Heterotic Coset Models and (0,2) String Vacua

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Abstract

A Lagrangian definition of a large family of (0,2) supersymmetric conformal field theories may be made by an appropriate gauge invariant combination of a gauged Wess–Zumino–Witten model, right–moving supersymmetry fermions, and left–moving current algebra fermions. Throughout this paper, use is made of the interplay between field theoretic and algebraic techniques (together with supersymmetry) which is facilitated by such a definition. These heterotic coset models are thus studied in some detail, with particular attention paid to the (0,2) analogue of the $\mathcal{N}=2$ minimal models, which coincide with the ‘monopole’ theory of Giddings, Polchinski and Strominger. A family of modular invariant partition functions for these (0,2) minimal models is presented. Some examples of $\mathcal{N}=1$ supersymmetric four dimensional string theories with gauge groups $E_6 \times \tilde{G}$ and $SO(10) \times \tilde{G}$ are presented, using these minimal models as building blocks. The factor $\tilde{G}$ represents various enhanced symmetry groups made up of products of $SU(2)$ and $U(1)$.

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1. Introduction

As is by now common knowledge, the presence of $N = 1$ supersymmetry in four-dimensional spacetime results in $(0,2)$ supersymmetry on the heterotic string worldsheet, together with the condition that states have odd integral right $U(1)$ charges\cite{1,2}. The study of superstring vacua with $N = 2$ worldsheet supersymmetry is quite a mature subject by now, but chiefly in the specialized area of $(2,2)$ vacua. This situation is largely due to the fact (observed in ref.\cite{3}) that generically, $(0,2)$ sigma models appeared to have dangerous worldsheet instanton effects. It was observed\cite{4} that $(0,2)$ models could nevertheless be shown to exist, but $(2,2)$ models were seen to be much easier to define and study, given the techniques available at that time.

It is certainly time to try to close the gap opened in our understanding of $(2,2)$ versus $(0,2)$ vacua, if we are ever going to honestly claim that we have some understanding of string theory in generic backgrounds. Indeed, the renaissance of $(0,2)$ models can already be said to have begun. Recent results have breathed new life into the study of $(0,2)$ models, primarily due to the invention of new techniques for defining and studying $N = 2$ models in general. The ‘linear $\sigma$–model’ approach to $N = 2$ models\cite{5} allowed for a completely new approach to understanding $N = 2$ string vacua, casting Calabi–Yau and Landau–Ginsburg formulations in the same framework and showing how $(0,2)$ models can arise simply as deformations of $(2,2)$ models. Continuing the development of these techniques, the work presented in ref.\cite{6} defined a much larger class of $(0,2)$ models (not only deformations of $(2,2)$'s), and refs.\cite{7,8,9,10} showed that the conditions for conformal invariance in $(0,2)$ models defined in this way are likely to be satisfied by a very large class of models.

So the situation suddenly looks much better for $(0,2)$ models, although we have not yet attained the level of understanding which we have of $(2,2)$ models. Another step in this direction would be to have a large class of exactly solvable conformally invariant models for use as a laboratory, as is traditional in almost any area of theoretical endeavour. The $N = 2$ minimal models and the rest of the Kazama–Suzuki models have played this role in the realm of $(2,2)$ models and have been of importance in the understanding of the moduli space of $(2,2)$ vacua. It would certainly be very pleasing to have the analogous family of building blocks with which to continue studying $(0,2)$ moduli space.

Generally, the search for modular invariant combinations of characters which define conformal field theories with such a highly non–diagonal structure is a daunting task if there is little physical guidance. However, there have been a number of successful searches using powerful algebraic techniques\cite{11,12}.

In ref.\cite{13}, in a study of magnetically charged four dimensional black hole solutions of string theory, a family of conformal field theories (CFT’s) were constructed as asymmetric orbifolds of affine $SU(2)$. These CFT's, when specialized to the heterotic string case, are examples of $(0,2)$ models and therefore have applications other than as $U(1)$ monopole backgrounds, as was noted in the discussion in ref.\cite{13}. In refs.\cite{14,15,16} it was shown that
this ‘monopole’ theory was an example of a larger class of \((0,2)\) CFT’s (‘Heterotic Coset Models’) which may be formulated directly by Lagrangian methods. These methods were used to generalize the CFT construction of magnetic black holes to dyonic black holes, Taub–NUT, and Kerr–Taub–NUT solutions of Heterotic String theory. These heterotic coset models are a non–trivial coupling of gauged Wess–Zumino–Witten models with world sheet fermions. It is immediately apparent that the models thus defined are very close in spirit and structure to the familiar \(N = 2\) minimal models and the rest of the Kazama–Suzuki models. In fact, those \((2,2)\) cases can be seen as special cases (but not, in general, deformations) of the \((0,2)\) models defined there. In this way, we now have a family of exactly solvable \((0,2)\) building blocks analogous to the one known for the \((2,2)\) case. This allows us to search for analogues of the Gepner points[17] in the moduli space of \((0,2)\) compactifications. Evidence for such points has recently been given in ref.[12].

This paper continues the work of ref.[15], by starting with the Lagrangian definition of the models presented there and proceeding to study their content. It is here that the power of field theory makes its presence felt, as a great deal can be written down quite readily about the partition function by using the field theory as a guide. Focusing on the direct analogue of the \(N = 2\) minimal models, which shall be referred to as the ‘\((0,2)\) minimal models’, we proceed to compute their elliptic genera in section 3, showing (in the spirit of ref.[18], and following the computation of ref.[19]) how the presence of supersymmetry coupled with a Lagrangian definition allows a great amount of information about the elliptic genera of the models to be extracted readily. In section 4 much further progress is made, by using more field theory techniques to motivate in great detail the form of the complete partition function and then completing the computation, checking modular invariance. Thus armed with a store of \((0,2)\) modular invariants we proceed in section 5 to study the massless spectrum of a handful of four dimensional string theories with \(SO(10) \times \tilde{G}\) and \(E_6 \times \tilde{G}\) gauge groups and \(N = 1\) spacetime supersymmetry. The factor \(\tilde{G}\) is an enhanced gauge symmetry group arising from the details of the internal conformal field theory, and generically takes the form of a product of \(SU(2)_6\) and \(U(1)\) factors. We conclude with a discussion in section 6.

2. Lagrangian definitions of Conformal Field Theories

2.1. (Super) Wess–Zumino–Witten Models

The starting point for a Lagrangian definition of a conformal field theory for the purposes of this paper is the Wess–Zumino–Witten (WZW) model[20] [21], based on a compact semi–simple Lie group \(G\) at level \(k\):

\[
I_{WZW}(g, k) =
- \frac{k}{4\pi} \int_{\Sigma} d^2 z \ \text{Tr}[g^{-1} \partial_z g \cdot g^{-1} \partial_{\Sigma} g] - \frac{ik}{12\pi} \int_B d^3 \sigma \ \epsilon^{ijk} \text{Tr}[g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g]
\] (2.1)
where the two dimensional surface $\Sigma = \partial B$ has coordinates $(z, \bar{z})$ and $g \in G$. This model has a ‘global’ $G_L \times G_R$ symmetry $g \rightarrow g_L(z)gg_R^{-1}(\bar{z})$ where $(g_L, g_R) \in (G_L, G_R)$. This results in the model’s huge success as a solvable system, due to the resulting affine Lie algebra satisfied by the currents\textsuperscript{22} [23]. Essentially, this model may be thought of as the conformal field theory of a string propagating on a group manifold $G[23]$.

There exists a supersymmetric extension of this model\textsuperscript{24}. One way that this may be discovered is by using a superfield construction of the supersymmetric WZW model. For our starting point, we shall simply write a supersymmetric WZW by putting in the free fermions (in the adjoint representation) immediately at the component level\textsuperscript{25}:

\begin{align*}
I^{(1,1)} &= I_{WZW}(g, k) + \frac{ik}{4\pi} \int_{\Sigma} d^2z \; \text{Tr}[\Psi_R \partial_z \Psi_R + \Psi_L \partial_z \Psi_L], \\
&\text{(2.2)}
\end{align*}

noting that supersymmetry is realized (on shell) simply as:

\begin{align*}
\delta g &= i\epsilon_R g \Psi_R + i\epsilon_L \Psi_L g \\
\delta \Psi_R &= \epsilon_R (g^{-1} \partial_z g - i\Psi_R \Psi_R) , \\
\delta \Psi_L &= \epsilon_L (\partial_z g g^{-1} + i\Psi_L \Psi_L) \\
&\text{(2.3)}
\end{align*}

2.2. Lagrangians for $(1,1)$ Coset Models

Coset models\textsuperscript{26} [27] are algebraic constructions of conformal field theories based on the current algebras of $G$ and a subgroup $H$. They may be given a Lagrangian realization by the use of gauged WZW models, where (naively) the picture of a string moving on a group manifold is replaced by one of having the string restricted to moving on the subspace given by the coset $G/H$. This is realized consistently as a conformal field theory by gauging away the unwanted degrees of freedom corresponding to movement outside the coset, constructing a gauge–invariant extension of the WZW model, by introducing non–propagating 2D gauge fields $A_a$. That this corresponds to the algebraic coset construction has been shown to a great extent over the last few years\textsuperscript{28} [29] [30], using a variety of methods.

The (naively) $N = 1$ supersymmetric Lagrangian for coset models was written down and studied in ref.\textsuperscript{31}. In the component form, we may write the action as follows:

\begin{align*}
I^{(2,2)} &= I_{WZW}(g) + I(g, A) + I_F(\Psi_L, \Psi_R, A) = \\
&= -\frac{k}{4\pi} \int_{\Sigma} d^2z \; \text{Tr}[g^{-1} \partial_z g \cdot g^{-1} \partial_{\bar{z}} g] \\
&- \frac{ik}{12\pi} \int_B d^3 \sigma \; \epsilon^{ijk} \text{Tr}[g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g] \\
&+ \frac{k}{2\pi} \int_{\Sigma} d^2z \; \text{Tr}[A_{\bar{z}} g^{-1} \partial_{\bar{z}} g - A_z \partial_z gg^{-1} + A_{\bar{z}} g^{-1} A_z g - A_z A_{\bar{z}}] \\
&+ \frac{ik}{4\pi} \int_{\Sigma} d^2z \; \text{Tr}[\Psi_R \mathcal{D}_z \Psi_R + \Psi_L \mathcal{D}_z \Psi_L] \\
&\text{(2.4)}
\end{align*}
where \( g \in G, \ A^a \in \text{Lie}(H), \) \( \Psi_L(R) \in \text{Lie}(G) - \text{Lie}(H), \) \( D_a = \partial_a + [A_a, .] \). The left and right moving ‘coset fermions’, \( \Psi_L \) and \( \Psi_R \), are minimally coupled to the gauge fields such that under:

\[
\begin{align*}
g \rightarrow hgh^{-1}, & \quad \Psi_L(R) \rightarrow h\Psi_L(R)h^{-1}, \\
A_a \rightarrow h\partial_a h^{-1} + hA_a h^{-1}, & \quad \text{where } h(z, \overline{z}) \in H,
\end{align*}
\]

the model is gauge invariant. It is gauge invariant precisely because \( I(g, A) \) is a gauge invariant extension of \( I_{WZW}(g) \), while the one–loop gauge anomalies that arise from the fermions cancel against each other, as the coupling of the left and right fermions to the gauge fields are identical in magnitude. The opposite chirality of the fermions then results in a relative minus sign.

This action has an \( N = 1 \) supersymmetry\(^{[25][31]} \):

\[
\begin{align*}
\delta g &= i\epsilon_R g \Psi_R + i\epsilon_L \Psi_L g \\
\delta \Psi_R &= \epsilon_R (1 - \Pi_0) \cdot (g^{-1}D_z g - i\Psi_R \Psi_R) \\
\delta \Psi_L &= \epsilon_L (1 - \Pi_0) \cdot (D_z g g^{-1} + i\Psi_L \Psi_L)
\end{align*}
\]

where \( \Pi_0 \) is the orthogonal projection of \( \text{Lie}(G) \) onto \( \text{Lie}(H) \).

2.3. \((2, 2)\) Supersymmetry from \((1, 1)\) Cosets

The model \((2.4)\) is also to be the definition for the \( N = 2 \) Kazama–Suzuki models\(^1\) for the reason that just as in the algebraic construction of Kazama and Suzuki\(^{[34]} \), an \( N = 2 \) supersymmetry arises from the \( N = 1 \) above \((2.6)\) when the space \( G/H \) is Kähler\(^{[25]} \): Taking \( T = \text{Lie}(G/H) \) as the complexification of the orthogonal complement of \( \text{Lie}(H) \) within \( \text{Lie}(G) \), the Kähler condition translates into the decomposition \( T = T_+ \oplus T_- \) where \( T_+, T_- \) are complex conjugate representations of \( H \), with \([T_+, T_+] \subset T_+, [T_-, T_-] \subset T_-\), and \( \text{Tr}(ab) = 0 \) for \( a, b \in T_+ \) or \( T_- \). The three pieces of the condition define first an almost complex structure, then integrability of this structure, and finally that the metric is a \((1, 1)\) quadratic form which is Kähler. Continuing to follow ref.\(^{[25]} \), we define \( \Pi_\pm \) as the orthogonal projections of \( T \) onto \( T_\pm \). The right (left)–moving ‘coset fermions’ decompose under this to \( \Psi_{R(L), \pm} = \Pi_\pm \Psi_{R(L)} \), and the fermion action becomes:

\[
I^F = \frac{ik}{2\pi} \int_{\Sigma} d^2z \, \text{Tr}[\Psi_{R,+} D_z \Psi_{R,-} + \Psi_{L,+} D_z \Psi_{L,-}].
\]
Now we can see that there is an R–symmetry (i.e. it does not commute with the supersymmetry) for each chirality which assigns the charge $\pm 1$ to quantities valued in $T_{\pm}$ and charge 0 to $g$ and the gauge fields. The $N = 1$ supersymmetry transformation (2.3) can be decomposed into terms of $\Delta R = \pm 1$, which will be our two supersymmetries giving $N = 2$, with parameters $\epsilon_{R(L),\pm}$:

\[
\begin{align*}
\delta g &= i\epsilon_{R,+} g \Psi_{R,-} + i\epsilon_{R,-} g \Psi_{R,+} \\
\delta \Psi_{R,+} &= \epsilon_{R,+} \Pi_+ \cdot \left[ D g g^{-1} - i (\Psi_{R,+} \Psi_{R,-} + \Psi_{R,-} \Psi_{R,+}) \right] - i\epsilon_{R,-} \Psi_{R,+} \Psi_{R,+} \\
\delta \Psi_{R,-} &= \epsilon_{R,-} \Pi_- \cdot \left[ D g g^{-1} - i (\Psi_{R,+} \Psi_{R,-} + \Psi_{R,-} \Psi_{R,+}) \right] - i\epsilon_{R,+} \Psi_{R,-} \Psi_{R,-} \\
\delta (\text{everything else}) &= 0,
\end{align*}
\]

and a similar set of $N = 2$ transformations on the left–moving side.

2.4. Lagrangians for $(0, 2)$ models

Recently, in ref.[14][15] it was noted that there are many more ways of combining the above ingredients to get gauge invariant models, and hence a larger class of conformal field theories. In particular, for the study of $(0, 2)$ conformal field theories it is possible to preserve the right moving structure of the Lagrangian, the couplings of the right moving fermions and the right action of the gauge group, and hence preserve the right supersymmetry. On the left hand side, fermions may be included with a priori arbitrary couplings, thus disallowing generally the possibility of a left supersymmetry. There is now the potential problem that the chiral gauge anomalies from the left and right do not cancel. This problem is circumvented by allowing the possibility to gauge as arbitrary a left action of the gauge group on $g$ as allowed by group theory. In general, an extension for $I_{WZW}(g)$ based on this resulting non–diagonal gauging of the WZW can be chosen so as to produce (classical) chiral anomaly terms. Simply requiring that the total anomaly from the three sectors vanishes restores gauge invariance.

This way of producing a $(0, 2)$ model by modifying the possible gaugings and the left fermion coupling allows great freedom in the type of left moving structures present in the models, as is evident in the prototype example of this type of construction, the ‘monopole’ theory of ref.[13], which was shown to be a heterotic coset in ref.[14]. In that model, there is an additional $SU(2)$ current algebra on the left, which is the world–sheet manifestation of spacetime rotational invariance. (This model was presented as the angular sector of a 4D spacetime extremal black hole with magnetic charge.) Similarly, such symmetries may be found in the heterotic coset realization of other spacetime backgrounds[14][16].

