TOPOLOGICAL COMPUTATION ANALYSIS OF
METEOROLOGICAL TIME-SERIES DATA

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Abstract. A topological computation method, called the MGSTD method, is applied to time-
series data obtained from meteorological measurement. The method gives decomposition of the
dynamics into invariant sets and gradient-like transitions between them, by dividing the phase space
into grids and representing the time-series as a combinatorial multi-valued map over the grids. Since
the time-series is highly stochastic, the multi-valued map is statistically determined by taking prefer-
able transitions between the grids into account. The time-series data are principal components of
pressure pattern in troposphere and stratosphere in the northern hemisphere. The application yields
some particular transitions between invariant sets, which leads to circular motion on the phase space
spanned by the principal components. The Morse sets and the circular motion are consistent with
the characteristic pressure patterns and the change between them that have been shown in preceding
meteorological studies.

Key words. Morse decomposition, time-series, noise, meteorology

AMS subject classifications. 37B30, 37B35, 37M10, 37N10

1. Introduction.

1.1. General background. Study of dynamics based on time-series data from
experiments or measurements of various nonlinear phenomena has been developed
since 1970s. The most fundamental idea is the method of delay-coordinates by Ruelle
and Packard, et al. [19], and it led to the mathematical theory of reconstructing
attractors from time-series data by Aeyels [3], Takens [22], and Sauer, Yorke, and
Casdagli [20]. The method and theories have been successfully applied to obtaining
dynamical information of a great variety of nonlinear phenomena.

In this paper, we study meteorological time-series data from the viewpoint of
dynamical systems, and illustrate an extension of such dynamical time-series analysis
which can capture not only attractors but also unstable dynamics by concatenating
a set of time-series data taken from scattered initial conditions in the phase space of
a dynamical system. More precisely, it yields a decomposition of the phase space of
the dynamics into finite numbers of isolated invariant sets called Morse sets that are
related in a gradient-like manner. The decomposition is called a Morse decomposition,
which may be considered as a crude but global representation of the entire dynamics
in its phase space. Recent development of computer-assisted methods for studying
dynamics [1, 8, 2] enabled us to understand various aspects of dynamics of concrete
nonlinear systems with the aid of computer. Among such methods, several people
including one of the authors of this paper have proposed a computational approach
[4, 6, 17] to obtain Morse decompositions together with a concise description of dynamics
of each Morse set (in terms of the Conley index) of a given dynamical system. This
method is mainly developed for iterated maps with parameters, but can also be applied
for ordinary differential equations [17].

The basic idea of obtaining Morse decomposition of a given dynamical system
is to set a finite grid decomposition on a domain of interest in the phase space of

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the dynamics, and then to construct a combinatorial multi-valued map over the finite grid elements, which is a rigorous outer approximation of the true time-evolution of the dynamical system. Such a combinatorial multi-valued map may be represented as a finite directed graph with nodes being grid elements and edges representing time-evolution from a grid element to a set of grid elements that intersect with (outer-approximation of) the true image of the given grid element. A strongly connected path component of this finite directed graph is indeed an isolated invariant set (or, more precisely, an isolating neighborhood whose maximal invariant subset is an isolated invariant set), and off of these strongly connected components the remaining part of the directed graph becomes gradient-like, since there is no more recurrent path in it.

Notice that this idea may work not only for numerically computed time-evolution of a dynamical system, but also for a set of sufficiently many time-series data obtained from experiments or measurements of a phenomenon driven by an unknown dynamics, if the set of time-series data is so abundant that the entire data can capture all the essential dynamical features occurring in the phase space. Once this last assumption may somehow be believed to be true, one may construct a combinatorial multi-valued map in a similar way from the time-series data, and hence one can obtain a Morse decomposition of the underlying dynamics directly obtained from the measurement of the phenomenon of interest, without relying on its mathematical models.

1.2. Morse decomposition of the dynamics from stochastic time-series data and its application to meteorological time-series data. When we study dynamics of real-world phenomena, the presence of noise is always an issue. Time-series data obtained from experiments or measurements of such phenomena inevitably contain noise. One may think of these time-series data as the true information of a time evolution governed by a deterministic dynamical system accompanied by some unknown noise. Therefore, in order to understand the dynamics of the phenomenon of interest, it is important and desirable to establish a reliable methodology that can lead to a meaningful conclusion about the dynamics of the phenomenon, say, by correctly removing the effect of noise for obtaining the deterministic part of the time-evolution, or by showing how the dynamics of a deterministically time-evolving system may change under the presence of noise, etc.

Two of the present authors and their collaborators are currently developing such a method [16], which we call the Morse graph method for stochastic time-series data (abbrev. MGSTD method) in this paper. Stochastic time-series in the phase space give an ensemble of transitions from one grid element to another. Extracting statistically preferred and relevant transition yields combinatorial multi-valued map. For thus determined multi-valued map, Morse decomposition is performed by simply following the already established procedure mentioned above. Applying the MGSTD method to time-series data generated from simple deterministic dynamical system models with stochastic terms applied in fact succeeds in reproducing stable and unstable invariant sets, as well as their connecting orbits, of the noise-less deterministic dynamical systems.

