Spin squeezing of high-spin, spatially extended quantum fields

Jay D Sau\textsuperscript{1,2}, Sabrina R Leslie\textsuperscript{1}, Marvin L Cohen\textsuperscript{1,3} and Dan M Stamper-Kurn\textsuperscript{1,3,4}

\textsuperscript{1} Department of Physics, University of California, Berkeley, CA 94720, USA
\textsuperscript{2} Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, MD 20742-4111, USA
\textsuperscript{3} Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA
E-mail: dmsk@berkeley.edu

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Abstract. Investigations of spin squeezing in ensembles of quantum particles have been limited primarily to a subspace of spin fluctuations and a single spatial mode in high-spin and spatially extended ensembles. Here, we show that a wider range of spin squeezing is attainable in ensembles of high-spin atoms, characterized by sub-quantum-limited fluctuations in several independent planes of spin-fluctuation observables. Further, considering the quantum dynamics of an \( f = 1 \) ferromagnetic spinor Bose–Einstein condensate, we demonstrate theoretically that a high degree of spin squeezing is attained in multiple spatial modes of a spatially extended quantum field and that such squeezing can be extracted from spatially resolved measurements of magnetization and nematicity, i.e. the vector and quadrupole magnetic moments, of the quantum gas. Taking into account several experimental limitations, we predict that the variance of the atomic magnetization and nematicity may be reduced as far as 20 dB below the standard quantum limits.

\textsuperscript{4} Author to whom any correspondence should be addressed.
1. Introduction

Spin squeezing describes a form of entanglement in a many-spin system for which the variance in measurements of particular collective spin variables is smaller than the minimum variance achievable in uncorrelated systems [1, 2]. Aside from representing a theoretically tractable form of entanglement in many-body quantum systems, spin squeezing also promises to improve measurement precision in a variety of applications such as chronometry, magnetometry and atom interferometry. The first realizations of this promise include the recent achievement of metrologically beneficial pseudo-spin squeezing in cold atomic gases on Zeeman-insensitive hyperfine transitions suitable for atomic clocks [3]–[5], and of squeezed states in a prototype atom interferometer [6].

The conventional form of spin squeezing involves the collective vector-spin operator, F, which is the equal-weight sum of the vector-spin operators for all of the ensemble’s constituent particles. Given an average ensemble spin oriented along the z-axis, the measurement variance of orthogonal spin projections obeys an uncertainty relation, \( \langle (\Delta F_x)^2 \rangle \langle (\Delta F_y)^2 \rangle \geq \langle F_z \rangle^2 \)/4. Ensemble states of uncorrelated particles are limited by the standard quantum limited variance, \( \langle (\Delta F_z)^2 \rangle_{\text{SQL}} = |\langle F_z \rangle|/2 \), achieved by partitioning the minimum measurement uncertainties equally between the two vector-spin projections.

This form of vector operator spin squeezing describes present experiments not only on pseudo-spin-1/2 atomic gases, i.e. where each atom is restricted to just two internal [3]–[5] or external [6, 7] states, but also on gases of higher-spin atoms (e.g. the \( f = 4 \) hyperfine-state manifold of Cs [8]). However, the description of higher-spin atoms by their vector-spin alone is incomplete. On the one hand, light–atom interactions that affect higher-order spin moments may impair efforts to create vector-spin squeezing in high-spin atomic gases [9, 10]. On the other hand, the deliberate addressing and measurement of higher-order spin moments [11] offers benefits for certain metrological applications, e.g. eliminating directional dependencies of atomic magnetometers [12].
Here, we consider theoretically the characterization and preparation of spin-squeezed states in $f = 1$ spinor Bose gases. We go beyond existing treatments of spin squeezing by evaluating such squeezing in an ensemble that is explicitly higher dimensional not only in its spin degrees of freedom, but also in its spatial degrees of freedom. Our investigation is motivated by two experimental results: the controlled amplification of spin noise in $^{87}$Rb quantum fluids [13]–[15] and the use of a spinor Bose–Einstein condensate as a spatially resolving magnetometer [16]. As we discuss herein, these two results, respectively, provide a complete means for inducing squeezing of both the spin vector and quadrupole moments of the $f = 1$ quantum gas, and also a means for applying the subsequent spatially extended non-classical quantum fluid toward spatially resolved magnetometry with sub-shot-noise sensitivity.

2. Single-mode squeezing for arbitrary spin

We begin by introducing a generalized treatment of spin squeezing that applies equally to ensembles of both spin-1/2 and higher-spin particles. Such squeezing will be defined with respect to a coherent spin state [2], in which all particles are prepared in an identical single-particle spin-$f$ state $|\psi\rangle$. Consider an orthogonal basis of the single-particle spin space that includes the state $|\psi\rangle$ and also the $2f$ states $|\xi_\alpha\rangle$. We now define many-body observables $M_a^{(1)} = \sum_\alpha (|\xi_\alpha\rangle\langle\psi| + |\psi\rangle\langle\xi_\alpha|)_{\alpha}$ and $M_a^{(2)} = \sum_\alpha i (|\xi_\alpha\rangle\langle\psi| - |\psi\rangle\langle\xi_\alpha|)_{\alpha}$, where the summation index $\alpha$ is taken over all particles in the $N$-particle ensemble. These spin-fluctuation operators describe the subspace of spin excitations orthogonal to the coherent spin state. With respect to this uncorrelated state, these observables satisfy the commutation relations $\{[M_a^{(1)}, M_b^{(2)}]\} = 2i\hbar\delta_{a,b}$. Thus, measurements of each pair $M_a^{(1,2)}$ independently obey an uncertainty relation $\langle(\Delta M_a^{(1)})^2\rangle\langle(\Delta M_a^{(2)})^2\rangle \geq N^2$. Following the treatment of squeezing for vector-spin operators, we identify one mode of spin fluctuations, labeled by $a$, to be spin squeezed when the measurement variance for one quadrature of the $M_a^{(1)}$–$M_a^{(2)}$ plane is below the standard quantum limit, $N$.

