Quasinormal resonances of a massive scalar field in a near-extremal Kerr black hole spacetime

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The fundamental resonances of near-extremal Kerr black holes due to massive scalar perturbations are derived analytically. We show that there exists a critical mass parameter, $\mu_c$, below which increasing the mass $\mu$ of the field increases the oscillation frequency $\Re(\omega)$ of the resonance. On the other hand, above the critical field mass increasing the mass $\mu$ increases the damping rate $\Im(\omega)$ of the mode. We confirm our analytical results by numerical computations.

I. INTRODUCTION

The uniqueness theorems [1–3] imply that the metric outside a newly born black hole should relax into a Kerr-Newman spacetime, characterized solely by the black-hole mass, charge, and angular momentum. The relaxation phase in the dynamics of perturbed black holes is characterized by `quasinormal ringing’, damped oscillations with a discrete spectrum (see e.g. [4, 5] for detailed reviews). These characteristic oscillations are then followed by late-time decaying tails [6, 7].

The black hole quasinormal modes (QNMs) correspond to solutions of the perturbations equations (the Teukolsky master equation [8]) with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [9]. Such boundary conditions single out a discrete set of black-hole resonances $\{\omega_n\}$ (assuming a time dependence of the form $e^{-i\omega t}$). In analogy with standard scattering theory, the QNMs can be regarded as the scattering resonances of the black-hole spacetime. They thus correspond to poles of the transmission and reflection amplitudes of a standard scattering problem in a black-hole spacetime.

Quasinormal resonances are expected to be excited by a variety of astrophysical processes involving black holes. Being the characteristic sound of the black hole itself, these free oscillations are of great importance from the theoretical [10, 11] and astrophysical point of view [4, 5]. They allow a direct way of identifying the spacetime parameters, especially the mass and angular momentum of the black hole. This has motivated a flurry of research during the last four decades aiming to compute the resonance spectrum of various types of black holes [4, 5].

It is worth nothing that in most cases of physical interest, the black-hole QNMs must be computed numerically by solving the perturbations equations supplemented by the appropriate physical boundary conditions. However, it has been shown [12–14] that the spectrum of quasinormal frequencies can be studied analytically in the near-extremal limit $a \to M$, where $M$ and $a$ are the mass and angular momentum per unit mass of the black hole, respectively.

The dynamics of scalar test fields in black-hole spacetimes is primarily of theoretical interest– it usually serves as a toy model for the analysis of gravitational black-hole perturbations. However, as pointed out in [15], the possible existence of boson stars could make scalar QNMs observationally relevant. Boson stars are assumed to be made up of self-gravitating massive scalar fields [15, 16]. If a boson star becomes unstable and collapses to form a black hole, it is expected to radiate scalar waves (along with gravitational waves) in the appropriate QNMs frequencies.

Former numerical investigations of massive QNMs (see e.g., [15, 17]) have found that increasing the mass $\mu$ of the field increases the oscillation frequency $\Re(\omega)$ of the mode. Below we shall provide an analytical explanation for this phenomena. Furthermore, we shall show that there exists a critical mass parameter, $\mu_c$, above which increasing the mass $\mu$ of the field actually increases the damping rate $\Im(\omega)$ of the mode.

II. DESCRIPTION OF THE SYSTEM

The physical system we consider consists of a massive scalar field coupled to a rotating Kerr black hole. The dynamics of a scalar field $\Psi$ of mass $\mu$ in the Kerr spacetime [18] is governed by the Klein-Gordon equation

$$ (\nabla^a \nabla_a - \mu^2) \Psi = 0 . $$

(1)

(It is worth emphasizing that $\mu$ stands for $\mathcal{M}G/hc$, where $\mathcal{M}$ is the mass of the scalar field. We use units in which $G = c = h = 1$.) One may decompose the field as

$$ \Psi_{lm}(t, r, \theta, \phi) = e^{im\phi} S_{lm}(\theta; a\omega) R_{lm}(r; a\omega) e^{-i\omega t} , $$

(2)
where \((t, r, \theta, \phi)\) are the Boyer-Lindquist coordinates \([18]\), \(\omega\) is the (conserved) frequency of the mode, \(l\) is the spheroidal harmonic index, and \(m\) is the azimuthal harmonic index with \(-l \leq m \leq l\). (We shall henceforth omit the indices \(l\) and \(m\) for brevity.) With the decomposition \([24]\), \(R\) and \(S\) obey radial and angular equations both of confluent Heun type coupled by a separation constant \(K(\omega)\) \([19,23]\). The sign of \(\omega\) determines whether the solution is decaying \((\omega < 0)\) or growing \((\omega > 0)\) in time.

