A Geometric Interpretation of the $\chi_y$ Genus on Hyper-Kähler Manifolds

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Abstract

The group $SL(2)$ acts on the space of cohomology groups of any hyper-Kähler manifold $X$. The $\chi_y$ genus of a hyper-Kähler $X$ is shown to have a geometric interpretation as the super trace of an element of $SL(2)$. As a by product one learns that the generalized Casson invariant for a mapping torus is essentially the $\chi_y$ genus.

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Contents

1 Introduction 1

2 The SL(2) Action on X 3

3 Proof of Theorem 1.1 3

References 4

1 Introduction

The $\chi_y$ genus of Hirzebruch is a very interesting and rather powerful invariant. There are three significant values for $y$. At $y = -1$ the $\chi_y$ genus is the Euler characteristic, at $y = 0$ it is the Todd genus while at $y = 1$ it is the signature. There seems to be, however, no geometric understanding of the genus away from these preferred values of $y$.

In this short note, I prove that for (compact) hyper-Kähler manifolds, there is, in fact, quite a clear geometric meaning to the genus.

For hyper-Kähler manifolds there is a natural $SL(2)$ action, associated with the holomorphic 2-form, on the cohomology groups $\bigoplus_p H^q(X, \Omega^p_X)$ which preserves $q$ and shifts $p$ by even integers. This means that $(-1)^{q+p}$ is preserved. One can, therefore, take the graded trace of an $SL(2)$ element, with the grading given by $(-1)^{p+q}$. Denote the graded trace of $U \in SL(2)$ by $\text{STr} U$.

The geometric meaning of the $\chi_y$ genus for hyper-Kähler $X$ is the content of the following

**Theorem 1.1** Let $X$ be an irreducible compact hyper-Kähler manifold of real dimension $4n$. Let $U \in SL(2)$ and $y$ an eigenvalue of $U$, in the two dimensional representation, then

$$\text{STr} U = \frac{\chi - y}{y^n}. \quad (1.1)$$

**Remarks:**

1) Note that, since $h_p^{(p,q)} = h^{(2n-p,q)}$, the right hand side is invariant under $y \to 1/y$ so that it does not depend on which eigenvalue one picks.

2) Once one expects that a result of this kind is true the proof turns out to be embarrassingly easy.

The motivation for this result comes from the study of 3-manifold invariants. Rozansky and Witten [RW] indicated how, given a hyper-Kähler manifold $X$, one could associate to the Mapping Torus $T_U$, the invariant $\text{STr} U$. In [I], I showed that one could perform
the associated path integral. The solution found there is, in fact, the Riemann-Roch formula for the \( \chi_y \) genus divided by \( y^n \). This motivated the above theorem, which can be proven without recourse to physics. However, one can now read the derivation in [1] as a path integral proof of the Riemann-Roch formula for the \( \chi_y \) genus.

That path integral calculation of \( \text{STr} \, U \) gave,

\[
\int_X \text{Todd} \left( TX_C \right) \text{Det} \left( U \otimes I - I \otimes e^R \right)^{1/2}.
\]  

(1.2)

Which can be re-written as

\[
\int_X \text{Todd} \left( TX_C \right) \prod_{i=1}^n \left( t - 2 \cosh x_i \right),
\]  

(1.3)

where \( t \) is the character of \( U \) in the 2-dimensional representation. The \( \chi_y \) genus is given by Riemann-Roch as [NR]

\[
\chi_{-y}(X) = \int_X \text{Todd} \left( TX_C \right) \prod_{i=1}^{2n} \left( 1 - ye^{-x_i} \right),
\]  

(1.4)

but since \( X \) is hyper-Kähler one has that \( x_{i+n} = -x_i \) for \( i \leq n \). This means that

\[
\chi_{-y}(X) = \int_X \text{Todd} \left( TX_C \right) \prod_{i=1}^n \left( (1 + y^2) - 2y \cosh(x_i) \right),
\]  

(1.5)

so that this suggests (1.1) on setting \( ty = 1 + y^2 \).

Consequently we have, in the notation of [T],

Corollary 1.2 The Rozansky-Witten invariant \( Z_{RW}^X[T_U] = \chi_{-y}/y^n \), for \( U \in SL(2, \mathbb{Z}) \).

Further Remarks:

1) The essential feature used here is the \( SL(2) \) action that is made available by the holomorphic 2-form. Hence this is not the same as thinking of \( X \) as a Kähler manifold and making use of the usual \( SL(2) \) action that comes from the symplectic 2-form (Lefschetz decomposition).

2) There is a rather more general formula that was suggested by the work of [RW]. If one considers a “mapping Riemann surface”, for a Riemann surface, \( \Sigma \), of genus \( g \), then the Rozansky-Witten invariant \( Z_{RW}^X[\Sigma_U] = \text{STr} \, U \) where \( U \in \text{Sp}(g) \) and this group acts on \( \bigoplus H^q \left( X, (\Omega_X^*)^\otimes g \right) \). In [T] a Riemann-Roch formula for this super trace was given which looks like a Riemann-Roch formula for a generalized \( \chi_y \) genus. That suggests that the corresponding generalized \( \chi_y \) can be rigorously shown to be the super trace. This has important implications for 3-manifold invariants.

