DE RHAM AND DOLBEAULT COHOMOLOGY OF SOLVMANIFOLDS WITH LOCAL SYSTEMS

HISASHI KASUYA

Abstract. Let $G$ be a simply connected solvable Lie group with a lattice $\Gamma$ and the Lie algebra $\mathfrak{g}$ and a representation $\rho : G \to GL(V_\rho)$ whose restriction on the nilradical is unipotent. Consider the flat bundle $E_\rho$ given by $\rho$. By using “many” characters $\{\alpha\}$ of $G$ and “many” flat line bundles $\{E_\alpha\}$ over $G/\Gamma$, we show that an isomorphism

$$\bigoplus_{\alpha} H^*(\mathfrak{g}, V_\alpha \otimes V_\rho) \cong \bigoplus_{E_\alpha} H^*(G/\Gamma, E_\alpha \otimes E_\rho)$$

holds. This isomorphism is a generalization of the well-known fact: “If $G$ is nilpotent and $\rho$ is unipotent then, the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho)$ holds”. By this result, we construct an explicit finite dimensional cochain complex which compute the cohomology $H^*(G/\Gamma, E_\rho)$ of solvmanifolds even if the isomorphism $H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho)$ does not hold. For Dolbeault cohomology of complex parallelizable solvmanifolds, we also prove an analogue of the above isomorphism result which is a generalization of computations of Dolbeault cohomology of complex parallelizable nilmanifolds. By this isomorphism, we construct an explicit finite dimensional cochain complex which compute the Dolbeault cohomology of complex parallelizable solvmanifolds.

1. Background and Main results

1.1. Background. We have nice theorem for de rham cohomology of nilmanifolds with local systems.

Theorem 1.1 (due to [13] or [16]). Let $N$ be a simply connected real nilpotent Lie group and $n$ the Lie algebra of $N$. Suppose $N$ has a lattice $\Gamma$. Let $\rho : N \to GL(V_\rho)$ be a finite dimensional unipotent representation. We define the flat bundle $E_\rho = (N \times V_\rho)/\Gamma$ given by the equivalent relation $(\gamma g, \rho(\gamma)v) \equiv (g, v)$ for $g \in N$, $v \in V_\rho$, $\gamma \in \Gamma$. Consider the cochain complex $\wedge n_\mathfrak{c}^* \otimes V_\rho$ of Lie algebra (see [14]) and the canonical inclusion

$$\wedge n_\mathfrak{c}^* \otimes V_\rho \to A^*(N/\Gamma, E_\rho).$$

Then this inclusion induces a cohomology isomorphism

$$H^*(n, V_\rho) \cong H^*(N/\Gamma, E_\rho).$$

Some researchers tried to extend Theorem 1.1 for solvmanifolds. In fact it is proved that for a simply connected solvable Lie group $G$ with the Lie algebra $\mathfrak{g}$ admitting a lattice $\Gamma$ and a representation $\rho : G \to GL(V_\rho)$, if:

(H) (16) The representation $\rho \oplus \text{Ad}$ is triangular or,
We consider the Dobeault complex (given by the equivalent relation $L$ the subcomplex of induces a cohomology isomorphism $GL$. Theorem 1.3. Let $\rho : G \to GL(V_\rho)$ be a finite dimensional holomorphic unipotent representation. We also consider the anti-holomorphic representation $\bar{\rho}$, and 1.2. These analogous each other. We consider the “many” characters of isomorphism theorems for solvmanifolds which are generalizations of Theorem 1.1 Main results. 1.2. HISASHI KASUYA

Theorem 1.2 (due to [17]). Let $N$ be a simply connected complex nilpotent Lie group and $n$ the Lie algebra (as a complex Lie algebra) of $N$. Suppose $N$ has a lattice $\Gamma$. Let $\sigma : N \to GL(V_\sigma)$ be a finite dimensional holomorphic unipotent representation. We also consider the anti-holomorphic representation $\bar{\sigma} : N \to GL(V_\sigma)$. Define the flat holomorphic vector bundle $L_\sigma = (N \times V_\sigma) / \Gamma$ over $G/\Gamma$ given by the equivalent relation $(\gamma g, \sigma(\gamma)v) \cong (g, v)$ for $g \in N$, $v \in V_\sigma$, $\gamma \in \Gamma$. We consider the Dobeault complex $(A^* (N/\Gamma, L_\sigma), \bar{\partial})$. We regard $\bigwedge \n^* \otimes V_\sigma$ as the subcomplex of $(A^0, (N/\Gamma, L_\sigma), \bar{\partial})$ which consists of the left-invariant "anti"-holomorphic forms with values in $L_\sigma$. Then the inclusion

$$\bigwedge \n^* \otimes V_\sigma \to A^0 (N/\Gamma, L_\sigma)$$

induces a cohomology isomorphism

$$H^*(n, V_\sigma) \cong H^0 (N, L_\sigma).$$

Hence since $N/\Gamma$ is complex parallelizable, we have an isomorphism

$$\bigwedge \mathbb{C}^{\dim N} \otimes H^* (n, V_\sigma) \cong H^*_{\bar{\partial}} (N, L_\sigma).$$

It is desired that Theorem 1.1 and 1.2 are generalized for solvmanifolds and we can compute the de Rham and Dolbeault cohomology of solvmanifolds even if the isomorphism $H^*(g, V_\rho) \cong H^* (G/\Gamma, E_\rho)$ (resp. $\bigwedge \mathbb{C}^{\dim N} \otimes H^* (n, V_\sigma) \cong H^*_{\bar{\partial}} (N, L_\sigma)$) does not hold.

1.2. Main results. The first purpose of this paper is to show new-type cohomology isomorphism theorems for solvmanifolds which are generalizations of Theorem 1.1 and 1.2. These analogous each other. We consider the ”many” characters of $G$ and ”many” line bundles over $G/\Gamma$. In this paper we prove:

Theorem 1.3. Let $G$ be a simply connected real solvable Lie group with a lattice $\Gamma$ and $\mathfrak{g}$ the Lie algebra of $G$. Let $N$ be the nilradical (i.e. maximal connected nilpotent normal subgroup) of $G$. Let $A_{(G,N)} = \{ \alpha \in \text{Hom}(G, \mathbb{C}^*) | \alpha|_{\mathfrak{n}} = 1 \}$ and $A_{(G,N)}(\Gamma)$ the set $\{ E_\alpha \}$ of all the isomorphism classes of flat line bundles given by $\{ V_\alpha \}_{\alpha \in A_{(G,N)}}$. Let $\rho : G \to GL(V_\rho)$ be a representation. For the nilradical $N$ of $G$, we assume that the restriction $\rho|_{\mathfrak{n}}$ is a unipotent representation. We consider the direct sum

$$\bigoplus_{\alpha \in A_{(G,N)}} \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho$$

of the Lie algebra cochain complexes. We also consider the direct sum

$$\bigoplus_{E_\alpha \in A_{(G,N)}(\Gamma)} A^*(G/\Gamma, E_\alpha \otimes E_\rho).$$

Then the inclusion

$$\bigoplus_{\alpha \in A_{(G,N)}} \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho \to \bigoplus_{E_\alpha \in A_{(G,N)}(\Gamma)} A^*(G/\Gamma, E_\alpha \otimes E_\rho).$$
induces a cohomology isomorphism
\[ \bigoplus_{\alpha \in \mathcal{A}(G,N)} H^*(\mathfrak{g}, V_\alpha \otimes V_\rho) \cong H^* \cong \bigoplus_{E_b \in \mathcal{A}(G,N)(\Gamma)} H^*(G/\Gamma, E_\alpha \otimes E_\rho). \]

We also prove:

**Theorem 1.4.** Let \( G \) be a simply connected complex solvable Lie group with a lattice \( \Gamma \) and \( \mathfrak{g} \) the Lie algebra (as a complex Lie algebra) of \( G \). Let \( N \) be the nilradical of \( G \). Let \( B_{(G,N)} = \{ \alpha \in \text{Hom}_\text{hol}(G, \mathbb{C}^*)|_{\alpha|_{\lambda}} = 1 \} \) and \( B_{(G,N)}(\Gamma) \) the set \( \{ L_\alpha \} \) of all the isomorphism classes of holomorphic line bundles given by \( \{ V_\alpha \}_{\alpha \in B_{(G,N)}} \). Let \( \sigma : G \to GL(V_\sigma) \) be a holomorphic representation. For the nilradical \( N \) of \( G \), we assume that the restriction \( \sigma|_{\lambda} \) is a unipotent representation. We consider the direct sum
\[ \bigoplus_{\alpha \in B_{(G,N)}} \bigwedge \mathfrak{g}^* \otimes V_\alpha \otimes V_\sigma, \]

of the Lie algebra cochain complexes. We also consider the direct sum
\[ \bigoplus_{L_\alpha \in B_{(G,N)}(\Gamma)} A^{0,*}(G/\Gamma, L_\alpha \otimes L_\sigma) \]

of Dolbeault complexes.