Applying this treatment to an ensemble of spin-1/2 particles, we recall that any coherent spin state can be written as the $m = +1/2$ eigenstate of a projection of the dimensionless vector-spin. Taking that projection to lie along $\hat{z}$, we identify the one mode of spin-fluctuation operators as the Pauli operators $\sigma_z$ and $\sigma_\phi$. More generally, for an ensemble of spin-$f$ particles prepared in the $|\psi\rangle = |m_z = f\rangle$ eigenstate of the $F_z$ spin operator, the spin-fluctuation operators defined by the state $|\xi_\phi\rangle = |m_z = f - 1\rangle$ act on $|\psi\rangle$ as $\sqrt{2f}/f$ times the spin-vector operators $F_x$ and $F_y$. For this one mode of spin-fluctuation operators, the commutation relation matches that of the spin-vector operators. Thus, we recover the conventional description of vector-spin squeezing as pertaining to just one mode of spin fluctuations atop a maximum-spin coherent spin state.

To illustrate further the possibility of squeezing in several spin-fluctuation modes, we consider spin squeezing of spin-1 particles prepared initially in the $|m_z = 0\rangle$ state. Our treatment is facilitated by working in a polar state basis, where $|\phi_e\rangle$ is the zero-eigenvalue state of $\vec{F} \cdot \vec{e}$, with $e \in \{x, y, z\}$. The pair of fluctuation operators defined according to the state $|\phi_e\rangle$ ($|\phi_0\rangle$) are identified as $M^{(1)} = -N_{xz} (-N_{yz})$ and $M^{(2)} = +F_y (-F_y)$, where the former is a component $^5$ Specifically, we use the basis $|\phi_e\rangle = (|m_z = 1\rangle - |m_z = -1\rangle)/\sqrt{2}$, $\phi_0 = i (|m_z = 1\rangle + |m_z = -1\rangle)/\sqrt{2}$ and $|\phi_y\rangle = |m_z = 0\rangle$. With this definition, the vector-spin operator takes the form $(F_a)_{bc} = -i\epsilon_{abc} |\phi_e\rangle (\phi_b)$, where the indices run over the Cartesian coordinates.

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of the quadrupole moment tensor\(^6\). The coherent spin state \(|m_z = 0\rangle\) is thus regarded as the uncorrelated vacuum state for independent spin fluctuations in the \(F_y - N_{yz}\) and \(F_z - N_{xz}\) planes, and correlated states may be spin squeezed with respect to vector-spin or quadrupole-spin components, or linear combinations thereof. Finally, we note that this unconventional form of spin squeezing, about the polar \(|m_z = 0\rangle\) state, may be mapped directly onto the more conventional vector-spin squeezing discussed above by a unitary transformation that rotates the polar basis states onto the proper \(F_z\) eigenstates; experimentally, such generic unitary transformations upon alkali-atom hyperfine spin states are produced by rf-pulses and quadratic Zeeman energy shifts [17].

3. Spin squeezing in a spatially extended quantum field

We now consider such multi-spin-mode squeezing in a spatially extended, ferromagnetic \(f = 1\) gaseous Bose–Einstein condensate where, as we show below, such squeezing is produced naturally by spin-dependent interactions\(^7\). As in recent experiments [13, 14], these condensed atoms (at rest) are prepared initially in the common spin state \(|m_z = 0\rangle\). Within the volume occupied by the condensate, we define position-space spin-fluctuation measurement densities

\[
M_a^{(1)}(\mathbf{r}) = (\phi_a^\dagger(\mathbf{r})\phi_z(\mathbf{r}) + \phi_z^\dagger(\mathbf{r})\phi_a(\mathbf{r})) ,
\]

\[
M_a^{(2)}(\mathbf{r}) = i(\phi_a^\dagger(\mathbf{r})\phi_z(\mathbf{r}) - \phi_z^\dagger(\mathbf{r})\phi_a(\mathbf{r})) ,
\]

where \(\phi_z(\mathbf{r})\) is now the position-space Bose field operator for the polar state \(|\phi_z\rangle\) \((e \in \{x, y, z\})\), and the index \(a \in \{x, y\}\) runs only over the transverse spin polarizations. These observables represent components of the local nematicity (rank-2 moments of the atomic spin) and magnetization (rank-1 moments of the atomic spin) of the quantum gas, and obey the commutation relation \(\{M_a^{(1)}(\mathbf{r}), M_b^{(2)}(\mathbf{r}')\} = 2i n(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{r}') \delta_{a,b}\), where \(n(\mathbf{r}) = \langle \phi_z^\dagger(\mathbf{r})\phi_z(\mathbf{r}) \rangle\), and expectation values are taken with respect to the polar condensate.

