices:

\[
\begin{align*}
\pi_5(\text{SU}(2)) & = \mathbb{Z}_2 \\
\uparrow & \\
\pi_4(\text{SU}(2)) & = \mathbb{Z}_2 \\
\uparrow & \\
\text{4D } & \eta\text{-invariant,}
\end{align*}
\] (1.2)

where the latter part is well known in Ref. [23] but the former homomorphism of $\pi_5(\text{SU}(2))$ and $\pi_4(\text{SU}(2))$ is not discussed in the literature (on physics). The two anomaly inflows finally meet at the junction of the domain walls and determine the perturbative and global anomalies of the Weyl fermion sitting there.

Then the anomaly-free condition is equivalent to requiring the 6-dimensional theory to be insensitive to both the U(1)$_A$ index and the exotic zero modes. Since these zero modes flip the sign of the fermion determinant, the bulk part of the anomaly-free set of fermion determinants becomes real, positive (at least in the continuum limit). The 4-dimensional boundary modes, on the other hand, can have their own complex phases.

Since our formulation is a Dirac fermion system with vector-like masses in the bulk, it is natural to assume that nonperturbative lattice regularization is available, using a simple Wilson Dirac operator. Putting the gauge fields on the junction of the domain walls, and extending it to the 5th and 6th directions using the Yang–Mills gradient flow, as proposed in Ref. [16], the 4-dimensional gauge invariance is maintained.

The rest of the paper is organized as follows. In Sect. 2, we explain how to distinguish the Dirac zero modes originating from the U(1)$_A$ anomaly and those related to the parity anomaly. Then we construct the 6-dimensional Dirac fermion system in the continuum theory and show how the two kinds of anomaly ladders are realized in Sect. 3. In Sect. 4, we discuss the anomaly-free condition. According to the more strict definition of the global anomaly (Refs. [21,22]), the anomaly-free condition is nontrivial for our target theory on a 4-dimensional torus (which is an essential requirement for lattice
regularizations). In Sect. 5 we discuss how to implement the gauge fields localized at the domain-wall junction and how to decouple the unwanted mirror fermions. In our formulation, there is an ambiguity in the choice of two domain walls, which is discussed in Sect. 6. Finally we propose how to regularize our formulation on a lattice in Sect. 7. Section 8 is devoted to a summary and discussions. Appendices A–C are given for technical details of our analysis.

2. Parity and U(1)A anomalies and related zero modes

We consider fermion determinants on a 6-dimensional Euclidean torus. We take the gamma matrices \( \gamma_i \) \((i = 1, \ldots, 6)\) to be Hermitian and to satisfy \( \{\gamma_i, \gamma_j\} = 2\delta_{ij} \). The 6-dimensional Dirac operator is denoted by \( D^{6D} = \sum_{i=1}^{6} \gamma_i \nabla_i \), where \( \nabla_i \) is the covariant derivative of a gauge group \( SU(N_c) \). Since we are interested in 4-dimensional gauge theory, we simply take the 5th and 6th components of the gauge fields to be zero, i.e., \( A_5(x) = A_6(x) = 0 \). Later we define the remaining four components of the gauge fields in the bulk by a two-parameter family of the 4-dimensional gauge fields localized at the domain-wall junction, but in this section we only require \( A_\mu (\mu = 1, 2, 3, 4) \) to be symmetric under \( x_5 \rightarrow -x_5 \).

Let us start with a ratio of determinants of a single Dirac fermion and a Pauli–Villars field,

\[
\exp(-W_{\text{periodic}}) \equiv \det \left( \frac{D^{6D} - M - i\mu \gamma_6 \gamma_7}{D^{6D} + M + i\mu \gamma_6 \gamma_7} \right),
\]

where \( \gamma_7 = i \prod_{i=1}^{6} \gamma_i \) is the chirality operator, \( M \) is the mass, and \( i\mu \gamma_6 \gamma_7 \) describes the constant axial vector current in the 6th direction,

\[
i\mu \bar{\psi} \gamma_6 \gamma_7 \psi,
\]

which is invariant under the \( U(1)_A \) rotation

\[
g^\theta \psi(x) = e^{i\theta \gamma_7} \psi(x), \quad \bar{\psi}(x)g^\theta = \bar{\psi}(x)e^{i\theta \gamma_7},
\]

where \( \theta \) is an arbitrary parameter. Note that the Pauli–Villars field has the opposite signs of the masses \( M \) and \( \mu \). Here we assume that the boundary condition is periodic in every direction in order to discuss the anomalies in the bulk 6-dimensions. In later sections, we introduce the domain walls to study the anomalies at the boundaries.

Next, we introduce a parity transformation in the 5th direction on the fermion fields:

\[
P' \psi(x) = i\gamma_5 R_5 \psi(x), \quad \bar{\psi}(x)P' = iR_5 \bar{\psi}(x) \gamma_5,
\]

where \( R_i \) denotes the reflection of the \( i \)th coordinate: \( R_if(x_i) = f(-x_i) \). Note that this parity is different from the conventional parity

\[
P \psi(x) = \gamma_1 \prod_{i \neq 1} R_i \psi(x), \quad \bar{\psi}(x)P = \prod_{i \neq 1} R_i \bar{\psi}(x) \gamma_1,
\]

where we take \( i = 1 \) to be the temporal direction. The main difference is that \( P'^2 = -1 \) while \( P^2 = 1 \). It is known that \( P \)-invariance exists only in even dimensions, while \( P' \) is allowed in any dimension. The massless Dirac fermion action

\[
S_F = \int d^6x \bar{\psi}(x) D^{6D} \psi(x),
\]
has both of \( P \) and \( P' \) symmetries, but the mass terms \( M \bar{\psi}(x)\psi(x) \) and \( i\mu \bar{\psi}(x)\gamma_6\gamma_7\psi(x) \) violate the \( P' \) symmetry.

As is well known in the literature (Refs. [35–37]), \( P' \) symmetry has an anomaly. Because of the anti-commutation relation \( \{D^6D, P' \} = 0 \), every eigenvalue \( i\lambda \) of \( D^6D \) has its complex-conjugate pair \( -i\lambda \), except for the zero modes. Therefore, under \( P' \), the massless fermion action is manifestly invariant, while the zero mode’s contribution to the fermion measure Jacobian is not, since \( P' \) flips its sign,

\[
D\bar{\psi}_0P'D\psi_0 = -D\bar{\psi}_0D\psi_0. \tag{2.7}
\]

Note that those from nonzero modes always cancel with their complex conjugates. Therefore, the \( P' \) transformation counts the number of zero modes \( I \).

Moreover, using \( P' \) and the axial \( U(1)_A \) rotation,\(^2\) with the angle \( \theta = \pi \)

\[
g^\pi \psi(x) = \exp(i\gamma_7\pi/2)\psi(x), \quad \bar{\psi}(x)g^\pi = \bar{\psi}(x)\exp(i\gamma_7\pi/2), \tag{2.8}
\]

we can decompose \( \mathcal{I} \) into two parts,

\[
\mathcal{I} = \mathcal{P} + \mathcal{I}, \tag{2.9}
\]

where \( \mathcal{P} \) denotes the conventional index (Ref. [38]) related to the \( U(1)_A \) anomaly, namely \( n_+ - n_- \) where \( n_\pm \) denote the number of zero modes with chirality \( \pm \). The other integer \( \mathcal{I} \) counts exotic zero modes, which possibly exist even when the \( U(1)_A \) anomaly is absent.\(^3\) As shown below, \( \mathcal{P} \) controls the perturbative gauge anomaly, while \( \mathcal{I} \) can be considered as the origin of global anomalies.

The fermion determinant Eq. (2.1) precisely reproduces this decomposition since

\[
\exp(-W_{\text{periodic}}) = \det \left( \frac{D^6D - M - i\mu\gamma_6\gamma_7}{D^6D + M + i\mu\gamma_6\gamma_7} \right) \times \det \left( \frac{D^6D + M + i\mu\gamma_6\gamma_7}{D^6D + M - i\mu\gamma_6\gamma_7} \right)
\]

\[
= \det \left( g^\pi \frac{D^6D - M - i\mu\gamma_6\gamma_7}{D^6D + M + i\mu\gamma_6\gamma_7} g^\pi \right)
\]

\[
\times \det \left( \frac{(g^\pi)^\dagger P'(D^6D + M - i\mu\gamma_6\gamma_7)P'(g^\pi)^\dagger}{D^6D + M + i\mu\gamma_6\gamma_7} \right)
\]

\[
= (-1)^\mathcal{P} \times (-1)^\mathcal{I}, \tag{2.10}
\]

where we have applied \( g^\pi \) rotation to the numerator of the former determinant, and \( P'(g^\pi)^\dagger \) to the latter. Note again that the \( \mu \) term is \( U(1)_A \) invariant.

We find that the above argument does not change on replacing the \( \mu \) term with

\[
i\mu \bar{\psi}\gamma_6\gamma_7R_5R_6\psi \quad \text{or} \quad i\mu \bar{\psi}\gamma_6\gamma_7R_6\psi. \tag{2.11}
\]

\(^2\) In Eq. (2.8), we have not used a naive operation \( \exp(i\gamma_7\pi/2) = i\gamma_7 \), since the (lattice) regularization should break this condition.

