The Computational Complexity of Estimating Convergence Time

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Abstract

An important problem in the implementation of Markov Chain Monte Carlo algorithms is to determine the convergence time, or the number of iterations before the chain is close to stationarity. For many Markov chains used in practice this time is not known. Even in cases where the convergence time is known to be polynomial, the theoretical bounds are often too crude to be practical. Thus, practitioners like to carry out some form of statistical analysis in order to assess convergence. This has led to the development of a number of methods known as convergence diagnostics which attempt to diagnose whether the Markov chain is far from stationarity. We study the problem of testing convergence in the following settings and prove that the problem is hard in a computational sense:

• Given a Markov chain that mixes rapidly, it is hard for Statistical Zero Knowledge (SZK-hard) to distinguish whether starting from a given state, the chain is close to stationarity by time $t$ or far from stationarity at time $ct$ for a constant $c$. We show the problem is in AM intersect coAM.

• Given a Markov chain that mixes rapidly it is coNP-hard to distinguish whether it is close to stationarity by time $t$ or far from stationarity at time $ct$ for a constant $c$. The problem is in coAM.

• It is PSPACE-complete to distinguish whether the Markov chain is close to stationarity by time $t$ or far from being mixed at time $ct$ for $c \geq 1$.

1 Introduction

Markov Chain Monte Carlo (MCMC) simulations are an important tool for sampling from high dimensional distributions in Bayesian inference, computational physics and biology and in applications such as image processing. An important problem that arises in the implementation is that if bounds on the convergence time are not known or impractical for simulation then one would like a method for determining if the chain is still far from converged.

A number of techniques are known to theoretically bound the rate of convergence time as measured by the mixing time of a Markov chain, see e.g. [11 11 14]. These have been applied with to problems such as volume estimation [15], Monte Carlo integration of log-concave functions [16], approximate counting of matchings [13] and estimation of partition functions from physics [12]. However, in most practical
applications of MCMC, there are no effective bounds on the convergence time so for example it may not be known if a chain on $2^{100}$ states mixes in time $1000$ or $2^{50}$. Even in the cases where rapid mixing is known, the bounds are often impractical since they are not tight especially since applications usually require multiple independent samples.

As a result, practitioners have focused on the development of a large variety of statistical methods, called convergence diagnostics which try to determine whether the Markov chain is far from stationarity (see e.g. surveys by [10, 4, 6, 5, 7, 17]). A majority of practitioners of the MCMC method run multiple diagnostics to test if the chains have converged. The two most popularly used public domain diagnostic software packages are CODA and BOA [18, 3]. The idea behind many of the methods is to use the samples from the empirical distribution obtained when running one or multiple copies of the chain, possibly from multiple starting states to compute various functionals and identify non-convergence.

While diagnostics are commonly used for MCMC, it has been repeatedly observed that they cannot guarantee convergence, see e.g. [6, 4, 2]. Here we formalize convergence to stationarity detection as an algorithmic problem and study its complexity in terms of the size of the description of the Markov chain, denoted by $n$. Our main contribution is showing that even in cases where the mixing time of the chain is known to be bounded by $n^C$ for some large $C$, the problem of distinguishing whether a Markov chain is close to or far from stationarity at time $n^C$ for $c$ much smaller than $C$ is “computationally hard”. In other words under standard assumptions in computational complexity the problem of distinguishing whether the chain is close to or far from stationarity cannot be solved in time $n^D$ for any constant $D$.

The strength of our results is in their generality as they apply to all possible diagnostics and in the weakness of the assumption - in particular in assuming that the mixing time of the chain is not too long and that the diagnostic is also given the initial state of the chain.

From the point of view of theoretical computer science, our results highlight the role of Statistical Zero Knowledge, AM, coAM and coNP in the computational study of MCMC.

### 2 Results

We begin by defining the mixing time which measures the rate of convergence to the stationary distribution.

Recall that the variation distance (or statistical distance) between two probability distributions $\mu$ and $\nu$ on state space $\Omega$ is given by $d_{tv}(\mu, \nu) = \frac{1}{2} \sum_{\omega \in \Omega} |\mu(\omega) - \nu(\omega)|$.

**Definition 1 (Mixing time)** Let $M$ be a Markov chain with state space $\Omega$, transition matrix $P$ and a unique stationary distribution $\pi$. The following measure of distance to stationarity will be convenient to define:

$$d(t) := \max_{x,y \in \Omega} d_{tv}(P^t(x, \cdot), P^t(y, \cdot)).$$

The $\varepsilon$-mixing time is defined to be,

$$\tau(\varepsilon) := \min\{t : d(t) \leq \varepsilon\}.$$

We refer to $\tau(1/4)$ as the mixing time. We also define the $\varepsilon$-mixing time starting from $x$:

$$\tau_x(\varepsilon) := \min\{t : d_{tv}(P^t(x, \cdot), \pi) \leq \varepsilon\}.$$

We note that $\tau_x(\varepsilon) \leq \tau(\varepsilon)$ for all $x$.

To formulate the problem, we think of the Markov chain as a “rule” for determining the next state of the chain given the current state and some randomness.
**Definition 2** We say that a circuit $C : \{0, 1\}^n \times \{0, 1\}^m \rightarrow \{0, 1\}^n$ specifies $P$ if for every pair of states $x, y \in \Omega$, $Pr_{r \sim \{0, 1\}^m}[C(x, r) = y] = P(x, y)$.

In this formalization, $x$ is the “current state”, $r$ is the “randomness”, $y$ is the “next state” and $C$ is the “rule”.

Next we formalize the notion of “Testing convergence”. We imagine the practitioner has a time $t$ in mind that she would like to run the Markov chain algorithm for. She would like to use the diagnostic to determine whether at time $t$:

- The chain is (say) within $1/4$ variation distance of stationarity
- or at least at distance $1/4$ away from it.

Requiring the diagnostic to determine the total variation at time $t$ exactly is not needed in many situations. Many practitioners will be happy with a diagnostic which will

- Declare the chain has mixed if it is within $1/8$ variation distance of stationarity at time $t$.
- Declare it did not mix if it is at least at distance $1/2$ away from it at time $t$.

An even weaker requirement for the diagnostic is to:

- Declare the chain has mixed if it is within $1/8$ variation distance of stationarity at time $t$.
- Declare it did not mix if it is at least at distance $1/2$ away from it at time $ct$, where $c \geq 1$.

Thus in the last formulation, the practitioner is satisfied with an approximate output of the diagnostics both in terms the time and in terms of the total variation distance. This is the problem we will study. In fact, we will make the requirement from the diagnostic even easier by providing it with a (correct) bound on the actual mixing time of the chain. This bound will be denoted by $t_{\text{max}}$.

