On the Cauchy Problem for the Cutoff Boltzmann Equation with Small Initial Data

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Abstract
We prove the global existence of the non-negative unique mild solution for the Cauchy problem of the cutoff Boltzmann equation for soft potential model $-1 \leq \gamma < 0$ with the small initial data in three dimensional space. Thus our result fixes the gap for the case $\gamma = -1$ in three dimensional space in the authors’ previous work (He and Jiang in J Stat Phys 168(2):470–481, 2017) where the estimate for the loss term was improperly used. The other gap in He and Jiang (2017) for the case $\gamma = 0$ in two dimensional space is recently fixed by Chen et al. (Arch Ration Mech Anal 240:327–381, 2021). The initial data $f_0$ is non-negative and satisfies that $\|\langle v \rangle^{\ell \gamma} f_0(x,v)\|_{L^3_{x,v}} \ll 1$ and $\|\langle v \rangle^{\ell \gamma} f_0\|_{L^{15/8}_{x,v}} < \infty$ where $\ell \gamma = 0$ when $\gamma = -1$ and $\ell \gamma = (1 + \gamma)^+$ when $-1 < \gamma < 0$. We also show that the solution scatters with respect to the kinetic transport operator. The novel contribution of this work lies in the exploration of the symmetric property of the gain term in terms of weighted estimate. It is the key ingredient for solving the model $-1 < \gamma < 0$ when applying the Strichartz estimates.

Keywords Boltzmann equation · Cutoff · Soft potential · Small initial data · Strichartz estimate

1 Introduction
We consider the Cauchy problem for the cutoff Boltzmann equation

$$\begin{cases}
\partial_t f + v \cdot \nabla_x f = Q(f, f) \\
f(0, x, v) = f_0(x, v)
\end{cases} \quad (1.1)$$

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in \((0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N, N = 2, 3\), where the initial data is small in \(L^N_{x,v}\) space. Recall that the collision operator \(Q(f, f)\) is given by

\[
Q(f, f)(v) = \int_{\mathbb{R}^N} \int_{\omega \in S^{N-1}} (f' f'_* - ff_*) B(v - v_*, \omega) \, d\omega \, dv_* ,
\]

and \(d\omega\) is the solid element in the direction of unit vector \(\omega\). We use the abbreviations \(f' = f(x, v', t), f'_* = f(x, v'_*, t), f_* = f(x, v, t)\), and the relation between the pre-collisional velocities of particles and after collision is given by

\[
v' = v - [\omega \cdot (v - v_*)] \omega, \quad v'_* = v_* + [\omega \cdot (v - v_*)] \omega, \quad \omega \in S^{N-1}.
\] (1.2)

In this paper, we consider the cutoff soft potential model, i.e., the collision kernel \(B\) being the product of kinetic part and angular part,

\[
B(v - v_*, \omega) = |v - v_*|^\gamma b(\cos \theta), \quad 0 \leq \theta \leq \pi/2 ,
\] (1.3)

where \(-N < \gamma < 0, \cos \theta = (\omega, (v - v_*)/(v - v_*))\),

and the angular function \(b(\cos \theta)\) satisfies the Grad’s cutoff assumption,

\[
\int_{S^{N-1}} b(\cos \theta) d\omega < \infty.
\] (1.4)

When \(\gamma = 0\), the kernel (1.3) is called the Maxwell molecules. When the cutoff condition (1.4) is satisfied, the collision operator \(Q\) can be split into the gain term \(Q^+\) and the loss term \(Q^-\).

It is useful to introduce the bilinear gain term

\[
Q^+(f, g)(v) = \int_{\mathbb{R}^N} \int_{S^{N-1}} f(v') g(v'_*) B(v - v_*, \omega) \, d\omega \, dv_* ,
\]

and the bilinear loss term

\[
Q^-(f, g)(v) = \int_{\mathbb{R}^N} \int_{S^{N-1}} f(v) g(v_*) B(v - v_*, \omega) \, d\omega \, dv_* .
\]

\[1.1\] Short Review

Let us briefly recall the progress on the Cauchy problem (1.1) for cutoff model with small initial data. To the best of our knowledge, Illner and Shinbrot [11] first showed the global existence of solutions for the Cauchy problem (1.1) for several cutoff models when the initial data has exponential decay in spatial variable and has suitable weight in velocity variable. They also discussed the asymptotic behavior of the solutions. The iteration method in [11] comes from the earlier work [14] of Kaniel and Shinbrot who designed it for the study of the initial boundary value problem in a bounded domain. Now it is called the Kaniel–Shinbrot iteration. Various of results about the small initial data Cauchy problem (1.1) for different models were then obtained during that decade by many authors through the same iteration or fixed point argument, see [4, 9] and reference therein for more details. Please note that the assumption that the initial data has exponential decay in spatial variable or in velocity variable is necessary to get these results.

With the first appearance of Strichartz estimates for the kinetic equation in the note of Castella and Perthame [7], this family of estimates seems to be a promising tool to solve (1.1) with initial data being small in Lebesgue space instead of decaying exponentially. Indeed,
with initial data small in Lebesgue assumption, Bournaveas et al. [6] used it to prove the existence of global weak solution for a nonlinear kinetic system modeling chemotaxis. For the Boltzmann equation, Arsénoï [2] considered a non-conventional collision kernel whose kinetic part is $L^p$ integrable for some $p$ depending on dimension, and then proved the existence of global weak solution for small data. But the uniqueness of the solution is unknown. The reason that the Strichartz estimates is not handy as one expects in solving small initial data Boltzmann equation lies in the fact that the loss term does not enjoy the same symmetry as the gain term does in the Lebesgue spaces. More precisely, the gain and loss terms both satisfy
\[
\|Q^\pm(f,g)\|_{L^r_x(R^N)} \leq C\|f\|_{L^p_x(R^N)}\|g\|_{L^q_x(R^N)},
\] (1.5)
where the norm is taking on the velocity variable and the exponents $p, q, r$ satisfying the scaling condition
\[
1/p + 1/q = 1 + \gamma/N + 1/r,
\] (1.6)
while the estimate for the loss term requires additional condition,
\[
1/p < 1/r,
\] (1.7)
which means that $f$ and $g$ need to be treated differently when dealing with the loss term. The estimate for the loss term in the authors’ previous work [10], i.e., the Lemma 2.4 in [10] includes the constraint (1.7) in the proof but not the statement of that Lemma. This constraint was then accidentally neglected when applying the Strichartz estimates to the loss term to solve the Cauchy problem. Therefore the result in [10], c.f. Proposition 2.4, holds for the gain term only Boltzmann equation instead of full equation (see also the paragraph before and after Lemma 2.7 below). More precisely, the work [10] pointed out that when the exponent of kinetic part equals $\gamma = 2 - N$, we can find suitable Strichartz spaces, c.f. (2.15), where the global mild solution for gain term only Boltzmann equation exists if the initial data $f_0$ is small in $L^N_x$. Very recently, Chen et al. [8] studied the full Boltzmann equation for the case $N = 2$, Maxwell molecules, by a different approach. Their idea can be sketched as follows. Using the fact that the kinetic transport equation can be converted to the free Schrödinger equation by Wigner transform and vice versa by inverse Wigner transform, they proved the spacetime estimates for the nonlinear Schrödinger equation to conclude the existence of mild solution for gain term only Boltzmann equation when the initial data is small enough in $L^2_{x,v}$. Due to the fact that kinetic transport operator, weight in velocity and differential operator for spatial variable are commuting with each other, they showed that if $f_0$ is small in $L^2_{x,v}$ and additionally $\|\langle \nu \rangle^{1/2} + (\nabla_x)^{1/2} f_0\|_{L^2_{x,v}}$ is finite, then the quantity $\|\langle \nu \rangle^{1/2} + (\nabla_x)^{1/2} f_+\|_{L^\infty_t L^2_{x,v}}$ remains finite where $f_+$ denotes the solution of the gain term only Boltzmann equation. The propagation of regularity and moment thus ensures the loss term is well-defined when plugging in $f_+$. Note the fact that if the initial $f_0$ is non-negative, then solution of the gain term only Boltzmann equation is also non-negative. Combining all the facts, using the solution of gain term only Boltzmann equation as an upper bound of the “beginning condition” of the Kaniel–Shinbrot iteration, one can ensure the global existence of the mild solution for the full equation. The uniqueness of the solution is not provided by Kaniel–Shinbrot iteration. Fortunately it can be saved by the fact that the solution of the gain term only Boltzmann equation is an upper bound of that for the full equation and the former lies in solution space already. The scattering and propagation of moment and regularity for the solution of the full equation are also proved by the similar idea.
1.2 Main Results

The main purpose of this paper is to solve the problem (1.1) for the case \( N = 3 \) when \( \gamma \) satisfying \(-1 \leq \gamma < 0\). Instead of using the correspondence between the Schrödinger equation and kinetic transport equation, the result of [10] for the gain term only Boltzmann equation will be our starting point, c.f. Proposition 2.4. On the other hand, we should adopt the strategy of [8] to recover the solution for the full equation from that for the gain term only Boltzmann equation. To get rid of the fact that the loss term is not symmetric, we also need an additional assumption about the initial data besides it is small in velocity-weighted \( L^3_{x,v} \) (no weight is needed when \( \gamma = -1 \)). We should assume that the initial data is also bounded in velocity-weighted \( L^{15/8}_{x,v} \) (no weight is needed when \( \gamma = -1 \)). Here the exponent 15/8 is just one of possible options. Please note that our additional assumption for the initial data does not require the additional regularity in spatial variable nor additional weight in velocity variable.

This difference on assumptions reflects the difference of the two methods. It is interesting and worth to explore more about this. First we note that the exponent \( \gamma = 2 - N \) of kinetic part of the collision kernel is special in the sense that it is scaling critical case in the content of dispersive equation. The spacetime \( L^2_{x,v} \) estimates in [8] is non-trivial since it is an end point estimate. Comparing \( L^2_{x,v} \) space with the \( L^r_x L^p_v \), \( r < N = 2, p > 2 \) in Proposition 2.4, it is not surprising that to recover the solution for full equation from that of gain term only equation in the \( L^2_{x,v} \) space, one needs to require the initial data has additional regularity in spatial variable as well as additional weight in velocity variable. If one follows the approach of [8] to study the case \( N = 3 \), both requirements seem to be unavoidable again. Also the other difficulty that will be encountered is that the exponent of kinetic part of the collision kernel under consideration is \( -1 \leq \gamma < 0 \) when \( N = 3 \) which is unlike \( \gamma = 0 \) when \( N = 2 \) as the latter is more convenient when applying the Fourier transform to the gain term of the collision operator.

