STRUCTURE OF THE EXTENDED SCHRÖDINGER-VIRASORO LIE ALGEBRA \( \tilde{\mathfrak{sv}} \)

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Abstract. In this paper, we study the derivations, the central extensions and the automorphism group of the extended Schrödinger-Virasoro Lie algebra \( \tilde{\mathfrak{sv}} \), introduced by J. Unterberger \[25\] in the context of two-dimensional conformal field theory and statistical physics. Moreover, we show that \( \tilde{\mathfrak{sv}} \) is an infinite-dimensional complete Lie algebra and the universal central extension of \( \tilde{\mathfrak{sv}} \) in the category of Leibniz algebras is the same as that in the category of Lie algebras.

1. Introduction

The Schrödinger-Virasoro Lie algebra \( \mathfrak{sv} \), originally introduced by M. Henkel in \[8\] during his study on the invariance of the free Schrödinger equation, is a vector space over the complex field \( \mathbb{C} \) with a basis \( \{ L_n, M_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z} \} \) and the Lie brackets:

\[
\begin{align*}
[L_m, L_n] &= (n-m)L_{m+n}, \quad [L_m, M_n] = nM_{m+n}, \quad [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}}, \\
[M_m, M_n] &= 0, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1}, \quad [M_m, Y_{n+\frac{1}{2}}] = 0,
\end{align*}
\]

for all \( m, n \in \mathbb{Z} \). It is easy to see that \( \mathfrak{sv} \) is a semi-direct product of the centerless Virasoro algebra (Witt algebra) \( \mathfrak{Vir}_0 = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \), which can be regarded as the Lie algebra that consists of derivations on the Laurent polynomial ring \[14\], and the two-step nilpotent infinite-dimensional Lie algebra \( \mathfrak{h} = \bigoplus_{m \in \mathbb{Z}} \mathbb{C}Y_{m+\frac{1}{2}} \bigoplus_{m \in \mathbb{Z}} \mathbb{C}M_m \), which contains the Schrödinger Lie algebra \( \mathfrak{s} \) spanned by \( \{ L_{-1}, L_0, L_1, Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0 \} \). Clearly \( \mathfrak{s} \) is isomorphic to the semi-direct product of the Lie algebra \( \mathfrak{sl}(2) \) and the three-dimensional nilpotent Heisenberg Lie algebra \( \langle Y_{-\frac{1}{2}}, Y_{\frac{1}{2}}, M_0 \rangle \). The structure and representation theory of \( \mathfrak{sv} \) have been extensively studied by C. Roger and J. Unterberger. We refer the reader to \[22\] for more details. Recently, in order to investigate vertex representations of \( \mathfrak{sv} \), J. Unterberger \[25\] introduced a class of new infinite-dimensional Lie algebras \( \tilde{\mathfrak{sv}} \) called the extended Schrödinger-Virasoro algebra (see section 2), which can be viewed as an extension of \( \mathfrak{sv} \) by a conformal current with conformal weight 1.

Keywords: Schrödinger-Virasoro algebra, central extension, derivation, automorphism.

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In this paper, we give a complete description of the derivations, the central extensions and the automorphisms for the extended Schrödinger-Virasoro Lie algebra \( \tilde{sv} \). The paper is organized as follows:

In section 2, we show that the center of \( \tilde{sv} \) is zero and all derivations of \( \tilde{sv} \) are inner derivations, i.e., \( H^1(\tilde{sv}, \tilde{sv}) = 0 \), which implies that \( \tilde{sv} \) is a complete Lie algebra. Recall that a Lie algebra is called complete if its center is zero and all derivations are inner, which was originally introduced by N. Jacobson in [10]. Over the past decades, much progress has been obtained on the theory of complete Lie algebras (see for example [13, 26, 20]). Note that since the center of \( sv \) is non-zero and \( \dim H^1(sv, sv) = 3 \), \( sv \) is not a complete Lie algebra.

In section 3, we determine the universal central extension of \( \tilde{sv} \). The universal covering algebra \( \hat{sv} \) contains the twisted Heisenberg-Virasoro Lie algebra, which plays an important role in the representation theory of toroidal Lie algebras [2, 3, 11, 24], as a subalgebra. Furthermore, in section 4, we show that there is no non-zero symmetric invariant bilinear form on \( \tilde{sv} \), which implies the universal central extension of \( \tilde{sv} \) in the category of Leibniz algebras is the same as that in the category of Lie algebras [9].

Finally in section 5, we give the automorphism groups of \( \tilde{sv} \) and its universal covering algebra \( \hat{sv} \), which are isomorphic.

Throughout the paper, we denote by \( \mathbb{Z} \) and \( \mathbb{C}^* \) the set of integers and the set of non-zero complex numbers respectively, and all the vector spaces are assumed over the complex field \( \mathbb{C} \).

2. The Derivation Algebra of \( \tilde{sv} \)

**Definition 2.1.** The extended Schrödinger-Virasoro Lie algebra \( \tilde{sv} \) is a vector space spanned by a basis \( \{ L_n, M_n, N_n, Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z} \} \) with the following brackets

\[
[L_m, L_n] = (n-m)L_{m+n}, \quad [M_m, M_n] = 0, \quad [N_m, N_n] = 0, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m-n)M_{m+n+1},
\]

\[
[L_m, M_n] = nM_{m+n}, \quad [L_m, N_n] = nN_{m+n}, \quad [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1-m}{2})Y_{m+n+\frac{1}{2}},
\]

\[
[N_m, M_n] = 2M_{m+n}, \quad [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}}, \quad [M_m, Y_{n+\frac{1}{2}}] = 0,
\]

for all \( m, n \in \mathbb{Z} \).

It is clear that \( \tilde{sv} \) is a perfect Lie algebra, i.e., \( [\tilde{sv}, \tilde{sv}] = \tilde{sv} \), which is finitely generated with a set of generators \( \{ L_{-2}, L_{-1}, L_1, L_2, N_1, Y_{\frac{1}{2}} \} \).

Define a \( \frac{1}{2} \mathbb{Z} \)-grading on \( \tilde{sv} \) by

\[
\deg(L_n) = n, \quad \deg(M_n) = n, \quad \deg(N_n) = n, \quad \deg(Y_{n+\frac{1}{2}}) = n + \frac{1}{2},
\]

for all \( n \in \mathbb{Z} \). Then

\[
\tilde{sv} = \bigoplus_{n \in \mathbb{Z}} \tilde{sv}_n = \bigoplus_{n \in \mathbb{Z}} \tilde{sv}_n \oplus \bigoplus_{n \in \mathbb{Z}} \tilde{sv}_{n+\frac{1}{2}},
\]
where \( \tilde{sv}_n = \text{span}\{L_n, M_n, N_n\} \) and \( \tilde{sv}_{n+\frac{1}{2}} = \text{span}\{Y_{n+\frac{1}{2}}\} \) for all \( n \in \mathbb{Z} \).

**Definition 2.2.** (5) Let \( G \) be a commutative group, \( g = \bigoplus_{g \in G} g \) a \( G \)-graded Lie algebra.

A \( g \)-module \( V \) is called \( G \)-graded, if

\[
V = \bigoplus_{g \in G} V_g, \quad g_h \subseteq V_{g+h}, \quad \forall g, h \in G.
\]

**Definition 2.3.** (5) Let \( g \) be a Lie algebra and \( V \) a \( g \)-module. A linear map \( D : g \to V \) is called a derivation, if for any \( x, y \in g \), we have

\[
D[x,y] = x.D(y) - y.D(x).
\]

If there exists some \( v \in V \) such that \( D : x \mapsto x.v \), then \( D \) is called an inner derivation.

Let \( g \) be a Lie algebra, \( V \) a module of \( g \). Denote by \( \text{Der}(g, V) \) the vector space of all derivations, \( \text{Inn}(g, V) \) the vector space of all inner derivations. Set

\[
H^1(g, V) = \text{Der}(g, V)/\text{Inn}(g, V).
\]

Denote by \( \text{Der}(g) \) the derivation algebra of \( g \), \( \text{Inn}(g) \) the vector space of all inner derivations of \( g \). We will prove that all the derivations of \( \tilde{sv} \) are inner derivations.

By Proposition 1.1 in [5], we have the following lemma.

**Lemma 2.4.**

\[
\text{Der}(\tilde{sv}) = \bigoplus_{n \in \mathbb{Z}} \text{Der}(\tilde{sv}_n),
\]

where \( \text{Der}(\tilde{sv}_n) \subseteq \tilde{sv}_{m+n} \) for all \( m, n \in \mathbb{Z} \).

\( \square \)

**Lemma 2.5.** \( H^1(\tilde{sv}_0, \tilde{sv}_\frac{n}{2}) = 0, \quad \forall n \in \mathbb{Z} \setminus \{0\} \).

**Proof.** We have to prove

\[
H^1(\tilde{sv}_0, \tilde{sv}_n) = 0, \quad \forall n \in \mathbb{Z} \setminus \{0\},
\]

\[
H^1(\tilde{sv}_0, \tilde{sv}_{n+\frac{1}{2}}) = 0, \quad \forall n \in \mathbb{Z}.
\]

(1) For \( m \neq 0 \), let \( \varphi : \tilde{sv}_0 \to \tilde{sv}_m \) be a derivation. Assume that

\[
\varphi(L_0) = a_1 L_m + b_1 M_m + c_1 N_m,
\]

\[
\varphi(M_0) = a_2 L_m + b_2 M_m + c_2 N_m,
\]

\[
\varphi(N_0) = a_3 L_m + b_3 M_m + c_3 N_m,
\]

where \( a_i, b_i, c_i \in \mathbb{C}, i = 1, 2, 3 \). Since

\[
\varphi[L_0, M_0] = [\varphi(L_0), M_0] + [L_0, \varphi(M_0)],
\]

