SYNTESABLE DIFFERENTIATION-INVARINT SUBSPACES

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ABSTRACT. We describe differentiation-invariant subspaces of $C^\infty(a,b)$ which admit spectral synthesis. This gives a complete answer to a question posed by A. Aleman and B. Korenblum. It turns out that this problem is related to a classical problem of approximation by polynomials on the real line. We will depict an intriguing connection between these problems and the theory of de Branges spaces.

1. INTRODUCTION AND MAIN RESULTS

Let $C^\infty(a,b)$ be the space of all infinitely differentiable functions on the interval $(a,b)$ equipped with the usual countably normed topology. A classical result of L. Schwartz [18] says that any closed linear subspace of $C^\infty(\mathbb{R})$ which is translation invariant is generated by the exponential monomials $x^k e^{ix}$ it contains and, thus, the structure of all translation invariant subspaces is well understood. This property is known as the spectral synthesis for translation invariant subspaces. In different contexts it was studied by J. Delsart, L. Schwartz, J.-P. Kahane, P. Malliavin, A.F. Leontiev, V.V. Napalkov, I.F. Krasichkov-Ternovskii.

In 2000s A. Aleman and B. Korenblum noticed that the description of differentiation-invariant subspaces (or simply $D$-invariant subspaces, where $Df = f'$) of $C^\infty(\mathbb{R})$ is much more complicated. First results in this direction were obtained by A. Aleman and B. Korenblum [4] who classified closed $D$-invariant subspaces $L$ of $C^\infty(a,b)$ in terms of the spectra of the restriction $D|_L$. Only three cases are possible: $\sigma(D|_L) = \emptyset$, $\sigma(D|_L) = \mathbb{C}$ and $\sigma(D|_L) = \Lambda$, where $\Lambda$ is a discrete subset of $\mathbb{C}$. In particular, it is shown in [4] that any closed $D$-invariant subspace with void spectrum is of the form

$$L_I = \{ f \in C^\infty(a,b) : f|_I \equiv 0 \},$$

where $I$ is some relatively closed subinterval of $(a,b)$ (with an obvious modification in the case when $I$ reduces to one point). Moreover, in the case $\sigma(D|_L) \neq \mathbb{C}$ there always exists the unique minimal interval $I$ such that $L_I \subset L$ (so-called residual interval for $L$).

In view of this it is natural to state the spectral synthesis problem for $D$-invariant subspaces as follows (all $D$-invariant subspaces are always assumed to be closed):

Is it true that any $D$-invariant subspace $L$ such that the spectrum $\sigma(D|_L)$ is discrete, satisfies

$$(1.1) \quad L = L_I + E(L),$$

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where $\mathcal{E}(L)$ is the linear span of exponential monomials contained in $L$ and $I$ is the residual interval for $L$?

This problem was posed in [4] where it was solved in the positive in the simplest situation when the set $\sigma(D|_L)$ is finite. It was further studied by A. Aleman and the authors in [3] where it was shown that the answer in general is negative. Surprisingly, the answer depends essentially on the relation between the Beurling–Malliavin density $D^{BM}(\Lambda)$, where $i\Lambda = \sigma(D|_L)$, and the size $|I|$ of the residual interval $I$.

**Theorem 1.1.** ([3, Theorems 1.1, 1.2]) If a $D$-invariant subspace $L$ has a compact residual interval $I$ and

$$2\pi D^{BM}(\Lambda) < |I|,$$

then the spectral synthesis property (1.1) holds. On the other hand, there exists a $D$-invariant subspace $L$ with $\Lambda$ of critical density (i.e., $2\pi D^{BM}(\Lambda) = |I|$) such that (1.1) fails.

In the case when residual interval is non-compact (i.e., $I = (a,c]$ or $I = [c,b)$) or void the spectral synthesis property always holds.

The Beurling–Malliavin density $2\pi D^{BM}(\Lambda)$ (for the definition see [9, 16]) appears naturally in this context since it is equal to the radius of completeness $r(\Lambda)$ of $\Lambda$,

$$r(\Lambda) := \sup \{a : \{e^{i\lambda t}\}_{\lambda \in \Lambda} \text{ is complete in } L^2(-a,a)\}.$$  

The proof of Theorem 1.1 is based on the methods developed by M. Mitkovski and A. Poltoratski in [4] (in connection with their solution of Pólya problem) and by the first two authors and A. Borichev in [5]. Another approach to Theorem 1.1 was suggested in [2].

Note that the case $2\pi D^{BM}(\Lambda) > |I|$ is impossible, since then any function in $C^\infty(I)$ can be approximated on $I$ by functions from $\mathcal{E}(L)$ together with its derivatives, and so $L = C^\infty(a,b)$.

The present paper is devoted to the most intriguing situation when $2\pi D^{BM}(\Lambda) = |I|$. We are able to classify completely the spectra of all synthesable $D$-invariant subspaces. It turns out that the answer depends on density of polynomials in some weighted spaces.

**Definition 1.2.** We will say that a discrete set $\Lambda = \{\lambda_n\} \subset \mathbb{C}$ is synthesable if the $D$-invariant subspace $L$ of $C^\infty(\mathbb{R})$ which has discrete simple spectrum $i\Lambda$ and non-trivial residual interval $I = [-r(\Lambda), r(\Lambda)]$ is unique (and, thus, coincides with $L_I + \mathcal{E}(L)$).

To avoid uninteresting technicalities we consider only the case of simple spectrum. All results trivially extend to the case of multiple spectrum and exponential monomials in place of exponentials.

For a nonnegative measurable function $W$ on $\mathbb{R}$ put

$$\mathcal{E}_0(W) = \{F : F \text{ is entire function of zero exponential type, } F \in L^2(W)\}.$$
The norm in the space $E_0(W)$ is inherited from $L^2(W)$. It is well-known that either $E_0(W)$ is dense in $L^2(W)$ or $E_0(W)$ is a (possibly zero) closed subspace in $L^2(W)$.

Now we are able to formulate the main result of the paper. Note that in the case when $E(L)$ is dense in $L^2(-r(\Lambda), r(\Lambda))$ or has finite codimension in this space it is not difficult to show that $\Lambda$ is synthesable (see Proposition 3.2 below). The interesting case is when the defect of $E(L)$ is infinite.

**Theorem 1.3.** Let $\Lambda \subset \mathbb{C}$ be a discrete set such that $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ has infinite codimension in $L^2(-r(\Lambda), r(\Lambda))$. Then $\Lambda$ is synthesable if and only if

(i) The product

\[ G(z) := \lim_{R \to \infty} \prod_{|\lambda| < R} \left(1 - \frac{z}{\lambda}\right) \]

converges to an entire function $G$ of exponential type;

(ii) $E_0(|G|^2)$ contains the set $\mathcal{P}$ of all polynomials and

\[ \dim \left( E_0(|G|^2) \ominus \text{Clos}_{E_0(|G|^2)} \mathcal{P} \right) \leq 1 \]

(the polynomials are dense in $E_0(|G|^2)$ up to codimension one).

This shows that a generic spectrum of critical density is not synthesable, in contrast to the non-critical density case. Even more surprising is that there exist two essentially different classes of synthesable spectra of critical density (see Examples 6.1 and 6.2) – those for which polynomials are dense in $E_0(|G|^2)$ and those for which polynomials are not dense, but have codimension 1. The second case is closely related to the solution of the long-standing spectral synthesis problem for exponential systems in $L^2(-\pi, \pi)$ obtained recently by the authors and A. Borichev [6].

One may ask why the spectrum is not synthesable when polynomials are non-dense with finite defect greater than 1. However, due to deep intrinsic properties of exponential systems, this situation can never happen.

In the case when the spectral synthesis holds there exists the unique $D$-invariant subspace with given residual interval and spectrum. Our second main result shows that even in the case when the spectral synthesis fails, the $D$-invariant subspaces with the same residual interval and spectrum have a chain structure.

**Theorem 1.4.** Let $L_1, L_2$ be two $D$-invariant subspaces of $C^\infty(a, b)$ with common residual subspace $L_{1,1} = L_{2,1}$ (or, which is the same, with common residual interval). If

\[ \sigma(D|_{L_1}) = \sigma(D|_{L_2}), \]

then either $L_1 \subset L_2$ or $L_2 \subset L_1$.

The crucial ingredient of the proofs is the de Branges theory of Hilbert spaces of entire functions [10]. Though, at first glance the studied problem is not directly related to the de Branges theory, there exist deep connections between them. Note that this is the case
with some other classical problems of harmonic analysis such that description of Fourier frames [15], Beurling–Malliavin type theorems [13], approximation by polynomials [1].

Section 2 is devoted to our toolbox from de Branges theory. In Sections 3 and 4 we prove, respectively, sufficiency and necessity parts of Theorem 1.3. In Section 5 Theorem 1.4 is proved. Finally, in Section 6 we give explicit examples of two cases when the spectral synthesis holds.

Notations. Given positive functions $U(x), V(x)$, the notation $U(x) \lesssim V(x)$ (or, equivalently, $V(x) \gtrsim U(x)$) means that there is a constant $C$ such that $U(x) \leq CV(x)$ holds for all $x$ in the set in question. We write $U(x) \eqsim V(x)$ if both $U(x) \lesssim V(x)$ and $V(x) \lesssim U(x)$.

For an entire function $F$ we denote by $Z_F$ the set of its zeros (no matter what their multiplicities are). By $D(z, r)$ and $\overline{D}(z, r)$ we denote the disc and the closed disc with center $z$ of radius $r$.

2. De Branges theory

In this section we briefly discuss some facts from the de Branges theory which are needed for the proof of Theorem 1.3. We do not pretend to give an overview of the whole theory or of its essential part. Here we only highlight the aspects of the theory which are important for us.

2.1. Classical theory. All results presented here can be found in Sections 19–35 of the de Branges monograph [10] (see, also, [17]). An entire function $E$ is said to be in the Hermite–Biehler class if $|E(z)| > |E^*(z)|$, $z \in \mathbb{C}^+$, where $E^*(z) = \overline{E(z)}$. With any such function we associate the de Branges space $H(E)$ which consists of all entire functions $F$ such that $F/E$ and $F^*/E$ restricted to $\mathbb{C}^+$ belong to the Hardy space $H^2 = H^2(\mathbb{C}^+)$. The inner product in $H(E)$ is given by

$$(F, G)_{H(E)} = \int_{\mathbb{R}} \frac{F(t)G(t)}{|E(t)|^2} dt.$$  

There exists an equivalent axiomatic description of de Branges spaces (see [10, Theorem 23]). A nontrivial Hilbert space $\mathcal{H}$ whose elements are entire functions is a de Branges space $H(E)$ (for some $E$) if and only if it satisfies the following axioms:

(A1) For every nonreal number $w$, the evaluation functional $F \mapsto F(w)$ is continuous;

(A2) Whenever $F$ is in the space and has a nonreal zero $w$, the function $F(z)\frac{z-w}{\overline{z-w}}$ is in the space and has the same norm as $F$;

(A3) The function $F^*$ belongs to the space whenever $F$ belongs to the space and has the same norm as $F$.

In what follows we require additional assumption that for any $w \in \mathbb{C}$ there exists $F \in \mathcal{H}$ such that $F(w) \neq 0$. This corresponds to the situation when $E$ has no real zeros.

The prime examples of de Branges spaces are Paley–Wiener spaces $\mathcal{PW}_a$, $a > 0$. Recall that $\mathcal{PW}_a$ is the space of all entire functions of exponential type at most $a$ whose restriction
Remark 2.3. Let $Z$ be a de Branges subspace of codimension 1. In what follows we always assume without loss of generality that the set $Z$ contains a subspace of codimension 1. In what follows we always assume without loss of generality that the set $Z$ contains a subspace of codimension 1.

The main results of de Branges theory says that this property is true for general de Branges spaces. For a de Branges space $\mathcal{H}(E)$ consider the set of all its de Branges subspaces

$$\text{Chain}(\mathcal{H}(E)) := \{ \mathcal{H} : \mathcal{H} \subset \mathcal{H}(E), \| \cdot \|_\mathcal{H} = \| \cdot \|_{\mathcal{H}(E)} \}.$$ 

Then subspaces in $\text{Chain}(\mathcal{H}(E))$ are ordered by inclusion.

**Theorem 2.1.** If $\mathcal{H}_1, \mathcal{H}_2 \in \text{Chain}(\mathcal{H}(E))$, then either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$.

