The η − η′ mixing mass term due to the derivative coupling $SU(3) \times SU(3)$ symmetry breaking term, produces an additional momentum-dependent pole term for processes with η′, but is suppressed in the η amplitude by a factor $m_\eta^2/m_{\eta'}^2$. Thus processes with η meson could then be described, to first order in $SU(3)$ breaking, by the usual momentum-independent η − η′ mixing angle, while processes with η′ could be effectively described by a new mixing angle as in the two-angle mixing scheme used in the parametrization of the pseudo-scalar meson decay constants in the current literature. In this paper we will obtain the new mixing angle by diagonalizing both the mixing mass term and the momentum-dependent mixing term. We find that the η − η′ system could be described by a meson field renormalization and a new mixing angle $\theta$ which differs from the usual mixing angle $\theta_P$ by a small momentum-dependent mixing $d$ term. Assuming nonet symmetry for the $\eta_0$ singlet amplitude, from the sum rules relating $\theta$ and $d$ to the measured vector meson radiative decays amplitudes, we obtain $\theta = -(13.99 \pm 3.1)^\circ$, $d = 0.12 \pm 0.03$ from $\rho \to \eta \gamma$ and $\eta' \to \rho \gamma$ decays, for $\omega$, $\theta = -(15.47 \pm 3.1)^\circ$, $d = 0.11 \pm 0.03$, and for $\phi$, $\theta = -(12.66 \pm 2.1)^\circ$, $d = 0.10 \pm 0.03$. It seems that vector meson radiative decays would favor a small η − η′ mixing angle as found in previous analysis.

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The η − η′ mixing angle used in the past to describe the η − η′ system is based on the assumption that the off-diagonal octet-singlet mixing mass term does not depend significantly on the energy of the state [1]. However, as with the derivative coupling $SU(3) \times SU(3)$ breaking terms used in the derivation of the $f_K/f_\pi$ ratio and the Callan-Treiman relation for the vector currents in $K^0_{l3}$ decays [2], recent works [3–6] show that a quadratic derivative off-diagonal octet-singlet mixing term could exist and requires two angles $\theta_8$ and $\theta_0$ to describe the pseudo-scalar meson decay constants. One could also describe the η − η′ system by the usual mixing angle $\theta_P$ with the additional off-diagonal derivative $SU(3)$ breaking mass term treated as a perturbation [7], the η and η′ physical states are still the usual linear combinations of the pure octet and singlet $SU(3)$ state with the momentum-independent mixing angle $\theta_P$, but the momentum-dependent off-diagonal mass term will give rise to
an additional contribution to processes involving $\eta$ and $\eta'$ by the quadratic momentum dependent pole term as in non-leptonic $K \to 3\pi$ decays, for which the $K$ meson pole term is suppressed relative to the pion pole term by a factor $m^2_\pi/m^2_K$. The $\eta$ pole contribution to the $\eta'$ amplitude is of a strength $d$, a first order $SU(3)$ breaking mixing term, like the $\sin\theta_P$ term, but the $\eta'$ pole contribution to the process with $\eta$ on the mass shell is of the strength $d (m^2_\eta/m^2_{\eta'})$, a second order $SU(3)$ breaking effect and is suppressed by a factor $m^2_\eta/m^2_{\eta'}$. Thus the quadratic momentum-dependent off-diagonal mixing mass term, while leaves the amplitude with $\eta$ almost unaffected, could enhance or suppress the $\eta'$ amplitude. This seems to be the origin of the two-angle description of the pseudo-scalar decay constants used in the literature as mentioned above. In this approach, $\theta_8$ is the usual mixing angle and, effectively describes the mixing of $\eta_0$ with $\eta_8$ to give the physical $\eta$ meson while $\sin\theta_0$ would effectively represent the admixture of $\eta_8$ in $\eta'$ due to both the usual mixing angle and the additional contribution from the momentum-dependent octet-singlet mixing term $d$ treated as a perturbation. To include higher order terms, we need to diagonalize both the momentum-independent and momentum-dependent mixing terms to put the Lagrangian in a canonical form. In this paper, we will show that in the presence of momentum-dependent mixing terms, the $\eta - \eta'$ system could be described by the new mixing angle and the renormalization of the $\eta$ and $\eta'$ meson fields. The new mixing angle contains the usual mixing angle and a small additional term coming from $d$. From the sum rules relating the pure octet and singlet vector meson radiative decay amplitudes to that for the measured decays amplitudes, and using nonet symmetry for pure octet and singlet amplitudes, we obtain solutions for the new mixing angle $\theta$ and the momentum-dependent mixing term $d$. For $\rho \to \eta\gamma$ and $\eta' \to \rho\gamma$ decays, $\theta = -(13.99 \pm 3.1)^\circ$, $d = 0.12 \pm 0.03$, for $\omega \to \eta\gamma$ and $\eta' \to \omega\gamma$ decays, $\theta = -(15.47 \pm 3.1)^\circ$, $d = 0.11 \pm 0.03$ and for $\phi \to \eta\gamma$ and $\phi \to \eta'\gamma$ decays, $\theta = -(12.66 \pm 2.1)^\circ$, $d = 0.10 \pm 0.03$. It is remarkable that these values are consistent with each other, to within experimental errors. After subtracting the $d$ terms, one would get a value of $-(8 - 10)^\circ$ for the usual mixing angle. It seems that vector meson radiative decays would favor a small $\eta - \eta'$ mixing angle as found in previous analysis, for example, a value between $-13^\circ$ and $-17^\circ$, or an average $\theta_P = -15.3^\circ \pm 1.3^\circ$ is obtained and $\theta_P \approx -11^\circ$ is obtained in 12, also a recent analysis 13, 14 using the more precise $V \to P\gamma$ measured branching ratios 10 found $\theta_P = -13.3^\circ \pm 1.3^\circ$.