Then the inclusion
\[ \bigoplus_{\alpha \in B_{(G,N)}} \bigwedge \mathfrak{g}^* \otimes V_\alpha \otimes V_\sigma \to \bigoplus_{L_\alpha \in B_{(G,N)}(\Gamma)} A^{0,*}(G/\Gamma, L_\alpha \otimes L_\sigma) \]

induces a cohomology isomorphism
\[ \bigoplus_{\alpha \in B_{(G,N)}} H^*(\mathfrak{g}, V_\alpha \otimes V_\sigma) \cong \bigoplus_{L_\alpha \in B_{(G,N)}(\Gamma)} H^{0,*}(G/\Gamma, L_\alpha \otimes L_\sigma). \]

**Remark 1.** The correspondence \( \mathcal{A}(G,N) \to \mathcal{A}(G,N)(\Gamma) \) (resp. \( B_{(G,N)} \to B_{(G,N)}(\Gamma) \)) is not 1 to 1. This remark is very important for the case the isomorphism \( H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho) \) (resp. \( H^*(\mathfrak{g}, V_\sigma) \cong H^{0,*}(G/\Gamma, L_\sigma) \)) does not hold.

The second purpose of this paper is to construct a explicit finite dimensional cochain complex which compute the de Rham cohomology \( H^*(G/\Gamma, E_\rho) \) and the Dolbeault cohomology \( H^{0,*}(G/\Gamma, L_\sigma) \) by using Theorem 1.3 and 1.4. We prove:

**Theorem 1.5.** Let \( G \) be a simply connected real (resp complex) solvable Lie group and \( \mathfrak{g} \) the Lie algebra of \( G \). Define \( \mathcal{A}(G,N) \) (resp. \( B_{(G,N)}(\Gamma) \)) as in Theorem 1.3 (resp Theorem 1.4). Let \( \rho : G \to GL(V_\rho) \) (resp. \( \sigma : G \to GL(V_\sigma) \)) be a representation with the assumption of Theorem 1.3 (resp Theorem 1.4). We consider the direct sum
\[ \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_\alpha^* \otimes V_\alpha \otimes V_\rho \]

(resp.
\[ \bigoplus_{\alpha \in B_{(G,N)}} \bigwedge \mathfrak{g}^* \otimes V_\alpha \otimes V_\sigma \]

) of the Lie algebra cochain complexes.

Then there exists a finite dimensional subcomplex
\[ A^* \subset \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_\alpha^* \otimes V_\alpha \otimes V_\rho \]
By Theorem 1.3 (resp Theorem 1.4), we have the inclusion

\[ \iota : A^* \to \bigoplus_{\alpha \in A_{G,N}(\Gamma)} A^*(G/\Gamma, E_{\alpha} \otimes E_\rho) \]

(resp.

\[ \iota : B^* \to \bigoplus_{\alpha \in A_{G,N}(\Gamma)} A^0(G/\Gamma, L_{\bar{\alpha}} \otimes L_{\bar{\sigma}}) \]

) inducing a cohomology isomorphism. Hence we have:

**Corollary 1.6.** Let \( A^* \) be a simply connected real abelian Lie group with a lattice \( \Gamma \) and \( a \) the Lie algebra of \( A \). We denote by \( A_s \) (resp. \( A_u \)) the semi-simple (resp. unipotent) part of \( A \) for the Jordan decomposition (see [9] for the definition). We will use the following facts.

**Lemma 2.1.** Let \( \rho: N \to GL(V_\varphi) \) be a representation. Then the map \( \rho': N \ni g \to (\varphi(g))_s \) is also a representation (see [2]). Since \( \varphi'(N) \) is connected nilpotent group and consists of semi-simple elements, the Zariski-closure of \( \varphi'(N) \) is an algebraic torus (see [9, Section 19]) and hence \( \varphi' \) is diagonalizable.

3. **Proof of Theorem 1.3**

### 3.1. Cohomology of tori.

Let \( \rho: A \to GL(V_\rho) \) be a representation. Suppose \( \rho = \beta \otimes \phi \) such that \( \beta \) is a character of \( A \) and \( \phi \) is a unipotent representation. Then we have:

If \( \beta \) is non-trivial, then we have

\[ H^*(a, V_\rho) = 0. \]

If the flat line bundle \( E_\beta \) is non-trivial, then we have

\[ H^*(A/\Gamma, E_\rho) = 0. \]
Proof. Suppose $\dim V_\rho = 1$. Then if $\beta$ is non-trivial, we can show $H^\ast(a, V_\rho) = H^\ast(a, V_\beta) = 0$ by simple computation and if $E_\beta$ is non-trivial, then we have $H^\ast(A/\Gamma, E_\rho) = H^\ast(\Gamma, E_\beta) = 0$ by [11] Lemma 2.1.

In case $\dim V_\sigma = n > 1$, by the triangulation of $\rho$, we have a $(n-1)$-dimensional $A$-submodule $V_\rho'$ such that $V_\rho' / V_\rho' = V_\beta$. Then by the long exact sequence of cohomology of Lie algebra or group (see [14]), the lemma follows inductively. □

Lemma 3.2. Let $\rho : A \to GL(V_\rho)$ be a representation. Then we have a basis of $V_\rho$ such that $\rho$ is represented by

$$\rho = \bigoplus_{i=1}^{k} \alpha_i \otimes \phi_i$$

for characters $\alpha_i$ of $G$ and unipotent representations $\phi_i$ of $G$.

Proof. For a character $\alpha$, we denote by $W_\alpha$ the subspace of $V_\rho$ consisting of the elements $w \in V_\rho$ such that for some positive integer $n$ we have $(\rho(a) - \alpha(a)I)^n w = 0$ for any $a \in A$. Since $A$ is abelian, we have a decomposition

$$V_\rho = W_{\alpha_1} \oplus \cdots \oplus W_{\alpha_k}$$

by generalized eigenspace decomposition of $\rho(a)$ for all $a \in A$. Let $\rho_i(a) = (\rho(a))|_{W_{\alpha_i}}$. Then we have $\rho = \rho_1 \oplus \cdots \oplus \rho_k$. We have $(\rho_i(a))_s = \alpha_i I$. Let $\phi_i(a) = (\rho_i(a))_u$. By Lemma 2.1, $\phi_i$ is a unipotent representation and we have $\rho_i(a) = (\rho_i(a))_s (\rho_i(a))_u = (\alpha_i \otimes \phi_i)(a)$. Hence the Lemma follows. □

Let $\{V_\alpha\}_{\alpha \in \text{Hom}(A, C^\ast)}$ be the set of all 1-dimensional representations of $A$ and $\mathcal{H}(A/\Gamma) = \{E_\beta\}$ the set of all the isomorphism classes of flat line bundles given by $\{V_\alpha\}_{\alpha \in \text{Hom}(A, C^\ast)}$. We notice that the correspondence $\{V_\alpha\}_{\alpha \in \text{Hom}(A, C^\ast)} \to \mathcal{H}(A/\Gamma)$ is not injective. We consider the direct sums

$$\bigoplus_{\alpha \in \text{Hom}(A, C^\ast)} \bigwedge \mathfrak{a}_C \otimes V_\alpha \otimes V_\rho$$

and

$$\bigoplus_{E_\alpha \in \mathcal{H}(A/\Gamma)} A^\ast(A/\Gamma, E_\alpha \otimes E_\rho).$$

Proposition 3.3. The inclusion

$$\bigoplus_{\alpha \in \text{Hom}(A, C^\ast)} \bigwedge \mathfrak{a}_C \otimes V_\alpha \otimes V_\rho \to \bigoplus_{E_\alpha \in \mathcal{H}(A/\Gamma)} A^\ast(A/\Gamma, E_\alpha \otimes E_\rho)$$

induces a cohomology isomorphism.

