THE SIZE OF THE JOINT-GIANT COMPONENT IN A BINOMIAL RANDOM DOUBLE GRAPH

MARK JERRUM∗,†

School of Mathematical Sciences
Queen Mary, University of London
Mile End Road
London E1 4NS

TAMÁS MAKAI∗,‡

Department of Mathematics G. Peano
University of Torino
Via Carlo Alberto, 10
10123, Torino, Italy

Abstract. We study the joint components in a random ‘double graph’ that is obtained by superposing red and blue binomial random graphs on \( n \) vertices. A joint component is a maximal set of vertices that supports both a red and a blue spanning tree. We show that there are critical pairs of red and blue edge densities at which a joint-giant component appears. In contrast to the standard binomial graph model, the phase transition is first order: the size of the largest joint component jumps from \( O(1) \) vertices to \( \Theta(n) \) at the critical point. We connect this phenomenon to the properties of a certain bicoloured branching process.

1. Introduction

In recent years there has been a growing interest in ‘multilayer networks’ as a model for large real-world structures [1]. Attention is focused on properties of a multilayer network that arise from interactions between the layers. In the language of graph theory, we can treat a multilayer network as a collection of graphs, all sharing a common vertex set. The simplest case is a double graph \( G = (V, E_1, E_2) \) formed by superposing two graphs \( G_1 = (V, E_1) \) and \( G_2 = (V, E_2) \) over the same vertex set. We refer to \( E_1 \) as the set of red edges and \( E_2 \) as the set of blue edges. We are particularly interested in the random double graph \( G(n, p_1, p_2) \) in which \( G_1 \) and \( G_2 \) are independent binomial (or Erdős-Rényi) random graphs on \([n]\), with edge probabilities \( p_1 \) and \( p_2 \), respectively. Thus a red edge is present between a given pair of vertices with probability \( p_1 \), independently of all the other potential red and blue edges, and similarly for the blue edges.

The most intensively studied phenomenon in the theory of random graphs is the emergence and growth of a ‘giant component’ in a binomial random graph as the edge probability increases [12, Chap. 5]. A giant component is a connected component that appears at a certain critical edge probability, specifically \( p = 1/n \). For \( p = c/n \) with \( c > 1 \), the giant component contains a constant proportion of the vertices, and is unique: all other connected components have size \( O(\log n) \) with high probability (whp). When \( c < 1 \) there is no giant component. This

E-mail addresses: m.jerrum@qmul.ac.uk, tamas.makai@unito.it.
∗ Supported by EPSRC grant EP/N004221/1.
† Supported by EPSRC grant EP/S016694/1.
‡ Supported by “Memory in Evolving Graphs” (Compagnia di San Paolo/Università degli Studi di Torino).
1 A property holds whp if the probability of it occurring tends to 1 as \( n \) tends to infinity.
phase transition phenomenon is now understood in great detail \[11\]. A natural extension of this line of work to double graphs is the following. A joint component is a maximal set of vertices that supports both a red and a blue spanning tree. (Note that the joint components form a partition of the vertex set of a double graph.) We ask whether the largest joint component — the potential joint-giant component — undergoes a phase transition and, if so, what is the nature of the transition. Note that the joint-giant component is not simply the intersection of the red and blue giant components considered in isolation, though it is contained in the intersection.

The question of the existence of a joint-giant component in a double graph was examined, in a slightly disguised form, by Buldyrev, Parshani, Paul, Stanley and Havlin \[5\]. The relevant scaling to use is \( p_1 = \lambda_1/n \) and \( p_2 = \lambda_2/n \), for constants \( \lambda_1, \lambda_2 \). Buldyrev et al. provided a heuristic argument that the joint-giant appears at certain critical values of the pair \((\lambda_1, \lambda_2)\), and confirmed this predicted behaviour experimentally. Independently, Molloy \[13\] provided a rigorous proof for the size of the joint-giant in the special case when \( \lambda_1 = \lambda_2 \), and stated what should be the generalisation to unequal edge densities and even to graphs formed from three or more distinguished edge sets. Both the heuristic and rigorous results approach the joint-giant from above, by repeatedly stripping vertices that cannot form part of it.

In this paper we take a very different approach to analysing the joint components of the double graph \( G(n, \lambda_1/n, \lambda_2/n) \). We show that whp any non-trivial joint component contains exactly two vertices or a linear fraction of the vertices. In addition, whp there can be at most one component of linear size, which we call the joint-giant. We establish the size of the joint-giant as a function of \( \lambda_1, \lambda_2 \). Interestingly, whereas the phase transition of a classical binomial random graph is second order, the phase transition in a double graph turns out to be first order. Thus, if we plot the size of the largest component (scaled by \( 1/n \)) in a binomial graph \( G(n, \lambda/n) \) as a function of \( \lambda \), the resulting curve is continuous; the phase transition is marked only by a discontinuity of the derivative at \( \lambda = 1 \). In contrast, for a double graph, the plot of the size of largest component is discontinuous at pairs \((\lambda_1, \lambda_2)\) lying on a curve \( C \) to be defined presently. For example, as noted by Molloy, there is a critical value \( \lambda^* = 2.4554 \) such that when \( \lambda_1 = \lambda_2 < \lambda^* \) there is no joint-giant component, and when \( \lambda_1 = \lambda_2 > \lambda^* \) there is a joint-giant of linear size, in fact containing about \( 0.5117 n \) vertices. The curve \( C \) defining the phase transition as a function of \( \lambda_1 \) and \( \lambda_2 \) is plotted in Figure 1. Above the curve, there is a unique joint-giant of linear size; below, the largest joint component has size at most 2, whp. These analytical results are consistent with numerical findings reported by Buldyrev et al. \[5\] and, of course, with the analytic result of Molloy \[13\].

![Figure 1](image)

**Figure 1.** The phase transition plotted as a function of \( \lambda_1 \) and \( \lambda_2 \). This is the curve \( C \) from Theorem 1.

A superficially similar percolation model is jigsaw percolation introduced by Brummitt, Chatterjee, Dey and Sivakoff \[4\]. This model is also defined on a double graph, however in this case a bottom-up approach is used. Initially every vertex is in its own partition and in every step of the process two partitions are merged if there is a red and a blue edge between them. Several
papers have been devoted to investigating when the process percolates i.e. when every vertex is contained in the same partition by the end of the process. So far the combination of various deterministic graphs with a binomial random graph \[\text{11}\text{8}\] and the combination of two binomial random graphs \[\text{3}\text{7}\] has been studied. In addition extensions to multi-coloured random graphs \[\text{0}\] and random hypergraphs \[\text{2}\] exist.

As hinted at earlier, one approach to locating the joint-giant is to repeatedly remove the vertices found in any small red and blue component of the graph. Molloy \[\text{13}\] analysed this process in order to establish the size of the joint-giant. Our method differs as we relate the size of the joint-giant in the double graph \(G(n, \lambda_1, \lambda_2)\) to a bicoloured branching process, where every particle in the process has \(\text{Po}(\lambda_1)\) red offspring and independently \(\text{Po}(\lambda_2)\) blue offspring.

A joint-giant exists precisely when there is a positive probability that such a branching process contains an infinite red-blue binary tree, i.e., one in which every particle has one red offspring and one blue offspring. In a sense, what we do is the opposite of the earlier approach, in that we are exploring the joint-giant component from within. We feel that this approach gives additional insight into the phase transition phenomenon. The two approaches mirror earlier work on the \(k\)-core of a random graph, with Pittel, Spencer and Wormald \[\text{14}\] approaching the \(\beta\)-core of a random graph, with Pittel, Spencer and Wormald \[\text{14}\] approaching the \(\alpha\)-core from above, and Riordan \[\text{15}\] from below.

Denote the coloured rooted unlabelled tree created by the above branching process by \(X_{\lambda_1, \lambda_2}\) and the associated probability distribution by \(P_{\lambda_1, \lambda_2}\). In order to state our result, we need to make some preliminary observations about \(X_{\lambda_1, \lambda_2}\). The root of the tree is \(v_0\). When we say that a particle \(x\) of the branching process has a certain property, we mean that the process consisting of \(x\) (as the new root) and its descendants has the property.

A binary red-blue tree of height \(d\) is a perfect binary tree of height \(d\), where every internal vertex has a red and a blue offspring. Let \(B_d\) be the event that \(X_{\lambda_1, \lambda_2}\) contains a binary red-blue tree of height \(d\) with \(v_0\) as the root, and let \(B = \lim_{d \to \infty} B_d\) be the event that \(X_{\lambda_1, \lambda_2}\) contains an infinite binary red-blue tree with root \(v_0\). Then \(P_{\lambda_1, \lambda_2}[B_0] = 1\). Also, each particle in the first generation of \(X_{\lambda_1, \lambda_2}\) has property \(B_d\) with probability \(P_{\lambda_1, \lambda_2}[B_d]\). As these events are independent for different particles, the number of red and blue offspring in the first generation with property \(B_d\) has a Poisson distribution with mean \(\lambda_1 P_{\lambda_1, \lambda_2}[B_d]\) and \(\lambda_2 P_{\lambda_1, \lambda_2}[B_d]\) respectively.

Thus, \(P_{\lambda_1, \lambda_2}[B_{d+1}] = P[\text{Po}(\lambda_1 P_{\lambda_1, \lambda_2}[B_d]) > 0] P[\text{Po}(\lambda_2 P_{\lambda_1, \lambda_2}[B_d]) > 0]\).

Since \(P[\text{Po}(\lambda x) > 0]\) is a continuous, increasing function of \(x\) on \([0, 1]\), it follows (e.g., from Kleene’s fixed point theorem) that \(P_{\lambda_1, \lambda_2}[B] = \lim_{d \to \infty} P_{\lambda_1, \lambda_2}[B_d]\) is given by the maximum solution \(\alpha\) to the equation

\[
\alpha = P[\text{Po}(\lambda_1 \alpha) > 0] P[\text{Po}(\lambda_2 \alpha) > 0].
\]

(Maximality comes from \(P_{\lambda_1, \lambda_2}[B_0] = 1\).)

