Global boundedness and decay property of a three-dimensional Keller–Segel–Stokes system modeling coral fertilization

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Abstract

This paper is concerned with the four-component Keller–Segel–Stokes system modeling the fertilization process of corals:

\begin{align*}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \nabla \cdot (\rho S(x, \rho, c) \nabla c) - \rho m, & (x, t) \in \Omega \times (0, T), \\
m_t + u \cdot \nabla m &= \Delta m - \rho m, & (x, t) \in \Omega \times (0, T), \\
c_t + u \cdot \nabla c &= \Delta c - c + m, & (x, t) \in \Omega \times (0, T), \\
u_t = \Delta u - \nabla P + (\rho + m) \nabla \phi, & \nabla \cdot u = 0, & (x, t) \in \Omega \times (0, T)
\end{align*}

subject to the boundary conditions \( \nabla c \cdot \nu = \nabla m \cdot \nu = (\nabla \rho - \rho S(x, \rho, c) \nabla c) \cdot \nu = 0 \) and \( u = 0 \), and suitably regular initial data \((\rho_0(x), m_0(x), c_0(x), u_0(x))\), where \( T \in (0, \infty] \), \( \Omega \subset \mathbb{R}^3 \) is a bounded domain with smooth boundary \( \partial \Omega \).

This system describes the spatio-temporal dynamics of the population densities of sperm \( \rho \) and egg \( m \) under a chemotactic process facilitated by a chemical signal released by the egg with concentration \( c \) in a fluid-flow environment \( u \) modeled by the incompressible Stokes equation. In this model, the chemotactic sensitivity tensor \( S \in C^2(\Omega \times [0, \infty[))^3 \times 3 \) satisfies \( |S(x, \rho, c)| \leq C_S(1 + \rho)^{-\alpha} \) with some \( C_S > 0 \) and \( \alpha \geq 0 \). We will show that for \( \alpha \geq \frac{1}{3} \), the solutions to the system are globally bounded and decay to a spatially homogeneous equilibrium exponentially as time goes to infinity. In addition, we will also...
show that, for any $\alpha \geq 0$, a similar result is valid when the initial data satisfy a certain smallness condition.

Keywords: Keller–Segel–Stokes, tensor-value sensitivity, global boundedness, decay property

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1. Introduction

Chemotaxis, the directed movement caused by the concentration of certain chemicals, is ubiquitous in biology and ecology, and has a significant effect on pattern formation in numerous biological contexts [14, 24]. The first mathematically rigorous studies of chemotaxis were carried out by Patlak [26] and Keller–Segel [18]. The latter work involves the derivation of a system of PDEs, now known as the Keller–Segel system, which, despite its simple structure, was proved to have a lasting impact as a theoretical framework describing the collective behavior of populations under the influence of a chemotactic signal produced by the populations themselves [2, 13, 33, 34]. In contrast to this well-understood Keller–Segel system, there seem to be few theoretical results on nontrivial behavior in situations where the signal is not produced by the population, such as in oxygenotaxis processes of swimming aerobic bacteria [29], or where the signal production occurs by indirect processes, such as in glycolysis reaction, tumor invasion and the spread of the mountain pine beetle [5, 7, 11, 16, 25, 27].

In this paper, we study a chemotaxis-fluid system modelling coral fertilization. Specifically, we are concerned with a Keller–Segel–Stokes system

\[
\begin{aligned}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \nabla \cdot (\rho S(x, \rho, c) \nabla c) - \rho m, \\
m_t + u \cdot \nabla m &= \Delta m - \rho m, \\
c_t + u \cdot \nabla c &= \Delta c - c + m, \\
\nabla \cdot u &= 0, \\
\nabla \cdot (\rho S(x, \rho, c) \nabla c) \cdot \nu &= \nabla m \cdot \nu = \nabla c \cdot \nu = 0, u = 0, \\
\rho(x, 0) &= \rho_0(x), m(x, 0) = m_0(x), c(x, 0) = c_0(x), u(x, 0) = u_0(x),
\end{aligned}
\]

where $T \in (0, \infty)$, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial \Omega$, the chemotactic sensitivity tensor $S(x, \rho, c) = (s_{ij}(x, \rho, c)) \in C^2(\Omega \times [0, \infty)^2)$, $i, j \in \{1, 2, 3\}$, and $\phi \in W^{2,\infty}(\Omega)$.

This PDE system describes the phenomenon of coral broadcast spawning [9, 10, 19, 20], where the sperm $\rho$ chemotactically moves toward the higher concentration of the chemical $c$ released by the egg $m$, while the egg $m$ is merely affected by random diffusion, fluid transport and degradation upon contact with the sperm. Meanwhile, the fluid flow vector $u$, modeling the ambient ocean environment, satisfies a Stokes equation, where $P = P(x, t)$ represents the associated pressure, and the buoyancy effect of the sperm and egg on the velocity, mediated through a given gravitational potential $\phi$, is taken into account. We note that the use of the Stokes equation instead of the Navier–Stokes equation is justified by the observation that the fluid flow is relatively slow compared with the movement of the sperm and egg. We further note that the sensitivity tensor $S(x, \rho, c)$ may take values that are matrices possibly containing nontrivial off-diagonal entries, which reflects that the chemotactic migration may not necessarily be oriented along the gradient of the chemical signal, but may rather involve rotational flux components (see [39, 40] for the detailed model derivation).
A two-component variant of (1.1) has been used in the mathematical study of coral broadcast spawning. Indeed, in [19, 20], Kiselev and Ryzhik investigated the important effect of chemotaxis on the coral fertilization process via the Keller–Segel type system of the form

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \chi \nabla \cdot (\rho \nabla c) - \mu \rho^\alpha, \\
ct + u \cdot \nabla c &= \Delta c + \rho \\
0 &= \Delta u + \nabla P + \rho \nabla \phi,
\end{align*}
\]

(1.2)

with a given regular solenoidal fluid flow vector \( u \). This model implicitly assumes that the densities of sperm and egg gametes are identical, and that the Péclet number for the chemical concentration \( c \) is small which allows us to ignore the effects of convection on \( c \). The authors showed that, for the Cauchy problem in \( \mathbb{R}^2 \), the total mass \( \int_{\mathbb{R}^2} \rho(x,t)dx \) can become arbitrarily small with increasing \( \chi \) in the case \( q > 2 \) of supercritical reaction, whereas in the critical case \( q = 2 \), a weaker but related effect within finite time intervals is observed. Recently, Ahn et al [1] established the global well-posedness of regular solutions for the variant model of (1.2) with \( c_t + u \cdot \nabla c = \Delta c - c + \rho \) instead of \( 0 = \Delta c + \rho \). They also proved that \( \int_{\mathbb{R}^d} \rho(x,t)dx \) (\( d = 2, 3 \)) asymptotically approaches a strictly positive constant \( C(\chi) \) which tends to 0 as \( \chi \to \infty \).

In [8], Espejo and Suzuki studied the three-component variant of (1.1)

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \chi \nabla \cdot (\rho \mathcal{S}(x,\rho,c)\nabla c) - \mu \rho^\alpha, \\
c_t + u \cdot \nabla c &= \Delta c - c + \rho, \\
u_t + \kappa(u \cdot \nabla)u &= \Delta u - \nabla P + \rho \nabla \phi, \\
\nabla \cdot u &= 0.
\end{align*}
\]

(1.3)

in the modeling of broadcast spawning when the interaction of chemotactic movement of the gametes and the surrounding fluid is not negligible. Here the coefficient \( \kappa \in \mathbb{R} \) is related to the strength of nonlinear convection. In particular, when the fluid flow is slow, we can use the Stokes instead of the Navier–Stokes equation, i.e. assume \( \kappa = 0 \) (see [6, 23]). It should be mentioned that the chemotaxis-fluid model with \( c_t + u \cdot \nabla c = \Delta c - c + \rho \) replacing the second equation in (1.3) has also been used to describe the behavior of bacteria of the species Bacillus subtilis suspended in sessile water drops [29]. From the viewpoint of mathematical analysis, this chemotaxis-fluid system compounds the known difficulties in the study of fluid dynamics with the typical intricacies in the study of chemotaxis systems. It has also been observed that when \( \mathcal{S} = \mathcal{S}(x,\rho,c) \) is a tensor, the corresponding chemotaxis-fluid system loses some energy-like structure, which plays a key role in the analysis of the scalar-valued case. Despite these challenges, some comprehensive results on the global-boundedness and large time behavior of solutions are available in the literature (see [3, 21, 22, 28, 30, 32, 36–38] for example). It has been shown that when \( \mathcal{S} = \mathcal{S}(x,\rho,c) \) is a tensor fulfilling

\[
|\mathcal{S}(x,\rho,c)| \leq \frac{C_\mathcal{S}}{(1 + \rho)^\alpha} \quad \text{for some } \alpha > 0 \text{ and } C_\mathcal{S} > 0,
\]

(1.4)

the three-dimensional system (1.3) with \( \mu = 0, \kappa = 0 \) admits globally bounded weak solutions for \( \alpha > 1/2 \) [30], which is slightly stronger than the corresponding subcritical assumption \( \alpha > 1/3 \) for the fluid-free system. As for \( \alpha \geq 0 \), when the suitably regular initial data \((\rho_0, c_0, u_0)\) fulfill a smallness condition, (1.3) with \( \mu = 0, \kappa = 1 \) possesses a global classical solution which decays to \((\rho_0, 0, 0)\) exponentially with \( \rho_0 = \frac{1}{|\Omega|} \int_\Omega \rho_0(x)dx \) [41].

Removing the presupposition that the densities of the sperm and egg coincide at each point, Espejo and Suzuki [9] looked at a simplified version of (1.1) in two dimensions, namely,
\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \chi \nabla \cdot (\rho \nabla c) - \rho m, \\
m_t + u \cdot \nabla m &= \Delta m - \rho m, \\
0 &= \Delta c + k_0 (m - \frac{1}{m} \int_{\Omega} mdx) \quad \text{with} \quad \int_{\Omega} cdx = 0,
\end{align*}
\] (1.5)

and showed that \( \int_{\Omega} \rho_0(x)dx \geq \int_{\Omega} m_0(x)dx \) implies that \( m(x,t) \) vanishes asymptotically, while \( \int_{\Omega} \rho(x,t)dx \to \frac{1}{m} (\int_{\Omega} \rho_0(x)dx - \int_{\Omega} m_0(x)dx) \) as \( t \to \infty \), provided that \( \chi \) is small enough and \( u \) is low. In two dimensions, Espejo and Winkler [10] have recently considered the Navier–Stokes version of (1.1):

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho &= \Delta \rho - \nabla \cdot (\rho \nabla c) - \rho m, \\
m_t + u \cdot \nabla m &= \Delta m - \rho m, \\
c_t + u \cdot \nabla c &= \Delta c - c + m, \\
u_t + \kappa (u \cdot \nabla) = \Delta u - \nabla P + (\rho + m)\nabla \phi, \quad \nabla \cdot u = 0,
\end{align*}
\] (1.6)

and established the global existence of classical solutions to the associated initial-boundary value problem, which tend towards a spatially homogeneous equilibrium in the large time limit.

Motivated by the above works, we shall consider the properties of solutions to the system (1.1) in the three-dimensional setting. In particular, we shall show that the corresponding solutions converge to a spatially homogeneous equilibrium exponentially as \( t \to \infty \) as well.

Throughout the rest of the paper, we shall assume that

\[
\begin{align*}
\rho_0 &\in C^0(\bar{\Omega}), \quad \rho_0 \geq 0 \quad \text{and} \quad \rho_0 \neq 0, \\
m_0 &\in C^0(\bar{\Omega}), \quad m_0 \geq 0 \quad \text{and} \quad m_0 \neq 0, \\
c_0 &\in W^{1,\infty}(\Omega), \quad c_0 \geq 0 \quad \text{and} \quad c_0 \neq 0, \\
u_0 &\in D(A^\beta) \quad \text{for all} \quad \beta \in (\frac{1}{2}, 1),
\end{align*}
\] (1.7)

where \( A \) denotes the realization of the Stokes operator in \( L^2(\Omega) \). Under these assumptions, we shall first establish the existence of global bounded classical solutions to (1.1):

\textbf{Theorem 1.1.} Suppose that (1.4), (1.7) hold with \( \alpha > \frac{1}{4} \). Then the system (1.1) admits a global classical solution \( (\rho, m, c, u, P) \), which is uniformly bounded in the sense that for any \( \beta \in (\frac{1}{2}, 1) \), there exists \( K > 0 \) such that for all \( t \in (0, \infty) \)

\[
\| \rho(\cdot, t) \|_{L^\infty(\Omega)} + \| m(\cdot, t) \|_{L^\infty(\Omega)} + \| c(\cdot, t) \|_{W^{1,\infty}(\Omega)} + \| A^\beta u(\cdot, t) \|_{L^2(\Omega)} \leq K.
\] (1.8)

Then, we establish the large time behavior of these solutions as follows:

\textbf{Theorem 1.2.} Under the assumptions of theorem 1.1, the solutions given by theorem 1.1 satisfy

\( \rho(\cdot, t) \to \rho_\infty, \quad m(\cdot, t) \to m_\infty, \quad c(\cdot, t) \to c_\infty, \quad u(\cdot, t) \to 0 \) in \( L^\infty(\Omega) \) as \( t \to \infty \).

Furthermore, when \( \int_{\Omega} \rho_0 \neq \int_{\Omega} m_0 \), there exist \( K > 0 \) and \( \delta > 0 \) such that

\[
\begin{align*}
\| \rho(\cdot, t) - \rho_\infty \|_{L^2(\Omega)} &\leq Ke^{-\delta t}, \\
\| m(\cdot, t) - m_\infty \|_{L^\infty(\Omega)} &\leq Ke^{-\delta t}, \\
\| c(\cdot, t) - c_\infty \|_{L^\infty(\Omega)} &\leq Ke^{-\delta t},
\end{align*}
\] (1.9) (1.10) (1.11)
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq Ke^{-\delta t},
\]

where \( \rho_\infty = \frac{1}{|\Omega|} \left\{ \int_\Omega \rho_0 - \int_\Omega \bar{m}_0 \right\}_+ \), \( m_\infty = \frac{1}{|\Omega|} \left\{ \int_\Omega \bar{m}_0 - \int_\Omega \rho_0 \right\}_+ \).

According to the result for the related fluid-free system, the subcritical restriction \( \alpha > \frac{1}{4} \) seems to be necessary for the existence of global bounded solutions. However, for \( \alpha \leq \frac{1}{4} \), inspired by [3, 4,1], we investigate the existence of global bounded classical solutions and their large time behavior under a smallness assumption imposed on the initial data, which can be stated as follows:

**Theorem 1.3.** Suppose that \( (1.4) \) hold with \( \alpha = 0 \) and \( \int_\Omega \rho_0 \neq \int_\Omega \bar{m}_0 \). Further, let \( N = 3 \) and \( p_0 \in (\frac{4}{3}, \infty), q_0 \in (N, \infty) \) if \( \int_\Omega \rho_0 > \int_\Omega \bar{m}_0 \); and \( p_0 \in (\frac{2N}{N+2}, \infty), q_0 \in (N, \infty) \) if \( \int_\Omega \rho_0 < \int_\Omega \bar{m}_0 \). Then there exists \( \varepsilon > 0 \) such that for any initial data \( (\rho_0, m_0, c_0, u_0) \) fulfilling (1.7) as well as

\[
\|\rho_0 - \rho_\infty\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \|m_0 - m_\infty\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \|\nabla c_0\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \|u_0\|_{L^\infty(\Omega)} \leq \varepsilon.
\]

(1.1) possesses a global classical solution \( (\rho, m, c, u, P) \). Moreover, for any \( \alpha_1 \in (0, \min\{\lambda_1, m_\infty + \rho_\infty\}) \), \( \alpha_2 \in (0, \min\{\alpha_1, \lambda_1', 1\}) \), there exist constants \( K_i \), \( i = 1, 2, 3, 4 \), such that for all \( t \geq 1 \),

\[
\|m(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t}, \quad \|\rho(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t},
\]

\[
\|c(\cdot, t) - m_\infty\|_{H^{\alpha_2}(\Omega)} \leq K_3 e^{-\alpha_2 t}, \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t}.
\]

Here \( \lambda_1' \) is the first eigenvalue of \( \Lambda \), and \( \lambda_1 \) is the first nonzero eigenvalue of \( -\Delta \) on \( \Omega \) under the Neumann boundary condition.

**Remark 1.1.** In theorem 1.3, we have excluded the case \( \int_\Omega \rho_0 = \int_\Omega \bar{m}_0 \). Indeed, in this case, some results of Cao and Winkler [4] suggest that exponential decay of solutions may not hold.

**Remark 1.2.** It is observed that the similar result to theorem 1.3 is also valid for the Navier–Stokes counterpart of (1.1) upon slight modification of the definition of \( T \) in (3.53) and (3.87).

