Towards a Unified Theory of Gauge and Yukawa Interactions

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Abstract. It is suggested to combine gauge and Yukawa interactions into one expression $\mathcal{H} = \frac{1}{2} \mathcal{D} \Psi \Psi$ where $\mathcal{D}$ is the generalized Dirac operator associated with a superconnection $\mathcal{D} = D + L$, $L$ being linked to the Higgs field (one doublet for simplicity). We advocate a version of the Minimal Standard Model where the Higgs field gives masses to the neutrinos and a CKM matrix to the leptons. Apart from a parameter $\mu \approx 80\text{ GeV}$ setting the mass scale, the (dimensionless) free parameters of three fermion generations are assembled in one operator $h$, invariant under gauge transformations. As we are free to choose $h$, the predictive power is rather limited. Still, the fine structure constant and the weak couplings remain unaffected: $\alpha^{-1} = \frac{128\pi}{3}$, $g = 1/2$, $g'/g = \sqrt{3}/5$. There are three relations that fix the masses of the $Z^0$, the $W^\pm$, and the Higgs, given the masses of all fermions. The present data are consistent with $m_H \approx 160\text{ GeV}$. Without these data, $m_H \geq \sqrt{2} m_W$ on general grounds.

1 Introduction

It is fair to say that, at present, the Standard Model belongs to the category of most thoroughly tested and best confirmed physical theories [1]. But the secret is that no one truly understands it. At least not in the way we understand QED or QCD. It appears that the vast number of free parameters, speculations about the number of Higgs doublets, and the ad-hoc definition of the Higgs potential defy easy analysis. With each new generation of fundamental fermions the unknowns have multiplied. The neutrinos may or may not have masses. It remains undecided whether nature provides another CKM matrix for the leptons. In the long run, the struggle for a better understanding will perhaps be resolved by resorting to string theory (M-theory or other oracles). In the meantime we might be content with modest explanations using constructions in ordinary (commutative) differential geometry. One concept, which convincingly illuminates the role of the Higgs field, comes under the heading superconnection. Implied is the concept of a
generalized Dirac operator which unites gauge and Yukawa couplings in one term.

In [2] we showed how the Higgs field fits into the framework of superconnections on some superbundle with structure group $U(n)$. We then proposed to take $n = 2$ to construct a model of gauge fields and two Higgs doublets. In [3] we added leptons to the model to see how gauge and Yukawa interactions can be combined by passing from a superconnection to the associated Dirac operator. Constructing such a model was a tentative step towards an understanding of the structure of more realistic theories. Of course, without quarks the leptonic model was not free of anomalies and thus called for an extension incorporating essential features of the Standard Model. A short account of such an attempt appeared in [4] where the gauge group $G$ was assumed to be a subgroup of $SU(5)$. Parallel to this work we opened a discussion in [5] and [6] on the mathematical background which should serve as a reference when we now resume the analysis of the Standard Model begun in [4]. Moreover, since we are dealing here with a chiral model, the question of consistency (absence of local nonabelian anomalies) arises. This problem has been dealt with and settled in [7].

The present exposition of the subject aims to provide motivation and practical tools rather than mathematical abstraction. Its goal is to arrive at predictions with minimal technical machinery.

2 The Gauge Group of the Standard Model

The Standard Model, extending of the earlier Weinberg-Salam model of electroweak interactions, is a gauge theory based on the Lie algebra

$$\text{Lie } G \cong \text{su}(3) \oplus \text{su}(2) \oplus \text{u}(1).$$

Though the gauge group $G$ is well defined locally, its global structure remains obscure unless we add further assumptions. Obviously, no restriction on the spectrum of the hypercharge $Y$ is to be expected on the basis of $u(1) = i\mathbb{R}$ alone. Nor would one be able to argue that the conditions $Q + Y \in \mathbb{Z}$ and $3Y \in \mathbb{Z}$ are satisfied where $Q$ denotes the electric charge. Still, in the vicinity of the unit, a possible difference between various choices of $G$ would not be felt at all, but globally it would: $G$ will dictate the subset of allowed representations of Lie $G$ and, therefore, the structure of particle multiplets admitted by the setup.

The present approach borrows from the idea that any grand unified theory, perhaps any theory beyond the Standard Model, ought to incorporate the gauge group $SU(5)$ in one way or another. As for the minimal version of the Standard Model, we require that $G$ be a subgroup of $SU(5)$ consistent with (4), i.e., we define

$$G = \{(u, v) \in U(3) \times U(2) \mid \det u \cdot \det v = 1\}$$

(2)
and let the embedding $G \rightarrow SU(5)$ be given by

$$(u,v) \mapsto \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$ 

It is easy to see that the Lie algebra $\text{Lie} G$ has indeed the required structure given by (1). The relation of the group $G$ to the color group $SU(3)$ and the electroweak group $U(2)$ is expressed by the following exact sequence

$$1 \rightarrow SU(3) \xrightarrow{j} G \xrightarrow{s} U(2) \rightarrow 1$$  \hspace{1cm} (3)

where $j(u) = (u,1)$ and $s(u,v) = v$. Inspite of the relationship (3), the group $G$ cannot be identified with the direct product $SU(3) \times U(2)$. It is still correct to say that the color group $SU(3)$ of quantum chromodynamics is embedded in $G$ as a subgroup. But the gauge group $U(2)$ of the Salam-Weinberg Theory is recovered here only as the quotient $G/SU(3)$. This fact deserves careful attention: it explains why the hypercharge $Y$ assumes fractional values. More specifically, the quotient structure accounts for the existence of values that are multiples of $1/3$.

The theory of leptons, gauge and Higgs particles is built around the assumption that the hypercharge $Y$ is the generator of $U(1)$, subgroup and center of $U(2)$. As these groups constitute proper symmetries (before spontaneous symmetry breaking takes place), the hypercharge is integer-valued for all leptonic states. When quarks are added, the picture changes. With quarks, the group $U(1)$ fails to be a subgroup of $G$ and hence cannot be regarded a symmetry though there is a related group $\tilde{U}(1)$ which can. To see more clearly the emergence of a fractional spectrum we consider the exact sequence

$$1 \rightarrow \mathbb{Z}_3 \xrightarrow{j} \tilde{U}(1) \xrightarrow{s} U(1) \rightarrow 1$$  \hspace{1cm} (4)

obtained from the groups in (3) by restricting to the centers. In more detail:

- The center $\tilde{U}(1)$ of $G$ consists of elements

  $$\text{diag}(e^{i\beta}, e^{i\beta}, e^{i\beta}, e^{i\alpha}, e^{i\alpha}) \in SU(5)$$

  satisfying $(e^{i\beta})^3(e^{i\alpha})^2 = 1$. It may conveniently be looked upon as a one-dimensional closed subgroup of the two-torus:

  $$\tilde{U}(1) = \{(e^{i\beta}, e^{i\alpha}) \mid 3\beta + 2\alpha = 0 \text{ mod } 2\pi\}.$$

- The cyclic group $\mathbb{Z}_3$ of order three is formed by the complex solutions of $z^3 = 1$. The injection $j$ takes $z$ into $(z,1) \in \tilde{U}(1)$.

- The surjection $s$ maps $(e^{i\beta}, e^{i\alpha})$ to $e^{i\alpha} \in U(1)$ where $U(1)$ relates to the hypercharge.
Viewed geometrically, the group $\tilde{U}(1)$ describes a closed curve on the 2-torus. Suppose we unwind the torus to obtain its covering plane with real coordinates $\alpha$ and $\beta$. Then the curve appears as a straight line with slope parameter $-2/3$:

The end points of the line have to be identified to form a closed curve on the torus. As we run once through this curve, the angle $\alpha$ assumes all values from 0 to $6\pi$. Phrased more formally, $\tilde{U}(1)$ is a threefold cover of the group $U(1) = \{e^{i\alpha}\}$. When it comes to particle multiplets, we must focus on the symmetry group $\tilde{U}(1)$ since its generator has an integer-valued spectrum. In unitary irreducible representations of the gauge group $G$, the hypercharge $Y$ assumes a constant value subject to the constraint

$$e^{i6\pi Y} = 1 \quad \text{oder} \quad 3Y \in \mathbb{Z},$$

as can be inferred from the behavior of the variable $\alpha$. In other words, the group $\tilde{U}(1)$ is not connected in a direct manner with $Y$ but rather with $3Y$. Phrased more formally, the covering map $s : \tilde{U}(1) \to U(1)$ admits a local inverse

$$s^{-1}(e^{i\alpha}) = (e^{-i2\alpha/3}, e^{i\alpha}).$$

and hence, at least locally (for small $\alpha$), the group $U(1)$ is represented by a phase factor $e^{-i\alpha Y}$ in any unitary irreducible representation of $G$, in such a way that $3Y$ becomes an integer.

In the leptonic sector, spontaneous symmetry breaking selects another one-parameter subgroup of $U(2)$ which remains unbroken and gives rise to the concept of electric charge. By convention, this subgroup is

$$U(1)_Q = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \right\}$$

Therefore, the charge $Q$ assumes integer values in this sector. Again, with quarks the situation changes. The group $U(1)_Q$ being no longer a symmetry
is replaced by its threefold cover,
\[ \tilde{U}(1)_Q = \left\{ (u, v) \mid u = e^{i\beta} I_3, \ v = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \ 3\beta + \alpha = 0 \mod 2\pi \right\} \quad (7) \]

Locally, we may put \( \beta = -\alpha/3 \) and \( \tilde{U}(1)_Q = e^{-i\alpha Q} \) such that \( 3Q \in \mathbb{Z} \).

