A CONSTRUCTION OF \( v \)-ADIC MODULAR FORMS

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Abstract. The classical theory of \( p \)-adic (elliptic) modular forms arose in the 1970’s from the work of J.-P. Serre \[Se1\] who took \( p \)-adic limits of the \( q \)-expansions of these forms. It was soon expanded by N. Katz \[Ka1\] with a more functorial approach. Since then the theory has grown in a variety of directions. In the late 1970’s, the theory of modular forms associated to Drinfeld modules was born in analogy with elliptic modular forms \[Go1\], \[Go2\]. The associated expansions at \( \infty \) are quite complicated and no obvious limits at finite primes \( v \) were apparent. Recently, however, there has been progress in the \( v \)-adic theory, \[Vi1\].

Also recently, A. Petrov \[Pe1\], building on previous work of \[Lo1\], showed that there is an intermediate expansion at \( \infty \) called the “A-expansion,” and he constructed families of cusp forms with such expansions. It is our purpose in this note to show that Petrov’s results also lead to interesting \( v \)-adic cusp forms à la Serre. Moreover the existence of these forms allows us to readily conclude a mysterious decomposition of the associated Hecke action.

1. Introduction

Let \( G_{2k}(z) \) be the classical elliptic modular Eisenstein series with \( q \)-expansion \( \sum c_n q^n \). It is well-known that the non-constant coefficients are of the form

\[
\sum_{d|n} d^{2k-1}.
\]

Let \( p \) be a fixed prime with weight space \( \mathbb{S}_p = \varprojlim_j \mathbb{Z}/(p^j - 1) \). Let \( s_p \in \mathbb{S}_p \) and let \( k_i \) be a collection of positive integers converging to \( s_p \). If \( d \mid n \) is prime to \( p \), then \( d^{2k_i - 1} \to d^{2s_p - 1} \). On the other hand, if \( p \mid d \) then the above powers of \( d \) converge to 0 \( p \)-adically. Thus the Eisenstein series very strongly suggest the existence of a good \( p \)-adic theory of modular forms.

In \[Pe1\], A. Petrov gives a powerful construction of cuspidal eigenforms which looks remarkably similar to the construction of Eisenstein series (see Remark \[2\] below). We exploit this analogy here to construct examples of \( v \)-adic modular forms, for finite primes \( v \) of \( \mathbb{F}_q[\theta] \). We further show how \( v \)-adic continuity allows one to decompose the Hecke action on Petrov’s forms.

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2. Review of the results of López and Petrov

In this section we briefly review the basic set-up of modular forms in finite characteristic and the results of B. López \[Lo1\], and A. Petrov \[Pe1\], whose exposition we follow (outside of a few small changes of notation which will be made clear below).

Let \( q = p^{m_0} \) and set \( A := \mathbb{F}_q[\theta] \) (N.B.: in \[Pe1\], the author uses “\( T \)” instead of “\( \theta \)”). Put \( K := \mathbb{F}_q(\theta) \) and \( K_\infty := \mathbb{F}_q((1/\theta)) \). So \( K_\infty \) is complete with respect to the absolute value.
If $|?|_\infty$ at $\infty$, and we let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\bar{K}_\infty$ of $K_\infty$ equipped with the canonical extension of $|?|_\infty$.

Let $\Omega$ be the Drinfeld upper half-plane; $\Omega$ is a connected 1-dimensional rigid analytic space. The set of $\mathbb{C}$ points of $\Omega$ is $\mathbb{C}_\infty \setminus \bar{K}_\infty$. Let $z$ be one such geometric point. Then, as in [Go1], we set

$$|z|_i := \inf_{x \in K} \{|z - x|_\infty\}. \quad (2)$$

If $\nu$ is in the value group of $\mathbb{C}_\infty$, then the set $\Omega_\nu := \{z \in \Omega \mid |z|_i \geq \nu\}$ is an admissible open subset of $\Omega$.

Let $\Gamma = \text{GL}_2(A)$ and let $k$ be a positive integer. Let $m$ be an integer with $0 \leq m < q - 1$. In analogy with the classical $\text{SL}_2(\mathbb{Z})$-theory, one can readily define the notion of modular forms of weight $k$ and type $m$ associated to $\Gamma$ (see [Go1], [Go2], [Ge1]). The space of such forms is denoted $M_k,m = M_{k,m}(\Gamma)$ and is finite dimensional. The subspaces of cusp forms and double-cusp forms are denoted $S_k,m = S_{k,m}(\Gamma)$ and $S^2_{k,m} = S^2_{k,m}(\Gamma)$.