As mentioned above, compared with the scalar sensitivity \( S \), the system (1.1) with rotational tensor loses a favorable quasi-energy structure. For example, we note that the integral

\[
\int_\Omega \rho \ln \rho + a \int_\Omega |\nabla c|^2 + b \int_\Omega |u|^2
\]

with appropriate positive constants \( a \) and \( b \) plays a favorable entropy-like functional in deriving the bounds of solution to (1.6). However, this will no longer be available in the present situation (see [10]). To overcome this difficulty, our approach underlying the derivation of theorem 1.1 will be based on the estimate of the functional

\[
\|\rho(\cdot, t)\|_{L^2(\Omega)}^2 + \|u(\cdot, t)\|_{W^{1,2}(\Omega)}^2 + \|\nabla c(\cdot, t)\|_{L^2(\Omega)}^2.
\]

In addition, the proof of the exponential decay results in theorem 1.2 relies on careful analysis of the functional

\[
G(t) := \int_\Omega (\rho - \bar{\rho})^2 + a \int_\Omega (m - \bar{m})^2 + b \int_\Omega (c - \bar{c})^2 + c \int_\Omega \rho m.
\]
with suitable parameters \(a, b, c > 0\). Indeed, it can be seen that \(G(t)\) satisfies the ODE: 
\[G'(t) + \delta_1 G(t) \leq 0\] 
for some \(\delta_1 > 0\), and thereby the convergence rate of solutions in \(L^2(\Omega)\) is established. At the same time, in comparison with the chemotaxis-fluid system considered in \([3, 41]\), due to

\[
\|e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_1 \left(1 + t^{-\frac{q}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}\right) e^{-\lambda t t}\|\omega\|_{L^p(\Omega)}
\]

for all \(\omega \in L^p(\Omega)\) with \(\int_\Omega \omega = 0\). 

\(-\rho m\) in the first equation of \((1.1)\) gives rise to some difficulty in mathematical analysis despite its dissipative feature. Accordingly, it requires a non-trivial application of the mass conservation of \(\rho(x, t) - m(x, t)\).

The plan of this paper is as follows: in section 2, we give a local existence result and some useful estimates. In section 3, in the case of \(S\) vanishing on the boundary, we investigate the existence and large time behavior of global bounded classical solutions under the assumption of either \(\alpha > \frac{1}{2}\) or smallness of the initial data. In the last section, on the basis of certain \textit{a} \textit{priori} estimates, we give the proofs of our main results.

2. Preliminaries

In this section, we provide some preliminary results that will be used in the subsequent sections. We begin by recalling the important \(L^p - L^q\) estimates for the Neumann heat semigroup on bounded domains ((31)).

\textbf{Lemma 2.1 (Lemma 1.3 of [31]).} Let \((e^{t\Delta})_{t \geq 0}\) denote the Neumann heat semigroup in the domain \(\Omega\) and \(\lambda_1 > 0\) denote the first nonzero eigenvalue of \(-\Delta\) in \(\Omega \subset \mathbb{R}^N\) under the Neumann boundary condition. There exists \(C_n, i = 1, 2, 3, 4\), such that for all \(t > 0\),

(i) If \(1 \leq q \leq p \leq \infty\), then for all \(\omega \in L^q(\Omega)\) with \(\int_\Omega \omega = 0\),

\[
\|e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_1 \left(1 + t^{-\frac{q}{2} \left(\frac{1}{q} - \frac{1}{p}\right)}\right) e^{-\lambda t t}\|\omega\|_{L^q(\Omega)};
\]

(ii) If \(1 \leq q \leq p < \infty\), then for all \(\omega \in L^q(\Omega)\),

\[
\|\nabla e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_2 \left(1 + t^{-\frac{p}{2} \left(\frac{1}{p} - \frac{1}{q}\right)}\right) e^{-\lambda t t}\|\omega\|_{L^q(\Omega)};
\]

(iii) If \(2 \leq q \leq p < \infty\), then for all \(\omega \in W^{1,q}(\Omega)\),

\[
\|\nabla e^{t\Delta} \omega\|_{L^p(\Omega)} \leq C_3 \left(1 + t^{-\frac{p}{2} \left(\frac{1}{p} - \frac{1}{q}\right)}\right) e^{-\lambda t t}\|\nabla \omega\|_{L^q(\Omega)};
\]

(iv) If \(1 \leq q \leq p < \infty\) or \(1 < q < \infty\) and \(p = \infty\), then for all \(\omega \in (L^q(\Omega))^N\),

\[
\|e^{t\Delta} \nabla \omega\|_{L^p(\Omega)} \leq C_4 \left(1 + t^{-\frac{1}{2} \left(\frac{1}{p} - \frac{1}{q}\right)}\right) e^{-\lambda t} t\|\omega\|_{L^q(\Omega)}.
\]

Next we introduce the Stokes operator and recall estimates for the corresponding semigroup. With \(L^p_0(\Omega) := \{\varphi \in L^p(\Omega) | \nabla \cdot \varphi = 0\}\) and \(P\) representing the Helmholtz projection of \(L^p(\Omega)\) onto \(L^p_0(\Omega)\), the Stokes operator on \(L^p_0(\Omega)\) is defined as \(A_p = -\Delta\) with domain \(D(A_p) := W^{2,p}(\Omega) \cap W_0^{2,p}(\Omega) \cap L^p_0(\Omega)\). Since \(A_p\) and \(A_{p_1}\) coincide on the intersection of their domains for \(p_1, p_2 \in (1, \infty)\), we will drop the index in the following.

\textbf{Lemma 2.2 (Lemma 2.3 of [3]).} The Stokes operator \(A\) generates the analytic semigroup \((e^{-tA})_{t \geq 0}\) in \(L^p_0(\Omega)\). Its spectrum satisfies \(\lambda_1^\nu = \inf \Re \sigma(A) > 0\) and we fix \(\mu \in (0, \lambda_1^\nu)\). For any such \(\mu\), we have

2820
(i) For any \( p \in (1, \infty) \) and \( \gamma \geq 0 \), there is \( C_5(p, \gamma) > 0 \) such that for all \( \phi \in L_p^p(\Omega) \),
\[
\|A^\gamma e^{-tA}\phi\|_{L_p^p(\Omega)} \leq C_5(p, \gamma)\|r^{-\gamma} e^{-\mu t}\|_{L_p^p(\Omega)};
\]
(ii) For any \( p, q \) with \( 1 < p \leq q < \infty \), there is \( C_\delta(p, q) > 0 \) such that for all \( \phi \in L_p^q(\Omega) \),
\[
\|e^{-tA}\phi\|_{L_p^q(\Omega)} \leq C_\delta(p, q)\|r^{-\frac{2}{p} + \frac{p}{q}} e^{-\mu t}\|_{L_p^q(\Omega)};
\]
(iii) For any \( p, q \) with \( 1 < p \leq q < \infty \), there is \( C_\gamma(p, q) > 0 \) such that for all \( \phi \in L_p^q(\Omega) \),
\[
\|\nabla e^{-tA}\phi\|_{L_p^q(\Omega)} \leq C_\gamma(p, q)\|r^{-\frac{2}{p} + \frac{p}{q}} e^{-\mu t}\|_{L_p^q(\Omega)};
\]
(iv) If \( \gamma \geq 0 \) and \( 1 < p < q < \infty \) satisfy \( 2\gamma - \frac{N}{q} \geq 1 - \frac{N}{p} \), there is \( C_\delta(\gamma, p, q) > 0 \) such that for all \( \phi \in D(A_q^\delta) \),
\[
\|\phi\|_{W^{1, p}(\Omega)} \leq C_\delta(\gamma, p, q)\|A^\gamma \phi\|_{L_p^p(\Omega)}.
\]

**Lemma 2.3 (Theorems 1 and 2 of [12]).** The Helmholtz projection \( P \) defines a bounded linear operator \( P: L^p(\Omega) \rightarrow L_p^p(\Omega) \); in particular, for any \( p \in (1, \infty) \), there exists \( C_\delta(p) > 0 \) such that \( \|P\omega\|_{L_p^p(\Omega)} \leq C_\delta(p)\|\omega\|_{L_p^p(\Omega)} \) for every \( \omega \in L^p(\Omega) \).

The following elementary lemma provides some useful information on both the short-time and the large-time behavior of certain integrals, which is used in the proof of theorem 1.3.

**Lemma 2.4 (Lemma 1.2 of [31]).** Let \( \alpha < 1 \) and \( \gamma \), \( \delta \) be positive constants such that \( \gamma \neq \delta \). Then there exists \( C_{10}(\alpha, \beta, \gamma, \delta) > 0 \) such that
\[
\int_0^\tau (1 + s^{\alpha-1})(1 + (t - s)^{-\beta}) e^{-\gamma t} e^{-\delta(\tau - s)} ds \leq C_{10}(\alpha, \beta, \gamma, \delta) \left(1 + \min\{\alpha, \beta, 1\}\right) e^{-\min\{\gamma, \delta\}t}.
\]

Next we recall the result on the local existence of classical solutions, which can be proved by a straightforward adaptation of well-known fixed point argument (see [32] for example).

**Lemma 2.5.** Suppose that \( (1.4), (1.7) \) and \( S(x, \rho, c) = 0, (x, \rho, c) \in \partial \Omega \times [0, \infty) \times [0, \infty) \)

hold. Then there exist \( T_{\text{max}} \in (0, \infty) \) and a classical solution \( (\rho, m, c, u, P) \) of \( (1.1) \) on \( (0, T_{\text{max}}) \). Moreover, \( \rho, m, c, u \) are nonnegative in \( \Omega \times (0, T_{\text{max}}) \), and \( T_{\text{max}} < \infty \), then for \( \beta \in \left(\frac{3}{2}, 1\right) \),
\[
\lim_{\tau \to T_{\text{max}}} \left(\|\rho(\cdot, \cdot)\|_{L_p^\infty(\Omega)} + \|m(\cdot, \cdot)\|_{L_p^\infty(\Omega)} + \|c(\cdot, \cdot)\|_{W^{1, 1}(\Omega)} + \|A^\beta u(\cdot, \cdot)\|_{L_p^\infty(\Omega)}\right) = \infty.
\]

This solution is unique, up to addition of constants to \( P \).

The following elementary properties of the solutions in lemma 2.5 are immediate consequences of the integration of the first and second equations in \( (1.1) \), as well as an application of the maximum principle to the second and third equations.

**Lemma 2.6.** Suppose that \( (1.4), (1.7) \) and \( (2.1) \) hold. Then for all \( t \in (0, T_{\text{max}}) \), the solution of \( (1.1) \) from lemma 2.5 satisfies
\[
\|\rho(\cdot, t)\|_{L_p^\infty(\Omega)} \leq \|\rho_0\|_{L_p^\infty(\Omega)}, \quad \|m(\cdot, t)\|_{L_p^\infty(\Omega)} \leq \|m_0\|_{L_p^\infty(\Omega)},
\]
\[
\int_0^t \|\rho(\cdot, s)m(\cdot, s)\|_{L_p^\infty(\Omega)} ds \leq \min\{\|\rho_0\|_{L_p^\infty(\Omega)}, \|m_0\|_{L_p^\infty(\Omega)}\},
\]
\[ \|\rho(\cdot, t)\|_{L^2(\Omega)} - \|m(\cdot, t)\|_{L^2(\Omega)} = \|\rho_0\|_{L^2(\Omega)} - \|m_0\|_{L^2(\Omega)}, \]  
(2.4)

\[ \|m(\cdot, t)\|_{L^2(\Omega)} + 2 \int_0^t \|\nabla m(\cdot, s)\|_{L^2(\Omega)}^2 \, ds \leq \|m_0\|_{L^2(\Omega)}, \]  
(2.5)

\[ \|m(\cdot, t)\|_{L^\infty(\Omega)} \leq \|m_0\|_{L^\infty(\Omega)}, \]  
(2.6)

\[ \|c(\cdot, t)\|_{L^\infty(\Omega)} \leq \max\{\|m_0\|_{L^\infty(\Omega)}, \|c_0\|_{L^\infty(\Omega)}\}. \]  
(2.7)

3. Proof of theorems for \( S = 0 \) on \( \partial \Omega \)

In this section, we shall consider the case in which besides (1.4), the sensitivity satisfies \( S = 0 \) on \( \partial \Omega \). Under this hypothesis, the boundary condition for \( \rho \) in (1.1) actually reduces to the homogeneous Neumann condition \( \nabla \rho \cdot \nu = 0 \).

3.1. Global boundedness for \( S = 0 \) on \( \partial \Omega \)

**Lemma 3.1.** Suppose that (1.4), (1.7) and (2.1) hold with \( \alpha > \frac{1}{4} \). Then for any \( \varepsilon > 0 \), there exists \( K(\varepsilon) > 0 \) such that, for all \( t \in (0, T_{\text{max}}) \), the solution of (1.1) satisfies

\[ \frac{d}{dt}\|\rho(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla \rho(\cdot, t)\|_{L^2(\Omega)}^2 \leq \varepsilon \|\Delta c(\cdot, t)\|_{L^2(\Omega)}^2 + K(\varepsilon). \]  
(3.1)

**Proof.** Multiplying the first equation of (1.1) by \( \rho \), we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_\Omega \rho^2 + \int_\Omega |\nabla \rho|^2 = \int_\Omega \rho S(x, \rho, c) \nabla \rho \nabla c - \int_\Omega \rho^2 m \leq \frac{1}{2} \int_\Omega |\nabla \rho|^2 + C_\varepsilon^2 \int_\Omega \frac{\rho^2}{(1 + \rho)^{2\alpha}} |\nabla c|^2. \]  
(3.2)

Now we estimate the term \( \frac{C_\varepsilon^2}{2} \int_\Omega \frac{\rho^2}{(1 + \rho)^{2\alpha}} |\nabla c|^2 \) in the right hand side of (3.2). In fact, if \( \alpha \geq \frac{3}{4} \),

\[ \frac{C_\varepsilon^2}{2} \int_\Omega \frac{\rho^2}{(1 + \rho)^{2\alpha}} |\nabla c|^2 \leq \varepsilon \int_\Omega |\nabla c|^4 + K(\varepsilon), \]  
(3.3)

while for \( \alpha \in (\frac{1}{4}, \frac{3}{4}) \),

\[ \frac{C_\varepsilon^2}{2} \int_\Omega \frac{\rho^2}{(1 + \rho)^{2\alpha}} |\nabla c|^2 \leq \frac{C_\varepsilon^2}{2} \int_\Omega \rho^{2-2\alpha} |\nabla c|^2 \leq \frac{C_\varepsilon^4}{16\varepsilon} \int_\Omega \rho^{4-4\alpha} + \varepsilon \int_\Omega |\nabla c|^4. \]  
(3.4)

On the other hand, by lemma 2.6 and the Gagliardo–Nirenberg inequality, we get

\[ \int_\Omega |\nabla c|^4 \leq C_{GN} \left\{ \|\Delta c\|_{L^2(\Omega)}^2 \|c\|_{L^\infty(\Omega)}^2 + \|c\|_{L^\infty(\Omega)}^4 \right\} \leq C_{GN} (\|\Delta c\|_{L^2(\Omega)}^2 + 1) \]  
(3.5)
and
\[
\int_{\Omega} |\rho|^{4-4\alpha} = \|\rho\|_{L^{2+\alpha}(\partial \Omega)}^{4-4\alpha} \leq C_{\text{GN}} \left\{ \|\nabla \rho\|_{L^{2+\alpha}(\partial \Omega)}^{(4-4\alpha)\lambda_1} \|\rho\|_{L^{2+\alpha}(\partial \Omega)}^{(4-4\alpha)(1-\lambda_1)} + \|\rho\|_{L^{2+\alpha}(\partial \Omega)}^{4-4\alpha} \right\}
\]
with \(\lambda_2 = \frac{6(3-4\alpha)}{5(4-4\alpha)}\). Due to \(\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)\), we have \((4-4\alpha)\lambda_2 < 2\) and thus
\[
\frac{C_{\text{S}}}{16\varepsilon} \int_{\Omega} |\rho|^{4-4\alpha} \leq \frac{1}{4} \int_{\Omega} |\nabla \rho|^2 + K_1
\]
by the Young inequality. Combining (3.2)–(3.6), we readily have (3.1).

