On The Sub-Mixed Fractional Brownian Motion

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Abstract

Let \( \{S^H_t, t \geq 0\} \) be a linear combination of a Brownian motion and of an independent sub-fractional Brownian motion with Hurst index \( 0 < H < 1 \). Its main properties are studied and it is shown that \( S^H \) can be considered as an intermediate process between a sub-fractional Brownian motion and a mixed fractional Brownian motion. Finally, we determine the values of \( H \) for which \( S^H \) is not a semi-martingale.

1 Introduction

Let \( \{B^H_t, t \in \mathbb{R}\} \) be a fractional Brownian motion (fBm) with Hurst index \( 0 < H < 1 \), i.e. a centered Gaussian process with stationary increments satisfying \( B^H_0 = 0 \), with probability 1, and \( \mathbb{E}(B^H_t)^2 = |t|^{2H}, t \in \mathbb{R} \). We obviously have for any real numbers \( t \) and \( s \)

\[
\text{cov}
\left
\begin{array}{c}
B^H_t \\
B^H_s
\end{array}
\right
= \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).
\]

Consider \( \{B_t, t \in \mathbb{R}\} \) an independent Brownian motion (Bm) and \((a, b)\) two real numbers such that \((a, b) \neq (0, 0)\).

The mixed-fractional Brownian motion (mfBm) is an extension of a Bm and a fBm. It was introduced in \cite{3} in order to solve some problems in mathematical finance, such as modelling some arbitrage-free and complete markets. The mfBm \( M^H = \{M^H_t(a, b); t \geq 0\} = \{M^H_t; t \geq 0\} \) of parameters \( a, b \) and \( H \) is defined as follows:

\[
\forall t \in \mathbb{R}_+, \quad M^H_t = M^H_t(a, b) = a B_t + b B^H_t.
\]

We refer also to \cite{5} and \cite{12} for further information on this process. Let us recall some of its main properties.

Lemma 1 The mfBm \( (M^H_t(a, b))_{t \in \mathbb{R}_+} \) satisfies the following properties:
• $M^H$ is a centered Gaussian process.

• $∀s ∈ \mathbb{R}_+, ∀t ∈ \mathbb{R}_+$,

\[
Cov \left( M^H_t(a, b), M^H_s(a, b) \right) = a^2(t ∧ s) + \frac{b^2}{2} \left( t^{2H} + s^{2H} - | t - s |^{2H} \right),
\]

where $t ∧ s = \frac{1}{2}(t + s - | t - s |)$.

• The increments of the mfBm are stationary.

In [2], the authors suggested a second extension of a Bm, called the sub-fractional Brownian motion (sfBm), that preserves most of the properties of the fBm, but not the stationarity of the increments. It is the stochastic process $\xi^H = \{\xi^H_t; t ≥ 0\}$, defined by:

\[
(1.2) \quad ∀t ∈ \mathbb{R}_+, \quad \xi^H_t = \frac{B^H_t + B^H_{-t}}{\sqrt{2}},
\]

This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition (see [2]). Let us state some results on the sfBm.

**Lemma 2** The sfBm $(\xi^H_t)_{t ∈ \mathbb{R}_+}$ satisfies the following properties:

• $\xi^H$ is a centered Gaussian process.

• $∀s ∈ \mathbb{R}_+, ∀t ∈ \mathbb{R}_+$,

\[
Cov \left( \xi^H_t, \xi^H_s \right) = s^{2H} + t^{2H} - \frac{1}{2} (s + t)^{2H} + | s - t |^{2H}.
\]

• The increments of the smfBm are not stationary.

We can easily remark that, when $H = 1/2$, $\xi^{1/2}$ is a Bm.

We refer to [2, 6, 11] for further information on this process.

In the spirit of [2] and [12], we introduce a new process, that we will call the sub-mixed fractional Brownian motion (smfBm). More precisely, the smfBm of parameters $a, b$ and $H$, is a process $S^H = \{S^H_t(a, b); t ≥ 0\} = \{S^H_t; t ≥ 0\}$, defined by:

\[
(1.3) \quad ∀t ∈ \mathbb{R}_+, \quad S^H_t(a, b) = \frac{a (B_t + B_{-t}) + b (B^H_t + B^H_{-t})}{\sqrt{2}} = a \xi_t + b \xi^H_t,
\]

where $\xi$ is a Bm, obviously independent of $\xi^H$.

When $a = 0$ and $b = 1$, $S^H = \xi^H$ is a sfBm. When $a = 1$ and $b = 0$, $S^H = \xi$ is a Bm.
So the smfBm is clearly an extension of the sfBm and the Bm. This is the flavor of this process. We will show first that it has the same properties as the sfBm. Then, we will prove that it has also some of the main properties of the mfBm, but that its increments are not stationary; they are more weakly correlated on non-overlapping intervals. Hence $S^H$ may be considered as being intermediate between the sfBm and the mfBm. This is why we call it the smfBm.

The aim of this paper is to study on one hand some key properties of the smfBm and on the other hand its martingale properties. The motivation of the authors is to measure the consequences of the lack of increments stationarity.

In section 2, the main properties of the smfBm are studied, namely:

- the mixed-self-similarity property (see [12]),
- the non Markovian property,
- the increments non stationarity property,
- the correlation coefficient and the influence of the parameters $a$ and $b$ on it,
- the comparison between the mfBm and the smfBm covariance properties.

Finally it is shown in section 3 that the smfBm is a semi-martingale if and only if

$$b = 0 \quad \text{or} \quad H \in \{1/2\} \cup ]3/4, 1[.$$  

2 Main properties

2.1 Basic properties

The following lemmas describe the basic properties of the smfBm.

**Lemma 3** The smfBm $(S^H_t(a, b))_{t \in \mathbb{R}_+}$ satisfies the following properties:

- $S^H$ is a centered Gaussian process.

- $\forall s \in \mathbb{R}_+, \forall t \in \mathbb{R}_+$,

$$\text{Cov} \left( S^H_t(a, b), S^H_s(a, b) \right) = a^2 \ (s \land t) \tag{2.1}$$

+ $b^2 \left( t^{2H} + s^{2H} - \frac{1}{2} \left( (s + t)^{2H} + | t - s |^{2H} \right) \right).$

- $\forall t \in \mathbb{R}_+$,

$$\mathbb{E} \left( (S^H_t(a, b))^2 \right) = a^2 t + b^2 \left( (2 - 2^{2H-1}) \ t^{2H} \right). \tag{2.2}$$
Proof. It is a direct consequence of lemma 2.

Notation. Let \((X_t)_{t \in \mathbb{R}^+}\) and \((Y_t)_{t \in \mathbb{R}^+}\) be two processes defined on the same probability space \((\Omega, F, \mathbb{P})\). The notation \(\{X_t\} \stackrel{\Delta}{=} \{Y_t\}\) will mean that \((X_t)_{t \in \mathbb{R}^+}\) and \((Y_t)_{t \in \mathbb{R}^+}\) have the same law.

Let us check the mixed-self-similarity property of the smfBm, which was introduced in \([12]\) in the mfBm case.

**Lemma 4** For any \(h > 0\), \(\{S^H_{ht}(a,b)\} \stackrel{\Delta}{=} \left\{S^H_t \left(ah^{1/2}, bh^H \right) \right\}\).

**Proof.** For fixed \(h > 0\), the processes \(\{S^H_{ht}(a,b)\}\), and \(\left\{S^H_t \left(ah^{1/2}, bh^H \right) \right\}\) are centered Gaussian. Therefore, one has only to prove that they have the same covariance function. We have for any \(s\) and \(t\) in \(\mathbb{R}^+\):

\[
Cov\left(S^H_{ht}(a,b), S^H_{ht}(a,b)\right) = (h^{1/2}a)^2 (s \wedge t) + (h^2b^2) \left( t^{2H} + s^{2H} - \frac{1}{2} \left( (s + t)^{2H} + |t - s|^{2H} \right) \right)
= Cov\left(S^H_t \left(ah^{1/2}, bh^H \right), S^H_t \left(ah^{1/2}, bh^H \right) \right).
\]

This ends the proof of the lemma.

**Lemma 5** For any \(H \in ]0, 1] \setminus \left\{ \frac{1}{2} \right\}, a \in \mathbb{R} \) and \(b \in \mathbb{R}^+, (S^H_t(a,b))_{t \in \mathbb{R}^+}\) is not a Markovian process.

