Supplementary Material: Analytical Description of Gain Depletion and Recovery in Quantum Dot Optical Amplifiers

Full analytical solution

Coefficients from the analytical solution from Eqs. (??)-(??)

- $p^{(3)}_{I,1}(t, \tau)$ valid for $t \geq 0$, $t \geq \tau$ and $\tau \geq 0$:
  
  \[ A_1^1(\tau) = c_0 \left( \sum_{l=0}^{p} c_{11}^l(\tau, T_2) - \eta_{110}(\tau) \right), \quad A_2^1(\tau) = c_0 c_{12}^l(\tau, T_2), \]
  
  \[ A_{3,4}^1(\tau) = -c_0 \eta_{111}(\tau), \quad A_{4,5}^1(\tau) = -c_0 \eta_{112}(\tau), \quad A_{5,6}^1(\tau) = -c_0 \eta_{113}(\tau). \]

- $p^{(3)}_{I,2}(t, \tau)$ valid for $t \geq 0$, $t \geq \tau$ and $\tau < 0$:
  
  \[ A_1^2(\tau) = c_0 \left( \sum_{l=0}^{p} c_{11}^l(\tau, T_2) - \eta_{120}(\tau) \right), \quad A_2^2(\tau) = -c_0 \eta_{121}(\tau). \]

- $p^{(3)}_{II}(t, \tau)$ valid for $t < 0$, $t < \tau$:
  
  \[ B_1^2(\tau) = c_0 c_{21}^l(\tau, T_2), \quad B_2^2(\tau) = -c_0 \eta_{2}^{l} \eta_{1}^{\prime}(\tau). \]

- $p^{(3)}_{III,1}(t, \tau)$ valid for $t \geq 0$, $t < \tau$ and $\tau \geq 0$:
  
  \[ C_1^1(\tau) = c_0 \sum_{l, \eta = 0, p} (c_{31}^0(\tau, T_2) - 2 \eta_{30}^{l} \eta_{30}^{\prime}(\tau)), \quad C_{12}^1(\tau) = -c_0 \eta_{34}(\tau), \]
  
  \[ C_{13}^1(\tau) = -c_0 \eta_{31}(\tau), \quad C_{14}^1(\tau) = -c_0 \eta_{33}(\tau), \quad C_{15}^1(\tau) = -c_0 \eta_{35}(\tau), \quad C_{16}^1(\tau) = -c_0 \eta_{36}(\tau), \quad C_{17}^1(\tau) = -c_0 \eta_{37}(\tau). \]

- $p^{(3)}_{III,2}(t, \tau)$ valid for $t < 0$, $t < \tau$ and $\tau \leq 0$:
  
  \[ C_{2,0}^1(\tau) = c_0 c_{32}^0(\tau, T_2) + \sum_{l=0, p} \eta_{36}^l(\tau), \quad C_{21}^1(\tau) = c_0 c_{33}^0(\tau, T_2) - \sum_{l=0, p} \eta_{37}^l(\tau), \]
  
  \[ C_{22}^1(\tau) = -c_0 \eta_{32}(\tau), \quad C_{23}^1(\tau) = -c_0 \eta_{33}(\tau), \quad C_{24}^1(\tau) = -c_0 \eta_{34}(\tau), \quad C_{25}^1(\tau) = -c_0 \eta_{35}(\tau), \quad C_{26}^1(\tau) = -c_0 \eta_{36}(\tau). \]
\* \( p_{III,2}^{(3)}(t, \tau) \) valid for \( t < 0, t \geq \tau \) and \( \tau < 0 \):

\[
C_{11}^2(\tau) = c_0 \sum_{l,l' = 0, p} (c_{41}^l(\tau, T_2) - \eta_{40}^{ll'}(\tau)), \quad C_{12}^2(\tau) = -c_0 \eta_{44}(\tau),
\]

