Homoclinic tangencies leading to robust heterodimensional cycles

Pablo G. Barrientos\textsuperscript{1} · Lorenzo J. Díaz\textsuperscript{2} · Sebastián A. Pérez\textsuperscript{3}

Received: 6 December 2021 / Accepted: 12 May 2022 / Published online: 26 June 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We consider $C^r$ $(r \geq 1)$ diffeomorphisms $f$ defined on manifolds of dimension $\geq 3$ with homoclinic tangencies associated to saddles. Under generic properties, we show that if the saddle is homoclinically related to a blender then the diffeomorphism $f$ can be $C^r$ approximated by diffeomorphisms with $C^1$ robust heterodimensional cycles. As an application, we show that the classic Simon–Asaoka’s examples of diffeomorphisms with $C^1$ robust homoclinic tangencies also display $C^1$ robust heterodimensional cycles. In a second application, we consider homoclinic tangencies associated to hyperbolic sets. When the entropy of these sets is large enough we obtain $C^1$ robust cycles after $C^1$ perturbations.

Keywords Blender · Cycles · Entropy · Heterodimensional cycle · Homoclinic tangency · Hyperbolic measure · Lyapunov exponent · Robust properties

Mathematics Subject Classification Primary 37C20; Secondary 37C29 · 37D20 · 37D30

Pablo G. Barrientos and Lorenzo J. Díaz were supported [in part] by CAPES finance code 001 (Brazil), CNPq Projeto Universal and CNPq research grants (Brazil). PGB was partially supported by FAPERJ grant JCNE and grant PID2020-113052GB-I00 funded by MCIN/AEI/10.13039/501100011033 (Spain). Lorenzo J. Díaz was partially supported by grant E-16/2014 INCT/FAPERJ (Brazil) and ANID PCI MEC 80190005 (Chile). Sebastián A. Pérez was partially supported by FONDECYT Iniciación No. 11220583 (Chile). The authors thank an anonymous referee for the careful reading of the manuscript and the helpful suggestions.

\textsuperscript{1} Instituto de Matemática e Estatística, Universidade Federal Fluminense, Gragoata Campus, Rua Prof. Marcos Waldemar de Freitas Reis, S/n–Sao Domingos, Niterói, RJ 24210-201, Brazil
\textsuperscript{2} Departamento de Matemática, PUC-Rio, Marquês de São Vicente 225, Gávea, Rio de Janeiro 22451-900, Brazil
\textsuperscript{3} Instituto de Matemáticas, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile
1 Introduction

Homoclinic tangencies and heterodimensional cycles associated to saddles are considered the main bifurcation mechanisms generating nonhyperbolic dynamics. Palis conjecture [41] claims that nonhyperbolic diffeomorphisms can be approached by systems displaying these bifurcations. The goal of this paper is to study the interplay between these two types of bifurcations. A saddle (i.e., a hyperbolic periodic point with nontrivial stable and unstable bundles) of a diffeomorphism has a homoclinic tangency if its stable and unstable manifolds have some nontransverse intersection. Two saddles with different u-indices (i.e., dimension of the unstable bundle) have a heterodimensional cycle if their stable and unstable manifolds intersect cyclically (by dimension deficiency, one of these intersections is necessarily nontransverse). Homoclinic tangencies may occur in manifolds of dimension two or higher, while the occurrence of heterodimensional cycles requires dimension at least three. In what follows, we will use the term cycle to refer either to a homoclinic tangency or to a heterodimensional cycle.

Note that cycles associated to saddles are breakable by small perturbations: by Kupka–Smale theorem, periodic points of generic diffeomorphisms are hyperbolic and their invariant manifolds are in general position (either they intersect transversely or are disjoint). Even more, cycles may be unbreakable if they are associated to nontrivial hyperbolic basic sets (i.e., not just periodic orbits). This leads to the notions of $C^r$ persistent and $C^r$ robust cycles. In the persistent case, there is a $C^r$ open set where the diffeomorphisms with cycles associated to saddles are $C^r$ dense. In the robust case, there is a hyperbolic set with a cycle that cannot be destroyed by small $C^r$ perturbations, see Definition 1.1.1. Bearing in mind the importance of robust dynamical properties, Bonatti proposed in [15] a stronger version of Palis conjecture: every nonhyperbolic diffeomorphism can be approached by diffeomorphisms with a robust cycle.

Palis conjecture holds for $C^1$ surface diffeomorphism [46] and there is some progress in higher dimensions, see [23, 24]. Note that there are no surface diffeomorphisms with $C^1$ robust homoclinic tangencies associated to basic sets, see [37]. In higher dimensions and in the $C^1$ topology, Bonatti conjecture is true for the so-called tame diffeomorphisms, [18]. For both conjectures, the $C^r$ case, $r \geq 2$, is widely open. The first reason for that is that they rely on the density of periodic points in the limit set, which is only known in the $C^1$ case, see [44, 45]. A second reason is the lack of connecting-like lemmas allowing intersections between the invariant manifolds of saddles in the presence of recurrences, see [16, 31]. In view of these obstacles, the following question (that can be posed in any regularity) provides a bridge between these conjectures and a step towards the one by Bonatti:

**Question 1** When does a cycle associated to saddles yield (by small) perturbations robust cycles?

The answer to this question depends on the type of cycle, the dimension of the ambience, and the required regularity. A strategy to find robust cycles is to see that a cycle can become robust when the saddle in the cycle belongs to (or when the unfolding of the cycle generates) some massive horseshoe. This vague term means thick horseshoes or blenders, according to the type of cycle. We observe that $C^1$ diffeomorphisms cannot display robustly thick horseshoes, [55], while blenders may occur in dimension three or higher in a $C^1$ robust way. This illustrates some disparities between the different settings. Blenders will play a key role in this paper and will be discussed in Sect. 2. For the discussion in this introduction, keep

---

1 Any robust cycle provides a persistent cycle. As far as we know, it is unknown the existence of persistent cycles that are not robust (meaning that the saddle belongs to a hyperbolic set with a robust cycle).
Homoclinic tangencies leading to robust heterodimensional…

in mind that a blender is a special type of hyperbolic set with some $C^1$ robust geometrical properties guaranteeing intersections between the local unstable manifold of the blender and a family of “strong stable” disks.

There are partial answers to Question 1. When $r \geq 2$, a homoclinic tangency of a $C^r$ diffeomorphism yields robust tangencies, see [39] in dimension two and [27, 42, 49] in higher dimensions. The generation of robust tangencies in the $C^1$ setting in higher dimensions depends crucially on the geometry of the tangency, see the discussion below. Finally, the question does a heterodimensional cycle yield robust heterodimensional cycles? has a positive answer in the coindex (i.e., the absolute value of the difference of the $u$-indices of the hyperbolic sets in the cycle) one case, see [18] for the $C^1$ case and [36] for the $C^r$ case, $r \geq 2$. The case of heterodimensional cycles of higher coindex is widely open. We observe that all robust cycles generated in this paragraph are of the same type as the ones in the initial cycle.

An issue related to Question 1 is to determine the interplay between heterodimensional cycles and homoclinic tangencies. This leads to the following refinement of Question 1:

**Question 2** (a) When does a homoclinic tangency associated to a saddle lead to robust heterodimensional cycles? (b) When does a heterodimensional cycle associated to a pair of saddles lead to robust homoclinic tangencies?

The answers to these questions involves global dynamical aspects. In this paper, we focus on item (a), providing settings where homoclinic tangencies associated to saddles lead to $C^1$ robust cycles after small $C^r$ perturbations, $r \geq 1$. Our constructions are motivated by the ones in [35], where heterodimensional cycles are obtained from homoclinic tangencies with some symmetry conditions. To describe our setting, we first comment some global aspects and restrictions concerning Question 2.

The presence of a homoclinic tangency implies the existence of an invariant space of dimension at least two that cannot be decomposed into one-dimensional invariant directions in a dominated way (see the appendix in Sect. 1 for the precise definitions). Thus partially hyperbolic diffeomorphisms having dominated splittings with one-dimensional bundles may display heterodimensional cycles but cannot exhibit homoclinic tangencies. This shows an obvious restriction to item (b). When considering homoclinic tangencies a key aspect is how they are embedded in the global dynamics. The following discussion illustrates this aspect and motivates our constructions and hypotheses.

To simplify the analysis, let us assume that the ambiance has dimension three and the saddle with the homoclinic tangency has $u$-index one. Under appropriate assumptions on the multipliers of the saddle, the unfolding of this tangency yields new hyperbolic periodic points with $u$-indices different from one. Once saddles of $u$-index two are generated and the resulting dynamics exhibits saddles of different $u$-indices, it is necessary to get cyclic heteroclinic relations between them.

The occurrence of such heteroclinic relations is an intricate question that involves geometrical constraints. For instance, the tangency may occur “inside” a normally contracting hyperbolic surface and the saddle with the homoclinic tangency may have a Jacobian whose restriction to that surface is bigger than one. In this case, due to the persistence of normally hyperbolic surfaces, the homoclinic tangency yields saddles contained in those surfaces with a normally contracting direction which are sources for the dynamics restricted to the surface. In this way, the tangency provides saddles of $u$-indices one and two. However, in the

---

2 Note that if $f \in \text{Diff}^r(M), r \geq 1$, then every $C^1$ neighbourhood of $f$ contains a $C^r$ neighbourhood of $f$. Hence, if $f$ has a $C^1$ robust cycle then it also has a $C^r$ robust cycle.
absence of further recurrences given by the global dynamics, there is no dynamical interaction between these saddles and heterodimensional cycles cannot occur. In rough terms, in the previous configuration, the dynamics can be “reduced” to a surface dynamics “multiplied” by a contraction, we will refer to it as “essentially two-dimensional bifurcations”.

We now discuss the setting and the hypotheses of our results. Recall that hyperbolic basic sets of surface diffeomorphisms cannot display $C^1$ robust homoclinic tangencies. Thus to get $C^1$ robust homoclinic tangencies in higher dimensions essentially two-dimensional dynamics must be avoided. To bypass this obstruction, in [19] it is considered a type of blender (called blender-horseshoe, see Remark 2.2) with a homoclinic tangency. Although [19, Theorem 1.2], see also [21, Theorem 1 and Corollary 2], are $C^1$ global results involving the recurrences provided by a homoclinic class (i.e., the closure of the set of transverse intersections between the invariant manifolds of a saddle) they lead to the following heuristic principle: a homoclinic class with robust index variability (i.e., containing saddles of different $u$-indices) and a nondominated “central” direction displays $C^1$ robust cycles (simultaneously, homoclinic tangencies and heterodimensional cycles). In the $C^1$ setting in [19, 21] the index variability provides blenders inside the homoclinic class as well as the following configuration: a saddle (satisfying some dissipative conditions) with a homoclinic tangency and homoclinically related to a blender. The tangency provides the nondomination condition and also (after its unfolding) the index variability. The blender leads to dynamics that are not essentially two-dimensional.

This sort of conditions motivated the hypotheses in Theorems A and B claiming the existence of $C^1$ robust heterodimensional cycles arising from homoclinic tangencies after small $C^r$ perturbations, $r \geq 1$. Theorem A considers the codimension one case (the saddle with the homoclinic tangency has $u$-index one) while Theorem B deals with the general setting (any $u$-index). The precise statements can be found in Sect. 1.1. The main difference between these two theorems is that in the codimension one case the homoclinic tangency provides only saddles of $u$-indices one and two. As a consequence, only heterodimensional cycles associated to these saddles can appear and hence these cycles have coindex one. In the general setting, the homoclinic tangency yields saddles of a wider range of $u$-indices, hence cycles of higher coindex may appear.

We also provide settings to which our results apply. For that, let us recall the first examples of $C^1$ robust homoclinic tangencies given by Simon [51] and Asaoka [2] built on the examples of nonhyperbolic diffeomorphisms in [1]. The key ingredient in these constructions is the existence of a hyperbolic set that is the local product of a nontrivial hyperbolic repeller (a DA-diffeomorphism in [51] and a Plykin repeller in [2]) and a contraction. We call this sort of hyperbolic set blenders of DA or Plykin type. A crucial property of this type of blender is that they contain their one-dimensional unstable manifolds. This property allows to get $C^1$ robust heterodimensional cycles in [1, 50] (although this terminology is not used there). The idea in [2, 51] (reviewed in Sect. 7) is to produce homoclinic tangencies associated to these blenders. In Corollary A (see also Remark 1.3) we prove that these examples satisfy the hypotheses of Theorem A. Consequently, for any $r \geq 1$, they lead to the simultaneous occurrence of $C^1$ robust homoclinic tangencies and heterodimensional cycles. Corollary A answers affirmatively a question posed in [19, Section 1.3] in a discussion about the $C^1$ coexistence of robust cycles and tangencies in the examples in [2].

Finally, we exhibit settings where entropy-like assumptions imply the existence of blenders. For that, we revisit the $C^1$ constructions in [4], showing that diffeomorphisms with hyperbolic measures whose entropy is sufficiently large exhibit blenders. These results are summarised in Theorem 1.4. Corollaries B and C translate our main results to that setting, providing simultaneous $C^1$ robust heterodimensional cycles of a variety of coindices.
We now state precisely our results.

1.1 Robust heterodimensional arising from homoclinic tangencies

In the sequel, given $r \geq 1$, we denote by $\text{Diff}^r(M^n)$ the space of $C^r$ diffeomorphisms of a compact boundaryless Riemannian manifold $M^n$ of dimension $n$ endowed with the $C^r$ topology. When we do not want to emphasise the dimension of the ambiance we will omit the superscript $n$.

The first ingredient in our results is the $cs$-blender. Blenders are hyperbolic sets with an additional geometric structure guaranteeing intersections between a family of “stable” disks and the local unstable manifold of the blender. Blenders are also $C^1$ robust sets. See Definition 2.1 for details.

Recall that two saddles of $f \in \text{Diff}^r(M)$ are homoclinically related if the invariant manifolds of their orbits intersect transversely and cyclically. To be homoclinic related defines an equivalence relation on the set of saddles of $f$. Two saddles that are homoclinically related have the same $u$-index. The homoclinic class of a saddle $P$ of $f$ is the closure of the saddles of $f$ that are homoclinically related to $P$. A homoclinic class is a transitive set (i.e., it contains a point whose orbit is dense in the set). As a blender $\Gamma$ of $f \in \text{Diff}^r(M)$ is a hyperbolic basic set (see Definition 2.1), its saddles form a dense subset of it and every pair of saddles of $\Gamma$ are homoclinically related. A saddle is homoclinically related to the blender if it is homoclinically related to some saddle of the blender, see (3) in Remark 2.3.

Recall that every hyperbolic set $\Lambda$ of a diffeomorphism $f$ has a well defined continuation for every $C^1$ nearby diffeomorphism $g$. We denote this continuation by $\Lambda_{g}$. If a hyperbolic set $\Lambda$ is transitive then the dimension of its unstable bundle does not depend on the point of $\Lambda$. This dimension is called the $u$-index of $\Lambda$.

Definition 1.1 (Robust cycles) Two transitive hyperbolic sets $\Lambda$ and $\Upsilon$ of $f \in \text{Diff}^r(M)$ with different $u$-indices form a heterodimensional cycle if their invariant stable and unstable sets intersect cyclically, that is,

$$W^s(\Lambda, f) \cap W^u(\Upsilon, f) \neq \emptyset \quad \text{and} \quad W^u(\Lambda, f) \cap W^s(\Upsilon, f) \neq \emptyset.$$ 

The coindex of this cycle is the absolute value of the difference of the $u$-indices of $\Lambda$ and $\Upsilon$. This cycle is $C^r$ robust if there is a $C^r$ neighbourhood $\mathcal{N}$ of $f$ such that the continuations $\Lambda_{g}$ and $\Upsilon_{g}$ have a heterodimensional cycle for every $g$ in $\mathcal{N}$.

A transitive hyperbolic set $\Lambda$ has a homoclinic tangency if its stable and unstable sets have some nontransverse intersection. This homoclinic tangency is $C^r$ robust if there is a $C^r$ neighbourhood of $f$ such that the continuation $\Lambda_{g}$ of $\Lambda$ has a tangency for every $g$ in that neighbourhood.

We now describe the class of homoclinic tangencies studied in this paper. Consider $f \in \text{Diff}^r(M^{m+n})$, with $r, m, n \geq 1$ and $m+n \geq 3$, and a saddle $P$ of $f$ of period $\pi$ and $u$-index $n$. Let $\lambda_1, \ldots, \lambda_m, \gamma_1, \ldots, \gamma_n$ be the eigenvalues of $Df^{\pi}(P)$ counted with multiplicity and ordered so that

$$|\lambda_m| \leq \cdots \leq |\lambda_1| = \lambda < 1 < \gamma = |\gamma_1| \leq \cdots \leq |\gamma_n|. \quad (1.1)$$

The multipliers of $P$ with modulus equal to $\lambda$ and $\gamma$ are called stable and unstable leading multipliers, respectively. We denote by $m_s = m_s(P)$ and $n_u = n_u(P)$ the number of stable and unstable leading multipliers, respectively, and say that $P$ is of type $(m_s, n_u)$. The saddle $P$ is simple if one of the following cases holds:
• $P$ is of type $(1, 1)$;
• $P$ is of type $(2, 1)$ and $\lambda_1$ and $\lambda_2$ are nonreal (and hence conjugate);
• $P$ is of type $(1, 2)$ and $\gamma_1$ and $\gamma_2$ are nonreal (hence conjugate);
• $P$ is of type $(2, 2)$, $\lambda_1$ and $\lambda_2$ are nonreal, and $\gamma_1$ and $\gamma_2$ are nonreal.

We define the leading Jacobian $J_P(f)$ of $P$ as the product of the moduli of all leading multipliers of $P$ counted with multiplicity,

$$J_P(f) \overset{\text{def}}{=} \lambda^m \gamma^n. \quad (1.2)$$

We are now ready to state our main results.

**Theorem A** Consider $f \in \text{Diff}^r(M^{m+1})$, $r \geq 1$, $m \geq 2$, with a homoclinic tangency associated to a simple saddle $P$ of u-index one such that:

1. $P$ is of type $(m_s, 1)$ with $m_s \in \{1, 2\}$,
2. $J_P(f) \geq 1$ and
3. $P$ is homoclinically related to a cs-blender $\Gamma$ of central dimension $m_s$.

Then there are diffeomorphisms $g$ arbitrarily $C^r$ close to $f$ having simultaneously $C^1$ robust heterodimensional cycles of coindex $i$ for every $1 \leq i \leq m_s$. These cycles are associated to a hyperbolic basic set containing $P_g$ and a hyperbolic basic set of u-index $1+i$.

Let us briefly comment on the hypotheses of Theorem A. Items (1) and (3) are compatibility conditions between the dominated splittings of the saddle and the blender. Condition (2) implies that the unfolding of the homoclinic tangency generates saddles of $u$-index greater than one, enabling the creation of heterodimensional cycles between these new saddles and the continuations of the saddle in the initial cycle.

The next result extends the previous theorem when the saddles exhibiting the homoclinic tangency have $u$-index $n > 1$. A key ingredient in this result is the notion of a double blender, see Definition 2.1.

**Theorem B** Consider $f \in \text{Diff}^r(M^{m+n})$, $r \geq 1$, $m, n \geq 2$, with a homoclinic tangency associated to a simple saddle $P$ of u-index $n$ that is

1. of type $(m_s, n_u)$ with $m_s, n_u \in \{1, 2\}$ and
2. homoclinically related to a double blender of central dimensions $(m_s, n_u)$.

