Abstract. In this paper, we study the deformation of the n-dimensional strictly convex hypersurface in $\mathbb{R}^{n+1}$ whose speed at a point on the hypersurface is proportional to $\alpha$-power of positive part of Gauss Curvature. For $\frac{1}{n} < \alpha \leq 1$, we prove that there exist the strictly convex smooth solutions if the initial surface is strictly convex and smooth and the solution hypersurfaces converge to a point. We also show the asymptotic behavior of the rescaled hypersurfaces, in other words, the rescaled manifold converges to a strictly convex smooth manifold. Moreover, there exists a subsequence whose the limit satisfies a certain equation.

1. Introduction

This paper concerns with the regularity of the $\alpha$-Gauss Curvature flow, describing the deformation of a n-dimensional compact strictly convex body $\Sigma$ in $\mathbb{R}^{n+1}$ which is subject to wear under impact from any random angle. An example can be a stone on a beach impacted by the sea, where the chance of impact at any point on the surface $\Sigma$ is in proportion to the $\alpha$-Gauss Curvature $K^\alpha$. Let $X(\cdot, \cdot) : \Sigma \times [0, T) \to \mathbb{R}^{n+1}$ be an embedding and set $\Sigma_t = X(\Sigma, t)$. Then the surface evolves by the following flow:

$$\frac{\partial X}{\partial t}(x, t) = -K^\alpha(x, t) \nu(x, t)$$

$$X(x, 0) = X_0(x)$$

where $\nu$ denotes the unit outward normal to $\Sigma_t$ and $\alpha > 0$.

1.1. Let $(0, T^*)$ be the maximal interval in which $vol(\Sigma_t)$ is nonzero. Then there are the known results for the case of $\alpha = 1$ in the flow following (1.1). If the initial surface in $\mathbb{R}^3$ is the smooth and strictly convex and has the central symmetry, then the solution $\Sigma_t$ converges to a point with a spherical shape [F]. Also Tso, [T], showed the existence and regularity of the solution when the initial hypersurface embedded in $\mathbb{R}^{n+1}$ is the the smooth and strictly convex. In the other words, the solution $\Sigma_t$ preserves the smoothness and the convexity for the time interval $(0, T^*)$. For a smooth, compact, and strictly convex initial surface in $\mathbb{R}^3$, the solution surface $\Sigma_t$ converges to a point and the rescaled solution surface $\tilde{\Sigma}_t$ approaches the round sphere with the normalized volume and for a non-smooth initial surface, the viscosity solution has $C^{1,1}$ regularity for time interval $(0, T^*)$ and $C^\infty$ regularity for
\[ t \geq t_0 \] where \( t_0 \) depends on the volume and diameter of the initial surface \( \Sigma_0 \) \[ A1 \]. For \( \alpha = \frac{1}{n+2} \), the solution, \( \Sigma_t \), is known as an affine normal flows. There exists a unique, smooth, convex solution such that the hypersurface \( \Sigma_t \) converges to a point and the rescaled solution converges to an ellipsoid when the initial hypersurface is a compact, smooth, and strictly convex \[ A5 \]. And for \( \frac{1}{n+2} < \alpha \leq \frac{1}{n} \) or \( 0 < \alpha \leq \frac{1}{n} \) under the assumption that the isoperimetric ratios are bounded, there exist a smooth, strictly convex solution converging to a point and a rescaled solution satisfying the certain equation \[ A2 \]. In addition, for \( \alpha = \frac{1}{n} \), the rescaled solution converges to a sphere and this holds for \( \alpha \geq \frac{1}{n} \) when the initial hypersurface is very close to a sphere \[ C1 \].

1.2. Main Theorem. Let us denote the rescaled \( \Sigma \) and a support function \( S \) by \( \tilde{\Sigma} \) and \( \tilde{S} \) respectively so that the volume enscribed becomes normalized. We state the main theorem.

**Theorem 1.1.**

Let \( \Sigma_0 = X(\Sigma, 0) \) be a compact, connected, strictly convex smooth manifold in \( \mathbb{R}^{n+1} \). Assume \( \frac{1}{n} < \alpha \leq 1 \). Then

(i) there exist a time \( T^* \) and strictly convex smooth solutions \( \{ \Sigma_t = X(\Sigma, t) \} \) satisfying \[ (1.1) \] for \( t \in [0, T^*) \) and \( \Sigma_t \) converges to a point as \( t \) approaches to \( T^* \).

(ii) And for any sequence \( \tau_i \to \infty \), there exists a subsequence \( \tau_{i_k} \) such that the rescaled manifold \( \tilde{\Sigma}_{\tau_{i_k}} \) converges to strictly convex manifold \( \tilde{\Sigma}_{T^*} \) uniformly in \( C^\infty \)-norm.

(iii) In addition, the limit, \( \tilde{S}(\cdot, \tau_{i_k}) \), of the volume normalized solution \( \tilde{S}(\cdot, \tau_{i_k}) \) satisfies the equation \( \tilde{K}_\alpha = \tilde{C}_\ast \tilde{S} \), a.e. for some positive constant \( \tilde{C}_\ast \), where \( \tilde{K}_\alpha \) is the gauss curvature of \( \tilde{\Sigma}_{T^*} \).

(iv) The principal curvatures of the rescaled hypersurfaces \( \tilde{\Sigma} \) have the uniform upper and lower bounds. In other words, let us denote the eigenvalues of \( (\tilde{h}_i^j) \) by \( \tilde{\lambda}_k \) for \( k = 1, \cdots, n \) and the smallest and largest one by \( \tilde{\lambda}_{\text{min}} \) and \( \tilde{\lambda}_{\text{max}} \), respectively. Then we have

\[
\frac{1}{M} \leq \tilde{\lambda}_{\text{min}} \leq \tilde{\lambda}_{\text{max}} \leq M
\]

for some constant \( 0 < M < \infty \).

1.3. Outline. Each sections in this paper will be organized as follow. In this section, we have introduced the known results for Gauss Curvature flow and our main theorem. In section 2, we also state the definitions of metric, second fundamental form and some curvatures, and their evolution equations. In addition, we obtain the evolution equations for the standard metric on sphere by introducing the support
function. In section 3, we prove the hypersurfaces preserve the strict convexity and also we get the uniform upper bounds of gauss curvature $K$ and the eigenvalues of the reverse second fundamental form. Then using these bounds we can get the uniform bound of curvatures of hypersurface $\Sigma$. This is proved in Section 5.2. Integral quantity plays the trigger role in getting the asymptotic behavior of hypersurface and $C^{1,1}$-regularity of the rescaled solution and the curvature bounds of the rescaled hypersurface will be introduced in section 4. In [1] and [C1], the authors showed that there exists the finite time $T^*$ such that the solution $\Sigma_t$ converges to a point as $t$ approaches to $T^*$.

Throughout the whole section, we consider the case of $\frac{1}{n} < \alpha \leq 1$ if there is no specific assumption for $\alpha$. We will also assume that $\Sigma_t$ is smooth whenever we prove a priori-estimates.

2. Evolution of the metric and curvature

2.1. Let $\{x_1, \cdots, x_n\}$ be the local coordinates of $\Sigma_t$ and $\nu$ be the outward unit normal vector to $\Sigma_t$. Then the induced metric and second fundamental form are defined by

$$g_{ij} = \left( \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right) \quad \text{and} \quad h_{ij} = -\left( \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right).$$

Also the Weingarten map $W_p : T_p M \to T_p M$ for the hypersurface $M \subset R^{n+1}$ can be given by

$$h^i_j = g^{ik} h_{kj},$$

where $(g^{ij})$ denotes the inverse matrix of $(g_{ij})$, and then $\sigma_k = \sum_{1 \leq k_1 < \cdots < k_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$, $H = \text{trace}(h^i_i) = \sigma_1 = \sum_{1 \leq i \leq n} \lambda_i$, $K = \det(h^i_i) = \sigma_n = \lambda_1 \lambda_2 \cdots \lambda_n$, and $|A|^2 = h_{ij} h^{ij} = \lambda_1^2 + \cdots + \lambda_n^2$ where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of the Weingarten map at $p$.

The evolutions of the metric, second fundamental form, and curvature are following. Throughout this paper, the symbol $\Box$ will be used in place of the operator $K^a (h^{-1})^{ij} \nabla_k \nabla_j$. For the detailed proof, the readers can refer to Chapter 2, [Z].

