Abstract. We formulate problems of tight closure theory in terms of projective bundles and subbundles. This provides a geometric interpretation of such problems and allows us to apply intersection theory to them. This yields new results concerning the tight closure of a primary ideal in a two-dimensional graded domain.

Introduction

The aim of this paper is to translate problems of tight closure theory in terms of projective bundles and subbundles in order to apply techniques of projective geometry such as intersection theory to them. This provides a geometric view on such problems and enables us also to work often characteristic free.

We describe shortly the construction of the projective bundles arising from tight closure. The most basic problem of tight closure theory is to decide whether $f_0 \in (f_1, \ldots, f_n)^*$, where $f_0, f_1, \ldots, f_n$ are elements in a Noetherian $K$-algebra $R$ ($K$ a field of positive characteristic $p$). This means by definition that there exists an element $c \in R$, not contained in any minimal prime, such that $cf_0^q \in (f_1^q, \ldots, f_n^q)$ for almost all powers $q = p^e$.

The starting point for our construction is the observation due to Hochster (see [16]) that for a local complete $K$-domain $(R, m)$ of dimension $d$ the containment $f_0 \in (f_1, \ldots, f_n)^*$ is equivalent to the property that $H^d_m(A) \neq 0$ holds, where $A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)$ is the so-called (generic) forcing algebra for the data $f_1, \ldots, f_n; f_0$. This characterization leads us to the study of the cohomological dimension of the open subset $D(mA) \subseteq \text{Spec } A$: if the ideal $(f_1, \ldots, f_n)$ is primary to $m$, then this open subset looks locally over $D(m) \subseteq \text{Spec } R$ like an affine space $K^{d-1}$, and the transition mappings are affine-linear.

In order to study the cohomological properties of this affine-linear bundle it is helpful to embedd it into a projective bundle. This is achieved in the following way: the spectrum $\text{Spec } R[T_0, T_1, \ldots, T_n]/(f_0T_0 + f_1T_1 + \ldots + f_nT_n)$ yields a geometric vector bundle $V'$ over $D(m)$, its sheaf of sections is given by the relations for the elements $f_0, f_1, \ldots, f_n$. The spectrum $\text{Spec } R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n)$ yields a closed subbundle $V \subseteq V'$ given by $T_0 = 0$. These vector bundles yield projective bundles $\mathbb{P}(V) \subset \mathbb{P}(V')$ and the complement $\mathbb{P}(V') - \mathbb{P}(V)$ is isomorphic to our affine-linear bundle.

If $R$ is a graded normal domain and if the $f_1, \ldots, f_n$ are homogeneous $R_+$-primary elements, we can go one step further and obtain a projective bundle together with a projective subbundle of codimension one (called forcing subbundle...
or forcing divisor) over Proj \( R \). The cohomological dimension of the complement of the forcing divisor is the same as the cohomological dimension of the affine-linear bundle over \( D(\mathfrak{m}) \), so we can work in an entirely projective setting, which is moreover smooth whenever \( R \) has an isolated singularity.

If \( R \) is a normal standard graded domain of dimension two then we are in a particularly manageable situation. The construction leads to projective bundles over smooth projective curves and the question whether \( f_0 \) belongs to the tight closure of \((f_1, \ldots, f_n)\) is equivalent to the question whether the complement of the forcing divisor is not an affine scheme (Proposition 3.9). This question is intimately related to the question whether the forcing divisor is ample.

This geometric interpretation provides in particular a tool to attack the following two problems of tight closure theory, which we will encounter here for several times and also in forthcoming papers (3,4).

The first problem is whether tight closure is the same as plus closure in positive characteristic. The plus closure of an ideal \( I \subseteq R \) in a Noetherian domain is just the contraction \( I^+ = R \cap IR^+ \), where \( R^+ \) is the integral closure of \( R \) in an algebraic closure of \( Q(R) \). A positive answer to this problem would imply the localization problem for tight closure.

If \( R \) is graded, then the question whether \( f_0 \in (f_1, \ldots, f_n)^{+\text{gr}} \) is in our geometric setting equivalent to the existence of projective subvarieties in \( \mathbb{P}(V') \) of \( \dim R - 1 \) and disjoined to the forcing divisor (3.10). Therefore we look at the relation between tight closure and plus closure as a relation between intersection-geometric properties of the forcing divisor and cohomological properties of the complement of it.

We will describe several situations in this paper where equality holds (the results 0.2, 13, 14 and 0.2 are also true in characteristic zero, whereas 3.1, 10.7 and 10.8 need positive characteristic). In [3] we will use our method to prove that the tight closure and the plus closure of a homogeneous \( R_+ \)-primary ideal in a normal homogeneous coordinate ring over an elliptic curve coincide in positive characteristic. The main ingredient for this result is the classification of vector bundles on elliptic curves due to Atiyah, which enables us to establish the same numerical criterion for both properties.

The second problem is the graded Briançon-Skoda-problem: What is the minimal number \( d_0 \) such that \( R_{\geq d_0} \subseteq (f_1, \ldots, f_n)^{+} \) holds, where \( f_1, \ldots, f_n \) are homogeneous \( R_+ \)-primary elements in a standard graded \( K \)-Algebra \( R \)? It is known that this containment is true for \( d_0 \geq d_1 + \ldots + d_n \), \( d_i = \deg f_i \) and that this number is a sharp bound in the parameter case, see [27] and [18] Theorem 2.9 and 6.1. However, this number is not much helpful in the general primary case.

Our interpretation suggests that in the two-dimensional situation the number \((d_1 + \ldots + d_n)/(n-1)\) should be an important bound for the degree, since the top self intersection number of the forcing divisor is \((d_1 + \ldots + d_n - (n-1)d_0) \deg O_Y(1)\). In this paper we show for \( n = 3 \) (Theorem 10.3, Theorem 10.7) that under some additional conditions \((d_1 + d_2 + d_3)/2 \) is the right bound. This gives for example that \( xyz \in (x^2, y^2, z^2)^{+} \) holds in \( K[x, y, z]/(x^3 + y^3 + z^3) \). In [4] we will show that \( R_{(d_1 + \ldots + d_n)/(n-1)} \subseteq (f_1, \ldots, f_n)^{+} \) holds under the condition that the relation bundle for \( f_1, \ldots, f_n \) is strongly semistable. This rests upon conditions for the inclusion in the tight closure in terms of the slopes of the corresponding bundles.

The content of this paper is as follows. In section 3 we discuss the characterization of tight closure via solid closure in terms of cohomological dimension and
forcing algebras due to Hochster [14]. Henceforth we shall work rather with solid closure than with tight closure. For two-dimensional rings this characterization leads to the problem of affineness of open subsets (Proposition 1.3).

The construction of the projective bundle, the forcing sequence and the forcing divisor associated to a tight closure problem, “is $f_0 \in (f_1, \ldots, f_n)^*$?” and its basic properties is given in section 3 and in section 8 for the graded case, yielding bundles over $\text{Proj } R$.

In section 4 we consider conditions for the forcing divisor to be ample, to be basepoint free and to be big. We give a geometric proof of the result of Smith [27] Theorem 2.2] that $f_0 \in (f_1, \ldots, f_n)^*$ implies $f_0 \in (f_1, \ldots, f_n)$ for $\deg f_0 \leq \deg f_i$, $i = 1, \ldots, n$ and show that it is also true for solid closure (Corollary 4.5).

The rest of this paper is devoted to the study of the tight closure of $R_+$-primary ideals in a two-dimensional normal standard graded algebra $R$. We show that the top self intersection number of the forcing divisor is $(d_1 + \ldots + d_n - (n - 1)d_0) \deg H$, where $H$ is the hyperplane section on $\text{Proj } R$, and that this number is very important for the affineness of the complement and hence for the tight closure question $f_0 \in (f_1, \ldots, f_n)^*$?

Sections 5-9 are concerned with the easiest case, the tight closure of a homogeneous parameter ideal $(f_1, f_2)$. Here our method brings rather new interpretations and proofs than new results. The construction yields ruled surfaces over the corresponding smooth projective curve together with a forcing section (Corollary 5.1). The tight closure problem becomes a question on the ampleness of this divisor (Theorem 5.3) and the number $(d_1 + d_2 - d_0) \deg H$ is the self intersection number of it. We recover the so-called vanishing theorem that $(f_1, f_2)^* = (f_1, f_2) + R_{\geq d_1 + d_2}$ holds for $p = 0$ or $p >> 0$ (Corollary 5.11).

In giving examples of ruled surfaces arising from forcing data we encounter Hirzebruch surfaces, incidence varieties, a classical construction of Serre of a Stein but non-affine variety and a new class of counterexamples to the hypersection problem (sections 3 and 5). This shows also that we can establish geometrically interesting properties from results of tight closure theory.

Section 8 deals with the plus closure and section 9 with primary relations and how they influence the $e$-invariant of the ruled surfaces.

Section 10 then deals with the tight closure of three primary homogeneous elements $f_1, f_2, f_3$ in a two-dimensional graded ring, yielding projective bundles of rank two over the curve. This is already a very subtle situation where new phenomena occurs, and a detailed study of the geometric situation is necessary to obtain results on tight closure. If the number $d_1 + d_2 + d_3 - 2d_0$ is $\leq 0$, then under some extra conditions on the $f_1, f_2, f_3$ concerning their relations, we show that $R_{\geq d_0} \subseteq (f_1, f_2, f_3)^*$ (Theorem 10.3 and also in the plus closure, Theorem 10.7). To mention just one example, it follows that for $R = K[x,y,z]/(x^d + y^d + z^d)$ we obtain $R_{\geq d} \subseteq (x^{d_1}, y^{d_2}, z^{d_3})^*$ for $d_1 + d_2 + d_3 = 2d$, $d_i < d$ (Example 10.9).

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1. **Forcing algebras and cohomological dimension**

Let $R$ denote a commutative ring and let $f_1, \ldots, f_n \in R$ and $f_0 \in R$ be elements. The $R$-algebra

$$A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0),$$

is the total ring of fractions of the ring $R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)$. Its non-zero closed subschemes are projective bundles over the projective curve $\text{Proj } (R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0))$. The image of the universal property of the total ring of fractions of the ring $R$ under the evaluation map $R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0) \to R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)$ is a projective bundle over the projective curve $\text{Proj } (R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0))$. The construction of the projective bundle, the forcing sequence and the forcing divisor associated to a tight closure problem, “is $f_0 \in (f_1, \ldots, f_n)^*$?” and its basic properties is given in section 3 and in section 8 for the graded case, yielding bundles over $\text{Proj } R$. To mention just one example, it follows that for $R = K[x,y,z]/(x^d + y^d + z^d)$ we obtain $R_{\geq d} \subseteq (x^{d_1}, y^{d_2}, z^{d_3})^*$ for $d_1 + d_2 + d_3 = 2d$, $d_i < d$ (Example 10.9).
is called the \((\text{generic})\) forcing algebra for the elements \(f_1, \ldots, f_n; f_0\). The forcing algebra forces that \(f_0 \in (f_1, \ldots, f_n)A\) and every other \(R\)-algebra with this property factors through \(A\). For studying tight closure problems in terms of forcing algebras, the cohomological properties of the subsets \(D(mA) \subseteq \text{Spec } A\) for the maximal ideals \(m \in \text{Spec } R\) are important. Recall that the cohomological dimension \(cd(U)\) of a scheme \(U\) is the maximal number \(i\) such that there exists a quasicoherent sheaf \(\mathcal{F}\) with \(H^i(U, \mathcal{F}) \neq 0\) (see [13] for this notion). For an ideal \(a \subseteq R\) we call the maximal number \(j\) such that there exists an \(R\)-module \(M\) with \(H^j_a(M) \neq 0\) the cohomological height, \(ch(a)\) (this is sometimes called the local cohomological dimension). For \(\dim R \geq 2\) we have \(ch(a) = cd(D(a)) + 1\), due to the long exact sequence of local cohomology.

We will not go back to the definition of tight closure but we recall the notion of solid closure in a form which is suitable for our purpose.

**Definition 1.1.** Let \(R\) be a Noetherian ring and let \(f_1, \ldots, f_n, f_0 \in R\). Then \(f_0\) belongs to the solid closure, \(f_0 \in (f_1, \ldots, f_n)^*\), if and only if for every local complete domain \(R' = R_m/\mathfrak{q}\) (where \(m\) is a maximal ideal of \(R\) and \(\mathfrak{q}\) is a minimal prime of \(R_m\)) we have \(H^d_{m'}(A') \neq 0\), where \(d = \dim R'\) and \(A'\) is the forcing algebra over \(R'\).

**Remarks 1.2.** This definition coincides with the definition given in [14, 1.2] due to [16] Corollary 2.4 and Proposition 5.3. The condition must only be checked for the maximal ideals \(m \supseteq (f_1, \ldots, f_n)\).

Suppose that \(R\) contains a field of characteristic \(p > 0\) and suppose furthermore that \(R\) is essentially of finite type over an excellent local ring or that the Frobenius endomorphism is finite. Then the tight closure of an ideal is the same as its solid closure, see [16] Theorem 8.6 (this is not true in characteristic 0 for \(\dim R \geq 3\)).

**Proposition 1.3.** Let \(R\) be a normal excellent domain. Let \(f_1, \ldots, f_n \in R\) be primary to a maximal ideal \(m\) of height \(d\), \(f_0 \in R\) and let \(A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)\) be the forcing algebra. Then the following hold.

(i) \(f_0 \in (f_1, \ldots, f_n)^*\) if and only if the cohomological height of the extended ideal \(mA\) is \(d\).

(ii) If \(d \geq 2\), then \(f_0 \in (f_1, \ldots, f_n)^*\) if and only if the cohomological dimension of \(W = D(mA) \subseteq \text{Spec } A\) is \(d - 1\).

(iii) If \(d = 2\), then \(f_0 \in (f_1, \ldots, f_n)^*\) if and only if \(D(mA)\) is not an affine scheme.

**Proof.** (i) Since the completion of a normal and excellent domain is again a domain the condition \(f_0 \in (f_1, \ldots, f_n)^*\) is equivalent to \(H^d_m(A') \neq 0\), where \(R'\) is the completion of \(R_m\). Since cohomology commutes under completion this is equivalent to \(H^d_m(A) \neq 0\). Since \(H^d_m(A) = H^d_{mA}(A)\) this implies that \(ch(mA) \geq d\), and equality must hold since the cohomological height of \(mA\) can not be bigger than \(ch(m) = d\). On the other hand, if \(H^d_m(A) = 0\), then this holds for every \(A\)-module \(M\), since \(A^{(J)} \to M \to 0\) and since \(H^d_{mA}(-) = 0\).

(ii) follows from (i) by the long exact sequence of local cohomology. (iii) follows from (ii) and the cohomological characterization of affine schemes. \(\square\)

**Lemma 1.4.** Let \(R\) be a Noetherian ring, let \(f_1, \ldots, f_n, f_0 \in R\) be elements and set \(A = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)\). Let \(W = D(mA) \subseteq \text{Spec } A\). Then the following hold.
(i) Suppose that $R, m$ is local of dimension $\dim R = d \geq 2$ and suppose that there exists another local Noetherian ring $R'$ of dimension $d$ and a ring homomorphism $R \rightarrow R'$ such that $V(mR') = V(mR)$ and $f_0 \in (f_1, \ldots, f_n)R'$ hold. Then the cohomological dimension of $W$ is $d - 1$ (and $f_0 \in (f_1, \ldots, f_n)^*$, if $R$ is normal and excellent).

