Quantum search for multiple items using parallel queries

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Abstract. In the quantum database search problem we are required to search for an item in a database. In this paper, we consider a generalization of this problem, where we are provided \(d\) identical copies of a database each with \(N\) items which we can query in parallel. Then, given \(k\) items, we are required to determine the locations where these items are stored. We show that any quantum algorithm for this task must perform \(\Omega\left(\sqrt{\frac{Nk}{d \min\{d,k\}}}\right)\) parallel queries. We also design a simple algorithm whose performance comes within a factor \(O(\log d)\) of this lower bound.

Our lower bound can be considered to be a generalization of a result of Zalka\textsuperscript{7} who considered the case \(k=1\) and \(d\) arbitrary. Our upper bound can be considered to be a generalization of the following two results: first, a result of Boyer, Brassard, Hoyer and Tapp\textsuperscript{2} who showed how to search for one of \(k\) items in a database, and second, a result of Heiligman\textsuperscript{4}, which showed that \(O(\sqrt{Nk})\) queries suffice for locating all items.

1 Introduction

In the database search problem, we are given an item and are required to find the location where it is stored. The goal is to perform the task making as few queries to the database as possible. The quantum database search algorithm of Grover\textsuperscript{3} shows that this task can be performed with \(O(\sqrt{N})\) queries to the database, where \(N\) is the size of the database. On the other hand it is well known that no classical (randomized) algorithm can perform this task with less than \(\Omega(N)\) queries.

Zalka\textsuperscript{7} showed that Grover’s algorithm for database search is optimal. Furthermore, he showed that if we are provided access to multiple databases, and allowed to query them in parallel, then the quantum algorithm that divides all locations equally among the databases and performs parallel independent searches cannot be improved. Thus, quantum search algorithms for searching one item using parallel queries is well understood.

The other extreme, when there are multiple items to be searched using queries to just one database has also been studied. Boyer, Brassard, Hoyer and Tapp\textsuperscript{2} showed that if it is known in advance that exactly \(k\) of the items are present in the database, then using \(O(\sqrt{N})\) queries, one can locate one of these items. Building on this, Mark Heiligman\textsuperscript{4} observed that one can determine the locations of all \(k\) items in time \(O(\sqrt{Nk})\).

The general problem of searching for multiple items using parallel queries does not seem to have been addressed before. In this paper, we consider the following problem (see
We are given \( d \leq \sqrt{N} \) copies of a database of \( N \) items. We are given \( k \leq \sqrt{N} \) items and promised that the database contains all of them. We are required to find the locations of all these items. We derive a lower bound for this general problem and show an algorithm with performance close to this lower bound. Let \( q(N, d, k) \) be the minimum number of parallel queries made by any quantum algorithm for the above task. Our results imply that there exists constants \( c_1 > 0 \) and \( c_2 > 0 \) such that for all large \( N \)

\[
\sqrt{\frac{Nk}{d \min\{k, d\}}} \leq q(N, d, k) \leq \sqrt{\frac{Nk \log d}{d \min\{k, d\}}}.
\]

In fact, for certain ranges of \( k \) and \( d \), the upper bound and lower bound differ only by a constant factor. See Theorem 3 below for the precise statements of our upper bound.

### 1.1 Background and notation

We assume that the reader is familiar with the basics of quantum circuits, especially the quantum database search algorithm of Grover [3] (see, for example, Nielsen and Chuang [6, Chapter 6]).

**Database search:** The database is modeled as a function \( f : \{0,1\}^n \rightarrow \{0,1\}^m \). The elements of \( \{0,1\}^n \) will be referred to as addresses, and will be identified this set with \([N] = \{0,1,2,\ldots,N-1\}\), where \( N = 2^n \). We refer to elements of \( \{0,1\}^m \) as items. We say that \( y \in \{0,1\}^m \) is in the database \( f \) if there is some \( x \in \{0,1\}^n \) such that \( f(x) = y \).

In the quantum model, this database is provided to us by means of an oracle unitary transformation \( T_f \) which acts on an \((n+m)\)-qubit space by sending the basis vector \(|x\rangle|z\rangle\) to \(|x\rangle|z \oplus f(x)\rangle\), where \( x \in \{0,1\}^n \) and \( z \in \{0,1\}^m \). Our database search can then be formulated as follows.

**Input:** We are allowed access to \( d \) copies of the database \( f \), via the unitary transformation

\[
T_f^{\otimes n} : |x_1\rangle|z_1\rangle|x_2\rangle|z_2\rangle\cdots|x_d\rangle|z_d\rangle \mapsto |x_1\rangle|z_1 \oplus f(x_1)\rangle|x_2\rangle|z_2 \oplus f(x_2)\rangle\cdots|x_d\rangle|z_d \oplus f(x_d)\rangle.
\]

We are given \( k \) distinct items \( y_1, y_2,\ldots,y_k \in \{0,1\}^m \) in \( k \) registers.

