SUBGROUP OF INTERVAL EXCHANGES
GENERATED BY TORSION ELEMENTS
AND ROTATIONS

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Abstract. Denote by $G$ the group of interval exchange transformations (IETs) on the unit interval. Let $G_{\text{per}} \subset G$ be the subgroup generated by torsion elements in $G$ (periodic IETs), and let $G_{\text{rot}} \subset G$ be the subset of 2-IETs (rotations).

The elements of the subgroup $G_1 = (G_{\text{per}}, G_{\text{rot}}) \subset G$ (generated by the sets $G_{\text{per}}$ and $G_{\text{rot}}$) are characterized constructively in terms of their Sah-Arnoux-Fathi (SAF) invariant. The characterization implies that a non-rotation type 3-IET lies in $G_1$ if and only if the lengths of its exchanged intervals are linearly dependent over $\mathbb{Q}$. In particular, $G_1 \subsetneq G$.

The main tools used in the paper are the SAF invariant and a recent result by Y. Vorobets that $G_{\text{per}}$ coincides with the commutator subgroup of $G$.

1. A GROUP OF IETs

Denote by $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ the sets of real, rational and natural numbers. By a standard interval we mean a finite interval of the form $X = [a, b) \subset \mathbb{R}$ (left closed - right open). We write $|X| = b - a$ for its length.

By an IET (interval exchange transformation) we mean a pair $(X, f)$ where $X = [a, b)$ is a standard interval and $f$ is a right continuous bijection $f: X \to X$ with a finite set $D$ of discontinuities and such that the translation function $\gamma(x) = f(x) - x$ is piecewise constant.

The map $f$ itself is often referred to as IET, and then $X = \text{domain}(f)$ and $D = \text{disc}(f)$ denote the domain (also the range) of $f$ and the discontinuity set of $f$, respectively.

Given an IET $f: X \to X$, the set $\text{disc}(f)$ partitions $X$ into a finite number of subintervals $X_k$ in such a way that $f$ restricted to each $X_k$ is a translation

$$f|_{X_k}: x \to x + \gamma_k,$$

so that the action of $f$ reduces to a rearrangement of the intervals $X_k$. The number $r$ of exchanged intervals $X_k$ can be specified by calling $f$ an $r$-IET.

Denote by $\mathcal{G}$ the set of IETs. Then

$$\mathcal{G} = \bigcup_{r \geq 1} \mathcal{G}_r,$$  

where $\mathcal{G}_r$ stands for the set of $r$-IETs:

$$\mathcal{G}_r = \{ f \in \mathcal{G} | \text{card}(\text{disc}(f)) = r - 1 \}.$$
For a standard interval $X = [a, b)$, the subset
\begin{equation}
G(X) = \{ f \in \mathcal{G} \mid \text{domain}(f) = X \}
\end{equation}
forms a group under composition (of bijections of $X$). Its identity is $\mathbf{1}_{G(X)} \in \mathcal{G}_1$, the identity map on $X$. Given two standard intervals $X$ and $Y$, there is a canonical isomorphism
\begin{equation}
\phi_{X,Y} : G(X) \to G(Y)
\end{equation}
defined by the formula
\begin{equation}
\phi_{X,Y}(f) = l \circ f \circ l^{-1} \in G(Y), \quad \text{for } f \in G(X),
\end{equation}
where $l = l_{X,Y}$ stands for the unique affine order preserving bijection $X \to Y$.

By the group of IETs we mean the group $G := G([0, 1))$. (It is isomorphic to $G(X)$, for any standard interval $X$).

The interval exchange transformations have been a popular subject of study in ergodic theory. (We refer the reader to the book [12] by Marcelo Viana which may serve a nice introduction and survey reference in the subject). Most papers on IETs study these as dynamical systems; they concern specific dynamical properties (like minimality, ergodicity, mixing properties etc.) the IETs may satisfy.

