EFFICIENT LINEARIZED LOCAL ENERGY-PRESERVING METHOD FOR THE KADOMTSEV-PETVIASHVILI EQUATION

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ABSTRACT. A linearized implicit local energy-preserving (LEP) scheme is proposed for the KPI equation by discretizing its multi-symplectic Hamiltonian form with the Kahan’s method in time and symplectic Euler-box rule in space. It can be implemented easily, and also it is less storage-consuming and more efficient than the fully implicit methods. Several numerical experiments, including simulations of evolution of the line-soliton and lump-type soliton and interaction of the two lumps, are carried out to show the good performance of the scheme.

1. Introduction. For over two centuries, many simplified models have been proposed for describing the motion of surface waves of an incompressible fluid. The Kandmtsev-Petviashvili (KP) equation

\[(u_t + u_{xxx} + 6uu_x)_x + \sigma u_{yy} = 0\]  

firstly introduced by Kadomtsev and Petviashvili in [19] is one of the most popular models for two-dimensional wave propagation. The equation describes the time evolution of a two-dimensional disturbance over the surface of a shallow basin by specifying a scalar function \(u(x, y, t)\) that can be interpreted as the displacement of the surface with respect to a reference level. In the case \(\sigma = -3\), Eq. (1) is usually called KPI equation, whereas in the case \(\sigma = 3\), the KPII equation. Both KPI and KPII are exactly integrable via the inverse scattering transformation. It

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is shown that the sign of $\sigma$ is critical with respect to the stability characteristics of line-solitons. In this paper, we mainly focus on developing an efficient numerical method for the KP equation.

The KP equation has garnered a great deal of interest in mathematical analysis [26, 1, 20] in the past years. Usually, it is regarded as the two-dimensional generalization of the KdV equation. However, the numerical analysis literature for the KP equation is relatively rare compared to the KdV equation. To the best of our knowledge, Katsis and Akylas [21] firstly studied the effect of slow rotation on the propagation of solitary internal waves in shallow fluids for the KP equation by the finite difference scheme (FDS). In [2], an explicit FDS suffering from a strict CFL condition was used to study the soliton phenomenon. Feng and Mitsui [13] proposed a linearized implicit FDS for investigating the collision of two lump-type solitary waves. The works in [1, 17, 30] are based on the pseudo-spectral method first introduced by Fornberg and Whitham [14]. In recent years, some methods combining Fourier (pseudo-)spectral method with discontinuous Galerkin method and operator splitting in time were proposed in [24, 32, 11]. The Fourier methods have zeros numerical dissipation and dispersion and high resolution in space, but they are only applicable to the problems with periodic boundary conditions and may over resolve the regions in which the solution changes slowly since their resolution is fixed throughout the domain. More related numerical methods for the KP equation can be referred to Refs. [34, 25, 12].

Based on the basic rule that numerical method should preserve the intrinsic properties of the original problems as much as possible, construction of structure-preserving methods for the conservative and Hamiltonian partial differential equations (PDEs) has been a hot topic. Since Marsden et al. [29] and Bridges and Reich [3] introduced the concepts of multi-symplectic Hamiltonian system and multi-symplectic integrator, many PDEs have been successfully solved by this method. The KPI equation belongs to the multi-symplectic Hamiltonian system and its multi-symplectic Hamiltonian formula was first written down in [28]. A fully implicit multi-symplectic Preissmann scheme was proposed in [28] and numerical results displayed its robustness. In [18], a linearized implicit multi-symplectic scheme was derived from the multi-symplectic Euler-box integrator [31].

Besides preservation of symplectic conservation law, multi-symplectic Hamiltonian system also conserves local energy and momentum conservation laws (LECL and LMCL). One may consider methods that preserve the discrete version of LECL/LMCL [35, 33, 27, 16]. These locally defined properties are independent of the choice of boundary conditions, giving the method conserving local energy/momentum an advantage over method preserving a global energy, especially since local conservation laws will always lead to global conservation laws whenever periodic/homogeneous boundary conditions are considered. Refs. [15, 4] presented two general frameworks on LEP integrators for the multi-symplectic Hamiltonian system. However, they are time-consuming when applied to KPI equation due to their fully implicit feature. In order to take into account preservation of LECL and computational efficiency, simultaneously, a good alternative may therefore be to design linearized implicit LEP scheme.

