Higher-Dimensional generalizations of Affine Kac-Moody and Virasoro Lie Algebras

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Abstract

We discuss the higher dimensional generalizations of the Virasoro and the Affine Kac-Moody Lie algebras. We present an explicit construction for a central extensions of the Lie Algebra $\text{Map}(X, g)$ where $g$ is a finite-dimensional Lie algebra and $X$ is a complex manifold that can be described as a "right" higher-dimensional generalization of $\mathbb{C}^*$ from the point of view of a corresponding group action. The constructed algebras have most of the good properties of finite dimensional semi-simple Lie algebras and are a new class of generalized Kac-Moody algebras. These algebras have description in terms of higher dimensional local fields.

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1 Introduction

The basic object of $d$-dimensional Quantum Field Theory (QFT) are $d$-dimensional fields (operator-valued distributions on $M^d$) and the representation of the Poincaré group in the Hilbert Space $H$ that behave accordingly to the Wightman axioms of QFT. In 1984 Belavin, Polyakov and Zamolodchikov ([BPZ]) initiated the study of two-dimensional Conformal Field Theory (CFT). In CFT we have a nice split of variables that leads to the notion of the "chiral part" of a conformal field theory, where fields are just operator-valued distributions on $\mathbb{C}^*$. This rises the huge interest to the theory of infinite-dimensional Lie algebras and their representations. It turns out that the symmetries of the most models of CFT are classified along the representations of Affine Kac-Moody Lie algebras and the Virasoro algebra.

A mathematical definition of the "chiral part" of a conformal field theory, called a Vertex algebra, was proposed by Borcherds ([B]). The axioms of Vertex algebras ([K2]) are mathematical description of the operator product expansion in CFT. The Vertex algebra approach simplifies the problem of classification of infinite dimensional symmetries in CFT. However, until now a classification of Vertex algebras seems to be far away. There is the solution to the classification problem only when the chiral algebra is generated by a finite number of quantum fields, closed under the operator product expansions (in a sense that only derivatives of the generating field may occur). Roughly speaking, in the finitely generated case there is noting but the Affine Kac-Moody algebras and the Virasoro Vertex algebra (Heisenberg algebra can be treated as a trivial Affine algebra). This fact was proven by Kac using the notion of a conformal algebra ([K2]), which is related to a chiral algebra in the same way as a Lie algebra is related to its universal enveloping algebra. On the space of fields (mutually local formal distributions in complex variable $z$) there is the action of the operator $\partial$, given by $\partial a(z) = \partial_z a(z)$ and the fields that appear in the right part of the commutation relations of two fields $a(z)$ and $b(w)$ can be viewed as $n$-products of these two fields.

The definition of conformal algebra is:

**Definition 1.1 (V.Kac)** A (Lie) conformal algebra is a $\mathbb{C}[\partial]$-module $R$, endowed with a family of $\mathbb{C}$-bilinear products $a^{(n)}b$, $n \in \mathbb{Z}_+$, satisfying axioms (C1)–(C4):

(C1) $a^{(n)}b = 0$ for $n > 0$,
(C2) $(\partial a)^{(n)}b = -na^{(n-1)}b$, $a^{(n)}\partial b = \partial(a^{(n)}b) - (\partial a)^{(n)}b$,
(C3) $a^{(n)}b = -\sum_{j=0}^{\infty} (-1)^{n+j} j! (\partial^j)^{(n)}(b^{(n+j)}a)$,
(C4) $a^{(m)}(b^{(n)}c) - b^{(n)}(a^{(m)}c) = \sum_{j=0}^{\infty} (\begin{pmatrix} m \\ j \end{pmatrix}) (a^{(j)}b)(n+j)c$,

where $\partial^j = \partial^j / j!$.

A conformal algebra $R$ is called finite if $R$ is a finitely generated $\mathbb{C}[\partial]$-module. The rank of conformal algebra $R$ is its rank as a $\mathbb{C}[\partial]$-module.

**Definition 1.2** Let $\mathfrak{g}$ be an arbitrary Lie algebra. A formal distribution Lie algebra $(\mathfrak{g}, F)$ is the space $F$ of all mutually local $\mathfrak{g}$-valued distributions in complex variable $z$.

It is very important that we have a functor from the category of formal distribution Lie
algebras to the category of conformal algebras as well as a functor in the opposite direction that canonically associates to a conformal algebra $R$ a formal distribution Lie algebra ($K2$). It means that when we use a very formal language of conformal algebras we do not loose the information about the physical origin. It is specially important to have it in mind when we would like to construct higher dimensional generalizations.

The approach to higher dimensional chiral algebras suggested by Beilinson and Drinfeld is based more on algebraic geometry than the representation theory. In (BD) they introduced the notion of ”Chiral algebra” as a quantization of what they call the ”coisson algebra” (a Poisson algebra on $X$ in the compound setting). A really challenging problem is to find out what is a chiral algebra on higher dimensional $X$ (the coisson algebras live in any dimension). More algebraic approach to the higher dimensional conformal algebras was suggested by Bakalov, D’Andrea, and Kac in (BDK, BDK1) introducing the notion of Lie pseudoalgebra. The basic idea is to replace the $\mathbb{C}[\partial]$ in the definition of conformal algebra $R$ by a Hopf algebra $H = U(g)$, where $g$ is a finite-dimensional Lie algebra and $U(g)$ is its universal enveloping algebra. A Lie pseudoalgebra is defined as an $H$-module $L$ endowed with an $H$-bilinear map

$$L \otimes L \rightarrow (H \otimes H) \otimes H L$$

subject to a certain skewsymmetry and Jacobi identity axioms. Unfortunately, for Lie pseudoalgebras the functor to the higher dimensional distribution Lie algebras was not explicitly constructed.

We suggest an approach to the higher dimensional CFT that preserves the connection between the higher dimensional distribution Lie algebras and a higher dimensional conformal algebras. The right idea is to replace the $\mathbb{C}[\partial]$ in the definition of conformal algebra $R$ by a universal enveloping algebra of some noncommutative Lie algebra $n$. We consider the case when $n$ is a nilpotent subalgebra of a simple Lie algebra $g$. In this situation we can construct a Lie algebra of formal local distributions on some complex manifold $M^g$ that has the most of the properties of two dimensional CFT. It means that we have the consistent definition of the OPE and the normal product of two fields in the dimension higher than two. The corresponding conformal algebra is a $U(n)$-module with the $e_n$-products subject to a certain skewsymmetry and Jacobi identity axioms. Here $\{e_n\}$ is a basis in some representation of $g$. When $g = \mathfrak{sl}_2$ our construction leads to the standard two dimensional situation, where $M^\mathfrak{g} = \mathbb{C}^*$ and the $n$-product of two fields corresponds to the $(e_n)$-product, where $\{e_n\}$ is a basis in some representation of $\mathfrak{sl}_2$.

As it was mentioned above, in the case of a finite conformal algebra, there are only two principal solutions to the system of axioms (C1)–(C4) (in this paper we do not discuss a conformal superalgebras, where we have more possibilities):

1. **Current conformal algebras** $\text{Cur}$ $g = \mathbb{C}[\partial] \otimes g$ associated to the Lie algebra $g$. The only non-trivial $n$-product is the 0-product: $a_{(0)} b = [a, b], a, b \in g$. We identify $g$ with the subspace of $\text{Cur} g$ spanned by elements $1 \otimes g, g \in g$. The corresponding formal distribution Lie algebra is $(\hat{g}, R)$ where $\hat{g} = g \otimes \mathbb{C}[t, t^{-1}]$. Formal distributions $g(z) = g \otimes \delta(t - z) = \sum_{n \in \mathbb{Z}} g \otimes t^n \otimes z^{-n-1}$, defined for every $g \in g$ and satisfy the
commutation relations:
\[ [g_1(z), g_2(w)] = [g_1, g_2](w)\delta(z - w). \]  

(1.2)

2. Virasoro conformal algebra \( \text{Conf} (\text{Vect} \ C^*) \). The centerless Virasoro algebra of algebraic vector fields on \( C^* \) is spanned by the vector fields \( L_n = -t^{n+1} \partial_t \). The \( C^* \)-valued formal distribution \( L(z) = \delta(t - z) \partial_t \) satisfies
\[ [L(z), L(w)] = \partial_w L(w) \delta(z - w) + 2L(w) \partial_w \delta(z - w). \]  

(1.3)

The conformal algebra \( \text{Conf} (\text{Vect} \ C^*) = C[\partial] \) associated to \( \{L(z)\} \) is defined by the \( n \)-products:
\[ L_{(0)} L = \partial L, \quad L_{(1)} L = 2L, \quad L_{(n)} L = 0 \quad \text{if} \quad n \geq 2. \]  

(1.4)

What makes these two cases so special? The answer is that they both are related to the action of the Lie algebra \( \mathfrak{sl}_2 \) on the space \( V^{\mathfrak{sl}_2} \) of regular functions on a subset \( M \simeq C^* \) of the flag manifold \( SL(2, \mathbb{C})/B \simeq \mathbb{CP}^1 \) and the corresponding fields are expressed in terms of the formal delta function \( \delta(t - w) = \sum_{n \in \mathbb{Z}} t^n w^{-n-1} \), associated with this space.

Our approach is based on the construction of a complex manifold \( M^0 \) that can be described as a "right" higher-dimensional generalization of \( C^* \) from the point of view of a corresponding group action. In sect.2 we define the space \( V^g \) for any simple Lie algebra \( g \) and the formal delta function associated with this space. In sect.3 we define the Lie algebra of formal distributions on \( V^g \) and the higher dimensional analogues of the \( n \)-products and \( \lambda \)-product for the conformal algebra associated with the Lie algebra of formal distributions on \( V^g \).

In sect.4 we discuss the basic ideas about the geometrical realization of the space \( V^g \) in general case for any simple Lie algebra \( g \), and in sect.5 we present an explicit realization of \( V^g \) for \( g = \mathfrak{sl}_3 \).

In sect.6 we define the 3-dimensional analogues of the Affine Kac-Moody algebras associated with the action of the Lie algebra \( \mathfrak{sl}_3 \). We call these algebras the Generalized Affine Kac-Moody algebras \( \mathfrak{g}^V \) associated with the space \( V^{\mathfrak{sl}_3} \). We think that it is especially important to consider with more details the \( \mathfrak{sl}_3 \)-case, because it is well known fact, that it is difficult to make a step from \( \mathfrak{sl}_2 \) to \( \mathfrak{sl}_3 \) and it is almost strait forward from \( \mathfrak{sl}_3 \) to go to the general case of of any simple Lie algebra (at least to \( \mathfrak{sl}_n \)). The constructed algebras have many of the good properties of "the generalized Affine Kac-Moody algebras". The basic idea of Borcherds is to think of generalized Affine Kac-Moody algebras as infinite dimensional Lie algebras which have most of the good properties of finite dimensional reductive Lie algebras (\( \mathfrak{B} \)). For instance, we construct the normalized invariant form \((, , )\) and the Cartan involution of \( \mathfrak{g}^V \).
In sect. 7 we discuss the higher dimensional version of the Virasoro conformal algebra associated with the space $V^{\mathfrak{sl}_3}$. The Virasoro algebra appears in many different contexts related to the Lie algebra $\mathfrak{sl}_2$ and contains $\mathfrak{sl}_2$ as a subalgebra. We will discuss one more context related to the conformal Virasoro algebra and similarly we define the generalized Virasoro conformal algebra associated with the space $V^{\mathfrak{sl}_3}$. This algebra contains $\mathfrak{sl}_3$ as a subalgebra and it is a rank one module over $U(n_+)$, where $n_+$ is the upper nilpotent subalgebra of $\mathfrak{sl}_3$. Also we have the semidirect sum of the generalized Virasoro conformal algebra and conformal algebra $g^V$. The generalized Virasoro conformal algebra (the root lattice Virasoro conformal algebra) associated to any simple Lie algebra $\mathfrak{g}$ will be constructed in \cite{G-K}. We would like to notice that our definition of the generalized Virasoro conformal algebras is different from the Virasoro pseudoalgebras defined in \cite{BDK, BDK1}.

In the future publications we shall present:
1. The explicit geometrical realization of the space $V^\mathfrak{g}$ for any simple Lie algebra $\mathfrak{g}$;
2. The representations of the Generalized Affine Kac-Moody algebras $\mathfrak{g}^V$ associated with the space $V^\mathfrak{g}$ and the generalized Virasoro conformal algebra associated with this space;
3. The axioms for the higher dimensional Vertex Algebras associated with the space $V^\mathfrak{g}$;
4. The higher dimensional generalizations of the $N = 1$ and $N = 2$ superconformal algebras.

