Operator product expansion of the energy momentum tensor in 2D conformal field theories on manifolds with boundary

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Abstract

Starting from the well-known expression for the trace anomaly we derive the $T \cdot T$ operator product expansion of the energy-momentum tensor in 2D conformal theories defined in the upper halfplane without making use of the additional condition of no energy-momentum flux across the boundary. The OPE turns out to be the same as in the absence of the boundary. For this result it is crucial that the trace anomaly is proportional to the Gauß-Bonnet density. Some relations to the σ - model approach for open strings are discussed.

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1 Introduction

In a conformal field theory defined on the two dimensional infinite plane the operator product expansion (OPE) of the energy-momentum tensor

\[ T_{++}(z)T_{++}(w) = \frac{\delta}{(z-w)^4} + \frac{2T_{++}(w)}{(z-w)^2} + \frac{\partial_+ T_{++}(w)}{z-w} + \text{fin. terms} \]  

plays a very fundamental role. There is a similar formula for \( T_{--} \cdot T_{--} \) obtained by \( z, w \rightarrow \bar{z}, \bar{w} \). The trace \( T_{+-} \) as well as \( T_{++} \cdot T_{--} \) vanish up to contact terms. Introducing moments of \( T_{++} \) and \( T_{--} \) one gets from (1) two independent Virasoro algebras with the same central charge \( c \). A further textbook fact relates the central charge to the Weyl anomaly of the theory generalized to curved 2D manifolds

\[ T_a^a(z) = \frac{c}{48\pi} R(z). \]  

Of course the trace vanishes in the flat limit \( R \rightarrow 0 \).

For manifolds with boundary there is no flux of energy-momentum across the boundary if there are no additional interactions localized at the boundary. In the following we concentrate ourselves to the simplest case, the upper half-plane. Then we have \( T_{12} = 0 \) on the real axis, i.e.

\[ T_{++}(z) = T_{--}(z), \quad \text{if Im}(z) = 0. \]  

The more general situation of present boundary interactions is relevant for instance in the \( \sigma \)-model picture of open strings [1, 2, 3, 4] as well as for surface critical behaviour in statistical mechanics [5]. Away from criticality, i.e. conformal invariance, energy-momentum flux across the boundary is allowed, of course. However, in the conformal limit the conserved current \( j_m = \epsilon^n T_{mn} \) for boundary conserving conformal transformations \( \epsilon \) has to be parallel to the boundary. This again forbids energy-momentum flux across the boundary. The consequences of (3) are well-known [4]. Using analyticity one can continue into the lower half-plane via \( T_{--}(z) = T_{++}(\bar{z}) \). Then there is only one independent component of the energy-momentum tensor and only one Virasoro algebra. The OPE has the same form (1) as in the absence of the boundary. The net result of this repetition can be summarized as follows: To find the conformal cases out of the generalized \( \sigma \)-models defined in the upper half-plane one has to look for solutions of

\[ T_a^a = 0, \quad \text{Re}(z) \geq 0 \text{ and } T_{++} = T_{--}, \quad \text{Im}(z) = 0. \]

Our paper will be a contribution towards the elimination of the condition (3) as an independent one. Although we are not showing that \( T_a^a = 0 \) implies (3), we derive (1) for the half-plane from the knowledge of the trace \( T_a^a \) alone. As mentioned already, in the standard derivation of (1) condition (3) is used as an essential input. Hence in some sense our note is in the tradition of the Curci-Paffuti theorem [4, 8, 9] which uncovers relations between otherwise independent coefficient functions in the trace of the energy-momentum tensor. Vanishing of the \( \beta_i \)-functions for \( i \neq \phi \) (\( \phi \) dilaton) results
in $\bar{\beta}_\phi = \text{const.}$ The constant value of $\bar{\beta}_\phi$ plays the role of the central charge in the corresponding conformal theory. The theorem has been extended to the case of $\sigma$-models on manifolds with boundary [10, 11]. The vanishing of all non-dilaton bulk as well as boundary $\bar{\beta}$-functions results in constant $\bar{\beta}_\phi$ and $\bar{\beta}_\hat{\phi}$ (dilaton coupling to the boundary).

Using Wess-Zumino type consistency relations one can show [11] that the constant values of $\bar{\beta}_\phi$ and $\bar{\beta}_\hat{\phi}$ are correlated. The trace $T_a^a$ for the extension to curved 2D manifolds has to be proportional to the Gauß-Bonnet density (see also ref. [12])

$$T_a^a(z) = \frac{c}{48\pi} R(z) + \frac{\hat{c}}{24\pi} \int ds \, \delta^{(2)}(z - z(s)) \, k(z(s))$$

with

$$c = \hat{c}.$$  \hspace{1cm} (5)

$k(z(s))$ is the geodesic curvature of the boundary at the point $z(s)$ and $s$ the length parameter measured with the 2D metric $g_{ab}$.