\[
C_{2,0}^2(\tau) = c_0(c_{43}^0(\tau, T_2) - \sum_{l = 0, p} \eta_{47}^l(\tau)), \quad C_{2,p}^2(\tau) = c_0(c_{42}^p(\tau, T_2) - \sum_{l = 0, p} \eta_{46}^l(\tau)),
\]

\[
C_{31}^2(\tau) = -c_0 \eta_{42}(\tau), \quad C_{32}^2(\tau) = -c_0 \eta_{41}(\tau),
\]

\[
C_{31}^2(\tau) = -c_0 \eta_{42}(\tau), \quad C_{32}^2(\tau) = -c_0 \eta_{41}(\tau),
\]

\[
C_{51}^2(\tau) = -c_0 \eta_{49}(\tau), \quad C_{52}^2(\tau) = -c_0 \eta_{45}(\tau),
\]

\[
C_{6,0}^2(\tau) = -c_0 \eta_{40}^0(\tau), \quad C_{6,p}^2(\tau) = -c_0 \eta_{47}^p(\tau),
\]

\[
C_{7}^2(\tau) = -c_0 \gamma_{41}(\tau) \Omega_p.
\]

Here, the abbreviation \( c_0 = -i(f_{0,e} + f_{0,h} - 1) \) has been introduced. The above coefficients consist of contributions stemming from different orders in the \( \chi^{(3)} \) expansion.

In the following the first order contributions (cp. Eq.(??)) are listed:

\[
c_{11}^0(\tau, T) = (\beta_{21}^0(T) - \beta_{11}^0(T)) \Omega_0, \quad c_{11}^p(\tau, T) = (\beta_{21}^p(T) - \beta_{11}^p(T)) e^{\hat{\tau} \Omega_p},
\]

\[
c_{12}^l(\tau, T) = \beta_{11}(T) \hat{\Omega}_l, \quad c_{21}^l(\tau, T) = \beta_{21}(T) \hat{\Omega}_l,
\]

\[
c_{31}^l(\tau, T) = (\beta_{21}(T) - \beta_{11}(T)) \Omega_i, \quad c_{32}^l(\tau, T) = \beta_{11}(T) \Omega_i, \quad c_{33}^l(\tau, T) = \beta_{21}(T) \Omega_i e^{-\frac{\pi}{4}},
\]

\[
c_{41}^l(\tau, T) = (\beta_{21}(T) - \beta_{11}(T)) \Omega_i e^{\frac{\pi}{4}}, \quad c_{42}^l(\tau, T) = \beta_{21}(T) \Omega_i, \quad c_{43}^l(\tau, T) = \beta_{11}(T) \Omega_i e^{\frac{\pi}{4}}
\]

with the pump and the probe part of the Rabi frequency: \( \Omega_p = \frac{d_{p} E_p}{\hbar} e^{-i\varphi_{p0}} \) and \( \Omega_0 = \frac{d_{p} E_0}{\hbar} e^{-i\varphi_{r0}} \), respectively. Furthermore, the abbreviations \( \hat{\Omega}_l = \Omega_l \left( e^{\frac{\pi i}{80} \delta_{l,0} + 1} \right) \) and \( \hat{\Omega}_l = \Omega_l \left( e^{-\frac{\pi i}{80} \delta_{l,0} + 1} \right) \) have been introduced.
The second order contributions read (cp. Eq. (??)):

