Entanglement in spin chains with gradients

Viktor Eisler\textsuperscript{1}, Ferenc Iglói\textsuperscript{2,3} and Ingo Peschel\textsuperscript{1}

\textsuperscript{1} Fachbereich Physik, Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
\textsuperscript{2} Research Institute for Solid State Physics and Optics, PO Box 49, H-1525 Budapest, Hungary
\textsuperscript{3} Institute of Theoretical Physics, Szeged University, H-6720 Szeged, Hungary
E-mail: eisler@physik.fu-berlin.de, igloi@szfki.hu and peschel@physik.fu-berlin.de

Received 28 October 2008
Accepted 2 December 2008
Published 3 February 2009

Abstract. We study solvable spin chains where either fields or couplings vary linearly in space and create a sandwich-like structure of the ground state. We find that the entropy of entanglement between two halves of a chain varies logarithmically with the interface width. After quenching to a homogeneous critical system, the entropy grows logarithmically in time in the $XX$ model, but quadratically in the transverse Ising chain. We explain this behaviour and indicate generalizations to other power laws.

Keywords: spin chains, ladders and planes (theory), entanglement in extended quantum systems (theory)

ArXiv ePrint: 0810.3788
1. Introduction

The entanglement between two parts of a quantum chain has been the topic of numerous recent studies [1]. For homogeneous systems, the entanglement entropy $S$ in the ground state has been found to be a quantity of order 1 if the system is non-critical, while it varies as $\ln L$ if it is critical. Here $L$ is the length of the subsystem, which can be either an interval in an infinite chain or half of a finite chain. The constant in front of the logarithm is proportional to the number of points of contact between the subsystems and to the central charge of the corresponding conformal field theory. In the vicinity of a phase transition, $S$ varies as $\ln \xi$, where $\xi$ is the correlation length, if $L \gg \xi$ [2]. There have also been studies of non-homogeneous systems. For example, if a defect separates the two subsystems, the prefactor of $\ln L$ varies with the defect strength in simple hopping models [3, 4], while for interacting electrons it scales either to zero or to the non-interacting value for large $L$ [5, 6]. Similarly, the central charge is modified in chains with aperiodic [7] or with random couplings [8]–[13].

In the present study we consider non-homogeneous systems of a different nature. We assume that one parameter in the Hamiltonian varies linearly along the chain and consider two cases. In an $XX$ model, we vary the strength of a magnetic field in the $z$-direction, while in a transverse Ising (TI) model we vary the couplings around the critical value. The first model constitutes a well-known problem, since it corresponds to free electrons hopping on a chain under the influence of a constant electric field. For a sufficiently large system, the central single-particle levels are then equidistant and form the famous Wannier–Stark ladder [14]–[18]. The situation has been realized experimentally in optical lattices subject to a constant acceleration [19, 20]. The second model was investigated recently with regard to its critical properties [21]. Physically, they have in common that the gradient terms introduce an interface into the ground state, and a length scale $\lambda$
which measures its width. In the hopping model, this interface separates regions where
the system is completely full and completely empty, respectively. In the TI model, it
separates ordered and non-ordered regions. This interface should have a strong influence
on the entanglement of the regions to the left and right of it. This is, indeed, what one
finds. The entanglement entropy becomes constant for large \( L \) and the asymptotic value is
determined by \( \ln(\lambda) \). Moreover, the deviation from the value in the homogeneous system
has a scaling form in the variable \( (L/\lambda) \).

We also discussed the time evolution that would ensue if the gradient was suddenly
removed. For the hopping model, it turns out that \( S(\lambda, t) \) depends only on the variable
\( (t^2 + \lambda^2) \) and thus can be obtained from the ground state entanglement with the field. This
means in particular that it varies logarithmically in time. For the TI model, the situation
is different and more interesting. Here one finds a quadratic increase of the entanglement
with time. Such a behavior has not been encountered before in such quenches, but we
show that it can be understood in a simple way using the quasiparticle picture of Calabrese
and Cardy [22].

In section 2 we review briefly the features of the Wannier–Stark problem. In section 3
we determine the correlation matrix from which \( S \) is calculated and present results for the
\( XX \) chain in its ground state. In section 4 the TI model is considered, again in its ground
state, while in section 5 results for the time evolution after the removal of the gradient
are given for both models. In section 6 we sum up our findings and in an appendix we
comment on more general spatial inhomogeneities and the derivation of the length scales.