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2 Generalized delta-function

In this section we introduce the notion of the generalized delta-function associated with a vector space $V^\mathfrak{g}$, where $\mathfrak{g}$ is a simple Lie algebra. First we recall the basic definitions of formal distributions and in particular the formal delta-function.

A formal distribution in the indeterminates $z, w, \ldots \in (\mathbb{C}^*)^N$ with values in a vector space $W$ is a formal expression of the form

$$\sum_{m,n,\ldots \in \mathbb{Z}} a_{m,n,\ldots} z^m w^n \ldots,$$

where $a_{m,n,\ldots}$ are elements of a vector space $W$. They form a vector space denoted by $W[[z, z^{-1}, w, w^{-1}]]$.

Given a formal distribution $a(z) = \sum_{m \in \mathbb{Z}} a_m z^m$, we define the trace (integral) by the usual formula

$$\text{Res}_z a(z) = a_{-1}. \quad (2.1)$$

Define the $\mathbb{C}$ - valued bilinear form on on the space of $\mathbb{C}$ - valued formal distribution by

$$< a(z), b(z) >= \text{Res}_z a(z)b(z) \quad (2.2)$$

Since $\text{Res}_z \partial a(z) = 0$, we have the usual integration by part:

$$< \partial a(z), b(z) >= -< a(z), \partial b(z)>, \quad (2.3)$$
where \( \partial a(z) = \frac{\partial}{\partial z} a(z) \).

We would like to remark, that the bilinear form (2.2) is invariant under the action of the Lie algebra \( \mathfrak{sl}_2 \) given by:

\[
X = \partial, \quad H = -2z\partial - 1, \quad Y = -z^2\partial - z, \tag{2.4}
\]

where \( X, H, Y \) is the standard basis in \( \mathfrak{sl}_2 \). This bilinear form defines the pairing between the Verma module of \( \mathfrak{sl}_2 \) with the highest weight \( -\frac{\alpha_0}{2} \) in the space \( V^+ \simeq \mathbb{C}[z] \) and the Verma module with the lowest weight \( \frac{\alpha_0}{2} \) in the space \( V^- \simeq \frac{1}{z} \mathbb{C}[\frac{1}{z}] \).

The action (2.2) results from the natural action of the group \( G = SL(2, \mathbb{C}) \) on the Flag manifold \( G/B_- \), where \( B_- \) is the lower Borel subgroup of \( G \). The space \( V^+ \) is isomorphic to the space of regular functions on the big cell \( U = N_+ \cdot [1] \in G/B_- \) and the space \( V^- \) is isomorphic to the space of regular functions on the dual cell \( U^* = N_- \cdot s_\alpha [1] \in G/B_- \) factorized by constants. Here \( s_\alpha \) denote the action of the generator of the Weyl group of \( \mathfrak{sl}_2 \) on the flag manifold. The space \( V^+ \) is the maximum isotropic subspace with respect to the bilinear form (2.2). The weight basis in \( V^+ \) is \( e_n = z^n, \ n \in \mathbb{Z}_+ \) and the dual basis in \( V^- \) is \( e_n^* = z^{-n-1}, \ n \in \mathbb{Z}_+ \). The complex torus \( \mathbb{C}^* \) is the intersection of \( U \) and \( U^* \) and the space of all regular functions on \( \mathbb{C}^* \) is \( \mathbb{C}[z, z^{-1}] = V^+ \oplus V^- \).

The subspaces \( V^+ \) and \( V^- \) are invariant under the multiplication and the action of \( \mathfrak{sl}_2 \), given by (2.4). We will denote the space \( \mathbb{C}[z, z^{-1}] \) by \( V_z^g \) for \( g = \mathfrak{sl}_2 \).

Recall that the formal delta function is the following formal distribution in \( z \) and \( w \) with values in \( \mathbb{C} \)

\[
\delta(z - w) = \sum_{n \in \mathbb{Z}} z^n \cdot w^{-n-1}. \tag{2.5}
\]

We can think about the formal delta function as an element of the space \( V_z^g \otimes V_w^g \) of the form

\[
\delta(z - w) = \sum_{n \in \mathbb{Z}_+} e_n \otimes e_n^* + \sum_{n \in \mathbb{Z}_+} e_n^* \otimes e_n = \delta(z - w)_+ + \delta(z - w)_-, \tag{2.6}
\]

where

\[
\delta(z - w)_- \in V_z^+ \otimes V_w^- \quad \text{and} \quad \delta(z - w)_+ \in V_z^- \otimes V_w^+. \tag{2.7}
\]

The well known properties of the formal delta-function (K1) result from (2.6). Let us mention some of them:

(a) For any formal distribution \( f(z) \in U[[z, z^{-1}]] \) one has: \( \text{Res}_z f(z) \delta(z - w) = f(w) \),

(b) \( \partial_z^j \delta(z - w) = (-\partial_w)^j \delta(z - w) \).

As we have mentioned before, the formal delta function is connected with the action of the Lie algebra \( \mathfrak{sl}_2 \) on functions on the Flag manifold and it is an element of the space \( V_z^g \otimes V_w^g \) (2.1).

The construction of the generalized formal delta functions is based on the same idea. First we will give a formal definition of the space \( V^g \) for a simple Lie algebra \( g \) and the generalized formal delta functions connected with this space. Then in the next sections we
will give the explicit construction for \( g = \mathfrak{sl}_3 \) and discuss the general construction for any simple Lie algebra \( g \).

Let \( g \) be a simple Lie algebra of rank \( l \). As a vector space, it has the triangular decomposition
\[
g = n_+ \oplus \mathfrak{h} \oplus n_- ,
\]
where \( \mathfrak{h} \) is a Cartan subalgebra and \( n_\pm \) are the upper and lower nilpotent subalgebras. Let
\[
b_\pm = h \oplus n_\pm
\]
be the upper and lower Borel subalgebras.

**Definition 2.1** A space \( V^g \) is a vector space endowed with \( \mathbb{C} \)-valued non-degenerated symmetric bilinear form
\[
< \cdot , \cdot > : V^g \otimes V^g \rightarrow \mathbb{C}
\]
such that the following axioms hold

(V1) \( V^g \) it a commutative associative algebra with unitary element \( 1 \) with respect to the multiplication. (We can think that \( V^g \) is a space of complex-valued functions on a complex manifold \( M \) or orbifold).

(V2) \( V^g \) is a \( g \)-module, such that the action of \( n_+ \) is a derivation of \( V^g \); it means that \( x(f \cdot g) = x(f) \cdot g + f \cdot x(g) \) for any \( x \in n_+ \) and any \( f, g \in V^g \). In particularly, it means that the unitary element of \( V^g \) is annihilated by all elements from \( n_+ \).

(V3) The bilinear form \( < \cdot , \cdot > \) is invariant under the multiplication in \( V^g \) and the action of the Lie algebra \( n_+ \):
\[
< f \cdot g , h > = < g , f \cdot h > + < f , x(g) > ,
\]
for any \( x \in n_+ \) and any \( f, g, h \in V^g \).

(V4) There is a maximal isotropic with respect to the bilinear form subspace \( V_+ \) in \( V^g \), such that \( V_+ \) is invariant under the multiplication and the action of \( n_+ \).

Then
\[
V^g = V_+ \oplus V_- ,
\]
where \( V_- \simeq (V_+)^* \) is the dual to the subspace \( V_+ \) in \( V^g \) with respect to the given bilinear form.

In the next section we will construct the explicit realization of the space \( V^g \) as the ring of complex-valued regular functions on a complex manifold \( M \). Because the vector space \( V^g \) is self-dual, we can identify \( V^g \) with \( (V^g)^* \). Let \( \{ e_\gamma \} \), \( \gamma \in \Gamma \) be a basis in \( V_+ \) and \( \{ e_\gamma^* \} \) is the dual basis in \( V_- \), where \( \Gamma \) is a discrete set that numerate the basis.

**Remark 2.1** In the previous definition instead of bilinear form we can postulate that the space \( V^g \) has a trace \( \text{Res} \) invariant with respect to the action of the nilpotent subalgebra \( n_+ \).
If such trace exists, the bilinear form on $V^\mathfrak{g}$ can be defined as

$$< f , g > = \text{Res} (f \cdot g).$$ \hfill (2.13)

In the other direction, if we have a bilinear form with the given properties on $V^\mathfrak{g}$, and $e_{\gamma_0} = 1$ for some $\gamma_0 \in \Gamma$, then the invariant trace can be defined as

$$\text{Res} (f) = < f, 1 > = \text{coefficient of } e_{\gamma_0}^*$$ \hfill (2.14)

in the decomposition with respect to the basis $\{e_\gamma\} \cup \{e_\gamma^*\}$, $\gamma \in \Gamma$.

**Definition 2.2** The generalized formal delta functions associated with the space $V^\mathfrak{g}$ is defined as an element of the space $V^\mathfrak{g} \otimes (V^\mathfrak{g})^* = V_+ \otimes V_- \oplus V_- \otimes V_+$ of the form

$$\delta_{V^\mathfrak{g}} = \sum_{\gamma \in \Gamma} e_\gamma \otimes e_\gamma^* \oplus e_\gamma^* \otimes e_\gamma.$$ \hfill (2.15)

Now suppose that the space $V^\mathfrak{g}$ is realized as the space $\text{Fun}(M)$ of formal distributions on $M$ and $z = (z_1, z_2, \ldots, z_n)$ are coordinates on $M$, then the space $V^\mathfrak{g} \otimes (V^\mathfrak{g})^*$ can be identified with the space of distributions in two sets of coordinates $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$. In this case we will use the notation

$$\delta_{V^\mathfrak{g}} (z - w),$$ \hfill (2.16)

or for short $\delta_V (z - w)$.

The action of the nilpotent subalgebra $n_+$ on $\text{Fun}(M)$ is given by vector fields. Thus, we have a Lie algebra homomorphism $n_+ \to \text{Vect} M$. So defined generalized formal delta function have the most of the properties of the standard delta function with respect to the trace $<2.2>$ and differentiations from $n_+$:

(a) for any formal distribution $f(z) \in V^\mathfrak{g}$ one has:

$$\text{Res}_z f(z) \delta_V(z - w) = f(w),$$ \hfill (2.17)

(b) for any element $a \in n_+$ one has:

$$(\xi_{a}^*)^j \delta(z - w) = (-\xi_{a}^w)^j \delta(z - w),$$ \hfill (2.18)

where $\xi_{a}^w$ is the image of $a$ in $\text{Vect} M$.

### 3 Local distributions on $V^\mathfrak{g}$

By the definition the space $V^\mathfrak{g} = V_+ \oplus V_-$ is a $U(n_+)$-module. We will think that $V^\mathfrak{g}$ is realized as a space of regular functions on some complex manifold $M^\mathfrak{g}$. Let $\{e_\gamma\}$, $\gamma \in \Gamma$ be a basis in $V_+$ and let $\{e_\gamma^*\}$ be the dual basis in $V_-$ constructed in the previous section.
Fix a triangular decomposition of \( g \):
\[
g = n_+ \oplus h \oplus n_-
\] (3.1)
where \( h \) is the Cartan subalgebra and \( n_{\pm} \) are the upper and the lower nilpotent subalgebras. Let
\[
b_{\pm} = h \oplus n_{\pm}
\] (3.2)
be the upper and lower Borel subalgebras. Let \( \Delta \) be the upper and lower Borel subalgebras. Let \( h_i = \alpha_i, i = 1, \ldots, l \), be the \( i \)th coroot of \( g \) and let \( l \) be the rank of \( g \). The set \( \{h_i\}_{i=1,\ldots,l} \) is a basis of \( h \). We choose a root basis of \( n_{\pm} \), \( \{e^a\}_{\alpha \in \Delta_{\pm}} \), where \( \Delta_{+} \) (\( \Delta_{-} \)) is the set of positive (negative) roots of \( g \), so that \([h, e^a] = \alpha(h)e^a\) for all \( h \in h \).

Let \( \xi_{\alpha_i} \) be the image of \( e^{\alpha_i}, \alpha \in \Delta_{+} \) in \( \text{Der}(V^g) \). Fix some ordering of \( \Delta_{+} \). Then we can write \( \partial_i \) instead of \( \xi_{\alpha_i} \) and \( (V^g) \) is a \( \mathbb{C}[\partial_1, \ldots, \partial_p] \)-module, where \( p = |\Delta_{+}| \). The \( \mathbb{C}[\partial_1, \ldots, \partial_p] \) has a Poincaré-Birkhoff-Witt basis of the form \( \{\partial_1^{k_1}\partial_2^{k_2} \ldots \partial_p^{k_p}\} \).