We denote this solution by \(\beta(\lambda_1, \lambda_2)\). Let \(C = \partial\{(\lambda_1, \lambda_2) \mid \beta(\lambda_1, \lambda_2) = 0\}\) be the boundary of the zero-set of \(\beta\). The main result of the paper is the following:

**Theorem 1.** For \((\lambda_1, \lambda_2) \in (\mathbb{R}^+)^2 \setminus C\) the number of vertices of \(G(n, \lambda_1/n, \lambda_2/n)\) in the joint-giant is \(\beta(\lambda_1, \lambda_2)n + o_p(n)\) as \(n \to \infty\).

In addition, we show that any non-trivial joint component\(^2\) of sublinear size contains exactly two vertices, i.e. it is a pair of vertices connected by a red and a blue edge.

**Theorem 2.** We have that whp no component of size \(k\) exists for any \(2 < k = o(n)\).

1.1. **Proof outline.** Our proof is based on a method introduced by Riordan \[\text{15}\] in order to determine the size of the \(k\)-core of a graph. The key idea is to define a pair of events, which depend only on the close neighbourhood of a vertex in the double graph, more precisely on vertices which are at distance \(o(\log n)\). The distance of two vertices in the double graph is defined as the distance in the graph \(G(V, E_1 \cup E_2)\). In addition whp every vertex for which the first event holds is contained in the joint-giant, however whp none of the vertices for which the

\(^2\)A trivial joint component has size 1
second event fails is found in the joint-giant. The result follows if the probabilities of the two events are close enough.

Local properties are chosen because there is an effective coupling between the random double graph in the close neighbourhood of a vertex and the branching process described above, allowing us to transfer results from the branching process to the random graph.

For the second event we will choose that either the neighbourhood of $v$ contains a short cycle, or $B_0$ holds for some appropriately chosen $s$. We show that any vertex, which does not have either of these properties is outside of any non-trivial joint component (Claim 15). An upper bound on the size of the joint-giant by providing an estimate on the expected number of these vertices and the second moment method.

The lower bound requires significantly more attention. In this case we define the event $A$, which is essentially a robust version of $B_0$. We show that whp many vertices have property $A$ and in addition every vertex with property $A$ is the root of a red-blue binary tree of depth $s$, where every leaf has property $A$ within the remainder of the graph (Lemma 12).

Now consider the graph spanned by the vertices found in the union of these trees. When applied to the $k$-core the previously described method already identifies almost every vertex within the $k$-core, as every edge is in the same graph. However this is not the case for joint-connectivity as even though every vertex is contained in a red and a blue subgraph, of size at least $s$, there is no guarantee that the set contains a red and a blue spanning tree. While small components may appear in the random graph, the previously described set, due to its special structure, no longer has this property, and thus any component within it must have size at least $n^{3/5}$ (Proposition 4). We complete the proof of Theorem 1 with a sprinkling argument to show that the graph spanned by this subset is connected in both the red and the blue graph.

Theorem 2 follows from a simple first moment argument.

1.2. Organisation of the paper. For a double graph $G$ on $n$ vertices let $U'(G)$ be the maximal subset of vertices of $G$ such that in the subgraph spanned by $U'$, denoted by $G[U']$, every vertex is found in both a red and a blue subgraph of size at least $n^{3/5}$. The key result for showing the lower bound on the size of the joint-giant is the following.

Proposition 3. For every $(\lambda_1', \lambda_2') \in (\mathbb{R}^+)^2 \setminus C$ we have

$$|U'(G(n, \lambda_1'/n, \lambda_2'/n))| \geq \beta(\lambda_1', \lambda_2') n + o_p(n).$$

Note that it is enough to consider $(\lambda_1', \lambda_2') \in (\mathbb{R}^+)^2 \setminus C$ with $\beta(\lambda_1', \lambda_2') > 0$ as otherwise the trivial lower bound 0 already implies the statement. Section 2 and 3 is devoted to proving this result for such a fixed pair $(\lambda_1', \lambda_2')$. Once this is done, Theorems 1 and 2 follow swiftly in Section 4.

2. A branching process

In this section we define and analyse a certain bicoloured branching process. This will form an idealised model of a random double graph. The model is adequate, since the random graph is locally tree-like. In analysing the branching process we rely heavily on ideas introduced by Riordan [15]. Later, in Section 3 we create a bridge from the branching process to random graphs.

Recall that in this section we consider $(\lambda_1', \lambda_2') \in (\mathbb{R}^+)^2 \setminus C$ with $\beta(\lambda_1', \lambda_2') > 0$.

Lemma 4. For $i = 1, 2$ we have $\lambda'_i \mathbb{P} \left[ \text{Po}(\lambda'_3, \beta(\lambda_1', \lambda_2')) > 0 \right] > 1$.

Proof. Without loss of generality let $i = 1$. Assume for contradiction that

$$\lambda'_1 \mathbb{P} \left[ \text{Po}(\lambda'_2, \beta(\lambda_1', \lambda_2')) > 0 \right] = \lambda'_1 \left( 1 - \exp(-\lambda'_2 \beta(\lambda_1', \lambda_2')) \right) \leq 1.$$  (1)
Note that $\lambda'_1 \neq 0$ as $\lambda'_1 = 0$ would imply $\beta(\lambda'_1, \lambda'_2) = 0$. Since $(\lambda'_1, \lambda'_2) \in (\mathbb{R}^+)^2 \setminus C$ we have
\[
\beta(\lambda'_1, \lambda'_2) = \mathbb{P}[\text{Po}(\lambda'_1 \beta(\lambda'_1, \lambda'_2)) > 0] \mathbb{P}[\text{Po}(\lambda'_2 \beta(\lambda'_1, \lambda'_2)) > 0] \\
= (1 - \exp(-\lambda'_1 \beta(\lambda'_1, \lambda'_2))) (1 - \exp(-\lambda'_2 \beta(\lambda'_1, \lambda'_2))) \\
\leq 1 - \exp(-\lambda'_1 \beta(\lambda'_1, \lambda'_2)) / \lambda'_1,
\]
or equivalently
\[
\exp(-\lambda'_1 \beta(\lambda'_1, \lambda'_2)) \leq 1 - \lambda'_1 \beta(\lambda'_1, \lambda'_2).
\]
Now this inequality holds only if $\lambda'_1 \beta(\lambda'_1, \lambda'_2) = 0$ leading to a contradiction, as neither $\lambda'_1$ nor $\beta(\lambda'_1, \lambda'_2)$ is equal to zero. \hfill \Box

By Lemma 4 and since $x < e^{x-1}$ holds when $x > 1$ we have
\[
\exp(\lambda'_i \mathbb{P}[\text{Po}(\lambda'_3-i \beta(\lambda'_1, \lambda'_2)) > 0]) > \varepsilon \lambda'_i \mathbb{P}[\text{Po}(\lambda'_3-i \beta(\lambda'_1, \lambda'_2)) > 0],
\]
for $i = 1, 2$. By continuity, there exists $\varepsilon_0 > 0$ such that
\[
\exp(\lambda'_i \mathbb{P}[\text{Po}(\lambda'_3-i \beta(\lambda'_1, \lambda'_2) - \varepsilon)) > 0]) > \varepsilon \lambda'_i \mathbb{P}[\text{Po}(\lambda'_3-i \beta(\lambda'_1, \lambda'_2)) > 0],
\]
for $i = 1, 2$ and all $\varepsilon \in (0, \varepsilon_0]$. Fix such an $\varepsilon_0$ and an $\varepsilon < \varepsilon_0$.

Recall that $\beta(\lambda_1, \lambda_2)$ is defined to be the maximum solution $\beta$ to
\[
\beta = (1 - e^{-\lambda_1 \beta})(1 - e^{-\lambda_2 \beta}). \tag{3}
\]

**Lemma 5.** The function $\beta$ is continuous in $(\mathbb{R}^+)^2 \setminus C$.

**Proof.** If $\beta(\lambda_1, \lambda_2) = 0$ then, since $(\lambda_1, \lambda_2) \notin C$, there is a neighbourhood of $(\lambda_1, \lambda_2)$ in which $\beta$ is zero; thus $\beta$ is certainly continuous at the point $(\lambda_1, \lambda_2)$.

If $\lambda_2 \leq 1$ then $\beta(\lambda_1, \lambda_2) = 0$. So fix $\lambda_2 > 1$. Note that $\beta = 0$ is always a solution of equation (3). If we assume that $\beta > 0$ then the relation (3) can be viewed as a function $g$ mapping $\beta$ to $\lambda_1$; explicitly,
\[
\lambda_1 = g(\beta) = -\frac{\ln(1 - \beta/(1 - e^{-\lambda_2 \beta}))}{\beta}. \tag{4}
\]

The function $g$ has domain $(0, \beta_{\text{max}})$, where $\beta = \beta_{\text{max}}$ is the positive solution to $\beta = 1 - e^{-\lambda_2 \beta}$, and has asymptotes $\beta = 0$ and $\beta = \beta_{\text{max}}$. Let $\beta^*$ be minimiser of $g$, and let $\lambda^* = g(\beta^*)$ be the corresponding value of the function. (For future reference, as $\lambda_1$ increases, $\lambda^*$ marks the phase transition where the joint-giant component appears, and $\beta^*$ is the fraction of vertices contained in the joint-giant.) Now, $\beta^*$ is a turning point of the function $g$. We will show that it is the unique such. It then follows that $g$ is monotone increasing in the range $[\beta^*, \beta_{\text{max}})$, and hence that the inverse function $g^{-1} : [\lambda^*, \infty) \rightarrow [\beta^*, \beta_{\text{max}}]$ is well defined, monotonically increasing and continuous. Thus, with $\lambda_2$ fixed, $\beta(\lambda_1, \lambda_2)$ is a continuous monotonically increasing function of $\lambda_1$ on $[\lambda^*, \infty)$ (and zero on $[0, \lambda^*)$). Since $\beta(\lambda_1, \lambda_2)$ is symmetric in $\lambda_1$ and $\lambda_2$, a similar statement holds for $\beta(\lambda_1, \lambda_2)$ viewed as a function of $\lambda_2$. Also, since $(\lambda_1, \lambda_2) \notin C$, these claims of continuity continue to hold in a neighbourhood of $(\lambda_1, \lambda_2)$.

