Overlap in odd dimensions.

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Abstract

In odd dimensions the lattice overlap formalism is simpler than in even dimensions. Masslessness of fermions can still be preserved without fine tuning and gauge invariance without gauge averaging can be maintained, although, sometimes, only at the expense of parity invariance. When parity invariance is enforced invariance under small gauge transformations can be maintained and continuum global gauge anomalies are reproduced.

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1. Introduction.

In even Euclidean dimensions, the overlap formalism [1] provides a way to put Weyl fermions interacting with gauge fields on the lattice. Based on an internal quantum mechanical supersymmetry [2], it has a mechanism of protecting masslessness of fermions. In this paper we deal with field theories in odd dimensional Euclidean space where chirality is absent. Still, protecting masslessness by the regularization is of potential interest and we are employing the overlap to do this. These issues were recently addressed in ref. [3].

1a. Continuum features.

Before we describe our objectives more specifically, let us briefly recall some basic continuum features. Unless stated otherwise, we work on a compact torus to avoid infrared issues. Consider, for example, a single massless Dirac $SU(2)$ doublet interacting with $SU(2)$ gauge fields in three Euclidean dimensions:

$$\mathcal{L} = \frac{1}{2g^2} Tr[F_{\mu\nu}F_{\mu\nu}] - \bar{\psi}_\sigma(\partial_\mu + iA_\mu)\psi.$$  \hspace{1cm} (1.1)

The $\sigma_\mu$ are Pauli matrices and $SU(2)$ and spinor indices are suppressed. Classically, the Lagrangian possesses both local gauge invariance and a global discrete symmetry usually referred to as “parity”. There also is fermion number conservation but this symmetry can be preserved by quantization and we shall ignore it henceforth. Under parity $A_\mu(x) \rightarrow -A_\mu(-x)$ and $\bar{\psi}(x), \psi(x) \rightarrow i\bar{\psi}(-x), i\psi(-x)$. A $\bar{\psi}\psi$ mass term for the fermions would switch sign under this parity. It is thought that not all classical symmetries can be preserved at the quantum level:\footnote{This view may be too dogmatic: see [3, 4].} If gauge invariance is kept, parity is broken and, if parity invariance is enforced, gauge invariance can be maintained only perturbatively, while “large” gauge transformations become anomalous [5, 6, 7, 8, 9]. The global gauge anomaly the theory is afflicted with is analogous to the Witten anomaly in four dimensions [10, 11, 12].

Examples for a global gauge anomaly can be found also for compact $U(1)$ with a unit charge Dirac fermion. In the $SU(2)$ case every gauge configuration can be paired with a gauge equivalent one in such a way that it is impossible to maintain equal weight for both. In the $U(1)$ case only some specific gauge orbits are anomalous, and if we consider non-compact $U(1)$ these orbits disappear. Thus, while there is an argument that parity invariant $SU(2)$ with a single doublet doesn’t exist, this argument does not fully extend to $U(1)$ with a single Dirac fermion.

If we replace the Dirac doublet in the $SU(2)$ case by a Dirac fermion in representation $j$ there will be a clash between parity and gauge invariance only when $I \equiv \frac{3}{2}j(j+1)(2j+1)$
is odd: \( j = \frac{1}{2}, j = \frac{5}{2}, j = \frac{9}{2}, \cdots \). When \( I \) is even a parity invariant and gauge invariant regularization is possible. Note that \( j = \frac{3}{2} \) falls into the category of “good” theories, so it is not only pseudoreality that matters. Also, for integral \( j \), the representation of the fermions is real and \( I \) is divisible by 4. Actually, for real representations one can consider Majorana fermions and then the role of \( I \) is played by \( \frac{I}{2} \). Still, since \( \frac{I}{2} \) is also even there are no problems with parity for Majorana fermions either. For Majorana fermions with \( j = 1 \) the continuum theory also has \( \mathcal{N} = 1 \) three dimensional supersymmetry. In the Majorana case fermion number is conserved only modulo 2 and the parity breaking mass terms are of Majorana type. Unlike in four dimensions, masslessness is not essential for supersymmetry. However, in conventional lattice approaches one would need to fine tune to make the gluon and gluino masses equal. Guaranteeing masslessness is one way to eliminate this fine tuning.

1b. Objective of paper.

Our main motivation is to see if and how the overlap reproduces the continuum features just reviewed. We would expect the overlap to fully capture at the regularized level some of the representation dependent differences, but the more subtle distinctions could become apparent only in the continuum limit. In other words, we wish to subject the basic overlap idea to a test in odd dimensions and see how close to the continuum it gets. It is also possible to envisage applications in three dimensions that employ the overlap [3,13] to other ends. As a test of the overlap, we are particularly interested in the following:

An example of global gauge anomaly has not been yet explicitly exhibited within the overlap formalism. There exist indirect arguments in favor of the overlap reproducing global gauge anomalies: In [14] the four dimensional Witten anomaly is shown to be reproduced in the continuum version of the domain wall approach [15] and this indicates that the lattice overlap will reproduce it also. Also, from [11,12] it is known that the Witten anomaly follows from the ordinary perturbative anomaly for bigger groups. That the perturbative anomaly is reproduced by the overlap is quite well established [16,17]; thus one should expect Witten’s anomaly to work correctly also. In this paper we look at the three dimensional analogue of the four dimensional Witten anomaly and wish to see directly how it gets reproduced on the lattice as a result of insisting on parity invariance.

One of the heaviest prices the overlap pays in even dimensions for dealing with Weyl fermions is the need to make a phase choice which breaks gauge invariance in fixed bosonic backgrounds. There exists a good phase choice and although it obeys many desired symmetries [1], it is neither unique (for example see [18]), nor completely natural. Gauge breaking must be allowed in general since for certain representations inevitable perturba-
tive anomalies are present. In odd dimensions, there are no perturbative anomalies and full gauge invariance can be preserved (as we saw above, sometimes at the expense of parity conservation). We wish to check whether the overlap is sufficiently flexible to reflect this by providing a more natural phase choice than available in even dimensions, namely a phase choice which restores full gauge invariance. We expect such a phase choice to exist since preserving full gauge invariance at the expense of parity is possible using more conventional lattice formulations than the overlap [7]. With the overlap we would like to see that also parity can be maintained in the subset of cases indicated by the continuum. To achieve parity and gauge invariance using conventional methods would require fine tuning.

The simplest example would be $SU(2)$ gauge theory with a triplet of Dirac fermions, but a more interesting one would be the case were the fermions are Majorana and the theory should become supersymmetric in the continuum. This would be an analogue of four dimensions where masslessness is essential and where the overlap could also be used to avoid fine tuning [19], as pointed out in [1]. To go from Dirac to Majorana in three dimensions should be just as easy as going from Weyl to Majorana-Weyl in 2 and 10 dimensions [20]. In other words, the fact that for real representations the Dirac theory admits a “perfect square root” in the continuum is something we expect the overlap to reproduce exactly on the lattice. As suggested in [20], employing dimensional reduction from $d = 10$ is a possible way to obtain lattice versions of other would be supersymmetric theories. Similarly [13], starting from other dimensions and using the appropriate fermions, more examples exist [21].

1c. Synopsis of results.

In [3] it was noted that using the same phase choices as in even dimensions, the so called “Brillouin-Wigner” convention, one could maintain exact parity invariance on the lattice and gauge invariance under “small” gauge transformations. Like in four dimensions one needs to rely on hopefully inoffensive gauge averaging to restore full gauge invariance in the continuum limit. The new points we make here are:

- We present examples of global gauge anomalies on the lattice where expected by the conventional, continuum arguments. We identify the underlying lattice mechanism causing these anomalies. Thus, we see no compelling evidence for the new universality classes, beyond those of [7], suggested in [3]. To us it seems more likely that gauge averaging will eliminate any interesting continuum limit whenever global anomalies are thought to exist in the continuum on all gauge orbits, like for $SU(2)$ with one doublet. For $U(1)$ with a single Dirac fermion, the simplest possibility is that gauge averaging eliminates some of the gauge orbits in the continuum limit, but leaves an
interacting theory which at infinite volume is indistinguishable from the non-compact case.

