CALCULUS ON SYMPLECTIC MANIFOLDS

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ABSTRACT. On a symplectic manifold, there is a natural elliptic complex replacing the de Rham complex. It can be coupled to a vector bundle with connection and, when the curvature of this connection is constrained to be a multiple of the symplectic form, we find a new complex. In particular, on complex projective space with its Fubini–Study form and connection, we can build a series of differential complexes akin to the Bernstein–Gelfand–Gelfand complexes from parabolic differential geometry.

1. Introduction

Throughout this article $M$ will be a smooth manifold of dimension $2n$ equipped with a symplectic form $J_{ab}$. Here, we are using Penrose’s abstract index notation [10] and non-degeneracy of this 2-form says that there is a skew contravariant 2-form $J^{ab}$ such that $J_{ab}J^{ac} = \delta_b^c$ where $\delta_b^c$ is the canonical pairing between vectors and co-vectors.

Let $\wedge^k$ denote the bundle of $k$-forms on $M$. The homomorphism

$$\wedge^k \to \wedge^{k-2}$$

given by $\phi_{abc...d} \mapsto J^{ab} \phi_{abc...d}$

is surjective for $2 \leq k \leq n$ with non-trivial kernel, corresponding to the irreducible representation

$$\begin{array}{ccccccc}
0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\
k^{th} \text{node}
\end{array}$$

of $\text{Sp}(2n, \mathbb{R}) \subset \text{GL}(2n, \mathbb{R})$.

Denoting this bundle by $\wedge^k_{\perp}$, there is a canonical splitting of the short exact sequence

$$0 \to \wedge^k_{\perp} \xrightarrow{\pi} \wedge^k \to \wedge^{k-2} \to 0$$

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and an elliptic complex \[2, 4, 6, 11, 13\]

\[
\begin{array}{c}
0 \to \wedge^0 \xrightarrow{d} \wedge^1 \xrightarrow{d} \wedge^2 \xrightarrow{d} \wedge^3 \xrightarrow{d} \cdots \xrightarrow{d} \wedge^n \\
\downarrow d_2 \\
0 \leftarrow \wedge^0 \leftarrow d_1 \leftarrow \wedge^1 \leftarrow d_1 \leftarrow \wedge^2 \leftarrow d_1 \leftarrow \wedge^3 \leftarrow \cdots \leftarrow d_1 \leftarrow \wedge^n
d\end{array}
\]

where

- \(d : \wedge^0 \to \wedge^1\) is the exterior derivative,
- for \(1 \leq k < n\), the operator \(d_\perp : \wedge^k \perp \to \wedge^{k+1} \perp\) is the composition
  \[
  \wedge^k \perp \leftrightarrow \wedge^k \xrightarrow{d} \wedge^{k+1} \xrightarrow{\pi} \wedge^{k+1} \perp,
  \]
a first order operator,
- \(d_\perp : \wedge^{k+1} \perp \to \wedge^k \perp\) are canonically defined first order operators, which may be seen as adjoint to \(d_\perp : \wedge^k \perp \to \wedge^{k+1} \perp\),
- \(d_2 : \wedge^n \perp \to \wedge^n \perp\) is the composition
  \[
  \wedge^n \perp \xrightarrow{d} \wedge^{n-1} \perp \xrightarrow{d} \wedge^n \perp,
  \]
a second order operator.

More explicitly, formulæ for these operators may be given as follows. Firstly, it is convenient to choose a symplectic connection \(\nabla_a\), namely a torsion-free connection such that \(\nabla_a J_{bc} = 0\), equivalently \(\nabla_a J^{bc} = 0\). As shown in [7], for example, such connections always exist and if \(\nabla_a\) is one such, then the general symplectic connection is

\[
\tilde{\nabla}_a \phi_b = \nabla_a \phi_b + J^{cd} \Xi_{abc} \phi_d \quad \text{where} \quad \Xi_{abc} = \Xi_{(abc)}.
\]

Then, for \(1 \leq k < n\), the operator \(d_\perp : \wedge^k \perp \to \wedge^{k+1} \perp\) is given by

\[
(2) \quad \phi_{def \ldots g} \mapsto \nabla_{[c} \phi_{def \ldots g]} - \frac{k}{2(n+1-k)} J^{ab} (\nabla_a \phi_{b[e \ldots f]} J_{cd}] \psi_{cdf \ldots g} \mapsto J^{bc} \nabla_b \psi_{cdf \ldots g}.
\]

Now suppose \(E\) is a smooth vector bundle on \(M\) and \(\nabla : E \to \wedge^1 \otimes E\) is a connection. Choosing any torsion-free connection on \(\wedge^1\) induces a connection on \(\wedge^1 \otimes E\) and, as is well-known, the composition

\[
\wedge^1 \otimes E \to \wedge^1 \otimes \wedge^1 \otimes E \to \wedge^2 \otimes E
\]
does not depend on this choice. (It is the second in a well-defined sequence of differential operators

\[
(4) \quad E \xrightarrow{\nabla} \wedge^1 \otimes E \xrightarrow{\nabla} \wedge^2 \otimes E \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \wedge^{2n-1} \otimes E \xrightarrow{\nabla} \wedge^{2n} \otimes E
\]
known as the coupled de Rham sequence.) In particular, we may define a homomorphism $\Theta : E \to E$ by
\[
J^{ab} \nabla_a \nabla_b \Sigma = \frac{1}{2n} \Theta \Sigma \quad \text{for} \quad \Sigma \in \Gamma(E).
\]
It is part of the curvature of $\nabla$ and if this is the only curvature, then
\[
(\nabla_a \nabla_b - \nabla_b \nabla_a) \Sigma = 2 J^{ab} \Theta \Sigma,
\]
and we shall say that $\nabla$ is symplectically flat. Looking back at (1), it is easy to see that there are coupled operators
\[
E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^{n-1} \otimes E \xrightarrow{\nabla} \Lambda^n \otimes E,
\]
explicit formulæ for which are just as in the uncoupled cases (2) and (3).

To complete the coupled version of (1) let us use
\[
\nabla^2 = \frac{2}{n} \Theta : \Lambda^n \otimes E \to \Lambda^n \otimes E
\]
for the middle operator. It is evident that
\[
E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E
\]
is a complex if and only if $\nabla$ is symplectically flat. The reason for the curvature term in (6) is that this feature propagates as follows.

Theorem 1. Suppose $E \xrightarrow{\nabla} \Lambda^1 \otimes E$ is a symplectically flat connection and define $\Theta : E \to E$ by (5). Then the coupled version of (1)
\[
0 \to E \xrightarrow{\nabla} \Lambda^1 \otimes E \xrightarrow{\nabla} \Lambda^2 \otimes E \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^{n-1} \otimes E \xrightarrow{\nabla} \Lambda^n \otimes E
\]
is a complex. It is locally exact except near the beginning where
\[
\ker \nabla : E \to \Lambda^1 \otimes E \quad \text{and} \quad \frac{\ker \nabla}{\im \nabla} : \Lambda^1 \otimes E \to \Lambda^2 \otimes E
\]
may be identified with the kernel and cokernel, respectively, of $\Theta$ as locally constant sheaves.

