ABC of Order Dependencies

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ABSTRACT

We enhance constrained-based data quality with approximate band conditional order dependencies (abcODs). Band ODs model the semantics of attributes that are monotonically related with small variations without there being an intrinsic violation of semantics. The class of abcODs generalizes band ODs to make them more relevant to real-world applications by relaxing them to hold approximately (abODs) with some exceptions and conditionally (bcODs) on subsets of the data. We study the problem of automatic dependency discovery over a hierarchy of abcODs.

First, we propose a more efficient algorithm to discover abODs than in recent prior work. The algorithm is based on a new optimization to compute a longest monotonic band (longest subsequence of tuples that satisfy a band OD) through dynamic programming to compute a longest monotonic band (longest subsequence of tuples that belong to the same band–this improves the performance significantly in practice without losing optimality). While unidirectional abcODs are most common in practice, for generality we extend our algorithms with both ascending and descending orders to discover bidirectional abcODs. Finally, we perform a thorough experimental evaluation of our techniques over real-world and synthetic datasets.

1. INTRODUCTION

1.1 Motivation

Modern data-intensive applications critically rely on high quality data to ensure that analyses are meaningful and do not fall prey to the garbage in, garbage out (GIGO) syndrome. In constraint-based data quality, dependencies are used to formalize data quality requirements. Previous work has focused mainly on functional dependencies (FDs) [12]. Several extensions to the notion of an FD have been studied, including order dependencies (ODs) [13, 14, 22], which express rules involving order.

We introduce a novel data dependency approximate band conditional OD (abcOD). Band ODs express order relationships between attributes with small variations causing rules involving order including ODs [14, 22], sequential dependencies [8] and denial constraints [5] to be violated without actual violation of application semantics. To match real world scenarios, we allow band ODs to hold either approximately (abODs) [16] with some exceptions, conditionally (bcODs) over subsets of the data, or both approximately and conditionally (abcODs) with a mix of ascending and descending orders (bidirectional abcODs).

Table 1 contains 22 sample releases of the Music dataset (Reprise records) from Discogs\(^1\) that are integrated from various sources. For tracking purposes music companies assign a catalog number (cat#) to each release of a particular label. When lexicographically ordered by attribute cat#, the release date (encoded using attributes year and month) is also approximately ordered over subsets of the data. Release dates are approximately ordered within subsets of the

\(^1\)www.discogs.com
tuples called series, i.e., \( \{t_1, t_2, t_3\} \) and \( \{t_4, t_5, t_6\} \) modulo ascending and descending orders.

Note that tuple \( t_3 \) has a smaller \( \text{cat#} \) than \( t_5 \) (CDW46012 < CDW46046), however, is released a few months later than tuple \( t_4 \) (1996/Feb > 1995/Oct; for month the sort order is according to the calendar ordering). This is common in the music industry as \( \text{cat#} \) is often assigned to a record before it is released at the production stage. Thus, tuples with delayed release dates will slightly violate an OD between \( \text{cat#} \) and (year, month). A permissible range to accommodate these small variations is called a band.

Attribute year has also a missing value (tuple \( t_{18} \)) and an erroneous value (tuple \( t_2 \)) that severely break the OD between \( \text{cat#} \) and (year, month), as the value of year for tuple \( t_2 \) is 1912 and for tuple \( t_{18} \) is 1996 despite the ascending trend within the series. We verified that the correct value of year for tuple \( t_2 \) should be 1995. (Table 2 shows statistics of violations.)

As another example, since vehicle identification numbers (VINs) for cars are assigned sequentially and independently by different manufacturing plants, attributes VIN and year in car datasets are conditionally ordered over subsets of data. There are small variations to the OD between these attributes as VINs are assigned to a car before it is manufactured and year denotes the time of the completion of the product. There are also actual errors to this band OD (as illustrated in Figure 1a), due to data quality issues. Fig. 1 plots a small sample of the real-world Car dataset\(^2\) and Music dataset series (separated by vertical lines). Series are identified by VIN and \( \text{cat#} \), respectively.

Data dependencies to identify data quality errors can be obtained manually through a consultation with domain experts, however, this is known to be an expensive, time consuming, and error-prone process [8, 12, 22]. Thus, automatic approaches to discover data dependencies to identify data quality issues are needed. The key technical problem that we study is how to automatically and efficiently discover data dependencies over a hierarchy of abcODs classes. While band ODs to permit small variations or approximate band ODs (abODs) to accommodate errors might be sufficient in some applications, there is also a need to study band conditional ODs (bcODs) and approximate band conditional ODs (abcODs) that hold over subsets of the data. Unidirectional abcODs are most common in practice (as we verified experimentally in Sec. 6), however, in some applications to understand the data one needs the additional semantics of ascending and descending orders that the bidirectional abcOD captures. Automatically discovered abcODs can be manually validated by domain experts, which is a much easier task than manual specification, and identified errors corrected subsequently.

### 1.2 Contributions

There are two variants of data dependency discovery algorithms. The first one is a global approach to automatically find all dependencies that hold in the data [12, 13, 14, 22]. The second one is a relativistic approach to find subsets of the data obeying the expected semantics [8, 9], which is laborious to do manually. We apply a hybrid approach to the discovery of abcODs that combines these two approaches.

To automatically identify candidates for embedded band ODs without human intervention, we use a cheaper global approach that finds all traditional ODs within an approximation ratio [13, 22]. The approach in [13, 22] is limited to identifying ODs that (i) do not permit small variations within a band, thus, we deliberately set the approximation ratio higher, and (ii) hold over the entire dataset rather than subsets of the data, thus, we separate the data into segments by using a divide-and-conquer approach. We use the identified traditional ODs, ranked by the measure of interestingness [13, 22], as candidate embedded band ODs to solve the problem of discovering abcODs.

We define the problem of abcOD discovery as an optimization problem desiring parsimonious number of segments that identify using conditions large fractions of the data (gain) each of which satisfies the embedded band OD with few violations (cost).

We make the following contributions in this paper.

1. **Hierarchy of abcOD classes.**
   - (a) We introduce a novel approximate band conditional OD (abOD) integrity constraint. Band ODs are based on small variations causing traditional ODs to be violated without an actual violation of application semantics. abcODs generalize band ODs to make them more applicable to real-world data by relaxing their requirements to hold approximately (abODs) [16] with some exceptions and conditionally (bcODs) on subsets of the data. abcODs do not consider a mix of ascending and descending orders, as bidirectional abcODs do.
   - (b) We develop methods to automatically compute the bandwidth to allow for small variations and candidate dependencies for the hierarchy of (bidirectional) abcODs to decrease the human burden of specifying them manually.

2. **Discovery over hierarchy of abcODs.**
   - (a) We improve on the recent work on the abODs discovery [16] based on the notion of a longest monotonic band (LMB) to identify longest subsequences of tuples that satisfy a band OD. We provide a new optimization to the dynamic programming algorithm to improve the LMB computation from \( O(n^2) \) to \( O(n \log n) \) time, where \( n \) is the number of tuples.
   - (b) We show that while the discovery of bcODs is relatively straightforward, there are codependencies between approximation and conditioning that introduce new challenges to the abcOD discovery problem. The naive solution to consider all possible segmentations of tuples is prohibitively expensive, as it leads to exponential time complexity. We

### Table 2: Statistics of top-5 music labels of Discogs.

| label            | \# total releases | \# missing years | \# incorrect years |
|------------------|-------------------|------------------|--------------------|
| Capitol Records  | 23075             | 1592             | 86                 |
| Reprise Records  | 9830              | 688              | 804                |
| Ninja Tune       | 2055              | 10               | 33                 |
| V2 Records       | 1551              | 13               | 15                 |
| BGO Records      | 598               | 47               | 15                 |

\(^2\)www.classicdriver.com

![Figure 1: Real-world series in Car and Music datasets.](image)
devise a dynamic programming algorithm based on LMBs that finds the optimal solution in $O(n^2 \log n)$ time. To further decrease the search space, we optimize the algorithm without losing optimality based on pieces, which are contiguous subsequences of tuples that satisfy band monotonicity. The optimized pieces-based algorithm is $O(n^2 \log n)$ time, where $m$ is the number of pieces. Since in practice pieces are large, hence, the number of pieces is much smaller than the number of tuples, the pieces-based algorithm is orders-of-magnitude faster.

(c) We extend our algorithms to account for ascending and descending orders. Based on this, we devise algorithms to discover bidirectional abcODs.

3. Experiments. We experimentally demonstrate the effectiveness and scalability of our solution, and compare our techniques with baseline methods on real-world and synthetic datasets.

We provide basic definitions in Section 2. In Sections 3–5, we study algorithms to discover abODs, abcODs and bidirectional abcODs, respectively. We discuss experimental results in Section 6 and related work in Section 7. We conclude in Section 8.

2. BACKGROUND

We use the following notational conventions.

- **Relations.** $R$ denotes a relation schema and $r$ denotes a specific table instance. Italic letters from the beginning of the alphabet $A, B, C$ denote single attributes. Also, $s$ and $t$ denote tuples in $r$ and $s.A$ denotes the value of an attribute $A$ in a tuple $s$. $dom(A)$ denotes the domain of an attribute $A$.

- **Lists.** Bold letters from the end of the alphabet $X, Y$ and $Z$ denote lists of attributes. $[A, B, C]$ denotes an explicit list of attributes. $dom(X) = dom(A) \cdot dom(B) \cdot dom(C)$ denotes the domain of $X$, where $X = [A, B, C]$. $s.X$ denotes the value of the list of attributes $X$ in the tuple $s$.