- We show that the overlap formalism simplifies significantly in odd dimensions allowing for gauge invariant phase choices. This simplification opens the door to the application of numerical methods to the study of those non-perturbative phenomena in three dimensions where massless fermions are essential. In some cases the computational cost seems not too daunting. The simplification we find has to do with the fact that in odd dimensions we typically don’t have to worry about fermion number non-conserving processes and therefore there is no a priori reason to prohibit discretizations of the relevant Dirac operators by finite matrices of fixed shape. In even dimensions the need to accommodate dynamically changing matrix shapes prohibits the level of explicitness achievable here. When Higgs fields in a coset space of the gauge group divided by a subgroup containing abelian factors are introduced the story changes since robust fermion zero modes can occur in three dimensions [22] in the infinite volume limit. The introduction of the Higgs fields in the overlap also spoils the applicability of the simplification, bringing us back to a formalism where fermion number violation cannot be ruled out a priori.

The plan of the paper is as follows: In section 2 we introduce the basic (simplified) formulae describing the overlap for Dirac and Majorana fermions in odd dimensions. In section 3 we present the simplest possible odd dimensional model: it has $d = 1$ and a $U(1)$ gauge group. Although this quantum mechanical model is trivial in itself, it contains mathematical ingredients that teach us useful lessons about higher odd dimensions, as has been noted recently in [23, 24, 25]. In odd dimensions the $d = 1$ model plays an illustrative role similar to the one played by $d = 2$ models [26, 28] in even dimensions. In section four we discuss the global anomaly in $d = 3$ dimensional models with gauge groups $U(1)$ and $SU(2)$. We end with our main conclusion that the overlap appears to work well in odd dimensions, and that further work might yield new results.

2. Basic formulae.

2a. Even to odd dimensional reduction.

Rather than follow [1] and derive the odd dimensional overlap from scratch [3], it is simpler to view the (odd) $d$-dimensional theory as a dimensionally reduced theory descending from one dimension higher, where the overlap has been worked out already. In

\[ f^{2}\] The index of any compact elliptic differential operator on an odd dimensional manifold is known to be zero. For nontrivial indices we need an open space in odd dimensions.
the higher dimension one direction is shrunk to a circle of vanishing radius. We only need to worry about the fermions in a fixed gauge background, so our dimensional reduction is quite trivial in the sense that it deals with a super-renormalizable bilinear action. The Weyl fermions in $d + 1$ dimensions become massless Dirac fermions in $d$ dimensions. We set the gauge field component in the shrunk direction to zero (later we shall set it to another constant and induce a mass term this way) and the fermion is assumed to obey periodic boundary conditions in the same direction. We choose the other components of the gauge field as independent of the shrinking coordinate and ignore the infinitely heavy Dirac fermions which make up parity doublets from the $d$ dimensional point of view.

This is the setup in continuum. On the lattice we pick a torus with sides consisting of $L$ lattice spacings in the extended directions and one side of minimal length (one lattice spacing) in the shrunk direction (all directions are compactified, but one is minimal in size). The link variables on the self-closing links are set to unity. The Weyl overlap for this degenerate $d + 1$ dimensional lattice should provide the overlap formulation for a massless Dirac fermion in $d$ dimensions. For definiteness we choose $d = 3$, but it is easy to generalize.

In four dimensions the chiral determinant is replaced at the regulated level by the overlap of two many-body states [1]. These are the ground states of two bilinear Hamiltonians,

$$\mathcal{H}_4^\pm = a^\dagger H_4^\pm a$$

with all indices suppressed. The matrices $H_4^\pm$ are obtained from

$$H_4(m) = \begin{pmatrix} B_4 + m & C_4 \\ C_4^\dagger & -B_4 - m \end{pmatrix}$$

with $H_4^+ = H_4(\infty)$, $H_4^- = H_4(-m_0)$ and $0 < m_0 < 2$. The infinite argument for $H_4^+$ can be replaced by any finite positive number, but the equations are somewhat simpler with our choice [27, 28, 29]. The matrices $C_4$ and $B_4$ are given below:

$$(C_4)_{x\alpha i, y\beta j} = \frac{1}{2} \sum_{\mu=1}^{4} \sigma^\mu_{\alpha\beta} \left[ \delta_{y,x+\hat{\mu}}(U_\mu(x))_{ij} - \delta_{x,y+\hat{\mu}}(U_\mu(y))_{ij} \right],$$

$$(B_4)_{x\alpha i, y\beta j} = \frac{1}{2} \delta_{\alpha\beta} \sum_{\mu=1}^{4} \left[ 2\delta_{xy}\delta_{ij} - \delta_{y,x+\hat{\mu}}(U_\mu(x))_{ij} - \delta_{x,y+\hat{\mu}}(U_\mu(y))_{ij} \right].$$

$x, y$ are sites on the lattice, $\alpha, \beta$ are Weyl spinor indices and $i, j$ are color indices. The $\sigma_\mu$ are Pauli matrices for $\mu = 1, 2, 3$ and $\sigma_4 = i$. Eliminating the dimension $\mu = 4$ we obtain

$$\mathcal{H}_3^\pm = a^\dagger H_3^\pm a, \quad H_3(m) = \begin{pmatrix} B_3 + m & C_3 \\ C_3^\dagger & -B_3 - m \end{pmatrix},$$
with \( C_3 \) and \( B_3 \) given by eq. (2.3), only the indices \( \mu \) run from 1 to 3 now and \( x \) and \( y \) label the sites on a three dimensional lattice. \( \alpha, \beta, i, j \) maintain their original ranges. The elimination of \( \sigma_4 \) implies \( C_3 = -C_3^\dagger \). The overlap is defined as \( <+|−> \). The bra and ket are, respectively, vacua for (2.1): \( \mathcal{H}_3^\pm |\pm> = E_{\text{vac}}^\pm |\pm> \) and there is a phase freedom we shall determine later.

2b. Gauge invariant / parity non-invariant form.

\( \mathcal{H}_3(m) \) can be brought into a more convenient form by unitary transformations acting in the block space made explicit in (2.2). Conjugating by \( \frac{1}{\sqrt{2}} e^{i\pi/4} (1 - i\sigma_3) \), we obtain:

\[
\mathcal{H}_3(m) \to \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \mathcal{H}_3(m) \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \sigma_3 \otimes (B_3 + m) - i\sigma_1 \otimes C_3.
\tag{2.5}
\]

The \( \sigma \) matrices in (2.5) operate in a different space from the ones in (2.3). We now rotate in the three dimensional space associated with the three \( \sigma \) matrices of (2.5) around the unit vector \( \frac{1}{\sqrt{3}} (1,1,1) \) by an angle of 120 degrees via conjugation by \( U = \frac{1}{2} [1 - i(\sigma_1 + \sigma_2 + \sigma_3)] \) and obtain (reusing the \( \mathcal{H}_3 \) symbol)

\[
\mathcal{H}_3(m) = \sigma_1 \otimes (B_3 + m) - i\sigma_2 \otimes C_3 = \begin{pmatrix} 0 & X \\ X^\dagger & 0 \end{pmatrix},
\tag{2.6}
\]

with \( X = B_3 + m + C_3 \).

This form is known to lead to an explicit formula for the overlap in terms of the determinant of some finite, fixed dimension matrices [1]. Making a natural phase choice one obtains:

\[
<+|−>_{D_c} = \det(\frac{1}{2} (1 + V)) \quad V \equiv X \frac{1}{\sqrt{X^\dagger X}} \quad V^\dagger V = 1.
\tag{2.7}
\]

Under a gauge transformation \( X \) changes by conjugation so \( <+|−>_{D_c} \) is fully gauge invariant. Gauge fields where \( \det(X) = 0 \) are assumed of zero measure and ignored. The subscript \( D_c \) stands for “Dirac fermions in a complex representation”. It is easy to check that in the free case \( \frac{1}{1+V} = \frac{1-V^\dagger}{V^\dagger-V} \) has only the right poles in momentum space, the doublers being canceled by zeros in the numerator.

