ON GREENBERG’S GENERALIZED CONJECTURE AND
(p, i)-REGULAR FIELDS

J. ASSIM\(^{(1)}\) AND Z. BOUGHADI\(^{(2)}\)
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\((1)\) Moulay Ismail University of Meknès, Morocco
Team work: Algebraic Theory and Application
e-mail: j.assim@umi.ac.ma
\((2)\) Moulay Ismail University of Meknès, Morocco
Team work: Algebraic Theory and Application
e-mail: z.boughadi@edu.umi.ac.ma

Abstract. For a number field \(F\) and an odd prime number \(p\), let \(\tilde{F}\) be the composite of all \(\mathbb{Z}_p\)-extensions of \(F\), \(\tilde{\Lambda}\) the associated Iwasawa algebra and \(X(\tilde{F})\) the Galois group over \(\tilde{F}\) of the maximal abelian unramified pro-\(p\)-extension of \(\tilde{F}\). In this paper we show that under the \(i\)-th twisted analogue of Leopoldt’s conjecture and a decomposition condition, the pseudo-nullity of the \(\tilde{\Lambda}\)-module \(X(\tilde{F}((\mu_p^{i−1}))\Delta)\) is implied by the existence of a \(\mathbb{Z}_p\)-extension \(F_\infty\) such that \(X_S^{(−i)}(F_\infty) := H_1(G_S(F_\infty), \mathbb{Z}_p(−i))\) is without torsion over the Iwasawa algebra associated to \(F_\infty\). This existence is fulfilled for \((p, i)\)-regular fields. In particular, when the integer \(i \equiv 1 \mod [F(\mu_p) : F]\) we have a sufficient condition for the validity of Greenberg’s generalized conjecture.

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1. Introduction

Let \(F\) be a number field and \(p\) an odd prime. The notation \(S\) stands for the set of \(p\)-adic and infinite primes of \(F\). Let \(F_S\) be the maximal extension over \(F\) which is unramified outside \(S\), and \(G_S(F)\) the Galois group \(\text{Gal}(F_S/F)\). For any \(\mathbb{Z}_p^d\)-extension \(L\) of \(F\) \((d \geq 1)\), we write \(\Gamma_L\) for its Galois group and \(\Lambda_L = \mathbb{Z}_p[\Gamma_L]\) the associated Iwasawa algebra to \(\Gamma_L\). As usual \(F^c\) will be the cyclotomic \(\mathbb{Z}_p\)-extension and \(\tilde{F}\) the compositum of all \(\mathbb{Z}_p\)-extensions of \(F\). Let \(\tilde{\Gamma}\) (resp. \(\Gamma_c\)) be the Galois group of the extension \(\tilde{F}/F\) (resp. \(F^c/F\)) and \(\tilde{\Lambda}\) (resp.
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\(\Lambda_c\) the associate Iwasawa algebra to \(\hat{\Gamma}\) (resp. \(\Gamma_c\)). It is well known that all \(\mathbb{Z}_p\)-extensions are unramified outside the set \(S\), thus \(F_S\) contains \(L\). We denote \(G_S(L)\) for the Galois group of the extension \(F_S/L\). For any integer \(i\) we write \(X_S^{(i)}(L)\) for the first homological group \(H_1(G_S(L), \mathbb{Z}_p(i))\). In the particular case \(i = 0\), \(X_S(L) := X_S^{(0)}(L) = H_1(G_S(L), \mathbb{Z}_p)\) is the Galois group of the maximal abelian pro-
\(p\)-extension of \(L\) which is unramified outside \(S\). The group \(X_S^{(i)}(L)\) has a structure of \(\Lambda_L\)-module, we will call it the twisted \(S\)-ramified Iwasawa module associated to \(L\). Denote by \(X(L)\) (resp. \(X'(L)\)) the Galois group over \(L\) of the maximal abelian unramified (respectively unramified and all \(p\)-primes split completely) pro-
\(p\)-extension of \(L\). The groups \(X(L)\) and \(X'(L)\) have a structure of \(\Lambda_L\)-module and they are \(\Lambda_L\)-torsion (e.g \([G73\, Theorem\, 1]\)). Furthermore, Greenberg proposed the following conjectures (c.f. \([G78]\) and \([G01]\)):

**Conjecture** (GC for short). The \(\Lambda_c\)-module \(X(F^c)\) is finite for any totally real number field \(F\).

**Conjecture** (GGC for short). For any number field \(F\) the \(\tilde{\Lambda}\)-module \(X(\tilde{F})\) is pseudo-null.

Both GC and GGC are equivalent when the number field \(F\) is totally real and satisfies Leopoldt’s conjecture. Only numerical results are known for GC, while many theoretical results have been proved for GGC. We cite Sharifi’s result on cyclotomic field \(\mathbb{Q}(\mu_p)\) (see \([Sh09]\) and Fujii \([F17]\)) for CM-field with \(p\) totally split, which is a generalization of Minardi \([M86]\) and Itoh \([I11]\) results. Recently, Nguyen Quang Do generalizes the work of Fujii \([F17]\). He showed that GGC can be detected early in a \(\mathbb{Z}_p^2\)-extension of the base field. Precisely, he proved the following theorem:

**Theorem 1.1.** \([N19\, Theorem\, 1.4]\). Let \(F\) be an imaginary field which satisfies Kuz’min-Gross conjecture. If \(F\) admits a \(\mathbb{Z}_p^2\)-extension \(F^{(2)}\) which is normal over \(\mathbb{Q}\) and contains \(F^c\) (such \(\mathbb{Z}_p^2\)-extension is called special in \([N19]\)) and \(X'(F^{(2)})\) is pseudo-null, then \(F\) verifies GGC.

To prove this result, Nguyen Quang Do shows the pseudo nullity of \(X'(\tilde{F})\) (GGC’ in \([N19]\)) and its equivalence to that of \(X(\tilde{F})\) (i.e. GGC) under the following condition

\((\text{Dec})\): All \(p\)-adic places has decomposition group in the extension \(\tilde{F}/F\) of \(\mathbb{Z}_p\)-rank at least 2.

Besides, while \([N19]\) relies on Kuz’min-Gross conjecture, here we’ll appeal to the twisted analogue of Leopoldt’s conjecture.
Conjecture \((LC_i \text{ for short})\). For any number field \(F\) and all integers \(i \neq 1\),
\[
H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0.
\]

The aim of this paper is to prove that under the conjecture \(LC_i\) and the condition \((\text{Dec})\), the pseudo-nullity of the \(\tilde{\Lambda}\)-module \(X(\tilde{F}(\mu_p))(i-1)^{\Delta}\) \((\Delta = \text{Gal}(\tilde{F}(\mu_p)/\tilde{F}))\) is implied by the existence of a \(\mathbb{Z}_p\)-extension for which the twisted \(S\)-ramified Iwasawa module is without torsion. More precisely, we prove the following theorem:

**Theorem 1.2.** Assume that the condition \((\text{Dec})\) holds for \(F\). Let \(i\) be an integer such that \(i \not\equiv 0 \mod \left[F(\mu_p) : F\right]\) and \(LC_i\) holds for \(F\). If \(F\) admits a \(\mathbb{Z}_p\)-extension \(F_\infty\) such that \(\text{Tor}_{\Lambda_\infty}(X_S^{(-i)}(F_\infty)) = 0\), then the \(\tilde{\Lambda}\)-module \(X(\tilde{F}(\mu_p))(i-1)^{\Delta}\) is pseudo-null.

Further, when the integer \(i \equiv 1 \mod \left[F(\mu_p) : F\right]\) we obtain a sufficient condition for the validity of Greenberg’s generalized conjecture.

**Corollary 1.3.** Assume that the condition \((\text{Dec})\) holds for \(F\). If there exists an integer \(i \equiv 1 \mod \left[F(\mu_p) : F\right]\) such that \(LC_i\) is true for \(F\) and a \(\mathbb{Z}_p\)-extension \(F_\infty\) satisfying \(\text{Tor}_{\Lambda_\infty}(X_S^{(-i)}(F_\infty)) = 0\). Then \(F\) satisfies Greenberg generalized conjecture.

We mention also that our methods give an other proof to Theorem 1.1 (see Remark 3.11).

To prove Theorem 1.2 we start with studying the \(\tilde{\Lambda}\)-module \(X_S^{(-i)}(\tilde{F})\). We use Greenberg’s methods \((\text{[G78]}\)) to determine its rank over \(\tilde{\Lambda}\) (Theorem 2.5) and prove that if \(i \not\equiv 0 \mod \left[F(\mu_p) : F\right]\), it is of projective dimension at most one. In section 3, we start by showing under the condition \((\text{Dec})\), two equivalent properties of the pseudo-nullity of \(X(\tilde{F}(\mu_p))(i-1)^{\Delta}\) (Theorem 3.4), namely

- \(X'(\tilde{F}(\mu_p))(i-1)^{\Delta}\) is pseudo-null.
- \(X_S^{(-i)}(\tilde{F})\) is without \(\tilde{\Lambda}\)-torsion.