Then there are diffeomorphisms $g$ arbitrarily $C^r$ close to $f$ having simultaneously $C^1$ robust heterodimensional cycles of coindex $i$ for every $1 \leq i \leq N$, where

$$N = m_s, \quad \text{if} \quad J_P(f) \geq 1 \quad \text{and} \quad N = n_u, \quad \text{if} \quad J_P(f) \leq 1.$$ 

These cycles are associated to hyperbolic basic sets containing $P_g$ and a hyperbolic basic set of $u$-index $n+i$.

In Theorem B, we obtain robust cycles associated to the continuations of the blender and new saddles of different $u$-indices. It is an open question if there are robust cycles associated to hyperbolic sets containing these new saddles.
1.2 Simultaneity of robust homoclinic tangencies and robust heterodimensional cycles

We now study the concurrence of robust cycles of different types. We start with an observation.

Remark 1.2 (Robust tangencies) Blender-horseshoes are a particular type of cs-blenders, see Remark 2.2. Under the assumptions of Theorem A or B, if the cs-blender $\Gamma$ of $f$ is a blender-horseshoe and the saddle $P$ is of type $(1, 1)$ then [19, Theorem 4.9] implies that there are diffeomorphisms $g$ arbitrarily $C^r$ close to $f$ having $C^1$ robust tangencies associated to a hyperbolic basic set containing $P_g$. This leads to $C^r$ open sets whose closure contains $f$ consisting of diffeomorphisms having simultaneously $C^1$ robust heterodimensional cycles and $C^1$ robust homoclinic tangencies. We do not know if under the assumptions of Theorem A or B and with the general definition of a cs-blender (see Definition 2.1) such a concurrence holds.

We observe that (in dimension three) there are several results assuring the simultaneous occurrence of the two types of robust cycles above. The results in [34] state conditions on the multipliers of a saddle-focus with a homoclinic tangency to generate simultaneously $C^1$ robust homoclinic tangencies and heterodimensional cycles by small $C^1$ perturbations. Similarly, in [5] it is proved that both types of robust cycles can be $C^1$ approximated by diffeomorphisms with heterodimensional cycles associated to saddles with nonreal multipliers. A nondominated $C^r$ setting, $r \geq 2$, with simultaneous occurrence of robust homoclinic tangencies and robust heterodimensional cycles was explored in [25].

We aim to get the simultaneity of robust cycles in more general settings. We apply Theorem A to the class of diffeomorphisms with robust homoclinic tangencies in [2, 51]. We see that these diffeomorphisms also display $C^1$ robust heterodimensional cycles. The key ingredient in this result are the cs-blenders of Plykin type obtained as a “local” product of a Plykin repeller by strong contractions, for details see Sect. 7.1. These cs-blenders have fixed points of $u$-index one. Roughly, the robust cycles in the next corollary are obtained bifurcating the homoclinic tangency of a fixed point in the blenders, see Fig. 1.

Corollary A Consider $f \in \text{Diff}^r(M^{m+1}, r \geq 1, m \geq 2$, with a cs-blender $\Gamma$ of Plykin type whose fixed point $P$ has leading multipliers $\lambda$ and $\sigma$ satisfying $\lambda \sigma > 1$. Suppose that $f$ has homoclinic tangency associated to $P$. Then there is a $C^r$ open set $\mathcal{C}$ whose closure contains $f$ such that every $g \in \mathcal{C}$ has simultaneously a $C^1$ robust homoclinic tangency and a $C^1$ robust heterodimensional cycle associated to the blender $\Gamma_g$.

Remark 1.3 An ipsis litteris version of Corollary A can be stated for cs-blenders of DA type. In this case, the construction is analogous, starting with a DA diffeomorphism defined on the two-dimensional torus, see for instance [47, Chapter 7.8]. This provides some constraints on the ambient manifold for the occurrence of blenders of DA type. Note that Plykin attractors are defined locally in a ball, therefore there are no topological constraints for their occurrence.

1.3 $C^1$ robust heterodimensional cycles arising from horseshoes with large entropy

We now use Theorems A and B to prove that homoclinic tangencies associated to horseshoes with sufficiently large entropy lead to $C^1$ robust heterodimensional cycles by small $C^1$ perturbations. For this, we use the results in [4] claiming that these horseshoes contain blenders.
We start with some preliminaries from ergodic theory. In what follows, all measures considered are probability ones. A measure $\mu$ of $f \in \text{Diff}^r(M^d)$ is called **hyperbolic** if it is $f$-invariant, ergodic, and all its Lyapunov exponents are different from zero. We list the Lyapunov exponents of $\mu$ (counted with multiplicity) as follows

$$
\chi_{-m}(\mu) \leq \cdots \leq \chi_{-1}(\mu) < 0 < \chi_{+1}(\mu) \leq \cdots \leq \chi_{+n}(\mu), \quad m + n = d,
$$

and say that $\mu$ has $u$-index $n$. To emphasise the nature of these exponents, we write

$$
\chi^{cs}_\mu = \chi_{-1}(\mu) \quad \text{and} \quad \chi^{cu}_\mu = \chi_{+1}(\mu).
$$

We also consider the stable and unstable Jacobians of $\mu$ defined by

$$
J^s_\mu \overset{\text{def}}{=} \exp \left( \sum_{i=1}^m \chi_{-i}(\mu) \right) \quad \text{and} \quad J^u_\mu \overset{\text{def}}{=} \exp \left( \sum_{i=1}^n \chi_{+i}(\mu) \right).
$$

The previous discussion applies to any ergodic measure supported on a horseshoe, in particular to its measure of maximal entropy. By a **horseshoe** of a diffeomorphism we mean a hyperbolic Cantor set that is transitive and locally maximal. In this case, the horseshoe $\Lambda$ has only one measure of maximal entropy $\mu_{\Lambda, \text{max}}$. We denote by $h_{\text{top}}(\Lambda, f)$ the topological entropy of $\Lambda$. Note that in the next two corollaries the required regularity is an integer, see also Theorem 1.4.

**Corollary B** Consider $f \in \text{Diff}^r(M^{m+1})$, with $r, m \geq 2$ and $r \in \mathbb{N}$, having a homoclinic tangency associated to a simple saddle $P$ of $u$-index one such that

1. $P$ is of type $(m_s, 1)$ with $m_s \in \{1, 2\},$
2. $J_P(f) \geq 1,$ and
3. $P$ is homoclinically related to a horseshoe $\Lambda$ satisfying

$$
h_{\text{top}}(\Lambda, f) > -\log J^s_{\mu_{\Lambda, \text{max}}} + \frac{1}{2r} \chi^{cs}_{\mu_{\Lambda, \text{max}}}.
$$

$$
W^u(\Gamma_g)
$$

**Fig. 1** Robust heterodimensional cycles in Corollary A
Then there are diffeomorphisms \( g \) arbitrarily \( C^1 \) close to \( f \) displaying simultaneously \( C^1 \) robust heterodimensional cycles of coindex \( i \) for every \( 1 \leq i \leq m_s \). These cycles are associated to \( \Lambda_g \) and a transitive hyperbolic set of \( u \)-index \( 1 + i \).

As the exponent \( \chi^{cs} \) is negative, the right-hand term in (1.6) increases with \( r \) and goes to \( -\log J^s_{\mu, \text{max}} \) as \( r \to \infty \). In particular, if the estimate holds for \( r_0 \) it also holds for any \( r > r_0 \). Therefore, this inequality is less restrictive when the regularity of the diffeomorphism increases. Similar comments hold for the lower bound of the topological entropy in Corollary C below.

Similarly, as a consequence of Theorem B, we obtain the following.

**Corollary C** Consider \( f \in \text{Diff}^r(M^{m+n}) \), with \( r, m \geq 2 \) and \( r \in \mathbb{N} \), having a homoclinic tangency associated to a simple saddle \( P \) of \( u \)-index one such that

1. \( P \) is of type \( (m_s, n_u) \) with \( m_s, n_u \in \{1, 2\} \),
2. \( P \) is homoclinically related to a horseshoe \( \Lambda \) with

\[
h_{\text{top}}(\Lambda, f) > \max \left\{ -\log J^s_{\mu, \text{max}} + \frac{1}{2r} \chi^{cs}_{\mu, \text{max}}, \log J^u_{\mu, \text{max}} - \frac{1}{2r} \chi^{cu}_{\mu, \text{max}} \right\}.
\]

Then there are diffeomorphisms \( g \) arbitrarily \( C^1 \) close to \( f \) having simultaneously \( C^1 \) robust heterodimensional cycles of coindex \( i \) for every \( 1 \leq i \leq N \), where

\[
N = m_s, \quad \text{if } J_P(f) \geq 1 \quad \text{and} \quad N = n_u, \quad \text{if } J_P(f) \leq 1.
\]

These cycles are associated to \( \Lambda_g \) and a transitive hyperbolic set of \( u \)-index \( n + i \).

We now state the variation of the results in [4] mentioned above (see Theorem 1.4). A first step of the proof of this variations involves linearisation and diagonalisation results (see [4, Theorems B and 3.2]). These results claim that given any diffeomorphism \( f \) with a horseshoe \( \Lambda \) there is a \( C^1 \) local perturbation \( g \) of \( f \) having an affine horseshoe \( \tilde{\Lambda} \) close to \( \Lambda \) in the Hausdorff distance and whose topological entropy is also close to the one of \( \Lambda \). Moreover, the set \( \tilde{\Lambda} \) has a dominated splitting consisting of one-dimensional bundles. A second step, also borrowed from [4], is that hyperbolic measures with “large entropy” (in the sense of Eq. (1.7), see also Remark 1.6) provide blenders, see [4, Theorems B’ and C].

Finally, recall that given any hyperbolic measure \( \mu \) of a diffeomorphism \( f \), Pesin theory provides stable and unstable manifolds, denoted by \( W^s_\mu(x, f), \ast = s, u \), for \( \mu \)-almost every point \( x \). Note that this requires \( f \) to be \( C^r \) with \( r > 1 \). We say that a transitive hyperbolic set \( \Lambda \) and a hyperbolic measure \( \mu \) are homoclinically related if there is a periodic orbit \( O \subset \Lambda \) such that \( W^s(O, f) \cap W^u_\mu(x, f) \neq \emptyset \) and \( W^u(O, f) \cap W^s_\mu(x, f) \neq \emptyset \) for \( \mu \)-almost every point \( x \in M \). Here the symbol \( \cap \) is used to denote the set of transverse intersection points between two submanifolds.

Note that Theorem 1.4 below it is required a regularity \( r \in \mathbb{N} \). This restriction is due to the fact that this result follows from [4, Theorem C] where this condition is an explicit requirement.

**Theorem 1.4** Consider \( f \in \text{Diff}^r(M^{m+n}) \), with \( n \geq 1 \), \( r, m \geq 2 \), and \( r \in \mathbb{N} \), and a hyperbolic measure \( \mu \) of \( f \) with \( u \)-index \( n \) such that

\[
h_\mu(f) > -\log J^s_\mu + \frac{1}{2r} \chi^{cs}_\mu.
\]

\[\text{Eq. (1.7)}\]

We thank S. Crovisier for explaining us this subtle point of the proofs.
Then there is a sequence \((g_k), g_k \in \text{Diff}^r(M^{m+n})\), of local perturbations of \(f\) supported on arbitrarily small neighbourhoods of the support of \(\mu\) and converging to \(f\) in the \(C^1\) topology such that every \(g_k\) has a cs-blender \(\Gamma_k\) such that

1. \(\Gamma_k\) has central dimension \(d_{cs} = m - 1\) and a dominated splitting consisting of one-dimensional subbundles,
2. \(\Gamma_k\) converges to the support of \(\mu\) in Hausdorff distance, and
3. \(h_{\text{top}}(\Gamma_k, g_k)\) converges to \(h_{\mu}(f)\).

Moreover, if \(f\) has a saddle \(P\) that is homoclinically related to the measure \(\mu\) then the sequence \((g_k)\) can be chosen such that the continuation \(P_k\) of \(P\) for \(g_k\) is homoclinically related to the blender \(\Gamma_k\). In addition, if \(P\) has a homoclinic tangency, then \(g_k\) can be chosen with a homoclinic tangency associated to \(P_k\).

**Remark 1.5** According to item (2) in Remark 2.3, the blenders \(\Gamma_k\) in Theorem 1.4 are cs-blenders of central dimension \(k\) for every \(k = 1, \ldots, m - 1\) (here \(m\) is the number of negative Lyapunov exponents of \(\mu\) counted with multiplicity).

**Remark 1.6** A result analogous to Theorem 1.4 holds under the assumption

\[
h_{\mu}(f) > \log J_{\mu}^u - \frac{1}{2r} \chi_{\mu}^{cu}\]

called in [4] *almost Pesin formula*. This condition leads to \(cu\)-blenders with similar properties as in Theorem 1.4. In particular, if \(\mu\) is a hyperbolic measure satisfying the Pesin entropy formula (that is, \(h_{\mu}(f) = \log J_{\mu}^u\)) then there are arbitrarily small \(C^1\) perturbations \(g\) of \(f\) having \(cu\)-blenders \(\Gamma_g\) satisfying conditions (1)–(3) in Theorem 1.4. Clearly, in item (1) the central dimension \(d_{cu}\) of the blender is \(k = 1, \ldots, n - 1\), where \(n\) is the \(u\)-index of \(\mu\).

**Remark 1.7** Double blenders (see Definition 2.1) with central dimensions \((m - 1, n - 1)\), where \(m\) and \(n\) are as in the previous remarks, and dominating splittings into one-dimensional subbundles are obtained when

\[
h_{\mu}(f) > \max \left\{ -\log J_{\mu}^s + \frac{1}{2r} \chi_{\mu}^{cs}, \log J_{\mu}^u - \frac{1}{2r} \chi_{\mu}^{cu} \right\}.
\]

**Termination 1.8** Throughout this paper, we will use the term *perturbation* only to refer to arbitrarily small ones.

**Organisation**

This paper is organised as follows. In Sect. 2, we introduce and discuss cs and cu-blenders. In Sect. 3, we study the generation of simple saddles with some prescribed domination properties. In Sect. 4, we analyse the return maps in a neighbourhood of the homoclinic tangency and explore the expanding-like properties derived from the expanding leading Jacobian hypothesis. Section 5 introduces unfolding families and uses them to get new saddles of different \(u\)-indices. Theorems A and B are proved in Sect. 6, for that we relate the saddles with different \(u\)-indices previously obtained with the blender. Section 7 is dedicated to Simon–Asaoka’s examples. Section 1 is an appendix dedicated to the proof of Theorem 1.4. Finally, Sect. 1 is another appendix about dominated splittings.
2 Blenders

There are several definitions of blenders, each one adapted to an appropriate context and emphasising some properties. We do not aim to discuss and compare these definitions. As mentioned above, blenders are hyperbolic sets with a geometric structure guaranteeing intersections between a family of “stable” disks and the local unstable manifold of the blender. A fundamental property of blenders is their $C^1$ robustness. A relevant point in our Definition 2.1 is the existence of a distinctive saddle. As far as we know, distinctive saddles exist in any known example of blenders, their existence is an explicit requirement in [11, Definition A1].

Our definition of a blender involves the concepts of hyperbolicity and partial hyperbolicity, stable and unstable bundles, strong invariant manifolds, and domination. For these concepts, see the appendix in Sect. 1. In what follows, we write dominated splittings $E_1 \oplus \cdots \oplus E_k$ with several bundles in such a way the bundle $E_i$ is more “contracting” than the bundle $E_{i+1}$.

We will use the following notation for the local invariant manifolds of a hyperbolic set $\Lambda$ of a diffeomorphism $f$ relative to a neighbourhood $U$ of $\Lambda$. We let

$$W^u_{U, \text{loc}}(\Lambda, f) \overset{\text{def}}{=} \{ x \in U : f^i(x) \in U \text{ for every } i \leq 0 \}. \quad (2.1)$$

The set $W^s_{U, \text{loc}}(\Lambda, f)$ is defined in the obvious way. A special case occurs when $\Lambda$ is a periodic orbit.

**Definition 2.1** (cs-, cu-blender, and double blender) Let $f \in \text{Diff}^1(M^d)$, $d \geq 3$. A hyperbolic and transitive set $\Gamma \subset M^d$ is a cs-blender if the following holds:

(a) (local maximality) There is an open neighbourhood $U$ of $\Gamma$ such that

$$\Gamma = \bigcap_{i \in \mathbb{Z}} f^i(\overline{U}).$$

(b) (partial hyperbolicity) There is a $Df$-invariant dominated splitting with three nontrivial bundles defined over $\Gamma$

$$T_{\Gamma}M^d = E^s \oplus E^c \oplus E^u,$$

where $E^s = E^{ss} \oplus E^c$ and $E^u$ are the stable and unstable bundles of $\Gamma$. We write

$$d_{cs} \overset{\text{def}}{=} \dim E^c \geq 1 \text{ and } d_{ss} \overset{\text{def}}{=} \dim E^{ss} \geq 1.$$

(c) (open set of embedded disks) There is an open set $\mathcal{O}^{ss}$ of $C^1$ embeddings of $d_{ss}$-dimensional disks such that:

(i) (distinctive saddle) There is a saddle $Q^* \in \Gamma$ whose strong stable manifold $W^{ss}(Q^*, f)$ contains a disk of $\mathcal{O}^{ss}$ containing $Q^*$ in its interior.

---

4 Let us summarise some settings where blenders play key roles. Blenders were initially defined as having central dimension one. Currently, there are versions (as the ones here) where the central dimensions are greater than one. Applications of blenders of central dimension one include: robustly transitive dynamics [17], robust heterodimensional cycles [18, 36], robust homoclinic tangencies [19], stable ergodicity [48], and construction of nonhyperbolic ergodic measures [14], among others. Blenders with larger central dimensions were introduced in [7, 38] to study instability problems in symplectic dynamics, in [9] to obtain robust heterodimensional cycles of large coindex, and in [3, 6] to get robust tangencies of large codimension. Blenders of large central dimension also appear in the study of ergodicity of conservative partial hyperbolic systems [4], of holomorphic dynamics [13, 26, 53], and of parametric families of maps (endomorphisms in [10, 12] and diffeomorphisms [8]).

5 Throughout this paper, $C^1$ embeddings are identified with their images in the ambient manifold.
(ii) (robust intersections) There is a $C^1$-neighbourhood $\mathcal{U}$ of $f$ such that for every $g \in \mathcal{U}$ the continuation $\Gamma_g$ of $\Gamma$ satisfies

$$W_{U, \text{loc}}^u(\Gamma_g, g) \cap D \neq \emptyset, \quad \text{for every } D \in \mathcal{D}^{ss}.$$ 

The set $U$ is called a reference domain of the blender. The sets $\mathcal{D}^{ss}$ and

$$B^{ss} \overset{\text{def}}{=} \text{int} \left( \bigcup_{D \in \mathcal{D}^{ss}} D \cap U \right)$$

are called superposition region and superposition domain of the blender, respectively. The point $Q^*$ is a distinctive saddle of the blender. We say that $d_{cs}$ is the central dimension of the blender.

A cu-blender of $f$ is cs-blender for $f^{-1}$. For a cu-blender, $d_{cu}$, $\mathcal{D}^{uu}$, and $B^{uu}$ are defined as above.

A double blender of central dimensions $(d_{cs}, d_{cu})$ is a hyperbolic set that is simultaneously a cs-blender of central dimension $d_{cs}$ and cu-blender of central dimension $d_{cu}$. In particular, a double blender has a pair of distinctive saddles $Q_{cs}^*$ and $Q_{cu}^*$ as well as $ss$- and $uu$-superposition regions and domains.

Note that cs-blenders with central dimension $d_{cs} \geq 1$ can occur only in dimension $1 + d_{cs} + 1 \geq 3$. Similarly, double blenders with central dimensions $(d_{cs}, d_{cu})$ can appear only in dimension $1 + d_{cs} + d_{cu} + 1 \geq 4$.