A natural problem of interest is to apply this MGSTD method not only to the time-series data generated from dynamical system models but also to those obtained from real experiment or measurement. The purpose of this paper is the application of the MGSTD method to time-series data from meteorological measurement. The reason that, among other stochastic phenomena, we focus on the meteorological dynamics is the following. First, as symbolized by the success of daily weather forecasting, we have already known a set of basic physical equations that governs
meteorological behavior, such as the Navier-Stokes equation with a rotational framework, the continuity equation for dry air and other materials, and radiative transfer equation [9]. This enables us not only to predict a future state of meteorological variables to some extent, but also to diagnose the past and present states. Second, artificial satellites’ highly-frequent, spatially-dense, global observation enables us to use reliable atmospheric data with sufficient spatial and temporal resolution for the period since 1980s. By applying data assimilation technique, the gridded data that are physically consistent can be made from observation and weather forecast data [13]. Finally, projecting meteorological time-series data onto a limited-dimensional phase space shows stable wandering around the climatology with non-Gaussian probability density function [14]. Although the existence of multiple equilibria as dynamically stable points was denied by a recent study [21], it has already been found that the moving tendency of the atmospheric state in the phase space indicates a preferable path there [15], which might be useful for a theoretical interpretation on atmospheric predictability.

1.3. Outline of the paper. The outline of the paper is the following. In Section 2, the MGSTD method will be explained. First we will review mathematical theory of combinatorial Morse decomposition of dynamical systems, and that of deterministic time-series data. Next we will review the method of constructing the multi-valued map from stochastic time-series data. We will also briefly present an example of application to a simple dynamical system model with noise applied to see how useful the MGSTD method is. Then we will explain the meteorological data to which the MGSTD method is applied. In Section 3, the result of our analysis based on the MGSTD method for datasets of troposphere and stratosphere will be shown. We will observe some circular motion in the phase space within highly stochastic time-series data. We will check the dependence of the result on the choice of values of parameters, and see the robustness of the result. Section 4, the last section, will be devoted for discussion and concluding remarks. We will discuss the relevance of the circular motion to the meteorologically known facts, and see that the result is consistent with, and complement to, former studies.

2. Method.

2.1. Morse decomposition of global dynamics from time-series data. The goal of this subsection is to introduce a method for obtaining a Morse graph from a given set of time-series data, which is expected to represent a Morse decomposition of an underlying dynamical system that generates the time-series data by observation. The first three sub-subsections are a brief summary of a computer-assisted method, developed in [4, 6], of obtaining a Morse decomposition of a dynamical system which is given in terms of a graph called a Morse graph. In §2.1.4, we apply this idea to time-series data to obtain a Morse graph of the underlying dynamical system.

2.1.1. Morse decomposition of a dynamical system. As explained above, the Morse decomposition of a dynamical systems is a decomposition of the phase space into recurrent part and gradient-like part. In this paper, we mainly consider discrete time dynamical systems, namely, an iterated map. Let $X$ be a compact metric space and $f : X \to X$ a continuous map.

Definition 2.1 (see [2]). A Morse decomposition of the map $f$ is a finite collection of disjoint isolated invariant sets $S_1, \ldots, S_n$ (called Morse sets) with a strict partial ordering $\prec$ on the index set $\{1, \ldots, n\}$ such that for every $x \in X \setminus \bigcup_{i=1}^n S_i$ and every complete orbit $\gamma = \{x_n\}_{n \in \mathbb{Z}}$ through $x_0 = x$, i.e. $x_{n+1} = f(x_n)$ for all $n \in \mathbb{Z}$,
there exist indices $i \prec j$ such that $x_n \rightarrow S_i$ and $x_{-n} \rightarrow S_j$ as $n \rightarrow +\infty$. (In this case, $\gamma$ is called a connecting orbit from $S_j$ to $S_i$.)

Here an isolated invariant set $S(\subset X)$ of $f$ means that it is an invariant set which has a compact neighborhood $N$ for which $S$ is its maximal invariant set in $N$ and sits in its interior, namely $S \subset \text{Int } N$. The neighborhood $N$ is called an isolating neighborhood of $S$.

Notice that Morse decomposition of a given dynamical system is not unique in general. The coarsest Morse decomposition of a map $f : X \rightarrow X$ consists of a single set $S$ which is the maximal invariant set of $f$ in $X$. If $i, j$ are such indices that $i \prec j$ but there is no other index $k$ such that $i \prec k \prec j$, then one can create a coarser Morse decomposition by replacing $S_i$ and $S_j$ with $S_i \cup S_j \cup C(i, j)$, where $C(i, j)$ denotes the union of all connecting orbits from $S_j$ to $S_i$.

For two Morse decompositions $\mathcal{S} = \{S_1, \ldots, S_n\}$ and $\mathcal{T} = \{T_1, \ldots, T_m\}$, we say $\mathcal{S}$ is a refinement of $\mathcal{T}$, if $n \geq m$ and if there is a surjective map $i : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$ such that $S_i \subset T_{i(i)}$ for any $i = 1, \ldots, n$. By definition, any connecting orbit $\gamma$ between $S_i$ and $S_j$ is also contained in $T_k$ if $i(i) = i(j) = k$.

A Morse decomposition can be represented in terms of a directed graph $G = (V, E)$ where $V = \{S_1, \ldots, S_n\}$ and $(S_j, S_i) \in E$ iff $i \prec j$. This graph is called a Morse graph. In order to represent the computed Morse decomposition in a compact way, it is convenient to plot a Morse graph whose edges are determined by the transitive reduction of the relation $\prec$ which is a minimal relation $\prec'$ whose transitive closure retrieves $\prec$. In what follows, such representation is used in this paper.