\(^3\) In the \( SU(2) \) theory, e.g., fermions cannot have nonzero \( n_+ - n_- \) in 6-dimensions since the \( U(1)_A \) anomaly is zero. Nevertheless, there exists the so-called mod-two index related to the homotopy group \( \pi_5(SU(2)) = \mathbb{Z}_2 \).
However, the nonlocal reflection operators $R_5$ or $R_6$ can make an unexpected cancelation of the physical phase which should be present in the 4-dimensional target theory.\footnote{We thank D. B. Kaplan for pointing out this problem in our original version of this paper, which was mainly analyzed with the operator $i\mu \bar{\psi} \gamma_6 \gamma R_5 R_6 \psi$.} Therefore, in the following analysis, we use the simple axial vector current background operator.

3. Two domain walls and anomaly inflow

3.1. Weyl fermion at the domain-wall junction

Let us now give domain-wall profiles to the two mass terms with $M$ and $\mu$:

$$\exp(-W_{2DW}) \equiv \det \left( \frac{D^{6D} + M \epsilon(x_6) + i\mu \epsilon(x_5) \gamma_6 \gamma_7}{D^{6D} + M + i\mu \gamma_6 \gamma_7} \right),$$

where $\epsilon(x) = x/|x|$ denotes the sign function. Since the fermion fields satisfy periodic boundary conditions, there also exist anti-domain walls in the determinant. Although the anti-domain walls do not appear in the expressions, we always assume that they are there, and will explicitly write them whenever it is necessary.

Decomposing the Dirac operator as

$$D^{6D} = D^{4D} + \gamma_5 \partial_5 + \gamma_6 \partial_6,$$

where we have set $A_5 = A_6 = 0$, we can obtain a solution of the Dirac equation

$$(D^{6D} + M \epsilon(x_6) + i\mu \epsilon(x_5) \gamma_6 \gamma_7) \psi(x) = 0,$$

localized at the domain-wall junction $x_5 = x_6 = 0$ as

$$\psi(x) = e^{-M|x_6|} e^{-\mu|x_5|} \phi(\bar{x}),$$

$$D^{4D} \phi(\bar{x}) = 0,$$

$$\gamma_6 \phi(\bar{x}) = \phi(\bar{x}),$$

$$i\gamma_5 \gamma_6 \gamma \phi(\bar{x}) = \phi(\bar{x}),$$

where $\bar{x} = (x_1, x_2, x_3, x_4)$ and we have assumed $M > 0$ and $\mu > 0$. Note that $\gamma_6$ commutes with $i\gamma_5 \gamma_6 \gamma_7$, and the two constraints by these operators force $\phi(\bar{x})$ to have positive chirality (see Appendix A). Namely, a Weyl fermion with positive chirality appears at the domain-wall junction. The Weyl fermion with the opposite chirality can be realized by flipping the signs of $M$ and $\mu$, which changes the boundary conditions to $\gamma_6 \phi(\bar{x}) = -\phi(\bar{x})$ and $i\gamma_5 \gamma_6 \gamma \phi(\bar{x}) = -\phi(\bar{x})$. As will be shown below, the appearance of the single Weyl fermion is not a coincidence, but required, to keep the gauge invariance of the total 6-dimensional theory.

The total determinant Eq. (3.1) becomes complex due to the sign function $\epsilon(x_5)$, which is odd under $P'$. We will see below that this complex phase that we denote by $\phi^{\text{total}}$ is almost localized at the 4-dimensional junction of the two domain walls, when the fermion contents are anomaly-free.
3.2. Anomaly inflow through the $M$ domain wall

In order to obtain the anomaly inflow through the $M$ domain wall, first we consider a simpler case with $\mu = 0$ and decompose the determinant into three parts,

$$
\det \left( \frac{D^{6D} + M \epsilon(x_6)}{D^{6D} + M} \right) = \det \left( \frac{D^{6D} + M + iM_2 \gamma_6 \gamma_7}{D^{6D} + M} \right)
\times \det \left( \frac{D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7}{D^{6D} + M + iM_2 \gamma_6 \gamma_7} \right)
\times \det \left( \frac{D^{6D} + M \epsilon(x_6)}{D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7} \right),
$$

(3.8)

and take the $M \gg M_2 \gg 0$ limit. In this limit, there is no doubt that the first determinant of the right-hand side converges to unity. It is also important to note that in this $\mu \to 0$ limit, the total determinant is real thanks to the Hermiticity, and the complex phase can be written as $\pi \Im$. From the second determinant, we obtain the axial $U(1)_A$ anomaly:

$$
\Im \ln \det \left( \frac{D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7}{D^{6D} + M + iM_2 \gamma_6 \gamma_7} \right)
= \Im \ln \det \left( \frac{e^{i\theta(x_6)\gamma_7} e^{-i\theta(x_6)\gamma_7} (D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7)}{D^{6D} + M + iM_2 \gamma_6 \gamma_7} e^{i\theta(x_6)\gamma_7} \right)
= \int d^6x \frac{1 - \epsilon(x_6)}{2} \frac{1}{6(4\pi)^3} e^{\mu_1 \cdots \mu_6} \text{tr} \left[ F_{\mu_1\mu_2} F_{\mu_3\mu_4} F_{\mu_5\mu_6} \right],
$$

(3.9)

where $\theta(x_6) = \pi (1 - \epsilon(x_6))/4$, and $\gamma_7^{\text{reg}}$ is the regularized chiral operator, e.g., with the heat-kernel method, and the standard Fujikawa method (Ref. [39]) is applied. Here, the $M_2 \gg 0$ limit removes the IR divergence coming from the massless boundary-localized modes. Since the integral in Eq. (3.9) counts the bulk instanton number in the region $x_6 < 0$, and gives a surface term at $x_6 = 0$, it can be decomposed as

$$
\pi \mathcal{P}_{x_6 < 0}^{6D} + \pi \text{CS},
$$

(3.10)

where $\mathcal{P}_{x_6 < 0}^{6D}$ is an integer and CS is the Chern–Simons term on the $M$ domain wall,

$$
\text{CS} \equiv - \int_{x_6 = 0} d^5x \frac{2}{3(4\pi)^3} e^{\mu_1 \cdots \mu_5} \text{tr} \left[ \frac{1}{2} A_{\mu_1} F_{\mu_2\mu_3} F_{\mu_4\mu_5} 
- i A_{\mu_1} A_{\mu_2} A_{\mu_3} F_{\mu_4\mu_5} - \frac{1}{5} A_{\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} A_{\mu_5} \right].
$$

(3.11)

In the third determinant of Eq. (3.8), only the boundary localized mode at the $M$ domain wall can contribute. Projecting the determinant to the one for the subspace which requires $\gamma_6 \psi = \psi$ and $(\gamma_6 \partial_6 + M \epsilon(x_6)) \psi = 0$, and rearranging the gamma matrices, one obtains

$$
\lim_{M \to \infty} \det \left( \frac{D^{6D} + M \epsilon(x_6)}{D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7} \right) = \det \left( \frac{\bar{D}^{5D}}{\bar{D}^{5D} + M_2} \right),
$$

(3.12)
where the determinant on the right-hand side is taken in the reduced space of $4 \times 4$ gamma matrices $\bar{\gamma}$ (see our notation in Appendix A), and the corresponding Dirac operator is given by $\bar{D}^{5D} = \sum_{i=1}^{5} \bar{\gamma}^i \nabla_i |_{x_6=0}$.

Let us denote the complex phase of the determinant (3.12) by

$$-i \frac{\eta^{5D}}{2} = i \text{Im} \det \left( \frac{\bar{D}^{5D}}{\bar{D}^{5D} + M_2} \right), \quad (3.13)$$

since $\eta^{5D}$ corresponds to a regularization of the so-called $\eta$-invariant (Refs. [24–26]):

$$\lim_{M_2 \to \infty} \eta^{5D} = \sum_{\lambda > 0} - \sum_{\lambda < 0}, \quad (3.14)$$

where $\lambda$ denotes the eigenvalues of $i\bar{D}^{5D}$. Therefore, we have obtained a formula

$$\mathcal{J} = \mathcal{P}_{x_6<0}^{6D} + \text{CS} - \frac{\eta^{5D}}{2}, \quad (3.15)$$

known as the Atiyah–Patodi–Singer index theorem (Refs. [40–42]).

Next, we turn to the $\mu$ domain wall and consider the limit $M \gg \mu \gg 0$. A similar decomposition to Eq. (3.8) is possible:

$$\det \left( \frac{D^{6D} + M \epsilon(x_6) + i\mu \epsilon(x_5)\gamma_6\gamma_7}{D^{6D} + M + i\mu \gamma_6\gamma_7} \right) = \det \left( \frac{D^{6D} + M \epsilon(x_6) + i\mu \epsilon(x_5)\gamma_6\gamma_7}{D^{6D} + M + i\mu \gamma_6\gamma_7} \right) \times \det \left( \frac{D^{6D} + \mu \epsilon(x_5)}{\bar{D}^{5D} + \mu} \right), \quad (3.16)$$

The first determinant in Eq. (3.16) gives the same contribution as the product of first and second ones in Eq. (3.8), i.e., they produce the same $\pi(\mathcal{P}_{x_6<0}^{6D} + \text{CS})$. This is consistent with the fact that the chiral anomaly term is insensitive to the $\mu$ domain wall, which is $U(1)_A$ invariant.

