Solitons of a simple nonlinear model on the cubic lattice

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Abstract

We study a simple nonlinear model defined on the cubic lattice. We propose a bilinearization scheme for the field equations and demonstrate that the resulting system is closely related to the well-studied integrable models, such as the Hirota bilinear difference equation and the Ablowitz–Ladik system. This result is used to derive the two sets of the N-soliton solutions.

Keywords: nonlinear lattice models, bilinear approach, explicit solutions, solitons

1. Introduction

In this paper we try to extend the area of application of the direct methods of the soliton theory. We show that the approaches developed in our previous works [1, 2] can be used to obtain a wide range of explicit solutions for nonlinear lattice models in three dimensions.

The model which we study seems to be new. We do not address the question of its integrability. Instead, we show that by means of elementary transformations one can reduce it to the well-studied integrable systems (the Ablowitz–Ladik hierarchy (ALH) [3] and the Hirota bilinear difference equation (HBDE) [4]). After that, we can use the standard techniques to derive solutions (or even use those already known) for our model, which are difficult to obtain by means of the straightforward approaches.

In this paper we restrict ourselves to the N-soliton solutions which are not only interesting in themselves, but also hint at (but surely do not prove) the integrability of the model.

2. Model

The model which we study in this paper describes the scalar fields defined at the vertices of the cubic lattice with the logarithmic interaction between the nearest neighbours,
\[
S = \sum_{n' \sim n''} \Gamma_{n'n''} \ln [1 + u(n') u(n'')]
\]
(2.1)

where \( n' \sim n'' \) means that vectors \( n' \) and \( n'' \) point to adjacent nodes of the lattice and \( \Gamma_{n'n''} \) are interaction constants that depend on the type (orientation) of the edge (see below). In more detail, we present the cubic lattice as

\[
\Lambda = \left\{ \mathbf{n} = \sum_{i=1}^{3} n_i \mathbf{e}_i, \quad n_i \in \mathbb{Z}, \quad \mathbf{e}_i \in \mathbb{R}^3 \right\}
\]
(2.2)

and define the model by

\[
S = \sum_{\mathbf{n} \in \Lambda} \sum_{i=1}^{3} \Gamma_i \ln \left[ 1 + u(\mathbf{n}) u(\mathbf{n} + \mathbf{e}_i) \right],
\]
(2.3)

where \( \Gamma_1, \Gamma_2 \) and \( \Gamma_3 \) are three constants restricted by

\[
\sum_{i=1}^{3} \Gamma_i = 0
\]
(2.4)

(we discuss this restriction in the conclusion).

The main object of this study are the ‘variational’ equations,

\[
\frac{\partial S}{\partial u(\mathbf{n})} = 0 \quad (\mathbf{n} \in \Lambda)
\]
(2.5)

which can be written as

\[
\sum_{i=1}^{3} \Gamma_i \left[ \frac{u(\mathbf{n} + \mathbf{e}_i)}{1 + u(\mathbf{n}) u(\mathbf{n} + \mathbf{e}_i)} + \frac{u(\mathbf{n} - \mathbf{e}_i)}{1 + u(\mathbf{n}) u(\mathbf{n} - \mathbf{e}_i)} \right] = 0.
\]
(2.6)

In what follows, we extensively use the fact that the cubic lattice is a bipartite graph and split \( \Lambda \) into the two sublattices, which we call ‘positive’ and ‘negative’:

\[
\Lambda = \Lambda^+ \cup \Lambda^-
\]
(2.7)

where

\[
\Lambda^+ = \left\{ \mathbf{n} = \sum_{i=1}^{3} n_i \mathbf{e}_i, \quad n_i \in \mathbb{Z} \quad \left| \sum_{i=1}^{3} n_i = 0 \mod 2 \right. \right\},
\]
\[
\Lambda^- = \left\{ \mathbf{n} = \sum_{i=1}^{3} n_i \mathbf{e}_i, \quad n_i \in \mathbb{Z} \quad \left| \sum_{i=1}^{3} n_i = 1 \mod 2 \right. \right\}.
\]
(2.8)

To compare the field equations of our model with already known systems, one can make the substitution

\[
u(n) = \begin{cases} 
\frac{w(n)}{(n \in \Lambda^+)} & (n \in \Lambda^+) \\
-1/\frac{w(n)}{(n \in \Lambda^-)} & (n \in \Lambda^-)
\end{cases}
\]
(2.9)

which transforms equation (2.6) into

\[
\sum_{i=1}^{3} \Gamma_i \left[ \frac{1}{w(n) - w(n + \mathbf{e}_i)} + \frac{1}{w(n) - w(n - \mathbf{e}_i)} \right] = 0.
\]
(2.10)

Written in this form, equation (2.6) can be viewed as a three-dimensional generalization of the already known integrable Toda-type and relativistic Toda-type lattices \([5, 9, 10, 12]\). Indeed,
in the case of $\Gamma_1 = -\Gamma_2, \Gamma_3 = 0$ one arrives at the Toda-type lattice that belongs to the list of theorem 4 of [5] while another reduction of (2.6), $w(n + e_j) = w(n + e_1 + e_2)$, leads to the equation that belongs to the list of the lattices of the discrete relativistic Toda type of theorem 3 of [5].

From the physical viewpoint, the action (2.1) or (2.3) describes an anharmonic lattice with the logarithmic interaction $V(u', u'') = \ln (1 + u' u'')$ which $i)$ in the small amplitude limit becomes the 'standard' harmonic one, $V(u', u'') \approx u' u''$, and $ii)$ is not new to the theory of integrable systems. As a bright example of its appearance we should mention the classical+ integrable analogue of the famous Heisenberg model of the quantum mechanics [6–8]). In this sense, the remarkable feature of equation (2.6) considered here is that it is one of a rather limited number of equations in multidimensions that, on the one hand, are the field equation of a nonlinear lattice model with a reasonable action/energy and, on the other hand, possess (as is shown below) multi-soliton solutions.