**Remark 2.2** (Blender-horseshoes) Blender-horseshoes were introduced in [19]. They have a one-dimensional central dimension. Their key property is that iterations of the disks in the superposition region provide new disks of that region. Every blender-horseshoe is a cs- or a cu-blender with central one-dimensional direction.

**Remark 2.3** (Properties of blenders) Consider $f \in \text{Diff}^r(M)$, $r \geq 1$, with a cs-blender $\Gamma$ of central dimension $d_{cs}$, reference domain $U$, partially hyperbolic splitting

$$T_{\Gamma} M^d = E^s \oplus E^c \oplus E^u, \quad d_{cs} = \text{dim } E^c, \quad d_{ss} = \text{dim } E^{ss}, \quad (2.2)$$

superposition region $\mathcal{D}^{ss}$, and distinctive saddle $Q^*$.

1. **Continuations of blenders**: Having a cs-blender is a $C^1$ open condition (and hence $C^r$ open). For every $g$ sufficiently $C^1$ close to $f$ the continuation $\Gamma_g$ of $\Gamma$ is also a cs-blender with the same reference domain $U$, the same superposition region $\mathcal{D}^{ss}$, and the same central dimension. Moreover, if $Q^*$ is a distinctive saddle of $\Gamma$ then its continuation $Q_{cs}^*$ is also a distinctive saddle of $\Gamma_g$.

2. **Central dimensions of a cs-blender**: The hyperbolicity of $\Gamma$ implies that if $D$ is a disk in $\mathcal{D}^{ss}$ and $X \in W_{U, \text{loc}}^u(\Gamma, f) \cap D$ then there is $Z \in \Gamma$ such that $X \in W_{U, \text{loc}}^u(Z, f) \cap D$. Moreover, the codimension of

$$T_{X} W_{U, \text{loc}}^u(Z, f) + T_{X} D$$

is at least $d_{cs}$ and is equal to $d_{cs}$ if and only if the intersection between $W_{U, \text{loc}}^u(Z, f)$ and $D$ at $X$ is quasi-transverse (i.e., $T_{X} W_{U, \text{loc}}^u(Z, f) \cap T_{X} D = \{0\}$).

Assume now that the bundle $E^c$ of $\Gamma$ splits in a dominated way as follows

$$E^c = E^c_1 \oplus \cdots \oplus E^c_k. \quad (2.3)$$

For each $1 \leq r \leq \ell \leq k$, consider the bundles and the dimensions

$$E_{r, \ell}^c \overset{\text{def}}{=} E_r^c \oplus \cdots \oplus E_\ell^c, \quad d_{r, \ell} \overset{\text{def}}{=} \text{dim } E_{r, \ell}^c.$$
We claim that $\Gamma$ is also a $cs$-blender of central dimension $d_{r,k}$ for each $r = 2, \ldots, k$. Using (2.3), for each $r \in \{2, \ldots, k\}$, we get a dominated splittings of $\Gamma$ of the form

$$T_\Gamma M^d = (E^{ss} \oplus E^c_{1,r-1}) \oplus E^c_{r,k} \oplus E^u,$$

(2.4)

where $E^{ss} \oplus E^c_{1,r-1}$ is a strong stable bundle and $E^c_{r,k}$ is a center stable bundle of dimension $d_{r,k}$.

To see that $\Gamma$ is a $cs$-blender of central dimension $d_{r,k}$ it is sufficient to consider the splitting in (2.4) and any family of disks $\mathcal{D}^{ss}$ of dimension $d_{ss} + d_{1,r-1}$ containing the initial family $\mathcal{D}^{ss}$ and such that some disk of this family contains $Q^*$ in its interior and is contained in the strong stable manifold $W^{ss}(Q^*)$ (now considered with respect the strong stable bundle $E^{ss} \oplus E^c_{1,r-1}$). By the observations above, the intersections properties in (c) hold.

In the special case when $E^c$ splits into one-dimensional bundles in a dominated way, we have that $\Gamma$ is a $cs$-blender of central dimensions $1, \ldots, d_{cs}$. This sort of blender was introduced in [9, 38] and called superblender in [4].

3 Homoclinic relations: Note that every pair of saddles of $\Gamma$ are homoclinically related. In particular, every saddle of $\Gamma$ is homoclinically related to any distinctive saddle of $\Gamma$. Thus any saddle homoclinically related to the blender is homoclinically related to any distinctive saddle of the blender.

The following remark is a standard consequence of the inclination lemma and the comments above.

**Remark 2.4** (Tangencies associated to distinctive saddles) Let $f \in \Diff^r(M)$ and consider two saddles $R$ and $S$ of $f$ which are homoclinically related. Suppose that $R$ has a homoclinic tangency. Then there is a $C^r$ perturbation $g$ of $f$ such that $S_g$ has a homoclinic tangency. We get the following consequence. Fix a $cs$-blender $\Gamma$ of $f$, a distinctive saddle $Q^*$ of $\Gamma$, and a saddle $P$ homoclinically related to $\Gamma$. Then if $P$ has a homoclinic tangency there is a $C^r$ perturbation $g$ of $f$ having a homoclinic tangency associated to the distinctive saddle $Q^*_g$ of the blender $\Gamma_g$.

A variation of the above assertion is the following. Consider a hyperbolic basic set $\Lambda$ of $f$ (i.e., locally maximal in an open neighbourhood of it) with a homoclinic tangency (not necessarily associated to a saddle). Given any periodic saddle $R \in \Lambda$ there is a $C^r$ perturbation $g$ of $f$ with a homoclinic tangency associated to $R_g \in \Lambda_g$.

3 Homoclinic relations of simple saddles with prescribed type

This section aims to prove Proposition 3.2 about the generation of simple saddles with some prescribed domination. The relevant cases lead to simple saddles of type (2, 1) and (2, 2).

We need to introduce some definitions needed to state Proposition 3.2. First note that if a saddle $P$ of a diffeomorphism $f$ has $u$-index one then $W^u_{loc}(P, f) \setminus \{P\}$ has two connected components. In this case, we say that the saddle $P$ is biaccumulated by its transverse homoclinic points if $W^u(P, f)$ transversely intersects both connected components of $W^u_{loc}(P, f) \setminus \{P\}$. Note that this definition does not depend on the choice of the local manifold. We will use this property in Sect. 4.4 to define return maps associated to a homoclinic tangency of a saddle with $u$-index one.

---

6 A local unstable manifold of $P$ is any disk contained in $W^u(P, f)$ of the same dimension as $W^u(P, f)$ that contains $P$ in its interior. A local stable manifold is similarly defined.
We now introduce the notion of transverse heteroclinic intersections adapted to a dominated decomposition. Consider a pair of saddles $P$ and $P'$ of the same index of a diffeomorphism $f$ such that there is a dominated splitting $E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$ defined on the orbits of $P$ and $P'$, where $E^{ss} \oplus E^{cs} = E^s$ (the stable bundle) and $E^{cu} \oplus E^{uu} = E^u$ (the unstable bundle). Consider now a transverse heteroclinic point $X \in W^u(P, f) \cap W^s(P', f)$.

Let us identify some invariant bundles defined along the orbit of $X$. First, using that $X \in W^s(P', f)$ and the splitting $E^{cu}(P') \oplus E^{uu}(P')$, we can define the bundle $E^{cu}(X)$ (the only bundle of $T_X W^u(P, f)$ of the same dimension of $E^{cu}$ that does not accumulate to $E^{uu}(P')$ as $n$ increases). In this way, we get a $Df$-invariant splitting $E^s(X) \oplus E^{cu}(X)$ defined along the orbit of $X$. The bundle $E^{ss}(X)$ is also defined. Similarly, using that $X \in W^u(P, f)$ and the splitting $E^{ss}(P) \oplus E^{cs}(P)$, considering negative iterates we can define the splitting $E^{cs}(X) \oplus E^u(X)$ along the orbit of $X$. The bundle $E^{uu}(X)$ is also defined.

The existence of an adapted dominated decomposition states some compatibility between the bundles above. The heteroclinic transverse point $X$ is adapted to the dominated splitting $E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$ of the orbits of $P$ and $P'$ if

- $E^{ss}(X)$ and $E^{cs}(X) \oplus E^{u}(X)$ are transverse,
- $E^{s}(X) \oplus E^{cu}(X)$ and $E^{uu}(X)$ are transverse.

Let us state a general transversality result:

**Remark 3.1** Consider $f \in \text{Diff}^r(M)$ having a pair of saddles $P$ and $P'$ of the same index such that there is a dominated splitting $E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$ defined on the orbits of $P$ and $P'$, where $E^{ss} \oplus E^{cs} = E^s$ and $E^{cu} \oplus E^{uu} = E^u$. Let $X \in W^u(P, f) \cap W^s(P', f)$ and $Y \in W^s(P, f) \cap W^u(P', f)$. Then there is $g$ arbitrarily close to $f$ preserving the orbits of $P$ and $P'$ such that $X \in W^u(P, g) \cap W^s(P', g)$ and $Y \in W^s(P, g) \cap W^u(P', g)$ are transverse heteroclinic points adapted to the splitting $E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$.

**Proposition 3.2** Consider $f \in \text{Diff}^r(M^{m+n})$, $r \geq 1$, $m \geq 2$, $n \geq 1$, with a pair of saddles $P$ and $P'$ of $u$-index $n$ such that $P$ is simple of type $(m_s, n_u)$ and the tangent space of the orbits of $P$ and $P'$ has a dominated splitting of the form $E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$, where $\dim(E^{cs}) = m_s$, $\dim(E^{cu}) = n_u$, $E^{cs}$ is contained in the stable bundle, and $E^{cu}$ is contained in the unstable bundle.

Suppose that there are transverse heteroclinic points $X \in W^s(P, f) \cap W^u(P', f)$ and $Y \in W^u(P, f) \cap W^s(P', f)$ which are adapted to the dominated splitting $E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}$. Then given any neighbourhood $V$ of $P'$ there is a $C^r$ perturbation $g$ of $f$ having a saddle $Q \in V$ such that

1. $Q$ is simple of type $(m_s, n_u)$ and $\log J_{P_g}(g) \cdot \log J_{P'_g}(g) > 0$,
2. $Q$ is homoclinically related to the continuations $P'_g$ and $P'_g$,
3. If $n = 1$ then $Q$ is biaccumulated by its transverse homoclinic points.

Finally, if $P$ has a homoclinic tangency then $Q$ can be taken with a homoclinic tangency.

For simplicity, in what follows, let us assume that the saddles have period one.

We will use this proposition to get saddles with nonreal central eigenvalues. Thus the relevant case occurs when $m_s$ or $n_u$ (or both) are different from one. In the literature many results provide conditions for the abundance of saddles with simple spectrum after perturbations. In particular, the case $m_s = n_u = 1$ is well-known (see, for instance, the linearisation result in [4, Theorem 3.2]). Thus we consider the cases when $m_s, n_u > 1$ and skip the case (1, 1).

Let us note that the condition on the existence of a saddle homoclinically related to $P$ with simple spectrum is not a restriction in our context. Indeed, if a saddle $P$ is homoclinically...
related to a saddle \( \tilde{P} \), given any neighbourhood \( V \) of \( \tilde{P} \) after a \( C^r \) perturbation of \( f \) we can assume that there is a saddle \( P' \in V \) with simple spectrum that is homoclinically related to \( P \) and \( \tilde{P} \).

The key step in the proof of Proposition 3.2 is the next lemma:

**Lemma 3.3** Consider \( f \in \text{Diff}^r (M^{n+n}) \) and saddles \( P \) and \( P' \) as in Proposition 3.2. Assume that \( P \) is of type \((2, n_u)\) and \( C^r \) linearisable in a neighbourhood \( U \). Given any neighbourhood \( V \) of \( P \) after a \( C^r \) perturbation of \( f \) we can assume that there is a saddle \( P' \in V \) with simple spectrum that is homoclinically related to \( P \) and \( \tilde{P} \).

The key step in the proof of Proposition 3.2 is the next lemma:

**Lemma 3.3** Consider \( f \in \text{Diff}^r (M^{n+n}) \) and saddles \( P \) and \( P' \) as in Proposition 3.2. Assume that \( P \) is of type \((2, n_u)\) and \( C^r \) linearisable in a neighbourhood \( U \). Given any neighbourhood \( V \) of \( P' \) there is a \( C^r \) perturbation \( g \) of \( f \) with a hyperbolic transitive set \( \Lambda_g \) containing \( P_g \), \( P'_g \), and a simple saddle \( Q \) of type \((2, n_u)\) such that the orbit of \( Q \) intersects \( U \) and \( V \) and satisfies

\[
\log J_{P_g}(g) \cdot \log J_{Q}(g) > 0.
\]

Before proving the lemma, let us give a simple two-dimensional geometrical argument leading to matrices with a pair of conjugate nonreal eigenvalues.

Consider a \( 2 \times 2 \) real matrix \( A \) with real eigenvalues \( \tau > \rho > 0 \) and the family of linear maps

\[
A_\varphi = A \circ R_\varphi, \quad \text{where} \quad R_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \quad \varphi \geq 0.
\]

**Claim 3.4** (Saddle-node bifurcations) There is \( \varphi_0 > 0 \) such that

- \( A_\varphi \) has different and positive real eigenvalues for every \( \varphi \in [0, \varphi_0) \),
- \( A_{\varphi_0} \) has a real eigenvalue of multiplicity two, and
- \( A_\varphi \) has a pair of nonreal eigenvalues for \( \varphi > \varphi_0 \) close enough to \( \varphi_0 \).

**Proof** Without loss of generality, we can assume that the eigenvectors of \( A \) are \( v = (1, 0) \) and \( w = (\cos \theta, \sin \theta) \), \( \theta \in (0, \pi) \), associated \( \tau > \rho > 0 \). Consider the circle map \( \tilde{A}_\varphi \) associated to \( A_\varphi \) given by

\[
\tilde{A}_\varphi(u) = \frac{A_\varphi(u)}{||A_\varphi(u)||}.
\]

Note that \( \tilde{A}_0 \) has the following hyperbolic fixed points: the “most expanding” eigenvector \( v \) of \( A \) is attracting for \( \tilde{A}_0 \) while the “most contracting” eigenvalue \( w \) of \( A \) is repelling for \( \tilde{A}_0 \). See Fig. 2. These hyperbolic points of \( \tilde{A}_0 \) in \( S^1 \) have continuations \( v_\varphi \) and \( w_\varphi \) for every small \( \varphi > 0 \). Note that for \( \varphi > 0 \) it holds \( \tilde{A}_\varphi(u) > u \), \( u = v, w \). Here locally we consider the ordering induced by the ordering in \( \mathbb{R} \). That is, the graph of \( \tilde{A}_\varphi \) moves up as \( \varphi \) increases. This implies that the fixed points \( v_\varphi \) and \( w_\varphi \) of \( A_\varphi \) “move up and down,” respectively, and hence \( v < v_\varphi < w_\varphi < w \). Thus there is some \( \varphi_0 \) (associated to a saddle-node bifurcation of the family \( (\tilde{A}_\varphi)_{\varphi} \)) such that these two points coincide. The parameter \( \varphi_0 \) corresponds to a real eigenvalue of multiplicity two of \( A_{\varphi_0} \). For \( \varphi > \varphi_0 \) close enough to \( \varphi_0 \), this leads to a pair of conjugate nonreal eigenvalues. An analytic proof can be found in this footnote.\(^7\)

We now recall a standard construction involving transverse homoclinic points and hyperbolic sets.

\(^7\) Since \( A \) has eigenvalues \( \tau > \rho > 0 \) it follows \( (\tau + \rho)^2 - 4\tau \rho > 0 \). Let

\[
\text{tr}(\varphi) = \text{trace}(A_\varphi) = (a_{11} + a_{22}) \cos \varphi + (a_{21} + a_{12}) \sin \varphi.
\]

Observe that \( A_\varphi \) is hyperbolic, parabolic, or elliptic, respectively, if and only if

\[
\text{tr}(\varphi)^2 - 4\tau \rho = \text{tr}(\varphi)^2 - 4 \det(A_\varphi)
\]
Remark 3.5 Consider \( f \in \text{Diff}^r(M^{m+n}) \) with a pair of saddles \( P \) and \( P' \) as in Proposition 3.2: the saddles have the same \( u \)-index, a dominated splitting \( E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu} \), and transverse homoclinic points \( X \in W^s(P, f) \cap W^u(P', f) \) and \( Y \in W^s(P', f) \cap W^u(P, f) \) adapted to this splitting. Consider small open sets \( U \ni P \) and \( V \ni P' \), we can assume that (recall the definitions in (2.1))
\[
\begin{align*}
f^{kX}(X) &\in W^s_{U,\text{loc}}(P, f), & f^{-kX}(X) &\in W^u_{V,\text{loc}}(P', f) \\
f^{kY}(Y) &\in W^s_{V,\text{loc}}(P', f), & f^{-kY}(Y) &\in W^u_{U,\text{loc}}(P, f)
\end{align*}
\]
for some constants \( k_X, k_Y > 0 \), see Fig. 3. Then for every pair of small neighbourhoods \( U_X \) of \( X \) and \( U_Y \) of \( Y \), the maximal invariant set \( \Lambda = \Lambda_{\varphi} \) of \( f \) in
\[
\tilde{U} = U \cup V \cup \tilde{U}_X \cup \tilde{U}_Y, \quad \text{where} \quad \tilde{U}_Z = \bigcup_{i=-k_Z}^{k_Z} f^i(U_Z), \quad Z = X, Y,
\]
is a transitive hyperbolic set. The fact that points \( X \) and \( Y \) are adapted to the dominated splitting implies that \( \Lambda \) has a dominated splitting
\[
E^{ss} \oplus E^{cs} \oplus E^{cu} \oplus E^{uu}, \quad E^s = E^{ss} \oplus E^{cs}, \quad E^u = E^{cu} \oplus E^{uu},
\]
extending the ones in the orbits of \( P \) and \( P' \).

We observe that there is a homoclinic version (i.e., with \( P \) having transverse homoclinic points and considering only one transverse homoclinic point) of the construction above.

Proof of Lemma 3.3 Let \( \Lambda = \Lambda_{\varphi} \) be the hyperbolic set provided by Remark 3.5 (applied to \( P, P', U, V \) as in the lemma, and the corresponding homoclinic points). We take a neighbourhood \( U_0 \) of \( P \) contained in \( U \), a transverse homoclinic point \( A \in \Lambda_{\varphi} \) of \( P \), and \( \ell > 0 \) such that the orbit \( O_f(A) \) of \( A \) satisfies
\[
\begin{align*}
O_f(A) \setminus \{ f(A), \ldots, f^{\ell-1}(A) \} &\subset U_0, \\
\{ f(A), \ldots, f^{\ell-1}(A) \} \cap U_0 &\neq \emptyset, \\
O_f(A) \cap V &\neq \emptyset.
\end{align*}
\]

Footnote 7 continued
is positive, zero, or negative. Note that \( \text{tr}(\varphi) \) has at most one critical point on \((0, \pi]\). As
\[
\begin{align*}
\text{tr}(0) = \tau + \rho &\geq 2\sqrt{\tau \rho} \quad \text{and} \quad \text{tr}(\pi) = -\text{tr}(0) = -(\tau + \rho).
\end{align*}
\]
there is a unique \( \varphi_0 \in (0, \pi] \) with \( \text{tr}(\varphi_0) = 2\sqrt{\tau \rho} \). Then \( \text{tr}(\varphi) > 2\sqrt{\tau \rho} \) if \( \varphi \in (0, \varphi_0) \) and \( \text{tr}(\varphi) < \sqrt{\tau \rho} \) if \( \varphi \in (\varphi_0, \pi] \). The former implies that \( A_{\varphi_0} \) is is hyperbolic if \( \varphi \in (0, \varphi_0) \) and elliptic for \( \varphi > \varphi_0 \) close enough to \( \varphi_0 \).

\[\square\] Springer
Fig. 3 Generation of saddles with prescribed itineraries

Take a small neighbourhood $U_A \subset U_0$ of $A$ such that $f^\ell(U_A) \subset U_0$, $f^i(U_A) \subset \widehat{U}$ for every $i \in \{0, \ldots, \ell\}$, and the sets $f(U_A), \ldots, f^{\ell-1}(U_A)$ are pairwise disjoint and disjoint from $U_0$. Let $f$ be the restriction of $f^\ell$ to $U_A$, that is, the transition map from $U_A$ to $U_0$ along the orbit of $A$, and $\mathcal{L}$ the restriction of $f$ to $U_0$ (a linear map). See Fig. 3. Observe that there is $k_0$ such that for every $k \geq k_0$ there is a periodic point $Q_k$ of period $k + \ell$ and $u$-index $n$

- $Q_k \in f^\ell(U_A),$

- $f^i(Q_k) \in U_0$ for every $i = 0, \ldots, k$ and $f^k(Q_k) \in U_A,$

- the eigenvalues of $Df^{k+\ell}(Q_k)$ satisfies (following the notation in (1.1))

$$|\lambda_3^{(k)}| < |\lambda_2^{(k)}| \leq |\lambda_1^{(k)}| < 1 < |\gamma_1^{(k)}| \leq |\gamma_2^{(k)}| < |\gamma_3^{(k)}|,$$

- $Q_k$ is homoclinically related to $P$.

