

Efficiency Calculation and Economic Analysis of Straight Fins with Various Shapes Based on Steady-state Heat Conduction

Jiying Chen¹, Shenghao Jiang², Jiayi Hu³ and Jiaxin Ji¹,*

¹College of New Energy, China University of Petroleum, Qingdao, Shandong, 266580, China
²School of Engineering and Applied Sciences, Harvard University, Cambridge, Massachusetts, 02138, America
³International School, Jiangxi University of Finance and Economics, Nanchang, Jiangxi, 330000, China

*Corresponding author’s e-mail: Ying1606050115@163.com

Abstract. Fins are an important means of heat transfer enhancement, which can increase the heat transfer area, reduce the heat resistance of convection heat transfer and enhance the heat transfer capacity of equipment. The analytical solutions of rectangular fins, trapezoidal fins and triangular fins in one dimension are given by using the direct solution method. Then the heat conduction of fins is discretized by the finite volume method (FVM). By solving the discrete linear equations, the numerical solutions of temperature distribution on the surface of fins with constant property and variable property are obtained. Based on the law of temperature distribution, the relationship between the length, the height and the width of fins and fin efficiency is explored, and its economic efficiency is discussed. The results show that the temperature gradient of rectangular fins is the smallest, and that of triangular fins is the most significant. For various shapes of fins, the most effective way to increase heat dissipation is to increase the width of the fin, and for the fin whose machining length is 1 unit, the efficiency of the first 20%~30% part of the maximum length changes significantly, which can be used as the focus of cost optimization.

1. Introduction

With the continuous progress of productivity, the energy problem in social development is becoming more and more serious. Energy conservation and emission reduction has become an important goal in industrial production. Although China is a country with large reserves of resources, there are still many problems to be solved urgently in energy utilization. Improving energy utilization rate is the key to production. Heat exchanger is the key equipment in the thermal system. Depending on its structural characteristics, heat transfer in different fluid media can be realized. Fins are widely used in heat exchangers, such as condensers, radiators and air heaters of refrigerating plant [1-4], which is a measure to enhance heat transfer by increasing the heat transfer area. Optimum design of fins with different shapes to improve the heat transfer efficiency of products is an effective measure to promote energy saving. The fins have various shapes. At present, the shapes of fins are mainly rectangular fins, trapezoidal fins, triangular fins and so on (Fig. 1). In special occasions, there will be circular finned tube and elliptical finned tube [5-6]. This study mainly focuses on the efficiency of rectangular fins, trapezoidal fins and triangular fins under reasonable thermodynamic assumptions in engineering.
2. Overview of physical model

Assuming that the fin is placed in the external environment with temperature $T_w$, the temperature at the end of the fin is constant $T_w$, as the first boundary condition; (2) the end of the fin is adiabatic (heat flux $q = 0$), as the second boundary condition. Because most of the fins are made of metal with good heat conduction, it is assumed that the temperature of the fins does not change in the direction of width and height, that is, the heat transfer resistance $1/h$ on the surface of the fins is much greater than the thermal-conduction resistance $\delta/\lambda$ in the fins. Therefore, the temperature of the fins can be considered uniform on any cross section. The total heat dissipation of fins can be calculated by accumulating the total heat dissipation $\phi$ of each infinitesimal element, and the heat dissipation of each infinitesimal element at the base-temperature of fins is $\phi_0$. Based on these assumptions, the research problem is simplified to one-dimensional steady-state heat conduction without internal heat source.

The formula for calculating fin efficiency $\eta$ is:

$$\eta = \frac{\phi}{\phi_0}$$  \hspace{1cm} (2-1)

3. Establishment of mathematical model

Figure 1. Side view of fins with various shapes

Figure 2. Schematic diagram of thermodynamic conditions for one-dimensional steady-state heat conduction

Figure 3. Analysis of infinitesimal element energy (rectangular, trapezoidal and triangular are all applicable. Here is the trapezoidal fin infinitesimal element.)
In order to establish a mathematical description of the temperature distribution of fins, the coordinates shown in Fig. 2 are established.

As shown in Fig. 3, under steady-state conditions, the infinitesimal element satisfies the following energy conservation:

$$\phi_x = \phi_i + \phi_{x+dx}$$  \hspace{1cm} (3-1)

By Fourier's law:

$$\phi_x = -\lambda A(x) \frac{d T}{d x}$$  \hspace{1cm} (3-2)

$$\phi_{x+dx} = \phi_x + \frac{d \phi_x}{d x}$$  \hspace{1cm} (3-3)

Where, $\lambda$: The thermal conductivity of fin materials; $A(x)$: The cross-sectional area of the fin at $x$.