Let $C$ be the Carlitz module and $\pi$ its period; the lattice $\Lambda_C$ associated to $C$ is then $A\pi$. Let $e_C(z)$ be the exponential of $C$ and put

$$u := u(z) = 1/e_C(\pi z). \quad (3)$$

(N.B.: in [Pe1], the author uses $t(z)$ where we have used $u(z)$.) The function $u$ is regular on $\Omega$. For $a \in A_+$ of degree $d$, we set

$$u_a := u(az). \quad (4)$$

It is very easy to see that $u_a = u^{q^d} + \{\text{higher terms in } u\}$.

Let $f(z) \in M_{k,m}$. Then $f$ has a unique expansion $f(z) = \sum_{i \geq 0} a_i u(z)^i$ converging to $f$ for $z \in \Omega_\nu$ with $\nu$ sufficiently large. In fact, as the space $\Omega$ is rigid-analytically connected, the above $u$-expansion uniquely determines the form $f(z)$.

Let $\Lambda \subset \mathbb{C}_\infty$ be an arbitrary $\mathbb{F}_q$-lattice; that is $\Lambda$ is a discrete (finite intersection with each ball around the origin), $\mathbb{F}_q$ submodule of $\mathbb{C}_\infty$. For instance, $\Lambda$ could be $\Lambda_C$ or the $g$-torsion points, $g \in A$, inside $\mathbb{C}_\infty$ of the Carlitz module. To such a lattice $\Lambda$ one associates the exponential function

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} (1 - z/\lambda), \quad (5)$$

which is easily seen to be an entire $\mathbb{F}_q$-linear function in $z$. As such, the derivative in $z$ of $e_\Lambda(z)$ is identically 1.

We set

$$t_\Lambda(z) := e'_\Lambda(s)/e_\Lambda(z) = 1/e_\Lambda(z) = \sum_{\lambda \in \Lambda} 1/(z + \lambda), \quad (6)$$

and, for a positive integer $k$,

$$S_{k,\Lambda}(z) := \sum_{\lambda \in \Lambda} 1/(z + \lambda)^k. \quad (7)$$

As in §3 of [Ge1], there is a monic polynomial, $G_{k,\Lambda}(X)$, of degree $k$, such that

$$G_{k,\Lambda}(t_\Lambda(z)) = S_{k,\Lambda}(z).$$

**Definition 1.** We set $G_n(z) := G_{n,\Lambda_C}(z)$.

It is easy to see that $G_n(z)$ has coefficients in $K$. 
Definition 2. Let \( f(z) \in M_{k,m} \) as above. We say that \( f(z) \) has an \( A \)-expansion if there exists a positive integer \( n \) and coefficients \( c_0, c_a, a \in A_+ \), such that
\[
f(z) = c_0(f) + \sum_{a \in A_+} c_a G_n(u_a) .
\]

Remarks 1. a. If such an expansion exists for a given \( n \), then the argument of López [Lo1] shows that it is unique.
b. If \( f \) is a Hecke eigenform, then Petrov [Pe1] establishes that \( n \) is uniquely determined by \( f \).
c. It is suspected, but not yet known, that all such expansions (8) are, in fact, uniquely determined by \( f \) for arbitrary forms \( f \).
d. The form of the polynomials \( G_n(X) \) we use is the correct one and the reader should simply ignore the normalization used in [Pe1].

Examples 1. The first class of examples of such expansions are the Eisenstein series: Let \( k \equiv 0 \pmod{q-1} \) be a positive integer and set
\[
G_k(z) := \sum (az + b)^{-k}
\]
where, as usual, we sum over all nonzero \((a, b) \in A \times A\). As in [Ge1], we see
\[
-G_k(z)/\pi^k = \sum_{a \in A_+} (\pi a)^{-k} + \sum_{a \in A_+} G_k(u_a).
\]

The second class of examples involve cusp forms. There are two fundamental cusp forms in the theory; \( \Delta \) and \( h \). Here \( \Delta \) is defined in analogy with its classical counterpart and \( h \) is a certain \( q-1 \)-st root of \( \Delta \). In [Lo1] López establishes the following \( A \)-expansions for these forms, thereby establishing that \( A \)-expansions also work for cusp forms.

Examples 2. We have
\[
\Delta = \sum_{a \in A_+} a^{q(q-1)} G_{q-1}(u_a) = \sum_{a \in A_+} a^{q(q-1)} u_a^{q-1},
\]
and
\[
h = \sum_{a \in A_+} a^q G_1(u_a) = \sum_{a \in A_+} a^q u_a.
\]

In [Pe1], Petrov builds on the above examples and constructs families of cusp forms with \( A \)-expansions which we now recall. Let \( \text{ord}_p(j) \) be the \( p \)-adic valuation of an integer \( j \).

