Abstract

The trigonometric Ruijsenaars-Schneider model is derived by symplectic reduction of Poisson-Lie symmetric free motion on the group $U(n)$. The commuting flows of the model are effortlessly obtained by reducing canonical free flows on the Heisenberg double of $U(n)$. The free flows are associated with a very simple Lax matrix, which is shown to yield the Ruijsenaars-Schneider Lax matrix upon reduction.

Mathematics Subject Classifications (2000): 37J35, 53D20.

Key words: Ruijsenaars-Schneider model, Poisson-Lie symmetry, symplectic reduction.
1 Introduction

The integrable many-body models of Calogero-Moser-Sutherland type \[4, 18, 12\] enjoy huge popularity in mathematical physics, for they possess a wide range of physical applications based on their intimate relationships to central areas of harmonic analysis, theory of special functions and symplectic geometry (see \[21\] for a review). One of the fruitful approaches to study the structure of these models is to represent them in terms of Hamiltonian reductions of ‘free’ systems with large symmetries. This philosophy originated from the celebrated papers of Olshanetsky-Perelomov \[15, 16\] and Kazhdan-Kostant-Sternberg \[9\] who, among others, derived the trigonometric Sutherland model by reduction of the free particle moving on the group \(U(n)\).

In this paper, we shall develop a Poisson-Lie generalization of the Kazhdan-Kostant-Sternberg reduction and we shall show that it yields the trigonometric Ruijsenaars-Schneider relativistic integrable system \[22\]. Thus our result corroborates the general expectation that the reduction translates the transition from the ordinary to the Poisson-Lie symmetric free systems into the transition from the non-relativistic to the relativistic integrable many-body models.

We here recall that various infinite dimensional generalizations of the Kazhdan-Kostant-Sternberg reduction have been studied in the nineties by Gorsky-Nekrasov and others (\[8, 13, 6\] and references therein). In particular, in the paper \[8\] a trigonometric Ruijsenaars-Schneider model was derived by reducing a Hamiltonian system on the magnetic cotangent bundle of the loop group of \(U(n)\). This reduction is intrinsically close to topological field theory and thus it should have a finite dimensional counterpart. In fact, Gorsky and Nekrasov \[8\] stated it as a problem to work out such a finite dimensional reduction in terms of a Heisenberg double. This motivated us, although we shall derive the usual trigonometric Ruijsenaars-Schneider model with Hamiltonian \([3, 21]\), which is different from the IIIb model \([20, 23]\) model obtained in \[8\].

Arguably, the crucial point in the Kazhdan-Kostant-Sternberg derivation of the trigonometric Sutherland model was the choice of a certain element \(\iota(x)\) of the Lie algebra \(u(n)\) of the group \(U(n)\), defined by

\[
\iota(x)_{jj} = 0, \quad \forall j, \quad \iota(x)_{jk} = ix, \quad \forall j \neq k,
\]

where \(x\) is a real non-zero parameter. Kazhdan, Kostant and Sternberg took for the unreduced phase space just the cotangent bundle \(T^*U(n)\), they picked as the unreduced Hamiltonian the one which induces the Killing geodesics on \(U(n)\), and they reduced using the adjoint action of \(U(n)\) on \(T^*U(n)\) by constraining the moment map \(J\) of this action to be equal to \(\iota(x)\):

\[
J(K) = \iota(x), \quad K \in T^*U(n).
\]

One may view the element \(\iota(x)\) as the key needed to unlock the room inside \(T^*U(n)\) in which the Sutherland model is stored, since starting from the constraint \((1.2)\) the standard symplectic reduction procedure yields the trigonometric Sutherland model \([9]\).

The Poisson-Lie analogue of the cotangent bundle \(T^*U(n)\) is the well-known Heisenberg double of \(U(n)\) constructed by Semenov-Tian-Shansky in 1985 \([17]\). What is somewhat less known is the correct Poisson-Lie analogue of the adjoint action of \(U(n)\) on \(T^*U(n)\), but this was made explicit by one of us in a recent paper \([10]\). The Heisenberg double is naturally equipped with a Hamiltonian that is invariant with respect to the ‘quasi-adjoint’ action of \([10]\) and

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\footnote{To be more precise, we shall deal with the standard one out of the possible physically different real forms \([20]\) of the complex trigonometric Ruijsenaars-Schneider model.}
generalizes the kinetic energy of the free geodesic motion. Thus the true problem is to identify
the Poisson-Lie analogue of the Kazhdan-Kostant-Sternberg element \( \iota(x) \) which will permit to
find the trigonometric Ruijsenaars-Schneider model inside the Heisenberg double. Our claim is
that this analogue is the upper-triangular \( n \times n \) matrix \( \nu(x) \) given by

\[
\nu(x)_{jj} = 1, \quad \forall j, \quad \nu(x)_{jk} = (1 - e^{-x})e^{(k-j)x/2}, \quad \forall j < k.
\]

