PAIR PRODUCTION
IN A TIME DEPENDENT MAGNETIC FIELD

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Abstract

The production of electron-positron pairs in a time-dependent magnetic field is estimated in the hypotheses that the magnetic field is uniform over large distances with respect to the pair localization and it is so strong that the spacing of the Landau levels is larger than the rest mass of the particles. This calculation is presented since it has been suggested that extremely intense and varying magnetic fields may be found around some astrophysical objects.

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1. Statement of the problem

Standard Q.E.D. clearly states that a static electric field of sufficient strength and extension can create pairs of opposite charges through a tunnel effect [1,2], but the same kind of production is not foreseen for magnetic fields in the same conditions, unless one would take into account the production of monopole pairs. The situation changes if the external fields vary in time and the effect of the field variation on the pair production rates has been discussed in detail in connection with the possibility of detecting these effects in experiments in optics [3], where the time dependence of the field is explicitly taken into account, with an essential negative answer, at least for the real pair production.* It may be possible that pairs are created in the presence of very large field strengths with time variation which is not too small; since it has been proposed that situations of this kind could take place in some phases of neutron star evolution[5,6], a preliminary investigation on the pair production in a very strong and non constant magnetic field is presented here. No detailed quantitative calculation is performed, but some order of magnitude estimates are given in order to show that the model calculation that will be presented can be well suited for the conditions referred to above.

These conditions correspond to a very intense magnetic, over $10^{10} T$, which remains constant over distances which are large in comparison with the dimensions involved in the electron production, during the time of variation the spatial uniformity is conserved, in this first investigation only a change of strength at constant direction is considered.

The rapidity of change cannot be too small in order for the production process not to be entirely negligible, but since it involves macroscopic objects it will be slow enough to allow the adiabatic treatment of the production process[7]. For the ratio between the magnetic and the rest energy, defined as $\Re = (2\hbar eB)/(m_e c)^2$, we obtain for the chosen values of $B$, $\Re > 4.5$, which allows some computational simplifications that will be used.

2. Determination of the production rate

The production rate is calculated here in the following external field configuration: the magnetic field grows at fixed direction, this could happen locally, in the case of a collapsing neutron star, when a compression of the magnetic field takes place[6].

* The effect of virtual pairs in external fields is a different issue, which is under investigation also experimentally[4], but it will not be discussed here.
In order to fix notations and conventions some well known features of the relativistic electron in static uniform magnetic field are rewritten here.

The Dirac equation\(^*\) \(H \psi = [\bar{\sigma} \cdot (\vec{p} + e\vec{A}) + \beta m] \psi = \epsilon \psi\) is written as:

\[
\begin{pmatrix}
m \\
\bar{\sigma} \cdot (\vec{p} + e\vec{A}) \\
-m \\
\end{pmatrix}
\begin{pmatrix}
\phi \\
\chi \\
\end{pmatrix}
= \epsilon
\begin{pmatrix}
\phi \\
\chi \\
\end{pmatrix}
\]

(1)

and the system is solved for \(\phi\) when \(\epsilon > 0\) and for \(\chi\) when \(\epsilon < 0\).

When the field becomes time dependent, the choice of the vector potential for the magnetic field becomes physically relevant, different choices yielding the same magnetic field give rise to different electric fields. So a choice is unavoidable: when the magnetic field is taken as constant in direction and varying in strength a cylindrical configuration, defined by \(\vec{A} = -\frac{1}{2} \vec{r} \wedge \vec{B}\), will be studied. Decomposing the variables into transverse and longitudinal components (the longitudinal direction is called \(z\)), the equation for \(\phi\) is

\[
[p_z^2 + p_T^2 + \frac{1}{4} e^2 r_T^2 B^2 + eB(L_z + \sigma_z)]\phi = [\epsilon^2 - m^2] \phi
\]

(2)

