CFT duals on extremal rotating NUT black hole

M. F. A. R. Sakti, A. Suroso, F. P. Zen

Theoretical Physics Lab., THEPI Division, Institut Teknologi Bandung, Jl. Ganesha 10 Bandung, 40132, Indonesia

Indonesia Center for Theoretical and Mathematical Physics (ICTMP), Institut Teknologi Bandung, Jl. Ganesha 10 Bandung, 40132, Indonesia

E-mail: m.fitrah@students.itb.ac.id, agussuroso@fi.itb.ac.id, fpzen@fi.itb.ac.id

Abstract: We investigate the Kerr-Newman-NUT black hole solution obtained from Plebański-Demiański solutions with several assumptions. The origin of the microscopic entropy of this black hole is studied using the conjectured Kerr/CFT correspondence which is first proposed for Kerr extremal black holes. The isometry of the near-horizon extremal Kerr-Newman-NUT black hole shows that the asymptotic symmetry group may be applied to compute the central charge of the Virasoro algebra. Furthermore, by assuming Frolov-Thorne vacuum, the temperatures can be obtained which then using Cardy formula, the microscopic entropy is obtained and agrees with the Bekenstein-Hawking entropy. We also assume the case when the lowest eigenvalue of the conformal operator $L_0$ is non-zero to find the logarithmic correction of the entropy of NHEKNUT black hole. At the limit $J \rightarrow 0$, the extremal Reissner-Nordström-NUT solution is produced and by adding the fibered coordinate we find the 5D solution. The second dual CFT is used to find the entropy and it still produces the area law of 5D black hole solution. So, the extremal Reissner-Nordström-NUT solution is also holographically dual to the CFT.

Keywords: black holes, NUT charge, space-time symmetries, Kerr/CFT correspondence

1Corresponding author.
1 Introduction

Based on the no-hair theorem, the Kerr-Newman solution is the most physical black hole solution with the spin $a$, mass $M$, and the electric charge $e$. However, theoretically there are many mathematical solutions that describe black holes, as one of the most common solutions, because they have various parameters other than those found in Kerr-Newman black hole, is Plebański-Demiański (PD) solution [1]. The PD solution has additional parameters such as magnetic charge $g$, acceleration $\alpha$, cosmological constant $\Lambda$, and NUT charge $l$. The presence of cosmological constants and acceleration in the solution adds the number of horizons to the black hole known as accelerated and cosmological horizons, respectively. A simpler solution that we will discuss in the following is the case when the cosmological constant and acceleration vanish. So the solution becomes Kerr-Newman-NUT (KNUT) black hole.
The KNUT solution is a more general solution compared to the Kerr-Newman one because there is an additional NUT charge which was first discovered by Taub [2] and later developed by Newman et. al. [3]. The interesting part of this solution is the presence of NUT charge in which the NUT is an abbreviation of Newman, Unti, and Tamburino that find the solution for larger manifold after Taub. The existence of a NUT charge causes the metric does not possess asymptotically flat solution. For the definition of the NUT charge itself, one can see it as the source that represents a gravomagnetic monopole parameter of the central mass [4] or a twist parameter of the surrounding space-time [5].

Studying the thermodynamics quantities of KNUT black holes seems to be an interesting and important thing because it can be one of the stages to study the merging of gravitational theory and the quantum field theory. The conjectured Kerr/CFT correspondence [6] has first demonstrated that the microscopic entropy of the extremal Kerr black hole matches the (macroscopic) Bekenstein-Hawking entropy. This shows that there is a relation between quantum gravity represented by AdS space-time and the conformal field theory (CFT). In other words, from this conjecture, the gravity is dual to the CFT. This correspondence has also been successfully used to obtain the entropy of various black hole solutions from microscopic point of view such as [7–18] or for the detailed review see [19]. We know that the entropy from the CFT is equal to the quarter of the black hole’s area and matches the Bekenstein-Hawking entropy but Kaul and Majumdar [20] find that there is a logarithmic correction of Bekenstein-Hawking entropy with certain formulations. However, from a microscopic point of view, Carlip [21] succeed to show that logarithmic correction is consistent with the result obtained by Kaul and Majumdar, i.e.

\[ S_{BH}^{corr} \sim S_{BH} - \frac{3}{2} \ln(S_{BH}) + \text{const.}, \]  

(1.1)

where \( S_{BH} = \text{Area}_4/4 \). \( \text{Area}_4 \) means the area of 4D black holes. We distinguish it because the 5D solution is also taken into account later. The corrections corresponding to Bekenstein-Hawking entropy further demonstrate the validity of Kerr/CFT correspondence.

In what follows, we use the Kerr/CFT correspondence for more generalized Kerr-Newman black holes, namely Kerr-Newman-NUT solution where this can be obtained from Plebański-Demiański solution. First we get the near-horizon geometry of the KNUT black hole in some coordinate systems for which all of the metric have the \( SL(2, R) \times U(1) \) isometry group. From this symmetry, it might be seen that asymptotic symmetry group (ASG), as proposed first in [22], can be applied to Near-Horizon-Extremal-Kerr-Newman-NUT (NHEKNUT) geometry to obtain the central charge. Therefore, there are some boundary conditions on the metric deviation that need to be determined in order to produce a finite charge and here we focus only on the left-moving part of the central charge so that there are the additional boundary conditions to the central charge. With the boundary condition we use, in the end this generates the central charge

\[ c_L = 12aM, \]  

(1.2)

which arise from Virasoro algebra. As we have said previously, here we assume that there is no contribution from the electromagnetic field to the central charge as well as the vanishing right-moving central charge.
After getting the associated central charge, the next step is to obtain the temperatures because these two quantities are required in Cardy formula for the entropy. In the assumptions we use, the temperature required in the computation of the entropy is the left-moving temperature. By using the Frolov-Thorne vacuum, we find that

$$T_L = \frac{M^2 + (a+l)^2}{4\pi a M}. \quad (1.3)$$

But actually besides that, we also obtain the temperatures which is the conjugate of the electric and magnetic charges from KNUT black hole. It will be useful for finding the entropy for Reissner-Nordström-NUT black hole in five dimensions. Finally, from Cardy formula, we find that the entropy of NHEKNUT black hole is

$$S_{KNUT} = \pi (2a^2 + q^2 + 2al), \quad (1.4)$$

which agrees with the Bekenstein-Hawking entropy.

When the angular momentum of the NHEKNUT geometry vanishes, it causes the central charge to vanishes too but the left-moving temperature becomes singular at this point but this should still produce the entropy corresponding to the Bekenstein-Hawking entropy. To solve the problem of central charge and temperature, we adopt the method used by Hartmann et.al. [7] which defines the second dual CFT. This is done by first adding a fibered coordinate that has a periodic property so that $z \sim z + 2\pi R_n$. In addition, the electromagnetic field obtained after adding the gauge transformation, because $J \to 0$, could be the part of the geometry so that a near-horizon metric of five dimensional extremal Reissner-Nordstrom-NUT solution can be obtained. Next, by using ASG and adding boundary conditions for the five dimensional solution, it produce the central charge of 5D geometry. Then by using the temperature that is the conjugate of the electric charge, the following entropy is obtained

$$S_{RNNUT} = \pi q^2. \quad (1.5)$$

This entropy still agrees with the Bekenstein-Hawking entropy in five dimensional geometry where $S_{BH} = \text{Area}_5/4G_5$.

