On the Optimisation of the GSACA Suffix Array Construction Algorithm

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Abstract. The suffix array is arguably one of the most important data structures in sequence analysis and consequently there is a multitude of suffix sorting algorithms. However, to this date the GSACA algorithm introduced in 2015 is the only known non-recursive linear-time suffix array construction algorithm (SACA). Despite its interesting theoretical properties, there has been little effort in improving the algorithm’s sub-par real-world performance. There is a super-linear algorithm DSH which relies on the same sorting principle and is faster than DivSufSort, the fastest SACA for over a decade. This paper is concerned with analysing the sorting principle used in GSACA and DSH and exploiting its properties in order to give an optimised linear-time algorithm. Our algorithm is not only significantly faster than GSACA but also outperforms DivSufSort and DSH.

Keywords: Suffix array · suffix sorting · string algorithms.

1 Introduction

The suffix array contains the indices of all suffixes of a string arranged in lexicographical order. It is arguably one of the most important data structures in stringology, the topic of algorithms on strings and sequences. It was introduced in 1990 by Manber and Myers for on-line string searches \[10\] and has since been adopted in a wide area of applications including text indexing and compression \[13\]. Although the suffix array is conceptually very simple, constructing it efficiently is not a trivial task.

When \(n\) is the length of the input text, the suffix array can be constructed in \(O(n)\) time and \(O(1)\) additional words of working space when the alphabet is linearly-sortable (i.e. the symbols in the string can be sorted in \(O(n)\) time) \[7911\]. However, algorithms with these bounds are not always the fastest in practice. For instance, DivSufSort has been the fastest SACA for over a decade although having super-linear worst-case time complexity \[38\]. To the best of our knowledge, the currently fastest suffix sorter is libsais, which appeared as source code in February 2021 on Github \[1\] and has not been subject to peer

\[1\] https://github.com/IlyaGrebnov/libsais last accessed: August 22, 2022
review in any academic context. The author claims that libsais is an improved implementation of the SA-IS algorithm and hence has linear time complexity $O(n)$. The only non-recursive linear-time suffix sorting algorithm GSACA was introduced in 2015 by Baier and is not competitive, neither in terms of speed nor in the amount of memory consumed. Despite the new algorithm’s entirely novel approach and interesting theoretical properties, there has been little effort in optimizing it. In 2021, Bertram et al. provided a much faster SACA DSH using the same sorting principle as GSACA. Their algorithm beats DivSufSort in terms of speed, but also has super-linear time complexity.

Our Contributions We provide a linear-time SACA that relies on the same grouping principle that is employed by DSH and GSACA, but is faster than both. This is done by exploiting certain properties of Lyndon words that are not used in the other algorithms. As a result, our algorithm is more than 11% faster than DSH on real-world texts and at least 46% faster than Baier’s GSACA implementation. Although our algorithm is not on par with libsais on real-world data, it significantly improves Baier’s sorting principle and positively answers the question whether the precomputed Lyndon array can be used to accelerate GSACA (posed in [4]).

The rest of this paper is structured as follows: Section 2 introduces the definitions and notations used throughout this paper. In Section 3, the grouping principle is investigated and a description of our algorithm is provided. In Section 4, our algorithm is evaluated experimentally and compared to other relevant SACAs. Finally, Section 5 concludes this paper and provides an outlook on possible future research.

2 Preliminaries

For $i,j \in \mathbb{N}_0$ we denote the set $\{k \in \mathbb{N}_0 : i \leq k \leq j\}$ by the interval notations $[i..j] = [i..j+1] = (i-1..j+1)$. For an array $A$ we analogously denote the subarray from $i$ to $j$ by $A[i..j] = A[i..j+1] = A(i-1..j+1) = A(i-1..j) = A(i..j+1) = A[i].A[i+1]...A[j]$. We use zero-based indexing, i.e. the first entry of the array $A$ is $A[0]$. A string $S$ of length $n$ over an alphabet $\Sigma$ is a sequence of $n$ characters from $\Sigma$. We denote the length $n$ of $S$ by $|S|$ and the $i$'th symbol of $S$ by $S[i-1]$, i.e. strings are zero-indexed. Analogous to arrays, we denote the substring from $i$ to $j$ by $S[i..j] = S[i..j+1] = S(i-1..j) = S(i-1..j+1) = S[i].S[i+1]...S[j]$. For $j > i$ we let $S[i..j]$ be the empty string $\varepsilon$. The suffix $i$ of a string $S$ of length $n$ is the substring $S[i..n]$ and is denoted by $S_i$. Similarly, the substring $S[0..i]$ is a prefix of $S$. A suffix (prefix) is proper if $i > 0$ ($i + 1 < n$). For two strings $u$ and $v$ and an integer $k \geq 0$ we let $uv$ be the concatenation of $u$ and $v$ and denote the $k$-times concatenation of $u$ by $u^k$.

We assume totally ordered alphabets. This induces a total order on strings. Specifically, we say a string $S$ of length $n$ is lexicographically smaller than another string $S'$ of length $m$ if and only if there is some $\ell \leq \min\{n,m\}$ such that
The algorithm consists of the following steps.

A non-empty string $S$ is a Lyndon word if and only if $S$ is lexicographically smaller than all its proper suffixes [14]. The Lyndon prefix of $S$ is the longest prefix of $S$ that is a Lyndon word. We let $L_i$ denote the Lyndon prefix of $S_i$. Note that a string of length one is always a Lyndon word, hence the Lyndon prefix of a non-empty string is also non-empty.

In the remainder of this paper, we assume an arbitrary but fixed string $S$ of length $n > 1$ over a totally ordered alphabet $\Sigma$ with $|\Sigma| \in O(n)$. Furthermore, we assume w.l.o.g. that $S$ is null-terminated, that is $S[n-1] = \$\$ and $S[i] > \$\$ for all $i \in [0..n-1]$.

The suffix array $SA$ of $S$ is an array of length $n$ that contains the indices of the suffixes of $S$ in increasing lexicographical order. That is, $SA$ forms a permutation of $[0..n]$ and $SA[0] <_{\text{lex}} SA[1] <_{\text{lex}} \ldots <_{\text{lex}} SA[n-1]$.

We assume the RAM model of computation, that is, basic arithmetic operations can be performed in $O(1)$ time on words of length $O(\log n)$, where $n$ is the size of the input. Reading and writing an entry $A[i]$ of an array $A$ can also be performed in constant time when $i$ and $A[i]$ have length in $O(\log n)$.

