RAPID DYNAMICAL CHAOS IN AN EXOPLANETARY SYSTEM

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Received 2012 June 20; accepted 2012 July 10; published 2012 July 27

ABSTRACT

We report on the long-term dynamical evolution of the two-planet Kepler-36 system, which consists of a super-Earth and a sub-Neptune in a tightly packed orbital configuration. The orbits of the planets, which we studied through numerical integrations of initial conditions that are consistent with observations of the system, are chaotic with a Lyapunov time of only ~10 years. The chaos is a consequence of a particular set of orbital resonances, with the inner planet orbiting 34 times for every 29 orbits of the outer planet. The rapidity of the chaos is due to the interaction of the 29:34 resonance with the nearby first-order 6:7 resonance, in contrast to the usual case in which secular terms in the Hamiltonian play a dominant role. Only one contiguous region of phase space, accounting for ~4.5% of the sample of initial conditions studied, corresponds to planetary orbits that do not show large-scale orbital instabilities on the timescale of our integrations (~200 million years). Restricting the orbits to this long-lived region allows a refinement of estimates of the masses and radii of the planets. We find that the long-lived region consists of the initial conditions that satisfy the Hill stability criterion by the largest margin. Any successful theory for the formation of this system will need to account for why its current state is so close to unstable regions of phase space.

Key words: celestial mechanics – planets and satellites: dynamical evolution and stability

1. INTRODUCTION

Despite the seeming regularity of the solar system, the planetary orbits are known to be chaotic with a Lyapunov time of ~5 million years (Laskar 1989; Wisdom & Sussman 1992). The hallmark of a chaotic system is sensitive dependence on initial conditions: two trajectories that start arbitrarily close to each other will diverge exponentially on a timescale known as the Lyapunov time. Chaos is seen not only in the orbits of the planets, but also among various satellites and minor bodies of the solar system (Wisdom 1985; Lecar et al. 2001; Goldreich & Rappaport 2002). However, few of the multiplanet systems that exist around other stars have orbits measured precisely enough to determine definitively if chaos is present. There is evidence that both the Kepler-11 and 55Cnc systems are chaotic (Gayon et al. 2008; Migaszewski et al. 2012), but in the case of 55Cnc, where the masses and inclinations are not well constrained, this conclusion is less certain.

The Kepler-36 system consists of a subgiant star of solar mass and the two transiting planets Kepler-36b and c, with orbital periods of 13.8 and 16.2 days and masses of 4.1 and 7.5 $M_{\oplus}$, respectively (Carter et al. 2012). All of the necessary parameters for integration—the bodies’ positions and velocities at a reference epoch, as well as their masses—have been measured precisely. This allows study of the true dynamical evolution of the system at a level of detail that has only been achieved for the solar system and a handful of exoplanetary systems (Gozdziewski 2005; Migaszewski et al. 2012).

This Letter is organized as follows. Section 2 describes our integration methods and the set of initial conditions we use. Our results on the chaotic behavior of the planets, including an explanation of its salient features, are given in Section 3. A stability analysis of the system is presented in Section 4. We discuss briefly in Section 5 how our results may constrain models of formation of the system.

2. NUMERICAL METHODS

The effect on the transit light curves of dynamical interactions between the planets was modeled in a Bayesian fashion using prior knowledge of the host star obtained through asteroseismic analysis. This resulted in a posterior joint probability distribution for the bodies’ masses and the planetary positions and velocities at a reference epoch (Carter et al. 2012). Initial conditions and masses drawn from this distribution form a representative set because they sample the possible configurations of the planets consistent with the data in a statistically appropriate way. All of the correlations between the uncertainties in these orbital parameters are naturally taken into account.

We studied the dynamical evolution of the Kepler-36 system through numerical integration of $10^4$ initial conditions and masses. Our primary integration scheme was a symplectic $n$-body mapping (Wisdom & Holman 1991). We implemented Chambers’ symplectic corrector to improve the accuracy of the integrator (Wisdom et al. 1996; Chambers & Migliorini 1997; Laskar & Robutel 2001; Wisdom 2006).

