TECHNICAL APPENDIX

Measuring real consumption and consumer price index bias under lockdown conditions

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Appendix A: Eurostat, IMF, UNECE and U.S. Bureau of Labor Statistics Advice

National and international statistical agencies are attempting to provide advice to national price statisticians on how to construct a CPI under lockdown conditions. Below, we give a sampling of this advice.

Here is the advice from Eurostat to European Union countries on how to calculate the EU’s Harmonized Index of Consumer Prices (HICP):

“The compilation of the HICP in the context of the COVID-19 crisis is guided by the following three principles:

- Stability of the HICP weights,
- Compilation of indices covering the full structure of the European version of the Classification of Individual Consumption According to Purpose (ECOICOP),
- Minimizing the number of imputed prices and sub-indices.

The first principle ensures that there will be no change in the sub-index weights used in the compilation of the HICP during the year, which is the standard practice. The HICP sub-indices are aggregated using weights reflecting the household consumption expenditure patterns of the previous year. The HICP weights are updated at the beginning of each year and are kept constant throughout the year. Thus, the weights will not change this year as a result of the impact of the COVID-19 on expenditures.

The second principle means that all sub-indices for the full ECOICOP structure will be compiled even when for some categories no products are available on the market. In such cases, prices do not exist and they should be replaced with imputed prices. Sub-indices consisting of both imputations and observed prices should be compiled and aggregated using the standard HICP compilation procedures.

Finally, the third principle underlines the idea that, whenever possible, missing price observations should be replaced by price quotes obtained from other sources. Price collection can fail because of restrictions that do not allow price collectors to visit sampled outlets, because outlets have been
closed down or because it is impossible to offer certain services (e.g. flights). Possible sources to replace the missing prices in case manual price collection activities are restricted are the following:

- Outlets’ websites,
- Telephone and email enquiries.

Some NSIs may also have access to scanner data that, although not yet integrated into the HICP production system, could be used for the replacement of the missing prices. However, the replacements using scanner data should be done with care as this could entail a change of outlet to a different category or market segment.” Eurostat (2020).

Thus the Eurostat advice boils down to carry on as if nothing has happened; i.e., keep the existing fixed basket methodology, use web scraped or telephone interview price data to replace previously personally collected data and if a consumption category disappears due to shut down restrictions, use carry forward imputed prices. This goes against the conclusions of this paper, but the Eurostat advice is understandable. In the short run, it is impossible for Eurostat to change their methodology to a variable basket approach. Thus the hope is probably that the lockdown period will be brief and when economies return to “normal”, the fixed basket approach will again be satisfactory and the use of the old fixed basket and imputation methods will enable price comparisons with the pre-lockdown period to be reasonably accurate.

Here is the advice from the UNECE:

“Price collection may be restricted due to closed outlets or price collectors may not be allowed to work or enter outlets. It may also be that outlets do not provide the usual set of prices through other channels (e.g. on paper or via e-mail) and/or there may be shortage of staff in the main office to receive and process the prices that are received. Alternative modes of price collection include telephone, e-mail, online prices and scanner data. However, it may be difficult to ensure a minimum coverage of all products (goods and services). In particular, this may be the case for products for which price collectors usually collect prices. This could, for example, be the case for
clothing and fresh food in many countries. In such cases the statistical office may have to rely on collecting a minimum of prices for the most important or the most representative products.

Compilation

For imputation of observations the general recommendation is to follow a bottom-up approach. This means that the first choice is to impute missing prices with observed price developments of similar products or products that are expected to have similar price developments. If such product prices are not available, the next choice will be to impute the missing prices with the average price development of the product group or the elementary aggregate to which the product belong. If these are not available, the closest available higher-level price index should be used for the imputation.

In some instances it may not be possible to collect prices for specific product groups or elementary aggregates or even indices above the elementary aggregate level. In such cases the price development of the product group or the elementary aggregate may be imputed by the price development of similar product groups or elementary aggregates. If this is not possible, the price development may be imputed by the higher-level index in which the product group or the elementary aggregate enters. However, imputation of a missing elementary aggregate by the overall CPI may also be justified. This corresponds to leaving the elementary aggregate out of the calculation of the CPI. This may be the preferred option if households' expenditures on an elementary aggregate is assessed to be zero or close to zero. In some countries this may be the case for e.g. international travels, domestic airline travels, child care and sports and cultural events.

These are general recommendations. National circumstances and knowledge of the developments for particular markets and products must be considered. In all cases, it is important to apply imputation methods that ensure the index reaches the correct level when again it becomes possible to collect prices and include them in the index.”

UNECE (2020)

The advice from the UNECE is similar to the advice from Eurostat but the last sentence in the above quotation provides an explicit explanation for the carry on as usual methodology; i.e., when things return to “normal”, the post lockdown CPI indexes will be comparable to the pre-lockdown CPI index.
Here is the advice from the International Monetary Fund:

“When facing increased numbers of missing prices, it is important to remind that all temporarily missing prices should be imputed using one of the methods described in Consumer Price Index Manual: Concepts and Methods. As noted in the Manual, carrying forward, or repeating the last available price, should be avoided as it introduces a downward bias into the index. The imputation techniques described in the Manual do not introduce bias into the index. Imputations are self-correcting, which means that once a price can be collected, the index returns to the correct level. This is important so that the CPI continues to provide a reliable estimate of price change. The CPI is a critical input to economic policy making, particularly during periods of economic uncertainty.

If an entire index is missing, it is recommended to use the next level up in aggregation as the basis for making the imputation. For example, if all prices for oranges are missing, the index for citrus fruits can be used as the basis for making the imputation. If all citrus fruits are missing, the index for fruits is used as the basis for making the imputation. If all fruits are missing, the index for fruits and vegetables is used. If fruits and vegetables are missing, the index for food is used. If the index for food is missing, the index for food and non-alcoholic beverages is used. Finally, if all food and non-alcoholic beverages is missing, the All Items index is used as the basis to impute.

... Users will continue to need data at the most detailed level. All indexes should continue to be published, even if they are imputed. As noted previously, all imputed indexes should be flagged and clearly noted for users. It is important for transparency that users are able to access the full set of data that are normally disseminated.”

IMF (2020)

The above advice is in line with the guidance provided by Eurostat and the UNECE. It is more explicit in one respect in that it rules out simple carry forward pricing and endorses inflation adjusted carry forward prices. We also endorse inflation adjusted carry forward prices over the use of carry forward prices.
Finally, here is some information on how the US Bureau of Labor Statistics is planning to produce its CPI under current conditions:

“How are prices collected for the CPI? Price data used to calculate the CPI are primarily provided by two different surveys that are administered continuously each month:

• Commodities and Services Pricing Survey, an establishment survey of businesses selling goods and services to consumers, used to provide the price data for the CPI.
• Housing Survey, a survey of landlords and tenants used to provide rent data for CPI’s shelter indexes.

Survey operations for CPI pricing surveys may be affected by limitations on data-collection staff, the availability of survey respondents, and the availability of items. Note that CPI data are collected throughout the entire month. Specifically, any given price in the CPI sample is collected in one of three defined pricing periods, corresponding roughly to the first 10, second 10, and final 10 days of the month. BLS uses several data-collection modes for CPI surveys that include telephone, internet, and automated electronic data capture. However, the majority of data are collected by personal visit. About 65 percent of CPI price data and 50 percent of CPI rent data are typically collected by personal visit. This type of collection has been suspended since March 16, 2020. (It was suspended on March 5th in the Seattle area.)

