Asymptotic properties for linear processes of functionals of reversible Markov Chains

by

Magda Peligrad

Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, Oh 45221-0025, USA

Abstract. In this paper we study the asymptotic behavior of linear processes having as innovations mean zero, square integrable functions of stationary reversible Markov chains. In doing so we shall preserve the generality of coefficients assuming only that they are square summable. In this way we include in our study the long range dependence case. The only assumption imposed on the innovations is the absolute summability of their covariances. Besides the central limit theorem we also study the convergence to fractional Brownian motion. The proofs are based on general results for linear processes with stationary innovations that have interest in themselves.

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1 Introduction

Let \((\xi_i)_{i \in \mathbb{Z}}\) be a stationary sequence of random variables on a probability space \((\Omega, \mathcal{K}, \mathbb{P})\) with finite second moment and zero mean \((E\xi_0 = 0)\). Let \((a_i)_{i \in \mathbb{Z}}\) be a sequence of real numbers such that \(\sum_{i \in \mathbb{Z}} a_i^2 < \infty\) and denote by

\[
X_k = \sum_{j=-\infty}^{\infty} a_{k+j} \xi_j, \quad S_n(X) = S_n = \sum_{k=1}^{n} X_k, \quad (1)
\]

\[
b_{n,j} = a_{j+1} + \ldots + a_{j+n} \quad \text{and} \quad b_n^2 = \sum_{j=-\infty}^{\infty} b_{n,j}^2.
\]

The linear process \((X_k)_{k \in \mathbb{Z}}\) is widely used in a variety of applied fields. It is properly defined for any square summable sequence \((a_i)_{i \in \mathbb{Z}}\) if and only if the stationary sequence of innovations \((\xi_i)_{i \in \mathbb{Z}}\) has a bounded spectral density. In general, the covariances of \((X_k)_{k \in \mathbb{Z}}\) might not be summable so that the linear process might exhibit long range dependence.

An important theoretical question with numerous practical implications is to prove stability of the central limit theorem under formation of linear sums.

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By this we understand that if \( \sum_{i=1}^{n} \xi_i / \sqrt{n} \) converges in distribution to a normal variable the same holds for \( S_n(X) / b_n \) properly normalized. This problem was first studied in the literature by Ibragimov (1962) who proved that if \((\xi_i)_{i \in \mathbb{Z}}\) are i.i.d. centered with finite second moments, then \( S_n(X) / b_n \) satisfies the central limit theorem (CLT). The extra condition of finite second moment was removed by Peligrad and Sang (2011). The central limit theorem for \( S_n(X) / b_n \) for the case when the innovations are square integrable martingale differences was proved in Peligrad and Utev (1997) and (2006-a), where an extension to generalized martingales was also given.

On the other hand, motivated by applications to unit root testing and to isotonic regression, a related question is to study the limiting behavior of \( S_{\lfloor nt \rfloor} / b_n \) (here and throughout the paper \( [x] \) denotes the integer part of \( x \)). The first results for i.i.d. random innovations go back to Davydov (1970), who established convergence to fractional Brownian motion. Extensions to dependent settings under certain protective criteria can be found for instance in Wu and Min (2005) and Dedecker et al. (2011), among others.

In this paper we shall address both these questions of CLT and convergence to fractional Brownian motion for linear processes with functions of reversible Markov chains innovations.

Kipnis and Varadhan (1986) considered partial sums \( S_n \) (where \( a_0 = 1 \), and 0 elsewhere) of an additive functional zero mean of a stationary reversible Markov chain and showed that the convergence of \( \text{var}(S_n) / n \) implies convergence of \( \{S_{\lfloor nt \rfloor} / \sqrt{n}, 0 \leq t \leq 1\} \) to the Brownian motion. There is a considerable number of papers that further extend and apply this result to infinite particle systems, random walks, processes in random media, Metropolis-Hastings algorithms. Among others, Kipnis and Landim (1999) considered interacting particle systems, Tierney (1994) discussed the applications to Markov Chain Monte Carlo. Liming Wu (1999) studied the law of the iterated logarithm.

Our first result will show that under the only assumption of absolute summability of covariances of innovations, the partial sums of the linear process \( S_n(X) / b_n \) satisfies the central limit theorem provided \( b_n \to \infty \). If we only assume the convergence of \( \text{var}(S_n) / n \) we can also treat a related linear process.

Furthermore, we shall also establish convergence to the fractional Brownian motion under a necessary regularity condition imposed to \( b_n^2 \). For a Hurst index larger than 1/2 we obtain a full blown invariance principle. This is not possible without imposing additional conditions for a Hurst index smaller than or equal to 1/2. However we can still get the convergence of finite dimensional distributions. For a Hurst index of 1/2 we shall also consider the short memory case, when the sequence of constants is absolutely summable, and obtain convergence to the Brownian motion.

In this paper, besides a condition on the covariances, no other assumptions such as irreducibility or aperiodicity are imposed.

The proofs are based on a result of Peligrad and Utev (2006-a) concerning the asymptotic behavior of a class of linear processes and spectral calculus. In addition, in Section 4.1 we develop several asymptotic results for a class of linear processes with stationary innovations, which is not necessarily Markov or
reversible. These results have interest in themselves and can be applied to treat other classes of linear processes.