Next, we show a going up theorem for the torsion triviality of the twisted \(S\)-ramified Iwasawa modules in a tower of multiple \(\mathbb{Z}_p\)-extensions, which start with a \(\mathbb{Z}_p\)-extension satisfying the twisted weak Leopoldt’s conjecture (Theorem 3.7). The last section is devoted to \((p, i)\)-regular number field. We prove that the \(\tilde{\Lambda}\)-module \(X(\tilde{F}(\mu_p))(i-1)^{\Delta}\) is pseudo-null for any \((p, i)\)-regular number fields \(F\) which satisfies the condition \((\text{Dec})\) (Theorem 4.3). In particular, Greenberg’s generalized conjecture is true for all \((p, 1)\)-regular field which verifies \((\text{Dec})\) (Corollary 4.4).
2. The twisted \( S \)-ramified Iwasawa module

Let \( F \) be a number field and \( p \) an odd prime number. We fix the following notations:

- \( \text{Gal}(N/K) \) the Galois group of an arbitrary Galois extension \( N/K \).
- \( L \) a \( \mathbb{Z}_p^d \)-extension of \( F \), \( d \geq 1 \).
- \( \Gamma_L \) the Galois group \( \text{Gal}(L/F) \).
- \( \Lambda_L \) the associated Iwasawa algebra to \( \Gamma_L \), \( \Lambda_L = \mathbb{Z}_p[[\Gamma_L]] \).
- \( F^c \) the cyclotomic \( \mathbb{Z}_p \)-extension of \( F \).
- \( \Gamma^c \) the Galois group \( \text{Gal}(F^c/F) \).
- \( \Lambda^c \) the associated Iwasawa algebra to \( \Gamma^c \).
- \( F^\infty \) an arbitrary \( \mathbb{Z}_p \)-extension of \( F \).
- \( \Gamma^\infty \) the Galois group \( \text{Gal}(F^\infty/F) \).
- \( \Lambda^\infty \) the associated Iwasawa algebra to \( \Gamma^\infty \).
- \( S \) the set of \( p \)-adic and infinite primes.
- \( A(F) \) the \( p \)-group of ideal classes.
- \( A'(F) \) the \( p \)-group of \((p)\)-ideal classes.
- \( X(F) \) the Galois group over \( F \) of the maximal abelian unramified \( p \)-extension of \( F \).
- \( X'(F) \) the Galois group over \( F \) of the maximal abelian unramified \( p \)-extension of \( F \) in which all \( p \)-primes split completely.
- \( F_S \) the maximal extension of \( F \) which is unramified outside \( S \).
- \( G_S(F) \) the Galois group \( \text{Gal}(F_S/F) \).
- \( G_S(L) \) the Galois group \( \text{Gal}(F_S/L) \).
- \( x_S^{(i)}(L) \) the first homological group \( H_1(G_S(L), \mathbb{Z}_p(i)) \).
- \((\cdot)^\vee\) the Pontryagin dual.
- \text{Ind} \text{\( \cdot \)} the induced module.
- \text{Coind} \text{\( \cdot \)} the coinduced module.
- \( r_1 := r_1(F) \) the number of real places of \( F \).
- \( r_2 := r_2(F) \) the number of complex places of \( F \).
- \( n_i = \begin{cases} r_2 & \text{if } i \text{ is even}, \\ r_1 + r_2 & \text{if } i \text{ is odd}. \end{cases} \)

For a module \( M \) over a ring \( R \), we denote by \( \text{Tor}_R(M) \) the \( R \)-torsion sub-module of \( M \) and \( \text{Fr}_R(M) \) the maximal quotient of \( M \) which is torsion free. We write \( \text{pd}_R M \) for the projective dimension of \( M \) over \( R \). If \( R \) is a domain, let \( K_R \) its field of fractions. We call the rank of \( M \) over \( R \) the dimension of the vector space \( M \otimes K_R \), and we denote it by \( \text{rank}_R M \). Recall that a finitely generated \( R \)-module \( M \) is called pseudo-null if \( M_p = 0 \) for all prime ideals \( p \) of height \( \text{ht}(p) \leq 1 \) or equivalently any prime ideal containing the annihilators \( \text{Ann}_R(M) \) is of height \( \geq 2 \). We say that two \( R \)-modules \( M \) and \( M' \) are pseudo-isomorph (and we denote \( M \sim M' \)) if there exists an \( R \)-homomorphism with pseudo-null kernel and cokernel.
Let $\tilde{F}$ be the composite of all $\mathbb{Z}_p$-extensions of $F$. Class field theory gives a description of the Galois group $\tilde{\Gamma} = \text{Gal}(\tilde{F}/F)$, namely
$$\tilde{\Gamma} \simeq \mathbb{Z}_p^{r_2 + 1 + \delta_F},$$
where $\delta_F$ is the Leopoldt defect conjecturally null (Leopoldt’s conjecture). A cohomological version of Leopoldt’s conjecture is the triviality of the second cohomology group $H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p)$. Generally, we have the following twisted analogue of Leopoldt’s conjecture (c.f. Greenberg, Schneider, ...)

**Conjecture (LC$_i$ for short).** For any number field $F$ and all integers $i \neq 1$,
$$H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0.$$

For $i = 0$ (i.e Leopoldt’s conjecture), the conjecture is true if $F/\mathbb{Q}$ is an abelian extension. Also, the conjecture LC$_i$ holds for any number field and any integer $i \geq 2$ by Soulé (c.f [S79]). Note that the conjectures LC$_i$ concern only number fields, thus as for the case $i = 0$ we define the weak version of LC$_i$ for an arbitrary $\mathbb{Z}_p^d$-extension $L$ of $F$.

**Conjecture (WLC$_i$ for short).** Let $L$ be a $\mathbb{Z}_p^d$-extension of the field $F$ and let $i$ be an integer. Then
$$H^2(G_S(L), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0.$$

**Proposition 2.1.** Let $L$ be a $\mathbb{Z}_p^d$-extension of $F$ and $i$ an integer. If $L$ contains a $\mathbb{Z}_p^{d-1}$-extension $L'$ for which WLC$_i$ holds, then $L$ satisfies WLC$_i$.

**Proof.** Let us denote by $\Gamma$ the Galois group of the extension $L/L'$. We have $\text{cd}(\Gamma) \leq 1$. Then the Hochschild-Serre spectral sequence
$$H^p(\Gamma, H^q(G_S(L), \mathbb{Q}_p/\mathbb{Z}_p(i))) \Longrightarrow H^{p+q}(G_S(L'), \mathbb{Q}_p/\mathbb{Z}_p(i))$$
gives the exact sequence
$$0 \rightarrow H^1(\Gamma, H^1(G_S(L), \mathbb{Q}_p/\mathbb{Z}_p(i))) \rightarrow H^2(G_S(L'), \mathbb{Q}_p/\mathbb{Z}_p(i)) \rightarrow H^2(G_S(L), \mathbb{Q}_p/\mathbb{Z}_p(i))^\Gamma \rightarrow 0.$$

Therefore, the vanishing of the group $H^2(G_S(L'), \mathbb{Q}_p/\mathbb{Z}_p(i))$ implies that of $H^2(G_S(L), \mathbb{Q}_p/\mathbb{Z}_p(i))$. □

**Remark 2.2.** (1) Using the same arguments as in the above proof and the Hochschild-Serre spectral sequence associated to the following groups extension
$$0 \rightarrow G_S(F_\infty) \rightarrow G_S(F) \rightarrow \Gamma_\infty \rightarrow 0,$$
we can prove that WLC$_i$ holds for any $\mathbb{Z}_p$-extension if $F$ satisfies LC$_i$. 
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(2) For any integer \(i\), the cyclotomic \(\mathbb{Z}_p\)-extension satisfies \(WLC_i\). Indeed, the Galois group \(G_S(F^c)\) acts trivially on \(\mathbb{Q}_p/\mathbb{Z}_p(k)\) if \(k \equiv 0 \mod [F(\mu_p) : F]\). Hence for any integer \(i\) if we choose \(j \geq 2\) such that \(i \equiv j \mod [F(\mu_p) : F]\), we have

\[H^2(G_S(F^c), \mathbb{Q}_p/\mathbb{Z}_p(i)) = H^2(G_S(F^c), \mathbb{Q}_p/\mathbb{Z}_p(j))(i - j)\]

Hence we conclude that the conjecture \(WLC_i\) holds for the cyclotomic \(\mathbb{Z}_p\)-extension at any integer \(i\), and thus for all \(\mathbb{Z}_p^d\)-extension containing \(F^c\).

In the rest of this section we study the \(\Lambda_L\)-module \(X^{(-i)}_S(L)\), using the methods of Greenberg ([G78]). We show that \(X^{(-i)}_S(L)\) is a finitely generated \(\Lambda_L\)-module and we determine its \(\Lambda_L\)-rank. While Greenberg uses Leopoldt’s conjecture, here we use \(LC_i\) or \(WLC_i\). Let’s start with the following helpful lemma.

**Lemma 2.3.** Let \(L\) be a \(\mathbb{Z}_p^d\)-extension \((d \geq 1)\) of \(F\) and let \(L'\) be a subfield of \(L\) such that \(\Gamma := \text{Gal}(L/L') \simeq \mathbb{Z}_p\). Then we have an exact sequence of \(\mathbb{Z}_p[[\text{Gal}(L'/F)]]\)-modules

\[0 \rightarrow (X^{(-i)}_S(L))_\Gamma \rightarrow X^{(-i)}_S(L') \rightarrow H_1(\Gamma, \mathbb{Z}_p(-i)G_S(L)) \rightarrow 0.\]  

**Proof.** The Hochschild-Serre spectral sequence associated to the groups extension

\[0 \rightarrow G_S(L) \rightarrow G_S(L') \rightarrow \Gamma \rightarrow 0,\]

gives rise to the following exact sequence

\[0 \rightarrow H^1(\Gamma, \mathbb{Q}_p/\mathbb{Z}_p(i)^{G_S(L)}) \rightarrow H^1(G_S(L'), \mathbb{Q}_p/\mathbb{Z}_p(i)) \rightarrow H^1(G_S(L), \mathbb{Q}_p/\mathbb{Z}_p(i))^\Gamma \rightarrow 0,
\]

since \(\text{cd}^{\Gamma} \leq 1\). Therefore we get the exact sequence of the lemma by applying Pontryagin duality. \(\square\)

Let \(F_\infty\) be a \(\mathbb{Z}_p\)-extension of \(F\) and \(F_n\) be the \(n\)-th layers of \(F_\infty/F\). Using [Sc79, Satz 6] we have for any \(i \neq 0\)

\[\text{rank}_{\mathbb{Z}_p} X^{(-i)}_S(F_n) = n_i p^n + \delta_{i,n},\]  

where \(\delta_{i,n} = \text{rank}_{\mathbb{Z}_p} H_2(G_S(F_n), \mathbb{Z}_p(-i))\) and

\[n_i = \begin{cases} r_2 & \text{if } i \text{ is even}, \\ r_1 + r_2 & \text{if } i \text{ is odd}. \end{cases}\]
Theorem 2.4. Let $i$ be an integer and let $F_\infty$ be a $\mathbb{Z}_p$-extension of $F$ satisfying WLC$^i$. Then

$$\text{rank}_{\Lambda_\infty} \mathfrak{x}_S^{(-i)}(F_\infty) = \begin{cases} r_2 & \text{if } i \text{ is even,} \\ r_1 + r_2 & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Notice that the $\mathbb{Z}_p$-module $\mathbb{Z}_p(-i)_{G_S(F_\infty)}$ is not finite if and only if one of the following two cases hold:

1. $i = 0$.
2. $F_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension and $i \equiv 0 \mod [F(\mu_p) : F]$.