Starting from (4) we will derive the $T \cdot T$ - OPE by adapting the technique of ref. [13] to the boundary case. The calculations will be done for general uncorrelated $c$, $\hat{c}$. At the end we will get (1) only for $c = \hat{c}$. As a by-product of this analysis we find, turning the argument around, that insisting on (3) as an independent input yields an alternative proof of $c = \hat{c}$.

## 2 Derivation of the OPE

We start from the conformal Ward identity [13] with the Weyl anomaly adapted in the sense of (4) to the case of a manifold with boundary

$$\frac{1}{2} \int d^2 z \sqrt{g} (P\epsilon)_{ab} \langle T^{ab}(z) \, \Phi_1(z_1) \ldots \Phi_N(z_N) \rangle =$$

$$= \left( \frac{c}{48\pi} \int d^2 z \sqrt{g} R \nabla_a e^a(z) \right) + \frac{\hat{c}}{24\pi} \int k(z(s)) \nabla_a e^a ds \langle \prod_j \Phi_j \rangle$$

$$- \sum_{k=1}^N \left( \epsilon^a(z_k) \frac{\partial}{\partial z_k} + \frac{d_k}{2} \nabla_a e^a(z_k) + i s_k \left( \frac{\epsilon_{ab}}{2} \nabla^b e^a + \omega_a e^a(z_k) \right) \right) \langle \prod_j \Phi_j \rangle.$$  \hspace{1cm} (6)

e$ is a diffeomorphism with $P\epsilon$ defined by

$$(P\epsilon)_{ab} = \nabla_a \epsilon_b + \nabla_b \epsilon_a - g_{ab} \nabla_c \epsilon^c ,$$

while $d_k$, $s_k$ are the dimensions and spins of the primary fields $\Phi_k(z_k)$. The identity (6) will be functionally differentiated with respect to $g_{mn}(w)$. This yields, among other terms, a double insertion of the energy-momentum tensor. By a suitable choice of $\epsilon(z)$ one can localize the expressions finally. This standard procedure [13, 14] in our case gets two new
aspects. The diffeomorphism has to respect the boundary and one has to handle the metric variation of the geodesic curvature \( k \) as well as the length parameter \( s \) on the boundary.

The energy-momentum tensor in (6) is defined by

\[
T^{ab} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{ab}} .
\]

Hence the natural position of indices is the upper one for \( T \) and \( \epsilon \) but the lower one for \( g, \partial \) and \( \nabla \). This means

\[
\delta \left( \sqrt{g}(P\epsilon)_{ab} \right) = \sqrt{g} \left[ \nabla_d (\epsilon^d \delta g_{ab} - \frac{1}{2} \epsilon^d g_{ab} g^{mn} \delta g_{mn}) + \frac{1}{2} g^{mn} \delta g_{mn} (\nabla_a \epsilon_b + \nabla_b \epsilon_a) + \delta g_{bd} \nabla_a \epsilon^d + \delta g_{ad} \nabla_b \epsilon^d - 2 \delta g_{ab} \nabla^d \epsilon^d \right] .
\]

(9)

Therefore, the flat limit of the derivative of the l.h.s. of (6) with respect to \( g_{--}(w) \) is

\[
\frac{\delta \text{l.h.s. of (6)}}{\delta g_{--}(w)} \bigg|_{\text{flat}} = \left\{ \begin{array}{l}
4 \int d^2 z T_{++}(w) \left[ \partial_- \epsilon^+(z) T_{++}(z) + \partial_+ \epsilon^-(z) T_{--}(z) \right] \\
+ i \int dz^1 \delta^{(2)}(z - w) (\epsilon^+(z) - \epsilon^-(z)) T_{++}(z) \\
+ 4 \partial_+ \epsilon^-(w) T_{+-}(w) - 4 \partial_+ \epsilon^+(w) T_{++}(w) - 2 \epsilon^d(w) \partial_d T_{++}(w) \prod_j \Phi_j \end{array} \right\} .
\]

(10)

We use the notation \( z = z^+ = z^1 + iz^2, \quad \bar{z} = z^- = z^1 - iz^2 \). The integral over \( z^1 \) has to be taken along the real axis.

