Weak integrability breaking: chaos with integrability signature in coherent diffusion

Marko Žnidarič
Physics Department, Faculty of Mathematics and Physics,
University of Ljubljana, 1000 Ljubljana, Slovenia
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We study how perturbations affect dynamics of an integrable many-body quantum system. Looking at spin transport in the Heisenberg chain with impurities, we find that in the thermodynamic limit transport gets diffusive already at an infinitesimal perturbation. Small extensive perturbations therefore cause an immediate transition from integrability to chaos. Nevertheless, there is a remnant of integrability encoded in the dependence of diffusion constant on the impurity density. At small densities it is proportional to the square root of the density, instead of to the density as would follow from Matthiessen’s rule. Results also highlight nontrivial interacting scattering on a single impurity.

At low temperatures one can often isolate particular degrees of freedom that experience only small “environmental” effects. Such situations can then be amenable to controlled experiments. With development of new theoretical and experimental techniques the last decade has seen a broad expansion of interest to genuine many-body systems (strong interactions) and nonequilibrium physics [11, 12], where one has to go beyond the ground state physics, which can be often described by integrable effective theories describing quasiparticles. A pertinent question is how is this integrability that is in practice (weakly) broken, reflected (if at all) in properties of generic states, as for instance in an out of equilibrium situation?

We study two questions: (i) breaking of integrability in a many-body system and, in particular, at what perturbation strength does one get full generic complexity as quantified by diffusive transport, and (ii) after integrability breaks and transport goes from non-diffusive (typical of integrable systems [8]) to diffusive, is there nevertheless some remaining signature of the parent integrability, or is it completely lost, making integrable systems a utterly singular notion that immediately goes into “featureless” diffusion in the thermodynamic limit (TDL)?

A lesson learned from single-particle quantum chaos [4], or few-degrees-of-freedom classical systems, is that the transition typically happens at a finite perturbation strength (for classical systems the KAM theorem makes that rigorous [5]). For many-body quantum systems the situation is less clear despite long history, e.g. [6, 9]. In particular, conflicting results are not clear on whether the transition from integrability to chaos happens at a finite or at an infinitesimal perturbation strength in the TDL. A bottleneck for numerical studies are small system sizes that make it impossible to assess weak perturbations where one expects emergence of long time- and length-scales. The fact that traditional criteria of single-particle quantum chaos employ spectral properties requiring full diagonalization also does not help. Note also that the nearest-neighbor level spacing, that deals with exponentially small energy scales (which are not observable) is not always the best indicator of complexity (chaos). For instance, small local perturbation suffices to make a system “chaotic” according to level spacing statistics [10], despite e.g. transport remaining ballistic [11]. It is therefore important to better understand integrability to chaos transition in many-body systems with experimentally relevant perturbations and observables, and at as large a system sizes as possible.

We do that by studying transport in a canonical many-body system, namely the Heisenberg spin-1/2 chain. Many of its high-energy and dynamical properties are by now relatively well understood. In particular, spin transport at zero magnetization (half-filling) in a gapless phase and at infinite temperature, a regime we focus on, goes from ballistic for anisotropy $\Delta < 1$ [12, 14] to superdiffusive at $\Delta = 1$ [15, 17]. We break integrability by magnetic fields (impurities) of strength $h$ at sites at distance $\lambda$ (Fig. 1). This gives us an opportunity to study three different kinds of integrability-breaking perturbations: the interaction $\Delta$, the impurity strength $h$, and the impurity density $1/\lambda$. The limit of low impurity density in particular is of high experimental relevance. Namely, the isotropic Heisenberg model is realized in a number of spin-chain materials, e.g. strontium cuprates. Very high heat conductivity measured at low-T is in fact attributed to ballistic transport along Heisenberg chains [18]. Because crystals are never perfect [19], or deliberately introducing impurities [20, 21], one invariably has to deal with an integrable system with low density of impurities. Transport in the Heisenberg model has also been studied in cold-atoms experiments [22, 24], promising even greater controllability in future. Very recently generalized hydrodynamics [25, 26] has been extended to study evolution in near-integrable systems [27, 29].