**Definition 1 (pss-tree [4]).** Let $pss[i]$ be the array such that $pss[i]$ is the index of the previous smaller suffix for each $i \in [0..n]$ (or -1 if none exists). Formally, $pss[i] := \max \{ j \in [0..n] : S_j <_{\text{lex}} S_i \} \cup \{-1\}$. Note that $pss$ forms a tree with -1 as the root, in which each $i \in [-1..n]$ is represented by a node and $pss[i]$ is the parent of node $i$. We call this tree the pss-tree. Further, we impose an order on the nodes that corresponds to the order of the indices represented by the nodes. In particular, if $c_1 < c_2 < \cdots < c_k$ are the children of $i$ (i.e. $pss[c_1] = \cdots = pss[c_k] = i$), we say $c_k$ is the last child of $i$.

Analogous to $pss[i]$, we define $nss[i] := \min \{ j \in [i..n] : S_j <_{\text{lex}} S_i \}$ as the next smaller suffix of $i$. Note that $S_n = \$\$ is smaller than any non-empty suffix of $S$, hence $nss$ is well-defined.

In the rest of this paper, we use $S = \text{acedebcecece}\$ as our running example. Fig.1 shows its Lyndon prefixes and the corresponding pss-tree.

**Definition 2.** Let $P_i$ be the set of suffixes with $i$ as next smaller suffix, that is

$$P_i = \{ j \in [0..i) : nss[j] = i \}$$

For instance, in our running example we have $P_4 = \{1,3\}$ because $nss[1] = nss[3] = 4$.

### 3 GSACA

We start by giving a high level description of the sorting principle based on grouping by Baier [12]. Very basically, the suffixes are first assigned to lexicographically ordered groups, which are then refined until the suffix array emerges. The algorithm consists of the following steps.
Initialisation: Group the suffixes according to their first character.

Phase I: Refine the groups until the elements in each group have the same Lyndon prefix.

Phase II: Sort elements within groups lexicographically.

Definition 3 (Suffix Grouping, adapted from [3]). Let $S$ be a string of length $n$ and $SA$ the corresponding suffix array. A group $G$ with group context $\alpha$ is a tuple $(\alpha, g_s, g_e)$ with group start $g_s \in [0..n)$ and group end $g_e \in [g_s..n)$ such that the following properties hold:

1. All suffixes in $SA[g_s..g_e]$ share the prefix $\alpha$, i.e. for all $i \in SA[g_s..g_e]$ it holds $S_i = \alpha S_{i+|\alpha|}$.
2. $\alpha$ is a Lyndon word.

We say $i$ is in $G$ or $i$ is an element of $G$ and write $i \in G$ if and only if $i \in SA[g_s..g_e]$. A suffix grouping for $S$ is a set of groups $G_1, \ldots, G_m$, where the groups are pairwise disjoint and cover the entire suffix array. Formally, if $G_i = (g_{s,i}, g_{e,i}, |\alpha_i|)$ for all $i$, then $g_{s,1} = 0, g_{e,m} = n - 1$ and $g_{s,j} = 1 + g_{e,j-1}$ for all $j \in [2..m]$. For $i, j \in [1..m]$, $G_i$ is a lower (higher) group than $G_j$ if and only if $i < j$ ($i > j$). If all elements in a group $G$ have $\alpha$ as their Lyndon prefix then $G$ is a Lyndon group. If $G$ is not a Lyndon group, it is called preliminary. Furthermore, a suffix grouping is Lyndon if all its groups are Lyndon groups, and preliminary otherwise.
With these notions, a suffix grouping is created in the initialisation, which is then refined in Phase I until it is a Lyndon grouping, and further refined in Phase II until the suffix array emerges. Fig. 2 shows a Lyndon grouping with contexts of our running example.

In Subsections 3.1 and 3.2 we explain Phases II and I, respectively, of our suffix array construction algorithm. Phase II is described first because it is much simpler. Subsection 3.3 describes how the data structures needed for Phase I are set up.

### 3.1 Phase II

In Phase II we need to refine the Lyndon grouping obtained in Phase I into the suffix array. Let $\mathcal{G}$ be a Lyndon group with context $\alpha$ and let $i, j \in \mathcal{G}$. Since $S_i = \alpha S_{i+|\alpha|}$ and $S_j = \alpha S_{j+|\alpha|}$, we have $S_i <_{\text{lex}} S_j$ if and only if $S_{i+|\alpha|} <_{\text{lex}} S_{j+|\alpha|}$. Hence, in order to find the lexicographically smallest suffix in $\mathcal{G}$, it suffices to find the lexicographically smallest suffix $p$ in $\{i + |\alpha| : i \in \mathcal{G}\}$. Note that removing $p - |\alpha|$ from $\mathcal{G}$ and inserting it into a new group immediately preceding $\mathcal{G}$ yields a valid Lyndon grouping. We can repeat this process until each element in $\mathcal{G}$ is in its own singleton group. As $\mathcal{G}$ is Lyndon, we have $S_{k+|\alpha|} <_{\text{lex}} S_k$ for each $k \in \mathcal{G}$. Therefore, if all groups lower than $\mathcal{G}$ are singletons, $p$ can be determined by a simple scan over $\mathcal{G}$ (by determining which member of $\{i + |\alpha| : i \in \mathcal{G}\}$ is in the lowest group). Consider for instance $\mathcal{G}_4 = \langle 3, 4, |ce| \rangle$ from Fig. 2. We consider $4 + |ce| = 6$ and $10 + |ce| = 12$. Among them, 12 belongs to the lowest group, hence $S_{10}$ is lexicographically smaller than $S_4$. Thus, we know that $SA[3] = 10$, remove 10 from $\mathcal{G}_4$ and repeat the same process with the emerging group $\mathcal{G}'_4 = \langle 4, 4, |ce| \rangle$. As 4 is the only element of $\mathcal{G}'_4$ we know that $SA[4] = 4$.

If we refine the groups in lexicographically increasing order (lower to higher) as just described, each time a group $\mathcal{G}$ is processed, all groups lower than $\mathcal{G}$ are singletons. (Note that we assume $S$ to be nullterminated, thus the lexicographically smallest group is always known to be the singleton containing $n - 1$.) However, sorting groups in such a way leads to a superlinear time complexity. Bertram et al. [3] provide a fast-in-practice $O(n \log n)$ algorithm for this, broadly following the described approach.

