Nonlinear Global Stabilization Control for the Underactuated WAcrobot System

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A WAcrobot is an underactuated nonlinear system that has three degrees of freedom (DOF) and two inputs. This paper discusses the global stabilization control problem for this 3-DOF underactuated system. A new control strategy is developed to solve this problem. The strategy first changes the 3-DOF WAcrobot system to be a 2-DOF reduced-order model in finite time. This transforms the stabilizing control of the WAcrobot system into that of the reduced-order model. After that, nonsingular control laws that globally stabilize the reduced-order model at the origin are designed. It guarantees the stabilizing control objective of the WAcrobot to be achieved. Finally, a simulation experimental example demonstrates the validity of the presented theoretical results. Simulation results show the advantage of our strategy over others.

1. Introduction

In our daily life, many natural systems essentially have nonlinear characteristics. It is meaningful to discuss the control problem for the nonlinear systems [1–5]. An underactuated mechanical system is a typical example of nonlinear systems, which has fewer control inputs than the number of systems’ degrees of freedom (DOF). Compared with the fully actuated system, the underactuated system has the characteristics of light weight, low energy consumption, and flexible movement because of the reduction of actuators. Such systems can be widely used in health care, space exploration, transportation, military, and other fields [6–9]. However, the reduction of inputs makes the system possess nonlinear constraints. And the constraints are usually second-order nonholonomic [10]. This means that the states of the system are in uncontrollable manifolds of the configuration space. The control design of an underactuated mechanical system is challenging in the nonlinear control area. Many researchers have devoted their efforts to the study on underactuated systems in the past few years [11–15].

Since the 1990s, the control problems for the simplest underactuated system (that is, the 2-DOF underactuated system) have been attracting much attention. In order to explore the nonlinear control strategy, many experimental models of 2-DOF underactuated systems have been presented in the laboratory. The models include Acrobot [16], Pendubot [17], TORA [18], and Pendulum-cart [19]. The stabilization control issue is a commonly addressed problem for these 2-DOF underactuated mechanical systems. And many stabilizing control methods have been developed, e.g., a partial feedback linearization method [20], an energy-based method [21], a nonsmooth Lyapunov function method [22], an equivalent-input disturbance method [23], and a trajectory tracking strategy [24].

With the deepening of the research work, the control problems presented by an n-DOF (n ≥ 3) underactuated system need to be further explored since multi-DOF systems are more consistent with the reality of natural systems.
However, this problem is not easy to solve because the nonlinear dynamics and constraint equations of underactuated systems become more complicated with the increase of the DOF. To solve the control problems for multi-DOF underactuated systems, it is necessary to study the simplest case, i.e., \( n = 3 \). Recently, some 3-DOF underactuated system models have been presented. Their dynamic analysis and control design have been intensively discussed [25–28]. Among them, a WAcrobot model (see Figure 1) is a typical example. This mechanical system has a wheel and a double inverted pendulum. The wheel rolls in a horizontal plane driven by an actuator. And the double pendulum freely rotates in a vertical plane driven by an actuator at the second joint. The first joint of the pendulum is passive. Note that the WAcrobot is a 3-DOF underactuated system, and it is the combination of a wheel and an Acrobot.

The WAcrobot has a good application prospect in the mobile robotic area. It is a perfect test bed for illustrating a nonlinear control algorithm for multi-DOF underactuated systems. So, it is meaningful to study the problems presented by this mechanical system. In [29], the swing-up and stabilization control of the WAcrobot were discussed. He global stabilization control problem for the WAcrobot as

\[
\begin{align*}
\dot{\theta}_1 & = \frac{1}{J_1} \tau_1 - \frac{1}{2J_1} \dot{\theta}_2 J_2 \sin \theta_2, \\
\dot{\theta}_2 & = \frac{1}{J_2} \tau_2 - \frac{1}{2J_2} \dot{\theta}_1 J_1 \sin \theta_1, \\
\dot{\theta}_3 & = \frac{1}{J_3} \tau_3 - \frac{1}{2J_3} \dot{\theta}_1 J_2 \sin \theta_1 - \frac{1}{2J_3} \dot{\theta}_2 J_1 \sin \theta_2.
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\end{align*}
\]
These dynamic equations can be rewritten in the following form:

\[ M(\theta)\ddot{\theta} + H(\theta, \dot{\theta}) + G(\theta) = \tau, \]  