This implies that

$$Q_k = f^{k+\ell}(Q_k) = \mathcal{F} \circ \mathcal{L}^k(Q_k) \in \Lambda$$

and that the orbit of $Q_k$ intersects both $U_0 \subset U$ and $V$. 

We first consider the case when $P$ is of type $(2, 1)$. If there are some large $k$ such that $\lambda_2^{(k)}$ and $\lambda_1^{(k)}$ are nonreal or equal, we are done. Thus, in what follows, we assume that $\lambda_2^{(k)}$ and $\lambda_1^{(k)}$ are both real and consider the two-dimensional central stable bundle $E^{cs}(Q_k)$ associated to them provided by Remark 3.5.

We write the linearising coordinates of $P$ and the matrix $Df(P)$ in the form

$$(x_{m-2}, x_c, x_n) \in \mathbb{R}^{m-2} \times \mathbb{R}^2 \times \mathbb{R}^n, \quad Df(Q) = \mathcal{L} = (L_{m-2}, \lambda R_0, L_n),$$

where $\lambda \in (0, 1)$, $R_0$ is as in (3.1), and $L_{m-2}$ and $L_n$ are linear maps. We consider linear maps of the form

$$\mathcal{L}_\alpha = (L_{m-2}, R_\alpha \circ (\lambda R_0), L_n) = (L_{m-2}, \lambda R_{\alpha+\theta}, L_n) \quad (3.3)$$

and the planes

$$\pi \overset{\text{def}}{=} \{(0^{m-2}, v_c, 0^n), v_c \in \mathbb{R}^2\}, \quad \tilde{\pi} \overset{\text{def}}{=} D\mathcal{F}^{-1}_A(\pi).$$

We first claim that we can assume that the determinant of $D\mathcal{F}_A|\tilde{\pi}$ is positive and $D\mathcal{F}_A|\tilde{\pi}$ preserves the orientation. Otherwise, we consider new homoclinic points corresponding to
two “loops” of \( A \), that is, a new transverse homoclinic point \( A’ \) of \( P \) nearby \( A \) whose orbit is as follows: there is large even \( i_0 \) such that
\[
A’ \in W^u_{U_0, \text{loc}}(P), \quad f^{\ell + i_0 + \ell}(A’) \in W^s_{U_0, \text{loc}}(P),
\]
\[
f^{\ell}(A'), \ f^{\ell + i_0}(A') \in U_A,
\]
\[
f^{\ell + i}(A') \in U_0 \quad \text{for every } i \in \{0, \ldots i_0\}.
\]

As for large \( i_0 \) the transition map \( \mathcal{F}' \) associated to \( f^{2\ell + i_0} \) at \( A’ \) is close to \( \mathcal{I} \circ \mathcal{L}'_0 \circ \mathcal{I} \) our claim follows. Note that for this new point \( A’ \) one must shrink the linearising neighbourhood in the next steps, but this just adds additional iterations by a linear map that does not change the central determinant.

To conclude the proof, consider local perturbations \( f_\alpha \) of \( f \) whose derivative in \( U_0 \) is \( \mathcal{L}_\alpha \). For each large \( k \) and \( \alpha \in \{0, 2\pi\} \) the diffeomorphism \( f_\alpha \) has a periodic point \( Q_{k, \alpha} \) that is the continuation of \( Q_k \). Note that if there is small \( \alpha \) and large \( k \) such that
\[
D(\mathcal{L}_\alpha \circ \mathcal{I}) \mathcal{F}^{-1}(Q_{k, \alpha}) = \mathcal{L}_\alpha \circ \mathcal{I}_{k, \alpha}, \quad \mathcal{I}_{k, \alpha} \overset{\text{def}}{=} D\mathcal{F}^{-1}(Q_{k, \alpha})
\]
has a nonreal central eigenvalue, we are done. Note that, independently of \( \alpha \),
\[
\hat{Q}_{k, \alpha} \overset{\text{def}}{=} \mathcal{F}^{-1}(Q_{k, \alpha}) \to A \quad \text{and} \quad \hat{\mathcal{I}}_{k, \alpha} \to D\mathcal{F}_A, \quad \text{as } k \to \infty. \tag{3.4}
\]

We also have
\[
\mathcal{L}_{k, \alpha}^0 = (L_{m-2}^{k}, R_{k, \alpha}^{k} R_{k, \theta}, L_{n}^{k}) = (\text{Id}_{m-2}, R_{k, \alpha}, \text{Id}_n) \circ \mathcal{L}_k,
\]
where \( \text{Id}_r \) denotes the identity map in \( \mathbb{R}^r \). Fix large \( k \) and the center eigenvectors of \( \nu_k^0 \) and \( \nu_k^0 \) of \( Df^{k+\theta}(\hat{Q}_{k, 0}) = \mathcal{L}_k \circ \hat{\mathcal{I}}_{k, 0} \). For the parameter \( \alpha = 0 \), we can chose these vectors such that
\[
u_k^0 = (\tau_k^\ell(u_k^0), u_k^0, \tau_k^u(u_k^0)) \in \mathbb{R}^{m-2} \times \mathbb{R}^2 \times \mathbb{R}^n, \quad u, v, w,
\]
where \( \tau_k^\ell : \mathbb{R}^2 \to \mathbb{R}^{m-2} \) and \( \tau_k^u : \mathbb{R}^2 \to \mathbb{R}^n \) go to the zero map as \( k \to \infty \) and \( \nu_k^\ell \) and \( \nu_k^u \) are unitary eigenvectors in \( \mathbb{R}^2 \) as in the proof of Claim 3.4. For each \( \alpha \in [0, 2\pi] \) we consider
\[
\hat{\nu}_k^\alpha = D\mathcal{F}^{-1}(u_k^0)(\nu_k^0) = (\tau_k^\ell(\hat{u}_\alpha^0), \hat{u}_\alpha^0, \tau_k^u(\hat{u}_\alpha^0)), \quad u, v, w,
\]
where the maps \( \tau_k^*, \hat{\tau}_k^* \) are defined similarly as above. Observe that, by (3.4), for each fixed \( \alpha \) the variation of these vectors goes to zero as \( k \to \infty \).

We now observe that the images of the vectors \( \nu_k^0 \) and \( \nu_k^0 \) by \( \mathcal{L}_k^\alpha \) are the rotation by the angle \( k\alpha \) in the central direction of their images by \( \mathcal{L}_k^\alpha \) (which are parallel to \( \hat{\nu}_k^0 \) and \( \hat{\nu}_k^0 \), respectively). Thus these vectors move as in the proof of Claim 3.4. This displacement is not altered by \( D\mathcal{F}_{Q_{k, \alpha}} \) since this map preserves the orientation in the central plane. We observe that, for appropriate small \( \alpha \) and big \( k \), the fractional part of \( k\alpha \) is greater than and close to \( \varphi_0 \), providing elliptic points. Now the conclusion follows as in Claim 3.4, implying that \( Q_{k, \alpha} \) is of type \((2, \ast)\), \( \ast \in \{1, 2\} \). The fact that if \( P \) is of type \((2, 1)\) then \( Q_{k, \alpha} \) is of type \((2, 1)\) follows from domination arguments.

In the case when \( P \) is of type \((2, 2)\) the argument is analogous. The only difference is that we need to add a rotation-like term in the center unstable direction. Equation (3.3) now reads as follows
\[
\mathcal{L}_{\alpha, \beta} = (L_{m-2}, \lambda R_{\alpha} \circ (\lambda R_{\beta} \circ (y R_{\varphi}), L_{n-2})
\]
\[
= (L_{m-2}, \lambda R_{\alpha + \theta}, y R_{\beta + \varphi} L_{n-2}),
\]
where $\gamma > 1$. This leads to a family depending on two independent parameters $\alpha$ and $\beta$. This allows us to deal with each two-dimensional central direction independently, simultaneously getting nonreal eigenvalues.

Finally, to get the property of the leading Jacobians, note that the number of iterates of the points $Q_{k,\alpha}$ in the neighbourhood $U_0$ can be chosen arbitrarily large compared to the length of the orbit. The assertion now follows using the domination splitting. This ends the proof of the lemma.

Remark 3.6 (The case $n = 1$) In our construction, the orbit of the resulting saddle $Q$ has most of its iterates in the linearising neighbourhood $U$ of $P$. Therefore, its local manifolds are large. This implies that $Q$ is homoclinically related to $P$ (and hence to $P'$). Consider now the case $n = 1$. As the saddle $P$ has transverse homoclinic points, the property of $Q$ having large unstable manifolds implies that both components of $W^u_{U,\text{loc}}(Q) \setminus \{Q\}$ transversely intersects the stable manifold of $P$ (and hence the stable manifold of $Q$, since $P$ and $Q$ are homoclinically related). This implies that $Q$ is biaccumulated by its transverse homoclinic points.

Remark 3.7 Assume that the saddle point $P$ in Lemma 3.3 has a homoclinic tangency. Then the “local” perturbation $g$ of $f$ can be taken preserving that tangency.

3.1 Proof of Proposition 3.2

Observe that after an arbitrarily small perturbation we can assume that the saddle $P$ is linearizable. We need to consider the following types $(2,1)$, $(2,2)$, and $(1,2)$ for $P$. We prove the proposition when $m_s = 2$ (in the case $(1,2)$ just consider $f^{-1}$). The existence of the saddle $Q$ of type $(2,n_u)$ follows from Lemma 3.3. The homoclinic relations and the biaccumulation property (when $n = 1$) follow from Remark 3.6. Remark 3.7 implies the preservation of the tangency of $P$. This ends the proof of the proposition.

4 Return maps at the homoclinic tangency

Throughout this section, we consider $f \in \text{Diff}^r(M^{m+n})$, $r, m \geq 2$ and $n \geq 1$, having a homoclinic tangency $Y$ associated to a simple saddle $P$ of type $(m_s, n_u)$ and $u$-index $n$. We first state in Sect. 4.1 generic conditions (C0)–(C5) at the homoclinic tangency. Thereafter, we fix a neighbourhood of the tangency and introduce first return maps to it (Sect. 4.2). We also see how the expanding leading Jacobian of the saddle $P$ (see Eq. (1.2)) implies expansion properties of the return maps (Sect. 4.3).

4.1 Generic conditions

For simplicity, in what follows, we assume that the period of $P$ is one. We also assume that $f$ is $C^r$ linearisable at $P$.

By Sternberg linearisation theorem [52], every $C^r$ neighbourhood of $f$ contains diffeomorphisms satisfying this condition.

We consider a small neighbourhood $W$ of $P$ with these linearising coordinates

$$(u, x, y, v) \in \mathbb{R}^{m-m_s} \times \mathbb{R}^{m_s} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n-n_u}.$$  

(4.1)
In these coordinates, \( P = (0^{m-m_s}, 0^{m_s}, 0^{n_u}, 0^{n-n_u}) \) and the map \( \Xi_0 \overset{\text{def}}{=} f|_W \) is of the form
\[
(u, x, y, v) \mapsto \Xi_0(u, x, y, v) \overset{\text{def}}{=} (Au, Bx, Cy, Dv),
\] (4.2)
here \( A, B, D, C \) are square matrices whose eigenvalues (using the notation in (1.1)) are \( \lambda_m, \lambda_{m-1}, \ldots, \lambda_{m_1+1} \) (the ones of \( A \)), \( \lambda_m, \ldots, \lambda_1 \) (the ones of \( B \)), \( \gamma_1, \ldots, \gamma_n \) (the ones of \( C \)), and \( \gamma_{n+1}, \ldots, \gamma_n \) (the ones of \( D \)). In the neighbourhood \( W \) the local invariant sets are (from now on we will omit the dependence on \( W \))
\[
W^\mu_{\text{loc}}(P, f) = \{(u, x, y, v) : u = 0^{m-m_s}, x = 0^{m_s}\} \subset W,
\]
\[
W^s_{\text{loc}}(P, f) = \{(u, x, y, v) : y = 0^{n_u}, v = 0^{n-n_u}\} \subset W,
\]
\[
W^{ss}_{\text{loc}}(P, f) = \{(u, x, y, v) : x = 0^{m_s}, y = 0^{n_u}, v = 0^{n-n_u}\} \subset W.
\] (4.3)

We select points \( Y^\pm \), with \( f^\ell(Y^-) = Y^+ \) for some \( \ell > 0 \), in the orbit of the tangency point \( Y \) such that
\[
Y^- = (0^{m-m_s}, 0^{m_s}, y^-, v^-) \in W^\mu_{\text{loc}}(P, f),
\]
\[
Y^+ = (u^+, x^+, 0^{n_u}, 0^{n-n_u}) \in W^s_{\text{loc}}(P, f),
\]
\[
\{f(Y^-), \ldots, f^{\ell-1}(Y^-)\} \cap W = \emptyset.
\] (4.4)

This can be obtained by shrinking \( W \).

To state the remaining conditions \((C1)-(C5)\), recall that if \( m > m_s \) there exists the strong stable foliation \( \mathcal{F}^{ss} \) of dimension \( m - m_s \) defined on \( W^s(P, f) \). This is the unique \( f \) invariant foliation of dimension \( m - m_s \) having the strong stable manifold \( W^{ss}(P, f) \) of \( P \) (tangent to \( E^{ss}(P) \)) as a leaf. Similarly, if \( n > n_u \) there is the strong unstable foliation \( \mathcal{F}^{uu} \) of dimension \( n - n_u \) defined on \( W^u(P, f) \). In the coordinates above, the (local) invariant foliations \( \mathcal{F}^{ss} \) on \( W^s_{\text{loc}}(P, f) \) and \( \mathcal{F}^{uu} \) on \( W^u_{\text{loc}}(P, f) \) are also straightened and have, respectively, the form
\[
\{(u, x, y, v) : x = \text{const.}, y = 0^{n_u}, v = 0^{n-n_u}\},
\]
\[
\{(u, x, y, v) : u = 0^{m-m_s}, x = 0^{m_s}, y = \text{const.}\}.
\]

Let \( E^{cu}(P) \) be the \( Df \)-invariant space of dimension \( m_s + n \) corresponding to the eigenspaces associated to the eigenvalues of modulus greater or equal than the stable leading multiplier of \( P \). Following [54] and based on [32], an extended center unstable manifold \( W^{cu}(P, f) \) of \( P \) is a locally invariant manifold of dimension \( m_s + n \) tangent to \( E^{cu}(P) \) at \( P \) containing \( W^u(P, f) \). This manifold always exists, but in general, it is not uniquely defined. Similarly, we can define an extended center stable manifold \( W^{cs}(P, f) \) of \( P \) of dimension \( m + n_u \) tangent to \( E^{cu}(P) \) at \( P \) and contains \( W^u(P, f) \).

Conditions \((C1)-(C5)\) are similar to the ones in [27, 28], translating the quasi-transverse conditions in [40] to our higher-dimensional context. For that, consider the subspace
\[
E_Y \overset{\text{def}}{=} T_Y W^s(P, f) \cap T_Y W^u(P, f).
\] (4.5)

The required conditions are:

\( (C1) \) \( \dim(E_Y) = (m+n) - \dim(T_Y W^s(P, f) + T_Y W^u(P, f)) = 1, \)
\( (C2) \) \( E_Y \) is not contained in \( T_Y \mathcal{F}^{ss}(Y) \), where \( \mathcal{F}^{ss}(W) \) denote the leaf of \( \mathcal{F}^{ss} \) through \( W \),
\( (C3) \) the contact between the stable and unstable manifolds at the tangency point \( Y \) is quadratic,
\( (C4) \) \( W^{cu}(P) \) is transverse to \( \mathcal{F}^{ss}(Y^+) \) at \( Y^+ \), and
\( (C5) \) \( W^{cs}(P) \) is transverse to \( \mathcal{F}^{uu}(Y^-) \) at \( Y^- \).
Conditions (C4) and (C5) are intrinsic as they do not depend on the choice of the extended center manifolds. Moreover, when \( n = 1 \) (and hence \( n_u = 1 \)) condition (C5) is vacuous. Finally, our conditions are \( C^r \) generic.

**Remark 4.1** Consider the \( f \)-invariant compact set \( \Lambda \) formed by the orbits of the saddle \( P \) and the tangency \( Y \). Under conditions (C1)–(C4), the set \( \Lambda \) has a dominated splitting of the form

\[
T_{\Lambda} M = E^{ss} \oplus E^{cu} \quad \text{with} \quad \dim(E^{cu}) = m_s + n,
\]

(4.6)

where \( E^{ss} \) is uniformly contracting. We consider cone fields

\[
X \in U \rightarrow \mathcal{C}^{ss}(X) \quad \text{and} \quad X \in U \rightarrow \mathcal{C}^{cu}(X)
\]

(4.7)

defined on a small neighbourhood \( U \) of \( \Lambda \) such that \( E^{ss} \subset \mathcal{C}^{ss} \) and \( E^{cu} \subset \mathcal{C}^{cu} \) as in Proposition 7.3. The set \( U \) is called a partially hyperbolic neighbourhood of \( \Lambda \).

A similar comment holds when (C4) is replaced by (C5).

**Remark 4.2** The \( C^r \) linearisation hypothesis of the saddle \( P \) implies that (locally in the linearisation neighbourhood \( W \)) the foliation \( \mathcal{F}^{ss} \) of \( W^{loc}_{ss}(P, f) \) is \( C^r \). Note also that the (local) strong stable manifold \( W^{ss}_{loc}(P, f) \) is a leaf of this foliation. As the saddle \( P \) is simple of type \( (m_s, n_u) \) (which is the weakest normal hyperbolicity condition) we have that \( |\lambda_{m_s+1}|/\lambda < 1 \), where \( \lambda \) is the modulus of the leading stable multipliers of \( P \). This implies that the foliation \( \mathcal{F}^{ss} \) can be extended to a \( C^1 \) foliation \( \mathcal{F}^{ss}_{\text{loc}} \) (defined on a neighbourhood of \( P \)) whose leaves are tangent to the strong stable cone field \( \mathcal{C}^{ss} \). See for instance [54, Equation 23], which is a reinterpretation of some results in [32]. However the obtained foliation may fail to be \( f \)-invariant. To guarantee the existence of an invariant extension of the strong stable foliation one needs a bunching condition, see [30, Section 2.3b of Chapter 3]. Actually, what we need of the extended foliation is its absolutely continuity, which does not demand bunching conditions.\(^8\)

Throughout this section, we assume that the saddle \( P \) is of type \( (m_s, n_u) \) and that \( P \) and its homoclinic tangency \( Y \) satisfy conditions (C0)–(C5).

