SPLITTING PROPERTIES OF LINEAR DIFFERENTIAL SYSTEMS WITH SMALL DELAYS

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Abstract. We investigate singular perturbation problems caused by small delays in the view of pseudo-exponential dichotomy. For a general linear non-autonomous retarded differential equation with small delay, previous works established the existence of a pseudo-exponential dichotomy. The main objective of this paper is to give a detailed analysis of three splitting properties of this dichotomy. By obtaining several new estimates and giving the explicit expressions of the bounds and the exponents associated with this dichotomy, we prove that as the delay tends to zero, the spectral gap approaches to infinity, and the angular distance and the separation index associated with this dichotomy are bounded from below by a positive constant which is independent of the delay.

1. Introduction

Singular perturbation problems widely arise from real world applications. They can be induced by small parameters, slow-fast diffusions, small-large delays and so on. There are a lot of good references on this topic. See, for instance, [16, 19, 20, 21, 22, 28, 32, 33] and the references therein.

As one of important singular perturbation problems, small-delay problems have been widely studied in the past decades. Spectral analysis and special solutions for linear differential equations with small delays were investigated in [11, 12, 15, 18]. Asymptotic behaviors and inertial manifolds for nonlinear equations were considered in [6, 11, 34]. Existence of periodic solutions for scalar differential equations with small delays were studied in [7, 8]. Recently, traveling waves arising from partial differential equations with small local (resp. nonlocal) delays [13, 29], and canard explosion and relaxation oscillations in differential systems with small delays [4, 27] also attracted many attentions. For many more dynamical behaviors induced by small delays, we refer to [28, Chapter 18]. Here we consider a general linear non-autonomous retarded differential equation with small delay in the view of exponential dichotomy, and study the effects of small delay on the spectral gap, the angular distance and the separation index [10, 30, 31].

Exponential dichotomy is an important concept in describing the hyperbolic property of linear differential equations. It is widely used in studying invariant manifolds, homoclinic/heteroclinic bifurcations, linearization and so on [2, 9, 19, 25]. As a generalization of exponential dichotomy, the concept of pseudo-exponential dichotomy was laid to establish the weak stable/unstable invariant manifolds (see, for instance, [26, 35]). Let $\mathcal{B}$ denote a Banach space. Assume that $\{T(t,s) : t \geq s\}$, a two-parameter family of bounded linear operators on $\mathcal{B}$, is a semigroup and strongly continuous in $s$ and $t$. Then the two-parameter family $\{T(t,s) : t \geq s\}$ is said to admit a pseudo-exponential dichotomy on an interval $J(\subset \mathbb{R})$ if for each $s \in J$, there exist a projection $P(s)$ on $\mathcal{B}$ and real constants $K$, $\alpha$ and $\beta$ with $K > 0$ and $\beta < \alpha$ such that the following properties hold:

(i) $T(t,s)P(s) = P(t)T(t,s)$ for $t \geq s$ in $J$.

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(ii) $T(t, s)|_{\mathcal{R}(P(s))}$ is an isomorphism from $\mathcal{R}(P(s))$ onto $\mathcal{R}(P(t))$, where $\mathcal{R}(P(s))$ is the range of $P(s)$. The inverse of $T(t, s)|_{\mathcal{R}(P(s))}$ is denoted by $T(s, t) : \mathcal{R}(P(t)) \to \mathcal{R}(P(s))$.

(iii) $|T(s, t)P(t)\phi| \leq Ke^{-\alpha(t-s)}|P(t)\phi|$ for $t \geq s$ in $J$ and $\phi \in \mathcal{B}$.

(iv) $|T(t, s)Q(s)\phi| \leq Ke^{\beta(t-s)}|Q(s)\phi|$ for $t \geq s$ in $J$ and $\phi \in \mathcal{B}$, where $Q(s) = I - P(s)$ and $I$ is the identity.

We refer to the constants $K$, $\alpha$ and $\beta$ as a bound, an upper exponent and a lower exponent of this pseudo-exponential dichotomy, respectively. The spectral gap associated with this dichotomy is defined by:

$$
\text{the spectral gap} = \sup \left\{ \alpha \in \mathbb{R} : \sup_{t \geq s} |T(s, t)P(t)|e^{\alpha(t-s)} < +\infty \right\} - \inf \left\{ \beta \in \mathbb{R} : \sup_{t \geq s} |T(t, s)Q(s)|e^{-\beta(t-s)} < +\infty \right\}.
$$

This constant plays an important role in the invariant manifold reduction (see, for instance, \cite{[3] [17] [25] [26]). Whenever there is no confusion, we always use $| \cdot |$ to denote the norms of the elements in a linear space endowed with a suitable norm.

Assume that a strongly continuous semigroup $\{T(t, s) : t \geq s\}$ admits a pseudo-exponential dichotomy. To describe the splitting properties of the dichotomy, we give the definitions of the angular distance and the separation index between the nonzero spaces $X_+ = \mathcal{R}(P(s))$ and $X_- = \mathcal{R}(I - P(s))$ for each $s \in J$. More information on the angular distance and the separation index can be found in \cite{[10] [30] [31]}. Before giving the definition for the angular distance between two nonzero subspaces, we first define the angular distance between two nonzero elements in the Banach space $X$ (see \cite{[10] [31]}). For each pair of nonzero elements $\xi_+$ and $\xi_-$ in $X$, the angular distance $\gamma(\xi_+, \xi_-)$ between $\xi_+$ and $\xi_-$ is defined by

$$
\gamma(\xi_+, \xi_-) := \left| \frac{\xi_+}{|\xi_+|} - \frac{\xi_-}{|\xi_-|} \right|.
$$

Then the angular distance $\gamma(X_+, X_-)$ between nonzero subspaces $X_+$ and $X_-$ is given by

$$
(1.1) \quad \gamma(X_+, X_-) := \inf \{ \gamma(\xi_+, \xi_-) : \xi_+ \in X_+/\{0\}, \, \xi_- \in X_-/\{0\} \}.
$$

The angular distance $\gamma(X_+, X_-)$ between subspaces $X_+$ and $X_-$ is closed related to the separation index $\text{dist}(X_+, X_-)$ (see \cite{[30]}), which is given by

$$
(1.2) \quad \text{dist}(X_+, X_-) := \inf_{\xi_+ \in X_+ \cap S} \left\{ \inf_{\xi_- \in X_-} |\xi_+ - \xi_-| \right\},
$$

where the set $S = \{ \xi \in X : |\xi| = 1 \}$ denotes the unit ball in the Banach space $X$. Both of them describe the geometric properties of the dichotomies for linear differential systems, and play important roles in the study of dynamical behaviors.