The angular functions \(S(\theta; \omega)\) are the spheroidal harmonics which are solutions of the angular equation \([8,21,24,23]\)

\[
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial S}{\partial \theta} \right) + \left[ K - a^2(\omega^2 - \mu^2) + a^2(\omega^2 - \mu^2) \cos^2 \theta - \frac{m^2}{\sin^2 \theta} \right] S = 0 .
\]

for the separation constants \(K_{lm}\). The expansion coefficients \(\{c_k\}\) are given in Ref. \([24]\).

The radial Teukolsky equation is given by \([8,25,26]\)

\[
\Delta \frac{d^2R}{dr^2} + \left( \Delta \frac{dR}{dr} + \left[ H^2 + \Delta(2ma\omega - K - \mu^2(r^2 + a^2)] \right) R = 0 ,
\]

where \(\Delta \equiv r^2 - 2Mr + a^2\) and \(H \equiv (r^2 + a^2)\omega - ma\). The zeroes of \(\Delta\), \(r_\pm = M \pm (M^2 - a^2)^{1/2}\), are the black hole (event and inner) horizons.

We are interested in solutions of the radial equation \([3]\) with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves at the black-hole horizon (as measured by a comoving observer) \([17]\). That is,

\[
R \sim \begin{cases} \frac{1}{2} e^{i\sqrt{\omega^2 - \mu^2}y} & \text{as } r \to \infty \quad (y \to \infty) ; \\ e^{-i(\omega - m\Omega)y} & \text{as } r \to r_+ \quad (y \to -\infty) , \end{cases}
\]

where the “tortoise” radial coordinate \(y\) is defined by \(dy = [(r^2 + a^2)/\Delta]dr\). These boundary conditions single out a discrete set of resonances \(\{\omega_n\}\) which correspond to the quasinormal resonances of the massive field \([17]\). (We note that, in addition to the QNMs resonances, the massive field is also characterized by a spectrum of bound states \([17,27,28]\) which tend to zero at spatial infinity.)

### III. THE QUASINORMAL RESONANCES

It is convenient to define new dimensionless variables

\[
x \equiv \frac{r - r_+}{r_+} ; \quad \tau \equiv \frac{r_+ - r_-}{r_+} ; \quad \varpi \equiv \frac{\omega - m\Omega}{2\pi T_{BH}} ; \quad k \equiv 2\omega r_+ .
\]

Here \(T_{BH} \equiv \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)}\) and \(\Omega \equiv \frac{a}{r_+^2 + a^2}\) are the temperature and angular velocity of the black hole, respectively. In terms of these dimensionless variables the radial equation becomes

\[
x(x + \tau) \frac{d^2R}{dx^2} + (2x + \tau) \frac{dR}{dx} + VR = 0 ,
\]

where \(V \equiv H^2/r_+ x(x + \tau) - K_{lm} + 2ma\omega - \mu^2r_+^2(x + 1)^2 + a^2\) and \(H = \frac{r_+^2}{2}\omega x^2 + r_+ kx + r_+ \varpi \tau/2\).

As we shall now show, the spectrum of massive quasinormal resonances can be studied analytically in the double limit \(a \to M\) and \(\omega \to m\Omega\) (see \([22]\) for the massless case). We first consider the radial equation \([8]\) in the far region \(x \gg \max\{\tau, M(m\Omega - \omega)\}\). Then Eq. \([8]\) is well approximated by

\[
x^2 \frac{d^2R}{dx^2} + 2x \frac{dR}{dx} + V_{far} R = 0 ,
\]
where \( V_{\text{int}} = (\omega^2 - \mu^2)r_+^2 x^2 + 2(\omega k - \mu^2 r_+)x + [-K_{lm} + 2m\omega + k^2 - \mu^2(r_+^2 + a^2)] \). A solution of Eq. (9) that satisfies the boundary condition (6) can be expressed in terms of the confluent hypergeometric functions \( M(a, b, z) \)