The focus of the present paper is different; we address certain questions on the group-theoretical structure of the group $G$ of IETs. (For recent results on this general area see [13] and [7]). In particular, we discuss possible generator subsets of the group $G$.

It was known for a while that the subgroup $G_{\text{per}}$ generated by periodic IETs forms a proper subgroup of $G$; in particular, $G_{\text{per}}$ contains no irrational rotations. (The SAF invariant introduced in the next section vanishes on $G_{\text{per}}$ but has non-zero value on irrational rotations). On the other hand, the set $G_{\text{per}}$ contains some uniquely ergodic, even pseudo-Anosov (self-similar) IETs (see [2]).

We show that the subgroup $G_1 = \langle G_{\text{per}}, G_{\text{rot}} \rangle$ generated by $G_{\text{per}}$ and the set of rotations $G_{\text{rot}} \subset G$ is still a proper subgroup of $G$. On the other hand, $G_1$ is large enough to contain all rank 2 IETs (see Section 4), in particular the IETs over quadratic number fields.

We present a constructive criterion (in terms of the SAF invariant) for a given IET to lie in $G_1$. It follows from this criterion that a 3-IET lies in $G_1$ if and only if the lengths of its exchanged subintervals are linearly independent. Note that 3-IETs generate the whole group of IETs. (More precisely, $\mathcal{G}_3 \cap G$ is a generating set for the group $G$).

### 2. The SAF invariant and subgroups of $G(X)$

Throughout the paper (whenever vector spaces or linear dependence/independence are discussed) the implied field (if not specified) is always meant to be $\mathbb{Q}$, the field of rationals.

Denote by $\mathbb{T}$ the tensor product of two copies of reals viewed as vector spaces (over $\mathbb{Q}$). Denote by $\mathbb{K}$ the skew symmetric tensor product of two copies of reals:
\begin{align*}
\mathbb{T} & := \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}; \\
\mathbb{K} & := \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R} \subset \mathbb{T}.
\end{align*}
Recall that $\mathbb{K}$ is the vector subspace of $\mathbb{T}$ spanned by the wedge products
\[ u \wedge v := u \otimes v - v \otimes u, \quad u, v \in \mathbb{R}. \]
The **Sah-Arnoux-Fathi (SAF) invariant** (sometimes also called the **scissors congruence invariant**) of $f \in G_r$ is defined by the formula

$$SAF(f) := \sum_{k=1}^{r} \lambda_k \otimes \gamma_k \in \mathbb{T},$$

where the vectors $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, $\vec{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_r) \in \mathbb{R}^r$ encode the lengths $\lambda_k = |X_k|$ of exchanged intervals $X_k$ and the corresponding translation constants $\gamma_k$, respectively (see (1.1)).

The SAF invariant was introduced independently by Sah [9] and Arnoux and Fathi [1]. The following lemma makes this invariant a useful tool in the study of IETs.

**Lemma 1.** Let $X$ be a standard interval. Then

(a) $SAF: G(X) \to \mathbb{T}$ is a group homomorphism;

(b) $SAF(G(X)) = K \subset \mathbb{T}$;

(c) $SAF: G(X) \to K$ is a surjective group homomorphism.

(This summarizes (a) and (b)).

2.1. **Subgroups of $G(X)$**. Let $X$ be a standard interval. An IET $f \in G(X)$ is called periodic if $f$ is an element of finite order in the group $G(X)$ (defined in (1.4)). Set

$$G_r(X) := G_r \cap G(X), \quad r \geq 1,$$

(see notation (1.2)). Note that $G_1(X) = \{1_{G(X)}\}$ is a singleton.

Consider the following subgroups of $G(X)$:

- $G'(X) = [G(X), G(X)]$ is the commutator subgroup of $G$ (generated by the commutators $f^{-1}g^{-1}fg$, with $f, g \in G(X)$);

- $G_{\text{per}}(X)$ is the subgroup of $G(X)$ generated by periodic IETs $f \in G(X)$;

- $G_0(X) = \{f \in G(X) \mid SAF(f) = 0\}$ (see (2.1));

- $G_{\text{rot}}(X) = G_1(X) \cup G_2(X) = G_2(X) \cup \{1_{G(X)}\}$ is the group of rotations on $X$.

