A Brooks’ Theorem for Triangle-Free Graphs

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Abstract. Let $G$ be a triangle-free graph with maximum degree $\Delta(G)$. We show that the chromatic number $\chi(G)$ is less than $67(1 + o(1))\Delta / \log \Delta$.

1 Introduction

A proper vertex coloring of a graph is an assignment of colors to all vertices such that adjacent vertices have distinct colors. The chromatic number $\chi(G)$ of a graph $G$ is the minimum number of colors required for a proper vertex coloring. Finding the chromatic number of a graph is NP-Hard [10]. Approximating it to within a polynomial ratio is also hard [15]. For general graphs, $\Delta(G) + 1$ is a trivial upper bound. Brooks’ Theorem [7] shows that $\chi(G)$ can be $\Delta(G) + 1$ only if $G$ has a component which is either a complete subgraph or an odd cycle.

A natural question is: can this bound be improved for graphs without large complete subgraphs? In 1968, Vizing [22] had asked what the best possible upper bound for the chromatic number of a triangle-free graph was. Borodin and Kostochka [6], Catalin [8], and Lawrence [19] independently made progress in this direction; they showed that for a $K_4$-free graph, $\chi(G) \leq 3(\Delta(G) + 2)/4$. On the other hand, Kostochka and Masurova [18], and Bollobás [5] separately showed that there are graphs of arbitrarily large girth (length of a shortest cycle) with $\chi(G)$ of order $\Delta(G)/\log \Delta$.

In 1995, Kim [16] proved that
$$\chi(G) \leq (1 + o(1))\frac{\Delta(G)}{\log \Delta(G)}$$
when $G$ has girth greater than 4. Later on, Johansson [12] showed that
$$\chi(G) \leq O\left(\frac{\Delta(G)}{\log \Delta(G)}\right)$$
when $G$ is a triangle-free graph (girth greater than 3). Alon, Krivelevich and Sudakov [3], and Vu [23] extended the method of Johansson to prove bounds on the chromatic number for graphs in which no subgraph on the set of all neighbors of a vertex has too many edges.

Both Kim and Johansson used the so-called semi-random method to show that the chromatic number of graphs with large girth is $O(\Delta(G)/\log \Delta(G))$. This technique, also known as the pseudo-random method, or the Rödl nibble, appeared first in Ajtai, Komlós and Szemerédi [2] and was applied to problems in hypergraph packings, Ramsey theory, colorings, and list colorings [9,13,14,17,21]. In general, given a set $S_1$, the goal is to show that there is an object in $S_1$ with a desired property $P$. This is done by locating a sequence of non-empty subsets $S_1 \supseteq \cdots \supseteq S_t$ with $S_t$ having property $P$. A randomized algorithm is applied to $S_t$, which guarantees that $S_{t+1}$ will be obtained with some non-zero (often small) probability. For upper bounds on chromatic number, the semi-random method is used to prove the existence of a proper coloring with a limited number of colors.

In this paper we prove that the chromatic number of a triangle-free graph $G$ is less than $67(1 + o(1))\Delta / \log \Delta$. As we will indicate in Section 2 our proof is derived from Kim’s proof of an upper bound

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to the chromatic number of graphs with girth greater than 4. We believe our technique is simpler than Johanssons’ which follows a different approach to that of Kims (see [20] for a comparison of both).

We give our proof by analyzing an iterative algorithm for graph coloring. To analyze this algorithm we identify a collection of random variables. The expected changes to these random variables after a round of the algorithm are written in terms of the values of the random variables before the round. We thus obtain a set of recurrence relations and prove that our random variables are concentrated around the solutions to the recurrence relations with some positive probability.

We describe our algorithm in Section 2. Section 2.1 contains motivation, which is followed by a formal description of the algorithm in Section 2.2. We give an outline of the analysis in Section 3. Section 4 contains some useful lemmas which we are used in Section 5 to give details of the analysis.

2 An Iterative Algorithm for Coloring a Graph

Our algorithm takes as input a triangle-free graph \( G \) on \( n \) vertices, its maximum degree \( \Delta \), and the number of colors to use \( \Delta/k \) where \( k \) is a positive number. It goes through rounds and assigns colors to more vertices each round. Initially all vertices are uncolored (no color assigned), at the end we have a proper vertex coloring of \( G \) with some probability.

**Definition 1.** Let \( t \) be a natural number. We define the following:
- \( G_t \) The graph induced on \( G \) by the vertices that are uncolored at the beginning of round \( t \).
- \( N_t(u) \) The set of vertices adjacent to vertex \( u \) in \( G_t \). That is, the set of uncolored neighbors of \( u \) at the beginning of round \( t \).
- \( S_t(u) \) The list of colors that may be assigned to vertex \( u \) in round \( t \), also called the palette of \( u \). For all \( u \) in \( V(G) \),
  \[ S_0(u) = \{1, \ldots, \Delta/k\}. \]
- \( D_t(u, c) \) The set of vertices adjacent to \( u \) that may be assigned color \( c \) in round \( t \). That is,
  \[ D_t(u, c) := \{v \in N_t(u) \mid c \in S_t(v)\}. \]

It will be useful to define variables for the sizes of the sets \( S_t(u) \) and \( D_t(u, c) \).