2. Wannier–Stark problem

The problem of lattice electrons in a homogeneous electric field has been the subject of
many investigations. In the form of a simple one-dimensional tight-binding model it was
studied, for example, in [23], [16]–[18]. The equivalent spin one-half \( XX \) chain was treated
in [15]. The corresponding eigenvalue equation appears also in the treatment of certain
reaction–diffusion models [24]. The Hamiltonian is, for a finite system of \( 2L \) sites with
open ends,

\[
H = -\frac{1}{2} \sum_{n=-L+1}^{L-1} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + h \sum_{n=-L+1}^{L} (n - 1/2)c_n^\dagger c_n. \tag{1}
\]

Here the linear field is chosen such that it goes through zero between sites 0 and 1 and
thus is odd under a reflection of the chain. We will always assume \( h \geq 0 \). The eigenvalue
equation for the single-particle states \( |k\rangle \) then is

\[
-\frac{1}{2}[\phi_k(n - 1) + \phi_k(n + 1)] + h(n - 1/2)\phi_k(n) = \omega_k\phi_k(n), \tag{2}
\]

and its general solution is given by a linear combination of the Bessel functions \( J_{n-\kappa}(1/h) \)
and \( Y_{n-\kappa}(1/h) \). The argument of these functions defines the characteristic length \( \lambda = 1/h \),
which will be of central importance in the following. For a finite system, \( \kappa \) and
\( \omega_k = h(\kappa - 1/2) \) follow from the boundary conditions. The resulting spectrum is shown
on the left of figure 1 for \( h = 0.05 \) and three values of \( L \). One can see a linear region of
equidistant levels with spacing \( h \) in the centre, while the level separation becomes larger at
the upper and lower ends. Pictures of the corresponding eigenfunctions were first shown
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Figure 1. Wannier–Stark problem. Left: single-particle eigenvalues $\omega_k$ for $h = 0.05$ and three values of $L$. Right: density profiles in the ground state for $L = 50$ and three values of the field.

by Saitoh [16]. As $L$ increases, the regions with non-linear dispersion are moved towards $\pm \infty$ and only the Wannier–Stark ladder with integer $\kappa = k$ remains. The eigenfunctions then are $\phi_k(n) = J_{n-k}(1/h)$ and concentrated near site $k$ of the chain. In the ground state, the single-particle levels with $k \leq 0$ are occupied. The resulting density profiles are shown on the right of figure 1 for three values of the field. One can see that the transition from high to low density takes place in a region of width $2\lambda$.

3. Entanglement in the $XX$ chain

In the following, we consider chains of $2L$ sites as in section 2 and study the entanglement between their left and right halves in the ground state. The corresponding entanglement entropy is $S = -\text{Tr} (\rho \ln \rho)$ where $\rho$ denotes the reduced density matrix of one of the subsystems, e.g. the right half. Both $\rho$ and $S$ follow [25, 26] from the correlation matrix

$$C_{mn} = \langle c^\dagger_m c_n \rangle = \sum_{k=-L+1}^{L} \phi_k(m)\phi_k(n)n_k,$$

where the $\phi_k(m)$ are the single-particle eigenfunctions appearing in (2) and $n_k$ the corresponding occupation numbers. In the ground state, these are one for $k \leq 0$ and zero otherwise. Restricting the matrix to the sites of the subsystem, $1 \leq m, n \leq L$, and calculating its eigenvalues $\zeta_l$, one obtains $S$ as

$$S = -\sum_l \zeta_l \ln \zeta_l - \sum_l (1 - \zeta_l) \ln (1 - \zeta_l).$$

The calculation of the matrix and the diagonalization are done numerically.

We first show that the length scale $\lambda$ introduced by the gradient appears directly in the single-particle eigenvectors of the correlation matrix. In figure 2 we have plotted, for fixed $L$, the eigenvector corresponding to the $\zeta_l$ which is closest to $1/2$ and thus gives the largest contribution to $S$. In the homogeneous case, it decays slowly from the boundary between the subsystems but extends through the whole interior. If the gradient is large enough,
however, it becomes confined to a region near the boundary and the amplitude effectively vanishes at a distance $\lambda$. This is similar to the situation in a homogeneous non-critical system, for example a dimerized hopping model [27]. In that case, the corresponding scale is the correlation length.