**Proof.** By lemma 3, \(S^H_t\) is a centered Gaussian process such that \(\mathbb{E} \left( S^H_t \right)^2 > 0\) for all \(t > 0\). Then, if \(S^H_t\) were a Markovian process, according to \([9]\), for all \(0 < s < t < u\) we would have:

\[
(2.3) \quad Cov\left(S^H_s, S^H_u\right) Cov\left(S^H_t, S^H_t\right) = Cov\left(S^H_s, S^H_t\right) Cov\left(S^H_t, S^H_u\right).
\]

We get by lemma 3:

\[
Cov\left(S^H_s, S^H_t\right) = a^2 s + b^2 s^{2H} + b^2 \left( t^{2H} - \frac{1}{2} (t + s)^{2H} - \frac{1}{2} (t - s)^{2H} \right),
\]
\[
Cov\left(S^H_t, S^H_t\right) = a^2 t + b^2 \left( 2 - 2^{2H-1} \right) t^{2H},
\]
\[
Cov\left(S^H_s, S^H_u\right) = a^2 u + b^2 u^{2H} + b^2 \left( u^{2H} - \frac{1}{2} (u + t)^{2H} - \frac{1}{2} (u - t)^{2H} \right),
\]
\[
Cov\left(S^H_s, S^H_u\right) = a^2 s + b^2 s^{2H} + b^2 \left( u^{2H} - \frac{1}{2} (u + s)^{2H} - \frac{1}{2} (u - s)^{2H} \right).
\]

Let \(s\) be fixed and set \(u = e^t\). When \(t \to +\infty\), Taylor expansions yield...
\[ t^{2H} - \frac{1}{2} (t + s)^{2H} - \frac{1}{2} (t - s)^{2H} = -H (2H - 1) \frac{s^2}{t^{2-2H}} + o \left( \frac{s^2}{t^{2-2H}} \right), \]

and

\[ u^{2H} - \frac{1}{2} (u + t)^{2H} - \frac{1}{2} (u - t)^{2H} = -H (2H - 1) \frac{t^2}{e^{(2-2H)t}} + o \left( \frac{t^2}{e^{(2-2H)t}} \right). \]

Therefore, for \((h, x) \in \{(s, t), (t, u), (s, u)\}\),

\[ \lim_{x \to \infty} \left( x^{2H} - \frac{1}{2} (x + h)^{2H} - \frac{1}{2} (x - h)^{2H} \right) = 0. \]

To verify (2.3), a necessary condition is that, when \(b \neq 0\),

\[ \lim_{t \to \infty} \left( \text{Cov} \left( S^H_t, S^H_u \right) \text{Cov} \left( S^H_s, S^H_t \right) - \text{Cov} \left( S^H_s, S^H_t \right) \text{Cov} \left( S^H_t, S^H_u \right) \right) = 0, \]

that is

\[ (a^2 s + b^2 s^{2H}) \lim_{t \to \infty} \left( (a^2 t + b^2 (2 - 2^{2H-1}) t^{2H}) - (a^2 t + b^2 t^{2H}) \right) = 0. \]

The last equality is satisfied when

\[ 2 - 2^{2H-1} = 1 \iff H = \frac{1}{2}. \]

The proof of lemma 5 is complete. \(\blacksquare\)

**Proposition 6** Second moment of increments:

We have for all \((s, t) \in \mathbb{R}^2_+, s \leq t\),

\[ (2.4) \]

\[ E \left( S^H_t(a, b) - S^H_s(a, b) \right)^2 = a^2(t - s) \]

\[ + \ b^2 \left( -2^{2H-1}(t^{2H} + s^{2H}) + (t + s)^{2H} + (t - s)^{2H} \right). \]

\[ (2.5) \]

\[ a^2(t - s) + b^2 \gamma(t - s)^{2H} \leq E \left( S^H_t(a, b) - S^H_s(a, b) \right)^2 \leq a^2(t - s) + b^2 \nu(t - s)^{2H}, \]

where

\[ \gamma = \begin{cases} 
2 - 2^{2H-1} & \text{if } H > \frac{1}{2}, \\
1 & \text{if } H \leq \frac{1}{2}.
\end{cases} \]
and

\[ \nu = \begin{cases} 
1 & \text{if } H \geq \frac{1}{2}, \\
2 - 2^{2H-1} & \text{if } H < \frac{1}{2}.
\end{cases} \]

**Proof.** Equality (2.4) is a direct consequence of equalities (2.1) and (2.2). So let us check the inequalities (2.5). Setting

(2.6) \[ A(s, t) = \left( \frac{t + s}{2} \right)^{2H} - \frac{t^{2H} + s^{2H}}{2}, \]

we can write

(2.7) \[ E \left( S^H_t(a, b) - S^H_s(a, b) \right)^2 - a^2(t - s) = b^2 \left( (t - s)^{2H} + 2^{2H} A(s, t) \right). \]

We get by convexity that, if \( H \leq \frac{1}{2} \), then \( A(s, t) \geq 0 \) and consequently

(2.8) \[ a^2(t - s) + b^2(t - s)^{2H} \leq E \left( S^H_t(a, b) - S^H_s(a, b) \right)^2, \]

and if \( H \geq \frac{1}{2} \), then \( A(s, t) \leq 0 \) and consequently

(2.9) \[ a^2(t - s) + b^2(t - s)^{2H} \geq E \left( S^H_t(a, b) - S^H_s(a, b) \right)^2. \]

To complete the proof of proposition 6, we need a technical lemma.

**Lemma 7** Consider, for any \( s > 0 \), the function \( f \) defined as follows

\[ f(x) = -2^{2H-1}((x + s)^{2H} + s^{2H}) + (x + 2s)^{2H} - (1 - 2^{2H-1}) x^{2H}, \quad x \geq 0. \]

If \( H < \frac{1}{2} \), \( f \) is a negative decreasing function, whereas, if \( H > \frac{1}{2} \), \( f \) is a positive increasing one.

**Proof.** (of lemma 7) It is clear that \( f(0) = 0 \). We get for \( x > 0 \)

\[ f'(x) = H x^{2H-1} g(x), \]

where

\[ g(x) = -2^{2H} \left( \frac{s}{x} + 1 \right)^{2H-1} + 2 \left( \frac{2s}{x} + 1 \right)^{2H-1} - (2 - 2^{2H}). \]

We have

\[ g'(x) = \frac{(2H - 1)s}{x^2} \left( 2^{2H} \left( \frac{s}{x} + 1 \right)^{2H-2} - 4 \left( \frac{2s}{x} + 1 \right)^{2H-2} \right). \]
Let us consider the two following cases:

**Case 1:** $H < \frac{1}{2}$. Since $2H - 1 < 0$, $2 - 2^{2H} > 0$ and consequently

$$\lim_{x \to 0^+} g(x) = -(2 - 2^{2H}) < 0 \quad \text{and} \quad \lim_{x \to +\infty} g(x) = 0.$$ 

Set

$$\ell(x) = \frac{s + x}{2s + x} = \frac{\frac{s}{x} + 1}{\frac{2s}{x} + 1}.$$ 

Since $\ell$ increases from $\frac{1}{2}$ to 1, $\ell^{2H-2}$ decreases from $2^{2-2H}$ to 1. Then $\ell(x)^{2H-2} \leq 2^{2-2H}$, which is equivalent to

$$2^{2H} \left( \frac{s}{x} + 1 \right)^{2H-2} - 4 \left( \frac{2s}{x} + 1 \right)^{2H-2} \leq 0,$$

and consequently $g'(x) \geq 0$. Since $g$ increases from $-(2 - 2^{2H})$ to 0, $g(x) \leq 0$ and therefore $f'(x) \leq 0$. Hence $f$ decreases and $f(x) \leq 0$.

**Case 2:** $H > \frac{1}{2}$. Following the same lines as in case 1, we get $g'(x) \leq 0$. Since the function $g$ decreases from $-(2 - 2^{2H})$ to 0, $f$ increases and $f(x) \geq 0$. This completes the proof of lemma 7.

Combining (2.8) and (2.9) with (2.7) and lemma 7 we complete the proof of proposition 6.

**Remark 8** As a consequence of proposition 6, we insist on the fact that the **smfBm** does not have stationary increments, but this property is replaced by inequalities (2.5).