Let $d$: $dom(X) \cdot dom(X) \to \mathbb{R}$ be a distance function defined on the domain of $X$. Distance function $d$ satisfies the following properties: anti-symmetry, triangle inequality and identity of indiscernibles. We consider $d(x_1, x_2) = ||x_2|| - ||x_1||$, where $||x||$ denotes the norm of the value list $x$. We model an order specification as a directive to sort a dataset in ascending or descending order.

**DEFINITION 2.1 (ORDER SPECIFICATION).** An order specification is a marked list of attributes, denoted as $\overline{Y}$. There are two ordering directions: $asc$ and $desc$, indicating ascending and descending ordering, respectively. As shorthand, $Y^\uparrow$ indicates $Y$ $asc$ and $Y^\downarrow$ indicates $Y$ desc.

**DEFINITION 2.2 (ORDER OPERATOR).** Let $\overline{Y}$ be a marked list of attributes in a relation $r$ and let $\Delta$ be a constant value. For two tuples $t, s \in r$, $t \preceq_\Delta s$ if

- $\overline{Y} = Y^\uparrow$ and $d(t.Y, s.Y) \geq -\Delta$ or
- $\overline{Y} = Y^\downarrow$ and $d(s.Y, t.Y) \leq \Delta$.

Let $t \preceq_\Delta s$ if $s \preceq_\Delta t$ but $s \not\preceq_\Delta t$ and let $t \preceq_\Delta s$ be the operator $t \preceq_\Delta s$, where $\Delta = 0$.

**DEFINITION 2.3 (BIDIRECTIONAL BAND OD).** Given a band-width $\Delta$, a list of attributes $X$, a marked list of attributes $\overline{Y}$ over a relation schema $R$, a bidirectional band order dependency (bidirectional band OD) denoted by $X \rightarrow_{\overline{Y}} \Delta \cdot 1 \cdot \overline{Y}$ holds over a table $r$, if $t \preceq_\Delta s$ implies $t \preceq_\Delta s$ for every tuple pair $t, s \in r$.

**EXAMPLE 2.4.** A bidirectional band OD $cat\# \rightarrow_{\Delta=1} \overline{Y}$ holds over tuples $\{t_1, t_3 - t_9\}$ in Table 1 with a band-width of one year. A bidirectional band OD $cat\# \rightarrow_{\Delta=12} \overline{Y}$ holds over tuples $\{t_1, t_3 - t_9\}$ in Table 1 with a band-width of 12 months.

Bidirectional band ODs specify that when tuples are ordered increasingly on the left-hand-side (cat# in Example 2.4), their right-hand-side (year in Example 2.4) must be ordered non-decreasingly (e.g., wrt series $S_1$ and $S_3$ in Example 2.6 discussed next) or non-increasingly (e.g., wrt the series $S_2$ in Example 2.6) within the specified band-width (e.g., $\Delta = 1$ in Example 2.4).

Since both ascending and descending trends are allowed, the consequent of the dependency is a list of marked attributes. We describe how to automatically compute band-width in Sec. 3.3. Note that traditional bidirectional ODs [14, 13, 22, 23, 24] are a special case of bidirectional band ODs, where $\Delta = 0$. We support other data types than numerical columns including categorical columns. For instance, months can be represented as strings as in the example in Section 1.1 over Table 1. Whenever the distance function can be preserved, the values of the columns are replaced with integers: $1, ..., n$, in a way that keeps the same ordering, i.e., higher values are replaced by larger integers. Computation over integers is more time and space efficient.

We consider a subclass of bidirectional band ODs, called simply band $OD$s for which bidirectionality is removed. We verified experimentally that unidirectional band ODs are most common in real-life applications in Section 6.

**DEFINITION 2.5 (BAND OD).** A bidirectional band OD is called a band order dependency (band OD) when a list of attributes within it is all marked as $asc$ or all as $desc$.

In real-world applications, (bidirectional) band ODs often hold approximately with some exceptions to accommodate errors and conditionally over subsets of the data (series). We call these dependencies (bidirectional) approximate conditional band $OD$s ((bidirectional) abcODs).

**EXAMPLE 2.6.** As shown in Figure 2, there are three series in Table 1: $S_1 = \{t_1 - t_{10}\}$ over bidirectional abcOD cat# $\rightarrow_{\Delta=1} \overline{Y}$, $S_2 = \{t_{10} - t_{14}\}$ over bidirectional abcOD cat# $\rightarrow_{\Delta=1} \overline{Y}$, and $S_3 = \{t_{15} - t_{22}\}$ over bidirectional abcOD cat# $\rightarrow_{\Delta=1} \overline{Y}$. There is a tuple with an erroneous year ($t_5$; correct year is 1995), and a tuple with a missing year ($t_{18}$; correct year is 1988).

We desire parsimonious segments that identify large subsets of data that satisfy a (bidirectional) band OD with few violations. We formally define the discovery problem over a hierarchy of bidirectional abcODs as an optimization problem in Sections 3–5.

3. DISCOVERY OF abODs

In order to reduce the search space of the data dependencies discovery problem, we use the notion of a longest monotonic band (LMB) to identify the longest subsequences of tuples that satisfy a band OD. We formally define LMBs in Sec. 3.1, then present key properties of LMBs, which lead to efficient calculation of LMBs in Sec. 3.2. We use LMBs in Sec. 3.3 to automatically compute the optimal band-width. (Thus, Sec. 3.3 is presented after Sections 3.1–3.2). The computation of LMBs and band-widths are used to discover abODs in Section 3.4 as well as to discover bcODs and (bidirectional) abcODs in Sections 4–5. Since unidirectional abcODs are most common in practice as shown in Table 5, we focus on them in Sections 3–4. Without loss of generality, we use ascending order as default, i.e., a marked list of attributes $\overline{Y}$ is cast to $Y^\uparrow$ when unidirectionality is expected. (We present how to extend our algorithms to cover bidirectionality in Section 5.)
3.1 Defining LMB

In contrast to previous work on identifying a longest monotonic subsequence [1, 4, 6, 8, 17] the definition of a longest monotonic band [16] allows for slight variations. Recall, that when we consider a band OD cat# → yr in Fig. 2 over Table 1, tuples t4 and t2 are considered to be correct (as they are within the band-width) and only tuple t2 is incorrect. LMBs are defined with respect to a band OD X → Y. In the remaining, T = {t1, ... , tn} denotes a sequence of tuples ordered lexicographically by X in ascending order.

**Definition 3.1 (Longest Monotonic Band).** Given a sequence of tuples T = {t1, t2, ..., tn}, a marked list of attributes Y and band-width Δ, a monotonic band (MB) is a subsequence of tuples M = {t1, ..., tk} over T, such that ∀i, j ∈ [1, ..., k]: ti < tj implies ti ⊆ Y tj (as they are within the band-width) and only tuple t2 is incorrect. LMBs are defined with respect to a band OD X → Y. In the remaining, T = {t1, ..., tn} denotes a sequence of tuples ordered lexicographically by X in ascending order.

**Definition 3.2.** Consider a band OD cat# → yr over Table 1. In T = {t1, ..., tn}, ordered by cat#, there is a LMB {t1, t3} with the values of year {1982, 1995, 1999, 2001, 2002}. Note that a LMB can have local decreases within the band-width and is also not necessarily a contiguous sequence of tuples.

**Definition 3.3 (Max & Min Tuples).** Given a sequence T = {t1, ..., tn} and list of attributes Y, tuple ti ∈ T is a maximal tuple, denoted as max(Y(t1, ..., tn), if ∀j ∈ [1, ..., n]: ti ⊆ Y tj and a minimal tuple denoted as min(Y(t1, ..., tn), if ∀j ∈ [1, ..., n]: ti ⊆ Y tj.

**Example 3.2.** Given T = {t1, t3} over Table 1, t3 is a minimal tuple and t6 is a maximal tuple over year.

3.2 Computing LMB

We proposed in prior work [16], an algorithm to compute a LMB in O(n²) time by reducing the problem to finding LMBs in subsequences. Let T[i] denote the prefix of a sequence T of length i, i.e., T[i] = {t1, t2, ..., ti}. Among all MBs that end at ti in T[i], the longest one is kept including its maximal tuple. The solution is based on the optimal substructure property that in order to find a LMB in sequence T[i+1], given those of T[i] til T[i], tuple ti+1 is verified whether it can extend the length of any existing LMBs in T[i] based on their maximal tuples. The MB with the longest length is kept. Once LMBs in subsequences are enumerated, the longest one is chosen as a LMB.

**Example 3.5.** Assume T = {t1, ..., t4} over Table 1, Y = [year] and Δ = 1. Initially, first tuple is a LMB in T[1] of length one with a maximal tuple t1. Since t1 ⊆Δ yr t2, t2 can extend the LMB in T[1] of length two in T[2] with a maximal tuple t2. Similarly, tuple t3 can extend the LMB in T[1], however not the one in T[2] as t3 ⊆Δ yr t2, hence, a LMB of length two in T[3] with the maximal tuple t3 is obtained. Finally, since tuple t4 can extend LMBs in T[1] and T[3], but the one based on T[3] is the longest, a LMB of length three in T is obtained with a maximal tuple t3.