we have

\[
a_2 mL_m + (b_2 + 2c_1) M_m + c_2 m N_m = 0.
\]
So $a_2 = 0$, $c_1 = -\frac{1}{2} b_2 m$, $c_2 = 0$ and $\varphi(M_0) = b_2 M_m$. Since
\[
\varphi[L_0, N_0] = [\varphi(L_0), N_0] + [L_0, \varphi(N_0)],
\]
we have
\[
a_3 m L_m + (b_3 m - 2b_1) M_m + c_3 m N_m = 0.
\]
Then $a_3 = 0, b_1 = \frac{1}{2} b_3 m, c_3 = 0$ and $\varphi(N_0) = b_3 M_m$. Therefore, we get
\[
\varphi(L_0) = a_1 L_m + \frac{1}{2} b_3 m M_m - \frac{1}{2} b_2 m N_m, \quad \varphi(M_0) = b_2 M_m, \quad \varphi(N_0) = b_3 M_m.
\]
Let $X_m = \frac{a_1}{m} L_m + \frac{1}{2} b_3 m M_m - \frac{1}{2} b_2 N_m$, we have
\[
\varphi(L_0) = [L_0, X_m], \quad \varphi(M_0) = [M_0, X_m], \quad \varphi(N_0) = [N_0, X_m].
\]
Then $\varphi \in \text{Inn}(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_m)$. Therefore,
\[
H^1(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_m) = 0, \quad \forall m \in \mathbb{Z} \setminus \{0\}.
\]
(2) For all $m \in \mathbb{Z}$, let $\varphi : \tilde{\mathfrak{g}}_0 \rightarrow \tilde{\mathfrak{g}}_{m + \frac{1}{2}}$ be a derivation. Assume that
\[
\varphi(L_0) = a Y_{m + \frac{1}{2}}, \quad \varphi(M_0) = b Y_{m + \frac{1}{2}}, \quad \varphi(N_0) = c Y_{m + \frac{1}{2}},
\]
for some $a, b, c \in \mathbb{C}$. Because $\varphi[L_0, M_0] = [\varphi(L_0), M_0] + [L_0, \varphi(M_0)]$, we have
\[
0 = [L_0, \varphi(M_0)] = [L_0, b Y_{m + \frac{1}{2}}] = b(m + \frac{1}{2}) Y_{m + \frac{1}{2}}.
\]
So $b = 0$ and $\varphi(M_0) = 0$. Since $\varphi[L_0, N_0] = [\varphi(L_0), N_0] + [L_0, \varphi(N_0)]$, we have
\[
0 = [a Y_{m + \frac{1}{2}}, N_0] + [L_0, c Y_{m + \frac{1}{2}}] = (c(m + \frac{1}{2}) - a) Y_{m + \frac{1}{2}}.
\]
Then $a = c(m + \frac{1}{2})$. Hence, we get
\[
\varphi(L_0) = c(m + \frac{1}{2}) Y_{m + \frac{1}{2}}, \quad \varphi(M_0) = 0, \quad \varphi(N_0) = c Y_{m + \frac{1}{2}}.
\]
Letting $X_{m + \frac{1}{2}} = c Y_{m + \frac{1}{2}}$, we obtain
\[
\varphi(L_0) = [L_0, X_{m + \frac{1}{2}}], \quad \varphi(M_0) = [M_0, X_{m + \frac{1}{2}}], \quad \varphi(N_0) = [N_0, X_{m + \frac{1}{2}}].
\]
Then $\varphi \in \text{Inn}(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_{m + \frac{1}{2}})$. Therefore,
\[
H^1(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_{m + \frac{1}{2}}) = 0, \quad \forall m \in \mathbb{Z}.
\]

\[\square\]

**Lemma 2.6.** $\text{Hom}_{\tilde{\mathfrak{g}}_0}(\tilde{\mathfrak{g}}_0, \tilde{\mathfrak{g}}_m^{\perp}) = 0$ for all $m, n \in \mathbb{Z}, \ m \neq n$. 


Proof. Let $f \in Hom_{\tilde{sv}}(\tilde{sv}_m, \tilde{sv}_n)$, where $m \neq n$. Then for any $E_0 \in \tilde{sv}_0, E_m \in \tilde{sv}_n$, we have

$$f([E_0, E_m]) = [E_0, f(E_m)].$$

Then $f([L_0, E_m]) = [L_0, f(E_m)]$, i.e.,

$$\frac{m}{2} f(E_m) = [L_0, f(E_m)] = \frac{n}{2} f(E_m).$$

So we have $f(E_m) = 0$ for all $m \neq n$. Therefore, we have $f = 0$. \[\square\]

By Lemma 2.5 and Proposition 1.2 in [3], we have the following Lemma.

**Lemma 2.7.** $\text{Der}(\tilde{sv}) = \text{Der}(\tilde{sv})_0 + \text{Inn}(\tilde{sv})$.

**Lemma 2.8.** For any $D \in \text{Der}(\tilde{sv})_0$, there exist some $a, b, c \in \mathbb{C}$ such that

$$D = \text{ad}(aL_0 - \frac{c}{2} M_0 + (b - \frac{a}{2}) N_0).$$

Therefore, $\text{Der}(\tilde{sv})_0 \subseteq \text{Inn}(\tilde{sv})$.

**Proof.** For any $D \in \text{Der}(\tilde{sv})_0$, assume that for all $m \in \mathbb{Z}$,

$$D(L_m) = a_{11}^m L_m + a_{12}^m M_m + a_{13}^m N_m,$$
$$D(M_m) = a_{21}^m L_m + a_{22}^m M_m + a_{23}^m N_m,$$
$$D(N_m) = a_{31}^m L_m + a_{32}^m M_m + a_{33}^m N_m,$$
$$D(Y_{m+\frac{1}{2}}) = b^{m+\frac{1}{2}} Y_{m+\frac{1}{2}},$$

where $a_{ij}^m, b^{m+\frac{1}{2}} \in \mathbb{C}$, $i, j = 1, 2, 3$. For any $E_m \in \tilde{sv}_m, E_n \in \tilde{sv}_n$, we have

$$D[E_m, E_n] = [D(E_m), E_n] + [E_m, D(E_n)].$$

In particular, $D[L_0, E_m] = [D(L_0), E_m] + [L_0, D(E_m)]$. Then

$$\frac{m}{2} D(E_m) = [D(L_0), E_m] + \frac{m}{2} D(E_m).$$

So

$$[D(L_0), E_m] = 0.$$ 

Since $[D(L_0), L_1] = 0, [D(L_0), M_0] = 0$ and $[D(L_0), N_0] = 0$, we can deduce that

$$a_{11}^0 = a_{12}^0 = a_{13}^0 = 0.$$ 

So $D(L_0) = 0$. Because $D[M_0, L_m] = [D(M_0), L_m] + [M_0, D(L_m)]$, we have

$$a_{21}^0 M_m - 2a_{13} M_m = 0.$$ 

Then

$$a_{21}^0 = 0, a_{13}^0 = 0, D(M_0) = a_{22}^0 M_0 + a_{23}^0 N_0.$$ 

By the fact that $D[M_0, N_m] = [D(M_0), N_m] + [M_0, D(N_m)]$, we have

$$a_{21}^m L_m + a_{22}^m M_m + a_{23}^m N_m = (a_{22}^0 + a_{33}^0) M_m.$$
Therefore,
\[ a_{21}^0 = 0, \quad a_{22}^0 = a_{33}^0 + a_{33}^m, \quad a_{23}^m = 0. \]

Then
\[ a_{33}^0 = 0, \quad D(M_m) = a_{22}^m M_m = (a_{22}^0 + a_{33}^m)M_m. \]

Since \( D[N_0, L_m] = [D(N_0), L_m] + [N_0, D(L_m)], \) we have
\[ a_{31}^0 M_m + 2a_{12}^m M_m = 0. \]

Then \( a_{31}^0 m = 0, 2a_{12}^m = 0 \) for all \( m \in \mathbb{Z}. \) Therefore,
\[ a_{31}^0 = 0, \quad a_{12}^m = 0, \quad D(N_0) = a_{32}^0 M_0. \]

According to \( D[N_m, N_n] = [D(N_m), N_n] + [N_m, D(N_n)], \) we obtain
\[ (a_{31}^m m - a_{31}^n m)N_{m+n} + 2(a_{32}^m - a_{32}^n)M_{m+n} = 0. \]

Then \( a_{31}^m n = a_{31}^n m, a_{32}^m = a_{32}^n \) for all \( m, n \in \mathbb{Z}. \) Hence, we have
\[ a_{31}^0 = a_{11}^1 m, \quad a_{32}^m = a_{33}^m, \quad \forall m \in \mathbb{Z}. \]

By the following two relations
\[
D[Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = [D(Y_{m+\frac{1}{2}}), Y_{n+\frac{1}{2}}] + [Y_{m+\frac{1}{2}}, D(Y_{n+\frac{1}{2}})],
\]
\[
D[L_m, L_n] = [D(L_m), L_n] + [L_m, D(L_n)],
\]
we have
\[
a_{22}^{m+n+1} = b^{m+\frac{1}{2}} + b^{n+\frac{1}{2}}, \quad m \neq n. \tag{2.1}
a_{11}^{m+n} = a_{11}^m + a_{11}^n, \quad m \neq n. \tag{2.2}
\]

It is easy to see that \( a_{11}^{-m} = -a_{11}^{m} \) for all \( m \in \mathbb{Z}. \) Let \( n = 1 \) in (2.2), then
\[
a_{11}^{m+1} = a_{11}^m + a_{11}^1, \quad m \neq 1.
\]

By induction on \( m \in \mathbb{Z}^+ \) and \( m \geq 3, \) we have
\[
a_{11}^m = a_{11}^2 + (m - 2)a_{11}^1, \quad m \geq 3.
\]

Let \( m = 4, n = -2 \) in (2.2), then we have \( a_{11}^2 = 2a_{11}^1. \) Therefore,
\[
a_{11}^m = ma_{11}^1, \quad \forall m \in \mathbb{Z}.
\]

Since \( D[L_m, M_n] = [D(L_m), M_n] + [L_m, D(M_n)] \) and
\[
D[L_m, N_n] = [D(L_m), N_n] + [L_m, D(N_n)],
\]
we get
\[
a_{22}^{m+n} = ma_{11}^1 + a_{22}^n, \quad n \neq 0, \tag{2.3}
a_{31}^1(m + n) = a_{31}^1(n - m), \quad a_{33}^{m+n} = ma_{11}^1 + a_{33}^n, \quad n \neq 0.
\]

Then we can deduce that
\[
a_{31}^1 = 0, \quad a_{33}^{m+1} = ma_{11}^1 + a_{33}^1.
\]
Note $a^0_{33} = 0$, then $a^1_{11} = a^1_{33}$. Therefore,

$$a^m_{33} = ma^1_{11}, \quad a^m_{22} = a^0_{22} + ma^1_{11}, \quad \forall m \in \mathbb{Z}.$$  

Because $D[L_m, Y_{n+\frac{1}{2}}] = [D(L_m), Y_{n+\frac{1}{2}}] + [L_m, D(Y_{n+\frac{1}{2}})]$, we have

$$b^{m+n+\frac{1}{2}} = ma^1_{11} + b^{n+\frac{1}{2}}, \quad n + \frac{1-m}{2} \neq 0.$$  

Let $m = 1$, then

$$b^{n+1+\frac{1}{2}} = a^1_{11} + b^{n+\frac{1}{2}}, \quad n \neq 0.$$  

By (2.1), we get

$$a^0_{22} + (m + n + 1)a^1_{11} = b^{n+\frac{1}{2}} + b^{n+\frac{1}{2}}, \quad m \neq n. \quad (2.4)$$  

Let $m = 0, n = 1$ in (2.4), then we have

$$a^0_{22} + a^1_{11} = 2b^{\frac{1}{2}}.$$  

Let $n = 0$ in (2.4), then we obtain $b^{m+\frac{1}{2}} = b^{\frac{1}{2}} + ma^1_{11}$ for all $m \in \mathbb{Z}$. Set $a^1_{11} = a, b^{\frac{1}{2}} = b, a^0_{32} = c$, then $a^0_{22} = 2b - a$ and

$$D(L_m) = maL_m, \quad D(M_m) = (2b - a + ma)M_m, \quad D(N_m) = cM_m + maN_m, \quad D(Y_{m+\frac{1}{2}}) = (b + ma)Y_{m+\frac{1}{2}}.$$  

for all $m \in \mathbb{Z}$. Then we can deduce that

$$D = ad(aL_0 - \frac{c}{2}M_0 + (b - \frac{a}{2})N_0).$$

From the above lemmas, we obtain the following theorem.

