OPTIMAL STOPPING AND CONTROL NEAR BOUNDARIES

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Abstract. We will investigate the value and inactive region of optimal stopping and one-sided singular control problems by focusing on two fundamental ratios. We shall see that these ratios unambiguously characterize the solution, although usually only near boundaries. We will also study the well-known connection between these problems and find it to be a local property rather than a global one. The results are illustrated by a number of examples.

1. Introduction

We consider an optimal stopping problem

\[ V(x) = \sup_\tau \mathbb{E}_x \left\{ e^{-r\tau} g(X_\tau) \right\}, \quad X_0 = x, \]

as well as a singular control problem

\[ W_Z(x) = \sup_Z \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X^Z_s) dZ_s \right\}, \quad X_0 = x, \]

where \( r > 0 \) and \( g : \mathbb{R}_+ \to \mathbb{R} \) is an upper semicontinuous function. Furthermore, the underlying process \( X_t \) is a linear, time-homogeneous Itô diffusion on \( \mathbb{R}_+ \); \( Z \) is a non-negative control process; and \( X^Z = X_t - Z_t \). We will elaborate on these assumptions in the sections to come.

1.1. Core of the study. We shall unambiguously determine the solution for these problems near the boundaries in quite a simple way. We will do this by focusing on two fundamental ratios. We use these ratios by embedding the Markovian methods and classical diffusion theory (see e.g. [20]) into the Beibel-Lerche method (see [4, 5]); for a similar approach, see also [9]. In the present study, the singular control case is especially interesting since one often has to solve control problems globally when applying the usual variational arguments. However, we are able to find inactive regions and the value function locally, i.e. near the boundary, without having to consider the problem on the whole state space. Moreover, this is the first time, that the author is aware of, that the Beibel-Lerche -method is used in a singular control situation.

Another interesting finding concerns the renowned connection between one-sided singular control and optimal stopping problems (see e.g. [7, 13, 14]). We will construct the classical associated optimal stopping problem whose value is the derivative of the value of
the given singular control problem. We will, however, show that this property only holds true locally, i.e. near the boundaries; as we operate on a level more general than is typical in the literature, this connection does not automatically hold on the whole space. We shall actually see an example where the value of a stopping problem can be the associated value for two different singular control problem on disjoint sets. Most interestingly, this means that we can interpret the renowned connection between the one-sided singular control and optimal stopping problem to be, in general, a local property rather than a global one. Moreover, we get an interesting necessary connection when this connection can hold on the whole state space.

1.2. Mathematical introduction. Consider the optimal stopping problem \( (1) \). Our study is strongly based on the ratios \( g/\psi \) and \( g/\varphi \), where \( \psi \) and \( \varphi \) are the increasing and decreasing fundamental solutions of ODE \( (A - r)u(x) = 0 \), where \( A \) is the infinitesimal generator of \( X \). The reason becomes clear after noticing that the stopping time \( \tau_z = \inf\{t \geq 0 \mid X_t = z\} \), the first hitting time to a state \( z \), provides a value (see e.g. II.10 in \([8]\))

\[
\mathbb{E}_x \left\{ e^{-r\tau_z} g(X_{\tau_z}) \right\} = \begin{cases} 
\frac{g(z)}{\psi(z)} \psi(x), & x \leq z \\
\frac{g(z)}{\varphi(z)} \varphi(x), & x > z.
\end{cases}
\]

The formulation above suggests that the ratios \( g/\psi \) and \( g/\varphi \) could strongly dictate the behaviour of \( V \). Indeed, one aspect of the paper is to show that the global maximum points of these ratios allow us to write down explicitly the value function near the boundaries, the upper semicontinuity of \( g \) being the only restricting requirement. The areas "near the boundaries" that we are interested in are in fact the intervals \((0, z^*)\) and \((y^*, \infty)\), where \( z^* \) and \( y^* \) are, respectively, the greatest and the smallest points that globally maximise \( g/\psi \) and \( g/\varphi \). In practice, this means that in every optimal stopping problem, as soon as one has found the global maximum points \( z^* \) and \( y^* \), the problem is completely solved near the boundaries. We shall also prove that the two sets, where the ratio \( \frac{g}{\psi} \) is increasing and where \( \frac{g}{\varphi} \) is decreasing belong to the continuation region. This is an observation that the author has not come across before.

Another aspect of the study is the following. From literature we know that under some weak assumptions \( \tau_S \), a hitting time to the set \( S = \{ x \in \mathbb{R}_+ \mid V(x) = g(x) \} \), is an optimal stopping time for \((1)\) whenever \( \tau_S < \infty \) a.s. (see e.g. Theorem 2.7 in \([18]\)). Furthermore, we know (Theorem 2.4 in \([18]\)) that \( \tau_S \leq \tau^* \) for any stopping time \( \tau^* \) that provides the value \( V(x) \). However, there have not been too many examples where \( \tau_S \neq \tau^* \). In the present paper we offer such examples.

There are naturally also other studies concerning the fundamental ratios \( g/\psi \) and \( g/\varphi \), which is no surprise as they play central roles in solving (at least one-sided) optimal stopping problems. In \([3]\) sufficient conditions are given under which the ratio \( g/\psi \) has exactly one global maximum point, after which \( g/\psi \) is decreasing, and it is shown that
the value function is then unambiguously given everywhere. (The analogous result holds for the ratio \( g/\varphi \).) Another direction is considered in \([9]\), where it is shown that the points which maximise the ratio \( g/h \) for some \( r \)-excessive function \( h \) are in the stopping region \( S \). This directly gives that the points that globally maximise \( g/\psi \) and \( g/\varphi \) are in the stopping region \( S \). Furthermore, in an extensive study \([16]\) on the optimal stopping of linear diffusions these ratios have also been inspected. In the aforementioned study (see also \([11, 10]\)), the finiteness of the ratios \( g/\psi \) and \( g/\varphi \) is linked to the finiteness of the value. Also, it is shown how these ratios agree on the boundaries to the ratios created by the value function. Additionally, in \([11]\), and later in \([10]\) in a more general setting, the optimal stopping time is characterised as a threshold rule relying on the ratios \( g/\psi \) and \( g/\varphi \) on the boundaries.

Taking advantage of the close relations between optimal stopping and singular control problems (see e.g. \([7, 13, 14]\)) we shall utilise similar techniques for singular control problems. We will show that the Beibel-Lerche based method also works in a (one-sided) singular control situation, where the ratios \( g/\psi' \) and \( g/\varphi' \) dictate the value. Maintaining the harmony between a singular control and optimal stopping scene, we see that the solution to a singular control problem is unambiguously characterised on the intervals \((0, z^{*})\) and \((y^{*}, \infty)\), though we now need some additional assumptions on the underlying diffusion. Here \( z^{*} \) and \( y^{*} \) are, respectively, the greatest and the smallest point that globally maximises \( g/\psi' \) and \( g/\varphi' \).

We will also link the two studied problems together by studying the well-known connection between the one-sided singular control problem and the optimal stopping problem on the intervals \((0, z^{*})\) and \((y^{*}, \infty)\). In the literature, one usually has sufficient assumptions to ensure this connection holds everywhere. However, we find that operating on a general level, this connection does not necessarily hold on the whole space, only near the boundaries.

The contents of this study are as follows. In Section 2 we will present the definitions and the optimal stopping problem in detail. This is followed in Section 3 by our main results. These results are then illustrated by several short examples in Section 4. Sections 5 and 6 are devoted to the singular stochastic control and its relationship to optimal stopping problems.

### 2. Optimal Stopping Problem and Definitions

Let \( X_t \) be a regular linear diffusion defined on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\), evolving on \( \mathbb{R}_+ := (0, \infty) \). The assumption that the state space is \( \mathbb{R}_+ \) is done for reasons of convenience — It could be any interval on \( \mathbb{R} \). We assume that \( X_t \) is given as the solution of the Itô equation

\[
\frac{dX_t}{dt} = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, 
\]
where $\mu : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are measurable mappings. We assume that for all $x \in \mathbb{R}_+$ there exists $\varepsilon > 0$ such that $\mu$ and $\sigma$ satisfy the condition $\int_{x-\varepsilon}^{x+\varepsilon} \frac{1+|\mu(y)|}{\sigma^2(y)} \, dy < \infty$. This ensures that (2) has a unique weak solution (see e.g. Chapter V in [15]). We assume that the boundaries 0 and $\infty$ are natural, exit, entrance, or killing (for a characterisation of boundaries, see e.g. Section II.1 in [8]). We understand that if the process hits an exit or a killing boundary, it is sent immediately to a cemetery state $\partial \notin \mathbb{R}_+$, where it stays for the rest of the time (cf. [12], Subsection 3.1). Consequently, boundaries cannot be used as stopping points for a diffusion in any circumstances.

We study an optimal stopping problem

\begin{equation}
V(x) = \sup_{\tau} \mathbb{E}_x \left\{ e^{-\tau r} g(X_{\tau}) \right\},
\end{equation}

where the supremum is taken over all $\mathcal{F}_\tau$-stopping times and $g : (0, \infty) \to \mathbb{R}$ is an upper semicontinuous reward function that attains positive values for some $x \in \mathbb{R}_+$ (if $g \leq 0$ always, it is never optimal to stop). Notice that as 0 and $\infty$ cannot be used as stopping points, $g$ is not necessarily defined on the boundaries and we further understand that $g(\partial) = 0$. We denote by $C := \{ x \mid V(x) > g(x) \}$ the continuation region and by $S := \{ x \mid V(x) = g(x) \}$ the stopping region of the considered problem. Furthermore, we follow the notations of [18] and define optimal stopping time to be any stopping time $\tau^*$ at which the supremum is attained.