The quasi-adjoint action has a so-called Poisson-Lie moment map which is a map \( \Lambda \) from the
Heisenberg double \( D \) to the group \( B \) of complex upper-triangular \( n \times n \) matrices with positive
real numbers on the diagonal. The Poisson-Lie reduction is then determined by requiring

\[
\Lambda(K) = \nu(x), \quad K \in D.
\]

As in the standard Kazhdan-Kostant-Sternberg case, the reduction is completely algorithmic,
although it is technically more involved to solve the constraint (1.4) and to identify the reduced
system. After doing this below, we shall also explain how we have found the constant \( \nu(x) \).

Readers not familiar with symplectic reduction based on Poisson-Lie symmetries may study
Lu’s work [11], but to understand our paper it really suffices to know that the reduction tool
operates in the same way as for ordinary symmetries if a ‘good moment map’ is available.

We organize the present paper as follows. In Section 2, we describe the Poisson-Lie symmetric
free motion on the group \( U(n) \). Then, in Section 3, we obtain the trigonometric Ruijsenaars-
Schneider model and its Lax matrix by reducing the free flows on the Heisenberg double.

\section{Poisson-Lie symmetric free motion on the group \( U(n) \)}

The Heisenberg double of \( U(n) \) is the group \( GL(n, \mathbb{C}) \) viewed as a real manifold. Every point
\( K \in GL(n, \mathbb{C}) \) admits two unambiguous Iwasawa decompositions

\[
K = b_L g_R^{-1} \quad \text{and} \quad K = g_L b_R^{-1} \quad \text{with} \quad b_{L,R} \in B, \ g_{L,R} \in U(n).
\]

There is a natural symplectic form \( \omega_+ \) on \( GL(n, \mathbb{C}) \), first described in [11],

\[
\omega_+ = \frac{1}{2} \text{tr}(d\Lambda_L \Lambda^{-1}_L \wedge d\Xi_L \Xi^{-1}_L) + \frac{1}{2} \text{tr}(d\Lambda_R \Lambda^{-1}_R \wedge d\Xi_R \Xi^{-1}_R).
\]

Here \( \Im z \) stands for the imaginary part of the complex number \( z \), \( \text{tr} \) is the ordinary matrix trace
and we use the Iwasawa maps \( \Lambda_{L,R} : GL(n, \mathbb{C}) \to B \) and \( \Xi_{L,R} : GL(n, \mathbb{C}) \to U(n) \) given by

\[
\Lambda_{L,R}(K) := b_{L,R} \quad \text{and} \quad \Xi_{L,R}(K) := g_{L,R}.
\]

Define now the Hermitian matrix-valued function \( L \) on \( GL(n, \mathbb{C}) \) as

\[
L(K) := (K^\dagger K)^{-1} = \Lambda_R(K) \Lambda_R(K)^\dagger, \quad K \in GL(n, \mathbb{C}).
\]
By Poisson-Lie symmetric free motion on $U(n)$ we mean a dynamical system the phase space of which is the Heisenberg double of $U(n)$ and the Hamiltonian of which is provided by

$$H_\mu(K) := \frac{1}{2} \sum_{j \neq 0} \frac{\mu_j}{j} \text{tr}(L(K)^j), \quad K \in GL(n, \mathbb{C}).$$

(2.5)

Whatever is the sequence of the real parameters $\mu_j$, the Hamiltonian $H_\mu$ is obviously invariant with respect to the following quasi-adjoint action $\triangleright$ introduced in [10]:

$$g \triangleright K := gK \Xi_R(gL(K)), \quad g \in U(n), \quad K \in GL(n, \mathbb{C}).$$

(2.6)

The flow induced by this Hamiltonian was explicitly identified in [24] and it reads

$$K_\mu(t) = b \exp \left(-it \sum_{j \neq 0} \mu_j (b^b)^{-j} \right)g^{-1}.$$  

(2.7)

Here the constant elements $b \in B$ and $g \in U(n)$ encode the choice of initial conditions. If we interpret $b_L$ in the decomposition (2.1) as ‘momentum’ and $g_R$ as ‘position’, then (2.7) says that the momentum is conserved and the position follows the standard Killing geodesics on $U(n)$. This fact justifies the terminology ‘free motion’.