3. Ansatz and bilinearization

In this section we present the main result of this paper. We bilinearize the field equation (2.6) and demonstrate the relationships of the resulting system with the already known integrable models.

The aim of this paper is to derive the soliton solutions, i.e. some particular solutions. To this end, we not only bilinearize the system (2.6) but also simplify it: we ‘split’ the seven-point equations into four-point ones. The procedure that we use is, for the most part, rather standard. However, to achieve our goals we have to apply some non-trivial, though very simple, tricks.

Below we discuss the main ideas behind the proposed substitutions and ansatz. Here we try to explain and ‘justify’ the non-standard moments of the construction used in what follows. A reader who considers that the only necessary justification of an ansatz is to check that it provides solutions for the equations in question may skip most of this section and proceed directly to proposition 3.1.

3.1. Three-leg reduction

The key idea behind the derivation of the soliton solutions for our model may be described as the three-leg reduction by analogy with the so-called three-leg representation of integrable systems on quad-graphs [10] and which is known, for example, for all equations of the Adler–Bobenko–Suris list [11]. Using the language of [14], we find the quad-equations for which action (2.3) provides the so-called weak Lagrangian formulation. The first manifestation of the three-dimensionality of our problem is that there is no clear way to build the system of polygons corresponding to our star equation. However a particular solution of this problem can be completed as follows.

Consider the four vectors $g_\ell$ ($\ell = 0, ..., 3$) given by

$$g_0 = \frac{1}{2} \sum_{i=1}^{3} e_i, \quad g_i = e_i - g_0 \quad (1 = 1, 2, 3) \quad (3.11)$$

related by

$$\sum_{\ell=0}^{3} g_\ell = 0. \quad (3.12)$$
Now, both \( u(n + e_i) = u(n + g_0 + g_i) \) and \( u(n - e_i) = u(n + g_j + g_k) \) \( \{i,j,k\} = \{1,2,3\} \) are obtained from \( u(n) \) by means of two translations which surmise the following substitution: if we can find the function \( \hat{u}(n) \) such that
\[
\frac{u(n + g_l + g_m)}{1 + u(n) u(n + g_l + g_l)} = \frac{\hat{u}(n + g_l) - \hat{u}(n + g_m)}{\lambda_l - \lambda_m}, \quad l \neq m, \ l, m \in \{0,1,2,3\},
\]
(3.13)
(the denominator in the right-hand side is introduced to preserve the symmetry with respect to the permutation \( l \leftrightarrow m \)) then equation (2.6) can be written as
\[
0 = \sum_{i=1}^{3} \hat{\Gamma}_i \left[ \hat{u}(n + g_i) - \hat{u}(n + g_0) \right]
\]
(3.14)
with constant \( \hat{\Gamma}_i \), and can be satisfied by making all \( \hat{\Gamma}_i \) equal to zero (we return to this question in what follows).

Of course, the function \( \hat{u}(n) \) is far from arbitrary: it has to meet various conditions following from the compatibility of the ansatz (3.13) together with its consistency with respect to translations. In particular, one can show that it must be a solution of certain Toda-type equation (we have some kind of duality here). So, to find suitable \( \hat{u} \) is not a trivial problem. However, one can hope to answer the arising questions in the framework of the four-point, or quad-equations. We do not discuss these problems now, because in what follows we will use, instead of ansatz (3.13), another, more general, one.

3.2. Bipartite three-leg reduction

For our purposes, ansatz (3.13) has a serious drawback, stemming not from the compatibility/consistency issues, but from the restriction (3.12). It turns out that when constructing explicit solutions such restrictions can be rather limiting. As we show in section 4, they can leave us with only one- and two-soliton solutions. Thus, it seems useful to relieve us of the condition (3.12), which can be done by elementary means.

First, we replace the vectors \( \{g_\ell\}_{\ell=0}^{3} \) with another set of vectors, \( \{\alpha_\ell\}_{\ell=0}^{3} \), that are related to \( \{e_i\}_{i=1}^{3} \) by
\[
e_i = \frac{1}{2} (\alpha_0 + \alpha_i - \alpha_j - \alpha_k).
\]
(3.15)
It is obvious that one can find infinitely many such quadruples. For example, one can take arbitrary \( \alpha_0 \) and then put \( \alpha_i = \alpha_0 - e_j - e_k \) \( \{i,j,k\} = \{1,2,3\} \). However, in our case this ambiguity is not a problem. As is shown below, these vectors appear in solutions that we are going to derive as parameters, hence different choices of \( \{\alpha_\ell\}_{\ell=0}^{3} \) just provide, in general, different solutions.

Now, one can present \( u(n \pm e_i) \) as
\[
\begin{align*}
u(n + e_i) &= u(n + \alpha_0 + \alpha_i - 2\delta) \\
u(n - e_i) &= u(n + \alpha_j + \alpha_k - 2\delta)
\end{align*}
\]
\( \{i,j,k\} = \{1,2,3\} \)
(3.16)
where
\[
\delta = \frac{1}{4} \sum_{\ell=0}^{3} \alpha_\ell.
\]
(3.17)
i.e. the translations by the vectors $\pm e_i$ cease to be sums of two translations, the fact that has been crucial for the ansatz (3.13). To restore this feature of $\{\alpha_i\}_{i=0}^3$, we use the bipartite property of the cubic lattice and introduce, instead of $u$, two functions, $q$ and $r$, by

$$ u(n) = \begin{cases} r(n - \delta) & (n \in \Lambda^+) \\ q(n + \delta) & (n \in \Lambda^-). \end{cases} \tag{3.18} $$