We now turn to the entanglement entropy. In figure 3 it is shown as a function of $L$ for three values of $h$. For $h = 0$ one has the well-known logarithmic increase [2], but in a finite gradient, $S$ bends over after an initial rise and saturates rapidly. The change in the behaviour takes place if $L \approx \lambda$. This saturation is easy to understand since the parts of the system outside the interface region are either full or empty and cannot contribute to the entanglement.
The saturation value, calculated numerically for sizes $L = 5\lambda$, is shown in figure 4 as a function of $\lambda$. It varies essentially logarithmically, with additional decaying oscillations. One can fit the data perfectly with the following form:

$$S_\infty(\lambda) = \frac{1}{6} \ln(2\lambda) + \frac{k}{2} + A \frac{\cos(2\lambda)}{\lambda},$$

where $A \approx 1/4$ and $k = 0.726$ is the non-universal constant appearing in the entanglement entropy of the homogeneous $XX$ chain of length $2L$, divided in the middle:

$$S_{\text{hom}} = \frac{1}{6} \ln(4L/\pi) + \frac{k}{2}.$$

The appearance of the constant $k$ in (5) is intriguing. One could make the two formulae identical by introducing an effective length $\lambda_{\text{eff}} = \pi \lambda/2$ in (5), but this length would not have the simple interpretation of an interface width. In section 5.1 it will be seen that there is also a close relation of $S_\infty(\lambda)$ to the time-dependent entropy after a certain quench.

One should mention that in the case $L \to \infty$ the correlation matrix can be written down analytically for any finite portion of the chain. As mentioned above, one then has to deal only with the Wannier–Stark ladder states, where the eigenfunctions are single Bessel functions of integer order. Thus

$$C_{mn} = \sum_{k=0}^{\infty} J_{k+m}(\lambda) J_{k+n}(\lambda).$$

Using a sum rule, (7) can be rewritten as a simple product of Bessel functions. Dropping the arguments $\lambda$ it reads

$$C_{mn} = \frac{\lambda}{2(m - n)} [J_{m-1} J_n - J_m J_{n-1}].$$
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Figure 5. Scaling behaviour of the difference $S(L, h) - S(L, 0)$ as a function of $Lh$ for four system sizes. The full line in the right half is the function (10).

In the limit $\lambda \to \infty$, this reduces to the well-known result

$$C_{mn} = \frac{\sin(\pi(m-n)/2)}{\pi(m-n)}$$

for the homogeneous system. For general values of $\lambda$ it can easily be evaluated numerically. Since the Bessel functions become exponentially small for large order, taking $L \gg \lambda$ practically coincides with the result in the thermodynamical limit. Comparing with the numerical results using the exact eigenfunctions of (1), one also finds excellent agreement.

We have also investigated the scaling limit, where the ratio of the length scales $L/\lambda = Lh$ is kept fixed while $L \to \infty$. We have calculated the entropy difference $\Delta S = S(L, h) - S(L, 0)$ for several values of $L$. The results are shown in figure 5.

In the case $Lh \gg 1$ we are in the Wannier–Stark limit discussed above, and for large $L$ the curves tend towards a scaling function $f(Lh)$ even for $Lh$ values only slightly above 1. This function is given by the difference of the asymptotic $\lambda \to \infty$ form of (5) and the formula (6) for the entropy of a half-chain. Taking the difference one finds

$$f(x) = \frac{1}{6} \ln \frac{\pi}{2x}$$

with $x = Lh$. The maximum is $\Delta S_{\text{max}} = f(1) = \frac{1}{6} \ln(\pi/2) = 0.0753$ in agreement with the value in the figure, and thus relatively small. It mirrors the slight bulge in the curve for $S(L)$ seen in figure 3 before the asymptotic value is reached.

For $Lh \leq 1$ the interface region fills the whole system, but the curves still show a nice scaling behaviour and $\Delta S$ approaches zero quadratically as $Lh \to 0$.