The variables \(p_z\) and \(\sigma_z\) are diagonalized with eigenvalues \(k\) and \(s\) with the position \(\phi = \frac{1}{\sqrt{Z}} e^{ikz} \phi_{k,s}\), we obtain

\[
[p_T^2 + \frac{1}{4} e^2 r_T^2 B^2 + eB(L_z + s)]\phi_{k,s} = (\epsilon^2 - k^2 - m^2) \phi_{k,s}
\]

(3)

For future use the solution of the remaining equation is expressed in term of the harmonic-oscillator operators. Following ref.[8] one can define:

\[
a_v = (p_v/b - ibv)/\sqrt{2} \quad \text{with} \quad v = x, y \quad b^2 = \frac{1}{2} eB
\]

\[
a_d = (a_x - ia_y)/\sqrt{2} , \quad a_g = (a_x + ia_y)/\sqrt{2} , \quad N_d = a_d^\dagger a_d , \quad N_g = a_g^\dagger a_g
\]

(4)

and express in this way the operators \(p_T\), \(r_T\), \(L_z\). The final result is:

\[
\epsilon^2 = m^2 + k^2 + eB(2N_d + 1 + s)
\]

(5)

The operators \(N_d\), \(N_g\) are standard number operators, we respectively call \(n\), \(\rho\) their eigenvalues and remark two known results relevant for the

\* here \(e > 0\) so the charge of the electron is \(-e\).
next developments: the quantum number \( \rho \) has no influence on the value of the energy; the lowest Landau level, defined by \( n = 0 \) and \( s = -1 \), has an energy independent of the strength of \( B \).

The second order equation for \( \chi \) is the same as the equation for \( \phi \); the normalized solutions of the original equation are:

\[
\psi_+ = \mathcal{N} \left( \frac{\phi}{w+m} \right), \quad \psi_- = \mathcal{N} \left( -\frac{\chi}{w+m} \right)
\]

\[
\hat{\mathcal{P}} = \hat{\sigma} \cdot (\vec{p} + e\vec{A}) \quad , \quad w = |\epsilon| \quad , \quad \mathcal{N} = \sqrt{(w + m)/2w}
\]

The second quantized electronic field is decomposed in terms of these solutions

\[
\Psi = \sum_j c_j \psi_{+,j} + \sum_j d_j^\dagger \psi_{-,j} \quad \Psi^\dagger = \sum_j c_j^\dagger \psi^\dagger_{+,j} + d_j \psi^\dagger_{-,j}
\]

so the standard representation of the (second quantized) Hamiltonian is produced:

\[
\mathcal{H} = \int d^3r \Psi^\dagger(\vec{r})[\hat{\alpha} \cdot (\vec{p} + e\vec{A}) + \beta m] \Psi(\vec{r}) = \sum_j w_j [c_j^\dagger c_j - d_j d_j^\dagger].
\]

The index \( j \) embodies all the relevant quantum numbers \( j \equiv (k, n, \rho, s) \), of course when they are read as positron quantum numbers, the sign of \( k \) and \( s \) must be reversed.

Now the variation of the Hamiltonian with time is given writing \( \vec{B} = B\vec{\kappa} \) and \( B = B_o + \dot{B}t \) and defining \( h = \partial H/\partial B \) and \( \alpha_{\pm} = \frac{1}{2} (\alpha_x \pm i\alpha_y) \).

In terms of the harmonic-oscillator operators, eq.(4), it results:

\[
h = -\frac{1}{2}e\hat{\alpha} \wedge \vec{\kappa} \cdot \vec{\kappa} = \frac{e}{2b} [-\alpha_+ a_d + \alpha_- a_d^\dagger - \alpha_- a_g + \alpha_+ a_g^\dagger],
\]

The derivative of the second quantized Hamiltonian is then:

\[
\frac{\partial \mathcal{H}}{\partial B} = \int d^3r \Psi^\dagger h \Psi = \sum_{j,j'} [c_j^\dagger c_{j'} \psi^\dagger_{+,j} h \psi_{+,j'} + c_j^\dagger d_{j'}^\dagger \psi^\dagger_{+,j} h \psi_{-,j'} + d_j c_{j'}^\dagger \psi^\dagger_{-,j} h \psi_{+,j'} + d_j d_{j'}^\dagger \psi^\dagger_{-,j} h \psi_{-,j'}].
\]