2.1.2. Graph representation of dynamics. In order to obtain a Morse decomposition of a map $f : X \rightarrow X$ with an aid of computer, we follow the idea of graph representation of dynamics using grid decomposition of the phase space, as given in [3]. In the case where $X$ is a compact domain in $\mathbb{R}^d$, let $Q$ be a cubical grid decomposition of $\mathbb{R}^d$ that covers $X$. If one can compute a rigorous outer approximation $[f(Q)]$ of the image of a grid element $Q \subset X$ under $f$ by using, say the interval arithmetic, let $\mathcal{F}(Q)$ be the set of all grid elements in $Q$ that intersects with $[f(Q)]$. This defines a multi-valued map $\mathcal{F}$ from $Q$ to itself. By definition, the union of all grid elements in $\mathcal{F}(Q)$ completely contains the true image $f(Q)$. In this sense, $\mathcal{F}$ can be considered as a rigorous outer-approximation of the map $f : X \rightarrow X$. We call $\mathcal{F}$ a combinatorial representation of $f$. Since it is a multi-valued map on $Q$, we use the notation $\mathcal{F} : Q \rightharpoonup Q$ in order to distinguish it from a usual map.

A combinatorial representation $\mathcal{F} : Q \rightharpoonup Q$ of $f$ can be equivalently represented by means of a directed graph $G = (V, E)$ where $V = Q$ and $(Q, Q') \in E$ iff $Q' \in \mathcal{F}(Q)$. The analysis of $G$ provides information on the asymptotic dynamics of $f$ represented by $\mathcal{F}$. For instance, each combinatorial invariant set defined as a set $S \subset Q$ for which $S \subset \mathcal{F}(S) \cap \mathcal{F}^{-1}(S)$ represents an isolating neighborhood $|S|$ with respect to $f$. Moreover, a combinatorial attractor defined as a set $A \subset Q$ such that $\mathcal{F}(A) \subset A$ represents an isolating neighborhood $|A|$ whose invariant part $A$ is stable in the sense of Conley [2]: Every forward orbit starting a point $x$ in some open neighborhood of $A$ (actually, in $\text{Int } |A|$) approaches $A$ ($\text{dist}(f^n(x), A) \rightarrow 0$ as $n \rightarrow \infty$). In particular, if there exist two combinatorial attractors for $\mathcal{F}$, then this implies the existence of two disjoint basins of attraction for $f$.

2.1.3. Combinatorial Morse decompositions. Extensive analysis of the dynamics via its combinatorial representation, or equivalently the directed graph $G$ can be obtained by computing the strongly connected components of $G$, that is, maximal sets of vertices $C \subset V$ such that for each $v, w \in C$ there exists a path from $v$ to $w$. 
This corresponds to the image of points \( \tilde{\Xi} = \pi(x_i) \) given by the observation function \( \pi \). Then, for each initial point \( x \in X \), we obtain a finite time-series \( \{y_i^j\}_{j=0,...,J} \) given by \( y_i^j = \pi(f^j(x)) \). For a finite set \( \Xi = \{x_i\}_{i=1,...,I} \) of initial conditions in \( X \) and a set of natural numbers \( \{J_i\}_{i=1,...,I} \), we obtain a set \( D \) of finitely many finite time-series data as follows:

\[
D = \{y_i^j = \pi(f^j(x_i)) \in \mathbb{R}^m | j = 1, \ldots, J_i, \ i = 1, \ldots, I\}
\]

This corresponds to the image of points \( \tilde{\Xi} = \{f^j(x_i)\}_{j=1,...,J_i, \ i=1,...,I} \) under the observation function \( \pi \). For later purpose, we also define

\[
D' = \{y_i^j = \pi(f^j(x_i)) \in \mathbb{R}^m | j = 1, \ldots, J_i - 1, \ i = 1, \ldots, I\}
\]

Let \( \mathcal{R} \) be a cubical grid decomposition of a domain in \( \mathbb{R}^m \) that covers \( \pi(X) \), and define \( \mathcal{V} = \{R \in \mathcal{R} | R \cap D \neq \emptyset\} \) and \( \mathcal{V}' = \{R \in \mathcal{R} | R \cap D' \neq \emptyset\} \). Then we define a combinatorial multi-valued map \( \mathcal{F} : \mathcal{V}' \rightarrow \mathcal{V} \) as follows: for \( R \in \mathcal{V} \) and \( R' \in \mathcal{V}' \), we define \( R \in \mathcal{F}(R') \) iff there exist \( y_i^j, y_i^{j+1} \in D \) such that \( y_i^j \in R' \) and \( y_i^{j+1} \in R \) hold.

Once \( \mathcal{F} \) is given, one can follow the same procedure as described in the previous sub-sections, and obtain a finite collection \( \{M_i : i = 1, \ldots, k\} \) of the strongly connected components of the equivalent directed graph representations of \( \mathcal{F} \), and hence the Morse graph of \( \mathcal{F} \). We call this method for obtaining a Morse graph from deterministic time-series data the Morse graph method for time-series data.