Denote by $\psi$ and $\varphi$ the increasing and decreasing fundamental solution to $(A-r)u(x) = 0$, where $r > 0$ and $A = \frac{1}{2}\sigma^2(x) \frac{d^2}{dx^2} + \mu(x) \frac{d}{dx}$ is the infinitesimal generator of the diffusion. We will analyse the behaviour of the solution applying the functions $g/\psi$ and $g/\varphi$. To prepare this, we denote by

- $M := \{ x \mid x = \arg\max\{g(x)/\psi(x)\} \}$ the set of global maximum points of $g/\psi$ and let $z^* = \sup\{M\}$ be the maximal element of that set; and
- $N := \{ x \mid x = \arg\max\{g(x)/\varphi(x)\} \}$ the set of global maximum points of $g/\varphi$ and let $y^* = \inf\{N\}$ be the minimal element of that set.

For a boundary point 0 we understand $g(0)/\psi(0) = \limsup_{x \to 0} g(x)/\psi(x)$ and $g(0)/\varphi(0) = \limsup_{x \to 0} g(x)/\varphi(x)$, and we use analogous interpretation for $\infty$. Especially we notice that with these interpretations $M, N \neq \emptyset$, and that $M$ and $N$ can include 0 and $\infty$ although $g$ is not necessarily defined on them. Moreover, by upper semicontinuity we have $z^* \in M$ and $y^* \in N$. Further, also by upper semicontinuity, it is true that $\frac{g(b)}{\varphi(b)} = \sup_z \left\{ \frac{g(z)}{\varphi(z)} \right\}$ and $\frac{g(a)}{\psi(a)} = \sup_y \left\{ \frac{g(y)}{\psi(y)} \right\}$ for all $b \in M$ and $a \in N$.

3. Optimal stopping near boundaries

The proofs for the results in this section are given only applying the ratio $g/\psi$, as the proofs with the ratio $g/\varphi$ are treated analogously with obvious changes.

3.1. Main results. Let us first show that there are possibly several stopping strategies that provide the values $\frac{g(z^*)}{\psi(z^*)}\psi(x)$ and $\frac{g(y^*)}{\varphi(y^*)}\varphi(x)$.
Lemma 3.1. (A) Let $a, b \in M$, $b \neq 0$, and $0 \leq a \leq b \leq z^* < \infty$ and let $x \leq b$. Then the following stopping times yield the same value $\frac{g(z^*)}{\psi(z^*)} \psi(x)$:

(i) $\tau_{z^*} = \inf\{t \geq 0 \mid X_t = z^*\}$.
(ii) $\tau_b = \inf\{t \geq 0 \mid X_t = b\}$.
(iii) $\tau_{(a,b)} = \inf\{t \geq 0 \mid X_t \in \{a, b\}\}$, for all $x \in [a, b]$.
(iv) $\tau_{M^+} = \inf\{t \geq 0 \mid X_t \in M^+\}$, where $M^+ \subset M$ is any subset that contains $z^*$.

(B) Let $a, b \in N$, $a \neq \infty$, and $0 < y^* \leq a \leq b \leq \infty$ and let $a \leq x$. Then the following stopping times yield the same value $\frac{g(y^*)}{\psi(y^*)} \psi(x)$:

(i) $\tau_{y^*} = \inf\{t \geq 0 \mid X_t = y^*\}$.
(ii) $\tau_a = \inf\{t \geq 0 \mid X_t = a\}$.
(iii) $\tau_{(a,b)} = \inf\{t \geq 0 \mid X_t \in \{a, b\}\}$, for all $x \in [a, b]$.
(iv) $\tau_{N^+} = \inf\{t \geq 0 \mid X_t \in N^+\}$, where $N^+ \subset N$ is any subset that contains $y^*$.

Proof. (A) (i), (ii) Since $b, z^* \in M$, we have for all $x \leq b$

$$\mathbb{E}_x \left\{ e^{-r\tau_{z^*}} g(X_{\tau_{z^*}}) \right\} = \frac{g(z^*)}{\psi(z^*)} \psi(x) = \frac{g(b)}{\psi(b)} \psi(x) = \mathbb{E}_x \left\{ e^{-r\tau_b} g(X_{\tau_b}) \right\}.$$ 

(iii) Let $a \leq x \leq b$. Then $\tau_{(a,b)}$ gives

$$\mathbb{E}_x \left\{ e^{-r\tau_{(a,b)}} g(X_{\tau_{(a,b)}}) \right\} = \mathbb{E}_x \left\{ e^{-r\tau_{(a,b)}} \frac{\psi(X_{\tau_{(a,b)}})}{\psi(x)} g(X_{\tau_{(a,b)}}) \right\} \psi(x)$$

$$= \mathbb{E}_x^\psi \left\{ \frac{g(X_{\tau_{(a,b)}})}{\psi(X_{\tau_{(a,b)}})} \right\} \psi(x),$$

where $\mathbb{E}_x^\psi$ is an expectation under a probability measure $\mathbb{P}_x^\psi(A) = \mathbb{E}_x \left\{ e^{-r\tau} \frac{\psi(X_t)}{\psi(x)} \right\}$. (so called Doob’s $\psi$-transform, see e.g. p. 34 in [3]). Using this representation we can calculate that

$$\mathbb{E}_x^\psi \left\{ \frac{g(X_{\tau_{(a,b)}})}{\psi(X_{\tau_{(a,b)}})} \right\} \psi(x) = \left[ \frac{g(a)}{\psi(a)} \mathbb{P}_x^\psi(X_{\tau_{(a,b)}} = a) + \frac{g(b)}{\psi(b)} \left( 1 - \mathbb{P}_x^\psi(X_{\tau_{(a,b)}} = a) \right) \right] \psi(x)$$

$$= \frac{g(b)}{\psi(b)} \psi(x),$$

where the last equality follows from the fact that $a, b \in M$.

(iv) As a diffusion is continuous, we must have $\tau_{M^+} = \tau_{(a,b)}$, where $b$ is the smallest point of $M^+$ such that $b \geq x$, and $a$ is the greatest point of $M^+$ such that $x \leq a$ (if no such $a$ exist, then $\tau_{M^+} = \tau_b$). Consequently the claim follows from part (iii) (or (ii)) \qed

Let us state our main result, which is greatly inspired by Theorem 2 in [5].

Proposition 3.2. (A) For $x \leq z^*$, the value function can be written as

$$V(x) = \frac{g(z^*)}{\psi(z^*)} \psi(x).$$

Moreover, $\tau_{M^+}$ is an optimal stopping time, where $M^+ \subset M$ is any subset that contains $z^*$. Lastly, $(0, z^*) \setminus M \subset C$, and $M \subset S$. 
For $x \geq y^*$, the value function can be written as

$$V(x) = \frac{g(y^*)}{\varphi(y^*)} \varphi(x).$$

Moreover, $\tau_{N^+}$ is an optimal stopping time, where $N^+ \subset N$ is any subset that contains $y^*$. Lastly $(y^*, \infty) \setminus N \subset C$, and $N \subset S$.

(C) If $z^* = 0$, then there is no $\varepsilon > 0$ such that the admissible stopping time $\tau_{(0, \varepsilon)} = \inf\{t \geq 0 \mid X_t \notin (0, \varepsilon]\}$ yields the value for all $x < \varepsilon$. Similarly if $y^* = \infty$, then there is no $H < \infty$ such that the admissible stopping time $\tau_{(H, \infty)} = \inf\{t \geq 0 \mid X_t \notin (H, \infty]\}$ yields the value for all $x > H$.

Proof. (A) Let $x \leq z^*$. Then, for all a.s. finite stopping times $\tau$, we have

$$\begin{align*}
\mathbb{E}_x \left\{ e^{-r\tau} g(X_{\tau}) \right\} &= \mathbb{E}_x \left\{ e^{-r\tau} \psi(X_{\tau}) \frac{g(X_{\tau})}{\psi(X_{\tau})} \right\} \\
&\leq \mathbb{E}_x \left\{ e^{-r\tau} \psi(X_{\tau}) \right\} \frac{g(z^*)}{\psi(z^*)} = \psi(x) \frac{g(z^*)}{\psi(z^*)}.
\end{align*}$$

By Lemma 3.11 this value is attained with a stopping rule $\tau_{M^+} = \inf\{t \geq 0 \mid X_t \in M^+\}$, where $M^+ \subset M$ is any subset containing $z^*$, and thus the proposed $V(x)$ is the value function.

Lastly, Theorem 2.1 in [9] says that a point $x \in \mathbb{R}_+$ is in the stopping set if and only if there exists a positive $\tau$-harmonic function $h$ such that $x \in \text{argmax}\{g/h\}$. This implies straightaway that $M \subset S$.

(C) Let $M = \{0\}$ and suppose, contrary to our claim, that there exists $\varepsilon > 0$ such that $\tau_{(0, \varepsilon)}$ is an optimal stopping time for $x \in (0, \varepsilon)$. Then we have

$$V(x) = \mathbb{E}_x \left\{ e^{-r\tau_{(0, \varepsilon)}} g(X_{\tau_{(0, \varepsilon)}}) \right\} = \frac{g(\varepsilon)}{\psi(\varepsilon)} \psi(x)$$

for all $x \in (0, \varepsilon)$. Since $\limsup\_{z \uparrow 0} g(z)/\psi(z)$ provides the maximum for $g/\psi$, there exists $\tilde{x} \in (0, \varepsilon)$ such that $g(\tilde{x})/\psi(\tilde{x}) > g(\varepsilon)/\psi(\varepsilon)$, whence for all $x \in (0, \tilde{x})$

$$\begin{align*}
\mathbb{E}_x \left\{ e^{-r\tau_{(0, \varepsilon)}} g(X_{\tau_{(0, \varepsilon)}}) \right\} = \frac{g(\tilde{x})}{\psi(\tilde{x})} \psi(x) > \frac{g(\varepsilon)}{\psi(\varepsilon)} \psi(x) = \mathbb{E}_x \left\{ e^{-r\tau_{(0, \varepsilon)}} g(X_{\tau_{(0, \varepsilon)}}) \right\} = V(x)
\end{align*}$$

which is contradicts the maximality of $V(x)$.