**Definition 2.**
- \( s_t(u) = |S_t(u)| \)
- \( d_t(u, c) = |D_t(u, c)| \)

Observe that for every round \( t \), vertex \( u \) in \( V(G) \), and color \( c \) in \( \{1, \ldots, \Delta/k\} \),

\[ d_t(u, c) \leq \Delta, \quad s_0(u) = \Delta/k, \quad s_t(u) \leq \Delta/k. \]

2.1 A Sketch of the Algorithm and the Ideas behind its Analysis

We say that a sequence \( x(n) \) is \( O(f(n)) \) if there is a positive number \( M \) such that \( |x(n)| \leq M|f(n)| \). All sequences in the big-oh are indexed by \( \Delta \), the maximum degree of graph \( G \). Remember that each occurrence of the big-oh comes with a distinct constant \( M \). We start by considering an algorithm that colors \( \Delta \)-regular graphs with girth greater than 4. The coloring produced is proper with positive probability.
Let \((d_t)\) and \((s_t)\) be sequences defined recursively as

\[
\begin{align*}
d_0 &:= \Delta \\
s_0 &:= \Delta/k \\
d_{t+1} &:= d_t(1 - c_1 \frac{a}{d_t})c_2 \\
s_{t+1} &:= s_tc_2
\end{align*}
\]

where \(c_1\) and \(c_2\) are constants between 0 and 1, which are determined by the analysis of the algorithm.

\[
\text{Repeat at every round } t, \text{ until } d_t/s_t < 1/2
\]

**Phase I - Coloring Attempt**

For each vertex \(u\) in \(G_t\):

- **Awake** vertex \(u\) with probability \(s_t/d_t\).
- If awake, assign to \(u\) a color chosen from \(S_t(u)\) uniformly at random.

**Phase II - Conflict Resolution**

For each vertex \(u\) in \(G_t\):

- If \(u\) is awake, uncolor \(u\) if an adjacent vertex is assigned the same color.
- Remove from \(S_t(u)\), all colors assigned to adjacent vertices.

\[S_{t+1}(u) = S_t(u)\]

end repeat

Permanently color each vertex \(u\) in \(V(G_t)\) with a color picked independently and uniformly at random from its palette \(S_t(u)\).

Observe that \(d_0/s_0 = k\) and

\[
\frac{d_{t+1}}{s_{t+1}} = \frac{d_t}{s_t} - c_1.
\]

In \(O(k)\) rounds \(d_t/s_t\) will be less than 1/2, and this marks the end of the repeat-until block.

The algorithm above is derived from Kim [16]. After some modifications, his analysis tells us that if graph \(G\) has girth greater than 4, then there are constants \(c_1\) and \(c_2\) less than 1 such that at each round \(t\), \(\forall u \in V(G_t), \forall c \in S_t(u)\)

\[
s_t(u) = s_t(1 + o(1)), \quad d_t(u, c) = d_t(1 + o(1))
\]

with probability greater than 0. The equations above imply that after the repeat-until block, if \(\Delta\) is large enough, then with positive probability \(s_t(u) > 2d_t(u, c)\) for all uncolored vertices \(u\) and colors \(c\) in their palette. Now, applying the result of Haxell [11], we find that the random assignment of colors to all uncolored vertices in the final step of the algorithm gives a properly colored graph with positive probability.

**The problem with cycles of length 4.** The analysis for the above algorithm is probabilistic and proves the property in equation (1) by induction, showing concentration of the variables around their expectations. It fails for graphs with 4-cycles. An example illustrates why: Consider a vertex \(u\) whose 2-neighborhood, the graph induced by vertices within distance 2 of \(u\), is the complete bipartite graph \(K_{\Delta,\Delta}\) with partitions \(X\) and \(Y\). Suppose that \(u\) and another vertex \(v\) are in \(X\). If \(v\) is colored with \(c\) in round 0 while \(u\) remains uncolored, then the set \(D_1(u, c) = \emptyset\); this violates equation (1) since \(d_1 \geq 1\) if for example \(k \geq 2\) and \(\Delta \geq 2/c_2\). So, when the graph has 4-cycles, \(d_{t+1}(u, c)\) is not necessarily concentrated around its expectation with positive probability, given the state of the algorithm at the beginning of round \(t\). We must modify the algorithm in two ways.

**First Modification: A technique for coloring graphs with 4-cycles.** While \(d_t(u, c)\) is not concentrated enough when the graph has 4-cycles, our analysis will show that the average of \(d_{t+1}(u, c)\) over all colors
in the palette of a vertex \( u \) is concentrated enough. How does this benefit us? Markov’s famous inequality may be interpreted as: a list of \( s \) positive numbers which average \( d \) has at most \( s/q \) numbers larger than \( qd \) for any positive number \( q \). We modify the algorithm so that at the end of each round \( t \), every vertex \( u \) removes from its palette every color \( c \) with \( d_{t+1}(u,c) > 2d_t \). Look at what happens in round \( t = 1 \). By a straightforward application of Markov’s inequality, instead of equation (1) we will have the less stringent property:

\[
\forall u \in V(G_t), \forall c \in S_t(u), s_t(u) \geq \frac{1}{2} s_t(1 - o(1)), \quad d_t(u,c) \leq 2d_t(1 + o(1)).
\]  

with positive probability. In fact, using a generalization of Markov’s inequality, the analysis will show that with a few more modifications our algorithm maintains, with positive probability, a slightly stronger property (still weaker than equation (1)).

Equation (1) implies that the \( s_t(u) \) and \( d_t(u) \) at all uncolored vertices \( u \) are about the same. It is a strong statement and helps in the proofs, but is too much to maintain on graphs with 4-cycles. Equation (2) is weaker and is obtained by our algorithm with positive probability. Moreover, it is sufficient to ensure that after the repeat-until block, with positive probability, for each uncolored vertex \( u \) and color \( c \) in its palette, \( s_t(u) \geq 2d_t(u,c) \). This is a key idea in our algorithm.

**Second Modification: Using independent random variables for easier analysis.** Instead of waking up a vertex with some probability, and then choosing a color from its palette uniformly at random; for each uncolored vertex \( u \) and color \( c \) in its palette, we will assign \( c \) to \( u \) independently with some probability. In case multiple colors remain assigned to the vertex after the conflict resolution phase, we will arbitrarily choose one of them to permanently color the vertex. This modification, adapted from Johansson [12], will make concentration of our random variables simpler.

