Derivative Expansion of the Effective Action for QED in 2+1 and 3+1 dimensions

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The derivative expansion of the one-loop effective action in QED$^3$ and QED$^4$ is considered. The first term in such an expansion is the effective action for a constant electromagnetic field. An explicit expression for the next term containing two derivatives of the field strength $F_{\mu\nu}$, but exact in the magnitude of the field strength, is obtained. The general results for both fermion and scalar electrodynamics are presented. The cases of pure electric and pure magnetic external fields are considered in detail. The Feynman technique for the perturbative expansion of the one-loop effective action in the number of derivatives is developed.

I. INTRODUCTION

Quantum electrodynamics is known to be the best studied example of quantum field theory. Mainly, this is due to the weakness of the fine structure (coupling) constant, $\alpha \approx 1/137$, which allows to perform many perturbative calculations as power series in $\alpha$ with an incredibly high accuracy. Despite the smallness of $\alpha$, even in the realm of quantum electrodynamics, there are some questions that theory has not answered yet. In this paper, in particular, we address the problem of derivation of the low-energy effective action which at present is solved only partially for QED.

The low-energy effective action in quantum electrodynamics describes the dynamics of the electromagnetic field, assuming that the production of the on shell fermions is absent or negligible. Apparently, such a description is self-consistent only if the fermions are massive and the characteristic photon energies are sufficiently small. The mentioned two conditions, as is clear, are necessary to suppress the process of the particle-antiparticle pair creation (on-shell).

Intuitively, the low-energy effective theory is obtained from quantum electrodymanics by “integrating out” the fermion field. After doing so, one arrives at a nonlinear theory that involves only the electromagnetic field degrees of freedom. In terms of the S-matrix language, one considers just those processes in QED which contain only photons among the asymptotic scattering states. The fermions, on the other hand, appear only through the internal loops by producing all kinds of photon vertices.

The problem of deriving the effective action is an old one. Its roots go back to the well known papers of Heisenberg and Euler $^{[1]}$, and Weisskopf $^{[2]}$. There, for the first time, the effective action in QED (for the case of a constant electromagnetic field) was derived. From the viewpoint of application, the derived effective action contains, for example, the information on the photon-photon scattering at the tree level. It was this scattering process, in fact, that motivated consideration of the problem in Ref. $^{[1,2]}$, in the first place. Later, some further progress was achieved by Schwinger $^{[3]}$ who, by using the proper time technique, rederived the result of Refs. $^{[1,2]}$ and, in addition, gave a nice interpretation to the imaginary part of the effective action in the case of a constant electric field.

Obviously, the next most natural step in deriving the low-energy effective action in QED would be to take into account the effect of small deviations from the constant configuration of the field. In other words, the problem is to obtain the effective action as an expansion in powers of derivatives of the field strength. It turns out, however, that the latter is very difficult to accomplish (see $^{[4,5]}$ for some early attempts in this direction) unless the weak field approximation is used. In this connection it is appropriate to mention that, in the weak field limit, the expansion is known up to four derivatives with respect to the field strength $^{[6]}$. Our approach, on the other hand, does not involve any assumptions about the weakness of the background field.

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A real progress in solving the problem started with the result of Ref. [3] where an elaborated method, which, in principle, leads to a general result for the derivative expansion in QED, was presented. Because of the complicated character of the method, however, the explicit expression applicable to the most general case of the electromagnetic field background was not presented there. Recently, the derivative expansion of the effective action was obtained in the case of (2 + 1)-dimensional QED [8]. This latter is a quite general result, containing all the terms quadratic in derivatives of the field strength with respect to the space-time coordinates. Finally, in our previous paper [9], we obtained a similar result for the effective action but in (3 + 1)-dimensional QED. As in (2 + 1)-dimensional case, it was given in a covariant form valid for the most general constant component of the electromagnetic field background what, as we will see later, is much more complicated problem than that in 2+1 dimensions.

For completeness, we mention that some related interesting results were obtained in Ref. [10] for QED and in Refs. [11,12] for non-Abelian gauge theories.

In this paper we extend our method, which was originally presented for the case of (3 + 1)-dimensional QED [3], to QED in 2 + 1 dimensions. In particular, we obtain the derivative expansion of the effective action which includes up to two space-time derivatives of the electromagnetic field and, further, we formulate the Feynman rules for the perturbative expansion of the one-loop effective action in the number of derivatives. We also derive the explicit expressions for the derivative corrections to the imaginary part of the effective action in an external electric field. And finally, as a byproduct, we resolve the controversy posed in [13] where a result different from that of [8] was presented.

The paper is organized as follows. In Sec. II we outline the general method developed in our previous paper [9]. Sec. II is devoted to solving some technical problems in dealing with functions of the matrix argument $F_{\mu \nu}$. Then, in Secs. IV and VII, we present the main results of our paper, namely, the derivative expansions for spinor and scalar QED, respectively. In Secs. IV and Sec. VII we calculate the derivative expansions for two particular cases of the external electromagnetic field, the purely magnetic and purely electric backgrounds, in both 2 + 1 and 3 + 1 dimensions. Finally, in Sec. VIII, we develop the Feynman diagram technique for generating the perturbative expansion in the number of derivatives. Four appendices contain different formulas used throughout the main text.

II. DERIVATIVE EXPANSION OF THE ONE-LOOP EFFECTIVE ACTION IN QED

Let us start from the general formalism which was originally developed in [3] for (3 + 1)-dimensional quantum electrodynamics. While doing so, we will notice that, to a great extent, the method does not depend on the dimension of the space-time. We will pay special attention to all those places where it does depend.

In this paper we restrict ourselves to the one-loop effective action. This is the same approximation which was used by Schwinger [3] in the case of a constant external electromagnetic field.

