On Kolmogorov Wave Turbulence in QCD*

A.H. Mueller\(^1\), A.I. Shoshi\(^2\) and S.M.H. Wong\(^1\)

\(^1\) Physics Department, Columbia University, New York, NY 10027, USA
\(^2\) Fakultät für Physik, Universität Bielefeld, D-33501 Bielefeld, Germany

Abstract

We investigate Kolmogorov wave turbulence in QCD or, in other words, we calculate the spectrum of gluons as a function of time, \(f_k(t)\), in the presence of a source which feeds in energy density in the infrared region at a constant rate. We find an early, an intermediate and a late time form for the gluon spectrum. Wave turbulence in QCD turns out to be somewhat different than the turbulence in the case of \(\phi^4\)-type theories studied by Zakharov, L’vov and Falkovich. The hope is that a good understanding of QCD wave turbulence might lead to a better understanding of the instability problem in the early stages of the evolution after a heavy ion collision.

Keywords: Kolmogorov wave turbulence, Kolmogorov spectra, Instability, Thermalization, Heavy Ion Collisions

1 Introduction

One of the key questions in heavy ion physics is how rapidly and by what mechanism equilibrium is reached after a collision. Phenomenological analyses of experimental data at RHIC suggest a rapid thermalization [1, 2] but, so far, no convincing theoretical picture has emerged which naturally gives such a quick approach to equilibrium.

The “bottom-up” picture of equilibration [3] is a detailed picture of how the initially produced hard gluons (at or near the saturation scale of the colliding ions) lose energy by radiating softer gluons. After they become sufficiently numerous

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these softer gluons equilibrate amongst themselves and continue to absorb energy from the initially produced hard gluons. After the hard gluons have lost all their energy the system has reached complete equilibration. Although difficult to evaluate precisely, the time for this equilibration was estimated to be on the order of 3 fm [5]. While it is an attractive picture the bottom-up scenario has a serious flaw in that the initially produced hard particles quickly acquire an instability as they spread out along the axis of the heavy ion collision as it was recently worked out by Arnold, Lenaghan and Moore [6] and was long ago advocated by Mrowczynski [7] (For other early discussions, see Refs. [8–10]). This instability then becomes the dominant mechanism for the creation of softer gluons [6], more important than the Bethe-Heitler radiation used in the bottom-up picture [3, 4]. The instability potentially speeds up equilibration, but so far it has been difficult to follow analytically how the QCD gluonic system evolves toward equilibration in the presence of the instability [11, 12]. Numerical studies [13–18] seem to indicate that the instability is effective at early times, but becomes less prominent in non-Abelian theories when the occupation number of low momentum modes becomes large. So far there is no analytic understanding of the numerical results.

The phase space spectrum found in numerical studies by Arnold and Moore [13] has a resemblance to that of the Kolmogorov spectrum in turbulence [19]. Indeed, the problem of wave turbulence with an infrared source of energy discussed by Zakharov, L’vov and Falkovich (ZLF) [20] (for a nice overview, see [21]) would seem to have much in common with the instability problem currently under discussion for a non-expanding QCD medium and where the source for the instability is a collection of hard particles having a fixed asymmetry in their momentum distribution. Wave turbulence is a somewhat easier problem to deal with since one can take a spherically symmetric source of inflow of energy in low momentum modes. One can hope that a good understanding of QCD wave turbulence will lead to a better understanding of the early stages of evolution after a high-energy heavy ion collision.

Wave turbulence in QCD appears to be somewhat different than the situation studied by Zakharov, L’vov and Falkovich [20]. The essence of the ZLF discussion is that waves, or particles, interact with each other locally in momentum. This is certainly the case, say, in a $\phi^4$ theory where the scattering cross section for a high momentum particle on a low momentum particle is very small. Soft quanta in $\phi^4$ theory interact very weakly with hard quanta because of the small overlap in their fields due to Lorentz contraction of the harder quanta. Soft gluons, on the other hand, interact with the currents of the harder gluons, and these currents do not decrease with the momentum of the harder gluons. In addition a gauge theory has a lower cutoff in frequency, the plasma frequency, which has no analog in a theory of the $\phi^4$ variety. Because of the strong interaction between soft and hard modes
along with the plasma frequency infrared cutoff QCD turbulence seems to be quite different than the ZLF problem.

In the problems considered by ZLF a source, say, of energy inserts energy at an infrared scale and a sink extracts that energy at a higher scale. The flow of energy then proceeds from a low momentum scale, through intermediate energy scales and exits at some high energy scale. A closely related process omits the high energy sink and has the energy flowing from low to high momentum through all intermediate scales. What is different in the QCD case is that energy can be absorbed by very high energy particles from a low energy source without the necessity of passing through intermediate scales. Particle number, however, cannot be directly transferred from low to high energy but must flow through intermediate scales, or be created at high energy through inelastic reactions. Thus in QCD there are important, even dominant, interactions which are not local in momentum.

In this paper we calculate the analytic form of the spectrum of gluons in the presence of a source which feeds in energy density at a constant rate $\dot{\epsilon}_0 = m_0^5/\alpha$. We suppose the energy comes into our system uniformly in space in the form of gluons which have a spherically symmetric momentum distribution and which are inserted uniformly in phase space just above the plasma frequency cutoff in a range $m < \omega < \bar{m}$ with $m$ the plasma frequency and $\bar{m}$ on the order of $m$. $m_0$ is the single dimensionful parameter in the discussion given in Sec.2 while in Sec. 3 we allow the incoming energy to be spread uniformly in phase space in a region $0 \leq k \leq k_0$ of momenta in which case $k_0$ is a separate dimensional parameter if we choose $k_0$ to be a scale larger than and independent of $m_0$. We always suppose $\alpha$, the gluonic coupling, to be small.

At late times after the source has become active, $m_0 t > (1/\alpha)^{9/5}$ in case the source energy is deposited in $m < \omega < \bar{m}$, the system of gluons is very close to thermal equilibrium. The incoming energy is transferred from the scale $m$ to the scale given by the temperature $T$ by direct absorption of soft gluons in an inelastic $3 \to 2$ process and, parametrically equally as important, by elastic scattering of soft gluons (scale $m$) on hard gluons (scale $T$) with the soft gluons losing energy to the hard gluons. Both $m$ and $T$ are slowly increasing with time.

In the time domain $(1/\alpha)^{7/5} < m_0 t < (1/\alpha)^{9/5}$ gluons having momentum much greater than $m$ are in thermal equilibrium while gluon occupation numbers are significantly above the thermal curve in the domain $m < \omega < \bar{m}$.

Finally, in the early time domain, $1 \ll m_0 t < (1/\alpha)^{7/5}$, the system is far from thermal equilibrium in both the high and low momentum regimes. If $p_0(t)$ is the maximum scale to which gluons have evolved, then $f_{p_0} \gg 1$ while $f_k = \frac{c(t)}{\alpha} \frac{m}{\omega}$ in the
domain \( m \ll \omega \ll p_0 \). This latter distribution is a sort of equilibrium distribution although it does not match onto a genuine equilibrium distribution at the scale \( p_0 \).

While intermediate momentum scales obey \( f_k \sim 1/k \) in all time domains, so long as \( m_0 t \gg 1 \), there is, nevertheless, a flow of energy and particle number from soft to hard scales through these intermediate scales. In the case of energy this flow is negligible compared to the direct transfer of energy from soft to hard scales, while in the case of particle number flow it is one of the dominant mechanisms of gluon number growth at the hard scale.

The dynamics which we use throughout this paper is the Boltzmann equation with a collision term consisting of \( 2 \leftrightarrow 2 \) and \( 2 \leftrightarrow 3 \) gluon processes. We have assumed the absence of important longrange coherent fields which, if present, could create problems for our approach.

### 2 The spectrum of gluons as a function of time

The problem we are concerned with here is very similar to the problem of wave turbulence discussed by Zakharov, L’vov and Falkovich [20]. However, Zakharov et al. consider scalar theories where interactions among particles are relatively local in momentum while the case of interest here, QCD, allows important interaction between low momentum and high momentum particles. In addition in a gluonic theory there is a minimum allowed frequency, the plasma frequency, at which waves can propagate while a corresponding phenomenon does not exist with purely scalar interactions.