Applications are given to a Metropolis Hastings Markov chain, to instantaneous functions of a Gaussian process and to random walks on compact groups.

Our paper is organized as follows: Section 2 contains the definitions, a short background of the problem and the results. Applications are discussed in Section 3. Section 4 is devoted to the proofs. The Appendix contains some technical results.

2 Definitions, background and results

We assume that \((\gamma_n)_{n \in \mathbb{Z}}\) is a stationary Markov chain defined on a probability space \((\Omega, \mathcal{F}, P)\) with values in a general state space \((S, \mathcal{A})\). The marginal distribution is denoted by \(\pi(A) = P(\gamma_0 \in A)\). Assume that there is a regular conditional distribution for \(\gamma_1\) given \(\gamma_0\) denoted by \(Q(x, A) = P(\gamma_1 \in A | \gamma_0 = x)\). Let \(Q\) also denotes the Markov operator acting via \((Qg)(x) = \int_S g(s)Q(x, ds)\).

Next, let \(L^2_0(\pi)\) be the set of measurable functions on \(S\) such that \(\int g^2 d\pi < \infty\) and \(\int gd\pi = 0\). If \(g, h \in L^2_0(\pi)\), the integral \(\int_S g(s)h(s)d\pi\) will sometimes be denoted by \(\langle g, h \rangle\).

For some function \(g \in L^2_0(\pi)\), let

\[\xi_i = g(\gamma_i), \quad S_n(\xi) = \sum_{i=1}^{n} \xi_i, \quad \sigma_n(g) = (E S_n^2(\xi))^{1/2}.\]  

(2)

Denote by \(\mathcal{F}_k\) the \(\sigma\)-field generated by \(\gamma_i\) with \(i \leq k\) and by \(\mathcal{I}\) the invariant \(\sigma\)-field.

For any integrable random variable \(X\) we denote \(E_kX = E(X|\mathcal{F}_k)\). With this notation, \(E_0\xi_1 = Qg(\gamma_0) = E(\xi_1|\gamma_0)\). We denote by \(||X||_p\) the norm in \(L^p(\Omega, \mathcal{F}, P)\).

The Markov chain is called reversible if \(Q = Q^*\), where \(Q^*\) is the adjoint operator of \(Q\). In this setting, the condition of reversibility is equivalent to requiring that \((\gamma_0, \gamma_1)\) and \((\gamma_1, \gamma_0)\) have the same distribution. Equivalently

\[\int_A Q(\omega, B)\pi(d\omega) = \int_B Q(\omega, A)\pi(d\omega)\]

for all Borel sets \(A, B \in \mathcal{A}\). The spectral measure of \(Q\) with respect to \(g\) is concentrated on \([-1, 1]\) and will be denoted by \(\rho_g\). Then

\[E(Q^n g(\gamma_0)Q^n g(\gamma_0)) =\langle Q^n g, Q^n g \rangle = \int_{-1}^{1} t^{n+m} \rho_g(dt)\]

Kipnis and Varadhan (1986) assumed that

\[\lim_{n \to \infty} \frac{\sigma_n^2(g)}{n} = \sigma^2_g\]  

(3)

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and proved that for any reversible ergodic Markov chain defined by (1) this condition implies

\[ W_n(t) = \frac{S_{[nt]}(\xi)}{\sqrt{n}} \Rightarrow |\sigma_g|W(t), \]

where \( W(t) \) is the standard Brownian motion, \( \Rightarrow \) denotes weak convergence.

As shown by Kipnis and Varadhan (1986, relation 1.1) condition (3) is equivalent to

\[ \int_{-1}^{1} \frac{1}{1-t^2} \rho_g(dt) < \infty, \]

and then

\[ \sigma_g^2 = \int_{-1}^{1} \frac{1 + t}{1-t^2} \rho_g(dt). \]

We shall establish the following central limit theorem:

**Theorem 1** Assume that \((\xi_j)_{j \in \mathbb{Z}}\) is defined by (2) and \(Q = Q^*\). Define \((X_k)_k\), \(S_n\) and \(b_n\) as in (1). Assume that \(b_n \to \infty\) as \(n \to \infty\) and

\[ \sum_{j \geq 0} |\text{cov}(\xi_0, \xi_j)| < \infty. \]

Then, there is a nonnegative random variable \(\eta\) measurable with respect to \(\mathcal{I}\) such that \(n^{-1} \mathbb{E}((\sum_{k=1}^{n} \xi_k)^2|\mathcal{F}_0) \to \eta\) in \(L_1\) as \(n \to \infty\) and \(\mathbb{E}\eta = \sigma_g^2\). In addition

\[ \lim_{n \to \infty} \frac{\text{Var}(S_n(X))}{b_n^2} = \sigma_g^2 \]

and

\[ \frac{S_n(X)}{b_n} \Rightarrow \sqrt{\eta} N \text{ as } n \to \infty, \]

where \(N\) is a standard normal variable independent on \(\eta\). Moreover if the sequence \((\xi_i)_{i \in \mathbb{Z}}\) is ergodic the central limit theorem in (7) holds with \(\eta = \sigma_g^2\).