When we are interested in the transition from the vacuum to the electron-positron pair configuration only the second of the four addenda is interesting and will therefore be calculated explicitly; all the addenda are important for higher order corrections.
Before going on with the calculation of the matrix element, the following observation (ref. [8]), is useful: the mean value of \( r_T^2 \) is found to be

\[
<n, \rho | r_T^2 | n, \rho > = (n + \rho + 1)/b^2
\]

In semiclassical states not having sharp eigenvalues the mean value becomes *

\[
<< n, \rho | r_T^2 | n, \rho >> \approx (\sqrt{\rho} - \sqrt{n})^2/b^2 .
\]

In both cases one observes that \( n \) cannot grow too much because it costs energy eq.(5), while \( \rho \) can grow without cost; so if the space at disposal of the system is large in most configurations we shall find

\[
<n, \rho | r_T^2 | n, \rho > \approx \rho/b^2 .
\]

It is, therefore, convenient to define \( \rho = b^2 R^2 \) and \( R \) may be interpreted as the mean position; the sum over \( \rho \) which at the end occurs, is converted into an integration according to \( \sum_\rho \approx b^2 \int dR^2 \). The region of integration will certainly be large, as compared with the microscopic lengths, so the terms containing \( R \) will be the most important and they will be calculated using the approximate representation \( a_g \approx b R e^{i \theta} , \ a_g^\dagger \approx b R e^{-i \theta} \). The conclusion is that the dominant contribution for macroscopic extension of the magnetic field may be extracted after reducing the expression of \( h \) to the much simpler form

\[
h_o = \frac{1}{2} i e R (e^{-i \theta} \alpha_+ - e^{i \theta} \alpha_- ) . \tag{11}
\]

The matrix element corresponding to the transition from the vacuum to a two-particle state is represented as

\[
\int d^3r \psi^\dagger_{+, j} h_o \psi_{-j'} = \frac{1}{2} i e R N_j N_{j'} M ,
\]

two quantum numbers are equal in \( j \) and \( j' \), they are \( k \) and \( \rho \). Using the relation:

\[
\hat{P} = \sigma_z k + b \sqrt{2}[\sigma_+ a_d + \sigma_- a_d^\dagger]
\]

the expression for \( M \) is found to be

\[
M = \left\{ (\sigma_+ e^{-i \theta} - \sigma_- e^{i \theta}) - \left[ -k^2 (\sigma_+ e^{-i \theta} - \sigma_- e^{i \theta}) + \sqrt{2} k b (\sigma_z a^\dagger d e^{-i \theta} - \sigma_z a d e^{i \theta}) + 2 b^2 (\sigma_+ a_d a_d^\dagger e^{i \theta} + \sigma_- a_d^\dagger a_d e^{-i \theta}) \right] \right\} . \tag{12}
\]

* this procedure is similar to the treatment of a condensed Bose system, where one puts \( a \approx a^\dagger \approx \sqrt{a^\dagger a} \).
From this form it appears that the two produced particles cannot have wholly equal quantum numbers, in particular it is impossible that both are produced in the ground state. There are eight relevant matrix elements which are listed as follows.

