THE ASYMPTOTICS OF THE $L^2$-CURVATURE AND THE SECOND VARIATION OF ANALYTIC TORSION ON TEICHMÜLLER SPACE

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ABSTRACT. We consider the relative canonical line bundle $K_{X/T}$ and a relatively ample line bundle $(L, e^{-\phi})$ over the total space $X \to T$ of fibration over the Teichmüller space by Riemann surfaces. We consider the case when the induced metric $\sqrt{-1\partial\bar\partial\phi}|_{X_y}$ on $X_y$ has constant scalar curvature and we obtain the curvature asymptotics of $L^2$-metric and Quillen metric of the direct image bundle $E^k = \pi_*(L^k + K_{X/T})$. As a consequence we prove that the second variation of analytic torsion $\tau_k(\bar\partial)$ satisfies

$$\partial\bar\partial \log \tau_k(\bar\partial) = o(k^{-1})$$

at the point $y \in T$ for any $l \geq 0$ as $k \to \infty$.

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1. Introduction

In the present paper we shall study the asymptotics of the second variation of analytic torsions for higher powers of a line bundle for a family of Riemann surfaces. Consider first a general holomorphic fibration $\pi : X \to T$ with fibers being $n$-dimensional compact manifolds, a relative ample line bundle $L$ and the relative canonical line bundle $K_{X/T}$ over $X$. Let $E^k = \pi_*(L^k + K_{X/T})$ be the direct image bundle over $T$. The holomorphic vector bundle $E^k$ is equipped with two natural metrics, the $L^2$-metric and Quillen metric (see Section 2). In
a recent preprint [18] we prove that the second variation of analytic torsion satisfies

\begin{equation}
\partial \bar{\partial} \log \tau_k(\bar{\partial}) = o(k^{n-1});
\end{equation}

see [18, Corollary 1.3]. This is done by comparing the curvatures of these two metrics, expanding the resolvent operator \((\Delta + k)^{-1}\) and by using Tian-Yau-Zelditch expansion of Bergman kernels on fibers. The same result can also be deduced from a recent paper of Finski [13]. It is then natural and interesting to ask whether the coefficients of the orders lower than \(k^{n-1}\) are zero. One important case of the above consideration is the fibration \(\mathcal{X}\) over Teichmüller space \(\mathcal{T}\) by Riemann surfaces, also called Teichmüller curve, [1]. It is now well-known that the Bergman kernel expansion for line bundles \(L^k\) over Riemann surfaces has only two terms, namely the linear term \(c_1k\) and constant term \(c_0\), the remaining term being exponentially decaying [14]. We may expect that the variation of the analytic torsion is also decaying exponentially for \(k \to \infty\). Indeed when the line bundle \(L\) is the relative canonical line bundle the exponential decaying property is proved in [12]; the analytic torsion in this case is expressed in terms of the Selberg zeta function [17] and can be studied by explicit computations. In the present paper we will consider a general line bundle \(L\) over the the Teichmüller curve and prove (1.1) holds for any order. In this case there is no explicit formula for the analytic torsion. We describe below more precisely our results and their proofs.

Let \(L\) be a relative ample line bundle, i.e. there exists a metric \(\phi\) such that its curvature

\[
\sqrt{-1} \partial \bar{\partial} \phi|_{\mathcal{X}_y} = \sqrt{-1} \phi_{\text{e.e}} dv \wedge d\bar{v} > 0
\]
on each fiber \(\mathcal{X}_y := \pi^{-1}(y)\) for any \(y \in \mathcal{T}\), and let \(K_{\mathcal{X}/\mathcal{T}} := K_{\mathcal{X}} - \pi^* K_{\mathcal{T}}\) denote the relative canonical line bundle. We will consider the following direct image bundle

\begin{equation}
E^k = \pi_*(L^k + K_{\mathcal{X}/\mathcal{T}})
\end{equation}

over the Teichmüller space \(\mathcal{T}\).

Let \(D_y = \partial_y + \partial_{\bar{y}}\) be the Dirac operator acting on \(A^{0,1}(\mathcal{X}_y, L^k + K_{\mathcal{X}/\mathcal{T}})\) of \((0,1)\)-forms. For any \(b > 0\), denote by \(D^{(b, +\infty)}\) the restriction of \(D\) on the sum of eigenspaces of \(A^{0,1}(\mathcal{X}_y, L^k + K_{\mathcal{X}/\mathcal{T}})\) for eigenvalues in \((b, +\infty)\). Then the (Ray-Singer) analytic torsion is defined by

\[
\tau_k(\bar{\partial}) = \tau_k(\bar{\partial}^{(b, +\infty)}) = \left(\det((D^{(b, +\infty)})^2)\right)^{1/2}
\]
and is a positive smooth function on \(\mathcal{T}\). Here \(b\) is a constant less than all positive eigenvalues of \(D\) (see Definition 2.2).

The analytic torsion and its second variation on Teichmüller space have been studied in details by [10, Theorem 3.10 and Theorem 5.8]. In this paper, we prove
Theorem 1.1. Let $\pi : X \to T$ be the holomorphic fibration over Teichmüller space $T$. Suppose that the induced metric $\sqrt{-1}\partial\bar{\partial}\phi|_{X_y}$ on $X_y$ has constant scalar curvature. Then

\begin{equation}
\partial\bar{\partial}\log \tau_k(\partial) = o(k^{-l})
\end{equation}

at any point $y \in T$ for any $l \geq 0$ as $k \to \infty$.

Here the asymptotic (1.3) denotes $(\partial\bar{\partial}\log \tau_k(\partial))(\zeta, \bar{\zeta}) = o(k^{-l})$ for any vector $\zeta \in T_yT$.

Recall Theorem 2.3 (see below) that the Quillen metric on $T$ is defined by the analytic torsion for the fibration. In the papers [6, 7, 8], J.-M. Bismut, H. Gillet and C. Soulé computed the curvature of Quillen metric for a locally Kähler family and obtained the differential form version of Grothendieck-Riemann-Roch Theorem. Moreover, they proved that as a holomorphic bundle,

\begin{equation}
\lambda_y \cong \bigotimes_{i \geq 0} \det H^i(X_y, L^k + K_{X/T})^{(-1)^{i+1}}.
\end{equation}

By Kodaira vanishing theorem, $H^i(X_y, K_{X/T} + L^k) = 0$ for all $i \geq 1$, thus

\begin{equation}
\lambda \cong (\det E^k)^{-1}.
\end{equation}

Let $\det \| \cdot \|_k$ denote the natural induced $L^2$-metric on line bundle $\lambda^{-1} = \det E^k$. Then it follows from (3.6) that

\begin{equation}
\det \| \cdot \|_k^2 = ((| \cdot |^b)^2)^*
\end{equation}

for $b > 0$ a sufficiently small constant, where $((| \cdot |^b)^2)^*$ denotes the dual of the $L^2$-metric $(| \cdot |^b)^2$.

The Chern forms of the $L^2$-metric has been studied intensively and Berndtsson [2, 3, 4] has found the curvature of the vector bundle,

\begin{equation}
\langle \sqrt{-1}\Theta^{E^k} u, u \rangle = \int_{X/M} kc(\phi)|u|^2 e^{-k\phi} + k((\Delta' + k)^{-1}i_{\mu_{\alpha}} u, i_{\mu_{\beta}} u)\sqrt{-1} \partial\bar{\partial} \phi,
\end{equation}

where the definitions of $c(\phi)$, $\mu_{\alpha}$ and $\Delta'$ are given in Theorem 2.5. To prove Theorem 1.1 we shall find the expansion of the curvature of $E^k$.

