A better lower bound on average degree of 4-list-critical graphs

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Abstract

This short note proves that every incomplete \( k \)-list-critical graph has average degree at least \( k - 1 + \frac{k-3}{k^2-2k+2} \). This improves the best known bound for \( k = 4, 5, 6 \). The same bound holds for online \( k \)-list-critical graphs.

1 Introduction

A graph \( G \) is \( k \)-list-critical if \( G \) is not \((k - 1)\)-choosable, but every proper subgraph of \( G \) is \((k - 1)\)-choosable. For further definitions and notation, see [5, 2]. Table I shows some history of lower bounds on the average degree of \( k \)-list-critical graphs.

Main Theorem. Every incomplete \( k \)-list-critical graph has average degree at least

\[
  k - 1 + \frac{k - 3}{k^2 - 2k + 2}.
\]

Main Theorem gives a lower bound of \( 3 + \frac{1}{10} \) for 4-list-critical graphs. This is the first improvement over Gallai’s bound of \( 3 + \frac{1}{13} \). The same proof shows that Main Theorem holds for online \( k \)-list-critical graphs as well. Our primary tool is a lemma proved with Kierstead [6] that generalizes a kernel technique of Kostochka and Yancey [8].

Definition. The maximum independent cover number of a graph \( G \) is the maximum \( \text{mic}(G) \) of \( \|I, V(G) \setminus I\| \) over all independent sets \( I \) of \( G \).

Kernel Magic (Kierstead and R. [6]). Every \( k \)-list-critical graph \( G \) satisfies

\[
  2 \|G\| \geq (k - 2) |G| + \text{mic}(G) + 1.
\]

The previous best bounds in Table I for \( k \)-list-critical graphs hold for \( k \)-Alon-Tarsi-critical graphs as well. Since Kernel Magic relies on the Kernel Lemma, our proof does not work for \( k \)-Alon-Tarsi-critical graphs. Any improvement over Gallai’s bound of \( 3 + \frac{1}{13} \) for 4-Alon-Tarsi-critical graphs would be interesting.
Table 1: History of lower bounds on the average degree $d(G)$ of $k$-critical and $k$-list-critical graphs $G$.

| $k$ | Gallai \[4\] $d(G) \geq$ | Kriv \[9\] $d(G) \geq$ | KS \[7\] $d(G) \geq$ | KY \[8\] $d(G) \geq$ | KS \[7\] $d(G) \geq$ | KR \[5\] $d(G) \geq$ | CR \[2\] $d(G) \geq$ | Here $d(G) \geq$ |
|---|---|---|---|---|---|---|---|---|
| 4 | 3.0769 | 3.1429 | — | 3.3333 | — | — | — | 3.1000 |
| 5 | 4.0909 | 4.1429 | — | 4.5000 | — | 4.0984 | 4.1000 | 4.1176 |
| 6 | 5.0909 | 5.1304 | 5.0976 | 5.6000 | — | 5.1053 | 5.1076 | 5.1153 |
| 7 | 6.0870 | 6.1176 | 6.0990 | 6.6667 | — | 6.1149 | 6.1192 | 6.1081 |
| 8 | 7.0820 | 7.1064 | 7.0980 | 7.7143 | — | 7.1128 | 7.1167 | 7.1000 |
| 9 | 8.0769 | 8.0968 | 8.0959 | 8.7500 | 8.0838 | 8.1094 | 8.1130 | 8.0923 |
| 10 | 9.0722 | 9.0886 | 9.0932 | 9.7778 | 9.0793 | 9.1055 | 9.1088 | 9.0853 |
| 15 | 14.0541 | 14.0618 | 14.0785 | 14.8571 | 14.0610 | 14.0864 | 14.0884 | 14.0609 |
| 20 | 19.0428 | 19.0474 | 19.0666 | 19.8947 | 19.0490 | 19.0719 | 19.0733 | 19.0469 |

2 The Proof

The connected graphs in which each block is a complete graph or an odd cycle are called Gallai trees. Gallai [4] proved that in a $k$-critical graph, the vertices of degree $k - 1$ induce a disjoint union of Gallai trees. The same is true for $k$-list-critical graphs ([1, 3]). For a graph $T$ and $k \in \mathbb{N}$, let $\beta_k(T)$ be the independence number of the subgraph of $T$ induced on the vertices of degree $k - 1$. When $k$ is defined in the context, put $\beta(T) := \beta_k(T)$.