What happens if BLS cannot collect CPI data? The percentage of prices in the CPI sample that may be unavailable, either because the outlet is closed or the item is out of stock, is expected to increase. When BLS cannot obtain a price either because of data-collection limitations or the item being unavailable, it will generally be considered “temporarily unavailable.” The CPI program has specific procedures for handling temporarily unavailable prices. Missing prices are generally imputed by the prices that are collected in the same or similar geographic area and item category. Essentially, the price movement of items that are not collected is estimated to be the same as those that are collected for a given item and geographic area.

Will data collection for CPI expenditure weights be affected? The Consumer Expenditure Survey (CE), a household survey capturing consumer spending data, is used to calculate relative importances (weights) of goods and services in the CPI market basket. CE in-person data collection ceased on March 19, 2020. CE data are collected by the U.S. Census Bureau through an agreement with BLS. The Census Bureau is transitioning to collecting these
data through telephone. Changes to CE survey operations will not have an immediate impact on CPI data, but may have long-term impacts. These weights are used in the chained CPI index (C-CPI-U). The March 2020 weights will be incorporated in the final March 2020 chained CPI indexes, which are released in February 2021. BLS also incorporates the CE weights in an annual weight update to the CPI-U and CPI-W indexes. These weight updates will be effective with the January 2022 indexes, released in February 2022. BLS is working on mitigation strategies to reduce measurement error of CPI weights caused by a potential loss of CE survey data.

**Under what circumstances would some data not be published?** A CPI index is not published if it fails a data-quality standard known as an adequacy ratio. Specifically, if BLS fails to collect at least one price in a geographic area that account for more than half the geographic weight of the index, the index is not published. Even in months without disruptions, some minor indexes with small samples occasionally fail this standard and are not published. (One example is Repair of household items.) Data-collection disruptions would have to be extremely severe for major CPI indexes not to be published based on this standard. Data-collection disruptions may be more severe in some area than others, and it is possible that some data for metro areas may fail data quality-standards and not be published. BLS will continue to monitor data-collection disruptions.”

The BLS approach to dealing with the COVID-19 pandemic is very much in line with the approach advocated in this paper.

**Appendix B: Measuring Real Consumption when there are only Two Commodities**

We show that the bias in a CPI that uses inflation adjusted carry forward prices will produce an inflation estimate for the first shut down period that is too low as compared to a cost of living index that uses reservation prices for the commodities that are not available. The companion estimate of real consumption (or welfare) that uses a Lowe index to deflate nominal household expenditures into an estimate for real consumption will be too high.
It may be useful for many readers to have a figure which can explain the underlying index number theory in a relatively simple way. In this Appendix we consider the case where \( M = 1 \) and \( N = 1 \); i.e., we have only one continuing commodity \( q \) and one unavailable commodity \( Q \).

We assume that the household or group of households have preferences that can be represented by the utility function, \( f(q,Q) \), which is linearly homogeneous, increasing and concave in \( q,Q \). The dual unit cost function for this utility function is \( c(p,P) \) where \( p \) and \( P \) are the positive prices for a unit of \( q \) and \( Q \) respectively.\(^1\)

The observed quantity data for periods 0 and 1 are \((q^0,Q^0)\) and \((q^1,Q^1)\). The observed price data for period 0 are \((p^0,P^0)\) but for period 1, only the price for a unit of the continuing commodity is observed, \( p^1 > 0 \); the price for the unavailable commodity is the Hicksian reservation price \( P^1* > 0 \). We assume that \( Q^1 \) is equal to 0:

\[(B1) \quad Q^1 = 0.\]

The period \( t \) utility level attained by the household is denoted by \( u^t \) for \( t = 0,1 \). We have the following definitions:

\[(B2) \quad u^0 = f(q^0,Q^0) > 0; \quad u^1 = f(q^1,0) > 0.\]

Denote the observed expenditure on consumer goods and services in period \( t \) by \( v^t \). We have the following definitions:

\[(B3) \quad v^0 = p^0q^0 + P^0Q^0;\]
\[(B4) \quad v^1 = p^1q^1 + P^1*Q^1 = p^1q^1 \quad \text{using assumption (B1)}.\]

\(^1\) The function \( c(p,P) \) is defined as \( c(p,P) = \min_{q \geq 0, P \geq 0} \{pq + PQ : f(q,Q) \geq 1 ; q \geq 0, Q \geq 0\} \). The unit cost function is also increasing, linearly homogeneous and concave in \( p \) and \( P \). We assume that it is also differentiable. For the early history of duality theory and its application to index number theory, see Diewert (1974) (1976) (2020a).
When the economic approach to index number theory is used, it is assumed that observed expenditures on consumer goods and services is equal to the minimum cost of achieving the utility level for the period under consideration. Using this approach we have the following equalities:

\[(B5) \quad v^0 = c(p^0, P^0)f(q^0, Q^0) = c(p^0, P^0)u^0;\]
\[(B6) \quad v^1 = c(p^1, P^{1*})f(q^1, Q^1) = c(p^1, P^{1*})f(q^1, 0) = c(p^1, P^{1*})u^1.\]

Let \(c_1(p, P)\) denote the partial derivative of \(c(p, P)\) with respect to \(p\); i.e., \(c_1(p, P) = \partial c(p, P)/\partial p\) and \(c_2(p, P) = \partial c(p, P)/\partial P\). The assumption of cost minimizing behavior on the part of households along with Shephard’s Lemma implies that the following relationships will hold:

\[(B7) \quad q^0 = c_1(p^0, P^0)u^0;\]
\[(B8) \quad Q^0 = c_2(p^0, P^0)u^0;\]
\[(B9) \quad q^1 = c_1(p^1, P^{1*})u^1;\]
\[(B10) \quad Q^1 = c_2(p^1, P^{1*})u^1 = 0 \quad \text{using assumption (B1)}.\]

Suppose the household spent all of its period t expenditure on consumer goods and services, \(v^t\), on purchases of the continuing commodity \(q\). Denote this period t hypothetical expenditure on purchases of \(q\) by \(q^{0e}\), the budgetary equivalent expenditure on the always available commodity. We have the following definitions:

\[(B11) \quad q^{0e} = v^0/p^0 = [p^0q^0 + P^0Q^0]/p^0 \quad \text{using (B3)}\]

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2 See Shephard (1953; 11) or Diewert (2020a; 11). Shephard’s Lemma implies that the consumer demand functions, \(q(u, p, P)\) and \(Q(u, p, P)\), regarded as functions of the consumer’s utility level \(u\) and the prices, \(p\) and \(P\) that the consumer faces, are equal to \(q(u, p, P) = c_1(p, P)u\) and \(Q(u, p, P) = c_2(p, P)u\). Hicks (1946; 311-331) introduced this type of demand function into the economics literature and so these functions are known as Hicksian demand functions. They can be estimated using econometric techniques; see Diewert and Feenstra (2019).
\[ q^0 + [P^0/p^0]Q^0. \]

(B12) \[ q^{1e} \equiv v^1/p^1 \]
\[ = p^1q^1/p^1 \]
\[ = q^1. \]

Since there are no purchases of the unavailable commodity in period 1, \( q^{1e} \) turns out to equal the observed consumption of the continuing commodity, which is \( q^1 \).