Theorem 1.2. For any vector $\zeta = \zeta^\alpha \frac{\partial}{\partial \phi^\alpha} \in T_yT$, we have

\begin{equation}
-\sqrt{-1}c_1(E^k, \| \cdot \|_k)(\zeta, \bar{\zeta}) = \frac{6k^2 - 6k\rho + \rho^2}{24\pi^2(-\rho)}\| \mu \|^2 + o(k^{-l})
\end{equation}

for any $l \geq 0$, where $\mu = -\partial_v(\phi_{\alpha\bar{v}}(\phi_{v\bar{v}})^{-1})\zeta^\alpha d\bar{v} \otimes \frac{\partial}{\partial v}$ and

\[\| \mu \|^2 = \int_{X_y} |\partial_v(\phi_{\alpha\bar{v}}(\phi_{v\bar{v}})^{-1})\zeta^\alpha|^2 \sqrt{-1}\partial\bar{\partial}\phi.\]
The Quillen metric $\| \bullet \|_Q$ on the determinant line $\lambda$ (see Definition 2.1) is patched by the $L^2$-metric $| \bullet |_b$ on $\lambda_b$ (see (2.1)) and the analytic torsion $\tau_k(\bar{\partial})$, i.e.

\[
\| \bullet \|_b = | \bullet |_b \tau_k(\bar{\partial}),
\]

where $b > 0$ is a sufficiently small constant. From [18, Proposition 3.9], we obtain the curvature of Quillen metric.

**Theorem 1.3.** For $\zeta \in T_y\mathcal{T}$, we have

\[
(\sqrt{-1}c_1(\lambda, \| \bullet \|_Q))(\zeta, \bar{\zeta}) = \frac{6k^2 - 6k\rho + \rho^2}{24\pi^2(-\rho)}\|\mu\|^2.
\]

Using (1.4) and (1.6) we have furthermore

\[
\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log(\tau_k(\bar{\partial}))^2 = -c_1(\lambda, \| \bullet \|_Q) - c_1(E^k, \| \bullet \|_k).
\]

Theorem 1.1 is then an immediate consequence of Theorem 1.2 and Theorem 1.3.

We mention finally that our result states that the second variation of analytic torsion decays faster than any $k^{-l}$, and it would be interesting to know if it is decaying exponentially as $e^{-kc}$.

This article is organized as follows. In Section 2, we fix the notation and recall some definitions and facts on analytic torsion, Quillen metric, Berndtsson’s curvature formula and Bergman kernel on Riemann surface. In Section 3, we find the expansion of $c_1(E^k, \| \bullet \|_k)$ and prove Theorem 1.2. We also give the expression of $-c_1(\lambda, \| \bullet \|_Q)$ and prove Theorem 1.3. By comparing with their expansions, we prove Theorem 1.1.

We would like to thank Bo Berndtsson for many insightful discussions on the curvature formula of direct image bundles and Miroslav Englis for careful explanation of Bergman kernel expansions on Riemann surfaces.

2. Preliminaries

2.1. Analytic torsion and Quillen metric. We start with the rather general setup of holomorphic fibrations and specify them later to the case of the fibration by Riemann surfaces over Teichmüller space. The definitions and results in this subsection can be found in [5, 6, 7, 8, 15, 16].

Let $\pi : \mathcal{X} \to M$ be a proper holomorphic mapping between complex manifolds $\mathcal{X}$ and $M$, $(F, h_F)$ a holomorphic Hermitian vector bundle on $\mathcal{X}$, $\nabla^F$ the corresponding Chern connection, and $R^F = (\nabla^F)^2$ its curvature. For any $y \in M$, let $\mathcal{X}_y = \pi^{-1}(y)$ be the fiber over $y$ with Kähler metric $\gamma^\mathcal{X}_y$ depending smoothly on $y$. The fibers are assumed to be compact.

The operator $D_y = \partial_y + \bar{\partial}_y$ acts on the fiber $A^{0,*}(\mathcal{X}_y, F)$. For $b > 0$, let $K_y^{b,p}$ be the sum of the eigenspaces of the operator $D_y^2$ acting on $A^{0,p}(\mathcal{X}_y, F)$.
for eigenvalues < b. Let \( U^b \) be the open set \( U^b = \{ y \in M; b \notin \text{Spec} \mathcal{D}_y \} \). Set
\[
K^{b,+} = \bigoplus_{p \text{ even}} K^{b,p}, \quad K^{b,+} = \bigoplus_{p \text{ odd}} K^{b,p}, \quad K^b = K^{b,+} \oplus K^{b,-}.
\]
Define the following line bundle \( \lambda^b \) on \( U^b \) by
\[
(2.1) \quad \lambda^b = (\det K^{b,0})^{-1} \otimes (\det K^{b,1}) \otimes (\det K^{b,2})^{-1} \otimes \cdots.
\]
For \( 0 < b < c \), if \( y \in U^b \cap U^c \), let \( K_y^{(b,c),p} \) be the sum of eigenspaces of \( D_y^p \) in \( E_y \) for eigenvalues \( \mu \) such that \( b < \mu < c \). Define similarly \( K_y^{(b,c),+}, K_y^{(b,c),-}, K_y^{(b,c)} \) and \( \lambda_y^{(b,c)} \). Let \( \partial^{(b,c)} \) and \( D^{(b,c)} \) be the restriction of \( \partial \) and \( D \) to \( K^{(b,c)} \), and \( D_y^{(b,c)} \) the restriction of \( D \) to \( K_y^{(b,c),\pm} \). The bundle \( \lambda_y^{(b,c)} \) has a canonical non-zero section \( T(\partial y^{(b,c)}) \) which is smooth on \( U^b \cap U^c \) (see (6, Definition 1.1)). For \( 0 < b < c \), over \( U^b \cap U^c \), we have the \( C^\infty \) identifications \( \lambda^c = \lambda^b \otimes \lambda_y^{(b,c)} \), which is given by the following \( C^\infty \) map
\[
(2.2) \quad s \in \lambda^b \mapsto s \otimes T(\partial y^{(b,c)}) \in \lambda^c.
\]
Definition 2.1 (8, Def. 1.1]). The \( C^\infty \) line bundle \( \lambda \) over \( M \) is \( \{(U^b, \lambda^b)\} \) with the transition functions (2.2) on \( U^b \cap U^c \).

The analytic torsion was introduced by Ray and Singer [16].

Definition 2.2. The analytic torsion \( \tau(\partial y^{(b,c)}) \) is defined as the positive real number
\[
\tau(\partial y^{(b,c)}) = \left( (\det(D_1^{(b,c)}))^2 (\det(D_2^{(b,c)}))^2 (\det(D_3^{(b,c)}))^3 \cdots \right)^{1/2},
\]
where \( D_p^{(b,c)} \) is the restriction of \( D \) to \( K^{(b,c),p} \), \( 1 \leq p \leq n \). If \( b \) is a small constant less than all positive eigenvalues of \( D_y^2 \), we denote \( \tau(\partial) := \tau(\partial^{(b,+,\infty)}) \).

Let \( \| \cdot \|^b \) denote the following metric on the line bundle \( (\lambda^b, U^b) \),
\[
(2.3) \quad \| \cdot \|^b = | \cdot |^{b \tau y(\partial^{(b,+,\infty)})},
\]
where \( | \cdot |^b \) is the standard \( L^2 \)-metric. The definition of Quillen metric \( \| \cdot \|_Q \) and its Chern form \( c_1(\lambda, \| \cdot \|_Q) \) are given by the following theorem.

