A new property of the Lovász number
and duality relations between graph parameters✩

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Abstract
We show that for any graph $G$, by considering “activation” through the strong product with another graph $H$, the relation $\alpha(G) \leq \vartheta(G)$ between the independence number and the Lovász number of $G$ can be made arbitrarily tight: Precisely, the inequality

$$\alpha(G \boxtimes H) \leq \vartheta(G \boxtimes H) = \vartheta(G) \vartheta(H)$$

becomes asymptotically an equality for a suitable sequence of ancillary graphs $H$.

This motivates us to look for other products of graph parameters of $G$ and $H$ on the right hand side of the above relation. For instance, a result of Rosenfeld and Hales states that

$$\alpha(G \boxtimes H) \leq \alpha^*(G) \alpha(H),$$

with the fractional packing number $\alpha^*(G)$, and for every $G$ there exists $H$ that makes the above an equality; conversely, for every graph $H$ there is a $G$ that attains equality.

These findings constitute some sort of duality of graph parameters, mediated through the independence number, under which $\alpha$ and $\alpha^*$ are dual to each other, and the Lovász number $\vartheta$ is self-dual. We also show duality of Schrijver’s and Szegedy’s variants $\vartheta^-$ and $\vartheta^+$ of the Lovász number, and explore analogous notions for the chromatic number under strong and disjunctive graph products.

Keywords: Graph, Lovász number, independence number, chromatic number, fractional packing number.

1. Independence number of a graph and its relaxations

In the present paper we consider graphs $G = (V,E)$, which throughout will be undirected and without loops $\square$. We shall be using the Lovász convention $\blacksquare$, writing $v \sim w$ to denote $vw \in E$ or $v = w$. We shall be concerned with various graph parameters, starting from the independence number (aka stability number or packing number)

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\[1\]

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where $I$ is called an independent (or stable) set if the induced graph $G|_I$ is a graph with no edges, i.e., the complement of the complete graph on the vertices $I$. Computing $\alpha$ is well-known to be NP-complete.

In the present paper we are interested in how the independence number behaves under product compositions of graphs $G = (V, E)$ and $H = (V', E')$. We will consider the strong product $G \boxtimes H$ and the disjunctive product $G \ast H$. These two products have as vertex set the Cartesian product $V \times V'$, while the corresponding edge sets are defined as follows:

$$(vv', uu') \in E(G \boxtimes H) \text{ iff } v = w \text{ and } v'u' \in E' \text{ or } vv \in E \text{ and } v'w' \in E'.$$

The two graph products are related by a de Morgan identity: $G \boxtimes H = G \ast \overline{H}$, which is why they are sometimes called "and" ($\boxtimes$) and "or" ($\ast$) product. They exhibit very different behaviour for the independence number:

$$\alpha(G \ast H) = \alpha(G) \alpha(H), \quad \text{but} \quad \alpha(G \boxtimes H) \geq \alpha(G) \alpha(H),$$

and the inequality is in general strict. E.g. for the five-cycle ("pentagon") $C_5$, we have $\alpha(C_5) = 2$ but $\alpha(C_5 \boxtimes C_5) = 5$.

The independence number and the strong graph product were studied as early as 1956, in Shannon’s seminal paper on zero-error communication [4], in particular the asymptotic behaviour of $\alpha(G^{\otimes n}) \sim \Theta(G)^n$, where $G^{\otimes n} = G \boxtimes G \boxtimes \cdots \boxtimes G$, giving rise to the zero-error (Shannon) capacity

$$\Theta(G) = \sup_n \left(\frac{\alpha(G^{\otimes n})}{n}\right)^{1/n}$$

of $G$. The strong graph product arises naturally in communication via noisy channels; indeed, if $G$ is the confusability graph of a channel, the confusability graph of $n$ independent uses of the channel is $G^{\otimes n}$.

In his paper, Shannon already introduced a useful upper bound on $\alpha$ and $\Theta$, which was to become known as the fractional packing number and denoted $\alpha^*$. This bound has then been also called Rosenfeld number in the literature, perhaps because its appearance in Shannon’s work was not fully appreciated. It is defined as

$$\alpha^*(G) = \max \sum_v t_v \text{ s.t. } t_v \geq 0 \forall v, \sum_{v \in C} t_v \leq 1 \forall \text{ cliques } C \subset V.$$  \hspace{1cm} (2)

Here, by a clique we mean a complete induced subgraph, i.e. $G|_C \simeq K_m, \ m = |C|$. Eq. (2) is a linear programme (LP), and hence efficiently computable once the cliques are known. To be precise, Shannon had defined it more generally for hypergraphs (cf. [1, 2]), which is more natural for an actual communication channel with inputs and outputs; the definition above, which is the one whose study Rosenfeld initiated [3], is obtained for the hypergraph of all cliques of $G$.

In fact, for the clique hypergraph of $G$, Shannon identified $\alpha^*(G)$ as the zero-error capacity assisted by instantaneous feedback of a channel with confusability graph $G$. In [2], it was shown that $\alpha^*(G)$ is also the zero-error capacity assisted by so-called “no-signalling” correlations. Both result extend to general channels and their hypergraphs, see [4, 5] for details. Shannon furthermore conjectured that $\log \alpha^*(G)$ equals the minimum of the usual Shannon capacity over all noisy channels with confusability graph $G$, which was proved later by Ahlswede [3]; see also [4] for an alternative proof. (Note that here the logarithm appears because in information theory the capacity is measured in bits per channel use, while in zero-error theory and combinatorics, it is defined via an $n$-th root.) All of these imply operational, information theoretic proofs of $\alpha(G) \leq \Theta(G) \leq \alpha^*(G)$. However, it can be seen also in elementary fashion, noticing that restricting the variables in eq. (2) to values $\{0, 1\}$ yields precisely the independence number, so $\alpha(G) \leq \alpha^*(G)$. To get the upper bound on $\Theta(G)$ as well, we use

$$\alpha^*(G \boxtimes H) = \alpha^*(G) \alpha^*(H),$$

which follows from the primal and dual LP characterizations of the fractional packing number (see Appendix A). In particular,

$$\alpha(G^{\otimes n}) \leq \alpha^*(G^{\otimes n}) = (\alpha^*(G))^n,$$
and the claim follows. For instance, \( \alpha^*(C_5) = \frac{5}{2} \) is an upper bound on \( \Theta(C_5) \), but it is not tight.

It took more than twenty years to improve this bound significantly, with the discovery of Lovász that a semidefinite programme (SDP) can emulate many of the nice properties of the fractional packing number:

\[
\vartheta(G) = \max \text{Tr} \,BJ \text{ s.t. } B \succeq 0, \, \text{Tr} \,B = 1, \, B_{vw} = 0 \, \forall vw \in E
\]

(where \( J \) is the all-ones matrix) is also an upper bound on \( \alpha(G) \) and is multiplicative:

\[
\vartheta(G \boxtimes H) = \vartheta(G \ast H) = \vartheta(G) \vartheta(H),
\]

hence \( \Theta(G) \leq \vartheta(G) \) \cite{2}. Returning to the pentagon, \( \vartheta(C_5) = \sqrt{5} = \Theta(C_5) \). For a selection of different characterizations of the Lovász number see Appendix A.

