Towards $m$-Cambrian Lattices

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Abstract. For positive integers $m$ and $k$, we introduce a family of lattices $C_k^{(m)}$ associated to the Cambrian lattice $C_k$ of the dihedral group $I_2(k)$. We show that $C_k^{(m)}$ satisfies some basic properties of a Fuss-Catalan generalization of $C_k$, namely that $C_k^{(1)} = C_k$ and $|C_k^{(m)}| = \text{Cat}^{(m)}(I_2(k))$. Subsequently, we prove some structural and topological properties of these lattices—namely that they are trim and EL-shellable—which were known for $C_k$ before. Remarkably, our construction coincides in the case $k = 3$ with the $m$-Tamari lattice of parameter 3 due to Bergeron and Préville-Ratelle. Eventually, we investigate this construction in the context of other Coxeter groups, in particular we conjecture that the lattice completion of the analogous construction for the symmetric group $S_n$ and the long cycle $(1 2 \ldots n)$ is isomorphic to the $m$-Tamari lattice of parameter $n$.

1. Introduction

For a finite Coxeter group $W$ and a Coxeter element $\gamma \in W$, Reading introduced in [15], and later in [17], the Cambrian lattice $C_\gamma$ as the sublattice of the weak order on $W$ induced by so-called $\gamma$-sortable elements. Remarkably, in the case where $W = S_n$ is the symmetric group and $\gamma_0 = (1 2 \ldots n)$ is the long cycle, the lattice $C_{\gamma_0}$ is isomorphic to the classical Tamari lattice $T_n$, introduced in [21]. Like noncrossing partitions, nonnesting partitions or clusters of a finite Coxeter group $W$, the $\gamma$-sortable elements of $W$ are counted by the generalized Catalan numbers of $W$ which were first defined in [4, Section 5.2] as

$$\text{Cat}(W) = \prod_{i=1}^n \frac{d_i + d_j}{d_i},$$

where $n$ is the rank of $W$, and $d_1 \leq d_2 \leq \cdots \leq d_n$ denote the degrees of $W$ in increasing order, see [16, Theorem 9.1]. In the special case where $W$ is the symmetric group, we recover the well-known Catalan numbers,

$$\text{Cat}(n) = \frac{1}{n} \binom{2n}{n-1},$$

which play an outstanding role in the field of combinatorics. In [18] Stanley has compiled a list of more than 200 mathematical objects which are counted by $\text{Cat}(n)$.

In [2], the numbers $\text{Cat}(W)$ have been further generalized. There, Athanasiadis investigated the number of regions in the fundamental chamber of the $m$-extended...
Shi arrangement associated to a crystallographic Coxeter group $W$. One of the main results of the mentioned article is that these regions are counted by the generalized $m$-Catalan numbers of $W$, defined as

$$\text{Cat}^{(m)}(W) = \prod_{i=1}^{n} \frac{md_i + d_i}{d_i},$$

where the numbers $d_i$ again denote the degrees of $W$ in increasing order, see [2, Corollary 1.3]. If $W$ is the symmetric group, then these numbers agree with the well-known Fuss-Catalan numbers

$$\text{Cat}^{(m)}(n) = \frac{1}{n} \binom{(m+1)n}{n-1},$$

which count for instance $m$-Dyck paths of height $n$, noncrossing set partitions of $\{1, 2, \ldots, n\}$ with block sizes divisible by $m$, or $(m+2)$-angulations of an $(mn+2)$-gon, see [8,9,24]. Many generalized Catalan objects have $m$-generalizations, for instance noncrossing partitions, nonnesting partitions, or clusters, and all these objects are counted by the numbers $\text{Cat}^{(m)}(W)$, see [1, Chapter 5] for a comprehensive overview.

A natural question is therefore raised: is it possible to generalize the $\gamma$-sortable elements of $W$ in an analogous way, for every Coxeter group $W$ and every Coxeter element $\gamma$?

Recall that the $m$-Tamari lattice $T_n^{(m)}$, introduced by Bergeron and Préville-Ratelle in [3], can be seen as an $m$-generalization of the Tamari lattice $T_n$ and thus of the special Cambrian lattice $C_{\gamma_0}$. Thus, it is desirable that such an $m$-generalization of the $\gamma$-sortable elements of $W$ comes with a partial order satisfying the following conditions.

**Condition 1.1.** Let $W$ be a Coxeter group, and let $\gamma \in W$ be a Coxeter element. If a poset $P^{(m)}$ is an $m$-generalization of $C_{\gamma}$, then $P^{(1)} = C_{\gamma}$.

**Condition 1.2.** Let $W$ be a Coxeter group, and let $\gamma \in W$ be a Coxeter element. If a poset $P^{(m)}$ is an $m$-generalization of $C_{\gamma}$, then $|P^{(m)}| = \text{Cat}^{(m)}(W)$.

In fact, it would be even more desirable to have the following third condition.

**Condition 1.3.** Let $W$ be a Coxeter group, and let $\gamma \in W$ be a Coxeter element. If a poset $P^{(m)}$ is an $m$-generalization of $C_{\gamma}$, then $P^{(m)}$ is a lattice.

In this paper, given positive integers $m$ and $k$, we construct a poset $C_k^{(m)}$ which is a subposet of the $m$-fold direct product of the Cambrian lattice of the dihedral group $I_2(k)$ with itself. We show that this family of posets satisfies Conditions 1.1–1.3. In particular, we prove the following theorem.

**Theorem 1.4.** For every $k, m \in \mathbb{N}_{>0}$, the lattice $C_k^{(m)}$ satisfies the following.

(i) $C_k^{(1)} = C_k$;

(ii) $|C_k^{(m)}| = \left(\frac{m^2 + m}{2}\right) k + m + 1 = \text{Cat}^{(m)}(I_2(k))$; and

(iii) for every $m' \in \mathbb{N}_{>0}$, with $m' < m$, the lattice $C_k^{(m')}$ is an interval of $C_k^{(m)}$. 


We go on with the investigation of some structural and topological properties of \( C_k^{(m)} \), and prove the following theorem.

**Theorem 1.5.** For every \( k, m \in \mathbb{N}_{>0} \), the lattice \( C_k^{(m)} \) is trim, and therefore EL-shellable.

Subsequently, we investigate whether this construction can be carried out for all finite Coxeter groups. If we consider the “Tamari case”, namely the Cambrian lattice \( C_{\gamma_0} \) of the symmetric group \( S_n \) with the long cycle \( \gamma_0 = (1 \ 2 \ \ldots \ n) \) as Coxeter element, then the poset \( C_{\gamma_0}^{(m)} \) is no longer a lattice, when \( n > 3 \). However, computer experiments suggest that its lattice completion \( \mathcal{L}(C_{\gamma_0}^{(m)}) \) is isomorphic to \( T_n^{(m)} \). Recall that the symmetric group \( S_3 \) is isomorphic to the dihedral group \( I_2(3) \). The following theorem is proven in [14].

**Theorem 1.6 ([14, Theorem 4.2]).** For every \( m \in \mathbb{N}_{>0} \) we have \( C_3^{(m)} \cong T_3^{(m)} \).

Theorems 1.5 and 1.6 suggest that an \( m \)-generalization of the \( \gamma \)-sortable elements of \( W \) can be achieved using \( m \)-multichains of \( C_\gamma \). However, computer experiments show that the approaches discussed so far cannot be generalized straightforwardly to other Coxeter elements of \( S_n \) or to other Coxeter groups of rank \( \geq 3 \).