The case (1) is Proposition 1 of [G78]. In the case (2) the group $G_S(F_\infty)$ acts trivially on $\mathbb{Z}_p(i)$, hence

$$\mathfrak{x}_S^{(-i)}(F_\infty) = \mathfrak{x}_S(F_\infty)(-i).$$

Then the theorem is deduced from Proposition 1 of [G78]. In the sequel we assume that we are out of these two cases. Let $\Gamma_{\infty,n} = \text{Gal}(F_\infty/F_n)$, the exact sequence [11] and the equality [2] give that

$$\text{rank}_{\mathbb{Z}_p} \mathfrak{x}_S^{(-i)}(F_\infty)_{\Gamma_{\infty,n}} = \text{rank}_{\mathbb{Z}_p} \mathfrak{x}_S^{(-i)}(F_n) = n_ip^n + \delta_{i,n}, \quad (3)$$

where $\delta_{i,n} = \text{rank}_{\mathbb{Z}_p} H_2(G_S(F_n), \mathbb{Z}_p(-i))$. In particular, this shows that $\mathfrak{x}_S^{(-i)}(F_\infty)$ is a finitely generated $\Lambda_\infty$-module. Let $\rho$ be the $\Lambda_\infty$-rank of $\mathfrak{x}_S^{(-i)}(F_\infty)$. By the structure theorem of the $\Lambda_\infty$-module $\text{Fr}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty))$ we have an exact sequence

$$0 \longrightarrow \text{Fr}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty)) \longrightarrow \Lambda_\infty^n \longrightarrow Z \longrightarrow 0, \quad (4)$$

where $Z$ is a finite $\Lambda_\infty$-module. Since $(\Lambda_\infty^n)_{\Gamma_{\infty,n}} = 0$, we obtain by applying the snack lemma to the exact sequence [4] that

$$\text{Fr}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty))_{\Gamma_{\infty,n}} = 0. \quad (5)$$

The exact sequence

$$0 \longrightarrow \text{Tor}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty)) \longrightarrow \mathfrak{x}_S^{(-i)}(F_\infty) \longrightarrow \text{Fr}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty)) \longrightarrow 0$$

and the snack lemma lead to the exact sequence

$$0 \longrightarrow \text{Tor}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty))_{\Gamma_{\infty,n}} \longrightarrow \mathfrak{x}_S^{(-i)}(F_\infty)_{\Gamma_{\infty,n}} \longrightarrow \text{Fr}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty))_{\Gamma_{\infty,n}} \quad (6)$$

$$0 \longleftarrow \text{Fr}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty))_{\Gamma_{\infty,n}} \longleftarrow \mathfrak{x}_S^{(-i)}(F_\infty)_{\Gamma_{\infty,n}} \longleftarrow \text{Tor}_{\Lambda_\infty}(\mathfrak{x}_S^{(-i)}(F_\infty))_{\Gamma_{\infty,n}}.$$
According to (5), the exact sequence (6) gives the exact sequence
\[ 0 \rightarrow (\text{Tor}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty)))_{\Gamma_{\infty,n}} \rightarrow \mathcal{X}_S^{(-i)}(F_\infty)_{\Gamma_{\infty,n}} \rightarrow (\text{Fr}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty)))_{\Gamma_{\infty,n}} \rightarrow 0. \]
(7)
and the isomorphism
\[ (\text{Tor}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty)))_{\Gamma_{\infty,n}} \simeq \mathcal{X}_S^{(-i)}(F_\infty)_{\Gamma_{\infty,n}}. \]
(8)
It is well known that (e.g. [W85, page 337])
\[ \text{rank}_{\mathbb{Z}}(\text{Tor}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty)))_{\Gamma_{\infty,n}} = \text{rank}_{\mathbb{Z}}(\text{Tor}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty)))_{\Gamma_{\infty,n}}. \]
(9)
Since \( H^2(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0 \) (by WLC) and \( \text{cd}(\Gamma_{\infty,n}) \leq 1 \), the Hochschild-Serre spectral sequence associated to the groups extension
\[ 0 \longrightarrow G_S(F_n) \longrightarrow G_S(F_\infty) \longrightarrow \Gamma_{\infty,n} \longrightarrow 0, \]
gives the following isomorphism
\[ H^2(G_S(F_n), \mathbb{Q}_p/\mathbb{Z}_p(i)) \simeq H^1(\Gamma_{\infty,n}, H^1(G_S(F_\infty), \mathbb{Q}_p/\mathbb{Z}_p(i))). \]
Then by the cohomology-homology duality we get
\[ H_2(G_S(F_n), \mathbb{Z}_p(-i)) \simeq \mathcal{X}_S^{(-i)}(F_\infty)_{\Gamma_{\infty,n}}. \]
From (8) and (9) it follows that
\[ \text{rank}_{\mathbb{Z}_p}(\text{Tor}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty)))_{\Gamma_{\infty,n}} = \delta_{i,n}. \]
(10)
The \( \Gamma_{\infty,n} \)-homology of the exact sequence (4) gives rise to the following exact sequence
\[ 0 \longrightarrow \mathbb{Z}_{\Gamma_{\infty,n}} \longrightarrow \text{Fr}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty))_{\Gamma_{\infty,n}} \longrightarrow (\Lambda_\infty)_{\Gamma_{\infty,n}} \longrightarrow \mathbb{Z}_{\Gamma_{\infty,n}} \longrightarrow 0. \]
Also we know that the \( \mathbb{Z}_p \)-rank of \( (\Lambda_\infty)_{\Gamma_{\infty,n}} \) is \( p^n \), since
\[ (\Lambda_\infty)_{\Gamma_{\infty,n}} \simeq \mathbb{Z}_p[[\Gamma_{\infty}/\Gamma_{\infty,n}]]. \]
Then, the \( \mathbb{Z}_p \)-rank of \( \text{Fr}_{\Lambda_{\infty}}(\mathcal{X}_S^{(-i)}(F_\infty))_{\Gamma_{\infty,n}} \) equals to \( \rho p^n \).
According to (3), the exact sequence (7) and the equality (10) we obtain that
\[ \rho = n_i. \]
\[ \square \]

The following result is an extension of the above theorem to multiple \( \mathbb{Z}_p \)-extensions.
Theorem 2.5. Let $L$ be a $\mathbb{Z}_p^d$-extension ($d \geq 1$) of $F$ which contains a $\mathbb{Z}_p$-extension $F_\infty$, satisfying WLC$^*_1$. Then

$$\text{rank}_{\Lambda_L} \mathcal{X}_S^{(-i)}(L) = \begin{cases} r_2 & \text{if } i \text{ is even,} \\ r_1 + r_2 & \text{if } i \text{ is odd.} \end{cases}$$

Proof. We proceed by induction on $d$. For $d = 1$, this is Theorem 2.4.

Let $d \geq 2$ and $L' \subset L$ be a $\mathbb{Z}_p^{d-1}$-extension of $F$ such that

- $L'$ contains $F_\infty$;
- If $\Gamma = \text{Gal}(L/L')$ and $\gamma$ is a topological generator of $\Gamma$ then $\gamma - 1$ is prime to the annihilator of $\text{Tor}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))$. Such choice is possible since the annihilator of $\text{Tor}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))$ is divisible by a finite number of primes (e.g. [G78, Lemma 2]).

Let $\Gamma_{L'} = \text{Gal}(L'/F)$ and $\Lambda_{L'}$ the associated Iwasawa algebra to $\Gamma_{L'}$. This choice of $L'$ implies that $\text{Tor}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))_{\Gamma} = \Lambda_{L'}$-torsion.

Let $x$ be an element of $\text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))$. We have

$$x \in \text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))_{\Gamma} \text{ if and only if } (\gamma - 1)x = 0.$$ Since $\gamma - 1$ is an element of $\Lambda_L$ and $\text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))$ is without $\Lambda_L$-torsion, we obtain that

$$H_1(\Gamma, \text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))) = \text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))^\Gamma = 0.$$ Therefore, the $\Gamma$-homology of the exact sequence

$$0 \longrightarrow \text{Tor}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L)) \longrightarrow \mathcal{X}_S^{(-i)}(L) \longrightarrow \text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L)) \longrightarrow 0$$
gives rise to the exact sequence

$$0 \longrightarrow \text{Tor}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))_{\Gamma} \longrightarrow \mathcal{X}_S^{(-i)}(L)_{\Gamma} \longrightarrow \text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L)_{\Gamma} \longrightarrow 0. \quad (11)$$

Recall that $\text{Tor}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))_{\Gamma}$ is $\Lambda_{L'}$-torsion hence follows that

$$\text{rank}_{\Lambda_L} \mathcal{X}_S^{(-i)}(L)_{\Gamma} = \text{rank}_{\Lambda_L} \text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))_{\Gamma} = \text{rank}_{\Lambda_L} \mathcal{X}_S^{(-i)}(L), \quad (12)$$

since $\text{Fr}_{\Lambda_L}(\mathcal{X}_S^{(-i)}(L))$ is without $\Lambda_L$-torsion and have the same $\Lambda_L$-rank as $\mathcal{X}_S^{(-i)}(L)$ (e.g. [G78, page 90]).

