On Nörlund summability of double Fourier series

Suresh Kumar Sahani$^{1,2,*}$, Vishnu Narayan Mishra$^3$ and Laxmi Rathour$^3$

$^1$ Department of Mathematics, MIT Campus, T.U, Janakpur, 45600, Nepal.
$^2$ Department of Mathematics, Rajarshi Janak Campus, T.U, Janakpur, Nepal.
$^3$ Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India.

* Correspondence: sureshkumarsahani35@gmail.com

Received: 12 March 2021; Accepted: 14 January 2022; Published: 21 June 2022.

Abstract: In this research paper, the authors studied some problems related to harmonic summability of double Fourier series on Nörlund summability method. These results constitute substantial extension and generalization of related work of Moricz [1] and Rhodes et al., [2]. We also constructed a new result on $(N,p_1^{(1)},p_2^{(2)})$ by regular Nörlund method of summability.

Keywords: Nörlund summability; Double Fourier series; Double matrix summability.

MSC: 40G05; 42B05; 42B08.

1. Introduction

Let $f(\alpha, \beta)$ be Lebesgue integral in the square $R(-\pi, \pi; -\pi, \pi)$ and be of period $2\pi$ in each of the variables $\alpha$ and $\beta$. Then the series

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \gamma_{mn} \left\{ r_{mn} \cos m\alpha \cos n\beta + S_{mn} \sin m\alpha \cos n\beta + t_{mn} \cos m\alpha \sin n\beta + q_{mn} \sin m\alpha \sin n\beta \right\}$$

is called the double Fourier series associated with the function $f(\alpha, \beta)$; (see [2,3]) where

$$\gamma_{mn} = \begin{cases} \frac{1}{4}, & \text{for } m = 0, n = 0; \\ \frac{1}{2}, & \text{for } m = 0, n > 0 \text{ or } m > 0, n = 0; \\ 1, & \text{for } m, n > 0, \end{cases}$$

$$r_{mn} = \frac{1}{\pi^2} \int_R f(\alpha, \beta) \cos m\alpha \cos n\beta \, d\alpha \, d\beta,$$

$$S_{mn} = \frac{1}{\pi^2} \int_R f(\alpha, \beta) \sin m\alpha \cos n\beta \, d\alpha \, d\beta,$$

$$t_{mn} = \frac{1}{\pi^2} \int_R f(\alpha, \beta) \cos m\alpha \sin n\beta \, d\alpha \, d\beta,$$

$$q_{mn} = \frac{1}{\pi^2} = \int_R f(\alpha, \beta) \sin m\alpha \sin n\beta \, d\alpha \, d\beta.$$

Also, we have

$$\chi(\alpha, \beta) = \chi_{x,y}(\alpha, \beta) = \frac{1}{4} \left\{ f(x + \alpha, y + \beta) + f(x - \alpha, y + \beta) + f(x + \alpha, y - \beta) + f(x - \alpha, y - \beta) - 4f(\alpha, \beta) \right\}.$$  

Definition 1. [4,5] Let $\{p_m^{(1)}\}$ and $\{p_n^{(2)}\}$ are two sequence of constants, real or complex. Let

$$\begin{cases} p_m^{(1)} = p_0^{(1)} + p_1^{(1)} + p_2^{(1)} + \ldots + p_m^{(1)}, \\ p_n^{(2)} = p_0^{(2)} + p_1^{(2)} + p_2^{(2)} + \ldots + p_n^{(2)}. \end{cases}$$

Open J. Math. Sci. 2022, 6, 99-107; doi:10.30538/oms2022.0180 https://pisrt.org/psr-press/journals/oms
We shall also consider a double Nörlund transform of \( \{a_{mn}\} \). Then the double Nörlund transform is

\[
V_{mn} = \frac{1}{p_m(1)p_n(2)} \sum_{l=0}^{m} \sum_{g=0}^{n} p_{m-l} p_{n-g} a_{lg}.
\]  

(9)

**Definition 2.** [4,5] The double sequence \( \{a_{lg}\} \) is said to be Nörlund summable to a limit \( v \) if

\[
V_{mn} \to V, (m,n) \to (\infty,\infty).
\]

(10)

It is also known as summable \((N, p_m(1), p_n(2))\).