This paper is organized as follows. In Section 2 we recall some terminology for partially ordered sets and briefly recall the definition of \( \gamma \)-sortable elements and Cambrian lattices. In Section 3 we introduce the lattice \( C_k^{(m)} \), and prove Theorems 1.4 and 1.5. Finally, in Section 4 we investigate this construction for other Coxeter groups.

## 2. Preliminaries

In this section, we recall the basic concepts needed in this article. For more information on partially ordered sets we refer to [19, Chapter 3]. For more information on Coxeter groups, we refer to [5] or [10], and for a more detailed introduction to the Cambrian lattices, we refer the reader to [17].

### 2.1. Partially Ordered Sets

Let \( \mathcal{P} = (P, \leq) \) be a finite partially ordered set (poset for short). We say that \( \mathcal{P} \) is bounded if it has a unique minimal and a unique maximal element, denoted by \( 0_\mathcal{P} \) and \( 1_\mathcal{P} \), respectively. We say that \( \mathcal{P} \) is a lattice if for every two elements \( x, y \in P \) there exists a least upper bound, which is called the join of \( x \) and \( y \) and is denoted by \( x \lor y \), and there exists a greatest lower bound, which is called the meet of \( x \) and \( y \) and is denoted by \( x \land y \). If \( x \ll y \) and there does not exist an element \( z \in P \) with \( x \ll z \ll y \), then we say that \( y \) covers \( x \), and we denote it by \( x \lhd y \). A set \( C = \{x_1, x_2, \ldots, x_s\} \subseteq P \) with \( x_1 \leq x_2 \leq \cdots \leq x_s \) is called a chain of \( \mathcal{P} \). We say that \( C \) is saturated if \( x_1 \ll x_2 \ll \cdots \ll x_s \). If \( x_1 = 0_\mathcal{P} \) and \( x_s = 1_\mathcal{P} \), then we say that \( C \) is maximal.

Let \( \mathcal{L} = (L, \leq) \) be a finite lattice with least element \( 0_L \) and greatest element \( 1_L \). An element \( x \in L \setminus \{0_L\} \) is called join-irreducible if for every \( X \subseteq L \) with \( \bigvee L X = x \) it follows that \( x \in X \). Hence, \( x \) cannot be expressed as a join of other elements in \( L \), or equivalently \( x \) covers exactly one element in \( L \). Denote the set of join-irreducible elements of \( \mathcal{L} \) by \( \mathcal{J}(\mathcal{L}) \). Dually, an element \( x \in L \setminus \{1_L\} \) is called
meet-irreducible if for every $X \subseteq L$ with $\bigwedge L X = x$ it follows that $x \in X$. Hence, $x$ cannot be expressed as a meet of other elements in $L$, or equivalently $x$ is covered by exactly one element in $L$. Denote the set of meet-irreducible elements of $L$ by $\mathcal{M}(L)$. An element $x \in L$ is called doubly irreducible if it is both, join- and meet-irreducible. The length of $L$, denoted by $\ell(L)$, is the maximal length of a saturated chain from $0_L$ to $1_L$. Then, $L$ is called extremal if $|\mathcal{F}(L)| = \ell(L) = |\mathcal{M}(L)|$, see [13].

Recall that an element $x \in L$ is called left-modular if for every $y \leq_L z$ we have
\begin{equation}
(y \vee_L x) \wedge_L z = y \vee_L (x \wedge_L z).
\end{equation}
If there exists a saturated maximal chain consisting of left-modular elements, then $L$ is called left-modular. If $L$ is extremal and left-modular, then $L$ is called trim, see [23].

Let $\mathcal{E}(L) = \{(x, y) \mid x \leq_L y\}$ denote the set of edges of the Hasse diagram of $L$. Given some other poset $(\Lambda, \leq_\Lambda)$, a map $\lambda : \mathcal{E}(L) \rightarrow \Lambda$ is called an edge-labeling of $L$. A chain of $L$ is called rising with respect to $\lambda$ if the sequence of edge labels of this chain is weakly increasing with respect to $\leq_\Lambda$. An edge-labeling is called EL-labeling if in every interval of $L$ there exists a unique rising maximal chain, and this chain is lexicographically first among all maximal chains in this interval. Then, $L$ is called EL-shellable if it admits an EL-labeling.

Suppose now that $L$ is left-modular with maximal saturated left-modular chain $0_L = x_0 \leq_L x_1 \leq_L \cdots \leq_L x_n = 1_L$. Then, define an edge-labeling of $L$ via
\begin{equation}
\psi(y, z) = \min\{i \mid y \vee_L x_i \wedge_L z = z\}.
\end{equation}
We have the following result.

**Theorem 2.1** ([12]). For a left-modular lattice $L$, the edge labeling $\psi$ is an EL-labeling of $L$.

2.2. Cambrian lattices. Let $W$ be a finite Coxeter group with set of simple reflections $S = \{s_1, s_2, \ldots, s_n\}$. It is well-known that $S$ generates $W$, and thus, every $w \in W$ can be written as a word in the alphabet $S$. Define the length of $w$ as $\ell_S(w) = \min\{k \mid w = s_{i_1}s_{i_2}\cdots s_{i_k}\}$. Further, define the weak order on $W$ as the partial order given by

$u \leq_S v$ if and only if $\ell_S(v) = \ell_S(u) + \ell_S(u^{-1}v)$.

Since $W$ is finite, there exists a longest element in $W$, denoted by $w_0$, and it is well-known that $(W, \leq_S)$ is in fact a lattice.

Now, let $\gamma = s_1s_2\cdots s_n \in W$ be a Coxeter element. Consider the half-infinite word

$\gamma^\infty = s_1s_2\cdots s_n|s_1s_2\cdots s_n|\cdots$.

The vertical bars in the representation of $\gamma^\infty$ are “dividers”, which have no influence on the structure of the word, but shall serve for a better readability. Clearly, every reduced word for $w \in W$ can be considered as a subword of $\gamma^\infty$. Among all reduced words for $w$, there is a unique reduced word, which is lexicographically first as a subword of $\gamma^\infty$. This reduced word is called the $\gamma$-sorting word of $w$. For $w \in W$, the $i$-th block of $w$ (with respect to $\gamma$) is the set of letters of the $\gamma$-sorting word of $w$ which belong to the $i$-th repetition of $\gamma$ in $\gamma^\infty$. Then, $w$ is
called \( \gamma \)-sortable if its blocks form a weakly decreasing sequence under inclusion. The set of all \( \gamma \)-sortable elements of \( W \), denoted by \( C_\gamma \), constitutes a sublattice of the weak order lattice on \( W \), see \([17, \text{Theorem 1.2}]\), and is called the \( \gamma \)-Cambrian lattice of \( W \), denoted by \( C_\gamma \). We remark that the Cambrian lattice depends on the choice of the Coxeter element.

3. \( m \)-Cambrian Lattices for the Dihedral Groups

In this section we define \( m \)-generalizations of the Cambrian lattice of the dihedral group \( I_2(k) \) of order \( 2k \), and prove Theorems 1.4 and 1.5. First, we briefly describe this group and its Cambrian lattice.

