Quantum state diffusion, measurement and second quantization

by

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Abstract

Realistic dynamical theories of measurement based on the diffusion of quantum states are nonunitary, whereas quantum field theory and its generalizations are unitary. This problem in the quantum field theory of quantum state diffusion (QSD) appears already in the Lagrangian formulation of QSD as a classical equation of motion, where Liouville’s theorem does not apply to the usual field theory formulation. This problem is resolved here by doubling the number of freedoms used to represent a quantum field. The space of quantum fields is then a classical configuration space, for which volume need not be conserved, instead of the usual phase space, to which Liouville’s theorem applies. The creation operator for the quantized field satisfies the QSD equations, but the annihilation operator does not satisfy the conjugate equation. It appears only in a formal role.
1. Introduction

Quantum measurement is a physical process by which the state of a quantum system influences the value of a classical variable. The meaning of quantum measurement here includes any such process, including laboratory measurements, but also other, very different, processes, such as the cosmic rays that produced small but detectable dislocations in mineral crystals during the Jurassic era, and the quantum fluctuations in the early universe that are believed to have caused today’s anisotropies in the universal background radiation and in galactic clusters [9].

Since Bohr and Einstein it has been recognized that it is difficult to represent quantum measurement as a dynamical process [10]. Quantum theories that attempt it have problems with unitarity. These include those theories which depend on quantum state diffusion (QSD), the principal subject of this letter [9, 2, 3, 4, 7].

According to Bohr, the result of a measurement is influenced by the conditions of the measurer. Dynamical theories of quantum measurement seek a dynamical process for this influence. Quantum measurement dynamics does not follow from the unitary dynamics of Schrödinger or Heisenberg, nor from the quantum dynamics of fields, strings or branes. In a unified physics, they must be reconciled.

The methods of quantum field theory have been used for measurement dynamics before, but this letter deals with the apparent contradiction between the principles upon which they are based. Quantum field theory is unitary, whereas quantum state diffusion is not. Further, it has long been known that quantum measurement is nonunitarity [6, 1].

Here we trace the problem to the classical dynamics of a de Broglie wave, considered as a classical field, in particular to the violation of Liouville’s theorem by the measurement process. This makes it necessary to reformulate the classical dynamics of the field differently, making quantum measurement dynamics a nonlinear field theory of a special type.

The experimental consequences of this theory are the same as the standard results of quantum state diffusion when applied to the dynamics of measurement, which is indistinguishable from the results of the usual interpretation of nonrelativistic quantum theory for past and current experiments, though
not necessarily for all future experiments [9]. But the classical field theory
of quantum state diffusion has some unusual features, in particular that the
de Broglie wave is defined by a point in a configuration space, not in a phase
space. There are canonical conjugate momenta, but their role is probably
formal, rather than physical, and they are not the same as the complex con-
jugate amplitudes. The reason for this violation of one of the basic principles
of field quantization is given in the next section.

This letter is confined to the nonrelativistic formulation of the dynamics of
the single-particle de Broglie waves of QSD, first as a classical field, then as
a quantized field.

2 Quantum state diffusion

QSD represents measurement dynamics as a continuous stochastic process, in
which the state vector is the solution of an Itô stochastic differential equation
[9]. This is expressed in terms of a complex differential stochastic fluctuation
$\xi$ with equal and independent fluctuations in its real and imaginary parts,
so that

$$
M \frac{d\xi}{dt} = 0, \quad M (d\xi)^2 = 0, \quad M |d\xi|^2 = dt.
$$

where $M$ represents the mean over an ensemble.