In general, gauging the following symmetry of the WZW model $g \to h_L g h_R^{-1}$ for $(h_L, h_R) \in (H_L, H_R) \subset (G_L, G_R)$ is anomalous. This simply means that one cannot write down an extension of the WZW model which promotes this symmetry to a local
invariance: There will always be terms which spoil gauge invariance. (This is because of the Wess–Zumino term; the ‘metric’ term may be simply minimally coupled.)

Knowing that we will get an anomaly, let us choose to write some gauge extension such that under gauge transformations the ‘anomalous’ piece does not depend upon the group element \( g \). This results in the anomalous piece taking the form of the standard 2D chiral anomaly. The unique\(^{[35]} \) action is:

\[
I_{GWZW}^G(g, A) = -\frac{k}{4\pi} \int d^2z \; \text{Tr}[g^{-1} \partial_z g \cdot g^{-1} \partial\bar{z} g] - \frac{ik}{12\pi} \int_B d^3\sigma \; \epsilon^{ijk} \text{Tr}[g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g] \tag{2.9}
\]

\[
+ \frac{k}{2\pi} \int d^2z \; \text{Tr}[A^R_z g^{-1} \partial_z g - A^L_z \partial\bar{z} g g^{-1} + A^R_z g^{-1} g - \frac{1}{2} \{ A^L_z A^L_{\bar{z}} + A^R_z A^R_{\bar{z}} \}],
\]

where \( A^{R(L)} = A^a t_{a,R(L)}. \) Under the infinitesimal variation

\[
g \rightarrow g + \epsilon^L g - g \epsilon^R \\
A^R_z \rightarrow A^R_z - \partial_z \epsilon^R - [A^R_z, \epsilon^R] \\
A^L_z \rightarrow A^L_z - \partial\bar{z} \epsilon^R - [A^L_z, \epsilon^R], \tag{2.10}
\]

for \( \epsilon^R = \epsilon^a t_{a,R(L)} \)

the variation is

\[
\delta I(g, A) = \frac{k}{4\pi} \int d^2z \epsilon^{(a)} \; \text{Tr}[t_{a,R} \cdot \Psi_R - t_{a,L} \cdot \Psi_R] \int d^2z \epsilon^{(a)} F^{(b)}_{\bar{z}z} \tag{2.11}
\]

where \( t_{a,L(R)} \in \text{Lie}(H) \), and \( F^{(b)}_{\bar{z}z} \equiv \partial_z \Psi^{(b)}_R - \partial\bar{z} \Psi^{(b)}_R \).

Notice in particular that for the popular diagonal gaugings of WZW models this variation is zero and the action reduces to the familiar one.

Turning to the right moving Majorana–Weyl fermions, it is sufficient to minimally couple them as coset fermions to the gauge fields:

\[
I^R_F(\Psi_R, A) = \frac{ik}{4\pi} \int \text{Tr}[\Psi_R \mathcal{D}\bar{z}\Psi_R] \tag{2.12}
\]

where \( \mathcal{D}\bar{z}\Psi_R = \partial\bar{z} \Psi_R + [A^R_{\bar{z}}, \Psi_R], \; \Psi_R \in \text{Lie}(G) - \text{Lie}(H). \)

This model is classically gauge invariant under:

\[
\Psi_R \rightarrow h_R \Psi_R h^{-1}_R \text{ and } A^R \rightarrow h_R d h^{-1}_R + h_R A^R h^{-1}_R \tag{2.13}
\]
There are $D = \dim(G) - \dim(H)$ fermions $\psi^i_R$ in $\Psi_R$, all coupled with charges derived from the generators $t_{a,R}$. The chiral anomalies appear at one loop and are:

$$\frac{1}{4\pi} \text{Tr}_A[t_{a,R} \cdot t_{b,R}] \int_\Sigma d^2 z \epsilon^{(a)} z^{(b)} F_{\Sigma z}^{(b)}.$$ (2.14)

(Note here the absence of $k$, which plays the role of $1/\hbar$. This really is a one loop effect.) Here $\text{Tr}_A$ means the trace in the adjoint representation.

It is a natural choice to add $D = \dim(G) - \dim(H)$ left moving Majorana–Weyl fermions with arbitrary values of the minimal couplings. To be precise, arrange them into a fundamental vector $\Lambda_L = \{\lambda^i_L\}$ of the group $SO(D)_L$ which acts on them as a global symmetry, and minimally couple them to the $H_L$ subgroup with generators $Q_{a,L}$ in this fundamental representation:

$$I_L^F(\lambda^i_L, A) = \frac{ik}{4\pi} \int_\Sigma d^2 z \Lambda^T_L (\partial z + \sum_a A^a Q_{a,L}) \Lambda^L_z.$$ (2.15)

Their chiral anomalies appear at one loop and are:

$$-\frac{1}{4\pi} \widetilde{\text{Tr}}[Q_{a,L} \cdot Q_{b,L}] \int_\Sigma d^2 z \epsilon^{(a)} z^{(b)} F_{\Sigma z}^{(b)}.$$ (2.16)

(Note again the absence of $k$. Also note the minus sign relative to (2.14), due to the opposite chirality. Here $\widetilde{\text{Tr}}$ is the trace in the fundamental representation of $SO(D)$.)

So if we add together the three actions (2.9), (2.12) and (2.15), we get a gauge invariant model if we ensure that all of the anomalies (classical and quantum) cancel:

$$k \text{Tr}[t_{a,R} \cdot t_{b,R} - t_{a,L} \cdot t_{b,L}] + \text{Tr}_A[t_{a,R} \cdot t_{b,R}] - \widetilde{\text{Tr}}[Q_{a,L} \cdot Q_{b,L}] = 0.$$ (2.17)

Note that (0, 2) supersymmetry is still present given that the parent $(N = 1)_R$ is preserved (see equation (2.8)), along with the structure of $G/H$ as seen by the right movers.

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2 Here and for the remainder of this paper, it is implicit that a consistent regularisation scheme has been chosen for calculation of the fermion anomalies, and such that the normalisation of the anomalies is chosen to be of this simple form.

3 It is important to realize that this choice is not necessary. Indeed, the most natural way to proceed beyond this would be to add more left moving fermions, in order to increase the total central charge of the left–moving sector. This would then immediately give rise to the possibility of smaller spacetime gauge groups for model building. For now, we shall merely note this in passing but hope to return to this important point in the not too distant future.
2.5. Bosonization

So far, the model as written is gauge invariant when we take into account the one–loop effects. To write a classically gauge invariant action it is necessary to bosonize the fermions. The bosonic action equivalent to $I^F_R + I^F_L$ is classically anomalous, and is easily seen to be\[15\] an $SO(D = \text{dim}G - \text{dim}H)$ WZW (at level 1) gauged anomalously with different embeddings of $H$ in $SO(D)$ on the left and on the right:

$$\tilde{g} \rightarrow \tilde{h}_L \tilde{g} \tilde{h}_R^{-1} \quad \text{for} \quad \tilde{g} \in SO(D) \quad \text{and} \quad (\tilde{h}_L, \tilde{h}_R) \in (H_L, H_R) \subset (SO(D)_L, SO(D)_R) \quad (2.18)$$

The $(H_L, H_R)$ are generated by $(Q_{a,L}, Q_{a,R})$. Choose the $Q_{a,R}$ such that when acting on the $\psi_i^R$’s in the fundamental representation of $SO(D)$ they are equivalent to the $t_{a,R}$ acting on the $\psi_i^R$ in the coset fermion $\Psi_R \in \text{Lie}(G) - \text{Lie}(H)$. This will ensure that the right moving fermions are correctly coupled and preserve the (now hidden) $N = 2$ on the right.

Then the bosonic action equivalent to the interacting fermions is just an action of the form (2.9) (with level 1), which yields the classical anomalies:

$$\frac{1}{4\pi} \widetilde{\text{Tr}}[Q_{a,R} \cdot Q_{b,R} - Q_{a,L} \cdot Q_{b,L}] \int_{\Sigma} d^2z \epsilon^{(b)} F^{(a)}_{zz}. \quad (2.19)$$

So canceling this against the anomaly of the $G/H$ bosonic model (and recalling from the above paragraph that $\widetilde{\text{Tr}}[Q_{a,R} \cdot Q_{b,R}] = \text{Tr}_{Ad}[t_{a,R} \cdot t_{b,R}]$), we recover (2.17) as the condition for a consistent model.

So finally we can write a classically gauge invariant analogue of (2.4) which realizes a $(0,2)$ conformal field theory written as the sum of two gauged Wess–Zumino–Witten models which are separately anomalous:

$$I^{(0,2)} = I_{GWZW}^{Gk}(g, A) + I^{SO(D)_1}_{GWZW}^{SO(D)}(\tilde{g}, A)$$

$$= I_{GWZW}^{Gk}(g) + I^{SO(D)_1}_{GWZW}^{SO(D)}(\tilde{g}) +$$

$$\frac{k}{2\pi} \int_{\Sigma} d^2z \text{Tr}[A^R_z \bar{g}^{-1} \partial_z \bar{g} - A^L_z \partial_{\bar{z}} \bar{g}^{-1} + A^R_z \bar{g}^{-1} A^L_z \bar{g} - \frac{1}{2} \{A^L_z A^L_{\bar{z}} + A^R_z A^R_{\bar{z}}\}]$$

$$+ \frac{1}{2\pi} \int_{\Sigma} d^2z \text{Tr}[A^R_z \tilde{g}^{-1} \partial_z \tilde{g} - A^L_z \partial_{\bar{z}} \tilde{g}^{-1} + A^R_z \tilde{g}^{-1} A^L_z \tilde{g} - \frac{1}{2} \{A^L_z A^L_{\bar{z}} + A^R_z A^R_{\bar{z}}\}], \quad (2.20)$$

where $D = \text{dim}(G) - \text{dim}(H)$ and so the heterotic coset is realized as: $[G_k \times SO(D)_1] / H$ with the gauged symmetry (2.10) and (2.18) and subject to (2.17).

2.6. The $(0,2)$ minimal models

For most of the sections in this paper, we will consider in detail the case $G = SU(2)$ with $H = U(1)$. These models are therefore the analogue of the $(2,2)$ minimal models.
and will accordingly be referred to as the $(0, 2)$ minimal models. Specializing some of the previous formulae to this case, we will use
\[ h_L = e^{i\epsilon \sigma_3/2}, \ h_R = e^{-i\epsilon \sigma_3/2}, \ \tilde{h}_L = e^{i\epsilon Q \sigma_2}, \ \tilde{h}_R = e^{-i\epsilon \sigma_2}, \ \text{and} \ \tilde{g} = e^{i\Phi \sigma_2}, \] (2.21)

and we have for the heterotic coset model in bosonic form:
\[
I_{\alpha, Q}^{(0,2)} = I_{ZW}(g \in SU(2), k) + \frac{k}{2\pi} \int d^2z \text{Tr} \left[ A_\tau^{-1} \partial_\tau g - \alpha A_\tau g^{-1} A_\tau g - \frac{1}{2}(1 + \alpha^2) A_\tau A_\tau^{-1} \right] + \frac{1}{4\pi} \int d^2z \left[ \partial_\tau \Phi \partial_\tau \Phi - 2A_\tau \partial_\tau \Phi - 2Q A_\tau \partial_\tau \Phi + (1 + Q)^2 A_\tau A_\tau^{-1} \right],
\] (2.22)

which is invariant under the following infinitesimal gauge transformations:
\[
g \to g + \frac{i}{2} \epsilon (\alpha \sigma_3 g + g \sigma_3), \ \Phi \to \Phi + (Q + 1) \epsilon, \ \text{and} \ A_a \to A_a + \partial_a \epsilon,
\] (2.23)

subject to the anomaly cancelation condition
\[
k(1 - \alpha^2) = 2(Q^2 - 1).
\] (2.24)

The theory of the $2\pi$ periodic boson $\Phi$ is the bosonised fermions and is equivalent to:
\[
I_F = \frac{i k}{4\pi} \int d^2z \left\{ \text{Tr}_{A_d} [\Psi_R D_R^T \Psi_R] + \Lambda_L^T (\partial_\tau + Q \cdot A_\tau) \Lambda_L \right\}.
\] (2.25)

The left–moving fermions are coupled to the $U(1)$ gauge fields with charge $Q$, whereas the right–movers are coset fermions with their charges determined by geometry. The cases $\alpha = \pm 1$, for which (due to (2.24)) $Q = \pm 1$ correspond to the familiar diagonal models, which is the family of $(2, 2)$ minimal models. The case $\alpha = 0$ is the charge $Q$ ‘monopole’ theory of ref.[13], and will henceforth be the focus of this paper. Other $\alpha$ are restricted to being integer, in order to furnish a faithful representation of a $U(1)$ subgroup of $SU(2)$. However, the anomaly equation prohibits such solutions.

The central charge (on both the left and right) of a $(0, 2)$ minimal model of level $k$ is given by the familiar formula
\[
c = \frac{3k}{k + 2}.
\] (2.26)

This follows from the simple fact that the gauging of a $U(1)$ subgroup effectively removes one bosonic degree of freedom, whose central charge is equal to that of the two fermions added for supersymmetry (on the right) or of current algebra fermions (on the left). So the central charge of the $SU(2)$ contribution to the model is all that remains.

\[4\] Note that the restrictions on analogous parameters in the case of heterotic cosets based upon non—compact groups are not so severe[14][15]. Also, the use of rational $\alpha$ for compact groups seems to give results consistent with conformal invariance, at least at one–loop[16].
3. Elliptic Genera for the \((0, 2)\) minimal models

Of all quantities which exist for supersymmetric models, those which correspond to
topological invariants of an associated geometry (when there is such a geometry), or more
generally an index of some type, are usually most accessible. Amongst such quantities is
the elliptic genus\(^{36-38}\). Given a \((0, 2)\) theory with at least a \(U(1)\) current algebra on
the left (which commutes with the \(N = 2\) supersymmetry), the partition function may be
written as follows:

\[
Z(q, \gamma_L, \gamma_R) = \text{Tr}_H \left[ (-1)^{F_L} q^{H_L} \exp \left( i \gamma_L J_{0, L} \right) (-1)^{F_R} q^{H_R} \exp \left( i \gamma_R J_{0, R} \right) \right].
\]

(3.1)

The elliptic genus arises when we consider the restriction of the Ramond sector partition
function to the case \(\gamma_R = 0\). Then the right–moving sector contains only the quantity
\(\text{Tr}(-1)^{F_R} q^{H_R}\). Due to the presence of supersymmetry, this quantity is only ever 1 or 0
when evaluated on sectors of the Hilbert space \(H\). By not grading states according to their
\(U(1)_R\) charge (i.e. putting \(\gamma_R = 0\)), the boson and fermion pairs at each non–zero mass
level contribute equally to the sum, but with a relative minus sign and thus contributions
from sectors with \(H_R \neq 0\) will sum to zero. Only the right moving ground state sectors
(those with \(H_R = 0\)) can contribute, due to the presence of unpaired states.

The elliptic genus is thus:

\[
Z(q, \gamma_L) = \text{Tr}_{H(H_R=0)} \left[ (-1)^{F_L} q^{H_L} \exp \left( i \gamma_L J_{0, L} \right) \right]
\]

(3.2)

which will encode for us all of the information about the states in the left moving sector
which couple to the right moving Ramond ground states.

It was in ref.\(^{18}\) that the elliptic genera for the \((A–series of the)\) \((2, 2)\) minimal models
was calculated in their Landau–Ginsburg formulation enabling comparison to those of their
algebraic formulation\(^{39}\). This provided further evidence to support the conjecture that
the Landau–Ginsburg models did indeed flow to the minimal models at their fixed point.
In ref.\(^{19}\), the elliptic genera of the \((A–series of the)\) \((2, 2)\) minimal models was calculated
using path integral techniques based upon the \((2, 2)\) supersymmetric gauged WZW models
discussed in the previous section. It is this approach which we will follow most closely
in order to calculate the elliptic genera of our \((0, 2)\) minimal models. The machinery
will need little modification to be applied here, as indeed our models are also based upon
supersymmetric gauged WZW models, and furthermore the right–moving sector is identical
to those of the \((2, 2)\) models.

3.1. Global \(U(1)_L\) symmetry

To begin with we need to identify the appropriate left–moving \(U(1)\) symmetry whose
global component we will use to grade all fields in the model, in preparation for the twisting
procedure. In the (2, 2) case, this was the $U(1)$ of the left $N = 2$, which commuted with the right $N = 2$. For this case we of course still need the latter constraint. This $U(1)$ will act upon the fermions of our model (which we have arranged into complex fermions $\lambda_L$ and $\psi_L$), and the bosonic fields $g$.