The rest of the paper is structured as follows. In Section 2 we show that \( \alpha \) and \( \vartheta \) can be made asymptotically equal by taking the strong product with suitable auxiliary graphs. Then, in Section 3 we recall (and prove) a result similar in spirit, due to Rosenfeld \cite{6} and Hales \cite{10}, which establishes a certain duality between \( \alpha \) and \( \alpha^* \). In Section 4, we go on to show a similar duality between Schrijver’s and Szegedy’s variants \( \vartheta^\pm \) of the Lovász number. Motivated by the Sandwich Theorem, Section 5 is devoted to an investigation of analogous questions with the chromatic number instead of the independence number. Throughout the text, various remarks offer reflections on our findings and highlight open problems. Finally, in Section 6 we conclude, discussing what we have learned and speculating on future directions.

2. Finite and asymptotic activation attaining the Lovász number

In general, \( \alpha(G) \) is strictly smaller than \( \vartheta(G) \) or indeed the integer part of the latter, and this persist even in the many-copy asymptotics: there are graphs with \( \Theta(G) < \vartheta(G) \) \cite{11,12}.

On the other hand, what we will show in this section is that going beyond graph products of the form \( G^{\boxtimes n} = G \boxtimes G \boxtimes \cdots \boxtimes G(n-1) \), and considering general products \( G \boxtimes H \), closes the gap between \( \alpha \) and \( \vartheta \). Indeed, Lovász \cite{2} already proved that for vertex-transitive \( G = (V,E) \), i.e. when the automorphism group of \( G \) maps any vertex to any other one,

\[
\vartheta(G \boxtimes \overline{G}) = \vartheta(G) \vartheta(\overline{G}) = |V| = \alpha(G \boxtimes \overline{G}).
\]

This begs the natural question whether for every graph \( G \), there exists another graph \( H \) such that

\[
\alpha(G \boxtimes H) = \vartheta(G \boxtimes H) = \vartheta(G) \vartheta(H) ?
\]

It turns out that by allowing \emph{weighted} graphs \((H, p)\), the answer is yes, even with \( H = \overline{G} \):

\begin{lemma}
For every graph \( G \), there exists a weight \( p \) on the vertices of the complementary graph \( H = \overline{G} \), such that

\[
\alpha(G \boxtimes (\overline{G}, p)) = \vartheta(G \boxtimes (\overline{G}, p)) = \vartheta(G) \vartheta(\overline{G}, p).
\]
\end{lemma}

Let us briefly recall the definition of weighted graphs and their graph invariants. A \emph{weighted graph} \((G, p)\) is a graph \( G \) equipped with a weight function \( p : V \to \mathbb{R}_+ \). The \emph{weighted independence number} \( \alpha(G, p) \) is the largest total weight of an independent set in \( G \), i.e. the largest sum of weights of the elements of an independent set. The \emph{weighted fractional packing number} of \((G, p)\) is likewise

\[
\alpha^*(G, p) = \max \sum_v p(v)t_v \text{ s.t. } t_v \succeq 0 \, \forall v, \, \sum_v t_v \leq 1 \, \forall \text{ cliques } C \subset V.
\]

Finally, the Lovász number of a weighted graph is defined as

\[
\vartheta(G, p) = \max \text{Tr} \,BH \text{ s.t. } B \succeq 0, \, \text{Tr} \,B = 1, \, B_{vw} = 0 \, \forall vw \in E,
\]
where the matrix $\Pi$ has entries $\Pi_{vw} = \sqrt{p(v)p(w)}$; cf. the definition for unweighted graphs \[1\].

Note that for the constant-1 weight, $p(v) = 1$ for all $v$, which we denote as $\mathbf{1}$, the graph invariants attain the values of their unweighted versions:

$$\alpha(G) = \alpha(G, \mathbf{1}), \quad \alpha^*(G) = \alpha^*(G, \mathbf{1}), \quad \vartheta(G) = \vartheta(G, \mathbf{1}), \quad etc.$$ 

We will also consider (strong and disjunctive) products of weighted graphs; their edge sets are the same as those of the unweighted versions, while the weights are multiplied pointwise: $(pp')(v, v') = p(v)p'(v')$.

**Dirac (bra-ket) notation.** In the rest of the paper we rely on the following useful conventional notation for linear algebra, called Dirac or bra-ket notation \[12\]: In a (real or complex) Hilbert space, the vectors are denoted $|\psi\rangle$, $|\phi\rangle$, etc. (“kets”), and the co-vectors — which are linear functions on the space — are $\langle \psi |$, $\langle \phi |$, etc. (“bras”), so that the inner product, denoted $\langle \phi | \psi \rangle$ is at the same time the application of the co-vector $\langle \phi |$ to the vector $|\psi\rangle$, and can also be read as the ordinary matrix product of the row vector $\langle \phi |$ with the column vector $|\psi\rangle$. This extends to other matrix products, such as $\langle \phi | M |\psi\rangle$ for a linear operator/matrix $M$, and to outer products $|\psi\rangle \langle \phi |$. In particular, the Hilbert space norm is $\| |\psi\rangle \|_2 = \sqrt{\langle \psi | \psi \rangle}$, and for a unit vector $|\psi\rangle$, $|\psi\rangle \langle \psi |$ is the projector onto the line spanned by $|\psi\rangle$. Note just one difference to usual mathematical convention: The inner product $\langle \phi | \psi \rangle$ is linear in the second argument, and conjugate linear in the first. In practice this difference will be unsubstantial for us, as the reader may assume real Euclidean spaces throughout.

**Proof of Lemma\[1\]** Let $G = (V, E)$ and let $\{ |\phi_v\rangle : v \in V \}$ be an orthonormal representation of $\overline{G}$, i.e. $\langle \phi_v | \phi_w \rangle = 0$ for all $vw \in E$, and $|h\rangle$ another unit vector (called the “handle” of the OR) such that $\vartheta(G) = \sum_{v \in V} \langle h | \phi_v \rangle^2$; this is another, equivalent characterization of the Lovász number \[2\], cf. Appendix A. Equip the graph with vertex weights $\rho(v) = \langle h | \phi_v \rangle^2$. Since the set $\{(v, v) : v \in V\}$ is an independent set in $G \boxtimes \overline{G}$, it follows that

$$\alpha(G \boxtimes (\overline{G}, \rho)) \geq \sum_{v \in V} 1 \cdot \rho(v) = \sum_{v \in V} \langle h | \phi_v \rangle^2 = \vartheta(G).$$

Hence,

$$\vartheta(G) \leq \alpha(G \boxtimes (\overline{G}, \rho)) \leq \alpha(G \boxtimes (\overline{G}, \rho)) = \vartheta(G) \vartheta(G, \rho).$$ \hspace{1cm} (7)

On the other hand, the first characterization of the Lovász number $\vartheta$ of a weighted graph given in \[14\], Sec. 5\] states that

$$\vartheta(\overline{G}, \rho) = \min_{\{ |\phi_v\rangle : h\rangle \}} \left( \max_{v \in V} \frac{\rho(v)}{\langle h | \phi_v \rangle^2} \right),$$

where the minimum is taken over all orthonormal representations and handles of $\overline{G}$. Since $\{ |\phi_v\rangle : v \in V \}$ with $|h\rangle$ is one candidate, a bound on the Lovász number of $(\overline{G}, \rho)$ is

$$\vartheta(\overline{G}, \rho) \leq \max_{v \in V} \frac{\rho(v)}{\langle h | \phi_v \rangle^2} = 1.$$

Hence, $\vartheta(G) \vartheta(G, \rho) \leq \vartheta(G)$ and the inequalities in (7) turn into equalities, i.e.

$$\alpha(G \boxtimes (\overline{G}, \rho)) = \vartheta(G) \vartheta(G, \rho) = \vartheta(G \boxtimes (\overline{G}, \rho)),$$ \hspace{1cm} (8)

as well as $\vartheta(\overline{G}, \rho) = 1$, concluding the proof. \hfill $\Box$

Now we come to our first main result of this paper; we show that \[11\] is attained asymptotically.