## 2 Kerr-Newman-NUT black holes

The no-hair theorem tells that the Kerr-Newman solution is the most physical black hole solution. However, from the theoretical point of view, there are many mathematical solutions that describe black holes, as one of the general solutions, because they have various parameters other than those found in Kerr-Newman black hole, is Plebański-Demiański (PD) solution [1]. This solution has the form [23].

\[
\begin{align*}
\text{ds}^2 &= \frac{1}{\Omega^2} \left[ -\frac{\Delta}{\Sigma} \left\{ \text{d}t - \left( a\sin^2\theta + 4l\sin^2\frac{\theta}{2} \right) \text{d}\phi \right\}^2 + \frac{\Sigma}{\Delta} \text{d}r^2 + \frac{\Sigma}{P} \text{d}\theta^2 \\
&\hspace{1cm} + \frac{P}{\Sigma} \sin^2\theta \left\{ a\text{d}t + \left( \text{r}^2 + (a+l)^2 \right) \text{d}\phi \right\}^2 \right], \quad (2.1)
\end{align*}
\]
A detailed derivation of the solution can be seen in the original paper [1]. The parameters $M, a, e, g, l, \alpha, \Lambda$ represent mass, spin, electric charge, magnetic charge, NUT charge, acceleration, and cosmological constant, respectively where we use the notation $k_B = G = h = c = 1$. Then $\omega$ is a free parameter that we can choose as a function of the previous seven parameters. In addition, the NUT charge can be defined as a gravomagnetic monopole parameter [4] or a twisting property of the surrounding spacetime [5]. Its name comes from the abbreviation of Newman, Unti, and Tamburino that find the solution for a twisting property of the surrounding spacetime [5]. Its name comes from the abbreviation of Newman, Unti, and Tamburino that find the solution for larger manifold [3] after Taub [2].

In what follows, we focus on some special cases since we want only look for the Kerr-Newman-NUT solution. To find the KNUT solution, we take the value of the cosmological constant $\Lambda$ and acceleration $\alpha$ to 0, so it results in $P_0 = P_1 = 0$. Then the circumstances $P_2 = 1, P_3 = l$, and $P_4 = 1$ are taken. Finally, we obtain

$$ds^2 = -\frac{\Delta}{\Sigma} \left\{ d\tilde{t} - \left( \alpha \sin^2 \theta + 4 \sin^2 \frac{\theta}{2} \right) d\phi \right\}^2 + \frac{\Sigma}{\Delta} d\tilde{r}^2 + \Sigma d\theta^2$$

$$+ \frac{\sin^2 \theta}{\Sigma} \left( a d\tilde{t} + \{ \tilde{r}^2 + (a + l)^2 \} d\phi \right)^2,$$

where

$$\Sigma = \tilde{r}^2 + (l + a \cos \theta)^2, \quad \Delta = (q^2 + a^2 - l^2) - 2M\tilde{r} + \tilde{r}^2.$$

Here, we have defined $q^2 = e^2 + g^2$. The solution (2.3) is equivalent to the following solution

$$ds^2 = -\frac{\Delta}{\Sigma} \left\{ d\tilde{t} - \left( \alpha \sin^2 \theta + 2l(1 - \cos \theta) \right) d\phi \right\}^2 + \frac{\Sigma}{\Delta} d\tilde{r}^2 + \Sigma d\theta^2$$

$$+ \frac{\sin^2 \theta}{\Sigma} \left( a d\tilde{t} + (\tilde{r}^2 + a^2 + l^2 + 2al) d\phi \right)^2.$$
The metric (2.4) has the event horizon, angular velocity, and Hawking temperature, respectively

\[ \hat{r}_+ = M + \sqrt{M^2 + l^2 - a^2 - q^2}, \]  
\[ \Omega_H = \frac{a}{\left[ \hat{r}_+^2 + (a + l)^2 \right]^2}, \]  
\[ T_H = \frac{\kappa}{2\pi} = \frac{r_+ - M}{2\pi \left[ \hat{r}_+^2 + (a + l)^2 \right]}, \]

in which the angular velocity is the same as with the accelerating black hole Kerr-Newman-NUT [24]. When its acceleration parameter vanishes, the Hawking temperature will be equal to the above. It deals with Bekenstein-Hawking entropy

\[ S_{BH}(T_H = 0) = \frac{\text{Area}}{4} = \pi(2a^2 + q^2 + 2al). \]  

However, Kaul and Majumdar find the logarithmic correction of the entropy of the black hole in general [20]. They find the correction has the form

\[ S_{corr}^{BH} \sim \frac{\text{Area}}{4} - \frac{3}{2} \ln \left( \frac{\text{Area}}{4} \right) + \text{const.} \]

Hence for the Kerr-Newman-NUT black hole, the corrected entropy is

\[ S_{corr}^{BH} \sim \pi(2a^2 + q^2 + 2al) - \frac{3}{2} \ln \left[ \pi(2a^2 + q^2 + 2al) \right] + \text{const.} \]

The metric (2.4) is related to the electromagnetic potential [23]

\[ A_\mu dx^\mu = -e\hat{r} \left[ a d\hat{t} - \left\{ (a + l)^2 - (l^2 + a^2\cos^2\theta + 2al\cos\theta) \right\} d\hat{\phi} \right] \]
\[ - \frac{g(l + a\cos\theta)}{a \left[ \hat{r}^2 + (l + a\cos\theta)^2 \right]} \left[ a d\hat{t} - \left\{ \hat{r}^2 + (l + a\cos\theta)^2 \right\} d\hat{\phi} \right]. \]

3 Near-horizon geometry of Kerr-Newman-NUT black holes

In this section, we wish to study the near-horizon geometry of extremal Kerr-Newman-NUT space-time where the extremal case occurs when \( M^2 = a^2 + q^2 = l^2 \). In order to do so, we change the coordinates by the following transformations [7, 19]

\[ \hat{r} = \hat{r}_+ + \lambda r_0 y, \]
\[ \hat{t} = \frac{r_0}{\lambda} \tau, \]
\[ \hat{\phi} = \varphi + \Omega_H \frac{r_0}{\lambda} \tau, \]

where we choose \( r_0 = (2a^2 + q^2 + 2al)/M \) and the parameter \( \lambda \) approaches zero. The metric (2.4) changes then to the near-horizon metric given by

\[ ds^2 = \chi(\theta) \left( -\frac{r_0^2}{M^2} y^2 d\tau^2 + \frac{dy^2}{y^2} + d\theta^2 \right) + \frac{r_0^2 M^2 \sin^2\theta}{\chi(\theta)} \left( d\varphi + \frac{2a}{M} y d\tau \right)^2, \]
where $\chi(\theta) = r_+^2 + (l + a \cos \theta)^2 = a^2(1 + \cos^2 \theta) + 2al\cos \theta + q^2$. This is the Near-Horizon Extremal Kerr-Newman-NUT (NHEKNUT) metric. In the near-horizon limit $\lambda \to 0$ or degenerate horizon, the electromagnetic potential is not regular. In this case, such problem can be circumvented by expanding the electromagnetic potential [25] but first we need to introduce the Coulomb electromagnetic potential

$$\Phi_H = -K^\mu A_\mu|_{\hat{r} = \hat{r} +} = \frac{e\hat{r} +}{\hat{r}^2 + (a + l)^2},$$

