ON THE DENSITY FUNCTION FOR THE VALUE-DISTRIBUTION OF AUTOMORPHIC L-FUNCTIONS

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Abstract. The Bohr-Jessen limit theorem is a probabilistic limit theorem on the value-distribution of the Riemann zeta-function in the critical strip. Moreover their limit measure can be written as an integral involving a certain density function. The existence of the limit measure is now known for a quite general class of zeta-functions, but the integral expression has been proved only for some special cases (such as Dedekind zeta-functions). In this paper we give an alternative proof of the existence of the limit measure for a general setting, and then prove the integral expression, with an explicitly constructed density function, for the case of automorphic L-functions attached to primitive forms with respect to congruence subgroups \( \Gamma_0(N) \).

1. Introduction

Let \( s = \sigma + it \) be a complex variable, \( \zeta(s) \) the Riemann zeta-function. Let \( R \) be a fixed rectangle in the complex plane \( \mathbb{C} \), with the edges parallel to the axes. By \( \mu_k \) we mean the \( k \)-dimensional usual Lebesgue measure. For \( \sigma > 1/2 \) and \( T > 0 \), define

\[
V_\sigma(T,R;\zeta) = \mu_1 \{ t \in [-T,T] \mid \log \zeta(\sigma + it) \in R \}.
\]

(The rigorous definition of \( \log \zeta(\sigma + it) \) will be given later, in Section 3.) In their classical paper [4], Bohr and Jessen proved the existence of the limit

\[
W_\sigma(R;\zeta) = \lim_{T \to \infty} \frac{1}{2T} V_\sigma(T,R;\zeta).
\]

This is now called the Bohr-Jessen limit theorem. Moreover they proved that this limit value can be written as

\[
W_\sigma(R;\zeta) = \int_R M_\sigma(z,\zeta) |dz|,
\]

where \( z = x + iy \in \mathbb{C}, |dz| = dx dy / 2\pi \), and \( M_\sigma(z,\zeta) \) is a continuous non-negative, explicitly constructed function defined on \( \mathbb{C} \), which we may call the density function for the value-distribution of \( \zeta(s) \).

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This work is a milestone in the value-distribution theory of $\zeta(s)$, and various alternative proofs and related results have been published; for example, Jessen and Wintner [8], Borchsenius and Jessen [5], Guo [6], and Ihara and the first author [7].

An important problem is to consider the generalization of the Bohr-Jessen theorem. The first author [11] proved that the formula (1.2) can be generalized to a fairly general class of zeta-functions with Euler products. However, (1.3) has not yet been generalized to such a general class. The reason is as follows.

The original proof of (1.2) and (1.3) by Bohr and Jessen depends on a geometric theory of certain "infinite sums" of convex curves, developed by themselves [3]. In later articles [8] and [5], the effect of the convexity of curves was embodied in a certain inequality due to Jessen and Wintner [8, Theorem 13]. Using this method, the Bohr-Jessen theory was generalized to Dirichlet $L$-functions (Joyner [9]) and Dedekind zeta-functions of Galois number fields (the first author [12]). These generalizations are possible because these zeta-functions have "convex" Euler products in the sense of [11, Section 5]. But this convexity cannot be expected for more general zeta-functions.

In [11], the first author developed a method of proving (1.2) without using any convexity, so succeeded in generalizing the theory. However, the method in [11] cannot give a generalization of (1.3).

So far, there is no proof of (1.3) or its analogues without using the convexity, or the Jessen-Wintner type of inequalities. For example, [7] gives a different argument of constructing the density functions for Dirichlet $L$-functions, but the argument in [7] also depends on the Jessen-Wintner inequality.

In [14] [15], the first author obtained certain quantitative results on the value-distribution of Dedekind zeta-functions of non-Galois fields and Hecke $L$-functions of ideal class characters, whose Euler products are not convex. But in these cases, they are "not so far" from the case of Dedekind zeta-functions of Galois fields. In fact, a simple generalization of the Jessen-Wintner inequality is proved ( [15, Lemma 2]) and is essentially used in the proof.

Actually, analyzing the proof of [8, Theorems 12, 13] carefully, we can see that the convexity of curves is not essential. The indispensable tool is the inequality of the Jessen-Wintner type. (However the convex property is probably of independent interest; see Section 8.)

It is the purpose of the present paper to obtain an analogue of (1.3) in the case of automorphic $L$-functions. The main result (Theorem 2.1) will be stated in the next section. The key is Proposition 7.3 which is an analogue of the Jessen-Wintner inequality for the automorphic case. The novelty of this proposition will be discussed in Section 6.

Except for the proof of this inequality, the argument can be carried out in more general situation. In Section 3 we will introduce a general class of zeta-functions, and in Sections 4 to 6 we will generalize the method in [12] to that
general class. Then in Section 7 we will prove the Jessen-Wintner inequality for the automorphic case to complete the proof of the main theorem.

2. Statement of the main result

Let $f$ be a primitive form of weight $\kappa$ and level $N$, that is a normalized Hecke-eigen new form of weight $\kappa$ with respect to the congruence subgroup $\Gamma_0(N)$, and write its Fourier expansion as

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{\kappa-1/2}e^{2\pi inz},$$

where the coefficients $\lambda_f(n)$ are real numbers with $\lambda_f(1) = 1$. Denote the associated $L$-function by

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n)n^{-s}.$$ 

This is absolutely convergent when $\sigma > 1$, and can be continued to the whole plane $\mathbb{C}$ as an entire function. We understand the rigorous meaning of $\log L_f(s)$ and of

$$V_\sigma(T, R; L_f) = \mu_1\{t \in [-T, T] \mid \log L_f(\sigma + it) \in R\}$$

in the sense explained in Section 3. The following is the main theorem of the present paper.

**Theorem 2.1.** For any $\sigma > 1/2$, the limit

$$W_\sigma(R; L_f) = \lim_{T \to \infty} \frac{1}{2T}V_\sigma(T, R; L_f)$$

exists, and can be written as

$$W_\sigma(R; L_f) = \int_R |M_\sigma(z, L_f)|dz,$$

where $M_\sigma(z, L_f)$ is a continuous non-negative function (explicitly given by (6.4) below) defined on $\mathbb{C}$.

The above function $M_\sigma(w, L_f)$ can be called the density function for the value-distribution of $L_f(s)$. The integral expression involving the density function is useful for quantitative studies; for example, in [12] [14] [15] we used such expressions to evaluate the speed of convergence of (3.4) below in the case of Dedekind zeta-functions and Hecke $L$-functions. Therefore we may expect that (2.2) can be used for quantitative investigation on the value-distribution of $L_f(s)$ (see also Remark 6.3).
Let \( \mathbb{P} \) be the set of all prime numbers. Since \( f \) is a common Hecke eigenform, \( L_f(s) \) has the Euler product
\[
L_f(s) = \prod_{p \in \mathbb{P}} \left(1 - \frac{\lambda_f(p)p^{-s}}{p} \right)^{-1} \prod_{p \nmid N} \left(1 - \frac{\alpha_f(p)p^{-s}}{p} - 2s\right)^{-1},
\]
where \( \alpha_f(p) + \beta_f(p) = \lambda_f(p), \beta_f(p) = \overline{\alpha_f(p)} \), and
\[
|\alpha_f(p)| = |\beta_f(p)| = 1.
\]
Also we know
\[
|\lambda_f(p)| \leq 1 \quad \text{(if } p|N) \]
(see [16, Theorem 4.6.17]).

It is known that, for any \( \varepsilon > 0 \), there exists a set of primes \( \mathbb{P}_f(\varepsilon) \) of positive density in \( \mathbb{P} \), such that the inequality
\[
|\lambda_f(p)| > \sqrt{2} - \varepsilon
\]
holds for any \( p \in \mathbb{P}_f(\varepsilon) \) (M. R. Murty [17, Corollary 2 of Theorem 4] in the full modular case, and M. R. Murty and V. K. Murty [18, Chapter 4, Theorem 8.6] for general \( \Gamma_0(N) \) case). This fact is used essentially in the course of the proof.