The exact sequence (11) shows that

$$\text{rank}_{\Lambda_L} \mathcal{X}_S^{(-i)}(L') = \text{rank}_{\Lambda_L} \mathcal{X}_S^{(-i)}(L)_{\Gamma} + \text{rank}_{\Lambda_L} H_1(\Gamma, \mathbb{Z}_p(-i)G_S(L)). \quad (13)$$
Observe that \( H_1(\Gamma, \mathbb{Z}_p(-i)_{\text{Gr}_S(L)}) = (\mathbb{Z}_p(-i)_{\text{Gr}_S(L)})^\Gamma \), thus \( H_1(\Gamma, \mathbb{Z}_p(-i)_{\text{Gr}_S(L)}) \) is either finite or isomorphic to \( \mathbb{Z}_p \). In particular, \( \text{rank}_{\Lambda_L} H_1(\Gamma, \mathbb{Z}_p(-i)_{\text{Gr}_S(L)}) \) is zero. Therefore, it follows from (12) and (13) that \( \text{rank}_{\Lambda_L} \mathfrak{A}_S^{(-i)}(L) = \text{rank}_{\Lambda_L} \mathfrak{A}_S^{(-i)}(L') \).

Now we focus on the projective dimension of the twisted \( S \)-ramified Iwasawa module, \( \mathfrak{A}_S^{(-i)}(L) \). As in above we regard firstly the case when \( L \) is a \( \mathbb{Z}_p \)-extension of \( F \).

**Proposition 2.6.** Let \( F_\infty \) be a \( \mathbb{Z}_p \)-extension of \( F \) for which WLC\(_i\) holds. Then \( \mathfrak{A}_S^{(-i)}(F_\infty) \) is of projective dimension at most 1.

**Proof.** It is known that \( \mathfrak{A}_S^{(-i)}(F_\infty) \) is of projective dimension at most 1 if and only if \( \mathfrak{A}_S^{(-i)}(F_\infty)^{\Gamma_{\infty,n}} \) is \( \mathbb{Z}_p \)-free for some \( n \), where \( \Gamma_{\infty,n} = \text{Gal}(F_\infty/F_n) \) and \( F_n \) is the \( n \)th layer of \( F_\infty \) (see \([W85, \text{Proposition 2.1}]\)). We claim that \( \mathfrak{A}_S^{(-i)}(F_\infty)^{\Gamma_{\infty,n}} \) is \( \mathbb{Z}_p \)-free.

Since \( \text{cd}(\Gamma_{\infty,n}) \leq 1 \), the homological Hoschilde-Serre spectral sequence associated to the groups extension

\[
0 \longrightarrow G_S(F_\infty) \longrightarrow G_S(F) \longrightarrow \Gamma_\infty \longrightarrow 0
\]

gives the following isomorphism

\[
H_1(\Gamma_{\infty}, \mathfrak{A}_S^{(-i)}(F_\infty)) \cong H_2(G_S(F), \mathbb{Z}_p(-i)). \quad (14)
\]

We know that \( \text{cd}(G_S(F)) \leq 2 \), so the \( G_S(F) \)-cohomology of the exact sequence

\[
0 \longrightarrow \mathbb{Z}_p(i) \longrightarrow \mathbb{Q}_p(i) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p(i) \longrightarrow 0
\]

gives that \( H^2(G_S(F), \mathbb{Q}_p/\mathbb{Z}_p(i)) \) is divisible, hence the \( \mathbb{Z}_p \)-module \( H_2(G_S(F), \mathbb{Z}_p(-i)) \) is \( \mathbb{Z}_p \)-free.

Recall that

\[
\mathfrak{A}_S^{(-i)}(F_\infty)^{\Gamma_{\infty,n}} = H_1(\Gamma_{\infty,n}, \mathfrak{A}_S^{(-i)}(F_\infty)).
\]

It follows then from the isomorphism (14) that \( \mathfrak{A}_S^{(-i)}(F_\infty)^{\Gamma_{\infty,n}} \) is \( \mathbb{Z}_p \)-free which means that \( \mathfrak{A}_S^{(-i)}(F_\infty) \) is of projective dimension at most 1. \( \square \)

For any module \( M \) over \( \mathbb{Z}_p[[\Gamma]] \), with \( \Gamma \simeq \mathbb{Z}_p \), we have the equivalence (e.g \([W85, \text{Proposition 2.1}]\))

\[
\text{pd}_{\mathbb{Z}_p[[\Gamma]]} M \leq 1 \iff M \text{ has no non trivial pseudo-null submodules}.
\]
However, this equivalence is not in general true when \( \Gamma \cong \mathbb{Z}_p^d \) with \( d \geq 2 \) (c.f \([P83\text{ Proposition 6 chap. I}])\). To extend Proposition 2.6 to multiple \( \mathbb{Z}_p \)-extensions we start with the study of pseudo-null submodules. For \( i = 0 \) we have the following result of Greenberg

**Proposition 2.7** \([G78\text{ Proposition 5}]). Assume that Leopoldt’s conjecture is valid for \( F \). Let \( L \) be a \( \mathbb{Z}_p^d \)-extension of \( F \) \((d \geq 1)\). Then \( \mathfrak{x}_S(L) \) contains no non trivial pseudo-null \( \Lambda_L \)-submodule.

The next proposition is a generalization of Greenberg’s result.

**Proposition 2.8.** Let \( i \) be an integer and suppose that \( F \) satisfies \( LC_i \). Let \( L \) be a \( \mathbb{Z}_p^d \)-extension \((d \geq 1)\) of \( F \). Then \( \mathfrak{x}_S^{(-i)}(L) \) contains no non trivial pseudo-null submodules.

**Proof.** (Sketch) The proof of this proposition is an adaptation of that of \([G78\text{ Proposition 5}]. It suffices to take \( \mathfrak{x}_S^{(-i)}(L) \) instead of \( \mathfrak{x}_S(L) \) and remark that the restriction map

\[
(\mathfrak{x}_S^{(-i)}(L))_{\Gamma} \longrightarrow \mathfrak{x}_S^{(-i)}(L'),
\]

is injective for any integer \( i \) and any \( \mathbb{Z}_p^{d-1} \)-subextension \( L' \) of \( L \) (see Lemma 2.3). Also we replace Leopoldt’s conjecture by \( LC_i \). \( \square \)

Now we can prove that the projective dimension of \( \mathfrak{x}_S^{(-i)}(L) \) over \( \Lambda_L \) is at most 1 when \( i \not\equiv 0 \mod [F(\mu_p) : F] \).

**Theorem 2.9.** Let \( i \) be an integer such that \( i \not\equiv 0 \mod [F(\mu_p) : F] \) and \( F \) satisfies \( LC_i \). For any \( \mathbb{Z}_p^d \)-extension \( L \) of \( F \), the \( \Lambda_L \)-module \( \mathfrak{x}_S^{(-i)}(L) \) is of projective dimension at most 1.

**Proof.** We prove this theorem by induction on \( d \). The case \( d = 1 \) is Proposition 2.6. For \( d \geq 2 \) we choose a subgroup \( \Gamma \cong \mathbb{Z}_p \) of \( \Gamma_L \) and denote by \( L' \) the subfield of \( L \) fixed by \( \Gamma \).

Recall that

\[
H_0(G_S(L), \mathbb{Z}_p(-i)) = 0 \text{ if and only if } i \not\equiv 0 \mod [F(\mu_p) : F],
\]

When the integer \( i \) satisfies \( i \not\equiv 0 \mod [F(\mu_p) : F] \), we then obtain by Lemma 2.3 the isomorphism

\[
(\mathfrak{x}_S^{(-i)}(L))_{\Gamma} \cong \mathfrak{x}_S^{(-i)}(L').
\]

By Proposition 2.8, the \( \Lambda_L \)-module \( \mathfrak{x}_S^{(-i)}(L) \) has no non trivial pseudo-null sub-modules. According to the proof of \([P83\text{ Lemme 5, chap. I}], we have

\[
\text{pd}_{\Lambda_L}(\mathfrak{x}_S^{(-i)}(L)) = \text{pd}_{\Lambda_L/q}(\mathfrak{x}_S^{(-i)}(L)/q \mathfrak{x}_S^{(-i)}(L)),
\]
for every prime ideal $q$ of height 1 such that $\Lambda_L/q$ is regular and $q$ does not divide the characteristic series of $\mathfrak{X}_S^{(-i)}(L)$.

Notice that $(\Lambda_L)_\Gamma \simeq \Lambda_{L'}$ which is a regular ring. Let $\gamma$ be a topological generator of $\Gamma$. Then if we choose $\Gamma$ such that $\gamma - 1$ does not divide the characteristic series $X_S^{(-i)}(L)$ (such a choice always exist, since the characteristic series of $X_S^{(-i)}(L)$ is divisible by a finite number of primes) we obtain that

$$\text{pd}_{\Lambda_L}(\mathfrak{X}_S^{(-i)}(L)) = \text{pd}_{\Lambda_{L'}}((\mathfrak{X}_S^{(-i)}(L))_\Gamma)$$

Therefore, it follows from \cite{15} that

$$\text{pd}_{\Lambda_L}(\mathfrak{X}_S^{(-i)}(L)) = \text{pd}_{\Lambda_{L'}}(\mathfrak{X}_S^{(-i)}(L'))$$.

\[ \square \]

**Remark 2.10.** When $i = 0$ and $L$ satisfies WLC$_i$, Nguyen Quang Do \cite{83, Proposition 3.4] shows that

- $\text{pd}_{\Lambda_L} \mathfrak{X}_S(L) \leq 1$ if $d \leq 2$ and,
- $\text{pd}_{\Lambda_L} \mathfrak{X}_S(L) = d - 2$ if $d > 2$.

Notice that if $i \equiv 0 \mod [F(\mu_p)/F]$ and $L$ contains the cyclotomic $\mathbb{Z}_p$-extension, the discussion in the proof of Theorem \cite{2.4} shows that

$$\mathfrak{X}_S^{(-i)}(L) = \mathfrak{X}_S(L)(-i).$$

Then by \cite{83, Proposition 3.4], if $i \equiv 0 \mod [F(\mu_p)/F]$ we have

- $\text{pd}_{\Lambda_L} \mathfrak{X}_S^{(-i)}(L) \leq 1$ if $d \leq 2$ and,
- $\text{pd}_{\Lambda_L} \mathfrak{X}_S^{(-i)}(L) = d - 2$ if $d > 2$.