(It contains all IETs $f \in G(X)$ with at most one discontinuity).

Observe that (a) in Lemma 1 implies immediately the inclusions

$$G'(X) \subset G_0(X); \quad G_{\text{per}}(X) \subset G_0(X),$$

as well as the fact that the set $G_0(X)$ forms a subgroup of $G(X)$.

An unpublished theorem of Sah [9] (mentioned by Veech in [11]) contains Lemma 1 and the equality $G'(X) = G_0(X)$. This equality has been recently extended by Vorobets [13] to also include $G_{\text{per}}(X)$ as an additional set:

$$G'(X) = G_0(X) = G_{\text{per}}(X).$$

(2.3)

We refer to Vorobets’s paper [13] for a nice self-contained introduction to the SAF invariant. In particular, the paper contains the proof of Lemma 1 and of the equality (2.3).

The results of the present paper concern the subgroup

$$G_1(X) = \langle G_{\text{per}}(X), G_{\text{rot}}(X) \rangle \subset G(X),$$

(2.4)
generated by periodic IETs and the rotations in \( G(X) \). The elements of \( G_1(X) \) are classified in terms of their SAF invariant, see Theorem 2. It is shown that “most” 3-IETs do not lie in \( G_1(X) \) (Lemma 3 and Theorem 4). In particular, it follows that \( G_1(X) \neq G(X) \).

3. Subgroup \( G_1 \) and its SAF invariant characterization.

For \( u \in \mathbb{R} \), denote 
\[
\mathbb{K}(u) = \{ u \wedge v \mid v \in \mathbb{R} \} \subset \mathbb{K} = \mathbb{R} \wedge \mathbb{Q} \mathbb{R}.
\]
\( \mathbb{K}(u) \) forms a vector subspace of \( \mathbb{K} \). If \( u \neq 0 \), \( \mathbb{K}(u) \) is isomorphic to \( \mathbb{R}/\mathbb{Q} \).

In the next lemma we compute the SAF invariant for 2-IETs. This is known and follows immediately from the definition (2.1), but is included for completeness.

Recall that \( G_r(X) = G_r \cap G(X) \) stands for the set of \( r \)-IETs on \( X \).

**Lemma 2.** Let \( X \) be a standard interval. Assume that \( f \in G_2(X) \) exchanges two subintervals of lengths \( \lambda_1 \) and \( \lambda_2 \) with \( \lambda_1 + \lambda_2 = |X| \). Then 
\[
\text{SAF}(f) = |X| \wedge \lambda_1 \in \mathbb{K}(|X|).
\]

**Proof.** For such \( f \) the translation constants are \( \gamma_1 = -\lambda_2 \) and \( \gamma_2 = -\lambda_2 + 1 = \lambda_1 \) (see (1.1)). It follows that 
\[
\text{SAF}(f) = \lambda_1 \otimes \gamma_1 + \lambda_2 \otimes \gamma_2 = \lambda_1 \otimes (-\lambda_2) + \lambda_2 \otimes \lambda_1 = \\
= \lambda_2 \wedge \lambda_1 = (\lambda_2 + \lambda_1) \wedge \lambda_1 = |X| \wedge \lambda_1.
\]

\[\square\]

**Lemma 3.** Let \( X \) be a standard interval. Let \( \beta \in \mathbb{K}(|X|) \). Then there exists \( f \in G_2(X) \) such that \( \text{SAF}(f) = \beta \).