3. The general formulation

A large part of the proof of our Theorem 2.1 can be carried out under a more general framework, that is, for general Euler products introduced in [11]. We begin with recalling the definition of those Euler products.

Let \( \mathbb{N} \) be the set of all positive integers, and \( g(n) \in \mathbb{N}, f(j,n) \in \mathbb{N} \) \((1 \leq j \leq g(n))\) and \( a_n^{(j)} \in \mathbb{C} \). Denote by \( p_n \) the \( n \)-th prime number. We assume
\[
g(n) \leq C_1 p_n^{\alpha}, \quad |a_n^{(j)}| \leq p_n^\beta
\]
with constants \( C_1 > 0 \) and \( \alpha, \beta \geq 0 \). Define
\[
\varphi(s) = \prod_{n=1}^{\infty} A_n(p_n^{-s})^{-1},
\]
where \( A_n(X) \) are polynomials in \( X \) given by
\[
A_n(X) = \prod_{j=1}^{g(n)} \left(1 - a_n^{(j)} X^{f(j,n)} \right).
\]
Then \( \varphi(s) \) is convergent absolutely in the half-plane \( \sigma > \alpha + \beta + 1 \) by (3.1). Suppose
(i) $\varphi(s)$ can be continued meromorphically to $\sigma \geq \sigma_0$, where $\alpha + \beta + 1/2 \leq \sigma_0 < \alpha + \beta + 1$, and all poles in this region are included in a compact subset of $\{ s \mid \sigma > \sigma_0 \}$,

(ii) $\varphi(\sigma + it) = O((|t| + 1)^C)$ for any $\sigma \geq \sigma_0$, with a constant $C > 0$,

(iii) It holds that

$$\int_{-T}^{T} |\varphi(\sigma_0 + it)|^2 dt = O(T).$$

(3.3)

We denote by $M$ the set of all $\varphi$ satisfying the above conditions.

**Remark 3.1.** Here we note that $L_f(s)$ defined in the preceding section belongs to $M$. In fact, the Euler product is given by (2.3). The condition (3.1) is satisfied with $\alpha = \beta = 0$ by (2.4), (2.5). It is entire, so (i) is obvious. Since it satisfies a functional equation, (ii) follows by using the Phragmén-Lindelöf convexity principle. Lastly, (iii) follows (with any $\sigma_0 > 1/2$) by Potter’s result [19].

Now let us define $\log \varphi(s)$. First, when $\sigma > \alpha + \beta + 1$, it is defined by the sum

$$\log \varphi(s) = -\sum_{n=1}^{\infty} \sum_{j=1}^{g(n)} \text{Log}(1 - a_n^{(j)} p_n^{-f(j,n)s}),$$

where $\text{Log}$ means the principal branch. Next, let

$$B(\rho) = \{ \sigma + i\Im \rho \mid \sigma_0 \leq \sigma \leq \Re \rho \}$$

for any zero or pole $\rho$ with $\Re \rho \geq \sigma_0$. We exclude all $B(\rho)$ from $\{ s \mid \sigma \geq \sigma_0 \}$, and denote the remaining set by $G(\varphi)$. Then, for any $s \in G(\varphi)$, we may define $\log \varphi(s)$ by the analytic continuation along the horizontal path from the right. Define

$$V_\sigma(T, R; \varphi) = \mu_1 \{ t \in [-T, T] \mid \sigma + it \in G(\varphi), \log \varphi(\sigma + it) \in \mathbb{R} \}.$$

Then, as a generalization of (1.2), the first author [11] proved the following

**Theorem 3.2.** (11) Let $\varphi \in M$. For any $\sigma > \sigma_0$, the limit

$$W_\sigma(R; \varphi) = \lim_{T \to \infty} \frac{1}{2T} V_\sigma(T, R; \varphi)$$

exists.

This theorem may be regarded as a result on weak convergence of probability measures, and Prokhorov’s theorem in probability theory is used in the proof given in [11].

In [12], the first author presented an alternative argument of proving such a limit theorem, again without using any convexity. This argument is based on Lévy’s convergence theorem. The method in [12] is more suitable to discuss the matter of density functions, so in the present paper we follow the method in [12].

In [12], only the case of Dedekind zeta-functions is discussed, but, as mentioned in [13], the idea in [12] can be applied to any $\varphi \in M$. Such a
generalization has, however, not yet been published, so we will give a sketch of the argument in the following Sections 4 and 5.

4. The method of Fourier transforms

Let \( \sigma > \sigma_0 \), and \( N \in \mathbb{N} \). The starting point of the argument is to consider the finite truncation of \( \varphi(s) \), that is

\[
\varphi_N(s) = \prod_{n \leq N} A_n(p_n^{-s})^{-1} = \prod_{n \leq N} \prod_{j=1}^{g(n)} \left( 1 - r_n^{(j)} p_n^{-if(j,n)t} \right)^{-1},
\]

where \( r_n^{(j)} = a_n^{(j)} p_n^{-f(j,n)\sigma} \). Then

\[
(4.1) \quad \log \varphi_N(s) = - \sum_{n \leq N} \sum_{j=1}^{g(n)} \log \left( 1 - r_n^{(j)} e^{-itf(j,n)\log p_n} \right).
\]

Note that

\[
|r_n^{(j)}| \leq |a_n^{(j)}| p_n^{-f(j,n)\sigma} \leq p_n^{\beta - \sigma} \leq p_n^{\beta - (\alpha + \beta + 1/2)} \leq p_n^{-1/2} \leq 1/\sqrt{2}.
\]

Let \( \mathbb{Z} \) be the set of all integers, \( \mathbb{R} \) the set of all real numbers, \( \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N \) be the \( N \)-dimensional unit torus, and define the mapping \( S_N : \mathbb{T}^N \to \mathbb{C} \), attached to \( \mathbb{1}.* \), by

\[
(4.2) \quad S_N(\theta_1, \ldots, \theta_N) = - \sum_{n \leq N} \sum_{j=1}^{g(n)} \log \left( 1 - r_n^{(j)} e^{2\pi if(j,n)\theta_n} \right).
\]

(Though \( S_N \) depends on \( \sigma \) and \( \varphi \), we do not write explicitly in the notation, for brevity. Similar abbreviation is applied to the notation of \( \lambda_N, \Lambda, K_n \) below.) We write \( z_n^{(j)}(\theta_n) = -\log(1 - r_n^{(j)} e^{2\pi if(j,n)\theta_n}) \) and \( z_n(\theta_n) = \sum_{j=1}^{g(n)} z_n^{(j)}(\theta_n) \). Then

\[
(4.3) \quad S_N(\theta_1, \ldots, \theta_N) = \sum_{n \leq N} z_n(\theta_n).
\]

For any Borel subset \( A \subset \mathbb{C} \), we define \( W_{N,\sigma}(A; \varphi) = \mu_N(S_N^{-1}(A)) \). Then \( W_{N,\sigma} \) is a probability measure on \( \mathbb{C} \).

Let \( R \subset \mathbb{C} \) be any rectangle with the edges parallel to the axes. The idea of considering the inverse image \( S_N^{-1}(R) \subset \mathbb{T}^N \) goes back to Bohr’s work (Bohr and Courant [2], Bohr [1], and Bohr and Jessen [4]). Also let \( E \) be any strip, parallel to the real or imaginary axis. We have the following two facts, whose proofs of these two facts are exactly the same as the proofs of [12] Lemma 1.

**Fact 1.** The sets \( S_N^{-1}(R) \), \( S_N^{-1}(E) \) are Jordan measurable.

**Fact 2.** For any \( \varepsilon > 0 \), there exists a positive number \( \eta \) such that, for any strip \( E \) whose width is not larger than \( \eta \), it holds that \( W_{N,\sigma}(E; \varphi) < \varepsilon \).