4. The transverse Ising chain

We now consider the inhomogeneous quantum Ising chain with Hamiltonian

$$H = - \sum_{n=-L+1}^{L-1} J_n \sigma_n^z \sigma_{n+1}^z - h \sum_{n=-L+1}^{L} \sigma_n^x$$

$$\text{doi:10.1088/1742-5468/2009/02/P02011}$$
where $\sigma_n^x$ and $\sigma_n^z$ are the components of a Pauli spin operator associated with site $n$, $J_n$ is the nearest-neighbour exchange interaction and $h$ the transverse field. The couplings are assumed to vary as

$$J_n = J[1 + gn].$$

(12)

The homogeneous chain with $g = 0$ has a quantum critical point at $J = h$ in the thermodynamic limit, $L \to \infty$. Thus setting $h = J$, the inhomogeneous system with $g > 0$ is undercritical in the left half and overcritical, i.e. in the ordered phase, on the right. The length characterizing the transition region can be obtained from a scaling argument given in the appendix [21]. This leads to $\lambda(g) = ag^{-\omega}$, for small $g$. Here $\omega = \nu/(1 + \nu)$ where $\nu$ is the correlation length exponent. It enters because the perturbation drives the system away from criticality. With $\nu = 1$ for the TI model and choosing the constant $a$ equal to 1, we have $\lambda(g) = g^{-1/2}$. The result for the $XX$ chain can also be obtained in this way by using $\nu = \infty$ for the marginal perturbation one has there.

The entropy of entanglement between the two halves of the chain in its ground state is calculated again from the correlation functions by writing (11) in terms of fermions. Here one can work either in terms of Majorana operators [26] or of Fermi operators [13]. For $S$ as a function of $L$ one finds the same overall behaviour as in the $XX$ model, i.e. it rises logarithmically for small $L$ and saturates then. The main difference is the absence of additional oscillations. This also holds for the asymptotic values which are plotted in figure 6 as a function of $g^{-1/2}$. The function $S_\infty$ can be fitted by a simple logarithm

$$S_\infty(\lambda) = \frac{1}{12} \ln(2\lambda) + k_1.$$

(13)

The factor $1/12$ instead of $1/6$ corresponds to the central charge $c = 1/2$ of the TI model. The constant is given by $k_1 = 0.297$. In contrast to the $XX$ case, there seems to be no relation to the constant $0.239$ appearing in the entropy of the homogeneous model.

Figure 6. Limiting value of the entanglement entropy in a TI chain as a function of $\lambda(g) = g^{-1/2}$. For comparison we have also plotted $1/12 \ln x + k_1$ with $k_1 = 0.297$. 

Here one can work either in terms of Majorana operators [26] or of Fermi operators [13]. For $S$ as a function of $L$ one finds the same overall behaviour as in the $XX$ model, i.e. it rises logarithmically for small $L$ and saturates then. The main difference is the absence of additional oscillations. This also holds for the asymptotic values which are plotted in figure 6 as a function of $g^{-1/2}$. The function $S_\infty$ can be fitted by a simple logarithm

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Figure 7. The entropy difference $S(L, g) - S(L, 0)$ as a function of $x = Lg^{1/2}$ for several values of $L$. The lines show the limiting behaviour; see the text.

With this information, we can again analyse the finite-size behaviour of the entropy, which is expected to be

$$S(L, g) - S(L, 0) = f(L/\lambda).$$

Indeed, for large $L$ the entropy difference approaches a universal function, as shown in figure 7. For small argument the scaling function behaves as $S(x) \sim x^4 \sim g^2$, which follows from the fact that the entropy must be an even function of $g$. For large arguments the asymptotic behaviour is $f(x) \approx -1/12 \ln x + \text{const}$ which is in agreement with the result in equation (13). For intermediate values of $x$, the scaling function has a maximum at $x_{\text{max}} \approx 4.6$ but, in contrast to the XX case in figure 5, is non-singular for all finite values of $x$.

Instead of having the gradient in the couplings, one can also put it into the transverse field. In this case, due to duality, the ordered and disordered sides of the chain are reversed, but the size of the interface $\lambda(g)$ is expected from scaling theory to vary in the same way as before. Calculating the entanglement entropy, it is seen that $S$ has again the form (13) and also the scaling function in (14) approaches for large $L$ a universal function with the same type of limiting behaviour.