3. Greenberg conjecture and torsion triviality

For any number field $F$, let $X(F)$ (respectively $X'(F)$) be the Galois group over $F$ of the maximal abelian unramified (respectively unramified and all $p$-primes split completely) $p$-extension of $F$. The groups $X(F)$ and $X'(F)$ are isomorphic to the $p$-group $A(F)$ of ideal classes and the $p$-group $A'(F)$ of $(p)$-ideal classes, respectively. Let $L$ be a multiple $\mathbb{Z}_p$-extension of $F$. We consider the two $\Lambda_L$-modules

$$X(L) = \lim_i X(E) \quad \text{and} \quad X'(L) = \lim_i X'(E),$$

where $E$ runs over finite sub-extensions of $L$ and the inverse limit is taken via the norm maps. It is well known that $X(L)$ (resp. $X'(L)$) is isomorphic to the maximal abelian unramified (respectively unramified and all $p$-primes split completely) pro-$p$-extension of $L$. 

In this section we show that for any number field $F$ satisfying $LC_i$, the pseudo-nullity of the $\tilde{\Lambda}$-module $X(\tilde{F}(\mu_p))(i-1)^{\Delta}$ provides the existence of a $\mathbb{Z}_p$-extension $F_\infty/F$ for which the associated twisted $S$-ramified Iwasawa module is without torsion over its associated Iwasawa algebra.

We have the following conjectures of Greenberg.

**Conjecture** (GC for short). The $\Lambda_c$-module $X(F^c)$ is finite for any totally real number field $F$.

**Conjecture** (GGC for short). For any number field $F$ the $\tilde{\Lambda}$-module $X(\tilde{F})$ is pseudo-null.

Also we find in the literature an étale version of GGC (c.f [N19])

**Conjecture** (GGC’ for short). For any number field $F$ the $\tilde{\Lambda}$-module $X'(\tilde{F})$ is pseudo-null.

By construction the $\tilde{\Lambda}$-module $X'(\tilde{F})$ is a quotient of $X(\tilde{F})$, thus the validity of GGC implies that of GGC’. Moreover, the other implication was considered by Nguyen Quang Do in [N19], precisely he shows that

$$GGC \text{ holds for } F \iff GGC' \text{ holds for } F,$$

(16) when the field $F$ satisfies the following condition:

$$(\text{Dec}) : \text{All } p\text{-adic places has decomposition group in the extension } \tilde{F}/F \text{ of } \mathbb{Z}_p\text{-rank at least } 2.$$  

This condition is known to be true for any imaginary Galois extension of $\mathbb{Q}$ and any number field which contains $\mu_p$ ([L-N, Theorem 3.2]).

The next result shows that the equivalence (16) remains true if we take the $e_i$-components of $X'(\tilde{F}(\mu_p))$ and $X(\tilde{F}(\mu_p))$

**Proposition 3.1.** Let $F$ be a number field which satisfies the condition (Dec). For any integer $i$, we have a pseudo-isomorphism

$$X(\tilde{F}(\mu_p))(i)^{\Delta} \sim X'(\tilde{F}(\mu_p))(i)^{\Delta},$$

where $\Delta$ is the Galois group of the extension $\tilde{F}(\mu_p)/\tilde{F}$.

**Proof.** For any finite subextension $E/F$ of $\tilde{F}(\mu_p)/F$ we have an exact sequence (e.g [B-N, Exact sequence (3μ)], [H-K, page 451])

$$0 \rightarrow U_E \rightarrow U'_E \rightarrow \bigoplus_{v \in S_p(E)} \mathbb{Z}_p \rightarrow A(E) \rightarrow A'(E) \rightarrow 0,$$
where $U_E$ (resp. $U'_E$) is the group of units (resp. $(p)$-units) of $E$ and $S_p(E)$ is the set of $p$-adic places in $E$. Taking the projective limit over all finite subextensions of $\tilde{F}(\mu_p)/F$ with respect to the norm maps we get the exact sequence

$$0 \rightarrow \lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p)}} U_E \rightarrow \lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p)}} U'_E \rightarrow \lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p) \atop v \in S_p(E)}} \mathbb{Z}_p \rightarrow X(\tilde{F}(\mu_p)) \rightarrow X'(\tilde{F}(\mu_p)) \rightarrow 0.$$  \hspace{1cm} \text{(17)}

For any finite subextension $E/F$ of $\tilde{F}(\mu_p)/F$ we denote $G_E = \text{Gal}(E/F)$ and $G_{E,v}$ the decomposition subgroup of $v$ in the extension $E/F$. Let $\tilde{\Gamma}_E = \text{Gal}(\tilde{F}(\mu_p)/E)$ and $\tilde{\Gamma}_{E,v}$ the decomposition subgroup at $v$ in the extension $\tilde{F}(\mu_p)/E$. Observe that

$$G_E = \tilde{\Gamma}_E/\tilde{\Gamma}_E \text{ and,}$$

$$G_{E,v} = \tilde{\Gamma}_{E,v}/\tilde{\Gamma}_{E,v}$$

$$= \tilde{\Gamma}_{F,v} \cap \tilde{\Gamma}_E$$

$$= \tilde{\Gamma}_{F,v} \tilde{\Gamma}_E/\tilde{\Gamma}_E.$$  

We have

$$\lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p) \atop v \in S_p(E)}} \bigoplus_{v \in S_p(E)} \mathbb{Z}_p = \lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p) \atop v \in S_p(F) \atop w \mid v}} \bigoplus_{v \in S_p(F)} \mathbb{Z}_p$$

$$= \bigoplus_{v \in S_p(F)} \lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p) \atop v \in S_p(F)}} \text{Ind}_{G_{E,v}}^{G_E} \mathbb{Z}_p$$

$$= \bigoplus_{v \in S_p(F)} \lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p) \atop v \in S_p(F)}} \text{Ind}_{\tilde{\Gamma}_{F,v} \tilde{\Gamma}_E/\tilde{\Gamma}_E}^{\tilde{\Gamma}_F/\tilde{\Gamma}_E} \mathbb{Z}_p$$

By \cite{R-Z} Theorem 6.10.8] and the definition of the induced module we have

$$\bigoplus_{v \in S_p(F)} \lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p) \atop v \in S_p(F)}} \text{Ind}_{\tilde{\Gamma}_{F,v} \tilde{\Gamma}_E/\tilde{\Gamma}_E}^{\tilde{\Gamma}_F/\tilde{\Gamma}_E} \mathbb{Z}_p = \bigoplus_{v \in S_p(F)} \text{Ind}_{\tilde{\Gamma}_{F,v}}^{\tilde{\Gamma}_F} \mathbb{Z}_p$$

$$= \bigoplus_{v \in S_p(F)} Z_p \hat{\otimes} Z_p[[\tilde{\Gamma}_{F,v}]]Z_p[[\tilde{\Gamma}_F]]$$

According to \cite{R-Z} Proposition 5.8.1] we get

$$\bigoplus_{v \in S_p(F)} Z_p \hat{\otimes} Z_p[[\tilde{\Gamma}_{F,v}]]Z_p[[\tilde{\Gamma}_F]] \simeq \bigoplus_{v \in S_p(F)} Z_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]].$$

Hence we conclude that

$$\lim_{\substack{\longrightarrow \\ F \subset E \subset \tilde{F}(\mu_p) \atop v \in S_p(E)}} \bigoplus_{v \in S_p(E)} \mathbb{Z}_p \simeq \bigoplus_{v \in S_p(F)} Z_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]].$$
After twisting $i$ times and taking the $\Delta$-invariant the exact sequence (17) becomes

$$0 \rightarrow \lim_{F \subset E \subset \tilde{F}(\mu_p)} U_E(i) \Delta \rightarrow \lim_{F \subset E \subset \tilde{F}(\mu_p)} U'_E(i) \Delta \oplus_{v \in S_p(F)} \mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i) \Delta \rightarrow 0$$

Let $\Delta_v$ be the decomposition subgroup of a fixed place of $\tilde{F}(\mu_p)$ above $v$ in the extension $\tilde{F}(\mu_p)/\tilde{F}$. Since $\Delta_v$ is a subgroup of $\tilde{\Gamma}_F$, it acts trivially on $\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]]$. Then for any $f \in \mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)$,

$$f \in \mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta_v} \Leftrightarrow \omega^i(\sigma)f = f \text{ for all } \sigma \in \Delta_v,$$

where $\omega$ is the Teichmuller character. Thus $\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta_v}$ is zero exactly when $\omega^i(\Delta_v) \neq 1$. We know that

$$\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta} \subset \mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta_v}.$$

Hence if $\omega^i(\Delta_v) \neq 1$, $\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta}$ is zero.

Suppose now that $\omega^i(\Delta_v) = 1$, then

$$\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta} = \mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta/\Delta_v}.$$

Let $\tilde{\Gamma}_v$ be the decomposition group of $v$ in the extension $\tilde{F}/F$. Notice that we have the following decomposition

$$\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v} = \Delta/\Delta_v \times \tilde{\Gamma}/\tilde{\Gamma}_v.$$

Then we obtain

$$\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta/\Delta_v} \simeq \mathbb{Z}_p[[\tilde{\Gamma}/\tilde{\Gamma}_v]].$$

Therefore, the pseudo-null of the $\tilde{\Lambda}$-module $\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta}$ is implied by the condition (Dec), in this case.

Then we conclude that under the condition (Dec), $\mathbb{Z}_p[[\tilde{\Gamma}_F/\tilde{\Gamma}_{F,v}]](i)_{\Delta}$ is pseudo-null over $\tilde{\Lambda}$.

Therefore, the last exact sequence gives the desired pseudo-isomorphism.