Let \( W \) be a vector space. Consider the space \( \text{End}W \otimes V^g \) of \( \text{End}W \)-valued formal distributions associated with the space \( V^g \). Any element \( a^V \in \text{End}W \otimes V^g \) is an expression of the form
\[
a^V = \sum_{\gamma \in \Gamma} a_{\gamma_{\gamma}} \otimes e_{\gamma} + \sum_{\gamma \in \Gamma} a_{\gamma} \otimes e_{\gamma}^* = a^V_+ + a^V_-,
\] (3.3)
where \( a_{\gamma_{\gamma}}, a_{\gamma} \in \text{End}W \). The coefficients in (3.3) are defined via the trace (2.14). For any \( a^V \in \text{End}W \otimes V^g \) and any \( f \in V^g \) we can define \( a^V_f \in \text{End}W \) as
\[
a^V_f = \text{Res}_{V^g}((1 \otimes f) \cdot a^V).
\] (3.4)

We have a natural action of \( \mathbb{C}[\partial_1, \ldots, \partial_p] \) on \( \text{End}W \otimes V^g \), induced by the action on \( V^g \), so that
\[
(\partial_i a)^V_f = -a^V_{\partial_i f},
\] (3.5)
or, more general:
\[
(\partial_1^{k_1}\partial_2^{k_2} \ldots \partial_p^{k_p} a)^V_f = (-1)^{k_1+k_2+\ldots+k_p} a^V_{\partial_1^{k_1}\partial_2^{k_2} \ldots \partial_p^{k_p} \partial_{k_1} \ldots \partial_{k_p} f}.
\] (3.6)

We will say that two formal distributions \( a^V \) and \( b^V \) are mutually local if they commutator can be expressed as a finite linear combination of the \( \delta_V \) and its derivatives:
\[
[a^V_+, \delta^V_+] = \sum_{k_1, \ldots, k_p} c^V_{(\partial_1^{k_1} \ldots \partial_p^{k_p} v^*_{\mu})} \partial_1^{k_1} \partial_2^{k_2} \ldots \partial_p^{k_p} \delta_{\mu}.
\] (3.7)
Since \( \delta_V \in V^g_+ \otimes (V^g_+)^* \) in the previous formula, we need to specify to which factor we apply the operator \( \partial_k \). \( \partial_{k,w} \) means that we apply it to the second factor in the tensor product.

The coefficient \( c^V_{(\partial_1^{k_1} \ldots \partial_p^{k_p} v^*_{\mu})} \in \text{End}W \otimes V^g_+ \) in this decomposition can be viewed as the \( v_{k_1, \ldots, k_p} \)-product of fields \( a^V \) and \( b^V \). The element \( v_{k_1, \ldots, k_p} \) is in the space \( V^g_{\mu} \), dual
to some representation of $\mathfrak{g}$ with a lowest weight $\mu$ and the lowest vector $v_\mu \in V_\mu$, and is dual to $\partial_1^{k_1} \partial_2^{k_2} \ldots \partial_p^{k_p}$ in the sense that

$$\partial_p^{k_p} \partial_{p-1}^{k_{p-1}} \ldots \partial_{1}^{k_1} v_{s_1 \ldots s_p} = (-1)^{k_1+k_2+\ldots+k_p} v_\mu^*.$$  \hfill (3.8)

In particular, $\partial_i v_\mu^* = 0$.

For any $f$ and $g \in V^\mathfrak{g}$ we can define $[a_f^V, b_g^V] \in \text{End} W \otimes V^\mathfrak{g}_z \otimes V^\mathfrak{g}_w$ as:

$$[a_f^V, b_g^V] = \text{Rez}_w \text{Rez}_z ([a^V, b^V] \cdot (1 \otimes f \otimes 1) \cdot (1 \otimes 1 \otimes g)).$$ \hfill (3.9)

To consider the space of formal distributions with the values in a Lie algebra $\mathfrak{a}$ we need to impose the skewsymmetry and the Jacobi identity axioms to the $a_f^V b_g^V$-products of two fields, $v \in V^\mathfrak{g}_s$. Denote by $R^\mathfrak{g}$ the set of all local fields (distributions) on $V^\mathfrak{g}$ with the values in $\mathfrak{a}$. The $v$-products define a $\mathbb{C}$-linear map

$$\mathcal{A} : V^\mathfrak{g}_s \otimes R^\mathfrak{g} \otimes R^\mathfrak{g} \longrightarrow R^\mathfrak{g}.$$ \hfill (3.10)

This map satisfies the following axioms, that are analogues of axioms (C1)-(C2) in the Def.1.1:

(H1) For any $a, b \in R^\mathfrak{g}$ the map $\mathcal{A} : V^\mathfrak{g}_s \otimes a \otimes b \longrightarrow R^\mathfrak{g}$ is non zero only on a finite dimensional subspace $V^\mathfrak{g}_s \subset V^\mathfrak{g}_s$,

(H2) $\mathcal{A} \circ (1 \otimes \partial_i \otimes 1) = \mathcal{A} \circ (\partial_i \otimes 1 \otimes 1)$ and $\mathcal{A} \circ (1 \otimes 1 \otimes \partial_i) = \partial_i \circ \mathcal{A} + \mathcal{A} \circ (1 \otimes \partial_i \otimes 1)$.

**Definition 3.1** $\Lambda(\text{Lie})$ generalized conformal algebra $R^\mathfrak{g}$, associated with the space $V^\mathfrak{g}_s$ is a $\mathbb{C}[\partial_1, \ldots, \partial_p]$-module, endowed with the map $\mathcal{A}$, satisfying axioms (H1)-(H2) as well as the skewsymmetry and the Jacobi identity axioms. A conformal algebra $R^\mathfrak{g}$ is called finite if $R^\mathfrak{g}$ is a finitely generated $\mathbb{C}[\partial_1, \ldots, \partial_p]$-module. The rank of conformal algebra $R^\mathfrak{g}$ is its rank as a $\mathbb{C}[\partial_1, \ldots, \partial_p]$-module.

Since $\mathfrak{n}_+$ is a non commutative algebra, it is difficult to write explicitly in terms of $\mathcal{A}$ the skewsymmetry and the Jacobi identity axioms. As for Conformal algebras these axioms have more simple form in terms of the so-called $\lambda$-bracket (see for reference [K2]) defined as:

$$[a_\lambda b] = \sum_{n \in \mathbb{Z}} \lambda^{(n)} a_{(n)} b.$$ \hfill (3.11)

We define a non-commutative analogue of the $\lambda$-bracket as follow. Consider a Poincaré-Birkhoff-Witt basis $\mathcal{B} = \{e_{a_1}^{k_1} \ldots e_{a_p}^{k_p}\} \in U(\mathfrak{n}_+)$. Let $\tilde{\lambda} = \lambda_1^{k_1} \ldots \lambda_p^{k_p}$ be the symbol of $e_{a_1}^{k_1} \ldots e_{a_p}^{k_p}$ in $\mathbb{C}[\lambda_1, \ldots, \lambda_p]$ and $\tilde{\partial} = \partial_1^{k_1} \ldots \partial_p^{k_p}$ its image in $\mathbb{C}[\partial_1, \ldots, \partial_p]$. Denote by $t^\lambda = (-\lambda_p)^{k_p} \ldots (-\lambda_1)^{k_1}$ the image of $\tilde{\lambda}$ under the transposition map. Define the $\lambda$-bracket of two elements $a, b \in R^\mathfrak{g}$ as:

$$[a_\lambda b] = \sum_{\partial_1^{k_1} \ldots \partial_p^{k_p} \in \mathcal{B}} \lambda_1^{(k_1)} \ldots \lambda_p^{(k_p)} a_{(\partial_1^{k_1} \ldots \partial_p^{k_p} v_{\mu})}, b.$$ \hfill (3.12)
The axioms (H1)-(H2) are rephrased as follows:

(H1) \[ [a \partial b] = \lambda [a, b], \quad (a \partial b) = (\partial + \lambda) [a, b], \]

(H2) \[ [\partial L] = \lambda [L], \quad (\partial L) = (\partial + \lambda) [L] \]

the skewsymmetry axiom:

(H3) \[ [a, b] = -[b, a] = -[b, a] \]

the Jacobi identity axiom:

(H4) \[ [a, b] = -[b, a] = -[b, a] \]

In this setting the axioms (C1)-(C4) of conformal algebra \( R \) in Def.1.1, applying to the affine Kac-Mody and Virasoro cases say that there are only two opposite cases:

(i) \( R/\partial R \) is a Lie algebra and \( V_\mu \) is one dimensional module. Then for \( a, b \in R/\partial R \) we define \( a_{(\mu)} b = [a, b] \). This is the Current conformal algebras case.

(ii) \( R/\partial R \) is one dimensional (rank one conformal algebra) and \( V_\lambda \) is a self-dual \( \mathbb{C}[\partial] \)-module that has the structure of Lie algebra.

More precisely in the second case we take \( \mu = -\alpha \) where \( \alpha \) is the root of \( \mathfrak{sl}_2 \). Then, \( V_\alpha \simeq \mathfrak{sl}_2 \) with \( v_\alpha = Y \) and the action \( \partial \) given by \( \partial = \text{ad}(X) \). Here \( X, H, Y \) is the standard basis of \( \mathfrak{sl}_2 \). Normalize this basis as \( v_\alpha = Y, H = \partial Y, -2X = \partial^2 Y \) and \( \partial X = 0 \). The dual basis is \( X, H/2, -Y/2 \). Take \( L = Y \) and define the products, corresponding to the elements of the dual basis as:

\[
L(0) L = L(X) L = \partial L, \quad L(1) L = L(H/2) L = [-H, L] = 2L.
\]

The axiom (H2) reads as \((\partial L)_{(e)} L = L(\partial e) L\), in particular \((\partial L)_{(H/2)} L = L(\partial H/2) L = -L(X) L\). These relations between the commutation relations of the \( \mathfrak{sl}_2 \) Lie algebra and n-products of the Virasoro conformal algebra is a one more manifestation of the link between Vir and \( \mathfrak{sl}_2 \) Lie algebras.

Given a generalized conformal algebra \( R^\theta \), a Lie algebra of \( V^\theta \)-local formal distributions associated to it is defined as follows. Fix a basis \( \{e_\gamma, e_\gamma^*\}, \gamma \in \Gamma \) in \( V^\theta \). Consider a vector space over \( \mathbb{C} \) with the basis

\[ a_{e_\gamma}, a_{e_\gamma}^* \], \quad \text{where} \ a^V \in R^\theta, \quad \text{Then the Lie algebra of} \ V^\theta \text{-local formal distributions is a quotient of this space by the} \ \mathbb{C} \text{-span of all elements of the form}

\[ (\lambda a^V + \mu b^V) e_\gamma - (\lambda a^V) e_\gamma - \mu (b^V) e_\gamma, \quad (\lambda a^V + \mu b^V) e_\gamma^* - (\lambda a^V) e_\gamma^* - \mu (b^V) e_\gamma^*; \]

\[ (\partial_\lambda a)^V e_\gamma + a_{\partial_\lambda e_\gamma}, \quad (\partial_\mu a)^V e_\gamma^* + a_{\partial_\mu e_\gamma}; \]

4 The space \( V^\theta \)

In this section we discuss the basic ideas about the realization of the space \( V^\theta \) in general case for any simple Lie algebra \( \mathfrak{g} \). We identify the space \( V^\theta \) with the space of regular function on an open subset (manifold) of the flag manifold \( B_- \setminus G \), where \( G \) is the simply-connected Lie group corresponding to the Lie algebra \( \mathfrak{g} \).
As a vector space, the Lie algebra $\mathfrak{g}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{n}_\pm$ are the upper and lower nilpotent subalgebras. Let

$$\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$$

be the upper and lower Borel subalgebras. Let $N_\pm$ (respectively, $B_\pm$) be the upper and lower unipotent subgroups (respectively, Borel subgroups) of $G$ corresponding to $\mathfrak{n}_\pm$ (respectively, $\mathfrak{b}_\pm$). Let $\{\alpha_i\}_{i=1,\ldots,l}$ be the root basis of $\mathfrak{g}$ and the $\{h_i\}_{i=1,\ldots,l}$, the coroot basis of $\mathfrak{h}$. We choose a root basis of $\mathfrak{n}_\pm$, $\{e^\alpha\}_{\alpha \in \Delta \pm}$, where $\Delta_+$ ($\Delta_-$) is the set of positive (negative) roots of $\mathfrak{g}$, so that $[h, e^\alpha] = \alpha(h)e^\alpha$ for all $h \in \mathfrak{h}$.