We then take a pullback of the sets \( \{|M_i| | i = 1, \ldots, k\} \), namely, define \( N_i = \pi^{-1}(|M_i|) \). Obviously, these are disjoint compact subsets of \( X \). Under suitable conditions which basically guaranteeing the abundance of the time-series data, one can show that these sets are indeed isolating neighborhoods and their maximal invariant sets \( M_i = \text{Inv}(N_i) \) give a Morse decomposition of the unknown dynamical system \( f : X \rightarrow X \). Since we do not intend to verify these conditions here, mainly because these are not practically verifiable, we omit the details of this statement, but just remark that there is some mathematical basis for this statement, even if it is not always practically meaningful for application problems.

### 2.1.4. Combinatorial Morse decompositions from deterministic time-series data

Supposing a set of time-series data is given from an unknown dynamical system, we consider a problem of obtaining a Morse decomposition of the underlying dynamical system from the set of time-series data. More precisely, let \( f : X \rightarrow X \) be a continuous map of a compact domain \( X \subset \mathbb{R}^d \), and let \( \pi : X \rightarrow \mathbb{R}^m \) be an observation map. Then, for each initial point \( x \in X \), we obtain a finite time-series \( \{y_i^j\}_{j=0,...,J} \) given by \( y_i^j = \pi(f^j(x)) \). For a finite set \( \Xi = \{x_i\}_{i=1,...,I} \) of initial conditions in \( X \) and a set of natural numbers \( \{J_i\}_{i=1,...,I} \), we obtain a set \( D \) of finitely many finite time-series data as follows:

\[
D = \{y_i^j = \pi(f^j(x_i)) \in \mathbb{R}^m | j = 1, \ldots, J_i, \ i = 1, \ldots, I\}
\]

This corresponds to the image of points \( \tilde{\Xi} = \{f^j(x_i)\}_{j=1,...,J_i, \ i=1,...,I} \) under the observation function \( \pi \). For later purpose, we also define

\[
D' = \{y_i^j = \pi(f^j(x_i)) \in \mathbb{R}^m | j = 1, \ldots, J_i - 1, \ i = 1, \ldots, I\}
\]

Let \( \mathcal{R} \) be a cubical grid decomposition of a domain in \( \mathbb{R}^m \) that covers \( \pi(X) \), and define \( \mathcal{V} = \{R \in \mathcal{R} | R \cap D \neq \emptyset\} \) and \( \mathcal{V}' = \{R \in \mathcal{R} | R \cap D' \neq \emptyset\} \). Then we define a combinatorial multi-valued map \( \mathcal{F} : \mathcal{V}' \rightarrow \mathcal{V} \) as follows: for \( R \in \mathcal{V} \) and \( R' \in \mathcal{V}' \), we define \( R \in \mathcal{F}(R') \) iff there exist \( y_i^j, y_i^{j+1} \in D \) such that \( y_i^j \in R' \) and \( y_i^{j+1} \in R \) hold.

Once \( \mathcal{F} \) is given, one can follow the same procedure as described in the previous sub-sections, and obtain a finite collection \( \{M_i : i = 1, \ldots, k\} \) of the strongly connected components of the equivalent directed graph representations of \( \mathcal{F} \), and hence the Morse graph of \( \mathcal{F} \). We call this method for obtaining a Morse graph from deterministic time-series data the Morse graph method for time-series data.

### 2.2. Multi-valued map from stochastic time-series data

In this sub-section, we review a method to construct a combinatorial multi-valued map from
Assume we are given a set $D$, as above, of $m$-dimensional time-series data. We set a domain $\Omega = [-L+\delta_1, L+\delta_1] \times [-L+\delta_2, L+\delta_2] \times \cdots \times [-L+\delta_m, L+\delta_m]$. $L > 0$ is set to be large enough to contain $D$; typically $L = 4$ is sufficient for the PCA data that we consider in this paper, associated to the fact that the PCA scores are each normalized to have zero mean and unity standard deviation. The parameters $\delta_1, \delta_2, \cdots, \delta_m \in \mathbb{R}$ are the shift of the origin of the domain $\Omega$, introduced to see the robustness to grid division. We put the grid decomposition $R$ on the domain $\Omega$ with grid size $h > 0$. Since $L$ is large enough, we may assume $-h/2 < \delta_i < h/2, i = 1, 2, \cdots, m$, without loss of generality.

We determine a multi-valued map from the time-series as follows. First, we directly obtain the number of data in a grid $i$, $\nu_i$. Next, by introducing a time delay $\tau$, we obtain the number of transitions from grids $i$ to $j$, $\mu_{i \rightarrow j}$. We then define the transition probability, or the conditional probability, from grids $i$ to $j$,

$$T_{i \rightarrow j} = \mu_{i \rightarrow j}/\nu_i$$  \hspace{1cm} (2.1)

We assign a map from $i$ to $j$ if the transition from $i$ to $j$ is superior to that from $j$ to $i$, while maps both from $i$ to $j$ and from $j$ to $i$ if the transition of one direction is not superior to the opposite. By introducing a parameter indicating the degree of superiority, $\rho \geq 1$, this is determined as,

$$i \rightarrow j \quad \text{if} \quad \rho \leq T_{i \rightarrow j}/T_{j \rightarrow i}$$  \hspace{1cm} (2.2a)

$$i \leftrightarrow j \quad \text{if} \quad \rho^{-1} \leq T_{i \rightarrow j}/T_{j \rightarrow i} < \rho$$  \hspace{1cm} (2.2b)

$$i \leftarrow j \quad \text{if} \quad T_{i \rightarrow j}/T_{j \rightarrow i} < \rho^{-1}$$  \hspace{1cm} (2.2c)