The minimum cost of achieving the utility level \( u^0 \) if the consumer faced the prices of period 1, \( p^1 \) and \( P^1* \), is \( c(p^1,P^1*)u^0 \). Since \( f(q^0,Q^0) \) equals \( u^0 \), we see that \((q^0,Q^0)\) is a feasible solution for this cost minimization problem and hence we have the following inequality:

(B13) \[ c(p^1,P^1*)u^0 \leq p^1q^0 + P^1*Q^0. \]

Similarly, the minimum cost of achieving the utility level \( u^1 \) if the consumer faced the prices of period 0, \( p^0 \) and \( P^0 \), is \( c(p^0,P^0)u^1 \). Since \( f(q^1,Q^1) = f(q^1,0) \) equals \( u^1 \), we see that \((q^1,0)\) is a feasible solution for this cost minimization problem and hence we have the following inequality:

(B14) \[ c(p^0,P^0)u^1 \leq p^0q^1 + P^0Q^1 = p^0q^1. \]

Now convert the hypothetical expenditures \( c(p^1,P^1*)u^0 \) into purchases of \( q \) using the price of the continuing commodity for period 1, \( p^1 \), to obtain the hypothetical quantity \( q^{0*} \):

(B15) \[ q^{0*} = c(p^1,P^1*)u^0/p^1 \]
\[ \leq [p^1q^0 + P^1*Q^0]/p^1 \]
\[ = q^0 + [P^1*/p^1]Q^0 \]
\[ = q^{0e*} \]
where $q_{0e^*}$ converts the period 0 purchases $Q^0$ of the disappearing commodity into equivalent amounts of the continuing commodity using the relative prices of period 1 and adds this amount to the period 0 actual purchases of the continuing commodity $q^0$.  

Convert the hypothetical expenditures $c(p^0,P^0)u^1$ into purchases of $q$ using the price of the continuing commodity for period 0, $p^0$, to obtain the hypothetical quantity $q_1^*$:

$$ (B16) \quad q_1^* = c(p^0,P^0)u^1/p^0 $$

$$ \leq p^0 q_1^*/p^0 \quad \text{using (B14)} $$

$$ = q_1. $$

Suppose that the consumption vectors, $(q^0, Q^0)$ and $(q^1, Q^1)$ are given for periods 0 and 1 along with a household utility function, $f(q, Q)$. Define the period 0 and 1 utility levels by $u^0 = f(q^0, Q^0)$ and $u^1 = f(q^1, Q^1)$. Finally, suppose that the reference prices $p > 0$ and $P > 0$ are given. The family of *Allen quantity indexes*, $Q_A(p, P, u^0, u^1)$, is defined as follows:

$$ (B17) \quad Q_A(p, P, u^0, u^1) = C(u^1, p, P)/C(u^0, p, P) $$

where $C(u, p, P)$ is the household’s minimum cost of achieving the utility level $u$ if it faces the prices $p, P$. Since we have assumed that the utility function is linearly homogeneous, the consumer’s total cost or expenditure function, $C(u, p, P)$, factors into the product of the unit cost function, $c(p, P)$ times the utility level $u$; i.e., we have $C(u, p, P) = c(p, P)u$. If we substitute this factorization of the cost function into definition (B17), we find that the Allen quantity index collapses down to the utility ratio, $u^1/u^0$; i.e., we have the following equality for all choices of the reference prices, $p$ and $P$:

$$ (B18) \quad Q_A(p, P, u^0, u^1) = C(u^1, p, P)/C(u^0, p, P) $$

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3 The hypothetical quantity $q_{0e^*}$ is useful when we define the Paasche quantity index; see Figure 1 below.

4 Each choice of $p$ and $P$ generates a possibly different Allen quantity index that measures aggregate quantity change between periods 0 and 1. The definition of the Allen (1949) quantity index provides a useful way to cardinalize a measure of consumer utility.

5 In the economics literature, the assumption of a linearly homogeneous utility function is sometimes called the assumption of homothetic preferences.
\[ c(p,P)u^1/c(p,P)u^0 = u^1/u^0. \]

Equations (B18) can be used to obtain alternative ways of estimating the Allen quantity index under our assumptions. These alternative expressions for \( u^1/u^0 \) can be illustrated by looking at Figure 1 below. Our first alternative way of expressing the utility ratio uses (B18) as follows:

\[
\begin{align*}
(B19) \quad u^1/u^0 &= c(p^0,P^0)u^1/c(p^0,P^0)u^0 \\
&= p^0 q^1/[p^0 q^0 + P^0 Q^0] \\
&= q^1/[q^0 + (P^0/p^0)Q^0] \\
&= q^1/q^{0e} 
\end{align*}
\]

using (B3), (B5) and (B16).

Our second alternative way of expressing the utility ratio uses (B18) as follows:

\[
\begin{align*}
(B20) \quad u^1/u^0 &= c(p^1,P^1)u^1/c(p^1,P^1)u^0 \\
&= p^1 q^1/p^1 q^{0*} \\
&= q^1/q^{0*}.
\end{align*}
\]

The alternative quantities of the continuing commodity defined above, \( q^0 \), \( q^1 \), \( q^{0*} \), \( q^{1*} \), \( q^{0c} \), \( q^{1c} \) and \( q^{0e*} \), are all illustrated in Figure 1.

The ingredients that go into the construction of the Laspeyres and Paasche quantity indexes can also be illustrated in Figure 1. The *Laspeyres quantity index*, \( Q_L \), is defined as follows:

\[
\begin{align*}
(B21) \quad Q_L &= [p^0 q^1 + P^0 Q^1]/[p^0 q^0 + P^0 Q^0] \\
&= p^0 q^1/[p^0 q^0 + P^0 Q^0] \\
&= s_q^0 [q^1/q^0] \\
&= q^1/[q^0 + (P^0/p^0)Q^0] \\
&= q^1/q^{0e} 
\end{align*}
\]

using (B1), \( Q^1 = 0 \) where \( s_q^0 = p^0 q^0/v^0 \) using (B15).
\[
q^*/q^{0e} \geq q^*/q^{0*} \quad \text{using (B16)}
\]
\[
u^1/u^0 \quad \text{using (B19)}.
\]

Hence the Allen quantity index (equal to the utility ratio, \(u^1/u^0\)) is bounded from above by the ordinary Laspeyres quantity index \(Q_L\). In Figure 1, \(Q_L\) is equal to \(q^1/q^{0e}\) and \(u^1/u^0\) is equal to \(q^1*/q^{0*}\). \(Q_L\) will typically have a large upward bias relative to the true Allen index under our assumptions.

The *Paasche quantity index*, \(Q_P^*\), is defined as follows:\(^6\)

\[
\begin{align*}
(B22) \quad Q_P^* &= [p^1q^1 + P^1*Q^1]/[p^1q^0 + P^1*Q^0] \\
&= p^1q^1/[p^1q^0 + P^1*Q^0] \quad \text{using (B1), } Q^1 = 0 \\
&= q^1/[q^0 + (P^1*/p^1) Q^0] \quad \text{using (B15)} \\
&= q^1/q^{0*} \quad \text{using (B15)} \\
&\leq q^1/q^{0*} \quad \text{using (B20).}
\end{align*}
\]

Hence the Allen quantity index is bounded from below by the ordinary Paasche quantity index \(Q_P^*\). In Figure 1, \(Q_P^*\) is equal to \(q^1/q^{0e*}\) and \(u^1/u^0\) is equal to \(q^1/q^{0*}\). It can be seen that \(Q_P^*\) is well below the true utility ratio, \(u^1/u^0\). Since \(q^{0e*}\) is well above \(q^{0*}\), it can also be seen that \(Q_L = q^1/q^{0e}\) is well above \(u^1/u^0 = q^1/q^{0*}\) since \(q^{0e}\) is well below \(q^{0*}\).\(^7\)

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\(^6\) We attach an asterisk to \(P^*_p\) because we require an estimate for the period 1 reservation price, \(P^1*\), in order to evaluate the index using observable data.