“Stepsize chaos” can be a source of error in symplectic integrators (Wisdom & Holman 1992), but it is negligible when the time step $\Delta t$ is smaller than the shortest physical timescale in the system by at least a factor of 10 or 20 (Rauch & Holman 1999). Each integration was carried out using a fixed time stepsize of 0.005 days. For each integration, the initial conditions were perturbed by a total of 5 parts in 100,000.

6 These initial conditions are published with Carter et al. (2012).

7 Stepsize chaos results from commensurabilities between the mapping frequency and physical frequencies of the system.
step of $\Delta t \lesssim 1\% P_b$. Similar results were obtained when using an even smaller time step ($\Delta t \lesssim 0.1\% P_b$) and when using a Bulirsch–Stoer integrator (Bulirsch & Stoer 1980).

Only Newtonian gravitational forces were considered; general relativity is unimportant due to the long timescale for relativistic precession ($\sim 10^8$ days) relative to the secular timescale of $3 \times 10^4$ days. Test integrations including the effect of general relativity, which we mimicked using a dipole potential that produces the correct precession rate (Nobili & Roxburgh 1986), confirmed this assumption.

### 3. RAPID DYNAMICAL CHAOS

A direct way of determining if a particular trajectory is chaotic is to measure how perturbations of that trajectory grow in time. This growth is governed by the tangent equations of the symplectic mapping, which we integrate in addition to the equations of motion for the system. The magnitude $d$ of perturbation of a chaotic trajectory will grow exponentially as

$$\lim_{t \to \infty} d(t) = d(0)e^{t/\tau},$$

where $\tau$ is the (shortest) Lyapunov time of the system. We estimate the Lyapunov time to be the value of $t / \log (d(t))$ at the end of the integration (with $d(0) = 1$).

The tangent equations were integrated forward by $10^7$ days for each trajectory. Over 99% of these initial conditions, including the orbit that best fits the data, are chaotic. The rest do not show exponential divergence on the timescale of the integrations and therefore may be quasi-periodic. Figure 1 shows the distribution of Lyapunov times for the entire set of initial conditions and for those belonging to the long-lived region (where we predict the true orbits must lie; see Section 4). The typical Lyapunov time for both groups is several thousand days. A more pertinent measure is the ratio of the Lyapunov time to the shortest orbital period. This ratio is $\sim 300$ for Kepler-36. In contrast, the Lyapunov time of the solar system is approximately $2 \times 10^7$ orbits of Mercury. One of the most rapid Lyapunov times observed within the solar system, that of the Saturnian moons Prometheus and Pandora, is $2000$ Promethean orbits (Farmer & Goldreich 2006).

Figure 1 shows that the estimated Lyapunov times range over two orders of magnitude, with two clusters centered at approximately 300 and 4000 days. This was unexpected, since chaotic zones in phase space are typically characterized by a single value of the minimum Lyapunov time (Henon 1983). Moreover, the estimated Lyapunov times change as the total integration time is increased, with a growing population centered on 300 days. This suggests that the initial conditions span two
nearly disconnected regions of the chaotic zone in phase space, characterized by different estimates of the local Lyapunov time (which is not the absolute shortest Lyapunov time, in the infinite limit), and that trajectories are moving between them. This will be returned to in Section 4; we turn now to understanding the rapidity of the chaotic behavior.

3.1. Origin of the Chaos

To gain an understanding of the chaos associated with a Lyapunov time of several thousand days, we sought evidence for resonant behavior. Mean motion resonances (MMRs) become important when the ratio of the planetary mean motions is close to a rational number, \( n_2/n_3 \sim (j + k)/j \), where \( j \) and \( k \) are mutually prime integers. Numerically we find that the angles characterizing the 29:34 eccentricity-type MMR, \( \theta_1 = 34\lambda_c - 29\lambda_b - 5\sigma_b \) and \( \theta_2 = 34\lambda_c - 29\lambda_b - 5\sigma_c \) (and the linear combinations of \( \theta_1 \) and \( \theta_2 \) appearing in the interaction Hamiltonian), to be behaving chaotically. Here, \( \lambda \) and \( \sigma \) refer to the mean longitude and longitude of periastron of the orbits. Chaotic alternations of these angles between circulation and libration are shown for a particular trajectory in Figure 2. It may be surprising that such a high-order (large \( k \)) MMR would be important, as resonance widths in phase space scale as \( k \) factors of eccentricities and inclinations (and therefore are usually negligible for nearly circular and coplanar orbits). However, the Laplace coefficients, which also factor into the widths, are large when the semimajor axis ratio of the planets is near unity, as is the case for Kepler-36. This allows the 29:34 resonance to be important.