In realistic settings it is natural to measure the running time of the diagnostics in relation to the running time of the chain itself as well as to the size of the chain. In particular it is natural to consider diagnostics that would run for time that is polynomial in $t$ and $t_{\text{max}}$. The standard way to formalize such a requirement is to insist that the inputs $t, t_{\text{max}}$ to the diagnostic algorithm to be given in unary form (note that if $t, t_{\text{max}}$ were specified as binary numbers, an efficient algorithm would be required to run in time poly-logarithmic in these parameters, a much stronger requirement). We continue with description of the different diagnostic problems and the statement of the hardness results.

### 2.1 Given Starting Point

The discussion above motivates the definition of the first problem below. Assume that we had a diagnostic algorithm. As input, it would take the the tuple $(C, x, 1^t, 1^{t_{\text{max}}})$, i.e., a description of the circuit which describes the moves of the Markov chain, an initial starting state for the chain, and the times $t$ and $t_{\text{max}}$, which are specified as unary numbers. The following theorems show that a diagnostic algorithm as described above is unlikely to exist under standard complexity-theoretic assumptions. We consider two versions of the convergence testing problem, one where the starting state of the Markov chain is specified (GapPolyTestConvergenceWithStart$_c,\delta$) and the other where it is arbitrary (GapPolyTestConvergence$_c$):

**Problem:** GapPolyTestConvergenceWithStart$_{c,\delta}$ (GPTCS$_{c,\delta}$).

**Input:** $(C, x, 1^t, 1^{t_{\text{max}}})$, where $C$ is a circuit specifying a Markov chain $P$ on state space $\Omega \subseteq \{0, 1\}^n$, where $x$ is the “current state”, $r$ is the “randomness”, $y$ is the “next state” and $C$ is the “rule”.
\[ x \in \Omega \text{ and } t, t_{\max} \in \mathbb{N}. \]

**Promise:** The Markov chain \( P \) is ergodic and \( \tau(1/4) \leq t_{\max}. \)

**YES instances:** \( \tau_x(1/4 - \delta) < t. \)

**NO instances:** \( \tau_x(1/4 + \delta) > ct. \)

Informally the input to this problem is the MC rule \( C \), a starting state \( x \), and times \( t, t_{\max} \). It is promised that the chain mixes by time \( t_{\max} \). The expectation from the diagnostic is to:

- Declare the chain has mixed if it is within \( 1/4 - \delta \) variation distance of stationarity at time \( t \).
- Declare it did not mix if it is at least at distance \( 1/4 + \delta \) away from it at time \( ct \), where \( c > 1 \).

Note again that the diagnostic is given room for error both in terms of the total variation distance and in terms of the time.

The following theorem refers to the complexity class \( \text{SZK} \), which is the class of all promise problems that have statistical zero-knowledge proofs with completeness \( 2/3 \) and soundness \( 1/3 \). It is believed that these problems cannot be solved in polynomial time.

**Theorem 1** Let \( c \geq 1. \)

- For \( 0 < \delta \leq 1/4 \), \( \text{GPTCS}_{c,\delta} \) is in \( \text{AM} \cap \text{coAM} \).
- For \( \sqrt{\frac{3}{2}} - 1.5 = 0.116025 \ldots < \delta \leq 1/4 \), \( \text{GPTCS}_{c,\delta} \) is in \( \text{SZK} \).
- Let \( 0 \leq \delta < 1/4 \). For
  \[ c < \frac{t_{\max}}{4t} \ln \left( \frac{2}{1 + 4\delta} \right), \]
  \( \text{GPTCS}_{c,\delta} \) is \( \text{SZK} \)-hard.

The most interesting part of the theorem is the last part which informally says that the problem \( \text{GPTCS}_{c,\delta} \) is \( \text{SZK} \)-hard. In other words, solving it in polynomial time will result in solving all the problems in \( \text{SZK} \) in polynomial time. The second part of the theorem states that for some values of \( \delta \) this is the “exact” level of hardness. The first part of the theorem states that without restrictions on \( \delta \) the problem belongs to the class \( \text{AM} \cap \text{coAM} \) (which contains the class \( \text{SZK} \)). The classes \( \text{AM} \) and \( \text{coAM} \) respectively contain the classes \( \text{NP} \) and \( \text{coNP} \) and it is believed that they are equal to them, but this is as yet unproven.

The restriction on the constant \( \delta \) in the second part of the result comes from the fact that the proof is by reduction to the \( \text{SZK} \)-complete problem \( \text{Statistical Distance} \) (SD, see Section 3 for precise definitions). Holenstein and Renner give evidence in [9] that SD is in \( \text{SZK} \) only when there is a lower bound on the gap between the completeness and soundness. We show that the restriction in Theorem 1 necessary since otherwise it would be possible to put SD in \( \text{SZK} \) for a smaller value of the completeness-soundness gap.

On the other hand, we can show a slightly weaker result and put \( \text{GPTCS}_{c,\delta} \) into \( \text{AM} \cap \text{coAM} \) without any restrictions on \( \delta \). To show this, we first prove that SD is in \( \text{AM} \cap \text{coAM} \) when no restriction is put on the gap between the completeness and soundness. This result may be interesting in its own right as it involves showing protocols for \( \text{Statistical Distance} \) that are new, to our knowledge.

### 2.2 Arbitrary Starting Point

So far we have discussed mixing from a given starting point. A desired property of a Markov chain is fast mixing from an arbitrary starting point. Intuitively, this problem is harder than the previous one since it
involves all starting points. This is consistent with our result below where we obtain a stronger hardness.

**Problem:** \textsc{GAPPolyTestConvergence}_{c,\delta} (GPTC_{c,\delta}).

**Input:** \((C, x, 1^t, 1^{t_{\text{max}}})\), where \(C\) is a circuit specifying a Markov chain \(P\) on state space \(\Omega \subseteq \{0,1\}^n\), \(x \in \Omega\) and \(t, t_{\text{max}} \in \mathbb{N}\).

**Promise:** The Markov chain \(P\) is ergodic and \(\tau(1/4) \leq t_{\text{max}}\).

YES instances: \(\tau(1/4 - \delta) < t\).

NO instances: \(\tau(1/4 + \delta) > ct\).

Note that the only difference between this and the previous problem is that the total variation distance is measured from the worst starting point instead of from a given starting point.

**Theorem 2** Let \(c \geq 1\).

- For \(0 < \delta \leq 1/4\), \(\text{GPTC}_{c,\delta} \in \text{coAM}\).
- Let \(0 \leq \delta < 1/4\). For
  \[
c < \frac{3/4 - \delta}{2\sqrt{t_{\text{max}}/t^2n^3}}
  \]
  it is coNP-hard to decide \(\text{GPTC}_{c,\delta}\).

Again the second part of the theorem is the more interesting part. It shows that the diagnostic problem is coNP hard so it is very unlikely to be solved in polynomial time. This hardness is stronger than \(\text{SZK}\)-hardness because \(\text{SZK}\) is unlikely to contain coNP-hard problems. If it did, this would imply that \(\text{NP} = \text{coNP}\) since \(\text{SZK} \subseteq \text{AM}\) and it is believed that \(\text{AM} = \text{NP}\). The first part of the theorem shows that the problem is always in coAM.