At time, \( t \), equal to zero we turn on a source of gluons which feeds a rate of increase of energy density \( \dot{\epsilon}_0 = \frac{dE}{dt} = \frac{m_0^3}{\alpha} \) which is constant in time and uniform in space. The energy is incoming isotropically in low momentum modes but as the density of gluons increases in time the source is modified so that the incoming gluons always have energy just above the plasma frequency, say \( m < \omega < \bar{m} \) where \( m \) is the plasma frequency and \( \bar{m} \) is of the same order as \( m \). The question we wish to answer concerns the occupaton number, \( f_\mathbf{k} \), of gluons as a function of time. As we shall see, the form of the spectrum changes over the course of time having an early time, an intermediate time and a late time form. The late time form of \( f_\mathbf{k} \) is the simplest and the most straightforward to describe so we begin there. Throughout we limit our discussion to the case of a fixed QCD coupling, \( \alpha \).
2.1 The spectrum for $m_0 t > (1/\alpha)^{9/5}$

At sufficiently large time it is apparent that the system will be extremely close to thermal equilibrium since the source can only be a very small perturbation on a system having a lot of energy. (The scaling solution for constant energy flow found by ZLF [20] (see also [21]), $f_k \sim (m/k)^{-5/3}$, connects an infrared source of energy to a quantum equilibrium solution of the Boltzmann equation which exists for $k > k_0$ whwere $f_{k_0} = 1$. The non equilibrium solution in the intermediate region of $k$ is necessary in a $\phi^4$-type theory considered by ZLF where energy flow is local in momentum. In the QCD case energy is transferred directly from the low momentum source to the high momentum modes allowing the intermediate momenta also to be in equilibrium. Indeed, we shall see throughout this paper that $f_k \sim m/k$ is required, in intermediate momentum regions, in order that the rate of emission and absorbtion of gluons of momentum $k$ by harder gluons exactly cancel.) Thus energy conservation gives

$$\epsilon(t) = \frac{m_0^5}{\alpha} \cdot t = g_E T^4$$

with $g_E = 2(N_c^2 - 1)\frac{\pi^2}{30}$, or

$$T = m_0 \left( \frac{m_0 t}{g_E \alpha} \right)^{1/4}.$$  \hspace{1cm} (2)

The corresponding plasma frequency is

$$\omega_p \equiv m = \sqrt{\frac{4\pi}{9} \alpha N_c T}.$$ \hspace{1cm} (3)

and is related to the Debye mass by $m = m_D/\sqrt{2}$. At large times the energy source inserts gluons into the system just above an energy $m$ which is far below the temperature of the system $T$. What we now wish to describe is how the energy gets transferred from the scale $m$ to the scale $T$ which is where almost all the energy is located. First we remark that, for very small $\alpha$, the spectrum is given by

$$f_k = \frac{1}{e^{\omega/T} - 1}$$ \hspace{1cm} (4)

which, when $\omega/T \ll 1$, can be approximated as

$$f_k \simeq \frac{T}{\omega}.$$ \hspace{1cm} (5)

When $k/m \gg 1$ we shall not distinguish between $\omega$ and $k$ in which case (5) could as well be written as

$$f_k \simeq \frac{T}{k}.$$ \hspace{1cm} (6)

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Of course if the system were in exact equilibrium there would be no transfer of energy and (6) would represent a system in complete equilibrium without flow of energy or particle number. Because we have a source of incoming energy the occupation number will be slightly higher than the equilibrium solution in the low momentum region. As we shall now demonstrate these low momentum gluons directly transfer energy to the hard gluons, having momentum $T$, with only a small fraction of energy flowing from the scale $m$ through an intermediate scale $k$, $m \ll k \ll T$, and then to the scale $T$. However, particle number, which must increase as $T^3$, cannot be transferred directly from low to high momentum but must pass through intermediate scales giving a flow of particles from infrared to ultraviolet in our spectrum.

Consider the elastic scattering of soft gluons, $q_1$ and $q_2$, with hard gluons having $p_1, p_2$ on the order of $T$. The process is illustrated in Fig. 1 and gives a flow of energy from soft to hard as

$$
\dot{\epsilon}^{\text{el}} = \frac{[2(N_c^2 - 1)]^2}{(2\pi)^{12}} \int_R d^3p_1 d^3p_2 d^3q_1 d^3q_2 \frac{1}{2\omega_1} \frac{1}{2\omega_2} (2\pi)^4 \delta^4(p_1 + q_1 - p_2 - q_2) \cdot |M|^2 \cdot \times [f_{q_1} f_{p_1}(1 + f_{q_2})(1 + f_{p_2}) - f_{q_2} f_{p_2}(1 + f_{q_1})(1 + f_{p_1})] (\omega_1 - \omega_2) \tag{7}
$$

where

$$
|M|^2 = \frac{64\pi^4}{N_c^2 - 1} \left( \frac{\alpha N_c}{\pi} \right)^2 \left[ 3 - \frac{ut}{s^2} - \frac{us}{t^2} - \frac{ts}{u^2} \right] \tag{8}
$$

with

$$
s = (p_1 + q_1)^2, t = (q_1 - q_2)^2, u = (p_1 - q_1)^2. \tag{9}
$$

The symbol $R$ in the integration in (7) restricts $\omega_{q_1}, \omega_{q_2}$ to be less than $\omega_{p_1}, \omega_{p_2}$. Now the hard particles $p_1$ and $p_2$ will have a distribution given by (4) so

$$
1 + f_p = f_p e^{E/T} \tag{10}
$$

allows one to write the bracket in (7) as

$$
[\ ] = e^{E_1/T} f_{p_1} f_{p_2} f_{q_1} f_{q_2} [(\omega_1 - \omega_2)/T + 1/f_{q_2} - 1/f_{q_1}] \tag{11}
$$

Figure 1: Elastic gluon interactions.
If \( f_{q_1} \) and \( f_{q_2} \) also took the form (4), or (5), we would get exactly zero as expected in exact equilibrium. However, the incoming flux of soft gluons will naturally increase \( f_q \) a little so that \([(\omega_1 - \omega_2)] \) is greater than zero corresponding to a flow of energy from soft to hard particles. We can estimate that flow of energy by rewriting (7) as

\[
\dot{\epsilon}^{el} = \frac{[2(N_c^2 - 1)]^2}{(2\pi)^8} \int_R \frac{d^3p_1 d^3q_1}{16E_1 E_2 \omega_1 \omega_2} d\Omega_{q_2} \frac{q_2^2}{d\omega_2/dq_2} |M|^2 (\omega_1 - \omega_2)[]. \tag{12}
\]

Taking \( \omega_1 - \omega_2 \) to be positive (Eq. (12) is even under \( q_1 \leftrightarrow q_2 \)) and

\[
|M|^2 \sim \alpha^2 \frac{(E\omega)^2}{m^4} \tag{13}
\]

\[
d^3p_1 \sim T^3 \tag{14}
\]

\[
d^3q_1 \sim m^3 \tag{15}
\]

\[
q_2^2 \sim m^2 \tag{16}
\]

\[
\omega_1 - \omega_2 \sim m \tag{17}
\]

along with (4) and (5) one gets

\[
\dot{\epsilon}^{el} \sim \frac{m_0^5}{\alpha} \left( \frac{m}{m_0} \right)^5 (\alpha f_{q_1})(\alpha f_{q_2}) [(\omega_1 - \omega_2) + T/f_{q_2} - T/f_{q_1}] \frac{1}{m}. \tag{18}
\]

Using

\[
m/m_0 \sim (\alpha m_0 t)^{1/4} \tag{19}
\]

from (2) and (3) and

\[
\alpha f_q \sim \alpha T/m \sim \sqrt{\alpha} \tag{20}
\]

from (2) and (5) one finds

\[
\dot{\epsilon}^{el} \sim \frac{m_0^5}{\alpha} \alpha^{9/4} (m_0 t)^{5/4} [(\omega_1 - \omega_2) + T/f_{q_2} - T/f_{q_1}] \frac{1}{m}. \tag{21}
\]

Thus so long as \( m_0 t \gg \alpha^{-9/5} \), \( f_q \) will be very close to the equilibrium distribution making the \([\ ]\) in (21) small and compensating \( \alpha^{9/4} (m_0 t)^{5/4} \).

Although energy is transferred mostly in a direct way from the incoming particles of the source to particles having \( p \sim T \) there is also interaction with particles having momentum \( k \) where \( m \ll k \ll T \). We can evaluate the rate of transfer of energy from (11) and (12) by the replacement \( p_1, p_2 \rightarrow k_1, k_2 \) along with the replacement \( d^3p_1 \rightarrow d^3k_1 \sim k^3 \) in (14). One easily finds

\[
\dot{\epsilon}_k^{el} \sim \left( \frac{m_0^5}{\alpha m} \right) \left( \frac{m}{m_0} \right)^5 (\alpha f_{q_1})(\alpha f_{q_2}) [(\omega_1 - \omega_2) + T/f_{q_2} - T/f_{q_1}] k/T \tag{22}
\]
after using (6). \( \dot{\epsilon}_k \) is the flow of energy from the particles coming from the source to those particles having momentum on the order of \( k \). As we have seen earlier with the convention \( \omega_1 > \omega_2 \), \([\ ]\) is positive so there is a positive flow of energy from the source to \( k \)-particles, but this flow is suppressed by a factor \( k/T \) compared to the flow to particles of momentum \( T \). However, since \( f_k \) changes only very slowly this flow of energy must correspond to a flow of particles toward higher momentum.