In addition, we avoid overestimating rare events; otherwise, for example, the transition probability for $\mu_{i \rightarrow j} = 10$ and $\nu_i = 20$ would be regarded equal to that for $\mu_{k \rightarrow l} = 1$ and $\nu_k = 2$, though the latter may occur just by chance. To this end, we introduce a threshold $\mu_*$ so that we take only maps $\mu_{i \rightarrow j}$ or $\mu_{j \rightarrow i} \geq \mu_*$ into account as the multi-valued map. In fact, as we will see later in FIG. 3.2 and FIG. 3.3 when $\mu_*$ is small, almost all the grids are strongly connected, leading to a Morse set so large as to cover all the relevant domain in the projection of the phase space; with increasing $\mu_*$, such a Morse set is divided into several Morse sets, some of which are robustly seen for various choices of the other parameters; when $\mu_*$ is large, even relevant transitions are removed, leaving only independent Morse sets without any transitions between them.

We thus have $5 + m$ control parameters: $m, h, \tau, \rho, \mu_*$, and $\delta_1, \delta_2, \cdots, \delta_m$. Among them, we set the dimension of the phase space $m = 2$ throughout the paper, according to the former studies [14, 15, 11, 10]. In addition, the time delay $\tau$ characterizes time scale inherent in dynamics, and should be so small that the mean free path of the trajectory does not exceed the grid size $h$; we set $\tau = 1 \text{day}$ throughout the paper. The other parameters left are $3 + m$. We are generally interested in a robust structure to the choice of these parameters. Some of resulted Morse sets and transitions between them may be commonly observed for substantially large region of the parameters while others may be not, and we focus only on the former kind of Morse sets. We set the grid size $h$ sufficiently large that a grid contains statistically enough number of data, whereas sufficiently small so that dynamics in the phase space is well expressed. We
choose the threshold $\mu_*$ still large that Morse sets are well divided, whereas still small so that there are transitions between them. The degree of superiority $\rho$ of transition probability is larger than or equal to unity so that accidental differences are considered equal, but not too much away from unity so that superior transitions are considered dominant. The origin of grid $\delta_i, i = 1, 2, \cdots, m$ should not affect the results, since the results need to be robust to the the manner of grid division. Our purpose is to obtain a most appropriate Morse graph which is hoped to represent an underlying dynamical system from the observed time-series data with noise by controlling these parameters. We call this method the Morse graph method for stochastic time-series data (abbrev. MGSTD method).

Before finishing the subsection, we briefly present the usefulness of the MGSTD method. We apply it to a simple noisy model, though the detail will appear in another paper [16]. We consider a stochastic differential equation (all variables and parameters are valid only in this paragraph),

$$\beta dx_t = -\frac{\partial V}{\partial x}(x_t)dt + \sigma dW_t$$

(2.3)

where $V(x)$ has the following double-well form,

$$V(x) = \frac{1}{2} \begin{cases} 
(x + 1)^2 & x < -1/2 \\
-x^2 + 1/2 & -1/2 \leq x < 1/2 \\
(x - 1)^2 & 1/2 \leq x 
\end{cases}$$
(2.4)

and $dW_t$ is a Wiener process. When no noise is applied ($\sigma = 0$), there are three invariant sets, one unstable fixed point $x = 0$ and two stable fixed points $x = \pm 1$, and the connecting orbits from $x = 0$ to $x = +1$ and from $x = 0$ to $x = -1$.

The Morse graph (left) and the corresponding Morse sets (right) when we do not (top) and do (bottom) apply the MGSTD method are shown in FIG. 2.1. The Morse sets are abbreviated as MS$i$, where $i = 0, 1, 2, \cdots$ is the index of the Morse set.
When we do not apply the MGSTD method, we obtain only one Morse set covering the relevant region of the phase space. When we do apply the MGSTD method, in contrast, we obtain Morse sets MS0 at around $x = 0$, and MS1 and MS2 at around $x = \pm 1$, respectively, and the Morse graph that has the edges from MS0 to MS1 and from MS0 to MS2. This in fact corresponds to the connecting orbits when no noise is applied.

2.3. Meteorological data. In this subsection, we explain the meteorological data to which the MGSTD method described above is applied. We obtained a set of time-sequence vectors based on a reanalysis dataset, in which the satellite and sounding observations are optimally mixed with the weather forecast, named JRA25/JCDAS archived by the Japan Meteorological Agency [18]. The dataset represents synoptic to global meteorological phenomena on a scale greater than several hundred kilometers, and covers a recent period with a 6-hour interval since many meteorological satellites were launched. The analysis period is restricted to three winter months, December, January, and February, from 1979/80 to 2009/10, and the analysis domain is the Northern Hemisphere extra-tropics north to 20°N. After subtracting the trivial seasonal cycle from the data, the low-pass filter extracting variations with a period longer than 10 days was taken for the geopotential height anomaly at a specific isobaric surface of 500 hPa for the tropospheric case and 10 hPa for the stratospheric case. The principal component analysis applied to the low-frequency variability (LFV) data eventually provided a set of time-series vectors: only the first and second modes that we used explain approximately 25% of the LFV variance for the tropospheric case and 65% of the LFV variance for the stratospheric case. The phase space is then spanned by two orthonormal bases of these first and second PC modes, just as [11, 10].

