A RANDOM MODEL OF PUBLICATION ACTIVITY

ÁGNES BACKHAUSZ AND TAMÁS F. MÓRI

Department of Probability Theory and Statistics,
Eötvös Loránd University
Pázmány P. s. 1/C, H-1117 Budapest, Hungary
E-mail address: agnes@cs.elte.hu, moritamas@ludens.elte.hu

Abstract. We examine a random structure consisting of objects with positive weights and evolving in discrete time steps. It generalizes certain random graph models. We prove almost sure convergence for the weight distribution and show scale-free asymptotic behaviour. Martingale theory and renewal-like equations are used in the proofs.

1. Introduction

In this paper we examine a dynamic model inspired by scientific publication activity and networks of coauthors. However, the model contains many simplifying assumptions that are not valid in reality. We still use the terminology of publications for sake of simplicity.

The model consists of a sequence of researchers. Each of them has a positive weight which is increasing in discrete time steps. The weights reflect the number and importance of the researcher’s publications. One can think of cumulative impact factor for instance.

We start with a single researcher having a random positive weight. At the $n$th step a new publication is born. The number of its authors is randomly chosen. Then we select the authors, that is, one of the groups of that size; the probability that a given group is chosen is proportional to the sum of the weights of its members. After that the weights of the authors of the new publication are increased by random bonuses. Finally, a new researcher is added to the system with a random initial weight.

This is a preferential attachment model; one can see that authors with higher weights have larger chance to be chosen and increase their weights when the new publication is born.

Date: 21 May 2012.

2000 Mathematics Subject Classification. 60G42, 05C80.

Key words and phrases. Scale free, random graphs, martingales, renewal equation.

The European Union and the European Social Fund have provided financial support to the project under the grant agreement no. TÁMOP 4.2.1./B-09/KMR-2010-0003.
We are interested in the weight distribution of the model. That is, for fixed \( t > 0 \), we consider the ratio of authors of weight larger than \( t \), and study the asymptotic behaviour of this quantity as the number of steps goes to infinity.

Our main results (Section 3) include the almost sure convergence of the ratio of authors of weight larger than \( t \) under suitable conditions; first, when all weights are integer valued, then assuming that these random variables have continuous distribution. In both cases we describe the limiting sequence or function and determine its asymptotics. They are polynomially decaying under suitable conditions, thus our model shows scale-free behaviour.

The proofs of the almost sure convergence are based on the methods of martingale theory, while the polynomial decay of the asymptotic weight distribution follows from the results of [1] about renewal-like equations. See Section 4 for the details.

This model generalizes some random graph models. To see this, assume that every publication has only one author, and at each step, when a publication is born, connect its author to the new one with an edge. We get a random tree evolving in time.

In the particular case where the initial weights and author’s bonuses are always equal to 1, we get the Albert–Barabási random tree [2]. The neighbour of the new vertex is chosen with probabilities proportional to the degrees of the old vertices. Similarly, if the initial weights and the bonuses are fixed, but they are not necessarily equal to each other, we get random trees with linear weights [8], sometimes called generalized plane oriented recursive trees. In these cases the asymptotic degree distribution is well-known.

2. Notations and assumptions

2.1. Notations. Let the label of the only researcher being present in the beginning be 0; the label of the researcher coming in the \( n \)th step is \( n \).

\( X_i \) is the initial weight of researcher \( i \) for \( i = 0, 1, \ldots \). We suppose that \( X_0, X_1, \ldots \) are independent, identically distributed positive random variables.

\( \nu_n \) is the number of coauthors at step \( n \). This is an integer valued random variable for each \( n \). Obviously \( \nu_n \leq n \) must hold for all \( n \geq 1 \). On the other hand, for technical reasons we also assume that \( \nu_n \geq 1 \) for all \( n \geq 1 \). Since the authors’ weights are not necessarily increased, this may be supposed without loss of generality.

Given that \( \nu_n = k \), a group of size \( k \) is chosen randomly from researchers \( 0, \ldots, n - 1 \). The probability that a given group is chosen is proportional to the total weight of the group. The selected researchers will be the authors of the \( n \)th paper.
Let \( Y_{n,1}, Y_{n,2}, \ldots, Y_{n,\nu_n} \) be nonnegative random variables. These are the authors’ bonuses at step \( n \). That is, the weight of the \( \nu \)th coauthor of the \( n \)th paper is increased by \( Y_{n,\nu} \). The order of the coauthors is the natural order of the labels.

Let \( Z_n \) be the total weight of the \( n \)th paper; that is, \( Z_n = Y_{n,1} + Y_{n,2} + \ldots + Y_{n,\nu_n} \) for \( n \geq 1 \).

\( W(n,i) \) denotes the weight of author \( i \) after step \( n \) for \( i = 0, \ldots, n \). This is equal to \( X_i \) plus the sum of all bonuses \( Y_{j,\ell} \) for which author \( i \) is the \( \ell \)th author of the \( j \)th paper (\( \ell = 1, \ldots, \nu_j, \ j = 1, 2, \ldots, n \)).

Let \( S_n \) be the total weight after \( n \) steps; namely,

\[
S_n = W(n,0) + \ldots + W(n,n) = X_0 + \ldots + X_n + Z_1 + \ldots + Z_n.
\]

\( X, \nu, Y, Z \) are random variables. \( X \) is equal to \( X_0 \) in distribution, and \( Y_n \) is equal to \( Y_{n,1} \) in distribution for \( n \geq 1 \). The other random variables will be determined later by the assumptions.

Finally, \( \mathcal{F}_n \) is the \( \sigma \)-algebra generated by the first \( n \) steps; \( \mathcal{F}_n^+ = \sigma \{ \mathcal{F}_n, \nu_{n+1} \} \).

Throughout this paper \( \mathbb{I}(A) \) denotes the indicator of event \( A \). We say that two sequences \( (a_n), (b_n) \) are asymptotically equal \( (a_n \sim b_n) \), if they are positive except finitely many terms, and \( a_n/b_n \to 1 \) as \( n \to \infty \). A sequence \( (a_n) \) is exponentially small if \( |a_n| \leq q^n \) holds for all sufficiently large \( n \in \mathbb{N} \) for some \( 0 < q < 1 \).