Let $f$ be a de Branges subspace of $\mathcal{H}(E)$. If codimension of $\mathcal{H}$ in $\mathcal{H}(E)$ is equal to $n \in \mathbb{N}$, then $\mathcal{H}$ is the closure of the domain of multiplication by $z^n$ in $\mathcal{H}(E)$,

$$\mathcal{H} = \text{Clos}_{\mathcal{H}(E)} \{ f \in \mathcal{H}(E) : z^n f \in \mathcal{H}(E) \}.$$ 

Moreover, in this case for any $m < n$ there exists a de Branges subspace of codimension $m$ given by $\text{Clos}_{\mathcal{H}(E)} \{ f \in \mathcal{H}(E) : z^m f \in \mathcal{H}(E) \}$.

In particular, if $\mathcal{H}$ is a proper de Branges subspace of $\mathcal{H}(E)$ and $\mathcal{H}(E)$ has no de Branges subspace of codimension 1, then $\dim(\mathcal{H}(E) \ominus \mathcal{H}) = \infty$.

**Remark 2.3.** Let $\mathcal{H}_0$ be a de Branges subspace of $\mathcal{H}(E)$ of codimension 1. Then, by [10] Problem 87, one can choose $E_0 = A_0 - iB_0$ such that $\mathcal{H}_0 = \mathcal{H}(E_0)$ and $A = A_0$, $B = zA_0 + B_0$. Note that in this case $A_0 \in \mathcal{H}(E)$.

One of the important notions in the de Branges theory is an associated function.

**Definition 2.4.** We will say that an entire function $G$ is an associated function for $\mathcal{H}(E)$ and write $G \in \text{Assoc}(\mathcal{H}(E))$ if, for any $w \in \mathbb{C}$ and $F \in \mathcal{H}(E)$,

$$\frac{F(z)G(w) - G(z)F(w)}{z - w} \in \mathcal{H}(E).$$

Of course, we have $\mathcal{H}(E) \subset \text{Assoc}(\mathcal{H}(E))$ and also $E \in \text{Assoc}(\mathcal{H}(E))$.

In what follows we will often use the representation $E = A - iB$, where $A = \frac{E + E^*}{2}$, $B = \frac{E - E^*}{2i}$ are entire functions which are real on $\mathbb{R}$. We will say that the corresponding function $A$ is $A$-function of the de Branges space $\mathcal{H}(E)$. Note that $\mathcal{H}(e^{i\alpha}E) = \mathcal{H}(E)$ for any $\alpha \in [0, 2\pi)$ and the $A$-function for $e^{i\alpha}E$ is given by $\cos \alpha A - \sin \alpha B$. Moreover, recall that the zero sets of the functions $\cos \alpha A - \sin \alpha B$ generate orthogonal bases of reproducing kernels for all $\alpha \in [0, 2\pi)$ except at most one. Such exceptional $\alpha$ exists if and only if $\mathcal{H}(E)$ contains a subspace of codimension 1. In what follows we always assume without loss of generality that the set $Z_A$ generates an orthogonal basis of reproducing kernels which is, up to normalization, of the form $\{ \frac{A}{z - \mu}, \mu \in Z_A \}$.

In what follows we will need the following technical lemma.
Lemma 2.5. Let $\mathcal{H}$ be a de Branges space and let $\tilde{\mathcal{H}}$ be its de Branges subspace of infinite codimension. Let $A$ and $\tilde{A}$ be the $A$-functions of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ respectively. If the function $A$ (and, hence, any element in $\mathcal{H}$) is of order at most, one then $|\tilde{A}(iy)/A(iy)| = o(|y|^{-N})$, $|y| \to \infty$, for any $N > 0$.

Proof. Note that both $A$ and $\tilde{A}$ are canonical products of genus 0 or 1 with real zeros. Therefore, if $t_n$ are zeros of $A$, then $|A(iy)|^2 = \prod_n (1 + y^2/t_n^2)$.

It follows from [10] Problem 93 that between each two zeros of $\tilde{A}$ there is at least one zero of $A$. Hence, if infinitely many intervals between two consecutive zeros of $\tilde{A}$ contain exactly one zero, then there exist $N$ such that $|\tilde{A}(iy)| \geq |y|^{-N}|A(iy)|$, $|y| \to \infty$. In this case [10] Theorem 26 implies that for any $F \in \mathcal{H}$ and any polynomial $P$ of degree $N+2$ which divides $F$, we have $F/P \in \tilde{\mathcal{H}}$. Thus, $\tilde{\mathcal{H}}$ contains the domain of multiplication by $z^{N+2}$ and so $\tilde{\mathcal{H}}$ has the codimension at most $N+2$ in $\mathcal{H}$, a contradiction.

2.2. Recent progress. In this subsection several facts recently discovered by the authors are collected. The next result was used in [1].

Theorem 2.6. Let $G$ be an entire function with simple zeros, $G^*/G$ be a ratio of two Blaschke products and $G \in \text{Assoc}(\mathcal{H}(E))$. Put

$$\mathcal{H} = \text{Span}\left\{ \frac{G(z)}{z-w}, w \in Z_G \right\}.$$ 

Then $\mathcal{H}$ is a de Branges subspace of $\mathcal{H}(E)$.

Proof. The proof follows easily from the axiomatic description of de Branges spaces. Indeed, it is clear that $\text{Span}\left\{ \frac{G(z)}{z-w}, w \in Z_G \right\}$ is closed under division by Blaschke factors, and so $\mathcal{H}$ is closed under division by Blaschke products. Since $G^*/G$ is a ratio of two Blaschke products, we conclude that the function

$$\frac{G^*(z)}{z-w} = \frac{G(z)}{z-w} \cdot \frac{z-w}{z-\bar{w}} \cdot \frac{G(z)}{G^*(z)}$$

is in $\mathcal{H}$ for any $w \in Z_G$, and so $\mathcal{H}$ is closed under the transform $F \mapsto F^*$.

Let $\{e^{i\lambda}\}_{\lambda \in \Lambda}$ be an incomplete system in $L^2(-\pi, \pi)$. In this case $\Lambda$ is a subset of the zero set of some nontrivial function in $\mathcal{P}W_\pi$ and so $\Lambda$ satisfies the Blaschke conditions in $\mathbb{C}^+$ and in $\mathbb{C}^-$. Fix some canonical product $G_\Lambda$ with simple zeros at $\Lambda$ such that $G_\Lambda^*/G_\Lambda$ is a ratio of two Blaschke products. Put

$$(2.1) \quad \mathcal{H}_{\Lambda,\pi} := \{ F : F \text{ is entire and } G_\Lambda F \in \mathcal{P}W_\pi \}.$$ 

Define the norm in $\mathcal{H}_{\Lambda,\pi}$ by the formula $\|F\|_{\mathcal{H}_{\Lambda,\pi}} = \|G_\Lambda F\|_{\mathcal{P}W_\pi}$. The following result was proved in [8] p. 217 using axiomatic description of de Branges spaces.

Theorem 2.7. The space $\mathcal{H}_{\Lambda,\pi}$ is a de Branges space.
As we will see below, the space $\mathcal{H}_{\Lambda,\pi}$ is closely connected with $D$-invariant subspaces (see Section 3). The space $\mathcal{H}_{\Lambda,\pi}$ is “generated by” the weight $|G_{\Lambda}|^2$. This produces some restrictions on the structure of Chain($\mathcal{H}_{\Lambda,\pi}$).

**Theorem 2.8.** If $\mathcal{H}$ is an infinite-dimensional de Branges subspace of $\mathcal{H}_{\Lambda,\pi}$, then there exists a subspace of $\mathcal{H}$ of infinite codimension in $\mathcal{H}$. Moreover, there exists $f \in \mathcal{H}$ such that $z^n f(z) \in \mathcal{H}$ for any $n \in \mathbb{N}$.

**Proof.** First, let us fix a function $F \in \mathcal{H}$ with infinite number of zeros. Let $\{s_n\}_{n=1}^{\infty}$ be such that $\{s_n\} \subset \mathbb{Z}_F$, $|s_{n+1}| > 10|s_n|$ and either $|\text{Im } s_n - 1| > \frac{1}{4}$, $n \in \mathbb{N}$, or $|\text{Im } s_n| > \frac{1}{4}$, $n \in \mathbb{N}$. Put

$$S(z) = \prod_n \left(1 - \frac{z}{s_n}\right).$$

The function $S$ satisfies either $|S(x)| \geq 1$, $x \in \mathbb{R}$, or $|S(x + i)| \geq 1$, $x \in \mathbb{R}$. Hence, $G_{\Lambda} F S^{-1} \in \mathcal{P}W_{\pi}$. Using [10] Theorem 26 we conclude that $FS^{-1} \in \mathcal{H}$. Moreover, by analogous arguments we can prove that $z^n F(z) S^{-1}(z) \in \mathcal{H}$ for any $n \in \mathbb{N}$.

Now assume that any subspace in Chain($\mathcal{H}$) has finite codimension. Then either the chain is finite or $\cap_{\mathcal{H}_1 \in \text{Chain(}\mathcal{H})} \mathcal{H}_1 = \emptyset$. On the other hand, $FS^{-1}$ is in the domain of multiplication by $z^n$ for any $n$ and it follows from Theorem 2.2 that $FS^{-1} \in \mathcal{H}_1$ for any de Branges subspace $\mathcal{H}_1$ of finite codimension in $\mathcal{H}$. We arrive to a contradiction. \qed

In what follows we denote by $k_{\Lambda} = k_{\Lambda}^{PW_{\pi}}$ the reproducing kernel of $\mathcal{P}W_{\pi}$, i.e., the cardinal sine function: $k_{\Lambda}(z) = \frac{\sin \pi(z-\Lambda)}{\pi(z-\Lambda)}$.

**Theorem 2.9.** If $\mathcal{H}$ is a de Branges subspace of $\mathcal{H}_{\Lambda,\pi}$, then

$$\dim(\mathcal{H}_{\Lambda,\pi} \ominus \mathcal{H}) \neq 2.$$

**Proof.** Assume the contrary. It is well known that existence of a de Branges subspace of codimension 1 is equivalent to the condition $\mu(\mathbb{R}) < \infty$, where $\mu$ is the measure from the Herglotz representation of $(E + e^{i\alpha}E^*)/(E - e^{i\alpha}E^*)$ for some $\alpha \in [0, 2\pi)$ (a so-called Clark measure). It is easy to prove that a de Branges space contains a de Branges subspace of codimension 2 if and only if $\int_{\mathbb{R}} |x| d\mu(x) < \infty$ (as such subspace one should take $\text{Clos}_{\mathcal{H}(E)} \{F : z^2 F \in \mathcal{H}(E)\}$).

Moreover, in this situation in $\mathcal{H}_{\Lambda,\pi}$ there exists a complete and minimal system of reproducing kernels $\{K_t\}_{t \in T}$ of the space $\mathcal{H}_{\Lambda,\pi}$ such that its biorthogonal system $\{G_{\Lambda}(z)\}_{t \in T}$ has codimension at least 2 (see [7, Proposition 9.1]). By Theorem 2.6, $\text{Span} \{\frac{G_{\Lambda}(z)}{z-t}\}_{t \in T}$ is a de Branges subspace of $\mathcal{H}_{\Lambda,\pi}$ of codimension 2. Then it is clear from the ordering theorem that $\text{Span} \{\frac{G_{\Lambda}(z)}{z-t}\}_{t \in T} = \mathcal{H}$.

Without loss of generality we can assume that $\Lambda \cap T = \emptyset$. Put

$$\mathcal{MS} := \{k_{\Lambda}^{PW_{\pi}}\}_{\Lambda \in \Lambda} \cup \left\{\frac{G_{\Lambda} G_T}{z-t}\right\}_{t \in T}.$$

This is a so-called mixed system in the Paley–Wiener space. First of all we note that the system $\mathcal{S} := \{k_{\Lambda}^{PW_{\pi}}\}_{\Lambda \in \Lambda \cup T}$ is complete and minimal in $\mathcal{P}W_{\pi}$. Indeed, if $H \in \mathcal{P}W_{\pi}$ \
{0} is orthogonal to $S$, then $H = G_A G_T S$ for some entire $S$. Hence, the function $G_T S$ belongs to $\mathcal{H}_{A,\pi}$ and is orthogonal to $\{K_t\}_{t \in T}$, a contradiction. Minimality of $S$ also follows immediately from minimality of $\{K_t\}_{t \in T}$.