**Proof.** Let \( \beta = |X| \wedge t \). Select a rational \( r \in \mathbb{Q} \) such that \( 0 < r|X| + t < |X| \). Set \( \lambda_1 = r|X| + t \) and \( \lambda_2 = |X| - \lambda_1 \). Take \( f \in G_2(X) \) which exchanges two intervals of lengths \( \lambda_1 \) and \( \lambda_2 \). Then, by Lemma 2
\[
\text{SAF}(f) = |X| \wedge \lambda_1 = |X| \wedge (r|X| + t) = |X| \wedge t,
\]
completing the proof. \[\square\]

**Corollary 1.** Let \( X \) be a standard interval. Then 
\[
\text{SAF}(G_{\text{rot}}(X)) = \text{SAF}(G_2(X)) = \mathbb{K}(|X|).
\]

**Proof.** Follows from Lemmas 2 and 3. \[\square\]

**Theorem 1.** Let \( X \) be a standard interval and let \( f \in G(X) \). Assume that \( \text{SAF}(f) \in \mathbb{K}(|X|) \). Then there exists a rotation \( g \in G_2(X) \) and \( h_1, h_2 \in G_{\text{per}}(X) \) such that 
\[
f = g \circ h_2 = h_1 \circ g.
\]

Note that in the above theorem \( G_{\text{per}}(X) \) can be replaced by \( G'(X) \) or \( G_0(X) \) (see (2.3)).

**Proof of Theorem 1** By Lemma 3 there exists \( g \in G_2(X) \) such that \( \text{SAF}(g) = \text{SAF}(f) \in \mathbb{K}(|X|) \). Take \( h_1 = f \circ g^{-1} \), \( h_2 = g^{-1} \circ f \). Then \( h_1, h_2 \in G_0(X) = G_{\text{per}}(X) \) in view of Lemma 1. \[\square\]
Theorem 2. Let $X$ be a standard interval and let $f \in G(X)$. Then

$$f \in G_1(X) \iff \text{SAF}(f) \in \mathbb{K}(|X|).$$

Proof. Direction $\Rightarrow$. In view of (2.4), it is enough to show the following two inclusions:

$$S_1 = \text{SAF}(G_{rot}(X)) \in \mathbb{K}(|X|); \quad S_2 = \text{SAF}(G_0(X)) \in \mathbb{K}(|X|).$$

Both are immediate: $S_1 = \mathbb{K}(|X|)$ by Corollary 1 and $S_2 = 0$.

Direction $\Leftarrow$. Follows from Theorem 1. $\square$

4. Rank 2 IETs lie in $G_1$

By the span of a subset $A \subset \mathbb{R}$ (notation: $\text{span}(A)$) we mean the minimal vector subspace of $\mathbb{R}$ (over $\mathbb{Q}$) containing $A$.

By the rank of an IET $f$ (notation $\text{rank}(f)$) we mean the dimension of the span of the set the lengths of the intervals exchanged by $f$:

$$\text{rank}(f) := \dim_{\mathbb{Q}}(L(f)),$$

where

$$L(f) := \text{span}(\{\lambda_1, \lambda_2, \ldots, \lambda_r\}) \quad (\text{if } f \in G_r).$$

The following implication is immediate: $\text{rank}(f) = 1 \implies f \in G_{\text{per}} = G_0$.

Theorem 3. For $f \in G(X)$, the following implication holds:

$$\text{rank}(f) \leq 2 \implies f \in G_1(X).$$

Proof. We may assume that $\text{rank}(f) = \dim(L(f)) = 2$. Since all $\lambda_k \in L(f)$, it follows that $|X| = \sum_{k=1}^r \lambda_k \in L(F)$. Let $B = \{|X|, u\}$ be a basis in $L(F)$. Then

$$\text{SAF}(f) \in L(f) \wedge_{\mathbb{Q}} L(f) = \{q |X| \wedge u \mid q \in \mathbb{Q}\} =$$

$$= \{|X| \wedge qu \mid q \in \mathbb{Q}\} \subset \mathbb{K}(|X|),$$

whence $f \in G_1(X)$, in view of Theorem 2. $\square$

5. Criterion for 3 IETs to lie in $G_1$

Let $f \in G_3(X)$. Then $f$ has 2 discontinuities, and $f$ acts by reversing the order of three subintervals, $X_1$, $X_2$ and $X_3$. Set $\lambda_i = |X_i|$. Then $X = \text{domain}(f) = [a, a + \lambda_1 + \lambda_1 + \lambda_3]$, with some $a \in \mathbb{R}$. Set for convenience $a = 0$ so that $X = [0, \lambda_1 + \lambda_1 + \lambda_3]$.