More precision and a proof of Theorem 1 will be provided in §2. Our next theorem yields some natural symplectically flat connections.

Theorem 2. Suppose $M$ is a $2n$-dimensional symplectic manifold with symplectic connection $\nabla_a$. Then there is a natural vector bundle $\mathcal{T}$ on $M$ of rank $2n + 2$ equipped with a connection, which is symplectically flat if and only if the curvature $R_{abcd}$ of $\nabla_a$ has the form
\[
R_{abcd} = \delta_a^c \Phi_{bd} - \delta_b^c \Phi_{ad} + J_{ad} \Phi_{be} J^{ce} - J_{bd} \Phi_{ae} J^{ce} + 2 J_{ab} \Phi_{de} J^{ce},
\]
for some symmetric tensor $\Phi_{ab}$.\]
In particular, the Fubini–Study connection on complex projective space is symplectic for the standard Kähler form and its curvature is of the form (7) for \( \Phi_{ab} = g_{ab} \), the standard metric. More generally, if the symplectic connection \( \nabla_a \) arises from a Kähler metric, then we shall see that (7) holds precisely in the case of constant holomorphic sectional curvature.

After proving Theorems 1 and 2 the remainder of this article is concerned with the consequences of Theorem 1 for the vector bundle \( T \) and those bundles, such as \( \bigotimes^k T \), induced from it. In particular, these consequences pertain on complex projective space where we shall find a series of elliptic complexes closely following the Bernstein-Gelfand-Gelfand complexes on the sphere \( S^{2n+1} \) as a homogeneous space for the Lie group \( \text{Sp}(2n+2, \mathbb{R}) \).

This article is based on our earlier work [6] but here we focus on the simpler case where we are given a symplectic structure as background. This results in fewer technicalities and in this article we include more detail, especially in constructing the BGG-like complexes in \( \S \) 5.

2. The Rumin–Seshadri complex

By the Rumin–Seshadri complex, we mean the differential complex (1) after [11]. However, the 4-dimensional case is due to R.T. Smith [12] and the general case is also independently due to Tseng and Yau [13]. In this section we shall derive the coupled version of this complex as in Theorem 1, our proof of which includes (1) as a special case. The following lemma is also the key step in [6].

**Lemma 1.** Suppose \( E \) is a vector bundle on \( M \) with symplectically flat connection \( \nabla : E \to \bigwedge^1 \otimes E \). Define \( \Theta : E \to E \) by (5). Then \( \Theta \) has constant rank and the bundles \( \ker \Theta \) and \( \text{coker} \Theta \) acquire from \( \nabla \), flat connections defining locally constant sheaves \( \ker \Theta \) and \( \text{coker} \Theta \), respectively. There is an elliptic complex

\[
\begin{align*}
E \xrightarrow{\nabla} \bigwedge^1 \otimes E & \xrightarrow{\nabla} \bigwedge^2 \otimes E & \xrightarrow{\nabla} \bigwedge^3 \otimes E & \xrightarrow{\nabla} \bigwedge^4 \otimes E & \cdots, \\
E & \xrightarrow{\ker \Theta} \bigwedge^1 \otimes E & \xrightarrow{\bigwedge^2 \otimes E} & \xrightarrow{\bigwedge^3 \otimes E} & \xrightarrow{\bigwedge^4 \otimes E} & \cdots,
\end{align*}
\]

where the differentials are given by

\[
\Sigma \xmapsto \begin{bmatrix} \nabla \Sigma \\ \Theta \Sigma \end{bmatrix}, \quad \begin{bmatrix} \phi \\ \eta \end{bmatrix} \xmapsto \begin{bmatrix} \nabla \phi - J \otimes \eta \\ \nabla \eta - \Theta \phi \end{bmatrix}, \quad \begin{bmatrix} \omega \\ \psi \end{bmatrix} \xmapsto \begin{bmatrix} \nabla \omega + J \wedge \psi \\ \nabla \psi + \Theta \omega \end{bmatrix}, \quad \cdots.
\]

It is locally exact save for the zeroth and first cohomologies, which may be identified with \( \ker \Theta \) and \( \text{coker} \Theta \), respectively.
Proof. From (5) the Bianchi identity for $\nabla$ reads
\[ 0 = \nabla_a (J_{bc} \Theta) = J_{[ab} \nabla_{c]} \Theta \]
and non-degeneracy of $J_{ab}$ implies that $\nabla_a \Theta = 0$. Consequently, the homomorphism $\Theta$ has constant rank and the following diagram with exact rows commutes
\[
\begin{array}{cccc}
0 & \rightarrow & \ker \Theta & \rightarrow & E & \overset{\Theta}{\rightarrow} & E & \rightarrow & \coker \Theta & \rightarrow & 0 \\
\downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla \\
0 & \rightarrow & \wedge^1 \otimes \ker \Theta & \rightarrow & \wedge^1 \otimes E & \overset{\Theta}{\rightarrow} & \wedge^1 \otimes E & \rightarrow & \wedge^1 \otimes \coker \Theta & \rightarrow & 0 \\
\end{array}
\]
and yields the desired connections on $\ker \Theta$ and $\coker \Theta$, which are easily seen to be flat. Ellipticity of the given complex is readily verified and, by definition, the kernel of its first differential is $\ker \Theta$.

To identify the higher local cohomology of this complex the key observation is that locally we may choose a 1-form $\tau$ such that $d\tau = J$ and, having done this, the connection
\[
\Gamma(E) \ni \tilde{\nabla} \Theta \rightarrow \nabla \Theta \rightarrow \nabla^2 \Theta \rightarrow \nabla^3 \Theta \rightarrow \nabla^4 \Theta \rightarrow \cdots .
\]
If needed, the details are in [6].

Proof of Theorem 1. In [6], the corresponding result [6, Theorem 4] is proved by invoking a spectral sequence. Here, we shall, instead, prove two typical cases ‘by hand,’ leaving the rest of the proof to the reader.