In this paper, we investigate the pseudo-exponential dichotomy for a linear retarded differential equation with small delay. More precisely, we consider a general linear non-autonomous retarded differential equation of the form

$$
(1.3) \quad \dot{x}(t) = L(t, x_t),
$$

where the linear operator $L(t, \cdot) : C[-r, 0] := C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous for each $t \in \mathbb{R}$, the constant $r$ is the delay and the section $x_t$ is defined by $x_t(\theta) := x(t + \theta)$ for $\theta \in [-r, 0]$. Let the space $C[-r, 0]$ of all continuous functions from $[-r, 0]$ into $\mathbb{R}^n$ be equipped with the supremum norm. Then by the Riesz Representation Theorem, the operators $L(t, \cdot)$ are
represented by
\begin{equation}
L(t, \phi) = \int_{-r}^{0} \eta(t, \theta) \phi(\theta), \quad t \in \mathbb{R}, \ \phi \in C[-r, 0],
\end{equation}
where the kernel \( \eta \) is an \( n \times n \) matrix-valued function on \( \mathbb{R} \times \mathbb{R} \), measurable in \( (t, \theta) \in \mathbb{R} \times \mathbb{R} \), and normalized so that \( \eta \) satisfies \( \eta(t, \theta) = \eta(t, -r) \) for \( \theta \leq -r \) and \( \eta(t, \theta) = 0 \) for \( \theta \geq 0 \), and \( \eta(t, \cdot) \) is continuous from the left on \( (-r, 0) \) and has bounded variation for each \( t \in \mathbb{R} \). Additionally, we assume that equation (1.3) satisfies the following hypothesis:

(H) there exists a positive constant \( M \) such that the total variation \( \text{Var}_{[-r,0]} \eta(t, \cdot) \) of \( \eta(t, \cdot) \) on \( [-r, 0] \) and the delay \( r \) respectively satisfy
\[ \text{Var}_{[-r,0]} \eta(t, \cdot) \leq M \text{ for each } t \in \mathbb{R} \text{ and } 0 < r < r_0 := 1/(Me). \]

By [19, Theorem 2.3, p.44], linear equation (1.3) with the initial value \( x_s = \phi \) has a unique solution \( x(\cdot, s, \phi) \). Define the solution operator \( T(t, s) : C[-r, 0] \to C[-r, 0] \) of equation (1.3) by
\begin{equation}
T(t, s) \phi := x_t(\cdot, s, \phi) \text{ for each } s, t \in \mathbb{R} \text{ with } t \geq s \text{ and } \phi \in C[-r, 0].
\end{equation}

Then the two-parameter family \( \{T(t, s) : t \geq s\} \) of the solution operators acting on the space \( C[-r, 0] \) is an evolutionary system and strongly continuous in \( t \) and \( s \) (see [19]).

Our goal is to study equation (1.3) with the hypothesis (H) in the view of pseudo-exponential dichotomy. We prove that the spectral gap corresponding to a dichotomy for equation (1.3) with (H) tends to infinity as the delay approaches to zero, and show that both the corresponding angular distance and the corresponding separation index are uniformly bounded from below for sufficiently small delay.

2. Related work and main results

In this section, we start by introducing some related work as preliminaries, and then state the main results in this paper.

Under the hypothesis (H), [12] established the existence of the so-called special matrix solution [12] for linear retarded differential equation (1.3), which is similar to the fundamental matrix solution of a linear nonautonomous ordinary differential equation. The special matrix solution for equation (1.3) with the hypothesis (H) has the following properties.

**Theorem 2.1** (Driver, 1976, Theorem 3). Assume that equation (1.3) satisfies the hypothesis (H). Then there exists a unique \( n \times n \) matrix-valued function \( \Phi \), which is defined on \( \mathbb{R} \times \mathbb{R} \) and satisfies that for each \( t_0 \in \mathbb{R} \):

(i) each column of \( \Phi(. , t_0) \) is a solution of equation (1.3) on \( \mathbb{R} \).

(ii) \( \Phi(t_0, t_0) = I \), where \( I \) is the identity.

(iii) \( \Phi(t, t_0) e^{(t-t_0)/r} \) is bounded for each \( t \leq t_0 \).

(iv) for each \( t \in \mathbb{R} \), the matrix \( \Phi(t, t_0) \) is nonsingular and \( \Phi(t, t_0)^{-1} = \Phi(t_0, t) \), and for each \( t_1, t_2 \in \mathbb{R} \), \( \Phi(t_2, t_1) \Phi(t_1, t_0) = \Phi(t_2, t_0) \).

(v) \( |\Phi(t, t_0)| \leq e^{\lambda_r |t-t_0|} \) for each \( t \in \mathbb{R} \), where the constant \( \lambda_r \) is the unique real root of equation \( Me^{\lambda} = \lambda \) in the interval \(( -1/r, 0) \).

Following the results stated in [11] [12], [1] proved the existence of a pseudo-exponential dichotomy and obtained the explicit expression of the corresponding projection by applying the formal adjoint equations associated with linear retarded differential equations (see [19], Chapter 6) and [23]. As shown in [1] Section 4, the existence of a pseudo-exponential dichotomy for equation (1.3) with the hypothesis (H) is summarized in the following theorem.
Theorem 2.2 (Arino & Pituk, 2001). Assume that equation (1.3) satisfies the hypothesis (H). Then \( \{T(t,s) : t \geq s\} \) defined by (1.5) admits a pseudo-exponential dichotomy. More precisely, there exist projections \( P(s), s \in \mathbb{R} \), and constants \( K_{1,r}, K_{2,r}, \alpha_r, \beta_r \) with \( \alpha_r > \beta_r \) such that for each \( s, t \in \mathbb{R} \) with \( t \geq s \), the following statements hold:

(i) \( T(t,s)P(s) = P(t)T(t,s) \).