(ii) Let $R$ be a normal domain over a field $K$ of characteristic zero. If there exists a finite extension $R \subseteq R'$ such that $f_0 \in (f_1, \ldots, f_n)R'$, then already $f_0 \in (f_1, \ldots, f_n)R$.

Proof. (i) The morphism $\text{Spec } R' \rightarrow \text{Spec } R$ lifts to a morphism $\varphi : \text{Spec } R' \rightarrow \text{Spec } A$ and $\varphi^{-1}(W) = D(mR')$. Thus we have an affine morphism $D(mR') \rightarrow W$ and the cohomological dimension of $D(mR')$ is $d - 1$.

(ii) This follows from the existence of the trace map, see [2, Remarks 9.2.4].

Remark 1.5. In positive characteristic it is sometimes possible to show that $f_0 \in (f_1, \ldots, f_n)^*$ by giving a finite extension $R \subseteq R'$ where $f_0 \in (f_1, \ldots, f_n)R'$ holds.

In fact it is a tantalizing question of Hochster whether this is always true – i.e. whether tight closure is the same as plus closure. A result of Smith [26] says that this is true for parameter ideals. In [3] we show that this is also true for homogeneous $R_+$-primary ideals in an affine normal cone over an elliptic curve.

The superheight of an ideal $a \subseteq R$ is the maximal height of $aR'$ in any Noetherian $R$-algebra $R'$. The superheight of an ideal is less or equal its cohomological height, and solid closure gives in characteristic $0$ examples that it may be less. In particular tight closure gives examples of open subsets $D(a)$ such that $a$ has superheight one, but $D(a)$ is not affine. For other examples see [24] and [2]. We will apply this in section 7 to give new counterexamples to the hypersection problem of complex analysis.

Corollary 1.6. Let $K$ be a field of characteristic zero and let $R$ be a normal excellent $K$-domain. Let $f_1, \ldots, f_n$ be primary to a maximal ideal $m$ of height $d$ and let $f_0 \in R$. Suppose that $f_0 \notin (f_1, \ldots, f_n)$, but $f_0 \in (f_1, \ldots, f_n)^*$. Then the cohomological height of $mA \subseteq A = R[T_1, \ldots, T_n]/(\sum f_iT_i + f_0)$ is $d$ and its superheight is $< d$.

Proof. Let $R'$ denote a local normal Noetherian domain of dimension $n \leq d$ and let $\varphi : A \rightarrow R'$ be a homomorphism such that $V((mRA)R') = V(mR')$. This gives a homomorphism $\psi : R \rightarrow R'$ such that $V(mR') = V(mR')$. If $n = d$, then $R \rightarrow R'$ would be finite (after enlarging the base field) and [3(ii)] would give $f_0 \in (f_1, \ldots, f_n)$. Hence the superheight is $< d$, but the cohomological height is $d$ due to [3(i)].

2. Homogeneous forcing algebras and projective bundles

Consider the mapping $\text{Spec } A \rightarrow \text{Spec } R$ over the open subset $U = D(I) = D(f_1, \ldots, f_n)$, where $A$ is the forcing algebra for $f_1, \ldots, f_n; f_0$. On $D(f_i)$, $i \geq 1$, one can identify $A_{f_i} = (R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0))_{f_i} \cong R_{f_i}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n]$. So this mapping looks locally like $D(f_i) \times \mathbb{K}^{n-1} \rightarrow D(f_i)$. The transition mapping on $D(f_i, f_j)$ is given by

$$R_{f_i, f_j}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n] \rightarrow R_{f_i, f_j}[T_1, \ldots, T_{j-1}, T_{j+1}, \ldots, T_n]$$
where $T_k \mapsto T_k$ for $k \neq i, j$ and $T_j \mapsto -1/f_j(\sum_{i \neq j} f_iT_i + f_0)$. This is an affine-linear mapping. Therefore we say that the forcing bundle $\text{Spec } A|_{D(I)}$ is an affine-linear bundle of rank $n - 1$. It is not a vector bundle in general.

We show how to associate to elements $f_1, \ldots, f_n; f_0$ a projective bundle over $D(I)$ together with a projective subbundle of codimension one such that the complement of the subbundle is the affine-linear bundle. This is more generally possible for every affine-linear bundle.

**Proposition 2.1.** Let $R$ be a commutative ring and let $f_1, \ldots, f_n$ and $f_0$ be elements and set $I = (f_1, \ldots, f_n)$, $U = D(I)$. The schemes

$$V = \text{Spec } R[T_1, \ldots, T_n]/(\sum_{i=1}^n f_iT_i)|_U \quad \text{and} \quad V' = \text{Spec } R[T_0, \ldots, T_n]/(\sum_{i=0}^n f_iT_i)|_U$$

are vector bundles on $U$. They are related by the short exact sequence of vector bundles

$$0 \longrightarrow V \longrightarrow V' \xrightarrow{T_0} \mathbb{A}_U^1 \longrightarrow 0.$$ 

The inclusion $V \subset V'$ yields a closed embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V')$ of projective bundles over $U$. Its complement $\mathbb{P}(V') - \mathbb{P}(V)$ is isomorphic to the forcing affine-linear bundle

$$\text{Spec } R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)|_U.$$ 

**Proof.** The bundle $V$ is on $D(f_i)$, $i = 1, \ldots, n$, isomorphic to

$$\text{Spec } R[f_i][T_1, \ldots, T_i-1, T_i+1, \ldots, T_n],$$

and the transition functions send $T_i \mapsto -1/f_i(f_1T_1 + \ldots + f_i^{-1}T_i-1 + f_{i+1}T_{i+1} + \ldots + f_nT_n)$, thus they are linear and $V$ (and $V'$) is a vector bundle on $U = \bigcup_{i=1}^n D(f_i)$.

The linear form $T_0$ is a global function on $V'$ which yields a linear mapping to $\mathbb{A}_U^1$. Its zero set is $V$. Looking at $D(f_i)$, the exactness of the sequence is clear.

$\mathbb{P}(V')$ is the projective bundle corresponding to the geometric vector bundle $V'$. The cone mapping $V' \dashrightarrow \mathbb{P}(V')$ maps $V(T_0 - 1)$ isomorphically onto $\mathbb{P}(V') - \mathbb{P}(V)$.

**Definition 2.2.** We call the short exact sequence in $\textbf{2.1}$ the forcing sequence and we call $\mathbb{P}(V')$ the projective bundle and $\mathbb{P}(V)$ the forcing projective subbundle or the forcing divisor associated to the elements $f_1, \ldots, f_n; f_0$.

**Remark 2.3.** The sections $\text{Spec } R \to \text{Spec } R[T_1, \ldots, T_n]/(\sum_{i=1}^n f_iT_i)$ are the relations for the ring elements $f_1, \ldots, f_n$. This is true for every open subset in Spec $R$.

We call this sheaf of sections the sheaf of relations $\mathcal{R} = \text{Rel}(f_1, \ldots, f_n)$. On $U = D(I)$, this is a locally free sheaf and we get the short exact forcing sequence of locally free sheaves

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}' \longrightarrow \mathcal{O}_U \longrightarrow 0.$$ 

These extensions are classified by $H^1(U, \mathcal{R}) = \text{Ext}^1(\mathcal{O}_U, \mathcal{R})$. The elements $f_0$ and $f_1, \ldots, f_n$ define the Čech-cocycle

$$(0, \ldots, -f_0/f_1, 0, \ldots, 0, f_0/f_j, 0, \ldots, 0) \in \Gamma(D(f_i, f_j), \mathcal{R}) \subseteq \Gamma(D(f_i, f_j), \mathcal{O}_U^\vee).$$

The dual sheaf $\mathcal{F} = \mathcal{R}'$ is the sheaf of linear forms for the vector bundle $V$, thus $V = \text{Spec } S(\mathcal{F})$ and $\mathbb{P}(V) = \text{Proj } S(\mathcal{F})$. Geometric vector bundles, their sheaf of relations and their sheaf of linear forms are essentially equivalent objects; in this paper we shall take mostly the viewpoint of geometric vector bundles, since in this form they appear starting from forcing algebras.
We gather together some characterizations of \( f_0 \in (f_1, \ldots, f_n) \) in terms of the geometric objects we consider.

**Lemma 2.4.** Let \( R \) be a commutative ring and let \( f_0, f_1, \ldots, f_n \in R \) be elements, \( U = D(f_1, \ldots, f_n) \). Let \( A = R[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n + f_0) \) be the forcing algebra. Then the following are equivalent.

(i) \( f_0 \in (f_1, \ldots, f_n) \).
(ii) There exists a section \( \text{Spec } R \to \text{Spec } A \).
(iii) The forcing algebra \( A \) is isomorphic to the algebra of relations 
\[
R[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n).
\]
Suppose furtheron that \( R = \Gamma(U, \mathcal{O}_X) \) (e.g. if \( R \) is normal and \( \text{ht } I \geq 2 \)). Then these statements are also equivalent with

(iv) The affine-linear bundle \( \text{Spec } A|_U \) has a section over \( U \).
(v) There exists a section \( U \to \mathbb{P}(V') \) which does not meet \( \mathbb{P}(V) \).
(vi) The forcing sequence splits.
(vii) The elements \( f_1, \ldots, f_n, f_0 \) define the zero element in \( H^1(U, R) \).

**Proof.** Suppose (i) holds, say \( -f_0 = \sum a_i f_i \). Then \( T_i \mapsto T_i + a_i \) is well defined and gives the isomorphism in (iii). On the other hand, a relation algebra has the zero section, thus the first three statements are equivalent.

(ii) \( \Rightarrow \) (iv) is a restriction, and (iv) \( \Rightarrow \) (ii) is true under the additional assumption.

(iv) and (v) are equivalent due to [2.1].

(i) gives also directly a section for \( V' \to A_U \to 0 \), thus we get (vi), which is equivalent with (vii). If the sequence splits, then \( V' = V \oplus \mathbb{A} \) on \( U \) and the complement of \( \mathbb{P}(V) \) is the vector bundle \( V \), which has the zero section. \( \square \)

### 3. The graded case: bundles on projective varieties

In order to use methods of projective geometry such as intersection theory to study the affineness of an open subset inside the spectra of a forcing algebra, we stick now to the graded case, where we get projective bundles over projective varieties.

Let \( K \) be a field and let \( R \) be a standard \( \mathbb{N} \)-graded \( K \)-algebra, i.e. \( R_0 = K \) and \( R \) is generated by finitely many elements of first degree. Let \( f_i \) be homogeneous elements of \( R \) of degrees \( d_i \). We say that the \( f_i \) are primary if \( D(R_+) \subseteq D(f_1, \ldots, f_n) \). We may find degrees \( e_i \) (possibly negative) for \( T_i \) such that the polynomials \( \sum_{i=1}^n f_i T_i, \sum_{i=0}^n f_i T_i \) and \( \sum_{i=1}^n f_i T_i + f_0 \) are homogeneous (for the last polynomial \( e_i = d_0 - d_i \) is the only choice).

Let \( A = R[(T_0), T_1, \ldots, T_n]/(P) \), where \( P \) is one of these polynomials. Then \( A \) is also graded and we have the following commutative diagram.

\[
\begin{array}{ccc}
\text{Spec } A \supset D(R_+) & \longrightarrow & D_+(R_+) \subset \text{Proj } A \\
\downarrow & & \downarrow \\
\text{Spec } R \supset D(R_+) & \longrightarrow & \text{Proj } R
\end{array}
\]

For \( Y = \text{Proj } R \) and a number \( m \) we set
\[
\mathcal{A}_Y(m) := D_+(R_+) \subset \text{Proj } R[T], \text{ deg } T = -m.
\]
This line bundle on \( Y \) is also \( \mathcal{A}_Y(m) = \text{Spec } S(\mathcal{O}_Y(m)) \) and its sheaf of sections is \( \mathcal{O}_Y(-m) \). Thus \( \mathcal{A}_Y(1) \) is the tautological bundle. (The algebras \( A \) may have
negative degrees, but \( \text{Proj } A \) can be defined as well, see \([3]\). The open subset \( D_+(R_+) \subset \text{Proj } A \) is the same as \( D_+(R_+) \subset \text{Proj } A_{\geq 0} \).

**Proposition 3.1.** Let \( R \) be a standard graded \( K \)-algebra and let \( f_1, \ldots, f_n \) be homogeneous primary elements. Let \( d_i = \text{deg } f_i \) and fix a number \( m \in \mathbb{Z} \) and set \( e_i = m - d_i \). Then the following hold.

(i) Set \( \text{deg } T_i = e_i \). Then

\[
\text{Proj } R[T_1, \ldots, T_n]/(\sum_{i=1}^{n} f_i T_i) \supset D_+(R_+) \longrightarrow \text{Proj } R
\]

is a vector bundle \( V_m \) of rank \( n - 1 \) over \( Y = \text{Proj } R \).

(ii) For this vector bundle \( V_m \) we have the exact sequence of vector bundles

\[
0 \longrightarrow V_m \longrightarrow \mathcal{O}_Y(-e_1) \times_Y \cdots \times_Y \mathcal{O}_Y(-e_n) \longrightarrow \mathcal{O}_Y(-m) \longrightarrow 0
\]

(iii) We have \( \text{Det } V_m \cong \mathcal{O}_Y(-\sum_{i=1}^{n} e_i + m) = \mathcal{O}_Y(\sum_{i=1}^{n} d_i - (n-1)m) \).

(iv) We have \( V_{m'} = V_m \otimes \mathcal{O}_Y(m - m') \).

(v) The projective bundle \( \mathbb{P}(V_m) \) does not depend on the chosen degree \( m \). For the relatively very ample sheaf \( \mathcal{O}_{\mathbb{P}(V_m)}(1) \) on \( \mathbb{P}(V_m) \) we have

\[
j^* \mathcal{O}_{\mathbb{P}(V_{m'})}(1) = \mathcal{O}_{\mathbb{P}(V_m)}(1) \otimes \pi^* \mathcal{O}_Y(m - m'),
\]

where \( j : \mathbb{P}(V_m) \to \mathbb{P}(V_m \otimes \mathcal{O}_Y(m - m')) \) is the isomorphism and \( \pi : \mathbb{P}(V_m) \to Y \) is the projection.

**Proof.** (i) and (ii). First note that the natural mapping \( (\text{deg } T_i = e_i) \)

\[
\text{Proj } R[T_1, \ldots, T_n]/(\sum_{i=1}^{n} f_i T_i) \supset D_+(R_+) \longrightarrow \mathcal{O}_Y(-e_1) \times_Y \cdots \times_Y \mathcal{O}_Y(-e_n)
\]

is an isomorphism. The ring homomorphism \( R[T] \to R[T_1, \ldots, T_n], T \mapsto \sum f_i T_i \) is homogeneous for \( \text{deg } T = m \). This gives the epimorphism of vector bundles, since the \( D_+(f_i) \) cover \( Y \). Its kernel is given by \( D_+(R_+) \subset \text{Proj } R[T_1, \ldots, T_n]/(\sum f_i T_i) \), thus this is also a vector bundle on \( Y \).

(iii) follows from (ii). If we tensorize the exact sequence for \( V_m \) with \( \mathcal{O}_Y(m - m') \) we get the sequence for \( V_{m'} \), hence (iv) follows.