**Theorem 2.9.** $\text{Der}(\tilde{sv}) = \text{Inn}(\tilde{sv})$, i.e., $H^1(\tilde{sv}, \tilde{sv}) = 0$.

**Lemma 2.10.** $C(\tilde{sv}) = \{0\}$, where $C(\tilde{sv})$ is the center of $\tilde{sv}$.

**Proof.** For any $E_{\frac{1}{2}} \in \tilde{sv}_{\frac{1}{2}}$, we have

$$[L_0, E_{\frac{1}{2}}] = \frac{n}{2} E_{\frac{1}{2}}.$$  

It forces $x \in \tilde{sv}_0$, for any $x \in C(\tilde{sv})$, since $[L_0, x] = 0$. Let $x = aL_0 + bN_0 + cM_0$, where $a, b, c \in \mathbb{C}$. Then

$$[x, L_1] = [aL_0 + bN_0 + cM_0, L_1] = [aL_0, L_1] = aL_1 = 0.$$  

So $a = 0$. By the following relations,

$$[x, Y_{\frac{1}{2}}] = [bN_0 + cM_0, Y_{\frac{1}{2}}] = bN_0 Y_{\frac{1}{2}} = bY_{\frac{1}{2}} = 0,$$

$$[x, N_0] = [cM_0, N_0] = -2cM_0 = 0,$$

we have $b = 0$ and $c = 0$. Therefore, $x = 0$. \qed

By Lemma 2.10 and Theorem 3.1 we have
Corollary 2.11. $\tilde{\mathfrak{sv}}$ is an infinite-dimensional complete Lie algebra.

3. The Universal Central Extension of $\tilde{\mathfrak{sv}}$

In this section, we discuss the structure of the universal central extension of $\tilde{\mathfrak{sv}}$. Let us first recall some basic concepts. Let $\mathfrak{g}$ be a Lie algebra. A bilinear function $\psi : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ is called a 2-cocycle on $\mathfrak{g}$ if for all $x, y, z \in \mathfrak{g}$, the following two conditions are satisfied:

$$
\psi(x, y) = -\psi(y, x),
\psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0.
$$

For any linear function $f : \mathfrak{g} \longrightarrow \mathbb{C}$, one can define a 2-cocycle $\psi_f$ as follows

$$
\psi_f(x, y) = f([x, y]), \quad \forall x, y \in \mathfrak{g}.
$$

Such a 2-cocycle is called a 2-coboundary on $\mathfrak{g}$. Denote by $C^2(\mathfrak{g}, \mathbb{C})$ the vector space of 2-cocycles on $\mathfrak{g}$, $B^2(\mathfrak{g}, \mathbb{C})$ the vector space of 2-coboundaries on $\mathfrak{g}$. Then the quotient space $H^2(\mathfrak{g}, \mathbb{C}) = C^2(\mathfrak{g}, \mathbb{C}) / B^2(\mathfrak{g}, \mathbb{C})$ is called the second cohomology group of $\mathfrak{g}$.

Theorem 3.1. $\dim H^2(\tilde{\mathfrak{sv}}, \mathbb{C}) = 3$.

Proof. Let $\varphi : \tilde{\mathfrak{sv}} \times \tilde{\mathfrak{sv}} \longrightarrow \mathbb{C}$ be a 2-cocycle on $\tilde{\mathfrak{sv}}$. Let $f : \tilde{\mathfrak{sv}} \longrightarrow \mathbb{C}$ be a linear function defined by

$$
\begin{align*}
f(L_0) &= -\frac{1}{2} \varphi(L_1, L_{-1}), \quad f(L_m) = \frac{1}{m} \varphi(L_0, L_m), \quad m \neq 0, \\
f(M_0) &= -\varphi(L_1, M_{-1}), \quad f(M_m) = \frac{1}{m} \varphi(L_0, M_m), \quad m \neq 0, \\
f(N_0) &= -\varphi(L_1, N_{-1}), \quad f(N_m) = \frac{1}{m} \varphi(L_0, N_m), \quad m \neq 0, \\
f(Y_{m+\frac{1}{2}}) &= \frac{1}{m+\frac{1}{2}} \varphi(L_0, Y_{m+\frac{1}{2}}), \quad \forall m \in \mathbb{Z}.
\end{align*}
$$

Let $\varphi_f = \varphi - \varphi_f$, where $\varphi_f$ is the 2-coboundary induced by $f$, then

$$
\varphi_f(x, y) = \varphi(x, y) - f([x, y]), \quad \forall x, y \in \tilde{\mathfrak{sv}}.
$$

By the known result on the central extension of the classical Witt algebra (see [1] or [15]), we have

$$
\varphi(L_m, L_n) = \alpha \delta_{m+n,0}(m^3 - m), \quad \forall m, n \in \mathbb{Z}, \quad \alpha \in \mathbb{C}.
$$

By the fact that $\varphi([L_0, L_m], M_n) + \varphi([L_m, M_n], L_0) + \varphi([M_n, L_0], L_m) = 0$, we get

$$
(m + n)\varphi(L_m, M_n) = n\varphi(L_0, M_{m+n}).
$$

So

$$
\varphi(L_m, M_n) = \frac{n}{m + n}\varphi(L_0, M_{m+n}), \quad m + n \neq 0. \quad (3.2)
$$

Then it is easy to deduce that

$$
\varphi(L_m, M_n) = 0, \quad m + n \neq 0.
$$
Furthermore,
\[ \mathfrak{u}(L_m, M_{-m}) = \varphi(L_m, M_{-m}) + mf(M_0) = \varphi(L_m, M_{-m}) - m\varphi(L_1, M_{-1}). \]  
(3.3)

Similarly, denote \( \mathfrak{u}(L_m, N_{-m}) = c(m) \) for all \( m \in \mathbb{Z} \), then we have
\[ \mathfrak{u}(L_m, N_n) = \delta_{m+n,0}c(m), \quad \mathfrak{u}(L_m, N_{-m}) = \varphi(L_m, N_{-m}) - m\varphi(L_1, N_{-1}). \]  
(3.4)

By (3.1), \( \varphi([L_0, L_m], Y_{n+\frac{1}{2}}) + \varphi([L_m, Y_{n+\frac{1}{2}}], L_0) + \varphi([Y_{n+\frac{1}{2}}, L_0], L_m) = 0 \), so
\[ (m + n + \frac{1}{2})\varphi(L_m, Y_{n+\frac{1}{2}}) = (n + \frac{1-m}{2})\varphi(L_0, Y_{m+n+\frac{1}{2}}). \]
Then for all \( m, n \in \mathbb{Z} \), we get
\[ \varphi(L_m, Y_{n+\frac{1}{2}}) = \frac{n + \frac{1-m}{2}}{m + n + \frac{1}{2}}\varphi(L_0, Y_{m+n+\frac{1}{2}}). \]

Consequently, we have
\[ \mathfrak{u}(L_m, Y_{n+\frac{1}{2}}) = 0, \quad m, n \in \mathbb{Z}. \]

From the relation that
\[ \varphi([N_0, Y_{m+\frac{1}{2}}], Y_{m-\frac{1}{2}}) + \varphi([Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}], N_0) + \varphi([Y_{m-\frac{1}{2}}, Y_{m+\frac{1}{2}}], N_0) = 0, \quad m \in \mathbb{Z}, \]
we have
\[ \varphi(Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}) = (m + \frac{1}{2})\varphi(N_0, M_0), \quad m \in \mathbb{Z}. \]  
(3.5)

By (3.1), \( \varphi([L_m, Y_{p+\frac{1}{2}}], Y_{q+\frac{1}{2}}) + \varphi([Y_{p+\frac{1}{2}}, Y_{q+\frac{1}{2}}], L_m) + \varphi([Y_{q+\frac{1}{2}}, L_m], Y_{p+\frac{1}{2}}) = 0 \), then
\[ (p + \frac{1-m}{2})\varphi(Y_{m+p+\frac{1}{2}}, Y_{q+\frac{1}{2}}) + (p-q)\varphi(M_{p+q+1}, L_m) - (q + \frac{1-m}{2})\varphi(Y_{m+q+\frac{1}{2}}, Y_{p+\frac{1}{2}}) = 0, \]
for all \( m, p, q \in \mathbb{Z} \). Let \( m = -p - q - 1 \), then for all \( p, q \in \mathbb{Z} \), we have
\[ (p+1+\frac{p+q}{2})\varphi(Y_{q-\frac{1}{2}}, Y_{q+\frac{1}{2}}) + (p-q)\varphi(M_{p+q+1}, L_{-p-q-1}) - (q+1+\frac{p+q}{2})\varphi(Y_{-q-\frac{1}{2}}, Y_{p+\frac{1}{2}}) = 0. \]  
(3.6)

Using (3.5), we get
\[ \varphi(L_{-p-q-1}, M_{p+q+1}) = \frac{p+q+1}{2}\varphi(N_0, M_0), \quad p \neq q. \]  
(3.7)

Letting \( p = -2, q = 0 \) in (3.7), we have
\[ \varphi(L_1, M_{-1}) = -\frac{1}{2}\varphi(N_0, M_0) = -\varphi(Y_{\frac{1}{2}}, Y_{-\frac{1}{2}}). \]  
(3.8)

It follows from (3.5) and (3.7) that
\[ \varphi(Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}) = -(2m+1)\varphi(L_1, M_{-1}), \quad m \in \mathbb{Z}, \]  
(3.9)
\[ \varphi(L_m, M_{-m}) = -m\varphi(Y_{\frac{1}{2}}, Y_{-\frac{1}{2}}) = m\varphi(L_1, M_{-1}), \quad m \in \mathbb{Z}. \]  
(3.10)

By (3.3) and (3.10), we have \( \mathfrak{u}(L_m, M_{-m}) = 0 \) for all \( m \in \mathbb{Z} \). Therefore,
\[ \mathfrak{u}(L_m, M_n) = 0, \quad \forall \ m, n \in \mathbb{Z}. \]
According to (3.1), \( \varphi([L_m, L_n], N_{m-n})+\varphi([L_n, N_{m-n}], L_m)+\varphi([N_{m-n}, L_m], L_n) = 0 \), then we have

\[
(n - m)\varphi(L_{m+n}, N_{m-n}) - (m + n)\varphi(N_m, L_m) + (m + n)\varphi(N_n, L_n) = 0.
\]

By (3.4), we get

\[
(n - m)c(m + n) + (m + n)c(m) = (m + n)c(n). 
\] (3.11)

Let \( n = 1 \) in (3.11) and note \( c(0) = c(1) = 0 \), then we obtain

\[
(m - 1)c(m + 1) = (m + 1)c(m).
\] (3.12)

Let \( n = 1 - m \) in (3.11), then

\[
c(m) = c(1 - m), \quad \forall m \in \mathbb{Z}.
\]