According to the formula of Newton's law of cooling, the heat dissipation through the side of the fin is as follows:

$$\phi_i = h F(T - T_f)$$  \hspace{1cm} (3-4)

Where, $h$: Local surface heat transfer coefficient of fluid with temperature $T_f$ and fin surface at location $x$; $F$: Side superficial area of infinitesimal element.

And because:

$$F = U ds = U(x) \sqrt{1 + f^2} dx$$  \hspace{1cm} (3-5)

Where, $U(x)$: The perimeter of infinitesimal element at $x$; $ds$: The molded line of infinitesimal element fins; $f$: The molded line equation of fins.

Substituting (3-5) into (3-4), the following formula is obtained:

$$\phi_i = h U(x) \sqrt{1 + f^2} (T - T_f) dx$$  \hspace{1cm} (3-6)

By substituting (3-2), (3-3) and (3-6) into the energy balance equation (3-1), it is concluded that:

$$\frac{d}{dx} \left( \lambda A(x) \frac{d T}{d x} \right) - h U(x) \sqrt{1 + f^2} (T - T_f) = 0$$  \hspace{1cm} (3-7)

By introducing excess temperature $\theta = T - T_f$, the complete mathematical model is as follows:

$$\left\{ \begin{array}{l}
\frac{d}{dx} \left( \lambda A(x) \frac{d \theta}{d x} \right) - h U(x) \sqrt{1 + f^2} \theta = 0 \\
\theta |_{x=0} = T_w - T_f \\
-\lambda \frac{d \theta}{d x} |_{x=L} = 0 
\end{array} \right.$$  \hspace{1cm} (3-8)

For rectangular fins, the reference analysis solution is [7]:

$$\theta = \theta_0 \frac{ch[m(x - H)]}{ch(mH)}$$  \hspace{1cm} (3-9)

Where:

$$ch(x) = \frac{e^x + e^{-x}}{2}.$$  \hspace{1cm}

The cosine function.

To calculate trapezoidal and triangular straight fins, simply replace all the molded line expressions:

$$U(x) = 2b + 4f(x)$$  \hspace{1cm} (3-10)

Substituting (3-10) into (3-8), the following formula is obtained:
\[ 2\lambda b \frac{d}{dx} \left[ f(x) \frac{d\theta}{dx} \right] = h(2b + 4f(x))\sqrt{1 + f^2 \theta} \]  
\hspace{1cm} (3-11)

Concluded as:
\[ f(x) \frac{d^2\theta}{dx^2} + f'(x) \frac{d\theta}{dx} = \frac{h(b + 2f(x))}{\lambda b} \sqrt{1 + f^2 \theta} \]  
\hspace{1cm} (3-12)

Assuming \( \theta = \theta_1 = e^{\int g(x)dx} \)

Making
\[ \phi(x) = \frac{h(b + 2f(x))}{\lambda b} \sqrt{1 + f^2} \]  
\hspace{1cm} (3-13)

Concluded as:
\[ g'(x) + \frac{f'(x)}{f(x)} g(x) + [g(x)]^2 = \frac{\phi(x)}{f(x)} \]  
\hspace{1cm} (3-14)

The solution of equation (2-14) must contain two linear independent special solutions. By using undetermined coefficient method and constant variation method, the general solution of (2-14) is obtained as follows:

\[ \theta(x) = (1 + C_1) e^{\int \frac{1}{f(x)} \left[ \phi(x) dx + C_2 \right]} \int \frac{1}{f(x)} \left[ \frac{dx}{\phi(x) dx + C_2} \right] dx \]  
\hspace{1cm} (3-15)

4. The numerical solution

4.1 Computational domain and equation discretization

The inner node method is used to discretize the computational domain, and the discretized object of the equation is the energy conservation equation of the infinitesimal element (3-7). Take the control volume as shown in Figure 4. Integrate the control volume \( P \):

![Figure 4. The selection of control volume](image)

\[ \int \frac{d}{dx} \left( \lambda A(x) \frac{dT}{dx} \right) dx = \int hU(x) \sqrt{1 + f^2 (T - T_f)} dx \]  
\hspace{1cm} (4-1)

Suppose that the \( T \) molded line is linear [8], then (4-1) on the left side is:
Assuming that $T$ is a stepped change in space 

$$
\int \frac{d}{dx} \left( \lambda A(x) \frac{dT}{dx} \right) dx = \left( \lambda A(x) \frac{dT}{dx} \right)_w - \left( \lambda A(x) \frac{dT}{dx} \right)_e = \lambda A(x) \left( T_E - T_P \right) \frac{1}{(\partial x)_e} - \lambda A(x) \frac{T_E - T_P}{(\partial x)_e} \quad (4-2)
$$