(3.3)

where $K = \partial_t + \Omega_H \partial_\phi$. The Coulomb electromagnetic potential will be used as the gauge transformation in the electromagnetic potential. By expanding the electromagnetic potential (2.11) in $r - r_+ = \lambda r_0y$ and add the gauge transformation (see appendix A), we obtain

$$A_\mu dx^\mu = f(\theta) \left( d\varphi + \frac{2a}{M} y d\tau \right) - \frac{e[M^2 - (a + l)^2]}{2aM} d\varphi,$$

(3.4)

where

$$f(\theta) = \left[ \frac{M^2 + (a + l)^2}{2aM} \right] \left[ 2gMl + e(M^2 - l^2) + a(2gM - 2el - aecos \theta)cos \theta \right].$$

(3.5)

Note that this metric (3.2) and the electromagnetic potential (3.4) will remain the same as the near-horizon geometry of Kerr-Newman space-time when the NUT charge vanishes.

We can see clearly that the metric (3.2) has time-like Killing vector, so we can remove the constant over $y^2 d\tau^2$ by using this scaling

$$y d\tau \to \frac{M}{r_0} y d\tau.$$  

(3.6)

Because of the scaling (3.6), the metric (3.2) is then

$$ds^2 = \chi(\theta) \left( -y^2 d\tau^2 + \frac{dy^2}{y^2} + d\theta^2 \right) + \frac{r_0^2 M^2 \sin^2 \theta}{\chi(\theta)} (d\varphi + k y d\tau)^2,$$

(3.7)

where $k = 2aM/(2a^2 + q^2 + 2al)$. Because of the scaling (3.6), the near-horizon electromagnetic potential (3.4) becomes

$$A_\mu dx^\mu = f(\theta) (d\varphi + k y d\tau) - \frac{c}{k} d\varphi,$$

(3.8)

where $c = e[M^2 - (a + l)^2]/[M^2 + (a + l)^2]$ and this is the general form of near-horizon electromagnetic field as the one discussed in [19], but the second term in the right hand side can be gauged away as in [7].

Besides the metric form (3.7), the NHEKNUT geometry can be represented in Poincare-type coordinates such used in some papers of the Kerr/CFT correspondence [6, 8]. The Poincare-type of metric of the NHEKNUT geometry can be obtained by using the following transformations

$$\hat{r} = \hat{r} + \frac{r_0 \lambda}{y},$$

$$\hat{t} = \frac{r_0 \lambda}{\hat{r}},$$

$$\hat{\phi} = \hat{\phi} + \Omega_H \frac{r_0 \lambda}{\hat{r}}.$$

(3.9)
and followed by the scaling
\[ \frac{d\hat{\tau}}{\hat{y}} \to \frac{M}{r_0} \frac{d\hat{\tau}}{\hat{y}}, \] (3.10)
to the metric (2.4). The Poincare-type of NHEKNUT geometry is then
\[ ds^2 = \chi(\theta) \left( -\frac{d\hat{\tau}^2 + d\hat{y}^2}{\hat{y}^2} + M^2 \frac{d\theta^2}{\chi(\theta)} \right) + r_0^2 M^2 \sin^2 \theta \left( d\hat{\phi} + k d\hat{\tau} \right)^2. \] (3.11)
The near-horizon electromagnetic potential now has the form
\[ A_\mu dx^\mu = f(\theta) \left( d\hat{\varphi} + \frac{kd\tau}{\hat{y}} \right) - \frac{c}{k} d\varphi. \] (3.12)
The metrics (3.2), (3.7), and (3.11) are not asymptotically flat space-times but asymptotically Anti-de Sitter (AdS). Those metrics cover only part of the NHEKNUT geometry. To cover the whole near-horizon geometry, we use the global coordinates. In order to find the global form, we just need to transform the metrics (3.7) or (3.11). The metric (3.11), by the following global coordinate transformation [6, 13, 26]
\[ \hat{y} = \frac{1}{r + \sqrt{1 + r^2 \cos^2 t}}, \quad \hat{\tau} = \hat{y} \sin t \sqrt{1 + r^2}, \quad \hat{\varphi} = \varphi + k \ln \left( \frac{\cos t + r \sin t}{1 + \sin t \sqrt{1 + r^2}} \right), \] (3.13)
will transform the Poincare-type of NHEKNUT geometry, such that we obtain
\[ ds^2 = \chi(\theta) \left[ -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + d\theta^2 \right] + \frac{r_0^2 M^2 \sin^2 \theta}{\chi(\theta)} \left( d\hat{\phi} + k r d\tau \right)^2. \] (3.14)
This global form can also be obtained from the metric (3.7) by the similar transformations such in [27]. The global near-horizon electromagnetic field is then
\[ A_\mu dx^\mu = f(\theta) \left( d\hat{\varphi} + k r d\tau \right), \] (3.15)
Note that we have added the gauge transformation before applying the global coordinate transformations to the near-horizon electromagnetic field.
For a fixed polar angle \( \theta \), the near-horizon geometry is a quotient of warped \( \text{AdS}_3 \) which the quotient arises from identification of \( \varphi \) coordinate. This near-horizon geometry of KNUT black hole has \( \text{SL}(2, R) \times U(1) \) isometry group where for the global NHEKNUT one, the subgroup \( U(1) \) is generated by the Killing vector
\[ \zeta_0 = -\partial_\varphi, \] (3.16)
and the subgroup \( \text{SL}(2, R) \) is generated by the three Killing vectors
\[ X_0 = 2 \partial_t, \]
\[ X_1 = 2 \sin t \frac{r}{\sqrt{1 + r^2}} \partial_t - 2 \cos t \sqrt{1 + r^2} \partial_r + \frac{2 \sin t}{\sqrt{1 + r^2}} \partial_\varphi, \]
\[ X_2 = -2 \cos t \frac{r}{\sqrt{1 + r^2}} \partial_t - 2 \sin t \sqrt{1 + r^2} \partial_r - \frac{2 \cos t}{\sqrt{1 + r^2}} \partial_\varphi. \] (3.17)
This is a hint that tells us, according to the conjectured Kerr/CFT correspondence, the near-horizon extremal black holes could be dual to the 2D CFT and the asymptotic symmetry group (ASG) might be applied.
4 Asymptotic Symmetry Group

We now employ the approach of Brown and Henneaux [22] to find the central charge of the holographic dual conformal field theory description of an extremal rotating black hole. Because the Kerr-Newman-NUT black hole is a solution of Einstein-Maxwell theory, it seems that there exists the non-vanishing contributions to the central charge from electromagnetic field besides the metric tensor. First, we assume the non-zero angular momentum $J$. Later in the section 7 we will see the case when $J = 0$ and the second dual CFT such used in [7] will be used to compute the entropy.