**Theorem 2.** For every graph $G$,

$$\sup_{H} \frac{\alpha(G \boxtimes H)}{\vartheta(G \boxtimes H)} = 1, \quad \text{or equivalently:} \quad \sup_{H} \frac{\alpha(G \boxtimes H)}{\vartheta(H)} = \vartheta(G).$$

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Before proving this, we recall the definition of blow-up of an integer-weighted graph, and a couple of auxiliary results from [13].

**Definition 3** (cf. Acín et al. [15, Def. A.2.9]). Let \((G, p)\) be a weighted graph with integer weights \(p(v) \in \mathbb{N}_{>0}\) for all \(v \in V\). Then the blow-up \(\Blup(G, p)\) is the unweighted graph with vertex set

\[
V(p) := \{(v,i) : v \in V, i \in \{1,\ldots,p(v)\}\},
\]

where \((v,i)\) and \((w,j)\) are adjacent in \(\Blup(G, p)\) if and only if \(vw\) is an edge in \(G\). In other words, each vertex \(v\) of \(G\) is “blown up” to an independent set \(K_{p(v)}\).

**Lemma 4** (Acín et al. [15, Lemma A.2.7]). Let \((G, p)\) be a weighted graph, \(q \geq 0\) and \(X \in \{\alpha, \Theta, \vartheta, \alpha^*\}\). Then,

\[
X(G, p) \leq X(G, r) \leq X(G, p + q1) \leq X(G, p) + q|V|, \tag{9}
\]

for any weight \(r\) with \(p(v) \leq r(v) \leq p(v) + q\) for all vertices \(v\) of \(G\). \(
\)

**Lemma 5** (Acín et al. [15, Lemma A.2.10]). For integer vertex weights \(p(v) \in \mathbb{N}_{>0}\),

1. \(\Blup(G_1 \boxtimes G_2, p_1 p_2) = \Blup(G_1, p_1) \boxtimes \Blup(G_2, p_2)\);
2. \(X(\Blup(G, p)) = X(G, p)\) for every \(X \in \{\alpha, \Theta, \vartheta, \alpha^*\}\).

**Proof of Theorem 2**. For any two graphs \(G\) and \(H\), Lovász’ fundamental inequality is \(\alpha(G \boxtimes H) \leq \vartheta(G \boxtimes H) = \vartheta(G) \vartheta(H)\), so only the achievability of the opposite inequality by a sequence of graphs \(H\) has to be demonstrated.

We use Lemma 1 giving us a weight \(p : V \rightarrow \mathbb{R}_{\geq 0}\) such that \(\alpha(G \boxtimes (\overline{G}, p)) = \vartheta(G \boxtimes (\overline{G}, p))\). Now, consider the sequence of graphs \(H_\ell := \Blup(\overline{G}, [\ell p])\); we claim that indeed, \(\alpha(G \boxtimes H_\ell) \sim \vartheta(G \boxtimes H_\ell)\) as required.

To see this, multiply every term in (5) by an integer \(\ell > 0\). Since the functions \(\alpha\) and \(\vartheta\) satisfy \(\ell X(G, p) = X(G, \ell p)\), it follows that

\[
\alpha(G \boxtimes (\overline{G}, \ell p)) = \vartheta(G) \vartheta(\overline{G}, \ell p). \tag{10}
\]

Now, by Lemma 4,

\[
\alpha(G \boxtimes (\overline{G}, \ell p)) \leq \alpha(G \boxtimes (\overline{G}, [\ell p])) \leq \alpha(G \boxtimes (\overline{G}, \ell p + 1)) \leq \alpha(G \boxtimes (\overline{G}, \ell p)) + |V|^2, \tag{11}
\]

and similarly

\[
\vartheta(\overline{G}, \ell p) \leq \vartheta(\overline{G}, [\ell p]) \leq \vartheta(\overline{G}, \ell p + 1) \leq \vartheta(\overline{G}, \ell p) + |V|. \tag{12}
\]

In addition, Lemma 5 implies that

\[
\alpha(G \boxtimes H_\ell) = \alpha(G \boxtimes (\overline{G}, [\ell p])),
\]

hence putting this together with eqs. (10), (11) and (12) we get

\[
\alpha(G \boxtimes H_\ell) \geq \alpha(G) \vartheta(\overline{G}, \ell p) \geq \vartheta(G) \left(\vartheta(H_\ell) - |V|\right) \geq \vartheta(G) \vartheta(H_\ell) - |V|^2.
\]

Since \(\vartheta(H_\ell) \rightarrow \infty\) with growing \(\ell\), the claim follows. \(
\)

**Remark** From the proof, we see that

\[
\sup_H \frac{\alpha(G \boxtimes H)}{\vartheta(H)} = \vartheta(G),
\]

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for any graph parameter $a(G \boxtimes H)$ in the numerator bounded between $\alpha(G \boxtimes H)$ and $\vartheta(G \boxtimes H)$, such as $\tilde{\alpha}(G \boxtimes H)$, the \textit{entanglement-assisted independence number} \cite{16,17}. Shannon’s original zero-error capacity $\Theta(G \boxtimes H)$, or Schrijver’s variant $\vartheta^-(G \boxtimes H)$ of the Lovász number — see Section \ref{sec:4} below.

This shows that the only upper bound on $\alpha$ that is (sub-)multiplicative under strong graph products, and is at least as good as $\vartheta$, is the Lovász number itself.

We can also give an information theoretic interpretation of Theorem \ref{thm:2} based on the recent discovery that $\vartheta(H)$ is precisely the zero-error capacity assisted by no-signalling correlations, of quantum channels with confusability graph $H$ \cite{18}. Hence the quotient $\frac{\alpha(G \boxtimes H)}{\vartheta(H)}$ is the ratio between how much we can communicate through $G$ with the aid of some $H$ that we “borrow”, and the “value” of that other channel.

\section{Duality of independence number and fractional packing number}

Taking inspiration from the second formulation of Theorem \ref{thm:2} we might wonder why we should have the Lovász number in the denominator. Perhaps more than one reader might object that it would be more natural to compare like with like, i.e. $\alpha$ with $\alpha$.

\begin{theorem} \text{(Rosenfeld \cite{6}, Hales \cite{10}).} \label{thm:6}
For every pair of graphs $G = (V,E)$ and $H = (V',E')$,
\begin{equation}
\alpha(G \boxtimes H) \leq \alpha^*(G) \alpha(H). \tag{13}
\end{equation}
Furthermore, this is tight for every $G$ and $H$ individually: Namely, there exist graphs $G'$ and $H'$ such that
\begin{align*}
\alpha(G \boxtimes H') &= \alpha^*(G) \alpha(H'), \tag{14} \\
\alpha(G' \boxtimes H) &= \alpha^*(G') \alpha(H). \tag{15}
\end{align*}
In other words, for all graphs $G$,
\begin{equation*}
\max_H \frac{\alpha(G \boxtimes H)}{\alpha(H)} = \alpha^*(G), \quad \max_H \frac{\alpha(G \boxtimes H)}{\alpha^*(H)} = \alpha(G).
\end{equation*}
\end{theorem}

\begin{proof}
All of this is (implicitly) included in the proof of \cite{6}, Thm. 2. We rephrase Rosenfeld’s proof in our terms, which seems slightly more direct to us and is more geared towards our objective.