### 4.2 Return maps

We take small closed neighbourhoods \( \Pi^\pm \subset W \) of \( Y^\pm \) (called reference boxes) such that \( f((\Pi^+) \cap \Pi^+) = \emptyset = f^{-1}(\Pi^-) \cap \Pi^- \). Following [29, Corollary 1] (see also [27, 28]), under conditions (C1)–(C5), we can consider the transition map \( \Xi_1 \) from \( Y^- \) to \( Y^+ \)

\[
\Xi_1 \equiv f^\ell|_{\Pi^-} : \Pi^- \rightarrow \Pi^+, \quad \Xi_1(\bar{u}, \bar{x}, \bar{y}, \bar{v}) = (\bar{u}, \bar{x}, \bar{y}, \bar{v}).
\]

(4.8)

When \( y, \tilde{y} \in \mathbb{R} \) the Taylor expansion (in crossed form) of \( \Xi_1 \) is given by

\[
\begin{align*}
\bar{u} - u^+ &= A_1 u + B_1 x + C_1 (y - y^-) + D_1 \bar{v} + \ldots, \\
\bar{x} - x^+ &= A_2 u + B_2 x + C_2 (y - y^-) + D_2 \bar{v} + \ldots, \\
\bar{y} &= A_3 u + B_3 x + C_3 (y - y^-)^2 + D_3 \bar{v} + \ldots, \\
\bar{v} - v^- &= A_4 u + B_4 x + C_4 (y - y^-) + D_4 \bar{v} + \ldots,
\end{align*}
\]

(4.9)

where the notation (\ldots) stands for higher order terms and \( A_i, B_i, C_i, D_i \) are matrices of the appropriate dimensions. Here \( B_3, C_2 \) and \( C_3 \) are nonzero matrices. When \( y, \tilde{y} \in \mathbb{R}^2 \) similar expressions can be obtained, see [29, Corollary 1].

\(^8\) Compare with the case of Anosov diffeomorphisms, see for instance [43, Section 7.1]
To study the returns associated to $\Sigma_1$, consider for sufficiently large $k$, say $k > k_0$, the strips

$$\Pi_+^k \overset{\text{def}}{=} \Sigma_0^{-k}(\Pi^-) \cap \Pi^+, \quad \Pi_-^k \overset{\text{def}}{=} \Sigma_0^k(\Pi_+^k) \subset \Pi^-.$$  \hspace{1cm} \text{(4.10)}

The strips $\Pi_+^k$ are depicted in Fig. 4 (in the codimension one case). We observe that the strips $\Pi_+^k$ have uniform “stable size” while the strips $\Pi_-^k$ have uniform “unstable size”.

The return map at the tangency point $Y^+$ is defined in a subset $\Sigma^+ \subset \Pi^+$ by

$$\mathcal{R}: \Sigma^+ \overset{\text{def}}{=} \bigcup_{k > k_0} \Pi_+^k \to \Pi^+, \quad \mathcal{R}|_{\Pi_+^k} = \mathcal{R}_k \overset{\text{def}}{=} \Sigma_1 \circ \Sigma_0^k.$$  \hspace{1cm} \text{(4.11)}

We start analysing suitable “first return maps” defined on a neighbourhood of the homoclinic tangency $Y^+$.

### 4.3 Expanding returns along center unstable cone fields

We now obtain “expansion of volume” properties (along the leading directions) derived from the expanding leading Jacobian of the saddle $P$, see Sect. 4.3.1. Additionally, in the codimension one case, we get expansion of suitable diameters, see Sect. 4.3.2.
4.3.1 Expansion of volume

Next lemma about expansion of volume of local submanifolds tangent to the cone field $C^{cu}$ in Remark 4.1 by the return maps $R_k$ is a variation of [35, Lemma 5]. To state it recall the definition of $J_P = J_P(f)$ in (1.2) and that a submanifold $S$ is tangent to $C^{cu}$ if $T_X S \subset C^{cu}(X)$ for every $X \in S$.

**Lemma 4.3** There is a constant $L > 0$ such that for every sufficiently large $k$ and every submanifold $S \subset \Pi_k^+$ of dimension $m_s + n$ tangent to $C^{cu}$ it holds

$$\text{vol}(R_k(S)) > L J_P^k \text{vol}(S),$$

where $\text{vol}()$ stands for the $(m_s + n)$-dimensional volume.

**Proof** We use the linearising coordinates $(u, x, y, v)$ of points on the set $W$ in (4.1). For points $X$ close to $Y$ the cone $C^{cu}(X)$ is an open Grassmannian neighbourhood of the subspace $\{0^{m-s}\} \times \mathbb{R}^{m_s} \times \mathbb{R}^n$. We can assume that $C^{cu}(X)$ does not depend on the point $X$, and hence it is constant in a neighbourhood of $Y$. Thus, every $(m_s + n)$-dimensional $C^1$ manifold $S \subset \Pi_k^+$ tangent to $C^{cu}$ can be seen as the graph of a $C^1$ map in the variables $(x, y, v)$. Moreover, we have that\(^9\)

$$\text{vol}(S) \asymp \text{vol}(\pi(S)).$$

where $\pi$ is the canonical projection from $W$ on $\{ u = 0^{m-s}\}$. For simplicity, we identify $\{ u = 0^{m-s}\}$ with $\mathbb{R}^{m_s} \times \mathbb{R}^n - u$ and write $\pi(u, x, y, v) = (x, y, v)$.

Since, by construction, $C^{cu}$ is $D\mathcal{R}_k$-invariant, we have that $\mathcal{R}_k(S)$ is also tangent to $C^{cu}$. Thus, we also have that

$$\text{vol}(\mathcal{R}_k(S)) \asymp \text{vol}(\pi(\mathcal{R}_k(S))).$$

Thus, to prove the lemma it is enough to get a constant $\tilde{K} > 0$ (independent of large $k$) such that

$$\text{vol}(\pi(\mathcal{R}_k(S))) \geq \tilde{K} J_P^k \text{vol}(\pi(S)). \quad (4.12)$$

As $\pi|_S : S \to \pi(S)$ is a $C^1$ diffeomorphism, the map

$$\tilde{\mathcal{R}}_k : \pi(S) \to \pi(\mathcal{R}_k(S)), \quad \tilde{\mathcal{R}}_k \overset{\text{def}}{=}= \pi \circ \mathcal{R}_k \circ (\pi|_S)^{-1}$$

is a $C^1$ diffeomorphism satisfying

$$\pi \circ \mathcal{R}_k = \tilde{\mathcal{R}}_k \circ \pi \quad \text{on} \quad S. \quad (4.13)$$

Thus, we have that

$$\text{vol}(\pi(\mathcal{R}_k(S))) = \text{vol}(\tilde{\mathcal{R}}_k(\pi(S))) = \int_{\pi(S)} \left| \det \frac{\partial \tilde{\mathcal{R}}_k}{\partial (x, y, v)} \right| dx \, dy \, dv.$$ 

To obtain the inequality (4.12) it is enough to get a constant $\tilde{K} > 0$ (independent of $k$) such that

$$\left| \det \frac{\partial \tilde{\mathcal{R}}_k}{\partial (x, y, v)} \right|_{\pi(S)} \geq \tilde{K} J_P^k. \quad (4.14)$$

\(^9\) The notation $f \asymp g$ means that there are constants $C, D > 0$ such that $C |g| < |f| < D |g|$. 

\[ Springer \]
To get the inequality in (4.14), recall that \( \mathcal{R}_k = \Sigma_1 \circ \Sigma_k \), where
\[
(u, x, y, v) \in \Pi_k^+ \mapsto \Sigma_k^0(u, x, y, v) = (u_k, x_k, y_k, v_k) \in \Pi^-,
\]
\[
(u_k, x_k, y_k, v_k) \in \Pi^- \mapsto \Sigma_1(u_k, x_k, y_k, v_k) = (\tilde{u}_k, \tilde{x}_k, \tilde{y}_k, \tilde{v}_k) \in \Pi^+.
\]
are defined by the Eqs. (4.2) and (4.8). In view of Eq. (4.13), we have that \( \widetilde{\mathcal{R}}_k = (\pi \Sigma_1 \pi^{-1}) \circ (\pi \Sigma_k \pi^{-1}) \) (here the maps \( \pi^{-1} \) are local inverses in the corresponding submanifolds) and thus
\[
\left. \frac{\partial \widetilde{\mathcal{R}}_k}{\partial (x, y, v)} \right|_{\pi(S)} = \left. \left( \frac{\partial (\tilde{x}_k, \tilde{y}_k, \tilde{v}_k)}{\partial (x_k, y_k, v_k)} \right)_{\pi \Sigma_0^k(S)} \left. \frac{\partial (x_k, y_k, v_k)}{\partial (x, y, v)} \right|_{\pi(S)} \right). \tag{4.15}
\]

Since, see (4.2),
\[
x_k = B^k x, \quad y_k = C^k y, \quad \text{and} \quad v_k = D^k v,
\]
where \( B, C, \) and \( D \) are square matrices, whose eigenvalues in absolute value are all equal to \( \lambda, \gamma, \) and strictly greater than \( \gamma, \) respectively. Thus, as
\[
| \det D | > \gamma^{n-n_u} > 1
\]
it follows
\[
\left| \left. \frac{\partial (x_k, y_k, v_k)}{\partial (x, y, v)} \right|_{\{u=0^{n-m_s}\}} \right| = J^k_\beta | \det D^k | \geq J^k_\beta. \tag{4.16}
\]

**Claim 4.4** There is \( \widetilde{K} > 0 \) such that for every \( k \) large enough it holds
\[
\left| \left. \left( \frac{\partial (\tilde{x}_k, \tilde{y}_k, \tilde{v}_k)}{\partial (x_k, y_k, v_k)} \right)_{\pi \Sigma_0^k(S)} \right| \right|_\pi(S) > \widetilde{K}.
\]

**Proof** First, observe that \( \det D \Sigma_1|_{\{u=0^{n-m_s}\}} (Y^-) \neq 0. \) Otherwise, the tangent space to \( W^{cu}(P) \) at \( Y^+ \) (which is the image of \( \{u = 0^{n-m_s}\} \) by \( D \Sigma_1 \)) cannot be transverse to \( F^{ss}(Y^+) \) at \( Y^+ \), contradicting assumption (C4). In particular, \( \det D \Sigma_1|_{\{u=0^{n-m_s}\}} \neq 0 \) in a neighbourhood \( V \) of the tangency point \( Y^- \). Note that \( S \subset \Pi_k^+ \) and \( \Sigma_0^k(\Pi_k^+) \subset \Pi^- \) converges to \( \{u = 0^{n-m_s}\} \) as \( k \to \infty \). Taking, if necessary, \( \Pi^- \) small and \( k \) large, we can guarantee that \( \Sigma_0^k(\Pi_k^+) \subset V \). Then
\[
\left| \left. \left( \frac{\partial (\tilde{x}_k, \tilde{y}_k, \tilde{v}_k)}{\partial (x_k, y_k, v_k)} \right)_{\pi \Sigma_0^k(S)} \right| \right|_\pi(S) \neq 0.
\]

This concludes the proof.

Equation (4.14) (and the lemma) follows from (4.15), (4.16), and Claim 4.4.

### 4.3.2 Expansion of diameters

We now restrict our attention to the case when the saddle \( P \) of \( f \) has \( u \)-index one (in particular, \( n = n_u = 1 \)). Note that in this case, the variables \( v, \tilde{v} \) in Eq. (4.1) do not appear.

Using the linearising coordinates in (4.1), consider the natural projections
\[
\pi : W \to \mathbb{R}^{m_s+1}, \quad \pi_c : \mathbb{R}^{m_s+1} \to \mathbb{R}^{m_s}, \quad \pi_u : \mathbb{R}^{m_s+1} \to \mathbb{R}. \tag{4.17}
\]
Given a subset \( A \subset (\{0^{n-m_s}\} \times \mathbb{R}^{m_s+1}) \cap W \) we define
\[
diam_i(A) \overset{\text{def}}{=} \text{diameter}(\pi_i(\pi(A))) \quad i = c, u. \tag{4.18}
\]
Lemma 4.5 There is a constant $K > 0$, depending only on the transition $\Sigma_1$, such that for every sufficiently large $k$ and every local submanifold $S \subset \Pi_k^+$ of dimension $m_s + n$ tangent to $C^{cu}$ it holds
\[
\diam_c(\mathcal{R}_k(S)) < K (\gamma^k \diam_u(S) + \lambda^k \diam_c(S)),
\]
where $\gamma > 1 > \lambda$ are the unstable and stable leading eigenvalues of $P$, respectively.

Proof Consider any $S$ as in the lemma and a point $X = (u, x, y) \in S \subset \Pi_k^+$. Let
\[
X_k = (u_k, x_k, y_k) \overset{\text{def}}{=} \Sigma_0^k(X) \in \Pi^- \quad \tilde{X}_k = \Sigma_1^k(X) = \mathcal{R}_k(X) \in \Pi^+.
\]
From (4.2), the coordinates $u_k, x_k$ are at most of order of $\lambda^k$, where $\lambda < 1$ is the leading stable eigenvalue of $P$ in (1.1). Thus, by the formula of $\Sigma_1$ in (4.9) (where now the coordinate $v$ does not appear) the $x$-coordinate of $\tilde{X}_k$ is of the form
\[
x^+ + A_2 u_k + B_2 x_k + C_2 (y_k - y^-) + \ldots, \tag{4.19}
\]
where $x^+$ is the $x$-coordinate of $Y^+$, $y^-$ is the $y$-coordinate of $Y^-$, see Eq. (4.4). Note that Eq. (4.19) implies that there is a constant $K > 0$, independent of $S$ and large $k$, such that
\[
\diam_c(\mathcal{R}_k(S)) < K \diam_u(\Sigma_0^k(S)) + K \lambda^k \diam_c(S)
\]
\[
\leq (\gamma^k \diam_u(S) + \lambda^k \diam_c(S)), \tag{4.20}
\]
where the last equality follows from the linearising hypothesis. This completes the proof of the lemma.

Remark 4.6 (Uniform estimates) The constants $K$ and $L$ in Lemmas 4.3 and 4.5 can be chosen uniform for perturbations. To see why this is so just note that the estimates in the proofs of these lemmas only depend on the center unstable cone field (which persists by perturbations).

4.4 Return maps at the homoclinic tangency: the codimension one case

We now refine the constructions in Sect. 4.2 in the codimension one case, when the saddle $P$ has $u$-index one and hence $n = n_u = 1$. Recall the choices of the tangency points $Y^-$ and $\Sigma_1(Y^-) = Y^+$ in (4.4), the reference boxes $\Pi^\pm$, and the strips $\Pi_k^\pm$ in (4.10).

Assuming that the saddle $P$ is biaccumulated by its transverse homoclinic points (recall the definition in the preamble of Sect. 3) we will “resize” the sets $\Pi_k^+$ to get additional dynamical control. We now go to the details. Our construction is inspired by [29, Section 1.4] (see also previous constructions in [39, 42]).

Lemma 4.7 Consider $f \in \text{Diff}^r(M^{n+1})$ with a saddle $P$ of $u$-index one that is biaccumulated by its transverse homoclinic points and has a homoclinic tangency. Assume that conditions (C0)–(C4) hold. Then the reference box $\Pi^-$ can be chosen such that there are sequences of points $(N_j^-)$ and $(N_j^+)$ and of $m$-dimensional submanifolds $(\Theta_j)$ with
\[
N_j^\pm \in W^s(P, f) \cap W^u_{\text{loc}}(P, f), \quad N_j^\pm \in \Theta_j \subset W^s(P, f) \cap \Pi^-,
\]
such that:

- the open “interval” $(N_j^-, N_j^+) \subset W^u_{\text{loc}}(P, f)$ contains $Y^-$ and satisfies
\[
\delta_j \overset{\text{def}}{=} \diam([N_j^-, N_j^+]) \to 0 \quad \text{as} \quad j \to \infty,
\]
• each submanifold $\Theta_j$ splits the set $\Pi^-$ into two connected components such that the component $C_j$ of $(\Pi^- \setminus \Theta_j)$ containing $Y^-$ satisfies $C_{j+1} \subset C_j$.

**Proof** The codimension one hypothesis implies that $W^s_{\text{loc}}(P, f)$ splits the box $\Pi^+$ into two connected components. So the same holds for $\Pi^- = \Sigma_1^{-1}(W^s_{\text{loc}}(P, f))$. By the biaccumulation property, both components contain disks of $W^s(P, f)$ accumulating to $\Sigma_1^{-1}(W^s_{\text{loc}}(P, f))$. The disks in one of these components also have two transverse intersections with $W^u_{\text{loc}}(P, f)$ accumulating to $Y^-$. This provides the sequences of submanifolds $\Theta_j$ and of transverse homoclinic points $N_j^\pm$.

Fix large $j$ and recall the definition of $\Pi^\pm_k$ in (4.10). For every $k$ large enough, the set $\Pi_k^- \setminus \Theta_j$ has three connected components. Denote by $\Pi^-_{j,k}$ the closure of the connected component intersecting $C_j$ and let

$$\Pi^+_{j,k} \overset{\text{def}}{=} \Sigma_0^{-k}(\Pi^-_{j,k}) \cap \Pi^+ \subset \Pi^+_k.$$  

(4.21)

Define the s- and u-boundaries of $\Pi^+_{j,k}$ by

$$\partial^s \Pi^+_{j,k} \overset{\text{def}}{=} \Sigma_0^{-k}(\Theta_j \cap \Pi^-_k) \cap \Pi^+ \quad \partial^u \Pi^+_{j,k} \overset{\text{def}}{=} \partial \Pi^+_j \setminus \partial^s \Pi^+_j.$$  

(4.22)

Observe that, by construction, $\partial^s \Pi^+_{j,k} \subset W^s(P, f)$.

Recall the definitions of $\gamma > 1$ in (1.1) and of the c- and u-diameters in (4.18). Next claim asserts that c-diameters of $\Pi^+_k$ are “independent” of $k$ while u-diameters exponentially shrink. The proof is straightforward and hence omitted.

**Claim 4.8** (c- and u-diameters of $\Pi^+_{j,k}$) There is $\rho > 0$ such that for every sufficiently large $j$ there is $k_0(j)$ such that for every $k \geq k_0(j)$ it holds

$$\text{diam}_c(\Pi^+_{j,k}) > \rho \quad \text{and} \quad \text{diam}_u(\Pi^+_{j,k}) < \gamma^{-k} \delta_j.$$  

**Remark 4.9** (Choice of quantifiers) Recall the constants $K, \delta_j,$ and $\rho$ in Lemmas 4.5 and 4.7 and Claim 4.8, respectively. There is large $j_\ast \geq 1$ such that

$$K \delta_{j_\ast} < \frac{\rho}{10}.$$  

As in what follows $j_\ast$ remains fixed, for notational simplicity, we write

$$\tilde{\Pi}^\pm_k \overset{\text{def}}{=} \Pi^\pm_{j_\ast,k}, \quad \text{where} \quad k > k_0(j_\ast).$$  

(4.23)

By construction, the homoclinic tangency point $Y^-$ belongs to the interior of $C_{j_\ast}$. Note that if $(Z^+_k)$ is a sequence of points with $Z^+_k \in \Pi^+_k$ with $Z^+_k \rightarrow Y^-$ then $Z^-_k \overset{\text{def}}{=} \Sigma_0^{-1}(Z^+_k) \in \Pi^-_k$ and $Z^-_k \rightarrow Y^-$. Therefore, $Z^-_k \in \tilde{\Pi}^-_k$ and $Z^+_k \in \tilde{\Pi}^+_k$ for every sufficiently large $k$. See Fig. 4.

