Overfitting Can Be Harmless for Basis Pursuit: Only to a Degree

Peizhong Ju∗ Xiaojun Lin∗ Jia Liu†
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Abstract

Recently, there have been significant interests in studying the generalization power of linear regression models in the overparameterized regime, with the hope that such analysis may provide the first step towards understanding why overparameterized deep neural networks generalize well even when they overfit the training data. Studies on min ℓ2-norm solutions that overfit the training data have suggested that such solutions exhibit the “double-descent” behavior, i.e., the test error decreases with the number of features \( p \) in the overparameterized regime when \( p \) is larger than the number of samples \( n \). However, for linear models with i.i.d. Gaussian features, for large \( p \) the model errors of such min ℓ2-norm solutions approach the “null risk,” i.e., the error of a trivial estimator that always outputs zero, even when the noise is very low. In contrast, we studied the overfitting solution of min ℓ1-norm, which is known as Basis Pursuit (BP) in the compressed sensing literature. Under a sparse true linear model with i.i.d. Gaussian features, we show that for a large range of \( p \) up to a limit that grows exponentially with \( n \), with high probability the model error of BP is upper bounded by a value that decreases with \( p \) and is proportional to the noise level. To the best of our knowledge, this is the first result in the literature showing that, without any explicit regularization in such settings where both \( p \) and the dimension of data are much larger than \( n \), the test errors of a practical-to-compute overfitting solution can exhibit double-descent and approach the order of the noise level independently of the null risk. Our upper bound also reveals a descent floor for BP that is proportional to the noise level. Further, this descent floor is independent of \( n \) and the null risk, but increases with the sparsity level of the true model.

1 Introduction

One of the mysteries of deep neural networks (DNN) is that they not only fit the training data nearly perfectly, but also generalize well to new test data (Zhang et al., 2017; Advani and Saxe, 2017). In classical statistical learning, it is well understood that there is a bias-variance trade-off (Bishop, 2006; Hastie et al., 2009). Thus, when the model has too many parameters (i.e., is overparameterized), it will produce large variance (despite low bias). As a result, although the model can be trained to nearly-zero training errors (which is said to have overfit the training data), it tends to produce large test errors. In order to manage this bias-variance trade-off, it is then essential to introduce regularization to limit the power of the model so that overfitting does not occur (Hoerl and Kennard, 1970; Donoho et al., 2005; Zhao and Yu, 2006; Yao et al., 2007; Meinshausen et al., 2009; Bickel et al., 2009). In contrast, modern DNNs have such a large number of layers and neurons that they can essentially fit arbitrary functions within a large class (Liang and Srikant, 2017). In that sense, they are powerful enough to “memorize” the training data. Indeed, even if the training data are perturbed with a significant amount of noise, DNNs have no problem in fitting such noisy data with zero training error (Zhang et al., 2017). It then becomes quite unexpected why empirically these networks can still produce good results on new test data.

Unfortunately, the performance of DNNs is difficult to analyze due to their non-linearity and non-convexity. As a first step towards understanding why overparameterization and overfitting may be harmless, a line of recent work has focused on linear regression models (Belkin et al., 2018, 2019; Bartlett et al., 2019; Hastie et al., 2019; Mei and Montanari, 2019; Muthukumar et al., 2019). Indeed, such results have demonstrated an interesting “double descent” phenomenon for linear models. Roughly speaking, let \( n \) be the number

∗School of Electrical and Computer Engineering, Purdue University. Email: {jup,linx}@purdue.edu
†Department of Computer Science, Iowa State University. Email: jialiu@iastate.edu
of training samples, and $p$ be the number of parameters of a linear regression model. As $p$ approaches $n$ from below, the test error of the model (that tries to best fit the training data) first decreases and then increases to infinity, which is consistent with the well-understood bias-variance trade-off. As $p$ further increases beyond $n$, overfitting starts to occur (i.e., the training error will always be zero). However, if one chooses the overfitting solution that minimizes the $\ell_2$ norm, the test error decreases again as $p$ increases. Such observations, combined with the fact that Stochastic Gradient Descent (SGD), which is often used to train DNNs, is known to choose the min $\ell_2$-norm solution for linear models (Azizan and Hassibi, 2018), seem to provide a hint why an overparameterized DNN may generalize well.

However, the analyses in these studies also reveal severe limitations of the min $\ell_2$-norm overfitting solution. That is, at least for i.i.d. Gaussian features (Belkin et al., 2019; Muthukumar et al., 2019), for large $p$ the model error of the min $\ell_2$-norm overfitting solution quickly approaches the so-called “null risk” (Hastie et al., 2019), i.e., the error incurred by a model that has all zero parameters and always predicts zero. Ideally, if the training data have lower noise, we would expect that the trained model would be more accurate and produce lower test errors, instead of approaching a model of zero estimates and producing a “null risk” that is independent of the noise level. Thus, it would be much more desirable to establish that the test error of the overfitting solution can be on the order of the noise level and independent of the null risk. Mei and Montanari (2019) studies a more general linear regression model for 2-layer neural networks with random (nonlinear) features, where the number of features $p$, the number of samples $n$, and the data dimension $d$ can all vary. (In contrast, the case of i.i.d. Gaussian features can be thought of as $p = d$.) The “ridgeless regression” setting in (Mei and Montanari, 2019) also corresponds to min $\ell_2$-norm overfitting solutions. The authors provide sharp asymptotes of the model error as functions of $p/d$ and $n/d$, when $p, d,$ and $n$ all approach infinity. These results suggest that, for certain ratios of $p/d$ and $n/d$, the asymptotic model error may approach the order of the noise level. However, when both $p$ and $d$ are much larger than $n$, the model error will still have a bias term that does not vanish, which is similar to the “null risk” discussed earlier. Muthukumar et al. (2019) studies variants of the min $\ell_2$-norm solutions and proves that their model errors can approach the noise level even when the models overfit. However, these variants of min $\ell_2$-norm solutions either require knowledge of the true model in advance, or require an explicit regularization step (which in turns requires knowledge of the sparsity and noise levels).

Given the above limitations of the min $\ell_2$-norm overfitting solution, in this paper we are interested in alternative forms of overfitting solutions that, without requiring any explicit regularization, can not only exhibit double-descent, but also produce test errors that are on the order of the noise level and are independent of the null risk. In particular, we focus on the overfitting solution with the minimum $\ell_1$-norm. This is known as Basis Pursuit (BP) in the compressed sensing literature (Chen et al., 2001). There are several reasons why we are interested in BP with overfitting. First, it does not involve any explicit regularization parameters, and thus can be used even if we do not know the sparsity level or the noise level. Second, it is well-known that using $\ell_1$-norm promotes sparse solutions (Donoho et al., 2005; Zhao and Yu, 2006; Meinshausen et al., 2009; Bickel et al., 2009), which is useful in the overparameterized regime. Third, it is known that the $\ell_1$-norm of the model is closely related to its “fat-shattering dimension,” which is also related to the Vapnik-Chervonenkis (V-C) dimension and the model capacity (Bartlett, 1998). Thus, BP seems to have the appealing flavor of “Occam’s razor” (Blumer et al., 1987), i.e., to use the simplest explanation that matches the training data. Although it is still unclear whether DNNs have some yet-to-be-understood inside trick to choose the simplest model that fits the training data, by first understanding BP we may shed lights on whether or not this line of thinking is a plausible direction to explore. Interestingly, the numerical results in (Muthukumar et al., 2019) suggest that, for a wide range of $p$, BP indeed exhibits double-descent and produces low test-errors. However, no analysis is provided in (Muthukumar et al., 2019). In the compressed sensing literature, test-error bounds for BP were provided for the overparameterized regime, see, e.g., (Donoho et al., 2005). However, the notion of BP there is different as it requires that the model does not overfit the training data. Hence, such results cannot be used to explain the “double-descent” of BP in the overfitting regime. Thus, to our knowledge, the performance analysis of BP in the overfitting regime remains an open problem.

In this paper, we provide new analytical bounds on the model error of BP in the overfitting regime. As in (Belkin et al., 2019), we consider a simple linear regression model with i.i.d. Gaussian features. We assume that the true model is sparse, and the sparsity level is $s$. BP is used to train the model by exactly fitting $n$ training samples. For a range of $p$ up to a value that grows exponentially with $n$, we show an upper
bound on the model error that decreases with \( p \), which confirms the “double descent” phenomenon observed for BP in the numerical results in (Muthukumar et al., 2019). To the best of our knowledge, this is the first analytical result in the literature establishing the double-descent of min \( \ell_1 \)-norm overfitting solutions. Further, our results reveal a key distinction between using \( \ell_1 \)-norm and using \( \ell_2 \)-norm. That is, when \( p \) is large, the model error of BP approaches a level on the order of the noise, with a multiplication factor that only depends on \( n \) and \( p \) but is independent of the magnitude of the signal (which equals to the null risk in our model, see Section 2). In contrast, the model error of the min \( \ell_2 \)-norm overfitting solution is usually a function of both the noise level and the null risk in similar settings (Belkin et al., 2019; Hastie et al., 2019).

Several new insights (and potential limitations) for BP are also revealed by our analytical bounds. First, for fixed \( p \), as the number of samples \( n \) increases, our analytical bound for the model error of BP also increases, suggesting that the requirement to overfit more data hurts the performance of BP. However, what is interesting (and perhaps fortunate) is that, no matter what the value of \( n \) is, there always exists a range of \( p \) (again growing exponentially with \( n \)) such that our model-error upper-bound descends to a level independent of \( n \). We refer to this as the “descent floor.” In this sense, an increasing level of overparameterization helps in a non-trivial manner that “cancels out” the additional error introduced by the requirement to overfit more data. We note that min \( \ell_2 \)-norm overfitting solutions also exhibit a descent floor when the noise is smaller than the signal (the latter again equals to the null risk). However, as we will elaborate further in Section 3 (see Corollary 2), when the noise is much smaller than the signal, the descent floor of BP tends to be significantly lower (because it does not depend on the null risk) and wider (because of the dependency on \( \log p \)) than that of min \( \ell_2 \)-norm solutions. Second, our upper bound will hold only for a range of \( p \) below a limit that grows exponentially with \( n \). When \( p \) grows even larger, eventually BP will pick the wrong feature and the model error will grow again, which is also observed in our numerical results. In other words, BP can only sustain “double-descent” to some degree (which is nonetheless large). Third, even at the right value of \( p \), the descent floor grows as a \( \sqrt{\sigma} \)-multiple of the noise level. While it is not surprising that less-sparse models are harder to learn, it is still somewhat disappointing that more samples (i.e., larger \( n \)) does not help to reduce this descent floor. This observation suggests that, due to overfitting, BP may not have used the larger number of samples efficiently.

A key step in the proof of our main results is to establish a connection between the model error of BP and the ability for a min \( \ell_1 \)-norm solution to fit only the noise. Intuitively, as the number of features increases, it should become increasingly easier to fit the noise, and thus the minimum \( \ell_1 \)-norm of such a noise-fitting solution should naturally become smaller. The same is also true for min \( \ell_2 \)-norm solutions. However, what distinguishes BP is that this ease of fitting noise directly translates into a smaller model error (see Proposition 3 in Section 4). In contrast, the model error of the min \( \ell_2 \)-norm solution may stay at the “null risk” even though the \( \ell_2 \)-norm of a noise-fitting solution diminishes to zero (Muthukumar et al., 2019). It would thus be interesting to study whether similar relationships between the generalization power and the ability to fit noise arise for other models, including DNNs.

Finally, our focus on the min \( \ell_1 \)-norm solution is also related to the recent mean-field studies of two-layer neural networks (Mei et al., 2018), where the first-layer weights are trained and the second layer performs a linear combining of the first-layer features but with fixed weights. Note that when the second-layer weights are fixed, one also bounds the \( \ell_1 \)-norm of the linear weights of all first-layer features. It would thus be of interest to revisit how controlling the \( \ell_1 \)-norm plays a role in the generalization power of such models (Bartlett, 1998).

## 2 Problem Setting

Consider a linear model as follows:

\[
y = x^T \beta + \epsilon, \tag{1}
\]

where \( x \in \mathbb{R}^p \) is a vector of \( p \) features, \( y \in \mathbb{R} \) denotes the output, \( \epsilon \in \mathbb{R} \) denotes the noise, and \( \beta \in \mathbb{R}^p \) denotes the regressor vector. We assume that each element of \( x \) follows \( i.i.d. \) standard Gaussian distribution, and \( \epsilon \) follows independent Gaussian distribution with zero mean and variance \( \sigma^2 \). Let \( s \) denote the sparsity of \( \beta \), i.e., \( \beta \) has at most \( s \) non-zero elements. Without loss of generality, we assume that all non-zero elements of \( \beta \) are in the first \( s \) elements. For any \( p \times 1 \) vector \( \alpha \) (such as \( \beta \)), we use \( \alpha_0 \) to denote the \( s \times 1 \) vector.
that consists of the first \( s \) elements of \( \alpha \), and use \( \alpha_1 \) to denote the \((p - s) \times 1\) vector that consists of the remaining elements of \( \alpha \). With this notation, we have \( \beta = \left[ \begin{array}{c} \beta_0 \\ 0 \end{array} \right] \).

Let \( \beta \) be the true regressor and let \( \hat{\beta} \) be an estimate of \( \beta \) obtained from the training procedure described below. Let \( w := \hat{\beta} - \beta \). According to our model setting, the expected test error satisfies

\[
E_{x,\epsilon} \left[ (x^T \hat{\beta} - (x^T \beta + \epsilon))^2 \right] = E_{x,\epsilon} \left[ (x^T w - \epsilon)^2 \right] = \|w\|_2^2 + \sigma^2. \tag{2}
\]

Since \( \sigma^2 \) is given, in the rest of the paper we will mostly focus on the model error \( \|w\|_2 \). Note that if \( \hat{\beta} = 0 \), we obtain the null risk (Hastie et al., 2019), which equals to \( \|\beta\|_2 \) and in turn equals to the \( \ell_2\)-norm of the signal \( x^T \beta \).

We next describe how BP computes \( \hat{\beta} \) from training data. The training data is given by \((X_{\text{train}}, Y_{\text{train}})\), where \(X_{\text{train}} \in \mathbb{R}^{n \times p}\) and \(Y_{\text{train}} \in \mathbb{R}^n\). Each row of \(X_{\text{train}}\) and \(Y_{\text{train}}\) corresponds to a (scaled) sample of \(x^T\) and \(y\), respectively. We use a \(p \times 1\) vector \(\epsilon_{\text{train}}\) to denote the noise in the training data. Then, the training data can also be written as

\[
Y_{\text{train}} = X_{\text{train}} \beta + \epsilon_{\text{train}}. \tag{3}
\]

Given \(X_{\text{train}}\) and \(Y_{\text{train}}\), our job is to find an estimator \(\hat{\beta}\) for \(\beta\) even though we do not know \(\epsilon_{\text{train}}\). As is common in compressed sensing (Donoho et al., 2005), we assume that each column of \(X_{\text{train}}\) is normalized. That is, we first divide both sides of Eq. (1) by \(\sqrt{n}\), i.e.,

\[
\frac{y}{\sqrt{n}} = \left( \frac{x}{\sqrt{n}} \right)^T \beta + \frac{\epsilon}{\sqrt{n}}. \tag{4}
\]

We then form a matrix \(H \in \mathbb{R}^{n \times p}\) so that each row is an \(i.i.d.\) sample of \((x/\sqrt{n})^T\). Writing \(H = [H_1 \ H_2 \ \cdots \ H_p]\), we then have \(E[\|H_i\|_2] = 1\), for all \(i \in \{1, 2, \cdots, p\}\). Now, let \(X_{\text{train}} = [X_1 \ X_2 \ \cdots \ X_p]\) be constructed in such a way that

\[
X_i = \frac{H_i}{\|H_i\|_2}, \quad \text{for all } i \in \{1, 2, \cdots, p\}, \tag{5}
\]

and let each row of \(Y_{\text{train}}\) and \(\epsilon_{\text{train}}\) be the corresponding values of \(y/\sqrt{n}\) and \(\epsilon/\sqrt{n}\) of the sample. Then, each column \(X_i\) will have a unit \(\ell_2\)-norm. Note that this normalization procedure distorts the value of \(\beta\) in Eq. (3) from the original value in Eq. (1). However, when \(n\) is large, the distortion is small. Since we focus on the model error \(\|w\|_2 = \|\beta - \hat{\beta}\|_2\), the normalization will not qualitatively affect the main conclusions of the paper.

In the rest of this paper, we focus on the situation of overparameterization, i.e., \(p > n\). Among many different estimators of \(\beta\), we are interested in those that perfectly fit (i.e., overfit) the training data, i.e.,

\[
X_{\text{train}} \hat{\beta} = Y_{\text{train}}. \tag{6}
\]

When \(p > n\), there are infinitely many \(\hat{\beta}\)'s that satisfy Eq. (6). In BP (Chen et al., 2001), \(\hat{\beta}\) is chosen by solving the following problem

\[
\hat{\beta}_{\text{BP}} := \arg \min_{\beta} \|\beta\|_1 \quad \text{subject to } X_{\text{train}} \hat{\beta} = Y_{\text{train}}. \tag{7}
\]

In other words, given \(X_{\text{train}}\) and \(Y_{\text{train}}\), BP finds the overfitting solution \(\hat{\beta}_{\text{BP}}\) with the minimal \(\ell_1\)-norm. Define \(w_{\text{BP}} := \hat{\beta}_{\text{BP}} - \beta\). In the rest of our paper, we will show how to estimate the model error \(\|w_{\text{BP}}\|_2\) of BP as a function of the system parameters such as \(n, p, s,\) and \(\sigma^2\). Note that (Donoho et al., 2005) also studies the model error of BP. However, the notion of BP there is different. In particular, the estimator \(\hat{\beta}\) only needs to satisfy \(\|Y_{\text{train}} - X_{\text{train}}\hat{\beta}\|_2 \leq \delta\). The main result there, i.e., Theorem 3.1 of (Donoho et al., 2005), requires \(\delta\) to be greater than the noise level \(\|\epsilon_{\text{train}}\|_2\) (and thus cannot be zero). Therefore, the result of (Donoho et al., 2005) does not capture the performance of BP for the overfitting setting stated in Eq. (6).
3 Main Results

Our main result is the following upper bound on the model error of BP with overfitting.