In terms of $q$ and $r$ equation (2.6) becomes

$$ 0 = \sum_{i=1}^3 \hat{\Gamma}_i \begin{pmatrix} -q(x_+ + \alpha_0 + \alpha_i) + q(x_+ + \alpha_j + \alpha_k) \\ 1 + q(x_+ + \alpha_0 + \alpha_i) r(x_+) + 1 + q(x_+ + \alpha_j + \alpha_k) r(x_+) \end{pmatrix} \quad (n \in \Lambda^+) $$

$$ 0 = \sum_{i=1}^3 \hat{\Gamma}_i \begin{pmatrix} r(x_- - \alpha_0 - \alpha_i) + q(x_- - \alpha_0 - \alpha_i) r(x_- - \alpha_0 - \alpha_i) \\ 1 + q(x_- - \alpha_0 - \alpha_i) r(x_- - \alpha_0 - \alpha_i) + 1 + q(x_- - \alpha_0 - \alpha_i) r(x_- - \alpha_0 - \alpha_i) \end{pmatrix} \quad (n \in \Lambda^-) \tag{3.19} $$

where $x_\pm = n \mp \delta$.

The original problem is discrete: all functions were defined on $\mathbb{Z}^3$. If we were using (3.13), we would actually pass to the body-centered cubic lattice, but the problem would remain discrete. However, now, after introducing the one-parametric family of vectors $\alpha$ and functions $q$ and $r$ that have different domains of definition, we make the next step and consider the above equations as defined on the whole $\mathbb{R}^3$,

$$ \begin{pmatrix} 0 = \sum_{i=1}^3 \hat{\Gamma}_i \begin{pmatrix} q(x + \alpha_0 + \alpha_i) - q(x + \alpha_j + \alpha_k) \\ r(x - \alpha_0 - \alpha_i) + q(x - \alpha_0 - \alpha_i) \end{pmatrix} + \begin{pmatrix} q(x + \alpha_j + \alpha_k) \\ r(x - \alpha_j - \alpha_k) \end{pmatrix} \\ 0 = \sum_{i=1}^3 \hat{\Gamma}_i \begin{pmatrix} q(x + \alpha_j + \alpha_k) \\ r(x - \alpha_j - \alpha_k) \end{pmatrix} \end{pmatrix} \quad (x \in \mathbb{R}^3). \tag{3.20} $$

In other words, we pass from discrete equations to difference ones, which is, of course, a reduction. However this reduction is rather typical for the ‘applied’ studies of discrete equations. Moreover, we repeat that we are looking for some particular solutions and hence can admit some reductions, provided they leave us with non-trivial residue. Looking at soliton solutions for various models one can see that, usually, they depend analytically on all arguments and parameters and can be obtained as solutions of corresponding difference equations. Finally, this question will be less significant when we finish the derivation of the soliton solutions. The final formulae are written in terms of $n$ with $\alpha$ being replaced with corresponding parameters and can be considered as solutions for the pure discrete problem.

To derive solutions for system (3.20), we make the following ansatz:

$$ \frac{q(x + \alpha_j + \alpha_m) - q(x + \alpha_i) - q(x + \alpha_m)}{1 + q(x + \alpha_j + \alpha_m) r(x)} = \frac{r(x - \alpha_j - \alpha_m)}{\lambda_i - \lambda_m} $$

$$ \frac{r(x - \alpha_j - \alpha_i)}{1 + q(x + \alpha_i) r(x - \alpha_j - \alpha_i)} = \frac{r(x - \alpha_i) - r(x - \alpha_m)}{\lambda_i - \lambda_m}. \tag{3.21} $$

for $l, m \in \{0, 1, 2, 3\}$ and $l \neq m$ (the simplest, self-dual, form of equation (3.13), discussed in the previous subsection), which, as is shown below, reduces the problem to the already known system (which also helps us to answer the questions about its compatibility) and, what is important for this work, leads to the N-soliton solutions.

Ansatz (3.21) reduces (3.20) to

$$ \begin{pmatrix} 0 = \sum_{i=1}^3 \hat{\Gamma}_i [q(x + \alpha_i) - q(x + \alpha_0)] \\ 0 = \sum_{i=1}^3 \hat{\Gamma}_i [r(x - \alpha_i) - r(x - \alpha_0)] \end{pmatrix} \tag{3.22} $$

5
with constants \( \hat{\Gamma}_i \) given by
\[
\hat{\Gamma}_i = \frac{\Gamma_i}{\lambda_i - \lambda_0} + \frac{\Gamma_j}{\lambda_i - \lambda_k} - \frac{\Gamma_k}{\lambda_i - \lambda_j} \quad \{i,j,k\} = \{1,2,3\}.
\] (3.23)

It easy to see that we can satisfy equation (3.22) without imposing additional conditions upon \( q \) and \( r \) by making all \( \hat{\Gamma}_i \) equal to zero. Solution of this elementary problem leads to the following restriction on the constants \( \lambda_\ell \):
\[
\lambda_0 = \sum_{i=1}^{3} \lambda_i - \frac{\sum_{i=1}^{3} \Gamma_i \lambda_i^2}{\sum_{i=1}^{3} \Gamma_i \lambda_i} \quad \text{(3.24)}
\]
(see appendix A for a proof).