(ii) \( T(t,s)P(s) \) is an isomorphism from \( \mathcal{R}(P(s)) \) onto \( \mathcal{R}(P(t)) \), where \( \mathcal{R}(P(s)) \) is the range of \( P(s) \) and the dimension \( \dim \mathcal{R}(P(s)) \) satisfies \( \dim \mathcal{R}(P(s)) = n \) for each \( s \in \mathbb{R} \). The inverse of \( T(t,s)P(s) \) is denoted by \( T(s,t) : \mathcal{R}(P(t)) \to \mathcal{R}(P(s)) \).

(iii) for each \( \phi \in C[{-r,0}] \),

\[
T(t,s)P(t)\phi \leq K_{1,r}e^{\alpha_r(s-t)}|P(t)\phi|,
\]

(2.1)

\[
|T(t,s)(I - P(s))\phi| \leq K_{2,r}e^{\beta_r(t-s)}|(I - P(s))\phi|.
\]

(2.2)

We remark that the representations of the projections \( \{P(s) : s \in \mathbb{R}\} \) are too complicated, thus they are omitted here and the readers are referred to [1, p.403].

As stated in Theorem 2.2, we see that the existence of a pseudo-exponential dichotomy for equation (1.3) with the hypothesis (H) is obtained. However, it does not give the explicit expressions of the bounds and the exponents for this dichotomy. We provide these expressions in the paper and further prove that the corresponding spectral gap approaches to infinity as the delay \( r \) tends to zero. More precisely, we have the following statements.

Theorem 2.3. Assume that equation (1.3) satisfies the hypothesis (H). Then for each \( s \in \mathbb{R} \), the following assertions hold:

(i) the constants \( \alpha_r, \beta_r \) and \( K_{i,r} \) in (2.1) and (2.2) can be as follows:

\[
\alpha_r = -\lambda_r, \quad \beta_r = \frac{\rho \ln(r \lambda_r)}{r} - \lambda_r,
\]

(2.3)

\[
K_{1,r} = 1, \quad K_{2,r} = -\frac{2e^{2+r\lambda_r}(r \lambda_r)^{1-2\rho}}{\rho \ln(r \lambda_r)},
\]

for every \( \rho \in (0, 1] \), where the constant \( \lambda_r \) is defined as in Theorem 2.1.

(ii) the norms \( |P(s)| \) and \( |I - P(s)| \) of the projections \( P(s) \) and \( I - P(s) \) have the following uniform bounds with respect with \( s \):

\[
\max\{|P(s)|, |I - P(s)|\} \leq 2e\gamma_r(K_r)^{2\gamma_r - 1},
\]

(2.4)

where the constants \( \gamma_r \) and \( K_r \) are respectively given by

\[
\gamma_r = (M - \beta_r) / (\alpha_r - \beta_r), \quad K_r = \max\{K_{1,r}, K_{2,r}\}.
\]

Furthermore, consider a family of linear retarded differential equations of the form (1.3), which satisfy the hypothesis (H) and are parameterized by the delay \( r \). Then we have the following:

(iii) as the delay \( r \to 0^+ \), the spectral gap \( \alpha_r - \beta_r \to +\infty \) and the constant

\[
\mathcal{L}_r := \frac{\alpha_r - \beta_r}{4(K_r)^2 \max\{|P(s)|, |I - P(s)|\}} \to +\infty.
\]

(2.5)

To show that the limit in (2.5) is useful in the invariant manifold reduction, we consider the following nonlinear retarded differential equation

\[
\dot{x}(t) = Lx_t + f(x_t),
\]

(2.6)

for each \( t \geq 0 \) and each \( x \in \mathbb{R}^n \), where \( L \) is the linear operator defined by (1.4) whose the kernel \( \eta \) is independent of \( t \), and the nonlinear term \( f : C[-r,0] \to \mathbb{R}^n \) satisfies the following hypothesis:
To construct the invariant manifolds in the neighbourhood of a singular point $x = 0$ for nonlinear system (2.3) with the hypothesis (H'), one usually assumes that the so-called spectral gap condition holds (see, for instance, [3, 19, 26]), that is,

$$L_r = \alpha_r - \beta_r 4(\lambda r)^2 \max\{|P(s)|, |I - P(s)|\} > K_f.$$  

Theorem 2.3 shows that the spectral gap condition for retarded differential equations can be easily satisfied if the delay is sufficiently small. This fact can be used in the study of complex oscillations arising from differential equations with small delays, such as canard explosion and relaxation oscillation [4, 14, 27].

Recall that the angular distance and the separation index associated with a splitting of the Banach space $C[-r, 0]$ are defined by (1.1) and (1.2). Based on Theorems 2.2 and 2.3, we further investigate the angular distance and the separation index for the obtained dichotomy.

**Theorem 2.4.** Consider a family of linear retarded differential equations of the form (1.3), which satisfy the hypothesis (H) and are parameterized by the delay $r$. For each $s \in \mathbb{R}$, let the subspaces $C_{\pm}$ be defined by

$$C_{\pm} = \mathcal{R}(P(s)), \quad C_{\pm} = \mathcal{R}(I - P(s)),$$

where $P(s)$ is the projection operator defined as in Theorem 2.2. Then there exist a sufficiently small $\tilde{r}_0$ with $0 < \tilde{r}_0 < 1/(Me)$ and a positive constant $\delta$ that is independent of $r$ in $(0, \tilde{r}_0)$ such that

$$0 < \delta \leq \text{dist}(C_{\pm}, C_{\mp}) \leq \gamma(C_{\pm}, C_{\mp}), \quad 0 < \delta \leq \text{dist}(C_{\pm}, C_{\pm}) \leq \gamma(C_{\pm}, C_{\mp}).$$  

This theorem shows that the angular distance $\gamma(C_{\pm}, C_{\mp})$ and the separation index $\text{dist}(C_{\pm}, C_{\mp})$ (resp. $\text{dist}(C_{\mp}, C_{\pm})$) for the obtained dichotomy are uniformly bounded from below for sufficiently small delay.