Next we provide a formal description of the algorithm we have just motivated.

### 2.2 A Formal Description of the Algorithm

Let \( (d_t) \) and \( (s_t) \) be sequences defined recursively as

\[
\begin{align*}
  d_0 &:= \Delta \\
  s_0 &:= \Delta/k \\
  d_{t+1} &:= d_t(1 - \frac{1}{16} e^{-1/2} \frac{s_t}{d_t}) e^{-1/2} \\
  s_{t+1} &:= s_t e^{-1/2}.
\end{align*}
\]  

For round \( t \), vertex \( u \), and color \( c \),

\[
\mathcal{F}_t(u,c) := \{ c \text{ is not assigned to any vertex adjacent to } u \text{ in round } t \}
\]  

is an event in the probability space generated by the random choices of the algorithm in round \( t \), given the state of all data structures at the beginning of the round.

Let

\[
\text{Desired } \mathcal{F}_t := e^{-1/2}.
\]
\textbf{Repeat at every round} $t$, until $d_t/s_t < 1/8$

\textit{Phase I - Coloring Attempt}
For each vertex $u$ in $G_t$, and color $c$ in $S_t(u)$:
\begin{itemize}
\item Assign $c$ to $u$ with probability $\frac{1}{d_t}$.
\end{itemize}

\textit{Phase II - Conflict Resolution}
For each vertex $u$ in $G_t$:
\begin{itemize}
\item Phase II.1
Remove from $S_t(u)$, all colors assigned to adjacent vertices.
\item Phase II.2
For each color $c$ in $S_t(u)$, remove $c$ from $S_t(u)$ with probability
$$1 - \min(1, \frac{\text{Desired } F_t(u,c)}{\text{Pr}(F_t(u,c))}).$$
\end{itemize}
If $S_t(u)$ has at least one color which is assigned to $u$,
then arbitrarily pick an assigned color from $S_t(u)$ to permanently color $u$.

\textit{Phase III - Cleanup (discard all colors $c$ with $d_{t+1}(u,c) \geq 2d_{t+1}$ from palette)}
For each vertex $u$ in $G_t$:
\begin{itemize}
\item $S_{t+1}(u) = S_t(u)$.
\item Let $\alpha = 1 - |S_{t+1}(u)|/s_{t+1}$.
If $\alpha < 0$, then $\alpha = 0$, otherwise if $\alpha > 1/2$, then $\alpha = 1/2$.
\item Let $\gamma$ be the smallest number in $[1, \infty)$ so that
$$\text{Average}_{c \in S_{t+1}(u)} d_{t+1}(u,c) \leq \frac{1 - 2\alpha}{1 - \alpha} \gamma d_{t+1}.$$
\end{itemize}
Remove all colors $c$ with $d_{t+1}(u,c) \geq 2\gamma d_{t+1}$ from $S_{t+1}(u)$.

\textbf{end repeat}

Permanently color each vertex $u$ in $V(G_t)$ with a color picked independently and uniformly at random from its palette $S_t(u)$.

\section{The Main Theorem}

\textbf{Theorem 1 (Main Theorem).} Given $67\Delta/\log \Delta$ colors and a triangle-free graph $G$ with maximum degree $\Delta$ large enough, our algorithm finds a proper coloring of the graph with positive probability.

We need some lemmas to prove the Main Theorem and before that we need the following definition.

\textbf{Definition 3.} $d_t(v) = \text{Average}_{c \in S_t(v)} d_t(v,c)$

\textbf{Lemma 1 (Main Lemma).} Given $\psi > 1$ and a triangle-free graph $G$ with maximum degree $\Delta$, there is a positive constant $\beta$ such that for the sequence $(e_t)$ defined by
\begin{equation}
e_0 = 0, \quad e_{t+1} = 3e_t + \beta(\sqrt[2]{\psi}) \quad \text{for } t > 0,
\end{equation}
if $s_t \gg \psi$ and $e_t \ll 1$ at round $t$, then
$$\forall u \in V(G_t), \exists \alpha \in [0, 1/2], \forall c \in S_t(u),$$
\begin{align*}
s_t(u) &\geq (1-\alpha)s_t(1-e_t) \\
d_t(u) &\leq \frac{1-2\alpha}{1-\alpha}d_t(1+e_t) \\
d_t(u, c) &\leq 2d_t(1+e_t)
\end{align*}
with positive probability.

We will prove the Main Lemma in Section 5 and assume it in this section. Using it we can immediately conclude the following.

**Corollary 1.** Given the setup of the Main Lemma(Lemma 1), if \( s_t \gg \psi \) and \( e_t \ll 1 \) at round \( t \), then
\[
\forall u \in V(G_t),
\begin{align*}
s_t(u) &\geq \frac{1}{2}s_t(1-e_t) \\
d_t(u) &\leq d_t(1+e_t)
\end{align*}
with positive probability.

**Lemma 2.** The repeat-until block finishes in \( 16e^{1/2}k \) rounds.

**Proof.** By the definition of sequences \((d_t)\) and \((s_t)\) in equation (3), we have
\[
\frac{d_{t+1}}{s_{t+1}} = \frac{d_t}{s_t}(1 - \frac{1}{16}\frac{s_t}{d_t}e^{-1/2}) = \frac{d_t}{s_t} - \frac{1}{16}e^{-1/2}.
\]
Since \( \frac{d_0}{s_0} = k \) we get \( \frac{d_t}{s_t} \leq \frac{1}{4} \) after \( 16e^{1/2}k \) rounds.

Let \( t_1 = 16e^{1/2}k \)

be the last round of the repeat-until block. Then the following lemma is a straightforward application of equation (3).