We begin by writing down the Lagrangian for the $\eta - \eta'$ system with the usual non-derivative mixing mass term $m^2_{\eta_8}$, the pure octet $\eta_8$ mass $m^2_8$, the singlet $\eta_0$ mass $m^2_0$, and the derivative $\eta_0 - \eta_8$ mixing term.
\[ \mathcal{L}_0 = \frac{1}{2} (\partial_\mu \eta_8 \partial^\mu \eta_8 + \partial_\mu \eta_0 \partial^\mu \eta_0 + m_8^2 \eta_8^2 + m_0^2 \eta_0^2) + d \partial_\mu \eta_8 \partial^\mu \eta_0 + m_{08}^2 \eta_8 \eta_0 \]  

(1)

To diagonalize this Lagrangian, we shall first make the substitution:

\[ \eta_8 = \frac{(\eta_0 - \eta_8)}{\sqrt{2}}, \quad \eta_0 = \frac{(\eta_0 + \eta_8)}{\sqrt{2}}, \]  

(2)

\[ \mathcal{L}_0 \] becomes,

\[ \mathcal{L}_1 = \frac{1}{2} (1 - d) \partial_\mu \eta_{81} \partial^\mu \eta_{81} + \frac{1}{2} (1 + d) \partial_\mu \eta_{01} \partial^\mu \eta_{01} + \frac{1}{2} (m_{81}^2 \eta_{81}^2 + m_{01}^2 \eta_{01}^2) + m_{081}^2 \eta_{81} \eta_{01} \]  

(3)

with

\[ m_{81}^2 = \frac{(m_0^2 + m_8^2 - 2m_{08}^2)}{2}, \quad m_{01}^2 = \frac{(m_0^2 + m_8^2 + 2m_{08}^2)}{2}, \quad m_{081}^2 = \frac{(m_0^2 - m_8^2)}{2}. \]  

(4)

To bring the kinetic term in \( \mathcal{L}_1 \) to the canonical form, we now perform a renormalization of \( \eta_{81} \) and \( \eta_{01} \) meson field operators:

\[ \eta_{81} = \frac{\eta_{82}}{\sqrt{1 - d}}, \quad \eta_{01} = \frac{\eta_{02}}{\sqrt{1 + d}}, \]  

(5)

and \( \mathcal{L}_1 \) becomes

\[ \mathcal{L}_2 = \frac{1}{2} \left( \partial_\mu \eta_{82} \partial^\mu \eta_{82} + \partial_\mu \eta_{02} \partial^\mu \eta_{02} + m_{82}^2 \eta_{82}^2 + m_{02}^2 \eta_{02}^2 \right) + \frac{m_{081}^2}{\sqrt{1 - d^2}} \eta_{82} \eta_{02} \]  