In this work, we propose an efficient algorithm to compute a LMB that improves the complexity to O(n log n) time. The new optimization to reduce the search space is based on maintaining only one monotonic band of each possible length with a minimal tuple among maximal tuples called a best tuple. This is based on the observation that tuple t_i+1 can extend at least one of the previous MBs of predefined length k in all subsequences in T[i] with best tuples denoted as s_k,i if s_k,i ⊆Δ Y t_i+1.

**Definition 3.6 (Best Tuple).** Given a sequence of tuples T, band-width Δ and list of attributes Y, for each k, i ∈ {1, ..., n}, s_k,i is a best MB of MBs of length k in T[i], if s_k,i is a minimal tuple among maximal tuples of all MBs with length k in T[i].

**Example 3.7.** Consider a sequence of tuples T = {t1 = 92, t2 = 12, t3 = 95} and band-width Δ = 1. There are three MBs of length one: t1, t2, and t3 in T, among which t1 is a best tuple. Similarly, there are two MBs of length two: {t1, t2} with a maximal tuple t2 and {t1, t3} with a maximal tuple t3. Thus, tuple t3 is a best tuple of MBs of length two in T.

The following theorem is helpful to decide if tuple t_i+1 extends previous MBs of length k in T[i] to MB with a best tuple of length k+1 in T[i+1] or a MB with a best tuple of length k+1 in T[i] remains the MB with best tuple of length k+1 in T[i+1].

**Theorem 3.1.** Given a band-width Δ, sequence of tuples T and list of attributes Y, let MB_k,i denote a MB with best tuple s_k,i among all MBs of length k in T[i]. If s_k,i ⊆Δ Y t_i+1, then there are two candidates for MB_k+1,i+1: MB_k+1,i+1 with s_k+1,i+1 being its maximal tuple; and a new MB_k,i+1 ∪ {t_i+1} with its maximal tuple over s_k,i+1 and t_i+1, i.e., max(Y(s_k,i+1, t_i+1)).

1. If s_k+1,i+1 is not a minimal tuple among {s_k,i, s_k+1,i, t_i+1}, then MB_k+1,i+1 = MB_k,i+1 ∪ {t_i+1} and s_k+1,i+1 = max(Y(s_k,i, t_i+1)).

2. Else, MB_k+1,i+1 = MB_k,i+1 and s_k+1,i+1 = s_k,i+1.

**Proof.** Consider first Theorem 3.1 case 1. Since d(t_i+1, Y, s_k,i, Y) ≤ Δ, tuple max(Y(s_k,i, t_i+1)) is the maximal tuple of a new MB with length k+1: MB_k+1,i+1. In addition, min(Y{s_k+1,i, max(Y(s_k,i, t_i+1)}) = max(Y(s_k,i, t_i+1)), therefore, max(Y{s_k+1,i, t_i+1}) is the best tuple among MBs with length k+1 in T[i+1]. Theorem 3.1 case 2 follows analogically.

Based on Lemma 3.8, best tuples for each possible k, i ∈ {1, ..., n} can be computed recursively.

**Lemma 3.8.** Let s_k,i Y equal to 0 for i ∈ {0, ..., n} and s_k,0 Y equal to 0 for k ∈ {0, ..., n}. A best tuple s_k+1,i+1 of a MB with length k+1 in a prefix T[i+1] satisfies the following recurrence, where u = min(Y{s_k+1,i, max(Y(t_i+1, s_k,i))}.

s_k+1,i+1 = \begin{cases} u & \text{if } s_k,i \subseteqΔ Y t_i+1 \\ s_k+1,i & \text{otherwise} \end{cases}
Example 3.9. Consider \( T = \{ t_1 - t_2 \} \) over Table 1, \( Y = \{ \text{year} \} \)
and \( \Delta = 1 \). Initially, \( s_{k,1}, Y \) is set to 0 and \( s_{k,0}, Y \) to \( \infty \) for \( i \in \{0, \ldots, 9\} \) and \( k \in \{1, \ldots, 9\} \). First tuple \( t_1 \) (with year ‘92) is checked, if it can extend any MB. Since \( s_{0,0} \preceq \Delta, Y \), a new MB of length one with a maximal tuple \( t_1 \) is found. \( s_{1,1} \) is set to \( \max_x(t_1, \text{year}, s_{0,0}) = '92 \). For each \( k \in \{1, \ldots, 8\}, s_{k,2} \preceq \Delta, Y \), \( s_{0,0} \). Thus, \( s_{k+1,1} \) is set to \( s_{k+1,0} = \infty \). Tuples \( \{t_2-t_9\} \) are processed accordingly.

Also, Lemma 3.10 states that best tuples are ordered by lengths of their monotonic bands.

Lemma 3.10. For each \( i \in [0, \infty] \), best tuples in \( T[i] \) are monotonically ordered, i.e., \( \forall k_1, k_2 \in [0, \infty], s_{k_2, i} \preceq s_{k_1, i} \).

Proof. In case of Theorem 3.1.1, \( s_{k,1} = \max_x(s_{k+1,1}, t_{i+1}) \), hence, \( s_{k,2} \preceq s_{k+1,1} \). In case of Theorem 3.1.2, given \( s_{k+1,1} = s_{k,1} \), we know \( s_{k,2} \preceq s_{k+1,1} \), therefore, \( s_{k,2} \preceq s_{k+1,1} = s_{k,1} \).

Thus, the exhaustive computation to store double-array of all best tuples can be avoided by calculating sorted single-array with best tuples of length \( k \) in \( T \) by keeping track of the shortest and longest length of MBs with a best tuple that ends at \( t_i \) in \( T[i] \). This is the crux of the solution to compute a LMB in \( O(n \log n) \) time with Algorithm 1 presenting the pseudo-code. Array \( B_{inc} \) stores a best tuple \( s_k \) for each \( k \in \{1, \ldots, n\} \) over the sequence, i.e., \( B_{inc}[k] = s_k \).

Initially, \( B_{inc}[0] = \text{max} \) with \( \infty \). \( Y = \infty \) for each \( k \in \{1, \ldots, n\} \). For each tuple \( t_i \) in \( T \), \( B_{inc} \) is updated by finding the \( \text{left-most position} \) of \( t_i \) in \( B_{inc} \), denoted by \( k_1 \) to \( k_2 \), as follows (Line 6).

- \( k_1 \) is the smallest index in \( B_{inc} \) that satisfies \( t_i \preceq Y, s_{k_1} \). It is the shortest length of MBs with a best tuple that ends at \( t_i \) in \( T[i] \).
- \( k_2 \) is the smallest index in \( B_{inc} \) that satisfies \( t_i \preceq \Delta, Y, s_{k_2} \). It is the longest length of MBs with a best tuple that ends at \( t_i \) in \( T[i] \).

\( P_{inc} \) is an array of size \( n \) that stores the set of lengths of shortest and longest MBs with best tuples ending at \( t_i \) for each \( i \in \{1, \ldots, n\} \), i.e., \( P_{inc}[i] = \{ k \mid k \in \{k_1, \ldots, k_2\} \} \). For each \( k \in \{k_1, \ldots, k_2\}, B_{inc}[k] \) is updated by \( \max_x(s_{k-1, i}, t_i) \) (Theorem 3.1, case 1) and adding \( [k_1, k_2] \) to \( P_{inc}[i] \) (Line 11). Given band-width \( \Delta \), each tuple \( t_i \) can contribute to maximally \( \Delta \) MBs, i.e., the sorted array can maximally be updated \( \Delta \) times for each tuple.

Example 3.11. Consider \( T = \{ t_1 - t_3 \} \) over Table 1, \( Y = \{ \text{year} \} \) and \( \Delta = 1 \).

Since, initially \( B_{inc}[k] = \infty \) and \( t_1 \preceq Y, s_{1} \) and \( t_1 \preceq \Delta, Y, s_{1} \), thus, \( k_1 = k_2 = 1 \) and \( B_{inc}[1] = \max(s_{0,1}, t_1) = '92 \).

Next, since \( t_2 \preceq Y, s_{2} \) and \( t_2 \preceq \Delta, Y, s_{2} \), hence, \( k_1 = k_2 = 2 \) and \( B_{inc}[2] = \max(s_{1,2}, t_2) = '92 \).

As \( t_3 \preceq Y, s_{3} \) and \( t_3 \preceq \Delta, Y, s_{3} \), thus, \( k_1 = k_2 = 2 \) and \( B_{inc}[2] = \max(s_{1,3}, t_3) = '96 \).

Therefore, \( B_{inc}[3] = \max(s_{1,4}, t_4) = '95 \).

The details of computing \( B_{inc} \) and \( P_{inc} \) for the following steps are reported in Fig. 3.

Next, we describe how to compute a LMB. The path of a LMB is constructed in a sequence of tuples \( T \) in reverse order by scanning the array \( P_{inc} \). Let \( k \in P_{inc}[i] \) be the largest value in \( P_{inc} \), i.e., there exists a LMB of the longest length \( k \) in a sequence of tuples \( T \); \( t_k \) is found as the \( k^{th} \) tuple in the LMB. Starting from \( P_{inc}[k-1] \) and \( k \), \( P_{inc} \) is scanned in reverse order until the first tuple \( P_{inc}[k-1] \) is found that contains \( k-1 \). Then, \( t_{k-1} \) is found, the \( k-1^{th} \) tuple in the LMB. \( P_{inc} \) is continued to be scanned until all \( k \) tuples in the LMB are found (Lines 13–16).

Example 3.12. Continuing Example 3.11, Fig. 3(f) reports the array \( P_{inc} \) for finding a LMB. \( P_{inc} \) is scanned to find the largest value \( 8 \) in \( P_{inc}[9] \).