**Lemma 2.1.** Let $\Sigma_0$ be strictly convex and $\Sigma_t = X(\Sigma, t)$ be smooth. For the $\alpha$-Gauss curvature flow, we have

(i) $\frac{\partial g_{ij}}{\partial t} = -2K^a h_{ij}$

(ii) $\frac{\partial \nu}{\partial t} = g^{ij} \frac{\partial K^a}{\partial x^i} \frac{\partial X}{\partial x^j} = \nabla^j K^a \frac{\partial X}{\partial x^j}$
\[ (iii) \frac{\partial h_{ij}}{\partial t} = \nabla_i \nabla_j K^\alpha - K^\alpha h_{jk} h_{ji}^k \\
= \alpha \Box h_{ij} + \alpha^2 K^\alpha (h^{-1})^k (h^{-1})^m \nabla_i h_{kl} \nabla_j h_{mn} - \alpha K^\alpha (h^{-1})^m (h^{-1})^n \nabla_i h_{mn} \nabla_j h_{kl} \\
+ \alpha K^\alpha h_{ij} - (1 + n\alpha) K^\alpha h_{ij} h_{ij}^k \]

\[ (iv) \frac{\partial K}{\partial t} = \alpha \Box K + \alpha (\alpha - 1) K^\alpha (h^{-1})^i \nabla_i K \]

\[ (v) \frac{\partial K^\alpha}{\partial t} = \alpha \Box K^\alpha + \alpha K^\alpha H \]

\[ (vi) \frac{\partial H}{\partial t} = \alpha \Box H + \alpha^2 K^\alpha \mathcal{E} \nabla_i K \nabla_j K - \alpha K^\alpha g^{ij} (h^{-1})^m \nabla_i h_{mn} \nabla j h_{kl} + \alpha K^\alpha H^2 \\
+ (1 - n\alpha) K^\alpha |A|^2 \]

\[ (vii) \frac{\partial |X|^2}{\partial t} = \Box |X|^2 - 2K^\alpha (h^{-1})^k g_{kl} + 2(n - 1) K^\alpha \langle X, v \rangle \]

2.2. The support function \( S(z, t) \) of the strictly convex surface is given as

\[ (2.1) \quad S(z) = \langle z, X(\nu^{-1}(z), t) \rangle, \quad \text{for} \quad z \in S^n. \]

Then \( X(z) \) can be written as \( X(z) = S(z)z + \nabla S(z) \) from the facts which are the definition of the support function \((2.1)\) and \( \nabla S(z) = \langle X(z), \nabla z \rangle \) for the connection of the standard metric \( \overline{g} \) on \( S^n \). And we have

\[ (2.2) \quad \frac{\partial z}{\partial x^i} = h_{ik} g^{kj} \frac{\partial X}{\partial x^l} \]

from the relationship between the tangent vector and the normal vector and the definition of the second fundamental form. In addition,

\[ (2.3) \quad h_{ij} = \nabla_i \nabla_j S + \overline{g}_{ij} \]

where \( \overline{g}_{ij} \) are the metric on \( S^n \), which this can obtain by taking covariant derivatives for \((2.1)\), \([Z]\).

Now we have the following relationships and the evolution equations (see Chapter 3 of \([Z]\) to understand the proof in detail).

**Lemma 2.2.** Let \( \Sigma_0 \) be strictly convex and \( X(\nu^{-1}(z), t) \) be smooth, where \( z \in S^n \). For the \( \alpha \)-Gauss curvature flow, we have

\[ (i) \quad \overline{g}_{ij} = h_{ik} g^{kj} h_{ij} \quad \text{and} \quad \overline{g}^{ij} = (h^{-1})^k g_{kl} (h^{-1})^l \]
\( h_{ij} = (h^{-1})^{ik} g_{kj} \)

\( H = g_{ik} (h^{-1})^{ij}, \quad |A|^2 = g^{ik} g_{ij}, \quad \text{and} \quad K = \det(h^i_j) = \frac{\det(g_{ij})}{\det(\nabla_i \nabla_j S + S g_{ij})} \)

(iv) Set \( S_k \) be the \( k \)-th symmetric polynomial of \( h_{ij} \) while \( \sigma_k \) is the \( k \)-th symmetric polynomial of \( h_{ij} \). Then \( S_n = K^{-1} \).

The following Lemma gives us the evolution equations of the support function, second fundamental form, and curvatures for the standard metric \( g_{ij} \) on \( S^n \).

**Lemma 2.3.** Let \( \Sigma_0 \) be strictly convex and \( X(\nu^{-1}(z), t) \) be smooth, where \( z \in S^n \). For the \( \alpha \)-Gauss curvature flow, we have

(i) \( \frac{\partial S}{\partial t} = -K^\alpha = -K^{-\alpha} \) or \( \left( -\frac{\partial S}{\partial t} \right) K^\alpha = 1 \), where \( K = K^{-1} \)

(ii) \( \frac{\partial h_{ij}}{\partial t} = -\nabla_i \nabla_j K^\alpha - K^\alpha g_{ij} = -\left( \nabla_i \nabla_j K^{-\alpha} + K^{-\alpha} g_{ij} \right) \)

(iii) \( \frac{\partial H}{\partial t} = g^{ij} \left( \nabla_i \nabla_j K^\alpha + K^\alpha g_{ij} \right) \)

(iv) \( \frac{\partial |A|^2}{\partial t} = 2g_{ij} h^i_j K^\alpha \)

(v) \( \frac{\partial K}{\partial t} = K(h^{-1})^{ij} \left( \nabla_i \nabla_j K^\alpha + K^\alpha g_{ij} \right) = K(h^{-1})^{ij} \nabla_i \nabla_j K^\alpha + K^{\alpha+1} H \)

(vi) \( \frac{\partial K^\alpha}{\partial t} = \alpha K^\alpha (h^{-1})^{ij} \nabla_i \nabla_j K^\alpha + \alpha K^{2\alpha} H \)

(vii) \( \frac{\partial K}{\partial t} = -K(h^{-1})^{ij} \left( \nabla_i \nabla_j K^{-\alpha} + K^{-\alpha} g_{ij} \right) \)

**Proof.** Taking the time derivative to (2.1) gives us

\[ \frac{\partial S}{\partial t} = \langle z, \nabla X \cdot \partial_{\nu^{-1}} + \partial X \rangle \]
and then (i) comes from (1.1). Also we can obtain (ii) and (iv) by the definitions of $h_{ij}$ and $|A|^2$, respectively. We know that

$$\frac{\partial}{\partial t} H = \mathcal{S}_{ij} (h^{-1})^k (h^{-1})^l \left( \nabla_k \nabla_l K^\alpha + \mathcal{S}_{kl} K^\alpha \right)$$

by (ii). From the evolution equation of the second fundamental form, we get the evolution equation of $K$:

$$\frac{\partial K}{\partial t} = -K \mathcal{S}_{im} h_{mj} (h^{-1})^n (h^{-1})^l \frac{\partial}{\partial t} h^{kl}$$

$$= K (h^{-1})^k \nabla_k \nabla_l K^\alpha + K^{\alpha+1} H,$$

which implies (vi). Also (vii) is obtained directly from the definition of $K$. 

\[\square\]

In addition, $S$ satisfies, as in [T],

$$-S_t (z, t) \left[ \det (\nabla_i \nabla_j S(z, t) + S(z, t) \delta_{ij}) \right]^{\alpha} = 1 \quad \text{for} \ (z, t) \in S^n \times (0, \tau^*),$$

which comes from Lemma 2.2 (iii) and Lemma 2.3 (i).

3. Curvature Estimate

We define the width, the inner radius and the outer radius of the convex hypersurface as follows:

- the inner radius $r_{\text{in}} = \sup \{r : B_r (y) \text{ is enclosed by } X \text{ for some } y \in \mathbb{R}^{n+1}\}$
- the outer radius $r_{\text{out}} = \inf \{r : B_r (y) \text{ encloses } X \text{ for some } y \in \mathbb{R}^{n+1}\}$

Now we shall show the strict convexity of $\Sigma_t$ will be preserved under the flow.

Lemma 3.1.

If $\Sigma_0$ is strictly convex, $\Sigma_t = X(\Sigma_t)$ is also strictly convex for $t > 0$ as long as it is smooth. And we have

$$\inf_{x \in \Sigma} K(x, t) \geq \inf_{x \in \Sigma} K(x, 0) > 0.$$

Proof. Let $Z(t) = \inf_{x \in \Sigma} K(x, t)$ and assume that the minimum is achieved at $X = X(x, t)$. Then, at $X$, we have

$$\nabla_i \nabla_j K \geq 0 \quad \text{and} \quad \nabla_i K = 0,$$
and then we get

\[
\frac{\partial K}{\partial t} = \alpha \Box K + \alpha (\alpha - 1) K^{\alpha - 1} (h^{-1})^j \nabla_i K \nabla_j K + K^{\alpha + 1} H \\
\geq K^{\alpha + 1} H.
\]  

(3.1)

Now \( H \geq n K^{1/n} \) imples

\[
\frac{\partial Z}{\partial t} \geq n Z^{\alpha + 1 + 1/n}.
\]

By the maximum principle, we can get \( Z(t) \geq Z(0) > 0 \) which gives the positive lower bound of \( K \) for \( t > 0 \), and then the strict convexity of \( \Sigma_t \).

\[\Box\]

We have the following lemma (cf. [A2]). We shall use the idea of Lemma 3.5 in [Z].