### 3. Proof of main results

In this section we give the detailed proof for main results stated in this paper. We start by the serval properties of an analytic function, which are useful in proving the last statement in Theorem 2.3.

**Lemma 3.1.** Let the function $g : \mathbb{R} \to \mathbb{R}$ be in the form

$$g(x) = Me^{rx} - x, \quad x \in \mathbb{R},$$

where the parameter $r$ satisfies $0 < r < r_0$ and the real constants $M, r_0$ are defined as in the hypothesis (H). Then the function $g$ has precisely two real zeros $\lambda_r$ and $\mu_r$ satisfying

$$0 < \lambda_r < \frac{1}{r_0} < \frac{1}{r} < \mu_r.$$  

Furthermore, the constant $\lambda_r$ satisfies the following limits:

$$r \lambda_r \to 0, \quad \lambda_r \to M, \quad \text{as} \quad r \to 0.$$
Figure 1. The graph of the function $h(x) = xe^{-x}$.

Proof. It is clear that $g'(x) = Mr e^{rx} - 1$ and $g''(x) = Mr^2 e^{rx} > 0$, then the derivative $g'$ of the function $g$ has a unique real zero

$$
\lambda^0 = -\frac{\ln(Mr)}{r} > -\frac{\ln(e^{-1})}{r} = \frac{1}{r},
$$

where the inequality is from the hypothesis (H). Moreover, we also obtain that the function $g$ has at most two real zeros. Since

$$
g(0) = M > 0, \quad g\left(\frac{1}{r}\right) = Me - \frac{1}{r} = \frac{Me - 1}{r} < 0, \quad \lim_{x \to +\infty} g(x) = +\infty,
$$

then by continuity there are real constants $\lambda_r$ and $\mu_r$ satisfying

$$
\lambda_r < 1/r < \mu_r, \quad g(\lambda_r) = 0, \quad g(\mu_r) = 0.
$$

Then by Rolle’s Theorem,

$$
\lambda_r < \frac{1}{r} < \lambda^0 < \mu_r.
$$

Note that

$$
\lambda_r = Me^{rx_1(r)} < Me = \frac{1}{r_0} < \frac{1}{r},
$$

then we get the inequalities in (3.1).

Since $g(\lambda_r) = 0$, then $rM = r\lambda_r e^{-r\lambda_r}$. It implies that

$$
r\lambda_r \to 0 \quad \text{as} \quad r \to 0,
$$

here we use the properties of the function $h(x) = xe^{-x}$ for $x \in \mathbb{R}$, see Figure 1. Then by $\lambda_r = Me^{\lambda_r}$, we obtain that $\lambda_r \to M$ as $r \to 0$. Therefore, the proof is now complete.

In the next lemma we prove a key inequality, which is useful in subsequent proof.

Lemma 3.2. Let $\varphi$ be a continuous function from $\mathbb{R}^+ := [0, +\infty)$ to $\mathbb{R}^+$ and satisfy

$$
(3.2) \quad c_1 := \max_{0 \leq t \leq r} \varphi(t) \quad \text{and} \quad \varphi(t) \leq c_2 \int_{t-r}^{t} \varphi(s) ds, \quad t \geq r,
$$
for a positive constant $c_2$ with $c_2r < 1$. Then the function $\varphi$ satisfies $\varphi(t) \leq c_1$ for each $t \geq 0$ and

$$\varphi(t) \leq \frac{c_1}{(c_2r)^{\rho}} \exp\left(\frac{\rho \ln(c_2r)}{r} t\right),$$

for each $t \in \mathbb{R}^+$ and each $\rho \in (0, 1]$.

Proof. To prove this lemma, we first claim that $\varphi(t) < c_1$ for $r \leq t \leq 2r$. Otherwise, suppose that $\max_{r \leq t \leq 2r} \varphi(t) \geq c_1$. Since

$$\varphi(r) \leq c_2 \int_0^r \varphi(s) ds \leq c_1 c_2 r < c_1,$$

then by continuity, there exist a constant $t_1 \in (r, 2r]$ such that

$$t_1 = \inf\{r \leq t \leq 2r : \varphi(t_1) = c_1\},$$

which implies that $\varphi(t) < c_1$ for $r < t < t_1$ and $\varphi(t_1) = c_1$. This is in contradiction with the fact that

$$\varphi(t_1) \leq c_2 \int_{t_1-r}^{t_1} \varphi(s) ds
= c_2 \left( \int_r^{t_1} \varphi(s) ds + \int_r^{t_1} \varphi(s) ds \right)
< c_2(c_1(2r - t_1) + c_1(t_1 - r))
= c_2c_1 r < c_1,$$

where we use the following inequalities

$$\int_r^{t_1-r} \varphi(s) ds \leq c_1(2r - t_1), \quad \int_r^{t_1} \varphi(s) ds \leq c_1(r - t_1).$$

Hence, the claim holds. It is not hard to prove that $\varphi(t) \leq c_1$ for all $t \geq 0$ by induction.

Let an auxiliary function $\psi$ from $\mathbb{R}^+$ to itself be defined by

$$\psi(t) = \sup_{s \geq t} \varphi(s) \quad \text{for} \quad t \geq 0.\tag{3.4}$$

By the above statements, the function $\psi$ is well-defined. By the conditions in (3.2),

$$\psi(t) = \sup_{s \geq t} \varphi(s) \leq \sup_{s \geq t} c_2 r \psi(s - r) = c_2 r \psi(t - r), \quad \text{for} \quad t \geq r.\tag{3.5}$$

Note that for each $t \geq r$ there exists a positive integer $j \geq 1$ such that $jr \leq t < (j + 1)r$, then by (3.4) and (3.5) we have

$$\varphi(t) \leq \psi(jr) \leq c_2 r \psi(jr - r) \leq c_1(c_2r)^j.$$  

Since $0 < \rho \leq 1$, $0 < c_2r < 1$ and $r \leq jr \leq t < (j + 1)r$, then we get

$$\varphi(t) \leq c_1(c_2r)^j \leq c_1 \exp(j \ln(c_2r))$$
$$\leq c_1 \exp((t/r - 1) \ln(c_2r))$$
$$\leq c_1 \exp(\rho(t/r - 1) \ln(c_2r)).$$

This, together with the facts that $\exp(\rho(t/r - 1) \ln(c_2r)) \geq 1$ and $\varphi(t) \leq c_1$ for $0 \leq t \leq r$, yields that the second statement holds. Therefore, the proof is now complete.$\square$
Similar estimates as these in Lemma 3.2 have been proved by [1, 12]. However, here we adopt a different method to get a new estimate. We also remark that to obtain the large spectral gap, we actually need the constant \( \rho \) to be in the interval \((0, 1/2)\).