The first part is identical to Hales’ proof of \cite{12,11} Thm. 4.2. Let $I \subset G \boxtimes H$ be an independent set of maximum size $\alpha(G \boxtimes H)$. Define, for vertices $v \in V$,
\begin{equation*}
f(v) := \frac{1}{\alpha(H)}|\{(v) \boxtimes H) \cap I|.
\end{equation*}

We claim that $f$ is a fractional packing of $G$. Indeed, for any clique $C \subset G$, $I_C := (C \boxtimes H) \cap I$ is an independent set of $C \boxtimes H$, which means that $I_C$ intersects each $C \boxtimes \{w\}$, $w \in V'$, in at most one point. Hence,
\begin{equation*}
J := \{w : \exists v \in C \ (v,w) \in I\}
\end{equation*}
is an independent set with $|I_C| = |J| \leq \alpha(H)$, and so
\begin{equation*}
\sum_{v \in C} f(v) = \frac{1}{\alpha(H)}|C \boxtimes H) \cap I| \leq 1.
\end{equation*}
But now,
\begin{equation*}
\alpha^*(G) \geq \sum_{v \in V} f(v) = \frac{1}{\alpha(H)}|I| = \frac{\alpha(G \boxtimes H)}{\alpha(H)},
\end{equation*}
proving the inequality \ref{eq:13}.

Eq. \ref{eq:15} is trivial with $G'$ any complete graph.
To prove eq. (14), consider an optimal fractional packing of $G$: $f(v) = \frac{n(v)}{N}$, with non-negative integers $N$ and $n(v)$; in particular, $\alpha^\ast(G) = \frac{1}{N} \sum_{v \in V} n(v)$. [Recall that the fractional packing number is an LP, hence it has an optimal solution consisting only of rational numbers.] Now let $H' = \text{Blup}(G, n)$, which we claim is the graph we are looking for. Indeed,

$$\alpha(H') = \max_{\mathcal{C} \subseteq G} \sum_{v \in \mathcal{C}} n(v) \leq N,$$

the first identity by the observation that the maximal independent sets are exactly the blow-ups of independent sets of $G$, the second inequality by the definition of a fractional packing. On the other hand, because the blown-up diagonal $\{(v, (v, \ell)) : v \in V, 1 \leq \ell \leq n(v)\}$ is an independent set in $G \boxtimes H'$, we have

$$\alpha(G \boxtimes H') \geq \sum_{v \in V} n(v) = N \alpha^\ast(G) \geq \alpha^\ast(G) \alpha(H').$$

As we know the opposite inequality already, this concludes the proof.

It may be instructive, or entertaining, to view Theorems 2 and 6 as some kind of tight combinatorial Hölder inequalities: The expression on the left hand side of eq. (13), which is a function of the graph product, is upper bounded by the product of functions of the factor graphs:

$$\alpha(G \boxtimes H) \leq b(G) c(H).$$

If for every graph $G$ (H) there exists an $H$ (G) making the above an equality, or an asymptotic equality, we call $b$ and $c$ (asymptotically) dual with respect to $a$, and the parameter $a$ the pivot of the duality. Rosenfeld’s Theorem 3 shows that $\alpha$ and $\alpha^\ast$ are dual with respect to $\alpha$, and Theorem 2 says that $\vartheta$ is asymptotically self-dual with respect to $\alpha$.

We are thus led to consider more general upper bounds on $\alpha(G \boxtimes H)$ in terms of products $b(G) c(H)$, with special attention to dual pairs. We do not know as of yet how to characterize all dual pairs for $\alpha$. However, in the next section we shall show a third example.

4. Duality of $\vartheta^-$ and $\vartheta^+$ with respect to $\alpha$

Schrijver’s variant $\vartheta^-$ [19, 20] and Szegedy’s variant $\vartheta^+$ [21] of the Lovász number are defined as follows:

$$\vartheta^-(G) = \max \text{Tr } BJ \text{ s.t. } B \geq 0, \text{ Tr } B = 1, B_{vw} \geq 0 \forall v, w, B_{vw} = 0 \forall vw \in E,$$

$$\vartheta^+(G) = \max \text{Tr } BJ \text{ s.t. } B \geq 0, \text{ Tr } B = 1, B_{vw} \leq 0 \forall vw \in E.$$

(See Appendix A for equivalent characterizations and more properties of these two parameters.) Then, we have

**Lemma 7.** For any two graphs $G = (V, E)$ and $H = (V', E')$,

$$\alpha(G \boxtimes H) \leq \vartheta^-(G \boxtimes H) \leq \vartheta^-(G) \vartheta^+(H) \leq \vartheta^+(G \ast H).$$

In particular, for a graph $G$ on $n$ vertices and its complement $H = \overline{G}$,

$$n \leq \alpha(G \boxtimes \overline{G}) \leq \vartheta^-(G) \vartheta^+(\overline{G}),$$

with equality if $G$ is vertex-transitive.

**Proof.** Schrijver and McEliece et al. proved $\alpha \leq \vartheta^-$ [19, 20]. The second and third inequality are proved via the primal and dual SDP characterizations of $\vartheta^\pm$.

$$\vartheta^-(G) \vartheta^+(H) \leq \vartheta^+(G \ast H):$$ We use the primal SDPs given above, according to which we choose feasible $B \geq 0$, $\text{Tr } B = 1$ and $C \geq 0$, $\text{Tr } C = 1$ for $\vartheta^-(G)$ and $\vartheta^+(H)$, respectively: $B_{vw} \geq 0$ for all $v, w$ and $B_{vw} = 0$...
Theorem 8. For every graph \( G = (V,E) \),

\[
\sup_H \frac{\alpha(G \boxtimes H)}{\vartheta^+(H)} = \vartheta^-(G), \tag{17}
\]

\[
\sup_H \frac{\alpha(G \boxtimes H)}{\vartheta^-(H)} = \vartheta^+(G). \tag{18}
\]

Proof. From Lemma 7 we know

\( \alpha(G \boxtimes H) \leq \vartheta^-(G) \vartheta^+(H) \),

hence the inequality “\( \leq \)” in both eqs. (17) and (18) follows. In the vertex-transitive case we have

\( \alpha(G \boxtimes G) = |V| = \vartheta^-(G) \vartheta^+(G) = \vartheta^+(G) \vartheta^-(G) \).

The general proof of “\( \geq \)” in eq. (17) is similar to Theorem 2. By Lemma 9 below, there exists a weight \( p : V \rightarrow \mathbb{R}_{\geq 0} \) such that

\( \alpha(G \boxtimes (G,p)) = \vartheta^-(G) \vartheta^+(G,p) \).

Now, letting \( H_\ell = \text{Blup}(G, [\ell \ p]) \) does the trick, observing that Lemmas 4 and 5 extend to \( \vartheta^\pm \).

Analogously, to prove eq. (18), we use Lemma 9 below once more, showing that there exists a weight \( q : V \rightarrow \mathbb{R}_{\geq 0} \) such that

\( \alpha(G \boxtimes (G,q)) = \vartheta^+(G) \vartheta^-(G,q) \).

As before, letting \( H_\ell = \text{Blup}(G, [\ell \ q]) \) does what we need, observing that Lemmas 4 and 5 extend to \( \vartheta^\pm \). \( \square \)

Lemma 9. For every graph \( G \), there exists a weight \( p \) on the vertices of the complementary graph \( H = \overline{G} \), such that

\( \alpha(G \boxtimes (\overline{G}, p)) = \vartheta^-(G) \vartheta^+(\overline{G}, p) \).