The second determinant in Eq. (3.16) in the $M \to \infty$ limit becomes

$$\det \left( \frac{\bar{D}^{5D} + \mu \epsilon(x_5)}{\bar{D}^{5D} + \mu} \right), \quad (3.17)$$

which needs a further decomposition into 5-dimensional bulk and 4-dimensional boundary contributions. While our target is the chiral fermion at the 4-dimensional junction at $x_5 = 0$, the standard Pauli–Villars regulator requires the opposite chiral mode as well, to construct a mass term. To this end, we explicitly write the anti-domain wall at $x_5 = L_5$, as was mentioned at the beginning of this section, and introduce a nonlocal coupling to the fermion there. More explicitly, we have

$$\det \left( \frac{\bar{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5)}{\bar{D}^{5D} + \mu} \right) = \text{Det} \left( \frac{\delta(x - x') \bar{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) + \mu_2^{x_5 x_5}}{\delta(x - x') \bar{D}^{5D} + \mu} \right) \times \text{Det} \left( \frac{\delta(x - x') (\bar{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5))}{\delta(x - x') \bar{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) + \mu_2^{x_5 x_5}} \right), \quad (3.18)$$
where $\text{Det}$ denotes the determinant in the \textit{doubled} space-time, so that we can insert a nonlocal mass term

$$
\mu_2^{x_5,x'_5} \equiv \mu_2 \left[ \delta(x_5)\delta(x'_5 - L_5) + \delta(x_5 - L_5)\delta(x'_5) \right].
$$

(3.19)

Note that this mass term violates the 5-dimensional gauge symmetry at $x_5 = 0$ and $x_5 = L_5$ boundaries. This term removes the contribution from the edge-localized modes in the first determinant of Eq. (3.18), while it plays the role of the UV cut-off in the second determinant, which represents our target Weyl fermion. In Appendix B, we present the details of this bulk/edge decomposition.

From the first determinant in Eq. (3.18), we obtain in its imaginary part another CS term restricted to the $x_5 < 0$ region (Ref. [18]):

$$
-\pi \text{CS}^{(x_5 < 0)} \equiv \pi \int_{x_5 = 0} d^5x \frac{4}{3(4\pi)^3} \frac{1 - e(x_5)\epsilon(L_5 - x_5)}{2} e^{\mu_1 \cdots \mu_5} \text{tr} \left[ \frac{1}{2} A_{\mu_1} F_{\mu_2 \mu_3} F_{\mu_4 \mu_5} - \frac{i}{2} A_{\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} A_{\mu_5} \right].
$$

(3.20)

The gauge invariance is violated at the boundaries in $\text{CS}^{(x_5 < 0)}$ due to the gauge noninvariant IR cut-off of the boundary modes:

$$
-\pi \delta \text{CS}^{(x_5 < 0)} = -\frac{i}{24\pi^2} \int_{x_6 = 0, x_5 = 0} d^4x e^{\mu_1 \cdots \mu_4} \text{tr} \left( A_{\mu_4} \partial_{\mu_3} A_{\mu_2} + i \frac{1}{2} A_{\mu_2} A_{\mu_3} A_{\mu_4} \right)(x) + \frac{i}{24\pi^2} \int_{x_6 = 0, x_5 = L_5} d^4x e^{\mu_1 \cdots \mu_4} \text{tr} \left( A_{\mu_4} \partial_{\mu_3} A_{\mu_2} + i \frac{1}{2} A_{\mu_2} A_{\mu_3} A_{\mu_4} \right)(x),
$$

(3.21)

where the gauge transformation is performed as $\delta A_\mu = -i \nabla_\mu \nu(x)$. This form exactly cancels the consistent anomaly (Refs. [43,44]) of the Weyl fermions localized at $x_5 = 0$ and $x_5 = L_5$, and their cancelation is essential to keep the overall gauge invariance of the theory.

Before taking the $M \gg \mu$ limit, the phase of the second determinant in Eq. (3.16) may receive contributions from exotic instantons, which are located in the region $x_5 < 0$. In the limit of $M \gg \mu$, if these instantons are condensed on the 5-dimensional $x_6 = 0$ plane, they could give an integer contribution $\pi F_M^{5D}$. For the second determinant in Eq. (3.18), only the boundary Weyl fermion modes with positive chirality at $x_5 = 0$ and negative chirality at $x_5 = L_5$ contribute, so that

$$
\text{lim}_{\mu \to \infty} \text{Det} \left( \frac{\delta(x - x') (\bar{D}^{5D} + \mu \epsilon(x)\epsilon(L_5 - x_5))}{\delta(x - x') (\bar{D}^{5D} + \mu \epsilon(x)\epsilon(L_5 - x_5)) + \mu_2^{x_5,x'_5}} \right) = \text{det} \frac{\mathcal{D}}{\mathcal{D} + \mu_2},
$$

(3.22)

holds, where $\mathcal{D}$ is defined as

$$
\mathcal{D} = P_-^5 \bar{D}^{4D} P_+^5 + P_+^5 \bar{D}^{4D} P_-^5,
$$

(3.23)

with $\bar{D}^{4D} = \sum_{i=1}^4 \tilde{\gamma}_i \nabla_i |_{x_6 = x_5 = 0}$ and $P_+^5 = (1 \pm \tilde{\gamma}_5)/2$. This determinant is not chiral gauge invariant and produces a consistent gauge anomaly which is precisely canceled by Eq. (3.21). As will be shown later, we define the bulk gauge fields from the 4-dimensional gauge fields at the junction in such a
way that the Dirac operator \( \bar{\psi} \nabla_i |_{x_6=0,3s=L_5} \) becomes that for an (almost) free fermion, so that the negative chiral mode at \( x_5 = L_5 \) is decoupled from the theory.

The complex phase of the determinant Eq. (3.22) is thus expressed by

\[
-\frac{i}{\pi} \frac{\pi}{2} \eta^{4D} + i\phi^{\text{anom}},
\]

where \( \eta^{4D} \) is the gauge-invariant part, while \( \phi^{\text{anom}} \) is the anomalous part.5

What we have obtained is the anomaly ladder

\[
\frac{\phi^{\text{total}}}{\pi} = T^{6D}_{x_6<0} + CS - \frac{\eta^{5D}}{2},
\]

\[
\frac{1}{2} \eta^{5D} = CS^{(x_5<0)} + T^{5D}_{M \gg \mu} + \frac{1}{2} \eta^{4D} - \frac{\phi^{\text{anom}}}{\pi},
\]

where \( T^{6D}_{x_6<0} \) denotes the 6-dimensional U(1)_A anomaly, CS and \( CS^{(x_5<0)} \) represent the 5-dimensional parity anomaly, and \( \phi^{\text{anom}} \) is the source for the consistent gauge anomaly. This result is consistent with the anomaly descent equations found by Stora (Ref. [27]) and Zumino (Refs. [29,31]), including the overall constant determined by Alvarez-Gaumé and Ginsparg (Ref. [32]) and Sumitani (Ref. [33]).

When the (perturbative) gauge anomaly is absent, as well as the number of fundamental fermions is even (to cancel the global anomaly), the formula reduces to

\[
\sum_f \phi^{\text{total}}_f = - \sum_f \frac{\pi}{2} \eta^{4D}_f ,
\]

mod 2\( \pi \), where we have added the subscript \( f \) to represent the flavor of different fermions. This means that the complex phase of the total fermion determinant essentially comes from the 4-dimensional edge modes, at least in the hierarchical limit of \( M \gg \mu \gg 0 \).

### 3.3. Anomaly inflow through the \( \mu \) domain wall

In order to trace the anomaly inflow via the \( \mu \) domain wall, let us take the limit \( \mu \gg M \gg 0 \). Ignoring \( M \), the fermion determinant can be decomposed into three parts,

\[
\det \left( \frac{D^{6D} + i\mu \epsilon (x_5) \gamma_6 \gamma_7}{D^{6D} + i\mu \gamma_6 \gamma_7} \right) = \det \left( \frac{D^{6D} + i\mu \gamma_6 \gamma_7 + \mu_2}{D^{6D} + i\mu \gamma_6 \gamma_7} \right) 
\]

\[
\times \det \left( \frac{D^{6D} + i\mu \epsilon (x_5) \gamma_6 \gamma_7 + \mu_2}{D^{6D} + i\mu \gamma_6 \gamma_7 + \mu_2} \right) 
\]

\[
\times \det \left( \frac{D^{6D} + i\mu \epsilon (x_5) \gamma_6 \gamma_7 + \mu_2}{D^{6D} + i\mu \gamma_6 \gamma_7 + \mu_2} \right),
\]

where we take the \( \mu \gg \mu_2 \gg 0 \) limit. Note that the first determinant converges to unity.

Unlike the \( M \) domain wall, the second determinant does not produce the axial U(1) anomaly. Due to the explicit violation of the SO(6) Lorentz symmetry by the axial vector background, the phase \( \phi^{6D} \) of the second determinant can be expanded in an SO(5) invariant series of 1/\( \mu \), except for

---

5 If we can nonperturbatively evaluate the 4-dimensional determinant phase, and perform a random gauge transformation on it, it is possible to smear out \( \phi^{\text{anom}} \) and extract the gauge-invariant phase \( \eta^{4D} \). It would, however, be practically difficult to separate \( \eta^{4D} \) and \( \phi^{\text{anom}} \) of a single Weyl fermion. Only the summation of \( \eta^{4D} \) for the anomaly-free combination would be possible to extract.
the nonperturbative zero mode’s contribution \( \pi T_{x_5<0}^{6D} \), which is located in the region \( x_5 < 0 \). More explicitly, we have

\[
\phi^{6D} = \pi T_{x_5<0}^{6D} + \mu \phi^{(1)} + \phi^{(2)}/\mu + \mathcal{O}(1/\mu^3),
\]

(3.28)

where, the leading-order contribution has the form of a Chern–Simons term,

\[
\phi^{(1)} = c_0 \pi \int d^6x \frac{4}{3(4\pi)^3} \frac{1 - \epsilon (x_5)}{2} \epsilon_{i_1 \cdots i_5} \begin{bmatrix} \frac{1}{2} A_{i_1} F_{i_2 i_3} F_{i_4 i_5} \\
- \frac{i}{2} A_{i_1} A_{i_2} A_{i_3} F_{i_4 i_5} - \frac{1}{5} A_{i_1} A_{i_2} A_{i_3} A_{i_4} A_{i_5} \end{bmatrix},
\]

(3.29)

and the next-to-leading-order contribution is

\[
\phi^{(2)} = \int d^6x \frac{1 - \epsilon (x_5)}{2} \epsilon_{i_1 \cdots i_5} \begin{bmatrix} c_1 D_4 F_{i_1} F_{i_2 i_3} F_{i_4 i_5} + c_2 D_6 F_{i_1} F_{i_2 i_3} F_{i_4 i_5} \\
+ c_3 D_6 F_{i_5} F_{i_6 i_3} F_{i_4 i_5} \end{bmatrix},
\]

(3.30)

where \( c_k \) are numerical constants\(^6\) determined by the representation of the fermion, and the summation of the indices is taken in 5 dimensions only. In the above computation, \( \mu_2 \) plays the role of an infrared cut-off, removing the contribution from the edge-localized modes.