Proof. Consider the decomposition

$$\sigma = \bigoplus_{i=1}^{k} \alpha_i \otimes \phi_i$$

as the above lemma. Then we have

$$\bigoplus_{\alpha \in \text{Hom}(A, C^\ast)} \bigwedge \mathfrak{a}_C \otimes V_\alpha \otimes V_\rho = \bigoplus_{\alpha \in \text{Hom}(A, C^\ast)} \bigwedge \mathfrak{a}^\ast \oplus \bigoplus_{i=1}^{k} V_{\alpha \alpha_i} \otimes V_{\phi_i}$$

and

$$\bigoplus_{E_\alpha \in \mathcal{H}(A/\Gamma)} A^\ast(A/\Gamma, E_\alpha \otimes E_\rho) = \bigoplus_{E_\alpha \in \mathcal{H}(A/\Gamma)} A^\ast(A/\Gamma, E_\alpha \otimes E_\rho).$$
and
\[ \bigoplus_{E\in\mathcal{H}(A/G)} A^*(A/\Gamma, E\otimes E) = \bigoplus_{E\in\mathcal{H}(A/G)} A^*(A/\Gamma, \bigoplus_{\alpha=1}^k E\otimes E_{\alpha} \otimes E_{\phi_i}). \]

By Theorem 1.1 and Lemma 3.1 we have
\[ H^*(\bigoplus_{E\in\mathcal{H}(A/G)} A^*(A/\Gamma, E\otimes E)) \cong H^*(A/G, \bigoplus_{\alpha=1}^k V_{\alpha} \otimes V_{\phi_i}). \]

By Lemma 3.1 we have
\[ H^*(\bigoplus_{\alpha\in\text{Hom}(A,C^*)} \bigwedge a_{\alpha}^* \otimes V_{\alpha} \otimes V_{\rho}) \cong H^*(A, \bigoplus_{i=1}^k V_{\phi_i}). \]

Hence the proposition follows. \[\square\]

3.2. Mostow bundle and spectral sequence. Let \( G \) be a simply connected solvable Lie group with a lattice \( \Gamma \) and \( \mathfrak{g} \) be the Lie algebra of \( G \). Let \( N \) be the nilradical of \( G \). It is known that \( \Gamma \cap N \) is a lattice of \( N \) and \( \Gamma/\Gamma \cap N \) is a lattice of the abelian Lie group \( G/N \) (see [16]). The solvmanifold \( G/\Gamma \) is a fiber bundle
\[ N/\Gamma \cap N = NT/\Gamma \rightarrow G/\Gamma \rightarrow G/N = (G/N)/(\Gamma/\Gamma \cap N) \]
over a torus with a nilmanifold \( N/\Gamma \cap N \) as fiber. We call this fiber bundle the Mostow bundle of \( G/\Gamma \). The structure group is \( NT/\Gamma_0 \) as left translations where \( \Gamma_0 \) is the largest normal subgroup of \( \Gamma \) which is normal in \( NT \) (see [18]).

Let \( \rho : G \rightarrow GL(V_{\rho}) \) be a representation such that the restriction \( \rho|_N \) is a unipotent representation. For the Mostow bundle \( p : G/\Gamma \rightarrow (G/N)/(\Gamma/\Gamma \cap N) \), we define the vector bundle
\[ \mathcal{H}^q(N/\Gamma \cap N) = \bigcup_{x\in(G/N)/(\Gamma/\Gamma \cap N)} \mathcal{H}^q(p^{-1}(x), E_{\rho}). \]

Over the torus \( (G/N)/(\Gamma/\Gamma \cap N) \). By Theorem 1.1 we have \( H^q(p^{-1}(x), E_{\rho}) \cong H^q(n, V_{\rho}). \) Hence let \( \Lambda_q : G/N \rightarrow GL(H^q(n, V_{\rho})) \) be the representation induced by the extension \( 1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1 \), then we can regard \( \mathcal{H}^q(N/\Gamma \cap N) \) as the flat bundle \( E_{\Lambda_q} \). We consider the filtration
\[ \bigwedge^{p+q} g^\mathbb{C} = \{ \omega \in \bigwedge^{p+q} g^\mathbb{C} | \omega(X_1, \ldots, X_{p+1}) = 0 \text{ for } X_1, \ldots, X_{p+1} \in n_{\mathbb{C}} \}. \]

This filtration gives the filtration of the de Rham complex \( \bigwedge g^\mathbb{C} \otimes V_{\rho} \) and the filtration of the de Rham complex \( A^*(G/\Gamma, E_{\rho}) \). We consider the spectral sequence \( E_{\ast}^{\ast}(g) \) of \( \bigwedge g^\mathbb{C} \otimes V_{\rho} \) and the spectral sequence \( E_{\ast}^{\ast}(G/\Gamma) \) of \( A^*(G/\Gamma, E_{\rho}) \). Set \( G/N = A \) and \( \Gamma/\Gamma \cap N = \Delta \) and \( a = \mathfrak{g}/n \). Then we have the commutative diagram
\[ \begin{array}{ccc}
E_{1}^{\ast}(g) & \longrightarrow & E_{1}^{\ast}(G/\Gamma) \\
\bigwedge a_{\ast}^* \otimes V_{\Lambda_q} & \cong & A^*(A/\Delta, E_{\Lambda_q})
\end{array} \]

(see [6, 116 Section 7]).
3.3. Proof of Theorem 1.3

Proof. Consider the spectral sequence \( E_1^{*,*}(g) \) of
\[
\bigoplus_{\alpha \in A(G,N)} \bigwedge g_C^* \otimes V_\alpha \otimes V_\rho
\]
and the spectral sequence \( E_1^{*,*}(G/\Gamma) \) of
\[
\bigoplus_{E_\alpha \in A(G,N)(\Gamma)} A^*(G/\Gamma, E_\alpha \otimes E_\rho).
\]
Set \( A = G/N \) and \( \Delta = \Gamma/\Gamma \cap N \) and \( a = g/n \). Since we can identify \( A(G,N) \) (resp. \( A(G,N)(\Gamma) \)) with Hom(\( A, C^* \)) (resp. \( H(A/\Delta) \)), we have the commutative diagram
\[
E_1^{*,*}(g) \longrightarrow E_1^{*,*}(G/\Gamma) \\
\bigoplus_{\alpha \in \text{Hom}(A, C^*)} a_C^* \otimes V_\alpha \otimes \Lambda a \longrightarrow \bigoplus_{E_\alpha \in H(A/\Delta)} A_{0, e}(A/\Delta, E_\alpha \otimes E_\Lambda).
\]
By Proposition 3.3, the homomorphism \( E_1^{*,*}(g) \rightarrow E_1^{*,*}(G/\Gamma) \) induces a cohomology isomorphism and hence we have an isomorphism \( E_2^{*,*}(g) \cong E_2^{*,*}(G/\Gamma) \). Hence the theorem follows. □

4. Proof of Theorem 1.4

4.1. Dolbeault cohomology of tori. First we prove Theorem 1.2 by Sakane’s Theorem [17].

Proof of Theorem 1.2. In case \( \dim V_\sigma = 1 \), \( \sigma \) is trivial and the theorem follows from Sakane’s Theorem [17].