\[ \gamma_{10}(\tau) = \sum_{l=0,p} \gamma_{l2}(\tau) + \sum_{b=e,h} \left\{ \left[ c_{31}^0 (\tau, T_2) \left( e^{\frac{\tau}{\Omega_0}} - 1 \right) \Omega_0 \beta_{11}^0 (T_{0,b}) + \right. \right. \]
\[ \left. \left. c_{31}^p (\tau, T_2) \left( e^{\frac{\tau}{\Omega_0}} - 1 \right) \Omega_p \beta_{21}^p (T_{0,b}) e^{\frac{\tau}{\Omega_p}} \right] - \sum_{l=0,p} \left( \gamma_{112}(\tau) e^{\frac{\tau}{\Omega_1(T_{1,b})}} + \gamma_{111}(\tau) e^{\frac{\tau}{\Omega_1(T_{0,b})}} \right) + \right. \]
\[ \left. \left[ c_{32}^0 (\tau, T_2) \left( e^{\frac{\tau}{\Omega_0}} - 1 \right) \Omega_0 \beta_{12}^0 (T_{1,b}) + c_{32}^p (\tau, T_2) \left( e^{\frac{\tau}{\Omega}} - 1 \right) T_{1,0} \Omega_p e^{\frac{\tau}{\Omega_p}} \right] + \right. \]
\[ \left. \left[ c_{33}^0 (\tau, T_2) \left( e^{\frac{\tau}{\Omega_0}} - 1 \right) \Omega_0 \beta_{22}^0 (T_{1,b}) e^{\frac{\tau}{\Omega_p}} + c_{33}^0 (\tau, T_2) \left( e^{\frac{\tau}{\Omega_1}} - 1 \right) T_{1,0} \Omega_0 \right] \right\}. \]

\[ \gamma_{20}(\tau) = \sum_{l=0,p} \gamma_{l2}(\tau) + \sum_{b=e,h} \left\{ \left[ c_{41}^0 (\tau, T_2) \left( 1 - e^{\frac{\tau}{\Omega_0}} \right) \Omega_0 \beta_{21}^0 (T_{0,b}) + \right. \right. \]
\[ \left. \left. c_{41}^p (\tau, T_2) \left( 1 - e^{\frac{\tau}{\Omega_0}} \right) \Omega_p \beta_{21}^p (T_{0,b}) e^{\frac{\tau}{\Omega_p}} \right] - \sum_{l=0,p} \left( \gamma_{111}(\tau) + \gamma_{112}(\tau) \right) + \right. \]
\[ \left. \left[ c_{42}^0 (\tau, T_2) \left( 1 - e^{\frac{\tau}{\Omega_0}} \right) \Omega_0 \beta_{22}^0 (T_{1,b}) + c_{42}^p (\tau, T_2) \left( 1 - e^{\frac{\tau}{\Omega_1}} \right) T_{1,0} \Omega_p e^{\frac{\tau}{\Omega_p}} \right] + \right. \]
\[ \left. \left[ c_{43}^0 (\tau, T_2) \left( 1 - e^{\frac{\tau}{\Omega_0}} \right) \Omega_0 \beta_{22}^0 (T_{1,b}) e^{\frac{\tau}{\Omega_p}} + c_{43}^0 (\tau, T_2) \left( 1 - e^{\frac{\tau}{\Omega_1}} \right) T_{1,0} \Omega_0 \right] \right\}. \]

\[ \gamma_{111}^l (\tau) = c_{11}^l (\tau, T_2) \sum_{b=e,h} c_{12}^l (\tau, T_{0,b}), \quad \gamma_{112}^l (\tau) = c_{12}^l (\tau, T_2) \sum_{b=e,h} c_{12}^l (\tau, \beta_{11}^l (T_{1,b})) \text{ with } T_{0,b} = \frac{T_{1,0} T_2}{T_2 - T_{1,0}}, \]
\[ \gamma_2(\tau) = c_{21}^l (\tau, T_2) \sum_{b=e,h} c_{21}^l (\tau, \beta_{11}^l (T_{1,b})), \quad \gamma_{30}(\tau) = \gamma_2(\tau) - \sum_{i=1}^6 \gamma_{3i}(\tau), \]
\[ \gamma_{31}(\tau) = c_{31}^0 (\tau, T_2) \sum_{b=e,h} \beta_{11}^0 (T_{0,b}) \Omega_0 e^{\frac{\tau}{\Omega_0}}, \quad \gamma_{32}(\tau) = c_{32}^0 (\tau, T_2) \sum_{b=e,h} \beta_{12}^0 (T_{1,b}) \Omega_0, \]
\[ \gamma_{33}(\tau) = c_{33}^0 (\tau, T_2) \sum_{b} T_{1,b} \Omega_0 e^{\frac{\tau}{\Omega_0}}, \quad \gamma_{34}(\tau) = c_{34}^0 (\tau, T_2) \sum_{b} \beta_{21}^0 (T_{0,b}) \Omega_p, \]
\[ \gamma_{35}(\tau) = c_{35}^0 (\tau, T_2) \sum_{b} T_{1,b} \Omega_p, \quad \gamma_{36}(\tau) = c_{36}^0 (\tau, T_2) \sum_{b} \beta_{21}^0 (T_{0,b}) \Omega_p e^{\frac{\tau}{\Omega_p}}. \]