Interestingly, if the set $M$ or $N$ contains more than one element, then by Proposition 3.2 there are several different optimal stopping times that provide the unique value function $V(x)$. Especially, the points of $M$ (and $N$) can now be interpreted as indifference points: For $x \leq z^*$, the decision maker receives the same value irrespectively whether she uses stopping time $\tau_b$, $\tau_{M^+}$, $\tau_{(a, b)}$, $\tau_M$, or $\tau_{z^*}$, where $a, b \in M$ are any points for which $a \leq x \leq b$. However, it is quite clear that $\tau_M$ is the smallest (a.s.) of all these stopping times (cf. Theorem 2.4 in [8]). In the sequel we wish to be unambiguous and thus we select $\tau_M$ to be our optimal stopping time on interval $(0, z^*)$ with a notion that there might also be others.

It should also be mentioned that the values $\limsup\_{x \to 0} \frac{g(x)}{\psi(x)}$ and $\limsup\_{x \to \infty} \frac{g(x)}{\psi(x)}$ are crucial when investigating the optimality of a stopping time $\tau_S = \inf\{t \geq 0 \mid X_t \in S\}$ on
the whole state space. This question has been treated quite comprehensively in [11, 10]. In addition, in Theorem 6.3(III) in [16] it is proven that \( \lim_{n \to \infty} \tau_S \wedge \tau(\alpha_n, \beta_n) \) provides the value function, where \( \alpha_n \) and \( \beta_n \) are suitable chosen sequences such that \( \alpha_n \to 0, \beta_n \to \infty \) when \( n \to \infty \). In our research this aspect is omitted, as the above mentioned analyses are quite exhaustive, and the present study does not bring any new insights to that subject.

3.2. Minor results. In short, Proposition 3.2 guarantees that if we can find the global maximum points of \( g/\psi \) and \( g/\varphi \), then the solution is unambiguously characterised near the boundaries. Usually the set \( M \) contains only one element, but this is not always the case, as will be illustrated with examples in Section 4 below. The proposition also gives rise to a handful of corollaries, which shall be presented in this subsection (proofs are given in the Subsection 3.4). These corollaries are more or less straight consequences of Proposition 3.2 (and hence of Theorem 2 from [5]), but they have not been written out explicitly anywhere.

Let us start with the ordering of \( z^* \) and \( y^* \).

**Corollary 3.3.** One has \( z^* \leq y^* \).

This means that the regions where the value is dictated by ratios \( g/\psi \) and \( g/\varphi \) are always separated.

Let us then present an easy, but surprisingly powerful corollary.

**Corollary 3.4.** (A) If \( z^* > 0 \). Then, for all \( x \leq z^* \), the value reads as in (4), \( \tau_M \) is the optimal stopping time and \( (0, z^*) \setminus M \subset C \).

(B) If \( y^* < \infty \). Then, for all \( x \geq y^* \), the value reads as in (5), \( \tau_N \) is the optimal stopping time, and \( (y^*, \infty) \setminus N \subset C \).

The power of this corollary lies in the fact that if one can check somehow that \( z^* > 0 \), the value is immediately characterised near the boundary 0; For example if \( g(0+) < 0 \) or \( \frac{d}{dx} g(0+) > 0 \), then we know at once that \( z^* > 0 \). In this way Corollary 3.4 can be used to quicken the standard Beibel-Lerche method: If \( z^* > 0 \), we immediately know the solution without needing more closely inspection. This corollary is also related to corollaries 7.2 and 7.3 in [16], where it is shown how the value function looks like if we find out that \( (0, z) \subset C \) and \( z \in S \) (and analogously for \( (y, \infty) \subset C \) and \( y \in S \)).

In the next two corollaries we will study more closely the situations that \( M \) or \( N \) include 0 or \( \infty \).

**Corollary 3.5.** If \( z^* = 0 \) and \( y^* = \infty \), then one of the following is true.

(i) It is optimal to stop immediately, i.e. \( S = \mathbb{R}_+ \) and \( g \) is r-excessive.

(ii) The optimal stopping rule is at least two-boundary rule (i.e. there are \( 0 < a < b < \infty \) such that \( (0, a] \cup [b, \infty) \subset S \)).

(iii) The optimal stopping time is not finite/admissible.
Lemma 3.8. (A) Denote by $C = \mathbb{R}_+ \setminus M$ and, for all $x \leq b^* := \sup\{M \setminus \{\infty\}\}$, the stopping time $\tau_M \setminus \{\infty\}$ is optimal. For $x \in (b^*, \infty)$ there does not exist an admissible stopping time that yields the value.

(B) Let $y^* = 0$. Then $V(x) = B\varphi(x)$, where $B = \limsup_{x \to 0} \left\{ \frac{g(x)}{\psi(x)} \right\}$. Especially, if $B = \infty$, then $V(x) \equiv \infty$.

Proof. (A) Since $g(x) = K\psi(x)$ for all $x \in (a,b)$, we can choose $K > 0$ on $(a,b)$. Theorem 1 in [5], Theorem 6.3(I) in [16], Proposition 5.10 in [11].

Let us then show that if $M$ (or $N$) includes an interval, then $g$ must equal to a fundamental solution on it.

Lemma 3.7. (A) If an interval $(a,b) \subset M$, then $g(x) = K\psi(x)$ for all $x \in (a,b)$ for some $K > 0$.

(B) If an interval $(a,b) \subset N$, then $g(x) = K\varphi(x)$ for all $x \in (a,b)$ for some $K > 0$.

Proof. (A) Since $(a,b) \subset M$ we know that $x = \text{argmax}\{g(y)/\psi(y)\}$ for all $x \in (a,b)$. In turns means that $\frac{g(x)}{\psi(x)} = K$ for some $K > 0$ on $(a,b)$. □

To end the subsection, we will show that using the ratios $g/\psi$ and $g/\varphi$ we can also say something about the continuation region outside the set $(0, z^*) \cup (y^*, \infty)$.

Lemma 3.8. (A) Denote by $C_\psi := \left\{ x \mid \frac{g(x)}{\psi(x)} \text{ is strictly increasing} \right\}$. Then $C_\psi \subset C$.

(B) Denote by $C_\varphi := \left\{ x \mid \frac{g(x)}{\varphi(x)} \text{ is strictly decreasing} \right\}$. Then $C_\varphi \subset C$.

Proof. (A) Let $x \in C_\psi$. Since $g/\psi$ is strictly increasing at $x$, we can choose $z > x$ such that $g(z)/\psi(z) > g(x)/\psi(x)$. Now, using a stopping time $\tau_z$, we get

$$V(x) \geq \mathbb{E}_x \left\{ e^{-r\tau_z} g(X_{\tau_z}) \right\} = \frac{g(z)}{\psi(z)}\psi(x) > \frac{g(x)}{\psi(x)}\psi(x) = g(x)$$

and so $x \in C$. □

3.3. Extensions.
3.3.1. **Involving an integral.** The problem setting can easily be extended to contain also integral term. To that end, let \( \pi : \mathbb{R}_+ \to \mathbb{R} \) be a measurable function such that 
\[
\mathbb{E}_x \left\{ \int_0^\infty e^{-rs}\pi(X_s)ds \right\} < \infty,
\]
and let us consider a problem 
\[
\sup_{\tau} \mathbb{E}_x \left\{ \int_0^\tau e^{-rs}\pi(X_s)ds + e^{-r\tau}g(X_\tau) \right\}.
\]
Using a strong Markov property, this can be rewritten as (cf. (1.13) in [16])
\[
(R_r\pi)(x) + \sup_{\tau} \mathbb{E}_x \left\{ e^{-r\tau} \left[ g(X_\tau) - (R_r\pi)(X_\tau) \right] \right\},
\]
where \((R_r\pi)(X_t) = \mathbb{E}_x \left\{ \int_t^\infty e^{-rs}\pi(X_s)ds \right\}\) is the resolvent of \(\pi\), or a cumulative net present value of \(\pi\). We see at once that all the results above hold for this problem with obvious changes and with the sets \(M = \{ x | x = \text{argmax} \left\{ \frac{g(z) - (R_r\pi)(z)}{\psi(z)} \right\} \}\) and \(N = \{ x | x = \text{argmax} \left\{ \frac{g(y) - (R_r\pi)(y)}{\varphi(y)} \right\} \}\).

3.3.2. **State dependent discounting.** Another fairly straightforward extension can be made by introducing a discounting function: Instead of a constant \(r > 0\), we can define \(r : \mathbb{R}_+ \to \mathbb{R}_+\) to be a continuous function, which is bounded away from zero, whence \(\int_0^t r(X_s)ds\) is the cumulative discounting from now until a time \(t\). As the ODE \((A - r(x))u(x) = 0\) has an increasing \(\psi_r\) and a decreasing \(\varphi_r\) as its two independent solutions, we see that all the previous results hold in this case.

3.3.3. **Other boundaries.** Assume that the boundaries are either absorbing or reflecting and that \(g\) is extended to be defined also on these boundaries.

For a reflecting boundary Lemma [3.1], Proposition [3.2][A],[B] and Corollary [3.6] can quite easily seen to hold as well as Lemmas [3.7] and [3.8]. However, as the boundaries can now be used as stopping points, Proposition [3.2][C] is no longer valid (for an counterexample, see Example 7 in Section [3.2][C] below), nor are its corollaries.

The process is trapped in an absorbing boundary once it hits there, and so we have to take into account the positiveness of the reward function at the boundaries. If \(g(0), g(\infty) \leq 0\), it is not worthwhile to stop at the boundaries. Consequently, in this case, absorbing boundaries behaves as killing ones and all the preceding results are valid with absorbing boundaries.

However, if \(g(0), g(\infty) > 0\) with absorbing boundaries, then the boundary points are always stopping points. This influences the analysis and consequently, of the introduced results, only Lemmas [3.7] and [3.8] hold true in the present formulation. (For a formulation of a value function near absorbing boundaries, see also Corollaries 7.2 and 7.3 in [16].)