In order to get a linear time complexity, we turn this approach on its head like Baier does [12]: Instead of repeatedly finding the next smaller suffix in a
\begin{algorithm}
\begin{algorithmic}
\State $A[0] \leftarrow n - 1$
\For{$i = 0 \rightarrow n - 1$}
\For{$j \in P_{A[i]}$}
\State Let $k$ be the start of the group containing $j$;
\State remove $j$ from its current group and put it in a new group $\langle k, k, |L_j| \rangle$
\State immediately preceding $j$’s old group;
\State $A[k] \leftarrow j$
\EndFor
\EndFor
\end{algorithmic}
\caption{Phase II of GSACA \cite{1,2}}
\end{algorithm}

Accordingly, Algorithm 1 correctly computes the suffix array from a Lyndon grouping. A formal proof of correctness is given in \cite{1,2}. Fig. 3 shows Algorithm 1 applied to our running example.

Note that each element $i \in [0..n - 1)$ has exactly one next smaller suffix, hence there is exactly one $j$ with $i \in P_j$ and thus $i$ is inserted exactly once into a new singleton group in Algorithm 1. Therefore, for each group from the Lyndon grouping obtained in Phase I, it suffices to maintain a single pointer to the current start of this group. In \cite{1}, these pointers are stored at the end of each group in $A$. This leads to them being scattered in memory, potentially harming cache performance. Instead, we store them contiguously in a separate array $C$, which improves cache locality especially when there are few groups.

Besides this minor point, there are two major differences between our Phase II and Baier’s, both are concerned with the iteration over a $P_i$-set.

The first difference is the way in which we determine the elements of $P_i$ for some $i$. The following observations immediately enable us to iterate over $P_i$.

\begin{lemma}
$P_i$ is empty if and only if $i = 0$ or $S_i \prec_{\text{lex}} S_{i-1}$. Furthermore, if $P_i \neq \emptyset$ then $i - 1 \in P_i$.
\end{lemma}
Since \( S \) is nullterminated, \( S[0] = n - 1 = 12 \). Hence we insert \( \mathcal{P}_{12} = \{0, 6, 10, 11\} \).

Next we have \( \mathcal{P}_{10} = \{7, 9\} \).

Next we have \( \mathcal{P}_8 = \{1, 3\} \).

The only remaining nonempty \( \mathcal{P} \) is \( \mathcal{P}_3 = \{2\} \) and \( \mathcal{P}_9 = \{8\} \), which are considered in that order. Inserting them gives the suffix array.

![Refining a Lyndon grouping for S = acedebceee$](image)

**Proof.** \( \mathcal{P}_0 = \emptyset \) by definition. Let \( i \in [1..n] \). If \( S_{i-1} >_{lex} S_i \) we have \( \text{nss}[i-1] = i \) and thus \( i - 1 \in \mathcal{P}_i \). Otherwise (\( S_{i-1} <_{lex} S_i \)), assume there is some \( j < i - 1 \) such that \( \text{nss}[j] = i \). By definition, \( S_j >_{lex} S_i \) and \( S_j <_{lex} S_k \) for each \( k \in (j..i) \). But by transitivity we also have \( S_j >_{lex} S_{i-1} \), which is a contradiction, hence \( \mathcal{P}_i \) must be empty.

**Lemma 3.** For some \( j \in [0..i) \), we have \( j \in \mathcal{P}_i \) if and only if \( j \)’s last child is in \( \mathcal{P}_i \), or \( j = i - 1 \) and \( S_j >_{lex} S_i \).

**Proof.** By Lemma 2 we may assume \( \mathcal{P}_i \neq \emptyset \) and \( j + 1 < i \), otherwise the claim is trivially true. If \( j \) is a leaf we have \( \text{nss}[j] = j + 1 < i \) and thus \( j \notin \mathcal{P}_i \) by definition. Hence assume \( j \) is not a leaf and has \( j' > j \) as last child, i.e. \( \text{pss}[j'] = j \) and there is no \( k > j' \) with \( \text{pss}[k] = j \). It suffices to show that \( j' \in \mathcal{P}_j \) if and only if \( j \in \mathcal{P}_i \). Note that \( \text{pss}[j'] = j \) implies \( \text{nss}[j] > j' \).

\( \iff \): From \( \text{nss}[j'] = i \) and thus \( S_j >_{lex} S_{j'} >_{lex} S_i \) (for all \( k \in (j'..i) \)) we have \( \text{nss}[j] \geq i \). Assume \( \text{nss}[j] > i \). Then \( S_j >_{lex} S_i \) and thus \( \text{pss}[i] = j \), which is a contradiction.

\( \iff \): From \( S_i <_{lex} S_j <_{lex} S_{j'} \) we have \( \text{nss}[j'] \leq i \). Assume \( \text{nss}[j'] < i \) for a contradiction. For all \( k \in (j'..i) \), \( \text{pss}[j'] = j \) implies \( S_k >_{lex} S_{j'} \). Furthermore, for all \( k \in (j'..\text{nss}[j']) \) we have \( S_k >_{lex} S_{\text{nss}[j']} \) by definition. In combination this implies \( S_k >_{lex} S_{\text{nss}[j']} \) for all \( k \in (j'..\text{nss}[j']) \). As \( \text{nss}[j] = i > \text{nss}[j'] \) we hence have \( \text{pss}[\text{nss}[j'] = j \), which is a contradiction.