Following ref. [19] we postulate the following changes under global $U(1)_L$:

$$\delta \lambda_L = i\epsilon c_L \lambda_L; \quad \delta \psi_R = i\epsilon c_R \psi_R; \quad \delta g = \frac{i}{2} \epsilon (x_L \sigma_3 g + x_R g \sigma_3)$$

and similarly for the complex conjugate fields $\bar{\psi}_R$ and $\bar{\lambda}_L$. The quantities $c_L, c_R, x_L$ and $x_R$ are charges to be determined. The requirement that this transformation commutes with the right supersymmetry fixes $-x_R = c_R$. Consistency with a left supersymmetry can no longer be a requirement here as in the (2, 2) case, because it is not present in a (0, 2) model. So we do not have the condition $x_L = c_L - 1$, which arose in ref. [19]. However it transpires that $c_L$ turns out to be free and can be chosen to set the overall normalisation of the charges of $U(1)_L$.

Now we wish for our complete action to be invariant under this global $U(1)_L$ symmetry. Structurally the situation is almost identical to the 2D gauge theory considerations we made earlier, when we constructed the coset: the bosonic sector ($I_{WZW}(g)$ and its gauge extension) is not invariant under (3.3), and under those transformations, produces the ‘classical anomaly’ $\epsilon k x_R$. Meanwhile, the fermions (2.25) are classically invariant under (3.3), but produce one–loop anomalies $2\epsilon (c_R - c_L Q)$. The anomaly cancelation equation is then $2(c_L Q - c_R) = k x_R$. We have four parameters ($x_L, x_R, c_L, c_R$) and two equations. It is convenient to use a gauge transformation to set $x_L = -x_R = x$, and putting $c_R = -x_R$ (determined above) gives $x = -2c_L Q/(k + 2)$. So $c_L$ is a free parameter here and is just a normalisation of the $U(1)_L$ charges, which we are always free to fix if there are no other constraints. If we rescale such that $c_L = 1/(2Q)$ then:

$$\delta \lambda_L = \frac{i\epsilon}{2Q} \lambda_L, \quad \delta \psi_R = \frac{i\epsilon}{(k + 2)} \psi_R, \quad \text{and} \quad \delta g = -\frac{i\epsilon}{2(k + 2)} (\sigma_3 g + g \sigma_3),$$

and similarly for the complex conjugates $\bar{\psi}_R$ and $\bar{\lambda}_L$. This represents a scale choice which produces the normalisation for the charges of the right–moving sector of the minimal models as set out in refs. [18] [19].

3.2. An Index and Deformation to Free Theory

The elliptic genus, equation (3.1) with $\gamma_R = 0$, may be regarded as a path integral evaluated on the torus with twisted boundary conditions. To be more precise, let us define our torus as usual starting with the lattice obtained by folding the $x^1 - x^2$ plane according to $x^1 \rightarrow x^1 + m$, $x^2 \rightarrow x^2 + n$ for $m, n \in \mathbb{Z}$. The torus with modular parameter $\tau$ is supplied
with complex coordinate via \( z = x^1 + \tau x^2 \). We will chose \( x^2 \) as the ‘time’ direction and \( x^1 \) as ‘space’.

The elliptic genus is then defined as the path integral

\[
Z_Q^{(0,2)}(\tau, \gamma) = \int_{\text{torus}} \mathcal{D}g \mathcal{D}\Psi_R \mathcal{D}\Lambda_L \mathcal{D}A_z \mathcal{D}A_\tau e^{-I_Q^{(0,2)}(g, \Psi_R, \Lambda_L)}
\]

with periodic boundary conditions in the space direction and for the time direction there is the \( U(1)_L \) twist:

\[
\begin{align*}
g(x^1, x^2 + 1) &= e^{-\frac{i\gamma \sigma_3}{2(k+2)}} g(x^1, x^2)e^{-\frac{i\gamma \sigma_3}{2(k+2)}} \\
\lambda_L(x^1, x^2 + 1) &= e^{\frac{i\gamma}{2\pi\sigma}} \lambda_L(x^1, x^2) \\
\psi_R(x^1, x^2 + 1) &= e^{(\pi i k/2)} \psi_R(x^1, x^2) \\
A_a(x^1, x^2 + 1) &= A_a(x^1, x^2)
\end{align*}
\]

Now as the elliptic genus is a supersymmetry index, there should exist smooth deformations of the system which preserve the supersymmetry and hence keep this quantity unchanged. Indeed it was this philosophy which was adopted in ref.[18] in order to calculate the elliptic genus for the Landau–Ginsburg models. By safely deforming the theory to the weak coupling regime, a successful computation could be carried out.

In ref.[19], this procedure was carried out for the gauged WZW formulation of the minimal models. The deformation appropriate to the problem was identified and the calculation reduced to a weak coupling problem. Once the \( U(1)_L \) charges of the constituent fields were identified, the problem was reduced to the free field computation of ref.[18]. Here, the same procedures follow. The structure of the right–moving sector is identical to that of ref.[19] and therefore the operator with which to deform the theory safely to weak coupling is the same. As we have identified the global \( U(1)_L \) charges in the previous section, there only remains the task of performing again the free–field computation of ref.[18], taking into account the contributions from the fermionic and bosonic modes in a Hilbert space approach.

This results in the following expression for the elliptic genera of the \((0,2)\) minimal models.

\[
Z_Q^{(0,2)}(q, \gamma, 0) = e^{-i\gamma \frac{k(Q-1)}{2(k+2)}} \cdot \frac{1 - e^{i\gamma k/2}}{1 - e^{i\gamma k/2}} \prod_{n=1}^{\infty} \frac{(1 - q^n e^{i\gamma k/2})(1 - q^n e^{-i\gamma k/2})}{(1 - q^n e^{i\gamma k/2})(1 - q^n e^{-i\gamma k/2})}.
\]

3.3. Properties of the Elliptic Genera

In this section we note some important properties of the elliptic genera which we computed above. This serves as a useful warmup for the case of the complete partition func-
tions, computed later in this paper. Indeed, some of the information extracted here will prove to be useful in those later computations.

First note that by expanding the expression (3.7) one can guess the following formula:

\[
Z_Q^{(0,2)}(q, \gamma, 0) = \sum_{r=0}^{Q-1} (-1)^r \mathcal{X}_{Q-1+2Qr;k}
\]

where \(\mathcal{X}_{l,k}\) is the character for the highest weight unitary irreducible representation with isospin \(l/2\) of the level \(k\) \(SU(2)\) affine Lie algebra. This relation is proven in Appendix A, using a convenient representation of the elliptic genera in terms of theta functions.

This explicitly shows the emergence of the ‘physical’ \(SU(2)\) affine Lie algebra on the left since the gauging leaves the \(SU(2)_L\) symmetry untouched: \(g \rightarrow g \exp(i\sigma_3/2)\). We have for the heterotic coset at level \(k\) a family of \(Q\) level \(k\) affine \(SU(2)\) characters on the left. Recall that the information content of the elliptic genus is precisely about all of the states from the left that couple the (supersymmetric) right moving sector’s Ramond ground states. In this case, the affine \(SU(2)\) encodes the spacetime rotation invariance of the ‘monopole’ (of charge \(Q\)) theory of \([13]\). There, the left moving fermions carry the spacetime \(U(1)\) monopole field of the heterotic string magnetic black hole background.

Turning to the formula (3.7) for the elliptic genera again, we see that in order to have a finite number of terms in the \(q^0\) level of the elliptic genus expansion, one must have that \(Q\) is an integer, as one would expect from the \(U(1)\) monopole interpretation of ref.[13]. This is of course a quite physical requirement from the CFT point of view, saying that we have a finite number, \(Q\), of highest weight vacuum states.

4. The Partition Functions of the \((0, 2)\) Minimal Models

In the previous section we calculated the elliptic genera for the \((0, 2)\) minimal models. In many formulations of a superconformal field theory, the calculation of this quantity is the closest one can get to the full partition function of the theory. In this case, it turns out that we can go much further, and calculate the full partition function for these models.

Our model is a manifestly left–right asymmetric combination of two gauged WZW models, and it is not obvious just how to make sense of the task of constructing its spectrum, given these unusual couplings, although a number of statements may be made given that there is a familiar \(SU(2)\) current algebra on the left, as we saw in the last section. Even with this knowledge, it is not an easy task to construct the modular invariant combination of characters which gives the partition function of these models when the asymmetry is present. It is therefore of great comfort to note that progress can be made by using our field theory intuition to study the path integral, and answer difficult algebraic questions.
4.1. A Change of Variables

Motivated by the gauge transformations under which the model (2.22) is invariant, a little thought suggests that the following changes of variables
\[
A_z \to \sqrt{2} \partial_z \phi_L, \quad A_\tau \to \sqrt{2} \partial_\tau \phi_R; \quad g \to ge^{-\frac{i\sigma_3}{2\sqrt{2}}\phi_R}, \quad \Phi \to \Phi + \sqrt{2}(Q\phi_L + \phi_R)
\]
might be interesting, for the simple reason that they would formally uncouple the action of the gauge symmetry from the original variables, and put them entirely on the \(\phi_L(R)\) fields:
\[
\delta \phi_L(R) = \epsilon/\sqrt{2}.
\]

The change of variables for all the fields except the gauge fields are harmless. However, those for the gauge fields require a non-trivial Jacobian to be computed, which we shall replace with an action for anticommuting ghosts in the standard way:
\[
\mathcal{D}A_z \mathcal{D}A_\tau = \mathcal{D}\phi_L \mathcal{D}\phi_R \det[\partial_z] \det[\partial_\tau] =
\]
\[
= \mathcal{D}\phi_L \mathcal{D}\phi_R \int \mathcal{D}b \mathcal{D}c \mathcal{D}\overline{b} \mathcal{D}\overline{c} \exp\left[i \int d^2z \left(b \partial_\tau c + \overline{b} \partial_\tau \overline{c}\right)\right]
\]  
(4.2)

After some algebra, and much strategic use of the anomaly equation to simplify expressions we find that the change of variables does indeed formally decouple the systems from one another:
\[
I_{\alpha,Q}^{(0,2)} = I_{WZW}^{SU(2)_k} + I_{WZW}^{SO(2)_1} + I_{WZW}^{SO(2)^{(k+2)}} + \int d^2z \left(b \partial_\tau c + \overline{b} \partial_\tau \overline{c}\right).
\]  
(4.3)

We have written the final result somewhat suggestively. Let us examine the terms. The first is just the pure \(SU(2)\) WZW, while the second term is simply the kinetic term for the free periodic boson (the bosonised fermions), \(\frac{1}{4\pi} \int \partial_z \Phi \partial_z \Phi\). The third term is actually the theory:
\[
I = -\frac{(k + 2)}{4\pi} \int d^2z (\partial_\tau \phi_L - \partial_\tau \phi_R)(\partial_\tau \phi_L - \partial_\tau \phi_R).
\]  
(4.4)

The idea is that we will fix a gauge by setting \(\partial_\tau \phi_L = 0\) (i.e. \(A_\tau = 0\)). This then gives us a level \(-(k + 2)\) \(SO(2)\) WZW theory in the variable \(\phi_R \equiv \phi\). (Notice that this level is

\[5\] Notice that this simple change of variables for the gauge fields fixes us to only considering gauge configurations which can be deformed to the identity, i.e. the trivial holonomy sector. Therefore we are really only working on a world sheet with the topology of the sphere. Later, we have to take other topologies and gauge configurations into account.

\[6\] Note that for subgroup \(U(1)\) the usual corrections proportional to the quadratic Casimir \(C_H\) due to the change of variables on the fermions and from the Jacobian of the covariant derivative do not appear.
somewhat arbitrary, except for the sign, but for internal consistency if the $\Phi$ theory has level 1 then this is the right interpretation for this theory.) This arbitrariness is lifted for non–abelian $H$. Notice the analogy with the diagonal case. Similar changes of variables have been found in that case which allows such a decoupling[28][11].

Note here that the central charge of the models can be computed giving the same result as before,[28][41], with slightly differing origins of the cancelations of the contributions which are not from $SU(2)_k$. This time, each $SO(2)$ theory contributes 1 to the central charge, while the ghosts contribute $-2$.

Of course, this decoupling into a sum of WZW’s does not complete the story. The gauge symmetry must still be present somehow, imposing conditions upon the complete theory in order to recover the coset. Such constraints will arise from the BRST symmetry of the system, which may be derived using the methods in ref.[11]. Using the following normalisation for WZW affine currents:

$$J_a = k\text{Tr}[t_ag^{-1}\partial_zg], \quad \mathcal{J}_a = k\text{Tr}[t_a\partial_zgg^{-1}]$$

for a generic model $g$ at level $k$, (for $SU(2)$ we use $t_3 = i\sigma_3/2$), and denoting the left (right) currents for the $\Phi$ and $\phi$ theories as $J$ ($\mathcal{J}$) and $I$ ($\mathcal{I}$) respectively, we derived the BRST currents:

$$\mathcal{J}_{BRST} = \frac{c}{2\pi} \left( J_3 + \mathcal{J} + \mathcal{I} \right), \quad J_{BRST} = \frac{c}{2\pi} (QJ - I).$$

These BRST currents give rise to the nilpotent BRST charge operators:

$$\bar{Q}_{BRST} = \oint \frac{dz}{2\pi i} :e^{(J_3 + \mathcal{J} + \mathcal{I})} : , \quad Q_{BRST} = -\oint \frac{dz}{2\pi i} :c(QJ - I) :$$

which allow us to define physical states via the cohomological problem

$$Q_{BRST}|\text{Phys}>=0; \quad |\text{Phys}>= |\text{Phys}>+Q_{BRST}|\text{Anything}>=.$$
4.2. From the Sphere to the Torus: ‘Continuous Orbifolds’

In the previous subsection we derived the powerful result that the non–diagonal action defining our models may be decoupled into a set of free WZW models (diagonal) together with a left–right asymmetric BRST system. The problem of identifying physical states then became one of projecting onto the gauge invariant subspace of the starting Hilbert space, which is just a product of states from WZW models. This is great progress, but it does not yet tell us anything about the consistent coupling between left and right sectors of the theory, as we have not yet put the theory on the torus.