The dihedral group \( I_2(k) \) is a Coxeter group of rank 2, thus it has exactly two canonical generators, which we denote by \( s_1 \) and \( s_2 \). Let \( \varepsilon \) denote the identity and let \( w_0 \) denote the longest element of \( I_2(k) \). The weak order on \( I_2(k) \), as well as its two Cambrian lattices are depicted in Figure 1. We see immediately that the two Cambrian lattices are mutually isomorphic, and thus it is justified that we speak of the Cambrian lattice of \( I_2(k) \). Hence, without loss of generality, we can set \( C_k = C_{s_1s_2} \). If we denote by \( \leq_k \) the restriction of the weak order to \( C_k \), then we can write \( C_k = (C_k, \leq_k) \). It is not hard to see that \( |C_k| = k + 2 \).

In the rest of this paper we use the following notation. Fix some \( m \in \mathbb{N} \) with \( m > 0 \). For \( u_i \in C_k \) with \( u_i \leq_k u_{i+1} \), consider an \( m \)-tuple

\[
(7) \quad w = \left( u_{l_1}, u_{l_1 + 1}, \ldots, u_{l_1 + 1}, u_{l_2}, u_{l_2 + 1}, \ldots, u_{l_2 + 1}, \ldots, u_{l_n}, u_{l_n + 1}, \ldots, u_{l_n + 1} \right),
\]

in \( (C_k)^m \), with \( l_i \in \mathbb{N} \) and \( \sum_{i=1}^n l_i = m \).
For brevity, we write \[ w = (u_1^{l_1}, u_2^{l_2}, \ldots, u_n^{l_n}), \]
if \( w \) contains \( l_i \) copies of \( u_i \), for \( i = 1, 2, \ldots, n \), in the order indicated in (7). We often omit the exponent if it is equal to 1.

**Definition 3.1.** Let \( C_k^{(m)} \) be the subset of \((C_k)^m\) consisting of all elements \( w \) of the form \( w = (e^0, u^1, v^2) \), where \( u \leq v \) and \( l_0 + l_1 + l_2 = m \).

For \( m \in \mathbb{N} \), we denote by \( C_k^{(m)} \) the induced subposet \((C_k^{(m)}, \leq_k)\) of \((C_k)^m, \leq_k\).

(By abuse of notation, we write \( \leq_k \) also for the componentwise order in \((C_k)^m\).

We will show that \( C_k^{(m)} \) is a lattice which satisfies the properties listed in Theorem 1.4, and thus Conditions 1.1–1.3.

**Example 3.2.** Consider the dihedral group \( I_2(4) \) (which is isomorphic to the Coxeter group \( B_2 \)) with simple generators \( s_1 \) and \( s_2 \), consider the Coxeter element \( \gamma = s_1s_2 \), and let \( m = 2 \). The corresponding Cambrian lattice is shown in Figure 2. The elements of \( C_4^{(2)} \) are precisely

\[
\begin{align*}
(e, e), & \quad (e, s_1), & \quad (e, s_2), & \quad (e, s_1s_2), \\
(e, s_1s_2s_1), & \quad (e, s_1s_2s_1s_2), & \quad (s_1, s_1), & \quad (s_1, s_1s_2), \\
(s_2, s_2), & \quad (s_2, s_1s_2s_1s_2), & \quad (s_1s_2, s_1s_2), & \quad (s_1s_2, s_1s_2s_1),
\end{align*}
\]

\[
\begin{align*}
(s_1s_2s_1, s_1s_2s_1s_2), & \quad (s_1s_2s_1s_2, s_1s_2s_1s_2).
\end{align*}
\]

We remark that the poset \( C_4^{(2)} \) is in fact a lattice, see Figure 3.

3.1. **Proof of Theorem 1.4.** First of all, it is immediately clear that \( C_k^{(1)} = C_k \), and thus Theorem 1.4(i) is trivially satisfied. The remaining parts of Theorem 1.4 are proven in the following lemmas.

**Lemma 3.3.** For \( k, m \in \mathbb{N}_{>0} \) we have

\[ |C_k^{(m)}| = \left( \frac{m^2 + m}{2} \right) k + m + 1. \]

**Proof.** First of all, we recall (for instance from Figure 1) that the Hasse diagram of the Cambrian lattice \( C_k \) is a cycle with \( k + 2 \) nodes. Thus, it has \( k + 2 \) edges,
which implies that there exist \( k + 2 \) covering relations in \( C_k \). Moreover, for every \( w \in C_k \) with \( w \neq \varepsilon \) and \( w \neq w_o \), there exists exactly one element \( w' \in C_k \) with \( w \lesssim_k w' \). We proceed by induction on \( m \).

For \( m = 1 \), we have \( C_k^{(1)} = C_k \), and the result follows from [16, Theorem 9.1]. Let now \( m = 2 \). Then, \( C_k^{(2)} \) consists of all pairs \((u, v)\) with either \( u = \varepsilon \), or \( u = v \), or \( u \lesssim_k v \). There are \( k + 2 \) pairs of the first type, \( k + 1 \) pairs of the second type (the pair \((\varepsilon, \varepsilon)\) is already considered in the first type), and \( k \) pairs of the third type (the pairs \((\varepsilon, s_1)\) and \((\varepsilon, s_2)\) are already considered in the first type). Thus, we obtain

\[
|C_k^{(2)}| = k + 2 + k + 1 + k = 3k + 3,
\]

as desired.

Now suppose that the formula is correct for \( C_k^{(m)} \). It is immediately clear from the definition that for every \((w_1, w_2, \ldots, w_m) \in C_k^{(m)} \), we have \((\varepsilon, w_1, w_2, \ldots, w_m) \in C_k^{(m+1)} \). Now let \( u \in C_k \) with \( u \neq \varepsilon, w_o \). The elements in \( C_k^{(m+1)} \) with \( u \) in the first component are of the form \((\varepsilon^i, u^j, v^l)\), where \( u \lesssim_k v \) and \( l_1 + l_2 = m + 1 \) and \( l_1 > 0 \). Thus, for fixed \( u \), there exist exactly \( m + 1 \) such elements, and since every \( u \neq \varepsilon, w_o \) has a unique upper cover in \( C_k \), these are the only elements with \( u \) in the first component, and we have \( k \) choices for \( u \). There exists one additional
Proof. By definition, we denote joins and meets in \( \mathcal{C}_k \). Hence, we have
\[
\begin{align*}
|C_k^{(m+1)}| &= |C_k^{(m)}| + k(m + 1) + 1 \\
&= \left( \frac{m^2 + m}{2} \right) k + m + 1 + k(m + 1) + 1 \\
&= \left( \frac{m^2 + m}{2} + m + 1 \right) k + (m + 1) + 1 \\
&= \left( \frac{(m+1)^2 + m+1}{2} \right) k + (m + 1) + 1,
\end{align*}
\]
as desired. \(\square\)

**Lemma 3.4.** The poset \( C_k^{(m)} \) is a sublattice of \( (C_k)^m \).

**Proof.** Since \( C_k \) is a lattice, it follows immediately that \( (C_k)^m \) is a lattice, and given two elements \( (w_1, w_2, \ldots, w_m), (w_1', w_2', \ldots, w_m') \in (C_k)^m \), their join and meet in \( (C_k)^m \) is given componentwise, and will be denoted by \( \vee_k \) and \( \wedge_k \), respectively. By abuse of notation, we denote joins and meets in \( C_k \) and \( C_k^{(m)} \) with the same symbol. Hence, we have
\[
\begin{align*}
(w_1, w_2, \ldots, w_m) \vee_k (w_1', w_2', \ldots, w_m') &= (w_1 \vee_k w_1', w_2 \vee_k w_2', \ldots, w_m \vee_k w_m'), \\
(w_1, w_2, \ldots, w_m) \wedge_k (w_1', w_2', \ldots, w_m') &= (w_1 \wedge_k w_1', w_2 \wedge_k w_2', \ldots, w_m \wedge_k w_m').
\end{align*}
\]