The novel contribution of this work is that the symmetric property of the gain term is explored further in terms of weighted estimate and this is the key step to study the model with \(-1 < \gamma < 0 \) when applying the Strichartz estimate to solve the problem. The gain term enjoys two different estimates based on two different scaling relations, i.e.,

\[
\| \langle v \rangle^{\ell} Q^+(f, g) \|_{L^q_x L^p_v(\mathbb{R}^N)} \leq C \| \langle v \rangle^{\ell} f \|_{L^p_v L^q_x(\mathbb{R}^N)} \| \langle v \rangle^{\ell} g \|_{L^q_v L^p_x(\mathbb{R}^N)},
\]

where \( 1/p + 1/q = 1 + \gamma/N + 1/\tau, \ \ell \geq 0 \), and

\[
\| \langle v \rangle^{\ell} Q^+(f, g) \|_{L^{\tau m}_x L^{p_m}_v} \leq C (p_m, \ell) \| \langle v \rangle^{\ell} f \|_{L^{p_m}_v L^{\tau m}_x} \| \langle v \rangle^{\ell} g \|_{L^{q_m}_v L^{p_m}_x},
\]

where \( 1/p_m + 1/q_m + 1/m = 1 + \gamma/N + 1/\tau_m, \ \ell > N/m \). On the other hand, the loss term only satisfies the second estimate above while the constraint (1.7) is unchanged. Please see Propositions 4.1 and 4.4 for more details. This new discovery on the property of gain term allows us to solve the Cauchy problem (1.1) for the soft potential model with exponent \(-1 < \gamma < 0 \) beyond \( \gamma = -1 \). It seems to us that this approach is more straight forward for the case \( N = 3 \), thus our argument is shorter than that in [8]. Unfortunately this method does not work for the case \( N = 2 \) since \( \gamma < 0 \) is below the critical case \( \gamma = 0 \) while \(-1 < \gamma < 0 \) is above critical case \( \gamma = -1 \) for \( N = 3 \).

To state the main results, let us introduce the mixed Lebesgue norm

\[
\| f(t, x, v) \|_{L^q_t L^r_x L^p_v},
\]
where the notation $L^q_i L^r_x L^p_v$ stands for the space $L^q(\mathbb{R}; L^r(\mathbb{R}^N; L^p(\mathbb{R}^N)))$. It is understood that we use $L^q_i(\mathbb{R}) = L^q_i((0, \infty))$ for the well-posedness problem which can be done by imposing support restriction to the inhomogeneous Strichartz estimates. We use $L^q_{x,v}$ to denote $L^q_s(\mathbb{R}^N; L^q_N(\mathbb{R}^N))$.

We also need to give a precise meaning of the scattering of the solution with respect to kinetic transport operator. Here we say that a global solution $f \in C([0, \infty), L^q_{x,v})$ scatters in $L^q_{x,v}$ as $t \to \infty$ if there exits $f_\infty \in L^q_{x,v}$ such that

$$\|f(t) - U(t)f_\infty\|_{L^q_{x,v}} \to 0$$

where $U(t)f(x, v) = f(x - vt, v)$ is the solution map of the kinetic transport equation

$$\partial_t f + v \cdot \nabla_x f = 0.$$

Please see the interesting discussion in [3] about scattering of the solution and its relation with the H-theorem.

For the purpose of clear representation, we should prove first the case $\gamma = -1$, then generalize the argument to the case $-1 < \gamma < 0$. Thus we first state the result for the case $\gamma = -1$ as follows.

**Theorem 1.1** Let $N = 3$ and assume the kernel $B$ in (1.3) has $\gamma = -1$ and satisfies (1.4). There exists a small number $\eta > 0$ such that if the initial data

$$f_0 \in \mathbb{B}_{\eta} = \{f_0|f_0 \geq 0, \|f_0\|_{L^3_{x,v}} < \eta, \|f_0\|_{L^3_{x,v}^{15/8}} < \infty\} \subset L^3_{x,v},$$

then the Cauchy problem (1.1) admits a unique and non-negative mild solution

$$f \in C([0, \infty), L^3_{x,v}) \cap L^q([0, \infty], L^r_x L^p_v),$$

where the triplet $(q, r, p)$ lies in the set

$$\left\{(q, r, p)| \frac{1}{q_1} = \frac{3}{p} - 1, \frac{1}{r} = \frac{2}{3} - \frac{1}{p}, \frac{1}{s} < \frac{1}{p} < \frac{4}{9}\right\}.$$  

The solution map $f_0 \in \mathbb{B}_{\eta} \to f \in L^q_i L^r_x L^p_v$ is Lipschitz continuous and the solution $f$ scatters with respect to the kinetic transport operator in $L^3_{x,v}$.

Next, we note that the local wellposedness result of Theorem 1.3 in [10] holds for gain term only Boltzmann equation instead of full equation due to the same reason mentioned above. The method of Theorem 1.1 can also fix the problem and we have the following result.

**Theorem 1.2** Let $N = 2$ or 3 and $B$ defined in (1.3) satisfies (1.4) and $-N < \gamma < 2 - N$. The Cauchy problem (1.1) is locally wellposed when the initial data lies in $B_R = \{f_0 \in L^{\alpha_s}_{x,v}(\mathbb{R}^N \times \mathbb{R}^N): f_0 \geq 0, \|f_0\|_{L^{\alpha_s}_{x,v}} < R, \|f_0\|_{L^m_{x,v}} < \infty\} \subset L^{\alpha_s}_{x,v}$

where $\alpha_s = 2N/(\gamma + N)$, $m = 2N/[(\gamma + N)(5\alpha - 1)]$ and $1/2 < \alpha < (N + 1)/(2N)$. More specially, for any $R > 0$ there exists a $T = T(r_s, p_s, R)$ such that for all $f_0 \in B_R$, the Cauchy problem (1.1) admits a unique mild solution

$$f \in C([0, T), L^{\alpha_s}_{x,v}) \cap L^q([0, T], L^r_x L^p_v),$$

where the triplet $(q_{s}, r_s, p_s)$ lies in the set

$$\left\{(1 - \frac{1}{q_s}, \frac{1}{r_s}, \frac{1}{p_s})| \frac{1}{q_s} = \frac{(2\alpha - 1)(\gamma + N)}{2N}, \frac{1}{r_s} = \frac{(1 - \alpha)(\gamma + N)}{N}, \frac{1}{p_s} = \frac{\alpha(\gamma + N)}{N}\right\}.$$
The solution map \( f_0 \in B_R \rightarrow f \in L^{s,v}([0, T]; L^{s,v}_x L^{p,v}_v) \) is Lipschitz continuous.

Finally, we state the result for the case \(-1 < \gamma < 0\). This part is not studied in the authors’ previous work [10], even for the gain term only Boltzmann equation. We use the notation \( \ell_\gamma^+ \) to denote \( \ell_\gamma + \varepsilon \) where \( \varepsilon > 0 \) is arbitrary small. The result is as follows.

**Theorem 1.3** Let \( N = 3 \) and assume the kernel \( B \) (1.3) has \(-1 < \gamma < 0\) and satisfies (1.4). Let \( \ell_\gamma = (1 + \gamma)^+ < 3/2 \). There exists a small number \( \eta > 0 \) such that if the initial data

\[
\| f_0 \|_{\ell_\gamma} \in \{ f_0 | f_0 \geq 0, \| \langle v \rangle^{\ell_\gamma} f_0 \|_{L^{3,v}_x} < \eta, \| \langle v \rangle^{\ell_\gamma} f_0 \|_{L^{15/8,v}_x} < \infty \} \subset L^{3,v}_x,
\]

then the Cauchy problem (1.1) admits a unique and non-negative mild solution

\[
\langle v \rangle^{\ell_\gamma} f \in C([0, \infty), L^{3,v}_x) \cap L^{4}(\gamma) \cap L^{s}_x L^{p,v}_v^T.
\]

where the triplet \((\varepsilon_1, r, \varepsilon_3)\) lies in the set (1.9). The solution map \( \langle v \rangle^{\ell_\gamma} f_0 \in \ell_\gamma \rightarrow \langle v \rangle^{\ell_\gamma} f \in L^{4}(\gamma) L^{s}_x L^{p,v}_v^T \) is Lipschitz continuous and the solution \( \langle v \rangle^{\ell_\gamma} f \) scatters with respect to the kinetic transport operator in \( L^{3}_x \).

### 1.3 Organization of the Paper

The proof of Theorem 1.1 is lengthy as it contains many parts. The proof of Theorem 1.2 is similar to that of Theorem 1.1. The proof of Theorem 1.3 is in the same spirit with that of Theorem 1.1 but it needs new weighted estimates. We thus organize the paper as follows.

In Sect. 2 we prove the global existence of solutions \( f_+ \) for the gain term only Boltzmann equation with small initial data in the suitable Strichartz spaces for the cases \( N = 3, \gamma = -1 \) and \( N = 2, \gamma = 0 \). In particular, the solution is non-negative if the initial data is non-negative. This part is mainly the reminiscence of [10]. Two useful estimates induced by the condition \( \| f_0 \|_{L^{15/8}_x} < \infty \) are also included. The result of this section is the preparation for the later analysis.

The Sect. 3 includes the proof of Theorem 1.1. It is composed of Subsects. 3.1, 3.2 and 3.3. We use \( h_1 = 0 \) and \( g_1 = f_+ \geq 0 \) (from Sect. 2) as the lower and upper bounds to build the beginning condition, \( 0 \leq h_1 \leq h_2 \leq g_2 \leq g_1 \), for the Kaniel–Shinbrot iteration. With the aid of \( \| f_0 \|_{L^{15/8}_x} < \infty \), the solution \( f_+ \) of Sect. 2 ensures the lose term \( Q^- (h_2, g_1) \) is well-defined in the sense that it lies in a suitable Strichartz space also. The same trick also makes sure that each term in the iteration process is well-defined, thus we can run the Kaniel–Shinbrot iteration to get the lower and upper solutions \( h \) and \( g \) of system (3.4). To close the iteration, we need to show \( g = h \). The argument requires again that the assumption that \( \| f_0 \|_{L^{15/8}_x} < \infty \).

To check the uniqueness of the solution, we consider the difference of the solutions for the full equation and the corresponding difference equation with zero initial data. The non-negativity of the solution helps us when using the continuity argument. The continuity in time, scattering of the solution and the solution map is Lipschitz continuous can be shown by the standard argument.

In the Subsect. 3.4, we include the proof of Theorem 1.2 by pointing out the main difference with that of Theorem 1.1.

The proof of Theorem 1.3 is included in Sect. 4 which generalizes the argument of Theorem 1.1 after building the weighted estimates for the gain and loss terms as well as weighted Strichartz estimates.
2 The Gain Term Only Boltzmann Equation

2.1 Global Existence for the Gain Term Only Boltzmann Equation

The main result, Proposition 2.4, is indeed included in [10]. To be self-contained, we will review the main strategy of the proof which is needed for the further analysis.

First we recall the Strichartz estimates for the kinetic transport equation,

\[
\begin{aligned}
\partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v) &= F(t, x, v), \quad (t, x, v) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N, \\
u(0, x, v) &= u_0(x, v).
\end{aligned}
\]  

(2.1)

To state the Strichartz estimates for (2.1), we need the following definition.