For our first case, let us suppose $n \geq 3$ and prove local exactness of
\[
\wedge^1 \otimes E \xrightarrow{\nabla_\perp} \wedge^2 \otimes E \xrightarrow{\nabla_\perp} \wedge^3 \otimes E \xrightarrow{\nabla_\perp} \wedge^4 \otimes E \rightarrow \cdots .
\]
Thus, we are required to show that if $\omega_{ab}$ has values in $E$ and
\[
\omega_{ab} = \omega_{[ab]} \quad J^{ab} \omega_{ab} = 0 \quad \nabla_{[c} \omega_{de]} = \frac{1}{n-1} J^{ab} (\nabla_a \omega_{bc}) J_{de],}
\]
then locally there is $\phi_a \in \Gamma(\wedge^1 \otimes E)$ such that
\[
\omega_{cd} = \nabla_{[c} \phi_{d]} - \frac{1}{2n} J^{ab} (\nabla_a \phi_b) J_{cd}.\]
If we set $\psi_c \equiv -\frac{1}{n-1} J^{ab} \nabla_a \omega_{bc}$, then
\[
\nabla_{[c} \omega_{de]} + J_{[cd} \psi_{e]} = 0
\]
and since $J \wedge - : \wedge^2 \rightarrow \wedge^4$ is injective it follows that
\[
\nabla_{[c} \psi_{d]} + \Theta \omega_{cd} = 0.
\]
In other words, we have shown that
\[
\nabla \omega + J \wedge \psi = 0 \\
\nabla \psi + \Theta \omega = 0 
\]
and Lemma 1 locally yields \( \phi_a \in \Gamma(\wedge^1 \otimes E) \) and \( \eta \in \Gamma(E) \) such that
\[
\nabla [a \phi_b] - J_{ab} \eta = \omega_{ab} \\
\nabla_a \eta - \Theta \phi_a = \psi_a 
\]
In particular,
\[
J^{ab} \nabla a \phi_b - 2n \eta = J^{ab} (\nabla a \phi_b - J_{ab} \eta) = J^{ab} \omega_{ab} = 0 
\]
and, therefore,
\[
\nabla [c \phi_d] - \frac{1}{2n} J^{ab} (\nabla a \phi_b) J_{cd} = \nabla [c \phi_d] - \eta J_{cd} = \omega_{cd}, 
\]
as required.

Our second case is more involved. It is to show that
\[
\wedge^n \nabla -\rightarrow \wedge^{n-1} \nabla \rightarrow 
\]
is locally exact. As regards \( \nabla \) : \( \wedge^n \otimes E \rightarrow \wedge^{n-1} \otimes E \), notice that
\[
J^{bc} \nabla b \psi_{cdef...g} = \frac{n+1}{2} J^{bc} \nabla_{[b \psi_{cdef...g]} 
\]
and that if \( \phi_{def...g} \in \Gamma(\wedge^k \otimes E) \), then
\[
J^{bc} J_{[bc \phi_{def...g]} = \frac{4(n-k)}{(k+1)(k+2)} J^{bc} J_{[bc \phi_{def...g]}J^{bc} 
\]
so if \( \phi_{def...g} \in \Gamma(\wedge^{n-1} \otimes E) \), then
\[
J^{bc} J_{[bc \phi_{def...g]} = \frac{4}{n(n+1)} \phi_{def...g}. 
\]
Therefore, \( \nabla \psi \in \Gamma(\wedge^{n-1} \otimes E) \) is characterised by
\[
J \wedge \nabla \psi = \frac{2}{n} \nabla \psi 
\]
as an equation in \( \wedge^{n+1} \otimes E \). In particular, in \( \wedge^{n+2} \otimes E \) we find
\[
J \wedge \nabla \nabla \psi = \nabla (J \wedge \nabla \psi) = \frac{2}{n} \nabla^2 \psi = J \wedge \Theta \psi = 0 
\]
whence \( \nabla \nabla \psi \) already lies in \( \wedge^n \otimes E \) and there is no need to remove the trace as in (2) to form \( \nabla^2 \psi \). Therefore, invoking (10) once again, the composition
\[
\wedge^n \otimes E \nabla \rightarrow \wedge^{n-1} \otimes E \nabla \rightarrow \wedge^n \otimes E \nabla \rightarrow \wedge^{n-1} \otimes E 
\]
is characterised by
\[
J \wedge \nabla^3 \psi = \frac{2}{n} \nabla \nabla^2 \psi = \frac{2}{n} \nabla^2 \nabla \psi = \frac{2}{n} J \wedge \Theta \nabla \psi = \frac{2}{n} J \wedge \nabla \Theta \psi 
\]
and, since \( J \wedge \Theta : \wedge^{n-1} \rightarrow \wedge^{n+1} \) is an isomorphism, we conclude that \( \nabla^3 \psi = \frac{2}{n} \nabla \Theta \psi \), equivalently that (8) is a complex.
Before proceeding, let us remark on another consequence of (9), namely that for
\[ \nu_{cdefg} \in \Gamma(\Lambda^n \otimes E), \]
\[ J_{[ab\nu_{cdefg}]} = 0 \iff J^{cd\nu_{cdefg}} = 0. \tag{11} \]
Now to establish local exactness, suppose \( \nu \in \Gamma(\Lambda^n \otimes E) \) satisfies \( \nabla_{\perp} \nu = 0 \). Equivalently, according to (10) and (11)
\[ \nu \in \Gamma(\Lambda^n \otimes E) \] satisfies \( \nabla \nu = 0 \) and \( J \wedge \nu = 0 \).
Lemma 1 implies that locally there are \( \phi \in \Gamma(\Lambda^n \otimes E) \)
\( \eta \in \Gamma(\Lambda^{n-1} \otimes E) \) such that
\[ \nabla \phi - J \wedge \eta = 0 \]
\[ \nabla \eta - \Theta \phi = \nu. \]
Since
\[ 0 \to \Lambda^{n-2} \xrightarrow{J^\wedge} \Lambda^n \to \Lambda^n_{\perp} \to 0 \]
is exact, we can write \( \phi \) uniquely as
\[ \phi = \psi + J \wedge \tau, \]
where \( \psi \in \Gamma(\Lambda^n \otimes E) \) and \( \tau \in \Gamma(\Lambda^{n-2} \otimes E) \). We conclude that
\[ \nabla \psi - J \wedge \hat{\eta} = 0 \]
\[ \nabla \hat{\eta} - \Theta \psi = \nu, \] (where \( \hat{\eta} = \eta - \nabla \tau \)).
However, as discussed above, these equations say exactly that
\[ \nabla^2 \psi - \frac{2}{n} \Theta \psi = \nu, \]
and exactness is shown. \( \square \)