These measurement density operators are used to construct operators corresponding to spatially resolved measurements of particular spin fluctuations of the quantum field. Defining the real measurement mode functions \(\{A(\mathbf{r}), B(\mathbf{r}), C(\mathbf{r}), D(\mathbf{r})\}\), we may define mode measurement operators as follows:

\[
\begin{pmatrix}
M_a^{(1)} & M_b^{(1)} & M_c^{(1)} & M_d^{(1)} \\
M_a^{(2)} & M_b^{(2)} & M_c^{(2)} & M_d^{(2)}
\end{pmatrix}
\begin{pmatrix}
A(\mathbf{r}) \\
B(\mathbf{r}) \\
C(\mathbf{r}) \\
D(\mathbf{r})
\end{pmatrix}
= \int d^3 \mathbf{r} \begin{pmatrix} A(\mathbf{r}) \\ B(\mathbf{r}) \\ C(\mathbf{r}) \\ D(\mathbf{r}) \end{pmatrix} \begin{pmatrix}
M_a^{(1)}(\mathbf{r}) \\
M_b^{(1)}(\mathbf{r}) \\
M_c^{(1)}(\mathbf{r}) \\
M_d^{(1)}(\mathbf{r})
\end{pmatrix} ,
\]

where the common polarization index is omitted for clarity. This definition accommodates measurements made on the extended quantum field where measurement results from different positions in the field, performed upon spatially varying linear combinations of the spin-fluctuation observables, are summed coherently to obtain the measurement result. Specifically, for \(M_{a,b,c,d}^{(1)}\) (and similarly for \(M_{a,b,c,d}^{(2)}\)), the local measurement weight is given as \(\sqrt{A^2 + B^2}\)

\(^6\) We adopt the definition \(N_{ab} = (F_x F_y + F_y F_z) - 4\delta_{ab}/3\), where \(a, b \in \{x, y, z\}\) and \(F_a\) denotes an angular momentum component operator in matrix form.

\(^7\) We emphasize that Bose–Einstein condensation, or even Bose–Einstein statistics, is not necessary for the existence of spin squeezing in an extended, multi-mode quantum field. Much of the notation developed in this work may be extended also to squeezing with more generic constituents. Here, we focus on spinor Bose–Einstein condensates not only for notational convenience, but also to describe the role of spin-dependent interactions in creating such squeezing.
and the locally measured quadrature is defined by the angle \(\tan^{-1}(B/A)\). The commutation relation
\[
\{[M^{(1)}_{A,B,C,D}, M^{(2)}_{A,B,C,D}]\} = 2i \int d^3 r (AD - BC)n(r)
\]
is non-zero to the extent that the two measurement operators probe non-parallel spin-fluctuation quadratures on a common population of atoms.

We see that our present treatment subsumes the commonly discussed single-mode description of spin squeezing in atomic ensembles. That is, the equal-weight sum of atomic spin operators is measured by spatial-mode operators defined with the uniform functions \(A = D = 1\) and \(B = C = 0\). However, our treatment easily allows for a multi-mode description of spin squeezing and collective spin measurements of a spatially extended quantum field. For example, spin squeezing in a spatially resolved measurement of the atomic spin, e.g. in the spinor-gas magnetometer of [16], may be treated by defining measurement mode functions corresponding to every resolved pixel in the magnetization-sensitive image. Our treatment applies also to optical measurements sensitive to spatially varying linear combinations of magnetization and nematicity, which are achieved by measuring different components of the linear optical susceptibility tensor [18].

4. Generation of spin squeezing in a ferromagnetic \(f = 1\) Bose–Einstein condensate

Now, let us consider the evolution of the paramagnetic condensate under the quantum-coherent spin dynamics produced by the spin-dependent contact interactions between atoms [19, 20]. As in previous experiments, we consider the condensate to be prepared in the uncorrelated polar spin dynamics produced by the spin-dependent contact interactions between atoms \([17, 18]\). As

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12 \text{ Generation of spin squeezing in a ferromagnetic } f = 1 \text{ Bose–Einstein condensate}
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operators (omitting the polarization index)

\[ Z_n = \int d^3 r \, \zeta_n(r) \mathcal{H}_0^{-1} Z(r), \]  
\[ P_n = \int d^3 r \, \zeta_n(r) \, P(r). \]  

These operators are found to be canonically conjugate, satisfying the equations of motion \( \hbar \dot{Z}_n = P_n \) and \( \hbar \dot{P}_n = -E_n^2 Z_n \) and commutation relations \([Z_n, P_m] = -i \delta_{n,m}\). The canonical evolution of magnon excitation mode \( n \) is akin to that of a harmonic oscillator with frequency \( \omega_n = \sqrt{E_n^2}/\hbar \), with \( E_n^2 > 0 \) defining stable magnon excitations and \( E_n^2 < 0 \) defining unstable modes.

The action of this canonical evolution on the initial fluctuations in each normal mode may be discerned by examining the scaled covariance matrix

\[ C_n(t) = \text{Re} \left\{ \left(\frac{\tilde{Z}_n(t)}{P_n(t)}\right) \left(\tilde{Z}_n(t) - \tilde{P}_n(t))\right) \right\}, \]

where the mode operators are scaled by their initial rms fluctuations. It is important to note that this scaling circularizes the initial fluctuations of the normal mode operators, but that these initial fluctuations do not generally have the minimum uncertainty. One finds, however, that the fluctuations do reach their minimum product for excitations of a homogeneous condensate and initial fluctuations do not generally have the minimum uncertainty. One finds, however, that the scaling circularizes the initial fluctuations of the normal mode operators, but that these initial fluctuations do not generally have the minimum uncertainty. One finds, however, that the

The Hermitian covariance matrix may be expressed as \( C_n(t) = R_n(t)S_n(t)R_n^\dagger(t) \) with \( S_n(t) \) being diagonal with eigenvalues

\[ S^\pm = 1 + 2 \kappa \sin^2 \omega_n t \pm 2 \sqrt{\kappa \sin^2 \omega_n t \left(1 + \kappa \sin^2 \omega_n t\right)}, \]

with \( \kappa = (\chi + \chi^{-1} - 2)/4 \) and \( \chi = E_n^2 \langle Z_n^2(0)\rangle/\langle P_n^2(0)\rangle \). We note that \( \kappa \sin^2 \omega_n t \) is positive for both stable and unstable magnon modes.