It should be noted that under the conditions of this theorem \(\sigma_g^2\) also has the following interpretation: the stationary sequence \((\xi_i)_{i \in \mathbb{Z}}\) has a continuous spectral density \(f(x)\) and \(\sigma_g^2 = 2\pi f(0)\).

In order to present the functional form of the CLT we introduce a regularity assumption which is necessary for this type of result. We denote by \(D([0,1])\) the space of functions defined on \([0,1]\) which are right continuous and have left hand limits at any point.

**Definition 2** We say that a positive sequence \((b_n^2)_{n \geq 1}\) is regularly varying with exponent \(\beta > 0\) if for any \(t \in [0,1]\),

\[ \frac{b_{nt}^2}{b_n^2} \to t^\beta \text{ as } n \to \infty. \]
We shall separate the case $\beta \in [1, 2]$ from the case $\beta \in [0, 1]$.

**Theorem 3** Assume that the conditions of Theorem 1 are satisfied and in addition $b_n^2$, defined by (1), is regularly varying with exponent $\beta$ for a certain $\beta \in [1, 2]$. Then, the process $\{b_n^{-1}S_{[nt]}, t \in [0, 1]\}$ converges in $D([0, 1])$ to $\sqrt{n}W_H$ where $W_H$ is a standard fractional Brownian motion independent of $\eta$ with Hurst index $H = \beta/2$.

The case $\beta \in [0, 1]$ is more delicate. For this case we only give the convergence of the finite dimensional distributions since there are counterexamples showing that the tightness might not hold without additional assumptions. As a matter of fact, for $\beta = 1$, it is known from counterexamples given in Wu and Woodroofe (2004) and also in Merlevède and Peligrad (2006) that the weak invariance principle may not be true for the partial sums of the linear process with i.i.d. square integrable innovations.

**Theorem 4** Assume that the conditions of Theorem 1 are satisfied and in addition $b_n^2$ is regularly varying with exponent $\beta$ for a certain $\beta \in [0, 1]$. Then the finite dimensional distributions of $\{b_n^{-1}S_{[nt]}, t \in [0, 1]\}$ converges to the corresponding ones of $\sqrt{n}W_H$, where $W_H$ is a standard fractional Brownian motion independent of $\eta$ with Hurst index $H = \beta/2$.

In the context of Theorems 3 and 4, condition (8) is necessary for the conclusion of this theorem (see Lamperti, 1962). This condition has been also imposed by Davydov (1970) for studying the weak invariance principle of linear processes with i.i.d. innovations.

The following theorem is obtained under condition (3).

**Theorem 5** Assume that $(\xi_j)$ is defined by (3) and condition (3) is satisfied. Define

$$X'_k = \sum_{j=-\infty}^{\infty} a_{k+j}(\xi_j + \xi_{j+1}), \quad S_n(X') = \sum_{k=1}^{n} X'_k, \quad (9)$$

Then the conclusion of Theorems 1, 3 and 4 hold for $S_n(X')$. In this case $\eta$ is identified as the limit $n^{-1}E(\sum_{k=1}^{n}(\xi_k + \xi_{k+1})^2|F_0) \to \eta$ in $L_1$ as $n \to \infty$. Furthermore, the stationary sequence $(\xi_k + \xi_{k+1})_{k \in \mathbb{Z}}$ has a continuous spectral density $h(x)$ and $E\eta = 2\pi h(0) = \lim_{n \to \infty} \text{Var}S_n(X')/b_n^2$.

We shall present next the short memory case:

**Theorem 6** Assume now $\sum_{i \in \mathbb{Z}} |a_i| < \infty$ and let $(X_k)_{k \geq 1}$ be as in Theorem 1. Assume that condition (3) is satisfied. Then the process $\{S_{[nt]}/\sqrt{n}, t \in [0, 1]\}$ converges in $D([0, 1])$ to $\sqrt{n}A/W$ where $W$ is a standard Brownian motion and $A = \sum_{i \in \mathbb{Z}} a_i$.

**Remark 7** It is easy to see that Theorems 4 extends Kipnis Varadhan result to linear processes. \(\blacksquare\)
We give a few examples of sequences \((a_n)\) satisfying the conditions of our theorems. In these examples the notation \(a_n \sim b_n\) means \(a_n/b_n \to 1\) as \(n \to \infty\).

**Example 1.** For the selection \(a_i \sim i^{-\alpha} \ell(i)\) where \(\ell\) is a slowly varying function at infinity and \(1/2 < \alpha < 1\) for \(i \geq 1\) and \(a_i = 0\) elsewhere, then, \(b^2_n \sim \kappa_{\alpha} n^{3-2\alpha} \ell^2(n)\) (see for instance Relations (12) in Wang et al. (2003)), where \(\kappa_{\alpha}\) is a positive constant depending on \(\alpha\). Clearly, Theorem 3 applies.