**Lemma 3.**
\[
s_{t_1} = \frac{\Delta}{k} \exp(-8e^{1/2}k) \text{ and, if } k \leq \frac{1}{9e^{1/2}} \log \Delta \text{ and } \Delta \text{ is large enough, then } s_{t_1} \gg 1.
\]

### 3.1 Bounding the Error Estimate in all Concentration Inequalities

Now we look at \( s_t \), which is used to bound the error term \( e_t \).

**Lemma 4.**
\[
e_t \leq 3'\mathcal{O}(\sqrt{\frac{k \exp(8e^{1/2}k)\psi}{\Delta}}).
\]

**Proof.** By Lemma 3 we have
\[
s_{t_1} = \frac{\Delta}{k} \exp(-8e^{1/2}k).
\]

Note that in equation (5), the recurrence for \( e_t \), the largest term is \( \mathcal{O}(\sqrt{\psi}/s_t) \). Since the sequence \((s_t)\) is decreasing, we use Lemma 3 to conclude that
\[
\mathcal{O}(\sqrt{\frac{\psi}{s_{t_1}}}) = \mathcal{O}(\sqrt{\frac{k \exp(8e^{1/2}k)\psi}{\Delta}}).
\]
is the maximum this term can be. Thus we can simplify the recurrence for \( e_t \) to

\[
e_{t+1} = 3e_t + \sqrt{\frac{k \exp(8e^{1/2}k)\psi}{\Delta}}.
\]

Since \( e_0 = 0 \), a simple upper bound for \( e_t \) is given by

\[
e_t \leq 3^t O\left(\sqrt{\frac{k \exp(8e^{1/2}k)\psi}{\Delta}}\right)
\]

where \( \alpha \) is some positive constant.

**Lemma 5.** Given \( \Delta/k \) colors where \( k \leq \frac{1}{\alpha^2} (\log \Delta) \) and a triangle-free graph \( G \) with maximum degree \( \Delta \), our algorithm reaches the end of the repeat-until block at round \( t_1 = O(k) \) with \( e_{t_1} \ll 1 \), and \( \forall u \in V(G_{t_1}) \), \( c \in S_{t_1}(u) \)

\[
s_{t_1}(u) \geq \frac{1}{2} s_{t_1}(1 - e_{t_1})
\]

\[
d_{t_1}(u, c) \leq 2d_{t_1}(1 + e_{t_1})
\]

with positive probability.

**Proof.** Let \( \psi = 3 \log \Delta \), and let \( t_1 \) be the number of rounds to reach the end of the repeat-until block. Using Lemma 3, we get

\[
e_{t_1} \leq 3^{t_1} O\left(\sqrt{\frac{k \exp(8e^{1/2}k)\psi}{\Delta}}\right).
\]

Using Lemma 3 it is straightforward to show that

\[
e_{t_1} \ll 1 \quad \text{and} \quad s_{t_1} \gg \psi
\]

if \( k \leq \frac{1}{\alpha^2} \log \Delta \). Applying Corollary 1 completes the proof.

We may now prove the Main Theorem.

**Proof (of the Main Theorem).** Using Lemma 5, we get

\[
\forall u \in V(G_{t_1}), c \in S_{t_1}(u) \quad s_{t_1}(u) \geq 2d_{t_1}(u, c)
\]

with positive probability. Now Haxell [11] shows that the final step of our algorithm finds (randomly coloring all uncolored vertices) finds a proper coloring with positive probability.

\[\square\]

### 4 Several Useful Inequalities

Now we look at some preliminaries which will be used in the proof details. The next lemma describes what happens to the average value of a finite subset of real numbers when large elements are removed. As shown in the statement of the lemma, it implies Markov’s Inequality [4].

**Lemma 6.** Consider a set of positive real numbers of size \( n \) and average value \( \mu \). If we remove an elements with value at least \( q\mu \) for some \( q > 1 \), then the remaining points have average

\[
\mu' \leq \mu \frac{1 - q\alpha}{1 - \alpha}
\]

In particular, \( \alpha \leq \frac{1}{q} \) since \( \mu' \geq 0 \).
Proof. The conclusion is obtained by a trivial manipulation of the following inequality which relates $\mu$ and $\mu'$.

$$q\mu \alpha + \mu'(1 - \alpha) \leq \mu$$

☐

The next lemma describes what happens when we add large elements to a finite subset of real numbers.

**Lemma 7.** Given the setup of Lemma 6, if we add $\alpha n$ points with value $q\mu$ to the sample, then the resulting larger sample has average

$$\mu' = \frac{1 + q\alpha}{1 + \alpha}$$

Proof. The conclusion is easily obtained from the following equation relating $\mu$ and $\mu'$.

$$\mu'(1 + \alpha) = \mu + q\mu \alpha$$

☐

We use the following lemma for computations with error factors.

**Lemma 8.** Let $(A_n)$ be a sequence such that $0 < A_n < c < 1$ (where $c$ is a constant), and let $(e_n)$ be another sequence. Then

$$1 - A_n(1 + e_n) = (1 - A_n)(1 + O(e_n))$$

Proof.

$$1 - A_n(1 + e_n) = (1 - A_n)(1 + e_n) - e_n$$

$$= (1 - A_n)(1 + e_n) - (1 - A_n) \frac{e_n}{1 - A_n}$$

$$= (1 - A_n)(1 + e_n) - (1 - A_n)O(e_n)$$

$$= (1 - A_n)(1 + O(e_n))$$

☐

We use the following version of Azuma’s inequality [20] to prove concentration of random variables.