The trajectories of thus obtained PC scores for troposphere (a) and stratosphere (b) are illustrated in FIG. 2.2 projected on the PC 1-2 plane, which in fact show highly stochastic dynamics. The probability density functions of the PC 1 and 2 are shown in FIG. 2.3. They take skewed, non-Gaussian forms. This is related to the existence of several known characteristic patterns of pressure field; for troposphere (FIG. 2.3 (a)), in particular, the ones at around (PC 1, PC 2) = (0, −1), (−1, 0), (−1, 1), and (1, 0) correspond to the ones called ZNAO, PNA, BNAO, and RNA, respectively [14].

Fig. 2.2. Trajectories projected on the PC 1-2 plane for troposphere (a) and stratosphere (b); different colors denote different years.
3. Result.

3.1. Troposphere. The Morse graph and the corresponding Morse sets for $\tau = 1$ day, $h = 0.25$, $\rho = 1.1$, and $\mu_* = 8$ are respectively shown in FIG. 3.1 (a) and (b). The Morse graph (a) shows a transition from MS0 (cyan) to MS1 (red) and then to MS4 (blue) (via MS2 (green) in between). By comparing with the Morse sets (b), the transition corresponds to a clockwise circular motion, from around $(1, 0)$ (MS0 (cyan)) to around $(0, -1)$ (MS1 (red)), and then to around $(-1, 0)$ (MS4 (blue)) (via one in between (MS2 (green))).

![Morse graph and Morse sets](image)

Fig. 3.1. Troposphere: Morse graph and corresponding Morse sets; $\tau = 1$ day, $h = 0.25$, $\rho = 1.1$, $\mu_* = 8$.

The rest of this subsection is devoted to show how the result obtained above is robust to the choice of parameters.

Firstly, we change the threshold of the number of map $\mu_*$. The tree in FIG. 3.2 illustrates the inclusion relation of Morse sets that are divided as we change the parameter $\mu_*$. The Morse graph and the corresponding Morse sets for various $\mu_*$ are shown in FIG. 3.3 with the other parameters fixed. Recall that the result shown in FIG. 3.1 is for $\mu_* = 8$. When $\mu_*$ is too small ($\mu_* = 7$), we obtain only Morse set so large as to cover all the phase space. This is because even rare events are taken into account. When $\mu_*$ is too large ($\mu_* = 11$), we obtain only small Morse sets without
transition from/to each other. This is because, to the contrary of our aim, even less rare, relevant transitions are also removed. In between, in a finite range of $\mu^*$, the clockwise circular motion is preserved; for $\mu^* = 9$, MS3 (orange) $\rightarrow$ MS4 (red) $\rightarrow$ MS5 (purple) $\rightarrow$ MS6 (blue) $\rightarrow$ MS7 (green), which is essentially the same motion as that for $\mu^* = 8$.

Secondly, we change the origin of grid ($\delta_1, \delta_2$). Recall that the result shown in FIG. 3.1 is for ($\delta_1, \delta_2$) = (0, 0). As FIG. 3.2 shows, the clockwise circular motion is essentially preserved for various origin of grid, each with a proper choice of $\mu^*$, with the other parameters fixed; for ($\delta_1, \delta_2$) = (+0.1, +0.1), MS0 (green) $\rightarrow$ MS2 (magenta) $\rightarrow$ MS3 (red) $\rightarrow$ MS4; for ($\delta_1, \delta_2$) = (+0.1, −0.1), MS0 (red) $\rightarrow$ MS2 (green) $\rightarrow$ MS3 (blue); for ($\delta_1, \delta_2$) = (−0.1, +0.1), MS5 (green) $\rightarrow$ MS6 (red) $\rightarrow$ MS7 (blue circle), MS8 (cyan), and MS9 (yellow circle) $\rightarrow$ MS10 (blue); for ($\delta_1, \delta_2$) = (−0.1, −0.1), MS3 (magenta) $\rightarrow$ MS4 (red) $\rightarrow$ MS6 (blue).

Thirdly, we change the grid size $h$, which is shown in FIG. 3.5. Recall that the result shown in FIG. 3.1 is for $h = 0.25$. When $h$ is large ($h = 0.5$), as is simply expected, the phase space does not have enough resolution, so that no significant transition is clearly observed, although we can see a rough trend: MS0 (red) $\rightarrow$ MS1 (green). When $h$ is small, $h = 0.1$, one may still see some traces of Morse sets obtained for $h = 0.25$, i.e., MS4 (red) and 7 (blue) The transitions between them are, however, no longer robust for such small grid size; in fact, when we change the origin of grid $\delta$ from (0, 0) to (+0.01, +0.01), the direction of transition can be converse. Thus too small grid size does not produce robust result.

