Mean time of archipelagos in 1D probabilistic cellular automata has phases

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Abstract

We study a non-ergodic one-dimensional probabilistic cellular automata, where each component can assume the states $\oplus$ and $\ominus$. We obtained the limit distribution for a set of measures on $\{\oplus, \ominus\}^\mathbb{Z}$. Also, we show that for certain parameters of our process the mean time of convergence can be finite or infinity. When it is finite we have showed that the upper bound is function of the initial distribution.

Keywords: particle process, phase transition, Birth and Death process.

1 Introduction

Generally the theoretical studies about probabilistic cellular automata or just PCA by simplicity, focuses attention to obtain condition under which the PCA is non-ergodic or ergodic\cite{1} i.e. the process can keep some knowledge about their initial condition forever; as opposed to ergodic ones which forget everything about their initial condition as $t \to \infty$. At another direction, when the PCA exhibits non-ergodicity we try to characterize the non-trivial invariant measure\cite{2}.

How long time one random processes can remember something about their initial conditions is a important characteristic. Let us denote this time by $\tau_\mu$ where $\mu$ is the initial distribution of our process(below we shall define this time at a more formal way). When the PCA is non-ergodic we have computational and theoretical works\cite{3, 4} which describe the expectation of $\tau_\mu$ at finite space.

Even at non-ergodic PCA, understand the behavior of the process for certain initial conditions is fruitful\cite{5, 6}. In this work, for a set of initial distributions, whose elements we call archipelagos, we have shown that our process converge, we exhibits the limit distribution and the expectation of $\tau_\mu$. Considering a subset of archipelagos, which we call archipelago of

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pluses (respectively archipelago of minuses) we get that the expectation of $\tau_\mu$ can be finite or infinity. On the first case, expectation of $\tau_\mu$ finite, we describe the upper bound this quantity. Also, we get proved that the upper bound is function of the initial distribution.

2 Definitions and Theorems

We study a random operators with one and the same configuration space $\Omega = \{\ominus, \oplus\}^\mathbb{Z}$ where $\mathbb{Z}$ is the set of integer numbers and $\ominus$ and $\oplus$ are called minus and plus respectively. A configuration is an bi-infinite sequence of minuses or pluses. The configuration space $\Omega$ is the set of configurations. Any configuration $x \in \Omega$ is determined by its components $x_i$ for all $i \in \mathbb{Z}$. The configuration, all of whose components are minuses, is called “all minuses”. Also, The configuration, all of whose components are pluses, is called “all pluses”.

Two configurations $x$ and $y$ are called close to each other if the set $\{i \in \mathbb{Z} : x_i \neq y_i\}$ is finite. A configuration is called an island of pluses if it is close to “all minuses”, we denote the set of island of pluses $\Delta_\oplus$. Respectively a configuration is called an island of minuses if it is close to “all pluses”, we denote the set of island of minuses $\Delta_\ominus$. If $x \in \Delta_\ominus$ there are positions $i < j$ such that $x_{i+1} = x_{j-1} = \ominus$ and $x_k = \ominus$ if $k \leq i$ or $j \leq k$ and for those same positions $i$ and $j$ we say that a island has length $j - i - 1$, we denote that quantity $\text{length}(x)$. If $y \in \Delta_\oplus$ there are positions $i < j$ such that $y_{i+1} = y_{j-1} = \oplus$ and $y_k = \oplus$ if $k \leq i$ or $j \leq k$ and for those same positions $i$ and $j$ we say that a island has length $j - i - 1$, we denote that quantity $\text{length}(y)$. We denote $\Delta = \Delta_\ominus \cup \Delta_\oplus$ the space of islands.

The normalized measures concentrated in the configuration “all minuses” and “all pluses” are denoted by $\delta_\ominus$ and $\delta_\oplus$ respectively. Also, given configuration $x$ we denote the normalized measure concentrated in $x$ by $\delta_x$.

We define cylinders in $\Omega$ in the usual way. By a thin cylinder we denote any set

$$\{x \in \Omega : x_{i_1} = a_1, \ldots, x_{i_k} = a_k\},$$

where $a_1, \ldots, a_k \in \{\ominus, \oplus\}$ are parameters. Thus defined thin cylinder is called a segment cylinder if the indices $i_1, \ldots, i_k$ form a segment in $\mathbb{Z}$. We denote by $\mathcal{M}$ the set of normalized measures on the $\sigma$-algebra generated by cylinders in $\Omega$. By convergence in $\mathcal{M}$ we mean convergence on all thin cylinders.
We denote by $A$, $A_{⊕}$ and $A_{⊖}$ the set of normalized measures on the $\sigma$-algebra generated by cylinders in $\Delta$, $\Delta_{⊕}$ and $\Delta_{⊖}$ respectively. Any $\mu \in A$ we call archipelago. Any $\mu \in A_{⊕}$ we call a archipelago of pluses and any $\mu \in A_{⊖}$ we call archipelago of minuses.

Any map $P : M \to M$ is called an operator. Given an operator $P$ and an initial measure $\mu \in M$, the resulting process is the sequence $\mu, \mu P, \mu P^2, \ldots$.

We say that a measure $\mu$ is invariant to $P$ if $\mu P = \mu$.

An arbitrary cellular automaton $P$ is determined by transition probabilities $\theta(b_k | a_{k-p}, \ldots, a_{k+q}) \in [0,1]$, where $p$, $q$ are non-negative integer numbers, provided for each $k$,

$$\sum_{b_k \in \{⊖, ⊕\}} \theta(b_k | a_{k-p}, \ldots, a_{k+q}) = 1.$$  \hspace{1cm} (1)

Thus a general operator $P$ is defined by (1).

Now, let us consider a probabilistic cellular automata in $\mathbb{Z}$, which we denote by $F$. Our operator is defined as follows: let $p = 0$ and $q = 1$, and transition probabilities

$$\theta(⊕|⊖⊖) = 0; \quad \theta(⊕|⊕⊖) = \beta; \quad \theta(⊕|⊖⊕) = \alpha; \quad \theta(⊕|⊕⊕) = 1.$$  \hspace{1cm} (2)

And $\theta(⊖|a_0a_1) = 1 - \theta(⊕|a_0a_1)$. Thus, we have defined our operator.

Evidently $\delta_{⊖}$ and $\delta_{⊕}$ are invariant measures of our process. Hence, for $\lambda \in [0,1]$, $\pi_{\lambda} = (1 - \lambda)\delta_{⊖} + \lambda\delta_{⊕}$ is invariant to our process.

Given $\mu \in A$, we define the random variable

$$\tau_{\mu} = \inf\{t \geq 0 : \mu F^t = \pi_{\lambda} \quad \text{for} \quad \lambda \in [0,1]\}.$$  

The infimum of the empty set is $\infty$.

If $\mu \in A$ we call giant of $\mu$ and we denote by $\text{giant}(\mu)$ the greatest length of those islands whose the $\delta-$measures of the convex combination of $\mu$ are
concentrated. (For the giant's definition we are using the result stated in lemma \[5\] that \(\Delta\) is countable). If there is not such greatest length, we say that \(\text{giant}(\mu)=\infty\).