Consider the flag manifold $B_- \backslash G$. It has a unique open $N_+$-orbit, the so-called big cell $U = [1] \cdot N_+ \subset B_- \backslash G$, isomorphic to $N_+$. Since $N_+$ is a unipotent Lie group, the exponential map $\mathfrak{n}_+ \to N_+$ is an isomorphism and we have $U \simeq \mathbb{C}^3$. From the action of $N_+$ on $U$ we can introduce a system $\{y_\alpha\}_{\alpha \in \Delta_+}$ of homogeneous coordinates on $U$. The homogeneous means that

$$h \cdot y_\alpha = -\alpha(h)y_\alpha$$

for all $h \in \mathfrak{h}$.

The action of $G$ on $B_- \backslash G$ gives us a map from $\mathfrak{g}$ to the Lie algebra of vector fields on $B_- \backslash G$, and hence on its open subset $U \simeq N_+$. Thus we obtain a Lie algebra homomorphism $\mathfrak{g} \to \text{ Vect } N_+$. With respect to this action the space $\text{ Fun } N_+$ of regular functions on $U$ has structure of the contragradient Verma module $M^*_\chi$ with lowest weight $0$ (for more details see [1]). We have a natural pairing $U(\mathfrak{n}_+) \times \text{ Fun } N_+ \to \mathbb{C}$, which maps $(P, A)$ to the value of the function $P \cdot A$ at the identity element of $N_+$, for any $A \in \text{ Fun } N_+$ and $P \in U(\mathfrak{n}_+)$. The vector $1 \in \text{ Fun } N_+$ is annihilated by $\mathfrak{n}_+$ and has weight $0$ with respect to $\mathfrak{h}$. Hence there is a non-zero homomorphism $\text{ Fun } N_+ \to M^*_\chi$ sending $1 \in \text{ Fun } N_+$ to a non-zero vector $v^*_\chi \in M^*_\chi$ of weight $0$. Since both $\text{ Fun } N_+$ and $M^*_\chi$ are isomorphic to $U(\mathfrak{n}_+)^\vee$ as $\mathfrak{n}_+$-modules, this homomorphism is an isomorphism.

It is known (1) that we can identify the Module $M^*_\chi$ with an arbitrary weight $\chi$ with $\text{ Fun } N_+$, where the latter is equipped with a modified action of $\mathfrak{g}$. Recall that we have a canonical lifting of $\mathfrak{g}$ to $\mathcal{D}_{\leq 1}(N_+)$ of differential operators on $U$ of order one, in a way that $a \to \xi_a$. The modified action is obtained by adding to each $\xi_a$ a function $\phi_a \in \text{ Fun } N_+$. The modified differential operators $\xi_a + \phi_a$ satisfy the commutation relations of $\mathfrak{g}$ if and only if the linear map $\mathfrak{g} \to \text{ Fun } N_+$, given by $a \to \phi_a$ is a one-cocycle of $\mathfrak{g}$ with coefficients in $\text{ Fun } N_+$.

If we impose the extra condition that the modified action of $\mathfrak{h}$ on $\text{ Fun } N_+$ remains diagonalizable, we get that our cocycle should be $\mathfrak{h}$-invariant: $\phi_{[h,a]} = \xi_h \cdot \phi_a$, for all $h \in \mathfrak{h}, a \in \mathfrak{g}$. The space of $\mathfrak{h}$-invariant one-cocycle of $\mathfrak{g}$ with coefficients in $\text{ Fun } N_+$ is canonically isomorphic to the first cohomology of $\mathfrak{g}$ with coefficients in $\text{ Fun } N_+$ (see [1]). By Shapiro’s lemma we have $H^1(\mathfrak{g}, \text{ Fun } N_+) = H^1(\mathfrak{g}, M^*_\chi) \simeq H^1(\mathfrak{b}, \mathbb{C}_0) = (\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}])^* \simeq \mathfrak{h}^*$. Thus, for each $\chi \in \mathfrak{h}$ we have an embedding $\rho_\chi : \mathfrak{g} \hookrightarrow \mathcal{D}_{\leq 1}(N_+)$ and the structure of $\mathfrak{h}^*$-graded $\mathfrak{g}$-module on $\text{ Fun } N_+$. 

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More detailed analyze shows that the action of \( n_+ \) on \( \text{Fun} N_+ \) is not modified and the action of \( h \in \mathfrak{h} \) is modified by \( h \rightarrow h + \chi(h) \). This comes from the reason that the weight of any monomial in \( \text{Fun} N_+ \) is equal to the sum of negative roots \([13]\) and the \( \mathfrak{h} \)-invariance of the one-cocycle \( \xi_h \cdot \phi_{e^\alpha} = \phi_{[h,e^\alpha]} = \alpha(h)\phi_{e^\alpha}, \ \alpha \in \Delta_+ \). Therefore \( \phi_{e^\alpha} = 0 \) for all \( \alpha \in \Delta_+ \). The vector \( 1 \in \text{Fun} N_+ \) is still annihilated ny \( n_+ \), but now it has weight \( \chi \) with respect to \( \mathfrak{h} \). Hence there is a non-zero homomorphism \( \text{Fun} N_+ \rightarrow M^\chi_\chi \) sending \( 1 \in \text{Fun} N_+ \) to a non-zero vector \( v^\chi_\chi \in M^\chi_\chi \) of weight \( \chi \). Since both \( \text{Fun} N_+ \) and \( M^\chi_\chi \) are isomorphic to \( U(n_+)^* \) as \( n_+ \)-modules, this homomorphism is an isomorphism.

For the reasons explained below we will consider the modified action of \( g \) on \( \text{Fun} N_+ \) with highest weight

\[
\chi = -\rho, \text{ where } \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha. \tag{4.4}
\]

**Proposition 4.1** With respect to the modified action of the Lie algebra \( g \) the space \( \text{Fun} N_+ \) has the structure of the Verma module \( M_{-\rho} \) with highest weight \( -\rho \).

The space \( \text{Fun} N_+ \) is closed under the multiplication and \( n_+ \) acts on this space by derivations. We will construct the bigger space with the same action of \( g \) that has all the properties of Definition 3.1.

We have the natural action of the Weyl group \( W \) of \( g \) on the flag manifold \( B_- \setminus G \). From the definition \( W \simeq N(T^l)/T^l \), where \( T^l \) is maximum torus in \( G \) and \( N(T^l) \) is its normalizer. We have a particular element \( w_0 \in W \) called the longest element (see \([W]\)), satisfying the following three conditions:

\[
(i) \ w_0(\Delta_+) = \Delta_-,
(ii) \ w_0 \cdot \rho = -\rho,
(iii) \ l(w_0) = |\Delta_+|,
\]

where \( \rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \). This element is uniquely defined, and satisfies \( w_0^2 = e \). For example, for type \( A_l \) and any simple positive root \( \alpha_i \), \( w_0 \alpha_i = -\alpha_{l+1-i} \), \( 1 \leq i \leq l \). It means that \( w_0 N_+ w_0^{-1} = N_- \).

Consider the big cell \( U^* \in B_- \setminus G \) dual to \( U = [1] \cdot N_+ \subset B_- \setminus G \). We have

\[
U^* = [1]w_0 \cdot N_- = [1]w_0 w_0 N_+ w_0 = Uw_0. \tag{4.5}
\]

Let \( \text{Fun} U^* \) be the ring of regular function on \( U^* \). We have \( U^* \simeq N_- \). From the action of \( N_- \) on \( U^* \) we can introduce a system \( \{x_\alpha\}_{\alpha \in \Delta_+} \) of homogeneous coordinates on \( U^* \). Homogeneous means that

\[
h \cdot x_\alpha = \alpha(h)x_\alpha \tag{4.6}
\]

for all \( h \in \mathfrak{h} \).

On the intersection \( U \cap U^* \) we can consider the change of variables on \( U \cap U^* \) from \( \{y_\alpha\} \) to \( \{x_\alpha\} \).

There is an ideal in \( \text{Fun} U^* \) that has a structure of the Verma module \( M^\vee_\rho \) with the lowest weight \( \rho \) with respect to the modified action of the Lie algebra \( g \).
Proposition 4.2 There is an element \( f \in \text{Fun } U \) that has the following properties:

(i) \( f \) has a weight \(-2\rho\) with respect to the action of \( \mathfrak{h} \),
(ii) \( \phi = (f)^{-1} \) is an element of \( \text{Fun } U^* \) of weight \( \rho \),
(iii) \( \phi \) is annihilated by \( n_- \),
(iv) the ideal \( V_\rho = \phi \cdot \text{Fun } U^* \) has the structure of the Verma module \( M'_\rho \) with the lowest weight \( \rho \) with respect to the modified action of the Lie algebra \( \mathfrak{g} \).

We have a natural pairing \(< \cdot, \cdot >\) between the Verma module \( M_{-\rho} \simeq \text{Fun } U \) with highest weight \(-\rho\) and the Verma module \( M'_\rho \simeq V_\rho \in \text{Fun } U^* \) with lowest weight \( \rho \).

Consider the space \( V_0 = \text{Fun } U \oplus \phi \cdot \text{Fun } U^* \). The element \( \phi \in V_0 \) define an invariant trace on \( V_0 \) in a way:

\[
\text{Res } f = < f, 1 >, \quad f \in V_0 \tag{4.7}
\]

This space have almost all properties of the space from the Definition 2.1, with exception that it is not closed under the multiplication. Now we will construct the bigger space \( V^\mathfrak{g} \), that contains \( V_0 \) as the subspace and closed under the multiplication.

Consider a complex submanifold \( M \in B_- \setminus G \) that is the intersection of \( U \) and \( U^* \)

\[
M = U \cap U^* \tag{4.8}
\]

As a set \( M \) is isomorphic to the complex space \( \mathbb{C}^n \setminus D_\phi \), where \( D_\phi \) is the divisor defined by \( \phi = \phi_1(y_{\alpha_1}, ..., y_{\alpha_n}) \cdot ... \cdot \phi_l(y_{\alpha_1}, ..., y_{\alpha_n}) = 0 \), \( i = 1, ..., l \) and each divisor \( D_{\phi_i} : \phi_i(y_{\alpha_1}, ..., y_{\alpha_n}) = 0 \) is a simple divisor.

Denote by \( V^\mathfrak{g} \) the space \( \text{Fun } M \) of regular functions on \( M \).

Theorem 4.1 The space \( V^\mathfrak{g} \) has all the properties listed in the Definition 2.1.

The proof to this theorem the Proposition 4.2 and the explicit construction for the manifold \( M = U \cap U^* \) for the Lie algebra \( \mathfrak{g} \) of one of the type \( A_n, B_n, C_n, D_n \) will be given in the future publications of the author. The case \( \mathfrak{g} = A_2 \) is discussed in the next section.

5 Explicit realization of the space \( V^\mathfrak{sl}_3 \)

In this section we give an explicit construction of the space \( V^\mathfrak{g} \) for \( \mathfrak{g} = \mathfrak{sl}_3 \) as the space of regular function on an open subset (manifold) of the flag manifold \( B_- \setminus G \), where \( G = SL(2, \mathbb{C}) \) is the simply-connected Lie group corresponding to \( \mathfrak{g} = \mathfrak{sl}_3 \). For this section we assume that \( \mathfrak{g} = \mathfrak{sl}_3 \). Fix a triangular decomposition

\[
\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \tag{5.1}
\]

where \( \mathfrak{h} \) is a Cartan subalgebra and \( \mathfrak{n}_\pm \) are the upper and lower nilpotent subalgebras. Let

\[
b_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm \tag{5.2}
\]
be the upper and lower Borel subalgebras. Let \( N_\pm \) (respectively, \( B_\pm \)) be the upper and lower unipotent subgroups (respectively, Borel subgroups) of \( G \) corresponding to \( n_\pm \) (respectively, \( b_\pm \)). Let \( h_i = \alpha_i^\vee \), \( i = 1, 2 \) be the \( i \)-th coroot of \( g \). The set \( \{ h_i \}_{i=1,2} \) is a basis of \( \mathfrak{h} \). We choose the root basis of \( n_\pm \) : \( \{ e^\alpha \} \alpha \in \Delta_\pm \), where \( \Delta_+ \) (\( \Delta_- \)) is the set of positive (negative) roots of \( \mathfrak{sl}_3 \), so that \( [h, e^\alpha] = \alpha(h)e^\alpha \) for all \( h \in \mathfrak{h} \).