Now define

\[
V_{N,\sigma}(T, R; \varphi) = \mu_1 \{ t \in [-T, T] \mid \log \varphi_N(\sigma + it) \in R \}.
\]
We see that $\log \varphi_N(\sigma + it) \in R$ if and only if
\[
\left( \left\{ -\frac{t}{2\pi} \log p_1 \right\}, \ldots, \left\{ -\frac{t}{2\pi} \log p_N \right\} \right) \in S_N^{-1}(R)
\]
(where $\{x\}$ means the fractional part of $x$). Since $\log p_1, \ldots, \log p_N$ are linearly independent over the rational number field $\mathbb{Q}$, in view of Fact 1, we can apply the Kronecker-Weyl theorem to obtain

**Proposition 4.1.** For any $N \in \mathbb{N}$, we have
\[
W_{N, \sigma}(R; \varphi) = \lim_{T \to \infty} \frac{1}{2T} V_{N, \sigma}(T, R; \varphi).
\] (4.4)

This is the "finite truncation" version of Theorem 3.2. Therefore, the remaining task to arrive at Theorem 3.2 is to discuss the limit $N \to \infty$. For this purpose, we consider the Fourier transform
\[
\Lambda_N(w) = \int_{\mathbb{C}} e^{i(z,w)} dW_{N, \sigma}(z; \varphi),
\]
where $(z,w) = \Re z \Re w + \Im z \Im w$. Our next aim is to show the following

**Proposition 4.2.** As $N \to \infty$, $\Lambda_N(w)$ converges to a certain function $\Lambda(w)$, uniformly in $\{ w \in \mathbb{C} | |w| \leq a \}$ for any $a > 0$.

**Proof.** The proof is quite similar to the argument in [12, Section 3]. It is easy to see that
\[
\Lambda_N(w) = \int_{T_N} e^{i(S_N(\theta_1, \ldots, \theta_N), w)} d\mu_N(\theta_1, \ldots, \theta_N),
\]
so in view of (4.3) we can write
\[
\Lambda_N(w) = \prod_{n \leq N} K_n(w)
\] (4.5)
with
\[
K_n(w) = \int_0^1 e^{i(z_n(\theta_n), w)} d\theta_n.
\]
Noting $|z_n^{(j)}(\theta_n)| \ll |r_n^{(j)}| \leq p_n^{\beta - \sigma}$ and (3.1), we have
\[
|z_n(\theta_n)|^2 = \sum_{j=1}^{g(n)} |z_n^{(j)}(\theta_n)|^2 \ll p_n^{2(\alpha + \beta - \sigma)}.
\]
Therefore, analogously to [12 (3.2)], we obtain
\[
|K_n(w) - 1| \ll |w|^2 p_n^{2(\alpha + \beta - \sigma)},
\] (4.6)
which implies
\[
|\Lambda_{n+1}(w) - \Lambda_n(w)| = |\Lambda_n(w)| \cdot |K_{n+1}(w) - 1| \ll |w|^2 p_n^{2(\alpha + \beta - \sigma)}.
\] (4.7)
Therefore, for $M > N$,

\begin{equation}
|\Lambda_M(w) - \Lambda_N(w)| \leq \sum_{n=N}^{M-1} |\Lambda_{n+1}(w) - \Lambda_n(w)|
\end{equation}

\[
\ll |w|^2 \sum_{n=N}^{M-1} p_{n+1}^{2(\alpha + \beta - \sigma)} \leq |w|^2 \sum_{n=N}^{\infty} p_{n+1}^{2(\alpha + \beta - \sigma)}.
\]

Since $\sigma > \sigma_0 \geq \alpha + \beta + 1/2$, the last sum tends to 0 as $N \to \infty$, uniformly in the region $|w| \leq a$. This implies the assertion of the proposition. \hfill \Box

From Proposition 4.2 in view of Lévy’s convergence theorem, we immediately obtain

**Corollary 4.3.** There exists a regular probability measure $W_\sigma(\cdot ; \varphi)$, to which $W_{N,\sigma}(\cdot ; \varphi)$ converges weakly as $N \to \infty$, and

\begin{equation}
\Lambda(w) = \int_\mathbb{C} e^{i\langle z, w \rangle} dW_\sigma(z; \varphi).
\end{equation}

Moreover, taking the limit $M \to \infty$ on (4.8), we obtain

\begin{equation}
|\Lambda(w) - \Lambda_N(w)| \ll |w|^2 \sum_{n=N}^{\infty} p_{n+1}^{2(\alpha + \beta - \sigma)}.
\end{equation}

5. Proof of Theorem 3.2

In this section we show how to prove Theorem 3.2 in the framework of our present method. The argument is very similar to that given in [12, Sections 3 and 4], so we omit some details.

First, using Fact 2 in Section 4, we can show (analogously to the argument in the last part of [12, Section 3]) that $R$ is a continuity set with respect to $W_\sigma$, and hence

\begin{equation}
W_\sigma(R; \varphi) = \lim_{N \to \infty} W_{N,\sigma}(R; \varphi).
\end{equation}

Now, following the method in [12, Section 4], we prove Theorem 3.2. Put

\begin{equation}
R_N(s; \varphi) = \log \varphi(s) - \log \varphi_N(s), \quad f_N(s; \varphi) = \frac{\varphi(s)}{\varphi_N(s)} - 1.
\end{equation}

When $\sigma > \alpha + \beta + 1$, since

\begin{equation}
R_N(s; \varphi) \ll \sum_{n>N} \sum_{j=1}^{g(n)} |a_n^{(j)}| p_n^{-f(j,n)\sigma} \ll \sum_{n>N} p_n^{\alpha + \beta - \sigma}
\end{equation}

which tends to 0 as $N \to \infty$, the assertion of the theorem directly follows from Proposition 4.1 and (5.1).

In the case $\sigma_0 < \sigma \leq \alpha + \beta + 1$, naturally we have to discuss more carefully. Let $\delta > 0$, and define

\[ K^\delta_N(T; \varphi) = \left\{ t \in [-T,T] \mid \sigma + it \in G(\varphi), \quad |\log \varphi(\sigma + it) - \log \varphi_N(\sigma + it)| \geq \delta \right\} , \]
and $k_N^\delta(T; \varphi) = \mu_1(K_N^\delta(T; \varphi))$. We will prove that $k_N^\delta(T; \varphi)$ is negligible, that is, for any $\varepsilon > 0$ we can choose $N_0 = N_0(\delta, \varepsilon)$ for which

\begin{equation}
\limsup_{T \to \infty} T^{-1} k_N^\delta(T; \varphi) \leq \varepsilon
\end{equation}

holds for any $N \geq N_0$.

Let $\alpha_0 = \sigma - \varepsilon$, $\alpha_1 = \sigma - 2\varepsilon$. We choose $\varepsilon$ so small that $\sigma_0 < \alpha_1 < \alpha_0 < \sigma$. For any $t_0 \in [-T, T]$, put

$$H(t_0) = \{ s \mid \sigma > \alpha_0, t_0 - 1/2 < t < t_0 + 1/2 \},$$

and define $\psi_N^\delta(t_0; \varphi) = 0$ if $H(t_0) \subset G(\varphi)$ and $|R_N(s; \varphi)| < \delta$ for any $s \in H(t_0)$, and $\psi_N^\delta(t_0; \varphi) = 1$ otherwise. Then clearly

\begin{equation}
k_N^\delta(T; \varphi) \leq \int_{-T}^{T} \psi_N^\delta(t_0; \varphi) dt_0.
\end{equation}

Using (5.2) we can find $\beta_0 = \alpha + \beta + 1 + C\delta^{-1}$ (with an absolute positive constant $C$) for which $|R_N(s; \varphi)| < \delta$ holds for any $s$ satisfying $\sigma \geq \beta_0$. Let $Q(t_0) = H(t_0) \cap \{ s \mid \sigma < \beta_0 \}$.