The three regimes of the system in a gradient, namely disordered (paramagnetic), interfacial and ordered (ferromagnetic) can also be probed by dividing the chain not in the middle but into two parts of length $\ell$ and $2L - \ell$ and calculating the corresponding entanglement entropy. In figure 8 we present results for the case where $g = 1/L$, which means that the coupling $J_n$ vanishes at the left boundary and equals 2 at the right one.

Both in the paramagnetic and in the ferromagnetic regime, the entropies approach a master curve as $L$ is increased. In the interface regime, the entropy has a maximum which is of the same order as the value for $\ell = L$. The shape of the curves is reminiscent of those one finds in a homogeneous system of different finite sizes, if one varies the coupling constant [2]. In fact, if one uses the coupling $J_{\text{hom}}(\ell) = J(1 - \ell/L)$ in the homogeneous system and calculates $S$ for a chain divided in the middle, the resulting curves are very
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Figure 8. Entanglement entropy for TI chains divided asymmetrically into two segments of length \( \ell \) and \( 2L - \ell \) for several system sizes and with \( g = 1/L \).

close to those shown in the figure, even in the interface region, provided the size of the homogeneous system is close to the size of that region. This equivalence can be understood as follows. The length scale in the gradient system is \( \lambda \); thus the correlation between two points having a distance larger that \( \lambda \) vanishes. Consequently, the contribution to the entropy at a given separation point \( x \) is coming from the sites in the range \([x - \lambda, x + \lambda]\). In this correlated domain, the couplings are varying only weakly and can be replaced by their average, which is just \( J_{\text{hom}}(\ell) \). Therefore the entropy is that of the homogeneous system with size \( \lambda \) and separation point in the middle.

5. Time evolution after a quench

The equilibrium results show that the main effect of the gradient is the introduction of a length scale, in terms of which the entropy still shows a logarithmic scaling. We now ask what happens if the gradient is suddenly switched off and a non-trivial time evolution of the state sets in.

5.1. XX chain

In the XX case, the fermionic operators evolve after the quench according to [28]

\[
    c_j(t) = \sum_m U_{jm}(t)c_m, \quad U_{jm}(t) = \sum_q \psi_q(j)\psi_q(m)e^{it\cos q},
\]

where the sum is over the allowed momenta \( q = \pi k/(2L + 1), k = 1, 2, \ldots, 2L \) for the homogeneous open chain, \( \psi_q(j) = L^{-1/2}\sin(q(j + L)) \) are the single-particle eigenfunctions and \( \omega_q = -\cos q \) the corresponding eigenvalues. Therefore the correlation matrix \( C(t) \) at time \( t \) is obtained by multiplying \( C(0) \) from the left and right by the matrices \( U^\dagger(t) \) and \( U(t) \), respectively. The entropy is then calculated from \( C(t) \) as before.

In figure 9 we show the resulting time evolution of \( S \) for various values of the initial gradient. For \( \lambda = 0 \) one is starting from a perfectly sharp domain wall. This situation has
Figure 9. Time evolution of the entanglement entropy in an $XX$ chain with $L = 150$ after switching off the gradient. The initial values were, from top to bottom, $\lambda = 50, 20, 10, 0$. The inset shows $S$ as a function of the variable $\tau = \sqrt{t^2 + \lambda^2}$.

already been studied with regard to the evolution of the density in [29, 30] and with respect to the particle-number fluctuations in [31]. The entanglement entropy was obtained in a DMRG calculation in [32], but not investigated further. The curve looks very much like the one in figure 4, on which we will comment presently. For larger $\lambda$ values, the initial entropy is higher, but the time evolution is also slower, and asymptotically all the curves seem to converge to the one with $\lambda = 0$. Moreover, introducing the new variable $\tau = \sqrt{t^2 + \lambda^2}$ one finds an exact collapse of the data, as shown in the inset.