**Theorem A (Azuma’s inequality)** Let $X$ be a random variable determined by $n$ trials $T_1, \ldots, T_n$, such that for each $i$, and any two possible sequences of outcomes $t_1, \ldots, t_i$ and $t_1, \ldots, t_{i-1}, t'_i$:

$$|E[X|T_1 = t_1, \ldots, T_i = t_i] - E[X|T_1 = t_1, \ldots, T_i = t'_i]| \leq \alpha_i$$

then

$$Pr(|X - E[X]| > t) \leq 2e^{-t^2/(\sum \alpha_i^2)}$$

We use the following version of the Lovasz Local Lemma [20]

**Theorem B (Lovasz Local Lemma)** Consider a set $\mathcal{E}$ of events such that for each $A \in \mathcal{E}$

- $Pr(A) \leq p < 1$, and
- $A$ is mutually independent of a set of all but at most $d$ of the other events.

If $4pd \leq 1$, then with positive probability, none of the events in $\mathcal{E}$ occur.
5 Proof of the Main Lemma

The following assumptions are repeatedly used in the lemmas of this section.

**Assumption 1 Assume**

\[ s_t \gg \psi, e_t \ll 1 \]

and with positive probability

\[ \forall u \in V(G_t), \forall c \in S_t(u), \exists \alpha \in [0, 1/2], \]

\[ s_t(u) \geq (1 - \alpha)s_t(1 - e_t) \]

\[ d_t(u) \leq \frac{1 - 2\alpha}{1 - \alpha} d_t(1 + e_t) \]

\[ d_t(u, c) \leq 2d_t(1 + e_t) \]

All events in this section are in the probability space generated by our randomized algorithm in round \( t+1 \), given the state of all data structures at the beginning of the round.

**Proof (of the Main Lemma).** The proof is by induction on the round number \( t \) using lemmas that follow. The base case, when \( t = 0 \), is trivially true. If we assume Assumption \( \square \) the induction hypothesis, for round \( t \) then for each vertex \( u \) in \( V(G_{t+1}) \), by Lemma \( \Box \) we have

\[ \Pr\{d_{t+1}(u) \leq d_{t+1}(1 + O(e_t + \sqrt{\psi s_t + 1/d_t})) \}
\]

\[ \geq 1 - e^{-\psi O(1)}. \]

For each \( c \) in \( S_{t+1}(u) \), by Lemma \( \square \) we have

\[ \Pr\{\exists \alpha \in [0, 1/2] \text{ such that} \}
\]

\[ s_{t+1}(u) \geq (1 - \alpha)s_{t+1}(1 - 3e_t + O(\sqrt{\psi s_t + 1/d_t})), \]

\[ d_{t+1}(u) \leq \frac{1 - 2\alpha}{1 - \alpha} d_{t+1}(1 + 3e_t + O(\sqrt{\psi s_t + 1/d_t})), \]

\[ d_{t+1}(u, c) \geq (1 - \alpha)2d_{t+1}(1 - 3e_t + O(\sqrt{\psi s_t + 1/d_t}))) \}
\]

\[ \geq 1 - e^{-\psi O(1)}. \]

Each of the events in the probabilities above is dependent on at most \( O(\Delta^2) \) other such events. If \( \psi \geq 3 \log \Delta \) and \( \Delta \) is large enough, then we use Theorem \( \Box \) to conclude that Assumption \( \Box \) holds for round \( t+1 \). \( \Box \)

The above proof of the Main Lemma required Lemmas \( \Box \) and \( \square \). The rest of this section will prove these lemmas. Next we consider the state of the palettes just before the cleanup phase of round \( t \).

**Definition 4.** Let \( \tilde{S}_t(u) \) be the list of colors in the palette of vertex \( u \) in round \( t \) just before the cleanup phase, and let \( \tilde{s}_t(u) \) be the size of \( \tilde{S}_t(u) \). That is, \( \tilde{S}_t(u) \) is obtained from \( S_t(u) \) by removing colors discarded in the conflict resolution phase.

**Lemma 9.** Given Assumption \( \Box \) for each vertex \( u \) in \( V(G_{t+1}) \) we have

\[ \Pr\{s_t(u)e^{-1/2}(1 - \frac{1}{2}e_t - O(\sqrt{\psi s_t})) \leq \tilde{s}_t(u) \leq s_t(u)e^{-1/2}(1 + O(\sqrt{\psi s_t})) \}
\]

\[ \geq 1 - e^{-\psi O(1)}. \]
Proof. Suppose \( u \) is an uncolored vertex at the beginning of round \( t \), and \( c \) a color in its palette.

\[
Pr\{c \text{ is removed from } S_t(u) \text{ in phase II.1}\} = 1 - \prod_{v \in D_t(u,c)} (1 - Pr\{v \text{ is assigned } c\})
\]

\[
= 1 - \prod_{v \in D_t(u,c)} \left(1 - \frac{1}{4} \frac{1}{d_t}\right)
\]

\[
\leq 1 - \left(1 - \frac{1}{4} \frac{1}{d_t}\right)^{d_t(u,c)}
\]

\[
\leq 1 - \left(1 - \frac{1}{4} \frac{1}{d_t}\right)^{2d_t(1+e_t)}
\]

\[
\leq 1 - e^{\log(1 - \frac{1}{4} \frac{1}{d_t})^{2d_t(1+e_t)}}
\]

\[
\leq 1 - e^{-1/2(1 - \frac{1}{2} e_t + O(\frac{1}{d_t}))}
\]

In phase II.2 of round \( t+1 \) we remove colors from the palette using an appropriate bernoulli variable, to get

\[
Pr\{c \notin \tilde{S}_t(u)\} = 1 - e^{-1/2(1 - \frac{1}{2} e_t + O(\frac{1}{d_t}))}.
\]

Using linearity of expectation

\[
E[\tilde{s}_t(u)] = s_t(u)e^{-1/2(1 - \frac{1}{2} e_t + O(\frac{1}{d_t}))}.
\]

