Finite Temperature Correlation Functions in Integrable QFT

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Abstract

Finite temperature correlation functions in integrable quantum field theories are formulated only in terms of the usual, temperature-independent form factors, and certain thermodynamic filling fractions which are determined from the thermodynamic Bethe ansatz. Explicit expressions are given for the one and two-point functions.
I. INTRODUCTION

Integrable quantum field theories in 1+1 space-time dimensions are fundamentally characterized by their factorizable S-matrix \cite{1}. The S-matrix along with certain bootstrap axioms in principle characterize the matrix elements of fields in the space of asymptotic multi-particle states (Form Factors) \cite{2,3}. Finally, the correlation functions at zero temperature can be characterized by infinite integral representations using the form factors. The S-matrix also governs the thermodynamics of the system through the Thermodynamic Bethe Ansatz (TBA) \cite{4}.

For many applications, especially in Condensed Matter Physics (see, for instance \cite{6}), the finite temperature correlation functions are of importance. In the Matsubara imaginary time formalism \cite{7}, finite temperature correlation functions can be viewed as correlation functions on an infinite cylindrical geometry, where the spacial coordinate $-\infty < x < \infty$ runs along the length of the cylinder and the euclidean time lives on a circle of radius $R = 1/T$, where $T$ is the temperature. In a picture where the hamiltonian evolution is along the circumference of the cylinder, let us call it the $R$-channel, the Hilbert space lives on the infinite line of the coordinate $x$, thus the states of the Hilbert space one has to sum over in performing thermodynamic averages have the usual multi-particle description in infinite volume. Deceptively simple as it may appear, this is an important observation, since it implies that in this picture, the matrix elements of operators do not depend on $R$, and can be expressed then by the usual, temperature-independent form factors. Formally, one has

\[
\langle \mathcal{O} \rangle_R = \frac{1}{Z} \text{Tr} \left( e^{-RH} \mathcal{O} \right) \quad (R \text{- channel}) \tag{1.1}
\]

where $Z$ is the partition function. In this paper we use the $R$-channel to formulate the finite temperature correlation functions, the main goal being to use the Thermodynamic Bethe Ansatz to make sense of (1.1). We show that the correlation functions can be characterized using only the usual form factors and some thermodynamic data, the so-called filling fractions, which are available from the TBA. We present explicit integral representations for the one and two point correlation functions.
Our formulation should be contrasted to the orthogonal picture, where hamiltonian evolution is viewed as along the length of the cylinder (\( x \)-direction). In this “L-channel”, the Hilbert space is on the finite volume \( R \), and the computation of correlation functions does not involve the sum over thermodynamic states but rather involves the notion of the ground state \(|0_R\rangle\) on the circle of radius \( R \)

\[
\langle O \rangle_R = \langle 0_R | O | 0_R \rangle \quad \text{(L – channel)}
\]

Presently, the precise structure of this ground state has not been identified and moreover, in this \( L \)-channel, matrix elements of operators may depend on \( R \) in a complicated unknown way. There has been however some progress in this direction by Smirnov. In fact, in a series of papers \[8\], he studied properties of such matrix elements in a semi-classical limit.

II. ONE-POINT FUNCTIONS

Since the main features of the finite temperature calculation of correlation functions are already present in the simplest case of the one–point function of a quantum field operator \( O \), let us start the discussion from the analysis of this case. Throughout this paper we let \( R = 1/T \), where \( T \) is the temperature. To simplify the following discussion, we assume that in the field theory in question there is one kind of particle \( A \) of mass \( m \). Generalization to other cases is quite easy. Multi-particle states are denoted as \(|\theta_1, \ldots, \theta_n\rangle\), where \( \theta \) is the rapidity parameterizing the energy and momentum

\[
e(\theta) = m \text{ch}\theta, \quad k(\theta) = m \text{sh}\theta.
\]

Using the resolution of the identity,

\[
1 = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \langle \theta_1, \ldots, \theta_n | \langle \theta_n, \ldots, \theta_1 \rangle.
\]

the one-point function of a local field \( O(x,t) \) at finite temperature has the formal representation.
\begin{equation}
\langle O(x,t) \rangle_R = \frac{1}{Z} \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \left( \prod_{i=1}^{n} e^{-e(\theta_i) R} \right) \langle \theta_n, \cdots, \theta_1 | O(0,0) | \theta_1, \cdots, \theta_n \rangle,
\end{equation}

where \( Z = \text{Tr} \left( e^{-RH} \right) \) is the partition function. From translation invariance, the above quantity does not depend on \( x, t \) and is only a function of the scaling variable \( mR \).

Our aim is to show that the previous expression can actually be organized in a way which better reveals its physical content and which moreover provides an efficient method for its actual computation. In addition, we will also show that the final form of the one-point functions may be regarded as a generalization of the formula obtained in the case of a free theory, once an appropriate dictionary between the interacting and free theories has been established. For this reason, let us initially consider the case of a free fermionic model\(^1\), i.e. an integrable model with S-matrix \( S = -1 \). This case was originally studied in [9]. For the following considerations, we need in particular to recall the main findings reported in the appendix A of [9].

**A. Main results in the free case**

In carrying out the sum over states it is important to include \( \delta \)-function contributions to the form factors as follows. Let \( A = \{ \theta_n \cdots \theta_1 \} \) and \( B = \{ \theta'_1 \cdots \theta'_m \} \) denote some ordered sets of rapidities. Then the form factor is expressed as a sum over all ways of breaking up \( A \) and \( B \) into two sets:

\begin{equation}
\langle A | O(x,t) | B \rangle = \sum_{A=A_1 \cup A_2; B=B_1 \cup B_2} S_{A,A_1} S_{B,B_1} \langle A_2 | B_2 \rangle \langle A_1 | O(x,t) | B_1 \rangle_{\text{conn}},
\end{equation}

where \( S_{A,A_1} \) are S-matrix factors required to bring \( | A \rangle \) into the order \( | A_2, A_1 \rangle \), i.e. \( \langle A | = S_{A,A_1} \langle A_2, A_1 | \), and similarly for \( | B \rangle \). The inner products \( \langle A_2 | B_2 \rangle \) are most easily evaluated by introducing free particle creation-annihilation operators \( | \theta_1 \cdots \theta_n \rangle = A^\dagger(\theta_1) \cdots A^\dagger(\theta_n)|0\rangle \),