The corresponding translation constants are easily computed:

$$\gamma_1 = f(0) - 0 = \lambda_2 + \lambda_3 = |X| - \lambda_1;$$
$$\gamma_2 = f(\lambda_1) - \lambda_1 = \lambda_3 - \lambda_1;$$
$$\gamma_3 = f(\lambda_1 + \lambda_2) - (\lambda_1 + \lambda_2) = 0 - (\lambda_1 + \lambda_2) = \lambda_3 - |X|.$$
Since $\lambda_2 = |X| - \lambda_1 - \lambda_3$, we can compute $\text{SAF}(X)$ in terms of linear combinations of the wedge products involving only $X$, $\lambda_1$ and $\lambda_3$:

\begin{equation}
\text{SAF}(X) = \sum_{k=1}^{3} \lambda_k \otimes \gamma_k = |X| \wedge (\lambda_1 - \lambda_3) - \lambda_1 \wedge \lambda_3.
\end{equation}

We need the following known basic fact (see e.g. [13, Lemma 3.1]).

**Lemma 4.** If $v_1, v_2, \ldots, v_n$ are $n \geq 2$ linearly independent real numbers, then the $\frac{n(n-1)}{2}$ wedge products $v_i \wedge v_j$, $1 \leq i < j \leq n$, are linearly independent.

**Corollary 2.** Let $v_1, v_2, v_3 \in \mathbb{R}$ be linearly independent. Then $v_1 \wedge v_2 \neq v_3 \wedge u$ for all $u \in \mathbb{R}$.

**Proof.** Assume to the contrary that

$$v_1 \wedge v_2 = v_3 \wedge u$$

for some $u \in \mathbb{R}$. If $u \notin \text{span} \{v_1, v_2, v_3\}$, then the set $\{u, v_1, v_2, v_3\}$ is linearly independent, contradicting Corollary 2.

And if $u \in \text{span} \{v_1, v_2, v_3\}$, then $u = q_1 v_1 + q_2 v_2 + q_3 v_3$ with some $q_i \in \mathbb{Q}$ whence

$$v_1 \wedge v_2 = q_1 v_3 \wedge v_1 + q_2 v_3 \wedge v_2,$$

a contradiction with Lemma 5 again. $\square$

**Lemma 5.** Let $f : X \to X$ be a 3-IET and assume that $\text{rank}(f) = 3$. (Equivalently, the set $\{\lambda_1, \lambda_1, \lambda_3\}$ is linearly independent). Then $f \notin G_1(X)$.

**Proof.** Assume to the contrary that $f \in G_1(X)$. Then, by Theorem 2, $\text{SAF}(f) \in \mathbb{K}(|X|)$. It follows from (5.1) that $\lambda_1 \wedge \lambda_3 \in \mathbb{K}(|X|)$, i.e. that

\begin{equation}
\lambda_1 \wedge \lambda_3 = |X| \wedge u,
\end{equation}

for some $u \in \mathbb{R}$. Since the numbers $\lambda_1, \lambda_3$ and $|X| = \lambda_1 + \lambda_2 + \lambda_3$ are linearly independent, (5.2) contradicts Corollary 2. $\square$

**Theorem 4** (Criterion for a 3-IET to lie in $G_1$). Let $f : X \to X$ be a 3-IET. Then

$$f \in G_1(X) \iff \text{rank}(f) \leq 2.$$  

**Proof.** The direction $\Rightarrow$ is a contrapositive restatement of Lemma 5. The direction $\Leftarrow$ is given by Theorem 3. $\square$

6. **Validation of the membership in classes $G_0(X)$ and $G_1(X)$**

Given $f \in G(X)$, we describe constructive procedures to decide whether $f \in G_{\text{per}}(X)$ and whether $f \in G_1(X)$. 