### 2.2 Study of the correlation coefficient of the smfBm increments

**Notation.** Let $X$ and $Y$ be two random variables defined on the same probability space $(\Omega, F, P)$ such that $V(X) \times V(Y) \neq 0$. We denote the correlation coefficient $\rho(X, Y)$ by:

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{V(X)\sqrt{V(Y)}}}.$$ 

**Lemma 9** We have for $a \in \mathbb{R}$, $b \in \mathbb{R}^*$, $s \in \mathbb{R}_+$, $t \in \mathbb{R}_+$ and $h \in \mathbb{R}_+$ such that $0 < h \leq t-s$,

$$\rho(S_{t+h}^H - S_t^H, S_{s+h}^H - S_s^H) = \frac{\gamma(s, t, h)}{\sqrt{(2 \frac{a^2}{b^2} h + \alpha(s, h))(2 \frac{a^2}{b^2} h + \alpha(t, h))}}.$$ 

where

$$\gamma(s, t, h) = \left( (t-s+h)^{2H} - 2(t-s)^{2H} + (t-s-h)^{2H} \right. \left. - (t+s)^{2H} + 2(t+s+h)^{2H} - (t+s+2h)^{2H} \right),$$

and $\alpha(s, h) = -2^{2H} \left( (s+h)^{2H} + s^{2H} \right) + 2 (2s+h)^{2H} + 2h^{2H}$. 
Proof. We have by equality (2.4)

\[
\mathbb{E}\left(S_{t+h}^H - S_t^H\right)^2 = a^2 h + b^2 \left(-2^{2H-1} \left((t+h)^{2H} + t^{2H}\right) + (2t + h)^{2H} + h^{2H}\right)
\]

(2.11)

\[
= a^2 h + \frac{b^2}{2} \alpha(t, h).
\]

Recall that a Bm has independent increments and that the processes \(\xi^H_t\) and \(\xi_t\) are independent. Then, we have

\[
\text{Cov} \left(S_{t+h}^H - S_t^H, S_{s+h}^H - S_s^H\right) = b^2 \text{Cov} \left(\xi_{t+h}^H - \xi_t^H, \xi_{s+h}^H - \xi_s^H\right),
\]

and we get by using lemma [2]

(2.12)

\[
\text{Cov} \left(S_{t+h}^H - S_t^H, S_{s+h}^H - S_s^H\right) = \frac{b^2}{2} \gamma(s, t, h).
\]

Combining (2.11) with (2.12), we complete the proof of lemma 9.

Corollary 10 Let \(a \in \mathbb{R}\) and \(b \in \mathbb{R}^*\). Then, the increments of \((S_t^H(a, b))_{t \in \mathbb{R}^+}\) are positively correlated for \(\frac{1}{2} < H < 1\), uncorrelated for \(H = \frac{1}{2}\), and negatively correlated for \(0 < H < \frac{1}{2}\).

Proof. Let us write the function \(\gamma\) given in (2.10) as

\[
\gamma(s, t, h) = f(t-s) - f(t + s + h),
\]

where \(f : x \mapsto (x+h)^{2H} - 2x^{2H} + (x-h)^{2H}\). We have for every \(x > 0\)

\[
f'(x) = 2H \left((x+h)^{2H-1} - 2x^{2H-1} + (x-h)^{2H-1}\right).
\]

The study of the convexity of the function \(x \mapsto x^{2H-1}\) enables us to determine the sign of \(f'\) and therefore the monotony of \(f\). This ends the proof of corollary 10.

As a direct consequence of lemma 9 we get the following corollary.

Corollary 11 Assume that \(b \neq 0\). Then, \(\left|\rho\left(S_{t+h}^H - S_t^H, S_{s+h}^H - S_s^H\right)\right|\) is a decreasing function of \(\frac{a^2}{b^2}\).

Thus, to model some phenomena, we can choose the parameters \(H, a\) and \(b\) in such a manner that \(\{S_t^H(a, b), t \geq 0\}\) yields a good model, taking the sign and the level of correlation of the phenomenon of interest into account. For example, let us assume that the parameters \(H\) and \(a\) are known with \(H > 1/2\), and \(b \neq 0\) is not known. Combining corollary 10 with corollary 11, we obtain that the correlation of the increments of \(S_H^H\) increases with \(|b|\).
2.3 Some comparisons between mfBm and smfBm

Set for any \( s, t > 0 \)

\[
R_H(s, t) = \text{Cov}\left(M_t^H(a, b), M_s^H(a, b)\right) \quad \text{and} \quad C_H(s, t) = \text{Cov}\left(S_t^H(a, b), S_s^H(a, b)\right).
\]

Let us compare \( R_H \) and \( C_H \).

Lemma 12

- \( C_H(s, t) \geq 0 \).
- If \( H > \frac{1}{2} \), \( C_H(s, t) < R_H(s, t) \).
- If \( H = \frac{1}{2} \), \( C_{1/2}(s, t) = R_{1/2}(s, t) \).
- If \( H < \frac{1}{2} \), \( C_H(s, t) > R_H(s, t) \).

Proof.

Let us show the first assertion. We have by equality (2.4)

\[
\frac{1}{2} \left( -2^{2H} (t^{2H} + s^{2H}) + 2(t + s)^{2H} + 2\mid t - s \mid^{2H} \right) = E\left(S_t^H(0, 1) - S_s^H(0, 1)\right)^2 \geq 0.
\]

Thus, we get for every \( 0 < s' < t' \)

\[
2(t' + s')^{2H} + 2(t' - s')^{2H} \geq 2^{2H} (t'^{2H} + s'^{2H}).
\]

By applying this inequality with \( t' = t + s \) and \( s' = t - s \), we obtain

\[
2(t^{2H} + s^{2H}) \geq (t + s)^{2H} + (t - s)^{2H}.
\]

This implies by equality (2.1) that \( C_H(s, t) \geq 0 \).

For the next three assertions, we observe that, by using the expressions of \( C_H \) and \( R_H \),

\[
C_H(s, t) - R_H(s, t) = \frac{b^2}{2} \left(t^{2H} + s^{2H} - (s + t)^{2H}\right).
\]

When \( H = \frac{1}{2} \), \( C_{1/2} = R_{1/2} \). When \( H \neq \frac{1}{2} \), set \( u = \frac{s}{t} \), \( 0 \leq u \leq 1 \). We get

\[
C_H(s, t) - R_H(s, t) = \frac{b^2}{2} t^{2H} g(u),
\]

where \( g(u) = 1 + u^{2H} - (1 + u)^{2H} \).

The study of the function \( g \) completes the proof of the lemma.

Let us turn to the expressions of the covariances of the mfBm and the smfBm increments on non-overlapping intervals. To this aim, denote for \( 0 \leq u < v \leq s < t \),

\[
R_{u, v, s, t} = \text{Cov}\left(M_v^H(a, b) - M_u^H(a, b), M_t^H(a, b) - M_s^H(a, b)\right)
\]

and

\[
C_{u, v, s, t} = \text{Cov}\left(S_v^H(a, b) - S_u^H(a, b), S_t^H(a, b) - S_s^H(a, b)\right).
\]

We deduce easily from lemma 1 and lemma 3 the following result.
Lemma 13 We have

\begin{align}
R_{u,v,s,t} &= \frac{b^2}{2} \left( (t-u)^{2H} + (s-v)^{2H} - (t-v)^{2H} - (s-u)^{2H} \right), \\
C_{u,v,s,t} &= \frac{b^2}{2} \left( (t+u)^{2H} + (t-u)^{2H} + (s+v)^{2H} + (s-v)^{2H} \\
& \quad - (t+v)^{2H} - (t-v)^{2H} - (s+u)^{2H} - (s-u)^{2H} \right).
\end{align}

Let us show that the covariances of the mfBm and the smfBm increments on non-overlapping intervals have the same sign but, those of the smfBm are smaller in absolute value than those of the mfBm.

Corollary 14 We have for \(0 \leq u < v \leq s < t\), that \(R_{u,v,s,t}\) and \(C_{u,v,s,t}\) are strictly positive (respectively strictly negative) for \(H > 1/2\) (respectively \(H < 1/2\)). Moreover, \(C_{u,v,s,t} < R_{u,v,s,t}\) (respectively \(>\)).

Proof. First, we have \(0 \leq u < v \leq s < t\)

\[ R_{u,v,s,t} = \frac{b^2}{2} \left( g_1(v) - g_1(u) \right), \]

where \(g_1(x) = (s-x)^{2H} - (t-x)^{2H}, u \leq x \leq v.\)

We have \(g'_1(x) = 2H \left( -(s-x)^{2H-1} + (t-x)^{2H-1} \right).\)

When \(H < 1/2\), \(g'_1 \leq 0\). Then \(g_1\) decreases and therefore \(R_{u,v,s,t} \leq 0\). When \(H > 1/2\), \(g'_1 \geq 0\). Then \(g_1\) increases and therefore \(R_{u,v,s,t} \geq 0\).