We say that our operator \(F\) is eroder of archipelago of pluses in mean linear time (respectively eroder of archipelago of minuses in mean linear time) if fixed \(\alpha\) and \(\beta\) there is constant \(k\) such that

\[
E(\tau_\mu) \leq k(1 + \text{giant}(\mu)),
\]

for all \(\mu \in A_\oplus\) (respectively for \(\mu \in A_\ominus\)) whose \(\text{giant}(\mu)\) is finite.

Now, we shall declare our main results.

**Theorem 1** Let \(\alpha > 0\), \(\beta < 1\) be. If \(\mu \in \mathcal{A}\), then exist \(\lambda \in [0, 1]\) such that

\[
\lim_{t \to \infty} \mu F^t = \pi_\lambda.
\]

In particular, if \(\mu \in A_\oplus\) (respectively \(\mu \in A_\ominus\)) then \(\lambda = 0\) i.e. \(\mu F^t\) goes to \(\delta_\oplus\) (\(\lambda = 1\) i.e. \(\mu F^t\) goes to \(\delta_\ominus\)) when \(t \to \infty\).

**Theorem 2** Let \(\beta < 1\), \(\mu \in A_\oplus\) and \(\text{giant}(\mu)\) finite.

(A) If \(\alpha < 1 - \beta\) then \(E(\tau_\mu) < \infty\);

(B) If \(\alpha \geq 1 - \beta\) then \(E(\tau_\mu) = \infty\).

**Theorem 3** Let \(\alpha > 0\), \(\mu \in A_\ominus\) and \(\text{giant}(\mu)\) finite.

(A) If \(\alpha > 1 - \beta\) then \(E(\tau_\mu) < \infty\);

(B) If \(\alpha \leq 1 - \beta\) then \(E(\tau_\mu) = \infty\).

**Theorem 4** Let \(\alpha > 0\) and \(\beta < 1\) be

(A) If \(\alpha < 1 - \beta\) then \(F\) is eroder of archipelago of pluses in mean linear time;

(B) If \(\alpha > 1 - \beta\) then \(F\) is eroder of archipelago of minuses in mean linear time.

### 3 Order

As very intuitive we shall assume \(\ominus \prec \oplus\). Now, let us introduce a partial order on \(\{\oplus, \ominus\}^\mathbb{Z}\) by saying that configuration \(x\) precedes configuration \(y\) or, what is the same, \(y\) succeeds \(x\) and writing \(x < y\) or \(y > x\) if \(x_i \leq y_i\) for all \(i \in \mathbb{Z}\).
Let us say that a measurable set $S \subset \{\oplus, \ominus\}$ is upper if
\[(x \in S \quad \text{and} \quad x < y) \implies y \in S.\]
Analogously, a set $S$ is lower if
\[(y \in S \quad \text{and} \quad x < y) \implies x \in S.\]
It is easy to check that a complement to an upper set is lower and vice versa.

We introduce a partial order on $\mathcal{M}$ by saying that a normalized measure $\mu$ proceeds $\nu$ (or $\nu$ succeeds $\mu$) if $\mu(S) \leq \nu(S)$ for any upper $S$ (or $\mu(S) \geq \nu(S)$ for any lower $S$, which is equivalent).

We call an operator $P : \mathcal{M} \rightarrow \mathcal{M}$ monotonic if $\mu \prec \nu$ implies $\mu P \prec \nu P$.

The lemma 1 was described in [7, 8] pages 28 and 81 respectively.

**Lemma 1**

Let $x, y$ two configuration. An operator $P$ on $\{\oplus, \ominus\}$ with transition of probabilities $\theta_k(\cdot, \cdot)$ is monotonic if only if
\[x < y \implies \theta_k(\oplus|x_{k-p} \ldots x_{k+q}) \leq \theta_k(\ominus|y_{k-p} \ldots y_{k+q}).\] (3)

**Lemma 2**

Our operator $F$ is monotonic.

**Proof.** It is enough use the lemma 1 and the definition (2).

4 Proof of theorem 1

We say that a configuration $x$ is a $$(\oplus, i)$$-jump if there is position $i$ such that $x_j = \oplus$ for all $j < i$ and $x_j = \ominus$ otherwise. We denote the measure concentrated in $$(\oplus, i)$$-jump by $\mathcal{J}^i_{\oplus \ominus}$. Analogously, We say that a configuration $x$ is a $$(\ominus, i)$$-jump if there is position $i$ such that $x_j = \ominus$ for all $j < i$ and $x_j = \oplus$ otherwise. We denote the measure concentrated in $$(\ominus, i)$$-jump by $\mathcal{J}^i_{\ominus \oplus}$.

**Lemma 3** For each position $j$, (i) If $\alpha > 0$, then
\[\lim_{t \to \infty} \mathcal{J}^j_{\ominus \oplus} F^t = \delta_\oplus.\]
(ii) If $\beta < 1$, then
\[\lim_{t \to \infty} \mathcal{J}^j_{\ominus \oplus} F^t = \delta_\ominus.\]
Proof. First, we will prove item (i). Let $L_1^\alpha$, $L_2^\alpha$, . . . be a sequence of random variable independent identically distributed, where

$$
\mathbb{P}(L_1^\alpha = 1) = \alpha \quad \text{and} \quad \mathbb{P}(L_1^\alpha = 0) = 1 - \alpha.
$$

Now, we get the simple fact, which can be verified by the Kolmogorov’s strong law\[10\]:

$$
\mathbb{P}\left(\lim_{t \to \infty} \frac{\sum_{N=1}^{t} L_N^\alpha}{t} = \alpha \right) = 1.
$$

(4)

At a informal way, note that by the definition of $F$ (see (2)), the random variable $\sum_{N=1}^{t} L_N^\alpha$ describe the number of new pluses that has appeared on $J_{\oplus \ominus} F_t$. Thus, (4) imply that the number of pluses goes to infinity almost surely and the only way that it can occur is when $J_{\oplus \ominus} F_t$ goes to $\delta_{\oplus}$ when $t \to \infty$. \textit{Thus, we conclude the proof of item (i).} To prove the item (ii) it is enough to consider $L_1^{1-\beta}$, $L_2^{1-\beta}$, . . . a sequence of random variable independent identically distributed, where

$$
\mathbb{P}(L_1^{1-\beta} = 1) = 1 - \beta \quad \text{and} \quad \mathbb{P}(L_1^{1-\beta} = 0) = \beta,
$$

and using analog arguments done to prove the item (i) we prove the item (ii). \textit{The lemma is proved.}

Lemma 4 \textit{Lets} $x \in \Delta_{\ominus}$, $y \in \Delta_{\ominus}$ and $\delta_x$ and $\delta_y$ your respective normalized measures.