Summarizing, the hypercharge and the electric charge are represented on \( \mathbb{C}^5 \) by the following traceless matrices:
\[ Y = \text{diag}(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, -1, -1) \]
\[ Q = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, -1) \]

Though many states, especially those invariant under the color group \( SU(3) \), carry integer charges, \( Q \) and \( Y \) assume fractional values in general, still obeying \( Q + Y \in \mathbb{Z} \).

## 3 The \( G \)-Supermodule of Fermions

By construction, there is a natural irreducible unitary action of the gauge group \( G \) on the space
\[ \mathbb{C}^5 = \mathbb{C}^3 \oplus \mathbb{C}^2 \]
with subspaces \( \mathbb{C}^3 \) and \( \mathbb{C}^2 \) carrying fundamental representations of the color group \( SU(3) \) and the weak-isospin group \( SU(2) \) respectively. But, passage to the \( \mathbb{Z}_2 \)-graded exterior algebra
\[ \Lambda^5 = \sum_{k=0}^{5} \Lambda^k \mathbb{C}^5 = \Lambda^+ \mathbb{C}^5 \oplus \Lambda^- \mathbb{C}^5, \quad \Lambda^\pm \mathbb{C}^5 = \sum_{(-1)^k = \pm 1} \Lambda^k \mathbb{C}^5 \]
is very essential if we want to let \( G \) act on a superspace. For a general discussion and details concerning the exterior algebra as a superspace we refer to [5]. It is apparent that the induced unitary representation \( \Lambda \) of \( G \) on \( \Lambda \mathbb{C}^5 \) is reducible. We take the view that \( \Lambda \mathbb{C}^5 \) is the basic \( G \)-module for fermions. Quarks and leptons of one generation will be grouped according to the irreducible constituents of the representation \( \Lambda \). In addition, there is another space \( \mathbb{C}^3 \), not a \( G \)-module, which describes the flavor degrees of freedom. While we refer to \( \Lambda \mathbb{C}^5 \) as the \( G \)-supermodule of fermions, we call the tensor product
\[ \Lambda \mathbb{C}^5 \otimes \mathbb{C}^3 \]
the inner space, because it incorporates all inner degrees of freedom.

We now turn to the structure of the \( G \)-supermodule. From (8) and the natural isomorphism (of vector spaces)
\[ \Lambda(\mathbb{C}^3 \oplus \mathbb{C}^2) \cong \Lambda \mathbb{C}^3 \otimes \Lambda \mathbb{C}^2, \]
where
\[ \bigwedge C^3 = \sum_{p=0}^{3} \bigwedge^p C^3, \quad \bigwedge C^2 = \sum_{q=0}^{2} \bigwedge^q C^3, \]
we obtain \( \bigwedge(u,v) = \bigwedge u \otimes \bigwedge v \) for \((u,v) \in G\) and hence
\[ \bigwedge^k(u,v) = \sum_{p+q=k} \bigwedge^p u \otimes \bigwedge^q v, \quad k = 0, \ldots, 5. \]
We call
\[ \kappa = (-1)^k = (-1)^{p+q} \]
the parity operator in \( \bigwedge C^5 \).
Within a generation of fermions, each multiplet (left- or right-handed) is associated with one of the following irreducible representations of \( G \),
\[ \bigwedge^{p,q} = \bigwedge^p \otimes \bigwedge^q \quad p = 0, 1, 2, 3, \quad q = 0, 1, 2 \]
whose dimension is \( \binom{3}{p} \binom{2}{q} \). To find its hypercharge we use Eq. (5),
\[ e^{-\alpha Y} = \bigwedge^{p,q}(s^{-1}(e^{i\alpha})) = \exp(-i2p\alpha/3 + iq\alpha), \]
and thus obtain the fundamental relation
\[ Y = \frac{2}{3}p - q. \]
We distinguish
- lepton fields : \( p = 0 \) or \( 3 \)
- quark fields : \( p = 1 \) or \( 2 \).
The electric charge then satisfies the formula of Gell-Mann-Nishijima
\[ Q = I_3 + \frac{1}{2}Y \]
where \( I_3 \) denotes the third component of the weak isospin. Clearly, \( I_3 = 0 \)
if \( q = 0, 2 \) and \( I_3 = \pm \frac{1}{2} \) if \( q = 1 \).
With each generation we associate a generalized Dirac field \( \psi \) having \( 2^5 = 32 \) elementary Weyl spinors as its components. Spinors that enter \( \psi \) are characterized by three different “parities” owing to the \( \mathbb{Z}_2 \)-gradings of \( \bigwedge C^5 \), \( \bigwedge C^3 \), and \( \bigwedge C^2 \). Their interpretation is as follows (recall that \( k = p + q \)):
\[ \begin{align*}
    k &= \text{even} : \text{right-handed} \\
    k &= \text{odd} : \text{left-handed} \\
    p &= \text{even} : \text{matter} \\
    p &= \text{odd} : \text{antimatter} \\
    q &= \text{even} : \text{singlets} \\
    q &= \text{odd} : \text{doublets}.
\end{align*} \]
Since charge conjugation acts on the inner space by complex conjugation, it passes from \([p, q]\) to \([3 - p, 2 - q]\). It thus interchanges left and right, matter and antimatter, and reverses the signs of \( Y \), \( I_3 \) and \( Q \), but takes singlets into singlets and doublets into doublets.
Choosing a Basis in $\bigwedge \mathbb{C}^5$

To describe the field $\psi$ in more conventional terms we need to construct a basis of eigenvectors in $\bigwedge \mathbb{C}^5$. Such a construction starts from a basis in $\mathbb{C}^5$. Let us emphasize: besides being a 5-dimensional complex linear space, $\mathbb{C}^5$ is endowed with a Hermitian structure and also comes with a distinguished orthonormal basis $(e_i)_{i=1}^5$ such that $(e_i)_j = \delta_{ij}$. We shall refer to it as the *standard basis* in $\mathbb{C}^5$.

An induced basis $e_I$ in $\bigwedge \mathbb{C}^5$ is then given by

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \bigwedge^k \mathbb{C}^5,$$

$I = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$, $0 \leq k \leq 5$

where $I$ runs over all subsets of $\{1, 2, 3, 4, 5\}$ including the empty set $\emptyset$. We simply have to identify the numbers $k, p, q$ previously introduced. Given any subset $I$,

- $k$ is the number of elements in $I$ taken from $\{1, 2, 3, 4, 5\}$,
- $p$ is the number of elements in $I$ taken from $\{1, 2, 3\}$,
- $q$ is the number of elements in $I$ taken from $\{4, 5\}$.

We shall also write $|I|$ in place of $k$, and the complement of $I$ in $\{1, 2, 3, 4, 5\}$ is denoted $I^c$.

The following table lists all 32 basis vectors and groups them according to their $p$ and $q$ values. For convenience, we write $I$ where we really mean $e_I$.

$$
\begin{array}{|c|c|c|c|}
\hline
I & q = 0 & q = 2 & q = 1 \\
\hline
p = 0 & \emptyset & 45 & 4 \\
\hline
p = 2 & 23 & 2345 & 234 \\
 & 13 & 1345 & 134 \\
 & 12 & 1245 & 124 \\
\hline
p = 1 & 1 & 145 & 14 \\
 & 2 & 245 & 24 \\
 & 3 & 345 & 34 \\
\hline
p = 3 & 123 & 12345 & 1234 \\
\hline
\end{array}
$$

The basis vectors of the first two columns have $I_3 = 0$ while those of the third and fourth column have $I_3 = \frac{1}{2}$ and $I_3 = -\frac{1}{2}$ respectively. By construction, each basis vector $e_I$ is an eigenvector of $Y$ and $Q$. 


The following two tables provide the hypercharges (left table) and the electric charges (right table) associated with the basis vectors:

| Y  | q = 0 | q = 2 | q = 1 |
|----|-------|-------|-------|
| p = 0 | 0   | -2   | -1   | -1   |
| p = 2 | 4/3 | -2/3 | 1/3  | 1/3  |
|       | 4/3 | -2/3 | 1/3  | 1/3  |
| p = 1 | 2/3 | -4/3 | -1/3 | -1/3 |
|       | 2/3 | -4/3 | -1/3 | -1/3 |
| p = 3 | 2   | 0    | 1    | 1    |

| Q  | q = 0 | q = 2 | q = 1 |
|----|-------|-------|-------|
| p = 0 | 0   | -1   | 0    | -1   |
| p = 2 | 2/3 | -1/3 | 2/3  | -1/3 |
|       | 2/3 | -1/3 | 2/3  | -1/3 |
| p = 1 | 1/3 | -2/3 | 1/3  | -2/3 |
|       | 1/3 | -2/3 | 1/3  | -2/3 |
| p = 3 | 1   | 0    | 1    | 0    |

After this assignment of charges, the basis vectors $e_I$ can be put in a 1:1 correspondence with the 32 Weyl spinors of the first generation:

$$
\begin{array}{cccc}
\text{1. gener.} & q = 0 & q = 2 & q = 1 \\
\hline
p = 0 & \nu_{eR} & e_R & \nu_{eL} & e_L \\
p = 2 & u_{1R} & d_{1R} & u_{1L} & d_{1L} \\
       & -u_{2R} & -d_{2R} & -u_{2L} & -d_{2L} \\
       & u_{3R} & d_{3R} & u_{3L} & d_{3L} \\
p = 1 & d_{1L}^c & u_{1L}^c & d_{1R}^c & -u_{1R}^c \\
       & d_{2L}^c & u_{2L}^c & d_{2R}^c & -u_{2R}^c \\
       & d_{3L}^c & u_{3L}^c & d_{3R}^c & -u_{3R}^c \\
p = 3 & e_L^c & \nu_{eL}^c & e_R^c & -\nu_{eR}^c \\
\end{array}
$$

(12)

Two more tables of this kind exist for the second and third generation. Some remarks are in order:

- The symbols (including their sign) stand for the components $\psi_I$ of the field $\psi$ where the appropriate subset $I \subset \{1, 2, 3, 4, 5\}$ is displayed in the table (11). For instance,

$$
\psi_{13} = -u_{2R} \quad \text{etc.}
$$

By assumption, all fermions are massless to begin with. Therefore, each entry also represents a particle or antiparticle. The upper half of
the table contains the particles ("matter") while the lower half contains
the antiparticles ("antimatter").