3.4. **Proofs to the corollaries.**
Proof of Corollary Suppose, contrary to our claim, that \( y^* < z^* \). Then for all \( x \in (y^*, z^*) \), by Proposition 3.2, both \( \tau_{y^*} \) and \( \tau_{z^*} \) would give the value, i.e.

\[
g(y^*)\varphi(y^*)(x) = \mathbb{E}_{x}\left\{ e^{-rt_{y^*}}g(X_{t_{y^*}}) \right\} = V(x) = \mathbb{E}_{x}\left\{ e^{-rt_{z^*}}g(X_{t_{z^*}}) \right\} = \frac{g(z^*)}{\psi(z^*)}\psi(x).
\]

From this we get

\[
\frac{\psi(x)}{\varphi(x)} = \frac{g(y^*)}{g(z^*)}\frac{\psi(z^*)}{\varphi(y^*)}.
\]

But as \( \psi \) is increasing and \( \varphi \) is decreasing, the ratio \( \psi/\varphi \) cannot be constant for all \( x \in (y^*, z^*) \). □

Proofs of Corollary 3.4 and 3.5. Straight consequences from Proposition 3.2. □

Proof of Corollary 3.6. (A) Since \( z^* = \infty \), we have \( A = \lim_{x \to \infty} \frac{g(x)}{\psi(x)} = \sup \{ \frac{g(x)}{\psi(x)} \} \). Clearly \( V_A(x) = A\psi(x) \) is an \( r \)-excessive majorant of \( g \), and consequently a candidate for the value.

Let us create an increasing sequence \( z_n \) such that \( \lim_{n \to \infty} z_n = z^* = \infty \), that the sequence \( \frac{g(z_n)}{\psi(z_n)} \) is increasing, and that \( \lim_{n \to \infty} \frac{g(z_n)}{\psi(z_n)} = A \). Fix \( x \in \mathbb{R}_+ \) and let \( n_x \in \mathbb{N} \) be such that \( z_n > x \) for all \( n > n_x \). Then the sequence of values

\[
V_n(x) = \mathbb{E}_{x}\left\{ e^{-rt_{z_n}}g(X_{t_{z_n}}) \right\} = \frac{g(z_n)}{\psi(z_n)}\psi(x)
\]

is increasing for all \( n > n_x \). Moreover, for each \( \varepsilon > 0 \) there exists \( n_\varepsilon \in \mathbb{N} \) such that

\[
V_A(x) - V_n(x) = \left( A - \frac{g(z_n)}{\psi(z_n)} \right) \psi(x) < \varepsilon, \quad \text{for all } n > n_\varepsilon.
\]

Therefore \( V_A(x) = \lim_{n \to \infty} V_n(x) \) is the limit of the increasing sequence of the values of the admissible stopping times \( \tau_{z_n} \), and thus it is the value.

(A) (i) The value reads as \( A\psi(x) \), but as \( \infty \) cannot be used as a stopping point, this value cannot be reached by any finite stopping time.

(A) (ii) By Proposition 3.2, for all \( x \in (0, b^*) \), \( \tau_{M^\delta\setminus\{\infty\}} \) provides the value. The rest follows from part (B)(i). □

4. Examples

Let us illustrate the results with a geometric Brownian motion on \( \mathbb{R}_+ \). Now \( X_t \) satisfies the stochastic differential equation

\[
dX_t = \mu X_t dt + \sigma X_t dW_t,
\]

where \( \mu \in \mathbb{R} \) and \( \sigma > 0 \), and the boundaries are natural. The fundamental solutions are \( \psi(x) = x^{\gamma^+} \) and \( \varphi(x) = x^{\gamma^-} \), where \( \gamma^+ \) is the positive root and \( \gamma^- \) the negative root of the characteristic equation

\[
\frac{1}{2}\sigma^2\gamma(\gamma - 1) + \mu\gamma - r = 0.
\]
We shall demonstrate the results of Section 3 numerically by choosing \( \mu = 0.1, \sigma = 0.2 \) and \( r = 0.24 \), so that \( \psi(x) = x^2 \) and \( \varphi(x) = x^{-6} \).

**Example 1.** We will present a simple example, where \( M \) have more than one point so that there are multiple different optimal stopping rules. By Proposition 3.2 this is true, if \( \frac{g}{\psi} \) have at least two maximum points. To get such a function, choose

\[
g_1(x) = \begin{cases} 
  x - 3, & x \leq 10 \\
  a_1 x - b_1, & x > 10,
\end{cases}
\]

where \( a_1 = 2 \frac{1}{3} \) and \( b_1 = 16 \frac{1}{3} \) are chosen so that \( g_1 \) is continuous and that \( M \) contains two points. With these choices \( M = \{6, 14\} \), \( \sup \{g(x)/\psi(x)\} = \frac{1}{12} \) and \( V(x) = \frac{1}{12} \psi(x) \) for all \( x \leq 14 \) (see Figure 1). For \( x < 6 \), the usually accepted optimal stopping time is \( \tau_6 = \inf \{t \geq 0 \mid g(x) = V(x)\} \), but also \( \tau_{14} \) is optimal despite the fact that \( \tau_{14} > \tau_6 \) a.s.

**Figure 1.** The functions \( V_1 \) and \( g_1 \). The points of \( M \) are denoted by black dots. The point 6 is a singleton stopping point, and it is an indifference point, where the value function coincides with the reward function.

**Example 2.** Extending the previous example, we now show \( M \) can be uncountable, so that there are uncountably many different optimal stopping rules. This is possible (by Proposition 3.2) if \( M \) contains an interval, which by Lemma 3.7 means that \( g(x) = K \psi(x) \) for some \( K \) on some interval. To that end, choose

\[
g_2(x) = \begin{cases} 
  x - 3, & x \leq 6 \\
  \frac{1}{12} x^2, & 6 < x \leq 10 \\
  \frac{1}{3} x - 8 \frac{1}{3}, & 10 < x \leq 14 \\
  3x - 27, & x > 14.
\end{cases}
\]

With this choice, we can calculate that \( \sup \{g(x)/\psi(x)\} = \frac{1}{12} \), and it is reached at the points \( [6, 10] \cup \{18\} = M \), so that \( V(x) = \frac{1}{12} \psi(x) \) for all \( x \leq 18 \) (see Figure 2). Further, by Proposition 3.2 for \( x < 18 \) any stopping time \( \tau_{\{a,18\}}, \ a \in [6,10] \) is optimal.
Figure 2. (A) The function $g_2/\psi$. (B) The functions $V_2$ and $g_2$. The points of $M$ are denoted by black dots. Now $g_2$ coincides with $\frac{1}{12}\psi$ on $(6, 10)$, and thus this whole interval maximizes $g_2/\psi$ and can be interpreted as indifference region for the decision maker.

Example 3. In this example we consider a case illustrating that $M$ can include $\infty$, and that the value function is not attainable although the stopping set $S \neq \emptyset$. Choose

$$g_3(x) = \begin{cases} 
  x - 3, & x < 12 \\
  \frac{1}{12}x^2 - \frac{x}{12} - b_3x^{-5}, & x \geq 12,
\end{cases}$$

where $b_3 \approx 742\,205$ is chosen so that $g_3$ is continuous. Now we see that $g_3(x)/\psi(x) \to \frac{1}{12}$ as $x \to \infty$, and so $M = \{6, \infty\}$, and that the value function $V_3(x) = \frac{1}{12}\psi(x)$, for all $x \in \mathbb{R}_+$, exists finitely. However, for $x > 6$ there is no admissible stopping time that provides this value (see Figure 3a). Notice that for $x \leq 6$, $\tau_6$ is an optimal stopping time. Worth observing is that $g_3$ does not satisfy the integrability condition $\mathbb{E}_x \{\sup_s \{e^{-r}s g_3(X_s)\}\} < \infty$, a sufficient condition for the existence of a finite solution. For a similar example, see Example 8.2 in [16].

Figure 3. (A) The functions $V_3$ and $g_3$. (B) The functions $V_4$ and $g_4$. All the stopping points are denoted by black dots. In (A), the only finite stopping point is 6 and in (B) there are infinite amount of stopping points.

Example 4. In this example we characterise a situation where $z^* = \infty$ and the value is nevertheless attained with a finite stopping time for all $x \in \mathbb{R}_+$. To that end, let us
choose
\[ g_4(x) = x^2 \sin(x). \]

Now clearly \( g_4(x)/\psi(x) = \sin(x) \), and thus \( M = \left\{ \pi/2 + k2\pi \right\}_{k \in \mathbb{N}} \), and \( z^* = \infty \). The value reads as \( V_4(x) = \psi(x) \) for all \( x \in \mathbb{R}_+ \), and it is attained with a stopping time \( \tau_M \), which is finite a.s. (see Figure 3b). In fact, by Proposition 3.2 for each \( x \) there are arbitrary many finite stopping times that provide the value \( V_4 \). Notice that, like in the previous example, the integrability condition \( \mathbb{E}_x \{ e^{-\tau_3 g_3(X_s)} \} < \infty \) does not hold in this one either.

**Example 5.** From Corollary 3.3 we know that \( z^* \leq y^* \). Let us now illustrate that \( z^* \) can equal to \( y^* \). For that end, let \( g_5(x) = \min\{\psi(x), \varphi(x)\} \). Now \( V_5(x) = g_5(x), M = (0, 1], N = [1, \infty) \), and consequently \( z^* = 1 = y^* \).

**Example 6.** In this example we show that if a boundary point can be used as a stopping point, previous results do not necessarily hold. Let the state space be \( [1, \infty) \) and let 1 be a reflecting boundary. Let
\[
g_6(x) = \begin{cases} 
1, & x = 1 \\
0, & x \in (1, 5) \\
1, & x \geq 5.
\end{cases}
\]

Then \( g_6 \) is an upper semicontinuous function, \( M = \{1\} \), and \( z^* = 1 \). It can be easily seen that \( \tau_{(1,5)} = \inf\{t \geq 0 \mid X_t \notin (1, 5)\} \) is an optimal stopping time. This differs from Proposition 3.2(C), which says that \( \tau_{(1,1+\varepsilon)} \) cannot provide optimal value for any \( \varepsilon > 0 \) for unattainable boundaries. Hence we see that as soon as a boundary can be used as a stopping point, Proposition 3.2(C) does not necessarily hold.