Summarising, $\beta$ is separately continuous and monotonically in $\lambda_1$ and $\lambda_2$ in a neighbourhood of $(\lambda_1, \lambda_2)$, which implies that $\beta$ is continuous at $(\lambda_1, \lambda_2)$.

It remains to show that $g(\beta)$ has just one turning point (i.e., that it is ‘U-shaped’). For this we check that for each $\lambda_1, \lambda_2$, equation (3) has at most two solutions for $\beta$ other than $\beta = 0$. We are thus interested in roots of $h(\beta) = (1 - e^{-\lambda_1 \beta})(1 - e^{-\lambda_2 \beta}) - \beta$ other than $\beta = 0$.

Differentiating twice,
\[
h''(\beta) = e^{-(\lambda_1 + \lambda_2) \beta} [(\lambda_1 + \lambda_2)^2 - \lambda_1^2 e^{\lambda_2 \beta} - \lambda_2^2 e^{\lambda_1 \beta}],
\]
which is positive up to a certain value of $\beta$ and then negative. Coupled with $h(0) = 0$ and $h'(0) = -1 < 0$, this implies that $h$ has at most two strictly positive roots. \hfill \Box
Figure 2 shows two plots of the function $h(\beta)$ in the subcritical (left) and supercritical (right) regimes.

Figure 2 shows two plots of the function $h$ from the proof of Lemma 5. In the first, $\lambda_1 = \lambda_2 = 2.4$ and there are no strictly positive roots, while in the second, $\lambda_1 = \lambda_2 = 2.5$ and there are two, the maximum possible. Between these two situations there is a critical value $\lambda^*$ such that setting $\lambda_1 = \lambda_2 = \lambda^*$ yields one positive root $\beta^*$. Thus Figure 2 provides an informal pictorial explanation of the first order phase transition: at $\lambda^*$ the maximum root jumps from 0 to $\beta^*$.

Lemma 5 implies that there exists a $\delta > 0$ such that

$$\beta(\lambda'_1 - \delta, \lambda'_2 - \delta) > \beta(\lambda'_1, \lambda'_2) - \varepsilon$$

and $\beta$ is continuous in the closed ball of radius $\delta$ centred at $(\lambda'_1, \lambda'_2)$, implying that for $i = 1, 2$ when $\xi \not> \delta$ we have

$$(\lambda'_i - \xi)\beta(\lambda'_1 - \xi, \lambda'_2 - \xi) \to (\lambda'_i - \delta)\beta(\lambda'_1 - \delta, \lambda'_2 - \delta) < \lambda'_i\beta(\lambda'_1 - \delta, \lambda'_2 - \delta).$$

Therefore there exists a $\xi \in (0, \delta)$ satisfying the following inequality for $i = 1, 2$

$$\lambda'_i\beta(\lambda'_1 - \delta, \lambda'_2 - \delta) > (\lambda'_i - \xi)\beta(\lambda'_1 - \xi, \lambda'_2 - \xi),$$

and fix such a $\delta$ and $\xi$.

At this point we have fixed a number of parameters, which we collect together here for future reference:

- $\lambda'_1, \lambda'_2 > 1$ satisfy $\beta(\lambda'_1, \lambda'_2) > 0$;
- $\varepsilon_0 > 0$ is such that inequality (2) holds for all $\varepsilon \in (0, \varepsilon_0]$;
- an $\varepsilon \in (0, \varepsilon_0]$;
- $\delta > 0$ satisfies inequality (5), and also that $\beta$ is continuous in the ball of radius $\delta$ centred at $(\lambda'_1, \lambda'_2)$;
- $\xi \in (0, \delta)$ satisfies inequality (6).

These settings will remain in force until the end of Section 3.

Initially we will work on the branching process and then integrate these results into the random graph model. If $\mathcal{E}_1$, $\mathcal{E}_2$ are properties of the branching process, with $\mathcal{E}_1$ depending only on the first $d$ generations, let $\mathcal{E}_1 \circ \mathcal{E}_2$ denote the event that $\mathcal{E}_1$ holds if we delete from $X$ all particles in generation $d$ that do not have property $\mathcal{E}_2$. For example, with $\mathcal{B}_1$ the property of having at least one red and blue offspring, as above, $\mathcal{B}_1 \circ \mathcal{B}_2 = \mathcal{B}_2$, the property of having a red and a blue offspring each with at least one red and one blue offspring.

Let $\mathcal{R}_k$ be the event that $\mathcal{B}_k$ holds in a robust manner, meaning that $\mathcal{B}_k$ holds even after any particle in generation $k$ is deleted. Since $\mathcal{B} = \mathcal{B}_k \circ \mathcal{B}$, the event $\mathcal{R}_k \circ \mathcal{B}$ is the event that $\mathcal{B}$ holds even after deleting an arbitrary particle in generation $k$ and all of its descendants.

Lemma 6. We have

$$\mathbb{P}_{\lambda'_1 - \delta, \lambda'_2 - \delta} [\mathcal{R}_k \circ \mathcal{B}] \nearrow \beta(\lambda'_1 - \delta, \lambda'_2 - \delta) = \mathbb{P}_{\lambda'_1 - \delta, \lambda'_2 - \delta} [\mathcal{B}]$$

as $k \to \infty$.

Proof. Fix $0 < \eta < 1$. Note that $X_{(1-\eta)(\lambda'_1 - \delta), (1-\eta)(\lambda'_2 - \delta)}$ can be obtained by constructing $X_{\lambda'_1 - \delta, \lambda'_2 - \delta}$, and then deleting each edge (of the rooted tree) independently with probability $\eta$, and taking for $X_{(1-\eta)(\lambda'_1 - \delta), (1-\eta)(\lambda'_2 - \delta)}$ the set of particles still connected to the root.
an upper bound on the probability that $\mathcal{B}$ holds for $X(1-\eta)(\lambda_1',\lambda_2')$, we use the above coupling and condition on $X_{\lambda_1',\lambda_2'}$.

If $\mathcal{B}$ does not hold for $X_{\lambda_1',\lambda_2'}$, it certainly does not hold for $X(1-\eta)(\lambda_1',\lambda_2')$. Furthermore, if $\mathcal{B} \setminus (\mathcal{R}_k \cap \mathcal{B})$ holds for $X_{\lambda_1',\lambda_2'}$, then there is a particle $v$ in generation $k$ such that if $v$ is deleted, then $\mathcal{B}$ no longer holds. The probability that $v$ is not deleted when passing to $X(1-\eta)(\lambda_1',\lambda_2')$ is $(1-\eta)^k$. The events $\overline{\mathcal{B}}$, $\mathcal{R}_k \cap \mathcal{B}$ and $\mathcal{B} \setminus (\mathcal{R}_k \cap \mathcal{B})$ exhaust the sample space, and hence

\[ \mathbb{P}(\lambda_1',\lambda_2',d) = \mathbb{P}_{\lambda_1',\lambda_2'}[\mathcal{R}_d \cap \mathcal{B}], \]

By Lemma 6, as $d \to \infty$, the sequence $\mathbb{P}_{\lambda_1',\lambda_2'}[\mathcal{R}_d \cap \mathcal{B}]$ is increasing. Taking the limit of the inequality above,

\[ \beta((1-\eta)(\lambda_1' - \delta), (1-\eta)(\lambda_2' - \delta)) = \mathbb{P}(\lambda_1',\lambda_2',d, \lambda_2') \leq \lim_{k \to \infty} \mathbb{P}_{\lambda_1',\lambda_2'}[\mathcal{R}_k \cap \mathcal{B}]. \]

Letting $\eta \to 0$, the lemma follows.

It will often be convenient to mark some subset of the particles in generation $d$. If $\mathcal{E}$ is an event depending on the first $d$ generations, then we write $\mathcal{E} \circ M$ for the event that $\mathcal{E}$ holds after deleting all unmarked particles in generation $d$. We write $\mathbb{P}_{\lambda_1',\lambda_2'}[\mathcal{E} \circ M]$ for the probability that $\mathcal{E} \circ M$ holds when, given $X_{\lambda_1',\lambda_2'}$, we mark the particles in generation $d$ independently with probability $\alpha$. We suppress $d$ from the notation, since it will be clear from the event $\mathcal{E}$.

Let

\[ r(\lambda_1,\lambda_2,d,\alpha) = \mathbb{P}_{\lambda_1',\lambda_2'}[\mathcal{R}_d \circ \mathcal{E}, \mathcal{M}], \]

**Lemma 7.** There exists a positive integer $d$ such that

\[ r(\lambda_1',\lambda_2',d,\alpha) > \beta(\lambda_1' - \delta, \lambda_2' - \delta). \]

**Proof.** Let $X'$ be a branching process where the root vertex has $\text{Po}(\delta - \xi)$ red and $\text{Po}(\delta - \xi)$ blue offspring, and the descendants of these offspring are as in $X_{\lambda_1',\lambda_2',\delta}$. Then the branching process $Y$ created by merging an independent copy of $X_{\lambda_1',\lambda_2',\delta}$ and $X'$ at the root provides a lower coupling on $X_{\lambda_1',\lambda_2',\xi}$, implying $r(\lambda_1',\lambda_2',d,\alpha) > \beta(\lambda_1' - \delta, \lambda_2' - \delta)$ is at least the probability that $\mathcal{R}_d \cap \mathcal{B}$ holds in $Y$.

When $\mathcal{R}_d \circ \mathcal{B}$ holds in $X'$ then $\mathcal{R}_d \circ \mathcal{B}$ holds as well. Now $\mathcal{R}_d \circ \mathcal{B}$ holds in $X'$ if the root of $X'$ has at least two red and two blue offspring, each having property $\mathcal{B}$. Since each offspring of the root has $\mathcal{B}$ with probability $\alpha = \beta(\lambda_1' - \delta, \lambda_2' - \delta)$, $X'$ has property $\mathcal{R}_d \circ \mathcal{B}$ with probability $\eta = \mathbb{P}[\text{Po}(\delta - \xi)\alpha] \geq 2^\xi > 0$.