In case \( \dim V_\sigma = n > 1 \), since \( \sigma \) is unipotent, we have a \((n - 1)\)-dimensional \( G \)-submodule \( V_\sigma' \subset V_\sigma \) such that \( V_\sigma/V_\sigma' \) is the trivial submodule. Then we have the spectral sequences
\[
0 \longrightarrow \bigwedge g^* \otimes V_\sigma' \longrightarrow \bigwedge g^* \otimes V_\sigma \longrightarrow \bigwedge g^* \otimes V_\sigma/V_\sigma' \longrightarrow 0
\]
and
\[
0 \longrightarrow A_0^0(G/\Gamma, L_{\bar{\sigma}'}) \longrightarrow A_0^0(G/\Gamma, L_{\bar{\sigma}}) \longrightarrow A_0^0(G/\Gamma, L_{\bar{\sigma}}/L_{\bar{\sigma}'}) \longrightarrow 0.
\]
We have the commutative diagram
\[
0 \longrightarrow \bigwedge g^* \otimes V_\sigma' \longrightarrow \bigwedge g^* \otimes V_\sigma \longrightarrow \bigwedge g^* \otimes V_\sigma/V_\sigma' \longrightarrow 0 \\
0 \longrightarrow A_0^0(G/\Gamma, L_{\bar{\sigma}'}) \longrightarrow A_0^0(G/\Gamma, L_{\bar{\sigma}}) \longrightarrow A_0^0(G/\Gamma, L_{\bar{\sigma}}/L_{\bar{\sigma}'}) \longrightarrow 0.
\]
Considering the long exact sequence of cohomologies, by the five lemma, the theorem follows inductively. □

Let \( A \) be a simply connected complex abelian group with a lattice \( \Gamma \) and \( a \) the Lie algebra of \( A \).
Lemma 4.1. Let \( \sigma : A \to GL(V_\sigma) \) be a holomorphic representation. Suppose \( \sigma = \beta \otimes \phi \) such that \( \beta \) is a character of \( A \) and \( \phi \) is a unipotent representation. Then we have:

If \( \beta \) is non-trivial, then we have

\[
H^*(a, V_\sigma) = 0.
\]

If the holomorphic line bundle \( L_{\bar{\beta}} \) is non-trivial, then we have

\[
H^{0,*}(A/\Gamma, L_{\bar{\sigma}}) = 0.
\]

Proof. In case \( \dim V_\sigma = 1 \), the lemma is proved in [11].

In case \( \dim V_\sigma = n > 1 \), by the triangulation of \( \sigma \), we have a \( (n-1) \)-dimensional \( A \)-submodule \( V_\sigma' \) such that \( V_\sigma/V_\sigma' = V_\beta \). Then we have the exact sequence

\[
0 \to A^{0,*}(A/\Gamma, L_{\bar{\sigma}}) \to A^{0,*}(A/\Gamma, L_{\bar{\sigma}}) \to A^{0,*}(A/\Gamma, L_{\bar{\sigma}}/L_{\bar{\sigma}'}) \to 0.
\]

Considering the long exact sequence of cohomologies, the lemma follows inductively. □

By similar proof of Lemma 4.2 we have the following lemma.

Lemma 4.2. Let \( \sigma : A \to GL(V_\sigma) \) be a holomorphic representation. Then we have a basis of \( V_\sigma \) such that \( \sigma \) is represented by

\[
\sigma = \bigoplus_{i=1}^k \alpha_i \otimes \phi_i
\]

for holomorphic characters \( \alpha_i \) and holomorphic unipotent representations \( \phi_i \).

Let \( \{V_\alpha\}_{\alpha \in \text{Hom}_{\text{hol}}(A, \mathbb{C}^*)} \) be the set of all 1-dimensional holomorphic representations of \( A \) and \( \mathcal{H}_{\text{hol}}(A/\Gamma) = \{L_\alpha\} \) the set of all the isomorphism classes of holomorphic line bundles given by \( \{V_\alpha\}_{\alpha \in \text{Hom}_{\text{hol}}(A, \mathbb{C}^*)} \). We notice that the correspondence \( \{V_\alpha\}_{\alpha \in \text{Hom}_{\text{hol}}(A, \mathbb{C}^*)} \to \mathcal{H}(A/\Gamma) \) is not injective. We consider the direct sums

\[
\bigoplus_{\alpha \in \text{Hom}_{\text{hol}}(A, \mathbb{C}^*)} \bigwedge a^* \otimes V_\alpha \otimes V_\sigma
\]

and

\[
\bigoplus_{L_\alpha \in \mathcal{H}_{\text{hol}}(A/\Gamma)} A^{0,*}(A/\Gamma, L_{\bar{\alpha}} \otimes L_{\bar{\sigma}}).
\]

Proposition 4.3. The inclusion

\[
\bigoplus_{\alpha \in \text{Hom}_{\text{hol}}(A, \mathbb{C}^*)} \bigwedge a^* \otimes V_\alpha \otimes V_\sigma \to \bigoplus_{L_\alpha \in \mathcal{H}_{\text{hol}}(A/\Gamma)} A^{0,*}(A/\Gamma, L_{\bar{\alpha}} \otimes L_{\bar{\sigma}})
\]

induces a cohomology isomorphism.

Proof. By using Theorem 1.2 and Lemma 4.1 and 4.2 we can prove the proposition by similar argument of the proof of Proposition 3.3 □
4.2. Mostow bundle and spectral sequence. Let \( G \) be a simply connected complex solvable Lie group with a lattice \( \Gamma \) and \( \mathfrak{g} \) be the Lie algebra of \( G \). Then the Mostow bundle

\[
N/\Gamma \cap N = N\Gamma /\Gamma \longrightarrow G/\Gamma \longrightarrow G/N = (G/N)/(\Gamma /\Gamma \cap N)
\]

is holomorphic.

Let \( \sigma : G \to GL(V_\sigma) \) be a representation such that the restriction \( \sigma|_N \) is a unipotent representation. For the Mostow bundle \( p : G/\Gamma \to (G/N)/(\Gamma /\Gamma \cap N) \), we define the vector bundle

\[
H^{0,q}(N/\Gamma \cap N) = \bigcup_{x \in (G/N)/(\Gamma /\Gamma \cap N)} H^{0,q}(p^{-1}(x), L_\sigma)
\]

over the torus \((G/N)/(\Gamma /\Gamma \cap N)\). By Theorem [12] we have \( H^{0,q}(p^{-1}(x), L_\sigma) \cong H^{0,q}(\mathfrak{n}, V_\sigma) \). Hence let \( \Lambda_\eta : G/N \to GL(H^q(\mathfrak{n}, V_\sigma)) \) be the representation induced by the extension \( 1 \to N \to G \to G/N \to 1 \), then we can regard \( H^{0,q}(N/\Gamma \cap N) \) as the flat holomorphic bundle \( L_\Lambda \). We consider the filtration

\[
F^p \bigwedge \mathfrak{g}^* = \{ \omega \in F^p \bigwedge \mathfrak{g}^* | \omega(X_1, \ldots, X_{p+1}) = 0 \text{ for } X_1, \ldots, X_{p+1} \in \mathfrak{n} \}.
\]

This filtration gives the filtration of the cochain complex \( \bigwedge \mathfrak{g}^* \otimes V_\sigma \) and the filtration of the Dolbeault complex \( A^{0,*}(G/\Gamma, L_\sigma) = C^\infty(G/\Gamma, L_\sigma) \otimes \bigwedge \mathfrak{g}^* \). We consider the spectral sequence \( \text{Dol} \mathcal{E}_1^{*,*}(\mathfrak{g}) \) of \( \bigwedge \mathfrak{g}^* \otimes V_\sigma \) and the spectral sequence \( \text{Dol} \mathcal{E}_1^{*,*}(G/\Gamma) \) of \( A^{0,*}(G/\Gamma, L_\sigma) \). Set \( G/N = A \) and \( \Gamma /\Gamma \cap N = \Delta \) and \( \mathfrak{a} = \mathfrak{g}/\mathfrak{n} \). By Borel’s result [2] Appendix 2], we have the commutative diagram

\[
\text{Dol} \mathcal{E}_1^{*,*}(\mathfrak{g}) \longrightarrow \text{Dol} \mathcal{E}_1^{*,*}(G/\Gamma)
\]

\[
\bigwedge \mathfrak{a}^* \otimes V_{\Lambda_\eta} \longrightarrow A^{0,*}(A/\Delta, L_{\bar{\Lambda}_\eta}).
\]