There also exists a weight \( q \) on \( H = \overline{G} \), such that

\( \alpha(G \boxtimes (\overline{G}, q)) = \vartheta^+(G) \vartheta^-(\overline{G}, q) \).
Proof. As one might expect, this goes very similar to the proof of Lemma 1 using the characterizations of $\vartheta^\pm$ in Appendix A.

For the first identity, according to eq. (A.10), we can find a non-negative orthonormal representation $|\phi_v\rangle$ of $G$ (meaning that $\langle \phi_v | \phi_w \rangle \geq 0$ for all vertices $v, w$) and a consistent unit vector $|h\rangle$ (meaning that $\langle h | \phi_v \rangle \geq 0$ for all $v$), such that $\vartheta^-(G) = \sum_{v \in V} |\langle h | \phi_v \rangle|^2$. On the other hand, this non-negative OR is feasible for $\vartheta^+$ of the complementary graph, according to eq. (A.11), and its weighted analogue. Hence, with $p(v) = |\langle h | \phi_v \rangle|^2$, we have $\vartheta^+(\overline{G}, p) \leq 1$. Now, as in the proof of Lemma 1 the diagonal $\{(v, v) : v \in V\}$ is an independent set in $G \boxdot (\overline{G}, p)$, with weight

$$\vartheta^-(G) = \sum_{v \in V} |\langle h | \phi_v \rangle|^2 \leq \alpha(G \boxdot (\overline{G}, p)) \leq \vartheta^-(G) \vartheta^+(\overline{G}, p) \leq \vartheta^-(G),$$

where we have used the weighted version of Lemma 7 and hence all of the above inequalities are identities.

For the second identity, we proceed very similarly. Indeed, according to eq. (A.15), we can find an obtuse representation $|\phi'_v\rangle$ of $G$ (meaning $\langle \phi'_v | \phi'_w \rangle \leq 0$ for all edges $vw \in E$) and a consistent unit vector $|h'|$, such that $\vartheta^+(G) = \sum_{v \in V} |\langle h' | \phi'_v \rangle|^2$. At the same time, this obtuse representation is feasible for $\vartheta^-$ of the complementary graph, according to eq. (A.9), and its weighted analogue. Hence, with $q(v) = |\langle h' | \phi'_v \rangle|^2$, we have $\vartheta^-(\overline{G}, q) \leq 1$. Now, as before, the diagonal $\{(v, v) : v \in V\}$ is an independent set in $G \boxdot (\overline{G}, q)$, with weight

$$\vartheta^+(G) = \sum_{v \in V} |\langle h' | \phi'_v \rangle|^2 \leq \alpha(G \boxdot (\overline{G}, q)) \leq \vartheta^-(G) \vartheta^+(\overline{G}, q) \leq \vartheta^+(G),$$

where we have used the weighted version of Lemma 7 and hence all of the above inequalities are identities.

$\square$

5. Analogues for the chromatic number as pivot

By the celebrated Sandwich Theorem, cf. [14],

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}) = \sigma(G),$$

where $\chi$ is the chromatic number and $\sigma$ the clique covering number of the graph $G$: $\chi(\overline{G}) = \sigma(G)$, because each valid colouring of a graph is a partitioning, or more generally covering, of its vertex sets by independent sets, which are precisely the cliques in the complementary graph. To avoid the awkward complements [observe $G \boxdot H = \overline{G * \overline{H}}$, so we have $\chi(G * H) = \sigma(G \boxdot \overline{H})$ and $\chi(G \boxdot H) = \sigma(\overline{G * \overline{H}})$], we will primarily present the following results in terms of the clique covering number, even though they may be better known or more attractive in their “chromatic” guise.

For all the other quantities introduced so far, there is a veritable “francesinha”:

$$\alpha(G) \leq \tilde{\alpha}(G) \leq \vartheta^-(G) \leq \vartheta(G) \leq \vartheta^+(G) \leq \alpha^*(G) \leq \sigma(G) = \chi(\overline{G}).$$

For the clique covering/chromatic number, both strong and disjunctive product yield interesting asymptotics; McEliece and Posner solved it for $\sigma(G \boxdot \overline{G})$ [22], and Witsenhausen initiated the study of $\sigma(G^{\oplus n})$ [23].

We start with the strong graph product, for which the older literature offers a duality between clique covering number and fractional packing/covering number:
**Theorem 10** (Cf. Hales [10], McEliece/Posner [22]). For every pair of graphs $G = (V, E)$ and $H = (V', E')$, 
\[ \sigma(G \boxtimes H) \geq \alpha^*(G) \sigma(H), \quad \text{i.e.} \quad \chi(G \ast H) \geq \alpha^*(G) \chi(H). \] (19)
Furthermore, this is (asymptotically) tight for every $G$ and $H$ individually. Namely, for all graphs $G$,
\[
\inf_{H} \frac{\sigma(G \boxtimes H)}{\sigma(H)} = \alpha^*(G), \quad \text{i.e.} \quad \inf_{H} \frac{\chi(G \ast H)}{\chi(H)} = \alpha^*(G),
\]
\[
\min_{H} \frac{\sigma(G \boxtimes H)}{\alpha^*(H)} = \sigma(G), \quad \text{i.e.} \quad \min_{H} \frac{\chi(G \ast H)}{\alpha^*(H)} = \chi(G).
\]

**Proof.** Hales’ proof of eq. (19) is quite similar to the proof of the Rosenfeld bound [13], now using the dual LP for $\alpha^*(G)$, eq. (A.10): Consider a minimal clique covering $C$ of $G \boxtimes H$, w.l.o.g. only using maximal cliques, which are of the form $C \boxtimes D$ for cliques $C \subset V$ and $D \subset V'$. Define
\[ g(C) := \frac{1}{\sigma(H)} |\{D : C \boxtimes D \in C\}|, \]
and confirm that it is a fractional covering of $G$. Indeed, for every vertex $v$ of $G$, the set
\[ \mathcal{D}(v) = \{D : \exists v \in C \text{ s.t. } C \boxtimes D \in C\} \]
is a clique covering of $H$, and so for all $v$,
\[ \sum_{C \ni v} g(C) = \frac{1}{\sigma(H)} |\mathcal{D}(v)| \geq 1. \]
On the other hand,
\[ \alpha^*(G) \leq \sum_{C \text{ clique}} g(C) = \frac{|C|}{\sigma(H)} = \frac{\sigma(G \boxtimes H)}{\sigma(H)}. \]

Regarding the asymptotic tightness, the second claim is trivial, taking any $H = \overline{K}_n$. For the first claim, recall the result of [22], which is the first step in the following:
\[ \alpha^*(G) = \inf_{n} \left( \frac{\sigma(G \boxtimes G^n)}{\sigma(G^n)} \right)^{1/n} \]
\[ = \inf_{n} \left( \prod_{k=0}^{n-1} \frac{\sigma(G \boxtimes G^{\left< k \right>})}{\sigma(G^{\left< k \right>})} \right)^{1/n} \]
\[ \geq \inf_{n} \min_{0 \leq k \leq n-1} \frac{\sigma(G \boxtimes G^{\left< k \right>})}{\sigma(G^{\left< k \right>})} \]
\[ = \inf_{n} \frac{\sigma(G \boxtimes G^{\left< n \right>})}{\sigma(G^{\left< n \right>})} \geq \inf_{H} \frac{\sigma(G \boxtimes H)}{\sigma(H)}, \]
and the latter we know already to be $\geq \alpha^*(G)$. \hfill \Box

**Remark** Comparing with Theorem 4 and its proof, only the Rosenfeld-Hales inequalities [13] and [19] are done in a similar fashion, but the achievability parts are very different. Indeed, for $\alpha$ we carefully construct a graph $H$ by blowing up the complement of $G$, attaining equality spot-on. For $\sigma$ instead we simply consider the sequence $H = G^{\left< k \right>}$ and get equality asymptotically.