As is known, the one-loop effective action in QED reduces to computing the fermion determinant

$$W^{(1)}(A) = \int d^n x \mathcal{L}^{(1)} = -i \ln \text{Det}(i\hat{D} - m) = -\frac{i}{2} \ln \text{Det} \left( \hat{D}^2 + \frac{e}{2} \sigma_{\mu \nu} F^{\mu \nu} + m^2 \right) =$$

$$= -\frac{i}{2} \int d^n x |x| tr \ln \left( \hat{D}^2 + \frac{e}{2} \sigma_{\mu \nu} F^{\mu \nu} + m^2 \right) |x|. \quad (1)$$

Here $\hat{D} = \gamma^\mu D_\mu$ and the covariant derivative is $D_\mu = \partial_\mu + i e A_\mu$. By definition, $\sigma_{\mu \nu} = i [\gamma_\mu, \gamma_\nu]/2$ and $tr$ refers to the spinor indices of the Dirac matrices $\gamma_\mu$. States $|x\rangle$ are the eigenstates of a self-conjugate coordinate operator $x_\mu$. Throughout the paper we use the Minkowski metric, i.e., $\eta_{\mu \nu} = (1, -1, -1, -1)$ or $\eta_{\mu \nu} = (1, -1, -1, -1)$, depending on the actual space-time dimension. And in both 2 + 1 and 3 + 1 dimensions, we work with the 4 × 4 representation of the Dirac $\gamma$-matrices.

For calculating the effective action in Eq. (1), we employ a version of the so-called worldline (or string-inspired) formalism developed in [14-16]. Such an approach to an ordinary field theory, based on the path integral over one-dimensional world lines, was extended to the evaluation of Feynman diagrams for Green functions in higher loop orders [17-19]. It has demonstrated its power reproducing known theoretical results in QED while allowing one to invoke new technique to study the theory’s behavior in strong coupling regime [20]. For some recent applications of the worldline formalism as well as for an extensive list of references see [21,22]. Note, however, that our method differs from the one commonly used in the literature by a choice of the worldline propagators, and is closer in spirit to the method used in [17-22].

With use of the formal identity $\ln(H + m^2) = -\int_0^\infty \exp[-i\tau(H + m^2)]d\tau/\tau$ for introducing the proper-time coordinate $\tau$, the effective Lagrangian can be represented through the diagonal matrix elements of the operator $U(\tau) = \exp(-i\tau H)$,
where the second order differential operator $H$ is given by

$$H = -\Pi_\mu \Pi^\mu + \frac{e}{2} \sigma_{\mu\nu} F^{\mu\nu}(x), \quad \Pi_\mu = -i D_\mu.$$  

(3)

The matrix elements $\langle x | \exp(-i\tau H) | x \rangle$ entering the right hand side of Eq. (2) may be interpreted as the matrix elements of the evolution operator of a spinning particle with $\tau$ and $H$ being the proper time and the Hamiltonian of the particle. The corresponding canonical momenta are $P_\mu$’s which obey the commutation relations $[x_\mu, P_\nu] = i\delta^\nu_\mu$ and are defined by $\langle x | P_\mu | y \rangle = -i\partial_\mu \delta(x - y)$ in coordinate representation. Following the standard approach [24], we represent the transition amplitude $\langle z | U(\tau) | y \rangle$ between points $x(0) = y$ and $x(\tau) = z$ in terms of a path integral over the real and Grassmann coordinates, $x(\tau)$ and $\psi(\tau)$, as

$$tr\langle z | U(\tau) | y \rangle = N^{-1} \int \mathcal{D}[x(\tau), \psi(\tau)] \exp \left\{ i \int_0^\tau dt [L_{bos}(x(t)) + L_{fer}(\psi(t), x(t))] \right\},$$  \n
(4)

where $N$ is a normalization factor, and

$$L_{bos}(x) = -\frac{1}{4} \frac{dx_\nu}{dt} \frac{dx^{\nu'}}{dt} - eA_\nu(x) \frac{dx^{\nu'}}{dt},$$  \n
(5)

$$L_{fer}(\psi, x) = \frac{i}{2} \psi_\nu \frac{d\psi^{\nu'}}{dt} - ie\psi^{\nu'} \psi^\lambda F_{\nu\lambda}(x).$$  \n
(6)

The integration in Eq. (4) goes over trajectories $x^\mu(t)$ and $\psi^\nu(t)$ parameterized by $t \in [0, \tau]$. The definition of the integration measure assumes the following boundary conditions

$$x(0) = y, \quad x(\tau) = z, \quad \psi(0) = -\psi(\tau).$$  \n
(7)

We choose a special gauge condition for the vector potential $A_\mu(x)$, namely the Fock-Schwinger gauge [25]

$$(x^\nu - y^\nu)A_\nu(x) = 0,$$  \n
(8)

which leads to the series

$$A_\nu(x) = \frac{1}{2} (x^\lambda - y^\lambda) F_{\lambda\nu}(y) + \frac{1}{3} (x^\lambda - y^\lambda)(x^\sigma - y^\sigma) \partial_\sigma F_{\lambda\nu}(y)$$

$$+ \frac{1}{8} (x^\lambda - y^\lambda)(x^\sigma - y^\sigma)(x^{\nu'} - y^{\nu'}) \partial_\sigma \partial_{\nu'} F_{\lambda\nu}(y) + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(x^\lambda - y^\lambda)(x^{\nu_1} - y^{\nu_1}) \ldots (x^{\nu_n} - y^{\nu_n})}{n!(n + 2)} \partial_{\nu_1} \partial_{\nu_2} \ldots \partial_{\nu_n} F_{\lambda\nu}(y).$$  \n
(9)

This choice of the gauge for the vector potential turns out to be very convenient for developing a perturbative theory in the number of the derivatives of the electromagnetic field with respect to the space-time coordinates.