The partition function of the sum of theories (4.3) may be written as (ignoring any \(U(1)\) symmetries which commute with the Hamiltonian)

\[
Z = \text{Tr}_{\mathcal{H}_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}
\]  

(4.9)

The big Hilbert space \(\mathcal{H}_0\) is of the big theory \(I_{WZW}(g) + I_{WZW}(\tilde{g}) + I_{WZW}(\phi) + I_{\text{ghosts}}\) and \(L_0, \bar{L}_0, c\) and \(\bar{c}\) refer to the appropriate quantities when acting in each subsector of the big Hilbert space \(\mathcal{H}_0\). We obtained physical state information on the sphere in the last section by projecting \(\mathcal{H}_0\) onto the subspace of states in the gauge–BRST cohomology. To begin to implement this at the level of partition functions we need to define a projection operator to insert into the Tr above. In a standard orbifold–like procedure, with a symmetry under a finite discrete group \(H = \{h_i\}\) to take into account, the projection operator would be

\[
P = \frac{1}{|H|} \sum_i h_i.
\]  

(4.10)

Clearly in our case, the appropriate operators would be\(^7\)

\[
P = \frac{1}{2\pi} \int d\theta e^{i\theta J_0}, \quad \bar{P} = \frac{1}{2\pi} \int d\bar{\theta} e^{i\bar{\theta} \bar{J}_0},
\]  

(4.11)

where \(J_0 = QJ_0 - I_0\), and \(\bar{J}_0 = \bar{J}_3 + J_0 + I_0\). However as we are studying string theory, we know that we must be much more careful. Considering the theory on the torus, the possibility of ‘twisted sectors’ should be taken into account. On the torus, we have the standard boundary conditions:

\[
[g, \Phi, \phi](x^1 + 1, x^2) = [g, \Phi, \phi](x^1, x^2)
\]

\[
[g, \Phi, \phi](x^1, x^2 + 1) = [g, \Phi, \phi](x^1, x^2).
\]  

(4.12)

\(^7\) Note that these two different projections would be tantamount to twisting in orthogonal directions on the world sheet. See later in the text.
(We use the notational triplet \([g, \Phi, \phi](x^1, x^2)\).) The twisted sectors are precisely the fields with the boundary conditions:

\[
\begin{align*}
[g, \Phi, \phi](x^1 + 1, x^2) &= \left[ e^{i \frac{\theta_1}{2}}, \Phi + (Q + 1) \theta_1, \phi + \theta_1 \right] (x^1, x^2) \\
[g, \Phi, \phi](x^1, x^2 + 1) &= \left[ e^{i \frac{\theta_2}{2}}, \Phi + (Q + 1) \theta_2, \phi + \theta_2 \right] (x^1, x^2),
\end{align*}
\]

where \(\theta_1, \theta_2\) are arbitrary elements of the gauge algebra. The twisted sector partition function is then:

\[
Z(\tau, \theta, \bar{\theta}) = \left\{ \begin{array}{l}
\text{Tr}_{H_{SU(2)_k}} q^{L_0} e^{-2\pi i \theta J_0} \bar{q}^{\bar{L}_0}, \\
\text{Tr}_{H_{SO(2)_1}} e^{2\pi i Q \theta J_0} q^{L_0} e^{-2\pi i \theta J_0} \bar{q}^{\bar{L}_0}, \\
\text{Tr}_{H_{SO(2)_-(k+2)}} e^{-2\pi i \theta I_0} q^{L_0} e^{-2\pi i \theta I_0} \bar{q}^{\bar{L}_0} \end{array} \right\}
\]

These type of twisted partition functions are going to be mapped into one another under modular transformations in the usual way. In the standard orbifold language we would construct the modular invariant partition function by summing over all of the twisted sectors. Here we integrate instead. By integrating over sectors twisted over both cycles, we see that we include both types of projections (4.11) naturally. Our result after doing the orbifold is then:

\[
Z^{(0,2)}_Q = \int d\theta d\bar{\theta} Z(\tau, \theta, \bar{\theta})
\]

Some comments pertaining to the relation to the field theory picture of last section are in order here. The twisted sectors of the partition function language here and the non–trivial holonomy sectors of the gauge fields which were (knowingly) ignored in the last section are of course related. The twisted sectors arise when it is taken into account that a field going around a cycle of the torus can return to a field which is related to the original field we thought of up to an arbitrary gauge transformation. When on the sphere in the last section we gauge fixed by fixing \(\phi_L\) to be a constant, but by parametrising the gauge fields in the way done in eqn. (4.1), we took into account only gauge configurations which may be connect locally, i.e. within each twisted sector. Therefore we only gauge fixed such transformations also. Nothing was done in the previous section about the possibility of jumping between gauges which are distinct because of the non–trivial topology of the torus. Gauge transformations characterized by (4.13), were still unaccounted for. This is reminiscent of taking into account the holonomy of the gauge field[29]. In this case of the minimal models, where we have that the symmetry group is abelian, the boundary conditions (4.13) are a consistent set of boundary conditions.

More generally, when the subgroup is not abelian, things become more complicated. We have to restrict to the maximal torus of the subgroup which in the field theory language
we can always do because the holonomy defines a map from the fundamental group of the torus to the subgroup, which can always be conjugated into the Cartan sub–algebra. Some care must be exercised to make sure that the conjugation and Weyl freedoms left over can be properly accounted for, if working directly with the holonomy, as in ref.\[29\]. In order to define the continuous orbifold (defined here) correctly, we note\[30\] that the appropriate projection operators generalising (4.11) contain only $J^0$, i.e. those currents in the Cartan sub–algebra of $H$, due to the requirement of compatibility with the BRST conditions (4.8). Therefore, the care needed with the holonomy sectors will again be carried out correctly by the methods in this section.

The doubly twisted partition function (4.14) naturally splits into a product of independent pieces, only related through the $U(1)$ twists:

$$Z(\tau, \theta, \bar{\theta}) = Z_{SU(2)_k}(\tau, \bar{\theta})Z_{SO(2)_1}(\tau, Q\theta, \bar{\theta})Z_{SO(2)_{-(k+2)}}(\tau, -\theta, \bar{\theta})Z_{\text{ghosts}}(\tau) .$$  \hspace{1cm} (4.16)

In the operator language, these components can be expressed in terms of characters of integrable representations for the different affine Lie algebras.

The $SU(2)_k$ WZW model component consists of a modular invariant combination of characters

$$Z_{SU(2)_k}(\tau, \bar{\theta}) = \sum_{L, \bar{L}=0}^k \chi_{L;k}(\tau, 0)N_{L\bar{L}} \chi_{\bar{L};k}(\tau, \bar{\theta}) ,$$  \hspace{1cm} (4.17)

where the integral matrices $N_{L\bar{L}}$ have been classified in ref.\[42\]. In general, for simply connected groups, the WZW path integral yields the diagonal modular invariant \[29\]. We notice, however that the ‘continuous orbifold’ considerations should work for more general invariants. We therefore shall work with more general $N_{L\bar{L}}$ than the diagonal case in what follows.

The $SO(2)_1$ component is the usual partition function for a compactified boson of radius one (see e.g. ref.\[43\])

$$Z_{SO(2)_1}(\tau, Q\theta, \bar{\theta}) = \sum_{m,n \in \mathbb{Z}} q^{\frac{1}{2}(m^2+n^2)} e^{2\pi Q\theta(m^2+n^2)} \frac{\eta(\tau)}{\eta(\tau)} .$$  \hspace{1cm} (4.18)

Finally, the non unitary boson and the ghosts yield a contribution

$$Z_{SO(2)_{-(k+2)}}(\tau, -\theta, \bar{\theta})Z_{\text{ghosts}}(\tau) = \sum_{\tilde{m}, \tilde{n}} q^{\frac{\tilde{m}^2}{k+2}} e^{-2\pi \theta \tilde{m}} \frac{\eta(\tau)}{\eta(\tau)} ,$$  \hspace{1cm} (4.19)

At this stage we do not have enough information about the non unitary theory to completely specify its spectrum $\tilde{m}, \tilde{n}$. We will adopt a two step strategy by first constraining the representations of the non unitary theory using the requirement of modular invariance of the final partition function, and then seeking agreement with the elliptic genus which we obtained using field theoretical methods in Section 3.
4.3. The (0,2) Modular Invariants

Before performing the integral over the twist, it is convenient to decompose the affine SU(2) characters so as to single out their U(1) dependence, according to

$$\mathcal{X}^{SU(2)}_{L; k}(\tau, \theta) = \sum_{M=0}^{2k-1} C^L_M(\tau) \Theta_M; k(\tau, \theta).$$

(4.20)

The SU(2) level k string functions $C^L_M(\tau)$ are defined in ref.[44]. They vanish for $L - M \not\in 2\mathbb{Z}$. The generalised $\Theta$–functions are defined as:

$$\Theta_{m; k}(\tau, z) = \sum_{n \in \mathbb{Z}} e^{2i\pi k(n + m/2k)} e^{2i\pi zk(n + m/2k)}.$$

(4.21)

The twist integral on the left–moving sector involves only the two bosons and is particularly simple

$$\frac{1}{2\pi} \int d\theta e^{2i\pi \theta [Q(\frac{m}{2} + n) - \tilde{m}]} = \delta_{Q(\frac{m}{2} + n) - \tilde{m}, 0}.$$

(4.22)

The right–moving sector integral is quite similar, but this time includes a contribution from the $\Theta$–function in the affine SU(2) character

$$\frac{1}{2\pi} \int d\tilde{\theta} e^{-2i\pi \tilde{\theta} [k(\tilde{\tau} + \frac{M}{2k}) + (\frac{m}{2} - n) + \tilde{n}]} = \delta_{k(\tilde{\tau} + \frac{M}{2k}) + (\frac{m}{2} - n) + \tilde{n}, 0}.$$

(4.23)

We choose to solve these two constraints in terms of the integers $m, n$. This means that the final partition function is non–vanishing only when a solution exists, that is when $\tilde{m}/Q, \tilde{n} + \tilde{m}/Q + M/2 \in 2\mathbb{Z}$. Assuming this is true, the partition function (4.15) is therefore of the form

$$Z(\tau) = \sum_{\tilde{m}, \tilde{n}} \sum_{L, L' = 0}^{k} \sum_{M=0}^{2k-1} \mathcal{X}_{L}(\tau, 0) N_{L L'} C^T_{M}(\tau) \Theta^T_{M(k+2) + k\tilde{n}; k(k+2)}(\tau/2, 0).$$

(4.24)

This expression is not too surprising since we expect to have an affine SU(2) algebra in the left–moving sector, and so we get SU(2) characters. We also expect an $N = 2$ superconformal symmetry for the right–moving sector, where we see that we have obtained expressions close to $N = 2$ characters[43], were it not for the constraints on $M$ deriving from the existence of a solution to (4.22)–(4.23).

Now we can impose the easier requirement of modular invariance, the $T$ invariance. The $q, \tilde{q}$ dependence in (4.24) is quickly found to be

$$\frac{L(L+1)}{4(k+2)} - \frac{1}{4} + \frac{\tilde{q}}{q^{4(k+2)}} - \frac{1}{2} + \frac{(\tilde{\tau}/Q)^2 - (\tilde{\tau}/Q)^2}{2} + 2\mathbb{Z}.$$

(4.25)
Since the part of the partition function which reflects the $SU(2)$ symmetry is modular invariant, we can neglect the $L, \overline{L}$ dependence in (4.25). Thus, $T$ invariance is satisfied when

$$\left( \frac{\tilde{m}}{Q} \right)^2 - \left( \frac{\tilde{n}}{Q} \right)^2 \in 2\mathbb{Z}. \quad (4.26)$$

So in the expression (4.24), the variables which we still need to properly constrain are $\tilde{m}, \tilde{n}$ but we also have a constraint on $\overline{M}$ coming from (4.23). Taking the $T$ invariance condition (4.26) into account, it is convenient then to introduce new unconstrained integers $m, a_m, b$ related to the old variables by

$$\frac{\tilde{m}}{Q} = \frac{m}{2}, \quad \frac{\tilde{n}}{Q} = \frac{m}{2} + a_m, \quad \overline{M} = 4b - m(1 + Q) - 2a_mQ, \quad (4.27)$$

where for even $m, a_m \in 2\mathbb{Z}$, and for odd $m, a_m \in \mathbb{Z}$. At this stage, this ensures the $T$ invariance of the partition function. The problem now is the $S$–invariance of

$$Z(\tau) = \sum_{m, a_m, L, \overline{L}} \sum_{b=0}^k \mathcal{X}_L(\tau) N_{LL} \overline{C}_L(4b - m(1 + Q) - 2a_mQ) \overline{\Theta}_{4b - mQ(2 - 1)}; kQ^2(\tau), \quad (4.28)$$

which we will achieve by carefully choosing the sums over $m, a_m$.

As the requirement of $S$ invariance is as usual much more difficult to study, the details of the computation are in Appendix B. Having imposed $S$ invariance, we found several modular invariants which can be written as follows:

For $Q$ odd,

$$Z_1(\tau) = \sum_{L, \overline{L}=0}^k \sum_{b=0}^{Q^2-2} \sum_{m=0}^{Q-1} \sum_{v=0}^{3} \mathcal{X}_L^{su(2)}(\tau) N_{LL} \overline{C}_L(4b)(\tau) \overline{\Theta}_{4bQ^2 - (4m-v)Q(Q^2-1); kQ^2(\tau)},$$

$$Z_2(\tau) = \sum_{L, \overline{L}=0}^k \sum_{b=0}^{Q^2-2} \sum_{m=0}^{Q-1} \mathcal{X}_L^{su(2)}(\tau) N_{LL} \overline{C}_L(\tau) \overline{\Theta}_{4bQ^2 - (4m-v)Q(Q^2-1); kQ^2(\tau)}$$

$$+ \sum_{v=1,3} \mathcal{X}_L^{su(2)}(\tau) N_{LL} \overline{C}_L(\tau) \overline{\Theta}_{(4b-2)Q^2 - (4m-v)Q(Q^2-1); kQ^2(\tau)}; \quad (4.29)$$

and for $Q$ even,

$$Z_\pm(\tau) = \sum_{L, \overline{L}=0}^k \sum_{b=0}^{Q^2-2Q-1} \sum_{m=0}^{Q-1} \sum_{v=0}^{3} \mathcal{X}_L^{su(2)}(\tau) N_{LL} \overline{C}_L(4b\mp v)(\tau) \overline{\Theta}_{(4b\mp v)Q^2 - (4m-v)Q(Q^2-1); kQ^2(\tau)}; \quad (4.30)$$

20
Here $N_{LL}$ are all the $SU(2)$ modular invariants of the $A$ and $D$ series, see equation $(B.8)$.

In order to extract some understanding of the physical content of our modular invariants, we need to assemble them into a form which facilitates comparison with the $(l, q, s)$ notation familiar in works on the $(2, 2)$ minimal models[17]. We concentrate on the $Q$ even case. Making use of $k + 2 = 2Q^2$ and (trivially) reversing a sign in $4m - v$ we extend the sum on $b$ up to $k - 1$, and change the level for the $\Theta$ functions, using the relations:

$$\Theta_{m;k}(\tau) = \Theta_{2m;2k}(\frac{\tau}{2}) \quad \text{and} \quad \Theta_{m;k}(\frac{\tau}{2}) = \Theta_{m;2k}(\tau) + \Theta_{m+2k;2k}(\tau).$$  \hfill (4.31)

After this, we get:

$$Z_{\pm}(\tau) = \sum_{L,\bar{L}} \sum_{b=0}^{k-1} \sum_{m=0}^{Q-1} \sum_{v=0}^{3} \chi_{LL}^{su(2)} N_{LL}^{\bar{L}}(4b \pm v)(k+2) - (4m + v)Qk;2k(k+2)(\tau)$$  \hfill (4.32)

We now compare the $\overline{C\Theta}$ combination with a character in[17]:

$$\chi_{l,q,s}^l = \sum_{b=0}^{k-1} \frac{\theta_{Q}}{Q(4b + q - s)} \Theta_{2q + (4b - s)(k+2);2k(k+2)}(\tau)$$  \hfill (4.33)

An obvious identification with our expressions is:

$$q - s = \pm v \text{ mod } 2k,$$

$$2q - s(k + 2) = \pm v(k + 2) - (4m + v)Qk \text{ mod } 4k(k + 2)$$  \hfill (4.34)

which gives:

$$s = vQ \mp v \text{ mod } 4, \quad q = (4m + v)Q \text{ mod } 2(k + 2).$$  \hfill (4.35)

Of course, $s$ is only defined modulo 4, since we can always absorb a factor $4\mathbb{Z}$ in the $b$ sum. As $Q$ is even, this means that $v = 0, 2$ describe the Neveu–Schwarz sector and $v = 1, 3$ the Ramond sector.

We therefore obtain the modular invariants $(4.32)$ in the $l, q, s$ notation:

$$Z_{\pm}(\tau) = \sum_{L,\bar{L}} \sum_{m=0}^{Q-1} \sum_{v=0}^{3} \chi_{LL}^{su(2)} N_{LL}^{\bar{L}}(4m + v)Q;vQ\mp v(\tau)$$  \hfill (4.36)

Recall that only even $\bar{L}$ occur in the NS sector, and only odd $\bar{L}$ in the R sector.

We have yet to discover the physical content of $Z_{\pm}$. In general, the sole requirement of modular invariance does not necessarily select a combination of characters that represents the partition function. There are two combinations of boundary conditions that are modular invariant:

$$\hat{Z}_{\pm} = \frac{1}{2}(A \quad A + P \quad A + A \quad P \mp P \quad P).$$  \hfill (4.37)
We use the convention that the Neveu–Schwarz (Ramond) sector is labeled by a $A$ ($P$) horizontally; the vertical $P$ denotes the additional insertion $(-1)^F$ in the trace. In particular, $\hat{Z}_\pm$ have the same NS sector, but a different R sector. The same feature is observed in (4.36) for $Z_{\pm}$.

The difference $\hat{Z}_+ - \hat{Z}_- = \text{Tr}_R (-1)^F$ is also a modular invariant, and corresponds to the elliptic genus (3.2), with $\gamma_L = 0$. The elliptic genus for our coset models has been computed in (3.8), and it is a straightforward exercise to verify that

$$Z_Q^{(0,2)}(q, 0, 0) = (-1)^{\frac{Q}{2}} (Z_-(\tau) - Z_+(\tau)).$$  (4.38)

The details are contained in Appendix C. This implies that for $Q \in 4\mathbb{Z} + 2$, the partition function of the coset models is $Z_+(\tau)$ in (4.36).