4.3. Proof of theorem.

Proof. Consider the spectral sequence \( \text{Dol} \mathcal{E}_1^{*,*}(\mathfrak{g}) \) of

\[
\bigoplus_{\alpha \in B_{(G,N)}} \bigwedge \mathfrak{g}^* \otimes V_\alpha \otimes V_\sigma
\]

and the spectral sequence \( \text{Dol} \mathcal{E}_1^{*,*}(G/\Gamma) \) of

\[
\bigoplus_{L_\alpha \in B_{(G,N)}(\Gamma)} A^{0,*}(G/\Gamma, L_\alpha \otimes L_\sigma).
\]

Set \( A = G/N \) and \( \Delta = \Gamma /\Gamma \cap N \) and \( \mathfrak{a} = \mathfrak{g}/\mathfrak{n} \). Since we can identify \( B_{(G,N)} \) (resp. \( B_{(G,N)}(\Gamma) \)) with \( \text{Hom}_{hol}(A, \mathbb{C}^*) \) (resp. \( \mathcal{H}_{hol}(A/\Delta) \)) as Section [14], we have the commutative diagram

\[
\text{Dol} \mathcal{E}_1^{*,*}(\mathfrak{g}) \longrightarrow \text{Dol} \mathcal{E}_1^{*,*}(G/\Gamma)
\]

\[
\bigoplus_{\alpha \in \text{Hom}_{hol}(A, \mathbb{C}^*)} \bigwedge \mathfrak{a}^* \otimes V_\alpha \otimes V_\Lambda \longrightarrow \bigoplus_{L_\alpha \in \mathcal{H}_{hol}(A/\Delta)} A^{0,*}(A/\Delta, L_{\bar{\alpha}} \otimes L_{\bar{\Lambda}}).
\]
By Proposition 4.3, the homomorphism \( \text{Dol}^1(E^*_{1}\mathfrak{g}) \to \text{Dol}^1(E^*_{1}\mathfrak{g}/\Gamma) \) induces a cohomology isomorphism and hence we have an isomorphism \( \text{Dol}^2(E^*_{2}\mathfrak{g}) \cong \text{Dol}^2(E^*_{2}\mathfrak{g}/\Gamma) \).

Hence the theorem follows. \( \square \)

5. CONSTRUCTION OF FINITE COCHAIN COMPLEX (DE RHAM CASE)

We will use the following proposition.

**Proposition 5.1.** ([2, Proposition 3.3]) Let \( G \) be a simply connected solvable Lie group and \( N \) the nilradical of \( G \). Then we have a simply connected nilpotent subgroup \( C \subset G \) such that \( G = C \cdot N \).

**Remark 2.** This proposition is given by the decomposition (not necessarily direct sum) \( \mathfrak{g} = \mathfrak{c} + \mathfrak{n} \) (see [3, Theorem 2.2]). Since this decomposition is compatible with any field (see [3, Theorem 2.2]), if \( G \) is complex Lie group we can take a subgroup \( C \) also complex.

Let \( G \) be a simply connected solvable Lie group and \( \mathfrak{g} \) be the Lie algebra of \( G \). Let \( N \) be the nilradical of \( G \). Let \( \rho : G \to GL(V_\rho) \) be a representation. Suppose the restriction \( \rho|_N \) is unipotent. We consider the direct sum \( \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho \)

Then we have the \( G \)-action on this cochain complex via \( \bigoplus \text{Ad} \otimes \alpha \otimes \rho \). Since this action is extension of the Lie derivation, the induced action on the cohomology is trivial. Consider the semi-simple part

\[
(( \bigoplus_{\alpha \in \mathcal{A}(G,N)} \text{Ad} \otimes \alpha \otimes \rho )(g))_s = \bigoplus_{\alpha \in \mathcal{A}(G,N)} (\text{Ad}_g)_s \otimes \alpha(g) \otimes (\rho(g))_s.
\]

Take a simply connected nilpotent subgroup \( C \subset G \) as Proposition 5.1. Since \( C \) is nilpotent, the map \( \Phi : C \ni c \mapsto \bigoplus_{\alpha \in \mathcal{A}(G,N)} (\text{Ad}_g)_s \otimes \alpha(g) \otimes (\rho(g))_s \in \text{Aut}(\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho) \)

is a homomorphism. We denote by

\[
\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)}
\]

the subcomplex consisting of the \( \Phi(C) \)-invariant elements.

**Lemma 5.2.** The inclusion

\[
\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)} \subset \bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho
\]

induces a cohomology isomorphism.

**Proof.** Since the induced \( G \)-action on the cohomology \( H^*(\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho) \) is trivial and \( \Phi(C) \)-action is semi-simple part of \( G \)-action, the induced \( \Phi(C) \)-action on the cohomology \( H^*(\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho) \) is also trivial and hence

\[
H^*(\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho)^{\Phi(C)} = H^*(\bigoplus_{\alpha \in \mathcal{A}(G,N)} \bigwedge \mathfrak{g}_C^* \otimes V_\alpha \otimes V_\rho).
\]
Since $\Phi$ is diagonalizable, we have

$$H^*(\bigoplus_{\alpha \in A(G,N)} g_C^* \otimes V_\alpha \otimes V_\rho)_{\Phi(C)} = H^*(\bigoplus_{\alpha \in A(G,N)} g_C^* \otimes V_\alpha \otimes V_\rho)_{\Phi(C)}.$$

Hence the lemma follows. \qed

The subcomplex $(\bigoplus_{\alpha \in A(G,N)} g_C^* \otimes V_\alpha \otimes V_\rho)_{\Phi(C)}$ is desired subcomplex $A^*_t$ as in Theorem 1.5. By using certain basis, we see that this complex is finite dimensional and write down the subcomplex $A^*_t$ as Corollary 1.6 explicitly.

We have a basis $X_1, \ldots, X_n$ of $g_C$ such that $(Ad_c)_s = \text{diag}(\alpha_1(c), \ldots, \alpha_n(c))$ for $c \in C$. Let $x_1, \ldots, x_n$ be the basis of $g_C^*$ which is dual to $X_1, \ldots, X_n$. We have a basis $v_1, \ldots, v_m$ of $V_\rho$ such that $(\rho(c))_s = \text{diag}(\alpha'_1(c), \ldots, \alpha'_m(c))$ for any $c \in C$. Let $v_\alpha$ be a basis of $V_\alpha$ for each character $\alpha \in A(G,N)$. By $G = C \cdot N$, we have $G/N = C/C\cap N$ and hence we have $A(G,N) = A_{C,C\cap N} = \{\alpha \in \text{Hom}(C, \mathbb{C}^*) | \alpha|_{C\cap N} = 1\}$.

For a multi-index $I = \{i_1, \ldots, i_p\}$ we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$, and $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$. We consider the basis

$$\{x_I \otimes v_\alpha \otimes v_k \}_{I \subseteq \{1, \ldots, n\}, \alpha \in A_{C,C\cap N}, k \in \{1, \ldots, m\}}$$

of $(\bigoplus_{\alpha \in A_{C,C\cap N}} g_C^* \otimes V_\alpha \otimes V_\rho)$. Since the action

$$\Phi : C \to \text{Aut}(\bigoplus_\alpha g^*_C \otimes V_\alpha \otimes V_\rho)$$

is the semi-simple part of $(\bigoplus \text{Ad} \otimes \alpha \otimes \rho)_{|C}$, we have

$$\Phi(a)(x_I \otimes v_\alpha \otimes v_k) = \alpha_{I}^{-1} \alpha_{I}^\prime x_I \otimes v_\alpha \otimes v_k.$$

Hence we have

$$(\bigoplus_\alpha g_C^* \otimes V_\alpha \otimes V_\rho)_{\Phi(C)}$$

$$= \langle x_I \otimes v_\alpha \otimes v_k \rangle_{I \subseteq \{1, \ldots, n\}, k \in \{1, \ldots, m\}}$$

$$= \bigwedge \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle \otimes \langle v_{\alpha_1}, \ldots, v_{\alpha_m} \rangle \otimes \langle v_1, \ldots, v_m \rangle.$$ 

Finally we construct a finite dimensional complex $A^*_t$ which computes the de Rham cohomology $H^*(G/\Gamma, E_\rho)$.