\(^1\)In this paper we consider only “fermionic” theories with \( S(\theta = 0) = -1 \). The formal extension to bosonic theories with \( S(\theta = 0) = 1 \) is a simple exercise.
and using \( \{ A(\theta), A^\dagger(\theta') \} = 2\pi \delta(\theta - \theta') \); they are sums of products of \( \delta \)-functions. The “connected” piece of the form factor \( \langle A_1 | \mathcal{O}(x, t) | B_1 \rangle_{\text{conn}} \) is defined to be the form factor with no overlap, i.e. \( \langle A_1 | B_1 \rangle = 0 \). The crossing relation is thus valid and one can define

\[
\langle \theta_n \cdots \theta_1 | \mathcal{O} | \theta'_1 \cdots \theta'_m \rangle_{\text{conn}} \equiv \text{FP} \left( \lim_{\eta_i \to 0} \langle 0 | \mathcal{O} | \theta'_1 \cdots \theta'_m, \theta_n - i\pi + i\eta_n, \cdots, \theta_1 - i\pi + i\eta_1 \rangle \right)
\]

(2.5)

where \( \text{FP} \) in front of the expression means taking its finite part, i.e. terms proportional to \((1/\eta_i)^p\), where \( p \) is some positive power, and also terms proportional to \( \eta_i/\eta_j, i \neq j \) are discarded in taking the limit. With this prescription the resulting expression is independent of the way in which the above limits are taken, and it is therefore the only quantity which has an unambiguous physical meaning. This way of taking the limit has been already used in the literature (see [10]).

There are two effects of the \( \delta \)-function terms coming from \( \langle A_2 | B_2 \rangle \) when \( A \neq A_1 \). The first effect is due to terms involving \( \delta(\theta - \theta')^2 \); these give rise to an overall factor which sums up to the partition function \( Z \), and therefore cancels out from the final result. The other effect of the \( \delta \)-functions is to give rise to a “filling-fraction” \( f(\theta) \) for integrations over rapidity. The final expression of the one–point functions is then given by

\[
\langle \mathcal{O}(x, t) \rangle_R = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \left( \prod_{i=1}^{n} f(\theta_i) e^{-\varepsilon(\theta_i)} \right) \langle \theta_n \cdots \theta_1 | \mathcal{O}(0) | \theta_1 \cdots \theta_n \rangle_{\text{conn}},
\]

(2.6)

where here \( \varepsilon = c(\theta) R \), the connected form factor is defined in (2.3), and

\[
f(\theta) = \frac{1}{1 + e^{-\varepsilon(\theta)}}.
\]

(2.7)

In the following it will be convenient to introduce the functions

\[
f_\sigma(\theta) = \frac{1}{1 + e^{-\sigma \varepsilon(\theta)}}.
\]

(2.8)

so that \( f = f_+ \) and \( f e^{-\varepsilon} = f_- \). As we will show below, the expression (2.6) for the one–point function is the one which generalizes to the interacting case.
B. Interacting case. Quasi–particle excitations

Let us now turn to the interacting case. Again, for simplicity, we assume the theory has a spectrum given by a single particle $A$ of mass $m$ with an S-matrix $S(\theta)$. We define

$$\sigma(\theta) = -i \log S(\theta), \quad \varphi(\theta) = -i \frac{d}{d\theta} \log S(\theta) .$$

The partition function at a finite temperature $T$ and on a volume $L$ (for $L \to \infty$) is determined by means of the Thermodynamic Bethe Ansatz equations as follows [4,5]. In a box of large volume $L$, $0 < x < L$, the quantization condition of the momenta is given by

$$e^{ik(\theta_i) L} \prod_{j \neq i} S(\theta_i - \theta_j) = 1,$$

which can be equivalently expressed as

$$mL \text{sh} \theta_i + \sum_{j \neq i} \sigma(\theta_i - \theta_j) = 2\pi n_i ,$$

where $n_i$ are integers. Introducing a density of occupied states per unit volume $\rho_1(\theta)$ as well as a density of levels $\rho(\theta)$, in the thermodynamic limit eq. (2.10) becomes

$$2\pi \rho = e + 2\pi \varphiLebesgue \rho_1 ,$$

where $(f * g)(\theta) = \int_{-\infty}^{\infty} d\theta' f(\theta - \theta') g(\theta')/2\pi$. Defining the pseudo-energy $\varepsilon(\theta)$ as

$$\frac{\rho_1}{\rho} = \frac{1}{1 + e^\varepsilon} ,$$

the minimization of the free-energy with respect to the densities of states leads to the integral equation

$$\varepsilon = eR - \varphi * \log(1 + e^{-\varepsilon}) ,$$

and the partition function is then given by

$$Z(L, R) = \exp \left[ mL \int \frac{d\theta}{2\pi} \text{ch} \theta \log \left(1 + e^{-\varepsilon(\theta)}\right) \right] .$$

The interesting point is that the above partition function can be interpreted as one of a free gas of fermionic particles but with energy given by $\varepsilon(\theta)/R$. Namely, there is a one–to–one correspondence between the above expression (2.14) and a partition function computed according to the following sum.
\[ Z(L, R) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \langle \theta_n \cdots \theta_1 | \theta_1 \cdots \theta_n \rangle \prod_{i=1}^{n} e^{-\varepsilon(\theta_i)} , \]  

(2.15)

where the scalar products of the states are computed by applying the standard free fermionic rules. To see this, let us define

\[ F(R) = \int \frac{d\theta}{2\pi} \text{ch} \theta \log \left( 1 + e^{-\varepsilon(\theta)} \right) , \]  

(2.16)

which admits the series expansion

\[ F(R) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} I_n(R) , \]  

(2.17)

where

\[ I_n(R) \equiv \int \frac{d\theta}{2\pi} \text{ch} \theta e^{-n\varepsilon(\theta)} . \]  

(2.18)

The partition function obtained by the TBA equation has the following expansion in powers of \((mL)\)

\[ Z(L, R) = 1 + (mL)F(R) + \frac{(mL)^2}{2!}(F(R))^2 + \cdots \frac{(mL)^n}{n!}(F(R))^n + \cdots \]  

(2.19)

Let us now compute the partition function by using the other expression (2.15), with a regularization of the squares of the \(\delta\)-functions appearing in \(\langle \theta_n \cdots \theta_1 | \theta_1 \cdots \theta_n \rangle\) as for a free fermionic theory on a sufficiently large volume \(L\):

\[ [\delta(\theta - \theta')]^2 \equiv \frac{mL}{2\pi} \text{ch}(\theta) \delta(\theta - \theta') . \]  

(2.20)

Hence

\[ Z(L, R) = 1 + Z_1 + Z_2 + \cdots Z_n + \cdots \]  