We can characterize this flow by a parameter, \( \dot{k}_{\text{flow}} \), giving the average change in a particle’s momentum with time in terms of which

\[
\dot{\epsilon}^{el}_k = 2 \left( \frac{N_c^2 - 1}{(2\pi)^3} \right) f_k 4\pi k^3 \dot{k}_{\text{flow}}. \tag{23}
\]

In turn we can view \( \dot{\epsilon}^{el}_k \) as the flow of particles past a given momentum \( k \) or

\[
\dot{\epsilon}^{el}_k = k\dot{\rho}_k \tag{24}
\]

where \( \dot{\rho}_k \) is the number of particles per unit volume which pass momentum \( k \), from lower toward higher momenta, in a unit of time

\[
\dot{\rho}_k = 2 \left( \frac{N_c^2 - 1}{(2\pi)^3} \right) f_k 4\pi k^2 \dot{k}_{\text{flow}}. \tag{25}
\]

Using (24) and (22) one sees that \( \dot{\rho}_k \) is independent of \( k \) so that there is a constant, in \( k \), flow of particles from lower to higher momenta. One can check, using (22) and (24) along with (25) and \([\omega_1 - \omega_2 + T/f_{q_1} - T/f_{q_2}] = m/(\alpha^{9/4}(m_0t)^{5/4})\) from (21), that \( \dot{\rho}_k \sim dT^3/dt \) so that the flow of particles is exactly the right amount to furnish the increase of the total number of particles of the system.

\[
\begin{align*}
\text{Figure 2: } 3 &\leftrightarrow 2 \text{ gluon processes.} \\
\end{align*}
\]

While the number of gluons in the system is growing with time \( \dot{\rho}_k \) is much less than the rate at which gluons enter the system, \( \frac{m_0^2}{\alpha m} \). Thus inelastic reactions are necessary in order to have chemical equilibrium. The dominant inelastic processes are those illustrated in Fig. 2 where gluons 1,2,3,4 have a momentum on the order of \( T \) and \( k \ll T \). Dimensionally these graphs give (see Appendix A)

\[
\dot{\epsilon}^{\text{inel}} \sim \frac{T^6 \omega \alpha^3}{m^2} \left[-f_{p_3} f_{p_4} (1 + f_{p_1}) (1 + f_{p_2}) (1 + f_k) + f_{p_1} f_{p_2} f_k (1 + f_{p_3}) (1 + f_{p_4})\right] \tag{26}
\]
or, using (2), (3) and (4)

\[ \dot{\epsilon}^{\text{inel}} \sim \frac{m_0^5}{\alpha^9} \alpha^{9/4} (m_0 t)^{5/4} \left[ \omega - T/f_k \right] \cdot \frac{1}{m} \]  

(27)

which is, parametrically, the same as (21). Again, when \( m < \omega < \bar{m} \) one needs \( f_k \) somewhat greater than \( T/\omega \) in order to transfer energy and lower the particle density to compensate for the gluons emerging from the source. We note that the LPM effect is small in the present case since the formation time of a gluon of momentum \( k \) is

\[ \tau_f(k) \sim \frac{2k}{k^2} \sim \frac{2k}{m^2 \cdot \tau_f/\lambda} \]  

(28)

or

\[ \left( \frac{\tau_f}{\lambda} \right)^2 \sim \frac{k}{m^2 \lambda} \sim \frac{k}{T} < 1 \]  

(29)

with \( \lambda \) the mean free path for the particle to undergo a collision of momentum transfer \( m \),

\[ \frac{1}{\lambda} \sim \frac{T^3 \alpha^2}{m^2}. \]  

(30)

Finally we evaluate the mean free path, \( \lambda_k \), for a particle of momentum \( k \) to undergo a collision of momentum transfer \( k \). Then, using (2),

\[ \frac{\lambda_k}{t} \sim \frac{1}{tk^3 f_k^2 \alpha^2 / k^2} \sim \left( \frac{k}{T} \right) \left[ \alpha^7 (m_0 t)^5 \right]^{-1/4}, \]  

(31)

so that we can expect the equilibrium spectrum \( f_k \simeq T/k \) for \( \omega > \bar{m} \) so long as \( m_0 t > \alpha^{-7/5} \). The case \( m < \omega < \bar{m} \) will be discussed in the next section.

### 2.2 The spectrum for \((1/\alpha)^{7/5} < m_0 t < (1/\alpha)^{9/5}\)

A major change occurs when \( m_0 t = (1/\alpha)^{9/5} \). We have seen that when \( m_0 t > (1/\alpha)^{9/5} \) the gluon system has a equilibrium solution although there is a constant, in momentum, flow of particles from the infrared region to the ultraviolet region. However, from (21) it is clear that when \( m_0 t < (1/\alpha)^{9/5} \) it is not possible to transfer the energy from the source in the region \( m < \omega < \bar{m} \) to the momentum region near \( T \) fast enough using a near equilibrium distribution in the soft region. What happens is physically clear: The gluons from the source “pile up” in the region \( m < \omega < \bar{m} \) until the occupation number becomes large enough to speed up the rate of transfer of energy from soft to hard modes so that the transfer can exactly compensate the rate of energy coming in from the source. For \( \omega > \bar{m} \) in the distribution of gluons
will be very close to the equilibrium distribution \( (4) \), however, when \( m < \omega < \bar{m} \) the distribution will be significantly altered. We now turn to a description of this altered distribution, illustrated in Fig. 3.

We suppose that the source creates gluons uniformly in phase space in the region \( m < \omega < \bar{m} \), with a constant inflow of energy \( \dot{\epsilon} = \frac{m_0^5}{\alpha} \) defining the parameter \( m_0 \). This corresponds to an increase in occupation number \( \dot{f}_k \sim \frac{m_0}{\alpha} \left( \frac{m}{m_0} \right)^5 \). Gluons having higher momentum, but located in the region \( m < \omega < \bar{m} \) will elastically scatter with gluons having momentum on the order of \( T \) and will lose energy to the harder gluons as given by (18). Thus a gluon at point 1 in Fig. 3 will move to point 3 further increasing \( \dot{f}_k \) while \( f_{k_1} \) will be determined by the competition between the incoming gluons causing \( \dot{f}_{k_1} \) to increase and the scattering with hard gluons causing \( \dot{f}_{k_1} \) to decrease. Thus \( f_k \) will increase rapidly as \( k \) decreases until \( f_k \) becomes large enough that gluons having momentum \( k \) are absorbed by hard gluons, according to the graphs in Fig. 2, at the same rate that are created from the external source and from gluons which arrive at \( k \) after having elastically scattered with hard gluons. We now give estimates of these rates.

Suppose \( k_3 \) (point 3 in Fig. 3) is the momentum at which the loss by inelastic absorption, the graphs in Fig. 2, exactly balances the rate of arrival of gluons coming directly from the source or coming from the source via higher momentum regions. Then the rate of arrival of gluons is

\[
\dot{j}_{source} \sim \frac{m}{\alpha} \left( \frac{m_0}{m} \right)^5 \left( \frac{m}{k_3} \right)^3
\]  

(32)

where \( \frac{m_0}{\alpha} \cdot \frac{1}{m} \) is the rate of the total number of gluons arriving over the whole phase space \( m < \omega < \bar{m} \) and \( \left( \frac{m}{k_3} \right)^3 \) expresses the fact that these gluons end up in
the limited region of phase space on the order of $k_3^3$. The rate at which gluons are absorbed by hard gluons, according to the graphs in Fig. 2 is

$$\dot{f}_{k_3}^{abs} \sim -\frac{T^6 \alpha^3}{m^5} [f_{p_1} f_{p_2} f_{k_3}(1 + f_{p_3})(1 + f_{p_4}) - f_{p_3} f_{p_4}(1 + f_{p_1})(1 + f_{p_2})(1 + f_{k_3})]$$

or, using thermal distributions for $f_{p_1}, f_{p_2}, f_{p_3}, f_{p_4}$

$$\dot{f}_{k_3}^{abs} \sim -m \sqrt{\alpha} f_{k_3} \left[ \frac{\omega}{T} - 1/f_{k_3} \right] T/m. \quad (33)$$

Dropping the $1/f_{k_3}$ term in (34) gives

$$\dot{f}_{k_3}^{abs} \sim -m \sqrt{\alpha} f_{k_3} \quad (34)$$

where we have anticipated $k_3/m \ll 1$ giving $\omega \simeq m$.