**Corollary 5.3.** Let $A^*_t$ be the subcomplex of $(\bigoplus_\alpha g_C^* \otimes V_\alpha \otimes V_\rho)_{\Phi(C)}$ defined as

$$A^*_t = \langle x_I \otimes v_\alpha \otimes v_k | (\alpha_I^\prime \alpha_k^\prime)^{-1} |_{\Gamma} = 1 \rangle.$$

Then we have an isomorphism

$$H^*(A^*_t) \cong H^*(G/\Gamma, E_\rho).$$

**Proof.** Consider the inclusion

$$\iota : (\bigoplus_{\alpha \in A_{C,N}} g_C^* \otimes V_\alpha \otimes V_\rho)_{\Phi(C)} \to \bigoplus_{E_\alpha \in A_{C(N)}(\Gamma)} A^*(G/\Gamma, E_\alpha \otimes E_\rho).$$

$$\iota(x_I \otimes v_\alpha \otimes v_k) \in A^*(G/\Gamma, E_\rho)$$

if and only if $(\alpha_I^\prime \alpha_k^\prime)^{-1} |_{\Gamma} = \rho |_{\Gamma}$. Hence we have

$$\iota^{-1}(A^*(G/\Gamma, E_\rho)) = A^*_t. \qed$$
Corollary 5.4. We consider the following conditions:

(i) For each multi-index $I = \{i_1, \ldots, i_p\}$ and $k \in \{1, \ldots, m\}$, the character $\alpha_I\alpha'^{-1}_k$ is trivial if and only if the restriction $(\alpha_I\alpha'^{-1}_k)|_{\mathbb{C}}$ is trivial.

(ii) For each multi-index $I = \{i_1, \ldots, i_p\}$ and $k \in \{1, \ldots, m\}$, the character $\alpha_I\alpha'^{-1}_k$ is trivial or non-unitary. If the condition (i) or (ii) holds, then we have an isomorphism

$$H^*(\mathfrak{g}, V_\rho) \cong H^*(G/\Gamma, E_\rho).$$

Proof. If the condition (i) holds, then we have $A^*_I = (\bigwedge^{*} \mathfrak{g}_C \otimes V_\rho)^{\Phi(C)}$. Hence we have

$$H^*(G/\Gamma, E_\rho) \cong H(\bigwedge^{*} \mathfrak{g}_C \otimes V_\rho)^{\Phi(C)} \cong H^*(\mathfrak{g}, V_\rho).$$

The condition (ii) is special case of the condition (i). Hence the corollary follows.

Remark 3. For a representation $\rho : G \to GL(V_\rho)$ such that the restriction $\rho|_{\mathfrak{n}}$ is trivial, the condition (M) (resp. (H)) in Section 1 is a special case of the condition (i) (resp (ii))

Remark 4. Let $\mathfrak{g}$ be the Lie algebra of $C$. Take a subvector $V \subset \mathfrak{g}$ (not necessarily Lie algebra) such that $\mathfrak{g} = V \oplus \mathfrak{n}$. Then we define the map

$$\text{ad}_x : \mathfrak{g} = V \oplus \mathfrak{n} \ni A + X \mapsto (\text{ad}_A)_x \in D(\mathfrak{g})$$

where $(\text{ad}_A)_x$ is the semi-simple part of $\text{ad}_A$ and $D(\mathfrak{g})$ is the Lie algebra of derivations of $\mathfrak{g}$. This map is a Lie algebra homomorphism and a diagonalizable representation (see [8] and [10]). Let $\text{Ad}_x : G \to \text{Aut}(\mathfrak{g})$ be the extension of $\text{ad}_x$. Then this map is identified with the map

$$G = C \cdot N \ni c \cdot n \mapsto (\text{Ad}_c) \in \text{Aut}(\mathfrak{g}).$$

We define the Lie algebra $\mathfrak{u}_G \subset D(\mathfrak{g}) \ltimes \mathfrak{g}$ as

$$\mathfrak{u}_G = \{X - \text{ad}_x X | X \in \mathfrak{g}\}.$$

Consider the above basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}_C$. Then in [10] the author showed that we have an isomorphism

$$\bigwedge \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle \cong \bigwedge (\mathfrak{u}_G \otimes \mathbb{C})^*.$$

(This fact gives the new developments of de Rham homotopy theory on solvmanifolds. See [10].) Hence we can regard

$$(\bigoplus_{\alpha \in \mathcal{A}_{G, N}} \bigwedge \mathfrak{g}_C \otimes V_\alpha \otimes V_\rho)^{\Phi(C)} = \bigwedge \langle x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n} \rangle \otimes \langle v_{\alpha_{i-1} \star \otimes v_1, \ldots, v_{\alpha_{m-1} \star \otimes v_m} \rangle$$

as the cochain complex of nilpotent Lie algebra of $\mathfrak{u}_G$ with values in some representation.

6. Construction of finite cochain complex (Dolbeault case)

In this case we can say almost same argument for de Rham case without difficulties. Let $G$ be a simply connected solvable Lie group and $\mathfrak{g}$ be the Lie algebra of $G$. Let $N$ be the nilradical of $G$. Let $\sigma : G \to GL(V_\sigma)$ be a holomorphic representation. Suppose the restriction $\sigma|_N$ is unipotent. We consider the direct sum

$$\bigoplus_{\alpha \in \mathcal{B}(G, N)} \mathfrak{g}^* \otimes V_\alpha \otimes V_\rho.$$
Then we have the $G$-action on this cochain complex via $\bigoplus\text{Ad} \otimes \alpha \otimes \rho$. Consider the semi-simple part

$$((\bigoplus_{\alpha \in B(G,N)} \text{Ad} \otimes \alpha \otimes \rho)(g))_s = \bigoplus_{\alpha \in B(G,N)} (\text{Ad}_g)_s \otimes \alpha(g) \otimes (\rho(g))_s.$$ 

Take a simply connected complex nilpotent subgroup $C \subset G$ as Proposition 5.1 and Remark 2. Since $C$ is nilpotent, the map

$$\Phi : C \ni c \mapsto \bigoplus_{\alpha \in B(G,N)} (\text{Ad}_g)_s \otimes \alpha(g) \otimes (\rho(g))_s \in \text{Aut}(\bigoplus_{\alpha \in B(G,N)} \bigwedge g^* \otimes V_\alpha \otimes V_\sigma)$$

is a homomorphism. We denote by

$$\bigwedge g^* \otimes V_\alpha \otimes V_\sigma)^\Phi(C)$$

the subcomplex consisting of the $\Phi(C)$-invariant elements. By similar proof of Lemma 5.2 we have:

**Lemma 6.1.** The inclusion

$$\bigwedge g^* \otimes V_\alpha \otimes V_\sigma)^\Phi(C) \subset \bigoplus_{\alpha \in B(G,N)} \bigwedge g^* \otimes V_\alpha \otimes V_\sigma$$

induces a cohomology isomorphism.