(2.21)

where for the first terms we have

\[ Z_1 = \int \frac{d\theta}{2\pi} \langle \theta | \theta \rangle e^{-\varepsilon(\theta)} = \int \frac{d\theta}{2\pi} d\theta' \delta(\theta - \theta') \langle \theta' | \theta \rangle e^{-\varepsilon(\theta)} = \]  

\[ = mL \int \frac{d\theta}{2\pi} \text{ch}(\theta) e^{-\varepsilon(\theta)} = (mL) I_1 ; \]
\[ Z_2 = \frac{1}{2} \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} <\theta_2 \theta_1 | \theta_1 \theta_2 > e^{-\varepsilon(\theta_1) - \varepsilon(\theta_2)} = \frac{1}{2} \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \left[(2\pi)^2 (\delta(\theta_1 - \theta_1)\delta(\theta_2 - \theta_2) + 
abla(\theta_1 - \theta_2)\delta(\theta_2 - \theta_1))\right] e^{-\varepsilon(\theta_1) - \varepsilon(\theta_2)} = \frac{1}{2} (mL)^2 I_1^2 - \frac{1}{2} (mL) I_2, \quad (2.23) \]

and for the next few

\[ Z_3 = \frac{(mL)^3}{3!} I_1^4 - \frac{(mL)^2}{2} I_1 I_2 + \frac{(mL)}{3} I_3, \quad (2.24) \]

\[ Z_4 = \frac{(mL)^4}{4!} I_1^4 - (mL)^3 I_1^2 I_2 + \frac{(ML)^2}{2} \left[ \frac{2}{3} I_1 I_3 + \frac{(I_2^2)}{2} \right] - \frac{(mL)}{4} I_4. \quad (2.25) \]

It is not difficult to carry on to higher order and to show that the series (2.13) precisely coincides with the one of eq. (2.19), the only difference being in the arrangement of the single terms. For instance, collecting the terms proportional to \((mL)\) in all \(Z_n\), one simply obtains for their sum \(F(R)\), whereas the sum of the terms proportional to \((mL)^n\) appearing in all \(Z_n\) gives rise to the higher power \((F(R))^n\).

The above remarkable fact can be interpreted as meaning that all physical properties of the system can be extracted by the quasi-particle excitations above the TBA thermal ground state. These excitations have dressed energy \(\tilde{\varepsilon} = \varepsilon(\theta)/R\) and dressed momenta \(\tilde{k}(\theta)\):

\[ \tilde{\varepsilon}(\theta) = \varepsilon(\theta)/R, \quad \tilde{k}(\theta) = k(\theta) + 2\pi(\sigma * \rho_1)(\theta). \quad (2.26) \]

In this context, the rapidity \(\theta\) plays then the role of a variable which simply parameterizes the dispersion relation of the quasi–particle excitations. This result was already understood in a non–relativistic situation by Yang and Yang [5] and a generalization of their result in our relativistic context is given for convenience in appendix A.

In the light of these considerations, let us return now to the sum (2.3). It is now clear that if we replace \(e(\theta)\) by \(\tilde{\varepsilon}(\theta)\) and proceed to evaluate the sum as in the free theory using (2.4), the partition function will properly be factored out and be canceled by the \(Z\) in the denominator, and furthermore the filling fractions will again be generated with \(\varepsilon\) now given by the TBA pseudo-energy. Thus, one obtains precisely the same formulae as (2.6, 2.7). We emphasize that the connected form factor appearing in (2.6) is the usual
(temperature-independent) form factor. Then the interpretation of eq. (2.6) is quite clear, i.e. each multiparticle state contributes to the thermal sum with its $T = 0$ form factor but each is weighted with the thermal probability of the occupation of that state, as determined self–consistently by the TBA equations.

A non-trivial check of the above claim is readily available for the trace of the energy momentum tensor $O = T_{\mu}^\mu$. The connected form-factors of this operator can be easily extracted and they depend only on the model through the function $\varphi$. For the first ones we have

$$\langle \theta | T_{\mu}^\mu | \theta \rangle_{\text{conn}} = 2\pi m^2 ;$$

$$\langle \theta_2, \theta_1 | T_{\mu}^\mu | \theta_1, \theta_2 \rangle_{\text{conn}} = 4\pi m^2 \varphi(\theta_1 - \theta_2) \text{ch}(\theta_1 - \theta_2) ,$$

and inductive application of the form factor residue equations leads to

$$\langle \theta_n \cdots \theta_1 | T_{\mu}^\mu | \theta_1 \cdots \theta_n \rangle_{\text{conn}} = 2\pi m^2 \varphi(\theta_{12}) \varphi(\theta_{23}) \cdots \varphi(\theta_{n-1,n}) \text{ch}(\theta_{1n})$$

+ permutations

(2.27)

where $\theta_{ij} = \theta_i - \theta_j$. For the aim of computing the thermal trace (i.e. considered as inserted into integral expressions over $\theta_i$), the last expression can be taken as

$$\langle \theta_n \cdots \theta_1 | T_{\mu}^\mu | \theta_1 \cdots \theta_n \rangle_{\text{conn}} = 2\pi m^2 n! \varphi(\theta_{12}) \varphi(\theta_{23}) \cdots \varphi(\theta_{n-1,n}) \text{ch}(\theta_{1n}) .$$

Finally, the vacuum expectation value of this operator is given by

$$< 0 | T_{\mu}^\mu | 0 > \equiv (T_{\mu}^\mu)_0 = \frac{\pi m^2}{2 \sum_i \sin \pi \alpha_i} ,$$

where $\alpha_i$ are the resonance angles entering the two–body scattering amplitude of the particle (in the more general case, one should consider the scattering amplitude of the particle with the lightest mass). Therefore we have

$$\langle T_{\mu}^\mu \rangle_R = (T_{\mu}^\mu)_0 + 2\pi m^2 \left( \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} f_-(\theta_i) \right) \varphi(\theta_{12}) \cdots \varphi(\theta_{n-1,n}) \text{ch}(\theta_{1n}) .$$

(2.31)

(2.32)

The $n$-th term of this series can be represented by the graph
where the dots are associated to the weights \( f_-(\theta_i) \), the thick lines between the dots to the functions \( \varphi(\theta_i - \theta_{i+1}) \) and the dotted line which links the ending dots to \( \text{ch}(\theta_{1n}) \).