2.2. Assumptions. Now we list the assumptions on the model.

Assumption 1. \( X_0, X_1, \ldots \) are independent, identically distributed. The initial weights \( X_n \), and the triplets \( (\mathcal{F}_{n-1}, (Y_{n,1}, \ldots, Y_{n,\nu_n}), \nu_n) \) are independent \( (n = 1, 2, \ldots) \).

Assumption 2. \( X \) has finite moment generating function.

Assumption 3. \( \nu_n \) and \( (Y_{n,1}, \ldots, Y_{n,\nu_n}) \) are independent of \( \mathcal{F}_{n-1} \) for \( n \geq 1 \).

Assumption 4. \( \nu_n \to \nu \) in distribution as \( n \to \infty \); in addition, \( \mathbb{E} \nu_n \to \mathbb{E} \nu < \infty \) and \( \mathbb{E} \nu_n^2 \to \mathbb{E} \nu^2 < \infty \) hold.

Recall that \( \nu_n \leq n \). Assumption 4 trivially holds if \( \nu \) is a fixed random variable with finite second moment, and the distribution of \( \nu_n \) is identical to the distribution of \( \min(n, \nu) \), or to the conditional distribution of \( \nu \) with respect to \( \{ \nu \leq n \} \).

Assumption 5. The conditional distribution of \( (Y_{n,1}, \ldots, Y_{n,\nu_n}) \), given \( \nu_n = k \), does not depend on \( n \). Moreover, the components are conditionally interchangeable, given \( \nu_n = k \).

Assumption 6. \( Z_n \) has finite expectation.

Now we know that \( (\nu_n, Y_n, Z_n) \to (\nu, Y, Z) \) in distribution as \( n \to \infty \), where \( Y \) and \( Z \) are random variables. We need that they also have finite moment generating functions, and they are not degenerate.
Assumption 7. $Y$ and $Z$ have finite moment generating functions.

Assumption 8. $X_n, Y_n, X$ and $Y$ are positive with positive probabilities for every $n = 1, 2, \ldots$. In addition, if $Y$ is integer-valued, then the greatest common divisor of the set $\{ i : \mathbb{P}(Y = i) > 0 \}$ is equal to 1.

The condition on the positivity of $X_n$ and $Y_n$ is not crucial. The positivity of $X$ and $Y$ implies that the same holds for $X_n$ and $Y_n$ if $n$ is large enough; we may assume this for all $n$ without loss of generality.

On the other hand, if $(Y_n)$ is identically equal to 0, that is, there are no bonuses at all, then the model only consists of the sequence of independent and identically distributed initial weights $X_n$, and the problem of empirical weight distribution becomes trivial. The last part of this assumption excludes periodicity.

There are two important particular cases satisfying all of our conditions. In the first one the weight of the paper is equally distributed among the authors. That is, $Z_1, Z_2, \ldots$ are independent identically distributed random variables, and $Y_{n,1} = \ldots = Y_{n,\nu_n} = Z_n / \nu_n$. The other option is that every author gets the total bonus, regardless the number of coauthors. More precisely, $Y_1, Y_2, \ldots$ are independent and identically distributed, and $Y_{n,1} = \ldots = Y_{n,\nu_n} = Y_n$, thus $Z_n = \nu_n Y_n$.

3. Main results

Discrete weight distribution. Suppose first that $X, Y_1, Y_2, \ldots$ are nonnegative integer valued random variables. Let $\xi_n(j)$ denote the number of researchers of weight $j$ after $n$ steps, that is,

$$\xi_n(j) = \left| \{ 0 \leq i \leq n : W(n, i) = j \} \right|, \quad j, n = 1, 2, \ldots .$$

The first theorem is about the almost sure behaviour of this quantity.

**Theorem 1.** $\xi_n(j) / n \rightarrow x_j$ almost surely as $n \rightarrow \infty$ with positive constants $x_j$, $j = 1, 2, \ldots$. The sequence $(x_j)$ satisfies the recursion

$$x_j = \sum_{i=1}^{j-1} \left[ \frac{(j-i)\mathbb{P}(Y = i)}{\mathbb{E}X + \mathbb{E}Z} + \mathbb{E}\left( (\nu-1)I(Y > i) \right) \right] + \mathbb{P}(X = j),$$

where $\alpha = \frac{\mathbb{P}(Y > 0)}{\mathbb{E}X + \mathbb{E}Z}$, $\beta = \mathbb{E}\left( (\nu-1)I(Y > 0) \right)$.

The second theorem describes the asymptotic behaviour of the sequence $(x_j)$.

**Theorem 2.** We have $x_j \sim C j^{-\gamma}$ as $j \rightarrow \infty$, where $C$ is a positive constant, and

$$\gamma = \frac{\mathbb{E}X + \mathbb{E}Z}{\mathbb{E}Y} + 1.$$
Continuous weight distribution. Now we assume that the distribution of $X$ and the conditional distributions of $Y_n \mid \nu_n = k$ are continuous for $k = 1, 2, \ldots, n, \ n = 1, 2, \ldots$. This implies that the distribution of $Y_n$ is continuous. Moreover, since the conditional distribution does not depend on $n$ according to Assumption 5, the distribution of $Y$ is also continuous.

Let $F(t) = \mathbb{P}(Y > t)$, $H(t) = \mathbb{E}((\nu - 1) \mathbb{I}(Y > t))$, and

$$L(t, s) = \frac{sF(s) + t(1 - F(s))}{\mathbb{E}X + \mathbb{E}Z} - H(s), \quad 0 \leq s \leq t.$$ 

It is clear that $L(t, s)$ is continuous, and, being the difference of two increasing functions, it is of bounded variation for fixed $t$.