\(^7\) Figure 1 uses the preferences of period 0 to illustrate the decline in real consumption. If we used the preferences of the household for period 1, then the change in real consumption can be measured by the utility ratio \(f(q^1,0)/f(q^0,0) = q^1/q^0\) using the linear homogeneity of \(f(q,Q)\). For the example illustrated in Figure 1, we see that \(q^1/q^0 > 1\). Thus using the post lockdown preferences, we have a utility increase whereas using the pre-lockdown preferences, we have \(u^1/u^0 = f(q^1,0)/f(q^0,0) < 1\), a utility decrease. This illustrates the need for more than one CPI and more than one estimate for real consumption growth as we transition from the pre-lockdown situation to the lockdown situation.
There are two sets of three parallel lines in Figure 1. The slope of the red straight line that is tangent to the period 0 indifference curve at \((q^0, Q^0)\) is equal to \(-P^0/p^0\) while the slope of the black straight line that is tangent to the period 1 indifference curve at \((q^1,0)\) is equal to \(-P^1*/p^1\). Note that \(P^1*/p^1 > P^0/p^0\), which reflects that the (imputed) market clearing price for the unavailable commodity in period 1, \(P^1*\), is much greater than the inflation adjusted carry forward price for the unavailable good, \((p^1/p^0)P^0\). However, if \(q\) and \(Q\) are perfect substitutes, then the period 0 indifference curve will coincide with the straight line (this line represents the period 0 budget constraint of the household) that is tangent to the period 0 indifference curve. In this case, it can be seen that \(q^{0e}\) will coincide with \(q^{0e}\) and
q^{0*} while q^1 will coincide with q^{1*}. In this case, the Allen quantity index, u^1/u^0 will be equal to Q_L, Q_P^* and Q_F^*.

In the perfect substitutes case, the household utility function is equal to the linear function, f(q,Q) = \alpha_q q + \alpha_Q Q where \alpha_q and \alpha_Q are positive constants that reflect the relative utility to the household for the consumption of a unit of each commodity. The Allen quantity index, u^1/u^0, is then equal to \frac{[\alpha_q q^1 + \alpha_Q Q^1]}{[\alpha_q q^0 + \alpha_Q Q^0]}.

The corresponding consumer price index is equal to \frac{[v^1/v^0]}{[u^1/u^0]} which in turn is equal to \frac{[v^1/(\alpha_q q^1 + \alpha_Q Q^1)]}{[v^0/(\alpha_q q^0 + \alpha_Q Q^0)]} = \frac{v^1}{v^0} where the period t quality adjusted unit value is defined as v^1 = \frac{v}{(\alpha_q q^1 + \alpha_Q Q^1)} for t = 0,1. Thus in the perfect substitutes case, the Allen quantity index is equal to a quality adjusted unit value index, which can readily be calculated provided that estimates for \alpha_q and \alpha_Q are available. A simple choice for the \alpha’s is to set them equal to the corresponding base period prices; i.e., set \alpha_q = p^0 and \alpha_Q = P^0. In this case, the quality adjusted unit value index collapses down to the Paasche price P_P which will be defined below by (B36).

The point of this digression into quality adjusted unit value indexes is that this type of index can be readily implemented by statistical agencies without having to do any econometric estimation and this type of index is perfectly acceptable even under lockdown conditions, provide that the group of products under consideration are close substitutes so that the household indifference surfaces for these products are close to being parallel planes. Thus the indifference curves in Figure 1 as drawn are applicable at higher levels of aggregation where the product groups under consideration are not close substitutes. In this case, it will be necessary to estimate reservation prices in order to

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8 Econometric estimation may lead to improved estimates for the \alpha’s. For example, suppose the statistical agency has information on the unit value prices for N highly substitutable commodities prevailing in a number of past periods. Denote the price vector for period t as \tilde{p}^t = [p_1^t, \ldots, p_N^t] for t = 1,\ldots,T. Then a reasonable estimator for the vector of quality adjustment factors is \alpha = (1/T) \Sigma_{t=1}^T (p_1^t)^T \tilde{p}^t; i.e., take the arithmetic average of past vectors of normalized product prices where the normalized vector of prices for period t is the period t vector of prices \tilde{p}^t divided by the price of the numeraire product 1. The numeraire product should be chosen to be the product with the largest average share of expenditures on the N products. As another example where econometrics can play an important role in estimating the \alpha’s, we note that most hedonic regression models can be interpreted as additive utility models; see Diewert (2020d).
approximate a cost of living index. If the products are close substitutes, then the indifference curves should be drawn as curves that are almost straight lines. In this case, quality adjusted unit values could be used instead of attempting to estimate preferences so that reservation prices can be calculated.

From (B21), it can be seen that \( Q_L \) is equal to the following observable expressions:

\[
(B23) \quad Q_L = \frac{q^1}{[q^0 + (P^0/p^0)Q^0]}
= \frac{q^1}{[q^0 + (P^0Q^0)/p^0]}
= \frac{q^1}{[v^0/p^0]}
= [v^1/p^1]/[v^0/p^0]
= [v^1/v^0]/[p^1/p^0].
\]

Thus period 1 aggregate Laspeyres real consumption can be set equal to \( q^1 \) and period 0 aggregate Laspeyres real consumption can be set equal to \( q^0 + (P^0/p^0)Q^0 \) which is equal to \( q^0 + v^0/p^0 \) which in turn is equal to period 0 real consumption of continuing commodities, \( q^0 \), plus the value of period 0 consumption of disappearing commodities due to shutdowns, \( v^0 \), deflated by the period 0 price level for continuing commodities, \( p^0 \). The final equality in (B23) is also useful; it shows that \( Q_L \) is equal to the value ratio, \( v^1/v^0 \), divided by the consumer price index for the continuing commodity, \( p^1/p^0 \).

From (B22), it can be seen that \( Q_{P^*} \) is equal to the following expression (which cannot be evaluated unless an estimate for the reservation price \( P^1^* \) is available):

\[
(B24) \quad Q_{P^*} = \frac{q^1}{[q^0 + (P^1^*/p^1)Q^0]}. \]

Note the similarity of this expression for \( Q_{P^*} \) to the decomposition of \( Q_L \) given by the first equality in (B23): the price ratio for soon to be unavailable products to the price of continuing products in period 0, \( P^0/p^0 \), that is used in (B22) is replaced by the reservation price for unavailable products to the price of continuing products in period 1, \( P^1^*/p^1 \).
The inflation adjusted carry forward price for unavailable products in period 1, $P_{1A}$, is defined as follows:

$$(B25) \quad P_{1A} = (p^1/p^0)P^0.$$  

The approximate Paasche quantity index that results if we use $P_{1A}$ in place of the true reservation price $P_{1*}$ is $Q_{P^A}$ defined as follows:

$$(B26) \quad Q_{P^A} = q^1/[q^0 + (P_{1A}/p^1)Q^0]$$

$$= q^1/[q^0 + (P^0/p^0)Q^0] \quad \text{using (B25)}$$

$$= Q_L \quad \text{using (B23)}$$

$$\geq u^1/u^0 \quad \text{using (B21)}.$$  

Thus using inflation adjusted prices in our highly simplified model leads to an approximation to the Paasche quantity index that is exactly equal to the observable Laspeyres quantity index defined by (B21). Typically, the strict inequality will hold in (B26) so both $Q_L$ and $Q_{P^A}$ will have upward biases as compared to the economic Allen index, $u^1/u^0$.