We have a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ such that $(\text{Ad}_c)_s = \text{diag}(\alpha_1(c), \ldots, \alpha_n(c))$ for $c \in C$. Let $x_1, \ldots, x_n$ be the basis of $\mathfrak{g}^*$ which is dual to $X_1, \ldots, X_n$. We have a basis $v_1, \ldots, v_m$ of $V_\sigma$ such that $(\sigma(c))_s = \text{diag}(\alpha'_1(c), \ldots, \alpha'_m(c))$ for any $c \in C$. Let $v_\alpha$ be a basis of $V_\alpha$ for each character $\alpha \in B(G,N)$. By $G = C \cdot N$, we have $G/N = C/C\cap N$ and hence we have $B(G,N) = B_{C,C\cap N} = \{\alpha \in \text{Hom}_{\text{hol}}(C, \mathbb{C}^*): \alpha|_{C\cap N} = 1\}$.

For a multi-index $I = \{i_1, \ldots, i_p\}$ we write $x_I = x_{i_1} \wedge \cdots \wedge x_{i_p}$, and $\alpha_I = \alpha_{i_1} \cdots \alpha_{i_p}$. We consider the basis

$$\{x_I \otimes v_{\alpha} \otimes v_k\}_{I \subset \{1, \ldots, n\}, \alpha \in A(C,C\cap N), k \in \{1, \ldots, m\}}$$

of $\bigoplus_{\alpha \in B_{C,C\cap N}} \bigwedge g^* \otimes V_\alpha \otimes V_\sigma$. Since the action

$$\Phi : C \rightarrow \text{Aut}(\bigoplus_{\alpha} \bigwedge g^* \otimes V_\alpha \otimes V_\sigma)$$

is the semi-simple part of $(\bigoplus \text{Ad} \otimes \alpha \otimes \sigma)|_C$, we have

$$\Phi(a)(x_I \otimes v_{\alpha} \otimes v_k) = \alpha^{-1}_I a \alpha_k x_I \otimes v_{\alpha} \otimes v_k.$$

Hence we have

$$\bigwedge g^* \otimes V_\alpha \otimes V_\sigma)^\Phi(C)$$

$$= \langle x_I \otimes v_{\alpha_1 \alpha_{k-1}^{-1} \otimes v_k} \rangle_{I \subset \{1, \ldots, n\}, k \in \{1, \ldots, m\}}$$

$$= \langle \bigwedge \{x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n}\} \rangle \otimes \langle v_{\alpha_1^{-1} \otimes v_1, \ldots, v_{\alpha_m^{-1} \otimes v_m} \rangle.$$

**Corollary 6.2.** Let $B^+_I$ be the subcomplex of $\{x_I \otimes v_{\alpha_1 \alpha_{k-1}^{-1} \otimes v_k} \}_{I \subset \{1, \ldots, n\}, k \in \{1, \ldots, m\}}$ defined as

$$B^+_I = \left\langle x_I \otimes v_{\alpha_1 \alpha_{k-1}^{-1} \otimes v_k} \left| \frac{\alpha_I \alpha_k^{-1}}{\alpha_I \alpha_k^{-1}} |_\rho = 1 \right. \right\rangle.$$
Then we have an isomorphism

\[ H^*(B^*_{1}) \cong H^{0,*}(G/\Gamma, L_\sigma). \]

**Proof.** It is known that we have the 1-1 correspondence between the isomorphism classes of flat holomorphic line bundles over a complex torus and the unitary characters of its lattice (see [15]). By this, for \( \alpha \in B(G,N) \), considering the unitary character \( \hat{\alpha} \), the holomorphic line bundle \( L_\alpha \) is trivial if and only if the restriction \( (\hat{\alpha}^-\hat{\alpha}^-)|_\Gamma \) is trivial. Hence

\[ \iota(x_1 \otimes v_{\alpha_1 \alpha_k^{-1}} \otimes v_k) \in A^*(G/\Gamma, L_\sigma) \]

if and only if the restriction \( (\frac{\partial}{\partial \alpha_k})|_\Gamma \) is trivial. Then we have \( \iota^{-1}(A^*(G/\Gamma, L_\sigma)) = B^*_1 \).

**Corollary 6.3.** We consider the following condition:

\((\ast)\) For each multi-index \( I = \{i_1, \ldots, i_p\} \) and \( k \in \{1, \ldots, m\} \), the character \( \alpha_{i_1} \alpha_k^{-1} \) is trivial if and only if the restriction \( (\frac{\partial}{\partial \alpha_k})|_\Gamma \) is trivial.

If the condition \( (\ast) \) holds, then we have an isomorphism

\[ H^*(g, V_\sigma) \cong H^{0,*}(G/\Gamma, L_\sigma). \]

**Proof.** Suppose the condition \( (\ast) \) holds. Then we have \( B^*_1 = (\bigwedge g^* \otimes V_\sigma)^{\Phi(C)} \).

Hence we have

\[ H^{0,*}(G/\Gamma, L_\sigma) \cong H^*(\bigwedge g^* \otimes V_\sigma)^{\Phi(C)} \cong H^*(g, V_\sigma). \]

**Remark 5.** We define the nilpotent Lie algebra \( u_G \) as Remark[4]. In complex case, \( u_G \) is also a complex Lie algebra. As similar to Remark[4] we have

\[ (\bigoplus_\alpha (\bigwedge g^* \otimes V_\alpha)^{\Phi(C)}) = \bigwedge (x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n}) \cong \bigwedge u_G^*. \]

Suppose \( G \) has a lattice \( \Gamma \). We consider the cochain complex

\[ B^*_1 = \left\{ x_1 \otimes v_{\alpha_1} \mid (\frac{\partial}{\partial \alpha_1})|_\Gamma = 1 \right\}. \]

Then we have an isomorphism \( H^{0,*}(B^*_1) \cong H^{0,*}(G/\Gamma) \) by Corollary 6.2. We consider the following condition.

\((\Box)\) For each \( 1 \leq i \leq n \), the restriction \( (\hat{\alpha}^-)|_\Gamma \) is trivial. If the condition \( (\Box) \) holds, then we have

\[ B^*_1 = \bigwedge (x_1 \otimes v_{\alpha_1}, \ldots, x_n \otimes v_{\alpha_n}) \cong \bigwedge u_G^*. \]

Let \( U_G \) be the simply connected complex Lie group with the Lie algebra \( u_G \). Then \( U_G \) is the nilradical of the semi-simple splitting of \( G \) (see [22]). It is known that if \( G \) has a lattice, then \( U_G \) has a lattice \( \Gamma' \) (see [1]).

Hence we have:

**Corollary 6.4.** Let \( G \) be a simply connected complex solvable Lie group with a lattice \( \Gamma \). If the condition \( (\Box) \) holds, then there exists a complex parallelizable nilmanifold \( U_G/\Gamma' \) such that we have an isomorphism

\[ H^{*,*}(G/\Gamma) \cong H^{*,*}(U_G/\Gamma'). \]
By this corollary we have some solvmanifolds whose Dolbeault cohomology is isomorphic to the Dolbeault cohomology of nilmanifolds.

7. Example

Let \( G = \mathbb{C} \ltimes \mathbb{C}^2 \) such that

\[
\phi(z) = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}.
\]

Then we have \( a + \sqrt{-1}b, c + \sqrt{-1}d \in \mathbb{C} \) such that \( \mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d) \) is a lattice in \( \mathbb{C} \) and \( \phi(a + \sqrt{-1}b) \) and \( \phi(c + \sqrt{-1}d) \) are conjugate to elements of \( SL(4, \mathbb{Z}) \) where we regard \( SL(2, \mathbb{C}) \subset SL(4, \mathbb{R}) \) (see [8]). Hence we have a lattice \( \Gamma = (\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)) \ltimes \phi \Gamma'' \) such that \( \Gamma'' \) is a lattice of \( \mathbb{C}^2 \).