(v) \( \mathbb{P}(V) \) does not change when \( V \) is tensorized with a line bundle. The relatively very ample sheaves behave like stated due to \([2], \text{Proposition } 4.1.4\). \( \square \)

**Remark 3.2.** We denote by \( \mathcal{R}(m) \) the locally free sheaf of sections in the vector bundle \( V_m \). This is the sheaf of relations of total degree \( m \), and \( \mathcal{R}(m) \otimes \mathcal{O}(m' - m) = \mathcal{R}(m') \) holds. The sequence in \([3])\) yields the short exact sequence

\[
0 \longrightarrow \mathcal{R}(m) \longrightarrow \oplus \mathcal{O}_Y(e_i) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.
\]

The sheaf of linear forms of total degree \( m \) is the dual sheaf \( \mathcal{F}(-m) = \mathcal{R}(m)^\vee \), thus \( V_m = \text{Spec } \mathcal{S}(\mathcal{F}(-m)) \) and \( \mathbb{P}(V) = \mathbb{P}(\mathcal{F}) \). The corresponding sequence is

\[
0 \longrightarrow \mathcal{O}_Y(-m) \longrightarrow \oplus \mathcal{O}_Y(-e_i) \longrightarrow \mathcal{F}(-m) \longrightarrow 0.
\]

The most important choice for \( m \) will be \( m = d_0 \), where \( d_0 \) is the degree of another homogeneous element \( f_0 \).
Remark 3.3. The sequence in (3.1) (ii) allows us to compute inductively the Chern classes of the vector bundles $V_m$ (or its sheaf of linear forms $F(-m)$). For the Chern polynomial $c_t(V_m) = \sum c_i(V_m)t^i$ we get the relation (let $H$ denote the hyperplane section of $Y$)

$$c_t(V_m)(1-mHt) = (1-c_1Ht)\cdots(1-e_nHt).$$

This yields $c_0(V_m) = 1$, $c_1(V_m) = (-e_1-\ldots-e_n+m)H$, $c_2(V_m) = (\sum e_ie_i - (e_1+\ldots+e_n-m)m)H$, etc.

Proposition 3.4. Let $R$ be a standard graded $K$-algebra, let $f_1,\ldots,f_n$ be homogeneous primary elements and let $f_0 \in R$ be also homogeneous. Let $d_i = \deg f_i$ and fix a number $m \in \mathbb{Z}$. Let $d_i = e_i = m - d_i$. Let

$$V_m = D_+(R+) \subset \Proj R[T_1,\ldots,T_n]/(\sum f_iT_i)$$

and

$$V_m' = D_+(R+) \subset \Proj R[T_0,\ldots,T_n]/(\sum f_iT_i)$$

be the vector bundles on $Y = \Proj R$ due to (3.3). Then the following hold.

(i) There is an exact sequence of vector bundles on $Y$,

$$0 \longrightarrow V_m \longrightarrow V_m' \xrightarrow{T_0} \mathbb{A}_Y(-e_0) \longrightarrow 0.$$

(ii) The embedding $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V')$ does not depend on the degree $m$ (and we skip the index $m$ inside $\mathbb{P}(V)$). The complement of $\mathbb{P}(V)$ is $\mathbb{P}(V') - \mathbb{P}(V) \cong D_+(R+) \subset \Proj R[T_1,\ldots,T_n]/(f_1T_1 + \ldots + f_nT_n + f_0)$.

(iii) Let $E$ be the Weil divisor (the hyperplane section) on $\mathbb{P}(V')$ corresponding to the relatively very ample invertible sheaf $O_{\mathbb{P}(V')}(1)$ (depending of the degree). Then we have the linear equivalence of divisors $\mathbb{P}(V) \sim E + e_0\pi^*H$, where $H$ is the hyperplane section of $Y$. If $e_0 = 0$, then $\mathbb{P}(V)$ is a hyperplane section.

(iv) The normal bundle for $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V')$ on $\mathbb{P}(V)$ is $\mathbb{A}_{\mathbb{P}(V')}(1) \otimes \pi^*\mathbb{A}_Y(-e_0)$.

Proof. (i). The homogeneous ring homomorphisms

$$R[T_0] \longrightarrow R[T_0,\ldots,T_n]/(\sum f_iT_i) \quad \text{(deg } T_0 = e_0 \text{) and}$$

$$R[T_0,\ldots,T_n]/(\sum f_iT_i) \longrightarrow R[T_1,\ldots,T_n]/(\sum f_iT_i), \quad T_0 \mapsto 0$$

induce the morphisms on $D_+(R+)$. The exactness is clear on $D(f_i), \ i = 1,\ldots,n$, and they cover $D_+(R+)$. The first statement in (ii) is clear, thus we assume $e_0 = 0$. The homogeneous ring homomorphism $R[T_0,\ldots,T_n]/(\sum f_iT_i) \to R[T_0,\ldots,T_n]/(\sum f_iT_i + f_0)$ where $T_0 \mapsto 1$ yields the closed embedding

$$\Proj R[T_0,\ldots,T_n]/(\sum f_iT_i + f_0) \supseteq D_+(R+) \hookrightarrow V_m' ,$$

where the image is given by $T_0 = 1$. But this closed subset $V_m(T_0 - 1) \subseteq V_m'$ is isomorphic to $\mathbb{P}(V') - \mathbb{P}(V)$ under the cone mapping $V_m' \longrightarrow \mathbb{P}(V')$. 


(iii). The mapping $T_0 : V'_m \to A_Y(-e_0)$ yields via the tautological morphism $A_{\mathbb{P}(V'_m)}(1) \to V'_m$ a morphism of line bundles on $\mathbb{P}(V'_m)$, $A_{\mathbb{P}(V'_m)}(1) \to \pi^*A_Y(-e_0)$. This corresponds to a section in the line bundle $A_{\mathbb{P}(V'_m)}(-1) \otimes \pi^*A_Y(-e_0)$ with zero set $\mathbb{P}(V_m)$. Thus $\mathbb{P}(V_m) \sim E + e_0 \pi^*H$.

(iv). Let $i : \mathbb{P}(V) \hookrightarrow \mathbb{P}(V')$ be the inclusion. Then $i^*(A_{\mathbb{P}(V')}(1) \otimes \pi^*A_Y(-e_0)) = A_{\mathbb{P}(V)}(-1) \otimes q^*A_Y(-e_0)$ is the normal bundle on $\mathbb{P}(V)$.

\textbf{Definition 3.5.} We call the sequence in \[3.4\] (i) again the \textit{forcing sequence} and we denote the situation $\mathbb{P}(V) \hookrightarrow \mathbb{P}(V')$ by $\mathbb{P}(f_1, \ldots, f_n; f_0)$. This is a projective bundle of rank $n - 1$ together with the forcing divisor $\mathbb{P}(V) = \mathbb{P}(f_1, \ldots, f_n)$ over $Y$.

\textbf{Remark 3.6.} Corresponding to the forcing sequence of vector bundles in \[3.4\] we have the exact sequence of relations $0 \to R(m) \to R'(m) \to \mathcal{O}_Y(e_0) \to 0$ of total degree $m$. For $e_0 = 0$ (or $m = d_0$) this extension corresponds to a cohomology class $c \in H^1(Y, R(m))$. The forcing sequence for the linear forms is $0 \to \mathcal{O}_Y(-e_0) \to \mathcal{F}'(-m) \to \mathcal{F}(-m) \to 0$.

The next results show that we can express the properties which are of interest from the tight closure point of view in terms of the projective bundles on $Y$.

\textbf{Lemma 3.7.} \textit{In the situation of \[3.4\] the following are equivalent.}

(i) $f_0 \in (f_1, \ldots, f_n)$.
(ii) There is a section $Y \to \mathbb{P}(V')$ disjoined to $\mathbb{P}(V) \subset \mathbb{P}(V')$.
(iii) The forcing sequence $0 \to V_m \to V'_m \to A_Y(-e_0) \to 0$ splits.
(iv) Let $e_0 = 0$. The corresponding cohomological class in $H^1(Y, R(m))$ vanishes.

\textbf{Proof.} Suppose that (i) holds and write $-f_0 = \sum_{i=1}^n a_i f_i$, where the $a_i$ are homogeneous. Set $e_0 = 0$. The $a_i$ define a homogeneous mapping

$$R[T_0, T_1, \ldots, T_n]/(\sum_{i=0}^n f_i T_i) \longrightarrow R \text{ by } T_0 \to 1, \ T_i \to a_i.$$  

The corresponding mapping $Y \to V'_m$ induces $Y \to \mathbb{P}(V')$ and its image is disjoint to $\mathbb{P}(V)$.

Suppose that (ii) holds. A section in $\mathbb{P}(V'_m)$ corresponds to a line bundle $L$ on $Y$ and an embedding $L \hookrightarrow V'_m$, see [13, Proposition 7.12]. Since the section is disjoined to $\mathbb{P}(V)$, the morphism $V_m \oplus L \to V'_m$ is an isomorphism, hence the sequence splits.

(iii). The splitting yields a section $A_Y(-e_0) \to V'_m$ and this means a homogeneous mapping $R[T_0, T_1, \ldots, T_n]/(\sum_{i=0}^n f_i T_i) \to R[T_0]$. For $T_0 = 1$ we get a solution for (i). (iii) and (iv) are equivalent. \qed

\textbf{Example 3.8.} Let $R$ denote a standard graded $K$-algebra and let $f_1, \ldots, f_n$ be homogeneous primary elements of degrees $d_i$. Let $f_0 = 0$. Then

$$R[T_0, \ldots, T_n]/(\sum_{i=0}^n f_i T_i) = R[T_1, \ldots, T_n]/(\sum_{i=1}^n f_i T_i)[T_0]$$

and we have the splitting forcing sequence

$$0 \longrightarrow V \longrightarrow V \oplus A_Y(-e_0) \longrightarrow A_Y(-e_0) \longrightarrow 0.$$  

Then $V \cong \mathbb{P}(V') - \mathbb{P}(V)$ and $\mathbb{P}(V')$ is just the projective closure of $V$.  

Proposition 3.9. Let \( R \) be a normal standard graded \( K \)-algebra of dimension \( d \geq 2 \), let \( f_1, \ldots, f_n \in R \) be primary homogeneous elements and let \( f_0 \) be another homogeneous element. Let \( V \) and \( V' \) be as in 3.4. Then the following are equivalent.

(i) \( f_0 \in (f_1, \ldots, f_n)^* \).

(ii) The cohomological dimension of \( \mathbb{P}(V') - \mathbb{P}(V) \) is \( d - 1 = \dim Y \).

In particular, if \( d = 2 \), then \( f_0 \in (f_1, \ldots, f_n)^* \) holds if and only if the open subscheme \( \mathbb{P}(V') - \mathbb{P}(V) \) is not affine.

Proof. We have \( \mathbb{P}(V') - \mathbb{P}(V) \cong D_+(R_+) \subseteq \text{Proj} \ R[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n + f_0) \). In general, every quasicoherent sheaf on an open subset \( D_+(a) \subseteq \text{Proj} \ S \) is quasicoherent extendible to \( \text{Proj} \ S \) and hence of type \( M \), where \( M \) is a graded \( S \)-module ([3, Propositions II.5.8 and II.5.15]). Therefore the cohomological dimensions of \( D_+(a) \) and of \( D(a) \) are the same. Hence the cohomological dimensions of \( \mathbb{P}(V') - \mathbb{P}(V) \) and of the forcing affine-linear bundle \( D(R_+) \subseteq \text{Spec} \ R[T_1, \ldots, T_n]/(f_1 T_1 + \ldots + f_n T_n + f_0) \) are the same, and the result follows from 3.4(ii) \( \Box \).

Lemma 3.10. Let \( R \) be a normal standard graded \( K \)-algebra of dimension \( d \geq 2 \), let \( f_1, \ldots, f_n \in R \) be primary homogeneous elements and let \( f_0 \) be another homogeneous element. Let \( V \) and \( V' \) be as in 3.4. Then the following are equivalent.

(i) \( f_0 \in (f_1, \ldots, f_n)^{+3} \), i.e. there exists a finite graded extension \( R \subseteq R' \) such that \( f_0 \in (f_1, \ldots, f_n)R' \).

(ii) There exists a finite surjective morphism \( g : Y' \to Y \) such that the pull back \( g^* \mathbb{P}(V') = \mathbb{P}(V') \times_Y Y' \) has a section not meeting \( g^* \mathbb{P}(V) = \mathbb{P}(V) \times_Y Y' \).

(iii) There exists a closed subvariety \( Y'' \subseteq \mathbb{P}(V') \) not intersecting \( \mathbb{P}(V) \), finite and surjective over \( Y \).

(iv) There exists a closed subvariety \( Y'' \subseteq \mathbb{P}(V') \) not intersecting \( \mathbb{P}(V) \) of dimension \( d - 1 = \dim Y \).

Proof. (i) \( \Leftrightarrow \) (ii). If \( R \subseteq R' \) is finite and graded such that \( f_0 \in (f_1, \ldots, f_n)R' \), then there exists a section \( Y' = \text{Proj} \ R' \to g^* \mathbb{P}(V') \) which does not meet \( g^* \mathbb{P}(V) \) due to 3.7. If \( g : Y' \to Y \) is such a morphism, then \( g^* \mathcal{O}_Y(1) \) is ample on \( Y' \) and this gives the homogeneous ring \( R' \).

Suppose that (ii) holds. Then the image of the section gives the closed subvariety \( Y'' \) finite over \( Y \). This gives (iii) and then (iv). Suppose that (iv) holds. The mapping \( Y'' \to \mathbb{P}(V') \to Y \) is projective and the fibers are zero-dimensional, since \( \mathbb{P}(V_y) \subseteq \mathbb{P}(V'_y) \) meets every curve, but \( Y'' \cap \mathbb{P}(V) = \emptyset \). Hence this mapping is finite and due to the assumption on the dimension it is surjective. So suppose that (iii) holds. The mapping \( g : Y' = Y'' \to \mathbb{P}(V') \to Y \) is finite and surjective, and the image of the section \( i \times id_{Y'} : Y' \to \mathbb{P}(V') \times_Y Y' = g^* \mathbb{P}(V') \) is disjoined to \( g^* \mathbb{P}(V) = \mathbb{P}(V) \times_Y Y' \) \( \Box \)

4. Ample and basepoint free forcing divisors

Let \( Z = \mathbb{P}(V) \subseteq \mathbb{P}(V') \) be the forcing divisor on \( Y = \text{Proj} \ R \) corresponding to homogeneous forcing data \( f_1, \ldots, f_n ; f_0 \in R \). When is \( Z \) ample and when is \( Z \) basepoint free? For \( c_0 = 0 \) the forcing divisor is a hyperplane section of \( \mathcal{O}_{\mathbb{P}(V')}(1) \), and the ampleness of this invertible sheaf is by definition the ampleness of the locally free sheaf \( F(-d_0) = \pi_* \mathcal{O}_{\mathbb{P}(V')}(1) \), see [3] III, §1 and [7] for further ampleness criteria for vector bundles and applications to tight closure problems.
Throughout this section we will assume that $K$ is algebraically closed. The following proposition shows that the ample property is interesting only in dimension two.

**Proposition 4.1.** Let $K$ be an algebraically closed field and let $R$ be a normal standard graded $K$-algebra of dimension $d$. Let $f_1, \ldots, f_n$ be homogeneous $R_+$-primary elements and let $f_0$ be another homogeneous element of degrees $d_i$. Let $V_m, V_m'$ be as in 3.4 and let $Z = \mathbb{P}(V)$ be the forcing divisor. Then the following hold.