Kahan’s method [22] proposed for quadratic ODEs is linearly implicit. Its general form was written down in [23] and it was applied to Hamiltonian ODEs with cubic Hamiltonian in [6, 7, 8, 9]. Eidnes and Li generalized the Kahan’s method to the multi-symplectic Hamiltonian system with cubic invariant to construct a linearized
implicit method [10]. The method can be applied to the KPI equation, but all the
terms of the given scheme have long stencils. In this work, inspired by Eidnes and
Li’s method and our previous work [4], we combine Kahan’s method and symplectic
Euler-box rule to construct a new linearly implicit scheme preserving a discrete
approximation to the LECL for the KPI equation.

The rest of this paper is organized as follows. In Section 2, we review the Kahan’s
method briefly. In Section 3, the multi-symplectic Hamiltonian form of KPI is first
presented, and then a linearized implicit LEP scheme is proposed. Some numerical
properties are analyzed in the same section. Numerical results are reported in
Section 4, and finally the conclusion is in Section 5.

2. Kahan’s method. For the quadratic ODE system
\[
\dot{y} = f(y) := Q(y,y) + By + c, \quad y \in \mathbb{R}^d,
\]
where \(Q\) is a \(\mathbb{R}^d\)-valued quadratic symmetric bilinear form, \(B \in \mathbb{R}^{d \times d}\) is a symmetric
constant matrix, and \(c \in \mathbb{R}^d\) is a constant vector, the Kahan’s method is defined by
the map
\[
y_n \rightarrow y_{n+1} : \quad \frac{y_{n+1} - y_n}{h} = Q(y_{n+1},y_n) + \frac{1}{2} B(y_{n+1} + y_n) + c,
\]
where \(h\) is the time step. Restrict (2) to be a Hamiltonian system on a Poisson
vector space with a constant Poisson structure
\[
\dot{y} = A \nabla H(y),
\]
where \(A\) is a constant skew-symmetric matrix and Hamiltonian \(H : \mathbb{R}^d \rightarrow \mathbb{R}\) is a
cubic polynomial function. Firstly, consider \(H\) to be homogeneous. Then it follows
from the result in Proposition 2.1 in [9], the Kahan’s method can be written as
\[
y_n \rightarrow y_{n+1} : \quad \frac{y_{n+1} - y_n}{h} = 3 A \bar{H}(y^n,y^{n+1},\cdot),
\]
where \(\bar{H}(\cdot,\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}\) is a symmetric 3-tensor satisfying
\(\bar{H}(y,y,y) = H(y)\). The symmetric 3-tensor \(\bar{H}(x,y,z)\) form can be obtained by polarization. The
polarization is a map from a homogeneous polynomial to a symmetric multi-linear
form in more variables [9], which can be used to generalize the Kahan’s method
to higher degree polynomial vector fields. Since the 3-tensor \(\bar{H}(x,y,z) = x^T Q(y) z\)
where \(Q(y) = \frac{1}{6} \nabla^2 H(y)\) with \(\nabla^2 H\) being the Hessian of \(H\), the Kahan’s method
(5) can be rewritten as
\[
y_n \rightarrow y_{n+1} : \quad \frac{y_{n+1} - y_n}{h} = 3 A \frac{\partial \bar{H}}{\partial x} \bigg|_{(y^n,y^{n+1},\cdot)},
\]
where \(\partial \bar{H}/\partial x\) represents the partial derivative with respect to the first argument of
\(\bar{H}\).

For the case that Hamiltonian \(H\) in (4) is cubic but non-homogeneous, i.e., the,
following general form
\[
H(y) = y^T Q(y)y + y^T By + c^T y + d,
\]
where \(6Q(y)\) is the linear part of \(\nabla^2 H\) and thus a symmetric matrix whose entries
are homogeneous linear polynomials, \(2B\) is the constant part of \(\nabla^2 H\) and thus
a symmetric constant matrix, \(c\) is a constant vector and \(d\) is a constant scalar.
Following the method suggested in [6], one can extend it to a homogeneous function
\(\tilde{H}(y_0, y_1, \ldots, y_d)\) such that \(\tilde{H}(1, y_1, \ldots, y_d) = H(y_1, \ldots, y_d)\), and extend \(A\) to \(\tilde{A}\) by
adding a zero initial row and column so that solve instead of (4) the equivalent, homogeneous cubic Hamiltonian system

\[ \dot{\tilde{y}} = A \nabla H(\tilde{y}). \]

By polarization, one gets the reformulation of Kahan’s method as (5) with

\[ \tilde{H}(x, y, z) = x^\top Q(y)z + \frac{1}{3}(x^\top By + y^\top Bz + z^\top Bx) + \frac{1}{3}c^\top(x + y + z) + d. \]

It is clear that \( \tilde{H}(x, y, z) \) is symmetric with respect to \( x, y \) and \( z \), and

\[ \frac{\partial \tilde{H}(x, y, z)}{\partial x} = Q(y)z + B(y + z) + \frac{c}{3}. \]

3. Linearized implicit LEP scheme for KPI equation. In this section, multi-symplectic Hamiltonian form of the KPI equation will be briefly presented, and then a novel linearized implicit LEP scheme will be developed.