Specifically, (if \( \mathcal{P}_i \) is not empty) we can iterate over \( \mathcal{P}_i \) by walking up the \( \text{pss} \)-tree starting from \( i - 1 \) and halting when we encounter a node that is not the
last child of its parent.\cite{Baier12} Baier \cite{Baier12} tests whether \( i - 1 \) (\( \text{pss}[j] \)) is in \( P_i \) by explicitly checking whether \( i - 1 \) (\( \text{pss}[j] \)) has already been written to \( A \). This is done by having an explicit marker for each suffix. Reading and writing those markers leads to bad cache performance because the accessed memory locations are hard to predict (for the CPU/compiler). Lemmata \cite{Olbrich15} and \cite{Olbrich15} enable us to avoid reading and writing those markers. In fact, in our implementation of Phase II, the array \( A \) is the only memory written to that is not always in the cache. Lemma \cite{Olbrich15} tells us whether we need to follow the \( \text{pss} \)-chain starting at \( i - 1 \) or not. Namely, this is the case if and only if \( S_{i - 1} > \text{lex} S_i \), i.e. \( i - 1 \) is a leaf in the \( \text{pss} \)-tree. This information is required when we encounter \( i \) in \( A \) during the outer for-loop in Algorithm \cite{Olbrich15} thus we mark such an entry \( i \) in \( A \) if and only if \( \text{pss}[i] = \emptyset \). Implementation-wise, we use the most significant bit (MSB) of an entry to indicate whether it is marked or not. By definition, we have \( S_{i - 1} > \text{lex} S_i \) if and only if \( \text{pss}[i] + 1 < i \). Since \( \text{pss}[i] \) must be accessed anyway when \( i \) is inserted into \( A \) (for traversing the \( \text{pss} \)-chain), we can insert \( i \) marked or unmarked into \( A \). Further, Lemma \cite{Olbrich15} implies that we must stop traversing a \( \text{pss} \)-chain when the current element is not the last child of its parent. We mark the entries in \( \text{pss} \) accordingly, also using the MSB of each entry. In the rest of this paper, we assume the \( \text{pss} \)-array to be marked in this way.

Consider for instance \( i = 6 \) in our running example. As \( 6 - 1 = 5 \) is a leaf (cf. Fig. \cite{Olbrich15}), we have \( 5 \in P_6 \). We can deduce the fact that \( 5 \) is indeed a leaf from \( \text{pss}[6] = 0 < 5 \) alone. If we had \( \text{pss}[6] = 5 \) instead, we would have \( \text{nss}[5] \neq 6 \) and thus \( 5 \notin P_6 \). Further, \( 5 \) is the last child of \( \text{pss}[5] = 4 \), so \( 4 \in P_6 \). Since \( 4 \) is not the last child of \( \text{pss}[4] = 0 \), we have \( P_6 = \{ 4, 5 \} \).

These optimisations unfortunately come at the cost of \( 2n \) additional bits of working memory for the markings. However, as they are integrated into \( \text{pss} \) and \( A \) there are no additional cache misses.

Let \( G[i] \) be the index of the group start pointer of \( i \)'s group in \( C \). Phase II with our first major improvement compared to Baier’s algorithm is shown in Algorithm \cite{Olbrich15}.

The second major change concerns the cache-unfriendliness of traversing the \( P_i \)-sets. This bad cache performance results from the fact that the next \( \text{pss} \)-value (and the group start pointer) cannot be fetched until the current one is in memory. Instead of traversing the \( P_i \)-sets one after another, we opt to traversing multiple such sets in a sort of breadth-first-search manner simultaneously. Specifically, we maintain a small (\( \leq 2^{10} \) elements) queue \( Q \) of elements (nodes in the \( \text{pss} \)-tree) that can currently be processed. Then we iterate over \( Q \) and process the entries one after another. Parents of last children are inserted into \( Q \) in the same order as the respective children. After each iteration, we continue to scan over the suffix array and for each encountered marked entry \( i \) insert \( i - 1 \) into \( Q \) until we either encounter an empty entry in \( A \) or \( Q \) reaches its maximum.

\footnote{Note that \( n - 1 \) is the last child of the artificial root \( -1 \). This ensures that we always halt before we actually reach the root of the \( \text{pss} \)-tree. Moreover, since the elements in \( P_i \) belong to different Lyndon groups (Corollary \cite{Olbrich15}), the order in which we process them is not important.}
Algorithm 2: Concrete implementation of Phase II of GSACA. The array $G$ maps each suffix to its Lyndon group and $C$ maps the Lyndon groups that resulted from Phase I to their current start. The correctness immediately follows from the correctness of Algorithm 1 and Lemmata 2 and 3.

capacity. This is repeated until the suffix array emerges. The queue size could be unlimited, but limiting it ensures that it fits into the CPU’s cache. Fig. 4 shows our Phase II on the running example and Algorithm 3 describes it formally in pseudo code. Note that this optimisation is only useful when the queue contains many elements, otherwise there is no time to prefetch the required data for an element in the queue while others are processed, and we effectively have Algorithm 1 with some additional overhead. Fortunately, in real world data this is usually the case and the small overhead for maintaining the queue is more than offset by the better cache performance (cf. Section 4).

Theorem 1. Algorithm 3 correctly computes the suffix array from a Lyndon grouping.

Proof. By Lemmata 2 and 3, Algorithms 1 and 3 are equivalent for a maximum queue size of 1. Therefore it suffices to show that the result of Algorithm 3 is independent of the queue size. Assume for a contradiction that the algorithm inserts two elements $i$ and $j$ with $S_i <_{lex} S_j$ belonging to the same Lyndon group with context $\alpha$, but in a different order as Algorithm 1 would. This can only happen if $j$ is inserted earlier than $i$. Note that, since $i$ and $j$ have the same Lyndon prefix $\alpha$, the $pss$-subtrees $T_i$ and $T_j$ rooted at $i$ and $j$, respectively, are isomorphic (see [4]). In particular, the path from the rightmost leaf in $T_i$ to $i$ has the same length as the path from the rightmost leaf in $T_j$ to $j$. Thus, $i$ and $j$ are inserted in the same order as $S_{i+|\alpha|}$ and $S_{j+|\alpha|}$ occur in the suffix array. Now the claim follows inductively.

3.2 Phase I

In Phase I, a Lyndon grouping is derived from a suffix grouping in which the group contexts have length (at least) one. That is, the suffixes are sorted and grouped by their Lyndon prefixes. Lemma 3 describes the relationship between
A ← (n - 1)⊥^{n-1}; // set A[0] = n - 1, fill the rest with "undefined"
Q ← queue containing only n - 1;
i ← 1; // current index in A
while Q is not empty do
    s ← Q.size();
    repeat s times // insert elements that are currently in the queue
        v ← Q.pop();
        if pss[v] is marked then // v is last child of pss[v]
            Q.push(pss[v]);
        end
        A[C[G[v]]] ← v; // insert v
        if pss[v] + 1 < v then mark A[C[G[v]]]; // v - 1 is leaf
            C[G[v]] ← C[G[v]] + 1; // increment start of v’s old group
        end
    end
    while Q.size() < w ∧ i < n ∧ A[i] ≠ ⊥ do // refill the queue
        if A[i] is marked then // A[i] - 1 is leaf
            Q.push(A[i] - 1);
        end
        i ← i + 1;
    end
end

**Algorithm 3:** Breadth-first approach to Phase II. The constant w is the maximum queue size.

the Lyndon prefixes and the \texttt{pss}-tree that is essential to Phase I of the grouping principle.