In [L-N], the authors show many formulations for Greenberg generalized conjecture (see also [H96, Section 2.2]). More precisely, Proposition 3.6 in [L-N] gives, for any number field $F$ satisfying the condition (Dec), the equivalence between the validity of $GGC'$ and the triviality of $\text{Tor}_{\tilde{\Lambda}}(\mathfrak{X}_S^{(-1)}(\tilde{F}(\mu_p)))^{\text{Gal}(F(\mu_p)/F)}$. Here we give a twisted version of this equivalence. We start with the following helpful result.
Proposition 3.2. Let $F$ be a number field. Then for any integer $i$ we have an isomorphism
\[
\lim_{\leftarrow v \in S(F)} \bigoplus_{i=1}^{\infty} H^2(E_v, \mathbb{Z}_p(i)) \cong \bigoplus_{v \in S(F)} \text{Ind}^F_{\Gamma_v} H^0(\bar{F}_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee,
\]
where the inverse limit is taken via the corestriction maps and $(\bar{F})_v$ is the completion of $\bar{F}$ at a fixed prime above $v \in S(F)$. In particular, if $F$ satisfies the condition (Dec), then $\lim_{\leftarrow v \in S(E)} H^2(E_v, \mathbb{Z}_p(i))$ is pseudo-null.

Proof. For any finite subextension $E/F$ of $\bar{F}/F$, let $G = \text{Gal}(E/F)$ and $G_v$ the decomposition subgroup of $v \in S$ in the extension $E/F$. We have
\[
\lim_{\leftarrow v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) = \lim_{\leftarrow v \in S(F)} \bigoplus_{w|v} H^2(E_w, \mathbb{Z}_p(i)) = \bigoplus_{v \in S(F)} \lim_{\leftarrow v \in S(F)} \text{Ind}^G_{G_v} H^0(E_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee.
\]
By local duality we have
\[
H^2(E_v, \mathbb{Z}_p(i)) \cong H^0(E_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee.
\]
Therefore
\[
\lim_{\leftarrow v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) \cong \bigoplus_{v \in S(F)} \lim_{\leftarrow v \in S(F)} \text{Ind}^G_{G_v} H^0(E_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee.
\]
(18)
Recall that
\[
\text{Ind}^G_{G_v} (H^0(E_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee) = (\text{Coind}^G_{G_v} H^0(E_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)))^\vee.
\]
Hence the isomorphism (18) becomes
\[
\lim_{\leftarrow v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) \cong \bigoplus_{v \in S(F)} (\lim_{\leftarrow v \in S(F)} \text{Coind}^G_{G_v} H^0(E_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)))^\vee.
\]
Let’s denote $\tilde{\Gamma} = \text{Gal}(\bar{F}/F)$, $\tilde{\Gamma}_E = \text{Gal}(\bar{F}/E)$, $\tilde{\Gamma}_v = \text{Gal}(\bar{F}_v/F_v)$ and $\tilde{\Gamma}_{v,E} = \text{Gal}(\bar{F}_v/E_v)$. Observe that
\[
G = \tilde{\Gamma}/\tilde{\Gamma}_E \text{ and } G_v = \tilde{\Gamma}_v/\tilde{\Gamma}_{v,E} = \tilde{\Gamma}_E \tilde{\Gamma}_v/\tilde{\Gamma}_E,
\]
and
\[
H^0(E_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) = H^0((\bar{F}_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee.
\]
Therefore, using [R-Z, Proposition 6.10.4] we conclude that
\[
\lim_{\leftarrow} \bigoplus_{v \in S_p(E)} H^2(E_v, \mathbb{Z}_p(i)) = \bigoplus_{v \in S_p(F)} \text{Coind}_{\hat{F}_v}^F H^0((\hat{F})_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee
\]
\[
= \bigoplus_{v \in S(F)} \text{Ind}_{\hat{F}_v}^F (H^0((\hat{F})_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee). 
\]

By the definition of the induced module we obtain that
\[
\lim_{\leftarrow} \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) = \bigoplus_{v \in S(F)} H^0((\hat{F})_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i))^\vee \otimes_{\mathbb{Z}_p[[\tilde{\Gamma}_v]]} \tilde{\Lambda}.
\]

Let \( v \in S(F) \), the local field \((\hat{F})_v\) contains the cyclotomic \(\mathbb{Z}_p\)-extension of \(F_v\). For such \( v \) there are two possible cases

1. \( i \equiv 1 \mod \left[ F_v(\mu_p) : F_v \right] \), then the absolute Galois group \( G_{(\hat{F})_v} \) act trivially on \( \mathbb{Q}_p/\mathbb{Z}_p(1-i) \). Hence
   \[
   H^0((\hat{F})_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) = \mathbb{Q}_p/\mathbb{Z}_p(1-i).
   \]

2. \( i \not\equiv 1 \mod \left[ F_v(\mu_p) : F_v \right] \), the action of \( G_{(\hat{F})_v} \) on \( \mathbb{Q}_p/\mathbb{Z}_p(1-i) \) is not trivial and the cohomological group \( H^0((\hat{F})_v, \mathbb{Q}_p/\mathbb{Z}_p(1-i)) \) is zero.

Thus we conclude that
\[
\lim_{\leftarrow} \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) = \bigoplus_{i \equiv 1 \mod \left[ F_v(\mu_p) : F_v \right]} \mathbb{Z}_p(i-1) \otimes_{\mathbb{Z}_p[[\tilde{\Gamma}_v]]} \tilde{\Lambda}.
\]

Hence using [R-Z, Proposition 5.8.1] we obtain
\[
\lim_{\leftarrow} \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) \simeq \bigoplus_{i \equiv 1 \mod \left[ F_v(\mu_p) : F_v \right]} \mathbb{Z}_p[[\tilde{\Gamma}/\tilde{\Gamma}_v]](i-1).
\]

Then if \( F \) satisfies the condition (Dec) the \( \tilde{\Lambda} \)-module \( \lim_{\leftarrow} \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) \) is pseudo-null. \( \square \)

The proof of Proposition \ref{3.2} leads to the following corollary.

**Corollary 3.3.** Let \( F \) be a number field and \( i \) an integer. If all \( p \)-adic place \( v \in S_p \) satisfies \( i \not\equiv 1 \mod \left[ F_v(\mu_p) : F_v \right] \). Then
\[
\lim_{\leftarrow} \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) = 0.
\]

Now we can generalize Proposition 3.6 in [L-N] to any twist \( i \not\equiv 0 \mod [F(\mu_p) : F] \).
Theorem 3.4. Let $i$ be an integer such that $i \not\equiv 0 \pmod{[F(\mu_p) : F]}$. For any number field $F$ satisfying $LC_i$ and the condition $(Dec)$, the following assertions are equivalent

1. $\text{Tor}_1(\mathfrak{A}^{(-i)}(\bar{F}^s)) = 0$.
2. $X'(\bar{F}(\mu_p))(i-1)^{\Delta}$ is pseudo-null.
3. $X(\bar{F}(\mu_p))(i-1)^{\Delta}$ is pseudo-null.

Proof. The equivalence between (2) and (3) is a consequence of Proposition 3.1, so we prove here only the equivalence between (1) and (2).

Let $E/F$ be a finite subextension of $\bar{F}/F$, the Poitou-Tate exact sequence writes

$$
0 \to \text{Ker}^2_S(E, \mathbb{Z}_p(i)) \to H^2(G_S(E), \mathbb{Z}_p(i)) \to \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) \to 0
$$

Taking the inverse limit over all finite subextensions of $\bar{F}/F$ with respect to the corestriction maps, we obtain the following exact sequence

$$
0 \to \varprojlim_{E} \text{Ker}^2_S(E, \mathbb{Z}_p(i)) \to \varprojlim_{E} H^2(G_S(E), \mathbb{Z}_p(i)) \to 0
$$

Proposition 3.2 shows that the $\bar{\Lambda}$-module $\varprojlim_{E} \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i))$ is pseudo-null, hence we have a pseudo isomorphism

$$
\varprojlim_{E} \text{Ker}^2_S(E, \mathbb{Z}_p(i)) \sim \varprojlim_{E} H^2(G_S(E), \mathbb{Z}_p(i)). \quad (19)
$$

Furthermore, it is well known that

$$
\varprojlim_{E} \text{Ker}^2_S(E, \mathbb{Z}_p(i)) \simeq X'(\bar{F}(\mu_p))(i-1)^{\Delta}.
$$

Then the pseudo isomorphism (19) becomes

$$
X'(\bar{F}(\mu_p))(i-1)^{\Delta} \sim \varprojlim_{E} H^2(G_S(E), \mathbb{Z}_p(i)). \quad (20)
$$

For any $\bar{\Lambda}$-module $M$ and any positive integer $k$, let (J89 Definition 1.4)

$$
E^k(M) := \text{Ext}^k_{\bar{\Lambda}}(M, \bar{\Lambda}).
$$
In particular, $E_1^i(M)$ is the adjoin of $M$ in Iwasawa theory and $E_0^0(M) := M^+ = \text{Hom}(M, \tilde{\Lambda})$ (P83 page 16).

