ON TITS’ CENTRE CONJECTURE FOR FIXED POINT SUBCOMPLEXES

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Abstract. We give a short and uniform proof of a special case of Tits’ Centre Conjecture using a theorem of J-P. Serre [8] and a result from [1]. We consider fixed point subcomplexes $X^H$ of the building $X = X(G)$ of a connected reductive algebraic group $G$, where $H$ is a subgroup of $G$.

1. Introduction

Let $G$ be a connected reductive linear algebraic group defined over an algebraically closed field $k$. Let $X = X(G)$ be the spherical Tits building of $G$, cf. [10]. Recall that the simplices in $X$ correspond to the parabolic subgroups of $G$, [8, §3.1]; for a parabolic subgroup $P$ of $G$, we let $x_P$ denote the corresponding simplex of $X$. The conjugation action of $G$ on itself naturally induces an action of $G$ on the building $X$, so the image of $G$ is a subgroup of the automorphism group of $X$. Given a subcomplex $Y$ of $X$, let $N_G(Y)$ denote the subgroup of $G$ consisting of elements which stabilize $Y$ (in this induced action).

Recall the geometric realization of $X$ as a bouquet of $n$-spheres. A subcomplex $Y$ of $X$ is called convex if whenever two points of $Y$ (in the geometric realization) are not opposite in $X$, then $Y$ contains the unique geodesic joining these points, [8, §2.1]. A convex subcomplex $Y$ of $X$ is contractible if it has the homotopy type of a point, [8, §2.2]. The following is a version due to J-P. Serre of the so-called “Centre Conjecture” by J. Tits, cf. [9, Lem. 1.2], [6, §4], [8, §2.4], [11]. This has been proved by B. Mühlherr and J. Tits for spherical buildings of classical type [5]. The simplex referred to in the conjecture is called a centre for $Y$.

**Conjecture 1.1.** Let $Y$ be a convex contractible subcomplex of $X$. Then there is a simplex in $Y$ which is stabilized by all automorphisms of $X$ which stabilize $Y$.

For a subgroup $H$ of $G$ let $X^H$ be the fixed point subcomplex of the action of $H$, i.e., $X^H$ consists of the simplices $x_P \in X$ such that $H \subseteq P$. Thus, if $H \subseteq K \subseteq G$ are subgroups of $G$, then we have $X^K \subseteq X^H$; observe that $X^H$ is always convex, cf. [8, Prop. 3.1]. Our main result, Theorem 3.1, gives a short, conceptual proof of a special case of Conjecture 1.1; namely, we consider subcomplexes of the form $Y = X^H$ for $H$ a subgroup of $G$, and we consider automorphisms only from $N_G(Y)$. The special case $G = \text{GL}(V)$ in Theorem 3.1 generalizes the classical construction of upper and lower Loewy series, see Remark 3.2(ii).

The initial motivation for Tits’ Conjecture 1.1 was a question about the existence of a certain parabolic subgroup associated with a unipotent subgroup of a Borel subgroup of $G$ (cf. [6, §4.1], [8, §2.4]). This existence theorem was ultimately proved by other means, [3, §3]. In Example 3.6 below we show that the existence of such a parabolic subgroup can be viewed as a special case of Theorem 3.1.

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2. Serre’s notion of complete reducibility

Following Serre [8, Def. 2.2.1], we say that a convex subcomplex $Y$ of $X$ is $X$-completely reducible ($X$-cr) if for every simplex $y \in Y$ there exists a simplex $y' \in Y$ opposite to $y$ in $X$. The following is part of a theorem due to Serre, [6, Thm. 2]; see also [8, §2] and [11].

**Theorem 2.1.** Let $Y$ be a convex subcomplex of $X$. Then $Y$ is $X$-completely reducible if and only if $Y$ is not contractible.

The notion of convexity for subcomplexes of $X$ has the following nice characterization in terms of parabolic subgroups due to Serre, [8, Prop. 3.1].

**Proposition 2.2.** Let $Y$ be a subcomplex of $X$. Then $Y$ is convex if and only if whenever $P, P'$, and $Q$ are parabolic subgroups in $G$ with $x_P, x_{P'} \in Y$ and $Q \supseteq P \cap P'$, then $x_Q \in Y$.