**Lemma 4.** Let \( c_1 < \cdots < c_k \) be the children of \( i \in [0..n) \) in the \texttt{pss}-tree as in Definition[1]. Then \( L_i \) is \( S[i] \) concatenated with the Lyndon prefixes of \( c_1, \ldots, c_k \).

More formally:

\[
L_i = S[i..\text{nss}[i]) = S[i]S[c_1..c_2] \cdots S[c_k..\text{nss}[i]) = S[i]L_{c_1} \cdots L_{c_k}
\]

**Proof.** By definition we have \( L_i = S[i..\text{nss}[i]) \). Assume \( i \) has \( k \geq 1 \) children \( c_1 < \cdots < c_k \) in the \texttt{pss}-tree (otherwise \( \text{nss}[i] = i + 1 \) and the claim is trivial). For the last child \( c_k \) we have \( \text{nss}[c_k] = \text{nss}[i] \) from Lemma[2] Let \( j \in [1..k) \) and assume \( \text{nss}[c_j] \neq c_{j+1} \). Then we have \( \text{nss}[c_j] < c_{j+1} \), otherwise \( c_{j+1} \) would be a child of \( c_j \). As we have \( S_{\text{nss}[c_j]} <_{\text{lex}} S_{c_j} \) and \( S_{c_j} <_{\text{lex}} S_{c_{j'}} \), for each \( j' \in [1..j) \) (by induction), we also have \( S_{\text{nss}[c_j]} <_{\text{lex}} S_{c_{j'}} \) for each \( i' \in (i..\text{nss}[c_j]). \) Since \( \text{nss}[i] > \text{nss}[c_j] \), \( \text{nss}[c_j] \) must be a child of \( i \) in the \texttt{pss}-tree, which is a contradiction.

We start from the initial suffix grouping in which the suffixes are grouped according to their first characters. From the relationship between the Lyndon prefixes and the \texttt{pss}-tree in Lemma[3] one can get the general idea of extending
The first step is the same as in Fig. 3. Note that \( P_0 = \emptyset \), hence 0 is not marked for further processing.

Now \( 6 - 1 = 5 \) and \( 10 - 1 = 9 \) are inserted into the queue and 6 and 10 are unmarked.

In the next step, the elements in the queue are inserted and replaced in the queue with their parents (if they are last children, which happens to be the case for 5 and 9). Note that they must be inserted in the same order as they appear in \( Q \).

Neither 4 nor 7 are the last child of their respective parent. However, we can advance the scan over \( A \) and insert \( 4 - 1 = 3 \) into \( Q \).

Next, 3 is inserted into \( A \). As 3 is the last child of 1, we insert 1 into \( Q \) and in the next step into \( A \). As 1 is not the last child of \( pss[1] = 0 \), \( Q \) is now empty.

We can continue the scan over \( A \) and insert \( 3 - 1 = 2 \) and \( 9 - 1 = 8 \) into \( Q \).

Finally, the elements in the queue can be inserted and the suffix array emerges.

Fig. 4: Refining a Lyndon grouping for \( S = abccabcbbc \) (see Fig. 2) into the suffix array using Algorithm 3. Marked entries are coloured blue while inserted but unmarked elements are coloured green. Note that the uncoloured entries are not actually present in the array \( A \) but only serve to indicate the current Lyndon grouping.

the context of a node’s group with the Lyndon prefixes of its children (in correct order) while maintaining the sorting \[1\]. Note that any node is by definition in a higher group than its parent. Also, by Lemma \[4\] the leaves of the \( pss \)-tree are already in Lyndon groups in the initial suffix grouping. Therefore, if we consider the groups in lexicographically decreasing order (i.e. higher to lower) and append the context of the current group to each parent (and insert them into new groups accordingly), each encountered group is guaranteed to be Lyndon \[1\]. Consequently, we obtain a Lyndon grouping. Fig. 5 shows this principle applied to our running example.

Formally, the suffix grouping satisfies the following property during Phase I before and after processing a group:
In the initial suffix grouping, the suffixes are grouped according to their first characters.

The first considered group contains the elements 2, 5, 8, 9 and 11 and has context e. The parents of the elements are 1, 4, 10 and 7, where the former three each have one child in the current group and the latter has two. All are in the group with context c. Thus, we first move 7 to a new group with context cee and then 1, 4 and 10 to a new group with context ce. Next the group with context d containing 3 is processed. The parent of 3 is 1 in a group with context ce, so it is moved to a new group with context ced. Note that 4 and 10 are now also in a Lyndon group (still with context ce).

The next processed group contains 7 and has context cee. The parent 6 is moved to a new group with context bcee. (As 6 is already in a singleton group, the actual grouping remains the same except for the context of 6's group.)

The next group again contains only one element, namely 1 with parent 0. Thus, 0 is put into a new group with context aced. Following that, the next group contains 4 and 10, hence their parents 0 and 6 are put into new groups with contexts acedc and bceee. Finally, the only remaining element with a non-root parent is 6 (with parent 0) in a group with context bceee. Hence, 0 is put into a Lyndon group with context acedcebceee. Afterwards, there is nothing more to do and we obtain the Lyndon grouping from Fig. 2.

Fig. 5: Refining the initial suffix grouping for $S = abccabcabc$ (see Fig. 2) into the Lyndon grouping. Elements in Lyndon groups are marked gray or green, depending on whether they have been processed already. Note that the applied procedure does not entirely correspond to our algorithm for Phase I; it only serves to illustrate the sorting principle.
Property 1. For any \( i \in [0..n] \) with children \( c_1 < \cdots < c_k \) there is \( j \in [0..k] \) such that

- \( c_1, \ldots, c_j \) are in groups that have already been processed,
- \( c_{j+1}, \ldots, c_k \) are in groups that have not yet been processed, and
- the context of the group containing \( i \) is \( S[i]L_{c_1} \cdots L_{c_j} \).

Furthermore, each processed group is Lyndon.

Additionally and unlike in Baier’s original approach, all groups created during our Phase I are additionally either Lyndon or only contain elements whose Lyndon prefix is different from the group’s context. This has several advantages which are discussed below.