This result can be derived analytically if one considers the limit $L \to \infty$. In this case, one can work with a ring instead of an open chain and the quantities $U_{jm}$ become Bessel functions. The equation for $C(t)$ then reads explicitly

$$C_{jl}(t) = i^{l-j} \sum_{m,n} i^{m-n} J_j(-m)(t)J_l(-n)(t)C_{mn}(0).$$  \hfill (16)

Furthermore the matrix $C(0)$ is given by (7) in this limit. Therefore one has sums of Bessel functions with two different arguments, $t$ and $\lambda$. Using their integral representations, the infinite sums over $m$ and $n$ can be carried out and one obtains

$$C_{jl}(t) = \sum_{k \geq 0} F_{j+k}^*(\lambda, t) F_{l+k}(\lambda, t),$$  \hfill (17)

where

$$F_n(\lambda, t) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{it\cos q - i\lambda \sin q + iqn}.$$  \hfill (18)

In this integral one can now rewrite the variables as $\lambda = \tau \cos \varphi$ and $t = \tau \sin \varphi$. The addition theorem for the trigonometric functions then yields

$$F_n(\tau, \varphi) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-i\tau \sin(q-\varphi)+i\varphi n} = e^{i\varphi n} J_n(\tau),$$  \hfill (19)

with $\varphi = \arctan(t/\lambda)$. 

doi:10.1088/1742-5468/2009/02/P02011
Thus, up to phase factors the quantities $F_n$ are Bessel functions with argument $\tau$, a fact which was not realized in [17]. At the level of the correlation matrix, the phase factors correspond to a simple unitary transformation $C \rightarrow U^\dagger C U$ and do not affect the entanglement entropy for which one finds the relation

$$S(\lambda, t) = S(0, \sqrt{t^2 + \lambda^2}).$$

(20)

The entropy thus depends only on the variable $\tau$ which is simply the distance from the domain-wall initial state in a space–time coordinate system where in the space direction we move to the equilibrium system with interface length $\lambda$ and then we further evolve this state in time. The lines of constant entropy are therefore circles in this quarter-plane. This explains in particular the result mentioned above that $S(\lambda, 0) = S(0, t)$ for $t \rightarrow \lambda$. Alternatively, for arbitrary time $t$ one could think of the evolving state as being effectively the ground state of a gradient problem with interface length $\lambda_{\text{eff}}(t) = \sqrt{\lambda^2 + t^2}$. Hence, one has a front propagating with a time-dependent speed $v(t) = d\lambda_{\text{eff}}/dt$.

Before closing this section we note that in finite systems one finds additional features at times $\tau = 2L, 4L, \ldots$, which are larger than those in figure 9. Then the entropy shows a step-like increase which can be attributed to the propagating fronts which return to the centre after being reflected at the open ends [33].

5.2. TI chain

In the TI chain, the calculation of the correlations using Majorana operators proceeds basically in the same way, but the time-dependent factors now contain the excitation energies $\omega_q = 2\sin(q/2)$ of the critical homogeneous system with $h = J = 1$. This gives a maximum velocity $v = 1$ for the excitations. In figure 10 the resulting entropy is shown for $g = 1/1024$ corresponding to $\lambda = 32$, and four different lengths which were all much larger than $\lambda$. One can see that $S$ increases up to $t \sim L$ and then drops again. The

Figure 10. Time evolution of the entanglement entropy in the TI chain after switching off a gradient $g = 1/1024$, for several values of $L$. The broken line is the fit using (21).
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increase is different from the linear law that one finds in homogeneous systems and can be fitted by a form

\[ S(t, g) = a(g)t^2 - b(g)t^3 + c(g), \]

(21)

where the cubic term represents a slight modification of the quadratic law. We have written \( a(g) \), \( b(g) \) and \( c(g) \), because for other \( g \) one finds the same time behaviour but with different coefficients. In particular \( a(g) \) is approximately linear in \( g \), i.e. \( a(g) = ag \) where \( a \approx 0.23 \).

The behaviour found above can be understood using the picture developed in [34] that at the quench, pairs of quasiparticles are emitted, which establish the entanglement between the parts of the system at later times. The number of these pairs must depend on the ‘distance’ of the initial state from the ground state of the final system. This has to be connected with the difference between the initial coupling constants and the critical value \( J = 1 \). The precise expression can be found by considering a quench from a homogeneous initial system with \( J \neq 1 \) and determining the coefficient \( \alpha \) in the linear law \( S(t) - S(0) = \alpha t \). This can be done numerically for our finite geometry, or by using the formula (2) in [35] for a segment in an infinite chain and dividing the result by 2. The latter approach gives \( \alpha \) as an integral over momenta

\[ \alpha(J) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} v_q H(y_q), \]