For stable magnon modes, the initial spin fluctuations rotate in the \( \tilde{Z}_n - \tilde{P}_n \) plane and also experience periodic, bounded dilatations and contractions. In contrast, for unstable magnon modes, the spin fluctuations become progressively squeezed along one quadrature axis and amplified along the other, with the squeezed/amplified axes defined by the rotation matrix \( R_n(t) \). The fluctuations and dynamical evolution of these mode operators relate directly to those of normalized spatial-mode measurement observables given as

\[ \tilde{M}_n^{(1)}(t) = \int d^3 r \, \frac{\mathcal{H}_0^{-1} \zeta_n(r)}{n^{1/2}(r) \langle Z_n^2(0)\rangle^{1/2}} M_n^{(1)}(r), \]

\[ \tilde{M}_n^{(2)}(t) = \int d^3 r \, \frac{\zeta_n(r)}{n^{1/2}(r) \langle P_n^2(0)\rangle^{1/2}} M_n^{(2)}(r), \]

corresponding to \( \tilde{Z}_n \) and \( \tilde{P}_n \), respectively, at \( t = 0 \). Spatially resolved spin measurements tailored to detecting the squeezed and amplified quadratures of the \( n \)th normal mode are defined as time-dependent linear combinations of \( \tilde{M}_n^{(1)} \) and \( \tilde{M}_n^{(2)} \) through the rotation matrix \( R_n(t) \).
4.1. Evolution of a homogeneous spinor condensate

For a homogeneous condensate, the evolution following a quench of the initially paramagnetic quantum gas is expressed simply. Translational symmetry allows one to identify normal modes \(a\ priori\) as possessing Fourier components only with magnitude \(k\). The normal-mode equation (6) is satisfied by the functions \(\xi_{k,\vartheta} = \sqrt{2/V} \cos(k \cdot r + \vartheta)\), with \(V\) being the condensate volume and with the magnon-excitation spectrum given as \(E_k^2 = (\epsilon_k + q)(\epsilon_k + q - q_0)\), where \(\epsilon_k = \hbar^2 k^2 / 2m\) [19]. Unstable modes exist for \(q\) below a critical value \(q_0 = -c_2 n\), specifically within the range \(|\epsilon_k + q - q_0/2| < q_0/2\).

The initial fluctuations of the normal mode operators defined with respect to the mode functions \(\xi_{k,\vartheta}\) are given as \(\langle Z_{k,\vartheta}^2 \rangle = (\epsilon_k + q)^{-2}\) and \(\langle P_{k,\vartheta}^2 \rangle = 1\), giving \(\kappa = q_0^2/(4E_k^2)\). The normalized covariance matrix for modes with wave vector \(k\) is given as

\[
C_k(t) = \begin{pmatrix}
1 - \frac{q_0}{\epsilon_k + q} \sin^2(\omega_k t) & \frac{q_0}{\hbar \omega_k} \cos(\omega_k t) \sin(\omega_k t) \\
\frac{q_0}{\hbar \omega_k} \cos(\omega_k t) \sin(\omega_k t) & 1 + \frac{q_0}{\epsilon_k + q - q_0} \sin^2(\omega_k t)
\end{pmatrix}.
\]

We see that the dynamically unstable momentum-space modes are progressively squeezed over time (figure 1). The maximally unstable mode, with \(\omega_k^2 = -(q_0/2\hbar)^2\), is accessed for \(q < q_0/2\). For this mode, the orientation of the squeezing axis is fixed at an angle \(\vartheta = -\pi/4\) in the \(M^{(1)}-M^{(2)}\) plane, and the greatest squeezing is achieved. Away from this maximally squeezed mode, the orientation of the squeezing/amplification axes varies in time. At long times, defined by \(\kappa \sin^2(\omega_k t) = \frac{q_0^2}{4\hbar^2 |\omega_k|} \sin^2(\omega_k |t|) \gg 1\), spin fluctuations at dynamically unstable wave vectors are squeezed by a factor that decreases exponentially in time and is given as

\[
S(-) \sim \frac{4\hbar^2 |\omega_k|^2}{q_0^2} e^{-2|\omega_k| |t|}.
\]

The orientation of the squeezed quadrature axis is given by the relation \(\tan \vartheta = -\hbar |\omega_k| / (\epsilon_k + q)\), with \(\vartheta\) subsequently varying between 0 and \(-\pi/2\).

This momentum-space treatment describes how the dynamical instabilities of a ferromagnetic \(f = 1\) spinor Bose–Einstein condensate may be used as a mode-by-mode parametric amplifier, both for the amplification of small spin fluctuations atop the polar condensate, as explored in [14, 15, 23], and for the generation of a spin-squeezed quantum field. This field may be manipulated further to suit a particular metrological application. For example, spin squeezing may first be prepared at a particular wave vector \(k\) by allowing the system to evolve for a given time at \(q = q_1\) for which \(\omega_k^2 < 0\). Thereafter, the quadratic shift may be adjusted to \(q = q_2\) at which the spin modes at wave vector \(k\) are dynamically stable. Under such stable evolution, the spin fluctuations undergo rotation in the \(M^{(1)}-M^{(2)}\) plane, so that squeezing can be generated along any desired quadrature axis. More general control of the \(k\)-dependent rotation angle could be realized by applying a \(k\)-dependent optically induced quadratic Zeeman shift, e.g. by taking advantage of the band structure of an appropriately tuned and polarized optical lattice.