• Quarks such as $u$ (up) and $d$ (down) come in three colors: $i = 1, 2, 3$.
  While quarks transform under the color group according to the representa-
tion $3$, antiquarks transform according to the representation $\bar{3}$.
  Since both $3$ and $\bar{3}$ are fundamental irreps of $SU(3)$, interchanging
  their role, as done here, has no physical effect.

• Together with each spinor the charged conjugate spinor (with upper
  index $c$) also enters the table (12) and so enters the field $\psi$. Since
  charge conjugation changes the chirality, we have to make precise what
  the symbols in (12) really mean. With the $d$-quark as an example our
  convention is:

  $d_L^c := (d^c)_L = (d_R^c)^c, \quad d_R^c := (d^c)_R = (d_L^c)^c$.

• Contrary to the traditional formulation of the Minimal Standard Model,
  there is room for a right-handed neutrino. Note the presence of the
  unconventional field $\nu_{eR}$ (together with its charge conjugate $\nu_{eL}^c$) in
  the table (12) and the fact that it transform trivially under the gauge
  group $G$. Thus, the right-handed neutrino does not couple to any
  gauge field whatsoever which makes it hard to detect it in experi-
  ments. It only couples to the Higgs field and so acquires a mass after
  symmetry breaking.

• Algebraic reasoning has led us to include certain minus signs in the
  table (12). One of the reasons is that we want the following condition
  to be satified:

  $\sigma_I \psi_I^c = \psi_{I^c}$ \hspace{1cm} (13)

where $\sigma_I$ is the sign of the permutation taking $\{I, I^c\}$ to its normal
order $\{1, 2, 3, 4, 5\}$. This in particular guarantees that, if $(\nu_e, e)_L$
trans-
forms as a $SU(2)$ doublet, so does the pair

$(e^c, -\nu_{eL}^c)_R = (e_L^c, -\nu^c_{eL})^c$

after charge conjugation.

5 The Concept of a Generalized Majorana Field

The field $\psi$ is thought of as some generalized Dirac field having sufficiently
many components $\psi_I$ to as to be able to describe all fundamental fermions
of one generation. For its mathematical construction we need to introduce
the spinor space $S$, basic to any Dirac field. Since its structure depends
merely on the choice of spacetime (of even dimension in any case), we call
$S$ the outer space. In the language of [4], $S$ is a Clifford supermodule, i.e., a linear space on which the $\gamma$ matrices act, carrying a $\mathbb{Z}_2$-grading

$$S = S^+ \oplus S^-$$

(14)

given by the chirality, the eigenvalues $\pm 1$ of $\gamma_5$. Field components taking values in $S^+$ ($S^-$) are said to be right-handed (left-handed).

In addition, we have assumed that there is another superspace, $\Lambda \mathbb{C}^5$, graded by the parity $\kappa$ of exterior powers. This space, specific to the Standard Model, is the same for all fermion generations. Since we wish to relate the chirality in $S$ to the parity in $\Lambda \mathbb{C}^5$, the field $\psi$ is required to take values in the even part of the tensor product $E = \Lambda \mathbb{C}^5 \otimes S$ which is

$$E^+ = (\Lambda \mathbb{C}^5 \otimes S)^+ = (\Lambda^+ \mathbb{C}^5 \otimes S^+) \oplus (\Lambda^- \mathbb{C}^5 \otimes S^-) .$$

(15)

In order to write $\psi$ in terms of its components $\psi_I$ we use the basis $e_I$ for the inner space as constructed in the previous section. With respect to the tensor product $\Lambda \mathbb{C}^5 \otimes S$, we decompose the field $\psi$ as

$$\psi(x) = \sum I e_I \otimes \psi_I(x) \in E^+ , \quad \psi_I(x) \in S .$$

(16)

The condition $\psi \in E^+$ translates into

$$\gamma_5 \psi_I = (-1)^{|I|} \psi_I .$$

(17)

Hence, $\psi_I$ is right(left)-handed depending on whether $|I|$ is even(odd).

Two operations of similar nature, one in $\Lambda \mathbb{C}^5$ and one in $S$, will play an important role:

- The Hodge operator $* : \Lambda \mathbb{C}^5 \to \Lambda \mathbb{C}^5$ is antilinear and acts on the basis as
  
  $$* e_I = \sigma_I e_I^c .$$

(18)

Recall that $\sigma_I$ is the sign of the permutation taking $\{I, I^c\}$ to its normal order $\{1, 2, 3, 4, 5\}$. Since $|I| + |I^c| = \text{odd}$, the Hodge operator is parity changing:

$$* \kappa* = - \kappa .$$

(19)

Note that $\sigma_{I^c} = \sigma_I$ (valid in odd dimensions) making the Hodge $*$ an involutive operator: $*^2 = 1$

- The charge conjugation $S \to S$, $s \mapsto s^c$, is antilinear, involutive and reverses the chirality: $S^\pm \to S^\mp$.

The identification

$$\text{parity in } \Lambda \mathbb{C}^5 = \text{chirality in } S$$
suggests to couple both operations, resulting in a single antilinear operator 
\( * : E^+ \rightarrow E^+ \) such that
\[
\psi(x) = \sum_I e_I \otimes \psi_I(x) \quad \Rightarrow \quad *\psi(x) = \sum_I *e_I \otimes \psi^*_I(x) .
\] (20)

Though the \(*\) operator now changes matter into antimatter and vice versa, it should not be confused with “charge conjugation” in the traditional sense,
\[
\psi_c(x) = \sum_I e_I \otimes \psi^*_I(x)
\]
which is not a symmetry. Using (13) and (18) we find
\[
*\psi(x) = \sum_I \sigma_I e_I^c \otimes \psi^*_I(x) = \sum_I e_I^c \otimes \sigma_I \psi^*_I(x) = \sum_I e_I \otimes \psi_I(x) = \psi(x)
\]

A generalized Dirac field \( \psi \) based on the \( G \)-module \( \bigwedge \mathbb{C}^5 \) is said to be selfdual or a generalized Majorana field if it satisfies the relation \( *\psi = \psi \).

Experimentally, three generations of fundamental fermions have been found: from the decays of the \( Z^0 \) boson one infers that there are exactly three generations (i.e., three is the number of neutrinos with masses below 45 GeV). It is a trivial matter to combine the Dirac fields \( \psi_f \) \( (f = 1, 2, 3) \) of three generations to a single master field:
\[
\Psi = \psi_1 \otimes \epsilon_1 + \psi_2 \otimes \epsilon_2 + \psi_3 \otimes \epsilon_3
\] (21)

where \( (\epsilon_i)_{i=1}^3 \) denotes the standard basis in \( \mathbb{C}^3 \), the flavor space. For consistency, the \(*\) operator must also act on \( \mathbb{C}^3 \) in an antilinear manner. We let it coincide with complex conjugation. Hence, if \( A \) is some matrix in \( \text{End} \mathbb{C}^3 \), then \( *A* \) stands for the complex conjugate matrix. The fundamental relation \( *\psi_f = \psi_f \), valid in each generation, is now equivalent to stating that \( *\Psi = \Psi \).

6 Analysis of Operators

The complex space \( \mathbb{C}^5 \) we started from not only provides a natural basis \( (e_i)_{i=1}^5 \) but is also equipped with a Hermitian structure given by the standard scalar product.