**Example 7.** Here we will study more closely the subtle issue about a continuation region and an optimal stopping time. Assume that \( z^* = 0 \). Then by Proposition 3.2(C) we know that \( \tau_{(0,\varepsilon)} \) is not an optimal stopping time for any \( \varepsilon > 0 \). From this one could wrongly conclude that \( (0, \varepsilon) \subset S \) for some \( \varepsilon > 0 \). However, contrary to this belief, we will illustrate that for some \( \varepsilon > 0 \) we can actually have \( (0, \varepsilon) \subset C \), in spite of the fact that \( \tau_{(0,\varepsilon)} \) is not an optimal stopping time.

Let
\[
g_7(x) = \begin{cases} 
x^{-6+x}, & x < 1 \\
-x^{-1}, & x \geq 1,
\end{cases}
\]

where we have chosen \( g_7 \) so that \( \lim_{a \to 0} \frac{g_7(a)}{\psi(a)} > 0 \). It is easy to check that \( M = \{0\} \) and \( N = \{\infty\} \) implying that \( z^* = 0 \) and \( y^* = \infty \). Then by Proposition 3.2(C) we know that \( \tau_{(0,\varepsilon)} \) is not an optimal stopping time for any \( \varepsilon > 0 \). However, it can be proven that (cf.
Theorem 6.3(III) and Corollary 7.2 in [16], for \( x \leq 0 \), say,
\[
V_7(x) = \sup_b \lim_{n \to \infty} \mathbb{E}_x \left\{ e^{-r(\tau_{(1/n,b)})} L_x(\tau_{(1/n,b)}) \right\}
\]
\[
= \sup_b \left\{ \frac{g_7(0)}{\varphi(0)} \varphi(x) + \left( \frac{g_7(b)}{\varphi(0)} \varphi(b) - \frac{g_7(0)}{\varphi(0)} \varphi(b) \right) \psi(x) \right\},
\]
which is maximized with \( b^* \approx 1.22 \). Thus \( V_7(0.1) \approx 10^6 > 794,328 = q_7(0.1) \). In other words, for \( x < b^* \) we have \( V_7(x) > q_7(x) \) so that \( (0,b^*) \subset C \). Moreover, the time \( T_\infty := \lim_{n \to \infty} \tau_{(1/n,b^*)} \) provides the greatest value, but now \( T_\infty \) is not an admissible stopping time.

We could have also used Lemma 3.8 to prove the point: Now \( g_7(x) \) is strictly decreasing on \( (0,0.37) \), meaning that \( (0,0.37) \subset C \).

**Example 8.** In all the preceding examples we have had \( V'(b) = g'(b) \) for all \( b \in M \). Here we will show that actually we do not need smooth fit in order to apply Proposition 3.2. To that end, let us take
\[
q_8(x) = |x|^2
\]
to be a step function, which is an upper semicontinuous function. In this case, \( M = \{1,2,3,\ldots\} \), \( z^* = \infty \) and \( V_8(x) = \psi(x) \). Moreover, now \( V_8(b) = q_8(b) \) for all \( b \in M \), but unlike in the preceding examples the smooth fit does not hold in this case. See Figure 4 for an illustration.

![Figure 4. The functions \( V_8 \) and \( q_8 \). The points of \( M \) are denoted by black dots.](image)

**5. Singular control problem**

In this section we shall show that the principles behind Beibel-Lerche -method can be used in a singular control situations resulting in a similar conclusions than in the previous section.

**5.1. Definitions.** Let us now consider a controlled diffusion
\[
X_t^Z = X_t - Z_t, \quad X_0^Z = X_0 = x,
\]
on $\mathbb{R}_+$, where $X_t$ is as previously and $Z_t$ is non-negative, non-decreasing, right-continuous, and $\mathcal{F}_t$-adapted. Consequently any admissible control has finite variation. Let $g : \mathbb{R}_+ \to \mathbb{R}$ be as previously and let us study a singular control problem

$$W_Z(x) = \sup_{Z} \mathbb{E}_x \left\{ \int_{0}^{\infty} e^{-rs} g(X_s^Z) dZ_s \right\}. \tag{6}$$

If $X_t^Z$ exits the state space at time $\zeta$, we understand $g \equiv 0$ for all $t \geq \zeta$. In this problem setting, we understand an optimal control to be any control $Z^*$ that produces the value $W_Z$. We denote by $C \subset \mathbb{R}_+$ and $S = \mathbb{R}_+ \setminus C$ the action region and action region, respectively, of an optimal control. Furthermore, we denote by $M' := \{ x \mid x = \arg\max\{g(z)/\psi'(z)\} \}$ the set of global maximum points of $g/\psi'$ and let $z'^* = \sup\{M'\}$ be the maximal element of that set. Moreover, we denote by $Z^b$ the control of reflecting downwards at the threshold $b$. It is known that, for $x < b$, the value applying $Z^b$ can be written as $\frac{\sigma(b)}{\psi'(b)} \psi(x)$ (see e.g. discussion below Lemma 3.1 in [17]).

To end this introductory subsection, let us present an auxiliary lemma, which is a consequence of Itô’s Lemma (cf. Lemma 3.1 from [17]). Before introducing the result, let us notice that an arbitrary control can be composed as $Z_t = Z_t^C + \sum_{0 \leq s \leq t} \Delta Z_s$, where $Z_t^C$ is the continuous part and $\Delta Z_s = Z_{s+} - Z_{s-}$ is the jump part of the control.

**Lemma 5.1.** Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a twice continuously differentiable function such that $\sigma(x)f'(x)$ is bounded on $(0, \varepsilon)$ for some $\varepsilon > 0$. Let $Z_t$ be an arbitrary admissible control such that $X_t^Z$ is bounded. Then

$$f(x) = \mathbb{E}_x \left\{ \int_{0}^{\infty} e^{-rs} \left( r(X_s^Z) - A \right) f(X_s^Z) ds + \int_{0}^{\infty} e^{-rs} f'(X_s^Z) dZ_s^C + \sum_{0 \leq s \leq t} e^{-rs} \Delta f(X_s^Z) \right\}. \tag{7}$$

*Proof.* Let us apply (generalised) Itô’s lemma to a mapping $e^{-rt} f(X_t^Z)$ to get

$$e^{-rt} f(X_t^Z) = f(x) + \mathcal{E}_t + \int_{0}^{t} e^{-rs} \left[ \left( A - r(X_s^Z) \right) f(X_s^Z) ds - f'(X_s^Z) dZ_s^C \right]$$

$$- \sum_{0 \leq s \leq t} e^{-rs} \Delta f(X_s),$$

where $\mathcal{E}_t = \int_{0}^{t} e^{-rs} \sigma(X_s^Z) f'(X_s^Z) dW_s$ is a local martingale (e.g. Theorem IV.30.7 in [19]). The boundedness of $X^Z$ implies that also $f(X_t^Z)$ is bounded so that $\lim_{t \to \infty} e^{-rt} f(X_t^Z) = 0$. Moreover, the boundedness of $X^Z$ together with the fact that $\sigma(x)f'(x)$ is bounded near 0, implies that $\mathbb{E}_x \{ \mathcal{E}_t \} = 0$. Therefore, the claim follows by taking expectation of both sides in (7) and letting $t \to \infty$. \hfill \square

### 5.2. The main results.

A use of a control $Z^b$ results into a value $\frac{\sigma(b)}{\psi'(b)} \psi(x)$, which clearly resembles the value of a one-sided optimal stopping problem, only now we have a ratio $\frac{\sigma}{\psi'}$ instead of $\frac{\sigma}{\psi}$. Hence it is no surprise that the analysis of singular control problem follows more or less analogous path to an optimal stopping problem.
Lemma 5.2. Assume that $z^* > 0$ and let $a,b \in M'$, $b \neq 0$, and $0 < a \leq b \leq z^* < \infty$ and let $x \leq b$. Then the following controls yield the same value $\frac{g(z^*)}{\psi(z^*)}\psi(x)$:

(i) Control $Z_x^{z^*}$.
(ii) Control $Z_x^b$.
(iii) If $x \in (a,b)$, wait until $X_t$ hits either $a$ or $b$ and then reflect downwards at a threshold it hits first. (I.e. wait time $\tau_{(a,b)}$ and then reflect downwards at a threshold $X_{\tau_{(a,b)}}$)
(iv) Wait until $X_t$ enters the set $M'^+$, where $M'^+ \subset M'$ is any subset that contains $z^*$, and use downward control with a reflecting threshold $X_{\tau_{M'^+}}$.

Proof. (i), (ii) Since $b,z^* \in M'$, we have, for $x < b$,

$$\mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^z) dZ_s^{z^*} \right\} = \frac{g(z^*)}{\psi'(z^*)} \psi(x) = \frac{g(b)}{\psi'(b)} \psi(x) = \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^Z) dZ_s^b \right\}.$$

(iii) Let $x \in (a,b)$. The proposed control gives value

$$\mathbb{E}_x \left\{ e^{-rs}; \tau_a < \tau_b \right\} \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^Z) dZ_s^a \right\} + \mathbb{E}_x \left\{ e^{-rs}; \tau_b < \tau_a \right\} \mathbb{E}_b \left\{ \int_0^\infty e^{-rs} g(X_s^Z) dZ_s^b \right\} = \frac{\psi(b)\varphi(x) - \varphi(b)\psi(x)}{\psi(b)\varphi(a) - \varphi(b)\psi(a)} \frac{g(a)}{\psi'(a)} \varphi(a) + \frac{\psi(x)\varphi(a) - \varphi(x)\psi(a)}{\varphi(a)} \frac{g(b)}{\psi'(b)} \psi(b) = \frac{\psi(b)\varphi(a) - \varphi(b)\psi(a)}{\psi(b)\varphi(a) - \varphi(b)\psi(a)} \varphi(x) + \frac{\psi(x)\varphi(a) - \varphi(x)\psi(a)}{\varphi(a)} \frac{g(b)}{\psi'(b)} \psi(b) \psi(x).$$

As $a,b \in M$, we see straight that the coefficient of $\varphi(x)$ vanishes, and further that the last term can be written as

$$\left( \frac{\psi(b)\varphi(a) - \varphi(b)\psi(a)}{\psi(b)\varphi(a) - \varphi(b)\psi(a)} \right) \frac{g(b)}{\psi'(b)} \psi(x) = \frac{g(b)}{\psi'(b)} \psi(x).$$

(iv) As a diffusion is continuous, we must have $\tau_{M'^+} = \tau_{(a,b)}$, where $b$ be the smallest point of $M'^+$ such that $b \geq x$, and $a$ is the greatest point of $M'^+$ such that $x \leq a$ (if no such $a$ exist, then $\tau_{M'^+} = \tau_b$). Consequently the claim follows from item (iii) (or (ii)) \qed

Let us state our main result. This utilises Lemma 5.1, and hence we shall make the following assumption.