Let us now indicate how in perturbation theory, that is for smooth gauge backgrounds, a Chern-Simons term with the expected coefficient is induced in the effective action. This result can be easily extracted from a calculation in [7], observing that \( X \) is precisely the Wilson Dirac lattice operator with a negative mass. The phase of \( \det(X) = \det(V)|\det(X)| \)
has been computed in [7] (to leading order and for \(0 < m_0 < 1\)) in a context in which it gives the \textit{infinite} mass limit answer of the continuum.\(^3\)

But, the phase of \(< +|− >_{D_c}\) in (2.7) is just half the phase of \(\det (V)\) (in perturbation theory) and this is the correct massless continuum result (more precisely, as explained in [7], it is one of the possible correct results).

We shall discuss parity in some detail in sub-section 2f below. For the time being we just assert that parity invariance would hold if the expression for the fermion determinant were real for all gauge backgrounds. One cannot find a phase to make \(< +|− >_{D_c}\) real and rescue parity invariance while preserving gauge invariance because one would then need to define \(\sqrt{\det (V)}\) for all backgrounds. Although \(V\) itself is sufficiently local, picking a branch of the square root would either amount to a nonlocal phase choice or violate gauge invariance. The previous paragraph only describes the leading polynomial part of the imaginary part of the logarithm of \(\det (\frac{1}{2}(1 + V))\). While the result of the perturbative computation is local, the numerical value of the coefficient implies also the existence of a nonlocal term if gauge invariance is invoked. The nonlocal term takes the discrete values 0 or \(i\pi\) depending on the gauge field.

\textit{2c. Massive fermions.}

Let us now introduce a mass term for the fermions. This is interesting, since in the infinite mass limit new phenomena may occur [7, 30]. As alluded before, to add a mass term we again dimensionally reduce from four dimensions, only this time keeping a constant but nonzero fourth component for the vector potential. On the lattice this produces a shift in our parameter \(m_0\), which we absorb, maintaining the new parameter \(m_0\) in the same range as before \((0 < m_0 < 2)\), and also a new term in \(H_3\). Going to the basis used in (2.6) the new Hamiltonian matrix becomes:

\[
H^\mu_3 (m) = \begin{pmatrix} \mu & X \\ X^\dagger & -\mu \end{pmatrix}.
\]

Note that the fermion mass is given by \(\mu > 0\) while \(m_0\) is an intrinsic mass to be interpreted as the mass of extra regulator fields. \(m_0\) goes to infinity when measured on physical scales

\(^3\) The conventions we adopted here become, in the notation of [7], \(s = 1, m < 0, \gamma_1 \gamma_2 \gamma_3 = i\). To use the result of [7] first make the inconsequential overall sign change \(D \rightarrow -D\). Therefore, we need the result of [7] for \(s = -1, m > 0, \gamma_1 \gamma_2 \gamma_3 = -i\). The result of [7] is derived assuming \(\gamma_1 \gamma_2 \gamma_3 = i\). To account for the different representation we chose, we need to take the complex conjugate of the answer of [7]. In the notation of [7] we end up with \(c_0 = \pi\). In our next sub-section we obtain \(c_\infty = 0\), agreeing with the \(n = 0\) case of equations (1.10-11) of [7].
in the continuum limit. The formula for the overlap in the complex Dirac case (2.7) changes only in that $V$ is replaced by $V^\mu$, which is no longer unitary:

$$V^\mu = \frac{1}{\sqrt{X^\dagger X + \mu^2 + \mu}}.$$  \hfill (2.9)

If we keep $\mu$ fixed and approach the continuum limit, the mass of the physical fermion becomes infinite. If we scale $\mu$ to zero with the gauge coupling constant instead, the fermions will acquire a finite mass. If we take the lattice-$\mu$ to infinity the physical fermion is infinitely massive. The large lattice-$\mu$ limit is obviously smooth. We conclude that at infinite physical fermion mass the overlap becomes gauge field independent because $V^\mu$ vanishes then. Thus, no Chern-Simons term is induced, nor are any of the special effects associated with it. Again, we obtained one of the possible continuum results according to [7], albeit the most uninteresting one.

2d. Adjoint Higgs Fields.

If the fourth component of the vector potential in the parent theory is an arbitrary function of the unreduced directions, we end up introducing a Higgs field in the adjoint representation. On the lattice we would write $U_4(x_1, x_2, x_3) = e^{iH(x_1, x_2, x_3)}$, expand to second order in $H$, and also add an invariant potential for $H$ to the rest of the action. Now, the parameter $\mu$ in (2.8) is replaced by a matrix which no longer commutes with $X$ because of the non-trivial $x$ dependence. As a result, the road to the more explicit expression (2.7) is blocked. This is just as well, since now fermion violation might become possible in fixed monopole backgrounds (at least in the infinite volume limit) and an expression for the fermionic path integral in terms of a finite matrix of fixed shape would at least require awkward explanations. The overlap is still potentially useful, but it has the same form as in even dimensions, and further simplification cannot be achieved. We shall not deal with Higgs fields in the rest of this paper. If we freeze the Higgs field, we go back to adding a mass term for the fermions, as in sub-section 2c.

2e. Real representations and Majorana fermions.

Assume the fermions are in a real (not complex and not pseudoreal) representation of the gauge group. Thus all the $U_\mu(x)$-matrices are real. It is easy to see then that:

$$\sigma_2 X \sigma_2 = X^T.$$  \hfill (2.10)

This implies that $\sigma_2 V$ is not only unitary but also antisymmetric. Thus, a special square root of $\det(V)$ exists: $\det(V) = (pf(\sigma_2 V))^2$. The pfaffian is analytic in the entries of $V$, just as the determinant is. It is also invariant under real gauge transformations. We end
up with a gauge invariant and parity invariant overlap, but only for real representations carried by fermions of Dirac type.

\[
< + | - >_{D_r} = \frac{|pf(\frac{1}{2}(\sigma_2 + \sigma_2 V))|^2}{pf(\sigma_2 V)}.
\]  

(2.11)

\(D_r\) stands for “Dirac fermions in a real representation”. The main factor in the overlap (representing the gauge invariant but parity violating version) is also a “perfect” square, directly implying a formula for the case that the representation is real and the fermions are of Majorana type:

\[
< + | - >_{M} = pf(\frac{1}{2}(\sigma_2 + \sigma_2 V)).
\]  

(2.12)

\(M\) stands for “Majorana fermions”. If we want a parity invariant formula we again face the problem of taking a square root of \(pf(\sigma_2 V)\) without violating locality or gauge invariance.

In summary, for Dirac fermions in a real representation we have complete gauge invariance and parity invariance on the lattice. For Majorana fermions, if we have exact parity invariance, gauge invariance cannot be fully maintained, although it could be recovered in the continuum limit by gauge averaging. We might wish to reverse the roles of gauge invariance and parity invariance, by preserving gauge invariance and adding an explicit counter-term to cancel the induced Chern-Simons term in the continuum limit and thus restore parity in the continuum. However, this requires a gauge invariant lattice construction of the exponent of the Chern-Simons action.

2f. Removal of parity breaking.

Three dimensional “parity” is a descendant of four dimensional parity. We know that in four dimensions [1] replacing \(\{U_\mu(x)\}\) by a parity image induces \(H_4(m) \rightarrow H'_4(m) = -\sigma_2 \otimes 1 \ H_4(m) \ \sigma_2 \otimes 1\). This implies \(H_3(m) \rightarrow H'_3(m) = H_3(m)|_{C_3 \rightarrow -C_3} \) or \(V \rightarrow V' = V^\dagger\). Therefore, parity is broken by the overlap being non-real. Although \(V\) is not strictly local there is no fundamental problem since \(X\) has a mass term. We have obtained a formulation quite similar to the conventional one [7], only that the fermions from the point of view of the overlap are massless and are expected to stay so even after the gauge dynamics is turned on. This expectation is based on an interpretation involving an infinite number of fermions [1, 3] and ought to be checked more directly.