\[ T_1 = \langle - , n | M | + , n \rangle = - e^{i\theta} - k^2 e^{i\theta} [(w_{+,n} + m)(w_{-,n} + m)]^{-1} \]
\[ T_2 = \langle + , n | M | - , n \rangle = e^{-i\theta} + k^2 e^{-i\theta} [(w_{+,n} + m)(w_{-,n} + m)]^{-1} \]
\[ T_3 = \langle + , n | M | + , n + 1 \rangle = \sqrt{2} k b e^{i\theta} \sqrt{n+1} [(w_{+,n} + m)(w_{+,n+1} + m)]^{-1} \]
\[ T_4 = \langle + , n + 1 | M | + , n \rangle = - \sqrt{2} k b e^{-i\theta} \sqrt{n+1} [(w_{+,n} + m)(w_{+,n+1} + m)]^{-1} \]
\[ T_5 = \langle - , n | M | - , n + 1 \rangle = - \sqrt{2} k b e^{i\theta} \sqrt{n+1} [(w_{-,n} + m)(w_{-,n+1} + m)]^{-1} \]
\[ T_6 = \langle + , n + 1 | M | + , n \rangle = \sqrt{2} k b e^{-i\theta} \sqrt{n+1} [(w_{-,n} + m)(w_{-,n+1} + m)]^{-1} \]
\[ T_7 = \langle + , n | M | - , n + 2 \rangle = 2 b^2 e^{i\theta} \sqrt{(n+1)(n+2)} [(w_{-,n+2} + m)(w_{+,n} + m)]^{-1} \]
\[ T_8 = \langle - , n + 2 | M | + , n \rangle = - 2 b^2 e^{-i\theta} \sqrt{(n+1)(n+2)} [(w_{+,n} + m)(w_{-,n+2} + m)]^{-1} \]

Further simplification is gained for strong magnetic fields, in fact in this case the excited Landau levels are very far from the fundamental one so the production of an electron-positron pair in their ground state would strongly be favoured, but it has just been seen that this transition is not allowed and at most one particle may be found in the ground state of its transverse motion. We may neglect the mass term in comparison with the magnetic energy and write:

\[ w_{+,n} = \sqrt{4b^2 (n+1) + k^2} \quad w_{-,n} = \sqrt{4b^2 n + k^2} \quad \text{if } n > 0 \]
\[ w_{-,0} = w_k = \sqrt{m^2 + k^2}. \]

The dominant terms for large \( b \) of the matrix elements are independent of \( b \) and they are explicitly:

\[ \tilde{T}_1 = - \tilde{T}_2^* \approx - e^{i\theta} \]
\[ \tilde{T}_3 = - \tilde{T}_4^* \to 0 \]
\[ \tilde{T}_5 = - \tilde{T}_6^* \approx - e^{i\theta} k/(w_k + m) \quad \text{if } n = 0 \quad \tilde{T}_5 = - \tilde{T}_6^* \to 0 \quad \text{if } n > 0 \]
\[ \tilde{T}_7 = - \tilde{T}_8^* \approx \frac{1}{2} e^{i\theta} \].

The same approximation is introduced into the normalization factors and for them it results \( \mathcal{N} \approx 1/\sqrt{2} \), but for \( \mathcal{N}_{-,0} = \sqrt{(w_k + m)/2w_k} \). The amplitude for the pair production is written in terms of the standard adiabatic expression \([7]\), with the following specifications: only the transition from the vacuum to the two particle state is considered; the reduction of
amplitude of the vacuum state is not taken into account; the initial instant
of the process is not pushed to \(-\infty\), but it is some finite time, \(t = 0\), in
which the field \(B(0) = B_o\) is already large. This is expressed writing the
adiabatic projection coefficient \([7]\) \(\gamma_{j,j'}\) as:

\[
\gamma_{j,j'}(t) = -\dot{B} \int_o^t d\tau \frac{\langle j,j' | \partial H / \partial B \rangle >}{\Delta E} \gamma_o \exp[-i \int_o^\tau \Delta Edt'] : 
\]

with \(\Delta E = w_j + w_{j'}\), eq (14).

From now on we refer to the matrix elements \(T_1, T_2\), in the other cases
slight modifications are required.