**Definition 2.1** We say that the exponent triplet \((q, r, p)\), for \(1 \leq p, q, r \leq \infty\) is KT-admissible if

\[
\frac{1}{q} = \frac{N}{2} \left( \frac{1}{p} - \frac{1}{r} \right) \quad (2.2)
\]

\[
1 \leq a \leq \infty, \quad p^*(a) \leq p \leq a, \quad a \leq r \leq r^*(a)
\]  

(2.3)

except in the case \(N = 1\), \((q, r, p) = (a, \infty, a/2)\). Here by \(a = \text{HM}(p, r)\) we have denoted the harmonic means of the exponents \(r\) and \(p\), i.e.,

\[
\frac{1}{a} = \frac{1}{2} \left( \frac{1}{p} + \frac{1}{r} \right)
\]  

(2.4)

Furthermore, the exact lower bound \(p^*\) to \(p\) and the exact upper bound \(r^*\) to \(r\) are

\[
\begin{aligned}
p^*(a) &= \frac{N a}{N + 1}, \quad r^*(a) = \frac{N a}{N - 1} \quad \text{if} \quad \frac{N a}{N + 1} \leq a \leq \infty, \\
p^*(a) &= 1, \quad r^*(a) = \frac{a^2}{N} \quad \text{if} \quad 1 \leq a \leq \frac{N + 1}{N}.
\end{aligned}
\]  

(2.5)

The triplets of the form \((q, r, p) = (a, r^*(a), p^*(a))\) for \(\frac{N + 1}{N} \leq a < \infty\) are called endpoints. The endpoint Strichartz estimate for the kinetic equation is false in all dimensions has been proved recently by Bennett et al. [5].

The mild solution of the kinetic equation (2.1) can be written as

\[
u = U(t)u_0 + W(t)F
\]  

(2.6)

where

\[
U(t)u_0 = u_0(x - vt, v), \quad W(t)F = \int_0^t U(t - s)F(s)ds.
\]  

(2.7)

The estimates for the operator \(U(t)\) and \(W(t)\) respectively in the mixed Lebesgue norm \(\| \cdot \|_{L_q^q L_r^r L_p^p}\) are called homogeneous and inhomogeneous Strichartz estimates respectively.

We record the estimates for the Eq. (2.6) in the following Proposition where \(p'\) denotes the conjugate exponent of \(p\) and so on.

**Proposition 2.2** [5, 17] Let \(u\) satisfies (2.1). The estimate

\[
\|u\|_{L_q^q L_r^r L_p^p} \leq C(q, r, p, N) \left( \|u_0\|_{L_{x,v}^q} + \|F\|_{L_{x,v}^{q'} L_{r'}^r L_{p'}^p} \right)
\]  

(2.8)

holds for all \(u_0 \in L_{x,v}^a\) and all \(F \in L_{x,v}^{q'} L_{r'}^r L_{p'}^p\) if and only if \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are two KT-admissible exponent triplets and \(a = \text{HM}(p, r)\), \(\tilde{a} = \text{HM}(\tilde{p}', \tilde{r}')\) with the exception of \((q, r, p)\) being an endpoint triplet.
Now we consider the Cauchy problem for the gain term only Boltzmann equation

\[
\begin{aligned}
\partial_t f_+ + v \cdot \nabla_x f_+ &= Q^+(f_+, f_+) \\
f_+(0, x, v) &= f_0(x, v).
\end{aligned}
\]  

(2.9)

We define the solution map by

\[
S f_+(t, x, v) = U(t) f_0 + W(t) Q^+(f_+, f_+).
\]

(2.10)

From (2.10) and Proposition 2.2, we will see that it holds the estimates

\[
\|S f_+\|_{L^q_t L^r_x L^p_v} \leq C_0 \|f_0\|_{L^q_{x,v}} + C_1 \|Q^+(f_+, f_+)\|_{L^q_t L^r_x L^p_v}^2
\]

\[
\leq C_0 \|f_0\|_{L^q_{x,v}} + C_2 \|f_+\|_{L^q_t L^r_x L^p_v}^2
\]

(2.11)

for suitable Strichartz spaces \(L^q_t L^r_x L^p_v\) and \(L^q_t L^r_x L^p_v\). Then the contraction mapping argument will work if the initial data is small in space \(L^q_{x,v}\). The key lies in the fact that if there exist admissible triplets \((q, r, p)\) and \((\tilde{q}', \tilde{r}', \tilde{p}')\) with \(HM(p, r) = HM(\tilde{p}', \tilde{r}')\) such that the estimate

\[
\|Q^+(f_+, f_+)\|_{L^q_t L^r_x L^p_v} \leq C \|f_+\|_{L^q_t L^r_x L^p_v}^2
\]

(2.12)

holds.

In order to prove the existence of such triplets, we need the estimates for the gain term in \(v\) variable. Indeed it is included in Theorem 1 and Theorem 2 of [1] by Alonso, Carneiro and Gamba. We collect what we need as follows.

**Proposition 2.3** [1] *Let* \(1 < p, q, r < \infty\) *with* \(-N < \gamma \leq 0\) *and*

\[
1/p + 1/q = 1 + \gamma/N + 1/r.
\]

(2.13)

*Assume the kernel* (1.3):

\[
B(v - v_*, \omega) = |v - v_*|^{\gamma} b(\cos \theta)
\]

with \(b(\cos \theta)\) satisfies Grad’s cutoff assumption (1.4). Then the bilinear operator \(Q^+(f, g)\) is a bounded operator from \(L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \to L^r(\mathbb{R}^N)\) via the estimate

\[
\|Q^+(f, g)\|_{L^r(\mathbb{R}^N)} \leq C \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.
\]

(2.14)

As we mention in the introduction that the main result of [10] actually holds for gain term only Boltzmann equation instead of full equation due to the negligence of constrain in the estimate for the loss term. More precisely, the proof of [10] infers (2.12) and the following result.

**Proposition 2.4** (cf. Theorem 1.1 in [10]) *Let* \(N = 2\) *or* \(3\) *and* \(B\) *defined in* (1.3) *satisfies* (1.4) *and* \(\gamma = 2 - N\). *The Cauchy problem* (2.9) *is globally wellposed in* \(L^N_{x,v}\) *when the initial data is small enough. More specially, there exists* \(\eta > 0\) *small enough such that for all* \(f_0\) *in the set

\[
B_\eta = \{ f_0 \in L^N_{x,v}(\mathbb{R}^N \times \mathbb{R}^N) : f_0 \geq 0 \text{ and } \|f_0\|_{L^N_{x,v}} < \eta \}
\]

there exists a globally unique mild solution

\[
f_+ \in C([0, \infty), L^N_{x,v}) \cap L^4([0, \infty), L^5_x L^p_v)
\]
where the triplet \((q, r, p)\) lies in the set
\[
\left\{(q, r, p)|\; \frac{1}{q} = \frac{N}{p} - 1, \; \frac{1}{r} = \frac{2}{N} - \frac{1}{p}, \; \frac{1}{N} < \frac{p}{N} < \frac{N+1}{N^2}\right\}. \tag{2.15}
\]

The solution map \(f_0 \in B_\eta \subset L_{x,v}^N \rightarrow f_+ \in L_{t}^q L_{x}^r L_{v}^p\) is Lipschitz continuous and the solution \(f_+\) scatters with respect to the kinetic transport operator in \(L_{x,v}^N\).

**Definition 2.5** We use the notation \((q, r, p)\) to address that it stems from the usual KT-admissible triplet \((q, r, p)\) and lies in the set (2.15) (i.e., (1.9)). We should call that \((q, r, p)\) is a solvable triplet. We say that \((\tilde{q}', \tilde{r}', \tilde{p}')\) is the conjugate triple of the solvable triplet \((q, r, p)\) if \(HM(p, r) = HM(\tilde{q}', \tilde{r}')\) and
\[
\|Q^+(f, f)\|_{L_t^q L_x^r L_v^p} \leq C \|f\|^2_{L_t^q L_x^r L_v^p}. \tag{2.16}
\]

Before we consider the full Boltzmann equation, we also need the following result.

**Corollary 2.6** Under the same conditions as Proposition 2.4, if we furthermore assume \(f_0 \geq 0\), then the solution \(f_+\) is also non-negative.

For the proof of Corollary 2.6 and later analysis, we include the portion of the proof of Proposition 2.4 which shows that if the admissible triplets \((q, r, p)\) lie in (2.15), there exist corresponding \((\tilde{q}', \tilde{r}', \tilde{p}')\) such that (2.12) holds, i.e., we can find solvable triple \((q, r, p)\) and its conjugate triplet \((\tilde{q}, \tilde{r}, \tilde{p})\).

**Proof of (2.16)** For \(v\) variable, we let \(r = \tilde{p}',\; p = q = p\) in (2.14), thus
\[
\frac{2}{p} = 1 + \frac{\gamma}{N} + \frac{1}{\tilde{p}'}.	ag{2.17}
\]
For \(x\) variables, the condition for being able to apply the Hölder inequality is
\[
2\tilde{p}' = r, \; r \geq 2. \tag{2.18}
\]
Furthermore the Strichartz inequality demands the relation of pairs \((p, r), (\tilde{p}', \tilde{r}')\),
\[
\frac{1}{p} + \frac{1}{r} = \frac{1}{\tilde{p}'} + \frac{1}{\tilde{r}'}.	ag{2.19}
\]
To apply the Hölder inequality to \(t\) variable, we need
\[
\frac{2}{q} = \frac{1}{q'} < 1, \tag{2.20}
\]
that is
\[
\frac{2}{q} + \frac{1}{q} = 1, \; \frac{1}{q} < \frac{1}{2}. \tag{2.21}
\]
Finally the KT-admissible conditions
\[
\frac{1}{q} = \frac{N}{2} \left(\frac{1}{p} - \frac{1}{r}\right) > 0, \tag{2.22}
\]
\[
\frac{1}{\tilde{q}} = \frac{N}{2} \left(\frac{1}{\tilde{p}} - \frac{1}{\tilde{r}}\right) > 0 \tag{2.23}
\]
must be fulfilled.
We note that once $\gamma$, $p$, $r$ are given, $q$, $\tilde{p}$, $\tilde{r}$, $\tilde{q}$ are determined. Thus we rewrite above conditions as

$$\begin{align*}
\frac{1}{p} + \frac{1}{r} &= 1 + \frac{\gamma}{N} & \text{from (2.17) and (2.18), (2.19)} \\
\frac{1}{p} + \frac{1}{r} &= \frac{2}{N} & \text{from (2.21) and (2.22), (2.19)} \\
0 < \frac{1}{p} - \frac{1}{r} < \frac{1}{N} & \text{from } 1/q < 1/2 \text{ in (2.21) and (2.22)} \\
0 < \frac{1}{p} - \frac{1}{r} < \frac{1}{2} \left(1 + \frac{\gamma}{N}\right) & \text{from (2.23) and (2.17)}
\end{align*}$$