If $\mu$ and $\mu'$ are two arbitrary sequences of real parameters then it follows easily from the formula (2.7) that the corresponding flows commute. In other words, we have

$$\{H_\mu, H_{\mu'}\}_+ = 0,$$

(2.8)

where $\{\cdot, \cdot\}_+$ is the Poisson bracket induced by the symplectic form $\omega_+$. Hence $L$ (2.4) can be interpreted as the Lax matrix of the commuting family of the dynamical systems $(GL(n, \mathbb{C}), \omega_+, H_\mu)$.

### 3 The reduction

In [10], the quasi-adjoint action $\triangleright$ was shown to admit the so-called Poisson-Lie moment map $\Lambda : GL(n, \mathbb{C}) \to B$ given by

$$\Lambda(K) = \Lambda_L(K)\Lambda_R(K), \quad K \in GL(n, \mathbb{C}).$$

(3.1)

This means, in particular, that for every $X \in u(n)$, every $K \in GL(n, \mathbb{C})$ and every function $f$ on $GL(n, \mathbb{C})$, we have

$$\frac{d}{ds} f(e^{sX} \triangleright K)|_{s=0} = \Im \text{tr}(X \{f, \Lambda\}_+(K)\Lambda(K)^{-1}).$$

(3.2)

The first step of the reduction of the Poisson-Lie symmetric free motion $(GL(n, \mathbb{C}), \omega_+, H_\mu)$ amounts to solving the moment map constraint

$$\Lambda(K) = \Lambda_L(K)\Lambda_R(K) = \nu(x), \quad K \in GL(n, \mathbb{C}),$$

(3.3)

with $\nu(x)$ defined in (1.3). The result is summarized in the following theorem.
Theorem 1. Denote by $\mathcal{C}$ the set of the regular elements of a Weyl alcove in the maximal torus $T_n \subset U(n)$, by $A$ the diagonal subgroup of $B$, by $N$ the group of complex upper-triangular matrices having 1 all along the diagonal, and by $G_x$ the isotropy group of $\nu(x)$, i.e.,

$$G_x := \{ g \in U(n) \mid g\nu(x)\nu(x)^\dagger g^{-1} = \nu(x)\nu(x)^\dagger \}. \quad (3.4)$$

Then every solution $K$ of the moment map constraint (3.3) can be written as

$$K = g \triangleright (\mathcal{N}(T)aT^{-1}), \quad (3.5)$$

where $T \in \mathcal{C}$, $a \in A$, $g \in G_x$ and $\mathcal{N}(T) \in N$ is given by

$$(\mathcal{N}(T))_{kl} = \prod_{m=1}^{l-k} \frac{e^\frac{i}{2}T_l - e^{-\frac{i}{2}}T_{k+m}}{T_l - T_{k+m-1}}, \quad \forall k < l. \quad (3.6)$$

Moreover, it holds that no two different points of the form $\mathcal{N}(T)aT^{-1}$ can be transformed into each other by the action of $G_x$.

The message of Theorem 1 is that the submanifold of $GL(n, \mathbb{C})$ defined as

$$S := \{ \mathcal{N}(T)aT^{-1} \mid T \in \mathcal{C}, a \in A \} \quad (3.7)$$

forms a global cross section of the orbits of $G_x$ in the inverse image of the moment map value $\nu(x)$. Therefore $S$ can serve as a model of the resulting reduced phase space.

In order not to break the line of the presentation, we postpone the proof of Theorem 1 for a while and, as the second step of the reduction, we evaluate the symplectic form (2.2) on the slice $S$ (3.7). The calculation of the Iwasawa maps (2.3) on the slice gives directly

$$\Lambda_L(\mathcal{N}(T)aT^{-1}) = \mathcal{N}(T)a, \quad \Xi_R(\mathcal{N}(T)aT^{-1}) = T, \quad (3.8)$$

$$\Lambda_R(\mathcal{N}(T)aT^{-1}) = Ta^{-1}\mathcal{N}(T)^{-1}T^{-1}, \quad \Xi_L(\mathcal{N}(T)aT^{-1}) = T^{-1}, \quad (3.9)$$

and, consequently, the reduced symplectic form reads

$$\omega_r(T, a) = \Im \text{tr}(T^{-1}dT \wedge a^{-1}da). \quad (3.10)$$

We choose the following parametrization of $T$ and $a$:

$$T := \text{diag}(e^{2i\zeta_1}, e^{2i\zeta_2}, ..., e^{2i\zeta_n}), \quad 0 \leq \zeta_k < \pi, \quad q_1 > q_2 > ... > q_n; \quad (3.11)$$

$$a := \text{diag}(e^{\zeta_1}, e^{\zeta_2}, ..., e^{\zeta_n}), \quad (3.12)$$

where

$$\zeta_k = -\frac{p_k}{2} - \frac{1}{4} \sum_{m<k} \ln \left(1 + \frac{\sinh^{2}\frac{\pi}{2}}{\sin^{2}(q_k - q_m)}\right) + \frac{1}{4} \sum_{m>k} \ln \left(1 + \frac{\sinh^{2}\frac{\pi}{2}}{\sin^{2}(q_k - q_m)}\right). \quad (3.13)$$

Then the reduced symplectic form $\omega_r$ becomes the Darboux one

$$\omega_r = \sum_k dp_k \wedge dq_k. \quad (3.14)$$
Hence we can identify the reduced phase space with the cotangent bundle $T^*C$ of the open Weyl alcove $C$.