It is convenient to separately treat the \(n = 0\) and the \(n > 0\) contribu-
tions. In this last case, in which the sum over all \(n\) will be performed, we
shall set \(n \approx n + 1\) and in so doing

\[
\int_o^\tau \Delta Edt' = \frac{2}{3eBn} [(2enB(t) + k^2)^{3/2} - (2enB_o + k^2)^{3/2}] ; 
\]

the second term gives rise to an irrelevant constant phase, in the calculation
of \(|\gamma_{j,j'}(t)| = \frac{1}{2} eR\dot{B} |J|\) the relevant integral is

\[
J = \int_o^t \frac{d\tau}{2\sqrt{2enB(\tau) + k^2}} \exp\left[ - \frac{2i}{3enB} (2enB(\tau) + k^2)^{3/2} \right] . 
\]

A convenient form for this expression is

\[
J = \frac{1}{2enB} \left[ f(S_o) \sqrt{2enB_o + k^2} + f(S(t)) \sqrt{2enB(t) + k^2} \right] 
\]

\[
f(q) = \int_1^\infty \exp[-iqy^3] dy 
\]

\[
S_o = \frac{2}{3enB} (2enB_o + k^2)^{3/2} 
\]

\[
S(t) = \frac{2}{3enB} (2enB(t) + k^2)^{3/2} . 
\]

The parameters \(S\) are very large in all the interesting situations. This is
better appreciated if we insert the powers of \(c, \hbar\) that are needed to come
back to standard units, for simplicity, when \(k = 0\)

\[
S = \frac{4c\sqrt{2e/\hbar}}{3B} \sqrt{nB^{3/2}} 
\]

then from the general asymptotic expansion of \(f\),

\[
f(q) \approx \frac{1}{3iq} e^{-iq} \sum_\ell \left( \frac{2}{3} \right)_\ell \left( \frac{i}{q} \right)^\ell 
\]
the first term is sufficient * and we obtain:

\[ J = -\frac{1}{4}i[(2enB_0 + k^2)^{-1}\exp[iS_o] - (2enB(t) + k^2)^{-1}\exp[iS(t)]] . \]

Moreover, since \( S(t) \) is very large, the term containing \( \cos[S_o - S(t)] \) which appears in \( |J|^2 \) is rapidly varying around 0 and it will be dropped in the following. This us allows to write:

\[ |J|^2 \approx \frac{1}{16(2enB_0 + k^2)^2} + \frac{1}{16(2enB(t) + k^2)^2} \quad (16) \]

The sum over the longitudinal quantum number is performed with the usual transition to the continuum variable \( \sum_k \to \frac{1}{2\pi} Z \int dk \), the sum over the transverse quantum number remains discrete and the final result is

\[ \sum_{k,n} |J|^2 \approx \frac{Z}{128} \zeta(3/2) \left[ \frac{1}{(2eB_0)^{3/2}} + \frac{1}{(2eB(t))^{3/2}} \right] . \quad (17) \]

The symbol \( \zeta \) represents the Riemann’s z-function [9].

For the estimate of the \( n = 0 \) term the difference between the two addenda (+) and (−) is important and we obtain:

\[ \int_0^\tau \Delta Edt' = k\tau + \frac{1}{3eBn}[(2enB(t) + k^2)^{3/2} - (2enB_0 + k^2)^{3/2}] = \Phi ; \]

The subsequent integral is more complicated:

\[ J_o = \int \frac{d\tau}{\Phi} e^{-i\Phi} , \]

but has the same qualitative features as in the previous case, \( \Phi \) is large and we consider the asymptotic expansion:

\[ J_o = i\frac{1}{\Phi^2} e^{-i\Phi} - 2\frac{\dot{\Phi}}{\Phi^4} e^{-i\Phi} + \cdots . \]

It may be verified that \( \dot{\Phi}/\Phi^2 \) is of the order \( 1/S \). Going on as in the previous case, we keep only the first term in the expansion, we drop the rapidly oscillating term and we obtain:

\[ |J_o|^2 \approx [k + \sqrt{2eB_0 + k^2}]^{-4} + [k + \sqrt{2eB(t) + k^2}]^{-4} \quad (18) \]