5. Locality in spin squeezing and measurement

It is illuminating to consider the evolution of a spatially extended \(f = 1\) quantum field in position space. The spin-dependent contact interactions to which we have ascribed the generation of a correlated quantum field act locally; thus, we would expect spatial
correlations to propagate at a finite speed across the spinor condensate. Yet, the above description for the homogeneous system (section 4.1) dealt with measurement observables that incorporate correlations between particles at infinite range. In contrast, given the local origin of spin correlations, we expect that spin squeezing should be effectively observed by local measurements of the collective atomic spin on a finite number of atoms.

One approach to tracing the spatial evolution of spin correlations is to consider correlations among the position-space spin-fluctuation measurement densities, via the covariance matrix defined as

\[
C(\mathbf{r}, \mathbf{r}'; t) = \text{Re} \left\langle \left( \frac{\Delta M_a^{(1)}(\mathbf{r})}{\Delta M_a^{(2)}(\mathbf{r})} \right) \left( \frac{\Delta M_a^{(1)}(\mathbf{r}')}{\Delta M_a^{(2)}(\mathbf{r}')} \right) \right\rangle, \tag{15}
\]

For the uncorrelated condensate prepared at time \( t = 0 \), we have \( C(\mathbf{r}, \mathbf{r}'; 0) = i n \delta^3(\mathbf{r} - \mathbf{r}') \), i.e. correlations are purely local. At later times, for the initially homogeneous condensate,

\[
C(\mathbf{r}, \mathbf{r}'; t) = \frac{n}{(2\pi)^3} \int d^3k e^{i \mathbf{k} \cdot (\mathbf{r} - \mathbf{r}')} C_k(t). \tag{16}
\]

Now, let us inquire as to the spatial range of spin squeezing produced by spin-dependent contact interactions by asking whether finite-volume spatial-mode operators may be defined

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**Figure 1.** The factor \( S^{(-)} \) by which spin fluctuations are reduced below the standard quantum limits, plotted versus evolution time \( t \) measured in units of \( \hbar / q_0 \). Solid curves are shown for momentum-space spin-fluctuation modes with either maximum gain, obtained with \( \epsilon_k + q = q_0 / 2 \) (black), or lower gain, with \( \epsilon_k + q = 7q_0 / 8 \) (gray). For the latter case, the long-time limit for \( S^{(-)} \) is shown as a dashed line. Insets show the corresponding spin fluctuations in the \( M^{(1)} - M^{(2)} \) plane (the \( Q \) representation) for the settings and times indicated by arrows. The long-time limits for the orientations of the squeezing and amplification quadratures are indicated in the insets by dashed gray lines.
that will capture the spin squeezing produced in a homogeneous spinor condensate. We consider two possibilities. First, we define ‘quadrature-tuned’ measurement mode functions through their Fourier transform as

\[
\begin{pmatrix}
A(k) & B(k) \\
C(k) & D(k)
\end{pmatrix} = f(k) R_k^\dagger(t),
\]

where \( f(k) \) is real. Now, following the evolution time \( t \), evaluating the covariance matrix with respect to these spin-fluctuation observables, we obtain simply

\[
C_{qt}(t) = \frac{nV}{(2\pi)^3} \int d^3k f^2(k) S_k(t).
\]

We consider also a second, simpler, ‘uniform quadrature’ mode function with Fourier transform

\[
\begin{pmatrix}
A(k) & B(k) \\
C(k) & D(k)
\end{pmatrix} = f(k) R_{k_{\text{max}}}^\dagger(t),
\]

where \( k_{\text{max}} \) is the wave vector for which maximal squeezing is achieved.

Now we are in a position to consider forms of the measurement operators appropriate for capturing squeezing achieved by the spinor Bose–Einstein condensate amplifier. We examine the long-time limit (equation (14)) and consider amplification under two conditions.

5.1. Shallow quench

Under the condition \( q_0/2 \leq q < q_0 \), the maximally unstable mode occurs at zero wave vector. Expanding about \( k = 0 \), we approximate

\[
|\omega_k| \simeq \sqrt{\frac{q(q_0 - q)}{\hbar}} + \left(\frac{\hbar}{m}\right) \left(\frac{(q_0 - 2q)}{4\sqrt{q(q_0 - q)}}\right) k^2 = \omega_{k=0} - D k^2.
\]

We note that \( D \), proportional to \( \hbar/m \), has the units of a diffusion constant. Let us now define the measurement operator through

\[
f(k) = \frac{8\pi^{3/2} \sigma^3}{\sqrt{V}} \ e^{-\sigma^2 k^2},
\]

so that the quantity \((AD - BC) > 0\) is a three-dimensional (3D) Gaussian with peak value of unity at the origin and with a volume of \( v = (2\pi \sigma^2)^{3/2} \).

Evaluating the correlation matrix, we now find that fluctuations of the \( M_{A,B,C,D}^{(1)} \) operator are squeezed with respect to their variance at \( t = 0 \). An estimate for the squeezing achieved using the quadrature-tuned mode function may be obtained analytically. If we assume that \( \sigma^2 > Dt \), then the reduction in the variance is given approximately as

\[
S_{qt}^{(-)} \simeq \frac{4q(q_0 - q)}{q_0^2} \left(1 - \frac{D t}{\sigma^2}\right)^{-3/2} e^{-2\omega_{k=0}t}.
\]

That is, the spin correlations produced in the quenched spinor Bose–Einstein condensate are indeed local, contained within a volume that grows diffusively in time.