3. Tractor bundles

For the rest of the article we suppose that we are given, not only a manifold \( M \) with symplectic form \( J_{ab} \), but also a torsion-free connection \( \nabla_a \) on the tangent bundle (and hence on all other tensor bundles) such that \( \nabla_a J_{bc} = 0 \). This is sometimes called a Fedosov structure \([7]\) on \( M \).
The curvature \( R_{ab}^{\ c} \) of \( \nabla_a \), characterised by
\[ (\nabla_a \nabla_b - \nabla_b \nabla_a)X^c = R_{ab}^{\ c}X^d, \]
satisfies
\[ R_{ab}^{\ c} = R_{[ab]}^{\ c} \]
\[ R_{[ab}^{\ c} d] = 0 \]
\[ R_{ab}^{\ c} dJ_{ce} = R_{ab}^{\ c} dJ_{cd} \]
and enjoys the following decomposition into irreducible parts
\[ R_{ab}^{\ c} = V_{ab}^{\ c} + \delta_a^{\ \ c}\Phi_{bd} - \delta_b^{\ \ c}\Phi_{ad} + J_{ad}\Phi_{be}J^{ce} - J_{bd}\Phi_{ae}J^{ce} + 2J_{ab}\Phi_{de}J^{ce}, \]
for some symmetric \( \Phi_{ab} \), where \( V_{ab}^{\ a} d = 0 \) (reflecting the branching
\[ \square = \square_{\perp} \oplus \square \]
of representations under $\text{GL}(2n, \mathbb{R}) \supset \text{Sp}(2n, \mathbb{R})$. Notice that
\begin{equation}
\Phi_{bd} = \frac{1}{2(n+1)} R_{ab}{}^{a} = \frac{1}{4(n+1)} J^{ae} R_{ae}{}^{b} J_{cd}.
\end{equation}
We define the \textit{standard tractor bundle} to be the rank $2n + 2$ vector bundle $\mathcal{T} \equiv \wedge^{0} \oplus \wedge^{1} \oplus \wedge^{0}$ with its \textit{tractor connection}
\begin{equation}
\nabla_{a} \begin{bmatrix} \sigma \\ \mu_{b} \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_{a} \sigma - \mu_{a} \\ \nabla_{a} \mu_{b} + J_{ab} \rho + \Phi_{ab} \sigma \\ \nabla_{a} \rho - \Phi_{ab} J^{bc} \mu_{c} + S_{a} \sigma \end{bmatrix}, \quad \text{where } S_{a} \equiv \frac{1}{2n+1} J^{bc} \nabla_{c} \Phi_{ab}.
\end{equation}
Readers familiar with conformal differential geometry may recognise the form of this connection as following the tractor connection in that setting \cite{[ref]}. If needs be, we shall write \textit{symplectic tractor connection} to distinguish the connection just defined from any alternatives. We shall need the following curvature identities.

\textbf{Lemma 2.} Let $Y_{abc} \equiv \frac{1}{2n+1} \nabla_{c} V_{ab}{}^{c}$. Then
\begin{equation}
Y_{abc} = 2 \nabla_{[a} \Phi_{b]c} - 2 J_{c[a} S_{b]} + 2 J_{ab} S_{c}
\end{equation}
and
\begin{equation}
J^{ad} \nabla_{a} Y_{bcd} = J^{ad} V_{bc}{}^{a} \Phi_{ed} + 4n (J^{ad} \Phi_{bc} \Phi_{ed} - \nabla_{b} S_{c}) + 2 J_{bc} J^{df} (\nabla_{a} S_{d} - J^{ef} \Phi_{ae} \Phi_{df}).
\end{equation}

\textbf{Proof.} Writing the Bianchi identity $\nabla_{[a} R_{bc]}{}^{d}{}_{e} = 0$ in terms of $V_{ab}{}^{c}$ and $\Phi_{ab}$ yields
\begin{equation}
\nabla_{[a} V_{bc]}{}^{d}{}_{e} = -2 \delta_{[b} ^{d} \nabla_{[a} \Phi_{c]}{}_{e} + 2 J^{df} J_{c[b} \nabla_{a} \Phi_{f]} - 2 J^{df} J_{[bc} \nabla_{a]} \Phi_{e}.
\end{equation}
and contracting over $a$ gives
\begin{equation}
\frac{1}{3} \nabla_{a} V_{bc}{}^{a} = \frac{4(n-1)}{3} \nabla_{[b} \Phi_{c]}{}_{e} + \frac{2}{3} \left[ \nabla_{[b} \Phi_{c]}{}_{e} - (2n + 1) J_{[b} S_{c]} \right] + \frac{1}{3} \left[ (2n + 1) J_{bc} S_{e} + 2 \nabla_{[b} \Phi_{c]}{}_{e} \right],
\end{equation}
which is easily rearranged as (13). For (14), firstly notice that
\begin{equation}
J^{ad} R_{ab}{}^{e}{}_{d} = J^{ed} R_{ab}{}^{a}{}_{d} = 2(n+1) J^{ad} \Phi_{bd}
\end{equation}
and the Bianchi symmetry may be written as $R_{a}{}^{e}{}_{c} = -\frac{1}{2} R_{bc}{}^{e}{}_{a}$. Thus,
\begin{equation}
J^{ad} \nabla_{a} \nabla_{b} \Phi_{cd} = \nabla_{b} J^{ad} \nabla_{a} \Phi_{ed} - J^{ad} R_{ab}{}^{e}{}_{c} \Phi_{ed} - J^{ad} R_{ab}{}^{e}{}_{d} \Phi_{ce} = -(2n+1) \nabla_{b} S_{c} - J^{ad} R_{ab}{}^{e}{}_{c} \Phi_{ed} + 2(n+1) J^{de} \Phi_{bd} \Phi_{ce}
\end{equation}
and so
\begin{equation}
J^{ad} \nabla_{a} \nabla_{[b} \Phi_{c]}{}_{d} = -(2n+1) \nabla_{[b} S_{c]} + \frac{1}{2} J^{ad} R_{bc}{}^{e}{}_{a} \Phi_{ed} + 2(n+1) J^{de} \Phi_{bd} \Phi_{ce}.
\end{equation}
From (13) we see that
\begin{equation}
J^{ad} \nabla_{a} Y_{bcd} = 2 J^{ad} \nabla_{a} \nabla_{[b} \Phi_{c]}{}_{d} + 2 \nabla_{[b} S_{c]} + 2 J_{bc} J^{ad} \nabla_{a} S_{d}.
\end{equation}
Therefore,
\[ J^{ad}\nabla_a Y_{bcd} = J^{ad}R_{bc}^\ e_a \Phi_{ed} - 4n\nabla_{[b} S_c] + 4(n+1)J^{de}\Phi_{ba}\Phi_{ce} + 2J_{bc}J^{ad}\nabla_a S_d. \]

Finally,
\[ J^{ad}R_{bc}^\ e_a \Phi_{ed} = J^{ad}V_{bc}^\ e_a \Phi_{ed} - 4J^{ad}\Phi_{ba}\Phi_{ed} - 2J_{bc}J^{ad}J^{ef}\Phi_{ae}\Phi_{df}, \]
so
\[ J^{ad}\nabla_a Y_{bcd} = J^{ad}V_{bc}^\ e_a \Phi_{ed} + 4nJ^{ad}\Phi_{ba}\Phi_{ed} - 2J_{bc}J^{ad}J^{ef}\Phi_{ae}\Phi_{df} - 4n\nabla_{[b} S_c] + 2J_{bc}J^{ad}\nabla_a S_d, \]
which may be rearranged as (14).