On the other hand, \( \langle T^\mu_\mu \rangle_R \) can be computed directly from the TBA [4] as

\[
\langle T^\mu_\mu \rangle_R - \langle T^\mu_\mu \rangle_0 = \frac{2\pi}{R} \frac{d}{dR}[RE(R)] ,
\]

where \( E(R) = -\log Z/L \). This can be expressed as

\[
\langle T^\mu_\mu \rangle_R - \langle T^\mu_\mu \rangle_0 = m \int d\theta \frac{e^{\varepsilon(\theta')}}{1 + e^{-\varepsilon}} \left( \partial_R \varepsilon \text{ch} \theta - \frac{1}{R} \partial_\theta \varepsilon \text{sh} \theta \right) ,
\]

where the functions \( \partial_R \varepsilon \) and \( \partial_\theta \varepsilon \) satisfy linear integral equations which can be easily solved. In fact, define the kernel

\[
\hat{K}(\theta,\theta') = \varphi(\theta,\theta') \frac{e^{-\varepsilon(\theta')}}{1 + e^{-\varepsilon}} .
\]

Then, by differentiating the integral equation (2.13) one obtains

\[
(1 - \hat{K}) \partial_R \varepsilon = e , \quad (1 - \hat{K}) \frac{1}{R} \partial_\theta \varepsilon = k ,
\]

where \( (\hat{K} \partial_R \varepsilon)(\theta) \equiv \int d\theta' \hat{K}(\theta,\theta') \partial_R \varepsilon(\theta')/2\pi \). Introducing the resolvent \( \hat{L} \) satisfying

\[
(1 + \hat{L})(1 - \hat{K}) = 1 ,
\]

we have

\[
\partial_R \varepsilon = (1 + \hat{L})e , \quad \frac{1}{R} \partial_\theta \varepsilon = (1 + \hat{L})k .
\]

Using \( 1 + \hat{L} = \sum_{n=0}^{\infty} \hat{K}^n \), one can easily express the above in an integral series:

\[
\partial_R \varepsilon = e + \varphi \ast \left( \frac{e^{-\varepsilon}}{1 + e^{-\varepsilon}} e \right) + ... \]

and similarly for \( \partial_\theta \varepsilon \). Substituting these series into (2.34) one then finds a perfect agreement with expression (2.32).
A model with one particle whose computation of one–point functions can be explicitly performed is provided by the Sinh-Gordon model. This is briefly discussed in the next section.

C. One–point functions in the Sinh–Gordon model

The Sinh-Gordon theory is defined by the action

\[ S = \int d^2 x \left[ \frac{1}{2} (\partial_\mu \Phi)^2 - \frac{m_0^2}{g^2} \cosh g \Phi(x) \right] . \] (2.40)

The model is invariant under a \( Z_2 \) symmetry \( \Phi \to -\Phi \). Its two-particle \( S \)-matrix is given by

\[ S(\theta, \alpha) = \frac{\tanh \frac{1}{2}(\theta - i\pi\alpha)}{\tanh \frac{1}{2}(\theta + i\pi\alpha)} , \] (2.41)

where \( \alpha \) is the so-called renormalized coupling constant \( \alpha(g) = \frac{g^2}{8\pi} \frac{1}{1 + \frac{g^2}{8\pi}} \). For real values of \( g \) the \( S \)-matrix has no poles in the physical sheet and hence there are no bound states. The function \( \varphi(\theta) \) of this model is explicitly given by

\[ \varphi(\theta) = \frac{2 \sin \alpha \cosh \theta}{\cosh^2 \theta - \cos^2 \pi \alpha} . \] (2.42)

We are interested, in particular, in computing the one–point functions of the exponential operators of this theory, \( \Psi_k(x) = e^{kg\Phi} \). To this aim, let us recall the general structure of the form-factors of the Sinh-Gordon model, as determined in \[11,12\], specializing at the end to the operators in question.

For scalar operators, the form–factors of the theory can be parameterized as follows

\[ F_n(\theta_1, \ldots, \theta_n) = H_n Q_n(x_1, \ldots, x_n) \prod_{i<j} \frac{F_{\min}(\theta_{ij})}{(x_i + x_j)} , \] (2.43)

where \( x_i \equiv e^{\theta_i} \) and \( \theta_{ij} = \theta_i - \theta_j \). \( F_{\min}(\theta) \) is an analytic function explicitly given by

\[ F_{\min}(\theta, \alpha) = \prod_{k=0}^{\infty} \frac{\Gamma \left( k + \frac{3}{2} + \frac{i\theta}{2\pi} \right) \Gamma \left( k + \frac{1}{2} + \frac{\alpha}{2} + \frac{i\theta}{2\pi} \right) \Gamma \left( k + 1 - \frac{\alpha}{2} + \frac{i\theta}{2\pi} \right)}{\Gamma \left( k + \frac{1}{2} + \frac{i\theta}{2\pi} \right) \Gamma \left( k + \frac{3}{2} - \frac{\alpha}{2} + \frac{i\theta}{2\pi} \right) \Gamma \left( k + 1 + \frac{\alpha}{2} + \frac{i\theta}{2\pi} \right)}^2 , \] (2.44)
where \( \hat{\theta} = i\pi - \theta \). It satisfies the functional equations

\[
F_{\min}(\theta) = F_{\min}(-\theta) S(\theta, \alpha),
\]

\[
F_{\min}(i\pi - \theta) = F_{\min}(i\pi + \theta),
\]

\[
F_{\min}(i\pi + \theta, \alpha) F_{\min}(\theta, \alpha) = \frac{\sinh \theta}{\sinh \theta + \sinh(i\pi \alpha)}.
\]

where the last equation is particularly important for the computation at a finite temperature.

\( H_n \) are normalization constants, which can be conveniently chosen as \( H_{2n+1} = H_1 \mu^{2n} \), \( H_{2n} = H_2 \mu^{2n-2} \), with \( \mu \equiv \left( \frac{4 \sin(\pi \alpha)}{f_{\min}(i\pi, \alpha)} \right)^{\frac{1}{2}} \). In absence of additional requirements on the form–factors, \( H_1, H_2 \) are two independent parameters. Finally, the functions \( Q_n(x_1, \ldots, x_n) \) are symmetric polynomials in the variables \( x_i \), which are fixed by the residue equations satisfied by the form factors and whose total and partial degrees depend on the specific operator. In general, they can be expressed in terms of the elementary symmetric polynomials \( \sigma_k^{(n)}(x_1, \ldots, x_n) \), defined by the generating function

\[
\prod_{i=1}^{n} (x + x_i) = \sum_{k=0}^{n} x^{n-k} \sigma_k^{(n)}(x_1, x_2, \ldots, x_n).
\]

