EXPONENTIAL SUMS ALONG $p$–ADIC CURVES

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ABSTRACT. Let $K$ be a $p$–adic field, $R$ the valuation ring of $K$, and $P$ the maximal ideal of $R$. Let $Y \subseteq \mathbb{R}^2$ be a non-singular closed curve, and $Y_m$ its image in $R/P^m \times R/P^m$, i.e. the reduction modulo $P^m$ of $Y$. We denote by $\Psi$ an standard additive character on $K$. In this paper we discuss the estimation of exponential sums of type $S_m(z, \Psi, Y, g) := \sum_{x \in Y_m} \Psi(zg(x))$, with $z \in K$, and $g$ a polynomial function on $Y$. We show that if the $p$-adic absolute value of $z$ is big enough then the complex absolute value of $S_m(z, \Psi, Y, g)$ is $O(q^m(1-\beta(f,g)))$, for a positive constant $\beta(f,g)$ satisfying $0 < \beta(f,g) < 1$.

1. Introduction

In this paper we shall discuss the estimation of exponential sums along $p$–adic curves. More precisely, let $K$ be a $p$–adic field, i.e. a finite algebraic extension of $\mathbb{Q}_p$, the field of $p$–adic numbers. Let $R$ be the valuation ring of $K$, $P$ the maximal ideal of $R$, and $\overline{K} = R/P$ the residue field of $K$. The cardinality of $\overline{K}$ is denoted by $q$, thus $\overline{K} = \mathbb{F}_q$. For $z \in K$, $v(z) \in \mathbb{Z} \cup \{+\infty\}$ denotes the valuation of $z$, and $|z| = q^{-v(z)}$ its absolute value. We fix a uniformizing parameter $\pi$ for $R$.

We set $X := R^n$ as $p$–adic analytic variety of dimension $n$, and $X_m := R/P^m \times R/P^m \times \ldots \times R/P^m$, ($n$ factors).

Let $Z \subseteq X$ be a closed subset. We denote by $Z_m$ the image of $Z$ in $X_m$; $Z_m$ is named the reduction modulo $P^m$ of $Z$.

Let $\Psi$ be an standard additive character on $K$. Thus for $z \in K$,

$$\Psi(z) = \exp(2\pi iT_{\overline{K}/\mathbb{Q}_p}(z)),$$

where $T_{\overline{K}}$ denotes the trace function.

Let $Y$ be a closed analytic subset of $X$, i.e. a closed subset which is locally the common locus of zeroes of a finite number of $K$–analytic functions. We define

$$S_m(z, \Psi, Y, g) := \sum_{x \in Y_m} \Psi(zg(x)),$$

where $z \in K$, $m = -v(z) \geq 1$, and $g(x)$ is a non-zero analytic function on $Y$.

In this paper we discuss the estimation of exponential sums of type (1.1), in the case of $Y$ is a plane non-singular curve. We note that the reduction modulo $P$ of
Y, i.e. $Y_1$, may be a singular curve. In this case, for $|z|$ big enough, we obtain an estimation of type

$$|S_m(z, \Psi, Y, g)|_C \leq A(K, f, g)q^{m(1-\beta(f, g))},$$

where $|x|_C$ denotes the absolute value of a complex number $x$, and $A(K, f, g)$, $\beta(f, g)$ are positive constants, with $0 < \beta(f, g) < 1$ (cf. theorem 3.1).}

Character sum estimates of Weil-type has a long history especially summing over points on a curve defined over a finite field \([9], [1]\). A generalization of the results of the Weil and Bombieri to $p$-adic lifting of points over finite fields has been obtained by Kumar-Helleseth-Calderbrank \([6]\), \([3]\), \([5]\), and Voloch-Walker \([8]\), \([7]\). More precisely, the mentioned works contain estimates of the form

$$|S(z, \Psi, Y, g)|_C \leq Bp^m q^{\frac{m}{2}},$$

(1.3)

where

$$S(z, \Psi, Y, g) := \sum_{x} \Psi(zg(x)),$$

(1.4)

$m = -v(z)$, $q = p^n$, and $B$ is a constant depending on the function field of $Y$. The above estimate is good when $n$ is large comparing $m$. Our main result considers exponential sums along the points of the reduction modulo $P^m$ of a non-singular curve $Y$ in the case in which $n$ is fixed and $m$ is big enough.

In the case of $Y = R^n$, Igusa developed a complete theory for the exponential sums of type \((1.1)\) (see e.g. \([3]\), \([2]\)). For other analytic varieties no known results are available, as far as the author knows.