**Lemma 5.1.** If $|f_N(s; \varphi)| < \delta/2$ for any $s \in Q(t_0)$, then $\psi_N^\delta(t_0; \varphi) = 0$.

This is a generalization of [12, Lemma 2], which further goes back to Bohr [1, Hilfssatz 5]. Bohr’s proof in [1] can be applied without change to the above general case, so we omit the proof.

Let $\beta_1 = 2\beta_0$, and let $P(t_0)$ be the rectangle given by $\alpha_1 \leq \sigma \leq \beta_1$, $t_0 - 1 \leq t \leq t_0 + 1$. Put

$$F_N(t_0; \varphi) = \iint_{P(t_0)} |f_N(s; \varphi)|^2 d\sigma dt.$$

(This can be defined only when $P(t_0)$ does not include a pole of $\varphi(s)$.) We use Lemma 5.1 and [12, Lemma 3] (which is [1, Hilfssatz 4]) to see that if $F_N(t_0; \varphi) < \pi(\varepsilon/2)^2(\delta/2)^2$

then $\psi_N^\delta(t_0; \varphi) = 0$. Therefore

\begin{equation}
\frac{1}{2T} \int_{-T}^{T} \psi_N^\delta(t_0; \varphi) dt_0 \leq b + \frac{\mu_1(S)}{2T},
\end{equation}

where $S$ is the set of all $t \in [-T, T]$ for which we can find a pole $s'$ of $\varphi(s)$ satisfying $|t - 3s'| \leq 2$, and

$$b = \frac{1}{2T} \mu_1 \left( \left\{ t_0 \in [-T, T] \mid F_N(t_0; \varphi) \geq \pi(\varepsilon/2)^2(\delta/2)^2 \right\} \right).$$

From the definition of $b$ we obtain

$$\pi(\varepsilon/2)^2(\delta/2)^2 b \leq \frac{1}{2T} \int_{t_0 \in [-T, T] \setminus S} F_N(t_0; \varphi) dt_0$$

$$= \frac{1}{2T} \int_{\alpha_1}^{\beta_1} \int_{-T}^{T+1} |f_N(s; \varphi)|^2 \int_{t_0}^{#} dt_0 dt d\sigma,$$
where the innermost integral (with the # symbol) is on \( t_0 \in [−T, T] \setminus \mathcal{S} \), \( t − 1 \leq t_0 \leq t + 1 \). This innermost integral is trivially \( \leq 2 \), and is equal to 0 if there exists a pole \( s' \) of \( \varphi(s) \) such that \( |t − \Im s'| \leq 1 \) (because then all \( t_0 \in [t − 1, t + 1] \) belongs to \( \mathcal{S} \)). Therefore

\[
π(ε/2)^2(δ/2)^2b ≤ \frac{1}{T}\int_{α_1}^{β_1} \int_{J(T+1)} |f_N(s; ϕ)|^2 \, dt \, dσ,
\]

where

\[
J(T) = \{ t \in [−T, T] \mid |t − \Im s'| > 1 \text{ for any pole } s' \text{ of } ϕ(s) \}.
\]

From (5.4), (5.5) and (5.6) we now obtain

\[
\frac{1}{2T}k_N^\delta(T; ϕ) ≤ \frac{1}{π(ε/2)^2(δ/2)^2T}\int_{α_1}^{β_1} \int_{J(T+1)} |f_N(s; ϕ)|^2 \, dt \, dσ + \frac{μ_1(S)}{2T}.
\]

On the double integral on the right-hand side, as an analogue of [12, Lemma 4], we can show the following lemma.

**Lemma 5.2.** For any \( η > 0 \), there exists \( N_0 = N_0(η) \), such that

\[
\frac{1}{T}\int_{α_1}^{β_1} \int_{J(T+1)} |f_N(s; ϕ)|^2 \, dt \, dσ < η
\]

for any \( N ≥ N_0 \) and any \( T ≥ T_0 \) with some \( T_0 = T_0(N) \).

**Proof.** Write the Dirichlet series expansion of \( ϕ(s) \) in the region \( σ > α + β + 1 \) as

\[
ϕ(s) = \sum_{k=1}^{∞} c_k k^{-s}.
\]

Then the Dirichlet series expansion of \( f_N(s) \) is

\[
f_N(s; ϕ) = \sum_k' c_k k^{-s},
\]

where the symbol \( \sum' \) means that the summation is restricted to \( k > 1 \) which is co-prime with \( p_1 p_2 \cdots p_N \). In [10, Appendix] it has been shown that, for any \( ε > 0 \), we can choose a sufficiently large \( N = N(ε) \) such that

\[
c_k = O(k^{α + β + ε})
\]

for all \( k \) co-prime with \( p_1 p_2 \cdots p_N \).

By (3.3) and the convexity principle we have

\[
\int_{J(T)} |ϕ(σ + it)|^2 \, dt = O(T)
\]

for any \( σ ≥ σ_0 \). On the other hand, using (4.1) we have

\[
ϕ_N(σ + it)^{-1} ≤ \exp \left( C \sum_{n ≤ N} \sum_j |a_n^{(j)}| p_n^{−f(j,n)σ} \right) \leq \exp \left( C' N^{α + β + 1 − σ} \right)
\]
Combining this estimate with (5.10) we obtain
\[
\frac{1}{T} \int_{J(T)} \left| f_N(\sigma + it; \phi) \right|^2 dt \ll \exp \left(2C' N^{\alpha + \beta + 1 - \sigma} \right),
\]
which is \( O(1) \) with respect to \( T \). Therefore by Carlson’s mean value theorem (see [20, Section 9.51])
\[
\lim_{T \to \infty} \frac{1}{T} \int_{J(T)} \left| f_N(\sigma + it; \phi) \right|^2 dt = \sum_{k} c_k^{2} k^{-2\sigma},
\]
uniformly in \( \sigma \). Using (5.9), we can estimate the right-hand side of (5.11) as
\[
\ll \sum_{k \geq pN+1} k^{2(\alpha + \beta + \varepsilon - \sigma)} \ll N^{1+2(\alpha + \beta + \varepsilon - \sigma)},
\]
whose exponent is negative for \( \sigma > \sigma_0 \) (if \( \varepsilon \) is sufficiently small). This immediately implies the assertion of the lemma.

Now, applying Lemma 5.2 with \( \eta = \pi \delta^2 \varepsilon^3 / 16 \) to (5.7), we arrive at (5.3).

The assertion of the theorem in the case \( \sigma_0 < \sigma \leq \alpha + \beta + 1 \) then follows by the same argument as in the last part of [12, Section 4].

6. The density function

In this section \( \sigma \) is any real number larger than \( \sigma_0 \). We discuss when it is possible to show that \( W_\sigma(\cdot; L_f) \) is absolutely continuous. Then by measure theory we can write
\[
W_\sigma(R; \phi) = \int_{R} M_\sigma(z, \phi) |dz|
\]
with the Radon-Nikodým density function \( M_\sigma(z; \phi) \).

For this purpose, we aim to show
\[
\Lambda_N(w) = O(|w|^{-(2+\eta)}) \quad (|w| \to \infty)
\]
uniformly in \( N \), with some \( \eta > 0 \).

If (6.2) is valid, then
\[
\int_{C} |\Lambda_N(w)||dw| < \infty.
\]
Therefore \( W_{N,\sigma} \) is absolutely continuous, and the Radon-Nikodým density function \( M_{N,\sigma}(z; \phi) \) is given by
\[
M_{N,\sigma}(z; \phi) = \int_{C} e^{-i(z,w)} \Lambda_N(w) |dw|
\]
and is continuous (see [8, p.53], [5, p.105]). Moreover, the above uniformity in \( N \) implies that the same estimate as (6.2) is valid for the limit function
\( \Lambda(w) \). Therefore \( W_\sigma \) is also absolutely continuous, hence (6.1) is valid with the continuous density function given by

\[
(6.4) \quad \mathcal{M}_\sigma(z; \varphi) = \int_C e^{-i(z,w)} \Lambda(w)|dw|.
\]

The following proposition reduces the problem to the evaluation of \( K_n(w) \):

**Proposition 6.1.** If there are at least five \( n \)'s, say \( n_1, \ldots, n_5 \), for which \( K_n(w) = O_n(|w|^{-1/2}) \) holds as \( |w| \to \infty \), then (6.2) is valid for any \( N \geq \max\{n_1, \ldots, n_5\} \), and so (6.1) and (6.3) are also valid.