(i) Suppose that $f_0$ is a unit and $d_i \geq 1$ for $i = 1, \ldots, n$. Then $Z$ is ample.

(ii) If $f_0$ is not a unit, then the cohomological dimension $\dim cd(\mathbb{P}(V') - Z) \geq d - 2$.

(iii) If $f_0$ is not a unit and $d \geq 3$, then $Z$ is not ample.

**Proof.** (i). We may assume that $f_0 = 1$. Then

\[ V_m' = D_+(R_+) \subset \text{Proj} \, R[T_0, T_1, \ldots, T_n]/(\sum_{i=1}^n f_i T_i + T_0) \cong \text{Proj} \, R[T_1, \ldots, T_n], \]

where $\deg T_i = e_i = -d_i + e_0$. Thus $V_m' \cong \mathcal{A}_Y(d_1 - e_0) \times_Y \ldots \times_Y \mathcal{A}_Y(d_n - e_0)$. For $m = d_0$ we see that $\mathcal{F}'(-d_0)$ is a sum of ample invertible sheaves, hence $\mathcal{F}'(-d_0)$ is ample due to [13, III, Corollary 1.8].

(ii). For $d = 0, 1$ there is nothing to show, so suppose $d \geq 2$. The zero set $V_+(f_0) \subset Y$ is a closed subset of dimension $\geq d - 2$. There exists a section $V_+(f_0) \to \mathbb{P}(V')$ which does not meet $Z$. Hence $\mathbb{P}(V') - Z$ contains a projective subvariety of dimension $d - 2$, thus the inequality holds for the cohomological dimension.

(iii). Due to (ii) the complement of $Z$ cannot be affine (it contains projective curves), hence $Z$ is not ample. \(\square\)

The forcing divisor $Z$ is basepoint free if and only if $\mathcal{O}_{\mathbb{P}((V))}(1)$ is generated by global sections for $e_0 = 0$ and this is true if and only if $\pi^* \mathcal{O}_{\mathbb{P}(V)}(1) = \mathcal{F}'(-d_0)$ is generated by global sections. A divisor $Z$ is called semiample ([23, Definition 2.1.14]) if $aZ$ is basepoint free for some $a \geq 1$. In this case there exists a (projective) morphism $\varphi : \mathbb{P}(V) \to \mathbb{P}^N$ such that $aZ = \varphi^{-1}(H)$, where $H$ is a hyperplane section in $\mathbb{P}^N$. Then $\mathbb{P}(V') - Z$ is projective over $\mathbb{P}^N - H$. Schemes which are proper over an affine scheme are called semiample and were studied in [10].

**Lemma 4.2.** Let $K$ be an algebraically closed field and let $R$ be a normal standard graded $K$-algebra. Let $f_1, \ldots, f_n$ be primary homogeneous elements and let $f_0$ be another homogeneous element. Suppose that $\mathbb{P}(V') - \mathbb{P}(V)$ is semiample. Then $f_0 \in (f_1, \ldots, f_n)^{+gr}$ if and only if $f_0 \in (f_1, \ldots, f_n)^{+gr}$.

**Proof.** Due to [10, Corollary 5.8] the cohomological dimension of a semiample scheme equals the maximal dimension of a closed proper subvariety. Thus $f_0 \in (f_1, \ldots, f_n)^*$ implies via [3.5] that there exists a projective subvariety $Y' \subset \mathbb{P}(V')$ of dimension $\dim Y$ which does not meet $\mathbb{P}(V)$. Therefore $f_0 \in (f_1, \ldots, f_n)^{+gr}$ due to [3.10]. \(\square\)

The condition in the following corollary is useful only for $\dim R = 2$. We will apply this in section [4].

**Corollary 4.3.** Let $K$ denote an algebraically closed field and let $R$ be a normal standard graded $K$-algebra. Let $f_1, \ldots, f_n$ be primary homogeneous elements and let $Z = \mathbb{P}(f_1, \ldots, f_n) \subset \mathbb{P}(f_1, \ldots, f_n; f_0)$ be the corresponding bundles on $Y = \text{Proj} \, R$. Suppose that the pull back $Z|_Z$ is ample. Then $f_0 \in (f_1, \ldots, f_n)^*$ if and only if $f_0 \in (f_1, \ldots, f_n)^{+gr}$. 

Proof. The theorem of Zariski-Fujita (see [21, Remark 2.1.18]) asserts that $Z$ is semiaffine. Then the complement of $Z$ is semiaffine and the result follows from 1.2.

Corollary 4.4. Let $R$ be a normal standard graded $K$-algebra and let $f_1, \ldots, f_n$ be primary homogeneous elements and let $f_0$ be another homogeneous element. Suppose that the locally free sheaf $\mathcal{F}'(-d_0)$ is generated by global sections. Then $f_0 \in (f_1, \ldots, f_n)^*$ if and only if $f_0 \in (f_1, \ldots, f_n)^{+\text{gr}}$.

Proof. Since $\mathcal{F}'(-d_0)$ is generated by global sections we know that the forcing divisor is basepoint free, hence $\mathbb{P}(V') - \mathbb{P}(V)$ is semiaffine. Hence the result follows from 1.3.

Note that the results 1.2 - 1.4 yield in characteristic zero in fact the stronger result that $f_0 \in (f_1, \ldots, f_n)^*$ holds if and only if already $f_0 \in (f_1, \ldots, f_n)$ holds. The following corollary was proved for tight closure in [21, Theorem 2.2] using differential operators in positive characteristic. Our version proves the same result for solid closure.

Corollary 4.5. Let $K$ denote an algebraically closed field and let $R$ be a normal standard graded $K$-algebra and let $f_1, \ldots, f_n$ be primary homogeneous elements of degrees $d_i$. Let $f_0$ be another homogeneous element of degree $d_0 \leq \min_i d_i$. Then $f_0 \in (f_1, \ldots, f_n)^*$ is only possible if $f_0 \in (f_1, \ldots, f_n)$.

Proof. Set $e_0 = 0$. Then $e_i = d_0 - d_i \leq 0$ and we have a surjection $\mathcal{O}_Y(-e_1) \oplus \cdots \oplus \mathcal{O}_Y(-e_n) \oplus \mathcal{O}_Y \to \mathcal{F}'(-d_0) \to 0$. Since the $\mathcal{O}_Y(a)$ for $a \geq 0$ are generated by global sections, we have also a surjection $\mathcal{O}_Y^k \to \mathcal{F}'(-d_0) \to 0$. Therefore $\mathcal{F}'(-d_0)$ is generated by global sections and we have a closed embedding $V' \to Y \times \mathbb{A}^k$.

Suppose that $f_0 \in (f_1, \ldots, f_n)^*$. Then by 4.4 we know that there exists a subvariety $Y' \subseteq \mathbb{P}(V')$ of dimension $\dim Y$ not meeting the forcing divisor $Z$. We may consider $Y' \subseteq V_+(T_0 - 1) \subseteq V'$, since $V_+(T_0 - 1)$ is isomorphic to $\mathbb{P}(V') - \mathbb{P}(V)$ via the cone mapping (see the proof of 3.4(ii)). All together we get a closed embedding $Y' \to Y \times \mathbb{A}^k$. Since $Y'$ is a projective variety, this factors through $Y \times \{P\}$, where $P \in \mathbb{A}^k$ is a closed point, and so $Y' \cong Y \times \{P\} \cong Y$, since $K$ is algebraically closed. Hence we get a section.

Even if the forcing divisor is not basepoint free, the existence of linearly equivalent effective divisors has consequences on the existence of closed subvarieties and hence on the existence of finite solutions (in the sense of 8.10 (iii) or (iv)) for the tight closure problem. See also Proposition 10.12.

Proposition 4.6. Let $R$ be a normal standard graded $K$-algebra such that $Y = \text{Proj} R$ is a smooth variety. Let $f_1, \ldots, f_n$ be homogeneous primary elements and let $f_0$ be another homogeneous element. Suppose that there exists a positive (effective $\neq 0$) divisor $L \subseteq Y$ such that for some $a \geq 1$ the divisor $a\mathbb{P}(V) - \pi^*L$ is linearly equivalent to an effective divisor. Then there exists a linearly equivalent effective divisor $D \sim a\mathbb{P}(V)$ with the property that the cohomological dimension of $\mathbb{P}(V') - \text{supp } D$ is smaller than the (cohomological) dimension of $Y$.

If $Y' \subseteq \mathbb{P}(V')$ is finite and surjective over $Y$ and disjoined to $\mathbb{P}(V)$ (as in 8.10(ii)), then $Y'$ must lie on the support of $D$.

Proof. Let $a\mathbb{P}(V) - \pi^*L \sim D'$ be effective, hence $a\mathbb{P}(V) \sim D = D' + \pi^*L$. The divisor $D'$ cuts out a hyperplane on every fiber, hence it is also a projective subbundle.

□
Since a projective bundle minus a dominant effective divisor is relatively affine over the base we see that \( \mathbb{P}(V') - \text{supp } D \) is affine over \( Y - \text{supp } L \). But the cohomological dimension of \( Y - \text{supp } L \) is smaller than the dimension of \( Y \) due to the theorem of Lichtenbaum ([13, Corollary 3.2]), hence this is also true for \( \mathbb{P}(V') - \text{supp } D \).

Now suppose that \( Y' \) is finite and surjective over \( Y \) and \( Y' \cap \mathbb{P}(V) = \emptyset \). Then we have from intersection theory the identities \( 0 = aY'.\mathbb{P}(V) = Y'(D' + \pi^*L) = Y'.D' + Y'.\pi^*L \). The second summand is a positive cycle, since \( Y' \) dominates \( Y \). Hence \( Y'.D' \) cannot be effective and the intersection of \( Y' \) and \( D' \) must be improper, so \( Y' \subset \text{supp } D' \subset \text{supp } D \). □

For the rest of this paper we will restrict to the situation where \( K \) is an algebraically closed field and \( R \) is a two-dimensional normal standard graded \( K \)-algebra. Then \( Y = \text{Proj } R \) is a smooth projective curve with hyperplane section \( H \). Homogeneous primary elements \( f_1, \ldots, f_n, f_0 \) yield the projective bundle \( \mathbb{P}(V') = \mathbb{P}(f_1, \ldots, f_n, f_0) \) of rank \( n - 1 \) over the curve (and of dimension \( n \)) together with the forcing divisor \( Z = \mathbb{P}(f_1, \ldots, f_n) \). Now \( f_0 \in (f_1, \ldots, f_n)^* \) holds if and only if the complement of the forcing divisor is not affine.

If the complement of the forcing divisor is affine (i.e. \( f_0 \not\in (f_1, \ldots, f_n)^* \)), then \( \Gamma(\mathbb{P}(V') - \mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V')}) \) is a finitely generated \( K \)-algebra of dimension \( n \). It follows that some multiple of the forcing divisor \( \mathbb{P}(V) \) defines a rational mapping to some projective space such that the dimension of the image is \( n \). This means by definition that \( \mathbb{P}(V) \) is big (has maximal Iitaka-dimension, see [21, Definition 2.2.1]).

It is sometimes possible to establish \( f_0 \in (f_1, \ldots, f_n)^* \) by showing that the forcing divisor is not big. The following proposition deals with equivalent conditions for bigness in our situation.

**Proposition 4.7.** Let \( R \) be a normal two-dimensional standard graded \( K \)-algebra. Let \( f_1, \ldots, f_n \) be homogeneous primary elements and let \( f_0 \) be another homogeneous element. Let \( Z = \mathbb{P}(V) \subset \mathbb{P}(V') \) denote the forcing divisor. Then the following are equivalent.

(i) There exists a positive divisor \( L \subset Y \) such that for some \( a \geq 1 \) the divisor \( aZ - \pi^*L \) is equivalent to an effective divisor.

(ii) There exists a linearly equivalent effective divisor \( D \sim aZ \) (\( a \geq 1 \)) such that \( \mathbb{P}(V') - \text{supp } D \) is affine.

(iii) The forcing divisor \( Z \) is big.

**Proof.** (i) \( \Rightarrow \) (ii) follows from [4, Théorème 4.5.2]. Suppose that (ii) holds, let \( X = \mathbb{P}(V) \) and let \( s \in \Gamma(X, \mathcal{O}_X(aZ)) \) be a section such that \( X_s \) is affine. The topology of \( X_s \) is generated by subsets \( X_t \subseteq X, t \in \Gamma(X, \mathcal{O}_X(bZ)), b \geq 1 \), see [4, Théorème 4.5.2]. Therefore the rational mapping defined by \( aZ \) is an isomorphism on \( X_s \) and the image has maximal dimension, hence \( Z \) is big (and (ii) \( \Rightarrow \) (iii)). On the other hand, if \( \emptyset \neq V \subset X_s \) is an affine subset which does not meet the fiber over a point \( P \in Y \), then there exists also \( t \in \Gamma(X, \mathcal{O}_X(bZ)) \) such that \( \emptyset \neq X_t \subseteq V \). Therefore \( bZ + (t) = \sum a_i D_i, a_i > 0 \) is an effective divisor and \( \mathbb{P}(V')_t \) is one of the \( D_i \) (hence (ii) \( \Rightarrow \) (i)). (iii) \( \Rightarrow \) (ii). If \( Z \) is big, then for some \( a \geq 1 \) the multiple \( aZ \) defines a mapping which is birational with its image. Therefore the mapping induces an isomorphism on an open affine subset \( X_s \cong D_+(s), s \in \Gamma(X, \mathcal{O}_X(aZ)) \). □

**Remark 4.8.** A numerically effective divisor \( Z \) is big if and only if its top self intersection number \( Z^n \) is \( > 0 \), see [21, Theorem VI.2.15] or [21, Theorem 2.2.14].
The top self intersection of the forcing divisor \( \mathbb{P}(f_1, \ldots, f_n) \subset \mathbb{P}(f_1, \ldots, f_n; f_0) \) corresponding to forcing data in a two dimensional normal graded domain \( R \) is \((d_1 + \ldots + d_n - (n-1)d_0) \deg H \), where \( H \) is the hyperplane section on \( Y = \text{Proj} R \). This follows from \( 3.1(iii) \).

5. Ruled surfaces and forcing sections

In this section \( K \) denotes an algebraically closed field and \( R \) denotes a standard graded two-dimensional \( K \)-algebra and we consider the tight closure of homogeneous parameters \( f_1 \) and \( f_2 \). The construction of projective bundles and subbundles from homogeneous elements described in section \( 3 \) leads in this setting to graded two-dimensional \( \mathbb{K} \). This follows from 3.1(iii).

### Corollary 5.1.

Let \( K \) denote an algebraically closed field and let \( R \) be a standard graded two-dimensional normal \( K \)-algebra, let \( f_1, f_2 \) be homogeneous parameters of degrees \( d_1 \) and \( d_2 \) and let \( f_0 \) be another element of degree \( d_0 \). Set \( l = d_1 + d_2 - d_0 \). Let \( e_1, e_2, e_0 \) be integers such that \( e_i + d_i = m \) and let \( V_m \) and \( V_m' \) as in \( 5.4 \) and set \( Y = \text{Proj} R \). Then the following hold.

(i) \( \mathbb{P}(V') \) is a ruled surface and \( \mathbb{P}(V) \subset \mathbb{P}(V') \) is a section, called the forcing section.