This time $\xi_n(t)$ denotes the number of researchers with weight more than $t$ after $n$ steps.

$$\xi_n(t) = \left| \{0 \leq i \leq n : W(n, i) > t\} \right|, \quad t > 0, \ n = 1, 2, \ldots.$$ 

**Theorem 3.** $\frac{\xi_n(t)}{n} \rightarrow G(t)$ almost surely, as $n \rightarrow \infty$, where $G(t)$ is the solution of the following integral equation.

$$G(t) = \frac{\int_0^t G(t - s) d_s L(t, s) + H(t) + \mathbb{P}(X > t)}{\mathbb{E}X + \mathbb{E}Z} + \mathbb{E}\nu$$

for $t > 0$, and $G(0) = 1$.

Adding some extra conditions we can obtain results on the asymptotic behaviour of $G$.

**Theorem 4.** Suppose that the distribution of $Y$ is absolutely continuous. Then we have $G(t) \sim Ct^{-\gamma}$ as $t \rightarrow \infty$, where $C$ is a positive constant, and

$$\gamma = \frac{\mathbb{E}X + \mathbb{E}Z}{\mathbb{E}Y}.$$ 

**Remark 1.** The difference of the exponents in the discrete and continuous cases is due to the difference in the definitions. Namely, in the first case $\xi_n$ denotes the weight distribution, while in the second case it stands for the complementary cumulative weight distribution function.

### 4. Proofs

First we prove some propositions we will often use in the sequel.

**Lemma 1.** Let $(\mathcal{F}_n)$ be a filtration, $(\xi_n)$ a nonnegative adapted process. Suppose that

$$\mathbb{E}((\xi_n - \xi_{n-1})^2 \mid \mathcal{F}_{n-1}) = O(n^{1-\delta})$$

(3)
holds with some \( \delta > 0 \). Let \((u_n)\), \((v_n)\) be nonnegative predictable processes such that \( u_n < n \) for all \( n \geq 1 \). Finally, let \((w_n)\) be a regularly
varying sequence of positive numbers with exponent \( \mu \geq -1 \).

(a) Suppose that
\[
\mathbb{E}(\xi_n \mid \mathcal{F}_{n-1}) \leq \left(1 - \frac{u_n}{n}\right)\xi_{n-1} + v_n,
\]
and \(\lim_{n \to \infty} u_n = u\), \(\lim sup_{n \to \infty} v_n/w_n \leq v\) with some random variables \(u > 0\), \(v \geq 0\). Then
\[
\limsup_{n \to \infty} \frac{\xi_n}{nw_n} \leq \frac{v}{u + \mu + 1} \quad \text{a.s.}
\]

(b) Suppose that
\[
\mathbb{E}(\xi_n \mid \mathcal{F}_{n-1}) \geq \left(1 - \frac{u_n}{n}\right)\xi_{n-1} + v_n,
\]
and \(\lim_{n \to \infty} u_n = u\), \(\lim inf_{n \to \infty} v_n/w_n \geq v\) with some random variables \(u > 0\), \(v \geq 0\). Then
\[
\liminf_{n \to \infty} \frac{\xi_n}{nw_n} \geq \frac{v}{u + \mu + 1} \quad \text{a.s.}
\]

This is a stochastic counterpart of a lemma of Chung and Lu [5]. We will often apply this proposition with the sequence \(w_n \equiv 1\) and \(\mu = 0\).

**Proof.** Suppose first that \(v\) is strictly positive. Let \(\mathcal{F}_0\) be the trivial \(\sigma\)-algebra, \(\xi_0 = 0\), and
\[
c_n = \prod_{i=1}^{n} \left(1 - \frac{u_i}{i}\right)^{-1}, \quad n \geq 1.
\]

We have
\[
\log c_n = \sum_{i=1}^{n} \frac{u_i}{i} \left(1 + o(1)\right) = u \sum_{i=1}^{n} \frac{1 + o(1)}{i}.
\]
Hence for all \(t > 1\) we get that \(\lim_{n \to \infty}(\log c_{[tn]} - \log c_n) = u \log t\). That is, \((c_n)\) is regularly varying with exponent \(u\). It is clear that
\[
(4) \quad \mathbb{E}(c_n\xi_n \mid \mathcal{F}_{n-1}) \leq c_{n-1}\xi_{n-1} + c_nv_n.
\]
Therefore \(c_n\xi_n\) is a submartingale. Consider the Doob decomposition \(c_n\xi_n = M_n + A_n\), where
\[
M_n = \sum_{i=1}^{n} (c_i\xi_i - \mathbb{E}(c_i\xi_i \mid \mathcal{F}_{i-1}))
\]
is a martingale, and
\[
A_n = \sum_{i=1}^{n} \left(\mathbb{E}(c_i\xi_i \mid \mathcal{F}_{i-1}) - c_{i-1}\xi_{i-1}\right).
\]
From inequality (4) it follows that
\[ A_n \leq \sum_{i=1}^{n} c_i v_i. \]

Consider the increasing process in the Doob decomposition of the square of the martingale \( M_n \). Using condition (3) we get that
\[
B_n = \sum_{i=1}^{n} \text{Var}(c_i \xi_i | F_{i-1}) = \sum_{i=1}^{n} \text{Var}(c_i (\xi_i - \xi_{i-1}) | F_{i-1}) \\
\leq \sum_{i=1}^{n} c_i^2 \mathbb{E}((\xi_i - \xi_{i-1})^2 | F_{i-1}) = O\left( \sum_{i=1}^{n} i^{1-\delta} c_i^2 \right).
\]

Since \( n^{1-\delta} c_n^2 \) is still regularly varying with exponent \( 2u + 1 - \delta \), it follows that \( B_n = O(n^{2-\delta} c_n^2) \) (see e.g. [3, 4]). Hence, by Propositions VII-2-3 and VII-2-4 of [6], we have
\[
M_n = O\left( B_n^{1/2 + \varepsilon} \right) = O\left( n^{(2-\delta)(1/2 + \varepsilon)} c_n^1 + 2\varepsilon \right) = o(nc_n) \quad \text{a.s.,}
\]
for all \( 0 < \varepsilon < \frac{\delta}{4(u+1)} \).