(i) If $\beta < 1$ then

$$
\lim_{t \to \infty} \delta_x F_t = \delta_{\ominus}.
$$

(ii) If $\beta = 1$ and $\alpha > 0$ then there is position $i$ such that

$$
\lim_{t \to \infty} \delta_x F_t = J_i^{\oplus \ominus}.
$$

(iii) If $\alpha > 0$ then

$$
\lim_{t \to \infty} \delta_y F_t = \delta_{\ominus}.
$$

(iv) If $\alpha = 0$ and $\beta < 1$ then there is position $i$ such that

$$
\lim_{t \to \infty} \delta_y F_t = J_i^{\ominus \oplus}.
$$

(v) If $\beta = 1$ and $\alpha = 0$ then $\delta_x F = \delta_x$ and $\delta_y F = \delta_y$. 

Proof. The items (ii), (iv) and (v) are simply. So, we will prove just the items (i) and (ii). Note that given \(x\) and \(y\) there is value \(j\) such that
\[
\delta_x < J^j_{⊕} \quad \text{and} \quad J^j_{⊕} < \delta_y.
\]
By the lemmas 2 and 3
\[
\lim_{t \to \infty} \delta_x F^t < \lim_{t \to \infty} J^j_{⊕} F^t = \delta_{⊕} \quad \text{for} \quad \beta < 1
\]
and
\[
\delta_{⊕} = \lim_{t \to \infty} J^j_{⊕} F^t < \lim_{t \to \infty} \delta_y F^t \quad \text{for} \quad \alpha > 0.
\]
As for all \(μ \in M\), \(δ_{⊕} < μ < δ_{⊕}\), we conclude the proof of lemma 4.

**Lemma 5** The \(Δ\) is countable.

**Proof.** By the \(Δ\)'s definition it is enough to prove that \(Δ_{⊕}\) is countable. It is what we will to do. Let us define
\[
I_n = \{x \in Δ_{⊕} : \text{length}(x) = n\},
\]
So,
\[
Δ_{⊕} = \bigcup_{n=1}^{∞} I_n.
\]
Of course that \(I_n\) is countable for all natural value \(n\), then \(Δ_{⊕}\) is countable too. Hence \(Δ\) is countable. We conclude the proof of the lemma 4.

**Comment:** The lemma 4 imply that any \(μ \in A\) is a finite or a countably infinite convex combination of \(δ\)–measures of elements of \(Δ\). So from now on, always that we get \(μ \in A\) we can write
\[
μ = \sum_{x \in Δ} k_x δ_x,
\]
where \(\sum_{x \in Δ} k_x = 1\) and for all \(x \in Δ\) we get \(k_x\) are non-negatives.

**Proof of Theorem 1.** We shall prove just the case where \(μ\) is a finite convex combination of \(δ\)–measures. The case when \(μ\) is a countably infinite convex combination of \(δ\)–measures is analog. Let \(x^1, \ldots, x^N\) islands(of pluses or of minuses) and \(δ_{x^1}, \ldots, δ_{x^N}\) its respective normalized measures. Also, we define \(Ω_{⊕}\) the set of island of pluses and \(Ω_{⊖}\) the set of island of minuses. So,
\[
μ = \sum_{x \in \{x^1, \ldots, x^N\}} k_x δ_x = \sum_{x \in \{x^1, \ldots, x^N\} \cap Ω_{⊕}} k_x δ_x + \sum_{x \in \{x^1, \ldots, x^N\} \cap Ω_{⊖}} k_x δ_x.
\]
where \( \sum_{x \in \{x^1, \ldots, x^N\}} k_x = 1 \) and \( k_x \geq 0 \) for \( x \in \{x^1, \ldots, x^N\} \). Using the linearity of \( F \) (see (1)) we get

\[
\mu F^t = \sum_{x \in \{x^1, \ldots, x^N\} \cap \Omega_\oplus} k_x (\delta_x F^t) + \sum_{x \in \{x^1, \ldots, x^N\} \cap \Omega_\ominus} k_x (\delta_x F^t).
\]

Using first the items (i) and (iii) from the lemma \( \Pi \) and after that

\[
\sum_{x \in \{x^1, \ldots, x^N\} \cap \Omega_\oplus} k_x = 1 - \sum_{x \in \{x^1, \ldots, x^N\} \cap \Omega_\ominus} k_x.
\]

We get, \( \mu F^t \) converge to \( \pi_\lambda \) when \( t \) goes to infinity, where \( \lambda = \sum_{x \in \{x^1, \ldots, x^N\} \cap \Omega_\ominus} k_x \).

To the particular cases, it is enough to observe that if \( \mu \) is archipelago of pluses, then

\[
\lambda = \sum_{x \in \{x^1, \ldots, x^N\} \cap \Omega_\ominus} k_x = 0.
\]

And if \( \mu \) is archipelago of minuses, then

\[
\lambda = \sum_{x \in \{x^1, \ldots, x^N\} \cap \Omega_\ominus} k_x = 1.
\]

we conclude the proof of the theorem \( \Pi \).

5 The processes \( X \) and \( Y \)

Let \( X = \{X_t\}_{t=0}^\infty \) assuming values in \( \{0, 1, 2, 3, \ldots\} \) where

\[
P(X_{t+1} = a + 1|X_t = a) = \begin{cases} 0 & \text{if } a = 0 \\ \alpha \beta & \text{if } a > 0. \end{cases}
\]

and

\[
P(X_{t+1} = a - 1|X_t = a) = \begin{cases} 0 & \text{if } a = 0 \\ (1 - \beta)(1 - \alpha) & \text{if } a > 0. \end{cases}
\]

So,

\[
P(X_{t+1} = a|X_t = a) = 1 - P(X_{t+1} = a - 1|X_t = a) - P(X_{t+1} = a + 1|X_t = a).
\]
Figure 1: The figure on the left side illustrate the results described on the lemma 4 about the limit of our process when we started at a measure concentrated at a island of pluses(items (i), (ii) and (v)). Similar illustration is obtained for the items (iii), (iv) and (v) when we started our process at a measure concentrated at a island of minuses. The figure on the right side illustrate the results described on the lemma 10 i.e. the behavior of $\mathbb{E}(\tau_x)$ when $\alpha < 1 - \beta$ and $\alpha \geq 1 - \beta$. The illustration of the lemma 11 is similar.
We shall denote
\[ \lim_{t \to \infty} \mathbb{P}(X_t \geq a | X_0 = n) \]
for every real value \( a \)
by
\[ \mathbb{P}(X_t \to \infty | X_0 = n). \]

Note that:
- If \( \alpha = 0 \) and \( 0 \leq \beta < 1 \) then for all \( \epsilon > 0 \), \( \mathbb{P}(X_t > \epsilon) \to 0 \) when \( t \to \infty \);
- If \( \beta = 0 \) and \( 0 < \alpha < 1 \) then for all \( \epsilon > 0 \), \( \mathbb{P}(X_t > \epsilon) \to 0 \) when \( t \to \infty \);
- If \( \alpha = 1 \) and \( 0 < \beta \leq 1 \) then \( \mathbb{P}(X_t \to \infty | X_0 > 0) > 0 \);
- If \( \beta = 1 \) and \( 0 < \alpha < 1 \) then \( \mathbb{P}(X_t \to \infty | X_0 > 0) > 0 \);
- If \( \alpha = 0 \) and \( \beta = 1 \) or \( \alpha = 1 \) and \( \beta = 0 \) then \( X_t = X_0 \) for all \( t > 0 \).