Using definition (B21), we have the following alternative way of expressing $Q_L$:

$$(B27) \quad Q_L = [p^0q^1 + P^0Q^1]/[p^0q^0 + P^0Q^0]$$

$$= p^0q^1/[p^0q^0 + P^0Q^0] \quad \text{using (B1), } Q^1 = 0$$

$$= p^0q^0[q^1/q^0]/v^0 \quad \text{using (B3)}$$

$$= s_q^0[q^1/q^0]$$

where $s_q^0 = p^0q^0/v^0$ is the expenditure share of the continuing commodity in period 0.

Using definition (B22), we have the following alternative way of expressing $Q_{P^*}$:

$$(B28) \quad Q_{P^*} = [p^1q^1 + P^1Q^1]/[p^1q^0 + P^1*Q^0]$$
\[
\frac{p_1 q_1}{[p_1 q_1^0 + P_1^* Q_0^0]}
\]
using \(Q_1 = 0\)

\[
\frac{p_1 q_1}{[(p_1/p_0)q_1^0 + (P_1^*/P_0^0)P_0 Q_0^0]}
\]

\[
\frac{[p_1 q_1/v_0^0][(p_1/p_0)q_1^0/v_0^0] + (P_1^*/P_0^0)(P_0^0 Q_0^0/v_0^0)]}{(p_1/p_0)s_q^0 + (P_1^*/P_0^0)s_Q^0}
\]

\[
\frac{[p_1 q_1/p_0 q_0^0][(p_1/p_0)s_q^0 + (P_1^*/P_0^0)s_Q^0]}{s_q^0 q_1^0/\{s_q^0 + s_Q^0[(P_1^*/P_0^0)/(p_1/p_0)]\}}
\]

where \(s_Q^0 = P_0^0 Q_0^0/v_0^0\) is the period 0 expenditure share of the commodity which will disappear in period 1. If \(P_1^*/P_0^0 > p_1/p_0\), then \(Q_0^* < s_q^0 q_1^0 / q_0^0 = Q_L^9\).

From (B21) and (B22), we see that relative to the true index, \(u_1^0/u_0^0\), \(Q_L\) has an upward bias and \(Q_{P^*}\) has a downward bias. This suggests that taking an average of \(Q_L\) and \(Q_{P^*}\) would lead to an index which could provide a closer approximation to the true index. Define the Fisher (1922) quantity index, \(Q_{F^*}\), as the geometric mean of \(Q_L\) and \(Q_{P^*}\):

\[
(Q_{F^*}) \equiv \left[Q_L Q_{P^*}\right]^{1/2}
\]

\[
= \left\{\left[\frac{s_q^0 q_1^0}{s_q^0 + s_Q^0[(P_1^*/P_0^0)/(p_1/p_0)]}\right]\right\}^{1/2}
\]

\[
= s_q^0 q_1^0/\{s_q^0 + s_Q^0[(P_1^*/P_0^0)/(p_1/p_0)]\}^{1/2}
\]

\[
= Q_L/\{s_q^0 + s_Q^0[(P_1^*/P_0^0)/(p_1/p_0)]\}^{1/2}
\]

\[
= Q_L/m(\rho)
\]

where

(B30) \(\rho = (P_1^*/P_0^0)/(p_1/p_0)\) and

(B31) \(m(\rho) = [s_q^0 + s_Q^0 \rho]^{1/2}\).

---

9 The economic approach to index number theory will always imply that \(P_1^*/P_0^0 \geq p_1/p_0\) or \(P_1^* = (p_1/p_0)P_0^0\) which is the inflation adjusted carry forward price for the disappearing product. The economic approach is consistent with \(P_1^* = (p_1/p_0)P_0^0\) but this equality will hold only if the consumer’s indifference curves are parallel straight lines so that \(q\) and \(Q\) are perfect substitutes after quality adjustment. The perfect substitutes condition is unlikely to hold under a widespread lockdown.
Thus $\rho$ is a relative inflation rate; it is the price index for locked out commodities, $P'^*/P^0$, divided by the price index for always available commodities, $p^*/p^0$, and $m(\rho)$ is the square root of the weighted mean of 1 and $\rho$, where the weights are the base period expenditure shares, $s^0_q$ for 1 and $s^0_Q$ for $\rho$. Thus if $\rho > 1$, then the weighted mean of 1 and $\rho$ will be greater than 1 and thus the square root of this weighted mean will also be greater than 1 and thus $Q^*_F$ will be less than the Laspeyres quantity index, $Q_L$. If $\rho = 1$, then $m(\rho) = 1$ and $Q^*_F = Q_L$.

It is useful to provide an additive approximation to $m(\rho)$. It can be seen that $m(\rho)$ can be interpreted as the geometric mean of 1 and the function $s^0_q + s^0_Q \rho$. An approximation to this geometric mean is the corresponding arithmetic mean. Thus we have the following approximation to $m(\rho)$:

$$m(\rho) \approx \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(s^0_q + s^0_Q \rho) = \left(\frac{1}{2}\right)[1 + s^0_q] + \left(\frac{1}{2}\right)s^0_Q \rho.$$

Using (B29) and (B32), we have the following approximation to the Fisher quantity index:

$$Q^*_F \approx s^0_q (q^1/q^0) / \left(\frac{1}{2}\right)\{[1 + s^0_q] + [s^0_Q (P'^*/P^0)/(p^1/p^0)]\}.$$

Up to this point, we have assumed that the household’s utility function, $f(q,Q)$ was linearly homogeneous, increasing and concave. If we are willing to assume that $f(q,Q)$ has a certain functional form, then it can be shown that the Fisher quantity index $Q^*_F$ defined by (B29) is exactly equal to the true Allen quantity index, $u^1/u^0$, under the assumption that the household utility function has this certain functional form and the first order Taylor series approximation to $m(\rho)$ around the point $\rho = 1$ is also equal to the right hand side of (A32). If $\rho = 1$, then the right hand side of (A32) is also equal to 1; if $\rho > 1$, then the right hand side of (A32) is also greater than 1.

This functional form is $f(q,Q) = [a_{11}q^2 + 2a_{12}qQ + a_{22}Q^2]^{1/2}$. The parameters for this function are the $a_{ij}$. These parameters must satisfy some restrictions in order to satisfy the concavity and monotonicity conditions for $f(q,Q)$. These conditions are described in Diewert (1976), Diewert and Hill (2010) and Diewert and Feenstra (2019). This functional form is flexible; i.e., it can approximate an arbitrary linearly
household is minimizing the cost of achieving the utility level \( u^1 = f(q^1, 0) \) in period 1 and the utility level \( u^0 = f(q^0, Q^0) \) in period 0. Thus under the assumption that \( f(q, Q) \) has the required functional form, we have the following exact equality:

\[
(B34) \quad u^1/u^0 = Q_F^* = [Q_L Q_P^*)^{1/2}.
\]

It is useful to develop formulae for the Laspeyres and Paasche price indexes that are counterparts to the formulae for the Laspeyres and Paasche quantity indexes given by (B27) and (B28).

