ON FEEBLY COMPACT SHIFT-CONTINUOUS TOPOLOGIES ON THE SEMILATTICE $\exp_n \lambda$

OLEG GUTIK AND OLEKSANDRA SOBOL

Abstract. We study feebly compact topologies $\tau$ on the semilattice $(\exp_n \lambda, \cap)$ such that $(\exp_n \lambda, \tau)$ is a semitopological semilattice and prove that for any shift-continuous $T_1$-topology $\tau$ on $\exp_n \lambda$ the following conditions are equivalent: (i) $\tau$ is countably pracompact; (ii) $\tau$ is feebly compact; (iii) $\tau$ is $d$-feebly compact; (iv) $(\exp_n \lambda, \tau)$ is an $H$-closed space.

Dedicated to the memory of Professor Vitaly Sushchansky

We shall follow the terminology of [6, 8, 9, 13]. If $X$ is a topological space and $A \subseteq X$, then by $\text{cl}_X(A)$ and $\text{int}_X(A)$ we denote the closure and the interior of $A$ in $X$, respectively. By $\omega$ we denote the first infinite cardinal and by $\mathbb{N}$ the set of positive integers.

A subset $A$ of a topological space $X$ is called regular open if $\text{int}_X(\text{cl}_X(A)) = A$.

We recall that a topological space $X$ is said to be

- quasiregular if for any non-empty open set $U \subset X$ there exists a non-empty open set $V \subset U$ such that $\text{cl}_X(V) \subseteq U$;
- semiregular if $X$ has a base consisting of regular open subsets;
- compact if each open cover of $X$ has a finite subcover;
- countably compact if each open countable cover of $X$ has a finite subcover;
- countably compact at a subset $A \subseteq X$ if every infinite subset $B \subseteq A$ has an accumulation point $x$ in $X$;
- countably pracompact if there exists a dense subset $A$ in $X$ such that $X$ is countably compact at $A$;
- feebly compact (or lightly compact) if each locally finite open cover of $X$ is finite [3];
- $d$-feebly compact (or $\text{DFCC}$) if every discrete family of open subsets in $X$ is finite (see [12]);
- pseudocompact if $X$ is Tychonoff and each continuous real-valued function on $X$ is bounded.

According to Theorem 3.10.22 of [8], a Tychonoff topological space $X$ is feebly compact if and only if $X$ is pseudocompact. Also, a Hausdorff topological space $X$ is feebly compact if and only if every locally finite family of non-empty open subsets of $X$ is finite [3]. Every compact space and every sequentially compact space are countably compact, every countably compact space is countably pracompact, and every countably pracompact space is feebly compact (see [2]), and every $H$-closed space is feebly compact too (see [10]). Also, it is obvious that every feebly compact space is $d$-feebly compact.

A semilattice is a commutative semigroup of idempotents. On a semilattice $S$ there exists a natural partial order: $e \leq f$ if and only if $ef = fe = e$. For any element $e$ of a semilattice $S$ we put

$$\uparrow e = \{ f \in S : e \leq f \}.$$ 

A topological (semitopological) semilattice is a topological space together with a continuous (separately continuous) semilattice operation. If $S$ is a semilattice and $\tau$ is a topology on $S$ such that $(S, \tau)$ is a topological semilattice, then we shall call $\tau$ a semilattice topology on $S$, and if $\tau$ is a topology on $S$ such that $(S, \tau)$ is a semitopological semilattice, then we shall call $\tau$ a shift-continuous topology on $S$.

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For an arbitrary positive integer \( n \) and an arbitrary non-zero cardinal \( \lambda \) we put

\[
\exp_n \lambda = \{ A \subseteq \lambda : |A| \leq n \}.
\]

It is obvious that for any positive integer \( n \) and any non-zero cardinal \( \lambda \) the set \( \exp_n \lambda \) with the binary operation \( \cap \) is a semilattice. Later in this paper by \( \exp_n \lambda \) we shall denote the semilattice \( (\exp_n \lambda, \cap) \).