### 2.3 Arbitrary mixing times

Finally we remove the restriction that the running time of the algorithm should be polynomial in the times \(t, t_{\text{max}}\). This corresponds to situations where the mixing time of the chain may be exponentially large in the size of the rule defining the chain. This rules out many situations of practical interest. However it is relevant in scenarios where analysis of the mixing time is of theoretical interest. For example there is an extensive research in theoretical physics on the rate of convergence of Gibbs samplers on spin glasses even in cases where the convergence rate is very slow (see [8] and follow up work). In such setups it is natural to define the problem as follows:

**Problem** \textsc{GapTestConvergence}_{c,\delta} (GTC_{c,\delta}).

**Input:** \((C, x, t)\), where \(C\) is a circuit specifying a Markov chain \(P\) on state space \(\Omega \subseteq \{0,1\}^n\), \(x \in \Omega\) and \(t \in \mathbb{N}\).

**Promise** The Markov chain \(P\) is ergodic.

YES instances: \(\tau(1/4 - \delta) < t\).

NO instances: \(\tau(1/4 + \delta) > ct\).

Note that the main difference is that in this problem the time \(t\) is given in binary representation. Thus, informally in this case the efficiency is measured with respect to the logarithm of \(t\). Additionally note that the mixing time of the chain itself does not put any restrictions on the diagnostic. We then prove the following result:
**Theorem 3** Let $1 \leq c \leq \exp(n^{O(1)})$.

- For $\exp(-n^{O(1)}) < \delta \leq 1/4$ it is in PSPACE to decide $GTC_{c,\delta}$.
- Let $0 \leq \delta < 1/4$, then, it is PSPACE-hard to decide $GTC_{c,\delta}$.

It is known that PSPACE hard problems are at least as hard as all the problem in polynomial time coNP, NP and all other problems in the polynomial hierarchy.

### 3 Protocols for statistical distance

Given a circuit $C: \{0,1\}^n \rightarrow \{0,1\}^n$, the probability distribution $p_C$ associated to $C$ assigns probability $p(\omega) = |C^{-1}(\omega)|/2^n$ to every $\omega \in \{0,1\}^n$. We will be interested in estimating the statistical distance between the distributions associated to a pair of circuits $C, C': \{0,1\}^n \rightarrow \{0,1\}^n$. Denote those distributions by $p$ and $p'$, respectively.

For a pair of constants $0 \leq s < c \leq 1$, $SD_{c,s}$ is defined to be the following promise problem. The inputs are pairs of circuits $C, C': \{0,1\}^n \rightarrow \{0,1\}^n$, the YES instances satisfy $d_{tv}(p, p') \geq c$, and the NO instances satisfy $d_{tv}(p, p') < s$.

Sahai and Vadhan [20] show that for every pair of constants $c, s$ the problem $SD_{c,s}$ is SZK-hard. They also show that when $c^2 > s$, $SD_{c,s}$ is in SZK. Our theorem yields a weaker conclusion, but covers a wider spectrum of parameters.

**Theorem 4** For any pair of constants $0 \leq s < c \leq 1$, $SD_{c,s}$ is in $AM \cap coAM$.

#### 3.1 An AM protocol

The following interactive protocol $P$ for $SD_{c,s}$ essentially appears in [20] but we rewrite it here for the precise parameters we need:

**V:** Flip a fair coin. If heads, generate a random sample from $C$. If tails, generate a random sample from $C'$. Send the sample $x$ to the prover.

**P:** Say if $x$ came from $C$ or from $C'$.

**V:** If prover is correct accept, otherwise reject.

**Claim 1** Protocol $P$ is an interactive proof for $SD_{c,s}$ with completeness $1/2 + c$ and soundness $1/2 + s$.

**Proof:** We prove soundness first. Let $T$ be the set of $x$s which the prover claims came from $C$. The accepting probability is

\[
\sum_{x \in T} \frac{p(x)}{2} + \sum_{x \not\in T} \frac{p'(x)}{2} = \frac{1}{2} \left( \sum_{x \in T} p(x) + \sum_{x \not\in T} p'(x) \right).
\]

No matter what $T$ is, we have that

\[
\frac{1}{2} \left( \sum_{x \in T} p(x) + \sum_{x \not\in T} p'(x) \right) \leq \frac{1}{2} \left( 1 - \sum_{x \not\in T} p(x) + \sum_{x \not\in T} p'(x) \right) \leq 1/2 + d_{tv}(p, p'),
\]

and so the accepting probability is at most $1/2 + s$.

To prove completeness, notice that the above inequality is tight when $T$ equals the set of those $x$ such that $p(x) > p'(x)$. So when the prover uses this strategy (say $C$ if $p(x) > p'(x)$ and $C'$ otherwise), the accepting probability becomes exactly $1/2 + d_{tv}(p, p') \geq 1/2 + c$.  

\[\Box\]
3.2 A coAM protocol

Showing that $SD_{c,s}$ is in coAM is a bit more involved. Such a protocol wants to accept when the statistical distance between $p$ and $p'$ is small, and reject when the statistical distance is large. To develop some intuition, let us first attempt to distinguish the cases when $p$ and $p'$ are the same distribution (i.e. $s = 0$) and the case when they are at some distance from one another (say $e = 1/2$).

Let’s forget for a moment that the verifier has to run in polynomial time. Suppose the verifier could get hold of the values

$$N(t) = ||\{\omega: |C^{-1}(\omega)| \geq t \text{ and } |C'^{-1}(\omega)| \geq t\}||$$

for every $t$ (which could potentially range between 0 and $2^n$). Then it can compute the desired statistical distance via the following identity which will be proven later:

$$\sum_{t=1}^{2^n} t \cdot (N(t) - N(t + 1)) = (1 - d_{tv}(p, p')) \cdot 2^n. \quad (1)$$

If we want the verifier to run in polynomial time, there are two issues with this strategy: First, the verifier does not have time to compute the values $N(t)$ and second, the verifier cannot evaluate the exponentially long summation in (1). If we only want to compute the statistical distance approximately, the second issue can be resolved by quantization: Instead of computing the sum on the left for all the values of $t$, the verifier chooses a small number of representative values and estimates the sum approximately. For the first issue, the verifier will rely on the prover to provide (approximate) values for $N(t)$. While the verifier cannot make sure that the values provided by a (cheating) prover will be exact, she will be able to ensure that the prover never grossly over-estimates the sum on the left by running a variant of the Goldwasser-Sipser protocol which we describe below. Since the sum on the left is proportional to one minus the statistical distance, it will follow that no matter what the prover’s strategy is, he cannot force the verifier to significantly underestimate the statistical distance without being detected.

We now give the details of this protocol, starting with a proof of (1).