Therefore

$$\gamma = 2 - N, \ a = N.$$ 

and by (2.3), (2.5) and (2.22),

$$\frac{1}{N} < \frac{1}{p} < \frac{N + 1}{N^2}, \ \frac{N - 1}{N^2} < \frac{1}{r} < \frac{1}{N}. $$

Thus we conclude that if the triplet $(q, r, p)$ satisfies (2.15), i.e.,

$$\left\{(q, r, p) \mid \frac{1}{q} = \frac{N}{p} - 1, \ \frac{1}{r} = \frac{2}{N} - \frac{1}{p}, \ \frac{1}{N} < \frac{1}{p} < \frac{N + 1}{N^2}\right\},$$

then (2.16) holds where the triplet $(\tilde{q}', \tilde{r}', \tilde{p}')$ is given by (2.17), (2.18), (2.19) and (2.20). \(\square\)

**Proof of Corollary 2.6** When $f_0 \geq 0$, we can see that the solution is non-negative by iterating Duhamel’s formula:

$$f_+(t) = U(t) f_0 + \int_0^t U(t - t_1) Q^+(U(t_1) f_0, U(t_1) f_0) dt_1$$

$$+ \int_0^t \int_0^{t_1} U(t - t_1) Q^+(U(t_1 - t_2) Q^+(U(t_2) f_0, U(t_2) f_0), U(t_1) f_0) dt_2 dt_1 + \cdots$$

Since each term in the right hand side is non-negative, it suffices to show the series converges. Using Strichartz estimates of Proposition 2.2 with solvable triplet repeatedly, we get that the Strichartz norm of $f_+$ for solvable triplets is bounded by a series of $L^N_{x,t}$ norm of $f_0$ which converges since initial data is small enough. \(\square\)

Now we explain the reason why the above approach cannot solve the full Boltzmann equation. First we record the estimate for the loss term whose proof can be obtained by dropping $\ell$ and $m$ and modifying the proof of Proposition 4.4.

**Lemma 2.7** Let $1 < p, q, r < \infty$ with $-N < \gamma < 0$, $1/p + 1/q = 1 + \gamma/N + 1/r$ and $1/p < 1/r$. Assume the kernel (1.3):

$$B(v - v_*, \omega) = |v - v_*|^{\gamma} b(\cos \theta)$$

with $b(\cos \theta)$ satisfies Grad’s cutoff assumption (1.4). Then the bilinear operator $Q^-$ is a bounded operator from $L^p(\mathbb{R}^N) \times L^q(\mathbb{R}^N) \to L^r(\mathbb{R}^N)$ via the estimate

$$\|Q^-(f, g)\|_{L^r(\mathbb{R}^N)} \leq C \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)}.$$
Proposition 2.8 Let $N = 3$ and $a_2 = 15/8$. Under the same assumptions as Proposition 2.4, if we further assume that $\| f_0 \|_{L^{a_2}_{x,v}} < \infty$, then solution $f_+$ in Proposition 2.4 also satisfies

$$\| f_+ \|_{L^{a_2}_{t} L^2_{x} L^2_{v}} < \infty,$$  \hfill (2.26)

where $(1/q_2, 1/r_2, 1/p_2) = (1/2, 11/30, 21/30)$ is a KT-admissible triplet with $1/p_2 + 1/r_2 = 2/a_2$.

Proof Due to the assumption that $\| f_0 \|_{L^{a_2}_{x,v}} < \infty$, we claim that there exist KT-admissible triplets $(q_2, r_2, p_2)$ and $(\tilde{q}_2, \tilde{r}_2, \tilde{p}_2)$ such that

$$\| Q^+(f_+, f_+) \|_{L^{q_2}_{t} L^2_{x} L^2_{v}} \leq C \| f_+ \|_{L^{q}_{t} L^2_{x} L^p_{v}} \| f_+ \|_{L^{q_2}_{t} L^2_{x} L^2_{v}},$$  \hfill (2.27)

and $a_2 = HM(p_2, r_2) = HM(\tilde{p}_2', \tilde{r}_2')$ where $\tilde{p}_2'$ means the conjugate of $\tilde{q}_2$ and so on. From this together with Strichartz estimate (2.8), we have

$$\| f_+ \|_{L^{q_2}_{t} L^2_{x} L^2_{v}} = \| S f_+ \|_{L^{q_2}_{t} L^2_{x} L^2_{v}} \leq C_0 \| f_0 \|_{L^{a_2}_{x,v}} + C_1 \| Q^+(f_+, f_+) \|_{L^{q_2}_{t} L^2_{x} L^2_{v}} \leq C_0 \| f_0 \|_{L^{a_2}_{x,v}} + C_2 \| f_+ \|_{L^{q}_{t} L^2_{x} L^p_{v}} \| f_+ \|_{L^{q_2}_{t} L^2_{x} L^2_{v}}.$$

The proof of Proposition 2.4 implies that $C_2 \| f_+ \|_{L^{q}_{t} L^2_{x} L^p_{v}} < 1$ where $(q_1, r, p)$ is a solvable triplet. Thus we have

$$\| f_+ \|_{L^{q_2}_{t} L^2_{x} L^2_{v}} \leq C_3 \| f_0 \|_{L^{a_2}_{x,v}} < \infty.$$  \hfill (2.26)

To prove (2.27), we define $\tilde{p}_2'$ and $\tilde{r}_2'$ as follows:

$$\frac{1}{\tilde{p}_2'} := \frac{1}{p_2} + \frac{1}{30}, \quad \frac{1}{\tilde{r}_2'} := \frac{1}{r_2} + \frac{1}{30}, \quad \frac{1}{q_2} = \frac{1}{2} + \frac{1}{q_1}.$$  \hfill (2.28)

By (2.14), it is easy to have

$$\frac{1}{p_2} + \frac{1}{r_2} = 2 + \frac{1}{\tilde{p}_2'} < \frac{1}{\tilde{p}_2'}, \quad \frac{1}{r_2} < \frac{1}{\tilde{p}_2'}, \quad \frac{1}{p_2} < \frac{1}{\tilde{p}_2'}, \quad \frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right),$$  \hfill (2.29a)

$$\frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right), \quad \frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right), \quad \frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right),$$  \hfill (2.29b)

$$\frac{1}{r_2} < \frac{1}{\tilde{p}_2'}, \quad \frac{1}{p_2} < \frac{1}{\tilde{p}_2'}, \quad \frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right),$$  \hfill (2.29c)

$$\frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right), \quad \frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right),$$  \hfill (2.29d)

$$\frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right), \quad \frac{1}{q_2} = \frac{3}{2} \left( \frac{1}{p_2} - \frac{1}{r_2} \right),$$  \hfill (2.29e)
and the proof of (2.27) is finished. □

**Remark 2.9** Our choice of \( p_2, r_2, \tilde{p}_2, \tilde{r}_2 \) in (2.28) is one of possible combinations which satisfies (2.29a, 2.29b, 2.29c, 2.29d, 2.29e). In fact, by (2.3), i.e., \( 1/r_2 < 1/p_2 < 2/r_2 \), and (2.29a), it is easy to find more options.

By the symmetry of the estimate for the gain term (2.14), the requirement \( 1/p_2 < 1/\tilde{p}_2 \) from (2.29a) can be removed from the estimate (2.27). However it is compulsory for the loss term due to Lemma 2.7. In summary, we have the following result.

**Corollary 2.10** Use the same notations as Proposition 2.8. Suppose \( f_1 \in L^q_{1} L^r_{x} L^p_{v} \) and \( f_2 \in L^q_{1} L^r_{x} L^p_{v} \), then \( Q^\pm (f_1, f_2) \in L^q_{1} L^r_{x} L^p_{v} \) and

\[
\| Q^- (f_1, f_2) \|_{L^q_{1} L^r_{x} L^p_{v}} \leq C \| f_1 \|_{L^q_{1} L^r_{x} L^p_{v}} \| f_2 \|_{L^q_{1} L^r_{x} L^p_{v}},
\]

\[
\| Q^+ (f_1, f_2) \|_{L^q_{1} L^r_{x} L^p_{v}} \leq C \| f_1 \|_{L^q_{1} L^r_{x} L^p_{v}} \| f_2 \|_{L^q_{1} L^r_{x} L^p_{v}},
\]

\[
\| Q^\mp (f_1, f_2) \|_{L^q_{1} L^r_{x} L^p_{v}} \leq C \| f_1 \|_{L^q_{1} L^r_{x} L^p_{v}} \| f_2 \|_{L^q_{1} L^r_{x} L^p_{v}}.
\]

### 3 Proof of Theorems 1.1 and 1.2

#### 3.1 Proof of Theorem 1.1

We will follow the idea of Chen et al. [8] to recover the solutions to the full Boltzmann equation from the solutions to the gain term only Boltzmann equation by making use of Kaniel and Shinbrot’s iteration [11, 14].

With the result of Sect. 2, Theorem 1.1 can be restated as the following.

**Proposition 3.1** Consider the Cauchy problem (1.1) with \( N = 3, \gamma = -1 \). Suppose the initial data

\[
f_0 \in B_{\eta} = \{ f_0 | f_0 \geq 0, \| f_0 \|_{L^3_{x,v}} < \eta, \| f_0 \|_{L^{15/8}_{x,v}} < \infty \} \subset L^3_{x,v},
\]

where \( \eta \) is chosen in Propositions 2.4. Then (1.1) admits a non-negative unique mild solution

\[
f \in C([0, \infty), L^3_{x,v}) \cap L^1([0, \infty), L^r_{x} L^p_{v})
\]

where the triple \((c_1, r, \beta)\) lies in the set (1.9). The solution map \( f_0 \in B_{\eta} \subset L^3_{x,v} \rightarrow f \in L^3_{1} L^r_{x} L^p_{v} \) is Lipschitz continuous and the solution \( f \) scatters with respect to the kinetic transport operator in \( L^3_{x,v} \).

**Proof** Let us denote the loss term \( Q^- (f_1, f_2) = f_1 L(f_2) \). First we recall that the Kaniel–Shinbrot iteration ensures that if there exist measurable functions \( h_1, h_2, g_1, g_2 \) which satisfy the beginning condition, i.e.,

\[
0 \leq h_1 \leq h_2 \leq g_2 \leq g_1,
\]

then the iteration \((n \geq 2)\)

\[
\partial_t g_{n+1} + v \cdot \nabla_x g_{n+1} + g_{n+1} L(h_n) = Q^+(g_n, g_n)
\]

\[
\partial_t h_{n+1} + v \cdot \nabla_x h_{n+1} + h_{n+1} L(g_n) = Q^+(h_n, h_n)
\]

\[
g_{n+1}(0) = h_{n+1}(0) = f_0
\]
will induce the monotone sequence of measurable functions
\[ 0 \leq h_1 \leq h_2 \leq h_{n+1} \leq g_{n+1} \leq g_n \leq g_1. \tag{3.3} \]
Thus the monotone convergence theorem implies the existence of the limits \( g, h \) with \( 0 \leq h \leq g \leq g_1 \) which satisfy
\[
\begin{align*}
\partial_t g + v \cdot \nabla_x g + gL(h) &= Q^+(g, g) \\
\partial_t h + v \cdot \nabla_x h + hL(g) &= Q^+(h, h) \tag{3.4}
\end{align*}
\]
Hence the Cauchy problem (1.1) is solved if one can further prove \( g = h = f \).