* it is possible to give a close expression for \( f \), in terms of the irregular confluent hypergeometric function \( U \) [9], although this is not very useful.
The sum over the longitudinal quantum number yields:

\[
\sum_k |\mathcal{J}_o|^2 \approx \frac{2Z}{15\pi} \left[ \frac{1}{(2eB_o)^{3/2}} + \frac{1}{(2eB(t))^{3/2}} \right].
\] (19)

Now we must simply collect the terms, i.e. the factor \(\frac{1}{4}(eRB)^2\), relating \(|\mathcal{J}|^2\) to \(|\gamma|^2\) and the integration \(\frac{1}{2}eB \int dR^2\), which performs the sum over the remaining quantum number \(\rho\), the final result is:

\[
I(t) = \frac{1}{16\sqrt{2}} \left[ \frac{1}{256} \zeta(3/2) + \frac{1}{15\pi} \right] ZR_M^4 e^{3/2} \dot{B}^2 \left[ \frac{B(t)}{B_o^{3/2}} + \frac{1}{\sqrt{B(t)}} \right].
\] (20)

The production rate in a cylinder of height \(Z\) and radius \(R_M\), in a uniform and very strong magnetic field, varying linearly with time is given approximately by \(dI/dt\). If we evaluate \(dI/dt\) we obtain a cubic dependence on \(\dot{B}\), built up by an original linear dependence of the transition amplitude, which is standard in the adiabatic processes and has to be squared, and by the dependence on \(B\) which is mainly related with the fact that the produced state extends over a transverse region becoming smaller and smaller as the magnetic field grows.

While the dependence on the longitudinal variable is trivial the dependence on the transverse one, i.e. the fourth power, is less obvious and it may be understood if we remember that the configuration is not translationally invariant in the transverse dimension. In fact, although the magnetic field is uniform, the electric field \(E\) grows linearly with \(R\) and this puts an evident limitation to the configurations one can consider: in fact too large values of \(R\) such that \(E = \frac{1}{2}BR > B\) cannot fit in this treatment. It is pleasant to find a numerical confirmation of the intuitive idea that the \(n = 0\) configuration must be important, in fact \(|\mathcal{J}_o| > |\mathcal{J}|\). The rest of the factors strongly depend on the details of the chosen configuration. In particular there must be a further and not very large factor which takes into account the other transition-matrix elements \(T_i, i > 2\).

### 3. Conclusions and outlook

A calculation of the pair production in a time-dependent magnetic field has been presented, in a simple situation where the already strong field is still growing in a fixed direction, \(\dot{B} > 0\). This process is suggested by the consideration of an imploding neutron star where the magnetic field is locally compressed [6]. The choice of a very strong field has a less dramatic effect on the production rate than one could have expected at first sight. In fact, to the growth of the state density in \(x\)-space there corresponds a growth in the distance among the energy levels and therefore a decrease
of density in momentum space. A stronger production rate could take place if both produced particles could be found in the ground state, but the dynamics does not allow this process. We may note, however, that as the distance of the Landau level grows there is an increasing number of longitudinal levels between two transverse levels, the convergence in the longitudinal momentum is always rapid, as $dk/k^4$, so the dependence on $k$ which has been neglected in the terms $T_i$ cannot play any important role.

A more formal and general observation can be done: the total angular momentum $\vec{J} = \int d^3r \vec{r} \wedge [\vec{E} \wedge \vec{B}]$ is zero in the considered volume and field configuration while an $e^- e^+$ pair in the lowest energy level has no orbital momentum and opposite spins, so the total angular momentum is 1, the transition is therefore impossible.

Another interesting configuration could be given by a magnetic field which changes its direction, like e.g. that one could find around a rotating neutron star [5]. The adiabatic formalism is certainly still useful, a first look at the problem shows that in this case some difference in the treatment is needed since the adiabatic states are, of course, time dependent, but the energy levels remain constant. Anyhow a more detailed investigation is required in order to obtain a definite answer.

Much more complicated is the possibility that an intense magnetic field may act together with the QCD vacuum leading to an enhanced production of quarks and so, finally, of hadrons. Similar effects have in fact been proposed [10], but any quantitative estimate is very uncertain.

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