What we find is that the faster-than-diffusive spin transport of the integrable XXZ spin chain upon integrability-breaking immediately goes into diffusion, with correspondingly diverging diffusion constant $D$ at

![FIG. 1. (Color online) XXZ chain (1) with magnetic field of amplitude $h$ at sites separated by distance $\lambda$ (shown is $\lambda = 3$).](image-url)
small perturbations (see Fig. 2). Standard argument would predict that for $\lambda \gg 1$ one should have $D \propto \lambda^2$ – namely, in diffusion resistances are additive, i.e., scatterings on different dilute impurities ($\sim 1/\lambda$ in number) are independent, the scattering rates $1/\tau_i$ additive, leading under simple kinetic (or Drude) picture to $D \sim \sigma^2 \tau \propto \lambda^{-1}$. This is indeed observed in the mentioned Heisenberg spin-chain materials [31] or dilute alloys [39]. What we find though is that this Matthiessen’s rule $D \propto \lambda^{-1}$, where $z$ is the dynamical exponent of the integrable model ($z = \frac{3}{2}$ in the isotropic Heisenberg chain). We also found other intriguing features: in the ballistic regime $\Delta < 1$ and for large $\lambda$ diffusion constant has a nontrivial dependence on $h$ explained by interacting scattering on a single impurity, and there is a regime of high impurity density where spin transport gets faster upon increasing the number of impurities.

Results.— We shall demonstrate everything on the anisotropic Heisenberg spin-1/2 chain [31] with periodic impurities,

$$
H = \sum_{r=0}^{L-1} \sigma^x r^x_{r+1} + \sigma^y r^y_{r+1} + \Delta \sigma^x r^x_{r+1} + h \sum_{k=1}^{M/L} \sigma^z [k, \pi/\lambda],
$$

where $M = L/\lambda$ is the number of impurities, $h$ the size of magnetic field, and $\lambda$ the distance between them (see Fig. 1) [32]. For $\Delta = 0$ the model is quadratic in fermionic variables and therefore trivially integrable, with spin transport being ballistic. For nonzero anisotropy (interaction) $\Delta$ and $h = 0$ one has instead interacting integrable model. We are going to check three different ways of breaking integrability: having small $h$, small interaction $\Delta$, or small density of impurities, $\lambda \gg 1$.

We note that if one has a single impurity (finite $L$ and $\lambda = \infty$), spin transport is the same [11] as for the clean integrable model. Previous studies of transport in the Heisenberg model under various (weak) perturbations include Refs. [33] [41].

