The Wasserstein geometry of non-linear $\sigma$ models and the Hamilton–Perelman Ricci flow

Mauro Carfora

Abstract. Non linear sigma models are quantum field theories describing, in the large deviations sense, random fluctuations of harmonic maps between a Riemann surface and a Riemannian manifold. Via their formal renormalization group analysis, they provide a framework for possible generalizations of the Hamilton–Perelman Ricci flow. By exploiting the heat kernel embedding introduced by N. Gigli and C. Mantegazza, we show that the Wasserstein geometry of the space of probability measures over Riemannian metric measure spaces provides a natural setting for discussing the relation between non–linear sigma models and Ricci flow theory. This approach provides a rigorous model for the embedding of Ricci flow into the renormalization group flow for non linear sigma models, and characterizes a non–trivial generalization of the Hamilton–Perelman version of the Ricci flow. We discuss in detail the monotonicity and gradient flow properties of this extended flow.

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5. A warped Gigli–Mantegazza heat kernel embedding
5.1. The induced G-M metric rescaling \((M, g) \mapsto (M, g_t^{(\omega)})\)
5.2. Warping \((M, g_t^{(\omega)})\) on \(M \times T^q\)
5.3. Harmonic energy rescaling and dilatonic action flow
6. The heat kernel embedding and Renormalization Group
6.1. A Wiener space associated to the heat kernel embedding
6.2. Deformed harmonic energy as a large deviation functional
7. Heat kernel embedding and Ricci flow
7.1. Geodesics and the Ricci curvature of \((M, g_t^{(\omega)})\)
7.2. Heat kernel induced Ricci flow
7.3. Monotonicity and gradient flow properties
Acknowledgment
References
1. Introduction

In 1982, inspired by the theory of harmonic maps, R. Hamilton published his landmark paper introducing Ricci flow [1]. Over the years, Hamilton’s work has been the point of departure and the motivating example for important developments in geometric analysis, most spectacularly in G. Perelman’s proof [2, 3, 4] of the Thurston geometrization program for three-manifolds [5, 6]. In the late 70’s and early 80’s, the Ricci flow independently appeared on the scene also in theoretical physics, in the framework of 2–dimensional Non-Linear sigma Model (\(NL\sigma M\)) theory. From the formal point of view of \(\infty\)–dimensional geometric analysis, \(NL\sigma M\)’s are quantum field theories describing, in the large deviations sense, random fluctuations of harmonic maps between a Riemannian surface \((\Sigma, \gamma)\) and a \(n\)–dimensional Riemannian manifold \((M, g)\). Prescient remarks of their connection with Ricci flow theory date back to Polyakov’s \(O(n)\) sigma model (1975) [7] and to J. Honerkamp’s Chiral multiloops paper (1972) [8]; they were made manifest in Daniel Friedan’s characterization [9] of the Ricci flow as the weak coupling limit of the renormalization group flow for \(NL\sigma M\). It must be noted that Friedan’s approach [9, 10, 11] was very geometric and exploited a sophisticated control over the interplay between the Ricci flow and the diffeomorphism group, actually introducing what later on came out to be known as the DeTurck version [12] of the Ricci flow. As recalled above, the rationale of the connection between \(NL\sigma M\) and Ricci flow may be traced back to the common roots in the geometry of harmonic maps. Attempts at a deeper understanding of this common structure are, however, very difficult to formalize since renormalization group techniques for non–linear quantum fields remain a challenge for mathematicians, (and physicists as well). We are dealing with a conceptual framework which connects the dynamical behavior of a physical theory to the analysis of flows in the space of its coupling parameters, and which must be adapted case by case, often with very different and sophisticated mathematical techniques. Speaking such different a language, one deeply rooted in geometrical analysis, the other more appropriate to the mathematical subtleties of quantum field theory, Ricci flow and renormalization group theory for non–linear \(\sigma\) model evolved independently until G. Perelman acknowledged the inspiring role played by the \(NL\sigma M\) effective action in his groundbreaking paper [2]. This has drawn renewed attention to the fact that the Ricci flow is the 1–loop approximation of the renormalization group (\(RG\)) flow for non–linear \(\sigma\) models, thus providing a framework for the Hamilton–Perelman theory which is open to generalizations [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. In spite of this renewed interest, real progress in obtaining viable extensions of the Ricci flow along this line of approach has been strongly hampered by the fact that the renormalization group flow for \(NL\sigma M\) is constructed perturbatively, under rather delicate geometrical and physical assumptions, often difficult to formalize. In particular, it is not clear how we should interpret the perturbative embedding of the Ricci flow in the renormalization group
of non-linear $\sigma$ models. As a matter of fact, the perturbative approach provides a hierarchy of truncated geometric flows, generated by powers of the Riemann tensor, which are very difficult to handle and to analyze when the flows develop singularities, (see however the recent papers [17], [21]). Moreover, the validity of this hierarchical extension of the Ricci flow is biased by the fact that it is not obvious in what sense a perturbative expansion approximates a full-fledged renormalization flow, (this is the linearization stability problem for the RG flow). The purpose of this paper is to address some of these problems in a geometrical analysis setting. In particular, we show that Wasserstein geometry and optimal transport offer an ideal framework for addressing the relation between Ricci flow and renormalization group from a novel point of view. We discuss the geometry of dilatonic non-linear $\sigma$ models in terms of maps between Riemannian surfaces $(\Sigma, \gamma)$ and Riemannian measure spaces $(M, g, d\omega)$, where the measure $d\omega$ describes the dilaton field. This formulation naturally induces a scale-dependent renormalization map of the theory provided by the heat kernel embedding of $(M, g, d\omega)$ into the Wasserstein space of probability measures $(\text{Prob}(M, g), d_{W}^{g})$ over $(M, g)$, where $d_{W}^{g}$ denotes the (quadratic) Wasserstein distance in $\text{Prob}(M, g)$. This construction provides a mathematical well-defined (toy) model for the RG flow for NL$\sigma$M theory. As we show explicitly, the geometrical deformation of $(M, g, d\omega)$ induced by the heat kernel embedding mimics a renormalization group flow for the corresponding dilatonic non-linear $\sigma$ model. Wasserstein geometry and optimal transport theory have recently drawn attention in attempts of extending the notion of Ricci curvature and Ricci flow to general metric spaces [24], [25], [26]. The analysis presented in this paper shows that Wasserstein metric and optimal transport also play a significant role in the rationale relating Ricci flow and renormalization group theory. In particular we prove that a natural extension of the (Hamilton–Perelman version of the) Ricci flow is defined by the RG scaling induced by the heat kernel embedding. This result extends in a non-trivial way a remarkable paper by Nicola Gigli and Carlo Mantegazza [27] who have been able to show that the heat kernel embedding in the Wasserstein space of probability measure has a tangent at the origin which is the Ricci flow. In our case we deal with a weighted heat kernel (the weight being associated with the dilatonic measure $d\omega$), but we found that the results in [27] can be naturally extended to this more general case. Even if the (weighted) heat kernel renormalization mimics many properties of the renormalization group flow for NL$\sigma$M, it should be clearly stressed that it is not the physical RG flow. The way this latter is perturbatively characterized suggests indeed that the actual RG flow is a singular perturbation of the flow defined by the heat kernel embedding. The situation is akin to that we encounter in quantum field theory where the quantum fluctuations of an interacting field can be perturbatively considered as a (singular) perturbation of the underlying formal Gaussian measure associated to the free field. To support this picture we discuss in detail the problem of characterizing a generalized Wiener measure (à la Gross) associated with our weighted
heat kernel embedding, drawing a parallel with the formal background field quantization of NLσM. Regardless of the status of the (weighted) heat kernel embedding as a toy (but full–fledged) renormalization group flow for NLσM, a result of this paper is that the heat kernel embedding of the metric measure space \((M, g, d\omega)\) in the Wasserstein space \((\text{Prob}(M, g), d_{\text{W}})\) emerges as a natural candidate for a geometric flow generalizing the (Hamilton–Perelman) Ricci flow. In particular we prove monotonicity and gradient flow properties of the heat kernel induced Ricci flow, and show that these are genuine extensions of the analogous properties in Ricci flow theory. The whole construction is based on a detailed analysis of the weighted heat kernel embedding and of its geometry in \((\text{Prob}(M, g), d_{\text{W}})\), and it suggests potential extensions of the theory to general metric measure spaces. Again, this is in line with the results of [27] which explores in detail the potentialities of the standard heat kernel embedding as the avatar of the Ricci flow in the non smooth setting. To what extent this is a viable alternative remains, however, a formidable open problem in geometrical analysis.

As for what concerns the structure of the paper, the table of contents is self explanatory. Note that when appropriate we shall often use the acronyms NLσM and RG for Non–Linear σ Model and Renormalization Group, respectively. To set basic notation, we let \(M\) denote a \(C^\infty\) compact \(n\)–dimensional manifold, \((n \geq 2)\), without boundary, and let \(\text{Diff}(M)\) and \(\text{Met}(M)\) respectively be the group of smooth diffeomorphisms and the open convex cone of all smooth Riemannian metrics over \(M\). For any \(g \in \text{Met}(M)\), we denote by \(\nabla_{(g)}\), (or \(\nabla\) if there is no danger of confusion), the Levi–Civita connection of \(g\), and let \(\mathcal{R}(g) = R_{klm} \partial_i \otimes dx^k \otimes dx^l \otimes dx^m\), \(\mathcal{Ric}(g) = R_{ab} dx^a \otimes dx^b\) and \(\mathcal{R}(g)\) be the corresponding Riemann, Ricci and scalar curvature operators, respectively.

2. Preliminaries: the geometrical setting

Non-linear σ-models are strictly related to harmonic map theory and to the geometry of the space of maps

\[
\phi : (\Sigma, \gamma) \longrightarrow (M, g)
\]

between a 2–dimensional smooth orientable surface without boundary \((\Sigma, \gamma)\), with Riemannian metric \(\gamma\), and a Riemannian manifold \((M, g)\). Any such map can be thought of as endowed with the minimal regularity allowing for the characterization of a (generalized) harmonic energy functional. This can make difficult to work in local charts on the target manifold \(M\), even at a physical level of rigor. A way out is to use the Nash embedding theorem [28], [29], according to which any compact Riemannian manifold \((M, g)\) can be isometrically embedded into some Euclidean space \(\mathbb{E}^m := (\mathbb{R}^m, \delta)\) for \(m\) sufficiently large, (e.g., \(m \geq \frac{1}{2} n(n+1) + n\) for free local isometric embeddings [28]). If \(J : (M, g) \hookrightarrow \mathbb{E}^m\), is any such an embedding we can define the Sobolev
B > 0 such that $D$ try. Explicitly, we assume that for every $\phi \in H$ and to carry out explicit computations, we further require that any $\phi \in H$.

The space of smooth maps $C^\infty(\Sigma, M)$ is dense $[31]$ in the Sobolev space $H^1(\Sigma, M)$, however maps of class $H^1(\Sigma, M)$ are not necessarily continuous, and to carry out explicit computations, we further require that any such $\phi \in H^1(\Sigma, M)$ is localizable, (cf. $[32]$, Sect. 8.4), and of bounded geometry. Explicitly, we assume that for every $x_0 \in \Sigma$ there exists a metric disks $D(x_0, \delta) := \{x \in \Sigma | d_g(x_0, x) \leq \delta\} \subset \Sigma$, of radius $\delta > 0$ and a metric ball $B(r, p) := \{z \in M | d_g(p, z) \leq r\} \subset (M, g)$ centered at $p \in M$, of radius $r > 0$ such that $\phi(D(x_0, \delta)) \subset B(r, p)$, with

$$r < r_0 := \min\left\{\frac{1}{3} \text{inj}(M), \frac{\pi}{6 \sqrt{\kappa}}\right\},$$

where $\text{inj}(M)$ and $\kappa$ respectively denote the injectivity radius of $(M, g)$, and the upper bound to the sectional curvature of $(M, g)$, (we are adopting the standard convention of defining $\pi/2 \sqrt{\kappa} = \infty$ when $\kappa \leq 0$). Under such assumptions, one can use local coordinates also for maps in $H^1(\Sigma, M)$. In particular for any $\phi \in H^1(D(x_0, \delta), M)$ we can introduce local coordinates $x^a$, for points in $(D(x_0, \delta), \Sigma)$, and $y^k = \phi^k(x)$, $k = 1, \ldots, n$, for the corresponding image points in $\phi(D(x_0, \delta)) \subset M$, and, by using a partition of unity, work locally in the smooth framework provided by the space of smooth maps

$$\text{Map}(\Sigma, M) := \{\phi : \Sigma \to M, x^a \mapsto y^k = \phi^k(x) \in C^\infty(D(x_0, \delta), M)\}.$$

(2.5)

Under these regularity hypotheses, we can introduce the pull–back bundle $\phi^{-1}TM$ whose sections $v \equiv \phi^{-1}V := V \circ \phi$, $V \in C^\infty(M, TM)$, are the vector fields over $\Sigma$ covering the map $\phi$. If $T^*\Sigma$ denotes the cotangent bundle to $(\Sigma, \gamma)$, then the differential $d\phi = \frac{\partial \phi^i}{\partial x^a} dx^a \otimes \frac{\partial}{\partial y^i}$ can be interpreted as a section of $T^*\Sigma \otimes \phi^{-1}TM$, and its Hilbert–Schmidt norm, in the bundle metric

$$\langle \cdot, \cdot \rangle_{T^*\Sigma \otimes \phi^{-1}TM} := \gamma^{-1}(x) \otimes g(\phi(x))(\cdot, \cdot),$$

(2.6)

is provided by, (see e.g. $[32]$),

$$\langle d\phi, d\phi \rangle_{T^*\Sigma \otimes \phi^{-1}TM} = \gamma^\ mu(x) \frac{\partial \phi^i(x)}{\partial x^\mu} \frac{\partial \phi^j(x)}{\partial x^\nu} g_{ij}(\phi(x)) = tr_\gamma(x) (\phi^* g).$$

(2.7)
The Wasserstein geometry of non-linear $\sigma$ models

The corresponding density

$$e(\phi) \, d\mu_\gamma := \frac{1}{2} \left< d\phi, d\phi \right>_{\Gamma \otimes \phi^{-1} T M} \, d\mu_\gamma = \frac{1}{2} \text{tr}_{\gamma(x)} (\phi^* g) \, d\mu_\gamma ,$$

(2.8)

where $d\mu_\gamma$ is the volume element of the Riemannian surface $(\Sigma, \gamma)$, is conformally invariant under two-dimensional conformal transformations

$$(\Sigma, \gamma_{\mu\nu}) \mapsto (\Sigma, e^{-\psi} \gamma_{\mu\nu}) , \quad \psi \in C^\infty(\Sigma, \mathbb{R}) , \quad (2.9)$$

and defines the harmonic map energy density associated with $\phi \in \text{Map}(\Sigma, M)$. In particular, the critical points of the functional

$$E[\phi, g]_{(\Sigma, M)} := \int_\Sigma e(\phi) \, d\mu_\gamma ,$$

(2.10)

are the harmonic maps of the Riemann surface $(\Sigma, [\gamma])$ into $(M, g)$, where $[\gamma]$ denotes the conformal class of the metric $\gamma$.

Remark 2.1. More generally, for $\phi \in H^1(\Sigma, M)$, the critical points of $E[\phi, g]_{(\Sigma, M)}$ define weakly harmonic maps $(\Sigma, \gamma) \to (M, g)$. However, for surfaces, weakly harmonic maps are harmonic.

In terms of the possible geometrical characterization of $(\Sigma, \gamma)$ and $(M, g)$, important examples of harmonic maps include harmonic functions, geodesics, isometric minimal immersions, holomorphic (and anti–holomorphic) maps of Kähler manifolds. It is worthwhile to observe that in such a rich panorama, also the seemingly trivial case of constant maps plays a basic role for the interplay between Ricci flow and (the perturbative quantization of) non–linear $\sigma$ models.

In order to make the paper self–contained, we conclude this preliminary part with a capsule of the physical framework relating non–linear $\sigma$ models to Ricci flow.

2.1. NL$\sigma$M and Ricci flow in a nutshell

In the above geometrical framework, let us consider on $\text{Map} (\Sigma, M)$ the action functional

$$S[\gamma, \phi; a, g, f] = a^{-1} \int_\Sigma \left[ \text{tr}_{\gamma(x)} (\phi^* g) + a \, f(\phi) \, K \right] \, d\mu_\gamma ,$$

(2.11)

where $a > 0$ is a parameter with the dimension of a length squared, $f : M \to \mathbb{R}$ is a function on $M$, and $K$ is the Gaussian curvature of $(\Sigma, \gamma)$. Geometrically the energy scale of the action $S[\gamma, \phi; a, g, f]$ is set by the dilaton coupling $[ f(\phi) \, K ]$ and by the length scale of the target space metric $g_{ab}$, i.e. $| Rm(g, y) | a$, where $| Rm(g, y) | := [R_{iklm} R^{iklm}]^{1/2}$. It follows that the background fields $f \in C^\infty(M, \mathbb{R})$ and $g \in \text{Met}(M)$ play the role of point dependent coupling parameters $\alpha$ on $M$,

$$\alpha := (a, g, f) ,$$

(2.12)

controlling the energetics of the action $S[\gamma, \phi; \alpha]$. In quantum theory this fiducial action, together with its possible deformations, describes a family of
2–dimensional QFTs known as (dilatonic) non–linear \(\sigma\)–models. They find applications ranging from condensed matter physics to string theory.

Remark 2.2. Further coupling terms can be added to the action \(S[\gamma, \phi; \alpha]\), (see e.g. \[34\]), in particular \(a^{-1} \int_{\Sigma} U(\phi) \, d\mu_{\gamma}\) and \(a^{-1} \int_{\Sigma} \phi^{*} \Xi\) where \(U \in C^{\infty}(M, \mathbb{R})\) and \(\Xi \in C^{\infty}(M, \wedge^{2} T^{*}M)\) is a 2–form on \(M\). In order to discuss the role that Wasserstein geometry plays in the interplay between Ricci flow and non–linear \(\sigma\) models we limit our analysis to the dilatonic field \(f\).

The (Euclidean) QFT associated with (2.11) is characterized by the measure on \(\text{Map}(\Sigma, M)\) formally defined by

\[
D_{\alpha}[\phi] \, e^{-S[\gamma, \phi; \alpha]},
\]

and by its moment generating function, (correlations). Here \(D_{\alpha}[\phi]\) is a (non–existant) functional measure on \(\text{Map}(\Sigma, M)\), possibly depending on the couplings \(\alpha\), and normalized so that (2.13) is a probability measure. The somewhat fanciful expression (2.13) hardly makes sense, even at a physical level of rigour, if we do not devise a way of controlling the spectrum of fluctuations of the fields \(\phi \in \text{Map}(\Sigma, M)\). Indeed, if we denote by \(\mathcal{C}\) the space in which the couplings \(\alpha\) are allowed to vary, then the fundamental problem concerning (2.13) is to introduce a filtration\(^1\) in \(\{\text{Map}(\Sigma, M) \times \mathcal{C}, e^{-S[\gamma, \phi; \alpha]} D_{\alpha}[\phi]\}\),

\[
\mathcal{RG}^{\tau} : [\text{Map}(\Sigma, M) \times \mathcal{C}] \longrightarrow [\text{Map}(\Sigma, M) \times \mathcal{C}]
\]

\[
(\phi, \alpha) \quad \mapsto \quad \mathcal{RG}^{\tau}(\phi, \alpha) = (\phi_{\tau}; \alpha(\tau)),
\]

which, as we vary the scale of distances \(\tau\) at which we probe the Riemannian surface \(\Sigma\), allows to tame the energetics of the fluctuations of the fields \(\phi : \Sigma \rightarrow M\) in terms of a renormalization of the couplings \(\alpha \mapsto \alpha(\tau)\). Such a filtration characterizes the renormalization group flow associated with the measure space \(\{\text{Map}(\Sigma, M) \times \mathcal{C}, e^{-S[\gamma, \phi; \alpha]} D_{\alpha}[\phi]\}\).

In order to describe this procedure in physical terms, select two scales of distances, say \(\Lambda^{-1}\) and \(\Lambda'^{-1}\), (one can equivalently interpret \(\Lambda\) and \(\Lambda'\) as the respective scales of momentum in the spectra of field fluctuations), with \(\Lambda'^{-1} > \Lambda^{-1}\). The general idea, central in K.G. Wilson’s analysis of the renormalization group flow, is to assume that, at least for \((\Lambda' \setminus \Lambda)\) small enough, we can put the \(\mathcal{RG}^{\tau}\) push–forward of the functional measure \(D_{\alpha}[\phi] \, e^{-S[\gamma, \phi; \alpha]}\), viz. \(\mathcal{RG}^{\Lambda'} \circ \mathcal{RG}^{\Lambda'}(D_{\alpha}[\phi]) \, e^{-\mathcal{RG}^{\Lambda'} S[\gamma, \phi; \alpha]}\), in the same form as the original functional measure, except for a small modification of the couplings \(\alpha\). Explicitly, let \(\Lambda' = e^{-\tau} \Lambda\), with \(0 < \tau < 1\), and assume that for every such \(\tau\) there exists a corresponding coupling \(\alpha + \delta \alpha\) such that the following identity holds

\[
\mathcal{RG}_{\tau}^{\Lambda'}(D_{\alpha}[\phi]) \, e^{-\mathcal{RG}^{\tau} S[\gamma, \phi; \alpha]} = D_{\alpha + \delta \alpha}[\phi] \, e^{-S[\gamma, \phi; \alpha + \delta \alpha]},
\]

\(^1\)We use here the term filtration in a rather loose sense. The relevant notion of progressively measurable maps over a non–decreasing family of sigma algebras in the appropriate functional space will be discussed in Section 6.
where we have denoted $\mathcal{RG}_\tau^\Lambda$ the push-forward action of the map $\mathcal{RG}^{\Lambda\Lambda'}$ for $\Lambda' = e^{-\tau} \Lambda$. In other words, we assume that an infinitesimal change in the cutoff can be completely absorbed in an infinitesimal change of the couplings. If this equation is valid at least to some order in $\tau$, we can iteratively use it to see how the couplings $\alpha$ are affected by a finite change of the cutoff. If the theory is, along the lines sketched above, renormalizable by a renormalization of the couplings, many of its properties can be obtained by the analysis of the $\beta$–flow vector field defined on the space of couplings $\mathcal{C}$ by

$$\beta(\alpha(\tau)) := -\frac{\partial}{\partial \tau} \alpha(\tau),$$

(2.16)

which formally appears as a natural geometrical flow on the measure space

$$\{\text{Map}(\Sigma, M) \times \mathcal{C}; D\alpha(\tau)[\phi]\}.$$  

According to (2.12), the space of couplings $\mathcal{C}$ for the dilatonic non-linear $\sigma$-model action (3.9) can be identified with the product of $C^\infty(M, \mathbb{R})$, (where the dilatonic coupling $f$ varies), with the infinite–dimensional stratified manifold of Riemannian structures on $M$, (parametrizing the $\text{Diff}(M)$-classes of metric couplings $g$), modulo overall length rescalings, (associated with the choice of the length parameter $a^{1/2}$), i.e.

$$\mathcal{C} = C^\infty(M, \mathbb{R}) \times \frac{\text{Met}(M)}{\text{Diff}(M) \times \mathbb{R}^+}$$  

(2.17)

where $\mathbb{R}^+$ denotes the group of rescalings defined by $a^{1/2} \mapsto \lambda a^{1/2}$, for $\lambda$ a positive number. As we have recalled, the true dimensionless coupling constants of the theory are the ratio of the length scale of the target space metric $g_{ab}(\phi(x))$ to $a$, and the dilatonic coupling $f(\phi(x))$. This remark has two important consequences:

(i) It implies that as long as the curvature of target Riemannian manifold $(M, g)$ is small as seen by $(\Sigma, \gamma(x))$, viz. $|\text{Rm}(g, \phi(x))| a << 1$, $\forall x \in \Sigma$, then the formal measure $D(g, f)[\phi] e^{-S[\phi; \alpha]}$ is concentrated around the minima of the fiducial action $S[\phi; \alpha_f = 0]$, i.e. the constant maps $x \mapsto \phi(x) = \phi_0$.

(ii) It also implies that the dilatonic coupling $f$ fluctuates around the constant value $f_0 := f(\phi_0)$. In such a framework one can control the nearly Gaussian fluctuations $\delta \phi$ of $\phi$, and one can address the (perturbative) renormalization group analysis described above [11, 35, 36, 37] so as to obtain a perturbative $\beta$–flow for the coupling fields $\alpha = (g, af)$. At leading order one gets

$$\frac{\partial}{\partial \tau} g_{ik}(\tau) = 2a \left( R_{ik}(\tau) + 2\nabla_i \nabla_k f(\tau) \right) + O\left(|\text{Rm}| a^2\right),$$

(2.18)

$$\frac{\partial}{\partial \tau} f(\tau) = c_0 - 2a \left( \frac{1}{2} \Delta f(\tau) - |\nabla f(\tau)|^2 \right) + O\left(|\text{Rm}| a^2\right),$$

(2.19)

where $R_{ik}(\tau)$ denotes the Ricci tensor of $(M, g(\tau))$, and where $c_0 := (\dim M - 26)/6$ is the central charge, playing in QFT the role of dimension. If we set $	ilde{f} := 2f - 2c_0 \tau$ and pass to the dimensionful variable $t := -a\tau$, then, as
10 Mauro Carfora

\( a \downarrow 0 \), the renormalization group flow \([2,18]\) reduces to Hamilton’s Ricci flow deformed \([12]\) by the action of the \(t\)-dependent diffeomorphism generated by the vector field \( W^i(t) := g^{ik} \nabla_k \hat{f}(t) \), i.e.,

\[
\frac{\partial}{\partial t} g_{ik}(t) = -2 \left( R_{ik}(t) + \nabla_i \nabla_k \hat{f}(t) \right),
\]

\[
\frac{\partial}{\partial t} \hat{f}(t) = \Delta \hat{f}(t) - |\nabla \hat{f}(t)|^2.
\]

(2.20)

(2.21)

Remark 2.3. If along the above RG flow we impose the (somewhat unphysical) dilatonic measure preservation constraint

\[
\frac{\partial}{\partial t} e^{-\hat{f}(t)} \ d\mu = 0 \rightarrow \left( |\nabla \hat{f}(t)|^2 - 2\Delta \hat{f}(t) - R(g(t)) \right) e^{-\hat{f}(t)} \ d\mu = 0.
\]

(2.22)

Inserted back in \(2.21\), this constraint gives rise to the Perelman version of the Ricci flow

\[
\frac{\partial}{\partial t} g_{ik}(t) = -2 \left( R_{ik}(t) + \nabla_i \nabla_k \hat{f}(t) \right),
\]

\[
\frac{\partial}{\partial t} \hat{f}(t) = -\Delta \hat{f}(t) - R(g(t)),
\]

(2.23)

(2.24)

which couples the forward Ricci flow with a backward parabolic evolution of the dilatonic potential.

3. The Wasserstein geometry of the dilatonic field

Let \((M, g, d\omega)\) be a \(n\)-dimensional compact Riemannian metric measure space \([38], [39]\), i.e. a smooth orientable manifold, without boundary, endowed with a Riemannian metric \(g\) and a positive Borel measure \(d\omega \ll d\mu_g\), absolutely continuous with respect to the Riemannian volume element, \(d\mu_g\).

Strictly speaking, \((M, g, d\omega)\) characterizes a weighted Riemannian manifold, (or Riemannian manifold with density), the corresponding metric measure space being actually defined by \((M, d_g(\cdot, \cdot), d\omega)\), where \(d_g(x, y)\) denotes the Riemannian distance on \((M, g)\). By a slight abuse of notation, we shall use \((M, g, d\omega)\) and \((M, d_g(\cdot, \cdot), d\omega)\) interchangeably. In such a framework, the two–dimensional dilatonic non–linear \(\sigma\) model is defined by a rather natural extension of the harmonic energy functional \(E[\phi, g]\) to \((M, g, d\omega)\). We start by recalling that the set \(\Meas(M)\) of all smooth Riemannian metric measure spaces can be characterized as

\[
\Meas(M) := \{ (M, g; d\omega) \mid (M, g) \in \Met(M), \ d\omega \in \mathcal{B}(M, g) \},
\]

(3.1)

where \(\mathcal{B}(M, g)\) is the set of positive Borel measure on \((M, g)\) with \(d\omega \ll d\mu_g\). Since in the compact–open \(C^\infty\) topology \(\Met(M)\) is contractible, the space \(\Meas(M)\) fibers trivially over \(\Met(M)\). In particular, the fiber \(\pi^{-1}(M, g)\) can be identified with the set of all (orientation preserving) measures \(d\omega \ll d\mu_g\) over the given \((M, g)\),

\[
\Meas(M, g) := \{ d\omega \in \Meas(M) : d\omega \ll d\mu_g \},
\]

(3.2)
endowed with the topology of weak convergence. There is a natural action of the group of diffeomorphisms $Diff(M)$ on the space $\text{Meas}(M)$, defined by

$$\text{Diff}(M) \times \text{Meas}(M) \longrightarrow \text{Meas}(M) \quad (3.3)$$

$$(\varphi; \ g, d\omega) \longmapsto (\varphi^* g, \ \varphi^* d\omega),$$

where $(\varphi^* g, \ \varphi^* d\omega)$ is the pull–back under $\varphi \in \text{Diff}(M)$. The Radon–Nikodym derivative $\varphi^* (g, d\omega) := \frac{d\omega}{d\mu_g}$ is a local Riemannian measure space invariant \[40\] under this action, i.e.

$$\varphi(\varphi^* g, \ \varphi^* (d\omega)) = \varphi^* \varphi(g, d\omega) = \frac{d\omega}{d\mu_g} \circ \varphi, \ \ \forall \varphi \in \text{Diff}(M), \quad (3.4)$$

and we can introduce the

**Definition 3.1.** The geometrical dilaton field associated with the Riemannian metric measure space $(M, g, d\omega)$ is defined by the map

$$f : \text{Meas}(M) \longrightarrow C^\infty(M, \mathbb{R}) \quad (3.5)$$

$$(M, g, d\omega) \longmapsto f(M, g, d\omega) := - \ln \left( V_g(M) \frac{d\omega}{d\mu_g} \right),$$

where $V_g(M) := \int_M d\mu_g$, and we can parametrize $\text{Meas}(M, g)$ according to

$$\text{Meas}(M, g) = \left\{ d\omega = e^{-f} \frac{d\mu_g}{V_g(M)} : f \in C^\infty(M, \mathbb{R}) \right\}. \quad (3.6)$$

**Remark 3.2.** In Riemann measure spaces, (and Ricci flow theory), sometimes it is necessary to restrict the action of $\text{Diff}(M)$ to the metric $g$ alone and leave the measure $d\omega$ fixed, (cf. section 5 in \[40\]). Absolute continuity of $d\omega$ with respect to $d\mu_g$ implies that in such a case $f$ is not a scalar and under the action of a diffeomorphism $\varphi \in \text{Diff}(M)$ it transforms as a density according to

$$e^{-f(\varphi^* g, d\omega)} = \left( \text{Jac}_{d\mu_g} \varphi^{-1} \right) e^{-f}, \quad (3.7)$$

where $\text{Jac}_{d\mu_g} \varphi^{-1}$ denotes the Jacobian of $\varphi^{-1}$ with respect to $d\mu_g$. However, a subtler interpretation is possible \[40\] which still preserves the invariant nature of the dilaton. To wit, according to (3.7) and Moser theorem \[41\], one can consider $\varphi \in \text{Diff}(M)$ as generating an active transformation on $\text{Meas}(M, g)$ mapping the measure $d\omega$ into another measure $d\omega' := (\varphi^{-1})^* (d\omega)$, (with the same total mass $\int_M d\omega$). By composing this mapping on $\text{Meas}(M, g)$ with the induced pull–back action on the resulting measure we get from (3.4) that the Radon–Nikodym derivative $\varphi(g, d\omega)$ transforms according to

$$\varphi(g, d\omega) \longmapsto \varphi(\varphi^* g, \ \varphi^* (d\omega')) = \varphi^* \varphi(g, d\omega') \quad (3.8)$$

$$= \varphi^* \varphi(g, (\varphi^{-1})^* (d\omega)) = \left( \text{Jac}_{d\mu_g} \varphi^{-1} \right) \frac{d\omega}{d\mu_g},$$

which is consistent both with (3.7) and the local Riemannian measure space invariant nature of the dilaton. If not otherwise stated we shall consider the natural action on $(M, g, d\omega)$ given by (3.3).
With these remarks along the way, we can geometrically characterize the action functional \((2.11)\) according to

**Definition 3.3.** (The dilatonic NL\(\sigma\)M action). If \(\phi \in \mathcal{H}^1(\Sigma, M)\) is a localizable map, then the associated non–linear \(\sigma\) model dilatonic action with coupling parameters \(a \in \mathbb{R}_{>0}\) and \((M, g, d\omega) \in \text{Meas}(M, g)\), is defined by

\[
(S, \gamma) \times \mathcal{H}^1(\Sigma, M) \times [\mathbb{R}_{>0} \times \text{Meas}(M, g)] \rightarrow \mathbb{R}
\]

\[
(\gamma, \phi; a, (M, g, d\omega)) \mapsto S[\gamma, \phi; a, g, d\omega] := a^{-1} \int_\Sigma \left[ tr_{\gamma(x)}(\phi^* g) - a K_\gamma \ln \phi^* \left( \frac{d\omega}{d\mu_g} V_g(M) \right) \right] d\mu_\gamma
\]

\[
:= \frac{2}{a} E[\phi, g]_{(\Sigma, M)} + \int_\Sigma K_\gamma f(\phi) d\mu_\gamma ,
\]

where \(K_\gamma\) is the Gaussian curvature of the Riemannian surface \((\Sigma, \gamma)\).

**Remark 3.4.** As already stressed in the introductory remarks, the parameter \(a > 0\) sets the (squared) length scale at which the pair \((\phi(\Sigma), S)\) probes the target metric measure space \((M, g, d\omega)\). Writing \(\ln \phi^* \left( \frac{d\omega}{d\mu_g} V_g(M) \right)\) in place of \(-f(\phi)\) may appear pedantic, however it emphasizes the often overlooked fact that the dilaton coupling \(f\) actually is the \((\text{Diff}(M)–\text{equivariant})\) assignment of a measure \(d\omega\) in \((M, g)\), and that we are dealing with a metric measure space and not just with a Riemannian manifold. In the definition, we also stressed the role of \(\text{Meas}(M, g)\) as the space of point–dependent coupling parameters for dilatonic non–linear \(\sigma\) models. As we have recalled, this latter aspect features prominently in the perturbative quantization of the model and its connection to Ricci flow. As we shall see, it plays a basic role also in motivating the use of Wasserstein geometry in the theory.

**Remark 3.5.** It is also important to recall again that, in stark contrast with the harmonic map energy \((2.10)\), the dilatonic term in \((3.9)\),

\[
- \int_\Sigma K_\gamma \ln \phi^* \left( \frac{d\omega}{d\mu_g} V_g(M) \right) d\mu_\gamma = \int_\Sigma K_\gamma f(\phi) d\mu_\gamma ,
\]

is not conformally invariant. As is well known, and as first stressed by E. Fradkin and A. Tseytlin [42], the role of this term is to restore the conformal invariance of \(E[\phi, g]\) which is broken upon quantization.

To discuss the role that Wasserstein geometry plays in non–linear \(\sigma\) model theory, we introduce the following characterization

**Definition 3.6.** If \(\text{Prob}(M)\) denote the set of all Borel probability measure on the manifold \(M\), then the space of probability–normalized dilaton fields over
The Wasserstein geometry of non-linear σ models \( (M,g) \in \mathcal{Met}(M) \), is the dense subspace of \( \text{Prob}(M) \) defined by

\[
\mathcal{DIL}_1(M,g) := \left\{ d\omega \in \text{Prob}(M) \mid d\omega := e^{-f} \frac{d\mu_g}{V_g(M)}, f \in C^\infty(M,\mathbb{R}) \right\}.
\] (3.11)

**Remark 3.7.** It is worthwhile stressing that the restriction to \( \mathcal{DIL}_1(M,g) \) is somewhat unphysical from the point of view of non–linear σ model theory, since it constrains a priori the dilaton field to be associated to a probability measure. As artificial as it may appear, this restriction plays a basic role in Perelman’s analysis of the Ricci flow, and it turns out to be appropriate also in the geometric analysis of the interaction between non–linear σ model, heat kernel embedding, and Ricci flow discussed here.