To understand why the 29:34 resonant angles should behave chaotically, we appeal to the resonance overlap criterion (Chirikov 1979). This states that chaos will ensue if the angles \( \theta_2 \) and \( \theta_1 \), when neglecting the interaction between them, are both analytically calculated to be librating in the same region of phase space. This can occur if the resonance widths, which are functions of the eccentricities, become large enough or if the resonance centers, determined to be where \( \theta_1 = 0 \) and \( \theta_2 = 0 \), coincide. We find that the vast majority of the initial conditions exhibit oscillations with a period of several thousand days in the angle \( \sigma_b - \sigma_c \), indicating that \( \sigma_b \sim \sigma_c \), i.e., that the resonant islands are overlapping.

The approximate equations of motion for resonant angles resemble those of coupled pendulums driven by the oscillations in the eccentricities and the periastra of the two planets. This correspondence can be used to show that the resulting chaotic behavior should have a Lyapunov time similar to the period of these oscillations (Holman & Murray 1996). For Kepler-36 this period is several thousand days, similar to the Lyapunov time, supporting this explanation of the origin of the chaos.

This explanation alone is not surprising, as the connection between chaotic behavior and MMR overlap is well known. Why then is the Lyapunov time (relative to the smallest orbital period) so short for this system? In previously known examples of MMR overlap, the periodic driving of the resonant angles by the eccentricity and periastra is governed by secular effects. In the Kepler-36 system, the driving is dominated by the effects of a nearby first-order MMR (\( P_b/P_c \sim 6/7 \)), which appear at a lower order (in the eccentricities) in the interaction potential. It can be shown that the proximity to the 6:7 resonance does not alter the qualitative character of the standard secular solution for the eccentricities and periastra but that it does affect the frequencies of the oscillations significantly. Following Malhotra et al. (1989), we estimate the shorter of the two modified secular timescales to be ~8000 days. We confirmed that the 6:7 terms are crucial for producing the correct behavior of the eccentricities and periastra numerically as well.

The 6:7 inclination-type terms appear in the Hamiltonian at the same order in the inclinations as the secular terms and hence should not be as crucial as the 6:7 eccentricity terms. Trajectories that have been projected onto the invariable plane remain chaotic with a Lyapunov time of several thousand days, confirming that a coplanar model is sufficient to explain the chaos.

A much smaller subset of the initial conditions shows chaotic behavior of the angles characterizing the 23:27 eccentricity-type MMR. Figure 3 delineates the chaotic zones (Lyapunov

**Figure 2.** Chaotic evolution of the resonant angles \( \theta_1 = 34\lambda_c - 29\lambda_b - 5\sigma_b \) and \( \theta_2 = 34\lambda_c - 29\lambda_b - 5\sigma_c \) for a randomly chosen trajectory from the long-lived region. The red overlaid points show a smoothed version of the black points to guide the eye.
3. Structure of the phase space near the Kepler-36 initial conditions. Red indicates the span of the long-lived trajectories on this plane and blue the entire sample. Black points correspond to trajectories that do not fit the data. Green lines indicate the nominal location of the 23:27 and 34:39 MMR. Only $a_b$ varies within each panel. Top: $(e_b, e_c) \sim (0.02, 0.0)$. Middle: $(e_b, e_c) \sim (0.021, 0.006)$. Bottom: $(e_b, e_c) \sim (0.036, 0.0)$.

Even if the planets never have close encounters, repeated weak interactions can lead to a second type of instability, in which the gradual exchange of angular momentum and energy between the planets results in drastic orbital variations. This is known as Lagrange instability. There is no known analytic criterion for Lagrange stability, but numerical integrations can demonstrate that a given trajectory does not show unstable behavior on the timescale of the integration and therefore can be considered to be “long-lived.”