Consider the flag manifold \( B_- \backslash G \). It has a unique open \( N_+ \)-orbit, the so-called big cell \( U = N_+ \cdot [1] \subset B_- \backslash G \), isomorphic to \( N_+ \), so that \( U \simeq \mathbb{C}^3 \). From the action of \( N_+ \) on \( U \) we can introduce a system \( \{ y_\alpha \}_{\alpha \in \Delta_+} \) of homogeneous coordinates on \( U \) in a following way. Let 

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix}
\]

be an element of \( N_+ \) and \( a, b, c \in \mathbb{C} \), then \( (a, b, c) \) are homogeneous coordinates on \( U \).

We have a Lie algebra homomorphism \( \mathfrak{sl}_3 \to \text{Vect} \mathcal{N}_+ \) With respect to this action the space of regular functions on \( U \) has structure of the contragradient Verma module \( M^*_0 \) with lowest weight \( 0 \). We have a natural pairing \( U(n_+) \times \text{Fun} \mathcal{N}_+ \to \mathbb{C} \), which maps \( (P, A) \) to the value of the function at the identity element of \( N_+ \), for any \( A \in \text{Fun} \mathcal{N}_+ \) and \( P \in U(n_+) \). This pairing define a isomorphism between the \( \text{Fun} \mathcal{N}_+ \) and \( M^*_0 \).

Consider the modified action of \( \mathfrak{sl}_3 \) on \( \text{Fun} \mathcal{N}_+ \) described in the previous section with the highest weight 

\[
\chi = -\rho = - (\alpha_1 + \alpha_2) = -\alpha_3,
\]

where \( \{ \alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2 \} \) are positive roots for \( \mathfrak{g} = \mathfrak{sl}_3 \). The corresponding basis in \( \mathfrak{sl}_3 \) is \( \{ e_1, e_2, e_3, h_1, h_2, f_1, f_2, f_3 \} \). The explicit formulas for the modified \( \mathfrak{sl}_3 \)-action in terms of homogeneous coordinates \( a, b, c \) on \( U \):

\[
\begin{align*}
\xi_{\alpha_1} &= \xi_{-\rho}(e_1) = \frac{\partial}{\partial a} = \partial_1, \\
\xi_{\alpha_2} &= \xi_{-\rho}(e_2) = \frac{\partial}{\partial b} + a \frac{\partial}{\partial c} = \partial_2, \\
\xi_{\alpha_3} &= \xi_{-\rho}(e_3) = \frac{\partial}{\partial c} = \partial_3, \\
\xi_{h_1} &= \xi_{-\rho}(h_1) = -2a \frac{\partial}{\partial a} + b \frac{\partial}{\partial b} - c \frac{\partial}{\partial c} - 1, \\
\xi_{h_2} &= \xi_{-\rho}(h_2) = a \frac{\partial}{\partial a} - 2b \frac{\partial}{\partial b} - c \frac{\partial}{\partial c} - 1, \\
\xi_{-\alpha_1} &= \xi_{-\rho}(f_1) = -a^2 \frac{\partial}{\partial a} - (c - ab) \frac{\partial}{\partial b} - ac \frac{\partial}{\partial c} - a, \\
\xi_{-\alpha_2} &= \xi_{-\rho}(f_2) = c \frac{\partial}{\partial a} - b^2 \frac{\partial}{\partial b} - b, \\
\xi_{-\alpha_3} &= \xi_{-\rho}(f_3) = -ac \frac{\partial}{\partial a} - b(c - ab) \frac{\partial}{\partial b} - c^2 \frac{\partial}{\partial c} - (2c - ab).
\end{align*}
\]
Proposition 5.1 With respect to the action of the Lie algebra \( \mathfrak{sl}_3 \) given by (5.4), the space \( \text{Fun} N_+ \) has the structure of the Verma module \( M_{-\rho} \) with highest weight \(-\rho\).

The space \( \text{Fun} N_+ \) is closed under the multiplication and \( n_+ \) acts on this space by derivations. We will construct the bigger space with the same action of \( \mathfrak{sl}_3 \) that has all the properties of Definition 3.1.

We have the natural action of the Weyl group \( W \) of \( \mathfrak{sl}_3 \) on the flag manifold \( B_- \setminus G \). From the definition \( W \simeq N(T) / T \), where \( T \) is maximum torus in \( G = SL(3, \mathbb{C}) \) and \( N(T) \) is its normalizer. In the case of \( \mathfrak{sl}_3 \) the Weyl group \( W \simeq S_3 \), where \( S_3 \) is a group of permutations. The longest element \( w_0 \in W \) corresponds to the permutation \((1, 3)\) via this identification.

Consider the big cell \( U^* \in B_- \setminus G \) dual to \( U = [1] \cdot N_+ \subset B_- \setminus G \). We have
\[
U^* = [1] w_0 \cdot N_- = U w_0.
\]
(5.5)

Let \( \text{Fun} U^* \) be the ring of regular function on \( U^* \). We have \( U^* \simeq N_- \). From the action of \( N_- \) on \( U^* \) we can introduce a system \( \{y_\alpha\}_{\alpha \in \Delta} \) of homogeneous coordinates on \( U^* \). Let
\[
n = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ t & r & 1 \end{pmatrix}
\]
be an element of \( N_- \) and \( s, r, t \in \mathbb{C} \). Then \( s, r, t \) are homogeneous coordinates on \( U^* \).

With respect to the action of the Cartan subalgebra (5.4) any monomial of the form \( s^n r^m t^k \) has the weight equal to the \(-\rho + n\alpha_1 + m\alpha_2 + k\alpha_3\).

The intersection \( U \cap U^* \) is an open subset in \( B_- \setminus G \). On the \( U \cap U^* \) the relations between the homogeneous coordinates \( a, b, c \) and \( s, r, t \) are given by:
\[
a = \frac{r}{t}, \quad b = -\frac{s}{t - rs}, \quad c = \frac{1}{t}, \quad \text{or} \quad s = -\frac{b}{c - ab}, \quad r = \frac{a}{c}, \quad t = \frac{1}{c}.
\]
(5.6)

We will show that there is an ideal in \( \text{Fun} U^* \) that has a structure of the Verma module \( M^{\vee}_\rho \) with the lowest weight \( \rho \) with respect to the action of the Lie algebra \( \mathfrak{g} \) given by (5.4).

Proposition 5.2 There is an element \( f(a, b, c) \in \text{Fun} U \) that has the following properties:

(i) \( f \) has a weight \(-2\rho\) with respect to \( \mathfrak{h} \),
(ii) \( \phi = f^{-1} \) is an element of \( \text{Fun} U^* \) of weight \( \rho \),
(iii) \( \phi \) is annihilated by \( n_- \),
(iv) the ideal \( V_\rho = \phi \cdot \text{Fun} U^* \in \text{Fun} U^* \) has the structure of the Verma module \( M^{\vee}_\rho \) with the lowest weight \( \rho \) with respect to the action of the Lie algebra \( \mathfrak{g} \) given by (5.4),
(v) the differential form \( \frac{da \wedge db \wedge dc}{c(c - ab)} \) is invariant under the transformation (5.6):
\[
\frac{da \wedge db \wedge dc}{c(c - ab)} \rightarrow \frac{dr \wedge ds \wedge dt}{t(t - rs)}.
\]

Proof. Consider a function \( f(a, b, c) = c(c - ab) = (t(t - rs))^{-1} \). Straightforward calculations show that this element have all the properties, listed above.
We have a natural pairing $\langle \cdot, \cdot \rangle$ between the Verma module $M_\rho \simeq \text{Fun} U$ with highest weight $-\rho$ and the Verma module $M'_\rho \simeq V_\rho \in \text{Fun} U^*$ with lowest weight $\rho$.

Consider the space $V_0 = \text{Fun} U \oplus \phi \cdot \text{Fun} U^*$. This space have almost all properties of the space from the Definition 2.1, with exception that it is not closed under the multiplication. Now we will construct the bigger space $V^\phi$, that is the algebraic closer (with respect to the multiplication) of the space $V_0$ and we will show that $V^\phi \simeq \text{Fun} U \cap U^*$.

We have a natural pairing $\langle \cdot, \cdot \rangle$ of the space $V$ and we will show that $\langle \cdot, \cdot \rangle$ for the modified action of $\mathfrak{g}$ on $\text{Fun} N_+$. Consider the space $\mathfrak{g}$.

Conceider a complex submanifold $M \in B_- \backslash G$ that is the intersection of $U$ and $U^*$

$$M = U \cap U^*. \quad (5.7)$$

Denote by $V^\phi$ the space $\text{Fun} M$ of regular functions on $M$. In order to obtain a nice description of $M$ and of the space $V^\phi$ we identify a point (a flag) in the flag manifold $B_- \backslash G$ with two orthogonal projective vectors:

$$\{(z_1 : z_2 : z_3), \ (w_1 : w_2 : w_3) \mid \sum_{i=1}^{3} z_i w_i = 0\}. \quad (5.8)$$

We have $[1] \in B_- \backslash G = \{(0, \ 0, \ 1); \ (1, \ 0, \ 0)\}$ and

$$U = \{(z_1 : z_2 : z_3), \ (w_1 : w_2 : w_3) \mid z_3 \neq 0, w_1 \neq 0\}. \quad (5.9)$$

In terms of homogeneous coordinates $(a, b, c)$ we have:

$$U = \{(-c + ab, -b, 1), \ (1, a, c), \ a, b, c \in \mathbb{C}\}. \quad (5.10)$$

The natural action of the Weyl group $W$ is just the corresponding permutation of coordinates. If $s \in W \simeq S_3$, then

$$U_s = \{(z_1 : z_2 : z_3), \ (w_1 : w_2 : w_3) \mid z_{s(3)} \neq 0, w_{s(1)} \neq 0\} \quad (5.11)$$

where $U_s$ is the big cell in $B_- \backslash G$, that is the image of $U$ under the action of the element $s \in W$. In particular, for the dual (opposite) sell we have $U^* = U_{w_0}$. Thus,

$$M = \{(z_1 : z_2 : z_3), \ (w_1 : w_2 : w_3) \mid \sum_{i=1}^{3} z_i w_i = 0, \text{ and } z_i \neq 0, w_i \neq 0, i = 1, 3\} \quad (5.12)$$

is the intersection of $\mathbb{C}^\ast \times \mathbb{C} \times \mathbb{C}^\ast \times \mathbb{C}$ with the hypersurface $\sum_{i=1}^{3} z_i w_i = 0$ in $\mathbb{C}P^2 \times \mathbb{C}P^2$.

If we fix homogeneous coordinates

$$(a, b, c), \ a = \frac{w_2}{w_1}, \ c = \frac{w_3}{w_1}, \ b = -\frac{z_2}{z_3} \quad (5.13)$$

on $U \simeq \mathbb{C}^3$, then $M$ is the total space of non trivial bundle with the base $(\mathbb{C})^2_{a, b}$ - the two-dimensional complex space and the fiber $\mathbb{C}^\ast \setminus \{c = ab\}$:

$$M \xrightarrow{\mathbb{C}^\ast \setminus \{c = ab\}} (\mathbb{C})^2_{a, b}. \quad (5.14)$$
The action (5.4) of the \( n_+ \) on \( \text{Fun} \ M \) in terms of projective coordinates \((z_1 : z_2 : z_3)\) and \((w_1 : w_2 : w_3)\) is:

\[
\xi_{\alpha_1} = -z_2 \frac{\partial}{\partial z_1} + w_1 \frac{\partial}{\partial w_2}, \quad \xi_{\alpha_2} = -z_3 \frac{\partial}{\partial z_2} + w_2 \frac{\partial}{\partial w_3}, \quad \xi_{\alpha_3} = -z_3 \frac{\partial}{\partial z_1} + w_1 \frac{\partial}{\partial w_3},
\]

and the element \( \phi \) from the Proposition 5.2 is

\[
\phi = \frac{z_3 \cdot w_1}{z_1 \cdot w_3}.
\]

This element defines \( n_+ \)-invariant trace on \( V^g \simeq \text{Fun} \ M \) in the following sense. The space \( V^g \) is the space of all regular function on \( M \) or all functions on \( \mathbb{C}^3 \) with the only poles at \( c = 0 \) and \( c = ab \). We can choose a basis \( \{e_{nml}; f_{nml}\} \) in \( V^g \) as:

\[
e_{nml} = a^n c^m (c - ab)^l, \quad n \geq 0, \quad m, l \in \mathbb{Z}; \quad f_{nml} = b^n c^m (c - ab)^l, \quad n > 0, \quad m, l \in \mathbb{Z}.
\]

**Proposition 5.3** Every regular function on \( M \) is a unique linear combination of the monomials of the basis (5.17).