On the other hand, using the fact \( u + \mu > -1 \), and the results of [3, 4] on regularly varying sequences we obtain that
\[
A_n \leq \sum_{i=1}^{n} c_i v_i \leq \left( 1 + o(1) \right) v \sum_{i=1}^{n} c_i w_i \sim v \frac{nc_n w_n}{u + \mu + 1}
\]
almost surely, as \( n \to \infty \). This implies that
\[
c_n \xi_n \leq \left( 1 + o(1) \right) \frac{v}{u + \mu + 1} nc_n w_n,
\]
thus the proof of part (a) is complete for positive \( v \).

The general case of nonnegative \( v \) can be deduced from the positive case by noticing that
\[
\mathbb{E}(\xi_n | F_{n-1}) \leq \left( 1 - \frac{u_n}{n} \right) \xi_{n-1} + \max(v_n, \varepsilon)
\]
for arbitrary \( \varepsilon > 0 \).

The proof of part (b) is similar. In this case
\[
A_n \geq \sum_{i=1}^{n} c_i v_i \sim \frac{v}{u + \mu + 1} nc_n w_n,
\]
a.s. on the event \( \{ v > 0 \} \). Hence, using \( c_n \xi_n \sim A_n \), we get that
\[
c_n \xi_n \geq \frac{v}{u + \mu + 1} nc_n w_n (1 + o(1)).
\]
On the event \( \{ v = 0 \} \) the inequality trivially holds. \( \square \)
Lemma 2. The conditional probability that an author of weight \( j \) is chosen, given \( \mathcal{F}_n^+ \) and \( \nu_{n+1} = k \), is equal to
\[
\frac{k - 1}{n} + \frac{n + 1 - k}{n} \cdot \frac{j}{S_n} = \frac{k - 1}{n} \left( 1 - \frac{j}{S_n} \right) + \frac{j}{S_n}.
\]

Proof. Consider those groups of size \( k \geq 2 \) that contain researcher \( i \) (\( 0 \leq i \leq n \)). There are \( \binom{n}{k-1} \) of them, because the total number of researchers is \( n + 1 \). Researcher \( i \) belongs to all of them, while the other researchers belong to \( \binom{n-1}{k-2} \) of those groups. Therefore the total weight of these groups can be obtained in the following way.

\[
\sum_{H \subseteq \{0, \ldots, n\}} \sum_{j \in H} W(n, j) = \binom{n}{k-1} W(n, i) + \sum_{j \neq i} \binom{n-1}{k-2} W(n, j)
\]

\[
= \binom{n-1}{k-1} W(n, i) + \binom{n-1}{k-2} S_n.
\]

On the other hand, the total weight of all groups of size \( k \) is given by
\[
\binom{n}{k-1} S_n.
\]

Hence the conditional probability that researcher \( i \) participates in the \((n+1)\)st paper given that it has \( k \) authors is equal to
\[
\frac{k - 1}{n} + \frac{n - k + 1}{n} \cdot \frac{W(n, i)}{S_n} = \frac{k - 1}{n} \left( 1 - \frac{W(n, i)}{S_n} \right) + \frac{W(n, i)}{S_n}.
\]

This obviously holds for \( k = 1 \) as well. \( \square \)

Proof of Theorem \( 1 \) Recall that in Theorem \( 1 \) we assumed that \( X, Y_1, Y_2, \ldots \) are integer valued random variables. Let us introduce
\[
H(i) = \mathbb{E} \left( (\nu - 1) \mathbb{I} (Y = i) \right),
\]
then \( \beta = \sum_{i=1}^{\infty} H(i) \).

We prove the theorem by induction on \( j \). The following argument is valid for all \( j = 1, 2, \ldots \). For \( j > 1 \) we will use the induction hypothesis.

At each step the number of authors of weight \( j \) may change due to the following events.

- A given author of weight \( j \) is chosen and he gets positive bonus.
- A given author of weight \( j - i \) is chosen and his bonus is equal to \( i \).
- The initial weight of the new author is \( j \).
Therefore Lemma 2 implies that

\begin{equation}
\mathbb{E}(\xi_n(j) \mid F_{n-1}^+) = \xi_{n-1}(j) \left[ 1 - \mathbb{P}(Y_n > 0 \mid F_{n-1}^+) \left( \frac{\nu_n - 1}{n - 1} + \frac{n - \nu_n}{n - 1} \cdot \frac{j}{S_{n-1}} \right) \right]
\end{equation}

\begin{equation}
+ \sum_{i=1}^{j-1} \xi_{n-1}(j-i) \mathbb{P}(Y_n = i \mid F_{n-1}^+) \left( \frac{\nu_n - 1}{n - 1} + \frac{n - \nu_n}{n - 1} \cdot \frac{j - i}{S_{n-1}} \right)
\end{equation}

+ \mathbb{P}(X_n = j).

Recall that \( \nu_n \geq 1 \) is assumed.

We introduce the time-dependent versions of the already defined quantities. Namely,

\( H_n(i) = \mathbb{E}(\nu_n - 1)I(Y_n = i) \); \( \beta_n = \sum_{i=1}^{n} H_n(i) = \mathbb{E}(\nu_n - 1)I(Y_n > 0) \).

Let us take conditional expectation given \( F_{n-1} \) in both sides of (5). Then we get that

\begin{equation}
\mathbb{E}(\xi_n(j) \mid F_{n-1}) = \xi_{n-1}(j) \left[ 1 - \frac{\beta_n}{n - 1} - \left( \mathbb{P}(Y_n > 0) - \frac{\beta_n}{n - 1} \right) \frac{j}{S_{n-1}} \right]
\end{equation}

\begin{equation}
+ \sum_{i=1}^{j-1} \xi_{n-1}(j-i) \left[ \frac{H_n(i)}{n - 1} + \left( \mathbb{P}(Y_n = i) - \frac{H_n(i)}{n - 1} \right) \frac{j - i}{S_{n-1}} \right]
\end{equation}

+ \mathbb{P}(X_n = j) \quad (j, n = 1, 2, \ldots).

We are going to apply Lemma 1 to the sequence \( (\xi_n(j)) \) with \( w_n \equiv 1 \) and \( \mu = 0 \). It is clear that \( |\xi_n(j) - \xi_{n-1}(j)| \leq \nu_n + 1 \), hence

\( \mathbb{E}(\xi_n(j) - \xi_{n-1}(j))^2 \mid F_{n-1} \) \leq \( \mathbb{E}(\nu_n + 1)^2 = O(1) \).