**Theorem 1.** When \( s \leq \sqrt{\frac{n}{
(16n)^4, \; \exp(\frac{n}{1792s^2})}\} \), then

\[
\frac{\|w^{BP}\|_2}{\|\epsilon_{train}\|_2} \leq 2 + 8 \left(\frac{7n}{\ln p}\right)^{1/4},
\]

with probability at least \( 1 - 6/p \).

Note that the assumption that \( s \leq \sqrt{\frac{n}{168 \ln(16n)}} \), which states that the true model is sufficiently sparse, implies that the interval \([(16n)^4, \; \exp(\frac{n}{1792s^2})]\) is not empty. The constants in Theorem 1 may be further optimized. Nonetheless, Theorem 1 reveals the following important insights:

1. The right-hand-side of Eq. (8) is monotone decreasing with respect to \( p \) (up to a value of \( p \) that grows exponentially in \( n \)). This is consistent with the “double descent” phenomenon for BP observed in recent numerical experiments in (Muthukumar et al., 2019).

2. According to Eq. (8), \( \|w^{BP}\|_2 \) is upper-bounded by a value proportional to \( \|\epsilon_{train}\|_2 \). Under the normalization procedure in Eq. (4) of Section 2, we have \( \|\epsilon_{train}\|_2^2 = \sigma^2 \). Further, when \( n \) is large, \( \|\epsilon_{train}\|_2 \) and \( \sigma \) are close (see Appendix A). Thus, our bound on the model error (and consequently the test error in Eq. (2)) is on the order of the noise level \( \sigma \). To the best of our knowledge, this is the first result in the literature showing that, without any explicit regularization in such setting where the number of features and the dimension of data are both much larger than the number of samples, a practical-to-compute overfitting solution can produce test errors approaching the order of the noise level independently of the null risk \( \|\beta\|_2 \).

3. The descent of \( \|w^{BP}\|_2 \) happens for a relatively large range of \( p \), when \( n \) is large and \( s \) is relatively small. However, the descent speed is logarithmic in \( p \), which is slow and again consistent with the empirical results in (Muthukumar et al., 2019).

4. For a fixed \( p \) inside the descent region, larger \( n \) makes the performance of BP worse. While this may be surprising, it is however reasonable. As \( n \) increases, the null space corresponding to Eq. (16) becomes smaller. Therefore, more data means that BP has to “work harder” to fit the noise in those data, and consequently BP introduces larger model errors.

5. Since the sparsity level \( s \) does not appear in Eq. (8), we conjecture that the descent speed of \( \|w^{BP}\|_2 \) is not sensitive to \( s \), as long as \( s \) still meets the assumption of Theorem 1 that \( s = O(\sqrt{n/\ln n}) \). However, \( s \) may affect the limit of the descent region, as we need \( p \leq O(\exp(n/s^2)) \).

6. Our bound only holds for \( p \in [(16n)^4, \; \exp(\frac{n}{1792s^2})]\), suggesting that for even larger \( p \) the descent will eventually stop. This is because, when there are too many spurious features, eventually some of them will look like the true features. BP may then pick those features and incur a large test error.

The upper limit of \( p \) in Theorem 1 thus suggests a descent floor for BP. This leads to the following corollary.

**Corollary 2.** If \( 1 \leq s \leq \sqrt{\frac{n}{168 \ln(16n)}} \), then by setting \( p = \left\lfloor \exp(\frac{n}{1792s^2}) \right\rfloor \), we have

\[
\frac{\|w^{BP}\|_2}{\|\epsilon_{train}\|_2} \leq 2 + 32\sqrt{14s}
\]

with probability at least \( 1 - 6/p \).

See Appendix E for the proof. Corollary 2 reveals a descent floor for BP, which is given by \( (2 + 32\sqrt{14s})\|\epsilon_{train}\|_2 \). Note that this descent floor is upper bounded by a value proportional to \( \sqrt{s} \) and the noise level \( \|\epsilon_{train}\|_2 \), but is independent of \( n \) and the null risk \( \|\beta\|_2 \). This corollary has the following implications.
1. On the positive side, while larger $n$ degrades the performance of BP (due to the requirement of fitting more noise in the training data), the larger $p$ (i.e., overparameterization) helps in a non-trivial way that cancels out the additional increase in model error, so that ultimately the descent floor is independent of $n$.

2. On the negative side, while the increase of the descent floor in $\sqrt{s}$ is not surprising (as less-sparse models are harder to learn), the fact that it does not decrease with $n$ (contrary to, e.g., LASSO (Zhao and Yu, 2006; Bickel et al., 2009; Meinshausen et al., 2009)) also suggests some inefficiency of BP. That is, due to its overfitting requirement, BP cannot achieve even lower model errors compared to other regularized models (such as LASSO) when more training samples are available.

3. Note that results in (Belkin et al., 2019; Hastie et al., 2019) suggest that min $\ell_2$-norm solutions also exhibit a descent floor when $\sigma$ is smaller than $||\beta||_2$. The descent floor occurs at $p/n = \frac{||\beta||_2}{p}\sigma$, and is at the level $\sqrt{2}||\beta||_2\sigma - \sigma^2$. Compared these two descent floors, we can see that, when $\sigma \ll ||\beta||_2$, the descent floor of BP can be orders-of-magnitude lower than that of min $\ell_2$-norm solutions. Further, the descent floor of BP is independent of the null risk $||\beta||_2$, which is not the case for min $\ell_2$-norm solutions. Finally, since the model error of BP decreases in $\log p$, we expect that the descent floor of BP to be significantly wider than that of min $\ell_2$-norm solutions, which will also be confirmed in our numerical results.

4 Main Ideas of the Proof

In this section, we present the main ideas behind the proof of Theorem 1, which also reveal additional insights for BP. We start with the following definition. Let

$$w^I := \arg\min_w \|w\|_1 \text{ subject to } X_{\text{train}}w = \epsilon_{\text{train}}, \ w_0 = 0. \quad (10)$$

In other words, $w^I$ is the regressor that fits only the noise $\epsilon_{\text{train}}$. Note that $w^I$ exists if and only if $p - s \geq n$, which is a little bit stricter than $p > n$. The rest of this paper is based on the condition that $w^I$ exists, i.e., $p - s \geq n$. Notice that in Theorem 1, the condition $p \geq (16n)^4$ already implies that $p \geq (16n)^4 \geq 2n \geq s + n$.

The first step (Proposition 3 below) is to relate the magnitude of $w^\text{BP}$ with the magnitude of $w^I$. The reason that we are interested in this relationship is as follows. Note that one potential way for an overfitting solution to have a small model error is that the solution uses the $(p-s)$ “redundant” elements of the regressor to fit the noise, without distorting the $s$ “significant” elements (that correspond to the non-zero basis of the true regressor). In that case, as $(p-s)$ increases, it will be easier and easier for the “redundant” elements of the regressor to fit the noise, and thus the model error may improve with respect to $p$. In other words, we expect that $\|w^I\|_1$ will decrease as $p$ increases. However, it is not always true that, as the “redundant” elements of the regressor better fit the noise, they do not distort the “significant” elements of the regressor. Indeed, $\ell_2$-minimization would be such a counter-example: as $p$ increases, although it is easier and easier for the regressor to fit the noise (Muthukumar et al., 2019), the “significant” elements of the regressor also go to zero (Belkin et al., 2019). This is precisely the reason why the min $\ell_2$-norm overfitting solution produces test errors approaching the null risk. In contrast, Proposition 3 below shows that this type of undesirable distortion will not occur for BP under suitable conditions.

Specifically, define

$$M := \max_{i \neq j} |X_i^T X_j|, \quad (11)$$

where $X_i$ and $X_j$ denote $i$-th and $j$-th columns of $X_{\text{train}}$, respectively. Thus, $M$ represents the largest absolute value of correlation (i.e., inner-product) between any two columns of $X_{\text{train}}$ (recall that the $\ell_2$-norm of each column is exactly 1). Further, let

$$K := \frac{1 + M}{sM} - 4. \quad (12)$$

We then have the following proposition that relates the model error $w^{\text{BP}}$ to the magnitude of $w^I$.
Proposition 3. When $K > 0$, we have
\[ \|w^{BP}\|_1 \leq \left(1 + \frac{8}{K} + 2\sqrt{\frac{1}{K}}\right)\|w^I\|_1 + \frac{2\|\epsilon_{\text{train}}\|_2}{\sqrt{KM}} \]  \tag{13} \]

See Appendix B for the proof. Proposition 3 shows that, as long as $\|w^I\|_1$ is small, $\|w^{BP}\|_1$ will also be small. Note that in Eq. (11), $M$ indicates how similar any two features (corresponding to two columns of $X_{\text{train}}$) are. As long as $M$ is much smaller than $1/s$, in particular if $M \leq 1/(8s)$, then the value of $K$ defined in Eq. (12) will be no smaller than 4. Then, the first term of Eq. (13) will be at most a constant multiple of $\|w^I\|_1$. In conclusion, $\|w^{BP}\|_1$ will not be much larger than $\|w^I\|_1$ as long as the columns of $X_{\text{train}}$ are not very similar.

Proposition 3 only captures the $\ell_1$-norm of $w^{BP}$. Instead, the test error in Eq. (2) is directly related to the $\ell_2$-norm of $w^{BP}$. Proposition 4 below relates $\|w^{BP}\|_2$ to $\|w^{BP}\|_1$.

Proposition 4. The following holds:
\[ \|w^{BP}\|_2 \leq \|\epsilon_{\text{train}}\|_2 + \sqrt{M}\|w^{BP}\|_1. \]

See Appendix C for the proof. Note that for an arbitrary vector $\alpha \in \mathbb{R}^p$, we can only infer $\|\alpha\|_2 \leq \|\alpha\|_1$. In contrast, Proposition 4 provides a much tighter bound for $\|w^{BP}\|_2$ when $M$ is small (i.e., the similarity between the columns of $X_{\text{train}}$ is low).

Combining Propositions 3 and 4 together, we immediately have the following corollary that relates $\|w^{BP}\|_2$ to $\|w^I\|_1$.

Corollary 5. When $K > 0$, we must have
\[ \|w^{BP}\|_2 \leq \left(1 + \frac{2}{\sqrt{K}}\right)\|\epsilon_{\text{train}}\|_2 + \sqrt{M}\left(1 + \frac{8}{K} + \frac{2}{\sqrt{K}}\right)\|w^I\|_1. \]

In order to bound $\|w^{BP}\|_2$, it only remains to bound $\|w^I\|_1$ and $M$. The following proposition gives an upper bound on $\|w^I\|_1$.

Proposition 6. When $n \geq 100$ and $p \geq (16n)^4$, we have
\[ \frac{\|w^I\|_1}{\|\epsilon_{\text{train}}\|_2} \leq \sqrt{1 + \frac{3n/2}{\ln p}}, \tag{14} \]
with probability at least $1 - 2e^{-n/4}$.

The proof of Proposition 6 is quite involved and will be explained in Appendix F. Proposition 6 shows that $\|w^I\|_1$ decreases in $p$ at the rate of $O(\sqrt{n/\ln p})$. This is also the reason that $n/\ln p$ shows up in the upper bound in Theorem 1. Further, $\|w^I\|_1$ is upper bounded by a value proportional to $\|\epsilon_{\text{train}}\|_2$, which, when combined with Corollary 5, implies that $\|w^{BP}\|_2$ is on the order of $\|\epsilon_{\text{train}}\|_2$. Note that the decrease of $\|w^I\|_1$ in $p$ trivially follows from its definition in (10) because, when $w^I$ contains more elements, the optimal $w^I$ in (10) should only have a smaller norm. In contrast, the contribution of Proposition 6 is in capturing the exact speed with which $\|w^I\|_1$ decreases with $p$, which has not been studied in the literature. When $p$ approaches $+\infty$, the upper bound in (14) becomes 1. Intuitively, this is because with an infinite number of features, eventually there are columns of $X_{\text{train}}$ that are very close to the direction of $\epsilon_{\text{train}}$. By choosing those columns, $\|w^I\|_1$ approaches $\|\epsilon_{\text{train}}\|_2$. Finally, the upper bound in (14) increases with the number of samples $n$. As we discussed earlier, this is because as $n$ increases, there are more constraints in (10) for $w^I$ to fit. Thus, the magnitude of $w^I$ increases.

Next, we present an upper bound on $M$ as follows.

Proposition 7. When $p \leq \exp(n/36)$, we have
\[ \Pr \left( \left\{ M \leq 2\sqrt{\frac{\ln p}{n}} \right\} \right) \geq 1 - 2e^{-\ln p} - 2e^{-n/144}. \]
See Appendix J for the proof. To understand the intuition behind, note that it is not hard to verify that for any \( i \neq j \), the standard deviation of \( X^T_j X_j \) equals to \( 1/\sqrt{n} \). Since \( M \) defined in Eq. (11) denotes the maximum over \( p \times (p - 1) \) such pairs of columns, it will grow faster than \( 1/\sqrt{n} \). Proposition 7 shows that the additional multiplication factor is of the order \( \sqrt{\ln p} \). As \( p \) increases, eventually we can find some columns that are close to each other, which implies that \( M \) is large. When some columns among the last \((p-s)\) columns of \( X_{\text{train}} \) are quite similar to the first \( s \) columns, \( M \) will be large and BP cannot distinguish the true features from spurious features. This is the main reason why the “double descent” will eventually stop when \( p \) is very large, and thus Theorem 1 only holds up to a limit of \( p \).

Combining Propositions 6 and 7, we can then prove Theorem 1. Please see Appendix D for details. Finally, we can establish matching lower bounds for Proposition 6 and Proposition 7, based on which we conjecture that the upper bound in Theorem 1 is reasonably tight. Please see Appendix K for discussions of those lower bounds.

5 Numerical Results

In this section, we provide numerical results that verify our analytic results in earlier sections. We simulate BP in the setting described in Section 2. The first set of numerical results verifies our bounds for \( \|w^f\|_1 \) and \( M \).

In Fig. 1, we draw the lower and upper bounds on \( \|w^f\|_1 \) as \((p-s)\) increases, for the setting of \( \|\epsilon_{\text{train}}\|_2 = 0.01 \) and \( n = 20 \). The blue curve “ub, Prop. 23” denotes our theoretical upper bound on \( \|w^f\|_1 \) (here we use a more detailed form that is tighter than Proposition 6, which is given in Proposition 23 in Appendix I). In the proof of Proposition 6, we use the idea that \( \|w^f\|_1 \) is mostly determined by the columns of \( X_{\text{train}} \) that are closest to the direction of \( \epsilon_{\text{train}} \). Specifically, among the last \((p-s)\) columns of \( X_{\text{train}} \), we order them based on the absolute values of their inner-products with \( \epsilon_{\text{train}} \), and choose \( 5n \) columns \( B_{(1)}, \cdots, B_{(5n)} \) with the largest values such that \( \|B_{(1)}^T \epsilon_{\text{train}}\| \geq \|B_{(2)}^T \epsilon_{\text{train}}\| \geq \cdots \geq \|B_{(5n)}^T \epsilon_{\text{train}}\| \). The upper bounds in Propositions 23 and 6 are derived based on the probability distribution of these columns. In contrast, using the exact values of these columns, we can construct even tighter bounds on \( \|w^f\|_1 \). Specifically, in Fig. 1, the yellow curve “ub, use \( B_{(5n)} \)” denotes a tighter upper bound on \( \|w^f\|_1 \) constructed by using the exact value of \( B_{(5n)} \) (see Corollary 12 in Appendix F). The green curve “ub, use \( B_{(1)} \) to \( B_{(5n)} \)” denotes yet another upper bound on \( \|w^f\|_1 \) constructed by using the exact values of \( B_{(1)}, \cdots, B_{(5n)} \) (see problem (33) in Appendix F). The red curve “real value” denotes the real value of \( \|w^f\|_1 \). The purple curve “lb, use \( B_{(1)} \)” denotes the lower bound in Lemma 30 of Appendix K using the exact value of \( B_{(1)} \). The first thing that we can verify is that \( \|w^f\|_1 \) decreases when \((p-s)\) increases. Further, the speed of decrease is relatively slow (notice that the x-axis is in log scale). In Fig. 1, we can see that all curves gets closer when \( p \) increases, which implies that our upper bound on \( \|w^f\|_1 \) is tight.

Further, we find that the upper bound using \( B_{(1)} \) to \( B_{(5n)} \) (green curve) almost overlaps with the exact value of \( \|w^f\|_1 \) (red curve), which verifies our intuition that these columns determine the value of \( \|w^f\|_1 \).

Figure 1: Bounds of \( \|w^f\|_1 \) and its exact value, where \( \|\epsilon_{\text{train}}\|_2 = 0.01, n = 20 \).

Figure 2: The value of \( M \) and its upper bounds (given in Proposition 7) for the cases of \( n = 300 \) and \( n = 1200 \).
In Fig. 2, we compare the upper bound on $M$ (given in Proposition 7) with its exact value. We show two cases, $n = 300$ (red curves) and $n = 1200$ (blue curves). For each case, we can see that the derived upper bound and the exact value of $M$ only differ by a constant factor, which verifies the tightness of our upper bound on $M$ in Proposition 7. The curve of the exact value of $M$ for $n = 1200$ also differs with that for $n = 300$ by a constant factor, which is close to $2 = \sqrt{1200/300}$. This figure thus validates our conclusion that $M$ is approximately proportional to $1/\sqrt{n}$ for a fixed $p$.

The next set of numerical results directly shows the performance of BP via the model error $\|w^{BP}\|_2$. In Fig. 3, we draw the value of $\|w^{BP}\|_2$ and its upper bound in Corollary 5 for cases of $s = 1$ and $s = 2$. All curves are computed using the exact value of $M$ and $\|w\|_1$, since we have already shown that our estimates on $M$ and $\|w\|_1$ are tight. We see that there is still a relatively large gap between our bound and the real value of $\|w^{BP}\|$ suggesting room for more refined analysis. Nonetheless, we do observe the same shape and trend between our bound and the exact value of $\|w^{BP}\|_2$, which suggests that our upper bound correctly captures how $\|w^{BP}\|_2$ changes with key parameters. Further, the exact values of $\|w^{BP}\|_2$ for $s = 1$ and $s = 2$ are very close, which is consistent with our conjecture that within the range of $p$ of Theorem 1, the descent speed is not very sensitive to $s$. This phenomenon can also be observed in Fig. 6 for larger values of $s$.