**Remark 6.2.** The proof of (6.2) in the above proposition is simple: just apply \( K_n(w) = O_n(|w|^{-1/2}) \) (for \( n_1, \ldots, n_5 \)) and the trivial estimate \( |K_n(w)| \leq 1 \) to the product formula (4.5). The result is (6.2) with \( \eta = 1/2 \), uniform in \( N \).

**Remark 6.3.** The existence of the density function is useful for quantitative studies. For instance, if there are at least ten \( n \)'s with \( K_n(w) = O(|w|^{-1/2}) \), then \( \Lambda_N(w) = O(|w|^{-5}) \) for large \( N \). This fact with (4.6), (4.10) leads the estimate

\[
(6.5) \quad |W_\sigma(R; L_f) - W_{N,\sigma}(R; L_f)| = O(\mu_2(R)N^{1+2(\alpha+\beta-\sigma)(\log N)^{2(\alpha+\beta-\sigma)}})
\]

for \( \sigma > \sigma_0 \), as an analogue of [12] (6.4)].

In [12], when \( \varphi = \zeta_K \) (the Dedekind zeta-function of a Galois number field \( K \)), the key estimate (6.2) was proved by using [8] Theorem 13]. In this case, \( \zeta_K \) has the Euler product of the form (3.2) with \( f(1, n) = \cdots = f(g(n), n) \) (= \( f(n) \), say, the inertia degree) and \( a_n^{(f)} = 1 \) (and hence \( r_n^{(1)} = \cdots = r_n^{(g(n))} = p_n^{-f(n)\sigma} \) (= \( r_n \), say)). Therefore

\[
z_n(\theta_n) = -g(n) \log(1 - r_n e^{2\pi i f(n)\theta_n}),
\]

which describes a curve when \( \theta_n \) moves from 0 to 1. This curve is convex, so the original Jessen-Wintner inequality ([8] Theorem 13]) can be directly applied. In this case we encounter only one type of curve, that is, the curve

\[
- \log(1 - \xi) \quad (\xi \in \mathbb{C}, \ |\xi| = r_n).
\]

When \( K \) is non-Galois, \( f(1, n), \ldots, f(g(n), n) \) are not necessarily the same as each other, so

\[
z_n(\theta_n) = -\sum_{j=1}^{g(n)} \log(1 - r_n^{(j)} e^{2\pi i f(j,n)\theta_n}).
\]

However, still in this case, the number of relevant types of curves

\[
- \sum_{j=1}^{g(n)} \log(1 - \xi^{f(j,n)}) \quad (\xi \in \mathbb{C}, \ |\xi| = p_n^{-\sigma})
\]

is finite, because there are only finitely many patterns of the decomposition of prime numbers into prime ideals in \( K \). Because of this finiteness, we can use [15] Lemma 2] (which is a simple generalization of [8] Theorem 13])
to show (6.2) in this case. The case of Hecke $L$-functions of ideal class characters can be treated in a similar way.

However in the automorphic case, we encounter infinitely many types of curves, because in this case $z_n(\theta_n)$ describes a curve
\begin{equation}
(6.6) \quad -\log(1-\alpha_f(p_n)\xi) - \log(1-\beta_f(p_n)\xi) \quad (\xi \in \mathbb{C}, |\xi| = p_n^{-\sigma}),
\end{equation}
which depends on $\alpha_f(p_n), \beta_f(p_n)$. Therefore we have to prove a new type of Jessen-Wintner inequality, suitable for the automorphic case. This will be done in the next section.

7. AN ANALOGUE OF THE JESSEN-WINTNER INEQUALITY FOR AUTOMORPHIC $L$-FUNCTIONS

Now we restrict ourselves to the case of automorphic $L$-functions. Except for the (finitely many) prime factors of $N$, the Euler factor of $L_f(s)$ is of the form
\begin{equation}
(1-\alpha_f(p_n)p_n^{-s})^{-1}(1-\beta_f(p_n)p_n^{-s})^{-1},
\end{equation}
so $z_n(\theta_n) = A_n(p_n^{-\sigma}e^{2\pi i \theta_n})$ with
\begin{equation}
A_n(X) = -\log(1-\alpha_f(p_n)X) - \log(1-\beta_f(p_n)X).
\end{equation}
When $\theta_n$ moves from 0 to 1, the points $z_n(\theta_n)$ describes a curve (6.6) on the complex plane, which we denote by $\Gamma_n$.

Let $x_n(\theta_n) = \Re z_n(\theta_n)$ and $y_n(\theta_n) = \Im z_n(\theta_n)$. Writing $w = |w|e^{i\tau}$ ($\tau \in [0, 2\pi]$) we have $w = |w|\cos \tau + i|w|\sin \tau$. Then
\begin{equation}
(7.1) \quad \langle z_n(\theta_n), w \rangle = |w|g_{\tau,n}(\theta_n),
\end{equation}
where
\begin{equation}
g_{\tau,n}(\theta_n) = x_n(\theta_n)\cos \tau + y_n(\theta_n)\sin \tau.
\end{equation}
Therefore
\begin{equation}
(7.2) \quad K_n(w) = \int_0^1 e^{i|w|g_{\tau,n}(\theta_n)}d\theta_n.
\end{equation}

**Lemma 7.1.** Let $n \in \mathbb{N}$ such that $p_n \nmid N$. For any fixed $\tau$, the function $g_{\tau,n}(\theta_n)$ (as a function in $\theta_n$) is a $C^\infty$-class function. Moreover, if $p_n \in \mathcal{P}_f(\varepsilon)$ and $n$ is sufficiently large, then $g''_{\tau,n}(\theta_n)$ has exactly two zeros on the interval $[0, 1]$.