**Lemma 3.2.** Under the assumptions of lemma 3.1, there exists a positive constant \(C = C(m_0, c_0)\) such that for all \(t \in (0, T_{\text{max}})\), the solution of (1.1) satisfies
\[
\frac{d}{dt} \|\nabla c(t)\|^2_{L^2(\Omega)} + 2 \|\nabla c(t)\|^2_{L^2(\partial \Omega)} + \|\Delta c(t)\|^2_{L^2(\Omega)} \leq K(\|\nabla u(t)\|^2_{L^2(\Omega)} + 1).
\]

**Proof.** Multiplying the \(c\)-equation of (1.1) by \(-\Delta c\) and by the Young inequality, we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla c|^2 + \int_{\Omega} |\Delta c|^2 + \int_{\Omega} |\nabla c|^2 \leq -\int_{\Omega} m \Delta c - \int_{\Omega} (u \cdot \nabla c) \Delta c
\]
\[
= -\int_{\Omega} m \Delta c - \int_{\Omega} \nabla c \cdot (\nabla u \cdot \nabla c) - \int_{\Omega} \nabla c \cdot (D^2 c \cdot u)
\]
\[
= -\int_{\Omega} m \Delta c - \int_{\Omega} \nabla c \cdot (\nabla u \cdot \nabla c)
\]
\[
\leq \int_{\Omega} |m|^2 + \frac{1}{4} \int_{\Omega} |\Delta c|^2 + (\int_{\Omega} |\nabla c|^4)^{\frac{1}{2}} (\int_{\Omega} |\nabla u|^2)^{\frac{1}{2}}
\]
\[
\leq \|m\|_{L^2(\Omega)}^2 + \frac{1}{4} \|\Delta c\|^2_{L^2(\Omega)} + \frac{1}{2\varepsilon} \|\nabla u\|^2_{L^2(\partial \Omega)} + \frac{\varepsilon}{2} \|\nabla c\|^4_{L^2(\Omega)},
\]
where the fact that \(u\) is solenoidal and vanishes on \(\partial \Omega\) is used to ensure \(\int_{\Omega} \nabla c \cdot (D^2 c \cdot u) = 0\).

By (3.5) and taking \(\varepsilon = \frac{1}{C_{\text{GN}}^2}\) in the above inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla c|^2 + \frac{1}{2} \int_{\Omega} |\Delta c|^2 + \int_{\Omega} |\nabla c|^2 \leq \|m\|_{L^2(\Omega)}^2 + C_{\text{GN}} \|\nabla u\|^2_{L^2(\partial \Omega)} + \frac{1}{4},
\]
which along with (2.5) readily ensures the validity of (3.7).

**Lemma 3.3.** Under the assumptions of lemma 3.1, the solution of (1.1) satisfies
\[
\frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\partial \Omega)} \leq K \left( \|\rho(t)\|^2_{L^2(\Omega)} + 1 \right),
\]
\[
\frac{d}{dt} \|\nabla u(t)\|^2_{L^2(\Omega)} + \|Au(t)\|^2_{L^2(\Omega)} \leq K \left( \|\rho(t)\|^2_{L^2(\Omega)} + 1 \right)
\]
for all \(t \in (0, T_{\text{max}})\) for a positive constant \(K\).
Proof. Testing the $u$-equation in (1.1) by $u$, using the Hölder inequality and Poincaré inequality, we can get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 = \int_{\Omega} \rho \nabla \cdot u$$

$$\leq \|\nabla \phi\|_{L^\infty(\Omega)} \|\rho + m\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$$

$$\leq \frac{1}{2} \|\nabla u\|^2_{L^2(\Omega)} + K_1(\|\rho\|^2_{L^2(\Omega)} + \|m\|^2_{L^2(\Omega)}),$$

which together with (2.5) yields (3.9). Applying the Helmholtz projection $P$ to the fourth equation in (1.1), testing the resulting identity by $Au$ and using the Young inequality, we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |Au|^2 + \int_{\Omega} |\nabla u|^2 = \frac{1}{2} \int_{\Omega} |Au|^2 + K_2(\int_{\Omega} \rho^2 + \int_{\Omega} m^2),$$

which yields (3.10), due to (2.5) and the fact that $\int_{\Omega} |\nabla u|^2 = \int_{\Omega} \nabla^2 u^2$. \hfill $\square$

Lemma 3.4. Under the assumptions of lemma 3.1, one can find $C > 0$ such that for all $t \in (0, T_{\text{max}})$, the solution of (1.1) satisfies

$$\|\rho(\cdot, t)\|_{L^2(\Omega)}^2 + \|\nabla c(\cdot, t)\|_{L^2(\Omega)}^2 + \|u(\cdot, t)\|_{W^{2,2}(\Omega)}^2 \leq K.$$  

Proof. By the Gagliardo–Nirenberg inequality

$$\|\rho\|_{L^2(\Omega)} \leq C_{GN} \left( \|\nabla \rho\|_{L^2(\Omega)}^{\frac{1}{2}} \|\rho\|_{L^1(\Omega)}^{\frac{1}{2}} + \|\rho\|_{L^2(\Omega)} \right)$$

and (3.1), for any $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that

$$\frac{d}{dt} \|\rho\|^2_{L^2(\Omega)} + \|\nabla \rho\|^2_{L^2(\Omega)} + \frac{1}{4} \|\nabla \rho\|^2_{L^2(\Omega)} \leq \varepsilon \|\Delta c\|_{L^2(\Omega)}^2 + K(\varepsilon). \quad (3.11)$$

Adding (3.9) and (3.10), and by the Poincaré inequality, one can find constants $K_i > 0$, $i = 2, 3, 4$, such that

$$\frac{d}{dt} \|\nabla c\|_{L^2(\Omega)}^2 + 2 \|\nabla c\|_{L^2(\Omega)} + \|\Delta c\|_{L^2(\Omega)}^2 \leq K_3 \left( \|\rho\|_{L^2(\Omega)}^2 + 1 \right)$$

$$\leq \frac{1}{8} \|\nabla \rho\|^2_{L^2(\Omega)} + K_4. \quad (3.12)$$

Recalling (3.7), we get

$$\frac{d}{dt} \|\nabla c\|_{L^2(\Omega)}^2 + 2 \|\nabla c\|_{L^2(\Omega)} + \|\Delta c\|_{L^2(\Omega)}^2 \leq K_5 \left( \|\nabla u\|^2_{L^2(\Omega)} + 1 \right). \quad (3.13)$$

Now combining the above inequalities and choosing $\varepsilon = \frac{K_5}{2K_3}$, one can see that there exists some constant $K_6 > 0$ such that

$$Y(t) := \|\rho(\cdot, t)\|_{L^2(\Omega)}^2 + \|u(\cdot, t)\|_{W^{2,2}(\Omega)}^2 + \varepsilon \|\nabla c(\cdot, t)\|_{L^2(\Omega)}^2.$$
satisfies \( Y'(t) + \delta Y(t) \leq K_6 \), where \( \delta = \min \{1, \frac{\beta}{4}\} \). Hence by an ODE comparison argument, we obtain \( Y(t) \leq K_7 \) for some constant \( K_7 > 0 \) and thereby complete the proof.

With all of the above estimates at hand, we can now establish the global existence result in the case \( S = 0 \) on \( \partial \Omega \).

**Proof of theorem 1.1 in the case \( S = 0 \) on \( \partial \Omega \).** To establish the existence of globally bounded classical solution, by the extensibility criterion in lemma 2.5, we only need to show that

\[
\|\rho(t)\|_{L^\infty(\Omega)} + \|m(t)\|_{L^\infty(\Omega)} + \|c(t)\|_{L^1(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \leq K_1
\]

for all \( t \in (0, T_{\text{max}}) \) with some positive constant \( K_1 \) independent of \( T_{\text{max}} \). This inequality can be found from lemmas 3.4 and 2.6, respectively.

From lemmas 3.4 and 2.5, we only need to show \( \rho \) satisfies

\[
\rho \leq K_8 \text{ and } m \leq K_4 \text{ and } c \leq K_4
\]

in \((0, T_{\text{max}})\). By the extensibility criterion in lemma 2.5, we only need to show

\[
\sup_{t \in (0, T_{\text{max}})} \|\nabla c\|_{L^\infty(\Omega)} \leq K_6
\]

for some positive constant \( K_6 \) independent of \( T_{\text{max}} \). This inequality is due to the Young inequality.

By the maximum principle, lemmas 2.1 and 2.5, we get

\[
\|\rho\|_{L^\infty(\Omega)} \leq \|e^{\Delta} \rho_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{-(t-s)\Delta} \nabla \cdot (\rho \nabla c + \rho u)\|_{L^\infty(\Omega)} \, ds
\]

\[
\leq \|\rho_0\|_{L^\infty(\Omega)} + C_4 \int_0^t (1 + (t-s)^{-\frac{3}{2}}) e^{-\lambda_1(t-s)} \|\rho \nabla c + \rho u\|_{L^1(\Omega)} \, ds
\]

\[
\leq \|\rho_0\|_{L^\infty(\Omega)} + K_9 \int_0^t (1 + (t-s)^{-\frac{3}{2}}) e^{-\lambda_1(t-s)} \|\rho\|_{L^1(\Omega)} \, ds
\]

\[
\leq \|\rho_0\|_{L^\infty(\Omega)} + K_9 \int_0^t (1 + (t-s)^{-\frac{3}{2}}) e^{-\lambda_1(t-s)} \|\rho\|_{L^1(\Omega)}^{\frac{1}{2}} \, ds
\]

\[
\leq \|\rho_0\|_{L^\infty(\Omega)} + K_{10} \sup_{t \in (0, T_{\text{max}})} \|\rho\|_{L^1(\Omega)}^{\frac{1}{2}}
\]

with \( K_{10} = K_9 \sup_{t \in (0, T_{\text{max}})} \|\rho\|_{L^1(\Omega)}^{\frac{1}{2}} \int_0^t (1 + s^{-\frac{3}{2}}) e^{-\lambda_1 s} \, ds \), where we have used \( \nabla \cdot u = 0 \). Taking supremum on the left side of the above inequality over \((0, T_{\text{max}})\), we obtain

\[
\sup_{t \in (0, T_{\text{max}})} \|\rho\|_{L^\infty(\Omega)} \leq \|\rho_0\|_{L^\infty(\Omega)} + K_{10} \sup_{t \in (0, T_{\text{max}})} \|\rho\|_{L^1(\Omega)}^{\frac{1}{2}},
\]

and thereby \( \sup_{t \in (0, T_{\text{max}})} \|\rho\|_{L^\infty(\Omega)} \leq K_{11} \) by the Young inequality. Finally, by a straightforward argument (see [10, lemma 3.10] or [29, page 340]), one can find \( K_{12} > 0 \) such that
ρm − ρc 2 − ρm 2 are nonnegative functions satisfying ρ(1 + cρ) c ≤ K12. The boundedness estimate (3.14) is now a direct consequence of the above inequalities and this completes the proof.

3.2. Large time behavior for $S = 0$ on $\partial \Omega$

This section is devoted to showing the large time behavior of global solutions to (1.1) obtained in the above subsection. In order to derive the convergence properties of solution with respect to the norm in $L^2(\Omega)$, we shall make use of the following lemma. In the sequel, we denote $\mathcal{J} = \frac{1}{|\Omega|} \int_{\Omega} f(x) \, dx$.

Lemma 3.5 (Lemma 4.6 of [10]). Let $\lambda > 0$, $C > 0$, and suppose that $y \in C^1([0, \infty))$ and $h \in C^0([0, \infty))$ are nonnegative functions satisfying $y'(t) + \lambda y(t) \leq h(t)$ for some $\lambda > 0$ and all $t > 0$. Then if $\int_0^\infty h(s) \, ds \leq C$, we have $y(t) \to 0$ as $t \to \infty$.

By means of the testing procedure and the Young inequality, we have

$$\frac{d}{dt} \int_{\Omega} (\rho - \bar{\rho})^2 = 2 \int_{\Omega} (\rho - \bar{\rho})(\Delta \rho - \nabla(\rho S(x, \rho, c) \nabla c) - u \cdot \nabla \rho - \rho m + \bar{m}) \tag{3.16}$$

$$= -2 \int_{\Omega} |\nabla \rho|^2 + 2 \int_{\Omega} \rho S(x, \rho, c) \nabla c \cdot \nabla \rho - 2 \int_{\Omega} (\rho - \bar{\rho})(\rho m - \bar{m})$$

$$\leq - \int_{\Omega} |\nabla \rho|^2 + K_1 \int_{\Omega} |\nabla c|^2 - 2 \int_{\Omega} (\rho - \bar{\rho}) \rho m,$$

$$\frac{d}{dt} \int_{\Omega} (m - \bar{m})^2 = 2 \int_{\Omega} (m - \bar{m})(\Delta m - u \cdot \nabla m - \rho m + \bar{m}) \tag{3.17}$$

$$= 2 \int_{\Omega} m(\Delta m - u \cdot \nabla m) - 2 \int_{\Omega} (m - \bar{m})(\rho m - \bar{m})$$

$$\leq -2 \int_{\Omega} |\nabla m|^2 - 2 \int_{\Omega} (m - \bar{m}) \rho m,$$

$$\frac{d}{dt} \int_{\Omega} (c - \bar{c})^2 = 2 \int_{\Omega} (c - \bar{c})(\Delta c - u \cdot \nabla c - (c - \bar{c}) + (m - \bar{m}))$$

$$= 2 \int_{\Omega} c(\Delta c - u \cdot \nabla c) - 2 \int_{\Omega} (c - \bar{c})^2 + 2 \int_{\Omega} (c - \bar{c})(m - \bar{m})$$

$$\leq -2 \int_{\Omega} |\nabla c|^2 - \int_{\Omega} (c - \bar{c})^2 + \int_{\Omega} (m - \bar{m})^2, \tag{3.18}$$

$$\frac{d}{dt} \int_{\Omega} |u|^2 = -2 \int_{\Omega} |\nabla u|^2 + 2 \int_{\Omega} (\rho + m) \nabla \phi \cdot u - 2 \int_{\Omega} \nabla P \cdot u$$

$$= -2 \int_{\Omega} |\nabla u|^2 + 2 \int_{\Omega} (\rho - \bar{\rho} + m - \bar{m}) \nabla \phi \cdot u$$

$$\leq -2 \int_{\Omega} |\nabla u|^2 + K_2 \left( \int_{\Omega} |\rho - \bar{\rho} + m - \bar{m}|^2 \right)^\frac{1}{2} \left( \int_{\Omega} |u|^2 \right)^\frac{1}{2}$$

$$\leq - \int_{\Omega} |\nabla u|^2 + K_3 \left( \int_{\Omega} |\rho - \bar{\rho}|^2 + \int_{\Omega} |m - \bar{m}|^2 \right),$$

where $\nabla \cdot u = 0$, $u \mid_{\partial \Omega} = 0$ and the boundedness of $u$, $\nabla \phi$ and $S$ are used.
Lemma 3.6. Under the assumptions of lemma 3.1,

\[
\| (\rho - \overline{\rho}) (\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty,
\]
\[
\| (m - \overline{m}) (\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty,
\]
\[
\| (c - \overline{c}) (\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty,
\]
\[
\| u (\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.
\]

Proof. From (3.16)–(3.19), it follows that

\[
\frac{d}{dt} \int_\Omega (\rho - \overline{\rho})^2 \leq -\int_\Omega |\nabla \rho|^2 + K_1 \int_\Omega |\nabla c|^2 + 2\overline{\rho} \int_\Omega \rho m, \quad (3.20)
\]
\[
\frac{d}{dt} \int_\Omega (m - \overline{m})^2 \leq -2 \int_\Omega |\nabla m|^2 + 2\overline{m} \int_\Omega \rho m, \quad (3.21)
\]
\[
\frac{d}{dt} \int_\Omega (c - \overline{c})^2 \leq -2 \int_\Omega |\nabla c|^2 - \int_\Omega (c - \overline{c})^2 + \int_\Omega (m - \overline{m})^2, \quad (3.22)
\]
\[
\frac{d}{dt} \int_\Omega |u|^2 \leq -\int_\Omega |\nabla u|^2 + K_3 \left( \int_\Omega |\rho - \overline{\rho}|^2 + \int_\Omega |m - \overline{m}|^2 \right). \quad (3.23)
\]

Since \( \int_\Omega |m - \overline{m}|^2 \leq C_p \| \nabla m \|_{L^2(\Omega)}^2 \) and \( \int_0^\infty \int_\Omega \rho m \leq K_2 \) by (2.3), an application of lemma 3.5 to (3.21) yields

\[
\| m (\cdot, t) - \overline{m}(t) \|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty. \quad (3.24)
\]

Since

\[
\int_0^\infty \int_\Omega |(m - \overline{m})|^2 ds \leq C_p \int_0^\infty \| \nabla m \|_{L^2(\Omega)}^2 ds \leq K_5, \quad (3.25)
\]

the application of lemma 3.5 to (3.22) also yields

\[
\| c (\cdot, t) - \overline{c}(t) \|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty \quad (3.26)
\]

and

\[
\int_0^\infty \| \nabla c \|_{L^2(\Omega)}^2 \leq \int_0^\infty \int_\Omega |m - \overline{m}|^2 + \int_\Omega |c_0 - \overline{c_0}|^2 \leq K_6. \quad (3.27)
\]

Furthermore, by (3.27), \( \int_\Omega |\rho - \overline{\rho}|^2 \leq C_p \| \nabla \rho \|_{L^2(\Omega)}^2 \) and \( \int_0^\infty \int_\Omega \rho m \leq K_4 \), lemma 3.5 implies that

\[
\| \rho (\cdot, t) - \overline{\rho}(t) \|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty, \quad (3.28)
\]

\[
\int_0^\infty \| \rho - \overline{\rho} \|_{L^2(\Omega)}^2 \leq C_p \int_0^\infty \| \nabla \rho \|_{L^2(\Omega)}^2 \leq K_7. \quad (3.29)
\]
Hence from (3.25), (3.29), \( \int_{\Omega} |u|^2 \leq C_p \| \nabla u \|^2_{L^2(\Omega)} \) and lemma 3.5, it follows that
\[
\| u(\cdot, t) \|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty
\] (3.30)
as well as \( \int_0^\infty \| \nabla u \|^2_{L^2(\Omega)} \leq K_8 \).