In the rest of this section, we will use Fig. 4, Fig. 5, and Fig. 6 to further verify how $\|w^{BP}\|_2$ changes with $n$, $\|\epsilon_{train}\|_2$, and $s$, respectively. First, we show how $n$ affects $\|w^{BP}\|_2$. In Fig. 4, we draw three curves of $\|w^{BP}\|_2$ for $n = 100$, $n = 250$, and $n = 500$, respectively. We set $\|\beta\|_2 = 1$, $\|\epsilon_{train}\|_2 = 0.01$, and $s = 1$. From Fig. 4, we find that when $p$ is relatively small (when all three curves are in the decreasing region), $\|w^{BP}\|_2$ is larger when $n$ increases. This is consistent with the trend predicted by our upper bound in Theorem 1. On the other hand, as $p$ increases, the lowest points of all three curves for different $n$ are very close. This observation verifies our conclusion in Corollary 2 that the descent floor is independent of $n$.

Next, we show how $\|\epsilon_{train}\|_2$ affects $\|w^{BP}\|_2$. In Fig. 5, we draw three curves of $\|w^{BP}\|_2$ with different $\|\epsilon_{train}\|_2$. The lowest values of those three curves are $0.005$, $0.023$, $0.081$ for $\|\epsilon_{train}\|_2 = 0.01$, $\|\epsilon_{train}\|_2 = 0.04$, and $\|\epsilon_{train}\|_2 = 0.16$, respectively. We find that the lowest values are nearly proportional to $\|\epsilon_{train}\|_2$.

In Fig. 6, we compare BP with the overfitting solution that minimizes the $\ell_2$-norm. We consider both the sparse ($s = 1$) and not sparse ($s = 100$) situations, as well as two different values of $\|\beta\|_2 = 1$ and $\|\beta\|_2 = 0.1$. We can observe the descent floors of both algorithms. However, comparing the curves with the same $s = 100$, we can see that the descent floors (the lowest points of the curves) of min $\ell_2$-norm solutions
Figure 6: Compare BP with min \( \ell_2 \)-norm, where \( \| \epsilon_{\text{train}} \|_2 = 0.01, n = 500 \).

vary significantly with \( \| \beta \|_2 \), while those of BP are insensitive to \( \| \beta \|_2 \). Further, comparing the curves with the same \( \| \beta \|_2 = 1 \), we can see that the descent floors of BP are significantly lower and wider than that of min \( \ell_2 \)-norm overfitting solutions (note that \( \| \epsilon_{\text{train}} \|_2 = 0.01 \ll \| \beta \|_2 = 1 \)). This result thus demonstrates the performance advantage of using BP in similar regimes. Finally, for the two curves for BP with \( \| \beta \|_2 = 1 \) but different sparsity levels \( s \), we can see that the descent floors (the lowest points) are nearly proportional to \( \sqrt{s} \), which confirms the result in Corollary 2.

6 Conclusions and Future Work

In this paper, we studied the generalization power of basis pursuit (BP) in the overparameterized regime when the model overfits the data. Under a sparse linear model with \textit{i.i.d.} Gaussian features, we show that the model error of BP not only exhibits “double descent,” but also approaches a descent floor that is on the order of the noise level and independent of the null risk. Further, our analysis reveals important insights on how the descent floor depends on the number of samples, the number of features, and the sparsity level of the true model.

There are several interesting directions for future work. First, it would be useful to make the discussion in the latter part of Appendix K rigorous, so that we can obtain matching lower bounds on the model error of BP. Second, we only study the \textit{i.i.d.} Gaussian features in this paper. It would be important to see if our main conclusions can also be generalized to other feature models (e.g., Fourier features (Rahimi and Recht, 2008)), models with mis-specified features (Belkin et al., 2019; Hastie et al., 2019), or even the 2-layer neural network models of (Mei et al., 2018). Finally, we hope that the difference between min \( \ell_1 \)-norm solutions and min \( \ell_2 \)-norm solutions reported here could help us understand the generalization power of overparameterized DNNs, or lead to training methods for DNNs with even better performance in such regimes.

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Yao, Y., Rosasco, L., and Caponnetto, A. (2007). On early stopping in gradient descent learning. Constructive Approximation, 26(2):289–315.
A An Estimate of $\|\epsilon_{\text{train}}\|^2$

**Lemma 8** (stated on pp. 1325 of (Laurent and Massart, 2000)). Let $U$ follow a chi-square distribution with $D$ degrees of freedom. For any positive $x$, we have

$$\Pr \left( \left\{ U - D \geq 2\sqrt{Dx + 2x} \right\} \right) \leq e^{-x},$$

$$\Pr \left( \left\{ D - U \geq 2\sqrt{Dx} \right\} \right) \leq e^{-x}.$$  

Notice that $n\|\epsilon_{\text{train}}\|^2/\sigma^2$ follows the chi-square distribution with $n$ degrees of freedom. We thus have

$$\Pr \left( \left\{ \|\epsilon_{\text{train}}\|^2 \leq 2\sigma^2 \right\} \right) = 1 - \Pr \left( \left\{ \frac{n\|\epsilon_{\text{train}}\|^2}{\sigma^2} \geq 2n \right\} \right) = 1 - \Pr \left( \left\{ \frac{n\|\epsilon_{\text{train}}\|^2}{\sigma^2} - n \geq n \right\} \right).$$

Now we use the fact that

$$2\sqrt{\frac{2 - \sqrt{3}}{2}n + 2 \cdot \frac{2 - \sqrt{3}}{2}n} = \sqrt{n^2(4 - 2\sqrt{3}) + (2 - \sqrt{3})n} = (\sqrt{3} - 1)n + (2 - \sqrt{3})n = n.$$  

We thus have

$$\Pr \left( \left\{ \|\epsilon_{\text{train}}\|^2 \leq 2\sigma^2 \right\} \right) = 1 - \exp\left( -\frac{2 - \sqrt{3}}{2}n \right) (\text{by Lemma 8 using } x = \frac{2 - \sqrt{3}}{2}n).$$

We also have

$$\Pr \left( \left\{ \|\epsilon_{\text{train}}\|^2 \geq \frac{\sigma^2}{2} \right\} \right) = 1 - \Pr \left( \left\{ \frac{n\|\epsilon_{\text{train}}\|^2}{\sigma^2} \leq \frac{n}{2} \right\} \right) = 1 - \Pr \left( \left\{ n - \frac{n\|\epsilon_{\text{train}}\|^2}{\sigma^2} \geq \frac{n}{2} \right\} \right) = 1 - \Pr \left( \left\{ n - \frac{n\|\epsilon_{\text{train}}\|^2}{\sigma^2} \geq 2\sqrt{\frac{n}{16}} \right\} \right) \geq 1 - \exp\left( -\frac{n}{16} \right) (\text{by Lemma 8 using } x = n/16).$$

In other words, when $n$ is large, $\|\epsilon_{\text{train}}\|^2$ should be close to $\sigma^2$. As a result, in the rest of the paper, we will use $\|\epsilon_{\text{train}}\|^2$ as a surrogate for the noise level.
B Proof of Proposition 3

Proof. Since we focus on $w^{BP}$, we rewrite BP in the form of $w^{BP}$. Notice that

$$\|\beta^{BP}\|_1 = \|w^{BP} + \beta\|_1 = \|w^{BP}_0 + \beta_0\|_1 + \|w^{BP}_1\|_1.$$  

Thus, we have

$$w^{BP} = \arg\min_{w} \|w_0 + \beta_0\|_1 + \|w_1\|_1$$

subject to $X_{train}w = \epsilon_{train}$. 

(15)

Define $G := X_{train}^T X_{train}$ and let $I$ be the $p \times p$ identity matrix. Let $| \cdot |$ denote the operation that takes the component-wise absolute value of every element of a matrix. We have

$$\|\epsilon_{train}\|_2^2 = \|X_{train}w^{BP}\|_2^2$$

$$= (w^{BP})^T G w^{BP}$$

$$= \|w^{BP}\|_2^2 + (w^{BP})^T (G - I) w^{BP}$$

$$\geq \|w^{BP}\|_2^2 - |w^{BP}|^T |G - I| |w^{BP}|$$

$$(a) \geq \|w^{BP}\|_2^2 - M |w^{BP}|^T |I - I| |w^{BP}|$$

$$= (1 + M)\|w^{BP}\|_2^2 - M \|w^{BP}\|_2^2,$$  

(16)

where in step (a) $I$ represents a $p \times p$ matrix with all elements equal to 1, and the step holds because $G$ has diagonal elements equal to 1 and off-diagonal elements no greater than $M$ in absolute value. Because $w^I$ also satisfies the constraint of (15), by the representation of $w^{BP}$ in (15), we have

$$\|w^{BP}_0 + \beta_0\|_1 + \|w^{BP}_1\|_1 \leq \|w^I_0 + \beta_0\|_1 + \|w^I_1\|_1.$$  

By definition (10), we have $w^I_0 = 0$ and $\|w^I_1\|_1 = \|w^I\|_1$. Thus, we have

$$\|w^{BP}_0 + \beta_0\|_1 + \|w^{BP}_1\|_1 \leq \|\beta_0\|_1 + \|w^I\|_1.$$  

By the triangle inequality, we have $\|\beta_0\|_1 - \|w^{BP}_0 + \beta_0\|_1 \leq \|w^{BP}_0\|_1$. Thus, we obtain

$$\|w^{BP}_1\|_1 \leq \|\beta_0\|_1 + \|w^{BP}_0 + \beta_0\|_1 + \|w^I\|_1$$

$$\leq \|w^{BP}_0\|_1 + \|w^I\|_1.$$  

(17)

We now use (16) and (17) to establish (13). Specifically, because $w^{BP}_0 \in \mathbb{R}^s$, we have

$$\|w^{BP}_0\|_2^2 \geq \frac{1}{s} \|w^{BP}_0\|_1^2.$$  

Thus, we have

$$\|w^{BP}\|_2^2 \geq \|w^{BP}_0\|_2^2 \geq \frac{1}{s} \|w^{BP}_0\|_1^2.$$  

(18)

Applying Eq. (17), we have

$$\|w^{BP}\|_1 = \|w^{BP}_1\|_1 + \|w^{BP}_0\|_1 \leq 2\|w^{BP}_0\|_1 + \|w^I\|_1.$$  

(19)

Substituting Eq. (18) and Eq. (19) in Eq. (16), we have

$$\frac{1 + M}{s} \|w^{BP}_0\|_1^2 - M (2\|w^{BP}_0\|_1 + \|w^I\|_1)^2 \leq \|\epsilon_{train}\|_2^2.$$  

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which can be rearranged into a quadratic inequality in $\|w_{BP}^0\|_1$, i.e.,
\[
\left(\frac{1+M}{s} - 4M\right)\|w_{BP}^0\|^2_1 - 4M\|w^f\|_1\|w_{BP}^0\|_1 \\
- (M\|w^f\|^2_1 + \|\epsilon_{\text{train}}\|^2_2) \leq 0.
\]

Since $K = \frac{1+M}{s} - 4 > 0$, we have the leading coefficient $\frac{1+M}{s} - 4M = KM > 0$. Solving this quadratic inequality for $\|w_{BP}^0\|_1$, we have
\[
\|w_{BP}^0\|_1 \leq \frac{4M\|w^f\|_1 + \sqrt{(4M\|w^f\|_1)^2 + 4KM (M\|w^f\|^2_1 + \|\epsilon_{\text{train}}\|^2_2)}}{2KM} \\
= \frac{2\|w^f\|_1 + \sqrt{4\|w^f\|^2_1 + K(\|w^f\|^2_1 + \frac{1}{M}\|\epsilon_{\text{train}}\|^2_2)}}{K}.
\]

Plugging the result into Eq. (19), we have
\[
\|w_{BP}\|_1 \leq \frac{4\|w^f\|_1 + 2\sqrt{4\|w^f\|^2_1 + 4\|\epsilon_{\text{train}}\|^2_2 + \|w^f\|_1^2 + \frac{1}{M}\|\epsilon_{\text{train}}\|^2_2}}{K} + \|w^f\|_1.
\]

This expression already provides an upper bound on $\|w_{BP}\|_1$ in terms of $M$ and $\|w^f\|_1$. To obtain an even simpler equation, combining $4\|w^f\|_1/K$ with $\|w^f\|_1$, and breaking the square root apart by $\sqrt{a + b + c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$, we have
\[
\|w_{BP}\|_1 \leq \frac{K + 4}{K} \|w^f\|_1 + \sqrt{\left(\frac{4\|w^f\|_1}{K}\right)^2 + \frac{4\|w^f\|^2_1}{K}} \\
+ \sqrt{\frac{4\|\epsilon_{\text{train}}\|^2_2}{MK}} \\
= \left(1 + \frac{4}{K} + 2\sqrt{\frac{1}{K}}\right) \|w^f\|_1 + \frac{2\|\epsilon_{\text{train}}\|^2_2}{\sqrt{KM}}.
\]

The result of the proposition thus follows. \qed

C Proof of Proposition 4

Proof. In the proof of Proposition 3, we have already proven Eq. (16). By Eq. (16), we have
\[
\|u_{BP}\|_2 \leq \sqrt{\frac{\|\epsilon_{\text{train}}\|^2_2 + M\|w_{BP}\|^2_1}{1 + M}} \\
\leq \sqrt{\|\epsilon_{\text{train}}\|^2_2 + M\|w_{BP}\|^2_1} \\
\leq \|\epsilon_{\text{train}}\|_2 + \sqrt{M\|w_{BP}\|_1}.
\]

D Proof of Theorem 1

The proof consists three steps. In step 1, we verify the conditions for Proposition 6 and get the estimation on $\|w^f\|_1$ by Proposition 6. In step 2, we verify the conditions for Proposition 7 and get the estimation on $M$ by Proposition 7. In step 3, we combine results in steps 1 and 2 to prove Theorem 1.

\footnote{Notice that in the proof of Proposition 3, to get Eq. (16), we do not need $K > 0$.}
Step 1
We first verify that the conditions for Proposition 6 are satisfied. Towards this end, from the assumption of Theorem 1 that
\[ p \in \left\{ (16n)^4, \exp\left( \frac{n}{1792s^2} \right) \right\}, \]
we have
\[ p \geq (16n)^4, \]
and
\[ p \leq \exp\left( \frac{n}{1792s^2} \right) \leq e^{n/1792} \] (since \( s \geq 1 \)).

Further, from the assumption of the theorem that \( s \leq \sqrt{\frac{n}{7168 \ln(16n)}} \), we have
\[ n \geq s^2 \cdot 7168 \ln(16n) \geq 7168 > 100 \] (since \( s \geq 1 \) and \( n \geq 1 \)).

Eq. (22) and Eq. (20) imply that the condition of Proposition 6 is satisfied. We thus have, from Proposition 6, with probability at least \( 1 - 2e^{-n/4} \),
\[ \|w_I\|_1 \leq \sqrt{1 + \frac{3n/2}{\ln p}} \|\epsilon_{\text{train}}\|_2. \]

From Eq. (21), we have
\[ p \leq e^{n/1792} \leq e^{n/2} \]
\[ \implies 1 \leq \frac{n/2}{\ln p}. \]

Therefore, we have
\[ \Pr \left( \left\{ \|w_I\|_1 \leq \sqrt{\frac{2n}{\ln p}} \|\epsilon_{\text{train}}\|_2 \right\} \right) \geq 1 - 2e^{-n/4}. \]

Step 2
Note that Eq. (21) implies that the conditions of Proposition 7 is satisfied. We thus have, from Proposition 7,
\[ \Pr \left( M \leq 2\sqrt{7} \sqrt{\frac{\ln p}{n}} \right) \geq 1 - 2e^{-\ln p} - 2e^{-n/144}. \]

Step 3
In this step, we will combine results in steps 1 and 2 and proof the final result of Theorem 1. Towards this end, notice that for any event \( A \) and any event \( B \), we have
\[ \Pr (\{A\} \cap \{B\}) = \Pr (\{A\}) + \Pr (\{B\}) - \Pr (\{A\} \cup \{B\}) \]
\[ \geq \Pr (\{A\}) + \Pr (\{B\}) - 1. \]

Thus, by Eq. (23) and Eq. (24), we have
\[ \Pr \left( \left\{ \|w_I\|_1 \leq \sqrt{\frac{2n}{\ln p}} \|\epsilon_{\text{train}}\|_2 \right\} \cap \left\{ M \leq 2\sqrt{7} \sqrt{\frac{\ln p}{n}} \right\} \right) \]
\[ \geq 1 - 2e^{-n/4} - 2e^{-\ln p} - 2e^{-n/144} \]
\[ \geq 1 - 6e^{-\ln p} \] (since \( \ln p \leq n/144 \leq n/4 \) by Eq. (21))
\[ = 1 - 6/p. \]
It remains to show that the event in (25) implies Eq. (8). Towards this end, note that from $M \leq 2\sqrt{7}\sqrt{\ln p}$, we have

$$K = \frac{1 + M}{sM} - 4 \text{ (by definition in Eq. (12))}$$

$$\geq \frac{1}{sM} - 4. \quad (26)$$

From the assumption of the theorem, we have

$$\exp\left(\frac{n}{1792s^2}\right) \geq p$$

$$\implies \frac{n}{1792s^2} \geq \ln p$$

$$\implies s \leq \sqrt{\frac{n}{1792 \ln p}} = \frac{1}{16\sqrt{7}} \sqrt{\frac{n}{\ln p}}. \quad (27)$$

Applying Eq. (27) to Eq. (26), we have

$$K \geq \frac{1}{\frac{1}{16\sqrt{7}} \sqrt{\frac{n}{\ln p}} \cdot 2\sqrt{7} \sqrt{\ln p}} - 4$$

$$= 8 - 4 = 4.$$ 

Applying

$$M \leq 2\sqrt{7}\sqrt{\ln p/n}, \|w^f\|_1 \leq \sqrt{\frac{2n}{\ln p}}\|\epsilon_{\text{train}}\|_2, \text{ and } K \geq 4.$$

to Corollary 5, we have

$$\|w^{\text{BP}}\| \leq 2\|\epsilon_{\text{train}}\|_2 + \sqrt{2\sqrt{7} \left(\frac{\ln p}{n}\right)^{1/4}} \cdot 4 \cdot \sqrt{\frac{2n}{\ln p}} \|\epsilon_{\text{train}}\|_2$$

$$= \left(2 + \frac{8(7n)}{\ln p} \right)^{1/4} \|\epsilon_{\text{train}}\|_2.$$

The result of Theorem 1 thus follows.