On the other hand $X_{\lambda_1',\lambda_2',\delta}$ has $\mathcal{R}_d \circ \mathcal{B}$ with probability $\mathbb{P}_{\lambda_1',\lambda_2',\delta}[\mathcal{R}_d \circ \mathcal{B}]$. Therefore the probability that $Y$ has $\mathcal{R}_d \circ \mathcal{B}$ is at least

\[ r_d = 1 - (1 - \eta)(1 - \mathbb{P}_{\lambda_1',\lambda_2',\delta}[\mathcal{R}_d \circ \mathcal{B}]). \]

By Lemma 6 as $d \to \infty$ we have $\mathbb{P}_{\lambda_1',\lambda_2',\delta}[\mathcal{R}_d \circ \mathcal{B}] \to \beta(\lambda_1' - \delta, \lambda_2' - \delta) = \alpha > 0$, so

\[ r_d \to 1 - (1 - \eta)(1 - \alpha) > \alpha, \]

and there is a $d$ with $r_d \geq \alpha$, completing the proof.

We fix the value of $d$, which satisfies Lemma 7 until the end of Section 5.

As the random graph model contains only a finite number of vertices, there is no equivalent for the event $\mathcal{B}$. In order to circumvent this we introduce an event $\mathcal{L}$, which depends only on the first $L$ generations of the branching process, such that conditional on $L$ the probability that $\mathcal{B}$ holds is close to one. For a non-negative integer $k$ let $X[k]$ denote the first $k$ generations of the branching process $X$.

**Lemma 8.** There exists a positive integer $L$ and an event $\mathcal{L}$ depending only on the first $L$ generations of the branching process satisfying

\[ \mathbb{P}_{\lambda_1',\lambda_2',\xi}[\mathcal{L}] > \beta(\lambda_1' - \delta, \lambda_2' - \delta). \]
and for every $X[L] \in \mathcal{L}$ we have

$$\mathbb{P}_{\lambda_1', \lambda_2' - \xi}[\mathcal{B} | X[L]] = \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}^{\beta(\lambda_1' - \xi, \lambda_2' - \xi)}[\mathcal{B} \circ M | X[L]] \geq 1 - 4^{-3d}.$$

**Proof.** By monotonicity we have $\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}^{\beta(\lambda_1' - \xi, \lambda_2' - \xi)}[\mathcal{B}] > \beta(\lambda_1' - \delta, \lambda_2' - \delta).$ Set

$$\eta = \min \left\{ \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B}] - \beta(\lambda_1' - \delta, \lambda_2' - \delta), 4^{-3d} \right\} > 0.$$

As $\mathcal{B}$ is measurable, there is an integer $L \geq d$ and an event $\mathcal{L}_1$ depending only on the first $L$ generations of the branching process such that $\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B} \triangle \mathcal{L}_1] \leq \eta^2/2$, where $\triangle$ denotes symmetric difference. Writing $1_\mathcal{E}$ for the indicator function of an event $\mathcal{E}$ we have

$$\eta^2/2 \geq \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B}^c \cap \mathcal{L}_1] = \mathbb{E}_{\lambda_1' - \xi, \lambda_2' - \xi}[1_{\mathcal{L}_1} \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B}^c | X[L]]],$$

where $\mathbb{E}_{\lambda_1' - \xi, \lambda_2' - \xi}$ is the expectation corresponding to $\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}$. Set

$$\mathcal{L} = \mathcal{L}_1 \cap \left\{ X[L] : \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B} | X[L]] \geq 1 - \eta \right\},$$

and note that the event $\mathcal{L}$ depends only on the first $L$ generations of $X$. Clearly the second inequality in the statement of the lemma holds for every element of $\mathcal{L}$ and thus we only need to verify the first inequality. Now if $X[L] \in \mathcal{L}_1 \setminus \mathcal{L}$ then $\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B}^c | X[L]] \geq \eta$ leading to

$$\mathbb{E}_{\lambda_1' - \xi, \lambda_2' - \xi}[1_{\mathcal{L}_1} \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B}^c | X[L]]] \geq \eta \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{L} \setminus \mathcal{L}].$$

Together with (7), this implies $\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{L}_1 \setminus \mathcal{L}] \leq \eta/2$ and hence

$$\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{L}] \geq \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{L}_1] \geq \eta/2 \geq \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{B}] - \eta^2/2 - \eta/2 \geq \beta(\lambda_1' - \delta, \lambda_2' - \delta),$$

completing the proof. \qed

Fix an integer $L$ and an event $\mathcal{L}$ which satisfies the previous lemma. Let $\mathcal{A}_0 = \mathcal{L}$, and for $t \geq 1$ set $\mathcal{A}_t = \mathcal{R}_d \circ \mathcal{A}_{t-1}$. Thus, $\mathcal{A}_t$ is a ‘recursively robust’ version of the event $\mathcal{B}_d \circ \mathcal{L}$.

**Lemma 9.** For any $t \geq 0$,

$$\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{A}_t] > \beta(\lambda_1' - \delta, \lambda_2' - \delta)$$

and for every $X[dt + L] \in \mathcal{A}_t$, we have

$$\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}^{\beta(\lambda_1' - \xi, \lambda_2' - \xi)}[\mathcal{B}_{dt + L} \circ M | X[dt + L]] \geq 1 - 4^{-2t + 1 + 2d}.$$

**Proof.** The proof is by induction on $t$. The statement holds for $t = 0$ by Lemma 8.

As the descendants of different particles in generation $d$ of the branching process are independent we have

$$\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{A}_t] = r(\lambda_1' - \xi, \lambda_2' - \xi, d, \mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{A}_{t-1}]).$$

By the induction hypothesis we have $\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}[\mathcal{A}_{t-1}] > \beta(\lambda_1' - \delta, \lambda_2' - \delta)$ and the first statement follows from Lemma 7. Now assume

$$\mathbb{P}_{\lambda_1' - \xi, \lambda_2' - \xi}^{\beta(\lambda_1' - \xi, \lambda_2' - \xi)}[\mathcal{B}_{dt(t-1) + L} \circ M | X[dt(t-1) + L]] \geq 1 - 4^{-(2t-1 + 2)d}.$$ 

Condition on $X[dt + L]$ and assume that $\mathcal{A}_t$ holds. Since $\mathcal{A}_t = \mathcal{R}_d \circ \mathcal{A}_{t-1}$, there is a smallest set $Y$ of particles in generation $d$ such that $\mathcal{A}_{t-1}$ holds for each $y \in Y$, and $\mathcal{R}_d \circ M$ holds if we mark only the particles in $Y$. Since any tree witnessing $\mathcal{R}_d$ contains a subtree witnessing $\mathcal{R}_d$ in which each particle has at most two red and two blue offspring, we have $|Y| \leq 4^d$.

Mark each particle in generation $dt + L$ independently with probability $\beta(\lambda_1' - \xi, \lambda_2' - \xi)$, and let $Y'$ be the set of particles $y \in Y$ for which $\mathcal{B}_{dt(t-1) + L} \circ M$ holds. By the induction hypothesis and because the descendants of different particles in generation $d$ of the branching process are independent each $y \in Y$ is included in $Y'$ independently with probability at least $1 - 4^{-(2t-1 + 2)d}$. Thus

$$\mathbb{P}(|Y \setminus Y'| \geq 2) \leq \binom{|Y|}{2} \left(4^{-(2t-1 + 2)d}\right)^2 \leq 4^{2d} \left(4^{-(2t-1 + 2)d}\right) \leq 4^{-(2t+2)d}.$$
By the definition of $R_d$, $B_d$ holds whenever $|Y \setminus Y'| \leq 1$ and we only keep the particles in $Y'$ in generation $d$. But then $B_d \circ B_{d(t-1)+L} \circ M = B_{dt+L} \circ M$ holds, proving the second statement. □

The final result we need for the branching process is that it is unlikely to become large quickly.

**Lemma 10.** For any positive integer $s = o(\log n)$ we have

$$
P_{\lambda'_1, \lambda'_2} \left[ |X[2s]| \geq n^{1/15} \right] = o(n^{-1}).
$$

**Proof.** Lemma 4 implies $\lambda'_1 + \lambda'_2 > 1$.

Let $f(x)$ be the probability generating function of $\text{Po}(\lambda'_1 + \lambda'_2)$, then $F_t(x)$, the probability generating function for $|X[t]|$, satisfies the following recursion $F_t(x) = x(f \circ F_{t-1})(x)$ with $F_0(x) = x$ (see e.g. [10]). Due to a well known property of probability generating functions we have $\mathbb{E}_{\lambda'_1, \lambda'_2}[|X[t]|^m] = (F_t \circ \exp)^{(m)}(0)$.

Set $c_1 = (\lambda'_1 + \lambda'_2)/(\lambda'_1 + \lambda'_2 - 1)$ and define

$$c_m = \left( \frac{2}{\lambda'_1 + \lambda'_2} \right)^m \max_{0 < k < m} \sum_{j=1}^{k} f^{(j)}(1) B_{k,j}(c_1, c_2, \ldots, c_{k-j+1}),$$

where $B_{k,j}$ denotes the Bell Polynomial

$$B_{k,j}(\alpha_1, \ldots, \alpha_{k-j+1}) = \sum_{i_1+i_2+\ldots+i_{k-j+1}=j} \left( \begin{array}{c} j \\ i_1, i_2, \ldots, i_{k-j+1} \end{array} \right) \prod_{r=1}^{k-j+1} \left( \frac{\alpha_r}{r!} \right)^{i_r}.$$

We show that for any positive integers $m$ and $t$ we have $(F_t \circ \exp)^{(m)}(0) \leq c_m (\lambda'_1 + \lambda'_2)^{mt}$. The proof is by induction on $m$ and $t$. Clearly $(F_0 \circ \exp)^{(m)}(0) = 1$ for every $m$ and also for $t \geq 1$ $(F_{t-1} \circ \exp)^{(1)}(0) = \mathbb{E}_{\lambda'_1, \lambda'_2}[|X[t-1]|] \leq (\lambda'_1 + \lambda'_2)^t/(\lambda'_1 + \lambda'_2 - 1)$.