In the calculations above, we have not explicitly taken into account that there are extra $U(1)$ symmetries which commute with the Hamiltonian. These correspond to the $U(1)$ of the $(N = 2)_R$ and the $U(1)$ of the $SU(2)_L$. We should label all of our states by their charges under these $U(1)$'s. In order to discover these labels, we could recompute our partition function as above but with particular attention paid to the combinations of currents $(J^3, J, I)$ which are orthogonal to the gauging currents. These will give rise to the $U(1)$'s we seek. Alternatively, we can simply note that a bonus of working with the $\Theta$–functions is that the $U(1)$ dependence may be extracted at any point due to the unique extension to generalized $\Theta$–functions defined for example in eqn.(4.21). This amounts to restoring the familiar $U(1)$ dependence of the $N = 2$ and $SU(2)$ characters in expressions (4.36).

5. Some Four Dimensional String Theories

5.1. Construction of Heterotic String Theories

With the $(0,2)$ minimal models' partition functions in place, we are now ready to investigate what we may learn about four dimensional string theories constructed from them. As usual, we would like to begin by taking the tensor product of various copies of the $(0,2)$ minimal models such that the total (internal) central charge is equal to 9, on both the left and right. With the $(2,2)$ minimal models (choosing the $A$–series) the number of ways of doing this is 168. With our $(0,2)$ minimal models alone, there is only one way! This is due to the fact that the level $k$, given by the anomaly equation $k = 2(Q^2 - 1) = 6, 16, 30, 48, 70, 96 ...$ grows rapidly because $Q$ is restricted to being an integer. So we may only construct the $(k = 6)^4$ model with four $(0,2)$ minimal models. However, it only suffices to include a single $(0,2)$ minimal model among a product of ordinary $(2,2)$ models to produce a $(0,2) c = 9$ compactification, and so the number of such
models we can make is considerably greater than one, using this procedure of ‘doping’ the (2,2) models with (0,2) models.

As the central charges of the left and right parts of the internal theories are the same, the procedure for constructing a heterotic string theory from minimal models is much the same as originally presented by Gepner[17]. The fact that we have written our lowest lying states in terms of $N = 2$, $(l,q,s)$ indices makes it straightforward to carry out the two important procedures: Aligning the boundary conditions in each theory and the generalized GSO projection. These result in a world sheet $(N = 2)_R$ theory with odd integral right $U(1)$ charges and hence $N = 1$ spacetime supersymmetry, the spacetime supercharge arising from the worldsheet $(N = 2)_R$ spectral flow operator. We worked in the light cone gauge, including therefore the two transverse bosons $\partial z X^i$ and their superpartners (which form affine $SO(2)$) on the right, for $cR = 12$, and on the left the two transverse bosons $\partial z X^i$ together with the 26 additional fermions (forming affine $E_8 \times SO(10)$) needed for $cL = 24$, giving us a modular invariant critical heterotic string theory.

In order to construct a modular invariant partition function for the tensor of the internal (0,2) minimal models which also preserves $N = 2$ super symmetry on the world sheet, we need to align the boundary conditions of the various theories. Following the approach in ref.[46], we have to identify the following components in the partition function:

$$NS^+ = A_A, \quad NS^- = P_A, \quad R^+ = A_P, \quad R^- = P_P.$$ (5.1)

We already know one of them, the elliptic genus $R^- = Z_+ - Z_-$ for $Q \in 4\mathbb{Z} + 2$. The other Ramond contribution is also easy to compute since $R^+ = (Z_+ + Z_-)_R$. To get the Neveu–Schwarz contributions, we make use of the $S$ modular transformation

$$(Z_+)_NS = \frac{1}{2}(NS^+ + NS^-) \overset{S}{\rightarrow} \frac{1}{2}(NS^+ + R^+)$$ (5.2)

from which we can deduce $NS^+$. $NS^-$ follows either by applying the $T$ modular transformation on $T : NS^+ \rightarrow NS^-$, or as $NS^- = 2(Z_+)_NS - NS^+$.

Performing these operations on (4.36), we get the buildings blocks for the (0,2) minimal

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8 The term is chosen because of the analogy with similar manipulations performed upon semiconductor materials to drastically change their band structure (spectrum) for the construction of novel electronic devices.
models partition function \((Q \in 4\mathbb{Z} + 2)\)

\[
NS^\pm = \sum_{\text{even } L} \chi_L N_{LT} \sum_{m=0}^{Q-1} \left( \sum_{\text{even } L} \chi_L^{4mQ;0} + \chi_L^{(4m+2)Q;2} \pm \chi_L^{4mQ;2} \pm \chi_L^{(4m+2)Q;0} \right)
\]

\[
R^\pm = \sum_{\text{odd } L} \chi_L N_{LT} \sum_{m=0}^{Q-1} \left( \sum_{\text{odd } L} \chi_L^{(4m+1)Q;Q-1} + \chi_L^{(4m+3)Q;Q-3} \right.
\]

\[
\pm \chi_L^{(4m+1)Q;Q-3} \pm \chi_L^{(4m+3)Q;Q-1} \right)
\]

From these expressions, the product partition function can be constructed as

\[
Z_{\text{prod}} = \frac{1}{2} \left( \prod_i NS^+_i + \prod_i NS^-_i + \prod_i R^+_i + \prod_i R^-_i \right).
\]

The product is over each component theory, including the \(SO(2)\) (spacetime fermions) and \(SO(10) \times E_8\) (current algebra fermions) contributions from the right and left respectively:

\[
NS^\pm_{g/ST} = (C_0 \pm C_2)(\overline{B}_0 \pm \overline{B}_2), \quad R^\pm_{g/ST} = (C_1 \pm C_3)(\overline{B}_1 \pm \overline{B}_3).
\]

where \(C_v\) stands for the character of \(SO(10) \times E_8\) representation \(C_v = (-1)^v D_v \cdot 1_{E_8}\). \(D_v\) is a \(SO(10)\) representation labeled by \((v) = (10, 16, 1, 16)\). \(B_v\) is the character of the \(SO(2)\) representation \((v) = (1, s, v, f)\).

After a little bit of algebra, the partition function can be expressed as (neglecting as usual the transverse bosons)

\[
Z_{\text{prod}} = \sum_{v=0}^{3} \sum_k \sum_{l_i, t_i = 0}^{Q-1} \sum_{d_i = 0}^{r_i + s_i, s_i > 2v} C_v \overline{B}_v + \sum_{r_i + s_i, s_i > 2v}^{\sum s_i} \prod_i^{\chi^{\text{ru}(2)} L_i L_i} N_{L_i L_i} \chi^{\text{ru}(2)} L_i \chi^{\text{ru}(2)} L_i \chi^{\text{ru}(2)} L_i + \chi^{\text{ru}(2)} L_i \chi^{\text{ru}(2)} L_i \chi^{\text{ru}(2)} L_i .
\]

The next step in constructing the heterotic string theory is to realize spacetime supersymmetry. This is done by projecting out all states which have other than odd integer right \(U(1)\) charge. Of course, this projection must be done in a modular invariant way, requiring the inclusion of twisted sectors. In particular this will build for us in spacetime the spin 3/2 gravitino, and simultaneously remove tachyons from the physical spectrum, leaving tachyons to contribute to the string theory only in loop amplitudes.

The three phenomena of modular invariance, odd–integer \(U(1)\) and spacetime supersymmetry are all crucially interlinked, of course, and are implemented by the familiar

\[9\] The \(E_8\) sector will only ever contribute a singlet here, playing its customary role as the ‘hidden’ gauge sector.
GSO projection. Under modular transformations, the characters of a spacetime supersymmetric model transform into sums of characters of a worldsheet $N = 2$ theory with only odd integrally charged $U(1)_R$ states present. This forms a unitary representation of the modular group, as can be checked explicitly. The most efficient way to carry out this orbifolding procedure is to work directly with the world sheet $N = 2$ spectral flow operator for the $c_R = 12$ system, noting that under its action relating the NS and $R$ sectors, the $U(1)$ charge changes by $c_R/6 = 2$. Starting with the modular invariant non–spacetime–supersymmetric partition function, one simply needs to generate all of the states which can be reached by the action of the flow operator (the twisted sectors of the orbifold) and project according to the charge condition in each sector. The fact that the right $U(1)$ charge changes by two under the action of the spectral flow guarantees that action of the spacetime supercharge is well defined on states in the theory.

The gauge symmetry arising from these models will arise from the left as usual as $E_8 \times SO(10) \times \tilde{G}$. All states in the model are singlets under the $E_8$, while $\tilde{G}$ is an enhanced gauge symmetry arising from the affine structures in the left part of the internal theory. Vertex operators for creation of spacetime vectors corresponding to gauge bosons of $\tilde{G}$ can be constructed as

$$V^\mu a = \langle 1 | J^a \cdot 1 \mid \psi^\mu \rangle \mid 1 \rangle .$$

(5.7)

The first contribution is the singlet from $SO(10)$, the second is a descendent of the vacuum of the internal theory under the affine current mode $J^a_{-1}$. This gives a state of left conformal weight 1. The third contribution $\psi^\mu$ is the $SO(2)$ vector with right conformal weight $1/2$ and charge 1, and the fourth is the NS vacuum of the internal theory. The operator $V^\mu a$ is thus an allowed massless vector in the theory, as $c_R = 12$ and $c_L = 24$. The action of spectral flow will fill out the enhanced gauge supermultiplet. Each constituent of the internal theory has either a $U(1)$ affine symmetry (if it is a $(2,2)$ minimal model) or an $SU(2)_k$ (if it is a $(0,2)$ minimal model at level $k$). Thus, the enhanced gauge symmetry group $\tilde{G}$ is a product of factors made up of $U(1)$’s and $SU(2)_k$’s. There is also the possibility of ‘accidental’ contributions to the enhanced gauge symmetry group occurring when there is a means of constructing a vertex operator for massless $SO(10)$ singlets which are spacetime vectors, as above, but now the descendants $J^a_{-1} \cdot 1 |$ are replaced instead by weight 1 states coming from the internal sector which are not descendents. We shall see this occurring in one of our examples.

5.2. Moving to $E_6$ Gauge Symmetry

There would seem to be the possibility that for a particular model with $c_L \text{int} = c_R \text{int} = 9$ the $SO(10)$ representations and representations of a diagonal $U(1)$ subgroup of $\tilde{G}$ might fill out complete $E_6$ representations, as happens for the $(2,2)$ models. For this to happen there must exist an operator in the theory on the left which acts as a ‘spectral flow’
operator, this time relating the various $SO(10) \times U(1)$ representations. The presence of such an operator is of course guaranteed in the case where the internal theory was built out of a $(2,2)$ model, as the internal $(N = 2)_L$ has a spectral flow operator from which such an object is built. (Indeed, the analogue of the gravitino in this case is the gaugino transforming as the anti–spinor $\bar{16}$ of $SO(10)$ with internal $U(1)$ charge $3/2$.) In the realm of $(0,2)$ models, it is not necessary that such an operator exists, as we see in half of the examples we present below. However, it is interesting to see that one can construct such an operator and use it to enhance $SO(10)$ times a diagonal $U(1)$ to $E_6$ if one so desired. One can choose a suitably normalised $U(1)$ subgroup of $\tilde{G}$ for this purpose, normalising the currents $J(z)$ such that $J(z)J(w) \sim 3/(z - w)^{-2}$. By bosonising this current according to $J = i\sqrt{3}\partial \varphi$ one can rewrite all fields $f_q$ in the theory with charge $q$ under this $U(1)$ as the product $f_q = f \cdot \exp(iq\varphi/\sqrt{3})$. The spectral flow operator we need is simply the action of $Q_{\text{int}} = \exp(i\sqrt{3}\varphi/2)$ in the internal sector which has conformal dimension $3/8$ and charge $3/2$. The total (weight 1) spectral flow operator for the left is made by multiplying this by the $\bar{16}$ from the $SO(10)$ theory, which has weight $5/8$ and $U(1)$ charge $1/2$. This state, the $\bar{16}_{3/2}$, and its conjugate $16_{-3/2}$ (after appropriate dressing from the right states) forms part of the gauge supermultiplet of $E_6$ via the decomposition $\mathbf{78} = \mathbf{1}_0 + \mathbf{16}_{3/2} + \mathbf{16}_{-3/2} + \mathbf{45}_0$, where $\mathbf{45}$ and $\mathbf{1}$ denote the adjoint and singlet of $SO(10)$. Notice that the action of the total spectral flow operator again changes the total $U(1)$ charge of a state by 2. To arrive at an $E_6$ model we can simply construct this operator and use it to project onto even integer $U(1)_L$ in an analogous procedure to that carried out on the right for spacetime supersymmetry.

In the examples we study here, we use our $SO(10) \times \tilde{G}$ models as a starting point. Each $SU(2)_6$ constituent of the left part of the internal theory has a $U(1)$ subgroup which contributes to the left spectral flow operator. Acting with the operator on the $SU(2)$ pieces has the effect of isolating the parafermion piece while modifying the $U(1)$ contribution from which $SU(2)$ is made. The total internal $U(1)$ current is given by $J_{\text{int}} = \prod_i J_i + 1/2 \prod_i J_i^3$, where the $(2,2)$ minimal models each contribute a $J$ and the $(0,2)$ minimal models each supply a $J^3$ from their Cartan subalgebra. The result of twisting the $SU(2)$ contribution from a $(0,2)$ minimal model can be summarised succinctly by decomposing the usual character (4.20) into (recall that we use $J^3/2$ for the $U(1)$)

$$\mathcal{X}_{l;\ell}(\tau, \frac{z}{2}) = \sum_{m=0}^{4k-1} C^l_m(\tau) \Theta_{2m;4k}(\tau, \frac{z}{4}) = \sum_{m=0}^{4k-1} Y^l_{m;n=2m}(\tau, z). \quad (5.8)$$

10 There are models in the literature where such an operator is present, however. See for example refs. 47, 6, 7, 12.

11 Projecting onto even integer of course, as we would like to retain for example the graviton, which has charge zero. Note that this projection is automatic in the $(2,2)$ case for the reasons described in the text.
This is similar to the decomposition of a full $N = 2$ character into $\mathcal{X}_{q,s}^l$ functions. States described by the $Y_{m;n}^l$ have dimensions and charges

$$\Delta = \frac{l(l + 2)}{4(k + 2)} - \frac{m^2}{4k} + \frac{n^2}{16k} + \mathbb{Z}$$

$$Q = \frac{n}{8} + 2\mathbb{Z}$$

(The familiar $SU(2)$ states are those for $n = 2m$). The action of the twist amounts simply to $Y_{m;n}^l \rightarrow Y_{m;n+3}^l$. It is easy to check that this twist (flow) action is to change the dimension and charge of a state by $q/2 + c/24$ and $c/6$ respectively, for each model, where $c = 9/4$, which is precisely the same as for the action of a spectral flow operator on an $N = 2$ state. Therefore for the action of the total flow operator, the $U(1)$ charge of this non–supersymmetric internal theory is again changed by $3/2$ as for an $N = 2$ supersymmetric model! Once we combine this operator with the current algebra sector, its action will again change the total $U(1)$ charge by $2$. Using this flow operator, we can obtain modular invariant partition functions for heterotic string theory with a linearly realised $E_6$ by a constructive method exactly analogous to that described above to realise spacetime supersymmetric models using the spectral flow operator of the right $N = 2$.

The result of projecting onto even $U(1)$ charge is that the $SU(2)$ factors from each theory get broken to $U(1)$. This is easy to see, as the descendents (discussed above) which make their gauge bosons all now have the wrong charge under the projecting $U(1)$. All that is, except the abelian contribution. In this way, we constructed a family of $E_6 \times U(1)^3$ models, presented below. This procedure, like any of those described above, is easily generalised to different situations, for example the numerous ‘doped’ models which can be made.

We now go on to describe the models which we studied using all of the methods we described above.

5.3. Four $SO(10) \times \tilde{G}$ examples

Let us start by considering the string theory where all the four factors in the internal theory are $A$–series $(0,2)$ minimal models at $k = 6$. We shall call this the (0000) model\textsuperscript{13}. The computation of the massless spectrum is obtained by the application of the procedures described above and the results of which can be found below. (For simplicity, in listing the massless matter content of a model we shall mention only the number of (spacetime) scalars

\textsuperscript{12} This of course will readily generalise to the case when the central charges of the internal theory are not equal, giving linear realisations of $SO(10)$ and $SU(5)$, for example\textsuperscript{17}.

\textsuperscript{13} Our notation shall be of the form $(WXYZ)$ where a letter is a ‘0’ for a $(0,2)$ minimal model and a ‘2’ for a $(2,2)$ minimal model. The pattern of constituent models can thus be easily read. We shall use $(WXYZ)_{E_6}$ to refer to the models which have $E_6$ gauge symmetry.
in each sector, specifically those which are the superpartners of (spacetime) right–moving fermions. \( N = 1 \) spacetime supersymmetry and CPT invariance are of course present in these consistent models, and therefore the reader may deduce the rest of the content of massless matter sector—superpartners and antiparticles—at their leisure. Consulting tables 5.1 will yield more details of the spectrum in each example.)