To numerically calculate transport properties at infinite temperature and zero magnetization we are going to couple the spin chain to magnetization reservoirs at first and last sites described by Lindblad operators $L_1 = \sqrt{\Gamma(1+\mu)} \sigma^+_0$, $L_2 = \sqrt{\Gamma(1-\mu)} \sigma^-_0$, and $L_3 = -\sqrt{\Gamma(1-\mu)} \sigma^-_{L-1}$, $L_4 = \sqrt{\Gamma(1+\mu)} \sigma^+_L$. Unless noted otherwise we use coupling strength $\Gamma = 1$ and driving strength $\mu = 0.1$. Evolution of the density matrix describing the system is described by the Lindblad master equation [42, 43] whose solution at long times converges to a nonequilibrium steady state (NESS). We are in particular interested in the NESS spin current $j = \text{tr}(\rho(2\sigma^x r^x_{k+1} - 2\sigma^y r^y_{k+1}))$, and magnetization at site $k$, $z_k = \text{tr}(\rho \sigma^z_k)$. Lindblad driving is such that it induces the NESS with magnetization varying from $+\mu$ to $-\mu$ along the chain (see Fig. 3 for an example), with the energy density being zero (infinite temperature). To represent $\rho(t)$ efficiently one uses a matrix product operator ansatz with matrices of size $\chi$, and then tDMRG method [44] to evolve $\rho(t)$ in time. The method has proved itself in the past, see e.g. [45] and references therein for more details, and allows at “easy” parameter values to simulate systems as large as $L \approx 2000$ sites. Critical parameter that determines the efficiency of the method is $\chi$. The largest $\chi$ we can afford to simulate is about $\chi \approx 100$ at $L \sim 1000$ ($\chi \approx 300$ for some smaller $L$) for which simulating relaxation up to $t \sim 10^3$ when NESS is reached can take of order a week of CPU time on $\approx 30$ Xeon cores. For parameters where truncation errors due to finite $\chi$ are larger we run simulations for different $\chi$ and use extrapolation to gain some accuracy (Fig. 3). Repeating such procedure for different $L$ we obtain $j(L)$ from which we then extract diffusion constant by asymptotic $j = -D \frac{2\mu}{L}$. Note that one must be careful to use $L$ larger than the scattering length; taking too small $L$ would lead to an incorrect superdiffusion claim.
diffusion, and therefore the scattering length should go as \( l \sim 1/h^2 \). In a system with dynamical exponent \( z \) distance and time scale as \( x^z \sim t \) (and current in the NESS as \( j \sim 1/L^{z-1} \)), e.g., \( z = 1 \) for ballistic, \( z = 2 \) for diffusion, and therefore the scattering length should go as \( l \sim 1/h^2/z \). For the isotropic model \( z = \frac{3}{2} \), predicting diffusion constant divergence \( D \sim 1/h^{3/3} \), similarly as for the disordered potential \( \Delta = 0 \). Numerical results in Fig. 4 agree with that scaling (we do note that the agreement is achieved only at very small \( h \lesssim 0.3 \), where numerics is hard because of large \( L \)). At larger \( h \) the scaling power \( \alpha \) of \( D \sim 1/h^\alpha \) becomes larger. From an experimental point of view we would in particular like to understand the case of dilute impurities, \( \lambda \gg 1 \). To that end we plot in Fig. 5 the scaling of \( D \) with \( \lambda \) for several values of \( h \). We see that \( D \) is not proportional to \( \lambda \). This is a consequence of non-ballistic transport between impurities and can be explained as follows. Focusing on a segment of length \( \lambda \) between two impurities, the magnetization difference across the segment is \( \delta z = 2\mu/M = 2\mu/\lambda L \) and will drive the current of size \( j \sim \delta z/\lambda^{z-1} \) through the segment. The last relation comes because an excitation needs time \( t \sim \lambda^2 \) to travel across the length \( \lambda \), resulting in a current \( \sim \lambda/\lambda^z \) (at fixed excitation density there are \( \sim \lambda \) excitations in a segment of length \( \lambda \)). The NESS current is the same in every segment and therefore scales as \( j \sim \frac{2\mu}{\lambda^3} \), resulting in

\[
D \sim \lambda^{2-z}.
\]  

For the isotropic model \( z = \frac{3}{2} \), predicting \( D \sim \sqrt{\lambda} \), agreeing within numerical errors with simulations in Fig. 4. For smaller \( h = 0.6, 0.3 \) the scattering length is so large that we would presumably need \( \lambda > 32 \), which is the largest \( \lambda \) we can reliably simulate. In Fig. 4 we plot magnetization profile across a chain, showing curious spikes at locations of impurities (spikes are typically stronger for smaller \( D \)).