**Definition 3.** [1,2,4–6] If

\[
\begin{align*}
p_m(1) &= 1, \quad \text{for } m = 0, 1, 2, \ldots; \\
p_n(2) &= 1, \quad \text{for } m = 0, 1, 2, \ldots,
\end{align*}
\]

(11)

then the double Nörlund transform reduces to double Cesàro transform of order one. This summability method is known as Cesàro summability \((C, 1, 1)\).

**Definition 4.** [1,2,5,6] If

\[
p_m(1) = \frac{1}{m+1}; m = 0, 1, 2, \ldots \quad \text{and} \quad p_n(2) = \frac{1}{n+1}, n = 0, 1, 2, \ldots,
\]

then the double Nörlund summability \((N, p_m(1), p_n(2))\) becomes Harmonic summability and is denoted by \((H, 1, 1)\).

**Definition 5.** [5] If for any \( \gamma \geq 1 \),

\[
V_{mn} \to V, (m,n) \to (-\infty, \infty)\]  

in such a manner that \( \gamma \geq \frac{m}{n}, \gamma \geq \frac{n}{m} \), then the sequence \( \{a_{lg}\} \) is said to be restrictedly summable \( N_p \) at \((x,y)\) to the same limit.

2. Known results

In 1953, Chow [7], for the first time studied Cesàro summability of double Fourier series. In 1958, Sharma [3] extended the results of Chow for \((H, 1, 1)\) summability which is weaker than \((C, 1, 1)\) summability of double Fourier series. There are several results on Nörlund summability of Fourier series [1–10]. This motivates us to study on the Nörlund summability of Fourier series in more generalized as particular cases. Therefore, in an attempt to make an advance in this research work, we study the double Fourier series and its conjugate series by Nörlund method. In 1963, Sing [11] proved the following theorem;

**Theorem 1.** If

\[
\int_{0}^{V} |\chi(y)| dy = O \left( \frac{v}{\log v - \Gamma} \right),
\]

(12)

where \( \chi(y) = f(v+y) + f(v-y) - 2f(y) \) as \( v \to 0 \), then the Fourier series of \( f(\cdot) \) at \( v=y \) is summable \((N, p_n)\) to \( f(y) \) where \( \{p_n\} \) is a real non-negative, non-increasing sequence such that

\[
\sum_{a=2}^{n} \left( \frac{p_a}{a \log a} \right) = O \left( P_n \right).
\]

(13)

Dealing with this topics, Pati [12] proved the following theorem;

**Theorem 2.** If \((N, p_n)\) be a regular Nörlund methods defined by a positive, monotonic decreasing sequences of coefficients \( \{p_n\} \) such that

\[
\log n = O(P_n), P_n \to \infty, \text{as } n \to \infty,
\]

(14)
then if
\[ \zeta(x) = \int_0^x |\chi(y)| \, dy = \left[ \frac{z}{P(\frac{1}{2})} \right] \]
(15)
as \( z \to 0^+ \), then the given Fourier series \( \chi(z) \) of \( z = y \) is summable \((N, p_n)\) to \( \chi(y) \).

3. Main Results

In this present research paper, we have established the following theorems which are the extended form of [2,11].

**Theorem 3.** If \((\alpha, \beta) \to (0, 0)\),
\[
\int_0^a \int_0^\beta |\chi(s,t)| \, ds \, dt = o \left( \frac{\alpha}{\log \alpha} \right), \quad \int_0^a \int_0^\beta |\chi(s,t)| \, dt \, ds = o \left( \frac{\beta}{\log \beta} \right),
\]
(16)
(17)
and
\[
\int_0^a \int_0^\beta |\chi(s,t)| \, ds \, dt = o \left( \frac{\alpha}{\log \alpha} \right), \quad (0 < \delta < \pi),
\]
(18)
then the double Fourier series of \( f(\alpha, \beta) \) at \( \alpha = x \) and \( \beta = y \) is summable \((N, p_n^{(1)}, p_n^{(2)})\) to the sum \( f(x, y) \) where \( \{p_n^{(v)}\}, \quad (v = 1, 2) \) are real non-negative, non-increasing sequence of constants such that
\[
\sum_{k=2}^n \left( \frac{p_k^{(v)}}{k \log k} \right) = o \left( \frac{p_n^{(v)}}{n} \right), \quad (v = 1, 2).
\]
(19)