4.1 Charges

To compute the charges associated with asymptotic symmetry group (ASG) of near horizon extremal Kerr-Newman-NUT black hole, we should consider all possible contributions from all different fields in the action and we can use the formalism in [28]. Asymptotic symmetries of this solution include diffeomorphisms $\xi$ such that

$$\delta_{\xi}A_{\mu} = L_{\xi}A_{\mu} = \xi^\mu (\partial_{\nu}A_{\mu}) + A_{\nu}(\partial_{\mu}\xi^\nu),$$

$$\delta_{\xi}g_{\mu\nu} = L_{\xi}g_{\mu\nu} = \xi^\sigma(\partial_{\sigma}g_{\mu\nu}) + g_{\mu\sigma}(\partial_{\nu}\xi^\sigma) + g_{\sigma\nu}(\partial_{\mu}\xi^\sigma),$$

as well as the following $U(1)$ gauge transformation $\Lambda$

$$\delta_{\Lambda}A_{\mu} = \partial_{\mu}\Lambda.$$

We denote the metric deviation and the electromagnetic field as $\delta_{\xi}A_{\mu} = a_{\mu}$ and $\delta_{\xi}g_{\mu\nu} = h_{\mu\nu}$. So there are two contributions to the associated charge of the asymptotic symmetry group from the Kerr-Newman-NUT solution, which is the contribution of the metric tensor and the electromagnetic field. So we have

$$Q_{\xi,\Lambda} = \frac{1}{8\pi} \int_{\partial\Sigma} \left( k^g_{\xi}[h; g] + k^A_{\xi,\Lambda}[h, a; g, A] \right),$$

where the integral is over the boundary of a spatial slice. The explicit expressions for the contribution of the metric tensor and electromagnetic field on the central charge respectively are

$$k^g_{\xi}[h; g] = -\frac{1}{4} \epsilon_{\rho\sigma\mu\nu} \left\{ \zeta^\nu D^\mu h - \zeta^\nu D_{\lambda}h^{\mu\lambda} + \frac{h}{2} D^\nu\zeta^\mu - h^{\nu\lambda} D_{\lambda} \zeta^\mu + \zeta_{\lambda} D^\nu h^{\mu\lambda} \right\} dx^\rho \wedge dx^\sigma,$$

$$k^A_{\xi,\Lambda}[h, a; g, A] = \left[ \epsilon_{\alpha\beta\mu\nu} \left\{ \frac{1}{8} \left( -\frac{1}{2} h F^{\mu\nu} + 2 F^{\mu\gamma} h_\gamma^\nu - \delta F^{\mu\nu} \right)(\zeta^\rho A_\rho + \Lambda) - \frac{1}{8} F^{\mu\nu} \zeta^\rho a_\rho \right\} \right] dx^\alpha \wedge dx^\beta,$$

where $\delta F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta}(\partial_{\alpha}a_{\beta} - \partial_{\beta}a_{\alpha})$. We should note that the last two terms in (4.5) as well as in (4.6) vanish for an exact Killing vector and an exact symmetry, respectively.
The charge $Q_{\zeta, \Lambda}$ generates symmetry through the Dirac brackets. The algebra of the asymptotic symmetric group is given by the Dirac bracket algebra of these charges

$$\{Q_{\zeta, \Lambda}, Q_{\bar{\zeta}, \bar{\Lambda}}\}_{DB} = \frac{1}{8\pi} \int \left( k^g_{\zeta} \left[ \mathcal{L}_{\zeta} g; g \right] + k^A_{\zeta} \left[ \mathcal{L}_{\zeta} g, \mathcal{L}_{\zeta} A + d\bar{\Lambda}; g, A \right] \right) = Q_{[\zeta, \Lambda], (\bar{\zeta}, \bar{\Lambda})} + \frac{1}{8\pi} \int \left( k^g_{\zeta} \left[ \mathcal{L}_{\zeta} \bar{g}; \bar{g} \right] + k^A_{\zeta} \left[ \mathcal{L}_{\zeta} \bar{g}, \mathcal{L}_{\zeta} \bar{A} + d\bar{\Lambda}; \bar{g}, \bar{A} \right] \right). \quad (4.7)$$

### 4.2 Boundary Conditions

To use ASG, we need to specify the boundary conditions of the metric and the electromagnetic potential deviations. The boundary conditions are imposed to produce finite charges for both gravitational and electromagnetic parts. Therefore, we adopt the boundary conditions such in most Kerr/CFT correspondence papers for both fields. For metric deviation, we impose the following boundary conditions

$$h_{\mu\nu} \sim \begin{pmatrix} \mathcal{O}(r^2) & \mathcal{O}(\frac{1}{r}) & \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(1) \\ \mathcal{O}(\frac{1}{r}) & \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(\frac{1}{r}) & \mathcal{O}(1) \\ \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(\frac{1}{r}) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}, \quad \text{in the basis } (t, r, \theta, \phi).$$

Then for the electromagnetic potential, we impose the following boundary conditions

$$a_\mu \sim (\mathcal{O}(r), \mathcal{O}(1/r^2), \mathcal{O}(1), \mathcal{O}(1/r)).$$

As in the [7], we append the additional boundary condition because we want to focus only on the central charge that comes from the left-moving Virasoro algebra. So we impose the boundary condition

$$Q_{\partial_t} = E_R = 0. \quad (4.10)$$

In this case, we want the total central charge due to the diffeomorphism transformation of the fields comes from the gravity only, as in the case of near-horizon extremal Kerr-Newman-AdS black holes. So another boundary condition is imposed, i.e.

$$Q_{\Lambda} = 0. \quad (4.11)$$

Hence as we have denoted, in the Dirac bracket calculation of the conserved charges (4.17), we consider the central term which is a result of the left-moving part only.

### 4.3 Central charge

The most general diffeomorphism symmetry that preserves such boundary conditions (4.8) is generated by the vector field

$$\zeta = \left\{ c_t + \mathcal{O}(r^{-3}) \right\} \partial_t + \left\{ -r\epsilon' (\phi) + \mathcal{O}(1) \right\} \partial_r + \mathcal{O}(r^{-1}) \partial_\theta + \left\{ \epsilon (\phi) + \mathcal{O}(r^{-2}) \right\} \partial_\phi, \quad (4.12)$$
where $c_t$ is an arbitrary constant and the prime (') denotes the derivative respect to $\phi$. This asymptotic symmetry group contains one copy of the conformal group of the circle generated by

$$\zeta = e(\phi)\partial \phi - r' e(\phi)\partial r,$$

(4.13)

that will be the part of the NHEKNU metric. We know that the azimuthal coordinate is periodic under the rotation $\phi \sim \phi + 2\pi$ hence we may define $\epsilon_n = -e^{-i\phi}r^2e^{-i\phi}$ and $\zeta = \zeta(\epsilon_n)$. Because of that, the vector field (4.13) becomes

$$\zeta = -e^{-i\phi}\partial \phi - i ne^{-i\phi}\partial r.,$$

(4.14)