So for the (0000) model we have

- 4 scalars in the 16 \( SO(10) \).
- 6 scalars in the 10 of \( SO(10) \).
- 13 scalars which are singlets of \( SO(10) \).

In addition there are the gauge degrees of freedom corresponding to the gauge symmetry \( SO(10) \times SU(2)^4 \).

Let us consider the \( SO(10) \) singlets. There are 13 scalars, together with their fermionic superpartners and all of the antiparticles. An interesting and important question is whether any of them are moduli, as with this knowledge we may begin to understand if there is any way to reach a sigma–model with spacetime geometrical interpretation from these models. To answer this, we need to check for the existence of exactly flat directions. As we know the partition function we could in principle compute various correlation functions to establish the presence of such exactly marginal deformations. However, the existence of an R–symmetry, the quantum symmetry appearing in the GSO projection in labeling the twisted sectors, sometimes helps us to argue for flatness of untwisted singlets\(^{14}\). Following \([48]\) we have \( R = \exp(2\pi i N/4) \) where the order of the discrete symmetry\(^{15}\) associated to the GSO projection is in this case, 4. As the spacetime superpotential transforms with charge \(-2 \mod 16\) we need to make sure that there are no couplings involving the \( SO(10) \) singlets \( \Phi_i, i = 1, ..., 13 \) of the form \( f(\Phi_i)\Phi_j^{s \neq i} \) for \( s = 0, 1 \) where \( f(\Phi_i) \) is a function of untwisted singlets only while \( \Phi_j \) is any \( SO(10) \) singlet. For this example, there is one singlet which has charge \(-2 \mod 4\), and its presence thus prevents us from arguing for the presence of flat directions preserving \( SO(10) \). However, by examining the charges of all of the fields under the enhanced gauge symmetry \( SU(2)^4 \), we see that there are 6 \( SO(10) \times SU(2) \) singlets from the untwisted sector which cannot form couplings in the superpotential which would spoil flatness. So we have six moduli which preserve \( SO(10) \times SU(2) \). Further analyses of this type, using R–symmetry and extended gauge symmetry may reveal further flat directions. It should be noted however, that once these methods are exhausted, it could well be that the possibility of couplings between \( SO(10) \) singlets that would ruin flatness

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\(^{14}\) This procedure has been used in arguing the flatness of \( E_6 \) singlets in certain Landau-Ginzburg orbifolds [7].

\(^{15}\) Note that for the ‘doped’ models it is the more familiar \( 2(k + 2) = 16 \). While a single \((2, 2)\) minimal model has a \( \mathbb{Z}_{k+2} \times \mathbb{Z}_2 \) symmetry associated with it, a single \((0, 2)\) minimal model has \( \mathbb{Z}_{2Q} \).
does not happen, simply because the couplings vanish of their own accord due to details of the conformal field theory. More work is needed to determine whether this is indeed the case. See table 5.1a for a summary of how the scalar singlets fall into the twisted sectors. (Note that the occupancy of the other twisted sectors can be deduced from supersymmetry.)

Although it was pointed out above there is only one choice of a \((k = 6)^4\) tensor product of purely \((0,2)\) minimal model factors, we have the possibility of replacing one (or more) of these by the standard \((2,2)\) minimal models, a procedure we referred to as ‘doping’. The result of all these computations, carried out in the same fashion as above, can be found in tables 5.1b–d at the end of this section. Notice that the number of singlets increases with the amount of doping by \((2,2)\) minimal models, as does the number of \(16\)’s and \(10\)’s. In all cases, there are singlets of charge \(-2\) under the \(\mathbb{Z}_{16}\) R–symmetry, but upon requiring preservation of some of the enhanced gauge symmetry, it is readily seen that at least some of those from the untwisted sector are moduli.

In addition there are massless vectors corresponding to the gauge symmetry \(SO(10) \times SU(2)^2 \times U(1)^2\) for the \((2200)\) model and \(SO(10) \times SU(2) \times U(1)^3\) for the \((2220)\) model. In the case of the \((2000)\) model, however, the gauge symmetry is \(SO(10) \times SU(2)^3 \times U(1)^3\). The two extra ‘accidental’ \(U(1)\)’s arise because there are two additional \((1,0)\) currents in this model which are not related to descendent states under the affine symmetry of any of the individual models.

5.4. Four \(E_6 \times U(1)^3\) examples

As described above, we took the models of the previous section and constructed four \(E_6\) \((0,2)\) string vacua from them, by acting with a left spectral flow operator. The particle content of these models is listed in table 5.2. The \((0000)_{E_6}\) model, which has a \(\mathbb{Z}_4\) R-symmetry, has

- 23 scalars in the \(27\) \(E_6\).
- 7 scalars in the \(\overline{27}\) of \(E_6\).
- 191 scalars which are singlets of \(E_6\).

For this model there are singlets of charge \(-2\) under the \(\mathbb{Z}_4\) R–symmetry requiring us to find more powerful methods of determining the moduli, as discussed in the previous subsection.

Turning to the ‘doped’ models, we find a number of interesting facts. First, the \((2000)_{E_6}\) model has the same spectrum as the \((0000)_{E_6}\) model! The projection to realise \(E_6\) has resulted in two equivalent models, thereby negating the effect of the doping. Examining the details of the spectrum, we see that \((2000)_{E_6}\) has naively a \(\mathbb{Z}_{16}\) R–symmetry. However, if we relabel the \(N\)th twisted sector as \(N\) mod 4, thus realising an \(\mathbb{Z}_4\) symmetry, then the two spectra are indeed identical.
The \((2200)_{E_6}\) model, which has \(\mathbb{Z}_{16}\) R–symmetry, has

- 49 scalars in the 27 of \(E_6\).
- 9 scalars in the \(\overline{27}\) of \(E_6\).
- 251 scalars which are \(E_6\) singlets.

Again, there are singlets of charge \(-2\) under the R–symmetry.

The \((2220)_{E_6}\) model, which also has \(\mathbb{Z}_{16}\) R–symmetry, has a spectrum which is \textit{identical} to that of the familiar \((2222)_{E_6}\) \((2,2)\) Gepner model! This time, the projection realising \(E_6\) has negated the doping. Indeed, upon examination of the explicit form of the partition function in terms of the various characters, (appropriately projected) we have been able to show that they are indeed equivalent models. The spectrum is:

- 149 scalars in the 27 of \(E_6\).
- 1 scalar in the \(\overline{27}\) of \(E_6\).
- 503 scalars which are singlets under \(E_6\).

There are some general features to remark upon. In all cases the gauge group is \(E_6 \times U(1)^3\), the \(SU(2)\) factors having been broken by the projection. Further to this is the increase with the degree of (effective) doping in the net number of chiral generations (number of 27’s minus number of \(\overline{27}\)’s) \((16, 40, 148)\) for \((0,2000)_{E_6}\), \((2200)_{E_6}\) and \((2220,2)_{E_6}\), respectively. Also increasing is the number of singlets, \((191,251,503)\). In addition, the number of singlets in the untwisted sector keeps increasing with respect to the number in any other single sector.

Whether these features will persist in other doping examples is not known. It is also not known whether they are of any significance. These questions will probably be answered when methods for determining the full set of moduli of these models are uncovered. The unexpected fact that in these examples, doping with either just one \((0,2)\) or just one \((2,2)\) minimal model has no effect after projecting to recover \(E_6\) is interesting. This feature appears to be present in other examples. However, it is not known what the general pattern is when there is more than one \((0,2)\) factor present, or if the levels are not all the same. Further investigation into these matters is continuing.
Table 5.1a: The spectrum for the massless matter fields (spacetime scalars which are lowest components of chiral superfields) in the (0000) model, in Nth twisted sector of the GSO projection. There are 6 $16$’s, 4 $10$’s and 13 $1$’s of $SO(10)$ respectively. The corresponding superpartners and antiparticles are obtained by spectral flow and CPT–conjugation respectively.

| $N$ | 10 | 16 | 1 |
|-----|----|----|---|
| 0   | 6  |    | 12|
| 1   |    |    |   |
| 2   |    | 1  |   |
| 3   |    | 3  |   |
| 4   |    | 3  |   |
| 6   |    |    | 1 |
| 8   | 3  | 6  |   |
| 11  |    | 1  |   |
| 12  | 3  | 6  |   |
| 14  |    | 1  |   |
| 15  |    | 4  |   |

Table 5.1b: The spectrum for the massless matter fields in the (2000) model. There are 12 $16$’s, 8 $10$’s and 30 $1$’s of $SO(10)$ respectively.

31
**Table 5.1c:** The spectrum for the massless matter fields in the (2200) model. There are 22 \textbf{10}'s, 16 \textbf{16}'s and 73 \textbf{1}'s of \textit{SO}(10) respectively.

| $N$ | \textbf{10} | \textbf{16} | \textbf{1} |
|-----|-------------|------------|----------|
| 0   | 16          | 38         |          |
| 2   |             | 5          |          |
| 3   |             | 2          |          |
| 4   | 2           | 10         |          |
| 6   |             | 1          |          |
| 7   |             | 1          |          |
| 8   | 1           | 4          |          |
| 12  | 3           | 8          |          |
| 14  |             | 7          |          |
| 15  | 13          |            |          |

**Table 5.1d:** The spectrum for the massless matter fields in the (2220) model. There are 54 \textbf{16}'s, 44 \textbf{10}'s and 207 \textbf{1}'s of \textit{SO}(10) respectively.

| $N$ | \textbf{10} | \textbf{16} | \textbf{1} |
|-----|-------------|------------|----------|
| 0   | 51          | 138        |          |
| 2   | 3           | 22         |          |
| 3   |             | 1          |          |
| 4   |             | 9          |          |
| 6   |             | 1          |          |
| 8   |             | 3          |          |
| 14  |             | 34         |          |
| 15  | 43          |            |          |
Table 5.2a: The spectrum for the massless matter fields in the (0000)\(E_6\) model. There are 23 \(27\)'s, 7 \(\overline{27}\)'s and 191 1's of \(E_6\) respectively.

| \(N\) | \(27\) | \(\overline{27}\) | 1 |
|-------|--------|-----------------|---|
| 0     | 6      | 6               | 60 |
| 1     |        |                 |    |
| 2     | 1      | 1               | 51 |
| 3     | 16     |                 | 80 |


Table 5.2b: The spectrum for the massless matter fields in the (2000)\(E_6\) model. There are 23 \(27\)'s, 7 \(\overline{27}\)'s and 191 1's of \(E_6\) respectively. Note that when we consider the contributions from the twisted sector \(N \mod 4\) we get exact agreement with Table 5.2a, eg there are 16 generations from \(N = 3 \mod 4\).

| \(N\) | \(27\) | \(\overline{27}\) | 1 |
|-------|--------|-----------------|---|
| 0     | 6      |                 | 36 |
| 2     | 1      |                 | 20 |
| 3     | 3      |                 | 50 |
| 4     |        | 3               | 9  |
| 6     |        | 1               | 19 |
| 7     |        |                 | 15 |
| 8     |        |                 | 6  |
| 10    |        |                 | 3  |
| 11    | 3      |                 |    |
| 12    | 3      | 9               |    |
| 14    |        | 9               |    |
| 15    | 10     |                 | 15 |
| $N$ | $27$ | $\overline{27}$ | $1$ |
|-----|------|----------------|-----|
| 0   | 16   | 0              | 52  |
| 1   | 6    |                | 26  |
| 2   | 7    | 2              | 45  |
| 3   |      |                | 75  |
| 4   | 1    |                | 10  |
| 5   | 2    |                | 14  |
| 6   | 1    |                | 9   |
| 7   |      |                | 1   |
| 8   | 2    |                | 6   |
| 11  |      |                | 9   |
| 12  |      |                | 1   |
| 14  |      |                | 5   |
| 15  | 19   |                |     |

**Table 5.2c:** The spectrum for the massless matter fields in the $(2200)_{E_6}$ model. There are $49\ 27$’s, $9\ \overline{27}$’s and $251\ 1$’s of $E_6$ respectively.

| $N$ | $27$ | $\overline{27}$ | $1$ |
|-----|------|----------------|-----|
| 0   |      | 452           |     |
| 2   |      | 35            |     |
| 4   |      | 16            |     |
| 5   |      | 1             |     |
| 15  | 149  |               |     |

**Table 5.2d:** The spectrum for the massless matter fields in the $(2220)_{E_6}$ and $(2222)_{E_6}$ models. There are $149\ 27$’s, $1\ \overline{27}$’s and $503\ 1$’s of $E_6$ respectively.
6. Discussion and Outlook

Our goal in this paper has been to describe how to use the (0,2) minimal models (presented herein) as building blocks in constructing exactly soluble (0,2) string compactifications, much as Gepner did years ago for (2,2) models\[17\]. The correspondence between Gepner models and Calabi–Yau (or more precisely Landau–Ginzburg) compactifications\[19\] is an illuminating example of the profound connection between conformal field theory and spacetime geometry in string theory. It is therefore tempting to speculate about a similar connection between exactly soluble (0,2) models and (0,2) Calabi–Yau compactifications. Indeed, the authors of ref.\[12\] have provided tantalizing hints of such a connection between their soluble models and the (0,2) Landau–Ginzburg orbifolds described in ref.\[6\]. It is our hope that detailed study of the class of models described in this paper will yield further hints in this direction, which may point the way towards a direct construction connecting the soluble models to specific Calabi-Yau compactifications as in ref.\[19\].

There are many interesting avenues of further study which present themselves in this paper. One is the computation of the modular invariant partition functions of the more general $G/H$ heterotic cosets, corresponding to families of (0,2) generalisation of the Kazama–Suzuki models. The closest analogue among these to the (0,2) minimal models studied in this paper would be the case where the symmetry $g \to gh$ is gauged, where $g \in G$ and $h \in H$ and $G/H$ is Kahler. The modular invariant (0,2) supersymmetric partition function arising from this construction (after coupling in fermions and canceling anomalies) would have states on the right coming from the $N=2$ Kazama–Suzuki series, assembled into the character $X^{N=2}$, and on the left, there would be a $X^G$ character, corresponding to the affine $G$-symmetry on the left, together with states associated with an $SO(\dim G − \dim H)/H$ coset, coming from the left–moving fermions. These models would again have $c_L = c_R$, and it would be very interesting to study the spectrum of string theories which can be constructed out of this. Generically they would have gauge group $SO(10)$ together with a factor coming from any affine symmetries present in the internal theory, as before. An $E_6$ gauge group would again be realisable by using a left spectral flow operator.

Another avenue of investigation is to compute the spectra of all of the possible ‘doped’ models which can be made from tensor products of (2,2) minimal models and at least one (0,2) minimal model. Here, we have studied only the $(k = 6)^4$ cases, (with and without $E_6$) which is only a small subset of the possible (0,2) exactly soluble string vacua which can now be made using the methods of this paper.

Of course, the focus in this paper has been on $c_L = c_R$ compactifications. It would be interesting to use our methods to attack the problem of finding exactly solvable $c_L > 9$ (0,2) vacua to complement those presented in ref.\[12\].

Furthermore, it is possible to extend even further the class of solvable (0,2) models
by considering extra orbifolds, as is done for $(2, 2)$ models. This naturally led to the understanding of mirror symmetry via the Greene–Plesser construction\cite{11} using Gepner models. Work is in progress on whether such issues can be addressed in the $(0, 2)$ context\cite{4}. Preliminary results show that there are indeed orbifold relations (involving the aforementioned $\mathbb{Z}_{2Q}$ symmetry) analogous to those found for the $(2, 2)$ models. Until we have a better understanding of how the spectra we have computed relates to a possible geometrical description, it is not yet clear what the interpretation of such results will ultimately be.

This leads us to the last (but not least) point. There is the problem of finding powerful arguments to determine moduli among the gauge singlets in all of these models, sidestepping the labour-intensive brute force calculation of all correlators directly from the partition function. In the examples presented, we were only able to see moduli which preserved the enhanced gauge symmetries as well as the $SO(10)$ or $E_6$ gauge group. It would be of interest to find methods which can determine the moduli which preserve only the generic $SO(10)$ or $E_6$ gauge group. These are likely to be the generic gauge groups arising in the region of moduli space connected to a possible sigma model geometrical description. Such methods are going to become absolutely indispensable if some understanding of where these models lie in the moduli space of $(0, 2)$ compactifications is to be gained.