Suppose a system with state $|\psi\rangle$ has Hamiltonian $H$, and the dynamical
variable $G$ with Hermitean operator $G$ is being measured. According to
QSD, measurement is a very rapid diffusion in state space, whose rate is
given by a real factor $c$ Then the quantum state diffusion equation is

$$
\frac{d|\psi(t)\rangle}{dt} = -(i/\hbar)H|\psi(t)\rangle - \frac{i}{2}c^2 G_\Delta^2 |\psi(t)\rangle + cG_\Delta |\psi(t)\rangle \frac{d\xi}{dt},
$$

where $G_\Delta = G - \langle \psi(t)|G|\psi(t)\rangle$ is the shifted $\psi$-dependent $G$-operator whose
expectation for the current state $|\psi(t)\rangle$ is zero. For simplicity, we will absorb
the constant $c$ into $G$. The equation is nonlinear, but the norm of $\psi(t)$ is
preserved. The stochastic coefficient $d\xi/dt$ is a highly singular function of
time, whose singular properties are handled by the Itô calculus, but they need
not concern us here. What is important is that it is a stochastic function of
time representing complex Gaussian white noise.

For laboratory experiments, the diffusion is so fast that the state appears to
jump between states on a time scale far shorter than the other time scales
of the system, in particular those of the Hamiltonian. However, according
to quantum state diffusion theory, this is a limiting case. For small isolated
quantum systems the diffusion is so slow that it has not been detected. This
is the other limiting case.

The details of quantum state diffusion as a theory of quantum measurement
are given in [9].

3 Classical measurement dynamics of the scalar field

For simplicity, first consider a free particle with no measurement, in a one-
dimensional box with a bounded energy, so that there is a finite number $N$
of discrete states. In energy representation, with energies $E_j = \hbar \omega_j$, the
corresponding complex amplitudes

$$\psi_j = \frac{1}{\sqrt{2}}(q_j + ip_j)$$

satisfy

$$i\dot{\psi}_j = \omega_j \psi_j,$$

so that

$$\dot{q}_j = \{q_j, H\} = \omega_j p_j, \quad \dot{p}_j = \{p_j, H\} = -\omega_j q_j, \quad \text{with} \quad H = \sum_j \omega_j \left(p_j^2 + q_j^2\right),$$

which are are Hamilton’s equations for $N$ oscillators with real canonically
conjugate configuration and momentum coordinates $q_j, p_j$. Schrödinger evolu-
tion of the wave produces a unitary transformation in the state space, a
generalized rotation on the unit sphere, identical to the motion of the phase
point in the phase space of the oscillators. The unit sphere is an energy shell
of a phase space, Liouville’s theorem is satisfied, so the phase space density
for a continuous distribution of systems is conserved. Second quantization of
the field amplitudes follows just as first quantization for the oscillators.

The same applies formally for the complex configuration coordinate and its
canonical conjugate momentum

$$\psi_j = \frac{1}{\sqrt{2}}(q_j + ip_j) \quad \text{and} \quad i\psi_j^* = \frac{1}{\sqrt{2}}(iq_j + p_j).$$

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In this representation, the equations of motion for the configuration and momentum coordinates are independent:

\[ \dot{\psi}_j = -i\omega_j \psi_j, \quad \dot{\psi}_j^* = i\omega_j \psi_j^* \quad \text{with} \quad H = \sum_j \omega_j \psi_j^* \psi_j = \sum_j (-i\omega_j)(i\psi_j^*)\psi_j. \]

The dynamics of quantum measurement is very different. The norm of the state is preserved, so it is confined to the unit sphere in state space, but the density of states on the surface of the unit sphere is not preserved. Measurement of the energy, for example, puts the system into one of the energy eigenstates, so a continuous distribution over the unit sphere of an ensemble of systems is reduced towards a set of at most \( N \) points. A uniform distribution over the unit sphere in this finite-dimensional state space evolves towards a set of equal \( \delta \)-distributions at each of the energy eigenstates. The total volume of the surface of the sphere is reduced towards zero. Liouville’s theorem is violated with a vengeance, so the space of quantum states of the particle cannot be the phase space of any classical Hamiltonian system.

However, the state space can be treated as a configuration space. There is no conservation of volume in this configuration space, so a Lagrangian or Hamiltonian measurement dynamics is possible. The equations of motion for the complex configuration coordinates of the oscillators are independent of the equations for the conjugate momenta, as they are when there is no measurement, but the conjugate momenta are no longer the same as the complex conjugates of the configuration coordinates.