Supposing \( V \) is any Hermitian vector space, one associates a multiplication operator \( \epsilon(v) \) to any \( v \in V \), acting on the exterior algebra,
\[
\epsilon(v)a = v \wedge a \quad (a \in \bigwedge V),
\]

\footnote{The definition of selfduality works for spaces \( \bigwedge \mathbb{C}^n \) where \( n \) is odd. If \( n = 1 \), the concept reduces to that of an ordinary Majorana field.}
and lets $\iota(v)$ be its adjoint with respect to the induced Hermitian structure in $\bigwedge V$. In the language of Fock spaces, these operators are said to be creation and annihilation operators respectively. It is evident that they are of odd type, i.e., they change the parity of elements in $\bigwedge V$. We state this property as

$$\epsilon(v), \iota(v) \in \text{End}^{-} \bigwedge V.$$  

With $V = \mathbb{C}^5$ we put $b_i = \epsilon(e_i)$ and $b_i^* = \iota(e_i)$ so as to obtain the usual anticommutation relations:

$$\{b_i, b_k^*\} = \delta_{ik}, \quad \{b_i, b_k\} = 0, \quad \{b_i^*, b_k^*\} = 0.$$  

The key observation is that lifting matrices $a = (a_{ik}) \in \text{End} \mathbb{C}^5$ to observables now admits an explicit description:

$$\theta(a) = \sum_{ik} a_{ik} b_i b_k^* \in \text{End}^+ \bigwedge \mathbb{C}^5. \quad (22)$$

To put it more formally, $\theta(a)$ is a derivation of the algebra $\bigwedge \mathbb{C}^5$. As for us, $\theta(a)$ is simply a linear operator of even type. Being a $G$-module, $\bigwedge \mathbb{C}^5$ carries a representation $\theta$ of Lie $G$. Note that the map $\theta$ constructed above extends the representation of the Lie algebra.

Some of the previously introduced observables receive a new description:

\[
    \begin{align*}
    Q &= \frac{1}{3} (b_1 b^*_1 + b_2 b^*_2 + b_3 b^*_3) - b_5 b^*_5 \\
    Y &= \frac{2}{3} (b_1 b^*_1 + b_2 b^*_2 + b_3 b^*_3) - b_4 b^*_4 - b_5 b^*_5 \\
    I_3 &= \frac{1}{2} (b_4 b^*_4 - b_5 b^*_5) \\
    p &= b_1 b^*_1 + b_2 b^*_2 + b_3 b^*_3 \\
    q &= b_4 b^*_4 + b_5 b^*_5.
    \end{align*}
\]

The significance of the operators $p$ and $q$ is that they generate the maximal algebra of operators, invariant under gauge transformations. This algebra, also referred to as the commutant $(\bigwedge G)',$ is abelian and 12-dimensional: there are 4 and 3 possible values for $p$ resp. $q$.

Global gauge transformations act on the operators $b_i$ as they would act on the basis $e_i$:

$$\bigwedge g b_i \bigwedge g^{-1} = \sum_j g_{ij} b_j \quad (g \in G).$$

Consequently,

$$\bigwedge g b_i^* \bigwedge g^{-1} = \sum_j (g^*)_{ij} b^*_j.$$  

The Hodge operator $*$ has already been seen to play a prominent role. We shall now elaborate on its properties a little further. Recall first that $*^2 = \mathbb{1}$. Second, we have

$$*p* = 3 - p, \quad *q* = 2 - q.$$
Third, the Hodge operator interchanges $b_i$ and $b_i^*$ apart from possible sign change: 

\[ *b_i * = \kappa b_i^*, \quad *b_i^* * = b_i \kappa. \]

See (11) for the definition of the parity operator $\kappa$. As a consequence, we get 

\[ *b_i b_j^* * = \kappa b_i^* b_j \kappa = b_i^* b_j = \delta_{ij} - b_j b_i^*. \]

Let $a \in \text{End} \mathbb{C}^5$ be arbitrary. From (22) and the antilinearity of $*$, 

\[ *\theta(a)^* = \sum_{ij} \bar{a}_{ij} * b_i b_j^* * \]

\[ = \sum_{ij} (a^*)_{ji} (\delta_{ij} - b_j b_i^*) = \text{tr} a^* - \theta(a^*). \]

In particular, 

\[ * \theta(a)^* = \theta(a), \quad a \in \mathfrak{su}(5) \quad (24) \]

owing to the relations $a^* = -a$ and $\text{tr} a = 0$, and ultimately: 

\[ * \bigwedge g^* = \bigwedge g, \quad g = e^a \in G. \quad (25) \]

This is to demonstrate that we cannot dispense with the trace condition $\text{tr} a = 0$, i.e., demanding that $G$ be a subgroup of $SU(5)$ (rather than of $U(5)$) if we want the consistency condition (22) to be satisfied.

While elements of the Lie algebra are unchanged under the $*$ operation, their Hermitian counterparts $i\theta(a)$ pick up a minus sign owing to antilinearity, and so do the charges:

\[ *Q^* = -Q, \quad *Y^* = -Y, \quad *I_3^* = -I_3. \]

This is in accord with the conception, formulated before, that the Hodge $*$ converts particles into antiparticles and vice versa.

7 The Higgs Field

There is no other way than to assume that particles receive their masses through the Higgs mechanism. As explained in [2] and [6], the mechanism works well only if the Higgs field is an odd operator on the internal $\mathbb{Z}_2$-graded space. In the present situation, this space is 

\[ V = \bigwedge \mathbb{C}^5 \otimes \mathbb{C}^3 \]

endowed with the obvious grading $V^\pm = \bigwedge^\pm \mathbb{C}^5 \otimes \mathbb{C}^3$. With one Higgs doublet, 

\[ \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \]

\footnote{Normally, the Higgs field is thought of as a $(Y = 1)$ doublet $(\phi_+, \phi_0)$. We prefer to work instead with $\phi_1 = \phi_0^*$ and $\phi_2 = -\phi_+^*$. Note that $\phi_1$ has zero electric charge while $\phi_2$ has the charge $Q = -1$. Note also that the scalar $\phi^* \phi = |\phi_1|^2 + |\phi_2|^2$ is gauge invariant.}
of hypercharge $Y = -1$, there is more than one choice for the Higgs field $\Phi$ (when written in terms of $\phi_1$ and $\phi_2$) if we want $\Phi$ to act as an operator on the (rather large) internal space $V$, merely requiring that $\phi$ transforms properly under the gauge group. As perceived by the founders of the Standard Model, this freedom of choice shows up in the appearance of a variety of undetermined Yukawa coupling constants, one for each elementary fermion field. Traditional thinking forbids to combine the elementary fermion fields into a single mathematical entity, as the constituents couple differently to the Higgs doublet $\phi$.

In essence, what we suggest here is to look at the freedom of fixing parameters (such as Yukawa couplings) from a different perspective. Recalling that the commutant $(\bigwedge G)' \subset \text{End} \bigwedge \mathbb{C}^5$ is nontrivial, we are free to choose some invariant operator

$$h \in (\bigwedge G)' \otimes \text{End} \mathbb{C}^3$$

and to define the Higgs field by

$$\Phi(x) = h^* \left( \phi_1(x)b_4 + \phi_2(x)b_5 \right)$$

so as to have some field acting on the inner space $\bigwedge \mathbb{C}^5 \otimes \mathbb{C}^3$. A gauge transformation, taking $\Phi$ into $\bigwedge g \Phi \bigwedge g^{-1}$, reveals that the complex scalar fields $\phi_i$ transform as desired.

In general, the invariant operator (26) is determined by providing 12 matrices

$$h(p, q) \in \text{End} \mathbb{C}^3 \quad (p = 0, \ldots, 3, \; q = 0, 1, 2)$$

The fact the matrices $h(p, 0)$ do not contribute to the Higgs field (27) reduces the number 12 to 8. A further reduction to 4 is achieved with help of the constraint

$$\star \Phi(x) \star = \Phi(x)^*$$

implying that particles couple to $\Phi$ in the same way as antiparticles couple to $\Phi^*$. The condition (29) is equivalent to

$$\star h \star = h^\dagger$$

where another invariant operator $h^\dagger$, associated with $h$, has been introduced so as to satisfy the equation $b_i h^\dagger = h^* b_i \kappa \; (i = 4, 5)$. Written out, the definition is

$$h^\dagger(p, q) = (-1)^{p+q} h(p, q + 1)^* \quad (p = 0, \ldots, 3, \; q = 0, 1).$$

Recall that $\star h(p, q) \star$ means complex conjugation of matrix elements in conjunction with the replacements $p \to 3 - p$ and $q \to 2 - q$. Therefore, another way to write the condition (30) is

$$h(3-p, 2-q) = (-1)^{p+q} h(p, q + 1)^T \quad (p = 0, \ldots, 3, \; q = 0, 1)$$
where $A^T$ denotes the transpose of a matrix $A \in \text{End} \mathbb{C}^3$. Therefore, among the matrices $h(p,q)$ ($q > 0$), only four have to be fixed in order to determine them all. Summarizing:

*We are working with a common Yukawa coupling set equal to unity and assemble all free parameters, such as the entries of the fermion mass matrix, in the operator $h$.*

Presently, there does not seem to exist a convincing theoretical argument that would settle the question as to the origin of $h$ and completely determine the matrices $h(p,q)$ entering the mass operator. We need first of all understand the mathematical origin of the flavor symmetry (when there is no Higgs condensate).