On account of (2.5), the reduced Hamiltonians are given by the formula

$$H_\mu(T, a) = \frac{1}{2} \sum_{j \neq 0} \mu_j \text{tr}(L(T, a)^j),$$  \hspace{1cm} (3.15)

where

$$L(T, a) = a^{-1}N(T)^{-1}(N(T)^\dagger)^{-1}a^{-1}.$$  \hspace{1cm} (3.16)

Since the commutativity (2.8) of the Hamiltonians is preserved by the reduction procedure, we may interpret $L(T, a)$ as the Lax matrix of the reduced system. The components of the Lax matrix $L$ in the Darboux variables can be directly evaluated from (3.6) and (3.16). This gives

$$L_{kl} = \frac{\Gamma_k \Gamma_l e^{\mu_k + \mu_l} \sinh \frac{x}{2}}{\sinh \left( \frac{x}{2} + iq_k - iq_l \right)} \prod_{m \neq k} \left( 1 + \frac{\sinh^2 \frac{x}{2}}{\sin^2 (q_k - q_m)} \right)^{\frac{1}{2}} \prod_{s \neq l} \left( 1 + \frac{\sinh^2 \frac{x}{2}}{\sin^2 (q_l - q_s)} \right)^{\frac{1}{2}},$$  \hspace{1cm} (3.17)

where the $U(1)$-valued quantities $\Gamma_k$ are defined as the phase factors of the following complex numbers:

$$e^{-iq_k} \prod_{m \neq k} \frac{e^{-\frac{x}{2}e^{-2iq_k} - e^{\frac{x}{2}e^{-2iq_m}}}}{e^{-2iq_k} - e^{-2iq_m}}.$$  \hspace{1cm} (3.18)

In the calculation of the Lax matrix, we have used that the inverse of the matrix $N(T)$ is

$$(N(T)^{-1})_{kl} = \prod_{m=1}^{l-k} \frac{e^{-\frac{x}{2} \hat{T}_k - e^{\frac{x}{2} \hat{T}_{k+m-1}}}}{\hat{T}_k - \hat{T}_{k+m}}, \hspace{1cm} \forall k < l.$$  \hspace{1cm} (3.19)

The role of the Lax matrix is to generate the commuting Hamiltonians by its eigenvalues. Thus the matrix obtained by the conjugation $L \rightarrow \Gamma^{-1}LT := L$, where $\Gamma$ is the diagonal matrix with the components $\Gamma_k$, serves equally well as a Lax matrix of the reduced system. We have

$$L_{kl} = \frac{\Gamma_k \Gamma_l \sinh \frac{x}{2}}{\sinh \left( \frac{x}{2} + iq_k - iq_l \right)} \prod_{m \neq k} \left( 1 + \frac{\sinh^2 \frac{x}{2}}{\sin^2 (q_k - q_m)} \right)^{\frac{1}{2}} \prod_{s \neq l} \left( 1 + \frac{\sinh^2 \frac{x}{2}}{\sin^2 (q_l - q_s)} \right)^{\frac{1}{2}},$$  \hspace{1cm} (3.20)

which is nothing but the standard trigonometric Ruijsenaars-Schneider Lax matrix [22] [20] [21].

The above arguments prove that our Hamiltonian reduction yields precisely the trigonometric Ruijsenaars-Schneider model. In particular, if we pick $\mu_\pm = \pm 1$ and all other $\mu_j$ vanishing, then we obtain from (3.15) the trigonometric Ruijsenaars-Schneider Hamiltonian [22] [20] [21],

$$H_{\text{trigo-RS}}(q, p) = \sum_{k=1}^{n} (\cosh p_k) \prod_{m \neq k} \left( 1 + \frac{\sinh^2 \frac{x}{2}}{\sin^2 (q_k - q_m)} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (3.21)