As will be discussed later, when the theory is free from both perturbative and global anomalies, the total complex phase of the determinants is

\[
\sum_f \phi_f^{6D} = \sum_f \phi_f^{(2)}/\mu + \mathcal{O}(1/\mu^3),
\]

(3.31)

which disappears in the limit \( \mu \rightarrow \infty \). Here the subscript \( f \) denotes the flavor index.

In the third determinant in Eq. (3.27), only the boundary localized modes on the \( \mu \) domain wall, which satisfy \( i\gamma_5 \gamma_6 \gamma_\gamma \psi = \psi \), can contribute. Here we see a significant difference from the previous subsection, where we obtained a single massless Dirac fermion determinant in Eq. (3.12). What we obtain here is not a single Dirac fermion but two (4-component) Dirac fermions having Pauli–Villars masses \( \pm \mu_2 \) with opposite signs, that are constrained to have the positive eigenvalue of the gamma matrix \( \gamma_5 \):

\[
\lim_{\mu \rightarrow \infty} \det \left( \frac{D^{6D} + i\mu \epsilon (x_5) \gamma_6 \gamma_\gamma + \mu_2}{D^{6D} + i\mu \epsilon (x_5) \gamma_6 \gamma_\gamma + \mu_2} \right) = \det \left( P_+ \frac{\hat{D}^{5D}}{\hat{D}^{5D} + \mu_2} P_+ + P_- \right) \times \det \left( P_+ \frac{\hat{D}^{5D}}{\hat{D}^{5D} - \mu_2} P_+ + P_- \right),
\]

(3.32)

where \( \hat{D}^{5D} = (\sum_{i=1}^4 \gamma_5 \nabla_i + \gamma_5 \partial_6) |_{x_5=0} \), and \( P_\pm \equiv (1 \pm \gamma_5)/2 \) is a Hermitian projection operator. In Appendix C we present the details of our computation.

\(^6\) We find that \( c_0 \) is logarithmically divergent since a single Pauli–Villars spinor is not enough to regularize the determinant. Modifying the second determinant of Eq. (3.27) to

\[
\det \left( \frac{D^{6D} + i\mu \epsilon (x_5) \gamma_6 \gamma_\gamma + \mu_2}{D^{6D} + i\mu \epsilon (x_5) \gamma_6 \gamma_\gamma + \mu_2} \right) \times \det \left( \frac{D^{6D} + i\lambda \gamma_6 \gamma_\gamma + \mu_2}{D^{6D} + i\lambda \epsilon (x_5) \gamma_6 \gamma_\gamma + \mu_2} \right)^{\mu/\lambda},
\]

we obtain a finite value of \( c_0 \) proportional to ln \( \Lambda \). Here, \( \Lambda \gg \mu \gg \mu_2 \gg 0 \) is assumed.
The determinant (3.32) is real and its phase can be defined as \( \pi T^{5D} \). This means that only the mod-two-type exotic index can communicate with the lower dimensions. No CS term appears, and therefore, no source of the perturbative gauge anomaly on the \( \mu \) domain wall.

Now, let us turn on the \( M \) domain wall and consider the limit \( \mu \gg M \gg 0 \):

\[
\det \left( \frac{D^{6D} + M \epsilon(x_6) + i \mu \epsilon(x_5) \gamma_6 \gamma_7}{D^{6D} + M + i \mu \gamma_6 \gamma_7} \right) = \det \left( \frac{D^{6D} + i \mu \epsilon(x_5) \gamma_6 \gamma_7 + M}{D^{6D} + i \mu \gamma_6 \gamma_7 + M} \right) \times \det \left( \frac{D^{6D} + i \mu \epsilon(x_5) \gamma_6 \gamma_7 + M \epsilon(x_6)}{D^{6D} + i \mu \epsilon(x_5) \gamma_6 \gamma_7 + M} \right),
\]

(3.33)

where the first determinant gives the same result as those in Eq. (3.27). Namely they produce the phase \( \phi^{6D} \).

The second determinant in Eq. (3.33) in the \( \mu \rightarrow \infty \) limit is

\[
\det \left( P_+ (\hat{D}^{5D} + M)^{-1} (\hat{D}^{5D} + M \epsilon(x_6)) P_+ + P_- \right) \times \det \left( P_+ (\hat{D}^{5D} - M)^{-1} (\hat{D}^{5D} - M \epsilon(x_6)) P_+ + P_- \right).
\]

(3.34)

This expression is \textit{almost} real, except for the domain wall \( x_6 = 0 \), since the complex phase comes from the noncommutativity of \( \hat{D}^{5D} \) and \( M \epsilon(x_6) \), which is proportional to \( \delta(x_6) \).

Let us further decompose Eq. (3.34) as

\[
\det \left( P_+ (\hat{D}^{5D} + M)^{-1} (\hat{D}^{5D} + M \epsilon(x_6) + M_2) P_+ + P_- \right) \times \det \left( P_+ (\hat{D}^{5D} - M)^{-1} (\hat{D}^{5D} - M \epsilon(x_6) - M_2) P_+ + P_- \right) \times \det \left( P_+ (\hat{D}^{5D} + M \epsilon(x_6) + M_2)^{-1} (\hat{D}^{5D} + M \epsilon(x_6)) P_+ + P_- \right) \times \det \left( P_+ (\hat{D}^{5D} - M \epsilon(x_6) - M_2)^{-1} (\hat{D}^{5D} - M \epsilon(x_6)) P_+ + P_- \right)
\]

(3.35)

and take the limit \( M \gg M_2 \gg 0 \). In this case, \( M_2 \) cannot completely separate the bulk and the edge modes, due to the projection operator \( P_+ \). Here we can only say that the total determinant is complex, whose phase is almost localized at \( x_6 = 0 \).

In the third and fourth determinants in Eq. (3.35), we observe an interesting dynamic. First of all, this combination of two determinants is real and positive. Therefore, the nontrivial complex phase resides in the first and second determinants, in a gauge-invariant way. Second, the Weyl fermions appear only in the third determinant, since only the positive chirality mode survives the projection \( P_+ \). The fourth determinant then contains the contribution from the bulk modes (and possibly from

---

7 The formula looks not only real but positive. The nontrivial phase \( \pi T^{5D} \), however, comes from the zero modes, where the determinant becomes ill defined. Therefore, in order to precisely compute \( \pi T^{5D} \), we need a careful massless limit from the massive determinant, as well as an appropriate regularization to count the number of exotic zero modes \( T^{5D} \). Since a good regularization should not break the complex conjugate pairs of nonzero modes, we can generally claim that the phase is \( \pi T^{5D} \).

8 As \( \hat{D}^{5D} \) and \( P_+ \) do not commute with \( M \epsilon(x_6) \), we make the order of the matrix operations explicit.
the doubler modes when we take a lattice regularization). Therefore, we are left with
\[
\lim_{M \to \infty} \det \left( P_+ (\hat{D}^{5D} + M \epsilon(x_6) + M_2)^{-1} (\hat{D}^{5D} + M \epsilon(x_6)) P_+ + P_- \right)
\times \det \left( P_+ (\hat{D}^{5D} - M \epsilon(x_6) - M_2)^{-1} (\hat{D}^{5D} - M \epsilon(x_6)) P_+ + P_- \right)
\propto \det \left( P^5_5 (\hat{D}^{4D} + M \epsilon(x_6)^2 + P_+ + P_-) \right) \times \exp(-i \phi^{an})
\]
(3.36)
where \( \hat{D}^{4D} = \sum_{i=1}^4 \tilde{y}_i \nabla_{i\xi_5=\xi_5=0} \), representing a single Weyl fermion determinant. Note that the phase \( \phi^{an} \) cannot be written as any local effective action in 4-dimensions. However, we already know its origin. It is the CS term on the \( M \) domain wall. It is hidden in the nonlocal phase \( \phi^{an} \) since we have integrated the bulk contribution in the 5th direction first.