**E Proof of Corollary 2**

*Proof.* For any $a \geq 1$, we have

$$|e^a| - e^{a/2} \geq e^a - e^{a/2} - 1 = e^{a/2}(e^{a/2} - 1) - 1$$

$$\geq \sqrt{e}(\sqrt{e} - 1) - 1 = e - \sqrt{e} - 1 \approx 0.0696.$$ 

It implies that $|e^a| \geq e^{a/2}$ for any $a \geq 1$. Taking logarithm at both sides, we have $\ln |e^a| \geq a/2$ for any $a \geq 1$. When $s \leq \sqrt{\frac{n}{168 \ln(16n)^2}}$, we have

$$\frac{n}{1792s^2} \geq 4 \ln(16n) \geq 1.$$

Thus, by the choice of $p$ in the corollary, we have

$$\ln p = \ln \left| \exp\left(\frac{n}{1792s^2}\right) \right| \geq \frac{n}{3584s^2}. \quad (28)$$

Substituting Eq. (28) into Eq. (8), we have

$$\frac{\|w^{\text{BP}}\|_2}{\|\epsilon_{\text{train}}\|_2} \leq 2 + 8 \left(7 \times 3584s^2\right)^{1/4}$$

$$= 2 + 32 \sqrt{14} \sqrt{s}.$$

□
Proof of Proposition 6: Bounding $\|w^I\|_1$

Recall that, by the definition of $w^I$ in Eq. (10), $w^I$ is independent of the first $s$ columns of $X_{train}$. For ease of exposition, let $A$ denote a $n \times (p-s)$ sub-matrix of $X_{train}$ that consists of the last $(p-s)$ columns, i.e.,

$$A := [X_{s+1} \ X_{s+2} \ \cdots \ X_p].$$

Thus, $\|w^I\|_1$ equals to the optimal objective value of

$$\min_{\alpha \in \mathbb{R}^{p-s}} \|\alpha\|_1 \text{ subject to } A\alpha = \epsilon_{train}. \quad (29)$$

Let $\lambda$ be a $n \times 1$ vector that denotes the Lagrangian multiplier associated with the constraint $A\alpha = \epsilon_{train}$. Then, the Lagrangian of the problem (29) is

$$L(\alpha, \lambda) := \|\alpha\|_1 + \lambda^T(A\alpha - \epsilon_{train}).$$

Thus, the dual problem is

$$\max_{\lambda} h(\lambda), \quad (30)$$

where the dual objective function is given by

$$h(\lambda) = \inf_{\alpha} L(\alpha, \lambda).$$

Let $A_i$ denote the $i$-th column of $A$. It is easy to verify that

$$h(\lambda) = \begin{cases} -\infty & \text{if there exists } i \text{ such that } |\lambda^T A_i| > 1, \\ -\lambda^T \epsilon_{train} & \text{otherwise}. \end{cases}$$

Thus, the dual problem (30) is equivalent to

$$\max_{\lambda} \lambda^T(-\epsilon_{train})$$

subject to $-1 \leq \lambda^T A_i \leq 1$ for all $i \in \{1, 2, \cdots, p-s\}. \quad (31)$

This dual formulation gives the following geometric interpretation. Consider the $\mathbb{R}^n$ space that $\lambda$ and $A_i$ stay in. Since $\|A_i\|_2 = 1$, the constraint $-1 \leq \lambda^T A_i \leq 1$ corresponds to the region between two parallel hyperplanes that are tangent to a unit hyper-sphere at $A_i$ and $-A_i$, respectively. Intuitively, as $p$ goes to infinity, there will be an infinite number of such hyperplanes. Since $A_i$ is uniformly random on the surface of a unit hyper-sphere, as $p$ increases, more and more such random hyperplanes "wrap" around the hyper-sphere. Eventually, the remaining feasible region becomes a unit ball. This implies that the maximum value of the problem (31) becomes $\|\epsilon_{train}\|_2$ when $p$ goes to infinity and the optimal $\lambda$ is attained when $\lambda^* = -\epsilon_{train}/\|\epsilon_{train}\|_2$. Our result in Proposition 6 is also consistent with this intuition that $\|w^I\|_1 \to \|\epsilon_{train}\|_2$ as $p \to \infty$. Of course, the challenge of Proposition 6 is to establish an upper bound of $\|w^I\|_1$ even for finite $p$, which we will study below.

Another intuition from this geometric interpretation is that, among all $A_i$’s, those “close” to the direction of $\pm \epsilon_{train}$ matter most, because their corresponding hyperplanes are the ones that wrap the unit hyper-sphere around the point $\lambda^* = -\epsilon_{train}/\|\epsilon_{train}\|_2$. Next, we construct an upper bound of (31) by using $q$ such “closest” $A_i$’s.

Specifically, for all $i \in \{1, 2, \cdots, p-s\}$, we define

$$B_i := \begin{cases} A_i & \text{if } A_i^T(-\epsilon_{train}) \geq 0, \\ -A_i & \text{otherwise.} \end{cases}$$

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Then, we sort $\mathbf{B}_i$ according to the inner product $\mathbf{B}_i^T (-\epsilon_{\text{train}})$. Let $\mathbf{B}_{(1)}, \ldots, \mathbf{B}_{(q)}$ be the $q < p - s$ vectors with the largest inner products, i.e,

$$\mathbf{B}_{(1)}^T (-\epsilon_{\text{train}}) \geq \mathbf{B}_{(2)}^T (-\epsilon_{\text{train}}) \geq \cdots \geq \mathbf{B}_{(q)}^T (-\epsilon_{\text{train}}) \geq 0.$$  (32)

We then relax the dual problem (31) to

$$\max_\lambda \lambda^T (-\epsilon_{\text{train}})$$
subject to $\lambda^T \mathbf{B}_{(i)} \leq 1$ for all $i \in \{1, 2, \ldots, q\}$.  (33)

Note that the constraints in (33) are a subset of those in (31). Thus, the optimal objective value of (33) is an upper bound on that of (31).

Figure 7: A 3-D geometric interpretation of Problem (33).

Figure 8: When all the points lie on some hemisphere, the objective value of Problem (35) can be infinity $\lambda$ takes the direction $\overrightarrow{OF}$.  

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Fig. 7 gives an geometric interpretation of (33). In Fig. 7, the gray sphere centered at the origin $O$ denotes the unit hyper-sphere in $\mathbb{R}^n$. The top (north pole) of the sphere $O$ is denoted by the point $A$. The north direction denotes the direction of $(-\epsilon_{\text{train}})$. The vector $\overrightarrow{OC}$ denotes some $B_{(i)}$, $i \in \{1, \cdots, q-1\}$. The green plane is tangent to the sphere $O$ at the point $C$. Thus, the space below the green plane denotes the feasible region defined by the constraint $\lambda^T B_{(1)} \leq 1$. The point $D$ denotes the intersection of the axis $\overrightarrow{OA}$ and the green plane. Similarly, the vector $\overrightarrow{OF}$ corresponds to $B_{(q)}$. Note that its corresponding hyperplane (not drawn in Fig. 7) intersects the axis $\overrightarrow{OA}$ at a higher point $E$. This suggests that, by replacing the vector $B_{(i)}$ in each of the constraints of (33) by another vector that has a smaller inner-product with $(-\epsilon_{\text{train}})$, the optimal objective value of (33) will be even higher. For example, in Fig. 7, the constraint corresponding to $\overrightarrow{OC}$ is replaced by that corresponding to $\overrightarrow{OD}$. This procedure is made precise below.

For each $i \in \{1, 2, \cdots, q\}$, we define

$$
C_{(i)} := \sqrt{\frac{1 - (B_{(q)}^T(-\epsilon_{\text{train}}))^2}{\|\epsilon_{\text{train}}\|^2}} \cdot \left( B_{(i)} - \frac{B_{(i)}^T(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|^2}(-\epsilon_{\text{train}}) \right) + \frac{B_{(q)}^T(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|^2}(-\epsilon_{\text{train}}).
$$

By the definition of $C_{(i)}$, it is easy to verify that $\|C_{(i)}\|_2 = 1$ and $C_{(i)}(-\epsilon_{\text{train}}) = B_{(q)}^T(-\epsilon_{\text{train}}) \leq B_{(i)}^T(-\epsilon_{\text{train}})$, for all $i \in \{1, \cdots, q\}$. Roughly speaking, $C_{(i)}$ is the point on the unit-hyper-sphere that is along the same (vertical) longitude as $B_{(i)}$, but at the same (horizontal) latitude as $B_{(q)}$.

Then, we can construct another problem as follows:

$$
\max_{\lambda} \lambda^T(-\epsilon_{\text{train}}) \quad \text{subject to} \quad \lambda^T C_{(i)} \leq 1, \text{ for all } i \in \{1, 2, \cdots, q\}.
$$

The following lemma shows that the solution to (35) is an upper bound on that of (33).

**Lemma 9.** The objective value of Problem (35) must be greater than or equal to that of Problem (33).

See Appendix G for the proof. We draw the geometric interpretation of the problem (35) in Fig. 8. Vectors $\overrightarrow{OD_1}$, $\overrightarrow{OD_2}$, and $\overrightarrow{OD_3}$ represent those vectors $C_{(i)}$. Since all $C_{(i)}$’s have the same latitude, points $D_1$, $D_2$, and $D_3$ locate on one circle centered at point $D$ (the circle is actually a hyper-sphere in $\mathbb{R}^{n-1}$). Therefore, tangent planes on those points have the same intersection point $E$ with the axis $\overrightarrow{OD}$.

We wish to argue that the vector $\overrightarrow{OE}$ is the optimal $\lambda$ for the problem (35). However, it is not always the case. Specifically, when all those $C_{(i)}$’s lie on some hemisphere in $\mathbb{R}^{n-1}$, we can find a direction $\lambda$ such that $\lambda^T(-\epsilon_{\text{train}})$ goes to infinity. For example, in Fig. 8, the direction $\overrightarrow{OE}$ corresponds to such a direction of $\lambda$ that $\lambda^T(-\epsilon_{\text{train}})$ goes to infinity. Fortunately, when $q$ is large enough, the probability that all $C_{(i)}$’s lie on some hemisphere in $\mathbb{R}^{n-1}$ is very small. Towards this end, we can utilize the following result from (Wendel, 1962).

**Lemma 10 (From (Wendel, 1962)).** Let $N$ points be scattered uniformly at random on the surface of a sphere in an $n$-dimensional space. Then, the probability that all the points lie on some hemisphere equals to

$$
2^{-N+1} \sum_{k=0}^{n-1} \binom{N-1}{k}.
$$

Applying Lemma 10 to all $q$ points $C_{(1)}, \cdots, C_{(q)}$ (represented by $D_1, D_2, D_3$ in Fig. 8) on the sphere in $\mathbb{R}^{n-1}$, we can quantify the probability that the situation in Fig. 8 does not happen, in which case we can then prove that the vector $\overrightarrow{OE}$ is the optimal $\lambda$ for the problem (35). Lemma 11 below summarizes this result.
Lemma 11. The problem (35) achieves the optimal objective value at

$$\lambda^* = \frac{\mathbb{B}_{(q)}(-\epsilon_{\text{train}})}{\mathbb{B}_{(q)}(-\epsilon_{\text{train}})}$$

with the probability at least

$$1 - 2^{-q+1} \sum_{i=0}^{n-2} \binom{q-1}{i} \geq 1 - e^{-(q/4-n)}.$$

See Appendix H for the proof. Letting $q = 5n$, and combining Lemmas 9 and 11, we have the following corollary.

Corollary 12. The following holds

$$\|\mathbf{w}_I\|_1 \leq \frac{\|\epsilon_{\text{train}}\|_2}{\mathbb{B}_{(5n)}(-\epsilon_{\text{train}})}$$

with probability at least $1 - e^{-n/4}$.

It only remains to bound $\mathbb{B}_{(i)}(-\epsilon_{\text{train}})$. Using the fact that each $\mathbb{B}_i$ is i.i.d. and uniformly distributed on the unit-hyper-hemisphere in $\mathbb{R}^n$, we have the following result.

Lemma 13. When $n \geq 100$ and $p \geq (16n)^4$, the following holds

$$\mathbb{B}_{(5n)}(-\epsilon_{\text{train}}) \geq \frac{\|\epsilon_{\text{train}}\|_2}{\sqrt{1 + \frac{3n}{2\ln p}}}$$

with probability at least $1 - e^{-5n/4}$.

See Appendix I for the proof. Combining Corollary 12 and Lemma 13, we then obtain Proposition 6.

G Proof of Lemma 9

The proof consists of two steps. In step 1, we will define an intermediate problem (36) below, and show that problem (33) is equivalent to the problem (36). In step 2, we will show that the any feasible $\lambda$ for the problem (36) is also feasible for the problem (35). The conclusion of Lemma 9 thus follows.

For step 1, the intermediate problem is defined as follows.

$$\max_{\lambda} \lambda^T(-\epsilon_{\text{train}}) \text{ subject to }$$

$$\lambda^T(-\epsilon_{\text{train}}) \geq \mathbb{B}_{(1)}(-\epsilon_{\text{train}}),$$

$$\lambda^T \mathbb{B}_{(i)} \leq 1 \text{ for all } i \in \{1, 2, \cdots, q\}. \quad (36)$$

In order to show that this problem is equivalent to (33), we use the following lemma.

Lemma 14. The value of the problem (33) is at least $\mathbb{B}_{(1)}(-\epsilon_{\text{train}})$.

Proof. Because $\|\mathbb{B}_{(1)} A_i\| \leq \|\mathbb{B}_{(1)}\|_2 \|\mathbb{B}_{(i)}\|_2 = 1$ for all $i \in \{1, \cdots, q\}$, $\mathbb{B}_{(1)}$ is feasible for the problem (33). The result of this lemma thus follows. \qed

By this lemma, we can add an additional constraint $\lambda^T(-\epsilon_{\text{train}}) \geq \mathbb{B}_{(1)}(-\epsilon_{\text{train}})$ to the problem (33) without affecting its solution. This is exactly problem (36). Thus, the problem (33) is equivalent to the intermediate problem (36), i.e., step 1 has been proven. Then, we move on to step 2. We will first use Lemma 15 to show that if $\mathbf{C}_{(i)}$ can be written in the form of

$$\mathbf{C}_{(i)} = \frac{\mathbf{B}_i + k\epsilon_{\text{train}}}{\|\mathbf{B}_i + k\epsilon_{\text{train}}\|_2}, \quad (37)$$
for some \( k > 0 \) and \( C^T_{(i)} \epsilon_{\text{train}} \leq 0 \), then any \( \lambda \) that satisfies \( \lambda^T B_i \leq 1 \) and \( \lambda^T (-\epsilon_{\text{train}}) \geq B^T_{(1)}(-\epsilon_{\text{train}}) \) must also satisfies \( \lambda^T C_{(i)} \leq 1 \). After that, we use Lemma 17 to show that all \( C_{(i)} \)'s indeed can be expressed in this form. The conclusion of step 2 then follows. Towards this end, Lemma 15 is as follows.

**Lemma 15.** For all \( i \in \{1, 2, \ldots, q\} \), for any \( \lambda \) that satisfy

\[
\lambda^T B_i \leq 1, \quad \lambda^T (-\epsilon_{\text{train}}) \geq B^T_{(1)}(-\epsilon_{\text{train}}),
\]

we must have

\[
\lambda^T \frac{B_i + k\epsilon_{\text{train}}}{\|B_i + k\epsilon_{\text{train}}\|_2} \leq 1,
\]

for any \( k \geq 0 \) that satisfies \( (B_i + k\epsilon_{\text{train}})^T \epsilon_{\text{train}} \leq 0 \).

**Proof.** We have

\[
\lambda^T B_i + \lambda^T k\epsilon_{\text{train}} \leq \lambda^T \frac{B_i + B_i^T k\epsilon_{\text{train}}}{\|B_i + k\epsilon_{\text{train}}\|_2} \leq \lambda^T B_i + k\epsilon_{\text{train}} \leq \|B_i + k\epsilon_{\text{train}}\|_2
\]

Here are reasons of each step: (i) By Eq. (32), we have \( \lambda^T (-\epsilon_{\text{train}}) \geq B^T_{(1)}(-\epsilon_{\text{train}}) \geq B^T_i(-\epsilon_{\text{train}}) \). Thus, we have \( \lambda^T k\epsilon_{\text{train}} \leq B^T_i k\epsilon_{\text{train}} \); (ii) \( \lambda^T B_i \leq 1 \) by the assumption of the lemma; (iii) \( B_i^T B_i = 1 \) by definition of \( B_i \); (iv) CauchyâŠŞSchwarz inequality; (v) \( \|B_i\|_2 = B_i^T B_i = 1 \).

Then, it only remains to prove that all \( C_{(i)} \)'s in Eq. (34) can be expressed in the specific form described above in Eq. (37). Towards the end, we need the following lemma, which characterizes important features of \( C_{(i)} \).

**Lemma 16.** For any \( i \in \{1, \ldots, q\} \), we must have \( \|C_{(i)}\|_2 = 1 \), and \( C^T_{(i)}(-\epsilon_{\text{train}}) = B^T_{(q)}(-\epsilon_{\text{train}}) \).

**Proof.** It is easy to verify that \( C^T_{(i)}(-\epsilon_{\text{train}}) = B^T_{(q)}(-\epsilon_{\text{train}}) \). Here we show how to prove \( \|C_{(i)}\|_2 = 1 \). Because

\[
\left( B_{(i)} - \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) \right)^T (-\epsilon_{\text{train}}) = 0,
\]

we know that the first and the second term on the right hand side (RHS) of Eq. (34) are orthogonal. Thus, we have

\[
\|C_{(i)}\|_2^2 = \|\text{1st term on the RHS of Eq. (34)}\|_2^2 + \|\text{2nd term on the RHS of Eq. (34)}\|_2^2.
\]

By Eq. (38), we also have

\[
\left\| \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) \right\|_2^2 + \left\| B_{(i)} - \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) \right\|_2^2 = \|B_{(i)}\|_2^2 = 1.
\]

Notice that

\[
\left\| \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) \right\|_2 = \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2}.
\]
Thus, we have
\[
\left\| B(i) - \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) \right\|_2 = \sqrt{1 - \left( \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2} \right)^2}.
\]

Thus, we have
\[
\|1\text{st term on the RHS of Eq. (34)}\|_2^2 = 1 - \left( \frac{B^T(q)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2} \right)^2,
\]
\[
\|2\text{nd term on the RHS of Eq. (34)}\|_2^2 = \left( \frac{B^T(q)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2} \right)^2.
\]

Applying those to Eq. (39), we then have \( \|C(i)\|_2 = 1 \).