Conventionally the \( \sigma_k^{(n)} \) with \( k > n \) and with \( n < 0 \) are zero. It is also convenient to introduce the symbol \( [n] \) defined by \( [n] \equiv \frac{\sin(n\pi \alpha)}{\sin \pi \alpha} \). As shown in [12], an interesting class of solutions of the recursive equations satisfied by the Form Factors (FF) is given by the following polynomials \( Q_n \):

\[
Q_n(k) = ||M_{ij}(k)||,
\]

where \( M_{ij}(k) \) is an \((n-1) \times (n-1)\) matrix with entries

\[
M_{ij}(k) = \sigma_{2i-j}[i - j + k],
\]

and the vertical lines denote the determinant of that matrix. These polynomials depend on an arbitrary number \( k \) and satisfy \( Q_n(k) = (-1)^{n+1} Q_n(-k) \). Form–factors of several

\(^2\)For simplicity the dependence of \( Q_n(k) \) on the variables \( x_i \) has been suppressed.
operators can be expressed in terms of the polynomials $Q_n(k)$. For instance, the whole set of FF of the elementary field $\Phi(x)$ are given by $Q_n(0)$, with the choice $H_1 = 1/\sqrt{2}$, $H_2 = 0$ whereas those of $T^\mu(x)$, are given by the even polynomials $Q_{2n}(1)$, with the choice $H_1 = 0$, $H_2 = 2\pi m^2$.

The form–factors of the exponential operators $\Psi_k(x) = e^{kg\Phi}$, are just given by the $Q_n(k)$, with the normalization constants fixed to be $H_1^k = \mu[k]$, $H_2^k = \mu^2[k]$. Since the one–point function of operators which are odd under the $Z_2$ symmetry $\Phi \to -\Phi$ is identically zero, the one–point correlators at finite temperature involving the exponential fields coincide with those of the even combination $\Psi_k + \Psi_{-k}$. Using the prescription (2.3) one finds

$$
\langle \cosh (kg\Phi) R \rangle_{(0)} = 1 + 4[k]^2 \sin \pi \alpha \int \frac{d\theta}{2\pi} f_-(\theta) + 4[k]^2 \sin \pi \alpha \times 
\times \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} f_-(\theta_1) f_-(\theta_2) \varphi(\theta_{12}) \left( [k]^2 \cosh \theta_{12} - \frac{[k - 1][k + 1]}{\cosh \theta_{12}} \right) + 
+ \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{d\theta_3}{2\pi} f_-(\theta_1) f_-(\theta_2) f_-(\theta_3) \left( 4 \sin \pi \alpha [k]^6 \varphi(\theta_{12}) \varphi(\theta_{23}) \cosh \theta_{13} + 
+ \mathcal{W}(\theta_{12}, \theta_{13}, \theta_{23}) \right) + \cdots
$$

where the function $\mathcal{W}(\theta_{12}, \theta_{13}, \theta_{23})$ appearing in the $3$–particle contribution can be expressed as

$$
\mathcal{W}(\theta_{12}, \theta_{13}, \theta_{23}) = \frac{4}{3} \varphi(\theta_{12}) \varphi(\theta_{13}) \varphi(\theta_{23}) \times 
\frac{1}{\cosh \theta_{12} \cosh \theta_{13} \cosh \theta_{23}} \left[ A(k) + B(k) \frac{\cosh \theta_{12} \cosh \theta_{13} \cosh \theta_{23}}{\cosh \theta_{12} \cosh \theta_{13} \cosh \theta_{23}} + 
+ C(k) \frac{\cosh(\theta_{13} - \theta_{32}) + \cosh(\theta_{12} - \theta_{23}) + \cosh(\theta_{21} - \theta_{13})}{\cosh \theta_{12} \cosh \theta_{13} \cosh \theta_{23}} \right],
$$

with the constants $A(k)$, $B(k)$ and $C(k)$ given by

$$
A(k) = \frac{7[k]^2}{2} \left( [k - 1]^2[k + 1]^2 - [k - 1][k]^2[k + 1] \right) + \frac{[k]^3}{2} \left( [k - 2][k + 1]^2 + [k - 1]^2[k + 2] + [k]^3[2]^2 \right) - \frac{[k]}{8} \left( [k - 2][k - 1][k + 1]^3 + [k - 2][k]^3[k + 2] + [k - 1]^3[k + 1][k + 2] - [k]^5(1 - [2]^2) \right)
$$

$$
B(k) = \frac{[k]}{16} \left( [k - 2][k]^2[k + 1]^2 + [k - 1]^2[k]^2[k + 2] + 2[k - 1]^2[k][k + 1]^2 + [k]^5 + 
- [k - 2][k - 1][k + 1]^3 - [k - 2][k]^3[k + 2] - [k - 1]^3[k + 1][k + 2] + 
- [k - 1][k]^3[k + 1] - 3[k - 2][k - 1][k][k + 1][k + 2]\right)
$$

$$
C(k) = \frac{[k]^3}{2} \left( [k - 2][k - 1][k + 1]^2 + [k - 1]^2[k]^2[k + 2] + [k - 1]^2[k][k + 1]^2 + [k]^3 + 
- [k - 2][k - 1][k + 1]^3 + [k - 2][k]^3[k + 2] - [k - 1]^3[k + 1][k + 2] + 
- [k - 1][k]^3[k + 2] - 3[k - 1][k][k + 1][k + 2] \right)
$$
The expressions for higher terms are rather complicated and are not given here. However, one can easily check that for $k = \pm 1$, the above expression coincides with the corresponding series of the trace of the stress–energy tensor. The plots of some of these quantities for some values of the coupling constant are shown in Figure 1. Notice that if the operator $\Psi_k$ is “on resonance” with the value of the coupling constant, namely if $k\alpha = n$, where $n$ is an integer, its one–point function coincides with its vacuum matrix element only. Moreover, varying $\alpha$, there may be a swapping of the profiles associated to the one–point functions of different $\Psi_k$.

D. Low and high–temperature limits of the one–point functions

Let us discuss in more detail the behavior of the one–point correlators (2.6) as functions of the temperature. First notice that if the anomalous dimension of the operator $\mathcal{O}$ is given by $2\Delta_{\mathcal{O}}$, its vacuum expectation value may be expressed as $\langle 0|\mathcal{O}|0 \rangle \equiv \langle \mathcal{O} \rangle_0 = Y m^{2\Delta_{\mathcal{O}}}$, with $Y$ a pure number. Hence, the series (2.3) relative to its one–point function at finite temperature can be cast in the scaling form

$$\frac{\langle \mathcal{O} \rangle_R}{\langle \mathcal{O} \rangle_0} = H(mR) \ ,$$

(2.51)

where $H(mR)$ is a function of the scaling variable $mR$.