Thus, condition (3) on the differences of the sequence \( \xi_n(j) \) is satisfied. Moreover, as \( n \to \infty \), we have

\( u_n = n \left[ \frac{\beta_n}{n - 1} + \left( \mathbb{P}(Y_n > 0) - \frac{\beta_n}{n - 1} \right) \frac{j}{S_{n-1}} \right] \to \beta + \alpha j. \)

Note that \( \alpha > 0 \) because of Assumption 5.

Though the random variables \( Z_1, Z_2, \ldots \) are not necessarily identically distributed, they satisfy the following conditions.

\[ \sum_{n=1}^{\infty} \frac{\text{Var}(Z_n)}{n^2} < \infty, \quad \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}Z_i = \mathbb{E}Z. \]

Therefore Kolmogorov's theorem (Theorem 6.7. in [7]) can be applied. We get that \( S_n \sim n(\mathbb{E}X + \mathbb{E}Z) \) almost surely as \( n \to \infty \). Using
this, and also the induction hypothesis when \( j > 1 \), we conclude that

\[
v_n = \sum_{i=1}^{j-1} \xi_{n-1}(j-i) \left[ \frac{H_n(i)}{n-1} + \left( \mathbb{P}(Y_n = i) - \frac{H_n(i)}{n-1} \frac{j-i}{S_{n-1}} \right) \right] + \mathbb{P}(X_n = j) \to \sum_{i=1}^{j-1} x_{j-i} \left[ H(i) + \mathbb{P}(Y = i) \frac{j-i}{\mathbb{E}X + \mathbb{E}Z} \right] + \mathbb{P}(X = j),
\]

as \( n \to \infty \).

From equations (5) and (6) one can see that \((u_n)\) and \((v_n)\) are non-negative predictable processes. Moreover, \( u_n < n \) if \( n \) is large enough, because then \( \nu_n < n \) and \( j < S_{n-1} \). We have also seen that the limit of \((u_n)\) is positive. Hence, by Lemma 1, the induction step and the proof of Theorem 1 is complete.

**Proof of Theorem 2.** Write recursion (1) in the following form.

\[
x_j = \sum_{i=1}^{j-1} w_{j,i} x_{j-i} + r_j,
\]

where for \( i, j \geq 1 \) we set

\[
w_{j,i} = \frac{((j-i)\mathbb{P}(Y = i) + \mathbb{E}((\nu-1)\mathbb{I}(Y = i)))}{\alpha j + \beta + 1},
\]

and

\[
r_j = \frac{\mathbb{P}(X = j)}{\alpha j + \beta + 1}.
\]

In order to apply Theorem 1 of [1] we try to find sequences \((a_i)\), \((b_i)\), \((c_{j,i})\) such that \( w_{j,i} = a_i + b_i + c_{j,i} \) holds, then we have to check that \( a_i, b_i, c_{j,i}, r_i \) satisfy the following conditions.

(i) \( a_i \geq 0 \) for \( i \geq 1 \), and the greatest common divisor of the set \( \{i : a_i > 0\} \) is 1;

(ii) \( r_i \) is nonnegative, and not identically zero;

(iii) there exists \( z > 0 \) such that

\[
1 < \sum_{i=1}^{\infty} a_i z^i < \infty, \quad \sum_{i=1}^{\infty} |b_i| z^i < \infty,
\]

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{i-1} |c_{j,i}| z^j < \infty, \quad \sum_{i=1}^{\infty} r_i z^i < \infty.
\]

Therefore we set

\[
a_i = \lim_{j \to \infty} w_{j,i} = \frac{\mathbb{P}(Y = i)}{\alpha(\mathbb{E}X + \mathbb{E}Z)} = \mathbb{P}(Y = i \mid Y > 0), \quad i = 1, 2, \ldots.
\]
then we define
\[ b_i = \lim_{j \to \infty} j(w_{j,i} - a_i) = \frac{1}{\alpha} \left[ H(i) - (\alpha i + \beta + 1)a_i \right]. \]

Finally, we introduce
\[ c_{j,i} = w_{j,i} - a_i - b_i = -b_i \cdot \frac{\beta + 1}{j(\alpha j + \beta + 1)}. \]

Since \((a_i)\) is a probability distribution, for (iii) it suffices to show that \((a_i), (b_i), (c_{j,i}), \) and \((r_i)\) are exponentially small.

According to Assumption 7, \(Y\) has finite moment generating function. This implies that \((a_i)\) is exponentially small. The same holds for \((b_i)\), because
\[ \sum_{i=1}^{\infty} H(i)e^{\varepsilon i} = \mathbb{E}((\nu - 1)e^{\varepsilon Y}) \leq \left[ \mathbb{E}(\nu - 1)^2 \mathbb{E}(e^{2\varepsilon Y}) \right]^{1/2} < \infty \]
if \(\varepsilon > 0\) is small enough. Finally,
\[ \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} |c_{j,i}|e^{\varepsilon i} = \sum_{j=1}^{\infty} \sum_{i=1}^{j-1} \left| b_i \right| \frac{\beta + 1}{j(\alpha j + \beta + 1)} e^{\varepsilon i} \leq \sum_{j=1}^{\infty} \frac{\beta + 1}{j(\alpha j + \beta + 1)} \sum_{i=1}^{\infty} \left| b_i \right| e^{\varepsilon i} < \infty. \]

The sequence \((r_j)\) is also exponentially small, because \(X\) has finite moment generating function by Assumption 7.

\(w_{j,i}, a_j, r_j\) are nonnegative. Assumption 8 guarantees that the greatest common divisor of the set \(\{j : a_j > 0\}\) is equal to 1, and \(r_j > 0\) for some \(j\).