Now we turn the above convergence in \( L^2(\Omega) \) into \( L^\infty(\Omega) \) with the help of the higher regularity of the solutions. Indeed, similar to the proof of \( \| c(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq K \) in theorem 1.1 in the case \( S = 0 \) on \( \partial \Omega \), \( \| m(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq K_{10} \) can be proved since \( \| \rho(\cdot, t) \|_{L^\infty(\Omega)} + \| m(\cdot, t) \|_{L^\infty(\Omega)} \leq K_9 \) for all \( t > 0 \) in (3.14). Hence from (3.14), there exists a constant \( K_{11} > 0 \) such that \( \| m(\cdot, t) - \overline{m}(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq K_{11} \), \( \| c(\cdot, t) - \overline{c}(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq K_{11} \), \( \| u(\cdot, t) \|_{W^{1,\infty}(\Omega)} \leq K_{11} \) for all \( t > 1 \). Therefore by (3.24), (3.26) and (3.30), the application of the interpolation inequality yields
\[
\| m - \overline{m} \|_{L^\infty(\Omega)} \leq C \left( \| m - \overline{m} \|^\frac{1}{4}_{L^\infty(\Omega)} \| m - \overline{m} \|^\frac{3}{4}_{L^2(\Omega)} + \| m - \overline{m} \|_{L^2(\Omega)} \right) \to 0 \quad \text{as} \quad t \to \infty,
\]
\[
\| c(\cdot, t) - \overline{c}(\cdot, t) \|_{L^\infty(\Omega)} \to 0, \quad \| u(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.
\]
In addition, similar to lemma 4.4 in [10] or lemma 5.2 in [3], there exist \( \vartheta \in (0, 1) \) and constant \( K_{12} > 0 \) such that \( \| \rho \|_{C^\vartheta([t_0, t+1])} \leq K_{12} \) for all \( t > 1 \), which along with (3.28) implies that \( \| \rho(\cdot, t) - \overline{\rho}(\cdot, t) \|_{C^\vartheta([t_0, t+1])} \to 0 \) as \( t \to \infty \) and then by the finite covering theorem, \( \| \rho(\cdot, t) - \overline{\rho}(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \) as \( t \to \infty \).

By a very similar argument as in lemma 4.2 of [10], we have

**Lemma 3.7.** Under the assumptions of lemma 3.1,
\[
\overline{\rho}(\cdot, t) \to \rho_\infty, \quad \overline{m}(\cdot, t) \to m_\infty, \quad \overline{c}(\cdot, t) \to m_\infty \quad \text{as} \quad t \to \infty
\]
with \( \rho_\infty = \{ \rho_0 - \overline{\rho} \}_+ \) and \( m_\infty = \{ m_0 - \overline{m} \}_+ \).

**Proof.** From (2.3) and (2.5), we have
\[
\int_{t_1}^t \| \rho m \|_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty, \quad (3.31)
\]
\[
\int_{t_1}^t \| \nabla m \|^2_{L^2(\Omega)} \to 0 \quad \text{as} \quad t \to \infty. \quad (3.32)
\]
On the other hand,
\[
\int_{t_1}^t \| \rho m \|_{L^2(\Omega)} = \int_{t_1}^t \int_{\Omega} \rho(m - \overline{m}) + \int_{t_1}^t \int_{\Omega} \rho \overline{m} \geq - \int_{t_1}^t \| \rho(\cdot, s) \|_{L^2(\Omega)} \| m - \overline{m} \|_{L^2(\Omega)} + |\Omega| \int_{t_1}^t \overline{\rho} \cdot \overline{m} \geq - K \int_{t_1}^t \| \nabla m \|_{L^2(\Omega)}^2 + |\Omega| \int_{t_1}^t \overline{\rho} \cdot \overline{m} \geq - K \left( \int_{t_1}^t \| \nabla m \|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} + |\Omega| \int_{t_1}^t \overline{\rho} \cdot \overline{m}.
\]
Inserting (3.31) and (3.32) into the above inequality, we obtain
\[ \int_{t-1}^{t} \overline{\rho} \cdot \overline{m} \to 0 \quad \text{as } t \to \infty. \tag{3.33} \]

Now if \( \overline{m_0} - m_0 \geq 0, (2.4) \) warrants that \( \overline{\rho} - \overline{m} \geq 0, \) which along with (3.33) implies that
\[ \int_{t-1}^{t} \overline{m}^2(s) \, ds \to 0 \quad \text{as } t \to \infty. \tag{3.34} \]

Noticing that \( \overline{m}(s) \geq \overline{m}(t) \) for all \( t \geq s, \) we have \( 0 \leq \overline{m}(t)^2 \leq \int_{t-1}^{t} \overline{m}^2(s) \, ds \to 0 \) as \( t \to \infty, \)
and thus \( \overline{\rho} \to \rho_{\infty} \) as \( t \to \infty \) due to (2.4). By very similar argument, one can see that \( \overline{\rho} \to 0 \) as \( t \to \infty \) and \( \overline{m} \to m_{\infty} \) as \( t \to \infty \) in the case of \( \overline{m_0} - m_0 < 0. \) Finally, it is observed that \( c(\cdot, t) \to m_{\infty} \) in \( L^2(\Omega) \) as \( t \to \infty \) is also valid (see lemma 4.7 of [10] for example) and thus \( \overline{\tau}(t) \to m_{\infty} \) as \( t \to \infty \) by the Hölder inequality.

Combining lemma 3.6 with lemma 3.7, we have

**Lemma 3.8.** Under the assumptions of lemma 3.1, we have
\[ \rho(\cdot, t) \to \rho_{\infty}, \quad m(\cdot, t) \to m_{\infty}, \quad c(\cdot, t) \to m_{\infty}, \quad u(\cdot, t) \to 0 \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty. \]

Now we proceed to estimate the decay rate of \( \| \rho(\cdot, t) - \rho_{\infty} \|_{L^\infty(\Omega)}, \| m(\cdot, t) - m_{\infty} \|_{L^\infty(\Omega)}, \| c(\cdot, t) - c_{\infty} \|_{L^\infty(\Omega)}, \text{ and } \| u(\cdot, t) \|_{L^\infty(\Omega)} \) when \( \int_{\Omega} \rho_0 \neq \int_{\Omega} m_0. \) To this end, we first consider its decay rate in \( L^2(\Omega) \) based on a differential inequality.

**Lemma 3.9.** Under the assumptions of lemma 3.1 and \( \int_{\Omega} \rho_0 \neq \int_{\Omega} m_0, \) for any \( \varepsilon > 0, \) there exist constants \( K(\varepsilon) > 0 \) and \( \varepsilon \) such that for \( t > t_\varepsilon, \)
\[ |\overline{\rho}(t) - \rho_{\infty}| + |\overline{m}(t) - m_{\infty}| \leq K(\varepsilon)e^{-(\rho_{\infty} + m_{\infty} - \varepsilon)t}, \tag{3.35} \]
\[ |\overline{\tau}(t) - m_{\infty}| \leq K(\varepsilon)e^{-\min\{1,(\rho_{\infty} + m_{\infty} - \varepsilon)\}t}. \tag{3.36} \]

**Proof.** For the case \( \int_{\Omega} \rho_0 > \int_{\Omega} m_0, \) we have \( \rho_{\infty} > 0 \) and \( m_{\infty} = 0. \) By lemma 3.8, there exists \( t_\varepsilon > 0 \) such that \( \rho(x, t) \geq \rho_{\infty} - \varepsilon \) for \( t > t_\varepsilon \) and \( x \in \Omega, \)
and thereby \( \frac{d}{dt} \int_{\Omega} m = -\int_{\Omega} \rho \cdot m \leq -((\rho_{\infty} - \varepsilon) \int_{\Omega} m) \quad \text{for } t > t_\varepsilon, \)
which implies that \( \overline{m}(t) \leq \overline{m_0}e^{-(\rho_{\infty} - \varepsilon)(t-t_\varepsilon)} \quad \text{for } t > t_\varepsilon. \)
Moreover, due to \( \overline{\rho} = \overline{m} + \rho_{\infty} \) by (2.4), we have \( |\overline{\rho}(t) - \rho_{\infty}| = |\overline{m}(t)| \leq \overline{m_0}e^{-(\rho_{\infty} - \varepsilon)(t-t_\varepsilon)} \quad \text{for } t > t_\varepsilon. \)
As for the case \( \int_{\Omega} \rho_0 < \int_{\Omega} m_0, \) similarly we can prove that \( |m(t) - m_{\infty}| = \overline{m} \leq \overline{m_0}e^{-(m_{\infty} - \varepsilon)(t-t_\varepsilon)} \quad \text{for } t > t_\varepsilon, \)
furthermore, by the third equation of (1.1), we have \( \frac{d}{dt} \int_{\Omega}(c - m_{\infty}) = \int_{\Omega}(m - m_{\infty}) - \int_{\Omega}(c - m_{\infty}), \)
and thereby \( |\overline{\tau}(t) - m_{\infty}| \leq K(\varepsilon)e^{-\min\{1,(m_{\infty} - \varepsilon)\}t}. \)

**Proof of theorem 1.2 in the case \( S = 0 \) on \( \partial\Omega. \)** By lemmas 3.6 and 3.8, we have
\[ \rho(\cdot, t) - \overline{\rho}(t) \to 0, \quad m(\cdot, t) - \overline{m}(t) \to 0, \quad \rho(\cdot, t) \to \rho_{\infty}, \quad m(\cdot, t) \to m_{\infty} \quad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty, \]
which implies that for any \( \varepsilon \in (0, \frac{\rho_{\infty} + m_{\infty}}{2}), \) there exists \( t_\varepsilon > 0 \) such that \( |\rho(\cdot, t) - \overline{\rho}(t)| < \varepsilon, \)
\[ |m(\cdot, t) - \overline{m}(t)| < \varepsilon, \quad \rho(\cdot, t) + m(\cdot, t) \geq \rho_{\infty} + m_{\infty} - \varepsilon \quad \text{for all } t > t_\varepsilon \text{ and } x \in \Omega. \]
Hence from (3.16)–(3.19), we have
\[ \frac{d}{dt} \int_{\Omega} (\rho - \overline{\rho})^2 + \int_{\Omega} |\nabla \rho|^2 \leq K_1 \int_{\Omega} |
abla c|^2 + 2\varepsilon \int_{\Omega} \rho \cdot m, \tag{3.37} \]
\[
\frac{d}{dt} \int_{\Omega} (m - \overline{m})^2 + 2 \int_{\Omega} |\nabla m|^2 \leq 2\varepsilon \int_{\Omega} \rho m,
\] (3.38)

\[
\frac{d}{dt} \int_{\Omega} (c - \overline{c})^2 + 2 \int_{\Omega} |\nabla c|^2 + \int_{\Omega} (c - \overline{c})^2 \leq \int_{\Omega} (m - \overline{m})^2,
\] (3.39)

\[
\frac{d}{dt} \int_{\Omega} |u|^2 + \int_{\Omega} |\nabla u|^2 \leq K_3 \left( \int_{\Omega} (\rho - \overline{\rho})^2 + \int_{\Omega} (m - \overline{m})^2 \right).
\] (3.40)

for \( t > t_\varepsilon \), as well as

\[
\frac{d}{dt} \int_{\Omega} \rho m = \int_{\Omega} \left[ \rho \left( \Delta m - u \cdot \nabla m - \rho m \right) + m(\Delta \rho - \nabla (\rho S(x, \rho, c) \nabla c) - u \cdot \nabla (\rho - \rho m)) \right]
\]

\[= -2 \int_{\Omega} \nabla \rho \nabla m - \int_{\Omega} \left( \rho u \cdot \nabla m + mu \cdot \nabla \rho \right) + \int_{\Omega} \rho S(x, \rho, c) \nabla c \cdot \nabla m - \int_{\Omega} \rho m^2 - \int_{\Omega} \rho^2 m
\]

\[\leq \int_{\Omega} |\nabla \rho|^2 + 2 \int_{\Omega} |\nabla m|^2 - \int_{\Omega} u \cdot \nabla (\rho m) + K_3 \int_{\Omega} |\nabla c|^2 - \int_{\Omega} \rho m(\rho + m)
\]

\[\leq \int_{\Omega} |\nabla \rho|^2 + 2 \int_{\Omega} |\nabla m|^2 + K_3 \int_{\Omega} |\nabla c|^2 - 2 \rho \rho(\rho + m) \int_{\Omega} \rho m,
\] (3.41)

where \( \nabla \cdot u = 0, u \mid_{\partial \Omega} = 0 \) and the boundedness of \( \rho \) are used.

On the other hand, by Poincaré’s inequality, there exists \( C_P > 0 \) such that

\[
\int_{\Omega} |\nabla \rho|^2 \geq C_P \int_{\Omega} (\rho - \overline{\rho})^2, \quad \int_{\Omega} |\nabla m|^2 \geq C_P \int_{\Omega} (m - \overline{m})^2,
\]

\[
\int_{\Omega} |\nabla c|^2 \geq C_P \int_{\Omega} (c - \overline{c})^2, \quad \int_{\Omega} |\nabla u|^2 \geq C_P \int_{\Omega} (u - \overline{u})^2.
\]

Therefore combining the above inequalities, and taking \( \varepsilon < \frac{a(\rho_\infty + m_\infty)C_P}{8 (K_3 + C_P)} \) with \( a = \min \{ \frac{1}{2}, \frac{K_3}{C_P}, \frac{K_1}{C_P} \} \), the functional

\[
G(t) := \int_{\Omega} (\rho - \overline{\rho})^2 + \int_{\Omega} (m - \overline{m})^2 + K_3 \int_{\Omega} (c - \overline{c})^2 + a \int_{\Omega} \rho m
\]

satisfies the ordinary differential inequality \( \frac{d}{dt} G(t) + \delta_i G(t) \leq 0 \) with \( \delta_i = \min \{ \frac{C_P}{2}, 1, \frac{2\rho_\infty + m_\infty}{C_P} \} \), which implies that

\[
\| \rho(\cdot, t) - \overline{\rho} \|_{L^2(\Omega)} + \| m(\cdot, t) - \overline{m} \|_{L^2(\Omega)} + \| c(\cdot, t) - \overline{c} \|_{L^2(\Omega)} \leq Ce^{-\frac{\delta_1 t}{2}}.
\] (3.42)

Moreover, by (3.42) and (3.40), \( \| u(\cdot, t) \|_{L^2(\Omega)} \leq Ce^{-\delta_2 t} \) for some \( \delta_2 > 0 \). At this position, combining (3.42) with lemma 3.9, we can find \( \delta_3 > 0 \) such that

\[
\| \rho(\cdot, t) - \rho_\infty \|_{L^2(\Omega)} + \| m(\cdot, t) - m_\infty \|_{L^2(\Omega)} + \| c(\cdot, t) - m_\infty \|_{L^2(\Omega)} \leq Ce^{-\delta_3 t}.
\] (3.43)

Hence as in the proof of lemma 3.6, we can obtain the decay estimates (1.9)–(1.12) by an application of the interpolation inequality, and thus the proof is complete.

### 3.3. Exponential decay under smallness condition

In this subsection, we give the proof of theorem 1.3 under the assumption that \( \mathcal{S} = 0 \) on \( \partial \Omega \). The proof thereof is divided into two cases (propositions 3.1 and 3.2).
3.3.1. The case $\int_{\Omega} \rho_0 > \int_{\Omega} m_0$. In this subsection we consider the case $\int_{\Omega} \rho_0 > \int_{\Omega} m_0$, i.e. $\rho_\infty > 0, m_\infty = 0$.