Carrying out the change of the variable $x(t)$ for $x'(t) = x(t) - y$ in the path integral in Eq. (4) (henceforth we omit the prime) and substituting Eq. (6) into Eq. (3), we obtain

$$tr\langle z | U(\tau) | y \rangle = N^{-1} \int \mathcal{D}[x(\tau), \psi(t)] \exp \left[ i \int_0^\tau dt \left( -\frac{1}{4} \frac{dx_\nu}{dt} \frac{dx^{\nu'}}{dt} - eA_\nu(x) \frac{dx^{\nu'}}{dt} + L_{bos}(x) \right) \right]$$

$$\times \exp \left[ i \int_0^\tau dt \left( \frac{i}{2} \psi_\nu \frac{d\psi^{\nu'}}{dt} - ie\psi^{\nu'} \psi^\lambda F_{\nu\lambda}(y) + L_{fer}^{int}(x, \psi) \right) \right].$$  \n
(10)

The new boundary conditions for $x(t)$ are $x(0) = 0$ and $x(\tau) = z - y$. Notice, that $F_{\mu\nu}$ in Eq. (10) does not depend on $x(t)$. As follows from Eqs. (5), (6) and (8), the expressions for the interacting terms, $L_{bos}^{int}(x)$ and $L_{fer}^{int}(x, \psi)$, containing derivatives of $F_{\mu\nu}$ with respect to coordinates, take the form
\[ L_{\text{bos}}^\text{int}(x) = \sum_{n=1}^{\infty} \frac{eF_{n\lambda\nu_1\nu_2...\nu_n+1}}{n!(n+2)} \frac{dx^{\nu_1}}{dt} x^{\nu_1}(t) \ldots x^{\nu_n+1}(t) \]
\[ = \frac{e}{3} F_{\nu\lambda,\sigma} \frac{dx^\nu}{dt} x^\lambda x^\sigma + \frac{e}{8} F_{\nu\lambda,\sigma\kappa} \frac{dx^\nu}{dt} x^\lambda x^\sigma x^\kappa + \ldots, \]
\[ L_{\text{fer}}^\text{int}(x, \psi) = -\sum_{n=1}^{\infty} i \frac{\hbar}{n!} eF_{\mu\nu_1...\nu_n} \bar{\psi}^\lambda(t) \psi^\mu(t) x^{\nu_1}(t) \ldots x^{\nu_n}(t) \]
\[ = -ieF_{\nu\lambda,\sigma} \bar{\psi}^\lambda x^\sigma - \frac{ie}{2} F_{\nu\lambda,\sigma\kappa} \bar{\psi}^\lambda x^\sigma x^\kappa + \ldots. \]

Here we use the conventional notation for the partial derivatives
\[ F_{\mu\nu_1...\nu_n}(x) = \partial_{\nu_1} \partial_{\nu_2} \ldots \partial_{\nu_n} F_{\lambda\mu}(x). \]

Now we see that the problem of obtaining the derivative expansion reduces to the evaluation of the path integral in Eq. (10) in the framework of the perturbative theory with an infinite number of interacting terms given in Eqs. (11) and (12). Fortunately, for computing the effective action that includes only a finite number of the derivatives, it is sufficient to consider only a finite number of the interacting terms. Later, we shall restrict ourselves to obtaining only the two-derivative terms in the action. So far, we continue developing the scheme for the most general case.

As usual, introducing real and Grassmann external sources, the matrix elements of the evolution operator can be represented as follows
\[ tr(z|U(\tau)|y) = \exp \left\{ i \int_0^\tau dt \left[ L_{\text{bos}}^\text{int} \left( \frac{1}{i} \frac{\delta}{\delta \psi(t)} \right) + L_{\text{fer}}^\text{int} \left( \frac{1}{i} \frac{\delta}{\delta \bar{\psi}(t)} - \frac{\delta}{\delta x(t)} \right) \right] \right\} \]
\[ \times Z_{\tau}[\eta, \xi](z;y) \bigg|_{\eta=0, \xi=0}, \]

where the generating functional is just the Gaussian path integral,
\[ Z_{\tau}[\eta, \xi](z;y) = N^{-1} \int D[x(t), \psi(t)] \exp \left[ i \int_0^\tau dt \left( -\frac{1}{2} \frac{dx_{\nu}}{dt} \frac{dx^\nu}{dt} - eF_{\lambda\nu}(y) \frac{dx^\nu}{dt} + 2\eta_{\nu} x^\nu \right) \right] \]
\[ \times \exp \left[ \frac{1}{2} \int_0^\tau dt \left( \bar{\psi}^\nu \frac{d\psi^\nu}{dt} - 2e\psi^\nu \bar{\psi}^\lambda F_{\lambda\nu}(y) + 2\xi_{\nu} \psi^\nu \right) \right]. \]

The calculation of this generating functional reduces to obtaining the “classical” trajectories for \( x_{\nu}(t) \) and \( \psi_{\nu}(t) \), satisfying the appropriate boundary conditions, and to computing the determinants of the one-dimensional differential operators,
\[ O_1 = \frac{\eta_{\mu\nu}}{2} \frac{d^2}{dt^2} - eF_{\mu\nu} \frac{d}{dt}, \quad \text{and} \quad O_2 = ieF_{\mu\nu} \frac{d}{dt} - 2ieF_{\mu\nu}, \]
defined on the interval \([0, \tau]\) with the periodic and antiperiodic boundary conditions for their eigenstates, respectively.

The “classical” trajectories are easily obtained by solving the equations of motion that the bosonic and Grassmanian worldline actions in Eq. (13) require. So, we arrive at
\[ x_{\mu}^\nu(t) = \left( \frac{e^{2eFt} - 1}{e^{2eF} - 1} \right)^{\mu\nu} (z - y)_\nu + \int_0^\tau dt' \left( \frac{e^{2eF(t-t')} - 1}{e^{2eF} - 1} \right) \theta(t - t') \left( \frac{e^{2eF(t-t') - 1}}{eF} \right)^{\mu\nu} \eta_{\nu}(t'), \]
and
\[ \psi_{\mu}^\nu(t) = \int_0^\tau dt' \left( e^{2eF(t-t')} \left( \theta(t - t') - \frac{1}{1 + e^{-2eF}} \right) \right)^{\mu\nu} \xi_{\nu}(t'). \]
Then, the result of the path integration in Eq. \((15)\) for the case of the coincident arguments \(z = y = x\) reads

\[
Z_{\tau}[\eta, \xi]|(x; x) = C_0 \sqrt{\frac{\text{Det}(O_2)}{\text{Det}'(O_1)}} \exp \left( i \frac{S_{\text{bos}}^{\text{cl}}[\eta]}{2} - \frac{1}{2} S_{\text{fer}}^{\text{cl}}[\xi] \right),
\]

where the normalization constant \(C_0\) should be determined by comparing the result with the Schwinger’s one, or by satisfying the normalization condition

\[
Z_{\tau=0}[\eta, \xi](z; y) = \delta(z - y),
\]

which is equivalent to the operator equality \(U(0) = 1\). The prime in Eq. \((19)\) denotes skipping a zero mode in the definition of the determinant. With our normalization convention for the determinants (see the next section), it is easy to check that the overall factor \(C_0 = -i/(2\pi)^2\) in \(3 + 1\) dimensions and \(C_0 = \exp[-i\pi/4]/[2(\pi)^{3/2}]\) in \(2 + 1\) dimensions.