Thus, a LMB with length \( 8 \) exists in \( T \) and \( t_8 \) is its eighth tuple.

A reverse scan (with marked arrows in Fig 3) from \( P_{inc}[9] \), the \( 7^{th} \) tuple \( t_8 \) is found. The operation is continued until all tuples in a LMB are found; i.e., \{ \{ t_1('92), t_2('96), t_4('95), t_5('99), t_6('00), t_7('99), t_8('01), t_9('02) \} \}.

Theorem 3.2. Algorithm 1 correctly finds a LMB in a sequence of tuples \( T \) of size \( n \) in \( O(n \log n) \) time and \( O(\Delta + 1) \) space.

Proof. To find a LMB in the sequence \( T \), best tuples are the key. Since a tuple \( B_{inc}[k] \) is updated by \( \max(s_{k-1, Y}, t_i) \), accordingly to Algorithm 1, where \( t_i \preceq Y, s_{k} \), the corresponding band MB \( B_{MB} \) is a MB with smallest maximal tuples that ends at tuple \( t_i \) in the sequence \( T[i] \). It is also a monotonic band with the shortest length, as \( k_1 \) is the smallest index in \( B_{inc} \). Similarly, \( B_{MB} \) is an MB of the longest length among MBs with the smallest maximal tuple that ends at \( t_i \) in \( T[i] \). For each \( t_i \) in \( T \), the lengths of MBs with the smallest maximal tuples that end at \( t_i \) fall into range \{ \{ k_1, k_2 \} \}. Therefore, the length of a LMB in \( T[i] \) is the maximal value in array \( P_{inc}[i] \).

For each tuple in \( T \) of size \( n \), it takes \( O(\log n) \) time to update arrays \( B_{inc} \) and \( P_{inc} \), since they are maintained sorted. Thus, Algorithm 1 takes \( O(n \log n) \) time to find a LMB in \( T \). For each tuple \( t_i \) maximally \( \Delta + 1 \) values are inserted into array \( P_{inc} \). Thus, Algorithm 1 takes \( O(\Delta + 1) \) space.

3.3 Band-Width and Candidate Dependencies

![Figure 3: Finding LMB; tuples in B_{inc} are represented by year.](image-url)
Our goal is to effectively identify outliers in a sequence of tuples, while being tolerant to tuples that slightly violate a traditional OD. Since band ODs may hold over subsets of data called series, to identify the correct band-width, we partition the entire sequence of tuples \( T \) (ordered by \( X \)) over a table \( r \) into contiguous subsequences of tuples \( S \). We identify contiguous subsequences of tuples by using divide-and-conquer method, such that tuples in \( S \) satisfy a traditional OD \( X \rightarrow Y \) within approximation ratio.

We would like to include a large number of tuples from each sequence \( S \) into a LMB by setting an optimal band-width \( \Delta \), such that the distances of outliers from a LMB are large. To capture this, we propose a method to automatically compute the optimal band-width based on LMBs. For a particular band-width \( \Delta \), \( d_{\Delta} \) denotes a distance of outliers from a LMB and \( a_{\Delta} \) denotes a distinctive degree of \( \Delta \) in a sequence of tuples \( S \).

\[
a_{\Delta} = \begin{cases} 0 & \text{if } \Delta = 0; \\ \frac{d_{\Delta - 1} - d_{\Delta}}{d_{\Delta}} & \text{otherwise}. \end{cases}
\]

For each outlier over a tuple \( t_j \) in \( S = \{t_1, \ldots, t_i, \ldots, t_n\} \), let \( t'_j, Y \) denote a repair of \( t_j, Y \). Let \( t_{eft}, Y \) denotes a maximal tuple in \( T[j - 1] \) that is part of a LMB in \( S \), and \( t_{ig} \) denotes a minimal tuple in \( T[j + 1, n] \) that is part of a LMB. We define the estimated repair \( t'_j, Y \) as \( (t_{eft}, Y + t_{ig}, Y) / 2 \).

**Example 3.13.** Consider \( S = \{t_1 - t_9\} \) in Table 1 and let \( \Delta = 1 \). Since the value 2012 of the tuple \( t_2 \) over attribute \( \text{year} \) is incorrect, the repair \( t'_2, \text{year} \) is calculated as \((1992 + 1995) / 2\), which is rounded to 1993.

The distance \( d(t, \text{LMB}) \) of tuple \( t \) from a LMB is computed as \( |d(t', Y, t, Y)| \). The distance \( d_{\Delta} \) of outliers from a LMB is calculated as the average distance i.e., \( d_{\Delta} = \sum_{t \notin \text{LMB}, j \in S} d(t, \text{LMB}) / \{|t : t \notin \text{LMB}, t \in S|\} \). The band-width \( \Delta \) is chosen that maximizes the distinctive degree \( a_{\Delta} \). Note that since entire sequence \( T \) is divided into contiguous subsequences \( S \), the band-width \( \Delta \) is the average aggregated value computed over all subsequences \( S \).

To identify candidate abcODs without human intervention, we use a global approach to find all traditional ODs within an approximation ratio [13, 22] to narrow the search space, as discovering traditional ODs is less computationally intensive. Since band ODs may hold over subsets of the data (with a mix of ascending and descending ordering), we separate an entire sequence of tuples into contiguous subsequences of tuples by using divide-and-conquer approach, such that tuples over contiguous subsequences satisfy a traditional OD within approximation ratio. Found traditional ODs ranked by the measure of interestingness [13, 22] are used as candidate embedded band ODs for the abOD, bcOD and abcOD discovery problems (Sections 3.4–5).

**Example 3.14.** Assume band-width is computed for an attribute \( \text{year} \) over Table 1 wrt an OD between \( \text{cat#} \) and \( \text{year} \) and an approximation ratio of 0.4 (set higher for traditional ODs as they do not take band-width into account). The divide-and-conquer method divides Table 1 into \( T_1 = \{t_1 - t_5\}, T_2 = \{t_6 - t_{10}\}, T_3 = \{t_{11} - t_{16}\} \) and \( T_4 = \{t_{17} - t_{22}\} \). Since distinctive degree value is the highest for band-width of 1 over \( T_1 \) and \( T_2, 2 \) over \( T_3 \) and 0 over \( T_4 \) the averaged band-width \( \Delta \) is 1 (rounded from \((1 + 2 + 0 + 1) / 4\)).

### 3.4 abcODs Discovery

In practice, band ODs may not hold exactly, due to errors in the data, but approximately with some exceptions. Given a band OD \( X \rightarrow Y \), the goal is to verify whether a band OD holds, such that inconsistent tuples that severely violate a band OD are few. As in prior work on functional dependency discovery [12], the minimum number of tuples are computed that must be removed from the given table for the band OD to hold. The problem of discovering approximate band ODs is defined as follows [16].

**Definition 3.15 (abOD Discovery).** Given a band OD \( \phi: X \rightarrow P \rightarrow Y \) and table \( r \), the approximate band OD discovery problem is to identify the minimal set of tuples that violate a band OD with an error ratio \( e(\phi) = \min |\{ t \in X : t \notin Y \} / |\{ t \in Y \} | \).

The measure \( e(\phi) \) has a natural interpretation as the fraction of tuples with inconsistencies affecting the dependency. Band ODs that hold approximately with some exceptions are called approximate band ODs (abODs). The approximate band OD discovery problem can be solved by finding a LMB in a sequence of tuples \( T \) over a table \( r \) ordered by \( X \). The minimal set of tuples that violate a band OD \( X \rightarrow Y \) are inconsistent tuples \( s \), such that \( s \in X, s \notin Y \).

**Lemma 3.16.** The approximate band OD discovery problem is solvable by finding a LMB with an error ratio \( e(\phi) = |s \notin Y \text{LMB}| / |s \notin X \text{LMB}| \).

**Proof.** The proof follows directly from the definition of LMBs (Definition 3.1).

**Example 3.17.** Consider the approximate band OD \( \phi: \text{cat#} \rightarrow \text{year} \rightarrow Y \) over a table with tuples \( \{t_1 - t_9\} \) in Table 1. Given Example 3.14, a LMB of length \( S \) is found, i.e., \( t_1(92), t_3(96), t_4(95), t_5(99), t_6(00), t_7(99), t_8(01), t_9(02) \). Thus, an error ratio \( e(\phi) = 1/9 \).

Since we provided a new optimization to the LMB computation (Theorem 3.2), based on Lemma 3.16, abOD discovery is improved from \( O(n^2) \) (as in prior work [16]) to \( O(n \log n) \).

**Theorem 3.3.** The abOD discovery problem can be solved in \( O(n \log n) \) time, where \( n \) is the number of tuples over a table.

### 4. DISCOVERY OF abcODs

To better understand the hierarchy of abcODs, initially, we consider the scenario when band ODs hold over subset of the data in Section 4.1 (bcODs). Both approximation and conditioning are studied in Sections 4.2–4.4 (abcODs).

#### 4.1 Discovery of bcODs

The band conditional OD discovery problem (without considering approximation) is defined as follows.

**Definition 4.1 (bcOD Discovery).** Let \( X \rightarrow Y \) be a band OD, \( T \) be a sequence of tuples ordered by \( X \) over a table \( r \). Let \( S \) denote all possible contiguous non-overlapping segmentations of \( T \). The band conditional OD (bcOD) discovery problem is to find the segmentation amongst \( S \in \mathcal{S} \) with minimal number of segments, where a band OD \( X \rightarrow Y \) holds in each segment.