3.1. Multi-symplectic Hamiltonian form. By introducing a series of auxiliary variables, one can rewrite the KPI equation into a standard multi-symplectic Hamiltonian form [28]

\[ Mz_t + Kz_x + Lz_y = \nabla_z S(z), \]

where the state variable \( z = (\phi, v, u, p, p^x, p^{xx}, p^{xy}, p^t, p^{xxx})^\top \in \mathbb{R}^{10} \), the state function

\[ S(z) = up + (p^{xxx})^2/2 + \sigma g^2/2 + u^3 - vp^x - up^{xx} - pp^t - gp^{xy}, \]

and the three skew-symmetric matrices

\[ M = \begin{pmatrix} 0 & M_1 \nabla \end{pmatrix}, \quad K = \begin{pmatrix} 0 & K_1 \nabla \end{pmatrix}, \quad L = \begin{pmatrix} 0 & L_1 \nabla \end{pmatrix} \]

with

\[ M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]

The system (9) admits the multi-symplectic conservation law

\[ \omega_t + \kappa_x + \chi_y = 0, \]

where

\[ \omega = dz \wedge M_+ dz, \quad \kappa = dz \wedge K_+ dz, \quad \chi = dz \wedge L_+ dz. \]

Here, matrices \( M_+ \), \( K_+ \) and \( L_+ \) represent the splitting of \( M \), \( K \) and \( L \), respectively, satisfying

\[ M = M_+ - M_+^T, \quad K = K_+ - K_+^T, \quad L = L_+ - L_+^T. \]

Additionally, it also possesses LECL

\[ E_t + F_x + \hat{F}_y = 0, \]

where

\[ E = S(z) + z_x^\top K_+ z + z_y^\top L_+ z, \quad F = -z_t^\top K_+ z, \quad \hat{F} = -z_t^\top L_+ z, \]

and \( x \)- and \( y \)-directional LMCLs which can be found in Ref. [4, 5]. For more details on the derivation of (9) for the KPI equation, please refer to Ref. [28].
3.2. Linearized implicit LEP scheme. First we introduce some notations which are helpful for constructing the scheme. For convenience, assume \((x, y) \in \Omega = (a_1, b_1) \times (a_2, b_2)\), the spacing of the grid points in the \(x\)-direction is uniform given by \(\Delta x\), and the spacing of grid points in \(y\)-direction, also uniform, by \(\Delta y\). We approximate the grid value \(u(x_1 + j \Delta x, a_2 + k \Delta y, u, n, \Delta t)\) by \(u^n_{j,k}\), \(j = 0, 1, \ldots, J\) and \(k = 0, 1, \ldots, K\). Define the finite difference operators

\[
\delta_x u^n_{j,k} = \frac{1}{\Delta x}(u^n_{j+1,k} - u^n_{j,k}), \quad \delta_x^+ u^n_{j,k} = \frac{1}{\Delta x}(u^n_{j+1,k} - u^n_{j,k}), \quad \delta_y u^n_{j,k} = \frac{1}{\Delta y}(u^n_{j,k+1} - u^n_{j,k}), \quad \delta_y^+ u^n_{j,k} = \frac{1}{\Delta y}(u^n_{j,k+1} - u^n_{j,k}),
\]

average operators

\[
A_t u^n_{j,k} = (u^n_{j,k+1} + u^n_{j,k})/2, \quad A_x u^n_{j,k} = (u^n_{j+1,k} + u^n_{j,k})/2, \quad A_y u^n_{j,k} = (u^n_{j,k+1} + u^n_{j,k})/2,
\]

and shift operators

\[
s_x^- u^n_{j,k} = u^n_{j-1,k}, \quad s_x^+ u^n_{j,k} = u^n_{j+1,k}.
\]