The expressions for \(S_{\text{bos}}^{\text{cl}}\) and \(S_{\text{fer}}^{\text{cl}}\) are quadratic forms in the external sources

\[
S_{\text{bos}}^{\text{cl}}[\eta] = \int_0^\tau dt_1 \int_0^\tau dt_2 \eta_\nu(t_1) D_\nu^{\nu}(t_1, t_2) \eta^\lambda(t_2),
\]

\[
S_{\text{fer}}^{\text{cl}}[\xi] = \int_0^\tau dt_1 \int_0^\tau dt_2 \xi_\nu(t_1) S_\nu^{\nu}(t_1, t_2) \xi^\lambda(t_2),
\]

where the Green functions are given in terms of functions of the matrix argument \(F_{\mu\nu}\)

\[
D(t_1, t_2) = \frac{1}{2eF} \left[ \epsilon(t_1 - t_2) \left( 1 - e^{2eF(t_1-t_2)} \right) + \coth(eF\tau) \left( 1 + e^{2eF(t_1-t_2)} \right) \right]
\]

\[
- \frac{e^{eF(\tau-2t_2)} + e^{eF(2t_1-\tau)}}{\sinh(eF\tau)},
\]

\[
S(t_1, t_2) = \frac{1}{2} \left[ \epsilon(t_1 - t_2) - \tanh(eF\tau) \right] e^{2eF(t_1-t_2)}.
\]

Substitution of Eqs. \((19)\), \((23)\) and \((24)\) into Eq. \((14)\) leads to the expression for \(\text{tr}|x; U|x\). After expanding the exponent in powers of the operator valued interacting terms, \(L_{\text{int}}^{\text{bos}}\) and \(L_{\text{int}}^{\text{fer}}\) (containing functional derivatives with respect to the sources \(\eta_\nu(t)\) and \(\xi_\nu(t)\)), one has to calculate the result of the derivative action on the generating functional. Starting from this point, we have to restrict ourselves to a specific finite number of the derivatives in the effective action. As we mentioned before, in this paper we are interested in the two-derivative terms (see Sec. \(X\) for some discussions on computing the higher order approximations). Therefore, we obtain

\[
\text{tr}|x|U(\tau)|x\rangle = \left( 1 + i \int_0^\tau dt \left[ V_2(t) + W_2(t) \right] - \frac{1}{2} \int_0^\tau dt_1 dt_2 \left[ V_1(t_1) V_1(t_2) + W_1(t_1) W_1(t_2) \right] \right. \]

\[
\left. - \int_0^\tau dt_1 dt_2 V_1(t_1) W_1(t_2) \right) \bigg|_{\eta=\xi=0} Z_{\tau}[\eta, \xi]|(x, x)\bigg|_{\eta=\xi=0},
\]

where, as follows from Eqs. \((11)\), \((12)\) and \((14)\), the vertex generating operators are

\[
V_1(t) = \frac{i}{3} e^{F_{\nu\lambda\rho \mu}} \lim_{t_0 \to t} \frac{d}{dt_0} \delta_\nu(t_0) \delta_\lambda(t) \delta_\mu(t);
\]

\[
V_2(t) = \frac{1}{8} e^{F_{\nu\lambda\rho \mu \kappa \lambda}} \lim_{t_0 \to t} \frac{d^3}{dt_0^3} \delta_\nu(t_0) \delta_\lambda(t) \delta_\mu(t) \delta_\kappa(t);
\]

\[
W_1(t) = -e^{F_{\nu\lambda\rho \mu \kappa \lambda}} \frac{\delta}{\delta \eta_\nu(t)} \delta_\kappa(t) \delta_\lambda(t) \delta_\mu(t);
\]

\[
W_2(t) = \frac{i}{2} e^{F_{\nu\lambda\rho \mu \kappa \lambda}} \delta_\nu(t) \delta_\lambda(t) \delta_\mu(t) \delta_\kappa(t).\]
Substituting the generating functional \((19)\) which depends on the Green functions \((23)\) and \((24)\), we rewrite Eq. \((25)\) in the form