\( \square \)

**Proposition 1.** The tractor connection \( \mathcal{T} \to \wedge^1 \otimes \mathcal{T} \) preserves the non-degenerate skew form
\[
\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_c \\ \hat{\rho} \end{bmatrix} \equiv \sigma \hat{\rho} + J^{bc}\mu_b\hat{\mu}_c - \rho\hat{\sigma}
\]
and its curvature is given by
\[
(\nabla_a \nabla_a - \nabla_b \nabla_a) \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ -Y_{abc}J^{cd}\mu_d + \frac{1}{2n}(J^{ed}V_{ab}^\ e_a \Phi_{de} - J^{ed}\nabla_c Y_{abd})\sigma \\ +2J_{ab}S_cJ^{cd}\mu_d + \frac{1}{2n}J^{cd}(\nabla_c S_d - J^{ef}\Phi_{ce}\Phi_{df})\sigma \end{bmatrix}.
\]

**Proof.** We expand
\[
\left\langle \nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_c \\ \hat{\rho} \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \nabla_a \begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_c \\ \hat{\rho} \end{bmatrix} \right\rangle
\]
to obtain
\[
(\nabla_a \sigma - \mu_a)\hat{\rho} + \sigma(\nabla_a \hat{\rho} - \Phi_{ab}J^{bc}\hat{\mu}_c + S_a\hat{\sigma}) + J^{bc}(\nabla_a \mu_b + J_{ab}\hat{\rho} + \Phi_{ab}\hat{\sigma})\mu_c + J^{bc}\mu_b(\nabla_a \hat{\mu}_c + J_{ac}\hat{\rho} + \Phi_{ac}\hat{\sigma}) =\]
\[
- (\nabla_a \rho - \Phi_{ab}J^{bc}\mu_c + S_a\rho)\hat{\sigma} - \rho(\nabla_a \hat{\sigma} - \hat{\mu}_a)
\]
in which all terms cancel save for
\[
(\nabla_a \sigma)\hat{\rho} + \sigma\nabla a \hat{\rho} + J^{bc}(\nabla_a \mu_b)\hat{\mu}_c + J^{bc}\mu_b\nabla a \hat{\mu}_c - (\nabla a \rho)\hat{\sigma} - \rho\nabla a \hat{\sigma},
\]
which reduces to
\[
\nabla a(\sigma \hat{\rho} + J^{bc}\mu_b\hat{\mu}_c - \rho\hat{\sigma}),
\]
as required. For the curvature, we readily compute
\[
\nabla_{[a} \nabla_b [\sigma \mu_d \rho] = \left[ \begin{array}{c} \nabla_{[a} \nabla_b \mu_d + J_{d[a} \Phi_{b]c} J^{ce} \mu_e - \Phi_{d[a} \mu_b] + T_{ab} \rho \\ J_{d[a} \Phi_{b]c} J^{ce} \mu_e - \Phi_{d[a} \mu_b] + T_{ab} \rho \\ \nabla_{[a} \nabla_b \rho - T_{abc} J^{cd} \mu_d + (\nabla_{[a} S_b] - J^{cd} \Phi_{ac} \Phi_{bd}) \end{array} \right],
\]
where \( T_{abc} \equiv \nabla_{[a} \Phi_{b]c} - J_{c[a} S_b] \). Lemma 2, however, states that
\[
T_{abc} = \frac{1}{2} Y_{abc} - J_{ab} S_c
\]
and
\[
4n(\nabla_{[a} S_b] - J^{cd} \Phi_{ac} \Phi_{bd}) = J^{cd} V_{ab} c e \Phi_{de} - J^{cd} \nabla_c Y_{abd} + 2 J_{ab} J^{cd} (\nabla_c S_d - J^{ef} \Phi_{ce} \Phi_{df}).
\]
Therefore,
\[
\nabla_{[a} \nabla_b [\sigma \mu_d \rho] = \left[ \begin{array}{c} \nabla_{[a} \nabla_b \mu_d + J_{d[a} \Phi_{b]c} J^{ce} \mu_e - \Phi_{d[a} \mu_b] + \frac{1}{2} Y_{abd} \rho \\ -S_d \sigma \\ \frac{1}{2} Y_{abc} J^{cd} \mu_d + \frac{1}{4n} (J^{cd} V_{ab} c e \Phi_{de} - J^{cd} \nabla_c Y_{abd}) \rho \\ + J_{ab} \left[ \begin{array}{c} 0 \\ -S_d \sigma \\ \frac{1}{2} Y_{abc} J^{cd} \mu_d + \frac{1}{4n} (J^{cd} V_{ab} c e \Phi_{de} - J^{cd} \nabla_c Y_{abd}) \rho \\ \frac{1}{2} Y_{abc} J^{cd} \mu_d + \frac{1}{4n} (J^{cd} V_{ab} c e \Phi_{de} - J^{cd} \nabla_c Y_{abd}) \rho \\ S_c J^{cd} \mu_d + \frac{1}{2n} J^{cd} (\nabla_c S_d - J^{ef} \Phi_{ce} \Phi_{df}) \sigma \\ \end{array} \right].
\]
Finally,
\[
R_{ab} c d \mu_c = V_{ab} c d \mu_c - 2 \Phi_{d[a} \mu_b] + 2 J_{d[a} \Phi_{b]c} J^{ce} \mu_e + 2 J_{ab} \Phi_{de} J^{ce} \mu_c,
\]
so
\[
\nabla_{[a} \nabla_b \mu_d + J_{d[a} \Phi_{b]c} J^{ce} \mu_e - \Phi_{d[a} \mu_b] = -\frac{1}{2} V_{ab} c d \mu_c - J_{ab} \Phi_{de} J^{ce} \mu_c
\]
whence
\[
\nabla_{[a} \nabla_b [\sigma \mu_d \rho] = \left[ \begin{array}{c} 0 \\ -S_d \sigma \\ \frac{1}{2} Y_{abc} J^{cd} \mu_d + \frac{1}{4n} (J^{cd} V_{ab} c e \Phi_{de} - J^{cd} \nabla_c Y_{abd}) \rho \\ + J_{ab} \left[ \begin{array}{c} \frac{1}{2} Y_{abc} J^{cd} \mu_d + \frac{1}{4n} (J^{cd} V_{ab} c e \Phi_{de} - J^{cd} \nabla_c Y_{abd}) \rho \\ S_c J^{cd} \mu_d + \frac{1}{2n} J^{cd} (\nabla_c S_d - J^{ef} \Phi_{ce} \Phi_{df}) \sigma \\ \end{array} \right],
\]
as required.

**Corollary 1.** The tractor connection is symplectically flat if and only if the curvature tensor \( V_{ab} c d \) vanishes.
4. Kähler geometry

Kähler manifolds provide a familiar source of symplectic manifolds equipped with a compatible torsion-free connection as in §3. In this case, the connection $\nabla_a$ is the Levi-Civita connection of a metric $g_{ab}$ and $J^b_a \equiv J_{ac}g^{bc}$ is an almost complex structure on $M$ whose integrability is equivalent to the vanishing of $\nabla_a J_{bc}$. In Kähler geometry, the Riemann curvature tensor decomposes into three irreducible parts:

$$R_{ab}{}^c{}_d = U_{ab}{}^c{}_d + \delta_a{}^c \Xi_{bd} - \delta_b{}^c \Xi_{ad} + g_{ad} \Xi^c_b - g_{bd} \Xi^c_a + J_{ab} \text{c}
abla_{bd} \text{c} + 2 J_{ab} \Sigma_{cd} + 2 J^c_d \Sigma_{ab} + \Lambda (\delta_a{}^c g_{bd} - \delta_b{}^c g_{ad} + J^c_a \Sigma_{bd} - J^c_b \Sigma_{ad} + 2 J^c_a \Sigma_{bd} + 2 J^c_b \Sigma_{ad} + 2 J^c_{ab} J^d_{cd}),$$

where indices have been raised using $g^{ab}$ and

- $U_{ab}{}^c{}_d$ is totally trace-free with respect to $g^{ab}$, $J^a{}_b$, and $J_{ab}$,
- $\Xi_{ab}$ is trace-free symmetric whilst $\Sigma_{ab} \equiv J_a{}^c \Xi_{bc}$ is skew.

