NOTE ON FROBENIUS MONOIDAL FUNCTORS

BRIAN DAY AND CRAIG PASTRO

ABSTRACT. It is well known that strong monoidal functors preserve duals. In this short note we show that a slightly weaker version of functor, which we call “Frobenius monoidal”, is sufficient.

The idea of this note became apparent from Prop. 2.8 in the paper of R. Rosebrugh, N. Sabadini, and R.F.C. Walters [4]. Throughout suppose that $\mathcal{A}$ and $\mathcal{B}$ are strict 1-monoidal categories.

Definition 1. A Frobenius monoidal functor is a functor $F : \mathcal{A} \to \mathcal{B}$ which is monoidal $(F, r, r_0)$ and comonoidal $(F, i, i_0)$, and satisfies the compatibility conditions

$$ir = (1 \otimes r)(i \otimes 1) : FA \otimes F(B \otimes C) \to F(A \otimes B) \otimes FC,$$

$$ir = (r \otimes 1)(1 \otimes i) : F(A \otimes B) \otimes FC \to FA \otimes F(B \otimes C),$$

for all $A, B, C \in \mathcal{A}$.

The compact case ($\otimes = \oplus$) of Cockett and Seely’s linearly distributive functors [2] are precisely Frobenius monoidal functors, and Frobenius monoidal functors with $ri = 1$ have been called split monoidal by Szlachányi in [5].

A dual situation in $\mathcal{A}$ is a tuple $(A, B, e, n)$, where $A$ and $B$ are objects of $\mathcal{A}$ and

$$e : A \otimes B \to I$$

$$n : I \to B \otimes A$$

are morphisms in $\mathcal{A}$, called evaluation and coevaluation respectively, satisfying the “triangle identities”:

$$1 \otimes n : A \otimes B \otimes A \to A \otimes B$$

$$\varepsilon \otimes 1 : A \otimes B \otimes A \to A$$

$$n \otimes 1 : B \otimes A \otimes B \to B$$

$$1 \otimes \varepsilon : B \otimes A \otimes B \to B.$$

Theorem 2. Frobenius monoidal functors preserve dual situations.

This theorem is actually a special case of the fact that linear functors (between linear bicategories) preserve linear adjoints [1].
Proof. Suppose that \((A, B, e, n)\) is dual situation in \(\mathcal{A}\). We will show that \((FA, FB, e, n)\), where \(e\) and \(n\) are defined as

\[
ed = \left( F(A \otimes FB) \xrightarrow{r} F(A \otimes B) \xrightarrow{Fe} FI \xrightarrow{i_0} I \right)
\]

\[
n = \left( I \xrightarrow{r_0} FI \xrightarrow{F_n} F(B \otimes A) \xrightarrow{i} FB \otimes FA \right),
\]

is a dual situation in \(\mathcal{B}\).

The following diagram proves one of the triangle identities.

\[
\begin{array}{cccc}
FA & \xrightarrow{1 \otimes r_0} & FA \otimes FI & \xrightarrow{1 \otimes F_n} & FA \otimes F(B \otimes A) & \xrightarrow{1 \otimes i} & FA \otimes FB \otimes FA \\
& \downarrow r & & \downarrow r & & \downarrow (\ddagger) & & \downarrow r \otimes 1 \\
F(A \otimes I) & & F(A \otimes B \otimes A) & & F(A \otimes B) \otimes FA & & F(1) \otimes FA \\
& \downarrow 1 & & \downarrow 1 & & \downarrow F(e \otimes 1) & & \downarrow F(e \otimes 1) \\
& & F(I \otimes A) & & F(I \otimes FA) & & F(I) \otimes FA \\
& & \downarrow i & & \downarrow i_0 \otimes 1 & & \downarrow i_0 \otimes 1 \\
& & FA & & & & &
\end{array}
\]

The square labelled by (\ddagger) requires the second Frobenius condition. We remark that to prove the other triangle identity is similar and requires the first Frobenius condition. \(\square\)

Proposition 3. Any strong monoidal functor is a Frobenius monoidal functor.