\[
\text{tr}(x|U(\tau)|x) = C_0 \sqrt{\frac{\text{Det}(O_2)}{\text{Det}'(G_1)}} \left\{ 1 - \frac{i}{8} e F_{\nu\lambda,\mu\kappa} \int_0^\tau dt \left[ \dot{D}^{\nu\lambda}(t, t) D^{\mu\kappa}(t, t) + \dot{D}^{\mu\kappa}(t, t) D^{\nu\lambda}(t, t) + 4 S^{\nu\lambda}(t, t) D^{\mu\kappa}(t, t) \right] \right. \\
\left. + \frac{i}{18} F_{\nu\lambda,\mu\kappa} F_{\sigma\kappa,\rho} \int_0^\tau dt_1 \int_0^\tau dt_2 \left[ 9 D^{\mu\kappa}(1, 2) (S^{\sigma\rho}(2, 2)) S^{\lambda\nu}(1, 1) - 2 S^{\kappa\lambda}(2, 1) S^{\nu\mu}(2, 1) \right] \right. \\
\left. + 6 S^{\sigma\kappa}(2, 2) \left( \dot{D}^{\nu\lambda}(1, 1) D^{\mu\kappa}(1, 2) + \dot{D}^{\mu\kappa}(1, 1) D^{\nu\lambda}(1, 2) + \dot{D}^{\nu\mu}(1, 2) D^{\lambda\kappa}(1, 1) \right) \right. \\
\left. + \dot{D}^{\nu\lambda}(1, 1) \dot{D}^{\sigma\kappa}(2, 2) D^{\mu\kappa}(1, 2) + 2 \dot{D}^{\nu\lambda}(1, 1) \left( \dot{D}^{\sigma\rho}(2, 2) D^{\mu\kappa}(1, 2) + \dot{D}^{\mu\rho}(2, 1) D^{\kappa\sigma}(2, 2) \right) \right. \\
\left. + \dot{D}^{\nu\mu}(1, 1) \dot{D}^{\sigma\rho}(2, 2) D^{\lambda\kappa}(1, 2) + 2 \dot{D}^{\nu\kappa}(1, 2) \left( \dot{D}^{\sigma\rho}(2, 2) D^{\mu\kappa}(1, 1) + \dot{D}^{\mu\rho}(2, 1) D^{\lambda\kappa}(1, 2) \right) \right. \\
\left. + \dot{D}^{\nu\sigma}(1, 2) \left( D^{\lambda\kappa}(1, 1) D^{\kappa\rho}(2, 2) + D^{\lambda\kappa}(1, 2) D^{\mu\rho}(1, 2) + D^{\lambda\kappa}(1, 2) D^{\mu\kappa}(1, 2) \right) \right) \right\}.
\]

Here the dotted functions are defined by the expressions

\[
\dot{D}^{\mu\nu}(1, 2) \overset{\text{def}}{=} \frac{\partial}{\partial t_1} D^{\mu\nu}(t_1, t_2),
\]

\[
\ddot{D}^{\mu\nu}(1, 2) \overset{\text{def}}{=} \frac{\partial^2}{\partial t_1 \partial t_2} D^{\mu\nu}(t_1, t_2),
\]

\[
\ddot{D}^{\mu\nu}(t, t) \overset{\text{def}}{=} \lim_{t_0 \rightarrow t} \frac{\partial}{\partial t_0} D^{\mu\nu}(t_0, t).
\]

Having the representation \((27)\) together with the Green functions \((23)\) and \((24)\), one is left with a need to perform the integrations over the proper time. This latter, however, may look like a rather complicated problem due to the necessity to disentangle the Lorentz indices while doing the integration. In the next section, we show how this problem can be solved.

### III. HOW TO DEAL WITH FUNCTIONS OF MATRIX ARGUMENT \(F_{\mu\nu}\)

In the previous section we developed the general method for calculation the derivative expansion in QED. However, there was not given an explicit final expression, since we needed a technique dealing with functions of the matrix argument \(F_{\mu\nu}\). Below we show, following the method of \((23)\), how to deal with those functions as well as how to calculate the determinants of the differential operators in Eq. \((10)\).

Let us begin by introducing notations that we are going to use below. When working with the electromagnetic field strength tensor, it is usually very convenient to introduce the invariants built of the field strength. In \((3 + 1)\)-dimensional theory, the standard choice of the two independent invariants reads

\[
\mathcal{F} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad \mathcal{G} = \frac{1}{8} \epsilon^{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa}.
\]

In our calculations, though, it will be more convenient to work with the following couple of invariants

\[
K_+ = \sqrt{\mathcal{F}^2 + \mathcal{G}^2 + \mathcal{F}}, \quad K_- = \sqrt{\mathcal{F}^2 + \mathcal{G}^2 - \mathcal{F}}.
\]

As for the \((2 + 1)\)-dimensional theory, there exists only one independent invariant built of the electromagnetic field strength, and it is given by the expression analogues to \(\mathcal{F}\) in Eq. \((31)\).

Now we proceed to the case of \((3 + 1)\)-dimensional QED. It is this case that was considered in \((26)\). The authors of that paper introduced the set of matrices \(A_{(j)}^{\nu\lambda}\) with \(j \in \{1, 2, 3, 4\},\)
\[ A_{(j)\mu} = \frac{-f_2^j \eta_{\mu\nu} + f_j F_{\mu\nu} + F_{\mu\nu}^2 - i f_j^* F_{\mu\nu}}{2(f_j^2 - f_j^2)}, \quad (33) \]

where

\begin{align*}
  f_{1,2} &= \pm i K_-, \quad f_{3,4} = \pm K_+; \\
  f_{1,2} &= \mp K_+, \quad f_{3,4} = \mp i K_-.
\end{align*} \tag{34} \tag{35}

The main property of the matrices \[ \{33 \} \] that we are interested in are their (left and right) contractions with the field strength tensor,

\[ F^{\nu\lambda} A_{(i)\lambda\mu} = A^{\nu\lambda}_{(i)\mu} F_{\nu\lambda} = f_i A_{(i)\mu}. \quad (36) \]

Other useful properties of these matrices that will be used below are

\[ \sum_j A_{(j)\mu}^\nu = \eta^{\mu\nu}, \quad A_{(j)\mu}^\nu = 1, \quad A_{(kj)}^\mu A_{(j)\nu\lambda} = \delta_{k\lambda} A_{(j)\nu}. \quad (37) \]

As follows from the property in Eq. \[ \{36 \} \], for any function \( \Phi(F) \) of the tensor argument \( F_{\mu\nu} \), we get

\[ \Phi(F)_{\mu\nu} = \sum_j A_{(j)\mu\nu} \Phi(f_{(j)}). \quad (38) \]

Matrices with similar properties can also be introduced for \( (2 + 1) \)-dimensional tensor \( F_{\mu\nu} \) as well. Indeed, the following set of matrices

\[ A_{(\pm 1)}^{\mu\nu} = \frac{1}{2} \left( \frac{(F^2)^{\mu\nu}}{2F} \pm \frac{F^{\mu\nu}}{\sqrt{2F}} \right), \quad A_{(0)}^{\mu\nu} = \eta^{\mu\nu} - \frac{(F^2)^{\mu\nu}}{2F} \quad (39) \]

in the \( (2 + 1) \)-dimensional case have properties similar to those in Eqs. \[ \{36 \} \] and \[ \{37 \} \]. As is easy to check directly, their eigenvalues are

\[ f_{\pm 1} = \pm \sqrt{2F}, \quad f_0 = 0. \quad (40) \]