Proposition 3.1. Suppose that (1.4) hold with $\alpha = 0$ and $\int_{\Omega} \rho_0 > \int_{\Omega} m_0$. Let $N = 3$, $\rho_0 \in (\frac{\rho_1}{N}, N)$, $q_0 \in (N, \frac{Nq_1}{\rho_1})$. There exists $\varepsilon > 0$ such that for any initial data $(\rho_0, m_0, c_0, u_0)$ fulfilling (1.7) as well as

$$\|\rho_0 - \rho_\infty\|_{L^p(\Omega)} \leq \varepsilon, \quad \|m_0\|_{L^p(\Omega)} \leq \varepsilon, \quad \|\nabla c_0\|_{L^q(\Omega)} \leq \varepsilon, \quad \|u_0\|_{L^2(\Omega)} \leq \varepsilon,$$

(1.1) admits a global classical solution $(\rho, m, c, u, P)$. In particular, for any $\alpha_1 \in (0, \min\{\lambda_1, \rho_\infty\})$, $\alpha_2 \in (0, \min\{\alpha_1, \lambda_1', 1\})$, there exist constants $K_i, i = 1, 2, 3, 4$, such that for all $t \geq 1$

$$\|m(\cdot, t)\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t},$$

(3.44)

$$\|\rho(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t},$$

(3.45)

$$\|c(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq K_3 e^{-\alpha_1 t},$$

(3.46)

$$\|u(\cdot, t)\|_{L^2(\Omega)} \leq K_4 e^{-\alpha_1 t}.$$  

(3.47)

Proposition 3.1 is the consequence of the following lemmas. In the proof of these lemmas, the constants $C_i > 0, i = 1, \ldots, 10$, refer to those in lemmas 2.1–2.4, respectively. We first collect some easily verifiable observations in the following lemma:

Lemma 3.10. Under the assumptions of proposition 3.1 and $\sigma = \int_0^\infty \left(1 + s^{-\frac{1}{\rho_1}}\right) e^{-\alpha_1 s} ds$, there exist $M_1 > 0, M_2 > 0$ and $\varepsilon > 0$ such that

$$C_3 + 2C_2 C_{10} e^{(1 + C_1 + C_i |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1}) \sigma} \leq \frac{M_2}{4}, \quad M_1 \varepsilon < 1,$$

(3.48)

$$12C_2 C_{10} (C_6 + 4C_6 C_9 C_{10}) \|\nabla \phi\|_{L^\infty(\Omega)} (M_1 + C_1 + C_i |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1} + 4C_1 + C_1 |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1}) \varepsilon < 1,$$

(3.49)

$$C_6 C_{10} (M_1 + C_1 + C_i |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1}) + \rho_\infty |\Omega|^{\frac{1}{\rho_1}} \leq \frac{M_1}{8},$$

(3.50)

$$3C_6 C_9 C_3 (M_1 + C_1 + C_i |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1}) + M_2 \varepsilon \leq \frac{M_1}{8},$$

(3.51)

$$3C_6 C_9 C_3 (M_1 + C_1 + C_i |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1}) (1 + 2C_6 C_{10}) \|\nabla \phi\|_{L^\infty(\Omega)} (M_1 + C_1 + C_i |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1} + 4\varepsilon (1 + C_1 + C_i |\Omega|^{\frac{1}{\rho_1}} - \frac{1}{\rho_1}) \varepsilon) \varepsilon \leq \frac{M_1}{4}.$$  

(3.52)

Let

$$T \triangleq \sup \left\{ \bar{T} \in (0, T_{\max}) \mid \begin{aligned} &\| (\rho - m)(\cdot, t) - e^{\Delta (\rho_0 - m_0)} \|_{L^p(\Omega)} \leq M_1 \varepsilon (1 + t^{-\frac{1}{\rho_1}} + \frac{1}{\rho_1}) e^{-\alpha_1 t} \text{ for all } \theta \in [q_0, \infty], t \in [0, \bar{T}]; \\
&\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq M_2 \varepsilon (1 + t^{-\frac{1}{\rho_1}}) e^{-\alpha_1 t} \text{ for all } t \in [0, \bar{T}). \end{aligned} \right\}$$

(3.53)
By (1.7) and lemma 2.5, \( T > 0 \) is well-defined. We first show \( T = T_{\max} \). To this end, we will show that all of the estimates mentioned in (3.53) is valid with even smaller coefficients on the right hand side. The derivation of these estimates will mainly rely on \( L^p - L^q \) estimates for the Neumann heat semigroup and the fact that the classical solutions on \( (0, T_{\max}) \) can be represented as

\[
(\rho - m)(\cdot, t) = e^{t\Delta} (\rho_0 - m_0) - \int_0^t e^{(t-s)\Delta} (\nabla \cdot (\rho S(x, \rho, c) \nabla c) + u \cdot \nabla (\rho - m))(\cdot, s) \, ds,
\]

(3.54)

\[
m(\cdot, t) = e^{t\Delta} m_0 - \int_0^t e^{(t-s)\Delta} (\rho m + u \cdot \nabla m)(\cdot, s) \, ds,
\]

(3.55)

\[
c(\cdot, t) = e^{(t-\Delta-1)c_0} + \int_0^t e^{(t-s)(\Delta-1)} (m - u \cdot \nabla c)(\cdot, s) \, ds,
\]

(3.56)

\[
u(\cdot, t) = e^{-t\Delta} u_0 + \int_0^t e^{-(t-s)\Delta} \mathcal{P}((\rho + m) \nabla \phi)(\cdot, s) \, ds
\]

(3.57)

for all \( t \in (0, T_{\max}) \) as per the variation-of-constants formula.

**Lemma 11.** Under the assumptions of proposition 3.1, for all \( t \in (0, T) \) and \( \theta \in [q_0, \infty] \),

\[
\|(\rho - m)(\cdot, t) - \rho_{\infty}\|_{L^p(\Omega)} \leq M_3 \varepsilon (1 + \frac{t^-\frac{\alpha}{\alpha - 2}}{\frac{\alpha}{\alpha - 2}}) e^{-\alpha t}.
\]

**Proof.** Since \( e^{t\Delta} \rho_{\infty} = \rho_{\infty} \) and \( \int_\Omega (\rho_0 - m_0 - \rho_{\infty}) = 0 \), the definition of \( T \) and lemma 2.1(i) show that

\[
\|(\rho - m)(\cdot, t) - \rho_{\infty}\|_{L^p(\Omega)} \leq \|(\rho - m)(\cdot, t) - e^{t\Delta}(\rho_0 - m_0)\|_{L^p(\Omega)} + \|e^{t\Delta}(\rho_0 - m_0 - \rho_{\infty})\|_{L^p(\Omega)}
\]

\[
\leq M_1 \varepsilon (1 + \frac{t^-\frac{\alpha}{\alpha - 2}}{\frac{\alpha}{\alpha - 2}}) e^{-\alpha t} + C_1 (1 + \frac{t^-\frac{\alpha}{\alpha - 2}}{\frac{\alpha}{\alpha - 2}}) (\|\rho_0 - \rho_{\infty}\|_{L^p(\Omega)} + \|m_0\|_{L^p(\Omega)}) e^{-\lambda t}
\]

\[
\leq M_3 \varepsilon (1 + \frac{t^-\frac{\alpha}{\alpha - 2}}{\frac{\alpha}{\alpha - 2}}) e^{-\alpha t}
\]

for all \( t \in (0, T) \) and \( \theta \in [q_0, \infty] \), where \( M_3 = M_1 + C_1 + C_1 |\Omega| \frac{\alpha}{\alpha - 2} \). \( \square \)

**Lemma 12.** Under the assumptions of proposition 3.1, for any \( k > 1 \),

\[
\|m(\cdot, t)\|_{L^p(\Omega)} \leq M_4 \|m_0\|_{L^p(\Omega)} e^{-\rho_{\infty} t} \quad \text{for all } t \in (0, T)
\]

(3.58)

with \( \sigma = \int_0^\infty (1 + s^{-\frac{\alpha}{\alpha - 2}}) e^{-\alpha t} \, ds \) and \( M_4 = e^{M_\sigma}. \)

**Proof.** Multiplying the \( m \)-equation in (1.1) by \( km^{k-1} \) and integrating the result over \( \Omega \), we get \( \frac{d}{dt} \int_\Omega m^k \leq -k \int_\Omega m^k \leq -k \int_\Omega m^k |\rho - m - \rho_{\infty}| \), since \( -\rho \leq |\rho - m - \rho_{\infty}| - m - \rho_{\infty} \leq -\rho_{\infty} + |\rho - m - \rho_{\infty}| \), lemma 3.11 yields

\[
\frac{d}{dt} \int_\Omega m^k \leq -k \rho_{\infty} \int_\Omega m^k + k \int_\Omega m^k |\rho - m - \rho_{\infty}|
\]

\[
\leq -k \rho_{\infty} \int_\Omega m^k + k \|\rho - m - \rho_{\infty}\|_{L^\infty(\Omega)} \int_\Omega m^k
\]

\[
\leq -k \rho_{\infty} \int_\Omega m^k + k M_3 \varepsilon (1 + \frac{t^-\frac{\alpha}{\alpha - 2}}{\frac{\alpha}{\alpha - 2}}) e^{-\alpha t} \int_\Omega m^k
\]
and thus \( f_{01} m^k \leq f_{01} m^k \exp(-k \rho_0 t) + k M_3 \int f_{01}(1 + s^{-\frac{\rho_0}{m}}) e^{-\alpha_1 s} ds \leq \|m_0\|_{L^p(\Omega)}^k e^{k(M_0, \sigma, \varepsilon, \rho, \omega_1)} \). The assertion (3.58) follows immediately.

**Lemma 3.13.** Under the assumptions of proposition 3.1, there exists \( M_3 > 0 \) such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq M_3 \varepsilon \left(1 + \frac{1}{t} \right) e^{-\alpha_1 t}
\]
for all \( t \in (0, T) \).

**Proof.** For any given \( \alpha_2 < \lambda_1' \), we fix \( \mu \in (\alpha_2, \lambda_1') \). By (3.57), lemmas 2.2 and 2.3, we obtain
\[
\|u(\cdot, t)\|_{L^0(\Omega)} \leq C_3 e^{-\frac{\mu}{2}(\cdot)} e^{-\mu} ||u_0||_{L^0(\Omega)} + \int_0^t ||e^{-((t-s)\lambda)}P((\rho + m) \nabla \phi)(\cdot, s)||_{L^0(\Omega)} ds
\]
(3.59)
\[
\leq C_3 e^{-\frac{\mu}{2}(\cdot)} e^{-\mu} ||u_0||_{L^0(\Omega)} + C_3 \int_0^t e^{-\mu((t-s))} ||P((\rho + m - \rho_0) \nabla \phi)(\cdot, s)||_{L^0(\Omega)} ds
\]
\[
\leq C_3 e^{-\frac{\mu}{2}(\cdot)} e^{-\mu} ||u_0||_{L^0(\Omega)} + C_3 \varepsilon \int_0^t e^{-\mu((t-s))} ||(\rho + m - \rho_0) \nabla \phi(\cdot, s)||_{L^0(\Omega)} ds
\]
where \( P((\rho + m) \nabla \phi) = \rho + \varepsilon P(\nabla \phi) = 0 \) is used. On the other hand, due to \( \alpha_1 < \rho_0 \), lemmas 3.11 and 3.12 show that
\[
\|\left(\rho + m - \rho_0\right)(\cdot, s)\|_{L^0(\Omega)} \leq \|\left(\rho - m - \rho_0\right)(\cdot, s) + 2(m - \rho_0)(\cdot, s)\|_{L^0(\Omega)} + 2\|m - \rho_0\|_{L^0(\Omega)}
\]
(3.60)
\[
\leq M_2 \varepsilon (1 + s^{-\frac{\rho}{m} - \frac{1}{m}}) e^{-\alpha_1 t}
\]
with \( M_2^* = M_3 + 4e^{M_0, \sigma, \varepsilon} \). Combining (3.59) with (3.60) and applying lemma 2.3, we have
\[
\|u(\cdot, t)\|_{L^0(\Omega)} \leq C_3 e^{-\frac{\mu}{2}(\cdot)} e^{-\mu} ||u_0||_{L^0(\Omega)} + C_3 C_B \|\nabla \phi\|_{L^0(\Omega)} M_2 \varepsilon \int_0^t e^{-\mu((t-s))} ||(\rho + m - \rho_0) \nabla \phi(\cdot, s)||_{L^0(\Omega)} ds
\]
\[
\leq C_3 e^{-\frac{\mu}{2}(\cdot)} e^{-\mu} ||u_0||_{L^0(\Omega)} + C_3 C_B C_{10} \|\nabla \phi\|_{L^0(\Omega)} M_2^* \varepsilon (1 + t^{-\frac{\rho}{m} - \frac{1}{m}}) e^{-\alpha_1 t}
\]
\[
\leq C_3 e^{-\frac{\mu}{2}(\cdot)} e^{-\mu} + 2C_3 C_B C_{10} \|\nabla \phi\|_{L^0(\Omega)} M_2^* \varepsilon e^{-\alpha_1 t}
\]
\[
\leq M_5 \varepsilon (1 + t^{-\frac{\rho}{m} - \frac{1}{m}}) e^{-\alpha_1 t}
\]
where \( M_5 = C_3 + 2C_3 C_B C_{10} \|\nabla \phi\|_{L^0(\Omega)} M_2^* \) and \( \frac{\rho}{m} - \frac{1}{m} < 1 \) is used.

**Lemma 3.14.** Under the assumptions of proposition 3.1, for all \( t \in (0, T) \),
\[
\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{M_5^2}{2} e^{\frac{1}{2} \varepsilon (1 + t^{-\frac{1}{4}})} e^{-\alpha_1 t}
\]

**Proof.** By (3.56) and lemma 2.1(iii), we have
\[
\||\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c(\cdot, t)\|_{L^\infty(\Omega)} + \int_0^t \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} ds
\]
\[
\leq C_3 (1 + t^{-\frac{1}{4}}) e^{-\lambda_{1-k} t} ||c(\cdot, t)\|_{L^\infty(\Omega)} + \int_0^t \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} ds + \int_0^t \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} ds.
\]
(3.61)
Now we estimate the last two integrals on the right-hand side of the above inequality. From lemmas 2.1(ii), 2.4 and 3.12 with $k = q_0$ and the fact that $q_0 > N$, it follows that

$$\int_0^t \| \nabla e^{(t-s)(\Delta-1)} m \|_{L^\infty(\Omega)} ds \leq C_2 \int_0^t \left( 1 + (t-s)^{\frac{1}{2} \frac{q}{2} - \frac{1}{4}} \right) e^{-\lambda_1(t-s)} \| m \|_{W^{1,q}(\Omega)} ds \leq C_2 M_4 \int_0^t \left( 1 + (t-s)^{\frac{1}{2} \frac{q}{2} - \frac{1}{4}} \right) e^{-\lambda_1(t-s)} e^{-\rho \alpha t} ds \leq C_2 C_{10} M_4 (1 + r^{\frac{1}{2}}) e^{-\alpha_0 t}.$$

On the other hand, by lemmas 2.1(ii), 2.4, 3.13 and the definition of $T$, we obtain

$$\int_0^t \| \nabla e^{(t-s)(\Delta-1)} u \cdot \nabla c \|_{L^\infty(\Omega)} ds \leq C_2 \int_0^t \left( 1 + (t-s)^{\frac{1}{2} \frac{q}{2} - \frac{1}{4}} \right) e^{-\lambda_1(t-s)} \| u \cdot \nabla c \|_{L^\infty(\Omega)} ds \leq C_2 (C_3 + 2C_{10} M_4 + 3C_{10} M_2 e \varepsilon^2) (1 + r^{\frac{1}{2}}) e^{-\alpha_0 t}.$$

From (3.61)–(3.63), it follows that

$$\| \nabla c \|_{L^\infty(\Omega)} \leq C \left( C_3 + 2C_{10} M_4 + 3C_{10} M_2 M_5 e \varepsilon^2 \right) (1 + r^{\frac{1}{2}}) e^{-\alpha_0 t} \leq M_2 \frac{C}{2} (1 + r^{\frac{1}{2}}) e^{-\alpha_0 t},$$

due to the choice of $M_1, M_2$ and $\varepsilon$ satisfying (3.48), (3.49), and thereby completes the proof. 