Next, we focus on \( \Delta < 1 \) where the integrable model is ballistic, first checking \( \lambda = 4 \). Here we have two possibilities, either taking small \( \Delta \), or taking small \( h \). Again, we find that small integrability breaking immediately leads to diffusion. For the two perturbation types Fermi’s golden rule gives scattering length scaling as \( l \sim 1/\lambda^2 \), or \( l \sim 1/h^2 \), leading to diffusion constant divergence \( D \sim 1/\lambda^2 \), or \( D \sim 1/h^2 \), respectively. This is confirmed numerically, see Appendix for data. Increasing \( \lambda \) at fixed \( h \) lead to some surprises in the isotropic case. Data in Fig. 5 show that for \( \Delta < 1 \), where the unperturbed dynamics is ballistic, eq. (2) leads to \( D \propto \lambda \) at \( \lambda \gg 1 \), agreeing with numerics. What is interesting is behavior at small \( \lambda \). Between \( \lambda = 2 \) and 4 diffusion constant increases by decreasing \( \lambda \). This means that transport gets faster when we add more impurities that scatter quasiparticles of the integrable model. The effect is more prominent at small \( \Delta \), and was also visible at \( h = 1 \) in the isotropic case (Fig. 4c). Let us now focus on \( \lambda \gg 1 \) and in particular on how \( D \) depends on other parameters like \( h \). Using the same argument as in deriving eq. (2) we can see that between rare impurities the profile will be flat, with a magnetization jump happening only at impurities (Fig. 6a). We also see that at \( \lambda \gg 1 \) it does not matter whether impurities are equidistant, like in

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**FIG. 4.** (Color online) Isotropic chain, \( \Delta = 1 \). (a) Dependence of \( D \) on \( h \) for \( \lambda = 4 \). (b) Diffusion constant scaling with \( \lambda \); for \( \lambda \gg 1 \) it is not proportional to \( \lambda \) (inverse impurity density) but is rather \( D \sim \lambda^{0.5} \). (c) Magnetization profile has spikes at impurities (\( h = 1, \lambda = 16, L = 512 \)).

**FIG. 5.** (Color online) Diffusion scaling for \( \Delta < 1 \) (\( h = 0.5 \)). For \( \lambda \gg 1 \) it is linear, with the prefactor given by single-impurity physics (Fig. 4), while in the shaded strip \( D \) counterintuitively increases by increasing perturbation.
FIG. 6. (Color online) Anisotropic XXZ with $\Delta = 0.6$. (a) Magnetization profile for $h = 0.5$ at every $\lambda = 32$ site in a chain with $L = 512$ (red), and for randomly placed $L/\lambda = 16$ impurities (dashed blue). Inset: scaling of the NESS current with $L$ giving $D \approx 140$. Blue points (overlapping with red squares for $\lambda = 32$) show current for the random case. (b) Scaling of $D(h)$ for $\lambda = 32$ (4 black circles). Red squares are obtained (no fitting parameters) from the single-impurity scattering in frame (c). Green squares is the exact $R_{\text{single}}$ noninteracting result. (c) Magnetization profile for a single impurity at the middle site ($\mu = 0.005$). The main plot shows results for $L = 128$ (zoom-in also for $L = 32$ and $h = 0.5$). Magnetization jump at the impurity, $dz := z_{L/2} - z_{L/2+1}$, and the NESS $j$ is used to plot the red squares in frame (b).

our simulations, or at random positions – $D$ is the same in both cases (the same holds at $\Delta = 1$). Besides the scaling $D \propto \lambda$ we should be able to get the full prefactor from studying the size of the jump at a single impurity. This is what we do in Fig. 6 for a particular $\Delta = 0.6$. Taking a single impurity at the middle of the chain, we study how the jump size $dz$ scales with $h$, and in particular how a single-impurity resistance $R_{\text{single}} = dz/j$ scales. We determine the jump $dz$ from 5 central sizes, for which the profile is independent of $L$ in the TDL. Numerics indicates that $R_{\text{single}} \sim h^{1.5}$ at small $h$ (Fig. 6). We stress that in the non-interacting case $\Delta = 0$ one can solve the corresponding Lindblad equation exactly (following e.g. [47]), obtaining odd-$L$ NESS values $j = 4\mu(1+1/\lambda)^2 \lambda^2 z_0,\ldots, (L-1)/2-1 = -z_{(L-1)/2+1},\ldots, L-2 = \mu(1+1/\lambda)^2 h^2 z_0 = -z_{L-1} = \mu(1+1/\lambda)^2 z_{L-1}/2 = 0$, giving $R_{\text{single}}(\Delta = 0) \approx h^2$ (scaling of current with $h$ in the single-impurity situation, including at $\Delta = 0$, was numerically studied in Ref. [11]). While this result is not useful for our situation at $\Delta = 0.6$, it carries two important messages: (i) $R_{\text{single}}(\Delta = 0)$ depends on $\Gamma$ – our $D$ is instead independent of $\Gamma$ in the TDL as also predicted theoretically [10] (see Appendix for data), (ii) the power $\approx 1.5$ we get at $\Delta = 0.6$ is non-perturbative and can not be obtained neither from $\Delta = 0$ limit, where it is 2, nor from small $h$ perturbation theory (see Appendix). It results from scattering in an interacting wire on a single impurity. Understanding that with e.g. the generalized hydrodynamics [25, 26] is an open problem. Using $R_{\text{single}}$ we can now predict that the diffusion constant for $\lambda \gg 1$ in a system that is ballistic without impurities, should be