In particular, for the Green functions \[ \{23 \} \] and \[ \{24 \} \], which are functions of the tensor argument \( F_{\mu\nu} \), we obtain the following representations,

\[ D^{\nu\lambda}(t_1, t_2) = \sum_j A_{(j)}^{\nu\lambda} \frac{1}{2 \epsilon f_j} \left[ \epsilon(t_1 - t_2) \left( 1 - e^{2\epsilon f_j (t_1 - t_2)} \right) + \coth(\epsilon f_j \tau) \left( 1 + e^{2\epsilon f_j (t_1 - t_2)} \right) \right. \]

\[ - \left. \frac{e^{\epsilon f_j (\tau - 2t_2)} + e^{\epsilon f_j (2t_1 - \tau)}}{\sinh(\epsilon f_j \tau)} \right], \quad (41) \]

\[ S^{\nu\lambda}(t_1, t_2) = \sum_j A_{(j)}^{\nu\lambda} \frac{1}{2} \left[ \epsilon(t_1 - t_2) - \tanh(\epsilon f_j \tau) \right] \exp[2\epsilon f_j (t_1 - t_2)]. \quad (42) \]

As is seen, in the case of vanishing field, the propagators \( D^{\nu\lambda}(t_1, t_2) \) and \( S^{\nu\lambda}(t_1, t_2) \) coincide with those used in \[ \{17 \} \] and \[ \{29 \} \].

Another problem is related to calculating the determinants of the operators \( \{16 \} \). The latter are nothing else but products of all eigenvalues of the operators. Once again, making use of the matrices in Eq. \[ \{33 \} \] or in Eq. \[ \{39 \} \] for \( (3 + 1) \)- or \( (2 + 1) \)-dimensional cases, respectively, we look for the eigenvectors of the operators \( O_1 \) and \( O_2 \) in the form

\[ x_{(j)\lambda}^\nu(t) = A_{(j)\lambda}^\nu \phi(t) \quad (43) \]

\[ \psi_{(j)\lambda}^\nu(t) = A_{(j)\lambda}^\nu \xi^\lambda \eta(t), \quad (44) \]

where \( \phi \) and \( \eta \) are constant nonzero vectors, \( \phi \) and \( \eta \) are scalar functions of \( t \). As a result, the problem of obtaining eigenvalues reduces to solving ordinary differential equations for the scalar functions \( \phi \) and \( \eta \) with appropriate boundary conditions.
Now, it is easy to check that, up to an unimportant constant, the corresponding determinants read (note that we skip a zero mode of the operator $O_1$)

\[ Det^{(3+1)}(O_1) = \frac{\sinh^2(e\tau K_+) \sinh^2(e\tau K_-)}{(e\tau K_+)^2 (e\tau K_-)^2}, \]

\[ Det^{(3+1)}(O_2) = \cosh^2(e\tau K_+) \cosh^2(e\tau K_-) \]  

in the case of QED in 3 + 1 dimensions, and

\[ Det^{(2+1)}(O_1) = \frac{\sinh^2(e\tau \sqrt{2F})}{(e\tau \sqrt{2F})^2}, \]

\[ Det^{(2+1)}(O_2) = \cosh^2(e\tau \sqrt{2F}) \]

in the case of QED in 2 + 1 dimensions. To obtain these results we used the following formulas for infinite products [27],

\[ \prod_{n=1}^{\infty} \left( 1 + \frac{x^2}{\pi^2 n^2} \right) = \frac{\sinh x}{x}, \quad \prod_{n=1}^{\infty} \left( 1 + \frac{4x^2}{\pi^2 (2n+1)^2} \right) = \cosh x, \]

and similar ones with replacement $x \to iy$.

### IV. GENERAL RESULT IN SPINOR QED

By making use of the results from the previous section, we can proceed with the calculation of (27).

After substituting the Green functions (41) and (42), as well as the explicit expressions for the determinants of the operators $O_1$ and $O_2$, a straightforward, though tedious computation gives the result for the diagonal matrix element of the $U(\tau)$,

\[
tr(x|U(\tau)|x) = tr(x|U(\tau)|x)_0 \times \left[ 1 - \frac{i}{8}e^{F_{\nu\lambda,\mu\kappa}} \sum_{j,l} \left( C^V(f_j, f_l) \left( A_{\nu j}^{\lambda j} A_{\mu j}^{\mu l} + 2A_{\nu j}^{\nu j} A_{\mu j}^{\lambda \lambda} \right) + 2C^W(f_j, f_l)A_{\nu j}^{\lambda j} A_{\mu j}^{\mu \kappa} \right) \right.
\]

\[
- \frac{i}{8}e^{2F_{\nu\lambda,\mu\kappa}} \sum_{j,l,k} \left( 9C^W(f_j, f_l, f_k)A_{\nu j}^{\lambda j} A_{\mu j}^{\mu \kappa} + 2A_{\nu j}^{\lambda j} A_{\mu j}^{\nu j} A_{\mu j}^{\lambda j} + 6C^W(f_j, f_l, f_k)A_{\nu j}^{\lambda j} A_{\mu j}^{\mu \rho} A_{\mu j}^{\mu \kappa} \right)
\]

\[
+ 6C^W(f_j, f_l, f_k)A_{\nu j}^{\sigma j} \left( A_{\nu j}^{\lambda j} A_{\mu j}^{\mu \kappa} + A_{\nu j}^{\nu j} A_{\mu j}^{\lambda j} \right) + 6C^W(f_j, f_l, f_k)A_{\nu j}^{\mu j} A_{\mu j}^{\sigma j} A_{\mu j}^{\mu \rho} + A_{\nu j}^{\sigma j} A_{\mu j}^{\nu j} A_{\mu j}^{\lambda j} + 2A_{\nu j}^{\lambda j} A_{\mu j}^{\sigma j} A_{\mu j}^{\mu \rho} \right)
\]