We have checked all conditions of Theorem 1 of [1]. Since \(X\) is not identically 0, there exists a \(k\) with \(x_k > 0\). On the other hand, by Assumption 8 \(P(Y = \ell) > 0\) for some \(\ell\). Now, one can see from the recursion that \(x_k, x_{k+1}, x_{k+2}, \ldots\) are all positive, hence the sequence \((x_n)\) has infinitely many positive terms. Therefore, applying the theorem we obtain that \(x_j \sim C j^{-\gamma}\) as \(j \to \infty\), where
\[ \gamma = -\frac{\sum_{i=1}^{\infty} b_i}{\sum_{i=1}^{\infty} ia_i}. \]
It is easy to see that
\[ \sum_{i=1}^{\infty} ia_i = \sum_{i=1}^{\infty} i \mathbb{P}(Y = i \mid Y > 0) = \frac{\mathbb{E}Y}{\mathbb{P}(Y > 0)}; \]
\[ -\sum_{i=1}^{\infty} b_i = \frac{\beta}{\alpha} + \sum_{i=1}^{\infty} ia_i + \frac{\beta + 1}{\alpha} = \frac{\mathbb{E}X + \mathbb{E}Z + \mathbb{E}Y}{\mathbb{P}(Y > 0)}. \]
Hence the statement of Theorem 2 follows. \qed
Proof of Theorem 3. We will use the results of the discrete part, namely, Theorem 1. Let \( h \) be sufficiently small positive number. We will consider limits as \( h \to 0 \).

Let \( F_n(t) = \mathbb{P}(Y_n > t) \) and \( H_n(t) = \mathbb{E}(\nu_n - 1 \mathbb{I}(Y_n > t)) \), as before. Furthermore, for a decreasing function \( \varphi \) let \( \Delta_h \varphi(t) = \varphi(t-h) - \varphi(t) \).

By Lemma 2 the conditional probability of the event that an author of weight between \( t - ih \) and \( t - (i-1)h \) is chosen, and his bonus is at least \((i-1)h\), given \( \mathcal{F}_{n-1}^+ \), is bounded from above by

\[
\mathbb{P}(Y_n > (i-1)h \mid \mathcal{F}_{n-1}^+) \leq \frac{t - (i-1)h}{S_{n-1}} + \left(1 - \frac{t - (i-1)h}{S_{n-1}}\right) \frac{\nu_n - 1}{n-1} \mathbb{P}(Y_n > (i-1)h \mid \mathcal{F}_{n-1}^+)
\]

Hence the conditional probability with respect to \( \mathcal{F}_{n-1} \) is at most

\[
u_i := \frac{t - (i-1)h}{S_{n-1}} F_n((i-1)h) + \frac{1}{n-1} \left(1 - \frac{t - (i-1)h}{S_{n-1}}\right) H_n((i-1)h).
\]

Note that \( \nu_i \) depends on \( n \), which is fixed at the moment. We get that

\[
\mathbb{E}(\xi_n(t) \mid \mathcal{F}_{n-1}) 
\leq \xi_{n-1}(t) + \sum_{i=1}^{\lceil t/h \rceil} \left[\xi_{n-1}(t - ih) - \xi_{n-1}(t - (i-1)h)\right] u_i + \mathbb{P}(X > t).
\]

After rearranging we obtain that

\[
\mathbb{E}(\xi_n(t) \mid \mathcal{F}_{n-1}) \leq \xi_{n-1}(t)(1 - u_1) + \sum_{i=1}^{\lceil t/h \rceil} \xi_{n-1}(t - ih)(u_i - u_{i+1}) + nu_{\lceil t/h \rceil + 1} + \mathbb{P}(X > t).
\]

Here

\[
u_1 = \frac{t}{S_{n-1}} + \frac{1}{n-1} \left(1 - \frac{t}{S_{n-1}}\right) \mathbb{E}(\nu_n - 1)
\]

\[
= \left(\frac{t}{\mathbb{E}X + \mathbb{E}Z} + \mathbb{E}\nu - 1\right) \frac{1 + o(1)}{n},
\]

and

\[
u_i - u_{i+1} = \frac{h}{S_{n-1}} F_n((i-1)h) + \frac{t - ih}{S_{n-1}} \Delta_h F_n(ih)
\]

\[
- \frac{1}{n-1} \frac{h}{S_{n-1}} H_n((i-1)h) + \frac{1}{n-1} \left(1 - \frac{t - ih}{S_{n-1}}\right) \Delta_h H_n(ih).
\]

This implies that

\[
n(u_i - u_{i+1}) \to \frac{h}{\mathbb{E}X + \mathbb{E}Z} F((i-1)h) + \frac{t - ih}{\mathbb{E}X + \mathbb{E}Z} \Delta_h F(ih) + \Delta_h H(ih),
\]
as \( n \to \infty \). Finally,

\[
nu_{\lceil t/h \rceil + 1} \leq \frac{n}{n - 1} \left( 1 + \frac{h}{S_{n-1}} \right) H_n(t),
\]

hence

\[
\limsup_{n \to \infty} nu_{\lceil t/h \rceil + 1} \leq H(t).
\]

Let

\[
G_u(t) = \limsup_{n \to \infty} \frac{\xi_n(t)}{n}
\]

(subscript \( u \) stands for “upper”). \( G_u(t) \) is a decreasing random function, and

\[
\limsup_{n \to \infty} \sum_{i=1}^{\lceil t/h \rceil} \xi_{n-1}(t - ih)(u_i - u_{i+1}) 
\]

\[
\leq \sum_{i=1}^{\lceil t/h \rceil} G_u(t - ih) \left[ \frac{F((i-1)h)}{\mathbb{E}X + \mathbb{E}Z} h + \frac{t - ih}{\mathbb{E}X + \mathbb{E}Z} \Delta_h F(ih) + \Delta_h H(ih) \right].
\]

Denote the sum on the right hand side by \( \Sigma_u(t, h) \). We want to apply Lemma 1 to the sequence \( \xi_n(t) \). It satisfies (8), and, similarly to the discrete case,

\[
\mathbb{E}(\xi_n(t) - \xi_{n-1}(t))^2 \mid \mathcal{F}_{n-1} \leq \mathbb{E}(\nu_n + 1)^2 = O(1)
\]

holds again. The other assumptions are also easy to check. Hence

\[
G_u(t) \leq \left[ \Sigma_u(t, h) + H(t) + \mathbb{P}(X > t) \right] \left[ \frac{t}{\mathbb{E}X + \mathbb{E}Z} + \mathbb{E} \nu \right]^{-1}.
\]