3.3. Bilinearization.

Finally, we bilinearize the system (3.21) by introducing the triplet of the tau-functions \( \sigma, \rho \) and \( \tau \):
\[
q = \frac{\sigma}{\tau}, \quad r = \frac{\rho}{\tau}. \quad \text{(3.25)}
\]
It is easy to check that \( q \) and \( r \) are solutions for (3.21) provided \( \sigma, \rho \) and \( \tau \) satisfy
\[
a_{l,m} \sigma(x + \alpha_l + \alpha_m) = \sigma(x + \alpha_l) \tau(x + \alpha_m) - \tau(x + \alpha_l) \sigma(x + \alpha_m) \quad \text{(3.26a)}
\]
\[
a_{l,m} \rho(x + \alpha_l + \alpha_m) = \tau(x + \alpha_l) \rho(x + \alpha_m) - \rho(x + \alpha_l) \tau(x + \alpha_m) \quad \text{(3.26b)}
\]
\[
b_{l,m} \tau(x + \alpha_l) \tau(x + \alpha_m) = \tau \tau(x + \alpha_l + \alpha_m) + \rho \sigma(x + \alpha_l + \alpha_m) \quad \text{(3.26c)}
\]
where \( l, m \in \{0,1,2,3\} \) and \( l \neq m \), the skew-symmetric constants \( a_{l,m} \) and symmetric constants \( b_{l,m} \) are related to \( \lambda_\ell \) by
\[
a_{l,m} b_{l,m} = \lambda_l - \lambda_m \quad \text{(3.27)}
\]
but are arbitrary apart from that.

To summarize, the main result of this paper can be presented as follows.

**Proposition 3.1.** Each solution of system (3.26) delivers a solution for the field equation (2.6), by means of (3.18) and (3.25), provided the parameters \( \{\lambda_\ell\}_{\ell=0}^{3} \) satisfy restrictions (3.24) and (3.27).

The proof of this statement is straightforward: system (3.26), together with (3.27), implies that the functions \( q \) and \( r \) given by (3.25) satisfy (3.20). As is shown in section 3.2, this and (3.24) guarantee that the functions \( u \) given by (3.18) solve (2.6).

3.4. Hirota–Ablowitz–Ladik system

The system (3.26) is an already known system that can be found in studies of a large number of integrable equations (see, e.g. [4, 15–19]). Probably, the most important appearance of (3.26) is in the theory of such well-studied integrable models, as the HBDE [4] and the ALH [3]. For example, an immediate consequence of (3.26a), or (3.26b), is the fact that \( \tau \) solves the HBDE, so both (3.26a) and (3.26b) can be viewed as linear problems from the zero-curvature representation of the HBDE. On the other hand, equations (3.26a) and (3.26b) describe the infinite chain of the Bäcklund transformations for the HBDE. It is also known that system (3.26) is closely
related to another integrable model, which is even ‘older’ than the HBDE: equation (3.26) describe the so-called Miwa shifts of the ALH. We do not discuss these questions here in detail referring the reader to section 4.1 (together with appendices A and B) of [1] and section 4 of [2].

The fact that we have reduced our problem to the well-studied system (3.26) has two advantages. First, we do not need to worry about the compatibility of the ansatz (3.21) or its consistency (see, e.g. [17]). Second, we can use the wide number of solutions already derived for (3.26). In this paper, we discuss the soliton ones (see the following section). However, we might obtain by the same ansatz the so-called finite-gap quasi-periodic solutions, for which system (3.26) is just the set of Fay identities, or various determinant solutions.

4. Soliton solutions

In what follows, we derive soliton solutions for our problem using the results of [20, 21], where we have presented a large number of identities for the matrices of a special type (soliton Fay identities).

In papers [20, 21] we describe two types of constructions that lead to the soliton solutions for various models. In the case of one spatial dimension the difference between these solitons is usually indicated by the words ‘bright’ and ‘dark’: bright solitons satisfy the zero boundary conditions while the dark ones (or their absolute values) tend to constants. In the multidimensional case, the situation is more complicated. For example, the simplest analogues of the one-dimensional bright solitons (one-soliton solutions) in multidimensions become the line solitons which decay in one direction (and its opposite) but are constant in orthogonal ones. Thus, ‘bright’ and ‘dark’ are not the best terms to classify the solitons in multidimensional models as the one of this paper. However, we will use them in what follows, just to distinguish the two types of solutions: solutions constructed from the matrices described in [20], that in a one-dimensional case lead to the dark solitons, and ones constructed from the matrices described in [21], that in a one-dimensional case lead to the bright solitons.

4.1. ‘Dark’ solitons.

Here we use one of the results of [20] which (after some simplification) can be formulated as follows: the determinants

$$\Omega = \det |1 + A|$$

(4.28)

of the matrices defined by

$$LA - AL^{-1} = |1\rangle\langle a|,$$

where $L = \text{diag} (L_1, \ldots, L_N)$, $|1\rangle$ is the $N$-column with all components equal to 1, $\langle a|$ is a $N$-component row that depends on the coordinates describing the model, satisfy

$$0 = (\xi - \eta) (T_{\xi\eta} \Omega) (T_{\zeta} \Omega) + (\zeta - \xi) (T_{\xi\zeta} \Omega) (T_{\eta} \Omega) + (\eta - \zeta) (T_{\eta\zeta} \Omega) (T_{\xi} \Omega).$$

(4.30)

Here, the shifts $T$ are defined as $T_{\xi} \Omega = \det |1 + T_{\xi} A|$, $T_{\xi\eta} \Omega = T_{\xi} T_{\eta} \Omega$ with

$$T_{\xi} A = AH_{\xi}$$

(4.31)

and

$$H_{\xi} = (\xi - L) (\xi - L^{-1})^{-1}.$$

(4.32)