Clearly \( \mathcal{DIL}_1(M,g) \approx \text{Prob}_{ac}(M,g) \), the set of absolutely continuous probability measures \( d\omega << d\mu_g \) on \( (M,g) \). We use this identification to characterize \( \mathcal{DIL}_1(M,g) \) as an infinite dimensional manifold locally modelled over the Hilbert space completion of the smooth tangent space

\[
T_\omega \text{Prob}_{ac}(M,g) \simeq \{ h \in C^\infty(M,\mathbb{R}), \int_M h \, d\omega = 0 \},
\] (3.12)

with respect to the Otto inner product on \( \text{Prob}_{ac}(M,g) \) defined, at the given \( d\omega = V_g^{-1}(M)e^{-f}d\mu_g \), by the \( L^2(M,d\omega) \) Dirichlet form \[\text{[43]}\]

\[
\langle \nabla \varphi, \nabla \zeta \rangle_{(g,d\omega)} \doteq \int_M \left( g^{ik} \nabla_k \varphi \nabla_i \zeta \right) \, d\omega,
\] (3.13)

for any \( \varphi, \zeta \in C^\infty(M,\mathbb{R})/\mathbb{R} \). We set

\[
T_f \mathcal{DIL}_1(M,g)
\] (3.14)

\[
\doteq \left\{ h \in C^\infty(M,\mathbb{R}), \int_M h \, d\omega = 0 \right\}^{L^2(M,d\omega)}.
\]

Under this identification, one can represent infinitesimal deformations of the dilaton field \( d\omega \), (thought of as vectors in \( T_f \mathcal{DIL}_1(M,g) \approx T_\omega \text{Prob}_{ac}(M,g) \)), in terms of the mapping

\[
T_f \mathcal{DIL}_1(M,g) \times \mathcal{DIL}_1(M,g) \longrightarrow C^\infty(M,\mathbb{R})/\mathbb{R},
\] (3.15)

\[
(h, d\omega = V_g^{-1}(M)e^{-f}d\mu_g) \quad \longrightarrow \quad \psi,
\]

where the function \( \psi \) associated to the given \( (h,d\omega) \) is formally determined on \( (M,g) \) by the elliptic partial differential equation

\[
- \nabla^i \left( e^{-f} \nabla_i \psi \right) = h \, e^{-f},
\] (3.16)

under the equivalence relation identifying any two such solutions differing by an additive constant. Recall that if \( \mathcal{L}_V d\omega \) denotes the Lie derivative of the volume form \( d\omega \) along the vector field \( V \in C^\infty(M,TM) \), then the weighted
divergence associated with the Riemannian measure space \((M, g, d\omega)\) is defined by
\[
\mathcal{L}_V \, d\omega = (\text{div}_\omega \, V) \, d\omega = \left[ e^f \, \nabla_i \left( e^{-f} \, V^i \right) \right] \, d\omega .
\]
(3.17)

It follows that the elliptic equation (3.16) can be equivalently written as
\[
\triangle_{\omega} \psi = -h ,
\]
(3.18)

where \(\triangle_{\omega} \) denotes the weighted Laplacian on \((M, g, d\omega)\) \cite{44}, \cite{45}, \cite{38},
\[
\triangle_{\omega} := \text{div}_\omega \, \nabla = \triangle_g - \nabla f \cdot \nabla .
\]
(3.19)

Let us denote by \(G_{\omega} : M \times M \rightarrow \mathbb{R} \) the Green function associated to the operator \(-\triangle_{\omega} \) acting on functions with vanishing \(d\omega\)–mean,
\[
\int_M \psi(y) \, d\omega(y) = 0,
\]
(3.20)

i.e. the solution of
\[
\triangle_{\omega,x} \, G_{\omega}(x,y) = -\delta_y + 1 ,
\]
(3.21)

where \(\delta_y \, d\omega\) is the Dirac measure at \(y \in (M, g, d\omega)\). On \(M \times M\) minus the diagonal \(\{x = y\}, \, G_{\omega}(x,y)\) is smooth and we can write
\[
\psi(x) = G_{\omega} h(x) := \int_M h(y) \, G_{\omega}(x,y) \, d\omega(y) ,
\]
(3.22)

whenever the integral makes sense. Conversely, given \(\varphi \in C^\infty(M, \mathbb{R})/\mathbb{R}\), we can formally define a vector
\[
\text{Prob}_{ac}(M, g) \rightarrow T_\omega \text{Prob}_{ac}(M, g)
\]
(3.23)

by its action on smooth functionals \(\mathcal{J} \in C^\infty(\text{Prob}_{ac}(M, g), \mathbb{R})\) according to
\[
(J_{\varphi} \mathcal{J})(d\omega) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \mathcal{J}(d\omega - \epsilon \triangle_{\omega} \varphi \, d\omega) ,
\]
(3.24)

3.1. Diffeomorphism group and metric measure spaces

According to the definition (3.14) and the above remarks, the tangent space \(T_\omega \text{Prob}_{ac}(M, g)\) at the given dilatonic measure \(d\omega \in \text{Prob}_{ac}(M, g)\) can be identified with the subspace of gradient vector fields in the set of all \(L^2(M, d\omega)\) vector fields over \((M, g, d\omega)\). This is deeply connected with the geometry of \(\text{Diff}(M)\). The link is well–known in the case of Riemannian manifolds \cite{47}, \cite{48}, \cite{49}, where the Remark 3.2 applies, (a very readable account is provided in \cite{50}). If one wants to stress the underlying structure of the Riemannian metric measure space \((M, g, d\omega)\) over which \(\text{Diff}(M)\) is acting, then there are some peculiarities which, for our purposes, it is useful to make explicit. For technical reasons \cite{51} let us consider the (topological) group, \(\text{Diff}^\ast(M)\), defined by the set of diffeomorphisms which, as maps \(M \rightarrow M\) are an open subset of the Sobolev space of maps \(\mathcal{H}^s(M, M)\), with \(s > \frac{n}{2} + 1\). As we mentioned above, for a given \((M, g, d\omega), \, M\) compact, and for any \(d\omega' \in \text{Prob}_{ac}(M, g)\)
The Wasserstein geometry of non-linear $\sigma$ models

$\text{Prob}_{ac}(M, g)$ Moser theorem \cite{[51]} implies that there is a diffeomorphism $\eta \in \text{Diff}^s(M), s > \frac{n}{2} + 1$, such that $d\omega' = \eta^*(d\omega)$. In more precise terms, there is a smooth map $\chi : \text{Prob}_{ac}(M, g) \rightarrow \text{Diff}^s(M)$, canonically defined for the given $(M, g, d\omega)$, such that the pull back–action

$$J_{\omega} : \text{Diff}^s(M) \rightarrow \text{Prob}_{ac}(M, g)$$

$$\eta \mapsto J_{\omega}(\eta) := \eta^*(d\omega) = d\omega'$$

satisfies $J_{\omega} \circ \chi = e$, where $e$ is the identity in $\text{Diff}^s(M)$.

**Remark 3.8.** This well–known result (cf. \cite{[52]}) is usually stated in terms of the reference measure provided by the (normalized) Riemannian volume element $V_g(M)^{-1} d\mu_g$. On a given Riemannian metric measure space $(M, g, d\omega)$ one can use as reference measure $d\omega$ as well.

If we denote by $J_{\omega}^{-1}(d\omega')$ the fiber of $J_{\omega}$ over $d\omega' \in \text{Prob}_{ac}(M, g)$ then we have $J_{\omega}^{-1}(d\omega') = \text{Diff}^s_{\omega}^{-1}(M)$, where

$$\text{Diff}^s_{\omega}^{-1}(M) := \{ \varphi \in \text{Diff}^s_{\omega}(M) \mid \varphi^*(d\omega') = d\omega' \}$$

is the group of $d\omega'$–preserving diffeomorphisms. Let us recall that the tangent space to $\text{Diff}^s(M)$ at the identity, $T_e \text{Diff}^s(M)$, consists of all $H^s(M, TM)$ vector fields on $M$. More generally, the tangent space $T_\vartheta \text{Diff}^s(M)$ at a diffeomorphism $\vartheta \in \text{Diff}^s(M)$ is given by

$$T_\vartheta \text{Diff}^s(M) : \{ U \in H^s(M, TM) \mid U \text{ covers } \vartheta \in \text{Diff}^s(M) \},$$

i.e. by the space of $H^s$ sections of the pull–back bundle

$$\vartheta^{-1}TM := \{ U = u \circ \vartheta, u \in C^\infty(M, TM) \}.$$ 

Hence, if $U \in T_\vartheta \text{Diff}^s(M)$ then $U \circ \vartheta^{-1} \in H^s(M, TM)$. By extending the standard approach \cite{[52]} to the Riemannian measure space $(M, g, d\omega)$, we introduce on $\text{Diff}^s(M)$ the weak $L^2(M, d\omega)$ inner product defined at $T_\vartheta \text{Diff}^s(M)$ by

$$\langle U, V \rangle_{(\omega, \vartheta)} := \int_M g(\vartheta^{-1}(u \circ \vartheta(x), v \circ \vartheta(x)))d\omega(\vartheta(x)),$$

for all $U, V \in T_\vartheta \text{Diff}^s(M)$. We can equivalently write \cite{[32]} as

$$\langle U, V \rangle_{(\omega, \vartheta)} = \int_{\vartheta^{-1}(M)} g(u(x), v(x)) (\vartheta^{-1})^*(d\omega)(x)$$

$$= \int_{\vartheta^{-1}(M)} g(u, v) \text{Jac}_{\omega}(\vartheta^{-1}) d\omega = \int_M g(u, v) \text{Jac}_{\vartheta}(\vartheta^{-1}) d\omega,$$

where we have used $\vartheta^{-1}(M) = M$ and denoted by $\text{Jac}_{\omega}(\vartheta^{-1})$ the Jacobian of $\vartheta^{-1}$ computed with respect to $d\omega$. Let $\eta \in \text{Diff}^s(M)$ and denote by

$$T_\vartheta \text{Diff}^s(M) \rightarrow T_{\vartheta \circ \eta} \text{Diff}^s(M)$$

$$U_{\vartheta} \mapsto (\mathcal{R}_\eta)_* U_{\vartheta} := U_{\vartheta} \circ \eta,$$
the push–forward of $U_\vartheta$ under the (smooth) right action defined on $Diff^s(M)$ by $R_\eta : \vartheta \mapsto \vartheta \circ \eta$. According to (3.29) one computes
\begin{equation}
\langle (R_\eta)_* U, (R_\eta)_* V \rangle_{(\omega, \vartheta \circ \eta)} = \int_M g(U \circ \eta(x), V \circ \eta(x)) \vartheta \circ \eta(x) \ d\omega ,
\end{equation}
\begin{equation}
= \int_{\eta^{-1}(M)} g(U(x), V(x)) \vartheta(x) (\eta^{-1})^*(d\omega) .
\end{equation}
If $\eta \in Diff^s_\omega(M)$ then $(\eta^{-1})^*(d\omega) = d\omega$ and (3.29) is right–invariant when restricted to the group of $d\omega$-volume preserving diffeomorphisms $Diff_\omega^s(M)$. In this connection, let us consider the weighted $L^2(M, d\omega)$ Helmholtz decomposition of vector fields which are of Sobolev class $H^s(M, TM)$, $s > n/2 + 1$, on $(M, g, d\omega)$
\begin{equation}
H^s(M, TM) \simeq T_\omega Diff^s(M) = Im \nabla \oplus Ker \nabla^* ,
\end{equation}
where $\nabla^* = \text{adjoint of the gradient } \nabla$ : $H^{s+1}(M, \mathbb{R}) \to H^s(M, TM)$. Hence, according to the Otto characterization of $T_\omega \text{Prob}_{ac}(M, g)$ we can write
\begin{equation}
T_\omega Diff^s(M) = T_\omega \text{Prob}_{ac}(M, g) \oplus T_\omega Diff_\omega^s(M) ,
\end{equation}
where we have identified $Ker \nabla^*$ with the tangent space to $Diff_\omega^s(M)$ at the identity map. Explicitly, for a given $v \in H^s(M, TM)$ we have
\begin{equation}
v = -\nabla \varphi \oplus v_\dagger ,
\end{equation}
where $v_\dagger$ is $\text{div } \omega$–free, and $\varphi$ is defined by the solution of the elliptic equation
\begin{equation}
\Delta_\omega \varphi = -\text{div } v ,
\end{equation}
which, according to (3.33), characterizes the correspondence $\text{div } v \leftrightarrow \varphi$. It is useful to rewrite (3.35) as
\begin{equation}
v = \Pi_\omega v \oplus (I - \Pi_\omega) v
\end{equation}
where
\begin{equation}
\Pi_\omega v(x) := \nabla_x (G_\omega \circ \text{div } \omega) v = \nabla_x \int_M G_\omega(x, y) \text{div } \omega, y v(y) d\omega(y) ,
\end{equation}
is the $L^2(M, d\omega)$–orthogonal projector onto $\text{Im } \nabla$ and $(I - \Pi_\omega)$ is the projector onto $Ker \nabla^* \simeq T_\omega Diff_\omega^s(M)$. We can extend $(I - \Pi_\omega)$ to the elements $V$ of the tangent space $T_\eta Diff^s$, $\eta \in Diff_\omega^s(M)$, by defining the right–invariant projector
\begin{equation}
(I - \Pi_\omega)\eta V := \left( (I - \Pi_\omega)(V \circ \eta^{-1}) \right) \circ \eta .
\end{equation}
Correspondingly we have the $L^2(M, d\omega)$ weak orthogonal splitting
\begin{equation}
T_\eta Diff^s \simeq (R_\eta)_* (T_\omega \text{Prob}_{ac}(M, g)) \oplus T_\eta Diff_\omega^s(M) ,
\end{equation}
where $(R_\eta)_* (T_\omega \text{Prob}_{ac}(M, g)) := (R_\eta)_* (\text{Im } \nabla)$ is the $\eta$–right translated spaces of gradient vector fields $\text{Im } \nabla \subset T_\omega Diff^s$. Note that the weak Riemannian metric (3.29) is right invariant on $Diff^s_\omega(M)$, and this allows to
The Wasserstein geometry of non-linear $\sigma$ models

consider the Otto inner product $\langle \cdot, \cdot \rangle_\omega$ as inducing on the space of smooth probability measure $\text{Prob}_{ac}(M,g) \simeq \text{Diff}^s(M)/\text{Diff}_\omega^s(M)$, $s > n/2 + 1$, a weak Riemannian structure such that the projection, (see (3.25)),

$$\mathcal{J}_\omega : \text{Diff}^s(M) \to \text{Prob}_{ac}(M,g) \quad \eta \mapsto \mathcal{J}_\omega(\eta) := (\eta^{-1})^* (d\omega) = d\omega'$$

is a Riemannian submersion. According to [52] one can exploit (3.31) to introduce on $\text{Diff}^s(M)$ a torsion–free connection which is compatible with the weak Riemannian structure defined by (3.29), viz.

$$\mathcal{T}_\eta \text{Diff}^s \ni \nabla_U V := (\mathcal{R}_\eta^*) (\nabla^{(g)}_u v), \quad (3.42)$$

or, more explicitly,

$$\left\langle \nabla_U V, W \right\rangle_{(\omega,\eta)} = \int_M g \left( \nabla^{(g)}_{u\circ \eta}(v \circ \eta), w \circ \eta \right) d\omega(\eta(x)),$$

where the tangent vectors $u, v, w, \in \mathcal{H}^s_t \simeq T_\eta \text{Diff}^{s+1}(M), U = u \circ \eta, V = v \circ \eta, W = w \circ \eta$, $\mathcal{T}_\eta \text{Diff}^s$, and $\nabla^{(g)}$ denotes the Levi–Civita connection of $(M,g,d\omega)$. The curvature of such a connection at $\eta \in \text{Diff}^s$ can be computed according to [48]

$$\tilde{R}(U, V) Z := (\mathcal{R}_\eta^*) (R^{(g)}(u, v) z), \quad (3.43)$$

where $Z = z \circ \eta$ and $R^{(g)}(u, v) z$ is the Riemann tensor of $(M,g,d\omega)$. Explicitly,

$$\left\langle \tilde{R}(U, V) Z, W \right\rangle_{(\omega,\eta)} = \int_M g \left( R^{(g)}(u \circ \eta, v \circ \eta) z \circ \eta, w \circ \eta \right) d\omega(\eta(x)),$$

provides the component of the curvature tensor associated with the weak Riemannian structure (3.29) at the point $\eta \in \text{Diff}(M)$. In particular, the components at the identity $e \in \text{Diff}(M)$ are given by the $d\omega$–average

$$\left\langle \tilde{R}(U, V) Z, W \right\rangle_{(\omega,e)} = \int_M g \left( R^{(g)}(u, v) z, w \right) d\omega(x). \quad (3.45)$$

In [48] this analysis is extended to the computation of the curvature of the space of probability measures $\text{Prob}_{ac}(M,g)$, a point that we address in the next section.

Remark 3.9. The Helmholtz decomposition can be seen as a linearization of Brenier’s polar factorization theorem [53], [54] and plays a subtle role in the Wasserstein geometry of the smooth $\infty$–dimensional manifold $\text{Prob}_{ac}(M,g)$. The above analysis, based on the geometry of $\text{Diff}(M)$ is somehow formal since it heavily relies on the smoothness of $\text{Prob}_{ac}(M,g)$ and of the Riemannian submersion map (3.41) which typically fails for general probability
measures. Nonetheless, as presciently shown by F. Otto [43], this formal Riemannian structure has many remarkable properties and provides the backbone for a fully rigorous description of the geometry of \((\operatorname{Prob}(\mathcal{M}), d^W_g)\), (see e.g. [55], [56], [57], [25], [58], [59]).

3.2. Wasserstein distance and calculus on dilaton space

Let \(\operatorname{Prob}(\mathcal{M} \times \mathcal{M})\) denote the set of Borel probability measures on the product space \(\mathcal{M} \times \mathcal{M}\). In order to compare any two distinct dilaton fields in \(\mathcal{DIL}(1)(\mathcal{M}, g)\), say \((\mathcal{M}, g, d\omega_1 = e^{-f_1} \mu_{g(M)})\) and \((\mathcal{M}, g, d\omega_2 = e^{-f_2} \mu_{g(M)})\), let us consider the set of measures \(d\sigma \in \operatorname{Prob}(\mathcal{M} \times \mathcal{M})\) which reduce to \(d\omega_1\) when restricted to the first factor and to \(d\omega_2\) when restricted to the second factor, i.e.

\[
\operatorname{Prob}(\omega_1, \omega_2)(\mathcal{M} \times \mathcal{M}) := \{ d\sigma \in \operatorname{Prob}(\mathcal{M} \times \mathcal{M}) \mid \pi_1^* d\sigma = d\omega_1, \pi_2^* d\sigma = d\omega_2 \},
\]

where \(\pi_1^*\) and \(\pi_2^*\) refer to the push–forward of \(d\sigma\) under the projection maps \(\pi_i\) onto the factors of \(\mathcal{M} \times \mathcal{M}\). Measures \(d\sigma \in \operatorname{Prob}(\omega_1, \omega_2)(\mathcal{M} \times \mathcal{M})\) are often referred to as couplings between \(d\omega_1\) and \(d\omega_2\). We shall avoid such a terminology since in our setting the term coupling has quite different a meaning.

Let us recall that given a (measurable and non–negative) cost function \(c : \mathcal{M} \times \mathcal{M} \to \mathbb{R}\), an optimal transport plan \(d\sigma_{\text{opt}} \in \operatorname{Prob}(\omega_1, \omega_2)(\mathcal{M} \times \mathcal{M})\) between the probability measures \(d\omega_1\) and \(d\omega_2\) in \(\operatorname{Prob}(\mathcal{M})\), (not necessarily in \(\operatorname{Prob}_{ac}(\mathcal{M}, g)\)), is defined by the infimum, over all \(d\sigma(x, y) \in \operatorname{Prob}(\omega_1, \omega_2)(\mathcal{M} \times \mathcal{M})\), of the total cost functional

\[
\int_{\mathcal{M} \times \mathcal{M}} c(x, y) d\sigma(x, y).
\]

On a Riemannian manifold \((\mathcal{M}, g)\), the typical cost function is provided \([24],[25],[33]\) by the squared Riemannian distance function \(d^2_g(\cdot, \cdot)\), and a major result of the theory \([61],[62],[33],[38]\), is that for any pair \(d\omega_1\) and \(d\omega_2\) \(\in \operatorname{Prob}(\mathcal{M})\), there is an optimal transport plan \(d\sigma_{\text{opt}}\), induced by a map \(\Upsilon_{\text{opt}} : \mathcal{M} \to \mathcal{M}\). The resulting expression for the total cost functional of the plan

\[
d^W_g(d\omega_1, d\omega_2) := \left( \inf_{d\sigma \in \operatorname{Prob}(\omega_1, \omega_2)(\mathcal{M} \times \mathcal{M})} \int_{\mathcal{M} \times \mathcal{M}} d^2_g(x, y) d\sigma(x, y) \right)^{1/2},
\]

characterizes the quadratic Wasserstein, (or more appropriately, Kantorovich-Rubinstein) distance between the two probability measures \(d\omega_1\) and \(d\omega_2\).

Remark 3.10. Note that there can be distinct optimal plans \(d\sigma_{\text{opt}}\) connecting general probability measures \(d\omega_1\) and \(d\omega_2\) \(\in \operatorname{Prob}(\mathcal{M})\), whereas on \(\operatorname{Prob}_{ac}(\mathcal{M}, g)\) the optimal transport plan is unique.

The quadratic Wasserstein distance \(d^W_g\) defines a finite metric on \(\operatorname{Prob}(\mathcal{M})\) and it can be shown that \((\operatorname{Prob}(\mathcal{M}), d^W_g)\) is a geodesic space, endowed with
the weak–* topology, (we refer to [55], [54], [59] for the relevant properties of Wasserstein geometry and optimal transport we freely use in the following). By an obvious dictionary, we identify the distance between the two dilaton fields \( f_1 \) and \( f_2 \) with the Wasserstein distance \( d^W_g (d\omega_1, d\omega_2) \) between the corresponding probability measures. This allows to characterize \((\mathcal{DIL}_1(M,g), d^W_g)\) as the (dense) subset, \((\text{Prob}_{ac}(M,g), d^W_g))\), of the (quadratic) Wasserstein space \((\text{Prob}(M), d^W_g)\). By way of illustration of the interplay between dilaton fields and Wasserstein geometry we have the

**Lemma 3.11.** Let \( f_1 \) and \( f_2 \) two distinct dilaton fields in \( \mathcal{DIL}_1(M,g) \), and let \( x \mapsto \exp_x (-\nabla \psi(x)) \) be the optimal map between the corresponding probability measures \( d\omega_1 = V^{-1}_g (M) e^{-f_1} d\mu_g \) and \( d\omega_2 = V^{-1}_g (M) e^{-f_2} d\mu_g \). Then the unique geodesic in \( (\mathcal{DIL}_1(M,g), d^W_g) \) connecting \( d\omega_1 \) and \( d\omega_2 \) is provided by the push–forward map,

\[
\Gamma : [0,1] \rightarrow \text{Prob}_{ac}(M,g) \quad \lambda \mapsto \Gamma(\lambda) := (\exp_x (-\lambda \nabla \psi(x)))_x^\ast d\omega_1 .
\]

**Proof.** This is the transcription in our setting of a well known basic result in optimal transport theory (cf. [63] for a very readable short introduction). For the convenience of the reader and for later use we give a brief outline of the proof. Let \( \exp_x^{(g)} \) denote the exponential map on \((M,g)\) based at \( x \in M \), then, as originally proven by R. McCann [53], there exists a function \( \psi : M \rightarrow \mathbb{R} \), the Kantorovich potential, such that, for \( d\omega_1 \)–almost all points \( x \in M \), we can define the map \( \aleph_\lambda : [0,1] \times M \rightarrow M \) \((3.50)\)

\[
(\lambda, x) \mapsto \aleph_\lambda(x) := \exp_x^{(g)} (-\lambda \nabla \psi(x)) ,
\]

with \( \aleph_\lambda|_{\lambda=1} = \aleph_{\text{opt}} \) providing the optimal transport map. The associated plan \( d\sigma_{\text{opt}} \in \text{Prob}_{\omega_1, \omega_2}(M \times M) \) is given by

\[
d\sigma_{\text{opt}} = (Id_M, \aleph_1)_x^\ast d\omega_1 , \quad (3.51)
\]

where \((Id_M, \aleph_1) : M \times M \rightarrow M \times M \) is defined by \((y, x) \mapsto (y, \aleph_1(x))\). The push–forward of \( d\omega_1 \) under \( \aleph_\lambda \),

\[
\Gamma : [0,1] \rightarrow \text{Prob}_{ac}(M,g) \quad \lambda \mapsto \Gamma(\lambda) := (\aleph_\lambda)_x^\ast d\omega_1 , \quad (3.52)
\]

generates the (unique) geodesic in \((\text{Prob}(M), d^W_g)\) connecting \( d\omega_1 = \Gamma(0) \) and \( d\omega_2 = \Gamma(1) \). Moreover, under the stated hypotheses, \( \Gamma(\lambda) << d\mu_g \), \( \lambda \in [0,1] \), thus the geodesic lies in \((\mathcal{DIL}_1(M,g), d^W_g)\). Note that we have the identity

\[
(d^W_g (d\omega_1, d\omega_2))^2 = \int_M |\nabla \psi|_g^2 d\omega_1 ,
\]

where \( |\nabla \psi|_g^2 := g^{ab} \nabla_a \psi \nabla_b \psi \). □
In other words, geodesics in $(\mathcal{DIL}(1)(M, g), d_g^W)$ are naturally associated with the transport of the corresponding dilatonic measures along the geodesics of the underlying Riemannian manifold $(M, g)$. We remark that the optimal transport map $\pi_{opt}$ associated with the optimal transport plan $d\sigma_{opt}$ is not generally smooth, (smoothness in optimal transport theory is related to rather sophisticated curvature assumptions [63]).

According to (3.35), the Wasserstein metric structure on $\mathcal{DIL}(1)(M, g) \approx \text{Prob}_{ac}(M, g)$ is induced by the Otto inner product $\langle \cdot, \cdot \rangle_{(g, d\omega)}$ defined by (3.14). This basic remark allows the introduction of a (weak) Riemannian calculus on the smooth Wasserstein space $(\text{Prob}_{ac}(M, g), d_g^W)$, which is related to the weak Riemannian geometry of the diffeomorphism group $\text{Diff}(M)$ discussed in section 3.1. In particular one can compute [46] the Riemannian curvature of $(\text{Prob}_{ac}(M, g), d_g^W)$. We stress once more, (cf. Remark 3.9), that the Riemannian structure and the attendant geometric analysis is quite more delicate in the general Wasserstein space $(\text{Prob}(M), d_g^W)$, where $\text{Prob}(M)$ has no smooth structure and hence one lacks any obvious notion of smoothness for vector fields. The nature of the difficulties and the development of a suitable weak Riemannian calculus on $(\text{Prob}(M), d_g^W)$ are presented, from different point of views, in [55], [56], [57], [25], [58], ([57] provides a thorough analysis with examples and useful interpretative remarks). Since we confine to the smooth Wasserstein space $(\text{Prob}_{ac}(M, g), d_g^W)$, the Riemannian interpretation can be implemented rigorously, and in what follows we shall make use of the following result proved by J. Lott, which we rephrase for the dilaton space $\mathcal{DIL}(1)(M, g)$, (for the last time we recall that we use $\mathcal{DIL}(1)(M, g)$ and $\text{Prob}_{ac}(M, g)$ interchangeably).

**Theorem 3.12. (J. Lott [46], Lemma 3)** Let $\nabla_{\psi(i)} : f \mapsto T_f \mathcal{DIL}(1)(M, g)$ be vector fields on $\mathcal{DIL}(1)(M, g)$ associated to potentials $\psi(i) \in C^\infty(M, \mathbb{R})/\mathbb{R}$ under the isomorphism (3.24). The Riemannian connection $\nabla$ of the Otto Riemannian metric $\langle \cdot, \cdot \rangle_{(g, d\omega)}$ is given by

$$
\langle \nabla_{\psi(i)} \psi(j), \psi(k) \rangle_{(g, d\omega)} = \int_M \nabla_{\psi(i)} \nabla^a \nabla^b \psi(j) \nabla_{\psi(k)} d\omega .
$$

(3.54)

In particular, we can define the Lie derivative of the inner product $\langle \cdot, \cdot \rangle_{(g, d\omega)}$ along the vector field $f \mapsto \nabla \psi \in T_f \mathcal{DIL}(1)(M, g)$ according to

$$
\mathcal{L}_{\nabla \psi} \langle \nabla_{\psi(i)}, \nabla_{\psi(k)} \rangle_{(g, d\omega)} = \langle \nabla_{\psi(i)}, \nabla_{\psi(k)} \rangle_{(g, d\omega)} + \langle \nabla_{\psi(i)}, \nabla_{\psi(i)} \rangle_{(g, d\omega)}
$$

(3.55)

$$
:= \langle \nabla_{\psi(i)} \nabla \psi, \psi \rangle_{(g, d\omega)} + \langle \nabla \psi, \nabla_{\psi(i)} \psi \rangle_{(g, d\omega)}
$$

$$
= 2 \int_M \nabla^a \psi(i) \nabla^b \psi(j) \psi \nabla_{\psi(k)} d\omega ,
$$

where $\text{Hess} \psi$ denotes the Hessian on $(M, g)$. Note that according to (3.34) and (3.35) the connection $\nabla$ at $d\omega$ is characterized by the gradient part of the
(M, g, d\omega) Helmholtz decomposition of the vector field \( \nabla_a \psi_{(i)} \nabla^a \nabla^b \psi_{(j)} \partial_b \), i.e. we can write

\[
\langle \nabla_{\psi_{(i)}} \land \psi_{(j)}, \land \psi_{(k)} \rangle (g, d\omega) = \int_M \Pi_\omega \left( \nabla_a \psi_{(i)} \nabla^a \nabla^b \psi_{(j)} \right) \nabla_b \psi_{(k)} \, d\omega, \tag{3.56}
\]

where the \( L^2(M, d\omega) \) projection operator \( \Pi_\omega \) is defined by \[\text{[3.38]}\]. Since the vector field \( \nabla \psi_{(i)} \cdot Hess \psi_{(j)} := \nabla_a \psi_{(i)} \nabla^a \nabla^b \psi_{(j)} \partial_b \) does not, in general, belong to \( T_\omega \text{Prob}_{ac}(M, g) \approx T_\ell \text{DIL}_{(1)}(M, g) \), we can interpret it as an element of \( T_\ell \text{Diff}(M) \), and for \( \eta \in \text{Diff}^s(M), s > n/2 + 1 \), consider

\[
\nabla \psi_{(i)} \cdot Hess \psi_{(j)} \circ \eta \in T_\eta \text{Diff}^s(M) \tag{3.57}
\]

as a section of the pull-back bundle \( \eta^{-1}TM \). A similar remark applies also to the commutator of two vector fields \( \land \psi_{(i)} \) and \( \land \psi_{(k)} \in T_\omega \text{Prob}_{ac}(M, g) \), defined for any \( \varphi \in C_0^\infty(M, \mathbb{R}) \) by \[\text{[3.16]}\]

\[
\langle [\land \psi_{(i)}, \land \psi_{(k)}], \land \varphi \rangle (g, d\omega) = \langle \nabla_{\psi_{(i)}} \land \psi_{(k)} - \nabla_{\psi_{(k)}} \land \psi_{(i)} \rangle (g, d\omega)
\]

\[
:= \int_M \left( \nabla_a \psi_{(i)} \nabla^a \nabla^b \psi_{(k)} \right) - \left( \nabla_a \psi_{(k)} \nabla^a \nabla^b \psi_{(i)} \right) \nabla_b \varphi \, d\omega, \tag{3.58}
\]

and we have \([\land \psi_{(i)}, \land \psi_{(k)}] \in T_\ell \text{Diff}(M) \). As already recalled we are in the typical situation of the Riemannian submersion described in section \[\text{3.1}\] To put this at work, one may proceed as in the classical O’Neill’s analysis of Riemannian submersion \[\text{[64], [65]}\], (see also Chap. 9 of \[\text{[66]}\]), and introduce the vector field, (actually a \((2,1)\) tensor field)

\[
d\omega \mapsto T_{\psi_{(i)}} \psi_{(j)} := \frac{1}{2} (I - \Pi_\omega) \left[ \land \psi_{(i)}, \land \psi_{(j)} \right] \in T_\ell \text{Diff}(M)
\]

\[
= (I - \Pi_\omega) \left( \nabla_a \psi_{(i)} \nabla^a \nabla^b \psi_{(j)} \partial_b \right) \in \text{Ker} \nabla^\ast \omega, \tag{3.59}
\]

defined by the \( \text{div}\omega \)-free part of \( \nabla_a \psi_{(i)} \nabla^a \nabla^b \psi_{(j)} \partial_b \). Note that in the last line of we have exploited the antisymmetry of \((I - \Pi_\omega) \left( \nabla_a \psi_{(i)} \nabla^a \nabla^b \psi_{(j)} \partial_b \right)\), (cf. Lemma 6 of \[\text{[3.16]}\]). The vector field \[\text{[3.59]}\] can be thought of as describing to what extent the distribution of gradient vector fields \( \land \psi_{(i)} \) fails to be integrable in \( \text{Diff}(M) \). With these remarks along the way one can characterize the Riemannian curvature of the Wasserstein space \( \text{DIL}_{(1)}(M, g)(M, g) \) according to the

**Theorem 3.13.** (J. Lott \[\text{[16], Th.1}\]) Let \( \land \psi_{(i)} \in T_\ell \text{DIL}_{(1)}(M, g) \) be given vector fields on \( \text{DIL}_{(1)}(M, g) \) associated to corresponding potentials \( \psi_{(i)} \in C^\infty(M, \mathbb{R})/\mathbb{R} \). The Riemannian curvature of the connection \( \overline{\nabla} \) is given, at \( d\omega \), by
\[
\langle R (\nabla \psi_i, \nabla \psi_j), \nabla \psi_k \rangle_{(g, d\omega)} \quad (3.60)
\]

\[
= \int_M \nabla^b \psi_i \nabla^c \psi_j \nabla^d \psi_k R^a_{b c d} \nabla_a \psi_h d\omega
\]

\[
- \int_M g_{a b} \left( 2 T^a_{\psi_i \psi_j} T^b_{\psi_k \psi_h} - T^a_{\psi_j \psi_k} T^b_{\psi_i \psi_h} + T^a_{\psi_i \psi_h} T^b_{\psi_j \psi_k} \right) d\omega.
\]

Remark 3.14. According to (3.45) we can interpret the \(d\omega\)-average of the Riemann tensor \(R^a_{b c d}\) in (3.60) as the curvature tensor of \(\langle \cdot, \cdot \rangle_{\omega, \eta}\),

\[
\left( \tilde{R} (\nabla \psi_i, \nabla \psi_j), \nabla \psi_k \right)_{(\omega, e)} ,
\]

evaluated at the identity map. Indeed, quite independently from optimal transport techniques, one can derive (3.60) as the formula computing the horizontal components of the Riemannian curvature associated with the Riemannian submersion

\[
J_\omega : Diff(M) \longrightarrow \text{Prob}_{ac}(M, g) \simeq Diff(M)/Diff_\omega(M) \quad (3.62)
\]

defined by (3.41), (cf. [48]). We emphasize once more that the characterization of curvature for the Wasserstein space \((\text{Prob}(M), d_W g)\), associated to general probability measures on \(M\), is quite more delicate. Details can be found in [57].

4. Constant maps localization and warping

Important insight into the structure of the Wasserstein geometry of the dilatonic non–linear \(\sigma\) model can be gained by the following analysis, showing how the theory behaves under a localization around constant maps.

To begin with, let \(\kappa\) denote the upper bound to the sectional curvature of \((M, g, d\omega)\), and let us consider a metric ball \(B(r, p) := \{ z \in M | d_g(p, z) \leq r \}\), centered at \(p \in M\), with radius \(r < r_0\), where \(r_0\), defined by (2.4) sets the length scale of the target \((M, g)\). For \(q \in \mathbb{N}\), let \(\{\phi_k\}_{k=1}^q \in \text{Map}(\Sigma, M)\) denote a collection of reference constant maps, (hence harmonic), taking values in the interior of \(B(r, p)\)

\[
\phi_k : \Sigma \rightarrow B(r, p) \setminus \partial B(r, p) \subset M \quad (4.1)
\]

\[
x \mapsto \phi_k(x) = y_k, \quad \forall x \in \Sigma, \quad k = 1, \ldots, q .
\]

We explicitly assume that \(r < \frac{\pi}{6\sqrt{\kappa}}\), \(inj(y) > 3r\) for all \(y \in B(r, p)\), and consider the center of mass [67] of the maps \(\{\phi_k\}\),

\[
\phi_{cm} \equiv cm \{ \phi(1), \ldots, \phi(q) \} , \quad (4.2)
\]
characterized as the minimizer of the function

\[ F(y; q) = \frac{1}{2} \sum_{k=1}^{q} d_g^2(y, y(k)), \]

where \( d_g^2(\cdot, \cdot) \) denotes the distance in \((M, g)\).

**Lemma 4.1.** Under the stated hypotheses the minimizer exists, is unique and \( cm \{ \phi(1), \ldots, \phi(q) \} \in B(2r, p) \).