Given the form of $h$, the next step is to allege that the Higgs field $\Phi$ enters the Dirac operator in a symmetrized form

$$
L = i(\Phi + \Phi^*) \in \text{End}^{-V(3)}
$$

so as to satisfy

$$
L^* = -L, \quad *L* = -L.
$$

Spontaneous symmetry breaking gives rise to a condensate,

$$
\Phi_c = m^* b_4, \quad m = r^{1/2} h, \quad r = |\langle \phi \rangle|^2
$$

and hence to a fermion mass operator

$$
M = -iL_c = m^* b_4 + b_4^* m, \quad *M* = M^*, \quad *m* = m^\dagger.
$$

Note the difference: while the symmetry breaking parameter $r$ is but an ordinary constant (to be obtained from the Higgs potential), the condensate $\Phi_c$ and the mass operator $M$ are still *operators of odd type*. In terms of the mass matrices $m(p,q) \in \text{End} \mathbb{C}^3$ ($q > 0$) the relation $*m* = m^\dagger$ may be written:

$$
m(3-p, 2-q) = (-1)^{p+q} m(p,q + 1)^T \quad (p = 0, \ldots, 3, \ q = 0, 1)
$$

The spectrum of $M^2$ describes the fermion masses (squared). The components $m(p,q)$ of $M$ refer to subgroups of particles, each group containing three particles of the same electric charge $Q$ but different flavors:

$$
m(0,1) : \ Q = 0 \quad m(0,2) : \ Q = -1
$$

$$
m(2,1) : \ Q = 3/2 \quad m(2,2) : \ Q = -1/3
$$

Here, we we have listed only those matrices $m(p,q)$ for which $p$ is even as the relation (37) says that matrices, for which $p$ is odd, are related to the former. At the end, the relation simply guarantees that matter fields ($p =$even) and antimatter fields ($p =$odd) receive equal masses.
Superconnections, Generalized Dirac Operators, and the Fermionic Action

Recall the interpretation of the gauge field $A$ as a connection 1-form, i.e., Euclidean spacetime is modelled by some four-dimensional Riemannian manifold $M$, and $A$ takes values in $T^* M \otimes \text{Lie} G$ where $T^* M$ stands for the cotangent bundle. Our main interest lies in the lifted field $\hat{A} = \theta(A)$ taking values in $T^* M \otimes \text{End}^+ \wedge \mathbb{C}^5$. In local coordinates,

$$A = dx^\mu \otimes A_\mu, \quad A_\mu(x) \in \text{Lie} G$$

$$\hat{A} = dx^\mu \otimes \hat{A}_\mu, \quad \hat{A}_\mu(x) \in \text{End}^+ \wedge \mathbb{C}^5.$$  

A superconnection (see [6] for details) extends the notion of a gauge connection and is given by some first-order differential operator of odd type,

$$\mathcal{D} = D + L, \quad D = d + \hat{A}, \quad (38)$$

acting on sections\footnote{Sections are referred to as $\wedge \mathbb{C}^5$-valued differential forms.} of the bundle $\wedge T^* M \otimes \mathbb{C}^5$, where $d$ denotes the exterior derivative, $D$ the covariant derivative, and $L$ the Higgs field. Generally speaking, $L$ could also include $n$-forms ($n \neq 1$) complying with the oddness\footnote{The property of being odd or even is defined with reference to the total $\mathbb{Z}_2$-grading of the space $\wedge T^* M \otimes \mathbb{C}^5$ on which these operators act.} of $\mathcal{D}$.

With an $2n$-dimensional Riemannian spin$^c$ manifold [6] one associates a Clifford bundle $C(M)$ and a spin bundle which locally coincides with $M \times S$, $S$ being the spinor space, a complex vector space of dimension $2^n$. At $x \in M$, there is an isomorphism $c : C(T^*_x M) \otimes \mathbb{C} \rightarrow \text{End} S$ and hence a way to construct $\gamma$ matrices,

$$\gamma^\mu = c(dx^\mu), \quad \{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu},$$

involving $g^{\mu\nu}$, the metric tensor. Spinor fields $\Psi(x)$ take values in the space

$$F = V \otimes S = \wedge \mathbb{C}^5 \otimes \mathbb{C}^3 \otimes S.$$  

To use the language of [6], $F$ is a twisted Clifford supermodule with $V$ the twisting space.

With a superconnection $\mathcal{D}$ one associates the generalized Dirac operator $\mathcal{D}$ which, roughly speaking, is obtained from $\mathcal{D}$ by replacing the basis elements $dx^\mu$ by $\gamma^\mu$ wherever they occur. Therefore, if $L$ is scalar,

$$\mathcal{D} = \hat{\phi} + \hat{A} + L = \gamma^\mu (\partial_\mu + \hat{A}_\mu) + L. \quad (39)$$
The Dirac operator acts on spinor fields such that
\[ \Psi(x) \in F^\pm \implies iD/\Psi(x) \in F^\mp. \]
Suppose the manifold \( M \) is orientable and \( \omega_0 \) is a volume form. Then the fermionic action, considered as a functional of the master field \( \Psi \), is
\[
S_F = \frac{1}{2} \int_M \nabla iD/\Psi \omega_0.
\] (40)

By construction, the integrand incorporates both gauge and Yukawa interactions. Following [2] we regard \( \Psi \mapsto \overline{\Psi} \) as an antilinear map into the dual space which reverses the chirality: \( (\Psi_L) = (\Psi)_R \). We also emphasize that, with regard to functional integration, the (Grassmann) variables \( \Psi \) and \( \overline{\Psi} \) are dependent, and a reasonable way to write the functional measure is
\[
d\Psi = \prod_{f,I} d\psi_{fI} \] (41)
with \( f = 1, 2, 3 \) and \( I \) running over the subsets of \{1, 2, 3, 4, 5\}.

Two important features characterize the ansatz (38) for the Dirac operator.

1. The operator \( p \) (but not \( q \)) commutes with \( D/ \):
\[
pD/ = D/p. \] (42)
One consequence is that the baryon number and the lepton number are preserved in interactions: leptons and quarks do not couple at vertices of a Feynman diagram. Another is that matter is not converted into antimatter. Though matter and antimatter may annihilate to yield gauge and Higgs particles.

2. The \( * \) operator anticommutes with \( D/ \) and hence commutes with \( iD/ \):
\[
* (iD/) = iD/.* \] (43)
This fact follows from \( *\gamma^\mu* = -\gamma^\mu \), (24) and (34). The property (43) of the Dirac operator implies that, if \( \Psi \) is a generalized Majorana field, so is \( \Psi' = iD/\Psi \):
\[
\Psi = *\Psi \implies \Psi' = *\Psi',
\]
and from
\[
*\Psi *\Psi' = \sum_I \overline{\psi}_I^c \psi_I'^c = \sum_I \overline{\psi}_{I'}^c \psi_I = \overline{\Psi}'\Psi,
\]
we obtain
\[
\overline{\Psi} iD/\Psi = iD/\overline{\Psi} \Psi. \] (44)

---

6To put a factor \( \frac{1}{2} \) in front is necessary because each elementary fermion field \( \psi_I \) enters the action together with its charge conjugate \( \psi_I^c \), both giving equal contributions.

7The relation \( *\gamma^\mu = -\gamma^\mu* \) is our way of stating that \((\gamma^\mu \psi)^c = -\gamma^\mu \psi^c\) in ordinary Dirac theory.
The two properties of the Dirac operator stated above suggest decomposing \( \Psi \) into (anti)matter fields by projecting onto the \( p = \text{even(odd)} \) parts:

\[
\psi_M + \psi_A = \Psi \\
\psi_M - \psi_A = (-1)^p \Psi .
\]

The fact that the \( * \) operator switches between matter and antimatter is now reflected by the relation \( *\psi_M = \psi_A \). The antimatter field is thus seen to be a redundant variable and, if desired, may be eliminated from the action functional using

\[
\psi_M iID /\psi = \psi_M iID /\psi + \psi_A iID /\psi = \psi_M iID /\psi + *\psi_M iID /\psi .
\]

The functional measure (41) now assumes the standard form \( d\psi_M d\bar{\psi}_M \).

The matter field may be decomposed even further so as to extract singlet and doublet components:

\[
\psi_{MS} + \psi_{MD} = \psi_M \\
\psi_{MS} - \psi_{AD} = (-1)^q \psi_M .
\]

Owing to the term \( L \), part of the Dirac operator and anticommuting with \( q \), the Dirac operator induces transitions \( S \to D \) and \( D \to S \). Also, since

\[
(-1)^{p+q}\psi_M = (-1)^q \psi_M ,
\]

the field \( \psi_{MS} \) is right-handed while \( \psi_{MD} \) is left-handed.

9 Currents

It is instructive to see how currents emerge from the ansatz (40). For this we need only evaluate \( \hat{A}/ = \theta(\hat{A}) \) in some basis. Let \( (-it_a)_{a=1}^{12} \) be any basis in \( \text{Lie} G \) (so that \( t_a^* = t_a \)). If the gauge field \( A \) has real components \( A_a^\mu \) given by \( iA_\mu = A_a^\mu t_a \), then

\[
i\hat{A} = A_a^\mu T_a \gamma^\mu , \quad T_a = \theta(t_a) .
\]

The representation of gauge couplings in terms of currents is an immediate result:

\[
\bar{\psi}_M i\hat{A}\psi_M = j^a_\mu A_a^\mu , \quad j^a_\mu = \bar{\psi}_M T_a \gamma^\mu \psi_M . \quad (45)
\]

Inspection shows that all currents preserve both \( p \) and \( q \) in the following sense:

\[
pT_a = T_a p , \quad qT_a = T_a q
\]
Each current can therefore be decomposed into constituents with a definite \((p,q)\) assignment,

\[ j_a^\mu = \sum_{p,q} j_a^\mu(p,q), \]

meaning that fermions with different \((p,q)\) assignments have no common vertex. There is no current for the right-handed neutrino: \( j_\mu^N(0,0) = 0 \).

Currents may be characterized according to their behavior with respect to chirality.