Now consider $(F_t \circ \exp)^{(m)}(0)$. By recursion and the generalised product rule we have

$$\left( F_t \circ \exp \right)^{(m)}(x) = \left( e^x(f \circ F_{t-1} \circ \exp)(x) \right)^{(m)} = e^{x} \sum_{k=0}^{m} \binom{m}{k} \left( F \circ F_{t-1} \circ \exp \right)^{(k)}(x). \quad (8)$$

Furthermore by Faà di Bruno’s Theorem:

$$\left( f \circ F_{t-1} \circ \exp \right)^{(k)}(x) = \sum_{j=1}^{k} f^{(j)}(1) B_{k,j} \left( (F_{t-1} \circ \exp)^{(1)}(x), \ldots, (F_{t-1} \circ \exp)^{(k-j+1)}(x) \right). \quad (9)$$

Now assume that for every $k < m$ we have $(F_{t-1} \circ \exp)^{(k)}(0) \leq c_k (\lambda'_1 + \lambda'_2)^{k(t-1)}$, then because the Bell polynomial is increasing in each of its parameters by [9] we have

$$(f \circ F_{t-1} \circ \exp)^{(k)}(0) \leq \sum_{j=1}^{k} f^{(j)}(1) B_{k,j} \left( (\lambda'_1 + \lambda'_2)^{t-1}, \ldots, c_{k-j+1}(\lambda'_1 + \lambda'_2)^{(k-j+1)(t-1)} \right)$$

$$= (\lambda'_1 + \lambda'_2)^{k(t-1)} \sum_{j=1}^{k} f^{(j)}(1) B_{k,j}(c_1, \ldots, c_{k-j+1}),$$
where the last equality follows from the fact that the sum in the Bell polynomial satisfies
\( i_1 + 2i_2 + (k - j + 1)k_{j+1} = k \). Together with [3] this implies
\[
(F_t \circ \exp)^{(m)}(0) \leq \sum_{k=0}^{m} \binom{m}{k} (\lambda'_1 + \lambda'_2)^{k(t-1)} \sum_{j=1}^{k} f^{(j)}(1) B_{k,j}(c_1, \ldots, c_{k-j+1}) \]
\[
\leq c_m \left( \frac{\lambda'_1 + \lambda'_2}{2} \right)^m \sum_{k=0}^{m} \binom{m}{k} (\lambda'_1 + \lambda'_2)^{k(t-1)} \]
\[
= c_m \left( \frac{\lambda'_1 + \lambda'_2}{2} \right)^m (1 + (\lambda'_1 + \lambda'_2)^{t-1})^m \lambda'_1 + \lambda'_2 \geq 1 \leq c_m (\lambda'_1 + \lambda'_2)^{tm}.
\]

Thus we have \( \mathbb{E}_{\lambda'_1, \lambda'_2}(|X[2s]|^{(0)}) = O((\lambda'_1 + \lambda'_2)^{60s}) = o(n) \) and the statement follows by Markov’s inequality. \( \square \)

3. FROM THE BRANCHING PROCESS TO RANDOM GRAPHS

Let \( T = T(n) \) satisfy \( T = o(\log n) \) and \( T/\log \log n \to \infty \). Set \( s = 1 + dT + L \) and consider
the random double graph \( \tilde{G} = \tilde{G}(n, 4s, \lambda'_1, \lambda'_2) \), whose distribution is that of \( G(n, \lambda'_1/n, \lambda'_2/n) \)
conditioned on the absence of any red-blue cycles of length at most \( 4s \), i.e., the girth of the
merged edge set is larger than \( 4s \).

Transferring from \( G \) to \( \tilde{G} \) has only a minor effect on the probability of local properties, as
sparse binomial random graphs are locally tree-like, and this also holds for the binomial double
digraph, when it is created from two sparse binomial random graphs. Roughly speaking this
means that when exposing the edges in \( \tilde{G} \) the probability that the next edge exposed is present
should be close to the probability that the edge is present in \( G(n, \lambda'_1/n, \lambda'_2/n) \), as long as the
number of exposed edges remains small. We prove this result in the following lemma.

**Lemma 11.** Let \( M_1, F_1 \) be disjoint sets of possible edges over \( V \) and \( M_2, F_2 \) be disjoint sets
of possible edges over \( V \), with \( |F_1|, |F_2| \leq n^{2/3} \) such that \( F_1 \cup F_2 \) contains no cycle of length at most \( 4s \). Let \( j \in \{1, 2\} \) and \( e = \{w_1, w_2\} \notin F_1 \cup M_j \). Then for large enough \( n \) we have
\[
\mathbb{P}\left[ e \in E_j(\tilde{G}) \mid F_1 \subseteq E_1(\tilde{G}) \subseteq M_1, F_2 \subseteq E_2(\tilde{G}) \subseteq M_2 \right] \leq \lambda'_j/n,
\]
if in addition \( w_2 \) is disjoint from any edge in \( F_1 \cup F_2 \) then
\[
(1 - n^{-1/4})\lambda'_j/n \leq \mathbb{P}\left[ e \in E_j(\tilde{G}) \mid F_1 \subseteq E_1(\tilde{G}) \subseteq M_1, F_2 \subseteq E_2(\tilde{G}) \subseteq M_2 \right].
\]

**Proof.** Without loss of generality assume \( j = 1 \). Let \( G = G(n, \lambda'_1/n, \lambda'_2/n) \). Let \( \mathcal{E} \) denote the
event \( F_1 \subseteq E_1(G) \subseteq M_1, F_2 \subseteq E_2(G) \subseteq M_2 \). In addition let \( \mathcal{D} \) be the event that no cycle of
length at most \( \ell = 4s \) is found in \( G \). Clearly
\[
\mathbb{P}\left[ e \in E_1(\tilde{G}) \mid F_1 \subseteq E_1(\tilde{G}) \subseteq M_1, F_2 \subseteq E_2(\tilde{G}) \subseteq M_2 \right] = \mathbb{P}[e \in E_1(G) \mid \mathcal{D}, \mathcal{E}].
\]

Note that conditional on \( \mathcal{E} \) for \( i = 1, 2 \) the edges in \( E_i(G) \) except those in \( F_i \) and \( M_i \) appear
independently not only of the other edges in \( E_i \), but also of the edges in \( E_{3-i} \). Since the event
\( e \in E_1(G) \) is increasing and the event \( \mathcal{D} \) is decreasing we have, by Harris’s Lemma ([4]),
\[
\mathbb{P}[e \in E_1(G) \mid \mathcal{D}, \mathcal{E}] = \frac{\mathbb{P}[e \in E_1(G), \mathcal{D} \mid \mathcal{E}]}{\mathbb{P}[\mathcal{D} \mid \mathcal{E}]} \leq \frac{\mathbb{P}[e \in E_1(G) \mid \mathcal{E}] \mathbb{P}[\mathcal{D} \mid \mathcal{E}]}{\mathbb{P}[\mathcal{D}] \mid \mathcal{E}]} = \frac{\lambda'_1}{n},
\]
proving the upper bound.

Now for the lower bound. Let \( \mathcal{P} \) be the event that there is no red-blue path between \( w_1 \) and
\( w_2 \), which consists of at most \( \ell - 1 \) edges. Note that conditional on \( \mathcal{D}, \mathcal{P}, \mathcal{E} \) the edge \( e \) is present
with probability $\lambda'_1/n$. Therefore
\[
\mathbb{P}[e \in E_1(G) | D, \mathcal{E}] \geq \mathbb{P}[e \in E_1(G), P | D, \mathcal{E}]
= \mathbb{P}[e \in E_1(G) | D, P, \mathcal{E}] \mathbb{P}[P | D, \mathcal{E}]
= \frac{\lambda'_1}{n} \mathbb{P}[P | D, \mathcal{E}].
\]

All that remains to show is that $\mathbb{P}[P | D, \mathcal{E}] \geq (1 - n^{-1/4})$. Note that both the event $P$ and $D$ are decreasing, therefore Harris’s Lemma implies $\mathbb{P}[P | D, \mathcal{E}] \geq \mathbb{P}[P | \mathcal{E}]$. Now consider a path of length at most $t - 1$ between $w_1$ and $w_2$. Any such path must contain a subpath, where none of the edges is contained in $F_1$ or $F_2$. In addition, for one of these subpaths, the first vertex of this subpath is $w_2$, while the last vertex of this subpath must be in the set $W$, the set of endpoints of edges in $F_1 \cup F_2 \cup \{w_1\}$. Now for $P$ to fail one such subpath must be present, but the probability of this event is at most
\[
\sum_{\ell=1}^{4s} |W|n^{\ell-1}((\lambda'_1 + \lambda'_2)/n)^{\ell} \leq \frac{5n^{2/3}}{n} \sum_{\ell=1}^{4s} (\lambda'_1 + \lambda'_2)^{\ell} \leq 5n^{-1/3}O(1) o(\log n) \leq n^{-1/4}
\]
if $n$ is large enough. Hence, the probability that such a subpath is present is at most $n^{-1/4}$, and so is the probability that $P$ fails, completing the proof.

For a double graph $G$ on $n$ vertices let $U(G)$ be the maximal subset of vertices of $G$ such that in the subgraph spanned by $U$ every vertex is found in both a red and a blue tadpole graph (a tadpole graph is a cycle and a path joined at a vertex, in our case the path may be empty).

Let $\tilde{G}_v[t]$ be the subgraph of $\tilde{G}$ formed by the vertices within distance $t$ of $v$, noting that for $t \leq s$ this graph is by definition a tree. For an event $\mathcal{E}$ of the branching process, which depends only on the first $t \leq 2s$ generations we say that a vertex $v \in V(\tilde{G})$ has property $\mathcal{E}$ if $\tilde{G}_v[2s]$ has property $\mathcal{E}$, when viewed as a branching process rooted at $v$.

Let $A = B_1 \circ A_T$ this event depends only on the first $s = 1 + dT + L$ generations of the branching process. In the following lemma we show that the number of vertices with property $A$ is large and every vertex with property $A$ is contained in $U$.