Computing the Ricci curvature from this decomposition, we find

$$R_{bd} \equiv R_{ab}{}^a{}_d = 2(n + 2) \Xi_{bd} + 2(n + 1) \Lambda g_{bd}$$

and therefore from (12) conclude that

$$\Phi_{ab} = \frac{n+2}{n+1} \Xi_{ab} + \Lambda g_{ab}.$$ 

Hence

$$J^a_c R_{ab}{}^c{}_d = J_a{}^c V_{bd} - J_d{}^c \Phi_{ad} - 2 J^a{}^c \Phi_{da} = J_a{}^c V_{bd} - \frac{n+2}{n+1} \Sigma_{bd} - 2(n + 1) \Lambda J_{bd}. $$

On the other hand, from (15) we find

$$J^a_c R_{ab}{}^c{}_d = -2(n + 2) \Sigma_{bd} - 2(n + 1) \Lambda J_{bd}$$

and, comparing these two expressions gives

$$J^a_c V_{bd} - \frac{n+2}{n+1} \Sigma_{bd} = -2(n + 2) \Sigma_{bd}$$

and we have established the following.

**Proposition 2.** Concerning the symplectic curvature decomposition on a Kähler manifold,

$$J^a_c V_{bd} = -\frac{n(n+2)}{n+1} \Sigma_{bd}. $$

**Corollary 2.** The symplectic tractor connection on a Kähler manifold is symplectically flat if and only if the metric has constant holomorphic sectional curvature.
Proof. According to Corollary 1, we have to interpret the constraint $V_{abcd} = 0$ in the Kähler case. From (15) it is already clear that $U_{abcd} = 0$ and Proposition 2 implies that also $\Sigma_{ab} = 0$ so (15) reduces to

$$R_{abcd} = \Lambda(\delta_a^c g_{bd} - \delta_b^c g_{ad} + J_a^c J_{bd} - J_b^c J_{ad} + 2J_{ab}J^c_d),$$

which is exactly the constancy of holomorphic sectional curvature. \[\square\]

5. BGG-like complexes on $\mathbb{CP}_n$

Fix a real vector space $g_{-1}$ of dimension $2n$, let $g_1$ denote its dual, and fix a non-degenerate 2-form $J_{ab} \in \Lambda^2 g_1$. The $(2n + 1)$-dimensional Heisenberg Lie algebra may be realised as

$$h = \mathbb{R} \oplus g_{-1},$$

where the first summand is the 1-dimensional centre of $h$ and the Lie bracket on $g_{-1}$ is given by

$$[X, Y] = 2J_{ab}X^a Y^b \in \mathbb{R} \hookrightarrow h.$$

We should admit right away that the reason for this seemingly arcane notation is that we shall soon have occasion to write

$$\text{sp}(2n + 2, \mathbb{R}) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \oplus \mathbb{R} \text{ as a Lie subalgebra of } \text{sp}(2n + 2, \mathbb{R}).$$

(a [2]-graded Lie algebra as in [3, §4.2.6]) and, in particular, regard $h = \mathbb{R} \oplus g_{-1} = g_{-2} \oplus g_{-1}$ as a Lie subalgebra of $\text{sp}(2n + 2, \mathbb{R})$. Be that as it may, let us suppose that $V$ is a finite-dimensional representation of $h$. The Lie algebra cohomology $H^r(h, V)$ may be realised as the cohomology of the Chevalley-Eilenberg complex

$$0 \to V \to h^* \otimes V \to \cdots \to \bigwedge^r h^* \otimes V \to \bigwedge^{r+1} h^* \otimes V \to \cdots$$

as, for example, in [3, Chapter IV]. We shall require, however, the following alternative realisation.

Lemma 3. There is a complex

$$0 \to V \xrightarrow{\partial} g_1 \otimes V \xrightarrow{\partial_1} \bigwedge^2 g_1 \otimes V \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_r} \bigwedge^r g_1 \otimes V$$

whose cohomology realises $H^r(h, V)$. Here, we are writing

$$\bigwedge^r g_1 \equiv \{\omega_{abc \cdots d} \in \bigwedge^r g_1 \mid J^{ab} \omega_{abc \cdots d} = 0\},$$

where $J^{ab} \in \bigwedge^2 g_{-1}$ is the inverse of $J_{ab} \in \bigwedge^2 g_1$ (let’s say normalised so that $J_{ab}J^{ac} = \delta_b^c$).
Proof. Notice that any representation \( \rho : \mathfrak{h} \to \text{End}(\mathcal{V}) \) is determined by its restriction to \( \mathfrak{g}_{-1} \subset \mathfrak{h} \). Indeed, writing \( \partial_a : \mathfrak{g}_{-1} \to \text{End}(\mathcal{V}) \) for this restriction, to say that \( \rho \) is a representation of \( \mathfrak{h} \) is to say that

\[
\begin{align*}
(\partial_a \partial_b - \partial_b \partial_a)v &= 2J_{ab} \theta v \\
(\partial_a \theta - \theta \partial_a)v &= 0
\end{align*}
\]

\[ \forall v \in \mathcal{V}, \]

where \( \theta \in \text{End}(\mathcal{V}) \) is \( \rho(1) \) for \( 1 \in \mathbb{R} \subset \mathfrak{h} \).

The splitting \( \mathfrak{h}^* = \mathfrak{g}_1 \oplus \mathbb{R} \) allows us to write (17) as

\[
\begin{align*}
\mathcal{V} \rightarrow \mathfrak{h}^* \otimes \mathcal{V} \rightarrow \wedge^2 \mathfrak{h}^* \otimes \mathcal{V} \rightarrow \wedge^3 \mathfrak{h}^* \otimes \mathcal{V} \rightarrow \cdots \\
\mathcal{V} \rightarrow \mathfrak{g}_1 \otimes \mathcal{V} \rightarrow \wedge^2 \mathfrak{g}_1 \otimes \mathcal{V} \rightarrow \wedge^3 \mathfrak{g}_1 \otimes \mathcal{V} \rightarrow \cdots,
\end{align*}
\]

where the differentials are given by

\[
\begin{align*}
v \mapsto \left[ \begin{array}{c}
\partial_a v \\
\theta v
\end{array} \right] \\
\phi \mapsto \left[ \begin{array}{c}
\partial_a \phi_b - J_{ab} \eta \\
\eta
\end{array} \right] \\
\omega \mapsto \left[ \begin{array}{c}
\omega_{bc} \\
J_{ab} \psi_{cd} + J_{ab} \psi_{cd}
\end{array} \right]
\end{align*}
\]

et cetera. In particular, notice that the homomorphisms

\[
\wedge^{r-1} \mathfrak{g}_1 \ni \psi \mapsto \pm J \wedge \psi \in \wedge^{r+1} \mathfrak{g}_1
\]

are

- independent of the representation on \( \mathcal{V} \),
- injective for \( 1 \leq r < n \),
- an isomorphism for \( r = n \),
- surjective for \( n < r \leq 2n - 1 \).