Definition 4 (Strongly preliminary group). We call a preliminary group \( G = \langle g_s, g_e, |\alpha| \rangle \) strongly preliminary if and only if \( G \) contains only elements whose Lyndon prefix is not \( \alpha \). A preliminary group that is not strongly preliminary is called weakly preliminary.

Lemma 5. For strings \( wu \) and \( vw \) over \( \Sigma \) with \( u <_{\text{lex}} wu \) and \( v >_{\text{lex}} vw \) we have \( wu <_{\text{lex}} vw \).

Proof. Note that there is no \( j \geq 1 \) such that \( \nu w = w^j \), since otherwise \( v \) would be a prefix of \( wv \) and thus \( v <_{\text{lex}} wv \). Hence, there are \( k \in \mathbb{N}, \ell \in [0..|w|], b \in \Sigma \) and \( m \in \Sigma^* \) such that \( wv = w^k w[0..\ell] bm \) and \( b > w[\ell] \). There are two cases:

- There is some \( j \geq 1 \) such that \( \nu w = w^j \).
  - If \( j|w| \leq k|w| + \ell \), then \( wu \) is a prefix of \( wv \).
  - Otherwise, the first different symbol in \( wu \) and \( wv \) is at index \( p = k|w| + \ell \) and we have \( (wu)[p] = w^j[p] = w[\ell] < b = (wv)[p] \).
- There are \( i \in \mathbb{N}, j \in [0..|w|], a \in \Sigma \) and \( q \in \Sigma^* \) such that \( wu = w^k w[0..j]aq \) and \( a < w[j] \).
  - If \( |w^k w[0..j]| \leq |w^k w[0..\ell]| \), the first different symbol is at index \( p = |w^k w[0..j]| \) with \( (wu)[p] = a < w[j] \leq (wv)[p] \).
  - Otherwise, the first different symbol is at index \( p = |w^k w[0..\ell]| \) with \( (wu)[p] = b > w[\ell] = (wv)[p] \).

In all cases, the claim follows.

Lemma 6. For any weakly preliminary group \( G = \langle g_s, g_e, |\alpha| \rangle \) there is some \( g' \in [g_s..g_e] \) such that \( G' = \langle g_s, g', |\alpha| \rangle \) is a Lyndon group and \( G'' = \langle g' + 1, g_e, |\alpha| \rangle \) is a strongly preliminary group. Splitting \( G \) into \( G' \) and \( G'' \) results in a valid suffix grouping.

Proof. Let \( G = \langle g_s, g_e, |\alpha| \rangle \) be a weakly preliminary group. Let \( F \subset G \) be the set of elements from \( G \) whose Lyndon prefix is \( \alpha \). By Lemma 4 we have \( S_i <_{\text{lex}} S_j \) for any \( i \in F, j \in G \setminus F \). Hence, splitting \( G \) into two groups \( G' = \langle g_s, g_s + |F| - 1, |\alpha| \rangle \) and \( G'' = \langle g_s + |F|, g_e, |\alpha| \rangle \) results in a valid suffix grouping. Note that, by construction, the former is a Lyndon group and the latter is strongly preliminary.
For instance, in Fig. 5 there is a group containing 1, 4 and 10 with context ce. However, 4 and 10 have this context as Lyndon prefix while 1 has ced. Consequently, 1 will later be moved to a new group. Hence, when Baier (and Bertram et al.) create a weakly preliminary group (in Fig. 5 this happens while processing the Lyndon group with context e), we instead create two groups, the lower containing 4 and 10 and the higher containing 1.

During Phase I we maintain the suffix grouping using the following data structures:

– An array $A$ of length $n$ containing the unprocessed Lyndon groups and the sizes of the strongly preliminary groups.
– An array $I$ of length $n$ mapping each element $s$ to the start of the group containing it. We call $I[s]$ the group pointer of $s$.
– A list $C$ storing the starts of the already processed Lyndon groups.

These data structures are organised as follows. Let $G = \langle gs, ge, |\alpha| \rangle$ be a group. For each $s \in G$ we have $I[s] = gs$. If $G$ is Lyndon and has not yet been processed, we also have $s \in A[gs .. ge]$ for all $s \in G$ and $A[gs] < A[gs + 1] < \cdots < A[ge]$. If $G$ is Lyndon and has been processed already, there is some $j$ such that $C[j] = gs$. If $G$ is (strongly) preliminary we have $A[gs] = ge + 1 - gs$ and $A[k] = 0$ for all $k \in (gs .. ge)$.

In contrast to Baier, we have the Lyndon groups in $A$ sorted and store the sizes of the strictly preliminary groups in $A$ as well [1,2]. The former makes finding the number of children a parent has in the currently processed group easier and faster. The latter makes the separate array of length $n$ used by Baier for the group sizes obsolete [1,2] and is made possible by the fact that we only write Lyndon groups to $A$.

As alluded above, we follow Baier’s approach and consider the Lyndon groups in lexicographically decreasing order while updating the groups containing the parents of elements in the current group.

$ge \leftarrow n - 1$;

while $ge \geq 0$ do

\[
gs \leftarrow I[A[gs]]; \\
\text{process group } \langle gs, ge, \bot \rangle; \\
gec \leftarrow gs - 1;
\]

end

Algorithm 4: Phase I: Traversing the groups [1,2]

Note that in Algorithm [3] $ge$ is always the end of a Lyndon group. This is due to the fact that a child is by definition lexicographically greater than its parent. Hence, when a group ends at $ge$ and all suffixes in $SA(gs .. n)$ have been processed, the children of all elements in that group have been processed and it consequently must be Lyndon. Thus, Algorithm [4] actually results in a Lyndon grouping. For a formal proof see [1].

Of course we have to explain how to actually process a Lyndon group. This is done in the rest of this section.

Let $G = \langle gs, ge, |\alpha| \rangle$ be the currently processed group and w.l.o.g. assume that no element in $G$ has the root $-1$ as parent (we do not have the root in the suffix
grouping, thus nodes with the root as parent can be ignored here). Furthermore, let \( A \) be the set of parents of elements in \( G \) (i.e. \( A = \{ pss[i] : i \in G, pss[i] > 0 \} \)) and let \( G_1 < \cdots < G_k \) be those (necessarily preliminary) groups containing elements from \( A \). For each \( g \in [1..k] \) let \( \alpha_g \) be the context of \( G_g \).