\[
R = C_1(2i\sqrt{\omega^2 - \mu^2 r_+})^{\frac{1}{2} + i\delta} x^{-\frac{1}{2} + i\delta} e^{-i\sqrt{\omega^2 - \mu^2 r_+}x} M\left(\frac{1}{2} + i\delta + ik, 1 + 2i\delta, 2i\sqrt{\omega^2 - \mu^2 r_+}x + C_2(\delta \to -\delta)\right),
\]

where \( C_1 \) and \( C_2 \) are constants. Here

\[
\kappa \equiv \frac{\omega k - \mu^2 r_+}{\sqrt{\omega^2 - \mu^2}},
\]

and

\[
\delta^2 \equiv k^2 + 2m\omega - K_{lm} - \frac{1}{4} - \mu^2(r_+^2 + a^2).
\]

The notation \((\delta \to -\delta)\) means “replace \( \delta \) by \(-\delta\) in the preceding term.”

We next consider the near horizon region \( x \ll 1 \). The radial equation is given by Eq. (8) with \( V \to V_{\text{near}} \equiv -K_{lm} + 2m\omega - \mu^2(r_+^2 + a^2) + (kx + \omega r/2)^2/x(x + \tau) \). The physical solution obeying the ingoing boundary conditions at the horizon is given by

\[
R = x^{-\frac{1}{2}} e^{\left(x^{-\frac{1}{2}} + 1\right)i\frac{1}{2}\omega x - k)} F_1(1, i\delta - ik, 1, 1 - i\omega; -x/\tau),
\]

where \( F_1(a, b; c; z) \) is the hypergeometric function.

The solutions (10) and (13) can be matched in the overlap region \( \max\{\tau, M(m\Omega - \omega)\} \ll x \ll 1 \). It is worth emphasizing that in order to have a non-trivial overlap region we must restrict our analytical solution to the regime of rapidly rotating \textit{near-extremal} black holes. In particular, the condition \( \tau \ll 1 \) is satisfied in the near-extremal limit. The \( x \ll 1 \) limit of Eq. (10) yields

\[
R \to C_1(2i\sqrt{\omega^2 - \mu^2 r_+})^{\frac{1}{2} + i\delta} x^{-\frac{1}{2} + i\delta} + C_2(\delta \to -\delta).
\]

The \( x \gg \tau \) limit of Eq. (13) yields

\[
R \to \tau^\frac{1}{2} e^{-i\delta - i\omega/2} F_1(1, i\delta - ik, 1, 1 - i\omega + i\omega) x^{-\frac{1}{2} + i\delta} + (\delta \to -\delta).
\]

By matching the two solutions in the overlap region one finds

\[
C_1 = \tau^\frac{1}{2} e^{-i\delta - i\omega/2} F_1(1, i\delta - ik, 1, 1 - i\omega + i\omega)(2i\sqrt{\omega^2 - \mu^2 r_+})^{-\frac{1}{2} - i\delta},
\]

\[
C_2 = x^\frac{1}{2} e^{-i\delta - i\omega/2} F_1(1, i\delta - ik, 1, 1 - i\omega + i\omega)(2i\sqrt{\omega^2 - \mu^2 r_+})^{-\frac{1}{2} + i\delta}.
\]

Approaching Eq. (10) for \( x \to \infty \) one gets

\[
R \to C_1(2i\sqrt{\omega^2 - \mu^2 r_+})^{i\delta} F_1(1 + 2i\delta, 1, 1 + i\delta + i\kappa) x^{-1 + i\kappa} + C_2(\delta \to -\delta) e^{i\sqrt{\omega^2 - \mu^2 r_+}x} + C_1(2i\sqrt{\omega^2 - \mu^2 r_+})^{-i\delta} F_1(1 + 2i\delta, 1, 1 + i\delta - i\kappa) x^{-1 - i\kappa} + C_2(\delta \to -\delta) e^{-i\sqrt{\omega^2 - \mu^2 r_+}x}.
\]

