Cosmological magnetic fields from photon coupling to fermions and bosons in inflation

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Abstract

We consider several gauge invariant higher dimensional operators that couple gravity, gauge fields and scalar or fermionic fields and thus break conformal invariance. In particular, we consider terms that break conformal invariance by the photon coupling to heavy and light fermions. While the coupling to heavy fermions typically do not induce significant magnetic fields, the coupling to light fermions may produce observable magnetic fields when there are a few hundred light fermions. Next we consider Planck scale modifications of the kinetic gauge terms of the form $f(\phi) F_{\mu\nu} F^{\mu\nu}$ and $h(\psi) F_{\mu\nu} \tilde{F}^{\mu\nu}$, where $f$ and $h$ are functions of scalar and pseudoscalar fields $\phi$, $\psi$, and $F_{\mu\nu}$, $\tilde{F}^{\mu\nu}$ and the gauge field strength and its dual, respectively. For a suitable choice of $f$ sufficiently strong magnetic fields may be produced in inflation to be potentially observable today. The pseudoscalar coupling may lead to birefringence in inflation, but no observable magnetic field amplification. Finally, we show that the photon coupling to metric perturbations produces by far too weak fields to be of cosmological interest.
1 Introduction

Inflation [1] is at this moment the only paradigm that offers explanation of structure formation consistent with current microwave background [2, 3] and large structure observations, as well as the observed homogeneity and isotropy of the Universe on large scales and the flatness, horizon and monopole problems. In this paper we consider several mechanisms in which conformal invariance of the gauge fields in inflation is broken and discuss under what conditions inflation can produce magnetic fields on cosmological scales. These fields may offer an explanation for the seeds of the micro-gauss magnetic fields observed in spiral arms of galaxies today [4], but also for the intergalactic magnetic fields that might influence the cosmological microwave background radiation (CMBR) and early structure formation. At the moment there is only an upper bound on cosmological magnetic fields of about $10^{-9}$ Gauss based on the current CMBR measurements [5]. In this paper we argue that one can obtain observable cosmological magnetic fields from inflation. Hence, magnetic fields should be included into the analyses of the upcoming high precision CMBR measurements, in particular in the measurements that include polarization. To support this point of view we note that there is now evidence for intergalactic micro-Gauss magnetic field [6], and for magnetic fields of strength $10^{-9}$ Gauss correlated on 1 Mpc [7] from statistical correlations between the positions of high-energy cosmic rays and BL Lacertae sources (quasars with jets).

It is well know that quantum loop corrections to general relativity induce corrections to Einstein’s gravity [8] in curved space-time backgrounds. Based on this observation Turner and Widrow [9] have argued that breakdown of conformal invariance in inflation may be responsible for large scale magnetic fields that are potentially observable. In particular they considered (i) the gravitational coupling $\mathcal{R}A^\mu A_\mu$, $\mathcal{R}^{\mu\nu}A_\mu A_\nu$, that induce a photon mass at the price of breaking gauge invariance, (ii) gauge invariant dimension six terms $\mathcal{R}_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} / m^2$, $\mathcal{R}^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} / m^2$, $\mathcal{R} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} / m^2$, and (iii) couplings of the photon field to other fields (scalars and axions). Here $\mathcal{R}^{\mu\nu\rho\sigma}, \mathcal{R}^{\mu\nu}, \mathcal{R}$ denote the curvature tensors and scalar, respectively, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the field strength, and $m$ is a mass parameter. The result of their analysis is that Type (i) models may lead to strong magnetic fields today, because the photon couples directly to the curvature, which is large in inflation, $\mathcal{R} \sim H^2$, and sufficiently small today so that it does not conflict with the current observational bounds on the photon mass. The analyses of Type (ii) and (iii) models were inconclusive.
In Ref. [10], Dolgov has shown that the trace anomaly due to the coupling of gauge fields to (massless) fermions may lead to gauge field amplification in inflation. Ratra [11] has considered the effect of the inflaton coupling to gauge fields of the form $e^{-\lambda\phi/M_P} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}$ and showed that one can obtain large scale magnetic fields in extended inflation. This idea was then reconsidered in the context of string cosmology by Gasperini, Giovannini and Veneziano [12] and by Lemoine and Lemoine [13], where the role of the inflaton is taken by the dilaton field. Garretson, Field and Carroll [14] have considered coupling of the electromagnetic field to a pseudoscalar boson; their conclusion was that it is difficult to obtain large scale magnetic fields in this mechanism. Mazzitelli and Spedalieri [15] have reconsidered Type (ii) models of Ref. [9] in the light of the Schwinger-DeWitt expansion [16] for a charged scalar field and argued that one cannot obtain large magnetic fields from these interactions. Calzetta, Kandus and Mazzitelli [17] have considered the charged scalar current in scalar electrodynamics as a source for large scale magnetic fields; their results have been contested by Giovannini and Shaposhnikov [18]. Magnetic field production from the dynamics of extra dimensions has been considered in [19]. Finally, Maroto [20] has recently suggested that coupling of gauge fields to metric perturbations may lead to significant gauge field amplification.

In Ref. [21] we have shown that the backreaction of superhorizon scalar fields breaks conformal invariance of gauge fields in inflation, leading to a gauge field generation mechanism. Alternatively, the photon may acquire a mass in inflation by a Higgs mechanism [22], when a charged scalar field gets an expectation value driven by a negative coupling term, $V_{\text{neg}} \sim -g s^2 \Phi^* \Phi$ to the inflaton field $s$. When applied to electromagnetism, this then results in magnetic field generation with the spectrum $B_\ell \propto \ell^{-1}$, where $\ell$ is the correlation length. The resulting field is sufficiently strong [21, 23, 24] to seed the galactic dynamo mechanism in flat universes with dark energy component. The galactic dynamo may be the mechanism that explains the micro-Gauss magnetic fields observed in many galaxies today [4]. An alternative is the Biermann battery [25] which may be operative on megaparsec scales at galaxy formation [26].

Magnetic fields may also be generated at cosmological phase transitions [27], as a consequence of hypermagnetic field amplification in presence of the right handed electrons through the Abelian triangle anomaly [28, 29], and in presence of a pseudoscalar field condensate [30].
These causal mechanisms create magnetic fields which are correlated typically on too small scales to be of relevance for galactic and super-galactic magnetic fields today \cite{31}. For a review of other models for cosmological magnetic field generation see \cite{32}.

In this paper we focus on mechanisms for magnetic field generation from terms induced by gauge field couplings to other quantum fields in presence of gravity. We restrict ourselves to terms that break conformal invariance in gauge invariant manner. The paper is organized as follows. In section 2 we consider the evolution of gauge fields is presence of heavy fermions in a curved space-time background. We make use of the local Drummond-Hathrell action \cite{8} and find that magnetic field amplification in inflation is typically very small, unless the number of massive fermions $N_F$ is very large, that is $N_F \gtrsim 10^5$. Next, in section 3 we consider the anomalous gauge field coupling to light fermions \cite{10} and find a potentially observable amplification of gauge fields in inflation, provided the number of light fermions is of the order a few hundred. In section 4 we then consider how modification of the gauge kinetic terms $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ and $-\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$ through the coupling to scalar and pseudoscalar fields may affect the dynamics of gauge fields in inflation. We find that the scalar coupling may result in significant amplification of gauge fields \cite{11}. On the other hand, the coupling to pseudoscalar fields leads to an interesting example of birefringence, but results in no observable amplification of magnetic fields. We then in section 5 show that gauge field coupling to scalar cosmological perturbations \cite{20} can lead only to a very minute amplification of gauge fields in inflation. In section 6 we briefly discuss preheating, and section 7 contains conclusions.

## 2 Photon coupling to gravity with heavy fermions

### 2.1 Drummond-Hathrell action

We now reconsider the effect of coupling of gauge fields to gravity, corresponding to Type (ii) models of Ref. \cite{9}, and argue that one cannot obtain strong magnetic fields from inflation. Drummond and Hathrell \cite{8} have shown that the scatterings of photons on gravitons in presence of massive fermions with a mass $M$ (shown in diagrams on figure 1) induce a correction to the Einstein-Hilbert action which, when computed in the $1/M^2$ expansion,
leads to the following effective action

\[
S = \int \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{4} g^{\mu
u} g^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \mathcal{L}_1 \right]
\]

\[
\mathcal{L}_1 = \frac{-\beta_e}{4M^2} \left[ b R g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + c R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + d R^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right]
\]

where \( G = (8\pi M_P^2)^{-1} \) denotes Newton’s constant, \( \beta_e = \alpha_e N_F/180\pi, \alpha_e = e^2/4\pi, \) where \( e \) is the photon coupling constant at the scale relevant in inflation, \( N_F \) the number of fermionic species with mass \( M \), \( b = -5 \), \( c = 26 \) and \( d = -2 \), \( R \) denotes the curvature scalar, \( R^{\mu\nu} \) and \( R^{\mu\nu\rho\sigma} \) the curvature tensors, \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) the gauge field strength, \( g = \det[g_{\mu\nu}] \) the determinant of the metric \( g_{\mu\nu} \), and \( M \) the fermion mass. In case when the gauge field is non-Abelian with an SU(N) symmetry, \( N_F \) should be replaced by \( N_F - 11N/2 \) \([10]\). Coupling to non-Abelian gauge fields thus reduces \( \beta_e \) and, as we shall show, damps the gauge field in inflation. This is contrary to fermionic loops, which induce amplification of gauge field. The action (1) is computed in the \( 1/M^2 \) expansion, and hence should be relevant for the photon field dynamics only when the curvature scale, which is characterized by the Hubble parameter \( H \), is small in comparison to the mass scale, \( |R| \sim H^2 \ll M^2 \). Moreover, the physical photon momentum \( k_{\text{phys}} \) should satisfy \( k_{\text{phys}}^2 \ll M^2, |R| \). When \( |R| \ll k_{\text{phys}}^2 \ll M^2 \) the term \( -(6\beta_e/M^2)g^{\mu\nu}D_{\rho}(g^{\nu\sigma}F_{\mu\rho})g^{\sigma\xi}D_{\xi}F_{\sigma\nu} \), induced by the one-loop vacuum polarization diagram in figure 2, may become relevant. For large physical momenta, \( k_{\text{phys}}^2 \gg M^2 \), the \( 1/M^2 \) expansion in (1) breaks down, and the fermions should be treated as massless. Dolgov \([10]\) has shown that in this case the fermion loop in figure 2 contributes to the stress-energy trace anomaly, which we consider in section 3. The problem of magnetic field growth in inflation in terms of the Schwinger-DeWitt expansion \([10]\) for scalar fields has been studied in \([13]\).

We shall now study the dynamics of gauge fields given by the Drummond-Hathrell action (1) in conformally flat space-times. Recall that inflation, radiation and matter era are all conformally flat space-times, whose metric can be parametrized as \( g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu} \), where \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \) denotes the Minkowski metric, and the scale parameter \( a = a(\tau) \) is a function of conformal time \( \tau \).

The equation of motion for the time-like field \( A_0 \) is nondynamical, and can be easily inferred from equations (1) and (96) :

\[
\nabla^2 A_0 - \partial_\tau \nabla \cdot \vec{A} = 0. \tag{2}
\]
Figure 1: The scattering diagrams that contribute to the Drummond-Hathrell action. The gravitons \( G_{\mu\nu} \) are represented by the dashed blue lines, the photons \( A_\mu \) by the wavy red lines and the fermions \( f \) by the solid green lines.

Figure 2: The one-loop diagram contributing to the higher order in derivatives dimension six operator of the Drummond-Hathrell action.

which can then be used to remove \( A_0 \) from the dynamical \( A_1 \)-equation. The resulting equation is purely transverse, and identical to one obtained in Coulomb gauge, \( A_0 = 0 = \nabla \cdot \vec{A} \). In Appendix A we show that the dynamical photon field equation can be recast as (cf. Eq. (101))

\[
\partial_\tau \left[ \left( 1 + 2\beta_e \frac{H^2}{M^2} (26 + 33w) \right) \partial_\tau A_i^T \right] - \left( 1 - 2\beta_e \frac{H^2}{M^2} (10 + 3w) \right) \nabla^2 A_i^T = 0, \tag{3}
\]

where \( w = p/\rho \), \( \beta_e = \alpha_e N_F / 180\pi \). In case when there are fermions with different masses, \( \beta_e / M^2 \) should be replaced by \( \sum_i \beta_i / M_i^2 \), where \( \beta_i = \alpha_e N_{F_i} / 180\pi \). For example, when the number of charged massive fermions is of the order \( N_F \sim 10^3 \), which is to be expected in a typical grand-unified theory, and taking \( \alpha_e \sim 1/30 \), we then have \( \beta_e \sim 1/15 \). Since typically \( H/M \ll 1 \), the terms that break conformal invariance are small. We shall now make this statement more quantitative.
2.2 Photon field in inflation

Note first that in de Sitter inflation \((w \equiv p/\rho = -1)\) Eq. (3) becomes conformally invariant

\[
1 - \frac{14\beta_e H^2}{M^2} \left( \partial^2 - \nabla^2 \right) A_i^T = 0,
\]

(4)

which is in agreement with Ref. [8]. Hence, in de Sitter inflation there is no photon field amplification. For power law inflation however conformal invariance is broken. Consider now the homogeneous mode, for which equation (3) is solved by

\[
\partial_\tau A_i^T = C_0 \left( 1 + 2\beta_e \left( \frac{H^2}{M^2} (26 + 33w) \right)^{-1} \right),
\]

(5)

where \(C_0 = C_0(\vec{x})\) characterizes the initial electric field in inflation. For \(w < -26/33\) the field \(\partial_\tau A_i^T\) grows in inflation. When \(2\beta_e (H^2 M^2)(26 + 33w) \approx -1\) a large amplification may result. In this case adiabaticity is broken for superhorizon modes, leading to amplification.