**Promise:** All the \( y_i \)'s are in the database \( f \).

**Goal:** To devise a quantum circuit with the minimum number of applications of \( T_f^{\otimes n} \) in order to determine the location of each item.

\*Fig. 1.* The problem

In the following, we assume that \( N \) is large and that \( d \) and \( k \) are much smaller than \( N \), say less than \( \sqrt{N} \). Also, when we say that a quantum algorithm solves a certain problem, we mean that it returns the correct answer with probability at least \( \frac{3}{4} \).
2 Lower bound

To present our lower bound argument, it will be convenient to reformulate the problem slightly so that standard lower bound techniques can be applied directly. For a function $f : \{0,1\}^n \rightarrow \{0,1\}^m$, let $f^{\otimes d} : \{0,1\}^{dn} \rightarrow \{0,1\}^{dm}$ be defined by

$$f^{\otimes d}(x_1, x_2, \ldots, x_d) = (f(x_1), f(x_2), \ldots, f(x_d)).$$

Thus, $f^{\otimes d}$ is a database with $dn$-bit addresses and $dm$-bit items, and we can associate as before the unitary transformation $T_{f^{\otimes d}}$ with it acting on the $(dn + dm)$-qubit space. Also, we have (with the natural reordering of coordinates) $T_{f^{\otimes d}} = T_{f^{\otimes d}}$. It is easy to see that if we have an efficient solution to the problem in Figure 1, then we have a solution with essentially the same complexity for the following problem.

**Input:** We are given a database $F : \{0,1\}^{dn} \rightarrow \{0,1\}^{dm}$ such that $F = f^{\otimes d}$ for some (unique) $f : \{0,1\}^n \rightarrow \{0,1\}^m$. We are given access to $F$ via the unitary transformation $T_F$. We are given $k$ distinct items $y_1, y_2, \ldots, y_k \in \{0,1\}^m$ in $k$ registers.

**Promise:** Either all $y_i$'s are in $f$ or exactly $k-1$ of them are in $f$.

**Goal:** To devise a quantum circuit with the minimum number of applications of $T_F$ in order to determine (with high probability) if all $y_i$'s are in $f$.

**Fig. 2. The reformulated problem**

With this formulation we can state our lower bound result.

**Theorem 1.** Let $k \leq 2^{m-1}$. Any quantum circuit for solving the problem in Figure 2 requires

$$\Omega\left(\sqrt{\frac{Nk}{d \min\{d, k\}}}\right).$$

applications of the transformation $T_F$.

We will make use of the following special case of a result of Ambainis [1] (see also Laplante and Magniez [5]).

**Theorem 2.** Let $G = (V_0, V_1, E)$ be a bipartite graph whose vertices are databases. The edges of this database have labels. The edge $(F_0, F_1)$ connecting databases is labeled by all addresses $a$ such that $F_0(a) \neq F_1(a)$. Let $\Delta_0$ be the minimum degree of a vertex in $V_0$ and $\Delta_1$ be the minimum degree of a vertex in $V_1$. Let $\ell_0$ be the maximum number of edges incident on a fixed vertex $F_0 \in V_0$ with the same address. Similarly, let $\ell_1$ be the maximum number of edges incident on a fixed vertex $F_1 \in V_1$ labeled with the same address. Suppose there is a quantum circuit that returns the value 0 with high probability for databases in $V_0$ and returns the value 1 with high probability for databases in $V_1$. Then, this circuit must contain

$$\Omega\left(\sqrt{\frac{\Delta_0 \Delta_1}{\ell_0 \ell_1}}\right).$$
applications of the unitary transform $T_F$.

With this, we are now ready to prove our lower bound.

Proof. (of Theorem 1) Fix any $k$ distinct items $y_1, y_2, \ldots, y_m \in \{0,1\}^m - \{0^m\}$. Let $V_0$ consist of all databases $f_0 \otimes d_0$ where $f_0$ ranges over databases that contain exactly $k - 1$ of the $y_i$’s and the other $2^m - k + 1$ locations contain the item $0^m$. Thus, there are exactly, $\binom{2^n}{k-1}k!$ vertices in $V_0$. Similarly, $V_1$ consists of databases of the form $f_1 \otimes d_1$ where $f_1$ has all $y_i$’s and the other locations contain $0^m$. The pair $(f_0 \otimes d_0, f_1 \otimes d_1)$ is an edge iff $f_0$ and $f_1$ differ in exactly one location.