Next we have for \(0 \leq u < v \leq s < t\)

\[ C_{u,v,s,t} = \frac{b^2}{2} \left( g_2(t) - g_2(s) \right), \]

where \(g_2(x) = -(x+v)^{2H} - (x-v)^{2H} + (x+u)^{2H} + (x-u)^{2H}, s \leq x \leq t.\)

We have \(g'_2(x) = 2H \left( g_3(u) - g_3(v) \right),\)

where \(g_3(y) = (x+y)^{2H-1} + (x-y)^{2H-1}, u \leq y \leq v.\)

We have \(g'_3(y) = (2H-1) \left( (x+y)^{2H-2} - (x-y)^{2H-2} \right).\)

When \(H < 1/2\), \(g'_3 > 0\). Since \(g_3\) increases, \(g'_2 < 0\) and therefore \(g_2\) decreases. Thus \(C_{u,v,s,t} \leq 0\). When \(H > 1/2\), \(g'_3 < 0\). Since \(g_3\) decreases, \(g'_2 > 0\) and therefore \(g_2\) increases. Thus \(C_{u,v,s,t} \geq 0\).
Finally let us denote by \( D(u, v, s, t) \) the quantity defined as follows

\[
D_{u,v,s,t} = C_{u,v,s,t} - R_{u,v,s,t}
\]

\[
= \frac{b^2}{2} \left( (t + u)^{2H} - (t + v)^{2H} + (s + v)^{2H} - (s + u)^{2H} \right)
\]

\[
= \frac{b^2}{2} \left( g_4(t) - g_4(s) \right).
\]

where \( g_4(x) = (x + u)^{2H} - (x + v)^{2H}, s \leq x \leq t. \)

Let us remark that, when \( H > 1/2 \), \( g_4 \) decreases, and when \( H < 1/2 \), \( g_4 \) increases.

This ends the proof of the lemma.

\[ \Box \]

Corollary 15 We have

- \( \lim_{s,t \to +\infty} R_{u,v,s,t} = 0 \) if and only if \( 0 < H \leq \frac{1}{2} \).
- For every \( 0 < H < 1 \), \( \lim_{s,t \to +\infty} C_{u,v,s,t} = 0 \).

\[ \Box \]

Proof. Combining (2.13) with Taylor expansions, we have as \( s, t \to +\infty \),

\[
R_{u,v,s,t} = b^2 H (v - u) \left( \frac{1}{t^{1-2H}} - \frac{1}{s^{1-2H}} \right) + o \left( \frac{1}{t^{1-2H}} \right) + o \left( \frac{1}{s^{1-2H}} \right),
\]

which proves the first assertion of the corollary.

Let us turn to \( C_{u,v,s,t} \). Combining (2.14) with Taylor expansions, we have as \( s, t \to +\infty \),

\[
C_{u,v,s,t} = b^2 H (2H - 1) (v^2 - u^2) \left( \frac{1}{t^{2-2H}} - \frac{1}{s^{2-2H}} \right) + o \left( \frac{1}{t^{2-2H}} \right) + o \left( \frac{1}{s^{2-2H}} \right),
\]

which completes the proof of corollary 15.

\[ \Box \]

In the next lemma, we will show that the increments of the smfBm on intervals \([u, u+r]\) and \([u+r, u+2r]\) are more weakly correlated than those of the mfBm.

Lemma 16 Assume \( H \neq 1/2 \). We have for \( u \geq 0 \) and \( r > 0 \),

\[
\left| \rho \left( S_{u+r}^{H} - S_{u}^{H}, S_{u+2r}^{H} - S_{u+r}^{H} \right) \right| \leq \left| \rho \left( M_{u+r}^{H} - M_{u}^{H}, M_{u+2r}^{H} - M_{u+r}^{H} \right) \right|.
\]

\[ \Box \]

Proof. Combining the definition of \( R_{u,v,s,t} \) with (2.13), we get

\[
\rho \left( M_{u+r}^{H} - M_{u}^{H}, M_{u+2r}^{H} - M_{u+r}^{H} \right) = \frac{R_{u,v,s,t}}{\sqrt{V(M_{u+r}^{H} - M_{u}^{H}) V(M_{u+2r}^{H} - M_{u+r}^{H})}}
\]

\[
= \frac{b^2 (2^{2H-1} - 1) r^{2H}}{\sqrt{V(M_{u+r}^{H} - M_{u}^{H}) V(M_{u+2r}^{H} - M_{u+r}^{H})}}.
\]
Moreover, we get by lemma 1

\[(2.18) \quad V(M_{u+r}^H - M_u^H) = V(M_{u+2r}^H - M_{u+r}^H) = V(M_r^H) = a^2 r + b^2 r^{2H}.\]

Then, combining (2.17) with (2.18), we have

\[(2.19) \quad \rho \left( M_{u+r}^H - M_u^H, M_{u+2r}^H - M_{u+r}^H \right) = \frac{b^2(2^{2H-1} - 1)r^{2H}}{a^2 r + b^2 r^{2H}}.\]

Let us turn to \( \rho \left( S_{u+r}^H - S_u^H, S_{u+2r}^H - S_{u+r}^H \right) \). We have

\[(2.20) \quad \rho \left( S_{u+2r}^H - S_u^H, S_{u+r}^H - S_u^H \right) = \frac{C_{u,u+r,u+2r}}{\sqrt{V(S_{u+2r}^H - S_{u+r}^H)V(S_{u+r}^H - S_u^H)}}.\]

Let us consider the two following cases.

Case 1. \( H < \frac{1}{2} \)

By using (2.19) and (2.20), we can rewrite inequality (2.16) as follows:

\[(2.21) \quad \frac{|C_{u,u+r,u+2r}|}{R_{u,u+r,u+2r}} \leq \frac{\sqrt{V(S_{u+2r}^H - S_{u+r}^H)V(S_{u+r}^H - S_u^H)}}{a^2 r + b^2 r^{2H}}.\]

Note that by corollary 14 and equality (2.15)

\[\frac{C_{u,u+r,u+2r}}{R_{u,u+r,u+2r}} = \frac{C_{u,u+r,u+2r}}{R_{u,u+r,u+2r}} = 1 + \frac{D_{u,u+r,u+2r}}{R_{u,u+r,u+2r}}.\]

Then, (2.21) can be rewritten as follows

\[(2.22) \quad 0 \leq 1 + \frac{D_{u,u+r,u+2r}}{R_{u,u+r,u+2r}} \leq \frac{\sqrt{V(S_{u+2r}^H - S_{u+r}^H)V(S_{u+r}^H - S_u^H)}}{a^2 r + b^2 r^{2H}}.\]

The second part of proposition 6 implies that

\[a^2 r + b^2 r^{2H} \leq \sqrt{V(S_{u+2r}^H - S_{u+r}^H)V(S_{u+r}^H - S_u^H)}.\]

Then, to prove (2.22), it suffices to show that

\[1 + \frac{D_{u,u+r,u+2r}}{R_{u,u+r,u+2r}} \leq 1.\]

By corollary 14 \( R_{u,u+r,u+2r} < 0 \) and \( D_{u,u+r,u+2r} > 0 \). The proof of case 1 is complete.

Case 2. \( H > \frac{1}{2} \)
Combining (2.19) with (2.20), we get
\[
\frac{\rho\left(S_{u+r}^H - S_u^H, S_{u+2r}^H - S_{u}^H\right)}{\rho\left(M_{u+r}^H - M_u^H, M_{u+2r}^H - M_{u+r}^H\right)} = \frac{C_{u,u+r,u+u+2r}}{\sqrt{V(S_{u+r}^H - S_u^H)V(S_{u}^H - S_{u+2r}^H)}} \frac{a^2 r + b^2 r^{2H}}{b^2(2^{2H-1} - 1)r^{2H}}.
\]

Recall that we have precise expressions of \(V(S_{u+r}^H - S_u^H)\) and of \(V(S_{u+2r}^H - S_{u+r}^H)\) by the first part of proposition [6] and of \(C_{u,v,s,t}\) by equality (2.14).