For concentration of \( \tilde{s}_t(u) \), suppose \( s_t(u) = m \). Let \( c_1, \ldots, c_m \) be the colors in \( S_t(u) \). Then \( \tilde{S}_t(u) \) may be considered a random variable determined by \( m \) trials \( T_1, \ldots, T_m \) where \( T_i \) is the set of vertices in \( G_t \) that are assigned color \( c_i \) in round \( t \). Observe that \( T_i \) affects \( \tilde{S}_t(u) \) by at most 1 given \( T_1, \ldots, T_{i-1} \). Now using Theorem [A] we get,

\[
Pr\{|\tilde{s}_t(u) - E[\tilde{s}_t(u)]| \geq \sqrt{\psi s_t(u)}\} \leq e^{-\psi O(1)}.
\]

We now focus on the sets \( D_t(u,c) \). The following two lemmas will help.

**Lemma 10.** Let \( u \) be an uncolored vertex, and \( c \) a color in its palette at the beginning of round \( t \). Then given Assumption [A] we have

\[
Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} = \frac{1}{4} \frac{1}{d_t} e^{-1/2(1 - \frac{1}{2} e_t + O(\frac{1}{d_t}))}.
\]

Proof.

\[
Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\} = Pr\{u \text{ is assigned } c\} Pr\{c \in \tilde{S}_t(u)\}
\]

\[
= \frac{1}{4} \frac{1}{d_t} e^{-1/2(1 - \frac{1}{2} e_t + O(\frac{1}{d_t}))}
\]

\[
\langle \text{Equation (??)}\rangle
\]

The following lemma is a consequence of the previous one.
Lemma 11. Let \( u \) be an uncolored vertex at the beginning of round \( t \). Then given Assumption 1, we have

\[
Pr\{u \text{ is colored}\} \geq \frac{1}{16} \frac{s_t}{d_t^{e^{-1/2}(1 - 3e_t + O(\frac{1}{d_t}))}}.
\]

**Proof.** Consider the event

\[
\{u \text{ is colored}\} = \bigcup_{c \in S_t(u)} \{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}.
\]

Since the events in the union on the right hand side of the equation above are independent,

\[
Pr\{u \text{ is colored}\} = 1 - \prod_{c \in S_t(u)} (1 - Pr\{u \text{ is assigned } c \text{ and } c \in \tilde{S}_t(u)\}).
\]

Now using Lemma 10, we get

\[
Pr\{u \text{ is colored}\} \geq 1 - \exp\left(-\frac{1}{8} s_t e^{-1/2}(1 - \frac{3}{2} e_t + O(\frac{1}{d_t}))\right)
\]

\[
= \frac{1}{16} \frac{s_t}{d_t^{e^{-1/2}(1 - \frac{3}{2} e_t + O(\frac{1}{d_t}))}}.
\]

\[\Box\]

We will need the following definitions.

**Definition 5.**

- Let \( \tilde{D}_t(u,c) \) be the set of uncolored vertices that have color \( c \) in their palettes and are uncolored in round \( t \), just before the cleanup phase. That is,

\[
\tilde{D}_t(u,c) = D_t(u,c) \setminus (\{v|c \notin \tilde{S}_t(v)\} \cup \{v|v \text{ is colored in round } t\}).
\]

- Let \( \tilde{d}_t(u,c) \) be the size of \( \tilde{D}_t(u,c) \).
- \( \tilde{d}_t(u) := \sum_{c \in \tilde{S}_t(u)} \tilde{d}_t(u,c) = \sum_{c \in S_t(u)} 1_{\{c \in \tilde{S}_t(u)\}} \tilde{d}_t(u,c) \)
- \( \tilde{d}_t(u) := \frac{\tilde{d}_t(u)}{\tilde{s}_t(u)} \)

**Lemma 12.** Given Assumption 1, for each vertex \( u \) in \( V(G_{t+1}) \) we have

\[
Pr\{\tilde{d}_t(u) \leq d_t(u)\}
\]

\[
\geq 1 - e^{-\psi O(1)}.
\]
Proof. Let \( u \) be an uncolored vertex at the beginning of round \( t \), and let \( c \) be a color in its palette. For a vertex \( v \) in \( D_t(u, c) \), Lemma 10 implies that \( Pr\{v \text{ is colored with } d\} = O(1/d_t) \) for any color \( d \) in \( S_t(v) \). Thus,

\[
Pr\{c \notin \tilde{S}_t(v) \cap \{v \text{ is colored}\} \} = \sum_{d \in S_t(v)} Pr\{c \notin \tilde{S}_t(v) \cap \{v \text{ is colored with } d\}\}
\]

\[
= \sum_{d \in S_t(v)} Pr\{c \notin \tilde{S}_t(v) | v \text{ is colored with } d\} Pr\{v \text{ is colored with } d\}
\]

\[
= Pr\{c \notin \tilde{S}_t(v)\}(1 + O(\frac{1}{d_t})) \sum_{d \in S_t(v)} Pr\{v \text{ is colored with } d\}
\]

\[
= Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is colored}\}(1 + O(\frac{1}{d_t})).
\]

A straightforward computation now shows that

\[
Pr\{c \notin \tilde{S}_t(v) \cap \{v \text{ is colored}\}\} = Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is colored}\}(1 + O(\frac{1}{d_t})) \tag{6}
\]

Now, \( v \) is removed from the set \( D_t(u, c) \) if either it is colored or color \( c \) is removed from its palette. This means that event

\[
\{v \notin \tilde{D}_t(u, c)\} = \{v \text{ is colored}\} \cup \{c \notin \tilde{S}_t(v) \cap \{v \text{ is not colored}\}\}.
\]