2. Preliminary results

In this section, to seek completeness, we collect some results that will be used in the next sections.

2.1. Serre’s measure. Suppose that $Y \subseteq X$ is non-singular closed analytic subset of dimension $d$ everywhere. The canonical immersion of $Y$ in $X$ induces on $Y$ a canonical measure $\mu_Y$, completely analog to volume measure of the real case. If

$$I = \{i_1, i_2, \ldots, i_d\},$$

with $i_1 < i_2 < \ldots < i_d$ is a subset of $d$ elements of $\{1, 2, \ldots, n\}$, we denote by $\omega_{Y,I}$ the differential form induced on $Y$ by $dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_d}$ and by $\alpha_{Y,I}$ the measure corresponding to $\omega_{Y,I}$. Then the canonical measure $\alpha_Y$ is defined as follows:

$$\alpha_Y := \sup_{I} \{\alpha_{Y,I}\},$$

where $I$ runs through all subsets of $d$ elements of $\{1, 2, \ldots, n\}$ (cf. \([1]\)).

If $Y$ is singular, i.e. if $Y$ is a closed analytic subset of $X$ of dimension $d$, we denote by $Y^{\text{reg}}$ the set of points on which $Y$ is smooth of dimension $d$. The canonical measure $\alpha_Y$ is defined on $Y^{\text{reg}}$ as a bounded measure (cf. \([1]\)). This allows us define the canonical measure in the singular case by

$$\alpha_Y(A) := \alpha_{Y^{\text{reg}}}(A \cap Y^{\text{reg}}),$$

for any $A \subseteq Y$. 

2 W.A. ZUNIGA-GALINDO
In [6] Serre showed that if $Y$ is closed non-singular analytic submanifold of dimension $d$ everywhere, then for $m$ big enough, it holds that

$$\text{Card}(Y_m) = \alpha_Y(Y)q^{md}. \quad (2.1)$$

Let $x_0$ be a fixed point of $Y_m$. The proof of (2.1) was accomplished by showing that

$$\alpha_Y(\{u \in Y \mid u \equiv x_0 \mod \pi^m\}) = \frac{1}{q^{md}}, \quad (2.2)$$

for $m$ big enough (cf. [6], page 347).

The following proposition follows directly from (2.2).

**Proposition 2.1.** Let $Y \subseteq X$ be a non-singular closed analytic curve, $z = \pi^{-m}u \in K, u \in R^\times$. Then for $|z|$ big enough, it holds that

$$S_m(z, \Psi, Y, g) = q^m \int_Y \Psi(zg(x))\alpha_Y(x). \quad (2.3)$$

For $Y$ singular, we do not know if there exists a relation between $S_m(z, \Psi, Y, g)$ and $\int_Y \Psi(zg(x))\alpha_Y(x)$. 

**2.2. Non-archimedean implicit function theorem.** A series 

$$g(x) = \sum_{i} c_i x^i \in K[[x]], x = (x_1, x_2, ..., x_n)$$

is named a special restricted power series, abbreviated as SRP, if $f(0) = 0$, and $c_i \in P^{i_1 + i_2 + \ldots + i_n},$ for all $i \neq 0$ in $\mathbb{N}^n$. This clearly implies that $f(x)$ in $R[[x]]$. Furthermore $f(x)$ is convergent at every $a \in R^n$.

**Lemma 2.1 ([3, Theorem 2.2.1]). (Implicit Function Theorem)** (i) If 

$$F_i(x, y) \in R[[x_1, \ldots, x_n, y_1, \ldots, y_m]],$$ 

$$F_i(0, 0) = 0 \text{ for all } i = 1, \ldots, m, F(x, y) = (F_1(x, y), \ldots, F_m(x, y))$$

and further 

$$\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}(0, 0) \neq 0 \text{ mod } \pi,$$

in which $\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}$ is the Jacobian determinant of the square matrix of size $m \times m$ with $\frac{\partial F_i}{\partial(y_j)}$ as its $(i,j)$-entry. Then there exists a unique $f(x) = (f_1(x), \ldots, f_m(x))$ with every $f_i(x)$ in $R[[x]]$ satisfying $f_i(0) = 0$, and $F(x, f(x)) = 0$.

(ii) If every $F_i(x, y)$ is an SRP in $x_1, \ldots, x_n, y_1, \ldots, y_m$, then every $f_i(x)$ is an SRP in $x_1, \ldots, x_n$. Furthermore if $a$ is in $R^n$, then $f(a)$ is in $R^m$ and $F(a, f(a)) = 0$, and if $(a, b)$ in $R^n \times R^m$ satisfies $F(a, b) = 0$, then $b = f(a)$. 