By the Lie bracket, the symmetry generator (4.14) satisfy the Virasoro algebra

$$i[\zeta_m, \zeta_n]_{LB} = (m - n)\zeta_{m+n},$$

(4.15)

without the central term because we do not define the quantum version yet. The non-zero metric deviations are

$$h_{tt} = 2i n r^2 \left[ \frac{\chi^2(\theta) - 4a^2M^2\sin^2\theta}{\chi(\theta)} \right] e^{-i\phi},$$

$$h_{rr} = -\frac{2i n \chi(\theta)}{(1 + r^2)^2} e^{-i\phi},$$

$$h_{r\phi} = -\frac{n^2 r \chi(\theta)}{1 + r^2} e^{-i\phi},$$

$$h_{\phi\phi} = \frac{2i n (2a^2 + q^2 + 2al)^2 \sin^2\theta}{\chi(\theta)} e^{-i\phi}.$$

(4.16)

The Dirac bracket of the conserved charges are now just the common forms of the Virasoro algebras with a central term

$$\{Q_\zeta, Q_{\bar{\zeta}}\}_{DB} = Q_{[\zeta, \bar{\zeta}]} + \frac{1}{8\pi} \int k^2 \left[ L_{\bar{\zeta}} \bar{\gamma}; \bar{\gamma} \right],$$

(4.17)

Then by defining

$$Q_\zeta \equiv L_n - \varrho \delta_{n,0},$$

(4.18)

where $\varrho = 3aM/2$ in this case, we obtain the conserved charges algebra in quantum version, such that

$$[L_m, L_n] = (m - n)L_{m+n} + aMm(m^2 - 1)\delta_{m+n,0}.$$  

(4.19)

From that algebra, we can read-off the value of the left-moving central charge, i.e.

$$c_L = 12aM = 12a\sqrt{a^2 + q^2 - l^2}.$$  

(4.20)
5 Temperature

After getting the central charge in the previous section, we need to find the temperature in order to use Cardy formula to obtain the entropy. To find the temperature, we use the analog of the Hartle-Hawking vacuum, i.e. Frolov-Thorne vacuum that have been used in the Kerr/CFT correspondence because the angular momentum is included within this vacuum. When the Hawking temperature is zero, this vacuum is a pure state. But here will be a little bit difference comparing to [6] because there is an emergence of the additional thermodynamics potentials such as electric and magnetic potentials. Both potentials are the conjugate of the electric and magnetic charges, respectively. Now, we will use the first law of the black hole thermodynamics

\[ T_H dS = dM - \Omega_H dJ - \Phi_e dQ_e - \Phi_g dQ_g, \]  

(5.1)

and the extremal condition that satisfies

\[ T_H^{\text{ex}} dS = dM - \Omega_H^{\text{ex}} dJ - \Phi_e^{\text{ex}} dQ_e - \Phi_g^{\text{ex}} dQ_g = 0. \]  

(5.2)

Hence we can obtain

\[ T_H dS = - \left[ (\Omega_H - \Omega_H^{\text{ex}}) dJ + (\Phi_e - \Phi_e^{\text{ex}}) dQ_e + (\Phi_g - \Phi_g^{\text{ex}}) dQ_g \right]. \]  

(5.3)

The electric potential and magnetic potential can be defined from (2.11), which are given by

\[ \Phi_e = \frac{e\hat{r}_+ + g(a + l)}{\hat{r}_+^2 + (a + l)^2}, \]  

(5.4)

\[ \Phi_g = \frac{-g(a + l)}{\hat{r}_+^2 + (a + l)^2}, \]  

(5.5)

where the extremal case means \( \hat{r}_+ = M \). The potentials (5.4) and (5.5) will be useful in the derivation of the temperatures related to the electromagnetic charges. For such constrained variations (5.3), we may write

\[ dS = \frac{dJ}{T_L} + \frac{dQ_e}{T_e} + \frac{dQ_g}{T_g}, \]  

(5.6)

We consider the quantum scalar field with eigenmodes of the asymptotic energy \( E \) and angular momentum \( J \), which are given by the following form

\[ \tilde{\Phi} = \sum_{E,J,s} \tilde{\phi}_{E,J,s} e^{-iE t + iJ \tilde{\phi} f_s(\hat{r}, \theta)}. \]  

(5.7)

for the Kerr black hole. In order to transform this to near-horizon quantities and take the extremal limit, we note that in the near-horizon coordinates (3.1) we have

\[ e^{-iE t + iJ \tilde{\phi}} = e^{-i(E - \Omega_H^{\text{ex}} J) \hat{\tau}_0 / \lambda + iJ \tilde{\phi}} = e^{-in \hat{\tau} + in \hat{\phi}}, \]  

(5.8)
where

\[ n_R = -(E - \Omega_{H}J) r_0 / \lambda, \quad n_L = J. \quad (5.9) \]

But this is only suitable when there is no contribution of the electromagnetic potential. So in our case, using the fact that there are potentials as the conjugates of the electric and magnetic potentials, we may extend Eq. (5.9) to

\[ n_R = -(E - \Omega^e_H J - \Phi^e_J Q_e - \Phi^g_J Q_g) r_0 / \lambda, \quad n_L = J. \quad (5.10) \]

Hence the density matrix in the asymptotic energy and angular momentum eigenbasis now has the Boltzmann weighting factor

\[ e^{-\left( e^{-n_H l} - \Phi^e_J Q_e - \Phi^g_J Q_g \right) \lambda} = e^{-n_R \lambda} - n_L \lambda - Q_e \lambda \lambda - Q_g \lambda \lambda. \quad (5.11) \]

If we take the trace over the modes inside the horizon, the Boltzmann weighting factor will be a diagonal matrix. We can compare the Eqs. (5.10) and (5.11) to obtain the definition of the temperatures of the CFT, such that

\[
\begin{align*}
T_R & = \frac{T_H r_0}{\lambda} \bigg|_{ex}, \\
T_L & = -\frac{\partial T_H / \partial \hat{\tau}^+}{\partial \Omega_H / \partial \hat{\tau}^+} \bigg|_{ex}, \\
T_e & = -\frac{\partial T_H / \partial \hat{\tau}^+}{\partial \Phi^e_J / \partial \hat{\tau}^+} \bigg|_{ex}, \\
T_g & = -\frac{\partial T_H / \partial \hat{\tau}^+}{\partial \Phi^g_J / \partial \hat{\tau}^+} \bigg|_{ex}. \end{align*} \quad (5.12)
\]

We know that for the extremal black holes, the Hawking temperature \( T_H \) will vanish but not all of the temperatures from the CFT will also vanish. We can see that only the right-moving temperature which is equal to zero in the extremal case and the others finally become

\[
\begin{align*}
T_L & = \frac{M^2 + (a + l)^2}{4\pi a M}, \\
T_e & = \frac{M^2 + (a + l)^2}{2\pi (2gM(a + l) + e \{ M^2 - (a + l)^2 \})}, \\
T_g & = -\frac{M^2 + (a + l)^2}{4\pi g M (a + l)}. \quad (5.13)
\end{align*}
\]