Theorem 1. Let \( k, n \) be two positive integers such that \( k - 2n \) is a positive multiple of \( q - 1 \) and \( n \leq p^{\text{ord}_p(k-n)} \). Then
\[
f_{k,n} := \sum_{a \in A_+} a^{k-n} G_n(u_a)
\]
is an element of \( S_{k,m} \) with \( n \equiv m \pmod{q-1} \).

It is easy to see that the \( u \)-expansion of \( f_{k,n} \) has coefficients in \( K \).

Remark 1. As Petrov points out in Remark 1.4 of [Pe1], the condition \( n \leq p^{\text{ord}_p(k-n)} \) is equivalent to having the \( p \)-adic expansion of \( k \) and \( n \) agree up to the \( \lfloor \log_p(n) \rfloor \)-th digit.

Petrov’s construction has a remarkable analog classically, but in the realm of Eisenstein series, not cusp forms, as the following remark makes clear.
Remark 2. Let \( G_{2k}(z) := \sum_{(0,0) \neq (m,n)} (mz + n)^{-2k} \) be classical Eisenstein series and set \( E_{2k}(z) := G_{2k}(z)/2\zeta(2k) \) where \( \zeta(z) \) is the Riemann zeta function. It is known that \( E_{2k} \) has the Lambert expansion

\[
E_{2k}(z) = 1 + \frac{2}{\zeta(1 - 2k)} \sum_{n=1}^{\infty} n^{2k-1} \frac{q^n}{1 - q^n}.
\] (14)

If we put \( q_n := q^n \) and \( G(x) := x/(1 - x) \), we can rewrite Equation (14) as

\[
E_{2k}(z) = 1 + \frac{2}{\zeta(1 - 2k)} \sum_{n=1}^{\infty} n^{2k-1} G(q_n),
\] (15)

and the analogy with Equation (13) is clear. Thus, in a sense, Petrov’s construction is both analogous to, and “orthogonal to” (i.e., lying in the space of cusp forms), the classical construction of Eisenstein series.

3. \( \mathfrak{v} \)-adic modular forms in the sense of Serre

In this section we interpolate the cusp forms of Theorem 1 at finite primes of \( A \). Let \( \mathfrak{v} \in \text{Spec}(A) \) be a fixed finite prime of degree \( d \) with completions \( A_{\mathfrak{v}} \), \( K_{\mathfrak{v}} \) respectively of \( A \) and \( K \). Let \( p_\mathfrak{v} \) be the monic generator of \( \mathfrak{v} \), and let \( A_{\mathfrak{v}}^+ \subset A_{\mathfrak{v}}^* \) be those monic elements also prime to \( \mathfrak{v} \).

Definition 3. We define the \( \mathfrak{v} \)-adic weight space \( S_\mathfrak{v} \) by

\[
S_\mathfrak{v} := \lim_{\substack{\longleftarrow \ni t \ni t}} \mathbb{Z}/((q^d - 1)p^t) = \mathbb{Z}/(q^d - 1) \times \mathbb{Z}_p.
\] (16)

If \( a \in A_{\mathfrak{v}}^+ \) then put \( a = a_0 a_1 \) where \( a_0 \in A_{\mathfrak{v}}^* \) is the \( q^d - 1 \)-st root of unity with \( a_0 \equiv a \mod \mathfrak{v} \) and \( a_1 \equiv 1 \mod \mathfrak{v} \). Thus, if \( s_\mathfrak{v} = (x_\mathfrak{v}, y_\mathfrak{v}) \in S_\mathfrak{v} \), then one defines \( a^{s_\mathfrak{v}} := a_0^{x_\mathfrak{v}} a_1^{y_\mathfrak{v}} \) in complete analogy with classical theory; the function \( s_\mathfrak{v} \mapsto a^{s_\mathfrak{v}} \) is readily seen to be continuous from \( S_\mathfrak{v} \) to \( A_{\mathfrak{v}}^* \).

Let \( s_\mathfrak{v} \in S_\mathfrak{v} \).

Definition 4. We set

\[
\hat{f}_{s_\mathfrak{v},n} := \sum_{a \in A_{\mathfrak{v}}^+} a^{s_\mathfrak{v}} G_n(u_a),
\] (17)

which is readily seen to converge to an element in \( A_{\mathfrak{v}}[[u]] \otimes K \).