6.1. **Does the inclusion** \( f \in G_{\text{per}}(X) \) **hold?** Since \( G_{\text{per}}(X) = G_0(X) \) (see (2.3)), one only has to test the equality

\[
\text{SAF}(f) := \sum_{k=1}^{r} \lambda_k \otimes \gamma_k = 0.
\]

We assume that the linear structure of \( f \) is known. By the linear structure of \( f \) we mean:

(a) a basis \( B = \{v_1, v_2, \ldots, v_n\} \) of the finite dimensional space

\[
L(f) := \text{span}(\{\lambda_1, \lambda_2, \ldots, \lambda_r\})
\]

(b) the (unique) linear representations \( \lambda_k = \sum_{i=1}^{n} q_{k,i} v_i, \) \( 1 \leq k \leq r, \) with all \( q_{k,i} \in \mathbb{Q} \).

Observe that all translation constants \( \gamma_k \) also lie in \( L(f) \), and their linear representation in terms of basis \( B \) can be computed. This way one can get presentation

\[
(6.1) \quad \text{SAF}(f) = \sum_{1 \leq i < j \leq n} p_{i,j} v_i \wedge v_j,
\]

with known \( p_{i,j} \in \mathbb{Q} \). By Lemma 4, \( \text{SAF}(f) = 0 \) if and only if all the constants \( p_{i,j} \) vanish.

6.2. **Does the inclusion** \( f \in G_1(X) \) **hold?** Again we assume that the linear structure of \( f \) is known. To answer the question, we proceed as follows.

First, we modify the basis \( B \) of \( L(f) \) to make \( v_1 = |X| \). Then we proceed just as before and get (6.1) with known \( p_{i,j} \in \mathbb{Q} \). By Theorem 2 and Lemma 4

\[
f \in G_1(X) \iff \text{SAF}(f) \in \mathbb{K}(|X|) \iff p_{i,j} = 0, \text{ for } 2 \leq i < j \leq n.
\]

7. **Class \( G_1 \) is not preserved under induction**

An IET \((X, f)\) is called minimal if its every orbit is dense in \( X \). For a sufficient condition for an IET to be minimal see [5].

Let \((X, f)\) be an \( r \)-IET and assume that \( Y \subset X \) is a standard subinterval. It is well known (see e.g. [4] or [5]) that the (first return) map \( f_Y : Y \to Y \) induced by \( f \) on \( Y \) is also an IETs (exchanging at most \( r+1 \) subintervals). It is known that, under the minimality assumption, the SAF invariant is preserved under induction.

**Proposition 1.** ([1], Part II, Proposition 2.13) Let \((X, f)\) be a minimal IET and assume that \( Y \subset X \) is a standard subinterval. Then

\[
\text{SAF}(f) = \text{SAF}(f_Y).
\]

**Corollary 3.** Let \((X, f)\) be a minimal IET and assume that \( Y \subset X \) is a standard subinterval. Then

\[
f \in G_{\text{per}}(X) \iff f_Y \in G_{\text{per}}(X)
\]

**Proof.** Follows from Proposition 1 and the fact that \( G_{\text{per}}(X) = G_0(X) \).

**Corollary 4.** Let \((X, f)\) be a minimal IET and assume that \( Y \subset X \) is a standard subinterval such that \( \frac{|Y|}{|X|} \in \mathbb{Q} \). Then

\[
f \in G_1(X) \iff f_Y \in G_1(X).
\]
Proof. Follows from Proposition 1 because \( K(\|X\|) = K(\|Y\|) \) assuming that \( \frac{\|Y\|}{\|X\|} \in \mathbb{Q} \). \( \square \)

**Theorem 5.** Let \((X, f)\) be a minimal IET. Then the following two assertions are equivalent:

(a) There exists a standard subinterval \( Y \subset X \) such that \( f_Y \in G_1(Y) \);

(b) \( \text{SAF}(X) = u \wedge v \), for some \( u, v \in \mathbb{R} \).