Theorem 2.3 ([6, 7, 8]). The metrics \( \| \cdot \|^b \) on \( (\lambda^b, U^b) \) patch into a smooth Hermitian metric \( \| \cdot \|_Q \) on the holomorphic line bundle \( \lambda \). The Chern form of Hermitian line bundle \( (\lambda, \| \cdot \|_Q) \) is
\[
(2.4) \quad c_1(\lambda, \| \cdot \|_Q) = -\left\{ \int_{\mathcal{X}/M} Td \left( \frac{-R^{T_{\mathcal{X}/M}}}{2\pi i} \right) \right. Tr \left. \left[ \exp \left( \frac{-R^F}{2\pi i} \right) \right] \right\}^{(1,1)}.
\]

The Knudsen-Mumford determinant is defined by
\[
\lambda y^{KM} = (\det R_{\pi_* F})^{-1}.
\]
On each fiber it is given by \( \lambda y^{KM} = \bigotimes_{i \geq 0} \det H^i(\mathcal{X}_y, F)^{(-1)^{i+1}} \).
Theorem 2.4 ([6, 7, 8]). Assume that \(\pi\) is locally Kähler. Then the identification of the fibers \(\lambda_y \cong \lambda_y^{KM}\) defines a holomorphic isomorphism of line bundles \(\lambda \cong \lambda^{KM}\). The Chern form of the Quillen metric on \(\lambda \cong \lambda^{KM}\) is given by (2.4).

Here locally Kähler means that there is an open covering \(\mathcal{U}\) of \(M\) such that if \(U \in \mathcal{U}\), \(\pi^{-1}(U)\) admits a Kähler metric.

2.2. Berndtsson’s curvature formula of \(L^2\)-metric. We refer [2, 3, 4] and references therein for the proof and background.

Let \(\pi : \mathcal{X} \to M\) be a holomorphic fibration with compact fibres and \(L\) a relative ample line bundle over \(\mathcal{X}\). We denote by \((z; v)\) a local admissible holomorphic coordinate system of \(\mathcal{X}\) with \(\pi(z; v) = z\), where \(z = \{z^\alpha\}_{1 \leq \alpha \leq \dim M}, v = \{v^i\}_{1 \leq i \leq \dim \mathcal{X} - \dim M}\) are the local coordinates of \(M\) and fibers, respectively.

Let \(\phi\) be a metric of \(L\) such that \(\sqrt{-1} \partial \bar{\partial} \phi\) is positive definite for any point \(y \in M\). Set

\[
\frac{\delta}{\delta z^\alpha} := \frac{\partial}{\partial z^\alpha} - \phi_{\alpha j} \phi^{\bar{j} k} \frac{\partial}{\partial v^k}, \quad 1 \leq \alpha \leq \dim M.
\]

Here \(\phi_{\alpha j} = \partial_{z^\alpha} \partial_{v^j} \phi\), \((\phi^{\bar{j} k})\) denotes the inverse of the matrix \((\partial_{k} \partial_{\bar{j}} \phi)\). The geodesic curvature \(c(\phi)\) is defined by

\[
c(\phi) = \left( \phi_{\alpha \bar{j}} - \phi_{\alpha j} \phi^{\bar{i} j} \phi_{i \bar{\bar{j}}} \right) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta,
\]

which is a well-defined real (1,1)-form on \(\mathcal{X}\). Let \(\{dz^\alpha; \delta v^k\}\) denote the dual frame of \(\{\frac{\partial}{\partial z^\alpha}; \frac{\partial}{\partial v^k}\}\). The form \(\sqrt{-1} \partial \bar{\partial} \phi\) has the following decomposition [11, Lemma 1.1]

\[
\sqrt{-1} \partial \bar{\partial} \phi = c(\phi) + \sqrt{-1} \phi_{ij} \delta v^i \wedge \delta \bar{v}^j.
\]

Consider the direct image bundle \(E := \pi_*(K_{\mathcal{X}/M} + L)\) with the natural \(L^2\)-metric, [2, 3, 4],

\[
\|u\|^2 := \int_{\mathcal{X}_y} |u|^2 e^{-\phi}.
\]

for any \(u = u' dv \otimes e \in E_y\), where \(e\) is a local holomorphic frame of \(L|_{\mathcal{X}}\), \(dv = dv^1 \wedge \cdots \wedge dv^n\). Here

\[
|u|^2 e^{-\phi} := (\sqrt{-1})^n |u'|^2 |e|^2 dv \wedge d\bar{v} = (\sqrt{-1})^n |u'|^2 e^{-\phi} dv \wedge d\bar{v}.
\]

We denote

\[
\mu_\alpha = - \frac{\partial}{\partial v^l} \left( \phi_{\alpha j} \phi^{\bar{i} j} \right) dv^l \otimes \frac{\partial}{\partial v^\alpha}.
\]

The following theorem was proved by Berndtsson in [4, Theorem 1.2], its proof can also be found in [11, Theorem 3.1].
Theorem 2.5 ([4]). For any \( y \in M \) the curvature \( \langle \Theta^E u, u \rangle \), \( u \in E_y \), of the Chern connection on the \( L^2 \)-metric is given by

\[
\langle \sqrt{-1} \Theta^E u, u \rangle = \int_{\chi_y} c(\phi) \vert u \vert^2 e^{-\phi} + ((1 + \Delta')^{-1} i_{\mu \alpha} u, i_{\mu \beta} u) \sqrt{-1} \, dz^\alpha \wedge d\bar{z}^\beta.
\]

Here \( \Delta' = \nabla'_* \nabla' + \nabla'_* \nabla \) is the Laplacian on \( L|_{\chi_y} \)-valued forms on \( \chi_y \) defined by the \( (1,0) \)-part of the Chern connection on \( L|_{\chi_y} \).

We replace now the Hermitian line bundle \( (L, e^{-\phi}) \) by \( (L^k, e^{-k\phi}) \), and consider the corresponding direct image bundle \( E^k := \pi_* (L^k + K_{\chi/M}) \). Let \( \nabla'^*_k \) (resp. \( \nabla'^*_k \)) be the adjoint operator of \( \nabla' \) with respect to \( (L^k, e^{-k\phi}) \) and \( (X, k\omega = k\sqrt{-1} \partial \bar{\partial} \phi) \) (resp. \( (X, \omega = \sqrt{-1} \partial \bar{\partial} \phi) \)). We have

\[
\nabla'^*_k = \sqrt{-1} [\Lambda_{k\omega}, \nabla'] = \frac{1}{k} \sqrt{-1} [\Lambda_{\omega}, \nabla'] = \frac{1}{k} \nabla'^*_k.
\]

Hence

\[
\Delta'_k = \nabla'^*_k \nabla' + \nabla' \nabla'^*_k = \frac{1}{k} \Delta'.
\]

From Theorem 2.5 and (2.9), the curvature of \( L^2 \)-metric (see (2.7)) on \( E^k \) is given by

\[
\langle \sqrt{-1} \Theta^{E^k_0} u, u \rangle = \int_{\chi_y} c(k\phi) \vert u \vert^2 e^{-k\phi} + ((1 + \Delta'_k)^{-1} i_{\mu \alpha} u, i_{\mu \beta} u) \sqrt{-1} \, dz^\alpha \wedge d\bar{z}^\beta
\]

\[
= \int_{\chi_y} k c(\phi) \vert u \vert^2 e^{-k\phi} + k((1 + \Delta')^{-1} i_{\mu \alpha} u, i_{\mu \beta} u) \sqrt{-1} \, dz^\alpha \wedge d\bar{z}^\beta
\]

for any element \( u \) of \( E^k_y \).