**Example 2.** Let us consider now the fractionally integrated processes since they play an important role in financial time series modeling and they are widely studied. Such processes are defined for \(0 < d < 1/2\) by

\[
X_k = (1-B)^{-d} \xi_k = \sum_{i \geq 0} a_i \xi_{k-i} \quad \text{with} \quad a_i = \frac{\Gamma(i+d)}{\Gamma(d) \Gamma(i+1)},
\]

(10)

where \(B\) is the backward shift operator, \(B \xi_k = \xi_{k-1}\). For this example, by the well known fact that for any real \(x\), \(\lim_{n \to \infty} \Gamma(n + x)/n^x \Gamma(n) = 1\), we have \(\lim_{n \to \infty} a_n/n^{d-1} = 1/\Gamma(d)\). Theorem 3 applies with \(\beta = 2d + 1\), since for \(k \geq 1\) we have \(a_k \sim \kappa_d k^{d-1}\) for some \(\kappa_d > 0\) and \(a_k = 0\) elsewhere.

**Example 3.** Now, if we consider the following selection of \((a_k)_{k \geq 0}\): \(a_0 = 1\) and \(a_i = (i+1)^{-\alpha} - i^{-\alpha}\) for \(i \geq 1\) with \(\alpha \in [0, 1/2]\) and \(a_i = 0\) elsewhere, then Theorem 3 applies. Indeed for this selection, \(b^2_n \sim \kappa_{\alpha} n^{1-2\alpha}\), where \(\kappa_{\alpha}\) is a positive constant depending on \(\alpha\).

**Example 4.** Finally, if \(a_i \sim i^{-1/2} (\log i)^{-\alpha}\) for some \(\alpha > 1/2\), then \(b^2_n \sim n^\alpha (\log n)^{1-2\alpha}/(2\alpha - 1)\) (see Relations (12) in Wang et al. (2003)). Hence (5) is satisfied with \(\beta = 2\).

3 Applications

3.1 Application to a Metropolis Hastings Markov chain.

In this subsection we analyze a standardized example of a stationary irreducible and aperiodic Metropolis-Hastings algorithm with uniform marginal distribution. This type of Markov chain is interesting since it can easily be transformed into Markov chains with different marginal distributions. Markov chains of this type are often studied in the literature from different points of view. See, for instance, Doukhan et al (1994) and Longla et al (2012) among many others.

Let \(E = [-1, 1]\) and let \(\nu\) be a symmetric atomless law on \(E\). The transition probabilities are defined by

\[
Q(x, A) = (1 - |x|) \delta_x(A) + |x| \nu(A),
\]

where \(\delta_x\) denotes the Dirac measure. Assume that \(\theta = \int_E |x|^{-1} \nu(dx) < \infty\). Then there is a unique invariant measure

\[
\pi(dx) = \theta^{-1} |x|^{-1} \nu(dx)
\]
and the stationary Markov chain \((\gamma_k)\) generated by \(Q(x, A)\) and \(\pi\) is reversible and positively recurrent, therefore ergodic.

**Theorem 8** Let \(g(-x) = -g(x)\) for any \(x \in E\) and assume
\[
\int_0^1 g^2(x)x^{-2}dv < \infty.
\]
Then, the conclusions of all our theorems in Section 2 hold for \((X_k)\) and \(S_n(X)\) defined by (1) with
\[
\eta = \sigma^2 = \theta^{-1}(\int_E g^2(x)|x|^{-1}v(dx) + 2 \int_E g^2(x)|x|^{-2}v(dx)).
\]

**Proof.** Since \(g\) is an odd function we have
\[
\mathbb{E}(g(\gamma_k)|\gamma_0) = (1 - |\gamma_0|)^k g(\gamma_0) \text{ a.s. (11)}
\]
Therefore, for any \(j \geq 0,\)
\[
\mathbb{E}(X_0X_j) = \mathbb{E}(g(\gamma_0)\mathbb{E}(g(\gamma_j)|\gamma_0)) = \theta^{-1} \int_E g^2(x)(1 - |x|)^j|x|^{-1}v(dx).
\]
Then,
\[
\sum_{j=1}^{k-1} |\mathbb{E}(X_0X_j)| \leq 2\theta^{-1} \sum_{j=1}^{k-1} \int_0^1 g^2(x)(1 - x)^jx^{-1}v(dx) \leq 2\theta^{-1} \int_0^1 g^2(x)x^{-2}v(dx)
\]
and therefore condition (6) is satisfied. \(\diamondsuit\)

### 3.2 Linear process of instantaneous functions of a Gaussian sequence

**Theorem 9** Let \((\xi_k)_{k \in \mathbb{Z}}\) be instantaneous functions of a stationary Markov Gaussian sequence \((\gamma_n)\), \(\xi_k = g(\gamma_n)\) where \(g\) is a measurable real function such that \(\mathbb{E}g(\gamma_n) = 0\) and \(\mathbb{E}g^2(\gamma_n) < \infty\). Define \(X_k\) and \(S_n(X)\) by (1). Then the conclusion of our theorems in Section 2 hold.

**Proof.** In order to apply our results, because \((\gamma_n)\) is reversible, we have only to check condition (6). Under our conditions \(g\) can be expanded in Hermite polynomials \(g(x) = \sum_{j \geq 1} c_j H_j(x)\), where \(\sum_{j=1} c_j^2 j! < \infty\).