Fourthly, we change the degree of superiority of transition probability $\rho$ (FIG. 3.6). Recall that the result shown in FIG. 3.1 is for $\rho = 1.1$. For the smallest $\rho$, i.e., unity, we observe essentially the same clockwise circulation: MS0 (cyan) $\rightarrow$ MS1 (red) $\rightarrow$ MS2 (green) $\rightarrow$ MS3 (blue). For large $\rho$ ($\rho = 3$), the transition from around (0, −1) to around (−1, 0) still survives (MS7 (magenta) and MS8 (red) $\rightarrow$ MS9 (yellow) $\rightarrow$ MS10 (blue)).
Fig. 3.3. Troposphere: Morse graph and corresponding Morse sets for various $\mu*$; $\tau = 1\text{day}$, $\rho = 1.1$, $h = 0.25$, $\delta_1 = \delta_2 = 0$. The ones for $\mu* = 8$ are identical to those in FIG. 3.1.
Fig. 3.4. Troposphere: Morse graph and corresponding Morse sets for various \((\delta_1, \delta_2)\); \(\tau = 1\text{day}, h = 0.25, \rho = 1.1\). Some small Morse sets are represented by filled circles instead of filled grids, due to the limitation of distinguishable colors.
3.2. Stratosphere. The Morse graph and the corresponding Morse sets for various $\mu_*$ are shown in Fig. 3.7 with the other parameters fixed. When $\mu_*$ is small ($\mu_* = 5$), there is a MS3 (grey) that covers a large part of the phase space; it has a loop in the first and fourth quadrant. With increasing $\mu_*$, the large Morse set is divided. When $\mu_* = 6$, there is a transition from MS1 (red) at around from $(1,0.5)$ to $(1,1)$, via MS2 (green) at around from $(0,0.5)$ to $(0,1)$, to the large MS4 (grey). When $\mu_* = 8$, there is a transition from MS0 (blue) and MS1 (cyan) at around $(-0.5,1)$
downward to the large MS3 (grey). When $\mu_s = 13$, the large MS is all divided to small ones, in which there are a transition downward from MS1 (red) via MS3 (magenta) to MS4 (cyan), and one right upward from MS4 to MS5 (yellow) at around $(0, -0.5)$.

We change the position of the origin $(\delta_1, \delta_2)$. The result for $(\delta_1, \delta_2) = (+0.1, +0.1)$ are shown in FIG. 3.8. When $\mu_s = 5$, there is a MS2 (grey) covering a large part of the phase space, which again has a loop in the first and fourth quadrant, and MS0 (red) at around $(1, 0.5)$. When $\mu_s = 6$, there is a transition from MS0 (green) at around $(0, 0.5)$ to $(0, 1)$ to the large MS2 (grey). When $\mu_s = 11$, there is a transition from MS1 (cyan) at around $(-0.5, 0.5)$, partially via MS3 (blue), downward to the large MS4 (grey). When $\mu_s = 15$, the large MS is all divided to small ones, in which there are a transition downward from MS2 (red) to MS3 (blue), and one left upward from MS1 (green) at around $(0, -0.5)$ to MS2 (red).

The result for $(\delta_1, \delta_2) = (+0.1, -0.1)$ are shown in FIG. 3.9. When $\mu_s = 4$, there is a MS2 (grey) covering a large part of the phase space; in this case loop is not clear. With increasing $\mu_s$, the large Morse set is divided. When $\mu_s = 10$, on the left side of the large MS1 (grey), there is a transition downward from MS3 (blue) to MS4 (magenta). This transition remains even when $\mu_s = 17$ and the large MS is all divided to small ones; from MS1 (green) to MS2 (blue).

The result for $(\delta_1, \delta_2) = (-0.1, +0.1)$ are shown FIG. 3.10. When $\mu_s = 3$, there is a MS2 (grey) covering a large part of the phase space, which again has a loop in the first quadrant. When $\mu_s = 5$, there is a transition from MS1 (green) at around $(0, 1)$ to the large MS3 (grey). When $\mu_s = 9$, there is a transition downward from MS2 (blue)
to the large MS4 (grey), and one left upward from MS1 (green) at around (0,0.5) to MS2 (blue). When $\mu_*=12$, the large MS is all divided to small ones, in which there are a transition downward from MS3 (magenta) via MS4 (cyan) to MS5 (yellow), and one upward from MS5 to MS6 (grey). In addition, the transition observed for $\mu_*=9$ left upward from MS2 (blue) at around (0,-0.5) to MS3 (magenta) still remains.

The result for $(\delta_1, \delta_2) = (-0.1, -0.1)$ are shown in FIG. 3.11. When $\mu_*=4$, there is a MS1 (grey) covering a large part of the phase space; in this case loop is not clear. When $\mu_*=5$, there is a transition from MS0 (red) at around from (1,0.5) to (1,1), via MS1 (green) at around (0,1), to the large MS3 (grey). With increasing $\mu_*$, the large Morse set is divided. When $\mu_*=15$, the large MS is all divided to small ones, in which there is a transition upward from MS2 (blue) to MS3 (magenta) at around (0,-0.5).

Next, we change the grid size $h$, which is shown in FIG. 3.12. When $h$ is large ($h=0.5$), like for troposphere, the phase space does not have enough resolution, so that no significant transition is clearly observed, although we can see a rough motion: downward from MS0 (red) to MS1 (green), and left upward from MS1 to MS2 (blue). When $h$ is small ($h=0.1$), we still see some similar Morse sets and some similar transitions between them; left upward from MS3 (magenta) to MS4 (cyan) for $(\delta_1, \delta_2) = (0,0)$, and similar one from MS6 (black) to MS8 (grey) for $(\delta_1, \delta_2) = (+0.03, +0.03)$. The motion downward in the second and the third quadrant is, however, not observed.