This paper is a continuation of \([11]\) where we study feebly compact topologies \( \tau \) on the semilattice \( \exp_n \lambda \) such that \( (\exp_n \lambda, \tau) \) is a semitopological semilattice. Therein, all compact semilattice \( T_1 \)-topologies on \( \exp_n \lambda \) were described. In \([11]\) it was proved that for an arbitrary positive integer \( n \) and an arbitrary infinite cardinal \( \lambda \) every \( T_1 \)-semitopological countably compact semilattice \( (\exp_n \lambda, \tau) \) is a compact topological semilattice. Also, there we construct a countably pracompact \( H \)-closed quasiregular non-semiregular topology \( \tau^2_k \) such that \( (\exp_2 \lambda, \tau^2_k) \) is a semitopological semilattice with the discontinuous semilattice operation and show that for an arbitrary positive integer \( n \) and an arbitrary infinite cardinal \( \lambda \) a semiregular feebly compact semitopological semilattice \( \exp_n \lambda \) is a compact topological semilattice.

In this paper we show that for any shift-continuous \( T_1 \)-topology \( \tau \) on \( \exp_n \lambda \) the following conditions are equivalent: (i) \( \tau \) is countably pracompact; (ii) \( \tau \) is feebly compact; (iii) \( \tau \) is \( d \)-feebly compact; (iv) \( (\exp_n \lambda, \tau) \) is an \( H \)-closed space.

The proof of the following lemma is similar to Lemma 4.5 of \([5]\) or Proposition 1 from \([11]\).

**Lemma 1.** Every Hausdorff \( d \)-feebly compact topological space with a dense discrete subspace is countably pracompact.

We observe that by Proposition 1 from \([11]\) for an arbitrary positive integer \( n \) and an arbitrary infinite cardinal \( \lambda \) every shift-continuous \( T_1 \)-topology \( \tau \) on \( \exp_n \lambda \) is functionally Hausdorff and quasiregular, and hence it is Hausdorff.

**Proposition 1.** Let \( n \) be an arbitrary positive integer and \( \lambda \) be an arbitrary infinite cardinal. Then for every \( d \)-feebly compact shift-continuous \( T_1 \)-topology \( \tau \) on \( \exp_n \lambda \) the subset \( \exp_n \lambda \setminus \exp_{n-1} \lambda \) is dense in \( (\exp_n \lambda, \tau) \).

**Proof.** Suppose to the contrary that there exists a \( d \)-feebly compact shift-continuous \( T_1 \)-topology \( \tau \) on \( \exp_n \lambda \) such that \( \exp_n \lambda \setminus \exp_{n-1} \lambda \) is not dense in \( (\exp_n \lambda, \tau) \). Then there exists a point \( x \in \exp_{n-1} \lambda \) of the space \( (\exp_n \lambda, \tau) \) such that \( x \notin \text{cl}_{\exp_n \lambda}(\exp_n \lambda \setminus \exp_{n-1} \lambda) \). This implies that there exists an open neighbourhood \( U(x) \) of \( x \) in \( (\exp_n \lambda, \tau) \) such that \( U(x) \cap (\exp_n \lambda \setminus \exp_{n-1} \lambda) = \emptyset \). The definition of the semilattice \( \exp_n \lambda \) implies that every maximal chain in \( \exp_n \lambda \) is finite and hence there exists a point \( y \in U(x) \) such that \( \uparrow y \cap U(x) = \{ y \} \). By Proposition 1(iii) from \([11]\), \( \uparrow y \) is an open-and-closed subset of \( (\exp_n \lambda, \tau) \) and hence \( \uparrow y \) is a \( d \)-feebly compact subspace of \( (\exp_n \lambda, \tau) \).

It is obvious that the subspace \( \uparrow y \) of \( \exp_n \lambda \) is algebraically isomorphic to the semilattice \( \exp_k \lambda \) for some positive integer \( k \leq n \). This and above arguments imply that without loss of generality we may assume that \( y \) is the isolated zero of the \( d \)-feebly compact semitopological semilattice \( (\exp_n \lambda, \tau) \).