**Proof of identity (1):** Let $f(\omega) = \min\{|C^{-1}(\omega)|, |C'^{-1}(\omega)|\}$. Then

$$\sum_{\omega \in \{0,1\}^n} f(\omega) = \sum_{t=1}^{2^n} t \cdot |\{\omega: f(\omega) = t\}| = \sum_{t=1}^{2^n} t \cdot (|\{\omega: f(\omega) \geq t\}| - |\{\omega: f(\omega) \geq t + 1\}|).$$

The right-hand side of this expression is exactly equal to the left-hand side of (1). For the left-hand size, using the formula $\min\{a, b\} = (a + b)/2 - |a - b|/2$ (where $a, b \geq 0$) we have

$$\sum_{\omega \in \{0,1\}^n} f(\omega) = \frac{1}{2} \sum_{\omega \in \{0,1\}^n} (|C^{-1}(\omega)| + |C'^{-1}(\omega)|) - \frac{1}{2} \sum_{\omega \in \{0,1\}^n} ||C^{-1}(\omega)| - |C'^{-1}(\omega)|| = 2^n - d_{tv}(p, p') \cdot 2^n$$

which equals the right-hand side of (1). ■

**A lower bound protocol for $N(t)$** We now show that a variant of the Goldwasser-Sipser lower bound protocol can be used to certify lower bounds on the quantities $N(t)$. More precisely, we design an AM protocol for the following problem:

**Input:** A pair of circuits $C, C': \{0,1\}^n \to \{0,1\}^n$, a number $1 \leq t \leq 2^n$, a target number $0 \leq \tilde{N} \leq 2^n$, and a fraction $0 < \delta \leq 1$ (represented in unary).

**Yes instances:** $(C, C', t, \tilde{N}, \delta)$ such that $N(t) \geq \tilde{N}$

**No instances:** $(C, C', t, \tilde{N}, \delta)$ such that $N((1 - \delta)t) < (1 - \delta)\tilde{N}$.
Here is a protocol for this problem. Here, $\delta_1, \delta_2$ are the largest values below $\delta$ that make the logarithms below integers. In the analysis, for simplicity we will assume that $\delta_1 = \delta_2 = \delta$.

**V:** Set $a = \log(\delta_1^2 \tilde{N}/54)$. Send a random hash function $g: \{0,1\}^n \rightarrow \{0,1\}^a$.

**P:** Let $c = \lceil (1 - \delta_1/2)(54/\delta_1^2) \rceil$. Send a set of values $\{\omega_1, \ldots, \omega_c\}$.

**V:** Set $b = \log(\delta_2^2 t/5000)$. Send a random hash function $h: \{0,1\}^n \rightarrow \{0,1\}^b$.

**P:** Let $d = \lceil (1 - \delta_2/2)(5000/\delta_4^2) \rceil$. For each $1 \leq i \leq c$, send sets $\{r_{i1}, \ldots, r_{ia}\}$ and $\{r'_{i1}, \ldots, r'_{ia}\}$.

**V:** If $g(\omega_i) = 0$ for all $i$ and $h(r_{ij}) = h(r'_{ij}) = 0$ and $C(r_{ij}) = C'(r'_{ij}) = \omega_i$ for all pairs $(i,j)$, accept, otherwise reject.

We first prove completeness: If $(C, C', t, \tilde{N}, \delta)$ is a yes instance, the protocol accepts with probability at least $2/3$. Let

$$S = \{\omega: |C^{-1}(\omega)| \geq t \text{ and } |C'^{-1}(\omega)| \geq t\}.$$  

The expected number of $\omega \in S$ with $g(\omega) = 0$ is at least $(54/\delta^2) \cdot (N(t)/\tilde{N})$. If $N(t) \geq \tilde{N}$, by Chebyshev’s inequality, the probability over $g$ of getting fewer than $c = (1 - \delta)/(54/\delta^2)$ such $\omega_i$s is at most $1/6$. Assuming all these $\omega_i$s exist, let’s fix one of them. We now look at the set $T_i = \{r: C(r) = \omega_i\}$. Since $\omega_i \in S$, $T_i$ has size at least $t$, so the expected number $r \in T_i$ such that $h(r) = 0$ is at least $5000/\delta_4^4$. By Chebyshev’s inequality, the probability of getting fewer than $d$ such $r_{ij}$s is at most $\delta^2/1248$. This bound holds for every $i$ and also for the sets $T'_i = \{r: C'(r) = \omega_i\}$. Taking a union bound over all $2c$ such sets we get that with probability at least $5/6$ over the choice of $h$, a sufficient number of $r_{ij}$s and $r'_{ij}$s exist for all values of $i$, so the verifier accepts.

We now prove soundness: If $(C, C', t, \tilde{N}, \delta)$ is a no instance, the protocol accepts with probability at most $2/3$. Now let

$$S = \{\omega: |C^{-1}(\omega)| \geq (1 - \delta)t \text{ and } |C'^{-1}(\omega)| \geq (1 - \delta)t\}.$$  

The expected number of $\omega \in S$ with $g(\omega) = 0$ is then at most $(1 - \delta)(54/\delta^2)$. In this case, $c$ is at least equal to $(1 + \delta/3)$ times this expected value. By Chebyshev’s inequality, the probability that there exist $c$ such $\omega_i$s is then less than $1/6$. If not, then the prover is forced to send at least one $\omega_i$ such that either $g(\omega_i) \neq 0$ or $\omega_i \notin S$. In the first case, the verifier rejects. In the second case, we let

$$T_i = \{r: C(r) = \omega_i\} \quad \text{and} \quad T'_i = \{r: C'(r) = \omega_i\}$$

so either $|T_i| < (1 - \delta)t$ or $|T'_i| < (1 - \delta)t$. Without loss of generality, let us assume the first case. Then the expected number of $r \in T$ such that $h(r) = 0$ is at most $(1 - \delta)(5000/\delta_4^4)$. We apply Chebyshev’s inequality again to conclude that with probability at least $5/6$, the prover is then forced to send some $r_{ij}$ such that either $h(r_{ij}) \neq 0$ or $C(r_{ij}) \neq \omega_i$. Thus the verifier accepts with probability at most $1/6 + 1/6 \leq 1/3$.

Repeating this protocol in parallel sufficiently many times, we have the following consequence, which we will use below:

**Claim 2** There is an AM lower bound protocol for $N(t)$ with completeness $1 - \delta/20n$ and soundness $\delta/20n$. 


A coAM protocol for statistical distance. We now give the coAM protocol for statistical distance. We begin with the observation that it is sufficient to handle the following special case of the problem:

**Input:** A pair of circuits $C, C': \{0,1\}^n \rightarrow \{0,1\}$ and a fraction $0 < \delta \leq 1/3$ (represented in unary).