\[ 5p[C] \gamma_{41}(\tau) = c_{41}^0 (\tau, T_2) \sum_{b=e,h} \beta_{21}^0 (T_{0,b}) \Omega_0, \quad \gamma_{42}(\tau) = c_{42}^0 (\tau, T_2) \sum_{b=e,h} \beta_{22}^0 (T_{1,b}) \Omega_0, \]
\[ \gamma_{43}(\tau) = c_{43}^0 (\tau, T_2) \sum_{b} T_{1,b} \Omega_0, \quad \gamma_{44}(\tau) = c_{44}^0 (\tau, T_2) \sum_{b} \beta_{11}^0 (T_{0,b}) \Omega_p e^{\frac{\tau}{\Omega_p}}. \]
\[ \gamma_5(\tau) = c_{42}^p(\tau, T_2) \sum_{b=e,h} T_{1,b} \Omega_p e^{\frac{\tau}{\alpha}}, \quad \gamma_6(\tau) = c_{43}^p(\tau, T_2) \sum_{b=e,h} \beta_{12}^p(T_{1,b}) \Omega_p e^{\frac{\tau}{\alpha}}, \]

\[ [5 pt] \gamma_{40}(\tau) = \sum_{b=e,h} \left\{ \sum_{l=0,p} \gamma_2^l(\tau) e^{\frac{\tau}{\beta_{22}^l(T_{1,b})}} - \gamma_4(\tau) e^{\frac{\tau}{\beta_{22}^l(T_{0,b})}} - \gamma_5(\tau) e^{\frac{\tau}{\beta_{22}^l(T_{1,b})}} - \gamma_6(\tau) e^{\frac{\tau}{\beta_{22}^l(T_{1,b})}} \right\} \]
Finally, the third order contributions (cp. Eq. (??)) read:

\[ \eta_{110}(\tau) = \sum_{l,l'=0,p} \eta^{l,l'}_{12}(\tau) + \sum_{b=e,h} \left\{ \eta_{31}(\tau)(e^{-\frac{\tau}{\beta_{21}(T_{0,b})}} - 1) + \eta_{32}(\tau)(e^{-\frac{\tau}{\beta_{11}(T_{0,b})}} - 1) + \eta_{33}(\tau)(e^{-\frac{\tau}{\omega_{0}}} - 1) + \eta_{34}(\tau) + \eta_{35}(\tau)(e^{\frac{\tau}{\beta_{25}(T_{2})}} - 1) + \eta_{36}^{0}(\tau)(e^{\frac{\tau}{\beta_{11}(T_{2})}} - 1) + \eta_{37}^{0}(\tau)(e^{\frac{\tau}{\beta_{21}(T_{2})}} - 1) + \eta_{38}(\tau)(e^{\frac{\tau}{\omega_{p}}} - 1) + \eta_{39}(\tau)(e^{\frac{\tau}{\omega_{p}}} - 1) - \sum_{l=0,p} (\eta_{111}^{l}(\tau)e^{-\frac{\tau}{\beta_{21}(T_{0,b})}} - \eta_{112}^{l}(\tau)e^{-\frac{\tau}{\omega_{0}}} - \eta_{113}^{l}(\tau)e^{\frac{\tau}{\beta_{13}(T_{2})}}) \right\}, \]