Based on the Proposition 2.4, it is natural to choose \( h_1 \equiv 0 \) and \( g_1 = f_+ \) where \( f_+ \geq 0 \) is the solution of gain term only Boltzmann equation (2.9) with initial data \( \| f_0 \|_{L_{t,v}^{1,2}} < \eta \). The Proposition 2.4 ensures
\[
f_+ \in C([0, \infty), L_{t,v}^3 \cap L^3([0, \infty], L_x^2 L_v^p)) \tag{3.5}
\]
where \((\varepsilon_1, \varepsilon, \tau)\) is a solvable triplet (i.e., satisfying (2.15)).

According to (3.2), we want to find \( h_2 \) and \( g_2 \) through
\[
\begin{align*}
\partial_t g_2 + v \cdot \nabla_x g_2 + g_2L(h_1) &= Q^+(g_1, g_1) \\
\partial_t h_2 + v \cdot \nabla_x h_2 + h_2L(g_1) &= Q^+(h_1, h_1) \tag{3.6}
\end{align*}
\]
Since \( h_1 \equiv 0 \), the first equation of (3.6) is exactly the gain term only Boltzmann equation. Hence we have \( g_2 = g_1 = f_+ \) by Proposition 2.4 and
\[
g_2(t) = U(t)f_0 + \int_0^t U(t-s)Q^+(g_1, g_1)(s)ds. \tag{3.7}
\]
Next we want to solve the second equation of (3.6) with the given \( g_1 \). More precisely, formally we have
\[
h_2(t) = U(t)f_0 e^{-\int_0^t U(t-s)L(g_1)(s)ds}. \tag{3.8}
\]
To ensure that \( L(g_1) \) is pointwisely a.e. well-defined when \( g_1 = f_+ \) satisfies (3.5), we recall that the assumption \( \| f_0 \|_{L_{t,v}^{15/8}} < \infty \) and Proposition 2.8 ensure
\[
g_1 = f_+ \in L^{q_2}([0, \infty], L_x^{r_2} L_v^{p_2}). \tag{3.9}
\]
Note that we are looking for a solution \( h_2 \in L^{q_2}([0, \infty], L_x^{r_2} L_v^{p_2}) \). From the estimate of Corollary 2.10,
\[
\| h_2L(g_1) \|_{L_x^{r_2} L_v^{p_2}} \leq C\| h_2 \|_{L_x^{r_2} L_v^{p_2}} \| g_1 \|_{L_x^{q_2} L_v^{p_2}},
\]
we know that if \( \phi \in L_x^{q_2} L_v^{p_2} \) then \( \langle h_2L(g_1), \phi \rangle \) is bounded. Since \( \langle h_2L(g_1), \phi \rangle = \langle L(g_1), h_2\phi \rangle \), we have \( L(g_1) \in L_x^{q_2} L_v^{p_2} \) where \( (1/q_2)' = 1/q_2 + 1/q_1 \) (1/\( \tau_2 \))' = 1/\( \tau_2 \) + 1/\( \tau_1 \) and (1/\( \tau_2 \))' = 1/\( \tau_2 \) + 1/\( \tau_1 \). Therefore \( L(g_1) \) is pointwisely a.e. well-defined.

Now we can compute \( h_2 \) by (3.8). It is easy to have \( h_1 \equiv 0 \leq h_2 \leq U(t)f_0 \leq g_2 \) by non-negativity of \( g_1 \). Therefore we conclude the beginning condition (3.1). From (3.9), we also have that
\[
h_2, g_2 \in L^{q_2}([0, \infty], L_x^{r_2} L_v^{p_2}).\]
Thus we can repeat the above argument to check that each term in (3.2) is well-defined. Therefore the method of Kaniel–Shinbrot ensures the existence of monotone sequence (3.3) and the limit functions $g, h$ satisfy (3.4).

We also note that from the monotone convergence theorem and (3.3), we have

$$g, h \in L^{q_1}([0, \infty), L^r_x L^p_y),$$

$$g, h \in L^{q_2}([0, \infty), L^r_x L^p_y),$$

$$Q^+(g, g), Q^+(h, h) \in L^r_t L^q_x L^p_y,$$

$$Q^-(g, g), Q^-(h, h) \in L^r_t L^q_x L^p_y.$$  \hspace{1cm} (3.10)

Thus we have a solution for the full Boltzmann equation if $g = h$.

To prove that $g = h$, we let $w = g - h \geq 0$. By (3.4) the difference $w$ satisfies the equation

$$\partial_t w + v \cdot \nabla_x w = Q^+(g, w) + Q^+(w, h) + Q^-(g, w) - Q^-(w, g)$$

with zero initial data. By Lemma 3.2 below we know that this equation has a unique solution $w \equiv 0$. Thus $g = h$ and we conclude the global existence of the non-negative mild solution for the full Boltzmann equation.

The uniqueness of the solution can be proved by a standard continuity argument and the fact that the solutions are non-negative which is given in Subsect. 3.2. Also the proof of the continuity in time, scattering of the solution and Lipschitz continuous of the solution map is included in the Subsect. 3.3. Thus the Proposition 3.1, i.e., Theorem 1.1 can be concluded by checking Subsects. 3.2 and 3.3. \hfill $\square$

**Lemma 3.2** Let $g, h$ be non-negative functions satisfy (3.10). Suppose that $w \geq 0$ is a mild solution of

$$\begin{cases}
\partial_t w + v \cdot \nabla_x w = Q^+(g, w) + Q^+(w, h) + Q^-(g, w) - Q^-(w, g) \\
w(0) = 0.
\end{cases} \hspace{1cm} (3.11)$$

Then $w \equiv 0$.

**Proof** Consider the given time interval $[0, T]$ and define

$$t_0 = \inf \left\{ t \in [0, T] \mid \|w(t)\|_{L^{q_2}([0, t], L^r_x L^p_y)} > 0 \right\}.$$

Then $w \equiv 0$ for $0 \leq t \leq t_0$. Let $t_0 \leq s \leq T$.

From (3.11), we have

$$w(s) = \int_0^s U(t - \tau) [Q^+(g, w) + Q^+(w, h) + Q^-(g, w) - Q^-(w, g)](\tau) d\tau.$$

Noting that $w \geq 0$, $0 \leq h \leq g$ and the operators $U(t), Q^+, Q^-$ are non-negative, we have

$$0 \leq w(s) \leq \int_0^s U(t - \tau) [Q^+(g, w) + Q^+(w, h) + Q^-(g, w)](\tau) d\tau. \hspace{1cm} (3.12)$$

Apply Strichartz estimates as Proposition 2.8 and use the estimate of Corollary 2.10, then we have

$$\|w\|_{L^{q_2}([t_0, s], L^r_x L^p_y)} \leq C \|g\|_{L^{q_1}([t_0, s], L^r_x L^p_y)} + \|h\|_{L^{q_1}([t_0, s], L^r_x L^p_y)} \|w\|_{L^{q_2}([t_0, s], L^r_x L^p_y)} \hspace{1cm} (3.13)$$

$$:= C(g, h, s) \|w\|_{L^{q_2}([t_0, s], L^r_x L^p_y)}.$$
Letting $s \to t_0^+$, clearly we have $C(g, h, s) < 1$ which means that
\[ \|w\|_{L^{p_0}([t_0, s]; L^{p_2}_w L^{p_2}_v)} = 0 \]
for some $s > t_0$. By continuity, we have $w|_{[0, T]} \equiv 0$ for any $T > 0$. \hfill \Box

### 3.2 Uniqueness of the Solution in Theorem 1.1

Assume that $g \geq 0$ and $h \geq 0$ both are mild solutions which satisfy (1.1). Let $w = g - h$. Comparing to Lemma 3.2, the function $w$ satisfies (3.11), but we do not have the property $w \geq 0$. For our convenience, we rewrite (3.11) as
\[ \begin{cases} \partial_t w + v \cdot \nabla_x w + wL(g) = Q^+(g, w) + Q^+(w, h) + Q^-(g, w) \\ w(0) = 0, \end{cases} \]
and want to show $w \equiv 0$. Since $g$, $h$ and thus $w$ satisfy (3.10), the term $L(g)$ is pointwisely a.e. well-defined. Thus the function $w$ satisfies
\[ w(t) = \int_0^t e^{-\int_\tau^t U(t - \tau)L(g) d\tau} \cdot U(t - s)[Q^+(g, w) + Q^+(w, h) + Q^-(g, w)](s)ds. \]
Using the fact that $L(g) \geq 0$ since $g \geq 0$, we have
\[ |w(t)| \leq \int_0^t U(t - s)[Q^+(g, |w|) + Q^+(|w|, h) + Q^-(g, |w|)](s)ds. \tag{3.14} \]
The Eq. (3.14) is in place of (3.12) for the proof of $w \equiv 0$, thus we conclude the uniqueness of the solution.

### 3.3 Continuity in $L^3_{x,v}$ and Scattering of the Solution in Theorem 1.1

Now we show that $f \in C([0, T], L^3_{x,v})$ for any $T \in [0, \infty]$. From the formula
\[ \int_0^t U(t - s)Q^-(f, f)(s)ds + f(t) = U(t)f_0 + \int_0^t U(t - s)Q^+(f, f)(s)ds \tag{3.15} \]
and the observation that each term in (3.15) is non-negative, we have
\[ 0 \leq f(t) \leq U(t)f_0 + \int_0^t U(t - s)Q^+(f, f)(s)ds. \tag{3.16} \]
It has been observed by Ovcharov [16] that $U(t)f_0 \in C(\mathbb{R}; L^N_{x,v})$, hence it suffices to show that $W(t)$ (see (2.7)) is also continuous. Let $0 \leq t \in (0, \infty]$. Applying inhomogeneous Strichartz with triplet $(\tilde{q}', \tilde{r}', \tilde{p}')$ used in (2.16), we see that
\[ \|W(t)Q^+(f, f)\|_{L^\infty([0, t]; L^3_{x,v})} = \int_0^t \|U(t - s)Q^+(f, f)\|_{L^3_{x,v}} ds \]
is bounded. Since $U(t)$ is continuous, we conclude that $W(t)$ is continuous from above expression. Also the solution map $f_0 \in B_q \subset L^3_{x,v} \to f \in L^q_t L^r_x L^p_v$ is Lipschitz continuous.