**Yes instances:** $(C, C', \delta)$ such that $d_{tv}(p,p') \leq \delta$

**No instances:** $(C, C', \delta)$ such that $d_{tv}(p,p') > 3\delta$.

We can reduce $SD_{c,s}$ for any pair of constants $0 \leq s < c \leq 1$ to the above problem via the XOR lemma of Sahai and Vadhan [20], which reduces $SD_{c,s}$ to $SD_{d,k,s^k}$ for an arbitrary constant $k$. When $k$ is chosen so that $(c/s)^k > 3$, the resulting instance can be handled by our protocol.

We now give the protocol for statistical distance:

**P:** Send claims $\tilde{N}_i$ for the values $N_i = N((1-\delta)^{-i})$, $0 \leq i \leq en/\delta$.

**P, V:** Run the AM lower bound protocol for $N_i$ on inputs $(C, C', (1-\delta)^{-i}, \tilde{N}_i, \delta)$ for every $1 \leq i \leq en/\delta$. If all of them pass accept, otherwise reject.

**V:** Accept if $\sum_{i=0}^{en/\delta} (\tilde{N}_i - \tilde{N}_{i+1})(1-\delta)^{-i} \geq (1-\delta)^2 \cdot 2^n$.

The soundness and completeness rely on the following approximation, which is a quantized version of (1):

$$\sum_{i=0}^{en/\delta} (N_i - N_{i+1})(1-\delta)^{-i} \leq (1 - d_{tv}(p,p'))2^n \leq \sum_{i=0}^{en/\delta} (N_i - N_{i+1})(1-\delta)^{-i}. \tag{2}$$

This is proved in a similar way as (1). For every $i$, we have the sandwiching inequality

$$(N_i - N_{i+1})(1-\delta)^{-i} \leq \sum_{\omega: f(\omega) \in [(1-\delta)^{-i}, (1-\delta)^{-(i+1)}]} f(\omega) \leq (N_i - N_{i+1})(1-\delta)^{-i},$$

which yields (2), after summing over all $i$ from 0 to $en/\delta$.

To prove completeness, consider an honest prover which claims $\tilde{N}_i = N_i$ for all $i$. By Claim 2 and a union bound, with probability at least $2/3$ none of the lower bounds protocols for $N_i$ reject. In this case, using (2), we get

$$\sum_{i=0}^{en/\delta} (\tilde{N}_i - \tilde{N}_{i+1})(1-\delta)^{-i} \geq (1-\delta)(1 - d_{tv}(p,p')) \cdot 2^n$$

establishing completeness. To prove soundness, assume now that the verifier accepts with probability at least $1/3$. By the soundness of the lower bound protocols for $N_i$ (Claim 2) and a union bound, there must exist at least one setting of the randomness of the verifier for which $N_{i-1} \geq (1-\delta)\tilde{N}_i$ for all $i$ (where $N_{-1} = N_0$) and the verifier accepts. Now (using the fact that the last value of $N_i$ is zero):

$$\sum_{i=0}^{en/\delta} (N_i - N_{i+1})(1-\delta)^{-i} = \frac{N_{i-1}}{1-\delta} + \sum_{i=0}^{en/\delta-1} N_i((1-\delta)^{-i} - (1-\delta)^{-i})$$

$$\geq \tilde{N}_0 + \sum_{i=0}^{en/\delta-1} (1-\delta)\tilde{N}_{i+1}((1-\delta)^{-i} - (1-\delta)^{-i})$$

$$= \delta \tilde{N}_0 + (1-\delta) \cdot \sum_{i=0}^{en/\delta} (\tilde{N}_i - \tilde{N}_{i+1})(1-\delta)^{-i}$$

$$\geq (1 - \delta)^3 \cdot 2^n$$

so from (2) we get that $1 - d_{tv}(p,p') \geq (1 - \delta)^3$, so $d_{tv}(p,p') \leq 1 - (1 - \delta)^3 \leq 3\delta$. 

4 Diagnosing Convergence for Polynomially Mixing Chains

The results of this section imply that even if the mixing time is restricted to being polynomial the diagnostic problem remains hard. The two cases we consider are the worst case start mixing time and the mixing time from a given starting state. Both hardness results are by reduction from a complete problem in the respective classes. We first prove Theorem 1.

Lemma 1 The problem GPTCS\(_c,\delta\) is in SZK for all \(c \geq 1\) and \(\sqrt{\frac{3-1.5}{2}} = .116025... < \delta \leq 1/4\).

Proof: The proof is by reduction to SD\(_c, s\) where \(c\) and \(s\) are chosen as follows. Choose \(k\) large enough such that

\[
\left(\frac{1}{4} + \delta - \frac{1}{k}\right)^2 > \frac{1}{4} - \delta + \frac{1}{k}.
\]

Let

\[
s = \frac{1}{4} - \delta + \frac{1}{k}
\]

and

\[
c = \frac{1}{4} + \delta - \frac{1}{k}.
\]

Suppose we are given an instance of GPTCS\(_c,\delta\) with input \((C, x, 1^t, 1^{t_{\text{max}}})\). Let \(\tau = \tau(1/k)\) be the time to come within \(1/k\) in variation distance of the stationary distribution. Let \(C\) output the distribution \(P^t(x, \cdot)\) over \(\Omega\). Let \(C'\) output the distribution \(P^\tau(x, \cdot)\) over \(\Omega\). In the YES case,

\[
|P^t(x, \cdot) - P^\tau(x, \cdot)| \leq \frac{1}{4} - \delta + \frac{1}{k}
\]

while in the NO case,

\[
|P^ct(x, \cdot) - P^\tau(x, \cdot)| > \frac{1}{4} + \delta - \frac{1}{k}.
\]

Since \(c \geq 1\), this implies that

\[
|P^t(x, \cdot) - P^\tau(x, \cdot)| > \frac{1}{4} + \delta - \frac{1}{k}.
\]

By (3), the constructed instance of SD\(_c, s\) is in SZK and the lemma follows. ■

Lemma 2 The problem GPTCS\(_c,\delta\) is in AM \(\cap\) coAM for all \(c \geq 1\) and \(0 < \delta \leq 1/4\).

This part of the result follows directly from Theorem 4 by reducing GPTCS\(_c,\delta\) to SD\(_c, s\) as above, without the restriction on the gap between \(c\) and \(s\). We can show that the gap for \(\delta\) in Lemma 1 is required for membership in SZK. Sahai and Vadhan [20] show that when \(c^2 > s\), SD\(_c, s\) is in SZK. Holenstein and Renner [9] show that this condition on the gap between \(c\) and \(s\) is in fact essential for membership in SZK.

Proposition 1 There exist \(c, s\) satisfying \(c^2 < s < c\) and \(c\) such that if there is an SZK protocol for an instance of GPTCS\(_c,\delta\) with a sufficiently small \(\delta\), then there is an SZK protocol for SD\(_c, s\).