By induction on \( n \), we deduce that \( c(-1) = c(2) \) determines all \( c(m) \) for \( m \in \mathbb{Z} \). On the other hand, \( c(m) = m^2 - m \) is a solution of equation (3.12). So

\[
c(m) = \beta(m^2 - m), \quad \beta \in \mathbb{C},
\]

is the general solution of equation (3.12). Therefore,

\[
\varphi(L_m, N_n) = \delta_{m+n,0}\beta(m^2 - m), \quad \forall m, n \in \mathbb{Z}, \quad \beta \in \mathbb{C}. 
\] (3.13)

By (3.1), we have

\[
\varphi([M_p, Y_{m+\frac{1}{2}}], Y_{n+\frac{1}{2}}) + \varphi([Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}], M_p) + \varphi([Y_{n+\frac{1}{2}}, M_p], Y_{m+\frac{1}{2}}) = 0.
\]

Then \((m-n)\varphi(M_{m+n+1}, M_p) = 0\). Let \( m+n+1 = q \), then we have \((2m+1-q)\varphi(M_q, M_p) = 0\). This means that

\[
\varphi(M_p, M_q) = 0, \quad \forall p, q \in \mathbb{Z}.
\]

Therefore, we have

\[
\varphi(M_m, N_n) = \varphi(M_n, N_m) = 0, \quad \forall m, n \in \mathbb{Z}. 
\] (3.14)

By (3.1), \( \varphi([L_0, M_m], N_n) + \varphi([M_m, N_n], L_0) + \varphi([N_n, L_0], M_m) = 0 \), we have

\[
(m + n)\varphi(M_m, N_n) = -2\varphi(L_0, M_{m+n}). 
\] (3.15)

Then for \( m + n \neq 0 \), we have

\[
\varphi(M_m, N_n) = \varphi(M_n, N_m) + \frac{2}{m + n}\varphi(L_0, M_{m+n}) = 0.
\]

On the other hand,

\[
\varphi(M_m, N_{-m}) = \varphi(M_m, N_{-m}) + 2f(M_0) = \varphi(M_m, N_{-m}) - 2\varphi(L_1, M_{-1}).
\]

By (3.1), \( \varphi([N_p, Y_{m+\frac{1}{2}}], Y_{n+\frac{1}{2}}) + \varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}], N_p) + \varphi([Y_{n+\frac{1}{2}}, N_p], Y_{m+\frac{1}{2}}) = 0 \), we have

\[
\varphi(Y_{p+m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) + (m-n)\varphi(M_{m+n+1}, N_p) - \varphi(Y_{p+m+\frac{1}{2}}, Y_{m+\frac{1}{2}}) = 0.
\]

Let \( n = -m - p - 1 \), then we have

\[
\varphi(Y_{p+m+\frac{1}{2}}, Y_{n-p-m-\frac{1}{2}}) + (2m + p + 1)\varphi(M_{-p}, N_p) - \varphi(Y_{m-\frac{1}{2}}, Y_{m+\frac{1}{2}}) = 0.
\]
By (3.9), we get 

$$\varphi(M_{-p}, N_p) = 2\varphi(L_1, M_{-1}), \quad \forall p \in \mathbb{Z}.$$ 

Therefore, 

$$\varphi(M_m, N_n) = 0, \quad \forall m, n \in \mathbb{Z}. \quad (3.16)$$

By (3.1), \(\varphi([L_0, M_m], Y_{n+\frac{1}{2}}) + \varphi([M_m, Y_{n+\frac{1}{2}}], L_0) + \varphi([Y_{n+\frac{1}{2}}, L_0], M_m) = 0\), we have 

$$(m + n + \frac{1}{2})\varphi(M_m, Y_{n+\frac{1}{2}}) = 0.$$ 

So \(\varphi(M_m, Y_{n+\frac{1}{2}}) = 0\) for all \(m, n \in \mathbb{Z}\). Therefore 

$$\varphi(M_m, Y_{n+\frac{1}{2}}) = \varphi(M_m, Y_{n+\frac{1}{2}}) - f([M_m, Y_{n+\frac{1}{2}}]) = 0.$$ 

By (3.1), \(\varphi([L_0, N_p], N_q) + \varphi([N_p, N_q], L_0) + \varphi([N_q, L_0], N_p) = 0\), we have 

$$(p + q)\varphi(N_p, N_q) = 0.$$ 

So \(\varphi(N_p, N_q) = \varphi(N_p, N_q) = \delta_{p+q,0}k(p)\), where \(\varphi(N_p, N_{-p}) = k(p)\). Then by (3.1), 

$$\varphi([L_{-p-q}, N_p], N_q) + \varphi([N_p, N_q], L_{-p-q}) + \varphi([N_q, L_{-p-q}], N_p) = 0,$$ 

we get 

$$pk(q) = qk(p).$$ 

Let \(q = 1\), then \(k(p) = pk(1)\). Set \(k(1) = k\), then we have 

$$\varphi(N_m, N_n) = k\delta_{m+n,0}m, \quad \forall m, n \in \mathbb{Z}.$$ 

By (3.1), \(\varphi([L_0, N_m], Y_{n+\frac{1}{2}}) + \varphi([N_m, Y_{n+\frac{1}{2}}], L_0) + \varphi([Y_{n+\frac{1}{2}}, L_0], N_m) = 0\), we have 

$$(m + n + \frac{1}{2})\varphi(N_m, Y_{n+\frac{1}{2}}) = \varphi(L_0, Y_{m+n+\frac{1}{2}}).$$ 

Then 

$$\varphi(N_m, Y_{n+\frac{1}{2}}) = \varphi(N_m, Y_{n+\frac{1}{2}}) - f(Y_{m+n+\frac{1}{2}})$$ 

$$= \varphi(N_m, Y_{n+\frac{1}{2}}) - \frac{1}{m + n + \frac{1}{2}}\varphi(L_0, Y_{m+n+\frac{1}{2}})$$ 

$$= 0, \quad \forall m, n \in \mathbb{Z}.$$ 

By (3.1), \(\varphi([L_0, Y_{m+\frac{1}{2}}], Y_{n+\frac{1}{2}}) + \varphi([Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}], L_0) + \varphi([Y_{n+\frac{1}{2}}, L_0], Y_{m+\frac{1}{2}}) = 0\), we have 

$$(m + n + 1)\varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) = (m - n)\varphi(L_0, Y_{m+n+\frac{1}{2}}). \quad (3.17)$$ 

For \(m + n + 1 \neq 0\), 

$$\varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) = \varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) - f([Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}])$$ 

$$= \varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) - (m - n)f(Y_{m+n+\frac{1}{2}})$$ 

$$= \varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) - \frac{m - n}{m + n + 1}\varphi(L_0, Y_{m+n+\frac{1}{2}}).$$ 

By (3.17), we have 

$$\varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) = 0, \quad m + n + 1 \neq 0.$$
On the other hand, for all \( m \in \mathbb{Z} \),
\[
\varphi(Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}) = \varphi(Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}) - f([Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}])
= \varphi(Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}) - (2m + 1)f(M_0)
= \varphi(Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}) + (2m + 1)\varphi(L_1, M_{-1}).
\]
By (3.9), we have
\[
\varphi(Y_{m+\frac{1}{2}}, Y_{m-\frac{1}{2}}) = 0, \quad \forall m \in \mathbb{Z}.
\]
So
\[
\varphi(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) = 0, \quad \forall m, n \in \mathbb{Z}.
\]
Therefore, \( \varphi \) is determined by the following three 2-cocycles
\[
\varphi_1(L_m, L_n) = \delta_{m+n,0} \frac{m^3 - m}{12}, \quad \varphi_2(L_m, N_n) = \delta_{m+n,0} (m^2 - m), \quad \varphi_3(N_m, N_n) = \delta_{m+n,0} n.
\]

**Remark 3.2.** It is referred in [25] that \( \tilde{\mathfrak{g}} \) has three independent classes of central extensions given by the cocycles
\[
c_1(L_m, L_n) = \delta_{m+n,0} \frac{m^3 - m}{12}, \quad c_2(L_m, N_n) = \delta_{m+n,0} m^2, \quad c_3(N_m, N_n) = \delta_{m+n,0} n.
\]
Here we prove in detail that \( \tilde{\mathfrak{g}} \) has only three independent classes of central extensions.

Let \( \mathfrak{g} \) be a perfect Lie algebra, i.e., \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\). \((\hat{\mathfrak{g}}, \pi)\) is called a central extension of \( \mathfrak{g} \) if \( \pi : \hat{\mathfrak{g}} \rightarrow \mathfrak{g} \) is a surjective homomorphism whose kernel lies in the center of the Lie algebra \( \hat{\mathfrak{g}} \). The pair \((\hat{\mathfrak{g}}, \pi)\) is called a covering of \( \mathfrak{g} \) if \( \hat{\mathfrak{g}} \) is perfect. A covering \((\hat{\mathfrak{g}}, \pi)\) is a universal central extension of \( \mathfrak{g} \) if for every central extension \((\hat{\mathfrak{g}}', \varphi)\) of \( \mathfrak{g} \) there is a unique homomorphism \( \psi : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}' \) for which \( \varphi \psi = \pi \). In [6], it is proved that every perfect Lie algebra has a universal central extension.

Let \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C} C_L \oplus \mathbb{C} C_{LN} \oplus \mathbb{C} C_N \) be a vector space over the complex field \( \mathbb{C} \) with a basis \( \{L_n, M_n, N_n, Y_{n+\frac{1}{2}}, C_L, C_{LN}, C_N \mid n \in \mathbb{Z}\} \) satisfying the following relations
\[
[L_m, L_n] = (n - m)L_{m+n} + \delta_{m+n,0} \frac{m^3 - m}{12} C_L,
[N_m, N_n] = n\delta_{m+n,0} C_N,
[L_m, N_n] = nN_{m+n} + \delta_{m+n,0} (m^2 - m) C_{LN},
[M_m, M_n] = 0, \quad [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = (m - n)M_{m+n+1},
[L_m, M_n] = nM_{m+n}, \quad [L_m, Y_{n+\frac{1}{2}}] = (n + \frac{1 - m}{2})Y_{m+n+\frac{1}{2}},
[N_m, M_n] = 2M_{m+n}, \quad [N_m, Y_{n+\frac{1}{2}}] = Y_{m+n+\frac{1}{2}}, \quad [M_m, Y_{n+\frac{1}{2}}] = 0,
[\tilde{\mathfrak{g}}, C_L] = [\tilde{\mathfrak{g}}, C_{LN}] = [\tilde{\mathfrak{g}}, C_N] = 0,
\]
for all \( m, n \in \mathbb{Z} \). Denote
\[
H = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} N_n \bigoplus \mathbb{C} C_N, \quad \mathfrak{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \bigoplus \mathbb{C} C_L, \quad H_{\mathfrak{Vir}} = H \bigoplus \mathfrak{Vir} \bigoplus \mathbb{C} C_{LN},
\]
\[
S = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} M_n \bigoplus \mathbb{C} Y_{n+\frac{1}{2}}, \quad \mathcal{H}_S = H \bigoplus S.
\]
They are all Lie subalgebras of \( \hat{\mathfrak{sv}} \), where \( H \) is an infinite-dimensional Heisenberg algebra, \( \mathfrak{Vir} \) is the classical Virasoro algebra, \( H_{\mathfrak{Vir}} \) is the twisted Heisenberg-Virasoro algebra, \( S \) is a two-step nilpotent Lie algebra and \( \mathcal{H}_S \) is the semi-direct product of the Heisenberg algebra \( H \) and \( S \). Then \( \hat{\mathfrak{sv}} \) is the semi-direct product of the twisted Heisenberg-Virasoro algebra \( H_{\mathfrak{Vir}} \) and \( S \), where \( S \) is an ideal of \( \hat{\mathfrak{sv}} \).