A free oscillations of the field (a quasinormal resonance) is characterized by a purely outgoing wave at spatial infinity. Thus, the coefficient of the exponent \( e^{-i\sqrt{\omega^2 - \mu^2 r_+}x} \) in Eq. (18) should vanish, see Eq. (6). Taking cognizance of Eqs. (16) and (18), one finds the resonance condition for the quasinormal modes of the massive field:

\[
\frac{\Gamma(2i\delta)\Gamma(1 + 2i\delta)(-2i\tau\sqrt{\omega^2 - \mu^2 r_+})^{-i\delta}}{\Gamma(\frac{1}{2} + i\delta - ik)\Gamma(\frac{1}{2} + i\delta - i\omega + ik)} + \frac{\Gamma(-2i\delta)\Gamma(1 - 2i\delta)(-2i\sqrt{\omega^2 - \mu^2 r_+})^{i\delta}}{\Gamma(\frac{1}{2} - i\delta - ik)\Gamma(\frac{1}{2} - i\delta - i\omega + ik)} = 0.
\]
The resonance condition (19) can be solved analytically in the regime $\tau \ll 1$ with $\omega \simeq m\Omega$. We first write it in the form
\[
\frac{1}{\Gamma(\frac{1}{2} - i\delta - i\varpi + ik)} = \mathcal{D} \times (-2i\tau \sqrt{\omega^2 - \mu^2 r_+})^{-2i\delta},
\] (20)
where $\mathcal{D} \equiv [\Gamma(2i\delta)]^2 \Gamma(\frac{1}{2} - i\delta - i\varpi + ik)/[\Gamma(-2i\delta)]^2 \Gamma(\frac{1}{2} + i\delta - ik)\Gamma(\frac{1}{2} + i\delta - i\varpi + ik)$. We note that $\mathcal{D}$ has a well defined limit as $a \to M$ and $\omega \to m\Omega$.

In the limit $\omega \to m\Omega$, where $\omega$ is almost purely real, one finds from Eq. (12) that $\delta^2$ is also almost purely real. If $\delta$ is almost purely real and larger than $\sim 1$, then one has $(-i)^{-2i\delta} = e^{(-i)^2(2i\delta)} = e^{-\pi \delta} \ll 1$. If $\delta$ is almost purely imaginary with a positive imaginary part, then one has $\tau^{-2i\delta} \to 0$ in the near-extremal $\tau \to 0$ limit. In both cases one therefore finds $\epsilon \equiv (2\pi \tau \sqrt{\omega^2 - \mu^2 r_+})^{-2i\delta} \ll 1$ on the r.h.s of Eq. (20).

Thus, a consistent solution of the resonance condition (20) may be obtained if $1/\Gamma(\frac{1}{2} - i\delta - i\varpi + ik) = O(\epsilon)$ (31). Suppose
\[
\frac{1}{2} - i\delta - i\varpi + ik = -n + \eta\epsilon + O(\epsilon^2),
\] (21)
where $n \geq 0$ is a non-negative integer and $\eta$ is a constant to be determined below. Then one has
\[
\Gamma(\frac{1}{2} - i\delta - i\varpi + ik) \simeq \Gamma(-n + \eta\epsilon) \simeq (-n)^{-1}\Gamma(-n + 1 + \eta\epsilon) \simeq \cdots \simeq \left[(-1)^n n!\right]^{-1}\Gamma(\eta\epsilon),
\] (22)
where we have used the relation $\Gamma(z + 1) = z\Gamma(z)$ (24). Next, using the series expansion $1/\Gamma(z) = \sum_{k=1}^{\infty} c_k z^k$ with $c_1 = 1$ [see Eq. (6.1.34) of (24)], one obtains
\[
1/\Gamma(\frac{1}{2} - i\delta - i\varpi + ik) = (-1)^n n!\eta\epsilon + O(\epsilon^2).
\] (23)
Substituting (23) into (20) one finds $\eta = \mathcal{D}/[(-1)^n n!]$.

Finally, substituting $\varpi \equiv (\omega - m\Omega)/2\pi T_{BH}$ and $k \equiv 2\omega r_+ = m + O(M T_{BH})$ [the last equality holds for $\omega = m\Omega + O(T_{BH})$] into Eq. (21), one obtains a simple formula for the quasinormal resonances of the massive field:
\[
\omega = m\Omega + 2\pi T_{BH}[m - \delta - i(n + \frac{1}{2})] + O(M T_{BH}^2, \epsilon T_{BH}).
\] (24)