Proof: The proof is by reduction from SD\(_c, s\) to GPTCS\(_c,\delta\). Let \((C, C')\) be an instance of SD\(_c, s\) where \(C\) and \(C'\) are circuits which output distributions \(\mu_1\) and \(\mu_2\) over \(\{0, 1\}^n\). Construct the Markov chain \(P\), whose state space is \([m] \times \{0, 1\}^n\) where \(m = p(n)\) is a polynomial in \(n\). The transitions of the chain are defined as follows. Let the current state be \((X_t, Y_t)\) where \(X_t \in [m]\) and \(Y_t \in \{0, 1\}^n\).
• If $X_t = 1$, choose $Y_{t+1}$ according to $\mu_1$.
• If $X_t = 2$, choose $Y_{t+1}$ according to $\mu_2$.
• Otherwise, set $Y_{t+1} = Y_t$.
• Choose $X_{t+1}$ uniformly at random from $[m]$.

The stationary distribution of the chain is given by $\pi(z, y) = \frac{1}{m}(\frac{1}{2}\mu_1(y) + \frac{1}{2}\mu_2(y))$. Take the starting state to be $x = (1, 0^n)$. In one step, the total variation distance from stationary can be bounded as

$$d_{tv}(P(x, \cdot), \pi) = \frac{1}{2}d_{tv}(\mu_1, \mu_2)$$

For $t > 1$, we have

$$P^t(x, \cdot) = U_{[m]} \times \left(1 - \left(\frac{m - 2}{m}\right)^{t-1}\right)\left(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2\right) + \left(\frac{m - 2}{m}\right)^{t-1}\mu_1$$  \hfill (4)

Hence, it can be verified that

$$d_{tv}(P^t(x, \cdot), \pi) = \frac{1}{2}\left(\frac{m - 2}{m}\right)^{t-1}d_{tv}(\mu_1, \mu_2)$$  \hfill (5)

Let $0 < \delta < (\sqrt{5}/2 - 1)/2$, $s = 1/2 - 2\delta$ and $c = 1/2 + 2\delta$ so that $c^2 < s < c$. Set $c = 1$, $t = 1$ and $t_{max} = m$.

In the YES case, $d_{tv}(\mu_1, \mu_2) < s$ and hence after one step,

$$d_{tv}(P(x, \cdot), \pi) < \frac{1}{2}s < \frac{1}{4} - \delta$$ \hfill (6)

In the NO case, $d_{tv}(\mu_1, \mu_2) > c$ and after one step,

$$d_{tv}(P(x, \cdot), \pi) \geq \frac{1}{2}c > \frac{1}{4} + \delta.$$ \hfill (7)

From (5) it can be seen that in both cases, $\tau(1/4) \leq m = t_{max}$. This completes the reduction since if there is an SZK protocol for $\text{GPTCS}_{c,\delta}$ with the above parameters, then it can be used to distinguish the YES and NO case of $\text{SD}_{c,s}$ for the above values of $c, s$.

We now complete the proof of Theorem

\begin{lemma}
Let $0 \leq \delta < 1/4$. For $1 \leq c < \frac{t_{max}}{4}\ln\left(\frac{2}{1+4\delta}\right)$, the problem $\text{GPTCS}_{c,\delta}$ is SZK-hard.
\end{lemma}

\begin{proof}
The proof uses the same reduction as in Proposition from $\text{SD}_{c,s}$. We recall that

$$d_{tv}(P^t(x, \cdot), \pi) = \frac{1}{2}\left(\frac{m - 2}{m}\right)^{t-1}d_{tv}(\mu_1, \mu_2)$$
Choose $m \geq 3$. Set $s = 1/4 - \delta$ and $c = 1$. Note that $c^2 > s$. In the YES case, $d_{tv}(\mu_1, \mu_2) < s$ and hence for any $t \geq 1$,
\[ d_{tv}(P^t(x, \cdot), \pi) < \frac{1}{2}s < \frac{1}{4} - \delta \]  
(8)
In the NO case, $d_{tv}(\mu_1, \mu_2) > c$ and hence
\[ d_{tv}(P^t(x, \cdot), \pi) \geq \frac{1}{2} \left( \frac{m - 2}{m} \right)^{ct-1} c \geq \frac{1}{2} \left( \frac{m - 2}{m} \right)^{ct-1} . \]  
(9)
Since $m \geq 3$, if $ct < \frac{m}{4} \ln \left( \frac{2}{1+4\delta} \right)$, then $d_{tv}(P^t(x, \cdot), \pi) > \frac{1}{4} + \delta$. Further, we see that in both the YES and NO case,
\[ \tau(1/4) \leq m \]  
(10)
We conclude the reduction by setting $t_{\text{max}} = m$. ■

Next we prove Theorem 2 and classify the complexity of diagnosing mixing from an arbitrary starting state given that the chain mixes in polynomial time. We will use the following result relating mixing time to the conductance.

**Definition 3 (Conductance, see e.g. [19])** Let $M$ be a Markov chain corresponding to the random walk on an edge weighted graph with edge weights $\{w_e\}$. Let $d_x$ denote the weighted degree of a vertex $x$. Define the conductance of $M$ to be $\Phi(M) := \min_{A \subseteq \Omega \neq \emptyset} \Phi_A(M)$ where
\[ \Phi_A(M) := \sum_{x \in A, y \in A^c} w_{xy} \sum_{x \in A} d_x \]  
(11)

**Theorem 5 (see [19])** Let $M$ be a Markov chain corresponding to the random walk on an edge weighted graph with edge weights $\{w_e\}$ as above. Let $\pi$ be the stationary distribution of the Markov chain.
\[ \tau(\varepsilon) \leq \frac{2}{\Phi^2(M)} \log \left( \frac{2}{\pi_{\text{min}} \varepsilon} \right) \]
where $\pi_{\text{min}}$ is the minimum stationary probability of any vertex.

**Lemma 4** For every $c \geq 1$, $0 < \delta \leq 1/4$, GPTC$_{c,\delta}$ is in coAM.

**Proof:** In the first step of the coAM protocol for GPTC$_{c,\delta}$ the prover sends a pair $x, y \in \Omega$ that maximizes $d_{tv}(P^t(x, \cdot), P^t(y, \cdot))$. Let $C_x$ be the circuit which outputs the distribution $P^t(x, \cdot)$ and let $C_y$ output the distribution $P^t(y, \cdot)$. In the YES case $\tau(1/4 - \delta) < t$ and for every $x, y$, $d_{tv}(P^t(x, \cdot), P^t(y, \cdot)) < 1/4 - \delta$. In the NO case, $\tau(1/4 + \delta) > ct$ and $c \geq 1$, therefore there must exist $x, y$ such that $d_{tv}(P^t(x, \cdot), P^t(y, \cdot)) > 1/4 + \delta$.