In the low–temperature limit $R \to \infty$, the pseudo–energy goes as $\epsilon(\theta) \sim mR \cosh \theta$ and therefore the $N$-th term of the series entering $H(mR)$ vanishes asymptotically as $e^{-NmR}$, so that the leading correction is given by

$$H(mR) \to 1 + \frac{\langle A|\mathcal{O}|A \rangle}{\pi} K_0(mR) + \cdots$$

(2.52)

where $K_0(x)$ is the usual Bessel function, and $|A\rangle = |\theta \rangle$.

3For the check of the 3–particle contribution one needs the trigonometric identity $[3] = -1 + [2]^2$. 

14
On the other hand, in the high-temperature limit \( R \to 0 \), the one-point correlators may become scaling invariant functions\(^4\) i.e.

\[
\langle \mathcal{O} \rangle_R \to \frac{1}{R^{\eta}} ,
\]

with a power-law exponent \( \eta \) ruled by the underlying Conformal Field Theory. This is the case, for instance, of massive integrable models obtained as a deformation of a Conformal Field Theory by a strong relevant field. Under this hypothesis, the limit \( R \to 0 \) can be controlled by means of a Conformal Perturbation Theory on the cylinder. Let

\[
\mathcal{A} = \mathcal{A}_{\text{CFT}} + \lambda \int \Phi(x) d^2 x ,
\]

be the action of the off-critical model, where \( \Phi(x) \) is the relevant operator which gives rise to the massive integrable theory, with \( \lambda \sim m^{2-2\Delta} \). The general structure of the perturbation series for the one-point functions is then given by

\[
\langle \mathcal{O} \rangle_R = \sum_{n=0}^{\infty} d_n ,
\]

where

\[
d_n = \frac{(-\lambda)^n}{n!} \int_{\text{cyl}} \langle \xi \mid \mathcal{O}(0) \Phi(X_1) \cdots \Phi(X_n) \mid \xi \rangle_{\text{conn}} d^2 X_1 \cdots d^2 X_n ,
\]

and \( X_i \) are points on the cylinder and the connected correlation functions are calculated in the unperturbed CFT. The field \( \xi \) entering eq. (2.56) is relative to the conformal operator of lowest anomalous dimension in the theory which plays the role of the vacuum state on the cylinder (for a unitary model, this field is the identity operator). Notice that the first term of the series is given by

\[
\langle \xi \mid \mathcal{O} \mid \xi \rangle = \left( \frac{2\pi}{R} \right)^{2\Delta_o} C_{\xi \mathcal{O} \xi} ,
\]

\(^4\)This is not always the case, because there are theories, like the thermal Ising model, with logarithmic behavior or models, like the Sinh-Gordon one, which present logarithmic corrections which may spoil the pure power law behavior given in the text.
and this will be different from zero only if the conformal structure constant $C_{\xi_1 \xi_2}$ does not vanish. In this case, the exponent $\eta$ in (2.53) will coincide with the anomalous dimension of the field $O$ itself, $\eta = 2\Delta_O$, otherwise the exponent $\eta$ will be given by $\eta = 2\Delta_O - q(2 - 2\Delta_{\Phi})$, where the integer $q$ is the first non-vanishing term in the series (2.53).

The above considerations open up the possibility to extract conformal data out of the high-temperature limit of the one-point function. This is easily checked by the analysis of some simple models which have only one massive particle in the spectrum.

1. Yang–Lee model

The $S$-matrix of the perturbed Yang–Lee model was determined in [13] to be

$$S(\theta) = \frac{\tanh \frac{1}{2}(\theta + i\frac{2\pi}{3})}{\tanh \frac{1}{2}(\theta - i\frac{2\pi}{3})}. \quad (2.58)$$

Therefore for $\varphi(\theta)$ we have

$$\varphi(\theta) = -\frac{\sqrt{3} \cosh \theta}{\cosh^2 \theta - \frac{1}{4}}. \quad (2.59)$$

The only off-critical primary operator of the theory coincides with the trace of the stress-energy tensor. In the conformal limit it goes to the field with the lowest anomalous dimension $2\Delta = -2/5$ which simultaneously plays the role of the vacuum of the cylinder. Hence, in this case we expect to find for the exponent $\eta = -2/5$. In order to extract this parameter, by taking the logarithm of both terms in (2.53)

$$-\eta \log R = \lim_{R \to 0} \langle O \rangle_R \quad (2.60)$$

$\eta$ is easily identified by isolating the terms proportional to $-\log R$ in the series of the right hand side. Observe that, factorizing the term $(T_\mu^\mu)_0$ in eq. (2.32), the logarithm of the series (2.32) can be expressed in terms of the usual cluster expansion

$$\log(T_\mu^\mu)_R - \log(T_\mu^\mu)_0 = \frac{2\pi m^2}{(T_\mu^\mu)_0} \left[ \int \frac{d\theta}{2\pi} f(\theta)e^{-\epsilon(\theta)} + \int \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} f(\theta_1)f(\theta_2)e^{-\epsilon(\theta_1) - \epsilon(\theta_2)} \times \left[ \varphi(\theta_1 - \theta_2)\cosh(\theta_1 - \theta_2) - \frac{1}{2} \left( \frac{2\pi m^2}{(T_\mu^\mu)_0} \right) \right] + \ldots \right] \quad (2.61)$$
where, for this model
\[
\frac{2\pi m^2}{(T^\mu_\mu)_0} = -2\sqrt{3}.
\] (2.62)