**Corollary 3.3.** \( \hat{\mathfrak{sv}} \) is the universal covering algebra of the extended Schrödinger-Virasoro algebra \( \hat{\mathfrak{s}v} \). □

Set \( deg(C_L) = deg(C_{LN}) = deg(C_N) = 0 \). Then there is a \( \frac{1}{2} \mathbb{Z} \)-grading on \( \hat{\mathfrak{sv}} \) by
\[
\hat{\mathfrak{sv}} = \bigoplus_{n \in \mathbb{Z}} \hat{\mathfrak{sv}}_{n\frac{1}{2}} = (\bigoplus \hat{\mathfrak{sv}}_n) \bigoplus (\bigoplus \hat{\mathfrak{sv}}_{n+\frac{1}{2}}),
\]
where
\[
\hat{\mathfrak{sv}}_n = \text{span}\{L_n, M_n, N_n\}, \quad n \in \mathbb{Z} \setminus \{0\},
\]
\[
\hat{\mathfrak{sv}}_0 = \text{span}\{L_0, M_0, N_0, C_L, C_{LN}, C_N\},
\]
\[
\hat{\mathfrak{sv}}_{n+\frac{1}{2}} = \text{span}\{Y_{n+\frac{1}{2}}\}, \quad \forall \ n \in \mathbb{Z}.
\]

**Lemma 3.4.** (cf. \[1\]) If \( g \) is a perfect Lie algebra and \( \hat{g} \) is a universal central extension of \( g \), then every derivation of \( g \) lifts to a derivation of \( \hat{g} \). If \( g \) is centerless, the lift is unique and \( \text{Der}(\hat{g}) \cong \text{Der}(g) \). □

It follows from Lemma 3.4 and Corollary 3.3 that
\[
D\big(C_L\big) = D\big(C_{LN}\big) = D\big(C_N\big) = 0,
\]
for all \( D \in \hat{\mathfrak{sv}} \). Therefore, \( \text{Der}(\hat{\mathfrak{sv}}) = \text{Inn}(\hat{\mathfrak{sv}}) \).

### 4. The Universal Central Extension of \( \hat{\mathfrak{sv}} \) in the Category of Leibniz Algebras

The concept of Leibniz algebra was first introduced by Jean-Louis Loday in \[16\] in his study of the so-called Leibniz homology as a noncommutative analog of Lie algebra homology. A vector space \( \mathcal{L} \) equipped with a \( \mathbb{C} \)-bilinear map \([\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}\) is called a Leibniz algebra if the following Leibniz identity satisfies
\[
[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad \forall \ x, y, z \in \mathcal{L}.
\]
(4.1)

Lie algebras are definitely Leibniz algebras. A Leibniz algebra \( \mathcal{L} \) is a Lie algebra if and only if \([x, x] = 0\) for all \( x \in \mathcal{L} \).
In [16], Jean-Louis Loday and Teimuraz Pirashvili established the concept of universal enveloping algebras of Leibniz algebras and interpreted the Leibniz (co)homology \( HL_* \) (resp. \( HL^* \)) as a Tor-functor (resp. Ext-functor). A bilinear \( \mathbb{C} \)-valued form \( \psi \) on \( \mathcal{L} \) is called a Leibniz 2-cocycle if
\[
\psi(x, [y, z]) = \psi([x, y], z) - \psi([x, z], y), \quad \forall \ x, y, z \in \mathcal{L}.
\] (4.2)
Similar to the 2-cocycle on Lie algebras, a linear function \( f \) on \( \mathcal{L} \) can induce a Leibniz 2-cocycle \( \psi_f \), that is,
\[
\psi_f(x, y) = f([x, y]), \quad \forall \ x, y \in \mathcal{L}.
\]
Such a Leibniz 2-cocycle is called trivial. The one-dimensional Leibniz central extension corresponding to a trivial Leibniz 2-cocycle is also trivial.

In this section, we consider the universal central extension of the extended Schrödinger-Virasoro algebra \( \tilde{sv} \) in the category of Leibniz algebras.

Let \( \mathfrak{g} \) be a Lie algebra. A bilinear form \( f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C} \) is called invariant if
\[
f([x, y], z) = f(x, [y, z]), \quad \forall \ x, y, z \in \mathfrak{g}.
\] (4.3)

**Proposition 4.1.** There is no non-trivial invariant bilinear form on \( \tilde{sv} \).

**Proof.** Let \( f : \tilde{sv} \times \tilde{sv} \to \mathbb{C} \) be an invariant bilinear form on \( \tilde{sv} \).

(1) For \( m \neq 0 \), we have
\[
f(L_m, L_n) = -\frac{1}{m}f([L_m, L_0], L_n) = -\frac{1}{m}f(L_m, [L_0, L_n]) = -\frac{n}{m}f(L_m, L_n),
\]
So
\[
f(L_m, L_n) = 0, \quad m + n \neq 0, m \neq 0.
\]
Similarly, we have
\[
f(L_m, N_n) = 0, \quad f(N_m, N_n) = 0, \quad m + n \neq 0, m \neq 0.
\]
For \( n \neq 0 \), we have
\[
f(L_0, L_n) = \frac{1}{n}f(L_0, [L_0, L_n]) = \frac{1}{n}f([L_0, L_0], L_n) = 0,
\]
\[
f(L_{-n}, L_n) = \frac{1}{3n}f([L_{-2n}, L_n], L_n) = \frac{1}{3n}f(L_{-2n}, [L_n, L_n]) = 0,
\]
Similarly, we can get
\[
f(L_0, N_n) = 0, \quad f(N_0, N_n) = 0, \quad f(L_{-n}, N_n) = 0, \quad f(N_{-n}, N_n) = 0.
\]
On the other hand,
\[
f(L_0, L_0) = \frac{1}{2}f([L_{-1}, L_1], L_0) = \frac{1}{2}f(L_{-1}, [L_1, L_0]) = -\frac{1}{2}f(L_{-1}, L_1) = 0.
\]
\[
f(L_0, N_0) = \frac{1}{2}f([L_{-1}, L_1], N_0) = \frac{1}{2}f(L_{-1}, [L_1, N_0]) = 0.
\]
\[
f(N_0, N_0) = f([L_{-1}, N_0], N_0) = f(L_{-1}, [N_1, N_0]) = 0.
\]
Therefore,
\[ f(L_m, N_n) = 0, \quad f(N_m, N_n) = 0, \quad f(L_m, L_n) = 0, \quad \forall \ m, n \in \mathbb{Z}. \]
Similarly, we obtain
\[ f(L_m, M_n) = 0, \quad f(M_m, M_n) = 0, \quad \forall \ m, n \in \mathbb{Z}. \]

(2) For all \( m, n \in \mathbb{Z} \), we have
\[ f(L_m, Y_{n+\frac{1}{2}}) = \frac{1}{n + \frac{1}{2}} f(L_m, [L_0, Y_{n+\frac{1}{2}}]) = \frac{1}{n + \frac{1}{2}} f([L_m, L_0], Y_{n+\frac{1}{2}}) = -\frac{m}{n + \frac{1}{2}} f(L_m, Y_{n+\frac{1}{2}}). \]
Then \( \frac{m + n + \frac{1}{2}}{n + \frac{1}{2}} f(L_m, Y_{n+\frac{1}{2}}) = 0. \) Obviously, \( f(L_m, Y_{n+\frac{1}{2}}) = 0 \) for all \( m, n \in \mathbb{Z} \).

(3) For all \( m, n \in \mathbb{Z} \), we have
\[ f(N_m, M_n) = \frac{1}{2} f(N_m, [N_0, M_n]) = \frac{1}{2} f([N_m, N_0], M_n) = 0, \]
\[ f(N_m, Y_{n+\frac{1}{2}}) = f(N_m, [N_0, Y_{n+\frac{1}{2}}]) = f([N_m, N_0], Y_{n+\frac{1}{2}}) = 0, \]
\[ f(M_m, Y_{n+\frac{1}{2}}) = \frac{1}{n + \frac{1}{2}} f(M_m, [L_0, Y_{n+\frac{1}{2}}]) = -\frac{1}{n + \frac{1}{2}} f([M_m, Y_{n+\frac{1}{2}}], L_0) = 0, \]
\[ f(Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}) = \frac{1}{m + \frac{1}{2}} f([L_0, Y_{m+\frac{1}{2}}], Y_{n+\frac{1}{2}}) = \frac{1}{m + \frac{1}{2}} f(L_0, [Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}]) \]
\[ = \frac{m - n}{m + \frac{1}{2}} f(L_0, M_{m+n+1}) = 0. \]

\( \square \)

**Remark 4.2.** In fact, it is enough to check that Proposition 4.1 holds for the set of generators \( \{L_{-2}, L_{-1}, L_1, L_2, N_1, Y_{\frac{1}{2}}\} \). By the proof of Proposition 4.1, we can see that the process of the computation is independent of the symmetry of the bilinear form. Similar to the method in section 4 in [9], we can deduce that
\[ HL^2(\tilde{\mathfrak{s}u}, \mathbb{C}) = H^2(\tilde{\mathfrak{s}u}, \mathbb{C}), \]
where \( HL^2(\tilde{\mathfrak{s}u}, \mathbb{C}) \) is the second Leibniz cohomology group of \( \tilde{\mathfrak{s}u} \). That is to say, the universal central extension of \( \tilde{\mathfrak{s}u} \) in the category of Leibniz algebras is the same as that in the category of Lie algebras.

5. The Automorphism Group of \( \tilde{\mathfrak{s}u} \)

Denote by \( Aut(\tilde{\mathfrak{s}u}) \) and \( I \) the automorphism group and the inner automorphism group of \( \tilde{\mathfrak{s}u} \) respectively. Obviously, \( I \) is generated by \( \exp(k \text{ad} M_m + l \text{ad} Y_{n+\frac{1}{2}}) \), \( m, n \in \mathbb{Z}, k, l \in \mathbb{C} \).

For convenience, denote
\[ L = \text{span}\{L_n \mid n \in \mathbb{Z}\}, \quad N = \text{span}\{N_n \mid n \in \mathbb{Z}\}, \]
\[ M = \text{span}\{M_n \mid n \in \mathbb{Z}\}, \quad Y = \text{span}\{Y_{n+\frac{1}{2}} \mid n \in \mathbb{Z}\}. \]

**Lemma 5.1.** Let \( \sigma \in \text{Aut}(\tilde{\mathfrak{u}}) \), then

\[ \sigma(M_n) \in M, \quad \sigma(Y_{n+\frac{1}{2}}) \in M + Y, \quad \sigma(N_n) \in M + Y + N, \]

for all \( n \in \mathbb{Z} \). In particular,

\[ \sigma(N_0) = \sum_{i=p}^{q} a_i M_i + N_0 + \sum_{j=s}^{t} b_j Y_{j+\frac{1}{2}}, \]

for some \( a_i, b_j \in \mathbb{C} \) and \( p, q, s, t \in \mathbb{Z} \).