For computing the covariances we shall apply the following well-known formula: if \(a\) and \(b\) are jointly Gaussian random variables, \(\mathbb{E}a = \mathbb{E}b = 0\), \(\mathbb{E}a^2 = \mathbb{E}b^2 = 1\), \(r = \mathbb{E}ab\), then
\[
\mathbb{E}H_k(a)H_l(b) = \delta(k, l)r^k k!.
\]
where \( \delta \) denotes the Kronecker delta. It follows that

\[
\text{cov}(\xi_0, \xi_k) = \mathbb{E} \sum_{j \geq 1} c_j^2 H_j(\gamma_0) H_j(\gamma_k) = \sum_{j \geq 1} c_j^2 r_{j,k}!.
\]

Clearly, because under our condition it is known that \( r_k = \exp(-\alpha k/2) \) for some \( \alpha > 0 \), then

\[
|\text{cov}(\xi_0, \xi_k)| \leq \exp(-\alpha k/2) \sum_{j \geq 1} c_j^2 j!
\]

and the result follows. ♦

For a particular class of weights of the form in Example 3, we mention that Breuer and Major (1983) studied this problem for Gaussian chains without Markov assumption.

### 3.3 Application to random walks on compact groups

In this section we shall apply our results to random walks on compact groups.

Let \( \mathcal{X} \) be a compact abelian group, \( \mathcal{A} \) a sigma algebra of Borel subsets of \( \mathcal{X} \) and \( \pi \) the normalized Haar measure on \( \mathcal{X} \). The group operation is denoted by +. Let \( \nu \) be a probability measure on \((\mathcal{X}, \mathcal{A})\). The random walk on \( \mathcal{X} \) defined by \( \nu \) is the stationary Markov chain having the transition function

\[
(x, A) \to Q(x, A) = \nu(A - x).
\]

The corresponding Markov operator denoted by \( Q \) is defined by

\[
(Qf)(x) = f \ast \nu(x) = \int_X f(x + y) \nu(dy).
\]

The Haar measure is invariant under \( Q \). We shall assume that \( \nu \) is not supported by a proper closed subgroup of \( \mathcal{X} \), a condition that is equivalent to \( Q \) being ergodic. In this context

\[
(Q^* f)(x) = f \ast \nu^*(x) = \int_X f(x - y) \nu(dy),
\]

where \( \nu^* \) is the image of measure \( \nu \) by the map \( x \to -x \). Thus \( Q \) is symmetric on \( L_2(\pi) \) if and only if \( \nu \) is symmetric on \( \mathcal{X} \), that is \( \nu = \nu^* \).

The dual group of \( \mathcal{X} \), denoted by \( \hat{\mathcal{X}} \), is discrete. Denote by \( \hat{\nu} \) the Fourier transform of the measure \( \nu \), that is the function

\[
g \to \hat{\nu}(g) = \int_X g(x) \nu(dx) \quad \text{with} \quad g \in \mathcal{X}.
\]

A function \( f \in L^2(\pi) \) has the Fourier expansion

\[
f = \sum_{g \in \hat{\mathcal{X}}} \hat{f}(g) g.
\]

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Ergodicity of $Q$ is equivalent to $\hat{\nu}(g) \neq 1$ for any non-identity $g \in \hat{X}$. By arguments in Borodin and Ibragimov (1994, ch. 4, section 9) and also Derriennic and Lin (2001, Section 8) condition (5) takes the form

$$\sum_{1 \neq g \in \hat{X}} \frac{|\hat{f}(g)|^2}{|1 - \hat{\nu}(g)|} < \infty. \quad (13)$$

Combining these considerations with the results in Section 2 we obtain the following result:

**Theorem 10** Let $\nu$ be ergodic and symmetric on $X$. Let $(\xi_i)$ be the stationary Markov chain with marginal distribution $\pi$ and transition operator $Q$. If for $g$ in $L_2(\pi)$ condition (13) is satisfied then the conclusions of Theorem 5 in Section 2 hold for $(X_k^r)$ and $S_n(X')$ defined by (9).

4 Proofs

4.1 Preliminary general results

This section contains some general results for linear processes of stationary sequences which are not necessarily Markov. We start by mentioning the following theorem which is a variant of a result from Peligrad and Utev (2006-a). See also Proposition 5.1 in Dedecker et al. (2011).

**Theorem 11** Let $(\xi_k)_{k \in \mathbb{Z}}$ be a strictly stationary sequence of centered square integrable random variables such that

$$\Gamma_j = \sum_{k=0}^{\infty} |E(\xi_{j+k} E_0 \xi_j)| < \infty \text{ and } \frac{1}{p} \sum_{j=1}^{p} \Gamma_j \to 0 \text{ as } p \to \infty. \quad (14)$$

For any positive integer $n$, let $(d_{n,i})_{i \in \mathbb{Z}}$ be a triangular array of numbers satisfying, for some positive $c$,

$$\sum_{i \in \mathbb{Z}} d_{n,i}^2 \to c^2 \text{ and } \sum_{j \in \mathbb{Z}} (d_{n,j} - d_{n,j-1})^2 \to 0 \text{ as } n \to \infty. \quad (15)$$