(ii) We have \( \text{Proj} R[T_1, T_2]/(f_1 T_1 + f_2 T_2) \supset D_+(R_+) = V_m \cong \mathbb{A}_Y(l - e_0) \). In particular, the exact forcing sequence is

\[
0 \rightarrow \mathbb{A}_Y(l - e_0) \rightarrow V_m' \rightarrow \mathbb{A}_Y(-e_0) \rightarrow 0.
\]

(iii) We have \( \text{Det} V' \cong \mathbb{A}_Y(l - 2e_0) \).

(iv) The normal bundle for the embedding \( Y \cong \mathbb{P}(V) \subset \mathbb{P}(V') \) is \( \mathbb{A}_Y(-l) \).

(v) The self intersection number of the forcing section \( Y \cong \mathbb{P}(V) \hookrightarrow \mathbb{P}(V') \) is \( l \deg H \), where \( H \) is the hyperplane section corresponding to \( \mathcal{O}_Y(1) \).

**Proof.** (i) follows from \( 3.4 \). (ii) The homomorphism \( R[T_1, T_2]/(f_1 T_1 + f_2 T_2) \rightarrow R[T] \) given by \( T_1 \mapsto f_2 T, T_2 \mapsto -f_1 T \) is homogeneous for \( \deg T = e_1 - d_2 = e_2 - d_1 \) and induces an isomorphism on \( D_+(R_+) \). Since \( \deg T = e_1 - d_2 = m - d_1 - d_2 = e_0 - l \) the corresponding line bundle is \( \mathbb{A}_Y(l - e_0) \). (iii) follows.

The normal bundle for the embedding on \( \mathbb{P}(V) \cong Y \) is \( N = \mathbb{A}_{\mathbb{P}(V)}(-1) \otimes \mathbb{A}_Y(-e_0) \) due to \( 3.4(iv) \). Furthermore, \( V_m = \mathbb{A}_Y(l - e_0) \) on \( Y \) and \( V_m = \mathbb{A}_{\mathbb{P}(V)}(1) \) is the tautological line bundle for \( \mathbb{P}(V) \cong Y \). This yields \( N = \mathbb{A}_Y(-l + e_0) \otimes \mathbb{A}_Y(-e_0) = \mathbb{A}_Y(-l) \). Its sheaf of sections is \( \mathcal{O}_Y(l) \) and its degree is the self intersection number, hence (iv) follows.

**Remark 5.2.** The corresponding sequence of sheaves are

\[
0 \rightarrow \mathcal{O}_Y(-e_0) \rightarrow \mathcal{F}'(-m) \rightarrow \mathcal{O}_Y(l - e_0) \rightarrow 0
\]

for the linear forms \( \mathcal{F}'(-m) \) and

\[
0 \rightarrow \mathcal{O}_Y(-l + e_0) \rightarrow \mathcal{R}'(m) \rightarrow \mathcal{O}_Y(e_0) \rightarrow 0
\]

for the relations \( \mathcal{R}'(m) \). These extensions are classified by \( H^1(Y, \mathcal{O}_Y(-l)) \) for \( e_0 = 0 \), where the elements \( f_1, f_2; f_0 \) correspond to the cohomology class \( f_0/f_1 f_2 \).
The following proposition gives a criterion for tight closure in the two-dimensional parameter case in terms of ampleness of the forcing divisor.

**Theorem 5.3.** Let $R$ be a two-dimensional standard graded normal $K$-algebra and let $Y = \text{Proj} \ R$. Let $f_1, f_2$ be homogeneous parameters and let $f_0$ be another homogeneous element. Let $s : Y \to \mathbb{P}(V')$ be the corresponding forcing section, $Z = \mathbb{P}(V) = s(Y)$. Then the following are equivalent.

(i) $f_0 \not\in (f_1, f_2)^*$.  
(ii) $\mathbb{P}(V') - Z$ is affine.  
(iii) The forcing divisor $Z$ on $\mathbb{P}(V')$ is ample.

*Proof.* We know the equivalence (i)$\iff$(ii) from proposition 3.9, so we have to show the equivalence (ii)$\iff$(iii). If $Z$ is ample, then its complement is affine. If $\mathbb{P}(V') - Z$ is affine, then it does not contain projective curves. Furthermore, there exist global functions on $\mathbb{P}(V') - Z$ which are not constant. Thus $aZ$, $a \geq 1$, is linearly equivalent with an effective divisor not containing $Z$. Hence the self-intersection number is positive and the criterion of Nakai yields that $Z$ is ample. \(\Box\)

**Corollary 5.4.** Let $K$ be an algebraically closed field and let $R$ be a normal two-dimensional standard graded $K$-algebra and let $f_1, f_2$ be homogeneous parameters of degrees $d_1, d_2$. Then $R_{d_1 + d_2} \subseteq (f_1, f_2)^*$. If the characteristic of $K$ is zero, then $(f_1, f_2)^* = (f_1, f_2) + R_{d_1 + d_2}$.

*Proof.* Let $\deg f_0 \geq d_1 + d_2$. Then $l \leq 0$ and the self intersection of $Z = \mathbb{P}(V) \subset \mathbb{P}(V')$ is not positive. Hence $Z$ is not ample and $f_0 \not\in (f_1, f_2)^*$ due to theorem 5.3.

Now let $f_0 \in (f_1, f_2)^*$, but $f_0 \not\in R_{d_1 + d_2}$. Then the self intersection is positive, but the forcing divisor $Z$ is not ample. Thus there must exist a curve $C \subset \mathbb{P}(V')$ disjoint to $Z$. By 3.10 it follows that $f_0 \in (f_1, f_2)^+$, so 1.4(ii) gives the result in characteristic 0. \(\Box\)

To prove the result of 5.4 also in positive characteristic, we need the notion of a normalized section and of the so-called $e$-invariant of a ruled surface. From the point of view of the forcing divisor it is technically convenient to introduce the normalizing number.

**Definition 5.5.** Let $W$ be a vector bundle over a smooth projective curve $Y$ and let $\pi : \mathbb{P}(W) \to Y$ be the projective bundle. Let $D$ be a divisor on $\mathbb{P}(W)$. We say that $D$ is *normalized* if $D$ has an effective representative, but for every divisor $D'$ of $Y$ of negative degree the divisor $D + \pi^*D'$ does not have an effective representative.

For any divisor $Z$ on $\mathbb{P}(W)$ we call the number $\nu$ characterized by the fact that there exists a divisor $D$ on $Y$ of degree $-\nu$ such that $Z + \pi^*D$ is normalized the *normalizing number* of $Z$.

If $Z = \mathbb{P}(V) \subset \mathbb{P}(V')$ is the forcing divisor of a tight closure problem in a two-dimensional normal standard graded $K$-algebra, then we call the normalizing number of $Z$ also the normalizing number of the problem or of the forcing data.

**Remarks 5.6.** Recall that a locally free sheaf $E$ on a smooth projective curve $Y$ is called normalized if $H^0(Y, E) \neq 0$, but $H^0(Y, E \otimes \mathcal{L}) = 0$ for every invertible sheaf $\mathcal{L}$ of negative degree. If $E$ is normalized, then a section $0 \neq s \in H^0(Y, E)$ yields a mapping $s^\vee : E^\vee \to \mathcal{O}_Y$, which must be surjective. Hence the kernel is locally free and we get a short exact sequence $0 \to \mathcal{O}_Y \xrightarrow{f} E \to F \to 0$ where $F$ is locally free and where the corresponding projective subbundle $\mathbb{P}(F) \subset \mathbb{P}(E)$ is normalized.
On the other hand, if $\mathbb{P}(\mathcal{F}) \subset \mathbb{P}(\mathcal{E})$ is normalized, then the sheaf $\mathcal{E} \otimes \ker(\mathcal{E} \to \mathcal{F})^\vee$ is normalized.

Suppose that $\mathbb{P}(W) = \mathbb{P}(\mathcal{E})$ is a ruled surface. Then a normalized subbundle of codimension one is the same as a normalized section. Recall that the $e$-invariant of a ruled surface is defined by $e = -C_0^2$, where $C_0$ is a normalized section, and that $e = -\deg \mathcal{E}$ for normalized $\mathcal{E}$.

If $\mathcal{F}^\prime(-m)$ is the sheaf of linear forms coming from a tight closure problem, then $\nu$ is also characterized by the property that there exists an invertible sheaf $\mathcal{L}$ of degree $e_0 \deg H - \nu$ such that $\mathcal{F}^\prime(-m) \otimes \mathcal{L}$ is normalized. If $H^0(Y, \mathcal{F}^\prime(-m) \otimes \mathcal{M}) = 0$ for every invertible sheaf $\mathcal{M}$ of negative degree, then $\nu \leq (m - d_0) \deg H$.

**Lemma 5.7.** Let $f_1, f_2, f_0 \in R$ as in \[15\]. Let $e$ denote the $e$-invariant of the ruled surface $\mathbb{P}(f_1, f_2; f_0)$, let $l = d_1 + d_2 - d_0$ and let $\nu$ be the normalizing number of $Z = \mathbb{P}(V)$. Then

$$e = 2\nu - l \deg H \quad \text{and} \quad \nu = \frac{l \deg H + e}{2}.$$  

**Proof.** Let $\mathfrak{d}$ be a divisor on $Y$ of degree $-\nu$ such that $Z + \pi^* \mathfrak{d}$ is normalized and let $C_0 \sim Z + \pi^* \mathfrak{d}$ be effective. Hence $C_0$ is a normalized section. Numerically we have $C_0 \sim Z \nu F$ ($F =$ fiber) and therefore $-e = C_0^2 = Z^2 - 2\nu = l \deg H - 2\nu$. □

Knowing the $e$-invariant of a ruled surface one may characterize the divisors which are ample. We recall this only for sections.

**Lemma 5.8.** Let $S$ be a ruled surface with $e$-invariant $e$ and let $D$ be a section. Then the following hold.

(i) Suppose that $e \geq 0$. Then $D$ is ample if and only if $D^2 > e$.

(ii) Suppose that $e < 0$ and that $\text{char} K = 0$ or $p >> 0$. Then $D$ is ample.

**Proof.** Let $C_0$ be a normalized section and write $D = C_0 + bF$ where $b \geq 0$. Then $D^2 = C_0^2 + 2b$, thus $b = 1/2(D^2 - C_0^2)$. If $e \geq 0$, then the criterion \[13\], Proposition V.2.20\] says $b > e$, thus $1/2(D^2 - C_0^2) > e = -C_0^2$ and this gives $D^2 > e$.

Let $e < 0$. If the characteristic is zero, then \[13\], Proposition 2.21\] gives the result. If the characteristic is positive, then \[15\], Exercise 2.14\] yields the condition $b > (e/2 + (g - 1)/p)$, and this is true for $p >> 0$. □

To apply this criterion on ruled surfaces arising from forcing equation, we need to know something about the $e$-invariant of them.

**Lemma 5.9.** Let $f_1, f_2, f_0 \in R$ be as in \[15\]. Let $\mathcal{E} = \mathcal{F}^\prime(-m)$ be the sheaf of linear forms of $V_{n_0}$. Then the following hold.

(i) If $e_0 \leq 0$, then $H^0(Y, \mathcal{E}) \neq 0$.

(ii) Let $e_0 \geq 0$. Let $\mathcal{L}$ be an invertible sheaf on $Y$ of negative degree $-k$. Then $H^0(Y, \mathcal{E} \otimes \mathcal{L}) = 0$ for $k > (l - e_0) \deg H$. If $f_0 \not\in (f_1, f_2)$, then this is also true for $k \geq (l - e_0) \deg H$.

(iii) Let $l \leq 0$ and $e_0 = 0$. Then $\mathcal{E}$ has global sections $\neq 0$, but $\mathcal{E} \otimes \mathcal{L}$ does not have for deg $\mathcal{L} < 0$ (i.e., $\mathcal{E} = \mathcal{F}^\prime(-d_0)$ is normalized).

**Proof.** (i). If $e_0 \leq 0$, then $0 \neq H^0(Y, \mathcal{O}_Y(-e_0)) \subseteq H^0(Y, \mathcal{E})$ by the forcing sequence and $\mathcal{E}$ has global sections $\neq 0$.

(ii) We consider the exact forcing sequence of the sheaf of linear forms

$$0 \longrightarrow \mathcal{O}_Y(-e_0) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y(l - e_0) \longrightarrow 0.$$
We tensorize with $\mathcal{L}$ and get the cohomology sequence
\[ 0 \to H^0(Y, \mathcal{O}_Y(-e_0) \otimes \mathcal{L}) \to H^0(Y, \mathcal{E} \otimes \mathcal{L}) \to H^0(Y, \mathcal{O}_Y(l - e_0) \otimes \mathcal{L}) \to \cdots. \]
Because of $e_0 \geq 0$ and $\deg \mathcal{L} < 0$ we have on the left hand side and because of $k > (l - e_0) \deg H$ we have on the right hand side an invertible sheaf of negative degree, thus the result follows.

If $k = (l - e_0) \deg H$, then on the right hand side we have an invertible sheaf of degree zero. If it is not trivial, then it has no global sections. Otherwise the sheaf is $\mathcal{O}_Y$ and the cohomology sequence is
\[ 0 \to H^0(Y, \mathcal{E} \otimes \mathcal{L}) \to H^0(Y, \mathcal{O}_Y) = K \to H^1(Y, \mathcal{O}_Y(-e_0) \otimes \mathcal{L}) \to \cdots. \]
Suppose that $H^0(Y, \mathcal{E} \otimes \mathcal{L}) \neq 0$. Then this maps surjective on $K$ and $K$ maps to zero. But then the short exact sequence splits, which contradicts the assumption $f_0 \notin (f_1, f_2)$.

(iii) follows from (i) and (ii).

Corollary 5.10. Let $f_1, f_2, f_0$ be as in 5.1 and let $e$ denote the $e$-invariant of $\mathbb{P}(f_1, f_2; f_0)$ and let $\nu$ denote the normalizing number. Then the following hold.

(i) We have $\nu \geq 0$ and $e \geq -l \deg H$.

(ii) Let $l > 0$. Then $\nu \leq l \deg H$ if $f_0 \notin (f_1, f_2)$ and $e \leq l \deg H$ if $f_0 \notin (f_1, f_2)$.

(iii) Let $l \leq 0$. Then $\nu = 0$ and $e = -l \deg H \geq 0$. Thus $\nu > 0$ or $e < 0$ implies that $l > 0$.

Proof. The statements on $e$ follow from the statements on $\nu$ by 5.7. Fix $e_0 = 0$ and let $\mathcal{E} = \mathcal{F}(-d_0)$ be the corresponding sheaf of linear forms. In determining $\nu$ we have to look for which invertible sheaves $\mathcal{L}$ there exist sections $0 \neq s \in \Gamma(Y, \mathcal{E} \otimes \mathcal{L})$. So the results follow all from the corresponding statements in 5.9 (ii). From 5.9 (ii) we get that $H^0(Y, \mathcal{F}(-d_0) \otimes \mathcal{L}) = 0$ for every invertible sheaf $\mathcal{L}$ of degree $-k$, $k > l \deg H$ (so $\mathcal{L}$ is automatically of negative degree), hence $\nu \leq l \deg H$.