By substituting (4-2) into (4-1), we get:

$$
\int \frac{h U(x)}{\sqrt{1 + f^2}} (T - T_i) dx = \int \frac{h U(x)}{\sqrt{1 + f^2}} (T - T_f) dx \quad (4-3)
$$

By substituting (4-2) into (4-1), we can get:

$$
\left( \frac{\lambda A_i}{(\partial x)_w} \right) + \frac{\lambda A_e}{(\partial x)_e} + \int \frac{h U(x)}{\sqrt{1 + f^2}} dx \frac{1}{(\partial x)_w} \left( T_p \right)_w + \frac{\lambda A_e}{(\partial x)_e} \left( T_p \right)_e + h \int \frac{h U(x)}{\sqrt{1 + f^2}} dx \frac{1}{(\partial x)_e} \left( T_f \right)_f \quad (4-4)
$$

Formula (4-4) is rearranged into

$$
a_{E_P} T_p = a_w T_w + a_E T_E + h \int \frac{h U(x)}{\sqrt{1 + f^2}} dx \quad (4-5)
$$

The temperature distribution can be obtained by solving the linear equations (4-5) obtained by the discrete method. Actual heat dissipation:

$$
\phi = \sum_{i=2}^{N-1} h F(x_i) \left( T_i - T_f \right) \quad (4-6)
$$

The heat dissipation of the whole fin at the base-temperature of fins:

$$
\phi_b = \phi = \sum_{i=2}^{N-1} h F(x_i) \left( T_i - T_w \right) \quad (4-7)
$$

5. Analysis of solution results

Assuming that the fins are adiabatic at the end and are placed in the air, the convection heat transfer coefficient on the surface of the fins is $10.55 \frac{W}{m^2 \cdot K}$, the ambient air temperature is $62^\circ C$, the length of the fins perpendicular to the vertical wall is $L=0.10m$, the width of the fins is $b=0.01m$, and the constant temperature at the joints is $T_w=80^\circ C$. Fig.5 shows the temperature of rectangular, trapezoidal and triangular fins varying with position.

![Figure 5. Temperature variation of rectangular, trapezoidal and triangular fins](image-url)
As can be seen from Fig. 5, under the same thermodynamic conditions, the temperature of rectangular fins decreases most slowly along the length of fins, while that of triangular fins decreases most quickly. And the temperature drop rate of triangular fins tends to increase. For components requiring rapid heat transfer, triangular fin is a more preferable choice.

Take the length of fins from zero to one meter as research object. Define the heat dissipation efficiency coefficient \( e_i \) corresponding to node \( i \) in numerical calculation:

\[
 e_i = \frac{\varphi_i}{\varphi_{\text{max}[1,2,3...n]}}
\]  

(5-1)

\( e_i \) denotes the ratio of the heat dissipation of corresponding control volume of each node to the maximum heat dissipation of all nodes. This method of comparing each value with regional maximum to normalize can be used to denote the heat conduction of the corresponding infinitesimal element relative to the whole fin after each segment of fin length is added. The data is plotted in Fig. 6.

From Fig. 6, it can be seen that the fin efficiency increases with the increase of fin length. To further illustrate this trend, the marginal cost theory in economics is more appropriate. In economics and finance, marginal cost refers to the cost that a manufacturer increases per unit of output. This concept indicates that the cost per unit of product is related to the total volume of product. Mathematically, marginal cost (MC) is expressed as partial derivative \( \frac{\partial T C}{\partial Q} \) of total cost (TC) and quantity (Q) [9-10]. For the research of fins, heat dissipation and fin efficiency are our concerned research objectives, and fin length is the "product" that we can increase or decrease freely. Therefore, the coefficient \( \zeta \) can be defined:

\[
\zeta = \frac{\Delta \varphi}{\Delta L}
\]  

(5-2)

Where: \( \varphi \) is the heat dissipation of infinitesimal element, and \( L \) is the length of fins. \( \zeta \) essentially represents the slope of the curve in Fig. 6.

Figure 6. Temperature variation of rectangular, trapezoidal and triangular fins
According to the change trend of image slope: increasing the length of fins at the initial stage can significantly increase heat dissipation and maintain high material efficiency. When the length is increased to about 0.3 m, the change of heat dissipation will slow down when the length is increased. In order to find the appropriate balance point between fin efficiency and heat dissipation, the length of fins should be selected between 0.2 and 0.4m in the image. The intersection point of heat dissipation and fin efficiency curve is also a reasonable choice in this case. The corresponding normalized heat dissipation and fin efficiency at the intersection point are both more than 70%. The relations between the fin efficiency of rectangular, trapezoidal and triangular fins and the width and height of the fins are shown in Fig. 7 and Fig. 8.