4 Lagrangian theory of free-field QSD

The configuration space trajectory for a time-independent dynamical system with two configuration coordinates \( q, q' \) is stationary for the action integral

\[ S = \int_{t_0}^{t_1} dt L(q, q'). \]

Equivalent Lagrangians have action integrals that give the same equations of motion.

Before treating the QSD equations, consider a simpler classical model. The equations of motion for a dynamical system with equivalent Lagrangians

\[ L = -q' \dot{q} + q' f(q), \quad L' = q q' + q' f(q) \]

(9)
are
\[ \dot{q} = f(q), \quad \dot{q}' = -q' \frac{\partial f(q)}{\partial q} \]  
(10)
and the momenta conjugate to \( q, q' \) are
\[ p = \frac{\partial L}{\partial \dot{q}} = -q', \quad p' = \frac{\partial L}{\partial \dot{q}'} = q. \]  
(11)
We can identify \(-q', q^*\) and \(p\), write them both as \(p\), and similarly we can write \(p'\) as \(q\). The primed coordinates are then no longer needed, as in the usual canonical theory of quantum fields. But a momentum then appears in the Lagrangian, as it does in quantum field theory, but which is not normally allowed in classical dynamics. The quantum theory of a complex amplitude of a free linear field has just this form with \( f(q) = i\omega q \), where \( q \) is complex, and \( q' = q^* \), its complex conjugate, which is treated as an independent canonical coordinate, giving
\[ L = -q' \dot{q} + i\omega q q', \quad L' = q \dot{q}' + i\omega q q'. \]  
(12)
We can now identify \(-q', q^*\) and \(p\) in the resultant Lagrange equations. This is consistent for this case because the equation of motion for \( q^* \) is just the complex conjugate of the equation of motion for \( q \). Identifying \(-q', q^*\) and \(p\) is a only a formal problem for the theory of quantum fields and is very convenient in practice.

However if \( f(q) \neq cq \), the equation of motion for \(-q' = q^*\) is not the complex conjugate of the equation for \( q \), so even if the identification is made at some initial time, it will no longer hold for later times. This is what happens for the Lagrangian theory of the wave \( \psi \) in QSD. For QSD we start with four independent configuration coordinates \( q, q', q^*, q'^* \). The starred coordinates \( q^*, q'^* \) are complex conjugates of the coordinates \( q, q' \), but the primed coordinates are not conjugate momenta. In QSD for a Schrödinger field, the configuration coordinates corresponding to \( q, q^* \) are \( \psi_j, \psi^*_j \).

The Lagrangian formulation of QSD for the measurement of a dynamical variable \( G \) of a particle follows from this approach. We derive the equations for the fields themselves, not the field components. It is convenient to express the total action, which is a function of the configuration coordinates \( \psi, \psi', \psi^*, \psi'^* \), as twice the real part of a complex Lagrangian \( L_c \), which depends on all the configuration coordinates except the last. Consequently \( L_c^* \)
is independent of $\psi'$ and makes no contribution to Lagrange's equation for $\psi$.

The action is

$$S = \int dt (L_c + L^*_c),$$

where

$$-iL_c = - \int d^3x \psi' \dot{\psi} - i \int d^3x \psi' H \psi + \int d^3x \psi' Q \psi$$

and

$$Q = Q(\psi^*, \psi) = -\frac{i}{2} \left( G - \int d^3x \psi^* G \psi \right)^2 + \left( G - \int d^3x \psi^* G \psi \right) d\xi/dt.$$ (15)

The Lagrangian of equation (14) is the form suitable for varying with respect to $\psi'$ and $\psi^*$. For the variation with respect to $\psi$ and $\psi^*$, we vary the equivalent Lagrangian obtained by partial integration.