To make this more quantitative, we now quantize the field as follows

\[
\vec{A}(\tau, \vec{x}) = a^{-\zeta} \sum_{T=1,2} \int \frac{d^3k}{(2\pi)^3} \left[ e^{i\vec{k} \cdot \vec{x}} A^{(1)}_k (\tau) \epsilon^T_k a^T_k + e^{-i\vec{k} \cdot \vec{x}} A^{(2)}_k (\tau) \epsilon^T_k a^\dagger_k \right],
\]

(6)

where \(A^{(i)}_k (i = 1, 2)\) denote the gauge field mode functions, \(\epsilon^T_k (T = 1, 2)\) are the transverse polarization vectors, and \(a^T_k\) and \(a^\dagger_k\) the photon field annihilation and creation operators, respectively, which satisfy \([a^T_k, a^\dagger_{k'}] = \delta_{TT'} \delta(k - k')\) and \([a^T_k, a^T_{k'}] = 0 = [a^\dagger_k, a^\dagger_{k'}]\). The field rescaling \(a^{-\zeta}\) in equation (6) can be anticipated as a consequence of broken conformal invariance. The gauge field quantization (6) is analogous to the conformal vacuum of the scalar field theory in inflation. In the case of a scalar field in conformal spacetime the procedure is quite straightforward. The kinetic term has a non-canonical form, \(\sqrt{-g} L_{\text{kin}}(\phi) = a^2 \eta^{\mu\nu}(\partial_{\mu}\phi)(\partial_{\nu}\phi)/2\). The following simple rescaling \(\partial_\tau \phi = \partial_\tau \varphi/a\) brings it to the canonical form, \(\sqrt{-g} L_{\text{kin}}(\varphi) \rightarrow \eta^{\mu\nu}(\partial_{\mu}\varphi)(\partial_{\nu}\varphi)/2\). Since the canonical momentum is now \(\pi_\varphi = \partial_\tau \varphi\), one can quantize \(\varphi\) by making use of the standard canonical quantization, \([\varphi(\tau, \vec{x}), \pi_\varphi(\tau, \vec{x}')] = i\delta(\vec{x} - \vec{x}')\). The Drummond-Hathrell action (1) on the other hand contains quite a nontrivial modification of the kinetic gauge term. How to determine the power \(\zeta\) in equation (3) is discussed in some detail below.

From the Friedmann equation (100) and \(\rho \propto a^{-3(w+1)}\), one can infer \(\partial_\tau a/a = 2/(1+3w)\tau\), \(a \propto \tau^{2/(1+3w)}\) (in de Sitter inflation \(\partial_\tau a/a = -1/\tau\) and \(a = -1/H\tau\)), so that equation (3)
can be recast as

$$\left( \partial^2_\tau - (1 - \delta_w) \nabla^2 - \frac{2\theta_w}{\tau} \partial_\tau \right) A^T_i = 0. \tag{7}$$

where

$$\delta_w = 2\beta_e (10 + 3w) \frac{H^2}{M^2}$$

$$\theta_w = \frac{3(1 + w)}{(1 + 3w)} \iota_w, \quad \iota_w = -2\beta_e (26 + 33w) \frac{H^2}{M^2} \tag{8}$$

are slowly varying functions of conformal time. Since $H^2 \propto \rho \propto \tau^{-6(1+w)/(1+3w)}$, this is indeed so in slow-roll inflation, for which $w \approx -1$. When the time dependence of the Hubble parameter is neglected, equation (7) can be reduced to the following Bessel equation

$$\left( \partial^2_\tau + \vec{k}^2 - \frac{\theta_w (1 + \theta_w)}{\tau^2} \right) A_{\vec{k}} = 0, \tag{9}$$

where the mode functions $A_{\vec{k}}$ are defined in equation (8), and we set $\delta_w \to 0$. This is legitimate for superhorizon modes, since the effect induced by $\delta_w$ is suppressed by $k^2$ when compared with the $\theta_w$-contribution. In Appendix A we also consider the scalar electrodynamics [15] and show that for the minimal coupling $\xi = 0$ and when $\xi < 1/6$ the gauge fields get damped in inflation. In the conformally coupled case ($\xi = 1/6$) there is essentially no effect, while in the case when $\xi > 1/6$ the gauge fields grow in inflation (cf. Eq. (106)).

In deriving equation (9), the requirement that the coefficient of the damping term $\partial_\tau A^T_i$ vanishes leads to the following conformal mode rescaling $a^{-\zeta} \propto (-\tau)^{\theta_w}$ in equation (8), and hence

$$a^{-\zeta} = a^{(1+3w)/2} \theta_w. \tag{10}$$

With this conformal rescaling the quantization rules for the photon field in (8) are canonical, that is $[a^T_{\vec{k}}, a^{T\dagger}_{\vec{k}'}] = \delta^{TT'} \delta(\vec{k} - \vec{k}')$ and the Wronskian $W[A^{(1)}_{\vec{k}}, A^{(2)}_{\vec{k}}] = i$.

The solution to equation (9) can be conveniently expressed in terms of Hankel functions as follows

$$A^{(j)}_{\vec{k}} = \frac{1}{2} \sqrt{-\pi\tau} H^{(j)}_{\nu}(-k\tau), \quad j = 1, 2, \quad \nu = \theta_w + \frac{1}{2} \tag{11}$$

1In fact both $1 - \delta_w$ and $\theta_w$ in Eq. (8) should be multiplied by $(1 - \iota_w)^{-1}$. This however generates terms that contribute to the effective action at higher orders in the $1/M^2$ expansion, so that they can be consistently omitted.
which, at early times in inflation when $\tau \to -\infty$, reduce to the conformal vacuum solutions
\begin{equation}
A^{(j)}_{\vec{k}} \xrightarrow{\tau \to -\infty} \frac{1}{\sqrt{2k}} e^{\mp i(k\tau + \pi \nu/2 + \pi/4)} + o((-k\tau)^{-1}), \quad j = 1, 2,
\end{equation}
with the standard Wronskian wave function normalization
\begin{equation}
W[A^{(1)}_{\vec{k}}, A^{(2)}_{\vec{k}}] \equiv A^{(1)}_{\vec{k}} \partial_\tau A^{(2)}_{\vec{k}} - \left( \partial_\tau A^{(1)}_{\vec{k}} \right) A^{(2)}_{\vec{k}} = i.
\end{equation}
When writing Eq. (11) we assumed that $\theta_w > -1/2$, so that $\nu > 0$, which holds in situations of physical interest.

We are primarily interested in computation of magnetic fields on cosmological scales today, which correspond to superhorizon scales at late time in inflation, for which $k_{\text{phys}} \ll H$. For these scales $k|\tau| \ll 1$, and one can use the small argument expansion for the Hankel functions $H^{(1,2)}_{\nu}$ in (11) to get
\begin{equation}
A^{(1)}_{\vec{k}} = A^{(2)}_{\vec{k}}^* = \frac{\sqrt{-\pi \tau}}{2} \left[ -i \frac{\Gamma(\nu)}{\pi} \left( -\frac{2}{k\tau} \right)^\nu + \frac{1}{\Gamma(\nu + 1)} \left( -\frac{k\tau}{2} \right)^\nu \left( 1 - o((k\tau)^2) \right) \right],
\end{equation}
where we assumed $k|\tau| \ll 1$, $\nu > 0$ and $\nu \neq \text{integer}$. The expansion of Hankel functions for $\nu = \text{integer}$ can be found for example in Ref. [33].

To model the gauge field evolution in radiation and matter eras, two important effects must be taken account of: (a) the energy density $\rho$ begins scaling away quickly with the expansion of the Universe, $\rho \propto 1/a^4 \propto 1/\tau^4$, and, (b) as a consequence of the inflaton decay into charged particles, the plasma conductivity grows large. Usually the transition to radiation era is fast, and typically takes less than one expansion time, while the inflaton decay to radiation is model dependent and may take many expansion times, so that one should consider these two effects separately. It is important to make this distinction because, as we shall see, a consequence of the inflation-radiation matching may be the photon field growth, while a large conductivity induces freeze-out of the magnetic field and decay of the electric field.

### 2.3 Photon field in radiation era

We shall now compute the gauge field in radiation era in the sudden transition approximation. That means that we neglect the effect of conformal invariance breakdown, which may play some role at early stages of radiation era. In addition we assume that the effect of
conductivity and preheating can be neglected. (For a heuristic account of the conductivity at preheating see Ref. [22].) The matching conditions then become simply

\[ \mathcal{A}_k^{(1)}(\tau_i) \equiv A_0 = \alpha_k A_k^{(+)}(\tau_r) + \beta_{-k}^* A_k^{(-)}(\tau_r) \]

\[ \partial_r A_k^{(1)}(\tau_i) + \frac{\nu - 1/2}{\tau_i} A_k^{(1)}(\tau_i) \equiv -\mathcal{E}_0 = \alpha_k \partial_r A_k^{(+)}(\tau_r) + \beta_{-k}^* \partial_r A_k^{(-)}(\tau_r), \]  

(15)

where \( A_k^{(\pm)} = (2k)^{-1/2} e^{\mp ik\tau} \) are the asymptotic (conformal) vacuum solutions in radiation era.

Determining the conformal times at the end of inflation (\( \tau = \tau_i \)) and the beginning of radiation era (\( \tau = \tau_r \)) requires some care. We start with the physical requirement, that the photon field is in the (Minkowski) vacuum for \( k_{\text{phys}} \equiv k/a \geq M \). We are free to set \( a = 1 \) at the conformal time when the vacuum changes from Minkowski to conformal, that is when the terms in Eqs. (1) and (3) that break conformal invariance become operative. The modes \( k = M \) then exit the horizon \( (k|\tau| = 1) \) at \( a = M/H \). This then implies the following functional form of the scale factor in inflation

\[ a(\tau) = \frac{M}{H} (-M\tau)^{2/(1+3w)}, \quad \tau \leq \tau_i = -\frac{1}{M}. \]  

(16)

The smooth matching \( a(\tau_i) = a(\tau_r) \) and \( \partial_\tau a(\tau_i) = \partial_\tau a(\tau_r) \) then implies for the scale factor in radiation

\[ a(\tau) = \frac{M \tau}{H \tau_r}, \quad \tau \geq \tau_r = -\frac{1 + 3w}{2} \frac{1}{M}. \]  

(17)

The nontrivial term \( (\nu - 1/2) A_k^{(1)}(\tau_i)/\tau_i \) in the second matching condition (15) comes from the scale prefactor in equation (13). In this case the coefficients \( \alpha_k \) and \( \beta_k^* \) correspond to the Bogoliubov coefficients because we are matching onto the conformal vacuum of radiation era, such that the \( k \)-mode amplitude amplification can be characterized by the number of produced particles \( N_k = |\beta_k|^2 \). The result for the matching coefficients is

\[ \alpha_k = \frac{A_0 - i\mathcal{E}_0/k}{2A_k^{(+)}(\tau_r)}, \quad \beta_{-k}^* = \frac{A_0 + i\mathcal{E}_0/k}{2A_k^{(-)}(\tau_r)}, \]  

(18)

and hence

\[ A_k(\tau) = A_0 \cos k(\tau - \tau_r) - \frac{\mathcal{E}_0}{k} \sin k(\tau - \tau_r). \]  

(19)

Now taking a small-argument expansion of the Hankel functions (14) and assuming \( \nu > 0 \), we obtain
\[ A_0 = -\frac{i}{2} \Gamma(\nu) (\pi M)^{-\frac{1}{2}} \left( \frac{2M}{k} \right)^{\nu}, \quad \nu > 0 \]

\[ E_0 = (\pi M)^{1/2} \left( 1 - \cot \pi \nu \right) \frac{\Gamma(\nu)}{\Gamma(\nu)} \left( \frac{k}{2M} \right)^{\nu}, \quad 1 > \nu > 0 \]

\[ E_0 = -i \Gamma(\nu + 1) \left( \frac{M}{\pi} \right)^{1/2} \left( \frac{k}{2M} \right)^{2-\nu}, \quad \nu > 1. \quad (20) \]

where we made use of \( \Gamma(1-z) \sin \pi z = \pi/\Gamma(z) \), such that these equations are valid for integer \( \nu \)'s as well. To get the photon field amplitude in radiation era, multiplication by a factor \( a^{-\zeta} = (H/M)^{-\frac{1+3w}{2}(\nu - \frac{1}{2})} = (H/M)^{-\frac{1+3w}{2}\theta_w} \) is required. When expressed in terms of \( \theta_w = \nu - \frac{1}{2} \), equations (20) become

\[ A_0 = -\frac{i}{2} \Gamma(\theta_w + 1/2) (\pi M)^{-\frac{1}{2}} \left( \frac{2M}{k} \right)^{\frac{1}{2}+\theta_w}, \quad \theta_w > -\frac{1}{2} \]

\[ E_0 = (\pi M)^{1/2} \left( 1 + i \tan \pi \theta_w \right) \frac{\Gamma(\theta_w + 1/2)}{\Gamma(\theta_w + 1/2)} \left( \frac{k}{2M} \right)^{\frac{1}{2}+\theta_w}, \quad \frac{1}{2} > \theta_w > -\frac{1}{2} \]

\[ E_0 = -i \Gamma(\theta_w + 3/2) \left( \frac{M}{\pi} \right)^{1/2} \left( \frac{k}{2M} \right)^{\frac{1}{2}-\theta_w}, \quad \theta_w > \frac{1}{2}. \quad (21) \]

From these and equation (19) we easily conclude that the contribution of \( A_0 \) dominates, and hence in this mechanism the electric field contribution \( E_0 \) can be neglected. For this to occur the term \((\nu - 1/2)A^{(1)}_k(\tau_i)/\tau_i\) in equation (19) plays an essential role, since it cancels out the leading contribution to \( E_0 \), making thus the \( E_0 \) term in equation (19) subleading. This means that all of the amplitude growth by the Drummond-Hathrell action is effected in inflation, and the magnetic field strength can be approximated by

\[ A_k(\tau) = A_0 \cos k(\tau - \tau_i) \quad (22) \]

and is independent of conductivity in radiation era. This is to be contrasted with the backreaction mechanism of Ref. [21], which corresponds to \( \theta_w + 1/2 \rightarrow M_A^2/H^2 \), where \( M_A \) is the photon mass in inflation, such that \( E_0/k \gg A_0 \) for superhorizon modes.