By the hypothesis \((H)\) and the inequality \((1.4)\) in [19, p.168], we can obtain the estimates for the solutions of equation \((1.3)\). More precisely, we have the next lemma.

**Lemma 3.3** (Hale & Verduyn Lunel, Theorem 1.1, p.168). Assume that equation \((1.3)\) satisfies the hypothesis \((H)\). Then the solution \(x(\cdot, t_0, \phi)\) of equation \((1.3)\) with the initial value \(x_{t_0} = \phi \in C[-r, 0]\) satisfies the following estimate

\[
|x_t(\cdot, t_0, \phi)| \leq e^{M(t-t_0)}|\phi| \quad \text{for} \quad t \geq t_0.
\]

By applying Theorem 2.1 and Lemmas 3.2 and 3.3, we can prove the following results, which are useful in giving the values of the constants \(K_{2x}\) and \(\beta_x\).

**Lemma 3.4.** For each \(\phi \in C[-r, 0]\), let \(x\) be the solution of equation \((1.3)\) with the initial value \(x_{t_0} = \phi\) and

\[
y(t) = \Phi(t_0, t)x(t) \quad \text{for} \quad t \geq t_0,
\]

where the matrix-valued function \(\Phi\) is defined as in Theorem 2.1. Then there exists a vector \(l(t_0, \phi) \in \mathbb{R}^n\) such that

\[
(3.6) \quad \lim_{t \to +\infty} y(t) = l(t_0, \phi).
\]

Furthermore, suppose that the function \(\phi\) satisfies \(|\phi| \leq 1\) and \(l(t_0, \phi) = 0\). Then the solution \(x\) satisfies the following estimate

\[
(3.7) \quad |x(t)| \leq \frac{2e^2}{\rho(r\lambda_r)^{\rho-1}\ln(r\lambda_r)} \exp \left( \frac{(\rho \ln(r\lambda_r) - \lambda_r)}{r} (t - t_0) \right),
\]

for each \(t \geq t_0\) and each \(\rho \in (0, 1]\).

**Proof.** By (iv) in Theorem 2.1, we see that the function \(y\) is well-defined. To prove the existence of the limit, we first give another property of the special matrix solution \(\Phi(\cdot, t_0) = (x^1(\cdot, t_0), ..., x^n(\cdot, t_0))\), that is,

\[
\Phi(t, t_0) = -\Phi(t_0, t)(L(t, x^1), L(t, x^2), ..., L(t, x^n))\Phi(t_0, t),
\]

where the derivative is with respect to \(t\). For each \(\xi \in \mathbb{R}^n\), by (iv) in Theorem 2.1, we see that

\[
\Phi(t_0, t)\Phi(t_0)\xi = \xi.
\]

Then we obtain that

\[
\dot{\Phi}(t_0, t)\Phi(t_0)\xi + \Phi(t_0, t)\Phi(t_0)\xi = 0,
\]

which yields that

\[
\Phi(t_0, t)\Phi(t_0)\xi = -\Phi(t_0, t)L(t, \Phi_t\xi) = -\Phi(t_0, t)(L(t, x^1), L(t, x^2), ..., L(t, x^n))\xi.
\]

By the arbitrariness of \(\xi\), we get

\[
\Phi(t_0, t)\Phi(t_0) = -\Phi(t_0, t)(L(t, x^1), L(t, x^2), ..., L(t, x^n)).
\]

This yields that \((3.8)\) holds. Then for \(t \geq t_0\),

\[
\dot{y}(t) = \Phi(t_0, t)x(t) + \Phi(t_0, t)\dot{x}(t)
= -\Phi(t_0, t)(L(t, x^1), L(t, x^2), ..., L(t, x^n))\Phi(t_0, t)x(t) + \Phi(t_0, t)L(t, x_t)
= -\Phi(t_0, t)L(t, \Phi_t y(t)) + \Phi(t_0, t)L(t, \Phi_t y_t),
\]

where the matrix-valued function \(\Phi\) is defined as in Theorem 2.1. Then there exists a vector \(\phi \in C[-r, 0]\) satisfies the following estimate
which yields
\[ \Phi(t, t_0) \dot{y}(t) = L(t, \Phi_t(y_0 - y(t))) \]
for \( t \geq t_0 \). Then by (v) in Theorem 2.1 we get that for \( t \geq t_0 + r \),
\[ |\Phi(t, t_0) \dot{y}(t)| \leq M \sup_{-r \leq \theta \leq 0} |\int_{t+\theta}^t \Phi(t+\theta, t_0) \dot{y}(\tau) d\tau| \]
\[ = M \sup_{-r \leq \theta \leq 0} |\int_{t+\theta}^t \Phi(t+\theta, \tau) \Phi(\tau, t_0) \dot{y}(\tau) d\tau| \]
\[ \leq M \sup_{-r \leq \theta \leq 0} |\int_{t+\theta}^t e^{r(\tau-t-\theta)} |\Phi(\tau, t_0) \dot{y}(\tau)| d\tau| \]
\[ \leq Me^{r\lambda_r} \int_{t-r}^t e^{r(\tau-t)} |\Phi(\tau, t_0) \dot{y}(\tau)| d\tau. \]
Since \( Me^{r\lambda_r} = \lambda_r \) and \( 0 < r \lambda_r < 1 \), then by Lemma 3.2 we have that for \( t \geq t_0 \),
\[ e^{r(t-t_0)} |\Phi(t, t_0) \dot{y}(t)| \leq \lambda_r \int_{t-r}^t e^{r(\tau-t_0)} |\Phi(\tau, t_0) \dot{y}(\tau)| d\tau \]
\[ \leq \frac{\kappa_1}{(r \lambda_r)^\rho} \exp \left( \frac{\rho \ln(r \lambda_r)}{r} (t - t_0) \right), \]
where the constant \( \kappa_1 \) is in the form
\[ \kappa_1 = \sup_{t_0 \leq t \leq t_0 + r} e^{r(t-t_0)} |\Phi(t, t_0) \dot{y}(t)|. \]
Hence, by Theorem 2.1 and (3.9) we obtain that for \( t \geq t_0 \),
\[ |\dot{y}(t)| \leq |\Phi(t_0, t)| |\Phi(t, t_0) \dot{y}(t)| \]
\[ \leq \frac{\kappa_1}{(r \lambda_r)^\rho} \exp \left( \frac{\rho \ln(r \lambda_r)}{r} (t - t_0) \right). \]
This together with \( r^{-1} \ln(r \lambda_r) < 0 \) yields that the limit of \( y(t) \) exists as \( t \to +\infty \). Thus, the first statement is proved.