Hence we assume that \( \tau \) is a \( d \)-feebly compact shift-continuous topology on \( \exp_n \lambda \) such that the zero \( 0 \) of \( \exp_n \lambda \) is an isolated point of \( (\exp_n \lambda, \tau) \). Next we fix an arbitrary infinite sequence \( \{ x_i \}_{i \in \mathbb{N}} \) of distinct elements of cardinal \( \lambda \). For every positive integer \( j \) we put

\[
a_j = \{ x_{n(j-1)+1}, x_{n(j-1)+2}, \ldots, x_{nj} \}.
\]

Then \( a_j \in \exp_n \lambda \) and moreover \( a_j \) is a greatest element of the semilattice \( \exp_n \lambda \) for each positive integer \( j \). Also, the definition of the semilattice \( \exp_n \lambda \) implies that for every non-zero element \( a \) of \( \exp_n \lambda \) there exists at most one element \( a_j \) such that \( a_j \in \uparrow a \). Then for every positive integer \( j \) by Proposition 1(iii) of \([11]\), \( a_j \) is an isolated point of \( (\exp_n \lambda, \tau) \), and hence the above arguments imply that \( \{ a_1, a_2, \ldots, a_j, \ldots \} \) is an infinite discrete family of open subset in the space \( (\exp_n \lambda, \tau) \). This contradicts the \( d \)-feebly compactness of the semitopological semilattice \( (\exp_n \lambda, \tau) \). The obtained contradiction implies the statement of our proposition.

\[\square\]
The following example show that the converse statement to Proposition [11] is not true in the case of topological semilattices.

**Example 1.** Fix an arbitrary cardinal \( \lambda \) and an infinite subset \( A \) in \( \lambda \) such that \( |\lambda \setminus A| \geq \omega \). By \( \pi : \lambda \rightarrow \exp_1 \lambda : a \mapsto \{ a \} \) we denote the natural embedding of \( \lambda \) into \( \exp_1 \lambda \). On \( \exp_1 \lambda \) we define a topology \( \tau_{dm} \) in the following way:

1. all non-zero elements of the semilattice \( \exp_1 \lambda \) are isolated points in \( (\exp_1 \lambda, \tau_{dm}) \); and
2. the family \( \mathcal{B}_{dm} = \{ U_B = \{ 0 \} \cup \pi(B) : B \subseteq A \text{ and } A \setminus B \text{ is finite} \} \) is the base of the topology \( \tau_{dm} \) at zero 0 of \( \exp_1 \lambda \).

Simple verifications show that \( \tau_{dm} \) is a Hausdorff locally compact semilattice topology on \( \exp_1 \lambda \) which is not compact and hence by Corollary 8 of [11] it is not feebly compact.

**Remark 1.** We observe that in the case when \( \lambda = \omega \) by Proposition 13 of [11] the topological space \( (\exp_1 \lambda, \tau_{dm}) \) is collectionwise normal and it has a countable base, and hence \( (\exp_1 \lambda, \tau_{dm}) \) is metrizable by the Urysohn Metrization Theorem [14]. Moreover, if \( |B| = \omega \) then the space \( (\exp_1 \lambda, \tau_{dm}) \) is metrizable for any infinite cardinal \( \lambda \), as a topological sum of the metrizable space \( (\exp_1 \omega, \tau_{dm}) \) and the discrete space of cardinality \( \lambda \).

**Remark 2.** If \( n \) is an arbitrary positive integer \( \geq 3 \), \( \lambda \) is any infinite cardinal and \( \tau^n_c \) is the unique compact semilattice topology on the semilattice \( \exp_n \lambda \) defined in Example 4 of [11], then we construct more stronger topology \( \tau^n_{dm} \) on \( \exp_n \lambda \) then \( \tau^n_c \) in the following way. Fix an arbitrary element \( x \in \exp_n \lambda \) such that \( |x| = n - 1 \). It is easy to see that the subsemilattice \( \uparrow x \) of \( \exp_n \lambda \) is isomorphic to \( \exp_1 \lambda \), and by \( h : \exp_1 \lambda \rightarrow \uparrow x \) we denote this isomorphism.