\[ \eta_{120}(\tau) = \sum_{l,l'=0,p} \eta^{l,l'}_{12}(\tau)e^{\frac{\tau}{\beta_{23}(T_{2})}} + \sum_{b=e,h} \left\{ \eta_{41}(\tau)(1 - e^{-\frac{\tau}{\beta_{11}(T_{0,b})}}) + \eta_{42}(\tau)(1 - e^{-\frac{\tau}{\beta_{11}(T_{0,b})}}) - \eta_{44}(\tau) + \eta_{43}(\tau)(1 - e^{\frac{\tau}{\omega_{0}}}) + \eta_{45}(\tau)(1 - e^{\frac{\tau}{\beta_{25}(T_{2})}}) + \eta_{46}^{0}(\tau)(1 - e^{\frac{\tau}{\beta_{11}(T_{2})}}) + \sum_{l=0,p} \eta_{47}(\tau)(1 - e^{\frac{\tau}{\beta_{11}(T_{2})}}) + \eta_{48}(\tau)(1 - e^{\frac{2\tau}{\omega_{p}}}) + \eta_{49}(\tau)(1 - e^{\frac{\tau}{\beta_{25}(T_{2})}}) - \sum_{l,l'=0,p} (\eta_{121}^{l}(\tau) - \eta_{112}^{l}(\tau) - \eta_{113}^{l}(\tau)) \right\}, \]

\[ \eta_{111}^{l}(\tau) = -\gamma_{110}(\tau)\tilde{\Omega}_{l} \sum_{b=e,h} \beta_{21}^{l}(T_{0,b}), \quad \eta_{112}^{l}(\tau) = -\frac{1}{2} \gamma_{111}(\tau)\tilde{\Omega}_{l} \alpha_{l}, \quad \eta_{113}^{l}(\tau) = \gamma_{112}(\tau)\beta_{22}^{l}(T_{2}), \]

\[ \eta_{121}^{l}(\tau) = -\gamma_{120}(\tau)\tilde{\Omega}_{l} \beta_{21}^{l}(T_{0,b}), \quad \eta_{22}^{l}(\tau) = \gamma_{22}(\tau)\beta_{22}^{l}(T_{2}), \quad \eta_{30}^{l}(\tau) = \eta_{22}^{l}(\tau)(T_{2}) - \sum_{i=1}^{9} \eta_{3i}(\tau) \]

\[ \eta_{31}(\tau) = -\gamma_{31}^{0}(\tau) \sum_{b=e,h} \beta_{21}^{0}(T_{0,b})\Omega_{0}, \quad \eta_{32}(\tau) = -\gamma_{32}^{0}(\tau)e^{\frac{\tau}{\omega_{p}}} \sum_{b=e,h} \beta_{11}^{0}(T_{0,b})\Omega_{p}, \]

\[ \eta_{33}(\tau) = -\frac{1}{2} \gamma_{31}(\tau)\alpha_{0}\Omega_{0}, \quad \eta_{34}(\tau) = \gamma_{34}(\tau)e^{\frac{\tau}{\omega_{0}}}\Omega_{0} + \gamma_{31}(\tau)\Omega_{p}, \quad \eta_{35}(\tau) = \gamma_{32}(\tau)\beta_{13}^{0}(T_{2})\Omega_{0}, \]

\[ \eta_{36}^{0}(\tau) = \gamma_{32}(\tau)e^{\frac{\tau}{\omega_{p}}}\Omega_{p}\beta_{11}^{0}(T_{2}), \quad \eta_{36}^{0}(\tau) = (\gamma_{33}(\tau) + \gamma_{35}(\tau))\Omega_{0}\beta_{11}^{0}(T_{2}), \]