### 2.4 Magnetic field spectrum

We shall now compute the magnetic field spectrum from the Drummond-Hathrell action in inflation and discuss it’s cosmological relevance. Since the field originates from the amplified
vacuum fluctuations in inflation, it is classical and stochastic on cosmological scales, so that
the field can be completely specified by the spectrum. By making use of the volume averaging
procedure [21] described in Appendix B, we find that the gauge field spectrum (21) – (22)
and equations (10) and (114) imply the following spectrum at the end of inflation
\[ B_{\ell, \theta_w} = \frac{2^\theta_w - 1/2}{\pi^{1/2}} \Gamma \left( \theta_w + 1/2 \right) b_{\theta_w + 1/2} \left( \frac{M}{H} \right)^{\frac{3}{2}(1+w)\theta_w} H^2(H\ell)^{\theta_w - 2} \]
\[ b_{\theta_w + 1/2}^2 = \frac{9 \times 2^{2\theta_w - 8}}{\pi^{3/2}} \frac{\Gamma (2 - \theta_w) \Gamma (\theta_w)}{\Gamma (\theta_w + 1/2) \Gamma (\theta_w + 2)}. \] (23)

To obtain the magnetic field today, we assume the field to be frozen in the plasma such
that \( B(T_0) = B(T_H)(T_H/T_0)^2 \), and the comoving scale \( \ell_c \) scales as \( \ell_c = (T_H/T_0) \ell \), where
\( T_0 = 2.75 \text{ K} \equiv 2.37 \times 10^{-13} \text{ GeV} \) and \( T_H \sim 10^{15} \text{ GeV} \) is the temperature corresponding to
the end of inflation when \( H \simeq 10^{13} \text{ GeV} \). With this the spectrum today can be written as
\[ B_{\ell, \theta_w} = \frac{2^\theta_w - 1/2}{\pi^{1/2}} \Gamma \left( \theta_w + 1/2 \right) b_{\theta_w + 1/2} \left( \frac{M}{H} \right)^{\frac{3}{2}(1+w)\theta_w} (T_0/T_H)^{\theta_w} \frac{H^{\theta_w}}{\ell_c^{2-\theta_w}}. \] (24)

When the effects of turbulence are included, a modest amplification of the amplitude on large
scales may result, provided the spectrum slope \( \theta_w - 2 < -1/2 \) (cf. figure 3 in Ref. [21]).
To evaluate the magnetic field strength on cosmological scales, the following conversions are
useful: \((1\text{GeV})^2 \equiv 1.44 \times 10^{19} \text{ Gauss} \) and \(10 \text{ kpc} \equiv 1.56 \times 10^{36} \text{ GeV}^{-1} \). In figure 3 we show
the spectra obtained for various \( \theta_w \) of physical interest. In particular, the vacuum spectrum,
which corresponds to the conformally invariant case, is recovered when \( \theta_w = 0 \), and can be
inferred from equation (118):
\[ B^\text{vac}_{\ell_c} = \frac{3}{16\pi} \ell_c^{-2} \approx 3 \times 10^{-55} \left( \frac{10 \text{ kpc}}{\ell_c} \right)^2 \text{Gauss} \quad \text{(vacuum)}. \] (25)

The thermal spectrum corresponds to \( \theta_w = 1/2 \) in equation (24) and reads
\[ B_{\ell_c, \frac{1}{2}} \approx 2 \times 10^{-44} \left( \frac{M}{H} \right)^{\frac{3}{2}(1+w)} \left( \frac{10 \text{ kpc}}{\ell_c} \right)^{\frac{3}{2}} \text{Gauss}. \] (26)

Since the lower bound on the seed field for the galactic dynamo is about \( B_{\text{seed}} \geq 10^{-35} \text{ Gauss} \)
at a comoving scale of \( \ell_c \sim 10 \text{ kpc} \) [21, 23], both of these spectra are too weak to be of
cosmological interest. When \( \theta_w = 1 \) however, equation (24) reduces to
\[ B_{\ell_c, 1} \approx 2 \times 10^{-33} \left( \frac{M}{H} \right)^{\frac{3}{2}(1+w)} \frac{10 \text{ kpc}}{\ell_c}, \] (27)
which is sufficiently strong to seed the galactic dynamo. In other words, to obtain the field of cosmological interest, the spectral index must be greater than about $-1$. An exception are models in which the photon becomes massive dynamically in inflation [21, 22]. Finally, when $\theta_w = 2$ the spectrum becomes scale invariant with the field strength (cf. Eq. (117)):

$$B_{c}^{\text{sc.inv.}} \approx 3 \times 10^{-11} \left(\frac{M}{H}\right)^{3(1+w)} \text{ Gauss}.$$  

(28)

Here we have not displayed the logarithmic enhancement $\sim (−\ln 2k_0\ell)^{1/2}$ in Eq. (117), where $k_0$ is the smallest momentum amplified in inflation. Note that for $M \sim H$ the field (28) is somewhat weaker than $B \sim 10^{-9}$ Gauss suggested in [7]. Furthermore, it conforms with the magnetic field bound from the CMBR considerations in [4]. In figure 3 we illustrate all of the spectra (25) – (28).

Figure 3: Magnetic-field spectra (24) from the Drummond-Hathrell action as a function of the comoving scale $\ell_c$ today. The vacuum spectrum (25) ($\theta_w = 0$) is shown in dot-dot-dashed green, the thermal spectrum (26) ($\theta_w = 1/2$) in dot-dashed gold, the $\theta_w = 1$ spectrum (27) in dashed red, and the scale invariant spectrum (28) ($\theta_w = 2$) in solid blue, where we assumed that the fermion mass $M \sim H$. We also show the relevant dynamo bounds $B_{\text{seed}} > 2 \times 10^{-27}$ Gauss for a universe with critical matter density, and $B_{\text{seed}} \gtrsim 2 \times 10^{-34}$ Gauss for a flat, low-density universe with dark energy component. Note that when $\theta_w \gtrsim 1$ the dynamo bound $B_{\text{seed}} \gtrsim 2 \times 10^{-34}$ Gauss at $\ell_c \sim 10$ kpc in a flat, low-density universe is amply satisfied.
An important question is of course whether one can obtain $\theta_w \sim 1$ from the Drummond-Hathrell action with a realistic choice of parameters. To investigate this, we write equation (8) as

$$N_F(\theta_w) = \frac{(1 + 3w)(26 + 33w) 30\pi M^2}{\alpha_e H^2 \theta_w}.$$  \hspace{1cm} (29)

To good approximation, $\theta_w$ in equation (8) maximizes when $w \simeq w_m = -59/66$ (for which $-(26 + 33w)(1 + w)$ is maximum), such that

$$N_F(\theta_w = 1) \simeq \frac{1665\pi}{\alpha_e} \frac{M^2}{H^2},$$  \hspace{1cm} (30)

and hence $N_F \sim 10^5$ charged fermions with a mass $M \sim H$ result in $\theta_w \sim 1$, too high to be expected in grand-unified theories. In this case $\iota_w \sim 30$, indicating break-down of the $1/M^2$ expansion. Further, in this mechanism the value $\theta_w = 1$ is not in any sense favoured, or natural. In conclusion, we have shown that is it quite unlikely that magnetic fields can be amplified by the Drummond-Hathrell action in inflation to a strength of interest for cosmology. Moreover, our result should be interpreted with caution, since large amplification is obtained in the region of parameter space when the $1/M^2$ expansion becomes unreliable. In order to address this question one would have to study systematically higher order corrections in the $1/M^2$ Schwinger-DeWitt expansion [16].

3 Photon coupling to gravity with light fermions

3.1 Trace anomaly

Dolgov [10] has argued that conformal invariance of gauge fields gets broken by the one-loop diagram shown in figure 1, even in presence of massless fermions. Taking account of the local contribution only, in a conformal space-time with the metric $g_{\mu\nu} = a^2 \eta_{\mu\nu}$, the following effective action

$$S_{\text{anom}} = \int d^4 x (-g)^{1+\frac{\kappa}{32\pi}} \left[ -\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right],$$  \hspace{1cm} (31)

where

$$\kappa_e(N_F) = \frac{2\alpha_e N_F}{3\pi}$$  \hspace{1cm} (32)

reproduces the equation of motion in Ref. [10]. This action manifestly breaks conformal invariance, since it is not invariant under the transformation $a \to \lambda a$. It is however invariant
under the following combination of conformal transformation and dilatation, $a \rightarrow \lambda a$, and $x^\mu \rightarrow \lambda' x^\mu$, with $\lambda' = \lambda^{-\kappa_e/2}$.

When non-Abelian gauge fields with an SU(N) symmetry are considered, the anomaly can be studied by $\kappa_e \rightarrow \kappa_e(N_F, N) = 2\alpha_e(N_F - 11N/2)/3\pi$. In the limit when the fermion mass $m^2 \ll |\mathcal{R}|$, where $\mathcal{R}$ is the curvature scalar, one can partially resum [34] terms involving the curvature scalar in the Schwinger-DeWitt series [35, 15]. For example, the effective action of a non-Abelian gauge theory [35] contains the following logarithmic term

$$S_{\text{ln}} = \int d^4x (-g)^{\frac{1}{2}} \left[ -\frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} \left( 1 - \frac{11}{12} \frac{\alpha_e C}{\pi} \ln \left( \frac{-\mathcal{R}}{24\pi \mu^2} \right) \right) \right], \quad (33)$$

where $\mu$ is a mass scale, $\alpha_e = e^2/4\pi$, $F^a_{\mu\nu}$ is the gauge field strength, and $C$ is the defined in terms of the structure constants by $\text{tr}[T^a T^b] = -\varepsilon^{a\cdots e\cdots d} F_{\mu\nu}^e F_{\rho\sigma}^d = -C \delta^{ab}$, such that $C = N$ for $SU(N)$. When the curvature in (33) is large, the effective gauge coupling becomes small, and one encounters a curvature induced asymptotic freedom. Now, since in inflation $\mathcal{R} = -(1 - 3w)\rho/M_P^2$ (cf. Eq. (99)) and $\rho \propto a^{3(1+w)}$, we have $\mathcal{R}^{-1} d\mathcal{R}/d\tau = -3(1 + w) da/d\tau$. This then implies that one can include the effect of the logarithmic curvature running from the non-Abelian loop corrections in inflation by the replacement $\kappa_e \rightarrow \kappa_e + (1 + w)(11\alpha_e C/4\pi)$ in Eq. (32) (cf. Eq. (34)). Similarly, for scalar electrodynamics [15] we have $\kappa_e \rightarrow \kappa_e + (1 + w)(\alpha_e N_s/8\pi)$, where $N_s$ is the number of light scalar particles ($m^2_s \ll |\mathcal{R}|$). With these remarks, the analysis presented in this section applies also to the resummed scalar curvature corrections. Note that in de Sitter inflation $w = -1$, and hence the logarithmic corrections in (33) do not lead to any amplification of gauge fields in inflation.