This raises two questions: First, whether for every $G$ there exists an $H$ with $\sigma(G \boxtimes H) = \alpha^*(G) \sigma(H)$? And second, whether
\[ \sup_{n} \frac{\alpha(G \boxtimes G^{\left< n \right>})}{\alpha(G^{\left< n \right>})} = \alpha^*(G)? \]
Or to determine the limit, if it converges to some smaller value $\geq \Theta(G)$.

Going to the disjunctive product, which has more edges, hence smaller clique covering numbers, than the strong product, we have the following relations involving Lovász $\vartheta$’s and variants:
Theorem 11. For any graphs \( G \) and \( H \),
\[
\sigma(G * H) \geq \vartheta(G) \vartheta(H), \quad \text{i.e.} \quad \chi(G \boxtimes H) \geq \vartheta(G) \vartheta(H),
\]
\[
\sigma(G * H) \geq \vartheta^+(G) \vartheta^+(H), \quad \text{i.e.} \quad \chi(G \boxtimes H) \geq \vartheta^+(G) \vartheta^+(H).
\]

As a consequence, for every graph \( G \),
\[
\inf_H \frac{\sigma(G * H)}{\vartheta(H)} \geq \vartheta(G),
\]
\[
\inf_H \frac{\sigma(G * H)}{\vartheta^+(H)} \geq \vartheta^+(G),
\]
with the obvious equivalent expressions in terms of the chromatic number.

Proof. According to the Sandwich Theorem, cf. [14],
\[
\sigma(G * H) \geq \vartheta(G) \vartheta(H).
\]
In fact, it is even known that [21]
\[
\sigma(G * H) \geq \vartheta^+(G) \vartheta^+(H),
\]
where we have invoked Lemma 7. \( \square \)

Remark. We do not know whether any of the infima in Theorem 11 is actually equal to the given lower bounds; but comparison with Theorems 2 and 8 suggests this as a distinct possibility.

However, intrinsically perhaps most interesting is the question of determining
\[
\zeta(G) := \inf_H \frac{\sigma(G * H)}{\vartheta(H)} = \inf_H \frac{\chi(G \boxtimes H)}{\chi(H)},
\]
of which we can trivially say that it is not larger than the Witsenhausen rate [23]
\[
R_W(G) = \inf_n (\sigma(G^n))^1/n = \inf_n (\chi(G^n))^1/n,
\]
by considering \( H = G^*k \). By analogy with Theorem 6, one might expect some kind of fractional combinatorial parameter, but we are not even aware of nontrivial lower bounds on (20).

6. Discussion

Many natural graph parameters arising as combinatorial optimization problems, such as independence number or chromatic number, are not generally multiplicative under graph products, but due to their nature retain super-multiplicativity (\( \alpha \) under the strong product) or sub-multiplicativity (\( \sigma, \chi \) under both the strong and disjunctive product), and this extends to numerical parameters such as \( \vartheta^{\pm} \). Some few, concretely the fractional packing and clique covering number, and the Lovász number miraculously turn out to be multiplicative (the first under strong products, the second under both strong and disjunctive products). For the others, there is the nontrivial problem of characterizing the regularizations
\[
\Theta(G) = \sup_n (\alpha(G^n))^1/n, \quad R_W(G) = \inf_n (\sigma(G^n))^1/n, \quad R^*(G) = \inf_n (\sigma(G^n))^1/n,
\]
only the last of which is known: McEliece and Posner showed it to equal the fractional packing number \( \alpha^*(G) \) [22].
In the present paper, we diverted from this consideration of the behaviour of graph parameters under the product of many copies of $G$, and looked more broadly how they are affected by products with a generic other graph $H$. After showing that the Lovász number is asymptotically attained by the independence number for every graph $G$ when activated by suitable graphs $H$, we embarked on a study of tight upper bounds on the independence number of graph products in terms of products of individual, “dual”, graph parameters. We could give some examples of such pairs, but have not been able to construct a general theory.

There are many questions left to be answered. For example, what are the pairs of dual graph parameters for $\alpha$, the entanglement-assisted independence number (beyond the self-dual $\vartheta$)?

Some of the most intriguing questions arise around dual pairs of which one is the same function as the pivot; already the determination of the other quantity in the bound, i.e. for example

$$\sup_H \frac{\Theta(G \boxtimes H)}{\Theta(H)} \quad \text{or} \quad \sup_H \frac{\tilde{\alpha}(G \boxtimes H)}{\tilde{\alpha}(H)},$$

is highly nontrivial. The first one is easily seen to be $\leq \alpha^*(G)$, so the question is whether there is a gap; for the second one we do not even have a target. In the same category falls the determination of $\zeta(G)$ in eq. (20). All these quantities are of the type of potential capacities – cf. [24], where they are studied in detail for the ordinary classical, quantum, private and other capacities of quantum channels.

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Appendix A. Fractional and semidefinite relaxations of the independence number

Here we collect several known, and a couple of new, useful characterizations of $\vartheta$, $\vartheta^-$ and $\vartheta^+$ as optimization problems, in particular SDPs. The graph will always be an unweighted graph $G = (V, E)$, although all of the formulas below have analogues with weights, cf. Knuth’s [14]. Recall the Lovász convention of denoting confusability of vertices as $v \sim w$, meaning equality ($v = w$) or an edge ($vw \in E$). An orthonormal representation (OR) of $G$ is an assignment of unit vectors $\{\phi_v\}$ in some (real) vector space to all vertices $v \in V$, such that $\langle \phi_v | \phi_w \rangle = 0$ for all $v \neq w$.

All of the following are from Lovász [2]:

$$\vartheta(G) = \max \| 1 + T \| \quad \text{s.t.} \quad 1 + T \succeq 0, \quad T_{vw} = 0 \ \forall v \sim w$$

(A.1)

$$= \max \text{Tr} BJ \quad \text{s.t.} \quad B \succeq 0, \quad \text{Tr} B = 1, \quad B_{vw} = 0 \ \forall vw \in E$$

(A.2)

$$= \min \lambda \quad \text{s.t.} \quad Z \succeq J, \quad Z_{vv} = \lambda \ \forall v, \quad Z_{vw} = 0 \ \forall v \neq w$$

(A.3)

$$= \min \left( \max_v \frac{1}{|\langle h | \phi_v \rangle|^2} \right) \quad \text{s.t.} \quad \{ |\phi_v \rangle \} \text{ is an OR of } G, \ |h\rangle \text{ unit vector.}$$

(A.4)
Here, $J$ is the all-ones matrix.

Observe that in eq. (A.1), $1 + T$ is precisely the Gram matrix $\langle \phi_v | \phi_w \rangle_{vw}$ of an orthonormal representation of $\mathcal{G}$, and by the definition of the operator norm,

$$\vartheta(G) = \|1 + T\| = \sum_v |\langle h | \phi_v \rangle|^2,$$

(A.5)

for an eigenvector $|h\rangle$ of the largest eigenvalue of $\sum_v |\phi_v \rangle \langle \phi_v |$, which has the same spectrum as $1 + T$.