Fix an arbitrary subset \( A \) in \( \lambda \) such that \( |\lambda \setminus A| \geq \omega \). For every zero element \( y \in \exp_n \lambda \setminus \uparrow x \) we assume that the base \( \mathcal{B}^n_{dm}(y) \) of the topology \( \tau^n_{dm} \) at the point \( y \) coincides with the base of the topology \( \tau^n_{c} \) at \( y \), and assume that \( \uparrow x \) is an open-and-closed subset and the topology on \( \uparrow x \) is generated by the map \( h : (\exp_2 \lambda, \tau^2_{c}) \rightarrow \uparrow x \). We observe that \( (\exp_n \lambda, \tau^n_{dm}) \) is a Hausdorff locally compact topological space, because it is the topological sum of a Hausdorff locally compact space \( \uparrow x \) (which is homeomorphic to the Hausdorff locally compact space \( (\exp_1 \lambda, \tau_{dm}) \) from Example [11] and an open-and-closed subspace \( \exp_n \lambda \setminus \uparrow x \) of \( (\exp_n \lambda, \tau^n_{dm}) \). It is obvious that the set \( \exp_n \lambda \setminus \exp_{n-1} \lambda \) is dense in \( (\exp_n \lambda, \tau^n_{dm}) \). Also, since \( \uparrow x \) is an open-and-closed subsemilattice with zero \( x \) of \( (\exp_n \lambda, \tau^n_{dm}) \), the continuity of the semilattice operations in \( (\exp_n \lambda, \tau^n_{dm}) \) and \( (\exp_n \lambda, \tau^n_{c}) \) and the property that the topology \( \tau^n_{dm} \) is more stronger than \( \tau^n_{c} \), imply that \( (\exp_n \lambda, \tau^n_{dm}) \) is a topological semilattice. Moreover, the space \( (\exp_n \lambda, \tau^n_{dm}) \) is not \( d \)-feebly compact, because it contains an open-and-closed non-\( d \)-feebly compact subspace \( \uparrow x \).

Arguments presented in the proof of Proposition [11] and Proposition 1(iii) of [11] imply the following corollary.

**Corollary 1.** Let \( n \) be an arbitrary positive integer and \( \lambda \) be an arbitrary infinite cardinal. Then for every \( d \)-feebly compact shift-continuous \( T_1 \)-topology \( \tau \) on \( \exp_n \lambda \) a point \( x \) is isolated in \( (\exp_n \lambda, \tau) \) if and only if \( x \in \exp_n \lambda \setminus \exp_{n-1} \lambda \).

**Remark 3.** We observe that the example presented in Remark 2 implies there exists a locally compact non-\( d \)-feebly compact semitopological semilattice \( (\exp_n \lambda, \tau^n_{dm}) \) with the following property: a point \( x \) is isolated in \( (\exp_n \lambda, \tau^n_{dm}) \) if and only if \( x \in \exp_n \lambda \setminus \exp_{n-1} \lambda \).

The following proposition gives an amazing property of the system of neighbourhood of zero in a \( T_1 \)-feebly compact semitopological semilattice \( \exp_n \lambda \).

**Proposition 2.** Let \( n \) be an arbitrary positive integer, \( \lambda \) be an arbitrary infinite cardinal and \( \tau \) be a shift-continuous feebly compact \( T_1 \)-topology on the semilattice \( \exp_n \lambda \). Then for every open neighbourhood \( U(0) \) of zero 0 in \( (\exp_n \lambda, \tau) \) there exist finitely many \( x_1, \ldots, x_m \in \lambda \) such that

\[
\exp_n \lambda \setminus cl_{\exp_n \lambda}(U(0)) \subseteq \uparrow x_1 \cup \cdots \cup \uparrow x_m.
\]
Proof. Suppose to the contrary that there exists an open neighbourhood $U(0)$ of zero in a Hausdorff feebly compact semitopological semilattice $(\exp n \lambda, \tau)$ such that

$$\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0)) \nsubseteq \uparrow x_1 \cup \cdots \cup \uparrow x_m$$

for any finitely many $x_1, \ldots, x_m \in \lambda$.