**Lemma 3.2.**

Let \( \Sigma_0 \) be convex, \( \Sigma_t = X(S^n, t) \) be smooth, and \( \alpha > 0 \). Also let us consider the sphere with radius \( r_{in}(T^* - \delta) \) and center at the origin contained in \( \Sigma_{T^* - \delta} \) and set \( \rho_0 = \frac{1}{2} r_{in}(T^* - \delta) \) where \( \delta \) is any positive constant satisfying \( \delta < T^* \). Then there is a constant \( C > 0 \) such that

\[
\sup_{z \in S^n, 0 \leq t \leq T^* - \delta} K^\alpha (z, t) \leq C = \max \left( \sup_{z \in S^n} K^\alpha (z, 0), \left( \frac{n \alpha + 1}{n \alpha \rho_0} \right)^{\alpha} \right).
\]

**Proof.** We consider the function \( \varphi = \frac{K^\alpha}{S - \rho_0} \), where \( S \) is support function. Here \( S(z, t) = (z, X(v^{-1}(z), t)) \) and then

\[
\frac{\partial S}{\partial t} = \left( z, \frac{\partial X}{\partial t} \right) = (z, -K^\alpha v) = -K^\alpha.
\]

Let us assume that \( \varphi \) has its maximum at \((z_0, t_0)\) for \( t_0 \leq T^* - \delta \). Then, at \((z_0, t_0)\), we get

\[\varphi_t \geq 0, \ \nabla_i \varphi = 0 \ \text{and} \ \nabla_i \nabla_j \varphi \leq 0.\]
Now we have \( 0 = \nabla_i \varphi = \frac{(S - \rho_0)K^\alpha - K^\alpha \nabla_i S}{(S - \rho_0)^2} = \frac{\nabla_i K^\alpha}{S - \rho_0} - \frac{K^\alpha \nabla_i S}{(S - \rho_0)^2} \) and then \( \nabla_i K^\alpha = \frac{K^\alpha \nabla_i S}{S - \rho_0}. \) Since

\[
0 \geq \nabla_i \nabla_j \varphi = \nabla_i \left( \frac{\nabla_j K^\alpha}{S - \rho_0} - \frac{K^\alpha \nabla_j S}{(S - \rho_0)^2} \right) = \frac{\nabla_j \nabla_i K^\alpha}{S - \rho_0} - \frac{\nabla_j \nabla_i \nabla_j S + K^\alpha \nabla_i \nabla_j S}{(S - \rho_0)^2} + \frac{2K^\alpha \nabla_i \nabla_j S}{(S - \rho_0)^3} = \frac{\nabla_j \nabla_i K^\alpha}{S - \rho_0} - \frac{K^\alpha \nabla_i \nabla_j S}{(S - \rho_0)^2},
\]
we also have \( \nabla_i \nabla_j K^\alpha \leq \frac{K^\alpha \nabla_i \nabla_j S}{S - \rho_0}. \) Therefore \( \varphi \) satisfies, at \((z_0, t_0),\)

\[
0 \leq \frac{\partial}{\partial t} \varphi = \frac{1}{S - \rho_0} \left( \alpha K^\alpha (h^{-1}) \nabla_i \nabla_j K^\alpha + \alpha K^{2\alpha} H + \frac{K^{2\alpha}}{S - \rho_0} \right) \leq \frac{1}{S - \rho_0} \left( \alpha K^\alpha (h^{-1}) \left( \frac{K^\alpha \nabla_i \nabla_j S}{S - \rho_0} \right) + \alpha K^{2\alpha} H + \frac{K^{2\alpha}}{S - \rho_0} \right).
\]
From Lemma 2.2, we can derive

\[
0 \leq \frac{\alpha K^\alpha (h^{-1}) \nabla_i \nabla_j S}{S - \rho_0} (h_{ij} - S g_{ij}) + \alpha K^{2\alpha} H + \frac{K^{2\alpha}}{S - \rho_0} = \frac{K^{2\alpha}}{S - \rho_0} (n\alpha - \alpha \rho_0 H + 1),
\]
which means

\[
0 \leq (n\alpha + 1) - \alpha \rho_0 H.
\]
If \( \frac{n\alpha + 1}{\alpha \rho_0} < H, \) that is, \( H > \frac{C}{\rho_0} > \frac{C}{r_{in}} > \frac{C}{r_{out}}, \) where \( C = \frac{n\alpha + 1}{\alpha}, \) then it’s contradiction. Therefore \( H \) is bounded, so \( K^\alpha \) is bounded since \( K^\alpha \leq n^{-\alpha} H \). Now we conclude

\[
\sup_{z \in S^a, \ 0 \leq t \leq T - \delta} K^\alpha(z, t) \leq C = \max \left( \sup_{z \in S^a} K^\alpha(z, 0), \left( \frac{n\alpha + 1}{n\alpha \rho_0} \right)^{\alpha} \right).
\]
\( \square \)

Now we consider the eigenvalues of the reverse second fundamental form.

Lemma 3.3.
Let \( \Sigma_0 \) be strictly convex, \( \Sigma_t = X(\Sigma, t) \) be smooth, and \( \frac{1}{n} < \alpha \leq 1 \). Also set \( \mathcal{H} = (h^{-1})^j_{ij} \). Then there is a constant \( C > 0 \) such that
\[
\sup_{x \in \Sigma} \mathcal{H} \leq C = \max \left( \frac{n\alpha - 1}{\alpha} K^{-1/n}, \sup_{x \in \Sigma} \mathcal{H}(x, 0) \right)
\]
for \( t > 0 \) as long as it is smooth.

**Proof.** First we have the evolution equation for \( \mathcal{H} \):
\[
\frac{\partial \mathcal{H}}{\partial t} = \alpha \Box \mathcal{H} - 2\alpha K^\alpha (h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{kl} \nabla_\rho h_{pq} \]
\[
- \alpha^2 K^\alpha (h^{-1})^j_{ij} (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{mn} \nabla_\rho h_{pq}
\]
\[
+ \alpha K^\alpha (h^{-1})^j_{ij} (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{mn} \nabla_\rho h_{pq} - \alpha K^\alpha H \mathcal{H} + n(1 + \alpha)K^\alpha - 2nK^\alpha
\]
since we can obtain
\[
(3.2) \quad \alpha \Box \mathcal{H} = -\alpha (h^{-1})^j_{ij} g_{ij} \Box h_{kl} + 2\alpha K^\alpha (h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{kl} \nabla_\rho h_{pq}
\]
from the second derivatives of \( \mathcal{H} \)
\[
(3.3) \quad \nabla_\gamma \nabla_\rho \mathcal{H} = - (h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} \nabla_\gamma h_{kl} g_{ij} + (h^{-1})^j_{ij} \nabla_\gamma \nabla_\rho g_{ij} + 2(h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{kl} \nabla_\rho h_{pq}
\]
\[
- (h^{-1})^j_{ij} \nabla_\rho h_{kl} \nabla_\lambda g_{ij} + \nabla_\gamma h_{kl} \nabla_\rho g_{ij}
\]
\[
= - (h^{-1})^j_{ij} \nabla_\gamma h_{kl} g_{ij} + 2(h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{kl} \nabla_\rho h_{pq}
\]
and we also have the Codazzi identity and symmetry of \( h_{ij} \). Then at the maximum point we get
\[
\frac{\partial \mathcal{H}}{\partial t} \leq -2\alpha K^\alpha (h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{mn} \nabla_\rho h_{pq}
\]
\[
+ \alpha K^\alpha (h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\gamma h_{mn} \nabla_\rho h_{pq}
\]
\[
- \alpha K^\alpha H \mathcal{H} + n(\alpha - 1)K^\alpha
\]
\[
\leq -(\alpha K^\alpha - n(\alpha - 1))K^\alpha
\]
\[
\leq -(anK^{1/n}H - n(\alpha - 1))K^\alpha.
\]
(3.4) \quad -2\alpha K^\alpha (h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\rho h_{pq} \]
\[
= -2\alpha K^\alpha (h^{-1})^j_{ij} \partial^i (h^{-1})^k_{kl} (h^{-1})^l_{ij} g_{ij} \nabla_\rho h_{pq}.
\]
and $H \geq n(K)^{1/n} \geq c_0 > 0$ for some positive constant $c_0$. On the other hand, we have a contradiction if $\mathcal{H} > \frac{n\alpha - 1}{\alpha K^{1/n}}$ at the maximum point. Hence $\mathcal{H} \leq \frac{n\alpha - 1}{\alpha} K^{-1/n}$. Then the result follows.

4. Integral Quantities and Asymptotic Behavior of Hypersurface

We shall define the volume $V(t)$ and the area $\mathcal{A}(t)$ enclosed by convex surface $\Sigma$ as follows:

- the volume function $V(t) = \frac{1}{n+1} \int_{\Sigma} \langle X, \nu \rangle d\sigma_{\Sigma} = \frac{1}{n+1} \int_{S^n} S d\sigma_S$
- the area function $\mathcal{A}(t) = \int_{\Sigma} d\sigma_{\Sigma} = \int_{S^n} \frac{1}{K} d\sigma_S$

**Lemma 4.1.** For the strictly convex and smooth solution $\Sigma_t = X(S^n, t)$ of $\alpha$-Gauss curvature flow (1.1), we have

$$\frac{\partial}{\partial t} V(t) = - \int_{S^n} \frac{1}{K^{1-\alpha}} d\sigma_{S^n}.$$  

**Proof.** First observe that from Lemma 2.2 and Lemma 2.3, we have

$$\int_{S^n} S K_t d\sigma_S = \int_{S^n} S K (h^{-1})^{ij} (\nabla_i \nabla_j S + S g_{ij}) d\sigma_S = \int_{S^n} S_t K (h^{-1})^{ij} h_{ij} d\sigma_S = n \int_{S^n} S_t K d\sigma_S$$

since $\nabla_i K (h^{-1})^{ij} = 0$. Hence we have

$$\frac{\partial}{\partial t} V(t) = \frac{1}{n+1} \int_{S^n} (KS_t + S K_t) d\sigma_S = \int_{S^n} KS_t d\sigma_S = - \int_{S^n} \frac{1}{K^{1-\alpha}} d\sigma_S.$$  

Now let us consider the rescaled solution

$$(4.1) \quad \tilde{X}(\tau) = \frac{X(t)}{V(t)^{1/(n+1)}}.$$
and also assume that the normalized volume \( \tilde{V}(\tau) = \frac{1}{n+1} \left( \int_{S^n} \frac{\tilde{S}}{\tilde{K}} d\sigma_{S^n} \right) = 1 \) where \( \tau(t) = -\log \left( \frac{V(t)}{V(0)} \right) \). For the rescaled solution, we have the rescaled metric, second fundamental form, and curvatures as follows:

- \( \tilde{\tilde{g}}_{ij} = V(t)^{-\frac{1}{n+1}} g_{ij} \) and \( \tilde{h}_{ij} = V(t)^{-\frac{1}{n+1}} h_{ij} \)
- \( \tilde{\tilde{H}}(t) = V(t)^{-\frac{1}{n+1}} H \) and \( \tilde{\tilde{K}} = V(t)^{-\frac{1}{n+1}} K \)
- \( \tilde{\tilde{S}} = V(t)^{-\frac{1}{n+1}} S \) and \( \tilde{\eta} = V(t)^{-\frac{n-1}{n+1}} \eta \) where \( \eta(t) = \int_{S^n} \frac{1}{K^{1-\alpha}} d\sigma_{S^n} \).