These approximations are confirmed by numerical calculations, the results of which are shown in figure 2. The quadrature-tuned spin-fluctuation measurement is indeed effective in capturing the spin squeezing produced in this system within a measurement volume that grows only slowly with evolution time. In contrast, a uniform-quadrature measurement requires larger
Figure 2. Degree of squeezing, $S^{(-)}$, obtained by local measurements after preparing an uncorrelated $|m_z = 0\rangle$ condensate and quenching it to a regime of dynamical instabilities with $q/q_0 = 3/4$. $S^{(-)}$ is calculated for variable $\sigma$ for quadrature-tuned measurement modes (solid lines) at evolution times $t q_0/\hbar = \{1, 2, 3, 4, 5\}$ and for uniform quadrature (dashed lines) at times $t q_0/\hbar = \{1, 3, 5\}$. The quadrature-tuned measurement mode is found to capture the maximal squeezing produced in the system within a measurement volume, parameterized by $\sigma$, that grows only slowly with time. Open circles highlight values of $\sigma$ at which $S^{(-)}$ is within 5% of the single-mode $k = 0$ value. As shown in the inset, the dimension $\sigma$ at which this condition is achieved (open circles) is approximated at late times $t$ according to equation (22) (solid line). In contrast, uniform-quadrature measurements must be made on larger volumes to observe the maximum attainable squeezing.

5.2. Deep quench

For a deep quench, with $q < q_0/2$ the maximum instability occurs on a sphere in $k$-space, with radius $k_{\text{max}}$ defined by $\epsilon_k + q = q_0/2$. About this shell, the gain is given approximately as

$$|\omega_k| \simeq \frac{q_0}{2\hbar} - \frac{\hbar}{m} \frac{q_0 - 2q}{q_0} (k - k_{\text{max}})^2 = \frac{q_0}{2\hbar} - D(k - k_{\text{max}})^2.$$  \hspace{1cm} (23)

To capture this squeezing, we define a measurement mode through the following relation:

$$f(k) = X e^{-\sigma^2 (k - k_{\text{max}})^2}.$$  \hspace{1cm} (24)
Figure 3. Degree of squeezing, $S^{(-)}$, obtained by local measurements following a deep quench to $q = -2q_0$. Quadrature-tuned measurements taken at evolution times $tq_0/\bar{h} = \{1, 2, 3, 4, 5\}$ evince the maximal single-mode spin squeezing for mode functions given by equation (24) with $\sigma$ growing only slowly with time (solid lines). Open circles highlight values of $\sigma$ at which $S^{(-)}$ is within 5% of the single-mode $k = k_{\text{max}}$ value. As shown in the inset, the dimension $\sigma$ at which this condition is achieved (open circles) is approximated at late times $t$ according to equation (25) (solid line). Uniform-quadrature mode functions, shown here for $tq_0/\bar{h} = \{1, 3\}$, require much larger $\sigma$ to capture the full spin squeezing in this system.

We conclude with a discussion of experimental prospects for realizing and detecting multi-mode spin squeezing in a quenched ferromagnetic $f = 1$ condensate. We focus on the case of $^{87}\text{Rb}$,
which has been shown experimentally [24]–[26] and theoretically [27, 28] to be ferromagnetic, with \( \Delta a = -1.4(3) \alpha_0 \) with \( \alpha_0 \) being the Bohr radius. The behavior of condensates prepared in the \( |m_z = 0 \rangle \) state and quenched to the regime of dynamic instabilities by a rapid change of \( q \) has been studied using \textit{in situ} magnetization-sensitive imaging. Pertinent to the present discussion, the amplification of spin fluctuations following quenches to different values of \( q \) was measured precisely in [14], and such amplification was found to be roughly consistent with quantum-limited amplification of quantum fluctuations in the initial state, with a gain, measured by the increase in the variance of spin fluctuations, as high as \(+30\,\text{dB} \).

In contrast to previous studies on the parametric amplification of spin fluctuations [14, 23], here we have considered the squeezing that may accompany such amplification. We assume experimental conditions similar to those of [13, 14], i.e. an optically trapped \( f = 1^{87}\text{Rb} \) condensate of \( N = 2.0 \times 10^6 \) atoms with peak density \( n = 2.6 \times 10^{14} \,\text{cm}^{-3} \). At this density, the characteristic time scale for spin mixing in the spinor condensate is \( h/q_0 = 8 \,\text{ms} \) and the characteristic length scale is \( (h^2/(2mq_0))^1/2 = 1.8 \,\mu\text{m} \). Let us estimate the maximum observable degree of spin squeezing, taking into account nonlinearities, atom loss and the spatial inhomogeneity of the condensate.

6.1. Single-mode nonlinearities and atom loss

So far, we have treated spin dynamics in a quenched \( f = 1 \) spinor condensate exclusively through the Bogoliubov approximation (equation (5)), i.e. neglecting high-order terms in the spin-fluctuation operators and also the depletion of the \( |m_z = 0 \rangle \) condensate due to spin mixing. Within this linear, undepleted-pump approximation, we obtain the unphysical result, expressed in equation (10), that the minimum spin-quadrature variance \( S^{(-)} \) in each unstable spin mode tends exponentially to zero. As in quantum-optical systems, sensible limits to squeezing are recovered by going beyond the linearized equations of motion and the undepleted-pump approximation.