\[
- C^V(f_j, f_l, f_k) \left( A_{\nu j}^{\lambda j} A_{\mu j}^{\mu \kappa} + A_{\nu j}^{\nu j} A_{\mu j}^{\lambda j} \right) + 2A_{\nu j}^{\lambda j} A_{\mu j}^{\sigma j} A_{\mu j}^{\mu \rho}
\]

\[
- 2C^V(f_j, f_l, f_k) \left( A_{\nu j}^{\lambda j} A_{\mu j}^{\sigma j} A_{\mu j}^{\lambda j} \right) + 2A_{\nu j}^{\lambda j} A_{\mu j}^{\lambda j} A_{\mu j}^{\lambda j}
\]

\[
- C^V(f_j, f_l, f_k) \left( A_{\nu j}^{\lambda j} A_{\mu j}^{\mu \kappa} + A_{\nu j}^{\nu j} A_{\mu j}^{\lambda j} \right)
\]

\[ ) \right], \]

(50)

where the explicit expressions for the coefficients $C^V \text{ or } C^W (\alpha, \beta, \gamma)$ (with $X, Y \in \{V, W\}$) are given in Appendix A and the diagonal matrix elements $tr(x|U(\tau)|x)_0$ correspond to the non-derivative case,

\[ tr(x|U(\tau)|x)_0^{(3+1)} = - \frac{i}{4\pi^2 \tau^2} (e\tau K_-)(e\tau K_+) \cot(e\tau K_-) \coth(e\tau K_+) \]

(51)

in 3 + 1 dimensions, and

\[ tr(x|U_0(\tau)|x)_0^{(2+1)} = \frac{\exp(-i\pi/4)}{2(\pi \tau)^{3/2}} (e\tau \sqrt{2F}) \coth(e\tau \sqrt{2F}) \]

(52)
in 2 + 1 dimensions.

Equation (30) (along with a similar one for scalar QED) is the main result of our paper. Note that the renormalization of the effective action formally reduces to (i) performing a subtraction (precisely the same as in the original Schwingert’s paper) of a term containing no derivatives of field strength with respect to coordinates, and (ii) changing all bare quantities for the renormalized ones, \( e \to e_R \) and \( A_\mu \to A_\mu^R \), defined as follows:

\[
e_R = Z_3^{1/2} e, \quad A_\mu^R = Z_3^{-1/2} A_\mu, \quad Z_3^{-1} = 1 + C e^2,
\]

where

\[
C^{(3+1)} = \frac{1}{12\pi^2} \int_1^{\infty} \frac{ds}{s} \exp(-s m^2),
\]

\[
C^{(2+1)} = \frac{1}{6\pi^{3/2}} \int_0^{\infty} \frac{ds}{\sqrt{s}} \exp(-s m^2) = \frac{1}{6\pi m}.
\]

and \( \Lambda \) is an ultraviolet cutoff in (3 + 1)-dimensional QED.

After subtraction and conversion to the renormalized quantities the effective action becomes finite in the limit \( \Lambda \to \infty \). Since the derivative part of the effective action depends on \( e \) and \( A_\mu \) only through the product \( e A_\mu = e_R A_\mu^R \) it does not change its form and no further renormalization is required to make the derivative part well defined (below we use only renormalized quantities, although we always omit the script “R” in their notation).

By using the asymptotic behavior of the coefficient functions (given in Appendix [3]), one easily finds the following expansion of \( tr\langle x|U(\tau)|x\rangle \) in powers of \( \tau \),

\[
tr\langle x|U(\tau)|x\rangle = tr\langle x|U(\tau)|x\rangle_0 \times \left[ 1 + \frac{ie \tau^3}{20} F^{\nu\lambda} F_{\nu\lambda,\mu} + \frac{ie \tau^3}{180} \left( \frac{7}{2} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} - F^{\nu\lambda} F_{\nu\lambda,\mu} \right) + \ldots \right].
\]

As is clear, this is the weak field limit of our general result in spinor QED. In the effective action, the given order in \( \tau \) results in the two-derivative corrections of the order \( 1/m^2 \):

\[
\mathcal{L}_{1/m^2}^{(3+1) \text{spin}} = \frac{\alpha}{720 \pi m^2} \left[ 18 F^{\nu\lambda} F_{\nu\lambda,\mu} + 7 F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} - 2 F^{\nu\lambda} F_{\nu\lambda,\mu} \right],
\]

in 3 + 1 dimensions, and of the order \( 1/m^3 \)

\[
\mathcal{L}_{1/m^3}^{(3+1) \text{spin}} = \frac{\alpha}{720 \pi m^3} \left[ 18 F^{\nu\lambda} F_{\nu\lambda,\mu} + 7 F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} - 2 F^{\nu\lambda} F_{\nu\lambda,\mu} \right],
\]

in 2 + 1 dimensions.

The expansion in Eq. (56) was obtained earlier in the heat kernel approach [4]. While the latter is a perfect tool for deriving the effective action in the weak field limit, it is not very useful when the field becomes strong. Our approach here, on the other hand, is free from such a limitation and the general result in Eq. (30) contains all the two derivative terms like \( \partial F \partial F (F/m^2)^n \) where \( n \) is an arbitrary positive integer and the Lorentz indices (not shown) are contracted in all possible ways. To substantiate this claim, we present the next to leading terms of the weak field expansion in Eq. (B1) in Appendix B.

As we saw above, the formal expansion in \( \tau \) corresponds to an expansion of the effective action in the inverse powers of the mass parameter. This means that, while making use of such an expansion, one cannot get any reliable results in the limit of the vanishing fermion mass. This, in particular, is the main reason why the authors of [3], who used an expression like (30), came to a wrong conclusion about the absence of corrections to the one-loop effective action coming from inhomogeneities of a static magnetic field when \( m \to 0 \). Such a conclusion “contradicts” the result of Ref. [8]. The latter, as we will see, completely agrees with our result for the derivative expansion.