The bcOD discovery problem is solvable by scanning the sequence of tuples and splitting it into contiguous segments, whenever an outlier tuple appears; i.e., the first tuple that violates a band OD with respect to the current segment represented by its best tuple. The violating tuple becomes the first tuple in the next segment.

**Theorem 4.1.** The bcOD discovery problem is solvable in \( O(n \log n) \) time and \( O(n) \) space over a sequence of tuples \( T \) of size \( n \).
The problem of abcODs discovery challenging (Sections 4.2–4.4) dependencies ordered by X, given a band OD. The bcOD discovery problem solution consists of two segments: {t1,t2} and {t2−t9}.

Whereas the discovery of bcODs is relatively straightforward, there exist dependencies between abODs and bcODs that make the problem of abcODs discovery challenging (Sections 4.2–4.4)

4.2 Discovery of abcODs Problem

To make band ODs relevant to real-world applications, we make them less strict to hold both approximately with some exceptions and conditionally on subsets of the data. We call these data dependencies approximate band conditional ODs (abcODs). Specifically, given a band OD \( X \rightarrow_{\Delta} Y \), where \( T \) is a sequence of tuples ordered by \( X \), our goal is to segment \( T \) into multiple contiguous, non-overlapping subsequences of tuples, called \( S \), such that (1) large fraction of tuples in each series satisfy a band OD (gain), and (2) outlier tuples that severely violate a band OD in each series are few and sparse (cost). We experimentally verified in Sec. 6 that in practice errors are few and sparse over real-world datasets.

**Definition 4.3 (Gain).** Let \( X \rightarrow_{\Delta} Y \) be a band OD, \( T \) be a sequence of tuples ordered by \( X \) over a table \( r \). \( |T_{nn}| \) denotes the number of non-null tuples in \( T \), and \( L_T \) denotes a LMB in \( T \). The gain \( g(T) \) is defined as the portions of \( T \) satisfying \( X \rightarrow_{\Delta} Y \).

\[
g(T) = |t : t \in L_T, t \in T| - |t : t \notin L_T, t \neq \text{null}, t \in T|
\]

**Example 4.4.** Following Example 3.12, where the LMB \( \{t_1, t_3−t_9\} \) of length 8 is found, and \( t_2 \) is the only outlier, the gain is computed as \( 8−1=7 \).

The outliers are penalized with the following cost function.

**Definition 4.5 (Cost).** Let \( X \rightarrow_{\Delta} Y \) be a band OD, \( T \) be a sequence of tuples ordered by \( X \) over a table \( r \). The cost \( e(T) \) is the maximum number of contiguous outliers that violate \( X \rightarrow_{\Delta} Y \) in \( T \).

\[
e(T) = \max_{k \in \{i−1, \ldots, j\}, 3 \leq i \leq |T|} \sum_{j=1}^{i-1} |t_k \notin L_T, t_k \in T|
\]

**Example 4.6.** Continuing with Example 4.4. Since \( t_2 \) is the only outlier in sequence \( \{t_1−t_9\} \), the cost is 1.

We define the abcOD discovery problem as a constrained optimization problem, aiming to maximize the gain function under the cost constraint.

**Definition 4.7 (abcOD Discovery Problem).** Let \( X \rightarrow_{\Delta} Y \) be a band OD, \( T \) be a sequence of tuples ordered by \( X \) over a table \( r \) and \( \varepsilon \) be an approximation error rate parameter. Let \( S \) denote all possible contiguous, non-overlapping segmentations of \( T \). The approximate band conditional OD (abcOD) discovery problem is to find the optimal segmentation among \( S \) in \( S \) that satisfies the following optimization function.

\[
\max_{S \in S} g(S) \quad \text{s.t. } e(S) \leq \varepsilon
\]

For each segment \( S \) in \( S \), let \( |S_{nn}| \) be the number of non-null tuples in \( S \). The gain \( g(S) \) is computed as sum of \( g(S) \) weighted by its length \( |S_{nn}| \).

\[
g(S) = \sum_{S \in S} g(S) \cdot |S_{nn}|
\]

the cost \( e(S) \) is computed as

\[
e(S) = \max_{S \in S} e(S)
\]

We call band ODs that hold conditionally over subsets of the data and approximately with some exceptions approximate band conditional ODs (abcODs). In Equation 5, a gain function rewards correct tuples weighted by the length of series \( |S_{nn}| \) excluding tuples with null values to achieve high recall. Otherwise small series would be ranked high with an extreme optimal case of all segments being individual tuples, which is obviously not desirable.

**Example 4.8.** Consider a band OD cat\# \( \rightarrow_{\Delta=1} \text{year} \) and an error rate \( \varepsilon = 1 \). Figure 4(e) visualizes two series based on sequence \( T = \{t_1−t_9, t_{15}−t_{22}\} \) in Table 1 with \( S_1 = \{t_1−t_9\} \), where \( t_2 \) is an outlier, and \( S_2 = \{t_{15}−t_{22}\} \), where \( t_{18} \) has a missing year. The segmentation \( \hat{S} = \{S_1, S_2\} \) maximizes the optimization function \( g(S) = (8−1) \cdot 9 + 7 \cdot 7 = 112 \) under a constraint \( \max(e(S_1), e(S_2)) = 1 \leq \varepsilon \).
Our problem of abcODs discovery is not a simple matter of finding splitting points. We study a technically challenging joint optimization problem motivated by real-life applications of finding splits, monotonic bands and approximation (to account for outliers), which is not easily obtained by simple visualization. Also, note that Fig. 1 presents only a small sample of the data extracted from the entire dataset to illustrate the intuition. In practice, the number of data series can be thousands over large datasets (see Table 5), thus, data cannot be split easily into a few segments. We argue that an automatic approach to discover abcODs is needed as formulating constraints manually requires domain expertise, is prone to human errors, and is excessively time consuming. Automatically discovered dependencies can be manually validated by domain experts, which is a much easier task than manual specification. (In our experimental evaluation in Sec. 6 it turned out that all the discovered series are true.) The purpose of our framework is to alleviate the cognitive burden of human specification.

4.3 Computing abcODs

A brute-force solution to the abcOD discovery problem is to consider all possible $2^n - 1$ segmentations of $n$ tuples over the dataset and select the optimal one as specified in Definition 4.7. This has an exponential time complexity, thus, we provide optimizations to compute abcODs more efficiently (in polynomial time). Next, we show that the solution to the abcOD discovery problem in a sequence of tuples contains optimal solutions in subsequences. Thus, the solution to the problem has an optimal substructure property.

**Theorem 4.2.** Let $\text{OPT}(j)$ denote an optimal solution to abcOD discovery problem in $T[j]$ and $T[i, j]$ denotes the subsequence $T[i, \ldots, j]$. The optimal solution $\text{OPT}(j), j \in \{1, \ldots, n\}$ in a prefix $T[j]$ contains optimal solutions to the subproblems in prefixes $T[1], T[2], \ldots, T[j-1]$.

**Proof.** For subsequence $T[j], i \in [1, j - 1]$ with optimal solution $\text{OPT}(i)$, if we force $T[i+1, j]$ to form a single series with gain $g(T[i+1, j])$, among all segmentations, where $T[i+1, j]$ forms a single series in $T[j]$, there does not exist any segmentations with greater gain than $\text{OPT}(i) + g(T[i+1, j])$.

| Table 3: Computing abcOD. |
|---|
| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $j$ | T | 92 | 12 | 1 | 12 | 96 | 95 | 99 | 90 | 01 | 02 | 03 | 04 | 05 | 06 | 07 | 08 | 09 |
| $X$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

For each $i \in [1, j - 1]$, a candidate solution with gain $\text{OPT}(i) + g(T[i+1, j])$ can be found; thus, the optimal solution in a prefix $T[j]$ can be selected among $j$ instead of $2^j - 1$ options. We develop a dynamic-programming algorithm (Algorithm 3) to solve the abcOD discovery problem. Two arrays are maintained of size $n$: array $G$ stores the overall gains of optimal solutions to the subproblems, i.e., $G[j] = \text{OPT}(j), j \in \{1, \ldots, n\}$; and array $X$ stores the corresponding series, i.e., $X[j]$ stores a segment ID $t$ that tuples $\{t_i, t_j\}$ belong to in a prefix $T[j], i \in [1, j]$. For each $i \in [1, j - 1]$, we consider $T[i+1, j]$ as a single series, compute its gain $g(T[i+1, j])$ and cost $e(T[i+1, j])$ by its LMB $\ell_{i+1,j}$ (Line 5–Line 6). If $g(T[i+1, j]) + g(T[j])$ is greater than existing $\text{OPT}(j)$ and its cost is less than the threshold $\epsilon$, we update $\text{OPT}(j)$ by $\text{OPT}(i) + g(T[i+1,j])$, and update segment ID $t$ in $X[j]$ (Line 9).

The optimal segmentation with IDs is collected by scanning segment IDs stored in $X$ in reverse order. Let $i = X[n]$; we split $T$ into subsequence $T[1, i - 1]$ and $T[i, n]$, and tuples $t_i, t_{i-1}$ forms a series. Then, subsequence $T[1, i - 1]$ is split in similar fashion until all tuples in $T$ are assigned to a segment (Line 13–Line 16).