From the above analysis, we may write the phase of the second determinant in Eq. (3.33) as
\[
\pi I_{5D} x_{5 < 0} - \frac{\pi}{2} \eta^{4D},
\]
up to some regularization-dependent term (which will be neglected below).9 Then the phase of the total 6D determinant can be decomposed as
\[
\phi^{total} = \pi I_{5D} x_{5 < 0} + \mu \phi^{(1)} + \phi^{(2)} \mu + \pi I_{5D} x_{6 < 0} - \frac{\pi}{2} \eta^{4D} + \mathcal{O}(1/\mu^3),
\]
(3.38)
where \( I_{5D}^{6D} \) and \( I_{5D}^{5D} \) are the exotic indices in the 6-dimensional bulk and the 5-dimensional \( \mu \) domain wall, respectively. When the theory has an anomaly-free combination of the fermion flavors, the total phase is
\[
\sum_f \phi^{total}_f = - \sum_f \frac{\pi}{2} \eta^{4D}_f + \mathcal{O}(1/\mu f),
\]
(3.39)
where \( f \) denotes the flavor index. Namely, the complex phase is determined by the fermion modes localized at the 4-dimensional junction in the \( \mu \to \infty \) limit, which is consistent with another \( M \gg \mu \gg 0 \) limit already seen in Eq. (3.26).

### 3.4. Domain-wall junction

In the previous two subsections, we traced two different anomaly inflows taking \( M \gg \mu \gg 0 \) and \( \mu \gg M \gg 0 \) limits. At finite \( M \) and \( \mu \), the situation can be more complicated but the nontrivial cancelation of anomalies among different dimensions should be maintained to keep the gauge invariance of the total theory. In the end, a single Weyl fermion always appears at the junction of the two domain walls.

When a small gauge transformation is performed at the 4-dimensional junction, the gauge current flows through the \( M \) domain wall, but never flows into the \( \mu \) domain wall, since there is no CS term that can absorb the gauge noninvariance. Instead, a large gauge transformation can create exotic instantons on the \( \mu \) domain wall and flip the sign of the partition function. Thus, we confirm that the perturbative anomaly inflow, which naturally exhibits the Stora–Zumino anomaly descent equations, is mediated by the \( M \) domain wall, while the inflow of the global anomaly goes through the \( \mu \) domain wall (see Fig. 1).

---

9 Since the \( M \gg \mu \gg 0 \) and \( \mu \gg M \gg 0 \) limits may give different regularizations of the low-energy effective theory, the remaining phase in the 4-dimensional junction may differ by \( f(\infty) - f(0) \), where \( f(M/\mu) \) is the regularization-dependent function.
4. Anomaly-free condition

Due to the topological obstructions of the U(1)\textsubscript{A} and P\textsuperscript{\prime} symmetries, a single Weyl fermion cannot be described by a 4-dimensional local field theory. If these anomalies are canceled among different flavors, the net anomaly inflow down to the 4-dimensional junction vanishes, and the chiral gauge current can be conserved, realizing a consistent 4-dimensional theory in the low-energy limit.

The cancelation of the U(1)\textsubscript{A} anomaly is assured if

\begin{equation}
\sum_{L} \text{tr} \left(T_{L}^{a} \{T_{L}^{b}, T_{L}^{c}\}\right) - \sum_{R} \text{tr} \left(T_{R}^{a} \{T_{R}^{b}, T_{R}^{c}\}\right) = 0,
\end{equation}

where \(T_{L/R}\) denote the gauge group generators in L/R representation of the corresponding left-/right-handed fermions. This is the well-known anomaly-free condition of the perturbative chiral gauge invariance. In our formulation, condition (4.1) guarantees the cancelation of the U(1)\textsubscript{A} anomaly as well as the CS term on the \(M\) domain wall, so that the gauge current never flows out of the 4-dimensional junction.

For global anomalies in 4-dimensions, it is usual to consider only the case with the SU(2) group. This is because the map from a 4-dimensional sphere \(S^{4}\) to the gauge group \(G: \pi_{4}(G)\) is only nontrivial for SU(2). In our formulation, this SU(2) anomaly is embedded as the phase of the 6-dimensional Dirac fermions through the APS(-like) index relation

\begin{equation}
\pi_{5}(SU(2)) = \mathbb{Z}_{2} \sim \pi_{4}(SU(2)) = \mathbb{Z}_{2}.
\end{equation}

This homomorphism is not found in the literature on physics. To cancel the SU(2) anomaly, we need an even number of fundamental fermions so that the gauge transformation never flips the sign of the total partition function.

However, the cancelation of the global anomalies is more nontrivial, as discussed in Refs. [21,22]. The global anomaly should be absent not only on a simple manifold like \(S^{4}\) or \(S^{5}\) but also on any compact manifold. Our setup on the 6-dimensional torus having domain-wall junctions of 4-dimensional tori, is already such a nontrivial example.

In fact, Lüscher found in the construction of a U(1) chiral gauge theory on the lattice (Ref. [7]), that a condition

\begin{equation}
\text{number fermions with odd charges} = \text{even}
\end{equation}

is necessary for the cancelation of the global anomaly.
is required to keep the nonperturbative chiral gauge invariance, although it was not clearly identified as one of the global anomalies.\textsuperscript{10} This is not surprising since on the 4-dimensional torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$, at least one cycle may develop a nontrivial map: $\pi_1(U(1)) = \mathbb{Z}$, even when the perturbative anomaly is absent.

In this paper, we do not try to extensively classify the global anomalies but just mention that if

\begin{equation}
\text{number fermions in the fundamental representation} = \text{even},
\end{equation}

after the irreducible decomposition, our 6-dimensional theory is free from the global anomalies that originate from the exotic index $I$. The standard model of particle physics satisfies the above condition if we identify $e/6$ as a unit charge of the hypercharge.

The above anomaly-free conditions are those that must be satisfied in the continuum limit. At a finite cut-off, we have to further control the remaining violation of the gauge invariance, since the anomaly cancelation is not perfect. This is due to the fact that the bulk determinant respects the 6-dimensional gauge invariance, which is not the one in our target 4-dimensional theory. As will be discussed in the next section we follow the strategy in Ref. [16] to use the Yang–Mills gradient flow in the 5th and 6th directions. The gradient flow realizes a kind of dimensional extension so that the fermions in the extra (flavor) space, share the same 4-dimensional gauge invariance.

One disadvantage of taking the gradient flow in both the 5th and 6th directions is that the role of the $\mu$ domain wall in detecting the global anomaly becomes obscure. Since the flowed gauge fields are invariant under any gauge transformations, it is unlikely to have nonzero index $I$ on the $\mu$ domain wall, which requires a nontrivial response to large gauge transformations. This means that the lattice formulation cannot detect inconsistencies of the gauge theory with odd number of flavors, which is anomalous under global gauge transformation. To circumvent this problem one should look for a better formulation which uses an extended gauge field sensitive to the global anomaly yet keeping the perturbative gauge invariance. We leave it as an open problem.

In this work, we take the following practical solution, which is similar in spirit to Ref. [22]. It is argued in Ref. [22] that some global anomalies cannot be detected on the mapping torus, which is a standard setup for discussing global anomalies, but can appear on other manifolds. In such theories, the mapping torus is in a sense an unlucky setup which cannot distinguish the anomalous and nonanomalous fermion contents. We may regard our setup using the Yang–Mills gradient flow as a similar unlucky example. Namely, to discuss both perturbative and global anomalies, we should use the general background of 6-dimensional gauge fields. Once the anomaly-free conditions are obtained in this general setup, then we may restrict the gauge fields using the gradient flow, to construct the target 4-dimensional gauge theory.

5. Decoupling of the mirror fermions

So far we have not discussed the effects of the anti-domain walls. In order to realize a single Weyl fermion, the massless modes at other domain-wall junctions must be decoupled from the theory.

First, we take the spatial extents in the 5th and 6th directions to be finite in the ranges $-L_5 < x_5 \leq L_5$ and $-L_6 < x_6 \leq L_6$. We take the fermion fields to satisfy periodic boundary conditions, which

\textsuperscript{10} Similar inconsistencies in U(1) chiral gauge theories on 2-dimensional tori were reported in Refs. [45] and [46]. It was also reported in Ref. [47] that the global anomaly cancelation in the SU(2) theory for $4n + 3/2$ representations is not trivial. The global anomalies for these cases may need to be reexamined on various manifolds.
requires (at least) one $M$ anti-domain wall at $x_6 = L_6(= -L_6)$ and one $\mu$ anti-domain wall at $x_5 = L_5(= -L_5)$. Our fermion determinant is now

$$\exp(-W_{2DW}) = \det \left( \frac{D^{6D} + M\epsilon(x_6)\epsilon(L_6 - x_6) + i\mu\epsilon(x_5)\epsilon(L_5 - x_5)\gamma_6\gamma_7}{D^{6D} + M + i\mu\gamma_6\gamma_7} \right),$$  \hspace{1cm} (5.1)$$

in which we have 4 domain-wall junctions. Two Weyl fermion modes with positive chirality appear at $(x_5, x_6) = (0, 0)$ and $(0, L_6)$, while those with negative chirality are localized at $(x_5, x_6) = (L_5, 0)$ and $(L_5, L_6)$.

Among these 4 junctions, only the one at $(x_5, x_6) = (0, 0)$ is needed to construct our world in 4-dimensions, and we would like the Weyl fermions at the other three junctions to be decoupled from the gauge fields. To achieve this, we use the profile of the gauge field in the 5th and 6th directions using the Yang–Mills gradient flow, following the idea in Ref. [16]. The gradient flow exponentially weakens the gauge fields with the flow time so that the Weyl fermions at $x_5 = L_5$ and $x_6 = L_6$ are decoupled from the gauge fields. As flowed gauge fields transform in the same way as the original fields, we can maintain the 4-dimensional gauge invariance of the total theory.