Assume that there exist two linearly independent elements $H_1, H_2$ in $\mathcal{H}_{A,\pi}$ such that $H_1, H_2 \perp H$ or, equivalently
\[
H_1, H_2 \perp \left\{ \frac{G_T(z)}{z - t} \right\}_{t \in T}.
\]
Put $F_1 = G_A H_2$, $F_2 = G_A H_2$. Then we have $F_{1,2} \perp MS$ in $\mathcal{P}W_\pi$. This contradicts Theorem 1.1 in [6] which says that if $S$ is a complete and minimal system of reproducing kernels in $\mathcal{P}W_\pi$, then the orthogonal complement to any mixed system of the form $MS$ is at most one-dimensional. □

3. Proof of Theorem 1.3: Sufficiency

3.1. Preliminary steps. Recall that continuous linear functionals on $C^\infty(a,b)$ are compactly supported distributions of finite order on the interval $(a,b)$, i.e., distributions of the form $\varphi = \sum_{k=0}^n c_k h_k^{(k)}$, where $c_k \in \mathbb{C}$, $h_k \in L^2(a,b)$ have compact supports $[c_k, d_k] \subset (a,b)$ and differentiation is understood in the sense of distributions. We denote this class of distributions by $S(a,b)$. Then the action of $\varphi \in S(a,b)$ on $f \in C^\infty(a,b)$ is given by
\[
\varphi(f) = \sum_{k=0}^n c_k (-1)^k \int_{c_k}^{d_k} h_k(t) f^{(k)}(t) \, dt.
\]
As usual we can define the Fourier transform of $\varphi \in S(a,b)$ by the formula
\[
\hat{\varphi}(z) = \varphi(e^{itz}).
\]
Since $\varphi$ has finite order, the function $\hat{\varphi}$ is an entire function of finite exponential type with at most polynomial growth on the real line. Note that in the case when $\varphi$ is a usual $L^2$ function (not a distribution) supported by the interval $[c, d] \subset (a,b)$ we will understand $\varphi(f)$ as the usual inner product (with $\overline{f}$):
\[
\varphi(f) = \int_c^d \varphi(t) f(t) \, dt = \int_{\mathbb{R}} \hat{\varphi}(x) \overline{\hat{f}}(x) \, dx,
\]
where $\hat{g}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} g(t) e^{itx} \, dt$ is the usual Fourier transform (we write $e^{itx}$ in place of $e^{-itx}$ to agree with the Fourier transform on $S(a,b)$).

In what follows we will assume that $L$ is a $D$-invariant subspace with the spectrum $i\Lambda$ such that $D^{BM}(\Lambda) = 1$ and with the residual interval $I = [-\pi, \pi]$.

We will sometimes use the following simple observation: if $L$ is a $D$-invariant subspace with spectrum $i\Lambda$, then $e^{it\gamma} L$ is also $D$-invariant with the spectrum $i(\Lambda + \gamma)$. Clearly, $L$ and $e^{it\gamma} L$ admit spectral synthesis or not simultaneously. In particular, we can always assume in what follows that $\Lambda \cap \mathbb{Z} = \emptyset$ (and even some weak separation similar to formula (3.9) below).
Let us fix some canonical product \( G_\Lambda \) with zero set \( \Lambda \) such that \( G_\Lambda(z)/G_\Lambda^*(z) \) is a ratio of two Blaschke products (since \( \Lambda \) has a finite completeness radius, it satisfies the Blaschke conditions in \( \mathbb{C}^+ \) and in \( \mathbb{C}^- \)). We will be mainly interested in the situation when the system \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) has infinite codimension in \( L^2(-\pi, \pi) \). In this case the space \( \mathcal{H}_{\Lambda, \pi} \) (defined by (2.1)) is nontrivial and, moreover, \( \dim(\mathcal{H}_{\Lambda, \pi}) = \infty \). The following lemma establishes the link between Theorem 1.3 and the constructions of Subsection 2.2.

**Lemma 3.1.** If \( \Lambda \) and \( G_\Lambda = G \) satisfy the conditions (i) and (ii) of Theorem 1.3, then \( \mathcal{E}_0(|G_\Lambda|^2) = \mathcal{H}_{\Lambda, \pi} \) as Hilbert spaces with equality of norms.

**Proof.** Since \( \mathcal{P} \subset \mathcal{E}_0(|G_\Lambda|^2) \), we conclude that \( G_\Lambda \in L^2(\mathbb{R}) \). Any function of order 1 representable as the principal value product (1.2) is of finite exponential type, and so \( G_\Lambda \) is in some Paley–Wiener space. In particular, \( G_\Lambda \) belongs to the Cartwright class (for the properties of this class of entire functions see, e.g., [11, 12]). Therefore, the Beurling–Malliavin density for \( \Lambda \) coincides with the usual density \( \lim_{r \to \infty} (2r)^{-1} \#(\Lambda \cap D(0, r)) \) and coincides with the type of \( G_\Lambda \) divided by \( \pi \). We conclude that the type of \( G_\Lambda \) equals \( \pi \) and its indicator diagram is the interval \( [-\pi i, \pi i] \).

Let \( F \in \mathcal{E}_0(|G_\Lambda|^2) \). Since \( G_\Lambda F \in L^2(\mathbb{R}) \) and \( F \) is of zero exponential type, \( G_\Lambda F \in \mathcal{PW}_\pi \). Thus, \( \mathcal{E}_0(|G_\Lambda|^2) \subset \mathcal{H}_{\Lambda, \pi} \).

Conversely, if \( G_\Lambda F \in \mathcal{PW}_\pi \), then \( F \) is of zero exponential type since \( G_\Lambda \) has the maximal (for \( \mathcal{PW}_\pi \)) indicator diagram. Hence, \( F \in \mathcal{E}_0(|G_\Lambda|^2) \).

Thus, in what follows we can replace the condition (ii) of Theorem 1.3 by the equivalent condition: polynomials belong to \( \mathcal{H}_{\Lambda, \pi} \) and are dense there up to codimension 1.

Now put

\[
L_0 = \overline{L_I + \mathcal{E}(L)},
\]

We consider the annihilators \( L^\perp \) and \( L_0^\perp \) in the dual space \( S(a, b) \). Since \( L_I \subset L_0 \), it is clear that any \( \varphi \in L_0^\perp \) should be supported by the interval \([ -\pi, \pi ]\). Also, \( \hat{\varphi}(\lambda) = \varphi(e^{i\lambda}) = 0 \) whenever \( e^{i\lambda} \). Hence,

\[
\widehat{L_0^\perp} = \{ F : F \in \mathcal{E}_\pi, \ F|_\Lambda = 0 \},
\]

where

\[
\mathcal{E}_\pi = \left\{ F : F = \sum_{k=0}^{n} z^k(F_k(z) + c_k), \ F_k \in \mathcal{PW}_\pi, \ c_k \in \mathbb{C} \right\}.
\]

By the Hahn–Banach Theorem, the equality \( L = L_0 \) is equivalent to the equality \( L^\perp = L_0^\perp \) or to \( \widehat{L^\perp} = \widehat{L_0^\perp} \), where \( \widehat{L^\perp} = \{ \hat{\varphi} : \varphi \in L^\perp \} \).

Since \( L \) is \( D \)-invariant, it is immediate from the duality that for any \( \varphi \in L^\perp \) and any \( n \in \mathbb{N} \), we have \( z^n \varphi(z) \in \widehat{L^\perp} \). It was shown in [4, Proposition 3.1] that the class \( \widehat{L^\perp} \) is also closed under dividing out zeros which are not in the spectrum:

if \( \varphi \in L^\perp \) and \( \hat{\varphi}(w) = 0 \), \( w \notin \Lambda \), then \( \frac{\hat{\varphi}(z)}{z - w} \in \widehat{L^\perp} \).

This observation plays a crucial role in what follows. Clearly, we can also divide by \( z - w \) if \( w \in \Lambda \), but multiplicity of the zero \( w \) is greater than its multiplicity in the spectrum.
Statement 2 of the following proposition was proved in [3]. We include the proof for the reader’s convenience.

**Proposition 3.2.** 1. Assume that \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) is complete in \( L^2(-\pi, \pi) \) or has finite codimension there. Then \( L = L_0 \).

2. Assume that \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) has infinite codimension in \( L^2(-\pi, \pi) \) and \( L \neq L_0 \). Also, assume that the set \( (\Re \Lambda) \mod \mathbb{Z} \) is infinite, i.e., \( \Re \Lambda \) is not contained in a finite union of progressions of the form \( \mathbb{Z} + \gamma, \gamma \in \mathbb{R} \). Then there exist entire functions \( E \) and \( T \) and a function \( F \in \mathcal{P} \mathcal{W}_\pi \) such that \( E \) has infinitely many zeros, \( \mathcal{Z}_E \cap \Lambda = \emptyset \), \( G_\Lambda E \in \mathcal{P} \mathcal{W}_\pi \cap \widehat{L}^\perp \), \( G_\Lambda T \in \mathcal{P} \mathcal{W}_\pi \), and

\[
\begin{align*}
\int_{\mathbb{R}} \frac{G_\Lambda(x)E(x)}{x-w} \overline{F(x)} \, dx &= 0, \quad w \in \mathcal{Z}_E, \\
\int_{\mathbb{R}} G_\Lambda(x)T(x)F(x) \, dx &\neq 0.
\end{align*}
\]

*Proof.* Assume that \( L \neq L_0 \) and let \( \varphi \in \mathcal{L}^\perp \setminus \{0\} \). Since \( \varphi \) annihilates exponentials \( e^{i\lambda t} \), we have \( \hat{\varphi}(\lambda) = 0, \lambda \in \Lambda \), whence for some entire function \( E \),

\[
\hat{\varphi} = G_\Lambda E \quad \text{and} \quad G_\Lambda E \in \mathcal{E}_\pi.
\]

Since the class \( \mathcal{L}^\perp \) is closed under dividing out zeros, we have

\[
\frac{G_\Lambda(z)E(z)}{z-w} \in \mathcal{L}^\perp, \quad w \in \mathcal{Z}_E.
\]

We will assume without loss of generality that \( \mathcal{Z}_E \cap \Lambda = \emptyset \) and \( E \) has no multiple zeros.

**Step 1: Proof of statement 1.** First we consider the case when any function \( E \) with the above properties is a polynomial. This is the case, in particular, when \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) is complete in \( L^2(-\pi, \pi) \) or has finite codimension there. Indeed, \( G_\Lambda E \in \mathcal{E}_\pi \). Therefore, if \( E \) has infinitely many zeros, then we have infinitely many linear independent elements of \( \mathcal{P} \mathcal{W}_\pi \) of the form \( G_\Lambda E/P \), where \( P \) is a polynomial with \( \mathcal{Z}_P \subset \mathcal{Z}_E \) of sufficiently large degree. This contradicts the fact that the system of reproducing kernels \( \{k_\lambda\}_{\lambda \in \Lambda} \) (the Fourier image of \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \)) is complete in \( \mathcal{P} \mathcal{W}_\pi \) or has a finite codimension there. Also, since \( G_\Lambda \) has completeness radius 1, it is impossible that \( E(z) = e^{\alpha z} \) for some \( \alpha \in \mathbb{C} \setminus \{0\} \).

If \( E \) is a polynomial, it follows from (3.2) that \( G_\Lambda \in \mathcal{L}^\perp \), whence \( z^kG_\Lambda \in \mathcal{L}^\perp \) for any \( k \in \mathbb{N} \).

Now let \( \psi \in \mathcal{L}^\perp_0 \). Then \( \hat{\psi} = G_\Lambda T \in \mathcal{E}_\pi \) for some entire function \( T \) and arguing as above we conclude that \( T \) is a polynomial. Thus, \( \mathcal{L}^\perp = \mathcal{L}^\perp_0 \) and so \( L = L_0 \).

**Step 2: Proof of statement 2.** If \( L \neq L_0 \), then by Step 1 there exists a function \( E \) as above which has infinitely many zeros. In view of (3.2) we can start with the function \( \frac{E(z)}{(z-w_1)\cdots(z-w_n)} \) in place of \( E \) where \( n \) is sufficiently large. Thus, we can assume that \( G_\Lambda E \in \mathcal{P} \mathcal{W}_\pi \). Denote by \( \varphi_w, w \in \mathcal{Z}_E \), the functional in \( \mathcal{L}^\perp \) such that \( \frac{G_\Lambda(z)E(z)}{z-w} = \varphi_w(z) \).