Let us finally evaluate the flows of the reduced Hamiltonians (3.15). For any curve $g_R(t)$ in the set of the regular elements in $G$, denote by $E[g_R(t)]$ its diagonalized form varying in the Weyl alcove $C \subset \mathbb{T}_n$. Suppose that the flow (2.7) starts on the slice $S$ (3.7) at time $t = 0$. We then immediately obtain that the Ruijsenaars-Schneider variable $T \in C$ develops according to

$$T(t) = e^{2iq(t)} = E[T(0)e^{it \sum_{j \neq 0} \mu_j (L(0))^{\dagger}_j}],$$  \hspace{1cm} (3.22)
i.e., $e^{2q(t)}$ moves along the ordered eigenvalues of the geodesic $e^{2q(0)}e^{it\sum_{j\neq 0}\mu_j(L(0))}$ defined by the initial values of the coordinates and the Lax matrix \(3.20\). This reproduces the well-known interpretation of the commuting Ruijsenaars-Schneider flows [22, 19, 20].

It is worth stressing that the Ruijsenaars-Schneider coupling constant arises from the arbitrary parameter \(x\) in the moment map value $\nu(x)$ (1.3). In (3.21) the velocity of light has been set to 1, and actually one can vary it arbitrarily by performing the reduction using an arbitrary multiple of the symplectic form $\omega_+$ (2.2) on the Heisenberg double.

4 The proof of Theorem 1

The proof of Theorem 1 will be based on the following three lemmas:

**Lemma 1.** If a vector $v \in \mathbb{R}^n$ with non-negative components, a real non-zero number $x \in \mathbb{R}$ and an element $\nu$ of the group $N$ satisfy the relation

$$\ln(\nu \nu^\dagger) = x(v v^\dagger - 1_n),$$

(4.1)

then it holds

$$\nu_{jk} = (1 - e^{-x})e^{(k-j)x/2}, \quad \forall j < k$$

(4.2)

and

$$v_k = \sqrt{n(e^x - 1)/1 - e^{-nx}e^{-kx/2}}, \quad \forall k.$$  

(4.3)

**Proof.** First of all, by exponentiating Eq. (4.1), we obtain

$$\nu \nu^\dagger = e^{-x} \left[ 1_n + \frac{e^{nx} - 1}{n} v v^\dagger \right].$$

(4.4)

Here we use the fact that $v^\dagger v = n$, which is an immediate consequence of the assumptions of the lemma.

Second, for an $n \times n$ matrix $M$, denote by $M_k$ the $(n-k) \times (n-k)$ matrix obtained by deleting the first $k$ rows and the first $k$ columns of $M$ (in particular $M_0 = M$). We find

$$\det(\nu \nu^\dagger)_{k-1} = e^{(k-1-n)x} \left[ 1 + \frac{e^{nx} - 1}{n} \sum_{m=k}^n |v_m|^2 \right]^2 \quad \forall k = 1, \ldots, n.$$  

(4.5)

We have calculated these determinants by using the identity

$$\det(1_m + uw^\dagger) = 1 + w^\dagger u, \quad \forall u, w \in \mathbb{C}^m,$$

(4.6)

where $u, w$ are column vectors and $1_m$ is the $m \times m$ unit matrix.

By comparing $\det(\nu \nu^\dagger)_k$ and $\det(\nu \nu^\dagger)_{k-1}$, we can evaluate the absolute values of all components of the vector $v$:

$$|v_k|^2 = \frac{n}{e^{nx} - 1} e^{(n-k)x} \left[ e^x \det(\nu \nu^\dagger)_{k-1} - \det(\nu \nu^\dagger)_k \right], \quad k = 1, \ldots, n - 1.$$  

(4.7)
The upper-triangularity of any \( \nu \in \mathbb{N} \) implies \( (\nu \nu^\dagger)_{k-1} = \nu_{k-1}^\dagger \nu_{k-1}^\dagger \) for \( k = 1, \ldots, n \), while the property \( \nu_{kk} = 1 \) implies \( \det(\nu_{k-1}) = \det(\nu_{k-1}^\dagger) = 1 \). Therefore \( \det(\nu \nu^\dagger)_{k-1} = 1 \) for each \( k = 1, \ldots, n \), and, from (4.7), we conclude

\[
|\nu_k|^2 = \frac{n(e^x - 1)}{1 - e^{-nx}} e^{-kx} \quad \forall k = 1, \ldots, n. \tag{4.8}
\]

So far we have proved the uniqueness of the non-negative real vector \( \nu \) given by (4.3). It is easy to check that \( \nu \) given by (4.2) verifies (4.1), or, equivalently, (4.4). The uniqueness of \( \nu \in \mathbb{N} \) then follows from the fact that the map \( b \mapsto bb^\dagger \) is a diffeomorphism between the group \( B \) and the space of positive definite Hermitian matrices. \( \text{Q.E.D.} \)

Notice that the element \( \nu \) (4.2) characterized by Lemma 1 is nothing but the constant \( \nu(x) \) given by Eq. (1.3).