Then we obtain the following Corollary.

**Corollary 4.2.** For the strictly convex and smooth rescaled solution \( \tilde{\Sigma}_\tau = \tilde{X}(S^n, \tau) \) of \( \alpha \)-Gauss curvature flow, we have the evolution equation of \( \tilde{X} \):

\[
\frac{\partial \tilde{X}}{\partial \tau} = -\frac{\tilde{K}^\alpha}{\tilde{\eta}} \tilde{v} + \frac{1}{n+1} \tilde{X} \quad \text{on} \quad S^n \times [0, +\infty),
\]

where \( \tilde{K} \) is the gauss curvature and \( \tilde{v} \) is the unit outward normal of \( \tilde{\Sigma}_\tau \).

**Proof.** Lemma 4.1 implies

\[
V(t) = V(0) - \int_0^t \eta(s) ds.
\]

Since \( \frac{\partial \tilde{X}}{\partial \tau} = \frac{\partial \tilde{X}}{\partial t} \frac{dt}{d\tau} = -\frac{K^\alpha V}{\eta V^{1/(\alpha+1)}} \tilde{v} + \frac{1}{n+1} \tilde{X} \), we get the result

\[
\frac{\partial \tilde{X}}{\partial \tau} = -\frac{\tilde{K}^\alpha}{\tilde{\eta}} \tilde{v} + \frac{1}{n+1} \tilde{X}.
\]

\(\square\)

Now we introduce some integral quantity to observe the asymptotic behavior of the rescaled hypersurface \( \tilde{\Sigma} \).
Lemma 4.3. Let us define the integral quantity \( \tilde{I} \) as follows:

\[
\tilde{I}(\tau) = \begin{cases} \\
\left( \int_{S^n} \frac{1}{S^{n-1}} \, d\sigma_S \right)^{\text{sign}(\alpha-1)} & \text{for } \alpha > 0 \text{ and } \alpha \neq 1 \\
\int_{S^n} \log \tilde{S} \, d\sigma_S & \text{for } \alpha = 1 \\
\end{cases}
\]

(4.3)

Then it satisfies

\[
\frac{\partial}{\partial \tau} \tilde{I}(\tau) \leq 0.
\]

(4.4)

Moreover, the equality holds if and only if \( \tilde{K}^\alpha = C\tilde{S} \text{ a.e. for some positive constant } C \).

Proof.

Case 1. Let us assume that \( 0 < \alpha < 1 \).

By the definition of the rescaled support function \( \tilde{S} \) and (4.2), we know that

\[
\tilde{K}^{-\alpha} \left( \frac{\partial \tilde{S}}{\partial \tau} - \frac{1}{n+1} \tilde{S} \right) = -\frac{1}{\tilde{\eta}}.
\]

(4.5)

Since multiplying both sides of the equation (4.5) by \( \tilde{S}^{-\beta} \) where \( \beta \) is expected to be chosen later implies

\[
\frac{1}{\tilde{S}^\beta} \left( \frac{\partial \tilde{S}}{\partial \tau} - \frac{1}{n+1} \tilde{S} \right) = -\frac{\tilde{K}^\alpha}{\tilde{\eta}^\beta},
\]

from the derivation of \( \tilde{I}(\tau) \) with respect to \( \tau \), we have

\[
\frac{\alpha}{1-\alpha} (\tilde{I}(\tau))^{-2} \frac{\partial}{\partial \tau} \tilde{I}(\tau) = \int_{S^n} \tilde{S}^\beta \, d\sigma_S = \frac{1}{n+1} \int_{S^n} \tilde{S}^{1-\beta} \, d\sigma_S - \int_{S^n} \frac{\tilde{K}^\alpha}{\tilde{\eta}^\beta} \, d\sigma_S \leq 0.
\]

(4.6)

Since \( \tilde{\eta}(\tau) \) is a positive, (4.6) is non-positive if

\[
\frac{1}{n+1} \left( \int_{S^n} \tilde{S}^{1-\beta} \, d\sigma_S \right) \left( \int_{S^n} \frac{1}{\tilde{K}^{1-\alpha}} \, d\sigma_S \right) \leq \int_{S^n} \frac{\tilde{K}^\alpha}{\tilde{S}^\beta} \, d\sigma_S,
\]

(4.7)

which implies non-positivity of the evolution equation of \( \tilde{I}(\tau) \). Hence it will suffice to show that the inequality above (4.7) holds. First, notice that we have

\[
\int_{S^n} \tilde{S}^{1-\beta} \, d\sigma_S = \int_{S^n} \left( \tilde{S}^{-\beta} \tilde{K}^\alpha \right)^{\frac{\beta-1}{\beta}} \left( \tilde{K}^{-\alpha} \right)^{\frac{\beta-1}{\beta}} \, d\sigma_S \\
\leq \left( \int_{S^n} \tilde{K}^\alpha \tilde{S}^{-\beta} \, d\sigma_S \right)^{\frac{\beta-1}{\beta}} \left( \int_{S^n} \frac{1}{\tilde{K}^{\alpha(\beta-1)}} \, d\sigma_S \right)^{\frac{1}{\beta}}
\]

(4.8)
for $\alpha(\beta - 1) = 1 - \alpha$, that is, $\beta = \frac{1}{\alpha}$ from Hölder inequality, which implies

$$\int_{S^n} \frac{1}{\tilde{S}^{\frac{1}{\alpha} - 1}} d\sigma_{S^n} \leq \left( \int_{S^n} \tilde{K}^{\alpha} \tilde{S}^{-\alpha} d\sigma_{S^n} \right)^{1-\alpha} \left( \int_{S^n} \frac{1}{\tilde{K}^{1-\alpha}} d\sigma_{S^n} \right)^{\alpha}. \tag{4.9}$$

We also have

$$\int_{S^n} \frac{1}{\tilde{K}^{1-\alpha}} d\sigma_{S^n} = \int_{S^n} \left( \frac{\tilde{S}}{\tilde{K}} \right)^{1-\alpha} d\sigma_{S^n}, \tag{4.10}$$

now from (4.8) and (4.10), we get

$$\left( \int_{S^n} \frac{1}{\tilde{S}^{\frac{1}{\alpha} - 1}} d\sigma_{S^n} \right) \left( \int_{S^n} \frac{1}{\tilde{K}^{1-\alpha}} d\sigma_{S^n} \right)^{1-\alpha} \leq \left( \int_{S^n} \tilde{K}^{\alpha} \tilde{S}^{-\alpha} d\sigma_{S^n} \right)^{1-\alpha} \left( \int_{S^n} \frac{1}{\tilde{K}^{1-\alpha}} d\sigma_{S^n} \right)^{\alpha}, \tag{4.11}$$

and then

$$\left( \int_{S^n} \frac{1}{\tilde{S}^{\frac{1}{\alpha} - 1}} d\sigma_{S^n} \right) \left( \int_{S^n} \frac{1}{\tilde{K}^{1-\alpha}} d\sigma_{S^n} \right) \leq \left( \int_{S^n} \tilde{K}^{\alpha} \tilde{S}^{-\alpha} d\sigma_{S^n} \right) \left( \int_{S^n} \frac{\tilde{S}}{\tilde{K}} d\sigma_{S^n} \right) = (n + 1) \left( \int_{S^n} \tilde{K}^{\alpha} \tilde{S}^{-\alpha} d\sigma_{S^n} \right), \tag{4.12}$$

since the normalized volume $\tilde{V}(\tau) = \frac{1}{n + 1} \left( \int_{S^n} \frac{\tilde{S}}{\tilde{K}} d\sigma_{S^n} \right) = 1$. After all, the last inequality (4.11) completes the proof of the desired result.

Case 2. Assume $\alpha = 1$.

Since $\tilde{S}$ satisfies the equation $\tilde{S}_\tau = \frac{\tilde{K}}{|S^n|} + \frac{1}{n+1} \tilde{S}$ where $|S^n|$ means the volume of $S^n$, $\frac{\partial I(\tau)}{\partial \tau} = \int_{S^n} \tilde{S} \tilde{S}_\tau d\sigma_{S^n} \leq 0$ is equivalent to

$$\frac{|S^n|^2}{n + 1} \leq \int_{S^n} \tilde{K} \tilde{S} d\sigma_{S^n}. \tag{4.13}$$

Then, we know that

$$|S^n| \leq \left( \int_{S^n} \frac{\tilde{K}}{\tilde{S}} d\sigma_{S^n} \right)^{\frac{1}{2}} \left( \int_{S^n} \frac{\tilde{S}}{\tilde{K}} d\sigma_{S^n} \right)^{\frac{1}{2}} = (n + 1)^{\frac{1}{2}} \left( \int_{S^n} \frac{\tilde{K}}{\tilde{S}} d\sigma_{S^n} \right)^{\frac{1}{2}}.$$
from Hölder inequality and $\tilde{V}(\tau) = 1$. And this implies (4.12) directly. In addition, the equality in (4.3) holds if and only if the equalities hold in (4.9) and (4.13), which implies the equation $\tilde{K}^\alpha = C\tilde{S}$ a.e. for some positive constant $C$. 