**Proof.** Let \( I \) be a nontrivial ideal of \( \tilde{\mathfrak{u}} \). Then \( I \) is a \( L_0 \)-module. Since the decomposition of eigenvalue subspace of \( L_0 \) is in concordance with the \( \frac{1}{2} \mathbb{Z} \)-grading of \( \tilde{\mathfrak{u}} \), we have

\[ I = \bigoplus_{n \in \mathbb{Z}} I_{n+\frac{1}{2}} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{u}_n. \]

Hence, there exists some \( n \in \mathbb{Z} \) such that \( aL_n + bM_n + cN_n \in I \) or \( Y_{n+\frac{1}{2}} \in I \), where \( a, b, c \in \mathbb{C} \) and not all zero. If \( aL_n + bM_n + cN_n \in I \), then

\[ [aL_n + bM_n + cN_n, M_0] = 2cM_n \in I, \quad [aL_n + bM_n + cN_n, N_0] = -2bM_n \in I. \]

If \( b = c = 0 \), then \( a \neq 0 \) and \( L_n \in I \). But \([L_n, \tilde{\mathfrak{u}}] = \tilde{\mathfrak{u}} \) for any \( n \in \mathbb{Z} \), then we have \( I = \tilde{\mathfrak{u}} \), a contradiction. So \( b \neq 0 \) or \( c \neq 0 \). Therefore, \( M_n \in I \), and we have \( aL_n + cM_n \in I \). Since \([aL_n + cM_n, N_1] = aN_{n+1} \in I \), we get \( N_{n+1} \in I \) if \( a \neq 0 \).

1. If there exists some \( M_n \in I \), by the fact that \([N_{m-n}, M_m] = 2M_m \) for all \( m \in \mathbb{Z} \), we obtain \( M \subseteq I \).
2. If there exists some \( N_n \in I \) and \( n \neq 0 \), then \( N \subseteq I \) since \([L_{m-n}, N_m] = nN_m \in I \) for all \( m \in \mathbb{Z} \). On the other hand, we have

\[ [N_0, M_m] = 2M_m, \quad [N_0, Y_{m+\frac{1}{2}}] = Y_{m+\frac{1}{2}}. \]

for all \( m \in \mathbb{Z} \). So \( M \subseteq I \), \( Y \subseteq I \), and therefore \( N \oplus M \oplus Y \subseteq I \).

If \( N_0 \in I \), according to the proof above, we have \( M \subseteq I \), \( Y \subseteq I \). In addition, \([L, N_0] = 0 \) and \([N, N_0] = 0 \), so \( \mathbb{C}N_0 \oplus M \oplus Y \subseteq I \).

3. If there exists some \( Y_{n+\frac{1}{2}} \in I \), we have \( Y \subseteq I \) since \([N_{m-n}, Y_{n+\frac{1}{2}}] = Y_{m+\frac{1}{2}} \) for all \( m \in \mathbb{Z} \). Moreover, \([Y_{m+\frac{1}{2}}, Y_{n+\frac{1}{2}}] = mM_{m+1} \) for all \( m \in \mathbb{Z} \) and \([Y_{n+\frac{1}{2}}, Y_{m+\frac{1}{2}}] = 2M_1 \), so \( M \subseteq I \).

Set \( \mathfrak{I}_1 = M, \quad \mathfrak{I}_2 = M \oplus Y, \quad \mathfrak{I}_3 = M \oplus \mathbb{C}N_0 \oplus Y, \quad \mathfrak{I}_4 = M \oplus N \oplus Y \). Then \( I = \mathfrak{I}_k \) for some \( k = 1, 2, 3, 4 \). Obviously, \( \mathfrak{I}_4 \) and \( \mathfrak{I}_2 \) both have infinite-dimensional center \( M \), while the center of \( \mathfrak{I}_3 \) and \( \mathfrak{I}_4 \) are zero, i.e.,

\[ C(\mathfrak{I}_4) = C(\mathfrak{I}_2) = M, \quad C(\mathfrak{I}_3) = C(\mathfrak{I}_4) = 0. \]

For any \( \sigma \in \text{Aut}(\tilde{\mathfrak{u}}) \), \( \sigma(I) \) is still a non-trivial ideal of \( \tilde{\mathfrak{u}} \) and \( \sigma(C(I)) = C(\sigma(I)) \). Then

\[ \sigma(\mathfrak{I}_i) = \mathfrak{I}_j, \quad i, j = 1, 2; \quad \sigma(\mathfrak{I}_k) = \mathfrak{I}_l, \quad k, l = 3, 4. \]
If \( \sigma(\mathfrak{J}_1) = \mathfrak{J}_2 \), then for every \( m \in \mathbb{Z} \), there exists unique \( x_m = \sum a_m M_m \in \mathfrak{J}_1 \) such that \( \sigma(x_m) = Y_{m+\frac{1}{2}} \). Then \( (m - n)M_{m+n+1} = 0 \) for all \( m, n \in \mathbb{Z} \), which is impossible. Therefore,

\[
\sigma(\mathfrak{J}_i) = \mathfrak{J}_i, \quad i = 1, 2.
\]

Moreover, we obtain

\[
\sigma(M_n) \in M, \quad \sigma(Y_{n+\frac{1}{2}}) \in M + Y. \tag{5.1}
\]

Assume that

\[
\sigma(N_0) = \sum a_i M_i + \sum b_j N_j + \sum c_k Y_{k+\frac{1}{2}},
\]

where \( a_i, b_j, c_k \in \mathbb{C} \). According to (5.1), \( \sigma(M_0) \in M \). So there exist some \( f(m) \in \mathbb{C}^* \) such that \( \sigma(M_0) = \sum f(m) M_m \). By \( \sigma[N_0, M_0] = [\sigma(N_0), \sigma(M_0)] \), we get

\[
\sum_m f(m) M_m = \sum_{m,j} b_j f(m) M_{m+j}. \tag{5.2}
\]

Set \( p = \min\{m \in \mathbb{Z} | f(m) \neq 0\} \), \( q = \max\{m \in \mathbb{Z} | f(m) \neq 0\} \). If \( j \neq 0 \), we have

\[
p + j < p \quad \text{if} \quad j < 0; \quad (\text{resp} \; q + j > q \quad \text{if} \quad j > 0).
\]

By (5.2), it is easy to see that \( b_j f(p) = 0 \) (resp. \( b_j f(q) = 0 \)). So \( b_j = 0 \) for all \( j \neq 0 \). Then by (5.2), \( b_0 = 1 \). Therefore,

\[
\sigma(N_0) = \sum a_i M_i + N_0 + \sum c_k Y_{k+\frac{1}{2}}.
\]

This forces that \( \sigma(\mathfrak{J}_k) = \mathfrak{J}_k, \quad k = 3, 4 \). \qed

**Lemma 5.2.** For any \( \sigma \in \text{Aut}(\mathfrak{su}) \), there exist some \( \pi \in \mathcal{I} \) and \( \epsilon \in \{\pm 1\} \) such that

\[
\tilde{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \epsilon \lambda N_{\epsilon n}, \tag{5.3}
\]

\[
\tilde{\sigma}(N_n) = a^n \epsilon N_{\epsilon n}, \tag{5.4}
\]

\[
\tilde{\sigma}(M_n) = \epsilon d^2 a^{n-1} M_{\epsilon(n-2\lambda)}; \tag{5.5}
\]

\[
\tilde{\sigma}(Y_{n+\frac{1}{2}}) = \epsilon d^n a^n Y_{\epsilon(n+\frac{1}{2}-\lambda)}, \tag{5.6}
\]

where \( \tilde{\sigma} = \pi^{-1} \sigma, \lambda \in \mathbb{Z} \) and \( a, d \in \mathbb{C}^* \). Conversely, if \( \tilde{\sigma} \) is a linear operator on \( \tilde{\mathfrak{su}} \) satisfying (5.3)-(5.6) for some \( \epsilon \in \{\pm 1\}, \lambda \in \mathbb{Z} \) and \( a, d \in \mathbb{C}^* \), then \( \tilde{\sigma} \in \text{Aut}(\mathfrak{su}) \).

**Proof.** By Lemma 5.1, for all \( \sigma \in \text{Aut}(\mathfrak{su}) \), \( \sigma(N_0) = \sum_{i=p}^{q} a_i M_i + N_0 + \sum_{j=s}^{t} b_j Y_{j+\frac{1}{2}} \) for some \( a_i, b_j \in \mathbb{C} \) and \( p, q, s, t \in \mathbb{Z} \). Let

\[
\pi = \prod_{j=s}^{t} \exp(-b_j ad Y_{j+\frac{1}{2}}) \prod_{i=p}^{q} \exp\left(-\frac{a_i}{2} ad M_i\right) \prod_{i,j=s}^{t} \exp\left(\frac{j-i}{4} b_i b_j ad M_{i+j+1}\right) \in \mathcal{I},
\]

then we can deduce that \( \sigma(N_0) = \pi(N_0) \), that is,

\[
\pi^{-1} \sigma(N_0) = N_0.
\]
Set \( \bar{\sigma} = \pi^{-1}\sigma \). By \([N_0, \bar{\sigma}(L_m)] = [N_0, \bar{\sigma}(N_m)] = 0 \) and \([N_0, \bar{\sigma}(Y_{m+\frac{1}{2}})] = \bar{\sigma}(Y_{m+\frac{1}{2}}) \) for all \( m \in \mathbb{Z} \), we get
\[
\bar{\sigma}(L_m) \in L + N, \quad \bar{\sigma}(N_m) \in N, \quad \bar{\sigma}(Y_{m+\frac{1}{2}}) \in Y.
\]
For any \( \bar{\sigma} \in \text{Aut}(\mathfrak{g}_\mathbb{R}) \), denote \( \bar{\sigma}|_L = \bar{\sigma}' \). By the automorphisms of the classical Witt algebra, \( \bar{\sigma}'(L_m) = \epsilon a^m L_{\epsilon m} \) for all \( m \in \mathbb{Z} \), where \( a \in \mathbb{C}^* \) and \( \epsilon \in \{\pm 1\} \). Assume that
\[
\bar{\sigma}(L_0) = \epsilon L_0 + \sum \lambda_i N_i, \quad \bar{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \sum \lambda(n_i) N_i, \quad n \neq 0,
\]
\[
\bar{\sigma}(N_n) = a^n \sum \mu(n_j) N_j, \quad \bar{\sigma}(M_n) = a^n \sum f(n_r) M_{n_r}, \quad \bar{\sigma}(Y_{n+\frac{1}{2}}) = a^n \sum h(n_t + \frac{1}{2}) Y_{n_t + \frac{1}{2}},
\]
where each formula is of finite terms and \( \mu(n_j), f(n_r), h(n_t + \frac{1}{2}) \in \mathbb{C}^* \), \( \lambda(i), \lambda(n_i) \in \mathbb{C} \).