Parity breaking is measured by the ratio

\[
\frac{< + | - >_{D_r}}{< + | - >_{D_r}^*} = \det V.
\]  

(2.13)

As already mentioned, taking square roots of quantities with no natural branch choices could amount to either loss of gauge invariance or to a nonlocal phase choice. We now
analyze the issue of “forbidden” square roots in the two problematic cases, Dirac with a complex representation and Majorana, in some greater detail. The main observation is that in both cases one can define the square of the desired object without violating gauge invariance, parity and locality (in the generalized sense). We append a prime to the overlap subscripts to indicate that the objects themselves aren’t yet defined, only their squares are, but the squares have the nice properties listed above:

\[(<+|→D'_c)^2 = \frac{(\det(\frac{1+V}{2}))^2}{\det(V)} = |\det(\frac{1+V}{2})|^2 = \det(\frac{1}{2}(1 + \frac{V + V^\dagger}{2})),\] (2.14)

\[(<+|→M'_c)^2 = \frac{\det(\frac{1}{2}(\sigma_2 + \sigma_2 V))}{\text{pf}(\sigma_2 V)}.\] (2.15)

In each case we deal with gauge invariant and real expressions. Gauge invariance implies that in a complete theory the gauge action will control the fluctuations of the squares. As the lattice β-gauge coupling is taken to infinity the behavior of the above squares would be determinable from the continuum theory. It is then essentially a question about the continuum whether the above squares have gauge invariant square roots or not. (We needn’t worry about renormalization effects, as we are really focusing only on the imaginary part of the effective action.) For SU(2) with Dirac fermions and \(j = \frac{1}{2}\) we expect problems, but for the Majorana case with \(j = 1\) we don’t. What kind of problems can occur? Clearly, the right hand side of (2.14) is non-negative. So, all that can go wrong with preserving the gauge invariance of the square root is that the square have odd rank zeros as we smoothly deform one link configuration into a gauge equivalent one. This is exactly how global gauge anomalies occur.

Based on the continuum, we expect to be able to safely take the square root (up to gauge configurations of vanishing weight) in the Majorana case, but we suspect that it is impossible to carry out this step in the Dirac case. If we are right about the Majorana case we obtain a quite elegant lattice regularization for supersymmetric YM in three dimensions. This has an analogue in four dimensions [1], where one could use the absolute value of the appropriate overlap, again relying on the continuum for locality, in particular on the recent claim that the gluino determinant has no odd rank zeros [31].

Some indirect evidence in favor of such a procedure can be found in the recent work testing the overlap in two dimensions [29]. There, a particular chiral model was investigated which is known to have a positive definite fermion determinant (when all species are combined) in the continuum. It was found numerically that carrying out gauge averaging (and thus respecting locality) gave continuum results indistinguishable within errors from a nonlocal procedure where gauge invariance was enforced by taking the absolute value of
the overlap. Apparently, if there is no conflict between the nonlocal expression and the continuum limit, the lattice nonlocality leaves no traces in the continuum limit.

2g. Parity invariant / gauge non-invariant Brillouin-Wigner phase choice.

We now consider the Brillouin-Wigner phase choice. With this phase choice a more direct comparison with [3] can be made and this is the phase choice we employ in even dimensions, so it is interesting to see how well it does here. For brevity, we discuss only a Dirac fermion in a complex representation. Let us first deal with the massless case, $\mu = 0$, since the main issue is reality, which is anyhow absent if $\mu \neq 0$. The Brillouin-Wigner convention amounts to fixing the phases of the $|\pm \rangle$ states by requiring they have positive overlaps with corresponding states in the absence of nontrivial gauge fields. We shall denote the appropriate matrices by the subscript $f$ for “free”. It is quite easy to see that:

$$<+|−>_D^{BW} = \det(\frac{1}{2}(1 + V)) z^{BW}, \quad z^{BW} = \frac{\det(\frac{1}{2}(1 + V^\dagger V_f))}{|\det(\frac{1}{2}(1 + V^\dagger V_f))|} \frac{1}{\sqrt{\det(V_f)}}. \quad (2.16)$$

The constant phase factor $\frac{1}{\sqrt{\det(V_f)}}$ is added for convenience. Actually, it is not hard to see that $\det(V_f) = 1$ for reasonable fermion boundary conditions. There is no gauge invariance and the expression is local in the sense that it involves no dangerous square roots. To see that the expression is real we compute its square:

$$(<+|−>_D^{BW})^2 = |\det(\frac{1}{2}(1 + V))|^2. \quad (2.17)$$

We find the same answer as in (2.14) and conclude that $<+|−>_D^{BW}$ is one possible candidate for $<+|−>_D^{c}$ and, since (2.16) is essentially local, it had to violate gauge invariance. The gauge violation is made explicit by the presence of the matrix $V_f$, which does not rotate under gauge transformations. However, the gauge violation is more minimal than it appears at first sight: equation (2.17) can be inverted to write $(z^{BW})^2 = \det^*(V)$ showing that $(z^{BW})^2$ is gauge invariant. For $V = V_f$, $z^{BW} = \frac{1}{\sqrt{\det(V_f)}} (= 1)$ and by continuity we see that there is gauge invariance for “small” gauge transformations. If the Brillouin-Wigner phase were completely well defined for any gauge background there would be no room for a problem. But it is not: there are exceptional configurations for which $VV_f^\dagger$ has $-1$ as an eigenvalue and $z^{BW}$ becomes ill defined. It is safe to ignore these configurations in the path integral – but their presence thwarts any continuity arguments beyond “small” gauge transformations. So gauge invariance is almost restored, but sign changes can occur under certain gauge transformations even in the continuum limit.

That this is the case will be argued in the next section. These sign changes are the lattice version of the global gauge anomaly. We have learned that the Brillouin-Wigner
phase choice is indeed a “good” one, in the sense that it goes as far as it only can, given the continuum results and moreover, it does not go beyond these, thus potentially respecting the inevitability of global gauge anomalies as long as only local counter-terms are allowed.

Consider two gauge equivalent configurations (A and B) that are known in the continuum to give opposite signs to the square roots when the latter are defined in a local way, by using the “doubling trick” of [10]. When A is connected to B by a smooth path in the continuum, we expect, on the lattice, the gauge invariant and parity non-invariant version (2.7) to have a total odd number of zeros along the path. We don’t expect $z^{BW}$ to have to vanish at all along the path, since we should be able to deform the path away from all the exceptional configurations, unless the endpoints are exceptional themselves. Thus, we expect (2.16) to have different signs for A and B. On the lattice, unlike in the continuum, A and B can be connected by a path contained entirely in a single gauge orbit. However, this path cannot have a continuous continuum image, no matter how we deform it inside the gauge orbit and, we expect it to be forced to pass through exceptional lattice configurations an odd number of times. This time the sign switches occur due to odd zeros in the numerator of the $z^{BW}$ factor in (2.16).

The inevitability of the sign switches can be heuristically understood as follows: There is no problem with gauge invariance as long as the expression is allowed to be complex since the fermion determinant, as a function of the parameter describing the path connecting A to B, can trace out a continuous closed curve in the complex plane even though it goes through zero an odd number of times. But, projecting the curve on the real axis (or any other smooth line going through the origin once) leaves no way for it to close if it crosses zero an odd number of times. Under the Brillouin-Winger phase choice gauge invariance is broken and the image of the curve on the real axis opens up, leaving $z^{BW}$ with opposite signs at A and B.

For $\mu \neq 0$ the $V$’s in (2.7) and (2.16) are replaced by the $V^\mu$ of (2.9). In sub-section 2c we noted that at infinite $\mu$ det$(1 + V^\mu) = 1$. The phase $z^{BW}$ also goes to one in the infinite $\mu$ limit. Therefore, in the infinite mass limit, no Chern-Simons term is induced in either one of the two definitions (2.7) and (2.16). Our $\mu$–mass term in (2.8) is identical to that of [3]. Our choice of $H_4^+ = H_4(\infty)$ makes our overlap definition somewhat different from the one in [3]. This difference should be insignificant in the continuum limit. Nevertheless, our conclusion about the infinite mass limit of the continuum theory differs from [3] and our fermion truly decouples when it becomes heavy.