$$D = \lambda/R_{\text{single}}.$$  (3)

From data in Fig. 6 for full many-impurity numerics we see that the agreement is good (due to numerical errors the accuracy of the fitted power 1.5 is about 10%).

Conclusion.– Using transport as an indicator we studied transition from integrability to chaos in the Heisenberg spin chain with impurities. By large-scale numerical simulations we find that one gets diffusion already for an infinitesimal perturbation strength, in line with simple Fermi’s golden rule. For the important case of dilute impurities we find that the diffusion constant scales as $D \sim h^{-2}$, where $z$ is the dynamical exponent of dilute impurities. Using transport as an indicator we can solve the corresponding Lindblad equation exactly (following e.g. [47]), obtaining odd-$L$ NESS values $j = 4\mu(1+1/\lambda)^2 \lambda^2 z_0,\ldots, (L-1)/2-1 = -z_{(L-1)/2+1},\ldots, L-2 = \mu(1+1/\lambda)^2 h^2 z_0 = -z_{L-1} = \mu(1+1/\lambda)^2 z_{L-1}/2 = 0$, giving $R_{\text{single}}(\Delta = 0) \approx h^2$ (scaling of current with $h$ in the single-impurity situation, including at $\Delta = 0$, was numerically studied in Ref. [11]). While this result is not useful for our situation at $\Delta = 0.6$, it carries two important messages: (i) $R_{\text{single}}(\Delta = 0)$ depends on $\Gamma$ – our $D$ is instead independent of $\Gamma$ in the TDL as also predicted theoretically [10] (see Appendix for data), (ii) the power $\approx 1.5$ we get at $\Delta = 0.6$ is non-perturbative and can not be obtained neither from $\Delta = 0$ limit, where it is 2, nor from small $h$ perturbation theory (see Appendix). It results from scattering in an interacting wire on a single impurity. Understanding that with e.g. the generalized hydrodynamics [25, 26] is an open problem. Using $R_{\text{single}}$ we can now predict that the diffusion constant for $\lambda \gg 1$ in a system that is ballistic without impurities, should be

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Appendix A: Additional data

Here we show original data used to determine diffusion constant $D$. In Fig. 7 we plot the NESS current expectation value $j$ for different $L$ and $h$, all for $\lambda = 4$ and $\Delta = 1$. In general larger $h$ require larger bond sizes $\chi$, making the method better suited to small perturbations $h$. However, at small $h$ the scattering length is larger and therefore one needs larger systems sizes to get into the asymptotic diffusive regime. To give a rough idea, at $h = 4$ we had to use $\chi = 200$ for $L = 144$ to get a bit less than 10% error in the NESS current. On the other hand for $h = 0.1$ bond size $\chi = 50$ suffices at $L = 100$ to get better than 1% precision, however large sizes are required, and at $L = 1800$ we could afford only $\chi = 50 - 80$ at which we estimate the error to be around 10%. In Fig. 8 we show a profile at different parameters, where spikes are not pronounced, than in the main text.

Going to the ballistic regime of $\Delta < 1$ we checked that the diffusion constant diverges as $\Delta \rightarrow 0$, shown in Fig. 9. We also see that the prefactor $a$ in $D \approx a/\Delta^2$ rapidly increases as $h$ gets smaller.