Let \( f(u) = \sum a_n u^n \in A_{\mathfrak{v}}[[u]] \otimes K \) be an arbitrary power series. We set

\[
\text{ord}_\mathfrak{v}(f) := \inf_n \{\text{ord}_\mathfrak{v}(a_n)\}.
\] (18)

Definition 5. We say that a power series \( f(u) \in A_{\mathfrak{v}}[[u]] \otimes K \) is a \( \mathfrak{v} \)-adic modular form in the sense of Serre if there exists a sequence \( f_i \in M_{k_i,m} \) such \( \text{ord}_\mathfrak{v}(f - f_i) \) tends to \( \infty \) with \( i \).

In other words, the power series \( f(u) \) is a \( \mathfrak{v} \)-adic limit of the \( u \)-expansions of true modular forms.

Now let \( f_{k,n} \) be as constructed in Theorem 1 and put \( \alpha := k - n \). As \( k - 2n \) is a positive multiple of \( q - 1 \), we find that \( \alpha \equiv n \mod (q - 1) \). Moreover, by Remark 2 we see that \( \alpha \) is divisible by a high power of \( p \). With this in mind, we write \( n q^d \)-adically as

\[
n = \sum_{e=0}^{t} n_e q^{de} \quad n_t \neq 0.
\] (19)
Definition 6. We let \( S_v(n) \subseteq S_v \) be the open subset consisting of those \( s_v = (x_v, y_v) \) such that \( x_v \equiv n \pmod{q - 1} \) and \( y_v \equiv 0 \pmod{q^{d(l+1)}} \).

Theorem 2. Let \( s_v \in S_v(n) \). Then \( \hat{f}_{s,v} \) is a \( v \)-adic modular form in the sense of Serre.

Proof. Let \( m_i \in S_v(n), i = 1, 2, \ldots \) be an increasing sequence of positive integer converging to \( s_v \) in \( S_v \). Put \( k_i = m_i + n \) so that \( k_i - 2n \) is divisible by \( (q - 1) \) and, for \( i \gg 1 \), is also positive. Note that if \( a \in v \) then \( a^{m_i} \to 0 \) in \( A_v \). It is now clear that, for \( i \gg 1 \), the forms \( f_{k_i,n} \) of Theorem 1 converge \( v \)-adically to \( \hat{f}_{s,v} \).

Definition 7. We call \( s_v + n \in S_v \) the weight of \( \hat{f}_{s,v} \). It’s type is given by the class of \( n \) modulo \( q - 1 \).

Remark 3. We do not know if those \( \hat{f}_{s,v} \), for \( s_v \) not in \( S_v(n) \), are modular forms in the sense of Serre. As of this writing, there is no reason to necessarily believe that they are.

4. HECKE OPERATORS

We show here how the existence of \( v \)-adic interpolations of Petrov’s forms \( f_{k,n} \) has striking implications for the action of the Hecke operators on any fixed form \( f_{k_0,n} \).

Let \( g \) be a monic prime of \( A \) of degree \( d \) and let \( \Lambda_g \subset \mathbb{C}_\infty \) be the \( \mathbb{F}_q \) submodule of \( g \)-division points. Let \( G_{k,\Lambda_g}(X) \) be constructed as in section 2. Let \( A_d \subset A \) be the \( \mathbb{F}_q \) submodule of polynomials of degree \( < d \), and let \( f \in M_{k,m} \) be an arbitrary element.

Definition 8. We set

\[
\hat{T}_g f(z) := \sum_{\beta \in A_d} f \left( \frac{a + \beta}{g} \right),
\]

and

\[
T_g f(z) := g^k f(gz) + \hat{T}_g f(z) = g^k f(gz) + \sum_{\beta \in A_d} f \left( \frac{a + \beta}{g} \right).
\]

One sees, as expected, that \( T_g f \) also belongs to \( M_{k,m} \) etc. If \( f \) has the \( u \)-expansion \( f = \sum_{n=0}^{\infty} a_n u^n \), then one has

\[
T_g f = g^k \sum_{n=0}^{\infty} a_n u_n^g + \sum_{k=0}^{\infty} a_k G_{k,\Lambda_g}(gu).
\]

As a consequence of Equation 22, we will view \( T_g \) and \( \hat{T}_g \) as operators on power series without referring to the original additive expansion exactly as in classical theory.

Remark 4. It is of fundamental importance that Petrov establishes that, for any \( k \) and \( n \), the cusp form \( f_{k,n} \) of Theorem 1 is a Hecke eigenform for any \( g \) with associated eigenvalue \( g^n \).