7.1. Twisted de Rham cohomology \( H^1(G/T, E_{Ad}) \). For a coordinate \( (w, z_1, z_2) \in \mathbb{C} \ltimes \phi \mathbb{C}^2 \) we have the basis \( \{v_1, \ldots, v_6\} \) of \( \mathfrak{g}_C \) such that

\[
v_1 = e^w \frac{\partial}{\partial z_1}, v_2 = e^\bar{w} \frac{\partial}{\partial \bar{z}_1}, v_3 = e^{-w} \frac{\partial}{\partial z_2}, v_4 = e^{-\bar{w}} \frac{\partial}{\partial \bar{z}_2}, v_5 = \frac{\partial}{\partial w}, v_6 = \frac{\partial}{\partial \bar{w}}.
\]

Consider the dual basis

\[
e^{-w}dz_1, e^{-\bar{w}}d\bar{z}_1, e^w dz_2, e^{\bar{w}} d\bar{z}_2, dw, d\bar{w}.
\]

As we consider \( \mathfrak{g}_C \) as a representation of \( \mathfrak{g} \) via \( Ad \), we have the cochain complex

\[
\bigoplus_\alpha \Lambda^\alpha \mathfrak{g}^* \otimes V_\alpha \otimes V_\rho \Phi(C)
\]

as section [5] where \( C = \mathbb{C} \). Then we have

\[
\bigoplus_\alpha \Lambda^\alpha \mathfrak{g}^* \otimes V_\alpha \otimes V_\rho \Phi(C) = \bigoplus_\alpha \langle e^{-w}dz_1 \otimes v_{e^w}, e^{-\bar{w}}d\bar{z}_1 \otimes v_{e^{\bar{w}}}, e^w dz_2 \otimes v_{e^{-w}}, e^{\bar{w}} d\bar{z}_2 \otimes v_{e^{-\bar{w}}}, dw, d\bar{w} \rangle \otimes \langle v_1 \otimes v_{e^{-w}}, v_2 \otimes v_{e^{\bar{w}}}, v_3 \otimes v_{e^w}, v_4 \otimes v_{e^{\bar{w}}}, v_5, v_6 \rangle.
\]

For any lattice \( \Gamma \) we have \( b_1(G/\Gamma) = b_1(\mathfrak{g}) = 2 \). But we will see that \( \dim H^1(G/\Gamma, \mathfrak{e}_{Ad}) \) varies for a choice of \( \Gamma \). If \( b, d \in \pi \mathbb{Z} \), then we have

\[
A^0 = \langle v_5, v_6 \rangle,
\]

\[
A^1 = \langle e^{-w}dz_1 \otimes v_1, e^{-\bar{w}}d\bar{z}_1 \otimes v_{e^w}, e^{-w}dz_2 \otimes v_{e^{-w}}, e^{-\bar{w}}d\bar{z}_2 \otimes v_{e^{-\bar{w}}}, dw \otimes v_5, \rangle,
\]

\[
de^{-w}dz_1 \otimes v_{e^w} \otimes v_1 \otimes v_{e^w}, e^{-\bar{w}}d\bar{z}_1 \otimes v_{e^{\bar{w}}}, e^w dz_2 \otimes v_{e^{-w}} \otimes v_4 \otimes v_{e^w}, e^{\bar{w}} d\bar{z}_2 \otimes v_{e^{-\bar{w}}}, e^w dz_2 \otimes v_{e^{-w}} \otimes v_3 \otimes v_{e^{-w}}, e^{\bar{w}} d\bar{z}_2 \otimes v_{e^{-\bar{w}}}.
\]

Hence we have \( \dim H^1(G/T, V_{Ad}) = \dim H^1(A^1) = 6 \).
On the other hand, if $b \notin \pi \mathbb{Z}$ or $d \notin \pi \mathbb{Z}$, then we have
\[
A_1^1 = \langle v_5, v_6 \rangle,
\]
\[
A_1^0 = \langle e^{-\bar{u}}d\bar{z}_1 \otimes v_1, e^{-\bar{u}}d\bar{z}_1 \otimes v_2, e^{\bar{u}}d\bar{z}_2 \otimes v_3, e^{\bar{u}}d\bar{z}_2 \otimes v_4, \rangle
\]
\[
dw \otimes v_5, dw \otimes v_6, d\bar{w} \otimes v_5, d\bar{w} \otimes v_6).
\]
Hence we have $\dim H^0(G/\Gamma, E_{\text{Ad}}) = \dim H^1(A_1^1) = 2$.

7.2. Dolbeault cohomology $H^*_\partial(G/\Gamma)$. For a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \times \mathbb{C}^2$, we consider the basis $(x_1, x_2, x_3) = (d\bar{z}_1, e^{-\bar{z}_1}d\bar{z}_2, e^{\bar{z}_1}d\bar{z}_3)$ of $g^*$. We consider $C = \mathbb{C} = \{(z_1)\}$ and $(\alpha_1, \alpha_2, \alpha_3) = (1, e^{z_1}, e^{-z_1})$ for $C$ and $(\alpha_1, \alpha_2, \alpha_3)$ as in Section 5. If $b \notin \pi \mathbb{Z}$ or $c \notin \pi \mathbb{Z}$, then (5) holds and hence we have $H^{*, *}(G/\Gamma) \cong \bigwedge \mathbb{C}^3 \otimes H^*(g)$.
If $b, d \in \pi \mathbb{Z}$, then the condition ($\Box$) holds and hence we have $H^{*, *}_\partial(G/\Gamma) \cong \bigwedge \mathbb{C}^3 \otimes \bigwedge \mathbb{C}^3$. There exists a lattice $\Gamma$ which satisfies the condition (5) or ($\Box$) (see [8]).

Acknowledgements.
The author would like to express his gratitude to Toshitake Kohno for helpful suggestions and stimulating discussions. This research is supported by JSPS Research Fellowships for Young Scientists.

REFERENCES

[1] L. Auslander, An exposition of the structure of solvmanifolds. I. Algebraic theory. Bull. Amer. Math. Soc. 79 (1973), no. 2, 227–261.
[2] K. Dekimpe, Semi-simple splittings for solvable Lie groups and polynomial structures. Forum Math. 12 (2000), no. 1, 77–96.
[3] K. Dekimpe, Solvable Lie algebras, Lie groups and polynomial structures, Compositio Math. 121 (2000), no. 2, 183–204.
[4] N. Dungey, A. F. M. ter Elst, D. W. Robinson, Analysis on Lie Groups with Polynomial Growth, Birkhäuser (2003).
[5] K. Hasegawa, Small deformations and non-left-invariant complex structures on six-dimensional compact solvmanifolds. Differential Geom. Appl. 28 (2010), no. 2, 220–227.
[6] A. Hattori, Spectral sequence in the de Rham cohomology of fibre bundles. J. Fac. Sci. Univ. Tokyo Sect. I 8 1960 289–331 (1960).
[7] F. Hirzebruch, Topological Methods in Algebraic Geometry, third enlarged ed., Springer-Verlag, 1966.
[8] K. Hasegawa, Small deformations and non-left-invariant complex structures on six-dimensional compact solvmanifolds. Differential Geom. Appl. 28 (2010), no. 2, 220–227.
[9] J. E. Humphreys, Linear algebraic groups. Springer-Verlag, New York 1981
[10] H. Kasuya, Minimal models, formality and hard Lefschetz properties of solvmanifolds with local systems. To appear in J. Differential Geometry. [http://arxiv.org/abs/1009.1940]
[11] H. Kasuya, Techniques of computations of Dolbeault cohomology of solvmanifolds. Math. Z DOI: 10.1007/s00209-012-1013-0 (online first). [arXiv:1107.4761 (preprint)]
[12] G. D. Mostow, Cohomology of topological groups and solvmanifolds. Ann. of Math. (2) 73 1961 20–48.
[13] K. Nomizu, On the cohomology of compact homogeneous spaces of nilpotent Lie groups. Ann. of Math. (2) 59, (1954). 531–538.
[14] A. L. Onishchik, E. B. Vinberg (Eds), Lie groups and Lie algebras II, Springer (2000).
[15] A. Polishchuk, Abelian Varieties, Theta Functions and the Fourier Transform. Cambridge University Press 2002.
[16] M.S. Raghunathan, Discrete subgroups of Lie Groups, Springer-Verlag, New York, 1972.
[17] Y. Sakane, On compact complex parallelisable solvmanifolds. Osaka J. Math. 13 (1976), no. 1, 187–212.
[18] N. Steenrod, The Topology of Fibre Bundles, Princeton University Press (1951).