From \([\bar{\sigma}(L_0), \bar{\sigma}(M_m)] = m \bar{\sigma}(M_m) \), we have
\[
\sum \epsilon m_r f(m_r) M_{m_r} + 2 \sum \lambda_i f(m_r) M_{i+m_r} = m \sum f(m_r) M_{m_r}.
\]
This forces that \( \lambda_i = 0 \) for \( i \neq 0 \) and \( \epsilon m_r + 2\lambda_0 = m \). So \( m_r = \epsilon(m - 2\lambda_0) \) and
\[
\bar{\sigma}(L_0) = \epsilon L_0 + \lambda_0 N_0, \quad \bar{\sigma}(M_n) = a^n f(\epsilon(n - 2\lambda_0)) M_{\epsilon(n - 2\lambda_0)},
\]
for all \( n \in \mathbb{Z} \). From \([\bar{\sigma}(L_n), \bar{\sigma}(M_0)] = 0 \), we get
\[
\lambda_0 M_{\epsilon n - 2\epsilon \lambda_0} = \sum \lambda(n_i) M_{n_i - 2\epsilon \lambda_0}.
\]
Then \( n_i = \epsilon n \) and \( \lambda(n_i) = \lambda_0 \) for all \( n \in \mathbb{Z} \). Therefore,
\[
\bar{\sigma}(L_n) = a^n \epsilon L_{\epsilon n} + a^n \lambda_0 N_{\epsilon n}, \quad \text{for all } n \in \mathbb{Z}.
\]
Since \([\bar{\sigma}(L_0), \bar{\sigma}(N_n)] = n \bar{\sigma}(N_n) \), we have \( \sum (en_j - n) \mu(n_j) N_{n_j} = 0 \). Obviously, \( n_j = \epsilon n \) and
\[
\bar{\sigma}(N_n) = a^n \mu(\epsilon n) N_{\epsilon n},
\]
for all \( n \in \mathbb{Z} \), where \( \mu(0) = 1 \). Comparing the coefficients of \( Y_{n_t + \frac{1}{2}} \) on the both sides of \([\bar{\sigma}(L_0), \bar{\sigma}(Y_{n_t + \frac{1}{2}})] = (n + \frac{1}{2}) \bar{\sigma}(Y_{n_t + \frac{1}{2}}) \), we obtain \( n_t + \frac{1}{2} = \epsilon(n + \frac{1}{2} - \lambda_0) \), which implies that \( \lambda_0 \in \mathbb{Z} \). So
\[
\bar{\sigma}(Y_{n_t + \frac{1}{2}}) = a^n h(\epsilon(n + \frac{1}{2} - \lambda_0)) Y_{\epsilon(n + \frac{1}{2} - \lambda_0)}, \quad \text{for all } n \in \mathbb{Z}.
\]
By \([\bar{\sigma}(N_n), \bar{\sigma}(M_m)] = 2\bar{\sigma}(M_{n+m}) \), we get
\[
\mu(\epsilon n) f(\epsilon(m - 2\lambda_0)) = f(\epsilon(m + n - 2\lambda_0)).
\]
Letting \( m = 2\lambda_0 \), we obtain
\[
f(\epsilon n) = f(0) \mu(\epsilon n).
\]
By the coefficients of \( Y_{\epsilon(m+n+\frac{1}{2}-\lambda_0)} \) on the both sides of \([\bar{\sigma}(N_m), \bar{\sigma}(Y_{n+\frac{1}{2}})] = \bar{\sigma}(Y_{m+n+\frac{1}{2}}) \), we have
\[
h(\epsilon(m + \frac{1}{2})) = \mu(\epsilon n) h(\frac{\epsilon}{2}).
\]
Similarly, comparing the coefficients of \(N_{\epsilon(m+n)}\) on the both sides of \([\tilde{\sigma}(L_n), \tilde{\sigma}(N_m)] = m\tilde{\sigma}(N_{m+n})\), we have \(m\mu(\epsilon m) = m\mu(\epsilon m + n)\) for all \(m, n \in \mathbb{Z}\). Then

\[
\mu(\epsilon m) = \mu(0) = 1,
\]

for all \(m \in \mathbb{Z}\). Therefore,

\[
f(\epsilon m) = f(0), \quad h(\epsilon (m + \frac{1}{2})) = h(\frac{\epsilon}{2}).
\]

Finally, we deduce that \(\epsilon h(\frac{\epsilon}{2})^2 = af(0)\) by comparing the coefficient of \(M_{\epsilon(m+n+1-2\lambda_0)}\) on the both sides of \([\tilde{\sigma}(Y_{m+\frac{1}{2}}), \tilde{\sigma}(Y_{n+\frac{1}{2}})] = (m - n)\tilde{\sigma}(M_{m+n+1})\). Let \(d = h(\frac{\epsilon}{2})\), then \(f(0) = \epsilon a^{-1}d^2\). Therefore,

\[
\begin{align*}
\tilde{\sigma}(L_n) &= a^n\epsilon Lcn + a^n\lambda_0 Ncn, \quad \tilde{\sigma}(N_n) = a^n Ncn, \\
\tilde{\sigma}(M_n) &= \epsilon a^2 a^{n-1} M_{(n-2\lambda_0)}, \quad \tilde{\sigma}(Y_{n+\frac{1}{2}}) = da^n Y_{(n+\frac{1}{2}-\lambda_0)}.
\end{align*}
\]

It is easy to check the converse part of the theorem.

\[\Box\]

Denote by \(\tilde{\sigma}(\epsilon, \lambda, a, d)\) the automorphism of \(\tilde{\mathfrak{sv}}\) satisfying (5.3)-(5.6), then

\[
\tilde{\sigma}(\epsilon_1, \lambda_1, a_1, d_1)\tilde{\sigma}(\epsilon_2, \lambda_2, a_2, d_2) = \tilde{\sigma}(\epsilon_1\epsilon_2, \lambda_1 + \lambda_2, a_1^2a_2, d_1d_2a_1^{\frac{\lambda_2-1}{2}}a_2^{\lambda_1-1}) \quad (5.7)
\]

and \(\tilde{\sigma}(\epsilon_1, \lambda_1, a_1, d_1) = \tilde{\sigma}(\epsilon_2, \lambda_2, a_2, d_2)\) if and only if \(\epsilon_1 = \epsilon_2, \lambda_1 = \lambda_2, a_1 = a_2, d_1 = d_2\).

Let

\[
\pi_{\epsilon} = \tilde{\sigma}(\epsilon, 0, 1, 1), \quad \tilde{\sigma}_d = (1, \lambda, 1, 1), \quad \tilde{\sigma}_{a,d} = (1, 0, a, d)
\]

and

\[
a = \{\pi_{\epsilon} \mid \epsilon = \pm 1\}, \quad t = \{\tilde{\sigma}_\lambda \mid \lambda \in \mathbb{Z}\}, \quad b = \{\tilde{\sigma}_{a,d} \mid a, d \in \mathbb{C}^*\}.
\]

By (5.7), we have the following relations:

\[
\tilde{\sigma}(\epsilon, \lambda, a, d) = \tilde{\sigma}(\epsilon, 0, 1, 1)\tilde{\sigma}(1, \lambda, 1, 1)\tilde{\sigma}(1, 0, a, d) \in atb,
\]

\[
\tilde{\sigma}(\epsilon, \lambda, a, d)^{-1} = \tilde{\sigma}(\epsilon, -\epsilon\lambda, a^{-\epsilon}, d^{-1}a^{\frac{\lambda^2}{2}+\lambda}),
\]

\[
\pi_{\epsilon_1}\pi_{\epsilon_2} = \pi_{\epsilon_1\epsilon_2}, \quad \tilde{\sigma}_{\lambda_1}\tilde{\sigma}_{\lambda_2} = \tilde{\sigma}_{\lambda_1+\lambda_2}, \quad \tilde{\sigma}_{a_1,d_1}\tilde{\sigma}_{a_2,d_2} = \tilde{\sigma}_{a_1a_2,d_1d_2},
\]

\[
\tilde{\sigma}_{-\epsilon}^{-1}\tilde{\sigma}_{\lambda}\tilde{\sigma}_{d} = \tilde{\sigma}_{\lambda}, \quad \tilde{\sigma}_{-\epsilon}^{-1}\tilde{\sigma}_{a,d}\tilde{\sigma}_{\epsilon} = \tilde{\sigma}_{a^*da^{-\frac{\epsilon^2}{2}}}, \quad \tilde{\sigma}_{-\epsilon}^{-1}\tilde{\sigma}_{a,d}\tilde{\sigma}_{\lambda} = \tilde{\sigma}_{a,da^{-\lambda}}.
\]

Hence, the following lemma holds.

**Lemma 5.3.** \(a, t\) and \(b\) are all subgroups of \(\text{Aut}(\tilde{\mathfrak{sv}})\) and

\[\text{Aut}(\tilde{\mathfrak{sv}}) = \mathcal{I} \rtimes ((a \ltimes t) \ltimes b),\]

where \(a \cong \mathbb{Z}_2 = \{\pm 1\}, t \cong \mathbb{Z}, b \cong \mathbb{C}^* \times \mathbb{C}^*\). □
Let $\mathbb{C}^\infty = \{ (a_i)_{i \in \mathbb{Z}} \mid a_i \in \mathbb{C}, \text{ all but a finite of the } a_i \text{ are zero } \}$, $\mathcal{I}_C$ a subgroup of $\mathcal{I}$ generated by $\{ \exp(kadM_n) \mid n \in \mathbb{Z}, k \in \mathbb{C} \}$ and $\mathcal{T} = \mathcal{I}/\mathcal{I}_C$ the quotient group of $\mathcal{I}$. Then $\mathbb{C}^\infty$ is an abelian group and $\mathcal{I}_C$ is an abelian normal subgroup of $\mathcal{I}$. As a matter of fact, $\mathcal{I}_C$ is the center of the group $\mathcal{I}$.

Note $(\alpha \exp Y_{i+\frac{1}{2}})^2 = \exp(2\alpha \exp Y_{i+\frac{1}{2}})$ is an isomorphism of group.


declare (ad $m$) is an abelian normal subgroup of $\mathcal{I}_C$.