Finally, the following lemma shows that \( C(i) \) can be written in the specific form in Eq. (37).

Lemma 17. Each \( C(i) \) defined in Eq. (34) satisfies that \( C(i)\epsilon_{\text{train}} \leq 0 \) and

\[
C(i) = \frac{B(i) + k(i)\epsilon_{\text{train}}}{\|B(i) + k(i)\epsilon_{\text{train}}\|_2},
\]

where
\[
k(i) = \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2} - \sqrt{1 - \left( \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2} \right)^2} \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2} \geq 0.
\]

Proof. Using Eq. (38) again, we decompose \( B(i) \) into two parts: one in the direction of \((-\epsilon_{\text{train}})\), the other orthogonal to \((-\epsilon_{\text{train}})\).

\[
B(i) = \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) + \left( B(i) - \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) \right).
\]

Thus, we have
\[
B(i) + k(i)\epsilon_{\text{train}} = \sqrt{1 - \left( \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2} \right)^2} \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}})
\]
\[
+ \left( B(i) - \frac{B^T(i)(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(-\epsilon_{\text{train}}) \right)\]
We then have
\[
\sqrt{1 - \left( \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2} \right)^2} \cdot (B_{(i)} + k_{(i)}\epsilon_{\text{train}})
\]
\[
\sqrt{1 - \left( \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2} \right)^2} \cdot \left( B_{(i)} - \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2}(\epsilon_{\text{train}}) \right)
\]
\[
+ \frac{B^T_{(q)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2^2}(\epsilon_{\text{train}})
\]

= C_{(i)}.

In other words, C_{(i)} and B_{(i)} + k_{(i)}\epsilon_{\text{train}} are along the same direction. Since \(\|C_{(i)}\|_2 = 1\), it must then also be equal to a normalized version of \(B_{(i)} + k_{(i)}\epsilon_{\text{train}}\), i.e.,
\[
\frac{B_{(i)} + k_{(i)}\epsilon_{\text{train}}}{\|B_{(i)} + k_{(i)}\epsilon_{\text{train}}\|_2} = C_{(i)}.
\]

This verifies (40). Note that \(C_{(i)}\epsilon_{\text{train}} = B_{(q)}\epsilon_{\text{train}} \leq 0\) by Lemma 16. It then only remains to prove \(k_{(i)} \geq 0\). Towards this end, because of Eq. (32), we have
\[
B^T_{(q)}(-\epsilon_{\text{train}}) \leq B^T_{(i)}(-\epsilon_{\text{train}})
\]
\[
\Rightarrow \sqrt{1 - \left( \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2} \right)^2} \leq 1.
\]

Thus, we have
\[
k_{(i)} \geq \frac{B^T_{(i)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2} - \frac{B^T_{(q)}(-\epsilon_{\text{train}})}{\|\epsilon_{\text{train}}\|_2} \geq 0.
\]

The result of the lemma thus follows.

Combining Lemma 15 and Lemma 17, we have proven that if \(\lambda^T(-\epsilon_{\text{train}}) \geq B^T_{(1)}\) and \(\lambda^TB_{(i)} \leq 1\), then \(\lambda^TC_{(i)} \leq 1\). Therefore, we have shown step 2, i.e., any feasible \(\lambda\) for the problem (36) is also feasible for the problem (35). The conclusion of Lemma 9 thus follows.

**H Proof of Lemma 11**

First, we show that \(\lambda_s\) defined in the lemma is feasible for the problem (35). Towards this end, note that because \(C^T_{(i)}(-\epsilon_{\text{train}}) = B^T_{(q)}(-\epsilon_{\text{train}})\) (see Lemma 16) for all \(i \in \{1, 2, \ldots, q\}\), we have \(\lambda^TC_{(i)} = 1\), which implies that \(\lambda_s\) is feasible for the problem (35). Then, it remains to show that \(\lambda_s\) is optimal for the problem (35) with probability at least \(1 - e^{-q/4-n}\).

Next, we will define an event \(\mathcal{E}\) with probability no smaller than
\[
1 - 2^{-q+1} \sum_{i=0}^{n-2} \binom{q-1}{i}.
\]
such that $\lambda^*$ is optimal whenever event $\mathcal{A}$ occurs. Towards this end, consider the null space of $-\epsilon_{\text{train}}$, which is defined as

$$\ker(-\epsilon_{\text{train}}) := \{\lambda \mid \lambda^T(-\epsilon_{\text{train}}) = 0\}.$$  

We then decompose all $C(i)$’s into two components, one is in the direction of $-\epsilon_{\text{train}}$, the other is in the null space of $-\epsilon_{\text{train}}$. Specifically, we have

$$C(i) = \left(C(i) - \frac{CT(i)(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}(-\epsilon_{\text{train}})\right) + \frac{CT(i)(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}(-\epsilon_{\text{train}})$$

$$= \left(C(i) - \frac{CT(q)(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}(-\epsilon_{\text{train}})\right) + \frac{CT(q)(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}(-\epsilon_{\text{train}}),$$

where in the last step we have used $CT(i)(-\epsilon_{\text{train}}) = CT(q)(-\epsilon_{\text{train}})$. For conciseness, we define

$$D(i) := C(i) - \frac{CT(q)(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}(-\epsilon_{\text{train}}).$$

Since $\|C(i)\|^2_2 = 1$ and $C(i)$ is orthogonal to $C(i) - D(i)$, we have

$$\|D(i)\|^2_2 = \sqrt{\|C(i)\|^2_2 - \|C(i) - D(i)\|^2_2} = \sqrt{1 - \left(\frac{CT(q)(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}\right)^2}.$$  

Thus, $D(i)$ has the same $\ell_2$-norm for all $i \in \{1, \cdots, q\}$. Therefore, $D(1), D(2), \cdots, D(q)$ can be viewed as $q$ points in a sphere in the space $\ker(-\epsilon_{\text{train}})$, which has $(n - 1)$ dimensions. By Lemma 17, we know that the projections of $C(i)$ and $B(i)$ to the space $\ker(-\epsilon_{\text{train}})$ have the same direction. Because $B(i)$’s are uniformly distributed on the hemisphere in $\mathbb{R}^n$, their projections to $\ker(-\epsilon_{\text{train}})$ are also uniformly distributed. Therefore, $D(i)$’s are uniformly distributed on a $(n - 1)$-dim sphere. By Lemma 10, with probability (41), there exists at least one of the vectors $D(1), D(2), \cdots, D(q)$ in any hemisphere. Let $\mathcal{A}$ denote this event with probability (41). Note that if we use a vector $\gamma \in \ker(-\epsilon_{\text{train}})$ to represent the axis of any such hemisphere in $\mathbb{R}^{n-1}$, then whether a vector $\zeta \in \ker(-\epsilon_{\text{train}})$ is on that hemisphere is totally determined by checking whether $\gamma^T \zeta > 0$. Thus, the event $\mathcal{A}$ is equivalent to, for any $\gamma \in \ker(-\epsilon_{\text{train}})$, there exists at least one of the vectors $D(1), D(2), \cdots, D(q)$ such that its inner product with $\gamma$ is positive.

We now prove the following statement that $\lambda^*$ is optimal whenever event $\mathcal{A}$ occurs. We prove by contradiction. Assume that event $\mathcal{A}$ occurs, suppose on the contrary that the maximum point is achieved at $\lambda = \mu \neq \lambda^*$ such that $\mu^T(-\epsilon_{\text{train}}) > (\lambda)^T(-\epsilon_{\text{train}})$. Since $\mu$ meets all constraints, we have

$$(\mu - \lambda^*)^T C(i) = \mu^T C(i) - 1 \leq 0 \text{ for all } i \in \{1, \cdots, q\}.$$  

Comparing the objective values at $\mu$ and $\lambda^*$, we have

$$(\mu - \lambda^*)^T(-\epsilon_{\text{train}}) > 0.$$  

(44)

Similar to the decomposition of $C(i)$ in Eq. (42), we decompose $(\mu - \lambda^*)$ into two components: one in the direction of $-\epsilon_{\text{train}}$ and the other in the null space of $-\epsilon_{\text{train}}$. Specifically, we have

$$(\mu - \lambda^*) = \left((\mu - \lambda^*) - \frac{(\mu - \lambda^*)^T(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}(-\epsilon_{\text{train}})\right) + \frac{(\mu - \lambda^*)^T(-\epsilon_{\text{train}})}{\epsilon_{\text{train}}^2}(-\epsilon_{\text{train}}).$$
Thus, we have
\[
(\mu - \lambda_*)^T C(i) = \left( (\mu - \lambda_*) - \frac{(\mu - \lambda_*)^T (\epsilon_{train})}{\|\epsilon_{train}\|_2^2} (-\epsilon_{train}) \right)^T 
\cdot \left( C(i) - \frac{C_i^T (\epsilon_{train})}{\|\epsilon_{train}\|_2^2} (-\epsilon_{train}) \right) 
+ \frac{1}{\|\epsilon_{train}\|_2^2} \left( (\mu - \lambda_*)^T (\epsilon_{train}) \right) \left( C_i^T (\epsilon_{train}) \right).
\]

For conciseness, we define
\[
\delta := (\mu - \lambda_*) - \frac{(\mu - \lambda_*)^T (\epsilon_{train})}{\|\epsilon_{train}\|_2^2} (-\epsilon_{train}).
\]

We then have
\[
(\mu - \lambda_*)^T C(i) = \delta^T D(i) + \frac{1}{\|\epsilon_{train}\|_2^2} (\mu - \lambda_*)^T (\epsilon_{train}) \left( C_i^T (\epsilon_{train}) \right) \geq \delta^T D(i),
\]

where the last inequality holds because \((\mu - \lambda_*)^T (\epsilon_{train}) > 0\) (by Eq. (44)) and \(C_i^T (\epsilon_{train}) = B_i^T (\epsilon_{train}) \geq 0\) (by Lemma 16 and Eq. (32)). Since \(\delta \in \ker(\epsilon_{train})\) and event \(A\) occurs, we can therefore find a \(D(k)\) such that \(\delta^T D(k) > 0\). Letting \(i = k\) in Eq. (45), we then have
\[
(\mu - \lambda_*)^T C(k) \geq \delta^T D(k) > 0,
\]

which contradicts Eq. (43). Therefore, \(\lambda^*\) must be optimal whenever event \(A\) occurs.

It only remains to show that the probability of event \(A\) given in Eq. (41) is at least \(1 - e^{-(q/4-n)}\), which is proven in the following Lemma 18.

**Lemma 18.**

\[
1 - 2^{-q+1} \sum_{i=0}^{n-2} \binom{q-1}{i} \geq 1 - e^{-(q/4-n)}. 
\]

The proof of Lemma 18 uses the following Chernoff bound.

**Lemma 19** (Chernoff bound for binomial distribution, Theorem 4(ii) in (Goemans, 2015)). Let \(X\) be a random variable that follows the binomial distribution \(B(m, \overline{p})\), where \(m\) denotes the number of experiments and \(\overline{p}\) denotes the probability of success for each experiment. Then
\[
\Pr(\{X \leq (1 - \delta)m\overline{p}\}) \leq \exp\left(-\frac{\delta^2 m\overline{p}}{2}\right)\quad \text{for all } \delta \in (0, 1).
\]

**Proof of Lemma 18:** Consider a random variable \(X\) with binomial distribution \(B(q - 1, 1/2)\). We have
\[
\Pr(\{X \leq n - 2\}) = 2^{-q+1} \sum_{i=0}^{n-2} \binom{q-1}{i}.
\]

Let
\[
\delta = 1 - \frac{2(n - 2)}{q - 1}, \quad \text{i.e.,} \quad 1 - \delta = \frac{2(n - 2)}{q - 1}.
\]

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Applying Chernoff bound stated in the Lemma 19, we have
\[
\Pr \left( \{ X \leq n - 2 \} \right) = \Pr \left( \left\{ X \leq (1 - \delta) \frac{q - 1}{2} \right\} \right)
\leq e^{-\delta^2(q-1)/4}.
\]
Also, we have
\[
\delta^2(q - 1)/4 = \frac{1}{4} \left( 1 - \frac{2(n - 2)}{q - 1} \right)^2 (q - 1)
\geq \frac{1}{4} \left( 1 - \frac{4(n - 2)}{q - 1} \right) (q - 1)
= \frac{1}{4} (q - 1 - 4(n - 2))
\geq \frac{q}{4} - n.
\]
Thus, we have
\[
1 - 2^{-q+1} \sum_{i=0}^{n-2} \left( \begin{array}{c}
q - 1 \\
i
\end{array} \right) = 1 - \Pr \left( \{ x \leq n - 2 \} \right)
\geq 1 - e^{-\delta^2(q-1)/4}
\geq 1 - e^{-(q/4 - n)}.
\]

I Proof of Lemma 13

The proof consists of three steps. Recall that \( B^{T}_{5n}(-\epsilon_{\text{train}}) \) ranks the \( 5n \)-th among all \( A^{T}_{i}(-\epsilon_{\text{train}}) \)'s and \( A^{T}_{i}\epsilon_{\text{train}} \)'s. In step 1, we first estimate the probability distribution about \( A^{T}_{i}(-\epsilon_{\text{train}}) \). In step 2, we use the result in step 1 to estimate \( B^{T}_{5n}(-\epsilon_{\text{train}}) \). In step 3, we relax and simplify the result in step 2 to get the exact result of Lemma 13. Without loss of generality\(^2\), we let \( \epsilon_{\text{train}} = [-\|\epsilon_{\text{train}}\|_{2} 0 \cdots 0]^{T} \). Thus, \( A^{T}_{i}(-\epsilon_{\text{train}}) = \|\epsilon_{\text{train}}\|_{2} A_{i1} \), where \( A_{ij} \) denotes the j-th element of the i-th column of \( A \).

Step 1

Notice that \( A_{i} \) (i.e., the i-th column of \( A \)) is a normalized Gaussian random vector. We use \( A'_{i} \) to denote the standard Gaussian random vector before the normalization, i.e., \( A'_{i} \) is a \( n \times 1 \) vector where each element follows i.i.d. standard Gaussian distribution. Thus, we have
\[
|A_{i1}| = \frac{|A'_{i1}|}{\|A'_{i}\|_{2}} = \frac{|A'_{i1}|}{\sqrt{(A'_{i1})^2 + \sum_{j=2}^{n}(A'_{ij})^2}}.
\]
For any \( k > 1 \), we then have
\[
\Pr \left( \left\{ \frac{1}{|A_{i}|} \leq k \right\} \right) = \Pr \left( \left\{ (A'_{i1})^2 \geq \frac{\sum_{j=2}^{n}(A'_{ij})^2}{k^2 - 1} \right\} \right).
\]

Notice that \( \sum_{j=2}^{n}(A'_{ij})^2 \) follows the chi-square distribution with \( (n - 1) \) degrees of freedom. When \( n \) is large, \( \sum_{j=2}^{n}(A'_{ij})^2 \) should be around its mean value. Further, \( A'_{i1} \) follows standard Gaussian distribution. Next, we use results of chi-square distribution and Gaussian distribution to estimate the distribution of \( A_{i1} \). The following lemma is useful for approximating a Gaussian distribution.

\(^2\)Rotating \( \epsilon_{\text{train}} \) around the origin is equivalent to rotating all columns of \( A \). Since the distribution of \( A_{i} \) is uniform on the unit hyper-sphere in \( \mathbb{R}^{n} \), such rotation does not affect the objective of the problem (29).
Lemma 20. When \( t \geq 0 \), we have
\[
\frac{\sqrt{2/\pi} \, e^{-t^2/2}}{t + \sqrt{t^2 + 4}} \leq \Phi_c(t) \leq \frac{\sqrt{2/\pi} \, e^{-t^2/2}}{t + \sqrt{t^2 + \frac{8}{\pi}}},
\]
where \( \Phi_c(\cdot) \) denotes the complementary cumulative distribution function (ccdf) of standard Gaussian distribution, i.e.,
\[
\Phi_c(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-u^2/2} \, du.
\]

Proof. By (7.1.13) in (Abramowitz and Stegun, 1972), we know that
\[
\frac{1}{t + \sqrt{t^2 + 2}} \leq e^{t^2/2} \int_0^\infty e^{-y^2} \, dy \leq \frac{1}{t + \sqrt{t^2 + \frac{4}{\pi}}} \quad (x \geq 0).
\]
Let \( x = t/\sqrt{2} \). We have
\[
\frac{1}{\sqrt{2} + \sqrt{2} + \frac{2}{\sqrt{2} + 2}} \leq e^{t^2/2} \int_0^\infty e^{-y^2} \, dy \leq \frac{1}{\sqrt{2} + \sqrt{2} + \frac{4}{\pi}}
\]
\[
\Rightarrow \frac{\sqrt{2/\pi} \, e^{-t^2/2}}{t + \sqrt{t^2 + 4}} \leq \frac{1}{\sqrt{2\pi} \sqrt{t + \sqrt{t^2 + 4}}} \leq \frac{\sqrt{2/\pi} \, e^{-t^2/2}}{t + \sqrt{t^2 + \frac{8}{\pi}}}
\]
\[
\Rightarrow \frac{\sqrt{2/\pi} \, e^{-t^2/2}}{t + \sqrt{t^2 + 4}} \leq \Phi_c(t) \leq \frac{\sqrt{2/\pi} \, e^{-t^2/2}}{t + \sqrt{t^2 + \frac{8}{\pi}}}.
\]

The result of this lemma thus follows. \( \square \)

The following lemma gives an estimate of the probability distribution of \( A_{i1} \).