Now we must also require that the rate of $k_3$-gluons cascading to even smaller momenta is not large compared to the rates given by (32) and (35). Suppose a $k_3$-gluon goes to a momentum $k_4$ via elastic scattering as given by Fig. 1, with the replacement $q_1 \rightarrow k_3, q_2 \rightarrow k_4$. Then (see Appendix B)

$$\dot{f}_{k_3}^{3\rightarrow4} \sim -T^3 \frac{\alpha^2}{m^3} \frac{m}{k_3} [f_{p_1} f_{p_2} f_{k_3}(1 + f_{p_2})(1 + f_{k_4}) - f_{p_2} f_{k_4}(1 + f_{p_1})(1 + f_{k_3})] \quad (36)$$

which gives

$$\dot{f}_{k_3}^{3\rightarrow4} \sim -m \sqrt{\alpha} f_{k_3} \sqrt{\alpha} f_{k_4} \left( \frac{k_4}{m} \right)^3 \frac{m}{k_3} \left[ \frac{\omega_3}{T} - \frac{\omega_4}{T} + 1/f_{k_4} - 1/f_{k_3} \right] \frac{T}{m}. \quad (37)$$

Dropping the $1/f_{k_3}$ and $1/f_{k_4}$ terms and using $\omega_3 - \omega_4 \sim -\frac{1}{m}(k_3^2 - k_4^2) \sim \frac{k_3^2}{m}$, since $k_4 \leq k_3 \ll m$, one gets

$$\dot{f}_{k_3}^{3\rightarrow4} \sim -m \sqrt{\alpha} f_{k_3} \sqrt{\alpha} f_{k_4} \left( \frac{k_4}{m} \right)^3 \left( \frac{k_3}{m} \right). \quad (38)$$

Since we are looking for the value of $k_3$ at which the cascading to lower momenta ceases it is natural to require that (32), (35) and (38) be of the same size when $k_4$ is of the same size as $k_3$. This gives, from (19), (32), (35) and (38)

$$k_3/m \sim \left[ \alpha^{9/5} m_0 t \right]^{5/4} \quad (39)$$

$$\sqrt{\alpha} f_{k_3} \sim \left[ \alpha^{9/5} m_0 t \right]^{-5}. \quad (40)$$

Finally, we give a calculation which shows that although $f_k$ increases sharply with decreasing $k$, in the region $m < \omega < \bar{m}$, it does not undergo a large change
as $\omega$ goes from above $\bar{m}$ to below $\bar{m}$. To that end we first argue that in the region $\omega > \bar{m}$ the occupancy $f_k$ lies not far from the thermal curve. Consider transitions $k_5 \rightarrow k_3$ by elastic scattering where $\omega_5$ is just above $\bar{m}$. Then, neglecting $1/f_{k_3}$ compared to $1/f_{k_5}$ (see Appendix B),

$$
\dot{f}^{5 \rightarrow 3}_{k_5} \sim -m\sqrt{\alpha f_{k_5}} \left[ \sqrt{\alpha f_{k_3}} \left( \frac{k_3}{m} \right)^3 \right] \left[ \frac{\omega_5 - m}{T} - 1/f_{k_5} \right] T/m. \tag{41}
$$

The term $\sqrt{\alpha f_{k_3}} \left( \frac{k_3}{m} \right)^3$ is very large so that $\dot{f}^{5 \rightarrow 3}_{k_5}$ can be small only if

$$
f_{k_5} \simeq \frac{T}{\omega_5 - m} \tag{42}
$$

which is close to the thermal curve. Now consider transitions, due to elastic scattering on hard gluons, where $k_1$ goes to $k_3$. One has (see Appendix B)

$$
\dot{f}^{1 \rightarrow 3}_{k_3} \sim m\sqrt{\alpha f_{k_1}} \sqrt{\alpha f_{k_3}} \left[ \frac{\omega_1 - m}{T} - 1/f_{k_1} \right] T/m. \tag{43}
$$

But the $\dot{f}^{1 \rightarrow 3}_{k_3}$ in (43) cannot dominate (35) which means $\sqrt{\alpha f_{k_1}}$ cannot be large. Since (42) means $\sqrt{\alpha f_{k_3}}$ is of order one, we see that there can be no strong change in $f_k$ as $\omega$ crosses the value $\bar{m}$, despite the fact that the source turns on abruptly for $\omega < \bar{m}$.

### 2.3 The spectrum for $m_0 t < (1/\alpha)^{7/5}$.

From (31) one sees that the mean free path divided by time for hard particles becomes less than one when $m_0 t < (1/\alpha)^{7/5}$ and thus that the majority of the gluons, and most of the energy, of the system are not in thermal equilibrium in the early time domain. Suppose, at a time $t$, the gluon spectrum has reached momentum $p_0(t)$. Then energy conservation gives

$$
\frac{m_0^5}{\alpha} t \sim f_{p_0} p_0^4 \tag{44}
$$

or, using

$$
m^2 \sim \alpha f_{p_0} p_0^2, \tag{45}
$$

one gets

$$
p_0 m \sim m_0^2 \sqrt{m_0 t}. \tag{46}
$$

For $k/m \gg 1$ one expects $f_k$ to have the form

$$
f_k \simeq \frac{c(t)m}{\alpha \omega}. \tag{47}
$$
The 1/k dependence of \( f_k \) will shortly be checked, but it naturally follows from the fact that the dominant interactions of gluons having momentum \( k \) are with gluons having momentum on the order of \( m \) and with momentum transfer of the order of \( m \) which is much less than \( k \). This means the Boltzmann equation will obey

\[
\frac{\partial}{\partial t} f_k(t) \sim m^3 \nabla_k^2 f_k(t) \tag{48}
\]

whose steady state solution is of the form (47). If we suppose that \( c(t) \) has a power dependence on \( m_0 t \),

\[
c(t) \sim \alpha^a (m_0 t)^b, \tag{49}
\]

then requiring \( f_k(t) \sim 1/\alpha \) when \( m_0 t \sim 1 \) with \( k \sim m \) and requiring \( f_k(t) \sim 1/\sqrt{\alpha} \) when \( m_0 t \sim \alpha^{-7/5} \) and \( k \sim m \), as given by (2) and (6), gives \( a = 0 \) and \( b = -5/14 \) so that

\[
c(t) \sim (m_0 t)^{-5/14}. \tag{50}
\]

Eqs. (45), (47) and (49) give

\[
\frac{m}{p_0} \sim c(t) \sim (m_0 t)^{-5/14} \tag{51}
\]

while (46) and (50) give

\[
m \sim m_0 (m_0 t)^{1/14}, \tag{52}
\]

\[
p_0 \sim m_0 (m_0 t)^{3/7}.
\]

It is now easy to check that the mean free path, \( \lambda_{p_0} \), for gluons having momentum \( p_0 \) obeys

\[
\frac{\lambda_{p_0}}{t} \sim \left[ t p_0^3 f_{p_0}^2 \alpha^2 / p_0^3 \right]^{-1} = \frac{p_0^3}{m^4 t} \sim 1 \tag{53}
\]

as one might expect. We can now see somewhat more sharply why (50) must hold. If \( c(t) \) were to be parametrically larger than that given by (50) then \( \lambda_{p_0}/t \) would be less than one and the distribution (47) would not be stable. If \( c(t) \) were to be much smaller than given by (50) then the maximum value of \( f_k \) which we would find, for very small values of \( k \), would significantly exceed that given in (67) and would lead to the right hand side of (68) being much bigger than one, which is not consistent.

Let us examine in a little more detail the consistency and stability of (47). In particular consider the emission or absorption of a gluon \( k \), where \( \bar{m} \ll k \ll p_0 \), by hard gluons as illustrated in Fig. 2. One has, as in (26),

\[
\dot{f}_k \sim - \frac{p_0^6 \alpha^3}{m^3 k^3} \left[ f_{p_1} f_{p_2} f_k (1 + f_{p_3}) (1 + f_{p_4}) - f_{p_3} f_{p_4} (1 + f_{p_2}) (1 + f_{p_3}) (1 + f_k) \right]. \tag{54}
\]
Now the $f$’s are all large so we can limit ourselves to terms with four factors of $f$ on the right-hand side of (54). This yields
\[
\dot{f}_k \sim \frac{p_0^6 \alpha^3}{m^2 k^3} \{ f_k [ f_{p_1} f_{p_2} ( f_{p_3} + f_{p_4} ) - f_{p_3} f_{p_4} ( f_{p_1} + f_{p_2} ) ] - f_{p_1} f_{p_2} f_{p_3} f_{p_4} \} \quad (55)
\]
Write
\[
E_3 = E_1 + \omega \lambda
\]
\[
E_4 = E_2 + \omega (1 - \lambda)
\quad (56)
\]
so that
\[
\dot{f}_k \sim \frac{p_0^6 \alpha^3}{m^2 k^3} \{ (1 - \lambda) f_{p_1}^2 ( f_{p_2} \omega f_k + f_{p_2}^2 ) + \lambda f_{p_2}^2 ( f_{p_1} \omega f_k + f_{p_1}^2 ) \} \quad (57)
\]
where $f'_{p_1} = \frac{\partial}{\partial E_k} f_{p_1}$, etc. In (54) we have done a dimensional analysis. The $p_0^6$ factor is a rough way of writing the phase space factors coming from, say, $d^3 p_1 d^3 p_2$. Thus the $f_{p_1}^2 ( f_{p_2} \omega f_k + f_{p_2}^2 )$ in (57) should be understood as averaged over $p_1$ and $p_2$. Since $p_1$ and $p_2$ come in symmetrically we can rewrite (57) more simply as
\[
\dot{f}_k \sim \frac{p_0^6 \alpha^3}{m^2 k^3} f_{p_1}^2 ( f_{p_2} \omega f_k + f_{p_2}^2 ) \quad (58)
\]
where $p_1, p_2$ are on the order of $p_0$. Now the two terms, $f_{p_1}^2 \omega f_k$ and $f_{p_2}^2$, are of the same size if $f_k$ has a $1/k$ behavior as indicated in (47). Other power laws for $f_k$ will not allow $\dot{f}_k$ in (58) to be zero. To see the stability of $\dot{f}_k = 0$ suppose $f_{p_2} \omega f_k + f_{p_2}^2$ were positive. Then $f_k$ would grow in time. But this would increase $f_{p_2} \omega f_k$ until it cancels $f_{p_2}^2$. Thus (47) is a stable solution with respect to emission and absorption of gluons of momentum $k$ by harder gluons. The exact value of $c$, in (47), is determined by the vanishing of $< f_{p_1}^2 \omega f_k + f_{p_2}^2 >$.