Therefore, we shall not consider those cases during the proofs of the lemma \( 6, 7, 8, \) and \( 9 \). Thus, from now on we will take \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \).

We denote the absorption probability of our process \( X \) hit the state 0 given that it started on the state \( i \) by \( h_i \). We note that \( h_0 = 1 \). The fundamental relationship among the \( h_i \)'s is the following (see [9]):
\[ \alpha \beta h_{i+1} - ((1 - \alpha)(1 - \beta) + \alpha \beta)h_i + (1 - \alpha)(1 - \beta)h_{i-1} = 0. \]  
(5)

For \( 0 < \alpha \leq 1 \) and \( 0 < \beta \leq 1 \) we define
\[ \gamma = \frac{(1 - \alpha)(1 - \beta)}{\alpha \beta}. \]  
(6)

**Lemma 6**  
(i) If \( \alpha \leq 1 - \beta \) then \( h_i = 1 \) for all \( i \);
(ii) If \( \alpha > 1 - \beta \) then \( h_i = \gamma^i \) for all \( i \);
(iii) If \( \alpha \leq 1 - \beta \) then
\[ \mathbb{P}(X_t \to \infty | X_0 = i) = 0. \]
(iv) If \( \alpha > 1 - \beta \) then
\[ \mathbb{P}(X_t \to \infty | X_0 = i) = 1 - h_i. \]
Proof. Considering $0 < \alpha < 1$ and $0 < \beta < 1$, we get that the general solution of (5)
\[
h_i = \begin{cases} 
A + B\gamma^i & \text{if } \alpha \neq 1 - \beta, \\
A + iB & \text{if } \alpha = 1 - \beta,
\end{cases}
\]
where $A$ and $B$ are constants. Using the facts that $h_0 = 1$, $0 \leq h_i \leq 1$ and the general solution of (5) we can conclude the proof of items (i) and (ii). The item (iii) is a right consequence of the item (i). Now we shall prove (iv).
Consider a process $X^N = \{X_t^N\}_{t=0}^{\infty}$ in $\{0, 1, \ldots, N\}$ where
\[
P(X_{t+1}^N = a|X_t^N = a) = \frac{\gamma^i}{i} \quad \text{and} \quad P(X_{t+1}^N = a-1|X_t^N = a) = \frac{1 - \gamma^i}{i-1}, \]

and
\[
P(X_{t+1}^N = N|X_t^N = N) = 1.
\]
Hence $X^N$ has two absorbing states, namely $\{0, N\}$. When we change the scale we have the same qualitative behavior of the process. Thus at $X^N$ for all $a \in \{0, 1, \ldots, N-1\}$ we take
\[
P(X_{t+1}^N = a+1|X_t^N = a) = \tilde{\alpha} \quad \text{and} \quad P(X_{t+1}^N = a-1|X_t^N = a) = \tilde{\beta}
\]
where
\[
\tilde{\alpha} = \frac{\alpha\beta}{\alpha\beta + (1-\alpha)(1-\beta)} \quad \text{and} \quad \tilde{\beta} = \frac{(1-\alpha)(1-\beta)}{\alpha\beta + (1-\alpha)(1-\beta)}
\]
So, $\tilde{\alpha} + \tilde{\beta} = 1$ and after rescaling $X^N$ became the well-known Gamblers’ ruin problem and is well-known that
\[
P(X_t^N = N|X_0^N = i) = \begin{cases} 
\frac{i}{N} & \text{if } \tilde{\alpha} = \tilde{\beta}; \\
\frac{i}{1-(\tilde{\beta}/\tilde{\alpha})^N} & \text{if } \tilde{\alpha} \neq \tilde{\beta}.
\end{cases}
\]

Thus, the probability of the gambler became infinitely rich, $P(X_t^N = N|X_0^N = i)$ when $N$ goes to $\infty$, is zero if $\tilde{\beta} \geq \tilde{\alpha}$ and is $1 - (\tilde{\beta}/\tilde{\alpha})^i$ if $\tilde{\beta} < \tilde{\alpha}$. But,
\[
\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)^i = \gamma^i = h_i \quad \text{and} \quad \tilde{\beta} < \tilde{\alpha} \Rightarrow 1 - \beta < \alpha.
\]

we conclude the proof of lemma 6.

Now, we define another process $Y = \{Y_t\}_{t=0}^{\infty}$ where
\[
P(Y_{t+1} = a+1|Y_t = a) = P(X_{t+1} = a-1|X_t = a)
\]
and
\[ \mathbb{P}(Y_{t+1} = a - 1|Y_t = a) = \mathbb{P}(X_{t+1} = a + 1|X_t = a). \]
Informally speaking, when \( X \) increase \( Y \) decrease and when \( X \) decrease \( Y \) increase.

Let us define the hitting time of state zero given that we stated at the state \( i \) by
\[ H_i^X = \inf\{t \geq 0 : X_t = 0 \quad \text{and} \quad X_0 = i\} \]
and
\[ H_i^Y = \inf\{t \geq 0 : Y_t = 0 \quad \text{and} \quad Y_0 = i\} \]
where the infimum of the empty set is \( \infty \). As common we denote \( \mathbb{E} \) the expectation. So, \( \mathbb{E}(H_i^X) \) and \( \mathbb{E}(H_i^Y) \) are the expected amount of time before the process \( X \) and \( Y \) respectively hits zero, conditioned that the process start at \( i \).

**Lemma 7** Let \( \beta < 1, \tilde{\alpha} = \alpha \beta, \tilde{\beta} = (1 - \beta)(1 - \alpha) \) and \( \gamma \) as defined at (6).

(i) If \( \alpha < 1 - \beta \) then
\[ \mathbb{E}(H_i^X) = \begin{cases} \frac{1}{\beta} + \frac{\gamma^{-2}}{\tilde{\alpha}(1 - \gamma^{-1})} & \text{if } i = 1; \\ \mathbb{E}(H_1^X) + \frac{\gamma^{-1}(i - 1)}{\tilde{\alpha}(1 - \gamma^{-1})} & \text{if } i > 1. \end{cases} \]

(ii) If \( \alpha \geq 1 - \beta \) then \( \mathbb{E}(H_i^X) = \infty \) for all \( i \geq 1 \).

**Proof.** As a direct consequence of lemma 6 items (ii) and (iv) we obtain the item (ii). The proof of item (i) is a particular case from the general result obtained to a birth and death process (see [11] pp:75-77). Lemma 7 is proved.

**Comment:** The lemma 6 show us if \( \alpha = 1 - \beta \) we get \( \mathbb{P}(X_t = 0|X_0 \geq 0) \to 1 \) when \( t \to \infty \), however lemma 7 show us that in this same case \( \mathbb{E}(H_i^X) = \infty \) for all \( i > 0 \). It can be explained by the fact that the convergence of \( X_t \) to zero when \( \alpha = 1 - \beta \) occur slowly than when \( \alpha < 1 - \beta \).