Now let \( w, w' \in C_k^{(m)} \). We are going to show that \( w \vee_k w', w \wedge_k w' \in C_k^{(m)} \) as well. By definition, \( w \) and \( w' \) can be written in the form
\[
w = (\ell_0, u_1', v_2) \quad \text{and} \quad w' = (\ell_3, (u')_4, (v')_5),
\]
for \( u, u', v, v' \in C_k \) with \( u \preceq_k u' \) and \( u' \preceq_k v' \), and \( l_i \in \mathbb{N} \) with \( l_0 + l_1 + l_2 = l_3 + l_4 + l_5 = m \). By construction, there are precisely two possibilities for \( w \wedge_k w' \), namely
\[
\begin{align*}
w \wedge_k w' &= (\ell_0, (u \wedge_k u')_{m_1}, (u \wedge_k v')_{m_2}, (v \vee_k v')_{m_3}), \quad \text{and} \\
w \wedge_k w' &= (\ell_3, (u \wedge_k u')_{m_4}, (v \wedge_k v')_{m_5}, (v \vee_k v')_{m_6}),
\end{align*}
\]
where \( m_i \in \mathbb{N} \) with
\[
\sum_{i=0}^{3} m_i = \sum_{i=4}^{7} m_i = m.
\]
We distinguish now two cases:

(i) The elements \( u \) and \( u' \) are comparable. Without loss of generality we may assume that \( u \leq_k u' \); hence \( u \wedge_k u' = u = u \wedge_k v' \) (since \( u \leq_k u' \leq_k v' \)). If \( v \leq_k u' \), then we obtain \( v \wedge_k u' = v = v \wedge_k v' \) (since \( v \leq_k u' \leq_k v' \)). Thus, we have
\[
\begin{align*}
w \wedge_k w' &= (\ell_0, u_{m_1+m_2}, v_{m_3}), \quad \text{or} \\
w \wedge_k w' &= (\ell_3, u_{m_4}, v_{m_5+m_6}),
\end{align*}
\]
which are contained in \( C_k^{(m)} \). If, on the other hand, \( v \nleq_k u' \), then \( v = s_2 \) and \( u = \varepsilon \) and it follows that \( v \wedge_k u' = \varepsilon \). Hence, we obtain
\[
\begin{align*}
w \wedge_k w' &= (\ell_0^{m_0+m_1+m_2}, (v \vee_k v')_{m_3}), \quad \text{or} \\
w \wedge_k w' &= (\ell_3^{m_4+m_5+m_6}, (v \wedge_k v')_{m_7}),
\end{align*}
\]
which, again, are contained in \( C_k^{(m)} \).
(ii) The elements \( u \) and \( u' \) are incomparable. Since we are in a dihedral group, we may assume that \( u = s_2 \) and \( v = w_0 \). Thus, we have \( u \land_k u' = \epsilon \) and \( v \land_k v' = \epsilon \). Then, we obtain
\[
\begin{align*}
    w \land_k w' &= (\epsilon^{m_0 + m_1 + m_2}, (v')^{m_3}), \quad \text{or} \\
    w \land_k w' &= (\epsilon^{m_4 + m_5}, (u')^{m_6}, (v')^{m_7}),
\end{align*}
\]
which are both in \( C_k^{(m)} \).

The proof for \( w \lor_k w' \) works analogously, which implies by definition that \( C_k^{(m)} \) is a sublattice of \( (C_k)^m \).

**Lemma 3.5.** For every \( k, m, m' \in \mathbb{N}_{>0} \) with \( m' \leq m \), the lattice \( C_k^{(m')} \) is an interval of \( C_k^{(m)} \).

**Proof.** This follows immediately from the definition of \( C_k^{(m)} \) and \( C_k^{(m')} \). \( \square \)

**Proof of Theorem 1.4.** This follows with Lemmas 3.3–3.5. \( \square \)

**Definition 3.6.** For \( k, m \in \mathbb{N}_{>0} \), the poset \( C_k^{(m)} = (C_k^{(m)}, \leq_k) \) is called the \( m \)-Cambrian lattice of \( I_2(k) \).

We conclude this section with the following open problem.

**Open Problem 3.7.** Construct bijections between \( C_k^{(m)} \) and the set of \( m \)-noncrossing partitions of \( I_2(k) \), the set of \( m \)-nonnesting partitions of \( I_2(k) \), or the set of \( m \)-clusters of \( I_2(k) \).

If we could solve Problem 3.7, then we could try to generalize this bijection to Coxeter groups \( W \) of rank \( \geq 3 \), and perhaps obtain an idea what \( m \)-sortable elements of \( W \) might look like.

3.2. **Proof of Theorem 1.5.** It was shown in [11, Theorem 4.17] that the Cambrian lattices associated to crystallographic Coxeter groups are trim. We show that the same holds for our \( m \)-Cambrian lattice of \( I_2(k) \). More precisely, we first show that \( C_k^{(m)} \) is extremal (Proposition 3.8), and then that it is left-modular (Proposition 3.15).

**Proposition 3.8.** For every \( k, m \in \mathbb{N}_{>0} \), the lattice \( C_k^{(m)} \) is extremal.

For the proof of Proposition 3.8 we need the following lemmas.

**Lemma 3.9.** Every element in \( C_k^{(m)} \) has at most two upper covers and at most two lower covers.

**Proof.** We will first show that every element in \( C_k^{(m)} \) has at most two upper covers. Since \( C_k^{(m)} \) is a sublattice of \( (C_k)^m \), the partial order on \( C_k^{(m)} \) is given component-wise. Let \( w \in C_k^{(m)} \) with \( w = (e^0, u^l, v^l) \), for \( u, v \in C_k \) with \( u \leq_k v \), and \( l_i \in \mathbb{N} \) with \( l_0 + l_1 + l_2 = m \). We distinguish several cases.

(i) If \( l_0, l_1, l_2 \geq 1 \), then there exist exactly two upper covers of \( w \) in \( C_k^{(m)} \), namely the elements \( w' = (e^{l_0-1}, u^{l_1+1}, v^{l_2}) \) and \( w'' = (e^{l_0}, u^{l_1-1}, v^{l_2+1}) \).
(ii) If \( l_0 = 0 \) and \( l_1, l_2 \geq 1 \), then \( w = (u^{l_1}, v^{l_2}) \) and there exists exactly one upper cover of \( w \) in \( C_k^{(m)} \), namely the element \( w' = (u^{l_1-1}, v^{l_2+1}) \).

(iii) If \( l_1 = 0 \) and \( l_0, l_2 \geq 1 \), then \( w = (e^0, v^{l_2}) \). Suppose further that \( v \neq w_0 \).