We will now show that, just as in the case of the Drummond-Hathrell action, in inflation the gauge fields are amplified in presence of massless fermions, and damped in presence of massless bosons. Making use of the variational principle, $\delta S_{\text{anom}}/\delta A_\nu = 0$, we arrive at the following equation of motion for the transverse gauge field

$$\left( \partial^2_\tau - \nabla^2 + \kappa_e \frac{\partial_\tau a}{a} \partial_\tau \right) A^T_i = 0. \quad (34)$$

We now quantize the field as in equation (8) and get the following mode equation

$$\left( \partial^2_\tau + \vec{k}^2 + \frac{\kappa_e}{2} \left( 1 - \frac{\kappa_e}{2} \right) \left( \frac{\partial_\tau a}{a} \right)^2 - \frac{\kappa_e}{2} \frac{\partial^2_\tau a}{a} \right) A^T_k = 0 \quad (35)$$

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with the conformal rescaling
\[ \zeta = \frac{\kappa_e}{2} \equiv \frac{\alpha_e N_F}{3 \pi}. \]  
(36)

Assuming a power law inflation with the scale factor \( a \propto \tau^{2/(1+3w)} \) (see Eq. (16)), this becomes
\[ \left( \partial^2 + \vec{k}^2 - \frac{\kappa'_e (1 + \kappa'_e)}{\tau^2} \right) \mathcal{A}_{k}^- = 0, \quad \kappa'_e = \frac{-\kappa_e}{1 + 3w}, \]  
(37)
such that \( \kappa'_e \) can be identified with \( \theta_w \) of equation (3), which is also consistent with equation (10). This then means that the mode functions in inflation are solved by equation (11):
\[ \mathcal{A}_{k}^{(j)} = \frac{1}{2} \sqrt{-\pi \tau} H^{(2)}_{\nu_i}(-k \tau), \quad j = 1, 2, \quad \nu_i = \frac{1}{2} + \kappa'_e, \]  
(38)
which satisfy the Wronskian (13). This is consistent with the canonical quantization:
\[ [A_i(\tau, \vec{x}), \Pi_j(\tau, \vec{y})] = i \delta_{ij} \delta(\vec{x} - \vec{y}), \]  
where \( \Pi_j = a^{e^*} \partial_\tau A_j \) denotes the canonical momentum.

### 3.2 Photon in radiation era

An important difference between the Dolgov anomaly and the Drummond-Hathrell action is that the anomaly also survives in radiation and matter eras, since it is does not dependent on the energy-density scaling. As a consequence the matching conditions on the mode functions now differ from Eqs. (15):
\[ A_{k}^{(1)}(\tau_i) \equiv A_0 = \alpha_{k}^- A_{k}^{(+)}(\tau_r) + \beta_{-k}^* A_{k}^{(-)}(\tau_r), \]  
\[ \partial_\tau A_{k}^{(1)}(\tau_i) \equiv -\mathcal{E}_0 = \alpha_{k}^- \partial_\tau A_{k}^{(+)}(\tau_r) + \beta_{-k}^* \partial_\tau A_{k}^{(-)}(\tau_r), \]  
(39)
where the radiation era modes are the following spherical Bessel functions
\[ A_{k}^{(+)}(\tau) = A_{k}^{(-)*}(\tau) = \frac{1}{2} \sqrt{2 \nu} \ H^{(2)}_{\nu_i}(-k \tau) \xrightarrow{\tau \to \infty} \frac{1}{\sqrt{2k}} e^{-i(k \tau - \nu_i/2 - \pi/4)}, \quad \nu_i = \frac{1 - \kappa_e}{2}. \]  
(40)
The following parametrization of the scale factor is convenient (cf. Eqs. (10) – (17))
\[ a(\tau) = \begin{cases} \left( -H \tau \right)^{2/(1+3w)}, & \tau \leq \tau_i = -1/H \\ \tau, & \tau \geq \tau_r = -\frac{1+3w}{2} \frac{1}{H}. \end{cases} \]  
(41)
We now make use of the Wronskian \( \mathbf{W}[A_{k}^{(+)}, A_{k}^{(-)}] = i \), and arrive at
\[ \alpha_{k}^- = -i \left[ A_{k}^{(1)}(\tau_i) \partial_\tau A_{k}^{(-)}(\tau_r) - (\partial_\tau A_{k}^{(1)}(\tau_i)) A_{k}^{(-)}(\tau_r) \right], \]  
\[ \beta_{-k}^* = i \left[ A_{k}^{(1)}(\tau_i) \partial_\tau A_{k}^{(+)}(\tau_r) - (\partial_\tau A_{k}^{(1)}(\tau_i)) A_{k}^{(+)}(\tau_r) \right]. \]  
(42)
which, with the help of equation (14), reduce to
\[
\alpha_{\vec{k}} = -\beta_{\vec{k}} = -\frac{1}{2} \left( -\frac{1 + 3w}{2} \right)^{\nu_i - 1/2} \Gamma(\nu_i) \left( \frac{1}{\Gamma(\nu_i)} + \frac{i \cos \pi \nu_i \Gamma(1 - \nu_i)}{\pi} \right) \left( \frac{2H}{k} \right)^{\nu_i - \nu_r} + o(k^{\nu_i - \nu_r}).
\] (43)

For \(\kappa_e > 0\) this is the dominant term since in this case \(\nu_i - \nu_r = -(1 - 3w)\kappa_e/2(1 + 3w) > 0\). Note that amplification maximizes in de Sitter inflation, when \(\alpha_{\vec{k}} = -\beta_{\vec{k}} \propto k^{-\kappa_e}\). After some algebra we obtain for the gauge field in radiation
\[
A_{\vec{k}}^{\text{rad}}(\tau) = A_0^{(\text{an})} \cos \left( k\tau - \frac{\pi}{2}(\nu_i - 1/2) \right)
\]
\[
A_0^{(\text{an})} = -\frac{i}{\sqrt{2k}} \left( -\frac{1 + 3w}{2} \right)^{\nu_i - \nu_r} \Gamma(\nu_i) \Gamma(1 - \nu_r) \left( \frac{2H}{k} \right)^{\nu_i - \nu_r} \Gamma(\nu_i),
\] (44)

where we used equations (40) and (43). For de Sitter inflation this reduces to
\[
A_{\vec{k}}^{\text{rad}}(\tau) \xrightarrow{\text{deSitter}} -\frac{i}{\sqrt{2k}} \Gamma \left( \frac{1 + \kappa_e}{2} \right)^2 \left( \frac{2H}{k} \right)^{-\kappa_e} \cos(k\tau + \pi\kappa_e/4).
\] (45)

When conductivity grows quickly in radiation era to a large value, the smooth matching (39) is not any more appropriate. In this case the field amplitude freezes-out to the value at the end of inflation (38) implying that, at the beginning of radiation era,
\[
A_{\vec{k}}^{\text{rad}}(\tau) \xrightarrow{\sigma \to \infty} -\frac{i}{\sqrt{2k}} \Gamma(\nu_i) \left( \frac{2H}{k} \right)^{\nu_i - 1/2}.
\] (46)

Note that this amplitude is smaller than (44), since \((\nu_i - \nu_r) - (\nu_i - 1/2) = \kappa_e/2 > 0\).

### 3.3 Magnetic field spectrum from the Dolgov anomaly

We first consider the simpler case (46), for which the magnetic field gets frozen in the plasma at the beginning of radiation era. The volume-averaged spectrum can be computed as in section 2.4, resulting in the following spectrum today
\[
B_{L_c,\kappa_e} = \frac{2^{(\kappa_e - 1)/2}}{\pi^{1/2}} \Gamma \left( \frac{\kappa_e + 1}{2} \right) b_{\kappa_e + 1} \left( \frac{T_0}{T_{\text{H}}} \right)^{\frac{4\pi}{3}} \frac{H^{2\pi}}{L_c^2}.
\]
\[
b_{\kappa_e + 1}^{2\kappa_e + 1} = \frac{9 \times 2^{\kappa_e - 8}}{\pi^{3/2}} \Gamma \left( 2 - \kappa_e/2 \right) \Gamma(\kappa_e/2) / \Gamma \left( (\kappa_e + 1)/2 \right) \Gamma \left( 2 + \kappa_e/2 \right),
\] (47)

where for simplicity we assumed de Sitter inflation, for which the resulting field is strongest. The scaling (36) introduces ambiguity to the field amplitude normalization. Our normalization corresponds to setting the scale factor \(a(\tau_i) = 1\) at the end of inflation, just as in
equation (41), such that at $\tau = \tau_i$ the horizon scale modes have the amplitude of the order the standard Minkowski vacuum amplitude. We believe that this justifies our normalization prescription. The analysis of the spectrum (17) is identical to that of the Drummond-Hathrell spectrum (24) discussed in equations (25) – (28), with the replacement $\theta_w \to \kappa_e/2$ and $M = H$.

The second case (44) – (45) applies when the conductivity is low in ‘radiation era’. This means that the inflaton oscillates for a long time before charged particles are produced that freeze-out the magnetic field. Denoting the freeze-out temperature by $T_{\text{freeze}} \ll T_H$, we then find that the gauge field dynamics in de Sitter inflation (45) induces the following the spectrum today

$$B_{\ell, \kappa_e} = \frac{2^{-\kappa_e-1/2}}{\pi} \Gamma \left( \frac{\kappa_e + 1}{2} \right)^2 b_{\kappa_e+\frac{1}{2}} \left( \frac{T_0}{T_{\text{freeze}}} \right)^{\kappa_e} \left( \frac{T_{\text{freeze}}}{T_H} \right)^{3\kappa_e/2} \frac{H^{\kappa_e}}{\ell_c^{2-\kappa_e}}$$

$$b_{\kappa_e+\frac{1}{2}}^2 = \frac{9 \times 2^{2\kappa_e-8}}{\pi^{3/2}} \frac{\Gamma (2 - \kappa_e) \Gamma (\kappa_e)}{\Gamma (\kappa_e + 1/2) \Gamma (\kappa_e + 2)}. \quad (48)$$

There is subtle difference between this spectrum and (24). The additional suppression here is due to the anomalous scaling $a_{\text{freeze}}^{-\kappa_e/2} = (T_{\text{freeze}}/T_H)^{\kappa_e/2}$ of the gauge field. Here $T_{\text{freeze}}$ denotes the temperature at which the modes freeze out due to a large conductivity. This expression is correct for the modes that enter horizon before the freeze-out. We do not discuss here the modes that grow for a while and freeze out before reentering horizon in radiation era. The dynamics of these modes is quite complex and their spectra lie in between those given in equations (17) and (48).

The physics of the inflaton decay may in fact be nonperturbative, and there is a large literature on the inflaton decay and preheating after inflation. Nevertheless, the gauge field dynamics at preheating is not yet fully understood [36]. A heuristic account of preheating with gauge fields is given in Ref. [22].

The magnetic field strength corresponding to the spectra (17) and (48) is very similar to those plotted in figure 3 and hence we do not plot them here. An interesting question is of course under what conditions (17) and (48) can result in spectra with a slope $\sim -1$, required to seed the galactic dynamo. To get the spectrum $B_\ell \propto \ell^{-1}$, equations (17) and (48) imply $1 \leq \kappa_e \leq 2$, or equivalently

$$\frac{3\pi}{2\alpha_e} \leq N_F \leq \frac{3\pi}{\alpha_e}. \quad (49)$$
For $\alpha_e \sim 1/40$ this gives the following bound

$$2 \times 10^2 \lesssim N_F \lesssim 4 \times 10^2$$

(50)

d for the number of charged fermions. In presence of bosonic fields amplification is reduced. For example, when the electromagnetism is embedded in an SU(N), then equation (19) should be read as a limit on $N_F - 11N/2$. When applied to the standard model, the photon field is amplified on scales smaller than the electroweak scale $M_W \sim 80$ GeV. On higher scales, the hypercharge field is amplified, and the relevant number of light fermions is $N_F = 12$ (above the top mass). Hence this mechanism results in weak magnetic field amplification if the standard model fermions are the only light fermions. To get strong amplification, one would need a few hundred of additional fermions with masses at some intermediate scale below inflation. It is not clear whether this can be made consistent with the grand unification.

The validity of the Dolgov anomaly equation (34) was questioned in [15]. Here we have solved Eq. (34) exactly to order $e^2$. To properly address the question of validity of this approach one would have to study the higher order corrections in the coupling constant expansion to the trace anomaly.

4 Photon coupling to scalar and pseudoscalar fields

4.1 Photon coupling to scalar field

Consider now the following action with gauge invariant coupling of the gauge field to a scalar field

$$S_\phi = \int d^4x \sqrt{-g} \left( -\frac{1}{4} f(\phi) g^{\mu\nu} g^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{2} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) + V(\phi) \right),$$

(51)

where $F_{\mu\nu}$ is the gauge field strength, $\phi$ is a scalar field $\phi$ and $V(\phi)$ is the corresponding potential. In general $\phi$ may couple to other fields as well. The ‘coupling’ function $f = f(\phi)$ we take to be of the form:

$$f = \sum_{n=0}^{\infty} f_n \left( \frac{\phi}{M_P} \right)^n,$$

(52)

where $f_n$ are some coupling coefficients and $M_P = (8\pi G)^{-1/2} \approx 2.4 \times 10^{18}$ GeV is the reduced Planck mass. For the standard gauge kinetic term, $f_0 = 1$, and in addition the potential $V(\phi)$ must be chosen such that today $\phi = 0$ and $V(\phi = 0) = 0$. A particular realization of this model has been considered by Ratra [11], where $f(\phi) \propto e^{-\lambda \phi/M_P}$, and $\phi$ was taken
to be the inflaton in an extended inflation with the potential $V(\phi) \propto e^{-\lambda' \phi/M_P}$. In this case $f_n = (-\lambda)^n/n!$. Ratra’s model has subsequently been reconsidered in the context of string-inspired cosmology [12, 13], where the role of $\phi$ is taken by the dilaton field. Here we study the conditions under which $f(\phi)$ and $V(\phi)$ can lead to significant amplification of the magnetic field in inflation. As a particular case we discuss Ratra’s model.