Now let’s investigate the stability with respect to elastic scattering $k_1 + p_1 \rightarrow k_2 + p_2$ as illustrated in Fig. 1, and where $\bar{m} \ll k_1, k_2 \ll p_0$. One has
\[
\dot{f}_{k_2}^{1+2} \sim \frac{p_0^2 \alpha^2}{m^2} \{ f_{p_1} f_{k_1} (1 + f_{p_2}) (1 + f_{k_2}) - f_{p_2} f_{k_2} (1 + f_{p_1}) (1 + f_{k_1}) \} \quad (59)
\]
where we include dimensional factors for the $p_i$ and $k_i$ phase space integrations, although we keep these momenta fixed in the Bose-factors to better understand the stability of the solution (47). Calling $\omega_{k_1} - \omega_{k_2} = \Delta \omega$, using (47) and writing
\[
f_{p_1} - f_{p_2} \simeq - f'_{p_1} \Delta \omega \quad (60)
\]
\[
f_{k_2} - f_{k_1} \simeq \frac{f_{k_1} f_{k_2}}{\omega f_{\omega}} \Delta \omega \quad (61)
\]
\[
14
\]
one can write (59) as
\[ \dot{j}_{k2}^{1\leftrightarrow2} \sim \frac{-p_0^2}{m^2} \frac{f_{k1} f_{k2} \Delta \omega}{\omega f_\omega} \left[ f^\prime_{p_1} \omega f_\omega + f^2_{p_1} \right]. \]  
(62)

Thus \( \dot{j}_{k2}^{1\leftrightarrow2} \) will be zero under exactly the same conditions on \( c(t) \), in (47), as we found for stability under emission and absorption as given in (58).

Finally, we turn to the region \( m < \omega < \bar{m} \). The situation here is very much like previously considered in Sec. 2.2. The rate at which gluons enter the system and then cascade down to a momentum \( k_3 \) is given as in (32)
\[ \dot{j}_{k3}^{\text{source}} \sim \frac{m}{\alpha} \left( \frac{m_0}{m} \right)^5 \left( \frac{m}{k_3} \right)^3 \]  
(63)
where now \( m \) is given by (52). Eq.(35) is now changed to
\[ \dot{j}_{k3}^{\text{abs}} \sim -m \sqrt{\alpha} f_{k3} \left[ \frac{1}{\sqrt{\alpha}} \frac{m}{p_0} \right] \]  
(64)
which is easily obtained from (58) by dropping \( f^2_{p_2} \) compared to \( f^\prime_{p_2} \omega f_k \). Finally (38) remains unchanged
\[ \dot{j}_{k3}^{3\rightarrow4} \sim -m \sqrt{\alpha} f_{k3} \sqrt{\alpha} f_{k4} \left( \frac{k_4}{m} \right)^3 \left( \frac{k_3}{m} \right), \]  
(65)
and where the explanation of the meaning of this term is exactly as in the previous section. Requiring that the \( \dot{j}_{k3} \) terms in (63), (64) and (65) be of the same size gives
\[ k_3/m \sim \left( \frac{m}{p_0} \right)^2 \left( \frac{m}{m_0} \right)^5 \sim (m_0 t)^{-5/14} \]  
(66)
and
\[ \dot{j}_{k3} \sim \frac{1}{\alpha} \left( \frac{p_0}{m} \right)^4 \left( \frac{m_0}{m} \right)^5 \sim \frac{1}{\alpha} (m_0 t)^{15/14} \]  
(67)
which now replace (39) and (40). We note that (66) and (67) agree with (39) and (40) when \( m_0 t = \alpha^{-7/5} \).

Finally, we note that the process where three soft gluons, having momenta on the order of \( k_3 \) as given in (66), annihilate into two harder gluons as illustrated in Fig. 4 with \( q_1, q_2, q_3 \sim k_3 \) gives an \( \dot{j}_{k3} \) of the same order as (63), (64) and (65). Indeed higher order processes where \( n \) gluons having momenta on the order of \( k_3 \) annihilate into 2 harder gluons also are of the same order (see Appendix C) since
\[ \alpha \left( \frac{k_3}{m} \right)^3 f_{k3} \sim 1. \]  
(68)
3 Fixing the source in terms of wavelengths

In the discussion of the previous section we have assumed that the source of incoming energy is spread uniformly in phase space in the frequency range $m < \omega < \bar{m}$ where $m$ and $\bar{m}$ are of the same order of magnitude. Now we are going to express the source in terms of wavelength rather than frequency. We again suppose that the rate of inflow of energy is

$$\dot{\epsilon} = \frac{m_0^5}{\alpha}, \quad \text{(69)}$$

but we here suppose that the gluons are uniformly deposited in phase space over the region

$$0 < k < k_0. \quad \text{(70)}$$

That is we take $f_{\text{source}}$ to be a constant, in both $\vec{k}$ and $t$, in the region (70) and where

$$\frac{2(N_c^2 - 1)}{(2\pi)^3} f_{\text{source}} \int_0^{k_0} 4\pi k^2 \omega_k dk = \frac{m_0^5}{\alpha}. \quad \text{(71)}$$

In the present situation we have two parameters $m_0$ and $k_0$ with which to specify the source. (By allowing $k_0$ to be time-dependent the current specification includes the source used in Sec. 2.) We shall, however, always suppose that $k_0/m_0 \geq 1$ as the situation where $k_0$ is less than $m_0$ seems to have an abnormally high rate of deposition of energy over a limited region of phase space. In the discussion that follows we deal with $m_0$ and $k_0$ as time-independent quantities, however, it will be easy to allow a slow time dependence of these quantities in our final results.

Much of the discussion of the last section is unchanged since the basic pattern of the three different time regimes is set by the basic structure of the problem. We begin with the early time regime where $m_0 t < (1/\alpha)^{7/5}$. From (52) and (66) we see that

$$\frac{k_3}{k_0} \lesssim (m_0 t)^{-2/7} \quad \text{(72)}$$
and thus that \( k_3/k_0 \ll 1 \). Thus the picture here is essentially that described in Sec. 2.3, except that, when \( k_0/m \ll 1 \), the region \( k_0 < k \lesssim m \) now has no incoming gluons from the source. However gluons can come into this region from the process of four gluons, having momentum on the order of \( k_3 \), going into a gluon, \( k \), and two harder gluons. (We note that the three soft gluon to two harder gluon process cannot give a gluon, \( k \), in the region \( k_0 < k \ll m \).)

We can evaluate more precisely the dependence of \( f_k \), when \( k_0 < k \ll m \), by requiring a cancellation in the rate of gluons of momentum \( k \) produced by the competing processes. First, the rate of production of gluons coming from \( q_1 + q_2 + q_3 + q_4 \rightarrow k + k_1 + k_2 \), with \( q_1, q_2, q_3 \), and \( q_4 \) of order \( k_3 \) while \( k_1 \) and \( k_2 \) are of order \( m \), is (see Appendix C)

\[
\dot{f}_{k}^{4 \rightarrow 3} \sim mf_k \alpha_f m
\]

where \( \alpha \left( \frac{k_3}{m} \right)^3 f_{k_3} \sim 1 \) has been used. Secondly the rate of gluons absorbed by hard gluons is, from (58), and after letting the \( 1/k^3 \) there become \( 1/m^3 \) reflecting the cutoff of the \( 1/k^3 \) singularity when \( k/m \ll 1 \),

\[
\dot{f}_{k}^{\text{abs}} \sim mf_k \frac{c^2}{\alpha} \left( 1/f_k - \frac{\omega \alpha}{mc} \right)
\]

where we have used

\[
< f_p c(t) m/\alpha + f_p^2 > = 0
\]

the stability condition determining \( c \) that we found in Sec.2.3. In arriving at (74) we have also used \( \left( \frac{m}{m} \right)^2 \alpha f_{p_0} \sim 1 \) and \( c = m/p_0 \). Finally, there is the rate at which gluons \( k \) convert to gluons of momentum around \( k_3 \) by elastic scattering with hard particles,

\[
\dot{f}_{k}^{k \leftrightarrow k_3} \sim -p_0^3 \frac{\alpha^2}{m^2} \left( \frac{k_3}{m} \right)^3 \frac{m}{k} \left[ f_{p_1} f_k (1 + f_{k_3})(1 + f_{p_2}) - f_{k_3} f_{p_2} (1 + f_k)(1 + f_{p_1}) \right].
\]

It is straightforward to get

\[
\dot{f}_{k}^{k \leftrightarrow k_3} \sim p_0^3 \frac{\alpha}{mk} \left[ f_k (\omega - m) f_p' + f_p^2 \right]
\]

which, after using (75), becomes

\[
\dot{f}_{k}^{k \leftrightarrow k_3} = -m f_k \frac{mc}{\alpha} \left[ \frac{\alpha(\omega - m)}{mc} - 1/f_k \right].
\]

If \( f_k \) were near \( \frac{\omega}{\alpha \omega} \), its normal size as given by (47), then (78) would dominate. (Recall \( c/\sqrt{\alpha} \sim \left[ \alpha^{7/5} m_0 t \right]^{-3/14} \) is a large quantity in the time domain we are currently considering.) This means that \( 1/f_k \) must be much smaller than “normal”, and the
The term can be dropped in (74). The remaining term in (74) is comparable to (73) so that one can get stability by requiring
\[-\omega \alpha / m \sim \frac{m}{k} \left[ \frac{\alpha (\omega - m)}{m c} - 1/f_k \right] \]
or
\[f_k \sim \frac{mc}{\alpha} \frac{1}{\omega - m + O([m_0 t]^{-5/14 k})} \tag{79}\]
so that $f_k$ begins to rise rapidly as $k$ decreases beyond $m$.