Corollary 5.11. Let $K$ be an algebraically closed field of characteristic $p$ and let $R$ be a two-dimensional normal standard graded $K$-algebra and let $f_1, f_2$ be homogeneous parameters of degrees $d_1, d_2$. Then for $p = 0$ and for $p >> 0$ we have $(f_1, f_2)^* = (f_1, f_2) + R_{\geq d_1} + d_2$.

Proof. We have proved the inclusion $\supseteq$ in 5.4. Thus suppose that $f_0 \notin (f_1, f_2)$ and $f_0 \notin R_{\geq d_1} + d_2$. Due to 5.3 we have to show that the forcing divisor $Z = \mathbb{P}(f_1, f_2) \subset \mathbb{P}(f_1, f_2; f_0)$ is ample. The self intersection number of $Z$ is positive due to the second assumption. If $e < 0$, then the statement follows from 5.10 (ii) for $p >> 0$. If $e \geq 0$, then 5.10 (ii) shows that $l \deg H > e$ and again 5.3 (i) gives ampleness.

6. Examples

We give some examples of ruled surfaces and their forcing sections arising from tight closure problems. Let $R$ be a standard graded normal two-dimensional $K$-algebra, $Y = \text{Proj} R$, and let $f, g$ be homogeneous parameters and $h$ homogeneous of degrees $d_1, d_2, d_0$. Set $l = d_1 + d_2 - d_0$.

Example 6.1. Suppose that $h \in (f, g)$. Then the forcing sequence splits and
\[ V'_m \cong \mathbb{A}_Y(l - e_0) \times \mathbb{A}_Y(-e_0) \]
This is normalized for $e_0 = \text{max}(0,l)$ and then isomorphic to $A_Y \times A_Y(-|l|)$ and $\mathbb{P}(V')$ is the projective closure of the line bundle $A_Y(-|l|)$. The forcing section is either the zero section of the line bundle or the closure section. The $e$-invariant is $|l| \deg H$.

**Example 6.2.** Suppose that $h = 1$. Then

$$\text{Proj } R[T_0, T_1, T_2]/(fT_1 + gT_2 + T_0) \cong \text{Proj } R[T_1, T_2].$$

and $V'_m = A_Y(-e_1) \times A_Y(-e_2)$ is decomposable. The forcing sequence is

$$0 \to A_Y(d_1 + d_2 - e_0) \to A_Y(d_1 - e_0) \times A_Y(d_2 - e_0) \to A_Y(-e_0) \to 0,$$

where $v \mapsto (gv, -fv)$ and $(a, b) \mapsto -(af + bg)$. The sequence is normalized for $e_0 = \text{max}(d_1, d_2)$. Let $d_1 \geq d_2$. Then the normalized sequence is

$$0 \to A_Y(d_2) \to A_Y \times A_Y(d_2 - d_1) \to A_Y(-d_1) \to 0.$$ 

The $e$-Invariant of the ruled surface is $e = |d_2 - d_1| \deg H$.

**Example 6.3.** Let $f$ and $g$ be of the same degree $d$ and let $h = 1$. This yields the trivial ruled surface $Y \times \mathbb{P}^1$ and the forcing section is the graph of the meromorphic function $-f/g$. For let $e_0 = d$. Then we have the forcing sequence

$$0 \to A_Y(d) \xrightarrow{g-f} A_Y \times A_Y \xrightarrow{fT_1 + gT_2} A_Y(-d) \to 0,$$

and $fT_1 + gT_2 = 0$ is equivalent with $T_1/T_2 = -g/f$. If $d \geq 1$, then $h \not\in (f, g)^*$, and so we see via tight closure that the complement of the graph of a non constant meromorphic function is affine.

**Example 6.4.** Let $R = K[x, y]$, thus $Y = \mathbb{P}^1_K$. The ruled surface $\mathbb{P}(f, g; h)$ must be a Hirzebruch surface $\mathbb{P}(A_{p_1} \times A_{p_2}^{(k)})$ and we have to determine the number $k \leq 0$.

Let $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ be a basis of homogeneous relations for all the relations for $(f, g, h)$. Let $e_i$ and $e'_i$ be its degrees and suppose that $e_i \geq e'_i$. Set $k = e'_i - e_i$. The homomorphism

$$K[x, y, T_1, T_2, T_3]/(fT_1 + gT_2 + hT_3) \xrightarrow{\psi} K[x, y, U, S]$$

given by $T_i \mapsto a_i U + b_i S$, $i = 1, 2, 3$, is well defined and is homogeneous for deg $T_i = e_i$, deg $U = 0$, deg $S = e_i - e'_i = -k$. It induces a mapping

$$\text{Proj } K[x, y, U, S] \supseteq D_+(x, y) = A_{p_1} \times A_{p_2}^{(k)} \to V'_m.$$ 

We claim that this is an isomorphism. Since $(-g, f, 0)$ is a relation, there exist $r, s \in R$ such that $(-g, f, 0) = r(a_1, a_2, a_3) + s(b_1, b_2, b_3)$. Since $(0, -h, g)$ and $(-h, 0, f)$ are also relations it follows that $(g, f) \subseteq (a_3, b_3)$. Hence $(a_3, b_3)$ is $R_+$-primary and $a_3$ and $b_3$ do not have a common divisor. Therefore there exists $t \in R$ such that $r = t b_3$, $s = -t a_3$, hence we may write $f = t(b_3 a_3 - a_3 b_2)$. The mapping $\psi$ is locally on $D(f)$ given by a linear transformation $R_f[T_2, T_3] \to R_f[U, S]$ and its determinant is $b_3 a_3 - a_3 b_2$, which is a unit in $R_f$. The same is true on $D(g)$, so the induced mapping is an isomorphism.

The forcing sequence is

$$0 \to A_{p_1}^{(l)}(l - e_3) \to A_{p_1} \times A_{p_2}^{(k)}(e'_3 - e_3) \xrightarrow{a_3 U + b_3 S} A_{p_2}^{(k)}(-e_3) \to 0.$$ 

It follows that $l = e_3 + e'_3$. 

Let \( f = x^2, g = y^2 \). For \( h = xy \) we have \( e_3 = e'_3 = 1 \), but for \( h = x^2 \) we have \( e_3 = 2, e'_3 = 0 \).

Let \( f, g, h \in R \) be homogeneous of the same degree \( d \) and suppose that \( f, g \) are parameters and that \( h \notin (f, g) \). Let \( E \subseteq \Gamma(Y, \mathcal{O}_Y(d)) \) denote the linear system spanned by \( f, g, h \) and set \( e_i = 0 \). Then the sequence from \( 3.1(ii) \) is
\[
0 \to V'_d \to \mathbb{A}^3_Y \to \mathbb{A}^*_{Y(-d)} \to 0.
\]
The mapping sends \((P, t_1, t_2, t_3) \mapsto t_1 f(P) + t_2 g(P) + t_3 h(P)\) and this is zero if and only if the point \( P \) lies on the divisor defined by the section \( t_1 f + t_2 g + t_3 h \).

Therefore
\[
\mathbb{P}(V') = \{(P, D) : P \in D, D \in E\}
\]
and the ruled surface \( \mathbb{P}(V') \) is the incidence variety to the linear system \( E \).

Now let \( Y = V_+(F) \subset \mathbb{P}^2_K = \text{Proj} K[x, y, z] \) be a smooth curve and consider the linear system of lines. Thus the ruled surface associated to the vector bundle
\[
V'_1 = D_+(x, y, z) \subset \text{Proj} K[x, y, z]/(F)[T_1, T_2, T_3]/(xT_1 + yT_2 + zT_3)
\]
\( (\deg T_i = 0) \) consists of the pairs \((P, L) \), where \( L \) is a line through \( P \in Y \). Suppose that \( x, y \) are parameters for the curve. Then the point \( Q = (0, 0, 1) \) does not belong to \( Y \). The forcing section maps a point \( P \in Y \) to the line passing through \( P \) and \( Q \), since this is the line given by \( T_3 = 0 \). The self intersection number of the forcing section is \( \deg F \). If \( \deg F \geq 2 \), then \( z \notin (x, y)^* \) and the forcing section is ample.

**Example 6.5.** Let \( F \in K[x, y, z] \) be a homogeneous polynomial of degree three such that \( Y = \text{Proj} K[x, y, z]/(F) \) is an elliptic curve and suppose that \( x, y \) are parameters. Consider \( \mathbb{P}(x, y, z) \) and set \( e_1 = e_2 = e_3 = 0 \), and let \( \mathcal{E} \) be the sheaf of linear forms for this grading. The global linear forms \( T_1, T_2, T_3 \) are a basis for \( H^0(Y, \mathcal{E}) \). A global linear form \( s = a_1 T_1 + a_2 T_2 + a_3 T_3 \) \( (a_i \in K) \) belongs to \( \Gamma(Y, \mathcal{E} \otimes \mathcal{O}_Y(-P)) \) if and only if \( s|_{V'_0} = 0 \), and this is the case if and only if \( (a_1, a_2, a_3) \) is a multiple of \( (x(P), y(P), z(P)) \). So this can happen at most at one point. Hence \( \dim H^0(Y, \mathcal{E} \otimes \mathcal{O}_Y(-P)) = 1 \) and \( H^0(Y, \mathcal{E} \otimes \mathcal{O}_Y(-P - Q)) = 0 \). Therefore \( \mathcal{E} \otimes \mathcal{O}_Y(-P) \) is normalized and the \( e \)-invariant of \( \mathbb{P}(x, y, z) \) is
\[
e = -\deg (\mathcal{E} \otimes \mathcal{O}_Y(-P)) = -3 + 2 = -1.
\]

7. Examples over the complex numbers \( \mathbb{C} \)

The ruled surfaces together with their forcing sections arising from tight closure problems yield analytically interesting examples over the field of complex numbers \( K = \mathbb{C} \).

**Corollary 7.1.** Let \( F \) be a homogeneous polynomial of degree three such that \( R = \mathbb{C}[x, y, z]/(F) \) defines an elliptic curve \( Y = \text{Proj} R \). Let \( f, g, h \) be homogeneous such that \( f \) and \( g \) are parameters, \( \deg h = \deg f + \deg g \) and \( h \notin (f, g) \). Then the complement of the forcing section \( \mathbb{P}(f, g) \) in the ruled surface \( \mathbb{P}(f, g; h) \) over \( Y \) is not affine, but it is a complex Stein space. The same is true for the open subset \( D(R+) \subset \text{Spec} R[T_1, T_2]/(fT_1 + gT_2 + h) \).

**Proof.** The condition on the degrees shows that \( h \in (f, g)^* \) and that the open complement is not affine. Since \( h \notin (f, g) \), the forcing sequence, which is an extension of \( \mathcal{O}_Y \) by \( \mathcal{O}_Y \), does not split. Hence due to \( \mathbb{P} \) the complement is Stein. The corresponding statement for the subset in the cone follows. \( \square \)
Example 7.2. Let $R = \mathbb{C}[x, y, z]/(x^3 + y^3 + z^3)$ and $f = x, g = y, h = z^2$. Then $z^2 \notin (x, y)$, but $z^2 \in (x, y)^*$. The open subset

$$D(x, y) \subseteq \text{Spec } \mathbb{C}[x, y, z][T_1, T_2]/(x^3 + y^3 + z^3, xT_1 + yT_2 + z^2)$$

is not affine, but it is a complex Stein space.

Remark 7.3. The first construction of a non-affine but Stein variety was given by Serre using non-split extensions of $O_Y$ by $O_Y$ on an elliptic curve, see [28] or [4]. Thus we may consider this classical construction as a construction using forcing algebras.

On the other hand we have to remark that tight closure takes into account the subtle difference between affine and Stein. This shows that tight closure is a conception of algebraic geometry, not of complex analysis.

It is also possible to construct new counterexamples to the hypersection problem. The first counterexample was given by Coltoiu and Diederich in [7]. For this and related problems in complex analysis see [8].

Proposition 7.4. Let $R$ be a standard graded normal two-dimensional $\mathbb{C}$-algebra, let $f, g, h$ be homogeneous elements in $R$ such that $V(f, g) = V(R_+)$, $h \notin (f, g)$ and $\deg f + \deg g - \deg h < 0$. Then

$$W = D(R_+) \subset \text{Spec } R[T_1, T_2]/(fT_1 + gT_2 + h)$$

is not Stein (considered as a complex space), but it fulfills the assumption in the hypersection problem, i.e. for every analytic surface $S \subset \text{Spec } R[T_1, T_2]/(fT_1 + gT_2 + h)$ the intersection $W \cap S$ is Stein.

Proof. Due to [1,0] the superheight of $W$ is one, and [3] Theorem 5.1 gives that the assumption of the hypersection problem holds. The self intersection number of the forcing section on the corresponding ruled surface is negative, thus due to [11] this section is contractible as a complex space and therefore its complement is not Stein. Hence the subset $W$ is also not Stein, because it is a $\mathbb{C}^\times$-bundle over this complement.

Example 7.5. Consider $R = \mathbb{C}[x, y, z]/(x^4 + y^4 + z^4)$ and the forcing algebra for the elements $x, y, z^3$, hence $A = \mathbb{C}[x, y, z, T_1, T_2]/(x^4 + y^4 + z^4, xT_1 + yT_2 + z^3)$. Then $z^3 \notin (x, y)$ in $R$, but $z^3 \in (x, y)^*$, for the degree of the self intersection is $-4$. Therefore $W = D(R_+) \subset \text{Spec } A$ is not Stein, but for every analytic surface $S \subset \text{Spec } A$ the intersection $W \cap S$ is Stein.

8. PLUS CLOSURE IN POSITIVE CHARACTERISTIC

The theorem of Smith [29], [18] Theorem 7.1] says that the tight closure and the plus closure of a parameter ideal are the same. In this section we will give a proof for this in the two-dimensional graded case within our setting. Let $K$ be an algebraically closed field of positive characteristic $p$ and let $f$ and $g$ be homogeneous parameters in a two-dimensional standard graded normal $K$-algebra $R$. Let $h$ be another homogeneous element. If the complement of the forcing divisor $Z \subset \mathbb{P}(f, g; h)$ is not affine, then we must find a curve on the ruled surface disjoined to $Z$.

Suppose first that $l = \deg f + \deg g - \deg h < 0$. Then the cohomology class $\frac{H}{H}$ has degree $-l$ and becomes after a Frobenius morphism $\left(\frac{H}{H}\right)^q \in H^1(Y, \mathcal{O}_Y(-ql))$,
but $H^1(Y, \mathcal{O}_Y(-ql)) = 0$ for $q = p^e$ sufficiently large. Therefore the forcing sequence splits after a Frobenius morphism and $h$ belongs to the Frobenius closure of $(f, g)$. Thus we have to consider the case $l = 0$.

**Proposition 8.1.** Let $K$ be an algebraically closed field of characteristic $p > 0$. Let $R$ be a standard graded normal two-dimensional $K$-algebra and let $f, g, h$ be homogeneous elements such that $f$ and $g$ are parameters and such that $\deg h = \deg f + \deg g$. Then there exists a composition of a Frobenius morphism and an Artin-Schreier extension of $Y = \text{Proj} R$ such that the image of the cohomology class $h/fg \in H^1(Y, \mathcal{O}_Y)$ vanishes.