Define the Laspeyres price index \( P_L^* \) as follows:

\[
(B35) \quad P_L^* \equiv \left[ p^1 q^0 + P_{1*} Q^0 \right]/\left[ p^0 q^0 + P^0 Q^0 \right]
\]

\[
= \left[ (p^1/p^0) p^0 q^0 + (P_{1*}/P^0) p^0 Q^0 \right]/v^0
\]

\[
= (p^1/p^0) s_q^0 + (P_{1*}/P^0) s_Q^0
\]

\[
= (p^1/p^0) \{s_q^0 + [(P_{1*}/P^0)/(p^1/p^0)] s_Q^0 \}.
\]

Define the Paasche price index \( P_P \) as follows:

\[
(B36) \quad P_P \equiv \left[ p^1 q^1 + P_{1*} Q^1 \right]/\left[ p^0 q^1 + P^0 Q^1 \right]
\]

\[
= p^1 q^1/p^0 q^1
\]

\[
= p^1/p^0.
\]

The Fisher (1922) price index \( P_F^* \) is defined as the geometric mean of \( P_L^* \) and \( P_P \):

\[
(B37) \quad P_F^* \equiv [P_L^* P_P]^{1/2}
\]

\[
= (p^1/p^0) \{s_q^0 + [(P_{1*}/P^0)/(p^1/p^0)] s_Q^0 \}^{1/2}
\]

\[
= (p^1/p^0) m(\rho)
\]

\[
= (p^1/p^0) \{(1/2)[1 + s_q^0 \} + (1/2) s_Q^0 [(P_{1*}/P^0)/(p^1/p^0)]\}
\]

homogeneous, twice continuously differentiable function to the second order around any point \((q, Q)\). This functional form has a linear utility function as a special case; see Diwert (2020a; 15).
where \( s_0^0 = P^0Q^0/v^0 \) is the period 0 expenditure share of the commodity which will disappear in period 1. If \( P^1^*/P^0 > p^1/p^0 \), then \( P_L^* > P_F^* > P_P \).

Using the above definitions, it is straightforward to show that the following well known equalities hold:

\[
(B38) \frac{v^1}{v^0} = P_L^* Q_P^* = P_P Q_L = P_F^* Q_F^*.
\]

In order to calculate \( Q_F^* \) defined by (B29) or \( P_F^* \) defined by (B37), we need an estimate for the reservation price \( P^1^* \). It is possible to generate an estimate for \( P^1^* \) if an estimate for the elasticity of demand for the products is known or could be estimated using econometric techniques. In order to accomplish this task, we first require some preliminary material.

Recall definitions (B7)-(B10) which defined the Hicksian demand functions\(^{12}\) for the two goods evaluated at the period 0 and 1 data for the household. For general positive prices, \( p \) and \( P \), and positive utility level \( u \), the Hicksian demand functions, \( q(u,p,P) \) and \( Q(u,p,P) \), can be defined in terms of the first order partial derivatives of the consumer’s unit cost function, \( c(p,P) \) as follows:

\[
(B39) q(u,p,P) = c_1(p,P)u ;
\]
\[
(B40) Q(u,p,P) = c_2(p,P)u
\]

where \( c_2(p,P) = \partial c(p,P)/\partial P \). The *Hicksian elasticity of demand* for \( Q \) as a function of \( P \) evaluated at \( u^0,p^0,P^0, \eta^0 \), is defined as follows:

\[
(B41) \eta^0 = \frac{[P^0/Q^0] \partial Q(u^0,p^0,P^0)/\partial P}{P^0 c_2(p^0,P^0) u^0/Q^0}
\]

differentiating both sides of (B40) with respect to \( P \) and evaluating the derivatives at \( (u^0,p^0,P^0) \).

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\(^{12}\) See Hicks (1946; 311-331).
\[= P^0 c_{22}(p^0, P^0)u^0/c_2(p^0, P^0)u^0 \text{ using (B8)}
\]
\[= P^0 c_{22}(p^0, P^0)/c_2(p^0, P^0) \leq 0\]

where the inequality follows from the concavity of the unit cost function in \(p\) and \(P\). We will assume that the inequality in (B41) is strict so that \(\eta^0 < 0\); i.e., we assume that as the price of \(Q\) goes up, consumers buy less of it and more of \(q\) in order to keep their utility or welfare level constant. Below, we will find it convenient to work with the negative of \(\eta^0\), which we define as \(\eta^0*\). Thus we have the following inequality:

\[(B42) \quad \eta^0* = -P^0 c_{22}(p^0, P^0)u^0/Q^0 > 0.\]

It turns out that it will be useful to find the period 0 reservation price for the disappearing commodity which we label as \(P^0*\). Thus \(P^0*\) is the price which will make the demand for \(q\) equal to 0 in period 0; i.e., \(P^0*\) is the solution to the following equation:

\[(B43) \quad 0 = Q(u^0, p^0, P^0*)
\]
\[= c_2(p^0, P^0*)u^0 \text{ using (B40)}.\]

The reason why it is useful to find the period 0 reservation price \(P^0*\) is due to the fact that if we can determine \(P^0*\), then we can determine the period 1 reservation price \(P^1*\); in fact, the following equation holds:

\[(B44) \quad P^1*/p^1 = P^0*/p^0.\]

Using (B43), we see that the following equation holds: \(^{13}\)

\[(B45) \quad 0 = c_2(p^0, P^0*)]^{13}\]

\(^{13}\) Since \(c(p, P)\) is linearly homogeneous in \((p, P)\), Euler’s Theorem on homogeneous functions implies that the first order derivatives of this function are homogeneous of degree 0 in \((p, P)\). Thus we have \(c_2(\lambda p, \lambda P) = c_2(p, P)\) for all \(\lambda > 0\). Choose \(\lambda = 1/p\) and we obtain the equation \(c_2(p, P) = c_2(1/P, p)\).
Thus if we set $p^1*$ equal to the inflation adjusted carry forward reservation price $(p^1/p^0)P^0*$, this will be the correct reservation price for period 1.