**Definition.** A current \( j_a^\mu = \bar{\psi}_M T^a \gamma^\mu \psi_M \) (for fixed index \( a \)) is said to be vectorlike if \( T_a^b \delta^4 = \delta^4 T_a^b \) and chiral otherwise. Likewise, the interaction \( j_a^\mu A^a_\mu \) is said to be vectorlike resp. chiral if the current is.

The reason for the above definition is the observation that the operators \( b_4 \) and \( b_4^* \) switch between left- and right-handed fields of the same kind. No matter what gauge group \( G \), vectorlike currents are abundant. They form a linear subspace of the space of all currents. While the total space is 12-dimensional, the subspace is 9-dimensional for our choice of \( G \). It is spanned by the electric current and eight currents pertaining to the color group. In other words, the interactions with the photon and the gluons are vectorlike.

There is no unique choice of generators \( T_a \) for the 3-dimensional quotient space. One choice could be:

\[
I_1 = \frac{1}{2} (b_5 b_4^* + b_4 b_5^*) \\
I_2 = \frac{i}{2} (b_5 b_4^* - b_4 b_5^*) \\
I_3 = \frac{1}{2} (b_4 b_4^* - b_5 b_5^*)
\]

(46)

It is indeed easily verified that \( I_a b_4 \neq b_4 I_a \ (a = 1, 2, 3) \). The observation made here corresponds to the fact that nature provides three basic parity violating interactions of matter corresponding to the exchange of \( W^\pm \) and \( Z^0 \) bosons. The coupling to the massive vector bosons specify the selection of \( T \)'s not commuting with \( b_4 \). In passing we mention that, with the group \( SU(5) \) replacing \( G \), the quotient space of chiral currents would be 9-dimensional, admitting further parity violating interactions.

**10 The Bosonic Action**

In the same way as the Yang-Mills connection\(^8\) \( D \) gives rise to the concept of curvature,

\[ F = D^2 = \frac{1}{2} dx^\mu \wedge dx^\nu F^a_{\mu\nu} (-iT_a), \]

the superconnection \( ID \) gives rise to a generalized curvature [6],

\[ IF = ID^2 = F + DL + L^2, \]

(47)

\(^8\)We use the terms connection and covariant derivative interchangeably.
where we have identified $DL$, the covariant derivative of the Higgs field, with the supercommutator $[D, L] = \{D, L\}$. Without the Higgs field the bosonic action would consist of nothing but the Yang-Mills term:

$$S_B = \frac{1}{2} \|F\|^2 = \frac{1}{2} \int_M |F|^2 \omega_0 \quad (L = 0).$$

The precise nature of the invariant $|F|^2$ better be such that it reduces to

$$|F|^2 = \frac{1}{2} \sum_{a, \mu \nu} (F^a_{\mu \nu})^2$$

in a flat Euclidean universe. Constructing an invariant $|F|^2$ for 2-forms $F$ with the required property is a well-known procedure. An extension to $\mathcal{IF}$, properly treating $p$-forms of arbitrary order, has been given and discussed in [2]. It requires various steps.

1. **Step.** A basis $-it_a$ in Lie $G$ is chosen such that

$$\text{Tr} T_a T_b = \delta_{ab} \quad (a, b = 1, \ldots, 12) \quad (48)$$

where $T_a = \theta(t_a)$ as before.

2. **Step.** Since any operator $A$ in $\wedge \mathbb{C}^5$ (like $T_a$) extends to an operator in

$$V = \wedge \mathbb{C}^5 \otimes \mathbb{C}^3,$$

we shall not distinguish between $A$ and its extended version, $A \otimes 1$. However when it comes to studying their traces, there would be a distinction unless ‘Tr’ in $\mathbb{C}^3$ is given another meaning:

$$\text{Tr} C = \frac{1}{3} \sum_{i=1}^{3} C_{ii}, \quad C = (C_{ik}) \in \text{End} \mathbb{C}^3.$$

The modified trace is but an average. We stipulate that traces, although written $\text{Tr} A$, always include an extra factor $\frac{1}{3}$ for operators $A$ in $V$. This precaution is necessary in order to preserve the validity of relations like (48) or $\text{Tr} q = 32$ and many others.

3. **Step.** There is no need for a differential form to be homogeneous (i.e., a $p$-form). As the construction of $\mathcal{IF}$ shows, we are dealing here with very general forms $B$ that are sections of the algebra

$$B = \wedge^* M \otimes \text{End} V.$$

---

Note that both $D$ and $L$ are odd operators.
Since both $\wedge^* T^* M$ and $\text{End} V$ are superalgebras, so is their product, and the symbol $\hat{\otimes}$ accounts for that fact: the tensor product is special for $\mathbb{Z}_2$-graded algebras [4]. Any element $B$ taking values in $\mathcal{B}$ has the following structure:\[B = \sum_I dx^I \otimes B_I, \quad B_I \in \text{End} V.\]

As $\dim M = 4$, the sum runs over the subsets $I \subset \{1, 2, 3, 4\}$ and 
\[dx^I = dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}\]
where \[I = \{\mu_1, \ldots, \mu_r\}, \quad \mu_1 < \cdots < \mu_r, \quad r = |I|.
\]

We let
\[|B|^2 = \sum_{I,J} g^{IJ} \text{Tr} B_I^* B_J\] (49)

where $g^{IJ} = (dx^I, dx^J)$, to be obtained from the Riemannian metric, and $g^{IJ} = \delta^{IJ}$ in a flat Euclidean universe.

Similar to the procedure in [2] we let the gauge-invariant bosonic action be given by
\[S_B = \int_M \frac{1}{2} |F|^2 + \mu^2 C|^2 \omega_0\] (50)

with $\mu$ some mass parameter and
\[C \in (\wedge^G) \otimes \text{End} \mathbb{C}^3, \quad C = *C* = C^*.\] (51)

Below we shall learn that it suffices to take $C = 1$. Without the shift operator $\mu^2 C$ the minimum of $S_B$ would be attained for a flat superconnection: $\mathcal{F} = 0$. By contrast, the ansatz (50) when taken as classical field theory predicts a constant curvature $\mathcal{F}$ in the ground state.

Thanks to the fact that $\mathcal{F}$ splits into $p$-forms with $p = 0, 1, 2$, the integrand above consists of three terms only,
\[\frac{1}{2} |\mathcal{F}|^2 + \mu^2 C|^2 = \frac{1}{2} |F|^2 + \frac{1}{2} |DL|^2 + \frac{1}{2} |L|^2 + \mu^2 C|^2\]
which are easily identified as the Yang-Mills term, the covariant kinetic term of the Higgs field, and the Higgs potential. In order to guarantee the correct behavior of the kinetic term, i.e.,
\[\frac{1}{2} |dL|^2 = g^{\mu\nu} (\partial_\mu \phi)^* (\partial_\nu \phi),\]
a normalization condition must be satisfied:
\[\frac{1}{2} \text{Tr} qhh^* = 1.\] (52)

As we shall see (in Section 13), this condition has important consequences.

\[\text{Again, we will be content here with a local description.}\]
11 The Higgs Potential and Symmetry Breaking

In this section, we shall work out the details of the Higgs potential

$$V(\phi) = \frac{1}{2} |L^2 + \mu^2 C|^2 = \frac{1}{2} \text{Tr} (L^2 + \mu^2 C)^2.$$ 

Starting from (27) and (33) we first evaluate some traces using *C* = C, *Φ* = Φ*, and the *-invariance of the trace:

- \( \text{Tr} L^4 = 2 \text{Tr} (\Phi \Phi^*)^2 = (\phi^* \phi)^2 \text{Tr} q(hh^*)^2 \)
- \( \text{Tr} L^2 C = - \text{Tr} (\Phi \Phi^* + \Phi^* \Phi) C = - \text{Tr} \Phi^*(C + \Phi \Phi) C \)

As a result of this calculation the Higgs potential may be written in terms of three constants:

$$V(\phi) = V_0 + \frac{\lambda}{4} (\phi^* \phi - r)^2.$$ 

Apart from the irrelevant \( V_0 \), the other two constants are

- the Higgs coupling \( \lambda = 2 \text{Tr} q(hh^*)^2 \)
- the condensate \( r = \mu^2 \text{Tr} qhh^* C / \text{Tr} q(hh^*)^2 \).

The invariant operator \( C \) is seen to enter the Higgs potential via two constants only, \( V_0 \) and \( r \). There will be no loss of generality when we decide to work \( C = I \) from now on. This fixes the parameter \( \mu \) setting the mass scale, and so

$$\lambda r = 4 \mu^2$$

owing to the normalization condition (52).

The techniques presented here are, for the most part, quite standard. However, we have decided to work with \( r = v^2/2 \) where \( v \) is, by convention, the symmetry breaking parameter used in most texts on the subject. Note that in the present framework the Higgs coupling constant \( \lambda \) can never be negative or zero. It cannot be arbitrarily small either. In fact,

$$\lambda \geq \frac{1}{4}.$$ 

The lower bound can be derived as follows. Define averages

$$\langle W \rangle = \text{Tr} qW/\text{Tr} q$$

for operators \( W \) acting on \( \bigwedge \mathbb{C}^5 \otimes \mathbb{C}^3 \). The obvious inequality \( \langle W^2 \rangle \geq \langle W \rangle^2 \) for \( W = hh^* \) together with \( \text{Tr} q = 32 \) and (52) leads to the lower bound for \( \lambda \).