The objective of this present work is to generalize the theorem B to a more general class of Nörlund summability for double Fourier series. We prove the following theorem:

**Theorem 4.** Let us suppose that the sequence \( \{p_b^{(1)}\} \) and \( \{p_a^{(2)}\} \) are positively and monotonically decreasing sequence of constants such that
\[
p_b^{(1)} = \sum_{c=0}^b p_c^{(1)} \quad \text{and} \quad p_a^{(2)} = \sum_{c=0}^a p_c^{(2)},
\]
(20)
\( p_b^{(1)} \to \infty, p_a^{(2)} \to \infty \) and if
\[
\chi(\alpha, \beta) = \int_0^a ds \int_0^\beta \chi(at) \, dt = O \left[ \frac{p_b^{(1)}}{p_b^{(1)}}, \frac{p_a^{(2)}}{p_a^{(2)}} \right],
\]
(21)
then double Fourier series (1) of the function \( f(x, y) \) is summable to the sum \( S \) at \( \alpha = m \) and \( \beta = n \), when \((N, p_b^{(1)}, p_a^{(2)})\) is regular Nörlund method.

The following lemmas are required in the proof of our theorems;

**Lemma 1.** If \( \{p_n\} \) is non-negative and non-increasing, then for \( 0 \leq a \leq b \leq \infty, 0 \leq t \leq \pi \) and for any \( n \), we have
\[
\left| \sum_{k=d}^b p_k e^{i(n-k)t} \right| \leq A P_{b-1}.
\]
(22)
Lemma 2. Under the condition of Lemma 1,
\[ \left| \sum_{k=0}^{n} \frac{p_k \sin \left( n - k + \frac{1}{2} \right) t}{\sin \frac{\pi}{2}} \right| = O \left( n P_n \right), \quad 0 \leq t \leq \frac{1}{n}. \] (23)

Lemma 3. Under the condition of Lemma 1,
\[ \left| \sum_{k=0}^{n} \frac{\sin \left( n - k + \frac{1}{2} \right) t}{\sin \frac{\pi}{2}} \right| = O \left( \frac{1}{t} P_{[t-1]} \right), \quad \text{for} \quad \frac{1}{n} \leq t \leq \delta. \] (24)

Lemma 4. Under the condition of Lemma 1,
\[ \left| \sum_{k=0}^{n} \frac{\sin \left( n - k + \frac{1}{2} \right) t}{\sin \frac{\pi}{2}} \right| = O(1), \quad \text{for} \quad 0 \leq \delta < t \leq \pi. \] (25)

they are uniformly in each of the intervals.

Proof of Theorem 3. Let \( U_{mn}(x, y; f) = U_{mn} \) denotes the rectangular \((m, n)^{th}\) partial sum of the series (1), then we must have
\[
U_{mn}(x, y; f) - f(x, y) = \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{\pi} \chi(\alpha, \beta) D^1_m(\alpha) D^2_n(\beta) \mathrm{d}\alpha \mathrm{d}\beta,
\] (26)
where
\[
D^1_m(\alpha) = \frac{\sin \left( m + \frac{1}{2} \right) \alpha}{2\sin \frac{\pi}{2}},
\] (27)
and
\[
D^2_n(\beta) = \frac{\sin \left( n + \frac{1}{2} \right) \beta}{2\sin \frac{\pi}{2}},
\] (28)
where \( D^1_m(\alpha) \) and \( D^2_n(\beta) \) are respectively denote the Dirichlet kernels.