Set \(x = \frac{2a}{r}\) and denote by \(A, B\) and \(C\) the functions defined as follows:
\[
A(x) = 2(x + 2)^{2H} + (2^{2H} - 2) - (x + 3)^{2H} - (x + 1)^{2H},
\]
\[
B(x) = 2 - x^{2H} - (x + 2)^{2H} + 2(x + 1)^{2H}
\]
and
\[
C(x) = 2 - (x + 2)^{2H} - (x + 4)^{2H} + 2(x + 3)^{2H}.
\]

Easy computations yield
\[
C_{u,u+r,u+u+2r} = \frac{b^2}{2} r^{2H} A(x)
\]
\[
V(S_{u+r}^H - S_u^H) = \frac{1}{2} \left(2a^2 r + b^2 r^{2H} B(x)\right)
\]
\[
V(S_{u+2r}^H - S_{u+r}^H) = \frac{1}{2} \left(2a^2 r + b^2 r^{2H} C(x)\right)
\]

Then, we have
\[
\frac{\rho\left(S_{u+r}^H - S_u^H, S_{u+2r}^H - S_{u}^H\right)}{\rho\left(M_{u+r}^H - M_u^H, M_{u+2r}^H - M_{u+r}^H\right)} = \frac{A(x) \left(a^2 r + b^2 r^{2H}\right)}{(2^{2H-1} - 1)^2 \sqrt{B(x) C(x) (2a^2 r + b^2 r^{2H} B(x))(2a^2 r + b^2 r^{2H} C(x))}}
\]

Since it has been proved in [?, see]p. 412\[TB,\] that
\[
\frac{A(x)}{(2^{2H-1} - 1) \sqrt{B(x) C(x)}} \leq 1,
\]
we get
\[
\frac{\rho\left(S_{u+r}^H - S_u^H, S_{u+2r}^H - S_{u}^H\right)}{\rho\left(M_{u+r}^H - M_u^H, M_{u+2r}^H - M_{u+r}^H\right)} \leq \frac{a^2 r + b^2 r^{2H}}{\sqrt{(2a^2 r / B(x) + b^2 r^{2H})(2a^2 r / C(x) + b^2 r^{2H})}}
\]

Therefore it suffices to show
\[
a^2 r + b^2 r^{2H} \leq 2a^2 r / B(x) + b^2 r^{2H} \quad \text{and} \quad a^2 r + b^2 r^{2H} \leq 2a^2 r / C(x) + b^2 r^{2H},
\]
that is
\[
0 < B(x) \leq 2 \quad \text{and} \quad 0 < C(x) \leq 2.
\]
Let us show the first double inequality. Since by lemma \[2\]

\[ b^2 r^{2H} B(x) = 2 V (b(\xi^H (u + r) - \xi^H (u))) , \]

\[ B(x) > 0. \] Moreover, since the function \( x \to x^{2H} \) is convex for \( H > 1/2, \) \( B(x) \leq 2. \) Similarly, we can establish \( 0 < C(x) \leq 2. \)

The proof of the lemma is complete.

[ ]

In \[12\], it was proved that the increments of the mfBm \( (M^H_t (a, b)) \) are short-range dependent if, and only if \( H < \frac{1}{2}. \) To end this subsection, let us show that for every \( H \in ]0, 1[, \) the increments of \( (S^H_t (a, b))_{t \in \mathbb{R}_+} \) are short-range dependent. For convenience, let us introduce the following notation 

\[ C(p, n) = C_{p,p+1,p+n,p+n+1}, \]

where \( p \) and \( n \) are integers with \( n \geq 1. \)

We get by \((2.14)\)

\[ C(p, n) = \frac{b^2}{2} \left( (n+1)^{2H} - 2n^{2H} + (n-1)^{2H} - (2p+n+2)^{2H} + 2(2p+n+1)^{2H} - (2p+n)^{2H} \right). \]

A third-order Taylor expansion enables us to state the following lemma.

**Lemma 17** For any \( 0 < H < 1 \) and \( p \in \mathbb{N}, \) we have when \( n \to +\infty \)

\[ C(p, n) = \left( 2(1 - H)H(2H - 1)(2p + 1)b^2 \right) n^{2H-3} + o(n^{2H-3}), \]

and consequently

\[ \sum_{n \geq 1} | C(p, n) | < +\infty. \]

### 3 Semi-martingale properties

In the sequel, we assume \( b \neq 0. \) For any process \( X, \) set

\[ \Delta^\nu_j X(t) = X(jt/n) - X((j - 1)t/n), \ j \in \{1, \ldots, n\}. \]

Denote by \( A_n \) the quantity defined as follows:

\[ A_n = \mathbb{E} \left( \sum_{j=1}^{n} \left( \Delta^\nu_j S(t) \right)^2 \right) = \sum_{j=1}^{n} \mathbb{E} \left( \Delta^\nu_j S(t) \right)^2. \]
Lemma 18  

• If $H < \frac{1}{2}$, then \( \lim_{n \to +\infty} A_n = +\infty \).

• If $H = \frac{1}{2}$, then $A_n = (a^2 + b^2) t$.

• If $H > \frac{1}{2}$, then \( \lim_{n \to +\infty} A_n = a^2 t \).

Proof. Since the processes $B$ and $B_H$ are independent, we have

\[
\mathbb{E} \left( \Delta_j^n S(t) \right)^2 = \frac{a^2}{2} \mathbb{E} \left( \Delta_j^n B(t) + \Delta_j^n B(-t) \right)^2 + \frac{b^2}{2} \mathbb{E} \left( \Delta_j^n B_H(t) + \Delta_j^n B_H(-t) \right)^2.
\]

Using equality (1.1), direct computations imply

\[
\mathbb{E} \left( \Delta_j^n S(t) \right)^2 = a^2 \frac{t}{n} + b^2 \frac{t^{2H}}{n^{2H}} + 2^{2H} b^2 \frac{t^{2H}}{n^{2H}} \left( \left( \frac{2j - 1}{2} \right)^{2H} - \frac{j^{2H} + (j - 1)^{2H}}{2} \right),
\]

and hence

\[
A_n = a^2 \frac{t}{n} + b^2 \frac{t^{2H}}{n^{2H}} n^{1-2H} + 2^{2H} b^2 \frac{t^{2H}}{n^{2H}} \sum_{j=1}^{n} \left( \left( \frac{2j - 1}{2} \right)^{2H} - \frac{j^{2H} + (j - 1)^{2H}}{2} \right).
\]

Let us consider the function $f$ defined as follows:

\[
f(x) = \left( \frac{2x - 1}{2} \right)^{2H} - \frac{x^{2H} + (x - 1)^{2H}}{2}, \quad x \geq 0.
\]

We deduce from convexity properties that, when $H < 1/2$, $f(x) > 0$, when $H = 1/2$, $f(x) = 0$ and when $H > 1/2$, $f(x) < 0$. We have also

\[
f'(x) = 2H \left( \left( \frac{2x - 1}{2} \right)^{2H-1} - \frac{x^{2H-1} + (x - 1)^{2H-1}}{2} \right), \quad x \geq 0.
\]

To determine \( \lim_{n \to +\infty} A_n \), we have to consider the following three cases.

Case 1. $H < 1/2$

Since $f' \leq 0$, for every $j \in \{1, \ldots, n\}$,

\[
f(j) \geq f(n) = \left( \frac{2n - 1}{2} \right)^{2H} - \frac{n^{2H} + (n - 1)^{2H}}{2} > 0.
\]

When $n$ is large enough, we get

\[
a^2 t + b^2 \frac{t^{2H}}{n} n^{1-2H} + 2^{2H} b^2 \frac{t^{2H}}{n} \left( -\frac{H(2H-1)}{4n} + o\left( \frac{1}{n} \right) \right) \leq A_n,
\]
and therefore, since $b \neq 0$,

$$\lim_{n \to +\infty} A_n = +\infty.$$  

**Case 2.** $H = 1/2$

We obviously have

$$A_n = (a^2 + b^2) t.$$  

**Case 3.** $H > 1/2$

Since $f' \geq 0$, $f$ increases from $f(1) = \frac{1}{2^{2H}} - \frac{1}{2}$ to

$$f(n) = \left(\frac{2n - 1}{2}\right)^{2H} - \frac{n^{2H} + (n - 1)^{2H}}{2} < 0.$$  

When $n$ is large enough, we get

$$a^2 t + b^2 t^{2H} n^{1-2H} + 2^{2H} b^2 t^{2H} n^{1-2H} \left(\frac{1}{2^{2H}} - \frac{1}{2}\right) \leq A_n \leq a^2 t + b^2 t^{2H} n^{1-2H} + 2^{2H} b^2 t^{2H} \left(-\frac{H(2H - 1)}{4n} + o\left(\frac{1}{n}\right)\right)$$  

and therefore

$$\lim_{n \to +\infty} A_n = a^2 t.$$  

This completes the proof of the lemma.