By Claim 1 there is an AM protocol $P$ for SD$_{1/4+\delta, 1/4-\delta}$ with completeness $3/4 + \delta$ and soundness $3/4 - \delta$. The prover and the verifier now engage in the AM protocol to distinguish whether the distance between the two distributions is large or small. The completeness and soundness follow from those of the protocol $P$. ■
Lemma 5 Let $0 \leq \delta < 1/4$. For $1 \leq c < 1/2\sqrt{t_{\text{max}}/t^2n^{2}(3/4 - \delta)}$, it is coNP-hard to decide $\text{GPTC}_{c,\delta}$.

Proof: The proof is by reduction from $\text{UNSAT}$, which is coNP hard. Let $\psi$ be an instance of $\text{UNSAT}$, that is, a CNF formula on $n$ variables. The vertices of the Markov chain are the vertices of the hypercube $H$, $V(H) = \{0, 1\}^n$ and edges $E(H) = \{(y_1, y_2) : |y_1 - y_2| = 1\}$. We set edge weights for the Markov chain as follows. Let $d$ be a parameter to be chosen later which is at most a constant.

- For each edge in $E(H)$ set the weight to be 1.
- If $\psi(y) = 0$ add a self loop of weight $n$ at $y$.
- If $\psi(y) = 1$ add a self loop of weight $n^d$ at $y$.

In the YES case, if $\psi$ is unsatisfiable, the Markov chain is just the random walk on the hypercube with probability $1/2$ of self loop at each vertex and it is well known that

$$\tau(1/4 - \delta) \leq C_\delta n \log n$$

where $C_\delta$ is a constant depending on $(1/4 - \delta)^{-1}$ polynomially.

In the NO case, where $\psi$ is satisfiable, we will lower bound the time to couple from a satisfying state $y$ and the state $\overline{y}$, obtained by flipping all the bits of $y$. Consider the distributions $X(t), Y(t)$ of the chain which are started at $y$ and at $\overline{y}$. We can bound the variation distance after $t$ steps as follows

$$d(t) \geq 1 - P[\exists s \leq t \text{ s.t. } X(s) \neq y] - P[\exists s \leq t \text{ s.t. } Y(s) = y]$$

In each step, the chain started at $y$ has chance at most $1/(n^{d-1} + 1)$ of leaving. On the other hand, the probability that the walk started from $\overline{y}$ hits $y$ in time $t$ is exponentially small. Therefore

$$d(t) \geq 1 - 2t/(n^{d-1} + 1)$$

which implies that

$$\tau(1/4 + \delta) > \frac{1}{2}n^{d-1}(3/4 - \delta).$$

Choose $d$ to be a large enough constant (which may depend polynomially on $\delta^{-1}$), such that

$$\frac{1}{2}n^{d-1}(3/4 - \delta) > cC_\delta n \log n. \quad (12)$$

On the other hand we can show a polynomial upper bound on the mixing time by bounding the conductance as follows. Let $M'$ be the Markov chain which is the random walk on the hypercube with self loop probabilities of $1/2$ (where the edge weights are as in the case where $\psi$ is unsatisfiable). We bound the conductance of $M$ by showing it is not too much smaller than the conductance of $M'$. We use the fact that for any vertex $x$, the weighted degree $d_x \leq (n^{d-1} + 1)d_x$. Let $A \subseteq V(H)$.

$$\Phi_A(M) = \sum_{x \in A, y \in A^c} \frac{w_{xy}}{d_x} = \sum_{x \in A, y \in A^c} \frac{w'_{xy}}{d_x} \geq \sum_{x \in A, y \in A^c} \frac{w'_{xy}}{(n^{d-1} + 1)\sum_{A \in \Omega} d_x} \geq \Phi_A(M') \geq \frac{1}{n^{d-1} + 1} \geq \frac{1}{n^d + 1}$$

where we are assuming the lower bound on the conductance of the hypercube is $1/n$.
We can lower bound $\pi_{\min}$ by $1/(n2^{n-1} + n^d2^n)$ and hence we have for large enough $n$,

$$\log(\pi_{\min})^{-1} \leq 2n$$

and hence by Theorem $\tau(1/4) \leq 32n^{2d+1}$.

The reduction can be completed by setting $x = 0^n$, the vector of all 0’s, $t_{\max} = 32n^{2d+1}$ and $t = C_\delta n \log n$. By (12), we see that $ct < 1/2 \sqrt{t_{\max}/n^3}(3/4 - \delta)$ as required.

5 Estimating Mixing Time for Arbitrary Markov Chains

In this section we prove Theorem 3, saying that the problem of testing convergence is PSPACE-complete. The idea of the hardness result is to simulate any PSPACE computation by a Markov chain so that if there is an accepting computation, the chain mixes quickly, while if the computation does not accept then the chain takes much longer to mix.

We also recall some standard complexity theory background.

**Definition 4** A problem $L$ is in PSPACE if there exists a Turing machine $M$ which on input $x$ of size $n$ uses a work tape with at most a polynomial $p(n)$ number of bits and outputs $M(x) = 1$ iff $x \in L$.

**Definition 5** A problem $L_1$ is polynomial time reducible to another problem $L_2$ if there exists a polynomial time computable function (i.e. there is a polynomial time TM which computes the output of $f$) $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that $x \in L_1$ iff $x \in L_2$.

**Definition 6** A problem $L$ is PSPACE-hard if any $A \in$ PSPACE is polynomial time reducible to it.

**Definition 7** A problem $L$ is in BP$_H$PSPACE if there is a probabilistic polynomial space Turing machine $M$ which on input $x$ can flip any number of coins that is a bounded function in $|x|$ (but can only store polynomially many of them), halts for every setting of the tosses, and satisfies

- If $x \in L$ then $P_r(M(x) = 1) \geq 2/3$.
- If $x \notin L$ then $P_r(M(x) = 1) < 1/3$.

The following result can be deduced from Savitch’s Theorem [22].

**Theorem 6** BP$_H$PSPACE = PSPACE

The proof of Theorem 3 now follows by the following two lemmas. The following lemma uses the fact that the $t$-step transition probabilities of a Markov chain can be approximated in BP$_H$SPACE (see for example [21]). We include all the details here for completeness.

**Lemma 6** For every $1 \leq c \leq \exp(n^{O(1)})$ and $\exp(-n^{O(1)}) < \delta \leq 1/4$, the problem GTC$_{c,\delta}$ is in BP$_H$PSPACE.

**Proof:** The proof is by showing that there is a randomized algorithm $A$ for GTC$_{c,\delta}$ with 2-sided error using at most a polynomial amount of space and exponentially many random bits. In particular, the algorithm $A$ on input $X = (C, \hat{x}, t)$ queries $C$ at most exponentially many times and
• If $\tau(1/4 - \delta) \leq t$, $P(A(X, r) = 1) \geq 2/3$.
• If $\tau(1/4 + \delta) > ct$, $P(A(X, r) = 1) < 1/3$.

We show below an algorithm to calculate a $\delta$ additive approximation $\hat{d}(t)$ to $d(t)$ with probability at least 2/3. The algorithm accepts if $\hat{d}(t) \leq 1/4$ and rejects otherwise.