Note that (2.17) does not hold in the massive case since $V^\mu$ is no longer unitary.
3. A one dimensional example.

3a. Definition and continuum solution.

The subtle interplay between parity and gauge invariance for odd dimensional Dirac operators extends to $d = 1$ [23,24]. The Lagrangian has no kinetic term for the gauge field and the fermions have a single spinor component. We take $U(1)$ as the gauge group since the nontriviality of $\pi_1(U(1))$ is essential.

$$\mathcal{L} = -\bar{\psi}(x)(\partial_x + iA(x))\psi(x) \equiv \bar{\psi}(x)D(A)\psi(x), \quad \psi(0) = -\psi(l), \quad \bar{\psi}(0) = -\bar{\psi}(l). \quad (3.1)$$

The eigenvalues of $D(A)$ are

$$\lambda_n = -ip_n, \quad p_n = \frac{2\pi}{l}(n - \frac{1}{2}) + \frac{\theta}{l}, \quad n \in \mathbb{Z}$$

where $\theta = -\int_0^l A(x)dx$ is the one dimensional Chern-Simons action and also the single purely bosonic gauge invariant variable in the model. To regulate the theory gauge invariantly, we introduce three Pauli Villars fields. Their masses and statistics are chosen to cancel the leading and subleading large $n$ behavior of the $p_n$. All the PV fields have action similar to $\psi$, only $D(A)$ is replaced by $D(A) + M$, where $M$ is a PV mass term. Two of the PV fields are bosonic with masses $m^b_{1,2}$ and one is fermionic with mass $m^f = m^b_1 + m^b_2$. The regulated determinant of $D$ is

$$Z_{\text{reg}}(A) = \mathcal{N} \prod_{n \in \mathbb{Z}} \frac{p_n(p_n + im^f)}{(p_n + im^b_1)(p_n + im^b_2)} \prod_{n=1}^{\infty} \left[ (1 + \frac{\gamma}{n + \alpha})(1 - \frac{\gamma}{n + \beta}) \right] \prod_{n=1}^{\infty} \left[ (1 + \frac{\gamma}{n - \alpha})(1 - \frac{\gamma}{n - \beta}) \right] \quad (3.2)$$

where

$$\alpha = \frac{\theta}{2\pi} - \frac{1}{2} + \frac{i m^b_2 l}{2\pi}, \quad \beta = \frac{\theta}{2\pi} - \frac{1}{2} + \frac{i m^b_1 l}{2\pi}, \quad \gamma = \frac{i m^b_1 l}{2\pi}. \quad (3.3)$$

The normalization $\mathcal{N}$ is chosen so that $Z_{\text{reg}}(0) = 1$. The infinite products can be done exactly and one ends up with:

$$Z_{\text{reg}}(A) = \mathcal{N} \frac{\sin(\pi(\beta - \gamma))\sin(\pi(\alpha + \gamma))}{\sin(\pi\alpha)\sin(\pi\beta)}. \quad (3.4)$$

Taking $m^b_{1,2}$ to positive infinity we obtain:

$$Z_{\text{reg}}(A) \to \frac{1 + e^{i\theta}}{2} = Z(A). \quad (3.5)$$

This agrees with [23, 24]. Under parity $\theta \to -\theta$ and $Z(A) \to Z^*(A)$. $Z(A)$ is gauge invariant because it is a periodic function of only the variable $\theta$.
Suppose we try to achieve parity invariance by “adding a counter-term” to extract the phase of $Z(A)$. The phase is $e^{i\theta}$ and is not periodic. Thus the real answer we are left with, $\cos(\frac{\theta}{2})$ isn’t periodic either. More precisely, the original periodicity $\theta \rightarrow \theta + 2\pi k$ has been reduced to $\theta \rightarrow \theta + 4\pi k$. Thus, the square of $Z(A)$ can be made both periodic and parity invariant. Taking the square root is a problem though.

One can phrase the problem of taking the square root in a language entirely analogous to the flow argument of [10]. Define a new theory whose path integral is directly the square $Z^2(A)$ by doubling the number of fermions. This is an example of the doubling trick mentioned in subsection 2g. In the new theory $D(A)$ is replaced by $D_2(A) = \sigma_1 \otimes D$. Since $\{D_2(A), \sigma_3 \otimes 1\} = 0$ and $D(A)$ is hermitian all eigenvalues of $D_2(A)$ occur in real pairs of opposite signs. We could tentatively try to define $\det(D(A)) = \sqrt{\det(D_2(A))}$ as the product of all positive eigenvalues of $D_2(A)$. Suppose we do that for the configuration $A(x) = 0$. We now deform this configuration to $A(x) = \frac{2\pi}{t}$ by $A(x;t) = t \frac{2\pi}{t}$, $0 \leq t \leq 1$. We wish to show that, as a function of $t$, one pair of eigenvalues of $D(A(x;t))$ cross each other at zero for some $t \in [0,1]$ and thus, if we follow the eigenvalues, we have a sign switch and end up with the $\sqrt{\det(D_2(A))}$ at $A = \frac{2\pi}{t}$ being the negative of the $\sqrt{\det(D_2(A))}$ at $A = 0$. To establish the crossing we introduce a two dimensional Dirac operator $\hat{D} = i\sigma_2 \otimes \frac{\partial}{\partial t} + D_2(A(x;t))$. The operator $\hat{D}$ has an index since it sees a two dimensional instanton. Therefore $\hat{D}$ has one normalizable (chiral) zero mode. Such a zero mode can exist only because there was one zero crossing in the spectrum of $D_2(A(x;t))$ as $t$ varied. Clearly, this argument is too complex for our simple problem, but it is an argument that generalizes to higher dimensions, where more direct tools cannot be applied. The main lesson for us is that the square root problem is indeed the image of the standard global anomaly as understood from the spectral flow argument.

In our simple example the problem is evident by inspection of the exact result. $Z(A)$ depends only on $\theta$ and as $\theta$ completes one cycle in the space of gauge orbits $Z(A)$ describes a circle in the complex plane centered at $1/2$ on the real axis and of radius $1/2$. The important feature of the circle is that it passes through the origin exactly once as $\theta$ goes over its cycle once. As a result, the phase of $Z(A)$ starts at $\frac{\pi}{2}$ and ends at $-\frac{\pi}{2}$. It is not a smooth image of the $\theta$ cycle since it doesn’t have integral winding. When $Z(A)$ goes through the origin the phase is ill defined and this is how periodicity is lost. Extracting the phase to obtain a real and hence parity invariant expression leaves a result that also violates periodicity: The parity invariant answer has a global gauge anomaly.

Actually, the simplest way to state the difficulty is that $e^{i\theta}$ winds an odd number of times as $\theta$ is taken from $0$ to $2\pi$ and therefore does not admit a smooth square root. In the general case, on the lattice, it is the phase of $\det(V)$ that winds. The phase of $\det(V)$,
in turn, is just the phase of \( \det(X) \). \( X \) is a simple and strictly local operator, the lattice Wilson-Dirac operator in odd dimensions with a negative mass term. We conclude that the simplest signature of the global anomaly would be a closed path in the space of gauge orbits along which the phase of \( \det(X) \) winds an odd number of times.

3b. Lattice overlap regularization.

We regulate the theory as in section 2, only \( \sigma_\mu = 1 \) and \( \mu \equiv 1 \). The spinorial index disappears. Since we work with \( U(1) \) there are no gauge indices either. We need the matrix \( X \) of (2.6) from which we should construct the matrix \( V \) of (2.7).

\[
(X(m))_{xy} = (m + B_1 + C_1)_{xy} = (1 + m)\delta_{xy} - \delta_{x,y+1}U(y)^+ \tag{3.6}
\]

The problem simplifies for \( m = -m_0 = -1 \). Although the main conclusions are true for all \( 0 < m_0 < 2 \) it is easier to derive them at this particular value and we keep \( m_0 = 1 \) for the time being. At this value of \( m \), \( X \) becomes unitary by itself and is the same as \( V \).