For small $h$ and $\Delta = 0.6$, where the unperturbed model is ballistic, the best fitting dependence in Fig. 10 gives $D \sim 1/h^{1.8}$. The power is not quite 2, as one would expect from Fermi’s golden rule. We note that something similar has been observed also in the case of disorder with random amplitude at every site in Ref. [45]. In our case we place impurity with the same amplitude $h$ at every 4 sites. What could play a role is that $\lambda = 4$ seems already quite close to a regime of $\lambda \gg 1$, see Fig. 5 where we know from the single-impurity scaling of $R_{\text{single}}$ that the power is close to 1.5. At $h = 0.5$ and $\Delta < 1$ we can see in Fig. 11 that large $L$ are required in order to reach the asymptotic diffusive spin transport.

We saw that for $\lambda \gg 1$ and $\Delta < 1$ we can predict the value of $D$ solely from a single-impurity situation. In the main text we used a fixed $\Delta = 0.6$, demonstrating that $R_{\text{single}} \approx 1.6/h^{1.5}$. We repeat that the power $\approx 1.5$ is non-perturbative. It is a consequence of the magnetiza-
The transition rate \( 1 / \tau \) from a single-particle eigenstate \( |k\) to another \( |k'\) is proportional to the matrix element \( |\langle k' | V | k \rangle|^2 \). Single-particle eigenstates \( |k\) are plane-waves and we can write wavenumber as \( k = \frac{2 \pi L}{h} p \) and \( k' = \frac{2 \pi L}{h} p' \), where \( p, p' \) are integers. Doing the calculation we get \( |\langle k' | V | k \rangle|^2 \sim \frac{h^2}{\lambda} \sin^2(\pi (p-p')) \). We see that (i) if \( p - p' \) is really an integer, like for \( \Delta = 0 \), there is no scattering, as it should be. Adding periodic potential to free particles does not modify ballistic transport. One needs interaction \( \Delta \neq 0 \), so that momenta are not integers anymore, if one wants to have scattering breaking ballistic transport. (ii) Just looking at the prefactor, the scattering rate \( 1 / \tau \sim h^2 \) which is the correct power 2 for the magnetization jump on a single-impurity only if \( \Delta = 0 \) (the case for which we anyway exactly solved the Lindblad equation). (iii) \( \tau \sim \lambda^2 \) would suggest \( D \sim \lambda^2 \), which is not correct. Factor \( 1 / \lambda^2 \) comes about simply due to the norm of \( V \); for larger \( \lambda \) there are simply less impurities in the system. And because in the Fermi’s golden rule we are summing amplitudes of scattering at different sites (different terms in \( V \)), we get a factor \( \lambda^2 \). According to Matthiessen’s rule one instead has to add rates (probabilities), which then gives the correct scaling \( D \sim \lambda \). Note though that neither of the two arguments gives the correct scaling in the anomalous isotropic model where \( D \sim \lambda^{2/3} \). In Fig. 12 we study how \( R_{\text{single}} \) depends on \( \Delta \), fixing \( h = 0.5 \). We can see that \( R_{\text{single}} \) increases by decreasing \( \Delta \), as it should. Beware that the limit \( \Delta = 0 \) is special in the sense that the model is ballistic there even with \( \Delta \neq 0 \) due to periodic impurities. In the figure we read \( dz \) from the jump at the middle three sites, regardless of \( \Delta \). At larger \( \Delta \) one should in fact take more than three sites because the width of the magnetization jump at the impurity depends on \( \Delta \). For instance, for \( \Delta = 0.6 \) used in the main text we determined that 5 sites is more appropriate. Such an adjustment would slightly lower the curve shown in Fig. 12 at larger \( \Delta \).