We can observe that $\tilde{I}$ is bounded below from $[F]$ for $\alpha = 1$ and $\tilde{I} \geq 0$ for $\alpha \neq 0$. Previous Lemma 4.3 for the evolution equation of $\tilde{I}$ gives us the following convergence.

**Corollary 4.4.** For the integral quantity $\tilde{I}(\tau)$ given by (4.3), we have
\[
\lim_{\tau \to \infty} \tilde{I}(\tau) = \tilde{I}_0
\]
for some constant $\tilde{I}_0$, moreover
\[
\lim_{\tau \to \infty} \frac{\partial}{\partial \tau} \tilde{I}(\tau) = 0.
\]

**Lemma 4.5.** Let us assume that $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are $n$-dimensional hypersurfaces embedded in $\mathbb{R}^{n+1}$ and monotone quantities of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are $\tilde{I}_1$ and $\tilde{I}_2$, respectively. If $\tilde{\Sigma}_1 \subset \tilde{\Sigma}_2$, then we have
\[
\tilde{I}_1 \leq \tilde{I}_2.
\]

**Proof.** Let us $\tilde{S}_1(z)$ and $\tilde{S}_2(z)$ be the support functions of $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$, respectively. We know that if $\tilde{\Sigma}_1 \subset \tilde{\Sigma}_2$, then $\tilde{S}_1(z) \leq \tilde{S}_2(z)$. Then by definition of $\tilde{I}$, we have
\[
\tilde{I}_{-1} \geq \int_{\mathbb{R}^n} \frac{1}{\tilde{S}_1^{\frac{1}{\alpha} - 1}} d\sigma_{\alpha} = \int_{\mathbb{R}^n} \frac{1}{\tilde{S}_2^{\frac{1}{\alpha} - 1}} d\sigma_{\alpha} = \int_{\mathbb{R}^n} \frac{1}{\tilde{S}_2^{\frac{1}{\alpha} - 1}} d\sigma_{\alpha} = \tilde{I}_{-1}^{-1} \quad \text{for} \quad z \in \mathbb{S}^n.
\]

Now we shall show that $\tilde{\Sigma}(\tau)$ has a finite width.

**Lemma 4.6.** Let us consider an ellipsoid $E(\tau)$ such that $r_{\min}(\tau) = a$ half of minor axis and $r_{\max}(\tau) = a$ half of major axis. Assume that $E(\tau)$ has the fixed volume $V(\tau)$. If $r_{\max}(\tau)$ goes to infinite, then $\tilde{I}(\tau)$ is also infinite.
Proof. Set \( r_1 \cdots r_{n+1} = C \) where \( C \) is some positive constant. The equation for ellipsoid is

\[
g : \frac{x_1^2}{r_1^2} + \cdots + \frac{x_{n+1}^2}{r_{n+1}^2} = 1
\]

where \( r_1 = r_{\text{min}}, \ r_{n+1} = r_{\text{max}} \) and \( r_1 \leq r_2 \leq \cdots \leq r_{n+1} \). Then an ellipsoid can be parameterized by:

\[
X = (r_1q_1, \cdots, r_{n+1}q_{n+1})
\]

where \( q = (q_1, \cdots, q_{n+1}) \in S^n \). We also can obtain a normal vector \( N = \frac{1}{2} \nabla g = (\frac{x_1}{r_1^2}, \cdots, \frac{x_{n+1}}{r_{n+1}^2}) \), a unit normal vector \( \nu = \frac{N}{||N||} \), and the support function \( \tilde{S} = \tilde{X} \cdot \tilde{\nu} = \frac{1}{||N||} \). Now we have \( \frac{x_i}{r_i^2} = N_i = ||N||v_i = ||N||z_i = \frac{1}{S}z_i \), and then

\[
\frac{x_i}{r_i} = \frac{r_i}{S}z_i.
\]

Since \( 1 = \frac{x_1^2}{r_1^2} + \cdots + \frac{x_{n+1}^2}{r_{n+1}^2} = \frac{r_1^2}{S^2}z_1^2 + \cdots + \frac{r_{n+1}^2}{S^2}z_{n+1}^2 \), we get

\[
\tilde{S}^2 = r_1^2z_1^2 + \cdots + r_{n+1}^2z_{n+1}^2
\]

and we also have

\[
\tilde{I}^{-1} = \int_{S^n} \frac{1}{\tilde{S}^{n-1}} d\sigma_{S^n} = \int_{S^n} \frac{1}{\sqrt{r_1^2z_1^2 + \cdots + r_{n+1}^2z_{n+1}^2}} d\sigma_{S^n}.
\]

We consider the following case in general: there is \( 1 \leq k \leq n+1 \) such that \( r_{n+1} \geq \cdots \geq r_k \gg r_{k+1} \geq \cdots \geq r_1 \) with \( r_1 \cdots r_{n+1} = C \), where \( C \) is some positive constant. Then we have

\[
C_1r_{n+1}^{\frac{1}{n+1}} \leq \int_{S^n \cap \{ \frac{1}{2} \leq z_{n+1} \leq 1 \}} \frac{1}{\sqrt{n+1} \sqrt{\frac{r_1^2}{n+1}z_1^2 + \cdots + \frac{r_{n+1}^2}{n+1}z_{n+1}^2}} d\sigma_{S^n} \leq \tilde{I}^{-1} \leq \int_{S^n} \frac{1}{\sqrt{r_1^2z_1^2 + \cdots + r_{n+1}^2z_{n+1}^2}} d\sigma_{S^n} \leq C_2r_{n+1}^{\frac{1}{n+1}}.
\]
where $C_1$ and $C_2$ are positive constants. Since $C r_{n+1}^{1-\frac{1}{n+1}}$ goes to zero for $\alpha < 1$ as $r_{\text{max}}(\tau)$ goes to infinity, $\tilde{I}(\tau)$ is also infinite. Similarly, for $\alpha = 1$, since 

$$c_1 \log r_{n+1} \leq \tilde{I} = \int_{S^n} \log \tilde{S} d\sigma_{S^n} \leq c_2 \log r_{n+1}$$

for some positive constant $c_1$ and $c_2$, $\tilde{I}(\tau) \to \infty$ as $r_{\text{max}}(\tau) \to \infty$. □

Now we shall introduce one theorem called John’s Theorem.

**Theorem 4.7** (John’s Theorem, [B]). Let $K$ be a convex body in $\mathbb{R}^n$. Then there exists an unique ellipsoid $E$ of maximal volume which is contained in each $K$. This ellipsoid $E$ is $B^*_n = \left\{ x \in \mathbb{R}^n : \sum_{1}^{n} x_i^2 \leq 1 \right\}$ if and only if the following conditions are fulfilled:

1. $B^*_n \subset K$
2. there are positive numbers $(c_i)_i^m$ and Euclidean unit vectors $(u_i)_i^m$ on the boundary of $K$ such that $\sum_{i=1}^{m} c_i u_i = 0$ and $\sum_{i=1}^{m} c_i (x, u_i)^2 = |x|^2$ for all $x \in \mathbb{R}^n$.

We define the width of the convex surface by the function $\bar{w}(z) = \tilde{S}(z) + \tilde{S}(-z)$ for $z \in S^n$ and let $\bar{w}_{\text{max}} = \max_{z \in S^n} \bar{w}(z)$ and $\bar{w}_{\text{min}} = \min_{z \in S^n} \bar{w}(z)$. Similarly, set $\tilde{S}_{\text{max}} = \max_{z \in S^n} \tilde{S}(z)$ and $\tilde{S}_{\text{min}} = \min_{z \in S^n} \tilde{S}(z)$. Then we have the following.

**Corollary 4.8.** For the rescaled hypersurface $\tilde{\Sigma}$ with the normalized volume, there exist some positive constants $0 < c \leq C < \infty$ such that

$$c \leq \bar{w}_{\text{min}} \leq \bar{w}_{\text{max}} \leq C$$

for all $\tau \in [0, \infty)$. □

**Proof.** We know that there exists an unique ellipsoid $E_n$ of maximal volume enclosed by the given convex body $\tilde{\Sigma}$ by Theorem 4.7. Thus we can set up $\tilde{\Sigma}$ between two ellipsoids by using an affine transformation, in other words,

$$E_n \subset \tilde{\Sigma} \subset \sqrt{n} E_n.$$ 

Then if the maximum radius of ellipsoid $E_n$ is infinite, the monotone quantity $\tilde{I}$ for $E_n$ is also infinite by Lemma 4.6. This fact and Lemma 4.5 give us $\tilde{\Sigma}$ doesn’t have the finite monotone quantity $\tilde{I}$. It is a contradiction to Corollary 4.4. Then this implies the desired conclusion for the rescaled hypersurface $\tilde{\Sigma}$ with the normalized volume. □
Corollary 4.9. For the rescaled hypersurface \( \tilde{\Sigma} \) with the normalized volume, we have

\[
\tilde{c} \leq \tilde{S}_{\text{min}} \leq \tilde{S}_{\text{max}} \leq \tilde{C}
\]

for some constants \( 0 < \tilde{c} \leq \tilde{C} < \infty \) and all \( \tau \in [0, \infty) \).

**Proof.** From Corollary 4.8 we get \( \tilde{S}_{\text{max}} \leq \tilde{C} \) for some positive constant \( \tilde{C} \), which implies \( \tilde{S}_{\text{min}} \geq \tilde{c} > 0 \) for some constant \( \tilde{c} \) since \( \tilde{V}(\tau) = 1 \).

\[ \square \]

Lemma 4.10.