The gauge invariant, but parity breaking regulated answer is

\[
Z^\text{reg}(\theta) = N\det\left(\frac{1}{2}(1 + X(-1))\right). \tag{3.7}
\]

The expression being gauge invariant, we can replace \( U(y)^+ \) in (3.6) by \( e^{i\theta} \), where \( -\pi < \theta \leq \pi \) is defined by \( \prod_{x=0}^{L-1} U(x) = e^{-i\theta} \) in analogy with the continuum variable. With antiperiodic boundary conditions imposed we have

\[
Z^\text{reg}(\theta) = \mathcal{N}\prod_{n=0}^{L-1} \frac{1 - z_ne^{i\theta}}{2}, \tag{3.8}
\]

where the \( z_n \) are all the distinct roots of \( z^L + 1 = 0 \). Therefore,

\[
Z^\text{reg}(\theta) = \frac{\mathcal{N}}{2L}e^{i\theta}(e^{-i\theta} + 1) = \frac{1 + e^{i\theta}}{2}. \tag{3.9}
\]

We have reproduced the continuum answer (3.5) exactly, without needing to take the cutoff \( L \) to infinity (\( \frac{L}{L} \) is our lattice spacing and should be taken to zero at fixed \( \theta \) and \( L \)). Had we chosen \( 0 < (m_0 \neq 1) < 2 \), there would have been some \( L \) dependence at finite \( L \) but the continuum limit is approached very rapidly as long as \( m_0 \) is not too close to the endpoints 0 or 2. Our discussion about windings in subsection 3a applies therefore on the lattice too.

The Brillouin-Wigner answer is just as easy to obtain. We find

\[
z^{\text{BW}} = \prod_{x=0}^{L-1} \frac{1 + U(x)}{|1 + U(x)|}, \tag{3.10}
\]
Note that everywhere the link variables are written excluding the phase factor that imposes the antiperiodic boundary conditions. The exceptional configurations are obvious now: they occur when any of the $U(x) = -1$. Suppose $U(x) \neq -1$ for all $x$. Write $U(x) = e^{ia(x)}$.

Avoidance of exceptional configurations implies one can choose $-\pi < a(x) < \pi$ (sharp inequalities at both ends) for all $x$ and therefore

$$z^{BW} = e^{\frac{i}{2} \sum_x a(x)}.$$  \hspace{1cm} (3.11)

The answer with the Wigner-Brillouin phase convention is, for unexceptional configurations,

$$Z_{BW}^{reg} = \frac{1}{2}(1 + e^{i\theta})z^{BW} = \cos \left[ \frac{1}{2} \sum_x a(x) \right],$$  \hspace{1cm} (3.12)

exhibiting the global gauge anomaly.

Take two gauge equivalent configurations: Configuration A has $U(x) = 1$ for all $x$ and configuration B has $U(x) = e^{i\frac{2\pi}{L}}$. One can connect the two configurations by a smooth path $U(x,t) = e^{it\frac{2\pi}{L}}$, $t \in [0,1]$. Assuming $L \geq 3$ we see that all exceptional configurations are avoided and $z^{BW}$ is well defined for all $t$. The product $Z_{BW}^{reg} = \cos(\pi t)$ is therefore also well defined and real along the path. It switches sign at $t = \frac{1}{2}$. This is the single zero crossing along the path. The zero occurs in the gauge invariant factor.

The same two configurations can be connected by another path: $a(x,t) = t\frac{2\pi}{L}$ for $x = 0, 1, \ldots L-2$ and $a(L-1,t) = t(\frac{2\pi}{L} - 2\pi)$ while $t \in [0,1]$ as before. This time the path is on a single gauge orbit since $\prod_x U(x,t)$ is $t$-independent. Therefore the gauge invariant factor in $Z_{BW}^{reg}$ stays constant (and equal to unity). The link variable set is forced to go through an exceptional configuration exactly once, at $t_* = \frac{L-1}{2L}$ where $z^{BW}$ is ill defined. For $t \in [0, t_*)$ $z^{BW} = 1$ but for $t \in (t_*, 1]$ $z^{BW} = -1$ producing the sign change.

There is an even simpler way to describe the problem. Consider the smooth path again. As the path is traversed $\det(V) = \det(X)$ must describe a closed circle in the complex plane. But

$$\det(X) = (-)^L \prod_{n=0}^{L-1} (z_n e^{it\frac{2\pi}{L}}).$$  \hspace{1cm} (3.13)

As $t$ goes from zero to one, the quantities $z_n e^{it\frac{2\pi}{L}}$, the eigenvalues of $V$, move on the unit circle such that at $t = 1$ we have $z_n$ taking the place previously held by $z_{n+1}$ if $n$ labels them cyclicly round the circle. Each eigenvalue replaces the one following it. Therefore, the sum of all the eigenvalue displacements is $2\pi$. In other words the determinant of $V$ wound around once. This is why it cannot have a smooth square root. Conversely, if we only knew that the determinant of $V$ winds round the origin once, we could conclude that
some eigenvalue crossed the negative real axis an odd number of times. Indeed, if this were not true the sum of all the eigenvalue motions could not have amounted to $2\pi$. Any time an eigenvalue of $V$ crosses $-1$, $\det(1 + V)$ has a simple zero and we can argue as above for the Brillouin-Wigner case.

The simpler argument from the previous paragraph has a significant advantage: Now the global anomaly can be seen from the behavior of $\det(V)$ so we may as well look at the phase of $\det(X)$, even when $V \neq X$ since $\det(V) = \det(X)/|\det(X)|$. $X$ is a simple, completely local operator, and its behavior is relatively easy to determine. For example, we see now without any effort why our major conclusions would not change when $m_0 \neq 1$. We also understand easily why $m_0$ is restricted to the interval $(0, 2)$.

The one dimensional example has taught us how we expect things to work out in any odd dimension. The rest of the paper is devoted to three dimensions.

4. Three dimensions.

4a. A path in orbit space for $U(1)$.

To detect a global anomaly for $U(1)$ we need two gauge equivalent configurations which have different Chern-Simons actions, so are connected by a large gauge transformation. The large gauge transformations are those that wind non-trivially as the location is taken round any of the cycles of the torus. For such a winding gauge transformation to have the effect of changing the Chern-Simons action, the flux through the two-torus made up of the other two directions must be non-vanishing. In a non-compact formulation with periodic boundary conditions on the vector potentials, such configurations will not occur in the continuum limit. On finite lattices, since the fermions only see compact link variables, no matter what the pure gauge action is, configurations of the above type are not ruled out. But, they become extinct as continuum is approached and one needn’t worry about them in a dynamical context (for non-compact $U(1)$). For our purpose here the pure gauge action is immaterial. We only want to see whether an orbit deemed globally anomalous by continuum arguments will also exhibit a global anomaly on the lattice.

We pick a simple configuration [25] with constant field strength $F_{12} \neq 0$ and uniform $A_3$. The total flux of $F_{12}$ through the two torus spanned by directions 1 and 2 at fixed $x_3$ is quantized and we pick the minimal uniform value, $F_{12} = \frac{2\pi}{l^2}$. As $A_3$ increases by $\frac{2\pi}{l}$ the Chern-Simons action changes by $2\pi$. Writing $A_3 = \frac{a+\theta}{l}$, we expect the fermion determinant to wind around the origin once as $\theta$ varies between 0 and $2\pi$, connecting the two gauge equivalent configurations $A_3 = \frac{a}{l}$ and $A_3 = \frac{2\pi+a}{l}$. The gauge invariant meaning of $a$ is in the phase factor $e^{ia}$, which can be thought of as a parameter labeling the different pairs of gauge equivalent configuration we could consider.
We put this set of configurations on the lattice in a straightforward manner. We pick 
\(-\pi < a \leq \pi\) and set the links in the third direction to \(e^{i\alpha}\). One can also think of \(a\) as a 
parameter interpolating between different boundary conditions in the third direction. By 
convention, \(a = 0\) corresponds to anti-periodic boundary conditions in the third direction. In 
addition, we implement boundary conditions of our choice in the two other directions. The “instanton” field in the 1,2 directions is realized as in [32]: The parallel transporters 
around all \(\{1, 2\}\) plaquettes are equal to \(e^{i2\pi L^2}\). For simplicity, all sides of the torus have \(L\) sites.