A preliminary stability analysis of the Kepler-36 system determined that $\sim9\%$ of the initial conditions did not satisfy the Hill stability criterion (Carter et al. 2012). It was left to future work to determine whether the initial conditions were Lagrange long-lived.

4.1. Lagrange Stability Analysis

We integrated all $10^4$ trajectories for $2.5 \times 10^9$ days and found that while only $30\%$ showed Lagrange unstable behavior during the first $\sim10^6$ days, this percentage had increased to $\sim75\%$ by
the end of the integrations. We classified Lagrange instability as variations in semimajor axis, eccentricity, and inclination that were different from their running average by greater than 10%. In general, the trajectories that were Lagrange unstable and those that did not show instability during these integrations were well mixed, but we could identify a single contiguous region in phase space, accounting for $\sim 4.5\%$ of the initial conditions, which had no unstable trajectories on these timescales. This region is characterized by lower eccentricities and inclinations, shows no systematic difference in the goodness of fit to the data, and contains five of the candidate quasi-periodic trajectories. One hundred randomly chosen initial conditions from this “long-lived core” were integrated for 200 million years ($> 5 \times 10^9 P_J$). Of these, only four exhibited instability; all came from the borders of the long-lived core.

The initial conditions belonging to the long-lived core satisfy

$$h > h_{\text{crit}} + \epsilon,$$

(3)

where the value of $\epsilon \sim 0.0007$ is determined numerically. In other words, orbits that are Lagrange long-lived must satisfy a criterion nearly identical to the Hill criterion, but with a slightly different critical value. This same relationship was found for Jupiter-mass planets (Barnes & Greenberg 2006). Our work suggests that the same link between Hill and Lagrange stability applies across orders of magnitude in the parameter $m_{\text{planet}}/m_{\text{star}}$, though the value of $\epsilon$ depends on this parameter. Contours of constant $d_L \equiv h - h_{\text{crit}}$ are indicated in Figure 4.

The initial conditions belonging to the long-lived region are those with the smallest values of the angular momentum deficit (AMD), a quantity that parameterizes how far the orbits are from purely circular and coplanar. It is straightforward to show that $d_L$ is a function of the AMD for nearly circular, coplanar orbits, explaining the observed dependence.

To explore the connection between $d_L$ and Lagrange instability, we focused on a group of 475 Hill stable trajectories with a uniform distribution in $d_L$. There is a sharp transition at $d_L = 0.0007$, where the trajectories change from being mostly Lagrange unstable to being entirely Lagrange long-lived. A smaller value of $d_L$ correlates with a shorter time to show

![Figure 4: Relationship between Hill and Lagrange stability. Upper trio of plots: the region of orbital element space where orbits can be long-lived. Lagrange unstable orbits are shown in red, orbits that have not undergone Lagrange stability in $2.5 \times 10^9$ days are in black. Lower trio of plots: grouping initial conditions based on their values of $d_L$ reveals the Lagrange long-lived core.](image)
Figure 5. Period ratio and eccentricity evolution ($e_b$ in black, $e_c$ in red) for a Lagrange unstable orbit. When the orbit begins to change qualitatively in an erratic way, the slope of $\log(d)$ changes, indicating a change in the estimated Lyapunov time. $\log(d)$ is the natural logarithm of the magnitude of the perturbation of the trajectory, as defined in Equation (1).

Lagrange instability and with smaller minimum approach distances between the two planets.

It is not obvious what longer integrations would reveal regarding the trajectories that appear to be long-lived on the timescale of our integrations but do not satisfy Equation (3) (i.e., are not in the long-lived core). Previous work indicates that unless these orbits are protected by some resonance mechanism they too will show Lagrange instability on relatively short timescales (Gladman 1993; Barnes & Greenberg 2007). We predict that future observations will show that the true orbits of the Kepler-36 planets belong to the long-lived core.