2.3. **Exponential sums along parametric curves.** If $A$ is open and compact set containing the origin and $h : A \to \mathbb{R}^2$ is an analytic mapping of the form

\[(2.4) \quad h(t) = (t, \eta(t)), \text{ with } \eta(t) = a_m t^m + \sum_{j=m+1}^{\infty} a_j t^j, \ m \geq 1, \text{ and } a_m \neq 0.\]

We call the set

\[(2.5) \quad V_{h,A}(\mathbb{R}) := \{(t, \eta(t)) \mid t \in A\},\]

a parametric curve at the origin.

Given a differential form $dx$, we denote by $|dx|$ the corresponding Haar measure, normalized such a way that the volume of $\mathbb{R}$ is 1.

If $V_{h,A}(\mathbb{R})$ is a parametric curve and $g$ a non-zero analytic function defined on an open set $U$ containing $A$, we define

\[\mu(h, g) := \text{ord}_t(g(t, \eta(t)) - g(0, 0)).\]

If $g(0, 0) = 0$, the number $\mu(h, g)$ is the intersection multiplicity at the origin of the analytic curves $V_{h,A}(\mathbb{R})$, and $V_{g,U}(\mathbb{R})$.

Given a real number $x$, we denote by $[x]$ the largest integer satisfying $[x] \leq x$.

**Lemma 2.2.** Let $V_{h,P^l}(\mathbb{R})$ be a parametric curve of type (2.3), $g(x, y)$ a non-constant analytic function such that $g(t, \eta(t)) = g(0, 0) + t^{\mu(h,g)} \alpha(t)$, $\alpha(0) = c_0 \in K^\times$, and $z = u\pi^{-m} \in K, u \in \mathbb{R}^\times$. If $l \geq \lceil \frac{m-\pi(c_0)}{\mu(h,g)} \rceil + 1$, then

\[|S_m(z, \Psi, V_{h,P^l}(\mathbb{R}), g)| \leq q^m (1 - \frac{1}{q^{l+1}})^{l+\frac{c_0}{\mu(h,g)}} \]

for $|z|$ big enough.

**Proof.** By proposition 2.1, for $|z|$ big enough

\[S_m(z, \Psi, V_{h,P^l}(\mathbb{R}), g)\]

can be expressed as an integral with respect to Serre’s measure as follows:

\[(2.6) \quad S_m(z, \Psi, V_{h,P^l}(\mathbb{R}), g) = q^m \int_{V_{h,P^l}(\mathbb{R})} \Psi(zg(x, y)) \alpha_{V_{h,P^l}}.\]

By using the fact that $V_{h,A}(\mathbb{R})$ is a parametric curve, we have that

\[(2.7) \quad S_m(z, \Psi, V_{h,P^l}(\mathbb{R}), g) = q^m \int_{P^l} \Psi(zg(t, \eta(t))) \mid dt \mid.\]

We put

\[(2.8) \quad g(t, \eta(t)) = g(0, 0) + t^{\mu(h,g)} \alpha(t),\]

with $\alpha(0) = c_0 \in K^\times$.

The integral in (2.7) admits the following expansion:
\[ S_m(z, \Psi, V_{h,p_\mu}(R), g) = q^m \Psi(zg(0,0)) \sum_{j=l}^{\infty} q^{-j} \int_{R^x} \Psi(z \pi^{j\mu(h,g)} \alpha(\pi^j t)) \, |dt| . \]

If \( l \geq \max \left[ \frac{m-v(c_0)}{\mu(h,g)} \right] + 1 \geq v(c_0) + 1 \), it holds that
\[ v(\alpha(\pi^j t)) = v(c_0), \text{ for every } t \in R^x, \text{ and } j \geq l, \]
and
\[ \Psi(z \pi^{j\mu(h,g)} \alpha(\pi^j t)) = 1, \text{ for every } t \in R^x, \text{ and } j \geq l. \]

Therefore from (2.9), (2.10), and (2.11), it follows that
\[ \left| S_m(z, \Psi, V_{h,p_\mu}(R), g) \right| \leq q^{m-l} \leq q^{m\left(1-\frac{1}{\mu(h,g)}}{\mu(h,g)} \right) + \frac{v(c_0)}{\mu(h,g)}. \]

3. Exponential sums along non-singular curves

In this section we discuss the estimation of exponential sums of type (1.1) along non-singular curves.