Proof. Recall that a strong monoidal functor is a monoidal functor and a comonoidal functor for which \(r = i^{-1}\) and \(r_0 = i_0^{-1}\). The commutativity of the following diagram proves one of the Frobenius conditions.

\[
\begin{array}{cccc}
F(A \otimes B) \otimes FC & \xrightarrow{i \otimes 1} & FA \otimes FB \otimes FC & \\
& \downarrow r & & \downarrow 1 \otimes r \\
F(A \otimes B \otimes C) & \xrightarrow{i} & FA \otimes F(B \otimes C) & \\
\end{array}
\]

The other is similar. \(\square\)

Proposition 4. The composite of Frobenius monoidal functors is a Frobenius monoidal functor.

Proof. Suppose that \(F : \mathcal{A} \rightarrow \mathcal{B}\) and \(G : \mathcal{B} \rightarrow \mathcal{C}\) are Frobenius monoidal functors. It is well known and easy to see that the composite of monoidal (resp. comonoidal) functors is monoidal (resp. comonoidal). We therefore need only
prove the Frobenius conditions, one of which follows from the commutativity of
\[ GF(A \otimes B) \otimes GFC \xrightarrow{r} G(F(A \otimes B) \otimes FC) \xrightarrow{Gr} GF(A \otimes B \otimes C) \]
where the square labelled by (‡) uses the Frobenius property of \( F \), and the square
labelled by ($) uses the Frobenius property of \( G \).

The other Frobenius condition follows from a similar diagram. \hfill \Box

It is not too difficult to see that a Frobenius monoidal functor \( F : 1 \rightarrow A \) is a
Frobenius algebra in \( A \). Therefore, we have the following corollary.

**Corollary 5.** Frobenius monoidal functors preserve Frobenius algebras. That is, if
\( R \) is a Frobenius algebra in \( A \) and \( F : A \rightarrow B \) is a Frobenius functor, then \( FR \)
is a Frobenius algebra in \( B \).

**Example 6.** Suppose that \( A \) is a braided monoidal category. If \( R \in A \) is a
Frobenius algebra in \( \mathcal{A} \), then \( F = R \otimes - : A \rightarrow A \) is a Frobenius monoidal
functor. The monoidal structure \((F, r, r_0)\) is given by
\[
\begin{align*}
    r_{A,B} &= \quad ( R \otimes A \otimes R \otimes B \quad \delta \otimes 1 \otimes 1 \quad R \otimes R \otimes A \otimes B \quad \mu \otimes 1 \otimes 1 \\
    r_0 &= \quad ( I \quad \eta \quad R )
\end{align*}
\]
and the comonoidal structure \((F, i, i_0)\) by
\[
\begin{align*}
    i_{A,B} &= \quad ( R \otimes A \otimes B \quad \epsilon \otimes 1 \otimes 1 \quad R \otimes R \otimes A \otimes B \quad 1 \otimes \epsilon \otimes 1 \\
    i_0 &= \quad ( R \quad \epsilon \quad I )
\end{align*}
\]
The Frobenius conditions now follow easily from the properties of Frobenius algebras.

This example shows that Frobenius monoidal functors generalize Frobenius algebras much in the same way that monoidal comonads, or comonoidal monads, generalize bialgebras.

The following proposition is a generalization of the fact that morphisms of Frobenius algebras (morphisms which are both algebra and coalgebra morphisms) are isomorphisms. It also generalizes the result that monoidal natural transformations between strong monoidal functors with (left or right) compact domain are invertible.