Suppose that \( l(t_0, \phi) = 0 \), then for \( t \geq t_0 \),
\[ x(t) = \Phi(t, t_0) y(t) \]
\[ = \Phi(t, t_0) \int_{-\infty}^t \dot{y}(\tau) d\tau \]
\[ = \int_{-\infty}^t \Phi(t, \tau) \Phi(\tau, t_0) \dot{y}(\tau) d\tau, \]
then by (3.9), (3.10) and (v) in Theorem 2.1 we obtain that
\[ |x(t)| \leq -\frac{\kappa_1 r}{\rho (r \lambda_r)^\rho \ln(r \lambda_r)} \exp \left( \frac{\rho \ln(r \lambda_r)}{r} - \lambda_r \right) (t - t_0), \quad t \geq t_0. \]
For \( |\phi| \leq 1 \) and \( t_0 \leq t \leq t_0 + r \), we observe that
\[ e^{r(t-t_0)} |\Phi(t, t_0) \dot{y}(t)| = e^{r(t-t_0)} |L(t, \Phi_t(y_t - y(t)))| \]
\[ \leq Me^{r\lambda_r} \left( \sup_{-r \leq \theta \leq 0} |\Phi_t y_t| + \sup_{-r \leq \theta \leq 0} |\Phi_t y(t)| \right). \]
By Lemma 3.3 we obtain

\[
\sup_{0 \leq t \leq r} \sup_{-r \leq \theta \leq 0} |\Phi_t y_t| = \sup_{-r \leq \theta \leq 0} |x(t)| = \max\{1, \sup_{0 \leq t \leq r} |x(t)|\}
\]

\[
\leq \max\{1, e^{Mr}\} = e^{Mr},
\]

\[
\sup_{0 \leq t \leq r} \sup_{-r \leq \theta \leq 0} |\Phi_t y(t)| = \sup_{0 \leq t \leq r} \sup_{-r \leq \theta \leq 0} |\Phi(t + \theta, t)\Phi(t, t_0) y(t)|
\]

\[
\leq e^{r\lambda_r + Mr}.
\]

Since $Me^{r\lambda_r} = \lambda_r$, 0 < $r\lambda_r$ < 1 and $Mer < 1$, then by (3.12-3.14) we obtain

\[
\kappa_1 \leq \lambda_r e^{Mr}(1 + e^{r\lambda_r}) \leq \lambda_r e^{Mr} e^{r\lambda_r}(1 + e^{-r\lambda_r}) < 2\lambda_r e^2.
\]

Substituting (3.15) into (3.11) yields the last statement. Therefore, the proof is complete. □

We remark that the results in Lemma 3.4 lead to a significantly different presentation from [12] Theorem 4], from which we can only obtain the limit in (3.6). However, to investigate the effects of small delay on the spectral gap, the angular distance and the separation index, it is necessary to obtain the estimate (3.7).

**Proof of Theorem 2.3.** By [1] Theorem 4.1], for each $t \in \mathbb{R}$ the projection $P(t)$ is in the form $P(t)\phi = \Phi_t(\cdot, t)l(t, \phi)$ for each $\phi \in C[-r, 0]$, where the vector $l(t, \cdot)$ is defined as in Lemma 3.4. By Theorem 2.1, we have that for $t \geq s$ and $\phi \in C[-r, 0],

\[
|T(s, t)P(t)\phi| = |T(s, t)\Phi_t(\cdot, t)l(t, \phi)| = |\Phi_s(\cdot, t)l(t, \phi)|
\]

\[
= \sup_{-r \leq \theta \leq 0} |\Phi(s + \theta, t)l(t, \phi)|
\]

\[
= \sup_{-r \leq \theta \leq 0} |\Phi(s + \theta, t + \theta)\Phi(t + \theta, t)l(t, \phi)|
\]

\[
\leq e^{\lambda_r|t-s|}|\Phi_t(\cdot, t)l(t, \phi)|
\]

\[
= e^{\lambda_r|t-s|}|P(t)\phi|.
\]

For each $s \in \mathbb{R}$ and each $\phi \in \mathcal{R}(I - P(s))$ with $|\phi| \leq 1$, let $x$ be the solution of equation (1.3) with $x_s = \phi$. Then $l(s, \phi) = 0$ and by Lemma 3.3 we have

\[
|x_t| \leq \frac{2e^2}{\rho(r\lambda_r)e^{1-1\ln(r\lambda_r)}} \exp\left(\left(\frac{\rho \ln(r\lambda_r)}{r} - \lambda_r\right)(t - s - r)\right), \quad t \geq s.
\]

Hence, (3.16) and (3.17) yield (2.3), that is,

\[
\alpha_r = -\lambda_r, \quad \beta_r = \frac{\rho \ln(r\lambda_r)}{r} - \lambda_r,
\]

\[
K_{1,r} = 1, \quad K_{2,r} = \frac{2e^{2+r\lambda_r}(r\lambda_r)^{1-2\rho}}{\rho \ln(r\lambda_r)}.
\]

Thus, (i) is obtained.