The condition $i \not\equiv 0 \mod [F(\mu_p) : F]$ implies that the action of $G_S(\tilde{F})$ over $\mathbb{Q}_p/\mathbb{Z}_p(i)$ is not trivial, so

$$H^0(G_S(\tilde{F}), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0.$$ 

Then the exact sequence (21) of [J14 Corollary 3] for the discrete $G_S(\tilde{F})$-module $\mathbb{Q}_p/\mathbb{Z}_p(i)$ writes

$$0 \to E^1(H^1(G_S(\tilde{F}), \mathbb{Q}_p/\mathbb{Z}_p(i))^\vee) \to \lim_{\rightarrow E} H^2(G_S(E), \mathbb{Z}_p(i))$$

$$0 \to E^2(H^1(G_S(\tilde{F}), \mathbb{Q}_p/\mathbb{Z}_p(i))^\vee) \to (H^2(G_S(\tilde{F}), \mathbb{Q}_p/\mathbb{Z}_p(i))^\vee)^+.$$ 

By the twisted weak Leopoldt conjecture we have

$$H^2(G_S(\tilde{F}), \mathbb{Q}_p/\mathbb{Z}_p(i)) = 0.$$ 

Thus the exact sequence (21) gives the following isomorphism

$$E^1(\mathcal{X}_S^{(-i)}(\tilde{F})) \simeq \lim_{\rightarrow E} H^2(G_S(E), \mathbb{Z}_p(i)).$$

This leads to an other isomorphism

$$E^1(E^1(\mathcal{X}_S^{(-i)}(\tilde{F}))) \simeq E^1(\lim_{\rightarrow E} H^2(G_S(E), \mathbb{Z}_p(i))).$$

On the one hand, by (20) $\lim_{\rightarrow E} H^2(G_S(E), \mathbb{Z}_p(i))$ is a torsion $\tilde{\Lambda}$-module, so using (iii) in [P83 Proposition 8, Chap. 1] we have

$$E^1(\lim_{\rightarrow E} H^2(G_S(E), \mathbb{Z}_p(i))) \sim \lim_{\rightarrow E} H^2(G_S(E), \mathbb{Z}_p(i)).$$

On the other hand, Theorem 2.9 shows that $\mathcal{X}_S^{(-i)}(\tilde{F})$ is a $\tilde{\Lambda}$-module of projective dimension at most 1. Then Theorem 1.6 and Lemma 1.8 of [J89] show that $E^1(E^1(\mathcal{X}_S^{(-i)}(\tilde{F})))$ is canonically isomorphic to the kernel of the natural map

$$\mathcal{X}_S^{(-i)}(\tilde{F}) \longrightarrow \mathcal{X}_S^{(-i)}(\tilde{F})^{++},$$

which is obviously $\text{Tor}_{\tilde{\Lambda}}(\mathcal{X}_S^{(-i)}(\tilde{F}))$. Hence we have

$$E^1(E^1(\mathcal{X}_S^{(-i)}(\tilde{F}))) \simeq \text{Tor}_{\tilde{\Lambda}}(\mathcal{X}_S^{(-i)}(\tilde{F})).$$

Therefore, the pseudo isomorphisms (21) and (22) give

$$\text{Tor}_{\tilde{\Lambda}}(\mathcal{X}_S^{(-i)}(\tilde{F})) \sim X'(\tilde{F}(\mu_p))(i - 1)^{\Delta}. $$
Since $\mathfrak{X}_S^{(-i)}(\tilde{F})$ has no non trivial pseudo-null submodules, the pseudo-nullity of $\text{Tor}_\Lambda(\mathfrak{X}_S^{(-i)}(\tilde{F}))$ is equivalent to its triviality. Hence we get the equivalence between (1) and (2) by this last pseudo-isomorphism.

\[ \square \]

**Remark 3.5.** 
(1) The hypotheses $i \neq 0 \mod [F(\mu_p) : F]$ insure that the cohomological group $H^0(G_S(\tilde{F}), \mathbb{Q}_p/\mathbb{Z}_p(i))$ is zero, which is needed to apply Corollary 3 in \[114\].

(2) If $F$ contains $\mu_p$, the equivalence between (1) and (2) is given in \[L-N\] Proposition 3.6 and for the other one (i.e (2) $\equiv$ (3)) see \[N19\] page 3.

Now we focus on the triviality of the $\tilde{\Lambda}$ -torsion of the twisted $S$-ramified Iwasawa module of $\tilde{F}$. Precisely, we show a going up property in a fixed tower of multiple $\mathbb{Z}_p$-extensions. Fix a tower of $\mathbb{Z}_p$-extensions

$F_\infty = F(1) \subset \cdots \subset F(d) \cdots \subset F(r) = \tilde{F},$

where $F_\infty$ is a $\mathbb{Z}_p$-extension of $F$ satisfying $WLC_i$ and $r = r_2 + 1 + \delta_F$. The choice of such a $\mathbb{Z}_p$-extension is always possible since the cyclotomic $\mathbb{Z}_p$-extension $F^c$ verifies $WLC_i$. For simplicity we will write $\mathfrak{X}_d^{(i)}$ instead of $\mathfrak{X}_S^{(i)}(F(d))$ and $\Lambda_d$ instead of $\Lambda_{F(d)}$. We start with the following helpful lemma (compare with \[P83\] 1. of Lemma 4).

**Lemma 3.6.** For any integer $1 \leq d \leq r$, $(\text{Tor}_{\Lambda_d}^{(i)}(\mathfrak{X}_d^{(i)}))_\Gamma$ is a $\Lambda_d$-torsion module, where $\Gamma$ is the Galois group of the extension $F^{(d+1)}/F^{(d)}$.

**Proof.** Remark that for all $d \geq 1$ the $\mathbb{Z}_p^d$-extension $F^{(d)}$ contains $F_\infty$. Thus according to Theorem \[2.5\] we know that $\mathfrak{X}_d^{(i)}$ is of $\Lambda_d$-rank equals to $n_i$. The exact sequence \[11\] for $L = F^{(d+1)}$ and $L' = F^{(d)}$ writes

$0 \rightarrow (\text{Tor}_{\Lambda_d}^{(i)}(\mathfrak{X}_d^{(i)}))_\Gamma \rightarrow (\mathfrak{X}_d^{(i)})_\Gamma \rightarrow (\text{Fr}_{\Lambda_d}^{(i)}(\mathfrak{X}_d^{(i)}))_\Gamma \rightarrow 0,$

Then we have

$\text{rank}_{\Lambda_d}(\mathfrak{X}_d^{(i)})_\Gamma = \text{rank}_{\Lambda_d}(\text{Fr}_{\Lambda_d}(\mathfrak{X}_d^{(i)}))_\Gamma + \text{rank}_{\Lambda_d}(\text{Tor}_{\Lambda_d}(\mathfrak{X}_d^{(i)}))_\Gamma$.

We know that $H_1(\Gamma, \mathbb{Z}_p(i)_{G_S(F^{(d+1)})})$ is $\Lambda_d$-torsion, hence by Lemma \[2.3\] we obtain

$\text{rank}_{\Lambda_d}(\mathfrak{X}_d^{(i)})_\Gamma = \text{rank}_{\Lambda_d}(\mathfrak{X}_d^{(i)}) = n_i = \begin{cases} r_2 & \text{if } i \text{ is even}, \\ r_1 + r_2 & \text{if } i \text{ is odd}. \end{cases}$

Since $\text{rank}_{\Lambda_d}(\text{Fr}_{\Lambda_d}(\mathfrak{X}_d^{(i)}))_\Gamma = \text{rank}_{\Lambda_d}(\mathfrak{X}_d^{(i)}(d+1)) = n_i$, the equality \[23\] implies that

$\text{rank}_{\Lambda_d}(\text{Tor}_{\Lambda_d}(\mathfrak{X}_d^{(i)}))_\Gamma = 0.$
Theorem 3.7. Let $F$ be a number field which admits a $\mathbb{Z}_p$-extension $F_\infty$ satisfying $WLC_i$. For all $1 \leq d < r$, if $\text{Tor}_{\Lambda(d)}(\mathfrak{X}_d^{-i-1}) = 0$ then $\text{Tor}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}) = 0$.

In particular, if $\text{Tor}_{\Lambda_\infty}(\mathfrak{X}_S^{-i-1}(F_\infty)) = 0$ then $\mathfrak{X}_d^{-i-1}$ is a $\Lambda_d$-torsion-free module for all $1 \leq d \leq r$.

Proof. Let $F_\infty = F^{(1)} \subset \cdots \subset F^{(d)} \cdots \subset F^{(r)} = \tilde{F}$ be a fixed tower of multiple $\mathbb{Z}_p$-extensions associate to the field $F$. Let $\Gamma = \text{Gal}(F^{(d+1)}/F^{(d)})$. The exact sequence

$$0 \rightarrow \text{Tor}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}) \rightarrow \mathfrak{X}_{d+1}^{-i-1} \rightarrow \text{Fr}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}) \rightarrow 0,$$

gives by the snack lemma the following exact sequence

$$(\text{Fr}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}))\Gamma \rightarrow (\text{Tor}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}))\Gamma \rightarrow (\mathfrak{X}_{d+1}^{-i-1})\Gamma \rightarrow \text{Fr}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1})\Gamma \rightarrow 0. \quad (24)$$

The $\Lambda_{d+1}$-module $\text{Fr}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1})$ is without torsion, hence

$$(\text{Fr}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}))\Gamma = 0.$$

Then the exact sequence (24) becomes

$$0 \rightarrow (\text{Tor}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}))\Gamma \rightarrow (\mathfrak{X}_{d+1}^{-i-1})\Gamma \rightarrow \text{Fr}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1})\Gamma \rightarrow 0. \quad (25)$$

The exact sequence (25) shows that if $\mathfrak{X}_{d}^{-i-1}$ is $\Lambda_d$-torsion free then $(\mathfrak{X}_{d+1}^{-i-1})\Gamma$ is also $\Lambda_d$-torsion free. Furthermore, Lemma 3.6 tells us that the module $(\text{Tor}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}))\Gamma$ is $\Lambda_d$-torsion. Therefore, the exact sequence (25) implies that $(\text{Tor}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}))\Gamma = 0$ which gives, by Nakayama’s Lemma, that

$$\text{Tor}_{\Lambda(d+1)}(\mathfrak{X}_{d+1}^{-i-1}) = 0.$$

□

Remark 3.8. It is clear from the proof that Theorem 3.7 remains true if we replace $F_\infty$ by a $\mathbb{Z}_p^d$-extension, $d \geq 1$.

As a consequence of Theorems 3.4 and 3.7, the pseudo-nullity of the $\tilde{\Lambda}$-module $X(\tilde{F}(\mu_p))^\Delta$ is implied by the existence of a $\mathbb{Z}_p$-extension for which the associated twisted $S$-ramified Iwasawa module is without torsion.
Theorem 3.9. Assume that the condition \((Dec)\) holds for \(F\). Let \(i\) be an integer such that \(i \not\equiv 0 \mod [F(\mu_p) : F]\) and the conjecture \(LC_i\) holds for \(F\). If \(F\) admits a \(\mathbb{Z}_p\)-extension \(F_\infty\) such that \(\text{Tor}_{\Lambda,\infty}(\mathfrak{A}^{(-i)}(F_\infty)) = 0\), then the \(\hat{\Lambda}\)-module \(X(\hat{F}(\mu_p))(i-1)^{\Delta}\) is pseudo-null. \(\square\)

In particular, if \(i \equiv 1 \mod [F(\mu_p) : F]\) we obtain a sufficient condition for the validity of Greenberg’s generalized conjecture.