Now turn to the region $m_0 t > (1/\alpha)^{7/5}$ where gluons having $\omega/m \gg 1$ are in thermal equilibrium. From (39)
\[k_3/k_0 \sim [\alpha^{9/5} m_0 t]^{5/4} m/k_0 . \tag{80}\]
So long as $k_3/k_0 \ll 1$, that is so long as $m_0 t \ll (1/\alpha)^{9/5}(k_0/m)^{4/5}$, the large peak in $f_k$ will still occur when $k \sim k_3$ and (39) and (40) should remain valid. Let us again investigate $f_k$ in the region $k_0 < k \lesssim m$ when $k_3/k_0 \ll 1$. Now the three relevant processes are
\[\dot{j}_k^{4+3} \sim mf_k \alpha f_m \left[ \frac{k_3}{m} \right]^4 \sim mf_k \alpha f_m \left[ \alpha^{7/5} m_0 t \right]^{-5} \tag{81}\]
while
\[\dot{j}_k^{k+k_3} \sim -\frac{m^2}{k} f_k \left[ \alpha^{7/5} m_0 t \right]^{-5/4} \left[ \frac{\omega - m}{T} - 1/f_k \right] T/m \tag{82}\]
and the process where $k$ is emitted or absorbed by thermal particle as illustrated in Fig. 2,
\[\dot{j}_k^{\text{abs}} \sim -mf_k \sqrt{\alpha} \left[ \frac{\omega}{T} - 1/f_k \right] T/m . \tag{83}\]
The dominant contributions now are (82) and (83). Requiring a cancellation to get a steady state gives
\[f_k \sim \frac{T}{m} \left[ (\omega - m)/m + O(k/m[\alpha^{9/5} m_0 t]^{5/4}) \right]^{-1} . \tag{84}\]
Thus $f_k$ grows as $k$ decreases below $m$. When $m_0 t$ approaches $(1/\alpha)^{9/5}(k_0/m)^{4/5}$ the picture starts to change as $k_3$ approaches $k_0$. At $m_0 t = (1/\alpha)^{9/5}(k_0/m)^{4/5}$ the occupancy $f_k$ begins to have little $k$-dependence for $k/k_0 < 1$. Requiring $\dot{j}_k^{\text{source}}$, as given by (71), to cancel with $\dot{j}_k^{\text{abs}}$, as given by (83), in the domain $0 < k < k_0$ gives
\[\sqrt{\alpha} f_k \sim \left[ \alpha^{9/5} m_0 t \right]^{-5/4} (m/k_0)^3 . \tag{85}\]
We note that (84), evaluated at $k = k_0$ is much smaller than (85). Thus for $m_0 t > (1/\alpha)^9/5 (k_0/m)^{4/5}$ there is a “discontinuity” in $f_k$ with

$$\sqrt{\alpha} f_k|_{k=k_0-\epsilon} - \sqrt{\alpha} f_k|_{k=k_0+\epsilon} \sim \left[\alpha^{9/5} m_0 t\right]^{-5/4} (m/k_0)^3.$$  \hspace{1cm} (86)

By the time $m_0 t \gg \alpha^{-9/5} (m/k_0)^{12/5}$ this jump in $f_k$ at $k_0$ has become very small, compared to the size of $f_k$, and complete “equilibration” is reached. Thus if $k_0$ is of the order of $m$ then complete thermalization occurs at a time $m_0 t \sim (1/\alpha)^9/5$ as in Sec. 2. If $k_0/m \ll 1$ then complete thermalization occurs when

$$m_0 t \sim (1/\alpha)^9/5 (m/k_0)^{12/5}.$$  \hspace{1cm} (87)

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**A  The derivation of Eq. (26)**

In this Appendix we consider the $2 \leftrightarrow 3$ gluon interaction where 4 gluons have momenta on the order of $T$ and the momentum of the remaining gluon is $k \ll T$. To derive the result given in Eq. (26), we follow Ref. [22] where the $2 \leftrightarrow 3$ process was calculated in all details. The most relevant process is the large angle gluon-gluon scattering with a small angle gluon radiation/absorption (gluon-gluon scattering with collinear emission/absorption). The inelastic $2 \leftrightarrow 3$ process is given by

$$\dot{N} = 2 \frac{[2(N_c^2 - 1)]^2}{(2\pi)^{15}} \int_{R} \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \frac{d^3 p_3}{2E_3} \frac{d^3 p_4}{2E_4} \frac{d^3 k}{2E_k} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4 - k) 
\times |M_{12\rightarrow 34k}|^2 [f_{p_1} f_{p_2} f_k (1 + f_{p_3}) (1 + f_{p_4}) - f_{p_3} f_{p_4} (1 + f_{p_1}) (1 + f_{p_2}) (1 + f_k)]$$  \hspace{1cm} (88)

where $\dot{N} \equiv 2(N_c^2 - 1) \int (d^3 k/(2\pi)^3) f_k$ and the symbol $R$ in the integration in (88) indicates that $k \ll p_1, p_2, p_3, p_4$. To evaluate (88) one supposes that the 5-gluon process
consists of a hard collision involving four gluons. Then the fifth gluon, \( k \), is collinear with one of the incoming or outgoing gluons, giving a factor of 4 which when divided by the symmetry factor, 2, explains the appearance of 2 in (88). Since all four possibilities lead to the same result, let us consider only the case where the gluons 4 and \( k \) are collinear. Collinear means that the invariant mass \( s_{4k} = (p_4 + k)^2 \) is significantly less than \( Q^2 = \min(|s|, |t|, |u|) \), where \( Q \) is the momentum scale of the hard scattering.

In Ref. [22] (see Eq. (20)) it has been shown that the following factorisation holds in the collinear limit

\[
\int \frac{d^3p_3}{(2\pi)^3E_3} \frac{d^3p_4}{(2\pi)^3E_4} \frac{d^3k}{(2\pi)^3E_k} (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4 - k) |M_{12\to34k}|^2 \\
\simeq \int \frac{d^3p_3}{(2\pi)^3E_3} \frac{d^3p_4}{(2\pi)^3E_4} (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) |M_{12\to34}|^2 \times \int \frac{dzds_{4k}}{s_{4k}} \frac{\alpha_s(s_{4k})}{2\pi} P_{gg}(z) \tag{89}
\]

where \( P_{gg}(z) \) is the unregularized \( g \to gg \) splitting function

\[
P_{gg}(z) = 2N_c \left( \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right) \tag{90}
\]

and \( |M_{12\to34}|^2 \) is the matrix element for the hard scattering given by (8). With (89) Eq. (88) becomes

\[
\dot{N} = 2 \left[ \frac{2(N_c^2 - 1)}{(2\pi)^3} \right]^2 \int \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} \frac{d^3p_3}{2E_3} \frac{d^3p_4}{2E_4} (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4) |M_{12\to34}|^2 \\
\times \int \frac{dzds_{4k}}{s_{4k}} \frac{\alpha_s(s_{4k})}{2\pi} P_{gg}(z) [f_{p_1}f_{p_2}f_k(1+f_{p_3})(1+f_{p_4}) - f_{p_3}f_{p_4}(1+f_{p_1})(1+f_{p_2})(1+f_k)] . \tag{91}
\]