Working in Coulomb gauge, $A_0 = 0$ and $\nabla \cdot \vec{A} = 0$ (cf. Eqs. (2-3)), the action (51) implies the following equation of motion in de Sitter inflation for the transverse gauge field

$$\left( \partial^2_\tau - \nabla^2 + \frac{d\phi}{d\tau} (\partial_\phi \ln f(\phi)) \partial_\tau \right) A^T_i = 0,$$

(53)

where we assumed that in inflation the field $\phi = \phi(\tau)$ is a function of time only. It is now clear that the coupling function $f$ is responsible for breakdown of conformal invariance, and hence its precise form determines whether the field gets amplified in inflation. In what follows we consider two types of functions: (A) quadratic coupling function

$$f = 1 + f_2 \frac{\phi^2}{M_P^2} + .. \quad \text{(Type A)},$$

(54)

and (B) exponential coupling function

$$f = e^{-\lambda \phi/M_P} \quad \text{(Type B)}.$$

(55)

Type A coupling functions are expected to be induced by the Planck physics. The coupling $f_1$ can be made to vanish in (54) by imposing the symmetry $\phi \to -\phi$. Similarly, by promoting $\phi$ to a complex field and imposing a $Z_N$ symmetry, one gets that $f_n = 0, \forall n < N$. Type B coupling functions may occur from string theory corrections to the gauge field dynamics.

4.1.1 Chaotic inflation

Making use of equation (123) in Appendix C.1 where we estimate $\partial_\tau \phi$ during slow-roll in chaotic inflation, where $V = m^2 \phi^2/2$, we can recast equation (53) as

$$\left( \partial^2_\tau - \nabla^2 + \frac{2M_P^2}{\phi} \frac{1}{\tau} (\partial_\phi \ln f(\phi)) \partial_\tau \right) A^T_i = 0.$$

(56)

For Type A coupling functions this implies

$$\left( \partial^2_\tau - \nabla^2 - \frac{2\theta_2}{\tau} \partial_\tau \right) A^T_i = 0, \quad \theta_2 = -2f_2,$$

(57)
where we neglected the higher order corrections. Similarly, for Type B coupling functions we obtain the following equation:

\[
\left( \partial^2 - \nabla^2 - \frac{2 \theta_{\lambda}}{\tau} \partial_{\tau} \right) A_i^T = 0, \quad \theta_{\lambda} = \lambda \frac{M_P}{\phi},
\]

(58)

where \( \phi = \phi_0 + (2M_P^2/\phi) \ln(-H_0 \tau) \). Since \( \theta_2 \) and \( \theta_\lambda \) are generally slowly varying functions of conformal time, equations (57-58) can be well approximated by the Bessel equation (3) analysed in detail in section 2.1. The corresponding magnetic field spectrum is calculated in (23-24), according to which the spectrum reads

\[
B_{\ell c} \propto \ell_c^{2+\theta}, \quad (\theta = \theta_2, \theta_{\lambda}),
\]

(59)

where \( \ell_c \) is the comoving scale. To get the amplitude one should set \( M = H \) in (23-24), which corresponds to the normalization to the Minkowski vacuum at the horizon crossing.

We assume that this normalization is effected by the local interactions in the de Sitter vacuum operative on subhorizon scales. With this the magnetic field spectrum can be simply read off from figure 3. To get an acceptable spectrum, we then have \( 2 \geq \theta_2, \theta_{\lambda} \geq 1 \), or

\[
\frac{1}{2} \leq -f_2 \leq 1, \quad \lambda \sim 30,
\]

(60)

where we took \( M_P/\phi \simeq 1/(2 \times 6^{1/4} N^{1/2}) \sim 1/20 \), and the number of e-folds \( N \sim 50 \). We have thus found that one can obtain magnetic field spectra with observable consequences with \( f_2 \) of order unity, which is a natural value. A generalization of Type A model contains higher order terms in the \( \phi/M_P \)-expansion. Since in chaotic inflation typically \( \phi \gg M_P \), the resulting magnetic field spectra of interest can be obtained with coupling coefficients \( |f_{2n}| \ll 1 \) (\( n = 2, 3, \ldots \)), where \( f_0 = 1, f_1 = f_2 = f_3 = 0 \). On the other hand, for Type B model the coupling \( \lambda \) is required to be unnaturally strong. Moreover, one gets a significant amplification in inflation only when the effective gauge coupling \( e^2(\phi) = e_0^2 e^{-\lambda \phi/M_P} \) starts very weak in inflation.

In the spectrum computation in section 2.4 we have neglected the conductivity in radiation era. A simple way of accounting for that effect is to assume that magnetic field freezes in when the conductivity becomes large. The spectrum calculation in this case can be performed analogously as in section 3.3 (cf. Eq. (17)). To get spectra of physical interest in this case, one requires by about a factor of two larger values of \( \theta_2 \) and \( \theta_{\lambda} \) from those indicated in (60).
4.1.2 Extended inflation

For the potential of extended inflation $V = V_0 e^{-\lambda \phi/M_P}$ equation (53) can be written as

$$\left( \frac{\partial^2}{\tau} - \nabla^2 - \frac{2 M_P}{\lambda' \left( \frac{2}{\lambda^2} - 1 \right)} \frac{1}{\tau} (\partial \phi \ln f(\phi)) \partial _\tau \right) A^T_\tau = 0. \quad (61)$$

where, to estimate $d\phi/d\tau$, we made use of equation (131) in Appendix C.2. For Type A model (54) this implies

$$\left( \frac{\partial^2}{\tau} - \nabla^2 - \frac{2 \theta'}{\lambda} \partial _\tau \right) A^T_\tau = 0, \quad \theta' = \frac{2 f_2}{\lambda' \left( \frac{2}{\lambda^2} - 1 \right) M_P}, \quad (62)$$

where $\phi$ is a slowly varying function of conformal time given in (132). Since $\phi \sim \phi_0 \sim -(M_P/\lambda') \ln(12/\lambda'^4)$, this immediately implies

$$1 \lesssim \theta' \lesssim 2 \implies f_2 \sim -\left( \ln \frac{12}{\lambda'^4} \right)^{-1} \left( \lambda' \ll 1 \right), \quad (63)$$

which is to be compared with the conditions in chaotic inflation (58) and (60). For Type B model (53) we get the following equation:

$$\left( \frac{\partial^2}{\tau} - \nabla^2 - \frac{2 \theta'}{\lambda} \partial _\tau \right) A^T_\tau = 0, \quad \theta' = -\frac{\lambda}{\lambda' \left( \frac{2}{\lambda^2} - 1 \right)}. \quad (64)$$

To get a magnetic field of observable strength we than have

$$1 \lesssim \theta' \lesssim 2 \implies \frac{2}{\lambda'} \lesssim \lambda \lesssim \frac{4}{\lambda'} \quad (0 < \lambda' \ll 1). \quad (65)$$

In this case the effective gauge coupling $e^2(\phi) = e^2_0 e^{-\lambda \phi/M_P}$ begins very weak at early stages in inflation ($\phi \ll -M_P$) and acquires today’s value at the end of inflation ($\phi \approx 0$). Precisely this feature of the pre-Big Bang model was used in [12] to get magnetic field amplification, where the gauge coupling constant is determined dynamically by the dilaton expectation value.

An interesting question is whether there is a natural spectrum predicted by inflation. In other words, is there a value of the spectrum slope $2 - \theta$ that is in any way singled out. Assume that at early stages of inflation $\theta > 2$, which implies a (tachyonic) instability resulting in a fast amplitude growth and spontaneous magnetization on superhorizon scales. This condensate may affect evolution of the fields in inflation and possibly single out the scale invariant spectrum. This question deserves further investigation.
4.2 Photon coupling to pseudoscalar field

We now consider the following coupling of a pseudoscalar field the Chern-Pontryagin density

\[ S_\psi = \int d^4x \sqrt{-g} \left( -\frac{1}{4} g^{\mu\nu} g^\rho\sigma F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4} h(\psi) g^{\mu\rho} g^\nu\sigma F_{\mu\nu} \tilde{F}_{\rho\sigma} + \frac{1}{2} g^{\mu\nu} (\partial_\mu \psi) (\partial_\nu \psi) + V(\psi) \right) \]  \hspace{1cm} (66)

where

\[ h = \sum_{n=0}^{\infty} h_n \left( \frac{\phi}{M_P} \right)^n \]  \hspace{1cm} (67)

is a coupling function of a pseudoscalar field \( \psi \), \( \tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\eta\rho} F_{\eta\rho} / 2 \) is the dual field strength and \( \epsilon_{\mu\nu\eta\rho} \) is the totally antisymmetric tensor defined by \( \epsilon_{0123} = 1 \). The case \( h(\psi) = h^2_2 \psi^2 / M_P^2 \) is mentioned in [3] as potentially interesting, but no attempt was made to solve it. There is an observational constraint on \( h \) coming from the magnetic dipole moment measurements of the neutron and electron, according to which \( |h_0| < \sim 10^{-9} (\psi = 0) \); for simplicity here we assume \( h_0 = 0 \) and \( \psi = 0 \). With \( \psi = 0 \) today there are no experimental constraints on \( h_n (n \geq 1) \). When applied to the quantum chromodynamics, the question of smallness of \( h \) is termed the strong CP-problem. Peccei and Quinn have suggested that \( h \) might be dynamically driven to zero by symmetry breaking of \( \psi \) at some high energy scale. The special case of this model was considered in [4], where the authors assumed that the axionic potential is of the form \( V(\psi) = V_0 [1 - \cos(\psi / \psi_0)] \), and found no significant amplification.

The action (66) implies the following equation of motion for the gauge field in inflation in Coulomb gauge (cf. Eq. (53)):

\[ \left( \partial_\tau^2 - \nabla^2 + \frac{d\psi}{d\tau} (\partial_\psi \ln h(\psi)) \nabla \times \right) A_i^T = 0, \hspace{1cm} (68)\]

where we used \( \eta^{\mu\rho} \partial_\mu \tilde{F}_{\rho\nu} = 0 \) and assumed that in inflation \( \psi = \psi(\tau) \). Conformal invariance is now broken by \( h = h(\psi) \), which then may lead to magnetic field amplification in inflation. It now follows from the analysis in section [4.1], that in inflation quite generically equation (68) can be rewritten as

\[ \left( \partial_\tau^2 - \nabla^2 - \frac{2\theta_\psi}{\tau} \nabla \times \right) A_i^T = 0, \hspace{1cm} (69)\]

where \( \theta_\psi \) is a slowly varying function of time in inflation, and \( \theta_\psi \approx 0 \) in radiation era. The specific forms of \( \theta_\psi \) for Type A and B models (54-55) in chaotic and extended inflation can be easily reconstructed from Eqs. (57-58), (62) and (64), provided \( f \rightarrow h \) and \( \phi \rightarrow \psi \).
Equation (69) can be written in terms of the mode functions (6) as follows
\[
\left( \partial^2 + k^2 \right) \epsilon_k^T A_k^T - \frac{2\theta_\psi}{T} i\vec{k} \times \epsilon_k^T A_k^T = 0. \tag{70}
\]
Since the circular polarization vectors satisfy \( i\vec{k} \times \epsilon_k^\pm = \pm \epsilon_k^\pm \) we can write (70) for the circularly polarized mode amplitudes as
\[
\left( \partial^2 + k^2 \mp \frac{2\theta_\psi}{T} \right) A_k^\pm = 0. \tag{71}
\]
Note that when \( \theta_\psi < 0 \ (\theta_\psi > 0) \) the mode \( A_k^+(A_k^-) \) gets amplified in inflation (where we took into account that \( \tau < 0 \) in inflation), such that in principle one can distinguish the magnetic fields produced by a pseudoscalar coupling to the Chern-Pontryagin density from other mechanisms discussed in this paper, where the field amplification is polarization independent. This is of course true provided amplification is strong enough to be observable.