2.3. Bergman Kernel on Riemann surface. Let \( M \) be an compact complex Kähler manifold with an ample line bundle \( L \) over \( M \). Let \( g \) be the Kähler metric on \( M \) corresponding to the Kähler form \( \omega_g = \text{Ric}(h) \) for some positive curvature Hermitian metric \( h \) on \( L \). The metric \( h \) induces a metric \( h_k \) on \( L^k \). Let \( \{S_0, \ldots, S_{d_k-1}\} \) be an orthonormal basis of the space \( H^0(M, L^k) \) with respect to the inner

\[
(S, T) = \int_M \langle S(x), T(x) \rangle_{h_k} dV_g,
\]

where \( d_k = \dim H^0(M, L^k) \). Then the diagonal of Bergman kernel is given by

\[
\sum_{i=0}^{d_k-1} \vert S_i(x) \vert^2_{h_k}.
\]

The Tian-Yau-Zelditch expansion of Bergman kernel has been extensively studied. For Riemann surfaces it has some particular nature in that the expansion has only two terms; more precisely we have the following
Theorem 2.6 ([14, Theorem 1.1]). Let $M$ be a regular compact Riemann surface and $K_M$ be the canonical line bundle endowed with a Hermitian metric $h$ such that the curvature $\text{Ric}(h)$ of $h$ defines a Kähler metric $g$ on $M$. Suppose that the metric $g$ has constant scalar curvature $\rho$. Then there is a complete asymptotic expansion:

$$
\sum_{i=0}^{d_k-1} \|S_i(x)\|_h^2 \sim k(1 + \frac{\rho}{2k}) + O\left(e^{-\frac{(\log k)^2}{8}}\right),
$$

where $\{S_0, \cdots, S_{d_k-1}\}$ is an orthonormal basis for $H^0(M, K_M^k)$ for some $k > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{\delta}, \sqrt{\frac{2}{|\rho|}}\}$, where $\delta$ is the injective radius at $x_0$.

We note that the expansion holds also for the bundle $L^m + K_M$ where $L$ is any ample line bundle such that its curvature gives a Kähler metric with constant scalar curvature. Indeed the same proof there works also for this case; alternatively one may argue abstractly that the expansion is determined by the curvature of $L$. (Presumably the above expansion can be proved using the more elementary method in [9].)

3. The second variation of analytic torsion

Let $\mathcal{T}$ be the Teichmüller space of Riemann surface of genus $g \geq 2$. Let $\pi : \mathcal{X} \to \mathcal{T}$ be the holomorphic fibration of the Teichmüller curve over $\mathcal{T}$, the fiber $\mathcal{X}_y := \pi^{-1}(y)$ being exactly the Riemann surface given by the complex structure $y \in \mathcal{T}$; see [1]. Let $L$ be a relative ample line bundle over $\mathcal{X}$, namely there exists a metric $\phi$ of $L$ such that the curvature $\sqrt{-1} \partial \bar{\partial} \phi|_{\mathcal{X}_y} > 0$, this implies that $\pi : \mathcal{X} \to \mathcal{T}$ is a local Kähler fibration. Denote

$$
\omega = \sqrt{-1} \partial \bar{\partial} \phi.
$$

We take $(z^1, \cdots, z^m, v)$ a local admissible coordinate system of $\mathcal{X}$ as in the Subsection 2.2. Then $\omega|_{\mathcal{X}_y} = \sqrt{-1} \partial \bar{\partial} \phi v du \wedge d\bar{u}$ gives a Kähler metric on $\mathcal{X}_y$, $\phi_{v\bar{v}} := \frac{\partial^2 \phi}{\partial v \partial \bar{v}}$. The scalar curvature is defined by

$$
\rho = -\frac{1}{\phi_{v\bar{v}}} \partial_v \partial_{\bar{v}} \log \phi_{v\bar{v}}.
$$

Now we assume that the scalar curvature $\rho$ is a constant. Up to a constant we can take $\phi$ such that

$$
e^{-\rho \phi} = \phi_{v\bar{v}}.
$$

In particular, $-\rho = \int_{\mathcal{X}_y} c_1(K_{\mathcal{X}_y})/\int_{\mathcal{X}_y} c_1(L)$ is a positive rational number. Let $K_{\mathcal{X}/\mathcal{T}} := K_{\mathcal{X}} - \pi^* K_{\mathcal{T}}$ denote the relative canonical line bundle. Consider the following direct image bundle over $\mathcal{T}$

$$
E^k := \pi_*(L^k + K_{\mathcal{X}/\mathcal{T}}),
$$

where $k$ is a positive integer.
for any integer \( k \geq 1 \). The operator \( D_y = \bar{\partial}_y + \bar{\partial}_y^* \) acts on \( A^{0, k}(X_y, L^k + K_{X/T}) \). Take a constant \( b > 0 \) smaller than all the positive eigenvalues of \( D_y \). Then \( K^{b,p}_y \approx H^p(X_y, K_{X_y} + L^k) \). Furthermore by Kodaira vanishing theorem,

\[
K^{b,0}_y \approx H^0(X_y, L^k + K_{X_y}) \cong \pi_*(L^k + K_{X/T})_y \quad K^{b,p}_y = 0, \quad \text{for } p \geq 1,
\]

consequently

(3.5) \[ \lambda^b = (\det \pi_*(L^k + K_{X_y}))^{-1}. \]

Since the metric \( \phi \) induces a metric \((\det \phi)^{-1} := (\phi_{\bar{\nu}})^{-1} \) on \( K_{X/T} \), by (2.7), we have

(3.6) \[ |u|^2 e^{-k\phi} = \sqrt{-1} |u|^2 e^{-k\phi} dv \wedge d\bar{v} = |u|^2 e^{-k\phi} (\det \phi)^{-1} \omega = |u|^2 \omega, \]

that is, the \( L^2 \)-metric \( || \cdot ||_k \) on \( \pi_*(L^k + K_{X/T}) \) given by (2.7) coincides with the standard \( L^2 \)-metric on \( \pi_*(L^k + K_{X/T}) \) induced by \((X_y, \omega|_y), (K_{X_y}, (\det \phi)^{-1}) \) and \((L, e^{-k\phi}) \). Thus the \( L^2 \)-metric \( (|| \cdot ||^b)^2 \) is dual to the determinant of the metric \( || \cdot ||^2 \). Using the definition (2.3) we have then

(3.7) \[ (|| \cdot ||^b)^2 = (|| \cdot ||^b)^2 (\partial k(\bar{\partial} (b, +\infty))) = (\det || \cdot ||^2)^{2}(\tau_b(\bar{\partial}))^2, \]

for \( b > 0 \) small enough, where \( \tau_b(\bar{\partial}) = \tau_b(\bar{\partial} (b, +\infty)) \) is the analytic torsion associated with \((X, \omega = \sqrt{-1} \partial \bar{\partial} \phi)) \) and \((L^k, e^{-k\phi}) \). Therefore,

(3.8) \[ \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log(\tau_b(\bar{\partial})) = -c_1(\lambda, || \cdot ||_2) - c_1(E^k, || \cdot ||_k). \]

3.1. The curvature of \( L^2 \)-metric. In this subsection we will find the expansion of the first Chern form \( c_1(E^k, || \cdot ||_k) \).

From (2.10), the curvature of \( L^2 \)-metric is

(3.9) \[ \langle \sqrt{-1} \Theta E^k u, u \rangle = \int_{X_y} kc(\phi)|u|^2 e^{-k\phi} + k((k + \Delta')^{-1} i_{\mu} u, i_{\bar{\mu}} u) \sqrt{-1} dz^\alpha \wedge d\bar{z}^\beta \]

for any element \( u \) of \( E^k_y \). For any tangent vector \( \zeta = \zeta^\alpha \frac{\partial}{\partial z^\alpha} \in T_y T \)

(3.10) \[ (-\sqrt{-1}) c_1(E^k, || \cdot ||_k)(\zeta, \bar{\zeta}) = \frac{1}{2\pi} \sum_{j=1}^{d_k} \langle \Theta E^k u_j, u_j \rangle (\zeta, \bar{\zeta}) \]

\[ = -\frac{\sqrt{-1}}{2\pi} \left( \int_{X_y} kc(\phi) \sum_{j=1}^{d_k} |u_j|^2 e^{-k\phi} \right) (\zeta, \bar{\zeta}) \]

where \( d_k := \dim H^0(X_y, (L^k + K_{X/T})|_{X_y}) \), \{ \bar{u}_j \}_{j=1}^{d_k} \) is an orthonormal basis of \( H^0(X_y, (L^k + K_{X/T})|_{X_y}) \) and