We can also write the left-moving temperature in the form \( T_L = 1/2\pi k \) where \( k \) is constant that is obtained in section 3. The Hartle-Hawking vacuum state is generalized around the extremal Kerr-Newman-NUT black hole with a density matrix given by

\[ \rho = e^{-\frac{l}{T_L} - \frac{Q_e}{T_e} - \frac{Q_g}{T_g}}. \quad (5.14) \]

Because of the boundary CFT is dual to the NHEKNUT black hole, the dual of this black hole is described by the CFT in the mixed state (5.14).
6 Entropy from CFT

6.1 Cardy formula

To compute the entropy, we use the famous Cardy formula that comes from the CFT. This entropy has the general form

\[ S = 2\pi \left( \sqrt{\frac{c_L E_L}{6}} + \sqrt{\frac{c_R E_R}{6}} \right), \]  

(6.1)

where \( E_L, E_R \) are the eigen energies of the operator \( L_0, \bar{L}_0 \). In order to obtain the entropy as a function of temperature as used in the Kerr/CFT correspondence, we use the thermodynamic relation \( dE = TdS \), such that

\[ S = \frac{\pi^2}{3} (c_L T_L + c_R T_R), \]  

(6.2)

For the near-horizon extremal Kerr-Newman-NUT solution, there are also \( T_e \) and \( T_g \) that actually have to be taken into account to the Cardy entropy. But in section 4, we have assumed that the charge from the electromagnetic field contribution is zero. Because the right-moving central charge is also zero and by using the central charge (4.20) and left-moving temperature in Eq. (5.13), the microscopic entropy of NHEKNUT black hole is then

\[ S = \frac{\pi^2}{3} c_L T_L = \pi (2a^2 + q^2 + 2al) = \frac{\text{Area}_4}{4}. \]  

(6.3)

It precisely agrees with the macroscopic Bekenstein-Hawking entropy (2.8).

6.2 Logarithmic correction

Here we show that in the entropy as shown in [20] especially for the NHKNNUT black hole. The general form of this logarithmic correction from the CFT is shown in [21] that we use here. In Cardy formula, the corrections come from the addition of the lowest eigenvalue \( E_{L0} \) of the conformal operator \( L_0 \), that often but not always has a zero value. The choice of boundary conditions actually affects the result, but in general as it is imposed in the section 4, the resulting Virasoro algebra generates the central charge and the eigenvalue of \( L_0 \) such as the following, respectively

\[ c_L = \frac{3\text{Area}_4 \beta}{2\pi \kappa}, \quad E_L = \frac{\text{Area}_4 \kappa}{16\pi \beta}, \]  

(6.4)

where \( \kappa \) is a surface gravity and \( \beta \) is an undetermined periodicity. It is clearly seen that we still only focus on the left-moving part and we have assumed \( E_{L0} << c_L \) in the derivation. Thus the corrected entropy is given by

\[ S^{\text{corr}} \sim S - \frac{3}{2} \ln S + \ln c_L + \text{const}. \]  

(6.5)

The central charge \( c_L \) is actually universal in the sense of being independent of the area of the black holes. We can make the third term above considered as arbitrary constant too, so
this will match (2.9) as the corrected Bekenstein-Hawking entropy. If we take into account
this logarithmic correction to NHEKNUT metric, we will have
\[
S_{\text{corr}} \sim \pi (2a^2 + q^2 + 2al)^{\frac{3}{2}} \ln \left[ \pi (2a^2 + q^2 + 2al) \right] + \text{const.} \quad (6.6)
\]

7 Reissner-Nordstrom-NUT solution in 5D

In the NHEKNUT metric we find, when the momentum angular \( J \) approaches 0, it causes
the central charge to vanish as well because its value is proportional to \( J \) and produces
the extremal Reissner-Nordström-NUT solution. While the left-moving temperature will be
singular because it is proportional to \( 1/J \). This is the same as what happened to
Kerr-Newman-AdS black hole. However, the resulting microscopic entropy also remains
in accordance with the Bekenstein-Hawking entropy because angular momentum values in
the central charge and temperature cancel each other. We do not want this to happen,
so we need another way to get the microscopic entropy from the finite central charge and
temperature.

As mentioned earlier, Hartman et.al. [7] face the same case and manage to obtain
the finite value of the microscopic entropy by using the second dual CFT. Here, we will
adopt the same method as they do, which is actually also done by Ghodsi and Garousi
[12]. First we need to add a new coordinate representing the fifth dimension of \( S^1 \) which
has property \( z \sim z + 2\pi R_n \) where \( R_n \) is an integer. The addition of this fibered coordinate
produces one additional Killing vector \( \partial_z \) which becomes \( U(1) \) symmetry in addition to
\( SL(2,R)_R \times U(1)_L \). Furthermore, we can also make the electromagnetic potential as a
component of the geometry as in the string theory where the electromagnetic potential can
be mapped to a geometric part [29]. This is actually a black holes solution that exists in
Kaluza-Klein theory [30] where dimensional reduction from 5D to 4D will produce dilaton
and electromagnetic fields in the four dimensional metric. The holographic dual of Kaluza-
Klein black hole is also studied in [31].

When the limit \( J \rightarrow 0 \), not only the left-moving temperature will be singular but also
the electromagnetic potential but this singularity can be eliminated by taking a certain
gauge transformation. To obtain the entropy of the five dimensional extremal Reissner-
Nordström-NUT black hole, we use the global form metric (3.14) included its electromag-
etic potential (3.15). For the electromagnetic potential in this solution, we choose the
following gauge transformation
\[
A \rightarrow A = A_{\mu} dx^\mu = \frac{2gMl + e(M^2 - l^2)}{2aM} d\phi. \quad (7.1)
\]
Hence when we take \( a = 0 \), the electromagnetic potential is then given by
\[
A = A_{\mu} dx^\mu = krdt + \frac{(gM - el)\cos\theta}{M} d\phi. \quad (7.2)
\]
where for Reissner-Nordström-NUT solution, the constant \( k = \frac{2gMl + e(M^2 - l^2)}{(M^2 + l^2)} \). This electromagnetic potential (7.2) can be the part of the geometry of the Reissner-
Nordström-NUT solution. Furthermore, the metric now is in the form
\[
ds_5^2 = ds_4^2 + (dz + A)^2, \quad (7.3)
\]
where the four dimensional metric, that we obtain from (3.14) and take \( a = 0 \), is
\[
\begin{align*}
  ds_4^2 &= \chi \left[ -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + d\theta^2 \right] + \frac{r_0^2 M^2 \sin^2 \theta}{\chi} d\phi^2, \\
  \text{(7.4)}
\end{align*}
\]
and herein the section \( r_0 = q^2/M \) and \( \chi = M^2 + \ell^2 \). As mentioned in the preceding few paragraphs, this method is used to obtain the extremal Reissner-Nordström-AdS geometry. Next, we will prove that the entropy of this black hole, using the second dual CFT, will correspond to the following Bekenstein-Hawking entropy
\[
  S_{BH} = \pi q^2. \quad \text{(7.5)}
\]

7.1 Central charge

Just like in the section 4, we do need to compute the central charge to find the entropy as well as the temperature KNUT one where the central charge comes from the five dimensional gravity only, so we can neglect the contribution of the electromagnetic potential. We have to note that in 5D gravity, the gravitational constant is \( G_5 = 2\pi G \) where we set the natural unit \( G = 1 \).