(6)

which can now be brought back to the octet-singlet basis by the transformation:

\[ \eta_{82} = \frac{(\eta_{03} - \eta_{83})}{\sqrt{2}}, \quad \eta_{02} = \frac{(\eta_{03} + \eta_{83})}{\sqrt{2}}. \]  

(7)

We have finally,

\[ \mathcal{L}_3 = \frac{1}{2} (\partial_\mu \eta_{83} \partial^\mu \eta_{83} + \partial_\mu \eta_{03} \partial^\mu \eta_{03} + m_{83}^2 \eta_{83}^2 + m_{03}^2 \eta_{03}^2) + m_{083}^2 \eta_{83} \eta_{03} \]  

(8)

with

\[ m_{82}^2 = \frac{(1 - \sqrt{1 - d^2})m_0^2 + (1 + \sqrt{1 - d^2})m_8^2}{2(1 - d^2)}, \quad m_{02}^2 = \frac{(1 + \sqrt{1 - d^2})m_0^2 + (1 - \sqrt{1 - d^2})m_8^2}{2(1 - d^2)}, \quad m_{082}^2 = \frac{m_0^2 - d(m_0^2 + m_8^2)/2}{(1 - d^2)}. \]  

(9)

Thus we have been able to bring the original Lagrangian of the pure octet \( \eta_8 \) and singlet \( \eta_0 \) mesons with the derivative coupling \( SU(3) \) symmetry breaking momentum-dependent \( \eta_8 - \eta_0 \) mixing term,
to the usual form with only the energy-independent mixing mass term with $L_3$ having the same form as $L_0$, except that the mass and mixing terms are modified by additional contributions from the momentum-dependent mixing term $d$ and the renormalization of the $\eta_8$ and $\eta_0$ meson fields, and in the limit of $d = 0$, we recover the usual mass term in $L_0$. In terms of $\eta_{83}$ and $\eta_{03}$ state, the pure $SU(3)$ octet and singlet state are then given by

$$
\eta_8 = \left( \frac{\sqrt{1 - d} + \sqrt{1 + d}}{2\sqrt{1 - d^2}} \right) \eta_{83} + \left( \frac{\sqrt{1 - d} - \sqrt{1 + d}}{2\sqrt{1 - d^2}} \right) \eta_{03},
$$

$$
\eta_0 = \left( \frac{\sqrt{1 - d} - \sqrt{1 + d}}{2\sqrt{1 - d^2}} \right) \eta_{83} + \left( \frac{\sqrt{1 - d} + \sqrt{1 + d}}{2\sqrt{1 - d^2}} \right) \eta_{03}.
$$

(10)

From the above expressions, we see that $\eta_{83}$ and $\eta_{03}$ states are mixture of the pure $\eta_8$ and $\eta_0$ and becomes the pure octet and singlet state in the limit of $d = 0$. This is an example of mixing caused by renormalization of the field operators due to the momentum-dependent derivative coupling $SU(3)$ breaking terms. The Lagrangian in Eq. (8) can now be brought to the diagonal form by writing $\eta_{83}$ and $\eta_{03}$ in terms of the physical $\eta$ and $\eta'$ states and the mixing angle $\theta$:

$$
\eta_{83} = \cos(\theta)\eta + \sin(\theta)\eta',
$$

$$
\eta_{03} = -\sin(\theta)\eta + \cos(\theta)\eta'.
$$

(11)

with $\theta$ given by:

$$
\tan(2\theta) = \frac{2m_{08}^2 - d(m_0^2 + m_8^2)}{(m_0^2 - m_8^2)\sqrt{1 - d^2}}
$$

(12)

or

$$
\sin(\theta) = \left( \frac{\cos(2\theta)}{\cos(\theta)} \right) \left( \frac{m_{08}^2 - d(m_0^2 + m_8^2)/2}{(m_0^2 - m_8^2)\sqrt{1 - d^2}} \right)
$$

(13)

which takes a simple form for $\theta$ small,

$$
\sin(\theta) = \left( \frac{m_{08}^2 - d(m_0^2 + m_8^2)/2}{(m_0^2 - m_8^2)\sqrt{1 - d^2}} \right)
$$

(14)