The dominant contribution to \( |M_{12\to34}| \) comes from the small-\( t \) region

\[
|M|^2 = \frac{64\pi^4}{(N_c^2 - 1)} \left( \frac{\alpha N_c}{\pi} \right)^2 \left[ -\frac{us}{t} \right] , \tag{92}
\]

where \( u \approx -s \) for small \( t = (p_1 - p_3)^2 \) due to \( s + u + t = 0 \). Using also \( s = (p_1 + p_2)^2 \approx 2E^2 \), the integration over \( p_4 \) in (91) gives

\[
\dot{N} = \frac{2(N_c^2 - 1)^2}{4(2\pi)^5} \left( \frac{\alpha N_c}{\pi} \right)^2 \int \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} \frac{d^3p_3}{2E_3} \frac{d^3p_4}{2E_4} \frac{1}{(p_1 - p_3)^2} \\
\times \int \frac{dzds_{4k}}{s_{4k}} \frac{\alpha_s(s_{4k})}{2\pi} P_{gg}(z) [f_{p_1}f_{p_2}f_k(1+f_{p_3})(1+f_{p_4}) - f_{p_3}f_{p_4}(1+f_{p_1})(1+f_{p_2})(1+f_k)] . \tag{93}
\]
With \( d^3p_3 = d^2p_{3\perp} dp_{3L} = d^2p_{3\perp} dE_3 \), where \( p_{3\perp} \) is the transverse and \( p_{3L} \) the longitudinal component of \( p_3 \), and \((p_1-p_3)^2 = p_{3\perp}^2 \), one can perform also the integration over \( dE_3 \) and obtains
\[
\dot{N} = \frac{|2(N_c^2-1)|}{4(2\pi)^5} \left( \frac{\alpha N_c}{\pi} \right)^2 \int_R \frac{d^3p_1}{2\pi} \frac{d^3p_2}{2\pi} \frac{d^3p_3}{2\pi} \frac{1}{p_{3\perp}^4} \cdot
\]
\[
\times \int_R \frac{dz ds_{3k}}{s_{4k}} \frac{\alpha_s(s_{4k})}{2\pi} \frac{\delta^4(p_1 + p_3 - p_2 - k_4)}{4(s_{4k}^2 - 1)} \frac{M^2}{T^2} P_{gg}(z) \left[ f_{p_1} f_{p_2} f_{k_1} (1 + f_{P_1}) (1 + f_{P_2}) - f_{P_1} f_{P_2} (1 + f_{P_1}) (1 + f_{P_2}) \right].
\]
\[
(94)
\]

Now, with \( p_{3\perp}^2 \sim m^2 \) because of screening, \( d^2p_{3\perp} \sim m^2 \), \( d^3p_1 \sim T^3 \), \( d^3p_2 \sim T^3 \) and the relation \( \dot{\epsilon} = \dot{N} \nu \), one obtains the parametrical estimate given in Eq. (26). Note that \( k \ll p_1, p_2, p_3, p_4 \) is satisfied by requiring \( z \) to take only values close to 0 in (94).

**B The derivation of Eqs. (36), (41) and (43)**

In this Appendix we show how to get the parametrical estimates given in Eqs. (36), (41) and (43). Let us first derive Eq. (36). The elastic scattering of soft gluons, \( k_3 \) and \( k_4 \) \((k_3 > k_4)\), with hard gluons having momenta \( p_1, p_2 \) on the order of \( T \), as shown in Fig. 1 with the replacement \( q_1 \rightarrow k_3 \) and \( q_2 \rightarrow k_4 \), is described by
\[
\dot{f}_{k_3}^{k_4} = - \frac{2(N_c^2-1)}{(2\pi)^3 \omega_3} \int \frac{d^3p_1}{2E_1} \frac{d^3p_2}{2E_2} \frac{d^3k_4}{2\omega_4} \delta^4(p_1 + k_3 - p_2 - k_4) |M|^2 \cdot
\]
\[
\times \left[ f_{k_3} f_{p_1} (1 + f_{k_4}) (1 + f_{p_2}) - f_{k_4} f_{p_2} (1 + f_{k_3}) (1 + f_{p_1}) \right].
\]
\[
(95)
\]
where \(|M|^2\) is given in Eq. (8). The hard momenta read \( p_i = (E_i, \vec{p}_i) \) with \( E_1 \approx |\vec{p}_i| \) while the soft momenta are \( k_j = (\omega_j, \vec{k}_j) \) with \( \omega_j^2 = m^2 + \vec{k}_j^2 \approx m^2 \).

It is the small angle scattering (\( t \) small) which gives the dominant contribution to \(|M|^2\),
\[
|M|^2 = \left( \frac{\alpha N_c}{\pi} \right)^2 \left[ -\frac{us}{t} \right].
\]
\[
(96)
\]
Taking into account that \( t \sim m^2 \) because of screening, \( u \approx -s \) due to \( s + u + t = 0 \), and \( s = (p_1 + k_3)^2 \approx 2E_1 \omega_3 \), one obtains
\[
|M|^2 \approx \left( \frac{\alpha N_c}{\pi} \right)^2 \left( \frac{2E m^2}{m^4} \right).
\]
\[
(97)
\]
Using also
\[
\delta^4(p_1 + k_3 - p_2 - k_4) = \delta^3(\vec{p}_1 + \vec{k}_3 - \vec{p}_2 - \vec{k}_4) \delta(E_1 + \omega_3 - E_2 - \omega_4)
\]
\[
(98)
\]
the integration over $p_2$ in Eq. (95) then gives

$$\tilde{j}^{3\rightarrow 4}_{k_3} \approx -\frac{16}{(2\pi)^3} \frac{(\alpha N_c)^2}{m^4} \int d^3 p_1 \ d^3 k_4 \ \delta(E_1 + \omega_3 - \sqrt{(\vec{p}_1 + \vec{k}_3 - \vec{k}_4)^2} - \omega_4)$$

$$\times [f_{k_3}f_{p_1}(1 + f_{k_4})(1 + f_{p_2}) - f_{k_4}f_{p_2}(1 + f_{k_3})(1 + f_{p_1})]$$

(99)

where $f_{p_2}$ within the brackets $[ ]$ is to be evaluated at $\vec{p}_2 = \vec{p}_1 + \vec{k}_3 - \vec{k}_4$.

The evaluation of the delta-function

$$\delta(E_1 + \omega_3 - \sqrt{(\vec{p}_1 + \vec{k}_3 - \vec{k}_4)^2} - \omega_4)$$

$$\approx \delta(|\vec{p}_1| - m + \frac{\vec{k}_3^2}{2m} - |\vec{p}_1| - \frac{\vec{p}_1}{|\vec{p}_1|}(\vec{k}_3 - \vec{k}_4) - m - \frac{\vec{k}_4^2}{2m})$$

$$\approx \delta(-|\vec{k}_3| \cos(\theta))$$

(100)

where $\theta$ is the angle between $\vec{p}_1$ and $\vec{k}_3$, allows us to perform also the angular integration, $d^3 p_1 \delta(-|\vec{k}_3| \cos(\theta)) = 2\pi p_1^2 d|\vec{p}_1|/|\vec{k}_3|$, which leads to

$$\tilde{j}^{3\rightarrow 4}_{k_3} \approx -\frac{16}{(2\pi)^2} \frac{(\alpha N_c)^2}{m^4} \int p_1^2 d|\vec{p}_1| \ d^3 k_4 \ \frac{1}{|\vec{k}_3|}$$

$$\times [f_{k_3}f_{p_1}(1 + f_{k_4})(1 + f_{p_2}) - f_{k_4}f_{p_2}(1 + f_{k_3})(1 + f_{p_1})]$$

(101)

Now, with $p_1^2 \sim T^2$ and $d|\vec{p}_1| \sim T$, is easy to see that Eq. (101) is parametrically the same as Eq. (36).

To get Eq. (41), one follows the same procedure as above, with the replacement $k_3 \rightarrow k_5$ and $k_4 \rightarrow k_3$, and gets

$$\tilde{j}^{5\rightarrow 3}_{k_5} \approx -\frac{16}{(2\pi)^2} \frac{(\alpha N_c)^2}{m^4} \int p_1^2 d|\vec{p}_1| \ d^3 k_3 \ \frac{1}{|\vec{k}_5|}$$

$$\times [f_{k_5}f_{p_1}(1 + f_{k_3})(1 + f_{p_2}) - f_{k_3}f_{p_2}(1 + f_{k_5})(1 + f_{p_1})]$$

(102)

Using (10) which allows us to write the bracket in (102) as

$$[ ] = e^{E_1/T} f_{p_1} f_{p_2} f_{k_3} f_{k_5} [(\omega_5 - \omega_3)/T + 1/f_{k_3} - 1/f_{k_5}]$$

(103)

and $p_1, p_2 \sim T$, $d^3 k_3 \sim k_3^3$ and (3), we get

$$\tilde{j}^{5\rightarrow 3}_{k_5} \sim -m \sqrt{\alpha} f_{k_5} \left( \sqrt{\alpha} f_{k_3} \left( \frac{k_3^3}{m} \right) \right) \left( \frac{m}{k_5^3} \right) \left[ \frac{\omega_5 - \omega_3}{T} + 1/f_{k_3} - 1/f_{k_5} \right] T/m$$

(104)

which for $m/k_5 \sim 1$, $w_3 \approx m$ and $1/f_{k_3} \ll 1/f_{k_5}$ reduces to the result in Eq. (41).
To get Eq. (43), one starts with
\[
\hat{f}^{1 \rightarrow 3}_{k_3} = \frac{2(N^2 - 1)}{(2\pi)^5 \omega_3} \int \frac{d^3 p_1}{2E_1} \frac{d^3 p_2}{2E_2} \frac{d^3 k_1}{2\omega_1} \delta^4(p_1 + k_1 - p_2 - k_3) |M|^2 \\
\times [f_{k_1} f_{p_1}(1 + f_{k_3})(1 + f_{p_2}) - f_{k_3} f_{p_2}(1 + f_{k_1})(1 + f_{p_1})] \tag{105}
\]
and follows the same steps as in above calculations which lead to
\[
\hat{f}^{1 \rightarrow 3}_{k_3} \sim m \sqrt{\alpha f_{k_1}} \sqrt{\alpha f_{k_3}} \left( \frac{k_1}{m} \right)^2 \left[ \frac{\omega_1 - m}{T} - 1/f_{k_3} \right] T/m, \tag{106}
\]
where the factor \((k_1/m)^2\) comes from the phase space \(d^3k_1\) and the angular integration \(d^3p_1 \delta(-|\vec{k}_1| \cos(\theta)) = 2\pi p_2^2 d|\vec{p}_2|/|\vec{k}_1|\). Since \(k_1\) is close to \(m\) in the case discussed, it is legitimate to set \((k_1/m)^2 \sim 1\) which gives the estimate in Eq. (43).