5 Unfolding the homoclinic tangency

Throughout this section, we consider $f \in \text{Diff}^r(M^{m+n}), r, m \geq 2$ and $n \geq 1$, with a simple saddle $P$ of type $(m, n_0)$ and $u$-index $n$ having a homoclinic tangency $Y$ satisfying (C0)–(C5). We will study how the unfolding of this tangency generates saddles with $u$-indices different from the one of $P$. 

\[\copyright\] Springer
5.1 Unfolding families and return maps

We embed the diffeomorphism $f$ into a family $(f_t)$, with $f_0 = f$, unfolding the homoclinic tangency. For that, we borrow the construction from [29, Section 1.2]. The number of parameters of the family $(f_t)$ depends on the type of the saddle $P$ (which is fixed for every $f_t$). We use the notation in Sect. 4.2. Note that transition map $\mathcal{T}_{1,t} \overset{\text{def}}{=} f_t^\ell$, see (4.9), is defined for every small $t$. Recall that $\lambda$ and $\gamma$ defined as in (1.1) are the modulus of the contracting and expanding leading multipliers of $P$.

Using the effective dimension, denoted by $d_e$, introduced in [54], there are three classes of simples saddles with leading Jacobian greater than one:

- $P$ has $d_e = 1$ if it is of type $(1,1)$ or $(1,2)$ and $\lambda \gamma > 1$;
- $P$ has $d_e = 2$ if it is of type $(1,2)$ with $\lambda \gamma < 1$ or of type $(2,1)$ or $(2,2)$ with $\lambda^2 \gamma > 1$;
- $P$ has $d_e = 3$ if it is of type $(2,2)$ and $\lambda^2 \gamma < 1$.

The effective dimension is the number of parameters considered in the unfolding of the tangency. Let us roughly explain the choice of these parameters. When $d_e = 3$ the parameter $t = (\alpha, \beta, \ell) \in [-\epsilon, \epsilon]^3$ (in cases $d_e = 2$ and $d_e = 1$ we consider $t = (\alpha, \beta) \in [-\epsilon, \epsilon]^2$ and $t = t \in [-\epsilon, \epsilon]$, respectively) is defined as follows. Let $\omega_t^u \overset{\text{def}}{=} W^u_{\text{loc}}(P, f_t) \cap \Pi^-$, then the distance between the folding point of $\mathcal{T}_{1,t}(\omega_t^u)$ and $W^s_{\text{loc}}(P, f_t)$ is $|t|$. To describe $\alpha$ note that in this case the leading multipliers of $P$ for $f$ are nonreal and equal to $\lambda e^{\pm i \theta}$ and $\gamma e^{\pm i \rho}$. The leading multipliers of $P$ for $f_t$ are $\lambda e^{\pm i(\theta + \alpha)}$ and $\gamma e^{\pm i(\rho + \beta)}$. We let $\mathcal{T}_0 = f_t|_W$ (here $W$ is the linearising neighbourhood in Sect. 4.1) and note that $\mathcal{T}_0$ is a “double rotation” of $\mathcal{T}_0$ by angles $\alpha$ and $\beta$.

Note that in the previous construction $P_{f_t} = P$ and that the leading Jacobian of $P$ is not modified:

$$J_P(f_t) = J_P(f).$$

(5.1)

Recall the construction of the return maps $\mathcal{R}_k = \mathcal{T}_1 \circ \mathcal{T}_0^k$ in Sect. 4.2. We now consider return maps $\mathcal{R}_k = \mathcal{T}_1 \circ \mathcal{T}_0^k$ similarly defined. For large $k > 0$, as in (4.10), we consider the strips

$$\Pi_{k,t}^+ \overset{\text{def}}{=} \mathcal{T}_{0,t}^{-k}(\Pi^-) \cap \Pi^+.$$

We now consider, for $k_0$ large enough, the corresponding return maps

$$\mathcal{R}_t : \Sigma_t^+ \overset{\text{def}}{=} \bigcup_{k \geq k_0} \Pi_{k,t}^+ \rightarrow \Pi^+, \quad \mathcal{R}_t|_{\Pi_{k,t}^+} \overset{\text{def}}{=} \mathcal{T}_1 \circ \mathcal{T}_0^k.$$  

(5.2)

A saddle of $f_t$ is said of single round type if it is a fixed point of $\mathcal{R}_t$.

Remark 5.1 (The codimension one case) Recall the definition of $\tilde{\Pi}_k^+ = \Pi_{j*,k}^+$ in (4.21) and (4.23). Observe that the $s$-boundary of $\tilde{\Pi}_k^+$ is not necessarily contained in the stable manifold of $P$ for $f_t$ when $t \neq 0$. However, the stable manifold of $P$ on $\Pi^+$ varies smoothly with $t$, for $|t|$ small enough. Then we can define continuations of $\tilde{\Pi}_{k,t}^+ \subset \Pi_{k,t}^+$ for small $|t|$ in such a way the $s$-boundaries are still contained $W^s(P, f_t)$ and the estimates for the $c$- and $u$-diameters in (4.8) hold.
5.2 Index variation

Our next goal is to prove the following proposition, which is essentially borrowed from [29, 49]. For the next proposition see the cone fields $C^u$ in Proposition 7.3 and recall the definition of the leading Jacobian in (1.2).

**Proposition 5.2** Consider $f \in \text{Diff}^r(M^{m+n})$, $r$, $m \geq 2$ and $n \geq 1$, with a simple saddle $P$ of $u$-index $n$ such that $J_P = J_P(f) > 1$. Assume that $P$ has a homoclinic tangency satisfying conditions (C0)–(C5).

Let $(f_t)$ be the unfolding family associated to $f$ in Sect. 5.1. Then there is a sequence of open sets of parameters $\Delta_k \to 0$ such that for every $t \in \Delta_k$ there are a contracting locally normally hyperbolic $C^r$ submanifold $M_{k,t}$ of dimension $n + d_e$ tangent to the cone field $C^u$ satisfying:

(i) If $d_e = 1$: there is a saddle $R_{k,t} \in M_{k,t}$ of $u$-index $n + 1$.
(ii) If $d_e = 2$:

- when $P$ is of type $(1, 2)$ and $\lambda_1 > 1$ there are saddles $R_{k,t}$, $S_{k,t} \in M_{k,t}$ of $u$-indices $n - 1$ and $n + 1$, respectively,
- when $P$ is of type $(2, 1)$ or $(2, 2)$ and $\lambda_2 > 1$ there are saddles $R_{k,t}$, $S_{k,t} \in M_{k,t}$ of $u$-indices $n - 1$ and $n + 2$, respectively.
(iii) If $d_e = 3$: there are saddles $Q_{k,t}$, $R_{k,t}$, $S_{k,t} \in M_{k,t}$ of $u$-indices $n - 1$, $n + 1$ and $n + 2$, respectively.

Moreover, these saddles are of single round type and have local unstable manifolds contained in $M_{k,t}$.

**Remark 5.3** By construction, each submanifold $M_{k,t}$ is contained in the set $\Pi_{k,t}^+$ in (4.10) for $k$ large enough and $t \in \Delta_k$. Also, according to the different cases, the saddles $Q_{k,t}$, $R_{k,t}$, $S_{k,t}$, $t \in \Delta_k$, belong to $\Pi_{k,t}^+$ and converge to the tangency point as $k \to \infty$.

In the codimension one case, by Remark 4.9, we have that $Q_{k,t}$, $R_{k,t}$, $S_{k,t} \in \tilde{\Pi}_{k,t}^+$. Therefore, by shrinking the submanifolds we can assume that $M_{k,t} \subset \tilde{\Pi}_{k,t}^+$ for every large $k$ and $t \in \Delta_k$.

**Proof** By [29, Theorem 3, Equation (1.2) in page 932 and Lemma 2] applied to the unfolding family $(f_t^{-1})$ (see also [49, Theorems A and C]), for sufficiently large $k$, there is a set $\Delta_k$ of parameters $t$, with $\Delta_k \to 0$ as $k \to \infty$, such that the return maps $R_{k,t}$ have a saddle $R_{k,t}$ of $u$-index $n + 1$. Besides, there are also saddles $S_{k,t}$ of $u$-index $n + 2$ (in cases (ii) and (iii)) and $Q_{k,t}$ of $u$-index $n - 1$ (in case (iii)). These saddles are of single round type.

Moreover, for the parameters $t \in \Delta_k$, there is also a contracting normally hyperbolic $C^r$ manifold $M_{k,t} \subset \Pi_{k,t}^+$ (resp. $\tilde{\Pi}_{k,t}^+$) which is locally $f_t$ invariant, tangent to the cone field $C^u$, and contains local unstable manifolds of $R_{k,t}$, $S_{k,t}$, $Q_{k,t}$ according to the cases. The dimension of $M_{k,t}$ is $n + d_e$. This completes the proof.

In the codimension one case we can obtain additional conclusions, compare with [35, Lemmas 11 and 12].

**Lemma 5.4** Under the assumption of Proposition 5.2, if the saddle $P$ has $u$-index one and it is biaccumulated by its transverse homoclinic points, then for every large $k$ and every $t \in \Delta_k$ it holds:

- $W^u(R_{k,t}, f_t) \cap \partial^s \tilde{\Pi}_{k,t}^+ \neq \emptyset$, in cases (i) and (ii) of Proposition 5.2.
\begin{itemize}
  \item \( W^u(S_{k,t}, f_t) \cap \overline{s^u \tilde{\Pi}_{k,t}^+} \neq \emptyset \), in case (ii) of Proposition 5.2.
\end{itemize}

In particular, as \( \partial^s \tilde{\Pi}_{k,t}^+ \subset W^s(P, f_t) \), it holds

\[
W^u(R_{k,t}, f_t) \cap W^s(P, f_t) \neq \emptyset \quad \text{and} \quad W^u(S_{k,t}, f_t) \cap W^s(P, f_t) \neq \emptyset.
\]

**Proof** Note that \( n = n_u = 1 \). Fix large \( k \) and \( t \in \Delta_k \). Let \( S \) be an \((m_s + 1)\)-dimensional submanifold contained in \( \tilde{\Pi}_{k,t}^+ \) tangent to the cone field \( C^{cu} \). This manifold is transverse to the foliation \( \tilde{F}^{ss} \) in Remark 4.2 and the angle between them is uniformly bounded. Recall the linearising neighbourhood \( W \) of \( f \) and the linearising coordinates \((u, x, y)\), see (4.1). Consider the projection \( \pi^{ss} : W \to [u = 0] \subset W \) along the leaves of \( \tilde{F}^{ss} \). Since the the foliation \( \tilde{F}^{ss} \) is \( C^1 \) then it is absolutely continuous. Thus there are constants \( c_1, c_2 > 0 \) (independent of the submanifold \( S \)) such that

\[
c_1 \text{vol}(S) < \text{vol}(\pi^{ss}(S)) < c_2 \text{vol}(S)
\]

(5.3)

where \( \text{vol}(\cdot) \) denotes \((m_s + 1)\)-volume.

Recall that the unfolding family \((f_t)\) preserves the leading Jacobian of \( P \). Using Lemma 4.3 and Remark 4.6, we get a constant \( L > 0 \) such that for every large \( k \geq 1 \) it holds

\[
\text{vol}(\mathcal{R}_{k,t}(S)) > L J_k^k \text{vol}(S).
\]

(5.4)

Thus, from (5.3) and (5.4), it hold that for every \( k \) large enough

\[
\text{vol}(\pi^{ss}(\mathcal{R}_{k,t}(S))) > c_1 c_2^{-1} L J_k^k \text{vol}(\pi^{ss}(S)) = \varrho_k \text{vol}(\pi^{ss}(S))
\]

(5.5)

where \( \varrho_k \overset{\text{def}}{=} c_1 c_2^{-1} L J_k^k > 1 \). As \( J_k > 1 \) we get that \( \varrho_k > 1 \) for every large \( k \). Arguing inductively, if \( \mathcal{R}_{k,t}^i(S) \subset \tilde{\Pi}_{k,t}^+ \) for every \( i = 0, \ldots, n - 1 \) we get

\[
\text{vol}(\pi^{ss}(\mathcal{R}_{k,t}^i(S))) > \varrho_k^i \text{vol}(\pi^{ss}(S)).
\]

(5.6)

We now focus on the saddles \( R_{k,t} \). We have the following facts about the sets \( \tilde{\Pi}_{k,t}^+ \). First recall Claim 4.8 about the “unstable size” of the \( \tilde{\Pi}_{k,t}^+ \) and the definitions of \( \partial^u \tilde{\Pi}_{k,t}^+ \) and \( \text{diam}_c \), in (4.22) and (4.18), respectively.

**Remark 5.5** Let \( \rho > 0 \) be the constant in Claim 4.8. Adjusting the “unstable sides” of \( \tilde{\Pi}_{k,t}^+ \), we can assume that the saddle \( R_{k,t} \) is \((\rho/10)\)-centered for every \( k \) sufficiently large: for every submanifold \( S \) of dimension \( m_s + 1 \) tangent to \( C^{cu} \), containing \( R_{k,t} \), and intersecting \( \partial^u \tilde{\Pi}_{k,t}^+ \) it holds

\[
\text{diam}_c(S) > \frac{\rho}{10}.
\]

The lemma (for \( R_{k,t} \)) follows immediately from the next claim:

**Claim 5.6** Consider a small disk \( W^u \subset W^u(R_{k,t}, f_t) \cap \mathcal{M}_{k,t} \) of dimension \((m_s + 1)\) centered at \( R_{k,t} \). Then there is \( m_0 \) such that \( \mathcal{R}_{k,t}^{m_0}(W^u) \cap \partial^s \tilde{\Pi}_{k,t}^+ \neq \emptyset \).

**Proof** By Eq. (5.6), there is a first \( m_0 \) such that \( \mathcal{R}_{k,t}^{m_0+1}(W^u) \cap \partial^s \tilde{\Pi}_{k,t}^+ \neq \emptyset \). If this intersection occurs in \( \partial^s \tilde{\Pi}_{k,t}^+ \) taking \( m_0 = n_0 + 1 \) we are done. Otherwise, the intersection occurs in \( \partial^u \tilde{\Pi}_{k,t}^+ \). We see that this gives a contradiction. Define

\[
W_0 \overset{\text{def}}{=} C(R_{k,t}, \mathcal{R}_{k,t}^{m_0}(W^u) \cap \tilde{\Pi}_{k,t}^+), \quad W_1 \overset{\text{def}}{=} \mathcal{R}_{k,t}(W_0),
\]
where \( C(A, \Upsilon) \) denotes the connected component of the set \( \Upsilon \) containing the point \( A \in \Upsilon \). Since \( W_1 \) intersects \( \partial^u \Pi_{k,t}^+ \), \( R_{k,t} \in W_1 \) (recall that the saddle is single round), and \( R_{k,t} \) is \( (\rho/10) \)-centered, Remark 5.5 implies that
\[
\text{diam}_c(W_1) > \frac{\rho}{10}.
\]
On the other hand, since \( W_0 \) is contained in \( \tilde{\Pi}_{k,t}^+ \), by item (b) in Claim 4.8 and recalling (4.23), it follows
\[
\text{diam}_u(W_0) < \gamma^{-k} \delta_{j_\ast}/2.
\]
Moreover, we can take \( k \) large enough with
\[
\lambda^k \text{diam}_c(W_0) < \delta_{j_\ast}/2.
\]
Using Lemma 4.5 and Remark 4.6, the previous inequalities, and the choice of quantifiers in Remark 4.9 we get
\[
\frac{\rho}{10} < \text{diam}_c(W_1) = \text{diam}_c(\mathfrak{N}_{k,t}(W_0)) < K(\gamma^k \text{diam}_u(W_0) + \lambda^k \text{diam}_c(W_0)) < K \delta_{j_\ast} < \frac{\rho}{10}.
\]
This contradiction implies the claim.

The proof of Lemma 5.4 for the saddle \( R_{k,t} \) is now complete. The proof of the lemma for the saddle \( S_{k,t} \) follows similarly and will be omitted.

### 6 Proof of Theorems A and B

Consider \( f \in \text{Diff}^r(M^{m+n}) \) with a homoclinic tangency associated to a saddle \( P \) of \( u \)-index \( n \) and a blender \( \Gamma \) as in Theorem A or B. This means that
- \( P \) is simple of type \((m_s, n_u)\), with \( m_s, n_u \in \{1, 2\} \),
- \( J_P(f) > 1 \), and
- \( P \) is homoclinically related to a blender \( \Gamma \).

In Theorem A we have that \( n = n_u = 1 \) and that \( \Gamma \) is a \( cs \)-blender of central dimension \( m_s \). In Theorem B we have that \( \Gamma \) is a double blender of central dimensions \((m_s, n_u)\).

Recalling Terminology 1.8, from now on we use the following nomenclature:

**Termination 6.1** (\( C^r \) perturbation) A \( C^r \) perturbation of \( f \) is a \( C^\infty \) diffeomorphism \( g \) that can be obtained arbitrarily \( C^r \) close to \( f \).

#### 6.1 Proof of Theorem A

The main step to prove Theorem A is the following proposition which summarises the previous constructions.

**Proposition 6.2** Consider \( f \) with a saddle \( P \) and a \( cs \)-blender \( \Gamma \) as in Theorem A. There is a \( C^r \) perturbation \( g \) of \( f \), \( r \geq 1 \), having a saddle \( Q \) such that

1. \( Q \) is simple of type \((m_s, 1)\) and \( J_Q(g) > 1 \),
(b) $Q$ is homoclinically related to the blender $\Gamma_g$ and to $P_g$.
(c) $Q$ is biaccumulated by its transverse homoclinic points,
(d) $Q$ has a homoclinic tangency in the superposition domain of $\Gamma_g$, and
(e) the superposition region of the blender $\Gamma_g$ contains a local strong stable manifold $W^{ss}_{loc}(Q, g)$.

Proof The proof of the proposition follows from Lemma 3.3 taking $P$ and $P’ = Q^*$ a distinctive point of the blender $\Gamma$ (which are homoclinically related to $P$) and a small neighbourhood $V$ of $Q^*$ contained in the superposition domain of $\Gamma$. Let $g$ and $Q$ be the diffeomorphism and the saddle provided by Lemma 3.3. Note that the saddle $Q$ satisfies (a)–(b) and its orbit intersects $V$ (thus, after replacing by some iterate, we can assume that $Q \in V$). Since we have $n = 1$, Remark 3.6 implies the biaccumulation property in (c). Moreover, by Remark 3.7, the saddle $Q$ can be taken with a homoclinic tangency, obtaining (d). Finally, item (e) also holds provided $V$ is sufficiently small (so that $W^{ss}_{loc}(Q, g)$ and $W^{ss}_{loc}(Q^*_g, g)$ are close enough) and the fact that the disks of the superposition domain form an open set. This completes the proof of the proposition.

We are now ready to conclude the other proof of Theorem A. Without lost of generality, after a $C^r$ perturbation, we can assume that the diffeomorphism $g$, the saddle $Q$, and its homoclinic tangency provided by Proposition 6.2 satisfy conditions (C0)–(C4). Using Proposition 5.2, we embed $g$ into an unfolding family $(g_1)$ (with one parameter if $m_s = 1$ and two parameters if $m_s = 2$), getting open sets of parameters $\Delta_k \rightarrow \emptyset$, submanifolds $\mathcal{M}^k_1$, and saddles $R^k_1$ of $u$-index two and $S^k_1$ (if $m_s = 2$) of $u$-index three, $t \in \Delta_k$, as in Proposition 5.2 and Remark 5.3. Fix $k$ large enough (so Lemma 5.4 is satisfied) and $t \in \Delta_k$.