As \( r > 0 \), we observe a breakdown of symmetry, and choosing the unitary gauge, we have

$$\phi_1 = 2^{-1/2} \varphi + r^{1/2}, \quad \phi_2 = 0$$

$$22$$
where \( \phi \) is the neutral scalar Higgs field. To extract its mass, we need only expand the Higgs potential up to second-order terms:

\[
V(\phi) = V_0 + \frac{1}{2}m_H^2 \phi^2 + O(\phi^3), \quad m_H = 2\mu.
\]

### 12 Masses and Coupling Constants of Vector Bosons

To give the particulars of vector bosons we must specify the basis \( -it_a \) in Lie \( G \) obeying (48):

\[
t_a = \begin{pmatrix}
\frac{1}{4} & 0 & 0 \\
0 & 0 & \sigma_a \\
\end{pmatrix}, \quad a = 1, 2, 3,
\]

\[
t_a = \frac{1}{4}\sqrt{\frac{3}{5}} \begin{pmatrix}
\frac{2}{3} & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -\frac{1}{2} \\
\end{pmatrix}, \quad a = 4,
\]

\[
t_a = \frac{1}{4} \begin{pmatrix}
\lambda_{a-4} & 0 \\
0 & 0 \\
\end{pmatrix}, \quad a = 5, \ldots, 12.
\]

We have written down the generators \( t_a \) as matrices acting on \( \mathbb{C}^3 \oplus \mathbb{C}^2 \) and made use of the Gell-Mann matrices \( \lambda_a \), which operate on \( \mathbb{C}^3 \), and the Pauli matrices \( \sigma_a \), which operate on \( \mathbb{C}^2 \). Checking the conditions (58) is facilitated by the relation [7]

\[
\text{Tr } T_a T_b = 8 \text{ tr } t_a t_b.
\]

With respect to the given basis, the mass matrix\(^{11}\) \( m^2 \) of the 12 vector bosons has matrix elements

\[
m_{ab}^2 = \text{Tr } [T_a, L_c][T_b, L_c]
\]

(see [2] for a derivation) with \( L_c \), the ‘condensate of \( L \)’, as in (36). By a straightforward computation using (52), we obtain

\[
m_{ab}^2 = r\{t_a, t_b\}_{44}, \quad (a, b = 1, \ldots, 12).
\]

where \( \{t_a, t_b\}_{ik} \) are the matrix elements of \( \{t_a, t_b\} \) as an operator on \( \mathbb{C}^5 \). By inspection, \( m_{aa}^2 = 0 \) if either \( a \geq 5 \) or \( b \geq 5 \) (gluons do not get masses) and we are left with some nontrivial \( 4 \times 4 \) matrix

\[
(m^2)_{a,b=1}^4 = \frac{r}{8} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & -\sqrt{3/5} \\
0 & 0 & -\sqrt{3/5} & 3/5 \\
\end{pmatrix}
\]

the eigenvalues of which provide the masses of the four vector bosons of the electroweak sector:

\[
\begin{align*}
\text{mass squared of the } W^\pm & : \quad m_W^2 = r/8 \\
\text{mass squared of the } Z^0 & : \quad m_Z^2 = r/5 \\
\text{mass squared of the } \gamma & : \quad 0
\end{align*}
\]

\(^{11}\) A matrix whose eigenvalues are the masses squared.
The eigenvalue $r/8$, when identified with $m_W^2$, together with (55) and (57), leads to an expression for the Higgs coupling constant,

$$\lambda = \frac{m_H^2}{8m_W^2},$$

and together with (56) to a remarkable inequality concerning the Higgs mass:

$$m_H \geq \sqrt{2} m_W.$$  (61)

The eigenvectors corresponding to the eigenvalues $r/5$ and 0 are

$$t_z = \cos \theta_W t_3 - \sin \theta_W t_4, \quad \cos \theta_W = \sqrt{5/8}$$

$$t_0 = \sin \theta_W t_3 + \cos \theta_W t_4, \quad \sin \theta_W = \sqrt{3/8}$$

with $\theta_W$ the Weinberg angle. The relation $\sin^2 \theta_W = 3/8$ is typical for a $SU(5)$-oriented gauge theory.

The change of basis leads to the photon field $A_{\mu}^0(x)$ and the field $Z_{\mu}(x)$ of the $Z^0$ particle given by $Zt_z + A_{\mu}^0 t_0 = A^3 t_3 + A^4 t_4$. Hence,

$$Z = \cos \theta_W A^3 - \sin \theta_W A^4$$

$$A^0 = \sin \theta_W A^3 + \cos \theta_W A^4$$

Let us now investigate the structure of the basis vector we associate with the photon:

$$t_0 = \sqrt{\frac{3}{8}} \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & \sigma_3 & 0 \end{pmatrix} + \sqrt{\frac{5}{8}} \begin{pmatrix} 1/4 & 3/8 \sigma_3 \\ 0 & -1/2 \end{pmatrix}$$

$$= \frac{1}{4} \sqrt{\frac{3}{2}} \text{diag}(1/3, 1, 1, 0, -1).$$

Physics takes place not in $\mathbb{C}^5$ but in $\wedge \mathbb{C}^5$. Letting $T_0 = \theta(t_0)$, we find a relationship with the operator of electric charge,

$$T_0 = \frac{1}{4} \sqrt{\frac{3}{2}} Q.$$  (62)

In the same manner we deduce, for $T_z = \theta(t_z)$,

$$T_z = \sqrt{\frac{2}{5}} \begin{pmatrix} 1 & 3/8 \\ 3/8 \sigma_3 & 0 \end{pmatrix}.$$  

Consult (23) and (46) for the structure of $Q$, $Y$, and $I_i$. Together with $2\sqrt{2} T_+ = b_4 b_5^*$ and $2\sqrt{2} T_- = b_5 b_4^*$ derived from

$$T_\pm = \frac{1}{\sqrt{2}} (T_1 \pm iT_2) = \frac{1}{2\sqrt{2}} (I_1 \pm iI_2)$$
we have thus obtained expressions for all currents of the electroweak sector:

\[ \bar{\psi}_M \gamma^\mu T_0 \psi_M \]  
\[ \text{electromagnetic current (vectorlike)} \]

\[ \bar{\psi}_M \gamma^\mu T_\pm \psi_M \]  
\[ \text{charged weak currents (chiral)} \]

\[ \bar{\psi}_M \gamma^\mu T_z \psi_M \]  
\[ \text{neutral weak current (chiral)} \]

Consider the fine structure constant \( \alpha = e^2/(4\pi) \) and recall the formula (62). To give the photon the coupling constant \( e \) is to say that

\[ e = \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} \text{ or } \alpha^{-1} = \frac{128\pi}{3} = 134.04 \ldots \]  
\[ (63) \]

Renormalization group equations suggest that \( \alpha^{-1} \) decreases slightly with energy. The value above is in accord with this conception: it lies halfway between \( \alpha^{-1}(m_Z) = 128 \) and \( \alpha^{-1}(0) = 137 \).

Sparked by the success, we proceed extracting the coupling constants \( g \) and \( g' \) of the Salam-Weinberg (SW) model from the relation\(^{12} \)

\[ \sum_{a=1}^{4} A^a T_a = \sum_{i=1}^{3} g A^i I_i + \frac{1}{2} g' A^4 Y \]

with the following result:

\[ g = \frac{1}{2}, \quad g' = \frac{1}{2} \sqrt{\frac{3}{5}}. \]  
\[ (64) \]

The value \( 1/2 \) obtained for \( g \) is consistent with the SW formula \( m_W = g v/2 \) taking the relations \( r = v^2/2 \) and \( m_W^2 = r/8 \) into account. The value for \( g' \) is consistent with the SW formula \( g'/g = \tan \theta_W \) provided \( \tan \theta_W = \sqrt{3/5} \).

### 13 Fermion Masses and CKM Matrices

Recall the structure of the fermion mass operator \( M = M^* \) from Section 7; \( M \) stays invariant under gauge transformations \( \Lambda g \) provided \( g \) is an element of the residual gauge group, leaving \( b_4 \) invariant. It consists of complex \( 3 \times 3 \) mass matrices \( m(p, q) \) where \( q > 0 \), each one pertaining to a group of three fermions having different flavors but same quantum numbers otherwise. The relation \( M = \ast M^\ast \), equivalently the relation (37), reflects the matter-antimatter symmetry of \( M \). A diagonalization can be achieved in the following sense. There is a unitary operator \( U \) on the Hermitian vector space \( V \), i.e. an element \( U \in SU(V) \), such that

- the transformed mass operator

\[ M' = U M U^* = m'^* b_4 + b_4^* m' \]

has diagonal mass matrices \( m' (p, q) \) for all \( p \) and \( q \),

\(^{12}\)Within the SW formalism, one writes \( B \) instead of \( A^4 \) and calls \( \frac{1}{2} Y \) the hypercharge.
the operator $U$ commutes with gauge transformations from the residual gauge group and may be represented as

$$U = U_0 b_4^* b_4 + U_1 b_4 b_4^*$$

with operators $U_i \in (\wedge G)^1 \otimes \text{End} \mathbb{C}^3$ where $\mathbb{C}^3$ is the flavor space,

- the relation $U = * U *$ is satisfied, and hence $U_1 = * U_0 *$,

- when replacing $T$ by $T' = U T U^*$, we observe that currents such as $\bar{\psi}_M \gamma^\mu T \psi_M$ remain unchanged for most generators $T$ except when $T = T_{\pm}$. The unitary transformation thus creates a flavor changing charged current associated with the transformed generator

$$T'_+ = U T_+ U^* = T_+ U_M, \quad U_M = U_1 U_0^*$$

where $U_M$ is known as the CKM matrix.