If \( \tilde{\Sigma}_0 \) is strictly convex, then there is a constant \( C > 0 \) such that

\[
\sup_{z \in S^n, 0 \leq \tau} \tilde{K}^\alpha(z, \tau) \leq C = \max \left( \sup_{z \in S^n} \tilde{K}^\alpha(z, 0), \left( \frac{n\alpha + 1}{n\tilde{\rho}_0} \right)^{\frac{1}{\alpha}} \right)
\]

where \( \tilde{\rho}_0 = \frac{1}{4} \tilde{w}_{\text{min}} \).

**Proof.** From the evolution equation of \( \tilde{K}^\alpha \), we have

\[
\frac{\partial \tilde{K}^\alpha}{\partial \tau} = \frac{\alpha}{\tilde{\eta}} \tilde{K}^\alpha + \frac{\alpha}{\tilde{\eta}} \tilde{K}^{2\alpha} \tilde{H} - \frac{n\alpha}{n + 1} \tilde{K}^\alpha
\]

where \( \tilde{\eta} = \tilde{K}^\alpha (\tilde{H}^{-1})^i \nabla_j \). By Corollary 4.8 we can consider \( \tilde{\rho}_0 = \frac{1}{4} \tilde{w}_{\text{min}} \) and then apply the maximum principle to the function \( \tilde{\phi} = \frac{\tilde{K}^\alpha}{\tilde{S} - \tilde{\rho}_0} \). Let us assume that the maximum of \( \tilde{\phi} \) is achieved at the interior point \( \tilde{P}_0 \) of \( \tilde{X} \). Then we have the following properties

\[
\tilde{\phi}_\tau \geq 0, \quad \nabla_i \tilde{\phi} = 0 \quad \text{and} \quad \nabla_i \nabla_j \tilde{\phi} \leq 0
\]

at \( \tilde{P}_0 \). Using the evolution equations of \( \tilde{K}^\alpha \) and \( \tilde{S} \) and calculating by the similar way to Lemma 3.2 implies

\[
0 \leq \frac{\alpha \tilde{K}^{2\alpha}(n - \tilde{S}\tilde{H})}{\tilde{\eta}(\tilde{S} - \tilde{\rho}_0)} + \frac{\alpha}{\tilde{\eta}} \tilde{K}^{2\alpha} \tilde{H} + \frac{\tilde{K}^{2\alpha}}{\tilde{\eta}(\tilde{S} - \tilde{\rho}_0)} - \frac{n\alpha \tilde{K}^\alpha}{n + 1} - \frac{\tilde{K}^\alpha \tilde{S}}{(n + 1)(\tilde{S} - \tilde{\rho}_0)}
\]

at \( \tilde{P}_0 \), which gives us the following

\[
0 \leq (n\alpha + 1) - \alpha \tilde{\rho}_0 \tilde{H} + 1.
\]

When we follow the same line of the last argument in Lemma 3.2 we can get

\[
\sup_{z \in S^n, 0 \leq \tau} \tilde{K}^\alpha(z, \tau) \leq C = \max \left( \sup_{z \in S^n} \tilde{K}^\alpha(z, 0), \left( \frac{n\alpha + 1}{n\alpha\tilde{\rho}_0} \right)^{\frac{1}{\alpha}} \right).
\]
Lemma 4.11. There is a uniform constant $0 < \Lambda < \infty$ such that

(i) $\frac{1}{\Lambda} \leq \tilde{S} \leq \Lambda$,

(ii) $\frac{1}{\Lambda} \leq \tilde{\eta} \leq \Lambda$ and

(iii) $\frac{1}{\Lambda^u} \leq \tilde{K} \leq \Lambda^u$.

Proof. (i) $\frac{1}{\Lambda_1} \leq \tilde{S} \leq \Lambda_1$ comes from Corollary 4.9 for some $\Lambda_1 > 0$.

(ii) From Lemma 4.10, we can derive that $\tilde{\eta}(\tau) \geq \frac{1}{\Lambda}$ for some positive constant $\Lambda_1 > C \frac{1}{\lambda \bar{a} \lambda} |S^n|^{-1}$, where $|S^n|$ is the volume of $S^n$ and $C$ is the upper bound of $\bar{K}^a$. In addition, by Hölder inequality and $\bar{V} = 1$, we have

\[
\tilde{\eta} = \int_{S^n} \bar{K}^{\alpha - 1}d\sigma_{S} \leq \left( \int_{S^n} \frac{1}{K} d\sigma_{S} \right)^{1-\alpha} |S^n|^{\alpha} \leq \left( (n+1) \Lambda_1 \right)^{1-\alpha} |S^n|^\alpha < \Lambda_u
\]

for some positive constant $\Lambda_u$. Then we get $\frac{1}{\Lambda_2} \leq \tilde{\eta} \leq \Lambda_2$ by selecting $\Lambda_2 = \max (\Lambda_1, \Lambda_u)$.

(iii) Let us consider the evolution of $\tilde{S} = \mu \tilde{S}$ for $\mu > 0$. Let $\bar{K}$ and $\bar{H}$ be Gauss curvature and Mean curvature of the hypersurface given by a support function $\tilde{S}$, respectively. Then $\bar{K} = \frac{1}{\mu^a} \tilde{K}$, $\bar{H} = \frac{1}{\mu} \tilde{H}$, $-(h^{-1})^{ij} = \frac{1}{\mu} (h^{-1})^{ij}$, and $\bar{\eta} = \mu^{(1-a)\alpha} \tilde{\eta}$. Let $\bar{Z}(\tau) = \inf_{z \in S^n} \bar{K}(z, \tau)$. Then we assume that the interior minimum of $\bar{Z}(\tau)$ is achieved at $P_0 = (z_0, \tau_0)$. From the evolution equation of $\bar{Z}(\tau)$, we have, at $P_0$,

\[
\frac{\partial \bar{Z}}{\partial \tau} = \frac{\alpha \mu^{n+1}}{\eta} \bar{Z}^{(h^{-1})^{ij}} \nabla_i \nabla_j \bar{Z} + \frac{\alpha (\alpha - 1) \mu^{n+1}}{\eta} \bar{Z}^{(h^{-1})^{ij}} \nabla_i \nabla_j \bar{Z} + \frac{\mu^{n+1}}{\eta} \bar{Z} \bar{H} - \frac{n}{n+1} \bar{Z}
\]

\[
\geq \frac{\mu^{n+1}}{\eta} \bar{Z} \bar{H} - \frac{n}{n+1} \bar{Z}
\]

\[
\geq \frac{n \mu^{n+1}}{\eta} \bar{Z}^{1+1/n} - \frac{n}{n+1} \bar{Z}
\]

\[
\geq n \bar{Z} \left( \frac{\mu^{n+1}}{\Lambda_2} \bar{Z}^{1/n} - \frac{1}{n+1} \right)
\]

where $\bar{\Lambda}_2 = \mu^{n(1-\alpha)} \Lambda_2$. Set $Q(\tau) = \frac{\mu^{n+1}}{\Lambda_2} \bar{Z}^{1/n}(\tau) - \frac{1}{n+1}$ and choose $\mu > \left( \frac{\Lambda_2}{n+1} \right)^{\frac{1}{n+1}} (\bar{Z}(0))^{-\frac{1}{n+1}}$ for $\bar{Z}(\tau) = \inf_{z \in S^n} \bar{K}(z, \tau)$ and $\bar{Z}(\tau) = \frac{1}{\mu} \bar{Z}(\tau)$, which tells us $Q(0) > 0$. Then the
evolution equation of $Q(\tau)$ is
\[
\frac{\partial Q}{\partial \tau} = \frac{\alpha \mu^{n+1}}{\eta} \nabla_i (\alpha h^{-1}) \nabla_j Q + \left( \alpha - \frac{1}{n} \right) \frac{\alpha n \Lambda_2}{\eta} \nabla_i Q \nabla_j Q
\]
\[
\geq \frac{\Lambda_2^{-1}}{\mu^{(n+1)(n-1)}} \left( Q + \frac{1}{n+1} \right)^n \left( \frac{\mu^{n+1}}{n\eta} H - \frac{1}{n+1} \right)
\]
\[
\geq \frac{\Lambda_2^{-1}}{\mu^{(n+1)(n-1)}} \left( Q + \frac{1}{n+1} \right)^n Q
\]
at the interior minimum point since $n \tilde{Z}^i \leq n \tilde{K}^i \leq H$. By the maximum principle, we have
\[
Q(\tau) \geq Q(0) > 0
\]
for all $\tau > 0$, which implies $\frac{\partial Z}{\partial \tau} > 0$ at $\tilde{P}_0$ and then it gives us contradiction. Hence we obtain
\[
\inf_{z \in S} \tilde{K}(z, \tau) \geq \inf_{z \in S} \tilde{K}(z, 0) > 0
\]
and also we have the desired result $\inf_{z \in S^m} \tilde{K}(z, \tau) \geq \inf_{z \in S^m} \tilde{K}(z, 0) > 0$ for all $\tau$. Combining Lemma 4.10 implies $\frac{1}{\Lambda^3} \leq \tilde{K} \leq \Lambda^3$ for some positive constant $\Lambda_3$. Now we select $\Lambda = \max_{i=1,2,3} \Lambda_i$.