We will show that there are $C^r$ robust heterodimensional cycles between each saddle $R^k_1$ and $S^k_1$ and the blender $\Gamma_t = \Gamma_c$, see Lemmas 6.3 and 6.4 below.

**Lemma 6.3** (Robust transverse intersections) Let $t \in \Delta_k$. The unstable manifolds $W^u(R^k_1, g_t)$ and $W^u(S^k_1, g_t)$ (if $m_s = 2$) transversely intersect the stable set $W^s(\Gamma_t, g_t)$. Therefore these intersections are $C^r$ robust.

**Proof** As the $u$-index of $Q$ is one ($n = n_u = 1$) it is either of type $(1, 1)$ or $(2, 1)$. First, if $Q$ is of type $(1, 1)$ then, by Lemma 5.4, for every $t \in \Delta_k$ it holds

$$W^u(R^k_1, g_t) \cap W^s(Q_t, g_t) \neq \emptyset.$$  

Second, if $Q$ is of type $(2, 1)$, again by Lemma 5.4, then for every $t \in \Delta_k$ it holds

$$W^u(R^k_1, g_t) \cap W^s(Q_t, g_t) \neq \emptyset \quad \text{and} \quad W^u(S^k_1, g_t) \cap W^s(Q_t, g_t) \neq \emptyset.$$  

In both cases, as $Q_t$ and the blender $\Gamma_t$ are homoclinically related, we get the corresponding intersections between the unstable manifolds of the saddles $R^k_1$ and $S^k_1$ and the stable set of $\Gamma_t$.

**Lemma 6.4** (Robust quasi-transverse intersections) Let $t \in \Delta_k$. There is a $C^r$ neighbourhood $\mathcal{U}$ of $g_t$ such that for every $h \in \mathcal{U}$ the manifolds and $W^{ss}(R^k_h, h)$ and $W^s(S^k_h, h)$ (if $m_s = 2$) contain disks in the superposition region of the blender $\Gamma_h$. As a consequence, for every $h \in \mathcal{U}$ it holds

$$W^s(S^k_h, h) \cap W^u_{loc}(\Gamma_h, h) \neq \emptyset \quad \text{and} \quad W^{ss}(R^k_h, h) \cap W^u_{loc}(\Gamma_h, h) \neq \emptyset.$$  

Springer
**Proof** By Remark 4.1, the set $\Lambda_g$ consisting of the orbits of $Q$ and the tangency point $Y$ is partially hyperbolic with a splitting of the form $E^{ss} \oplus E^{ca}$, where $\dim E^{ss} = m - m_s$. Remark 4.2 provides the foliation $\tilde{F}^{ss}_{loc}(Q)$ defined on a neighbourhood of $\Lambda_g$ and tangent to the cone field $C^{ss}$. The leaves of this foliation have uniform size (depending only on the neighbourhood of $g$) and restricted to the set $\Lambda_g$ is invariant. This, in particular, implies that $\tilde{F}^{ss}_{loc}(Q)$ is a local strong stable manifold $W^{ss}_{loc}(Q, g)$ of $Q$. Note that considering some backward iteration of this leaf we get a sufficiently large local manifold. Therefore, by item (e) of Proposition 6.2, the set $W^{ss}_{loc}(Q, g)$ is a disk in the region of superposition $\mathcal{G}^{ss}$ of the blender $\Gamma_g$.

Note that for $k$ large enough the orbit of $R^k \Gamma$ can be chosen with iterates arbitrarily close to the orbit of $Y$, in particular, close to $Q$. Thus, after replacing by some iterate, we can assume that $R^k \Gamma$ is close enough to $Q$ so that its local strong stable manifold $W^{ss}_{loc}(R^k \Gamma, g)$ is close to $W^{ss}_{loc}(Q, g)$. Since the superposition domain $B^{ss}$ of the blender $\Gamma_g$ is an open set and the (local) strong stable manifold of $R^k \Gamma$ varies $C^1$ continuously in a neighbourhood of $g$, we have that (provided that $\mathcal{U}$ is small) $W^{ss}_{loc}(R^k \Gamma, h)$ is also a disk in $\mathcal{G}^{ss}$. By item (1) in Remark 2.3 about continuations of blenders, the superposition domain $B^{ss}$ is also a superposition domain of the continuation $\Gamma_h$ of $\Gamma_g$. The robust intersection property of blenders (item (c) (ii) in Definition 2.1) implies that $W^{ss}_{loc}(R^k \Gamma, h) \cap W^{uu}_{loc}(\Gamma_h, h) \neq \emptyset$, proving the lemma for the saddle $R^k \Gamma$. The proof for the saddle $S^k_h$ is analogous and omitted.

Note that if $m_2 = 2$, we get robust cycles of coindex one (associated to $R^k \Gamma$ and $\Gamma_h$) and two (associated to $S^k_h$ and $\Gamma_h$). This ends the proof of Theorem A. $\Box$

### 6.2 Proof of Theorem B

The proof of Theorem B is similar to the one of Theorem A. The robust quasi-transverse intersections are obtained exactly as in Theorem A (see Lemma 6.7). However, since the biaccumulation property is not available in the general case, the proof of the existence of transverse intersections must be different. For that, we use the geometrical properties of double blenders (see Lemma 6.6).

The main step to prove Theorem B is the following proposition (analogous to Proposition 6.2) summarising the constructions in previous sections when $n > 1$.

**Proposition 6.5** Consider $f$ with a saddle $P$ and a double blender $\Gamma$ as in Theorem B. There is a $C^r$ perturbation $g$ of $f$, $r \geq 1$, having a saddle $Q$ such that

(a) $Q$ is simple of type $(m_s, n_u)$ and $J_Q(g) > 1$,

(b) $Q$ is homoclinically related to the double blender $\Gamma_g$ and to $P_g$,

(c) $Q$ has a homoclinic tangency in the $ss$-superposition domain of $\Gamma_g$, and

(d) the uu-superposition region of the double blender $\Gamma_g$ contains a local strong unstable manifold of some point of the orbit of $Q$.

**Proof** Recall that the double blender $\Gamma$ has a pair of distinctive saddles $Q^*_{cs}$ and $Q^*_{cu}$ and $ss$- and uu-superposition regions ($\mathcal{G}^{ss}$ and $\mathcal{G}^{uu}$) and domains ($B^{ss}$ and $B^{uu}$), see Definition 2.1.

Let $U$ and $V$ be neighbourhoods of $Q^*_{cs}$ and $Q^*_{cu}$ such that $U \subset B^{ss}$ and $V \subset B^{uu}$. By hypothesis, $Q^*_{cu}$ is homoclinically related to $Q^*_{cs}$ and $P$. Having this in mind, a slight variation of Lemma 3.3 applied $P$ and $P' = Q^*_{cs}$, we get a diffeomorphism $g$ with a saddle $Q$ satisfying (a)–(b) such that $Q \in U$ and has some iterate $Q' \in V$. Taking $V$ sufficiently small, the continuous dependence of the local invariant manifolds (with respect to the point and the diffeomorphism) implies that $W^{ss}(Q, g)$ contains a disk in $\mathcal{G}^{ss}$ and $W^{uu}(Q', g)$ contains a
disk in $\mathcal{D}^{uu}$. Finally, by Remark 3.7, the saddle $Q$ can be taken with a homoclinic tangency, obtaining (c)-(d). This completes the proof of the proposition.

We are now ready to complete the proof of Theorem B. Without loss of generality, after a $C'$ perturbation, we can assume that the saddle $Q$ and its homoclinic tangency provided by Proposition 6.5 satisfy conditions (C0)–(C5). As in the proof of Theorem A, we apply Proposition 5.2 to $Q$ and $g$ to get an unfolding family $(g_t)$ and sets of parameters $\Delta_k \to 0$ such that for each $t \in \Delta_k$

(i) if $m_s = 1$ then there exists a saddle $R^k_t$ of $u$-index $n + 1$, and
(ii) if $m_s = 2$ then there exist two saddles $R^k_t$ and $S^k_t$ of $u$-indices $n + 1$ and $n + 2$, respectively.

Now we are going to verify the existence of robust cycles stated in the theorem.

**Lemma 6.6** (Robust transverse intersections) Let $t \in \Delta_k$. The unstable manifolds $W^u(R^k_t, g_t)$ and $W^u(S^k_t, g_t)$ (if $m_s = 2$) transversely intersect the stable set $W^s(\Gamma_1, g_t)$. Therefore these intersections are $C'$ robust.

**Proof** By construction, the orbits of $R^k_t$ and $S^k_t$ follow the orbit of $Q$, thus they intersect the open set $V$. Thus, taking $V$ small enough, there are points $R^k_t \in \mathcal{O}(R^k_t)$ and $S^k_t \in \mathcal{O}(S^k_t)$ such that both strong unstable manifolds $W^{uu}(\hat{R}^k_t, g_t)$ and $W^{uu}(\hat{S}^k_t, g_t)$ contain a disk in $\mathcal{D}^{uu}$. The robust intersection property of the cu-blender $\Gamma_1$, see (c)(ii) in Definition 2.1, now implies that the intersections

$$W^u(\hat{R}^k_t, g_t) \cap W^s(\Gamma_1, g_t) \neq \emptyset \quad \text{and} \quad W^u(\hat{S}^k_t, g_t) \cap W^s(\Gamma_1, g_t) \neq \emptyset$$

are $C'$-robust. Finally, note that since the $u$-indices of $\Gamma_1, R^k_t,$ and $S^k_t$ (when $m_s = 2$) are $n$, $n + 1$, and $n + 2$, respectively, the intersections above can be done transverse after a perturbation.

**Lemma 6.7** (Robust quasi-transverse intersections) Let $t \in \Delta_k$. There is a $C'$ neighbourhood $\mathcal{U}$ of $g_t$ such that for every $h \in \mathcal{U}$ the manifolds and $W^{ss}(R^k_h, h)$ and $W^{ss}(S^k_h, h)$ (if $m_s = 2$) contain disks in the superposition region of the blender $\Gamma_h$. As a consequence, for every $h \in \mathcal{U}$ it holds

$$W^s(S^k_h, h) \cap W^u_{\text{loc}}(\Gamma_h, h) \neq \emptyset \quad \text{and} \quad W^s(R^k_h, h) \cap W^u_{\text{loc}}(\Gamma_h, h) \neq \emptyset.$$

**Proof** As in the proof of Theorem A, the strong stable manifolds $W^{ss}(R^k_t, g_t)$ (in both cases i) and ii)) and $W^{ss}(S^k_t, g_t)$ contain a disk in $\mathcal{D}^{ss}$. The robust intersection property of the cs-blender $\Gamma_t$ provides the $C'$-robust intersections

$$W^s(R^k_t, g_t) \cap W^u(\Gamma_t, g_t) \neq \emptyset \quad \text{and} \quad W^s(S^k_t, g_t) \cap W^u(\Gamma_t, g_t) \neq \emptyset,$$

proving the lemma.

Summarising, the sets $\mathcal{O}(R^k_t)$ and $\Gamma_t$ form a $C^1$ robust heterodimensional cycles of coindex one and the sets $\mathcal{O}(S^k_t)$ and $\Gamma_t$ form a $C'$-robust heterodimensional cycles of coindex two. The proof of Theorem B is now complete.

**Remark 6.8** When the effective dimension is three, we can also obtain robust cycles of coindex one associated to the continuation of the double blender and saddles of $u$-index $n - 1$. 

\[ Springer \]
7 Simultaneity of robust cycles: Simon–Asaoka’s examples

In this section, we use Theorem A to prove Corollary A, claiming that the diffeomorphisms with robust homoclinic tangencies in [2, 51] also display $C^1$ robust heterodimensional cycles. We start by introducing the cs-blenders of Plykin type.

7.1 cs-blenders of Plykin type

Consider a two-dimensional diffeomorphism $h$ with a Plykin repeller $\Sigma$ inside the unitary disk $D \subset \mathbb{R}^2$. For a detailed description of this repeller, see [47, Chapter 8.6], for instance. The key property of the repeller $\Sigma$ is that the disk $D$ is foliated by local unstable manifolds $W^u_{\text{loc}}(p, h)$ of points $p \in \Sigma$. The repeller $\Sigma$ has a fixed point, say the origin $(0, 0)$, such that

$$Dh(0, 0) = \begin{pmatrix} \lambda & 0 \\ 0 & \sigma \end{pmatrix}, \quad 0 < \lambda < 1 < \sigma. \quad (7.1)$$

Consider $U \overset{\text{def}}{=} (-1, 1)^{n-2} \times D$ and a local diffeomorphisms of the form

$$\varphi: U \to \mathbb{R}^{n-2} \times D \quad \varphi(x, y, z) = (yx, h(y, z))$$

where

$$0 < \gamma < \inf_{(y, z) \in D} \|Dh(y, z)\|.$$ 

Thus $\Gamma \overset{\text{def}}{=} \{0^{n-2}\} \times \Sigma$ is a hyperbolic set of $\varphi$ and $P = (0^n) \in \Gamma$ is a fixed point of $\varphi$ of $u$-index one. Note also that $\Gamma$ is the maximal invariant set of $\varphi$ in $U$. See Fig. 5.

Given now any compact manifold $M^n, n \geq 3$, we can embed the local diffeomorphism $\varphi$ into a diffeomorphism $f_\varphi = f \in \text{Diff}^r(M^n)$. With a slight abuse of notation, we denote by $\Gamma$ and $P$ the corresponding hyperbolic set and saddle of $f$.

**Remark 7.1** By construction, the surface $S \overset{\text{def}}{=} \{0^{n-2}\} \times D$ is normally hyperbolic. Thus there is a $C^r$ neighbourhood $\mathcal{U}$ of $f$ such that for every $g \in \mathcal{U}$ there are well defined continuations
\(\Gamma_g\) and \(S_g\) of \(\Gamma\) and \(S\), respectively, with \(\Gamma_g \subset S_g\). As \(\mathbb{D} \subset \mathbb{R}^2\) is foliated by the (local) unstable manifolds \(W^u_{\mathbb{D}, \text{loc}}(p, h), p \in \Sigma\), by normal hyperbolicity of \(S_g\), see [32], we have that \(S_g\) is also foliated by local unstable manifolds of \(\Gamma_g\).

Next lemma is just a reformulation of the discussion in [15, Section 4.2].

**Lemma 7.2**  The set \(\Gamma\) is a cs-blender.

We say the blender in Lemma 7.2 is of Plykin type. As a consequence of item (1) in Remark 2.3, for every \(g\) sufficiently \(C^1\) close to \(f\), the continuation \(\Gamma_g\) of \(\Gamma\) is also cs-blender (of Plykin type).

**Proof**  Since having a cs-blender is a \(C^1\) robust property, it is enough to prove the lemma for \(f\). Properties (a) of local maximality of \(\Gamma\) and (b) partial hyperbolicity of \(\Gamma\) follows considering any neighbourhood \(V = (-\delta, \delta)^{n-2} \times \mathbb{D}\) (small \(\delta > 0\)) of \(\Gamma\) and the (dominated) splitting \(T_V M^n = E^{ss} \oplus E^{cs} \oplus E^u\), where \(E^{ss}\) is \(\mathbb{R}^{n-2} \times \{0^2\}\) and \(T_{\Sigma} \mathbb{R}^2 = E^{cs} \oplus E^u\) is the hyperbolic splitting of \(\Sigma\).

To define the family of disks \(\mathcal{D}^{ss}\) of the superposition region of \(\Gamma\), consider first a thin cone field \(C^{ss}\) associated to the strong stable direction \(E^{ss}\) of \(\Gamma\). We extend \(C^{ss}\) to a small neighbourhood \(W\) of \(V\) (maybe after shrinking \(\delta\)) and, for simplicity, denote this extension also by \(C^{ss}\). Recall that a \((n - 2)\)-dimensional disk \(\Delta \subset V\) is tangent to the cone field \(C^{ss}\) if \(T_p \Delta \subset C^{ss}(p)\) for every \(p \in \Delta\). The family \(\mathcal{D}^{ss}\) consists of all \((n - 2)\)-dimensional disks \(\Delta\) containing in \(W\) and tangent to \(C^{ss}\) whose intersection with the boundary of \(V\) has two connected components. This defines an open family of disks.

Given any saddle \(Q \in \Gamma\), consider its local strong stable manifold relative to \(U W^{ss}_{U, \text{loc}}(Q, f)\). This set contains a disk of \(\mathcal{D}^{ss}\) containing \(Q\) in its interior. Thus, any periodic point of \(\Gamma\) can be chosen as a distinctive saddle of the blender, proving property (c-i) in the definition of a cs-blender.

To get property (c-ii) about robust intersections, just observe that \(W^u_{U, \text{loc}}(\Gamma, f) = S\) and that for every disk \(D\) of \(\mathcal{D}^{ss}\) it holds \(S \cap D \neq \emptyset\). The proof of the lemma is now complete.

### 7.2 Local deformations: generation of robust tangencies

To generate a robust homoclinic tangency associated to the blender \(\Gamma\) we argue as in [2, 51]. Let us sketch the main arguments there. Recall that \(P\) is the fixed point of \(\Gamma\). Consider a small \((n - 1)\)-dimensional disk \(\Delta \subset W^s(P, f)\) disjoint from the closure of \(V\), hence disjoint from \(\Gamma\). Through a smooth isotopy supported outside the closure of \(V\), we carry \(\Delta\) close to \(\Gamma\) in such a way that the following properties hold:

(i) \(T \overset{\text{def}}{=} \Delta \cap V\) is a ss-tube: \(T\) is a manifold diffeomorphic to \((-1, 1)^{n-2} \times S^1\), foliated by disks of \(\mathcal{D}^{ss}\).

(ii) the transverse intersection between \(T\) and \(S\) is a simple closed curve \(\alpha\),

(iii) the curve \(\alpha\) (and thus the manifold \(T\)) is tangent to \(W^u_{\text{loc}}(P, f)\).

This isotopy produces a new diffeomorphism whose restriction to \(V\) coincides with \(f\). Hence \(\Gamma\) is a cs-blender of this new diffeomorphism. With a slight abuse of notation, we continue calling the resulting diffeomorphism by \(f\). We claim that for every \(g\) sufficiently \(C^r\) close to \(f\) the set \(\Gamma_g\) has a homoclinic tangency and thus \(\Gamma\) has a \(C^r\) robust tangency. Indeed, consider the continuations \(T_g\) of \(T\) and \(S_g\) of \(S\) for \(g\) close to \(f\). Note that \(T_g \subset W^s(P_g, g)\) is also a ss-tube and \(\alpha_g = T_g \cap S_g\) is also a simple closed curve. On the other hand, as
$S_g = W^u_{U, \text{loc}}(\Gamma_g, g)$, there is $P_0 \in \Gamma_g$ such that $W^u_{\text{loc}}(P_0, g)$ and $\alpha_g$ are tangent (this is a consequence of Remark 7.1).

We denote by $\mathcal{T}$ the $C^r$ neighbourhood of $f$ obtained above.