Following the TBA analysis \[4\], it is known that in the limit \(R \to 0\), the pseudo–energy \(\varepsilon(\theta)\) tends to a plateau of value \(\varepsilon_0 = \log \left[\sqrt{\frac{\pi+1}{2}}\right]\) in a region \(|\theta| < 2 \log(2/R)\). Therefore the first estimate of \(\eta\) is obtained by the first term of the right hand side of (2.61)
\[
K_1 = \frac{2\pi m^2}{(T^\mu_\mu)_0} \int d\theta \frac{f(\theta)}{2\pi} e^{-\varepsilon(\theta)} \simeq \frac{2\pi m^2}{(T^\mu_\mu)_0} \int_{-\log(\frac{2}{R})}^{\log(\frac{2}{R})} f(\theta) e^{-\varepsilon(\theta)} = \frac{2\pi m^2}{(T^\mu_\mu)_0} \frac{1}{1 + e^{\varepsilon_0} \frac{1}{\pi}} \log \left(\frac{2}{R}\right),
\] (2.63)
i.e.
\[
\eta \simeq -\frac{1}{\pi} \frac{4\sqrt{3}}{3 + \sqrt{5}} = -0.421178...
\] (2.64)
which is within 5% to the actual value \(\eta = -0.4\). The next correction is obtained by isolating the term proportional to \(-\log R\) in the next term of the cluster expansion (2.61), which is equivalent to computing the quantity
\[
\frac{3}{\pi^2} \left(\frac{1}{3 + \sqrt{5}}\right)^2 \int_{-\infty}^{+\infty} d\theta \frac{1}{\cosh^2 \theta - \frac{1}{4}} = 0.0268126...
\] (2.65)
so that the estimate of \(\eta\) improves to the values
\[
\eta \simeq -0.394365...
\] (2.66)
which differs from the actual value by 1%.

The main result of these computations can be summarized by saying that the exponent \(\eta\) will be expressed by a series (of alternating signs, for this model) whose terms involve, in addition to the ratio (2.62), inverse powers of \(\pi\) and the plateau value \(\varepsilon_0\).

\[2. \text{The non–unitary model } \mathcal{M}_{3,5}\]

Another integrable model with only one massive excitation to which we can easily apply the above consideration is the off–critical “thermal” deformation of the minimal model
\( M_{3,5} \), which has been previously studied in \([13, 17]\). The S–matrix of the model is given by
\[
S(\theta) = -i \tanh \frac{1}{2} (\theta - i \frac{\pi}{2})
\]
and therefore \( \varphi(\theta) = 1/\cosh \theta \). In the conformal limit, the trace
of the stress–energy tensor tends to the conformal field \( \mathcal{O} \) of anomalous dimension \( 2\Delta = 2/5 \)
and this field has a non–vanishing structure constant with the field \( \xi \) of anomalous dimension
\( 2\Delta \xi = -1/10 \) which plays the role of vacuum state of the cylinder. Hence we expect that the
exponent \( \eta \) of this theory should be equal to \( 2/5 \). By using the same arguments as above,
with the additional data
\[
\frac{2\pi m^2}{(T_\mu^\mu)_0} = 2 , \quad \epsilon_0 = - \log \left[ \frac{\sqrt{5} + 1}{2} \right] ,
\]
for the first estimate of the exponent \( \eta \) we have
\[
\eta \sim \frac{2 \frac{1}{\pi} + \frac{\sqrt{5}}{3}}{\frac{1}{\pi} + \frac{\sqrt{5}}{3}} = 0.393453...
\]
which is within 1% of the actual value \( \eta = 0.4 \). In this model, the fact that the first term of
the cluster series gives a better approximation to \( \eta \) seems to be related to the vanishing of
the next leading correction in the cluster expansion.

**III. TWO-POINT FUNCTIONS**

Having understood the main features of the one-point functions, the same reasoning
applies to the higher point functions. In this section we focus on the two-point functions.
We begin with the double summation over states
\[
\langle \mathcal{O}(x, t) \mathcal{O}(0) \rangle_R = \frac{1}{Z} \sum_{\psi, \psi'} e^{-E_{\psi R}} \langle \psi | \mathcal{O}(x, t) | \psi' \rangle \langle \psi' | \mathcal{O}(0) | \psi \rangle .
\]
As explained in \([9]\), for a free theory the terms in \([2, 4]\) with \( A_1 \neq A \) give rise to three kinds
of contributions:
(i). Terms that diverge in infinite volume involving \( [\delta(\theta - \theta')]^2 \) which can be arranged into
an overall factor of \( Z \).
(ii). Terms that give rise to factors of the one-point function \( (\langle \mathcal{O} \rangle_R)^2 \).
(iii). Terms that modify the integration over rapidity for both the $\psi$ and $\psi'$ states by the same filling fraction $f$.

The remaining contributions are only expressed in terms of connected form-factors for which crossing symmetry is valid. As for the one-point function, the only difference between the free and interacting theory is that $e, k$ should be replaced by the TBA dressed quantities $\tilde{e}, \tilde{k}$. A further justification of this will be given below.

Introduce an index $\sigma = 1$ for the $\psi'$ particles and $\sigma = -1$ for the $\psi$ particles. For the $\psi'$ particles the integrations $\int d\theta$ are accompanied by $f(\theta)$ defined in (2.7), whereas for the $\psi$ particles one has $f(\theta)e^{-\varepsilon(\theta)}$ due to the $e^{-E_\psi R}$ in (3.1). These factors can be expressed as $f_\pm(\theta)$ as defined in (2.8). By using the crossing symmetry, the connected form factors appearing in (3.1) are

$$\langle \psi' | \mathcal{O}(0) | \psi \rangle_{\text{conn}} = \langle \psi + i\pi, \psi' | \mathcal{O}(0) | 0 \rangle,$$

where $\psi + i\pi$ denotes all rapidities shifted by $i\pi$. We can describe this by defining

$$\langle 0 | \mathcal{O}(\theta_1, \ldots, \theta_N)_{\sigma_1, \ldots, \sigma_N} = \langle 0 | \mathcal{O}(\theta_1 - i\pi\bar{\sigma}_1, \ldots, \theta_N - i\pi\bar{\sigma}_N) \rangle,$$

where $\bar{\sigma} = (\sigma - 1)/2 \in \{0, 1\}$. Finally,

$$\langle \mathcal{O}(x, t) \mathcal{O}(0) \rangle_R = \langle \mathcal{O}(0) \mathcal{O}(x, t) \rangle_R.$$

Shifting $t$ by $R$ in (3.3) gives the factor $e^{-\varepsilon_t}$. Noting that $f_\sigma e^{-\sigma\varepsilon_t} = f_{-\sigma}$, we simply relabel the particles $\sigma \to -\sigma$ in the sum. Since $\langle \mathcal{O}(0) \mathcal{O}(x, t) \rangle = \langle \mathcal{O}(-x, -t) \mathcal{O}(0) \rangle$, then (3.4) holds as long as

$$|\langle 0 | \mathcal{O}(\theta_1, \ldots, \theta_N)_{\sigma_1, \ldots, \sigma_N} |^2 = |\langle 0 | \mathcal{O}(\theta_1, \ldots, \theta_N)_{-\sigma_1, \ldots, -\sigma_N} |^2.$$
The latter identity can be proven as follows. Let $|\vec{\theta} - i\pi \vec{\sigma}\rangle$ denote $|\theta_1 - i\pi \sigma_1, \ldots, \theta_N - i\pi \sigma_N\rangle$ and $\langle \vec{\theta} + i\pi \vec{\sigma}|$ denote $\langle \theta_N + i\pi \sigma_N, \ldots, \theta_1 + i\pi \sigma_1|$. Then,

$$\langle 0|\mathcal{O}(0)|\theta_1 \cdots \theta_N\rangle_{\sigma_1 \cdots \sigma_N}|^2 = \langle 0|\mathcal{O}|\vec{\theta} - i\pi \vec{\sigma}\rangle \langle \vec{\theta} + i\pi \vec{\sigma}|\mathcal{O}|0\rangle .$$  \hspace{1cm} (3.6)