**Lemma 12.** The number of vertices of $\tilde{G}$ with property $A$ is whp at least $\beta(\lambda'_1 - \delta, \lambda'_2 - \delta)n$. In addition whp every vertex with property $A$ is in $U(G)$.

**Proof.** Let $v$ be a vertex of $\tilde{G}$, and explore its neighbourhood, until distance $2s$ in the following way. Initially we set $v$ ‘active’ and all other vertices ‘untested’. In each step we pick an ‘active’ vertex $w$ closest to $v$, and expose one by one the edges between $w$ and the ‘untested’ vertices. The newly discovered neighbours of $w$ is set as ‘active’, while the state of $w$ is changed to ‘tested’. Note that at the end of each step there are no edges incident to ‘untested’ vertices. Therefore by Lemma 11 as long as we have reached at most $n^{2/3}$ vertices, conditional on everything so far each red edge is present with probability $(1 + O(n^{-1/4}))\lambda'_1/n$, while each blue edge is present with probability $(1 + O(n^{-1/4}))\lambda'_2/n$. As the number of untested vertices is $n - O(n^{2/3})$, we may couple the number of new red and blue neighbours of $w$ found with a Poisson distribution with mean $\lambda'_1$ and $\lambda'_2$ respectively so that the two numbers agree with probability $1 - O(n^{-1/4})$.

By Lemma 10 whp, $X_{\lambda_1', \lambda_2'}[2s]$ contains at most $n^{1/15}$ particles, implying that whp $\tilde{G}_v[2s]$ also contains at most $n^{1/15}$ vertices. The previous argument implies that $\tilde{G}_v[2s]$ and $X_{\lambda_1', \lambda_2'}[2s]$ can be coupled as to agree in the natural sense whp. Therefore $v$ has property $A$ with probability $\mathbb{P}_{\lambda_1', \lambda_2'}[A] + o(1)$.

Write $A$ for the set of vertices with property $A$. Clearly $\mathbb{E}[|A|] = \mathbb{P}_{\lambda_1', \lambda_2'}[A] n + o(n)$. Note that the probability that two vertices are within distance $2s$ is at most
\[
\sum_{i=1}^{2s} n^{i-1}(p_1 + p_2)^i = o(1),
\]
implying \( \text{Var}(|A|) = o(n^2) \). Hence (by Chebyshev’s inequality), \(|A|/n\) converges in probability to

\[
\mathbb{P}_{X_1, X_2}[A] = \mathbb{P}\left[ \mathbb{P}\left( X_1^t \mathbb{P}(X_1, X_2[T]) > 0 \right) \geq \mathbb{P}\left( X_2^t \mathbb{P}(X_1, X_2[T]) > 0 \right) \right] \\
= \beta(\lambda_1 - \delta, \lambda_2 - \delta).
\]

(10)

All that is left to show is that what every vertex with property \( A \) is in \( U \). Similarly as before reveal \( \tilde{G}_v[s] \) and based on this we can decide whether \( A \) holds. Let us suppose that at most \( n^{1/15} \) vertices have been examined, an event of probability \( 1 - o(n^{-1}) \) by Lemma \[10\]. Condition on \( v \) having property \( A \). Let \( u_1, \ldots, u_k \) be all the offspring of \( v \) with property \( A_T \). Since \( v \) has property \( A \) we know that \( k \geq 2 \) and that there is at least one red offspring and one blue offspring within \( u_1, \ldots, u_k \). We do not commit ourselves to a particular choice or red or blue offspring at this stage, as we need some flexibility later. Choose any offspring \( u \) and let \( S \) be the set of vertices in generation \( s - 1 \) in the subtree rooted at \( u \).

We now examine the vertices \( w \in S \) one by one. For each \( w \) we explore its descendants for the next \( s \) generations to test whether, in the branching process rooted at \( v \), the particle \( w \) has property \( A \). The exploration process for a given vertex is abandoned if the number of its descendants reaches \( n^{1/15} \). Should the test be successful, we mark \( w \). Since \( |S| \leq n^{1/15} \) exploration processes are run and each exploration process exposes at most \( n^{1/15} \) edges, and because each exploration process is abandoned with probability \( o(n^{-1}) \), similarly to the argument above the descendants of \( w \) can be coupled to the branching process \( X_{V_i}^t \) with probability \( 1 - o(1) \). Hence, for large enough \( n \), the probability that we mark \( w \) is at least

\[
\mathbb{P}_{X_1^t, X_2^t}[A] - o(1) = \mathbb{P}\left[ \mathbb{P}(X_1^t \mathbb{P}_1, X_2^t[T]) > 0 \right] \mathbb{P}\left[ \mathbb{P}(X_2^t \mathbb{P}_1, X_2^t[T]) > 0 \right] - o(1)
\]

(9)

In summary, ignoring an error probability of \( 1 - o(n^{-1}) \), we can view each \( w \in S \) as marked independently with probability (at least) \( \beta(\lambda_1 - \delta, \lambda_2 - \delta) \).

Now, in the tree \( \tilde{G}_v[s] \) rooted at \( v \), every offspring \( u \) has property \( A_T \), and every vertex at depth \( dT + L \) starting from \( u \) has property \( A \) with probability at least \( \beta(\lambda_1 - \delta, \lambda_2 - \delta) \). Since \( T/\log \log n \rightarrow \infty \), we have \( 4^{-2(2T+2d)} \leq n^{-3} \) if \( n \) is large enough. Hence, from Lemma \[9\] with probability \( 1 - o(n^{-1}) \) all of the vertices \( u_1, \ldots, u_k \) have property \( B_{dT+L} \circ A \). Note that the failure probability is small enough that this situation obtains whp uniformly over vertices \( v \) satisfying \( A \). Suppose that this is the case. Then, certainly, \( v \) has property \( B_{1+dT+L} \circ A \). More than that, the fact that \( v \) has property \( A \) is witnessed by any choice of a red offspring \( u \) and blue offspring \( u_j \). For each \( i, j \in [k] \) with \( u_i \) red and \( u_j \) blue, choose a minimal subtree of \( \tilde{G} \) witnessing that \( v \) has property \( B_{1+dT+L} \circ A \). (A minimal subtree is a balanced binary red-blue tree of depth \( 1 + dT + L \).) Let the collection of all such subtrees, ranging over feasible \( i, j \) in \( [k] \), be denoted \( T_v \). Note that if \( \tau \in T_v \) is any such subtree, then each leaf of \( \tau \) has property \( A \) (in the graph \( \tilde{G} \) excluding the vertices of \( \tau \)).

Let \( U^o = \bigcup_{\tau \in T_v} V(\tau) \), where the first union is over all vertices \( v \) with property \( A \). With probability \( 1 - o(n^{-1}) = 1 - o(1) \), we have that \( |T_v| > 0 \) for every vertex \( v \) with property \( A \). It follows that all vertices with property \( A \) are contained in \( U^o \). On the other hand, it is not too difficult to see that \( U^o \subseteq \mathbb{U}(\tilde{G}) \). Take any vertex \( u \in U^o \). By definition of \( U^o \) we must have \( u \in V(\tau) \) for some \( \tau \in T_v \). Trace a red path in \( \tau \) from \( u \) to a leaf \( w \) of \( \tau \). Now pick a suitable tree from \( T_w \) (i.e., one that shares only vertex \( w \) with \( \tau \)) and trace a red path in it from \( w \) to a leaf \( w' \). Note that we have included sufficiently many trees in \( T_w \) that we can avoid using the
parent of \( w \). Then repeat, tracing a red path from \( w' \), etc. This process will terminate when the path intersects itself, at which point we have a red tadpole. Clearly, the same construction works for blue tadpoles.

Lemma 12 implies that \( A \), the set of vertices with property \( A \) is a subset of \( U \). Next we will show that \( U \subseteq B_s \), where \( B_s \) is the set of vertices with property \( B_s \).

Claim 13. For every \( v \in U \) we have that \( v \) has property \( B_s \).

Proof. Consider the subgraph \( \tilde{G}[U] \) of \( \tilde{G} \) induced by \( U \). The result follows if we can show that \( v \) has property \( B_s \) already within \( \tilde{G}[U] \). Every vertex \( v \) in \( U \) lies within a red tadpole; choose a tadpole for each \( v \) and form the union of tadpoles over all vertices \( v \in U \). The resulting graph has the property that each of its connected components has at least one cycle. We can therefore choose a red subgraph \( (U, F_r) \) of \( \tilde{G} \) such that every connected component of \( (U, F_r) \) is unicic. Orient the edges in each cycle of \( (U, F_r) \) consistently, and orient all other other edges towards the unique cycle in their component. Repeat the process to obtain a blue subgraph \( (U, F_b) \) together with an orientation.

Let \( v \) be any vertex in \( U \). There is a unique oriented red edge \((v, u_r)\) leaving \( v \) and a unique oriented blue edge \((v, u_b)\) leaving \( v \). Make \( u_r \) and \( u_b \) the offspring of \( v \). There are unique red and blue oriented edges leaving \( u_r \) and \( u_b \), so the process can be repeated. The choice of orientations for the edges of \( (U, F_r) \) and \( (U, F_b) \) ensures that the process never gets stuck and that, up to depth \( s \), no cycles are created. Thus there is a complete red-blue binary tree of depth \( s \) rooted at \( v \), witnessing the fact that \( v \) has property \( B_s \).

Proposition 3 follows if we show that every red component and every blue component within \( U \) has size at least \( n^{3/5} \). Consider a red component within \( U \). Any such component must contain at least as many edges as vertices, as every vertex is found within a red tadpole graph. In addition every vertex within the component must have a blue neighbour in \( U \), but may not have a red neighbour in \( U \), other than in the component it is contained in, as this would contradict the maximality of a component. Now a vertex within the component may be incident to red edges leading outside of \( U \), however no such red neighbour may have a blue neighbour within \( U \) as this would contradict the maximality of \( U \). In fact, since every vertex in \( U \) has a blue neighbour in \( U \), the last two conditions can be replaced by the following: no vertex in the component may have a red neighbour outside the component that has a blue neighbour in \( U \).