There are analogous formulas for Schrijver’s $\vartheta^-$, the second and third are from [19, 20], the fourth is due to de Carli Silva and Tunçel [25, Cor. 4.2]; see also [26]:

$$\vartheta^-(G) = \max_{|\beta\rangle} \|1 + T\| \text{ s.t. } 1 + T \succeq 0, \ T_{vw} \geq 0 \ \forall v, w, \ T_{vw} = 0 \ \forall v \sim w \tag{A.6}$$

$$= \max_{|\beta\rangle} \text{Tr} BJ \text{ s.t. } B \succeq 0, \ \text{Tr} B = 1, \ B_{vw} \geq 0 \ \forall v, w, \ B_{vw} = 0 \ \forall vw \in E \tag{A.7}$$

$$= \min \lambda \text{ s.t. } Z \succeq J, \ Z_{vw} = \lambda \ \forall v, \ Z_{vw} \leq 0 \ \forall v \neq w \tag{A.8}$$

$$= \min \left( \max_v \frac{1}{|\langle h | \phi_v \rangle|^2} \right) \text{ s.t. } \{|\phi_v\rangle\} \text{ is an obtuse rep. of } G, \ |h\rangle \text{ consistent unit vector}. \tag{A.9}$$

Here, an obtuse representation of $G$ is an assignment of unit vectors $\{|\phi_v\rangle\}$ to all vertices $v \in V$ such that $\langle \phi_v | \phi_w \rangle \leq 0$ for all $v \neq w$, and a unit vector $|h\rangle$ is called consistent if $\langle h | \phi_v \rangle \geq 0$ for all $v \geq 25$. The first relation, eq. (A.6), is proved by equating it with eq. (A.7). Namely, observe that

$$\|1 + T\| = \max_{|\beta\rangle} \langle \beta | (1 + T) | \beta\rangle = \max_{|\beta\rangle} \text{Tr} |\beta\rangle \langle \beta | (1 + T) = \max_{|\beta\rangle} \text{Tr} (|\beta\rangle \langle \beta | \circ (1 + T) J,$$

where the maximization is over unit vectors $|\beta\rangle$ with components $\beta_v, |\beta\rangle |\beta\rangle$ is the projection onto $\mathbb{C} |\beta\rangle$, and $\circ$ is the Schur/Hadamard (entry-wise) product of matrices. To attain the maximum, w.l.o.g. all vector components $\beta_v \geq 0$, so all entries $\beta_v \beta_w$ of $|\beta\rangle |\beta\rangle$ are non-negative, and so $B = |\beta\rangle \langle \beta | \circ (1 + T)$ is feasible for eq. (A.7). Conversely, any such $B$ we can write as $B = |\beta\rangle \langle \beta | \circ (1 + T)$ with a unit vector $|\beta\rangle$ with non-negative components, and a matrix $T$ feasible for eq. (A.7).

Note that in eq. (A.6), $1 + T$ is precisely the Gram matrix $\langle \phi_v | \phi_w \rangle_{vw}$ of a non-negative orthonormal representation of $\mathcal{G}$, i.e. $\langle \phi_v | \phi_w \rangle \geq 0$ for all $v$ and $w$ and $\langle \phi_v | \phi_w \rangle = 0$ for $vw \in E$. By the definition of the operator norm,

$$\vartheta^-(G) = \|1 + T\| = \sum_v \langle h | \phi_v \rangle^2 \tag{A.10}$$

for an eigenvector $|h\rangle$ of the largest eigenvalue of $\sum_v |\phi_v \rangle \langle \phi_v |$, which has the same spectrum as $1 + T$. Furthermore, one may assume $\langle h | \phi_v \rangle \geq 0$ for all $v \in V$. This is due to the Perron-Frobenius theorem [27], which guarantees that the Gram matrix $\langle \phi_v | \phi_w \rangle_{vw}$ has a unit eigenvector $|\mu\rangle = \sum_v \mu_v |v\rangle$ with non-negative entries $\mu_v$ for the largest eigenvalue $\theta = \|1 + T\|$. 

$$\|1 + T\| = \sum_{vw} \mu_v \mu_w \langle \phi_v | \phi_w \rangle = \langle X | X \rangle,$$

with $|X\rangle = \sum_v \mu_v |\phi_v\rangle = \sqrt{\theta} |h\rangle$. By construction, $\langle h | \phi_v \rangle \geq 0$, and one can check by direct calculation that

$$\left( \sum_v |\phi_v \rangle \langle \phi_v | \right) |h\rangle = \theta |h\rangle.$$

For Szegedy’s $\vartheta^+$ [21], instead, we have:

$$\vartheta^+(G) = \max \|1 + T\| \text{ s.t. } 1 + T \succeq 0, \ T_{vw} \leq 0 \ \forall v \sim w \tag{A.11}$$

$$= \max \text{Tr} BJ \text{ s.t. } B \succeq 0, \ \text{Tr} B = 1, \ B_{vw} \leq 0 \ \forall vw \in E \tag{A.12}$$

$$= \min \lambda \text{ s.t. } Z \succeq J, \ Z_{vw} = \lambda \ \forall v, \ Z_{vw} \geq 0 \ \forall v, w, \ Z_{vw} = 0 \ \forall v \neq w, \tag{A.13}$$

$$= \min \left( \max_v \frac{1}{|\langle h | \phi_v \rangle|^2} \right) \text{ s.t. } \{|\phi_v\rangle\} \text{ is a non-negative OR of } G, \ |h\rangle \text{ consistent unit vector}. \tag{A.14}$$
the above inner products are 0 by our choice of \( |\psi\rangle \).

Let \( \psi \neq 0 \) and \( x \) be arbitrary. We prove that

\[
\langle \psi | x \rangle = \sum_v \langle \psi_v | v \rangle = \sum_v \langle \psi | x \rangle_v = \sum_v \langle \psi | x \rangle_v \geq 0,
\]

as desired. To prove this, we can use the fact that \( \langle \psi | x \rangle_v \) is positive for all \( v \) by using the equivalence proof with \( (A.13) \).}

First, if \( |v \rangle \) is an edge, then \( \langle \psi | x \rangle_v \) is positive.

By Lemma 12 below, the value at least \( \text{Tr} |BJ\rangle \).

Note that in eq. (A.11), \( \langle |h\rangle | \) is positive.

Thus, all entries \( \beta_v \beta_w \) of the matrix \( |B\rangle \) are non-negative, and so \( |B| = |B\rangle \circ (\mathbb{1} + T) \) is feasible for eq. (A.12). Conversely, any such \( B \) we can write as \( B = |B\rangle \circ (\mathbb{1} + T) \) with a unit vector \( |\beta\rangle \) with non-negative components, and a matrix \( T \) feasible for eq. (A.11).

In Section 4, we need the following formulation of \( \vartheta^+ \):

\[
\vartheta^+(G) = \max \sum_v \langle |h\rangle | \vartheta_v \rangle^2 \text{ s.t. } \{ |\vartheta_v \rangle \} \text{ is an obtuse rep. of } G, \text{ } |h\rangle \text{ consistent unit vector.} \tag{A.15}
\]

Proof. Let \( B \) be an optimal solution in eq. (A.12) for \( G \). Since \( B \) is positive semidefinite, there exist vectors \( |\psi_v \rangle \) for \( v \in V \), such that \( B_{vw} = \langle \psi_v | \psi_w \rangle \). Let \( |\Psi\rangle = \sum_v |\psi_v \rangle \) and note that \( \langle \Psi | \Psi \rangle = \sum_v B_{vw} = \text{Tr} BJ \) is the objective function value of the solution \( B \). Furthermore, \( \langle \psi_v | \Psi \rangle \) is the \( v \)-th row sum of \( B \). Let \( |\phi_v \rangle = \frac{1}{\|v\|_{\psi_v}} |\psi_v \rangle \). (If the numerator is 0, let \( |\phi_v \rangle \) be a unit vector orthogonal to all others, by moving to a higher dimension if necessary.) Also, let \( |h\rangle = \frac{1}{\|\Psi\|_{\psi_v}} | \Psi \rangle \). We will show that this is a solution for (A.15) of value at least \( \text{Tr} BJ \).