**Proposition 7.** Suppose that \( F, G : A \rightarrow B \) are Frobenius monoidal functors and
that \( \alpha : F \rightarrow G \) is a monoidal and comonoidal natural transformation. If \( A \in A \)
is part of a dual situation, i.e., \((A, B, e, n)\) or \((B, A, e, n)\) is a dual situation, then
\( \alpha_A : FA \rightarrow GA \) is invertible.
Proof. We shall assume that \( A \) is part of the dual situation \((A, B, e, n)\). The other case is treated similarly. The component \( \alpha_B : FB \to GB \) has mate

\[
GA \longrightarrow GA \otimes FB \otimes FA \longrightarrow GA \otimes GB \otimes FA \longrightarrow GA \otimes GB \otimes FA \longrightarrow GA
\]

which we will show is the inverse to \( \alpha_A \).

If \( \alpha \) is both monoidal and comonoidal then the diagrams

\[
\begin{align*}
F A \otimes F B & \xrightarrow{\alpha_A \otimes \alpha_B} GA \otimes GB \\
F(A \otimes B) & \xrightarrow{\alpha_A \otimes \alpha_B} G(A \otimes B) \\
F I & \xrightarrow{F \alpha} GI
\end{align*}
\]

\[
\begin{align*}
F B \otimes FA & \xrightarrow{\alpha_A \otimes \alpha_B} GB \otimes GA \\
F(B \otimes A) & \xrightarrow{\alpha_A \otimes \alpha_B} G(B \otimes A) \\
F I & \xrightarrow{F \alpha} GI
\end{align*}
\]

commute. The following diagrams prove that \( \alpha_A \) is invertible. The first diagram above says exactly that the triangle labelled by \((£)\) below commutes. The second diagram above that the triangle labelled by \((¥)\) below commutes.

\[
\begin{align*}
F A \otimes F B \otimes FA & \xrightarrow{\alpha \otimes 1 \otimes 1} GA \otimes FB \otimes FA \\
F A \otimes F B \otimes FA & \xrightarrow{1 \otimes n} GA \otimes FB \otimes FA \\
F A \otimes F B \otimes FA & \xrightarrow{\alpha \otimes 1 \otimes 1} GA \otimes FB \otimes FA \\
F A \otimes F B \otimes FA & \xrightarrow{e \otimes 1} GA \otimes GB \otimes FA \\
F A & \xrightarrow{1 \otimes n} GA \otimes FB \otimes FA \\
F A & \xrightarrow{\alpha} FA \\
F A & \xrightarrow{1 \otimes n} GA \otimes FA \\
F A & \xrightarrow{\alpha} FA
\end{align*}
\]

Denote by \( \text{Frob}(\mathcal{A}, \mathcal{B}) \) the category of Frobenius monoidal functors from \( \mathcal{A} \) to \( \mathcal{B} \) and all natural transformations between them.

**Proposition 8** (cf. [4] Prop. 2.10). If \( \mathcal{B} \) is a braided monoidal category, then \( \text{Frob}(\mathcal{A}, \mathcal{B}) \) is a braided monoidal category with the pointwise tensor product of functors.

Proof. Consider the pointwise tensor product of Frobenius monoidal functors \( F, G : \mathcal{A} \to \mathcal{B} \). That is,

\[
(F \otimes G)A = FA \otimes GA.
\]

It is obviously an associative and unital tensor product with unit \( I(A) = I \) for all \( A \in \mathcal{A} \).
We may define morphisms as follows:

\[ r = (r \otimes r)(1 \otimes c^{-1} \otimes 1) : (F \otimes G) A \otimes (F \otimes G) B \rightarrow (F \otimes G)(A \otimes B) \]

\[ r_0 = r_0 \otimes r_0 : I \rightarrow (F \otimes G) I \]

\[ i = (1 \otimes c \otimes 1)(i \otimes i) : (F \otimes G)(A \otimes B) \rightarrow (F \otimes G) A \otimes (F \otimes G) B \]

\[ i_0 = i_0 \otimes i_0 : (F \otimes G) I \rightarrow I. \]

That these morphisms provide a monoidal and a comonoidal structure on \( F \otimes G \) is not too difficult to show, and is omitted here. The following diagram proves the first Frobenius condition, where the \( \otimes \) symbol has been removed as a space spacing mechanism.