When written in terms of the variable \( u = 2ik\tau \), equation (71) reduces to the following Whittaker equation [33]:
\[
\left( \partial^2_u - \frac{1}{4} \pm \frac{i\theta_\psi}{u} \right) A_k^\pm = 0. \tag{72}
\]
The solutions can be expressed in terms of the Whittaker functions \( W_{\frac{1}{2},\lambda}(u) \) as
\[
A_k^\pm(\tau) = \frac{e^{\mp \frac{\pi}{2} \theta_\psi}}{\sqrt{2k}} W_{\frac{1}{2}, \pm i\theta_\psi}(2ik\tau), \tag{73}
\]
with the Wronskian normalization \( \mathbf{W}[A_k^+(\tau), A_k^-(-\tau)] = i \), where \( A_k^-(-\tau) \) denotes the (second) linearly independent solution. The asymptotic form of (73) is
\[
A_k^\pm(\tau) \xrightarrow{k|\tau| \to \infty} \frac{1}{\sqrt{2k}} e^{-ik\tau \pm i\theta_\psi \ln(-2k\tau)}, \tag{74}
\]
such that it approaches the Minkowski vacuum at \( \tau \to -\infty \). The only effect of the \( \psi \)-field at a distant past is a time-dependent phase shift in the mode functions. For superhorizon modes close to the end of inflation the following asymptotic form is useful
\[
A_k^\pm \xrightarrow{k|\tau| \to 0} \frac{1}{\sqrt{2k}} \frac{e^{\mp \frac{\pi}{2} \theta_\psi}}{\Gamma(1 \pm i\theta_\psi)}. \tag{75}
\]
Now matching this smoothly onto the radiation vacuum solutions \( A_k^{\pm\text{rad}} = (2k)^{-1/2} e^{\mp i k \tau} \) results in the vacuum spectrum of a modified amplitude. The circularly polarized modes \( A_k^\pm \) get amplified in inflation by a factor
\[
\left| \frac{A_k^\pm}{A_k^{\pm\text{rad}}} \right| \simeq e^{\mp |\theta_\psi| \mp \frac{\pi}{2} \theta_\psi}, \tag{76}
\]
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where we made use of the Stirling formula, $|\Gamma(1 \pm i\theta)| \simeq (2\pi|\theta|)^{1/2}e^{-\pi|\theta|/2}$ ($|\theta| \geq 1$).
This means that, when $\theta < 0$ ($\theta > 0$), the mode $A_k^+ (A_k^-)$ gets amplified by a factor $\sim e^{\pi|\theta|}$, while the amplitude of $A_k^- (A_k^+)$ remains unchanged. This is an extreme case of birefringence, where the photon of one polarization corresponds to a tachyonic particle, and the other to a massive vector particle.

We have thus shown that a pseudoscalar field coupling to the Chern-Pontryagin density cannot change the vacuum spectrum $B_\ell \propto \ell^{-2}$ of gauge fields in inflation. In this case conformal invariance is broken in inflation such that the vacuum amplitude changes and an unobservable phase shift to the gauge mode functions is induced. The resulting amplification of the vacuum spectrum is typically too weak to be observable today. Moreover, thermal processes in radiation era are expected to create magnetic field with the spectrum $B_\ell^{\text{th}} \propto \ell^{-3/2}$, which dominates over the vacuum spectrum on scales of interest in cosmology.

Even though this model is not relevant for cosmological magnetic fields, it may be of interest for baryogenesis. An explicit CP-violation in the pseudoscalar sector may result in a nonperturbative amplification of the CP-violating phase of a complex pseudoscalar field at preheating. When the field decays into the standard model fermions, a net baryon number may be produced. This is reminiscent of the Affleck-Dine mechanism, with an important difference however: the $\psi$-field is a pseudoscalar field which is not charged under the baryon number, so that baryon violating processes are needed to get a nonzero baryon production. In a related work the dynamics of a pseudoscalar field in radiation era and its relevance for baryogenesis has been considered in [29, 30].

5 Photon coupling to scalar cosmological perturbations

An interesting proposal has been recently put forward by Maroto, who considered the effect of the photon field coupling to the scalar metric perturbations. We now reconsider this mechanism (without making a perturbative expansion of the mode amplitudes). In presence of the scalar cosmological perturbations the metric gets modified as follows

$$g^{\mu\nu} = a^{-2}\left((1 - 2\Phi)d\tau^2 - (1 + 2\Phi)d\vec{x}^2\right), \quad (77)$$
where $\Phi = \Phi(\eta, \vec{x})$ denotes the (gauge invariant) scalar gravitational potential. The potential $\Phi$ is to good approximation scale invariant. The CMBR measurements and the large scale structure formation imply that $\Phi \sim 10^{-5}$.

The gauge field equation of motion is simply

$$\partial_\mu \left( \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \right) = 0,$$

which, when written in components and working to linear order in $\Phi$ ($|\Phi| \ll 1$), reads

$$\partial_i \left[ (1 - 2\Phi) (\partial_i A_0 - \partial_\tau A_i) \right] = 0$$

and

$$\partial_\tau \left[ (1 - 2\Phi) (\partial_i A_0 - \partial_\tau A_i) \right] + \partial_j \left[ (1 + 2\Phi) (\partial_j A_i - \partial_i A_j) \right] = 0.$$  

Equation (79) can be also written as

$$\nabla^2 A_0 - 2 \nabla \Phi \cdot \nabla A_0 = \partial_\tau \nabla \cdot \vec{A} - 2 (\nabla \Phi) \cdot \partial_\tau \vec{A},$$

where we used $\nabla \ln(1 - 2\Phi) \approx -2 \nabla \Phi$. Setting $A_0 = 0$ leads to the following generalized Coulomb gauge

$$\partial_\tau \nabla \cdot \vec{A} = 2 (\nabla \Phi) \cdot \partial_\tau \vec{A}.$$  

It turns out that this modification becomes unimportant on superhorizon scales, where $k|\tau| \ll 1$. We can now write equation (80) in the gauge (82) as

$$\left( \partial_\tau^2 - \nabla^2 \right) A_i - 2 (\partial_\tau \Phi) \partial_\tau A_i + \partial_i \nabla \cdot A = 2 (\partial_j \Phi) (\partial_j A_i - \partial_i A_j) = 0,$$

where we neglected $\Phi$ in favour of one in $1 - 2\Phi$ and $1 + 2\Phi$, which is a controlled truncation when working to leading order in $\Phi$. The last two terms in (83) can be consistently neglected, since formally they are a small correction to the $\nabla^2 A_i$-term. This follows from equation (82) and $\Phi \ll 1$. Moreover, it is easy to see that the term $2(\partial_\tau \Phi) \partial_\tau A_i$ (which potentially leads to gauge field amplification) dominates on superhorizon scales ($k|\tau| \ll 1$) over the terms we have just argued can be neglected. With this equation (83) simplifies to

$$\left( \partial_\tau^2 - \nabla^2 \right) A_i = 2 (\partial_\tau \Phi) \partial_\tau A_i = 0.$$  

We shall now solve this equation in the mean field approximation

$$\Phi \to \Phi_L \equiv \langle \Phi \rangle_L,$$
where \( \langle \Phi \rangle_L \) denotes a spatial average over some scale \( L \). Since \( \Phi \) is almost scale invariant in inflation, we expect \( \Phi_L \) to be a slow (logarithmic) function of time and \( L \). Taking for example \( \Phi_L \sim \Phi_0 \ln(-\tau/L) \) (\( \Phi_0 \sim 10^{-5} \ll 1 \)), we find \( \partial_\tau \Phi \rightarrow \Phi_0/\tau \). With this we can write equation (84) in the mean field approximation as the following Bessel equation

\[
\left( \partial_\tau^2 - \nabla^2 - \frac{2\Phi_0}{\tau} \partial_\tau \right) A_i = 0.
\]

(86)

In section 2.1 we study in detail this equation (cf. Eq. (7)) and show that the solutions for the mode functions can be written in terms of the Hankel functions (11) with the index \( \nu = \frac{1}{2} + \Phi_0 \), and that the resulting mode amplification \( A_\vec{k} \propto (-k\tau)^{\nu-1/2} \) is small when \( |\nu - 1/2| = |\Phi_0| \ll 1 \). Since in radiation era \( \partial_\tau \Phi = 0 \), the mode functions are simply \( A^{\pm}_{\vec{k} \rightarrow \vec{k}'} = (2k)^{-1/2} e^{\pm ik\tau} \), so that the calculation in section 2.4 leading to the spectrum (23) applies, and hence we get an almost vacuum spectrum today \( B_\ell \propto \ell^{2+\Phi_0} \) (\( \Phi_0 \sim 10^{-5} \)).

We shall now argue that the mean field approximation represents an upper bound on the amplification on superhorizon scales. This approximation captures the effect of a spatially averaged gravitational potential, but does not account for inelastic scatterings. Since inelastic scatterings are in general dissipative, their inclusion can lead only to suppression of gauge fields. Moreover, the sign of \( \Phi_L \) in (85) can be both positive and negative, suggesting an additional suppression when compared to (86). To show that this is indeed so, we now study Eq. (84) by making an \textit{Ansatz} for the momentum dependence of the mode functions. Upon performing a Fourier transform on (84) we get

\[
\left( \partial_\tau^2 + k^2 \right) A_\vec{k} - 2 \int \frac{d^3k'}{(2\pi)^3} (\partial_\tau \Phi_{\vec{k}'}) \partial_\tau A_{\vec{k} - \vec{k}'} = 0,
\]

(87)

where we suppressed the polarization indices. The mean field approximation consists of neglecting inelastic scatterings that lead to nontrivial momentum exchange, \textit{i.e.} it amounts to \( A_{\vec{k} - \vec{k}'} \rightarrow A_\vec{k} \).

The spectrum of the potential \( \Phi \) is typically close to scale invariant, which means that

\[
\Phi_\vec{k} = \phi_0(\tau) \Lambda^{\beta - 3} k^{-\beta}, \quad k < \Lambda,
\]

(88)

where \( \beta \) is the spectral index for superhorizon modes \( (k < \Lambda) \), and

\[
\phi_0 = C_0 \frac{1}{a} \frac{d}{d\tau} \left( \frac{1}{a} \int a^2 d\tau \right)
\]

(89)
is a slowly varying function of $\tau$, and $C_0 \sim 10^{-5}$ is a normalization constant. For a scale-invariant spectrum $\beta \equiv 5/2 - n = 3/2$. On large cosmological scales the CMBR measurements constrain $|\beta - 3/2| \lesssim 0.2$; on smaller scales a similar constraint results from large scale structure formation. The cutoff $\Lambda$ in (88) is of the order the horizon $H$ at the end of inflation, such that on scales $k_{\text{phys}} = k/a > \Lambda \sim H$ the vacuum spectrum $|\Phi_k| \sim |\phi_0|/\sqrt{k}$ is recovered. Similarly for the gauge field we take

$$A_k = A_0(\tau)k^{-\alpha}, \quad k < \Lambda,$$

(90)

where the scale invariant spectrum for the magnetic field corresponds to $\alpha = 5/2$, and $A_k \propto k^{-1/2}$ for $k \geq a\Lambda$. Now inserting the Ansätze (88-90) into (87) results in

$$\left( \partial_{\tau}^2 + k^2 - \frac{\partial_x \phi_0}{2\pi^2(2-\beta)} \left( \frac{k}{\Lambda} \right)^{3-\beta} \mathcal{I}_{\alpha\beta}(x_0, x_1) \partial_x \right) A_0 = 0,$$

(91)

where

$$\mathcal{I}_{\alpha\beta}(x_0, x_1) = \int_{x_0}^{x_1} dx x^{1-\alpha} \left[ (1 + x)^{2-\beta} - |1 - x|^{2-\beta} \right].$$

(92)

Figure 4: The unshaded region corresponds to the phase space where $\mathcal{I}_{\alpha\beta}(0, x_1 = \Lambda/k)$ is dominated by the ultraviolet cutoff, and where the only self-consistent solutions of (91) lie (marked by the thin vertical line $\alpha \approx 1/2$, $|\beta - 3/2| \leq 0.2$). In the light blue shaded triangle the integral $\mathcal{I}_{\alpha\beta}(0, \infty)$ in (92) is convergent.

solution of (91) is obtained when when $\alpha + \beta < 3$. In this case the ultraviolet cutoff $x_1$

\footnote{For the special case $\beta = 2$ the integral $\mathcal{I}_{\alpha\beta}(x_0, x_1)/(2-\beta) \rightarrow \int_{x_0}^{x_1} dx x^{1-\alpha} \ln[(1 + x)/|1 - x|]$.}
dominates the integral (92), and equation (91) becomes
\[ \left( \partial^2_\tau + k^2 - \frac{d\phi_0/d\tau}{\pi^2(3 - \alpha - \beta)} \left( \frac{k}{\Lambda} \right)^\alpha a^{\alpha + \beta - 3} \partial_\tau \right) A_k = 0. \] (93)

Since in inflation \( d\phi_0/d\tau \sim C_0/\tau \), and \( a \simeq -1/H\tau \), Eq. (93) can be simplified to
\[ \left( \partial^2_\tau + k^2 + 2\gamma_\tau(k) \partial_\tau \right) A_k = 0, \quad \gamma_\tau(k) \sim \frac{C_0}{(-H\tau)^{\alpha + \beta - 2}} \frac{1}{2\pi^2(3 - \alpha - \beta)} \left( \frac{k}{\Lambda} \right)^\alpha. \] (94)

When \( \alpha + \beta \approx 2, \gamma_\tau \sim \gamma_0 \) is almost time independent, and equation (94) is self-consistently solved by
\[ A_k^\pm \sim \begin{cases} \frac{1}{\sqrt{2k}} e^{-\gamma_0 \tau \pm i\omega_0 \tau} & \text{when } \gamma_0 < k \\ \frac{1}{\sqrt{2k}} e^{-\gamma_\pm \tau} & \text{when } \gamma_0 > k \end{cases} \] (95)
where \( \omega_0 = \sqrt{k^2 - \gamma_0^2} \) and \( \gamma_\pm = \gamma_0 \pm \sqrt{\gamma_0^2 - k^2} > 0 \). The critical case \( k_c = \gamma_0(k_c) \) corresponds to \( k_c \sim (C_0^2/4\pi^4)(H^2/\Lambda) \ll H \). In all cases in (95) the spectrum on large superhorizon scales is to good approximation the Minkowski vacuum, and the gauge field actually decays slowly in inflation. This suggests that the mean field equation (86) can only overestimate amplification, and in addition it indicates that the vacuum \( \alpha = 1/2 \) approximates well the actual solution. The solution to (94) is represented by the thin vertical line in figure 4, and it is the only self-consistent solution of (91). Indeed, when \( \alpha > 3 \) the integral (92) is dominated by the infrared cutoff. In this case \( I_{\alpha\beta}(x_0, \infty)/(2 - \beta) \approx (2/\alpha)(k/k_0)^{\alpha - 3} \), which cannot be made consistent with (87). Finally, when \( \alpha, \beta < 3 \) and \( \alpha + \beta > 3 \) the integral \( I_{\alpha\beta}(0, \infty) \) converges (in the light blue shaded triangle in figure 4) and can be evaluated [33] to yield \( I_{\alpha\beta}(0, \infty) = B(2 - \alpha, \alpha + \beta - 4) - B(3 - \beta, 2 - \alpha) - B(\alpha + \beta - 4, 3 - \beta) \), where \( B(\mu, \nu) = \Gamma(\mu)\Gamma(\nu)/\Gamma(\mu + \nu) \) denotes the beta function. Now since \( |\beta - 3/2| \sim 0.2 \), Eq. (91) can yield only a tiny amplification of the vacuum spectrum, such that \( \alpha \approx 1/2 \), inconsistent with \( \alpha + \beta > 3 \).