We can find an approximation to $P^0*$ by equating the first order Taylor series approximation to $Q(u^0,p^0,P)$ around the point $P^0$ to zero. Call this approximation to $P^0*$, $P^{0**}$. Thus we need to solve the following equation for $P^{0**}$:

\[(B46) \quad 0 = Q(u^0,p^0,P^0) + \left[ \partial Q(u^0,p^0,P^0)/\partial P \right][P^{0**} - P^0] \]

\[= Q^0 + c_{22}(p^0,P^0)u^0[P^{0**} - P^0] \]

where the second equality follows by differentiating $Q(u^0,p^0,P^0)$ defined by $(B8)$. Solving the equation $(B46)$ for $P^{0**}$ leads to the following estimate for $P^{0*}$:

\[(B47) \quad P^{0**} - P^0 = -Q^0/c_{22}(p^0,P^0)u^0 \]

\[= P^0/\eta^{0*} \quad \text{using definition (B42)}. \]

Divide both sides of $(B47)$ through by $P^0$ and we obtain the following expression for $P^{0**}/P^0$:

\[(B48) \quad P^{0**}/P^0 = 1 + 1/\eta^{0*} > 1. \]

\[\text{**The approximation methodology used here is similar to the methodology used by Hausman (1981) (2003), Diewert (1998), Diewert and Feenstra (2019), Diewert, Fox and Schreyer (2019) and Brynjolfsson, Collis, Diewert, Eggers and Fox (2020).} \]
The inequality in (B48) follows because we assumed \( \eta^0 > 0 \). From (B44), we see that \( P^1 = (p^1/p^0)P^0 \). Approximate \( P^0 \) by \( P^{0*} \) and define the approximate reservation price for period 1 as:

\[
(B49) \quad P^{1**} = (p^1/p^0)P^{0**} = (p^1/p^0)P^0(1 + 1/\eta^{0*}) \quad \text{using (B48)}.
\]

The last equation in (B49) can be rearranged to give us the following approximation to the term \( (P^1*/P_0)/(p^1/p^0) \) which appears in the formulae for \( Q_P^*, Q_F^*, P_L^* \) and \( P_F^* \):\(^{15}\)

\[
(B50) \quad \rho^* = (P^{1**}/P^0)/(p^1/p^0) = 1 + 1/\eta^{0*}.
\]

The above analysis shows that if an estimate for the elasticity of demand \( \eta^{0*} \) can be obtained, then estimates for the bias in the Laspeyres quantity index \( Q_L \) defined by (B27) relative to the Paasche and Fisher quantity indexes defined by (B28) and (B29) can be calculated. Similarly, if an estimate for the elasticity of demand \( \eta^{0*} \) exists, then estimates for the bias in the Paasche price index \( P_P \) defined by (B36) relative to the Laspeyres and Fisher quantity indexes defined by (B35) and (B37) can be calculated.

To illustrate the possible magnitudes of the differences between the Laspeyres quantity index for consumption defined by (B27) relative to the Fisher quantity index defined by (B37), let \( s_q^0 = 1/2, s_Q^0 = 1/2 \) (so that 1/2 of all consumption producing industries are shut down), \( p^1/p^0 = 1 \) (so that there is no inflation in the prices of continuing commodities) and \( \eta^{0*} = 1/2 \). In this case, \( \rho^* = (P^{1**}/P^0)/(p^1/p^0) = 1 + 1/\eta^{0*} = 1 + 2 = 3 \) and \( m(\rho^*) = [s_q^0 + s_Q^0 \rho^*]^{1/2} = [1/2 + (1/2)3]^{1/2} = 2^{1/2} = 1.414. \)\(^{16}\) Then \( Q_L/Q_F^* = m(\rho^*) = 1.414 \). Hence the Laspeyres quantity index will overstate real consumption (from a welfare perspective) by about 40%.

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\(^{15}\) See (A28), (A29), (A35) and (A37).

\(^{16}\) If we use the approximation to \( m(\rho) \) defined by the right hand side of (A32), we find that this approximate estimate is equal to 1.5 as compared to the true estimate equal to \( 2^{1/2} \approx 1.414 \).

The bias in the Paasche price index (which is equal to the Laspeyres and Paasche price indexes for just the continuing commodities in our simple model) relative to the Fisher index will go in the other direction and understate cost of living inflation. Thus \( P_F/P_F^* = 1/m(\rho^*) = 1/1.414 = 0.707 \). Thus the Paasche price index (which is also the carry forward price index in our simple model) will understate cost of living inflation by about 30%. Carry forward basket type indexes which have a basket which is approximately proportional to the period 0 quantity vector will have biases (relative to a welfare or cost of living perspective) similar to the biases in the Laspeyres quantity index and in the Paasche price index. Our simple model analysis indicates that the bias in fixed basket type price indexes will be very large if the pre-shutdown share of expenditure for commodities that are ultimately unavailable during lockdown periods is large.

**Appendix C: Defining Reservation Prices Under Lockdown Conditions**

We use the same notation as was used in section 2 of the main text with one exception: we now assume that some components of the \( Q^1 \) vector could be positive. We also assume that the analysis here is applied to a single household. An example where the quantity consumed of a lockdown affected commodity is positive in a lockdown period is rental housing where the tenant is given a rent holiday. Thus in this case, if the first commodity in Group 2 is rental housing, then \( Q^1_1 > 0 \) but the observed price is \( P^1_1 = 0 \). However, the utility value of its free rent to the household is not 0; it is a positive reservation price, \( P^*_1 > 0 \). Our problem here is to find a way to estimate this reservation price. Once we have found this price, it is appropriate to add \( P^*_1 Q^1_1 \) to the observed expenditures, \( p^1 \cdot q^1 \), in order to better approximate the value of actual consumer expenditure in period 1. Hence this Appendix has some relevance to the construction of the national accounts during lockdown periods as well as being relevant for the construction of a cost of living index.

Restating the notation used in section 2, denote the period \( t \) price and quantity vectors for always available products by \( p^t = [p^t_1, ..., p^t_M] >> 0_M \) and \( q^t = [q^t_1, ..., q^t_M] >> 0_M \) for \( t = 0, 1 \). The lockdown affected price and quantity vectors for period 0 are \( P^0 = [P^0_1, ..., P^0_N] >> 0_N \).
and $Q^0 = [Q^0_1,...,Q^0_N] >> 0_N$. The observed price for the locked down commodities in period 1 is $P^1 = 0_N$, and the corresponding observed quantity vector is $Q^1 = [Q^1_1,...,Q^1_N] > 0_N$. Most of the components of $Q^1$ will be zero but some components could be positive; i.e., the government may be providing some goods and services free of charge or it may legislate rent holidays for tenants which will lead to 0 prices. We provide a methodology for finding estimates for the vector of reservation prices $P^{1*}$.

We first consider the consumer’s period 0 cost minimization problem. Suppose the household utility function is $u = f(q,Q)$ where $f(q,Q)$ is nonnegative, increasing and concave in the components of the vectors $q$ and $Q$. Define the period 0 utility level as $u^0 = f(q^0,Q^0) > 0$. We assume that $(q^0,Q^0)$ solves the following period 0 cost minimization problem:

(C1) $C(u^0,p^0,P^0) = \min_{q,Q \geq 0} \{ p^0 \cdot q + P^0 \cdot Q : f(q,Q) \geq u^0; q \geq 0_M; Q \geq 0_N \}$

where $C(u,p,P)$ is the consumer’s cost function.\(^{17}\)

Define the consumer’s conditional cost function, $C^*(u,p,Q)$, for $u \geq 0$, $p >> 0_M$ and $Q \geq 0_N$ as follows:

(C2) $C^*(u,p,Q) = \min_{q \geq 0M} \{ p \cdot q : f(q,Q) \geq u; q \geq 0_M \}$.

When solving the cost minimization problem defined by (C2), the consumer minimizes the cost associated with the purchase of always available goods and services, $p \cdot q$, subject to achieving at least the utility level $u$, conditional on having on hand, the vector of sometimes available goods and services $Q$.\(^{18}\) We note that if the consumer’s utility

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\(^{17}\) The cost function is used to define true cost of living indexes such as $C(u^0,p^1,P^{1*})/C(u^0,p^0,P^0)$ and $C(u^1,p^0,P^{1*})/C(u^1,p^0,P^0)$.