By restricting the initial conditions to the long-lived core we refine the system parameters. This results in median values and 84.2% and 15.8% confidence intervals for the masses and radii of the planets of $m_b = 4.32 M_{\oplus} \pm 0.19$, $m_c = 7.84 M_{\oplus} \pm 0.33$, $R_b = 1.49 R_{\oplus} \pm 0.035$, and $R_c = 3.68 R_{\oplus} \pm 0.055$. The best-fit values are $m_b = 4.11 M_{\oplus}$, $m_c = 7.46 M_{\oplus}$, $R_b = 1.46 R_{\oplus}$, and $R_c = 3.59 R_{\oplus}$. The mutual inclination is constrained to be less than 1°.

Although these planets seem to be on the brink of instability, the orbits are stable to small perturbations. In particular, we confirmed that a hypothetical third planet, with a period of $2.6 P_c$ and $M \leq 5 M_{\oplus}$, need not disrupt the two planet system on seven million year timescales.

4.2. Transition to Lagrange Instability

We now return to the evolution of the distribution of Lyapunov times shown in Figure 1. We find that the onset of Lagrange unstable behavior occurs when the trajectory moves between the two peaks of the distribution (at $\sim 300$ and $\sim 4000$ days). A typical Lagrange unstable trajectory is shown in Figure 5. The eccentricities and semimajor axes remain nearly constant for about $10^5$ days, after which the orbital elements vary dramatically. At the same time, the estimated Lyapunov time changes from roughly 15,000 days to 180 days.

The ability for trajectories to cross between the two nearly disconnected regions of the chaotic zone (which are characterized by different estimates of the local Lyapunov time and also are chaotic for different reasons) can be understood as a consequence of resonance overlap (Mardling 2008; Murray & Holman 1997). As Figure 3 shows, separate MMRs merge at higher eccentricities. Our hypothesis is that chaotic diffusion of eccentricities to higher values results in MMR overlap, at which point a bottleneck between the two regions is formed. Once this occurs, trajectories can enter the Lagrange unstable region where they explore a broader range of period ratios. This chaotic diffusion may occur for all of the chaotic trajectories, in
which case the distribution of Lyapunov times will approach an asymptotic form with all trajectories belonging to the 300 day peak as longer integrations are performed.

5. DISCUSSION

We have presented a dynamical analysis of the Kepler-36 system which has yielded several surprising results. The orbits are chaotic with an extremely short Lyapunov time, yet some still manage to be long-lived. The closeness of this system to instability is an intriguing feature of Kepler-36. In particular, did the planets form with orbits contained in the long-lived core? Alternatively, did some dissipative process drive them into this long-lived configuration?

It would seem that tidal dissipation is negligible for this system, despite the proximity of the planets to the star. If tidal dissipation were important then the planets would have been on more eccentric orbits in the past, and, assuming the orbital separation was not very different at that time, it is unlikely such a configuration would have been stable.

Alternatively, the planets could have been closer together if they were protected by a resonance. A commonly discussed mechanism for forming compact multiplanet systems is convergent migration in the gaseous protoplanetary disk or the disk of remnant planetesimals (Terquem & Papaloizou 2007). Perhaps the Kepler-36 planets formed at large orbital distances, and migrated until being locked into a 6:7 resonance. After migration ended, subsequent tidal evolution could have driven the planets apart to their current configuration. However, we recognize that it may be challenging to create closely packed resonant systems through conventional migration mechanisms (Rein et al. 2012), though other studies find that it is possible to produce systems like Kepler-36 (Ida & Lin 2010; Ogihara et al. 2010; Pierens et al. 2011). Therefore, this system may contain important clues about the relevance of convergent migration to the formation and dynamical evolution of planetary systems.

We thank J. Wisdom for sharing with us his code that calculates coefficients in the disturbing function, as well as other members of the Kepler TTV team and our anonymous reviewer for helpful comments on the text. E.A. acknowledges support for this work which was provided by NSF Career grant AST-0645416.