By Lemma 3.1, we have that $r\lambda_r \to 0$ and $\lambda_r \to M$ as $r \to 0+$, then as $r \to 0+$,

\[
\alpha_r - \beta_r = -\lambda_r - \left(\frac{\rho \ln(r\lambda_r)}{r} - \lambda_r\right)
\]

\[
= -\frac{\rho \ln(r\lambda_r)}{r}
\]

\[
= -\frac{\rho \lambda_r \ln(r\lambda_r)}{r\lambda_r} \to +\infty.
\]
To prove (ii), for each $s \in \mathbb{R}$ we denote $q_+ := |P(s)|$ and $q_- := |I - P(s)|$. Then from (2.1) and (2.2) it follows that for each $\tau > 0$,

\begin{align}
|T(s + \tau, s)P(s)| &\geq (K_{1,r})^{-1}e^{\alpha_r \tau}q_+ , \\
|T(s + \tau, s)(I - P(s))| &\leq K_{2,r}e^{\beta_r \tau}q_- .
\end{align}

Let the function $\Psi$ be defined by

$$
\Psi(\tau) = (K_{1,r})^{-1}e^{\alpha_r \tau} - K_{2,r}e^{\beta_r \tau} \quad \text{for } \tau > 0.
$$

Since $\alpha_r > \beta_r$, then we can compute that the function $\Psi$ satisfies that $\Psi(\tau_0) = 0$ and $\Psi(\tau) > 0$ for $\tau > \tau_0$, where the constant $\tau_0$ is in the form

$$
\tau_0 := \frac{\ln(K_{1,r}K_{2,r})}{\alpha_r - \beta_r} > 0.
$$

Applying Lemma 3.3 and (3.19) yields that for each $\tau > \tau_0 > 0$,

\begin{align}
\Psi(\tau) &\leq ||T(s + \tau, s)P(s)||q_+^{-1} - ||T(s + \tau, s)(I - P(s))||q_-^{-1} \\
&\leq |T(s + \tau, s)P(s)q_+^{-1} + T(s + \tau, s)(I - P(s))q_-^{-1}| \\
&\leq e^{M^*}q_+^{-1}P(s) + q_-^{-1}(I - P(s))| \\
&\leq e^{M^*}q_+^{-1}I + (q_+^{-1} - q_-^{-1})P(s) \\
&\leq e^{M^*}q_-^{-1}1 + |q_+^{-1} - q_-^{-1}|q_+ \\
&\leq e^{M^*}q_-^{-1}(1 + |q_+ - q_-|) \\
&\leq 2e^{M^*}q_-^{-1},
\end{align}

where we use the fact that

$$
|q_+ - q_-| = ||P(s)|| - |I - P(s)|| \leq |P(s) + (I - P(s))| = 1.
$$

Recall that the constant $\gamma_r$ is in the form

$$
\gamma_r = \frac{M - \beta_r}{\alpha_r - \beta_r}.
$$

Since $M > \alpha_r$ and $\alpha_r > \beta_r$, then we can check that $\gamma_r > 1$. Let the function $\tilde{\Psi}$ be defined by

$$
\tilde{\Psi}(\tau) := 2e^{M^*}(\Psi(\tau))^{-1}, \quad \tau > \tau_0.
$$

Then we can compute that the function $\tilde{\Psi}$ reaches the minimum value in $(\tau_0, +\infty)$ at

$$
\tau_1 := \frac{\ln(K_{1,r}K_{2,r}(M - \beta_r)) - \ln(M - \alpha_r)}{\alpha_r - \beta_r}.
$$

Substituting $\tau = \tau_1$ into $\tilde{\Psi}(\tau)$ yields that

$$
\min_{\tau > \tau_0} \tilde{\Psi}(\tau) = \tilde{\Psi}(\tau_1) = \frac{2K_{1,r}}{(K_{1,r}K_{2,r}(M - \beta_r))^{1 - \gamma_r} - K_{1,r}K_{2,r}(M - \alpha_r)^{-\gamma_r}} \\
= 2\gamma_r K_{1,r} \left( \frac{K_{1,r}K_{2,r}(M - \beta_r)}{M - \alpha_r} \right)^{\gamma_r - 1} \\
= 2\gamma_r K_{1,r} \left( 1 + \frac{1}{\gamma_r - 1} \right)^{\gamma_r - 1} (K_{1,r}K_{2,r})^{\gamma_r - 1}.
$$
where we use the fact that 
\[ 1 + \frac{1}{\gamma_r - 1} = \frac{M - \beta_r}{M - \alpha_r}. \]

Thus, by (3.20) we have
\[
q_- \leq 2\gamma_r K_{1,r} \left( 1 + \frac{1}{\gamma_r - 1} \right)^{\gamma_r - 1} (K_{1,r}K_{2,r})^{\gamma_r - 1}.
\]

Since \( K_r = \max\{K_{1,r}, K_{2,r}\} \), \( \gamma_r > 1 \) and
\[
\left( 1 + \frac{1}{\gamma_r - 1} \right)^{\gamma_r - 1} < e,
\]
then by (3.21) we have that
\[
q_- \leq 2e\gamma_r (K_r)^{2\gamma_r - 1}.
\]

Similarly, the above inequalities also hold for \( q_+ \). Thus, (ii) is proved.

By Lemma 3.1, for each \( \rho \in (0, 1/2) \) we have that as \( r \to 0^+ \), the following limits hold:
\[
\beta_r = \frac{\rho \ln(r \lambda_r)}{r} - \lambda_r \to -\infty,
\]
\[
\gamma_r = \frac{M - \beta_r}{\alpha_r - \beta_r} \to 1,
\]
\[
\frac{\alpha_r - \beta_r}{(K_{1,r})^{2\gamma_r - 1}} = \alpha_r - \beta_r \to +\infty,
\]
\[
\frac{\alpha_r - \beta_r}{(K_{2,r})^{2\gamma_r - 1}} = \frac{\lambda_r \rho^{2(\gamma_r + 1)}}{(2 \exp(2 + r \lambda_r))^{2\gamma_r + 1}} \frac{(\ln(r \lambda_r))^{2(\gamma_r + 1)}}{(r \lambda_r)(1-2\rho)(2\gamma_r + 1)} \to +\infty.
\]

By (3.21) and the definition of the constant \( L_r \) given in (2.5), we obtain
\[
L_r \geq \frac{\alpha_r - \beta_r}{8e\gamma_r (K_r)^{2\gamma_r - 1}},
\]
then by (3.22) the limit in (2.5) holds. Thus, (iii) is proved. Therefore, the proof is now complete.