**Lemma 3.15.** Under the assumptions of proposition 3.1, for all $\theta \in [q_0, \infty]$ and $t \in (0, T)$,

$$\| (\rho - m)(\lambda, t) - e^{\Delta} (\rho_0 - m_0) \|_{L^p(\Omega)} \leq \frac{M_1}{2} (1 + r^{\lambda - \frac{\lambda}{2} \frac{q}{2} - \frac{1}{4}}) e^{-\alpha_0 t}.$$

**Proof.** According to (3.54), lemma 2.1(iv), we have

$$\| (\rho - m)(\lambda, t) - e^{\Delta} (\rho_0 - m_0) \|_{L^p(\Omega)} \leq \int_0^t \| e^{(t-s)\Delta} (\nabla \cdot (\rho S(x, \rho, c) \nabla c) + u \cdot \nabla (\rho - m)) (\lambda, s) \|_{L^p(\Omega)} ds \leq \int_0^t \| e^{(t-s)\Delta} \nabla \cdot (\rho S(x, \rho, c) \nabla c) (\lambda, s) \|_{L^p(\Omega)} ds + \int_0^t \| e^{(t-s)\Delta} \nabla \cdot ((\rho - m - \rho_\infty) u)(\lambda, s) \|_{L^p(\Omega)} ds \leq C_4 C_S \int_0^t \left( 1 + (t-s)^{-\frac{1}{2} \frac{q}{2} - \frac{1}{4}} \right) e^{-\lambda_1(t-s)} \| \rho(\lambda, s) \|_{L^p(\Omega)} \| \nabla c(\lambda, s) \|_{L^\infty(\Omega)} ds + C_4 \int_0^t \left( 1 + (t-s)^{-\frac{1}{2} \frac{q}{2} - \frac{1}{4}} \right) e^{-\mu(t-s)} \| u(\rho - m - \rho_\infty)(\lambda, s) \|_{L^p(\Omega)} ds = I_1 + I_2.$$
Now we need to estimate $I_1$ and $I_2$. Firstly, from lemmas 3.11 and 3.12, we obtain
\[
\|\rho(\cdot, s)\|_{L^0(\Omega)} \leq \|\rho - m - \rho_\infty(\cdot, s)\|_{L^0(\Omega)} + \|m(\cdot, s)\|_{L^0(\Omega)} + \|\rho_\infty\|_{L^0(\Omega)} \\
\leq M_3 e^{(1 + s^{-\frac{2}{3}}(\frac{1}{4} - \frac{1}{6}))} e^{-\alpha_1 t} + M_6
\]  
(3.64)
with $M_6 = e^{(1 + C_1 + C_1|\Omega|^{\frac{1}{2}} - \frac{1}{6})} t + \rho_\infty|\Omega|^{\frac{1}{6}}$, which together with lemmas 3.14 and 2.1 implies that
\[
I_1 \leq C_4 C_5 M_7 \int_0^T (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}) e^{-\lambda_1 (t-s)} \|\nabla \psi\|_{L^\infty(\Omega)} ds \\
+ M_7 e^{\int_0^T (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}) e^{-\lambda_1 (t-s) (1 + s^{-\frac{2}{3}}(\frac{1}{4} - \frac{1}{6}))} e^{-\alpha_1 t} \|\nabla \psi\|_{L^\infty(\Omega)} ds} \\
\leq C_4 C_5 M_7 e^{\int_0^T (1 + s^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}) e^{-\lambda_1 (t-s) (1 + s^{-\frac{2}{3}}(\frac{1}{4} - \frac{1}{6}))} e^{-\alpha_1 t} \|\nabla \psi\|_{L^\infty(\Omega)} ds} \\
+ 3M_7 e^{\int_0^T (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}) e^{-\lambda_1 (t-s) (1 + s^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}))} e^{-2\alpha_1 t} e^{-\lambda_1 (t-s)} ds} \\
\leq C_4 C_5 M_7 e^{\int_0^T (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}) e^{-\alpha_1 t})} \\
\leq \frac{M_7}{4} e^{(1 + t^{-\frac{2}{3}}(\frac{1}{4} - \frac{1}{6}))} e^{-\alpha_1 t}
\]  
(3.65)
with $M_7 := C_4 C_5 M_7$, where we have used (3.50) and (3.51) and $\frac{1}{p_0} - \frac{1}{q_0} < \frac{1}{N}$. On the other hand, from lemmas 3.11 and 3.13, it follows that
\[
I_2 = C_4 \int_0^T (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}) e^{-\mu (t-s)} \|\rho - m - \rho_\infty\|_{L^\infty(\Omega)} \|\psi\|_{L^\infty(\Omega)} ds \\
\leq 3C_4 M_8 e^{\int_0^T (1 + (t-s)^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}) e^{-\mu (t-s) (1 + s^{-\frac{1}{2}}(\frac{1}{4} - \frac{1}{6}))} e^{-(\alpha_1 + \alpha_2) t} ds} \\
\leq 3C_4 M_8 C_10 e^{\int_0^T (1 + t^{-\min(0, \theta(\frac{1}{4} - \frac{1}{6}))}) e^{-\min(\mu_0, \alpha_1 + \alpha_2) t} ds} \\
\leq \frac{M_7}{4} e^{(1 + t^{-\frac{2}{3}}(\frac{1}{4} - \frac{1}{6}))} e^{-\alpha_1 t},
\]  
(3.66)
where we have used (3.52) and $\frac{1}{p_0} - \frac{1}{q_0} < \frac{1}{N}$. Hence combining the above inequalities leads to our conclusion immediately.

**Proof of theorem 3.13 in the case $S = 0$ on $\partial \Omega$, part 1 (proposition 3.1).** First we claim that $T = T_{\text{max}}$. In fact, if $T < T_{\text{max}}$, then by lemmas 3.14 and 3.15, we have
\[
\|\nabla \psi(\cdot, t)\|_{L^\infty(\Omega)} \leq M_6 e^{(1 + 1 - \frac{2}{3}) e^{-\alpha_1 t}}
\]
and
\[
\|\rho(\cdot, t) - e^{\Delta (\rho_0 - m_0)}\|_{L^p(\Omega)} \leq M_8 e^{(1 + t^{-\frac{2}{3}}(\frac{1}{4} - \frac{1}{6})) e^{-\alpha_1 t}}
\]
for all $\theta \in [q_0, \infty]$ and $t \in (0, T)$, which contradicts the definition of $T$ in (3.53). Next, we show that $T_{\text{max}} = \infty$. In fact, if $T_{\text{max}} < \infty$, we only need to show that as $t \to T_{\text{max}}$, $\|\rho(\cdot, t)\|_{L^\infty(\Omega)} + \|m(\cdot, t)\|_{L^\infty(\Omega)} + \|c(\cdot, t)\|_{W^{1, \infty}(\Omega)} + \|A^\beta u(\cdot, t)\|_{L^2(\Omega)} \to \infty$
according to the extensibility criterion in lemma 2.5.
Let $t_0 := \min\{1, \frac{t_{\text{max}}}{2}\}$. Then from lemma 3.12, there exists $K_1 > 0$ such that for $t \in (t_0, T_{\text{max}})$,
\[
\|m(\cdot, t)\|_{L^\infty(\Omega)} \leq K_1 e^{-\rho_{\infty} t}.
\]  
(3.67)
Moreover, from lemma 3.11 and the fact that
\[
\|\rho(\cdot, t) - \rho_{\infty}\|_{L^\infty(\Omega)} \leq \|\rho(\cdot) - m\|_{L^\infty(\Omega)} + \|m(\cdot, t)\|_{L^\infty(\Omega)},
\]  
it follows that for all $t \in (t_0, T_{\text{max}})$ and some constant $K_2 > 0$,
\[
\|\rho(\cdot, t) - \rho_{\infty}\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha t}.
\]  
(3.68)
Furthermore, lemma 3.14 implies that there exists $K'_3 > 0$ such that
\[
\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq K'_3 e^{-\alpha t} \quad \text{for all } t \in (t_0, T_{\text{max}}).
\]  
(3.69)
On the other hand, we can conclude that $\|c(\cdot, t)\|_{L^\infty(\Omega)} + \|A^\beta u(\cdot, t)\|_{L^2(\Omega)} \leq C$ for $t \in (t_0, T_{\text{max}})$. In fact, we first show that there exists a constant $M_0 > 0$ such that
\[
\|A^\beta u(\cdot, t)\|_{L^2(\Omega)} \leq M_0 e^{-\alpha t}
\]  
(3.70)
for $t_0 < t < T_{\text{max}}$. By (3.57), we have
\[
\|A^\beta u(\cdot, t)\|_{L^2(\Omega)} \leq \|A^\beta e^{-\alpha t_0} u_0\|_{L^2(\Omega)} + \int_0^t \|A^\beta e^{-(t-s)\lambda} P((\rho + m - \rho_{\infty}) \nabla \phi)(\cdot, s)\|_{L^2(\Omega)} ds.
\]  
According to lemma 2.2, $\|A^\beta e^{-\alpha t} u_0\|_{L^2(\Omega)} \leq C_2 e^{-\alpha t} \|A^\beta u_0\|_{L^2(\Omega)}$ for all $t \in (0, T_{\text{max}})$. On the other hand, from lemmas 2.2, 2.3 and and 3.11, it follows that there exists $\hat{M} > 1$ such that
\[
\int_0^t \|A^\beta e^{-(t-s)\lambda} P((\rho + m - \rho_{\infty}) \nabla \phi)(\cdot, s)\|_{L^2(\Omega)} ds
\]  
\[
\leq C_8 C_5 \|\nabla \phi\|_{L^\infty(\Omega)} \hat{M} \int_0^t e^{-\mu(t-s)} (t-s)^{-\beta} \|\rho(\cdot) - m\|_{L^\infty(\Omega)} + 2 \|m(\cdot, s)\|_{L^\infty(\Omega)} ds
\]  
\[
\leq C_8 C_5 \|\nabla \phi\|_{L^\infty(\Omega)} \hat{M} \int_0^t e^{-\mu(t-s)} (t-s)^{-\beta} (1 + s^{-\frac{2}{(1-\beta)} + \frac{\beta}{1-\beta}}) e^{-\alpha t} ds
\]  
\[
\leq C_8 C_5 C_10 \|\nabla \phi\|_{L^\infty(\Omega)} \hat{M} \int_0^t e^{-\mu(t-s)} (t-s)^{-\beta} (1 + t_0^{-\frac{2}{(1-\beta)} + \frac{\beta}{1-\beta}}) e^{-\alpha t} ds
\]  
\[
\leq C_8 C_5 C_10 \|\nabla \phi\|_{L^\infty(\Omega)} \hat{M} e^{-\alpha t} (1 + t_0^{-\frac{2}{(1-\beta)} + \frac{\beta}{1-\beta}})
\]  
for $t_0 < t < T_{\text{max}}$. Hence combining the above inequalities, we arrive at (3.70).

Since $D(A^{\beta}) \hookrightarrow L^\infty(\Omega)$ with $\beta \in (\frac{N}{2}, 1)$, we have
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha t} \quad \text{for some } K_4 > 0 \text{ and } t \in (0, T_{\text{max}}).
\]  
(3.71)
Now we turn to show that there exists $K_5' > 0$ such that
\[
\|c(\cdot, t)\|_{L^\infty(\Omega)} \leq K_5' e^{-\alpha t} \quad \text{for all } t \in (0, T_{\text{max}}).
\]  
(3.72)
Indeed, from (3.56), it follows that
\[
\|e\|_{L^\infty(\Omega)} \leq \|e^{(\Delta)^{-1}}c_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta)^{-1}}(m - u \cdot \nabla e)\|_{L^\infty(\Omega)} \, ds \\
\leq e^{-\alpha t}\|c_0\|_{L^\infty(\Omega)} + \int_0^t \|e^{(t-s)(\Delta)^{-1}}m(\cdot, s)\|_{L^\infty(\Omega)} \, ds \\
+ \int_0^t \|e^{(t-s)(\Delta)^{-1}}u \cdot \nabla e(\cdot, s)\|_{L^\infty(\Omega)} \, ds.
\]
(3.73)

An application of (3.58) with \(k = \infty\) yields
\[
\int_0^t \|e^{(t-s)(\Delta)^{-1}}m(\cdot, s)\|_{L^\infty(\Omega)} \, ds \leq \int_0^t e^{-(t-s)}\|m(\cdot, s)\|_{L^\infty(\Omega)} \, ds \\
\leq \|m_0\|_{L^\infty(\Omega)} M_4 \int_0^t e^{-(t-s)}e^{-\rho \alpha s} \, ds \\
\leq M_4 C_{10} e^{-\alpha t}.
\]
(3.74)

On the other hand, from (3.71) and (3.69), we can see that
\[
\int_0^t \|e^{(t-s)(\Delta)^{-1}}u \cdot \nabla e\|_{L^\infty(\Omega)} \, ds \leq \int_0^t e^{-(t-s)}\|u\|_{L^\infty(\Omega)} \|\nabla e\|_{L^\infty(\Omega)} \, ds \\
\leq K_3 K_4 \int_0^t e^{-2\alpha s}e^{-(t-s)} \, ds \\
\leq K_3 K_4 C_{10} e^{-\alpha t}.
\]
(3.75)

Hence, inserting (3.74), (3.75) into (3.73), we arrive at the conclusion (3.72). Therefore we have \(T_{\max} = \infty\), and the decay estimates in (3.44)–(3.47) follow from (3.67)–(3.72), respectively.

3.3.2. The case \(\int_\Omega \rho_0 < \int_\Omega m_0\). In this subsection we consider the case \(\int_\Omega \rho_0 < \int_\Omega m_0\), i.e. \(m_\infty > 0, \rho_\infty = 0\).

**Proposition 3.2.** Suppose that (1.4) hold with \(\alpha = 0\) and \(\int_\Omega \rho_0 < \int_\Omega m_0\). Let \(N = 3\), \(p_0 \in (\frac{2N}{N - 2}, N)\), \(q_0 \in (N, \frac{Np_0}{2N - p_0})\). Then there exists \(\varepsilon > 0\) such that for any initial data \((\rho_0, m_0, c_0, u_0)\) fulfilling (1.7) as well as
\[
\|\rho_0\|_{L^p(\Omega)} \leq \varepsilon, \quad \|m_0 - m_\infty\|_{L^q(\Omega)} \leq \varepsilon, \quad \|\nabla c_0\|_{L^{q'}(\Omega)} \leq \varepsilon, \quad \|u_0\|_{L^q(\Omega)} \leq \varepsilon,
\]
(1.1) admits a global classical solution \((\rho, m, c, u, P)\). Furthermore, for any \(\alpha_1 \in (0, \min\{\lambda_1, m_\infty\})\), \(\alpha_2 \in (0, \min\{\alpha_1, \lambda_1, 1\})\), there exist constants \(K_i > 0, i = 1, 2, 3, 4\), such that
\[
\|m(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t},
\]
(3.76)
\[
\|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_2 t},
\]
(3.77)
\[
\|c(\cdot, t) - m_\infty\|_{W^{1, \infty}(\Omega)} \leq K_3 e^{-\alpha_2 t},
\]
(3.78)
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_2 t}.
\]
(3.79)
The proof of proposition 3.2 proceeds in a parallel fashion to that of proposition 3.1. However, due to differences in the properties of $\rho$ and $m$, there are significant differences in the details of their proofs. Thus, for the convenience of the reader, we will give the full proof of proposition 3.2.

The following can be verified easily:

**Lemma 3.16.** Under the assumptions of proposition 3.2, it is possible to choose $M_1 > 0, M_2 > 0$ and $\varepsilon > 0$ such that

\[ C_3 \lesssim \frac{M_2}{6}, \quad C_2 C_{10} (1 + C_1 + C_1 |\Omega|^\frac{1}{n} - \frac{1}{n} + M_1) \lesssim \frac{M_2}{6}, \tag{3.80} \]

\[ 18C_2 C_9 C_{10} (1 + 2C_9 C_{10} (1 + C_1 + C_1 |\Omega|^\frac{1}{n} - \frac{1}{n} + 2M_1) \| \nabla \phi \|_{L^\infty(\Omega)} \varepsilon \lesssim 1, \] \tag{3.81}

\[ 2C_1 + (\min \{1, |\Omega|\}) \frac{1}{n} \lesssim \frac{M_1}{8}, \quad 24C_2 C_9 C_{10} M_2 \varepsilon < 1, \tag{3.82} \]

\[ 24C_4 C_{10} C_6 (1 + 2C_9 C_{10} (1 + C_1 + C_1 |\Omega|^\frac{1}{n} - \frac{1}{n} + 2M_1) \| \nabla \phi \|_{L^\infty(\Omega)} \varepsilon < 1, \] \tag{3.83}

\[ 24C_4 C_{10} (1 + C_1 + C_1 |\Omega|^\frac{1}{n} - \frac{1}{n} + M_1) \varepsilon < 1, \tag{3.84} \]

\[ 12C_2 C_9 C_{10} M_1 M_2 \varepsilon < 1, \tag{3.85} \]

\[ C_{10} C_6 C_4 (1 + C_1 + C_1 |\Omega|^\frac{1}{n} - \frac{1}{n} (1 + 2C_9 C_{10} (1 + C_1 + C_1 |\Omega|^\frac{1}{n} - \frac{1}{n} + 2M_1) \| \nabla \phi \|_{L^\infty(\Omega)} \varepsilon < \frac{1}{24}. \tag{3.86} \]

Similar to the proof of proposition 3.1, we define

\[ T \triangleq \sup \left\{ \bar{T} \in (0, T_{\text{max}}) ; \begin{aligned} &\| (m - \rho)(\cdot, t) - e^{t\Delta} (m_0 - \rho_0) \|_{L^p(\Omega)} \lesssim \varepsilon (1 + t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}) e^{-\alpha_1 t}, \\ &\| \rho(\cdot, t) \|_{L^q(\Omega)} \lesssim M_1 \varepsilon (1 + t^{-\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}) e^{-\alpha_1 t}, \forall \theta \in [q_0, \infty], \\ &\| \nabla c(\cdot, t) \|_{L^\infty(\Omega)} \lesssim M_2 \varepsilon (1 + t^{-\frac{n}{2}}) e^{-\alpha_1 t} \text{ for all } t \in [0, \bar{T}). \end{aligned} \right\} \tag{3.87} \]

By lemma 2.5 and (1.7), $T > 0$ is well-defined. As in the previous subsection, we first show $T = T_{\text{max}}$, and then $T_{\text{max}} = \infty$. To this end, we will show that all of the estimates mentioned in (3.87) are valid with even smaller coefficients on the right hand side than appearing in (3.87).