Proof follows immediately from the description of \( M \) in terms of projective coordinates (5.12).

The element

\[
\phi = \frac{z_3 \cdot w_1}{z_1 \cdot w_3} = e_{0,-1,-1}
\]

is from this basis and for any regular function \( f \) on \( M \) we can define a trace:

\[
\text{Res} f = \text{coefficient at } \phi \text{ in the decomposition with respect to this basis.}
\]

This trace is invariant with respect to the action of \( n_+ \) in a sense that

\[
\text{Res} (\xi_{\alpha_i}(f)) = 0, \text{ for } i = 1, 2, 3.
\]

This invariant trace defines a bilinear invariant form on \( V^g \simeq \text{Fun} \ M \) as:

\[
<f, g > = \text{Res} f \cdot g \text{ for } f, g \in \text{Fun} \ M.
\]

**Remark 5.1** This invariant form, being restricted to the subspace \( V_0 = \text{Fun} U \oplus \phi \cdot \text{Fun} U^* \subset V^g \) is the natural pairing between the Verma module \( M_{-\rho} \simeq \text{Fun} U \) with highest weight \(-\rho\) and the Verma module \( M'_\rho \simeq V_\rho \subset \text{Fun} U^* \) with lowest weight \( \rho \).

Consider the subspace \( V_+ \) in \( \text{Fun} \ M \) that consists of all regular functions on \( M_+ = (\mathbb{C})^3_{a,b,c} \setminus D_{\{c=0\}} \). As a linear space \( M_+ \) is spanned by the elements \( \{e_{nml}, n, l \geq 0, m \in \mathbb{Z}; f_{nml}, n > 0, m \geq 0, m \in \mathbb{Z}\} \) of the basis (5.17). For simplicity, we will denote the basis in \( V_+ \) by \( \{e_\gamma\}, \gamma \in \Gamma \), where \( \Gamma \) is the set of all indexes of \( \{e_{nml}, f_{nml}\} \). We assume that \( e_0 = 1 \). We have

\[
<e_\gamma, \phi > = 0 \text{ for } \gamma \neq 0
\]

and

\[
<e_0, \phi > = 1.
\]
Proposition 5.4  (i) The space $V_+$ contains the unit function $1$ and it is closed under the multiplication.
(ii) $V_+$ is invariant under the action (5.15) of $n_+$.
(iii) The bilinear form (5.21) is non degenerated on $V^0$.
(iv) $V_+$ is the maximal isotropic subspace in $V^0$ with respect to the bilinear form (5.21).

Let $V_ - \in \text{Fun}_M$ be the subspace dual to $V_+$ in $V^0$ with respect to the bilinear form (5.21) and let $\{e_\gamma^*\}$, $\gamma \in \Gamma$ be the basis in $V_-$ dual to $\{e_\gamma\}$, $\gamma \in \Gamma$. We have $e_0^* = \phi$.

The space $V^{\text{sl}_3}$ of all regular functions on $M$ is the direct sum of two mutually dual subspaces $V^{\text{sl}_3} = V_+ \oplus V_-$. The space $V^{\text{sl}_3}$ satisfies to all properties (V1)-(V2) of the Definition 2.1. So we can introduce the generalized formal delta function associated with the space $V^0$ as an element of the space

$$V^0 \otimes (V^0)^* = V_+ \otimes V_- \oplus V_- \otimes V_+$$

(5.24)

in a way described in the sec.2:

$$\delta_{V^0} = \sum_{\gamma \in \Gamma} e_\gamma \otimes e_\gamma^* + e_\gamma^* \otimes e_\gamma.$$  

(5.25)

The space $V^0$ is realized as the space of function $\text{Fun}_M$ or, more general, as the space of formal distributions on $M$ and $z = (z_1, z_2, z_3)$ are global coordinates on $M$. Here $z_1 = a$, $z_2 = b$, $z_3 = c$, where $(a, b, c)$ are homogeneous coordinates on $M$. The space $V^0 \otimes (V^0)^*$ can be identified with the space of distributions in two sets of coordinates $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3)$. In this case we will use the notation

$$\delta_{V^0} (z - w),$$  

(5.26)

or for short, $\delta_V (z - w)$. We also can define $\delta_V (z - w)_{\pm}$ as

$$\delta_V (z - w)_+ = \sum_{e_\gamma^* \in V_+} e_\gamma^* \otimes e_\gamma \in V^0 \otimes V^{\pm}_+$$

(5.27)

and

$$\delta_V (z - w)_- = \sum_{e_\gamma \in V_+} e_\gamma \otimes e_\gamma^* \in V^{\pm}_+ \otimes V^0.$$  

(5.28)

So defined generalized formal delta function has the most of the properties of the standard delta function with respect to the trace (2.14) and differentiations from $n_+$: (a) For any formal distribution $f(z) \in V^0$ one has:

$$\text{Res}_z f(z) \delta(z - w) = f(w),$$  

(5.29)

$$z_i - w_i \delta(z - w) = 0;$$  

(5.30)

(b) for any element $a \in n_+$ one has:

$$\xi_a^i \delta(z - w) = (-\xi_a^w)^i \delta(z - w),$$ 

(5.31)
where $\xi_{a z}$ is the image of $a$ in $\text{Vect} \, M$.

We would like to remark that for the Lie algebra $\mathfrak{sl}_2$ the complex manifold $M \simeq U \cap U^* \simeq \mathbb{C}^*$ is contactable to the one dimensional real manifold $S^1$. The trace is the integral over the $S^1 : |z| = 1$:

$$\text{Res}_z a(z) = \oint_{|z|=1} a(z) \, dz$$

and the formal Cauchy formula can be written in the form

$$\text{Res}_z a(z) \partial_z^{(k)} \frac{1}{z - w} = (-1)^k \partial_z^{(k)} a(w)_+, \quad \text{for} \quad |z| > |w|$$

$$\text{Res}_z a(z) \partial_z^{(k)} \frac{1}{z - w} = (-1)^{k+1} \partial_z^{(k)} a(w)_-, \quad \text{for} \quad |z| < |w|$$

Amazingly, we have very similar situation for the space $M_{\mathfrak{sl}_3}$. The expression of the trace in terms of the integral over a real 3-dim. cycle as well as higher dimensional version of the formal Cauchy formula will be given in [G-K1].

**Remark 5.2** For the Lie algebra $\mathfrak{sl}_2$ we have only two orbits of the Weyl group in the Flag manifold. The space $\mathbb{C}^*$ is the intersection of two existing dual orbit. When we came to $\mathfrak{sl}_3$ we have two possibilities for generalization. One is to define the space $M^0$ as an intersection of two mutually dual orbits, what we did above, and the another one is to define $M^0$ as

$$\tilde{M} = \bigcap_{w \in W} U_w.$$  

the intersection of all orbits of the Weyl group in $B_\infty \setminus G$. Since the action of $s \in W \simeq S_3$ is just the corresponding permutation of coordinates:

$$U_s = \{(z_1 : z_2 : z_3), \ (w_1 : w_2 : w_3) \mid z_{a(3)} \neq 0, w_{s(1)} \neq 0\}.$$  

Thus,

$$\tilde{M} = \{(z_1 : z_2 : z_3), \ (w_1 : w_2 : w_3) \mid \sum_{i=1}^3 z_i w_i = 0, \text{and} \ z_i \neq 0, \ w_i \neq 0, \ i = 1, 2, 3\}$$

is the intersection of four-dimensional complex torus with the hypersurface $\sum_{i=1}^3 z_i w_i = 0$ in $\mathbb{CP}^2 \times \mathbb{CP}^2$.

In terms of homogeneous coordinates

$$(a, b, c), \ a = \frac{w_2}{w_1}, \ c = \frac{w_3}{w_1}, \ b = -\frac{z_2}{z_3}$$

on $U_1 \simeq \mathbb{C}^3$, $\tilde{M}$ can be described as the total space of non trivial bundle with the base $(\mathbb{C}^*)^2_{a b}$, the two-dimensional complex torus, and the fiber $\mathbb{C}^* \setminus \{c = ab\}$:

$$\tilde{M} \xrightarrow{(c=ab)} (\mathbb{C}^*)^2_{a b}.$$  

By the construction this space is invariant under the action of the Weyl group $W$.

One of reason for the first choice is motivated by the following theorems.
Theorem 5.1 The cohomology group

\[ H^*_3(M^{sl_3}, \mathbb{Z}) \simeq \mathbb{Z}, \quad (5.40) \]

and

\[ H^*_3(\tilde{M}, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}. \quad (5.41) \]

6 Generalized Affine Kac-Moody algebras associated with the space \( V^{sl_3} \)

Let \( g \) be a simple Lie algebra over \( \mathbb{C} \). Consider the formal loop algebra \( Lg = g((t)) \). The affine algebra \( \hat{g} \) is defined as the central extension of \( Lg \). As a vector space, \( \hat{g} = Lg \oplus \mathbb{C}K \), and the commutation relations are:

\[ [K, \cdot] = 0, \quad \text{and} \quad [A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) + (A, B)\text{Res}_t(f \cdot g')K \quad (6.1) \]

where \( \langle \cdot, \cdot \rangle \) is an invariant bilinear form on \( g \), normalized as in [K1] by the requirement that \( \langle \alpha_{max}, \alpha_{max} \rangle = 2 \).

Let \( \{J^a\}_{a=1, \ldots, \dim g} \) be a basis of \( g \). Denote by \( J^a_n = J^a \otimes t^n \in Lg \), then \( J^a_n, n \in \mathbb{Z} \), and \( K \) form a topological basis for \( \hat{g} \). Consider a field (generating function):

\[ J^a(z) = \sum_{n \in \mathbb{Z}} J^a_n \otimes z^{-n-1} = J^a \otimes \delta(t - z). \quad (6.2) \]

For any element \( a \in g \) we will define the corresponding field as

\[ a(z) = a \otimes \delta(t - z). \quad (6.3) \]

Here \( \delta(t - z) \) is the formal delta function associated with the space \( \text{Fun} \mathbb{C}^* \simeq V^{sl_2} \). For any \( a, b \in g \) the fields \( a(z) \) and \( b(z) \) are mutually local. The commutation relations in terms of the fields are:

\[ [a(z), b(w)] = [a, b](w) \delta(z - w) + K \cdot (a, b) \partial_w \delta(z - w). \quad (6.4) \]

In the same way we can define the affine Lie algebra, associated with the space \( V^{sl_3} \simeq \text{Fun} M \). Let

\[ \delta_V = \delta_{V^{sl_3}}(t - z) \quad (6.5) \]

be the delta function associated with the space \( V^{sl_3} \), defined as [5.25].

For any element \( a \in g \) we can define a field \( a^{sl_3}(z) \) on \( M = U \cap U^* \) as:

\[ a^{sl_3}(z) = a \otimes \delta_V(t - z). \quad (6.6) \]

Consider the formal loop algebra \( L^{sl_3}g = g \otimes \text{Fun} M \), associated with the manifold \( M \) with the obvious commutation relations.
Let $\xi_i^z = \xi_{\alpha_i}^z$ be the image of $e_i$, $i = 1, 2, 3$ in $\text{Vect} \, M$. Then $\partial_i^z = \xi_i^z$ for each $i = 1, 2, 3$ define a non trivial cocycle $c_i$ on $L^s t g$ as:

$$c_i(A \otimes f(t), B \otimes g(t)) = (A, B) \text{Res}_t (f \cdot \partial_i^z(g)).$$  \hfill (6.7)

**Remark 6.1** The Lie algebra $L^s t g$ has infinite dimensional central extension. Let $f_i \in V^s t_i$, $i = 1, 2, 3$ be functions such that $\sum_{i=1}^3 f_i \partial_i = 0$, then the vector field $\sum_{i=1}^3 f_i \partial_i$ defines a non trivial central extension of $L^s t g$ as:

$$c_{f_1, f_2, f_3}(A \otimes f(t), B \otimes g(t)) = (A, B) \text{Res}_t (f \cdot \sum_{i=1}^3 f_i \partial_i^z(g)).$$  \hfill (6.8)

We can define the $g^V$ as an infinite dimensional central extension of the Lie algebra $L^s t g$.

Not all these central extensions are equally good from the point of view of the conformal dimension. The left hand side and the right hand side of (6.4) have the same conformal dimensions. Multiplication by $\partial_k w \delta(z - w)$ add $k + 1$ to the conformal dimension $\Delta$ of the field $a(z)$. The conformal dimension is defined from the action of the Virasoro operators, in particular, for the field $a(z)$ of the conformal dimension $\Delta$ we have

$$[L_0, a(z)] = (z \frac{d}{dz} + \Delta) a(z).$$  \hfill (6.9)

For the Affine algebras all fields have conformal dimension $\Delta = 1$ same as the delta-function. For the Generalized Affine Kac-Moody algebras associated with the space $V^s t_i$ we would like to consider the central extensions compatible with the conformal dimension.