To differentiate the r.h.s. of (6) we first note, that the term under the sum does not contribute to singularities at \( z = w \). Hence we need only

\[
\delta(\nabla_a \epsilon^a) = \frac{1}{2} \epsilon \nabla^b \delta g_{ab}
\]

(11)

\[
\delta(\sqrt{g} R) = \sqrt{g} (\nabla^a \nabla^b \delta g_{ab} - g^{ab} \nabla^2 \delta g_{ab})
\]

(12)

and the variations of \( k \) and \( s \). The variation of \( s \) is trivial. From \( ds^2 = g_{ab} dz^a dz^b \) we get \( (t \) denotes an arbitrary parameter for the boundary)

\[
\delta(ds) = \frac{1}{2|z|} \delta g_{ab} z^a z^b dt .
\]

(13)

For \( k \) we start from ('denotes differentiation with respect to \( s \))

\[
k^2 = \nabla(s) z^a \nabla(s) z^b g_{ab} = g_{ab}(z'^a + z'^m z^m \Gamma^a_{mn})(z'^b + z'^j z^j \Gamma^b_{jl}) .
\]

(14)
The sign in $\delta k = \pm \frac{\delta k^2}{2K}$ we fix at the end by comparison with the well-known expression for $k$ in conformal gauge. To figure out correctly the variation of $z'$ and $z''$ one has to keep in mind $\frac{d}{ds} = (z^a z^b g_{ab})^{-1} \frac{d}{d\varphi}$. With some algebra we find

$$
(\delta k)_{\text{flat}} = -k_{\text{flat}} (z^a z^b - \frac{1}{2} n^a n^b) \delta g_{ab} + n^a z^b z^c \partial_b \delta g_{ac} - \frac{1}{2} z^a z^b \partial_n \delta g_{ab} .
$$

(15)

After variation the flat limit has been taken. $n$ is the inward unit normal vector, $\partial_n$ denotes differentiation in the direction of $n$. $k_{\text{flat}}$ is the curvature of the boundary measured with $g_{ab} = \delta_{ab}$.

Specializing to the half plane geometry eqs. (12)-(13), (15) lead to

$$
\frac{\delta \text{r.h.s. of (6)}}{\delta g_{--}(w)} |_{\text{flat}} = \left \langle \prod_j \Phi_j \left ( \frac{c - \hat{c}}{48\pi} \int dz \partial_+ \partial_+ \delta^{(2)}(z - w) \delta a \epsilon^a(z) 
- \frac{\hat{c}}{48\pi} \int dz \partial_+ \partial_+ \delta^{(2)}(z - w) \right )
- 2i \delta^{(2)}(z - w) \partial_+ \partial_+ \epsilon^a(z) 
+ \text{ terms irrelevant for the OPE of } T \cdot T ,
\right \rangle .
$$

(16)

In the case without boundary $\epsilon^+(z) = \frac{1}{z-v}$ and $\epsilon^- = 0$, due to $\partial_+ \frac{1}{z-v} = \pi \delta^{(2)}(z-v)$, yields a localized $T \cdot T$ product built out of $T_{++} T_{++}$ alone [13]. Since our diffeomorphism has to keep invariant the real axis we must require

$$
\epsilon^+(z) = \epsilon^-(z) \quad \text{if } \quad z = \bar{z} .
$$

(17)

Therefore, the best we can do is to choose

$$
\epsilon^+(z) = \frac{1}{z-v} + \frac{1}{z-\bar{v}} ; \quad \epsilon^-(z) = \frac{1}{z-v} + \frac{1}{\bar{z}-v} .
$$

(18)

This leads to

$$
\partial_- \epsilon^+(z) = \pi \delta^{(2)}(z-v) = \partial_+ \epsilon^-(z) .
$$

(19)

At this place we should make a comment on the distributions arising as part of the derivatives of the diffeomorphism. Since $\epsilon^a(z)$ in eq. (8) has to be defined globally, this
Ward identity as it stands is valid for regularized versions of \cite{18} only. The simplest variant is

\[ \epsilon^+(z) = \frac{z - \bar{v}}{|z - v|^2 + \delta^2} + \frac{z - \bar{v}}{|z - \bar{v}|^2 + \delta^2}, \quad \epsilon^-(z) = \epsilon^+(\bar{z}) . \quad (20) \]

Only afterwards one can study the limit \( \delta \to 0 \). This remark is helpful in making sense out of some questionable expressions one gets in calculating with \cite{18} directly. For instance, on the formal level instead of \cite{18} one finds \( \partial_- \epsilon^+(z) = \pi \delta^2(z-v)^2 + \pi \delta^2(z-\bar{v})^2 \). For \( v \neq \bar{v} \) the second term can be dropped, for \( v = \bar{v} \) it seems to double the coefficient in front of \( \delta^2(z-v) \). But for \( v = \bar{v} \) the \( z \)-integration in the vicinity of \( v \) is restricted to a halfcircle. These two effects conspire in such a way that the \( \delta \to 0 \) limit of \( \partial_- \epsilon^+ \) is indeed \cite{18} with \( \delta^2(z-v) \) understood as the \( \delta \)-function of the halfplane, i.e. \( \int d^2 z \delta^2(z-v)f(z) = f(v) \) for all \( v \) with \( Im(v) \geq 0 \).