**Example 4.9.** Consider abcOD discovery problem over $T = \{t_1, \ldots, t_{15}\} \in Table 1$ given a band OD cat$\# \to$cat$\# \to$year$^+$ and an error rate $\epsilon = 1$. First, $T[1] = \{t_1\}$ is examined. It forms a singleton series with the gain $\text{OPT}(1) = 1 = (G[1] = 1)$. Next, subsequence $T[2] = \{t_1, t_2\}$ is considered. Tuple $t_2$ can either form its own series with the gain equal to 1 (with the overall gain $1 + \text{OPT}(1) = 2$) or be merged into the same series with $t_1$ with the gain $2^2 = 4$. Thus, $t_2$ and $t_1$ are merged as well as $G[2] = 4$ and $X[2] = 1$. Wrt tuple $t_3$, there are three candidates to the optimal solution in $T[3]: (1) t_3$ forms a single series $\text{OPT}(t_3) = 1$ and optimal solution $\{t_1, t_2\}$ to $T[2]$ remains $g(T[2]) = 4$, thus, the total gain is $1 + 4$; (2) $t_2, t_3$ forms a series $\text{OPT}(t_3) = 4$ = $g(t_2, t_3) = 2^2 = 4$ and optimal solution $\{t_1\}$ to $T[1]$ remains $g(T[1]) = 1$, in which case the total gain is also $4 + 1 = 5$; and (3) $\{t_1, t_2\}$ forms a series, where $t_2$ ($t_1$ $t_2$) is an outlier; hence, the total gain is $2 + 1 = 3$. Among the above candidates option (1) (or 2) is chosen with maximal gain, and $G[3] = 5$; $X[3] = 2$ (or $X[3] = 3$). The rest of tuples are processed accordingly with the results reported in Table 3. To output all series, the array $X$ is checked in reverse order. Given that $X[22] = 15$, an optimal solution is achieved by a series consisting of $t_{15} - t_{22}$ and an optimal solution in $T[9]$; given $X[9] = 1$, optimal solution in $T[9]$ is achieved by a series consisting also of $\{t_1 - t_2\}$.

**Theorem 4.3.** Algorithm 3 solves the abcOD discovery problem optimally in $O(n^2 \log n)$ time in a sequence $T$ of size $n$.

**Proof.** Algorithm 3 applies dynamic programming to solve Equation 7, which is proved in Theorem 4.2. The recurrence in Equation 7 specifies that the optimal solution in subsequence $T[i]$ is selected among $i$ alternative options: (1) a singleton series consisting of $t_i$, and the optimal solution in subsequence $T[1 - i]; (2)$ a series of length 2 consisting of $\{t_i, t_{i+1}\}$, and the optimal solution in subsequence $T[i - 2]$, etc.; and finally, a series of length $i$ consisting of all tuples in subsequence $T[i]$. It requires $O(n)$ iteration to process subsequence $T[i], i \in [1, n]$, where each iteration takes time $O(n \log n)$ based on Theorem 3.2. Thus, Algorithm 3 solves the abcOD discovery problem in time $O(n^2 \log n)$.
4.4 Pruning with Pieces

To further prune the search space, we develop another optimization for the abcOD discovery solution based on the idea of splitting and segmenting. A sequence of tuples is split into pieces, which are contiguous subsequences of tuples that are monotonic within band-width $\Delta$. Then, abcOD discovery segmentation is computed based on pieces to speed up the performance without sacrificing optimality. To split a sequence $T$ into pieces, we first introduce the notion of pre-pieces. To distinguish outliers and boundary tuples between adjacent series, we consider bidirectional orders in the definition of pre-pieces, since they can potentially overlap even for case of discovery of unidirectional abcODs.

Definition 4.10 (Pre-Piece). Given a sequence $T$ of $n$ tuples and a marked list of attributes $\mathbf{Y}$, a contiguous subsequence $T'$ of $\{t_i, t_{i+1}, \ldots, t_j\}$ is a pre-piece (PP) if (1) $\forall_{k \in \mathbb{N}, t \leq t_k \leq \Delta} t_m$ or $t_i, Y = \text{null}$, and (2) $T'$ cannot be extended without violating the property (1). $T'$ is called an increasing pre-piece (IP) if $\mathbf{Y} = \uparrow\uparrow\uparrow$ and a decreasing pre-piece (DP) if $\mathbf{Y} = \downarrow\downarrow\downarrow$.

Example 4.11. Let band-width $\Delta = 1$ and $\mathbf{Y} = \uparrow\uparrow\uparrow$ or $\mathbf{Y} = \downarrow\downarrow\downarrow$. Consider a sequence of tuples $T = \{t_1 - t_6, t_6 - t_2\}$ over Table 1 ordered by an attribute $\text{cat}##$. There are five pre-pieces in $T$, i.e., $\{t_1-t_2\}, \{t_2-t_4\}, \{t_6, t_6, t_15\}$ and $\{t_15-t_22\}$ as marked with dashed lines in Fig. 4(a).

A pairwise-disjoint set of pieces is obtained by separating the intersections of all pre-pieces.

Definition 4.12 (Piece). A piece $P$ is a subsequence in a sequence of tuples $T$ such that: (1) non-overlapping tuples from a pre-piece with other pre-pieces create a separate piece, and (2) overlapping tuples between pre-pieces create a separate piece.

Example 4.13. Continuing Example 4.11 Fig. 4(b) illustrates pieces in $T = \{t_1 - t_6, t_6 - t_2\}$: $P_1 = \{t_1\}, P_2 = \{t_2\}, P_3 = \{t_3, t_4\}, P_4 = \{t_5-t_7\}, P_5 = \{t_6, t_6, t_15\}, P_6 = \{t_15\}$ and $P_7 = \{t_16-t_22\}$.

The pseudo-code of the algorithm to compute pieces is presented in Algorithm 4. Pre-pieces are constructed by scanning iteratively tuples in a sequence $T$. Tuple $t \in T$ can both extend existing IPs and (or) DPs, and form a new IP and (or) DP. To make sure each PP cannot extend existing IP of length 2 that is longer than existing DP with length 1; i.e., a PP $\{t_1 - t_2\}$ is found. $\text{M}_{\text{inc}}[3] = 1$ is updated and the pre-piece boundary indexes 1, 3 is inserted into array $L$. The rest of the tuples are processed accordingly (Figure 5) with pieces boundaries obtained by scanning $L$ reported in Figure 4(b).

Lemma 4.15. Algorithm 4 takes $O((\Delta + 1) \cdot n)$ time to compute pieces in a sequence $T$ of size $n$.

Algorithm 4: Compute Pieces

| input : $T = \{t_1, t_2, \ldots, t_n\}, \Delta$ |
| output : the set of pieces $P$ in $T$ |
| 1 $\text{M}_{\text{inc}} \leftarrow \emptyset, \text{M}_{\text{dec}} \leftarrow \emptyset; P \leftarrow \emptyset; L \leftarrow$ array of size $n$ |
| 2 $i \leftarrow 0$ |
| 3 for each $j$ $\leftarrow \lceil 1, n \rceil$ do |
| 4 $l_{\text{dec}} \leftarrow$ indices in $\text{M}_{\text{dec}}$ updated by $j - 1$ |
| 5 $l_{\text{inc}} \leftarrow$ indices in $\text{M}_{\text{inc}}$ updated by $j - 1$ |
| 6 $l_{\text{dec}} \leftarrow$ maximal value in $\text{M}_{\text{dec}}$ updated by $j - 1$ |
| 7 $l_{\text{dec}} \leftarrow$ maximal value in $\text{M}_{\text{dec}}$ updated by $j - 1$ |
| 8 for each $i \in l_{\text{dec}}$ do |
| 9 $\text{M}_{\text{inc}}[j] \leftarrow \max(\text{M}_{\text{inc}}[j], 1)$ |
| 10 if $l_i \leq \Delta$ or $(t_j)$ then |
| 11 $l_{\text{max}} \leftarrow \max(l_{\text{inc}}, l_j)$ |
| 12 update $\text{M}_{\text{inc}}[\text{max}]$ by $\max(\text{M}_{\text{inc}}[\text{max}], \text{M}_{\text{inc}}[i] + 1)$ |
| 13 else if $l_{\text{inc}} > l_{\text{dec}}$ then |
| 14 $\text{M}_{\text{dec}}[\text{inc}] \leftarrow \max(\text{M}_{\text{dec}}[\text{inc}], 1)$ |
| 15 for each $j \in l_{\text{inc}}$ do |
| 16 $\text{M}_{\text{dec}}[\text{dec}] \leftarrow \max(\text{M}_{\text{dec}}[\text{dec}], 1)$ |
| 17 if $l_i \leq \Delta$ or $(t_j)$ then |
| 18 $l_{\text{min}} \leftarrow \min(l_{\text{inc}}, l_j)$ |
| 19 update $\text{M}_{\text{dec}}[\text{min}]$ by $\max(\text{M}_{\text{dec}}[\text{min}], \text{M}_{\text{dec}}[i] + 1)$ |
| 20 else if $l_{\text{dec}} > l_{\text{inc}}$ then |
| 21 $\text{M}_{\text{dec}}[\text{dec}] \leftarrow \max(\text{M}_{\text{dec}}[\text{dec}], 1)$ |
| 22 while $i < n$ do |
| 23 $P \leftarrow \text{add}(\{t_{ij} - t_{ij+1}\})$ |
| 24 $i \leftarrow i + 1$ |
| 25 return $P$ |

Pieces are used to prune the search space for the abcODs discovery (Algorithm 5). Instead of processing each tuple individually, the sequence is processed piece by piece (Line 6–Line 9). Alg. 5 extends Alg. 3 by additionally keeping tuples in the same piece always within the same segment (Line 11). Note that multiple pieces can belong to the same series.