where

\[
\begin{align*}
H(\theta, \dot{\theta}) &= 
\begin{bmatrix}
H_1(\theta, \dot{\theta}) \\
H_2(\theta, \dot{\theta}) \\
H_3(\theta, \dot{\theta})
\end{bmatrix}, \\
G(\theta) &= 
\begin{bmatrix}
G_2(\theta) \\
G_3(\theta)
\end{bmatrix}, \\
\tau &= 
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix},
\end{align*}
\]

\[
\begin{align*}
H_1(\theta, \dot{\theta}) &= -\alpha_3 \dot{\theta}_2^2 \sin \theta_2 - \alpha_3 (\dot{\theta}_2 + \dot{\theta}_3)^2 \sin(\theta_2 + \theta_3), \\
H_2(\theta, \dot{\theta}) &= -\alpha_6 (2\dot{\theta}_2 + \dot{\theta}_3) \theta_2 \sin \theta_1, \\
H_3(\theta, \dot{\theta}) &= \alpha_5 \theta_2^2 \sin \theta_3, \\
G_2(\theta) &= -\beta_1 \sin \theta_2 - \beta_2 \sin(\theta_2 + \theta_3), \\
G_3(\theta) &= -\beta_2 \sin(\theta_2 + \theta_3).
\end{align*}
\]

(6)

Denote \( K(\theta, \dot{\theta}) + P(\theta) \) to be the total energy of the WAcrobot. From (5), we easily obtain

\[ \ddot{\mathcal{E}}(\theta, \dot{\theta}) = \dot{\theta}^T \tau = \dot{\theta}_1 \tau_1 + \dot{\theta}_2 \tau_2. \]

(7)

Let \( x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2, z_1 = \theta_3, z_2 = \dot{\theta}_3, \) \( x = [x_1, x_2, x_3, x_4]^T, \) and \( z = [z_1, z_2]^T. \) Then, it is not difficult to obtain the state space form of (5) as

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= Y_1(x, z) + \Phi_1(x, z)\tau_1 + \Psi_1(x, z)\tau_2, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= Y_2(x, z) + \Phi_2(x, z)\tau_1 + \Psi_2(x, z)\tau_2, \\
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= Y_3(x, z) + \Phi_3(x, z)\tau_1 + \Psi_3(x, z)\tau_2,
\end{align*}
\]

where

\[
\begin{align*}
Y_1(x, z) &= M^{-1}(\theta) (1 - H_1(\theta, \dot{\theta}) - H_2(\theta, \dot{\theta}) - G_2(\theta) \tau_1) - H_3(\theta, \dot{\theta}) - G_3(\theta) \tau_2, \\
Y_2(x, z) &= M^{-1}(\theta) (\Phi_1(x, z) - \Phi_2(x, z) \tau_1 - \Phi_3(x, z) \tau_2), \\
Y_3(x, z) &= M^{-1}(\theta) (\Phi_1(x, z) - \Phi_2(x, z) \tau_1 - \Phi_3(x, z) \tau_2) - \Phi_2(x, z) \tau_1 - \Phi_3(x, z) \tau_2.
\end{align*}
\]

(9)

It follows from (9) that both \( \Psi_1(x, z) \) and \( \Phi_1(x, z)\Psi_1(x, z) - \Phi_2(x, z)\Psi_2(x, z) - \Phi_3(x, z)\Psi_3(x, z) \) are order principal determinants of \( M^{-1}(\theta) \). Since \( M^{-1}(\theta) \) is a positive definite matrix, we get

\[
\begin{align*}
\Psi_3(x, z) > 0, \\
\Phi_1(x, z)\Psi_1(x, z) - \Phi_2(x, z)\Psi_2(x, z) - \Phi_3(x, z)\Psi_3(x, z) > 0.
\end{align*}
\]

(10)

3. Reduced-Order Model of the WAcrobot System

The control objective discussed in this paper is to design controllers \( r_1 \) and \( r_2 \) to globally stabilize WAcrobot system (8) at \([x, z]^T = 0\). Note that the nonlinear dynamics of (8) is very complicated. In order to simplify the structure of system (8), we design

\[
\tau_2 = \frac{W(z_1, z_2) - Y_3(x, z) - \Phi_3(x, z)\tau_1}{\Psi_3(x, z)},
\]

(11)

where

\[
W(z_1, z_2) = -r_2 (z_1^{5/3} + r_1^{5/3}z_1) \frac{1}{5/3},
\]