**Proof.** The Frobenius morphism $\Phi$ acts on $H^1(Y, \mathcal{O}_Y)$ $p$-linear yielding the so called Fitting decomposition $H^1(Y, \mathcal{O}_Y) = V_s \oplus V_n$ such that $\Phi|_{V_s}$ is bijective and $\Phi|_{V_n}$ is nilpotent [14, III §3]. Thus we may write $c = c_1 + c_2$, where $c_2$ becomes zero after applying a certain power of the Frobenius. Thus we may assume that $c = c_1 \in V_s$. Consider the Artin-Schreier sequence

$$0 \rightarrow \mathbb{Z}/(p) \rightarrow \mathcal{O}_Y \xrightarrow{\Phi - id} \mathcal{O}_Y \rightarrow 0,$$

which is exact in the étale topology. It yields the exact sequence

$$0 \rightarrow H^1_{et}(Y, \mathbb{Z}/(p)) \rightarrow H^1(Y, \mathcal{O}_Y) \xrightarrow{\Phi - id} H^1(Y, \mathcal{O}_Y) \rightarrow \ldots .$$

There exists a basis $c_j$ of $V_s$ such that $\Phi(c_j) = c_j$, see [23, §14]. Thus we may assume that $\Phi(c) - c = 0$ and we consider $c \in H^1_{et}(Y, \mathbb{Z}/(p))$. Hence $c$ represents an Artin-Schreier extension $Y'$ of $Y$ and the cohomology class $c$ vanishes on $Y'$, see [22, III §4].

**Remark 8.2.** We describe the Artin-Schreier extension appearing in the last proof explicitly. Let $c = h/f_1f_2$ and suppose that $c^p - c = 0$ in $H^1(Y, \mathcal{O}_Y)$. This means that $c^p - c = a_2 - a_1$ where $a_i \in \Gamma(U_i, \mathcal{O}_Y)$, $U_i = D_+(f_i) = \text{Spec} R_i$. Let $U_i' = \text{Spec} R_i[T_i]/(T_i^p - T_i + a_i)$, $i = 1, 2$. The transition function $T_1 \mapsto T_2 + c$ is due to

$$(T_2 + c)^p - (T_2 + c) + a_1 = T_2^p - T_2 + c^p - c + a_1 = T_2^p - T_2 + a_2$$

well defined and $U_1'$ and $U_2'$ glue together to a scheme $Y' \rightarrow Y$. The cohomology class in $Y'$ is $c = h/f_1f_2 = T_1 - T_2$, $T_i \in \Gamma(U_i', \mathcal{O}_{Y'})$, so that $c = 0$ in $H^1(Y', \mathcal{O}_{Y'})$.

**Remark 8.3.** Let $Y = \text{Proj} R$ as in [8, §4]. The $p$-rank of $Y$ is $\dim V_s \leq g(Y)$. This is the same as the $p$-rank of the jacobian of $Y$, see [23, §15]. The $p$-rank is 0 if and only if the plus closure (=tight closure) of any homogeneous parameter ideal is the same as its Frobenius closure.

Suppose now that $Y = \text{Proj} K[x, y, z]/(F)$ is an elliptic curve, thus $H^1(Y, \mathcal{O}_Y) \cong K$. An elliptic curve with $p$-rank 0 is called supersingular (or is said to have Hasse invariant 0). The criterion [13, Proposition IV.4.21] says that $Y$ is supersingular if and only if the coefficient of $(xyz)^{p-1}$ in $F^{p-1}$ is 0, or equivalently $F^{p-1} \in (x^p, y^p, z^p)$.

On the other hand, if the Hasse invariant is 1, then $F^{p-1} \notin (x^p, y^p, z^p)$, and the criterion of Fedder [18, Theorem 3.7] tells us that $K[x, y, z]/(F)$ is Frobenius pure.
9. Primary Relations and $e$-invariant

We give some further estimates of the $e$-invariant of a ruled surface arising from forcing parameter data which depend upon the existence of homogeneous primary relations of some total degree. We will apply this to tight closure problems of higher rank in the next section.

**Lemma 9.1.** Let $R$ be a normal standard graded $K$-algebra. Let $f_1, \ldots, f_n \in R$ be homogeneous of degrees $d_i$ and such that the $f_i, i \neq j$ are primary for every $j$. Let $g_i$ be a homogeneous primary relation for $f_i$ of total degree $k$. Then there exists a sequence $0 \to \mathcal{O}_Y \to R(k) \to \mathcal{L} \to 0$ such that $\mathcal{L}$ is locally free.

**Proof.** The relation $(g_1, \ldots, g_n)$ is a global element $\neq 0$ in $\Gamma(Y, R(k))$ and yields a subsheaf $\mathcal{O}_Y \subseteq \mathcal{R}(k)$. We show that the quotient is locally free, and consider $R_g, g = g_1$ (the $D_+(g_i)$ cover $Y$, since they are primary). $\Gamma(D_+(g), R(k))$ is the kernel of the mapping

$$(R_g)_{e_1} \oplus \ldots \oplus (R_g)_{e_n} \sum f_i \to (R_g)_k$$

$(d_i + e_i = k)$. If $(h_1, \ldots, h_n)$ is an element of the kernel, then

$$(h_1, \ldots, h_n) = \frac{h_1}{g_1}(g_1, \ldots, g_n) + (0, h_2 - \frac{h_1}{g_1}g_2, \ldots, h_n - \frac{h_3}{g_1}g_n).$$

The second summand is a relation for $(f_2, \ldots, f_n)$. Because these are also primary, $\mathcal{L}$ is locally free on $Y$. \hfill \square

**Proposition 9.2.** Let $R$ be a normal two-dimensional standard graded $K$-algebra. Let $f_1, f_2, f_3 \in R$ be homogeneous elements which are pairwise primary of degrees $d_1, d_2, d_3$. Suppose that $g_1, g_2, g_3 \in R$ is a primary homogeneous relation of total degree $k$. Set $a = \max(k - d_3, d_1 + d_2 - k)$. Then for the normalizing number $\nu$ and the $e$-invariant of $\mathbb{P}(f_1, f_2, f_3)$ we have

$$\nu \leq a \deg H \quad \text{and} \quad e \leq |2k - d_1 - d_2 - d_3| \deg H.$$ 

If $2k = d_1 + d_2 + d_3$, then the sheaf of linear forms $\mathcal{F}(-k)$ is normalized and the $e$-invariant of $\mathbb{P}(f_1, f_2, f_3)$ is 0.

**Proof.** From [1.1] we have the sequence $0 \to \mathcal{O}_Y \to R(k) \to \mathcal{L} \to 0$, where $\mathcal{L}$ is an invertible sheaf. We know that $\det \mathcal{R}(k) \cong \mathcal{O}_Y(-d_1 - d_2 - d_3 + 2k)$ from [5.2] and therefore $\mathcal{L} = \mathcal{O}_Y(-d_1 - d_2 - d_3 + 2k)$. The dual sequence for the linear forms is then $0 \to \mathcal{O}_Y(d_1 + d_2 + d_3 - 2k) \to \mathcal{F}(-k) \to \mathcal{O}_Y \to 0$. If $d_1 + d_2 - k \leq k - d_3$, then the sheaf on the left has degree $\leq 0$, hence $\mathcal{F}(-k) \otimes \mathcal{M}$ has no global sections $\neq 0$ whenever $\mathcal{M}$ has negative degree. Hence $\nu \leq (k - d_3) \deg H$.

If $d_1 + d_2 - k > k - d_3$, we tensorize with $\mathcal{O}_Y(-d_1 - d_2 - d_3 + 2k)$ and now $\mathcal{F}(k - d_1 - d_2 - d_3)$ is in between two invertible sheaves of degree $\leq 0$. Hence $\nu \leq (d_1 + d_2 - k) \deg H$. Thus by lemma [5.7] the $e$-invariant is $e = 2\nu - l \deg H \leq (2a - l) \deg H$, which gives the result.

If $d_1 + d_2 + d_3 = 2k$, then we have the forcing sequence $0 \to \mathcal{O}_Y \to \mathcal{F}(-k) \to \mathcal{O}_Y \to 0$, thus the sheaf $\mathcal{F}(-k)$ has sections $\neq 0$ and $\mathcal{F}(-k)$ is normalized with $\deg \mathcal{F}(-k) = 0$. \hfill \square

**Corollary 9.3.** Let $f_1, f_2, f_3 \in K[x, y, z]$ be homogeneous of degrees $d_1, d_2, d_3$ such that $d_1 + d_2 + d_3$ is even. Let $g_1, g_2, g_3 \in K[x, y, z]$ be homogeneous of degrees $e_1, e_2, e_3$ and such that $k = e_1 + e_2 = (d_1 + d_2 + d_3)/2$. Let $F = f_1g_1 + f_2g_2 + f_3g_3$
and set \( R = K[x, y, z]/(F) \). Suppose that the \( f_i \) are pairwise parameters for \( R \), that \( R \) is normal and that \( V(g_1, g_2, g_3) = V(x, y, z) \). Then the \( e \)-invariant of the ruled surface \( \mathbb{P}(f_1, f_2, f_3) \) over \( \text{Proj} \ R \) is \( e = 0 \).

**Proof.** All the conditions in proposition \( 9.2 \) are fulfilled. \( \square \)

**Example 9.4.** Consider a Fermat polynomial \( x^m + y^m + z^m \in K[x, y, z] \) and let \( R = K[x, y, z]/(x^m + y^m + z^m) \). Let \( f = x^{d_1}, g = y^{d_2}, h = z^{d_3} \) such that \( d_i < m \). If \( d_1 + d_2 + d_3 = 2m \), then the \( e \)-invariant of \( \mathbb{P}(x^{d_1}, y^{d_2}, z^{d_3}) \) is 0. Just take \((x^{m-d_1}, y^{m-d_2}, z^{m-d_3})\) as a primary relation.

**10. Projective bundles of higher rank over a curve**

In this last section we consider again a two-dimensional standard graded normal \( K \)-algebra \( R \) over an algebraically closed field \( K \), but now we look at the tight closure of three homogeneous primary elements \( (f_1, f_2, f_3) \). A forth element \( f_4 \) gives the projective bundle \( \mathbb{P}(f_1, f_2, f_3; f_4) \) of rank two over the smooth base curve \( Y = \text{Proj} \ R \) together with the forcing subbundle \( Z = \mathbb{P}(f_1, f_2, f_3) \), which is itself a ruled surface over \( Y \). We will need properties of these ruled surfaces to obtain results on \( \mathbb{P}(f_1, f_2, f_3; f_4) \). The third self intersection number of the forcing divisor is \( Z^3 = (d_1 + d_2 + d_3 - 2d_4) \deg H \), where \( H \) is the hyperplane section on \( Y \).

**Lemma 10.1.** Let \( R \) be a normal standard graded two-dimensional \( K \)-algebra. Let \( f_1, \ldots, f_4 \) be homogeneous elements of degrees \( d_i \) such that \( f_1 \) and \( f_2 \) are parameters. Set \( l = d_1 + d_2 + d_3 - 2d_4 \). Suppose that the \( e \)-invariant of the forcing subbundle \( Z = \mathbb{P}(f_1, f_2, f_3) \) is \( e \geq 0 \). If \( l \deg H \geq e \), then \( Z \) is a numerically effective divisor. If \( l \deg H > e \), then the pull back \( Z|_Z \) is ample.

**Proof.** The intersection of \( Z \) with a curve \( C \not\subseteq Z \) is \( \geq 0 \). The intersection of a curve \( C \) may be computed with the pull back of \( Z \) on \( Z \). We have \((Z|_Z)^2 = Z^3 = l \deg H \). Then the result follows from \( 5.3 \) (The ample criterion of \( 5.3 \) is true even if \( D = C_0 + bF \) is a priori not effective and a similar argument shows that \( D^2 \geq 0 \) is equivalent with \( D \) numerically effective). \( \square \)

**Corollary 10.2.** Let \( K \) be an algebraically closed field and let \( R \) be a normal standard graded two-dimensional \( K \)-algebra. Let \( f_1, \ldots, f_4 \) be homogeneous elements such that \( f_1 \) and \( f_2 \) are parameters and such that \( l \deg H > e \geq 0 \), where \( e \) denotes the \( e \)-invariant of the forcing subbundle \( Z = \mathbb{P}(f_1, f_2, f_3) \). Then \( f_4 \in (f_1, f_2, f_3)^* \) if and only if \( f_4 \in (f_1, f_2, f_3)^{+g\mathcal{F}} \).

**Proof.** This follows from \( 1.3 \) and \( 10.1 \). \( \square \)

**Theorem 10.3.** Let \( R \) be a normal standard graded two-dimensional \( K \)-algebra. Let \( f_1, f_2, f_3 \) be homogeneous elements such that \( f_1, f_2 \) are parameters and such that the \( e \)-invariant of \( \mathbb{P}(f_1, f_2, f_3) \) is \( e = 0 \). Let \( d_1 + d_2 + d_3 \) be even and let \( m = (d_1 + d_2 + d_3)/2 \). Then \( R_{\geq m} \subseteq (f_1, f_2, f_3)^* \).

**Proof.** Let \( f_4 \) be homogeneous of degree \( m \). Due to \( 10.1 \) we know that the forcing divisor \( Z \subset \mathbb{P}(f_1, f_2, f_3; f_4) \) is numerically effective. On the other hand the third self intersection number of \( Z \) is zero. Due to \( 1.8 \) the forcing divisor is not big and the complement is not affine. \( \square \)

**Corollary 10.4.** Let \( R \) be a normal two-dimensional standard graded \( K \)-algebra. Let \( f_1, f_2, f_3 \in R \) be homogeneous elements, which are pairwise primary of degrees
Suppose that \(d_1 + d_2 + d_3\) is even and set \(m = (d_1 + d_2 + d_3)/2\). Suppose that there exists a primary homogeneous relation of total degree \(m\) for \(f_1, f_2, f_3\). Then \(R_{\geq m} \subseteq (f_1, f_2, f_3)^*\).

**Proof.** This follows from [12] and [10.3].

**Example 10.5.** Let \(F = x^m + y^m + z^m\) and \(R = K[x, y, z]/(F)\). Then \(R_{\geq m} \subseteq (x^{d_1}, y^{d_2}, z^{d_3})^*\), where \(d_1 + d_2 + d_3 = 2m\) and \(d_i < m\).

For instance we get \(xyz \in (x^2, y^2, z^2)^*\) modulo \(x^3 + y^3 + z^3 = 0\). This was stated in [13] as an elementary example of what is not known in tight closure theory. The first proof was given in [23].

In the present case of a projective bundle of rank two we cannot characterize \(f_4 \not\in (f_1, f_2, f_3)^*\) by the ampleness of the forcing divisor, as the following example shows. The first example of an affine open subset in a three-dimensional smooth projective variety with no ample divisor on the complement was given by Zariski and described in [9].

**Example 10.6.** Let \(R = K[x, y]\) and consider on \(\mathbb{P}^1 = \text{Proj} R\) the projective bundle of rank two defined by the forcing data \(x^4, y^4, x^3y^3\). The third self intersection number of the forcing subbundle \(Z\) is zero, hence \(Z\) is not ample. But the complement of \(Z\) is affine, since \(x^2y^3 \not\in (x^4, y^4) = (x^4, y^4)^* = (x^4, y^4, x^3)^*\) in the regular ring \(K[x, y]\). Therefore \(Z\) is also big.