Let \( \{ V_{mn}(x, y) \} \) denote the double Nörlund transform of the sequence \( \{ V_{mn} - f(x, y) \} \), then
\[
V_{mn}(x, y) = \frac{1}{p^{(1)}_m p^{(2)}_n} \sum_{l=0}^{m} \sum_{g=0}^{n} p^{(1)}_{m-l} p^{(2)}_{n-g} \left\{ U_{lg} - f(x, y) \right\}
\]
\[
= \frac{1}{p^{(1)}_m p^{(2)}_n} \sum_{l=0}^{m} \sum_{g=0}^{n} p^{(1)}_{m-l} p^{(2)}_{n-g} \frac{1}{2\pi} \int_{0}^{\pi} \int_{0}^{\pi} \chi(\alpha, \beta) D^1_l(\alpha) D^2_g(\beta) \mathrm{d}\alpha \mathrm{d}\beta
\]
\[
= \int_{0}^{\pi} \int_{0}^{\pi} \chi(\alpha, \beta) \left\{ \frac{1}{2\pi p^{(1)}_m} \sum_{l=0}^{m} p^{(1)}_{m-l} \frac{\sin \left( l + \frac{1}{2} \right) l}{\sin \frac{\pi}{2}} \right\} \cdot \left\{ \frac{1}{2\pi p^{(2)}_n} \sum_{g=0}^{n} p^{(2)}_{n-g} \frac{\sin \left( g + \frac{1}{2} \right) g}{\sin \frac{\pi}{2}} \right\} \mathrm{d}\alpha \mathrm{d}\beta
\] (29)
where
\[
N^{(1)}_m(\alpha) = \frac{1}{2\pi p^{(1)}_m} \sum_{l=0}^{m} p^{(1)}_{m-l} \frac{\sin \left( m - l + \frac{1}{2} \right) \alpha}{\sin \frac{\pi}{2}},
\] (30)
\[ N_n^{(2)}(\beta) = \frac{1}{2\pi p_n^{(2)}} \sum_{g=0}^{n} p_{g-n}^{(2)} \sin \left( n - g + \frac{1}{2} \right) \beta \sin \frac{\beta}{\pi}. \]  
(31)

Also Eq. (29) can be written as

\[ U_{mn}(x, y) - f(x, y) = \int_0^\pi \int_0^\pi \chi(\alpha, \beta)N_m^{(1)}(\alpha)N_n^{(2)}(\beta) \, d\alpha \, d\beta 
= \left( \int_0^\pi \int_0^\pi + \int_0^\delta \int_0^\pi + \int_\delta^\pi \int_0^\pi + \int_\delta^\pi \int_\delta^\pi \right) \chi(\alpha, \beta)N_m^{(1)}(\alpha) . N_n^{(2)}(\beta) \, d\alpha \, d\beta 
= I_1 + I_2 + I_3 + I_4, \text{ (say).} \]  
(32)

By hypothesis and using the results of (23) and (24), we easily obtain

\[ |I_4| = \left| \int_\delta^\pi \int_\pi \chi(\alpha, \beta) . N_m^{(1)}(\alpha) . N_n^{(2)}(\beta) \right| 
= O \left( \frac{1}{p_m^{(1)} p_n^{(2)}} \int_\delta^\pi \int_\pi \chi(\alpha, \beta)N_m^{(1)}(\alpha) . N_n^{(2)}(\beta) \, d\alpha \, d\beta \right) 
= O \left( \frac{1}{p_m^{(1)} p_n^{(2)}} \int_0^\pi \int_0^\pi \chi(\alpha, \beta) \, d\alpha \, d\beta \right) \quad \text{(As } N_n^{(2)}(\beta), N_m^{(1)}(\alpha) \text{ are even function)} \n= O(1). \]  
(33)

Also, for \( I_3 \), we have

\[ I_3 = \int_\delta^\pi N_m^{(1)}(\alpha) \, d\alpha \int_0^\pi \chi(\alpha, \beta)N_n^{(2)}(\beta) \, d\beta 
= \int_\delta^\pi N_m^{(1)}(\alpha) \, d\alpha \left\{ \int_0^{\frac{\pi}{2}} + \int_\frac{\pi}{2}^{\delta} \right\} \chi(\alpha, \beta).N_n^{(2)}(\beta) \, d\beta 
= I_{3,1} + I_{3,2}, \text{ (say).} \]  
(34)