C \hspace{1cm} n \rightarrow 2 \hspace{1cm} gluon merging

Here we show that the process where \(n\) soft gluons, having momenta on the order of \(k_3\) as given in Eq. (66), merge into two harder gluons (with \(k_1, k_2 \sim m\)) gives \(\hat{f}_{k_3}\) on the same order as (63), (64) and (65). It is instructive to begin with a simpler process where only three soft gluons annihilate into two harder gluons as shown in Fig. 4, with \(q_1, q_2, q_3 \sim k_3\), and then go over to the more general case. For the \(3 \rightarrow 2\) process we have
\[
\hat{f}^{3 \rightarrow 2}_{k_3} = \frac{2(N^2 - 1)}{(2\pi)^5 \omega_3} \int \frac{d^3 q_1}{2\omega_1} \frac{d^3 q_2}{2\omega_2} \frac{d^3 k_1}{2\omega_1} \frac{d^3 k_2}{2\omega_2} \delta^4(q_1 + q_2 + q_3 - k_1 - k_2) |M|^2 \\
\times [f_{q_1} f_{q_2} f_{q_3}(1 + f_{k_1})(1 + f_{k_2}) - f_{k_1} f_{k_2}(1 + f_{q_1})(1 + f_{q_2})(1 + f_{q_3})] \tag{107}
\]
and after the dimensional estimate
\[
|M|^2 \sim \alpha^3 \frac{1}{m^2} \tag{108}
\]
and the integration over \(k_2\)
\[
\hat{f}^{3 \rightarrow 2}_{k_3} \sim \frac{1}{m^2} \int \frac{d^3 q_1}{m^2} \frac{d^3 q_2}{m^2} \frac{d^3 k_1}{m^2} \delta(\omega_{q_1} + \omega_{q_2} + \omega_{q_3} - \omega_{k_1} - \omega_{k_2}) \\
\times [f_{q_1} f_{q_2} f_{q_3}(1 + f_{k_1})(1 + f_{k_2}) - f_{k_1} f_{k_2}(1 + f_{q_1})(1 + f_{q_2})(1 + f_{q_3})] \tag{109}
\]
Doing another dimensional estimate
\[
d^3 k_1 \delta(\omega_{q_1} + \omega_{q_2} + \omega_{q_3} - \omega_{k_1} - \omega_{k_2}) \sim m^2, \tag{110}
\]
keeping only the leading contribution in the brackets
\[
[f_{q_1} f_{q_2} f_{q_3}(1 + f_{k_1})(1 + f_{k_2}) - f_{k_1} f_{k_2}(1 + f_{q_1})(1 + f_{q_2})(1 + f_{q_3})] = [f_{k_3}^3 f_m] + \text{corrections} \quad (111)
\]
and using \(d^3q_1, d^3q_2 \sim k_3^3\), we find
\[
\dot{f}_{k_3}^{3 \rightarrow 2} \sim m f_m \alpha f_k \left[ \frac{k_3}{m} \right] f_{k_3}^3 \quad (112)
\]
which with (68) reduces to
\[
\dot{f}_{k_3}^{3 \rightarrow 2} \sim m f_m \alpha f_k \quad (113)
\]
and is of the same order as (63), (64) and (65).

Following the same procedure, it is easy to show for the \(n \rightarrow 2\) process that
\[
\dot{f}_{k_3}^{n \rightarrow 2} \sim m f_m \alpha f_k \left[ \frac{k_3}{m} \right] f_{k_3}^n \quad (114)
\]
which again, when using (68), becomes of the same order as (63), (64) and (65).

References

[1] P. F. Kolb, U. W. Heinz, P. Huovinen, K. J. Eskola and K. Tuominen, Nucl. Phys. A 696 (2001) 197; P. Huovinen, P. F. Kolb, U. W. Heinz, P. V. Ruuskanen and S. A. Voloshin, Phys. Lett. B 503 (2001) 58; P. F. Kolb, P. Huovinen, U. W. Heinz and H. Heiselberg, Phys. Lett. B 500 (2001) 232; P. F. Kolb, J. Sollfrank and U. W. Heinz, Phys. Rev. C 62 (2000) 054909.

[2] D. Teaney, Phys. Rev. C 68 (2003) 034913; D. Teaney, J. Lauret and E. V. Shuryak, arXiv:nucl-th/0110037; Phys. Rev. Lett. 86 (2001) 4783.

[3] R. Baier, A. H. Mueller, D. Schiff and D. T. Son, Phys. Lett. B 502 (2001) 51.

[4] S. M. H. Wong, Phys. Rev. C 54 (1996) 2588.

[5] R. Baier, A. H. Mueller, D. Schiff and D. T. Son, Phys. Lett. B 539 (2002) 46.

[6] P. Arnold, J. Lenaghan and G. D. Moore, JHEP 0308 (2003) 002.

[7] S. Mrowczynski, Phys. Lett. B 214 (1988) 587; Phys. Lett. B 314 (1993) 118; Phys. Rev. C 49 (1994) 2191; Phys. Lett. B 393 (1997) 26; S. Mrowczynski and M. H. Thoma, Phys. Rev. D 62 (2000) 036011; J. Randrup and S. Mrowczynski, Phys. Rev. C 68 (2003) 034909.
[8] E. S. Weibel, Phys. Rev. Lett. 2, 83 (1959).

[9] O. Buneman, Phys. Rev. Lett. 1, 8 (1958).

[10] U. W. Heinz, Nucl. Phys. A 418 (1984) 603C; Y. E. Pokrovsky and A. V. Selikhov, JETP Lett. 47 (1988) 12 [Pisma Zh. Eksp. Teor. Fiz. 47 (1988) 11]; Sov. J. Nucl. Phys. 52 (1990) 146 [Yad. Fiz. 52 (1990) 229]; Sov. J. Nucl. Phys. 52 (1990) 385 [Yad. Fiz. 52 (1990) 605]; O. P. Pavlenko, Sov. J. Nucl. Phys. 55 (1992) 1243 [Yad. Fiz. 55 (1992) 2239].

[11] A. H. Mueller, A. I. Shoshi and S. M. H. Wong, Phys. Lett. B 632 (2006) 257; “A modified 'bottom-up' thermalization in heavy ion collisions,” arXiv:hep-ph/0512045.

[12] D. Bodeker, JHEP 0510 (2005) 092.

[13] P. Arnold and G. D. Moore, Phys. Rev. D 73 (2006) 025006.

[14] P. Romatschke and M. Strickland, Phys. Rev. D 68 (2003) 036004; P. Romatschke and M. Strickland, Phys. Rev. D 70 (2004) 116006.

[15] A. Rebhan, P. Romatschke and M. Strickland, Phys. Rev. Lett. 94 (2005) 102303;

[16] A. Dumitru and Y. Nara, Phys. Lett. B 621 (2005) 89; A. Dumitru, Y. Nara and M. Strickland, “Ultraviolet avalanche in anisotropic non-Abelian plasmas,” arXiv:hep-ph/0604149.

[17] P. Romatschke and R. Venugopalan, Phys. Rev. Lett. 96 (2006) 062302; “The unstable Glasma,” arXiv:hep-ph/0605045; P. Romatschke and A. Rebhan, “Plasma instabilities in an anisotropically expanding geometry,” arXiv:hep-ph/0605064.

[18] S. Mrowczynski, Acta Phys. Polon. B 37 (2006) 427.

[19] P. Arnold and G. D. Moore, Phys. Rev. D 73 (2006) 025013.

[20] V. Zakharov, V. L’vov, and G. Falkovich, “Kolmogorov Spectra of Turbulence, Wave Turbulence”, Springer-Verlag, Berlin, 1992.

[21] R. Micha and I. I. Tkachev, Phys. Rev. D 70 (2004) 043538.

[22] S. M. H. Wong, “Out-of-equilibrium collinear enhanced equilibration in the bottom-up thermalization scenario in heavy ion collisions,” arXiv:hep-ph/0404222.