Note that many subcomplexes which arise naturally in the building are fixed point subcomplexes. For example, the apartments of $X$ are the subcomplexes $X^T$ for maximal tori $T$ of $G$ and, more generally, the smallest convex subcomplex containing two simplices $x_P$ and $x_{P'}$ is $X^{P \cap P'}$.

Following Serre [8], we say that a (closed) subgroup $H$ of $G$ is $G$-completely reducible ($G$-cr) provided that whenever $H$ is contained in a parabolic subgroup $P$ of $G$, it is contained in a Levi subgroup of $P$; for an overview of this concept see for instance [7] and [8]. In the case $G = \text{GL}(V)$ ($V$ a finite-dimensional $k$-vector space) a subgroup $H$ is $G$-cr exactly when $V$ is a semisimple $H$-module, so this faithfully generalizes the notion of complete reducibility from representation theory. An important class of $G$-cr subgroups consists of those that are not contained in any proper parabolic subgroup of $G$ at all (they are trivially $G$-cr). Following Serre, we call them $G$-irreducible ($G$-ir), [8]. As before, in the case $G = \text{GL}(V)$, this concept coincides with the usual notion of irreducibility. If $H$ is a $G$-completely reducible subgroup of $G$, then $H^0$ is reductive, [7, Property 4].

Since $X^H$ is a convex subcomplex of $X = X(G)$ for any subgroup $H$ of $G$, Theorem 2.1 applies in this case and we have the following result (see [7, p19], [8, §3]):

**Theorem 2.3.** Let $H$ be a subgroup of $G$. Then $H$ is $G$-completely reducible if and only if the subcomplex $X^H$ is not contractible.

**Remark 2.4.** By convention, the empty subcomplex of $X$ is not contractible. This is consistent with Theorem 2.1 because $H$ is $G$-ir if and only if $X^H = \emptyset$, and a $G$-ir subgroup is $G$-cr.

Our next result [11, Thm. 3.10] gives an affirmative answer to a question by Serre, [7, p. 24]. The special case when $G = \text{GL}(V)$ is just a particular instance of Clifford Theory.

**Theorem 2.5.** Let $N \subseteq H \subseteq G$ be subgroups of $G$ with $N$ normal in $H$. If $H$ is $G$-completely reducible, then so is $N$.

3. Tits’ Centre Conjecture for fixed point subcomplexes

Here is the main result of this note.

**Theorem 3.1.** Let $Y$ be a convex, contractible subcomplex of $X$. Suppose that $Y$ is of the form $Y = X^H$ for a subgroup $H$ of $G$. Then there is a simplex in $Y$ which is stabilized by all elements in $N_G(Y)$.
Proof. Let $M$ be the intersection of all parabolic subgroups of $G$ corresponding to simplices in $Y$. Since $H \subseteq M$, we have $X^M \subseteq X^H$. But every parabolic subgroup containing $H$ contains $M$, by definition of $M$. Hence $X^M = X^H$. Set $K := N_G(Y)$. It is clear that $M$ contains $K$, by definition of $K$. Hence $X^K \subseteq X^M$, so it suffices to show that $X^K \neq \emptyset$. Now $Y = X^M$ is contractible, so Theorem 2.3 implies that $M$ is not $G$-cr. Thus, by Theorem 2.5, it follows that $K$ is not $G$-cr and again by Theorem 2.3 that $X^K$ is contractible. In particular, $X^K$ is non-empty, by Remark 2.4. Thus $K$ stabilizes a simplex in $X^M$, as claimed.

Remarks 3.2. (i). Let $H \subseteq K \subseteq G$ be subgroups of $G$ with $H$ normal in $K$. Suppose that $X^H$ is contractible. Since $H$ is normal in $K$, the latter permutes the simplices in $X^H$, and so $K \subseteq N_G(X^H)$. It thus follows from Theorem 3.1 that $K$ fixes a simplex in $X^H$.

(ii). Observe that Theorem 3.1 can be viewed as a generalization of the classical construction of upper and lower Loewy series in representation theory (for definitions, see e.g., [1]). Let $V$ be a finite-dimensional $k$-vector space. Let $H \subseteq K \subseteq GL(V)$ be subgroups of $GL(V)$ with $H$ normal in $K$ and suppose that $V$ is not $H$-semisimple. Then the upper and lower Loewy series of the $H$-module $V$ are proper $K$-stable flags in $V$, and so they provide “natural centres” for the action of $K$ on the complex $X(V)^H$, where $X(V)$ is the flag complex of $V$.