We fix an arbitrary $y_1 \in \lambda$ such that $(\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0))) \cap \uparrow y_1 \neq \emptyset$. By Proposition 1(iii) of \cite{11} the set $\uparrow y_1$ is open in $(\exp n \lambda, \tau)$ and hence the set $(\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0))) \cap \uparrow y_1$ is open in $(\exp n \lambda, \tau)$ too. Then by Proposition \cite{11} there exists an isolated point $m_1 \in \exp n \lambda \setminus \exp n-1 \lambda$ in $(\exp n \lambda, \tau)$ such that $m_1 \in (\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0))) \cap \uparrow y_1$. Now, by the assumption there exists $y_2 \in \lambda$ such that

$$(\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0))) \cap (\uparrow y_2 \cup \uparrow y_1) \neq \emptyset.$$ Again, since by Proposition 1(iii) of \cite{11} both sets $\uparrow y_1$ and $\uparrow y_2$ are open-and-closed in $(\exp n \lambda, \tau)$, Proposition \cite{11} implies that there exists an isolated point $m_2 \in \exp n \lambda \setminus \exp n-1 \lambda$ in $(\exp n \lambda, \tau)$ such that

$$m_2 \in (\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0))) \cap (\uparrow y_2 \cup \uparrow y_1).$$

Hence by induction we can construct a sequence $\{y_i : i = 1, 2, 3, \ldots\}$ of distinct points of $\lambda$ and a sequence of isolated points $\{m_i : i = 1, 2, 3, \ldots\} \subset \exp n \lambda \setminus \exp n-1 \lambda$ in $(\exp n \lambda, \tau)$ such that for any positive integer $k$ the following conditions hold:

(i) $(\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0))) \cap (\uparrow y_k \setminus (\uparrow y_1 \cup \cdots \cup \uparrow y_{k-1})) \neq \emptyset$; and

(ii) $m_k \in (\exp n \lambda \setminus \cl_{\exp n \lambda}(U(0))) \cap (\uparrow y_k \setminus (\uparrow y_1 \cup \cdots \cup \uparrow y_{k-1}))$.

Then similar arguments as in the proof of Proposition \cite{11} imply that the following family

$\{\{m_i\} : i = 1, 2, 3, \ldots\}$

is infinite and locally finite, which contradicts the feebly compactness of $(\exp n \lambda, \tau)$. The obtained contradiction implies the statement of the proposition. \qed

Proposition 1(iii) of \cite{11} implies that for any element $x \in \exp n \lambda$ the set $\uparrow x$ is open-and-closed in a $T_1$-semitopological semilattice $(\exp n \lambda, \tau)$ and hence by Theorem 14 from \cite{3} we have that for any $x \in \exp n \lambda$ the space $\uparrow x$ is feebly compact in a feebly compact $T_1$-semitopological semilattice $(\exp n \lambda, \tau)$. Hence Proposition \cite{2} implies the following proposition.

**Proposition 3.** Let $n$ be an arbitrary positive integer, $\lambda$ be an arbitrary infinite cardinal and $\tau$ be a shift-continuous feebly compact $T_1$-topology on the semilattice $\exp n \lambda$. Then for any point $x \in \exp n \lambda$ and any open neighbourhood $U(x)$ of $x$ in $(\exp n \lambda, \tau)$ there exist finitely many $x_1, \ldots, x_m \in \uparrow x \setminus \{x\}$ such that

$$\uparrow x \setminus \cl_{\exp n \lambda}(U(x)) \subseteq \uparrow x_1 \cup \cdots \cup \uparrow x_m.$$ The main results of this paper is the following theorem.

**Theorem 1.** Let $n$ be an arbitrary positive integer and $\lambda$ be an arbitrary infinite cardinal. Then for any shift-continuous $T_1$-topology $\tau$ on $\exp n \lambda$ the following conditions are equivalent:

(i) $\tau$ is countably pracompact;

(ii) $\tau$ is feebly compact;

(iii) $\tau$ is d-feebly compact;

(iv) the space $(\exp n \lambda, \tau)$ is $H$-closed.