Next we want to show that the solution $f$ scatters, i.e., there exists a function $f_\infty \in L^3_{x,v}$ such that
\[ \|f(t) - U(t)f_\infty\|_{L^3_{x,v}} \to 0 \text{ as } t \to \infty. \]
The above statement is equivalent to proving that
\[ \|U(-t)f(t) - f_\infty\|_{L^3_{x,v}} \to 0 \quad \text{as} \ t \to \infty, \quad (3.17) \]
since \( U(t) \) preserves the \( L^3_{x,v} \) norm.

By the Duhamel formula, we have
\[ U(-t)f(t) = f_0 + \int_0^t U(-s)Q(f, f)(s)ds. \quad (3.18) \]

Hence the scattering of \( f(t) \) is confirmed if we have the convergence of the integral
\[ \int_0^\infty U(-t)Q(f, f)(t)dt \]
in \( L^3_{x,v} \). In this case \( f_\infty \) is given by
\[ f_\infty = f_0 + \int_0^\infty U(-t)Q(f, f)(t)dt. \quad (3.19) \]

We rewrite (3.18) as
\[ U(-t)f(t) + \int_0^t U(-s)Q^- (f, f)(s)ds = f_0 + \int_0^t U(-s)Q^+(f, f)(s)ds. \]

Since each term of above equation is non-negative, thus we have
\[ \int_0^t U(-s)Q^-(f, f)(s)ds \leq f_0 + \int_0^t U(-s)Q^+(f, f)(s)ds \]
\[ \leq f_0 + \int_0^\infty U(-s)Q^+(f, f)(s)ds. \quad (3.20) \]

By monotone convergence theorem, it holds that
\[ \int_0^\infty U(-s)Q^-(f, f)(s)ds \leq f_0 + \int_0^\infty U(-s)Q^+(f, f)(s)ds. \]

Then we are reduced to prove the right hand side of (3.20) is bounded in \( L^3_{x,v} \).

Let \( U^*(t) \) be the adjoint operator of \( U(t) \), it is clearly that \( U^*(t) = U(-t) \). Let \((\tilde{q}, \tilde{r}, \tilde{p})\) be the KT-admissible triplet chosen in the proof of (2.16) and recall that \( 1/\tilde{p}' + 1/\tilde{r}' = 2/3 \).

By duality, the homogeneous Strichartz estimate
\[ \|U(t)g\|_{L^\tilde{q}'_t L^\tilde{r}'_x L^\tilde{p}'_v} \leq C \|g\|_{L^\tilde{q}^2_{x,v}} \]
implies
\[ \left\| \int_0^\infty U^*(t)Q^+(f, f)dt \right\|_{L^3_{x,v}} \leq C \|Q^+(f, f)\|_{L^\tilde{q}_t L^\tilde{r}_x L^\tilde{p}_v} \leq C \|f\|_{L^3_{x,v}}^2 \]
where the second inequality holds as before. Thus we conclude that \( f \) scatters.
3.4 Proof of Theorem 1.2

Proof As before, we apply the Strichartz estimate to the gain term only Boltzmann equation

\[ \| Sf_+(t, x, v) \|_{L^q([-T, T]; L^r_x L^p_v)} \leq C \left( \| f_0 \|_{L^{ax}_v} + \| Q(f_+, f_+) \|_{L^q([-T, T]; L^r_x L^p_v)} \right). \]

To show that

\[ \| Sf_+(t, x, v) \|_{L^q([-T, T]; L^r_x L^p_v)} \leq C_1 \| f_0 \|_{L^{ax}_v} + C_2 T^\beta \| f(t, x, v) \|_{L^q([-T, T]; L^r_x L^p_v)}^2 \]

with \( \beta > 0 \), we need to find KT-admissible triplets \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) which satisfy

\[ \frac{2}{p} = 1 + \frac{\gamma}{N} + \frac{1}{\tilde{p}}, \quad \frac{2}{r} = \frac{1}{\tilde{r}}, \quad \frac{1}{p} + \frac{1}{r} = \frac{1}{\tilde{p}} + \frac{1}{\tilde{r}} \]

and

\[ \frac{2}{q} + \beta = \frac{1}{\tilde{q}}. \]

It is already found in [10] that the set (1.10) is the collection of all possible \((q, r, p)\) where

\[ \beta = \frac{(2 - N) - \gamma}{2} > 0. \]

To recover the solution for the full equation, we need another pair of KT-admissible triplets \((q_2, r_2, p_2)\) and \((\tilde{q}_2, \tilde{r}_2, \tilde{p}_2)\). It is straightforward to check that the following choice works:

\[ \frac{1}{p_2} = 2\alpha \left( \frac{\gamma + N}{N} \right), \quad \frac{1}{r_2} = (3\alpha - 1) \left( \frac{\gamma + N}{N} \right), \quad \frac{1}{\tilde{p}_2} = (3\alpha - 1) \left( \frac{\gamma + N}{N} \right), \quad \frac{1}{\tilde{r}_2} = 2\alpha \left( \frac{\gamma + N}{N} \right). \]

\[ \square \]

4 Theorem 1.3: The Case \(-1 < \gamma < 0\)

In this section, we give the proof for Theorem 1.3. It is in the same spirit as that of the case \(\gamma = -1\) except that we need the weighted estimates for the gain and loss terms as well as the weighted Strichartz estimates.

4.1 Weighted Estimates

Let \( \langle v \rangle = (1 + |v|^2)^{1/2} \). To prove the weighted estimates for the gain term, we consider the quantity

\[ \langle \langle v \rangle^\ell \rangle^+ Q^+ (f, g), \langle v \rangle^{-\ell} \psi \rangle, \quad \ell > 0. \]  

(4.1)

When \( \ell = 0 \), Alonso et al. [1] introduce a bilinear operator to give (4.1) two representations which are used to prove the estimates collected in Proposition 2.3 (see the upcoming proof of Proposition 4.1 for two representations). In what follows, we first prove that the quantity (4.1) with \( \ell \neq 0 \) can also be bounded by the formulas with the same representations as \( \ell = 0 \). Then the desired estimate follows.
Proposition 4.1 Let $\ell \geq 0, 1 < p, q, r < \infty$ with $-N < \gamma \leq 0$ and
\[ \frac{1}{p} + \frac{1}{q} = 1 + \frac{\gamma}{N} + \frac{1}{r}. \] (4.2)
Assume the kernel (1.3):
\[ B(v - v_*, \omega) = |v - v_*|^\gamma b(\cos \theta) \]
with $b(\cos \theta)$ satisfies Grad’s cutoff assumption (1.4). Then the bilinear operator $Q^+(f, g)$ satisfies
\[ \| \langle v \rangle^\ell Q^+(f, g) \|_{L^r_{2N}(\mathbb{R}^N)} \leq C \| \langle v \rangle^\ell f \|_{L^p_{2N}(\mathbb{R}^N)} \| \langle v \rangle^\ell g \|_{L^q_{2N}(\mathbb{R}^N)}. \] (4.3)
If $\ell > N/m$ and $1 < p_m, q_m, m, r_m < \infty$ satisfy
\[ \frac{1}{p_m} + \frac{1}{q_m} = 1 + \frac{\gamma}{N} + \frac{1}{r_m}, \] (4.4)
and
\[ \frac{1}{p_m} + \frac{1}{q_m} + \frac{1}{m} = 1, \] (4.5)
then we have
\[ \| \langle v \rangle^\ell Q^+(f, g) \|_{L^{r_m}_{2N}(\mathbb{R}^N)} \leq C\langle p_m, \ell \rangle \| \langle v \rangle^\ell f \|_{L^{p_m}_{2N}(\mathbb{R}^N)} \| \langle v \rangle^\ell g \|_{L^{q_m}_{2N}(\mathbb{R}^N)}. \] (4.6)

Remark 4.2 Note that we use different notations $p, q, r$ and $p_m, q_m, r_m$ to tell (4.3) from (4.6) for their exponents satisfying different relations.

Proof of Proposition 4.1 First of all, we need to adopt the notations used in [1]. Let
\[ \hat{u} = u/|u|, \ u = v - v_*. \]
It is well known that the pre–post collision velocity relation (1.2) is equivalent to
\[ v' = v - \frac{1}{2}(u - |u|\sigma), \quad v'_* = v_* + \frac{1}{2}(u - |u|\sigma), \] (4.7)
and, see [18],
\[ d\omega = \frac{1}{(2\cos \theta)^{N-2}} d\sigma, \]
where $\theta$ is the angle between $\omega$ and $u = v - v_*$. In [1], the collision kernel is denoted by
\[ B(|u|, \hat{u} \cdot \sigma) = |u|^\gamma b(\hat{u} \cdot \sigma). \]
(Note that $\gamma$ above is denoted by $\lambda$ in [1].) They also define the bilinear operator
\[ \mathcal{P}(\psi, \phi)(u) := \int_{S^{N-1}} \psi(u^-)\phi(u^+)b(\hat{u} \cdot \sigma) d\sigma, \] (4.8)
where the variables $u^+$ and $u^-$ are defined by
\[ u^- := \frac{1}{2}(u - |u|\sigma) \quad \text{and} \quad u^+ := u - u^- = \frac{1}{2}(u + |u|\sigma). \]
We note that the vector $\omega$ is often used in the occurrence (1.2) and $\sigma$ in (4.7), while the vector $\sigma$ in (4.7) is denoted by $\omega$ in [1]. However this difference of notations clearly does not affect the proof of any related result. They also use $\tau$ and $\mathcal{R}$ to denote the translation and reflection operators
\[ \tau_v \psi(x) := \psi(x - v), \quad \mathcal{R} \psi(x) := \psi(-x). \]
Use above notations, the representations
\[
\int_{\mathbb{R}^N} Q^+(f, g)(v) \psi(v)dv = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(v)g(v-u)\mathcal{P}(\tau_v R \psi, 1)(u)|u|^{\gamma}dudv \\
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(u + v)g(v)\mathcal{P}(1, \tau_{-v} R \psi)(u)|u|^{\gamma}dudv
\]

(4.9)

are used to prove the estimates collected in Proposition 2.3. More precisely, when \( \gamma = 0 \), Alonso et al. used the first line of (4.9) (Eq. (4.1) in [1]) as a starting point to show (4.3) with \( \ell = 0 \). When \(-N < \gamma < 0\), both lines of (4.9) (Eqs. (5.1) and (5.12) in [1]) are used to show the case \( \ell = 0 \) of (4.3).

On the other hand, we follow the observation of Lions [15] to write
\[
\langle Q^+(f, g), \psi \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{S^{N-1}} f(v)g(v_*)\psi(v')B(v - v_*, \omega)d\omega dv_* dv
\]

(4.10)

Here \( T \) is a Radon transform
\[
T\psi(x) = |x|^{\gamma} \int_{\omega \in S^{N-1}_+} b(\cos \theta) \psi(x - (x \cdot \omega)\omega)d\omega,
\]

with \( \cos \theta = (x \cdot \omega)/|x|, x \neq 0, x = |x|(0, 0, 1) \) and \( \omega = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta), 0 \leq \theta \leq \pi/2 \). The regularizing effect of \( T \) is first studied by Lions [15], then studied by several authors, see [12, 13] for more details.