One can readily verify that \( \Sigma_u(t, h) \) converges to

\[
\frac{1}{\mathbb{E}X + \mathbb{E}Z} \left[ \int_0^t G_u(t - s) F(s) \, ds - \int_0^t G_u(t - s)(t - s) \, dF(s) \right]
\]

\[
- \int_0^t G_u(t - s) \, dH(s) = \int_0^t G_u(t - s) \, d_s L(t, s)
\]

as \( h \to 0 \), since the Riemann–Stieltjes integrals in the expression exist. This implies that

\[
(9)
\]

\[
G_u(t) \leq \left[ \int_0^t G_u(t - s) \, d_s L(t, s) + H(t) + \mathbb{P}(X > t) \right] \left[ \frac{t}{\mathbb{E}X + \mathbb{E}Z} + \mathbb{E} \nu \right]^{-1}.
\]

Therefore the solution of the corresponding integral equation (2) with initial condition \( G_u(0) = 1 \) is an upper bound for \( G_u(t) \). That is, \( G_u(t) \leq G(t) \), where \( G(t) \) is the deterministic function given in the theorem.

Now we give lower bounds by analogous argumentation.
We estimate from below the conditional probability that an author with weight between \( t - ih \) and \( t - (i - 1)h \) is chosen and his bonus is at least \( ih \), given \( F_{n-1}^+ \). Similarly to (7), we have that it is greater than or equal to

\[
\left[ \frac{t - ih}{S_{n-1}} + \left( 1 - \frac{t - ih}{S_{n-1}} \right) \nu_{n-1} - 1 \right] \mathbb{P}(Y_n > ih \mid F_{n-1}^+).
\]

Hence the lower bound of the conditional probability with respect to \( F_{n-1} \) is the following.

\[
\ell_i := \frac{t - ih}{S_{n-1}} F_n(ih) + \frac{1}{n - 1} \left( 1 - \frac{t - ih}{S_{n-1}} \right) H_n(ih).
\]

We obtain that

\[
\mathbb{E}(\xi_n(t) \mid F_{n-1}) \geq \xi_{n-1}(t) + \sum_{i=1}^{\lfloor t/h \rfloor} \left[ \xi_{n-1}(t - ih) - \xi_{n-1}(t - (i - 1)h) \right] \ell_i + \mathbb{P}(X > t).
\]

After rearranging we get a formula similar to (8).

\[
E(\xi_n(t) \mid F_{n-1}) \geq \xi_{n-1}(t)(1 - \ell_1) + \sum_{i=1}^{\lfloor t/h \rfloor} \xi_{n-1}(t - ih)(\ell_i - \ell_{i+1}) + n\ell_{\lfloor t/h \rfloor + 1} + \mathbb{P}(X > t).
\]

Here

\[
\ell_1 = \frac{t - h}{S_{n-1}} F_n(h) + \frac{1}{n - 1} \left( 1 - \frac{t - h}{S_{n-1}} \right) H_n(h)
\]

\[
= \left( \frac{t - h}{EX + EZ} F(h) + H(h) \right) \frac{1 + o(1)}{n},
\]

and

\[
\ell_i - \ell_{i+1} = \frac{h}{S_{n-1}} F_n(ih) + \frac{t - (i + 1)h}{S_{n-1}} \Delta_h F_n((i + 1)h)
\]

\[
- \frac{1}{n - 1} H_n(ih) + \frac{1}{n - 1} \left( 1 - \frac{t - (i + 1)h}{S_{n-1}} \right) \Delta_h H_n((i + 1)h).
\]

This implies that \( n(\ell_i - \ell_{i+1}) \) converges to

\[
\frac{h}{EX + EZ} F(ih) + \frac{t - (i + 1)h}{EX + EZ} \Delta_h F((i + 1)h) + \Delta_h H((i + 1)h)
\]

as \( n \to \infty \). Finally,

\[
n\ell_{\lfloor t/h \rfloor + 1} \geq - \frac{2nh}{S_{n-1}} + H_n(t + 2h),
\]
therefore
\[ \liminf_{n \to \infty} n \ell_{[t/h]+1} \geq -\frac{2h}{\mathbb{E}X + \mathbb{E}Z} + H(t + 2h). \]

Let
\[ G_\ell(t) = \liminf_{n \to \infty} \frac{\xi_n(t)}{n}; \]
then \( G_\ell(t) \) is also a decreasing random function. On the right hand side of (10) we have
\[ \liminf_{n \to \infty} \sum_{i=1}^{[t/h]} \xi_{n-1}(t - ih)(\ell_i - \ell_{i+1}) \geq \Sigma_\ell(t, h), \]
where
\[ \Sigma_\ell(t, h) = \sum_{i=1}^{[t/h]} G_\ell(t - ih) \left[ \frac{F(ih)}{\mathbb{E}X + \mathbb{E}Z} h + \frac{t - (i + 1)h}{\mathbb{E}X + \mathbb{E}Z} \Delta_h F((i + 1)h) + \Delta_h H((i + 1)h) \right]. \]

Applying Lemma 1 we get that
\[ G_\ell(t) \geq \left[ \Sigma_\ell(t, h) - \frac{2h}{\mathbb{E}X + \mathbb{E}Z} + H(t + 2h) + \mathbb{P}(X > t) \right] \times \left[ \frac{t - h}{\mathbb{E}X + \mathbb{E}Z} F(h) + H(h) + 1 \right]^{-1}. \]

Let \( h \) go to zero again. The sum \( \Sigma_\ell(t, h) \) converges to the same Riemann–Stieltjes integral as \( \Sigma_u(t, h) \) does. Thus the right hand side of the inequality above converges to the right hand side of (10). Hence we obtain that \( G_\ell(t) \geq G(t) \). This, together with the estimation for \( G_u(t) \), implies the statement of the theorem.