**Lemma 2.** For every \( g \in U(n) \) and \( K \in GL(n, \mathbb{C}) \), it holds

\[
\Lambda(g \triangleright K)\Lambda(g \triangleright K)^\dagger = g\Lambda(K)\Lambda(K)^\dagger g^{-1}. \tag{4.9}
\]

**Proof.** We remark that

\[
\Lambda(g \triangleright K) = \Lambda_L(g \triangleright K)\Lambda_R(g \triangleright K) = \Lambda_L(gK)\Lambda_R(K\Xi_R(g\Lambda_L(K))). \tag{4.10}
\]

Hence we have,

\[
\Lambda(g \triangleright K)\Lambda(g \triangleright K)^\dagger = \Lambda_L(gK)\Xi_R(g\Lambda_L(K))^\dagger K^{-1}K^{-1\dagger}\Xi_R(g\Lambda_L(K))\Lambda_L(gK)^\dagger = g\Lambda_L(K)K^{-1}K^{-1\dagger}\Lambda_L(K)^\dagger g^{-1} = g\Lambda_L(K)\Lambda_R(K)\Lambda_R(K)^\dagger\Lambda_L(K)^\dagger g^{-1} = g\Lambda(K)\Lambda(K)^\dagger g^{-1}. \tag{4.11}
\]

\( \text{Q.E.D.} \)

**Lemma 3.** Every element \( K \in GL(n, \mathbb{C}) \) can be written as

\[
K = g \triangleright (bT^{-1}), \tag{4.12}
\]

where \( g \in U(n) \), \( b \in B \) and \( T \) is in the closure of the Weyl alcove, i.e., \( T \in \mathcal{C} \). Moreover, for every fixed non-zero \( x \in \mathbb{R} \), the element \( g \) can be chosen to satisfy the condition \( (g^{-1}v)_k \geq 0 \) for all \( k = 1, \ldots, n \), where \( v \) is the vector defined by Eq. (4.3).

**Proof.** For every \( h \in U(n) \) and every \( K \in GL(n, \mathbb{C}) \), the following identity follows from the definition (2.3) of the Iwasawa maps:

\[
\Xi_R(\Xi_R^{-1}(h\Lambda_L^{-1}(K))\Lambda_L(K)) = h^{-1}. \tag{4.13}
\]

We infer the surjectivity of the map \( \eta_K : U(n) \to U(n) \) defined for fixed \( K \in GL(n, \mathbb{C}) \) by

\[
\eta_K(g) = \Xi_R(g^{-1}\Lambda_L(K)). \tag{4.14}
\]

Then the existence of \( g \in U(n) \), such that the matrix \( T := \Xi_R(g^{-1} \triangleright K) \) is in \( \mathcal{C} \), is a consequence of the relation

\[
\Xi_R(g^{-1} \triangleright K) = \Xi_R(g^{-1}\Lambda_L(K))^{-1}\Xi_R(K)\Xi_R(g^{-1}\Lambda_L(K)). \tag{4.15}
\]
Let \( \tau \) be any diagonal element of \( U(n) \). Then we have
\[
g \triangleright (bT^{-1}) = (g\tau) \triangleright (\tau^{-1}b\tau T^{-1}).
\] (4.16)

We thus see that the representation \( [4.12] \) of the element \( K \in GL(n, \mathbb{C}) \) is not unique because instead of the triple \( g \in U(n), b \in B, T \in \bar{C} \) we can take a triple \( g\tau \in U(n), \tau^{-1}b\tau \in B, T \in \bar{C} \) for any \( \tau \). In particular, we can arrange the phases in \( \tau \) in such a way that the components of the vector \( \tau^{-1}g^{-1}v \) become real and non-negative. \textit{Q.E.D.}

**Proof of Theorem 1.** Fix a real non-zero \( x \) and parametrize \( K \in GL(n, \mathbb{C}) \) as in Lemma 3. By virtue of Lemma 2, the moment map constraint
\[
\Lambda(K) = \Lambda(g \triangleright (bT^{-1})) = \nu(x)
\] (4.17)
can be rewritten as
\[
g\Lambda(bT^{-1})\Lambda(bT^{-1})^\dagger g^{-1} = \nu(x)\nu(x)^\dagger,
\] (4.18)
or, as
\[
\ln(\Lambda(bT^{-1})\Lambda(bT^{-1})^\dagger) = g^{-1}\ln(\nu(x)\nu(x)^\dagger)g = x((g^{-1}v)(g^{-1}v)^\dagger - 1_n).
\] (4.19)
Note that \( \Lambda(bT^{-1}) \) lies in the group \( N \), for it holds
\[
\Lambda(bT^{-1}) = \Lambda_L(bT^{-1})\Lambda_R(bT^{-1}) = bTb^{-1}T^{-1} \in N.
\] (4.20)
Moreover, following Lemma 3, all components of the vector \( g^{-1}v \) are real non-negative. Lemma 1 then says that
\[
\Lambda(bT^{-1}) = \nu(x);
\] (4.21)
\[
g^{-1}v = v.
\] (4.22)
The condition \( [4.21] \) and the relation \( [4.18] \) imply that \( g \in G_x \), hence, we have so far proved that every solution of the moment map constraint \( [3.3] \) can be written as \( g \triangleright (bT^{-1}) \) where \( g \in G_x \) and \( bT^{-1} \) satisfies the condition \( [4.21] \).