These limits are partly elucidated in the full dynamics of a single spin mode. We consider therefore the dynamics of a ferromagnetic spinor condensate in the single-mode approximation. Such a treatment would be appropriate for a spinor condensate spanning a volume \( v \lesssim (h^2/2mq_0)^{3/2} \), realized experimentally with an atom number of \( nv \lesssim 1000 \). The full Hamiltonian is now given as

\[
\mathcal{H} = \sum_{\phi \in \{x,y\}} \left\{ -\frac{C_2}{2v} \left( \phi_a^{\dag 2} \phi_z^2 + \phi_z^{\dag 2} \phi_a^2 \right) + q \phi_a^{\dag} \phi_a \right\} + \frac{C_2}{v} \phi_x^{\dag} \phi_x \phi_y^{\dag} \phi_y - \frac{C_2}{2v} \left( \phi_x^{\dag} \phi_y - \phi_y^{\dag} \phi_x \right)^2, \tag{26}
\]

where the Bose operators \( \phi_x \) annihilate particles in polar state \( |\phi_x \rangle \) and in the single spatial mode of the condensate.

In this expression, the single-polarization term \( \propto (\phi_a^{\dag 2} \phi_z^2 + \phi_z^{\dag 2} \phi_a^2) \) matches the two-axis counter-twisting Hamiltonian described in [2]. On its own, this number-conserving Hamiltonian produces squeezing that saturates with time due to depletion of the pump (the population \( \phi_x^{\dag} \phi_x \)), reaching the Heisenberg limit with \( S^{(-)} \propto 1/N \) [29]. However, the Hamiltonian contains additional terms: the externally imposed and pump-dependent quadratic Zeeman shifts and inter-polarization coupling. These nonlinear terms restrict spin squeezing to sub-Heisenberg scaling, with \( S^{(-)} \propto 1/\sqrt{N} \). Nevertheless, the expected single-mode squeezing is significant. We have performed exact single-mode calculations for a 1000-atom sample and find a reduction of spin fluctuations by as much as 20 dB.
The single-mode limit allows us to discuss an additional limitation arising from the decay of atoms from the trap. Atom loss can be modeled by adding a noise source and a density-dependent loss rate in the coherent evolution of the Bose fields \[29\]. We have performed numerical calculations including such terms, representing quantum noise in the evolution of a classical field as in the truncated Wigner approximation (TWA) \[30\]. In accord with experimental observations, we assume an initial per-atom loss rate of \(1 \text{s}^{-1}\). We find that squeezing in an \(N = 1000\) atom sample can still be the level of 20 dB, a result that is not surprising given the very short spin-mixing time as compared to the lifetime of trapped atoms.

### 6.2. Effects of inhomogeneity and multi-mode nonlinearities

Experimental realizations of spinor Bose–Einstein condensates also differ from our idealized treatment in that the condensates are inhomogeneous and, thus, the adequacy of a linearized normal-mode description to long-time dynamical evolution is less clear than in the homogeneous case. To assess the influence of mixing between normal modes and also atom losses for this situation, we have performed numerical simulations of the evolution of a quasi-1D quenched spinor Bose–Einstein condensate. Such a condensate is assumed to be sufficiently tightly confined in two radial dimensions that the spinor wavefunction is radially uniform, while being more weakly confined in the third dimension, along which the wavefunction is allowed to vary. Specifically, we consider a condensate with Thomas–Fermi radii of 3 and 200 \(\mu\)m along the radial and axial dimensions, respectively. The central, radially averaged condensate density \(n_0\) determines the spin mixing interaction strength to be \(2|c_2|n_0 = h \times 15\) Hz.

The quantum evolution of this spinor condensate is calculated using the TWA to simulate the effects of quantum noise and squeezing \[23\]. This approximation involves adding random noise to the initial mean-field condensate wavefunction, with the noise variance determined, in this case, by the 1D condensate density. Here, the initial zero-temperature mean-field solution is assumed to have perfect long-range coherence, neglecting the possible influence on coherence of 1D confinement as seen experimentally in extremely elongated, narrowly confined gases \[31\]–\[33\]; rather, the 1D case is considered to provide a simple illustration of the effects of confinement. Temporal evolution is thereafter considered according to classical wave equations, which include the full nonlinear dynamics given by spin-mixing Hamiltonian (the nonlinear Schrödinger equation) and also the effects of atom loss (through additional noise and loss terms, described above). Expectation values, variances and correlations for various measurement operators are quantified by repeating the calculations for many instances of the initial noise. These calculations were found to be convergent at a temporal resolution of 3.5 \(\mu\)s and a spatial resolution of 0.5 \(\mu\)m.

We present the results of a numerical simulation of an initially polar, trapped, \(^{87}\text{Rb}\) spinor condensate quenched at \(t = 0\) to a quadratic Zeeman shift of \(q = h \times 2.5\) Hz. The results for several numerical runs, i.e. for three random instances of initial noise evaluated according to the TWA, are presented in figure 4(a). Dynamical instabilities following the quench to low \(q\) result in amplified fluctuations in the transverse components of the magnetization and the related components of the nematicity; only one polarization of the observables, \(F_x\) and \(N_{yz}\), is shown in the figure. The strong local correlation between these observables is clearly evident, reflecting the \(\theta \simeq \pi/4\) angle of the amplified quadrature axis for the maximally unstable normal modes.

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Figure 4. Numerical simulations of amplification and squeezing of spin fluctuations in a quenched, trapped spinor Bose–Einstein condensate. The evolution of quantum noise is simulated via the TWA. (a) The results of simulation runs, each with different instances of random initial noise, are shown at $t = 65$ ms after a quench to the regime of dynamical instabilities (see text for details). One polarization of local spin-fluctuation observables (components of magnetization (red solid line) and nematicity (blue dashed line)) is shown, scaled to unity for a fully spin-polarized condensate at its center. Strong local correlation of the amplified spin fluctuations is evident. (b) Squared values of the amplified (solid symbols) and squeezed (closed symbols) normal-mode quadrature measurement operators, evaluated for runs 1 (circles), 2 (squares) and 3 (diamonds), scaled to their initial statistical variance, indicate a reduction of initial spin fluctuations by 20–30 dB.