The defined operators satisfy the discrete Leibniz rules

\[
\begin{align*}
\delta_t (u^n_{j,k} v^n_{j,k}) &= \delta_t u^n_{j,k} A_t v^n_{j,k} + A_t u^n_{j,k} \delta_t v^n_{j,k},
\delta_x (u^n_{j-1,k} v^n_{j,k}) &= u^n_{j,k} \delta_x v^n_{j,k} + v^n_{j,k} \delta_x u^n_{j,k},
\delta_y (u^n_{j,k-1} v^n_{j,k}) &= u^n_{j,k} \delta_y v^n_{j,k} + v^n_{j,k} \delta_y u^n_{j,k}.
\end{align*}
\]

**Lemma 3.1.** For the grid function \(z^n_{j,k} \in \mathbb{R}^{d_1}\) and matrix \(B \in \mathbb{R}^{d_1 \times d_1}\), there exist the following identities

\[
\begin{align*}
(\delta_t z^n_{j,k})^T B \delta_w A_t z^n_{j,k} - (\delta_w A_t z^n_{j,k})^T B \delta_t z^n_{j,k} \\
= \delta_w \left( (\delta_t s_w^- z^n_{j,k})^T B A_t z^n_{j,k} \right) - \delta_t \left( (\delta_w z^n_{j,k})^T B z^n_{j,k} \right), \quad (12)
\end{align*}
\]

and

\[
\begin{align*}
(\delta_t z^{n+1}_{j,k})^T B \delta_w A_t z^{n+1}_{j,k} - (\delta_w A_t z^{n+1}_{j,k})^T B \delta_t z^{n+1}_{j,k} \\
+ (\delta_t z^{n+1}_{j,k})^T B \delta_w A_t z^{n+1}_{j,k} - (\delta_w A_t z^{n+1}_{j,k})^T B \delta_t z^{n+1}_{j,k} \\
= \delta_w \left( (\delta_t s_w^- z^{n+1}_{j,k})^T B A_t z^{n+1}_{j,k} \right) + \delta_w \left( (\delta_t s_w^- z^{n+1}_{j,k})^T B A_t z^{n+1}_{j,k} \right) \\
- \delta_t \left( (\delta_w z^{n+1}_{j,k})^T B z^{n+1}_{j,k} \right) - \delta_t \left( (\delta_w z^{n+1}_{j,k})^T B z^{n+1}_{j,k} \right), \quad (13)
\end{align*}
\]

where \(w = x, y\).

**Proof.** Combining the commutative laws and the discrete Leibniz rules of the difference operators, the left-hand term of (12) can be reformulated as

\[
(\delta_t z^n_{j,k})^T B \delta_w A_t z^n_{j,k} - (\delta_w A_t z^n_{j,k})^T B \delta_t z^n_{j,k} \\
= (\delta_t z^n_{j,k})^T B \delta_w A_t z^n_{j,k} - \delta_t \left( (\delta_w z^n_{j,k})^T B z^n_{j,k} \right) + (\delta_t z^n_{j,k})^T B \delta_w A_t z^n_{j,k} - \delta_t \left( (\delta_w z^n_{j,k})^T B z^n_{j,k} \right).
\]

The result (13) can be derived, analogously. This completes the proof.

Our aim is to develop linearized implicit LEP scheme for the KPI equation, so we discretize Eq. (9) by using the symplectic Euler-rule in space and Kahan’s method in time; that is,

\[
M \delta_t z^n_{j,k} + K \delta_x A_t z^n_{j,k} = -K \delta_x A_t z^n_{j,k} + L \delta_y A_t z^n_{j,k} = \frac{\partial S}{\partial x} \bigg|_{(z^n_{j,k}, t^{n+1})}.
\]

(14)
Here, $K_+$ and $L_+$ represent, respectively, the upper triangular matrices of $K$ and $L$, and $S(\cdot, \cdot, \cdot)$ is a polarized homogeneous cubic polynomial function according to the general cubic polynomial $S$. The expression of $\hat{S}$ can be found in (8).