Combining (4.9) and (4.10), we have
\[
\int Q^+(f, g)(v) \psi(v)dv = \int \int f(v)g(v-u)\mathcal{P}(\tau_v R \psi, 1)(u)|u|^{\gamma}dudv \\
= \int \int f(u + v)g(v)\mathcal{P}(1, \tau_{-v} R \psi)(u)|u|^{\gamma}dudv \\
= \int \int f(v)g(v_*)\tau_{-v_*} \circ T \circ \tau_{v_*} \psi(v)dv_* dv
\]

(4.11)

With these preparations, we are ready to estimate the quantity \( \langle Q^+(f, g), \psi \rangle \). From the conservation of energy, either \( \langle v' \rangle \leq 2 \langle v \rangle \) or \( \langle v' \rangle \leq 2 \langle v_* \rangle \) has to be true. Hence for any \( \ell \geq 0 \) we have either
\[
\langle v \rangle^{-\ell} \leq \langle v' \rangle^{-\ell} \text{ or } \langle v_* \rangle^{-\ell} \leq \langle v' \rangle^{-\ell}.
\]

(4.12)

We define \( \Psi(v, v_*) := (\tau_{-v_*} \circ T \circ \tau_{v_*})\psi(v) \) and \( \psi_{-\ell}(v) = \langle v \rangle^{-\ell} \psi(v) \). From (4.12), we know that one of the followings estimates is true:
\[
|\Psi(v, v_*)| \leq \int_{S^{N-1}} |\psi(v')|B(v - v_*, \omega)d\omega \\
\leq \langle v \rangle^{\ell} \int_{S^{N-1}} |\psi'(v')|B(v - v_*, \omega)d\omega \\
= \langle v \rangle^{\ell} (\tau_{-v_*} \circ T \circ \tau_{v_*})|\psi_{-\ell}|(v),
\]

(4.13)
or

\[
|\Psi(v, v_*)| \leq \int_{S^{N-1}} |\psi(v')| B(v - v_*, \omega) d\omega
\]

\[
\leq \langle v_* \rangle^\ell \int_{S^{N-1}} |\langle v' \rangle^{-\ell} \psi(v')| B(v - v_*, \omega) d\omega
\]

\[
= \langle v_* \rangle^\ell \langle \tau_{-v_*} \circ T \circ \tau_{v_*} \rangle |\psi_{-\ell}||(v).
\]

Denote \( f_\ell(v) = \langle v \rangle^\ell f(v) \) and \( g_\ell(v_*) = \langle v_* \rangle^\ell g(v_*) \). Combining (4.11), (4.13) and (4.14), we have

\[
\left| \int Q^+(f, g)(v) \psi(v) dv \right| \\
\leq \int \int \left\{ |f_\ell(v)g(v_*)| + |f(v)g_\ell(v_*)| \right\} \langle \tau_{-v_*} \circ T \circ \tau_{v_*} \rangle |\psi_{-\ell}||(v)dv_* dv.
\]

Using the formulas of (4.11) to the right hand side of (4.15), we have

\[
\left| \int Q^+(f, g)(v) \psi(v) dv \right| \\
\leq \int \int \left\{ |f_\ell(v)g(v - u)| + |f(v)g_\ell(v - u)| \right\} \mathcal{P}(\tau_vR|\psi_{-\ell}|, 1)(u)|u|^{\gamma} du dv
\]

and

\[
\left| \int Q^+(f, g)(v) \psi(v) dv \right| \\
\leq \int \int \left\{ |f_\ell(u + v)g(v)| + |f(u + v)g_\ell(v)| \right\} \mathcal{P}(1, \tau_{-v}R|\psi_{-\ell}|)(u)|u|^{\gamma} du dv
\]

We note that the right hand sides of (4.16) and (4.17) still preserve the form of the representations in (4.9). Following the proofs of [1], we have

\[
\left| \int Q^+(f, g)(v) \psi(v) dv \right| \\
\leq C \left( \|f_\ell\|_{L^p\left(\mathbb{R}^N\right)} \|g\|_{L^q\left(\mathbb{R}^N\right)} + \|f\|_{L^p\left(\mathbb{R}^N\right)} \|g_\ell\|_{L^q\left(\mathbb{R}^N\right)} \right) \|\psi_{-\ell}\|_{L^r_m\left(\mathbb{R}^N\right)}
\]

\[
\leq C \|f_\ell\|_{L^p_m\left(\mathbb{R}^N\right)} \|g_\ell\|_{L^q_m\left(\mathbb{R}^N\right)} \|\psi_{-\ell}\|_{L^r_m\left(\mathbb{R}^N\right)},
\]

thus we conclude (4.3) by duality.

The proof of estimate (4.3) is an easy consequence of above argument which can be done by revising (4.18). Let \(1/a_1 = 1/p_m + 1/m\) and \(1/a_2 = 1/q_m + 1/m\) and note that we then have

\[
\frac{1}{a_1} + \frac{1}{q_m} = 1 + \frac{\gamma}{N} + \frac{1}{\tau_m} \quad \text{or} \quad \frac{1}{p_m} + \frac{1}{a_2} = 1 + \frac{\gamma}{N} + \frac{1}{\tau_m}
\]

which is exactly (4.2). Parallel to (4.18), we have

\[
\left| \int Q^+(f, g)(v) \psi(v) dv \right| \\
\leq C \left( \|f_\ell\|_{L^p_m\left(\mathbb{R}^N\right)} \|g\|_{L^q_m\left(\mathbb{R}^N\right)} + \|f\|_{L^p_m\left(\mathbb{R}^N\right)} \|g_\ell\|_{L^q_m\left(\mathbb{R}^N\right)} \right) \|\psi_{-\ell}\|_{L^r_m\left(\mathbb{R}^N\right)}
\]

\[
\leq C \|f_\ell\|_{L^p_m\left(\mathbb{R}^N\right)} \|g_\ell\|_{L^q_m\left(\mathbb{R}^N\right)} \|\psi_{-\ell}\|_{L^r_m\left(\mathbb{R}^N\right)},
\]
where we used the condition $\ell m > N$ in the last inequality. By duality, we conclude (4.6). □

To prove the weighted estimate for the loss term. We begin with the following.

**Proposition 4.3** (The Hardy–Littlewood–Sobolev inequality) If $x, y \in \mathbb{R}^N$, $1 < p, q < \infty$, $-N < \gamma < 0$ and $\frac{1}{p} + \frac{1}{q} + \frac{\gamma}{N} = 2$, then we have

$$\left| \iint f(x)|x-y|^{\gamma} g(y) dxdy \right| \leq C_{N, \gamma, p} \|f\|_{L^p} \|g\|_{L^q}.$$  

**Proposition 4.4** Assume $B(v - v_s, \omega)$ defined in (1.3) satisfies $-N < \gamma < 0$ and (1.4). If $\ell > N/m$ and $1 < p_m, q_m, m, r_m < \infty$ satisfy

$$\frac{1}{p_m} + \frac{1}{r_m} < 1, \quad \frac{1}{m} < 1,$$  

and

$$\frac{1}{p_m} + \frac{1}{q_m} + \frac{1}{m} = 1 + \frac{\gamma}{N} + \frac{1}{r_m},$$  

then we have

$$\|\langle v \rangle^\ell Q^-(f, g)\|_{L^{r_m}} \leq C(p_m, \ell) \|\langle v \rangle^\ell f\|_{L^{p_m}} \|\langle v \rangle^\ell g\|_{L^{q_m}}.$$  

Here we note that the first inequality of (4.20) is equivalent to $1/p_m < 1/r_m$.

**Proof** Assume $1 < p_m, q_m, m, r_m < \infty$ satisfy (4.4) and (4.2). Let $1 < a_1, a_2 < \infty$ be defined by

$$\frac{1}{a_1} := \frac{1}{p_m} + \frac{1}{r_m}, \quad \frac{1}{a_2} := \frac{1}{q_m} + \frac{1}{m}. \quad (4.23)$$

Then we have $1 < a_1, a_2 < \infty$ and

$$\frac{1}{a_1} + \frac{1}{a_2} = 2 + \frac{\gamma}{N}. \quad (4.24)$$

By Hardy–Littlewood–Sobolev inequality and Hölder inequality, we have

$$\left| \langle Q^-(f, g), h \rangle \right| = \left| \iint \int f(v)g(v_s)h(v)B(v - v_s, \omega) d\omega dv_s dv \right|$$

$$= \left| C \iint \int f(v)g(v_s)h(v)|v - v_s|^{\gamma} dv_s dv \right|$$

$$\leq C \|f \cdot h\|_{L^a_1} \|g\|_{L^a_2} \quad (4.25)$$

$$\leq C \|\langle v \rangle^\ell f\|_{L^{p_m}} \|\langle v \rangle^{\ell - \ell} h\|_{L^{r_m}} \|\langle v \rangle^{\ell} g\|_{L^{q_m}} \|\langle v \rangle^{-\ell}\|_{L^m}$$

$$\leq C \|\langle v \rangle^\ell f\|_{L^{p_m}} \|\langle v \rangle^{\ell - \ell} h\|_{L^{r_m}} \|\langle v \rangle^{\ell} g\|_{L^{q_m}},$$

where the last inequality holds when $\ell > N/m$. Then we conclude that (4.22) holds. □

**Remark 4.5** Our proof also includes the estimate

$$\|\langle v \rangle^\ell Q^-(f, g)\|_{L^{r_m}} \leq C(p_m, \ell) \|\langle v \rangle^\ell f\|_{L^{p_m}} \|g\|_{L^{q_m}}.$$  

where $1/a_2 = 1/q_m + 1/m$ and

$$\frac{1}{p_m} + \frac{1}{a_2} = 1 + \frac{\gamma}{N} + \frac{1}{r_m}.$$
We also need to build the weighted Strichartz estimates. We consider the weight \( \langle v \rangle^\ell, \ell \in \mathbb{R} \) as the multiplication operator. Using the notations of (2.7), we note that the following communication relations hold:

\[
\langle v \rangle^\ell U(t)u_0 = U(t)\langle v \rangle^\ell u_0, \quad \langle v \rangle^\ell W(t)F = W(t)\langle v \rangle^\ell F.
\]  

(4.26)

Combining above facts with Proposition 2.2, we have the following result.

**Corollary 4.6** Let \( u \) satisfy the kinetic transport equation (2.1). The estimate

\[
\|\langle v \rangle^\ell u\|_{L_t^q L_x^p L_v^r} \leq C(q, r, p, N) \left(\|\langle v \rangle^\ell u_0\|_{L_{x,v}^3} + \|\langle v \rangle^\ell F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}} L_v^{\tilde{r}}}\right)
\]

holds for all \( u_0 \in L_{t,x,v}^1 \) and all \( F \in L_t^{\tilde{q}} L_x^{\tilde{r}} L_v^{\tilde{r}} \) if and only if \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are two KT-admissible exponents triplets and \( a = \text{HM}(p, r) = \text{HM}(\tilde{p}, \tilde{r}) \) with the exception of \((q, r, p)\) being an endpoint triplet.