Then, since \( v \neq e \), there is exactly one element \( v' \in C_k \) that covers \( v \). Thus \( w \) has exactly two upper covers, namely \( w' = (e^0, v^{l_2-1}, v') \) and \( w'' = (e^0-1, v^{l_2+1}) \). Let now \( v = w_0 \) and let \( u_1, u_2 \) be the two lower covers of \( w_0 \) in \( C_k \). Then, \( w \) has two upper covers, namely \( w' = (e^{0-1}, u_1, w_0^{l_2}) \) and \( w'' = (e^{0-1}, u_2, w_0^{l_2}) \).

(iv) If \( l_0 = l_1 = 0 \) and \( l_2 \geq 1 \), then \( w = (v^m) \) for some \( v \neq e \). If \( v = w_0 \), then \( w \) is the greatest element of \( C_k^{(m)} \), and therefore, it has no upper cover. If on the other hand \( v \neq w_0 \), then there is a unique element \( v' \in C_k \) with \( v \preceq v' \), and it follows that \( w \) has a unique upper cover as well, which is the element \( w' = (v^{m-1}, v') \).

(v) If \( l_0 = l_2 = 0 \) and \( l_1 \geq 1 \), then the reasoning is analogous to (iv).

(vi) If \( l_1 = l_2 = 0 \) and \( l_0 \geq 1 \), then \( w = (e^m) \) and \( w \) is the minimum element of \( C_k^{(m)} \). Since \( e \) has exactly two upper covers in \( C_k \), namely the reflections \( s_1 \) and \( s_2 \), it follows that \( w \) has exactly two upper covers as well, which are the elements \( w' = (e^{m-1}, s_1) \) and \( w'' = (e^{m-1}, s_2) \).

The dual statement follows analogously. \( \square \)

Lemma 3.10. For every \( k, m \in \mathbb{N}_{>0} \), we have \( |J(C_k^{(m)})| = mk = |M(C_k^{(m)})| \).

Proof. Let \( w \in C_k^{(m)} \). If \( w \) is meet-irreducible, then it has precisely one upper cover. It follows from the proof of Lemma 3.9 that it must be of the form \( w = (u^l, v^{m-l}) \), for \( u \preceq v, u \neq e, w_o \), and \( l \in \{1, 2, \ldots, m\} \). (The unique upper cover of \( w \) is then \( w' = (u^{l-1}, v^{m-l+1}) \).) Hence, \( |M(C_k^{(m)})| = mk \).

If \( w \) is join-irreducible, then it has precisely one lower cover. Again, it follows from the proof of Lemma 3.9 that it must be of the form \( w = (e^l, w^{m-l}) \) for \( w \neq e, w_o \), and \( l \in \{0, 1, \ldots, m-1\} \). (The unique lower cover of \( w \) is then \( w' = (e^l, w^{m-l-1}) \), where \( w' \) is the unique lower cover of \( w \) in \( C_k \).) Since \( |C_k \setminus \{e, w_o\}| = k \), we obtain \( |J(C_k^{(m)})| = mk \). \( \square \)

Corollary 3.11. An element \( w \in C_k^{(m)} \) is doubly-irreducible if and only if \( w \) is of the form \((w^m)\) for \( w \neq e, w_o \).

Lemma 3.12. The number of cover relations in \( C_k^{(m)} \) is \( km^2 + 2m \).

Proof. Let \( w \in C_k^{(m)} \), and let \( u, v \in C_k \) with \( u \neq e, w_o \), and \( u \preceq v \). If \( w = (u^m) \), then Corollary 3.11 implies that \( w \) has a unique upper cover in \( C_k^{(m)} \), and there are \( k \) choices for \( w \). If \( w = (u^l, v^{m-l}) \), then we can conclude from the proof of Lemma 3.9 that \( w \) has a unique upper cover in \( C_k^{(m)} \), and we have \( k(m-1) \) choices for \( w \). Thus, the previous cases yield \( km \) edges. If \( w = (e^m) \), we obtain two more edges. If \( w = (e^l, w_o^{m-l}) \), then the proof of Lemma 3.9 implies that \( w \) has two upper covers, and we obtain \( 2(m-1) \) new edges. Finally, if \( w = (e^0, u^l, v^2) \) with \( l_0, l_1 > 0 \), then we obtain \( km(m-1)/2 \) elements with two upper covers each. Hence, in total, we have

\[ km + 2 + 2(m-1) + km(m-1) = km^2 + 2m \]
edges in $C_k^{(m)}$.  

\[ \ell(C_k^{(m)}) = mk. \]

**Lemma 3.13.** For every $k, m \in \mathbb{N}_{>0}$, we have $\ell(C_k^{(m)}) = mk$.

**Proof.** First of all recall, for instance from Figure 1, that $\ell(C_k) = k$. Thus, by definition of the direct product of lattices, we have $\ell((C_k)^m) = mk$, and the length of a sublattice of $(C_k)^m$ cannot exceed this value.

Now we explicitly construct a maximal chain in $C_k^{(m)}$ which has length $mk$. For that, write $x_1 = s_1, x_2 = s_1s_2, x_3 = s_1s_2s_1, \ldots, x_{k-1} = s_1s_2s_1 \ldots$ and set $x_0 = \varepsilon$, and $x_k = w_{\ell}$. Then, the chain $\varepsilon \leq_k x_1 \leq_k x_2 \leq_k \cdots \leq_k x_{k-1} \leq_k w_{\ell}$ is a maximal chain of $C_k$. For $i \in \{1,2,\ldots,k\}$ and $j \in \{1,2,\ldots,m\}$, define $x_{i,j} = (x_{i-1}^{m-j},x_i^j)$.

Clearly, $x_{i,j} \in C_k^{(m)}$. Since $C_k^{(m)}$ is a sublattice of $(C_k)^m$, we have $x_{i,j} \leq_k x_{i,j+1}$ for $i \in \{1,2,\ldots,k\}$ and $j \in \{1,2,\ldots,m-1\}$. If we additionally define $x_{0,0} = (\varepsilon^m)$, then we find that $x_{0,0} \leq_k x_{1,0} \leq_k x_{1,1} \leq_k \cdots \leq_k x_{1,m} \leq_k x_{2,1} \leq_k \cdots \leq_k x_{k,m}$ is a maximal chain in $C_k^{(m)}$ having length $mk$, which proves the claim.  

**Proof of Proposition 3.8.** It follows from Lemmas 3.10 and 3.13.  

**Example 3.14.** Consider the lattice $C_4^{(2)}$ shown in Figure 3. We have

\[
\begin{align*}
x_0 &= \varepsilon, & x_1 &= s_1, & x_2 &= s_1s_2, & x_3 &= s_1s_2s_1, & x_4 &= s_1s_2s_1s_2.
\end{align*}
\]

Further, we have

\[
\begin{align*}
x_{0,0} &= (\varepsilon,\varepsilon), & x_{1,1} &= (\varepsilon,s_1), & x_{1,2} &= (s_1,s_1), & x_{2,1} &= (s_1,s_1s_2), & x_{2,2} &= (s_1s_2,s_1s_2), & x_{3,1} &= (s_1s_2,s_1s_2s_1), & x_{3,2} &= (s_1s_2s_1,s_1s_2s_1s_2), & x_{4,1} &= (s_1s_2s_1,s_1s_2s_1s_2), & x_{4,2} &= (s_1s_2s_1s_2,s_1s_2s_1s_2).
\end{align*}
\]

which is a maximal chain in $C_4^{(2)}$.  