**Proof.** Implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial and implication (iii) $\Rightarrow$ (i) follows from Proposition 1 of \cite{11}, Lemma \cite{1} and Proposition \cite{11}.

Implication (iv) $\Rightarrow$ (ii) follows from Proposition 4 of \cite{10}.

(ii) $\Rightarrow$ (iv) We shall prove this implication by induction.

By Corollary 2 from \cite{11} every feebly compact $T_1$-topology $\tau$ on the semilattice $\exp 1 \lambda$ such that $(\exp 1 \lambda, \tau)$ is a semitopological semilattice, is compact, and hence $(\exp 1 \lambda, \tau)$ is an $H$-closed topological space.
Next we shall show that if our statements holds for all positive integers $j < k \leq n$ then it holds for $j = k$. Suppose that a feebly compact $T_1$-semitopological semilattice $(\exp_k \lambda, \tau)$ is a subspace of Hausdorff topological space $X$. Fix an arbitrary point $x \in X$ and an arbitrary open neighbourhood $V(x)$ of $x$ in $X$. Since $X$ is Hausdorff, there exist disjoint open neighbourhoods $U(x) \subseteq V(x)$ and $U(0)$ of $x$ and zero $0$ of the semilattice $\exp_k \lambda$ in $X$, respectively. Then $\text{cl}_X(U(0)) \cap U(x) = \emptyset$ and hence by Proposition 2 there exists finitely many $x_1, \ldots, x_m \in \lambda$ such that

$$\text{exp}_k \lambda \cap U(x) \subseteq \uparrow x_1 \cup \cdots \cup \uparrow x_m.$$ 

But for any $x \in \lambda$ the subsemilattice $\uparrow x$ of $\text{exp}_k \lambda$ is algebraically isomorphic to the semilattice $\exp_{k-1} \lambda$. Then by Proposition 1(iii) of [11] and Theorem 14 from [3], $\uparrow x$ is a feebly compact $T_1$-semitopological semilattice, and the assumption of our induction implies that $\uparrow x_1, \ldots, \uparrow x_m$ are closed subsets of $X$. This implies that

$$W(x) = U(x) \setminus (\uparrow x_1 \cup \cdots \cup \uparrow x_m)$$

is an open neighbourhood of $x$ in $X$ such that $W(x) \cap \text{exp}_k \lambda = \emptyset$. Thus, $(\text{exp}_k \lambda, \tau)$ is an $H$-closed space. This completes the proof of the requested implication.

The following theorem gives a sufficient condition when a $d$-feebly compact space is feebly compact.

**Theorem 2.** Every quasiregular $d$-feebly compact space is feebly compact.

*Proof.* Suppose to the contrary that there exists a quasiregular $d$-feebly compact space $X$ which is not feebly compact. Then there exists an infinite locally finite family $\mathcal{U}_0$ of non-empty open subsets of $X$.

By induction we shall construct an infinite discrete family of non-empty open subsets of $X$.

Fix an arbitrary $U_1 \in \mathcal{U}_0$ and an arbitrary point $x_1 \in U_1$. Since the family $\mathcal{U}_0$ is locally finite there exists an open neighbourhood $U(x_1) \subseteq U_1$ of the point $x_1$ in $X$ such that $U(x_1)$ intersects finitely many elements of $\mathcal{U}_0$. Also, the quasiregularity of $X$ implies that there exists a non-empty open subset $V_1 \subseteq U(x_1)$ such that $\text{cl}_X(V_1) \subseteq U(x_1)$. Put

$$\mathcal{U}_1 = \{U \in \mathcal{U}_0 : U(x_1) \cap U = \emptyset\}.$$ 

Since the family $\mathcal{U}_0$ is locally finite and infinite, so is $\mathcal{U}_1$. Fix an arbitrary $U_2 \in \mathcal{U}_1$ and an arbitrary point $x_2 \in U_2$. Since the family $\mathcal{U}_1$ is locally finite, there exists an open neighbourhood $U(x_2) \subseteq U_2$ of the point $x_2$ in $X$ such that $U(x_2)$ intersects finitely many elements of $\mathcal{U}_1$. Since $X$ is quasiregular, there exists a non-empty open subset $V_2 \subseteq U(x_2)$ such that $\text{cl}_X(V_2) \subseteq U(x_2)$. Our construction implies that the closed sets $\text{cl}_X(V_1)$ and $\text{cl}_X(V_2)$ are disjoint and hence so are $V_1$ and $V_2$. Next we put