Note that \( \wedge^{r+1} \mathfrak{g}_1 \) is complementary to the image of (21) for \( 1 \leq r < n \). Also note the isomorphisms

\[
\wedge^{2n+1-r} \mathfrak{g}_1 \xrightarrow{J \wedge J \wedge \cdots \wedge J} \wedge^{r-1} \mathfrak{g}_1, \quad \text{for} \ n \leq r \leq 2n + 1,
\]

under which the kernel of (21) may be identified with

\[
\wedge^{2n+1-r} \mathfrak{g}_1, \quad \text{for} \ n \leq r \leq 2n - 1.
\]

Diagram chasing in (20) (or the spectral sequence of a filtered complex) finishes the proof. \( \square \)

Remark. Evidently, the equations (19) are algebraic versions of

\[
\begin{align*}
(\nabla_a \nabla_b - \nabla_b \nabla_a) \Sigma &= 2J_{ab} \Theta \Sigma \\
(\nabla_a \Theta - \Theta \nabla_a) \Sigma &= 0
\end{align*}
\]

\[ \forall \Sigma \in \Gamma(\mathcal{E}), \]

which hold for a symplectically flat connection \( \nabla_a \) on smooth vector bundle \( \mathcal{E} \) on \( M \). Also (20) is the evident algebraic counterpart to the differential complex of Lemma 1. It follows that explicit formulæ for
the operators $\partial_1$ in the complex (18) follow the differential versions (2) and (3) with $\wedge^n g \otimes V \to \wedge^n g \otimes V$ being given by $\partial_1^2 = \frac{2}{n} \partial_1$.

Let us now consider the tractor connection on $\mathbb{C}P^n$. According to Theorem 2, the remarks following its statement, and the discussions in §3, this is the connection on $T = \wedge^0 \oplus \wedge^1 \oplus \wedge^0$ given by

$$
\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + J_{ab} \rho + g_{ab} \sigma \\ \nabla_a \rho - J_a^b \mu_b \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma \\ \nabla_a \mu_b + g_{ab} \sigma \\ \nabla_a \rho - J_a^b \mu_b \end{bmatrix} + \begin{bmatrix} -\mu_a \\ J_{ab} \rho \\ 0 \end{bmatrix}.
$$

The induced operator $\nabla : \wedge^1 \otimes T \to \wedge^2 \otimes T$ is

$$
\begin{bmatrix} \sigma_b \\ \mu_{bc} \\ \rho_b \end{bmatrix} \mapsto \begin{bmatrix} \nabla_{[a \sigma_b]} \\ \nabla_{[a \mu_{bc}]} + g_{[a \sigma_b]} \\ \nabla_{[a \rho_b]} - J_{[a \rho_b]} \end{bmatrix} + \begin{bmatrix} \mu_{[ab]} \\ -J_{[a \rho_b]} + \frac{1}{2n} J_{cd} \mu_{cd} J_{ab} \\ 0 \end{bmatrix}.
$$

but Corollary 2 says the tractor connection on $\mathbb{C}P^n$ is symplectically flat so we should contemplate $\nabla_1^* : \wedge^1 \otimes T \to \wedge^2 \otimes T$ from Theorem 1 viz.

$$
\begin{bmatrix} \sigma_b \\ \mu_{bc} \\ \rho_b \end{bmatrix} \mapsto \begin{bmatrix} \nabla_{[a \sigma_b]} - \frac{1}{2n} J^{cd} \nabla_{e \sigma_d} J_{ab} \\ \cdots \\ \cdots \end{bmatrix} + \begin{bmatrix} \mu_{[ab]} - \frac{1}{2n} J_{cd} \mu_{cd} J_{ab} \\ -J_{[a \rho_b]} - \frac{1}{2n} \rho_{[a \rho_b]} J_{[a \rho_b]} \\ 0 \end{bmatrix}.
$$

From these formulæ, let us focus attention on the homomorphisms

$$
0 \to T \to \wedge^1 \otimes T \to \wedge_1^2 \otimes T \to \cdots
$$

(22)

$$
\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} -\mu_a \\ J_{ab} \rho \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} \sigma_b \\ \mu_{bc} \\ \rho_b \end{bmatrix} \mapsto \begin{bmatrix} \mu_{[ab]} - \frac{1}{2n} J_{cd} \mu_{cd} J_{ab} \\ -J_{[a \rho_b]} - \frac{1}{2n} \rho_{[a \rho_b]} J_{[a \rho_b]} \\ 0 \end{bmatrix}.
$$

It is evident that this is a complex and that its cohomology so far is $\wedge^0$ in degree 0 and $\wedge^2 \wedge^1$ in degree 1.

On the other hand, one may check that the defining representation of the Lie algebra $\mathfrak{sp}(2n + 2, \mathbb{R})$ on $\mathbb{R}^{2n+2} = \mathbb{R} \oplus \mathbb{R}^{2n} \oplus \mathbb{R}$ restricts via (16) to a representation of the Heisenberg Lie algebra $\mathfrak{h} = \mathbb{R} \oplus \mathfrak{g}_{-1}$, given explicitly by

$$
\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} -\mu_a \\ J_{ab} \rho \\ 0 \end{bmatrix}.$$

$$
\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} \sigma \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} -\mu_a \\ J_{ab} \rho \\ 0 \end{bmatrix}.
$$
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(noticing that equations (19) hold, as they must). We may also find \( \theta \) as part of the curvature of the tractor connection on \( \mathbb{C}P_n \). Specifically, the formula from Proposition 1 reduces to

\[
(\nabla_a \nabla_a - \nabla_b \nabla_a) \begin{bmatrix} \sigma \\ \mu_d \\ \rho \end{bmatrix} = 2J_{ab} \begin{bmatrix} \rho \\ J_d^e \mu_e \\ -\sigma \end{bmatrix}
\]

and we find \( \theta \) as the top component of \( \Theta \) : \( \mathcal{T} \to \mathcal{T} \) where \( \Theta \) is defined by (5).