Since \( L \neq L_0 \), there exist \( \psi \in \mathcal{L}^\perp_0 \) and \( f_1 \in C^\infty([-\pi, \pi]) \) such that

\[
\varphi(f_1) = 0, \quad \varphi \in \mathcal{L}^\perp, \quad \text{but} \quad \psi(f_1) \neq 0.
\]
In particular, \( \phi_w(f_1) = 0, \ w \in \mathcal{Z}_E. \) Since \( G_\Lambda E \in \mathcal{P}W_\pi, \) this condition can be rewritten as
\[
\int_{\mathbb{R}} \frac{G_\Lambda(x)E(x)}{x-w} F_1(x) dx = 0, \quad w \in \mathcal{Z}_E,
\]
where \( F_1 = \hat{f}_1. \) However, for \( \psi \in L^1_0 \) we know only that \( \hat{\psi} \in \mathcal{E}_\pi \) and so \( \hat{\psi} = PG_\Lambda T \) where \( G_\Lambda T = \bar{h} \in \mathcal{P}W_\pi \) and \( P \) is a polynomial of some degree \( N. \) If we write \( P(z) = \sum_{k=0}^N c_k z^k, \) then
\[
\psi = \sum_{k=0}^N c_k h^{(k)}.
\]

The functionals \( \phi_w \) and \( \psi \) annihilate \( e^{i\lambda t}, \ \lambda \in \Lambda. \) Therefore, we can replace \( f_1 \) by a function \( f_2 \) such that \( \bar{f}_2(t) = \bar{f}_1(t) - \sum_{j \in J} c_j e^{i\lambda_j t} \) for any finite set \( \{\lambda_j\}_{j \in J} \subset \Lambda. \) Since the set \( (\text{Re} \ \Lambda) \mod \mathbb{Z} \) is infinite, we can choose \( \lambda_j \) so that \( f_2(\pm \pi) = 0, \ 0 \leq \ell \leq N. \) Therefore, for \( F_2 = \bar{f}_2 \) we have \( (-iz)^{j} F_2 = f_2^{(j)} \in \mathcal{P}W_\pi, \ 0 \leq j \leq N, \) and so \( P^* F_2 \in \mathcal{P}W_\pi. \) Hence,
\[
\psi(f_1) = \psi(f_2) = \sum_{k=1}^N (-i)^k c_k \int_{-\pi}^\pi h(t) f_2^{(k)}(t) dt
\]
\[
= \sum_{k=1}^N (-i)^k c_k \int_{\mathbb{R}} G_\Lambda(x)T(x)(-ix)^k F_2(x) dx = \int_{\mathbb{R}} G_\Lambda(x)T(x)P^*(x)F_2(x) dx
\]
and we conclude that \( \int_{\mathbb{R}} G_\Lambda T P^* F_2 \neq 0. \)

On the other hand, dividing \( E \) by sufficiently many factors of the form \( z - w_t, \ w_t \in \mathcal{Z}_E, \) we can assume that \( z^k G_\Lambda(\frac{E(z)}{z-w}) \in \mathcal{P}W_\pi, \ k = 0, 1, \ldots, N. \) Since \( z^k G_\Lambda(\frac{E(z)}{z-w}) \in \mathcal{L}^1, \) we can write \( P(z) \frac{G_\Lambda(\frac{E(z)}{z-w})}{z-w} = \tilde{\eta}_w(z), \) where \( \eta_w \in L^1. \) Then we have
\[
\int_{\mathbb{R}} \frac{G_\Lambda(x)E(x)}{x-w} P^*(x)F_2(x) dx = \int_{\mathbb{R}} P(x) \frac{G_\Lambda(x)E(x)}{x-w} F_2(x) dx
\]
\[
= \int_{\mathbb{R}} \tilde{\eta}_w(x) F_2(x) dx = \eta_w(\bar{f}_2) = \eta_w(\bar{f}_1) = 0, \quad w \in \mathcal{Z}_E.
\]

Thus, the functions \( E, T \) and \( F = P^* F_2 \) satisfy (3.1). \( \square \)

Note that we also have \( G_\Lambda(z) E(z) z^n \in \mathcal{L}^1. \) So, for \( E \) and \( F \) from Proposition 3.2, we have
\[
\begin{cases}
\int_{\mathbb{R}} \frac{G_\Lambda(x)E(x)}{x-w} F(x) dx = 0, & w \in \mathcal{Z}_E,
\int_{\mathbb{R}} G_\Lambda(x)T(x) F(x) dx \neq 0,
\end{cases}
\]
for any \( n \in \mathbb{N}_0 \) such that \( G_\Lambda(z) E(z) z^n \in \mathcal{P}W_\pi. \)

There is one technical problem in the above proof in the case when the set \( (\text{Re} \ \Lambda) \mod \mathbb{Z} \) is finite (i.e., \( \text{Re} \ \Lambda \) is contained in a finite union of progressions of the form \( \mathbb{Z} + \gamma). \) Indeed, in this case the exponentials \( e^{i\lambda t} \) may take only finite number of values at \( \pm \pi \) and we cannot guarantee the existence of their linear combination \( \sum_{j \in J} c_j e^{i\lambda_j t} \) which coincides at \( \pm \pi \) with \( f_1 \) up to derivative of order \( N. \) This difficulty can be overcome by a simple perturbation argument.
In what follows we say that a sequence \( \{z_n\}_{n \in \mathbb{N}} \) is \textit{lacunary} if \( \lim \inf_{n \to \infty} |z_{n+1}|/|z_n| > 1 \). By a \textit{lacunary canonical product} we will mean a zero genus canonical product with a lacunary zero set.

**Proposition 3.3.** Let the set \((\Re \Lambda) \mod \mathbb{Z}\) be finite Assume that \( \{e^{i\lambda}\}_{\lambda \in \Lambda}\) has infinite codimension in \( L^2(-\pi, \pi) \) and \( L \neq L_0 \). Then there exists a sequence \( \tilde{\Lambda} \) such that for the corresponding function \( G_\Lambda \) we have

\[
|G_\Lambda(z)| \asymp |G_\tilde{\Lambda}(z)|, \quad \text{dist}(z, \mathbb{R}) \geq 1/10,
\]

and entire functions \( E, T \) and \( F \in \mathcal{PW}_\pi \) such that \( E \) has infinitely many zeros, \( Z_E \cap \Lambda = \emptyset \), \( G_\Lambda E, G_\Lambda T \in \mathcal{PW}_\pi \), and

\[
\begin{cases}
\int_{\mathbb{R}} G_\Lambda(x)E(x)F(x)dx = 0, & w \in Z_E, \\
\int_{\mathbb{R}} G_\Lambda(x)T(x)F(x)dx \neq 0.
\end{cases}
\]

**Proof.** Arguing as in the beginning of the proof of Proposition 3.2, we can find \( E \) and \( F_1 \in \mathcal{PW}_\pi, F_1 = \hat{f}_1 \), such that

\[
\varphi_w(f_1) = \int_{\mathbb{R}} \frac{G_\Lambda(x)E(x)}{x-w}F_1(x)dx = 0, \quad w \in Z_E, \quad \text{but} \quad \psi(f_1) \neq 0.
\]

We need to replace \( \Lambda \) by a perturbed sequence \( \tilde{\Lambda} \) preserving these properties.

Without loss of generality, \( \text{dist}(\Lambda, \mathbb{Z}) > 0 \) and \( Z_E \cap \mathbb{Z} = \emptyset \). Let us expand \( F_1 \) into the series with respect to the basis of cardinal sine functions, \( F_1 = \sum_{n \in \mathbb{Z}} \tilde{a}_n k_n \). Then the equation

\[
\int_{\mathbb{R}} \frac{G_\Lambda(x)E(x)}{x-w}F_1(x)dx = 0
\]

is equivalent to

\[
\sum_{n \in \mathbb{Z}} \frac{G_\Lambda(n)E(n)a_n}{n-w} = 0.
\]

Now put

\[
G_{\tilde{\Lambda}} = G_\Lambda + \delta \frac{Q_1}{Q_2} G_\Lambda,
\]

where \( \delta > 0 \) and \( Q_1, Q_2 \) are two lacunary canonical products such that \( Z_{Q_2} \subset \Lambda \) and \( |Q_1(n)/Q_2(n)| = o(|n|^{-M}), n \in \mathbb{Z}, n \to \infty \), for any \( M > 0 \). E.g., one can take \( Z_{Q_2} = \{\lambda_{n_k}\}_{k \in \mathbb{N}} \) to be some lacunary subsequence of \( \Lambda \) and put

\[
Q_1(z) = \prod_k \left( 1 - \frac{z}{\lambda_{n_k} + 2^{-k}} \right).
\]

In this case it is clear that \( G_{\tilde{\Lambda}} \) has a unique zero near each of the points \( \lambda_{n_k} \) which is different from \( \lambda_{n_k} \). If we define \( \tilde{\Lambda} \) as the zero set \( G_{\tilde{\Lambda}} \), then the set \((\Re \tilde{\Lambda}) \mod \mathbb{Z}\) is infinite.

The property (3.4) is obvious. Therefore, for any entire function \( U \) such that \( G_\Lambda U \in \mathcal{PW}_\pi \) we have \( G_{\tilde{\Lambda}} U \in \mathcal{PW}_\pi \). Indeed, by (3.4) \( G_{\tilde{\Lambda}} U \in L^2(\mathbb{R} + i) \) and also \( G_{\tilde{\Lambda}} U \) is of finite exponential type.
Now we put $F_2 = \sum_n \bar{b}_n k_n$, where
\[\bar{b}_n = \frac{a_n G_\Lambda(n)}{G_\Lambda(n)}\]
Since $|G_\Lambda(n)| \asymp |G_\Lambda(n)|$, $n \in \mathbb{Z}$, we have $F_2 \in PW_\pi$ and the equation (3.6) can be rewritten as
\[
\int_{\mathbb{R}} G_\Lambda(x) E(x) \overline{F_2(x)} dx = \sum_{n \in \mathbb{Z}} G_\Lambda(n) E(n) \frac{b_n}{n - w} = 0.
\]
Also it is clear that if $F_2 = \hat{f}_2$, $f_2 \in L^2(-\pi, \pi)$, then $f_2 - f_1 \in C^\infty[-\pi, \pi]$. Denote by $N$ the order of the distribution $\psi \in L^1_0$ such that $\psi(\hat{f}_1) \neq 0$. Choosing a sufficiently small $\delta$ we can achieve that the norms $\|f_2 - f_1\|_{L^2[-\pi, \pi]}$, $j = 0, 1, \ldots, N$, are sufficiently small and so $\psi(\hat{f}_2) \neq 0$.

Since the set $(\text{Re } \Lambda) \mod \mathbb{Z}$ is infinite, we may continue as in the proof of Proposition 3.2 to obtain (3.5).

3.2. Sufficient conditions for synthesability. The following two propositions will play the key role in the proof of Theorem 1.3.

Proposition 3.4. Let the sequence $\Lambda$ be such that $r(\Lambda) = 2\pi$ and the system $\{k_\lambda^{PW_\pi}\}_{\lambda \in \Lambda}$ has infinite codimension in $PW_\pi$. If the polynomials belong to the de Branges space $H_{\Lambda, \pi}$ and are dense there, then $\Lambda$ is synthesable.

Proof. Assume the contrary. Then there exists a $D$-invariant subspace $L$ with $\sigma(D|_L) = \Lambda$ and $I = [-\pi, \pi]$ such that $L \neq L_0$.

Assume first that the set $(\text{Re } \Lambda) \mod \mathbb{Z}$ is infinite and let $E$ be the function from Proposition 3.2. From (3.1) we conclude that the mixed system
\[\mathcal{MS} := \{k_\lambda^{PW_\pi}\}_{\lambda \in \Lambda} \cup \left\{ \frac{G_\Lambda E}{z - w} \right\}_{w \in \mathbb{Z}}\]
is not complete in $PW_\pi$. Indeed, the two parts of the system $\mathcal{MS}$ are mutually orthogonal. The first condition in (3.1) means that $F$ is orthogonal the system $\left\{ \frac{G_\Lambda E}{z - w} \right\}_{w \in \mathbb{Z}}$ while, by the second condition, $F \notin \text{Span}\{k_\lambda\}_{\lambda \in \Lambda}$.

Let $H \in PW_\pi$ be a nonzero function such that $H \perp \mathcal{MS}$. Then $H = G_\Lambda H_1$ for some function $H_1 \in H_{\Lambda, \pi}$ and using (3.3) we get
\[
\int_{\mathbb{R}} |G_\Lambda(x)|^2 \frac{E(x)}{x - w} \overline{H_1(x)} dx = 0, \quad w \in \mathbb{Z}_E.
\]
Put
\[\tilde{H} = \text{Span}_{H_{\Lambda, \pi}} \left\{ \frac{E(z)}{z - w} \right\}.
\]
Then, by Theorem 2.6, $\tilde{H}$ is a de Branges subspace of the space $H_{\Lambda, \pi}$. Since $E$ has infinitely many zeros, we have $\dim \tilde{H} = \infty$. By the hypothesis, $\text{Ch}(H_{\Lambda, \pi}) = \{P_n : n \in \mathbb{N}_0\} \cup \{H_{\Lambda, \pi}\}$. Therefore, $\mathcal{H} = H_{\Lambda, \pi}$. Since $H_1 \perp \tilde{H}$ in $H_{\Lambda, \pi}$, we have $H_1 = 0$, a contradiction.
Now consider the case when the set \((\Re \Lambda) \bmod Z\) is finite. In this case, by Proposition 3.3 there exists a perturbed sequence \(\hat{\Lambda}\) such that (3.3) is satisfied. Note that \(F \in \mathcal{PW}_\pi\) if and only if \(F(z+i) \in \mathcal{PW}_\pi\) and \(\|F\|_{L^2(\mathbb{R})} \simeq \|F\|_{L^2(\mathbb{R}+i)}\), \(F \in \mathcal{PW}_\pi\). Therefore, by (3.4), for an entire function \(U\) we have \(G_\Lambda U \in \mathcal{PW}_\pi\) if and only if \(G_\Lambda U \in \mathcal{PW}_\pi\), and so \(H_{\Lambda,\pi} = H_{\hat{\Lambda},\pi}\) as sets with equivalence of norms. It follows that the polynomials are dense in \(H_{\hat{\Lambda},\pi}\). As above, this leads to a contradiction with (3.5). □

**Proposition 3.5.** Let the sequence \(\Lambda\) be such that \(r(\Lambda) = 2\pi\) and the system \(\{L^{\mathcal{PW}_\pi}_\lambda\}_{\lambda \in \Lambda}\) has infinite codimension in \(\mathcal{PW}_\pi\). If de Branges space \(H_{\Lambda,\pi}\) has a de Branges subspace of codimension 1 which contains polynomials as a dense subset, then \(\Lambda\) is synthesable.