(12)

and \( r_1 \) and \( r_2 \) are positive constants. This is a relationship between control torques \( r_1 \) and \( r_2 \). Substituting (11) into (8) yields

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= Y_1(x, z) + \frac{\Psi_1(x, z) [W(z_1, z_2) - Y_3(x, z)]}{\Psi_3(x, z)} + \frac{\Phi_1(x, z) - \frac{\Psi_1(x, z) \Phi_4(x, z)}{\Psi_3(x, z)}}{r_1}, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= Y_2(x, z) + \frac{\Psi_2(x, z) [W(z_1, z_2) - Y_3(x, z)]}{\Psi_3(x, z)} + \frac{\Phi_2(x, z) - \frac{\Psi_2(x, z) \Phi_4(x, z)}{\Psi_3(x, z)}}{r_1}, \\
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= W(z_1, z_2) = -r_2 (z_2^{5/3} + r_1^{5/3}z_1) \frac{1}{5/3}.
\end{align*}
\]

(13a)

(13b)
It is clear that (13a) and (13b) are a nonlinear cascade system, and the state variable $z$ is separated from others. According to the results in [30], the finite-time stabilization of subsystem (13b) at $z = 0$ is achieved. As a result, system (13a) and (13b) becomes the following form in finite time

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= f_1(x) + g_1(x)r_1, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= f_2(x) + g_2(x)r_1,
\end{align*}
$$

(14)

where

$$
\begin{align*}
f_1(x) &= \frac{Y_1(x,0)\Psi_3(x,0) - Y_3(x,0)\Psi_1(x,0)}{\Psi_3(x,0)}, \\
f_2(x) &= \frac{Y_2(x,0)\Psi_3(x,0) - Y_3(x,0)\Psi_2(x,0)}{\Psi_3(x,0)}, \\
g_1(x) &= \frac{\Phi_1(x,0)\Psi_3(x,0) - \Phi_3(x,0)\Psi_1(x,0)}{\Psi_3(x,0)}, \\
g_2(x) &= \frac{\Phi_2(x,0)\Psi_3(x,0) - \Phi_3(x,0)\Psi_2(x,0)}{\Psi_3(x,0)}.
\end{align*}
$$

(15)

From (10), it is easy to obtain $g_1(x) > 0$ for $x \in \mathbb{R}^4$. In fact, nonlinear system (14) is the zero dynamics of cascade system (13a) and (13b). In other words, (14) is a reduced-order system of the WAcrobot. The physical model of (12) is shown in Figure 2. In order to achieve the control objective of the WAcrobot, it is necessary to design a controller $r_1$ such that system (14) is stabilized at $x = 0$.

4. Design of the Stabilizing Controller for the Reduced-Order Model

This section discusses the design of a stabilizing controller $r_1$ for reduced-order system (14) under the condition $z = 0$.

Let $E(x) = \Xi(\theta, \dot{\theta})|_{z=0}$. From (1), (2), and (7), it is easy to get $E(0) = \beta_1 + \beta_2$ and $E(\dot{x}) = x_2r_1$. A Lyapunov function for system (14) is constructed to be

$$
V(x) = \frac{\mu_1 E_2}{2}(x) + \frac{\mu_2}{2}x_1^2 + \frac{\mu_3}{2}x_2^2,
$$

(16)

where $E(x) = E(x) - E(0)$ and $\mu_i > 0 (i = 1, 2, 3)$ are constants. The derivative of $V(x)$ is

$$
\frac{dV(x)}{dt} = \mu_1 \dot{E}(x)\dot{E}(x) + \mu_2 x_1\dot{x}_1 + \mu_3 x_2\dot{x}_2
= \mu_1 \dot{E}(x)x_2r_1 + \mu_2 x_1x_2 + \mu_3 x_2[f_1(x) + g_1(x)r_1]
= \left[ \mu_1 \dot{E}(x) + \mu_3 g_1(x) \right] r_1 + \mu_2 x_1 + \mu_3 f_1(x) x_2.
$$

(17)

We design the control law $r_1$ to be

$$
\tau_1 = -\gamma z_2 - \mu_2 x_1 - \mu_3 f_1(x) \frac{\mu_1 \dot{E}(x) + \mu_3 g_1(x)}{\mu_1 \dot{E}(x)},
$$