The variation of $S$ with respect to $\psi'$ is straightforward, the $L^*_c$ term does not contribute, and $L_c$ gives the QSD equation for $\psi$, and the derivative of $L_c$ with respect to $\dot{\psi}$ gives the definition of the momentum $p_\psi$. Together they make Hamilton’s equations for $\psi$:

$$\dot{\psi} = -iH\psi + Q\psi, \quad \text{(QSD)}, \quad p_\psi = -i\psi'$$ (16)

The corresponding operations with $\psi^*$ and $\dot{\psi}^*$ give Hamilton’s equations for $\psi^*$ and $\psi^*$ and as the action is real, the equations are just the complex conjugates of those for $\psi$ and $\psi'$.

$$\dot{\psi}^* = iH\psi^* + Q^* \psi^*, \quad \text{(QSD)}, \quad p_{\psi^*} = i\psi'^*.$$ (17)

Since these are complex conjugate equations, $\psi^*$ remains the complex conjugate wave for all time.

The equation for $\dot{\psi}'$ is given by the variation with respect to $\psi$. It is not nearly so simple as the QSD equation, as it involves both the $L^*_c$ and $L_c$ terms. The additional terms ensure that even if initially $\psi' = \psi^*$, it does not remain so for later times, as it does in the absence of measurement. In this way the (probably nonphysical) momentum space of $\psi', \psi'^*$ can carry away the phase space volume that is lost by the motion of the state in the configuration space of $\psi, \psi^*$. 

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5 Second quantization

Before going into the analysis, we note an essential difference between this theory and the usual theory of second quantization. In both, the most important operator is the field creation operator $\psi^\ast$.

In the usual classical field theory, (we ignore the factors $i$), the complex conjugate of the field amplitude $\psi^\ast$ is the canonical conjugate of the amplitude $\psi$, and both are important for quantization. The operator $\psi$ is the field annihilation operator, whose properties come from its commutation relations, and these in turn are derived from the commutation relations for the conjugate momentum. $\psi^\ast$ is also the Hermitian conjugate operator of $\psi$. Both these roles are held by the same operator, which greatly simplifies the theory.

For the quantization of the QSD equations, these roles are separated. The annihilation operator corresponding to the creation operator $\psi^\ast$ is the conjugate momentum operator $\psi^\ast'$, which is not the same as the Hermitian conjugate operator $\psi$. For convenience we denote the annihilation operator by

$$\psi^{\ast'} = \psi^0.$$  \hfill (18)

The quantum state of a Schrödinger field with $n$ particles is given by operating $n$ times on the vacuum with the creation operator $\psi^\ast(t)$. The Heisenberg equations of motion for $\psi^\ast(t)$ are the same as Hamilton’s equations (17) above. But because of the decoupling between the coordinate and momentum equations, only the first of these equations is important. As far as we know, the second Heisenberg equation has no physical significance. The annihilation operator $\psi^0(t)$ has an important formal role in deriving Heisenberg equations, just as in ordinary quantum field theory, but since only the creation operators are needed to obtain a field from the vacuum, the annihilation operators appear to have no other physical significance than this.

Because of the decoupling of the coordinate and momentum equations of motion in the Heisenberg equations, the relatively complicated Heisenberg equation for $\psi^0$ is not needed to get the physical field. However, without the momenta $\psi'$ and $\psi^0$, there would be no Liouville theorem for the classical formulation of QSD, and consequently no unitarity for the quantized field. The price of unitarity is additional fields that are not physical, as far as we know.
6 Discussion

There are two possible approaches to the quantum field theory of QSD. In the first, which is treated here, the QSD equations for the de Broglie wave are derived from a classical Lagrangian, as a prelude to the second quantization. In the second, a quantum state diffusion term is added to the second quantized equations of motion for the particle.

The physical difference between these two approaches, is that in the first approach, the Liouville’s equation is satisfied in the extended phase space of the de Broglie wave, so the quantized theory can be unitary. This is consistent with the unitarity of standard quantum field theory, but the price is the introduction of conjugate momenta that appear to play no physical role. In the second approach, any diffusion terms will destroy the unitarity of the field theory, which makes it very difficult or impossible to reconcile with the modern theory of fields, strings and branes.

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