As noted in Fig. 5, we have to consider the number of children an element in \( A \) has in \( G \). Namely, if a node has multiple children with the same Lyndon prefix, of course all of them contribute to its Lyndon prefix. This means that we need to move two parents in \( A \), which are currently in the same group, to different new groups if they have differing numbers of children in \( G \).

Let \( A_\ell \) contain those elements from \( A \) with exactly \( \ell \) children in \( G \). Maintaining Property [1] requires that, after processing \( G \), for some \( g \in [1..k] \) the elements in \( A_\ell \cap G_g \) are in groups with context \( \alpha_g \alpha' \). Note that, for any \( \ell < \ell' \), we have \( \alpha_g \alpha' <_{lex} \alpha_g \alpha'' \). Consequently, the elements in \( A_\ell \cap G_g \) must form a lower group than those in \( A_{\ell'} \cap G_g \) after \( G \) has been processed [12]. To achieve this, first the parents in \( A_{G_g} \) are moved to new groups, then those in \( A_{G_g−1} \) and so on [12].

We proceed as follows. First, determine \( A \) and count how many children each parent has in \( G \). Then, sort the parents according to these counts using a bucket sort. Because the elements of yet unprocessed Lyndon groups must be sorted in \( A \), this sort must be stable. Further, partition each bucket into two sub-buckets, one containing the elements that should be inserted into Lyndon groups and the other containing those that Then, for the sub-buckets (in the order of decreasing count; for equal counts: first strongly preliminary then Lyndon sub-buckets) move the parents into new groups [3]. These steps will now be described in detail.

For brevity, we refer to those elements in \( A \) which have their last child in \( G \) as finalists. Partition \( A \) into \( F_\ell \) and \( N_\ell \), such that the former contains finalists and the latter the non-finalists.

In order to determine the aforementioned sub-buckets, we associate a key with each element in \( A \) such that (stably) sorting according to these keys yields the desired partitioning. Specifically, for a fixed \( \ell \), let \( key(s) = 2\ell \) for each \( s \in F_\ell \) and \( key(s) = 2\ell + 1 \) for each \( s \in N_\ell \).

As we need to sort stably, the bucket sort requires an additional array \( B \) of length \(|G|\), and another array for the bucket counters.

Finding parents is done using the same \( pss \) array as in Phase II. Since \( A[g_s .. g_e] \) is sorted by increasing index, children of the same parent are in a contiguous part of \( A[g_s .. g_e] \). Hence, we determine \( A \) and the keys within one scan over \( A[g_s .. g_e] \). Since in practice most elements have no sibling in the same Lyndon group, we treat those explicitly. Specifically, we move \( F_1 \) to \( A[g_s .. g_s + |F_1|] \) and \( N_1 \) to \( B[|G| − |N_1| .. |G|] \). Parents with keys larger than two are written with their keys interspersed to \( B[1 .. 2(|A| − |A_1|)] \). Interspersing the keys is done to improve the cache-locality and thus performance.

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3 Note that Baier broadly follows the same steps (determine parents, sort them, move them to new groups accordingly) [12]. However, each individual step is different because of our distinction between strongly preliminary, weakly preliminary and Lyndon groups.
Fig. 6: Shown is the memory layout during the bucket sort that is applied during the processing of a Lyndon group. The data in grey areas is irrelevant. $p_1 < \cdots < p_m$ are the elements in $\mathcal{P} \setminus \mathcal{P}^1$ and $k_i = \text{key}(p_i)$.

Then we copy $N_1$ to $A[g_s \ldots g_e \mid A_1]$. Note that $A_1$ is now correctly sorted in $A[g_s \ldots g_e + |A_1|]$. Then we sort $A \setminus A_1$ by the keys. That is, we count the frequency of each key, determine the end of each key’s bucket and insert the elements into the correct bucket in $A[g_s + |A_1| \ldots g_s + |A|]$. Fig. 6 shows how the data is organised during the sorting.

Reordering parents into Lyndon groups Let $A'$ be a sub-bucket of length $k$ containing only parents which will now be moved to a Lyndon group and whose context is extended by $\alpha^q$ for some fixed $q$. Note that the bucket sort ensures that $A'$ is sorted. Within each current preliminary group $G' = \langle g_s, g_e, |\beta| \rangle$, the elements in $A'$ must be moved to a new Lyndon group following $G' \setminus A'$. For each element $s$ in $A'$, we decrement $A[I[s]]$ (i.e. the size of the group currently containing $s$) and write $s$ to $A[I[s] + A[I[s]]]$. Afterwards, the new group start must be set (iff $G'$ is now not empty) to $I[s] + A[I[s]]$ (the start of the old group plus the remaining size of the old group). To determine whether $G'$ is now not empty, we mark inserted elements in $A$ using the MSB. If $A[I[s]]$ has the MSB set, we do not need to change the group pointer $I[s]$.

Reordering parents into strongly preliminary groups Let $A'$ be a sub-bucket of length $k$ containing only parents which are now moved to a strongly preliminary group. The general procedure is similar to the reordering into Lyndon groups, but simpler. First, we decrement the sizes of the old groups. In a second scan over $A'$, we set the new group pointer as above, and in a third scan we increment the sizes of the new groups.

Note that in the reordering step, we iterate two and three times, respectively, over the elements in a sub-bucket and that in each scan the group pointers are required. Furthermore, the group pointers are updated only in the last scan. As the group pointers are generally scattered in memory, it would be inefficient to fetch them individually in each scan for two reasons. Firstly, a single group pointer could occupy an entire cache-line (i.e. we mostly keep and transfer irrelevant data in the cache). Secondly, the memory accesses are unnecessarily unpredictable. In order to mitigate these problems, we pull the group pointers into the temporary array $B$, that is, we set $B[i] \leftarrow I[A[i + g_s]]$ for each $i \in [0..|A|]$. Of course, in
this fetching of group pointers we have the same problems as before, but during the actual reordering the group pointers can be accessed much more efficiently.

Note that in contrast to [1], we do not compute the parents on the fly during Phase I but instead use the very fast algorithm by Bille et al. [4] to compute \( pss \) in advance and then mark the last children. There are two reasons for this, namely determining the parents on the fly as done in [1] requires a kind of pointer jumping that is very cache unfriendly and hence slow; and secondly it is not clear how to efficiently determine on the fly whether a node is the last child of its parent.

Another difference that is speeding up the algorithm is that we only write Lyndon groups to \( A \). This way we do not have to rearrange elements in weakly preliminary groups when some of their elements are moved to new groups. Furthermore, it is possible to have the elements in Lyndon groups sorted in \( A \) which makes determining the parents and their corresponding keys easier and faster.