More explicitly, we take

$$A_\mu(x, x_5, x_6) = A_{\mu}^{x|x_5|+|x_6|}(\tilde{x}) \quad (\mu = 1, \ldots, 4), \quad A_5 = A_6 = 0,$$  \hspace{1cm} (5.2)$$

where $\tilde{x} = (x_1, x_2, x_3, x_4)$ is the coordinate of the 4-dimensional torus, while $A_\mu^t$ denotes the solution of the Yang–Mills gradient flow at a flow time $t$,

$$\frac{\partial}{\partial t}A_\mu^t = \frac{\xi(t)}{M}D_\nu^t F_{\nu\mu}^t \quad (\mu, \nu = 1, \ldots, 4),$$  \hspace{1cm} (5.3)$$

where $D_\nu$ and $F_{\nu\mu}^t$ are the covariant derivatives and field strengths with respect to the flowed gauge field $A_\mu^t$, respectively. The term $\xi$ is an arbitrary constant of order 1. Here $A_\mu^0 \equiv A_\mu(\tilde{x})$ is the physical dynamical variable over which we integrate in the path integral. Our finite volume setup is shown in Fig. 2.

Recently, Okumura and Suzuki (Ref. [48]) found that the mirror fermions in 4-dimensional effective theory (Refs. [49,50]) using the Yang–Mills gradient flow in a 5-dimensional domain wall setup are not completely decoupled from the gauge fields. This can be seen by the exact conservation of the total fermion numbers of physical and mirror fermions, which implies that the mirror fermions are sensitive to the topology of the original gauge fields even after the gradient flow, and the resulting theory should have nonlocal properties due to these remnants of mirror fermions.

This problem of nonlocality is inherited by our 6-dimensional model, unless we give up employing the Yang–Mills gradient flow. Since the procedure of fixing the 6-dimensional gauge fields using the 4-dimensional configuration itself is already nonlocal in terms of 6-dimensional quantum field theory, it might be safer if we can achieve a mechanism of decoupling mirror fermions in a local and dynamical way in 6-dimensional field theory setups. However, we have not found any such formulation realizing the localization of gauge fields at the domain-wall junction.

6. The choice of $\mu$ domain-wall operator

In this work, we have chosen the axial vector back ground $i\mu\epsilon(x_5)\gamma_6\gamma_7$, which is insensitive to $U(1)_A$ to realize the $\mu$ domain wall. This choice is, however, not the unique solution for having chiral modes at the domain-wall junction. For example, we find that for the operators

$$i\mu\epsilon(x_5)\gamma_6\gamma_7 R_6 \quad \text{or} \quad i\mu\epsilon(x_5)\gamma_6\gamma_7 R_5 R_6,$$  \hspace{1cm} (6.1)$$
the 4-dimensional localized solution in Eq. (3.3) is unchanged. The structure of the anomalies is, however, different among these operators. In particular, the use of \( i\mu\epsilon(x_5)\gamma_6\gamma_7R_5R_6 \) makes the total fermion determinant real, even when the theory is anomalous. It seems that the nonlocality induced by the reflection operators \( R_5 \) and \( R_6 \) makes an unwanted cancelation of the complex phase, including the phase that should survive in the continuum limit.

It is unclear whether the \( \mu \) domain wall and associated \( P' \) anomaly necessarily and sufficiently classify the global anomalies. For lower dimensions than 6, we find only mod-two-type indices as in the SU(2) anomaly, and our \( \mu \) domain wall looks appropriately detecting them. In higher dimensions, however, we have more nontrivial indices, e.g., \( \pi_6(\text{SU}(2)) = \mathbb{Z}_{12} \). We do not understand what happens if we extend our formulation to 8 dimensions or higher. A more mathematically precise treatment of our system would be required to fully understand this.

Another interesting possibility is to use a simple pseudoscalar operator, which was studied in a previous work by Neuberger (Ref. [54]),

\[ i\mu\epsilon(x_5)\gamma_7, \]  

which is a twisted mass under \( U(1)_A \) rotation. The fermion determinant

\[
\det \left( \frac{D^{6D} + M\epsilon(x_6) + i\mu\epsilon(x_5)\gamma_7}{D^{6D} + M + i\mu\gamma} \right) 
\]  

(6.3)

has a single Weyl fermion mode in the low-energy limit, too. However, as the pseudoscalar operator is odd in either \( P' \) and (\( \pi \) rotation of) \( U(1)_A \), both of the two domain walls produce the CS terms and the relation to the global anomaly is unclear.

The detailed mechanism of possible unphysical cancelations of the complex phase of the fermion determinant, and how to choose the appropriate domain-wall operators, need a further investigation.

7. A proposal for lattice regularization

Since our formulation is based on a massive Dirac fermion in 6 dimensions, it is natural to assume that a nonperturbative lattice regularization using the Wilson fermion is available, as it shares the same symmetries as in the continuum formulation. Here we give just a simple proposal for how to regularize our 6-dimensional Dirac fermion system on a lattice. Detailed analysis of the locality of the resulting 4-dimensional theory, decoupling the doublers, modified chiral gauge symmetry, etc. will be discussed elsewhere.

First we pick up a set of link variables \( \{U_\mu(\bar{x})\} (\mu = 1, \ldots, 4) \) on the 4-dimensional junction at \((x_5, x_6) = (0, 0)\). Then we solve the lattice version of the Yang–Mills gradient flow equation,

\[
\frac{\partial}{\partial t}U^I_\mu(\bar{x}) = -\left\{ \partial_{\bar{x}\mu}S_G(U^I) \right\} U^I_\mu(\bar{x}), \]  

(7.1)

using \( U^0_\mu(\bar{x}) = U_\mu(\bar{x}) \) as the initial condition, where \( \partial_{\bar{x}\mu}S_G(U^I) \) denotes the Lie derivative of the gauge action \( S_G(U^I) \) with respect to \( U^I_\mu(\bar{x}) \), to define

\[
U_\mu(\bar{x}, x_5, x_6) = U^{\lfloor |x_5|+|x_6| \rfloor}(\bar{x}). \]  

(7.2)

Here we always set \( U_5 = U_6 = \text{unity} \). Note that the resulting link variables \( U_\mu(\bar{x}, x_5, x_6) \) are symmetric under \( x_5 \to -x_5 \) and \( x_6 \to -x_6 \).
Fig. 2. Schematic view of our 6-dimensional finite space. The ± symbols show the Weyl modes with positive and negative chiralities, localized at each of the four domain-wall junctions. The case with $M > 0$ and $\mu > 0$ is shown. Our target Weyl fermion with positive chirality is localized at the origin, while the other three Weyl fermions are decoupled from the gauge fields by the gradient flow.

We are now ready to define the 4-dimensional path integral of anomaly-free theory with Weyl fermions. Together with the gauge part of the action $S_G([U_\mu(\vec{x})])$, we define

$$\int DU_\mu(\vec{x}) \exp (-S_G([U_\mu(\vec{x})])) \prod_i \exp \left[ -W^i_{\text{lat}}([U_\mu(\vec{x})]) \right], \quad (7.3)$$

where

$$\exp \left[ -W^i_{\text{lat}}([U_\mu(\vec{x})]) \right] = \det \left( \begin{array}{cc} D_{W}^{6D R_l} + M_l \epsilon(x_6 - a/2)\epsilon(L_6 - x_6 - a/2) + i \mu_i \epsilon(x_5 - a/2)\epsilon(L_5 - x_5 - a/2)\gamma_6\gamma_7 \\ D_{W}^{6D R_l} + M_l + i \mu_i \gamma_6\gamma_7 \end{array} \right), \quad (7.4)$$

where $D_{W}^{6D R_l}$ denotes the Wilson Dirac operator in the $R_l$ representation of the gauge group, and $M_l$ and $\mu_i$ are chosen to be positive/negative for positive/negative chiral modes. Note that the Wilson term has to have the opposite sign to $M_l$ and $\mu_i$. These mass parameters are to be of the order of the lattice cut-off $1/a$. However, to avoid contamination from the doubler modes, $M_l$ and $\mu_i$ should have upper bounds, too.

In the above formula, the argument of the sign functions is shifted by $-a/2$ with the lattice spacing $a$ so that it is well defined on integer values of coordinates on the lattice. We always assume that the set of fermion flavors satisfy the anomaly-free conditions (4.1) and (4.4).

As a final remark for this section, we note that the full chiral gauge symmetry will not be satisfied until we take the $L_5 = L_6 = \infty$ limits.

8. Summary and discussion

We have proposed a 6-dimensional regularization of chiral gauge theories in 4 dimensions. Using the two different kinds of domain walls, we have succeeded in localizing a single Weyl fermion at the junction of the domain walls. One domain wall is made giving a kink mass in the 6th direction to the fermions, while another domain wall is made by giving a kink structure in the 5th direction to a background operator which is insensitive to the U(1)$_A$ rotation.

The conventional $M$ domain wall mediates the perturbative anomaly inflow and naturally exhibits the chain of the 6-dimensional U(1)$_A$, 5-dimensional parity, and 4-dimensional gauge anomalies,
known as the descent equations, found by Stora (Ref. [27]) and Zumino (Refs. [29,31]). On another
domain wall, the fermions are forced to form an (almost) real representation and only mediates
the mod-two type anomaly, which we have assumed to be the source of the global anomalies.

The anomaly-free condition of the target 4-dimensional gauge theory is translated to the one for
the set of 6-dimensional Dirac fermion determinants to keep the axial $U(1)$ and $P'$ symmetries.
Using the Yang–Mills gradient flow in the 5th and 6th directions, we can control the remnant of
gauge noninvariance due to the finite cut-offs, and decouple the Weyl fermions at the junctions of
anti-domain walls. As our formulation is nothing but a massive vector-like theory, we expect that a
nonperturbative regularization on a lattice is possible, using standard Wilson Dirac fermions.