Under the transformation $\sigma \to -\sigma$, one has $\vec{\sigma} \to -\vec{\sigma} - 1$. For local operators, the form factor is invariant under a shift of all rapidities by the same constant, thus

$$\langle 0|\mathcal{O}(0)|\theta_1 \cdots \theta_N\rangle_{-\sigma_1 \cdots -\sigma_N}|^2 = \langle 0|\mathcal{O}|\vec{\theta} + i\pi \vec{\sigma}\rangle \langle \vec{\theta} - i\pi \vec{\sigma}|\mathcal{O}|0\rangle$$

$$= \langle 0|\mathcal{O}|\vec{\theta} - i\pi \vec{\sigma}\rangle \langle \vec{\theta} + i\pi \vec{\sigma}|\mathcal{O}|0\rangle ,$$  \hspace{1cm} (3.7)

where we have used crossing symmetry in the second line. The orders of rapidities in (3.7) can be brought to the order in (3.6) using $\langle 0|\mathcal{O}|\cdots \theta_i, \theta_j, \cdots\rangle = S(\theta_i - \theta_j) \langle 0|\mathcal{O}|\cdots \theta_j, \theta_i, \cdots\rangle$.

In doing this one always encounters the pair of factors $S(\theta - i\pi), S(-\theta - i\pi)$. Using crossing $S(\theta) = S(i\pi - \theta)$ and unitarity $S(\theta)S(-\theta) = 1$, the latter factor equals 1. Thus, (3.6) holds.

**IV. CONCLUSIONS**

We have proposed explicit integral representations for the one and two-point correlation functions at finite temperature involving form factors and the TBA pseudo-energy-momentum. Several checks were performed, in particular we showed that for the trace of the energy momentum tensor, our expression for the one-point function coincides precisely with the TBA result. We also checked that the Kubo-Martin-Schwinger relation holds for the two-point function.

We close by listing some possible applications of our formalism that deserve further investigation. By extending our results to boundary field theories at finite temperature, as was done for the free case in [9], it should be possible to study the finite temperature properties of conductivities in quantum wires with impurities. Another application is to the study of finite temperature crossovers in the vicinity of a zero temperature quantum phase transition (for a review see [18]). Finally there is the issue of spin-diffusion in quantum spin
chains, which would require the study of the infinite temperature limit of the continuum version of a spin-spin two-point correlation function [19].

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Appendix A. TBA Dressed Energy and Momentum

Let \((n_j, \theta_j)\) and \((n'_j, \theta'_j)\) be two Bethe states both satisfying (2.10), where \(n'_j = n_j\) except for \(j = \alpha\). Subtracting the two equations one obtains

\[
mL \left( \text{sh}\theta'_j - \text{sh}\theta_j \right) = \sum_i \sigma(\theta_j - \theta_i) - \sigma(\theta'_j - \theta'_i) .
\] (4.1)

\((j \neq \alpha)\). Since \(\theta'_j \approx \theta_j\) we can introduce a function \(\chi(\theta)\) and write

\[
L \left( \text{sh}\theta'_j - \text{sh}\theta_j \right) \approx \chi(\theta_j) \text{ch}\theta_j .
\] (4.2)

In this thermodynamic limit, (4.1) can be written as

\[
2\pi(1 - \tilde{K})(\rho\chi) = \sigma(\theta - \theta_\alpha) - \sigma(\theta - \theta'_\alpha) ,
\] (4.3)

where \(\rho\) is the previously introduced density of levels and \(\tilde{K}\) is defined in (2.35). One also has

\[
\sigma(\theta - \theta_\alpha) - \sigma(\theta - \theta'_\alpha) = \int_{\theta_\alpha}^{\theta'_\alpha} d\theta' \tilde{K}(\theta, \theta')(1 + e^{\varepsilon(\theta')}) .
\] (4.4)

Using the resolvent operator, as defined in (2.37), one has

\[
\rho\chi(\theta) = \int_{\theta_\alpha}^{\theta'_\alpha} d\theta' \frac{\hat{L}(\theta, \theta')}{2\pi}(1 + e^{\varepsilon(\theta')}) .
\] (4.5)

Now consider the difference in energy \(\Delta E\) between the two Bethe states:

\[
\Delta E = m \text{ch}\theta'_\alpha - m \text{ch}\theta_\alpha + m \int d\theta \text{ sh}\theta \frac{\chi(\theta)\rho(\theta)}{1 + e^{\varepsilon(\theta)}} .
\] (4.6)

Substituting (4.3), and using the property \(\hat{L}(\theta, \theta')(1 + e^{\varepsilon(\theta')}) = (1 + e^{\varepsilon(\theta)})\hat{L}(\theta', \theta)\), along with (2.38) one finds

\[
\Delta E = \frac{1}{R} (\varepsilon(\theta'_\alpha) - \varepsilon(\theta_\alpha)) .
\] (4.7)

The dressed momentum proceeds similarly:

\[
\Delta P = \sum_j m \text{sh}\theta'_j - m \text{sh}\theta_j =
\] (4.8)

\[
= m \text{sh}\theta'_\alpha - m \text{sh}\theta_\alpha + m \int d\theta \text{ ch}\theta \frac{\chi(\theta)\rho(\theta)}{1 + e^{\varepsilon(\theta)}} .
\]
Again substituting (4.3) and using (2.38) one finds

\[ \Delta P = \tilde{k}(\theta'_\alpha) - \tilde{k}(\theta_\alpha) \]  

where \( \tilde{k} \) is defined in (2.26).
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Figure 1a. One–point correlators of $e^{kg\Phi}$ as functions of $mR$ for $\alpha = 1/20$.

Figure 1b. One–point correlators of $e^{kg\Phi}$ as functions of $mR$ for $\alpha = 1/\sqrt{7}$. 