3) Similar, though not identical, path integral formulae are available for general holomorphic symplectic manifolds.

4) Justin Sawon [S] has made use of the weight system in [RW] in an ingenious way to get constraints on the Chern numbers of \( X \).
2 The $\text{SL}(2)$ Action on $X$

The $SL(2, \mathbb{C})$ action on the cohomology groups of $X$, that we are interested in, is perhaps best explained at the level of the Lie algebra, $\text{Lie} \ SL(2) := sl(2)$. Let $L_\epsilon : H^q(X, \Omega^p_X) \to H^q\left(X, \Omega^{p+2}_X\right)$ be the map given by the cup-product with the holomorphic 2-form $\epsilon$. Let $\iota_\epsilon : H^q(X, \Omega^p_X) \to H^q\left(X, \Omega^{p-2}_X\right)$ be contraction with respect to $\epsilon$. To fix conventions we note that in local holomorphic coordinates if $\omega \in \Omega^{(p,q)}(X)$, then, suppressing the anti-holomorphic factors, (the Einstein summation convention is in force)

$$\omega = \omega_{I_1,\ldots,I_p} dz^{I_p} \wedge \cdots \wedge dz^{I_1}, \quad (2.1)$$

and

$$\iota_\epsilon \omega = \frac{p(p-1)}{2} \omega_{I_1,I_2,I_3,\ldots,I_p} \epsilon^{I_1I_2} dz^{I_3} \wedge \cdots \wedge dz^{I_p}. \quad (2.2)$$

The algebra satisfied by these operators is, by a straightforward computation,

$$[\iota_\epsilon, L_\epsilon] = (n-p) \quad (2.3)$$

understood as a map $H^q(X, \Omega^p_X) \to H^q(X, \Omega^p_X)$. The generators of $sl(2)$ are then realized as

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \sim L_\epsilon \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \sim \iota_\epsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim (n-p). \quad (2.4)$$

The following is taken from the survey by Huybrechts [H] (but see also the original work by Fujiki [F]). Let,

$$H^q(X, \Omega^p_X) _\epsilon := \ker L_{\epsilon}^{n-p+1}, \quad (2.5)$$

then the Lefschetz decomposition theorem tells us that

$$H^q(X, \Omega^p_X) \bigoplus_{(p-l) \geq \max(p-n,0)} L_{\epsilon}^{p-l} H^q(X, \Omega^{2l-p}_X) _\epsilon. \quad (2.6)$$

One thinks of $L_\epsilon$ as a raising operator, and the $H^q(X, \Omega^p_X) _\epsilon$, for $0 \leq p \leq n$, are the highest weight vectors of the $n-p+1$ dimensional irreducible representations of $SL(2, \mathbb{C})$. One also has, by a straightforward count, that

$$\dim \mathbb{R} H^q(X, \Omega^p_X) _\epsilon := h^{(p,q)}_\epsilon = h^{(p,q)} - h^{(p-2,q)}. \quad (2.7)$$

3 Proof of Theorem 1.1

The proof is by direct computation.
Let $t_r$ be the character of $U$ in the $r$ dimensional irreducible representation of $SL(2,\mathbb{C})$ and set $t_2 = t$. Note that $t_1 = 1$, and I use the convention that $t_r = 0$ for $r \leq 0$, as well as $h^{(p,q)} = 0$ if $p < 0$. Then

$$\text{STr } U = \sum_{q=0}^{2n} \sum_{p=0}^{n} (-1)^{p+q} t_{n-p+1} h^{(p,q)}. \quad (3.1)$$

One can re-write this expression as

$$\text{STr } U = \sum_{q=0}^{2n} \sum_{p=0}^{n} (-1)^{p+q} h^{(p,q)} (t_{n-p+1} - t_{n-p-1}) . \quad (3.2)$$

Now notice that, on making use of Serre duality, which implies that $h^{(p,q)} = h^{(2n-p,q)}$, that the $\chi_y$ genus satisfies,

$$\frac{\chi_y}{y^n} = \sum_{q=0}^{2n} \sum_{p=0}^{n-1} (-1)^{p+q} h^{(p,q)} (y^{p-n} + y^{n-p}) + \sum_{q=0}^{2n} (-1)^{q} h^{(n,q)}. \quad (3.3)$$

A comparison of (3.2) and (3.3) shows us that they agree if we can set

$$t_{r+1} - t_{r-1} = y^r + y^{-r} \quad r > 0. \quad (3.4)$$

For $r = 1$ this reads as

$$ty = y^2 + 1, \quad (3.5)$$

which is simply the characteristic polynomial for the two-dimensional representation of $U$, where $y$ is an eigenvalue and $t$ is the trace. We make this identification, then (3.4) is a standard relationship between characters and eigenvalues for $SL(2)$.

\[\square\]

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