Assumption 5.3. Assume that $\sigma(x)\psi'(x)$ is bounded on $(0,\varepsilon)$ for some $\varepsilon > 0$ and let us consider only controls $Z$ such that $X^Z$ is bounded from above.

Proposition 5.4. (A) Let Assumption 5.3 hold. For $x \leq z^*$, the value can be written as

$$W^*_x(x) = \frac{g(z^*)}{\psi'(z^*)} \psi(x).$$

Moreover, an admissible optimal control exists and it is any reflecting downwards -control that leads to this value (cf. Lemma 5.2).

(B) If $z^* = 0$, then there is no threshold $\varepsilon > 0$ such that a downwards reflecting control $Z^\varepsilon$ would yield the maximal value for $x \leq \varepsilon$. 
Proof. (A) Let \( x \leq z^* \) and let \( Z_t \) be an arbitrary admissible control such that \( X_t^Z \) is bounded from above. Then we have (when we understand \( dZ_t = dZ_t^C + \Delta Z_t \))

\[
\mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^Z) dZ_s \right\} = \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^Z) dZ_s^C + \sum_{0 \leq s} e^{-rs} \int_{X_s^-}^{X_s^+} g(u) du \right\}
\]

\[
= \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} \frac{g(X_s^Z)}{\psi'(X_s^Z)} \psi'(X_s^Z) dZ_s^C + \sum_{0 \leq s} e^{-rs} \int_{X_s^-}^{X_s^+} \frac{g(u)}{\psi'(u)} \psi'(u) du \right\}
\]

\[
\leq \frac{g(z^*)}{\psi'(z^*)} \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} \psi'(X_s^Z) dZ_s \right\} = \frac{g(z^*)}{\psi'(z^*)} \psi(x),
\]

where the last equality follows from Lemma 5.1. For \( x \leq z^* \) this value is attained by applying a reflecting control \( Z^z \), and so the proposed \( W_Z(x) \) is the value function. Furthermore, any control from Lemma 5.2 produces this maximal value.

(B) Let \( M' = \{0\} \) and suppose, contrary to our claim, that there exists \( \varepsilon > 0 \) such that, for \( x \in (0, \varepsilon) \), a singular control \( Z^\varepsilon \) provides the maximal value, so that \( W_Z(x) = \frac{g(z^*)}{\psi'(z^*)} \psi(x) \). Since \( \limsup_{x \to 0^+} \frac{g(x)}{\psi'(x)} \) is the maximum for \( g(\cdot)/\psi'(\cdot) \), there exists \( \hat{x} \in (0, \varepsilon) \) such that \( \frac{g(\hat{x})}{\psi'(\hat{x})} > \frac{g(\varepsilon)}{\psi'(\varepsilon)} \), whence for all \( x \in (0, \hat{x}) \)

\[
\mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^Z) dZ_s \right\} = \frac{g(\hat{x})}{\psi'(\hat{x})} \psi(x) > \frac{g(\varepsilon)}{\psi'(\varepsilon)} \psi(x)
\]

\[
= \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^Z) dZ_s \right\} = W_Z(x),
\]

contradicting the maximality of \( W_Z(x) \).

We see that the optimal control is not uniquely determined if \( M' \) has at least two members. In fact we could use following kind of control: Choose \( M''_1^+ \) to be an arbitrary sequence of subsets of \( M' \) that includes \( z^* \). In the first step wait time \( \tau_{M''_1^+} \) and then reflect downwards at \( X_{\tau_{M''_1^+}} \) until \( X_t \notin M''_1^+ \). In the second step wait time \( \tau_{M''_1^+} \) and then reflect downwards at \( X_{\tau_{M''_1^+}} \) until \( X_t \notin M''_1^+ \), etc. This control also leads to a value \( \frac{g(z^*)}{\psi'(z^*)} \psi(x) \).

However, in the sequel we wish to be unambiguous, and consistent with the optimal stopping scene, and hence we select \( M' \) to be our optimal action region on interval \((0, z^*)\) with a notion that there might also be others.

5.3. Minor results. The main proposition for control problem gives similar corollaries that we got for optimal stopping problem.

If \( g(0+) < 0 \) or \( \frac{d}{d \varepsilon} \frac{g(0+)}{\psi'(0+)} > 0 \), then \( z^* > 0 \) and we can directly apply the following corollary.
Corollary 5.5. Let Assumption 5.3 hold. If \( z^* > 0 \), then the value reads as \( \frac{g(z^*)}{\psi(z^*)} \psi(x) \) and \( (0, z^*) \setminus M' \subset C \).

Proof. Straight consequence of Proposition 5.4. □

The special cases that 0 or \( \infty \) is contained in \( M' \) are handled in the following two corollaries.

Corollary 5.6. Let \( 0 \in M' \).

(A) One can never use control \( Z^0 \), i.e. one cannot reflect downwards at 0.

(B) If \( z^* = 0 \), then one of the following is true.

(i) It is optimal to drive the process instantaneously (i.e. infinitely fast) to the boundary 0, whence \( S = \mathbb{R}_+ \).

(ii) There exists \( \varepsilon > 0 \) such that \( (0, \varepsilon) \subset S \) and \( C \neq \emptyset \).

(iii) The optimal control is something else than an admissible reflecting control satisfying Assumption 5.3.

Proof. (A) If 0 is unattainable, it is never attained at a finite time and thus \( Z^0 \) cannot be used as a control. If 0 is attainable (i.e. exit or killing), then the process is terminated immediately at 0, before \( Z^0 \) is activated, and thus \( Z^0 \) cannot be used.

(B) Straight consequence of Proposition 5.4(B). □

Corollary 5.7. Let Assumption 5.3 hold and assume that \( z^* = \infty \). Then \( W(x) = A \psi(x) \), where \( A = \limsup_{x \to \infty} \left\{ \frac{g(x)}{\psi(x)} \right\} \). Especially, if \( A = \infty \), then \( W(x) \equiv \infty \). Moreover,

(i) if \( M' = \{ z^* \} = \{ \infty \} \), then \( C = \mathbb{R}_+ \setminus \{ \infty \} \) and there is no admissible optimal control;

(ii) if there is at least one other element in \( M' \), then there exists an admissible optimal control for \( x \leq b^* := \sup\{ M' \setminus \{ \infty \} \} \) and no admissible optimal control for \( x > b^* \), and \( C = \mathbb{R}_+ \setminus M' \).

Proof. Let \( A = \limsup_{x \to \infty} \left\{ \frac{g(x)}{\psi(x)} \right\} \). Using the arguments from the proof of Proposition 5.4, we known that for all admissible controls \( Z \) under Assumption 5.3 it is true that

\[
\mathbb{E}_x \left\{ \int_{0}^{\infty} e^{-rs} g(X_s^Z) dZ_s \right\} \leq A \psi(x).
\]

Since \( z^* = \infty \), we can choose an increasing sequence \( z_n \in \mathbb{R}_+ \), such that \( \lim_{n \to \infty} z_n = z^* \), \( \frac{g(z_n)}{\psi(z_n)} \) is increasing, and \( \lim_{n \to \infty} \frac{g(z_n)}{\psi(z_n)} = A \). Then a sequence of controls \( Z^{z_n} \) gives an increasing sequence of values \( \frac{g(z_n)}{\psi(z_n)} \psi(x) \). As this converges to \( A \psi(x) \), it must be the maximal value.

(i) If \( \infty \) is unattainable, then it is never reached and there is no control that provides the optimal value \( A \psi(x) \). If \( \infty \) is attainable (i.e. exit or killing), then the process is terminated immediately at \( \infty \) before a control \( Z^\infty \) is activated.

(ii) Straight consequence from Proposition 5.4 and part (i). □
Notice that by Corollary 5.7 the condition $z^* = \infty$ alone is not enough to guarantee that there are no admissible controls that provide the optimal value — If $M' \setminus \{\infty\} \neq \emptyset$, then there are admissible optimal controls which give the maximal value, at least for some $x \in \mathbb{R}_+$.

**Lemma 5.8.** If an interval $(a, b) \subset M$, then $g(x) = K\psi'(x)$ for all $x \in (a, b)$ for some $K > 0$.

**Proof.** Since $(a, b) \subset M$ we know that $x = \arg\max\{g(z)/\psi'(z)\}$ for all $x \in (a, b)$. This in turns means that $\frac{g(x)}{\psi'(x)} = K$ for some $K > 0$ on $(a, b)$. □

5.4. **Differences to optimal stopping.** The main difference to the optimal stopping case is the fact that in the singular control problem only the ratio $g/\psi'$ has meaning; the ratio $g/\varphi'$ leads to values that cannot be attained with downward control. It should be mentioned that this cannot be fixed easily by adding an upward control to the problem. This is because when both downward and upward controls are present, the solution can rarely be identified with a one-sided control, and as a consequence the analysis of the simple ratios $g/\psi'$ and $g/\varphi'$ is not adequate.

We also notice that we found more corollaries in the optimal stopping scene. This can be seen due to a rather simple local characterisation of stopping region/continuation region in optimal stopping problems: if $V(x) > g(x)$, then $x$ is in the continuation region. On the other hand, in control problems action regions/inaction regions are rarely found locally; More often than not, they are found globally applying variational inequalities. Most noticeably this is seen in the fact that in the control scene the set $C_{\psi'} = \left\{x \mid \frac{g(x)}{\psi'(x)} \text{is strictly increasing}\right\}$ is not necessarily part of the inaction region, as we shall see in Example 10.