The derivation of these estimates will mainly rely on $L^p - L^q$ estimates for the Neumann heat semigroup and the corresponding semigroup for Stokes operator, and the fact that the classical solutions of (1.1) on $(0, T)$ can be represented as

\[ (m - \rho)(\cdot, t) = e^{t\Delta} (m_0 - \rho_0) + \int_0^t e^{(t-s)\Delta} (\nabla \cdot (\rho S(x, \rho, c) \nabla c) - u \cdot \nabla (m - \rho))(\cdot, s) ds, \tag{3.88} \]

\[ \rho(\cdot, t) = e^{t\Delta} \rho_0 - \int_0^t e^{(t-s)\Delta} (\nabla \cdot (\rho S(x, \rho, c) \nabla c) + u \cdot \nabla \rho + \rho m)(\cdot, s) ds, \tag{3.89} \]

\[ c(\cdot, t) = e^{(\Delta-1) c_0} + \int_0^t e^{(t-s)(\Delta-1)} (m - u \cdot \nabla c)(\cdot, s) ds. \tag{3.90} \]
\[ u(\cdot, t) = e^{-\alpha t}u_0 + \int_0^t e^{-(t-s)\lambda} P((\rho + m)\nabla \phi)(\cdot, s)\, ds. \] (3.91)

**Lemma 3.17.** Under the assumptions of proposition 3.2, we have
\[ \|(m - \rho)(\cdot, t) - m_\infty\|_{L^p(\Omega)} \leq M_3 (1 + t^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})}) e^{-\alpha t} \]
for all \( t \in (0, T) \) and \( \theta \in [q_0, \infty] \).

**Proof.** Since \( e^{\lambda t}(m_0 - \rho_0) = m_\infty \) and \( \int_{\Omega} (m_0 - \rho_0 - m_\infty) = 0 \), from the definition of \( T \) and lemma 2.1(i), we get
\[ \|(m - \rho)(\cdot, t) - m_\infty\|_{L^p(\Omega)} \leq \| e^{\lambda t}(m_0 - \rho_0)\|_{L^p(\Omega)} + \| e^{\lambda t}(m_0 - \rho_0) - e^{\lambda t} m_\infty\|_{L^p(\Omega)} \]
\[ \leq e^{\lambda t} \| e^{\lambda t}(m_0 - \rho_0)\|_{L^p(\Omega)} + C_1 (1 + t^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})}) \| m_0 - m_\infty\|_{L^p(\Omega)} e^{-\alpha t} \]
\[ \leq (1 + C_1 + C_1 |\Omega|^{-\frac{1}{p'}}) (1 + t^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})}) e^{-\alpha t} \]
for all \( t \in (0, T) \) and \( \theta \in [q_0, \infty] \). This lemma is proved for \( M_3 = 1 + C_1 + C_1 |\Omega|^{-\frac{1}{p'}} \). \( \square \)

**Lemma 3.18.** Under the assumptions of proposition 3.2, we have
\[ \|(m\cdot, t) - m_\infty\|_{L^p(\Omega)} \leq M_4 (1 + t^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})}) e^{-\alpha t} \]
for all \( t \in (0, T) \), \( \theta \in [q_0, \infty] \).

**Proof.** From lemma 3.17 and the definition of \( T \), it follows that
\[ \|(m\cdot, t) - m_\infty\|_{L^p(\Omega)} \leq \| (m - \rho - m_\infty)(\cdot, t)\|_{L^p(\Omega)} + \| \rho(\cdot, t)\|_{L^p(\Omega)} \]
\[ \leq (M_3 + M_1) (1 + t^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})}) e^{-\alpha t}. \]
The lemma is proved for \( M_4 = M_3 + M_1 \). \( \square \)

**Lemma 3.19.** Under the assumptions of proposition 3.2, there exists \( M_5 > 0 \) such that
\[ \| u(\cdot, t)\|_{L^p(\Omega)} \leq M_5 (1 + t^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})}) e^{-\alpha t} \]
for all \( t \in (0, T) \).

**Proof.** For any given \( \alpha_3 < \lambda'_1 \), we can fix \( \mu \in (\alpha_3, \lambda'_1) \). By (3.91), lemma 2.2, 2.3 and \( P(\nabla \phi) = 0 \), we obtain that
\[ \| u(\cdot, t)\|_{L^p(\Omega)} \leq C_0 e^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})} \| u_0\|_{L^p(\Omega)} + \int_0^t e^{-(t-s)\lambda} P((\rho + m)\nabla \phi)(\cdot, s)\|_{L^p(\Omega)} ds \]
\[ \leq C_0 e^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})} \| u_0\|_{L^p(\Omega)} + C_0 C_9 \int_0^t e^{-\mu(t-s)} \| \rho(\cdot, s)\|_{L^p(\Omega)}\| \nabla \phi\|_{L^\infty(\Omega)} ds \]
\[ \leq C_0 e^{-\frac{\varepsilon}{2}(\frac{1}{p} - \frac{1}{2})} \| u_0\|_{L^p(\Omega)} + C_0 C_9 \| \nabla \phi\|_{L^\infty(\Omega)} \int_0^t e^{-\mu(t-s)} \| \rho(\cdot, s)\|_{L^p(\Omega)} ds. \]

2839
By lemma 3.18 and the definition of $T$, we get
\[
\|\rho + m - m_\infty\|_{L^\infty(\Omega)} = \|m - m_\infty\|_{L^\infty(\Omega)} + \|\rho\|_{L^\infty(\Omega)} \leq (M_4 + M_1)e^{(1 + s^{-\frac{1}{2}}(\frac{\alpha}{\theta} - \frac{1}{\theta}))e^{-\alpha_1 t}}.
\] (3.93)

Inserting (3.93) into (3.92), and noting $\frac{\alpha}{\theta} - \frac{1}{\theta} < 1$, we have

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_6 e^{-\alpha_1 t} u_0(\Omega) + C_6 C_9 (M_4 + M_1)\|\nabla \phi\|_{L^\infty(\Omega)} \int_0^t (1 + s^{-\frac{1}{2}}(\frac{\alpha}{\theta} - \frac{1}{\theta}))e^{-\alpha_1 t} e^{-\mu_1 s} ds
\]
\[
\leq C_6 e^{-\alpha_1 t} u_0(\Omega) + C_6 C_9 C_{10} (M_4 + M_1)\|\nabla \phi\|_{L^\infty(\Omega)} (1 + t^{\min(0, 1 - \frac{1}{2}(\frac{\alpha}{\theta} - \frac{1}{\theta}))})e^{-\alpha_1 t}
\]
\[
= M_5 e^{(1 + t^{-\frac{1}{2}}(\frac{\alpha}{\theta} - \frac{1}{\theta})\alpha_1 t)}
\]

with $M_5 = C_6 + C_6 C_9 C_{10} (M_4 + M_1)\|\nabla \phi\|_{L^\infty(\Omega)}$. \hfill \suit

Lemma 3.20. Under the assumptions of proposition 3.2, we have

\[
\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq \frac{M_2}{2} \varepsilon (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t} \text{ for all } t \in (0, T).
\]

Proof. From (3.90) and lemma 2.1(iii), we have

\[
\|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq \|c^{(\Delta^{-1})}\nabla c_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla c^{(\Delta^{-1})(\Delta^{-1})} (m - u \cdot \nabla c)(\cdot, s)\|_{L^\infty(\Omega)} ds
\]
\[
\leq C_1 (1 + t^{-\frac{1}{2}}) e^{-\alpha_1 t} \|\nabla c_0\|_{L^2(\Omega)} + \int_0^t \|\nabla c^{(\Delta^{-1})(\Delta^{-1})} (m - m_\infty)(\cdot, s)\|_{L^\infty(\Omega)} ds
\]
\[
+ \int_0^t \|\nabla c^{(\Delta^{-1})} u \cdot \nabla c(\cdot, s)\|_{L^\infty(\Omega)} ds.
\] (3.94)

In the second inequality, we have used $\nabla c^{(\Delta^{-1})(\Delta^{-1})} m_\infty = 0$.

From lemmas 2.1, 2.4 and 3.18, it follows that

\[
\int_0^t \|\nabla c^{(\Delta^{-1})(\Delta^{-1})} (m - m_\infty)(\cdot, s)\|_{L^\infty(\Omega)} ds
\]
\[
\leq C_2 \int_0^t (1 + (t - s)^{-\frac{1}{2}}(\frac{\alpha}{\theta} - \frac{1}{\theta})\alpha_1 t) e^{-\alpha_1 t} ds
\]
\[
\leq C_2 M_1 \varepsilon \int_0^t (1 + (t - s)^{-\frac{1}{2}}(\frac{\alpha}{\theta} - \frac{1}{\theta})\alpha_1 t) e^{-\alpha_1 t} ds
\]
\[
\leq C_2 C_{10} M_2 \varepsilon (1 + t^{\min(0, 1 - \frac{1}{2}(\frac{\alpha}{\theta} - \frac{1}{\theta}))}) e^{-\alpha_1 t}.
\]

On the other hand, by lemmas 2.1(ii), 2.4 and the definition of $T$, we obtain
\[
\begin{align*}
\int_0^t \| \nabla e(t-s)(\Delta - m_\infty) u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
\leq C_2 \int_0^t \left[ 1 + \left( t - s \right)^{-\frac{1}{2}} \right] e^{-\lambda_1(t-s)} \| u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
\leq C_2 \int_0^t \left[ 1 + \left( t - s \right)^{-\frac{1}{2}} \right] e^{-\lambda_1(t-s)} \| u(\cdot, s) \|_{L^\infty(\Omega)} \| \nabla c(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
\leq C_2 M_\varepsilon^2 \int_0^t \left[ 1 + \left( t - s \right)^{-\frac{1}{2}} \right] e^{-\lambda_1(t-s)} \left( 1 + s^{-\frac{1}{2}} \right) e^{-\left( \alpha_1 + \alpha_2 \right)t} \, ds \\
\leq C_2 M_\varepsilon^2 \int_0^t e^{-\lambda_1(t-s)} e^{-\left( \alpha_1 + \alpha_2 \right)t} \left( 1 + \left( t - s \right)^{-\frac{1}{2}} \right) e^{-\left( \alpha_1 + \alpha_2 \right)t} \, ds \\
\leq C_2 M_\varepsilon^2 \int_0^t e^{-\left( \alpha_1 + \alpha_2 \right)t} \left( 1 + \left( t - s \right)^{-\frac{1}{2}} \right) e^{-\left( \alpha_1 + \alpha_2 \right)t} \, ds \\
\leq C_2 M_\varepsilon^2 \int_0^t e^{-\left( \alpha_1 + \alpha_2 \right)t} \, ds.
\end{align*}
\]

Hence combining above inequalities with (3.80) and (3.81), we arrive at the conclusion. \(\square\)

**Lemma 3.21.** Under the assumptions of proposition 3.2, we have
\[
\| \rho(\cdot, t) \|_{L^\infty(\Omega)} \leq \frac{M_1}{2} \varepsilon \left( 1 + \left( t - \frac{\varepsilon}{2} \right)^{-\frac{1}{2}} \right) e^{-\alpha_1 t} \quad \text{for all } t \in (0, T), \theta \in [\theta_0, \infty].
\]

**Proof.** By the variation-of-constants formula, we have
\[
\rho(\cdot, t) = e^{t \left( \Delta - m_\infty \right)} \rho_0 - \int_0^t e^{(t-s) \left( \Delta - m_\infty \right)} \left( \nabla \cdot \left( \rho S(\cdot, \rho, c) \nabla c \right) - u \cdot \nabla \rho \right)(\cdot, s) \, ds \\
+ \int_0^t e^{(t-s) \left( \Delta - m_\infty \right)} \rho(m_\infty - m)(\cdot, s) \, ds.
\]

By lemma 2.1, the result in section 2 of [15] and \(\alpha_1 < \min \{ \lambda_1, m_\infty \} \), we obtain
\[
\| \rho(\cdot, t) \|_{L^\infty(\Omega)} \\
\leq e^{-m_\infty t} \left( \left\| \rho_0 - \bar{\rho}_0 \right\|_{L^\infty(\Omega)} + \left\| \bar{\rho}_0 \right\|_{L^\infty(\Omega)} \right) + \int_0^t \left\| e^{(t-s) \left( \Delta - m_\infty \right)} \nabla \cdot \left( \rho S(\cdot, \rho, c) \nabla c \right) - u \cdot \nabla \rho \right)(\cdot, s) \, ds \\
+ \int_0^t \left\| e^{(t-s) \left( \Delta - m_\infty \right)} \rho(m_\infty - m)(\cdot, s) \right\|_{L^\infty(\Omega)} \, ds \\
\leq C_1 \left( 1 + t^{-\frac{1}{2}} \right) e^{-\left( \lambda_1 + m_\infty \right)t} \| \rho_0 - \bar{\rho}_0 \|_{L^\infty(\Omega)} + \left( \min \{ 1, |\Omega| \} \right)^{\frac{1}{2}} e^{-m_\infty t} \varepsilon \\
+ C_4 \int_0^t \left[ 1 + \left( t - s \right)^{-\frac{1}{2}} \right] e^{-\left( \lambda_1 + m_\infty \right)(t-s)} \| \rho \|_{L^\infty(\Omega)} \| \nabla c \|_{L^\infty(\Omega)} \, ds \\
+ \int_0^t \left\| e^{(t-s) \left( \Delta - m_\infty \right)} \nabla \cdot (\rho u)(\cdot, s) \right\|_{L^\infty(\Omega)} \, ds \\
+ \int_0^t \left\| e^{(t-s) \left( \Delta - m_\infty \right)} \rho(m_\infty - m)(\cdot, s) \right\|_{L^\infty(\Omega)} \, ds \\
\leq \left( 2C_1 + \left( \min \{ 1, |\Omega| \} \right)^{-\frac{1}{2}} \right) \left( 1 + \frac{t^{-\frac{1}{2}}}{\left( \lambda_1 + m_\infty \right)^{-1}} \right) e^{-\alpha_1 t} \\
+ C_4 \int_0^t \left( 1 + \left( t - s \right)^{-\frac{1}{2}} \right) e^{-\left( \lambda_1 + m_\infty \right)(t-s)} \| \rho \|_{L^\infty(\Omega)} \| \nabla c \|_{L^\infty(\Omega)} \, ds \\
+ C_4 \int_0^t \left( 1 + \left( t - s \right)^{-\frac{1}{2}} \right) e^{-\left( \lambda_1 + m_\infty \right)(t-s)} \| \rho \|_{L^\infty(\Omega)} \| \nabla c \|_{L^\infty(\Omega)} \, ds \\
+ C_4 \int_0^t \left( 1 + \left( t - s \right)^{-\frac{1}{2}} \right) e^{-\left( \lambda_1 + m_\infty \right)(t-s)} \| \rho \|_{L^\infty(\Omega)} \| m - m_\infty \|_{L^\infty(\Omega)} \, ds \\
= \left( 2C_1 + \left( \min \{ 1, |\Omega| \} \right)^{-\frac{1}{2}} \right) \left( 1 + t^{-\frac{1}{2}} \right) e^{-\alpha_1 t} + I_1 + I_2 + I_3.
\]
By the definition of $T$, lemmas 3.20, 2.4 and (3.82), we get

$$I_1 \leq 3C_2C_3M_1M_2\varepsilon^2 \int_0^t (1 + (t - s)^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\lambda_1(t-s)} e^{-2\alpha_1(t)} (1 + s^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) ds$$

$$\leq 3C_2C_3M_1M_2\varepsilon^2 (1 + t^{\min(0, -\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}}))}) e^{-\min\{\lambda_1, 2\alpha_1\} t}$$

$$\leq \frac{M_1}{8} \varepsilon (1 + t^{-\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\alpha_1 t}.$$

Similarly, by (3.84) and (3.85), we can also get

$$I_2 \leq 3C_2C_1M_1M_2\varepsilon^2 \int_0^t (1 + (t - s)^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\lambda_1(t-s)} e^{-2\alpha_1(t)} (1 + s^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) ds$$

$$\leq 3C_2C_1M_1M_2\varepsilon^2 (1 + t^{\min(0, -\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}}))}) e^{-\min\{\lambda_1, 2\alpha_1\} t}$$

$$\leq \frac{M_1}{8} \varepsilon (1 + t^{-\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\alpha_1 t},$$

$$I_3 \leq 3C_2C_1M_1M_2\varepsilon^2 \int_0^t (1 + (t - s)^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\min(t-s)} e^{-2\alpha_1(t)} (1 + s^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) ds$$

$$\leq 3C_2C_1M_1M_2\varepsilon^2 (1 + t^{\min(0, -\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}}))}) e^{-\min\{t-s, 2\alpha_1\} t}$$

$$\leq \frac{M_1}{8} \varepsilon (1 + t^{-\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\alpha_1 t},$$

respectively, where the fact that $q_0 \in (N, \frac{Np_0}{\max(0, N-p_0)})$ warrants $-\frac{N}{p_0} + \frac{N}{\min(p_0)} > -1$ is used. Hence the combination of the above inequalities yields $\|\rho(t)\|_{L^p(\Omega)} \leq \frac{M_1}{8} \varepsilon (1 + t^{-\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\alpha_1 t}$.

\[\square\]