There are still a lot of open issues to be investigated. There is an arbitrariness in the choice of
the $\mu$ domain-wall operator to realize a single Weyl fermion at the domain-wall junction. It is also
unclear whether the $\mu$ domain wall and associated $P'$ anomaly necessarily and sufficiently classify
the global anomalies.

In even dimensions, the $P'$ symmetry and its anomaly are usually neglected. Our work, however,
suggests its relation to the global anomalies in lower dimensions. If we can formulate the $P'$ anomaly
on a lattice, the lattice Dirac operator could have a modified $P'$ symmetry, analogous to the modified
chiral symmetry (Ref. [52]) through the Ginsparg–Wilson relation (Ref. [53]). It is an interesting
question whether the modified Dirac operator realizes the exotic mod-$n$ index theorems, identifying
explicit link variable configurations that give nontrivial indices on the lattice.

In our formulation, we have switched off the gauge fields in the directions of extra dimensions and
use the Yang–Mills gradient flow to maintain 4-dimensional gauge invariance. One concern is that
this treatment of the gauge fields is nonlocal in the extra-dimensions and may not fully decouple the
mirror fermions, which has already been discussed in Ref. [16]. It is then an interesting question
whether our formulation can be extended to a model with physical extra dimensions in the gauge
sector also. Such a direction may be linked to studies of higher dimensions beyond the standard
models (Ref. [51]).

Our formulation suggests that there is the possibility of doubly gapped topological insulators
in 4 dimensions, having a conducting mode on 2-dimensional edges, which may be realized in
condensed-matter systems.

Finally, it would be great if we could incorporate the Higgs field in our 6-dimensional lattice and
give a nonperturbative definition of the standard model, which is also an interesting subject for
further study.

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Appendix A. Gamma matrices
Although our results do not depend on the basis of the gamma matrices, we summarize here the most
convenient one to make our analysis simple.
For Euclidean 4-dimensional gamma matrices, we use the so-called chiral representation:

\[
\tilde{\gamma}_{i=1,2,3} = \begin{pmatrix} -i\sigma_i \\ i\sigma_i \end{pmatrix}, \quad \tilde{\gamma}_4 = \begin{pmatrix} \mathbb{I} \\ \mathbb{I} \end{pmatrix},
\]

(A1)

where \(\sigma_i\) denotes the Pauli matrices and \(\mathbb{I}\) is the 2 \(\times\) 2 identity matrix.

In this paper, we also introduce another set of the gamma matrices,

\[
\tilde{\gamma}'_{i=1,2,3,4} = i\tilde{\gamma}_5\tilde{\gamma}_i, \quad \tilde{\gamma}'_5 = \tilde{\gamma}_5.
\]

(A2)

Note that the matrices \(\tilde{\gamma}'_i\) satisfy the same Clifford algebra as \(\tilde{\gamma}_i\).

For the 8 \(\times\) 8 gamma matrices in 6-dimensions, we use

\[
\gamma_i = \begin{pmatrix} \tilde{\gamma}_i \\ -\tilde{\gamma}_i \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} i\mathbb{I} \\ -i\mathbb{I} \end{pmatrix}, \quad \gamma_6 = \begin{pmatrix} \mathbb{I} \\ \mathbb{I} \end{pmatrix},
\]

(A3)

where \(\mathbb{I}\) is the 4 \(\times\) 4 identity matrix.

With these gamma matrices, the chiral operators are given as

\[
\tilde{\gamma}_5 = \begin{pmatrix} \mathbb{I} \\ -\mathbb{I} \end{pmatrix}, \quad \gamma_7 = \begin{pmatrix} \tilde{\gamma}_5 \\ -\tilde{\gamma}_5 \end{pmatrix}.
\]

(A4)

It is also useful to note that \(i\gamma_5\gamma_6\gamma_7\) is represented by

\[
i\gamma_5\gamma_6\gamma_7 = \begin{pmatrix} \tilde{\gamma}_5 \\ \tilde{\gamma}_5 \end{pmatrix},
\]

(A5)

so that one can easily confirm that the constraints \(\gamma_6 = \pm 1\) and \(i\gamma_5\gamma_6\gamma_7 = \pm 1\) on the 6-dimensional spinor lead to \(\tilde{\gamma}_5 = \pm 1\) on the 4-dimensional spinor.

Appendix B. Bulk/edge decomposition of the 5-dimensional domain-wall fermion determinant

It was shown a long ago by Callan and Harvey (Ref. [18]) that the 5-dimensional domain-wall fermion determinant can be decomposed into a bulk part, which produces the CS term, and an edge part, which converges to the Weyl fermion determinant, canceling gauge noninvariance with each other. However, there has been no explicit formula for the decomposition, except for the one at one-loop level (Ref. [20]). Here we propose a nonperturbative method for the bulk/edge decomposition.

The difficulty in the decomposition is in the fact that we have to introduce a gauge nonsymmetric regulator to separate the bulk and edge modes. For example, if we introduce a simple mass \(\mu_2\) for this,

\[
\det \left( \frac{\tilde{D}^{5D} + \mu (x_5) \epsilon (L_5 - x_5)}{\tilde{D}^{5D} + \mu} \right) = \det \left( \frac{\tilde{D}^{5D} + \mu (x_5) \epsilon (L_5 - x_5) + \mu_2}{\tilde{D}^{5D} + \mu} \right) \times \det \left( \frac{\tilde{D}^{5D} + \mu (x_5) \epsilon (L_5 - x_5) + \mu_2}{\tilde{D}^{5D} + \mu (x_5) \epsilon (L_5 - x_5) + \mu_2} \right),
\]

(B1)

we end up with a Weyl fermion determinant, which produces the so-called covariant anomaly. This means that the decomposition is not complete, but the high-energy modes still have a part of the boundary effective action, which compensates the difference between the consistent and covariant anomalies.
Here we introduce a mass term that breaks the gauge symmetry only at the boundaries \( x_5 = 0 \) and \( x_5 = L_5 \):

\[
\mu_2 [\bar{\psi}(\bar{x}, 0)\psi(\bar{x}, L_5) + \bar{\psi}(\bar{x}, L_5)\psi(\bar{x}, 0)],
\]

where \( \bar{x} = (x_1, x_2, x_3, x_4) \). Note that this is the conventional mass term used in the domain-wall fermions in vector-like theories. The fermion action with this mass term is rewritten as

\[
S_F = \int d^5x \int d^5\bar{x} \bar{\psi}(\bar{x}, x_5) \left[ \delta(x - x') \left\{ \tilde{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) \right\} + \mu_2^{x_5 x'_5} \right] \psi(x'),
\]

where

\[
\mu_2^{x_5 x'_5} = \mu_2 \left[ \delta(x_5)\delta(x'_5 - L_5) + \delta(x_5 - L_5)\delta(x'_5) \right],
\]

and our target fermion determinant with the Pauli–Villars fields can be decomposed as

\[
\det \left( \frac{\tilde{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5)}{\tilde{D}^{5D} + \mu} \right) = \det \left( \frac{\delta(x - x')\left( \tilde{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) \right) + \mu_2^{x_5 x'_5}}{\delta(x - x')\left( \tilde{D}^{5D} + \mu \right)} \right)
\times \det \left( \frac{\delta(x - x')\left( \tilde{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) \right)}{\delta(x - x')\left( \tilde{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) \right) + \mu_2^{x_5 x'_5}} \right),
\]

where the determinant \( \det \) is taken in the doubled space of \( x \) and \( x' \).

To the second determinant, only boundary Weyl fermion modes with positive chirality at \( x_5 = 0 \) and negative chirality at \( x_5 = L_5 \) contribute, so that

\[
\lim_{\mu \to \infty} \det \left( \frac{\delta(x - x')\left( \tilde{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) \right)}{\delta(x - x')\left( \tilde{D}^{5D} + \mu \epsilon(x_5)\epsilon(L_5 - x_5) \right) + \mu_2^{x_5 x'_5}} \right) = \frac{\mathcal{D}}{\mathcal{D} + \mu_2}
\]

holds, where \( \mathcal{D} \) is defined as

\[
\mathcal{D} = \bar{p}_+^{5\tilde{D}^{4D}p}_+^{5\tilde{D}^{4D}} + p_+^{5\bar{\tilde{D}}^{4D}p}_,
\]

with \( \tilde{D}^{4D} = \sum_{i=1}^4 \tilde{\gamma}_i \nabla_i |_{x_5=0} \), \( \bar{\tilde{D}}^{4D} = \sum_{i=1}^4 \tilde{\gamma}_i \nabla_i |_{x_5=0, x_5=L_5} \), and \( p_+^5 = (1 \pm \gamma_5)/2 \). This form of the fermion determinant with Pauli–Villars is known to correctly produce consistent anomaly. This justifies a naive computation of the imaginary part of the first determinant in Eq. (B5), which leads to \( \pi \text{CS}(x_5 < 0) \).

**Appendix C. Fermion determinant on the \( \mu \) domain wall**

In this appendix, we give the details of the computation in Eqs. (3.32) and (3.34).