**Proof.** This is a standard property of the center of mass [67]. See for instance [68] (Chap.4, p.175).

We denote by \( \{ d_g(\phi_{cm}, \phi(k)) \} \) the distances between the maps \( \{ \phi(k) \} \) and their center of mass \( \phi_{cm} \). The overall strategy for introducing the constant maps \( \{ \phi(k) \} \) is to use the distances \( \{ d_g(\phi_{cm}, \phi(k)) \} \) and the dilatonic measure \( d\omega \) to set the scale at which \((\Sigma, \gamma)\) probes the geometry of \((M, g)\). To this end, we localize \( \phi \in \text{Map}(\Sigma, M) \), around the center of mass of \( \{ \phi(k) \}_{k=1}^{q} \), by choosing \( d\omega \) according to

\[ d\omega(z; q) := C_r^{-1}(q) e^{-F(z; q) 2r^2} \frac{d\mu_g(z)}{V_g(M)}, \quad z \in M, \]

where \( F(z; q) \) is the center of mass function (4.3), \( V_g(M) \) is the Riemannian volume of \((M, g)\), and \( C_r(q) \) denotes a normalization constant such that \( \int_M d\omega(z; q) = 1 \). Since \( F(z; q) \) attains its minimum at \( \phi_{cm} \), the measure \( d\omega \) is concentrated around the center of mass of the \( \{ \phi(k) \} \)'s, and, as \( r \rightarrow 0^+ \), weakly converges to the Dirac measure \( \delta_{cm} \) supported at \( \phi_{cm} \). Note that according to (4.3) we can factorize the density \( d\omega/d\mu_g(z) \) as

\[ \frac{d\omega(z; q)}{d\mu_g(z)} = V_g(M)^{-1} \prod_{k=1}^{q} e^{-\frac{d_g^2(z, \phi(k))}{4r^2} - \frac{\ln C_r(q)}{q}}. \]

This latter remark suggests to interpret the distances \( d_g(\phi_{cm}, \phi(k)), k = 1, \ldots, q \) as coordinates \( \{ \xi(k) \} \) in a \( q \)-dimensional flat torus \( \mathbb{T}_q^{cm} \) of unit volume, and consider the product manifold

\[ N^{n+q} := M \times (\omega) \mathbb{T}_q^{cm}, \]

endowed with the warped product measure

\[ d\mu_N(z, \xi) := d\mu_g(z) \prod_{k=1}^{q} e^{-\frac{d_g^2(z, \phi(k))}{4r^2} - \frac{\ln C_r(q)}{q}} d\xi(k), \quad \{ \xi(i) \} \in \mathbb{T}^q, \]
with total volume

\[ \int_{N^{n+q}} d\mu_N(z, \xi) = \int_{N^{n+q}} d\mu_g(z) \prod_{k=1}^{q} e^{-\frac{d_{\phi(k)}^2(z)}{4 r^2} - \frac{\ln C_r(q)}{q}} d\xi_k \]

\[ = C_r^{-q}(q) \int_M e^{-\frac{F(z, q)}{2 r^2}} d\mu_g(z) \int_T q \prod_{k=1}^{q} d\xi_k \]

\[ = V_g(M) \int_M d\omega(z; q) = V_g(M) . \]

The measure (4.7) is clearly Riemannian, being induced on \( N^{n+q} := M \times \mathbb{T}^q \)

by the warped metric

\[ h^{(q)}(y, \xi) := g(y) + \left( \frac{d\omega(y; q)}{d\mu_g(y)} \right) V_g(M)^{\frac{2}{q}} \sum_{i=1}^{q} d\xi_{i(\xi_i)}, \quad \xi_i \in [0, 1], , (4.9) \]

according to

\[ d\mu_{h^{(q)}}(y, \xi) = d\mu_g(y) \left( \frac{d\omega(y; q)}{d\mu_g(y)} \right) V_g(M) \prod_{i=1}^{q} d\xi_{i(\xi_i)} \]

\[ = d\mu_g(y) \prod_{k=1}^{q} e^{-\frac{d_{\phi(k)}^2(y)}{4 r^2} - \frac{\ln C_r(q)}{q}} d\xi_k = d\mu_N(y, \xi) . \]

**Remark 4.2.** Trading the Riemannian metric measure space \((M, g, d\omega)\) with

the warped Riemannian manifold

\((N^{n+q} := M \times \mathbb{T}^q, h^{(q)})\)

is a standard procedure in the Riemannian measure space setting, (cf. [40],

and [69] for the application to Perelman’s reduced volume). This construction

can be naturally extended to the injection of \((M, g, d\omega)\) in the \(\infty\)-dimensional

fibering \((M \times \mathbb{T}^\infty, h^{(\infty)})\).

As a function of the distances from the constant maps \(\{\phi(k)\}\), the probability

measure \(d\omega \in \text{Prob}_{ac}(M, g) \approx DL_1(1)(M, g)\) defined by (4.4) is Lipschitz on

\((M, g)\) and smooth on \(M \setminus \bigcup_{k=1}^{q} \{\phi(k), \text{Cut}(\phi(k))\}\), where \(\text{Cut}(\phi(k))\) denotes

the cut locus of each \(\phi(k)\). In terms of the associated dilaton field we have

**Lemma 4.3.** The dilaton field associated to the measure \(d\omega\),

\[ f(z; q) := -\ln \left( \frac{d\omega(z; q)}{d\mu_g(z)} V_g(M) \right) \]

\[ = \frac{1}{4 r^2} \sum_{k=1}^{q} d_{\phi(k)}^2(z) + \ln C_r(q) , \]
has a gradient $\nabla f$ which exists a.e. on $(M, g)$ and $f \in H^1(M, \mathbb{R})$.

Proof. As a sum of squared distances from the constant maps $\{\phi(k)\}$, $f$ is smooth on $M \setminus \bigcup_{k=1}^q \{\phi(k), \text{Cut}(\phi(k))\}$, and Lipschitz on $(M, g)$. Thus the first part of the lemma is a direct consequence of Rademacher’s theorem. In particular the distributional gradient of $f$ with respect to the Riemannian metric, $\nabla f$ is an $L^\infty$ vector field on $(M, g)$, that is $f \in W^{1, \infty}(M, \mathbb{R})$ where the Sobolev norm $\|f\|_{W^{1, \infty}}$ is defined by $\|f\|_{W^{1, \infty}} := \esssup_{M} (|f| + |\nabla f|)$, (cf. Chap.4, Th. 5 of [70]). Since $M$ is compact, it follows [71], (Chap. 11, Corollary 11.4), that $f \in L^2(M, \mathbb{R})$ and $\nabla f \in L^2(M, TM)$, i.e., $f \in H^1(M, \mathbb{R})$. Let us observe that $\nabla f$ can be integrated by parts over $(M, g)$ against locally Lipschitz vector field $v$, in the sense that $\int_M f \nabla_i v^i d\mu_g = \int_M v^i \nabla_i f d\mu_g$, (cf. Chap.9, lemma. 7.113 of [68] for a proof of this well–known property of Lipschitz functions). Better global regularity cannot be expected since, if we compute the Laplacian of $f$, we get

$$\Delta_g f(z; q) = \frac{1}{4 r^2} \sum_{k=1}^q \Delta_g d_g^2(z, \phi(k)),$$

which, from the standard properties of the distance function, implies that $\Delta_g f(z; q)$ is a distribution with a singular part supported on the cut locus $\bigcup_{k=1}^q \text{Cut}(\phi(k))$ of $(M, g, \{\phi(k)\})$. □

If we localize $f$ to the geodesic ball $B(2r, p)$ the situation is simpler, and we have

**Lemma 4.4.** For $z \in B(2r, p)$, $r < \frac{\pi}{6\sqrt{\kappa}}$, the dilaton field $f(z; q)$ is a strictly convex function. In particular, if $[0, 1] \ni s \mapsto \lambda^{(k)}(s)$, $k = 1, \ldots, q$, are minimal geodesics connecting $\lambda^{(k)}(0) = z$ to $\lambda^{(k)}(1) = \phi(k)$, and $\exp_z : T_z M \rightarrow M$ denotes the exponential mapping based at $z$, then we can write

$$f(z; q) = \frac{1}{4 r^2} \sum_{k=1}^q \int_0^1 g(\dot{\lambda}^{(k)}, \dot{\lambda}^{(k)}) ds - \ln C_r(q) \quad (4.14)$$

$$= \frac{1}{4 r^2} \sum_{k=1}^q \left| \exp_z^{-1} \phi(k) \right|^2 - \ln C_r(q),$$

where $\exp_z^{-1} \phi(k) \in T_z M$, $\sum_{k=1}^q \exp_{z, \text{cm}}^{-1} \phi(k) = 0$, and

$$\left| \exp_z^{-1} \phi(k) \right| := \left[ (\exp_z^{-1} \phi(k))^a (\exp_z^{-1} \phi(k))^b \delta_{ab} \right]^{1/2} = d_g(\phi(k), z). \quad (4.15)$$

In particular, away from the cut locus $\text{Cut}(\phi(k))$ of each $\phi(k)$, the dilaton field $f$ is differentiable on $M \setminus \bigcup_{k=1}^q \{\phi(k), \text{Cut}(\phi(k))\}$, and one computes

$$\frac{\partial}{\partial z^i} f(z; q) = -\frac{1}{2 r^2} \sum_{k=1}^q \dddot{\lambda}^{(k)}(s) \Big|_{s=1} = -\frac{1}{2 r^2} g_{ih}(z) \sum_{k=1}^q (\exp_z^{-1} \phi(k))^i \quad (4.16).$$

The Wasserstein geometry of non-linear $\sigma$ models
Moreover, if \( \phi \) in our case see also Section 4.1 below). Hence we have

\[
\phi \text{ (cf. Eells–Fuglede [72] th. 9.1 pp. 153-154; for a general metric space setup)}
\]

simplexwise smooth Riemannian metric \( (X, \gamma) \) concerning harmonic maps

\[
\phi \text{ may be not available. However, if the map }\phi \text{ is Lipschitz on } (M, g) \text{. In general, the composition between a map }\phi \in H^1(\Sigma, M) \text{ and a Lipschitz function } f : M \to \mathbb{R} \text{ does not map naturally in } H^1(\Sigma, \mathbb{R}) \text{ since the distributional gradient } \nabla \phi(x) \text{ does not necessarily vanish almost everywhere for } \phi(x) \in \cup_{k=1}^q \{ \text{Cut}(\phi(\Omega)) \}, \text{ and a chain rule for differentiating weakly } f \circ \phi \text{ may be not available. However, if the map } \phi \in H^1(\Sigma, M) \text{ is of finite harmonic energy } E[\phi, g|_{\Sigma, M}] \text{ then the function } d_g(\phi(x), y) \text{ is, for any point } y \in M \text{ of class } H^1(\Sigma, \mathbb{R}) \text{. This is a particular case of a more general result concerning harmonic maps } L^2(X, Y) \text{ between a Riemannian polyhedron, with simplexwise smooth Riemannian metric } (X, \gamma) \text{, and a metric space } (Y, d_Y) \text{. (cf. Eells–Fuglede [72] th. 9.1 pp. 153-154; for a general metric space setup see also Section 4.1 below). Hence we have } f \circ \phi \in H^1(\Sigma, \mathbb{R}) \text{. Moreover, if } \phi \in H^1(\Sigma, M) \text{ is localizable, we can introduce local coordinates } \{ D_{(a)}, x^a \} \text{ in } (\Sigma, \gamma) \text{, and } y^k = \phi^k(x), \ k = 1, \ldots, n, \text{ for the corresponding image points in } \phi(D_{(a)}) \subset M, \text{ and work locally in the smooth framework provided by the space of smooth maps } \text{Map}(\Sigma, M \times (f) \mathbb{T}^q). \]
As an elementary consequence of the duality between the metric warping \((4.17)\) and the map warping \((4.19)\) we get

**Lemma 4.7.** The harmonic energy functional associated with the map \(\Phi(q)\) is provided by

\[
E[\Phi(q), h^{(q)}]_{(\Sigma, N^{n+q})} = E[\phi, g]_{(\Sigma, M)} \quad (4.20)
\]

\[
+ \frac{F(\phi_{cm}; q)}{2} \mathcal{D}[q^{-1} f(\phi; q)]_{(\Sigma, R)},
\]

where \(\mathcal{D}[q^{-1} f(\phi; q)]_{(\Sigma, R)}\) is the Dirichlet energy

\[
\mathcal{D}[q^{-1} f(\phi; q)]_{(\Sigma, R)} := \frac{1}{2} \int_{\Sigma} \left| \frac{df(\phi(x); q)}{q} \right|^2 d\mu_\gamma(x) \quad (4.21)
\]

associated to the map \(q^{-1} f \circ \phi : (\Sigma, \gamma) \rightarrow \mathbb{R},\) and \(F(\phi_{cm}; q)\) is the minimum of the center of mass function \((4.3)\).

**Proof.** Since the map \(\Phi(q)\) is localizable and \(f \circ \phi \in H^1(\Sigma, \mathbb{R})\), a direct computation in local charts provides

\[
E[\Phi(q), h^{(q)}]_{(\Sigma, N^{n+q})} := \frac{1}{2} \int_{\Sigma} \gamma^{\mu \nu} \partial \Phi^a_{(q)}(x, \xi) \partial \Phi^b_{(q)}(x, \xi) h_{ab}(\phi) d\mu_\gamma \quad (4.22)
\]

\[
+ \frac{1}{2} \sum_{k=1}^{q} d^2_g(\phi_{cm}, \phi(k)) \int_{\Sigma} \left| \frac{df(\phi(x); q)}{q} \right|^2 d\mu_\gamma
\]

\[
+ \frac{F(\phi_{cm}; q)}{2} \mathcal{D}[q^{-1} f(\phi; q)]_{(\Sigma, R)},
\]

where \(a, b = 1, \ldots, n + q, \left| df(\phi; q) \right|^2 := \gamma^{\mu \nu} \partial f(\phi; q) \partial f(\phi; q),\) and where we exploited the relation

\[
\gamma^{\mu \nu} \frac{\partial \Phi^l_{(q)}(x, \xi)}{\partial x^\mu} \frac{\partial \Phi^m_{(q)}(x, \xi)}{\partial x^\nu} \delta_{lm} = \frac{1}{4} \left| \frac{df(\phi; q)}{q} \right|^2 \epsilon^2 \sum_{k=1}^{q} d^2_g(\phi_{cm}, \phi(k)), \quad (4.23)
\]

for \(l, m = n + 1, \ldots, q.\)

This extended harmonic map set–up is interesting in many respects. In particular, if we choose the given surface \((\Sigma, \gamma)\) to be topologically the 2–torus \(T^2\) endowed with a conformally flat metric associated to the dilaton field, then the functional \(E[\Phi(q), h^{(q)}]\) can be directly connected to the non–linear \(\sigma\) model dilatonic action \((3.9).\) The underlying formal procedure is well–known, but in our setting the center of mass localization makes its role rather subtle and far reaching.
Proposition 4.8. Let \((\Sigma \simeq \mathbb{T}^2, \delta)\) be a flat 2–torus. For \(\phi \in \mathcal{H}^1(\mathbb{T}^2, M)\) a localizable map taking values in \(M \setminus \bigcup_{k=1}^q \text{Cut}(\phi(k))\), we denote by \(f \circ \phi\) the induced dilaton field over \(\phi(\mathbb{T}^2)\). If we endow \(\mathbb{T}^2\) with the conformally flat metric 

\[
\gamma_{\mu\nu} = e^{f(\phi;q)} \delta_{\mu\nu},
\]

then in the resulting conformal gauge \((\mathbb{T}^2, \gamma)\) we can write

\[
S[\gamma, \phi; F(\phi_{cm};q), g, dw] = \frac{2q}{q} E[\Phi(q), h(q)](\Sigma, N^{n+q}),
\]

Proof. Under the stated hypotheses, \(\phi(\mathbb{T}^2) \subseteq M \setminus \bigcup_{k=1}^q \text{Cut}(\phi(k))\) so that \(f\), restricted to \(\phi(\mathbb{T}^2)\), is smooth. Since \(\phi\) is localizable, we can use local charts and compute (in the weak sense) the Gaussian curvature of \((\mathbb{T}^2, \gamma)\) according to 

\[
K_f = -\frac{1}{2q} \Delta_f f(\phi;q).
\]

By integrating \(K_f f(\phi;q)\) over the surface \(\Sigma \simeq \mathbb{T}^2\) we get

\[
\int_{\Sigma} f(\phi;q) K_f d\mu_\gamma = -\frac{1}{2q} \int_{\Sigma} f(\phi;q) \Delta_f f(\phi;q) d\mu_\gamma
\]

\[
= \frac{1}{2q} \int_{\Sigma} |d f(\phi;q)|^2 \gamma d\mu_\gamma = q \mathcal{D}[g^{-1} f(\phi;q)](\Sigma, \mathbb{R}),
\]

where we have integrated by parts, and where the underset \((f)\) stresses the fact that the relation holds in the dilaton induced conformal gauge \((4.24)\). Thus, \((4.20)\) can be rewritten as

\[
E[\Phi(q), h(q)](\Sigma, N^{n+q}) = \frac{F(\phi_{cm};q)}{2q} \int_{\Sigma} f(\phi;q) K_f d\mu_\gamma,
\]

and if we identify the non–linear \(\sigma\) model coupling constant \(a\) with

\[
a \equiv \frac{F(\phi_{cm};q)}{q},
\]

then \((4.25)\) follows from the definition \((3.9)\) of the dilatonic action. \(\square\)

Remark 4.9. If in the statement of Prop. 4.8 we do not restrict \(\phi(\mathbb{T}^2)\) to \(M \setminus \bigcup_{k=1}^q \text{Cut}(\phi(k))\), then the distribution \(\Delta_f f(\phi;q)\) would acquire a singular part supported on the inverse image of the cut locus \(\bigcup_{k=1}^q \text{Cut}(\phi(k))\) of \((M, g, \{\phi(k)\})\). This singular part contributes to \(K_f\) with conical–metric singularities, as in the case of polyhedral surfaces. It is not difficult to extend the above results to this more general setting, by considering from the very outset polyhedral surfaces \((\Sigma \simeq \mathbb{T}^2)\) supporting a piecewise flat metric with conical singularities. We do not belabor on this point, since it does not add much to the analysis that follows.

Remark 4.10. Note that \(q^{-1} F(\phi_{cm};q)\) is the average squared distance between the constant maps \(\{\phi(k)\}\) and their center of mass \(\phi_{cm}\). According to the assumptions leading to lemma 4.1, the center of mass is contained in a
metric ball \(B(p, 2r)\) of radius \(2r\) with \(r < \frac{\kappa}{6 \sqrt{n}}\), where \(\kappa\) is the upper bound to the sectional curvature of \((M, g)\). This implies that the average squared distance between the \(\{\phi_{(k)}\}'s\) and their center of mass \(\phi_{cm}\) is bounded by the (squared) diameter of \(B(p, 2r)\). In particular, under the identification (4.28), the non–linear \(\sigma\) model coupling constant \(a\) satisfies
\[
a |\kappa| \leq \frac{4}{9} \pi^2 ,
\]
which is weaker than the bound \(a |\kappa| < < 1\) characterizing the point–like limit in which the renormalization group for non–linear \(\sigma\) model yields the Ricci flow.

4.1. Harmonic energy as a center of mass functional

The presence of the center of mass \(q^{-1} F(\phi_{cm}; q)\) as a dilatonic coupling in the above analysis may appear incidental to the particular set up we have concocted. Actually, it is a manifestation of the fact that the center of mass plays a basic role in characterizing harmonic map functionals in a general metric space setting [72], [32], [73]. It is important to make this role explicitly available also for non–linear \(\sigma\) models, since it provides the rationale for the use of the heat kernel embedding in Wasserstein space.

We start by characterizing the metric space structure associated with the warped manifold \((N^{n+q} := M \times_{(f)} \mathbb{T}^q, h^{(q)})\). If \(\lambda : [0, 1] \rightarrow N^{n+q}, \ s \mapsto \lambda(s) = (y(s), \xi(s))\) is a curve in \((N^{n+q}, h^{(q)})\), and \(\{s_j\} := 0 = s_0 < \ldots < s_m = 1\) is a partition of \([0, 1]\), then the length of \(\lambda\) in \((N^{n+q}, h^{(q)})\) is defined by, (see e.g. [74]),
\[
L_N(\lambda) := \lim_{\{s_j\} \rightarrow \rightarrow} \sum_{i=1}^m \left[ d_g^2(y(s_{i-1}), y(s_i)) + e^{-\frac{2f(y(s_{i-1}))}{q}} d_{T_\mathbb{T}^q}^2(\xi(s_{i-1}), \xi(s_i)) \right]^{\frac{1}{2}},
\]
where the limit is with respect to the refinement of ordering of the partitions \(\{s_j\}\) of \([0, 1]\), and
\[
d_{T_\mathbb{T}^q}(\xi(s_{i-1}), \xi(s_i)) := \inf_{m_k \in \mathbb{Z}^q} \sum_{k=1}^q \left| \bar{\xi}_{(k)}(s_{i-1}) - \bar{\xi}_{(k)}(s_i) + m_k \right|^2
\]
is the squared distance in \(\mathbb{T}^q\), \(\bar{\xi}_{(k)} \in \mathbb{R}^q\) being the representative components of \(\xi \in \mathbb{T}^q\). If we denote by \(\Gamma_{(Y, Z)}\) the set of all (piecewise) smooth curves connecting two points \(Y = (y, \xi)\) and \(Z = (z, \zeta)\) in \(N^{n+q}\), then their distance
\[
d_h(Y, Z) := \inf_{\lambda \in \Gamma_{(Y, Z)}} L_N(\lambda),
\]
characterizes the metric space \((N^{n+q}, d_h)\) associated with the warped Riemannian manifold \((N^{n+q}, h^{(q)})\).

With these preliminary remarks along the way, fix \(x \in \Sigma\), and for \(0 < \epsilon < 1\) small enough, let \(D(x, \epsilon)\) be the disk of radius \(\epsilon\) in the tangent space \(T_x \Sigma\),
and denote by \( \exp_x^{(\gamma)} : T_x \Sigma \rightarrow (\Sigma, \gamma) \) the exponential mapping based at \( x \). We assume that \( \exp_x^{(\gamma)}(D(x, \epsilon)) \) is convex and that \( D(x, \epsilon) \) is endowed with the pull–back measure \( d\tilde{\mu}_\gamma := \left( \exp_x^{(\gamma)} \right)^* d\mu_\gamma \). Let us consider the map

\[
\tilde{\Phi}(q) : D(x, \epsilon) \subset T_x \Sigma \rightarrow N^{n+q}
\]

\[
v \mapsto \tilde{\Phi}(q)(v) := \Phi(q) \left( \exp_x^{(\gamma)}(v) \right),
\]

and define, for almost all \( x \in \Sigma \), the center of mass function

\[
\Sigma \rightarrow \mathbb{R}_{>0}
\]

\[
x \mapsto G[\Phi(q)(x), D(x, \epsilon)]
\]

\[
= \frac{1}{|D_\delta(\epsilon)|} \int_{D(x,\epsilon)} d^2_h \left( \Phi(q)(x), \Phi(q) \left( \exp_x^{(\gamma)}(v) \right) \right) d\tilde{\mu}_\gamma(v),
\]

where \( |D_\delta(\epsilon)| = 4\pi \epsilon^2 \) is the Euclidean area of the disk \( D(x, \epsilon) \subset T_x \Sigma \), and \( d_h \) is the distance \([4.32]\).

**Remark 4.11.** Note that \( \Phi(q)(x) \) is an actual center of mass of the subset \( \tilde{\Phi}(q)(D(x, \epsilon)) \), with respect to the push–forward measure \( \left( \tilde{\Phi}(q) \right)_# d\tilde{\mu}_\gamma \), if the point \( \Phi(q)(x) \in N^{n+q} \) minimizes the function

\[
Y \mapsto G[Y, D(x, \epsilon)]
\]

\[
= \frac{1}{|D_\delta(\epsilon)|} \int_{D(x,\epsilon)} d^2_h \left( Y, \Phi(q) \left( \exp_x^{(\gamma)}(v) \right) \right) d\tilde{\mu}_\gamma(v).
\]

In particular, we shall say that the map \( \Phi(q) \in \mathcal{H}^1(\Sigma, N^{n+q}) \) is an \( \epsilon \)–center of mass in the above sense if \( \Phi(q)(x) \) minimizes \([4.35]\) for almost every \( x \in \Sigma \).

From the very structure of \([4.34]\) it follows that, by integrating the center of mass function \( G[D(x, \epsilon)] \) over \( (\Sigma, \gamma) \), we can characterize an approximate energy functional for maps of low regularity according to

**Definition 4.12.** Let \( \Phi(q) \in L^2(\Sigma, N^{n+q}) \) be a square summable map \( \Phi(q) : (\Sigma, \gamma) \rightarrow (N^{n+q}, d_h) \), and let

\[
E_\epsilon[\Phi(q), d_h] := \frac{1}{2} \int_{\Sigma} \frac{G[\Phi(q)(x), D(x, \epsilon)]}{\epsilon^2} d\mu_\gamma(x),
\]

denote the \( \epsilon \)–approximate energy associated with the center of mass function \( x \mapsto G[D(x, \epsilon)] \), then the \( L^2 \)–energy of the map \( \Phi(q) \) is defined by

\[
\mathcal{E}[\Phi(q), d_h] := \lim_{\epsilon \rightarrow 0} E_\epsilon[\Phi(q), d_h] \in \mathbb{R} \cup \{+\infty\},
\]

where the limit is in the sense of weak convergence of measures.

**Remark 4.13.** It is elementary to prove, (as in \([32]\), Lemma 8.4.1), that the \( \epsilon \)–approximate energy functional \( E_\epsilon[\Phi(q), d_h] \) is minimized iff \( \Phi(q)(x) \) is a center of mass for almost all \( x \in \Sigma \).
The basic observation in this general setting is that the center of mass functional \( E\left[\Phi(q)\right] \) reduces to the standard harmonic map energy \( E\left[\Phi(q), h^{(q)}\right] \) whenever the map \( \Phi(q) \) is regular enough. In particular we have

**Proposition 4.14.** If \( \Phi(q) \) is a localizable map \( \in H^1(\Sigma, N^{n+q}) \), then

\[
\lim_{\epsilon \to 0} E_\epsilon \left[\Phi(q), d_h\right] = E \left[\Phi(q), h^{(q)}\right].
\] (4.38)

**Proof.** This is a particular case of a more general result, (for instance one can adapt, with obvious modifications, Theor. 8.4.1 of [32]), stating that the equality (4.38) holds as soon as \( \Phi(q) \in L^2(\Sigma, N^{n+q}) \) is localizable and the functional \( E\left[\Phi(q), h^{(q)}\right] \) can be defined, (in the local chart associated to the localization adopted). \( \square \)

It is worthwhile to note that we can express the center of mass functional \( E_\epsilon \left[\Phi(q), d_h\right] \) in terms of the metric geometry of the factor manifolds \( (M, d_g) \) and \( \mathbb{T}^q \) defining \( N^{n+q} \). Explicitly, according to the definition of the map \( \Phi(q) \), and of the characterization (4.30) and (4.32) of the distance function \( d_h \), we can write

\[
d_h^2 \left(\Phi(q)(x), \Phi(q,x)(v)\right) = d_g^2 \left(\phi(x), \phi(\exp_x^{(\gamma)}(v))\right)
\] (4.39)

\[+ \frac{q^2}{4} e^{-\frac{2f(\phi(x))}{q}} \sum_{k=1}^{q} d_g^2(\phi_{cm}, \phi(k)) \left| e \frac{f(\phi(x))}{q} - e \frac{f(\phi(\exp_x^{(\gamma)}(v)))}{q} \right|^2,
\]

which, inserted in (4.36), provides

\[
E_\epsilon \left[\Phi(q), d_h\right] = \frac{1}{2 D_\delta(\epsilon)} \int_{\Sigma} \left( \int_{D(x,\epsilon)} d_g^2 \left(\phi(x), \phi(\exp_x^{(\gamma)}(v))\right) \frac{d\widehat{\mu}_\gamma(v)}{\epsilon^2} \right) d\mu_\gamma(x)
\]

\[+ \frac{q^2}{e^2 |D_\delta(\epsilon)|} \sum_{k=1}^{q} \frac{d_g^2(\phi_{cm}, \phi(k))}{8} \times
\]

\[\times \int_{\Sigma} \left( \int_{D(x,\epsilon)} e^{-\frac{2f(\phi(x))}{q}} \left| e \frac{f(\phi(x))}{q} - e \frac{f(\phi(\exp_x^{(\gamma)}(v)))}{q} \right|^2 \frac{d\widehat{\mu}_\gamma(v)}{\epsilon^2} \right) d\mu_\gamma(x).
\]

In analogy with (4.37), we define the harmonic map energy functional for
maps $\phi$ in $L^2(\Sigma, M)$ according to

$$
E[\phi, d_g] := \lim_{\epsilon \to 0} \frac{1}{D_\delta(\epsilon)} \int_\Sigma \left( \int_{D(x, \epsilon)} d_g^2 \left( \phi(x), \phi(\exp_x^{(\gamma)}(v)) \right) \frac{d\mu_\gamma(v)}{\epsilon^2} \right) d\mu_\gamma(x),
$$

and the Dirichlet energy for functions $f \circ \phi$ in $L^2(\Sigma, \mathbb{R})$

$$
\mathcal{D}[f(\phi; q)] := \lim_{\epsilon \to 0} \int_\Sigma \left( \int_{D(x, \epsilon)} q^2 e^{-\frac{2f(\phi(x))}{q}} \frac{e^{-\frac{2f(\phi(x))}{q}}}{2} \frac{e^{-\frac{2f(\phi(\exp_x^{(\gamma)}(v)))}{q}}}{2} \right) d\mu_\gamma(v) d\mu_\gamma(x).
$$

We have

$$
E[\Phi(q), d_h] := E[\phi, d_g] + \frac{F(\phi_{cm}; q)}{2} \mathcal{D}[f(\phi; q)],
$$

with the obvious proviso that for square summable maps each one of the (lower semicontinuous) functionals $E[\Phi(q); h(q)]$, $E[\phi, d_g]$, and $\mathcal{D}[f(\phi; q)]$ defined above can be infinite. Again, for localizable Sobolev maps $\Phi(q)$ one can prove that $E[\phi, d_g]$, and $\mathcal{D}[f(\phi; q)]$ can be identified with their corresponding functional $E[\phi, g]$, and $\mathcal{D}[f(\phi; q)]$, appearing in the factorization (4.20).

**Remark 4.15.** The approximation scheme underlying (4.36) and (4.40) naturally appears, somewhat in disguise, when considering the cut–off action functional for the regularized lattice versions of quantum non–linear $\sigma$ models, (see e.g. [34]).

### 4.2. Heat Kernel embedding in Wasserstein space

To put things in context, let us consider the isometric embedding of $(M, d_g)$ into $(\text{Prob}(M), d_g^W)$ defined by

$$(M, d_g) \hookrightarrow (\text{Prob}(M), d_g^W)
$$

$$
y \mapsto \delta_y,
$$

where $\delta_y$ is the Dirac measure supported at the generic $y \in (M, g)$. It is easy to prove that this is indeed an isometry since one directly computes $d_g^W(\delta_y, \delta_z) = d_g(y, z)$, by using the obvious optimal plan $d\sigma(u, v) = \delta_y(u) \otimes \delta_z(v)$ in (3.48). This isometry allows to represent the center of mass coupling as

$$a := q^{-1} F(\phi_{cm}; q) = \frac{1}{2q} \sum_{k=1}^q \left[ d_g^W(\delta_{(k)}, \delta_{cm}) \right]^2,
$$

where $\delta_{(k)}$ and $\delta_{cm}$ are the Dirac measures supported at $\phi_{(k)}$ and $\phi_{cm}$, respectively. The advantage of the, otherwise rather formal, Wasserstein representation (4.45) is that we can view $\delta_{(k)}$ and $\delta_{cm}$ as heat sources. This suggests
a natural heat kernel technique for the geometrical analysis of the scaling
properties of $q^{-1}F(\phi_{cm}; q)$ and, as we shall see, of the functionals $E[\phi, g]$, and
$S[\gamma, \phi; a, g, d\omega]$.

We start exploiting the Wasserstein representation (4.45) and discussthe
deformation of the center of mass coupling $q^{-1}F(\phi_{cm}; q)$ by extending (4.44)
to a heat kernel embedding of the constant maps $\{\phi_{(k)}\}$. To this end, let $t$ be
a running (squared) length scale and denote by $p_{Mt}^{M} : M \times M \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$
the heat kernel on $(M, g)$, i.e. the (minimal positive) solution of the heat
equation
\[
\left( \frac{\partial}{\partial t} - \Delta_{(g, y)}^{M} \right) p_{M}^{M}(y, z) = 0 , \tag{4.46}
\]
\[
\lim_{t \searrow 0^{+}} p_{M}^{M}(y, z) d\mu_{g}(z) = \delta_{z} ,
\]
where $\delta_{z}(y)$ and $\Delta_{(g, y)}^{M}$ denote the Dirac measure at $z \in M$ and the Laplace–
Beltrami operator on $(M, g)$, respectively. Also let
\[
p_{M}^{M}(y, \phi_{(k)}) = \int_{M} p_{t-s}^{M}(y, z) p_{s}^{M}(\phi_{(k)}, z) d\mu_{g}(z) , \quad t > s > 0 , \tag{4.47}
\]
and
\[
p_{M}^{M}(y, \phi_{cm}) = \int_{M} p_{t-s}^{M}(y, z) p_{s}^{M}(\phi_{cm}, z) d\mu_{g}(z) , \quad t > s > 0 , \tag{4.48}
\]
be the heat kernels with sources the Dirac measures $\delta_{(k)}$ and $\delta_{cm}$, respectively.
Note that, away of the cut locus $\text{Cut } (z)$, $p_{M}^{M}(y, z)$ can be differentiated any
number of times and its spatial derivatives commute with the $t \searrow 0^{+}$ limit,
(this is an elementary consequence of a theorem by Malliavin and Strook [75]–see also [76]). In particular, it is useful to recall the following asymptotics
related to the geometry of heat diffusion on Riemannian manifolds.

**Theorem 4.16.** [77], (see also [78], [79]). There are smooth functions $\Xi_{k}(y, z)$
defined on $(M \times M)/\text{Cut } (z)$, with $\Upsilon_{0}(y, z) = 1$, such that the asymptotic expansion
\[
p_{t}^{M}(y, z) d\mu_{g}(y) - \left( \frac{1}{4\pi t} \right)^{n/2} e^{-\frac{d_{g}^{2}(y, z)}{4t}} \sum_{k=0}^{N} \Xi_{k}(y, z) t^{k} d\mu_{g}(y) = \mathcal{O} \left( t^{N+1-\frac{n}{2}} \right) \tag{4.49}
\]
holds uniformly as $t \searrow 0^{+}$ on compact subsets of $(M \times M)/\text{Cut } (z)$.

**Theorem 4.17.** (T. H. Parker [80]). Let $\text{Cut}_{L} \subset (M \times M)$ be the set of points
$(y, z)$ such that $z$ is conjugate to $y$ along some geodesics $\gamma$ with length at most $L$. If $D_{\gamma}$ denotes the (absolute value of the) Jacobian of the exponential map
along $\gamma$, then
\[
p_{t}^{M}(y, z) = \left( \frac{1}{4\pi t} \right)^{n/2} \sum_{\gamma} e^{-\frac{1}{4} \int_{0}^{1} |\gamma'(s)|^{2} ds} D_{\gamma}^{-\frac{1}{2}} \left( 1 + \mathcal{O}(t) \right) , \tag{4.50}
\]
where the convergence is uniform as $\lim_{t \uparrow 0^+}$ on compact subsets of $(M \times M)/\text{Cut}_t$, and where the summation is over all locally minimal geodesics $\gamma$ in $(M, g)$ connecting $y$ to $z$.

Remark 4.18. The former is the well–known heat kernel asymptotics, the latter is a small–time asymptotics recently obtained by T. H. Parker [80], showing quite explicitly that heat tends to diffuse instantaneously along geodesics.

In particular, (4.50) can be heuristically identified with the semiclassical approximation to the (Euclidean) path integral representation of $p_t^M(y, z)$ provided by

$$p_t^M(y, z) = \int_{P_{yz}} e^{-\frac{1}{4} \int_0^t |\gamma(s)|^2 \, ds} \mathcal{D}[\gamma],$$  

(4.51)

where $\mathcal{D}[\gamma]$ is a functional measure on the space $P_{yz}$ of all paths $\gamma$ connecting, in time $t$, the points $y$ and $z$.

Since $\frac{1}{4} \int_0^t |\gamma(s)|^2 \, ds = \frac{d^2(y, z)}{4t}$ along a minimal geodesic $\gamma$, both these asymptotic expansions provide the heuristics of the celebrated Varadhan’s large deviation formula [81],

$$- \lim_{t \uparrow 0^+} t \ln \left[ p_t^M(y, z) \right] = \frac{d^2(y, z)}{4},$$  

(4.52)

which, together with the expression (4.3) for the center of mass function, (evaluated at $\phi_{cm}$), allows us to provide yet another formal representation for the coupling,

$$F(\phi_{cm}; q) = - \lim_{t \uparrow 0^+} 2t \ln \prod_{k=1}^q p_t^M(\phi(k), \phi_{cm}).$$  

(4.53)

The convergence in (4.52), and hence in (4.53) is uniform, and together with (4.45), the expression (4.53) suggests the possibility of deforming $q^{-1} F(\phi_{cm}; q)$ by using the heat kernel flow $(\delta(k), t) \mapsto p_t^M$, $t \in (0, \infty)$ in $(\text{Prob}(M), d^W_g)$. This is tantamount to extending the isometry (4.44) to the injective embedding defined, for any fixed length scale $t \in [0, \infty)$, by the map

$$(M, d_g) \times \mathbb{R}_{\geq 0} \leftrightarrow (\text{Prob}(M), d^W_g)$$  

(4.54)

$$(y, t) \quad \mapsto \quad p_t^M(\cdot, y) \, d\mu_g(\cdot),$$

which associates to each point $y \in (M, g)$ the corresponding heat kernel measure with source at $y$.

Remark 4.19. The injectivity of the heat kernel embedding in quadratic Wasserstein space is discussed at length in [27]. Below we shall analyze (4.54) in detail in the case of the weighted heat kernel associated with $(M, g, d\omega)$.