Let us now recall the Bichteler-Dellacherie theorem [?, see]section VIII.4]DELL.

**Theorem 19** Assume that a filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, 0 < T < +\infty$, satisfies the usual assumptions, i.e. it is right-continuous, $\mathcal{F}_T$ is complete and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_T$. An a.s. right-continuous, $\mathcal{F}$-adapted stochastic process $\{X_t, 0 \leq t \leq T\}$ is a $\mathcal{F}$-semi-martingale if and only if

$$I_X\left(\beta(\mathcal{F})\right)$$  

is bounded in $L^0$, where

$$\beta(\mathcal{F}) = \left\{ \sum_{j=0}^{n-1} f_j \mathbf{1}_{[t_j, t_{j+1}]} \mid n \in \mathbb{N}, 0 \leq t_0 \leq \ldots \leq t_n \leq T, \right.$$  

$$\forall j, f_j \text{ is } \mathcal{F}_{t_j} \text{ measurable and } |f_j| \leq 1, \text{ with probability } 1 \right\},$$  

and

$$I_X(\theta) = \sum_{j=0}^{n-1} f_j (X_{t_{j+1}} - X_{t_j}), \quad \theta \in \beta(\mathcal{F}).$$  

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Following the same lines as those of [3], we introduce two definitions.

**Definition 20** A stochastic process \( \{X_t, 0 \leq t \leq T\} \) is a weak semi-martingale with respect to a filtration \( \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \) if \( X \) is \( \mathcal{F} \)-adapted and \( I_X(\beta(\mathcal{F})) \) is bounded in \( L^0 \).

We insist on the fact that if a process \( X \) is not a weak semi-martingale with respect to its own filtration, then it is not a weak semi-martingale with respect to any other filtration.

**Definition 21** Let \( \{X_t, 0 \leq t \leq T\} \) be a stochastic process. We call \( X \) a weak semi-martingale if it is a weak semi-martingale with respect to its own filtration \( \mathcal{F}^X = (\mathcal{F}^X_t)_{0 \leq t \leq T} \). We call \( X \) a semi-martingale if it is a semi-martingale with respect to the smallest filtration that contains \( \mathcal{F}^X \) and satisfies the usual assumptions.

Let us determine now the values of \( H \) for which the smfBm is not a semi-martingale.

**Corollary 22** If \( 0 < H < \frac{1}{2} \), then the smfBm \( S^H(a, b) \) is not a weak semi-martingale.

**Proof.** A direct consequence of lemma 18 is that since \( 0 < H < \frac{1}{2} \) and \( b \neq 0 \), the quadratic variation of the smfBm is infinity. To complete the proof of the corollary, it suffices to apply proposition 2.2 of [3, pp. 918-919].

The study of the case \( H > 3/4 \) is based on a result of [1, p. 348]. We insist on the fact that this method is different from the one which was used in [3].

**Proposition 23** For every \( T > 0 \), \( H \in ]\frac{3}{4}, 1[ \), and \( a \neq 0 \), the smfBm

\[
S^H(a, b) = \{S^H_t(a, b), t \in [0, T]\}
\]

is a semi-martingale equivalent in law to \( a \times B_t \), where \( \{B_t, t \in [0, T]\} \) is a Bm.

**Proof.** The smfBm \( S^H \) can be rewritten as follows

\[
\forall t \in \mathbb{R}^+ \quad S^H_t(a, b) = a\left(\xi_t + \frac{b}{a} \xi_t^H\right)
\]

where \( \xi \) and \( \xi^H \) have been introduced by equation (1.3). Recall that the processes \( \xi \) and \( \xi^H \) are independent.

The covariance function of the Gaussian process \( \frac{b}{a} \xi^H \)

\[
R(s, t) = \frac{b^2}{a^2}\left(t^{2H} + s^{2H} - \frac{1}{2}(s + t)^{2H} + |t - s|^{2H}\right)
\]

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is twice continuously differentiable on $[0, T]^2 \setminus \{(s, t); t = s\}$.

According to [1, p. 348], it suffices to verify $\frac{\partial^2 R}{\partial s \partial t} \in L^2([0, T]^2)$, in order to show that the process

$$\{\xi_t + \frac{b}{a} \xi_t^H, t \in [0, T]\}$$

is a semi-martingale equivalent in law to a Bm.

We have for any $(s, t) \in [0, T]^2 \setminus \{(s, t); t = s\}$

$$\frac{\partial^2 R(s, t)}{\partial s \partial t} = \frac{b^2}{a^2} H(2H - 1) \left( |t - s|^{2H-2} - (s + t)^{2H-2}\right).$$

It is easy to check that if $H > \frac{3}{4}$, then $\frac{\partial^2 R}{\partial s \partial t} \in L^2([0, T]^2)$. This completes the proof of the proposition.

To study the case $H \in ]1/2, 3/4]$, we follow the same lines as those of [3]. But many technical results have to be proved. Let us first recall the definition of a quasi-martingale.

**Definition 24** A stochastic process $\{X_t, 0 \leq t \leq T\}$ is a quasi-martingale if $X_t \in L^1$ for all $t \in [0, T]$, and

$$\sup_{\tau} \sum_{j=0}^{n-1} \left\| \mathbb{E} \left( X_{t_{j+1}} - X_{t_j} \mid \mathcal{F}_{t_j} \right) \right\|_1 < + \infty,$$

where $\tau$ is the set of all finite partitions

$$0 = t_0 < t_1 < ... < t_n = T \text{ of } [0, T].$$

In the following key lemma, we will specify the relation between quasi-martingale and weak semi-martingale in the case of our process $S^H$.

**Lemma 25** If $S^H$ is not a quasi-martingale, then it is not a weak semi-martingale.

**Proof.**

Let us assume that $S^H$ is a weak semi-martingale. Then, by theorem 1 of [10], we have

$$I_{S^H} \left( \beta(\mathcal{F}^{S^H}) \right),$$

which was defined in theorem [19] is bounded in $L^2$, and therefore in $L^1$.

But, for any partition $0 = t_0 < t_1 < .. < t_n = T$,

$$\sum_{j=0}^{n-1} \text{sgn} \left( \mathbb{E} \left( S^H_{t_{j+1}} - S^H_{t_j} \mid \mathcal{F}^{S^H}_{t_j} \right) \right) 1_{[t_j, t_{j+1}]} \in \beta(\mathcal{F}^{S^H}),$$

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and
\[
\left\| I_{S^t} \left( \sum_{j=0}^{n-1} \text{sgn} \left( \mathbb{E} \left( S^t_{t_j+1} - S^t_{t_j} \mid \mathcal{F}^S_{t_j} \right) \right) 1_{[t_j, t_{j+1})} \right) \right\|_1
\]
\[
= \left\| \sum_{j=0}^{n-1} \text{sgn} \left( \mathbb{E} \left( S^t_{t_j+1} - S^t_{t_j} \mid \mathcal{F}^S_{t_j} \right) \right) \left( S^t_{t_j+1} - S^t_{t_j} \right) \right\|_1
\]
\[
\geq \sum_{j=0}^{n-1} \mathbb{E} \left( \text{sgn} \left( \mathbb{E} \left( S^t_{t_j+1} - S^t_{t_j} \mid \mathcal{F}^S_{t_j} \right) \right) \left( S^t_{t_j+1} - S^t_{t_j} \right) \right)
\]
\[
= \sum_{j=0}^{n-1} \mathbb{E} \left( \text{sgn} \left( \mathbb{E} \left( S^t_{t_j+1} - S^t_{t_j} \mid \mathcal{F}^S_{t_j} \right) \right) \mathbb{E} \left( \left( S^t_{t_j+1} - S^t_{t_j} \right) \mid \mathcal{F}^S_{t_j} \right) \right)
\]
\[
= \sum_{j=0}^{n-1} \left\| \mathbb{E} \left( S^t_{t_j+1} - S^t_{t_j} \mid \mathcal{F}^S_{t_j} \right) \right\|_1.
\]

Then, \( S^t \) is a quasi-martingale. The proof of the lemma is complete.

The following lemmas deal with the two last cases \( 1/2 < H < 3/4 \) and \( H = 3/4 \).

**Proposition 26** If \( H \in \left[ \frac{1}{2}, \frac{3}{4} \right] \), then the smfBm \( S^H(a, b) \) is not a quasi-martingale.