To observe the regularity of the solution around the maximal time $T^*$, let us consider the evolution equation (4.2). Then the evolution equation for $\tilde{S}$ is
\[
(4.15) \quad \frac{\partial \tilde{S}}{\partial \tau} = -\frac{\tilde{K}^i}{\tilde{\eta}} + \frac{1}{n+1} \tilde{S}
\]
and also follows
\[
(4.16) \quad \left( \frac{1}{n+1} \tilde{S} - \tilde{S}_i \right) \left[ \det (\nabla_i \nabla_j \tilde{S} + \tilde{S} \delta_{ij}) \right]^{1/3} = \frac{1}{\tilde{\eta}}.
\]
Now we shall derive $C^{1,1}$-estimate for the solution of (4.16) as in [GH] and [GS].

**Lemma 4.12.** Suppose that $\tilde{S} \in C^4$ is a solution of the equation (4.16). Then we have
\[
|\nabla_2 \tilde{S}| \leq C \quad \text{on} \quad S_\eta \times [0, \infty)
\]
where $C$ is a positive constant depending on $\tilde{S}$ and the first derivative of $\tilde{S}$ in time and space.
In addition, we get
\[ (4.17) \]
where \( \mu \) and \( \rho \) are positive constants. If the maximum of \( v \) achieved at the initial time, we have done. So we assume that \( v \) has its space-time maximum at some interior point \( P_0 = (z_0, \tau_0) \) and for some unit vector \( \zeta \). We can assume \( \zeta = (1, 0, \ldots, 0) \) by choosing an orthonormal frame about \( z_0 \) and the matrix \( \{ \nabla_i \nabla_j \tilde{S}(z_0) \} \) is diagonal. Then
\[ (4.18) \]
\[
\begin{align*}
\nabla_i \nabla_j \tilde{S}(z_0) &= \left( \frac{1}{2} \mu \left| \nabla_i \tilde{S}(z_0) \right|^2 - \rho \tilde{S}(z_0) \right), \\
\end{align*}
\]

Let \( \mathcal{L} \) be the linearized operator at \( P_0 \)
\[
\mathcal{L} = \frac{1}{(\tilde{S} \tau - \frac{1}{n+1} \tilde{S})(P_0)} \frac{\partial}{\partial \tau} + a F_{ij} (\nabla_k \nabla_l \tilde{S} + \tilde{S} \delta_{kl}) \nabla_i \nabla_j.
\]

Then \( \{ \nabla_k \nabla_l \tilde{S} + \tilde{S} \delta_{kl} \} \) is diagonal.
We know that for \( F(M) = \log(\det M) \) where \( M \) is a positive definite matrix,
\[
(F_{ij}) = \frac{\partial F}{\partial M_{ij}} = M^{-1} \quad \text{and} \quad \frac{\partial^2 F}{\partial M_{ij} \partial M_{kl}} = F_{ij,kl} = -F_{ik} F_{jl}.
\]

Let
\[ w = \log v(z, \tau) = \log |\tilde{S}(z, \tau)| + \log \left( \nabla_1 \nabla_1 \tilde{S}(z, \tau) + \tilde{S}(z, \tau) \right) + \frac{1}{2} \mu \left| \nabla_1 \tilde{S}(z, \tau) \right|^2 - \rho \tilde{S}(z, \tau) \]
which also attains its maximum at \( P_0 \), so \( \nabla \nabla w(P_0) = 0, \nabla_i \nabla_j w(P_0) \leq 0, \) and \( \nabla v(P_0) \geq 0 \).
Since \( \nabla_i \nabla_j \tilde{S} + \tilde{S} \delta_{ij} > 0 \) from the strictly convexity of \( \tilde{S} \), \( (F_{ij}((\nabla_i \nabla_j \tilde{S} + \tilde{S} \delta_{ij})(z_0))) \) is diagonal, so
\[
\mathcal{L}(w)(P_0) = \frac{1}{(\tilde{S} \tau - \frac{1}{n+1} \tilde{S})(P_0)} \frac{\partial w}{\partial \tau}(P_0) + a F_{ii} ((\nabla_i \nabla_i \tilde{S} + \tilde{S} \delta_{ii})(P_0)) \nabla_i \nabla_j w(P_0) \leq 0.
\]

From now on, we will use the notation \( \nabla_{ij} \) in place of \( \nabla_i \nabla_j \) for convenience.
We have that at \( P_0 \)
\[
(4.17) \quad \nabla_i \nabla_i w = \frac{\nabla_i \tilde{S}}{\tilde{S}} + \frac{\nabla_i \nabla_i \tilde{S} + \nabla_i \tilde{S}}{\nabla_{ii} \tilde{S} + \tilde{S}} - \rho \nabla_i \tilde{S} = 0 \quad \text{for } i = 2, \ldots, n.
\]

In addition, we get
\[ (4.18) \]
\[
\begin{align*}
\nabla_i \nabla_i \nabla_i \nabla_i w &= \frac{\nabla_i \tilde{S}}{\tilde{S}} - \frac{\left( \frac{\nabla_i \tilde{S}}{\tilde{S}} \right)^2}{\tilde{S}^2} + \frac{\nabla_i \nabla_i \tilde{S} + \nabla_i \tilde{S}}{\nabla_{ii} \tilde{S} + \tilde{S}} - \frac{\left( \nabla_i \nabla_i \tilde{S} + \nabla_i \tilde{S} \right)^2}{(\nabla_{ii} \tilde{S} + \tilde{S})^2} + \mu (\nabla_{ii} \tilde{S})^2 + \mu \nabla_i \tilde{S} \nabla_{ii} \tilde{S} - \rho \nabla_i \tilde{S} \\
&\leq 0 \quad \text{for all } i,
\end{align*}
\]
After using the formulas (4.21)-(4.24) and the following properties

(4.22)

\[ \frac{1}{n+1} \nabla_{11} \hat{S} - \nabla_{11} \hat{S}_\tau \quad \text{at } P_0. \]

We use the properties of covariant derivatives:

(4.23)

\[ \nabla_{kji} \hat{S} = \nabla_{ijk} \hat{S} + \delta_{ik} \nabla_{j} \hat{S} - \delta_{ij} \nabla_{k} \hat{S} \]

and

(4.24)

\[ \nabla_{lkji} \hat{S} = \nabla_{lijk} \hat{S} + 2 \delta_{kl} \nabla_{j} \hat{S} - 2 \delta_{ij} \nabla_{lk} \hat{S} + \delta_{il} \nabla_{jk} \hat{S} - \delta_{kj} \nabla_{lk} \hat{S}. \]

After using the formulas (4.21)-(4.24) and the following properties

\[ \frac{\alpha(\nabla_{ij} \hat{S} + \delta_{ij} \nabla_i \hat{S})^2}{(\nabla_i \hat{S} + \hat{S})(\nabla_j \hat{S} + \hat{S})} = \frac{\alpha(\nabla_{i1} \hat{S} + \nabla_i \hat{S})^2}{(\nabla_i \hat{S} + \hat{S})(\nabla_1 \hat{S} + \hat{S})} + \sum_{i=1}^{n} \sum_{j=2}^{n} \frac{\alpha(\nabla_{ij} \hat{S})^2}{(\nabla_i \hat{S} + \hat{S})(\nabla_j \hat{S} + \hat{S})} \]
and \( \frac{\alpha \mu (\nabla_{11} S + \bar{S}) \nabla_1 S \delta_{i1} \nabla_1 S}{\nabla_i S + \bar{S}} = \alpha \mu (\nabla_1 S)^2 \) and several computations, (4.20) will be simplified by

\[
(4.25)
\]
\[
0 \geq - \frac{\tilde{S}_\tau (\nabla_{11} \tilde{S} + \bar{S})}{\tilde{S} (\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau)} - \frac{\tilde{S}_\tau + \nabla_{11} \tilde{S}}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{(\frac{1}{n+1} \nabla_{11} S - \nabla_{11} \tilde{S}_\tau)^2}{(\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau)^2} + \frac{\alpha(\nabla_{11} S - \nabla_{1i} S)}{\nabla_{1i} S + \bar{S}}
\]
\[
+ \sum_{i=1}^{n} \sum_{j=2}^{n} \frac{\alpha(\nabla_{ij} S)^2}{(\nabla_{ij} S + \bar{S})(\nabla_{ij} S + \bar{S})} - \frac{\alpha(\nabla_{ij} S)(\nabla_{1i} S + \bar{S})}{\nabla_{ij} S + \bar{S}} - \frac{\alpha(\nabla_{ij} S)}{\nabla_{ij} S + \bar{S}}
\]
\[
- \frac{\mu(\nabla_{11} S + \bar{S})(\nabla_{1i} S)^2}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{\rho(\nabla_{11} S + \bar{S}) \tilde{S}_\tau}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{\alpha \rho(\nabla_{11} S + \bar{S})(\nabla_{1i} S)^2}{\nabla_{1i} S + \bar{S}} - \frac{\alpha \rho(\nabla_{11} S + \bar{S}) \nabla_{1i} S}{\nabla_{1i} S + \bar{S}}.
\]