**Proof.** We consider the case when the set \((\Re \Lambda) \bmod Z\) is infinite. In the case when \((\Re \Lambda) \bmod Z\) is finite, one should use Proposition 3.3 (in place of Proposition 3.2) together with the fact that \(H_{\Lambda,\pi} = H_{\hat{\Lambda},\pi}\) with equivalence of norms and so polynomials are dense in \(H_{\hat{\Lambda},\pi}\) up to codimension 1 as well.

**Step 1.** Assume that \(L\) is not synthesable. Let \(E\) be the entire function from Proposition 3.2 and consider the linear space

\[
\mathcal{H}_{\mathrm{alg}} = \text{Span}(\left\{ \frac{E(z)}{z-w} : w \in \mathbb{Z}_E \right\} \cup \left\{ z^n E(z) : z^n G_\Lambda E \in \mathcal{PW}_\pi \right\}).
\]

Then \(G_\Lambda \mathcal{H}_{\mathrm{alg}} \subset \hat{L}^\perp\). By Proposition 3.2 and formulas (3.3), there also exist a function \(F = \hat{f}\), \(f \in C^\infty[-\pi,\pi]\), and an entire function \(T\) such that \(G_\Lambda T \in \mathcal{PW}_\pi\) and

\[
\begin{align*}
\int_{\mathbb{R}} G_\Lambda(x) \cdot V(x) \cdot \overline{F(x)} \, dx = 0, \quad V \in \mathcal{H}_{\mathrm{alg}}, \\
\int_{\mathbb{R}} G_\Lambda(x) T(x) \overline{F(x)} \, dx \neq 0.
\end{align*}
\]

Moreover, we can assume that \(|F(x)| = o((1+|x|)^{-10})\), \(x \in \mathbb{R}\), \(|x| \to \infty\). Indeed, otherwise in place of \(f\) we can choose a function of the form \(g(t) = f(t) - \sum_{j \in J} c_j e^{i\lambda_j t}\) for some finite set \(J\) such that \(g^\ell(\pm \pi) = 0\), \(0 \leq \ell \leq 10\).

**Step 2.** Note that the chain \(\text{Chain}(\mathcal{H}_{\Lambda,\pi})\) contains only two infinitely dimensional subspaces: \(\mathcal{H}_{\Lambda,\pi}\) itself and its subspace of codimension 1. Put

\[
\mathcal{H} = \overline{\text{Span}} \mathcal{H}_{\Lambda,\pi} \mathcal{H}_{\mathrm{alg}}.
\]

Using the axiomatic description of de Branges spaces it is easy to show (as in Theorem 2.6) that \(\mathcal{H}\) is a de Branges space. Since \(E\) has infinitely many zeros, \(\dim \mathcal{H} = \infty\). Let \(H\) be the projection of \(F\) to \((\overline{\text{Span}}\{k_\lambda : \lambda \in \Lambda\})^\perp\). Then, by (3.7), \(H \perp G_\Lambda \mathcal{H}_{\mathrm{alg}}\) and \(H \neq 0\). We can write \(H = G_\Lambda H_1\) and so \(H_1 \in \mathcal{H}_{\Lambda,\pi}\). Then \(H_1 \perp \mathcal{H}\) and we conclude that \(\mathcal{H}\) has codimension 1 in \(\mathcal{H}_{\Lambda,\pi}\).

**Step 3.** Let \(A\) be a de Branges \(A\)-function which corresponds to the space \(\mathcal{H}\). Since \(\mathcal{H}\) has codimension 1 in \(\mathcal{H}_{\Lambda,\pi}\), the function \(A\) can be chosen so that \(A \in \mathcal{H}_{\Lambda,\pi}\) by Remark
By Theorem 2.9, $\mathcal{H}$ has no de Branges subspace of codimension 1, and so $Z_A$ is a uniqueness for $\mathcal{H}$. The functions $A(z)w, w \in Z_A$, are in $\mathcal{H}$ and, by (3.7),

$$
\begin{align*}
\int_{\mathbb{R}} G_A(x)A(x) F(x) dx &= 0, \\
\int_{\mathbb{R}} G_A(x)T(x) F(x) dx &\neq 0,
\end{align*}

(3.8)

$$

where $G_A T \in PW_\pi$. We may assume that $T$ is not a polynomial (otherwise we can replace $T$ by $T + A(z)w_0$).

**Step 4.** Now we expand $G_A T$ and $F$ with respect to the orthonormal basis $\{k_n\}_{n \in \mathbb{Z}}$ of cardinal sine functions, $k_n(z) = \frac{\sin \pi (z-n)}{\pi (z-n)}$:

$$
G_A T = \sum_{n \in \mathbb{Z}} b_n k_n, \quad F = \sum_{n \in \mathbb{Z}} a_n k_n, \quad \{a_n\}, \{b_n\} \in \ell^2.
$$

Moreover, since $|F(x)| = o((1 + |x|)^{-10}), x \in \mathbb{R}, |x| \to \infty$, we have $\{a_n\} \in \ell^1$ and $\{G_A(n)A(n)a_n\} \in \ell^1$.

We can assume that $\Lambda \cap \mathbb{Z} = \emptyset, a_n \neq 0$ and $b_n \neq 0$ for any $n \in \mathbb{Z}$. Otherwise, we can use a shifted basis $\{k_{n+\gamma}\}_{n \in \mathbb{Z}}$ with any $\gamma \in [0, 1)$ and find a $\gamma$ such that the set $\mathbb{Z} + \gamma$ will have these properties. Moreover, we can find $\gamma$ such that, for some $c > 0$,

$$
\text{dist}(w, \mathbb{Z} + \gamma) \geq \frac{c}{|w|^2}, \quad w \in Z_T, w \neq 0.
$$

(3.9)

Indeed, if we take $c$ so small that

$$
2c \sum_{w \in Z_T, w \neq 0} |w|^{-2} < 1/2,
$$

then the intervals $[\text{Re} w - c|w|^{-2}, \text{Re} w + c|w|^{-2}], w \neq 0$, considered mod $\mathbb{Z}$ will not cover the interval $[0, 1]$. We will assume without loss of generality that $\gamma = 0$.

The first equation in (3.8) becomes

$$
\sum_{n \in \mathbb{Z}} \frac{G_A(n)A(n)a_n}{w - n} = 0, \quad w \in Z_A.
$$

So, for some entire function $S_1$, we have

$$
\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{G_A(n)A(n)a_n}{z - n} = \frac{A(z)S_1(z)}{\sin \pi z}.
$$

(3.10)

On the other hand,

$$
\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{b_n(-1)^n}{z - n} = \frac{G_A(z)T(z)}{\sin \pi z}.
$$

(3.11)

Comparing the residues at integer points we get

$$
S_1(n)T(n) = (-1)^na_nb_n.
$$

Let $W$ be a function from $PW_\pi$ such that $W(n) = (-1)^n a_nb_n, n \in \mathbb{Z}$. Then there exists an entire function $U$ such that

$$
S_1(z)T(z) - W(z) = U(z) \sin \pi z.
$$
Since, by (3.10), \( AS_1 \in \mathcal{P}W_\pi \) and \( G\Lambda A \in \mathcal{P}W_\pi \), we have \( AS_1G\Lambda T - G\Lambda AW \in \mathcal{P}W_{2\pi} \). Hence, \( G\Lambda AU \sin \pi z \in \mathcal{P}W_{2\pi} \) and, finally, \( G\Lambda AU \in \mathcal{P}W_\pi \) and \( AU \in \mathcal{H}_{\Lambda,\pi} \).

**Step 5.** Let us show that the function \( U \) is constant. Since \( A \) is of maximal growth in \( \mathcal{H} \), it is easy to see that \( U \) is of zero exponential type. Assume that \( U \) has at least one zero \( u_0 \). Since \( AU \in \mathcal{H}_{\Lambda,\pi} \), the function \( \frac{AL}{z-u_0} \) belongs to the domain of multiplication by \( z \) in \( \mathcal{H}_{\Lambda,\pi} \) and so \( \frac{AU}{z-u_0} \in \mathcal{H} \) and vanishes on \( \mathcal{Z}_A \), a contradiction to the fact that \( \mathcal{Z}_A \) is a uniqueness set for \( \mathcal{H} \).

**Step 6.** We have seen that \( U = c \) is a constant function and so
\[
\frac{S_1(z)T(z)}{\sin \pi z} = U(z) + \frac{W(z)}{\sin \pi z} = c + \sum_{n \in \mathbb{Z}} \frac{a_nb_n}{z - n}.
\]
In addition we know that \( \kappa := (G\Lambda T, F)_{\mathcal{P}W_\pi} = \sum_n a_nb_n \neq 0 \), and \( a_n = (-1)^n \overline{F(n)} = o(n^{-10}) \). Then it is easy to see that for \( |z - l| < 1/2 \) we have
\[
c + \sum_{n \in \mathbb{Z}} \frac{a_nb_n}{z - n} = c + \frac{\kappa}{l} + \frac{a_kb_l}{z - l} + o\left(\frac{1}{|l|}\right).
\]
Therefore, for sufficiently big \( l \in \mathbb{Z} \), there exists a unique \( z_l \) such that \( |z_l - l| < 1/2 \),
\[
c + \sum_n \frac{a_nb_n}{z_l - n} = 0
\]
and
\[
z_l = l - \frac{la_kb_l}{cl + \kappa + o(1)} = l + o(|l|^{-9}). \tag{3.12}
\]
By (3.9), such \( z_l \) can not be a zero of \( T \) when \( l \) is sufficiently large. Hence, \( S_1 \) has zeros \( z_l \) of the form (3.12) for all sufficiently large \( l \). Since \( S_1T \in \mathcal{P}W_\pi + z\mathcal{P}W_\pi \), we conclude, by simple estimates of canonical products, that \( T \) has at most finite number of zeros and is of zero type. Thus, \( T \) is a polynomial, a contradiction.

3.3. **End of the proof of sufficiency part in Theorem 1.3.** Now we are ready to prove the sufficiency of conditions (i) and (ii) of Theorem 1.3. Let \( G\Lambda = G \). By Lemma 3.1 we have \( \mathfrak{e}_0(|G\Lambda|^2) = \mathcal{H}_{\Lambda,\pi} \). Hence, \( \mathcal{P} \subset \mathcal{H}_{\Lambda,\pi} \) and \( \mathcal{P} \) is dense in \( \mathcal{H}_{\Lambda,\pi} \) or has codimension one there. In either of the cases \( \Lambda \) is synthesable by Proposition 3.4 or by Proposition 3.5 respectively. \( \square \)

4. **Proof of Theorem 1.3: Necessity**

By the hypothesis, the system \( \{e^{i\Lambda t}\}_{\lambda \in \Lambda} \) has infinite defect in \( L^2(-\pi, \pi) \). Let \( G\Lambda \) be some canonical product with zero set \( \Lambda \) such that \( G\Lambda/G\Lambda^* \) is a ratio of two Blaschke products. Then the corresponding space \( \mathcal{H}_{\Lambda,\pi} \) is infinite-dimensional. Moreover, multiplying, if necessary by \( e^{bz} \), \( b \in \mathbb{R} \), we can achieve that the indicator function of \( G\Lambda \) satisfies \( h_{G\Lambda}(0) = h_{G\Lambda}(\pi) \). Assume that it is not true that the polynomials are dense in \( \mathcal{H}_{\Lambda,\pi} \) up to
codimension 1 (it is possible, in particular, that the set of all polynomials is not contained in $H_{A,\pi}$). We will show that in this case $\Lambda$ is not synthesable.