**Proof.** Hereafter, for brevity, we write $p_n = p, p_n^{-\sigma} = q, 2\pi \theta_n = \theta, z_n(\theta_n) = z(\theta), g_{\tau,n}(\theta_n) = g_{\tau}(\theta), x_n(\theta_n) = x(\theta)$, and $y_n(\theta_n) = y(\theta)$. Since the Taylor expansion of $A_n(x)$ is given by
\begin{equation}
A_n(x) = \sum_{j=1}^{\infty} a_j x^j \quad \text{with} \quad a_j = \frac{1}{j}(\alpha_f(p)^j + \beta_f(p)^j),
\end{equation}
we have
\begin{equation}
z(\theta) = \sum_{j=1}^{\infty} a_j q^j e^{ij\theta}.
\end{equation}
Therefore, putting $b_j = \Re a_j$ and $c_j = \Im a_j$, we have
\[
x(\theta) = \sum_{j=1}^{\infty} q^j u_j(\theta), \quad y(\theta) = \sum_{j=1}^{\infty} q^j v_j(\theta),
\]
where
\[
u_j(\theta) = b_j \cos(j\theta) - c_j \sin(j\theta), \quad v_j(\theta) = b_j \sin(j\theta) + c_j \cos(j\theta).
\]
Differentiate these series termwisely with respect to $\theta$; for example
\[
x'(\theta) = -\sum_{j=1}^{\infty} jq^j v_j(\theta), \quad y'(\theta) = \sum_{j=1}^{\infty} jq^j u_j(\theta)
\]
and so on. From (2.4) we have $|a_j| \leq 2/j$, so
\[(7.3) \quad |b_j| \leq 2/j, \quad |c_j| \leq 2/j.
\]
Noting these estimates and $q < 1$, we see that these differentiated series are convergent absolutely. Therefore $x(\theta), y(\theta)$ are belonging to the $C^\infty$-class, and so is $g_r(\theta)$. In particular the above termwise differentiation is valid, and we have
\[(7.4) \quad g'_r(\theta) = -\sum_{j=1}^{\infty} jq^j v_j(\theta) \cos \tau + \sum_{j=1}^{\infty} jq^j u_j(\theta) \sin \tau
\]
\[= -qv_1(\theta) \cos \tau + qu_1(\theta) \sin \tau + E_1(q; \theta, \tau),
\]
where $E_1(q; \theta, \tau)$ denotes the sum corresponding to $j \geq 2$, and
\[(7.5) \quad |E_1(q; \theta, \tau)| \leq 2 \sum_{j \geq 2} jq^j (|b_j| + |c_j|) \leq 2 \sum_{j \geq 2} jq^j \left( \frac{2}{j} + \frac{2}{j} \right)
\]
\[= 8 \sum_{j \geq 2} q^j = \frac{8q^2}{1 - q}.
\]
Since $q = p_n^{-\sigma} \leq 2^{-1/2} = 1/\sqrt{2}$, we find that $E_1(q; \theta, \tau) = O(q^2)$ as $q \to 0$ (that is, $n \to \infty$), where the implied constant is absolute. Therefore from (7.4) we have
\[
g'_r(\theta) = -qb_1 \sin(\theta - \tau) - qc_1 \cos(\theta - \tau) + O(q^2).
\]
Write $\gamma_1 = \arg a_1$. Then $b_1 = |a_1| \cos \gamma_1$, $c_1 = |a_1| \sin \gamma_1$, and so
\[(7.6) \quad g'_r(\theta) = -q|a_1|((\cos \gamma_1 \sin(\theta - \tau) + \sin \gamma_1 \cos(\theta - \tau)) + O(q^2)
\]
\[= -q|a_1| \sin(\gamma_1 + \theta - \tau) + O(q).
\]
Similarly, one more differentiation gives
\[(7.7) \quad g''_r(\theta) = -\sum_{j=1}^{\infty} j^2 q^j u_j(\theta) \cos \tau - \sum_{j=1}^{\infty} j^2 q^j v_j(\theta) \sin \tau
\]
\[= -q|a_1| \cos(\gamma_1 + \theta - \tau) + E_2(q; \theta, \tau),
\]
where $E_2(q; \theta, \tau)$, the sum corresponding to $j \geq 2$, satisfies

$$
|E_2(q; \theta, \tau)| \leq 2 \sum_{j \geq 2} j^2 q^j (|b_j| + |c_j|) \leq 2 \sum_{j \geq 2} j^2 q^j \left( \frac{2}{j} + \frac{2}{j} \right)
$$

$$
= 8 \sum_{j \geq 2} j q^j = \frac{8q^2(2 - q)}{(1 - q)^2}.
$$

(The proof of the last equality: Put $J = \sum_{j \geq 2} jq^j$, and observe that $J = \sum_{j \geq 1} (j + 1)q^{j+1} = q \sum_{j \geq 1} jq^j + \sum_{j \geq 1} q^{j+1} = q^3 + qJ + q^2/(1 - q)$. Therefore $E_2(q; \theta, \tau) = O(q^2)$ with an absolute implied constant (by using again $q \leq 1/\sqrt{2}$), and hence

$$
g''_r(\theta) = -q(|a_1| \cos(\gamma_1 + \theta - \tau) + O(q)).
$$

Furthermore

$$
g'''_r(\theta) = q|a_1| \sin(\gamma_1 + \theta - \tau) + E_3(q; \theta, \tau)
$$

with

$$
|E_3(q; \theta, \tau)| \leq 2 \sum_{j \geq 2} j^3 q^j (|b_j| + |c_j|) \leq 8 \sum_{j \geq 2} j^2 q^j
$$

$$
= 8q^2 \left( \frac{3}{1 - q} + \frac{1}{1 - q^2} + \frac{2q(2 - q)}{(1 - q)^2} \right) = O(q^2)
$$

with an absolute implied constant. (The evaluation of $\sum_{j \geq 2} j^2 q^j$ can be done similarly to the last equality of (7.8).)

Now we assume that $p_n \in \mathbb{P}_f(\varepsilon)$, where $\varepsilon$ is a small positive number. Recall $a_1 = \alpha_f(p) + \beta_f(p) = \lambda_f(p)$. Therefore from (2.6) we have $|a_1| > \sqrt{2} - \varepsilon$. On the other hand, the term $O(q)$ can be arbitrarily small when $n$ is sufficiently large. Therefore from (7.9) we find that, for sufficiently large $n$, if $\theta = \theta_0$ is a solution of $g''_r(\theta) = 0$, then $\cos(\gamma_1 + \theta_0 - \tau)$ is to be close to 0. That is, writing $\theta = \theta_1, \theta_2$ be two solutions of $\cos(\gamma_1 + \theta - \tau) = 0$ in the interval $0 \leq \theta < 2\pi$, we see that $\theta_0$ is close to $\theta_1^c$ or $\theta_2^c$.

Now consider $g'''_r(\theta)$. From (7.10) and (7.11) we have

$$
g'''_r(\theta) = q(|a_1| \sin(\gamma_1 + \theta - \tau) + O(q)).
$$

Since

$$
|\sin(\gamma_1 + \theta_1^c - \tau)| = |\sin(\gamma_1 + \theta_2^c - \tau)| = 1,
$$

we see that $g'''_r(\theta) \neq 0$ around $\theta = \theta_j^c$ ($j = 1, 2$), if $p_n \in \mathbb{P}_f(\varepsilon)$ and $n$ is sufficiently large. This implies that $g''_r(\theta)$ is monotone around $\theta = \theta_j^c$. Therefore we conclude that there is at most one solution $\theta = \theta_0$ of $g''_r(\theta) = 0$ around $\theta_j^c$.

Moreover, from (7.9) we see that $g''_r(\theta)$ is negative around the value of $\theta$ satisfying $\cos(\gamma_1 + \theta - \tau) = 1$, and is positive around the value of $\theta$ satisfying $\cos(\gamma_1 + \theta - \tau) = -1$. Therefore $g'''_r(\theta)$ changes its sign twice in the interval $0 \leq \theta < 1$, so that the above solution $\theta_0$ indeed exists both around $\theta_1^c$ and
around $\theta_2$. We denote those solutions by $\theta_1^\prime$ and $\theta_2^\prime$, respectively. That is, $g_\tau'(\theta) = 0$ has exactly two solutions in the interval $0 \leq \theta < 2\pi$.

**Remark 7.2.** By the same reasoning as above, we can show that, if $p_n \in \mathbb{P}_f(\varepsilon)$ and $n$ is sufficiently large, $g_\tau'(\theta) = 0$ also has exactly two solutions $\theta_1^\epsilon$ and $\theta_2^\epsilon$ in the interval $0 \leq \theta < 2\pi$. In fact, there exists two solutions $\theta = \theta_1^\epsilon, \theta_2^\epsilon$ of $\sin(\gamma_1 + \theta - \tau) = 0$ in the interval $0 \leq \theta < 2\pi$, and $\theta_j^\epsilon$ is close to $\theta_j (j = 1, 2)$. (We can further show that, for any $l \in \mathbb{N}$, there exist exactly two solutions of $g_\tau^{(l)}(\theta) = 0$.)

Now we can prove an analogue of the Jessen-Wintner inequality for automorphic $L$-functions. In the rest of this section, we follow the argument in the proof of [8, Theorem 12]. We use the notation defined in the proof of Lemma 7.1 and in Remark 7.2. The integral (7.2) can be rewritten as

\begin{equation}
K_n(w) = \frac{1}{2\pi} \int_0^{2\pi} e^{i|w|g_\tau'(\theta)} d\theta.
\end{equation}

**Proposition 7.3.** If $p_n \in \mathbb{P}_f(\varepsilon)$ and $n$ is sufficiently large, we have

$$K_n(w) = O\left(\frac{1}{q^{1/2}|w|^{1/2}} + \frac{1}{q|w|}\right),$$

with the implied constant depending only on $\varepsilon$.