Note that in analog way as proved the lemmas 6 and 7, we can prove the lemmas 8 and 9.

**Lemma 8** Let the absorption probability of our process \( Y \) hit the state 0 given that it started on the state \( i \) by \( \hat{h}_i \).

(i) If \( \alpha \geq 1 - \beta \) then \( \hat{h}_i = 1 \) for all \( i \);
(ii) If $\alpha < 1 - \beta$ then $\hat{h}_i = \gamma^{-i}$ for all $i$;
(iii) If $\alpha \geq 1 - \beta$ then
\[ P(Y_t \to \infty | Y_0 = i) = 0. \]
(iv) If $\alpha < 1 - \beta$ then
\[ P(Y_t \to \infty | Y_0 = i) = 1 - \hat{h}_i. \]

**Lemma 9**

Let $\alpha > 0$, $\tilde{\alpha} = \alpha \beta$, $\tilde{\beta} = (1 - \beta)(1 - \alpha)$ and $\gamma$ as defined at (6)

(i) If $\alpha \leq 1 - \beta$ then $\mathbb{E}(H^Y_1) = \infty$ for all $i \geq 1$.
(ii) If $\alpha > 1 - \beta$ then
\[ \mathbb{E}(H^Y_1) = \begin{cases} \frac{1}{\tilde{\alpha}} + \frac{\gamma^2}{\tilde{\beta}(1 - \gamma)} & \text{if } i = 1; \\ \mathbb{E}(H^Y_1) + \frac{\gamma(i - 1)}{\tilde{\beta}(1 - \gamma)} & \text{if } i > 1. \end{cases} \]

6 **Proof of theorems 2, 3 and 4**

Given $x \in \Delta_\oplus$, we denote the minimum value $i$ such that $x_i = \oplus$ by $i_{\min}$ and the maximum value $i$ such that $x_i = \ominus$ by $i_{\max}$. Thus, we shall define the following configurations
\[ (\underline{x})_i = \begin{cases} \oplus & \text{if } i = i_{\max}; \\ \ominus & \text{otherwise}. \end{cases} \quad \text{and} \quad (\overline{x})_i = \begin{cases} \oplus & \text{if } i_{\min} \leq i \leq i_{\max}; \\ \ominus & \text{otherwise}. \end{cases} \]

Note that $\overline{x}$, $\underline{x} \in \Delta_\oplus$ and $\underline{x} \prec x \prec \overline{x}$.

We will consider island of pluses where $x = \overline{x}$. Thus, there are positions $i < j$ such that $x_k = \oplus$ if $i < k < j$ and $x_k = \ominus$ otherwise and for those same positions $i$ and $j$ we get $\text{length}(x) = j - i - 1$. If $n = 1$ then $\underline{x} = x = \overline{x}$. Take $x = \overline{x}$, we will associate our process acting in $\delta_x$ with $X$.

Give a island of pluses $x$ where $x = \overline{x}$ and respective normalized measure concentrated in $x$, $\delta_x$. There are positions $i_0 < j_0$ such that $x_{i_0} = x_{j_0} = \ominus$ and $x_k = \oplus$ if $i_0 < k < j_0$. We assume $X_0 = j_0 - i_0 - 1$, note that $X_0 = \text{length}(x)$,
what is the number of consecutive pluses between the positions $i_0$ and $j_0$. Also we define $X_t = j_t - i_t - 1$, where the random variables $i_t$ and $j_t$, ($i_t < j_t$), are defined as follows (see Figure 2):

\[
\begin{align*}
\mathbb{P}(i_t = i_{t-1} - 1, j_t = j_{t-1}) &= \begin{cases} 
0 & \text{if } j_{t-1} = i_{t-1} + 1; \\
\theta(\oplus|\ominus\oplus)\theta(\ominus|\ominus\ominus) & \text{otherwise.}
\end{cases} \\
\mathbb{P}(i_t = i_{t-1} - 1, j_t = j_{t-1} - 1) &= \begin{cases} 
0 & \text{if } j_{t-1} = i_{t-1} + 1; \\
\theta(\ominus|\ominus\oplus)\theta(\ominus|\ominus\ominus) & \text{otherwise.}
\end{cases} \\
\mathbb{P}(i_t = i_{t-1}, j_t = j_{t-1}) &= \begin{cases} 
1 & \text{if } j_{t-1} = i_{t-1} + 1; \\
\theta(\ominus|\ominus\ominus)\theta(\ominus|\ominus\ominus) & \text{otherwise.}
\end{cases} \\
\mathbb{P}(i_t = i_{t-1}, j_t = j_{t-1} - 1) &= \begin{cases} 
0 & \text{if } j_{t-1} = i_{t-1} + 1; \\
\theta(\ominus|\ominus\ominus)\theta(\ominus|\ominus\ominus) & \text{otherwise.}
\end{cases}
\end{align*}
\]

\(\theta(\cdot|\cdot)\) is the probability transitions of our process \([2]\). Note that $i_t$ and $j_t$ describe the probability of the length of the island of pluses: increase, decrease or stay. If $x$ is a island of pluses $\delta_x F^t$ will be a measure concentrated at an island of pluses for each natural value $t$.

Now, it is easy to conclude:

\[
\begin{align*}
\mathbb{P}(X_t = a + 1|X_{t-1} = a) &= \mathbb{P}(i_t = i_{t-1} - 1, j_t = j_{t-1}); \\
\mathbb{P}(X_t = a - 1|X_{t-1} = a) &= \mathbb{P}(i_t = i_{t-1}, j_t = j_{t-1} - 1); \\
\mathbb{P}(X_t = a|X_{t-1} = a) &= \mathbb{P}(i_t = i_{t-1}, j_t = j_{t-1}) + \mathbb{P}(i_t = i_{t-1} - 1, j_t = j_{t-1} - 1).
\end{align*}
\]

Where $a = j_{t-1} - i_{t-1} - 1$. Thus, we have conclude the task to associate our process acting in $\mathcal{F}$ with $X$.

```
... ⊕ ⊕ ⊕ ⊕ ⊕ ⊕... X_4 = 2
... ⊕ ⊕ ⊕ ⊕ ⊕ ⊕... X_3 = 2
... ⊕ ⊕ ⊕ ⊕ ⊕ ⊕... X_2 = 2
... ⊕ ⊕ ⊕ ⊕ ⊕ ⊕... X_1 = 2
... ⊕ ⊕ ⊕ ⊕ ⊕ ⊕... X_0 = 2
```

Figure 2: Here we illustrate a fragment of our process, which occur with positive probability. The initial configuration is a island of pluses, $x$, where $x = \overline{x}$, the length of the island is 2 and $i_{\text{min}} = 4$ and $i_{\text{max}} = 5$. Also, $i_0 = 3$ and $j_0 = 6$; $i_1 = 2$ and $j_1 = 5$; $i_2 = 1$ and $j_2 = 5$; $i_3 = 1$ and $j_3 = 4$ and $i_4 = 1$ and $j_4 = 4$. Also, on the right side we show the correspondent values assumed by the $X$ process.
Let $x \in \Delta_{\oplus}$ and $y \in \Delta_{\ominus}$, we define the random variables

$$
\tau_x = \inf \{ t \geq 0 : \delta_x F^t = \delta_{\ominus} \} \quad \text{and} \quad \tau_y = \inf \{ t \geq 0 : \delta_y F^t = \delta_{\ominus} \}
$$

the infimum of the empty set is $\infty$.