The condition of being a member of \( U \), being a global one, is difficult to deal with. Therefore we look first at an event that is closely related to the one just described, but which refers only to local conditions. For \( W \subseteq V \) let \( C_r(W) \) be the event that

- \( W \) contains a red monocyclic spanning subgraph;
- every vertex in \( W \) has a blue neighbour in \( B_s \);
- no vertex in \( W \) has a red neighbour in \( V \setminus W \) with a blue neighbour in \( A \).

The event \( C_b(W) \) is defined analogously for blue components, i.e. all the colours are swapped. Set \( \mathcal{C} \) as the event that for every \( W \subseteq V \) with \( T \leq |W| \leq n^{3/5} \) neither \( C_r(W) \) nor \( C_b(W) \) holds.

Lemma 14. The event \( \mathcal{C} \) holds whp in \( \tilde{G} = \tilde{G}(n, \lambda_1, \lambda_2) \).

Proof. We will examine the probability of the events in the definition of \( \mathcal{C}_r(W) \) one by one when \( W \) contains \( k \) vertices. Denote by \( \mathcal{R}(W) \) the event that \( W \) contains a red monocyclic spanning subgraph. The number of connected monocyclic graphs on \( k \) vertices is at most \( k^k \), as there are \( k^{k-2} \) ways to select a spanning tree on \( k \) vertices and fewer than \( k^2 \) ways to select an additional edge. Therefore by Lemma 11 the probability that \( \mathcal{R}(W) \) holds is at most

\[
\mathbb{P}[\mathcal{R}(W)] \leq k^k \left( \frac{\lambda_1}{n} \right)^k.
\]

Note that until this point we have only exposed red edges in \( W \).

Let \( \mathcal{E} \) be the event that in \( X_{\lambda_1, \lambda_2} \) the particle \( v_0 \) has a blue offspring with property \( B_s \) and \( \mathcal{F} \) be the event that it does not have a red offspring that has a blue offspring with property \( A \).
Note that these two events are independent. Since \( s \to \infty \) we have \( \mathbb{P} [B_s] = \beta (\lambda'_1, \lambda'_2) + o(1) \). Therefore
\[
\mathbb{P}_{\lambda'_1, \lambda'_2} [\mathcal{E}] = \mathbb{P} \left[ \text{Po}(\lambda'_2 \beta (\lambda'_1, \lambda'_2)) > 0 \right] + o(1),
\] (11)
and
\[
\mathbb{P}_{\lambda'_1, \lambda'_2} [\mathcal{F}] = \mathbb{P} \left[ \text{Po} \left( \lambda'_2 \mathbb{P}_{\lambda'_1, \lambda'_2} [A] \right) > 0 \right] = 0,
\] (10)
\[
\leq \exp \left( -\lambda'_1 \mathbb{P} (\text{Po}(\lambda'_2 \beta (\lambda'_1 - \delta, \lambda'_2 - \delta)) > 0) \right).
\] (12)

For each \( w \in W \) we explore its neighbourhood for the next \( s + 2 \) generations. In order to achieve this we run the exploration process described in Lemma 12 after marking every vertex in \( W \) ‘active’ and all other vertices ‘untested’. We abandon the exploration process associated to a given \( w \) if we reach \( n^{1/15} \) vertices in this exploration and by Lemma 10 this occurs with probability \( o(n^{-1}) \).

Denote by \( T_w \) the graph discovered during the exploration process associated to \( w \). (If the exploration is abandoned, \( T_w \) is whatever has been discovered at the point of abandonment.) Let \( \mathcal{E}_w \) and \( \mathcal{F}_w \) be the events that \( T_w \) has property \( \mathcal{E} \) and \( \mathcal{F} \) respectively, when viewed as a branching process rooted at \( w \). In addition denote by \( \mathcal{O}_w \) the event that the exploration process associated to \( w \) was abandoned.

Note that from Lemma 11 the argument above shows that we may couple the trees \( T_w \) for \( w \in W \) with \( |W| \) independent branching processes \( X_{\lambda'_1, \lambda'_2} \) such that \( T_w \) agrees with \( X_{\lambda'_1, \lambda'_2} \) with probability \( 1 - o(1) \) for each \( w \in W \). Thus for any \( W' \subseteq W \)
\[
\mathbb{P} \left[ \bigcap_{w \in W'} \left( (\mathcal{E}_w \cup \mathcal{O}_w) \cap \mathcal{F}_w \right) \right] \leq \left( \mathbb{P}_{\lambda'_1, \lambda'_2} [\mathcal{E}] \mathbb{P}_{\lambda'_1, \lambda'_2} [\mathcal{F}] + o(1) \right)^{|W'|}.
\] (13)

The event that \( w \) does not have a red neighbour in \( V \setminus W \) that in turn has a blue neighbour in \( A \) is contained in \( \mathcal{F}_w \). On the other hand the vertex \( w \) can have a blue neighbour in \( B_s \) even if \( \mathcal{E}_w \cup \mathcal{O}_w \) does not hold. However this is only possible if the following event \( \mathcal{D}_w \) holds: for some \( w' \in W \setminus \{w\} \) there is an edge, other than a possible red edge \( \{w, w'\} \), between \( T_w \) and \( T_{w'} \). The discussion so far is summarised in the following inclusion:
\[
\mathcal{C}_r(W) \subseteq \mathcal{R}(W) \cap \bigcap_{w \in W} \left( (\mathcal{E}_w \cup \mathcal{O}_w \cup \mathcal{D}_w) \cap \mathcal{F}_w \right).
\] (14)

Now consider an auxiliary graph \( H \) with vertex set \( W \), where two vertices are connected if there is an edge between the corresponding exploration processes other than a red edge connecting the roots.

Note that the edges we consider between \( T_w \) and \( T_{w'} \) are either not present or have not been exposed. Therefore by Lemma 11 the probability that there is an edge between \( T_w \) and \( T_{w'} \) is at most \( n^{2/15} (\lambda'_1 / n + \lambda'_2 / n) \leq n^{-1/5} \). Therefore \( G(W, n^{-4/5}) \) provides an upper coupling on \( H \) independently of the result of the exploration processes.

For some \( W' \subseteq W \) denote by \( \mathcal{D}(W') \) the event that in \( G(W, n^{-4/5}) \) the set of vertices with positive degree corresponds to \( W' \). In light of the coupling just described, we have
\[
\bigcap_{w \in W} \left( (\mathcal{E}_w \cup \mathcal{O}_w \cup \mathcal{D}_w) \cap \mathcal{F}_w \right) \subseteq \bigcup_{W' \subseteq W} \mathcal{D}(W') \cap \bigcap_{w \in W \setminus W'} \left( (\mathcal{E}_w \cup \mathcal{O}_w) \cap \mathcal{F}_w \right).
\] (15)

Note that \( \mathcal{D}(W') \) is independent of \( \bigcap_{w \in W \setminus W'} \left( (\mathcal{E}_w \cup \mathcal{O}_w) \cap \mathcal{F}_w \right) \). Therefore
\[
\mathbb{P} \left[ \mathcal{D}(W') \cap \bigcap_{w \in W \setminus W'} \left( (\mathcal{E}_w \cup \mathcal{O}_w) \cap \mathcal{F}_w \right) \right] = \mathbb{P} [\mathcal{D}(W')] \mathbb{P} \left[ \bigcap_{w \in W \setminus W'} \left( (\mathcal{E}_w \cup \mathcal{O}_w) \cap \mathcal{F}_w \right) \right].
\]
Now $D(W')$ is contained in the event that $G(W', n^{-4/5})$ contains at least $|W'|/2$ edges which has probability at most

$$\binom{|W'|/2}{2} \left( n^{-4/5} \right)^{|W'|/2} \leq \left( e|W'|n^{-4/5} \right)^{|W'|/2} \leq n^{-|W'|/12}. $$

Therefore by \([11]\), \([12]\) and \([13]\) we have

$$\Pr \left[ \bigcup_{W' \subseteq W} \left( D(W') \cap \bigcap_{w \in W \setminus W'} ((E_w \cup C_w) \cap F_w) \right) \right] \leq \sum_{k=0}^{\ell} \binom{k}{\ell} n^{-\ell/12} \left( \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2)) > 0 \right] \exp \left(-\lambda'_1 \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2) > 0) \right] + o(1) \right) \right)^{k-\ell}$$

$$= \left( \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2)) > 0 \right] \exp \left(-\lambda'_1 \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2) > 0) \right] + o(1) \right) \right)^k,$$

(16)

where in the last equality we use $n^{-1/12} = o(1)$ and the binomial theorem.

Putting together \([14]\), \([15]\), and \([16]\), and recalling that the event $R(W)$ is independent of the other events,

$$\sum_{W' \subseteq V \atop T' \leq |W'| \leq n^{3/5}} \Pr [C_r(W)]$$

$$\leq \sum_{k=T}^{n^{3/5}} \binom{n}{k} k^k \left( \frac{\lambda'_1}{n} \right)^k \left( \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2)) > 0 \right] \exp \left(-\lambda'_1 \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2) > 0) \right] + o(1) \right) \right)^k$$

$$\leq \sum_{k=T}^{n^{3/5}} \left( e\lambda'_1 \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2)) > 0 \right] \exp \left(-\lambda'_1 \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2) > 0) \right] + o(1) \right) \right)^k$$

(17)

where the last inequality follows from $\beta(\lambda'_1 - \delta, \lambda'_2 - \delta) \geq \beta(\lambda'_1, \lambda'_2) - \varepsilon$. By \([2]\) we have

$$e\lambda'_1 \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2)) > 0 \right] \exp \left(-\lambda'_1 \Pr \left[ \text{Po}(\lambda'_2, \beta(\lambda'_1, \lambda'_2) - \varepsilon) > 0 \right] \right) < 1,$$

and thus \([17]\) is $o(1)$ when $n$ is large enough A similar bound holds for the sum of $\Pr [C_b(W)]$. Since

$$\overline{C} = \bigcup_{W' \subseteq V \atop T' \leq |W'| \leq n^{3/5}} C_r(W) \cup C_b(W)$$

the result follows. \(\square\)

Now we have everything needed to prove Proposition \(3\).