### 4.2 Gain Term Only Equation for \( N = 3 \) and \(-1 < \gamma \leq 0\)

**Proposition 4.7** Let \( N = 3 \) and collision kernel \( B \) defined in (1.3) satisfies (1.4) and \(-1 < \gamma \leq 0\). Let \( \ell_\gamma = (1 + \gamma)^+ < 3/2 \). There exists a small number \( \eta > 0 \) such that if the initial data \( f_0 \) is in the set

\[
B_\eta^{\ell_\gamma} = \{ f_0 \in L_{t,x,v}^3(\mathbb{R}^3 \times \mathbb{R}^3) : \|\langle v \rangle^{\ell_\gamma} f_0\|_{L_{3,x,v}^3} < \eta \} \subset L_{3,x,v}^3,
\]

then the Cauchy problem (2.9) admits a globally unique mild solution

\[
\langle v \rangle^{\ell_\gamma} f_+ \in C([0, \infty), L_{3,x,v}^3) \cap L_{3,x,v}^3
\]

where the triplet \((q, r, p)\) lies in the set

\[
\left\{ (q, r, p) \mid \frac{1}{q} = \frac{3}{p} - 1, \frac{1}{r} = \frac{2}{p} - \frac{3}{r}, \frac{1}{p} < 4 \right\}.
\]

(4.28)

The solution map \( \langle v \rangle^{\ell_\gamma} f_0 \in B_\eta^{\ell_\gamma} \to \langle v \rangle^{\ell_\gamma} f_+ \in L_{q,x,v}^3 L_{x,v}^p \) is Lipschitz continuous and the solution \( \langle v \rangle^{\ell_\gamma} f_+ \) scatters with respect to the kinetic transport operator in \( L_{3,x,v}^3 \).

**Proof** The proof can be done by exactly the same argument as that for Proposition 2.4 except that the estimates there need to be replaced by weighted ones. Parallel to (2.11), we claim that the following key estimates hold:

\[
\|S(\langle v \rangle^{\ell_\gamma} f_+)^\|_{L_t^{q_1} L_x^{r_1} L_v^{p_1}} \leq C_0\|\langle v \rangle^{\ell_\gamma} f_0\|_{L_{q,x,v}^3} + C_1\|\langle v \rangle^{\ell_\gamma} Q^+(f_+, f_+)\|_{L_t^{q_1} L_x^{r_1} L_v^{p_1}} \leq C_0\|\langle v \rangle^{\ell_\gamma} f_0\|_{L_{q,x,v}^3} + C_2\|\langle v \rangle^{\ell_\gamma} f_+\|^2_{L_t^{q_1} L_x^{r_1} L_v^{p_1}}.
\]

(4.29)

where we choose the same triplets as Proposition 2.4 (see also Definition 2.5). Then the first inequality follows from Corollary 4.6. To show the second inequality is equal to show

\[
\|\langle v \rangle^{\ell_\gamma} Q^+(f_+, f_+)\|_{L_t^{q_1} L_x^{r_1} L_v^{p_1}} \leq C\|\langle v \rangle^{\ell_\gamma} f_+\|^2_{L_t^{q_1} L_x^{r_1} L_v^{p_1}}.
\]

(4.30)

which is parallel to (2.16). Please note that the only change occurs at \( v \) variable and at which we should verify.
When \( \gamma = 2 - N = -1 \), the relation (2.13) and the proof of (2.16) require
\[
\frac{1}{p} + \frac{1}{p} \neq \frac{2}{3} + \frac{1}{p'}.
\] (4.31)

When \(-1 < \gamma \leq 0\), we can rewrite (4.31) as
\[
\frac{1}{p} + \frac{1}{p} + \frac{1 + \gamma}{3} = 1 + \frac{\gamma}{3} + \frac{1}{p'}.
\] (4.32)

and note that this is the form of (4.5) with \(1/m = (1 + \gamma)/3 \leq 1/3\). Since \(1/3 < 1/p < 4/9\), the relation (4.4) is satisfied. Let \(\ell_\gamma = (3/m)^+ = (1 + \gamma)^+ > N/m = (1 + \gamma)\), then we have (4.6), i.e., (4.30).

Also we need a weighted version of Proposition 2.8.

**Proposition 4.8** Let \(N = 3\) and \(a_2 = 15/8\). Under the same assumptions as Proposition 4.7, if we further assume \(\|\langle v \rangle^{\ell_\gamma} f_0\|_{L^2_{\gamma} L^2_v} < \infty\), then solution \(f_+\) in Proposition 4.7 also satisfies
\[
\|\langle v \rangle^{\ell_\gamma} f_+\|_{L^2_{\gamma} L^2_{\gamma} L^2_v}^2 < \infty
\] (4.33)

where \((1/q_2, 1/r_2, 1/p_2) = (1/2, (12 + \gamma)/30, (20 - \gamma)/30)\) is a KT-admissible triplet with \(1/p_2 + 1/r_2 = 2/a_2\).

**Proof** Follow the idea of Proposition 2.8, it suffices to show that
\[
\|\langle v \rangle^{\ell_\gamma} Q^+(f_+, f_+)\|_{L^2_{\gamma} L^2_{\gamma} L^2_v} \leq C \|\langle v \rangle^{\ell_\gamma} f_+\|_{L^2_{\gamma} L^2_{\gamma} L^2_v} \|\langle v \rangle^{\ell_\gamma} f_+\|_{L^2_{\gamma} L^2_{\gamma} L^2_v}.
\] (4.34)

Using the same trick of rewriting (4.31) as (4.32), we rewrite (2.29a), i.e., \(1/p + 1/p_2 = 2/3 + 1/p_2^\prime\) as
\[
\frac{1}{p} + \frac{1}{p_2} + \frac{1 + \gamma}{3} = 1 + \frac{\gamma}{3} + \frac{1}{p_2^\prime}.
\] (4.35)

and apply (4.6) of Proposition 4.1. \(\square\)

Using again the argument of Propositions 4.8 and 4.4 and the result of Proposition 4.1, we have the following weighted version of Corollary 2.10.

**Corollary 4.9** Use the same notations as Proposition 4.8 but \(\gamma \neq 0\). Suppose \(\langle v \rangle^{\ell_\gamma} f_1 \in L^q_{\gamma} L^r_x L^p_v\) and \(\langle v \rangle^{\ell_\gamma} f_2 \in L^q_{\gamma} L^r_x L^p_v\). Then \(\langle v \rangle^{\ell_\gamma} Q^\pm(f_1, f_2) \in L^q_{\gamma} L^r_x L^p_v\) and
\[
\|\langle v \rangle^{\ell_\gamma} Q^-(f_1, f_2)\|_{L^q_{\gamma} L^r_x L^p_v} \leq C \|\langle v \rangle^{\ell_\gamma} f_1\|_{L^q_{\gamma} L^r_x L^p_v} \|\langle v \rangle^{\ell_\gamma} f_2\|_{L^q_{\gamma} L^r_x L^p_v},
\]
\[
\|\langle v \rangle^{\ell_\gamma} Q^+(f_1, f_2)\|_{L^q_{\gamma} L^r_x L^p_v} \leq C \|\langle v \rangle^{\ell_\gamma} f_1\|_{L^q_{\gamma} L^r_x L^p_v} \|\langle v \rangle^{\ell_\gamma} f_2\|_{L^q_{\gamma} L^r_x L^p_v},
\]
\[
\|\langle v \rangle^{\ell_\gamma} Q^+(f_1, f_2)\|_{L^q_{\gamma} L^r_x L^p_v} \leq C \|\langle v \rangle^{\ell_\gamma} f_2\|_{L^q_{\gamma} L^r_x L^p_v} \|\langle v \rangle^{\ell_\gamma} f_1\|_{L^q_{\gamma} L^r_x L^p_v}.
\]

**4.3 Proof of Theorem 1.3** As we did for the case \(\gamma = -1\), we use the solution \(f_+\) in Proposition 4.7 to construct the beginning condition of the the Kaniel–Shinbrot iteration, i.e. Let \(g_1 = f_+\) and \(h_1 \equiv 0\). Then the system (3.6) gives \(g_2 = f_+ = g_1\). By Corollary 4.9 and the argument after (3.9), we have \(\langle v \rangle^{\ell_\gamma} L(g_1) \in L^q_{\gamma} L^r_x L^p_v\) where \((1/q_2)\prime = 1/q_2 + 1/q_1, (1/r_2)\prime = 1/r_2 + 1/r_1\) and...
\[1/r + 1/x \text{ and } (1/p_2)(q_2, r_2, p_2) \text{ is given by Proposition 4.8}. \]

Therefore \(L(g_1)\) is pointwisely a.e. well-defined. Then we can compute \(h_2\) by (3.6) and have the beginning condition \(0 \leq h_1 \leq h_2 \leq g_2 \leq g_1\). Hence the limit functions \(g, h\) of iteration exist and we have

\[
\begin{align*}
&\langle v \rangle^{\ell \gamma} g, \quad \langle v \rangle^{\ell \gamma} h \in L^{\gamma}((0, \infty], L^2_x L^p_v), \\
&\langle v \rangle^{\ell \gamma} g, \quad \langle v \rangle^{\ell \gamma} h \in L^{\gamma}((0, \infty], L^2_x L^p_v), \\
&\langle v \rangle^{\ell \gamma} Q^+(g, g), \quad \langle v \rangle^{\ell \gamma} Q^+(h, h) \in L^\gamma L^2_x L^p_v, \\
&\langle v \rangle^{\ell \gamma} Q^-(g, g), \quad \langle v \rangle^{\ell \gamma} Q^-(h, h) \in L^\gamma L^2_x L^p_v.
\end{align*}
\]

Replacing the estimates from Lemma 3.2 to the end of Sect. 3 by their weighted versions, we see that the remaining part of the proof follows. For example, (3.13) is replaced by

\[
\begin{align*}
&\|\langle v \rangle^{\ell \gamma} w\|_{L^{\gamma}((0, s], L^2_x L^p_v)} \\
&\leq C\|\langle v \rangle^{\ell \gamma} g\|_{L^{\gamma}((0, s], L^2_x L^p_v)} + \|\langle v \rangle^{\ell \gamma} h\|_{L^{\gamma}((0, s], L^2_x L^p_v)}\|\langle v \rangle^{\ell \gamma} w\|_{L^{\gamma}((0, s], L^2_x L^p_v)} \\
&:= C(g, h, s)\|\langle v \rangle^{\ell \gamma} w\|_{L^{\gamma}((0, s], L^2_x L^p_v)},
\end{align*}
\]

and the others are similar.

\[\square\]

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