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ERRATUM: “RAPID DYNAMICAL CHAOS IN AN EXOPLANETARY SYSTEM” (2012, ApJL, 755, L21)

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Received 2013 July 30; published 2013 August 21

Online-only material: color figures

SUMMARY

We have found an error in the one of the symplectic codes used in the Letter “Rapid Dynamical Chaos in an Exoplanetary System” (2012, ApJL, 755, L21). This error only affected our estimate of the Lyapunov time; all long-term stability tests were carried out using a different symplectic mapping. The incorrect version of the code used for Lyapunov time estimation was only used for this work. We have re-performed all numerical tests involving Lyapunov times with a corrected code and find that none of the conclusions of the Letter regarding chaos in the Kepler 36 system have changed.

In particular, over 99% of the Kepler 36 initial conditions exhibit rapid dynamical chaos characterized by median Lyapunov time of \( \sim 5 \times 10^3 \) days. Before publication, we had confirmed the presence of fast chaos independently using a Bulirsch-Stoer code, and so this main result could not have been affected by the error in the symplectic integrator.

Figure 1 shows the distribution of Lyapunov times (based on different length integrations, for the long-lived core of initial conditions and the entire sample of 10,000) that appeared in the original Letter and a corrected version. In both cases, longer integrations reveal

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Figure 2. Structure of the phase space near the Kepler-36 initial conditions. A: the corrected version, B: the old, incorrect version. Red indicates the span of the long-lived trajectories on this plane (only in B) and blue the entire sample. Black points correspond to trajectories nearby in phase space (which do not fit the data of the Kepler 36 system). Green lines indicate the nominal location of the 23:27 and 34:39 MMR. Only \(e_b\) varies within each panel. Top: \((e_b, e_c) \sim (0.02, 0.0)\). Middle: \((e_b, e_c) \sim (0.021, 0.006)\). Bottom: \((e_b, e_c) \sim (0.036, 0.0)\). At higher eccentricities, nearby mean motion resonances overlap, forming a large chaotic zone and allowing trajectories to explore a large range of period ratios. The two plots only differ in minor ways, without changing the overall structure. In the upper set of plots, the chaotic zones near the 34:29 and 23:27 nominal resonance locations appear to line up better in the corrected version.

(A color version of this figure is available in the online journal.)

that the distribution of estimated Lyapunov times becomes bimodal, with a growing peak at several hundred days. This arises because the estimated Lyapunov time of an initial condition changes to shorter times when the orbit begins to exhibit instabilities (as in Figure 5 of the original Letter). The same behavior is still seen in the corrected integrations, as evidenced by the growing peak at \(10^2–10^3\) days in Figure 1, but only after a longer amount of time—in other words, these trajectories are still unstable, but the typical instability time is longer when using the correct code compared to the incorrect code.

In fact, this is the only significant difference we find between the published results regarding Lyapunov times and the corrected results: the incorrect version of the code caused some initial conditions to show instabilities on timescales of \(10^5–10^7\) days rather than \(10^7–10^9\) days. Both timescales are short compared to the age of the system.

We still find that the initial conditions, when projected onto the invariant plane of the planets, are chaotic with a median Lyapunov time of about 10,000 days. Therefore, our explanation of the rapid chaos, motivated analytically by the coplanar 34:29 resonant angles, still applies. Lastly, Figure 2 shows the map of the chaotic zones associated with different mean motion resonances around the Kepler 36 initial conditions. Again, we show both the original, published version and a corrected version. There is no significant difference between the two that affects our conclusions or our interpretation of the transition to instability.

DESCRIPTION OF THE ERROR

Our estimates of the Lyapunov time were carried out using a symplectic integrator in canonical heliocentric coordinates (e.g., Wisdom 2006). In these coordinates, the Hamiltonian consists of three pieces: \(H_K\), a Keplerian Hamiltonian for each of the planets, \(H_I\), the standard interaction potential between the planets, and \(H_c\), which is proportional to \(m_{\text{planet}}/m_*\) and when applied shifts the positions of all the planets by an equal amount. We inadvertently set the coefficient of \(H_c\) to zero. We were correctly integrating the equations of motion and the tangent equations of a Hamiltonian system which differed from the true planetary Hamiltonian by a set
of small terms. We believe that the reason why this Hamiltonian gave such nearly correct results was because it still consisted of a dominant Keplerian piece plus the (correct form) of the interaction between the planets.

We thank Matthew Payne for helping us find this bug.

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