**Proof of Theorem 4.** Let the density function of \( Y \) be denoted by \( f \). From the absolute continuity of \( F \) the same follows for \( H \). Let \( h \) be defined by
\[ H(t) = \int_1^\infty h(s) \, ds. \]
Differentiating \( L \) with respect to \( s \) we obtain that
\[ \frac{\partial}{\partial s} L(t, s) = \frac{F(s) - sf(s) + tf(s)}{\mathbb{E}X + \mathbb{E}Z} + h(s) \quad (0 \leq s \leq t). \]
Hence equation (2) may be written in the following form.
\[ G(t) = \int_0^t G(t - s) w_{t,s} \, ds + r(t), \]
where
\[
 w_{t,s} = \frac{F(s) + (t-s)f(s)}{\mathbb{E}X + \mathbb{E}Z} + \frac{h(s)}{t + (\mathbb{E}X + \mathbb{E}Z)\mathbb{E}\nu} ;
\]
\[
 r(t) = \frac{H(t) + \mathbb{P}(X > t)}{\mathbb{E}X + \mathbb{E}Z + \mathbb{E}\nu} .
\]

In order to apply Theorem 2 of [1] write \( w_{t,s} \) in the following form.
\[
 w_{t,s} = f(s) + \frac{F(s) - (s + (\mathbb{E}X + \mathbb{E}Z)\mathbb{E}\nu) f(s) + h(s)(\mathbb{E}X + \mathbb{E}Z)}{t + (\mathbb{E}X + \mathbb{E}Z)\mathbb{E}\nu} ;
\]
\[
 = f(s) + \frac{b(s)}{t + d} ,
\]

where
\[
 b(s) = F(s) - (s + (\mathbb{E}X + \mathbb{E}Z)\mathbb{E}\nu) f(s) + h(s)(\mathbb{E}X + \mathbb{E}Z) ;
\]
\[
 d = (\mathbb{E}X + \mathbb{E}Z)\mathbb{E}\nu .
\]

Next we check that all assumptions required in [1] hold. Since \( f \) is a probability density function, \( G \) is clearly decreasing and \( w \) is nonnegative, all we need is the following three facts.

(i) \( d \) is a positive constant,
(ii) \( r \) is a nonnegative, continuous function,
(iii) there exists \( z > 1 \) such that
\[
 \int_0^\infty f(t) z^t dt < \infty , \quad \int_0^\infty |b(t)| z^t dt < \infty ,
\]
and \( r(t) z^t \) is directly Riemann integrable on \([0, \infty)\).

Here [1] follows from Assumption [8]. From the continuity of \( F \) and \( H \) the same follows for \( r \). Finally, the first part of condition \([1][3]\) easily follows from Assumptions 2 and 7. In addition, using that \( r \) is monotonically decreasing we get that
\[
 \sum_{n=1}^\infty \sup_{0 \leq \theta \leq \tau} r(t + n\tau + \theta) z^{t+n\tau+\theta} \leq \sum_{n=1}^\infty [r(t + n\tau) z^{t+n\tau}] z^\tau
\]
for \( z > 1 \). The right hand side is finite for almost all \( t \), because \( \int_0^\infty r(s) z^s ds \) is finite. Therefore \( r(t) z^t \) is directly Riemann integrable.

Thus Theorem [1][4] follows from Theorem 2 of [1]. Using the continuity of \( G \) and the method of the discrete case it is easy to see that \( G \) is not
identically 0 for large $t$, thus it is polynomially decaying. What is left
is to determine the exponent, that is,

$$\gamma = -\frac{\int_0^\infty b(s) \, ds}{\int_0^\infty sf(s) \, ds}.$$  

The denominator is equal to $\mathbb{E}Y$. In the numerator we have

$$\int_0^\infty b(s) \, ds$$

$$= \int_0^\infty \left( F(s) - (s + (\mathbb{E}X + \mathbb{E}Z) \mathbb{E}\nu) f(s) + h(s) (\mathbb{E}X + \mathbb{E}Z) \right) ds$$

$$= \mathbb{E}Y - \mathbb{E}Y - (\mathbb{E}X + \mathbb{E}Z) \mathbb{E}\nu + H(0) (\mathbb{E}X + \mathbb{E}Z)$$

$$= -(\mathbb{E}X + \mathbb{E}Z) \mathbb{E}\nu + \mathbb{E}(\nu - 1) (\mathbb{E}X + \mathbb{E}Z)$$

Therefore we got that

$$\gamma = \frac{\mathbb{E}X + \mathbb{E}Z}{\mathbb{E}Y},$$

and the proof of Theorem 4 is complete.  

\[ \square \]

References

[1] Á. Backhausz, T. F. Móri, Asymptotics of a renewal-like recursion and an integral equation. Manuscript. arXiv:1104.1027v4 [math.CA]. http://arxiv.org/abs/1104.1027v4

[2] A-L. Barabási, R. Albert, Emergence of scaling in random networks, Science 286 (1999), 509–512.

[3] N. H. Bingham, C. M. Goldie, J. L. Teugels, Regular variation, Encyclopedia of Mathematics and its Applications, 27, Cambridge Univ. Press, Cambridge, 1987. MR0898871 (88i:26004)

[4] R. Bojanic, E. Seneta, Slowly varying functions and asymptotic relations, J. Math. Anal. Appl., 34 (1971), 302–315. MR0274676 (43 #438)

[5] F. Chung, L. Lu, Complex graphs and networks, CBMS Regional Conference Series in Mathematics, 107, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006. MR2248695 (2007i:05169)

[6] J. Neveu, Discrete-parameter martingales. North-Holland Publishing Co., New York, 1975. MR0402915

[7] V. V. Petrov, Limit theorems of probability theory, Oxford Univ. Press, New York, 1995. MR1353441 (96h:60048)

[8] B. Pittel, Note on the heights of random recursive trees and random $m$-ary search trees, Random Struct. Algorithms 5 (1994), 337–348.

Department of Probability Theory and Statistics, Eötvös Loránd University, Pázmány P. s. 1/C, H-1117 Budapest, Hungary

E-mail address: agnes@cs.elte.hu

Department of Probability Theory and Statistics, Eötvös Loránd University, Pázmány P. s. 1/C, H-1117 Budapest, Hungary

E-mail address: moritamas@ludens.elte.hu