**Proof.** When $\theta$ moves between $\theta_i^\epsilon$ and $\theta_j^\epsilon$ (1 \leq i, j \leq 2) (mod 2\pi), the values of $\sin(\gamma_1 + \theta - \tau)$ and $\cos(\gamma_1 + \theta - \tau)$ varies continuously and monotonically, and there exists a unique value $\theta = \theta_{ij}$ between $\theta_i^\epsilon$ and $\theta_j^\epsilon$ at which

$$|\sin(\gamma_1 + \theta_{ij} - \tau)| = |\cos(\gamma_1 + \theta_{ij} - \tau)| = 1/\sqrt{2}$$

holds.

We split the interval $0 \leq \theta < 1$ (mod 2\pi) into four subintervals at the values $\theta_{ij}$ (1 \leq i, j \leq 2). Then on two of those subintervals (which we denote by $I_A$ and $I_B$) the inequality $|\sin(\gamma_1 + \theta - \tau)| \geq 1/\sqrt{2}$ holds, while on the other two subintervals (which we denote by $I_C$ and $I_D$) the inequality $|\cos(\gamma_1 + \theta - \tau)| \geq 1/\sqrt{2}$ holds.

Since $p_n \in \mathbb{P}_f(\varepsilon)$ and $n$ is sufficiently large, we can again use the facts $|a_1| > \sqrt{2} - \varepsilon$ and the term $O(q)$ is small. Therefore from (7.6) we find

\begin{equation}
|g_\tau'(\theta)| \geq q((\sqrt{2} - \varepsilon)(1/\sqrt{2}) - \varepsilon) \geq q(1 - 2\varepsilon)
\end{equation}

for $\theta \in I_A \cup I_B$. Similarly from (7.9) we find that, for sufficiently large $n$,

\begin{equation}
|g_\tau''(\theta)| \geq q(1 - 2\varepsilon)
\end{equation}

for $\theta \in I_C \cup I_D$.

The number $\theta_2$ is included in $I_A$ or $I_B$, say $I_A$. Then $\theta_2^\epsilon \in I_B$. Therefore also $\theta_1^\epsilon \in I_A$ and $\theta_2^\epsilon \in I_B$. We split $I_A$ into two subintervals at $\theta = \theta_1^\epsilon$. Then in the both of those subintervals, $g_\tau'(\theta)$ is monotone. Therefore, applying
the first derivative test (Titchmarsh \[21\] Lemma 4.2) with \(\ell_i, j\) to those subintervals we have
\[
\left| \int_{I_A} e^{i |w|} g_r(\theta) d\theta \right| \leq 2 \cdot \frac{4}{\min\{|w|g_r'(\theta)|\}} \leq \frac{8}{q|w|(1-2\varepsilon)},
\]
and the same inequality holds for the integral on \(I_B\).

As for the integrals on the intervals \(I_C\) and \(I_D\), we use the second derivative test (\[21\] Lemma 4.4). The monotonicity is not required for the second derivative test, so we need not divide \(I_C\) into subintervals. Using (7.14), we have
\[
\left| \int_{I_C} e^{i |w|} g_r(\theta) d\theta \right| \leq \frac{8}{\sqrt{|w|(1-2\varepsilon)}},
\]
and the same for \(I_D\). Collecting these inequalities, we obtain the assertion of the proposition. \(\square\)

Proposition 7.3 implies that
(7.15) \[K_n(w) = O_n,\varepsilon(|w|^{-1/2}) \quad (|w| \to \infty)\]
if \(p_n \in \mathbb{P}_f(\varepsilon)\) and \(n\) is sufficiently large. The set \(\mathbb{P}_f(\varepsilon)\) is of positive density, especially it includes infinitely many elements (so surely includes five elements). Therefore we can obviously apply Proposition 6.1 to \(\varphi(s) = L_f(s)\), and the proof of Theorem 2.1 is now complete.

8. The Convexity

In our proof of Theorem 2.1, the convexity of relevant curves plays no role. However the geometric property of the curve \(\Gamma_n\) is of independent interest. We conclude this paper with the following

**Proposition 8.1.** If \(p_n \in \mathbb{P}_f(\varepsilon)\) for a small positive number \(\varepsilon\) and \(n\) is sufficiently large, the curve \(\Gamma_n\) is a closed convex curve.

**Remark 8.2.** Using \[8\] Theorem 13 we have that each curve \(\Gamma_n\) is convex if |\(\xi\)| is sufficiently small. But their theorem does not give any explicit bound of \(|\xi\|\) (which may depend on \(n\)), so we cannot deduce the above proposition from their theorem.

**Proof of Proposition 8.1.** Assume \(p_n \in \mathbb{P}_f(\varepsilon)\) and \(n\) is large. Then
\[
u_1(\theta)^2 + \nu_1(\theta)^2 = \nu_1^2 = |\alpha_f(p) + \beta_f(p)|^2 > (\sqrt{2} - \varepsilon)^2
\]
by (2.6). Therefore at least one of \(|\nu_1(\theta)|^2\) and \(|\nu_1(\theta)|^2\) is larger than \((\sqrt{2} - \varepsilon)^2 / 2\), that is, at least one of \(|\nu_1(\theta)|\) and \(|\nu_1(\theta)|\) is larger than \((\sqrt{2} - \varepsilon) / \sqrt{2} > 1 - \varepsilon\). Let
\[
\Theta(u_1,n) = \{\theta \in [0,2\pi) \mid |u_1(\theta)| > 1 - \varepsilon\},
\]
\[
\Theta(v_1,n) = \{\theta \in [0,2\pi) \mid |v_1(\theta)| > 1 - \varepsilon\}.
\]
Then \(\Theta(u_1,n) \cup \Theta(v_1,n) = [0,2\pi)\).

First consider the case when \(\theta \in \Theta(v_1,n)\). The curve \(\Gamma_n\) consists of the points \(z(\theta) = x(\theta) + iy(\theta)\). We identify \(\mathbb{C}\) with the \(\mathbb{R}^2\)-space \(\{(x,y) \mid x, y \in \mathbb{R}\}\). ...
\( \mathbb{R} \), and identify \( z(\theta) \) with \( (x(\theta), y(\theta)) \). We study the behavior of the tangent line of the planar curve \( \Gamma_n \) at \( z(\theta) \), when \( \theta \) varies. By \( \Xi(\theta) \) we denote the tangent of the angle of inclination of the tangent line at \( z(\theta) \). Then

\[
\Xi(\theta) = \frac{y'(\theta)}{x'(\theta)} = -\left( \sum_{j=1}^{\infty}jq_j^j u_j(\theta) \right) / \left( \sum_{j=1}^{\infty}jq_j^j v_j(\theta) \right).
\]

It is to be noted that the denominator is \( qv_1(\theta) + O(q^2) \), so this is non-zero for sufficiently small \( q \) (that is, sufficiently large \( n \)), because now we assume \( \theta \in \Theta(v_1, n) \).