In addition assume

$$\sup_{j \in \mathbb{Z}} |d_{n,j}| \to 0 \text{ as } n \to \infty. \quad (16)$$

Then $\sum_{j \in \mathbb{Z}} d_{n,j} \xi_j$ converges in distribution to $\sqrt{\eta}cN$ where $N$ is a standard Gaussian random variable independent of $\eta$. The variable $\eta$ is measurable with respect to the invariant sigma field $\mathcal{I}$ and $n^{-1}E((\sum_{k=1}^{n} \xi_k)^2|\mathcal{F}_0) \to \eta$ in $L_1$ as $n \to \infty$. Furthermore $(\xi_i)_{i \in \mathbb{Z}}$ has a continuous spectral density $f(x)$ and $E\eta = 2\pi f(0)$. If the sequence $(\xi_i)_{i \in \mathbb{Z}}$ is ergodic we have $\eta = 2\pi f(0)$. 
**Proof.** The proof follows the lines of Theorem 1 from Peligrad and Utev (2006-a). We just have to repeat the arguments there with \( b_{n,i} / b_n \) replaced by \( d_{n,i} \) and take into account that the properties (15) and (16) are precisely all is needed to complete the proof.  

Next we shall establish the convergence of finite dimensional distributions.

**Theorem 12** Define \((X_k)\) and \(S_n\) by (11) and assume condition (14) is satisfied. Then \(S_n/b_n\) converges in distribution to \( \sqrt{n} N \) where \( N \) and \( \eta \) are as in Theorem 11. If we assume in addition that condition (8) is satisfied, then the finite dimensional distributions of \\{\(W_n(t) = b_n^{-1} S_{[nt]}, t \in [0,1]\)\} converge to the corresponding ones of \( \sqrt{n}W_H \), where \( W_H \) is a standard fractional Brownian motion independent of \( \eta \) with Hurst index \( H = \beta/2 \).

**Proof.** The central limit theorem part requires just to verify the conditions of Theorem 11 for \( d_{n,j} = b_{n,j} / b_n \) and \( c = 1 \). Condition (16) was verified in Peligrad and Utev (1997, page 448-449) while condition (15) was verified in Lemma A.1. in Peligrad and Utev (2006-a).

We shall prove next the second part of the theorem. Notice that if we impose (8), for each \( t \) fixed

\[
\text{var}(W_n(t)) \to 2\pi f(0)t^\beta
\]

and \( W_n(t) \Rightarrow nt^{\beta/2}N. \)

Let \( 0 \leq t_1 \leq \ldots \leq t_k \leq 1 \). By Cramèr-Wold device, in order to find the limiting distribution of \((W_n(t_j))_{1 \leq j \leq k}\) we have to study \( V_n = \sum_{i=1}^k u_i W_n(t_i) \) where \( u_i \) is a real vector. Let us compute its limiting variance. To find it, let \( 0 \leq s \leq t \leq 1 \). By using the fact that for any two real numbers \( a \) and \( b \) we have \( a(a - b) = (a^2 + (a - b)^2 - b^2)/2 \), we obtain the representation:

\[
\text{cov}(W_n(t), W_n(s)) = \text{var}(W_n(s)) + \text{cov}(W_n(s), W_n(t) - W_n(s)) = \text{var}(W_n(s)) + 1/2[\text{var}(W_n(t) - W_n(s)) + \text{var}(W_n(t)) - \text{var}(W_n(s))].
\]

By stationarity,

\[
\text{var}(W_n(t) - W_n(s)) = \text{var}(W_n[t-\lfloor ns\rfloor]),
\]

and by (17) and the fact that \( b_n \to \infty \) we obtain

\[
\lim_{n \to \infty} \text{cov}(W_n(t), W_n(s)) = \pi f(0)(s^\beta + t^\beta - |t - s|^\beta).
\]

So,

\[
\lim_{n \to \infty} \frac{1}{2\pi f(0)}\text{var}(V_n) = \sum_{i=1}^k u_i^2 t_i^\beta + \sum_{i=1}^{k-1} \sum_{j=i+1}^k u_i u_j (t_i^\beta + t_j^\beta - (t_j - t_i)^\beta) = B_k.
\]

Writing now

\[
V_n = \sum_{i=1}^k u_i W_n(t_i) = \sum_{j \in \mathbb{Z}} d_{n,j}(k) \xi_j,
\]
where \( d_{n,j}(k) = \sum_{i=1}^{k} u_i b_{[nt_i,j],j} / b_n \), we shall apply Theorem 11. The second part of (15) and (16) were verified in Peligrad and Utev (1996 and 2006-a). It remains to verify the first part of condition (15). By the point (iii) of Lemma 14 in the Appendix we obtain

\[
\text{var}(V_n) / \sum_{j \in Z} d^2_{n,j}(k) \rightarrow 2 \pi f(0),
\]

which combined with (19) implies that the first part of (15) is verified with \( c^2 = \lim_{n \to \infty} \sum_{j \in Z} d^2_{n,j}(k) = B_k \). In other words, the finite dimensional distributions are convergent to those of a fractional Brownian motion with Hurst index \( \beta/2 \).