By going back to the regularized form \( \delta > 0 \) one can proof that the second line of the r.h.s. of \cite{11} and the first term in the third line of the r.h.s. of \cite{11} vanish. Altogether after the usual redefinition \( T \to \frac{T}{2\pi} \) and the extraction of operator relations from Green functions one gets with \cite{3}, \cite{10}, \cite{10}, \cite{20}

\[ T_{++}(w)(T_{++}(v) + T_{--}(v)) = \frac{c}{2} \left( \frac{1}{|w-v|^4} + \frac{1}{|w-\bar{v}|^4} \right) - \frac{\pi c}{48} \Delta \delta^2(w-v) \]

\[ + 2 \left( \frac{1}{|w-v|^2} + \frac{1}{|w-\bar{v}|^2} \right) T_{++}(w) + \left( \frac{1}{v-w} + \frac{1}{\bar{v}-w} \right) \partial_+ T_{++}(w) \]

\[ + \left( \frac{1}{v-\bar{v}} + \frac{1}{v-w} \right) \partial_- T_{++}(w) + 2\pi \delta^2(w-v)T_{+-}(w) \]

\[ + (c - \hat{c}) \left[ \frac{i}{12} \delta(w^2) \left( \frac{2}{(w^1-v^1)^3} + \frac{2}{(w^1-\bar{v}^1)^3} + \pi \delta^2(w^1-v^1) \delta(v^2) \right) \right. \]

\[ - \frac{1}{24} \delta'(w^2) \left( \frac{1}{(w^1-v^1)^2} + \frac{1}{(w^1-\bar{v}^1)^2} \right] + \text{reg. terms} . \quad (21) \]

This is our main result. In most cases one is interested in on shell relations only, then \( \partial_- T_{++} = T_{+-} = 0 \) and contact terms can be dropped. Therefore, from \cite{21} we find

\[ T_{++}(w)(T_{++}(v) + T_{--}(v)) = \frac{c}{2} \left( \frac{1}{|w-v|^4} + \frac{1}{|w-\bar{v}|^4} \right) + 2 \left( \frac{1}{|w-v|^2} + \frac{1}{|w-\bar{v}|^2} \right) T_{++} \]

\[ + \left( \frac{1}{v-w} + \frac{1}{\bar{v}-w} \right) \partial_+ T_{++}(w) + \text{reg. terms} . \quad (22) \]

For the line of arguments presented in the introduction it is crucial, that the factor multiplying \( (c - \hat{c}) \) in eq. \cite{21} contains also non-contact terms.

Eq. \cite{22} is just what one usually gets for \( T_{++} T_{++} + T_{++} T_{--} \) making use of the continuation in the whole 2D plane via \( T_{--}(z) = T_{++}(\bar{z}) \). The last property is a consequence of analyticity and \( T_{++} = T_{--} \) \textit{on the boundary}. Within our approach we have no thorough
method to disentangle the two products, but it seems naturally that the terms containing $v$ or $\bar{v}$ are due to the first or second product, respectively. There is necessary an investigation of the tensor structure of $T_{ab}T_{cd}$ using only general covariance and $T^a_a = 0$.

3 Concluding remarks

As anticipated in the introduction the main conclusion of our analysis is that the standard $T \cdot T$ - OPE for the half plane can be derived without making use of the condition of vanishing energy-momentum flux across the boundary. All one needs is the Weyl anomaly. Thus our discussion illustrates a further aspect of the fact that in two dimensions boundary critical behaviour is determined by the bulk properties only [13, 17].

In the $\sigma$-model picture of open strings in general background fields one obtains equations of motion for these background fields by the requirement of vanishing $T^a_a$ in a distributional sense. This means $\bar{\beta}_i = 0$ in the bulk for fields $i$ coupling in the bulk and $\bar{\beta}_j = 0$ on the boundary for fields $j$ coupling only on the boundary. As argued in ref. [15] the finite jump of $\bar{\beta}_i$ in going from the bulk to the boundary is irrelevant and yields no unwanted additional condition for the background fields. What concerns equations for these target space fields, the role of (3) is a different one. While the $\bar{\beta}_i = 0$ conditions refer to the target space only, condition (3) necessarily contains 2D derivatives of the string position. It is a (boundary) condition on the 2D surface. Therefore, even if beyond the $T \cdot T$- OPE problem the condition (3) should remain an independent one, there would arise no additional equation for the target space fields.

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