From this identity, together with the ‘duality’ property of the matrices $H_{\xi}$,
which implies
\[ T_\xi T_1/\xi = T_0 \]  
(4.34)

it is easy to derive

**Proposition 4.1.1.** Functions
\[ \tau = \Omega, \quad \sigma = -F^{-1}T_0^{-1}\Omega, \quad \rho = FT_0\Omega \]  
(4.35)

where \( F \) is defined by
\[ T_\alpha F = -\alpha F \]  
(4.36)

satisfy the system
\[ a(\xi, \eta) \tau (T_\xi \eta \sigma) = (T_\xi \sigma) (T_\eta \tau) - (T_\xi \tau) (T_\eta \sigma), \]  
(4.37a)
\[ a(\xi, \eta) \rho (T_\xi \eta \tau) = (T_\xi \tau) (T_\eta \rho) - (T_\xi \rho) (T_\eta \tau), \]  
(4.37b)
\[ b(\xi, \eta) (T_\xi \eta \tau) (T_\eta \tau) = \tau (T_\xi \eta \tau) + \rho (T_\xi \eta \sigma), \]  
(4.37c)

with
\[ a(\alpha, \beta) = \alpha - \beta, \quad b(\alpha, \beta) = 1 - \frac{1}{\alpha \beta}, \]  
(4.38)

(see appendix B for a proof).

It is clear that (4.37) is exactly (3.26), provided we identify the translations \( x \rightarrow x + \alpha \xi \) with the action of \( T_\alpha \xi \), where \( \{\alpha_\xi\}_{\xi = 0}^3 \) is a fixed set of constants, and put \( a_{\alpha m} = a(\alpha, \alpha m) \) and \( b_{\alpha m} = b(\alpha, \alpha m) \), which implies
\[ \lambda_\xi = \alpha_\xi + \alpha^{-1}_\xi \]  
(4.39)

(up to a non-essential constant).

Thus, we have all that is necessary to write down the ‘dark’-soliton solutions for the field equations of our model.

Using the construction described in section 3.2, we write the relation between the translations by \( e_i \) and the action of the shifts \( T_\alpha \xi \) as
\[ u(n + e_i) = T^{1/2}_{\alpha_0} T^{1/2}_{\alpha_1} T^{1/2}_{\alpha_2} T^{1/2}_{\alpha_3} u(n) \quad \{i, j, k\} = \{1, 2, 3\} \]  
(4.40)

or
\[ u(n + e_i) = T_\alpha T^{-1}_* u(n), \quad T_* = [T^{-1}_{\alpha_0} T_\alpha T^{-1}_{\alpha_2} T_\alpha T^{-1}_{\alpha_3}]^{1/2}. \]  
(4.41)

Then, we introduce the matrices \( X_i \) \((i = 1, 2, 3)\)
\[ X_i = H_{\alpha_i} H^{-1}_{\alpha_i}, \quad H_* = [H^{-1}_{\alpha_0} H_{\alpha_1} H_{\alpha_2} H_{\alpha_3}]^{1/2} \]  
(4.42)

describing the \( n \)-dependence,
\[ A(n + e_i) = A(n) X_i \]  
(4.43)

as well as two matrices, \( M_0 \) and \( M_1 \) which describe the action of \( T_\delta^{-1} \) and \( T_0 T_\delta^{-1} \), where \( T_\delta \) is the shift corresponding to the translation \( x \rightarrow x + \delta \), \( T_\delta = \left( \prod_{\xi = 0}^3 T_\alpha \right)^{1/4}, \)
\[ M_0 = \mathbb{H}^{-1}, \quad M_1 = L^2 \mathbb{H}^{-1} \] (4.44)

where
\[ \mathbb{H} = \left[ \prod_{\ell=0}^{3} H_{\alpha_{\ell}} \right]^{1/4}. \] (4.45)

It should be noted that calculating the action of the shift \( T_\delta \) on the matrices \( A \) and the function \( F \) one has to raise the products \( \prod_{\ell=0}^{3} H_{\alpha_{\ell}} \) and \( \prod_{\ell=0}^{3} \alpha_{\ell} \) to the power 1/4. This leads to some restrictions on the parameters \( \alpha_{\ell} \) and \( L_n \):
\[ \prod_{\ell=0}^{3} \alpha_{\ell} > 0, \] (4.46a)
which should be considered together with equation (3.24) and
\[ \prod_{\ell=0}^{3} \left[ L_n + L_n^{-1} - (\alpha_{\ell} + \alpha_{\ell}^{-1}) \right] > 0 \quad (n = 1, ..., N). \] (4.46b)

As the result we can formulate

**Proposition 4.1.2.** The 'dark' soliton solutions can be presented as
\[ u(n) = \pm u_0^{ \pm} \frac{\det \left[ 1 + A(n) M_0^{\pm1} \right]}{\det \left[ 1 + A(n) M_0^1 \right]} \quad (n \in \Lambda^\pm) \] (4.47)

where \( u_0 = (\alpha_0 \alpha_1 \alpha_2 \alpha_3)^{-1/4} \),
\[ F(n) = \prod_{i=1}^{3} (\alpha_{i}/\alpha_*)^n_i, \quad \alpha_* = (\alpha_1 \alpha_2 \alpha_3/\alpha_0)^{1/2}, \] (4.48)
\[ A(n) = C X(n), \quad X(n) = \prod_{i=1}^{3} X_i^n_i \] (4.49)

(\( n_i \) are the components of \( n \), \( n = \sum_{i=1}^{3} n_i e_i \)), with the matrices \( X_i \) being given by (4.42) and
\[ C = \left( \frac{c_n}{L_m L_n - 1} \right)_{m,n=1,...,N}. \] (4.50)

Here, \( \alpha_0 = \alpha_0 (\alpha_1, \alpha_2, \alpha_3) \) is a solution of
\[ \alpha_0 + \alpha_0^{-1} = \sum_{i=1}^{3} (\alpha_i + \alpha_i^{-1}) - \frac{\sum_{i=1}^{3} \Gamma_i (\alpha_i^2 + \alpha_i^{-2})}{\sum_{i=1}^{3} \Gamma_i (\alpha_i + \alpha_i^{-1})} \] (4.51)

and \( c_n \), \( L_n \) \( (n = 1, ..., N) \) and \( \alpha_i \) \( (i = 1, 2, 3) \) are arbitrary (up to the restrictions (4.46)) constants.