After this last step, we arrive at the Lagrangian:

$$
\mathcal{L} = \frac{1}{2} (\partial_\mu \eta \partial^\mu \eta + \partial_\mu \eta' \partial^\mu \eta' + m_\eta^2 \eta^2 + m_{\eta'}^2 \eta'^2)
$$

(15)

with $m_\eta^2$ and $m_{\eta'}^2$ given by:

$$
m_\eta^2 = \frac{(m_0^2 + m_8^2 - 2d m_{08}^2)}{2(1 - d^2)} - \frac{(m_0^2 - m_8^2)\cos(2\theta)}{2\sqrt{1 - d^2}} + \frac{(d(m_0^2 + m_8^2) - 2m_{08}^2)\sin(2\theta)}{2(1 - d^2)}
$$

$$
m_{\eta'}^2 = \frac{(m_0^2 + m_8^2 - 2d m_{08}^2)}{2(1 - d^2)} + \frac{(m_0^2 - m_8^2)\cos(2\theta)}{2\sqrt{1 - d^2}} - \frac{(d(m_0^2 + m_8^2) - 2m_{08}^2)\sin(2\theta)}{2(1 - d^2)}
$$

(16)
The pure octet $\eta_8$ and singlet $\eta_0$ can now be expressed terms of $\eta$ and $\eta'$. From Eqs. (10,11), we have:

$$\eta_8 = C_{8\eta} \eta + C_{8\eta'} \eta', \quad \eta_0 = C_{0\eta} \eta + C_{0\eta'} \eta'.$$

(17)

with

$$C_{8\eta} = \left( \frac{(\sqrt{1 - d} - \sqrt{1 + d}) \sin(\theta)}{2\sqrt{(1 - d^2)}} + \frac{(\sqrt{1 - d} + \sqrt{1 + d}) \cos(\theta)}{2\sqrt{(1 - d^2)}} \right),$$

$$C_{8\eta'} = \left( \frac{(\sqrt{1 - d} - \sqrt{1 + d}) \cos(\theta)}{2\sqrt{(1 - d^2)}} + \frac{(\sqrt{1 - d} + \sqrt{1 + d}) \sin(\theta)}{2\sqrt{(1 - d^2)}} \right),$$

$$C_{0\eta} = \left( \frac{-(\sqrt{1 - d} + \sqrt{1 + d}) \sin(\theta)}{2\sqrt{(1 - d^2)}} + \frac{-(\sqrt{1 - d} - \sqrt{1 + d}) \cos(\theta)}{2(1 - d^2)} \right),$$

$$C_{0\eta'} = \left( \frac{(\sqrt{1 - d} + \sqrt{1 + d}) \cos(\theta)}{2\sqrt{(1 - d^2)}} + \frac{(\sqrt{1 - d} - \sqrt{1 + d}) \sin(\theta)}{2\sqrt{(1 - d^2)}} \right).$$

(18)

For $d = 0$, we recover the usual expression given in Eq. (11).

To first order in $d$, we have,

$$\eta_8 = \left( d \sin(\theta)/2 + \cos(\theta) \right) \eta + \left( -d \cos(\theta)/2 + \sin(\theta) \right) \eta',$$

$$\eta_0 = \left( -\sin(\theta) - d \cos(\theta)/2 \right) \eta + \left( \cos(\theta) - d \sin(\theta)/2 \right) \eta'.$$

(19)

Consider now the $d$ terms in Eq. (19). The contribution to $\eta'$ amplitude from the pure $\eta_8$ term is proportional to $(-d \cos(\theta)/2 + \sin(\theta))$ which gives $-d/2$ from the first term, while another $-d/2$ from the $\sin(\theta)$ term. Similarly, the $d$ term in the $\eta$ amplitude coming from the pure singlet $\eta_0$ term $(\sin(\theta) + d \cos(\theta)/2)$ cancels out ( $\sin(\theta)$ having the same $d$ term with opposite sign). More precisely, to first order in $d$, and neglecting also $\sin(\theta)/2$ term in $\cos(\theta)$, we have from Eq. (14):

$$\eta_8 = \left( d \sin(\theta)/2 + \cos(\theta) \right) \eta + \left( \sin(\theta_0) + \frac{d m_0^2}{(m_0^2 - m_8^2)} \right) \eta',$$

$$\eta_0 = \left( -\sin(\theta_0) + \frac{d m_0^2}{(m_0^2 - m_8^2)} \right) \eta + \left( \cos(\theta) - d \sin(\theta)/2 \right) \eta'.$$

(20)

where $\theta_0$ is the mixing angle for $d = 0$ (the usual mixing angle).