**Proof of Proposition \(3\)** Note that the event $|U'(G)| \geq x$ is monotone increasing for any $x \geq 0$. Since not having a cycle of length at most 4s is a decreasing event, by Harris’s Lemma we have

$$\Pr \left[ |U'(\overline{G}(n, \lambda'_1/n, \lambda'_2/n))| \geq x \right] \leq \Pr \left[ |U'(G(n, \lambda'_1/n, \lambda'_2/n))| \geq x \right].$$

For $W \subseteq V$ let $C_r(W)$ be the event that

- $W$ contains a red monocyclic spanning subgraph;
- every vertex in $W$ has a blue neighbour in $U$;
- no vertex in $W$ has a red neighbour in $V \setminus W$ with a blue neighbour in $U$.

15
In addition let $C'_b(W)$ be the analogous events for blue components, i.e. all the colours are swapped. Let $C'$ be the event that for every $W \subseteq U$ with $T \leq |W| \leq n^{3/5}$ neither $C'_b(W)$ nor $C'_r(W)$ holds. Noting that every component in $U(\tilde{G})$ has size at least $T$, event $C'$ implies $U'(\tilde{G}) \supseteq U(\tilde{G})$.

By Lemma 12 and Claim 13 whp, $A \subseteq U(\tilde{G}) \subseteq B_\varepsilon$. These inclusions imply $C'_r(W) \subseteq C_r(W)$ and $C'_b(W) \subseteq C_b(W)$, which in turn imply $C' \supseteq C$. By Lemma 14 $C$ holds whp, and hence $C'$ also holds whp.

Lemma 12 states that whp $|U(\tilde{G})| \geq \beta(\lambda_1 - \delta, \lambda_2 - \delta)n$, from which $|U'(G)| \geq |U(G)| \geq \beta(\lambda_1 - \delta, \lambda_2 - \delta)n \geq \beta(\lambda_1, \lambda_2)n - \varepsilon n$, whp, and the result follows as this inequality holds for arbitrary $0 < \varepsilon < \varepsilon_0$. □

4. PROOFS OF THEOREM 1 AND 2

Let $G_v[t]$ be the subgraph of $G$ formed by vertices at distance at most $t$ from $v$.

Claim 15. If $v$ is in a joint component of size larger than one then for every $t > 0$ either $G_v[t]$ contains a cycle, or $G_v[t]$ is a tree and $G_v[t]$ has property $B_t$ when viewed as a branching process rooted at $v$.

Proof. Let $J$ be the subgraph of $G$ spanned by the joint component of $v$. The result follows if we can show that if $J_v[t]$ is a tree then $J_v[t]$ has property $B_t$, when viewed as a branching process rooted at $v$. We will give a procedure for marking vertices in the tree $J_v[t]$ that terminates with a complete red-blue binary subtree in $J_v[t]$ of depth $t$ being marked. This subtree is a witness to $v$ having property $B_t$.

First mark $v$. Since $J$ is connected in the red graph, there must be a red path from $v$ to some leaf of $J_v[t]$. Let the first vertex on this path be $u_r$. Similarly, let $u_b$ be the first vertex in some blue path from $v$ to a leaf of $J_v[t]$. Mark $u_r$ and $u_b$. By construction, there is a red path from $u_r$ to a leaf of $J_v[t]$ lying completely in the subtree of $J_v[t]$ rooted at $u_r$. Also, since $J$ is connected in the blue graph, there must also be a path from $u_b$ to a leaf of $J_v[t]$, and this path necessarily lies within the subtree of $J_v[t]$ rooted at $u_r$. A similar argument applies to the vertex $u_b$. The situation at $u_r$ and $u_b$ replicates the situation that existed at the root $v$ of $J_v[t]$, so we can continue the marking process until we reach the leaves. The result is a witness to $v$ having property $B_t$. □

Proof of Theorem 1. For the lower bound our aim is to show that for every $\varepsilon > 0$ whp the size of the joint-giant is at least $\beta(\lambda_1, \lambda_2)n - 2\varepsilon n$. If $\beta(\lambda_1, \lambda_2) = 0$ there is nothing to prove. Recall that $\beta(\lambda_1, \lambda_2)$ is continuous, therefore for every $\varepsilon > 0$ there exists a $\delta$ such that $\beta(\lambda_1 - \delta, \lambda_2 - \delta) > \beta(\lambda_1, \lambda_2) - \varepsilon$ and $\beta(\lambda_1 - \delta, \lambda_2 - \delta) \in \mathbb{R}^2 \setminus C$.

We will expose the edges of $G(n, \lambda_1/n, \lambda_2/n)$ in two rounds. First we expose the edges of $G(n, (\lambda_1 - \delta)/n, (\lambda_2 - \delta)/n)$ and then merge this graph with a copy of $G(n, \delta/n, \delta/n)$, which provides a lower coupling for $G(n, \lambda_1/n, \lambda_2/n)$.

Expose all the edges in $G(n, (\lambda_1 - \delta)/n, (\lambda_2 - \delta)/n)$. Applying Proposition 3 to $G(n, (\lambda_1 - \delta)/n, (\lambda_2 - \delta)/n)$ we have whp that

$$|U'(G(n, (\lambda_1 - \delta)/n, (\lambda_2 - \delta)/n))| \geq \beta(\lambda_1 - \delta, \lambda_2 - \delta)n - o(n) \geq \beta(\lambda_1, \lambda_2) - 2\varepsilon)n,$$

for large enough $n$.

Recall that every red component in $U'(G(n, (\lambda_1 - \delta)/n, (\lambda_2 - \delta)/n))$ has size at least $n^{3/5}$ and note that there are at most $n$ such components. Therefore the probability that there exists a pair of red components in $U'(G(n, (\lambda_1 - \delta)/n, (\lambda_2 - \delta)/n))$ with no red edge between these components in $G(n, \delta/n, \delta/n)$ is at most

$$n^2(1 - \delta/n)n^{6/5} \leq n^2 \exp\left(-\delta n^{1/5}\right) = o(1).$$

An analogous proof for the blue graph completes the proof of the lower bound.
Now for the upper bound. Claim \([15]\) implies that if \(v\) is in the joint-giant then either \(G_v[s]\) contains a cycle, which occurs with probability \(o(1)\) or \(G_v[s]\) is cycle-free and when viewed as a branching process rooted at \(v\) it has property \(B_s\), which occurs with probability at most \(\beta(\lambda_1, \lambda_2) + o(1)\). Write \(N\) for the number of vertices satisfying one of these two conditions. Then \(\mathbb{E}[N] = \beta(\lambda_1, \lambda_2) n + o(n)\). Recall that the probability that a pair of vertices are within distance \(2s\) is \(o(1)\), implying that \(\text{Cov}(N) = o(n^2)\), and thus by Chebyshev’s inequality the number of such vertices is concentrated around its expectation providing the upper bound. \(\square\)

**Proof of Theorem 2.** Note that any jointly-connected component must contain both a red and a blue spanning tree. The probability that there exists a component of size between \(3\) and \(\varepsilon n\) is at most

\[
\sum_{k=3}^{\varepsilon n} \binom{n}{k} k^{4k-4} \left(\frac{\lambda_1}{n}\right)^{k-1} \left(\frac{\lambda_2}{n}\right)^{k-1} \leq \sum_{k=3}^{\varepsilon n} \frac{n^2}{\lambda_1 \lambda_2 k^4} \left(\frac{\varepsilon n}{k}\right)^k \left(\frac{\lambda_1}{n}\right)^k \left(\frac{\lambda_2}{n}\right)^k \leq \sum_{k=3}^{\varepsilon n} \frac{n^2}{\lambda_1 \lambda_2 k^4} \left(\frac{e\lambda_1 \lambda_2}{n}\right)^k = o(1)
\]

when \(\varepsilon < (e\lambda_1 \lambda_2)^{-1}\). \(\square\)

**References**

[1] G. Bianconi. *Multilayer Networks*. Oxford University Press, 2018.

[2] B. Bollobás, O. Cooley, M. Kang, and C. Koch. Jigsaw percolation on random hypergraphs. *J. Appl. Probab.*, 54(4):1261–1277, 2017.

[3] B. Bollobás, O. Riordan, E. Slivken, and P. Smith. The threshold for jigsaw percolation on random graphs. *Electron. J. Combin.*, 24(2):Paper 2.36, 14, 2017.

[4] C. D. Brummitt, S. Chatterjee, P. S. Dey, and D. Sivakoff. Jigsaw percolation: what social networks can collaboratively solve a puzzle? *Ann. Appl. Probab.*, 25(4):2013–2038, 2015.

[5] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin. Catastrophic cascade of failures in interdependent networks. *Nature*, 464(7291):1025–1028, April 2010.

[6] O. Cooley and A. Gutiérrez. Multi-coloured jigsaw percolation on random graphs. Submitted. arXiv:1712.00992.

[7] O. Cooley, T. Kapetanopoulos, and T. Makai. The sharp threshold for jigsaw percolation in random graphs. Submitted. arXiv:1809.01907.

[8] J. Gravner and D. Sivakoff. Nucleation scaling in jigsaw percolation. *Ann. Appl. Probab.*, 27(1):395–438, 2017.

[9] T. E. Harris. A lower bound for the critical probability in a certain percolation process. *Proc. Cambridge Philos. Soc.*, 56:13–20, 1960.

[10] T. E. Harris. *The theory of branching processes*. Die Grundlehren der Mathematischen Wissenschaften, Bd. 119. Springer-Verlag, Berlin; Prentice-Hall, Inc., Englewood Cliffs, N.J., 1963.

[11] S. Janson, D. E. Knuth, T. Łuczak, and B. Pittel. The birth of the giant component. *Random Structures Algorithms*, 4(3):231–358, 1993. With an introduction by the editors.

[12] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[13] M. Molloy. Sets that are connected in two random graphs. *Random Structures Algorithms*, 45(3):498–512, 2014.

[14] Boris Pittel, Joel Spencer, and Nicholas Wormald. Sudden emergence of a giant \(k\)-core in a random graph. *J. Combin. Theory Ser. B*, 67(1):111–151, 1996.

[15] O. Riordan. The \(k\)-core and branching processes. *Combin. Probab. Comput.*, 17(1):111–136, 2008.