Of course, it is far from clear that one should expect the models described herein (and others in this class) to arise as special points in Calabi–Yau moduli spaces. Indeed, there are known examples of $N = 1$ supersymmetric compactifications where the internal space is taken to be an asymmetric orbifold \cite{50} which one does not expect to be related by smooth deformation to a more conventional geometry. Thus is could equally well be that our models are describing analogues of such asymmetric geometries, this time based on interacting conformal field theories instead of free field theory. These questions will be answered when we learn of more powerful ways of determining the moduli of the models.

**Note added**

After the appearance of this paper, we were informed by Ralph Blumenhagen and Andreas Wissskirchen that they have been able to reproduce the spectra of our $E_6$ examples using their simple current program.

\footnote{Results akin to this in the context of $(0, 2)$ linear sigma models have been presented in ref.\cite{52}.}
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Appendix A. Elliptic Genus as a sum of SU(2) affine characters

We show here that the elliptic genus (3.7) is

\[ Z_Q(q, \gamma, 0) = \sum_{r=0}^{Q-1} (-1)^r X_{Q-1+2Qr,k}(\tau, z), \quad (A.1) \]

where the complete affine SU(2) characters at level \( k \) are

\[ X_{l;k}(\tau, z) = \frac{\Theta_{l+1;k}(\tau, z) - \Theta_{l-1;k}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} = \sum_{m=0}^{2k-1} C_m^l(\tau) \Theta_{m;k}(\tau, z), \quad (A.2) \]

and the generalised \( \Theta \)-functions are defined in eqn. (4.21). The proof follows closely the one in ref. [39] for a similar situation in the (2,2) models. The goal is achieved by first expressing the elliptic genus (3.7) as a ratio of Jacobi theta functions, then showing that the ratio ‘sum over characters’ divided by ‘elliptic genus’ is an elliptic function and finally proving that this ratio is a modular invariant function, actually equal to one.

We rewrite the elliptic genus using the first Jacobi theta function [53]

\[ \Theta_1(\nu|\tau) = [2q^{1/4} \prod_{1}^{\infty} (1 - q^{2n})] \sin \pi \nu \prod_{1}^{\infty} (1 - q^n e^{2i\pi \nu})(1 - q^n e^{-2i\pi \nu}). \quad (A.3) \]

If we put respectively

\[ \nu = \frac{\gamma}{2\pi} \frac{1}{2Q}, \quad \nu = \frac{\gamma}{2\pi} \frac{1}{k + 2}, \quad (A.4) \]

then the numerator and denominator of the elliptic genus are nothing but theta functions (the square brackets cancel) and we get

\[ Z_\alpha(q, \gamma, 0) = \frac{\Theta_1 \left( \frac{\gamma}{2\pi} \frac{1}{2Q} | \tau \right)}{\Theta_1 \left( \frac{\gamma}{2\pi} \frac{1}{k + 2} | \tau \right)}. \quad (A.5) \]

We define the new variable

\[ u = \frac{\gamma}{2\pi} \frac{1}{k + 2} = \frac{\gamma}{2\pi} \frac{1}{2Q^2}, \quad (A.6) \]

and the elliptic genus (A.5) becomes

\[ Z(\tau, u) = \frac{\Theta_1(Qu|\tau)}{\Theta_1(u|\tau)}. \quad (A.7) \]

For later convenience, define the function \( K(\tau, z) \) as the sum of characters

\[ K(\tau, z) = \sum_{r=0}^{Q-1} (-1)^r X_{Q-1+2Qr;k}(\tau, z). \quad (A.8) \]
Now we look at the periodicity in the $u, z$ variables of the $Z, K$ functions. From standard properties of the Jacobi theta functions, $Z$ behaves as

\[ Z(\tau, u + 2) = Z(\tau, u), \quad Z(\tau, u + 2\tau) = e^{-4i\pi(Q^2 - 1)(\tau + u)}Z(\tau, u). \]  

(A.9)

Similarly, due to the properties of the affine $SU(2)$ characters, $K$ behaves as

\[ K(\tau, z + 2) = K(\tau, z), \quad K(\tau, z + 2\tau) = e^{-4i\pi(Q^2 - 1)(\tau + z)}K(\tau, z). \]  

(A.10)

Therefore the ratio

\[ R(\tau, z) = \frac{K(\tau, 2z)}{Z(\tau, 2z)} = A \prod_i \frac{\Theta_1(z - a_i|\tau)}{\Theta_1(z - b_i|\tau)} \]  

is an elliptic function of $z$ (of periods 1 and $\tau$), and as such can be expressed as a ratio of Jacobi theta functions, where $a_i, b_i$ are respectively the zeroes and poles of $R$ in the fundamental domain.

Next we show that this elliptic function $R(\tau, z)$ is modular invariant. From ref. [39], we have

\[ Z(\tau + 1, 2z) = Z(\tau, 2z), \quad Z(-1/\tau, -2z/\tau) = e^{4i\pi(Q^2 - 1)z^2/\tau}Z(\tau, 2z). \]  

(A.12)

Since the $SU(2)$ characters satisfy $X_{n,k}(\tau + 1, z) = X_{n,k}(\tau, z)$, we get

\[ K(\tau + 1, 2z) = K(\tau, 2z). \]  

(A.13)

For the $S$ transformation, recall that the $k + 1$ characters span a linear representation of the modular group

\[ X_{n,k}(-1/\tau, -z/\tau) = e^{i\pi k z^2/2\tau} \sqrt{2} \sum_{n' = 0}^{k} \sin \pi \frac{(n + 1)(n' + 1)}{k + 2} X_{n',k}(\tau, z). \]  

(A.14)

We have to show that our particular sum of characters (A.8) transforms into itself, up to an overall factor as in (A.12). After inserting the proper values for $k = 2(Q^2 - 1)$ and $n = Q - 1 + 2Qr$ in (A.14), one can prove that

\[ \sum_{r=0}^{Q-1} (-1)^r \sin \pi \frac{(1 + 2r)(n' + 1)}{2Q} = Q \sum_{r' \in \mathbb{Z}} (-1)^{r'} \delta_{n',Q-1+2Qr'}. \]  

(A.15)
So the $K$ transformation is
\[
K(-1/\tau, -2z/\tau) =
\]
\[
e^{4i\pi (Q^2 - 1)z^2/\tau} \sum_{n' = 0}^{2(Q^2-1)} \sum_{r' \in \mathbb{Z}} (-1)^{r'} \delta_{n',Q-1+2Qr'} X_{n';k}(\tau, 2z) \]
\[
= e^{4i\pi (Q^2 - 1)z^2/\tau} \sum_{r' = 0}^{Q-1} (-1)^{r'} X_{Q-1+2Qr';k}(\tau, 2z) \]
\[
= e^{4i\pi (Q^2 - 1)z^2/\tau} K(\tau, 2z). \quad (A.16)
\]
This is the same factor as for the $Z$ function, and consequently $R(\tau, z)$ is a modular invariant.

The last step uses the lemma of ref. [39] which states that an elliptic function, which is also modular invariant, has to be a constant. We are left to show that this constant is one, which is achieved by taking the $q \to 0$ limit in both $Z, K$.

From the elliptic genus expression we get (putting $y = \exp(i\gamma/2Q^2) = \exp(4i\pi z)$)
\[
Z(q = 0, 2z) = y^{-(Q-1)/2} \frac{1 - y^Q}{1 - y} = y^{-(Q-1)/2} \sum_{n=0}^{Q-1} y^n, \quad (A.17)
\]
and exactly the same result from the expansion of the $SU(2)$ characters at $q = 0$. Therefore the constant is one and the relation (A.1) is proven.

**Appendix B. Modular Invariance of (0, 2) models.**

In the partition function (4.24), the parameters at our disposal are $\tilde{m}, \tilde{n}$ but we also have a constraint on $\overline{M}$ coming from (4.23). Taking the $T$ invariance condition (1.20) into account, it is convenient then to introduce new unconstrained integers $m, a_m, b$ related to the old variables by
\[
\frac{\tilde{m}}{Q} = \frac{m}{2}, \quad \frac{\tilde{n}}{Q} = \frac{m}{2} + a_m, \quad \overline{M} = 4b - m(1 + Q) - 2a_mQ, \quad (B.1)
\]
where for even $m, a_m \in 2\mathbb{Z}$, and for odd $m, a_m \in \mathbb{Z}$. At this stage, this ensures the $T$ invariance of the partition function. The problem now is the $S$–invariance of this object
\[
Z(\tau) =
\]
\[
\sum_{m, a_m} \sum_{L, L = 0}^{Q^2-2} \sum_{b=0}^{Q^2-2} \mathcal{X}_L(\tau) N_{L,L} \overline{\Theta}_{4b-m(1+Q)-2a_mQ}(\tau) \Theta_{(4b-m)Q^2-Q(m+2a_m);kQ^2}(\tau) \]
\[
= \sum_{L, L = 0}^{k} \mathcal{X}_L(\tau) N_{L,L} R_{\overline{\Gamma}}(\tau) \quad (B.2)
\]
which we will achieve by carefully choosing the sums over \( m, a_m \). (See (A.2) and (4.21) for the definition of the SU(2) characters, string functions and the generalized theta functions.) Their range are determined by the various properties of the level \( k \) string function \( C_{m}^{L} = C_{m+2kZ}^{L} = C_{k-m}^{L} \) and theta function \( \Theta_{m+2kZ,k} = \Theta_{m,k} \). Together with (B.2), this implies that \( 0 \leq m < 4Q(Q-1) \) and \( 0 \leq a_m < 2Q(Q^2-1) \). Note also that the non vanishing of the string functions puts some restrictions on the allowed spins \( L \equiv m(1 + Q) \text{ mod } 2 \). When \( Q \) is odd, only even spins may occur.

Since the string functions and theta functions have a different level (and hence different index periodicity), it is convenient to rewrite

\[
m = m_0 + m_1 4(Q - 1), \quad 0 \leq m_0 \leq 4(Q - 1) - 1, \quad 0 \leq m_1 \leq Q - 1, \quad (B.3)
\]

and one sees that only the theta function index depends on \( m_1 \). Thus, \( a_m \) is constrained only by the value of \( m_0 \) and will be denoted \( a_0 \). Next we introduce the new variables

\[
v = m_0 + 2a_0, \quad u = m_0(1 + Q) + 2a_0Q = m_0 + vQ, \quad (B.4)
\]

which gives

\[
R_{L}(\tau) = \sum_{u,v,m_1} \sum_{b=0}^{Q^2-2} C_{(4b-u)Q^2-(4m_1-v)Q(Q^2-1);kQ^2}(\tau).
\]

Both \( u \) and \( v \) can be defined modulo 4, due to the sum over \( 4b \), the presence of \( m_1 \) and the periodicity of the theta function. Note, however, that they are not completely independent due to their very definition; we will return to this point when we study the various cases more explicitly.

In this appendix, we will show that the following ansatz

\[
Z(\tau) = \sum_{L,L=0}^{k} \sum_{b=0}^{Q^2-2} \sum_{m_1=0}^{Q-1} \sum_{u,v=0}^{3} X_{L}(\tau)N_{LL} C_{(4b-u)( treating the \( u \) and \( v \) sums, which depend on the parity of \( Q \), in order to ensure modular invariance of \( Z \). Originally, this Ansatz was found by first performing a \( S \) modular transformation on (B.2), then computing the \( b \) sum and adjusting the various sums in order to go back to the original expression (B.2).

A first simplification in (B.6) occurs thanks to the identity

\[
\sum_{a=0}^{Q-1} \Theta_{x-a4(Q^2-1);k;0} = \Theta_{x;k}(\tau,0), \quad (B.7)
\]
so that

\[ R_{T'}(\tau) = \sum_{u,v} \sum_{b=0}^{Q^2-2} \frac{C_{4b-u}(\tau)\Theta_{(4b-u)Q+v(Q^2-1);k}(\tau)}{b} . \]  

(B.8)

For the next steps, we will need the transformation properties under the \( S \) modular transformation of the \( SU(2) \) characters \( \chi_{L;k} \), theta functions \( \Theta_{n;p} \) and string functions \( C_{m;k}^L \), respectively,

\[
\chi_{L;k}(\frac{-1}{\tau}) = \sum_{L'=0}^{k} S_{LL'} \chi_{L';k}(\tau),
\]

\[
\Theta_{n;k}(\frac{-1}{\tau}) = \sqrt{-i\tau} \sum_{n'=0}^{2k-1} s_{nn'} \Theta_{n';k}(\tau),
\]

\[
C_{L+2m;k}(\frac{-1}{\tau}) = \frac{1}{\sqrt{-i\tau}} \sum_{L'=0}^{k} \sum_{m'=0}^{k-1} S_{LL'} C_{L'+2m';k}(\tau) s^*_{L'+2m',L+2m},
\]

where

\[
S_{LL'} = \sqrt{\frac{2}{k+2}} \sin \pi \frac{(L+1)(L'+1)}{k+2}, \quad s_{nn'} = \frac{1}{\sqrt{2k}} e^{-i\pi \frac{nn'}{k}}.
\]

(B.10)

Note that the \( s \) matrix depends on the level of the string or theta function.

Let us first study the \( SU(2) \) modular invariance. Recall that \( C_{M}^L \) vanishes unless \( L-M = 0 \mod 2 \). We cannot in general split the problem into its \( SU(2) \) part (\( L \) index) and its \( C\Theta \) part (\( M \) index), since the \( M \) index sum depends on the parity of the \( L \) index sum, as a consequence of (B.9). We will return to this as we study the individual cases in more detail. For the moment, it is useful to compute some partial sums on the \( SU(2) \) side.

From [12] we have for the \( A \) and \( D \) invariants,

\[
A_{k+1} : \sum_{l=0}^{k} \chi_l \chi_{l},
\]

\[
D_{k/2+2} : \sum_{l=0}^{k/4-1} \left| \chi_{2l} + \chi_{k-2l} \right|^2 + 2 \left| \chi_{k/2} \right|^2, \quad k \in 4\mathbb{Z}
\]

(B.11)

\[
D_{k/2+2} : \sum_{l=0}^{k/2-1} \left| \chi_{2l} \right|^2 + \sum_{l=0}^{k/2-1} \chi_{2l+1} \chi_{k-2l-1}, \quad k \in 4\mathbb{Z} + 2
\]

We see that the \( N_{L,T} \) are essentially of the form \( \delta_{L,T} \) or \( \delta_{k-L,T} \). Applying (B.9) to (B.0)
and summing separately over $L = 2l, 2l + 1$ we find (where $\epsilon = 0, 1$)

\[
\sum_{l=0}^{k/2-\epsilon} S_{L', 2l+\epsilon} S^*_{2l+\epsilon, \tau} = \frac{1}{2} (\delta_{L', \tau} + (-1)^\epsilon \delta_{k-L', \tau}) ,
\]

(B.12)

The next step is to study the behaviour of (B.8) under $S$, looking at the $M$ index part only

\[
R_L (\frac{-1}{\tau}) = \frac{1}{2kQ} \sum_{u,v} \sum_{b=0}^{Q^2-2} \sum_{M',N'=0}^{2k-1} e^{-i\pi b (4bQ^2 - u)} e^{i\pi N'(4b - u)Q + v(Q^2 - 1))} C_{M', \tau} (\tau) \Theta_{N'; k}(\tau) .
\]

(B.13)

Note that we have used that $C_{4b} = C_{4bQ^2}$ due to the periodicity $2k = 4(Q^2 - 1)$ of $C$. We first perform the sum over $b$,

\[
\sum_{b=0}^{Q^2-2} e^{-i\pi 4b (M'Q^2 - N' Q)} = (Q^2 - 1) \sum_{c=0}^{4Q^2-1} \delta_{N', M'Q+c(Q^2-1)/Q} .
\]

(B.14)

Since $Q^2 - 1$ and $Q$ are relatively prime, it must be that $\bar{c} = \overline{c} Q$. Next, by rewriting $\overline{c}$ as

\[
\bar{c} = \overline{c}_0 + 4\overline{c}_1 , \quad 0 \leq \overline{c}_0 \leq 3 , \quad 0 \leq \overline{c}_1 \leq Q - 1 ,
\]

(B.15)

neither the phase factor in (B.13) nor the theta function depend on $\overline{c}_1$, and performing the sum over $\overline{c}_1$ just yields a multiplicative factor of $Q$. We obtain

\[
R_L (\frac{-1}{\tau}) = \frac{1}{4} \sum_{u,v} \sum_{M'=0}^{2k-1} \sum_{\overline{c}_0=0}^{3} \left\{ \frac{\Theta_{M'Q + \overline{c}_0(Q^2-1)}}{\Theta_{M'Q + \overline{c}_0}} \Theta_{M'\tau + \overline{c}_0\tau} (\tau) \times \right.
\]

\[
e^{-i\pi \frac{\bar{c}}{2} (M'Q + \overline{c}_0(Q^2 - 1))} e^{-i\pi \frac{\bar{c}}{2} (M' + \overline{c}_0Q)} \left\} ,
\]

(B.16)

Note the close resemblance between (B.16) and (B.8). In order to continue we need to specify the type of modular $SU(2)$ modular invariant which is used as well as the parity of $Q$.