**Lemma 4.1.** Assume that it is not true that the polynomials are dense in $H_{A,\pi}$ up to codimension 1. Then there exists a de Branges subspace in the chain $\text{Chain}(H_{A,\pi})$ which is both infinite-dimensional and of infinite codimension in $H_{A,\pi}$.

**Proof.** Note that any finite-dimensional subspace of an arbitrary de Branges space is of the form $P_n e^S$ for some function $S$ which is real on $\mathbb{R}$. In the case of $H_{A,\pi}$ it follows that $e^SG_\Lambda \in P_{\pi}$ by the choice of $G_\Lambda$. Thus, if $H$ is a finite-dimensional subspace of $H_{A,\pi}$, it is of the form $P_n$.

Now let $H$ be some infinite-dimensional subspace from $\text{Chain}(H_{A,\pi})$. If it has an infinite codimension, we are done. If it has finite codimension, then by Theorems 2.2 and 2.9, its codimension can be only one and also any proper de Branges subspace of $H$ has infinite codimension in $H$. Thus, any infinite-dimensional subspace of $H$ would do the job. Finally, note that if all de Branges subspaces of $H$ are finite-dimensional, then $H = \text{Clos}_{H_{A,\pi}} P$. Since $\dim(H_{A,\pi} \ominus H) = 1$, we arrive to a contradiction. $\square$

From now on we assume that $\tilde{H} \in \text{Chain}(H_{A,\pi})$ is both infinite-dimensional and of infinite codimension. Put $\tilde{H}_0 = \{ f \in \tilde{H} : fP \subset \tilde{H} \}$.

Note that $\tilde{H}_0 \neq \emptyset$ by Theorem 2.8.

Now we are looking for a function $T \in H_{A,\pi}$ such that $T \notin \tilde{H}$ and $PT \subset H_{A,\pi}$. Let $A$ and $\tilde{A}$ be the $A$-functions of $H_{A,\pi}$ and $\tilde{H}$. By Lemma 2.5 $|\tilde{A}(iy)/A(iy)|$ tends to zero faster that any polynomial as $|y| \to \infty$. Then there exists a lacunary canonical product $S$ with $Z_S \subset Z_A$ such that $T := A/S \in H_{A,\pi}$ and $|T| \gtrsim |\tilde{A}|$ on $i\mathbb{R}$ whence $T \notin \tilde{H}$. So,

$$G_\Lambda T \notin G_\Lambda \tilde{H}.$$  

Hence, there exists $F \in PW_\pi$ such that

$$\begin{cases} 
\int_\mathbb{R} G_\Lambda(x)V(x)\overline{F(x)}dx = 0 & \text{for any } V \in \tilde{H}, \\
\int_\mathbb{R} G_\Lambda(x)T(x)\overline{F(x)}dx \neq 0. 
\end{cases}$$  

(4.1)

**4.1. Key lemma.** For the proof we need to find a function $F$ satisfying equations of the form (4.1) such that $F = \hat{f}$ for some $f \in C^\infty[-\pi, \pi]$.

In what follows we will use the following simple observation.

**Remark 4.2.** Assume that $zf \in PW_\pi$. Then

$$\int_\mathbb{R} f(x) \sin \pi x dx = \sum_{n \in \mathbb{Z}} (-1)^n f(n) = 0.$$  

(4.2)

Indeed, $\int_\mathbb{R} f(x) \sin \pi x dx = (zf, \pi k_0) = 0$. The second equality follows from the fact that $f \in PW_\pi$ and $\int_\mathbb{R} |f| < \infty$, whence $g(z) = e^{i\pi z}f$ is in the Hardy space $H^1$ in the upper
half-plane. Then \(|g(iy)| = o(y^{-1}), y \to \infty\). On the other hand, \(f(z) = \frac{\sin \pi z}{\pi} \sum_n \frac{(-1)^n f(n)}{z-n}\), and so \(|g(iy)| = e^{-\pi y}|f(iy)| \to \left| \sum_n (-1)^n f(n) \right|/(2\pi) as y \to \infty\).

**Lemma 4.3.** There exist an entire function \(T_0\) and a function \(F_0 \in PW_\pi\) such that \(PT_0 \subset H_{\Lambda, \pi}\),

\[
|F_0(n)| = o(|n|^{-N}), \quad |n| \to \infty,
\]

for any \(N > 0\) and

\[
\begin{align*}
\int_{\mathbb{R}} G_A(x)V(x)F_0(x)dx &= 0 \quad \text{for any } V \in \hat{H}, \\
\int_{\mathbb{R}} G_A(x)T_0(x)F_0(x)dx &\neq 0.
\end{align*}
\]

**Proof.** Without loss of generality we assume that \((\Lambda \cup Z_\Lambda) \cap \mathbb{Z} = \emptyset\) (see the discussion in Subsection 2.1).

**Step 1.** Let \(F \in PW_\pi\) be a function satisfying (4.1). Recall that the functions \(\{G_{\Lambda, \hat{A}}\}_{\mu \in Z_{\hat{A}}}\) form an orthogonal basis in \(\hat{H}\). Hence, \(\int_{\mathbb{R}} G_A(x)\frac{\hat{A}(x)}{x-\mu}F(x)dx = 0\) for any \(\mu \in Z_{\hat{A}}\). Let \(F = \sum_{n \in \mathbb{Z}} \tilde{a}_n k_n\), where \(\tilde{a}_n = F(n) \in \ell^2\), be the expansion of \(F\) with respect to the orthonormal basis of cardinal sine functions. Then

\[
\sum_{n \in \mathbb{Z}} \frac{G_A(n)\hat{A}(n)\tilde{a}_n}{\mu - n} = 0, \quad \mu \in Z_{\hat{A}}.
\]

Therefore, there exists an entire function \(S\) such that

\[
\frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} \frac{G_A(n)\hat{A}(n)\tilde{a}_n}{z-n} = \hat{A}(z)S(z).
\]

Comparing the values at \(n \in \mathbb{Z}\), we get \(S(n) = (-1)^n G_A(n)\tilde{a}_n\) and so

\[
\sum_{n \in \mathbb{Z}} \left| \frac{S(n)}{G_A(n)} \right|^2 < \infty.
\]

The idea is to divide \(S\) by some lacunary product \(U\) and to show that the function \(\tilde{F} = \sum_{n \in \mathbb{Z}} \tilde{d}_n k_n\), where \(\tilde{d}_n = a_n/U(n)\) has the required properties. This can be easily done if there exist a sequence \(\{s_k\}_{k \in \mathbb{N}} \subset Z_S\) and \(N > 0\) such that

\[
\text{dist}(s_k, Z) \gtrsim |s_k|^{-N},
\]

since in this case \(|U(n)| grows faster than any polynomial. If such \(s_k\) exist, one can go directly to the Step 5. Otherwise, we first need to modify the functions \(F\) and \(S\).

**Step 2.** Choose an increasing sequence \(\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}\) such that \(\text{dist}(\{n_k\}, Z_S) \geq 1/10, k \in \mathbb{N}\). Such sequence exists since, otherwise, there will be a point of \(Z_S\) in \(D(n, 1/10)\) for all \(n \in \mathbb{Z}\) except a finite number. On the other hand, it follows from (4.5) that \(A S \in PW_\pi + zPW_\pi\). Thus, \(\hat{A}\) is at most a polynomial, a contradiction to the choice of \(\hat{H}\).
Shifting $\Lambda$ if necessary, we can assume without loss of generality that $\text{dist}(n, \Lambda) \gtrsim (n^2 + 1)^{-1}$, $n \in \mathbb{Z}$ (see the proof of (3.9)). Since $S$ and $G_\Lambda$ are entire functions of order at most 1 and $\text{dist}(n_k, Z_S \cup \Lambda) \gtrsim n_k^{-2}$, there exists $N > 2$ such that

$$
|S(z)| \asymp |S(n_k)|, \quad |G_\Lambda(z)| \asymp |G_\Lambda(n_k)|, \quad z \in D_k := \overline{D}(n_k + \frac{1}{n_k}, \frac{1}{10n_k^2}),
$$

with the constants independent on $k$. This follows from standard estimates of canonical products. Therefore, by (4.6), $|S(z)| = o(|G(z)|)$, $z \in D_k$, $k \to \infty$. From now on we assume $N$ to be fixed.

**Step 3.** Let $P$ be a polynomial such that $Z_P \subset Z_\mathcal{S} \setminus \mathbb{Z}$. Consider the function $F_1 = \sum_{n \in \mathbb{Z}} \bar{b}_n k_n$, where $b_n = a_n / P(n)$. Let us show that $F_1 \perp \{ \frac{G_\Lambda}{z^\mu} \}_{\mu \in \mathbb{Z}}$ and so $F_1 \perp G_\Lambda H$. As in Step 1, this orthogonality is equivalent to the interpolation formula

$$
\frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} G_\Lambda(n) \tilde{A}(n) b_n = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n \tilde{A}(n) S(n) \frac{P(n)(z - n)}{P(z)} = \tilde{A}(z) S(z),
$$

which is obviously true.

We have $F_1(n) = F(n) / P(n) = F(n) / P^*(n)$, $n \in \mathbb{Z}$, where $P^*(z) = \overline{P(z)}$. Hence, there exists an entire function $W$ such that $P^*F_1 - F = W \sin \pi z$. Since $F, F_1 \in \mathcal{P}W_\pi$ we conclude that $W$ is a polynomial.

Since $zf \in \mathcal{P}W_\pi$, $\int_{\mathbb{R}} f(x) \sin \pi x \, dx = 0$ by (11.2). Thus,

$$
\int_{\mathbb{R}} G_\Lambda T\overline{F} = \int_{\mathbb{R}} G_\Lambda T\overline{P^*F_1} - \int_{\mathbb{R}} G_\Lambda T W^* \sin \pi x = \int_{\mathbb{R}} G_\Lambda T\overline{F_1},
$$

since $zG_\Lambda TW^* \in \mathcal{P}W_\pi$. We conclude that $F_1 \perp G_\Lambda V, V \in \mathcal{H}$, but, for $T_1 = TP$ we have $\int_{\mathbb{R}} G_\Lambda T_1 \overline{F_1} \neq 0$. Note also that we still have $G_\Lambda T_1 \mathcal{P} \subset \mathcal{P}W_\pi$.

**Step 4.** According to Step 3, for any polynomial $P$ we can construct a new function $F_1 = \sum_{n \in \mathbb{Z}} \bar{b}_n k_n$ such that for the function $S_1 = S/P$ we have

$$
\frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} G_\Lambda(n) \tilde{A}(n) b_n = \tilde{A}(z) S_1(z).
$$

Also, if $K = \deg P$, then

$$
|S_1(z)| = o(|z|^{-K}|G_\Lambda(z)|), \quad |z| \to \infty, \quad z \in D_k,
$$

uniformly with respect to $z \in D_k$.

Now let us choose two lacunary canonical products $Q_1$ and $Q_2$ with the following properties:

(a) $Z_{Q_1} \subset \{ n_k + \frac{1}{n_k} \}$, $Z_{Q_2} \subset \Lambda$;

(b) $\left| \frac{Q_1(n)}{Q_2(n)} \right| \lesssim \frac{1}{n^2 + 1}$, $n \in \mathbb{Z}$, and $\left| \frac{Q_1(z)}{Q_2(z)} \right| \lesssim \frac{1}{|z|^2}$, $\text{dist}(z, Z_{Q_2}) \gtrsim 1$;

(c) $\left| \frac{Q_1(z)}{Q_2(z)} \right| > \frac{1}{|z|^K}$ when $\left| z - \left( n_k + \frac{1}{n_k} \right) \right| = \frac{1}{10n_k^N}$, $k \in \mathbb{N}$.