(22)

where \( v_q = \cos(q/2) \) is the velocity of the final quasiparticles,

\[ y_q = \frac{(J + 1) \sin(q/2)}{\sqrt{(J - 1)^2 + 4J \sin^2(q/2)}} \]

(23)

and

\[ H(y) = -\left[ \frac{1 + y}{2} \ln \left( \frac{1 + y}{2} \right) + \frac{1 - y}{2} \ln \left( \frac{1 - y}{2} \right) \right]. \]

(24)

Using partial integrations, this can be evaluated in closed form and gives

\[ \alpha(J) = \frac{|J - 1|}{\pi \sqrt{J}} \left[ \frac{\pi}{2} - \arctan \left( \frac{|J - 1|}{2\sqrt{J}} \right) \right] - \frac{(J - 1)^2}{2\pi J} \ln \left| \frac{J + 1}{J - 1} \right|. \]

(25)

For small \( |J - 1| \) this varies linearly, \( \alpha \approx |J - 1|/2 \), while for \( J \to \infty \) (or for \( J \to 0 \)) it approaches the saturation value \( 1/\pi \). The last term in (25) is non-analytic at \( J = 1 \), but a good approximation in the range \( |J - 1| \lesssim 0.2 \) is

\[ \alpha(J) = 0.475|J - 1| - 0.55(J - 1)^2, \]

(26)

which we will use for simplicity. A similar result follows from the direct calculation in the finite geometry.

The quantity \( \alpha/2 \) can now be used as the density of emitted pairs in the phenomenological formula of Calabrese and Cardy, which gives \( S(t) \), in a continuum approximation, as

\[ S(t) - S(0) = \frac{1}{2} \int_{-t}^{t} dx \alpha(J) = \alpha(J)t, \]

(27)

doi:10.1088/1742-5468/2009/02/P02011
where the integral counts all the pairs which end up in different halves of the system up to time \( t \). In an inhomogeneous system, the obvious generalization of this formula is

\[
S(t) - S(0) = \frac{1}{2} \int_{-t}^{t} dx \alpha(J(x)).
\]  

(28)

If the initial couplings vary along the chain as \( 1 + gn^\theta \), this gives

\[
S(t) - S(0) = \frac{0.475}{\theta + 1} t^{\theta+1} - \frac{0.55}{2\theta + 1} g^2 t^{2\theta+1}.
\]  

(29)

In the case \( \theta = 1 \), this has exactly the form (21) and moreover the coefficient of the \( gt^2 \) term equals 0.24 which is very close to the value 0.23 found by fitting. The agreement is also good for the cubic term where \( b(g)/g^2 \) equals 0.18 in (29) and 0.12 in the fit. Hence this formula describes the increase of \( S \) very well. In order to check it further, we have also studied the case \( \theta = 2 \), where the couplings increase quadratically from the centre. Then \( S(t) \) looks very similar, but a closer analysis shows that it varies basically as \( gt^3 \), which is again the prediction of (29). Also the coefficient has the correct value. Nevertheless one should point out that there is a region of small times where the numerical \( S(t) \) is rather flat and not so well described by the formulæ. This seems to hold up to \( t \sim \lambda \) and would mean that inside the interface region the picture has to be modified. Another remark concerns the form of equation (28). To leading order, it is an integral over \( |J - 1| \), which can also be interpreted as the (local) energy gap in the initial state. Such an expression was also used in a recent field theoretical treatment of inhomogeneous quenches; see equation (76) in [36]. In our approach we also obtain the exact prefactor on the lattice. However, the integral (22) from which it follows does not permit us to read off the result directly.

Finally, let us comment on the decrease of \( S(t) \) beyond \( t = L \). This is a feature of the finite geometry which one also finds in homogeneous quenches. There a linear increase of \( S \) is followed by an almost linear decrease and a zigzag variation of \( S(t) \) results for larger times. It can also be understood in the quasiparticle picture. Due to the open ends, the quasiparticles moving outwards are reflected at the ends and follow their inward moving partners with a certain delay. As soon as they cross the middle, their contribution to the entanglement between left and right vanishes. This effect sets in when the quasiparticles from the ends arrive at the centre, because their partners follow immediately.