4b. Argument for the winding of \(\det(V)\).

On the basis of the previous section we wish to argue that as \(\theta\) is varied \(\det(X)\) 
winds round the origin once. This will happen if and only if, as \(\theta\) covers its range, an odd 
number of \(V\)-eigenvalues cross the point \(-1\) on the unit circle. In turn, this is equivalent 
to counting how many real negative eigenvalues of \(X\) one finds as \(\theta\) traces out the loop of 
orbits. Since \(tr(X - 3 - m)^{2k+1} = 0\) the eigenvalues of \(X\) are paired in \((\lambda, 6 + 2m - \lambda)\) 
pairs, but this is dependent on the specifics of our discretization and we shall not make use 
of it. Clearly, \(a\) can be absorbed in \(\theta\), so we ignore \(a\) in what follows and set the boundary 
condition to antiperiodic in all directions for definiteness.

Our lattice configuration has translational invariance in the \(x_3\) direction so we go to 
momentum space (labeled by \(p_3\)) in that direction. Because of the antiperiodic boundary 
conditions \(p_3\) takes the values \(\frac{2\pi}{L}(n - \frac{1}{2})\) with \(n = 0, .., L - 1\). \(X\) is block-diagonal and the 
block labeled by \(p_3\) is given by:

\[
X_3(p_3) = 1 - m_0 - \cos(p_3 + \frac{\theta}{L}) + i\sigma_3 \sin(p_3 + \frac{\theta}{L}) + B_2 + C_2. \tag{4.1}
\]

\(B_2\) and \(C_2\) are the same as in (2.3) only \(\mu\) is restricted to 1,2 now. We are looking for 
real eigenvalues of \(X_3(p_3)\) for any \(\theta \in (0, 2\pi]\) and any \(p_3\) defined by an \(0 \leq n \leq L - 1\). If 
\(X_3(p_3)\psi = \lambda\psi\) with real \(\lambda\), \(\psi^\dagger\sigma_3 X_3(p_3)\psi\) must be real and hence

\[
\sin(p_3 + \frac{\theta}{L}) = 0. \tag{4.2}
\]

Both \(p_3\) and \(\theta\) are fixed by this equation to either \(\frac{\theta}{L} = -p_3 = \frac{\pi}{L}\) or \(\frac{\theta}{L} = \pi - p_3 = \frac{2\pi}{L} (\frac{L+1}{2} - \lfloor \frac{L+1}{2} \rfloor)\) where \([x]\) denotes the largest integer smaller or equal to \(x\). In those cases we have

\[
X_3(p_3) = 1 - m_0 - \epsilon + B_2 + C_2 \tag{4.3}
\]

with \(\epsilon = 1\) in the first case and \(\epsilon = -1\) in the second. We are searching for a negative 
eigenvalue \(\lambda\) of \(X_3(p_3)\). Define \(m_2 = 1 - m_0 - \epsilon - \lambda\). We are looking simultaneously for
solutions $\psi$ (with $\psi^\dagger \psi > 0$) and negative numbers $\lambda$ such that

$$\sigma_3(X_3(p_3) - \lambda)\psi = H_2(m_2)\psi = 0.$$  \hfill (4.4)

Here $H_2(m_2)$ is given by an expression similar to (2.2), only in two dimensions, and the gauge field is that of a two dimensional instanton. For $\epsilon = -1$, $m_2 > 0$, and we know from [1, 32] that $H_2(m_2)$ has a gap and no zero energy solutions. The single remaining possibility is $\epsilon = 1$. To obtain $\lambda = -m_0 - m_2 < 0$, taking into account that $0 < m_0 < 2$ and that $m_2 < 0$, we can search only in the window $-2 < m_2 < 0$. But, precisely in this range, we know, again from previous work [1], that there does exist a unique $-2 < m_2^z < 0$, which approaches zero as $L$ is taken to infinity, where $H_2(m_2)$ has a zero energy eigenstate. As function of $-2 < m_2 < 0$ there is exactly one crossing of zero in the spectral flow of $H_2(m_2)$. Let the state of zero energy be denoted by $\psi^z$. For $L$ large enough that $m_0 > -m_2^z$ we have

$$X_3(p_3)\psi^z = \lambda^z\psi^z$$  \hfill (4.5)

with $\lambda^z = -m_2^z - m_0 < 0$.

Our search was exhaustive. We learn that as a function of $\theta$ one eigenvalue of $X$ will cross the negative real axis once. Thus, $\det(V)$ will circle the origin once as theta is varied over its range and there is no smooth square root and there is a global gauge anomaly.

Our argument relied on two dimensional properties that were established partly numerically. Since the argument is a bit involved, we have carried out a direct numerical check. Moreover, setting up the numerical check is worthwhile because we also wish to check explicitly that indeed the Brillouin-Wigner definition reproduces the global anomaly. For the gauge non-invariant – parity invariant case the above does not constitute a full proof since it could be that $z^{BW}$ changes sign an odd number of times as $\theta$ is varied, thus annulling the gauge invariant effect of $\det(\frac{1}{2}(1 + V))$ going through its unique zero.

Note that the above argument about the phase of $\det(X)$ essentially gives the coefficient of the induced Chern-Simons action as computed in [7] perturbatively. We have determined the $\theta$ dependence of $\det(X)$ by picking a set of gauge configurations for which the three dimensional Chern-Simons action becomes identical to the one dimensional Chern-Simons action.

If we invert the argument, we could say that the perturbative computation in [7] of the coefficient of the Chern-Simons term is an indication that in two dimensions the spectral flow should have the properties desired to reproduce instanton effects on the lattice. Thus, not only does the induced Chern-Simons term indicate that the exact anomalies will be found in one dimension lower [15], but it also provides evidence that instantons should work correctly, which is already well known [1].
Also note that had we taken \( m > 0 \) we would have found no solutions with \( \lambda^z < 0 \) for \( L \) large enough.

4c. Numerical result for \( U(1) \).

\[
\begin{align*}
\text{Figure 1} & \quad \text{Path of } \det(V) \text{ in the complex plane as } \theta \text{ varies from } 0 \text{ to } 2\pi \text{ at } a = -\frac{\pi}{5}. \quad \text{A and B are gauge equivalent and the beginning and end of the path. Points along the path are connected to guide the eye and are equally spaced in } \theta. \quad \text{The arrow emanating from A indicates the sense in which the path is traversed for decreasing } \theta.
\end{align*}
\]

We checked on the computer that indeed the phase of \( \det(V) \) winds round the origin as argued above. Figure 1 shows the path of \( \det(V) \) in the complex plane for \( L = 8, m_0 = .8 \).

Each value of \( a \) defines one pair of gauge equivalent configurations \( A \) and \( B \) where \( B \) has \( a \) shifted by \( 2\pi \). As \( \theta \) is varied over one period, \( A \) and \( B \) are connected by a path of gauge in-equivalent intermediary configurations. We expect \( \det(1 + V) \) to go through zero once. We pick \( a = -\frac{\pi}{5}, L = 8 \) and \( m_0 = .8 \) and trace the evolution of \( \det(1 + V) \) in Figure 2. We see the expected zero crossing and that the dependence on \( \theta \) is similar to what we found in the one dimensional model. We also follow the \( z^{BW} \) phase along the
Figure 2  Path of $\frac{\det(1+V)}{\det(1+V_f)}$ and of $\frac{\det(V+V_f)}{|\det(V+V_f)|}$ in the complex plane for the same set of configurations as in Figure 1. We see the single zero crossing at O by the gauge invariant factor (the circle). The Brillouin-Wigner phase factor is a semi-circle making the weights of configurations A and B differ by a sign. Note that the shape of $\det(\frac{1}{2}(1 + V))$ looks close to being $\propto (1 + e^{i\theta})$, the one dimensional answer. Points along the paths are connected to guide the eye. The arrow emanating from A indicates the sense in which the path is traversed for decreasing $\theta$. A few corresponding points on the two paths are connected by lines ending in arrows.

path. It describes a smooth semi-circle, playing the role of a square root of $\det(V)$. The configurations A and B have weights of opposite sign therefore.