### 3.3 Initialisation

In the initialisation, the \( pss \)-array with markings must be computed. We use an implementation by Bille et al. [4] and then add the markings in a right-to-left scan over \( pss \) (for \( i = n - 1 \rightarrow 0 \), if \( pss[pss[i]] \) is unmarked, mark \( pss[i] \) and \( pss[pss[i]] \), otherwise unmark \( pss[i] \)). Further, the initial suffix grouping needs to be constructed. For each \( c \in \Sigma \) let the \textit{leaf-c-bucket} be the interval in \( SA \) containing those \( i \) with \( S[i] = c \) and \( S_i >_{\text{lex}} S_{i+1} \) and let the \textit{inner-c-bucket} be the interval in \( SA \) containing those \( i \) with \( S[i] = c \) and \( S_i <_{\text{lex}} S_{i+1} \). We compute the leaf and inner bucket sizes in a right-to-left scan over \( S \) (determining for some \( i \) whether \( S_i <_{\text{lex}} S_{i+1} \) can be done in constant time during a right-to-left scan [8,12]). Computing the prefix sums over the bucket sizes yields the bucket boundaries. In a second right-to-left scan over \( S \) the references (in the array \( I \)) to the group starts (i.e. bucket starts) are set and the leaves are inserted into the respective leaf-buckets.

### 4 Experiments

Our implementation \texttt{FGSACA} of the optimised GSACA is publicly available\footnote{https://gitlab.com/qwerzuiop/lfgsaca}.

We compare our algorithm with the GSACA implementation by Baier [1,2], and the \texttt{double sort} algorithms \texttt{DS1} and \texttt{DSH} by Bertram et al. [3]. The latter two also use the grouping principle but employ integer sorting and have super-linear time complexity. \texttt{DSH} differs from \texttt{DS1} only in the initialisation: in \texttt{DS1} the suffixes are sorted by their first character while in \texttt{DSH} up to 8 characters are considered. We further include \texttt{DivSufSort} 2.0.2 and \texttt{libsais} 2.7.1 since the former is used by Bertram et al. as a reference [3] and the latter is the currently fastest suffix sorter known to us.

All experiments were conducted on a Linux-5.4.0 machine with an AMD EPYC 7742 processor and 256GB of RAM. All SACAs were compiled with...
GCC 10.3.0 with flags `-O3 -funroll-loops -march=native -DNDEBUG`. Each algorithm was executed five times on each test case and we use the mean as the final result.

We evaluated the algorithms on data from the Pizza & Chili corpus. From the set of real texts (in the following `PC-Real`) we included `english` (1GiB), `dna` (386MiB), `sources` (202MiB), `proteins` (1.2GiB) and `dblp.xml` (283MiB). From the set of real repetitive texts (`PC-Rep-Real`) we included `cere` (440MiB), `einstein.en.txt` (446MiB), `kernel` (247MiB) and `para` (410MiB). Furthermore, from the artificial repetitive texts (`PC-Rep-Art`) we included `fib41` (256MiB), `rs.13` (207MiB) and `tm29` (256MiB).

In order to test the algorithms on large inputs for which 32-bit integers are not sufficient, we also use a dataset containing larger texts (Large), namely the first $10^{10}$ bytes of the English Wikipedia dump from 01.06.2022 (9.4GiB) and the human DNA concatenated with itself (6.3GiB).

For each of the datasets and algorithms, we determined the average time and memory used, relative to the input size. The results are shown in Fig. 7.

All algorithms were faster on the more repetitive datasets, on which the differences between the algorithms were also smaller. On all datasets, our algorithm is between 46% and 60% faster than GSACA and compared to DSH about 2% faster on repetitive data, over 11% faster on `PC-Real` and over 13% faster on `Large`.

Especially notable is the difference in the time required for Phase II: Our Phase II is between 33% and 50% faster than Phase II of DSH. Our Phase I is also faster than Phase I of DS1 by a similar margin. Conversely, Phase I of DSH is much faster than our Phase I. However, this is only due to the more elaborate construction of the initial suffix grouping as demonstrated by the much slower Phase I of DS1. Our initialisation requires constructing and marking the `psa`-array and is thus much slower than the initialisation of GSACA and DS1. (Note that in GSACA the unmarked `psa`-array is also computed, but on the fly during Phase I.) For DSH, the time required for the initialisation is much more dependent on the dataset, in particular it seems to be faster on repetitive data. Compared to FGSACA, `libsais` is 46%, 34%, 31% and 3% faster on `Large`, `PC-Real`, `PC-Rep-Real` and `PC-Rep-Art`, respectively.

Memory-wise, for 32-bit words, FGSACA uses about 8.83 bytes per input character, while DS1 and DSH use 8.94 and 8.05 bytes/character, respectively. GSACA always uses 12 bytes/character. On `Large`, FGSACA expectedly requires about twice as much memory. For DS1 and DSH this is not the case, mostly because they use 40-bit integers for the additional array of length $n$ that they require (while we use 64-bit integers). DivSufSort requires only a small constant amount of working memory and `libsais` never exceeded 21kiB of working memory on our test data.

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5 [http://pizzachili.dcc.uchile.cl/](http://pizzachili.dcc.uchile.cl/)
6 [https://dumps.wikimedia.org/enwiki/20220601/](https://dumps.wikimedia.org/enwiki/20220601/)
7 [https://www.ncbi.nlm.nih.gov/assembly/GCF_000001405.40](https://www.ncbi.nlm.nih.gov/assembly/GCF_000001405.40)
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Fig. 7: Normalised running time and working memory averaged for each category. The horizontal red line indicates the time for *libsais*. For **Large** we did not test GSACA because Baier’s reference implementation only supports 32-bit words.

5 Concluding Remarks

We only considered single threaded suffix array construction. As modern computers gain their processing power more and more through parallelism, it may be worthwhile to spend effort on trying to parallelise our algorithm. For instance, while processing a final group, all steps besides the reordering of the parents are entirely independent of other groups.

Implementation-wise, in the case that 32-bit integers are not sufficient it may be worthwhile to use 40-bit integers instead of 64-bit integers for our auxiliary data structures.

Also, the results for DSH and DS1 indicate that it may be useful to use an already refined suffix grouping as a starting point for our Phase I, as this enables us to skip many refinement steps.

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