Lemma 21.
\[
\Pr \left( \left\{ \frac{1}{|A_{i1}|} \leq k \right\} \right) \geq 2 \left( 1 - \frac{1}{\sqrt{e}} \right) \sqrt{\frac{2}{\pi} \frac{e^{-t^2/2}}{t + \sqrt{t^2 + 4}}}, \quad (47)
\]
where
\[
t = \sqrt{\frac{n + \sqrt{2n - 1}}{k^2 - 1}}.
\]

Proof. For any \( m > 0 \), we have
\[
\Pr \left( \left\{ \frac{1}{|A_{i1}|} \leq k \right\} \right) = \Pr \left( \left\{ (A_{i1})^2 \geq \frac{\sum_{j=2}^n (A_{ij})^2}{k^2 - 1} \right\} \right)
\]
\[
\geq \Pr \left( \left\{ (A_{i1})^2 \geq \frac{n - 1 + 2\sqrt{(n-1)m + 2m}}{k^2 - 1} \right\} \right)
\]
\[
\cdot \Pr \left( \left\{ \sum_{j=2}^n (A_{ij})^2 \leq n - 1 + 2\sqrt{(n-1)m + 2m} \right\} \right) \quad \text{(since all } A_{ij} \text{'s are i.i.d.)}
\]
Notice that $\sum_{j=2}^{n}(A'_{ij})^2$ follows chi-square distribution with $(n-1)$ degrees freedom. Applying Lemma 8, we have

$$\Pr\left(\left\{ \frac{1}{|A_{i1}|} \leq k \right\} \right) \geq \Pr\left(\left\{ (A'_{i1})^2 \geq \frac{n-1+2\sqrt{(n-1)m+2m}}{k^2-1} \right\} \right) \cdot (1 - e^{-m})$$

$$= 2(1 - e^{-m})\Phi_c\left( \frac{\sqrt{n-1+2\sqrt{(n-1)m+2m}}}{k^2-1} \right)$$

(since the distribution of $A_{i1}$ is symmetric with respect to 0).

We now let $m = 1/2$ in Eq. (48). Then

$$\sqrt{\frac{n-1+2\sqrt{(n-1)m+2m}}{k^2-1}} = \sqrt{\frac{n+\sqrt{2(n-1)}}{k^2-1}} = t.$$  

Applying Lemma 20, the result of this lemma thus follows. \(\square\)

**Step 2**

Next, we estimate the distribution of $B^T_{(5n)}(-\epsilon_{\text{train}})$. We first introduce a lemma below, which will be used later.

**Lemma 22.** If $t \geq 0.5$, then $t + \sqrt{t^2 + 4} < e^{t+0.5}$.

**Proof.** Let $f(t) = e^{t+0.5} - (t + \sqrt{t^2 + 4})$. Then $f(0.5) \approx 0.157 > 0$. We only need to prove that $df/dt \geq 0$ when $t \geq 0.5$. Indeed, when $t \geq 0.5$, we have

$$\frac{df(t)}{dt} = e^{t+0.5} - 1 - \frac{t}{\sqrt{t^2 + 4}} \geq e - 1 - 1 \geq 0 \text{ (notice that } t \leq \sqrt{t^2 + 4} \text{ for any } t).$$

\(\square\)

Now, we estimate $B^T_{(5n)}(-\epsilon_{\text{train}})$ by the following proposition.

**Proposition 23.** Let

$$C = \frac{1}{5} \left(1 - \frac{1}{e^2}\right) \sqrt{\frac{2}{\pi}} \approx 0.063.$$  

(49)

When $p - s \geq ne^{9/8}/C$, the following holds.

$$\frac{\|\epsilon_{\text{train}}\|_2}{B^T_{(5n)}(-\epsilon_{\text{train}})} \leq \sqrt{1 + \frac{n + \sqrt{2\sqrt{n-1}}}{\sqrt{2 \ln \left( \frac{C(p-s)}{n} \right)^2} - 1}},$$  

(50)

with probability at least $1 - e^{-5n/4}$.

(Notice that, by applying this proposition in Corollary 12, Eq. (50) already suggests an upper bound of $\|w^T\|_1$.)

**Proof.** For conciseness, we use $\rho(n, k)$ to denote the right-hand-side of Eq. (47), i.e.,

$$\rho(n, k) = 10C \frac{e^{-t^2/2}}{t + \sqrt{t^2 + 4}} \bigg|_{t = \sqrt{\frac{n+2\sqrt{2\sqrt{n-1}}}{k^2-1}}}.$$  

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Let $k$ take the value of the RHS of Eq. (50). Then, we have

\[
    t = \sqrt{n + \frac{2(n-1)}{k^2-1}}
\]

\[
    = \sqrt{\frac{n + \frac{2(n-1)}{k^2-1}}{1 + \frac{n + \frac{2(n-1)}{k^2-1}}{2\ln \frac{n}{C(p-s)}}}} - 1
\]

\[
    = \sqrt{2\ln \frac{C(p-s)}{n}} - 1. \quad (51)
\]

Because $p - s \geq ne^{9/8}/C$, we have $t \geq 0.5$. By Lemma 22, we have $t + \sqrt{t^2 + 4} < e^{t + 0.5}$. Thus, we have

\[
    \rho(n, k) \geq 10C \exp\left(-\frac{t^2}{2} - t - 0.5\right)
\]

\[
    = 10C \exp\left(-\frac{1}{2}(t + 1)^2\right)
\]

\[
    = 10C \frac{n}{C(p-s)} \quad \text{(using Eq. (51))}
\]

\[
    = \frac{10n}{p-s}. \quad (52)
\]

By the definition of $B_{(5n)}$ and Eq. (32), we have

\[
    \Pr (\{\text{Eq. (50)}\}) = \Pr \left(\#\{i \mid i \in \{1, 2, \ldots, p-s\}, \frac{1}{|A_{i1}|} \leq k \geq 5n\}\right). \quad (53)
\]

Consider a random variable $x$ following the binomial distribution $B(p-s, \rho(n, k))$. Since $A_{i1}$’s are i.i.d. and $\Pr \left(\frac{1}{|A_{i1}|} \leq k\right) \geq \rho(n, k)$, we must have

\[
    \text{Eq. (53)} \geq \Pr (\{x \geq 5n\}) = 1 - \Pr (\{x \leq 5n-1\}) \geq 1 - \Pr (\{x \leq 5n\}).
\]

It only remains to show that $\Pr (\{x \leq 5n\}) \leq e^{-5n/4}$. Applying Lemma 19, we have

\[
    \Pr (\{x \leq 5n\}) = \Pr (\{x \leq (1 - \delta)(p-s)\rho(n, k)\}) \leq e^{-\delta^2(p-s)\rho(n, k)/2}, \quad (54)
\]

where

\[
    \delta = 1 - \frac{5n}{(p-s)\rho(n, k)} \quad \text{(so } 5n = (1 - \delta)(p-s)\rho(n, k)).
\]

Since $(p-s)\rho(n, k) \geq 10n$ by Eq. (52), we must have $\delta \geq 0.5$. Substituting into Eq. (54), we have $\Pr (\{x \leq 5n\}) \leq \exp(-0.5^2 \cdot (10n)/2) = e^{-5n/4}$.

\[
\]

**Step 3**

Notice that by utilizing Proposition 23 and Corollary 12, we already have an upper bound on $\|w^I\|_1$. To get the simpler form in Lemma 13, we only need to use the following lemma to simplify the expression in Proposition 23.

**Lemma 24.** When $n \geq 100$ and $p \geq (16n)^4$, we must have

\[
    \text{RHS of Eq. (50)} \leq \sqrt{1 + \frac{3n/2}{\ln p}}.
\]

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Proof. Because $n > 100$ and $p \geq (16n)^4$, we have $p \geq 10^{12}$. Thus, we have
\[
\ln p \geq 25 \quad \text{(since $\ln 10 \approx 2.3 > 25/12$)}
\]
\[
\implies \sqrt{\ln p} - 2 \geq 3
\]
\[
\implies \sqrt{\ln p} - 2 \geq \sqrt{3 \ln 2 + 6} \quad \text{(since $\ln 2 < 1$)}
\]
\[
\implies \frac{1}{2} \left( \sqrt{\ln p} - 2 \right)^2 \geq \frac{3}{2} \ln 2 + 3
\]
\[
\implies \frac{3}{2} (\ln p - \ln 2) \geq \ln p + 2 \sqrt{\ln p + 1} \quad \text{(by expanding the square and rearranging terms)}
\]
\[
\implies \sqrt{\ln p} + 1 \leq \sqrt{\frac{3}{2} \ln p - \ln 2} \quad \text{(by taking square root on both sides)}.
\]
Because $s \leq n$ and $p \geq (16n)^4 \geq 2n$, we have $\ln(p - s) \geq \ln(p - n) \geq \ln(p/2)$. Thus, we have
\[
\sqrt{\ln p} + 1 \leq \sqrt{\frac{3}{2} \ln(p - s)}.
\] (55)

We still use $C$ defined in Eq. (49). We have
\[
p \geq (16n)^4 \implies p \geq \left( \frac{n}{C} \right)^4 + n + \left( (16n)^4 - \left( \frac{n}{C} \right)^4 - n \right).
\] (56)

Note that
\[
(16n)^4 - \left( \frac{n}{C} \right)^4 - n = n \left( n^3 \left( 16^4 - \left( \frac{1}{C} \right)^4 \right) - 1 \right)
\]
\[
\geq n (n^3 - 1) \quad \text{(because $16^4 - \left( \frac{1}{C} \right)^4 \approx 16^4 - \left( \frac{1}{0.063} \right)^4 > 1$)}
\]
\[
\geq 0 \quad \text{(because $n \geq 1$)}.
\]

Applying it in Eq. (56), we have
\[
p - n \geq \left( \frac{n}{C} \right)^4
\]
\[
\implies p - s \geq \left( \frac{n}{C} \right)^4 \quad \text{(because $s \leq n$)}
\]
\[
\implies (p - s)^{-3} \left( \frac{C}{n} \right)^4 (p - s)^4 \geq 1
\]
\[
\implies -3 \ln(p - s) + 4 \ln \frac{C(p - s)}{n} \geq 0
\]
\[
\implies 2 \ln \frac{C(p - s)}{n} \geq \frac{3}{2} \ln(p - s)
\]
\[
\implies 2 \ln \frac{C(p - s)}{n} \geq (\sqrt{\ln p} + 1)^2 \quad \text{(by Eq. (55))}
\]
\[
\implies \left( \sqrt{2 \ln \frac{C(p - s)}{n} - 1} \right)^2 \geq \ln p.
\] (57)

When $n \geq 100$, we always have
\[
n - 1 \leq \frac{n^2}{8}
\]
\[
\implies \sqrt{2 \ln n - 1} \leq \frac{n}{2}.
\] (58)

Substituting Eq. (57) and Eq. (58) into the RHS of Eq. (50), the conclusion of this lemma thus follows. \qed
J Proof of Proposition 7

For conciseness, we define $G_{ij} := X_i^T X_j$. According to the normalization in Eq. (5), we have

$$G_{ij} := \frac{H_i^T H_j}{\|H_i\|_2 \|H_j\|_2}.$$ 

Our proof consists of four steps. In step 1, we relate the tail probability of any $|G_{ij}|$ (where $i \neq j$) to the tail probability of $H_i^T H_j$. In step 2, we estimate the tail probability of $H_i^T H_j$. In step 3, we use union bound to estimate the cdf of $M$, so that we can get an upper bound on $M$ with high probability. In step 4, we simplify the result derived in step 3.

Step 1: Relating the tail probability of $|G_{ij}|$ to that of $H_i^T H_j$.

For any $i \neq j$, we have

$$\Pr\left(\{|G_{ij}| > a\}\right) = \Pr\left(\{|G_{ij}| > a, \|H_i\|_2 \geq \sqrt{\frac{n}{2}}, \|H_j\|_2 \geq \sqrt{\frac{n}{2}}\}\right) + \Pr\left(\{|G_{ij}| > a, (\|H_i\|_2 < \sqrt{\frac{n}{2}} \text{ or } \|H_j\|_2 < \sqrt{\frac{n}{2}})\}\right).$$  \hspace{1cm} (59)

The first term can be bounded by

$$\Pr\left(\{|G_{ij}| > a, \|H_i\|_2 \geq \sqrt{\frac{n}{2}}, \|H_j\|_2 \geq \sqrt{\frac{n}{2}}\}\right) \leq \Pr\left(\{|H_i^T H_j| > \frac{na}{2}\}\right),$$

because

$$|G_{ij}| > a, \|H_i\|_2 \geq \sqrt{\frac{n}{2}}, \|H_j\|_2 \geq \sqrt{\frac{n}{2}} \implies |H_i^T H_j| > \frac{na}{2}.$$  

Thus, we have, from Eq. (59),

$$\Pr\left(\{|G_{ij}| > a\}\right) \leq \Pr\left(\{|H_i^T H_j| > \frac{na}{2}\}\right) + \Pr\left(\{|H_i\|_2 < \sqrt{\frac{n}{2}}\}\right)$$

$$\quad + \Pr\left(\{|H_j\|_2 < \sqrt{\frac{n}{2}}\}\right)$$

$$= 2 \Pr\left(\{|H_i^T H_j| > \frac{na}{2}\}\right) + 2 \Pr\left(\{|H_i\|_2 < \sqrt{\frac{n}{2}}\}\right),$$  \hspace{1cm} (60)

where the last equality is because the distribution of $H_i^T H_j$ is symmetric around 0, and $H_j$ has the same distribution as $H_i$. Notice that $\|H_i\|_2^2$ follows chi-square distribution with $n$ degrees of freedom. By Lemma 8 (using $x = n/16$), we have

$$\Pr\left(\{|H_i\|_2 < \sqrt{\frac{n}{2}}\}\right) = \Pr\left(\{|H_i\|_2^2 < \frac{n}{2}\}\right) \leq e^{-n/16}.$$

Thus, we have

$$\Pr\left(\{|G_{ij}| > a\}\right) \leq 2 \Pr\left(\{|H_i^T H_j| > \frac{na}{2}\}\right) + 2e^{-n/16}.$$  \hspace{1cm} (61)
Step 2: Estimating the tail probability of $H_i^TH_j$.

Notice that $H_i^TH_j$ is the sum of product of two Gaussian random variables. We will use the Chernoff bound to estimate its tail probability. Towards this end, we first calculate the moment generating function (M.G.F) of the product of two Gaussian random variables.

**Lemma 25.** If $X$ and $Y$ are two independent standard Gaussian random variables, then the M.G.F of $XY$ is

$$E[e^{tXY}] = \frac{1}{\sqrt{1-t^2}},$$

for any $t^2 < 1$.

**Proof.**

$$E[e^{tXY}] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{txy} e^{-\frac{x^2+y^2}{2}} \, dx \, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(y-xt)^2}{2}} \, dy \right) \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx$$

$$= \frac{1}{\sqrt{1-t^2}}.$$

We introduce the following lemma that helps in our calculation later.

**Lemma 26.** For any $x > 0$,

$$\arg \max_{t \in (0, 1)} \left( tx + \frac{n}{2} \ln(1-t^2) \right) = \frac{-n + \sqrt{n^2 + 4x^2}}{2x}.$$

**Proof.** Let

$$f(t) = tx + \frac{n}{2} \ln(1-t^2), \quad t \in (0, 1).$$

Then, we have

$$\frac{df(t)}{dt} = x - \frac{nt}{1-t^2}.$$

Letting $df(t)/dt = 0$, we have exactly one solution in $(0, 1)$ given by

$$t = \frac{-n + \sqrt{n^2 + 4x^2}}{2x}.$$

Notice that $df(t)/dt$ is monotone decreasing with respect to $t$ and thus $f(t)$ is concave on $(0, 1)$. The result of this lemma thus follows.

We then use the Chernoff bound to estimate $H_i^TH_j$ in the following lemma.

**Lemma 27.**

$$\Pr \left( \left\{ H_i^TH_j > \frac{na}{2} \right\} \right) \leq \exp \left( -\frac{n}{2} \left( at + \ln \frac{2t}{a} \right) \right),$$

where

$$t = \frac{-1 + \sqrt{1 + a^2}}{a}.$$
Proof. Notice that
\[
H_i^T H_j = \sum_{k=1}^{n} H_{ik} H_{jk} = \sum_{k=1}^{n} Z_k,
\]
where \( Z_k := H_{ik} H_{jk} \). Using the Chernoff bound, we have
\[
\Pr \left( \{ H_i^T H_j > x \} \right) \leq \min_{t>0} e^{-tx} \prod_{k=1}^{n} \mathbb{E}[e^{tZ_k}]
\]
Since each \( Z_k \) is the product of two independent standard Gaussian variable, using Lemma 26, we have, for any \( x > 0 \),
\[
\Pr \left( \{ H_i^T H_j > x \} \right) \leq \min_{t\in(0,1)} e^{-tx} (1 - t^2) \left( \frac{n+\sqrt{n^2+4x^2}}{2x} \right)^{n/2} (\text{by Lemma 26})
\]
where the last equality is because \( t = \left( -n + \sqrt{n^2+4x^2} \right) / 2x \) is one solution of the quadratic equation in \( t \) that \( xt^2 + nt - x = 0 \) (which implies \( 1 - t^2 = nt/x \)).

Letting \( x = \frac{na}{2} \), we get \( t = (-1 + \sqrt{1+a^2}) \), and
\[
\exp \left( -tx - \frac{n}{2} \ln(nt/x) \right) = \exp \left( -\frac{na}{2} - \frac{n}{2} \ln \frac{2t}{a} \right) = \exp \left( -\frac{n}{2} \left( at + \ln \frac{2t}{a} \right) \right).
\]
The result of this lemma thus follows.

\[\square\]

Step 3: Estimating the distribution of \( M \).

Since \( M \) is defined as the maximum of all \( |G_{ij}| \) for \( i \neq j \), we use the union bound to estimate the distribution of \( M \) in the following proposition.

Proposition 28.
\[
\Pr \left( \left\{ M \leq 2\sqrt{6} \sqrt{\frac{\ln p}{n} \left( \frac{6 \ln p}{n} + 1 \right)} \right\} \right) \geq 1 - 2e^{-\ln p} - 2e^{-n/16+2\ln p}.
\]

To prove Proposition 28, we introduce a technique lemma first.

Lemma 29. For any \( x > 0 \), we must have
\[
\ln x \geq 1 - \frac{1}{x}.
\]

Proof. We define a function
\[
f(x) := \ln x - (1 - \frac{1}{x}), \quad x > 0.
\]
It suffices to show that \( \min f(x) = 0 \). We have
\[
\frac{df(x)}{dx} = \frac{1}{x} - \frac{1}{x^2} = \frac{x-1}{x^2}.
\]
Thus, \( f(x) \) is monotone decreasing in \((0, 1)\) and monotone increasing in \((1, \infty)\). Thus, \( \min f(x) = f(1) = 0 \).

The conclusion of this lemma thus follows. \[\square\]
We are now ready to prove Proposition 28.