(18)

where $\gamma > 0$ is a constant. Combining (17) and (18) gives

$$
\frac{dV(x)}{dt} = -\gamma x_2^2 \leq 0.
$$

(19)

From Figure 2, it is easy to deduce that system (14) has the same dynamic properties as the Pendubot. As a result, following the same analytical procedure given in Section 4 of [22] yields that $E(x) \rightarrow E(0), x_1 \rightarrow 0$, and $x_2 \rightarrow 0$ based on (16) and (19). This means that the control law (18) drives system (14) to converge the set

$$
\Omega = \{x | x_1 = 0, x_2 = 0, E(x) = E(0)\}
= \{x | x_1 = 0, x_2 = 0, \omega(x_3, x_4) = 0\},
$$

(20)

where

$$
\omega(x_3, x_4) = M_{22}(0)x_3^2 - 2(\beta_1 + \beta_2)[1 - \cos x_3].
$$

(21)

Since $x_3 = \dot{x}_1$, we have that $\omega(x_3, x_4) = 0$ which describes a homoclinic orbit around $x_3 = x_4 = 0$. Thus, the state variable $x$ in $\Omega$ can enter any arbitrarily small neighbourhood area of $x = 0$. In this area, system (14) approximates the following linear system:

$$
\dot{x} = Ax + Br_1,
$$

(22)

where

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & A_1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & A_2 & 0
\end{bmatrix},
$$

(23)

$$
B = \begin{bmatrix}
0 \\
B_1 \\
B_3 \\
B_2
\end{bmatrix}.
$$
Remember that the control law is stabilized linear system (22) at \( x = 0 \). From the above statements, we get that the use of the control law \( \tau_1 \) in (18) and (27) in sequence guarantees the global stabilization of reduced-order system (14) at \( x = 0 \).

Remark 1. Note that the control law \( \tau_1 \) in (18) is singular when \( \mu_1 \tilde{E}(x) + \mu_3 g_1(x) = 0 \). This case occurs only when \( E(x) < \tilde{E}(0) \) (i.e., \( \tilde{E}(x) < 0 \)) because \( \mu_1 > 0 \), \( \mu_3 > 0 \), and \( g_1(x) > 0 \). When \( E(x) < \hat{E}(0) \), it follows from (1) and (2) that

\[
0 > \tilde{E}(x) \geq (\beta_1 + \beta_2) \cos x_2 - E(0) \geq -2E(0).
\]

So, we have

\[
\mu_1 \tilde{E}(x) + \mu_3 g_1(x) \geq \mu_3 g_1(x) - 2\mu_1 E(0).
\]

From (29), it is easy to obtain that the singularity of \( \tau_1 \) in (18) does not occur if

\[
\mu_3 g_1(x) - 2\mu_1 E(0) > 0.
\]

We select the control parameters such that

\[
\mu_3 > \frac{2\mu_1 E(0)}{\rho},
\]

\[
\rho = \min[g_1(x)].
\]

This condition guarantees (30) to hold. In other words, the control law in (18) is not singular under condition (31).

Remark 2. There are eight control parameters in this paper, i.e., \( r_1 \) and \( r_2 \) in (12); \( \mu_1, \mu_2, \) and \( \mu_3 \) in (16); \( \gamma \) in (19); and \( R \) and \( Q \) in (26). In order to make the control design process clear, a set of control parameters for these parameters is given as follows:

1. An optimal set of \( r_1 \) and \( r_2 \) is selected for (12) that makes the stabilization time of (13b) to be smallest.
2. For fixed \( \gamma \), a set of \( \mu_1, \mu_2, \) and \( \mu_3 \) is chosen to make reduced-order system (14) enter an arbitrarily small neighborhood area of \( x = 0 \).
3. A set of \( R \) and \( Q \) is selected for the control law \( \tau_1 \) that stabilizes system (22) at the origin.