![Figure 7. The relationship between fin efficiency and the width of fins](image1)

![Figure 8. The relationship between fin efficiency and the height of fins](image2)

It can be seen from Fig. 7 and Fig. 8 that the increase of the height and width of fins is accompanied by the increase of thermal efficiency. Compared Fig. 6, Fig. 7 and Fig. 8, the effect of increasing the height of fins on efficiency is the most significant. Similar to the variation law of the length of fins, increasing the height and width of fins also has the phenomenon of marginal cost decreasing. Especially in the study of fin width, after experiencing short and steep efficiency
fluctuation, increasing fin width has little effect on the fin efficiency of all kinds of fins. The efficiency of rectangular fins is the highest under the same fin height compared with the other two.

6. Conclusion

1) The temperature of the fins decreases with the length of the fins, and the speed of the decrease varies with the shape of fins. Compared with rectangular and trapezoidal fins, triangular fins have the fastest temperature drop rate. When the fin is long enough, the temperature of the fin will drop to the fluid temperature. By analyzing the molded line equation, it can be further summarized: with the increase of the absolute value of the slope of fin molded line, the temperature of fins decreases faster.

2) With the increase of the length of fins, the fin efficiency decreases, and the decreasing speed is getting slower and slower. For different shapes of fins, changing the length of fins, the decreasing speed of the fin efficiency increases with the increase of absolute value of fin molded line. From Fig. 7, it can be seen that with the increase of the width of fins, the fin efficiency increases rapidly and finally approaches one. For the same width of fins, the efficiency of rectangular fin is the highest, trapezoidal fin is the second, and triangular fin is the smallest. Fig. 8 shows that with the increase of the height of fins, the fin efficiency will increase rapidly, and finally approaches one. For the same height of fins, the efficiency of triangular fin is less than that of rectangular fin.

3) From Figs. 6, 7 and 8, it can be seen that the heat dissipation of fins increases with the increase of the length, height and width of fins, but the fin efficiency decreases with the increase of the length of fins. Compared Figs. 7 and Fig. 8, the most effective way to increase the heat dissipation of both rectangular fins and triangular fins is to increase the width of fins.

References

[1] W.T. Ji., C.Y. Zhao, J. Lofton., Z.Y. Li, D.C., Zhang, Y.L. He, W.Q. Tao (2018) Condensation of r134a and r22 in shell and tube condensers mounted with high-density low-fin tubes. Journal of Heat Transfer, 140:091503

[2] N. Lewpiriyawong, K. L. Khoo, C. Sun, P. S. Lee. (2019) Thermal and hydraulic analysis of aluminium oblique-tube condenser coils with plain fins manufactured by controlled atmosphere brazing. International Journal of Refrigeration, 101:81-89

[3] Priyam A, Chand P (2019) Experimental investigations on thermal performance of solar air heater with wavy fin absorbers. Heat and Mass Transfer, 55:2651–2666

[4] Sathish Kumar, T. R., Jegadheeswaran, S., Chandramohan, P. (2018). Performance investigation on fin type solar still with paraffin wax as energy storage media. Journal of Thermal Analysis and Calorimetry, 136:101–112

[5] Abdulkerim O., Ali P., Ali B. Olcay, M., Hilmi A (2018) An experimental, computational and flow visualization study on the air-side thermal and hydraulic performance of louvered fin and round tube heat exchangers. International Journal of Heat and Mass Transfer, 121:153-169

[6] S. S., Yogesh, A.S. Selvaraj., D.K. Ravi., T.K.R. Rajagopal (2018) Heat transfer and pressure drop characteristics of inclined elliptical fin tube heat exchanger of varying ellipticity ratio using CFD code. International Journal of Heat and Mass Transfer, 2018, 119:26-39

[7] Shiming Yang, Wenquan Tao(2006)Heat Transfer. Higher Education Press, Beijing

[8] Payam H., Hassan K. B., Mehdi M., Navid B., Emad H. M.(2019) Numerical modeling of nanofluid flow and heat transfer in a quartered gearwheel-shaped heat exchanger using FVM. Chinese Journal of Physics, 59:591-605

[9] Siverskog, J., Henriksson, M. (2019). Estimating the marginal cost of a life year in sweden’s public healthcare sector. The European Journal of Health Economics.20: 1-12.

[10] Bryon J.P, Allen M.F. (2019) A comparison of parametric and nonparametric estimation methods for cost frontiers and economic measures. Journal of Applied Economics 22:59-84