Under the action of (4.54), the constant maps $\{\phi(k)\}$ and their center of mass $\phi_{cm}$ are injected in $(\text{Prob}(M), d^W_g)$ according to

$$(\phi(k), t) \quad \mapsto \quad (\delta(k), t) \mapsto p_t^M(\cdot, \phi(k)) \, d\mu_g(\cdot)$$  

(4.55)

$$(\phi_{cm}, t) \quad \mapsto \quad (\delta_{cm}, t) \mapsto p_t^M(\cdot, \phi_{cm}) \, d\mu_g(\cdot).$$
The Wasserstein geometry of non-linear $\sigma$ models

The coupling $a := q^{-1} F(\phi_{cm}; q)$ is correspondingly deformed into

$$ (a, t) \mapsto a_t $$

(4.56)

$$ := \frac{1}{2q} \sum_{k=1}^{q} \left[ d_W^g \left( p_t^M (\cdot, \phi_{(k)}) d\mu_g (\cdot), p_t^M (\cdot, \phi_{cm}) d\mu_g (\cdot) \right) \right]^2 , $n
with

$$ \lim_{t \to 0} a_t = \frac{1}{2q} \sum_{k=1}^{q} \left[ d_W^g (\delta_{(k)}, \delta_{cm}) \right]^2 = \frac{F(\phi_{cm}; q)}{q} = a .
$$

(4.57)

The induced scaling in passing from $a$ to $a_t$ is geometrically controlled by the

**Proposition 4.20.** If $K$ denote the lower bound of the Ricci curvature of $(M, g)$, (i.e. $Ric_g (v, v) \geq K |v|^2$, $\forall v \in TM$), then the coupling $a$ scales under heat kernel deformation of the constant maps $\{ \phi_{(k)} \}$ according to

$$ a_t \leq e^{-2Kt} a . $$

(4.58)

**Proof.** (4.58) directly follows from the basic inequality [58], [82], [26],

$$ d_W^g \left( p_t^M (\cdot, y) d\mu_g (\cdot), p_t^M (\cdot, z) d\mu_g (\cdot) \right) ^2 \leq e^{-Kt} d_W^g (\delta_y, \delta_z) = e^{-Kt} d_g (y, z) , \quad \forall t > 0 , ,
$$
governing the Wasserstein geometry of heat diffusion on a Riemannian manifold.  

The general role of the center of mass, discussed in Section 4.1 indicates that we can analyze the scaling behavior of the harmonic energy functional $E[\Phi_{(q)}, h^{(q)}]$, and hence of the dilatonic action $S[\gamma, \phi; a, g, d\omega]$, along the same lines described above. To this end, we need to characterize the heat kernel injection of $(\Phi_{(q)} (\Sigma), h^{(q)})$ in the Wasserstein space $(\text{Prob}(N), d_W^h)$ of Borel probability measures on $N^{n+q}$.

**5. A warped Gigli–Mantegazza heat kernel embedding**

The structure of $(M \times \omega \mathbb{T}^q, h^{(q)})$ allows us to model a significant part of our analysis on the results discussed in a terse paper by N. Gigli and C. Mantegazza [27]. They analyze in detail the geometry of the heat kernel embedding associated to the Laplace–Beltrami heat semigroup, connecting it to the Ricci flow. In our case, the relevant heat semigroup is the one generated by a weighted Laplacian on $(M, g, d\omega)$, and we provide an extension of their results to the warped Riemannian manifold $(M \times \omega \mathbb{T}^q, h^{(q)})$. This allows us to discuss the scaling behavior of $E[\Phi_{(q)}, h^{(q)}]$ and connect it to the Hamilton–Perelman version of the Ricci flow.

As a preliminary step, we need to define a heat kernel adapted to the warped product structure of $(M \times \omega \mathbb{T}^q, h^{(q)})$. To this end, let $Y^a := (y^i, v^a)$, with
\(a = 1, \ldots, n+q, \quad i = 1, \ldots, n, \quad \text{and} \quad \alpha = 1, \ldots, q\) denote coordinates adapted to \(M \times_{\omega} T^q\). Under such a splitting, a standard computation for the Laplacian \(\Delta_h\) on \((M \times_{\omega} T^q, h^{(q)})\) provides the relation

\[
\begin{align*}
\Delta_h &= \frac{1}{\sqrt{\det h}} \frac{\partial}{\partial Y^a} \left( \sqrt{\det h} h^{ab} \frac{\partial}{\partial Y^b} \right) \\
&= \frac{e^f}{\sqrt{\det g}} \frac{\partial}{\partial y^i} \left( \sqrt{\det g} e^{-f} g^{ik} \frac{\partial}{\partial y^k} \right) + e^q \frac{\partial}{\partial \nu^\alpha} \left( \delta^{\alpha\beta} \frac{\partial}{\partial \nu^\beta} \right) \\
&= \Delta_g + e^{2f} \Delta_T - \nabla f \cdot \nabla,
\end{align*}
\]

where \(\Delta_g\) and \(\Delta_T\) respectively denote the Laplace–Beltrami operator on \((M, g)\), and on \((T^q, \delta)\), and where \(\nabla f \cdot \nabla = g^{ik} \nabla_i f \nabla_k\) is the distributional directional derivative along the gradient of the Lipschitz function \(f\). According to (3.17) the operator \(\Delta_g - \nabla f \cdot \nabla\) appearing in (5.1) is the weighted Laplacian on \((M, g, d\omega)\), (cf. (3.17)),

\[
\Delta_\omega := \text{div}_\omega \nabla = \Delta_g - \nabla f \cdot \nabla, \quad (5.2)
\]

and we can write

\[
\Delta_h = \Delta_\omega + e^{2f} \Delta_T. \quad (5.3)
\]

The properties of the heat kernel associated to \(\Delta_\omega\) are provided by [38] Theorem 5.1. The weighted Laplacian \(\Delta_\omega\) is symmetric with respect to the defining measure \(d\omega\), and can be extended to a self–adjoint operator in \(L^2(M, d\omega)\) generating the heat semigroup \(e^{t \Delta_\omega}\), \(t \in \mathbb{R}_{>0}\). The associated heat kernel \(p_t(\omega)(\cdot, z)\) is defined as the minimal positive solution of

\[
\begin{align*}
\left( \frac{\partial}{\partial t} - \Delta_\omega \right) p_t(\omega)(y, z) &= 0, \\
\lim_{t \to 0^+} p_t(\omega)(y, z) d\omega(z) &= \delta_z,
\end{align*}
\]

with \(\delta_z\) the Dirac measure at \(z \in (M, d\omega)\). The heat kernel \(p_t(\omega)(y, z)\) is \(C^\infty\) on \(\mathbb{R}_{>0} \times M \times M\), is symmetric \(p_t(\omega)(y, z) = p_t(\omega)(z, y)\), satisfies the semigroup identity \(p_t(\omega)(y, z) = \int_M p_t(\omega)(y, x) p_\omega(x, z) d\omega(x)\), and \(\int_M p_t(\omega)(y, z) d\omega(z) = 1\). Moreover, Varadhan’s large deviation formula holds

\[
- \lim_{t \to 0^+} t \ln \left[ p_t(\omega)(y, z) \right] = \frac{d^2_g(y, z)}{4}, \quad (5.5)
\]

where the convergence is uniform over all \((M, g, d\omega)\).

\textbf{Proof.} This characterization of the heat kernel \(p_t(\omega)(\cdot, \cdot)\) is a direct consequence of the properties of the weighted Laplacian \(\Delta_\omega\). For a more detailed discussion see Grigoryan [38] Th.7.13, and § 7.5, Th. 7.20, for the smoothness property of the weighted heat kernel \(p_t(\omega)(\cdot, \cdot)\). \(\square\)
Remark 5.2. From the point of view of heat theory, the operator $\triangle_h$ generates a diffusion in the base manifold $(M, g, d\omega)$ driven by $\triangle_h$, weakly coupled to an independent heat propagation in the torus fibers $T_y$ via a thermal diffusivity given by $\exp\{2f(y)/q\}$. Since we do not need to locally vary the geometry of the torus fibers, (up to a fiberwise rescaling), we can freeze the heat diffusion in each $T_y$, and consider only the heat kernel embedding induced by $p_t(\omega)(y, z)$.

Lemma 5.3. Let $(\text{Prob}(N), d_{W}^{h})$, be the Wasserstein spaces of probability measures over the warped manifold $(N \simeq M \times_\omega T^q, h^{(q)})$. If $p_t^{(\omega)}(\cdot, z) d\omega(\cdot) \otimes \delta_\zeta^{T^q}$ denotes the tensor product between the weighted heat kernel on $(M, g, d\omega)$ and the Dirac measure supported at $\zeta \in T^q$, then the map

$$\Upsilon_t : (M \times_\omega T^q, h^{(q)}) \hookrightarrow (\text{Prob}(N), d_{h}^{W})$$

$$(z, \zeta) \longmapsto \Upsilon_t(z, \zeta) := p_t^{(\omega)}(\cdot, z) d\omega(\cdot) \otimes \delta_\zeta^{T^q},$$

is, for any $t \geq 0$, injective.

Proof. We have the obvious inclusion

$$\left(\text{Prob}(M, g), d_{g}^{W}\right) \times \left(\text{Prob}(T^q), d_{T}^{W}\right) \hookrightarrow (\text{Prob}(N), d_{h}^{W})$$

$$(p_t^{(\omega)}(\cdot, z) d\omega(\cdot), \delta_\zeta^{T^q}) \longmapsto p_t^{(\omega)}(\cdot, z) d\omega(\cdot) \otimes \delta_\zeta^{T^q}. \quad (5.7)$$

The map $\Upsilon_t$ restricted to the torus fibers, i.e. $\zeta \longmapsto \delta_\zeta^{T^q}$, is an isometric embedding of $T^q$ into $(\text{Prob}(T^q), d_{T}^{W})$. By adapting to the heat semigroup generated by $\triangle_h$ the analysis of the injectivity of the Laplace–Beltrami heat flow, (cf. Th. 2.3 and Proposition 5.16 of [27]), it follows that the restriction, $z \mapsto p_t^{(\omega)}(\cdot, z) d\omega(\cdot)$, of $\Upsilon_t$ to $(M, g, d\omega)$ is an injective embedding of $(M, g, d\omega)$ into $(\text{Prob}(M), d_{g}^{W})$. Hence $\Upsilon_t$ injects in $(\text{Prob}(N), d_{h}^{W})$. \qed

To discuss the properties of the map $\Upsilon_t$ defined by $(5.6)$, let us use coordinates $Z^\alpha := (z^i, \zeta^\alpha)$, with $a = 1, \ldots, n + q$, $i = 1, \ldots, n$, and $\alpha = 1, \ldots, q$, adapted to the product structure of the manifold $N := M \times_\omega T^q$. In the corresponding coordinate bases $\{\partial_i\}, \{\partial_\alpha\}$, let us consider vector fields $U^i_\perp \partial_i \in C^\infty(M, TM)$ and $U^\alpha_\parallel \partial_\alpha \in C^\infty(T^q, T T^q)$, and the associated vector field $U \in C^\infty(N, TN)$

$$U = U^a(z, \zeta)\partial_\alpha := U^i_\perp(z)\partial_i + U^\alpha_\parallel(\zeta)\partial_\alpha. \quad (5.8)$$

For any $t > 0$, we can naturally associate to the vector field $U_\perp$ a corresponding element in $T_{p_t(d\omega)}\text{Prob}(M, g)$, the tangent space to $\text{Prob}(M, g)$ at $p_t(d\omega) := p_t^{(\omega)}(\cdot, z) d\omega(\cdot)$.
Lemma 5.4. Let \( t \mapsto p_t^{(\omega)}(\cdot, z) \, d\omega(\cdot), t \in (0, \infty), \) be the flow of probability measure in \( \text{Prob}_{ac}(M, g) \) defined by the weighted heat kernel \( p_t^{(\omega)}(\cdot, z) \). Then, the map
\[
TM \times (0, \infty) \longrightarrow \text{C}^\infty(M, \mathbb{R})
\]
(5.9)

\[ (z, U_\perp(z); t) \longmapsto U_\perp^t(z) \nabla_i^{(z)} \ln p_t^{(\omega)}(\cdot, z), \]
defines, for each \( t \in (0, \infty) \), a tangent vector in \( T_{p_t(d\omega)} \text{Prob}(M, g) \).

Note that the superscript \( (z) \) in (5.9) signifies that the differentiation is applied to the indicated variable.

Remark 5.5. In the notation for the tangent space \( T_{p_t(d\omega)} \text{Prob}(M, g) \) we used the shorthand \( p_t(d\omega) \) to denote the probability measure on \( (M, g) \) associated with the heat kernel distribution \( p_t^{(\omega)}(\cdot, z) \) evaluated at time \( t \) and with a fixed source \( \delta_z \). This notation may become ambiguous when considering on \( T_{p_t(d\omega)} \text{Prob}(M, g) \) the associated \( L^2 \) inner product spaces, since these will functionally depend on the heat source. For instance, \( \psi \in T_{p_t(d\omega)} \text{Prob}(M, g) \), has a natural point–dependent \( L^2 \) norm evaluated, at fixed heat source \( \delta_z \), according to
\[
\| \psi \|^2_{L^2(p_t(d\omega, z))} := \int_M |\psi(y)|^2 \, p_t^{(\omega)}(y, z) \, d\omega(y) .
\]
(5.10)

We use the notation \( L^2(p_t(d\omega, z)) \) to emphasize, whenever necessary, the location of the fixed heat source, and the simpler notation \( L^2(p_t(d\omega)) \) if there is no danger of confusion.

Proof. To prove lemma 5.4 let us observe that for \( t > 0 \) the function \( y \mapsto U_\perp^t(z) \nabla_i^{(z)} \ln p_t^{(\omega)}(y, z) \) is smooth. By integrating over \( (M, p_t^{(\omega)} d\omega) \) we get
\[
\int_M \left( U_\perp^t(z) \nabla_i^{(z)} \ln p_t^{(\omega)}(y, z) \right) p_t^{(\omega)}(y, z) \, d\omega(y)
\]
(5.11)
\[
= \int_M U_\perp^t(z) \nabla_i^{(z)} p_t^{(\omega)}(y, z) \, d\omega(y)
\]
\[
= U_\perp^t(z) \nabla_i^{(z)} \int_M p_t^{(\omega)}(z, y) \, d\omega(y) = 0 ,
\]
where, in the last passage, we have exploited the symmetry of the heat kernel, \( p_t^{(\omega)}(z, y) = p_t^{(\omega)}(y, z) \), and \( \int_M p_t^{(\omega)}(z, y) \, d\omega(y) = 1 \). It follows that, for each \( t \in (0, \infty), \) \( 5.9 \) defines, according to (5.11), a tangent vector in \( T_{p_t(d\omega)} \text{Prob}(M, g) \).

Remark 5.6. By proceeding similarly, and using the map
\[
(z, U_\perp(z); t) \longmapsto U_\perp^t(z) \nabla_i^{(z)} p_t^{(\omega)}(\cdot, z)
\]
(5.12)
we can interpret \( U_\perp^t(z) \nabla_i^{(z)} p_t^{(\omega)}(\cdot, z) \) as an element of the tangent space \( T_{d\omega} \text{Prob}(M, g) \).
Remark 5.7. Roughly speaking, the vector field $U_i^\perp(z) \nabla_i^{(z)} \ln p_t^{(\omega)}(\cdot, z)$, (or, equivalently, $U_i^\perp(z) \nabla_i^{(z)} p_t^{(\omega)}(\cdot, z)$), can be thought of as describing the perturbation in the probability measure $p_t^{(\omega)}(\cdot, z)$ as we vary the heat source $\delta_z$ in the direction $U_\perp(z)$.

These remarks imply that we can exploit Otto’s parametrization (cf. [3,18]), and represent $U_i^\perp(z) \nabla_i^{(z)} p_t^{(\omega)}(\cdot, z)$, or $U_i^\perp(z) \nabla_i^{(z)} \ln p_t^{(\omega)}(\cdot, z)$, as (the gradient of) a scalar potential, $\psi_{(t,z,U_\perp)} \in C^\infty(M, \mathbb{R})$, according to the

**Proposition 5.8.** For each fixed $t > 0$, and for any $U_\perp \in C^\infty(M,TM)$, the elliptic partial differential equation,

$$
\text{div}_\omega \left( p_t^{(\omega)}(y, z) \nabla^{(y)} \psi_{(t,z,U_\perp)}(y) \right) = -U_\perp(z) \cdot \nabla^{(z)} p_t^{(\omega)}(y, z),
$$

(5.13)

admits a unique solution $\psi_{(t,z,U_\perp)} \in C^\infty(M, \mathbb{R})$, with $\int_M \psi_{(t,z,U_\perp)} d\omega = 0$, smoothly depending on the data $t, z, U_\perp$, and such that $\nabla^{(y)} \psi_{(t,z,U_\perp)}(y) \not\equiv 0$ if $U_\perp \not\equiv 0$.

**Proof.** This is a rather obvious adaptation of a similar statement in [27], Prop. 3.1. For its relevance in what follows, and for the convenience of the reader, we outline the proof in our case. According to the definition (3.17) of the weighted divergence and the positivity of the heat kernel we have

$$
\Delta^{(y)} \psi_{(t,z,U_\perp)}(y) - \nabla^{(y)} \left( f(y) - \ln p_t^{(\omega)}(y, z) \right) \cdot \nabla^{(y)} \psi_{(t,z,U_\perp)}(y)
$$

$$
= -U_\perp(z) \cdot \nabla^{(z)} \ln p_t^{(\omega)}(y, z).
$$

(5.14)

It follows that (5.13) can be equivalently rewritten as

$$
\Delta^{(y)}_{p_t^{(\omega)}} \psi_{(t,z,U_\perp)}(y) = -U_\perp(z) \cdot \nabla^{(z)} \ln p_t^{(\omega)}(y, z),
$$

(5.15)

where

$$
\Delta^{(y)}_{p_t^{(\omega)}} := \Delta^{(y)} - \nabla^{(y)} \left( f(y) - \ln p_t^{(\omega)}(y, z) \right) \cdot \nabla^{(y)}
$$

(5.16)

is the weighted Laplacian associated with the measure $p_t^{(\omega)}(y, z)d\omega(y)$. Solutions $\psi_{(t,z,U_\perp)}$ of (5.15) are naturally defined modulo an additive constant. To remove this redundancy, we normalize $\psi_{(t,z,U_\perp)}$ by requiring

$$
\int_M \psi_{(t,z,U_\perp)}(y) d\omega(y) = 0.
$$

(5.17)

For any given $t > 0$, denote by $\mathcal{H}^1(M, \mathbb{R}; p_t(d\omega))$ the Sobolev space of functions which together their gradients are square summable with respect to the heat kernel measure $p_t(d\omega)$, (see remark 5.5). Assume that $\psi_{(t,z,U_\perp)} \in$
\( \mathcal{H}^1(M, \mathbb{R}; p_t(d\omega)) \), and write (5.13) distributionally as
\[
\int_{M_y} \left( \nabla_{\gamma}^{(y)} \hat{\psi}(t,z,\omega_y)(y) \nabla^{\gamma} \chi(y) \right) p_t^{(\gamma)}(y, z) d\omega(y) = \int_{M_y} \left( U_\perp(z) \cdot \nabla^{(z)} \ln p_t^{(\gamma)}(y, z) \chi(y) \right) p_t^{(\gamma)}(y, z) d\omega(y)
\] (5.18)
for any test function \( \chi \in C_0^\infty(M, \mathbb{R}) \subset W_0^{1,2}(M, \mathbb{R}) \), (by density). According to (5.11), \( U_\perp(z) \cdot \nabla^{(z)} \ln p_t^{(\gamma)}(y, z) \) is \( L^2(M, \mathbb{R}; p_t(d\omega)) \)-orthogonal to the constant functions and by the Fredholm alternative it follows that (5.18), and hence (5.13), has a unique solution \( \hat{\psi}_{(t,z,\omega_y)} \) in \( \mathcal{H}^1(M, \mathbb{R}; p_t(d\omega)) \). Standard elliptic regularity then implies that \( \hat{\psi}_{(t,z,\omega_y)} \in C^\infty(M, \mathbb{R}) \), with a smooth dependence on the data \( t, z, U_\perp \). In order to prove that if \( U_\perp \neq 0 \) then \( \nabla_{\gamma}^{(y)} \hat{\psi}_{(t,z,\omega_y)}(y) \neq 0 \) we exploit an induced heat equation associated to the elliptic problem (5.13). From
\[
\frac{\partial}{\partial t} \left( U_\perp(z) \cdot \nabla^{(z)} p_t^{(\gamma)}(y, z) \right) = U_\perp(z) \cdot \nabla^{(z)} \Delta^{(y)} p_t^{(\gamma)}(y, z) \quad (5.19)
\]
and (5.13) we get that \( \text{div}_{\omega}^{(y)} \left( p_t^{(\gamma)}(y, z) \nabla^{(y)} \hat{\psi}_{(t,z,\omega_y)}(y) \right) \) satisfies the heat equation
\[
\left( \frac{\partial}{\partial t} - \Delta^{(y)} \right) \left[ \text{div}_{\omega}^{(y)} \left( p_t^{(\gamma)}(y, z) \nabla^{(y)} \hat{\psi}_{(t,z,\omega_y)}(y) \right) \right] = 0 \; , \quad (5.20)
\]
with
\[
\lim_{t \searrow 0} \text{div}_{\omega}^{(y)} \left( p_t^{(\gamma)}(y, z) \nabla^{(y)} \hat{\psi}_{(t,z,\omega_y)}(y) \right) = \text{div}_{\omega}^{(y)} U_\perp(z) \; ,
\]
in the weak sense. Hence, if \( \nabla^{(y)} \hat{\psi}_{(t,z,\omega_y)} \equiv 0 \) then, by the (backward) uniqueness of the heat flow, it follows that we must have \( \text{div}_{\omega}^{(y)} U_\perp(z) \equiv 0 \), \( \forall U_\perp \in C^\infty(M, TM) \). This necessarily implies \( U_\perp \equiv 0 \). \( \square \)

If we denote by
\[
\mathcal{H}_{t, z}(TM) := \left\{ \nabla \hat{\psi} \in C^\infty(M, TM) : \hat{\psi} \in C^\infty(M, \mathbb{R}) \right\}^{L^2(p_t(d\omega, z))}, \quad (5.21)
\]
the Hilbert space of gradient vector fields obtained by completion with respect to the Otto \( L^2(p_t(d\omega, z)) \) norm
\[
\| \nabla \hat{\psi} \|^2_{\mathcal{H}_{t, z}} := \int_M \left| \nabla^{(y)} \hat{\psi} \right|^2_{g(y)} p_t^{(\gamma)}(y, z) d\omega(y) \; , \quad (5.22)
\]
then we have
Lemma 5.9. The map

\[ T_z M \rightarrow T_{p_t(\omega)} \text{Prob}(M, g) \xrightarrow{\sim} \mathcal{H}_{t, z}(TM) \]  

is, for any \( t \in (0, \infty) \), an injection.

Proof. This is an immediate consequence of Proposition 5.8. \( \square \)

The heat kernel parametrization (5.23) can be applied to any adapted vector field \( W = W_\perp + W_\parallel \), and we shall set

\[ W^\alpha_t(y, v) := \left( \nabla^i(y) \psi(t, z, W_\perp)(y), W_\parallel^\alpha(v) \right), \]  

with an obvious notation.

5.1. The induced G-M metric rescaling \((M, g) \mapsto (M, g_t(\omega))\)

According to (5.23), and in the spirit of Otto’s formal Riemannian calculus [83], we can interpret \( \nabla \psi(t, z, U_\perp) \) as the push–forward of \( U_\perp \in T_z M \) to the tangent space \( T_{p_t(\omega)} \text{Prob}(M, g) \), under the heat kernel embedding map (5.6). This remark motivated a basic observation by N. Gigli and C. Mantegazza which we extend to the weighted heat kernel embedding according to Proposition 5.10.

Proposition 5.10. (cf. Def. 3.2 and Prop. 3.4 of [27]). For any \( t > 0 \), \( z \in M \), and \( U_\perp, W_\perp \in T_z M \), let us denote by \( \nabla(y) \psi(t, z, U_\perp) \) and \( \nabla(y) \psi(t, z, W_\perp) \) the corresponding vector fields defined by the weighted heat kernel injection map (5.23). Then, the symmetric bilinear form

\[ z \mapsto g_t(\omega)(U_\perp(z), W_\perp(z)) \]  

defines a scale–dependent metric tensor on \( M \), varying smoothly in \( 0 < t < \infty \).

Proof. We briefly outline the proof, an obvious adaptation of Prop. 3.4 of [27]. To begin with, let us observe that as a consequence of the uniqueness property of the defining pde (5.13), the functions \( \psi(t, z, U_\perp) \) and \( \psi(t, z, W_\perp) \) depend linearly from the vectors \( U_\perp \) and \( W_\perp \). This implies that the expression (5.25) is bilinear, besides being manifestly symmetric and non–negative. Moreover, the smoothness of the weighted heat kernel \( p_t(\omega) d\omega \), and the smooth dependence of \( \psi(t, z, U_\perp) \) and \( \psi(t, z, W_\perp) \) from the data \((t, z, U_\perp, W_\perp)\), imply that \( g_t(\omega)(U_\perp(z), W_\perp(z)) \) is smooth in its arguments. Since \( p_t(\omega) d\omega > 0 \), if we assume that \( g_t(\omega)(U_\perp(z), U_\perp(z)) = 0 \) then we necessarily get \( \nabla(y) \psi(t, z, U_\perp) \equiv 0 \).
0 and hence, according to proposition 5.8 \( U_\perp(z) \equiv 0 \). It follows that the bilinear form (5.25) is a smooth metric on \( M \).

To characterize geometrically the metric (5.25) let us consider a one–parameter family of locally Lipschitz diffeomorphisms \( \Theta_\lambda \) in \( M \),

\[
\Theta_\lambda : [0,1) \times M \longrightarrow M
\]

\[
(\lambda, x) \longmapsto c_\lambda(x),
\]

where \( c_\lambda(x) \) is the point in \( M \) reached at time \( \lambda \) along the \( \Theta_\lambda \)–trajectory issued from \( x \), and where \( \Theta_0 = id_M \). Let \( c'_\lambda(x) \) be the \( \lambda \)–dependent velocity field associated, for almost every \( \lambda \), with the trajectories \( (\lambda, x) \longmapsto c_\lambda(x) \) of \( \Theta_\lambda \). Locally Lipschitz diffeomorphisms map null sets to null sets, and the behavior of the heat kernel embedding along \( \Theta_\lambda \) is described by the

**Lemma 5.11.** (cf. Th.2.5 in [27]). Let \( \delta_{c_\lambda(z)} \) denote the heat source at the point \( c_\lambda(z) \in M \) reached at time \( \lambda \) along the \( \Theta_\lambda \)–trajectory issued from \( z \). Let \( c_\lambda(z) \longmapsto p_t^{(\omega)}(\cdot, c_\lambda(z))d\omega(\cdot) \) be the corresponding heat embedding map. Then, for almost every \( \lambda \in [0,1] \), there exists, along the path \( p_t^{(\omega)}(\cdot, c_\lambda(z))d\omega(\cdot), \lambda \in [0,1], \) in \( \text{Prob}_{ac}(M,g) \), a tangent velocity field \( \mathbf{v}_\lambda \in \mathcal{H}_{t,c_\lambda(z)}(TM) \) such that the relation

\[
c'_\lambda(z) \cdot \nabla_{c_\lambda(z)} p_t^{(\omega)}(y, c_\lambda(z)) + \text{div}^{(y)}(\mathbf{v}_\lambda(y) p_t^{(\omega)}(y, c_\lambda(z))) = 0,
\]

holds in the sense of distributions.

**Proof.** Since \( \Theta_\lambda \) is locally Lipschitz, the curve of measures, (at fixed \( t \)),

\[
\lambda \longmapsto p_t^{(\omega)}(\cdot, c_\lambda(z))d\omega(\cdot)
\]

is absolutely continuous with respect to the Wasserstein distance. By a result of Ambrosio, Gigli and Savaré (cf. Th. 8.3.1 in [55] and Th. 13.8 in [59]), for almost every \( \lambda \) the path \( p_t^{(\omega)}(\cdot, c_\lambda(z))d\omega(\cdot), \lambda \in [0,1], \) in \( \text{Prob}_{ac}(M,g) \) admits a tangent velocity field \( y \longmapsto \mathbf{v}_\lambda(y) \in \mathcal{H}_{t,c_\lambda(z)}(TM) \), where \( \mathcal{H}_{t,c_\lambda(z)}(TM) \) is the Hilbert space (5.24) associated with \( c_\lambda(z) \). Moreover the continuity equation

\[
\mathcal{L}_{(\frac{\partial}{\partial \lambda} + \mathbf{v}_\lambda)} p_t^{(\omega)}(y, c_\lambda(z)) \, d\omega(y) \, d\lambda = 0
\]

holds in the distributional sense on \( M \times [0,1] \), i.e.

\[
\int_M \int_{[0,1]} \left( \frac{\partial}{\partial \lambda} \varphi(\lambda, y) + \mathbf{v}_\lambda(y) \cdot \nabla^{(y)} \varphi(\lambda, y) \right) p_t^{(\omega)}(y, c_\lambda(z)) \, d\omega(y) \, d\lambda = 0
\]

for all \( \varphi \in C^\infty_0(M \times [0,1]) \). Here we have denoted by \( \mathcal{L}_{(\frac{\partial}{\partial \lambda} + \mathbf{v}_\lambda)} \) the (weakly defined) Lie derivative in the direction of the \( M \times [0,1] \)–vector field \( \frac{\partial}{\partial \lambda} + \mathbf{v}_\lambda \).
Hence, for almost every $\lambda$, we can write
\begin{align}
\mathcal{L}_{(\frac{\partial}{\partial \lambda} + v_\lambda)} p_t^{(\omega)}(y, c_\lambda(z)) d\omega(y) d\lambda &= \frac{\partial}{\partial \lambda} p_t^{(\omega)}(y, c_\lambda(z)) d\omega(y) d\lambda + \mathcal{L}_{v_\lambda} \left( p_t^{(\omega)}(y, c_\lambda(z)) d\omega(y) d\lambda \right) \\
&= \left[ \frac{\partial}{\partial \lambda} p_t^{(\omega)}(y, c_\lambda(z)) + div_{c_\lambda(z)} p_t^{(\omega)}(y, c_\lambda(z)) \right] d\omega(y) d\lambda = 0.
\end{align}

Along the curve $\lambda \mapsto c_\lambda(z)$, we have
\begin{align}
\frac{\partial}{\partial \lambda} p_t^{(\omega)}(y, c_\lambda(z)) &= c'_\lambda(z) \cdot \nabla_{c_\lambda(z)} p_t^{(\omega)}(y, c_\lambda(z)),
\end{align}
which inserted in (5.30), implies that the velocity vector $v_\lambda$ satisfies (5.27), as stated.

As a direct consequence of Lemma 5.11 we have the following result (cf. Prop. 3.5 of [27]), which can be interpreted as a form of equivariance of the heat kernel embedding under (Lipschitzian) diffeomorphisms,

**Lemma 5.12.** For almost every $\lambda$ the velocity field $y \mapsto v_\lambda(y)$ can be represented as the $L^2([0, 1] \times M, \nu_\lambda \otimes d\lambda)$ vector field $(\lambda, z) \mapsto \nabla^{(y)} \tilde{\psi}_{(t, c_\lambda(z), c'_\lambda)}(y)$ covering the curve $\lambda \mapsto \nu_\lambda := p_t^{(\omega)}(\cdot, c_\lambda(z))d\omega(\cdot)$ in $\text{Prob}_{ac}(M, g)$.

**Proof.** If we compare (5.27) with the elliptic PDE (5.13), characterizing the scalar potential $\tilde{\psi}_{(t, c_\lambda(z), c'_\lambda)}$ associated with the vector $U_{\perp}(z) \equiv c'_\lambda(z)$, then, for almost every $\lambda$, we get
\begin{align}
div_{c_\lambda(z)} p_t^{(\omega)}(y, c_\lambda(z)) &= c'_\lambda(z) \cdot \nabla_{c_\lambda(z)} p_t^{(\omega)}(y, c_\lambda(z)) \\
&= \frac{\partial}{\partial \lambda} p_t^{(\omega)}(y, c_\lambda(z)) \tilde{\psi}_{(t, c_\lambda(z), c'_\lambda)}(y).
\end{align}
Since the solution of (5.13) is unique, we have that for almost every $\lambda$ we can write $\tilde{\psi}_{(t, c_\lambda(z), c'_\lambda)}(y) = v_\lambda(y)$, as stated.

The above results imply the following property that, in line with Lemma 3.11, extends to $(M, g, d\omega)$ a basic observation of Gigli–Mantegazza.

**Proposition 5.13.** (cf. Prop. 3.5 of [27]). The heat kernel induced metric (5.25) can be identified with the (squared) norm of the Wasserstein metric speed of the absolutely continuous curves of measures $\lambda \mapsto \nu_\lambda := p_t^{(\omega)}(\cdot, c_\lambda(z))d\omega(\cdot) \in \left( \Upsilon_t(M), d_W^g \right)$.
\[ \begin{align*}
g_t^\omega (c_\lambda'(z), c_\lambda'(z)) &:= \left\langle \nabla \tilde{\psi}_{(t, c_\lambda(z), U_{\perp})}, \nabla \tilde{\psi}_{(t, c_\lambda(z), U_{\perp})} \right\rangle_{(g, p_t(d\omega))} \\
abla \tilde{\psi}_{(t, c_\lambda(z), U_{\perp})} &\left( g, p_t(d\omega) \right) = \left| \frac{d\nu_\lambda(z, t)}{d\lambda} \right|^2 := \left[ \lim_{\epsilon \to 0} \frac{d^W_g (\nu_{\lambda+\epsilon}(z, t), \nu_\lambda(z, t))}{\epsilon} \right]^2,
\end{align*} \]

where \( \Upsilon_t(M) \) denotes the image of \((M, g)\) in \((\text{Prob}_{ac}(M, g), d^W_g)\) under the heat kernel embedding map \((5.6)\).

**Proof.** Let \( K_f \in \mathbb{R} \) denote the lower bound of the Bakry–Emery Ricci curvature of \((M, g, d\omega)\),

\[ \text{Ric}_g(v, v) + \text{Hess}_f(v, v) \geq K_f g(v, v), \quad \forall v \in TM. \quad (5.34) \]

According to a result of D. Bakry, I. Gentil, and M. Ledoux, \((84)\) Corollary 4.2), we have, for any given \( t > 0 \),

\[ d^W_g \left( p_t^\omega(\cdot, c_\lambda(z)), p_t^\omega(\cdot, c_{\lambda}(z)) \right) \]

\[ \leq e^{-K_f t} d^W_g (\delta_{c_\lambda(z)}, \delta_{c_{\lambda}(z)}) = e^{-K_f t} d_g (c_\lambda(z), c_{\lambda}(z)). \]

Since the trajectories of \( \Theta_\lambda \) are absolutely continuous, (the family of diffeomorphisms \( \Theta_\lambda \) is locally Lipschitz), \((5.35)\) implies that the curves in \( \text{Prob}_{ac}(M, g) \) defined by

\[ \lambda \mapsto \nu_\lambda(z, t) := p_t^\omega(\cdot, c_\lambda(z))d\omega(\cdot) \quad (5.36) \]

are locally absolutely continuous in the metric topology induced by the Wasserstein distance \( d^W_g \), and we can define, for almost every \( \lambda \in [0, 1] \), the metric derivative \((55)\) of \( \nu_\lambda \),

\[ \left| \frac{d\nu_\lambda(z, t)}{d\lambda} \right| := \lim_{\epsilon \to 0} \frac{d^W_g (\nu_{\lambda+\epsilon}(z, t), \nu_\lambda(z, t))}{\epsilon}. \quad (5.37) \]

Absolute continuity of the \( \Theta_\lambda \) trajectories, also implies that the vector field \( \nabla \tilde{\psi}_{(t, c_\lambda(z), c_\lambda'(z))} \) is uniquely determined by \((5.29)\), and its norm in the formal Riemannian structure defined on \( T_{\nu_\lambda} \text{Prob}_{ac}(M, g) \) by the \( L^2(M, \nu_\lambda) \) Otto inner product is provided by

\[ \left| \frac{d\nu_\lambda(z, t)}{d\lambda} \right|^2 = \int_M \left| \nabla_y \tilde{\psi}_{(t, c_\lambda(z), c_\lambda'(z))}(y) \right|^2 g_t^\omega (y, c_\lambda(z)) d\omega(y) \quad (5.38) \]

\[ = g_t^\omega (c_\lambda'(z), c_\lambda'(z)), \quad (5.39) \]

as stated. \( \square \)
Let us consider the curves of measures
\[ \lambda \mapsto \nu_{\lambda}(c, t) := p_{t}^{(\omega)}(\cdot, c_{\lambda})d\omega(\cdot), \] (5.40)
associated to the set of all (absolutely continuous) curves, \( \{c_{(z, x)}\} := \{[0, 1] \ni \lambda \mapsto c_{\lambda} \in M\} \), \( c_{0} = z, \ c_{1} = x \), connecting the points \( z \) and \( x \) in \((M, g)\). If we let
\[ d_{g_{t}^{(\omega)}}(x, z) := \inf_{\{c_{(z, x)}\}} \int_{0}^{1} \sqrt{g_{t}^{(\omega)}(c'_{\lambda}, c'_{\lambda})} d\lambda \] (5.41)
denote the Riemannian distance in \((M, g_{t}^{(\omega)})\), then (5.36) implies
\[ d_{g_{t}^{(\omega)}}(x, z) := \inf_{\{c_{(z, x)}\}} \int_{0}^{1} \left| \frac{dv_{\lambda}(c, t)}{d\lambda} \right| d\lambda, \] (5.42)
which characterizes metrically the manifold \((M, g_{t}^{(\omega)})\).