In addition, since

\[
(4.26)
\]
\[
\sum_{i=1}^{n} \sum_{j=2}^{n} \frac{\alpha(\nabla_{ij} S)^2}{(\nabla_{ij} S + \bar{S})(\nabla_{ij} S + \bar{S})} = \sum_{i=2}^{n} \sum_{j=2}^{n} \frac{2 \alpha \rho(\nabla_{1i} S) \nabla_i S}{\nabla_{ij} S + \bar{S}} - \sum_{i=2}^{n} \frac{\alpha \rho(\nabla_{1i} S)(\nabla_{ij} S)^2}{\nabla_{ij} S + \bar{S}} - \frac{\alpha(\nabla_{ij} S)^2}{\nabla_{ij} S + \bar{S}}
\]

from (4.23) and (4.17), we have

\[
(4.27)
\]
\[
0 \geq - \frac{\tilde{S}_\tau (\nabla_{11} \tilde{S} + \bar{S})}{\tilde{S} (\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau)} - \frac{\tilde{S}_\tau + \nabla_{11} \tilde{S}}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{(\frac{1}{n+1} \nabla_{11} S - \nabla_{11} \tilde{S}_\tau)^2}{(\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau)^2} + \frac{\alpha(\nabla_{11} S - \nabla_{1i} S)}{\nabla_{1i} S + \bar{S}}
\]
\[
+ \sum_{i=2}^{n} \frac{2 \alpha \rho(\nabla_{1i} S) \nabla_i S}{\nabla_{ij} S + \bar{S}} - \sum_{i=2}^{n} \frac{\alpha \rho(\nabla_{1i} S)(\nabla_{ij} S)^2}{\nabla_{ij} S + \bar{S}} - \frac{\alpha(\nabla_{ij} S)^2}{\nabla_{ij} S + \bar{S}} - \frac{\alpha \mu(\nabla_{1i} S)^2}{\nabla_{1i} S + \bar{S}}
\]
\[
- \frac{\mu(\nabla_{11} S + \bar{S})(\nabla_{1i} S)^2}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{\rho(\nabla_{11} S + \bar{S}) \tilde{S}_\tau}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{\alpha \rho(\nabla_{11} S + \bar{S}) \nabla_{1i} S}{\nabla_{1i} S + \bar{S}} - \frac{\alpha \rho(\nabla_{11} S + \bar{S}) \nabla_{1i} S}{\nabla_{1i} S + \bar{S}}.
\]
Let \( \gamma_i = \nabla_{\tilde{\mathcal{H}}} \tilde{S} + \tilde{S} \). Then (4.27) will be

\[
0 \geq -\alpha - \frac{\tilde{S}_\tau \gamma_1}{\tilde{S} \left( \frac{1}{n+1} \tilde{S} - \tilde{S}_\tau \right)} - \frac{\gamma_1}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{\alpha \gamma_1}{\tilde{S}} + \sum_{i=2}^{n} \alpha \rho \left( \nabla_{\tilde{\mathcal{H}}} \tilde{S} \right)^2 \frac{\gamma_1}{\gamma_i} \left( \rho - \frac{2}{\tilde{S}} \right) - \frac{\alpha \left( \nabla_{\tilde{\mathcal{H}}} \tilde{S} \right)^2}{\tilde{S}^2}
\]

\[
- \alpha \mu \left( \nabla_{\tilde{\mathcal{H}}} \tilde{S} \right)^2 - \frac{\mu \tilde{S}_\tau \gamma_1}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \frac{\rho \tilde{S}_\tau \gamma_1}{\frac{1}{n+1} \tilde{S} - \tilde{S}_\tau} + \alpha \mu \gamma_1^2 - 2 \alpha \mu \tilde{S} \gamma_1 + \alpha \mu \tilde{S} - \alpha \rho \gamma_1 + \frac{\alpha \rho \tilde{S} \gamma_1}{\gamma_i}
\]

at \( P_0 \). We obtained the lower and upper bounds of \( \tilde{S} \) on \([0, \infty)\) in Lemma 4.11 and \( |\nabla \tilde{S}| \) also is bounded for \( i = 1, \cdots, n \) since \( \tilde{\mathcal{H}} \) is a strictly convex. In addition, since \( \frac{1}{n+1} \tilde{S} - \tilde{S}_\tau \) has the positive lower bound from Lemma 4.11, choosing \( \mu \) and \( \rho \) implies that

\[
0 \geq A \gamma_1^2 + B \gamma_1 + C_1,
\]

where \( A \) is a positive constant and \( B \) and \( C_1 \) are some constants, give us the desired result.

\[ \Box \]

**Corollary 4.13.** There exist some positive constants \( C \) such that

\[
\sup_{x \in \tilde{\mathcal{H}}, t \geq 0} \tilde{\mathcal{H}} \leq C.
\]

Moreover, \( \tilde{\lambda}_{\min} \geq C_1 > 0 \) for some constant \( C_1 \). Here \( \tilde{\lambda}_{\min} = \tilde{\lambda}_1 \leq \cdots \leq \tilde{\lambda}_n = \tilde{\lambda}_{\max} \) where \( \tilde{\lambda}_i \)'s are the eigenvalues of \( \tilde{H}' \).

Furthermore, combining Lamma 4.10 and Corollary 4.13 implies the following Corollory.

**Corollary 4.14.** All curvatures on the rescaled hypersurface \( \tilde{\Sigma} \) are bounded above and below by the uniform constants. In other words there exists some constant \( 0 < M < \infty \) such that

\[
\frac{1}{M} \leq \tilde{\lambda}_{\min} \leq \tilde{\lambda}_{\max} \leq M.
\]

5. **Existence of Solutions and Proof of Main Theorem**

5.1. **Short time existence.** Let us assume that \( \Sigma_t \) is smooth. Then we get the uniform \( C^{1,1} \) estimates of the coefficient of our equation (2.4) and this equation becomes uniformly parabolic. Thus the regularity theory of uniform parabolic equations and the application of implicit function theorem give us the short time existence as in [L].
5.2. **Long time existence.** Let \( \lambda_i \) be the eigenvalues of \( (h^i) \). We know that \( \lambda_i \) is positive by the strictly convexity. Also we have \( K = \lambda_1 \cdots \lambda_n \leq C_1 \) and \( H = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq C_2 \) from Lemma 3.2 and Lemma 3.3 where \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) and \( C_1 \) and \( C_2 \) are some positive constants. These give us, for each \( i = 1, \cdots, n, \)

\[
0 < \frac{1}{C_2} \leq \lambda_i
\]

from \( \frac{1}{\lambda_i} < \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \leq C_2 \) and also

\[
0 < \lambda_i \leq \frac{C_1}{\Pi_{j\neq i} \lambda_j} \leq C_1 C_2^{n-1},
\]

which imply there are \( 0 < \lambda \leq \Lambda < \infty \) satisfying

\[
\lambda|\xi|^2 \leq K^\alpha (h^{-1})^{ij} \xi_i \xi_j \leq \Lambda|\xi|^2.
\]

Then we know that the support function \( S(z, t) \) satisfies a uniformly parabolic equation in \( \Sigma_t \). Hence \( S(z, t) \) is \( C^{2,\gamma} \) and then \( C^\infty \) in \( \Sigma_t \) through the standard bootstrap argument using the Schauder theory. If there is a \( 0 < T_1 < T^* \) such that \( \Sigma_t \) is smooth on \([0, T_1)\) but it is not smooth after \( T_1 \). The uniform \( C^{2,\gamma} \)-estimate for \( S(z, t) \) implies \( \Sigma_{T_1} \) is \( C^{2,\gamma} \) and then we have the smooth solution \( \Sigma_{T_1} \). From the short time existence and uniqueness, \( \Sigma_t \) is \( C^\infty \) on \([0, T_1 + \delta)\). It is a contradiction. Therefore \( T_1 = T^* \) for some small \( \delta > 0 \) and there is a smooth solution \( \Sigma_t \) on \([0, T^*)\). Also the solution \( \Sigma_t \) will be strictly convex by Lemma 3.1.

**Proof of Theorem 1.1.** We have the uniform bounds of curvature and all of the higher derivatives of the second fundamental form to the rescaled manifold by Corollary 4.14 and then the equation (4.16) will be uniformly parabolic. In addition, we have \( C^{1,1} \)-regularity of the solution \( \tilde{S} \) from Lemma 4.12. By applying the Harnack inequality to the linearized equation satisfied by \( \tilde{S}_{\tau} \), we obtain that \( \tilde{S}_{\tau} \) is Hölder continuous through the similar argument as in [GH]. We can apply Evans-Krylov theorem and Schauder estimates (see [CC]) to the concave operator obtained by taking exponent \( \frac{1}{ma} \) to the equation (4.16), which implies \( C^{2,\gamma} \)-regularity of \( \tilde{S} \) for \( 0 < \gamma < 1 \). And then we have the smooth and strictly convex rescaled solution by the standard bootstrap argument using Schauder theory and Corollary 4.13. In other words, for every sequence of \( \tau_k \rightarrow \infty \), we can find a subsequence \( \tau_{k_i} \) such that
\( \tilde{S}(\cdot, \tau_k) \rightarrow \tilde{S}_*(\cdot) \). Also the integral quantity

\[
\tilde{I}(\tau) \begin{cases} 
\left( \int_{S^n} \frac{1}{\tilde{S}^{n+1}} \, d\sigma_{S^n} \right)^{\text{sign}(\alpha-1)} & \text{for } \alpha > 0 \text{ and } \alpha \neq 1, \\
\int_{S^n} \log \tilde{S} \, d\sigma_{S^n} & \text{for } \alpha = 1
\end{cases}
\]

satisfies the monotonicity \( \frac{d}{d\tau} \tilde{I}(\tau) \leq 0 \) and the equality holds if and only if \( \tilde{K}^\alpha = C \tilde{S} \) for some positive constant \( C \) holds for a choice of the origin. For the limit manifold \( \tilde{\Sigma}^*_T \) of the volume rescaled manifold \( \tilde{\Sigma}_{\tau_k} \), following the same argument as Theorem 16, \( \tilde{I}(\tau) \rightarrow -\infty \) if \( \tilde{\Sigma}^*_T \) doesn’t satisfy \( \tilde{K}^\alpha = C \tilde{S} \), a.e. for some positive constant \( C \), which implies a contradiction. Therefore the proof is complete.

□

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