**Proposition 3.15.** For every $k, m \in \mathbb{N}_{>0}$, the lattice $C_k^{(m)}$ is left-modular.

**Proof.** We show that the maximal chain constructed in the proof of Lemma 3.13 consists of left-modular elements. Let $i \in \{1,2,\ldots,k\}$ and $j \in \{1,2,\ldots,m\}$, and consider the element $x_{i,j} = (x_{i-1}^{m-j},x_i^j)$, where $x_{i-1}^j = s_1s_2s_1 \ldots$ and $x_i^j = s_1s_2s_1 \ldots$.

Now let $w = (\ell^{l_1},u_1,l_2) \in C_k^{(m)}$ and $w' = (\ell^{l_1'},u_1',l_2') \in C_k^{(m)}$, where $u \leq_k u'$ and $u' \leq_k u''$. Suppose that $w \leq_k w'$. Write $w(l)$ shortly for the $l$-th component of $w$. Since $C_k^{(m)}$ is a sublattice of $(C_k)^m$, joins and meets in $C_k^{(m)}$ are given componentwise. Thus, it is sufficient to check condition (5) for each component. That means, we need to check whether $(w(l) \lor_k x_{i,j}(l)) \land_k w'(l) = w(l) \lor_k (x_{i,j}(l) \land_k w'(l))$ holds for all $l \in \{1,2,\ldots,m\}$. We notice that this equality is satisfied whenever $w(l) = w'(l)$, using the absorption law of lattices. We distinguish several cases.

(i) If $w(l) = \varepsilon$, $w'(l) = \varepsilon$, then $(\varepsilon \lor_k x_{i,j}(l)) \land_k \varepsilon = x_{i,j}(l) = (\varepsilon \land_k x_{i,j}(l)) \lor_k \varepsilon$.  

(ii) If $w(l) = \varepsilon$, $w'(l) = u'$, then $(\varepsilon \lor_k x_{i,j}(l)) \land_k u' = x_{i,j}(l) \land_k u' = \varepsilon \land_k (x_{i,j}(l) \land_k u')$.  


(iii) Let \( w(l) = u, w'(l) = u' \). Suppose that \( u <_k u' \). Then, since \( u \neq e \), and \( u' <_k v' \), neither \( u \) nor \( u' \) can be \( s_2 \). Then, we have either \( u <_k u' \leq_k x_{i,j}(l) \), or \( u \leq_k x_{i,j}(l) \leq_k u' \), or \( x_{i,j}(l) \leq_k u <_k u' \). We can check easily that the condition holds in each of these three subcases.

(iv) Let \( w(l) = u, w'(l) = v' \). Suppose that \( u <_k v' \). If \( v' = s_2 \), then \( u = e \) which contradicts the assumption. If \( u \neq s_2 \), then the result follows analogously to (iii).

Suppose now that \( u = s_2 \) (and thus \( v' = w_o \)). Then we have \( (s_2 \vee_k x_{i,j}(l)) \wedge_k w_o = w_o = s_2 \vee_k (x_{i,j}(l) \wedge_k w_o) \).

(v) If \( w(l) = v, w'(l) = u' \), then this works analogously to (iii).

(vi) If \( w(l) = v, w'(l) = v' \), then this works analogously to (iv). \( \square \)

Proof of Theorem 1.5. Propositions 3.8 and 3.15 imply that \( C_k^{(m)} \) is trim, and with Theorem 2.1 follows that \( C_k^{(m)} \) is EL-shellable. \( \square \)

3.3. Topology of \( C_k^{(m)} \). It is well-known that the order complex of an EL-shellable poset is homotopy equivalent to a wedge of spheres, see for instance [6, Theorem 5.9]. In fact, using the trimness of \( C_k^{(m)} \), we can say a bit more. Recall that an interval \([x,y]\) in a lattice is called nuclear if \( y \) can be expressed as the join of the atoms in \([x,y]\). We have the following result.

**Theorem 3.16** ([23, Theorem 7]). Let \( \mathcal{L} \) be a finite lattice. If \( \mathcal{L} \) is trim and nuclear, then its order complex is homomorphic to a sphere, whose dimension is 2 less than the number of atoms of \( \mathcal{L} \). If \( \mathcal{L} \) is trim but not nuclear, then its order complex is contractible.

Then, we have the following.

**Proposition 3.17.** Let \( \mu \) denote the Möbius function of \( C_k^{(m)} \). For \( w, w' \in C_k^{(m)} \), with \( w \leq_k w' \), we have

\[
\mu(w, w') = \begin{cases} 
1, & \text{if } [w, w'] \text{ is nuclear and has two atoms,} \\
-1, & \text{if } w <_k w' \text{ or} \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** Recall from Lemma 3.9 that every element of \( C_k^{(m)} \) has at most two upper covers. Hence, if \([w, w']\) is nuclear, then we have either \( w <_k w' \) or \( w' \) is the join of the two atoms in \([w, w']\), and hence we obtain \( \mu(w, w') = \pm 1 \) as desired. If \([w, w']\) is not nuclear, then Theorem 3.16 implies that the associated order complex is contractible, and hence has reduced Euler characteristic 0. Proposition 3.8.6 in [19] implies that the Möbius function of an interval in a poset takes the same value as the reduced Euler characteristic of the associated order complex, and the result follows. \( \square \)

4. Other Coxeter groups

4.1. The Symmetric group. In this section we focus on the Coxeter group \( W = A_{n-1} \), which is isomorphic to the symmetric group \( S_n \). It is well-known that \( S = \{s_1, s_2, \ldots, s_{n-1}\} \) with \( s_i = (i \ i+1) \) for \( i \in \{1, 2, \ldots, n-1\} \) can be taken as a canonical generating set for \( W \). The element \( \gamma_0 = s_1s_2 \cdots s_{n-1} = (1 \ 2 \ \ldots \ n) \) is a
Coxeter element of $W$, and it follows from [15, Theorem 6.4] and [7, Theorem 9.6] that $C_{\gamma_0}$ and $T_n$ are isomorphic.

Since $A_2 = \Theta_3$ is isomorphic to the dihedral group $I_2(3)$, we have currently two $m$-generalizations of the $\gamma_0$-Cambrian lattice of $A_2$. On the one hand, regarding $C_{\gamma_0}$ as the Tamari lattice $T_3$, we can consider the $m$-Tamari lattice $T_3^{(m)}$ of Bergeron and Préville-Ratelle, see [3, Section 5]. On the other hand, regarding $A_2$ as the dihedral group $I_2(3)$, we can consider the $m$-Cambrian lattice $C_3^{(m)}$ from Section 3. Remarkably, these two $m$-generalizations of $C_{\gamma_0}$ are isomorphic, as Theorem 1.6 shows.

For $n > 3$ though, the poset $C_{\gamma_0}^{(m)}$ is no longer a lattice. However, if we consider the Dedekind-MacNeille completion $\mathcal{L}(C_{\gamma_0}^{(m)})$ of $C_{\gamma_0}^{(m)}$, namely the smallest lattice containing $C_{\gamma_0}^{(m)}$ as a subposet, then computer experiments suggest the following conjecture.

**Conjecture 4.1.** Let $W = \Theta_n$, and let $\gamma_0 = (1 \ 2 \ldots \ n)$. For every $m, n > 0$ we have $\mathcal{L}(C_{\gamma_0}^{(m)}) \cong T_n^{(m)}$.