**Lemma 1.** If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k - 1$, then

$$2\|T\| \leq (k - 2)|T| + 2\beta(T).$$

**Proof.** Suppose the lemma is false and choose a counterexample $T$ minimizing $|T|$. Plainly, $T$ has more than one block. Let $A$ be an endblock of $T$ and let $x$ be the unique cutvertex of $T$ with $x \in V(A)$. Consider $T' := T - (V(A) \setminus \{x\})$. By minimality of $|T|$, we have

$$2\|T\| - 2\|A\| \leq (k - 2)(|T| + 1 - |A|) + 2\beta(T').$$

Since $T$ is a counterexample, $2\|A\| > (k - 2)(|A| - 1)$. So, if $k > 4$, then $A = K_{k-1}$ and if $k = 4$, then $A$ is an odd cycle. In both cases, $d_T(x) = k - 1$. Consider $T^* := T - V(A)$. By minimality of $|T|$, we have

$$2\|T\| - 2\|A\| - 2 \leq (k - 2)(|T| - |A|) + 2\beta(T^*).$$

Since $T$ is a counterexample, $2\|A\| + 2 > (k - 2)|A| + 2(\beta(T) - \beta(T^*))$. In $T^*$, all of $x$’s neighbors have degree at most $k - 2$. But $d_T(x) = k - 1$, so some vertex in $\{x\} \cup N(x)$ is in a maximum independent set of degree $k - 1$ vertices in $T$. Hence $\beta(T^*) \leq \beta(T) - 1$, which gives

$$2\|A\| > (k - 2)|A|,$$

a contradiction since $k \geq 4$. 

\[\square\]
Proof of Main Theorem. Let \( G \neq K_k \) be a \( k \)-list-critical graph. The theorem is trivially true if \( k \leq 3 \), so suppose \( k \geq 4 \). Let \( \mathcal{L} \subseteq V(G) \) be the vertices with degree \( k - 1 \) and let \( \mathcal{H} = V(G) \setminus \mathcal{L} \). Put \( \|\mathcal{L}\| := \|G[\mathcal{L}]\| \) and \( \|\mathcal{H}\| := \|G[\mathcal{H}]\| \). By Lemma \( \text{II} \),

\[
2 \|\mathcal{L}\| \leq (k - 2)|\mathcal{L}| + 2\beta(\mathcal{L})
\]

Hence,

\[
2 \|G\| = 2 \|\mathcal{H}\| + 2 \|\mathcal{H}, \mathcal{L}\| + 2 \|\mathcal{L}\|
\]

\[
= 2 \|\mathcal{H}\| + 2((k - 1)|\mathcal{L}| - 2\|\mathcal{L}\|) + 2 \|\mathcal{L}\|
\]

\[
= 2 \|\mathcal{H}\| + 2(k - 1)|\mathcal{L}| - 2\|\mathcal{L}\|
\]

\[
\geq 2 \|\mathcal{H}\| + k|\mathcal{L}| - 2\beta(\mathcal{L}),
\]

which is

\[
\beta(\mathcal{L}) \geq \|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\|. \quad (1)
\]

Let \( M \) be the maximum of \( \|I, V(G) \setminus I\| \) over all independent sets \( I \) of \( G \) with \( I \subseteq \mathcal{H} \). Then

\[
\text{mic}(G) \geq M + (k - 1)\beta(\mathcal{L}).
\]