\(^{18}\) Under our regularity conditions on $f(q,Q)$, it can be shown that $C^*(u,p,Q)$ has the following properties: it is increasing in $u$ for fixed $p$ and $Q$, it is nondecreasing, concave and linearly homogeneous in $p$ for fixed $u$. 

function \( f(q,Q) \) is estimated using available econometric techniques for data in the pre-lockdown period, then an empirical estimate for the conditional cost function \( C^*(u,p,Q) \) defined by (C2) can be obtained.

We assume that the observed period 1 consumption vector \( q^1 \) solves the conditional cost minimization problem (C2) when \((u,p,Q)\) equals \((u^1,p^1,Q^1)\). Using this assumption, we have the following equality:

\[
(C3) \; p^1 q^1 = C^*(u^1,p^1,Q^1).
\]

Assuming that \( C^*(u^1,p^1,Q) \) is differentiable with respect to the components of \( Q \) at the point \( Q = Q^1 \), define the vector of *period 1 reservation prices for the goods and services* subject to lockdown restrictions, \( P^1^* \), as the negative of the vector of first order partial derivatives of \( C^*(u^1,p^1,Q^1) \) with respect to the components of \( Q \); i.e., define \( P^1^* \) as follows:

\[
(C4) \; P^1^* \equiv - \nabla_Q C^*(u^1,p^1,Q^1) \geq 0_N
\]

where the inequality in (C4) follows from the fact that \( C^*(u,p,Q) \) is nonincreasing in the components of \( Q \).

We use the reservation prices defined by (C4) as the prices for the lockdown affected goods and services for period 1. Consider the following period 1 (regular) cost minimization problem where \( u^1 \equiv f(q^1,Q^1) \):

\[
(C5) \; C(u^1,p^1,P^1^*) \equiv \min_{q,Q} \{ p^1 \cdot q + P^1^* \cdot Q : f(q,Q) \geq u^1; \; q \geq 0_M; \; Q \geq 0_N \}
\]

\[
= \min_{Q} \{ P^1^* \cdot Q + \min_{q} \{ p^1 \cdot q : f(q,Q) \geq u^1; \; q \geq 0_M \} : Q \geq 0_N \}
\]

\[
= \min_{Q} \{ P^1^* \cdot Q + C^*(u^1,p^1,Q) : Q \geq 0_N \}
\]

using definition (C2).

and \( Q \), it is convex in \((u,Q)\) for fixed \( p \) and it is nonincreasing in \( Q \) for fixed \( p \) and \( u \). If in addition \( f(q,Q) \) is linearly homogeneous in \( q,Q \), then \( C^*(u,p,Q) \) is linearly homogeneous in \((u, Q)\).
The first order necessary conditions for $Q^1$ to solve the final cost minimization problem\(^\text{19}\) in (C5) are the following conditions:

\[(C6) \quad P^1 + \nabla Q^*(u^1,p^1,Q^1) = 0_N.\]

But conditions (C6) are equivalent to equations (C4), which were used to define the reservation prices $P^1$. Hence we have the following:

\[(C7) \quad C(u^1,p^1,P^1) = \min_Q \{P^1 Q + C^*(u^1,p^1,Q) : Q \geq 0_N\}\]
\[= P^1 Q^1 + C^*(u^1,p^1,Q)\]
\[= P^1 Q^1 + p^1 q^1\]

The above algebra shows that if the consumer had the “income” $p^1 q^1 + P^1 Q^1$ to spend on the commodities in scope for the lockdown period 1 and faced the prices $p^1$ and $P^1$, then the consumer would minimize the cost of achieving the actual utility level $u^1 \equiv f(q^1,Q^1)$ by choosing the observed consumption vector, $(q^1,Q^1)$. Thus the observed lockdown restricted consumption vector would be freely chosen by the consumer facing the price vectors $(p^1,P^1)$ in order to minimize the cost of achieving the actual period 1 utility level, $u^1$.

Our conclusion at this point is that if the target CPI is a cost of living index, then the reservation price vector $P^1$ should be used to value period 1 locked down products in place of the observed vector of zero prices, $P^1 = 0_N$. Laspeyres, Paasche and Fisher indexes can be formed using the reservation prices $P^1$.

If we make the further assumption that the utility function $f(q,Q)$ is the square root of a quadratic form in the components of $q$ and $Q$, then since this functional form is exact for

\(^\text{19}\) These conditions are also sufficient for $Q^1$ to solve the last minimization problem since $C^*(u,p,Q)$ is a convex function of $Q$ under our regularity conditions.
the Fisher index using the reservation prices, the utility ratio, \( u^1/u^0 \), would be equal to this Fisher index.

There is only one aspect of the reservation price methodology presented here which is not standard: the methodology above shows how theoretical reservation prices can be obtained for products that are supplied at zero prices during the lockdown period but the corresponding quantities held by consumers may be positive instead of being set equal to zero. If there are such products so that \( Q^1 > 0 \), then \( P^1\cdot Q^1 > 0 \) and this imputed expenditure on lockdown affected products should be added to the expenditures on always available goods and services during period 1, \( p^1\cdot q^1 \), to form overall consumer expenditures or “income”.

This has implications for the System of National Accounts: actual consumer expenditures on goods and services during lockdown periods, which are equal to \( p^1\cdot q^1 \), need to be augmented by imputed expenditures on lockdown affected goods and services that are provided at zero prices but have marginal utilities that are above zero, \( P^1\cdot Q^1 \). For lockdown period products \( n \) that are simply not available, the corresponding quantity \( Q^1_n \) will equal 0 so \( P^1_n\cdot Q^1_n \) will also equal 0 and there is no need to add it to actual expenditures. But for some government services provided at zero cost and for rental housing that is temporarily supplied at a zero price, we need to add these imputed expenditures to actual expenditures to get an accurate picture of actual real consumption and welfare.

There is an example in Diewert and Feenstra (2019) that shows how the Fisher functional form can be estimated econometrically and it also calculates the resulting reservation prices for missing products in a grocery store example. So it is possible to estimate reservation prices, but the estimation of reservation prices at scale is not possible at the present time. The actual reservation prices will have to be approximated by inflation adjusted carry forward prices or by simple unadjusted carry forward prices. The analysis in Appendix A above shows that inflation adjusted carry forward prices can have a substantial downward bias relative to their true reservation prices.
Finally, the above methodology can be modified to deal with a related problem associated with the measurement of housing rents. Many urban areas have some form of rent control imposed on price increases for rental housing. If these rent controls have been in place for long periods of time, the actual price paid for a rented dwelling unit in period $t$, say $P_n^t$, could be well below a current market price $P_n^{t*}$, which would likely approximate the appropriate period $t$ reservation price. Thus in order to calculate a cost of living index, the actual price paid $P_n^t$ should be replaced by its higher opportunity cost price $P_n^{t*}$ and the resulting Laspeyres, Paasche and Fisher indexes should be calculated using these higher prices.\footnote{This replacement should only be done if there are local rent controls that have led to artificially low rental prices.} In the national accounts, the actual rental income and expenditure, $P_n^t Q_n^t$, will be recorded in both the production and household accounts but the imputed housing expenditures above the recorded expenditures, $P_n^{t*} Q_n^t - P_n^t Q_n^t > 0$, needs to be added to household expenditures on consumer goods and services.