Thus,

\[ |I_{3,1}| = O \left( \frac{n}{p_m^{(1)}} \int_0^{\pi} \int_0^{\frac{\pi}{2}} \chi(\alpha, \beta) \, d\beta \right) 
= O \left( \frac{n}{p_m^{(1)}} \right). O \left( \frac{1}{\log n} \right) = O(1). \]  
(35)

Again, by (23) and (24) and hypothesis,

\[ |I_{3,2}| = O \left( \frac{1}{p_m^{(1)}} \int_0^\pi \int_\delta^\pi \chi(\alpha, \beta) \, d\alpha \int_0^{\frac{\pi}{2}} \frac{p_n^{(2)}}{\beta} \, d\beta \right) 
= O \left( \frac{1}{p_m^{(1)}} \int_\delta^\pi \int_\delta^\pi \frac{p_n^{(2)}}{\beta} \chi(\alpha, \beta) \, d\alpha \int_0^{\delta} \chi(\alpha, \beta) \, d\beta \right) 
= (|I_{3,2,1}| + o(|I_{3,2,2}|), \text{ (say).} \]  
(36)

where

\[ \chi_1(\alpha, \beta) = \int_0^\beta |\chi(\alpha, \beta)| \, d\omega, \]  
(37)
and \( I_{3,2,1} \) stands for two inner intervals. Thus,

\[
|I_{3,2,1}| = O \left( \frac{1}{p_{m(1)} P_n} \int_{\delta}^{\pi} \frac{1}{r-1} \phi_1(a, \tau) - n P_n^{(2)} \phi_1(a, 1 - n^{-1}) \right)
\]

\[
= O \left( \frac{1}{p_{m(1)} P_n} \int_{\delta}^{\pi} \frac{1}{r-1} \int_{0}^{\delta} |\chi(\alpha, \beta)| d\beta \right) + O \left( \frac{n}{p_{m(1)}} \int_{\delta}^{\pi} \frac{1}{r-1} \int_{0}^{\gamma} |\chi(\alpha, \beta)| d\beta \right)
\]

\[
= O(1) + O \left( \frac{n}{p_{m(1)}} \frac{1}{\log n} \right)
\]

(38)

and

\[
|I_{3,2,2}| = O \left( \frac{1}{p_{m(1)} P_n^{(2)}} \int_{\delta}^{\pi} \frac{1}{r-1} \phi_1(\alpha, \beta) \right)
\]

\[
= O \left( \frac{1}{p_{m(1)} P_n^{(2)}} \int_{\delta}^{\pi} \frac{1}{r-1} \int_{0}^{\delta} |\chi(\alpha, \beta)| d\beta \right)
\]

\[
= O \left( \frac{1}{p_{m(1)} P_n^{(2)}} \int_{\delta}^{\pi} \frac{1}{r-1} \int_{0}^{\delta} |\chi(\alpha, \beta)| d\beta \right)
\]

(39)

Also,

\[
\int_{\frac{1}{2}}^{\delta} \frac{\beta}{\log(\frac{1}{p})} d \left( P_{\beta^{-1}}^{(2)} \right) = \int_{\frac{1}{2}}^{\delta} \frac{\beta}{\log(\frac{1}{p})} d \left( y P_{y[y]}^{(2)} \right)
\]

(40)

for

\[
\int_{j}^{j+1} \frac{1}{y \log y} d \left( y P_{y[y]}^{(2)} \right) < \frac{1}{j \log j} \int_{j}^{j+1} d \left( y P_{y[y]}^{(2)} \right) = \frac{1}{j \log j} \left[ y P_{y[y]}^{(2)} \right]_{j}^{j+1}
\]

\[
= \frac{1}{j \log j} \left\{ (j + 1) P_{j+1}^{(2)} - k P_{j}^{(2)} \right\}
\]

\[
< \frac{1}{j \log j} \left\{ P_{j+1}^{(2)} + P_{j}^{(2)} \right\}
\]