In order to solve the condition \( [4.21] \), we represent \( b \) as \( b = Na \), where \( a \) is diagonal and \( N \) is in the group \( N \). We find immediately that Eq. \( [4.21] \) can be rewritten as
\[
\Lambda(bT^{-1}) = bTb^{-1}T^{-1} = NTN^{-1}T^{-1} = \nu(x),
\] (4.23)
which means that \( a \) can be arbitrary and \( N \) has to satisfy the condition
\[
N = \nu(x)TN^{-1}.
\] (4.24)
Writing the condition \( [4.24] \) in components (with the help of Eq. \( [1.3] \)) gives
\[
(1 - T_jT_{j+k}^{-1})N_{j,j+k} = \sum_{s=1}^{k}(1 - e^{-x})e^{xT_j}N_{j+s,j+k}T_{j+k}^{-1}.
\] (4.25)
With the replacement \( j \to j - 1 \) and \( k \to k + 1 \), Eq. \( [4.25] \) becomes
\[
(1 - T_{j-1}T_{j+k}^{-1})N_{j-1,j+k} = \sum_{s=1}^{k+1}(1 - e^{-x})e^{xT_{j-1+s}}N_{j-1+s,j+k}T_{j+k}^{-1} =
\]
\[
eq e^{xT_j}N_{j,j+k} + (e^{xT_j} - e^{-x})T_jT_{j+k}^{-1}N_{j,j+k} =
\]
\[
eq e^{xT_j}N_{j,j+k} + (e^{xT_j} - e^{-x})T_jT_{j+k}^{-1}N_{j,j+k} =
\] (4.26)
This equation implies that \( T_j \neq T_{j+k} \) for all \( j \) and \( k \), which means that \( T \) must be a regular element of the Weyl alcove. In other words, the strict inequalities must hold in (3.11). Indeed, by setting \( k = 0 \) we that see that \( T_{j-1} \neq T_j \) (and \( N_{j-1,j} \neq 0 \)) because the r.h.s. is non-zero (\( N_{jj} = 1 \)). Then for \( k = 1 \) we see that \( T_{j-1} \neq T_{j+1} \) (and \( N_{j-1,j+1} \neq 0 \)) because the r.h.s. is non-zero, and so on. From Eq. (4.26) we can also infer that

\[
N(T)_{kl} = \prod_{m=1}^{t-k} \frac{e^{\frac{2}{m} T_1 - e^{-\frac{2}{m} T_1} + m}}{T_1 - T_1 + m - 1}, \quad \forall k < l.
\]

(4.27)

It remains to prove the last sentence of the statement of Theorem 1. Suppose that there exist \( T_1, T_2 \in C, a_1, a_2 \in A \) and \( g \in G_x \) such that

\[
N(T_1) a_1 T_1^{-1} = g \triangleright (N(T_2) a_2 T_2^{-1}).
\]

(4.28)

From Eq. (4.28), we have obviously

\[
\Lambda_L(g \triangleright (N(T_2) a_2 T_2^{-1})) = \Lambda_L(g N(T_2) a_2) = \Lambda_L(N(T_1) a_1 T_1^{-1}) = N(T_1) a_1.
\]

(4.29)

Moreover, from Eqs. (1.15) and (1.28), we obtain

\[
\Xi_R(g \triangleright (N(T_2) a_2 T_2^{-1})) = \Xi_R(g N(T_2) a_2)^{-1} T_2 \Xi_R(g N(T_2) a_2) = \Xi_R(N(T_1) a_1 T_1^{-1}) = T_1.
\]

(4.30)

Since both \( T_1 \) and \( T_2 \) are in the open Weyl alcove, it follows that \( \Xi_R(g N(T_2) a_2) := \Theta \) is in the maximal torus \( \mathbb{T}_n \) and \( T_1 = T_2 \). Thus, with the help of Eq. (4.29), we have

\[
g N(T_2) a_2 = \Lambda_L(g N(T_2) a_2) \Xi_R(g N(T_2) a_2)^{-1} = N(T_1) a_1 \Theta^{-1}.
\]