**Proof of Proposition 28**: Applying Lemma 27 to Eq. (61), we have

\[
\Pr (\{|G_{ij}| > a\}) \leq 2 \exp \left( -\frac{n}{2} \left( at + \ln \frac{2t}{a} \right) \right) + 2e^{-n/16},
\]

(62)

where

\[
t = -1 + \frac{\sqrt{1 + a^2}}{a}.
\]

(63)

Since \( M = \max_{i \neq j} |G_{ij}| \), we have

\[
\Pr (\{M \leq a\}) = 1 - \Pr \left( \bigcup_{i \neq j} \{|G_{ij}| > a\} \right)
\]

\[
\geq 1 - \sum_{i \neq j} \Pr (\{|G_{ij}| > a\}) \quad \text{(by the union bound)}
\]

\[
= 1 - p(p-1) \Pr (\{|G_{ij}| > a\}) \quad \text{(since all } G_{ij} \text{ has the same distribution)}
\]

\[
\geq 1 - e^{2\ln p} \Pr (\{|G_{ij}| > a\})
\]

\[
\geq 1 - e^{-n/16+2\ln p} - 2 \exp \left( -\frac{n}{2} \left( at + \ln \frac{2t}{a} - \frac{4 \ln p}{n} \right) \right) \quad \text{(by Eq. (62))}.
\]

(64)

Let

\[
a = 2 \sqrt{6} \left( \frac{\ln p}{n} \left( \frac{6 \ln p}{n} + 1 \right) \right).
\]

(65)

Substituting Eq. (65) into Eq. (63), we have

\[
at = -1 + \sqrt{1 + a^2}
\]

\[
= -1 + \sqrt{1 + \frac{24 \ln p}{n} + \left( \frac{12 \ln p}{n} \right)^2}
\]

\[
= -1 + \sqrt{\left( \frac{12 \ln p}{n} + 1 \right)^2}
\]

\[
= \frac{12 \ln p}{n}.
\]

(66)

Thus, we have

\[
\frac{2t}{a} = \ln \frac{2at}{a^2} = \ln \frac{2 \cdot \frac{12 \ln p}{n}}{24 \cdot \frac{\ln p}{n} \left( \frac{6 \ln p}{n} + 1 \right)} = \ln \frac{1}{\frac{6 \ln p}{n} + 1}
\]

\[
\geq 1 - \left( \frac{6 \ln p}{n} + 1 \right) \quad \text{(by Lemma 29)}
\]

\[
= -\frac{6 \ln p}{n}.
\]

(67)

By Eq. (66) and Eq. (67), we have

\[
-\frac{n}{2} \left( at + \ln \frac{2t}{a} - \frac{4 \ln p}{n} \right) \leq -\frac{n}{2} \left( \frac{12 \ln p}{n} - \frac{6 \ln p}{n} - \frac{4 \ln p}{n} \right) = -\ln p.
\]

Substituting into Eq. (64), the result of this proposition follows.  

\[\blacksquare\]
Step 4: Simplifying the expression in Proposition 28.

By the assumption of Proposition 7 that \( p \leq \exp(n/36) \), we have
\[
\frac{6 \ln p}{n} + 1 \leq \frac{7}{6}.
\]

Thus, we have
\[
2\sqrt{6} \sqrt{\frac{\ln p}{n} \left(\frac{6 \ln p}{n} + 1\right)} \leq 2\sqrt{7} \sqrt{\frac{\ln p}{n}}.
\]

We also have
\[
\frac{-n}{16} + 2 \ln p \leq \frac{-n}{16} + 2 \cdot \frac{n}{36} = -\frac{n}{144}.
\]

Applying Eq. (68) and Eq. (69) to Proposition 28, we then get Proposition 7.

K Lower bounds

We next discuss how tight the upper bounds in the earlier subsections are. Note that \( \|w^I\|_1 \) and \( M \) play key roles in the upper bound in Corollary 5. Below, we first show that the upper bounds for \( \|w^I\|_1 \) (in Proposition 6) and for \( M \) (in Proposition 7) are quite tight.

A trivial lower bound on \( \|w^I\|_1 \) is \( \|w^I\|_1 \geq \|\epsilon_{\text{train}}\|_2^2 \). To realize that, letting \( w^I_{(i)} \) denote the \( i \)-th element of \( w^I \), we have
\[
\|\epsilon_{\text{train}}\|_2^2 = \|X_{\text{train}}w^I\|_2 = \left\| \sum_{i=1}^{p} w^I_{(i)}X_i \right\|_2 \\
\leq \sum_{i=1}^{p} |w^I_{(i)}| \cdot \|X_i\|_2 = \|w^I\|_1 \quad \text{(notice } \|X_i\|_2 = 1)\).
\]

Even by this trivial lower bound, we immediately know that our upper bound on \( \|w^I\|_1 \) in Proposition 6 is accurate when \( p \to \infty \). Indeed, we can do more than this trivial lower bound, as shown in Proposition 31 below.

Following the construction of Problem (33), it is not hard to show that \( B(1) \), i.e., the vector that has the largest inner-product with \((-\epsilon_{\text{train}})\), defines a lower bound for \( \|w^I\|_1 \).

**Lemma 30.**
\[
\|w^I\|_1 \geq \frac{\|\epsilon_{\text{train}}\|_2^2}{B(1)(-\epsilon_{\text{train}})}
\]

**Proof.** Let
\[
\lambda_* = \frac{(-\epsilon_{\text{train}})}{B(1)(-\epsilon_{\text{train}})}.
\]

By the definition of \( B(1) \), for any \( i \in \{1, 2, \ldots, p-s\} \), we have
\[
|\lambda_*^T A_i| = \left| \frac{A_i^T \epsilon_{\text{train}}}{|B(1)_{\text{train}}|} \right| \leq 1.
\]

In other words, \( \lambda_* \) satisfies all constraints of the problem (31), which implies that the optimal objective value of (31) is at least
\[
\lambda_*^T (-\epsilon_{\text{train}}) = \frac{\|\epsilon_{\text{train}}\|_2^2}{B(1)(-\epsilon_{\text{train}})}.
\]

The result of this lemma thus follows. \(\square\)
By bounding $B^P_{(1)}(-\epsilon_{\text{train}})$, we can show the following result.

**Proposition 31.** When $p \leq e^{(n-1)/16}/n$ and $n \geq 17$, then

$$\frac{\|w^I\|_1}{\|\epsilon_{\text{train}}\|_2} \geq \sqrt{1 + \frac{n}{9 \ln p}}$$

with probability at least $1 - 3/n$.

The proof is available in Appendix L. Comparing Proposition 6 with Proposition 31, we can see that, with high probability, the upper and lower bounds of $\|w\|_1$ differ by at most a constant factor.

For $M$, we have the following lemma.

**Proposition 32.** Assume $p > n$. For any $\delta > 0$, there exists a threshold $\tilde{n}$ such that for any $n \geq \tilde{n}$ and $e^{4\delta n} < |p/2| < \exp\left(\frac{2-\sqrt{3}}{4} n\right)$, the following holds.

$$\Pr\left(\left\{ M \geq \sqrt{\frac{2}{8} \sqrt{\ln p/n}} \right\} \right) \geq 1 - \exp\left(-\sqrt{p}\right).$$

Comparing Proposition 7 and with Proposition 32, we can see that when $n$ and $p$ are large, with high probability the upper and lower bounds on $M$ only differ by a constant factor.

The previous discussion suggest that our bounds on $\|w^I\|_1$ and $M$ are quite tight. For Corollary 5, however, we do not know how to obtain a matching lower bound on $\|w^{BP}\|_2$. Still, we conjecture that our lower bound on $\|w^{BP}\|_2$ may be reasonably tight for the following reasons. First, in Proposition 3, the first term of the upper bound on $\|w^{BP}\|_1$ is at most a constant multiple of $\|w^I\|_1$ when $K$ is larger than 4. Intuitive, $w^I$ is a special case of $w^{BP}$ when the intended signal (i.e., $\beta$) is zero. Thus, we expect that $\|w^{BP}\|_1$ should be at least on the same order of $\|w^I\|_1$. This suggests that our upper bound on $\|w^{BP}\|_1$ may be reasonably tight.

**Lemma 33.** There are at most $(n + s)$ non-zero elements in $w^{BP}$.

**Proof.** If we prove that

$$w^{BP}_1 = \arg\min_{w_1} \|w_1\|_1 \text{ subject to } X_{\text{train}} \begin{bmatrix} w^{BP}_0 \\ w_1 \end{bmatrix} = \epsilon_{\text{train}},$$

then the result of this lemma follows, since by (70), there are at most $n$ non-zero elements in $w^{BP}_1$. We now prove (70) by contradiction. Suppose in contrary that there exist one $w^*_1 \neq w^{BP}_1$ such that

$$w^*_1 = \arg\min_{w_1} \|w_1\|_1 \text{ subject to } X_{\text{train}} \begin{bmatrix} w^{BP}_0 \\ w_1 \end{bmatrix} = \epsilon_{\text{train}}.$$  

Thus, we have

$$\|w^*_1\|_1 \leq \|w^{BP}_1\|_1 \\
\implies \|w^{BP}_0 + \beta_0\|_1 + \|w^*_1\|_1 \leq \|w^{BP}_0 + \beta_0\|_1 + \|w^{BP}_1\|_1.$$  

Notice that

$$X_{\text{train}} \begin{bmatrix} w^{BP}_0 \\ w^*_1 \end{bmatrix} = \epsilon_{\text{train}}.$$  

Therefore, by (15), we have

$$\begin{bmatrix} w^{BP}_0 \\ w^*_1 \end{bmatrix} = w^{BP} = \begin{bmatrix} w^{BP}_0 \\ w^{BP}_1 \end{bmatrix},$$

which implies that $w^*_1 = w^{BP}$, which contradicts our assumption that $w^*_1 \neq w^{BP}_1$. \qed
By Lemma 33, we then have the following lower bound on \( \|w_{\text{BP}}\|_2 \):

\[
\|w_{\text{BP}}\|_2 \geq \frac{\|w_{\text{BP}}\|_1}{\sqrt{n+s}} \geq \frac{\|w_{\text{BP}}\|_1}{\sqrt{2n}}. \tag{71}
\]

By Propositions 4 and 7, our upper bounds roughly correspond to

\[
\|w_{\text{BP}}\|_2 = \|\epsilon_{\text{train}}\|_2 + O\left(\sqrt{\frac{\ln p}{n}}\right) \|w_{\text{BP}}\|_1. \tag{72}
\]

Comparing it with (71), we can see that there is still significant difference between the orders of the factors in front of \( \|w_{\text{BP}}\|_1 \). Therefore, we cannot rigorously prove that the result in Proposition 4 is tight. However, we conjecture that it is reasonably tight for the following reason. Although there can be \((n+s)\) non-zero elements in \(w_{\text{BP}}\), we conjecture that the first \(s\) elements tend to have more weight than the rest. Therefore, it is likely that

\[
\|w_{\text{BP}}\|_2 \geq \|w_{\text{BP}}^0\|_2 \geq \frac{\|w_{\text{BP}}^0\|_1}{\sqrt{s}} = \Omega\left(\frac{1}{\sqrt{s}}\right) \|w_{\text{BP}}\|_1. \tag{73}
\]

By Eq. (27), we have

\[
s \leq \frac{1}{16\sqrt{7}} \sqrt{n \ln p} \Rightarrow \frac{1}{\sqrt{s}} = \Omega\left(\frac{s}{\ln p/n}\right).
\]

If Eq. (73) is true, we will then have

\[
\|w_{\text{BP}}\|_2 = \Omega\left(\frac{s}{\sqrt{n \ln p}}\right) \|w_{\text{BP}}\|_1,
\]

which is comparable to Eq. (72), and suggests that Proposition 4 may also be reasonably tight. How to rigorously establish Eq. (73) would be an interesting direction for future work.

L  Proof of Proposition 31

To prove Proposition 31, we will prove a slightly stronger result in Proposition 34 given below.

**Proposition 34.** When \((p-s) \leq e^{(n-1)/16}/n\) and \(n \geq 17\), the following holds.

\[
\frac{\|w^f\|_1}{\|\epsilon_{\text{train}}\|_2} \geq \sqrt{1 + \frac{n-1}{4\ln n + 4\ln(p-s)}}, \tag{74}
\]

with probability at least \(1 - 3/n\).

To prove Proposition 34, we introduce a technical lemma first.

**Lemma 35.** For any \(x \in [0,1)\), we have

\[
\ln(1-x) \geq \frac{-x}{\sqrt{1-x}}. \tag{75}
\]

**Proof.** Let

\[
f(x) = \ln(1-x) + \frac{x}{\sqrt{1-x}}.
\]

...
Note that \( f(0) = 0 \). Thus, it suffices to show that \( df(x)/dx \geq 0 \) when \( x \in [0,1) \). Indeed, we have

\[
\frac{df(x)}{dx} = -\frac{1}{1-x} + \frac{\sqrt{1-x} - x^{-1}}{2\sqrt{1-x}}
= -\frac{\sqrt{1-x} + 1 - x + x/2}{(1-x)^{3/2}}
= \frac{2 - x - 2\sqrt{1-x}}{2(1-x)^{3/2}}
= \frac{(1 - \sqrt{1-x})^2}{2(1-x)^{3/2}}
\geq 0.
\]

The result of this lemma thus follows.

We are now ready to prove Proposition 34.

**Proof of Proposition 34:** Because of Lemma 30, we only need to show that

\[
\|\epsilon_{\text{train}}\|_2 B_{(1)}^T(-\epsilon_{\text{train}}) \geq \sqrt{1 + \frac{n-1}{4\ln n + 4\ln(p-s)}},
\]

with probability at least \( 1 - 3/n \). Similar to what we do in Appendix I, without loss of generality, we let \( \epsilon_{\text{train}} = [-\|\epsilon_{\text{train}}\|_2, 0, \cdots, 0]^T \). Thus,

\[
\|\epsilon_{\text{train}}\|_2 B_{(1)}^T(-\epsilon_{\text{train}}) = \max_i |\mathbf{A}_{i1}|.
\]

We use the following two steps in order to get an upper bound of \( 1/\max_i |\mathbf{A}_{i1}| \). Step 1: estimate the distribution of \( 1/|\mathbf{A}_{i1}| \) for any \( i \in \{1, \cdots, p-s\} \). Step 2: utilizing the fact that all \( \mathbf{A}_{i1} \)'s are independent, we estimate \( 1/\max_i |\mathbf{A}_{i1}| \) base on the result in Step 1.

The Step 1 proceeds as following. For any \( i \in \{1, \cdots, p-s\} \) and any \( k \geq 0 \), we have

\[
\Pr \left( \frac{1}{|\mathbf{A}_{i1}|} \geq k \right)
= \Pr \left( \left( \mathbf{A}_{i1}^T \right)^2 \leq \frac{\sum_{j=2}^{n} (\mathbf{A}_{ij}^T)^2}{k^2 - 1} \right) \quad \text{ (by Eq. (46)).}
\]

Therefore, for any \( m > 0 \), we have

\[
\Pr \left( \left( \mathbf{A}_{i1}^T \right)^2 \leq \frac{n-1-2\sqrt{(n-1)m}}{k^2 - 1} \right)
\]

\[
\geq \Pr \left( \sum_{j=2}^{n} (\mathbf{A}_{ij}^T)^2 > n-1-2\sqrt{(n-1)m} \right) \quad \text{ (because all } \mathbf{A}_{ij}^T \text{'s are independent)}
\]

\[
\geq \left( 1 - 2\Phi \left( \sqrt{\frac{n-1-2\sqrt{(n-1)m}}{k^2 - 1}} \right) \right)
\cdot \left( 1 - e^{-m} \right) \quad \text{ (by Lemma 8).}
\]

Let \( m = (n-1)/16 \) and define

\[
t := \sqrt{\frac{(n-1)/2}{k^2 - 1}}.
\]

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We have
\[ \sqrt{\frac{n-1-2\sqrt{(n-1)m}}{k^2-1}} = t. \] (78)

Substituting Eq. (78) and \( m = (n-1)/16 \) to Eq. (76), we have
\[
\Pr \left( \left\{ \frac{1}{|A_{i1}|} \geq k \right\} \right) \geq \left( 1 - e^{-(n-1)/16} \right) \left( 1 - 2\Phi(t) \right)
\]
\[
\geq \left( 1 - e^{-(n-1)/16} \right) \left( 1 - \frac{2\sqrt{2/\pi}e^{-t^2/2}}{t + \sqrt{t^2 + 8/\pi}} \right) \quad \text{(by Lemma 20)}
\]
\[
\geq \left( 1 - e^{-(n-1)/16} \right) \left( 1 - e^{-t^2/2} \right) \quad \text{(since } t \geq 0 \implies t + \sqrt{t^2 + 8/\pi} \geq 2\sqrt{2/\pi}).
\]

Now, let \( k \) take the value of the RHS of Eq. (74), i.e.,
\[ k = \sqrt{1 + \frac{n-1}{4 \ln n + 4 \ln(p-s)}}. \]

By Eq. (77), we have
\[
t^2 = \frac{(n-1)/2}{k^2-1}
\]
\[
= \frac{(n-1)/2}{\left( \sqrt{1 + \frac{n-1}{4 \ln n + 4 \ln(p-s)}} \right)^2 - 1}
\]
\[
= 2 \ln n + 2 \ln(p-s),
\]
which implies that
\[ e^{-t^2/2} = \frac{1}{n(p-s)}. \]

Thus, we have
\[
\Pr \left( \left\{ \frac{1}{|A_{i1}|} \geq k \right\} \right) \geq \left( 1 - e^{-(n-1)/16} \right) \left( 1 - \frac{1}{n(p-s)} \right). \] (79)
Next, in Step 2, we use Eq. (79) to estimate \(1/ \max_i |A_{i1}| \). Since all \(A_{i1} \)'s are independent, we have

\[
\Pr \left( \left\{ \frac{1}{\max_i |A_{i1}|} \geq k \right\} \right) = \prod_{i=1}^{p-s} \Pr \left( \left\{ \frac{1}{|A_{i1}|} \geq k \right\} \right) \quad \text{(since all \(A_{i1} \) are independent)}
\]

\[
\geq \left( \left(1 - e^{-(n-1)/16}\right) \left(1 - \frac{1}{n(p-s)} \right) \right)^{p-s} \quad \text{(by Eq. (79))}
\]

\[
= \exp \left( (p-s) \ln(1 - e^{-(n-1)/16}) \right) \cdot \exp \left( (p-s) \ln(1 - \frac{1}{n(p-s)}) \right)
\]

\[
\geq \exp \left( \frac{(p-s)e^{-(n-1)/16}}{\sqrt{1 - e^{-(n-1)/16}}} \right) \exp \left( \frac{1}{\sqrt{1 - \frac{1}{n(p-s)}}} \right)
\]

(by Lemma 35)

\[
= \exp \left( \frac{1}{\sqrt{1 - e^{-(n-1)/16}}} \right) \exp \left( \frac{1}{\sqrt{1 - e^{-(n-1)/16}}} \right)
\]

(because \(e^x \geq 1 + x\))

\[
\geq \left( 1 - \frac{1}{n\sqrt{1 - e^{-(n-1)/16}}} \right) \left( 1 - \frac{1}{n\sqrt{1 - 1/17}} \right)
\]

(because of the assumption of the proposition, i.e., \(p - s \leq e^{(n-1)/16} / n \) and \(n(p-s) \geq n \geq 17\))

\[
\geq \left( 1 - \frac{1}{n\sqrt{1 - e^{-(n-1)/16}}} \right) \left( 1 - \frac{1}{n\sqrt{1 - 1/17}} \right) \quad \text{(because \(n \geq 17\))}
\]

\[
= \frac{1}{n\sqrt{1 - 1/e}} - \frac{1}{n\sqrt{1 - 1/17}} + \frac{1}{n\sqrt{1 - 1/e}} \cdot \frac{1}{n\sqrt{1 - 1/17}}
\]

\[
\geq 1 - \frac{2}{\sqrt{1 - 1/e}} \cdot \frac{1}{n} \quad \text{(because \(17 > e\))}
\]

\[
\geq 1 - \frac{3}{n} \quad \text{(because \(e \geq 9/5\)).}
\]

The result of this proposition thus follows. □

Finally, we use the following lemma to simplify the expression in Proposition 34. The result of Proposition 31 thus follows.