**Theorem 3.2.** The scheme is a linearly implicit LEP method, which preserves the discrete LECL

$$\delta_t F^n_{j,k} + \delta^+_x F^{n+1/2}_{j,k} + \delta^+_y \hat{F}^{n+1/2}_{j,k} = 0,$$

(15)

where

$$E^n_{j,k} = \hat{S}(z^n_{j,k}, z^n_{j,k}, z^{n+1}_{j,k})$$

$$+ \frac{1}{3} \left( (\delta_x z^n_{j,k})^T K_+ z^n_{j,k} + (\delta_x z^n_{j,k})^T K_+ z^{n+1}_{j,k} + (\delta_x z^n_{j,k})^T K_+ z^n_{j,k} \right)$$

$$+ \frac{1}{3} \left( (\delta_y z^n_{j,k})^T L_+ z^n_{j,k} + (\delta_y z^n_{j,k})^T L_+ z^{n+1}_{j,k} + (\delta_y z^n_{j,k})^T L_+ z^n_{j,k} \right),$$

$$E^{n+1/2}_{j,k} = \frac{1}{3} \left( (\delta_s^+ z^n_{j,k})^T K_+ A_{t, z^n_{j,k}} + (\delta_s^+ z^n_{j,k})^T K_+ A_{t, z^{n+1}_{j,k}} + (\delta_s^+ z^n_{j,k})^T K_+ A_{t, z^n_{j,k}} \right),$$

$$E^{n+1/2}_{j,k} = \frac{1}{3} \left( (\delta_s^+ z^n_{j,k})^T K_+ A_{t, z^n_{j,k}} + (\delta_s^+ z^n_{j,k})^T K_+ A_{t, z^{n+1}_{j,k}} + (\delta_s^+ z^n_{j,k})^T K_+ A_{t, z^n_{j,k}} \right).$$

**Proof.** Left-multiplying $(\delta z^n_{j,k})^T$ on both sides of Eq. (14) together with the skew-symmetric property of matrix $M$

$$3(\delta z^n_{j,k})^T \frac{\partial \hat{S}}{\partial x} \bigg|_{(z^n_{j,k}, z^{n+1}_{j,k})} = (\delta z^n_{j,k})^T K_+ \delta^+_x A_{t, z^n_{j,k}} - (\delta_x A_{t, z^n_{j,k}})^T K_+ \delta z^n_{j,k}$$

$$+ (\delta z^n_{j,k})^T L_+ \delta^+_y A_{t, z^n_{j,k}} - (\delta_y A_{t, z^n_{j,k}})^T L_+ \delta z^n_{j,k}.$$  

(16)

And also taking the inner product with $\delta z^{n+1}_{j,k}$ on both sides of Eq. (14), $\delta z^n_{j,k}$ on both sides of Eq. (14) for the next time step, and then adding them together with the vanishment of $(\delta z^n_{j,k})^T M \delta z_{j,k} + (\delta z^n_{j,k})^T M \delta z^{n+1}_{j,k}$ reads

$$\left( (\delta z^{n+1}_{j,k})^T K_+ \delta^+_x A_{t, z^n_{j,k}} - (\delta_x A_{t, z^n_{j,k}})^T K_+ \delta z^{n+1}_{j,k} + (\delta z^{n+1}_{j,k})^T L_+ \delta^+_y A_{t, z^n_{j,k}} - (\delta_y A_{t, z^n_{j,k}})^T L_+ \delta z^{n+1}_{j,k} \right)$$

$$+ (\delta z^n_{j,k})^T L_+ \delta^+_y A_{t, z^{n+1}_{j,k}} - (\delta_y A_{t, z^{n+1}_{j,k}})^T L_+ \delta z^n_{j,k}$$

$$= \left( (\delta z^n_{j,k})^T \frac{\partial \hat{S}}{\partial x} \bigg|_{(z^n_{j,k}, z^{n+1}_{j,k})} + (\delta z^n_{j,k})^T \frac{\partial \hat{S}}{\partial x} \bigg|_{(z^{n+1}_{j,k}, z^{n+2}_{j,k})} \right).$$

Noting that $\hat{S}(\cdot, \cdot, \cdot)$ is a homogenous cubic polynomial function, one has

$$\left( (\delta z^n_{j,k})^T \frac{\partial \hat{S}}{\partial x} \bigg|_{(z^n_{j,k}, z^{n+1}_{j,k})} \right) = \frac{1}{\Delta t} \left( \hat{S}(z^{n+1}_{j,k}, z_{j,k}, z^n_{j,k}) - \hat{S}(z^n_{j,k}, z_{j,k}, z^{n+1}_{j,k}) \right),$$

$$\left( (\delta z^{n+1}_{j,k})^T \frac{\partial \hat{S}}{\partial x} \bigg|_{(z^{n+1}_{j,k}, z^{n+2}_{j,k})} \right) = \frac{1}{\Delta t} \left( \hat{S}(z^{n+2}_{j,k}, z_{j,k}, z^{n+1}_{j,k}) - \hat{S}(z^{n+1}_{j,k}, z_{j,k}, z^n_{j,k}) \right),$$