IV. NUMERICAL CONFIRMATION

We shall now verify the validity of the analytically derived formula (24) for the massive resonances. The black-hole quasinormal frequencies can be computed using standard numerical techniques, see (17) for details. We present here results for the case $l = m = 2$. Substituting in Eq. (21), $a \to M$ and $M\omega = 1 + O(M T_{BH})$ [see Eq. (21) for $a \to M$], one finds $K_{22} \simeq 6\pi - \frac{9}{2} M^2 \mu^2$, where we have used the expansion coefficients $c_1 = \frac{1}{3}$ and $c_2 = -\frac{3}{10\pi}$ from (24). Next, substituting this value of $K_{22}$ into (12), one obtains
\[
\delta_{22}^2 = \frac{25}{28} - \frac{1}{7} M^2 \mu^2 + O(M T_{BH}).
\] (25)
One therefore finds that $\delta_{22}$ is real for $\mu < \mu_c = \sqrt{25/32} M^{-1}$ and imaginary for larger values of the field mass. (It is worth emphasizing that in full units $M \mu$ stands for the dimensionless ratio $GM \mu/hc = \mu M/M_{\text{Planck}}$.)

Taking cognizance of Eqs. (24)-(25), one finds that for small mass values ($\mu < \mu_c$, where $\delta_{22}$ is real), increasing the mass $\mu$ of the field increases the oscillation frequency $\mathcal{R}(\omega)$ of the resonance. On the other hand, for $\mu > \mu_c$ (where $\delta_{22}$ is imaginary) increasing the mass $\mu$ of the field increases the damping rate $\Im(\omega)$ of the mode.

In Table I we present a comparison between the analytically derived massive resonances, Eq. (24), and the numerically computed frequencies (17). We find an almost perfect agreement between the two. Table I demonstrates the fact that the agreement between the numerical data and the analytical formula (24) is quite good already at $a/M = 0.9$. This is quite surprising since the assumption $\tau \ll 1$ breaks down for this value of the rotation parameter.
TABLE I: Massive scalar quasinormal resonances of a near-extremal Kerr black hole with $a/M = 0.995$. The data shown is for the mode $l = m = 2$, see also [17]. We display the ratio between the analytically derived frequencies, $\omega_{\text{ana}}$, and the numerically computed values, $\omega_{\text{num}}$. The numerically computed frequencies of the massive field agree with the analytical formula (24) to within 2%.

| $a/M$ | $\Re \omega_{\text{ana}} / \Re \omega_{\text{num}}$ | $\Im \omega_{\text{ana}} / \Im \omega_{\text{num}}$ |
|-------|---------------------------------|---------------------------------|
| 0.0   | 1.003                           | 0.983                           |
| 0.1   | 1.003                           | 0.983                           |
| 0.2   | 1.003                           | 0.983                           |
| 0.3   | 1.004                           | 0.983                           |

TABLE II: Massive scalar quasinormal resonances of a near-extremal Kerr black hole. The data shown is for the mode $l = m = 2$ with $M\mu = 0.1$, see also [17]. We display the ratio between the analytically derived frequencies, $\omega_{\text{ana}}$, and the numerically computed values, $\omega_{\text{num}}$. The agreement between the numerical data and the analytical formula (24) is quite good already at $a/M = 0.9$.

| $a/M$ | $\Re \omega_{\text{ana}} / \Re \omega_{\text{num}}$ | $\Im \omega_{\text{ana}} / \Im \omega_{\text{num}}$ |
|-------|---------------------------------|---------------------------------|
| 0.9   | 1.007                           | 1.095                           |
| 0.95  | 1.010                           | 1.053                           |
| 0.99  | 1.005                           | 0.995                           |
| 0.995 | 1.003                           | 0.983                           |

V. SUMMARY

In summary, we have studied analytically the quasinormal mode spectrum of massive fields in the spacetime of near-extremal rotating black holes. It was shown that the fundamental resonances can be expressed in terms of the black-hole physical parameters: the temperature $T_{BH}$ and the angular velocity $\Omega$. Furthermore, we have shown that there exists a critical mass parameter, $\mu_c(l, m)$, below which increasing the mass $\mu$ of the field increases the oscillation frequency $\Re(\omega)$ of the resonance. On the other hand, above the critical field mass increasing the mass $\mu$ of the field increases the damping rate $\Im(\omega)$ of the mode. We confirmed our analytical results by numerical computations.

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