5. Numerical Example

A numerical example is presented in this section in order to verify the validity of our presented theoretical results. In the simulation experiment, the physical parameters of the WAcrobot were chosen to be [29]

\[
\begin{align*}
\mu_1 &= 1.22 \text{ kg}, \\
\mu_2 &= 0.28 \text{ kg}, \\
\mu_3 &= 0.72 \text{ kg}, \\
L_1 &= 0.05 \text{ m}, \\
L_2 &= 0.15 \text{ m}, \\
L_3 &= 0.45 \text{ m}, \\
L_{cc} &= 0.075 \text{ m}, \\
L_{cc} &= 0.225 \text{ m}, \\
J_1 &= 0.00153 \text{ kg} \cdot \text{m}^2, \\
J_2 &= 0.000598 \text{ kg} \cdot \text{m}^2, \\
J_3 &= 0.013138 \text{ kg} \cdot \text{m}^2.
\end{align*}
\]

The initial condition was selected to be

\[
\begin{bmatrix}
\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3
\end{bmatrix}^T = [0, \pi, 0, 0, 0, 0.1]^T.
\]

The physical meaning of (33) is that the double inverted pendulum of the WAcrobot begins to move from the straight-down position with a small velocity, while the wheel is stationary at the origin. The control parameters in (12) were selected to be \( r_1 = 1 \) and \( r_2 = 5 \). The simulation results of subsystem (13b) are shown in Figure 3. Note that \( \theta_1 \) and \( \theta_3 \) converge to zero in 0.25 seconds. It means that WAcrobot system (8) changes to reduced-order system (14) quickly. This demonstrates the validity of the presented theoretical results in Section 3.

To show the validity of the stabilizing controller for reduced-order model (14), the parameters in (16) and (18) were chosen as

\[
\begin{align*}
\alpha_1 &= \frac{\mu_1 L_1}{2m_1} \cos x_2 - \mu_1 E(0), \\
\alpha_2 &= \frac{\mu_1 L_2}{2m_1} \cos x_2 - \mu_1 E(0), \\
\alpha_3 &= \frac{\mu_1 L_3}{2m_1} \cos x_2 - \mu_1 E(0), \\
\beta_1 &= \frac{\mu_1 L_1}{2m_1} \cos x_2 - \mu_1 E(0), \\
\beta_2 &= \frac{\mu_1 L_2}{2m_1} \cos x_2 - \mu_1 E(0), \\
\beta_3 &= \frac{\mu_1 L_3}{2m_1} \cos x_2 - \mu_1 E(0).
\end{align*}
\]
\[
\begin{align*}
\mu_1 &= 350, \\
\mu_2 &= 200, \\
\mu_3 &= 15, \\
\gamma &= 50, \\
R &= 0.5, \\
Q &= 0.01I_4,
\end{align*}
\]  
\tag{34}

where \( I_4 \) is a \( 4 \times 4 \) identity matrix. A simple calculation gives \( \rho = 141.25 \) and \( 2\mu_1 E(0)/\rho = 14.147 \). Thus, condition (31) is satisfied. Figure 4 shows the simulation results for \( x(t) \) by controllers (11) and (18). It is clear that \( \dot{\theta}_1 \) and \( \ddot{\theta}_1 \) are stabilized at zero while \( \dot{\theta}_2 \) and \( \ddot{\theta}_2 \) are in a homoclinic orbit. This shows the validity of the theoretical results in Section 4.

When \( x(t) \) enters the small neighbourhood area of \( x(t) = 0 \) at \( t = 30 \) s, the controller \( \tau_1 \) switches to (27) from (18). The simulation results of controllers and \( x(t) \) are shown in Figures 5 and 6, respectively. Note that the switch of \( \tau_1 \) from (18) to (27) causes small sudden changes in \( \dot{\theta}_1 \) and \( \ddot{\theta}_1 \) at \( t = 30 \) s. Moreover, it is clear that the WAcrobot can be quickly stabilized at the origin in 40 seconds, and the maximal value of inputs is less than 1.1 Nm. Compared with the results in [29], the value of inputs in this paper is more smaller, and the motion process of the WAcrobot system is more smooth and natural. This provides guarantee for the safe and stable operation of the control system. All these
demonstrate the advantages of our presented control strategy.

6. Conclusion

This paper addressed the global stabilization of a 3-DOF underactuated WAcrobot system. A novel stabilizing control strategy was developed. First, the 3-DOF WAcrobot was changed to be a 2-DOF reduced-order model in finite time. This makes the stabilizing control of the WAcrobot be easy to handle. And then, the stabilizing control laws were designed that globally stabilize the reduced-order system at the origin. And the condition that avoids the singularity in the stabilizing control law was also presented. Finally, the numerical simulation example showed the validity of the proposed theoretical results. In the future, we will extend the new strategy to the stabilization of other multi-DOF underactuated systems [31] and will also further explore other stabilizing control methods for the WAcrobot system.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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