(3.11) \[ \mu = -\partial_v (\phi_{\bar{\alpha}}(\phi_{\bar{\nu}})^{-1}) \zeta^\alpha d\bar{v} \otimes \frac{\partial}{\partial v}. \]

We shall now find expansion of the two terms in \( \text{RHS of (3.10)} \) in \( k \).
For any \( y \in T \), \( X_y \) is a Riemann surface and the metric \( \omega|_{X_y} \) is a Kähler metric with constant scalar curvature \( \rho \). By (3.3), the curvature operator \( R^* \) of cotangent bundle is
\[
R^* := R_{\bar{v}v}^i \frac{\partial}{\partial v^i} dv \wedge d\bar{v} \wedge i \frac{\partial}{\partial v^i}
:= (-\partial_v \partial_{\bar{v}} (\phi_{v\bar{v}})^{-1} + \partial_{\bar{v}} (\phi_{v\bar{v}})^{-1} \partial_v (\phi_{v\bar{v}})^{-1} \phi_{v\bar{v}}) i \frac{\partial}{\partial v} dv \wedge d\bar{v} \wedge i \frac{\partial}{\partial v}
= -e^{\rho \phi} \partial_v \partial_{\bar{v}} \log e^{\rho \phi} \cdot \text{Id}
= -\rho \cdot \text{Id}
\]
when acting on the space \( A^{0,1}(X_y, L^k) \). Thus by [18, Lemma 3.2], for any \( \alpha \in A^{0,1}(X_y, L^k) \),
\[
(k + \rho - \nabla^* \nabla') \alpha = (k - R^* - \nabla^* \nabla') \alpha
= (dv)^* \nabla'(\nabla_v \alpha)
= \phi_{v\bar{v}}^{-1} i \frac{\partial}{\partial v^i} (\partial - k \partial \phi) \nabla_v \alpha
= e^{\rho \phi} \nabla_v \nabla_v \alpha.
\]
Here \( \nabla'^* \) denotes the adjoint operator of the \( (1, 0) \)-part \( \nabla' \) of Chern connection, and we have used the following notations:
\[
\nabla_{l\bar{v}} := \partial_{\bar{v}} + l \rho \phi_{v\bar{v}}, \quad \nabla_v := \partial_v - k \phi_v
\]
for any integer \( l \).

**Lemma 3.1.** For any \( a, b \in \mathbb{R} \), we have
\[
e^{\rho \phi} [\partial_{\bar{v}} - a \phi_{v\bar{v}}, \partial_v - b \phi_v] = a - b.
\]

**Proof.** By a direct computation we have
\[
e^{\rho \phi} [\partial_{\bar{v}} - a \phi_{v\bar{v}}, \partial_v - b \phi_v] = e^{\rho \phi} (-b \phi_{v\bar{v}} + a \phi_{v\bar{v}}) = a - b,
\]
where the last equality follows from (3.3). \( \square \)

Using the lemma we have then
\[
e^{\rho \phi} [\nabla_{l\bar{v}}, \nabla_v] = e^{\rho \phi} [\partial_{\bar{v}} + \rho \phi_{v\bar{v}}, \partial_v - k \phi_v] = -\rho - k,
\]
and
\[
[\nabla_{l\bar{v}}, e^{\rho \phi} \nabla_v] = [\nabla_{l\bar{v}}, e^{\rho \phi}] \nabla_v + e^{\rho \phi} [\nabla_{l\bar{v}}, \nabla_v]
= e^{\rho \phi} \rho \phi_{v\bar{v}} \nabla_v + (-l \rho - k)
= e^{\rho \phi} \nabla_v \rho \phi_{v\bar{v}} + (-l + 1) \rho - k).
\]
Combining (3.14) and (3.17) we find
\[
\nabla_v e^{\rho \phi} \nabla_v \nabla_{\bar{v}} = (e^{\rho \phi} \nabla_v \nabla_{\bar{v}} + e^{\rho \phi} \nabla_v \rho \phi_{v\bar{v}} + (-2 \rho - k)) \nabla_{\bar{v}}
= e^{\rho \phi} \nabla_v \nabla_{2\bar{v}} \nabla_{\bar{v}} + (-2 \rho - k) \nabla_{\bar{v}}.
\]
By induction we get

\begin{equation}
\nabla_{l_0} \cdots \nabla_{l_\theta} (e^{\rho \phi} \nabla_v) \nabla_{l_\theta} = (e^{\rho \phi} \nabla_v) \nabla_{(l_1)_{0}} \cdots \nabla_{l_\theta} + \left( -k + \frac{l + 3}{2} \rho \right) l \nabla_{l_0} \cdots \nabla_{l_\theta}.
\end{equation}

for any \( l \geq 1 \). In fact for \( l = 1 \) this is exactly (3.18). So we assume that (3.19) holds for \( 1, \cdots, l - 1 \). By (3.17)

\begin{align*}
\nabla_{l_0} \cdots \nabla_{l_\theta} (e^{\rho \phi} \nabla_v) \nabla_{l_\theta} &= \nabla_{l_0} \left( (e^{\rho \phi} \nabla_v) \nabla_{l_0} \cdots \nabla_{l_\theta} + \left( -k - (l + 1) \rho \right) (l - 1) \nabla_{(l - 1)_0} \cdots \nabla_{l_\theta} \right) \\
&= (e^{\rho \phi} \nabla_v) \nabla_{(l_1)_0} \cdots \nabla_{l_\theta} + \left( -k - (l + 1) \rho \right) \nabla_{(l - 1)_0} \cdots \nabla_{l_\theta},
\end{align*}

completing the proof of (3.19).

For later convenience we set

\begin{equation}
k = \frac{m - \rho}{2}, \quad A_n = \left( -k - \frac{n + 3}{2} \rho \right) n = \left( -\frac{m}{2} - \frac{n + 2}{2} \rho \right) n
\end{equation}

and

\begin{equation}
\Box_n := (e^{\rho \phi} \nabla_v) \cdots (e^{\rho \phi} \nabla_v) \nabla_{(n)_0} \nabla_{(n - 1)_0} \cdots \nabla_{l_\theta}.
\end{equation}

The formula (3.19) can now be written as

\begin{equation}
\Box_n \Box_1 = \Box_{n+1} + A_n \Box_n, \quad \Box_1 = (e^{\rho \phi} \nabla_v) \nabla_{l_\theta}.
\end{equation}

**Lemma 3.2.** For any \( N \in \mathbb{N}_+ \), the operator \( \Box_{N+1} \) is self-adjoint and

\begin{equation}
(\Box_{N+1})^2 = \sum_{n=N+1}^{2N+2} b_n \Box_n
\end{equation}

for some constants \( b_n \) with \( |b_n| = O(k^{N+1}) \) as \( k \to \infty \).