(41)

For

\[
P_{k+1}^{(2)} \leq P_{k}^{(2)} \text{ and } k P_{k}^{(2)} \leq P_{k}^{(2)} \leq \frac{2 P_{j}^{(2)}}{j \log j} + \frac{P_{j}^{(2)}}{j \log j}
\]

(42)

thus,

\[
\int_{\frac{1}{2}}^{\delta} \frac{1}{y \log y} d \left( x P_{x[x]}^{(2)} \right) < A + \sum_{j=\infty}^{n} \left( \frac{2 P_{j}^{(2)}}{j \log j} + \frac{P_{j}^{(2)}}{j \log j} \right) = O \left( P_{n}^{(2)} \right)
\]

(43)

Now, by hypothesis (19) and using the Eqs (43) and (29), we get

\[
|I_{3,2,1}| = O(1)
\]

(44)

Combining (34), (35), (36), (38), (39), (43) and (44), we get

\[
|I_3| = O(1)
\]

(45)
Similarly, we can show that

\[ |I_2| = O(1). \]  \hfill (46)

Now, for \( I_1 \),

\[
I_1 = \int_0^\delta \int_0^\tau \chi(\alpha, \beta) N^{(1)}_m(\alpha) N^{(2)}_n(\beta) d\alpha d\beta \\
= \left( \int_0^\frac{1}{2} \int_0^\frac{1}{2} + \int_0^1 \int_0^\delta + \int_0^\delta \int_0^\frac{1}{2} + \int_0^\delta \int_\frac{1}{2}^\tau \right) \chi(\alpha, \beta) N^{(1)}_m(\alpha) N^{(2)}_n(\beta) d\alpha d\beta \\
= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} \quad \text{(say)}.
\]  \hfill (47)

Then by (23) and (24),

\[
|I_{1,1}| = O \left( \int_0^\frac{1}{2} \int_0^\frac{1}{2} |\chi(\alpha, \beta)| mn \, d\alpha d\beta \right) \\
= O(mn)O \left( \frac{\frac{1}{m}}{\log m} \cdot \frac{1}{\pi} \right) \\
= O(1). \]  \hfill (48)

Similarly,

\[
\begin{align*}
|I_{1,2}| &= O(1), \\
|I_{1,3}| &= O(1).
\end{align*} \]  \hfill (49)

and

\[
\int_\frac{1}{\alpha}^\delta \int_\frac{1}{\beta}^\tau |\chi(\alpha, \beta)| \frac{p^{(1)}_{\alpha-1}}{\alpha} \frac{p^{(2)}_{\beta-1}}{\beta} \, d\alpha d\beta \\
= \chi(\delta, \tau) \frac{1}{\delta} p^{(1)}_{\delta-1} \frac{1}{\tau} p^{(2)}_{\tau-1} - \frac{1}{\tau} p^{(2)}_{\tau-1} \int_\frac{1}{\alpha}^\delta \Phi(\alpha, \tau) d\left( \frac{p^{(1)}_{\alpha-1}}{\alpha} \right) - \frac{1}{\delta} p^{(1)}_{\delta-1} \int_\frac{1}{\beta}^\tau \chi(\alpha, \beta) d\left( \frac{p^{(2)}_{\beta-1}}{\beta} \right). \]  \hfill (50)

Thus,

\[
|I_{1,4}| = O \left( \int_\frac{1}{\alpha}^\delta \int_\frac{1}{\beta}^\tau |\chi(\alpha, \beta)| \frac{1}{P^{(1)}_m P^{(2)}_n} \frac{p^{(2)}_{\alpha-1}}{\alpha} \frac{p^{(2)}_{\beta-1}}{\beta} \, d\alpha d\beta \right) \\
= O(1) + O \left( \frac{1}{P^{(1)}_m P^{(2)}_n} (C_1 + C_2 + C_3) \right), \]  \hfill (51)

where \( O(1) \) corresponds to the integrated part in (49) and \( C_1, C_2 \) and \( C_3 \) are repetitively denote the remaining there integrals. Thus following the estimates in \( I_{3,2} \), we have

\[
\begin{align*}
C_2 &= O(1), \\
C_3 &= O(1).
\end{align*} \]  \hfill (52)