$$\mathcal{U}_2 = \{U \in \mathcal{U}_1 : U(x_2) \cap U = \emptyset\}.$$ 

Also, we observe that it is obvious that $U(x_1) \cap U = \emptyset$ for each $U \in \mathcal{U}_1$.

Suppose for some positive integer $k > 1$ we construct:

(a) a sequence of infinite locally finite subfamilies $\mathcal{U}_1, \ldots, \mathcal{U}_{k-1}$ in $\mathcal{U}_0$ of non-empty open subsets in the space $X$;

(b) a sequence of open subsets $U_1, \ldots, U_k$ in $X$;

(c) a sequence of points $x_1, \ldots, x_k$ in $X$ and a sequence of their corresponding open neighbourhoods $U(x_1), \ldots, U(x_k)$ in $X$;

(d) a sequence of disjoint non-empty subsets $V_1, \ldots, V_k$ in $X$

such that the following conditions hold:

(i) $\mathcal{U}_i$ is a proper subfamily of $\mathcal{U}_{i-1}$;

(ii) $U_i \in \mathcal{U}_{i-1}$ and $U_i \cap U = \emptyset$ for each $U \in \mathcal{U}_j$ with $i \leq j \leq k$;

(iii) $x_i \in U_i$ and $U(x_i) \subseteq U_i$;

(iv) $V_i$ is an open subset of $U_i$ with $\text{cl}_X(V_i) \subseteq U(x_i)$,

for all $i = 1, \ldots, k$, and

(v) $\text{cl}_X(V_1), \ldots, \text{cl}_X(V_k)$ are disjoint.
Next we put
\[ \mathcal{U}_k = \{ U \in \mathcal{U}_{k-1} : U(x_1) \cap U = \ldots = U(x_k) \cap U = \emptyset \}. \]

Since the family \( \mathcal{U}_{k-1} \) is infinite and locally finite, there exists a subfamily \( \mathcal{U}_k \) in \( \mathcal{U}_{k-1} \) which is infinite and locally finite. Fix an arbitrary \( U_{k+1} \in \mathcal{U}_k \) and an arbitrary point \( x_{k+1} \in U_{k+1} \). Since the family \( \mathcal{U}_k \) is locally finite, there exists an open neighbourhood \( U(x_{k+1}) \subseteq U_{k+1} \) of the point \( x_{k+1} \) in \( X \) such that \( U(x_{k+1}) \) intersects finitely many elements of \( \mathcal{U}_k \). Since the space \( X \) is quasiregular, there exists a non-empty open subset \( V_{k+1} \subseteq U(x_{k+1}) \) such that \( \text{cl}_X(V_{k+1}) \subseteq U(x_{k+1}) \). Simple verifications show that the conditions \((i) - (iv)\) hold in the case of the positive integer \( k+1 \).

Hence by induction we construct the following two infinite countable families of open non-empty subsets of \( X \):
\[ \mathcal{U} = \{ U_i : i = 1, 2, 3, \ldots \} \quad \text{and} \quad \mathcal{V} = \{ V_i : i = 1, 2, 3, \ldots \} \]
such that \( \text{cl}_X(V_i) \subseteq U_i \) for each positive integer \( i \). Since \( \mathcal{U} \) is a subfamily of \( \mathcal{U}_0 \) and \( \mathcal{U}_0 \) is locally finite in \( X \), \( \mathcal{U} \) is locally finite in \( X \) as well. Also, above arguments imply that \( \mathcal{V} \) and
\[ \overline{\mathcal{V}} = \{ \text{cl}_X(V_i) : i = 1, 2, 3, \ldots \} \]
are locally finite families in \( X \) too.