\(Z\) is not numerically effective: set \(e_4 = 0\) (and \(e_1 = e_2 = e_3 = 2\)), then \(Z\) is a hyperplane section. The pull back of \(Z\) on \(Z\) yields the hyperplane section on the ruled surface \(Z = \mathbb{P}(x^4, y^4, x^4)\) for this grading. Let \(E = Z|_Z\) denote this hyperplane section, let \(C = \mathbb{P}(x^4, y^4) \subseteq Z\) be the forcing section and let \(L\) be a disjoined section corresponding to \(x^4 \in (x^4, y^4)^*\). Then we know that \(C \sim E + 2\pi^*H (3.4\text{(iii)})\) and therefore \(E.L = C.L - 2\pi^*H.L = -2\pi^*H.L < 0\).

The divisor \(Z\) is also not semiample: We know that there exists a curve \(L\) such that \(Z.L < 0\). Let \(P \in L\) and suppose there exists an effective divisor \(D \sim aZ\) such that \(P \not\in D\). Then \(L \not\subseteq D\) yields a contradiction.

The next results deal with the plus closure in positive characteristic.

**Theorem 10.7.** Let \(K\) denote an algebraically closed field of characteristic \(p > 0\) and let \(R\) be a normal two-dimensional standard graded \(K\)-algebra. Let \(f_1, f_2, f_3 \in R\) be homogeneous elements which are pairwise primary. Let \(g_1, g_2, g_3 \in R\) be a primary homogeneous relation of total degree \(k\). Then for \(m = \max (k, d_1 + d_2 + d_3 - k)\) we have \(R_{\geq m} \subseteq (f_1, f_2, f_3)^{+p}\).

**Proof.** Let \(f_4 \in R_m\), and set \(e_4 = 0\). The forcing sequence for the relations, \(0 \to R(m) \to R'(m) \to O_Y \to 0\), corresponds to an element \(c \in H^1(Y, R(m))\). We have to show that there exists a smooth projective curve \(Y\) and a finite morphism \(\psi : Y \to Y\) such that \(\psi^*(c) \in H^1(Y, O_Y)\) is 0. The primary relation yields an exact sequence \(0 \to O_Y \to R(k) \to L \to 0\) due to [3.1]. Hence \(\det R(k) = O_Y(-d_1 - d_2 - d_3 + 2k) \cong L\).

Now \(-d_1 - d_2 - d_3 + 2k \geq 0\) and only if \(k \geq (d_1 + d_2 + d_3 - k)\). If \(k < (d_1 + d_2 + d_3 - k)\), we tensorize this sequence with \(O_Y(d_1 + d_2 + d_3 - 2k)\) and get \(0 \to O_Y(d_1 + d_2 + d_3 - 2k) \to R(d_1 + d_2 + d_3 - k) \to O_Y \to 0\).

In both cases we have an exact sequence \(0 \to O_Y(a) \to R(m) \to O_Y(b) \to 0\) such that \(a, b \geq 0\). Due to [3.1] and the preceding remarks there we know that
Proof. The condition $D$ be written as $D$ is a closed subscheme. Hence, if $\sim$ gives the result.

Hence, we may consider $c \in H^1(Y', \mathcal{O}_{Y'}(a))$ and again this vanishes after a finite mapping. □

Corollary 10.8. Let $K$ be an algebraically closed field of positive characteristic, let $f_1, f_2, f_3 \in K[x,y,z]$ be homogeneous elements of degrees $d_1, d_2, d_3$ and let $g_1, g_2, g_3 \in K[x,y,z]$ be primary homogeneous elements of degrees $e_1, e_2, e_3$ such that $m = e_1 + d_1$ and $2m \geq d_1 + d_2 + d_3$. Set $F = f_1g_1 + f_2g_2 + f_3g_3$ and $R = K[x,y,z]/(F)$ and suppose that $R$ is normal and that the $f_i$ are pairwise primary in $R$. Then $R_{\geq m} \subseteq (f_1, f_2, f_3)^{+\sigma}$. Proof. This follows from 10.7, since $m \geq d_1 + d_2 + d_3 - m$. □

Example 10.9. Let $F = x^m + y^m + z^m$. Then $R_{\geq m} \subseteq (x^{d_1}, y^{d_2}, z^{d_3})^{+\sigma}$ for $d_1 + d_2 + d_3 \leq 2m$ and $d_i < m$.

Example 10.10. Let $F \subseteq K[x,y,z]$ be a homogeneous equation for an elliptic curve. Then $\nu > 0$. Then the forcing divisor $D$ is big and there exists a linearly equivalent effective divisor $D \sim \nu D$ such that its complement is affine, see 4.7. The intersection of $\nu D$ and $D$ contains a lot of subtle information for the tight closure problem.

Proposition 10.12. Let $R$ be a normal two-dimensional standard graded $K$-algebra. Let $f_1, f_2, f_3$ be homogeneous primary elements and let $f_0$ be another homogeneous element. Suppose that $\nu > 0$. Then there exists an effective divisor $D$, $Z \sim D = H + F$, where $H$ is the horizontal component and $F$ the fiber components. Moreover, the following hold.

(i) If $H - Z \cap H$ is not affine, then $f_0 \notin (f_1, f_2, f_3)^{+\sigma}$.

(ii) If $H - Z \cap H$ is affine (this is fulfilled when the pull back $Z|_H$ is ample or when $H \cap Z$ contains components which lie in a fiber), then there does not exist a finite graded solution for the tight closure problem, i.e. $f_0 \notin (f_1, f_2, f_3)^{+\sigma}$.

Proof. The condition $\nu > 0$ means that there exists a positive divisor $L \subseteq Y$ such that there exists an effective divisor $D' \sim Z - \pi^*L$. Then $Z \sim D = D' + \pi^*L$ may be written as $D = H + F$, where $H$ is a projective subbundle and $F$ consists of fiber components.

We look at the intersection $Z \cap H$. (i) $H - Z \cap H = H \cap (\mathbb{P}(V') - Z) \subseteq \mathbb{P}(V') - Z$ is a closed subscheme. Hence, if $H - Z \cap H$ is not affine, then also $\mathbb{P}(V') - Z$ is not affine and $f_0 \notin (f_1, f_2, f_3)^{+\sigma}$.

(ii) We have to show that the forcing divisor $Z$ intersects every curve $C \not\subseteq Z$ positively. Due to 4.6 we only have to consider curves on $H$. Then the assumption gives the result. □
Example 10.13. Let \( Y = \mathbb{P}^1_K = \text{Proj} \ K[x, y] \) and consider the projective bundles corresponding to the forcing data \( x, y, 1; 1 \). Then \( \nu > 0 \) and \( Z \) is big. \( Z^3 = 2 > 0 \) and \( Z \) is numerically effective, but there exists a disjoined curve to \( Z \) (the solution section corresponding to \( 1 \in (x, y, 1) \)) and its complement is not affine.

We set \( e_1 = 0 \), thus \( e_1 = e_2 = -1 \). Eliminating \( T_3 \) in the forcing equations yields the splitting forcing sequence
\[
0 \rightarrow \mathbb{A}_Y(1) \times \mathbb{A}_Y(1) \rightarrow \mathbb{A}_Y(1) \times \mathbb{A}_Y(1) \times \mathbb{A}_Y \rightarrow \mathbb{A}_Y \rightarrow 0.
\]
The forcing subbundle is \( Z \cong \mathbb{P}^1 \times \mathbb{P}^1 \) given by the equation \( T_4 = 0 \). We have \( \mathcal{F}'(0) = \mathcal{O}_Y(1) \oplus \mathcal{O}_Y(1) \oplus \mathcal{O}_Y \) and also \( \mathcal{F}'(0) \otimes \mathcal{O}_Y(-1) \) has sections \( \neq 0 \). Therefore \( \nu > 0 \) and \( Z \) is big. A section is for example \( xT_1 \), thus a divisor \( D \) linearly equivalent to \( Z \) is given by \( D = H + F \), where \( H = \{ T_1 = 0 \} \) and \( F = \{ x = 0 \} \). \( H \) is a Hirzebruch surface \( \mathbb{P}(\mathcal{A}_Y \times \mathcal{A}_Y(-1)) \) (the blowing up of a projective plane). \( H \cap Z \) is a (horizontal) fiber on \( Z \cong \mathbb{P}^1 \times \mathbb{P}^1 \) and a line on \( H \) not meeting the exceptional divisor (which is also the solution section). The self intersection number of \( Z \cap H \) on \( H \) is
\[
(Z|_H)^2 = Z^2. H = Z^2(Z - F) = Z^3 - Z^2.F = 2 - 1 = 1,
\]
hence \( Z \) is numerically effective. (The self intersection of \( Z \cap H \) on \( Z \) is 0.)

Example 10.14. Let \( K \) be an algebraically closed field of positive characteristic \( p \geq 3 \) and consider
\[
R = K[x, y, z]/(x^4 + ay^4 + bz^4 + cxz^3 + dyz^3)
\]
where \( a, b, c, d \neq 0 \) are chosen such that \( Y = \text{Proj} R \) is smooth. We want to show that both cases described in [10.12] do actually occur depending on the coefficients. \( y \) and \( z \) are parameters and we consider the elements \( x^4, y^4, z^4 \) and \( xy^2z^3 \). First, let the homogeneous forcing algebra
\[
A = R[T_1, T_2, T_3, T_4]/(x^4T_1 + y^4T_2 + z^4T_3 + xy^2z^3T_4)
\]
be graded by \( e_1 = 1 \) (\( e_1 = e_2 = e_3 = 3 \)). From the curve equation and the homogeneous forcing equation we get
\[
z^3\left( -yz + cz + dys \right)T_1 + zT_3 + xy^2T_4 = y^4(aT_1 - T_2).
\]
This gives us the global linear form (of total degree 7) given by \( G \)
\[
-bz + cx + dy \quad \frac{y^4}{y^4}T_1 + \frac{z}{y^4}T_3 + \frac{x}{y^4}T_4 \ \text{on} \ \mathcal{D}_+(y) \ \text{and} \ \frac{a}{z^3}T_1 - \frac{1}{z^3}T_2 \ \text{on} \ \mathcal{D}_+(z),
\]
showing that \( \nu > 0 \). We change the grading and set \( e_4 = 0 \) and we consider the linear form \( zG \) of total degree 6. We have then \( \mathbb{P}(V) = \mathcal{V}_{+}(T_4) \sim \mathcal{V}_{+}(G) + \mathcal{V}_{+}(z) = H + F \) on \( \mathbb{P}(V') \) and we are in the situation of [10.12].

When does the intersection \( \mathbb{P}(V) \cap \mathcal{V}_{+}(z) \) have fiber components? If \( z \neq 0 \), then the equation for \( G \) on \( \mathcal{D}_+(y) \) does not vanish, thus there cannot be fiber components. So look at \( z = 0 \). Then \( x^4 + ay^4 = 0 \) and the equation for \( G \) becomes just \( \frac{cx + dy}{y^4}T_1 = 0 \). Thus there exists a fiber component if and only if \( cx + dy = 0 = x^4 + ay^4 \) has a solution, and this means \( (d/c)^4 = -a \).

Consider the equation \( x^4 - y^4 + z^4 + xz^3 + yz^3 = 0 \). This yields a smooth curve and the intersection has fiber components, hence every curve intersects the forcing divisor and therefore \( xy^2z^3 \not\in (x^4, y^4, z^4) + \mathcal{F} \). Does it belong to the tight closure?

The equation \( x^4 + y^4 + z^4 + xz^3 + yz^3 = 0 \) yields also a smooth curve, and here the intersection does not have a fiber component. Hence the intersection is
a section and its self intersection number on $H = V_+(G)$ is negative. Therefore the complement of it cannot be affine, hence the complement of the forcing divisor $P(V)$ is not affine, thus $xy^2z^3 \in (x^4, y^4, z^4)^*$. Does it also belong to the (graded) plus closure? (We have the primary relation $(z, az, bz + cx + cy)$ of total degree 5, hence we only know that $R_7 \subset (x^4, y^4, z^4)^{+gr}$ due to [0.7].

References

[1] J. Bingener: Holomorph-prävollständige Resträume zu analytischen Mengen in Steinschen Räumen. J. reine angew. Math. 285, 149-171, 1976.
[2] H. Brenner, On superheight conditions for the affineness of open subsets, J. Algebra 247 (2002), 37-56.
[3] H. Brenner, Tight closure and plus closure for cones over elliptic curves, submitted.
[4] H. Brenner, Slopes of vector bundles on projective curves and applications to tight closure problems, To appear in Trans. Am. Math. Soc.
[5] H. Brenner and S. Schröer, Ample families, multihomogeneous spectra, and algebraization of formal schemes, to appear in Pac. J. Math.
[6] W. Bruns and J. Herzog, “Cohen-Macaulay-Rings,” Cambridge Univ. Press, Cambridge, UK, 1993.
[7] Coltoiu, M., Diederich, K.: Open sets with Stein hypersurface sections in Stein spaces. Ann. Math. 145 (1997), 175-182.
[8] K. Diederich, Some aspects of the Levi problem: Recent Developments. In Geometric Complex Analysis (edited by Junjiro Noguchi et. al.), 1996, 163-181.
[9] J. E. Goodman, Affine open subsets of algebraic varieties and ample divisors, Ann. of Math. 89 (1969), 160-183.
[10] J. E. Goodman and A. Landman, Varieties proper over affine schemes, Inv. Math. 20 (1973), 267-312.
[11] H. Grauert, Über Modifikationen and exceptionelle analytische Mengen, Math. Ann. 146 (1962), 331-368.
[12] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique II. Inst. Hautes Études Sci. Publ. Math. 8 (1961).
[13] R. Hartshorne, Cohomological dimension of algebraic varieties, Ann. of Math. 88 (1968), 403-450.
[14] R. Hartshorne, “Ample Subvarieties of Algebraic Varieties,” Springer-Verlag, Berlin/-Heidelberg/New York, 1970.
[15] R. Hartshorne, “Algebraic Geometry,” Springer-Verlag, New York, 1977.
[16] M. Hochster, Solid closure, Contemp. Math. 159 (1994), 103-172.
[17] C. Huneke, “Tight Closure and Its Application,” AMS, 1996.
[18] C. Huneke, Tight Closure, Parameter Ideals, and Geometry, in Six Lectures on Commutative Algebra, Birkhäuser, Basel, 1998.
[19] C. Huneke, K. Smith, Tight closure and the Kodaira vanishing theorem, J. reine angew. Math. 484 (1997), 127-152.
[20] J. Kollar, “Rational Curves on Projective Varieties,” Springer-Verlag, Berlin, 1995.
[21] R. Lazarsfeld, “Positivity in Algebraic Geometry” (Preliminary Draft), 2001.
[22] J. S. Milne, “Étale Cohomology,” Princeton University Press, Princeton, New Jersey, 1980.
[23] D. Mumford, “Abelian Varieties.” Oxford University Press, Bombay, 1970.
[24] A. Neeman, Steins, affines and Hilbert’s fourteenth problem, Ann. of Math. 127 (1988). 229-244.
[25] A. Singh, A computation of tight closure in diagonal hypersurfaces, J. Algebra 203, No. 2 (1998), 579-589.
[26] K. E. Smith, Tight closure of parameter ideals, Inv. Math. 115 (1994), 41-60.
[27] K. E. Smith, Tight closure in graded rings, J. Math. Kyoto Univ 37, No 1 (1997), 35-53.
[28] H. Umemura, La dimension cohomologique des surfaces algébriques, Nagoya Math. J. 47 (1972), 155-160.

Mathematische Fakultät, Ruhr-Universität, 44780 Bochum, Germany

E-mail address: brenner@cobra.ruhr-uni-bochum.de