This agrees with the perturbation treatment of the derivative $SU(3) \times SU(3)$ symmetry breaking terms given in [16], except for the $d \sin(\theta)$ term which is second order in $SU(3)$ breaking.

Using Eq. (12) to express $m_{08}^2$ in terms of $\tan(2\theta)$, the expressions for $\eta$ and $\eta'$ masses in Eq. (16) are then:

$$m_\eta^2 = \left( \frac{m_0^2 + m_8^2}{2} \right) - \frac{(m_0^2 - m_8^2)}{2\sqrt{(1 - d^2)} \cos(2\theta)} - \frac{(d \tan(2\theta))(m_0^2 - m_8^2)}{2\sqrt{(1 - d^2)}}$$

$$m_{\eta'}^2 = \left( \frac{m_0^2 + m_8^2}{2} \right) + \frac{(m_0^2 - m_8^2)}{2\sqrt{(1 - d^2)} \cos(2\theta)} - \frac{(d \tan(2\theta))(m_0^2 - m_8^2)}{2\sqrt{(1 - d^2)}}$$

(21)
which now depend only on \( m_0^2, m_S^2 \) and \( d \). By taking the mass difference \( m_\eta^2 - m_8^2 \) and \( m_{\eta'}^2 - m_8^2 \), we obtain:

\[
m_\eta^2 - m_8^2 = \frac{(m_0^2 - m_S^2)}{2} \left( -1 + \frac{1}{\sqrt{(1 - d^2)\cos(2\theta)}} - \frac{d \tan(2\theta)}{\sqrt{(1 - d^2)}} \right)
\]

\[
m_{\eta'}^2 - m_8^2 = \frac{(m_0^2 - m_S^2)}{2} \left( 1 + \frac{1}{\sqrt{(1 - d^2)\cos(2\theta)}} - \frac{d \tan(2\theta)}{\sqrt{(1 - d^2)}} \right) \tag{22}
\]

This implies,

\[
m_\eta^2 - m_8^2 = R (m_{\eta'}^2 - m_8^2). \tag{23}
\]

with \( R \) given by:

\[
R = \left( -1 + \sqrt{(1 - d^2)\cos(2\theta) - d \sin(2\theta)} \right) \left( 1 + \sqrt{(1 - d^2)\cos(2\theta) - d \sin(2\theta)} \right)^{-1} \tag{24}
\]

As \( d \) is a small \( SU(3) \times SU(3) \) breaking parameter, putting \( d = \sin(\alpha) \) and \( \sqrt{1 - d^2} = \cos(\alpha) \), the above expression Eq. (24) takes a simple form,

\[
R = -\tan(\theta + \alpha/2)^2 \tag{25}
\]

For small \( d, \alpha \approx \sin(\alpha) = d, \theta + \alpha/2 \approx \theta_P \), and \( R \) is essentially the usual relation \( R = -\tan(\theta_P)^2 \) which is not affected by the presence of a momentum-dependent mixing term.

With our Lagrangian in the diagonal form, we shall now try to determine \( \theta \) and \( d \) using the sum rules [7], obtained by equating the vector meson radiative decay matrix element for the pure octet \( \eta_8 \) and singlet \( \eta_0 \) with the expressions for these quantities extracted from the measured matrix elements with \( \eta \) and \( \eta' \) given by Eq. (17). Defining, as in [7], the electromagnetic form factor \( V \rightarrow P \) by:

\[
< P(p_P)|J^em_\mu|V(p_V) > = \epsilon_{\mu PP_V\epsilon_V} g_{VP}\gamma
\]

where \( g_{VP}\gamma \) is the on-shell \( VP\gamma \) coupling constant with dimension the inverse of energy. We have, for the radiative decay rates [15]

\[
\Gamma(V \rightarrow P\gamma) = \frac{\alpha}{24} g_{VP}\gamma^2 \left( \frac{m_V^2 - m_P^2}{m_V} \right)^3
\]

\[
\Gamma(P \rightarrow V\gamma) = \frac{\alpha}{8} g_{VP}\gamma^2 \left( \frac{m_P^2 - m_V^2}{m_P} \right)^3 \tag{27}
\]