Fix \( v \) as before and now also fix \( f := f_{k_0,n} = \sum a_j u^j \) where \( k_0 \) and \( n \) satisfy the hypotheses of Theorem 1. We decompose \( f \) as \( f_{0,v} + f_{1,v} = f_0 + f_1 \) where

\[
f_0 := \sum_{a \in \Lambda_{v,+}} a^{k_0-n} G_n(u_a),
\]

and

\[
f_1 := \sum_{a \in v \cap A_+} a^{k_0-n} G_n(u_a).
\]
Set \( f_i = \sum a^{(i)}_j v_j^i \) for \( i = 0, 1 \). From the definition of \( f \) one easily deduces that \( f_1 = p_{k_0}^{a_0} \sum a_j v_j^{a_0} \).

Example 1. Let \( v = (0) \) and \( h = \sum_{a \in A^+} a^q u_a \) as before. Thus \( h_{0,v} = \sum_{a \in A^+, a(0) \neq 0} a^q u_a \) and \( h_{1,v} = \sum_{a \in A^+, a(0) = 0} a^q u_a \). Let \( g \not\in v \) be a monic irreducible.

Theorem 3. With the above notation, both \( f_0 \) and \( f_1 \) are separately eigenforms for \( T_g \) with eigenvalue \( g^n \).

Proof. Let \( k_i \) be an infinite sequence of positive integers converging to \( k_0 \) in the \( v \)-adic weight space \( S_v \). Then, as in the proof of Theorem 2, the expansions of \( f_{k_i} \) converge to that of \( f_0 \). Note further that each such cusp form is an eigenform for \( T_g \) with eigenvalue \( g^n \). Note further that, by definition, the action of \( T_g \) on \( f_{k_i} \) converges to the action of \( T_g \) on \( f_0 \). Thus, \( T_g f_0 = g^n f_0 \). Writing \( f_1 = f - f_0 \) then gives the statement for \( f_1 \).

Remarks 2. a. Petrov has pointed out that the techniques of [Pe1] allow for a direct proof of Theorem 3.

b. Petrov also points out that the decomposition \( f = f_0 + f_1 \) is actually a decomposition of \( v \)-adic modular forms. Indeed, one sees that \( f_1 \) is a modular form for \( \Gamma_0(v) \) and so is also a \( v \)-adic form by Theorem 6.2 of [Vi1].

Suppose now that \( g = p_v \).

Theorem 4. We have

\[ \hat{T}_g f_0 = g^n f_0 \] (25)

Proof. Noting that \( g^{k_i} \) tends to 0 \( v \)-adically, the result follows as in the proof of Theorem 3.

Corollary 1. We have

\[ T_g f_1 = g^n f_1 - g^{k_0} \sum a_n^{(0)} u_n^g. \] (26)

Proof. We have \( T_g f = T_g f_0 + T_g f_1 = g^n f = g^n f_0 + g^n f_1 \). On the other hand, \( T_g f_0 = g^{k_0} \sum a_n^{(0)} u_n^g + \hat{T}_g f_0 \) which, by Theorem 3 equals \( g^{k_0} \sum a_n^{(0)} u_n^g + g^n f_0 \). The result follows directly.

One can further decompose \( f_0 \) and \( f_1 \) by using a prime \( v_1 \neq v \) etc.

References

[Ge1] E.-U. GEKELER: On the coefficients of Drinfeld modular forms, Invent. Math. 93 (1988), 667-700.
[Go1] D. GOSS: \( \pi \)-adic Eisenstein series for function fields, Compos. Math. 41 (1980), 3-38.
[Go2] D. GOSS: The algebrist’s upper half-plane, Bull. Amer. Math. Soc. 2 (1980), no. 3, 391-415.
[Pe1] A. PETROV: \( A \)-expansions of Drinfeld modular forms, J. Number Theory 133 (2013), 2247-2266.
[Ka1] N.M. KATZ: \( p \)-adic properties of modular schemes and modular forms Modular forms of one variable, III, L.N.M. #350, (1973), 69-190.
[Lo1] B. LÓPEZ: A non-standard Fourier expansion for the Drinfeld discriminant function, Arch. Math. 95 (2010), 143-150.
[Lo2] B. LÓPEZ: Action of Hecke operators on two distinguished Drinfeld modular forms, Arch. Math. 97 (2011), 423-429.
[Se1] J.-P. SERRE: Formes modulaires et fonctions zêta \( p \)-adiques, Modular forms of one variable, III, L.N.M. #350, (1973), 191-268.
[Vi1] C. Vincent: Drinfeld modular forms modulo $p$ and Weierstrass points on Drinfeld modular curves, *Thesis: Univ. of Wisc.* (2012).

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