One can show that if (b) of the above theorem holds and \( \text{SAF}(X) \neq 0 \) then \( u, v \in L(f) \).

Proof. (a)\( \Rightarrow \) (b). \( \text{SAF}(X) = \text{SAF}(Y) \in K([Y]) \) whence \( \text{SAF}(X) = |Y| \wedge v \), for some \( v \in \mathbb{R} \).

(b)\( \Rightarrow \) (a). Let \( \text{SAF}(X) = u \wedge v \), for some \( u, v \in \mathbb{R} \). Without loss of generality, both \( u, v \) can be selected positive (using the identities \( u \wedge v = (-v) \wedge u \) and \( 1 \wedge 1 = 0 \)). Select \( q \in \mathbb{Q} \) so that \( 0 < qu < |X| \). Select any standard subinterval \( Y \subset X \) of length \( |Y| = qu \). Then \( \text{SAF}(f_Y) = u \wedge v = |Y| \wedge (q^{-1}v) \in \text{SAF}(|Y|) \), and hence \( f_Y \in G_1(Y) \). \( \square \)

Let \( K \subset \mathbb{R} \) be a subfield of reals. An IET \( f \) is said to be over \( K \) if \( L(f) \subset K \) (see (4.1)), i.e. if all (lengths of exchanged intervals) \( \lambda_k \) lie in \( K \).

We complete the paper by the following result.

**Theorem 6.** Let \( K \) be a real quadratic number field and let \((X, f)\) be a minimal IET over \( K \). Let \( Y \subset X \) be a standard subinterval. Then

\[ f_Y \in G_1(Y) \iff |Y| \in K. \]

Proof. Since \( \text{rank}(f) = \dim(L(f)) \leq \dim(F) = 2 \), and \( \text{rank}(f) \neq 1 \) (because \( f \) is not periodic), we conclude that \( \text{rank}(f) = 2 \) and \( L(f) = K \).

**Proof of the \( \Leftarrow \)** implication. Select a basis \( B = \{|Y|, u\} \) in \( K \). Then

\[ \text{SAF}(f_Y) = \text{SAF}(f) \in K \wedge K = \{q|Y| \wedge u \mid q \in \mathbb{Q}\} \subset K(|Y|). \]

By Theorem 2, \( f_Y \in G_1(Y) \).

**Proof of the \( \Rightarrow \)** implication. It has been proved in [3] that minimal rank 2 IETs must be uniquely ergodic. Thus \( f \) is uniquely ergodic. By McMullen theorem [6 Theorem 2.1], \( \text{SAF}(f) \neq 0 \). (McMullen uses the “Galois flux” invariant for the IETs over a quadratic number field which, in his setting, is equivalent to the SAF invariant).

Select a basis \( B = \{|X|, u\} \) in \( K \). Then

\[ \text{SAF}(f_Y) = \text{SAF}(f) \in K \wedge K = \{q|X| \wedge u \mid q \in \mathbb{Q}\}. \]

Since \( \text{SAF}(f) \neq 0 \), \( \text{SAF}(f_Y) = q|X| \wedge u \), with some \( q \in \mathbb{Q}, q \neq 0 \).

By Theorem 2, \( f_Y \in G_1(Y) \) implies that \( q|X| \wedge u = |Y| \wedge v \), for some \( v \in \mathbb{R} \). This is incompatible with the assumption \( Y \notin K \) in view of Corollary 2, completing the proof. \( \square \)
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