\( \diamond \)

**Discussion on tightness.** As we mentioned above, for \( \beta \leq 1 \) the conditions of Theorem 12 are not sufficient to imply tightness. However for \( \beta > 1 \) we can obtain tightness in \( D([0,1]) \) endowed with Skorohod topology. By the point (i) of Lemma 14 in Appendix we have the inequality

\[
E|S_k|^2 \leq \left( E[\xi_0^2] + 2 \sum_{k \in Z} |E(\xi_k)| \right) \sum_{j \in Z} b^2_{k,j}.
\]

Therefore, by using (14) and (8), the conditions of Lemma 2.1 p. 290 in Taqqu (1975) are satisfied when \( \beta > 1 \), and the tightness follows. \( \diamond \)

To treat the short memory case we mention the following result in Peligrad and Utev (2006-b).

**Theorem 13** Assume that \( X_k \) and \( S_n \) are defined by (1) and \( \sum_{i \in Z} |a_i| < \infty \). Moreover assume that for some \( c_n > 0 \) the innovations satisfy the invariance principle

\[
c_n^{-1} S_{[nt]}(\xi) \Rightarrow \eta W(t),
\]

where \( \eta \) is \( \mathcal{I} \)-measurable and \( W \) is a standard Brownian motion on \([0,1]\) independent on \( \mathcal{I} \). In addition assume that the following condition holds:

\[
E \max_{1 \leq j \leq n} |S_j(\xi)| \leq C c_n.
\]

where \( C \) is a positive constant. Then, the linear process also satisfies the invariance principle, i.e. \( c_n^{-1} S_{[nt]}(X) \Rightarrow \eta |A| W(t) \) as \( n \to \infty \) where \( A = \sum_{i \in Z} a_i \).

**4.2 Normal and reversible Markov Chains**

In this subsection we give the proofs of the theorems stated in Section 2. The goal is to verify condition (14) that will assure that all the results in the sub-section 4.1 are valid.
We start by applying the general results to normal Markov chains, for which \(QQ^* = Q^*Q\). For this case condition (14) is implied by
\[
\sum_{k \geq 0} ||Q^k g||_2^2 < \infty. \tag{21}
\]
Indeed, we start by rewriting (14) in operator notation:
\[
|\mathbb{E}[\xi_{j+k} \mathbb{E}(\xi_j | \mathcal{F}_0)]| = |\mathbb{E}(\mathbb{E}_0 \xi_{k+j} \mathbb{E}_0 \xi_j)| = |< Q^{k+j} g, Q^j g >| = |< Q^{[k/2]+j} g, (Q^*)^{k-[k/2]} Q^j g >| \leq ||Q^{[k/2]+j} g||_2 ||(Q^*)^{k-[k/2]} Q^j g||_2.
\]
For normal operator, by using the properties of conditional expectation, we have
\[
||(Q^*)^{k-[k/2]} Q^j g||_2 = ||Q^j (Q^*)^{k-[k/2]} g||_2 \leq ||(Q^*)^{k-[k/2]} g||_2.
\]
Since for all \(\varepsilon > 0\), and any two numbers \(a\) and \(b\) we have \(|ab| \leq a^2/2\varepsilon + \varepsilon b^2/2\), by the above considerations we easily obtain
\[
\sum_{k \geq 0} |\mathbb{E}[\xi_{j+k} \mathbb{E}(\xi_j | \mathcal{F}_0)]| \leq \sum_{k \geq 0} ||Q^{[k/2]+j} g||_2 ||Q^{k-[k/2]} g||_2 \\
\leq \frac{1}{\varepsilon} \sum_{k \geq j} ||Q^k g||_2^2 + \varepsilon \sum_{k \geq 0} ||Q^k g||_2^2,
\]
condition (14) is verified under (21), by letting \(j \to \infty\) followed by \(\varepsilon \to 0\).

In terms of spectral measure \(\rho_g(dz)\), condition (21) is implied by
\[
\int_D \frac{1}{1-|z|} \rho_g(dz) < \infty,
\]
where \(D\) is the unit disk. Note that this condition is stronger than the condition needed for the validity of CLT for the partial sums (i.e. the case \(a_1 = 1, a_i = 0\) elsewhere), which requires only the condition \(\int_D \frac{1}{1-|z|} \rho_g(dz) < \infty\) (see Gordin and Lifshitz (1981), or in Ch. IV in Borodin and Ibragimov (1994)).

For the reversible Markov chains just notice that
\[
\mathbb{E}[\xi_{j+k} \mathbb{E}(\xi_j | \mathcal{F}_0)] = \int_{-1}^1 i^{2j+k} \rho_g(dz) = \text{cov}(\xi_0, \xi_{2j+k})
\]
and then, condition (14) is verified under (6) because
\[
\sum_{k \geq 0} |\mathbb{E}[\xi_{j+k} \mathbb{E}(\xi_j | \mathcal{F}_0)]| = \sum_{k \geq 2j} |\text{cov}(\xi_0, \xi_k)| \to 0 \text{ as } j \to \infty.
\]

Theorems 1, 3 and 4 follow as simple applications of the results in subsection 4.4.