In the YES case, $d(t) \leq 1/4 - \delta$ and therefore with probability at least 2/3, $\hat{d}(t) \leq 1/4$ and the algorithm will accept. In the NO case, $d(ct) > 1/4 + \delta$. Since the distance $d$ is non-increasing (see e.g. Chapter 4 in \cite{1}), $d(t) > 1/4 + \delta$. Therefore, with probability at least 2/3, $\hat{d}(t) > 4$ and the algorithm will reject.

The algorithm to compute $\hat{d}(t)$ is as follows. Note first that it is possible to enumerate over all elements of the state space of the chain $\Omega$ using at most a polynomial amount of space. It is enough to check for each state whether it is reachable from $\hat{x}$ which can be done in PSPACE once we can enumerate all the adjacencies $y$ for a vertex $v$. But this can be done in PSPACE by running over all possible random strings and checking if for some $r$, $C(v, r) = y$.

For $x \in \Omega$ the algorithm runs the chain for $t$ steps $N$ times, starting at $x$ each time, and sets $f_{x,z}$ to be the fraction of times the chain stops at $z$. The estimates $f_{x,z}$ and $f_{y,z}$ can be computed with a polynomial amount of space in this way. Let

$$M_{xy}^t = \frac{1}{2} \sum_{z \in \Omega} |f_{x,z} - f_{y,z}|.$$ 

$M_{xy}^t$ can be computed with a polynomial amount of space by running over all $z$. Let

$$\hat{d}(t) = \max_{x,y} M_{xy}^t.$$ 

There are two sources of error in the estimate for $P^t(x, z)$. The first is due to using only a polynomial amount of space, whereas the $t$-step probabilities may be doubly exponentially small. The size of the error is inversely exponential in the space we use. This (additive) error can be bounded by $\delta_a = \delta/4$ using a polynomial amount of space since $\delta$ is always at least $\exp(-n^{O(1)})$. The second source of error $\delta_r$ is random and can be bounded by $\delta/4$ by Chernoff bounds for an overall error of at most $\delta/2$. Thus, if the number of runs $N$ is at least $48n\delta^{-2}$, by Chernoff bounds,

$$P(|P^t(x, z) - f_{x,z}| > \delta/2) \leq 2^{-3n-2}.$$ 

Therefore, for every $x, y$, taking union over all $z$,

$$P(|M_{xy}^t - d_{tv}(P^t(x, \cdot), P^t(y, \cdot))| > \delta) \leq 2^n2^{-3n-2} \leq 2^{-2n-2}.$$ 

Therefore, we have

$$P(|\hat{d}(t) - d(t)| > \delta) \leq P(\exists x, y \text{ s.t. } |M_{xy}^t - d_{tv}(P^t(x, \cdot), P^t(y, \cdot))| > \delta) \leq \frac{1}{4}$$

where the last inequality follows by taking the union over all $x, y$.

\begin{lemma}
For every $1 \leq c \leq \exp(n^{O(1)})$ and $0 \leq \delta < 1/4$, it is PSPACE-hard to decide GTC$_{c,\delta}$.
\end{lemma}

\begin{remark}
In fact, the conclusion holds even if the Markov chain is restricted to be reversible.
\end{remark}
The circuit tion of the circuit is at most polynomial in $G$ of states of the machine and a constant number of bits on the tape.) This implies in particular that the graph vertices.

Secondly, because the TM reads and writes to only a small number of bits, we only have to check for a few

Next, we show bounds on the mixing time with the edge weights as defined above. For this we observe the following.

**Claim 3** In the YES case the graph $G$ is connected and $\tau(1/4 - \delta) \leq 10D^32^{3n}/(1 - 4\delta)$.

**Proof:** Note that by the assumption on the Turing machine, all states are connected by $w$ edges to either $s_{\text{acc}}$ or $s_{\text{rej}}$. Since $s_{\text{rej}}$ is connected by a $w$-edge to $s_{\text{start}}$ and $s_{\text{start}}$ is connected to $s_{\text{acc}}$ since we are in the YES case it follows that the graph $G$ is connected.

We now use the conductance bound on the mixing time from Theorem 5 in the following way. For every set but the empty set or the complete graph, there is weight at least $w$ from the set to its complement. Furthermore - the total weight of each set is at most $D^2w$. Therefore the conductance $\Phi \geq D^{-1}2^{-n}$ and hence we conclude that the mixing time $\tau(\varepsilon)$ is at most

$$2D^22^{2n}\log(2/\tau_{\text{min}}\varepsilon).$$
\( \pi_{\text{min}} \) is the minimum probability of any state in the space which we can lower bound by \( w/(D2^n) = D^{-1/2-n} \) and \( \varepsilon = 1/4 - \delta \). The proof follows since
\[
\log 2 \leq 1, \quad \log(1/\pi_{\text{min}}) \leq \log(D2^n) \leq D2^n, \quad \log(1/\varepsilon) = \log(1/(1/4 - \delta)) \leq 4/(1-4\delta).
\]

**Claim 4** In the NO case the graph \( G \) is not connected. Moreover \( d(t) \geq 1 - 2t/w \) for all \( t \) and
\[
\tau(1/4 + \delta) \geq \tau(1/2) \geq w/4.
\]

**Proof:** We first note that the bound \( \tau(1/4 + \delta) \geq \tau(1/2) \geq w/4 \) immediately follows from \( d(t) \geq 1 - 2t/w \).
In order to show that the graph is not connected we note that \( s_{\text{start}} \) and \( s_{\text{acc}} \) are not in the same component.
This follows from the fact that all edges of \( G \) are legal transitions of the machine. The only other edges of the Markov chain are loops or the edge connecting \( s_{\text{rej}} \) to \( s_{\text{start}} \). Consider in the graph \( G \) the component of \( s_{\text{start}} \) and of \( s_{\text{acc}} \) denoted by \( C_{\text{start}} \) and \( C_{\text{acc}} \) respectively.
In order to bound \( d(t) \) we look at the distributions \( X(t), Y(t) \) of the chain started at \( s_{\text{start}} \) and at \( s_{\text{acc}} \). Note that
\[
d(t) \geq 1 - P[\exists s \leq t \text{ s.t. } X(s) \in C_{\text{acc}}] - P[\exists s \leq t \text{ s.t. } Y(s) \in C_{\text{start}}]
\]
We note that the only way to move between the components is by following the edge 1 weight and the probability of taking this edge at any step (conditioned on the past) is at most \( w^{-1} \). It therefore follows that:
\[
d(t) \geq 1 - 2t/w,
\]
as needed.

By the claims 3 and 4
\[
10D^32^{3n}/(1-4\delta) \geq 4(1-4\delta) \geq c.
\]

To complete the reduction, let \( t = 10D^32^{3n}/(1-4\delta) \) and set the starting state \( x = s_{\text{start}} \).

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