5.5. **Extensions.** We can easily extend the results from this section to concern also a running payoff case. To that end let $\pi : \mathbb{R}_+ \to \mathbb{R}$ be once continuously differentiable function for which $\mathbb{E}_x \left\{\int_0^\infty |\pi(X_s)|ds\right\} < \infty$, and let us study a problem

\begin{equation}
\sup_Z \mathbb{E}_x \left\{\int_0^\infty e^{-rs}\pi(X_s^Z)ds + \int_0^\infty e^{-rs}g(X_s^Z)dZ_s\right\}.
\end{equation}

As the resolvent $(R_r\pi)(x)$ solves the ordinary differential equation $(A - r)u(x) = -\pi(x)$, a straight consequence of Lemma 5.1 is that

\begin{equation}
(R_r\pi)(x) = \mathbb{E}_x \left\{\int_0^\infty e^{-rs}\pi(X_s^Z)ds + \int_0^\infty e^{-rs}(R_r\pi)'(X_s^Z)dZ_s\right\}.
\end{equation}

Hence the problem (9) can be re-written as

\begin{equation}
(R_r\pi)(x) + \sup_Z \mathbb{E}_x \left\{\int_0^\infty e^{-rs} \left(g(X_s^Z) - (R_r\pi)'(X_s^Z)\right)dZ_s\right\},
\end{equation}

provided that the conditions of Lemma 5.1 are satisfied.
It follows at once that all the results from this section hold for this problem with obvious changes and with the set \[ M' = \left\{ x \mid x = \arg\max \left\{ g(z) - R \psi'(z) \right\} \right\}. \]

5.6. **Controlling upwards.** Until now we have only controlled the diffusion downwards. Let us now introduce an upward control defining

\[ X_t^Y = X_t + Y_t, \quad X_0^Y = x, \]

on \( \mathbb{R}_+ \), where \( X_t \) is as previously and \( Y_t \) is a non-negative, non-decreasing, right-continuous, and \( \mathcal{F}_t \)-adapted. For a function \( g \), defined as previously, we define a singular control problem

\[ W_Y(x) = \sup_Y \mathbb{E}_x \left\{ \int_0^\infty e^{-rs} g(X_s^Y) dY_s \right\}. \]

(10)

It is known that, for \( x > a \), the value applying \( Y_t^a \), i.e. a control that reflects upwards at a threshold \( a \), can be written as \( -g(a) / \varphi'(a) \) (see e.g. discussion below Lemma 3.1 in [17]). It is now quite clear that all the results from this section hold true for the problem (10) near the upper boundary with obvious changes; instead of \( g/\psi' \) we have \( -g/\varphi' \), instead of \( M' \) we have \( N' = \left\{ x \mid x = \arg\max \{-g(y)/\varphi'(y)\} \right\} \), instead of \( z^* \) we have \( y^* = \inf\{N'\} \), etc.

5.7. **Examples.** Let \( X_t \) be as in Section 4, i.e. \( X_t \) is a geometric Brownian motion for which \( \psi(x) = x^2 \) and \( \varphi(x) = x^{-6} \).

**Example 9.** Here we show that both an impulse control and a singular control can yield the maximal value. Let

\[ g_9(x) = \begin{cases} 
16 (\sqrt{x} - 2), & x \leq 16 \\
2x, & x \in (16, 25) \\
20 (\sqrt{x} - 2.5), & x \geq 25,
\end{cases} \]

so that \( g_9 \) is continuous, increasing, \( \sup \{ \frac{g_9(x)}{\varphi'(x)} \} = 1 \), and \( M' = [16, 25] \). Consequently, by Proposition 5.4 for \( x \leq 25 \), a control \( Z_{25}^x \) (with \( M'^+ = \{25\} \)) provides the solution and the value reads as \( W_{Z}(x) = \psi(x) \).

On the other hand, taking \( M' = [16, 25] \) to be the action region, we get a control that coincides with an impulse control \( (\tau_{25}; \zeta) \) with \( \tau_{25} = \inf\{t \geq 0 \mid X_t = 25\} \) and \( \zeta = 9 \). That is, every time we hit the state 25, we jump to the state 16. It can be easily calculated that, for \( x \leq 25 \), this impulse control also results into a value \( \psi(x) \).

**Example 10.** In this example we will see that \( C_{\psi'} = \{ x \mid \frac{g(x)}{\psi'(x)} \text{ is strictly increasing} \} \) does not necessarily belong to the inaction region. Take

\[ g_{10}(x) = \begin{cases} 
\sqrt{x} - 1, & x < 6.25 \\
1.4\sqrt{x} - 2, & x \geq 6.25.
\end{cases} \]
Now $M' = \{ z^* \} = \{ 4 \}$, and $C_{\psi'} = (0,4) \cup (6.25,8.16)$. However, it can be shown (cf. Lemma 1 in [1] for a verification result) that the action region is $[4,\infty)$ and that the optimal control is $Z^4$. In other words, contrary to what Lemma 3.8 from optimal stopping scene would suggest, the region $C_{\psi'}$ do not necessarily belong to the inaction region in singular control problems.

6. Connection between singular control and optimal stopping

6.1. Introducing the associated optimal stopping problem. It is well known that a singular stochastic control problem is closely related to an optimal stopping problem (see e.g. [7,13,14,6,2]). In the most analysis there are some growth or positiveness restrictions on the payoff $g$, or there are some remarkable restrictions for the diffusion process. In that sense the connection revealed here is more general than the previous studies as we let the payoff $g$ to be any upper semicontinuous function that attains positive values somewhere on $\mathbb{R}_+$, and $X_t$ to be any diffusion satisfying Assumption 5.3.

To introduce this connection in the present case, let us consider the control problems (6) and (10) and let $X, X^Z, X^Y, Z, Y, W^Z, W^Y$ and $g$ be as in the previous section.

Let us define an associated diffusion

$$ d\hat{X}_t = \left( \mu(\hat{X}_t) + \sigma'(\hat{X}_t)\sigma(\hat{X}_t) \right) dt + \sigma(\hat{X}_t)dW_t $$

and let us consider an optimal stopping problem

$$ \hat{V}(x) = \sup_{\tau} \mathbb{E}_x \left\{ e^{-\int_0^\tau (r-\mu'(\hat{X}_s))ds} g(\hat{X}_\tau) \right\}. $$

(The infinitesimal generator $\hat{A} - (r - \mu'(x))$ can be seen as a derivative of the operator $A - r$.) It is quite clear that if $\psi$ and $\varphi$ are convex, then their derivatives $\psi'$ and $-\varphi'$ are the non-negative increasing and decreasing fundamental solutions to $(\hat{A} - (r - \mu'(x)))u(x) = 0$. This in turn ensures that in the convex case, for $\tau_b = \inf\{ t \geq 0 | X_t = b \}$,

$$ \mathbb{E}_x \left\{ e^{-\int_0^{\tau_b} (r-\mu'(\hat{X}_s))ds} \right\} = \begin{cases} \frac{\psi'(x)}{\varphi'(b)}, & x \leq b \\ \frac{\varphi'(x)}{\psi'(b)}, & x > b \end{cases} $$

In the following we state sufficient conditions guaranteeing that the fundamental solutions $\psi$ and $\varphi$ are convex.

Lemma 6.1. (A) Assume that a transversality condition $\lim_{t \to \infty} \mathbb{E}_x \{ e^{-rt}X_t \} = 0$ holds and that $r > \mu'(x)$ for all $x \in \mathbb{R}_+$. Further assume that either 0 is unattainable or $\lim_{x \to 0} (rx - \mu(x)) \geq 0$. Then $\psi(x)$ is convex.

(B) Assume that one or the other of the following hold.

(i) $\mu(x) \geq 0$ for all $x \in \mathbb{R}_+$; or
(ii) a transversality condition holds, $\infty$ is unattainable and $r > \mu'(x)$ for all $x \in \mathbb{R}_+$. 

Then \( \varphi \) is convex.

**Proof.** Item (A) and the latter part of item (B) follow from similar deduction to Corollary 1 from \[3\]. The former part of item (B) follows after noticing that
\[
\frac{1}{2} \sigma^2(x) \varphi''(x) = r \varphi(x) - \mu(x) \varphi'(x).
\]
\[\square\]

6.2. The connection between singular control and optimal stopping. Assuming that the condition (12) holds, we can straightforwardly apply Proposition 3.2 to the associated stopping problem (11) with the sets \( \hat{M} = \{ x \mid x = \arg\max \{ \frac{g(z)}{\psi(z)} \} \} \) and \( \hat{N} = \{ x \mid x = \arg\max \{ \frac{g(y)}{\varphi'(y)} \} \} \). On the other hand, at the same time we can apply Proposition 5.4 to the control problems (6) and (10) with the sets \( M' = \hat{M} \) and \( N' = \hat{N} \).

These facts give the following proposition, which is the celebrated connection between a singular control problem and the associated optimal stopping problem in our case.

**Proposition 6.2.** Assume that \( \psi \) and \( \varphi \) satisfy (12) and that Assumption 5.3 hold.

(A) One has \( W'_Z(x) = \hat{V}(x) \) for all \( x \leq z' \).
(B) One has \( W'_Y(x) = \hat{V}(x) \) for all \( x \geq y' \).

We see that the associated stopping problem carries, potentially, more information than a single singular control problem; Although \( g/\varphi' \) played no role in a downward controlled singular control problem, it has a well defined meaning in the associated stopping problem. This is illustrated in Example 11 in Subsection 6.4. Moreover, the proposition reveals that there is no guarantee, \textit{a priori}, that we can connect a one-sided singular control problem to its associated stopping problem on \textit{the whole state space}. This observation gives us the following, quite interesting, necessary condition under which this connection can hold everywhere.

**Corollary 6.3.** Assume that \( \psi \) and \( \varphi \) satisfy (12) and that Assumption 5.3 hold.

(A) We can have \( W'_Z(x) = \hat{V}(x) \) for all \( x \in \mathbb{R}_+ \) only if \( y' = \infty \).
(B) We can have \( W'_Y(x) = \hat{V}(x) \) for all \( x \in \mathbb{R}_+ \) only if \( z' = 0 \).