Corollary 3.10. Assume that the condition \((Dec)\) holds for \(F\). If there exist an integer \(i \equiv 1 \mod [F(\mu_p) : F]\) such that

1. The conjecture \(LC_i\) is true for \(F\).
2. There exists a \(\mathbb{Z}_p\)-extension \(F_\infty\) satisfying \(\text{Tor}_{\Lambda,\infty}(\mathfrak{A}^{(-i)}(F_\infty)) = 0\).

Then \(F\) satisfies Greenberg generalized conjecture. \(\square\)

Remark 3.11. (1) Theorem 3.9 and Corollary 3.10 remain true if we replace \(F_\infty\) by a \(\mathbb{Z}_d\)-extension, \(d \geq 1\).

(2) In [N19], Nguyen Quang Do gives an inductive process for the pseudo-nullity of the unramified totally split Iwasawa module in a tower of multiple \(\mathbb{Z}_p\)-extensions. Precisely, he proves that GGC holds for any number field \(F\) (see [N19, Theorem 1.4]) satisfying the following conditions:

- \(F\) verifies Kuz’mín-Gross conjecture (i.e \(X'(F^c)^{\Gamma_c}\) is finite).
- \(F\) admits a \(\mathbb{Z}_p^2\)-extension \(F^{(2)}\) which is normal over \(\mathbb{Q}\), contains \(F^c\) and \(X'(F^{(2)})\) is pseudo-null.

Under these conditions, Lemma 1.2 of [N19] proves that the decomposition group of any \(p\)-place of \(F^{(2)}\) has \(\mathbb{Z}_p\)-rank \(2\).

By choosing \(i \geq 2\) and \(i \equiv 1 \mod [F(\mu_p) : F]\), Theorem 3.4 gives an equivalence between the pseudo-nullity of \(X'(F^{(2)})\) and the triviality of \(\text{Tor}_{\Lambda,(2)}(\mathfrak{A}^{(-i)}(F^{(2)}))\). Then, by considering a tower of multiple \(\mathbb{Z}_p\)-extensions \(F^c \subset F^{(2)} \subset \cdots \subset \hat{F}\) the same proof of Theorem 3.7 implies that \(\text{Tor}_{\hat{\Lambda}}(\mathfrak{A}^{(-i)}(\hat{F})) = 0\). Using again Theorem 3.4 we obtain Nguyen Quang Do’s result.

4. \((p, i)\)-regular number fields

The aim of this section is to show the triviality of the \(\hat{\Lambda}\)-torsion of the twisted \(S\)-ramified Iwasawa module of the compositum of all \(\mathbb{Z}_p\)-extensions of a \((p, i)\)-regular number field (see Definition 4.1). As consequence we obtain that for \((p, i)\)-regular fields which satisfy the condition \((Dec)\) the \(\hat{\Lambda}\)-module \(X(\hat{F}(\mu_p))(i-1)^{\Delta}\) is pseudo-null.
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The \((p, i)\)-regular number fields were introduced in [A95], they are a twisted generalization of \(p\)-rational fields (see [M90] and [M-N]). A number field \(F\) is called \((p, i)\)-regular if the second cohomological group \(H^2(G_S(F), \mathbb{Z}_p/p(i))\) is trivial. There are many equivalent properties to the \((p, i)\)-regularity we recall here Definition 4.1 of [N19] which quotes most of this equivalences. For the proof of the equivalences between the assertions from (1) to (4) we refer to [A95] and the last one can be found in [N19].

**Definition 4.1.** Let \(i\) be an integer, a number field \(F\) is called \((p, i)\)-regular if and only if it satisfies one of the following equivalent conditions:

1. \(H^2(G_S(F), \mathbb{Z}_p/p(i)) = 0\).
2. The \(\mathbb{Z}_p\)-module \(\mathfrak{X}_S^{(−i)}(F^c(\mu_p)))_{G_\infty}\) is free, where \(G_\infty = \text{Gal}(F^c(\mu_p)/F)\).
3. If \(F\) contains \(\mu_p\) or \(i \equiv 1 \pmod{[F(\mu_p) : F]}\), then \(F\) admits only one \(p\)-place and \(A'(F)\) is null. If \(F\) does not contain \(\mu_p\) and \(i \equiv 1 \pmod{[F(\mu_p) : F]}\), then \(A'(F(\mu_p))(i - 1)\text{Gal}(F(\mu_p)/F)\) is null, and for any \(p\)-place \(v\) of \(F\), the local degree \([F_v(\mu_p) : F_v]\) does not divide \((i - 1)\).

If \(i \neq 1\), there is also equivalence with any of the following:

4. \(H^2(G_S(F), \mathbb{Z}_p(i)) = 0\).
5. The module \(\mathfrak{X}_S^{(−i)}(F^c)\) is \(\Lambda_c\)-free.

For any integer \(i \neq 1\), the \((p, i)\)-regularity of a number field \(F\) is equivalent to the freeness of the \(\Lambda_c\)-module \(\mathfrak{X}_S^{(−i)}(F^c)\). In particular, for \(i \neq 1\) any \((p, i)\)-regular number field \(F\) satisfies \(\text{Tor}_{\Lambda_c}(\mathfrak{X}_S^{(−i)}(F^c)) = 0\).

Also, notice that we have the following periodicity, for \(i \equiv j \pmod{[F(\mu_p) : F]}\)

\[ F \text{ is } (p, i)\text{-regular } \iff \ F \text{ is } (p, j)\text{-regular}. \]

If \(F\) is \((p, 1)\)-regular and \(i \equiv 1 \pmod{[F(\mu_p) : F]}\),

\[ \text{Tor}_{\Lambda_c}(\mathfrak{X}_S^{(−1)}(F^c))) = \text{Tor}_{\Lambda_c}(\mathfrak{X}_S^{(−i)}(F^c))(i - 1) \ (i \geq 2) = 0. \]

Then as a consequence of Theorem 3.7 we get

**Proposition 4.2.** Let \(i\) be an integer. For any \((p, i)\)-regular number field \(F\) we have

\[ \text{Tor}_{\Lambda}(\mathfrak{X}_S^{(−i)}(\tilde{F})) = 0. \]

The above proposition is a twisted generalization of Theorem 1 in [F09], since the conditions of [F09, Theorem 1] imply the \(p\)-rationality.

Notice that \((p, i)\)-regular fields satisfy \(LC_i\) (Proposition 1.2 in [A95]), hence we have the following consequence of Theorem 3.4 and Proposition 4.2.
Theorem 4.3. Let $i \not\equiv 0 \mod [F(\mu_p) : F]$ be an integer. For any imaginary $(p, i)$-regular number field which satisfies the condition (Dec), the $\tilde{\Lambda}$-module $X(\tilde{F}(\mu_p))(i-1)^{\Delta}$ is pseudo-null.

In [N19], the author shows that the étale $e_{i-1}$-part of GGC' holds (i.e. $X'(\tilde{F}(\mu_p))(i-1)^{\Delta}$ is pseudo-null) for all $(p, i)$-regular number fields having a "special $\mathbb{Z}_p^2$-extension" (see [N19, Theorem 4.2]) and satisfying Kuz’min-Gross conjecture. In Theorem 4.3 we need only the condition (Dec).

In particular, when the integer $i \equiv 1 \mod [F(\mu_p) : F]$ we get the following

Corollary 4.4. Greenberg’s generalized conjecture holds for all imaginary $(p, 1)$-regular number field which satisfies the condition (Dec).

The next proposition shows that when $i \not\equiv 0, 1 \mod [F(\mu_p) : F]$ we can omit the condition (Dec).

Proposition 4.5. The $\tilde{\Lambda}$-module $X(\tilde{F}(\mu_p))(i-1)^{\Delta}$ is pseudo-null for any $(p, i)$-regular number field $F$ with $i \not\equiv 0, 1 \mod [F(\mu_p) : F]$.

Proof. The second part of assertion 3) of Definition 4.1 gives that for any $p$-place $v$ of $F$ we have

$$i \not\equiv 1 \mod [F_v(\mu_p) : F_v].$$

Then using Corollary 3.3 we obtain that

$$\lim_{\leftarrow} \bigoplus_{v \in S(E)} H^2(E_v, \mathbb{Z}_p(i)) = 0.$$

By the same proof as in Theorem 3.4 we can show (without assuming Condition (Dec)) in this case the following equivalence

$$\text{Tor}_{\tilde{\Lambda}}(\mathcal{X}^{(-i)}_S(\tilde{F})) = 0 \iff X(\tilde{F}(\mu_p))(i-1)^{\Delta} \text{ is pseudo-null.}$$

Further, by Proposition 4.2 we have

$$\text{Tor}_{\tilde{\Lambda}}(\mathcal{X}^{(-i)}_S(\tilde{F})) = 0.$$

Thus the $\tilde{\Lambda}$-module $X(\tilde{F}(\mu_p))(i-1)^{\Delta}$ is pseudo-null.

The condition $i \not\equiv 0 \mod [F(\mu_p) : F]$ means that Theorem 4.3 does not concern $p$-rational number fields. Thus we can ask the following question:

Does $p$-rational number fields satisfy Greenberg’s generalized conjecture?

Note that if the field $F$ contains $\mu_p$ we know that the $(p, 1)$-regularity is equivalent to the $p$-rationality and the condition (Dec) is satisfied. Then GGC holds for any $p$-rational field which contains $\mu_p$. 
For totally real number field \( F \), it is known that \( \mathfrak{X}_S(F^c) \) is a \( \Lambda_c \)-torsion module. Then the following surjection
\[
\xrightarrow{\text{surjection}}
\]
implies that \( X(F^c) = 0 \) if \( F \) is \( p \)-rational. This means that the above question still positive in this case.

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