It is clear that, for a fixed $N$ and sufficiently large $K$, this can be achieved.
Now consider the function \( S_2 = S_1 + \frac{Q_1}{Q_2}G_A \). By (4.7), property (c) and the Rouché theorem, \( S_2 \) has exactly one zero in the disk \( D_k \) when \( n_k + \frac{1}{n_k} \in \mathbb{Z}Q_1 \) and \( k \) is sufficiently large. Put \( F_2 = \sum_{n \in \mathbb{Z}} c_n k_n \), where
\[
c_n = b_n + (-1)^n \frac{Q_1(n)}{Q_2(n)}.
\]
We show that \( F_2 \perp G_A \tilde{H} \), but \( \int_{\mathbb{R}} G_A T_1 F_2 \neq 0 \). For the first property we need to show that \( F_2 \perp \{ \frac{G_A(z)}{z - \mu} \}_{\mu \in \mathbb{Z}} \). Since \( F_1 \perp G_A \tilde{H} \), this is equivalent to
\[
\sum_{n \in \mathbb{Z}} \frac{(-1)^n G_A(n) \tilde{A}(n) Q_1(n)}{(\mu - n) Q_2(n)} = 0, \quad \mu \in \mathbb{Z}.
\]
This equation would follow from the interpolation formula
\[
(4.8) \quad \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{(-1)^n G_A(n) \tilde{A}(n) Q_1(n)}{Q_2(n)(z - n)} = \frac{G_A(z) \tilde{A}(z) Q_1(z)}{Q_2(z) \sin \pi z}.
\]
Clearly, the residues at \( n \) in the left-hand side and in the right-hand side coincide and so the difference is an entire function which is, by (b), \( o(1) \) as \( |z| \to \infty \) and \( \text{dist}(z, \mathbb{Z}) \geq 1/10 \). Therefore, the interpolation formula (4.8) is true. Finally,
\[
\int G_A T_1 F_2 = \sum_{n \in \mathbb{Z}} G_A(n) T_1(n) F_2(n)
= \sum_{n \in \mathbb{Z}} G_A(n) T_1(n) b_n + \sum_{n \in \mathbb{Z}} \left( \frac{(-1)^n G_A(n) T_1(n) Q_1(n)}{Q_2(n)} \right).
\]
Note that \( z G_A T_1 Q_1/Q_2 \in \mathcal{P}W_\pi \). Hence, by (4.12), \( \sum_{n \in \mathbb{Z}} \left( \frac{(-1)^n G_A(n) T_1(n) Q_1(n)}{Q_2(n)} \right) = 0 \) and so
\[
\int G_A T_1 F_2 = \sum_{n \in \mathbb{Z}} G_A(n) T_1(n) F_1(n) = \int G_A T_1 F_1 \neq 0.
\]
**Step 5.** At Step 4 we constructed a function \( F_2 = \sum_{n \in \mathbb{Z}} c_n k_n \) and the corresponding function \( S_2 \) such that \( S_2(n) = (-1)^n G_A(n) c_n \) and
\[
\sin \pi z \sum_{n \in \mathbb{Z}} G_A(n) \tilde{A}(n) c_n = \tilde{A}(z) S_2(z).
\]
By the construction, there is a sequence \( \{ s_k \} \) of zeros of \( S_2 \) with the property
\[
(4.9) \quad \text{dist}(s_k, \mathbb{Z}) \gtrsim |s_k|^{-N}.
\]
Since \( G_A T_1 P \subset \tilde{H} \), there exists a lacunary entire function \( U_0 \) such that \( G_A T_1 U_0 P \subset \mathcal{P}W_\pi \) and \( x^n = o(|U_0(x)|), \ |x| \to \infty, \ x \in \mathbb{R} \), for any \( n > 0 \). Then it is clear that we can choose another lacunary product \( U \) such that \( \mathbb{Z}_U \subset \{ s_k \} \) and \( G_A T_1 U P \subset \mathcal{P}W_\pi \).

Put \( F_0(z) = \sum_{n \in \mathbb{Z}} d_n k_n \), where \( d_n = c_n/U(n) \). Note that, by (4.9), \( |U(n)| \) tend to infinity super-polynomially and so (4.3) is satisfied. Let us show that \( F_0 \) satisfies (4.4), that is, \( F_0 \perp G_A \tilde{H} \), but \( \int_{\mathbb{R}} G_A T_0 F_0 \neq 0 \), where \( T_0 = T_1 U \).
For the proof of the first property we use again the interpolation formula argument and show that
\[
\frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{G_{\Lambda}(n)\hat{A}(n)d_n}{z - n} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{(-1)^n \hat{A}(n)S_2(n)}{U(n)(z - n)} = \frac{\hat{A}(z)S_2(z)}{U(z) \sin \pi z}.
\]
The first equality follows from the fact that \(G_{\Lambda}(n)d_n = (-1)^n S_2(z)/U(n)\). To prove the second equality we use again the fact that the difference of the left-hand side and the right-hand side is an entire function which is \(o(1)\) as \(|z| \to \infty\), \(\text{dist} (z, \mathbb{Z}) \geq 1/10\), and thus is identically zero. It follows that \(F_0 = 0\).

It remains to prove that \(\int_{\mathbb{R}} G_{\Lambda}T_0F_0 \neq 0\). Since \(F_0(n) = F_2(n)/U^*(n), \ n \in \mathbb{Z}\), we have \(U^*F_0 - F_2 = W \sin \pi z\) for some entire function \(W\). Since \(F_2\) and \(F_0\) are in \(\mathcal{P}W_\pi\) we have the estimate
\[
|W(z)| \lesssim 1 + |U^*(z)|, \quad \text{dist} (z, \mathbb{Z}) \gtrsim 1.
\]
Recall that \(G_{\Lambda}T_1U\pi \subset \mathcal{P}W_\pi\) and so \(G_{\Lambda}T_1U\pi \subset \mathcal{P}W_\pi\), whence \(\int G_{\Lambda}T_1U\pi \sin \pi x = 0\). Then
\[
\int G_{\Lambda}T_1U\pi F_0 = \int G_{\Lambda}T_1F_2 + \int G_{\Lambda}T_1U\pi \sin \pi x = \int G_{\Lambda}T_1F_2 \neq 0.
\]
Thus, \(F_0\) satisfies (4.4) with \(T_0 = T_1U\).

4.2. End of the proof. Recall that we assume that the system \(\{e^{i\lambda t}\}_{\lambda \in \Lambda}\) has infinite defect in \(L^2(-\pi, \pi)\) and that it is not true that the polynomials are dense in \(\mathcal{H}_{\Lambda, \pi}\) up to codimension 1. To arrive to a contradiction, we need to construct a non-synthesable \(D\)-invariant subspace.

Let \(\hat{\mathcal{H}}\) and \(\hat{\mathcal{H}}_0\) be the same as above. Put
\[
M = \{f \in L^2(-\pi, \pi) : \hat{f} \in G_{\Lambda}\hat{\mathcal{H}}_0\}.
\]
Each element \(f \in M\) defines a continuous linear functional on \(C^\infty(-2\pi, 2\pi)\) defined by
\[
\varphi_f(h) = \int_{-\pi}^{\pi} h(t)f(t)dt = \int_{\mathbb{R}} \hat{h}[t, -\pi, \pi](\hat{f})(-x)dx, \quad h \in C^\infty(-2\pi, 2\pi).
\]
We use the fact that \(\hat{f}(x) = \hat{f}(-x)\).

Now let
\[
L = M^\perp = \{h \in C^\infty(-2\pi, 2\pi) : \varphi_f(h) = 0, \ f \in M\}.
\]
By the construction, \(L\) is a closed subspace of \(C^\infty(-2\pi, 2\pi)\) and \(\{h \in C^\infty(-2\pi, 2\pi) : h|[-\pi, \pi] \equiv 0\} \subset L\). Cleary, \(\varphi_f(e^{i\lambda t}) = \hat{f}(\lambda) = 0, \ f \in M, \) and so \(e^{i\lambda t} \in L\). Since the set of common zeros of \(\{\hat{f} : \ f \in M\}\) coincides with \(\Lambda\), we have \(\sigma(D|_L) = i\Lambda\).

Let us show that \(L\) is \(D\)-invariant which is a consequence of the fact that functions in \(\hat{\mathcal{H}}_0\) can be multiplied by polynomials. We need to show that \(\int_{-\pi}^{\pi} h'(t)f(t)dt = 0\) whenever \(h \in L, \ f \in M\). Since \(f\) vanishes outside \([-\pi, \pi]\), the integral depends only on the values of \(h\) inside this interval. Thus we may assume without loss of generality that \(supp h \subset \)**
If $H = \hat{h}$ are rapidly decaying on $\mathbb{R}$ and we have
\[
\int_{-\pi}^{\pi} h'(t)f(t)dt = -i \int_{\mathbb{R}} xH(x)F(-x)dx.
\]
We have $F \in G_{A,\hat{H}}$ and, by definition of $\hat{H}$, we have $xF(x) \in G_{A,\hat{H}}$. Thus, $-ixF(-x) = \frac{\hat{f}}{f_1(x)}$ for some $f_1 \in M$. Hence,
\[
\int_{-\pi}^{\pi} h'(t)f(t)dt = \int_{-\pi}^{\pi} h(t)f_1(t)dt = 0.
\]

Now we find a function in $L \setminus L_0$. Let $F_0$ be the function constructed in Lemma 1.3. Recall that $F_0 = \sum_{n \in \mathbb{Z}} d_n k_n$, where $|d_n| = o(|n|^{-N})$, $|n| \to \infty$, for any $N > 0$. Then $F_0 = \hat{f}_0$ where $f_0 = \sum_{n \in \mathbb{Z}} d_n e^{-int}$. By the condition on the coefficients $d_n$, the function $f_0$ can be continued as a $2\pi$-periodic function which is in $C^\infty(\mathbb{R})$.

Since $F_0$ satisfies (1.4), we conclude that the function $f_0$ is annihilated by any functional from $M$ and so $f_0 \in L$. Indeed, if $f \in M$, we have $\hat{f} \in G_{A,\hat{H}}$ and so
\[
\varphi_f(f_0) = \int_{-\pi}^{\pi} f(t)f_0(t)dt = \int_{\mathbb{R}} \hat{f}(x)F_0(x)dx = 0.
\]

At the same time, $G_{A}T_0 \in PW_\pi$ and so there exists $g \in L^2(-\pi, \pi)$ such that $G_{A}T_0 = \hat{g}$. It is clear that the functional $\varphi_g(h) = \int_{-\pi}^{\pi} h(t)g(t)dt$ annihilates $L_0$. However, we have $\varphi_g(f_0) = \int_{\mathbb{R}} G_{A}T_0 \mathcal{F}_{C} \neq 0$, whence $f_0 \notin L_0$.

Thus, we have shown that if the system $\{e^{\lambda t}\}_{\lambda \in \Lambda}$ has infinite codimension in $PW_\pi$ and $\Lambda$ is synthesable, then polynomials belong to $\mathcal{H}_{A,\pi}$ and are dense there up to codimension 1. Hence, $G_{A} \in PW_\pi$ and, in particular, $G_{A}$ is in the Cartwright class. Therefore, $G_{A}$ can be represented as the principal value product (1.2) up to a constant (there is no additional exponential factor since $G_{A}^*/G_{A}$ is a ratio of two Blaschke products). Moreover, $G_{A}$ is of exponential type $\pi$ and its indicator diagram is given by $[-\pi i, \pi i]$. Therefore, $\mathcal{E}_{0}(|G_{A}|^2) = \mathcal{H}_{A,\pi}$ (see the proof of Lemma 3.1). Thus, polynomials are dense in $\mathcal{E}_{0}(|G_{A}|^2)$ up to codimension 1. This completes the proof of Theorem 1.3.

5. Proof of Theorem 1.4

Without loss of generality we can assume that $[-\pi, \pi] \subset (a, b)$ and $I_1 = I_2 = [-\pi, \pi]$. Put
\[
\mathcal{H}_{j,0} = \{f \text{ entire} : G_{A}f \in PW_\pi \cap L^1_j\}, \quad j = 1, 2.
\]
Then $\mathcal{H}_{j,0} \subset \mathcal{H}_{A,\pi}$. Put $\mathcal{H}_{j} = \text{Span}_{\mathcal{H}_{A,\pi}} \mathcal{H}_{j,0}$.

We would like to show that either $\mathcal{H}_1 \subset \mathcal{H}_2$ or $\mathcal{H}_2 \subset \mathcal{H}_1$. This would be true, if we could show that $\mathcal{H}_1$ and $\mathcal{H}_2$ are de Branges subspaces of $\mathcal{H}_{A,\pi}$. The possibility of division by a Blaschke factor follows from the corresponding property for $\mathcal{L}_j$. It is not clear, however, whether $\mathcal{H}_j$ are closed under $*$-transform.

To overcome this difficulty, we use the following variant of de Branges Ordering Theorem: If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two closed subspaces of a de Branges space $\mathcal{H}$ which are invariant under
division by Blaschke factors, then there exists \( a \in \mathbb{R} \) such that either \( e^{iaz} \mathcal{H}_1 \subset \mathcal{H}_2 \) or \( e^{iaz} \mathcal{H}_2 \subset \mathcal{H}_1 \).