**Lemma 36.** If \(n \geq 17\), then

\[
\sqrt{1 + \frac{n-1}{4\ln p + 4\ln(p-s)}} \geq \sqrt{1 + \frac{n}{9\ln p}}.
\]

**Proof.** Because \(n \geq 17\), we have

\[
\frac{n-1}{n} = 1 - \frac{1}{n} \geq 1 - \frac{1}{17} \geq \frac{8}{9}.
\]

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Therefore, we have
\[
\begin{align*}
n - 1 & \geq \frac{4}{9} + \frac{4}{9} \\
\implies \frac{n - 1}{n} & \geq \frac{4 \ln p + 4 \ln(p - s)}{9 \ln p} \\
\implies \frac{n - 1}{n} & \geq \frac{4 \ln p + 4 \ln(p - s)}{9 \ln p} \\
\implies \frac{n - 1}{4 \ln p + 4 \ln(p - s)} & \geq \frac{n}{9 \ln p} \\
\implies \sqrt{1 + \frac{n - 1}{4 \ln p + 4 \ln(p - s)}} & \geq \sqrt{1 + \frac{n}{9 \ln p}}.
\end{align*}
\]

\[\square\]

M Proof of Proposition 32

To prove Proposition 32, we use similar steps and settings as those in Appendix J. In step 1, we relate the cumulative distribution of \(|G_{ij}|\) to the tail probability of \(H_i^T H_j\). In step 2, we estimate the tail probability of \(H_i^T H_j\). In step 3, we derive a lower bound on \(M\) with high probability. In step 4, we simplify the result derived in step 3.

Step 1: Relating the cumulative distribution of \(|G_{ij}|\) to the tail probability of \(H_i^T H_j\).

We have
\[
\Pr \left( \{ |G_{ij}| < a \} \right) = \Pr \left( \{ |G_{ij}| < a, \|H_i\|_2 < \sqrt{2n}, \|H_j\|_2 < \sqrt{2n} \} \right)
\]
\[
+ \Pr \left( \{ |G_{ij}| < a, \|H_i\|_2 \geq \sqrt{2n} \text{ or } \|H_j\|_2 \geq \sqrt{2n} \} \right)
\]
\[
\leq \Pr \left( \{ \|H_i^T H_j\| < 2na \} \right) + \Pr \left( \{ \|H_i\|_2 > \sqrt{2n} \} \right) + \Pr \left( \{ \|H_j\|_2 > \sqrt{2n} \} \right)
\]
\[
= 1 - 2 \Pr \left( \{ \|H_i^T H_j\| > 2na \} \right) + 2 \Pr \left( \{ \|H_i\|_2 > \sqrt{2n} \} \right). \tag{80}
\]

Since \(\|H_i\|_2^2\) follows chi-square distribution with \(n\) degrees of freedom, we will now use Lemma 8 to bound \(\Pr \left( \{ \|H_i\|_2 > \sqrt{2n} \} \right)\). Towards this end, notice that
\[
2 \sqrt{n} \left( \frac{2 - \sqrt{3}}{2} \right) + 2 \cdot \frac{2 - \sqrt{3}}{2} n
\]
\[
= \left( \sqrt{4 - 2\sqrt{3} + 2 - \sqrt{3}} \right) n
\]
\[
= \left( \sqrt{\sqrt{3} - 1}^2 + 2 - \sqrt{3} \right) n
\]
\[
= n. \tag{81}
\]
We then have
\[
\Pr \left( \left\| \mathbf{H}_i \right\|_2 \geq \sqrt{2n} \right) \\
= \Pr \left( \left\{ \left\| \mathbf{H}_i \right\|_2 \geq 2n \right\} \right) \\
= \Pr \left( \left\{ \left\| \mathbf{H}_i \right\|_2^2 - n \geq 2 \left( n \left( \frac{2 - \sqrt{3}}{2n} \right) + 2 \cdot \frac{2 - \sqrt{3}}{2n} \right) \right\} \right) \quad \text{(by Eq. (81))}
\leq \exp \left( -\frac{2 - \sqrt{3}}{2n} \right) \quad \text{(by Lemma 8 using } x = \frac{2 - \sqrt{3}}{2n}).
\]

Applying it in Eq. (80), we have, for any \( i \neq j \),
\[
\Pr (\{|G_{ij}| < a\}) \geq 1 - 2 \Pr (\{|H_i^T H_j| > 2na\}) + 2 \exp \left( -\frac{2 - \sqrt{3}}{2n} \right).
\tag{82}
\]

**Step 2: Estimating the tail probability of** \( H_i^T H_j \).

We first introduce a lemma as follows, which will be used in estimating the tail probability of \( H_i^T H_j \).

**Lemma 37.** When \( x > 0 \),
\[
\frac{-1 + \sqrt{1 + 4x^2}}{2x} = \arg \max_{\lambda \in (-1,1)} \left( \lambda x - \ln \frac{1}{\sqrt{1 - \lambda^2}} \right).
\]

**Proof.** Let
\[
f(\lambda) := \lambda x - \ln \frac{1}{\sqrt{1 - \lambda^2}} = \lambda x - \frac{1}{2} \ln(1 - \lambda^2), \quad \lambda \in (-1,1).
\]

We have
\[
\frac{df(\lambda)}{d\lambda} = x - \frac{\lambda}{1 - \lambda^2} = \frac{-x\lambda^2 - \lambda + x}{1 - \lambda^2}.
\]

Notice that on \( \lambda \in (-1,1) \), the sign of \( df/d\lambda \) is the same as that of \( g_x(\lambda) := -x\lambda^2 - \lambda + x \), which is a quadratic form in \( \lambda \). We have \( g_x(-1) = 1 \) and \( g_x(1) = -1 \). Therefore, there must exists exactly one \( \lambda^* \in (-1,1) \) such that \( g(\lambda^*) = 0 \), which is given by
\[
\lambda^* = \frac{-1 + \sqrt{1 + 4x^2}}{2x} \in (-1,1).
\]

Considering that \( g_x(\lambda) \) is the quadratic form of \( \lambda \), we have \( g_x(\lambda) > 0 \) for \( \lambda \in (-1,\lambda^*) \), and \( g_x(\lambda) < 0 \) for \( \lambda \in (\lambda^*,1) \). Thus, \( f(\lambda) \) achieves the maximum at \( \lambda^* \).

The following lemma estimates the tail probability of \( H_i^T H_j \).

**Lemma 38.** Fix \( a > 0 \), for any \( \delta > 0 \), there exists a threshold \( \bar{n} \) such that, for all \( n > \bar{n} \), we have
\[
\Pr \left( \{|H_i^T H_j| > 2na\} \right) \geq \exp \left( -n \left( \frac{t - \ln(t/2 + 1) + \delta}{2} \right) \right),
\]
\[
\text{where}
\]
\[
t = \sqrt{1 + 16a^2} - 1.
\]
Proof. Let $Z_k$ be the distribution that corresponds to the product of two random \textit{i.i.d.} standard Gaussian random variables. Then, we know that $\mathbf{H}_i^T \mathbf{H}_j$ are the sum of $n$ independent samples of such $Z_k$'s. According to Cramér’s Theorem (Theorem 2.2.3 in (Dembo and Zeitouni, 2009)), for any $\delta > 0$, there exists a threshold $\tilde{n}$ such that for all $n > \tilde{n}$, we have

$$
\Pr\left( \left\{ \frac{1}{n} \mathbf{H}_i^T \mathbf{H}_j > 2a \right\} \right) \geq e^{-n(R + \delta)},
$$

where

$$
R := \inf_{x > 2a} \sup_{\lambda \in (-1, 1)} \{ \lambda x - \ln \mathbb{E}[e^{\lambda Z_k}] \}
= \inf_{x > 2a} \frac{1}{2} \left( \sqrt{1+4x^2} - 1 - \ln \frac{2x^2}{\sqrt{1+4x^2} - 1} \right) \quad \text{(by Lemma 25)}.
= \inf_{x > 2a} \frac{1}{2} \left( \sqrt{1+4x^2} - 1 - \ln \frac{\sqrt{1+4x^2}+1}{2} \right) \quad \text{(by Lemma 37)}
\leq \frac{1}{2} \left( \sqrt{1+4x^2} - 1 - \ln \frac{\sqrt{1+4x^2}+1}{2} \right) \bigg|_{x=2a} \quad \text{(according to the definition of infimum)}
= \frac{1}{2} \left( t - \ln(t/2 + 1) \right) \quad \text{(by the choice of } t \text{ in the lemma)}.
$$

The conclusion of this lemma thus follows. \hfill \Box

\textbf{Step 3: Estimating the distribution of } M. \\

We now estimate the distribution of $M$ in the following proposition.

\textbf{Proposition 39.} For any $\delta > 0$, there exists a threshold $\tilde{n}$ such that for any $n \geq \tilde{n}$ and $p \in \{ x \mid \frac{\ln|x/2|}{n} > 2\delta \}$,

$$
\Pr \left( \left\{ M \geq \frac{1}{4} \sqrt{\left( \frac{\ln|p/2|}{n} - 2\delta \right) \left( \left( \frac{\ln|p/2|}{n} - 2\delta \right) + 2 \right)} \right\} \right) 
\geq 1 - \exp \left( 2 \exp \left( - \frac{2 - \sqrt{3}}{2} n + \ln|p/2| \right) - 2 \exp \left( \frac{1}{2} (\ln|p/2|) \right) \right).
$$

(83)
Proof. For any $a > 0$ and $\delta > 0$, let $\bar{n}_1$ be the threshold in Lemma 38. Then for any $n > \bar{n}_1$, we have

$$
\Pr (\{M \geq a\})
= 1 - \Pr \left( \bigcap_{i \neq j} \{|G_{ij}| < a\} \right)
\geq 1 - \Pr \left( \bigcap_{i \in \{1, \ldots, \lfloor p/2 \rfloor\}, j = i+\lfloor p/2 \rfloor} \{|G_{ij}| < a\} \right)
= 1 - \Pr (\{|G_{1,1+\lfloor p/2 \rfloor}| < a\})^{\lfloor p/2 \rfloor} \quad \text{(because all $G_{1,i+\lfloor p/2 \rfloor}$'s are independent)}
\geq 1 - \left( 1 - 2 \Pr \left( \{H_j^T H_j > 2na\} \right) + 2 \exp \left( -\frac{2 - \sqrt{3}}{2} n \right) \right)^{\lfloor p/2 \rfloor} \quad \text{(by Eq. (82))}
\geq 1 - \left( 1 + 2 \exp \left( -\frac{2 - \sqrt{3}}{2} n \right) - 2 \exp \left( -n \left( \frac{1}{2}(t - \ln(t/2 + 1)) + \delta \right) \right) \right)^{\lfloor p/2 \rfloor}
\quad \text{(by Lemma 38, with $t = \sqrt{1 + 16a^2} - 1$ and $n > \bar{n}_1$)}
\geq 1 - \exp \left( \frac{1}{2} \left( 2 \exp \left( -\frac{2 - \sqrt{3}}{2} n + \ln|p/2| \right) - 2 \exp \left( -n \left( \frac{1}{2}(t - \ln(t/2 + 1)) + \delta \right) \right) \right) \right)
\quad \text{(since $1 + x \leq e^x$)}
\geq 1 - \exp \left( 2 \exp \left( -\frac{2 - \sqrt{3}}{2} n + \ln|p/2| \right) - 2 \exp \left( -n \left( \frac{\ln|p/2|}{n} - \frac{t}{2} + \frac{1}{2} \ln(t/2 + 1) - \delta \right) \right) \right)
\geq 1 - \exp \left( 2 \exp \left( -\frac{2 - \sqrt{3}}{2} n + \ln|p/2| \right) - 2 \exp \left( -n \left( \frac{\ln|p/2|}{n} - \frac{t}{2} - \delta \right) \right) \right) \quad \text{(84)}
\quad \text{(since $\ln(t/2 + 1) \geq 0$)}.
$$

Now, we choose

$$
a = \frac{1}{4} \sqrt{\left( \frac{\ln|p/2|}{n} - 2\delta \right) \left( \frac{\ln|p/2|}{n} - 2\delta \right) + 2}. \quad \text{(85)}
$$

we have

$$
t = \sqrt{1 + 16a^2} - 1
= \sqrt{1 + \left( \frac{\ln|p/2|}{n} - 2\delta \right) \left( \frac{\ln|p/2|}{n} - 2\delta \right) + 2} - 1
= \sqrt{\left( \frac{\ln|p/2|}{n} - 2\delta + 1 \right)^2} - 1
= \frac{\ln|p/2|}{n} - 2\delta.
$$

Thus, we have

$$
\frac{n \left( \frac{\ln|p/2|}{n} - \frac{t}{2} - \delta \right) = \frac{\ln|p/2|}{2}}{n \left( \frac{\ln|p/2|}{n} - \frac{t}{2} - \delta \right) = \frac{\ln|p/2|}{2}}. \quad \text{(86)}
$$

Substituting Eq. (85) and Eq. (86) into Eq. (84), the result of this proposition thus follows. 

\qed
Step 4: Simplifying the expression in Proposition 39.

Now we simplify the result in Proposition 39 by using the additional assumption of $e^{4δn} < |p/2| < \exp\left(\frac{2 - \sqrt{3}}{4}n\right)$ in Proposition 32. We have

$$-\frac{2 - \sqrt{3}}{2}n + \ln|p/2| \leq -\frac{2 - \sqrt{3}}{4}n.$$  

Thus, the RHS of Eq. (83) in Proposition 39 can be relaxed and simplified as follows.

$$1 - \exp\left(2\exp\left(-\frac{2 - \sqrt{3}}{2}n + \ln|p/2|\right) - 2\exp\left(\frac{1}{2}\ln|p/2|\right)\right) \geq 1 - \exp\left(2\exp\left(-\frac{2 - \sqrt{3}}{4}n\right) - 2\exp\left(\frac{1}{2}\ln|p/2|\right)\right)$$  

$$\geq 1 - \exp\left(2\exp\left(-\frac{2 - \sqrt{3}}{4}n\right) - 2\exp\left(\frac{1}{2}\ln\frac{p}{2}\right)\right)$$  

$$= 1 - \exp\left(2\exp\left(-\frac{2 - \sqrt{3}}{4}n\right) - \sqrt{2(p - 1)}\right).$$

As $n$ increases, $\exp\left(-\frac{2 - \sqrt{3}}{4}n\right)$ becomes very small. Also, since $p > n$, by the assumption of the lemma, we can easily find a threshold $\tilde{n}_2$ such that, for all $p > n \geq \tilde{n}_2$, the following holds:

$$2\exp\left(-\frac{2 - \sqrt{3}}{4}n\right) - \sqrt{2(p - 1)} \leq -\sqrt{p}.$$  

Therefore, the RHS of Eq. (83) in Proposition 39 can be further relaxed as $1 - \exp(-\sqrt{p})$. Now, we simplify the bound of $M$ in Proposition 39. We have

$$\frac{1}{4} \sqrt{\left(\frac{\ln|p/2|}{n} - 2δ\right)\left(\frac{\ln|p/2|}{n} - 2δ\right) + 2} \geq \frac{1}{4} \sqrt{\frac{\ln|p/2|}{2n} + 2}$$  

(because $e^{4δn} < |p/2|$)

$$\geq \frac{1}{4} \sqrt{\frac{\ln|p/2|}{2n} + 2} \geq 2.$$  

(87)

Since $p > n$, we can easily find a threshold $\tilde{n}_3$ such that, for all $p > n \geq \tilde{n}_3$, the following holds:

$$\ln(p - 1) - \ln 2 \geq \frac{\ln p}{2},$$

which implies that

$$\frac{\ln p}{2} \leq \ln \frac{p - 1}{2} \leq \ln \frac{|p/2|}{2}.$$  

Applying it in Eq. (87), we have, when $n \geq \tilde{n}_3$,

$$\frac{1}{4} \sqrt{\left(\frac{\ln|p/2|}{n} - 2δ\right)\left(\frac{\ln|p/2|}{n} - 2δ\right) + 2} \geq \frac{1}{4} \sqrt{\frac{\ln|p/2|}{n}} \geq \frac{\sqrt{2}}{8} \sqrt{\frac{\ln p}{n}}.$$
Thus, by choosing \( \hat{n} = \max\{\tilde{n}_1, \tilde{n}_2, \tilde{n}_3\} \), we have, for any \( n \geq \hat{n} \),

\[
\Pr \left( \left\{ M \geq \frac{\sqrt{2}}{8} \sqrt{\ln \frac{p}{n}} \right\} \right) \\
\geq \Pr \left( \left\{ M \geq \frac{1}{4} \sqrt{\left( \frac{\ln |p/2|}{n} - 2\delta \right) \left( \frac{\ln |p/2|}{n} - 2\delta \right) + 2} \right\} \right) \\
\geq 1 - \exp \left( -\sqrt{p} \right).
\]

The conclusion of Proposition 32 thus follows.