Let \( f : U \to K \) be an analytic function on an open and compact neighborhood \( U \subseteq K^2 \) of a point \((x_0, y_0) \in K^2\). By an \( K \)-analytic curve at \((x_0, y_0)\), we mean an analytic set of the form
\[ V_{f,U}(K) = \{(x, y) \in U \mid f(x, y) = 0\}. \]
We set \( V_{f,U}(R) := V_{f,U}(K) \cap R^2 \).

Let \( W \) be an open and compact set containing the origin, and \( V_{f,W}(K) \) an analytic curve at the origin, such that the origin is a smooth point, i.e.
\[ f(x, y) = ax + by + (\text{higher order terms}), \quad a \neq 0 \text{ or } b \neq 0. \]
Since \( f(x, y) \) is a convergent series, by multiplying it by a non-zero constant \( c_0 \in R \), we may suppose that \( f(x, y) \in R[[x, y]] \setminus P[[x, y]] \).

We define
\[ L(f, (0,0)) := \min\{v(a), v(b)\}, \]
with \( a, b \) as in (3.2). The Jacobian criteria implies that \( L(f, (0,0)) = 0 \) if and only if the origin is a non-singular point of the reduction modulo \( \pi \) of \( f(x, y) \).
If \( L(f, (0,0)) = v(b) \neq 0 \), by making a change of coordinates of the form \( x = \pi^{v(b)+1}x', y = \pi^{v(b)+1}y' \), we get that \( f(\pi^{v(b)+1}x', \pi^{v(b)+1}y') = \pi^{2v(b)+1}f(x', y') \) with \( L(f, (0,0)) = 0 \). Furthermore \( f(x', y') \) is an SRP.
Lemma 3.1. Let $V_{f, p^l, p^l}(R)$ be an analytic curve at the origin, such that the origin is a smooth point, and $g$ an analytic function such that $g |V_{f, p^l, p^l}(R) \neq 0$, with $g(0,0) = 0$. There exist constants $C(K, f, g), \gamma(f, g)$ such that if $l \geq \left\lfloor \frac{m-\gamma(f,g)}{\mu(f,g)} \right\rfloor + 1$, then

$$|S_m(z, \Psi, V_{f, p^l, p^l}(R), g)| \leq C(f, g)q^{m(1-\frac{1}{\mu(f,g)})},$$

for $|z| \text{ big enough.}$

Proof. We assume without loss of generality that $f(x,y) \in R[[x,y]] \setminus P[[x,y]]$, and that $L(f, (0,0)) = v(b) \geq 1$. Suppose that and

$$g(x,y) = g_D(x,y) + \text{ (higher order terms)},$$

with $g_D(x,y)$ a homogeneous polynomial of degree $D$. By proposition 2.1, for $|z|$ big enough

$$S_m(z, \Psi, V_{f, p^l, p^l}(R), g)$$

can be expressed as an integral with respect to Serre’s measure as follows:

$$S_m(z, \Psi, V_{f, p^l, p^l}(R), g) = \int_{V_{f, p^l}(R)} \Psi(zg(x,y))\alpha_{V_{f, p^l}, f, g}.$$

By making a change of coordinates of the form $x = \pi^{v(b)+1}x', y = \pi^{v(b)+1}y'$ in (3.4), and assuming that $l \geq m + 1 \geq L(f, (0,0) + 1$, we have that

$$S_m(z, \Psi, V_{f, p^l, p^l}(R), g) = q^{-v(b)-1}S_m(z \pi^{D(v(b)+1)}, V_{f^*, p^{1-v(b)-1} \times p^{1-v(b)-1}}(R), g^*),$$

where

$$f^*(x', y') = \pi^{-2v(b)+1}f(\pi^{v(b)+1}x', \pi^{v(b)+1}y'),$$

is an SRP, and

$$g^*(x', y') = \pi^{-D(v(b)+1)}g(\pi^{v(b)+1}x', \pi^{v(b)+1}y').$$

Since

$$L(f^*, (0,0)) = 0,$$

it holds that

$$\frac{\partial f^*(x', y')}{\partial y'}(0,0) \neq 0 \text{ mod } \pi.$$

Thus, by implicit function theorem (see lemma 2.1) $V_{f^*, p^{1-v(b)-1} \times p^{1-v(b)-1}}(R)$ is a parametric curve, i.e. there exists an SRP function $h(t)$ such that

$$V_{f^*, p^{1-v(b)-1} \times p^{1-v(b)-1}}(R) = \{(t, h(t)) | t \in P^{1-v(b)-1}\}.$$ 