**B.1. $Q$ odd, non–diagonal invariant**

This particular $SU(2)$ invariant, $D_{k+2}$, involves only even spins, and there is only the even sum in (B.11). Furthermore, the sum over even spins only is enough to have $SU(2)$
invariance,
\[
\sum_{l,\overline{L}=0}^{k/2} S_{L',2l-1} D_{2l,2l'} \bar{S}_{2l,\overline{L}'} = D_{L',\overline{L}'} .
\] (B.17)

Since the left-moving SU(2) part was invariant by itself, we just have to show that the right-moving C\Theta part is also invariant, independently of the value of \(\overline{L}'\).

First we should see what the ranges for \(u,v\) are; this depends on \(Q\) but not on the SU(2) modular invariant chosen. For \(Q\) odd choose the following two possibilities
\[
\begin{cases}
0 \leq v \leq 3 & \text{and} \\
u = 0 & \text{and} \\
v = 1, 3
\end{cases}
\]
\[
\begin{cases}
0 \leq u \leq 3 & \text{and} \\
v = 0 & \text{and} \\
u = 1, 3
\end{cases}
\] (B.18)

Of course this is in agreement with the restrictions on \(u,v\) from \(m_0, a_0\), see (B.4).

With the first choice in (B.18), we have only the sum over \(v\) in (B.16) (for \(Q\) odd, \(Q^2 - 1 \in 8\mathbb{Z}\) and the phase is independent of \(\bar{c}_0\))
\[
\sum_{v=0}^{3} e^{i\pi \frac{v}{2}} M' Q = 4\delta_{M',4} \mathbb{Z} = 4\delta_{M',4} \mathbb{Z}
\] (B.19)
since \(Q\) is odd. Therefore we get from (B.16)
\[
R_{L'}(\frac{-1}{\tau}) = \sum_{m'=0}^{k/2-1} \sum_{\overline{c}_0=0}^{3} C_{4m'}(\tau) \overline{C}_{4m'Q + \overline{c}_0(Q^2-1); k(\tau)}
\] (B.20)
which is exactly (B.8) with this particular choice of \(u,v\).

For the second choice in (B.18), we do separately the sums over even and odd \(v\). One finds that \(M = 2m'\) with \(m'\) and \(\overline{c}_0\) simultaneously odd or even. Then we get from (B.16)
\[
R_{L'}(\frac{-1}{\tau}) = \sum_{m'=0}^{k/2-1} \sum_{\overline{c}_0=0,2}^{3} C_{4m'}(\tau) \overline{C}_{4m'Q + \overline{c}_0(Q^2-1); k(\tau)}
\] (B.21)
\[
+ \sum_{m'=0}^{k/2-1} \sum_{\overline{c}_0=1,3}^{3} C_{4m'-2}(\tau) \overline{C}_{(4m'-2)Q + \overline{c}_0(Q^2-1); k(\tau)}
\]
which is the same as (B.8) for this second choice of \(u,v\) in (B.18).

**B.2. Q odd, diagonal invariant**

As above the constraint \(\overline{L} \equiv m(1+Q) \mod 2\) eliminates odd spins. The major difference now is that the sum over even spins is not enough to transform the diagonal invariant into
itself. Instead, using (B.12) we get
\[
Z(\frac{-1}{\tau}) = \sum_{L',L'=0}^{k} X_{L'}(\tau) \frac{1}{2} (\delta_{L',L'} + \delta_{L',k-L'}) \mathcal{R}_{L'}(\frac{-1}{\tau}).
\] (B.22)

For each value of \( L' \) we have on the right–moving side exactly the same expression as (B.8). The first choice in (B.18) implies that \( M' = L' + 2m' \in 4\mathbb{Z} \), which implies also that \( L' \) is even, hence there is no odd spins! We get thus
\[
Z(\frac{-1}{\tau}) = \sum_{l=0}^{k/4} \left\{ \frac{1}{2} X_{2l}(\tau) \overline{C}_{4b}(\tau) \overline{Q}(Q^2 - 1); k(\tau) \right. \right.
\]
\[
+ \frac{1}{2} X_{2l}(\tau) \overline{C}_{4b}^{k-2l}(\tau) \overline{Q}(Q^2 - 1); k(\tau) \left. \right\} \] (B.23)

where \( \tau \) runs from 0 to 3. The first term is fine, but we must work on the second term, to remove the \( k-2l \). We use the symmetry of the string function \( C_{k+m} = C_{k+m}^{k-2l} \), the periodicity of the theta function and the fact that \( k = 2(Q^2 - 1) \in 16\mathbb{Z} \subset 4\mathbb{Z} \) to show that
\[
\overline{C}_{4b}^{k-2l}(\tau) \overline{Q}(Q^2 - 1); k(\tau) = \overline{C}_{4b}(\tau) \overline{Q}(Q^2 - 1); k(\tau)
\] (B.24)

where \( 4b' = 4b + k \) and \( \overline{\tau} = \tau - 2 \). Since we are summing over all values of \( b \) allowed by the periodicity of the string and theta functions, we can absorb the shift by \( k \). The same is true for the shift in \( \overline{\tau} \), since we sum over all allowed values for \( \overline{\tau} \). Therefore
\[
\sum_{b=0}^{Q^2 - 2} \sum_{\overline{\tau}=0}^{3} \overline{C}_{4b}^{k-2l}(\tau) \overline{Q}(Q^2 - 1); k(\tau) = \sum_{b=0}^{Q^2 - 2} \sum_{\overline{\tau}=0}^{3} \overline{C}_{4b}(\tau) \overline{Q}(Q^2 - 1); k(\tau)
\] (B.25)

and the modular invariance of (B.8) for the first choice in (B.18) has been proven.

For the second choice in (B.18), we first do the sums over even and odd \( v \) separately, as for the non-diagonal \( SU(2) \) invariant. As above, we can then rewrite the \( \overline{C}_{4b}^{k-2l} \) as \( \overline{C}_{4b}^{2l} \) which finally leads to the invariance
\[
Z(\frac{-1}{\tau}) = \sum_{\overline{\tau}=0,2} X_{2l}(\tau) \overline{C}_{4b}^{2l}(\tau) \overline{Q}(Q^2 - 1); k(\tau)
\]
\[
+ \sum_{\overline{\tau}=1,3} X_{2l}(\tau) \overline{C}_{4b-2}(\tau) \overline{Q}(Q^2 - 1); k(\tau) = Z(\tau). \] (B.26)

\[B.3. \ Q \ even, \ diagonal \ invariant\]

We start by reexamining the allowed values for \( u, v \). Carefully looking at (B.4) we find at least two allowed possibilities which can be expressed compactly as \( u = \pm v \).

45
In comparison with the earlier cases, odd \( Q \), there are two important differences. First, we have to split the sum over \( L \) (depending on its parity) in taking the \( S \) transformation, which yields, after solving the \( SU(2) \) part with the help of (B.12)

\[
Z\left(-\frac{1}{\tau}\right) = \sum_{L, L' = 0}^{k} \mathcal{X}_L(\tau) \frac{1}{2} (\delta_{L, L'} + \delta_{k-L, L'}) R_{LL'}^{even}(-\frac{1}{\tau})
\]

\[
+ \sum_{L, L' = 0}^{k} \mathcal{X}_L(\tau) \frac{1}{2} (\delta_{L, L'} - \delta_{k-L, L'}) R_{LL'}^{odd}(-\frac{1}{\tau}).
\]

(B.27)

The notation \( R^{even} \) (\( R^{odd} \)) means that this term contains only a sum over even (odd) values of \( u = \pm v \), as it corresponds to even (odd) spin sums, and \( u \equiv L \mod 2 \) from the non-vanishing of the string function in (B.8). Second, the phase factor in (B.16) can be simplified, using the fact that \( Q^2 \in 4\mathbb{Z} \).

We can immediately add up the first and the third term together, and the \( v \) sums complement each other to give \((\overline{c}_0 \text{ is } \overline{c} \text{ now})\)

\[
\sum_{v=0}^{3} e^{i\pi \frac{v}{2}} (M' (Q \mp 1) - \overline{c}(1 \pm Q)) = 4\delta_{M' (Q \mp 1) - \overline{c}(1 \pm Q), 4\mathbb{Z}} = 4\delta_{M', 4\mathbb{Z} \pm \overline{c}}.
\]

(B.28)

Therefore we get a first contribution to (B.27)

\[
Z\left(-\frac{1}{\tau}\right)^{LL} = \frac{1}{2} \sum_{L=0}^{k} \sum_{\overline{c}=0}^{3} \mathcal{X}_L(\tau) \overline{C}_{4b \mp \overline{c}}(\tau) \overline{\mathcal{C}}_{(4b \mp \overline{c})Q + \overline{c}(Q^2 - 1); k}(\tau).
\]

(B.29)

For the two \( \delta_{k-L', L} \) terms, there is a relative minus sign, which can be expressed as \((-1)^v\), so the only change we have to do in recombining these terms is to insert this sign in (B.28). This gives

\[
\sum_{v=0}^{3} e^{i\pi \frac{v}{2}} (M' (Q \mp 1) - \overline{c}(1 \pm Q) + 2) = 4\delta_{M', 4\mathbb{Z} \pm 2 \pm \overline{c}}
\]

(B.30)

and a second contribution to (B.27)

\[
Z\left(-\frac{1}{\tau}\right)^{L,k-L} = \frac{1}{2} \sum_{L=0}^{k} \sum_{\overline{c}=0}^{3} \mathcal{X}_L(\tau) \overline{C}_{4b + 2 \mp \overline{c}}^{k-L}(\tau) \overline{\mathcal{C}}_{(4b + 2 \mp \overline{c})Q + \overline{c}(Q^2 - 1); k}(\tau).
\]

(B.31)

Now we use the symmetry of \( C \) as in (B.24) with the difference that \( k = 2(Q^2 - 1) \in 8\mathbb{Z} - 2 \subset 4\mathbb{Z} + 2 \), so that \( 4b + 2 + k \in 4\mathbb{Z} \), and also \( Qk \in 2k\mathbb{Z} \), and

\[
\overline{C}_{4b \pm 2 \mp \overline{c}}^{k-L}(\tau) \overline{\mathcal{C}}_{(4b \pm 2 \mp \overline{c})Q + \overline{c}(Q^2 - 1); k}(\tau) = \overline{C}_{4b' \mp \overline{c}}^L(\tau) \overline{\mathcal{C}}_{4b'Q + \overline{c}(Q^2 - 1); k}(\tau).
\]

(B.32)
Thus,

\[
\sum_{b=0}^{Q^2-2} \sum_{\tau=0}^{3} \overline{C}^{k-L}_{4b+2+\tau} (\tau) \overline{\Theta}_{4b+2+\tau} Q + \tau (Q^2-1); k(\tau) = \sum_{b=0}^{Q^2-2} \sum_{\tau=0}^{3} \overline{C}^{L}_{4b+\tau} (\tau) \overline{\Theta}_{4b} Q + \tau (Q^2-1); k(\tau) \tag{B.33}
\]

which combines with (B.29) to give us (4.30).

### B.4. Q even, non–diagonal invariant

From (B.11) this $SU(2)$ invariant naturally splits into odd and even spins and is

\[
Z(\tau) = \sum_{b=0}^{Q^2-2} \sum_{v=0}^{3} \left\{ \mathcal{X}_{2l} \overline{C}^{2l}_{4b+\tau} \overline{\Theta}_{4b+\tau} Q + v (Q^2-1); k(\tau) \right. \\
+ \mathcal{X}_{2l+1} \overline{C}^{k-2l-1}_{4b+\tau} \overline{\Theta}_{4b+\tau} Q + v (Q^2-1); k(\tau) \left. \right\} \tag{B.34}
\]

The even sum is the same as in the diagonal case, but the odd sum has an extra overall $(-1)^{L'}$ due to the $k-2l-1$ in $C$ (compare with (B.27)). Therefore, when $L'$ is even there is no difference with the diagonal case and summing both the $LL$ and $L, k-L$ contributions yields

\[
Z^{even}(\frac{-1}{\tau}) = \sum_{b=0}^{Q^2-2} \sum_{\tau=0}^{3} \mathcal{X}_{2l}(\tau) \overline{C}^{2l}_{4b+\tau}(\tau) \overline{\Theta}_{4b+\tau} Q + \tau (Q^2-1); k(\tau) \tag{B.35}
\]

which is fine.

For the odd $L'$ spin, the extra $-1$ kills the term from $\delta_{k-L',\tau}$, and the two contributions add up with no relative sign, giving

\[
Z^{odd}(\frac{-1}{\tau}) = \sum_{b=0}^{Q^2-2} \sum_{\tau=0}^{3} \mathcal{X}_{2l+1}(\tau) \overline{C}^{k-2l-1}_{4b+\tau}(\tau) \overline{\Theta}_{4b+\tau} Q + \tau (Q^2-1); k(\tau) \tag{B.36}
\]

which is also fine. Here we chose to express the $\overline{C}^{2l+1}$ in terms of $\overline{C}^{k-2l-1}$ in order to match the original expression.

### Appendix C. The Elliptic Genus and the $(0,2)$ Modular Invariants

We show that the elliptic genus (3.2) is proportional to $Z' = Z_+ - Z_-$. Taking the
SU(2) diagonal modular invariant for simplicity, it is suggestive to write it as

\[ Z' = \sum_{\text{odd } L} X_L N_{LT} \sum_{m=0}^{Q-1} \left\{ \left( X_{(4m+1)Q;Q-1} - X_{(4m+1)Q;Q-3} \right) \right. \\
+ \left. \left( X_{(4m+3)Q;Q-3} - X_{(4m+3)Q;Q-1} \right) \right\}. \]  

(C.1)

Since the quantities in brackets have the same \( L, q \) values but \( s \) different by 2, they represent \( \text{Tr}(-1)^F q^{L_0-c/24} \) taken in the complete \( N = 2 \) representation. Due to the unbroken supersymmetry in the Ramond sector, this is non zero only for those representations containing a Ramond ground state. In the parametrisation where \( 0 \leq l \leq k, -k - 1 \leq q \leq k + 2, -1 \leq s \leq 2 \), the ground states appear in \( X_{l+1;1}^l \) or \( X_{l-1;-1}^l \). We need to find for which values of \( l \) these characters occur in (C.1).

Consider first \( 0 \leq m < Q/2 \) (so that \( 0 < q < k+2 \)). In that case, we must look for \( X_{l+1;1}^l \). If \( Q - 1 \equiv 1 \), it occurs for \( l = (4m + 1)Q - 1 \) with a plus sign, and for \( l = (4m + 3)Q - 1 \) with a minus sign. For \( Q - 3 \equiv 1 \), the same values of \( l \) are selected, but the signs are reversed. In short, we get the values

\[ \sum_{n=0}^{Q-1} \delta_{l,(2n+1)Q-1} (-1)^{n+\frac{Q}{2}+1} \]  

(C.2)

For the other range \( Q/2 \leq m \leq Q - 1 \), we should subtract \( 2(k + 2) = 4Q^2 \) to \( q \) for it to fall in the proper domain, and we must look for the occurrence of \( X_{l-1;-1}^l \). When \( Q - 3 \equiv -1 \), this happens for \( l = (4(Q - m - 1) + 3)Q - 1 \) with a minus sign, and for \( l = (4(Q - m - 1) + 1)Q - 1 \) with a plus sign. When \( Q - 1 \equiv -1 \), the signs are reversed. This range gives the contribution

\[ \sum_{n=0}^{Q-1} \delta_{l,(2n+1)Q-1} (-1)^{n+\frac{Q}{2}+1} \]  

(C.3)

Combining the two, we find that the result is

\[ Z_+ - Z_- = 2(-1)^{\frac{Q}{2}+1} \sum_{n=0}^{Q-1} (-1)^n X_{(2n+1)Q-1;k}^{su(2)} = 2(-1)^{\frac{Q}{2}+1} Z_Q^{(0,2)}(q, 0, 0). \]  

(C.4)

Since the set of allowed values for \( l \) is symmetric under \( l \rightarrow k - l \), this proof is also valid for the other types of SU(2) modular invariants.

A proof along the same lines is valid for the \( Q \) odd case.

48
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