Furthermore, we get

\[
\exp(b_m \exp Y_{i+1/2}) = 1 + \beta \exp Y_{i+1/2} + \frac{1}{2} \beta^2 (ad Y_{i+1/2})^2,
\]

\[
\exp(\alpha \exp Y_{i+1/2}) = \exp(\beta \exp Y_{i+1/2}) + \alpha \exp Y_{i+1/2},
\]

for all $\alpha, \beta \in \mathbb{C}$. Therefore, we get

\[
\exp(b_m \exp Y_{i+1/2}) \exp(b_m \exp Y_{i+1/2}) \cdots \exp(b_m \exp Y_{i+1/2}) = 1 + \sum_{k=1}^{t} b_m \exp Y_{i+1/2} + \frac{1}{2} \sum_{k=1}^{t} \sum_{1 \leq i \leq j \leq t} b_m \exp Y_{i+1/2} \exp Y_{j+1/2}
\]

\[
= \exp(\sum_{k=1}^{t} b_m \exp Y_{i+1/2}) + \frac{1}{2} \sum_{1 \leq i \leq j \leq t} b_m \exp Y_{i+1/2} \exp Y_{j+1/2} - \exp(\sum_{k=1}^{t} b_m \exp Y_{i+1/2}) \exp(\sum_{k=1}^{t} b_m \exp Y_{i+1/2}),
\]

\[
= \exp(\sum_{k=1}^{t} b_m \exp Y_{i+1/2}) \exp(\sum_{1 \leq i \leq j \leq t} \frac{m_i - m_j}{2} b_m \exp Y_{i+1/2} \exp Y_{i+1/2}),
\]

for all $m_k \in \mathbb{Z}, b_m \in \mathbb{C}, 1 \leq k \leq t$. Therefore,

\[
\exp(b_m \exp Y_{i+1/2}) \exp(b_m \exp Y_{i+1/2}) \cdots \exp(b_m \exp Y_{i+1/2}) \mathcal{I}_C = \exp(\sum_{k=1}^{t} b_m \exp Y_{i+1/2}) \mathcal{I}_C.
\]

(5.8)

**Lemma 5.4.** $\mathcal{I}_C$ and $\mathcal{T}$ are isomorphic to $\mathbb{C}^\infty$.

**Proof.** Define $f : \mathcal{I}_C \rightarrow \mathbb{C}^\infty$ by

\[
f(\prod_{i=1}^{s} \exp(\alpha_{k_i} \exp Y_{k_i})) = (a_p)_{p \in \mathbb{Z}},
\]

where $a_{k_i} = a_{k_i}$ for $1 \leq i \leq s$, and the others are zero, $k_i \in \mathbb{Z}$ and $k_1 < k_2 < \cdots < k_s$. Since every element of $\mathcal{I}_C$ has the unique form of $\prod_{i=1}^{s} \exp(\alpha_{k_i} \exp Y_{k_i})$, it is easy to check that $f$ is an isomorphism of group.

Similar to the proof above, we have $\mathcal{T} \cong \mathbb{C}^\infty$ via (5.8). □
Theorem 5.5. \( Aut(\widetilde{sv}) = (I_C \times \mathbb{Z}) \rtimes ((a \rtimes t) \rtimes b) \cong (\mathbb{C}^\infty \rtimes \mathbb{C}^\infty) \rtimes ((\mathbb{Z}_2 \rtimes \mathbb{Z}) \rtimes (\mathbb{C}^* \rtimes \mathbb{C}^*)) \).

Lemma 5.6. (cf. [21]) Let \( g \) be a perfect Lie algebra and let \( \widehat{g} \) be its universal covering algebra of \( g \). Then every automorphism \( \sigma \) of \( g \) admits a unique extension \( \widetilde{\sigma} \) to an automorphism of \( \widehat{g} \). Furthermore, the map \( \sigma \mapsto \widetilde{\sigma} \) is a group monomorphism.

We will use Lemma 5.6 to obtain all the automorphisms of \( \widehat{sv} \) from those of \( \widetilde{sv} \).

For a perfect Lie algebra \( g \), its universal covering algebra is constructed as follows in [21]. Let \( V = \Lambda^2 g/J \), where

\[ J = \text{span}\{ x \wedge [y, z] + y \wedge [z, x] + z \wedge [x, y] \mid x, y, z \in g \} \]

is a subspace of \( \Lambda^2 g \). Then there is a natural Lie algebra structure in the space \( \tilde{g} = g \oplus V \) with the following bracket

\[ [x + u, y + v] = [x, y] + x \wedge y, \]

for all \( x, y \in g, u, v \in V \), where \( x \wedge y \) is the image of \( x \wedge y \) in \( V \) under the canonical morphism \( \Lambda^2 g \rightarrow V \). Then the derived algebra \( \widehat{g} = [\tilde{g}, \tilde{g}] \) of \( \tilde{g} \) is the universal central extension of \( g \). In fact, given \( x \in g \) there exists \( c \in V \) such that \( x + c \in \widehat{g} \). Then the canonical map \( \widehat{g} \rightarrow g \) is onto with kernel \( c \subset V \) and the resulting central extension

\[ \{0\} \rightarrow c \rightarrow \widehat{g} \rightarrow g \rightarrow \{0\} \]

of \( g \) is universal in the sense that there exists a unique morphism from it into any other given central extension of \( g \).

For any \( \theta \in Aut(g) \), \( \theta \) induces an automorphism \( \theta_V \) of \( V \) via

\[ \theta_V(x \wedge y) = \theta(x) \wedge \theta(y), \]

for all \( x, y \in g \). Obviously, \( \theta \) extends to an automorphism \( \theta_c \) of \( \tilde{g} \) by

\[ \theta_c(x + v) = \theta(x) + \theta_V(v), \]

for all \( x \in g, v \in V \). By restriction, \( \theta_c \) induces an automorphism \( \tilde{\theta} \) of \( \widehat{g} \).

In the following section, we will describe the automorphism group of the universal central extension of \( \widehat{sv} \) using the above method. Firstly, we have the following lemmas.
Lemma 5.7. In $V = \Lambda^2(\mathfrak{s}\mathfrak{u})/J$, we have the following relations for all $m, n \in \mathbb{Z}$:

\[
L_m \vee L_n = \frac{n - m}{m + n}L_0 \vee L_{m+n}, \quad m + n \neq 0;
\]

\[
L_m \vee L_{-m} = \frac{m^3 - m}{6}L_2 \vee L_{-2};
\]

\[
L_m \vee N_n = \frac{n}{m + n}L_0 \vee N_{m+n}, \quad m + n \neq 0;
\]

\[
L_m \vee N_{-m} = \frac{m^2 + m}{2} (L_1 \vee N_{-1} + L_{-1} \vee N_1) - mL_{-1} \vee N_1;
\]

\[
L_m \vee M_n = \frac{n}{2}N_0 \vee M_{m+n}; \quad L_m \vee Y_{n+\frac{1}{2}} = (n + \frac{1 - m}{2})N_0 \vee Y_{m+n+\frac{1}{2}};
\]

\[
N_m \vee N_n = m\delta_{m+n,0}N_1 \vee N_{-1}; \quad N_m \vee M_n = N_0 \vee M_{m+n}; \quad N_m \vee Y_{n+\frac{1}{2}} = N_0 \vee Y_{m+n+\frac{1}{2}};
\]

\[
M_m \vee M_n = M_m \vee Y_{n+\frac{1}{2}} = 0; \quad Y_{m+\frac{1}{2}} \vee Y_{n+\frac{1}{2}} = \frac{m - n}{2}N_0 \vee M_{m+n+1}.
\]

\[\square\]

Using Lemma 5.7, we have the following result.

Lemma 5.8. The universal central extension of $\mathfrak{s}\mathfrak{u}$, denoted by $\widehat{\mathfrak{s}\mathfrak{u}}$, has a basis \( \{L'_n, M'_n, N'_n, Y''_{n+\frac{1}{2}}, C_L, C_{LN}, C_N \mid n \in \mathbb{Z} \} \) with the following products:

\[
[L'_m, L'_n] = (n - m)L'_{m+n} + \delta_{m+n,0}\frac{m^3 - m}{12}C_L,
\]

\[
[N'_m, N'_n] = n\delta_{m+n,0}C_N,
\]

\[
[L'_m, N'_n] = nN'_{m+n} + \delta_{m+n,0}(n^2 - n)C_{LN},
\]

\[
[M'_m, M'_n] = 0; \quad [Y''_{m+\frac{1}{2}}, Y''_{n+\frac{1}{2}}] = (m - n)M'_{m+n+1},
\]

\[
[L'_m, M'_n] = nM'_{m+n}; \quad [L'_m, Y''_{n+\frac{1}{2}}] = (n + \frac{1 - m}{2})Y''_{m+n+\frac{1}{2}},
\]

\[
[N'_m, M'_n] = 2M'_{m+n}; \quad [N'_m, Y''_{n+\frac{1}{2}}] = Y''_{m+n+\frac{1}{2}}; \quad [M'_m, Y''_{n+\frac{1}{2}}] = 0,
\]

\[
[\mathfrak{s}\mathfrak{u}, C_L] = [\mathfrak{s}\mathfrak{u}, C_{LN}] = [\mathfrak{s}\mathfrak{u}, C_N] = 0,
\]

where

\[
L'_0 = L_0, \quad N'_0 = N_0 + L_{-1} \vee N_1;
\]

\[
L'_m = L_m + \frac{1}{m}L_0 \vee L_m, \quad N'_m = N_m + \frac{1}{m}L_0 \vee N_m, \quad m \neq 0;
\]

\[
M'_m = M_m + \frac{1}{2}N_0 \vee M_n, \quad Y''_{n+\frac{1}{2}} = Y''_{n+\frac{1}{2}} + N_0 \vee Y''_{n+\frac{1}{2}}, \quad n \in \mathbb{Z};
\]

\[
C_L = 2L_2 \vee L_{-2}; \quad C_{LN} = \frac{1}{2}(L_1 \vee N_{-1} + L_{-1} \vee N_1), \quad C_N = N_{-1} \vee N_1.
\]

\[\square\]
Lemma 5.9. For any $\tilde{\theta} \in \text{Aut}(\hat{\mathfrak{sv}})/\mathcal{I}$, we have

$$
\tilde{\theta}(L'_n) = a^n \epsilon L'_{en} + a^n \lambda N'_{en} - \lambda \delta_{n,0} C_{LN} + \frac{\epsilon}{2} \lambda^2 \delta_{n,0} C_N,
$$  (5.9)

$$
\tilde{\theta}(N'_n) = a^n N'_{en} + (\epsilon - 1) \delta_{n,0} C_{LN} + \epsilon \lambda \delta_{n,0} C_N,
$$  (5.10)

$$
\tilde{\theta}(M'_n) = \epsilon d^2 a^{n-1} M'_{e(n-2\lambda)},
$$  (5.11)

$$
\tilde{\theta}(Y'_{n+\frac{1}{2}}) = d a^n Y'_{e(n+\frac{1}{2}-\lambda)},
$$  (5.12)

$$
\tilde{\theta}(C_L) = \epsilon C_L,
$$  (5.13)

for all $n \in \mathbb{Z}$, where $\epsilon \in \{\pm 1\}$, $\lambda \in \mathbb{Z}$, $a, d \in \mathbb{C}^*$. Conversely, if $\tilde{\theta}$ is a linear operator on $\hat{\mathfrak{sv}}$ satisfying (5.9)-(5.13) for some $\epsilon \in \{\pm 1\}$, $\lambda \in \mathbb{Z}$, $a, d \in \mathbb{C}^*$, then $\tilde{\theta} \in \text{Aut}(\hat{\mathfrak{sv}})$.

From the above lemmas and Theorem 5.5, we obtain the last main theorem.

**Theorem 5.10.** $\text{Aut}(\tilde{\mathfrak{sv}}) \cong \text{Aut}(\hat{\mathfrak{sv}})$.

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