To find the non-trivial diffeomorphisms, we need to set some consistent boundary conditions of the metric deviation as in 4D solution. Here we adopt the same boundary condition such in [7]
\[
  h_{\mu\nu} \sim \begin{pmatrix}
  \mathcal{O}(r^2) & \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(\frac{1}{r}) & \mathcal{O}(1) \\
  \mathcal{O}(\frac{1}{r}) & \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(\frac{1}{r}) & \mathcal{O}(\frac{1}{r}) \\
  \mathcal{O}(\frac{1}{r^2}) & \mathcal{O}(1) & \mathcal{O}(\frac{1}{r}) & \mathcal{O}(\frac{1}{r}) \\
  \mathcal{O}(\frac{1}{r}) & \mathcal{O}(1) & \mathcal{O}(1) & \mathcal{O}(1)
\end{pmatrix},
\]
in the basis \((t, r, \theta, \phi, z)\). Hence the most general diffeomorphisms are of the form
\[
  \zeta = \{ b_t + \mathcal{O}(r^{-3}) \} \partial_t + \{ -r\epsilon(z) + \mathcal{O}(1) \} \partial_r + \mathcal{O}(r^{-1}) \partial_\theta \\
  + \{ b_\phi + \mathcal{O}(r^{-2}) \} \partial_\phi + \{ \epsilon(z) + \mathcal{O}(r^{-2}) \} \partial_z,
\]
where \( \epsilon(z) = -e^{-inz} \) and the prime (′) denotes the derivative respect to \( z \). Here \( b_t, b_\phi \) are arbitrary constants. We can see that the boundary conditions (7.6) allow
\[
  \zeta_z = \epsilon(z) \partial_z - r\epsilon(z) \partial_z,
\]
but do not allow \( \zeta_\epsilon \) such the case of 4D NHEKNUT geometry. Furthermore, to compute the central charge, we may follow the same steps such in section 4 but for five dimensional gravity [12], we have
\[
  k^\beta_\zeta[h; g] = -\frac{1}{2} \frac{1}{3!} \xi_{\rho\gamma\mu\nu} \left\{ \zeta^\nu D^\mu h - \zeta^\nu D_\lambda h^{\mu\lambda} + \frac{h}{2} D^\nu \zeta^\mu - h^{\nu\lambda} D_\lambda \zeta^\mu + \zeta_\lambda D^\nu h^{\mu\lambda} \\
  + \frac{h}{2} (D^\mu \zeta_\lambda + D_\lambda \zeta^\mu) \right\} dx^\rho \wedge dx^\sigma \wedge dx^\tau,
\]
\[
\text{(7.9)}
\]
where the last two terms vanish for the exact Killing vector. Finally, the central charge is given by

$$c_z = 6k \chi, \quad (7.10)$$

that is associated to $\zeta_z$.

### 7.2 Temperature

We are left only by the temperatures conjugate to electric and magnetic charges. Hence from the first law of black hole thermodynamics, we have

$$dS = \frac{dQ_e}{T_e} + \frac{dQ_g}{T_g}. \quad (7.11)$$

As the second dual CFT for extremal Reissner-Nordström-AdS solution, the magnetic charge is held fixed. So now the vacuum is a pure state. The remaining temperature is given by

$$T_e = \frac{1}{2\pi k} = \frac{M^2 + l^2}{2\pi [2gML + e(M^2 - l^2)]}. \quad (7.12)$$

This can easily be obtained by setting $a = 0$ in the temperature (5.12).

### 7.3 Entropy

Finally, we can compute the microscopic entropy of the extremal Reissner-Nordström-NUT black hole by using Cardy formula

$$S_{RN\text{NUT}} = \frac{\pi^2}{3} c_z T_e = \pi (M^2 + l^2) = \pi q^2. \quad (7.13)$$

In the five dimensional gravity theory, the entropy of the Reissner-Nordström-NUT black hole solution admits this relation

$$S_{BH} = \frac{\text{Area}_5}{4G_5} = \frac{\text{Area}_4 L}{4(2\pi)} = \pi (M^2 + l^2) = \pi q^2, \quad (7.14)$$

where $L$ is the length of the fifth (fibered) coordinate which is $2\pi$. So the entropy from the CFT for this 5D solution also matches the Bekenstein-Hawking entropy. If we take into account the logarithmic correction, the entropy is then

$$S_\text{corr} \sim \pi q^2 - \frac{3}{2} \ln (\pi q^2) + \text{const.} \quad (7.15)$$

### 8 Concluding Remarks

In this letter, we have studied the duality between the NHEK NUT black hole and the CFT on its boundary by using the Kerr/CFT correspondence. We choose this black hole solution because it has NUT charge that makes the solution is not asymptotically flat and can be considered as the gravomagnetic monopole parameter or a twist parameter of the
surrounding space-time and it is more general than the Kerr-Newman solution. In order to study the duality, first we have obtained the NHEKNUT solution in three types of coordinate which all of this solutions have the same isometry, i.e. $SL(2, R) \times U(1)$, so ASG can be applied to find the central charge. Before computing the microscopic entropy, the Frolov-Thorne vacuum is considered and extended to add the additional contribution of the electromagnetic potential to obtain the temperatures. Finally, it is found that the entropy of NHEKNUT black hole,

$$S = \pi(2a^2 + q^2 + 2al),$$

still agrees with the Bekenstein-Hawking entropy and we have proved once again that the Kerr/CFT correspondence is true at this point. We have showed also the logarithmic correction of the entropy coming from the CFT for this black hole.

A special case occurs when the angular momentum $J$ vanishes. It makes NHEKNUT solution becomes the extremal Reissner-Nordström-NUT solution and produces finite entropy but with the singular left-moving temperature and zero central charge. It makes us use the second dual CFT to prove that it still have to produce finite and the non-zero central charge and temperature to obtain the same entropy with the Bekenstein-Hawking one. We add the fifth fibered coordinate to extend the solution to 5D geometry and make the use of the electromagnetic potential as the part of the geometry too. After getting the 5D solution, we use the same method as we use from NHEKNUT to find the central charge and the temperature to compute the entropy. At the end, we prove that the microscopic entropy,

$$S_{RN\text{NUT}} = \pi q^2,$$

that comes from the second dual CFT, matches the Bekenstein-Hawking entropy in general. The origin of NUT charge is still interesting to be understood and herein we know that Kerr/CFT correspondence do succeed again to study the microscopic origin of the entropy of the rotating NUT black holes.

As future work, we want to study the Kerr/CFT correspondence for the rotating Kiselev black hole solution. This solution adds a new parameter that represents the existence of quintessential dark energy around the black hole. We think that it maybe useful in astrophysics since there exist the dark energy.