(18)

Adding Eqs. (16) and (17) together, and then using the results (18) together with the Lemma 3.1 and the symmetry of $\hat{S}(\cdot, \cdot, \cdot)$ with respect to the three arguments, one can complete the proof.
The component-wise scheme according to the matrix form (14) is

\[
\begin{aligned}
&\delta^+_t A_t(p^n\tau^j)_{j,k} = 0, \\
&\frac{3}{2} \delta^+_t A_t(p^x\tau^j)_{j,k} + \delta^+_t A_t(p^y\tau^j)_{j,k} = -A_t(p^\tau)^n_{j,k}, \\
&\delta^+_x A_t(p^xx\tau^j)_{j,k} = A_t p^n_{j,k} + 3u^n_{j,k} u^{n+1}_{j,k} - A_t(p^xx\tau^j)_{j,k}, \\
&\delta^+_y A_t(p^xy\tau^j)_{j,k} = \sigma A_t g^n_{j,k}, \\
&\delta^+_x A_t v^n_{j,k} = A_t u^n_{j,k}, \\
&\frac{2}{3} \delta^+_y A_t v^n_{j,k} = A_t g^n_{j,k}, \\
&\delta^+_x A_t u^n_{j,k} = -A_t(p^xx\tau^j)_{j,k}.
\end{aligned}
\]

(19)

Firstly, we remove the operator \( A_t \) from the equations in the first, fourth, fifth and the last lines of (19), and then eliminate the auxiliary variables, and finally get a linearized implicit scheme only in term of the variable \( u \), that is

\[
\delta_t A_u \delta^+_x u^n_{j,k} + (\delta^+_x \delta^+_x)^2 A_t u^n_{j,k} + 3\delta^+_x \delta^+_x (u^n_{j,k} u^{n+1}_{j,k}) + \sigma \delta^+_y \delta^+_x A_t u^n_{j,k} = 0.
\]

(20)

**Remark 1.** Difference splitting matrices \( K_+ \) and \( L_+ \) may result in different linearized implicit scheme. For example, one can take \( K_+ = K/2 \) and \( L_+ = L/2 \).

**Theorem 3.3.** When the KPI equation is subject to the periodic or zero Dirichlet boundary conditions, the linearized implicit LEP scheme (14) preserves a discrete global energy

\[
\mathcal{E}^n = \mathcal{E}^{n-1} = \cdots = \mathcal{E}^0,
\]

(21)

where

\[
\mathcal{E}^n = \frac{\Delta x \Delta y}{6} \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \left( 6(u^n_{j,k})^2 u^{n+1}_{j,k} - (\delta^+_x u^n_{j,k})^2 - 2\delta^+_x u^n_{j,k} \delta^+_y u^{n+1}_{j,k} \right) + \sigma (\delta^+_y v^n_{j,k})^2 + 2\sigma (\delta^+_y v^n_{j,k} \delta^+_y v^{n+1}_{j,k}).
\]

Here, \( v^n_{j,k} \) and \( v^{n+1}_{j,k} \) are determined by (19).

**Proof.** The polarized \( \bar{S}(x, z, y) \) of \( S(z) \) is

\[
\bar{S}(x, z, y) = x^T Q(z) y + \frac{1}{3} (x^T B z + z^T B y + y^T B x),
\]

(22)

where

\[
Q(z) = \begin{pmatrix} Q_1(z) & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} B_1 & -B_2 \\ -B_2^T & B_3 \end{pmatrix},
\]

with

\[
Q_1(z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \sigma \end{pmatrix},
\]

\[
B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Substitute (22) into the discrete local energy conservation law (15), and then sum the obtained equation over all spatial induces. After some tedious calculations together with the vanishement of the numerical fluxes and the identities...
Table 1. Numerical results for the KPI equation obtained by the present scheme: \( \Omega = [0, 40] \times [0, 2] \), \( r = (\Delta t, \Delta x, \Delta y) = (0.1, 0.2, 0.2) \) and \( t = 1 \).