**Proof.** Using (3.22) we have the following factorization

\begin{equation}
\Box_{N+1} = (\Box_1 - A_N) \Box_N = \prod_{n=0}^{N} (\Box_1 - A_n).
\end{equation}

The identity (3.13) implies that

\begin{equation}
\Box_1 = k + \rho - \nabla^{\tau} \nabla^l
\end{equation}

is self-adjoint. Combining this with (3.24) we see that \( \Box_{N+1} \) is also self-adjoint.
We compute \((□_{N+1})^2\) using (3.24) and (3.22),

\[
(□_{N+1})^2 = \prod_{n=0}^{N} (□_1 - A_n) □_{N+1}
\]

\[
= \prod_{n=0}^{N-1} (□_1 - A_n) ((□_1 - A_{N+1}) + (A_{N+1} - A_N)) □_{N+1}
\]

\[
= \prod_{n=0}^{N-1} (□_1 - A_n) □_{N+2} + (A_{N+1} - A_N) \prod_{n=0}^{N-1} (□_1 - A_n) □_{N+1}
\]

\[
= \prod_{n=0}^{N-1} (□_1 - A_n) □_{N+2} + O(k) \prod_{n=0}^{N-1} (□_1 - A_n) □_{N+1},
\]

where the last equality follows from (3.20), \(|A_n| = O(k)|. In other words \(\prod_{n=0}^{N} (□_1 - A_n) □_{N+1}\) is a linear combination of \(\□_{N+2}\) and \(\□_{N+1}\) with the coefficients being of the form \(\prod_{n=0}^{N-1} (□_1 - A_n)\). Repeating the same procedure we can reduce the coefficients to be constants, namely there exist constants \(b_n\) with \(|b_n| \in O(k^{N+1}), N + 1 \leq n \leq 2N + 2\) such that

\[
(□_{N+1})^2 = \sum_{n=N+1}^{2N+2} b_n □_{n}.
\]

\[\square\]

**Lemma 3.3.** For any \(N \in \mathbb{N}_+\) and \(m > (-\rho)N\) the following identity holds as an operator on the space \(A^{k,1}(X_y, L^k)\),

\[
(k + \Delta')^{-1}(1 - a_N □_{N+1}) = \sum_{n=0}^{N} a_n □_{n},
\]

where \(□_0 := Id\) and

\[
a_n = \frac{2^{n+1}}{(n+2)!(m+\rho)n \cdots (m+\rho)m}.
\]

**Proof.** By (3.13) and (3.22) we have

\[
k - \Delta' + \rho = k - \nabla'^* \nabla' + \rho = e^{\rho \phi} \nabla_v \nabla_v = □_1.
\]

Recalling the notation in (3.20) this becomes

\[
k + \Delta' = 2k + \rho - □_1 = m - □_1.
\]
A direct computation using (3.22) gives

\[
(\sum_{n=0}^{N} a_n \Box_n) (k + \Delta') = \sum_{n=0}^{N} a_n \Box_n (m - \Box_1)
\]

\[
= \sum_{n=0}^{N} ma_n \Box_n - \sum_{n=0}^{N} a_n (\Box_{n+1} + A_n \Box_n)
\]

\[
= ma_0 + \sum_{n=1}^{N} ((m - A_n)a_n - a_{n-1}) \Box_n - a_N \Box_{N+1}
\]

\[
= 1 - a_N \Box_{N+1},
\]

where the last equality holds since \(a_0 = \frac{1}{m}\) and

\[
(m - A_n)a_n - a_{n-1} = \frac{1}{2} (n + 2)(m + \rho n)a_n - a_{n-1} = 0.
\]

Denote

\[(3.31) \|D^n \mu\|^2 := \langle \nabla_n \nabla_{(n-1)0} \cdots \nabla_{\theta} \mu, e^{\rho \phi} \nabla_{n0} \nabla_{(n-1)0} \cdots \nabla_{\theta} \mu \rangle \]

for \(n \geq 1\), and \(\|D^0 \mu\|^2 := \|\mu\|^2\). Then

**Lemma 3.4.** For \(n \geq 0\) and \(\mu = -\partial_{\theta}(\phi_{\alpha \theta}(\phi_{\theta \theta})^{-1}) \zeta^\alpha \omega \partial_{\theta} \otimes \frac{\partial}{\partial x^s}\) (see also (3.11)) we have the following identity

\[(3.32) \|D^n \mu\|^2 = (\rho)^n \frac{n!(n+3)!}{3 \cdot 2^{n+1}} \|\mu\|^2.\]

**Proof.** We observe first that the adjoint \((e^{\rho \phi} \nabla_{\theta})^* = -e^{\rho \phi} (\partial_{\theta} - \rho \phi_{\theta})\), and with some abuse of notation we introduce temporarily \(D_{-\nu} = \partial_{\theta} - \rho \phi_{\theta}\). Thus

\[
\|D^n \mu\|^2 = \langle \nabla_n \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu, e^{\rho \phi} \nabla_{n\theta} e^{(n-1)\rho \phi} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu \rangle
\]

\[
= -\langle e^{\rho \phi} D_{-\nu} \nabla_{n\theta} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu, e^{(n-1)\rho \phi} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu \rangle.
\]

We use Lemma 3.1 repeatedly and find

\[(3.33) \|D^n \mu\|^2 = (\rho)(n + 1) \langle \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu, e^{(n-1)\rho \phi} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu \rangle
\]

\[
- \langle e^{\rho \phi} \nabla_{n\theta} D_{-\nu} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu, e^{(n-1)\rho \phi} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu \rangle
\]

\[
= (\rho)(n + 1 + n + \cdots + 2) \|D^{n-1} \mu\|^2
\]

\[
- \langle e^{\rho \phi} \nabla_{n\theta} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} D_{-\nu} \mu, e^{(n-1)\rho \phi} \nabla_{(n-1)\theta} \cdots \nabla_{\theta} \mu \rangle
\]

\[
= (\rho) \frac{n(n+3)}{2} \|D^{n-1} \mu\|^2,
\]

where the last equality holds by a direct checking and using (3.3), namely

\[(3.34) D_{-\nu} \mu = (\partial_{\theta} - \rho \phi_{\theta}) (\partial_{\theta}(\phi_{\alpha \theta}(\phi_{\theta \theta})^{-1}) \zeta^\alpha) = 0.
\]
Hence
\[(3.35) \quad \|\nabla^n \mu\|^2 = (-\rho)^n \frac{n!(n+3)!}{3 \cdot 2^{n+1}} \|\mu\|^2.\]

We shall also need the following elementary identity involving the Gamma function \(\Gamma(x)\).

**Lemma 3.5.** Let \(0 < a < \frac{1}{N+1}\). The following summation formula holds
\[
\sum_{n=0}^{N} (-1)^n n!(n+3)\Gamma\left(\frac{1}{a} - n\right) = \frac{\Gamma\left(\frac{1}{a}\right)(5a + 3)}{(2a + 1)(a + 1)}
\]
\[
+ \frac{(-1)^{N+1}(N + 1)!\Gamma\left(\frac{1}{a} - (N + 1)\right)((N + 6)a + (N + 4))(a(N + 1) - 1)}{(2a + 1)(a + 1)}.
\]

**Proof.** This is simply a consequence of the following identity
\[(2a + 1)(a + 1)(n + 3)
\]
\[
= -(n + 5)a + (n + 3)(an - 1) - ((n + 6)a + (n + 4))(a(n + 1) - 1)\frac{n + 1}{a - (n + 1)}.
\]

Indeed
\[
(2a + 1)(a + 1) \sum_{n=0}^{N} (-1)^n n!(n+3)\Gamma\left(\frac{1}{a} - n\right)
\]
\[
= \sum_{n=0}^{N} (-1)^n n!\Gamma\left(\frac{1}{a} - n\right)((n + 5)a + (n + 3))(an - 1)
\]
\[
+ \sum_{n=0}^{N} (-1)^{n+1} n!\Gamma\left(\frac{1}{a} - n\right)((n + 6)a + (n + 4))(a(n + 1) - 1)\frac{n + 1}{a - (n + 1)}
\]
\[
= -\sum_{n=0}^{N} (-1)^n n!\Gamma\left(\frac{1}{a} - n\right)((n + 5)a + (n + 3))(an - 1)
\]
\[
+ \sum_{n=0}^{N} (-1)^{n+1}(n + 1)!\Gamma\left(\frac{1}{a} - (n + 1)\right)((n + 6)a + (n + 4))(a(n + 1) - 1)
\]
\[
= \Gamma\left(\frac{1}{a}\right)(5a + 3)
\]
\[
+ \frac{(-1)^{N+1}(N + 1)!\Gamma\left(\frac{1}{a} - (N + 1)\right)((N + 6)a + (N + 4))(a(N + 1) - 1)}{(2a + 1)(a + 1)},
\]
completing the proof.