The bottom left square commutes by the Frobenius condition, and the others by properties of the braiding. The second Frobenius condition follows from a similar diagram. So, \( F \otimes G \) is a Frobenius monoidal functor.

The braiding \( c_{F,G} : F \otimes G \rightarrow G \otimes F \) is given on components by

\[ (c_{F,G})_A = c_{F,G,A} : F A \otimes F A \rightarrow F A \otimes F A. \]

Corollary 9. If \( \mathcal{B} \) is a braided monoidal category and \( \mathcal{A} \) is a self-dual compact category, meaning that for any object \( A \in \mathcal{A} \), \( (A, A, e, n) \) is a dual situation in \( \mathcal{A} \), then \( \text{Frob}(\mathcal{A}, \mathcal{B}) \) is a self-dual braided compact category.

Proof. By Theorem 2 Frobenius monoidal functors preserve duals, and therefore, for any \( A \in \mathcal{A} \), \( (FA, FA, e, n) \) is a dual situation in \( \mathcal{B} \). □

Recall that, if \( \mathcal{A} \) is a small monoidal category, and if small colimits exist and commute with the tensor product in \( \mathcal{B} \), then the equations

\[ F * G = \int^{A,B} \mathcal{A}(A \otimes B, -) \cdot FA \otimes FB \]

\[ J = \mathcal{A}(I, -) \cdot I, \]

where \( \cdot \) denotes copower, describe the convolution monoidal structure on the functor category \( \mathcal{A}(\mathcal{B}) \) (cf. [2]). Then we have:

Theorem 10. If \( \mathcal{A} \) is a small monoidal category and \( \mathcal{B} \) is a monoidal category having all small colimits commuting with tensor, then any Frobenius monoidal functor
$F : \mathcal{A} \rightarrow \mathcal{B}$ for which the canonical evaluation morphism

$$\int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \rightarrow F$$

is an isomorphism, becomes an algebra with a comultiplication which satisfies the Frobenius identities in the convolution functor category $[\mathcal{A}, \mathcal{B}]$.

Note that, by the Yoneda lemma, the equation (2) is satisfied by all the functors $F : \mathcal{A} \rightarrow \mathcal{B}$ if $\mathcal{A}$ is a closed monoidal category and the canonical evaluation morphism

$$\int^{B,C} \mathcal{A}(A, B \otimes C \otimes [B \otimes C, -]) \rightarrow \mathcal{A}(A, -)$$

is an isomorphism for all $A \in \mathcal{A}$.

Before we prove Theorem 10 we will need the following lemma.

**Lemma 11.** Assuming equation (2) in Theorem 10, we may also derive the two variable version, that is, that the canonical evaluation morphism

$$\int^{A,B} \mathcal{A}(A \otimes B, -) \cdot F(A \otimes B) \rightarrow F$$

is an isomorphism.

**Proof.** The canonical evaluation morphism

$$\int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \rightarrow \int^{A,B} \mathcal{A}(A \otimes B, -) \cdot F(A \otimes B)$$

is a retraction of (either of the canonical morphisms in the opposite direction), say, $k$. We may compose the canonical morphism

$$\int^{A,B} \mathcal{A}(A \otimes B, -) \cdot F(A \otimes B) \rightarrow F$$

with the isomorphism

$$F \cong \int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C)$$

of our assumption to get a morphism

$$\int^{A,B} \mathcal{A}(A \otimes B, -) \cdot F(A \otimes B) \rightarrow \int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C),$$
which makes the diagram

\[
\begin{array}{c}
\int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \\
\downarrow h
\\
\mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \rightarrow \int^{A,B} \mathcal{A}(A \otimes B, -) \cdot F(A \otimes B)
\\
\downarrow
\\
\int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C)
\end{array}
\]

commute. Therefore \(lh = 1\). We have \(hl = hlhk = hk = 1\) so \(l\) is an isomorphism, hence the canonical evaluation morphism

\[
\int^{A,B} \mathcal{A}(A \otimes B, -) \cdot F(A \otimes B) \rightarrow F
\]

is an isomorphism. \(\square\)