**Proof.** (A) By Proposition 6.2 we have \( \hat{V}(x) = \frac{g(y)}{\varphi'(y)} \varphi'(x) \), for all \( x \geq y' \). As a value \( \frac{g(y)}{\varphi'(y)} \varphi(x) \) cannot be reached by applying a downward control, we can have \( W'_Z(x) = \hat{V}(x) \) for all \( x \in \mathbb{R}_+ \) only if \( y' = \infty \). Part (B) follows analogously. \[\square\]

6.3. When fundamental solutions can be concave. Earlier we have seen that the Laplace transform of the hitting time in (12) holds if \( \psi \) and \( \varphi \) were convex. Here we generalise this to a case where \( \psi \) and \( \varphi \) are concave near the boundaries.

The following theorem (Theorem 1 in \[3\]) reveals that the fundamental solutions are concave near the boundaries, if \( r - \mu' < 0 \) there.
**Theorem 6.4.** Assume that the transversality condition holds and that 0 is unattainable. Then for all $x \in \mathbb{R}_+$
\[
\sigma^2(x) \frac{\psi''(x) - \psi''(x)}{S'(x)} = 2r \int_0^x \psi(y) (\theta(x) - \theta(y)) m'(y) dy, \quad \text{and}
\]
\[
\sigma^2(x) \frac{\varphi''(x)}{S'(x)} = 2r \int_x^\infty \varphi(y) (\theta(y) - \theta(x)) m'(y) dy,
\]
where $\theta(x) = rx - \mu(x)$.

The following sums up the Laplace transform of the hitting time in this concave case.

**Lemma 6.5.** (A) Assume that the transversality condition holds and that 0 is unattainable for a diffusion $X_t$. Further, assume that there exists $\varepsilon > 0$ such that $r - \mu'(x) < 0$ for all $x \in (0, \varepsilon)$ (it may be negative also elsewhere). Let $b > 0$ and $x \in (0, b)$. Then
\[
\mathbb{E}_x \left\{ e^{-\int_0^\tau_a (r - \mu'(\hat{X}_s)) ds} \right\} = \frac{\psi'(x)}{\psi'(b)},
\]
where $\tau_a = \inf\{ t \geq 0 \mid \hat{X}_t = b \}$ is the first hitting time to a state $b$.

(B) Assume that the transversality condition holds and that $\infty$ is unattainable for a diffusion $X_t$. Further, assume that there exists $H < \infty$ such that $r - \mu'(x) < 0$ for all $x \in (H, \infty)$ (it may be negative also elsewhere). Let $a < \infty$ and $x \in (a, \infty)$. Then
\[
\mathbb{E}_x \left\{ e^{-\int_0^\tau_a (r - \mu'(\hat{X}_s)) ds} \right\} = \frac{\varphi'(x)}{\varphi'(a)},
\]
where $\tau_a = \inf\{ t \geq 0 \mid \hat{X}_t = a \}$ is the first hitting time to a state $a$.

**Proof.** The proof follows quite closely that of Theorem 9 in [2].

(A) Now $\psi''(x) \psi'(x) - \psi''(x) \varphi'(x) = 2r B S'(x) > 0$, where $\hat{S}'(x) = S'(x)/\sigma^2(x)$ is the scale derivative of the process $\hat{X}$. Because of this, we see that the ratio $\varphi'(x)/\psi'(x)$ is increasing. Moreover, as $\psi'$ and $\varphi'$ are two independent solutions to $\left( \hat{A} - (r - \mu'(x)) \right) u(x) = 0$, any solution to it can be expressed as $c_1 \psi'(x) + c_2 \varphi'(x)$ for some $c_1, c_2 \in \mathbb{R}$. It is now an easy exercise in linear algebra to demonstrate that if $x \in (a, b)$, then
\[
G(x; a, b) := \mathbb{E}_x \left\{ e^{-\int_0^{\tau_{(a,b)}} (r - \mu'(\hat{X}_s)) ds} \right\} = \frac{\varphi'(x) - \frac{\varphi'(a)}{\varphi'(b)} \psi'(x)}{\varphi'(a) - \frac{\varphi'(a)}{\varphi'(b)} \psi'(a)} + \frac{\psi'(x) - \frac{\psi'(a)}{\varphi'(a)} \varphi'(x)}{\psi'(b) - \frac{\psi'(a)}{\varphi'(a)} \varphi'(b)},
\]
where $\tau_{(a,b)} = \inf\{ t \geq 0 \mid \hat{X}_t \notin (a, b) \}$ is the first exit time of $\hat{X}$ from an open interval $(a, b)$. Invoking the alleged boundary conditions of $X$ implies that
\[
\frac{\psi'(x) - \frac{\psi'(a)}{\varphi'(a)} \varphi'(x)}{\psi'(b) - \frac{\psi'(a)}{\varphi'(a)} \varphi'(b)} = \frac{\psi'(x) - \frac{\psi'(a)/S'(a)}{\varphi'(a)/S'(a)} \varphi'(x)}{\psi'(b) - \frac{\psi'(a)/S'(a)}{\varphi'(a)/S'(a)} \varphi'(b)} \rightarrow \frac{\psi'(x)}{\psi'(b)} \text{ as } a \rightarrow 0.
\]
Consider now the first term on the right-hand side of (13). We want to show that it converges to 0 as $a$ approaches 0. To that end, we firstly notice that it can be written
as

\[
\frac{\psi'(x) \frac{\varphi'(x)}{\psi'(x)} - \frac{\varphi'(b)}{\psi'(b)}}{\psi'(a) \frac{\varphi'(a)}{\psi'(a)} - \frac{\varphi'(b)}{\psi'(b)}} \geq 0,
\]

where the inequality follows from the fact that \(\varphi'/\psi'\) is increasing and \(\psi' > 0\). Secondly, from Theorem 6.4 we see at once that \(\psi\) is concave on \((0, \varepsilon)\), whence we can approximate

\[
\frac{\psi'(x) \frac{\varphi'(x)}{\psi'(x)} - \frac{\varphi'(b)}{\psi'(b)}}{\psi'(a) \frac{\varphi'(a)}{\psi'(a)} - \frac{\varphi'(b)}{\psi'(b)}} = \frac{\psi'(x) \psi'(\varepsilon) \frac{\varphi'(x)}{\psi'(x)} - \frac{\varphi'(b)}{\psi'(b)}}{\psi'(a) \psi'(\varepsilon) \frac{\varphi'(a)}{\psi'(a)} - \frac{\varphi'(b)}{\psi'(b)}} \leq \frac{\psi'(x) \psi'(\varepsilon) \frac{\varphi'(x)}{\psi'(x)} - \frac{\varphi'(b)}{\psi'(b)}}{\psi'(a) \psi'(\varepsilon) \frac{\varphi'(a)}{\psi'(a)} - \frac{\varphi'(b)}{\psi'(b)}},
\]

for all \(a < \varepsilon\).

Letting \(a \to 0\) and invoking again the alleged boundary conditions of \(X\) we get

\[
\frac{\psi'(x) \frac{\varphi'(x)}{\psi'(x)} - \frac{\varphi'(b)}{\psi'(b)}}{\psi'(a) \frac{\varphi'(a)}{\psi'(a)} - \frac{\varphi'(b)}{\psi'(b)}} \to \frac{\psi'(x) \frac{\varphi'(x)}{\psi'(x)} - \frac{\varphi'(b)}{\psi'(b)}}{\psi'(a) \frac{\varphi'(a)}{\psi'(a)} - \frac{\varphi'(b)}{\psi'(b)}} \leq 0,
\]

indicating that the first term on the right hand side of (13) tends to zero as \(a\) tends to zero. Consequently

\[
\lim_{a \to 0} G(x; a, b) = \mathbb{E}_x \left\{ e^{-\int_0^\tau (r - \mu'(\dot{X}_s)) ds} \right\} = \frac{\psi'(x)}{\psi'(b)}.
\]

(B) Proof is analogous to the part (A). \(\square\)

6.4. Examples. Again, let \(X_t\) be a geometric Brownian motion for which \(\psi(x) = x^2\) and \(\varphi(x) = x^{-6}\). Let us present perhaps the most striking example of the paper.

Example 11. In this example we will illustrate that the associated stopping problem can be associated problem to a two different singular control problem at a same time. Let

\[
\Pi(x) = \begin{cases} x^2, & x \leq 1 \\ x^{-8}, & x > 1. \end{cases}
\]

It is quite straightforward to show that

\[
\tilde{\Pi}(x) = \begin{cases} \frac{2}{3} \psi'(x), & x \leq 1 \\ \frac{2}{3} \varphi'(x), & x > 1. \end{cases}
\]

Moreover, now \(M' = \{1\}\) and \(\sup \{\Pi(x) \psi'\} = \frac{1}{2}\). Thus we can say that for \(x \leq 1\) we have \(W_Z(x) = \frac{1}{2} \psi(x)\) and \(\tilde{V}_{\Pi}(x) = W_Z'(x)\). (Actually, applying Theorem 1 from [1], we can say that \(Z^1\) is the optimal control on \(\mathbb{R}_+\) and that \(W_Z(x) = \int_1^x g_{11}(z)dz + \frac{1}{2} \psi(1)\) for \(x \geq 1\).)

On the other hand, now we also have \(N' = \{1\}\) and \(\sup \{\Pi(x) \varphi'\} = \frac{1}{6}\), so that \(W_Y(x) = \frac{1}{6} \varphi(x)\) and \(\tilde{V}_{\Pi}(x) = -\frac{1}{6} \varphi'(x)\) for \(x \geq 1\). (Again we can prove that, for \(x \leq 1\), \(W_Y(x) = \int_x^1 g_{11}(y)dy + \frac{1}{2} \psi(1)\))

It follows that the value \(\tilde{V}_{\Pi}\) of the associated optimal stopping problem is, in fact, the associated value function for two different singular control problems on separate regions...
(0, 1) and (1, ∞). This means that the connection between the one-sided singular control and optimal stopping problem is in general a local property rather than global.

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