**Lemma 10**  
Let $\beta < 1$, $x \in \Delta_{\oplus}$ and $\delta_x$ your respective normalized measures.

(i) If $\alpha < 1 - \beta$ then $\mathbb{E}(\tau_x) < \infty$;

(ii) If $\alpha \geq 1 - \beta$ then $\mathbb{E}(\tau_x) = \infty$;

**Proof**. For any island of pluses, $x$, we get

$$
\delta_{\ominus} \prec \delta_x \prec \delta_{\oplus}.
$$

So, by the lemma [2] for any natural value $t$

$$
\delta_{\ominus} F^t \prec \delta_x F^t \prec \delta_{\oplus} F^t. \quad (8)
$$

Hence,

$$
\mathbb{E}(\tau_\ominus) \leq \mathbb{E}(\tau_x) \leq \mathbb{E}(\tau_{\oplus}). \quad (9)
$$

We showed previously in this section the association between the process $X$ and the evolution of the length of the island. That association we shall use in this proof. Now, we shall prove the item (ii). Note that

$$
\mathbb{E}(\tau_\ominus) = \mathbb{E}(H_{\oplus}^X).
$$

By the lemma [7] item (ii) we get if $\alpha \geq 1 - \beta$ then $\mathbb{E}(H_{\oplus}^X) = \infty$, then $\mathbb{E}(\tau_x) = \infty$. Thus, using (9) we get $\mathbb{E}(\tau_x) = \infty$. We conclude the proof of (ii). Now, we shall prove the item (i). Let $x$ be a island of plus whose length($\tau$) = $n$, so

$$
\mathbb{E}(\tau_{\ominus}) = \mathbb{E}(H_{\ominus}^X).
$$

By the lemma [7] item (i) we get if $\alpha < 1 - \beta$ then $\mathbb{E}(H_{\ominus}^X)$ is finite, then $\mathbb{E}(\tau_x)$ is finite. Thus, using (9) we conclude that $\mathbb{E}(\tau_x)$ is finite. The lemma [10] is proved.

**Lemma 11**  
Let $\alpha > 0$, $y \in \Delta_{\ominus}$ and $\delta_y$ your respective normalized measures.

(i) If $\alpha > 1 - \beta$ then $\mathbb{E}(\tau_y) < \infty$;

(ii) If $\alpha \leq 1 - \beta$ then $\mathbb{E}(\tau_y) = \infty$;
Proof. The proof is analog to the proof of lemma 10. It is enough associate our process with $Y$, what can be done naturally by take a island of minuses. The $Y$ associated in this way will describe the probability of the length of the island: decrease, increase and stay the same.

Also, we need to define: given a island of minus, $y$, we denote the minimum value $i$ such that $y_i = \ominus$ by $i_{\text{min}}$ and the maximum value $i$ such that $y_i = \ominus$ by $i_{\text{max}}$. Thus, we shall denote the following configurations

$$(y)_i = \begin{cases} \ominus & \text{if } i = i_{\text{max}}; \\ \oplus & \text{otherwise}. \end{cases}$$

and $$(\bar{y})_i = \begin{cases} \ominus & \text{if } i_{\text{min}} \leq i \leq i_{\text{max}}; \\ \oplus & \text{otherwise}. \end{cases}$$

Therefore, $\bar{y} < y < y$.

Thus,

$$\mathbb{E}(\tau_{y}) \leq \mathbb{E}(\tau_{\bar{y}}) \leq \mathbb{E}(\tau_{\bar{y}}).$$

So, using the lemma 9 we conclude the lemma 11.

On the cases when $\mathbb{E}(\tau_{x})$ and $\mathbb{E}(\tau_{y})$ are finite, through the lemmas 7 and 9 we can obtain a estimation of the mean time for the island “disappear”. In this direction, we will prove the lemma 12.

Lemma 12 Let $\gamma$ as defined at (6). Given $\alpha$ and $\beta$ there are constants $k_1$ and $k_2$ such that:

(i) Let $\beta < 1$ be. If $\alpha < 1 - \beta$ then

$$k_1 + k_2 \gamma \leq \mathbb{E}(\tau_{x}) \leq k_1 + (\text{length}(x) - 1)k_2 \quad \text{for all } \quad x \in \Delta_{\oplus}.$$  

(ii) Let $\alpha > 0$ be. If $\alpha > 1 - \beta$ then

$$k_2 + k_1 \gamma \leq \mathbb{E}(\tau_{y}) \leq k_2 + (\text{length}(y) - 1)k_1 \quad \text{for all } \quad y \in \Delta_{\ominus}.$$  

Proof. We shall prove (i). By the lemma 7

$$\mathbb{E}(H_n^x) = \begin{cases} \frac{1}{(1-\beta)(1-\alpha)} + \frac{\gamma^{-2}}{\alpha \beta (1-\gamma^{-1})} & \text{for } \quad n = 1; \\ \frac{1}{(1-\beta)(1-\alpha)} + \frac{(n-1)\gamma^{-1}}{\alpha \beta (1-\gamma^{-1})} & \text{for } \quad n > 1. \end{cases}$$

So,

$$k_1 + \gamma k_2 = \mathbb{E}(H_1^x) \quad \text{and} \quad \mathbb{E}(H_n^x) = k_1 + (n-1)k_2 \quad \text{(10)}.$$
where \( k_1 = ((1 - \beta)(1 - \alpha))^{-1} \) and \( k_2 = \gamma^{-1}/(\alpha\beta(1 - \gamma^{-1})) \).

Now let us consider \( x \) a island of pluses, of course that \( x \) and \( \overline{x} \) are islands with the same length. Using (9) we get

\[
\mathbb{E}(\tau_x) \leq \mathbb{E}(\tau_{\overline{x}}) \leq \mathbb{E}(\tau_{x}).
\]

Also, we know that \( \mathbb{E}(\tau_x) = \mathbb{E}(H_1^X) \) and for \( \text{length}(\overline{x}) = n \) we get \( \mathbb{E}(\tau_{\overline{x}}) = \mathbb{E}(H_n^X) \). Thus using (10) we conclude the proof of item (i). The proof of (ii) is analog. The lemma 12 is proved.

We say that our operator \( F \) is eroder of island of pluses in mean linear time (respectively eroder of island of minuses in mean linear time) if fixed \( \alpha \) and \( \beta \) there is constant \( k \) such that

\[
\mathbb{E}(\tau_x) \leq k(1 + \text{length}(x)) \quad \text{for all} \quad x \in \Delta_{\oplus}.
\]

(Respectively for all \( x \in \Delta_{\ominus} \)). Here we are using the name eroder different of that used at [1, 12, 13]. There the name eroder was used for deterministic operators.