To prove the last theorem, we introduce several results on the angular distance and the separation index, which show the relationships between these geometric properties of a splitting and the corresponding projections. These results are summarized in the following lemma.

**Lemma 3.5.** Let the Banach space \( C[-r, 0] \) have a splitting \( C[-r, 0] = C_1 \oplus C_1 \), where \( C_j \) are nonzero closed subspaces of \( C[-r, 0] \), and \( P_1 \) and \( P_2 = I - P_1 \) are the corresponding projections with \( \mathcal{R}(P_1) = C_1 \) and \( \mathcal{R}(P_2) = C_2 \). Then the following statements hold:

(i) the angular distance \( \gamma(C_1, C_2) \) defined as in (1.1) satisfies the following estimates:
\[
(\gamma(C_1, C_2))^{-1} \leq |P_j| \leq 2(\gamma(C_1, C_2))^{-1}.
\]

(ii) the separation indices \( \text{dist}(C_1, C_2) \) and \( \text{dist}(C_2, C_1) \) defined as in (1.2) satisfy that
\[
\text{dist}(C_1, C_2) = (|P_1|)^{-1}, \quad \text{dist}(C_2, C_1) = (|P_2|)^{-1}.
\]

The statement (3.23) is obtained by [10, Lemma 1.1, p.156], and the statement (3.24) is given in [30, Lemma 3]. Thus the detailed proof is omitted.

By applying these statements in Lemma 3.5 and Theorem 2.4, we can given the proof for the final theorem.
Proof of Theorem 2.4. For each $s \in \mathbb{R}$, let the subspaces $\mathcal{C}_\pm$ be in the form

$$\mathcal{C}_+ = \mathcal{R}(P(s)), \quad \mathcal{C}_- = \mathcal{R}(I - P(s)),$$

where $P(s)$ is the projection operator defined as in Theorem 2.2. Take $\rho$ with $\rho \in (0, 1/2)$ in (2.3). Then the fact that $r\lambda_r \to 0$ as $r \to 0^+$, which is obtained in Lemma 3.1, yields that

$$K_{2,r} = -\frac{2e^{2r+\lambda_r} (r\lambda_r)^{1-2\rho}}{\rho \ln(r\lambda_r)} \to 0, \quad \text{as} \quad r \to 0^+.$$

This implies that $K_r = \max\{K_{1,r}, K_{2,r}\} = 1$ for sufficiently small $r$. As stated in (3.22), we have the limit:

$$\gamma_r = \frac{M - \beta_r}{\alpha_r - \beta_r} \to 1, \quad \text{as} \quad r \to 0^+.$$

Then the right side of (2.4) satisfies that

$$2e\gamma_r(K_r)^{2\gamma_r - 1} \to 2e, \quad \text{as} \quad r \to 0^+.$$

Thus by continuity there exists a sufficiently small $\bar{\tau}_0$ with $0 < \bar{\tau}_0 < 1/(Me)$ and a positive constant $\delta$ that is independent of $r$ in $(0, \bar{\tau}_0]$ such that

$$2e\gamma_r(K_r)^{2\gamma_r - 1} \leq \delta^{-1}. \quad (3.25)$$

Applying (2.4), (3.25) and (3.24) in Lemma 3.5, we obtain that

$$\text{dist}(\mathcal{C}_+, \mathcal{C}_-) \geq \delta > 0 \quad (3.26)$$

for each $r \in (0, \bar{\tau}_0]$. By (3.23) and (3.24) in Lemma 3.5, we obtain that $\gamma(\mathcal{C}_+, \mathcal{C}_-)$ and $\text{dist}(\mathcal{C}_+, \mathcal{C}_-)$ satisfy

$$\gamma(\mathcal{C}_+, \mathcal{C}_-) \geq \text{dist}(\mathcal{C}_+, \mathcal{C}_-).$$

This together with (3.26) yields

$$0 < \delta \leq \text{dist}(\mathcal{C}_+, \mathcal{C}_-) \leq \gamma(\mathcal{C}_+, \mathcal{C}_-).$$

Similarly, we can obtain

$$0 < \delta \leq \text{dist}(\mathcal{C}_-, \mathcal{C}_+) \leq \gamma(\mathcal{C}_+, \mathcal{C}_-).$$

Therefore, the proof is now complete.

4. Concluding remarks

We have investigated the effects of small delays on linear non-autonomous retarded differential equations in the view of pseudo-exponential dichotomy. The existence of special matrix solutions and pseudo-exponential dichotomies for linear retarded differential equations with small delays were established in [12] and [1], respectively. This paper has developed the results in the aforementioned works. Based on several new estimates obtained in this paper, we give the explicit expressions of the bounds and the exponents associated with this dichotomy. By applying these estimates and viewing the delay as a parameter, we show that if the delay is sufficiently small, the spectral gap corresponding to this dichotomy can be large enough, and the angular distance and the separation index are uniformly bounded from below. We hope that these results stated in this paper will be used to study canard explosion and relaxation oscillation arising from differential equations with small delays, and we refer the readers to several recent works [4, 14, 27] and a good literature [28].

Compared to linear autonomous retarded differential equations, the semigroups generalized by the solution operators of neutral differential equations have not only point spectrum but also continuous spectrum [20, 24]. This causes a big obstacle in establishing the existence of pseudo-exponential dichotomies. One of the remaining problems is to prove the existence of a pseudo-exponential dichotomy for a general neutral differential equations with small delay.
Recently, large spectral gaps induced by small delays for neutral differential equations were considered in [5], where the proof is finished by the Banach Fixed Point Theorem. By the estimates obtained in [5], it is also likely to get how the angular distance and the separation index associated with a dichotomy for a general neutral differential equations with small delay vary as the delay tends to zero.

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