However, the picture above is not reproduced for all $a$. As $L$ gets larger it seems that the global anomaly is seen on the lattice for all $a \in (-2\pi, 0)$, but for $a \in (0, 2\pi)$ the path goes through an exceptional configuration, and the overall sign change cancels, leaving us with identical weights for the relevant configurations A and B. The exceptional configuration occurs at $a = 2\pi$. There might be a simple explanation for this configuration,
but we have not found one yet. By changing the boundary conditions of the fermions to periodic in all three direction, the range of $a$ (resetting the point $a = 0$ to the antiperiodic case, so we can compare) where the global anomaly is seen moves to $a \in (-\pi, \pi)$. However, exceptional configurations appear now at other values of $a$. Therefore, it appears that the Brillouin-Wigner phase choice, for the set of configurations we considered here, does not behave exactly as a continuum definition obtained by using the doubling trick would.\footnote{There is some similarity between this phenomenon and specific singular gauge transformations in two dimensional $U(1)$ gauge theories. In two dimensions the effect of these gauge transformations cannot be ignored [33].}

Of course, our configurations are quite special.

There is little point to try to quantify how often such cancelations will happen for more general configurations since we do not expect global anomalies to play such a central role for $U(1)$ in the continuum anyhow. The important lesson from the above examples is the possibility to reproduce global gauge anomalies for some configurations that are smooth and therefore cannot be ignored in the continuum limit. Also, one should keep in mind that it is not necessary for the Brillouin-Wigner phase convention to become identical to the definition obtained from the doubling trick in the continuum. It could be that the doubling trick minimizes in some sense the amount of gauge breaking while the Brillouin-Wigner phase convention does not. This does not deter from the validity of the Brillouin-Wigner phase convention when there are no anomalies present. However, had we found that all global anomalies from the continuum are wiped out by the Brillouin-Wigner phase convention there would have been a reason to worry about it, or else, we would have been forced to conclude that global gauge anomalies found in the continuum are not to be taken seriously.

4d. An $SU(2)$ example.

The true analogue of $d = 1$, $U(1)$ is $d = 3$, $SU(2)$ since $\pi_1(U(1)) = Z$ and $\pi_3(SU(2)) = Z$. The $d = 3$, $U(1)$ analysis above only helped us understand how global anomalies work on the lattice. Also, since the matrices involved are smaller for $U(1)$, numerical computations can be done faster. Unlike for $U(1)$, for $SU(2)$ one can observe the global gauge anomaly even in a background that has zero field strength everywhere. Moreover all loops winding round the tori can be made trivial too, so the configuration is just a gauge transform of the trivial one, with all link variables set to the unit matrix. In a regularization employing the doubling trick, the global anomaly would assign relative weight one (with some normalization) to this configuration (A) and relative weight minus one to a configuration (B) which is obtained by gauge transforming the trivial configuration.
by an element of the nontrivial homotopy class of maps from the three torus to \(SU(2)\).

An example of such an element is found in [7] where it appeared in momentum space in the Feynman diagram computation of the induced Chern-Simons action. It is trivial to take the continuum expression, view it now in real space and discretize it to put it on the lattice. Actually, \(V_f\) for a single component Dirac fermion provides the map with regular spin being viewed as isospin and regular space-time being viewed as momentum space. Let us call the image of \(V_f\), with a trivial factor in what is now spin space included, \(G\). The mass parameter \(m_0\) in \(G\) controls the approximate smoothness of the configuration on the lattice and we pick \(m_0 \equiv m_0^G = 1\).

\[
(G)_{\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{x,y} \frac{-m_0^G \delta_{ij} + \sum_{\mu}[1 - \cos(\frac{2\pi}{L} x_{\mu})]\delta_{ij} + i \sum_{\mu}\sigma_{\mu,ij}\sin(\frac{2\pi}{L} x_{\mu})}{\sqrt{[-m_0^G + \sum_{\mu}[1 - \cos(\frac{2\pi}{L} x_{\mu})]^2 + \sum_{\mu}[\sin(\frac{2\pi}{L} x_{\mu})]^2}}. \tag{4.6}
\]

To check for the global anomaly within the Brillouin-Wigner convention all we need to do now is to compare \(z_{BW}\) for the two configurations, both gauge equivalent to a trivial background. This simplifies the numerical work significantly.

\[
\frac{z_{BW}(B)}{z_{BW}(A)} = \frac{\det(V_f G + GV_f)}{\det(V_f G + GV_f)} \frac{1}{\det(V_f)}. \tag{4.7}
\]

The parameter \(m_0\) in \(V_f\) controls how many degrees of freedom represent massless fermions [2]. This number is small when \(m_0 \equiv m_0^f\) is close to zero.

For \(m_0^G = 1\) and \(m_0^f = .8\) and \(L = 8\) we obtained \(-1\) for the ratio in (4.6), the desired result. If we decrease either of these 3 parameters the sign switch can be made to disappear. The masses can be made smaller without loosing the sign switch if \(L\) is increased. In the continuum limit \(L\) is taken to \(\infty\) at fixed masses and the sign switch will stay. We therefore established that the global anomaly occurs on the trivial orbit for a large but smooth gauge transformation.

We checked the stability of the finding on the trivial gauge orbit. First, we perturbed the gauge transformation \(G\) by sizable, but limited, random perturbations. The sign switch relative to \(A\) remained. Then we made the perturbations completely random. This wiped out the sign switch. Once the gauge transformation is random it does not matter whether one multiplies by \(G\) also, or not. There is no sign switch relative to configuration \(A\). However, if the gauge transformations were random but limited, following it up by \(G\) produced a sign switch, while without \(G\) the sign was the same as for \(A\). Limited and unlimited fluctuating gauge transformation (but with no \(G\)-factor) have been observed not to give sign switches also in the case studied in [3]. There the \(|+>\) states were not taken at infinite mass, so we do not have to be in agreement, but we are. Of course, all
statements about configurations with a certain amount of randomness in them are of a statistical nature.

Similarly to the $U(1)$ case, not all sign switches mandated by continuum arguments occur on the lattice. It is difficult to guess what exactly happens to the theory if one carries out gauge averaging with the Brillouin-Wigner phase convention. The configurations that are typical of the continuum seem to cancel out because of the global anomaly, but something is left over. The problem is similar to the question what happens to an anomalous theory in four dimensions if regulated by the overlap and gauge averaged with the Brillouin-Wigner phase choice. Since in three dimensions the gauge violation is somewhat simpler, it might be easier to discover the fate of the anomalous theories of this paper. A necessary first step would be to investigate other orbits, with non-trivial gauge invariant content. We would like to find out what kind of effective action for the gauge invariant background is induced by the gauge transformations left over after cancelations due to the continuum global gauge anomaly have taken effect. Any understanding we would reach would have to be checked against variations of the Brillouin-Wigner phase convention. After all, the fate of an anomalous gauge theory might very well be non-universal.

5. Summary.

The overall inter-dimensional relationships we see in this paper indicate an intrinsic consistency of the overlap formalism across different dimensions. We have seen examples where the overlap reproduces global continuum anomalies. We therefore hope that the overlap needs not only perturbative, but also global anomaly cancelation to produce acceptable continuum theories after gauge averaging. There are several applications of our formalism we could think of. An interesting application (albeit a costly one) we are proposing is to $\mathcal{N} = 1$ supersymmetric gauge theories in five dimensions where one could search for nontrivial fixed points [34].

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