We return now to (3.10). From (3.2) we have \((-\rho)c_1(L|\mathcal{X}_y) = c_1(K_{\mathcal{X}_y})\), and then \(\rho < 0\) since both \(L|\mathcal{X}_y\) and \(K_{\mathcal{X}_y}\) are ample. Let \(h\) be a smooth metric on \(K_{\mathcal{X}_y}\) such that its curvature \(R(h) = \sqrt{-1}(-\rho)\phi_{\mu}d\nu \wedge d\bar{\nu}\), which gives a Kähler metric on \(\mathcal{X}_y\). Moreover, by a direct calculation, its scalar curvature is \(-1\).
Using (3.6) and Theorem 2.6 we get the following TYZ expansion associated with the line bundle $(L^k + K_{X/T})|_{X_y}$,

\[
\sum_{j=1}^{d_k} |u_j|^2 e^{-k\phi} = (-\rho) \left( -\frac{k}{\rho} + 1 + \frac{1}{2} + O \left( e^{-\frac{\left(\log \left( 1 - \frac{k}{\rho} \right) \right)^2}{8}} \right) \right) \frac{\omega|_{X_y}}{2\pi}
\]

(3.36)

Now we prove Theorem 1.2.

\[\text{Proof.} \] For any \( l \geq 0 \), we fix an integer \( N \in \mathbb{N}_+ \) such that \( N - \frac{1}{2} > l \), and take \( m \) large such that \( m > (-\rho)(N + 1) \).

The second term in the RHS of (3.10) is by Lemma 3.3

\[
\sum_{j=1}^{d_k} k\langle (k + \Delta')^{-1} i_{\mu}u_j, i_{\mu}u_j \rangle = \sum_{j=1}^{d_k} k\langle \sum_{n=0}^{N} a_n \Box_n i_{\mu}u_j, i_{\mu}u_j \rangle + \sum_{j=1}^{d_k} k\langle (k + \Delta')^{-1} a_N \Box_{N+1} i_{\mu}u_j, i_{\mu}u_j \rangle.
\]

(3.37)

We compute the first term in the RHS of the above formula as

\[
\sum_{j=1}^{d_k} k\langle \sum_{n=0}^{N} a_n \Box_n i_{\mu}u_j, i_{\mu}u_j \rangle = \sum_{n=0}^{N} \sum_{j=1}^{d_k} k a_n (-1)^n \langle i_{\nabla_{n \bar{e}} \ldots \nabla_{\bar{e}}} u_j, (e^{\rho \phi} \nabla_{\bar{e}})^n i_{\mu}u_j \rangle
\]

(3.38)

where we use (3.21) and Stoke’s theorem in the first equality, a direct computation and (3.14) in the second equality. This combined with (3.36) gives

\[
\sum_{j=1}^{d_k} k\langle \sum_{n=0}^{N} a_n \Box_n i_{\mu}u_j, i_{\mu}u_j \rangle = \sum_{n=0}^{N} k a_n (-1)^n \int_{X/T} |\nabla^n u_j|^2 e^{-k\phi},
\]

(3.39)
Disregarding the remainder terms and using Lemma 3.4 and (3.28) we get

\[
\sum_{n=0}^{N} k a_n (-1)^n \| \nabla^i \mu \|^{2} \frac{1}{2\pi} \left( k + \frac{-\rho}{2} \right)
\]

\[
= \sum_{n=0}^{N} k \frac{2^{n+1}(-1)^n}{(n+2)!(m+\rho_m)\cdots(m+\rho_m)} \cdot (-\rho)^n n!(n+3)! \frac{1}{3 \cdot 2^{n+1}} \frac{1}{2\pi} \left( k + \frac{-\rho}{2} \right)
\]

(3.40)

\[
= \frac{m}{24\pi} \sum_{n=0}^{N} (-1)^n \frac{(-\rho)^n}{m} \left( 1 + \frac{m}{m-\rho} \right) \cdots \left( 1 + m \right) \frac{1}{\Gamma(-\rho)} \sum_{n=0}^{N} (-1)^n n!(n+3)\Gamma \left( \frac{m}{-\rho} - n \right).
\]

From Lemma 3.5 and noting \( \frac{m}{-\rho} > N+1 \), the above can be written as

(3.41)

\[
\sum_{n=0}^{N} k a_n (-1)^n \| \nabla^i \mu \|^{2} \frac{1}{2\pi} \left( k + \frac{-\rho}{2} \right) = \frac{m}{24\pi} (3 + 5\frac{\rho}{m})
\]

\[
+ \frac{m}{24\pi} (-1)^N+1 (N+1) \left( (N+4) + (N+6)(\frac{-\rho}{m}) \right) \frac{1}{\Gamma(-\rho)} \frac{1}{m} \sum_{n=0}^{N} (-1)^n n!(n+3)\Gamma \left( \frac{m}{-\rho} - n \right)
\]

\[
= \frac{m}{24\pi} (3 + 5\frac{\rho}{m}) + O \left( \frac{1}{m^N} \right)
\]

\[
= \frac{3k - \rho}{12\pi} + O \left( \frac{1}{kN} \right).
\]

On the other hand, it is easy to see that

(3.42)

\[
\sum_{n=0}^{N} k a_n (-1)^n \| \nabla^i \mu \|^{2} O \left( e^{-\left( \frac{\log \left( \frac{k}{k+1} \right)^2}{8} \right)}\right) \frac{1}{2\pi} = o \left( \frac{1}{m^N} \right) = o \left( \frac{1}{k^N} \right).
\]

Substituting (3.41) and (3.42) into (3.39), we get

(3.43)

\[
\sum_{j=1}^{d_k} k \sum_{n=0}^{N} a_n \delta_n i_{\mu} u_j, i_{\mu} u_j = \frac{3k - \rho}{12\pi} \| \mu \|^{2} + O \left( \frac{1}{kN} \right).
\]

For the second term in the RHS of (3.37) we use Cauchy-Schwarz inequality,

\[
\sum_{j=1}^{d_k} k \langle (k + \Delta)^{-1} a_N \delta_{N+1} i_{\mu} u_j, i_{\mu} u_j \rangle
\]

\[
\leq \sum_{j=1}^{d_k} k |a_N| \| (k + \Delta)^{-1} \delta_{N+1} i_{\mu} u_j \| \| i_{\mu} u_j \|
\]

\[
\leq \sum_{j=1}^{d_k} |a_N| \| \delta_{N+1} i_{\mu} u_j \| \| i_{\mu} u_j \|,
\]
where the second inequality holds because the eigenvalues of $k + \Delta'$ are greater than or equal to $k$. By Lemma 3.2, $\Box_{n+1}$ is self-adjoint, so that
\[
\|\Box_{N+1} i_\mu u_j\| = |(\Box_{N+1})^2 i_\mu u_j, i_\mu u_j|^{1/2} = \left( \sum_{n=N+1}^{2N+2} b_n \langle \Box_n i_\mu u_j, i_\mu u_j \rangle \right)^{1/2}
\]
(3.45)
\[
\leq \left( \sum_{n=N+1}^{2N+2} \left| b_n \max_{X_y} |\nabla^{-1} \mu|^2 \|u_j\|^2 \right| \right)^{1/2} = O \left( k^{N+1} \right),
\]

where the second equality follows from Lemma 3.2, the third equality holds by the same proof as (3.39), the last equality holds since $|b_n| = O(k^{N+1})$, $\|u_j\| = 1$ and $\max_{X_y} \nabla^{-1} \mu|^2$ is independent of $k$. By Lemma 3.2, we have
\[
\sum_{j=1}^{d_k} k((k + \Delta')^{-1} a_N \Box_{N+1} i_\mu u_j, i_\mu u_j) \leq \sum_{j=1}^{d_k} |a_N| |i_\mu u_j| O \left( k^{N+1} \right) \leq |a_N| d_k \max_{X_y} |\mu|^2 O \left( k^{N+1} \right)
\]
(3.46)
\[
= O \left( \frac{1}{k^{N+1}} \right),
\]
where the last equality follows from $|a_N| = O \left( \frac{1}{k^{N+1}} \right)$, $d_k = O(k)$, and $\max_{X_y} |\mu|^2$ is independent of $k$. This settles the second term in the RHS of (3.37).