For convenience, we give in Table. I the measured radiative branching ratios together with the extracted coupling constant \( g_{VP}\gamma \) in unit of GeV\(^{-1}\) and its theoretical value derived either from
TABLE I: Theoretical values for $V \rightarrow P\gamma$ with $\theta_P = 0$, $k=0.85$ together with the measured branching ratios and the extracted $g_{VP\gamma}$, taken from Ref. [10].

| Decay          | $g_{VP\gamma}$, $\theta_P = 0$, $k=0.85$ | $g_{VP\gamma}$ (exp.) | BR (exp) [10] |
|----------------|------------------------------------------|------------------------|---------------|
| $\rho^0 \rightarrow \pi^0\gamma$ | $1/3 g_u$                                    | $0.72 \pm 0.04$          | $(4.5 \pm 0.5) \times 10^{-4}$ |
| $\rho^0 \rightarrow \pi^0\gamma$ | $1/3 g_u$                                    | $0.83 \pm 0.05$          | $(6.0 \pm 0.8) \times 10^{-4}$ |
| $\rho^0 \rightarrow \eta\gamma$  | $0.58 g_u (f_\pi/f_{\eta\gamma})$           | $1.59 \pm 0.06$          | $(3.00 \pm 0.20) \times 10^{-4}$ |
| $\omega \rightarrow \pi^0\gamma$ | $0.99 g_u$                                    | $2.29 \pm 0.03$          | $(8.28 \pm 0.28)\%$          |
| $\omega \rightarrow \eta\gamma$  | $0.17 g_u (f_\pi/f_{\eta\gamma})$           | $0.45 \pm 0.02$          | $(4.6 \pm 0.4) \times 10^{-4}$ |
| $\phi \rightarrow \pi^0\gamma$  | $0.06 g_u$                                    | $0.13 \pm 0.003$         | $(1.27 \pm 0.06) \times 10^{-3}$ |
| $\phi \rightarrow \eta\gamma$   | $0.47 g_u (f_\pi/f_{\eta\gamma})$           | $0.71 \pm 0.01$          | $(1.309 \pm 0.024)\%$        |
| $\phi \rightarrow \eta'\gamma$  | $-0.31 g_u (f_\pi/f_{\eta'\gamma})$          | $-(0.72 \pm 0.01)$       | $(6.25 \pm 0.21) \times 10^{-5}$ |
| $\eta' \rightarrow \rho^0\gamma$ | $0.82 g_u (f_\pi/f_{\eta\gamma})$           | $1.35 \pm 0.02$          | $(29.1 \pm 0.5)\%$          |
| $\eta' \rightarrow \omega\gamma$ | $0.29 g_u (f_\pi/f_{\eta\gamma})$           | $0.44 \pm 0.02$          | $(2.75 \pm 0.23)\%$          |
| $K^{*\pm} \rightarrow K^\pm\gamma$ | $0.38 g_u (f_\pi/f_{K^\pm\gamma})$          | $0.84 \pm 0.04$          | $(9.9 \pm 0.9) \times 10^{-4}$ |
| $K^{*0} \rightarrow K^0\gamma$  | $-0.62 g_u (f_\pi/f_{K^0\gamma})$           | $-(1.27 \pm 0.05)$       | $(2.46 \pm 0.22) \times 10^{-3}$ |

an SU(3) effective Lagrangian with nonet symmetry for the $V \rightarrow \eta_0\gamma$ amplitude or from the quark counting rule with the coupling constant $g_{VP\gamma}$ given in terms of the quark coupling constant $g_q$, $(q = u, d, s)$ for the magnetic transition $(q\bar{q})(1^-) \rightarrow (q\bar{q})(0^-)\gamma$ [11, 13, 15]. More details on the theoretical values for $V \rightarrow \eta_0\gamma$ and $V \rightarrow \eta_0\gamma$ can be found in Ref. [11].