**Acknowledgments**

We gratefully acknowledge support from Ministry of Research, Technology, and Higher Education of the Republic of Indonesia and PMDSU Research Grant 2017. M.F.A.R.S. also thanks all members of Theoretical Physics Laboratory, Institut Teknologi Bandung for the valuable support.
The electromagnetic potential for the degenerate horizon

For the degenerate horizon, the electromagnetic potential is not regular. To solve it, we can expand the electromagnetic potential such that

\[
A = A_\mu dx^\mu = A_t dt + A_\phi d\phi
\]

\[
= \left( A_t \big|_{\hat{r}=\hat{r}_+} + \frac{\partial A_t}{\partial \hat{r}} \big|_{\hat{r}=\hat{r}_+} \lambda r_0 y \right) \frac{r_0}{\lambda} d\tau + \left( A_\phi \big|_{\hat{r}=\hat{r}_+} + \frac{\partial A_\phi}{\partial \hat{r}} \big|_{\hat{r}=\hat{r}_+} \lambda r_0 y \right) \left( d\phi + \Omega_H \frac{r_0}{\lambda} d\tau \right)
\]

\[
= -\Phi_H \frac{r_0}{\lambda} d\tau + \left( \frac{\partial A_\phi}{\partial \hat{r}} \big|_{\hat{r}=\hat{r}_+} + \Omega_H \frac{\partial A_\phi}{\partial \hat{r}} \big|_{\hat{r}=\hat{r}_+} r_0^2 y d\tau + \left( A_\phi \big|_{\hat{r}=\hat{r}_+} + \frac{\partial A_\phi}{\partial \hat{r}} \big|_{\hat{r}=\hat{r}_+} \lambda r_0 y \right) d\phi. \tag{A.1}
\]

So if we want to take the limit \( \lambda \to 0 \), we have to use the gauge transformation

\[
A \to A + \Phi_H \frac{r_0}{\lambda} d\tau, \tag{A.2}
\]

to remove the singularity.

References

[1] J. F. Plebanski and M. Demianski, Rotating, charged, and uniformly accelerating mass in general relativity, Ann. Phys. 98 (1976) 98.

[2] A. H. Taub, Empty space-times admitting a three parameter group of motions, Ann. Math. 53 (1951) 472.

[3] E. Newman, L. Tamburino, and T. Uitti, Empty-space generalization of the Schwarzschild metric, J. Math. Phys. 4, (1963) 915.

[4] D. Lynden-Bell and M. Nouri-Zonoz, Classical monopoles: Newton, NUT space, gravomagnetic lensing, and atomic spectra, Rev. Mod. Phys. 70 (1998) 427 [arXiv:gr-qc/9612049].

[5] A. Al-Badawi and M. Halilsoy, On the physical meaning of the NUT parameter, Gen. Relativ. Gravit. 38 (2006) 1729.

[6] M. Guica, T. Hartman, W. Song and A. Strominger, The Kerr/CFT correspondence, Phys. Rev. D 80 (2009) 124008 [arXiv:0809.4266].

[7] T. Hartman, K. Murata, T. Nishioka and A. Strominger, CFT duals for extreme black holes, JHEP 04 (2009) 019 [arXiv:0811.4393].

[8] A. M. Ghezelbash, Kerr/CFT correspondence in the low energy limit of heterotic string theory, JHEP 08 (2009) 045 [arXiv:0901.1670v2].

[9] H. Lü, J. Mei and C.N. Pope, Kerr-AdS/CFT correspondence in diverse dimensions, JHEP 04 (2009) 054 [arXiv:0811.2225].

[10] R. Li and JR Ren, Holographic dual of linear dilaton black hole in Einstein - Maxwell - dilaton - axion gravity, JHEP 09 (2010) 039 [arXiv:1009.3139].

[11] D. Anninos and T. Hartman, Holography at an extremal de Sitter horizon, JHEP 03 (2010) 096 [arXiv:0910.4587].
A. Ghodsi and M.R. Garousi, The RN/CFT correspondence, Phys. Lett. B 687 (2010) 79 [arXiv:1009.3139].

A. M. Ghezelbash, Kerr-Bolt spacetimes and Kerr/CFT correspondence, Mod. Phys. Lett. A 27 (2012) 1250046 [arXiv:0902.4662].

M. Astorino, Microscopic entropy of the magnetised extremal Reissner-Nordstrom black hole, JHEP 10 (2015) 016 [arXiv:1507.04347].

M. Astorino, Magnetised Kerr/CFT correspondence, Phys. Lett. B 751 (2015) 96 [arXiv:1508.01583v4].

H. M. Siahaan, Magnetized Kerr/CFT correspondence, Class. Quantum Grav. 33 (2016) 155013 [arXiv:1508.01152].

M. Astorino, CFT duals for accelerating black holes, Phys. Lett. B 760 (2016) 393 [arXiv:1605.06131v3].

M. Sinamuli and R. B. Mann, Super-entropic black holes and the Kerr-CFT correspondence, JHEP 08 (2016) 148 [arXiv:1512.07597].

G. Compère, The Kerr/CFT correspondence and its extensions, Living Rev. Relativ. 20 (2017) 1 [arXiv:1203.3561v4].

R. K. Kaul and P. Majumdar, Logarithmic correction to the Bekenstein-Hawking entropy, Phys. Rev. Lett. 84 (2000) 5255 [arXiv:gr-qc/0002040v3].

S. Carlip, Logarithmic corrections to black hole entropy, from the Cardy formula, Class. Quantum Grav. 17 (2000) 4175 [arXiv:gr-qc/0005017v3].

J.D. Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity, Commun. Math. Phys. 104 (1986) 207.

J. Podolský and H. Kadlecová, Radiation generated by accelerating and rotating charged black holes in (anti-)de Sitter space, Class. Quantum Grav. 26 (2006) 105007 [arXiv:0903.3577v1].

K. Jan and H. Gohar, Hawking radiation of scalars from accelerating and rotating black holes with NUT parameter, Astrophys. Space. Sci. 350 (2014) 279 [arXiv:1304.5963v3].

J. Bičák and F. Hejda, Near-horizon description of extremal magnetized stationary black holes and Meissner effect, Phys. Rev. D 92 (2015) 104006 [arXiv:1510.01911v2].

J. Bardeen and G. T. Horowitz, The extreme Kerr throat geometry: a vacuum analog of AdS2×S2, Phys. Rev. D 60 (1999) 104030 [arXiv:hep-th/9905099].

J. Mei, The entropy for general extremal black holes, JHEP 04 (2010) 005 [arXiv:1002.1349].

G. Barnich and F. Brandt, Covariant theory of asymptotic symmetries, conservation laws and central charges, Nuc. Phys. B 633 (2002) 3 [arXiv:hep-th/0111246v2].

M. Guica and A. Strominger, Wrapped M2/M5 duality, JHEP 10 (2009) 036 [arXiv:hep-th/0701011].

G.T. Horowitz and T. Wiseman, General black holes in Kaluza-Klein theory, arXiv:1107.5563v2.

T. Azeyanagi, N. Ogawa and S. Terashima, Holographic duals of Kaluza-Klein black holes, JHEP 04 (2009) 061 [arXiv:0811.4177v5].