| Mesh   | \( r \)    | \( r/2 \)    | \( r/2^2 \)    | \( r/2^3 \)    | \( r/2^4 \)    |
|--------|------------|-------------|-------------|-------------|-------------|
| \( e_\infty \) | 4.4921e-2 | 1.1041e-2 | 2.8764e-3 | 7.1999e-4 | 1.6418e-4 |
| Rate   | –          | 2.0248     | 1.9405     | 1.9982     | 2.1327     |
| \( e_2 \) | 5.8460e-2 | 1.3052e-2 | 3.2214e-3 | 8.0842e-4 | 1.9908e-4 |
| Rate   | –          | 2.1632     | 2.0185     | 1.9945     | 2.0218     |

\[
\sum_{j=0}^{J-1} \sum_{k=0}^{K-1} (u^n_{j,k} \delta w v^n_{j,k}) = - \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} (v^n_{j,k} \delta w u^n_{j,k}), \quad w = x, y, \text{one completes the proof.}
\]

4. Numerical experiments. In our simulations, we mainly focus on the numerical performance of the proposed scheme, including accuracy of the solution, convergence rate of the solution and preservation of the energy. The solution error at \( n \)th time level is scaled by the maximal error norm \( e_\infty = \max_{j,k} |u(x_j, y_k, t_n) - u^n_{j,k}| \) and the average error norm \( e_2 = \left( \Delta x \Delta y \sum_{j,k} |u(x_j, y_k, t_n) - u^n_{j,k}|^2 \right)^{1/2} \). The preservation of energy is tested by \( |\mathcal{E}^n - \mathcal{E}^0| \).

**Example 1:** (Line-soliton solution) The KPI equation admits the line-soliton solution

\[
u(x, y, t) = 2\gamma^2 \text{sech}^2 \left( \gamma (x + \lambda y - (4\gamma^2 - 3\lambda^2)t - x_0) \right), \quad -\infty < x < +\infty \tag{23}
\]

which means the soliton propagates with velocity \( 4\gamma^2 - 3\lambda^2 \) in the direction with the angle of \( \tan^{-1}(\lambda^{-1}) \) to the positive \( x \)-axis and its amplitude is \( 2\gamma^2 \).

In our simulations, we consider the initial condition \( u(x, y, 0) = 2\text{sech}^2(x - \sqrt{\gamma^2}y) \), \( \Omega = [0, 40] \times [0, 2] \), and the exact boundary conditions for \( y = b_1, b_2 \) and periodic boundary conditions for \( x = a_1, a_2 \). With these initial values, the equation has an \( x \)-periodic solution \( u(x, y, t) = 2\text{sech}^2 \left( \text{mod}(x - \sqrt{\gamma^2}y - \frac{\lambda}{2}t, 40) - 20 \right) \).

Firstly, we test convergence rate of the scheme. Table 1 reports the maximal and average errors in solution at time \( t = 1 \) and the corresponding convergence rates. The results indicate that the solution of the present scheme converges to the exact one with order 2 in both maximal and average error norms.

Secondly, we carry out a long-term simulation with \( \Delta t = 1e-2 \) and \( \Delta x = \Delta y = 1e-1 \) till time \( t = 12 \). The variation of solution errors as time evolves and profile of the soliton at \( t = 12 \) are illustrated with Fig. 1. One can see that both the variations of \( e_\infty \) and \( e_2 \) have quasi periodicity. The variation of \( e_\infty \) is similar to that in [18], but the present scheme shows a bit more solution accuracy. The right graph in Fig. 1 shows that the line-soliton moves approximately 30 units from \( t = 0 \) to \( t = 12 \). Therefore, the speed of the numerical solution is agreement with the theoretical speed \( 5/2 \).

Finally, we show the efficiency of the proposed scheme by comparing its CPU time with the fully implicit multi-symplectic scheme [28]. The computations are done on MacBook (macOS Catalina, 1.4GHz, Intel Core i5) with \( \Delta t = 1e-2 \) and \( \Delta x = \Delta y = 1e-1 \) till time \( t = 12 \). The maximal errors in solution of the present scheme and the one proposed in [28] are 3.589e-2 and 3.562e-2, respectively, while their CPU times are, respectively, 34.218s and 226.475s.
Example 2: (Lump type solution) For the KPI equation, there exists a lump type soliton solution which can be expressed as

$$u(x, y, t) = 4 \frac{-(x - x_0 - 3\mu^2 t)^2 + \mu^2(y - y_0)^2 + 1/\mu^2}{((x - x_0 - 3\mu^2 t)^2 + \mu^2(y - y_0)^2 + 1/\mu^2)^2}. \quad (24)$$