(iii). In [8, Prop. 2.11], J-P. Serre showed that Theorem 2.5 is a consequence of Tits’ Centre Conjecture [1,1]. So, Theorem 3.1 is just the reverse implication of Serre’s result [8, Prop. 2.11] in the special case when Theorem 2.5 applies.

(iv). Let $k_0$ be any field and let $k$ be the algebraic closure of $k_0$. Suppose that $G$ is defined over $k_0$. One can define what it means for a subgroup $H$ defined over $k_0$ to be $G$-completely reducible over $k_0$, cf. [1, Sec. 5], [8, Sec. 3]. In [1, Thm. 5.8], it is proved that if $k_0$ is perfect, then a subgroup $H$ is $G$-cr over $k_0$ if and only if it is $G$-cr. Using this, one can show that the proof of Theorem 3.1 goes through for buildings of the form $X = X(G(k_0))$. In particular, this includes many finite spherical buildings attached to finite groups of Lie type.

(v). In the Centre Conjecture [1,1], one considers all automorphisms of the building. If $X = X(G)$, then in many cases, $Aut X$ is generated by inner and graph automorphisms of $G$, together with field automorphisms (cf. [10, Intro.]). We will consider graph and field automorphisms in the setting of Theorem 3.1 in future work (see [2, Sec. 6]).

Our final result gives a characterization of subcomplexes of $X$ of the form $X^H$ for a subgroup $H$ of $G$.

Proposition 3.3. Let $Y \subseteq X$ be a subset of simplices of $X$. Then $Y$ is a subcomplex of $X$ of the form $Y = X^H$ for some subgroup $H$ of $G$ if and only if for every $n \in \mathbb{N}$, the following condition holds:

\[(3.4) \quad \text{if } P_1, \ldots, P_n, Q \text{ are parabolic subgroups with } x_{P_i} \in Y \text{ and } Q \supseteq \bigcap_{i=1}^n P_i, \text{ then } x_Q \in Y.\]

Proof. First suppose that $Y = X^H$ for some subgroup $H$ of $G$. Let $n \in \mathbb{N}$ and let $x_{P_1}, \ldots, x_{P_n} \in Y$. If $Q$ is a parabolic subgroup of $G$ containing $\cap_{i=1}^n P_i$, then $Q$ contains $H$, because each $P_i$ does, so $x_Q \in Y$.

Conversely, suppose that condition (3.4) holds for all $n \in \mathbb{N}$. Let $H$ be the intersection of all $P$ such that $x_P \in Y$. By the descending chain condition, we have $H = \cap_{i=1}^m P_i$ for some $m \in \mathbb{N}$ and some $P_i$ with $x_{P_i} \in Y$. It follows from condition (3.4) for $n = m$ that for any parabolic subgroup $P$ containing $H$, $x_P \in Y$, so $X^H \subseteq Y$. It is clear from the definition of $H$ that $Y \subseteq X^H$. \[\square\]
Remark 3.5. Note that $Y$ is a subcomplex of $X$ precisely when condition (3.4) holds for $n = 1$. Further, by Proposition 2.2, $Y$ is convex if and only if condition (3.4) holds for $n = 2$.

As indicated in the Introduction, a fundamental theorem of Borel and Tits on unipotent subgroups of Borel subgroups of $G$ [3] §3 yields a key example for Theorem 3.1.

Example 3.6. Let $U$ be a non-trivial unipotent subgroup of $G$ contained in a Borel subgroup $B$ of $G$. Let $Y = X^U$. Note that $U$ is not $G$-cr; for if $U$ is contained in a Borel subgroup $B^-$ opposite to $B$, then $U$ is contained in the maximal torus $B^- \cap B$ of $G$, which is absurd. So $Y$ is contractible, by Theorem 2.3. Thus, by Theorem 3.1, $N_G(U)$ stabilizes a simplex in $Y$, i.e., there is a parabolic subgroup $P$ of $G$ containing $N_G(U)$. Now, the construction of Borel and Tits in [3] yields such a parabolic subgroup $P$ which enjoys additional properties; for example, it is stabilized by automorphisms of $G$ which stabilize $U$. The framework for $G$-complete reducibility developed in [11] and subsequent papers allows one to associate such canonical parabolic subgroups to all non-$G$-cr subgroups of $G$, see [2] Sec. 5.

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