To prove that \((t, g) \mapsto g_{t}^{(\omega)}\) is a variation (and possibly a geometrical deformation) of the original Riemannian structure we need to show that
\[ \lim_{t \downarrow 0} g_{t}^{(\omega)} = g. \] This is a singular limit and requires some care. For the standard heat kernel embedding the argument is tersely presented in [27], and can be easily adapted to our particular case according to

**Lemma 5.14.** (cf. Prop. 3.7 of [27]).
\[ \lim_{t \downarrow 0} g_{t}^{(\omega)}(c'_{\lambda}, c'_{\lambda}) = g(c'_{\lambda}, c'_{\lambda}) \] (5.43)

**Proof.** From (5.35) and (5.37) we get
\[ \left| \frac{dv_{\lambda}(c, t)}{d\lambda} \right| := \lim_{\epsilon \to 0} d_{g}^{W}(v_{\lambda+\epsilon}, c_{\lambda}) - d_{g}(c_{\lambda}, c_{\lambda}) \] (5.44)
\[ \leq e^{-K_{f}t} \lim_{\epsilon \to 0} d_{g}(c_{\lambda+\epsilon}, c_{\lambda}) \] (5.45)
which, according to (5.42), implies
\[ d_{g_{t}^{(\omega)}}(x, z) \leq e^{-K_{f}t} d_{g}(x, z). \] (5.46)

Let us observe that the right member of (5.42) provides the intrinsic Wasserstein distance between \( p_{t}^{(\omega)}(\cdot, z)d\omega \) and \( p_{t}^{(\omega)}(\cdot, x)d\omega \) on the image \( \Upsilon_{t}(M) \subset \text{Prob}_{ac}(M, g) \) of \((M, g, d\omega)\). In general, this is larger than the actual Wasserstein distance between \( p_{t}^{(\omega)}(\cdot, z)d\omega \) and \( p_{t}^{(\omega)}(\cdot, x)d\omega \) in \( \text{Prob}_{ac}(M, g) \) which is defined by
\[ d_{g}^{W}\left(p_{t}^{(\omega)}(\cdot, z)d\omega, p_{t}^{(\omega)}(\cdot, x)d\omega\right) := \inf_{\{\tilde{c}_{(z, x)}\}} \int_{0}^{1} \left| \frac{d\tilde{c}_{(z, x)}}{d\lambda} \right| d\lambda, \] (5.47)
where the \( \inf \) is over all absolutely continuous curves of probability measure, \( \lambda \mapsto \tilde{c}_{(z, x)}(\gamma) \in \text{Prob}_{ac}(M, g) \), connecting \( p_{t}^{(\omega)}(\cdot, z)d\omega \) to \( p_{t}^{(\omega)}(\cdot, x)d\omega \). Hence, from (5.42) and (5.46), we get
\[ d_{g}^{W}\left(p_{t}^{(\omega)}(\cdot, z)d\omega, p_{t}^{(\omega)}(\cdot, x)d\omega\right) \leq d_{g_{t}^{(\omega)}}(x, z) \leq e^{-K_{f}t} d_{g}(x, z). \] (5.48)
The upper bound in (5.48) immediately provides \( \limsup_{t \to 0} g_t^{(\omega)} \leq g \). A similar control on \( \liminf_{t \to 0} g_t^{(\omega)} \) is less direct since the lower bound in (5.48) is expressed in terms of the secant Wasserstein distance between \( p_t^{(\omega)}(\cdot, z)d\omega \) and \( p_t^{(\omega)}(\cdot, x)d\omega \). To circumvent this, one equivalently characterizes the metric \( g_t^{(\omega)} \),

\[
g_t^{(\omega)}(c_\lambda', c_\lambda) = \int_M |\nabla_y \hat{\psi}(t, c_\lambda, c_\lambda')(y)|^2 g p_t^{(\omega)}(y, c_\lambda) \, d\omega(y),
\]

by going to its variational description defined by [27]

\[
g_t^{(\omega)}(c_\lambda', c_\lambda) = \sup_{\varphi \in C_0^\infty(M, \mathbb{R})} \left\{ 2 \int_M \nabla_y \varphi \cdot \nabla_y \hat{\psi}(t, c_\lambda, c_\lambda')(y) p_t^{(\omega)}(y, c_\lambda) \, d\omega(y) - \int_M |\nabla_y \varphi(y)|^2 g p_t^{(\omega)}(y, c_\lambda) \, d\omega(y) \right\},
\]

which implies

\[
g_t^{(\omega)}(c_\lambda', c_\lambda) \geq 2 \int_M \nabla_y \varphi \cdot \nabla_y \hat{\psi}(t, c_\lambda, c_\lambda')(y) p_t^{(\omega)}(y, c_\lambda) \, d\omega(y) - \int_M |\nabla_y \varphi(y)|^2 g p_t^{(\omega)}(y, c_\lambda) \, d\omega(y),
\]

for any \( \varphi \in C_0^\infty(M, \mathbb{R}) \). Writing, as usual, \( \hat{\psi}(t) := \hat{\psi}(t, c_\lambda, c_\lambda') \) for ease of notation we have, by an obvious transposition of the argument provided in [27], Prop. 3.7,

\[
\int_M \nabla_y \varphi \cdot \nabla_y \hat{\psi}(t)(y) p_t^{(\omega)}(y, c_\lambda) \, d\omega(y) = \int_M \varphi(y) \, \text{div}^{(g)} \left( \nabla_y \hat{\psi}(t)(y) \right) p_t^{(\omega)}(y, c_\lambda) \, d\omega(y) = \int_M \varphi(y) c'_\lambda \cdot \nabla c_\lambda p_t^{(\omega)}(y, c_\lambda) \, d\omega(y),
\]

where, in the last line, we exploited the pde (5.13) defining the potential \( \hat{\psi}(t, c_\lambda, c_\lambda') \) corresponding to the tangent vector \( c'_\lambda \). The relation (5.52), can be equivalently rewritten as

\[
\int_M \nabla_y \varphi \cdot \nabla_y \hat{\psi}(t)(y) p_t^{(\omega)}(y, c_\lambda) \, d\omega(y) = c'_\lambda \cdot \nabla c_\lambda \int_M \varphi(y) p_t^{(\omega)}(y, c_\lambda) \, d\omega(y) = c'_\lambda \cdot \nabla c_\lambda \varphi_t(c_\lambda),
\]
where \( \varphi_t(c_\lambda) \) is the solution of the heat equation \( \left( \frac{\partial}{\partial t} - \Delta^{(c_\lambda)}_\omega \right) \varphi_t = 0 \) with \( \lim_{t \searrow 0} \varphi_t = \varphi \). From (5.51) and (5.53) we get

\[
g_t^{(\omega)} (c'_\lambda, c_\lambda) \geq 2 c'_\lambda \cdot \nabla_{c_\lambda} \varphi_t (c_\lambda) - \int_M |\nabla_y \varphi(y)|^2 g_t^{(\omega)}(y, c_\lambda) \, d\omega(y),
\]

which, by choosing a \( \varphi \in C^\infty_0(M, \mathbb{R}) \) with \( \nabla \varphi = c'_\lambda \), yields

\[
\liminf_{t \to 0} g_t^{(\omega)} (c'_\lambda, c_\lambda) \geq g(c'_\lambda, c_\lambda).
\]

Together with (5.46), this implies

\[
\lim_{t \searrow 0} g_t^{(\omega)} (c'_\lambda, c_\lambda) = g(c'_\lambda, c_\lambda),
\]

as required.

\[
\square
\]

5.2. Warping \((M, g_t^{(\omega)})\) on \(M \times \mathbb{T}^q\)

To extend the Gigli–Mantegazza construction to the warped manifold \(M \times \mathbb{T}^q\), let us consider the solution \((t, f) \mapsto \exp[-2 f_t^{(\omega)}(z)]\) of the heat equation

\[
\left( \frac{\partial}{\partial t} - \Delta^{(z)}_\omega \right) e^{-2 f_t^{(\omega)}(z)} = 0, \quad t \in (0, \infty),
\]

associated to the warping factor \(e^{-2 f_t^{(\omega)}(z)}\) in the metric (4.17). Note that we can equivalently write (5.57) as

\[
\frac{\partial}{\partial t} f_t^{(\omega)} = \Delta^{(z)}_\omega f_t^{(\omega)} - \frac{2}{q} |\nabla f_t^{(\omega)}|_g^2
\]

\[
= \Delta^{(z)}_\omega f_t^{(\omega)} - \nabla^i f \nabla_i f_t^{(\omega)} - \frac{2}{q} |\nabla f_t^{(\omega)}|_g^2,
\]

\[
\lim_{t \searrow 0} f_t^{(\omega)}(z) = f(y).
\]

We have

**Proposition 5.15.** For any \(t > 0, (z, \zeta) \in M \times \mathbb{T}^q\) and \(U, W \in T_{(z,\zeta)} M \times \mathbb{T}^q\), let \(U_t(y, v)\) and \(W_t(y, v)\) denote the vector fields defined by the map \((5.24)\). Then

\[
h_t(U(z, \zeta), W(z, \zeta)) = g_t^{(\omega)}(U_\perp(z), W_\perp(z)) + e^{-2 f_t^{(\omega)}(z)} \delta_{\alpha\beta} U^\alpha(\zeta) W^\beta(\zeta),
\]

provides a scale–dependent metric tensor on \(M \times \mathbb{T}^q\), varying smoothly with \(t \in (0, \infty)\).
Proof. In analogy with (5.25), let us consider the symmetric bilinear form
\[ h_t(U(z, \zeta), W(z, \zeta)) \]  
\[ := \int_{M \times T^n} h_{ab}^{(q)}(y, v) U_t^\alpha(y, v) W_t^\beta(y, v) p_t^{(\omega)}(y, z) \, d\omega(y) \otimes \delta_\zeta^{T^n}, \] 
which can be seen as the pull–back, under the heat kernel embedding (5.6), of the Otto inner product in \( T_{P_t(d\omega) \otimes \delta} \text{Prob}(M \times T^n, h) \), (see (3.13)). If we decompose the warped metric \( h(\Phi(q)) \) according to (4.17), then we can write
\[ h_t(U(z, \zeta), W(z, \zeta)) \] 
\[ = \int_{M \times T^n} g_{ik}(y) \nabla_i(y) \tilde{\psi}(t, z, U_\perp) \nabla_j(y) \tilde{\psi}(t, z, W_\perp) p_t^{(\omega)}(y, z) \, d\omega(y) \otimes \delta_\zeta^{T^n} \] 
\[ + \int_{M \times T^n} e^{-\frac{2f(y)}{q}} \delta_{\alpha\beta} U_\parallel^{\alpha}(v) W_\parallel^{\beta}(v) p_t^{(\omega)}(y, z) \, d\omega(y) \otimes \delta_\zeta^{T^n} \] 
\[ = \int_{M} g_{ik}(y) \nabla_i(y) \tilde{\psi}(t, z, U_\perp) \nabla_j(y) \tilde{\psi}(t, z, W_\perp) p_t^{(\omega)}(y, z) \, d\omega(y) \] 
\[ + \delta_{\alpha\beta} U_\parallel^{\alpha}(\zeta) W_\parallel^{\beta}(\zeta) \int_{M} e^{-\frac{2f(y)}{q}} p_t^{(\omega)}(y, z) \, d\omega(y) \] 
where we have exploited \( \int_{T^n} \delta_\zeta^{T^n} = 1 \), and \( \int_{T^n} \delta_{\alpha\beta} U_\parallel^{\alpha}(v) W_\parallel^{\beta}(v) \delta_\zeta^{T^n} = \delta_{\alpha\beta} U_\parallel^{\alpha}(\zeta) W_\parallel^{\beta}(\zeta) \). We immediately recognize in the last line the \( t \)-dependent metric tensor \( g_t^{(\omega)} \) defined by (5.25). To cast in a more explicit form also the term quadratic in \( U_\parallel^{\alpha} \) and \( W_\parallel^{\beta}(\zeta) \), let us exploit the symmetry of the heat kernel \( p_t^{(\omega)}(y, z) = p_t^{(\omega)}(z, y) \) to compute, for any \( t > 0 \),
\[ \int_{M} e^{-\frac{2f(y)}{q}} p_t^{(\omega)}(y, z) \, d\omega(y) = \int_{M} e^{-\frac{2f(y)}{q}} p_t^{(\omega)}(z, y) \, d\omega(y) \] 
\[ = e^{t \Delta_\omega} \left( e^{-\frac{2f(z)}{q}} \right) := e^{-\frac{2f_t^{(\omega)}(z)}{q}}, \] 
where \( f_t^{(\omega)}(z) \) is the solution of the heat equation (5.57). It follows that the symmetric bilinear form (5.61) can be written as
\[ h_t(U(z, \zeta), W(z, \zeta)) \] 
\[ = g_t^{(\omega)}(U_\perp(z), W_\perp(z)) + e^{-\frac{2}{q} f_t^{(\omega)}(z)} \delta_{\alpha\beta} U_\parallel^{\alpha}(\zeta) W_\parallel^{\beta}(\zeta), \] 
as stated. Finally, as in proposition 5.10, the properties of the weighted heat kernel (5.41), and the smooth dependence of the potential \( \tilde{\psi}(t, z, U_\perp) \) from the
The Wasserstein geometry of non-linear $\sigma$ models

data $(t, z, U_\perp)$, imply that $h_t$ is indeed a metric on $M \times T^q$, which according to (3.43) varies smoothly with the scale $t \in [0, \infty)$. □

5.3. Harmonic energy rescaling and dilatonic action flow

The deformation induced in $(M, g)$ and in $M \times T^q$ by the heat kernel embedding generates a corresponding geometric rescaling of the Harmonic energy functionals (2.10), (4.20) and of the associated dilatonic NL$\sigma$M action (3.9). To describe the nature of such rescaling we shall consider explicitly the functional $E[\phi, g]$. The extension to the warped map $\Phi(q) : (\Sigma, \gamma) \to (M \times T^q, h^{(q)})$ is a straightforward, and we limit ourselves to state how the relevant results generalize to $E[\Phi(q), h^{(q)}].$

Let us consider the Hilbert bundle $(p_t(\omega) \circ \phi)^{-1}T_{p_t(d\omega)}\text{Prob}(M, g)$, covering the map

$\Sigma \simeq \mathbb{T}^2 \to (M, g, d\omega) \to (\text{Prob}_{ac}(M, g), d^W_g)$

and whose fiber over $x \in \Sigma$ is given by the Hilbert space (5.21) evaluated at $z = \phi(x)$,

$$(p_t(\omega) \circ \phi)^{-1}T_{p_t(d\omega)}\text{Prob}(M, g)\big|_x \simeq \mathcal{H}_{t, \phi(x)}(TM).$$

We have

**Lemma 5.16.** For any given $x \in \Sigma$, we can associate to the vector $v = v^i \frac{\partial}{\partial \phi^i(x)} \in (\phi^{-1}TM)_x$ the vector field over $M$ given by

$$\widehat{\Psi}_{t,\phi} : \mathbb{R}_{\geq 0} \times (\phi^{-1}TM)_x \to \mathcal{H}_{t, \phi(x)}(TM)$$

$$(t, v^i \frac{\partial}{\partial \phi^i(x)}) \mapsto \widehat{\Psi}_{t,\phi}(v) := \nabla^i_{(y)}\widehat{\psi}(t, \phi(x), v)(y) \frac{\partial}{\partial y^i},$$

where $y \mapsto \widehat{\psi}(t, \phi(x), v)(y)$ denotes the Otto potential associated to $v = v^i \frac{\partial}{\partial \phi^i(x)}$, (cf. (5.13)).

**Proof.** This immediately follows by recalling that the map

$$T_{\phi(x)}M \to T_{p_t(d\omega)}\text{Prob}(M, g) \simeq \mathcal{H}_{t, \phi(x)}(TM)$$

is, according to (5.23), an injection. □

To handle the situation where $x$ varies smoothly over $\Sigma$, we denote by $\phi^{-1}TM \boxtimes TM$ the vector bundle over $\Sigma \times M$ whose fiber over $(x, y) \in \Sigma \times M$ is given by

$$\phi^{-1}TM \boxtimes TM\big|_{(x, y)} = \phi^{-1}TM\big|_x \otimes TM_y.$$
With this notation along the way, let us consider the section of $\phi^{-1}TM \boxtimes TM$ associated to the push-forward $\phi_* \partial_\alpha = \frac{\partial \phi^i}{\partial x^\alpha} \frac{\partial}{\partial \phi^i}$ of the $x^\alpha$–basis vector,

\[
\Sigma \times M \longrightarrow \phi^{-1}TM \boxtimes TM \quad (5.70)
\]

\[
(x, y) \longmapsto \left(\phi_* \partial_\alpha, \tilde{\Psi}_{t, \phi}(\phi_* \partial_\alpha)\right), \quad \alpha = 1, 2.
\]

Since is clear from the notation adopted which $\phi^{-1}TM$–vector we are talking about, we shall write (5.70) as $\tilde{\Psi}_{t, \phi}(\phi_* \partial_\alpha)$.

**Remark 5.17.** If we fix the index $\alpha = 1, 2$, and let $x^\alpha \in [0, 1]$, then we can consider $\phi_* \partial_\alpha$ as the tangent vector covering the coordinate curve $\phi_\alpha : [0, 1] \longrightarrow M, \ x^\alpha \longmapsto \phi(x^\alpha, x^\beta = 0), \ \beta \neq \alpha$, issued from $\phi(0)$. In particular, we can think of $\tilde{\Psi}_{t, \phi}(\phi_* \partial_\alpha)$ as the tangent vector in $(\text{Prob}_{ac}(M, g), d^\nu)$ to the absolutely continuous curve of probability measures

\[
[0, 1] \ni x^\alpha \longmapsto \nu(x^\alpha) := p_t(\cdot, \phi_\alpha(x)) d\omega(\cdot). \quad (5.71)
\]

According to (5.68), for any $t > 0$, we can associate to $d\phi = dx^\alpha \otimes \frac{\partial \phi^i}{\partial x^\alpha} \frac{\partial}{\partial \phi^i}$ the heat kernel deformed differential

\[
\Sigma \times M \longrightarrow T^*\Sigma \otimes (\phi^{-1}TM \boxtimes TM) \quad (5.72)
\]

\[
(x, y) \longmapsto d\tilde{\Psi}_{t, \phi} := dx^\alpha \otimes \tilde{\Psi}_{t, \phi}(\phi_* \partial_\alpha)
\]

\[
= dx^\alpha \otimes \nabla^i_{(y)\psi(t, \phi(x), \phi_* \partial_\alpha)} \frac{\partial}{\partial y^i}.
\]

The corresponding space of sections $\left\{d\tilde{\Psi}_{t, \phi} \in T^*\Sigma \otimes (\phi^{-1}TM \boxtimes TM)\right\}$ is endowed with the $T^*\Sigma \otimes L^2(p_t(d\omega, \phi(x))$ inner product

\[
\left(d\tilde{\Psi}_{t, \phi}(u), d\tilde{\Psi}_{t, \phi}(v)\right)_{p_t(d\omega)}(x) := \gamma^{-1}(dx^\alpha, dx^\beta)(x) \quad (5.73)
\]

\[
\otimes \int_M g_{km}(y) \nabla^k_y \psi(t, \phi(x), u(x)) \nabla^m_y \psi(t, \phi(x), v(x)) p_t^{(\omega)}(y, \phi(x)) d\omega(y),
\]
in terms of which we can define the pre-Hilbertian $L^2(p_t(d\omega) \otimes d\mu_\gamma)$ norm

$$\| d\hat{\Psi}_{t,\phi} \|^2_{p_t(d\omega) \otimes d\mu_\gamma} = \left\langle d\hat{\Psi}_{t,\phi}, d\hat{\Psi}_{t,\phi} \right\rangle_{p_t(d\omega) \otimes d\mu_\gamma}$$

$$= \int_{\Sigma} d\mu_\gamma(x) \gamma^{\alpha\beta}(x) \int_M g\left(\hat{\Psi}_{t,\phi}(\phi_* \partial_\alpha), \hat{\Psi}_{t,\phi}(\phi_* \partial_\beta)\right) p^\omega_t(y, \phi(x)) d\omega(y)$$

$$= \int_{\Sigma} d\mu_\gamma(x) \gamma^{\alpha\beta}(x)$$

$$\times \int_M g_{km}(y) \nabla^k_y \hat{\psi}(t, \phi(x), \phi_* \partial_\alpha) \nabla^m_y \hat{\psi}(t, \phi(x), \phi_* \partial_\beta) p^\omega_t(y, \phi(x)) d\omega(y).$$

(see (5.70)). We shall often write $\| d\hat{\Psi}_{t,\phi} \|^2_t$ if there is no danger of confusion.

Let

$$\mathcal{H}_{t,\phi}(\Sigma, M) := \{ C^\infty(\Sigma \times M, T^*\Sigma \otimes (\phi^{-1}TM \boxtimes TM)) \} L^2(p_t(d\omega) \otimes d\mu_\gamma),$$

(5.75)

be the Hilbert space of sections of $T^*\Sigma \otimes (\phi^{-1}TM \boxtimes TM)$ obtained by completion with respect to the norm (5.74), and define the associated energy functional

$$\mathcal{E} : \mathcal{H}_{t,\phi}(\Sigma, M) \rightarrow \mathbb{R},$$

$$d\hat{\Psi}_{t,\phi} \quad \mapsto \quad \mathcal{E}[\hat{\Psi}_{t,\phi}] := \frac{1}{2} \| d\hat{\Psi}_{t,\phi} \|^2_t.$$

The geometrical meaning of $\mathcal{E}[\hat{\Psi}_{t,\phi}]$ is provided by

**Proposition 5.18.** The energy functional $\mathcal{E}[\hat{\Psi}_{t,\phi}]$ is a generalized harmonic map functional over the Wasserstein metric space

$$\Upsilon_t \left( (M, g, d\omega) \cap (\text{Prob}_{ac}(M, g), d^w_g) \right),$$

(5.77)

where $\Upsilon_t$ is the weighted heat kernel injection map (5.6). $\mathcal{E}[\hat{\Psi}_{t,\phi}]$ can be identified with the smooth deformation of $\mathcal{E}[\phi, g]_{(\Sigma, M)}$ generated by the flow of metrics $(t, g) \mapsto g_t^\omega$. In particular, for $t \in (0, \infty)$, we can write

$$\mathcal{E}[\hat{\Psi}_{t,\phi}] := \frac{1}{2} \| d\hat{\Psi}_{t,\phi} \|^2_t$$

(5.78)

$$= \frac{1}{2} \int_{\Sigma} \gamma^{\mu
u} \frac{\partial \phi^i}{\partial x^\mu} \frac{\partial \phi^j}{\partial x^\nu} (g_t^\omega)_{ij}(\phi) d\mu_\gamma := \mathcal{E}[\phi, g_t^\omega]_{(\Sigma, M)},$$

with

$$\lim_{t \searrow 0} \mathcal{E}[\hat{\Psi}_{t,\phi}] = \mathcal{E}[\phi, g]_{(\Sigma, M)}.$$  

(5.79)
Proof. Since $\mathcal{E}[\Psi_t,\phi]$ is conformally invariant and $\Sigma \simeq \mathbb{T}^2$, we can assume that $\gamma$ is the flat metric $\delta^{\alpha\beta}$ and write (5.76) as

$$ \mathcal{E}[\Psi_t,\phi] = \frac{1}{2} \sum_{\alpha=1,2} \int_{\Sigma} d\mu_{\gamma}(x) \int_M \left| \nabla_y \psi(t,\phi(x),\phi,\partial_{\alpha}) \right|^2 g p_t^{(\omega)}(y,\phi(x)) d\omega(y). $$

According to (5.38), we have, for almost every $x^\alpha \in [0,1]$,

$$ \int_M \left| \nabla_y \psi(t,\phi(x),\phi,\partial_{\alpha}) \right|^2 g p_t^{(\omega)}(y,\phi(x)) d\omega(y) = \left| \frac{d\nu(x^\alpha)}{dx^\alpha} \right|^2, $$

where

$$ \left| \frac{d\nu(x^\alpha)}{dx^\alpha} \right| = \lim_{\epsilon \to 0} \frac{d^W g(\nu(x^\alpha + \epsilon), \nu(x^\alpha))}{\epsilon} $$

is the metric speed, in $\text{Prob}_{ac}(M,g), d^W_E$, of the curves (5.71), (see (5.37)). By absolute continuity of $x^\alpha \mapsto \nu(x^\alpha)$ we can write

$$ \mathcal{E}[\Psi_t,\phi] = \frac{1}{2} \sum_{\alpha=1,2} \int_{\Sigma} d\mu_{\gamma}(x) \left| \frac{d\nu_\alpha(\phi(x))}{dx^\alpha} \right|^2 $$

$$ = \lim_{\epsilon \to 0} \frac{1}{2} \sum_{\alpha=1,2} \int_{\Sigma} d\mu_{\gamma}(x) \left[ \frac{d^W g(\nu_\alpha + \epsilon, \nu_\alpha)}{\epsilon^2} \right]^2, $$

from which it follows that $\mathcal{E}[\Psi_t,\phi]$ has the structure of a generalized harmonic map functional over the Wasserstein metric space $\mathcal{Y}_t ((M,g,d\omega)) \cap \text{Prob}_{ac}(M,g), d^W_E$, (see Section 4.1). On the other hand, according to the definition (5.75) of the deformed metric $g_t^{(\omega)}$ and of the vector fields $\hat{\psi}(t,\phi(x),\phi,\partial_{\alpha})$, we can write the inner product (5.73) as

$$ \left( d\Psi_t, d\Psi_t \right)_{p_t(d\omega)} (x) = \gamma^{\alpha\beta} g_t^{(\omega)} \left( \frac{\partial \phi^i}{\partial x^\alpha}, \frac{\partial \phi^i}{\partial x^\beta}, \frac{\partial \phi^j}{\partial x^\alpha}, \frac{\partial \phi^j}{\partial x^\beta} \right)(x) $$

$$ = \gamma^{\alpha\beta}(x) \frac{\partial \phi^i}{\partial x^\alpha} \frac{\partial \phi^j}{\partial x^\beta} (g_t^{(\omega)})_{ij}(\phi(x)), $$

which implies that (5.76) can be rewritten as the harmonic energy functional $E[\phi, g_t^{(\omega)}]|_{\Sigma,M}$ associated with the deformed $(M,g_t^{(\omega)})$,

$$ \mathcal{E}[\Psi_t,\phi] := \frac{1}{2} \left\| d\Psi_t,\phi \right\|^2_i $$

$$ = \frac{1}{2} \int_{\Sigma} \gamma^{\mu\nu} \frac{\partial \phi^i}{\partial x^\mu} \frac{\partial \phi^j}{\partial x^\nu} (g_t^{(\omega)})_{ij}(\phi) d\mu_{\gamma} = E[\phi, g_t^{(\omega)}]|_{\Sigma,M}, $$

as stated. Moreover, from Lemma 5.14 we get

$$ \lim_{t \downarrow 0} E[\phi, g_t^{(\omega)}]|_{\Sigma,M} = E[\phi, g]|_{\Sigma,M}, $$

(5.87)
from which it follows that $\mathcal{E}[\hat{\Psi}_{t,\phi}]$ can be pulled-back to $\text{Map}(\Sigma, M)$ as a (smooth) deformation of the harmonic map functional $E[\phi, g]_{(\Sigma, M)}$. □

It is straightforward to extend the above analysis to the warped map $\Phi(q) \in \text{Map} \left( M^n \times_f \mathbb{T}^n \right)$, (see (4.19)), and characterize a deformation of the harmonic energy functional (4.22) according to

**Lemma 5.19.** The heat kernel embedding (5.6) generates the scale-dependent harmonic energy functional on the warped manifold $M^n \times_f \mathbb{T}^n$ given by

$$E[\Phi(q), h_t^{(q)}]_{(\Sigma, N^{n+q})} := \frac{1}{2} \int_{\Sigma} \gamma^{\mu\nu} \frac{\partial \Phi^a (x, \xi)}{\partial x^\mu} \frac{\partial \Phi^b (x, \xi)}{\partial x^\nu} (h_t)_{ab} (\phi) \, d\mu_\gamma$$

$$= \frac{1}{2} \int_{\Sigma} \gamma^{\mu\nu} \frac{\partial \phi^i}{\partial x^\mu} \frac{\partial \phi^j}{\partial x^\nu} (g)^{i,j}_{(\omega)} (\phi) \, d\mu_\gamma$$

$$+ \frac{1}{8} \sum_{k=1}^{q} \frac{d^2_{g}(\phi_{cm}; \phi(k))}{2} \int_{\Sigma} \left| df^t_{(\phi(x); q)} \right|^2 \gamma_\gamma \, d\mu_\gamma,$$

where $(t, h) \mapsto h_t$, $t \in (0, \infty)$, is the flow of metrics defined by (5.64).

This directly implies the

**Proposition 5.20.** If $t \mapsto (\gamma_t)_{\mu\nu} = e^{f^t_{(\phi)} (\phi)} \delta_{\mu\nu}$, $t \in (0, \infty)$, denotes the family of conformally flat metrics on $\Sigma \simeq \mathbb{T}^2$ associated with $(t, f) \mapsto f^t_{(\omega)}$, then in the conformal gauge $(\Sigma, \gamma_t)$ the harmonic energy functional (5.88) provides a scale-dependent family of dilatonic actions

$$S_M \left[ \gamma_t, \phi; F(\phi_{cm}; q), f^t_{(\omega)}, g^t_{(\omega)} \right]$$

$$\left( f_t \right) := \frac{2}{F(\phi_{cm}; q)} E[\Phi(q), h_t^{(q)}]_{(\Sigma, N^{n+q})},$$

such that

$$\lim_{t \searrow 0} S_M \left[ \gamma_t, \phi; F(\phi_{cm}; q), f^t_{(\omega)}, g^t_{(\omega)} \right] = S_M \left[ \gamma, \phi; F(\phi_{cm}; q), f, g \right].$$

**Proof.** In the conformal gauge, (see (4.24)),

$$\left( \Sigma_t, (\gamma_t)_{\mu\nu} = e^{f^t_{(\phi)} (\phi)} \delta_{\mu\nu} \right), \quad t \in [0, \infty),$$

(5.91)
the harmonic energy functional \((5.88)\) can be rewritten as
\[
E[\Phi(q), h_t^{(q)}]_{(\Sigma, N^{n+q})} = E[\phi, g_t^{(\omega)}]_{(\Sigma, M)}
\]
\[
+ \frac{F(\phi_{cm}; q)}{2} \int_{\Sigma} f_t^{(\omega)}(\phi; q) K_f^{(\omega)} d\mu_{\gamma_t} .
\]
If we define \(S_M[\gamma_t, \phi; F(\phi_{cm}; q), f_t^{(\omega)}, g_t^{(\omega)}]\) according to \((5.89)\), then the statement follows from \((5.87)\) and the definition \((4.25)\) of the dilatonic action \(S_M[\gamma, \phi; F(\phi_{cm}; q), f, g]\). □

6. The heat kernel embedding and Renormalization Group

The results of the previous section imply that along the heat kernel embedding \(Y_t\) we get the induced flow
\[
[0, \infty) \ni t \mapsto S_M[\gamma_t, \phi; F(\phi_{cm}; q), f_t^{(\omega)}, g_t^{(\omega)}],
\]
deforming the dilatonic action \(S_M[\gamma, \phi; F(\phi_{cm}; q), f, g]\) in the direction of the non–trivial geometric rescaling \((t, g, f) \mapsto g_t^{(\omega)}, f_t^{(\omega)}\) of the coupling \((M, g, d\omega)\). This strongly suggests a connection between heat kernel embedding and the circle of ideas and techniques related to renormalization group.

As discussed in the introductory section 2.1, the strategy of the renormalization group analysis of the non–linear \(\sigma\) model \([10], [11]\) is to discuss the scaling behavior of the (quantum) fluctuations of the maps \(\phi : \Sigma \rightarrow M\) around the background average field \(\phi_{cm}\), defined by the distribution of the center of mass of a large \((q \rightarrow \infty)\) number of randomly distributed independent copies \(\{\phi(j)\}_{j=1}^{q}\) of \(\phi\) itself. This background field technique can be seen as a natural extension of the constant map localization described in Section 4. To formulate it within our framework, we must provide, in a suitable abstract Wiener space, a Borel functional measure whose properties reflect the heat kernel–induced flow for the harmonic map functional \(E[\Phi(q), h_t^{(q)}]\).

For technical reasons, (existence of a unique center of mass), we assume that the maps \(\{\phi(j)\}_{j=1}^{q}\) all take values in a convex ball \(B(p,r)\) with \(r < \min\left\{\frac{1}{3} \text{inj}(M), \frac{\pi}{6\sqrt{\kappa}}\right\}\); (see \([2.4]\), and section 4 for notation). Under these hypotheses, for any given \(x \in \Sigma\) there is a unique center of mass in \(B(p,2r)\) for the corresponding collection of points \(\{\phi(j)(x)\}_{j=1}^{q} \in B(p,r)\), (see section 4). Thus,
\[
\phi_{cm} : \Sigma \rightarrow M
\]
\[
x \mapsto \phi_{cm}(x) := \inf_{y \in B(p,2r)} \left[ \frac{1}{2} \sum_{j=1}^{q} d_g^2(y, \phi(j)(x)) \right],
\]
\[\text{See [34] for a general review, and [10] for a renormalization group analysis in the Ricci flow setting.}\]
is well–defined and provides the center of mass map generated by the set of maps \( \{ \phi_{(i)} \}_{i=1}^q \). Let \( \phi_{cm}^{-1} TM \) be the corresponding pull–back bundle over \( \Sigma \). Since \( B(p, 2r) \) is assumed convex, we can introduce \( q \) sections \( v_{(i)} : \Sigma \to \phi_{cm}^{-1} TM, \ x \mapsto v_{(i)}(x) = v_{(i)}^k(x) \frac{\partial}{\partial \phi_{cm}(x)} \), and parametrize the maps \( \{ \phi_{(i)} \}_{i=1}^q \) according to

\[
\phi_{(i)}(x) = \exp_{cm}(x)(v_{(i)}(x)) ,
\]

where \( \exp_{cm}(x) : T_{\phi_{cm}(x)}M \to B(p, 2r) \) denotes the exponential map based at \( \phi_{cm}(x) \), and where the center of mass constraint takes the form \( \sum_{j=1}^q v_{(j)}(x) = 0 \). More generally, for \( v \in \phi_{cm}^{-1} TM \) we shall write \( \phi_{(v)}(x) = \exp_{cm}(x)(v(x)) \).

As recalled above, the strategy in the (perturbative) renormalization group analysis of non–linear \( \sigma \) model is to let the sections \( v_{(i)} \) fluctuate around a classical background defined by the center of mass map \( \phi_{cm} \). The fluctuations are generated by assuming that the fields \( v_{(i)} \), subject to the constraint \( \sum_{j=1}^q v_{(j)}(x) = 0 \), are vector valued random variables distributed on \( Map(\Sigma, \phi_{cm}^{-1} TM) \) according to a (non–existing) infinite-dimensional probability measure \( \mathbb{P}[D v_{(i)}] \). It is customary to formally write

\[
\mathbb{P}[D v_{(i)}] := Z^{-1} e^{-S_{cm}[v_{(i)}; a]} D v_{(i)} ,
\]

where \( S_{cm}[v_{(i)}; a] := \frac{2}{a} E[\phi_{(i)}, g] \) is the non–linear \( \sigma \) model action associated to the harmonic map functional \( E[\phi_{(i)}, g] \), and where the normalization factor

\[
Z := \int_{Map(\Sigma, \phi_{cm}^{-1} TM)} e^{-S_{cm}[v_{(i)}; a]} D v_{(i)}
\]

is the partition function of the theory. In such a setting, the completion of \( Map(\Sigma, \phi_{cm}^{-1} TM) \) under the energy norm associated to \( S_{cm}[v_{(i)}; a] \), is considered to be the physical Hilbert space of choice. It is well–known that, from the point of view of (\( \infty \)-dimensional) geometrical analysis, the existence of the distribution measure \( \mathbb{P}[D v_{(i)}] \) on such a space is obstructed by the Borel–Cantelli lemma, according to which \( S_{cm}[v_{(i)}; a] \) is \( \mathbb{P} \)–almost surely divergent.

The way out from this impasse is to extract from \( S_{cm}[v_{(i)}; a] \), by perturbative techniques, a (dimensionally regularized) Gaussian measure with respect to which \( \prod_{j=1}^q e^{-S_{cm}[v_{(j)}; a]} D v_{(j)} \) can be formally interpreted as generating the fluctuations of \( v_{(j)} \) in the large deviations sense.

**Remark 6.1.** Typically, the Gaussian measure in question is generated by a rough Laplace-Beltrami operator \( \triangle_\Sigma \) associated to the pulled back Levi–Civita connection on \( \phi_{cm}^{-1} TM \). Since \( \Sigma \) is two–dimensional, the Green function of \( \triangle_\Sigma \) is logarithmically divergent and needs to be regularized in order to be used as the covariance function of a Gaussian measure. Dimensional regularization is often the choice.

The regularization procedure, necessary for defining the reference Gaussian measure, introduces a running length scale in the theory. By renormalization group techniques, this allows to control the behavior of the measure.
$e^{-S_{cm}[v(i); a]} D v(i)$ under fluctuations, and gives rise to a perturbative rescaling of the geometrical couplings $(M, g, d\omega)$ which, as recalled in section 2.1 connects to the Ricci flow [16], [10], [11], [34].

To provide a mathematical well-defined functional representation of the heat kernel embedding of the non-linear $\sigma$ model, which to some extent conveys the ideas of the physics path integral and of renormalization group, we need to relax on considering the energy norm completion of $Map(\Sigma, \phi_{cm}^{-1} TM)$ as the physical Hilbert space. This can be done by constructing a Gaussian measure on an abstract Wiener space associated with the heat kernel embedding. The resulting picture gives rise to a Gaussian renormalization group action, a toy model of NL$\sigma$M renormalization which nonetheless conveys many of the relevant features of the physical RG flow.