We shall now comment on these items. Passage to the diagonal form of all mass matrices means that we construct the mass eigenstates:

- $m'(0,1) = \text{diag}(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau})$
- $m'(0,2) = \text{diag}(m_e, m_\mu, m_\tau)$
- $m'(2,1) = \text{diag}(m_u, m_c, m_t)$
- $m'(2,2) = \text{diag}(m_d, m_s, m_b)$

To keep this list short we confined attention to $p = \text{even}$. The structure of $U$ follows at once from the splitting

$$V \cong V_0 \oplus V_1, \quad V_0 \cong V_1$$

corresponding to the eigenvalues 0 and 1 of the operator $b_4 b_4^*$. This allows us to write

$$b_4 \cong \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad b_4^* \cong \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad U \cong \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix}, \quad M \cong \begin{pmatrix} 0 & m^* \\ m & 0 \end{pmatrix}$$

from which all results can be inferred at ease. The operators $U_0$ and $U_1$ have components

$$U_0(p,q) = \begin{cases} \text{unitary} & \text{if } q = 0, 1 \\ 0 & \text{if } q = 2 \end{cases}, \quad U_1(p,q) = \begin{cases} \text{unitary} & \text{if } q = 1, 2 \\ 0 & \text{if } q = 0 \end{cases}$$

and the unitary transformation $M' = U M U^*$ diagonalizes each individual mass matrix $m(p,q)$ by means of a biunitary transformation:

$$m'(p,q) = U_0(p,q-1) m(p,q) U_1^*(p,q) \quad (q > 0).$$

Performing these transformations eliminates many irrelevant parameters from $M$ and hence from the input operator $h$.

The analysis above also reveals the structure of the CKM matrix:
1. $U_M$ is thought of as an element of $(\bigwedge G) \otimes \text{End} \mathbb{C}^3$, having components

$$U_M(p,q) = U_0(p,q)U_1^*(p,q) = \begin{cases} \text{unitary} & q = 1 \\ 0 & q = 0,2 \end{cases}$$

2. As the CKM matrix satisfies the relation $U_M^* = *U_M^*$, we have

$$U_M(3-p,1) = U_M(p,1)^T, \quad p = 0, \ldots, 3.$$

3. There are precisely two independent unitary $3 \times 3$ matrices which completely determine $U_M$:

$$U_M(0,1) = \text{CKM matrix of the leptons}$$

$$U_M(2,1) = \text{CKM matrix of the quarks}$$

They enter the charged currents and thus induce flavor changing interactions:

$$j^\mu_+ = \bar{\psi}_M \gamma^\mu T_+ \psi_M^I, \quad j^\mu_- = \bar{\psi}_M^I \gamma^\mu T_- \psi_M, \quad \psi_M = U_M \psi_M.$$

There are no flavor changing neutral currents in such a theory.

From the fact that traces are invariant under unitary transformations we obtain the following trace formulas

$$\text{Tr } m(0,1)m(0,1)^* = \frac{1}{3}(m^2_{\nu_e} + m^2_{\nu_\mu} + m^2_{\nu_\tau})$$

$$\text{Tr } m(0,2)m(0,2)^* = \frac{1}{3}(m^2_{\nu_e} + m^2_{\nu_\mu} + m^2_{\nu_\tau})$$

$$\text{Tr } m(2,1)m(2,1)^* = \frac{1}{3}(m^2_{\nu_e} + m^2_{\nu_\mu} + m^2_{\nu_\tau})$$

$$\text{Tr } m(2,2)m(2,2)^* = \frac{1}{3}(m^2_{\nu_e} + m^2_{\nu_\mu} + m^2_{\nu_\tau}).$$

Consequently, taking (37) into account,

$$\text{Tr } M^2 = \text{Tr } q m m^* = \sum_{p,q} q(\psi_3) (\psi_2) \text{Tr } m(p,q)m(p,q)^*$$

$$= \frac{4}{3}(m^2_{\nu_e} + m^2_{\nu_\mu} + m^2_{\nu_\tau} + m^2_e + m^2_\mu + m^2_\tau)$$

$$+ 4(m^2_u + m^2_c + m^2_t + m^2_d + m^2_s + m^2_b).$$  \hspace{1cm} (65)

In the same manner, one finds

$$\text{Tr } M^4 = \text{Tr } q(m m^*)^2 = \sum_{p,q} q(\psi_3) (\psi_2) \text{Tr } (m(p,q)m(p,q)^*)^2$$

$$= \frac{4}{3}(m^4_{\nu_e} + m^4_{\nu_\mu} + m^4_{\nu_\tau} + m^4_e + m^4_\mu + m^4_\tau)$$

$$+ 4(m^4_u + m^4_c + m^4_t + m^4_d + m^4_s + m^4_b).$$  \hspace{1cm} (66)

Two equations relate the $W$ mass and the Higgs mass to the fermion masses:

$$16m^2_W = \text{Tr } M^2$$  \hspace{1cm} (67)

$$4m^2_H m^2_W = \text{Tr } M^4.$$  \hspace{1cm} (68)
The first equality uses (65), (35), (52), and (60):
\[ \text{Tr} \ M^2 = \text{Tr} \ q mm^* = r \ Tr \ q hh^* = 2r = 16m_W^2, \]
while the second uses (66), (35), (54), (60), (55), and (57):
\[ \text{Tr} \ M^4 = \text{Tr} \ q(m m^*)^2 = r^2 \text{Tr} \ q(h h^*)^2 = \frac{1}{2}r^2 \lambda = 4\lambda r m_W^2 = 4m_H^2 m_W^2. \]

Empirically [8], the mass of the top quark is dominant among the fermion masses, and so
\[ \text{Tr} \ M^2 \approx 4m_t^2, \quad \text{Tr} \ M^4 \approx 4m_t^4. \]
The relations (67) and (68) therefore tell us that
\[ m_H \approx m_t \approx 2m_W. \]

Prior to its observation, the value of the top quark mass has been predicted [9] on theoretical grounds within a 10% error bracket, certainly one of the greatest triumphs of the Standard Model. Again, the value 2m_W for the top mass obtained above is off the empirical value (174 ± 7) GeV by 10%. A value for the Higgs mass around 2m_W has already been predicted in [2]. The fact that m_H depends strongly on the top quark mass, suggesting m_H ≈ 158 GeV, has also been noted by Okumura [10] who argues on the basis of noncommutative geometry. Similarly, Pirogov and Zenin [11], using the renormalization group approach, find that a cutoff equal to the Planck scale would give the Higgs boson a mass around 160 GeV. Surprisingly, the Higgs mass values offered in the literature, though dependent on very different schemes, all seem to to converge.

References

1. S. Catani et.al.: The QCD and the Standard Model Working Group: Summary Report, hep-ph/0005114
   M.W. Grunewald, Phys.Rept. 322 (1999), 125
   W. Hollik, Acta Phys.Polon. B30 (1999), 1787
   C. Dionisi, Nucl.Phys.Proc.Suppl. 38 (1995), 125
   S.J. Brodsky: Precision Tests of QCD and the Standard Model, hep-ph/9506322
   P.H. Chankowski: Precision Tests of the MSSM, hep-th/9505304

2. G. Roepstorff: Superconnections and the Higgs Field, hep-th/9801040 and J.Math.Phys. 40 (1999) 2698

3. G. Roepstorff: Superconnections and Matter, hep-th/9801045
4. G. Roepstorff: Superconnections: an Interpretation of the Standard Model, [hep-th/9907221](http://arxiv.org/abs/hep-th/9907221), Electronic Journal of Diff. Equ., Conf. 04, 2000, pp.165-174

5. G. Roepstorff and Ch. Vehns: An Introduction to Clifford Supermodules, [math-ph/9908029](http://arxiv.org/abs/math-ph/9908029)

6. G. Roepstorff and Ch. Vehns: Generalized Dirac Operators and Superconnections, [math-ph/9911006](http://arxiv.org/abs/math-ph/9911006)

7. G. Roepstorff: A Class of Anomaly-Free Gauge Theories, [hep-th/0005079](http://arxiv.org/abs/hep-th/0005079)

8. CDF and DO Collaboration (G. Brooijmans for the collaboration): Top Quark Mass Measurements at the Tevatron, [hep-ex/0005030](http://arxiv.org/abs/hep-ex/0005030)

9. E. Laenen, J. Smith, and W. van Neerven, Phys.Lett. B321 (1994), 254
   E.L. Berger, H. Contopanagos, Phys.Rev. D54 (1996), 3085
   S. Catani, M.L. Mangano, P. Nason, L. Trentadue, Phys.Lett. B378 (1996), 329

10. Y. Okumura: An estimation of the Higgs boson mass in the two loop approximation in a noncommutative differential geometry, [hep-ph/9707350](http://arxiv.org/abs/hep-ph/9707350)

11. Yu.F. Pirogov and O.V. Zenin: Two-loop renormalization group profile of the standard model and a new generation, [hep-ph/9808414](http://arxiv.org/abs/hep-ph/9808414)