Finally, we change the degree of superiority of transition probability $\rho$, which is shown in FIG. 3.13. For the smallest $\rho$, i.e., unity, no significant transition is clearly observed, although we can see a rough motion: MS0 (red) at around (0,1) to the large MS1 (green) for $\mu_*=10$. For large $\rho$ ($\rho=3$), we observe essentially the same motion as for $\rho=1.1$: transition from MS6 (red), via MS7 (green) and MS8 (blue circle) at around (0,1), to the large MS11 (grey) for $\mu_*=6$, and one in the second and the third quadrant downward from MS3 (magenta) to MS5 (yellow) for $\mu_*=17$. The motion right upward from third quadrant to the origin is, however, not observed.

4. Discussion and concluding remarks. For troposphere, we have observed mainly three Morse sets at around (1,0), (0,-1), and (-1,0). Noting that the choice of signs of PC scores is arbitrary, by comparing the probability density FIG. 2.3(a) with that in the former paper [14], these three Morse sets respectively correspond to RNA, ZNAO, and PNA; the last may include BNAO, as it extends to around (-1,1). Moreover, we detected the clockwise circular dynamics from around (1,0) to (0,-1) and then to (-1,0). This corresponds to the circular dynamics form RNA to ZNAO and then to PNA (and BNAO), which is consistent with the circular motion discovered in past studies [15] [11].

The results for stratosphere is summarized as follows. First, there is a transition leftward from a Morse set at around from (1,0.5) to (1,1), via a Morse set at around (0,1), to the second or third quadrant. Second, there is a transition downward from the second to the third quadrant. Third, in the third quadrant, there is a transition right upward towards the origin. Fourth, there is a transition left upward from the origin towards the second quadrant. To sum up the above three, we obtain a large counterclockwise circulation from the first via the second to the third quadrant, and a small counterclockwise circulation inside the third quadrant. Comparing to the meteorological knowledge, the Morse set at around from (1,0.5) to (1,1) may be related to the final stage of a stratospheric sudden warming (SSW) event. The transition leftward from the Morse set as well as that left upwards from the origin mean the po-
Fig. 3.7. Stratosphere: Morse graph and corresponding Morse sets for various $\mu_\ast$; $\tau = 1$ day, $\rho = 1.1$, $h = 0.25$, $\delta_1 = \delta_2 = 0$. 

\[ \mu_\ast = 5 \]

\[ \mu_\ast = 6 \]

\[ \mu_\ast = 8 \]

\[ \mu_\ast = 13 \]
Fig. 3.8. Stratosphere: Morse graph and corresponding Morse sets for various $\mu_\ast$; $\tau = 1$ day, $\rho = 1.1$, $h = 0.25$, $\delta_1 = +0.1$, $\delta_2 = +0.1$. 

| $\mu_\ast$ | Morse Graph | Morse Sets |
|------------|-------------|------------|
| 5          | ![Morse Graph](image1) | ![Morse Sets](image1) |
| 6          | ![Morse Graph](image2) | ![Morse Sets](image2) |
| 11         | ![Morse Graph](image3) | ![Morse Sets](image3) |
| 15         | ![Morse Graph](image4) | ![Morse Sets](image4) |
Fig. 3.9. Stratosphere: Morse graph and corresponding Morse sets for various $\mu_*$; $\tau = 1$ day, $\rho = 1.1$, $h = 0.25$, $\delta_1 = +0.1$, $\delta_2 = -0.1$. 
Fig. 3.10. Stratosphere: Morse graph and corresponding Morse sets for various $\mu_*$; $\tau = 1$ day, $\rho = 1.1$, $h = 0.25$, $\delta_1 = -0.1$, $\delta_2 = +0.1$. 

Morse graph

| $\mu_*$ | Morse sets |
|---------|------------|
| 3       | [MS0, MS1, MS2] |
| 5       | [MS0, MS1, MS2] |
| 9       | [MS0, MS1, MS2] |
| 12      | [MS0, MS1, MS2] |
Fig. 3.11. Stratosphere: Morse graph and corresponding Morse sets for various $\mu_*; \tau = 1 \text{day}$, $\rho = 1.1$, $h = 0.25$, $\delta_1 = -0.1$, $\delta_2 = -0.1$. 
Fig. 3.12. Troposphere: Morse graph and corresponding Morse sets for various $h$; $\tau = 1$ day, $\rho = 1.1$. Some small Morse sets are represented by filled circles instead of filled grids, due to the limitation of distinguishable colors.

lar vortex amplification that typically occurs after the SSW. Moreover, the transition downward from the second to third quadrant is associated with a wave-energy charge by the vertical propagation of Rossby waves with their zonal wavenumber one. The transition right upwards towards the origin corresponds to the reset of the characteristic pattern, as PC1 and PC2 both become near zero. These are consistent with the previous studies [11].

Previous studies with the use of stochastic differential equation [11, 10] have given averaged vector field, in an Eulerian point of view, but they have not shown a trajectory corresponding to circular motion. The present result, by contrast, indicates the
Fig. 3.13. Troposphere: Morse graph and corresponding Morse sets for various $\rho; \tau = 1$ day, $h = 0.25$. Some small Morse sets are represented by filled circles instead of filled grids, due to the limitation of distinguishable colors.
existence of the averaged trajectory, by connecting the transitions from Morse sets to others, in a Lagrangian point of view. The results of these approaches complementarily support the existence of the circular motion.

In conclusion, we applied the MGSTD method to the time-series data from meteorological measurement, and obtained the characteristic atmospheric states and the preferable transitions between them as the Morse sets and the transitions between them, which are relevant to the previous studies. The result supports the usefulness of the present method not only for time-series from the model dynamical systems but also for that from real measurements and experiments.

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