**Example 4.2.** Figure 4 shows the Tamari lattice $T_4$ realized as a Cambrian lattice $C_{\gamma_0}$ for $W = \Theta_4$ and $\gamma_0 = (1 \ 2 \ 3 \ 4)$. The elements of $C_{\gamma_0}^{(2)}$ are precisely:

- $(e, e)$,
- $(e, s_1)$,
- $(e, s_2)$,
- $(e, s_3)$,
- $(e, s_1 s_2)$,
- $(e, s_1 s_3)$,
- $(e, s_2 s_3)$,
- $(e, s_1 s_2 s_3)$,
- $(e, s_1 s_2 s_3 s_1)$,
- $(e, s_1 s_2 s_3 s_2)$,
- $(e, s_1 s_2 s_3 s_1 s_2)$,
- $(e, s_1 s_2 s_3 s_1 s_2 s_1)$,
- $(s_1, s_1 s_2)$,
- $(s_1, s_1 s_3)$,
- $(s_1, s_2)$,
- $(s_2, s_2 s_3)$,
- $(s_2, s_2 s_1)$,
- $(s_3, s_3)$.
Now we notice that for instance the pairs \((s_1s_3, s_1s_2s_3s_2)\) and \((s_1s_2, s_1s_2s_3)\) do not have a meet in \(\langle C^2_4, \leq \rangle\), since
\[
(\varepsilon, s_1s_2s_3) \leq \gamma (s_1s_3, s_1s_2s_3s_2), (s_1s_2, s_1s_2s_3), \quad \text{and}
(s_1, s_1s_2) \leq \gamma (s_1s_3, s_1s_2s_3s_2), (s_1s_2, s_1s_2s_3),
\]
but \((\varepsilon, s_1s_2s_3)\) and \((s_1, s_1s_2)\) are incomparable. The meet of \((s_1s_3, s_1s_2s_3s_2)\) and \((s_1s_2, s_1s_2s_3)\) in \(\langle C^2_4, \leq \rangle\) is \((s_1s_3, s_1s_2s_3s_2)\). If we now successively add all the missing meets, then we can check that we have to include the following ten elements:
\[
(s_1s_2, s_1s_2s_3s_1), \quad (s_2, s_1s_2s_3s_1), \quad (s_1, s_1s_2s_3s_1), \quad (s_1s_2s_3, s_1s_2s_3s_1s_2),
(s_1s_2s_3s_2, s_1s_2s_3s_1s_2), \quad (s_2s_3, s_1s_2s_3s_1s_2), \quad (s_3, s_1s_2s_3s_1s_2),
(s_1s_2s_3s_1s_2, s_1s_2s_3s_1s_2s_1), \quad (s_2s_3s_1s_2, s_1s_2s_3s_1s_2s_1), \quad (s_3s_1s_2s_3s_1s_2s_1),
\]
and these 55 elements form a lattice which is indeed isomorphic to \(\mathcal{T}^{(2)}_4\), see Figure 5. In this figure, we used the following identification of labels:

\[\begin{align*}
0 & \leftrightarrow \varepsilon, & 1 & \leftrightarrow s_3, & 2 & \leftrightarrow s_1, & 3 & \leftrightarrow s_1s_2, \\
4 & \leftrightarrow s_1s_3, & 5 & \leftrightarrow s_2, & 6 & \leftrightarrow s_1s_2s_1, & 7 & \leftrightarrow s_2s_3, \\
8 & \leftrightarrow s_2s_3s_2, & 9 & \leftrightarrow s_1s_2s_3, & 10 & \leftrightarrow s_1s_2s_3s_2, & 11 & \leftrightarrow s_1s_2s_3s_1, \\
12 & \leftrightarrow s_1s_2s_3s_1s_2, & 13 & \leftrightarrow s_1s_2s_3s_1s_2s_1.
\end{align*}\]

### 4.2. Computer Experiments.

In this paragraph we present computer experiments to check how our construction from Section 3 behaves with respect to other Coxeter groups and other Coxeter elements. We notice that the cardinality of \(\mathcal{C}^m_\gamma\) does not depend on the choice of \(\gamma\) which comes from the fact that the number of edges in \(\mathcal{C}_\gamma\) does not depend on \(\gamma\) either. However, computer experiments suggest that the cardinalities of \(\mathcal{L}(\langle \mathcal{C}^m_\gamma, \leq \rangle)\) coincide with the corresponding Fuss-Catalan number only in the case \(W = S_n\), where \(\gamma = \gamma_0 = (1\ 2 \ldots\ n)\) or \(\gamma = (n\ n-1 \ldots\ 1)\). The posets \(\langle \mathcal{C}^m_\gamma, \leq \rangle\) were created using SAGE and SAGE-COMBINAT [20,22], and the cardinality of their lattice completion was determined using ConExp [25].

**Example 4.3.** Let \(W = A_3\) with Coxeter diagram \(s_1 \cdots s_2 \cdots s_3\). The table in Figure 6 lists the cardinalities of \(\mathcal{L}(\langle \mathcal{C}^m_\gamma, \leq \rangle)\) for \(m \in \{1, 2, 3, 4\}\). The sequence of \(m\)-Catalan numbers for \(A_3\) starts with
\[14, 55, 140, 285, \ldots.\]
Figure 5. The lattice $\mathcal{L}(C^{(2)}_\gamma)$ is indeed isomorphic to $T^{(2)}_4$. The orange elements indicate the elements not present in $C^{(2)}_\gamma$.

| $\gamma$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ |
|----------|-------|-------|-------|-------|
| $s_1s_2s_3$ | 14   | 55   | 140   | 285   |
| $s_1s_3s_2$ | 14   | 59   | 162   | 355   |
| $s_2s_1s_3$ | 14   | 56   | 146   | 305   |
| $s_3s_2s_1$ | 14   | 55   | 140   | 285   |

$|C^{(m)}_\gamma|$ | 14   | 45   | 94    | 161   |

Figure 6. The cardinalities of $\mathcal{L}((C^{(m)}_\gamma, \leq_\gamma))$ for $W = A_3$ and $m \in \{1, 2, 3, 4\}$. 
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$\gamma$ & $m = 1$ & $m = 2$ & $m = 3$ \\
\hline
$s_1s_2s_3s_4$ & 42 & 273 & 969 \\
$s_1s_2s_4s_3$ & 42 & 305 & 1211 \\
$s_1s_3s_2s_4$ & 42 & 308 & 1252 \\
$s_1s_4s_3s_2$ & 42 & 305 & 1211 \\
$s_2s_3s_1s_4$ & 42 & 282 & 1045 \\
$s_2s_4s_3s_1$ & 42 & 308 & 1252 \\
$s_3s_2s_1s_4$ & 42 & 282 & 1045 \\
$s_4s_3s_2s_1$ & 42 & 273 & 969 \\
\hline
\end{tabular}
\caption{The cardinalities of $L((C_{\gamma}^{(m)}, \leq_\gamma))$ for $W = A_4$ and $m \in \{1, 2, 3, 4\}$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
$\gamma$ & $m = 1$ & $m = 2$ & $m = 3$ \\
\hline
$s_1s_2s_3$ & 20 & 85 & 226 \\
$s_1s_3s_2$ & 20 & 97 & 292 \\
$s_2s_1s_3$ & 20 & 89 & 249 \\
$s_3s_2s_1$ & 20 & 88 & 242 \\
\hline
\end{tabular}
\caption{The cardinalities of $L((C_{\gamma}^{(m)}, \leq_\gamma))$ for $W = B_3$ and $m \in \{1, 2, 3, 4\}$.}
\end{table}