This statement follows by a simple modification of the argument from the proof of [10, Theorem 35]. Let us show that in our case \( a \) must be zero. Indeed, assume that \( e^{iaz} \mathcal{H}_1 \subset \mathcal{H}_2 \). Then for any functional \( \varphi \in L_1^+ \) such that \( \hat{\varphi} \in \mathcal{P}_W \) we have

\[
e^{iaz} \hat{\varphi} \in e^{iaz} G_\Lambda \mathcal{H}_1 \subset G_\Lambda \mathcal{H}_2 \subset \mathcal{P}_W.
\]

However, we can choose \( \varphi \in L_1^+ \) so that \( \text{conv supp } \varphi = [−\pi, \pi] \) since \( I = [−\pi, \pi] \) is the minimal interval for which \( L_I \subset L_1 \). Therefore \( e^{iaz} \hat{\varphi} \notin \mathcal{P}_W \) for any \( a \neq 0 \). We conclude that

\[
(5.1) \quad \mathcal{H}_1 \subset \mathcal{H}_2 \quad \text{or} \quad \mathcal{H}_2 \subset \mathcal{H}_1.
\]

Now assume that \( L_1 \not\subset L_2 \) and \( L_2 \not\subset L_1 \). Then there exist two functions \( f_1, f_2 \in C^\infty(a, b) \) and functionals \( \psi_j \in L_j^+ \), \( j = 1, 2 \), such that \( \varphi(f_j) = 0 \) for any \( \varphi \in L_1^+ \), but \( \psi_2(f_1) \neq 0 \), and, similarly, \( \varphi(f_2) = 0 \) for any \( \varphi \in L_2^+ \), but \( \psi_1(f_2) \neq 0 \). Arguing as in the proof of Proposition 3.2, we can find entire functions \( T_j, F_j \), \( j = 1, 2 \), such that

\[
\int_{\mathbb{R}} G_\Lambda f F_j = 0, \quad f \in \mathcal{H}_j, \quad \int_{\mathbb{R}} G_\Lambda T_j F_j \neq 0.
\]

Consider the projections of \( F_j \) onto \( (\overline{\text{span}}\{k_\lambda : \lambda \in \Lambda\})^\perp \), they are of the form \( G_\Lambda H_j \) with \( H_j \in \mathcal{H}_\Lambda,\pi \). Thus, we get

\[
\int_{\mathbb{R}} G_\Lambda f \overline{G_\Lambda H_1} = 0, \quad f \in \mathcal{H}_1, \quad \int_{\mathbb{R}} G_\Lambda T_2 \overline{G_\Lambda H_1} \neq 0
\]

and

\[
\int_{\mathbb{R}} G_\Lambda f \overline{G_\Lambda H_2} = 0, \quad f \in \mathcal{H}_2, \quad \int_{\mathbb{R}} G_\Lambda T_1 \overline{G_\Lambda H_2} \neq 0.
\]

Hence, \( T_j \in \mathcal{H}_j, j = 1, 2 \), and \( T_1 \notin \mathcal{H}_2, T_2 \notin \mathcal{H}_1 \). This contradicts (5.1).
6. Examples

In this section we give examples of synthesable subspaces.

We say that an entire function $U$ of zero exponential type (which is not a polynomial) belongs to the Hamburger class if it is real on $\mathbb{R}$, has only real and simple zeros $\{s_k\}$, and for any $M > 0$, $|s_k|^M = o(|U'(s_k)|)$, $s_k \to \infty$.

**Example 6.1.** Let $U$ be a Hamburger class function such that $Z_U \subset Z$ and the polynomials belong to the space $L^2(\mu_U)$, where $\mu_U = \sum_{n \in \mathbb{Z}} U'(n)^2 \delta_n$, and are dense there. Put $\Lambda = \mathbb{Z} \setminus Z_U$. Then the polynomials are dense in $H_{\Lambda, \pi}$.

Note that there are many such functions $U$, e.g., $U(z) = \prod_{n \in \mathbb{N}} \left(1 - z/[n^\alpha]\right)$, $\alpha > 2$.

**Proof.** Let $F \in H_{\Lambda, \pi}$ with $G_{\Lambda}(z) = \frac{\sin \pi z}{\pi U(z)}$. Then $FG_{\Lambda} \in P_{W_{\pi}}$ and so

$$F(z)G_{\Lambda}(z) = \frac{F(z) \sin \pi z}{\pi U(z)} = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} \frac{F(n)}{U'(n)} \cdot \frac{1}{z - n},$$

by the classical Whittaker–Shannon–Kotelnikov formula. Hence,

$$\|F\|_{H_{\Lambda, \pi}}^2 = \left\| \frac{F \sin \pi z}{\pi U} \right\|_{P_{W_{\pi}}}^2 = \sum_{n \in \mathbb{Z}} \left| \frac{F(n)}{U'(n)} \right|^2.$$

Since $U$ is of Hamburger class, we have $G_{\Lambda} P \in P_{W_{\pi}}$ for any polynomial $P$. Hence, $P \subset H_{\Lambda, \pi}$ and for any polynomial $P \in P$,

$$(F, P)_{H_{\Lambda, \pi}} = \sum_{n \in \mathbb{Z}} \frac{F(n) \overline{P(n)}}{|U'(n)|^2} = (F, P)_{L^2(\mu_U)}.$$

Thus, polynomials are dense in $H_{\Lambda, \pi}$. \qed

The next example shows that “Codimension One Case” is also possible. This situation is much more subtle and, surprisingly, it is related to the recent result from [6] which says that the spectral synthesis for exponential systems in $L^2(-\pi, \pi)$ always holds up to one-dimensional defect.

**Example 6.2.** There exists $\Lambda$ such that polynomials belong to $H_{\Lambda, \pi}$ and have codimension one there.

**Proof.** Step 1. We will use the construction of a nonhereditarily complete system from [6]: there exist two disjoint sets $\Lambda_1$, $\Lambda_2$ such that the system $\{k_\lambda\}_{\lambda \in \Lambda_1 \cup \Lambda_2}$ is complete and minimal in $P_{W_{\pi}}$, while the mixed system

$$\mathcal{MS} = \{k_\lambda\}_{\lambda \in \Lambda_2} \cup \left\{ \frac{G_1(z)G_2(z)}{z - \lambda} \right\}_{\lambda \in \Lambda_1}$$

has codimension one in $P_{W_{\pi}}$ (and the defect of such mixed systems is always at most 1). Here $G_1, G_2$ are two canonical products with zero sets $\Lambda_1$ and $\Lambda_2$ respectively, such that $G = G_1G_2$ is the generating function of the system $\{k_\lambda\}_{\lambda \in \Lambda_1 \cup \Lambda_2}$. 


It is shown in [6] that $\mathcal{MS}$ is not complete if and only if there exist entire functions $S_1$ and $S_2$ and a sequence $\{a_n\} \in \ell^2$ such that two interpolation equations hold:

$$
\begin{align*}
G_1(z)S_1(z) &= \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{G_1(n)G_2(n)z}{z-n}a_n, \\
G_2(z)S_2(z) &= \frac{1}{\pi} \sum_{n \in \mathbb{Z}} (-1)^n a_n, \\
\end{align*}
$$

In this case the function $G_2S_2$ will belong to the orthogonal complement to the system $\mathcal{MS}$.

By construction in [6], the functions $G_1$ and $S_2$ are lacunary canonical products with real zeros. Since $G_2S_2 \in \mathcal{PW}_\pi$, it follows that $G_2 \in \mathcal{PW}_\pi$ and $G_2$ decays faster than any polynomial on $\mathbb{R}$. So, polynomials belong to the space $\mathcal{H}_{\Lambda_2, \pi}$. Also note that multiplying the first equation in (6.1) by $\frac{z^{k+1}}{G_1(z)}$ we get

$$
\frac{z^{k+1}S_1(z)}{\sin \pi z} = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \frac{G_2(n)n^{k+1}a_n}{z-n}.
$$

Letting $z = 0$ we obtain $\sum_{n \in \mathbb{Z}} G_2(n)n^k a_n = \sum_{n \in \mathbb{Z}} G_2(n)n^k \bar{G}_2(n) S_2(n) = 0$, $k \in \mathbb{N}_0$, whence $h := G_2S_2 \perp z^k G_2$ in $\mathcal{PW}_\pi$. Thus the polynomials are not dense in $\mathcal{H}_{\Lambda_2, \pi}$.

**Step 2.** Let $s_0$ be an arbitrary zero of $G_1$. We claim that

$$
\frac{G_1(z)}{(z-s_0)(z-s)} \in \text{Clos}_{\mathcal{H}_{\Lambda_2, \pi}} \mathcal{P}, \quad s \in \Lambda_1, \ s \neq s_0.
$$

Assume that (6.2) is proved. Then we can show that

$$
\dim \left( \mathcal{H}_{\Lambda_2, \pi} \ominus \text{Span}\left\{ \frac{G_1(z)}{(z-s_0)(z-s)} : \ s \in \Lambda_1, s \neq s_0 \right\} \right) = 1.
$$

Indeed, if $F \perp \left\{ \frac{G_1(z)}{(z-s_0)(z-s)} : s \in \Lambda_1, s \neq s_0 \right\}$ in $\mathcal{H}_{\Lambda_2, \pi}$, then

$$
G_2F \perp \left\{ k_\lambda \right\}_{\Lambda_2} \cup \left\{ \frac{G(z)}{(z-s_0)(z-s)} : s \in \Lambda_1, s \neq s_0 \right\}
$$

in $\mathcal{PW}_\pi$. Since the codimension of $\mathcal{MS}$ is 1, it is clear that the latter system has codimension at most 2 in $\mathcal{PW}_\pi$. Hence, the dimension of

$$
\mathcal{H}_{\Lambda_2, \pi} \ominus \text{Span}\left\{ \frac{G_1(z)}{(z-s_0)(z-s)} : s \in \Lambda_1, s \neq s_0 \right\}
$$

is either 1 or 2. By Theorem 2.9, it can be only 1.

From (6.2) and (6.3) we immediately get that $\text{Clos}_{\mathcal{H}_{\Lambda_2, \pi}} \mathcal{P}$ has codimension 1 in $\mathcal{H}_{\Lambda_2, \pi}$. By Proposition 3.5 $\Lambda_2$ is synthesizeable.

**Step 3.** It remains to prove (6.2). We will use the fact that $\tilde{G}_1 = \frac{G_1}{z-s_0}$ is a lacunary canonical product with zeros $\{s_k\}_{k=1}^\infty$. Put

$$
H_N(z) = \prod_{k=1}^N \left(1 - \frac{z}{s_k}\right), \quad N \geq 1.
$$
It is sufficient to show that the polynomials \( \frac{H_N(z)}{z - s_l}, \) \( N \geq l, \) tend to \( \frac{\hat{G}_1(z)}{z - s_l} \) in \( \mathcal{H}_{\mathbb{A}_2} \) for any \( l, \) or, equivalently, that \( \frac{G_2 H_N}{z - s_l} \) tend to \( \frac{G}{(z - s_0)(z - s_l)} \) in \( \mathcal{P} W_\pi. \) Note that, for \( N > l \) and \( x \in \mathbb{R}, \)

\[
\left| \frac{H_N(x + i)}{x + i - s_l} \right| = \left| \frac{\hat{G}_1(x + i)}{x + i - s_l} \right| \cdot \prod_{k > N} \left| 1 - \frac{x + i}{s_k} \right|^{-1} \leq \left| \frac{\hat{G}_1(x + i)}{x + i - s_l} \right| \cdot \frac{|s(x)|}{|x - s_l| + 1},
\]

where \( s(x) \) is the point from \( \{ s_k \} \) closest to \( x. \)

Since \( \left| \frac{H_N(x + i)}{x + i - s_l} \right| \lesssim |\hat{G}_1(x + i)| \) (with the constant depending on \( l, \) but uniformly with respect to \( N > l \)) and \( \hat{G}_1 G_2 \in \mathcal{P} W_\pi, \) we conclude that

\[
\int_{|x| > A} \left| \frac{H_N(x + i)}{x + i - s_l} - \frac{\hat{G}_1(x + i)}{x + i - s_l} \right|^2 |G_2(x + i)|^2 dx \lesssim \int_{|x| > A} |\hat{G}_1(x + i) G_2(x + i)|^2 dx \to 0, \quad A \to \infty.
\]

Since \( \frac{H_N(x + i)}{x + i - s_l} - \frac{\hat{G}_1(x + i)}{x + i - s_l} \) converges to zero uniformly on compact subsets, it follows that choosing first \( A \) and then a sufficiently large \( N, \) we can make the integral

\[
\int_\mathbb{R} \left| \frac{H_N(x + i)}{x + i - s_l} - \frac{\hat{G}_1(x + i)}{x + i - s_l} \right|^2 |G_2(x + i)|^2 dx
\]

as small as we wish. Since \( \| F \| _{L^2(\mathbb{R})} \asymp \| F \| _{L^2(\mathbb{R} + i)}, \) \( F \in \mathcal{P} W_\pi, \) we conclude that the polynomials \( \frac{H_N(z + i)}{z + i - s_l} \) converge to \( \frac{G_1(z)}{(z - s_0)(z - s)} \) which completes the proof of (6.2). \( \square \)

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