The simplest, 1-soliton solution can be rewritten as
\[ u(n) = \pm u_0^{ \pm} e^{\pm f(n)} \frac{\cosh (h(n) \pm \delta_1)}{\cosh (h(n) \pm \delta_0)} \quad (n \in \Lambda^\pm) \] (4.52)
where \( u_* = 1/\bar{\alpha} \),
\[
\begin{align*}
\varphi &= \sum_{i=1}^3 \varphi_i e_i, & \varphi_i &= \ln \frac{\alpha_i}{\bar{\alpha}}; \\
\chi &= \sum_{i=1}^3 \chi_i e_i, & \chi_i &= \frac{1}{2} \ln \frac{H_i}{H_*}.
\end{align*}
\] (4.53)

with arbitrary \( f_0 \) and \( h_0 \) and
\[
\begin{align*}
\varphi &= \sum_{i=1}^3 \varphi_i e_i, & \varphi_i &= \ln \frac{\alpha_i}{\bar{\alpha}}; \\
\chi &= \sum_{i=1}^3 \chi_i e_i, & \chi_i &= \frac{1}{2} \ln \frac{H_i}{H_*}.
\end{align*}
\] (4.54)

Here, \( H_i = (\alpha_i - L)/(\alpha_i - 1/L) \), and \( L \) are scalars replacing the matrices \( H_{\alpha_i} \) and \( L \), instead of the matrices \( M_{0,1} \) we use the constants \( \delta_{0,1} \) given by,
\[
\begin{align*}
\delta_0 &= -\frac{1}{2} \ln H, & \delta_1 &= \frac{1}{2} \ln \left( L^2/H \right)
\end{align*}
\] (4.55)

with
\[
\begin{align*}
\bar{\alpha} &= (\alpha_0 \alpha_1 \alpha_2 \alpha_3)^{1/4}, & H &= (H_0 H_1 H_2 H_3)^{1/4}
\end{align*}
\] (4.56)

and
\[
\begin{align*}
\alpha_* &= (\alpha_1 \alpha_2 \alpha_3/\alpha_0)^{1/2}, & H_* &= (H_1 H_2 H_3/H_0)^{1/2}.
\end{align*}
\] (4.57)

It is easy to see that we have a real analogue of the dark soliton of the complex models like the nonlinear Schrödinger or the Ablowitz–Ladik equations: the plane wave \( \pm u_* e^{i f(n)} \) (in the complex case \( f(n) \) is pure imaginary) modulated by the factor which in the \( |n| \rightarrow \infty \) limit tends to one of the two constants, \( L \) or \( 1/L \), depending on the direction \( n/|n| \) in which we approach the infinity (that determines the sign of \( \chi, n \)).

4.2. ‘Bright’ solitons

To derive the second type of soliton solutions one does not need any additional calculations but can use the soliton Fay identities from [21] which were obtained for the tau-functions
\[
\begin{align*}
\tau &= \det |1 + AB|, \\
\sigma &= \tau \langle a | (1 + BA)^{-1} | 1 \rangle, \\
\rho &= \tau \langle b | (1 + AB)^{-1} | 1 \rangle
\end{align*}
\] (4.58)

where \( A \) and \( B \) are solutions of
\[
\begin{align*}
LA - AR &= |1 \rangle \langle a |, \\
RB - BL &= |1 \rangle \langle b |.
\end{align*}
\] (4.59)

Here, like in the previous section, \( L \) and \( R \) are constant diagonal \( N \times N \)-matrices, \( L = \text{diag} (L_1, ..., L_N) \) and \( R = \text{diag} (R_1, ..., R_N), | 1 \rangle \) is the \( N \)-column with all components equal to 1, \( \langle a | \) and \( \langle b | \) are \( N \)-component rows that depend on the coordinates describing the model.

The shifts \( T_\xi \) are defined, in this case, by
\[
\begin{align*}
T_\xi | a \rangle &= | a \rangle (R - \xi)^{-1}, \\
T_\xi | b \rangle &= | b \rangle (L - \xi)
\end{align*}
\] (4.60)

or, as a consequence, by
\[
\begin{align*}
T_\xi A &= A (R - \xi)^{-1}, \\
T_\xi B &= B (L - \xi).
\end{align*}
\] (4.61)
The simplest soliton Fay identities, which are equations (3.12)–(3.14) of [21], are exactly equation (4.37) with
\[ a(\xi, \eta) = \xi - \eta, \quad b(\xi, \eta) = 1 \]
which implies
\[ \lambda_\ell = \alpha_\ell. \] (4.63)
Thus, the only thing that we have to do to obtain solutions for our equations is to gather all formulae describing \( u^{(n)} \) in terms of the tau-functions and the shifts \( T_{\alpha_\ell} \), with a fixed set of constants \( \{ \alpha_\ell \}_{\ell=0}^3 \), which correspond, as in the previous subsection, to the translations \( x \to x + \alpha_\ell \).