### 6.1. A Wiener space associated to the heat kernel embedding

Let $d\phi(i)(x) = dx^\alpha \otimes (\phi(i)_*) \partial_\alpha$, $v(i) \in \phi_{cm}^{-1} TM$, denote the differential of the map $\phi(i)(x) = \exp_{cm}(x)(v(i)(x))$, (see (6.3)). According to (5.72), we define the corresponding heat kernel deformed section by

$$
\Sigma \times M \longrightarrow T^* \Sigma \otimes (\phi_{cm}^{-1} TM \boxtimes TM)
$$

(6.7)

$$(x, y) \mapsto d\tilde{\Psi}_t(i) := dx^\alpha \otimes \tilde{\Psi}_{t, \phi_{cm}}((\phi(i)_*) \partial_\alpha),$$

where

$$
\tilde{\Psi}_{t, \phi_{cm}}((\phi(i)_*) \partial_\alpha) = \nabla^k_{(y)} \tilde{\psi}(t, \phi_{cm}(x), (\phi(i)_*) \partial_\alpha) \frac{\partial}{\partial y^k}.
$$

(6.8)

**Remark 6.2.** Note that the section $dx^\alpha \otimes \tilde{\Psi}_{t, \phi_{cm}}((\phi_{cm})_* \partial_\alpha)$, associated to the differential $d\phi_{cm}$ of the center of mass map $\phi_{cm}$, corresponds to $\sum_{j=1}^q v(i) = 0$. Hence we set

$$
d\tilde{\Psi}_{t, cm} := dx^\alpha \otimes \tilde{\Psi}_{t, cm}((\phi_{cm})_* \partial_\alpha) := \sum_{j=1}^q d\tilde{\Psi}_{t, (i)}.
$$

(6.9)

The natural framework for the discussing the energetics of the fluctuations of the maps $d\tilde{\Psi}_{t, (i)}$ around their background $d\tilde{\Psi}_{t, cm}$, is the Hilbert space of sections

$$
\mathcal{H}_{t, \phi_{cm}}(\Sigma, M)
$$

(6.10)

$$
:= \left\{ C^\infty \left( \Sigma \times M, T^* \Sigma \otimes (\phi_{cm}^{-1} TM \boxtimes TM) \right) \right\}^{L^2(p_t(d\omega) \otimes d\mu_\gamma)},
$$

obtained by completion with respect to the $L^2(p_t(d\omega) \otimes d\mu_\gamma)$-norm

$$
\| d\tilde{\Psi}_{t, \phi_{cm}} \|^2_{p_t(d\omega) \otimes d\mu_\gamma} = \left\langle d\tilde{\Psi}_{t, \phi_{cm}}, d\tilde{\Psi}_{t, \phi_{cm}} \right\rangle_{p_t(d\omega) \otimes d\mu_\gamma},
$$

(6.11)
defined by (5.74) and (5.75), (for \( \phi = \phi_{cm} \)).

As hinted in the introductory remarks, \( \mathcal{H}_{t, \phi_{cm}}(\Sigma, M) \) is too small to support a Gaussian measure randomizing the distribution of the \( d\widehat{\Psi}_{t, \phi_{cm}}^{(j)} \) and giving them finite (harmonic) energy norm. The mathematical strategy to circumvent this difficulty is due to L. Gross \[85\], and is to modify the Hilbertian norm so as to characterize a Banach space large enough to host the desired Borel measure, (various geometrical aspects of Gross theory are discussed in \[86\], \[87\], \[88\], \[89\], \[90\]). In our case the required extension is naturally suggested by the properties of the heat kernel embedding. For any fixed \( t > 0 \), and \( \eta \in \mathbb{R} \), let us consider the inner product defined, on sections of the bundle (6.7), by

\[
\langle d\widehat{\Psi}_{t,u}, d\widehat{\Psi}_{t,v} \rangle_{(t,-\eta)} \quad (6.12)
\]

\[
:= \frac{a^{-1}}{\int_{\Sigma} d\mu_{\gamma}(x) \int_{M} \left( g_{km}(y) \widehat{\Psi}^{k}_{t, \phi_{cm}} ((\phi(u))_{*} \partial_{\alpha}) \right) \times \left( 1 - a \Delta_{pt}(d\omega) \right)^{\eta} \widehat{\Psi}^{m}_{t, \phi_{cm}} ((\phi(u))_{*} \partial_{\beta}) \right] p_{t}^{(\omega)}(y, \phi_{cm}(x)) d\omega(y),
\]

where \( \Delta_{pt}(d\omega) \) denotes the weighted (rough) Laplacian \[6.16\] acting on the vector fields \( M \ni y \mapsto \nabla_{y} \widehat{\Psi}_{t, \phi_{cm}}((\phi(u))_{*} \partial_{\alpha}) \in TM \), and where, for later convenience, we have inserted the dimensional coupling constant \( a \), (see (3.9)), to make (6.12) explicitly dimensionless.

**Remark 6.3.** For \( \eta = 0 \), (6.12) induces the energy norm on \( \mathcal{H}_{t, \phi_{cm}}(\Sigma, M) \),

\[
\langle d\widehat{\Psi}_{t,u}, d\widehat{\Psi}_{t,u} \rangle_{(t,-\eta)} \bigg|_{\eta=0} = \frac{2}{a} \mathcal{E}[\widehat{\Psi}_{t, \phi_{cm}}] \quad (6.13)
\]

\[
= \sum_{\alpha=1,2} \int_{\Sigma} d\mu_{\gamma}(x) \int_{M} \left| \nabla_{y} \widehat{\Psi}_{t, \phi_{cm}}((\phi(u))_{*} \partial_{\alpha}) \right|^{2} g_{p}^{(\omega)}(y, \phi(x)) d\omega(y).
\]

Similarly, from the relation

\[
\int_{M} \left( \nabla^{i} \widehat{\Psi}_{t} \Delta_{pt}(d\omega) \nabla^{i} \widehat{\Psi}_{t} \right) p_{t}^{(\omega)}(y, \phi(x)) d\omega(y) \quad (6.14)
\]

\[
= - \int_{M} \left( \nabla^{k} \nabla^{i} \widehat{\Psi}_{t} \nabla^{k} \nabla^{i} \widehat{\Psi}_{t} \right) p_{t}^{(\omega)}(y, \phi(x)) d\omega(y),
\]
where we have set \(H\) representative dense as a dense subset into \(H\). We can inject \(H\) that \(H\) ded as a dense subset into \(H\).

Proof. Continuous inclusions of dense subsets.

Lemma 6.4. If \(H_{\psi}^\eta (\Sigma, M)\) denotes the topological dual of \(H_{\psi}^{-\eta} (\Sigma, M)\) then, for \(\eta > 0\),

\[
H_{\psi}^\eta (\Sigma, M) \ derechos H_{\psi} (\Sigma, M) \ derechos H_{\psi}^{-\eta} (\Sigma, M),
\]

are continuous inclusions of dense subsets.

Proof. The duality between the distribution–valued differentials \(d\phi_{(u)}^{(\eta)} = dx^\beta \otimes (\phi_{(u)}^{(\eta)}, \partial_\beta) \in H_{\psi}^{-\eta} (\Sigma, M)\) and the sections \(d\Phi_{(u)}^{(\eta)} \in H_{\psi}^\eta (\Sigma, M)\), is defined by the pairing

\[
\left< d\Phi_{(u)}^{(\eta)}, d\phi_{(u)}^{(\eta)} \right>_{(t, \eta)} := a^{-1} \int_{\Sigma} d\mu(x) \gamma^{\alpha \beta}(x) \times \int_{M} \left( \widehat{\nu}_{x}^{\eta}(\phi_{(u)}, \partial_\alpha) \right) \circ _{g(y)} \left( 1 - a \Delta_{p_t(d\omega)} \right) (\phi_{(u)}^{(\eta)}, \partial_\beta) p_t^{(\omega)}(y, \phi) d\omega(y),
\]

where \(\circ_{g(y)}\) denotes the \((M, g)\) inner product. Since \(H_{\psi} (\Sigma, M)\) is embedded as a dense subset into \(H_{\psi}^{-\eta} (\Sigma, M)\), it follows by topological duality that \(H_{\psi}^\eta (\Sigma, M)\) is densely embedded into \(H_{\psi} (\Sigma, M)\). In particular, we can inject \(H_{\psi}^\eta (\Sigma, M)\) in \(H_{\psi} (\Sigma, M)\) by identifying \(d\Phi_{(u)}^{(\eta)} \) with its representative \(d\Phi_{(u)}^{(\eta)} \) in \(H_{\psi} (\Sigma, M)\), via the Riesz representation theorem

\[
\left< d\Phi_{(u)}^{(\eta)}, d\Xi_{t} \right>_{(t, \eta)} = \left< d\Phi_{(u)}^{(\eta)}, d\Xi_{t} \right>_{p_t(d\omega) \otimes d\mu}, \quad \forall d\Xi_{t} \in H_{\psi} \cap H_{\psi}^\eta,
\]

where we have set \(H_{\psi}^\eta \equiv H_{\psi}^\eta (\Sigma, M)\) and \(H_{\psi} \equiv H_{\psi} (\Sigma, M)\).
The norm (6.12) is concocted in such a way that the dual space $\mathcal{H}_{t,\phi}^{-\eta}(\Sigma, M)$ is large enough to support a Gaussian measure $Q_{H_t}$ for which the triple

$$\left(\mathcal{H}_{t,\phi}(\Sigma, M), \mathcal{H}_{t,\phi}^{-\eta}(\Sigma, M), Q_{H_t}\right)$$

is an abstract Wiener space. Explicitly, we have

**Proposition 6.5.** Set $\mathcal{H}_{t}^{-\eta} \equiv \mathcal{H}_{t,\phi}^{-\eta}(\Sigma, M)$ and let $\mathcal{E}^{-\eta}$ be the Borel field for $\mathcal{H}_{t}^{-\eta}$, defined as the smallest $\sigma$-algebra with respect to which the maps

$$d\phi_i^{(-\eta)} \mapsto \left\langle d\tilde{\Psi}(\eta), d\phi_i^{(-\eta)} \right\rangle_{(t, \eta)}$$

are measurable. Then, for $\eta > \frac{\dim \Sigma \times M}{2}$, $\mathcal{H}_{t,\phi}^{-\eta}(\Sigma, M)$ is endowed with a finite Borel measure $Q_{H_t}$, characterized by its Fourier transform according to

$$\int_{\mathcal{H}_{t}^{-\eta}} e^{-\sqrt{-1} \left\langle d\tilde{\Psi}(\eta), d\phi^{(-\eta)} \right\rangle_{(t, \eta)}} Q_{H_t} \left[ d\phi^{(-\eta)} \right] = e - \frac{1}{2} \left\langle d\tilde{\Psi}, d\tilde{\Psi} \right\rangle_{\text{restriction to } \mathcal{E}^{(\eta)}(\mathcal{H}_{t}^{-\eta})} = e - \frac{1}{a} \mathcal{E}[\tilde{\Psi}, \mathcal{E}]$$

with $Q_{H_t} \left( \mathcal{H}_{t,\phi}^{-\eta}(\Sigma, M) \right) = 1$, and where $\mathcal{E}[\tilde{\Psi}, \mathcal{E}]$ is the energy norm (5.78).

**Proof.** By the Sobolev embedding theorem, if $\eta > \frac{\dim \Sigma \times M}{2}$, the space $\mathcal{H}_{t}^{\eta}$ is continuously embedded into the set of continuous and (bounded) sections $C_{\Sigma \times M}^{0} := C^{0}(\Sigma \times M, T^{*} \Sigma \otimes (\phi_{cm}^{-1} TM \boxtimes TM))$. Let

$$\mathcal{CY}(\mathcal{H}_{t}^{\eta}) := \{ T \in \mathcal{L}(\mathcal{H}_{t}^{\eta}, H) | \dim H < \infty, T(\mathcal{H}_{t}^{\eta}) = H \}$$

denote the set of linear maps from $\mathcal{H}_{t}^{\eta}$ onto finite dimensional Hilbert spaces $H \subset \mathcal{H}_{t}$. Let $d\lambda_{T}(\mathcal{H}_{t}^{\eta})$ be the Lebesgue measure on $T(\mathcal{H}_{t}^{\eta})$, and denote by $\left\langle \cdot, \cdot \right\rangle_{T(\mathcal{H}_{t}^{\eta})}$ the restriction to $T(\mathcal{H}_{t}^{\eta})$ of the $\mathcal{H}_{t}^{\eta}$ inner product. With this notation along the way, it is easily checked that the family of Gaussian measures

$$Q_{T(\mathcal{H}_{t}^{\eta})}$$

$$:= (2\pi)^{-\frac{\dim(T(\mathcal{H}_{t}^{\eta}))}{2}} e^{-\frac{1}{2} \left\langle d\tilde{\Psi}(\eta), d\tilde{\Psi}(\eta) \right\rangle_{T(\mathcal{H}_{t}^{\eta})}} d\lambda_{T}(\mathcal{H}_{t}^{\eta}), \quad T \in \mathcal{CY}(\mathcal{H}_{t}^{\eta})$$

defines a cylinder set measure on $\mathcal{H}_{t}^{\eta} \cap C_{\Sigma \times M}^{0}$. The linear injective map with dense range $\mathcal{H}_{t,\phi}(\Sigma, M) \hookrightarrow \mathcal{H}_{t,\phi}^{-\eta}(\Sigma, M)$ defined by the embedding (6.17), pushes forward $Q_{T(\mathcal{H}_{t}^{\eta})}$ to a Gaussian, white noise, measure $Q_{H_{t}}$ on $\mathcal{H}_{t,\phi}^{-\eta}(\Sigma, M)$. By exploiting the extended Bochner–Minlos theorem [86], [88], [90] for the triple of Hilbert spaces (6.17), the measure $Q_{H_{t}}$ can be characterized via its Fourier transform according to (6.22).

**Remark 6.6.** Note that from the general theory of Gaussian measures on Banach spaces, (cf. [91], Chap.III), we have

$$\mathcal{E}[\tilde{\Psi}, \mathcal{E}] = \frac{1}{2} \int_{\mathcal{H}_{t}^{-\eta}} \left\langle d\tilde{\Psi}(\eta), d\phi^{(-\eta)} \right\rangle_{(t, \eta)}^{2} Q_{H_{t}} \left[ d\phi^{(-\eta)} \right].$$

(6.25)
Moreover, according to Fernique theorem \cite{92}, applied to the measurable and sub-additive function $\| \cdot \|_{(t, \eta)}$, there exists $\rho \in (0, \infty)$ such that
\[
\int_{\mathcal{H}_{t}^{-\eta}} e^{\rho \| d\phi_{(u)}^{(-\eta)} \|^{2}_{(t, \eta)}} Q_{\mathcal{H}_{t}} [d\phi_{(u)}] < \infty .
\] (6.26)

6.2. Deformed harmonic energy as a large deviation functional

Let us consider (6.22) for the center of mass section
\[
d\hat{\Psi}_{t, cm} := \sum_{j=1}^{q} d\hat{\Psi}_{t, (j)} ,
\] (6.27)
associated to the fields (6.3). We write the collection of vector fields $\{v_{(j)}\}_{j=1}^{q}$, generating the maps $\phi_{(j)}$, as $v_{(j)} = \exp_{cm}^{-1} \phi_{(j)}$, and consider them, for $q \to \infty$, as a sequence of independent, identically distributed random variables on $(\mathcal{H}_{t}^{-\eta}, \mathcal{B}_{t}^{-\eta}, Q_{t})$. The associated normalized partial sums $\frac{1}{q} \sum_{j=1}^{q} v_{(j)} (\phi_{(j)})$ are distributed under $\prod_{j=1}^{q} Q_{t} [d\phi_{(j)}]$. Denote by
\[
Q_{(t,q)} (d\phi_{cm})
\] (6.28)
the distribution of
\[
(\phi_{(1)}, \ldots, \phi_{(q)}) \mapsto \frac{1}{q} \sum_{j=1}^{q} \exp_{cm}^{-1} \phi_{(j)}
\] (6.29)
under $\prod_{j=1}^{q} Q_{t} [d\phi_{(j)}]$. According to the strong law of large numbers we have that, as $q \to \infty$,
\[
\frac{1}{q} \sum_{j=1}^{q} \exp_{cm}^{-1} \phi_{(j)} \to 0 ,
\] (6.30)
or, equivalently
\[
\frac{1}{q} \sum_{j=1}^{q} d\phi_{(j)}^{(-\eta)} \to d\phi_{cm}^{(-\eta)} ,
\] (6.31)
for $Q_{(t,q)} (d\phi_{cm})$–almost every $d\phi_{(j)}$. The deviant behavior of $\frac{1}{q} \sum_{j=1}^{q} d\phi_{(j)}^{(-\eta)}$, with respect to an exponential rate of convergence to the center of mass $d\phi_{cm}^{(-\eta)}$, is governed by the large deviations of the family of distributions
\[
\{ Q_{(t,q)} (d\phi_{cm}) \mid q > 1 \}
\] (6.32)
according to the following result

**Proposition 6.7.** Let $J : \mathcal{H}_{t} \hookrightarrow \mathcal{H}_{t}^{-\eta}$ denote the inclusion map defined by Lemma 6.4. For $d\phi_{cm}^{(-\eta)} \in J (\mathcal{H}_{t}) \cap \mathcal{H}_{t}^{-\eta}$ let $d\hat{\Psi}_{t, cm} = J^{-1} \left( d\phi_{cm}^{(-\eta)} \right)$, then
\[
\Gamma_{t} (d\phi_{cm}^{(-\eta)}) := \frac{1}{a} \mathcal{E} [d\hat{\Psi}_{t, cm}] \quad \text{if} \quad d\phi_{cm}^{(-\eta)} \in J (\mathcal{H}_{t}) \cap \mathcal{H}_{t}^{-\eta}
\] (6.33)
\[
:= \infty \quad \text{if} \quad d\phi_{cm}^{(-\eta)} \notin J (\mathcal{H}_{t}) \cap \mathcal{H}_{t}^{-\eta}
\]
is the rate functional governing the large deviations for the family of distributions $\{ Q_{(t,q)} (d\phi_{cm}) \mid q > 1 \}$. 

60 Mauro Carfora
Proof. Let us assume that \( d\phi^{(-\eta)}_{cm} = J(d\tilde{\Psi}_{t,cm}) \), then the rate functional governing the large deviations of \( \{ Q_{(t,q)}(d\phi_{cm}) \mid q > 1 \} \) is provided by the Legendre transform: \(^4\)

\[
\Gamma_t \left( d\phi^{(-\eta)}_{cm} \right) = \sup \left\{ \left\langle \zeta^{(\eta)}, J \left( d\tilde{\Psi}_{t,cm} \right) \right\rangle_{(t,\eta)} - \frac{1}{2} \| J^* \left( \zeta^{(\eta)} \right) \|_{H_t}^2 \mid \zeta^{(\eta)} \in \mathcal{H}^0_t \right\},
\]

where \( J^* : \mathcal{H}^0_t \rightarrow \mathcal{H}_t \) is the adjoint map to \( J \), and where, (see (6.22) for notation),

\[
\| J^* \left( \zeta^{(\eta)} \right) \|_{H_t}^2 := \left\langle J^* \left( \zeta^{(\eta)} \right), J^* \left( \zeta^{(\eta)} \right) \right\rangle_{p_t(d\omega)\otimes d\mu_\gamma}.
\]

According to Lemma 6.4, \( J^* (\mathcal{H}^0_t) \) is dense in \( \mathcal{H}_t \), hence, there is \( d\tilde{\Psi}_t \in \mathcal{H}_t \) such that we can write

\[
\left\langle \zeta^{(\eta)}, J \left( d\tilde{\Psi}_{t,cm} \right) \right\rangle_{(t,\eta)} = \left\langle J^* \left( \zeta^{(\eta)} \right), d\tilde{\Psi}_{t,cm} \right\rangle_{(t,\eta)} = \left\langle d\tilde{\Psi}_t, d\tilde{\Psi}_{t,cm} \right\rangle_{(t,\eta)},
\]

and (6.34) reduces to

\[
\Gamma_t \left( d\phi^{(-\eta)}_{cm} \right) = \sup \left\{ \left\langle d\tilde{\Psi}_t, d\tilde{\Psi}_{t,cm} \right\rangle_{(t,\eta)} - \frac{1}{2} \| d\tilde{\Psi}_t \|_{H_t}^2 \mid d\tilde{\Psi}_t \in \mathcal{H}_t \right\}
= \frac{1}{2} \| d\tilde{\Psi}_{t,cm} \|_{H_t}^2 = \frac{1}{a} \mathcal{E}[\tilde{\Psi}_{t,cm}].
\]

To complete the proof we need to show that the assumption \( \Gamma_t (d\phi^{(-\eta)}_{cm}) < \infty \) necessarily implies that \( d\phi^{(-\eta)}_{cm} = J(d\tilde{\Psi}_{t,cm}) \). Set \( \Gamma_t (d\phi^{(-\eta)}_{cm}) \leq C^2 / 2 \) for some \( C > 0 \). Since on the unit ball \( \| J^* (\zeta^{(\eta)}) \|_{H_t}^2 = 1 \), we have

\[
\Gamma_t \left( d\phi^{(-\eta)}_{cm} \right) \geq \frac{1}{2} \left\langle \zeta^{(\eta)}, d\phi^{(-\eta)}_{cm} \right\rangle_{(t,\eta)}^2, \quad \zeta^{(\eta)} \in \mathcal{H}^0_t,
\]

and we can bound \( \left\langle \zeta^{(\eta)}, d\phi^{(-\eta)}_{cm} \right\rangle_{(t,\eta)} \) according to

\[
\left| \left\langle \zeta^{(\eta)}, d\phi^{(-\eta)}_{cm} \right\rangle_{(t,\eta)} \right| \leq C \| J^* (\zeta^{(\eta)}) \|_{H_t}.
\]

The density of \( J^* (\mathcal{H}^0_t) \) in \( \mathcal{H}_t \) and Riesz representation theorem then easily imply that there is a unique \( d\tilde{\Psi}_{t,cm} \in \mathcal{H}_t \) such that \( d\phi^{(-\eta)}_{cm} = J(d\tilde{\Psi}_{t,cm}) \). \( \Box \)

\(^4\)Large deviation theory for Gaussian measure over a generalized Wiener space is nicely discussed in Chap. 3 of [91].
This result characterizes the deformed harmonic energy $\frac{1}{4} \mathcal{E} [\hat{\Psi}_{t,\text{cm}}]$ as the large deviation functional governing the $\mathcal{O}(q)$ fluctuations of the distribution of the maps $d\phi_{ij}$, around the center of mass $d\phi_{cm}$, as compared to the $\mathcal{O}(\sqrt{q})$ Gaussian fluctuations sampled by the central limit theorem. Note that in the path integral approach $\Gamma_t (d\phi_{cm}^{-\eta})$ plays the role of the effective action of the non-linear $\sigma$ model formally derived, from the physical Hilbert space measure $e^{-S_{cm}[v(t)]:a} D [v(t)]$, by expressing the asymptotic series of associated Feynman amplitudes in terms of 1–Particle Irreducible diagrams \cite{34}.

According to (5.78), and (5.79) we have that for $t \searrow 0$ $d\hat{\Psi}_{t,v} \rightarrow d\phi_{(v)}$, $\mathcal{E} [\hat{\Psi}_{t,v}] \rightarrow E [\phi_{(v)}]$, and (6.22) correspondingly extends to $t = 0$ as

$$
\int_{\hat{\mathcal{H}}_{t=0}^{-\eta}} e^{\sqrt{-t} \left\langle \hat{d}\Psi_{t=0,v}^{(n)} d\phi_{(u)}^{-\eta} \right\rangle (t=0, \eta)} \mathcal{Q}_{t=0} [d\phi_{(u)}] = e^{-\frac{1}{a} E [\phi_{(u)}]} .
$$

(6.40)

If we consider the non–decreasing family of sub–$\sigma$–algebras $\{ B_t^{-\eta} \}_{t \geq 0}$ generated by requiring that $\left\langle \hat{d}\Psi_{t,v}^{(n)} d\phi_{(u)}^{-\eta} \right\rangle (t, \eta)$ are $\{ B_t^{-\eta} : t \in [0, \infty) \}$–progressively measurable maps, then we can interpret the curve of measures

$$
t \mapsto (\mathcal{H}_t, \mathcal{H}_t^{-\eta}, \mathcal{Q}_{\mathcal{H}_t}) , \quad t \geq 0 ,
$$

(6.41)

as describing the $\mathcal{Q}_{\mathcal{H}_t}$–distribution of the fields $d\phi^{-\eta}$ as we deform the geometry of $(M, g, dw)$ along the heat kernel embedding. In this sense (6.41) provides a natural framework for a renormalization group analysis of the heat kernel embedding of the non–linear $\sigma$ model. The filtration $(\mathcal{H}_t, \mathcal{H}_t^{-\eta}, \mathcal{Q}_{\mathcal{H}_t})$, $t \geq 0$, is described by the family $t \mapsto \mathcal{E} [\hat{\Psi}_{t,v}]$ of associated characteristic functions. According to (5.78) and Lemma 5.19, $t \mapsto \mathcal{E} [\hat{\Psi}_{t,v}]$ is characterized by the running metric $t \mapsto h_t^{(q)}$, i.e. by the coupling between the running metric $g_t^{(w)}$ and the running dilaton $f_t^{(w)}$ describing the behavior of the warped harmonic energy functional $E [\Phi_{(q)}, h_t^{(q)}]$ as the distances in $(M^q \times \times (f, \mathbb{T}^q)$ are rescaled by the heat kernel embedding. The associated tangent vector, the so called beta–function,

$$
\beta (h_t^{(q)}) := \frac{d}{dt} h_t^{(q)} ,
$$

(6.42)

plays a major role in renormalization group theory. Typically, in field theory one is able to compute the beta function only up to a few leading order terms in the perturbative expansion of the effective action. Notwithstanding this limitation, the resulting truncated flow, associated with (6.42), can be exploited to study both the validity of the perturbative expansion and the nature of the possible fixed points of the renormalization group action \cite{34}.

In our case, we have the exact expression for the effective action $\mathcal{E} [\hat{\Psi}_{t,v}]$, (or, equivalently for $E [\Phi_{(q)}, h_t^{(q)}]$), however $(\mathcal{H}_t, \mathcal{H}_t^{-\eta}, \mathcal{Q}_{\mathcal{H}_t})_{t \geq 0}$ is not the natural physical renormalization group filtration of the dilatonic non–linear $\sigma$ model. Nonetheless, the beta function (6.42) computed along the heat kernel...
embedding turn out to be remarkably similar to the ones obtained, to leading order, by the formal perturbative path integral approach.

7. Heat kernel embedding and Ricci flow

According to (6.42), the beta function associated with the heat kernel embedding is provided by a time derivative of the flow of metrics \( (h, t) \mapsto h_t^{(q)} \), i.e.

\[
\frac{d}{dt} h_t = \frac{d}{dt} g_t^{(\omega)} - \frac{2}{q} e^{-\frac{2f_t^{(\omega)}}{q}} \delta_{\mathcal{E}_0} \frac{dt}{df_t^{(\omega)}}.
\]

(7.1)

Since the whole geometry of \( (M \times \mathbb{T}^q, h_t) \) is controlled by \( (M, g_t^{(\omega)}) \), we start with a detailed computation of the Riemannian curvatures associated with \( (M, g_t^{(\omega)}) \) in the smooth case, induced by the heat kernel embedding

\[
(M, d_g) \times \mathbb{R}_{>0} \leftrightarrow (\text{Prob}_{\text{ac}}(M, g), d_{\mathcal{W}})
\]

(7.2)

\[
(z, t) \mapsto p_t^{(\omega)}(\cdot, z)d\omega(\cdot), \quad t > 0.
\]

7.1. Geodesics and the Ricci curvature of \( (M, g_t^{(\omega)}) \)

Let us consider, for \( t > 0 \), the curve of measures

\[
\lambda \mapsto \nu(c(\lambda), t) := p_t^{(\omega)}(\cdot, c(\lambda))d\omega(\cdot), \quad t > 0,
\]

(7.3)

associated with an absolutely continuous curve

\[
c : [-\epsilon, \epsilon] \ni \lambda \mapsto c(\lambda) \in M, \quad c(\lambda = 0) = z, \quad \epsilon > 0,
\]

(7.4)

passing through the point \( z \in M \). We have the following result locally characterizing the geodesics of \( (M, g_t^{(\omega)}) \).

**Proposition 7.1.** For any given \( t > 0 \), let \( \tilde{\nabla}^{(t)} \) denote the Levi–Civita connection of \( (M, g_t^{(\omega)}) \). If the curve \( c \) is a geodesic for \( \tilde{\nabla}^{(t)} \) then the vector field \( \tilde{\psi}_{(t, c(\lambda), c'(\lambda))} \in T_{\nu(\lambda)} \text{Prob}_{\text{ac}}(M, g) \), covering the curve \( \lambda \mapsto \nu(\lambda) := p_t^{(\omega)}(\cdot, c(\lambda))d\omega(\cdot) \), satisfies the Hamilton–Jacobi equation

\[
\frac{\partial}{\partial \lambda} \tilde{\psi}_{(t, c(\lambda), c'(\lambda))}(y) + \frac{1}{2} \left| \nabla^{(y)} \tilde{\psi}_{(t, c(\lambda), c'(\lambda))}(y) \right|_{g}^2 = 0,
\]

(7.5)

where \( \cdot \mid_{g} \) and \( \nabla \) respectively denote the Riemannian norm and the Levi–Civita connection in the original \( (M, g) \).
Proof. If $c$ is a geodesic for $\nabla^{(t)}$ then, according to (5.25) and Lemma 5.12 we have
\[
\frac{d}{d\lambda} g_t^{(\omega)} (c'(\lambda), c'(\lambda)) = 2 g_t^{(\omega)} \left( \nabla_{c'}(\lambda)c'(\lambda), c'(\lambda) \right) = 0 \quad (7.6)
\]

\[
= \frac{d}{d\lambda} \int_M \left| \nabla(y) \hat{\psi}_{\lambda,t}(y) \right|^2_g p_t^{(\omega)}(y, c(\lambda)) \, d\omega(y)
\]

\[
= 2 \int_M \left( \nabla^i \hat{\psi}_{\lambda,t} \frac{\partial}{\partial x} \nabla^i \hat{\psi}_{\lambda,t} \right) p_t^{(\omega)}(y, c(\lambda)) d\omega(y)
\]

\[
+ \int_M \left| \nabla(y) \hat{\psi}_{\lambda,t}(y) \right|^2_g \frac{\partial}{\partial x} p_t^{(\omega)}(y, c(\lambda)) d\omega(y)
\]

where we set for notational simplicity $\hat{\psi}_{\lambda,t} = \hat{\psi}_{(t,c(\lambda),c'(\lambda))}$. According to Lemma 5.11 and 5.12 along the curve of measures $\lambda \mapsto p_t^{(\omega)}(\cdot, c(\lambda))d\omega(\cdot)$ we can write, for almost every $\lambda \in (-\epsilon, \epsilon)$,
\[
\frac{\partial}{\partial \lambda} p_t^{(\omega)}(y, z) = -\text{div} G^{(\omega)} \left( \nabla \hat{\psi}_{\lambda,t}(y) p_t^{(\omega)}(y, c(\lambda)) \right) . \quad (7.7)
\]

By exploiting this relation, which we interpret in the distributional sense, and integrating by parts (7.6), we get
\[
\int_M \nabla^i \hat{\psi}_{\lambda,t} \left( \frac{\partial}{\partial \lambda} \nabla^i \hat{\psi}_{\lambda,t} + \frac{1}{2} \nabla^i \left| \nabla \hat{\psi}_{\lambda,t} \right|^2_g \right) p_t^{(\omega)}(y, c(\lambda)) d\omega(y)
\]

\[
= \int_M \nabla^i \hat{\psi}_{\lambda,t} \nabla^i \left( \frac{\partial}{\partial \lambda} \hat{\psi}_{\lambda,t} + \frac{1}{2} \left| \nabla \hat{\psi}_{\lambda,t} \right|^2_g \right) p_t^{(\omega)}(y, c(\lambda)) d\omega(y)
\]

\[
= -\int_M \Delta p_t^{(\omega)} \hat{\psi}_{\lambda,t} \left( \frac{\partial}{\partial \lambda} \hat{\psi}_{\lambda,t} + \frac{1}{2} \left| \nabla \hat{\psi}_{\lambda,t} \right|^2_g \right) p_t^{(\omega)}(y, c(\lambda)) d\omega = 0 .
\]

According to (5.15) we have
\[
\Delta p_t^{(\omega)} \hat{\psi}_{(t,c(\lambda),c'(\lambda))} = -c'(\lambda) \cdot \nabla^{c(\lambda)} \ln p_t^{(\omega)}(y, c(\lambda)) , \quad (7.9)
\]

and the last line of (7.8) becomes
\[
\int_M \left( \frac{\partial}{\partial \lambda} \hat{\psi}_{\lambda,t} + \frac{1}{2} \left| \nabla \hat{\psi}_{\lambda,t} \right|^2_g \right) c'(\lambda) \cdot \nabla^{c(\lambda)} p_t^{(\omega)}(y, c(\lambda)) d\omega = 0 . \quad (7.10)
\]

This relation must hold for any (germ of) geodesic $(-\epsilon, \epsilon) \ni \lambda \mapsto c(\lambda)$ in $(M, g_t^{(\omega)})$ passing through $z$, i.e. for any $c'(\lambda) \neq 0$. According to Proposition 5.8 this implies
\[
\frac{\partial}{\partial x} \hat{\psi}_{\lambda,t} + \frac{1}{2} \left| \nabla \hat{\psi}_{\lambda,t} \right|^2_g = 0 . \quad (7.11)
\]

As an immediate consequence of this result and of the characterization of the geodesics of $(\text{Prob}_{\text{ac}}(M, g), d_g^{W})$ with respect of the Otto inner product we have
Let us consider the tensor field $T$ of $\mathrm{Prob}_{ac}(M,g)$ and by F. Otto and C. Villani [93], according to which if a vector field $\hat{\psi}_{t(c(\lambda),c'(\lambda))} \in T_{c(\lambda)}\mathrm{Prob}_{ac}(M,g)$, tangent to a smooth curve of measures $\lambda \mapsto p_t^{(\omega)}(\cdot,c(\lambda))d\omega(\cdot)$, satisfies the Hamilton–Jacobi equation (7.15), then the curve in question is a geodesic in the smooth probability space $(\mathrm{Prob}_{ac}(M,g),\nabla)$. Since in the smooth setting minimizing geodesics are unique [53], (cf. Lemma 3.11), the stated result follows.

According to Lemma 7.2 the Levi–Civita connection $\tilde{\nabla}$ of $(M,g_t^{(\omega)})$ can be identified with the induced $\nabla$ connection on $\mathcal{Y}_t(\mathcal{M},g)$, in particular we can compute the curvature of $(M,g_t^{(\omega)})$ by restricting to $\mathcal{Y}_t(\mathcal{M},g)$ the Riemannian curvature of the Wasserstein space $(\mathrm{Prob}_{ac}(M,g),\langle\cdot,\cdot\rangle_{p_t(d\omega)})$. As a preliminary result, let us consider the tensor field $T_{\psi(U)}\psi(V)$ defined by (3.59) and associated to the measure $p_t^{(\omega)}d\omega$. Since $\mathcal{Y}_t(\mathcal{M},g)$ is a totally geodesic submanifold we have

**Lemma 7.3.** Let $U_\perp,V_\perp \in T_z(M)$ and let $\hat{\psi}(U) := \hat{\psi}_{t,z,U(z)}$ and $\hat{\psi}(V) := \hat{\psi}_{t,z,V(z)}$ be the corresponding potentials solutions of (5.13). Then, for any $t > 0$ and $\forall U_\perp,V_\perp \in T_z(M)$,

$$T_{\psi(U)}\psi(V) := (I - \Pi_{p_t(d\omega)}) \left( \nabla a \hat{\psi}(U) \nabla b \hat{\psi}(V) \right) = 0,$$

(7.14)

where $\Pi_{p_t(d\omega)}$ is the projection operator (2.35) associated with $p_t(d\omega)$.