Next we shall show that the family \( \mathcal{V} \) is discrete in \( X \). Indeed, since the family \( \overline{\mathcal{V}} \) is locally finite in \( X \), by Theorem 1.1.11 of [8] the union \( \bigcup \overline{\mathcal{V}} \) is a closed subset of \( X \), and hence any point \( x \in X \setminus \bigcup \overline{\mathcal{V}} \) has an open neighbourhood \( O(x) = X \setminus \bigcup \overline{\mathcal{V}} \) which does not intersect the elements of the family \( \mathcal{V} \). If \( x \in \text{cl}_X(V_i) \) for some positive integer \( i \), then our construction implies that \( U(x_i) \) is an open neighbourhood of \( x \) which intersects only the set \( V_i \in \mathcal{V} \). Hence \( X \) has an infinite discrete family \( \mathcal{V} \) of non-empty open subsets in \( X \), which contradicts the assumption that the space \( X \) is \( d \)-feebly compact. The obtained contradiction implies the statement of the theorem.

We finish this note by some simple remarks about dense embedding of an infinite semigroup of matrix units and a polycyclic monoid into \( d \)-feebly compact topological semigroups which follow from the results of the paper [5].

Let \( \lambda \) be a non-zero cardinal. On the set \( B_\lambda = (\lambda \times \lambda) \cup \{0\} \), where \( 0 \notin \lambda \times \lambda \), we define the semigroup operation “.” as follows
\[ (a, b) \cdot (c, d) = \begin{cases} (a, d), & \text{if } b = c; \\ 0, & \text{if } b \neq c. \end{cases} \]
and \((a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0\) for \( a, b, c, d \in \lambda \). The semigroup \( B_\lambda \) is called the semigroup of \( \lambda \times \lambda \)-matrix units (see [7]).

The bicyclic monoid \( B(p, q) \) is the semigroup with the identity 1 generated by two elements \( p \) and \( q \) subject only to the condition \( pq = 1 \) [7]. For a non-zero cardinal \( \lambda \), the polycyclic monoid \( P_\lambda \) on \( \lambda \) generators is the semigroup with zero given by the presentation:
\[ P_\lambda = \left\langle \{ p_i \}_{i \in \lambda}, \{ p_i^{-1} \}_{i \in \lambda} \mid p_i p_j^{-1} = 1, p_i p_j^{-1} = 0 \text{ for } i \neq j \right\rangle \]
(see [5]). It is obvious that in the case when \( \lambda = 1 \) the semigroup \( P_1 \) is isomorphic to the bicyclic semigroup with adjoined zero.

By Theorem 4.4 from [5] for every infinite cardinal \( \lambda \) the semigroup of \( \lambda \times \lambda \)-matrix units \( B_\lambda \) does not densely embed into a Hausdorff feebly compact topological semigroup, and by Theorem 4.5 from [5] for arbitrary cardinal \( \lambda \geq 2 \) there exists no Hausdorff feebly compact topological semigroup which contains the \( \lambda \)-polycyclic monoid \( P_\lambda \) as a dense subsemigroup. These theorems and Lemma [1] imply the following two corollaries.

**Corollary 2.** For every infinite cardinal \( \lambda \) the semigroup of \( \lambda \times \lambda \)-matrix units \( B_\lambda \) does not densely embed into a Hausdorff \( d \)-feebly compact topological semigroup.

**Corollary 3.** For arbitrary cardinal \( \lambda \geq 2 \) there exists no Hausdorff \( d \)-feebly compact topological semigroup which contains the \( \lambda \)-polycyclic monoid \( P_\lambda \) as a dense subsemigroup.
The proof of the following corollary is similar to Theorem 5.1(5) from [4].

**Corollary 4.** There exists no Hausdorff topological semigroup with the d-feeably compact square which contains the bicyclic monoid \(C(p, q)\) as a dense subsemigroup.

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Faculty of Mechanics and Mathematics, National University of Lviv, Universytetska 1, Lviv, 79000, Ukraine

E-mail address: o_gutik@franko.lviv.ua, ovgutik@yahoo.com, olesyasobol@mail.ru