To make the final formulae clearer we change the notation: we write \( L^\pm \) instead of \( L \) and \( R \),
\[ L^+ = R, \quad L^- = L, \] (4.64)
and slightly modify the definition of the rows \( \langle a \rangle \) and \( \langle b \rangle \) and of the matrices \( A \) and \( B \): we use in what follows the new rows
\[ \langle a^+ \rangle = \langle b | L^{-1} | \rangle, \quad \langle a^- \rangle = \langle a | R^{-1} | \rangle \] (4.65)
and the new matrices
\[ A^+ = B \Gamma^{-1}, \quad A^- = A \Gamma^{-1} \] (4.66)
where \( \Gamma \) and \( \Gamma \) are the matrices corresponding to the \( T_\delta \) (i.e. \( \delta \)-translations),
\[ T_\delta \langle a \rangle = \langle a | R^{-1} | \rangle, \quad T_\delta \langle b \rangle = \langle b | \Gamma \rangle \] (4.67)
and are given by
\[ \Gamma = \left[ \prod_{\ell=0}^3 \left( R - \alpha_\ell \right) \right]^{1/4}, \]
\[ \Gamma = \left[ \prod_{\ell=0}^3 \left( L - \alpha_\ell \right) \right]^{1/4}. \] (4.68)
These matrices and rows satisfy
\[ L^\pm A^\pm - A^\pm L^\mp = | \langle a^\pm \rangle | \] (4.69)
and their dependence on \( n \) can be presented as
\[ \langle a^\pm(n) \rangle = \langle e^{\pm} | X^\pm(n) \rangle, \]
\[ A^\pm(n) = C^\pm X^\pm(n) \] (4.70)
where
\[ X^\pm(n) = \prod_{i=1}^3 \left( X_i^\pm \right)^{n_i} \] (4.71)
with
\[ X_i^\pm = (L^\mp - \alpha_i)^{\pm 1} \left( L_i^\mp \right)^{\mp 1} \] (4.72)
and
Again, as in the ‘dark’-soliton case, the appearance of the fractional powers in the above formulae leads to some restrictions on the parameters $L_\pm$ ($n = 1, ..., N$):

$$\prod_{\ell=0}^{3} (L_\pm^\ell - \alpha_\ell) > 0 \quad (n = 1, ..., N).$$

(4.74)

The analysis of these inequalities is easier than that of the corresponding ones, (4.46), from the previous section. The simplest (but not the only) solution is to calculate $\alpha_0$ from (3.24) and then take all $L_\pm^n$ greater than $\max_{\ell=0,...,3} \alpha_\ell$.

Finally, we can formulate the main result of this section as follows.

**Proposition 4.2.1.** The ‘bright’-soliton solutions can be presented as

$$u(n) = \langle a^\pm(n) | 1 + U^\pm(n) \rangle^{-1} | 1 \rangle \quad (n \in \Lambda^\pm)$$

(4.75)

where

$$U^\pm(n) = A^\pm(n) M^\pm A^\pm(n)$$

(4.76)

with

$$M^\pm = \left[ \prod_{\ell=0}^{3} (L_\pm^\ell - \alpha_\ell) \right]^{1/2}$$

(4.77)

and

$$\langle a^\pm(n) | = \langle c^\pm | X^\pm(n),$$

$$A^\pm(n) = C^\pm X^\pm(n)$$

(4.79)

where $\langle c^\pm |$ are constant $N$-rows, $\langle c^\pm | = (c^\pm_1, ..., c^\pm_N)$ and $C^\pm$ are constant matrices given by

$$C^\pm = \left( \frac{c^\pm_m}{L^\pm_m - L^\pm_\ell} \right)_{m,n=1,...,N}$$

(4.80)

with the matrices $X^\pm(n)$ being defined in (4.71) and (4.72).

Here, $\alpha_0 = \alpha_0(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$\alpha_0 = \alpha_0(\alpha_1, \alpha_2, \alpha_3) = \frac{\sum_{i=1}^{3} \alpha_i }{\sum_{i=1}^{3} \Gamma_i \alpha_i}$$

(4.81)

and $c^\pm_n$, $L^\pm_n$ ($n = 1, ..., N$) and $\alpha_i$ ($i = 1, 2, 3$) are arbitrary (up to the restrictions (4.74)) constants.
The simplest, 1-soliton solution can be presented as
\[ u(n) = \pm u_* \frac{e^{\pm f(n)}}{\cosh (h(n) \pm \delta)} \quad (n \in \Lambda^\pm) \] (4.82)

where
\[ u_* = \frac{|L^+ - L^-|}{2\sqrt{L^+ L^-}}, \]
\[ f(n) = f_0 + (\varphi, n), \quad h(n) = h_0 + (\chi, n) \] (4.83)

with arbitrary \( f_0 \) and \( h_0 \) and
\[ \varphi = \sum_{i=1}^{3} \varphi_i e_i, \quad \varphi_i = \frac{1}{3} \ln \frac{L^+ - \alpha_i}{L^- - \alpha_i}, \]
\[ \chi = \sum_{i=1}^{3} \chi_i e_i, \quad \chi_i = \frac{1}{3} \ln \frac{L^+ - \alpha_0}{L^- - \alpha_0}. \] (4.84)

Here, the two constants \( L^\pm \) replace the matrices \( L^{\pm} \), the constant \( \delta \) which is used instead of the matrices \( M^{\pm} \) is given by \( \delta = \frac{1}{2} \ln \frac{L^+}{L^-} \) with the ‘averages’ \( L^\pm \) and \( L^\pm_* \) defined as
\[ L^\pm = \left[ \prod_{\ell=0}^{3} (L^\pm - \alpha_\ell) \right]^{1/4}, \quad L^\pm_* = \left[ \prod_{\ell=1}^{3} (L^\pm - \alpha_0) \right]^{1/2}. \] (4.85)

Again, as in the ‘dark’-soliton case, we have a real analogue of the bright soliton of the complex models: the plane wave \( \pm u_* e^{\pm f(n)} \) modulated by the typical soliton factor (this time the \( \text{sech} \)-factor) which vanishes in the \( |n| \to \infty \) limit, except in the cases when we approach the infinity along the directions perpendicular to \( \chi \).