For this purpose, it is enough to consider

\[
\lim_{\mu \to \infty} \det \left( \frac{D^{6D} + i \mu \epsilon(x_5)\gamma_6\gamma_7 + M_1}{D^{6D} + i \mu \epsilon(x_5)\gamma_6\gamma_7 + M_2} \right)
\]

(C1)
in the $\mu \to \infty$ limit with arbitrary masses $M_1$ and $M_2$. It receives contributions only from the boundary localized modes, which are constrained to satisfy

$$ \gamma_5 (\partial_5 + i \mu \epsilon (x_5) \gamma_5 \gamma_6 \gamma_7) \psi = 0, \quad (C2) $$

whose solution is given by

$$ \psi = e^{-\mu|x_5|} \psi', \quad i \gamma_5 \gamma_6 \gamma_7 \psi' = \psi'. \quad (C3) $$

The operator $i \gamma_5 \gamma_6 \gamma_7$ has a $4 \times 4$ block-diagonal form so that its projection operator can be expressed as

$$ \hat{P}_6^\pm \equiv (1 \pm i \gamma_5 \gamma_6 \gamma_7) / 2 = \begin{pmatrix} P_\pm & \tilde{P}_\pm \\ \tilde{P}_\pm & -P_\pm \end{pmatrix}, \quad (C4) $$

where $P_\pm \equiv (1 \pm \gamma_5) / 2$ are projection operators for 4-component spinors.

With the above constraint, multiplying by $-i \gamma_5$, and defining $D^{5D} = (\sum_{i=1}^4 \gamma_i \nabla_i + \gamma_6 \partial_6)|_{x_5=0}$, the determinant Eq. (C1) can be rewritten as

$$ \det \left[ \hat{P}_6^+ ( -i \gamma_5 D^{5D} - i \gamma_5 M_2) ^{-1} ( -i \gamma_5 D^{5D} - i \gamma_5 M_1) \hat{P}_6^+ + \hat{P}_6^- \right]. \quad (C5) $$

Inserting the two unitary operators

$$ Q_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \gamma_5 & -i \\ \gamma_5 & i \end{pmatrix}, \quad Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & -i \gamma_5 \\ -\gamma_5 & i \end{pmatrix}, \quad (C6) $$

we obtain a $4 \times 4$ block-diagonalized form

$$ \det \left[ \hat{P}_6^+ Q_1^\dagger Q_2^\dagger ( -i \gamma_5 D^{5D} - i \gamma_5 M_2) ^{-1} Q_2 Q_1 ( -i \gamma_5 D^{5D} - i \gamma_5 M_1) Q_1^\dagger \hat{P}_6^+ + \hat{P}_6^- \right] $n = \det \left[ \hat{P}_6^+ Q_1^\dagger \begin{pmatrix} (\hat{D}^{5D} - M_2)^{-1} (\hat{D}^{5D} - M_1) \\ (\hat{D}^{5D} + M_2)^{-1} (\hat{D}^{5D} + M_1) \end{pmatrix} Q_1 \hat{P}_6^+ + \hat{P}_6^- \right], \quad (C7) $$

where $\hat{D}^{5D} = (\sum_{i=1}^4 \gamma_i \nabla_i + \gamma_5 \partial_6)|_{x_5=0}$. Since $Q_1$ commutes with $\hat{P}_6^+$, we can factorize the determinant as

$$ \det \left[ P_+(\hat{D}^{5D} + M_2)^{-1} (\hat{D}^{5D} + M_1) P_+ + P_- \right] \times \det \left[ P_+(\hat{D}^{5D} - M_2)^{-1} (\hat{D}^{5D} - M_1) P_+ + P_- \right]. \quad (C8) $$

When $M_1$ and $M_2$ both commute with $\hat{D}^{5D}$, this determinant is not only real but positive.

References

[1] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B 185, 20 (1981); 195, 541 (1982) [erratum].
[2] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B 193, 173 (1981).
[3] D. B. Kaplan, Phys. Lett. B 288, 342 (1992) [arXiv:hep-lat/9206013] [Search INSPIRE].
[4] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B **414**, 785 (1994) [arXiv:hep-lat/9308004] [Search INSPIRE].

[5] H. Neuberger, Phys. Lett. B **417**, 141 (1998) [arXiv:hep-lat/9707022] [Search INSPIRE].

[6] H. Neuberger, Phys. Lett. B **427**, 353 (1998) [arXiv:hep-lat/9801031] [Search INSPIRE].

[7] M. Luscher, Nucl. Phys. B **549**, 295 (1999) [arXiv:hep-lat/9811032] [Search INSPIRE].

[8] Y. Kikukawa and Y. Nakayama, Nucl. Phys. B **597**, 519 (2001) [arXiv:hep-lat/0005015] [Search INSPIRE].

[9] R. Narayanan and H. Neuberger, Nucl. Phys. B **443**, 305 (1995) [arXiv:hep-th/9411108] [Search INSPIRE].

[10] W. Bock, M. F. L. Golterman, and Y. Shamir, Phys. Rev. Lett. **80**, 3444 (1998) [arXiv:hep-lat/9709154] [Search INSPIRE].

[11] O. Bar and I. Campos, Nucl. Phys. B **581**, 499 (2000) [arXiv:hep-lat/0001025] [Search INSPIRE].

[12] M. Luscher, Nucl. Phys. B **568**, 162 (2000) [arXiv:hep-lat/9904009] [Search INSPIRE].

[13] H. Suzuki, Nucl. Phys. B **585**, 471 (2000) [arXiv:hep-lat/0002009] [Search INSPIRE].

[14] Y. Kikukawa, Phys. Rev. D **65**, 074504 (2002) [arXiv:hep-lat/0105032] [Search INSPIRE].

[15] D. M. Grabowska and D. B. Kaplan, Phys. Rev. Lett. **116**, 211602 (2016) [arXiv:1511.03649 [hep-lat]] [Search INSPIRE].

[16] C. G. Callan, Jr. and J. A. Harvey, Nucl. Phys. B **250**, 427 (1985).

[17] S. G. Naculich, Nucl. Phys. B **296**, 837 (1988).

[18] S. Chandrasekhar, Phys. Rev. D **49**, 1980 (1994) [arXiv:hep-th/9311050] [Search INSPIRE].

[19] X. z. Dai and D. S. Freed, J. Math. Phys. **35**, 5155 (1994); **42**, 2343 (2001) [erratum] [arXiv:hep-th/9405012] [Search INSPIRE].

[20] E. Witten, [arXiv:1508.04715 [cond-mat.mes-hall]] [Search INSPIRE].

[21] E. Witten, Phys. Lett. B **117**, 324 (1982).

[22] L. Alvarez-Gaumé, S. Della Pietra, and V. Della Pietra, Phys. Lett. B **166**, 177 (1986).

[23] D. B. Kaplan and M. Schmaltz, Phys. Lett. B **368**, 44 (1996) [arXiv:hep-th/9510197] [Search INSPIRE].

[24] R. Stora, *Progress In Gauge Field Theory* (Nato Advanced Study Institute, Cargese, France, 1983).

[25] B. Zumino, *Chiral Anomalies And Differential Geometry: Lectures Given At Les Houches*, August 1983, Conference: C83-06-27.1, pp. 1291–1322.

[26] B. S. DeWitt and R. Stora, *Relativity, Group and Topology II: proceedings* (North-Holland, Amsterdam, 1984), p. 1322.

[27] B. Zumino, Y. S. Wu, and A. Zee, Nucl. Phys. B **239**, 477 (1984).

[28] L. Alvarez-Gaumé and P. H. Ginsparg, Nucl. Phys. B **243**, 449 (1984).

[29] T. Sumitani, J. Phys. A **17**, L811 (1984).

[30] M. F. Atiyah and I. M. Singer, Proc. Nat. Acad. Sci. **81**, 2597 (1984).

[31] A. N. Redlich, Phys. Rev. D **29**, 2366 (1984).

[32] A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).

[33] L. Alvarez-Gaumé, S. Della Pietra, and G. W. Moore, Ann. Phys. **163**, 288 (1985).

[34] M. F. Atiyah and I. M. Singer, Bull. Am. Math. Soc. **69**, 422 (1963).

[35] K. Fujikawa, Phys. Rev. Lett. **42**, 1195 (1979).

[36] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Math. Proc. Cambridge Philos. Soc. **77**, 43 (1975).

[37] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Math. Proc. Cambridge Philos. Soc. **78**, 405 (1975).

[38] M. F. Atiyah, V. K. Patodi, and I. M. Singer, Math. Proc. Cambridge Philos. Soc. **79**, 71 (1976).

[39] J. Wess and B. Zumino, Phys. Lett. B **37**, 95 (1971).

[40] W. A. Bardeen and B. Zumino, Nucl. Phys. B **244**, 421 (1984).

[41] R. Narayanan and H. Neuberger, Nucl. Phys. B **477**, 521 (1996) [arXiv:hep-th/9603204] [Search INSPIRE].

[42] T. Iizubuchi and J. Nishimura, J. High Energy Phys. **9910**, 002 (1999) [arXiv:hep-lat/9903008] [Search INSPIRE].
[47] O. Bar, Nucl. Phys. B 650, 522 (2003) [arXiv:hep-lat/0209098] [Search INSPIRE].
[48] K. i. Okumura and H. Suzuki, arXiv:1608.02217 [hep-lat] [Search INSPIRE].
[49] D. M. Grabowska and D. B. Kaplan, arXiv:1610.02151 [hep-lat] [Search INSPIRE].
[50] H. Makino and O. Morikawa, arXiv:1609.08376 [hep-lat] [Search INSPIRE].
[51] T. Asaka, W. Buchmuller, and L. Covi, Phys. Lett. B 523, 199 (2001) [arXiv:hep-ph/0108021] [Search INSPIRE].
[52] M. Luscher, Phys. Lett. B 428, 342 (1998) [arXiv:hep-lat/9802011] [Search INSPIRE].
[53] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25, 2649 (1982).
[54] H. Neuberger, arXiv:hep-lat/0303009 [Search INSPIRE].