Again for \( C_4 \)

\[
C_4 = O \left( \int_\frac{1}{\alpha}^\delta \frac{\alpha}{\log(\frac{1}{\alpha})} d\left( \frac{p^{(1)}_{\alpha-1}}{\alpha} \right) \cdot \int_\frac{1}{\beta}^\tau \frac{\beta}{\log(\frac{1}{\beta})} \left[ \frac{p^{(2)}_{\beta-1}}{\beta} \right] \right) \\
= O \left( \frac{p^{(1)}_m p^{(2)}_n}{\log(\frac{1}{\alpha}) \log(\frac{1}{\beta})} \right). \]  \hfill (53)
As in (43), using the estimates (52) and (53), we get from (51) that

$$|I_{1,4}| = O(1). \tag{54}$$

Thus,

$$|I_1| = O(1). \tag{55}$$

Combining (33), (44), (45), (55), we get (32). Which competes the proof of the theorem. □

**Proof of Theorem 4.** Let $U_{b,a}$ denote the rectangular $(b,a)^{th}$ partial sums of the series (1) Then,

$$U_{b,a} = \int_{0}^{\pi} \int_{0}^{\pi} \chi(\alpha, \beta) N_b^{(1)}(\alpha) N_a^{(2)}(\beta) \, d\alpha \, d\beta$$

$$= \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi} \chi(\alpha, \beta) N_b^{(1)}(\alpha) N_a^{(2)}(\beta) \, d\alpha \, d\beta$$

$$= I_1 + I_2 + I_3, \quad \text{(say).} \tag{56}$$

where $0 < \delta < \pi$ and $0 < \theta < \pi$. Now,

$$\left| N_b^{(1)}(\alpha) \right| \leq \frac{1}{2\pi P_b^{(1)}} \sum_{l=0}^{b} \frac{P_b^{(1)}}{\sin^{2} \frac{l+\frac{1}{2}}{2}}, \tag{57}$$

and

$$\left| N_a^{(2)}(\beta) \right| \leq \frac{1}{2\pi P_a^{(2)}} \sum_{g=0}^{a} \frac{P_a^{(2)}}{\sin^{2} \frac{a-g+\frac{1}{2}}{2}} = O(a). \tag{58}$$

Thus

$$I_1 = O \left[ b \int_{0}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{2}} \chi(\alpha, \beta) \, d\alpha \, d\beta \right]$$

$$= O \left( b \frac{P_b^{(1)}}{P_b^{(1)}} \left( \frac{a P_a^{(2)}}{P_a^{(2)}} \right) \right)$$

$$= O \left( \frac{P_b^{(1)}}{P_b^{(1)}} \frac{P_a^{(2)}}{P_a^{(2)}} \right)$$

$$= O(1), \quad \text{(because } b P_b^{(1)} \leq P_b^{(1)} \text{ and } a P_a^{(2)} \leq P_a^{(2)} \text{).} \tag{59}$$

Also, by Lemma 1, we have

$$I_2 = \left[ \frac{1}{P_b^{(1)} P_a^{(2)}} \int_{\frac{\pi}{2}}^{\delta} \int_{\frac{\pi}{2}}^{\theta} \chi(\alpha, \beta) \times \frac{P_b^{(1)} P_a^{(2)}}{P_b^{(1)} P_a^{(2)}} \right]$$

$$= O \left( \frac{1}{P_b^{(1)} P_a^{(2)}} \chi(\alpha, \beta) \frac{P_b^{(1)} P_a^{(2)}}{P_b^{(1)} P_a^{(2)}} \right) \left[ \int_{\frac{\pi}{2}}^{\delta} \int_{\frac{\pi}{2}}^{\theta} \chi(\alpha, \beta) \times \frac{P_b^{(1)} P_a^{(2)}}{P_b^{(1)} P_a^{(2)}} \, d\alpha \, d\beta \right] + O(1)$$