Example 4.4. Let $W = A_4$ with Coxeter diagram $s_1 - s_2 - s_3 - s_4$. The table in Figure 7 lists the cardinalities of $L((C_{\gamma}^{(m)}, \leq_\gamma))$ for $m \in \{1, 2, 3, 4\}$. The sequence of $m$-Catalan numbers for $A_4$ starts with

$$42, 273, 969, 2530, \ldots$$

Example 4.5. Let $W = B_3$ with Coxeter diagram $s_1 \overset{4}{\longrightarrow} s_2 \overset{5}{\longrightarrow} s_3$. The table in Figure 8 lists the cardinalities of $L((C_{\gamma}^{(m)}, \leq_\gamma))$ for $m \in \{1, 2, 3, 4\}$. The sequence of $m$-Catalan numbers for $B_3$ starts with

$$20, 84, 220, 455, \ldots$$

Example 4.6. Let $W = B_4$ with Coxeter diagram $s_1 \overset{4}{\longrightarrow} s_2 \overset{5}{\longrightarrow} s_3 \overset{6}{\longrightarrow} s_4$. The table in Figure 9 lists the cardinalities of $L((C_{\gamma}^{(m)}, \leq_\gamma))$ for $m \in \{1, 2, 3, 4\}$. The sequence of $m$-Catalan numbers for $B_4$ starts with

$$70, 495, 1820, 4845, \ldots$$
Example 4.7. Let \( W = D_4 \) with Coxeter diagram \( s_1 \rightarrow s_3 \rightarrow s_2 \). The table in Figure 10 lists the cardinalities of \( \mathcal{L}((C^{(m)}_\gamma, \leq \gamma)) \) for \( m \in \{1, 2, 3, 4\} \). The sequence of \( m \)-Catalan numbers for \( D_4 \) starts with

\[
50, 336, 1210, 3185, \ldots
\]

Example 4.8. Let \( W = F_4 \) with Coxeter diagram \( s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \). The table in Figure 11 lists the cardinalities of \( \mathcal{L}((C^{(m)}_\gamma, \leq \gamma)) \) for \( m \in \{1, 2, 3, 4\} \). The sequence of \( m \)-Catalan numbers for \( F_4 \) starts with

\[
105, 780, 2926, 7875, \ldots
\]
\[ m = 1 \quad m = 2 \quad m = 3 \quad m = 4 \]

| \( \gamma = s_1 s_2 s_3 s_4 \) | 105 | 960 | 4497 | 15062 |
| \( \gamma = s_1 s_2 s_4 \) | 105 | 1192 | 6666 | 25687 |
| \( \gamma = s_1 s_3 s_2 s_4 \) | 105 | 1218 | 7280 | 30545 |
| \( \gamma = s_1 s_4 s_3 s_2 \) | 105 | 1192 | 6666 | 25687 |
| \( \gamma = s_2 s_3 s_1 s_4 \) | 105 | 1065 | 5754 | 22561 |
| \( \gamma = s_2 s_4 s_3 s_1 \) | 105 | 1218 | 7280 | 30545 |
| \( \gamma = s_3 s_2 s_1 s_4 \) | 105 | 1065 | 5754 | 22561 |
| \( \gamma = s_3 s_3 s_2 s_1 \) | 105 | 960 | 4497 | 15062 |

\[ |C^{(m)}_\gamma| \quad 105 \quad 415 \quad 931 \quad 1653 \]

**Figure 11.** The cardinalities of \( L((C^{(m)}_\gamma, \leq \gamma)) \) for \( W = F_4 \) and \( m \in \{1,2,3,4\} \).

| \( \gamma = s_1 s_2 s_3 \) | 32 | 152 | 436 | 975 |
| \( \gamma = s_1 s_3 s_2 \) | 32 | 184 | 622 | 1598 |
| \( \gamma = s_2 s_1 s_3 \) | 32 | 165 | 515 | 1248 |
| \( \gamma = s_3 s_2 s_1 \) | 32 | 165 | 506 | 1195 |

\[ |C^{(m)}_\gamma| \quad 32 \quad 108 \quad 229 \quad 395 \]

**Figure 12.** The cardinalities of \( L((C^{(m)}_\gamma, \leq \gamma)) \) for \( W = H_3 \) and \( m \in \{1,2,3,4\} \).

**Example 4.9.** Let \( W = H_3 \) with Coxeter diagram \( s_1 \xrightarrow{s_2} s_3 \). The table in Figure 12 lists the cardinalities of \( L((C^{(m)}_\gamma, \leq \gamma)) \). The sequence of \( m \)-Catalan numbers for \( H_3 \) starts with

\[ 32, 143, 384, 805, \ldots \]

**Example 4.10.** Let \( W = H_4 \) with Coxeter diagram \( s_1 \xrightarrow{s_2} s_3 \xrightarrow{s_4} s_3 \). The table in Figure 13 lists the cardinalities of \( L((C^{(m)}_\gamma, \leq \gamma)) \) for \( m \in \{1,2,3,4\} \). The sequence of \( m \)-Catalan numbers for \( H_4 \) starts with

\[ 280, 2232, 8602, 23485, \ldots \]

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\[ \gamma = s_1s_2s_3s_4 \]
\[ \gamma = s_1s_2s_4s_3 \]
\[ \gamma = s_1s_3s_2s_4 \]
\[ \gamma = s_1s_4s_3s_2 \]
\[ \gamma = s_2s_3s_1s_4 \]
\[ \gamma = s_2s_4s_3s_1 \]
\[ \gamma = s_3s_2s_1s_4 \]
\[ \gamma = s_3s_4s_2s_1 \]

\[ |C^{(m)}_{\gamma}| \]
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- \( m = 1 \) 280 3676 24213 111275
- \( m = 2 \) 280 4264 35759 159646
- \( m = 3 \) 280 4630 37444 136802
- \( m = 4 \) 280 4148 30861 117284

\[ \begin{array}{|c|c|c|c|c|}
\hline
m &= 1 & m &= 2 & m &= 3 & m &= 4 \\
\hline
\gamma = s_1s_2s_3s_4 &= 280 & 3676 & 24213 & 111275 \\
\gamma = s_1s_2s_4s_3 &= 280 & 4264 & 30813 & 149662 \\
\gamma = s_1s_3s_2s_4 &= 280 & 4630 & 37529 & 205236 \\
\gamma = s_1s_4s_3s_2 &= 280 & 4792 & 37544 & 191441 \\
\gamma = s_2s_3s_1s_4 &= 280 & 4148 & 30861 & 136802 \\
\gamma = s_2s_4s_3s_1 &= 280 & 4074 & 2659 & 117284 \\
\gamma = s_3s_2s_1s_4 &= 280 & 4034 & 28649 & 136802 \\
\gamma = s_3s_4s_2s_1 &= 280 & 4074 & 2659 & 117284 \\
\hline
\end{array} \]

\textbf{Figure 13.} The cardinalities of \( \mathcal{L}(\{C^{(m)}_{\gamma}, \leq_{\gamma}\}) \) for \( W = H_4 \) and \( m \in \{1, 2, 3, 4\} \).

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