We evaluate \( \Xi'(\theta) \). First, by differentiation we have

\[
\Xi'(\theta) = X_1(\theta) + X_2(\theta) + X_3(\theta) + X_4(\theta),
\]

say, where

\[
X_1(\theta) = qv_1(\theta) / \left( \sum_{j=1}^{\infty}jq_j^j v_j(\theta) \right),
\]

\[
X_2(\theta) = \left( \sum_{j=2}^{\infty}j^2q_j^j v_j(\theta) \right) / \left( \sum_{j=1}^{\infty}jq_j^j v_j(\theta) \right),
\]

\[
X_3(\theta) = (qu_1(\theta))^2 / \left( \sum_{j=1}^{\infty}jq_j^j v_j(\theta) \right)^2,
\]

and

\[
X_4(\theta) = \left( \sum_{j,k \in \mathbb{N}} jk^2q_j^j u_j(\theta)u_k(\theta) \right) / \left( \sum_{j=1}^{\infty}jq_j^j v_j(\theta) \right)^2.
\]

We write

\[
\sum_{j=1}^{\infty}jq_j^j v_j(\theta) = qv_1(\theta)(1 + Y(\theta)),
\]

where

\[
Y(\theta) = \sum_{j=2}^{\infty}jq_j^{j-1}v_j(\theta)/v_1(\theta).
\]

Since \( |v_1(\theta)| > 1 - \varepsilon \), using (7.3) we have

\[
|Y(\theta)| \leq \frac{4}{1 - \varepsilon} \sum_{j=2}^{\infty} q^{j-1} = \frac{4q}{(1 - \varepsilon)(1 - q)} = O(q).
\]
(noting \( q \) is small). Therefore

\[
\left( \sum_{j=1}^{\infty} j q^j v_j(\theta) \right)^{-1} = \frac{1}{qv_1(\theta)} \left( 1 - \frac{Y(\theta)}{1 + Y(\theta)} \right) \\
= \frac{1}{qv_1(\theta)} + O \left( \frac{1}{q(1 - \epsilon)} \frac{|Y(\theta)|}{1 - |Y(\theta)|} \right) = \frac{1}{qv_1(\theta)} + O(1).
\]

This implies

\[
X_1(\theta) = 1 + O(q). \tag{8.5}
\]

The numerator of \( X_2(\theta) \) can be evaluated, as in (7.8), by \( O(q^2) \). Therefore

\[
X_2(\theta) = O(q^2 \cdot q^{-1}) = O(q). \tag{8.6}
\]

As for \( X_3(\theta) \), again using \(|v_1(\theta)| > 1 - \epsilon \) and (8.4) we obtain

\[
X_3(\theta) = \frac{u_1(\theta)^2}{v_1(\theta)^2} \left( 1 - \frac{Y(\theta)}{1 + Y(\theta)} \right)^2 = \frac{u_1(\theta)^2}{v_1(\theta)^2} + O(q). \tag{8.7}
\]

Lastly, we have

\[
X_4(\theta) \ll \sum_{j,k \in \mathbb{N}} kq^{j+k} \cdot q^{-2} \ll q, \tag{8.8}
\]

because

\[
\sum_{j,k \in \mathbb{N}, j+k \geq 3} kq^{j+k} = \sum_{j \geq 1} q^j \sum_{k \geq \max\{1, j\}} kq^k = q \sum_{k \geq 2} kq^k + \sum_{j \geq 2} q^j \sum_{k \geq 1} kq^k
\]

\[
= qJ + (q + J) \sum_{j \geq 2} q^j = O(q^3)
\]

(where \( J \) was defined just after (7.8)). Collecting (8.2), (8.5), (8.6), (8.7) and (8.8), we obtain

\[
\Xi'(\theta) = 1 + \frac{u_1(\theta)^2}{v_1(\theta)^2} + O(q). \tag{8.9}
\]

Note that all the implied constants in the above formulas are absolute. When \( n \) is large, \( O(q) \) becomes small, so (8.9) implies that \( \Xi'(\theta) > 0 \). That is, if \( p_n \in \mathbb{P}_f(\epsilon) \), \( n \) is sufficiently large, and \( \theta \in \Theta(v_1, n) \), then \( \Xi(\theta) \) is monotonically increasing.

In the case when \( \theta \in \Theta(u_1, n) \), we change the roles of the axes. That is, now we identify \( z(\theta) \in \mathbb{C} \) with \((-y(\theta), x(\theta)) \in \mathbb{R}^2 \). Instead of \( \Xi(\theta) \), we consider \( \Xi^*(\theta) = x'(\theta)/y'(\theta) \). (The denominator \( y'(\theta) \) is non-zero for large \( n \) because \( \theta \in \Theta(u_1, n) \).) Then \(-\Xi^*(\theta)\) is the tangent of the angle of
inclination of the tangent line, under this new choice of the axes. We can proceed similarly, and obtain, analogously to (8.9),

\[
(-\Xi^*(\theta))' = 1 + \frac{v_1(\theta)^2}{u_1(\theta)^2} + O(q),
\]

hence \(-\Xi^*(\theta)\) is monotonically increasing when \(\theta \in \Theta(u_1, n)\). Therefore the tangent of the angle of inclination is always increasing, which implies that the curve \(\Gamma_n\) is convex.

\[\square\]

References

[1] H. Bohr, Zur Theorie der Riemann'schen Zetafunktion im kritischen Streifen, Acta Math. 40 (1915), 67-100.
[2] H. Bohr and R. Courant, Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunktion, J. Reine Angew. Math. 144 (1914), 249-274.
[3] H. Bohr and B. Jessen, Om Sandsynlighedsfordelinger ved Addition af konvekse Kurver, Dan. Vid. Selsk. Skr. Nat. Math. Afd. (8)12 (1929), 1-82.
[4] H. Bohr and B. Jessen, Über die Werteerteilung der Riemannschen Zetafunktion, I, Acta Math. 54 (1930), 1-35; II, ibid. 58 (1932), 1-55.
[5] V. Borchsenius and B. Jessen, Mean motions and values of the Riemann zeta function, Acta Math. 80 (1948), 97-166.
[6] C. R. Guo, The distribution of the logarithmic derivative of the Riemann zeta function, Proc. London Math. Soc. (3)72 (1996), 1-27.
[7] Y. Ihara and K. Matsumoto, On certain mean values and the value-distribution of logarithms of Dirichlet \(L\)-functions, Quart. J. Math. (Oxford) 62 (2011), 637-677.
[8] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. 38 (1935), 48-88.
[9] D. Joyner, Distribution Theorems of \(L\)-functions, Longman Sci.&Tech., 1986.
[10] R. Kačinskaïte and K. Matsumoto, Remarks on the mixed joint universality for a class of zeta functions, Bull. Austral. Math. Soc. 95 (2017), 187-198.
[11] K. Matsumoto, Value-distribution of zeta-functions, in "Analytic Number Theory", Proc. Japanese-French Sympos. held in Tokyo, K. Nagasaka and E. Fouvy (eds.), Lecture Notes in Math. 1434, Springer-Verlag, 1990, pp.178-187.
[12] K. Matsumoto, Asymptotic probability measures of zeta-functions of algebraic number fields, J. Number Theory 40 (1992), 187-210.
[13] K. Matsumoto, Asymptotic probability measures of Euler products, in "Proceedings of the Amalfi Conference on Analytic Number Theory", E. Bombieri et al. (eds.), Univ. Salerno, 1992, pp.295-313.
[14] K. Matsumoto, On the speed of convergence to limit distributions for Hecke \(L\)-functions associated with ideal class characters, Analysis 26 (2006), 313-321.
[15] K. Matsumoto, On the speed of convergence to limit distributions for Dedekind zeta-functions of non-Galois number fields, in "Probability and Number Theory — Kanazawa 2005", S. Akiyama et al. (eds.), Adv. Stud. Pure Math. 49, Math. Soc. Japan, 2007, pp.199-218.
[16] T. Miyake, Modular Forms, Springer, 1989.
[17] M. Ram Murty, Oscillations of Fourier coefficients of modular forms, Math. Ann. 262 (1983), 431-446.
[18] M. Ram Murty and V. Kumar Murty, Non-Vanishing of \(L\)-Functions and Applications, Progr. in Math. 157, Birkhäuser, 1997.
[19] H. S. A. Potter, The mean values of certain Dirichlet series I, Proc. London Math. Soc. 46 (1940), 467-478.
[20] E. C. Titchmarsh, The Theory of Functions, 2nd ed., Oxford, 1939.
[21] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, Oxford, 1951.

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