Clearly, the quantum circuit returns the answer 0 with high probability for databases in $V_0$ and returns the answer 1 with high probability for databases in $V_1$. To get our lower bound, it remains only to compute the items $\Delta_0, \Delta_1, \ell_0$ and $\ell_1$ for this graph. It is easy to check that $\Delta_0 = N - k + 1$ (there are these many locations in $[N]$ where we can introduce the missing item) and $\Delta_1 = k$ (there are these many ways to delete a item).

To determine $\ell_0$, fix a database $f_0 \otimes d_0 \in V_0$ and a location $(x_1, x_2, \ldots, x_d)$, suppose the missing item in $f_0$ is $y_1$. Let the item stored at this location be $(v_1, v_2, \ldots, v_d)$. In an adjacent database, $f_1 \otimes d_1$, if the contents of this location are different, then one of the $v_i$’s ($d$ possibilities) must change from $0^m$ to $y_1$. It is easy to verify that there are at most $d$ choices for $f_1 \otimes d_1$.

To determine $\ell_0$ fix a database $f_0 \otimes d_0 \in V_1$ and a location $(x_1, x_2, \ldots, x_d)$. Let the item stored at this location be $(v_1, v_2, \ldots, v_d)$. In an adjacent database, $f_0 \otimes d_0$, if the contents of this location are different, then one of the $v_i$’s must change from being a $y_i$ and become $0^m$. It is easy to verify that we have at most $\min\{d, k\}$ choices for $f_0 \otimes d_0$.

Our claim now follows immediately from Theorem 2.

3 The algorithm

In this section, we design a quantum algorithm with performance close to the lower bound proved in the previous section.

Lemma 1. Given one copy of the database of size $N$, and a promise that at most $t$ items out of $y_1, y_2, \ldots, y_k$ are in the database, we can determine the location of all these $t$ items using $O(\sqrt{Nt})$ queries.

Proof. The algorithm of Figure 3 makes a total of $O(\sqrt{Nt})$ queries.

Let $Y = \{y_1, y_2, \ldots, y_k\}$. Perform the following step for $i = 1, 2, \ldots, t$:

Step $i$: Search for an item from $Y$ in the database assuming that there are $t - i + 1$ of them in the database. If an item $y$ is found, set $Y \leftarrow Y \setminus \{y\}$. This requires $O\left(\sqrt{\frac{N}{k-i+1}}\right)$ queries to the database.
We can now state our upper bound.

**Theorem 3.** 1. If $k \leq \sqrt{d}$, then the locations of all $k$ items can be determined with $O\left(\sqrt{\frac{N}{d}}\right)$ parallel queries.

2. If $\sqrt{d} < k \leq d\lg d$, then the locations of all $k$ items can be determined with

$$O\left(\sqrt{\frac{Nk\lg d}{d\min\{k,d\}}}\right)$$

parallel queries.

3. If $d\lg d < k$, then the location of all $k$ items can be determined with

$$O\left(\sqrt{\frac{Nk}{d}}\right)$$

parallel queries.

**Proof.** If $d$ is a small constant, Lemma 1 already implies our theorem. So, we assume that $d$ is large. The idea is to ensure that different databases are used to search different parts of the addresses. However, to balance the load on the different copies, we assign random subsets of addresses to the different databases. More formally, we randomly partition the address set $[N]$ into $d$ disjoint sets each of size $N/d$. Then, we search these sets in parallel dedicating one database for each. The probability that any set has more than $t$ items is at most

$$\binom{k}{t} \left(\frac{1}{d}\right)^t.$$  \hspace{1cm} (1)

We will choose the value of $t$ so that this quantity is much less than $1/d$, so that with constant probability each set has at most $t$ items. Then, we can apply Lemma 1 and obtain an algorithm that makes $O(\sqrt{Nt/d})$ parallel queries, and with constant probability determines the location of all items. We repeat the algorithm several times to reduce the probability of error.

$k \leq \sqrt{d}$: Take $t = 2$, and conclude from (1) that the probability that any one set has at least two items is at most $1/(2d)$. Since, there are at most $d$ sets, with probability $\frac{1}{2}$ all sets have at most 1 item.

$\sqrt{d} < k \leq d\lg d$: In this case, take $t = 5\lg d$ and conclude from (1) that the probability that some set has at least $t$ elements is at most $\frac{1}{d}$. Thus with probability at least $\frac{1}{2}$ all sets have fewer than $5\lg d$ elements.

$d < k \leq d\lg d$: In this case, take $t = \frac{5k\lg d}{d}$. The rest of the arguments is the same as before.

$k > d\log d$: In this case, take $t = \frac{2k}{d}$. The rest of the argument is the same as before.

It thus follows that in all cases our claim holds.
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