Consider two grading operators on $V^s t_i$:

$$L_c = -c \partial_3 - a \partial_1, \quad L_{c-ab} = -(c - ab) \partial_3 - b \partial_2.$$  \hfill (6.10)

Here $z = (a, b, c)$ where $(a, b, c)$ are homogeneous coordinates on $U$.

**Proposition 6.1** Operators $L_c$ and $L_{c-ab}$ commute and define $\mathbb{Z} \times \mathbb{Z}$ grading on $V^s t_i$:

$$V^s t_i = \bigoplus_{(n_1, n_2) \in \mathbb{Z}} V_{n_1 n_2}$$  \hfill (6.11)

where $V_{n_1 n_2} = \{ f \in V^s t_i | -L_c(f) = n_1 f; \quad -L_{c-ab}(f) = n_2 f \}$.

Any element of $V^s t_i$ is a linear combination of the elements of the basis (5.17), and every element from this basis have a form

$$a^n b^m c^k (c - ab)^l$$  \hfill (6.12)

for some $n, m \in \mathbb{Z}_+$ and $k, l \in \mathbb{Z}$. From the explicit formulas for $\partial_1, \partial_2, \partial_3$ (5.4) we have

$$a^n b^m c^k (c - ab)^l \in V_{n+k+l, m+k+l}.$$  \hfill (6.13)
This compleat the proof.

The grading operators $L_c$ and $L_{c-ab}$ are analogues of the energy operator $L_0$ that is a part of the Virasoro algebra

$$Vir = \bigoplus_{n \in \mathbb{Z}} C L_n \oplus C c.$$  \hfill (6.14)

It is well known that that there is close relations between the Virasoro and Affine Lie Algebras in many aspects including the representation theory. In the next section we construct a natural higher dimensional analogue of the Virasoro algebra associated with the space $V_{sl_3}$.

We have the commutation relations:

$$[L_c, a^{sl_3}(z)] = (c \partial_3 + a \partial_1 + 2) a^{sl_3}(z); \quad [L_{c-ab}, a^{sl_3}(z)] = ((c - ab) \partial_3 + b \partial_2 + 2) a^{sl_3}(z).$$  \hfill (6.15)

It means that the field $a^{sl_3}(z)$ has conformal dimension $\Delta = (2, 2) \in \mathbb{Z} \times \mathbb{Z}$ The generalized delta function $\delta_V(z - w)$ as well has conformal dimension $(2, 2)$. From this we have the following proposition:

**Proposition 6.2** Consider two cocycles on $L^{sl_3} g = g \otimes \text{Fun} M$ defined as:

$$c_i(A \otimes f(t), B \otimes g(t)) = (A, B) \text{Res}_t (f \cdot L_i^t(g)),$$

where $t = (a, b, c)$ and $L_1 = \frac{1}{1(c-ab)} L_c$, $L_2 = \frac{1}{1(c-ab)} L_{c-ab}$. Then the central extension defined by each of these cocycle has the right conformal dimension with respect to the action of the grading operators $L_c$ and $L_{c-ab}$.

**Remark 6.2** We have two Affine Kac-Moody subalgebras $g \otimes \mathbb{C}[c, c^{-1}]$ and $g \otimes \mathbb{C}[(c - ab), (c - ab)^{-1}]$ naturally imbedded in $L^{sl_3} g$. The restriction of each of these two cocycles to these subalgebras gives the standard Kac-Moody cocycle as in (6.4).

**Definition 6.1** The Generalized Affine Kac-Moody algebra $g^V$ associated with the space $V^{sl_3}$ is defined as a two-dimensional central extension of $L^{sl_3} g$. As a vector space, $g^V = L^{sl_3} g \oplus \mathbb{C} K_1 \oplus \mathbb{C} K_2$, and the commutation relations: $[K_i, \cdot] = 0$, $i = 1, 2$ and

$$[a^{sl_3}(z), b^{sl_3}(w)] = [a, b]^{sl_3}(w) \delta_V(z - w) + \sum_{i=1}^2 K_i \cdot (a, b) L_i^w \delta_V(z - w).$$  \hfill (6.17)

The commutations relations (6.17) give the description of generalized Affine Kac-Moody algebras associated with the space $V^{sl_3}$ in terms of local fields. Here ”local” means that the commutator is annihilated by some polynomial function of $(z_i - w_i)$.

It is not completely appropriate to call the constructed algebras the generalized Affine Kac-Moody algebras. These algebras have many of the properties of ”the generalized Affine Kac-Moody algebras”. The basic idea of R. Borcherds is to think of generalized Affine Kac-Moody algebras as infinite dimensional Lie algebras which have most of the good properties of finite dimensional reductive Lie algebras.
The original definition of generalized Affine Kac-Moody algebras, given by R. Borcherds \cite{B1}, is:

Consider a Lie algebra $G$ that has the following properties:

1. $G$ has an invariant symmetric bilinear form $(, )$.
2. $G$ has a (Cartan) involution $\omega$.
3. $G$ is graded as $G = \bigoplus_{n \in \mathbb{Z}} G_n$ with $G_n$ finite dimensional and with $\omega$ acting as $-1$ on the "Cartan subalgebra" $G_0$.
4. $(g, \omega(g)) > 0$ if $g \in G_n$, $g \neq 0$ and $n \neq 0$.

The Generalized Affine Kac-Moody algebras $g^V$ associated with the space $V^{sl_3}$ are defined by the same conditions with one small change: we replace condition 3 by

3*. $G$ is graded as $G = \bigoplus_{n \in \mathbb{Z}} G_n$ with $G_n$ not necessarily finite dimensional and with $\omega$ acting as $-1$ on the "Cartan subalgebra" $G_0$. Here $k = 2$ is the rank of $sl_3$.

In order to prove this statement we need to construct an invariant symmetric bilinear form $(, )$ and a (Cartan) involution $\omega$ on $g^V$.

The normalized invariant form $(, )$ on $g^V$ can be described as follows. Take the normalized invariant form $(, )$ on $g$ and extend $(, )$ to the whole $g^V$ by

$$(A \otimes f(t), B \otimes g(t)) = (A, B)(\text{Res}_t \phi \cdot f \cdot g); \quad (K_1, K_j) = 0; \quad (L^{sl_3} g, C K_1 \oplus C K_2 \oplus C K_3) = 0.$$ \hspace{1cm} (6.18)

To define a Cartan involution of $g^V$ consider the Cartan involution $\tilde{\omega}$ of $g$ and the transformation $w_0$ on $M$ given by

$$a \rightarrow -\frac{b}{c - ab}, \quad b \rightarrow \frac{a}{c}, \quad c \rightarrow \frac{1}{c},$$ \hspace{1cm} (6.19)

where $t = (a, b, c)$ are homogeneous coordinates on $U$. This transformation is defined by the change from the homogeneous coordinates on $U$ to the homogeneous coordinates on $U^*$, given by \textcircled{56}. The Cartan involution of $g^V$ can be expressed as:

$$\omega(A \otimes f(t) + \lambda_1 K_1 + \lambda_2 K_2) = \tilde{\omega}(A) \otimes f(w_0(t)) - \lambda_1 K_1 - \lambda_2 K_2.$$ \hspace{1cm} (6.20)

It is not difficult to see that for any element $g = A \otimes f(t) \in L^{sl_3} g$ we have $(g, \omega(g)) > 0$.

In QFT it is important to consider the correlations functions of two or more fields. In 2-dimensional conformal field theory the correlator of two fields

$$\langle v^* | A(z) B(w) | v \rangle$$ \hspace{1cm} (6.21)

is a rational function of $z$ and $w$ in the domain $|z| > |w|$ with poles only at hyperplanes $z = 0$, $w = 0$, and $z = w$. This important property in quantum field theory in terms of OPE (operator product expansion) means that the product of two fields at nearby points can be expanded in terms of other fields and the small parameter $z - w$. In terms of commutator the OPE can be expressed as

$$A(z)B(w) = [A_-(z), B(w)] + :A(z)B(w):,$$ \hspace{1cm} (6.22)

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where \([A_-(z), B(w)]\) is a singular part of the OPE and

\[ : A(z) B(w) := A(z)_+ B(w) + B(w) A(z)_- \]  \hspace{1cm} (6.23)

is the normal ordered product of two fields. The singular part of the OPE of two local fields \(A(z)\) and \(B(w)\), given by

\[ [A_-(z), B(w)] = \sum_{n=0}^{N} A_{(n)}(w) \partial_{(n)}(z - w)_+, \]  \hspace{1cm} (6.24)

is a finite linear combination of other fields \(A_{(n)}(w)\) from the theory. Here \(\partial_{(n)} = \frac{1}{n!} \partial^n\). The \(\delta(z - w)_+\) is a regular function with respect to \(w\) in the domain \(|z| > |w|\) and has the properties: \(\partial_w \delta(z - w)_+ = -\partial_z \delta(z - w)_+\) and \(\delta(z - w)_+|_{w = 0} = 1/z\). From these properties we have

\[ \delta(z - w)_+ = \frac{1}{z - w} \text{ in the domain } |z| > |w|. \]  \hspace{1cm} (6.25)

For the regular delta function we don’t need these properties to find the sum

\[ \delta(z - w)_+ = \sum_{n=0}^{\infty} z^{-n-1} w^n = \frac{1}{z - w}, \]  \hspace{1cm} (6.26)

but for the generalized delta function this can give us some idea.

In order to have the consistent definition of the correlator of two fields in the dimension higher than 2, we need to introduce the OPE and the normal product of two fields on \(M\).

Consider two fields \(a^{s_k}(z), b^{s_k}(w) \in \mathfrak{g}^V\). Define \(a^{s_k}(z)_\pm\) as

\[ a^{s_k}(z)_\pm = a \otimes \delta_V (z - w)_\pm, \]  \hspace{1cm} (6.27)

where \(\delta_V (z - w)_\pm\) are defined by (5.27). Then the OPE can be defined as:

\[ a^{s_k}(z) b^{s_k}(w) = (a, b) \left( \sum_{i=1}^{2} K_i \cdot L_i^w \delta_V (z - w)_+ \right) + [a, b]^{s_k}(w) \delta_V (z - w)_+ + : a^{s_k}(z) b^{s_k}(w) : \]  \hspace{1cm} (6.28)

where

\[ : a^{s_k}(z) b^{s_k}(w) : = a^{s_k}(z)_+ b^{s_k}(w) + b^{s_k}(w) a^{s_k}(z)_-. \]  \hspace{1cm} (6.29)

It would be nice to have the formula similar to (6.25) for the \(\delta_V (z - w)_+,\) defined as (5.27). The space \(V^{s_k}\) is a sum of two dual subspaces \(V^g = V_+ \oplus V_-\). Here \(g = s_k\). The positive part of delta function is defined as

\[ \delta_V (z - w)_+ = \sum_{e_\gamma \in V_+} e_\gamma^* \otimes e_\gamma \in V_+^z \otimes V_+^w. \]  \hspace{1cm} (6.30)

The space \(V_+\) of all regular functions on \(M_+\) is the direct sum of two subspaces \(V_+ = V_{++} \oplus V_{+-}\) where \(V_{++}\) is a space of all regular functions on \(U \simeq \mathbb{C}^3\). For the dual space
This form of OPE is very similar to what we have in 2-dimensional conformal field theory. According to this decompositions we can write
\[
\delta_V (z - w)_+ = \delta_V (z - w)_{++} + \delta_V (z - w)_{+-}. \tag{6.31}
\]
The distribution
\[
\delta_V (z - w)_{++} \in V^w_{--} \otimes V^w_{++} \tag{6.32}
\]
is a regular function with respect to \(w = (w_1, w_2, w_3)\) in the domain \(U \simeq \mathbb{C}^3\). Thus, this function is well defined, when \(w_i = 0\). From the construction of the basis \(5.17\), we have
\[
\delta_V (z - w)_{++}|_{w_1=0, w_2=0, w_3=0} = \phi(z_1, z_2, z_3) = \frac{1}{z_3(z_3 - z_1 z_2)}. \tag{6.33}
\]
Additionally, we have
\[
\partial^w_i \delta_V (z - w)_{++} = -\partial^w_i \delta_V (z - w)_{++}, \quad i = 1, 2, 3. \tag{6.34}
\]
This equality we understand at the level of distributions.