\[ \eta_{37}^{0}(\tau) = (\gamma_{33}(\tau) + \gamma_{35}(\tau))e^{\frac{\tau}{\omega_{p}}}\Omega_{p}\beta_{21}^{0}(T_{2}), \quad \eta_{37}^{0}(\tau) = \gamma_{36}(\tau)\Omega_{0}\beta_{21}^{0}(T_{2}), \quad \eta_{38}(\tau) = \frac{1}{2} \gamma_{34}(\tau)e^{\frac{\tau}{\omega_{p}}}\Omega_{p}\alpha_{p}, \quad \eta_{39}(\tau) = \gamma_{36}(\tau)\beta_{23}^{0}(T_{2})e^{\frac{\tau}{\omega_{p}}}\Omega_{p}, \]
\[ \eta_{41}(\tau) = -\gamma_{40}(\tau) \sum_{b=e,h} \beta_{11}^0(T_{0,b}) \Omega_0, \quad \eta_{42}(\tau) = -\gamma_{40}(\tau) e^{\frac{\tau}{\alpha_p}} \sum_{b=e,h} \beta_{21}^p(T_{0,b}) \Omega_p, \]

\[ \eta_{43}(\tau) = \frac{1}{2} \gamma_{41}(\tau) \alpha_0 \Omega_0, \quad \eta_{44}(\tau) = \gamma_{41}(\tau) \Omega_p + \gamma_{44}(\tau) e^{\frac{\tau}{\alpha_0}} \Omega_0, \quad \eta_{45}(\tau) = \gamma_{42}(\tau) \beta_{23}^0(T_2) \Omega_0, \]

\[ \eta_{46}^p(\tau) = \gamma_{42}(\tau) e^{\frac{\tau}{\alpha_p}} \Omega_p \beta_{21}^p(T_2), \quad \eta_{46}^0(\tau) = (\gamma_{43}(\tau) + \gamma_{45}(\tau)) \Omega_0 \beta_{21}^0(T_2), \]

\[ \eta_{47}(\tau) = \gamma_{46}(\tau) \Omega_0 \beta_{11}^0(T_2), \quad \eta_{47}^p(\tau) = (\gamma_{43}(\tau) + \gamma_{45}(\tau)) e^{\frac{\tau}{\alpha_p}} \Omega_p \beta_{11}^p(T_2), \]

\[ \eta_{48}(\tau) = -\frac{1}{2} \gamma_{44}(\tau) e^{\frac{\tau}{\alpha_p}} \Omega_p \alpha_p, \quad \eta_{49}(\tau) = \gamma_{46}(\tau) \beta_{13}^p(T_2) e^{\frac{\tau}{\alpha_p}} \Omega_p, \]

\[ \eta_{40}(\tau) = \sum_{l,l'=0,p} \eta_{l,l'}^2(\tau) e^{\frac{\tau}{\beta_{21}^0(T_2)}} + \sum_{b=e,h} \left\{ -\eta_{41}(\tau) e^{-\frac{\tau}{\beta_{11}^0(T_{0,b})}} - \eta_{42}(\tau) e^{-\frac{\tau}{\beta_{21}^p(T_{0,b})}} - \eta_{43}(\tau) e^{\frac{2\tau}{\alpha_0}} - \eta_{44}(\tau) \tau ight. \\
\left. - \eta_{45}(\tau) e^{\frac{2\tau}{\alpha_p}} - \sum_{l=0,p} \left( \eta_{46}^l(\tau) e^{\frac{\tau}{\beta_{21}^0(T_2)}} - \eta_{47}^l(\tau) e^{\frac{\tau}{\beta_{11}^0(T_2)}} \right) - \eta_{48}(\tau) e^{-\frac{2\tau}{\alpha_p}} - \eta_{49}(\tau) e^{\frac{2\tau}{\beta_{13}^p(T_2)}} \right\}. \]