**Lemma 13** Let \( \alpha > 0 \) and \( \beta < 1 \) be.

(i) If \( \alpha < 1 - \beta \) then \( F \) is eroder of island of pluses in mean linear time;

(ii) If \( \alpha > 1 - \beta \) then \( F \) is eroder of island of minuses in mean linear time;

**Proof**. Straight from the lemma 12.

On the theorems 2, 3 and 4, we shall prove just the case where \( \mu \) is a finite convex combination of \( \delta \)-measures. The case when \( \mu \) is a countably infinite convex combination of \( \delta \)-measures is analog.

**Proof of the theorem 2**

Let \( \mu \) a archipelago of pluses. So,

\[
\mu = \sum_{x \in \{x^1, \ldots, x^N\}} k_x \delta_x,
\]

where \( \sum_{i=1}^N k_{x^i} = 1 \); \( k_{x^1}, \ldots, k_{x^N} \) are positives and \( x^1, \ldots, x^N \) are island of pluses. Note that by the theorem [1] and \( \mu \) definition

\[
\tau_{\mu} = \inf\{t \geq 0 : \mu F^t = \delta_{\oplus}\} = \inf\{t \geq 0 : k_{x^1}(\delta_{x^1} F^t) + \ldots + k_{x^N}(\delta_{x^N} F^t) = \delta_{\oplus}\} = \inf\{t \geq 0 : (\delta_{x^1} F^t) = \ldots = (\delta_{x^N} F^t) = \delta_{\oplus}\}.
\]
By the lemma 10 if $\alpha \geq 1 - \beta$ we get
$$\mathbb{E}(\tau_{x^i}) = \infty \quad \text{for all} \quad i = 1, \ldots, N,$$
what imply $\mathbb{E}(\tau_{\mu}) = \infty$. Also, using the lemma 10 if $\alpha < 1 - \beta$ we get
$$\mathbb{E}(\tau_{x^i}) < \infty \quad \text{for all} \quad i = 1, \ldots, N,$$
what imply $\mathbb{E}(\tau_{\mu}) < \infty$. Thus we have conclude the proof of the theorem 2.

**Proof of theorem 3.** The proof is analog to the proof of the theorem 2. We just need to use the lemma 11 and that $\tau_{\mu} = \inf\{t \geq 0 : \mu F^t = \delta_{\ominus}\}$.

**Proof of theorem 4.** First we shall prove (A.4). If $\mu$ is a archipelago of pluses then
$$\mu = \sum_{x \in \{x^1, \ldots, x^N\}} k_x \delta_x,$$
where $\sum_{i=1}^N k_{x^i} = 1$; $k_{x^1}, \ldots, k_{x^N}$ are positives and $x^1, \ldots, x^N$ are island of pluses. By the theorem 1 and $\mu$ definition
$$\tau_{\mu} = \inf\{t \geq 0 : \mu F^t = \delta_{\ominus}\} = \inf\{t \geq 0 : (\delta_{x^1} F^t) = \ldots = (\delta_{x^N} F^t) = \delta_{\ominus}\}.$$
Hence,
$$\mathbb{E}(\tau_{\mu}) \leq \max\{\mathbb{E}(\tau_{x^i}), i = 1, \ldots, N\}.$$
By the lemma 13 item (i) given $\alpha$ and $\beta$ such that $\alpha < 1 - \beta$ there is constant $k$ such that
$$\mathbb{E}(\tau_{x^i}) \leq k(1 + \text{length}(x^i)) \quad \text{for all} \quad i = 1, \ldots, N,$$
therefore for that same constant $k$
$$\max\{\mathbb{E}(\tau_{x^i}), i = 1, \ldots, N\} \leq k(1 + \max\{\text{length}(x^i) : i = 1, \ldots, N\}) = k(1 + \text{giant}(\mu)).$$
Thus, we have conclude the proof of (A.4). The proof of (B.4) is analog.

**Lemma 14** Let $\alpha > 0$ and $\beta < 1$ be. If $\mu \in A \setminus (A_{\ominus} \cup A_\ominus)$ i.e when $\mu$ is not a archipelago of pluses or minuses.

**Lemma 14** Let $\alpha > 0$ and $\beta < 1$ be. If $\mu \in A \setminus (A_{\ominus} \cup A_\ominus)$ then $\mathbb{E}(\tau_{\mu}) = \infty$. 

Proof. As considered on the theorems 2, 3 and 4 we will prove just for the case when \( \mu \) is a finite convex combination of \( \delta \)-measures. If \( \mu \in A \setminus (A_\oplus \cup A_\ominus) \) then

\[ \mu = \mu_x + \mu_y, \]

where

\[ \mu_x = \sum_{x \in \{x^1, \ldots, x^i\}} k_x \delta_x, \quad \mu_y = \sum_{y \in \{y^{i+1}, \ldots, y^N\}} k_y \delta_y, \]

\( x^1, \ldots, x^i \) belongs to \( \Delta_\oplus \) and \( y^{i+1}, \ldots, y^N \) belongs to \( \Delta_\ominus \). Note that

\[ \tau_\mu = \inf\{t \geq 0 : \delta_{x^1} F^t = \ldots = \delta_{x^i} F^t = \delta_\oplus \text{ and } \delta_{y^{i+1}} F^t = \ldots = \delta_{y^N} F^t = \delta_\ominus\}. \]

\[ \geq \inf\{t \geq 0 : \delta_{y^{i+1}} F^t = \ldots = \delta_{y^N} F^t = \delta_\ominus\} = \tau_{\mu_y}. \]

Also

\[ \tau_\mu \geq \inf\{t \geq 0 : \delta_{x^1} F^t = \ldots = \delta_{x^i} F^t = \delta_\ominus\} = \tau_{\mu_x} \]

So, \( \mathbb{E}(\tau_\mu) \geq \mathbb{E}(\tau_{\mu_y}) \) and \( \mathbb{E}(\tau_\mu) \geq \mathbb{E}(\tau_{\mu_x}) \). By the lemmas 10 and 11 if \( \alpha < 1 - \beta \) then \( \mathbb{E}(\tau_{\mu_j}) = \infty \) for all \( j = i + 1, \ldots, N \). So \( \mathbb{E}(\tau_{\mu_y}) = \infty \), thus \( \mathbb{E}(\tau_{\mu}) = \infty \). Another hand, if \( \alpha \geq 1 - \beta \) then \( \mathbb{E}(\tau_{\mu_x}) = \infty \) for all \( j = 1, \ldots, i \). So \( \mathbb{E}(\tau_{\mu_x}) = \infty \), thus \( \mathbb{E}(\tau_{\mu}) = \infty \). We conclude the proof of lemma 14.