Proof of Theorem 5
In order to prove this theorem, we shall also apply Theorem 12 along to the tightness discussion at the end of Section 3. We denote \( \gamma_j = \xi_j + \xi_{j+1} \) and verify condition (14) for this sequence of innovations. We have

\[
|E(\gamma_{k+j}E_0\gamma_j)| = |< Q^{k+j}g + Q^{k+j+1}g, Q^jg + Q^{j+1}g >|
\]

and by spectral calculus

\[
\sum_{k \geq 0} |< Q^{k+j}g + Q^{k+j+1}g, Q^jg + Q^{j+1}g >| = \sum_{k \geq 0} \int_{-1}^{1} t^{k+2j}(1+t)^2d\rho_g.
\]

We divide the sum in 2 parts, according to \( k \) even or odd. When \( k = 2u \) the sum has positive terms and it can be written as

\[
\sum_{u \geq 0} \int_{-1}^{1} t^{2u+2j}(1+t)^2d\rho_g \leq \int_{-1}^{1} \frac{t^{2j}}{1-t^2}(1+t)^2d\rho_g = \int_{-1}^{1} \frac{t^{2j}(1+t)}{1-t}d\rho_g.
\]

When \( k \) is odd

\[
\sum_{k \geq 1, k \text{ odd}} |\int_{-1}^{1} t^{k+2j}(1+t)^2d\rho_g| \leq \int_{-1}^{1} \sum_{k \geq 1, k \text{ odd}} |t^{k+2j}(1+t)^2|d\rho_g
\]

\[
\leq \int_{-1}^{1} \sum_{k \geq 1, k \text{ odd}} |t^{k+2j}(1+t)^2|d\rho_g \leq \sum_{u \geq 0} |t^{2u+2j}(1+t)^2|d\rho_g,
\]

and we continue the computation as for the case \( k \) even. It follows that

\[
\frac{1}{m} \sum_{j=1}^{m} \sum_{k \geq 0} |E(\gamma_{k+j}E_0\gamma_j)| \leq \frac{2}{m} \sum_{j=1}^{m} \int_{-1}^{1} \frac{t^{2j}(1+t)}{1-t}d\rho_g.
\]

Note that (15) implies that \( \rho_g(1) = 0 \). We also have \( m^{-1} \sum_{j=1}^{m} t^{2j}(1+t) \) is convergent to 0 for all \( t \in [-1,1] \). Furthermore, \( m^{-1} \sum_{j=1}^{m} t^{2j}(1+t) \) is dominated by 2 and in view of (15) and Lebesgue dominated convergence theorem we have

\[
\lim_{m \to \infty} \int_{-1}^{1} \frac{1}{m} \sum_{j=1}^{m} \frac{t^{2j}(1+t)}{1-t}d\rho_g = 0,
\]

and therefore condition (14) is satisfied. ♦

**Proof of Theorem 6.**

Theorem 6 follows by combining Theorem 13 with the invariance principle in Kipnis and Varadhan (1997). We have only to verify condition (20). It is known that the maximal inequality required by condition (20) holds for partial sums of functions of reversible Markov chains. Indeed, we know from Proposition 4 in Longla et al. (2012) that

\[
E(\max_{1 \leq i \leq n} S_i^2) \leq 2E(\max_{1 \leq i \leq n} X_i^2) + 22 \max_{1 \leq i \leq n} E(S_i^2) \tag{22}
\]

and then, condition (3) and stationarity implies condition (20) with \( c_n = \sqrt{n} \). ♦

13
5 Appendix

Facts about spectral densities. In the following lemma we combine a few facts about spectral densities, covariances, behavior of variances of sums and their relationships. The first two points are well known. They can be found for instance in Bradley (2007, Vol 1, 0.19-0.21 and Ch.8). The point (iii) was proven in Peligrad and Utev (2006-a).

Lemma 14 Let \((\xi_i)_{i \in \mathbb{Z}}\) be a stationary sequence of real valued variables with \(E\xi_0 = 0\) and finite second moment. Let \(F\) denotes the spectral measure and \(f\) denotes its spectral density (if exists) i.e.

\[
E(\xi_0 \xi_k) = \int_{-\pi}^{\pi} e^{-ikt} dF(t) = \int_{-\pi}^{\pi} e^{-ikt} f(t) dt.
\]

(i) For any positive integer \(n\) and any real numbers \(a_1, \ldots, a_n\),

\[
E \left( \sum_{k=1}^{n} a_k \xi_k \right)^2 = \int_{-\pi}^{\pi} \left| \sum_{k=1}^{n} a_k e^{ikt} \right|^2 f(t) dt \leq 2\pi \| f \|_\infty \sum_{k=1}^{n} a_k^2.
\]

(ii) Assume \(\sum_{k=1}^{\infty} |E(\xi_0 \xi_k)| < \infty\). Then, \(f\) is continuous.

(iii) Assume that the spectral density \(f\) is continuous, and let \((d_{n,j})_{j \in \mathbb{Z}}\) be a double array of real numbers with \(d_n^2 = \sum_{j \in \mathbb{Z}} d_{n,j}^2 < \infty\) that satisfies the condition

\[
\frac{1}{d_n^2} \sum_{j \in \mathbb{Z}} |d_{n,j} - d_{n,j-1}|^2 \to 0. \tag{23}
\]

Then,

\[
\lim_{n \to \infty} \frac{1}{d_n^2} E \left( \sum_{j \in \mathbb{Z}} d_{n,j} \xi_j \right)^2 = 2\pi f(0). \tag{24}
\]

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