**Proof.** For \( n \in \mathbb{N} \) and \( j \in \{1, 2, ..., n\} \), let us denote
\[
\Delta_j^n S^t = S^t_{\frac{j}{n}} - S^t_{\frac{j-1}{n}}.
\]
Since conditional expectation is a contraction with respect to the \( L^1 \)-norm, we have for all \( n \in \mathbb{N} \) and all \( j = 1, ..., n-1 \),
\[
\left\| \mathbb{E} \left( \Delta_{j+1}^n S^t \mid \Delta_j^n S^t \right) \right\|_1 \leq \left\| \mathbb{E} \left( \Delta_{j+1}^n S^t \mid \mathcal{F}_{\frac{T_j}{n}} \right) \right\|_1.
\]
Moreover, since \( \mathbb{E} \left( \Delta_{j+1}^n S^t \mid \Delta_j^n S^t \right) \) is a centered Gaussian random variable,
\[
\left\| \mathbb{E} \left( \Delta_{j+1}^n S^t \mid \Delta_j^n S^t \right) \right\|_1 = \sqrt{\frac{2}{\pi}} \left\| \mathbb{E} \left( \Delta_{j+1}^n S^t \mid \Delta_j^n S^t \right) \right\|_2.
\]

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Consequently,
\[
\sum_{j=1}^{n-1} \left\| E(\Delta_{j+1}^n S^H | F_{j+1}^n) \right\|_1 \geq \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| E(\Delta_{j+1}^n S^H | \Delta_j^S T_j n) \right\|_2 \geq \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \frac{Cov(\Delta_{j+1}^n S^H, \Delta_j^S T_j n)}{\sqrt{Cov(\Delta_j^S T_j n, \Delta_j^S T_j n)}} = \sqrt{\frac{2}{\pi}} I_n.
\]

We have by lemma 13
\[
Cov(\Delta_{j+1}^n S^H, \Delta_j^S T_j n) = C \frac{T_{j+1} T_j T_j T_{j+1}}{n} = \frac{b^2 T^{2H}}{2 n^{2H}} \left(2^{2H} (2j^{2H} + 1) - 2 - (2j + 1)^{2H} - (2j - 1)^{2H}\right).
\]

Combining proposition 6 with the fact that \(2H > 1\), we get
\[
Cov(\Delta_j^S T_j n, \Delta_j^S T_j n) \leq \frac{1}{n} \left(a^2 T + b^2 T^{2H} \left(-2^{2H-1}(n^{2H} + (n-1)^{2H}) + (2n-1)^{2H} + 1\right)\right).
\]

Then,
\[
I_n \geq \frac{b^2 T^{2H}}{2 n^{2H-\frac{1}{2}}} \sum_{j=1}^{n-1} \frac{u_j}{v_j} = \frac{b^2 T^{2H}}{2 n^{2H-\frac{1}{2}}} \times \left(1 \sum_{j=1}^{n-1} \frac{u_j}{v_j}\right),
\]

where we have for any \(n \in \mathbb{N}^*\),
\[
u_n = 2^{2H} (2n^{2H} + 1) - 2 - (2n + 1)^{2H} - (2n - 1)^{2H}
\]

and
\[
v_n = \sqrt{a^2 T + b^2 T^{2H} \left(-2^{2H-1}(n^{2H} + (n-1)^{2H}) + (2n-1)^{2H} + 1\right)}.
\]

Since
\[
\lim_{n \to +\infty} \frac{u_n}{v_n} = \frac{2^{2H} - 2}{\sqrt{a^2 T + b^2 T^{2H}}}.
\]
we have by Césaro theorem that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n-1} u_j = \frac{2^{2H} - 2}{\sqrt{a^2 T + b^2 T^{2H}}}.$$  

Hence, since $\frac{1}{2} < H < \frac{3}{4}$ and $\frac{2^{2H} - 2}{\sqrt{a^2 T + b^2 T^{2H}}} > 0$, we have $\lim_{n \to \infty} I_n = +\infty$. Then, we get by using (3.1) that

$$\sup_{\tau} \sum_{j=0}^{n-1} \left\| \mathbb{E} \left( S^{H}_{t_{j+1}} - S^{H}_{t_j} | \mathcal{F}_{t_j} \right) \right\|_1 = +\infty.$$ 

This completes the proof of the lemma.

Proposition 27 The smfBm $S^{\frac{3}{4}}(a, b)$ is not a quasi-martingale.

Proof. Since conditional expectation is a contraction with respect to the $L^1$—norm, we have for all $n \in \mathbb{N}$ and all $j = 1, ..., n - 1$,

$$\left\| \mathbb{E} \left( \Delta^n_j S^{\frac{3}{4}} | \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right) \right\|_1 \leq \left\| \mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \mathcal{F}_{\frac{S^{\frac{3}{4}}}{n}} \right) \right\|_1.$$ 

Moreover, since $\mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right)$ is a centered Gaussian random variable,

$$\left\| \mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right) \right\|_2 = \sqrt{\frac{2}{\pi}} \left\| \mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right) \right\|_2.$$ 

Consequently,

$$\sum_{j=1}^{n-1} \left\| \mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \mathcal{F}_{\frac{S^{\frac{3}{4}}}{n}} \right) \right\|_1 \geq \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| \mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right) \right\|_2,$$

and the lemma is proved if we show that

(3.2) $\lim_{n \to \infty} \sum_{j=1}^{n-1} \left\| \mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right) \right\|_2 = +\infty.$

For $n \in \mathbb{N}$ and $j = 1, ..., n - 1$,

$$\left( \Delta^n_{j+1} S^{\frac{3}{4}}, \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right)$$

is a Gaussian vector. Therefore,

(3.3) $\mathbb{E} \left( \Delta^n_{j+1} S^{\frac{3}{4}} | \Delta^n_j S^{\frac{3}{4}}, ..., \Delta^n_1 S^{\frac{3}{4}} \right) = \sum_{k=1}^{j} b_k \Delta^n_k S^{\frac{3}{4}},$
where the vector \( b = \begin{pmatrix} b_1 \\ \vdots \\ b_j \end{pmatrix} \) solves the system of linear equations

\[(3.4) \quad m = Ab,\]

in which \( m \) is a \( j \)-vector whose \( k \)-th component \( m_k \) is

\[
Cov\left(\Delta_{j+1}^n S^{\frac{3}{4}}, \Delta_k^n S^{\frac{3}{4}}\right)
\]

and \( A \) is the covariance matrix of the Gaussian vector

\[
\left(\Delta_1^n S^{\frac{3}{4}}, \ldots, \Delta_j^n S^{\frac{3}{4}}\right).
\]

Note that \( A \) is symmetric and, since the random variables

\[
\Delta_1^n S^{\frac{3}{4}}, \ldots, \Delta_j^n S^{\frac{3}{4}}
\]

are linearly independent, \( A \) is also positive definite. It follows from \((3.3)\) and \((3.4)\) that

\[(3.5) \quad \left\| E\left(\Delta_{j+1}^n S^{\frac{3}{4}} | \Delta_k^n S^{\frac{3}{4}}, \ldots, \Delta_1^n S^{\frac{3}{4}}\right)\right\|^2 = b^T Ab = m^T A^{-1} m \geq \|m\|^2 \lambda^{-1},\]

where \( \lambda \) is the largest eigenvalue of the matrix \( A \). Set \( I \) the identity matrix and \( C = (C_{i,k})_{1 \leq i, k \leq j} \) the covariance matrix of the increments of the sfBm with index \( 3/4 \). We have

\[
A = \frac{a^2 T}{n} I + b^2 C,
\]

and consequently

\[(3.6) \quad \lambda = \frac{a^2 T}{n} + b^2 \mu,\]

where \( \mu \) is the largest eigenvalue of the matrix \( C \). We deduce also from lemma \( 13 \)

\[
C_{ik} = \frac{T^{3/2}}{2 n^{3/2}} \left( \left( \frac{|k - i| + 1}{2} \right)^{3/2} - 2 |k - i|^{3/2} + \| k - i \| - 1 \right) \left( \frac{|k - i| + 1}{2} \right)^{3/2}
\]

\[
+ 2 (k + i - 1)^{3/2} - (k + i)^{3/2} - (k + i - 2)^{3/2} \right)
\]

\[
= \frac{T^{3/2}}{2 n^{3/2}} (E_{ik} + F_{ik}),
\]

where

\[
E_{ik} = 2 \left( \frac{|k - i| + 1}{2} \right)^{3/2} + \| k - i \| - 1 \right) \left( \frac{|k - i| + 1}{2} \right)^{3/2}
\]

\[
- |k - i|^{3/2}
\]

and

\[
F_{ik} = 2 \left( (k + i - 1)^{3/2} - \frac{(k + i)^{3/2} + (k + i - 2)^{3/2}}{2} \right).
\]
Note that the convexity of the function $x \rightarrow x^{3/2}$, $x \geq 0$, implies that $E_{ik} \geq 0$ and $F_{ik} \leq 0$. Moreover, since $H = 3/4 > 1/2$, corollary 14 yields $C_{ik} \geq 0$.