To conclude this section, we would like to note that using equation (3.13), without introducing the \( \alpha \)-vectors, one arrives at the restrictions \( \prod_{\ell=0}^{3} (L - g_\ell) = \prod_{\ell=0}^{3} (R - g_\ell) = 1 \) where \( \{g_\ell\}_{\ell=0}^{3} \) are parameters corresponding to \( \{g_\ell\}_{\ell=0}^{3} \). Thus, for a given set of the \( g \)-parameters, one has to construct two diagonal matrices of only four roots of the fourth-order equation, which leads to general 2-soliton solutions or degenerate solutions with one soliton in one component (say, \( L \propto 1 \)) and three solitons in another one, whereas proposition 4.2.1, resulting from (3.15) gives \( N \)-soliton solutions for arbitrary \( N \).

5. Conclusion

As one can see from the above presentation, the procedure of deriving the \( N \)-soliton solutions was mostly the reduction to the already known equations: we have established the links between the field equations (2.6) and the Hirota–Ablowitz–Ladik system (3.26).

We would like to note once more that there were two non-trivial steps in our algorithm. First was the two-sublattice representation of \( u(n) \) given by (3.18) which has been used in our previous works [1, 2] and which may be viewed as the most straightforward way to introduce the Ablowitz–Ladik triplet of the tau-functions.

The second moment was the introduction of the frame \( \{\alpha_\ell\}_{\ell=0}^{3} \). In this paper we have used it just to split the field equations and, actually, as a way to resolve the restriction imposed on translations (for example, that translations corresponding to \( e_i \) and \( -e_i \) are mutually inverse). However, this construction, whose geometric importance has been discussed by various
authors (see, for example, [13]), can be generalized to connect other Hirota-type (star-type) equations that are typical for physical applications with the cell-type (defined, for example, on a cube or an octahedron) equations that usually appear in the mathematical works devoted to such questions as classification, integrability, geometric content etc.

Considering the restriction (2.4) we would like to note that it was crucial for the procedure we have used to derive the presented solutions. However, we cannot claim that it is necessary for the existence of the soliton-like solutions or the integrability of the model. Restrictions of this type often appear in the studies of integrable models. If we consider, for example, the HBDE, the restriction similar to (2.4) is present in the most of the works devoted to this system (including the original paper [4]). However, as it has been demonstrated in, for example, [22], the HBDE is integrable even without it (the widespread opinion now is that it is required for the existence of Hirota-form soliton solutions). At the same time, there are many known situations when the integrability, and hence the existence of the solitons, are related to some restrictions on the constants of a model. So, the role of the restriction (2.4) remains an open question.

The fact that model (2.3) possesses the N-soliton solutions is strong evidence (but surely not a proof) of its integrability. Thus, a straightforward continuation of this work is to look for the zero-curvature representation, the conservation laws, the Bäcklund transformations etc, i.e. to analyze the standard set of problems that arises in connection with any integrable system. However, these questions are out of the scope of the present paper and surely deserve separate studies.

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Appendix A. Proof of (3.24)

By simple algebra, one can derive from (3.23) the identity

\[ \tilde{\Gamma}_i \prod_{i \neq j}^3 (\lambda_i - \lambda_j) = C_i G_0 + (\lambda_0 - L) G_1 + G_2 \]  

(A.1)

where

\[ L = \sum_{i=1}^3 \lambda_i, \quad G_n = \sum_{i=1}^3 \Gamma_i \lambda_i^n \quad (n = 0, 1, 2) \]  

(A.2)

and

\[ C_i = \lambda_i^2 - \lambda_0 \lambda_i + \lambda_j \lambda_k \quad \{i, j, k\} = \{1, 2, 3\}. \]  

(A.3)

In our case, due to the restriction (2.4), \( G_0 = 0 \). Thus, the right-hand side of (A.1) (and hence all \( \tilde{\Gamma}_i \)) vanishes when

\[ \lambda_0 = L - G_2 / G_1 \]  

(A.4)

which proves (3.24).
Appendix B. Proof of proposition 4.1.1

To prove the fact that functions $\tau$ and $\rho$ defined in proposition 4.1.1 satisfy (4.37b) one has just to rewrite (4.30) with $\zeta = 0$

$$0 = (\xi - \eta) \left( T_{\xi \eta} \Omega \right) - \xi \left( T_{\xi 0} \Omega \right) + \eta \left( T_{\eta 0} \Omega \right)$$

(B.1)

and to express $\Omega$ and $T_{0} \Omega$ in terms of $\tau$ and $\rho$.

Applying $T_{0}^{-1}$ to (B.1) and expressing $\Omega$ and $T_{0}^{-1} \Omega$ in terms of $\tau$ and $\sigma$ one can see that $\tau$ and $\sigma$ satisfy (4.37a) with $a(\xi, \eta) = \xi - \eta$.

Finally, to prove (3.26c), we rewrite (B.1) with $\eta$ replaced with $1/\eta$:

$$0 = (\xi \eta - 1) \left( T_{\xi T_{1/\eta} \Omega} \right) - \xi \eta \left( T_{\xi 0} \Omega \right)$$

(B.2)

After application of $T_{1/\eta}^{-1}$, which is equal, due to (4.34), to $T_{\eta} T_{0}^{-1}$, this identity becomes

$$0 = (\xi \eta - 1) \left( T_{\xi \Omega} \right) - \xi \eta \left( T_{\xi \eta} \Omega \right)$$

(B.3)

Replacing $\Omega$ and $T_{0}^{-1} \Omega$ with $\tau$, $\rho$ and $\sigma$ one arrives at (4.37c) with $b(\xi, \eta) = 1 - 1/\xi \eta$.

This concludes the proof of proposition 4.1.1.

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