Since \( G \) is triangle-free, \( u \) and \( v \) do not have any common neighbors. This implies that

\[
Pr\{v \notin \tilde{D}_t(u, c) | c \in \tilde{S}_t(u)\}
\]

\[
= Pr\{v \notin \tilde{D}_t(u, c)\}(1 + O(\frac{1}{d_t}))
\]

\[
= (Pr\{v \text{ is colored}\} + Pr\{c \notin \tilde{S}_t(v) \cap \{v \text{ is not colored}\}\})(1 + O(\frac{1}{d_t}))
\]

\[
= (Pr\{v \text{ is colored}\} + Pr\{c \notin \tilde{S}_t(v)\} Pr\{v \text{ is not colored}\})(1 + O(\frac{1}{d_t})) \tag{equation 11}
\]

\[
= (Pr\{v \text{ is colored}\} + (1 - e^{-1/2})(1 - Pr\{v \text{ is colored}\}))(1 + O(\frac{1}{d_t})) \tag{equation 12}
\]

\[
= (1 - (1 - Pr\{v \text{ is colored}\})e^{-1/2})(1 + O(\frac{1}{d_t}))
\]

\[
\geq (1 - (1 - \frac{1}{16} s_t (e^{-1/2}))e^{-1/2})(1 + 2e_t + O(\frac{1}{d_t})) \tag{Lemma 11}
\]

Using linearity of expectation

\[
E[\tilde{d}_t(u) | c \in \tilde{S}_t(u)] = E[\tilde{d}_t(u, c)](1 + O(\frac{1}{d_t})) \leq d_t(u, c)(1 - \frac{1}{16} s_t (e^{-1/2})e^{-1/2})(1 + 2e_t + O(\frac{1}{d_t})). \tag{7}
\]

Now using the above bound

\[
E[\tilde{d}_t(u)] = \sum_{c \in S_t(u)} Pr\{c \in \tilde{S}_t(u)\} E[\tilde{d}_t(u, c) | c \in \tilde{S}_t(u)]
\]

\[
\leq e^{-1/2} \sum_{c \in S_t(u)} d_t(u, c)(1 - \frac{1}{16} s_t (e^{-1/2})e^{-1/2})(1 + 2e_t + O(\frac{1}{d_t}))
\]

\[
\leq e^{-1/2} s_t(u) d_t(u)(1 - \frac{1}{16} s_t (e^{-1/2})e^{-1/2})(1 + 2e_t + O(\frac{1}{d_t}))
\]

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For concentration of \( \tilde{d}_t(u) \), suppose \( s_t(u) = m \). Let \( c_1, \ldots, c_m \) be the colors in \( S_t(u) \). Then \( \tilde{d}_t(u) \) may be considered a random variable determined by the random trials \( T_1, \ldots, T_m \), where \( T_i \) is the set of vertices in \( G_t \) that are assigned color \( c_i \) in round \( t \). Observe that \( T_i \) affects \( d_t(u) \) by at most \( d_t(u, c_i) \).

Thus \( \sum \alpha_t^2 \) in the statement of Theorem A is less that \( \sum_{c \in S_t(u)} d_t^2(u, c) \). This upperbound is maximized when the \( d_t(u, c) \) take the extreme values of \( 2d_t \) and \( 0 \) subject to \( d_t(u) = \frac{1}{s_t(u)} \sum_{c \in S_t(u)} d_t(u, c) \). Thus

\[
\sum \alpha_t^2 \leq O((d_t)^2d_t(u)s_t(u)/d_t) \leq O(s_t(u)d_t^2(u))
\]

Using Theorem A we get

\[
Pr\{ \tilde{d}_t(u) - e^{-1/2}s_t(u)d_t(u)(1 - \frac{1}{16} \frac{s_t}{d_t} e^{-1/2})e^{-1/2}(1 + 2e_t + O(\frac{1}{d_t})) \geq O(\sqrt{\psi s_t(u)d_t(u)}) \} \leq e^{-\psi(1)}.
\]

Lemma D says that

\[
Pr\{ s_t(u)e^{-1/2}(1 - \frac{1}{2} + O(\sqrt{\frac{\psi}{s_t}})) \leq \tilde{d}_t(u) \leq s_t(u)e^{-1/2}(1 + O(\sqrt{\frac{\psi}{s_t}})) \} \geq 1 - e^{-\psi(1)}.
\]

Combining the above two inequalities we have

\[
Pr\{ \frac{\tilde{d}_t(u)}{s_t(u)} - d_t(u)(1 - \frac{1}{16} \frac{s_t}{d_t} e^{-1/2})e^{-1/2}(1 + 2e_t + O(\frac{1}{d_t} + \sqrt{\frac{\psi}{s_t}})) \geq O(\sqrt{\frac{\psi d_t d_t(u)}{s_t}}) \} \leq e^{-\psi(1)}.
\]

Therefore

\[
Pr\{ \frac{\tilde{d}_t(u)}{s_t(u)} \geq d_t(u)(1 - \frac{1}{16} \frac{s_t}{d_t} e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}})) \} \leq e^{-\psi(1)}.
\]

\( \square \)

Note that \( \frac{\tilde{d}_t(u)}{s_t(u)} \) is the average \( |\tilde{D}_t(u, c)| \) at a vertex \( u \) at the end phase II. Phase III only brings this average down by removing colors with large \( d_{u,c} \). Thus we get the next lemma almost immediately.