To conclude, in this section we have reconsidered the coupling of gauge fields to the scalar metric perturbations [24] and showed that there is no significant amplification of gauge fields in inflation. We first analysed the problem in the mean field approximation (which overestimates the effect), and then we estimated the effect of dissipation from inelastic scatterings by employing a simple approximation for the spectra of \( \Phi \) and \( A_i \). The disagreement with Ref. [24] is probably due to the use of the time dependent perturbation theory in [24], which often leads to spurious secular effects when applied in large time limit.
6 Photon at preheating

Since at the moment there is no satisfactory account of both the resonant inflaton decay and plasma conductivity at preheating, in the present work most of the times we make the conservative assumption that no amplification occurs at preheating epoch. For an heuristic account of this problem see for example Ref. [22], and for a numerical work see [36]. Since explosive particle production is quite typical for many models of preheating, abundant production of charged particles occurs quite generically immediately after inflation. The sudden matching approximation [11] may thus be appropriate, according to which the conductivity becomes large quickly at the beginning of radiation epoch such that the electric field generated in inflation decays, while the magnetic field gets frozen in. In this work we adopt this simple approximation. For a recent reformulation of the matching conditions appropriate for treatment of superhorizon charged scalar fluctuations created in inflation [17] in terms of more sophisticated field theoretical techniques see Ref. [18].

7 Discussion and concluding remarks

We have considered various effects that break conformal invariance of gauge fields in inflation. We first analyse the Drummond-Hathrell action in section 2 which describes a local one-loop modification to the gauge field action in presence of massive fermions, and find only a very modest amplification of gauge fields. On the other hand in section 3 we consider the Dolgov trace anomaly of gauge fields in presence of massless fermions and find a significant gauge field amplification in inflation, provided the number of light fermions is of the order a few hundred. We have then in section 4.1 considered modified kinetic gauge terms by coupling to a scalar field and found that under quite reasonable conditions on the couplings one gets significant amplification of gauge fields in inflation, leading to potentially observable magnetic fields on cosmological scales. In section 4.2 we consider coupling of a pseudoscalar field to the Chern-Pontryagin density, and find only a mild amplification of the gauge field amplitude in inflation, while the vacuum spectrum remains unchanged. We find in this case a nice example of birefringence induced by gravity, where one photon polarization exhibits tachyonic growth in inflation, while the other, which corresponds to a massive excitation, remains of a constant amplitude. Finally, in section 5 we consider gauge field coupling
to scalar metric perturbations \[20\] and find that the effect is by far too weak to produce observable magnetic fields.

When this work was nearing completion, Ref. \[43\] appeared which overlaps in part with section \[4.1\]. The results of section \[4.1\] were presented earlier at the workshop Beyond the Standard Model in Bad Honnef (March 2001).

Acknowledgements

I wish to thank Anne Davis, Kostantinos Dimopoulos, and Ola Törnkvist for collaboration on related issues and for useful comments. I thank Gert Aarts for helpful discussions. I am indebted to Ola Törnkvist for discussions of the gauge field amplification from the Dolgov anomaly analysed in section \[3\].

Appendix A

Here we present derivation of the photon field equation of motion starting with the Drummond-Hathrell action (1). In a conformally flat space-time the equation of motion can be recast as

$$\eta^{\mu\rho} \partial_\mu F_{\rho\nu} + \frac{\beta_e}{M^2} \partial_\mu \left[ b \eta^{\mu\rho} R F_{\rho\nu} + \frac{c}{2} a^2 (R^\mu{}^\rho + R^\rho{}^\mu) F_{\rho\nu} + da^3 \eta_{\tau\nu} R^\mu{}^\gamma{}^\rho{}^\sigma F_{\rho\sigma} \right] = 0.$$  \hspace{1cm} (96)

The metric of a conformally flat space-time can be in general written as $g_{\mu\nu} = a^2(\eta) \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric, and the scale factor $a = a(\tau)$ is a function of conformal time $\tau$. The nonvanishing components of the Riemann tensor have the following simple form

$$R_{0i0}^j = -R_{i00}^j = -\eta_i^j \left( \frac{\partial^2 a}{a} - \left( \frac{\partial_r a}{a} \right)^2 \right)$$

$$R_{i0j}^0 = -R_{0ij}^0 = -\eta_{ij} \left( \frac{\partial^2 a}{a} - \left( \frac{\partial_r a}{a} \right)^2 \right)$$

$$R_{ijk}^l = \left( \eta_i^l \eta_j^k - \eta_i^k \eta_j^l \right) \left( \frac{\partial_r a}{a} \right)^2$$  \hspace{1cm} (97)

such that the curvature tensor components are

$$R_{00} = R_{00} = -3 \left( \frac{\partial^2 a}{a} - \left( \frac{\partial_r a}{a} \right)^2 \right)$$

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\[ R_{ij} = \mathcal{R}^0_{i0j} + \mathcal{R}^l_{ij} = -\eta_{ij} \left( \frac{\partial^2 a}{a} + \left( \frac{\partial a}{a} \right)^2 \right), \]  

(98)

and the Ricci curvature scalar reads

\[ \mathcal{R} = -6 \frac{\partial^2 a}{a^3} = -8\pi G \rho (1 - 3w), \]  

(99)

where \( w = p/\rho \) \((-1 \leq w < 1)\). To get the last equality we used Einstein’s equation

\[ R_{\mu\nu} - \mathcal{R}g_{\mu\nu}/2 = 8\pi GT_{\mu\nu} \]  

which, in a conformal space time with the metric \( g_{\mu\nu} = a^2 \eta_{\mu\nu} \), read

\[ H^2 \equiv \left( \frac{\partial a}{a} \right)^2 = \frac{8\pi G}{3} \rho \quad \text{(Friedmann)} \]

\[ \frac{\partial^2 a}{a} = \frac{\partial^2 a}{a^3} - \left( \frac{\partial a}{a^2} \right)^2 = -\frac{4\pi G}{3} (\rho + 3p), \]  

(100)

where we used the ideal fluid stress energy tensor \( T^\mu_\nu = \delta(\rho, -p, -p, -p) \), where \( \rho \) and \( p \) denote the (total) energy density and pressure, respectively. When \( \rho + 3p < 0 \) \((-1 \leq w < -1/3)\) the Universe undergoes an accelerating expansion (inflation); the limiting case \( w = -1 \) corresponds to de Sitter space-time, while matter and radiation era are obtained when \( w = 0 \) and \( w = 1/3 \). More generally, \( \rho \propto a^{-3(1+w)} \), such that upon integrating the Friedmann equation one obtains \( a \propto \tau^{2/(1+3w)} \) (in de Sitter inflation \( a = -1/H \tau \)).

Now making use of equations (97 – 99) and working in Coulomb gauge in which \( A_0 = 0 = \nabla \cdot \vec{A} \), we simplify equation (99) to get

\[ \left( \partial^2 - \nabla^2 \right) A^T_i + \frac{\beta_e}{M^2} \left[ (6b + c) \frac{\partial^2 a}{a^3} + (c + 2d) \left( \frac{\partial a}{a^2} \right)^2 \right] \nabla^2 A^T_i 
\]

\[ + \frac{\beta_e}{M^2} \partial_\tau \left[ - (6b + 3c + 2d) \frac{\partial^2 a}{a^3} \partial_\tau A^T_i + (3c + 2d) \left( \frac{\partial a}{a^2} \right)^2 \partial_\tau A^T_i \right] = 0, \]  

(101)

where we used \( F_{0i} = \partial_\tau A^T_i \) and \( \partial_j F_{ji} = \nabla^2 A^T_i \), and \( A^T_i = (\delta_{ij} - \partial_i \partial_j/\nabla^2) A_j \) denotes the transverse photon field. Making use of Einstein’s equations (3), equation (99) can be recast as

\[ \left( \partial^2 - \nabla^2 \right) A^T_i + \frac{\beta_e}{M^2 M_\rho^2} \left[ \frac{1}{6} (6b + 3c + 4d) \rho - \frac{1}{2} (6b + c) \right] \nabla^2 A^T_i 
\]

\[ + \frac{\beta_e}{M^2 M_\rho^2} \partial_\tau \left[ \left( - \frac{1}{6} (6b - 3c - 2d) \rho + \frac{1}{2} (6b + 3c + 2d) \right) \partial_\tau A^T_i \right] = 0, \]  

(102)

where \( M_\rho \equiv (8\pi G)^{-1/2} = 2.4 \times 10^{18} \text{GeV} \) is the reduced Planck mass. With \( H^2 = \rho/3M_\rho^2 \), \( p = w\rho \), and \( b = -5, c = 26 \) and \( d = -2 \), we finally arrive at

\[ \partial_\tau \left[ \left( 1 + 2\beta_e \frac{H^2}{M^2} (26 + 33w) \right) \partial_\tau A^T_i \right] - \left( 1 - 2\beta_e \frac{H^2}{M^2} (10 + 3w) \right) \nabla^2 A^T_i = 0. \]  

(103)
In section 2 we use this equation to study the time evolution of the photon field in inflation.

In order to make a comparison with the photon coupling to a scalar field studied in Ref. [15], first note that equation (102) yields

\[ \partial_\tau \left[ \left( 1 + \frac{15}{16} \beta_e \frac{H^2}{M_s^2} (1 - 6 \xi) (1 - 3w) \right) \partial_\tau A_i^T \right] - \left( 1 + \frac{3}{16} \beta_e \frac{H^2}{M_s^2} [7 - 30 \xi - (13 - 90 \xi)w] \right) \nabla^2 A_i^T = 0, \]  

(104)

where we made use of the Schwinger-DeWitt coefficients \( b_s = (-5/16) + (15/8)\xi, \ c_s = 1/4 \) and \( d_s = -3/8, \) and \( M_s \) is the scalar field mass. This case is quite different from the fermionic case studied in section 2 in that for \( \xi \leq 1/6 \) the photon field is damped in inflation. For a conformally coupled scalar field Eq. (104) reduces to

\[ \partial_\tau^2 A_i^T - \left( 1 + \frac{3}{8} \beta_e \frac{H^2}{M_s^2} (1 + w) \right) \nabla^2 A_i^T = 0, \quad \xi = \frac{1}{6} \]  

(105)

and no significant amplification occurs in inflation. On the other hand when \( \xi > 1/6 \) the photon coupled to a scalar field the amplitude grows in inflation. In this case the gauge field amplification can be obtained by effecting the following replacement in Eq. (8):

\[ \tau_w \rightarrow \tau_{\xi w} = \frac{15}{16} \beta_e (6 \xi - 1) (1 - 3w) \frac{H^2}{M_s^2}, \quad \xi > \frac{1}{6}. \]  

(106)

When both massive fermionic and scalar fields are present, one should add contributions from both fields. In de Sitter inflation \( (w = -1 \) and \( H = \text{const}) \) Eq. (104) simplifies to

\[ \left( 1 + \frac{15}{4} \beta_e \frac{H^2}{M_s^2} (1 - 6 \xi) \right) \left( \partial_\tau^2 - \nabla^2 \right) A_i^T = 0, \]  

(107)

and thus conformal invariance is recovered, just like in the fermionic case (4).