If we now consider the entire complex from Theorem 1, with filtration induced by

\[
\bigwedge^0 \subset \bigwedge^1 \oplus \bigwedge^0 \subset \bigwedge^0 \oplus \bigwedge^1 \oplus \bigwedge^0 = \mathcal{T}
\]

of \( \mathcal{T} \), then the associated spectral sequence (or corresponding diagram chasing) yields (22) continuing as in (18) including the middle operator \( \nabla^2_{\perp} - \frac{2}{n} \theta : \bigwedge^n_{\perp} \to \bigwedge^n_{\perp} \). The same reasoning pertains for any Fedosov structure with \( V_{abcd} = 0 \) as in Corollary 1. Evidently, this sequence of vector bundle homomorphisms is induced by the complex (18) and, together with Lemma 3, the spectral sequence of a filtered complex (or the appropriate diagram chasing) immediately yields the following.

**Theorem 3.** Suppose \( \nabla_a \) is a torsion-free connection on a symplectic manifold \( (M, J_{ab}) \), such that \( \nabla_a J_{bc} = 0 \) and so that the corresponding curvature tensor \( V_{ab}^c \) vanishes. Fix a finite-dimensional representation \( E \) of \( \text{Sp}(2n+2, \mathbb{R}) \) and let \( E \) denote the associated ‘tractor bundle’ induced from the standard tractor bundle and the representation \( E \) (so that the standard representation of \( \text{Sp}(2n+2, \mathbb{R}) \) on \( \mathbb{R}^{2n+2} \) yields the standard tractor bundle). In accordance with Corollary 1, the induced ‘tractor connection’ \( \nabla : E \to \bigwedge^1 \otimes E \) is symplectically flat and we may define \( \Theta : E \to E \) by (5). Having done this, there are complexes of differential operators

\[
0 \to H^0(\mathfrak{h}, E) \to H^1(\mathfrak{h}, E) \to H^2(\mathfrak{h}, E) \to \cdots \to H^n(\mathfrak{h}, E)
\]

\[
0 \leftarrow H^{2n+1}(\mathfrak{h}, E) \leftarrow H^{2n}(\mathfrak{h}, E) \leftarrow H^{2n-1}(\mathfrak{h}, E) \leftarrow \cdots \leftarrow H^{n+1}(\mathfrak{h}, E)
\]

where \( H^r(\mathfrak{h}, E) \) denotes the tensor bundle on \( M \) that is induced by the cohomology \( H^r(\mathfrak{h}, E) \) as a representation of \( \text{Sp}(2n, \mathbb{R}) \). This complex is locally exact except near the beginning where

\[
\ker : H^0(\mathfrak{h}, E) \to H^1(\mathfrak{h}, E) \quad \text{and} \quad \ker : H^1(\mathfrak{h}, E) \to H^2(\mathfrak{h}, E)
\]

\[
\text{im} : H^0(\mathfrak{h}, E) \to H^1(\mathfrak{h}, E)
\]
may be identified with the locally constant sheaves \( \ker \Theta \) and \( \text{coker} \Theta \), respectively. In particular, for \( \mathbb{CP}^n \) with its Fubini–Study connection, these sheaves vanish and the complex is locally exact everywhere.

**Proof.** It remains only to observe that for the Fubini–Study connection we see from (23) that \( \Theta : \mathcal{T} \to \mathcal{T} \) is an isomorphism. \( \square \)

The main point about Theorem 3, however, is that if the representation \( E \) of \( \text{Sp}(2n + 2, \mathbb{R}) \) is irreducible, then the representations \( H^r(\mathfrak{h}, E) \) of \( \text{Sp}(2n, \mathbb{R}) \) are also irreducible and are computed by a theorem due to Kostant [9]. Specifically, if we denote the irreducible representations of \( \text{Sp}(2n + 2, \mathbb{R}) \) and \( \text{Sp}(2n, \mathbb{R}) \) by writing the highest weight as a linear combination of fundamental weights and recording the coefficients over the corresponding nodes of the Dynkin diagrams for \( C_{n+1} \) and \( C_n \), as is often done, then Kostant’s Theorem says that

\[
\begin{align*}
H^0(\mathfrak{h}, \begin{array}{cccccc}
b & c & d & e & f \\
\end{array}) &= \begin{array}{cccccc}
b & c & d & e & f \\
\end{array} \\
H^1(\mathfrak{h}, \begin{array}{cccccc}
b & c & d & e & f \\
\end{array}) &= \begin{array}{cccccc}
a+b+1 & c & d & e & f \\
\end{array} \\
H^2(\mathfrak{h}, \begin{array}{cccccc}
b & c & d & e & f \\
\end{array}) &= \begin{array}{cccccc}
a & b+c+1 & d & e & f \\
\end{array} \\
H^3(\mathfrak{h}, \begin{array}{cccccc}
b & c & d & e & f \\
\end{array}) &= \begin{array}{cccccc}
a & b & c+d+1 & e & f \\
\end{array} \\
&\vdots \\
H^n(\mathfrak{h}, \begin{array}{cccccc}
b & c & d & e & f \\
\end{array}) &= \begin{array}{cccccc}
a & b & c & e+f+1 \\
\end{array}
\end{align*}
\]

and for \( r \geq n + 1 \), there are isomorphisms \( H^r(\mathfrak{h}, E) = H^{2n+1-r}(\mathfrak{h}, E) \).

Using the same notation for the bundles \( H^r(\mathfrak{h}, E) \), the complexes of Theorem 3 become

\[
\begin{array}{cccccc}
b & c & d & e & f \\
\end{array} \xrightarrow{\nabla^{a+1}} \begin{array}{cccccc}
a+b+1 & c & d & e & f \\
\end{array} \xrightarrow{\nabla^{b+1}} \begin{array}{cccccc}
a & b+c+1 & d & e & f \\
\end{array} \xrightarrow{\nabla^{c+1}} \begin{array}{cccccc}
a & b & c+d+1 & e & f \\
\end{array} \xrightarrow{\nabla^{e+1}} \begin{array}{cccccc}
a & b & c & e+f+1 \\
\end{array} \xrightarrow{\nabla^{2f+2}} \begin{array}{cccccc}
a & b & c & e+f+1 \\
\end{array} \xrightarrow{\nabla^{e+1}} \ldots \\
\end{array}
\]

for arbitrary non-negative integers \( a, b, c, d, \ldots, e, f \). When all these integers are zero, this is the Rumin–Seshadri complex. Just the first three terms in this complex, in the special case when only \( a \) is non-zero, are already essential in [3]. For example, if \( a = 1 \), then the first two
differential operators are
\[ \sigma \mapsto \nabla_a \nabla_b \sigma + \Phi_{ab} \sigma \quad \text{and} \quad \phi_{bc} \mapsto (\nabla_a \phi_{bc} - \nabla_b \phi_{ac})_\perp \]
where \( \phi_{bc} \) is symmetric and \( (\ )_\perp \) means to take the trace-free part with respect to \( J_{ab} \). From the curvature decomposition and Bianchi identity we find that their composition is
\[ \sigma \mapsto -V_{ab}^c \nabla_d \sigma + Y_{abc} \sigma, \]
which vanishes in case \( V_{abcd} = 0 \). In case \( \Theta \) is invertible, as for the Fubini–Study connection, we conclude that this sequence of differential operators is locally exact.

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