First, if \(uvw \in E\) is an edge, then

\[
\langle \psi_v | \psi_w \rangle = \frac{\langle \psi_v | \psi_w \rangle}{\|v\|_{\psi_v} \|v\|_{\psi_w}} = \frac{B_{vw}}{\|v\|_{\psi_v} \|v\|_{\psi_w}} \leq 0,
\]

so we have an obtuse representation. Second,

\[
\langle \psi_v | \psi_w \rangle = \frac{\langle \psi_v | \psi_w \rangle}{\|v\|_{\psi_v} \|v\|_{\psi_w}} \geq 0,
\]

since \( \langle \Psi | \psi_v \rangle \) is the \( v \)-th row sum of \( B \) which is nonnegative by Lemma 12 below. Thus, \( |h\rangle \) is a consistent vector for the obtuse representation. Note that we should be a bit careful and point out that for \( |\psi_v \rangle = 0 \) the above inner products are 0 by our choice of \( |\psi_v \rangle \). Third, the objective function: Let \( S \) be the set of indices of the nonzero rows of \( B \), i.e. \( S = \{ v : |\psi_v \rangle \neq 0 \} \). We now have, using \( B_{vw} \geq 0 \),

\[
\sum_v \langle |h\rangle | \vartheta_v \rangle^2 = \sum_{v \in S} \frac{\|v\|_{\psi_v} \|v\|_{\psi_w}}{\|v\|_{\psi_v} \|v\|_{\psi_w}} = \text{Tr} BJ \sum_{v \in S} B_{vw} \frac{\langle \Psi | \psi_v \rangle}{B_{vw}}^2.
\]

Noting furthermore \( \sum_v B_{vw} = 1 \), we can use Jensen’s inequality to the convex function \( x^2 \), to obtain

\[
\sum_v \langle |h\rangle | \vartheta_v \rangle^2 \geq \frac{1}{\text{Tr} BJ} \left( \sum_{v \in S} B_{vw} \frac{\langle \Psi | \psi_v \rangle}{B_{vw}} \right)^2 = \frac{1}{\text{Tr} BJ} \left( \sum_{v \in S} \langle \Psi | \psi_v \rangle \right)^2 = \frac{1}{\text{Tr} BJ} \langle \Psi \rangle^2 = \text{Tr} BJ.
\]

This proves that the optimal solution to (A.15) is at least as large as the optimal solution to (A.12).
Lemma 12. If \( |\langle h, \phi_v \rangle| \) for \( v \in V \) be a solution for (A.15) of value \( \theta := \sum_v |\langle h, \phi_v \rangle|^2 \). Define

\[
|\psi_v| = \frac{1}{\sqrt{\theta}} \langle h, \phi_v | \phi_v \rangle |h\rangle,
\]

and let \( B \) be the Gram matrix of these \( |\psi_v| \), i.e. \( B_{vw} = \langle \psi_v | \psi_w \rangle \). We will show that \( B \) is a solution for (A.12) of value at least \( \theta \). To start, as a Gram matrix, it is positive semidefinite.

First, for an edge \( vw \in E \), we have that

\[
B_{vw} = \langle \psi_v | \psi_w \rangle = \frac{1}{\theta} \langle h, \phi_v | \phi_v \rangle |h\rangle |\phi_w \rangle |\phi_w \rangle = 0,
\]
since \( \langle h, \phi_v \rangle \geq 0 \) for all \( v \) and \( \langle \phi_v | \phi_w \rangle \leq 0 \) for \( vw \in E \). Second,

\[
\text{Tr} B = \sum_v \langle \psi_v | \psi_v \rangle = \frac{1}{\theta} \sum_v |\langle h, \phi_v \rangle|^2 = 1.
\]

Finally, letting \( M = \sum_v |\phi_v | \phi_v | \phi_v | \), we have

\[
\text{Tr} BJ = \sum_v \langle \psi_v | \psi_w \rangle = \frac{1}{\theta} \sum_v |\langle h, \phi_v | \phi_v \rangle |\phi_w \rangle |\phi_w \rangle | h\rangle = \frac{1}{\theta} |\langle h | M^2 | h \rangle| \geq \frac{1}{\theta} |\langle h | M | h \rangle | |\langle h | M | h \rangle| = \frac{1}{\theta} \left( \sum_v |\langle h, \phi_v \rangle|^2 \right)^2 = \theta,
\]

where in the third line we have used \( |\langle h | h \rangle| \leq 1 \), hence \( M^2 = MM \geq M |h\rangle |h\rangle |M \). This proves that the optimal solution to (A.12) is at least as large as the optimal solution to (A.15), concluding the proof. \( \square \)

Lemma 12. If \( B \) is an optimal solution to (A.12), then the row sum of any nonzero row of \( B \) is positive.

Proof. Suppose, by contradiction, that \( B \) is an optimal solution to (A.12) and that the \( v \)-th row of \( B \) is nonzero and has non-positive row sum. Since \( B \succeq 0 \), we have that \( B_{vv} \succeq 0 \), with equality if and only if the \( v \)-th row is all zero. This implies that \( B_{vv} > 0 \) and thus \( \sum_{v \neq v} B_{vv} \leq -B_{vv} < 0 \). Therefore, changing both the \( v \)-th row and column to zeros strictly increases the sum of the entries of \( B \), while decreasing the trace. Note that this change keeps \( B \) positive semidefinite, since it is equivalent to changing a vector in the Gram representation of \( B \) to the zero vector. Therefore we can positively scale this new matrix to have trace 1 and greater sum of all entries, giving us a better solution to (A.12), a contradiction. \( \square \)

In this paper, we also looked at weighted Lovász numbers \( \vartheta(G, p) \) and variants \( \vartheta^\pm(G, p) \). These are defined by replacing the all-ones matrix \( J \) in eqs. (A.2), (A.7) and (A.12) by the weights matrix \( \Pi = \left[ \sqrt{p(v)} p(w) \right]_{vw} \). The other formulas are changed accordingly; in particular eqs. (A.4), (A.9) and (A.14) simply receive the weight \( p(v) \) in the numerator.

Finally, we record here the mutually dual LPs of fractional packing and fractional clique covering:

\[
\alpha^*(G) = \max \sum_v t_v \text{ s.t. } t_v \geq 0 \forall v, \sum_{v \in C} t_v \leq 1 \forall \text{ cliques } C \subset V
\]

\[
= \min \sum_C s_C \text{ s.t. } s_C \geq 0 \forall \text{ cliques } C \subset V, \sum_{C \cap v} s_C \geq 1 \forall v.
\]

In particular, \( \alpha(G) \leq \alpha^*(G) \leq \sigma(G) \).

Cf. the very nice book [3] for details on these, where it is also discussed that a natural notion of fractional colouring and fractional chromatic number leads to the same LP.

One thing we can check easily is the multiplicativity of \( \alpha^* \) under strong graph products: \( \alpha^*(G \boxtimes H) = \alpha^*(G) \alpha^*(H) \). Indeed, since the product of primal feasible solutions for \( \alpha^*(G) \) and \( \alpha^*(H) \) is feasible for \( \alpha^*(G \boxtimes H) \), we obtain “\( \geq \)”. Likewise, “\( \leq \)” follows by observing that the product of dual feasible solutions of the two graphs is dual feasible for \( \alpha^*(G \boxtimes H) \).