**Lemma 3.22.** Under the assumptions of proposition 3.2, we have

$$\|(m - \rho)(\cdot, t) - e^{\Delta}(m_0 - \rho_0)\|_{L^p(\Omega)} \leq \frac{\varepsilon}{2} (1 + t^{-\frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\alpha_1 t} \text{ for } \theta \in [q_0, \infty), t \in (0, T).$$

**Proof.** From (3.88) and lemma 2.1(iv), it follows that

$$\|(m - \rho)(\cdot, t) - e^{\Delta}(m_0 - \rho_0)\|_{L^p(\Omega)}$$

$$\leq \int_0^t \|e^{\alpha_1\Delta}(\nabla \cdot (\rho S(\cdot, \rho, c)) - u \cdot \nabla (m - \rho)))(\cdot, s)\|_{L^p(\Omega)} ds$$

$$\leq \int_0^t \|e^{\alpha_1\Delta} \nabla \cdot (\rho S(\cdot, \rho, c)))(\cdot, s)\|_{L^p(\Omega)} ds + \int_0^t \|e^{\alpha_1\Delta} \nabla \cdot ((m - \rho - m_\infty)u)(\cdot, s)\|_{L^p(\Omega)} ds$$

$$\leq C_4 C_5 \int_0^t (1 + (t-s)^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\alpha_1(t-s)} \|\rho(\cdot, s)\|_{L^p(\Omega)} \|\nabla c(\cdot, s)\|_{L^\infty(\Omega)} ds$$

$$+ C_4 \int_0^t (1 + (t-s)^{-\frac{1}{4} - \frac{3}{8} (\frac{1}{\theta} + \frac{1}{\hat{\theta}})}) e^{-\alpha_1(t-s)} \|u(m - \rho - m_\infty)(\cdot, s)\|_{L^p(\Omega)} ds$$

$$= I_1 + I_2.$$
From the definition of $T$ and (3.86), we have
\[
I_1 \leq C_4 C_5 M_1 M_2 e^{2} \int_0^t (1 + (t - s)^{-\frac{1}{2}} (s^2 + 1)) e^{-\lambda_1(t-s)} (1 + s^{-\frac{1}{2}} (s^2 + 1)) e^{-2\alpha_{1}t} ds
\]
\[
\leq 3 C_4 C_5 M_1 M_2 e^{2} (1 + t^{\min(0, \frac{1}{2} (s^2 + 1))}) e^{-\min(\lambda_1, \lambda_1) t_{\infty}} e^{-\alpha_{1}t} ds
\]
\[
\leq \frac{e}{4} (1 + t^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_1 t}.
\]

On the other hand, from lemmas 3.17, 3.19 and (3.87), it follows that
\[
I_2 = C_4 \int_0^t (1 + (t - s)^{-\frac{1}{2}} (s^2 + 1)) e^{-\lambda_1(t-s)} \|m - \rho - m_{\infty}\|_{L^{\infty}(\Omega)} \|u\|_{L^{\infty}(\Omega)} ds
\]
\[
\leq 2 C_4 C_5 M_1 M_2 e^{2} \int_0^t (1 + (t - s)^{-\frac{1}{2}} (s^2 + 1)) e^{-\lambda_1(t-s)} (1 + s^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_{1}t (1 + s^{-\frac{1}{2}} (s^2 + 1))} e^{-2\alpha_{1}t} ds
\]
\[
\leq 6 C_4 C_5 M_1 M_2 e^{2} (1 + t^{\min(0, \frac{1}{2} (s^2 + 1))}) e^{-\lambda_1(t-s)} e^{-\alpha_{1}t} (1 + t^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_{1}t} ds
\]
\[
\leq \frac{e}{4} (1 + t^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_1 t}.
\]

Combining the above inequalities, we arrive at \(\|(\rho - m)(\cdot, t) - e^{\Delta_2}(\rho_0 - m_0)\|_{L^{p}(\Omega)} \leq \frac{e}{4} (1 + t^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_1 t}\), and thus complete the proof of this lemma.

By the above lemmas, we can claim that $T = T_{\max}$. Indeed, if $T < T_{\max}$, by lemmas 3.22, 3.21 and 3.20, we have
\[
\|m - \rho\|_{L^{p}(\Omega)} \leq \frac{e}{4} (1 + t^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_{1}t},
\]
\[
\|\rho_0\|_{L^{p}(\Omega)} \leq \frac{e}{4} (1 + t^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_{1}t},
\]
\[
\|\nabla c\|_{L^{p}(\Omega)} \leq \frac{e}{4} (1 + t^{-\frac{1}{2}} (s^2 + 1)) e^{-\alpha_{1}t},
\]
for all $\theta \in [\theta_0, \infty]$ and $t \in (0, T)$, which contradict the definition of $T$ in (3.87). Next, the further estimates of solutions are established to ensure $T_{\max} = \infty$.

**Lemma 3.23.** Under the assumptions of proposition 3.2, there exists $M_6 > 0$ such that
\[
\|A^{\delta} u(\cdot, t)\|_{L^{2}(\Omega)} \leq e M_6 e^{-\alpha_{1} t} \text{ for } t \in (t_0, T_{\max}) \text{ with } t_0 = \min\left\{\frac{T_{\max}}{6}, 1\right\}.
\]

**Proof.** For any given $\alpha_2 < \lambda_1$, we can fix $\mu \in (\alpha_2, \lambda_1)$. From (3.91), it follows that
\[
\|A^{\delta} u(\cdot, t)\|_{L^{2}(\Omega)} \leq \|A^{\delta} e^{-\Delta_0} u_0\|_{L^{2}(\Omega)} + \int_0^t \|A^{\delta} e^{-(\cdot-s)A_P} ((\rho + m - m_{\infty}) \nabla \phi)(\cdot, s))\|_{L^{2}(\Omega)} ds.
\]
In the first integral, we apply lemma 2.2, which gives
\[
\|A^{\delta} e^{-\Delta_0} u_0\|_{L^{2}(\Omega)} \leq C_3 |\Omega| \int_0^{\min(\lambda_1, \lambda_{\infty})} t^{-\beta} e^{\alpha_{1}t} \|u_0\|_{L^{p}(\Omega)} \leq C_3 |\Omega| \int_0^{\min(\lambda_1, \lambda_{\infty})} t^{-\beta} e^{\alpha_{1}t} \varepsilon.
\]
for all $t \in (0, T)$. Next by lemmas 2.3, 3.17 and 3.21, we have

$$
\int_0^t \| A^{\alpha} e^{-(t-s)A} \mathcal{P}((\rho + m - m_{\infty}) \nabla \phi)(\cdot, s) \|_{L^2(\Omega)} \, ds \\
\leq C_5 C_5 \| \nabla \phi \|_{L^2(\Omega)} \| \Omega \|^\frac{\alpha - 2}{\alpha} \int_0^t e^{-\mu(t-s)}(t-s)^{-\beta} \| (m(\cdot, s) - \rho(\cdot, s) - m_{\infty}) \|_{L^2(\Omega)} + 2 \| \rho(\cdot, s) \|_{L^2(\Omega)} \| \omega \|_{L^2(\Omega)} \, ds \\
\leq M_6 \int_0^t e^{-\mu(t-s)}(t-s)^{-\beta}(1 + s^{-\frac{2}{\alpha}} + \frac{t}{t-s}) e^{-\alpha \beta s} \, ds \\
\leq M_6 C_10 (1 + t^{-1}) e^{-\alpha \beta s},
$$

where $M_6 = (M_3 + M_1)C_5 C_5 \| \nabla \phi \|_{L^2(\Omega)} \| \Omega \|^\frac{\alpha - 2}{\alpha}$. Therefore there exists $M_6 > 0$ such that

$$
\| A^{\alpha} u(\cdot, t) \|_{L^2(\Omega)} \leq C_6 e^{-\alpha \beta t} \text{ for } t \in (t_0, T_{\max}).
$$

Lemma 3.24. Under the assumptions of proposition 3.2, there exists $M_7 > 0$ such that

$$
\| c(\cdot, t) - m_{\infty} \|_{L^\infty(\Omega)} \leq M_7 e^{-\alpha \beta t} \text{ for all } (t_0, T_{\max}) \text{ with } t_0 = \min \{ \frac{T_{\max}}{6}, 1 \}.
$$

Proof. From (3.90) and lemma 2.1, we have

$$
\| (c - m_{\infty})(\cdot, t) \|_{L^\infty(\Omega)} \leq C_1 e^{-\beta t} \| c_0 - m_{\infty} \|_{L^\infty(\Omega)} + \int_0^t \| e^{(t-s)(\Delta - 1)} (m - m_{\infty})(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
+ \int_0^t \| e^{(t-s)(\Delta - 1)} u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} \, ds. \tag{3.97}
$$

By lemmas 2.4, 3.18, we obtain

$$
\int_0^t \| e^{(t-s)(\Delta - 1)} (m - m_{\infty})(\cdot, s) \|_{L^\infty(\Omega)} \, ds \leq C_1 \int_0^t (1 + (t-s)^{-\frac{2}{\alpha}}) e^{-\beta (t-s)} \| (m - m_{\infty})(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
\leq C_1 C_{10} M_4 e^{-\alpha \beta t}. \tag{3.98}
$$

On the other hand, by lemmas 2.4, 3.19 and 3.20, we get

$$
\int_0^t \| e^{(t-s)(\Delta - 1)} u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} \, ds \leq C_1 \int_0^t (1 + (t-s)^{-\frac{2}{\alpha}}) e^{-\beta (t-s)} \| u \cdot \nabla c(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
\leq C_1 \int_0^t (1 + (t-s)^{-\frac{2}{\alpha}}) e^{-\beta (t-s)} \| u(\cdot, s) \|_{L^\infty(\Omega)} \| \nabla c(\cdot, s) \|_{L^\infty(\Omega)} \, ds \\
\leq 6C_1 M_3 M_2 C_10 e^{2\alpha \beta t} e^{-\alpha \beta t}. \tag{3.99}
$$

Therefore combining the above equalities, we arrive at the desired result. 

Proof of theorem 1.3 in the case $S = 0$ on $\partial \Omega$, part 2 (proposition 3.2). We now come to the final step to show that $T_{\max} = \infty$. According to the extensibility criterion in lemma 2.5, it remains to show that there exists $C > 0$ such that for $t_0 := \min \{ \frac{T_{\max}}{6}, 1 \} < t < T_{\max}$

$$
\| \rho(\cdot, t) \|_{L^\infty(\Omega)} + \| m(\cdot, t) \|_{L^\infty(\Omega)} + \| c(\cdot, t) \|_{W^{1, \infty}(\Omega)} + \| A^{\alpha} u(\cdot, t) \|_{L^2(\Omega)} < C.
$$

From lemmas 3.18 and 3.21, there exists $K_i > 0$, $i = 1, 2, 3$, such that
\[\|m(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t}, \quad \|\rho(\cdot, t)\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t}, \quad \|\nabla c(\cdot, t)\|_{L^\infty(\Omega)} \leq K_3 e^{-\alpha_1 t}\]

for \(t \in (t_0, T_{\text{max}})\). Furthermore, lemma 3.24 implies that \(\|c(\cdot, t) - m_\infty\|_{W^{1, \infty}(\Omega)} \leq K'_3 e^{-\alpha_2 t}\) with some \(K'_3 > 0\) for all \(t \in (t_0, T_{\text{max}})\). Since \(D(A^\beta) \hookrightarrow L^\infty(\Omega)\) with \(\beta \in \left(\frac{3}{2}, 1\right)\), it follows from lemma 3.23 that \(\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_1 t}\) for some \(K_4 > 0\) for all \(t \in [t_0, T_{\text{max}}]\). This completes the proof of proposition 3.2.

Before we move to the next section, we remark that the following result is also valid by suitably adjusting \(\varepsilon > 0\) for the larger values of \(p_0\) or \(q_0\).

**Corollary 3.1.** Let \(N = 3\) and \(\int_\Omega \rho_0 \neq \int_\Omega m_0\). Further, let \(p_0 \in (\frac{N}{2}, \infty), q_0 \in (N, \infty)\) if \(\int_\Omega \rho_0 > \int_\Omega m_0\), and \(p_0 \in (\frac{N}{2}, \infty), q_0 \in (N, \infty)\) if \(\int_\Omega \rho_0 < \int_\Omega m_0\). There exists \(\varepsilon > 0\) such that for any initial data \((\rho_0, m_0, c_0, u_0)\) fulfilling (1.7) as well as

\[
\|\rho_0 - \rho_\infty\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \|m_0 - m_\infty\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \|\nabla c_0\|_{L^\infty(\Omega)} \leq \varepsilon, \quad \|u_0\|_{L^p(\Omega)} \leq \varepsilon,
\]

(1.1) admits a global classical solution \((\rho, m, c, u, P)\). Moreover, for any \(\alpha_1 \in (0, \min\{\lambda_1, \lambda_\infty + \rho_\infty\})\), \(\alpha_2 \in (0, \min\{\alpha_1, \lambda'_1, \alpha'_1\})\), there exist constants \(K_i, i = 1, 2, 3, 4, \) such that for all \(t \geq 1\)

\[
\|m(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t}, \quad \|\rho(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t},
\]

\[
\|c(\cdot, t) - m_\infty\|_{W^{1, \infty}(\Omega)} \leq K_3 e^{-\alpha_2 t}, \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_1 t}.
\]

**4. Proof of main results for general \(S\)**

In this section, we give the proof of our results for the general matrix-valued \(S\). This is accomplished by an approximation procedure. In order to make the previous results applicable, we introduce a family of smooth functions \(\rho_\eta \in C_0^\infty(\Omega)\) and \(0 \leq \rho_\eta(x) \leq 1\) for \(\eta \in (0, 1)\), \(\lim_{\eta \to 0} \rho_\eta(x) = 1\) and let \(S_\eta(x, \rho, c) = \rho_\eta(x)S(x, \rho, c)\). Using this definition, we regularize (1.1) as follows

\[
\begin{aligned}
&\left(\rho_\eta\right)_t + u_\eta \cdot \nabla \rho_\eta = \Delta \rho_\eta - \nabla \cdot (\rho_\eta S_\eta(x, \rho, c)) \nabla c_\eta - \rho_\eta m_\eta, \\
&(m_\eta)_t + u_\eta \cdot \nabla m_\eta = \Delta m_\eta - \rho_\eta m_\eta, \\
&(c_\eta)_t + u_\eta \cdot \nabla c_\eta = \Delta c_\eta - c_\eta + m_\eta, \\
&(u_\eta)_t = \Delta u_\eta - \nabla P_\eta + (\rho_\eta + m_\eta) \nabla \varphi, \quad \nabla \cdot u_\eta = 0,
\end{aligned}
\]

where

\[
\begin{aligned}
\frac{\partial u_\eta}{\partial \nu} = \frac{\partial m_\eta}{\partial \nu} = \frac{\partial c_\eta}{\partial \nu} = 0, \quad u_\eta = 0.
\end{aligned}
\]

with the initial data

\[
\rho_\eta(x, 0) = \rho_0(x), \quad m_\eta(x, 0) = m_0(x), \quad c(x, 0) = c_0(x), \quad \text{and} \quad u_\eta(x, 0) = u_0(x), \quad x \in \Omega.
\]

It is observed that \(S_\eta\) satisfies the additional condition \(S = 0\) on \(\partial \Omega\). Therefore based on the discussion in section 3, under the assumptions of theorems 1.1 and 1.3, the problem (4.1)–(4.2) admits a global classical solution \((\rho_\eta, m_\eta, c_\eta, u_\eta, P_\eta)\) that satisfies

\[
\|m_\eta(\cdot, t) - m_\infty\|_{L^\infty(\Omega)} \leq K_1 e^{-\alpha_1 t}, \quad \|\rho_\eta(\cdot, t) - \rho_\infty\|_{L^\infty(\Omega)} \leq K_2 e^{-\alpha_1 t},
\]

\[
\|c_\eta(\cdot, t) - m_\infty\|_{W^{1, \infty}(\Omega)} \leq K_3 e^{-\alpha_2 t}, \quad \|u_\eta(\cdot, t)\|_{L^\infty(\Omega)} \leq K_4 e^{-\alpha_1 t}.
\]

for some constants \(K_i, i = 1, 2, 3, 4, \) and \(t \geq 0\). Applying a standard procedure such as in lemmas 5.2 and 5.6 of [3], one can obtain a subsequence of \(\{\eta_j\}_{j \in \mathbb{N}}\) with \(\eta_j \to 0\) as \(j \to \infty\) such that \(\rho_j \to \rho, \quad m_j \to m, \quad c_j \to c, \quad u_j \to u\) in \(C^0_{\text{loc}}(\Omega) \times (0, \infty)\) as \(j \to \infty\) for some \(\vartheta \in (0, 1)\). Moreover, by the arguments as in lemmas 5.7 and 5.8 of [3], one can also show that
\((\rho, m, c, u, P)\) is a classical solution of (1.1) with the decay properties asserted in theorems 1.2 and 1.3. The proofs of theorems 1.1–1.3 are thus complete.

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