**Proof.** Let us consider $U_\perp,V_\perp,W_\perp \in T_z(M)$ as elements of $T_z\mathrm{Diff}(M)$, the tangent space to $\mathrm{Diff}(M)$ at the identity map, then we can introduce
the connection $\nabla$ associated to the weak Riemannian structure induced on $Diff(M)$ by the $L^2(p_t(d\omega))$ inner product according to

$$\left\langle \nabla_{U_\perp} V_\perp, W_\perp \right\rangle_{p_t(\omega)} = \int_M g \left( \nabla_{U_\perp}^g V_\perp, W_\perp \right) p_t(\omega)(y, z) \, d\omega(y), \quad (7.15)$$

(see [3.43]). By considering the restriction of $\nabla$ to $\text{Prob}(M, g)$ we have

$$U_\perp \hookrightarrow \nabla \hat{\psi}(U) \quad (7.16)$$

$$V_\perp \hookrightarrow \nabla \hat{\psi}(V)$$

$$\nabla_{U_\perp} V_\perp \hookrightarrow \nabla \hat{\psi}(U) \nabla \hat{\psi}(V)$$

$$\Pi_{p_t(d\omega)} \left( \nabla^a \hat{\psi}(U) \nabla^b \hat{\psi}(V) \right) \hookrightarrow \nabla \hat{\psi}(U) \nabla \hat{\psi}(V) ;$$

in the weak sense, hence we can formally rewrite (7.14) as

$$T_{\hat{\psi}(U)\hat{\psi}(V)} = \nabla \hat{\psi}(U) \nabla \hat{\psi}(V) - \nabla \hat{\psi}(U) \nabla \hat{\psi}(V) \quad (7.17)$$

Thus, $T_{\hat{\psi}(U)\hat{\psi}(V)}$ can be interpreted as the second fundamental form of the immersion $[Y_t((M, g)), \nabla] \hookrightarrow \text{Prob}_{\text{ac}}(M, g) \subset Diff(M)$. Since for any $t > 0$ the immersion is totally geodesic, $T_{\hat{\psi}(U)\hat{\psi}(V)} = 0$ follows. On less formal ground, we can prove the stated result by introducing normal geodesic coordinates with respect to $(M, g_t(\omega))$, centered at $z \in M$. To this end, let $\{e(i)(z)\}$ denote a basis in $T_z(M)$ and let us consider a set of curves in $(M, g_t(\omega))$, issuing from $z \in M$ with tangent vector $c'(i)(0) := \{e(i)(z)\}$,

$$c(i) : [-\epsilon, \epsilon] \ni \lambda \mapsto c(i)(\lambda) \in M, \quad c(i)(0) = z, \quad i = 1, \ldots, n \quad (7.18)$$

According to Proposition 7.1, Lemma 5.13 and 5.12, these curves are geodesics of $(M, g_t(\omega))$, for a given $t > 0$, iff the corresponding curves of probability measures and potentials

$$\lambda \mapsto p_t(\omega)(y, c(i)(\lambda)) \, d\omega(y) \quad (7.19)$$

$$\lambda \mapsto \hat{\psi}^{(i)}_{(t, \lambda)} := \hat{\psi}^{(i)}_{(t, c(i)(\lambda), c'(i)(\lambda))}$$

evolve according to the relations, (defining the optimal transport map),

$$\frac{\partial}{\partial \lambda} p_t(\omega)(y, c(i)(\lambda)) + \text{div}_\omega^g \left( p_t(\omega)(y, c(i)(\lambda)) \nabla^g(y) \hat{\psi}^{(i)}_{(t, \lambda)}(y) \right) = 0 \quad (7.20)$$

$$\frac{\partial}{\partial \lambda} \hat{\psi}^{(i)}_{(t, \lambda)}(y) + \frac{1}{2} \left| \nabla^g(y) \hat{\psi}^{(i)}_{(t, \lambda)}(y) \right|_g^2 = 0 ,$$

(see [5.30] and [7.5]), with the initial conditions $p_t(\omega)(y, c(i)(\lambda = 0)) = p_t(\omega)(y, z)$ and $\hat{\psi}^{(i)}_{(t, \lambda=0)} := \hat{\psi}^{(i)}_{(t, z, c'(i)(0))}$. In such a framework, normal geodesic coordinates at $z$ can be characterized by: (i) Choosing the initial $\hat{\psi}^{(i)}_{(t, \lambda=0)}$, \ldots
(hence the $c'_{(i)}(0)$), in such a way that the connection coefficients (7.13) vanish at $z$; (ii) Propagating the chosen $\hat{\psi}^{(i)}_{(t,\lambda=0)}$ according to the Hamilton–Jacobi equation in (7.20), (see (7.5)), so as to obtain the curve of potentials $\lambda \mapsto \hat{\psi}^{(i)}_{(t,\lambda)}$; (i) Use the potentials so obtained to evolve the initial density $p_t^{(\omega)}(y, z)$ by means of the continuity equation in (7.20). Since

$$\frac{\partial}{\partial \lambda} p_t^{(\omega)}(y, c_{(i)}(\lambda)) = c'_{(i)}(\lambda) \cdot \nabla_{c_{(i)}(\lambda)} p_t^{(\omega)}(y, c_{(i)}(\lambda)),$$

(7.21)

the procedure described implies that along (7.20) we recover the relation

$$\text{div}_\omega \left( p_t^{(\omega)}(y, c_{(i)}(\lambda)) \nabla^{(y)} \hat{\psi}^{(i)}_{(t,\lambda)}(y) \right) = - c'_{(i)}(\lambda) \cdot \nabla_{c_{(i)}(\lambda)} p_t^{(\omega)}(y, c_{(i)}(\lambda)),$$

(7.22)

connecting the tangent vector $c'_{(i)}(\lambda)$ to the potential $\hat{\psi}^{(i)}_{(t,\lambda)}$, and hence identifying them with $\hat{\psi}^{(i)}_{(t,c_{(i)}(\lambda),c'_{(i)}(\lambda))}$. For our purposes it is sufficient to consider the step (i). To this end we set $\hat{\psi}^{(i)} := \hat{\psi}^{(i)}_{(t,\lambda=0)}$ for notational simplicity, and require that the $\{ \hat{\psi}^{(i)} \}_{i=1,\ldots,n}$ are such that

$$\langle \nabla \hat{\psi}^{(i)} \nabla \psi_{(k)}, \nabla \zeta \rangle_{pr_t(d\omega)}(z)$$

(7.23)

$$= \int_M \nabla_a \hat{\psi}^{(i)}(y) \nabla^a \nabla^b \hat{\psi}^{(k)}(y) \nabla_b \zeta(y) \, p_t^{(\omega)}(y, z) \, d\omega(y)$$

$$= \int_M \Pi_{pr_t(d\omega)} \left( \nabla_a \hat{\psi}^{(i)}(y) \nabla^a \nabla^b \hat{\psi}^{(k)}(y) \right) \nabla_b \zeta(y) \, p_t^{(\omega)}(y, z) \, d\omega(y) = 0,$$

for any choice of $\zeta \in C^\infty(M, \mathbb{R})/\mathbb{R}$. Since $\nabla$ is the Levi–Civita connection for $(M, p_t^{(\omega)})$, (7.23) is symmetric under the exchange $\nabla \psi_{(i)} \leftrightarrow \nabla \psi_{(k)}$ and we can write

$$\langle \nabla \hat{\psi}^{(i)} \nabla \psi_{(k)}, \nabla \zeta \rangle_{pr_t(d\omega)}(z)$$

(7.24)

$$= \frac{1}{2} \int_M \Pi_{pr_t(d\omega)} \left( \nabla^b \left( \nabla_a \hat{\psi}^{(i)} \nabla^a \hat{\psi}^{(k)} \right) \right) \nabla_b \zeta \, p_t^{(\omega)}(y, z) \, d\omega(y)$$

$$= \frac{1}{2} \int_M \nabla^b \left( \nabla_a \hat{\psi}^{(i)} \nabla^a \hat{\psi}^{(k)} \right) \nabla_b \zeta \, p_t^{(\omega)}(y, z) \, d\omega(y),$$

where we have exploited the fact that $\Pi_{pr_t(d\omega)}$ projects onto the gradient vector fields. Integrating by parts with respect to $p_t^{(\omega)}(y, z) \, d\omega(y)$ we get

$$\langle \nabla \hat{\psi}^{(i)} \nabla \psi_{(k)}, \nabla \zeta \rangle_{pr_t(d\omega)}(z)$$

(7.25)

$$= - \frac{1}{2} \int_M \Delta_{pr_t(d\omega)} \left( \nabla_a \hat{\psi}^{(i)} \nabla^a \hat{\psi}^{(k)} \right) \zeta(y) \, p_t^{(\omega)}(y, z) \, d\omega(y),$$
so that the condition (7.23) reduces to

$$\Delta_{p_t(d\omega)} g \left( \nabla \hat{\psi}^{(i)}, \nabla \hat{\psi}^{(k)} \right) = 0 \quad (7.26)$$

in the distributional sense. Thus, almost everywhere \( g \left( \nabla \hat{\psi}^{(i)}, \nabla \hat{\psi}^{(k)} \right) = C_{ik} \), for some constant \( C_{ik} \). We normalize \( C_{ik} \) and choose the geodesic coordinate initial data \( \hat{\psi}^{(i)} := \hat{\psi}_{(\ell,\lambda=0)}^{(i)} \) according to

$$g \left( \nabla \hat{\psi}^{(i)}, \nabla \hat{\psi}^{(k)} \right) = \delta_{ik} \quad . \quad (7.27)$$

In particular we get that, in a suitable weak sense, each \( \hat{\psi}^{(i)} \) should be a solution of the eikonal equation

$$\left| \nabla \hat{\psi}^{(i)} \right|_{g}^2 = 1 \quad , \quad (7.28)$$

on \((M,g)\). An explicit characterization of the solutions of the eikonal equation can be obtained in terms of the distance function of \((M,g)\), (see e.g. [94], [95], and [96], [97] for a detailed analysis). To apply the theory in our case, let \( \mathcal{U}_z \) be a star–shaped open set of \( 0 \in T_zM \) and let \( \exp_z^g : \mathcal{U}_z \cap T_zM \longrightarrow (M,g) \) denote the exponential map of \((M,g)\) at \( z \in M \). Let \( E_{(i)}(z), \ i = 1, \ldots, n \), be an orthonormal basis of \((T_zM,g)\). For \( y \in M \) within the cut locus \( \text{Cut}(z,g) \) of \( z \) in \((M,g)\), let \( \{z^i(y) := g(E_{(i)}, (\exp_z^g)^{-1}(y))\} \) be its normal geodesic coordinates in \( T_zM \). For \( y \in M \setminus (\text{Cut}(z,g) \cup \{z\}) \) the distance function \( d_g(z,y) \) is \( C^\infty \), and if we denote by \( d_g(0,(\exp_z^g)^{-1}(y)) \) its pull–back in \((T_zM,g)\) we have

\[
\begin{align*}
d^2_g \left( 0, (\exp_z^g)^{-1}(y) \right) &= \sum_{i=1}^{n} d^2_g \left( 0, g \left( E_{(i)}, (\exp_z^g)^{-1}(y) \right) \right) \\
&= \sum_{i=1}^{n} (z^i(y))^2 \; . \quad (7.29)
\end{align*}
\]

Hence we compute \( \frac{\partial}{\partial z^i} d_g \left( 0, g \left( E_{(i)}, (\exp_z^g)^{-1}(y) \right) \right) = \delta^k_{(i)} \) and we can write \( \left| \nabla^{(y)} d_g \left( 0, g \left( E_{(i)}, (\exp_z^g)^{-1}(y) \right) \right) \right|^2_\ast = 1, \) in \( M \setminus \text{Cut}(z,g) \cup \{z\} \). Hence \( \hat{\psi}^{(i)}(y) := d_g \left( 0, g \left( E_{(i)}, (\exp_z^g)^{-1}(y) \right) \right) \) naturally appears as a candidate solution of the eikonal equation in the open set \( M \setminus \text{Cut}(z,g) \cup \{z\} \). To handle the presence of the singular domain \( \text{Cut}(z,g) \cup \{z\} \) where the distance function is not differentiable we can interpret the solution of (7.28) in the viscosity sense [94], [95]. Thus, if we normalize \( \hat{\psi}^{(i)} \) to a vanishing \( p_t^{(\omega)} d\omega \) average, we have that for any \( t > 0 \) the functions

\[
\begin{align*}
\hat{\psi}^{(i)}(y) &= d_g \left( 0, g \left( E_{(i)}, (\exp_z^g)^{-1}(y) \right) \right) \\
&- \int_{(\exp_z^g)^{-1}(M)} d_g \left( 0, g \left( E_{(i)}, (\exp_z^g)^{-1}(x) \right) \right) \left( (\exp_z^g)^{\ast} (p_t^{(\omega)}) d\omega \right)
\end{align*}
\]
generate normal geodesic coordinates
\[ g_t^{(ω)} \left( c'(i)(0), c'(k)(0) \right) (z) = δ_{ik} \] (7.31)

\[ \langle \nabla\nabla_{ψ(i)} ψ(κ), \nablaζ \rangle p_t(δω)(z) = 0, \]
in \((M \setminus (Cut(z, g) \cup \{z\}))\), \(g_t^{(ω)}\) via the steps \((ii)\) and \((iii)\) described above. In such normal coordinates we compute \(\nabla^a \nabla^b ψ(j) = 0\) almost everywhere in \(M \setminus (Cut(z, g) \cup \{z\})\), hence
\[ T_{ψ(i)ψ(j)} = (I - Π(p_t(δω))) \left( \nabla^a ψ(i) \nabla^b ψ(j) \right) = 0, \] (7.32)
in the weak sense, and the stated lemma follows. □

With these preliminary results along the way a direct application of Theorem 3.13 provides

**Proposition 7.4.** For any fixed \(t > 0\), \(z \in M\), and \(U_⊥, V_⊥, W_⊥, Z_⊥ \in T_z(M)\), let us denote by \(\nablâψ(t,U) := \nabla(y)̂ψ(t,z,U_⊥)\), \(\nablâψ(t,V) := \nabla(y)̂ψ(t,z,V_⊥)\), \(\nablâψ(t,W) := \nabla(y)̂ψ(t,z,W_⊥)\), and \(\nablâψ(t,Z) := \nabla(y)̂ψ(t,z,Z_⊥)\) the corresponding vector fields defined by the weighted heat kernel injection map \((7.23)\). Then the Riemannian curvature operator \(\tilde{Rm}^{(t)}(U_⊥, V_⊥) W_⊥\) of \((M, g_t^{(ω)})\) at the point \(z \in M\) is given by
\[ g_t^{(ω)} \left( \tilde{Rm}^{(t)}(U_⊥, V_⊥) W_⊥, Z_⊥ \right) (z) \]
\[ = \int_M g \left( Rm \left( \nablâψ(t,U), \nablâψ(t,V) \right), \nablâψ(t,W), \nablâψ(t,Z) \right) (y) Π_t^{(ω)}(y, z) dω(y), \]
where \(Rm(ω, ω) \circ \circ\) denotes the Riemann tensor of \((M, g)\).

By exploiting this result and the above characterization of the normal coordinates in \((M, g_t^{(ω)})\) we can readily compute the Ricci tensor associated to \((M, g_t^{(ω)})\) according to

**Proposition 7.5.** For any \(t > 0\), \(z \in M\), and \(U_⊥, W_⊥ \in T_z M\), the Ricci curvature of \((M, g_t^{(ω)})\) in the direction of the 2–plane spanned by the vectors \(U_⊥, W_⊥ \in T_z M\) is provided by
\[ \tilde{Ric}^{(t)}(U_⊥, W_⊥)(z) := \text{trace} \left( e_{(i)} \mapsto \tilde{Rm}^{(t)} \left( e_{(i)}, U_⊥ \right) W_⊥ \right) (z) \]
\[ = \int_M \text{Ric} \left( \nablâψ(t,U), \nablâψ(t,W) \right) (y) Π_t^{(ω)}(y, z) dω(y), \]
where \(\text{Ric}\) denotes the Ricci tensor of \((M, g)\) and where \(\{e_{(i)}(z) := c'_{(i)}(0)\}\) denotes the \(\{g_t^{(ω)}\}–\text{orthonormal}\) basis of \(T_z M\) associated with the potentials \(\{ψ_{(j)}\}\) defining normal coordinates in \((M, g_t^{(ω)}),\) centered at \(z \in M\).
Proof. By tracing the expression (7.33) for the Riemann curvature operator $\tilde{R}m(t)$ and by exploiting the orthonormality conditions $g_t(\omega) (e(i), e(k)) (z) = \delta_{ik}$, and $g \left( \nabla \tilde{\psi}(i), \nabla \tilde{\psi}(k) \right) = \delta_{ik}$, (see (7.27), which hold in normal geodesic coordinates one immediately gets (7.34). □

7.2. Heat kernel induced Ricci flow

We have set the stage for discussing the scaling dependence of the metric $(M, g_t(\omega))$ for $t \in (0, \infty)$ and compute the associated beta function. To this end, for $t > 0$, $z \in M$ and $U_\perp \in T_z(M)$, let $\tilde{\psi}(t, U) := \tilde{\psi}(t, z, U_\perp)$ be the corresponding potential defined by the weighted heat kernel injection map (5.23). Let us consider for $t > 0$, and for $U_\perp(z)$ varying in $T_z(M)$, the smooth functional

$$U_\perp \mapsto P(U)(z, t) := \int_M \tilde{\psi}(t, U)(y) p_t(\omega)(y, z) d\omega(y), \quad (7.35)$$

providing the coordinate representation in $(C^\infty(M, \mathbb{R}))^*$ of $p_t(\omega)(\cdot, z) d\omega(\cdot)$ at fixed heat source $\delta_z$. We define a time–dependent tangent vector $X(t, z) \in T_z M$ by

$$g_t(\omega)(X(t, z), U_\perp(z)) = -\frac{d}{dt} P^\phi(t)(z). \quad (7.36)$$

Explicitly one computes

$$\frac{d}{dt} \int_M \tilde{\psi}(t, U)(y) p_t(\omega)(y, z) d\omega(y) \quad (7.37)$$

$$= \int_M \left( \tilde{\psi}(t, U)(y) \frac{\partial}{\partial t} \ln p_t(\omega)(y, z) \right) p_t(\omega)(y, z) d\omega(y)$$

$$= \int_M \left( \tilde{\psi}(t, U)(y) \Delta_p(\omega) \ln p_t(\omega)(y, z) \right) p_t(\omega)(y, z) d\omega(y)$$

$$= -\int_M \left( \nabla \ln p_t(\omega)(y, z) \cdot \nabla \tilde{\psi}(t, U)(y) \right) p_t(\omega)(y, z) d\omega(y), \quad \forall U_\perp \in T_z M ,$$

where we exploited the relation $(p_t(\omega))^{-1} \Delta_\omega p_t(\omega) = \Delta_p(\omega) \ln p_t(\omega)$, which holds pointwise for $t > 0$. Hence $X(t, z)$ is provided by

$$g_t(\omega)(X(t, z), U_\perp(z)) \quad (7.38)$$

$$= \int_M \left( \nabla \ln p_t(\omega)(y, z) \cdot \nabla \tilde{\psi}(t, U)(y) \right) p_t(\omega)(y, z) d\omega(y).$$

We have the following
Lemma 7.6. The Lie derivative of the metric tensor $g_t^{(\omega)}$ along the $t$–dependent vector field $z \mapsto X_{(t,z)}$ is provided by

$$\mathcal{L}_{X_{(t,z)}} g_t^{(\omega)} (U_\perp, U_\perp) = 2 \int_M \left( \nabla^a \psi_{(t,U)} \text{Hess}_{ab} \ln p_t^{(\omega)}(y, z) \nabla^b \psi_{(t,U)} \right) p_t^{(\omega)}(y, z) \, d\omega(y) .$$  

(7.39)

Proof. Let $\nabla_{\psi_{(t)}}$ denote the vector $\in T_{p_t(d\omega)} \text{Prob}_{ac}(M, g)$ associated with $X_{(t,z)} \in T_z M$. According to the characterization of the Lie derivative $\overline{\mathcal{L}}$ on $\text{Prob}(M, g), \text{(cf. (3.55))}$, we have

$$\overline{\mathcal{L}}_{\nabla_{\psi_{(t)}}} g_t^{(\omega)} (U_\perp, U_\perp) = 2 \int_M \left( \nabla^a \psi_{(t,U)} \text{Hess}_{ab} \ln p_t^{(\omega)}(y, z) \nabla^b \psi_{(t,U)} \right) p_t^{(\omega)}(y, z) \, d\omega(y) ,$$

(7.40)

where we have exploited the definition (3.54) of the Riemannian connection on $\text{Prob}(M, g)$ and $\nabla_{\varphi} = \nabla(y) \ln p_t^{(\omega)}(y, z)$, (see (7.38)), to write, $\forall \varphi \in C^\infty(M, \mathbb{R})$,

$$\langle \overline{\nabla}_{\nabla_{\psi_{(t)}}} \nabla_{\nu_{(t)}}, \nabla_{\nu_{(t)}} \rangle_{(g, p_t(d\omega))} = \int_M \left( \nabla_{\nabla \varphi}(y) \cdot \nabla \ln p_t^{(\omega)}(y, z) \cdot \nabla_{\varphi}(y) \right) p_t^{(\omega)}(y, z) \, d\omega(y) .$$

(7.41)

Since, according to Lemma 7.2, the Levi–Civita connection $\overline{\nabla}$ of $(M, g_t^{(\omega)})$ can be identified with the induced $\overline{\nabla}$ connection on $\Upsilon_t ((M, g))$, we have that the Lie derivative $\overline{\mathcal{L}}_{\nabla_{\psi_{(t)}}}$ on $\text{Prob}_{ac}(M, g)$ when restricted to $\Upsilon_t ((M, g))$ reduces to $\mathcal{L}_{X_{(t,z)}}$ and the Lemma follows. \hfill $\Box$

Remark 7.7. Note that the Lie derivative (7.39) captures heat concentration phenomena related to the heuristics of heat propagation along geodesics expressed by the asymptotics (4.50), (cf. Th. 4.17). In line with (4.49) and Varadhan’s large deviation formula (5.5), for $y$ in compact subset $M \setminus (\text{Cut}(z) \cup \{ z \})$, we have the uniform limit

$$- 4 \lim_{t \searrow 0^+} t \text{Hess} \left( \ln p_t^{(\omega)}(y, z) \right) = \text{Hess} d_g^2(y, z) ,$$

(7.42)

whereas for $y \in \text{Cut}(z)$, the hessian $\text{Hess} \left( \ln p_t^{(\omega)}(y, z) \right)$ diverges to $- \infty$ faster than $t^{-1}$ as $t \searrow 0$. This implies that $-t \ln p_t^{(\omega)}(y, z)$ plays the role of a smooth mollifier of the (squared) distance function $d_g^2(y, z)$, and the limit of $-t \text{Hess} (\ln p_t^{(\omega)}(y, z))$ for $t \searrow 0$ computes $\text{Hess} d_g^2(y, z)$ in the sense of distributions. In particular it can be shown that the singular part of $\text{Hess} d_g^2(y, z)$ is concentrated, for $\dim M \geq 2$, on $\text{Cut}(z)$ and
it is absolutely continuous with respect to the \( (n - 1) \)-dimensional Hausdorff measure \( \mathcal{H}^{n-1} \) on \( \text{Cut}(z) \).

The explicit computation of the beta function \( \frac{d}{dt} g_t^{(\omega)}(U_\perp, U_\perp) \) can be carried out along the lines of [27], with some obvious modifications due to the presence of the weighted Laplacians \( \triangle_\omega \) and \( \triangle_{p_t(\omega)} \). We start with some preliminary remarks. Let us recall that the standard Bochner-Weitzenböck formula for the Laplace–Beltrami operator \( \triangle_g \) on \( (M, g) \) reads

\[
2 \nabla^i \phi \nabla_i \triangle_g \phi = \triangle_g |\nabla \phi|^2_g - 2 |\text{Hess} \phi|^2_g - 2 R^{ik} \nabla_i \phi \nabla_k \phi ,
\]

for any \( \phi \in C^\infty(M, \mathbb{R}) \). From (7.43) and

\[
2 \nabla^i \phi \nabla_i \triangle_\omega \phi = 2 \nabla^i \phi \nabla_i (\triangle_g \phi - \nabla_k f \nabla^k \phi) ,
\]

and

\[
2 \nabla^i \phi \nabla_i \triangle_{p_t(\omega)} \phi = 2 \nabla^i \phi \nabla_i \left( \triangle_g \phi - \nabla_k \left( f - \ln p_t(\omega) \right) \nabla^k \phi \right) ,
\]

a direct computation extends (7.43) to \( (M, g, d\omega) \) and to \( (M, g, p_t(\omega) d\omega) \) according to

\[
2 \nabla^i \phi \nabla_i \triangle_\omega \phi = \triangle_\omega |\nabla \phi|^2_g - 2 |\text{Hess} \phi|^2_g
\]

\[
- 2 \left( R_{ik} + \text{Hess}_{ik} f \right) \nabla^i \phi \nabla^k \phi ,
\]

and

\[
2 \nabla^i \phi \nabla_i \triangle_{p_t(\omega)} \phi = \triangle_{p_t(\omega)} |\nabla \phi|^2_g - 2 |\text{Hess} \phi|^2_g
\]

\[
- 2 \left( R_{ik} + \text{Hess}_{ik} \left( f - \ln p_t(\omega)(y, z) \right) \right) \nabla^i \phi \nabla^k \phi ,
\]

respectively. Let us observe that from (5.15) it easily follows that (5.20) can be rewritten as

\[
\left( \frac{\partial}{\partial t} - \triangle_\omega(y) \right) \left[ p_t^{(\omega)}(y, z) \triangle_{p_t(\omega)}^{(y)} \hat{\psi}_{(t, z, U_\perp)}(y) \right] = 0 ,
\]

or equivalently as

\[
\left( \frac{\partial}{\partial t} - \triangle_{p_t(\omega)}^{(y)} - \nabla(y) \ln p_t^{(\omega)} \cdot \nabla(y) \right) \triangle_{p_t(\omega)}^{(y)} \hat{\psi}_{(t, z, U_\perp)} = 0 .
\]

Also note that by differentiating both members of (5.13) with respect to \( t \) we get, for any \( t > 0 \), (cf. [27] Prop. 4.1 for a similar computation for the standard heat kernel),

\[
div_\omega^{(y)} \left[ \left( \frac{\partial}{\partial t} p_t^{(\omega)}(y, z) \right) \nabla(y) \hat{\psi}_{(t, z, U_\perp)}(y) + p_t^{(\omega)}(y, z) \nabla(y) \frac{\partial}{\partial t} \hat{\psi}_{(t, z, U_\perp)}(y) \right]
\]

\[
= - U_\perp(z) \cdot \nabla(z) \frac{\partial}{\partial t} p_t^{(\omega)}(y, z) ,
\]

Since \( \nabla(z) \frac{\partial}{\partial t} p_t^{(\omega)}(y, z) = \nabla(z) \triangle_\omega^{(y)} p_t^{(\omega)}(y, z) \), and the weighted laplacian

\[
\triangle_{p_t(\omega)}^{(y)} \hat{\psi}_{(t, z, U_\perp)} = 0 ,
\]

and

\[
\triangle_{p_t(\omega)}^{(y)} \hat{\psi}_{(t, z, U_\perp)} = 0 .
\]
\( \Delta^{(y)}_{\omega} \) acts with respect to the variable point \( y \), we can write, after an obvious rearrangement of terms,

\[
div_{\omega}^{(y)} \left[ p_{t}^{(\omega)}(y, z) \nabla^{(y)} \frac{\partial}{\partial t} \hat{\psi}_{(t, z, U_{\perp})}(y) \right] = - \Delta_{\omega}^{(y)} \left[ U_{\perp}(z) \cdot \nabla^{(z)} p_{t}^{(\omega)}(y, z) \right] - div_{\omega}^{(y)} \left[ \nabla^{(y)} \hat{\psi}_{(t, z, U_{\perp})}(y) \Delta_{\omega}^{(y)} p_{t}^{(\omega)}(y, z) \right].
\] (7.51)

By inserting the defining pde (5.13) in \( \Delta_{\omega}^{(y)} \left[ U_{\perp}(z) \cdot \nabla^{(z)} p_{t}^{(\omega)}(y, z) \right] \), we eventually get

\[
div_{\omega} \left[ p_{t}^{(\omega)}(y, z) \nabla \frac{\partial}{\partial t} \hat{\psi}_{(t, z, U_{\perp})}(y) \right] = \Delta_{\omega} \left[ div_{\omega} \left( p_{t}^{(\omega)}(y, z) \nabla \hat{\psi}_{(t, z, U_{\perp})}(y) \right) \right] - div_{\omega} \left[ \nabla \hat{\psi}_{(t, z, U_{\perp})}(y) \Delta_{\omega} p_{t}^{(\omega)}(y, z) \right],
\] (7.52)

where we dropped the superscript \( (y) \) since the operators \( \nabla, div_{\omega}, \) and \( \Delta_{\omega} \) all act with respect to \( y \).

With these preliminary remarks along the way we get

**Lemma 7.8.** (cf. Prop. 4.1 in [27]). For \( t \in (0, \infty) \) we have

\[
\frac{d}{dt} g_{t}^{(\omega)}(U_{\perp}, U_{\perp}) = -2 \int_{M} \left[ Hess \hat{\psi}_{(t, U)} \right]_{g}^{2} + (R_{ik} + Hess_{ik} f) \nabla^{i} \hat{\psi}_{(t, U)} \nabla^{k} \hat{\psi}_{(t, U)} \right] p_{t}^{(\omega)}(y, z) d\omega(y).
\] (7.53)

**Proof.** According to (5.4) we have

\[
\frac{d}{dt} g_{t}^{(\omega)}(U_{\perp}, U_{\perp}) = 2 \int_{M} \left( \nabla \hat{\psi}_{(t, U)} \cdot \nabla \frac{\partial}{\partial t} \hat{\psi}_{(t, U)} \right) p_{t}^{(\omega)}(y, z) d\omega(y) + \int_{M} \left| \nabla \hat{\psi}_{(t, U)} \right|_{g}^{2} \Delta_{\omega} p_{t}^{(\omega)}(y, z) d\omega(y).
\] (7.54)
The Green formula for the weighted Laplacian $\triangle_\omega$, (which we shall use repeatedly also for the heat weighted Laplacian $\triangle_p t(\omega)$),

$$\int_M \varphi \triangle_\omega \psi \, d\omega = \int_M \varphi \text{div}_\omega \cdot \nabla \psi \, d\omega$$  \( (7.55) \)

which holds pointwise for any $\varphi, \psi \in C^\infty(M, \mathbb{R})$, allows to compute the first term in (7.54) according to

$$\int_M \left( \nabla \hat{\psi}(t,U) \cdot \nabla \frac{\partial}{\partial t} \hat{\psi}(t,U) \right) p_t^{(\omega)}(y,z) \, d\omega(y)$$  \( (7.56) \)

$$= - \int_M \hat{\psi}(t,U) \text{div}_\omega \left[ \frac{\partial}{\partial t} \hat{\psi}(t,U) \right] d\omega(y)$$

$$= - \int_M \hat{\psi}(t,U) \Delta_\omega \left[ \text{div}_\omega \left( p_t^{(\omega)}(y,z) \nabla \hat{\psi}(t,U) \right) \right] d\omega(y)$$

$$+ \int_M \hat{\psi}(t,U) \text{div}_\omega \left[ \Delta_\omega p_t^{(\omega)}(y,z) \nabla \hat{\psi}(t,U) \right] d\omega(y)$$

$$= \int_M \left[ \nabla \hat{\psi}(t,U) \cdot \nabla \Delta_\omega \hat{\psi}(t,U) - \Delta_\omega \left| \nabla \hat{\psi}(t,U) \right|^2_g \right] p_t^{(\omega)}(y,z) \, d\omega(y)$$

where (7.52) refer to the relation used in the computation, and where in the last line we have integrated by parts. Inserted into (7.54), the expression (7.56) eventually yields

$$\frac{d}{dt} g_t^{(\omega)}(U_\perp, U_\perp)$$  \( (7.57) \)

$$= 2 \int_M \left[ \nabla \hat{\psi}(t,U) \cdot \nabla \Delta_\omega \hat{\psi}(t,U) - \frac{1}{2} \Delta_\omega \left| \nabla \hat{\psi}(t,U) \right|^2_g \right] p_t^{(\omega)}(y,z) \, d\omega(y)$$

$$= - 2 \int_M \left[ \left| Hess \hat{\psi}(t,U) \right|^2_g \right] \left( R_{ik} + Hess_{ik} f \right) \nabla^i \hat{\psi}(t,U) \nabla^k \hat{\psi}(t,U) \right] p_t^{(\omega)}(y,z) \, d\omega(y)$$

where we have exploited the weighted Bochner–Weitzenböck formula (7.46). \qed

From Lemma 7.8 and the expression (7.34) of the the Ricci tensor of $(M, g_t^{(\omega)})$ we directly get

**Theorem 7.9.** (The heat kernel induced Ricci flow). Along the weighted heat kernel embedding $(0, \infty) \times M \ni (t,z) \mapsto p_t^{(\omega)}(\cdot, z) d\omega(\cdot) \in \text{Prob}_{uc}(M, g)$, the beta function $\beta(g_t^{(\omega)})$ associated to the scale dependent metric $(M, g_t^{(\omega)})$ is
provided by

\[
\frac{d}{dt} g_t^{(\omega)}(U_\perp, W_\perp) = -2 \tilde{Ric}^{(t)}(U_\perp, W_\perp) \tag{7.58}
\]

\[
- 2 \int_M \left( \nabla \psi_{(t,U)} \cdot \nabla \nabla f \cdot \nabla \psi_{(t,W)} \right) p_t^{(\omega)}(y, z) d\omega(y)
\]

\[
- 2 \int_M \left( \text{Hess} \, \tilde{\psi}_{(t,U)} \cdot \text{Hess} \, \tilde{\psi}_{(t,W)} \right) p_t^{(\omega)}(y, z) d\omega(y),
\]

where \( \tilde{Ric}^{(t)} \) denotes the Ricci curvature of the evolving metric \((M, g_t^{(\omega)})\).

**Proof.** According to Lemma 7.8, we have (cf. (7.53)),

\[
\frac{d}{dt} g_t^{(\omega)}(U_\perp, W_\perp)(z) = -2 \int_M \left[ \text{Hess} \, \tilde{\psi}_{(t,U)} \cdot \text{Hess} \, \tilde{\psi}_{(t,W)} \right. \\
+ R_{ab} \nabla^a \tilde{\psi}_{(t,U)} \nabla^b \tilde{\psi}_{(t,W)} \left. \right] p_t^{(\omega)}(y, z) d\omega(y) \tag{7.59}
\]

\[
-2 \int_M \nabla^a \tilde{\psi}_{(t,U)} \nabla^b \tilde{\psi}_{(t,W)} p_t^{(\omega)}(y, z) d\omega(y).
\]

For \( t > 0 \) we can apply Proposition 7.5 which computes the Ricci curvature of \((M, g_t^{(\omega)})\) at \( z \) in the direction of the 2–plane spanned by the vectors \((U_\perp, W_\perp) \in T_z M\). \(\square\)

The above result indicates a striking connection between the beta function for the scale dependent flow \([t, (M, g)] \mapsto (M, g_t^{(\omega)}) \), \( t > 0 \), and (a generalized version of) the DeTurck–Hamilton version of the Ricci flow. This is further supported by the behavior of (7.58) in the singular limits \( t \searrow 0 \) and \( q \nearrow \infty \). The former controls how the curve of heat kernel embeddings \((0, \infty) \times M \ni (t, z) \mapsto p_t^{(\omega)}(\cdot, z) d\omega(\cdot) \in \text{Prob}(M, g)\) approaches the isometric embedding of \((M, g)\) in the non–smooth \( \text{Prob}(M) \supset \text{Prob}_{\text{ac}}(M, g)\). The latter is related to our choice \( d\omega(q) \) of the dilatonic measure \( d\omega(q) \) localizing the \( \text{NL} \Sigma M \) maps \( \phi : \Sigma \longrightarrow (M, g, d\omega) \) around the center of mass of the constant maps \( \{ \phi_k \}_{k=1}^q \) (see Section 4). The two limits clearly interact since the heat kernel \( p_t^{(\omega)} \) explicitly depends on the choice of the measure \( d\omega(q) \). Since for any given \( d\omega(q) \) we have a well–defined flow (7.58), it is geometrically natural to consider the large \( q \) limit as defined by

\[
\lim_{q \nearrow \infty} \left\{ \lim_{t \searrow 0} \frac{d}{dt} g_t^{(\omega(q))} \right\}_{q \in \mathbb{N}}. \tag{7.60}
\]

With these remarks along the way we have

**Proposition 7.10.** Let \([0, 1] \ni s \mapsto \gamma_s, \gamma(0) \equiv z\) denote a geodesic in \((M, g)\). Then the beta function (7.58) associated with the weighted heat kernel embedding \((M, g, d\omega) \longrightarrow (\text{Prob}_{\text{ac}}(M, g), d_g^W)\) is tangent for \( t = 0 \) and \( q \nearrow \infty \)
to the perturbative beta functions for the dilatonic non–linear \( \sigma \) model

\[
\left. \frac{d}{dt} g_t^{(\omega)}(\hat{\gamma}_s, \hat{\gamma}_s) \right|_{t=0} \bigg|_{q \to \infty} = -2 \left[ \text{Ric}_g(\hat{\gamma}_s, \hat{\gamma}_s) + \text{Hess}_f(\hat{\gamma}_s, \hat{\gamma}_s) \right],
\]

(7.61)

\[
\left. \frac{d}{dt} f_t^{(\omega)} \right|_{t=0} \bigg|_{q \to \infty} = \bigtriangleup_g f - |\nabla f|_g^2,
\]

(7.62)

where the equality holds for almost every \( s \in [0,1] \). 

**Proof.** By directly adapting a basic result (Th. 4.6) of [27], one can control the \( t \searrow 0 \) limit in (7.53) and get

\[
\left. \frac{d}{dt} \int_0^1 g_t^{(\omega)}(\hat{\gamma}_s, \hat{\gamma}_s) \, ds \right|_{t=0} = -2 \int_0^1 \left[ \text{Ric}_g(\hat{\gamma}_s, \hat{\gamma}_s) + \text{Hess}_f(\hat{\gamma}_s, \hat{\gamma}_s) \right] \, ds,
\]

(7.63)

and almost everywhere for \( s \in [0,1] \) we have

\[
\left. \frac{d}{dt} g_t^{(\omega)}(\hat{\gamma}_s, \hat{\gamma}_s) \right|_{t=0} = -2 \left[ \text{Ric}_g(\hat{\gamma}_s, \hat{\gamma}_s) + \text{Hess}_f(\hat{\gamma}_s, \hat{\gamma}_s) \right].
\]

(7.64)

If we couple this expression with the flow (5.58) for the function \( f_t^{(\omega)} \), evaluated for \( t = 0 \),

\[
\frac{\partial}{\partial t} f_t^{(\omega)} \bigg|_{t=0} = \bigtriangleup_g f - \frac{2+q}{q} |\nabla f|_g^2,
\]

(7.65)

we immediately get the stated result in the \( q \nearrow \infty \) limit. It is worthwhile stressing that the actual proof of (7.64) in [27], on which the above result heavily relies, is rather technical. By contrast, the underlying rationale is simple. To wit, since the (Kantorovich) potential \( \hat{\psi}(t, \hat{\gamma}_s) \) in (7.51), (for \( U_\perp \equiv \hat{\gamma}_s \)), is generated by the tangent to the geodesic \( s \mapsto \gamma_s \), one expects that \( \left| \text{Hess}_\hat{\psi}(t, \hat{\gamma}_s) \right|_{t=0} \) vanishes at \( \delta_z \) and that it should remain small for \( 0 < t < \varepsilon \), with \( \varepsilon \) small enough. As reasonable as it appears, the vanishing of \( \text{Hess}_\hat{\psi}(t, \gamma_s) \) for \( t \searrow 0 \) is deceptively difficult to prove, and commands a technical tour de force, (cf. Lemma 4.4 and Prop. 4.5 in [27]).