6.1 Finite space

Any cellular automaton may have infinite space \( \mathbb{Z} \) or finite space \( \mathbb{Z}_n \)-the set of remainders modulo \( n \), where \( n \) is an arbitrary natural number. In this case we have a finite Markov chain which is ergodic except degenerate cases. But the speed of convergence may be very different for different values of parameters. Note that our process at finite space is closer to computer simulation.

To our finite cellular automata, let us consider the set of states \( \Omega_n = \{\oplus, \ominus\}^{\mathbb{Z}_n} \). Elements of \( \Omega_n \) we call circulars. The circulars are finite sequences of pluses \( \oplus \) and minuses \( \ominus \), but now we imagine these sequences to have circular form. We denote by \( |C| \) the number of components in a circular \( C \) (see figure 3 where \( |C| = n \)). Also we we shall denote the circular whose all the components are equal to \( \oplus \), \( C_\oplus \) and the circular whose all the components are equal to \( \ominus \), \( C_\ominus \).

Note that \( \Omega_n \) has \( 2^n \) circulars. We denote by \( \mathcal{M}_{\Omega_n} \) the set of distributions in \( \Omega_n \). The circular obtained at time \( t \) was denoted by \( C^t \) and its \( i \)-th components were denoted by \( C_i^t \), where \( i = 0, \ldots, |C^t| - 1 \). We consider the circulars
$C^t$ as representations of measures $\mu^t \in \mathcal{M}_{\Omega_n}$, so the sequence $C^0, C^1, C^2, \ldots,$ is a trajectory of some random process $\mu^0, \mu^1, \mu^2, \ldots$.

Note that different of infinity space, our process with finite space if started with a configuration $C$ different of $C_\oplus$ and $C_\ominus$ we get

$$P(\exists t_0: t > t_0 \text{ imply } C^t = C_\oplus) > 0$$

and

$$P(\exists t_0: t > t_0 \text{ imply } C^t = C_\ominus) > 0.$$  

Given $C$, whose $C^0 = C$ and $|C| = n$, we define

$$\tau^n_C = \inf\{t \geq 1: C^t = C_\ominus \text{ or } C^t = C_\oplus\}.$$  

Let $X^n$ the process defined to obtain (7) (Gambler’s ruin problem at \{0, \ldots, n\}) we define

$$H^n_i = \inf\{t \geq 0: X^n_t \in \{0, n\} \text{ and } X^n_0 = i\}.$$  

Using theorem 1.3.5 in \cite{9} and the probabilities transition of $X^n$ we get

$$\mathbb{E}(H^n_i) = \begin{cases} \frac{n(1 - \gamma^i) + i(\gamma^n - 1)}{(1 - \gamma^n)(1 - \alpha - \beta)} & \text{for } \gamma \neq 1; \\ \frac{in - i^2}{2(1 - \alpha)\alpha} & \text{for } \gamma = 1. \end{cases}$$  

where $\gamma = ((1 - \alpha)(1 - \beta))/\alpha\beta$. Hence, for $0 < i < n$

$$\mathbb{E}(H^n_i) \leq \mathbb{E}(H^n_{n-i}) \leq \begin{cases} n & \text{for } \gamma \neq 1; \\ n^2 & \text{for } \gamma = 1. \end{cases}$$  

To prove the lemma \cite{10} we associate the process $X$ with the evolution of island of pluses. At similar way we can associate $X^n$ with the evolution of
two kinds of circulars, $B \notin \{C_\ominus, C_+\}$, defined as follows: there are $0 \leq i < j \leq n - 1$ such that (i) $B_k = \oplus$ for all $i < k < j$ and $B_k = \ominus$ otherwise or (ii) $B_k = \ominus$ for all $i < k < j$ and $B_k = \ominus$ otherwise. We will call $B$ blocks and we will denote the number of pluses in $B$, $l(B)$ (see Figure 4).

Note that using the association between any block $B$ and $X_n$,

$$
\mathbb{E}(H_1^{X_n}) \leq \mathbb{E}(\tau_B) = \mathbb{E}(H_{l(B)}^{X_n}).
$$

Of course that any measure $\mu \in \mathcal{M}_{\Omega_n}$ is a finite convex combination of $\delta$--measures concentrated at circulars, $\delta_C$. Thus, if

$$
\mu = \sum_{B \in \Omega_n} k_B \delta_B,
$$

where $B$ are blocks, $\sum_{B \in \Omega_n} k_B = 1$ and $k_B$ are non-negative real values and

$$
\tau_\mu^n = \inf\{t \geq 0 : \mu^t = C_\ominus \text{ or } \mu^t = C_\oplus\},
$$

then

$$
\mathbb{E}(\tau_\mu^n) = \max\{\mathbb{E}(\tau_\mu^n) : k_B > 0\} \leq \max\{\mathbb{E}(H_i^{X_n}) : i = 0, \ldots, n\}.
$$

Therefore, if $\mu$ is of the form (11), then $\mathbb{E}(\tau_\mu^n)$ is of the order $O(n)$ for $\gamma \neq 1$ and $O(n^2)$ for $\gamma = 1$. It means that when we perform a computer simulation of this process, assuming initially a circular whose $|B| = n$, we will wait at average, no more that $n$ time steps for $\gamma \neq 1$ and $n^2$ time steps for $\gamma = 1$ to our process achieve one absorption state. At another side,

$$
\mathbb{E}(\tau_\mu^n) \geq \min\{\mathbb{E}(H_i^{X_n}) : i = 0, \ldots, n\} \geq \mathbb{E}(H_1^{X_n}).
$$


\footnote{As well-known $f(n) = O(g(n))$ if only if there is constants $c$ and $n_0$ such that $|f(n)| \leq c|g(n)|$ for all $n > n_0$.}
But,

\[ \mathbb{E}(H_1^{X_n}) = \begin{cases} 
  k_1^{1}(\alpha,\beta)n - k_2^{2}(\alpha,\beta) & \text{if } \gamma \neq 1; \\
  k_\alpha(n - 1) & \text{if } \gamma = 1.
\end{cases} \]

where \( k_\alpha = 1/(2(1 - \alpha)\alpha) \), \( k_1^{1}(\alpha,\beta) = (1 - \gamma)/((1 - \gamma^n)(1 - \alpha - \beta)) \) and \( k_2^{2}(\alpha,\beta) = 1/(1 - \alpha - \beta) \). Thus, we can conclude

\[ k_1^{1}(\alpha,\beta)n - k_2^{2}(\alpha,\beta) \leq \mathbb{E}(\tau_n^\mu) \leq n \quad \text{if } \gamma \neq 1 \]

and

\[ k_\alpha(n - 1) \leq \mathbb{E}(\tau_n^\mu) \leq n^2 \quad \text{if } \gamma = 1. \]

Hence, given values \( \alpha \) and \( \beta \) if \( \gamma \leq 1 \) i.e. \( \alpha \geq 1 - \beta \) then \( \mathbb{E}(\tau_n^\mu) \) goes to infinity when \( n \to \infty \). What agree with the results on the theorem 2. If \( \gamma > 1 \) we can not conclude anything about the behavior of \( \mathbb{E}(\tau_n^\mu) \) when \( n \to \infty \).

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