Applying Kernel Magic and using (1) gives

\[
2 \|G\| \geq (k - 2)|G| + M + (k - 1)\beta(\mathcal{L}) + 1
\]

\[
\geq (k - 2)|G| + M + (k - 1) \left( \|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\| \right) + 1
\]

\[
= (k - 2)|G| + M + (k - 1)\|\mathcal{H}\| + \frac{k(k - 1)}{2}|\mathcal{L}| - (k - 1)\|G\| + 1.
\]

Hence

\[
(k + 1)\|G\| \geq (k - 2)|G| + M + (k - 1)\|\mathcal{H}\| + \frac{k(k - 1)}{2}|\mathcal{L}| + 1 \quad (2)
\]

Let \( \mathcal{C} \) be the components of \( G[\mathcal{H}] \). Then \( \alpha(C) \geq \frac{|C|}{\chi(C)} \) for all \( C \in \mathcal{C} \). Whence

\[
M + (k - 1)\|\mathcal{H}\| \geq \sum_{C \in \mathcal{C}} k\frac{|C|}{\chi(C)} + (k - 1)\|C\|. \quad (3)
\]

If \( \mathcal{L} = \emptyset \), then \( G \) has average degree at least \( k \geq k - 1 + \frac{k - 3}{k - 2} \). So, assume \( \mathcal{L} \neq \emptyset \). Then \( G[\mathcal{H}] \) is \((k - 1)\)-colorable by \( k\)-list-criticality of \( G \). In particular, \( \chi(C) \leq k - 1 \) for every \( C \in \mathcal{C} \). For every \( C \in \mathcal{C} \),

\[
k\frac{|C|}{\chi(C)} + (k - 1)\|C\| \geq \left( k - \frac{1}{2} \right)|C|. \quad (4)
\]

To see this, first suppose \( C \in \mathcal{C} \) is not a tree. Then \( \|C\| \geq |C| \) and hence \( k\frac{|C|}{\chi(C)} + (k - 1)\|C\| \geq k\frac{|C|}{k - 1} + (k - 1)|C| \geq (k - \frac{1}{2})|C| \). If \( C \) is a tree, then \( \chi(C) \leq 2 \) and hence
\[
\frac{k|C|}{\chi(C)} + (k - 1)\|C\| \geq k\frac{|C|}{\chi(C)} + (k - 1)(|C| - 1) \geq (k - \frac{1}{2})|C| \text{ unless } |C| = 1. \text{ This proves (1) since the bound is trivially satisfied when } |C| = 1.
\]

Now combining (2), (3) and (4) with the basic bound
\[
|\mathcal{L}| \geq k|G| - 2\|G\|,
\]
gives
\[
(k + 1)\|G\| \geq (k - 2)|G| + \left(k - \frac{1}{2}\right)|\mathcal{H}| + \frac{k(k - 1)}{2}|\mathcal{L}| + 1
\]
\[
= \left(2k - \frac{5}{2}\right)|G| + \frac{k^2 - 3k + 1}{2}|\mathcal{L}| + 1
\]
\[
\geq \left(2k - \frac{5}{2}\right)|G| + \frac{k^2 - 3k + 1}{2}(k|G| - 2\|G\|) + 1.
\]

After some algebra, this becomes
\[
2\|G\| \geq \left(k - 1 + \frac{k - 3}{k^2 - 2k + 2}\right)|G| + \frac{2}{k^2 - 2k + 2}.
\]
That proves the theorem. \[\Box\]

The right side of equation (4) in the above proof can be improved to \(k|C|\) unless \(C\) is a \(K_2\) where both vertices have degree \(k\) in \(G\). If these \(K_2\)'s could be handled, the average degree bound would improve to \(k - 1 + \frac{k - 3}{(k - 1)^2}\).

**Conjecture.** Every incomplete (online) \(k\)-list-critical graph has average degree at least
\[
k - 1 + \frac{k - 3}{(k - 1)^2}.
\]

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