$$= O(1) + \frac{1}{P_b^{(1)} P_a^{(2)}} \left( \int_{\frac{\pi}{2}}^{\delta} \int_{\frac{\pi}{2}}^{\theta} \chi(\alpha, \beta) \times \frac{P_b^{(1)} P_a^{(2)}}{P_b^{(1)} P_a^{(2)}} \, d\alpha \, d\beta \right)$$

$$= O(1). \tag{60}$$
Similarly,
\[ J_3 = \int_0^\pi \int_0^\pi \chi(x, \beta) N_a^{(1)}(a) N_a^{(2)}(\beta) \times da \, d\beta = O(1). \] (61)

Thus, combining (57), (58), (59), (60), (61), we obtain (56). This completes the proof of the theorem. \(\square\)

4. Conclusion

The negligible set of conditions has been obtained for the finite series in this paper. We also discuss how Nörlund summability includes a broader range of functions that can be approximated. The approximation is a broad field having a widespread application in signals. Approximation treats signal one variable system and image as to the variable system.

Author Contributions: All authors contributed equally in this paper. All authors read and approved the final version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Data Availability: All data required for this research is included within this paper.

Funding Information: No funding is available for this research.

Acknowledgments: The author would like to thank the referee for his/her valuable comments which lead to an improvement of the original manuscript.

References

[1] Móricz, F., & Rhoades, B. E. (1992). Summability of double Fourier series by Nörlund methods at a point. *Journal of Mathematical Analysis and Applications*, 167(1), 203-215.

[2] Nigam, H. K., & Sharma, K. (2012). On double summability of double conjugate Fourier series. *International Journal of Mathematics and Mathematical Sciences*, 2012, Article ID: 104592. https://doi.org/10.1155/2012/104592.

[3] Sharma, P. L. (1958). On the harmonic summability of double Fourier series. *Proceedings of the American Mathematical Society*, 9(6), 979-986.

[4] Tripathi, L. M., & Singh, A. P. (1981). A study of double Fourier series by Nörlund summability. In *Indagationes Mathematicae (Proceedings)*, 84(1), 139-143.

[5] Herriot, J. G. (1942). Nörlund summability of double Fourier series. *Transactions of the American Mathematical Society*, 52(1), 72-94.

[6] Lal, S., & Tripathi, V. N. (2003). On the study of double Fourier series by double matrix summability method. *Tamkang Journal of Mathematics*, 34(1), 1-16.

[7] Chow, Y. S. (1953). On the Cesaro summability of double Fourier series. *Tohoku Mathematical Journal, Second Series*, 5(3), 277-283.

[8] Mishra, V. N., Khatri, K., & Mishra, L. N. (2012). Approximation of functions belonging to Lip $\xi(t), r$ Lip $(\xi(t), r)$ class by $(N, p_n)(E, q)(N, p_n)(E, q)$ summability conjugate series of Fourier series. *Journal of Inequalities and Applications*, 2012, Article No 296. https://doi.org/10.1186/1029-242X-2012-296.

[9] Mishra, L. N., Mishra, V. N., & Khatri, K. (2014). On the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L_r, \xi(t))(r \geq 1)$-class by matrix $(C_1-Np)$ operator of conjugate series of its Fourier series. *Applied Mathematics and Computation*, 237(2), 252-263.

[10] Mishra, V. N., Khatri, K., & Mishra, L. N. (2012). Product summability transform of conjugate series of Fourier series. *International Journal of Mathematics and Mathematical Sciences*, 2012, Article ID: 298923. https://doi.org/10.1155/2012/298923.

[11] Singh, T. (1963). On Nörlund summability of Fourier series and its conjugate series. In *Proceedings of the National Academy of Sciences, India*, 29, 65-73.

[12] Pati, T. (1961). A generalization of theorem of Iyenger on the harmonic summability of Fourier series. *Indian Journal of Mathematics*, 3, 85-90.

© 2022 by the authors; licensee PSRP, Lahore, Pakistan. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC-BY) license (http://creativecommons.org/licenses/by/4.0/).