Example 4.16. Consider abcOD discovery problem over $T = \{t_1 - t_6, t_6 - t_2\}$ in Table 1 given $\text{cat}## \rightarrow \Delta \rightarrow \uparrow\uparrow\uparrow$ and an error rate $\varepsilon = 1$. Fig. 4(b) illustrates pieces $P_1 - P_7$ (Example 4.14). A piece $P_1 = \{t_1\}$ examined first forms a singleton series with the gain $g(P_1) = 1$. Piece $P_2$ can either form its own series with overall gain equal to 2, or be merged with $P_1$ with gain equal to 4, which is the optimal solution in subsequent $P_2, P_2, \ldots$. The rest of pieces are processed accordingly with results reported in Table 4, where the optimal solution is highlighted in each subsequence. Figure 4(c) illustrates the series over pieces $\{P_1, P_2\}$.

Figure 5: Details for pieces computation.
Table 4: Computing abcODs with Pieces.

| g(P₀) | g(P₁) | g(P₀, P₁) | g(P₂) | g(P₀, P₂) | g(P₁, P₂) | g(P₀, P₁, P₂) |
|-------|-------|-----------|-------|-----------|-----------|---------------|
| 4     | 10    | 11        | 8     | 63        | 35        | 63            |
| 8     | 10    | 14        | 3      | 35        | 35        | 63            |
| 35    | 25    | 30        | 35    | 35        | 35        | 63            |
| 63    | 46    | 52        | 34    | 44        | 44        | 64            |
| -     | -     | -         | -     | -         | -         | 112           |
| -     | -     | -         | -     | -         | -         | 100           |

Algorithm 5: Computing Series with Pieces

input : T = \{t₁, t₂, \ldots, tₙ\}, \Delta, \epsilon
output : segmentation S in T
1. P ← \{P₁, P₂, \ldots, Pₙ\} by ComputePieces(T, \Delta)
2. X, G ← two arrays of size m + 1; S ← ∅
3. X ← ∅; G ← \{0, \ldots, 0\}
4. for j ← [1, m] do
5. for i ← [1, j − 1] do
6. \(L_{i+1,j} ← ComputeLMB(P[i+1, j], \Delta)\)
7. \(\epsilon_{i+1,j} ← cost(\epsilon[P[i+1, j]]; g_P[i+1, j])\)
8. if \(\epsilon_{i+1,j} ≤ \epsilon\) and \(G[i+1, j] > G[j]\) then
9. \[G[i+1, j] ← G[i] + g_P[i+1, j]; \Delta \] = \(i\)
10. for j ← n to 1 do
11. \(\Delta \leftarrow \Delta[j]\); add S = \{P_i \} into S
12. if j = 1 then return S

In practice, pieces are large, hence, the number of pieces is small (i.e., \(m \ll n\)), which leads to an efficient and optimal algorithm for abcOD discovery.

**Theorem 4.4.** Algorithm 5 finds optimal solution for abcOD discovery problem in \(O(m^2 \log n)\) time, where \(m\) is the number of pieces in \(T\), and \(n\) is the number of tuples in \(T\).

**Proof.** Assume \(T[i]\) ends at piece \(P_i = \{t_{i-m+1}, \ldots, t_i\}\) of length \(m\). Every tuple in \(P_i\) belongs to the same sets of pre-pieces, there are no outliers that violate a LMB in \(P_i\), i.e., \(g(T[i−m+1, i]) = m^2\). If the algorithm does not find the optimal solution, then there exists a tuple \(t_{i-k} \in P_i\), \(1 \leq k \leq m - 1\) that splits \(P_i\) into two series: \(\{t_{i-m+1}, \ldots, t_{i-k}\}\) and \(\{t_{i-k+1}, \ldots, t_i\}\), where the profit of \(OPT(-k) + k^2\). By contradiction this assumption does not hold, i.e., \(OPT(i) < OPT(i−k) + k^2\). The length of \(t_{i-j}\) is the first tuple in the last series \(S_{i-m}\) of the optimal solution \(OPT(i)\), where the length of a LMB in series \(S_{i-m}\) is \(l\), i.e., \(j \geq m + 1, j > 0\); and the maximal number of consecutive outliers in \(T[i−m] = g\). According to Theorem 3.1, \(\{t_{i-m+1}, \ldots, t_i\}\) extends the length of \(S_{i-m}\) by \(m\) without increasing \(q\). That is, \(OPT(i) = OPT(i−j) + (l + m)^2\). Similarly, \(OPT(i−k) = OPT(i−j) + (l + m − k)^2\). This means that \(OPT(i) − OPT(i−k) = (l + m)^2 − (l + m − k)^2 = 2k(l + m) > k^2\), which leads to the contradiction.

Algorithm 5 first finds all pieces in the sequence \(T\) of length \(n\), which takes time \(O(4n)\). Assume the number of pieces is \(m\), the algorithm applies dynamic programming on \(m\) pieces, similarly as Algorithm 3, which takes time \(O(m^2 \log n)\), as computing LMBs takes time \(O(m^2 \log n)\). Therefore, the overall time complexity is \(O(m^2 \log n)\).

5. **Bidirectional Discovery**

Unidirectional abcODs are most common in practice (as we verified experimentally in Sec. 6), however, in some applications one needs the additional semantics of ascending and descending orders. Thus, for generality we extend our formalism to bidirectional abcODs discovery. To deal with bidirectionality, we introduce the concepts of (longest) increasing and decreasing bands (following Definition 3.1).

**Definition 5.1.** ([Increasing and Decreasing Bands])

A monotonic band is called an increasing band (IB) if \(T = Y^\uparrow\) (and a longest IB (LIB) if it is the longest amongst IBs) and a decreasing band (DB) if \(T = Y^\downarrow\) (and a longest DB (LDB) if it is the longest amongst DBs).

**Example 5.2.** Consider a band OD cat# \(\rightarrow \Delta=1\) year over Table 1 ordered by cat#.

A LMB over \(T\) consisting of tuples \(\{t_{10−t_{14}}\}\) forms a LDB.

To compute a LMB both a LIB (Section 3) and a LDB need to be computed, and the one with longer length is chosen as a LMB. A LDB in \(T\) is symmetrical as calculating a LIB, where the best tuple of DBs of length \(k + 1\) in \(T[i−1]\), denoted as \(l_{k+1,i−1}\), satisfies the following recurrence,\(\upsilon = \max\{l_{k+1,i}, min\{l_{k+1,i−1}, l_{k+1,i−2}\}\} l_{k+1,i−1} = \left\{\begin{array}{ll} \upsilon \quad k_{k+1,i} \geq \Delta \quad Y_{i−1} \\
\upsilon \quad \text{otherwise} \end{array}\right.\)

The adapted Algorithm 3 to consider symmetry between IBs and DBs remains optimal for discovery of bidirectional abcODs.

**Theorem 5.1.** Extended Algorithm 3 solves bidirectional abcOD discovery problem optimally in \(O(n^3 \log n)\) time over a sequence of tuple \(T\) of size \(n\).

However, pieces-based algorithm (Algorithm 5) may produce sub-optimal solutions over datasets with bidirectional abcODs, when adjacent increasing and decreasing pre-pieces are near symmetric with erroneous values on the borders.

**Example 5.3.** Consider a sequence of tuples \(T = \{t_{i−t_{13}}\}\) in Fig. 6, where attribute cat\# corresponds to sequence index over a bidirectional abcOD cat# \(\rightarrow \Delta\) year. Let \(\Delta = 1\) and \(\epsilon = 1\). As shown in Fig. 6, there are two pre-pieces in \(T\) (denoted with dash lines), and thus, three pieces: \(P_1 = \{t_{i−t_{10}}\}, P_2 = \{t_{i−t_{7}}\}, P_3 = \{t_{i−t_{13}}\}\). Alg. 5 finds two non-optimal solutions in \(T\) with the same gain of 85: 1) \(S_1 = \{t_{i−t_{7}}\} \quad \text{and} \quad S_2 = \{t_{i−t_{13}}\}\); and 2) \(S_1 = \{t_{i−t_{10}}\} \quad \text{and} \quad S_2 = \{t_{i−t_{13}}\}\).

But the solution by Alg. 3 proven to be optimal for discovery of bidirectional abcODs (Theorem 5.1): \(S_1 = \{t_{i−t_{10}}\} \quad \text{and} \quad S_2 = \{t_{i−t_{13}}\}\) with a higher gain equal to 89.
The above case is rare in real-world applications. Experimental results in Sec. 6.4 illustrate that the pieces-based algorithm does not sacrifice precision in practice over bidirectional abcODs.

6. EXPERIMENTAL EVALUATION

6.1 Data Characteristics and Settings

Datasets. We use two real-world datasets for experiments (1) Music (Footnote 1), and (2) Car (Footnote 2). The collected Music dataset has 1M tuples. It contains information about music releases over 100 years including attributes label, title, country, artists, genres, cat#, format, year and month. The Car dataset has 362 tuples. It contains information about second-hand cars including attributes year, vehicle identification number (VIN), orderid, description, model, link, blockid and cartype. Whenever it is not stated otherwise, we report the results with respect to (bidirectional) abcODs cat# $\rightarrow_\Delta \text{ONO}$ over the Music dataset and VIN $\rightarrow_\Delta \text{ONO}$ over Car dataset that we automatically identified as candidates for embedded band ODs (as described in Section 4.2).