7.3 Proof of Corollary A

The first item of the proposition was obtained above. The second item follows applying Theorem A to $f$ and $P$. Note that $P$ is a simple saddle of type $(1, 1)$ with leading Jacobian $J_P(f) = \lambda \sigma > 1$ and homoclinically related to $\Gamma$. After shrinking $\mathcal{T}$, we can assume that these conditions hold for $P_g$ and $\Gamma_g$ for every $g \in \mathcal{T}$. By the second part of Remark 2.4, there is a dense subset $\mathcal{D}$ of $\mathcal{T}$ of diffeomorphisms $g$ having a homoclinic tangency associated to $P_g$. The diffeomorphisms in $\mathcal{D}$ satisfy the hypotheses of Theorem A. Thus for each $g \in \mathcal{D}$ there is an open set $C_g$ whose closure contains $g$ consisting of diffeomorphisms with robust heterodimensional cycles as above. Taking now

$$C \overset{\text{def}}{=} \bigcup_{g \in \mathcal{D}} C_g,$$

the corollary follows. \qed

Appendix: Horseshoes with large entropy and blenders. Proof of Theorem 1.4

In this appendix, we explain how Theorem 1.4 follows from the results in [4].

By the assumption, the hyperbolic measure $\mu$ is such that there is $\epsilon > 0$ with

$$h_\mu(f) > -\log J^s_\mu(f) + \frac{1}{2r} \chi^c_\mu(f) + \epsilon. \quad (7.2)$$

Applying [4, Theorem B'] to the hyperbolic measure $\mu$, given any $\delta > 0$ we get $C^1$ perturbation $h \in \text{Diff}^r(M)$ of $f$ with an affine horseshoe $\Gamma_\delta$ whose hyperbolic splitting $E^S_\delta \oplus E^u_\delta$ satisfies

- $\Gamma_\delta$ has a constant linear part $A_\delta$ which is a diagonal matrix with distinct real positive eigenvalues,
- The Lyapunov exponents of $\mu$ are $\delta$-close to the Lyapunov exponents of $A_\delta$. In particular,
  $$\text{Jac}_{E^S_\delta}(A_\delta) \to J^s_\mu \quad \text{and} \quad \chi^c_\mu(A_\delta) \to \chi^c_\mu \quad \text{as} \ \delta \to 0,$$
  where $\text{Jac}_{E^S_\delta}(A_\delta) \overset{\text{def}}{=} \det A_\delta|_{E^S_\delta}$ and $\chi^c_\mu(A_\delta)$ is the negative Lyapunov exponent of $A_\delta$ closest to zero. From this convergence, taking $\delta > 0$ small enough, we have
  $$\frac{1}{2r} \chi^c_\mu + \frac{\epsilon - \delta}{2} > -\log \text{Jac}_{E^S_\delta}(A_\delta) \quad \text{and}$$
  $$\frac{1}{2r} \chi^c_\mu + \frac{\epsilon - \delta}{2} > \frac{1}{2r} \chi^c(A_\delta), \quad (7.3)$$
- $\Gamma_\delta$ is $\delta$-close to the support of $\mu$ in Hausdorff distance,

$\Lambda$ horseshoe $\Lambda$ is said affine if there are a neighbourhood $U$ of $\Lambda$ and a chart $\varphi : U \to \mathbb{R}^d$ such that $\varphi \circ f \circ \varphi^{-1}$ is locally affine in $U$. If the chart $\varphi$ can be chosen such that $D(\varphi \circ f \circ \varphi^{-1})(x)$ is a matrix $A$ independent of $x$, we say that $\Lambda$ has constant linear part $A$. 

\copyright Springer
\( h_{\text{top}}(\Gamma_\delta, h_\delta) > h_\mu(f) - \delta. \)

The last inequality implies that

\[
    h_{\text{top}}(\Gamma_\delta, h_\delta) > h_\mu(f) - \delta \geq -\log J_\mu + \frac{1}{2r} \chi_{cs}^{\text{cs}}(A_\delta) + \epsilon - \delta
    \geq -\log \text{Jac}_{\delta}^{\text{cs}}(A_\delta) + \frac{1}{2r} \chi_{cs}^{\text{cs}}(A_\delta).
\]

Observe that the inequality in (7.4) allows us to apply [4, Theorem C] to the affine horseshoe \( \Gamma_\delta \) of \( h_\delta \), getting a perturbation \( g_\delta \) of \( h_\delta \) supported in a small neighborhood of \( \Gamma_\delta \) such that the continuation \( \Gamma_{g_\delta} \) of \( \Gamma_\delta \) is a cs-blender of central dimension \( d_{cs} = m - 1 \) with a dominated splitting whose bundles are one-dimensional. Taking now a sequence \( (\delta_k) \to 0 \), for each large \( k \) we get diffeomorphisms \( h_k = h_{\delta_k} \) and perturbations \( g_k \) of \( h_k \) with blenders \( \Gamma_k \) as before. These blenders satisfy conditions (1), (2), and (3) in the theorem.

To conclude the proof of the theorem, it remains to see that if the saddle \( P \) is homoclinically related to \( \mu \) then we can chose the perturbations \( g_k \) such that the continuations \( P_k \) of \( P \) and \( \Gamma_k \) of \( \Gamma \) are homoclinically related. To see this, we need to review the steps in the proof of [4, Theorem B'], which is a combination of [4, Theorem B] and Katok's approximation theorem [33] and its extensions in [4, Theorem 3.3] and [22, Theorem 2.12].

The first step is to apply Katok's approximation theorem to the hyperbolic measure \( \mu \) of \( f \) to obtain a horseshoe \( \Lambda \) of \( f \) such that:

(a) \( \Lambda \) is homoclinically related to \( \mu \),
(b) \( \Lambda \) is close to the support of \( \mu \) in the Hausdorff distance,
(c) the topological entropy of \( \Lambda \) is close to \( h_\mu(f) \),
(d) the \( f \)-invariant measures of \( \Lambda \) are close to \( \mu \) in the weak* topology, and
(e) the Lyapunov exponents of any ergodic measure of \( \Lambda \) are close to the Lyapunov exponents of \( \mu \).

By hypothesis, the saddle \( P \) is homoclinically related to \( \mu \). Thus (a) and the \( \lambda \)-lemma imply that \( P \) and \( \Lambda \) are homoclinically related. We can assume that \( P \notin \Lambda \). Otherwise, if \( P \in \Lambda \), we extract a subhorseshoe \( \tilde{\Lambda} \) of \( \Lambda \) such that \( P \notin \tilde{\Lambda} \) and \( h_{\text{top}}(\tilde{\Lambda}, f) \) is close to \( h_{\text{top}}(\Lambda, f) \). In particular, properties (a)–(e) also hold for \( \tilde{\Lambda} \).

The second step in the proof of [4, Theorem B'] is to apply [4, Theorem B] to the horseshoe \( \Lambda \) to get affine horseshoes with constant linear part. Since the perturbation in the latter result is done in a small neighborhood of \( \Lambda \) and \( P \notin \Lambda \), we obtain a \( C^1 \) perturbation \( h \in \text{Diff}^1(M) \) of \( f \) with \( P_h = P \) and such that the continuation \( \Lambda_h \) of \( \Lambda \) contains an affine horseshoe \( \Gamma_h \) with constant linear part. In particular, \( P_h \) is homoclinically related with \( \Gamma_h \).

Finally, when \( P \) has a homoclinic tangency, the constructions above can be done preserving that tangency. This concludes the proof of the theorem.

### Appendix: Dominated splittings

In this appendix, we review the notion of a dominated splitting.

Given an invariant set \( \Lambda \) of \( f \in \text{Diff}^1(M) \), a splitting \( T_\Lambda M = E \oplus F \) over \( \Lambda \) is called dominated if it is \( Df \) invariant and there is \( \ell \in \mathbb{N} \) such that for every \( x \in \Lambda \) and every pair of unit vectors \( u \in E(x) \) and \( v \in F(x) \) it holds

\[
    \frac{||Df^\ell(x)(u)||}{||Df^\ell(x)(v)||} \leq \frac{1}{2},
\]

where the norms are taken in the direction of the stable and unstable bundles.
here $E(x)$ and $F(x)$ are the bundles at $x$ and $|| \cdot ||$ is the norm. In this case, we say that $F$ dominates $E$.

A $Df$ invariant bundle $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ with $k$ bundles, $k \geq 3$, is dominated if the bundles $T_\Lambda M = (E_1 \oplus \cdots \oplus E_i) \oplus (E_{i+1} \oplus \cdots \oplus E_k)$ are dominated for every $i \in \{1, \ldots, k-1\}$.

The bundle $E$ over $\Lambda$ is uniformly contracting if there are constants $C > 0$ and $\kappa \in (0, 1)$ such that

$$||Df^n(x)(u)|| \leq C \kappa^n ||u||$$

for every $x \in \Lambda$, $n \geq 0$, and every vector $u \in E(x)$. Similarly, the bundle $F$ is uniformly expanding if it is uniformly contracting for $f^{-1}$. A dominated splitting $E \oplus F$ over $\Lambda$ is partially hyperbolic if either $E$ is uniformly contracting or $F$ is uniformly expanding. When $E$ is uniformly contracting and $F$ is uniformly expanding the splitting is hyperbolic. An invariant set $\Lambda$ is (partially) hyperbolic if it has a (partially) hyperbolic splitting.

Given a splitting $E \oplus F$ and $\alpha > 0$, the cone field of size $\alpha > 0$ around the bundle $E$, denoted by $C_{\alpha,F}$, consist of all vectors $v = v_E + v_F$, $v_E \in E$ and $v_F \in F$, such that the norm of $v_F$ is larger or equal than the norm of $\alpha v_E$. Below, for simplicity, we omit the dependence on $\alpha$ of the cone fields.

The following proposition is a well-known result about dominated splittings. We refer to [20, Chapter B.2] for details.

**Proposition 7.3** Let $f \in Diff^1(M)$ and $\Lambda \subset M$ be a compact $f$-invariant set with a dominating splitting $T_\Lambda M = E \oplus F$. Then there are neighbourhoods $U$ of $\Lambda$ in $M$ and $\mathcal{N}$ of $f$ in $Diff^1(M)$ such that for every $g \in \mathcal{N}$ the maximal invariant set $\Lambda_g$ of $g$ in $U$ has a dominated splitting $T_{\Lambda_g} = E_g \oplus F_g$ with $\dim(E_g) = \dim(E)$ such that

1. the bundles of the dominated splitting depend continuously with the point $x$ and the map $g$,
2. there are continuous open cone fields $C_E$ and $C_F$ defined on $U$ with $E(x) \subset C_E(x)$ and $F(x) \subset C_F(x)$ such that for every $g \in \mathcal{N}$ it holds:
   - $Dg^{-1}(C_E(x)) \subset C_E(g^{-1}(x))$ if $x, g^{-1}(x) \in U$ and
   - $Dg(C_F(x)) \subset C_F(g(x))$ if $x, g(x) \in U$.

**References**

1. Abraham, R., Smale, S.: Nongenericity of $\Omega$-stability. In: Global Analysis (Proc. Sympos. Pure Math., vol. XIV, Berkeley, Calif., 1968), pp. 5–8. Amer. Math. Soc., Providence (1970)
2. Asaoka, M.: Hyperbolic sets exhibiting $C^1$-persistent homoclinic tangency for higher dimensions. Proc. Am. Math. Soc. 136(2), 677–686 (2008)
3. Asaoka, M.: Stable intersection of Cantor sets in higher dimension and robust homoclinic tangency of the largest codimension. Trans. Am. Math. Soc. 375(2), 873–908 (2022)
4. Avila, A., Crovisier, S., Wilkinson, A.: $C^1$ density of stable ergodicity. Adv. Math., vol. 379, Paper No. 107496, 68 (2021)
5. Barrientos, P.G.: Historic wandering domains near cycles. Nonlinearity 35(6), 3191–3208 (2022)
6. Barrientos, P.G., Raibekas, A.: Robust tangencies of large codimension. Nonlinearity 30(12), 4369–4409 (2017)
7. Barrientos, P.G., Raibekas, A.: Robustly non-hyperbolic transitive symplectic dynamics. Discrete Contin. Dyn. Syst. 38(12), 5993–6013 (2018)
8. Barrientos, P.G., Raibekas, A.: Robust degenerate unfoldings of cycles and tangencies. J. Dyn. Differ. Equ. 33(1), 177–209 (2021)
9. Barrientos, P.G., Ki, Y., Raibekas, A.: Symbolic blender-horseshoes and applications. Nonlinearity 27(12), 2805–2839 (2014)
10. Berger, P.: Generic family with robustly infinitely many sinks. Invent. Math. 205(1), 121–172 (2016)
11. Berger, P.: Generic family displaying robustly a fast growth of the number of periodic points. Acta Math. 227(2), 205–262 (2021)
12. Berger, P., Crovisier, S., Pujals, E.: Iterated functions systems, blenders, and parablenders. In: Conference of Fractals and Related Fields, pp. 57–70. Springer (2015)
13. Biebler, S.: Newhouse phenomenon for automorphisms of low degree in $C^3$. Adv. Math. 361, 106952, 39 (2020)
14. Bochi, J., Bonatti, C., Díaz, L.J.: Robust criterion for the existence of nonhyperbolic ergodic measures. Commun. Math. Phys. 344(3), 751–795 (2016)
15. Bonatti, C., Crovisier, S.: Récurrence et généricité. Invent. Math. 158(1), 33–104 (2004)
16. Bonatti, C., Díaz, L.J.: Persistent nonhyperbolic transitive diffeomorphisms. Ann. Math. (2) 143(2), 357–396 (1996)
17. Biebler, S.: Newhouse phenomenon for automorphisms of low degree in $C^3$. Adv. Math. 361, 106952, 39 (2020)
18. Bochi, J., Bonatti, C., Díaz, L.J.: Robust criterion for the existence of nonhyperbolic ergodic measures. Commun. Math. Phys. 344(3), 751–795 (2016)
19. Bonatti, C., Díaz, L.J.: Abundance of $C^1$-robust homoclinic tangencies. Trans. Am. Math. Soc. 364(10), 5111–5148 (2012)
20. Bonatti, C., Díaz, L.J., Viana, M.: Dynamics beyond uniform hyperbolicity, Encyclopaedia of Mathematical Sciences, vol. 102. A global geometric and probabilistic perspective. Mathematical Physics, III. Springer, Berlin (2005)
21. Bonatti, C., Crovisier, S., Díaz, L.J., Gourmelon, N.: Internal perturbations of homoclinic classes: non-dominance, cycles, and self-replication. Ergod. Theory Dyn. Syst. 33(3), 739–776 (2013)
22. Buzzi, J., Crovisier, S., Sarig, O.: Measures of maximal entropy for surface diffeomorphisms. Ann. Math. (2) 195(2), 421–508 (2022)
23. Crovisier, S., Pujals, E.R.: Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. Invent. Math. 201(2), 385–517 (2015)
24. Crovisier, S., Sambarino, M., Yang, D.: Partial hyperbolicity and homoclinic tangencies. J. Eur. Math. Soc. (JEMS) 17(1), 1–49 (2015)
25. Díaz, L.J., Pérez, S.A.: Nontransverse heterodimensional cycles: stabilisation and robust tangencies. Trans. Amer. Math. Soc. arXiv:2101.08926v1 (2020) (preprint, to appear)
26. Dujardin, R.: Non-density of stability for holomorphic mappings on $P^k$. J. École Polytech. Math. 4, 813–843 (2017)
27. Gonchenko, S.V., Shilnikov, L.P., Turaev, D.V.: On the existence of Newhouse regions in a neighborhood of systems with a structurally unstable homoclinic Poincaré curve (the multidimensional case). Dokl. Akad. Nauk 329(4), 404–407 (1993)
28. Gonchenko, S.V., Turaev, D.V., Shilnikov, L.P.: Dynamical phenomena in multidimensional systems with a structurally unstable homoclinic Poincaré curve. Dokl. Akad. Nauk 330(2), 144–147 (1993)
29. Gonchenko, S.V., Shilnikov, L.P., Turaev, D.V.: On dynamical properties of multidimensional diffeomorphisms from Newhouse regions. I. Nonlinearity 21(5), 923–972 (2008)
30. Hasselblatt, B., Katok, A.: Principal structures. In: Handbook of Dynamical Systems, vol. 1A, pp. 1–203. North-Holland, Amsterdam (2002)
31. Hayashi, S.: Connecting invariant manifolds and the solution of the $C^1$ stability and $\Omega$-stability conjectures for flows. Ann. Math. (2) 145(1), 81–137 (1997)
32. Hirsch, M.W., Pugh, C.C., Shub, M.: Invariant Manifolds. Lecture Notes in Mathematics, vol. 583. Springer, Berlin (1977)
33. Katok, A.: Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. Inst. Hautes Études Sci. Publ. Math. 51, 137–173 (1980)
34. Li, D.: Homoclinic bifurcations that give rise to heterodimensional cycles near a saddle-focus equilibrium. Nonlinearity 30(1), 173–206 (2017)
35. Li, D., Turaev, D.: Persistent heterodimensional cycles in periodic perturbations of Lorenz-like attractors. Nonlinearity 33(3), 971–1015 (2020)
36. Li, D., Turaev, D.: Persistence of heterodimensional cycles. arXiv:2105.03739v1 (2021) (preprint)
37. Moreira, C.G.: There are no $C^1$-stable intersections of regular cantor sets. Acta Math. 206(2), 311–323 (2011)
38. Nassiri, M., Pujals, E.R.: Robust transitivity in Hamiltonian dynamics. Ann. Sci. Éc. Norm. Supér. (4) 45(2), 191–239 (2012)
39. Newhouse, S.E.: The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms. Inst. Hautes Études Sci. Publ. Math. 50, 101–151 (1979)
40. Newhouse, S., Palis, J., Takens, F.: Bifurcations and stability of families of diffeomorphisms. Inst. Hautes Études Sci. Publ. Math. 57, 5–71 (1983)
41. Palis, J.: A global view of dynamics and a conjecture on the denseness of finitude of attractors. Astérisque 261, 339–351 (2000)
42. Palis, J., Viana, M.: High dimension diffeomorphisms displaying infinitely many periodic attractors. Ann. Math. (2) 140(1), 207–250 (1994)
43. Pesin, Y.B.: Lectures on partial hyperbolicity and stable ergodicity. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich (2004)
44. Pugh, C.C.: An improved closing lemma and a general density theorem. Am. J. Math. 89, 1010–1021 (1967)
45. Pugh, C.: Against the $C^2$ closing lemma. J. Differ. Equ. 17, 435–443 (1975)
46. Pujals, E.R., Sambarino, M.: Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. Ann. Math. (2) 151(3), 961–1023 (2000)
47. Robinson, C.: Dynamical systems. Studies in Advanced Mathematics, 2nd edn. Stability, symbolic dynamics, and chaos. CRC Press, Boca Raton (1999)
48. Rodriguez Hertz, F., Rodriguez Hertz, M.A., Tahzibi, A., Ures, R.: New criteria for ergodicity and nonuniform hyperbolicity. Duke Math. J. 160(3), 599–629 (2011)
49. Romero, N.: Persistence of homoclinic tangencies in higher dimensions. Ergod. Theory Dyn. Syst. 15, 735–757 (1995)
50. Simon, C.P.: A 3-dimensional Abraham–Smale example. Proc. Am. Math. Soc. 34, 629–630 (1972)
51. Simon, C.P.: Instability in $\text{Diff}^r(T^3)$ and the nongenericity of rational zeta functions. Trans. Am. Math. Soc. 174, 217–242 (1972)
52. Sternberg, S.: Local contractions and a theorem of Poincaré. Am. J. Math. 79, 809–824 (1957)
53. Taffin, J.: Blenders near polynomial product maps of $\mathbb{C}^2$. J. Eur. Math. Soc. (JEMS) 23(11), 3555–3589 (2021)
54. Turaev, D.: On dimension of non-local bifurcational problems. Int. J. Bifurc. Chaos Appl. Sci. Eng. 6(5), 919–948 (1996)
55. Ures, R.: Abundance of hyperbolicity in the $C^1$ topology. Ann. Sci. École Norm. Sup. (4) 28(6), 747–760 (1995)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.