Less evident in these position-space plots is the sharp reduction in the initial fluctuations of the normal-mode measurement observables. To reveal this reduction, we identify the normal dynamical modes based on the initial condensate distribution. The temporal gain determined for the first 41 unstable modes is shown in the inset of figure 5, sorted by decreasing order of temporal gain ($|\omega_n|$) and labeled by an integer mode index. Normalized spatial-mode measurement operators are then formed as defined in equations (11) and (12), and the values of these operators at different evolution times $t$ are evaluated for an ensemble of 600 numerical runs. From these values, we evaluate the scaled covariance matrices $C_n$ numerically and determine therefrom the quadrature axes that are amplified and squeezed, and also the measurement variance along those axes.

The squares of the measurement outcomes along the amplified and squeezed quadrature axes for the three simulation runs discussed above are shown in figure 4(b). Aside from pronounced variations according to the initial random values assigned to the quantum noise in these simulations, these data indicate a suppression of spin fluctuations between 20 and 30 dB.
Figure 5. Suppression of spin fluctuations in a quenched, trapped spinor Bose–Einstein condensate, evaluated for normal dynamical modes (labeled by mode index \( n \)) at several times of evolution (labeled on the left of the graph). The calculated early-time gain of these unstable modes is shown in the inset. The measurement variance in the squeezed quadrature mode \( S^{(-)} \) diminishes exponentially in time, saturating at an evolution time of \( t = 65 \) ms to a minimum value of \( 3.6 \times 10^{-3} \) for the highest-gain mode and then growing due to saturation and nonlinear effects.

The ensemble averaged reduction \( S^{(-)} \) in the initial spin fluctuations, determined for each mode \( n \) and at different evolution times, is shown in figure 5. We find that the temporal evolution of the dynamically unstable modes causes the measurement variance of the squeezed quadrature operators to diminish exponentially in time for evolution times up to about 60 ms, after which the variance saturates and then begins to grow. The minimum variance of about \( 3.6 \times 10^{-3} \) or \(-24\) dB is achieved for the unstable modes with the highest gain (lowest mode index) at \( t = 65 \) ms. This maximum degree of squeezing matches well with the estimated effects of nonlinearities and atom loss presented above.

7. Conclusion

We have presented a formalism for the evaluation of spin squeezing in ensembles of high-spin atoms. This formalism identifies several independent planes of spin fluctuations atop an arbitrary coherent spin state, revealing a greater resource for applications of spin squeezing in quantum information and metrology than for ensembles of spin-1/2 particles. Moreover, we developed an understanding of spin squeezing in spatially extended quantum fields, where squeezing in many spatial modes can be separately generated and measured. This formalism is applicable to many contemporary experiments on spin squeezing of atomic gases or of other distributed quantum objects.

Applying this formalism, we consider a source of spin squeezing in an extended spinor Bose–Einstein condensate. Referring to recent experiments studying the evolution of \(^{87}\)Rb condensates quenched to a regime of dynamical instabilities [13]–[15], we show that the strong
amplification of spin fluctuations demonstrated in those experiments is also accompanied by spin squeezing, i.e. that the dynamically unstable normal modes act as quantum parametric amplifiers of spin fluctuations. Treating experimental realities such as atom losses and the effects of confinement, we apply analytic and numerical tools to determine that measurable spin squeezing of better than 20 dB may be achieved in such systems.

This treatment is relevant also to the question of symmetry breaking in a quantum many-body system. An initially polar-state spinor Bose–Einstein condensate preserves the SO(2) rotational symmetry about the alignment axis. The quantum quench considered in this work, a rapid reduction in the quadratic Zeeman shift to induce dynamical instabilities and parametric amplification of spin fluctuations, traverses the atomic system across a symmetry-breaking quantum phase transition, where the spin-dependent contact interactions favor broken-symmetry states of maximum magnetization [13]. In a closed noiseless quantum system, bona fide breaking of symmetry does not occur; rather, the system evolves coherently to a symmetric superposition of broken-symmetry states. The spin-squeezed states described in this work indeed represent such a coherent superposition.

The metrological significance of spin-squeezed quantum fields, as well as the question of symmetry breaking in quantum systems, provides strong motivation for the detection and characterization of squeezing in quenched spinor Bose gases. For this purpose, spatially resolved measurements of the magnetization of optically trapped condensates have been demonstrated [16], and optical means of measuring the components of both the magnetization and the nematicity have been suggested [18]. However, sensitivity below the atomic shot noise level, required for the direct detection of squeezing, is experimentally challenging and was not achieved in those measurements.

Alternatively, we propose that one could use the condensate itself as a pre-measurement amplifier. Following the generation of inhomogeneous spin squeezing with the quadratic shift at $q_1$, one may quickly increase the quadratic Zeeman shift. The previously unstable dynamical modes now evolve stably as the squeezed and amplified spin-fluctuation quadratures rotate in their quadrature spaces. After their rotation, the spin fluctuations may be parametrically amplified again by returning the quadratic shift to $q_1$. Spin squeezing produced by the first stage of amplification would result in a reduction of spin fluctuations observed after the second amplification stage, as compared to a single-stage amplification of vacuum spin noise. We have performed TWA-based numerical simulations that confirm that this technique is applicable to trapped spinor Bose–Einstein condensates at experimentally accessible settings.

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