The solution indicates that the lump pulse will move to the positive $x$-direction with velocity $3\mu^2$. In the following simulation, we adopt the initial condition $u_0(x, y) = u(x, y, 0)$, $\Omega = (-10, 10)^2$ where $\mu = 1.0$, $x_0 = 0$ and $y_0 = 0$, and periodic boundary conditions for $x = a_1, a_2$ and exact boundary conditions for $y = b_1, b_2$. In this case, the equation admits an $x$-periodic solution

$$u(x, y, t) = 4 \frac{-(\text{mod}(x - 3\mu^2 t + 10, 20) - 10)^2 + \mu^2y^2 + 1/\mu^2}{((\text{mod}(x - 3\mu^2 t + 10, 20) - 10)^2 + \mu^2y^2 + 1/\mu^2)^2}. \quad (25)$$

Our simulations are done with $\Delta t = 1e-2$, $\Delta x = \Delta y = 1e-1$ up to $t = 5$. Fig. 2 shows the numerical solutions at different times and contours of the exact solutions. Stable propagation of the lump type solitary wave can be observed without any deformation. Also one can see that the obtained lump type solitary wave moves along the positive $x$-axis at the beginning, and then goes out the right boundary and comes back into computational domain from the left boundary. By comparing the graphs for contours, one sees that the numerical solution is agreement with the exact solution.

To display the long-term stability of the present scheme, we conduct the simulation up to $t = 20$. The results in Fig. 3 verify its good long-time stability and excellent preservation of the waveform except a small phase difference between the numerical and exact solutions.

Example 3: (Collision of two lump type solitary waves) We adopt the following initial conditions

$$u(x, y, t) = 4 \sum_{i=1}^{2} \frac{-(x - x_i)^2 + \mu_i^2(y - y_i)^2 + 1/\mu_i^2}{((x - x_i)^2 + \mu_i^2(y - y_i)^2 + 1/\mu_i^2)^2}, \quad \Omega = (0, 50) \times (0, 40), \quad (25)$$

with the parameters $x_1 = 6$, $x_2 = 16$, $y_1 = 20$, $y_2 = 20$, $\mu_1 = 1$ and $\mu_2 = 0.7$, and zero Dirichlet boundary conditions. Recalling the lump type solution (24), the higher lump on the left will move with velocity 3 along the positive $x$-direction, while the lower lump on the right moving in the same direction with velocity 1.47.
Figure 2. The lump type solitary wave at time $t = 0$ (top), $t = 0.5$ (the second row), $t = 1$ (the third row) and $t = 5$ (bottom), respectively. Left: the profiles of solution; Middle: contours of numerical solution; Right: contours of exact solution.

Figure 3. The lump type solitary wave at time $t = 20$. Left: the profile of solution; Middle: contours of numerical solution; Right: contours of exact solution.
Hence, there will be ‘direct collision’ between the two lumps because the centers of the two pulses are situated on the same line with $y = 20$. This can also be confirmed by the graphs in Fig. 4. As it is seen from graphs, the lower lump splits into two solitary waves (see the top-right graph) at the beginning of the collision, and then two pulses with almost equal amplitudes are generated after their collision, and finally, the two formed solitary waves move in the opposite direction along $y$-axis, while keeping the total momentum be zero as before. The presented results indicate that collision of the two lumps is inelastic. The plot in the last graph indicates good preservation of the global energy throughout computations.

5. Conclusion. KPI equation is a famous model for two-dimensional wave propagation with wide applications in the modern physics of nonlinear waves. multisymplectic Hamiltonian form has excellent local conservation laws which are independent of the boundary conditions and produce rich information than global conservation laws, so in this work, we first recast the KPI equation into a multisymplectic Hamiltonian formulation, and then develop a linearized implicit scheme preserving a discrete LECL for the equation. The method is linearized implicit so that it could be easier to implement, less storage-consuming and more efficient than the fully implicit methods. Some numerical results on the evolution of two types of solitons and collision of two lump-type solitary waves are reported to display the good performance of our scheme.
In this work, the linearized implicit LEP scheme (14) is applied to the KPI equation. Next, a simple numerical simulation is carried out to display the performance of the scheme (14) for the KPII equation. By taking the initial value \[ u(x, y, 0) = 8 \exp(2x)/(1+\exp(2x))^2, \quad \Omega = (-40, 40) \times (0, 10) \] and periodic boundary conditions, one can see the good simulation of the motion of soliton from Fig. 5.

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