6. Summary

We have studied particular inhomogeneous systems where a power-law variation of some parameter introduces an interface with a certain width \( \lambda \). Beyond that region the ground state approaches a product form. This suggests that instead of the real length of the system, only \( \lambda \) should enter the entanglement properties. In fact, we found that both for the \( XX \) model and the TI model the entropy in large systems is given by conformal expressions where \( \lambda \) appears in the logarithms. This result is also plausible because one knows that the entanglement is connected with the interface between the two subsystems that one considers. One finds the same \( \ln \lambda \) behaviour in the \( q \)-symmetric \( XXZ \) Heisenberg chain, where the interface is produced by boundary fields [37] and the reduced density matrix is known explicitly [38]. It is also somewhat similar to the situation

doi:10.1088/1742-5468/2009/02/P02011

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for non-critical states, where the correlation length appears in the formulae. However, in our case, there is no translational invariance. For finite lengths $L$ we have also shown that the entanglement entropy has a scaling form in the variable $L/\lambda$.

The time evolution after a removal of the gradient turned out to be particularly interesting. In the $XX$ case, it is logarithmic, as found usually in local quenches [28, 22, 33]. Moreover, it displays a particular space–time symmetry relating static and dynamic entanglement. Formally, this results because both the single-particle states in the Wannier–Stark problem and the time evolution in the final homogeneous system are given by Bessel functions. Thus if one can treat, for example, the quench from a sharp interface, one has found the solution for all gradients and all times. However, one cannot apply the CFT approach of [39] since the walls there are of a different nature and lead to a linear behaviour of $S(t)$.

In the TI model, the time evolution after the quench turned out to be quadratic, a result not encountered before. We were able to explain this in the simple quasiparticle picture of [34] and could even give the numerical constants. We mentioned that more general power-law variations of the couplings lead to analogous results which can be understood in the same way. In a sense, this explanation works better than one might expect, because the assumption that quasiparticles are only emitted from nearby sites is not well founded near a critical point. Thus a more direct derivation following the lines of [36] would certainly be useful and interesting.

**Acknowledgments**

FI is indebted to the Freie Universität Berlin for hospitality during the starting period of this research. His work was supported by the Hungarian National Research Fund under grant No. OTKA TO48721, K62588 and K75324.

**Appendix**

Although our main concerns in this study were linearly varying parameters, similar results are obtained for other power laws. Consider a variation of the couplings in the transverse Ising model of the form

$$J_n = J[1 + g|n|^\theta].$$ (A.1)

This leads to a length scale $\lambda$ which can be estimated as follows. The typical deviation of the couplings from the critical value is $\Delta(\lambda) \sim Jg\lambda^\theta$, which leads to a length scale $\xi \sim \Delta(\lambda)^{-\nu}$. Since in the problem there is only one length scale, the width of the interface, we have $\xi \sim \lambda$ from which one obtains the self-consistency equation

$$\lambda \sim [g\lambda^\theta]^{-\nu},$$ (A.2)

with the solution:

$$\lambda = a g^{-\nu/(\theta+1)}.$$ (A.3)

Using $\nu = 1$ the exponent is $\omega = 1/((\theta + 1)$ which gives $1/2$ for the linear variation considered in the main text.

As an example, let us consider a quadratic variation, $\theta = 2$. Physically, this means that for $J = h$ and $g \geq 0$ the system is critical in the centre and ordered more and more as
one moves towards the ends. Thus one has a kind of sandwich structure with the thickness of the central part varying as $\lambda \sim g^{-1/3}$. Calculating the entropy, one finds again that it saturates for large $L$ and the asymptotic value varies as in (13). The constant now has the value $k_2 = 0.548$ if one sets $a = 1$ in (A.3). The difference $S(L, g) - S(L, 0)$ is shown in figure A.1.

The scaling function resembles that of the $XX$ model in that it also has a cusp separating two different regimes. For small arguments it varies as $x^3$, i.e. it is proportional to $g$. This behaviour is possible here since $g < 0$ and $g > 0$ are not equivalent. For large arguments it is given by $-1/12 \ln x + \text{const.}$

The case $\theta = 1$ corresponding to a linear ‘trapping potential’ for the central part gives very similar results. The length is now $\lambda = g^{1/2}$ as in section 4, but the scaling function looks qualitatively as in figure A.1. However, for small arguments the behaviour is quadratic.

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