In terms of $g_{VP\gamma}$, the sum rules read:

$$S(V \rightarrow \eta\gamma) = g_{VP\gamma} C_{8\eta} + g_{VP\eta'} C_{8\eta'} = \frac{g_{VP\eta\gamma}}{g_{VP\pi^0\gamma}} g_{VP\pi^0\gamma}$$

$$S(\eta' \rightarrow V\gamma) = g_{VP\eta'} C_{0\eta} + g_{VP\eta'} C_{0\eta'} = \frac{g_{VP\eta\gamma}}{g_{VP\pi^0\gamma}} g_{VP\pi^0\gamma}$$

and similarly for other vector meson radiative decays. Thus with the updated values of the measured values for $g_{VP\gamma}$ in Table. [1] we have, for $\rho$ meson radiative decay:

$$S(\rho \rightarrow \eta\gamma) = 1.59 C_{8\eta} + 1.35 C_{8\eta'} = 1.12$$

$$S(\eta' \rightarrow \rho\gamma) = 1.59 C_{0\eta} + 1.35 C_{0\eta'} = 1.63$$

for $\omega$ meson:

$$S(\omega \rightarrow \eta\gamma) = 0.45 C_{8\eta} + 0.44 C_{8\eta'} = 0.29$$

$$S(\eta' \rightarrow \omega\gamma) = 0.45 C_{0\eta} + 0.44 C_{0\eta'} = 0.53$$
and for $\phi$ meson:

$$S(\phi \rightarrow \eta \gamma) = 0.71C_{8\eta} - 0.72C_{8\eta'} = 0.88$$

$$S(\phi \rightarrow \eta' \gamma) = 0.71C_{0\eta} - 0.72C_{0\eta'} = -0.59$$

From the sets of the above equations, we obtain the following solutions for $\theta$ and $d$:

$$\theta = -(13.99 \pm 3.1)^{\circ}, \quad d = 0.12 \pm 0.03, \quad \text{for } \rho$$

$$\theta = -(15.48 \pm 3.1)^{\circ}, \quad d = 0.11 \pm 0.03, \quad \text{for } \omega$$

$$\theta = -(12.66 \pm 2.1)^{\circ}, \quad d = 0.10 \pm 0.03, \quad \text{for } \phi$$

Thus, by treating exactly the derivative coupling mixing term with our Lagrangian in a diagonal form we have found a small mixing angle in vector meson radiative decays which also give small mixing angle found in previous works [11–14] mentioned above. Also, by subtracting the $d$ term in $\theta$, we obtain a value $-(8 - 10)^{\circ}$ for the usual mixing angle. This value is smaller by a few degrees than the values we obtained in our previous work [7]. This could be due to the exact treatment of the momentum-dependent mixing term in our Lagrangian.

Since $SU(3)$ breaking is due mainly to the factor $f_\pi/f_{\eta_s}$ in $\rho \rightarrow \eta \gamma$ and $\omega \rightarrow \eta \gamma$ decays, the values for $\theta$ and $d$ obtained from $\rho \rightarrow \eta \gamma$ and $\omega \rightarrow \eta \gamma$ decays suffer from less theoretical uncertainties than the values obtained from $\phi \rightarrow \eta \gamma$ decay, which could be sensitive to $SU(3)$ breaking effect for the $s$ quark magnetic coupling $g_s$. The coupling $g_s = k g_u$, with $k = 0.85$ given in Table II is in fact consistent with the value for $k$ obtained from the new measured branching ratio for $K^{*0} \rightarrow K^0\gamma$ from which we obtain $k = 0.83 \pm 0.04$. It is remarkable that the value for $\theta$ and $d$ from $\phi$ meson radiative decays are consistent with the ones for $\rho$ and $\omega$. This indicates that $SU(3)$ breaking for $\phi$ meson radiative decays is correctly given by $K^* \rightarrow K\gamma$ decays.

In conclusion, we have diagonalized both the mass term and the momentum-dependent mixing term in the $\eta - \eta'$ Lagrangian and shown that the $\eta - \eta'$ system can be described by two parameters, the meson field renormalization and a new $\eta - \eta'$ mixing angle which differs from the usual mixing angle by a small momentum-dependent mixing term. Using the measured vector meson radiative decays, we obtain consistent solutions for the mixing angle and the momentum-dependent mixing term. The small mixing angle we found is consistent with previous determinations. It seems that
vector meson radiative decays would favor a small $\eta - \eta'$ mixing angle.

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