Finally, in Fig. 13 we check that asymptotically at large

![Figure 11](image1.png)

**FIG. 11.** (Color online) Top: Raw data for XXZ with \( h = 0.5 \) from Fig. 6. Bottom: Data used in Fig. 6b, where \( \lambda = 32 \), \( \Delta = 0.6 \). Another way is to try to use Fermi’s golden rule work as it would result in an integer power, like 2 for \( h \). One might be therefore tempted to obtain it from a small-dependent value that is the same as for the clean model. (i) If \( \Delta = 0 \), taking periodic \( \Delta = 0 \). Another way is to try to use Fermi’s golden rule work as it would result in an integer power, like 2 for \( h \). One might be therefore tempted to obtain it from a small-dependent value that is the same as for the clean model. (ii) Just looking at the prefactor, the scattering rate \( 1 / \tau \sim h^2 \) which is the correct power 2 for the magnetization jump on a single-impurity only if \( \Delta = 0 \) (the case for which we anyway exactly solved the Lindblad equation). (iii) \( \tau \sim \lambda^2 \) would suggest \( D \sim \lambda^2 \), which is not correct. Factor \( 1 / \lambda^2 \) comes about simply due to the norm of \( V \); for larger \( \lambda \) there are simply less impurities in the system. And because in the Fermi’s golden rule we are summing amplitudes of scattering at different sites (different terms in \( V \)), we get a factor \( \lambda^2 \). According to Matthiessen’s rule one instead has to add rates (probabilities), which then gives the correct scaling \( D \sim \lambda \). Note though that neither of the two arguments gives the correct scaling in the anomalous isotropic model where \( D \sim \lambda^{2/3} \). In Fig. 12 we study how \( R_{\text{single}} \) depends on \( \Delta \), fixing \( h = 0.5 \). We can see that \( R_{\text{single}} \) increases by decreasing \( \Delta \), as it should. Beware that the limit \( \Delta = 0 \) is special in the sense that the model is ballistic there even with \( \Delta \neq 0 \) due to periodic impurities. In the figure we read \( dz \) from the jump at the middle three sites, regardless of \( \Delta \). At larger \( \Delta \) one should in fact take more than three sites because the width of the magnetization jump at the impurity depends on \( \Delta \). For instance, for \( \Delta = 0.6 \) used in the main text we determined that 5 sites is more appropriate. Such an adjustment would slightly lower the curve shown in Fig. 12 at larger \( \Delta \).

![Figure 12](image2.png)

**FIG. 12.** (Color online) Single-impurity resistance. Dependence of \( R_{\text{single}} = j / dz \) (where \( dz \) is magnetization jump at the middle three sites) on \( \Delta \) for fixed \( h = 0.5 \). Inset: Profiles for two values of \( L \) show that, while at smaller \( \Delta \) one indeed has overlap on 3 central sites, at larger \( \Delta \) one would have to take \( dz \) on more than 3 points (driving strength is \( \mu = 0.005 \)).

![Figure 13](image3.png)

**FIG. 13.** (Color online) Comparing bath coupling strength \( \Gamma = 1 \) used throughout the paper (red) with \( \Gamma = 3 \) (blue). Boundary jumps due to larger boundary resistance are fairly large for \( L = 128 \), however, at larger \( L \) the boundary jump gets small. This is in-line with the NESS currents being the same in the TDL (inset). Parameters are \( \lambda = 32 \), \( \Delta = 6 \), \( h = 0.75 \), and driving \( \mu = 0.1 \).
the value of $D$ does not depend in the Lindblad coupling strength $\Gamma$. In all other simulations we used $\Gamma = 1$, while here we also show $\Gamma = 3$. The value of $\Gamma$ essentially determines the boundary resistance. Magnetization jump at the boundary will be of size $dz_{\text{boundary}} \sim R_{\text{boundary}} j$. If one has diffusion then the current scales as $j \sim 1/L$ in the TDL, and therefore the jump will go to zero and will not affect $D$. In the thermodynamic limit the coupling strength $\Gamma$ plays a role only if the transport is ballistic, $z = 1$. As soon as one is sub-ballistic ($z > 1$) $\Gamma$ does not matter in the TDL (in practice, large or small $\Gamma$ could be a numerical nuisance).