Combining the above estimate with (3.43) we obtain the estimates for (3.37),
\[
\sum_{j=1}^{d_k} k((k + \Delta')^{-1} i_\mu u_j, i_\mu u_j) = \frac{3k - \rho}{12\pi} \|\mu\|^2 + O \left( \frac{1}{k^{N+1}} \right).
\]
(3.47)

On the other hand by [18, (3.46)] and (2.6) we have
\[
\Delta(c(\phi)(\zeta, \bar{\zeta})) = \phi_v^{-1} \frac{\partial^2}{\partial v \partial \bar{v}} (c(\phi))_{\alpha \beta} \zeta^\alpha \bar{\zeta}^\beta = (\partial \bar{\partial} \log \phi_v)(\zeta^\alpha \frac{\delta}{\delta z^\alpha}, \bar{\zeta}^\beta \frac{\delta}{\delta \bar{z}^\beta}) - |\mu|^2
\]
(3.48)
\[
= (-\rho) \partial \bar{\partial} \phi (\zeta^\alpha \frac{\delta}{\delta z^\alpha}, \bar{\zeta}^\beta \frac{\delta}{\delta \bar{z}^\beta}) - |\mu|^2
\]
\[
= (-\rho)(-\sqrt{-1}) c(\phi)(\zeta, \bar{\zeta}) - |\mu|^2.
\]

By integrating along fibers and using Stoke’s theorem, we get
\[
(-\sqrt{-1}) \left( \int_{X_y} c(\phi) \omega \right)(\zeta, \bar{\zeta}) = \frac{1}{-\rho} \|\mu\|^2.
\]
(3.49)
From above equality, the first term in the RHS of (3.10) is

\[- \frac{\sqrt{-1}}{2\pi} \left( \int_{\mathcal{X}_g} kc(\phi) \sum_{j=1}^{d_x} |u_j|^2 e^{-k\phi} \right) (\zeta, \bar{\zeta}) \]

\[= - \frac{\sqrt{-1}}{2\pi} \left( \int_{\mathcal{X}_g} c(\phi) \omega \right) (\zeta, \bar{\zeta}) k \frac{1}{2\pi} \left( k \frac{1}{\rho^2} + \frac{1}{2(-\rho)} + O \left( e^{-(\log(\frac{k}{8} + 1))^2} \right) \right) \]

(3.50)

\[= \frac{1}{2\pi - \rho} \frac{\|\mu\|^2}{\lambda} \left( k + \frac{-\rho}{2} + O \left( e^{-(\log(\frac{k}{8} + 1))^2} \right) \right) \]

\[= \frac{k\rho - 2k^2}{8\pi^2 - \rho} \|\mu\|^2 + O \left( ke^{-\frac{(\log(\frac{k}{8} + 1))^2}{\rho}} \right). \]

Finally using (3.47) and (3.50) we get the an expansion for (3.10) as

\[(- \sqrt{-1} a_1 (E^k, \|\bullet\|_k)(\zeta, \bar{\zeta}) \]

(3.51)

\[= 3k - \rho \|\mu\|^2 + O(1) + \frac{k\rho - 2k^2}{8\pi^2 - \rho} \|\mu\|^2 + O \left( ke^{-\frac{(\log(\frac{k}{8} + 1))^2}{\rho}} \right) \]

\[= 6k^2 - 6k\rho + \rho^2 \frac{\|\mu\|^2}{24\pi^2 - \rho} + O \left( \frac{1}{\sqrt{k} + 2} \right). \]

This completes the proof of Theorem 1.2 since \(\frac{\sqrt{-1}}{2} > l. \] \(\square\)

3.2. The curvature of Quillen metric. We prove now Theorem 1.3.

Proof. From [18, Proposition 3.9], the curvature of Quillen metric is

(3.52)

\[(\sqrt{-1} a_1 (\lambda, \|\bullet\|_q))(\zeta, \bar{\zeta}) \]

\[= \frac{k^2}{(2\pi)^2} \int_{\mathcal{X}_g} (-\sqrt{-1})c(\phi)(\zeta, \bar{\zeta}) \omega + \frac{k}{(2\pi)^2} \int_{\mathcal{X}_g} \left( \frac{1}{2} |\mu|^2 - \frac{\rho}{2} (-\sqrt{-1}) c(\phi)(\zeta, \bar{\zeta}) \right) \omega \]

\[+ \frac{1}{(2\pi)^2} \int_{\mathcal{X}_g} (-\sqrt{-1}) c(\phi)(\zeta, \bar{\zeta}) \left( -\frac{1}{6} \Delta \rho + \frac{1}{24} (|R|^2 - 4|Ric|^2 + 3\rho^2) - \frac{\rho}{4} |\mu|^2 \right) \omega \]

\[+ \frac{1}{(2\pi)^2} \left( \frac{1}{12} \|\mu\|^2_{Ric} + \frac{1}{12} \|\nabla^\mu\|^2 - \frac{1}{4} \|\tilde{\nabla}^\mu\|^2 \right), \]

where \(\|\mu\|^2_{Ric} = \int_{\mathcal{X}_g} (\mu^i\overline{\mu}^j R_{i\ell}(\phi^k\phi^\ell\phi^j\phi^k)) \omega^n. \)

The Chern curvature tensor \(R\) is given by

(3.53)

\[R_{v\bar{v}v\bar{v}} = -\partial_v \partial_{\bar{v}} \phi_{v\bar{v}} + \phi_{v\bar{v}}^{-1} \partial_v \phi_{v\bar{v}} \partial_{\bar{v}} \phi_{v\bar{v}} = \rho(\phi_{v\bar{v}})^2, \]

and then

(3.54)

\[|R|^2 = |R_{v\bar{v}v\bar{v}}|^2 (\phi_{v\bar{v}})^{-4} = \rho^2, \quad Ric_{v\bar{v}} = (\phi_{v\bar{v}})^{-1} R_{v\bar{v}v\bar{v}} = \rho \phi_{v\bar{v}}. \]

Hence

(3.55)

\[\|\mu\|^2_{Ric} = \rho \|\mu\|^2, \quad |Ric|^2 = \rho^2. \]
Using (3.34) we have
\[
\nabla' \mu = (\partial_v - \rho \phi_v) \mu dv = D_{-v} \mu dv = 0,
\]
consequently
\[
\bar{\partial}^* \mu = -\sqrt{-1} \Lambda \nabla' \mu = 0.
\]
Substituting (3.54), (3.55), (3.56) and (3.57) into (3.52), and using (3.49), we obtain
\[
(\sqrt{-1} c_1(\lambda, \| \cdot \|_Q))(\zeta, \bar{\zeta}) = \frac{6k^2 - 6k \rho + \rho^2}{24\pi^2 (\rho)} ||\mu||^2,
\]
completing the proof of Theorem 1.3. □

3.3. The proof of Theorem 1.1.
Proof. We estimate (3.8) using (3.51) and (3.52),
\[
\frac{1}{2\pi} \partial \bar{\partial} \log(\tau_k(\bar{\partial}))^2(\zeta, \bar{\zeta}) = (\sqrt{-1}) (c_1(\lambda, \| \cdot \|_Q) - c_1(E_k, \| \cdot \|)) (\zeta, \bar{\zeta}) = O \left( \frac{1}{k^{\frac{1}{2}}} \right) = o(k^{-1}).
\]
□

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