A consequence of Lemma 11 is that we may write

\[
F^* F = \int^{X,C} \mathcal{A}(X \otimes C, -) \cdot FX \otimes FC
\]

\[
\cong \int^{X,C} \mathcal{A}(X \otimes C, -) \cdot \left( \int^{A,B} \mathcal{A}(A \otimes B, X) \cdot F(A \otimes B) \right) \otimes FC
\]

\[
\cong \int^{X,A,B,C} \left( \mathcal{A}(X \otimes C, -) \times \mathcal{A}(A \otimes B, X) \right) \cdot (F(A \otimes B) \otimes FC)
\]

\[
\cong \int^{X,A,B,C} \left( \int^{X} \mathcal{A}(X \otimes C, -) \times \mathcal{A}(A \otimes B, X) \right) \cdot (F(A \otimes B) \otimes FC)
\]

\[
\cong \int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B) \otimes FC,
\]

(Yoneda)

and similarly,

\[
F^* F \cong \int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot FA \otimes F(B \otimes C).
\]

Proof of Theorem 10. Using the isomorphisms of equation (5) and Lemma 11 one of the Frobenius equations may be written as

\[
\int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B) \otimes FC \xrightarrow{1 \otimes i \otimes 1} \int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot FA \otimes FB \otimes FC
\]

\[
\int 1 \otimes r
\]

\[
\int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \xrightarrow{1 \otimes i} \int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot FA \otimes F(B \otimes C).
\]

\[
\int 1 \otimes r
\]
This diagram is seen to commute as $F$ is a Frobenius monoidal functor. The other Frobenius equation follows from a similar diagram.

To prove the second part of the theorem, assume that $\mathcal{A}$ is a closed monoidal category and that equation (♯) holds. The following calculation verifies the claim.

\[
\int^{A,B,C} \mathcal{A}(A \otimes B \otimes C, -) \cdot F(A \otimes B \otimes C) \\
\cong \int^{A,B,C} \mathcal{A}(C, [A \otimes B, -]) \cdot F(A \otimes B \otimes C) \quad (\mathcal{A} \text{ closed}) \\
\cong \int^{A,B} F(A \otimes B \otimes [A \otimes B, -]) \quad \text{(Yoneda)} \\
\cong \int^{X,A,B} \mathcal{A}(X, A \otimes B \otimes [A \otimes B, -]) \cdot FX \quad \text{(Yoneda)} \\
\cong \int^{X} \mathcal{A}(X, -) \otimes FX \quad \text{(♯)} \\
\cong F \quad \text{(Yoneda)}
\]

□

References

[1] J. R. B. Cockett, J. Koslowski and R. A. G. Seely. Introduction to linear bicategories, Math. Struct. Comp. Science 10 no. 2 (2000): 165–203.
[2] J. R. B. Cockett and R. A. G. Seely. Linearly distributive functors, J. Pure Appl. Algebra 143 (1999): 155–203.
[3] Brian Day. On closed categories of functors, in Reports of the Midwest Category Seminar IV, Lecture Notes in Mathematics 137 (1970): 1–38.
[4] R. Rosebrugh, N. Sabadini and R.F.C. Walters. Generic commutative separable algebras and cospans of graphs, Theory Appl. Categories 15 (2005): 164–177.
[5] Kornél Szlachányi. Finite quantum groupoids and inclusions of finite type, Fields Inst. Comm. 30 (2001): 393–407.

Department of Mathematics, Macquarie University, New South Wales 2109 Australia

E-mail address: craig@ics.mq.edu.au