(4.31)

Because \( \Theta \) is diagonal, we have also

\[
\Theta^{-1} = \Xi_L(N(T_1) a_1 \Theta^{-1}) = \Xi_L(g N(T_2) a_2) = g,
\]

(4.32)

hence Eq. (4.31) can be rewritten as

\[
\Theta^{-1} N(T_2) a_2 = N(T_1) a_1 \Theta^{-1}.
\]

(4.33)

This implies the desired statement \( a_1 = a_2 \); and it is also worth observing that \( U(n) \) factorized by its center acts freely on the constrained manifold defined by (3.3). \textit{Q.E.D.}

\textbf{Remark.} We can now explain how we have found the Poisson-Lie analogue \( \nu(x) \) (1.3) of the Kazhdan-Kostant-Sternberg element \( \iota(x) \) (1.1). First, we wanted to proceed similarly to [9], going to a gauge where \( g_R \) is diagonal. We found that the moment map constraint (4.23) can be then satisfied only if \( \nu(x)_{kk} = 1 \) for all \( k \). Second, we wanted that the dimension of the isotropy group \( G_x \) (3.4) be the same as in the standard Kazhdan-Kostant-Sternberg construction. These two requirements lead to the choice of \( \nu(x) \) given in (1.3).

\section{Discussion and outlook}

As was already mentioned in the Introduction, it had been known previously that trigonometric Ruijsenaars-Schneider models can be obtained by reduction based on \textit{infinite-dimensional ordinary} symmetry [8]. In our approach, we have achieved essentially the same goal by means
of reduction based on finite-dimensional Poisson-Lie symmetry. Apart from avoiding analytical subtleties of infinite-dimensional manifolds, one of the advantages of our finite-dimensional approach seems to be the fact that the geometric picture relying on the Heisenberg double automatically comes together with the integration algorithm for the commuting Ruijsenaars-Schneider flows. It appears to us that this cannot be obtained in a similarly simple and direct manner in the infinite-dimensional approach.

There exists also a finite-dimensional Hamiltonian reduction treatment of the complex trigonometric Ruijsenaars-Schneider model utilizing a certain holomorphic, Poisson-Lie symmetric phase space [7]. It is not clear to us whether it is possible to consider a ‘real form’ of that construction (especially before the reduction) in a way which would make contact with our reduction. We expect, however, that the search for a real form variant of the construction of [7] might be a fruitful approach to arrive at a geometric understanding of the hyperbolic Ruijsenaars-Schneider model. We plan to study this problem in the future.

Related further papers treat rational and elliptic Ruijsenaars-Schneider models as well, applying various versions of classical Hamiltonian reduction, but none of the works published to date use compact, finite-dimensional Poisson-Lie groups to obtain directly the real trigonometric model, which we achieved here. The reader may consult [2, 3, 13, 6] and the references therein.

Since our principal guideline in writing this paper was an effort to address also the readers who are not experts in Poisson-Lie geometry, we have not worked out here several further issues that would require a deeper preliminary exposition of the general theory of Poisson-Lie symmetry. To fill this gap, we are preparing a continuation of this article. We shall present there a geometric interpretation of the Ruijsenaars duality [19, 20, 21] that links together two different real forms of the complex trigonometric Ruijsenaars-Schneider model. We shall treat the duality in terms of two gauge slices (of group theoretic origin) in the Poisson-Lie reduction, generalizing the picture put forward in [13, 6] to understand the duality between the trigonometric Sutherland model and a certain real form of the complex rational Ruijsenaars-Schneider model [20]. As a byproduct, we shall directly obtain the dual Lax matrix and the dual commuting flows as well. Finally, we shall give an account of the non-relativistic limit of the trigonometric Ruijsenaars-Schneider model in the geometric perspective. Thereby we shall explain how the Poisson-Lie reduction described in this paper can be viewed not only as a conceptual generalization of the standard Kazhdan-Kostant-Sternberg reduction but also as its natural one-parameter deformation.

We are aware of the fact that there exists a large literature also on the quantum group aspects of the quantum mechanical trigonometric Ruijsenaars-Schneider model, but to discuss this is outside the scope of the present work. See, for example, [21, 5, 14] and references therein. The relationship to the quantization of our Poisson-Lie reduction may be worth further investigation.

**Acknowledgements.** L.F. was supported in part by the Hungarian Scientific Research Fund (OTKA grant T049495) and by the EU network ‘ENIGMA’ (contract number MRTN-CT-2004-5652). He is also grateful to the IML for an invitation, and thanks J. Balog for discussions.
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