Real-world Datasets. We categorize the real-world data into five groups by sampling the datasets. Table 5 shows statistics for the sampled datasets with real-world errors; SS denotes series size, MV missing values and IV incorrect values. Inc denotes percentage of tuples in unidirectional (ascending) series. Note that over the entire Car dataset all series are ascending and over the entire Music-Full dataset 92.1% of tuples belong to ascending series, thus, unidirectional abcODs are most common in practice.

- Music-Full is the full Music dataset with 1M tuples.
- Music-Random is a random sample of the above dataset by providing incomplete information from each series.
- Music-IncDec has series with ascending & descending orders.
- Music-Inc contains music series with only ascending orders.
- Music-Simple all tuples belong to a single series.
- Car contains vehicle information from multiple brands.

CER Datasets. Although the real-world Music dataset has real errors, we also randomly modify this dataset for some experiments with synthetic errors to control the error rate by replacing original values. We denote the perturbed datasets with a controlled error rate as CER datasets. To evaluate the robustness, we vary the missing and erroneous values in the range of 5% to 35%.

Gold Standard. We verify the ground truth as follows.

- Real-world Datasets: For all real-world datasets except Music-Full, we manually verify the correctness of series wrt (bidirectional) abcODs of all variations. For Music-Full the values provided are estimates based on our algorithms, and annotations in the original data. This is summarized in Table 5.
- CER datasets: We use manually-verified ground truth of series with respect to (bidirectional) abcODs over real-world datasets for CER-datasets.

Algorithms. We developed the following discovery algorithms in Java summarized in Table 6.

6.2 Quality of abcOD Discovery

Real-world Data. Table 7 presents the results of the (bidirectional) abcOD discovery on the real-world datasets. We made the following observations on the Music datasets. GAP achieves high recall over all datasets with a large loss in precision. As the algorithm relies on big “gaps” in cat# to discover series and most catalog numbers in the same series are close enough, it only splits series occasionally. Thus, due to its simplicity, the GAP algorithm has a high recall, however, the “gaps” of cat# between consecutive series are not always large, which causes the algorithm to merge series unnecessarily and leads to low precision. MONOSCALE has a high precision and the lowest recall among all algorithms, since it does not take into account outliers and tends to split series when the outliers occur. To overcome the flaw, we implemented a version of the algorithm called A-MONOSCALE that iteratively removes single outliers to discover series. As shown in Table 7, the adapted method increases recall over all datasets. LMS is tolerant to outliers in each series, however, does not handle small violations to monotonicity (treating them as outliers), hence, it achieves high precision by splitting series (due to many consecutive outliers detected). Finally, our SD-PIE approach (and thus, also our SD approach) overcomes the problems of other techniques. It dominates other approaches (F-measure above 0.93 and improved by up to 17% over other methods) and also achieves high accuracy and recall in all datasets (significantly better than other methods). We made similar observations over the results for the Car dataset (Table 7). Based on Theorem 4.4 SD-PIE achieves the same accuracy as SD over Car as all series in this dataset are ascending.

CER Datasets: Fig. 7 illustrates the quality results of abcOD discovery on the Music CER datasets. We observe analogous behaviors of all approaches with the controlled error rate as on the real datasets. The algorithm achieves high F-measure (above .82) for a reasonable amount of noise (up to 15%).

6.3 Band-Width Variations

Note that we manually specify band-width parameter only in this subsection to evaluate the effect of the parameter variations. We
6.4 Efficiency and Effectiveness

We evaluate the scalability of the different discovery algorithms over 500K tuples fraction of the Music-Full dataset divided into 10 random portions in Fig. 10. We observe that (1) the piece-based SD-PIE algorithm significantly reduces the runtime over the SD algorithm on average by two orders-of-magnitude, however, without sacrificing the accuracy over bidirectional abcODs as illustrated in Fig. 9 over the Music-IncDec dataset divided into 10 portions. The runtime is a consequence of the complexity of the (bidirectional) abcOD discovery problem, which for SD is $O(n^3 \log n)$ in the number of tuples (Theorems 4.3 and 5.1). We developed pruning strategies in the SD-PIE algorithm that is $O(n \log n)$ in the number of tuples ($O(n^2)$ in the number of pieces ($m$). Note that in practice pieces are large, hence, the number of pieces is small (i.e., $m \ll n$); (2) SD-PIE has smaller runtime than that of LMS because it generates a smaller number of pieces as LMS does not allow for small variations; (3) MONOSCALE (not shown in Fig. 10) has comparable runtime to SD-PIE; while GAP is faster than SD-PIE, due to its simplicity by relying on large gaps, it has a much worse accuracy as reported in Section 6.2.

The SD-PIE discovery algorithm runs over the Music-Full dataset with 1M tuples for 31.5 minutes. Note again that SD-PIE achieves the same accuracy as SD over Car as all series in this dataset are increasing (Theorem 4.4).

6.5 Discovery over Multiple Attributes

To measure the effectiveness and efficiency of the (bidirectional) abcOD discovery over multiple attributes, we use both the Music and Car datasets to generate the following data.

- 2-Attributes: Attribute year is split into centuries and years (e.g., 1993 is 19 and 93).
- 4-Attributes: Attribute year is split into: centuries, centuries, decades and years (e.g., 1993 is 1, 9, 9 and 3).

Table 8 shows the results of the (bidirectional) abcOD discovery. We observe that our solution over multiple attributes obtains similar F-measure as over a single attribute (i.e., year) in both datasets. Furthermore, our solution over four attributes has slightly lower F-measure, because the distance function leads to slightly different value when year is split into four attributes.

Running (bidirectional) abcOD discovery over multiple attributes takes as expected more time (however, reasonably more) than that on a single attribute (Figure 11). We made similar observations by considering other attributes that cannot be computed from year over the Music dataset, such as the categorical attribute month, over a band OD cat# $\rightarrow \Delta$ [year, month]. The time overhead over a band OD cat# $\rightarrow \Delta$ year is around 40% on average.

6.6 Candidate Generation

We measured that the divide-and-conquer approach (described in Section 3.3) based on traditional approximate ODs to identify candidates for embedded bandODs leads to an increased number of reported “errors” without there often being an actual violation. The error rates for the Music dataset are 15% and 20% with the (bidirectional) abcOD and traditional OD discovery, respectively (the error rates for the Car dataset are 15% and 23%, respectively).
7. RELATED WORK

Integrity constraints that specify attribute relationships, are commonly used to characterize data quality. Functional dependencies (FDs) are one of the oldest and most popular type of integrity constraints [12]. FDs have found application in a number of areas, such as schema design, data integration, query optimisation, data cleaning, and security [7]. In practice, dependencies may not hold exactly, due to errors in the data. Thus, approximate FDs have been defined that hold with some exceptions controlled by the number of tuples to be removed from the given table for the dependency to be satisfied. To effectively identify data quality rules different techniques to discover approximate FDs [12, 19], and conditional FDs that hold over subsets of the data, have been developed [9, 7].

A number of extensions to the classical notion of FDs have been proposed to express monotonicity including order dependencies (ODs) [14, 23, 24, 22, 13] that subsume FDs. While discovery of a specified OD can be performed in linear time in the number of tuples, the discovery of a predefined approximate OD raises the complexity to quadratic in the number of attributes for both unidirectional ODs and bidirectional ODs that consider a mix of ascending and descending order [13]. However, the prior work on discovery of approximate ODs [12, 22, 13] does not consider discovering conditional dependencies that hold on subsets of the data (and small variations) as in our work, which is an involved process.

Different variations of ODs have been studied including sequential dependencies (SDs) [8]. SDs specify that when tuples have consecutive antecedent values, their consequents must be within a specified range. The discovery problem was studied for approximate and conditional SDs [8]. However, in contrast to abcODs, SDs do not allow for small variations controlled by band-width, and a mix of ascending and descending trends. Denial Constraints (DCs) [5], a universally quantified first-order logic formalism, are more expressive than ODs [24]. DCs subsume ODs as they allow six operators for comparison of tuples \(<,\leq,\geq,=,\neq\). The authors in [5] study the discovery of approximate DCs without considering conditional dependencies. Also, abcODs express order with small variations causing DCs to be violated without actual violation of application semantics. Differential dependencies (DDs) [21] require that if the values of the attribute from antecedent of the DD satisfy a distance constraint, then the values of attributes from the consequent of the DD satisfy a distance constraint. The discovery problem for approximate and conditional DDs that hold approximately over subsets of the data have not been studied.

Our problem is relevant to a sequence segmentation [25] into non-overlapping partitions characterized by a model (e.g., mean and median), a general data mining problem for analyzing sequential data. Solutions to a sequence segmentation fall into two categories [25]: (1) fast heuristic algorithms, including top-down [15, 26], bottom-up [18, 25] and randomized [11] greedy and (2) approximation algorithms [10, 25]. Existing sequence segmentation solutions do not consider approximate monotonic segments and allowing for small variations.

8. CONCLUSIONS

We devise techniques to efficiently and effectively discover a novel data quality rule in the form of (bidirectional) abcODs. In future work, we plan to adapt sampling techniques used for functional dependency and key discovery [19] and utilize distributed computing as in previous work on data discovery that includes order operators for ODs [20] to further improve the efficiency of our discovery algorithm. We are also interested in studying the properties for abcODs including axiomatization and inference [22, 23].

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