**Appendix B**

We now outline computation of a volume averaged magnetic field strength for a given gauge field spectrum of the form

\[ A_k = C_0 V^{\frac{1}{2}} k^{-\alpha} \cos(k \tau + \varphi_0), \]  

(108)

where \( \alpha = \theta_w + 1/2 \) for the particular spectrum (21) – (22), and we displayed explicitly the volume \( V \) dependence, which was for simplicity ignored in the Wronskian normalization (13).
Following Ref. [21] we define the volume averaged magnetic field correlated on a scale $\ell$ as:

$$B_\ell^2 = \langle B_i(\ell, \vec{x})B_i(\ell, \vec{x}) \rangle - \langle B_i(\ell, \vec{x})B_i(\ell, \vec{x}) \rangle_{\text{vac}},$$

$$B_i(\ell, \vec{x}) = \frac{3}{4\pi \ell^3} \int_{|\vec{y} - \vec{x}| \leq \ell} d^3 y B_i(\vec{y}),$$

(109)

where the average $\langle \cdot \rangle$ is taken over Fock space as well as the position $\vec{x}$, and we subtracted the (divergent) vacuum contribution $\langle \cdot \rangle_{\text{vac}}$. The definition (109) corresponds to a sharp ball window function $w(\vec{x} - \vec{y}, \ell) = 1$ for $|\vec{x} - \vec{y}| \leq 1$, and zero otherwise.

Ignoring for the moment the vacuum contribution, and making use of the Fourier transform

$$B_i(\vec{y}, \tau) = \int d^3 k' (2\pi)^3 e^{i\vec{k}' \cdot \vec{y}} i\epsilon_{ijl} k'_j A_i(k', \tau), \quad A_i(k', \tau) = \epsilon^T_i A_{k', \tau},$$

(110)

where $\epsilon^T_i$ denotes the unit transverse polarization 3-vector. To evaluate the incurring integrals it is convenient to introduce the following new variables: $\vec{r}_1 = \vec{y} - \vec{x}$, $\vec{r}_2 = \vec{z} - \vec{x}$, $\vec{K} = (\vec{k}' - \vec{k}'')/2$ and $\vec{k} = (\vec{k}' - \vec{k}'')/2$, such that after some algebra we obtain

$$B_\ell^2 = \left( \frac{3}{4\pi \ell^2} \right)^2 \frac{1}{8V} \int_{k_0}^{\Lambda} \frac{dk}{k^2} \langle A_k A_{-k} \rangle (2k\ell \cos 2k\ell - \sin 2k\ell)^2,$$

(111)

where we introduced the infrared and ultraviolet cut-offs, $k_0$ and $\Lambda$, respectively, which can be sent to infinity at the end of calculation. Keeping finite cutoffs is useful for discussion of the divergent limits, $\alpha_{\text{vac}} = 1/2$ (vacuum spectrum) and $\alpha_{\text{sc.inv.}} = 5/2$ (scale-invariant spectrum). Equation (111) can be further simplified

$$B_\ell^2 = \frac{9 \times 2^{2\alpha-7}}{\pi^2} C_0^2 \nu_\alpha, \quad \nu_\alpha(x_0, x_1) = \int_{x_0}^{x_1} \frac{dx}{x^{2(\alpha-1)}} j_1^2(x),$$

(112)

where $x_0 = 2k_0\ell$, $x_1 = 2\Lambda\ell$, $j_1 = d j_0/dx = (\cos x - \sin x/x)/x$ denotes the spherical Bessel function, and we used the long time-average $\langle \cos^2(k\tau + \varphi_0) \rangle_\tau = 1/2$. When $1/2 < \alpha < 5/2$ the integral $\nu_\alpha(x_0 \to 0, x_1 \to \infty)$ converges and can be evaluated exactly (cf. Eqs. 6.576.2 and 9.122.1 in Ref. [33]). The result is

$$\nu_\alpha(0, \infty) = \frac{\sqrt{\pi} \Gamma \left( \frac{5}{2} - \alpha \right) \Gamma \left( \alpha - \frac{1}{2} \right)}{4 \Gamma(\alpha) \Gamma \left( \alpha + \frac{3}{2} \right)}, \quad \frac{1}{2} < \alpha < \frac{5}{2}.$$  

(113)

Putting everything together, we obtain

$$B_\ell = b_\alpha C_0 \ell^{\alpha - \frac{1}{2}}, \quad b_\alpha^2 = \frac{9 \times 2^{2\alpha-9}}{\pi^{3/2}} \frac{\Gamma \left( \frac{5}{2} - \alpha \right) \Gamma \left( \alpha - \frac{1}{2} \right)}{\Gamma(\alpha) \Gamma \left( \alpha + \frac{3}{2} \right)}, \quad \frac{1}{2} < \alpha < \frac{5}{2},$$

(114)
such that \( b_1 = \sqrt{3}/(8\sqrt{\pi}) \) for the thermal spectrum and \( b_3 = 3/8\pi \) for the spectrum \( B_\ell \propto \ell^{-1} \).

For the vacuum spectrum (\( \alpha = 1/2 \)) the integral \( \iota_{3/2} \) diverges in the UV. Upon introducing an UV cut-off, the leading-log contribution to the integral equals \( \iota_{3/2} = (1/2) \ln 2\Lambda \ell \), and hence the spectrum reads

\[
B_\ell^{\text{vac}} = \frac{3}{8\pi \sqrt{2}} C_0 (\ln 2\Lambda \ell)^{\frac{3}{2}} \ell^{-2}, \quad \alpha = \frac{1}{2}.
\]  

The question is of course what is the natural value for the UV cut-off. At a first sight it may seem reasonable to choose the Hubble scale \( \Lambda \sim H \), or the Planck scale \( \Lambda \sim M_P \). We believe however that it is more natural to take \( \Lambda \sim e/2\ell \), implying that measurements at some scale \( \ell \) are mostly sensitive to the field excitations at that scale. With this equation (115) becomes cut-off independent:

\[
B_\ell^{\text{vac}} \sim \frac{3}{8\pi \sqrt{2}} C_0 \ell^{-2}, \quad \alpha = \frac{1}{2}.
\]  

For the scale invariant spectrum \( \alpha = 5/2 \), the integral \( \iota_\alpha \) becomes infrared dominated, suggesting an infrared cut-off, \( k_0 \). To leading-log accuracy \( \iota_\alpha (x_0, \infty) = -(1/9) \ln x_0 (x_0 = 2k_0\ell) \), so that

\[
B_\ell^{\text{sc.inv.}} = \frac{1}{2\pi} C_0 (-\ln 2k_0\ell)^{\frac{5}{2}}, \quad \alpha = \frac{5}{2}.
\]  

In this case the natural cut-off is the lowest momentum that gets amplified in inflation, implying that information about the beginning of inflation is present in the spectrum. This should not be taken very seriously however, because in practice it may be very difficult to distinguish a scale invariant spectrum from an almost scale invariant spectrum with \( \alpha \lesssim 5/2 \) in which the dependence on the initial conditions in inflation drops out. For this reason, when plotting the scale-invariant spectrum we take \( k_0 = e/2\ell \), for which equation (117) becomes truly scale invariant, \( B_\ell^{\text{sc.inv.}} \sim C_0/2\pi \).

**Appendix C**

Here we consider two simple inflationary models: chaotic inflation and extended inflation.

**C.1 Chaotic inflation with \( V(\phi) = m^2\phi^2/2 \)**

For simplicity we assume that \( \phi \) is the inflaton with the potential \( V(\phi) = m^2\phi^2/2 \), where the cosmic microwave background radiation (CMBR) measurements constrain \( m \approx 1.8 \times 10^{13} \) GeV. The considerations in this section can be easily generalized to any power-law
potential $V_n(\phi) = (\lambda_n/n!)(\phi/M_P)^n$, with $\lambda_4 \approx 5 \times 10^{-13}$, etc. The equation of motion for the inflaton $\phi$ reads
\[ \frac{d^2\phi}{dt^2} + 3H \frac{d\phi}{dt} + m^2 \phi = 0, \] (118)
where $H^2 = \rho/3M_P^2$, and $\rho$ is the total energy density. Equation (118) can be easily solved in the slow-roll approximation \[ in which \frac{d^2\phi}{dt^2} \ll 3H \frac{d\phi}{dt}, \frac{\partial V}{\partial \phi} \text{ and } (\frac{d\phi}{dt})^2 \ll 2V \] such that
\[ H \equiv \frac{1}{a} \frac{da}{dt} = \frac{m\phi}{\sqrt{6}M_P}. \] (119)
The result is
\[ \phi = \phi_0 - \sqrt{\frac{2}{3}} M_P m t, \] (120)
where $\phi_0$ can be related to the number of e-folds before the end of inflation, $\phi_0(N) \approx 4\sqrt{6}M_P^2N$. Integrating (119) once we then get for the scale factor
\[ \ln \frac{a}{a_0} = \frac{m\phi_0 t}{\sqrt{6}M_P} - \frac{m^2 t^2}{6} \approx Ht. \] (121)
The time dependence of $H$ becomes important only at late times in inflation. This then immediately implies
\[ a \approx -\frac{1}{H\tau} \] (122)
and
\[ \frac{d\phi}{d\tau} \approx a \frac{d\phi}{dt} \approx \frac{2M_P^2}{\phi} \frac{1}{\tau} \] (123)
where $\phi^2(\tau) \approx 4\sqrt{6}M_P^2N$.

C.2 Extended inflation

Extended inflation \[ is realised by an exponential potential $V(\phi) = M_P^4 e^{-\lambda' \phi/M_P}$, and it corresponds to the Jordan-Brans-Dicke theory of gravity rewritten in the Jordan frame. For an alternative formulation see \[. In its original version the model has limitations related to the graceful exit problem and the CMBR constraints. The spectral slope $n$ of scalar

\[ ^3 \text{For more formal treatment of the slow-roll conditions see for example \[.} \]
\[ ^4 \text{An alternative definition is the following time averaged expansion rate: } \bar{H} = H_0 - \frac{m^2 t^2}{6} \approx H_0 - (m^2/6H_0) \ln(-H_0\tau), \text{ where } H_0 = \frac{m\phi}{\sqrt{6}M_P}. \text{ This definition gives the exact result for the scale factor \[ when expressed in terms of cosmic time $t$.} \]
cosmological perturbations $n - 1 = -\lambda^2$, and the fraction of perturbations in gravitational radiation $r = 5\lambda^2$ are constrained to be much smaller than unity \cite{39}. Here we assume that the former problem is solved by matching at the end inflation onto a power law potential, while the latter can be resolved by an appropriate choice of the coupling constant $\lambda'$. We discuss extended inflation primarily for its simplicity, and focus on the attractor solution \cite{42,41} because it corresponds to more generic initial conditions than the slow-roll case.

The inflaton equation of motion reads

$$
\frac{d^2 \phi}{dt^2} + 3H \frac{d\phi}{dt} - \lambda' \frac{V_0}{M_P} e^{-\lambda' \phi / M_P} = 0, 
$$

where

$$
H^2 = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V(\phi).
$$

It is quite easy to show that equations (124-125) are solved by the following attractor solution:

$$
\frac{d\phi}{dt} = \frac{2M_P}{\lambda' t},
$$

$$
\phi = \frac{2M_P}{\lambda'} \left( \ln \sqrt{V_0} \frac{t}{M_P} - \frac{1}{2} \ln \left[ \frac{2}{\lambda'^2} \left( \frac{6}{\lambda'^2} - 1 \right) \right] \right),
$$

such that

$$
V = \frac{2}{\lambda'^2} \left( \frac{6}{\lambda'^2} - 1 \right) \frac{M_P^2}{t^2},
$$

and hence

$$
H \equiv \frac{1}{a} \frac{da}{dt} = \frac{2}{\lambda'^2 t}.
$$

Integrating this once gives

$$
a = \left( M_0 t \right)^{2/\lambda'^2},
$$

where $M_0 < M_P$ is a mass scale that characterises the beginning of inflation, $t_0 \sim 1/M_0$. This immediately implies that $\lambda' < \sqrt{2}$ is required to get a superluminal expansion. On the other hand to get a nearly scale-invariant spectrum of perturbations, $\lambda' \ll 1$. Since inflation is assumed to terminate at $\phi \approx 0$, in order to get more than 60 $e$-folds (required to solve the standard cosmological problems), the potential $V_0$ should be sufficiently small, such that at the beginning of inflation we have $V(\phi_0) \ll M_P^2$. Now making use of $ad\tau = dt$ we can reexpress (129) as

$$
a = \left( - \left( \frac{2}{\lambda'^2} - 1 \right) M_0^2 \tau \right)^{-2/\lambda'^2},
$$

36
so that

\[ \frac{d\phi}{d\tau} = a \frac{d\phi}{dt} = -\frac{2M_P}{\lambda'\left(\frac{2}{\lambda^2} - 1\right)} \frac{1}{\tau}. \]  

Finally, equation (126) can be expressed as a function of conformal time as follows:

\[ \phi = \phi_0 - \frac{2\lambda'}{2 - \lambda'^2} M_P \ln \left[ -\left(\frac{2}{\lambda^2} - 1\right) M_0 \tau \right] \]

\[ \phi_0 = -\frac{M_P}{\lambda'} \ln \left[ \frac{2}{\lambda'^2} \left(\frac{6}{\lambda^2} - 1\right) \frac{M_P^2 M_0^2}{V_0} \right]. \]

Equations (123) and (131) are used in section [1] for computation of magnetic field amplification in inflation.

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