**Lemma 13.** Given Assumption D for each \( u \) in \( V(G_{t+1}) \) we have

\[
Pr\{d_{t+1}(u) \leq d_t(u)(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t} + \frac{1}{d_t}})) \} \geq 1 - e^{-\psi(1)}.
\]

**Proof.** Let \( u \) be a vertex in \( V(G_{t+1}) \). By Lemma 12

\[
Pr\{ \tilde{d}_t(u) \leq d_t(u)(1 - \frac{1}{16} \frac{s_t}{d_t} e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}})) \} \geq 1 - e^{-\psi(1)}.
\]

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Now
\[ d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}})) \]
\[ = d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t}) + O(\sqrt{\frac{\psi d_t(u)}{s_t}})) \]
\[ = d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) + d_t O(\sqrt{\frac{\psi d_t(u)}{s_t d_t}}) \]
\[ \leq d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}})). \]

Thus the event
\[ \{ \frac{\tilde{d}_t(u)}{s_t(u)} \geq d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \} \]
\[ \subseteq \{ \frac{\tilde{d}_t(u)}{s_t(u)} \geq d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}})) \}. \]

Therefore
\[ e^{-\psi}O(1) \]
\[ \leq Pr\{ \frac{\tilde{d}_t(u)}{s_t(u)} \geq d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})) \} \]
\[ \leq Pr\{ \frac{\tilde{d}_t(u)}{s_t(u)} \geq d_t(u)(1 - \frac{1}{16} s_t e^{-1/2})e^{-1/2}(1 + 2e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t} + \sqrt{\frac{\psi d_t}{s_t d_t(u)}})) \}. \]

\[ \square \]

Next we show that in the cleanup phase of round $t$, a vertex discards so many colors that its palette size in round $t + 1$ becomes less than $\frac{1}{2} s_{t+1}(1 - e_{t+1})$ with a very small probability.

**Lemma 14.** Given Assumption [H] for each vertex $u$ in $V(G_{t+1})$ we have
\[ Pr\{ \exists \alpha \in [0, \frac{1}{2}] \text{ such that } \forall c \in S_{t+1}(u) \]
\[ s_{t+1}(u) \geq (1 - \alpha)s_{t+1}(1 - \frac{3}{2} e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \]
\[ d_{t+1}(u) \leq \frac{1 - 2\alpha}{1 - \alpha} d_{t+1}(1 + \frac{3}{2} e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \]
\[ d_{t+1}(u, c) \leq 2d_{t+1}(1 + \frac{3}{2} e_t + O(\sqrt{\frac{\psi}{s_t}} + \frac{1}{d_t})), \]
\[ \geq 1 - e^{-\psi}O(1). \]

**Proof.** Consider vertex $u \in V(G_{t+1})$. Using Assumption [H] at round $t$, $\exists \alpha \in [0, \frac{1}{2}]$ such that $s_t(u) \geq (1 - \alpha)s_t(1 - e_t)$ and $d_t(u) \leq \frac{1 - 2\alpha}{1 - \alpha} d_t(1 + e_t)$. By Lemma [H] we get
\[ Pr\{ s_t(u)e^{-1/2}(1 - \frac{1}{2} e_t + O(\sqrt{\frac{\psi}{s_t}})) \leq \tilde{s}_t(u) \leq s_t(u)e^{-1/2}(1 + \frac{1}{2} e_t + O(\sqrt{\frac{\psi}{s_t}})) \}
\[ \geq 1 - e^{-\psi}O(1). \]
Now

\[ \hat{s}_t(u) = s_t(u)e^{-1/2}(1 + \frac{1}{2}e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t})) \]

\[ \geq (1 - \alpha)s_t e^{-1/2}(1 + \frac{1}{2}e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t})) \]

\[ \geq (1 - \alpha)s_{t+1}(1 + \frac{1}{2}e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t})). \]

By Lemma 12 we get

\[ \Pr\{\hat{d}_t(u) \leq d_t(u)(1 - \frac{1}{16}st e^{-1/2})e^{-1/2}(1 + \frac{3}{2}e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t} + \sqrt{s_d d_t})\} \]

\[ \geq 1 - e^{-\psi}O(1). \]

Now

\[ \hat{d}_t(u) \leq d_t(u)(1 - \frac{1}{16}st e^{-1/2})e^{-1/2}(1 + \frac{3}{2}e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t})) \]

\[ \leq 1 - 2\alpha \frac{1}{1 - \alpha} d_t(1 - \frac{1}{16}st e^{-1/2})e^{-1/2}(1 + \frac{3}{2}e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t})) \]

\[ \leq 1 - 2\alpha \frac{1}{1 - \alpha} \gamma d_{t+1}. \]

where \( \gamma \) is the smallest number in \([1, \infty]\) for which the above inequality is true. Combining the preceding inequalities, we get

\[ \gamma = 1 + 3e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t}). \]

In the cleanup phase of our algorithm (given in Section 2.2), the change in palette is equivalent to the following process.

1. Add \( \frac{1}{1 - \alpha} \hat{s}_t(u) \) arbitrary colors to \( u \)'s palette, with \( \hat{d}_t(u, c) = 2\gamma d_{t+1} \). This adjusts the palette size to \( \hat{s}_t(u) \geq s_{t+1}(1 + 3e_t + O(\sqrt{\psi / s_{t+1}} + 1/d_t) \). Lemma 7 ensures that the adjusted new average is \( \hat{d}_t(u) \leq \gamma d_{t+1} \)
2. Remove all the colors with \( d_t(u, c) \geq 2\gamma d_{t+1} \).

Now we use Lemma 8, setting \( \mu \) to \( \gamma d_{t+1} \) and \( q\mu \) to \( 2\gamma d_{t+1} \), to get

\[ s_{t+1}(u) \geq (1 - \alpha)s_{t+1}(1 + 3e_t + O(\sqrt{\psi / s_{t+1}} + \frac{1}{d_{t+1}})) \]

and

\[ d_{t+1}(u) \leq \frac{1 - 2\alpha}{1 - \alpha} d_{t+1}(1 + 3e_t + O(\sqrt{\psi / s_t} + \frac{1}{d_t})). \]

The result is obtained using Lemmas 9 and 12.

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