So, using the Gershgorin circle theorem [7] we obtain

$$\mu \leq \max_{k=1,\ldots,j} \sum_{k=1}^{j} |C_{ik}| \leq \frac{T^{3/2}}{2n^{3/2}} \max_{k=1,\ldots,j} \sum_{k=1}^{j} E_{ik},$$

and consequently

$$\mu \leq \frac{T^{3/2}}{n^{3/2}} \sum_{k=1}^{j} \left(2 \left(\frac{|k-1|+1}{2}^{3/2} + |k-1|-1 \right)^{3/2} - |k-1|^{3/2}\right)$$

$$= \frac{T^{3/2}}{n^{3/2}} \sum_{k=1}^{j} \left(|k-1|+1 \right)^{3/2} + |k-1|-1 |k-1|^{3/2} - 2 |k-1|^{3/2}$$

$$= \frac{T^{3/2}}{n^{3/2}} \sum_{k'=0}^{j-1} \left((k'+1)^{3/2} - 2 k^{3/2} + |k'-1|^{3/2}\right)$$

$$= \frac{T^{3/2}}{n^{3/2}} \left(1 + j^{3/2} - (j-1)^{3/2}\right)$$

$$\leq \frac{T^{3/2}}{n^{3/2}} \left(\frac{1}{n^{3/2}} + \frac{1}{n^{3/2}} \max_{j-1 \leq x \leq j} \frac{d(x^{3/2})}{dx}\right)$$

$$\leq \frac{5}{2n} T^{3/2}.$$

Hence combining equality (3.6) with the above result, we obtain

$$\lambda^{-1} \geq \alpha n,$$

where $\alpha = \frac{2}{T(2a^2 + 5b^2 T^{1/2})}$.

Next, let us determine a suitable lower bound of $\|m\|_2^2$. From the lemma 13 we have

$$\|m\|_2^2 = \sum_{k=1}^{j} \left(Cov\left(\Delta_{j+1}^{n} S_{k+1}^{4}, \Delta_{k}^{n} S_{k}^{4}\right)\right)^2$$

$$= \frac{T^{3/2} b^4}{4n^3} \sum_{k=1}^{j} \left(f_1(k) - f_2(k)\right)^2,$$

where

$$f_1(k) = (j-k+2)^{3/2} - 2(j-k+1)^{3/2} + (j-k)^{3/2}$$

and

$$f_2(k) = (j+k+1)^{3/2} - 2(j+k)^{3/2} + (j+k-1)^{3/2}.$$

The functions $f_1$ and $f_2$ satisfy three properties, which we shall use at the end of the proof. We will state them in the following technical lemma.
Lemma 28  For any \( k \in \{1, \ldots, j\} \)

- \( f_1(k) \geq 0 \) and \( f_2(k) \geq 0 \),
- \( f_1(k) - f_2(k) > 0 \),

\[
(3.9) \quad f_1(k) - f_2(k) \geq \frac{3}{4} \left( (j - k + 1)^{-1/2} - (j + k - 1)^{-1/2} \right) \geq 0.
\]

Proof. (of lemma 28) The first assertion of the lemma is due to the fact that the function \( x \mapsto x^{3/2} \) is convex on the interval \([0, +\infty[\). Now, let us prove the second assertion of the lemma. Consider the function \( g \) defined by

\[
g(x) = (x + 1)^{3/2} - 2x^{3/2} + (x - 1)^{3/2}, \quad x \geq 1.
\]

Since the function \( x \mapsto x^{1/2} \) is concave on \([1, +\infty[\), \( g \) decreases on this interval and consequently

\[
f_1(k) = g(j - k + 1) > g(j + k) = f_2(k).
\]

Finally, let us prove inequality \((3.9)\). For every \( a \geq 1 \), let us consider the function \( g_a \) defined by

\[
g_a(x) = (a + x)^{3/2} - 2a^{3/2} + (a - x)^{3/2}, \quad 0 \leq x \leq 1 \leq a.
\]

We have \( g_a(0) = 0, g_a'(x) = \frac{3}{2} ((a + x)^{1/2} - (a - x)^{1/2}) \) and therefore \( g_a'(0) = 0 \).

On the other hand, by Taylor-Lagrange theorem, we get that there exists \( c \in ]0, 1[ \) such that

\[
g_a(1) = g_a(0) + g_a'(0) + \frac{1}{2} g_a''(c) = \frac{1}{2} g_a''(c),
\]

where

\[
g_a''(x) = \frac{3}{4} \left( (a + x)^{-1/2} + (a - x)^{-1/2} \right).
\]

Next, it is easy to check that the function \( g_a'' \) increases, and consequently

\[
g_a''(0) \leq g_a''(c) \leq g_a''(1).
\]

So, we have

\[
\frac{3}{4} a^{1/2} \leq g_a(1) \leq \frac{3}{4} (a - 1)^{1/2},
\]

and therefore

\[
f_1(k) = g_{j-k+1}(1) \geq \frac{3}{4} (j - k + 1)^{-1/2} \quad \text{and} \quad f_2(k) = g_{j+k}(1) \leq \frac{3}{4} (j + k - 1)^{-1/2},
\]

which ends the proof of the lemma.

\[\blacksquare\]
Let us turn back to the proof of proposition 27. Combining (3.8) with (3.9), we get

\begin{equation}
\|m\|_2^2 \geq \frac{9 b^4 T^3}{64 n^3} \sum_{k=2}^{j} \left( (j-k+1)^{-1/2} - (j+k-1)^{-1/2} \right)^2.
\end{equation}

For every integer \( j \geq 1 \), let us consider the function

\[ f_j(x) = (j - x + 1)^{-1/2} - (j + x - 1)^{-1/2}, \quad 1 \leq x \leq j. \]

Since \( f_j \) increases, we have

\begin{equation}
\sum_{k=2}^{j} \left( (j-k+1)^{-1/2} - (j+k-1)^{-1/2} \right)^2 \geq \int_{1}^{j} f_j(x)^2 \, dx.
\end{equation}

But

\begin{align*}
\int_{1}^{j} f_j(x)^2 \, dx &= \int_{1}^{j} \left( \frac{1}{j-x+1} + \frac{1}{j+x-1} - 2 \frac{1}{\sqrt{j^2 - (x-1)^2}} \right) \, dx \\
&= \ln(2j-1) - 2 \int_{1}^{j} \frac{1}{\sqrt{j^2 - (x-1)^2}} \, dx \\
&= \ln(2j-1) + 2 \arccos \left( \frac{j-1}{j} \right) - \pi.
\end{align*}

Hence, combining (3.7) with (3.10), (3.11) and (3.12), we get

\begin{equation}
\|m\|_2^2 \lambda^{-1} \geq \frac{\beta}{n^2} \left( \ln(2j-1) + 2 \arccos \left( \frac{j-1}{j} \right) - \pi \right),
\end{equation}

where \( \beta = \frac{\alpha}{64} (9 T^3 b^4). \)

Combining (3.13) with (3.5), we have,

\begin{align*}
\sum_{j=1}^{n-1} \| E \left( \Delta_{j+1}^{n} s_{3/4} \mid \Delta_{j}^{n} s_{3/4}, \ldots, \Delta_{1}^{n} s_{3/4} \right) \|_2 \\
&\geq \frac{\sqrt{\beta}}{n} \sum_{j=1}^{n-1} \sqrt{\ln(2j-1) + 2 \arccos \left( \frac{j-1}{j} \right) - \pi}.
\end{align*}

Since \( \lim_{n \to \infty} \sqrt{\ln(2n-1) + 2 \arccos \left( \frac{n-1}{n} \right) - \pi} = +\infty \), we have by Cesaro theorem

\[ \lim_{n \to \infty} \frac{\sqrt{\beta}}{n} \sum_{j=1}^{n-1} \sqrt{\ln(2j-1) + 2 \arccos \left( \frac{j-1}{j} \right) - \pi} = +\infty, \]

which completes the proof of proposition 27.
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