Consider the function of two sets of variables \(z = (z_1, z_2, z_3)\) and \(w = (w_1, w_2, w_3)\) given by
\[
F(z, w) = \frac{z_1 z_2 w_1 w_2}{(z_1 - w_1)(z_2 - w_2)(z_3 - w_3)(z_3 - z_1 z_2) - (w_3 - w_1 w_2)} = \frac{z_1}{(z_1 - w_1)(z_2 - w_2)(z_3 - z_1 z_2 - (w_3 - w_1 w_2))} + \frac{w_2}{(z_2 - w_2)(z_3 - w_3)(z_3 - z_1 z_2) - (w_3 - w_1 w_2)}. \tag{6.35}
\]

**Proposition 6.3** In the domain \(D_1: \vert z_1 \vert > \vert w_1 \vert > 0, \vert z_2 \vert > \vert w_2 \vert > 0, \vert z_3 \vert > \vert w_3 \vert, \vert z_3 - z_1 z_2 \vert > \vert w_3 - w_1 w_2 \vert\) the function \(F(z, w) = \delta_V (z - w)_{++}\) in a sense that:
\[
\text{Res}_z F(z, w) \cdot f(z) = f(z)_{++}. \tag{6.36}
\]
and in the domain
\[
D_2: \vert z_1 \vert > \vert w_1 \vert > 0, \vert z_2 \vert > \vert w_2 \vert > 0, \vert z_3 \vert < \vert w_3 \vert, \vert z_3 - z_1 z_2 \vert > \vert w_3 - w_1 w_2 \vert\] the function \(-F(z, w) = \delta_V (z - w)_{+-}\) in a sense that:
\[
\text{Res}_z F(z, w) \cdot f(z) = -f(z)_{+-}. \tag{6.37}
\]

The proof is based on a realization of the trace as an integral over the 3-dimensional cycle \(S\) and the corresponding formal power series expansion for \(F(z, w)\) on \(S\) \((G-K1)\).

Then, the OPE expansion of two fields \(a(z), b(w) \in g^V\) can be written in terms of the function \(F(z, w)\) if we replace in the OPE \((6.28)\) the \(\delta_V (z - w)_+\) by the sum
\[
F(z, w) = \sum_{\vert z_1 \vert > \vert w_1 \vert > 0, \vert z_2 \vert > \vert w_2 \vert > 0, \vert z_3 \vert > \vert w_3 \vert, \vert z_3 - z_1 z_2 \vert > \vert w_3 - w_1 w_2 \vert} \text{Res}_z F(z, w) \cdot f(z)_{++} - \sum_{\vert z_1 \vert > \vert w_1 \vert > 0, \vert z_2 \vert > \vert w_2 \vert > 0, \vert z_3 \vert < \vert w_3 \vert, \vert z_3 - z_1 z_2 \vert > \vert w_3 - w_1 w_2 \vert} \text{Res}_z F(z, w) \cdot f(z)_{+-}.
\]

This form of OPE is very similar to what we have in 2-dimensional conformal field theory.
7 Generalized Virasoro algebra associated with the space $V^{\mathfrak{sl}_3}$

In section 3 we constructed the $n$-products of the Virasoro conformal algebra from the commutation relations of $\mathfrak{sl}_2$ \[3.13\]. Now we will apply the same ideas to construct a generalized Virasoro conformal algebra associated to the Lie algebra $\mathfrak{sl}_3$.

For $\mathfrak{g} = \mathfrak{sl}_3$ positive roots are $\{\alpha_1, \alpha_2, \alpha_3 = \alpha_1 + \alpha_2\}$ and the corresponding weight basis in $\mathfrak{sl}_3$ is $\{e_1, e_2, e_3, h_1, h_2, f_1, f_2, f_3\}$. Denote by $\partial_i = ad(e_i)$. Lie algebra $\mathfrak{sl}_3$ has a structure of the $U(n_+)$-module with the lowest weight $-\alpha_3$. Via the Poincaré-Birkhoff-Witt basis we can identify $U(n_+) \simeq \mathbb{C}[\partial_1, \partial_2, \partial_3]$. Denote by $L = f_3$ the lowest vector with respect to this action, then $\mathfrak{sl}_3 \simeq \mathbb{C}[\partial_1, \partial_2, \partial_3]L$ is the rank one $\mathbb{C}[\partial_1, \partial_2, \partial_3]$-module. Consider a basis in $\mathfrak{sl}_3$ of the form

$$L = f_3, \quad \partial_1 L = -f_2, \quad \partial_2 L = f_1; \quad \partial_1 \partial_2 L = h_1, \quad \partial_3 L = h_1 + h_2;$$

$$\partial_1^2 \partial_2 L = -2e_1, \quad \partial_2 \partial_3 L = -e_2, \quad \partial_1 \partial_2 \partial_3 L = -e_3.$$  \hspace{1cm} (7.1)

The dual basis (with respect to the normalized bilinear form) is

$$e_3, \quad -e_2, \quad e_1; \quad \frac{1}{3} h_1 - \frac{1}{3} h_2, \quad \frac{1}{3} h_1 + \frac{2}{3} h_2; \quad -\frac{1}{2} f_1, \quad -f_2, \quad -f_3.$$ \hspace{1cm} (7.2)

For any element $e$ of the dual basis we define a bilinear operation $L(e)L$ product as:

$$L((f_3^*)L = [e_3, L] = \partial_3 L, \quad L((\partial_1 f_3^*)L = [-e_2, L] = -\partial_2 L,$$

$$L((\partial_2 f_3^*)L = [e_1, L] = \partial_1 L, \quad L((\partial_3 f_3^*)L = [-2h_1 - h_2, L] = 3L.$$ \hspace{1cm} (7.3)

The other products are 0 because $[f_i, f_3] = 0$ for $i = 1, 2, 3$.

Let

$$\delta_V = \delta_{V^{\mathfrak{sl}_3}}(t - z)$$ \hspace{1cm} (7.4)

be a delta function associated with the space $V^{\mathfrak{sl}_3}$, defined in \[5.25\]. Denote by $L^\theta(z)$ the field associated with the space $V^{\mathfrak{sl}_3} \simeq \text{Fun} M$ with the commutation relation defined by products \(7.3\) :

$$[L^\theta(z), L^\theta(w)] = \partial_3 L^\theta(w) \delta_V(z - w) - \partial_2 L^\theta(w) \partial_1^w \delta_V(z - w) + \partial_1 L^\theta(w) \partial_2^w \delta_V(z - w) +$$

$$3L^\theta(w) \partial_3^w \delta_V(z - w)$$ \hspace{1cm} (7.5)

**Theorem 7.1** The defined above commutation relation satisfies to the skewsymmetry and Jacobi identity axioms
where

\[ [L^\theta(w), L^\theta(z)] = \partial_3 L^\theta(z) \delta_V(z - w) - \partial_2 L^\theta(z) \partial_1^\theta \delta_V(z - w) + \partial_1 L^\theta(z) \partial_2^\theta \delta_V(z - w) + \]
\[3 \partial_3 L^\theta(w) \delta_V(z - w) = \partial_3 L^\theta(w) \delta_V(z - w) + \partial_2 L^\theta(w) \partial_1 \delta_V(z - w) + \partial_1 \partial_2 L^\theta(w) \delta_V(z - w) - \partial_2 \partial_1 L^\theta(w) \delta_V(z - w) - \]
\[ 3 \partial_3 L^\theta(w) \partial_2^\theta \delta_V(z - w) - 3 \partial_3 L^\theta(w) \delta_V(z - w) - [-L^\theta(z), L^\theta(w)]. \tag{7.6} \]

Define a field on \( V^{sl}_3 \) by
\[ L^\theta(z) = -\delta_V(t - z) \partial_3^\theta + \partial_1^\theta \delta_V(t - z) \partial_2^\theta - \partial_2^\theta \delta_V(t - z) \partial_1^\theta. \tag{7.7} \]
The straightforward calculations show that this field satisfies to the commutation relation (7.6).

For any function \( f \in V^{sl}_3 \) define an operator \( L_f \) as \( L_f^\theta = Res_z (f(z)L^\theta(z)) \) so that
\[ L_f^\theta = -f \partial_3^\theta + \partial_1 f \partial_2^\theta - \partial_2 f \partial_1^\theta. \tag{7.8} \]
From the direct calculations we have
\[ [L_f^\theta, L_g^\theta] = L_{\{f, g\}}^\theta, \tag{7.9} \]
where \( \{f, g\} \) define a Lie bracket on the space \( V^{sl}_3 \) given by
\[ \{f, g\} = g \partial_3 f - f \partial_3 g + \partial_1 f \partial_2 g - \partial_1 g \partial_2 f. \tag{7.10} \]
This bracket is skew-symmetric and satisfies the Jacobi identity
\[ \{h, \{f, g\}\} + \{f, \{g, h\}\} + \{g, \{h, f\}\} = 0 \tag{7.11} \]
for any \( h, f, g \in V^\theta \), so it is a Lie bracket. This completes the proof.

**Proposition 7.1** The Lie bracket defined in (7.10) corresponds to the contact Lie bracket on the real space \( M_3 = V^\theta_3 \) (we now suppose that \( (a, b, c) \) are real coordinates). The operators \( L_f^\theta \) are contact vector fields on \( M_3 \) with the contact Hamiltonian \( f \). The corresponding contact structure is defined by a 1-form \( \beta \), such that \( \beta(\partial_3) = -1, \beta(\partial_1) = 0, \beta(\partial_2) = 0, \beta \wedge d\beta = 0 \) and \( \beta \wedge d\beta \neq 0 \).

**Proof.** Let \( \beta \) be a contact 1-form on 3-dimensional real manifold \( M : \beta \wedge d\beta \neq 0 \). Vector field \( \xi_f \) on \( M \) is contact with the contact Hamiltonian \( f \in Fun M \) if \( L_{\xi_f} \) preserves the foliation \( \beta = 0 \) and \( \beta(\xi_f) = f \). Then \( \xi_f = f \cdot \alpha + \theta(f) \) where \( \alpha \) is the vector field defined by \( \beta(\alpha) = 1 \) and \( d\beta(\alpha, \cdot) = 0 \) and \( \theta \) is the bivector field on \( M \), inverse to the 2-form \( d\beta \). The contact Lie bracket on \( Fun M \) is defined by \( \{\xi_f, \xi_g\} = \xi_{\{f, g\}} \) or more explicitly:
\[ \{f, g\} = f\alpha(g) - g\alpha(f) + \theta(f, g). \tag{7.12} \]
Take \( \alpha = -\partial_3, \theta = \partial_1 \wedge \partial_2 \) and \( \beta = -dc + adb \). This completes the proof.
Proposition 7.2 The Lie algebra \(V^{sl_3}\) with the bracket (7.10) contains the Lie subalgebra isomorphic to \(sl_3\).

Proof. For simplicity we will use the homogeneous coordinates \((a, b, c)\) on \(M^g\). The action of \(sl_3\) in terms of these coordinates is given by (5.4). Operators

\[
L_1, \quad L_a, \quad L_b, \quad L_c, \quad L_{c-ab}, \quad L_{ac}, \quad L_{-b(c-ab)}, \quad L_{c(c-ab)}
\] (7.13)

are closed with respect to the Lie bracket (7.9) and form the Lie algebra isomorphic to \(sl_3\). The Cartan subalgebra generators \(L_c = -c \partial_3 - a \partial_1, \quad L_{c-ab} = -(c - ab) \partial_3 - b \partial_2\) are grading operators from the previous section and they are analogues of the energy operator \(L_0\) in two dimensional CFT.

We constructed a natural higher dimensional analogues of the Virasoro algebra and the Virasoro conformal algebra associated with the space \(V^{sl_3}\). We suggest to denote this algebra by \(Vir^{sl_3}\).

We also define a semidirect sum of \(Vir^{sl_3}\) and the generalized affine Kac-Moody algebra \(g^V\) associated with the space \(V^{sl_3}\). The \(Vir^{sl_3}\) acts on \(g^V\) via its action as derivations of \(V^{sl_3}\). In terms of fields this action is given by

\[
[L^g(z), a^{sl_3}(w)] = \partial_3 a^{sl_3}(w) \delta_V(z - w) - \partial_2 a^{sl_3}(w) \partial_1^{\omega} \delta_V(z - w) + \partial_1 a^{sl_3}(w) \partial_2^{\omega} \delta_V(z - w) + 2a^{sl_3}(w) \partial_3^{\omega} \delta_V(z - w)
\] (7.14)

where \(a^{sl_3}(w) \in g^V\).

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