Now, we set $g^* (h(t)) = t^{v(f,g)}\tilde{\beta}(t)$, with $\tilde{\beta}(0) := \gamma(f, g) \in K^\times$. The result follows from (3.3), and (3.6) by lemma 2.2.
We note that the previous lemma is valid if \( g(0, 0) \neq 0 \). Indeed, if \( g(x, y) = g(0, 0) + g'(x, y) \), with \( g'(0, 0) = 0 \), then
\[
S_m(z, \Psi, V_f, p^i \cdot p^i(R), g) = \Psi(zg(0, 0))S_m(z, \Psi, V_f, p^i \cdot p^i(R), g').
\]

Given a non-constant polynomial \( f(x, y) \in R[x, y] \), we define for every non-singular point \( P \in V_f(R) = \{(x, y) \in R^2 \mid f(x, y) = 0\} \), the number
\[
L(f, P) := \min\{v(\frac{\partial f}{\partial x}(P)), v(\frac{\partial f}{\partial y}(P))\}.
\]

This definition generalizes that given in (3.3).

**Proposition 3.1.** Let \( f(x, y) \in R[x, y] \) be a non-constant polynomial, such that \( V_f(R) \) does not have singular points on \( R^2 \). There is a positive constant \( c(f) \), depending only on \( f \) such that
\[
L(f, P) \leq c(f), \text{ for all } P \in V_f(R).
\]

**Proof.** By contradiction, we suppose that \( L(f, P) \) is not bounded on \( V_f(R) \). So there is a sequence \( (Q_i)_{i \in \mathbb{N}} \) of points of \( V_f(R) \) satisfying \( \lim L(f, Q_i) \to \infty \), when \( i \to \infty \). Since \( V_f(R) \) is a compact set, the sequence \( Q_i \) has a limit point \( \overline{Q} \in V_f(R) \). But \( \overline{Q} \) is a singular point of \( V_f(R) \), contradiction.

The numbers \( L(f, P) \) were introduced by A. Néron and they appear naturally in the computation of several p-adic integrals (see [10], [11] and the references therein).

Given two non-zero polynomial functions \( f(x, y), g(x, y) \) with coefficients in \( R \), such that \( g(0, 0) = f(0, 0) = 0 \), we define
\[
\sigma_f(g) := \sup_{P \in V_f(R)} \{\text{mult}_P(f, g)\} \in \mathbb{N} \setminus \{0\}.
\]

**Theorem 3.1.** Let \( f(x, y), g(x, y) \in R[x, y], \) be non-zero polynomials, with \( g(0, 0) = f(0, 0) = 0 \). If the curve \( V_f(R) \) is non-singular, then there exists a constant \( A(K, f, g) \) such that
\[
|S_m(z, \Psi, V_f(R), g)| \leq A(K, f, g) q^{m(1 - \frac{1}{\pi(q)})}.
\]

for \( |z| \) big enough.

**Proof.** We set a positive integer \( l = m + 1 \geq c(f) + 1 \), where \( c(f) \) is the constant whose existence was established in proposition 3.1.

The compact set \( V_f(R) \) admits the following covering:
\[
V_f(R) = \bigcup_{i=0}^{r} V_{f, D_i}(R),
\]
with
\[
V_{f, D_i}(R) := V_f(R) \cap (x_i, y_i) + P^l \times P^l,
\]
and \( Q_i = (x_i, y_i) \in V_f(R), \) \( i = 1, \ldots, r \). Because each \( V_{f, D_i}(R) \) is an analytic curve at \( (x_i, y_i) \), it follows from covering (3.8) that
By applying lemma 3.1 in (3.10), we have

\[ |S_m(z, \Psi, V_f(R), g)| \leq \sum_{i=1}^{r} |S_m(z, \Psi, V_{f_i}(R), g)|. \]  

The result follows from (3.11) by observing that \( \max_i \left \{ 1 - \frac{1}{\sigma(f)} \right \} \leq 1 - \frac{1}{\sigma(f)} \).

Let

\[ f(x) = \alpha_0 \left( \prod_{i=1}^{r} (x - \alpha_i)^{e_i} \right) Q(x) \in K[x] \]

be a non-constant polynomial in one variable, and \( Q(x) \) an irreducible polynomial of degree greater or equal two.

We put

\[ \sigma(f) := \max_i \{ e_i \}. \]  

The following corollary follows immediately from theorem 3.1.

**Corollary 3.1.** Let \( f(x) \) be a non-constant polynomial in one variable. Then with the above notation, it holds for \( |z| \) big enough that

\[ \left| \sum_{x \in R_m} \Psi(\alpha_0) \right| \leq A(K, f, g)q^{m(1 - \frac{1}{\sigma(f)})}. \]  

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