It is also important to observe that we can connect the singular limit \( t \searrow 0, q \nearrow \infty \) of the beta function (7.58) to the Hamilton–Perelman Ricci flow. To this end we couple the evolutions of the rescaled dilaton \( f_t^{(\omega)} \) and of the metric \( g_t^{(\omega)} \) via the probability measure \( d\omega \). In particular if we require that \( \left. \frac{d}{dt} d\mu_{h_t^{(\omega)}} \right|_{t=0} = 0 \), (recall that \( d\mu_{h_t^{(\omega)}} \bigg|_{t=0} = d\omega \)), we have

**Proposition 7.11.** Under the constraint

\[
\left. \frac{d}{dt} d\mu_{h_t^{(\omega)}} \right|_{t=0} = 0
\]

(7.66)
the beta function (7.58) is tangent, for \( t = 0 \) and \( q \to \infty \), to the generators of the Hamilton–Perelman Ricci flow according to

\[
\frac{d}{dt} g_t^{(\omega)}(\gamma_s, \dot{\gamma}_s) \bigg|_{t=0} \bigg|_{q \to \infty} = -2 \left[ \text{Ric}_g(\dot{\gamma}_s, \dot{\gamma}_s) + \text{Hess}_f(\dot{\gamma}_s, \dot{\gamma}_s) \right],
\]

(7.67)

\[
\frac{d}{dt} f_t^{(\omega)} \bigg|_{t=0} \bigg|_{q \to \infty} = -\Delta_g f - R^{(g)},
\]

(7.68)

where the equality holds for almost every \( s \in [0,1] \).

Proof. To prove this result, it is useful to recast (7.67) in terms of the metric \( h_t \) defined by (4.17). Denote by \( \{y^i\}_{i=1}^n \) local (geodesic) coordinates on \((M, g_t)\), and \( \{\zeta^\alpha\}_{\alpha=1}^q \) local coordinates on \((T^\ast q, \delta)\). Then, according to a standard formula for warped product metrics (cf. e.g. p.46 of [68]), we have

\[
R^{(h)}_{ij} = R^{(g)}_{ij} + \nabla_i \nabla_j f - \frac{1}{q} \nabla_i f \nabla_j f,
\]

(7.69)

\[
R^{(h)}_{\alpha\beta} = \frac{1}{q} e^{-\frac{2f(y)}{q}} \delta_{\alpha\beta} \left( \Delta_g f - |\nabla f|^2_g \right),
\]

(7.70)

\[
R^{(h)}_{i\alpha} = 0, \quad i = 1, \ldots, n, \quad \alpha = 1, \ldots, q,
\]

(7.71)

where \( R^{(g)}_{ij} \) denotes the components of the Ricci tensor of \((M, g)\), (and similarly the subscript \( g \) refers to gradient, hessian, and Laplacian relative to the metric \( g \)). From

\[
\frac{d}{dt} h_t = \frac{d}{dt} g_t^{(\omega)} - \frac{2}{q} e^{-\frac{2f^{(\omega)}}{q}} \delta_{\gamma q} \frac{d}{dt} f_t^{(\omega)},
\]

(7.72)

and (7.65), we get

\[
\frac{d}{dt} h_t \bigg|_{t=0} = -2 \text{Ric}^{(h)} - \frac{2}{q} \nabla f \otimes \nabla f.
\]

(7.73)

If we impose, at \( t = 0 \), the preservation along (7.73) of the probability measure density \( d\mu_h \), (note that \( d\mu_{h_t}|_{t=0} = d\omega \)), then we get

\[
\frac{d}{dt} d\mu_{h_t} \bigg|_{t=0} = \frac{d}{dt} e^{-\frac{2f^{(\omega)}}{q}} d\mu_{g_t} \bigg|_{t=0} = 0.
\]

(7.74)

From (7.73) we compute

\[
\frac{d}{dt} d\mu_{h_t} \bigg|_{t=0} = - \left[ R^{(h)} + \frac{1}{q} |\nabla f|^2_g \right] d\mu_h = 0,
\]

(7.75)

where

\[
R^{(h)} = R^{(g)} + 2\Delta_g f - \frac{q+1}{q} |\nabla f|^2_g,
\]

(7.76)

denotes the scalar curvature of the metric \( h \) expressed in terms of the scalar curvature \( R^{(g)} \) of \((M, g)\). Thus,

\[
\frac{d}{dt} d\mu_{h_t} \bigg|_{t=0} = 0 \Rightarrow |\nabla f|^2_g = R^{(g)} + 2\Delta_g f,
\]

(7.77)
which, introduced in (7.65), provides the backward heat equation
\[
\frac{d}{dt} f_t^{(\omega)} \bigg|_{t=0} = -\Delta g f - R^{(g)} - \frac{2}{q} |\nabla f|^2 g.
\]
(7.78)

It follows that the flow \( t \mapsto h_t \), defined by (7.53) by enforcing the volume density preservation (7.74), has a tangent which, at \( t = 0 \), and for \( q \nearrow \infty \), reduces to the tangent vector defining the Hamilton–Perelman flow. □

The strong similarity between (7.58) and the (DeTurck version of the) Ricci flow, and the tangency conditions described in Propositions 7.10 and 7.11 may suggest that (7.58) is indeed the Ricci flow in disguise. In the case of the standard heat kernel embedding [27], the induced flow on the distance function, tangential to the Ricci flow for \( t = 0 \), is well defined for any \( t \geq 0 \), and with strong control on the topology of \( M \) and good continuity properties with respect to (measured) Gromov–Hausdorff convergence. As argued in [27], these properties strongly contrast with the typical behavior of the Ricci flow, characterized by the development of curvature singularities and by a poor control on Gromov–Hausdorff limits of sequences of Ricci evolved manifolds. The explicit expression (7.58) for the heat kernel induced flow \((t,g) \mapsto g_t^{(\omega)}\), and in particular the presence of the norm–contracting term \( \int_M |Hess \psi(t,U)|^2 p_t^{(\omega)}(y,z) d\omega(y) \), (cf. (7.58) for \( U_\perp = W_\perp \)), indicates that along the flow \((t,g) \mapsto g_t^{(\omega)}\) there is a strong control of the metric geometry of \( g_t^{(\omega)} \). To provide evidence in this direction without belaboring on the subtle aspects of measured Gromov–Hausdorff convergence we describe the monotonicity properties of (7.58). Not surprisingly, they turn to be somehow stronger than those in Ricci flow theory.

### 7.3. Monotonicity and gradient flow properties

From Lemma 7.6 and 7.8 we get

**Proposition 7.12.** Let

\[
(0, \infty) \ni t \mapsto \varphi(t) \in Diff(M)
\]
(7.79)

\[
\frac{\partial}{\partial t} \varphi(t) = X_{(t,z)}, \quad \lim_{t \searrow 0} \varphi(t) = Id_M
\]

be the curve of diffeomorphisms generated by the vector field \( X_{(t,z)} \) defined by (7.36). Then, for any \( U_\perp, V_\perp \in T_z M \),

\[
\left( \varphi^{-1}(t) \right)^* \frac{d}{dt} \left( \varphi^* g_t^{(\omega)} \right) (U_\perp, V_\perp) \quad (7.80)
\]

\[
= -2 \int_M \hat{\psi}_{(t,U)} \Delta_{p_t(d\omega)} \hat{\psi}_{(t,V)} P_t^{(\omega)}(y,z) d\omega(y),
\]
In particular, the pull-back \( (\varphi^* g_t^{(\omega)}) (U_\perp, U_\perp) \) is a monotonically decreasing function of \( t \),

\[
(\varphi^{-1}(t))^* \frac{d}{dt} (\varphi^* g_t^{(\omega)}) (U_\perp, U_\perp) = -2 \int_M \left( \Delta p_t (d\omega) \right) \psi_{(t,U)}^2 p_t^{(\omega)} (y,z) d\omega(y) < 0 ,
\]

and moreover we have

\[
\frac{d}{dt} \left( (\varphi^{-1}(t))^* \frac{d}{dt} (\varphi^* g_t^{(\omega)}) \right) (U_\perp, U_\perp) = 4 \int_M \left| \nabla (\varphi^* (t) \cdot \varphi^* g_t^{(\omega)}) \right|^2 p_t^{(\omega)} (y,z) d\omega(y) .
\]

**Proof.** We can factor out the \( \varphi(t) \) in (7.81) by exploiting the familiar DeTurck argument [12], (cf. [99] for details). Explicitly, for \( 0 < t < \epsilon \), one computes

\[
\frac{d}{dt} \left( \varphi^* g_t^{(\omega)} \right) = \left. \frac{d}{ds} \left( \varphi^* (t + s) g_t^{(\omega)} \right) \right|_{s=0} = \varphi^* (t) \left( \frac{d}{dt} g_t^{(\omega)} \right) + \frac{d}{ds} \left( \varphi^* (t + s) g_t^{(\omega)} \right)_{s=0} = \varphi^* (t) \left( \frac{d}{dt} g_t^{(\omega)} \right) + \varphi^* (t) \left( \mathcal{L}_{X(t,z)} g_t^{(\omega)} \right) .
\]

Hence, we get

\[
(\varphi^{-1}(t))^* \frac{d}{dt} \left( \varphi^* (t) g_t^{(\omega)} \right) = \frac{d}{dt} g_t^{(\omega)} + \mathcal{L}_{X(t,z)} g_t^{(\omega)} ,
\]

which can be identified with the convective derivative of \( g_t^{(\omega)} \) along the curve of \( t \)-dependent diffeomorphisms \((7.79)\) associated with the flow \( (t, \delta_z) \mapsto (t, p_t^{(\omega)} (\cdot, z) d\omega(\cdot)) , t \in (0, \infty) \). According to Lemma [7.6] we have

\[
(\varphi^{-1}(t))^* \frac{d}{dt} \left( \varphi^* (t) g_t^{(\omega)} \right) (U_\perp, V_\perp) = \frac{d}{dt} g_t^{(\omega)} + \mathcal{L}_{X(t,z)} g_t^{(\omega)} ,
\]

\[
= \frac{d}{dt} \int_M g_{ik} (y) \nabla^i (\psi_{(t,U)} \nabla^k (\psi_{(t,V)} p_t^{(\omega)} (y,z) d\omega(y) ,
\]

\[
+ 2 \int_M \text{Hess} s (y) \ln p_t^{(\omega)} (y,z) \nabla^i (\psi_{(t,U)} \nabla^k (\psi_{(t,V)} p_t^{(\omega)} (y,z) d\omega(y) \nabla^i (\psi_{(t,U)} \nabla^k (\psi_{(t,V)} p_t^{(\omega)} (y,z) d\omega(y) \nabla^i (\psi_{(t,U)} \nabla^k (\psi_{(t,V)} p_t^{(\omega)} (y,z) d\omega(y) ,
\]

\[
= -2 \int_M \left[ \text{Hess} \psi_{(t,U)} \cdot \text{Hess} \psi_{(t,V)} + \left( R_{ik} + \text{Hess} s (f - \ln p_t^{(\omega)} \right) \nabla^i \psi_{(t,U)} \nabla^k \psi_{(t,V)} \right] p_t^{(\omega)} (y,z) d\omega(y) ,
\]
where we have taken into account (7.53). On the other hand for any $t > 0$ and the identities
\[
\begin{align*}
\Delta_{p_t(d\omega)} \left( \nabla \hat{\psi}(t,U) \cdot \nabla \hat{\psi}(t,V) \right) &= \nabla \hat{\psi}(t,U) \cdot \Delta_{p_t(d\omega)} \nabla \hat{\psi}(t,V) \\
&+ \nabla \hat{\psi}(t,V) \cdot \Delta_{p_t(d\omega)} \nabla \hat{\psi}(t,U) + 2 Hess \hat{\psi}(t,U) \cdot Hess \hat{\psi}(t,V),
\end{align*}
\]
(7.86)
and
\[
\begin{align*}
\nabla \hat{\psi}(t,U) \cdot \nabla \left( \Delta_{p_t(d\omega)} \hat{\psi}(t,V) \right) &= \Delta_{p_t(d\omega)} \left( \nabla \hat{\psi}(t,U) \cdot \nabla \hat{\psi}(t,V) \right) \\
&- \nabla \hat{\psi}(t,U) \cdot \Delta_{p_t(d\omega)} \nabla \hat{\psi}(t,V) - 2 Hess \hat{\psi}(t,U) \cdot Hess \hat{\psi}(t,V) (7.87)
\end{align*}
\]
(7.88)
\[

\nabla^i \hat{\psi}(t,U) \left( R_{ik} + Hess_{ik} \left( f - \ln p_t^{(\omega)} \right) \right) \nabla^k \hat{\psi}(t,V),
\]
(7.89)

By taking into account (7.85), this proves (7.80). If we set $U = V$ in the above relations then we get the integrated weighted Bochner–Weitzenböck formula, (cf. (7.47)),
\[
\begin{align*}
\int_M \hat{\psi}(t,U) \Delta_{p_t(d\omega)} \hat{\psi}(t,V) p_t^{(\omega)}(y, z) d\omega(y) &= 0
\end{align*}
\]
which together with (7.85), (for $U = V$), proves (7.81). To show that the monotonicity result in (7.81) is strict, let us observe that the vanishing of \( \int_M (\Delta_{p_t(\omega)} \hat{\psi}(t,U))^2 p_t^{(\omega)}(y,z) \, d\omega(y) \) would necessarily imply \( \Delta_{p_t(\omega)} \hat{\psi}(t,U) = 0 \), hence according to proposition 5.8 and (5.15) the corresponding vanishing of \( U_\perp(z) \), contradicting the stated hypotheses. Finally, in order to prove (7.82) let us set for notational convenience

\[
A(z,t) := \int_M \left(\Delta_{p_t(\omega)} \hat{\psi}(t,z,U_\perp)(y)\right)^2 \, p_t^{(\omega)}(y,z) \, d\omega(y).
\]

Since the heat kernel \( p_t^{(\omega)}(y,z) \) and the associated weighted Laplacian \( \Delta_{p_t(\omega)} \) are smooth for \( t > 0 \), we compute

\[
\frac{d}{dt} A(z,t) = 2 \int_M \left(\Delta_{p_t(\omega)} \hat{\psi}(t,U)\right) \frac{\partial}{\partial t} \left(\Delta_{p_t(\omega)} \hat{\psi}(t,U)\right) \, p_t^{(\omega)}(y,z) \, d\omega(y)
+ \int_M \left(\Delta_{p_t(\omega)} \hat{\psi}(t,U)\right)^2 \frac{\partial}{\partial t} p_t^{(\omega)}(y,z) \, d\omega(y)
= \int_M \left(\Delta_{p_t(\omega)} \hat{\psi}(t,U)\right)^2 \, \Delta_{\omega} p_t^{(\omega)}(y,z) \, d\omega(y)
+ \int_M \nabla p_t^{(\omega)}(y,z) \cdot \nabla \left(\Delta_{p_t(\omega)} \hat{\psi}(t,U)\right)^2 \, d\omega(y)
+ \int_M \left(\Delta_{p_t(\omega)} \hat{\psi}(t,U)\right)^2 \, \Delta_{\omega} p_t^{(\omega)}(y,z) \, d\omega(y)
= -2 \int_M \left| \nabla^{(y)} \left(\Delta_{p_t(\omega)} \hat{\psi}(t,z,U_\perp)(y)\right) \right|^2 \, p_t^{(\omega)}(y,z) \, d\omega(y),
\]

where, in the last passage, we have integrated by parts both with respect to the \( p_t^{(\omega)}(y,z) \, d\omega(y) \) and the \( d\omega(y) \) measures. \( \Box \)

Note that (7.81) can be rewritten as

\[
\frac{d}{dt} g_t^{(\omega)}(U_\perp,U_\perp)(z) = -\mathcal{L}_{X(t,z)} g_t^{(\omega)}(U_\perp,U_\perp) - 2 \int_M \left(\Delta_{p_t(\omega)} \hat{\psi}(t,U)\right)^2 \, p_t^{(\omega)}(y,z) \, d\omega(y),
\]

(7.93)
which shows that the evolution of $g_t^{(\omega)}(U_\perp, U_\perp)(z)$ results from a balance between the $-2 \int_M (\Delta p_t (d\omega) \hat{\psi}_{(t,U)})^2 p_t^{(\omega)} d\omega$ term, which tends to contract the $g_t^{(\omega)}$–norm of vectors, and the term

$$\mathcal{E}_{\psi_{(t,z)}} g_t^{(\omega)}(U_\perp, U_\perp)$$

(7.94)

$$= 2 \int_M \text{Hess } \ln p_t^{(\omega)}(y, z) \left( \nabla \hat{\psi}_{(t,U)}, \nabla \hat{\psi}_{(t,U)} \right) p_t^{(\omega)}(y, z) d\omega(y),$$

which, as already stressed, computes $\text{Hess } d_g^2(y, z)$ in the sense of distributions and, by the local convexity of $d_g^2(y, z)$, tends to contrast this contraction.

It is not difficult to show that the pulled back flow (7.81) is a gradient flow. Let $\{E^{(\omega)}(z)\}_{a=1}^2$ denote a basis in $T_z M$, orthonormal with respect to the given $(M, g)$, and for any $t > 0$ consider the functional

$$\mathcal{F} \left( (M, g_t^{(\omega)}) \right)$$

(7.95)

$$:= \int_{\varphi(z,t)(M)} d\omega(z) g^{ab}(z) \int_M \hat{\psi}_{(t,a)} \Delta p_t (d\omega) \hat{\psi}_{(t,b)} p_t^{(\omega)}(y, z) d\omega(y),$$

where $\varphi(z,t)$ denotes the curve of diffeomorphisms defined by (7.79).

**Remark 7.13.** In $\mathcal{F} \left( (M, g_t^{(\omega)}) \right)$ one may consider more natural tracing with respect to $g_t^{(\omega)}(z)$ rather than with respect to $g(z)$. In such a case, the corresponding functional can be identified (modulo the action of the curve of diffeomorphisms $t \mapsto \varphi(t)$) with the time–derivative of the Riemannian volume of $(M, g_t^{(\omega)})$. As a consequence, this apparently more general functional, has a non–trivial $L^2(d\omega)$ gradient only along the scalar variation of $g_t^{(\omega)}$, and does not capture all possible tensorial variations (and deformation) of $g_t^{(\omega)}$, variations which are fully described by the $L^2(d\omega)$–gradient of $\mathcal{F} \left( (M, g_t^{(\omega)}) \right)$.

Indeed, we have

**Lemma 7.14.** Let $z \mapsto \frac{1}{2} \chi_{ab}(z) \in \otimes^2_{\text{sym}} T^*_z M$ be a smooth symmetric bilinear form on $(M, g_t^{(\omega)})$ thought of as acting fiberwise as a tangent bundle endomorphism. For $0 < \epsilon < 1$ sufficiently small, let us consider the variation of basis vectors $\{E^{(\omega)}(z)\} \in T_z M$ defined by $E^{(\omega)} \mapsto E^{(\epsilon)} := (\delta^a_b + \frac{\epsilon}{2} \chi^a_b(z)) E^{(a)}$. If we let $\hat{\psi}^{(\epsilon)}_{(t,b)}$ denote the induced variation in the potentials $\hat{\psi}^{(\epsilon)}_{(t,b)} := \hat{\psi}_{(t,E^{(\epsilon)})}$, then for any $t > 0$

$$\hat{\psi}^{(\epsilon)}_{(t,b)}(y) = (\delta^a_b + \frac{\epsilon}{2} \chi^a_b(z)) \hat{\psi}_{(t,a)}(y),$$

(7.96)

and the corresponding linearization of the functional $\mathcal{F} \left( (M, g_t^{(\omega)}) \right)$ in the direction of the variation defined by the bilinear form $\chi$ is provided by
\[ \mathcal{D} \mathcal{F} \left( (M, g_t^{(\omega)}) \right) \circ \chi \]

(7.97)

\[ = \int_{\hat{\varphi}_{(s,t)}(M)} \left( \int_M \hat{\psi}_{(t,a)} \Delta_{p_t(d\omega)} \hat{\psi}_{(t,b)} p_t^{(\omega)}(y, z) \, d\omega(y) \right) \chi^{ab}(z) \, d\omega(z). \]

Proof. By the linearity of the defining elliptic PDE (5.13) it immediately follows that the potential \( \hat{\psi}^{(e)}_{(t,b)} \) associated to the rescaled basis vector \( E^{(e)}_{(b)} := (\delta^a_b + \frac{\epsilon}{2} \chi^a_b(z))^E_{(a)} \) is provided by (7.96). Similarly, from the very definition of the metric \( g_t^{(\omega)}(z) \), (cf. (5.25)), we directly have

\[ g_t^{(\omega,e)}(E_{(c)}, E_{(d)}) := g_t^{(\omega)}(E_{(c)}, E_{(d)}) \]

(7.98)

\[ := \int_M g_{ik}(y) \nabla^i(y) \hat{\psi}^{(e)}_{(t,c)} \nabla^k(y) \hat{\psi}^{(e)}_{(t,d)} p_t^{(\omega)}(y, z) \, d\omega(y) \]

\[ = (\delta^a_c + \frac{\epsilon}{2} \chi^a_c(z)) (\delta^b_d + \frac{\epsilon}{2} \chi^b_d(z)) \times \int_M g_{ik}(y) \nabla^i(y) \hat{\psi}_{(t,a)} \nabla^k(y) \hat{\psi}_{(t,b)} p_t^{(\omega)}(y, z) \, d\omega(y). \]

Hence, to leading order in \( \epsilon \), we get

\[ g_t^{(\omega,e)}(E_{(c)}, E_{(d)}) = g_t^{(\omega)}(E_{(c)}, E_{(d)}) + \epsilon \chi_{cd}(z) + O(\epsilon^2), \]

as expected. In particular, by letting \( \chi \) vary in the space of all symmetric bilinear form \( C^\infty(M, \otimes_2^{\text{sym}} T^* M) \) we can interpret \( \chi \) as describing the generic metric variation of \( g_t^{(\omega)} \). From

\[ g^{cd}(z) \int_M \hat{\psi}^{(e)}_{(t,c)} \Delta_{p_t(d\omega)} \hat{\psi}^{(e)}_{(t,d)} p_t^{(\omega)}(y, z) \, d\omega(y) \]

(7.99)

\[ = g^{cd}(z) (\delta^a_c + \frac{\epsilon}{2} \chi^a_c(z)) (\delta^b_d + \frac{\epsilon}{2} \chi^b_d(z)) \]

\[ \times \int_M \hat{\psi}_{(t,a)} \Delta_{p_t(d\omega)} \hat{\psi}_{(t,b)} p_t^{(\omega)}(y, z) \, d\omega(y) \]

\[ = \left( g_{ab}(z) + \epsilon \chi_{ab}(z) + \frac{\epsilon^2}{4} \chi_{ad}(z) \chi^d_b(z) \right) \]

\[ \times \int_M g^{ac}(y) g^{bd}(y) \hat{\psi}_{(t,c)} \Delta_{p_t(d\omega)} \hat{\psi}_{(t,d)} p_t^{(\omega)}(y, z) \, d\omega(y), \]
we easily compute

\[
D \mathcal{F} \left( (M, g^{(\omega)}_t) \right) \circ \chi := \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \mathcal{F} \left( (M, g^{(\omega,\epsilon)}_t) \right)
\]  

(7.100)

\[
= \int_{\varphi(z,t)(M)} \left( \int_M g^{ac}_{\omega} g^{bd}_{\omega} \psi_{(t,c)} \Delta_{p_t(d\omega)}^{2} \psi_{(t,d)} \right.
\]

\[
\times p_t^{(\omega)}(y,z) d\omega(y) \right) \chi_{ab}(z) d\omega(z).
\]

According to (7.99) the bilinear form \( \chi \in C^\infty(M, \otimes^2_{\text{sym}} T^*M) \) induces the generic metric variation of \( g^{(\omega)}_t \), hence we have

**Theorem 7.15.** The heat induced Ricci flow (7.80) is the gradient flow of the functional \( \mathcal{F} \left( (M, g^{(\omega)}_t) \right) \) with respect to the \( L^2(d\omega) \) inner product on the space of metrics \( \text{Met}(M) \).

Along the same lines is easy to prove that the flow (7.58) is the gradient flow of the functional obtained from \( \mathcal{F} \left( (M, g^{(\omega)}_t) \right) \) by undoing the action of the diffeomorphism \( \varphi_{(t,z)} \), i.e.

\[
\hat{\mathcal{F}} \left( (M, g^{(\omega)}_t) \right)
\]

(7.101)

\[
:= \int_M d\omega(z) g^{ab}_{(\omega)}(z) \int_M \psi_{(t,a)} \Delta_{p_t(d\omega)}^{2} \psi_{(t,b)} p_t^{(\omega)}(y,z) d\omega(y)
\]

\[
+ \frac{1}{2} \int_M g^{ab}_{(\omega)}(z) \mathcal{L}_{X_{(t,z)}} g^{(\omega)}_t(E_{(a)}, E_{(b)}) d\omega(z),
\]

(cf. (7.39) for the definition of the Lie derivative along \( X_{(t,z)} \)). In particular we have

**Lemma 7.16.**

\[
\frac{d}{dt} \hat{\mathcal{F}} \left( (M, g^{(\omega)}_t) \right) = D \hat{\mathcal{F}} \left( (M, g^{(\omega)}_t) \right) \circ \frac{dg_t^{(\omega)}}{dt} \leq 0.
\]  

(7.102)

**Proof.** By the chain rule we have, for \( 0 < t < \epsilon \),

\[
\frac{d}{dt} \hat{\mathcal{F}} \left( (M, g^{(\omega)}_t) \right) = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} \hat{\mathcal{F}} \left( (M, g^{(\omega)}_{t+\epsilon}) \right)
\]

(7.103)

\[
= D \hat{\mathcal{F}} \left( (M, g^{(\omega)}_t) \right) \circ \frac{dg_t^{(\omega)}}{dt}.
\]
The Wasserstein geometry of non-linear $\sigma$ models

From (7.90) and (7.39) we get

$$\hat{F}((M, g_t^{(\omega)})) = \int_M d\omega(z) g^{ab}(z) \left( \int_M \left[ \text{Hess} \hat{\psi}_{(t,a)} \cdot \text{Hess} \hat{\psi}_{(t,ab)} + \nabla^i \hat{\psi}_{(t,a)} \left( R_{ik} + \text{Hess}_{ik} f \right) \nabla^k \hat{\psi}_{(t,b)} \right] p_t^{(\omega)}(y, z) d\omega(y) \right),$$

whose linearization in the direction $\chi := \frac{d}{dt} g_t^{(\omega)}$ can be easily computed, (as in (7.99)), to be

$$D \hat{F}((M, g_t^{(\omega)})) \circ \frac{dg_t^{(\omega)}}{dt} = \int_M d\omega(z) \frac{dg_t^{(\omega)}}{dt} (g_t^{(\omega)})_{ab}(z) \left( \int_M g^{ac} g^{bd} \left[ \text{Hess} \hat{\psi}_{(t,c)} \cdot \text{Hess} \hat{\psi}_{(t,d)} + \nabla^i \hat{\psi}_{(t,c)} \left( R_{ik} + \text{Hess}_{ik} f \right) \nabla^k \hat{\psi}_{(t,d)} \right] p_t^{(\omega)}(y, z) d\omega(y) \right).$$

The lemma follows by inserting the expression (7.58) for the heat kernel induced Ricci flow.

The geometrical meaning of $\hat{F}((M, g_t^{(\omega)}))$ is quite natural since we get

\section*{Proposition 7.17.} For $t > 0$ the functional $\hat{F}((M, g_t^{(\omega)}))$ is a deformation of the Perelman $F$–energy functional and

$$\lim_{t \searrow 0} \hat{F}((M, g_t^{(\omega)})) = \int_M \left( R(g) + |\nabla f|^2_g \right) e^{-f(z)} d\mu_g(z),$$

in the weak sense.

\textit{Proof.} Let us assume that the measure $d\omega$ is localized in a ball $B(r, z) := \{ q \in M \mid d_g(q, z) \leq \frac{1}{3} \text{inj} (M, g) \}$, (otherwise introduce in the integrals defining $\hat{F}$ a smooth radial cutoff function $\zeta(d_g(q, z)) := 1$ for $d_g(q, z) \leq \frac{1}{3} \text{inj} (M, g)$, and $\zeta(d_g(q, z)) := 0$ for $d_g(q, z) \geq \frac{1}{2} \text{inj} (M, g)$). We let $\{ E_{(a)}(z) \}_{a=1}^n$ denote a basis in $T_z M$, orthonormal with respect to $(M, g)$, and let $[0, 1] \ni s \mapsto \gamma_{(a)}$, $\gamma_{(a)}(0) \equiv z$, $\gamma_{(a)}(0) \equiv E_{(a)}(z)$ denote the corresponding geodesics in $(M, g)$. Again, from (7.90) and (7.39) we compute
\[
\sum_{a=1}^{n} \int_{M} \left( \Delta_{p_t} (d\omega) \hat{\psi}_{(t,z,E_{(a)})} \right)^2 p_t^{(\omega)}(y,z) \, d\omega(y) \]  
\tag{7.107}
\]

+ \frac{1}{2} \sum_{a=1}^{n} \mathcal{L}_{X_{(t,z)}} g^{(\omega)}(E_{(a)}, E_{(a)})

\]

= \sum_{a=1}^{n} \left( \int_{M} \left| Hess \hat{\psi}_{(t,a)} \right|_{g}^2 

\right.

+ (R_{ik} + Hess_{ik} f) \nabla^{i} \hat{\psi}_{(t,a)} \nabla^{k} \hat{\psi}_{(t,a)} \right) p_{t}^{(\omega)}(y,z) \, d\omega(y) \bigg) .

According to Proposition 7.10, we have \( |Hess \hat{\psi}_{(t,a)}|_{g}^2 \to 0 \) almost everywhere as \( t \searrow 0 \). For \( 0 < t \) small enough \( p_{t}^{(\omega)} \) is localized around \( z \) and the trace over the \( \{ \nabla \hat{\psi}_{(t,a)} \} \) reduces to the trace over the orthonormal vectors \( \{ \gamma'_{(a)}(0) \equiv E_{(a)}(z) \} \). Abusing notation, (i.e., tracing over \( \{ \gamma'_{(a)}(0) \} \) before taking the \( \searrow 0 \) limit), we can write

\[
\sum_{a=1}^{n} \left( \int_{M} \left( \Delta_{p_t} (d\omega) \hat{\psi}_{(t,z,E_{(a)})} \right)^2 p_t^{(\omega)}(y,z) \, d\omega(y) \right)
\tag{7.108}
\]

+ \frac{1}{2} \sum_{a=1}^{n} \mathcal{L}_{X_{(t,z)}} g^{(\omega)}(E_{(a)}, E_{(a)})

\]

= \int_{M} \left( R_{g}(y) + \Delta_{g} f \right) p_{t}^{(\omega)}(y,z) \, d\omega(y) + O(t^{3/2})

= \int_{M} \left( R_{g}(y) + \Delta_{\omega} f + |\nabla_{g} f|_{g}^2 \right) p_{t}^{(\omega)}(y,z) \, d\omega(y) + O(t^{3/2}) ,

where \( R_{g} \) and \( \Delta_{g} \) respectively denote the scalar curvature and the Laplace–Beltrami laplacian on \((M,g)\), where we exploited the relation \( \Delta_{g} f = \Delta_{\omega} f + |\nabla_{g} f|_{g}^2 \). Hence, we can write

\[
\hat{F} \left( (M, g^{(\omega)}) \right)
\tag{7.109}
\]

= \int_{M} \left( \int_{M} \left( R_{g}(y) + \Delta_{\omega} f + |\nabla_{g} f|_{g}^2 \right) p_{t}^{(\omega)}(y,z) \, d\omega(y) \right) \, d\omega(z) + O(t^{3/2}) .
By the symmetry of the heat kernel and letting $t \to 0$, we compute
\[
\lim_{t \to 0} \hat{F}((M, g_t^{(\omega)})) = \int_M \left( R_g(z) + \Delta \omega f(z) + |\nabla f|^2_{g}(z) \right) d\omega(z) = \int_M \left( R_g(z) + |\nabla f|^2_{g}(z) \right) e^{-f(z)}d\mu_g(z),
\]
which is the expression for the Perelman $\mathcal{F}$–energy, (at $t = 0$), associated with the function $f$ on $(M, g)$. It follows that $\hat{F}((M, g_t^{(\omega)}))$ can be seen as a deformation of Perelman’s $\mathcal{F}$–energy.

This is a suitable point at which we should come back to our sponsor, the interplay between Ricci flow and NL$\sigma$M. We do so by observing that it would be appealing to associate $\hat{F}((M, g_t^{(\omega)}))$, or a variant thereof, to Zamolodchikov’s $c$–theorem [101]. Recall that this result, (established for two dimensional quantum field theories, but rather conjectural for higher dimensional theories), concerns the existence of a functional of the coupling constants of the theory, (i.e., $(M, g, d\omega)$ in the dilatonic NL$\sigma$M case), which is non–increasing along the RG flow and stationary at the fixed points of the flow, where it takes the value of the central charge $c$ of the conformal field theory described by the fixed point of the RG action. The relation between Ricci flow and the renormalization group for the non–linear $\sigma$ model has suggested [21], [22], [23] that Perelman’s $\mathcal{F}$–energy may be a natural candidate for such a functional. As natural as it appears, this identification is rather delicate since it involves a strong extrapolation of the underlying perturbative regime governing the relation between Ricci and RG flow. If we factor out the unphysical Perelman type constraint, fixing the dilatonic measure $d\omega$ along the Ricci–Perelman flow, the functional $\mathcal{F}$ appears in NL$\sigma$M theory as a spacetime action generating the (conformally invariant) fixed points of the 1–loop RG flow. It is clear that the perturbative nature of this characterization makes difficult if not impossible to prove that $\mathcal{F}$ plays indeed the role of a full $c$–functional, (see however [23]). In this connection, the RG avatar defined by the heat kernel embedding has an obvious advantage since, as we have shown in Propositions 5.18 and 6.7, we can associate to it the non–perturbative effective action $\mathcal{E}[\hat{\Psi}_{t, \phi_{cm}}]$ defined by the deformed harmonic map functional. This suggests the following characterization. Let $\{\phi(j)\}_{j=1}^g \to M$ denote the collection of maps fluctuating according to the Gaussian measure $Q_t[d\phi(j)]$ around the classical background provided by their center of mass $\phi_{cm}$, (cf. Section 6). We can associate with this background the natural modification of (7.101) obtained by localizing (the heat source $z$ dependence in) $\hat{F}_{\phi_{cm}}(\Sigma)((M, g_t^{(\omega)}))$ to $\phi_{cm}(\Sigma) \subseteq M$, i.e.

\footnote{There have been recent indications [100] to a proof in dimension 4.}
\[ \mathcal{F}_{\phi_{cm}(\Sigma)} \left( (M, g_t(\omega)) \right) \]
\[ := \int_{\phi_{cm}(\Sigma)} \omega(z) g^{ab}(z) \int_{M} \psi(t,a) \Delta^{2}_{p_{t}(d\omega)} \psi(t,b) p^{(\omega)}_{t}(y, z) \omega(y) \]
\[ + \frac{1}{2} \int_{\phi_{cm}(\Sigma)} g^{ab}(z) \mathcal{L}_{X_{(t,z)}} g^{(\omega)}_{t}(E(a), E(b)) \omega(z) . \]

Since \( \mathcal{E}[\hat{\Psi}_{t,\phi_{cm}}] \) is the large deviation functional associated with the distribution \( \Pi_{j=1}^{q} Q_{t}[d\phi(j)] \), the functional \( \mathcal{F}_{\phi_{cm}(\Sigma)} \) may be heuristically interpreted as describing the average deformation in \( \mathcal{E}[\hat{\Psi}_{t,\phi_{cm}}] \) induced, along the flow \( (M, g) \mapsto (M, g_{t}(\omega)) \), by the fluctuating \( \{\phi(j)\}_{j=1}^{q} \). According to Lemma 7.16 and Proposition 7.17, the functional \( \mathcal{F}_{\phi_{cm}(\Sigma)} \) is monotonically decreasing along \( (t, g) \mapsto g_{t}(\omega), t \in [0, \infty) \), and reduces to Perelman’s \( \mathcal{F} \)-functional, (localized to \( \phi_{cm}(\Sigma) \subseteq M \)), as \( t \to 0 \). Its role as a candidate \( c \)-functional for the heat kernel induced RG action on NL\( \sigma \)M rests on the characterization of the nature of the fixed points for the generalized Ricci flow evolution \( (t, g) \mapsto g_{t}(\omega), t \in [0, \infty) \). This is clearly an open and very difficult problem, and perhaps an appropriate point at which to end this long analysis.

In Ricci flow theory, the true added value of Wasserstein geometry and of its connection to optimal transport theory lies in the observation that the properties of \( d_{W}^{g} \) are deeply related to the Ricci curvature of the underlying metric measure space \( (M, g, d\omega) \), a fact that has been independently noticed and exploited by various authors [102, 24, 25, 103]. The deep connections among NL\( \sigma \)M theory, Ricci flow, and Wasserstein geometry explored here add a further perspective to this rich interplay.

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Mauro Carfora
Dipartimento di Fisica, Università' degli Studi di Pavia
and
Istituto Nazionale di Fisica Nucleare, Sezione di Pavia
via A. Bassi 6, I-27100 Pavia, Italy
e-mail: mauro.carfora@pv.infn.it