V. SPINOR QED IN 2+1 DIMENSIONS

Let us consider the case of the purely magnetic field background to which a special attention was paid in [8]. To proceed with analyzing this case, note that the electromagnetic field strength tensor takes the following form,
where $B(x)$ is a pseudoscalar function coinciding with the magnetic field strength and $F^{\mu\nu}$ is a constant matrix with the only nonzero components $F_{12} = -F_{21} = 1$. As is seen it satisfies the following normalization condition: $F^{\mu\nu}F_{\mu\nu} = 2$.

To reduce the general result presented in Eq. (59) for the particular choice of the field given in Eq. (53), we have to use the properties of $A_{\mu}^{\nu}$’s presented in Sec. III. Just to get feeling how they work, let us consider an example,

\[
F^{\mu\nu}(x) = B(x)F^{\mu\nu},
\]

(59)

where $B(x)$ is a pseudoscalar function coinciding with the magnetic field strength and $F^{\mu\nu}$ is a constant matrix with the only nonzero components $F_{12} = -F_{21} = 1$. As is seen it satisfies the following normalization condition: $F^{\mu\nu}F_{\mu\nu} = 2$.

To reduce the general result presented in Eq. (59) for the particular choice of the field given in Eq. (53), we have to use the properties of $A_{\mu}^{\nu}$’s presented in Sec. III. Just to get feeling how they work, let us consider an example,

\[
F_{\nu\lambda,\mu\kappa} \sum_{j,l} C^{W}(f_{j}, f_{l}) A_{(j)}^{\nu\lambda} A_{(l)}^{\mu\kappa} = \frac{\partial_{\mu} B}{B} \sum_{j,l} C^{W}(f_{j}, f_{l}) A_{(j)}^{\mu\kappa}
\]

\[
= 2\frac{\partial_{\mu} B}{B} \sqrt{2F} \left[ C^{W}(\sqrt{2F}, 0) A_{(0)}^{\mu\kappa} + C^{W}(\sqrt{2F}, \sqrt{2F}) \left( A_{(-1)}^{\mu\kappa} + A_{(+1)}^{\mu\kappa} \right) \right]
\]

\[
= -2iC^{W}(\sqrt{2F}, \sqrt{2F}) (F^{2})^{\mu\nu} \partial_{\mu} B = -2iC^{W}(\sqrt{2F}, \sqrt{2F}) \sum_{i=1}^{2} \partial_{i} B.
\]

(60)

In this derivation, we made use of the Bianchi identity. We recall that the latter should be satisfied since the electromagnetic field was introduced in the theory through the vector potential by minimal coupling. The identity itself reads $A_{(0)}^{\mu} \partial_{\mu} B = 0$. The direct consequence of it is the independence of the magnetic field, for the particular choice (59), on the time coordinate. By noticing that the matrix $A_{(0)}^{\mu}$, as well as any other from the set, does not depend on $B(x)$ we obtain the secondary identity, $A_{(0)}^{\mu\nu} \partial_{\mu} \partial_{\nu} B = 0$, by differentiating the original one. It is this last form of the Bianchi identity that was actually used in our derivation in Eq. (60).

The other expressions, similar to that in Eq. (60), along with the functions like $C^{W}(\sqrt{2F}, \sqrt{2F})$ are listed in Appendix C.

The final result for the derivative part of the diagonal matrix element (53), for the particular choice of the field configuration in Eq. (59), reads

\[
tr(x|U(\tau)|x)^{(2+1)}_{der} = \frac{i e^{2} (\partial_{\tau} B)^{2}}{(4\pi e|B|)^{3/2}} \frac{1}{\sqrt{\omega}} (3\omega^{2} Y^{4} - 3\omega Y^{3} - 4\omega^{2} Y^{2} + 3\omega Y + \omega^{2})
\]

\[
= \frac{i e^{2} (\partial_{\tau} B)^{2}}{4(4\pi e|B|)^{3/2}} \frac{1}{\sqrt{\omega}} d^{3} (\omega \coth \omega),
\]

(61)

where $\omega = \tau|eB|$, $Y = \coth \omega$, and $(\partial_{\tau} B)^{2} = \sum_{i=1}^{2} \partial_{i} B \partial_{i} B$. Substituting the last expression into Eq. (2), we come to the integral representation for the derivative part of the effective Lagrangian (we perform the change of the integration variable $\tau$ for $\omega = \tau|eB|$),

\[
\mathcal{L}_{(2+1) spin}^{der}(B) = -\frac{e^{2} (\partial_{i} B)^{2}}{4(4\pi e|B|)^{3/2}} \int_{0}^{\infty} \frac{d\omega}{\sqrt{\omega}} \exp \left( \frac{-m^{2}}{2|eB|} \omega \right) d^{3} (\omega \coth \omega).
\]

(62)

The last expression coincides with the result presented in (62) (note that in notation of (62) $\partial_{i} B \partial_{i} B = 4\partial B \partial B$). One can be convinced that the integrand in (62) is a negative function what means that inhomogeneities of the magnetic field background, in approximation under consideration (one-loop and two derivatives), lead to the reduction of vacuum energy density for any value of the ratio $m^{2}/|eB|$. The latter situation does not, however, prove that a spontaneous generation of a non-homogeneous magnetic field happens in QED since the sign of the two derivative term in the expansion of the effective action is not a sufficient argument for making a conclusion of that kind (62).

We would like also to give another representation for the derivative part of the Lagrangian in terms of special functions. To get it, we need to perform the integration in (62) by parts (see Eq. (D2) in Appendix F). Here is such a representation,

\[
\mathcal{L}_{(2+1) spin}^{der}(B) = \frac{e^{2} (\partial_{i} B)^{2}}{\sqrt{2\pi (4\pi e|B|)^{3/2}}} \left[ 5\zeta \left( \frac{3}{2} \right) + \frac{1}{2} + \frac{m^{2}}{2|eB|} \right] - 9 \frac{m^{2}}{2|eB|} \zeta \left( \frac{3}{2} \right) \left( 1 + \frac{m^{2}}{2|eB|} \right)
\]

\[
+ 3 \left( \frac{m^{2}}{2|eB|} \right)^{2} \zeta \left( \frac{3}{2} \right) \left( 1 + \frac{m^{2}}{2|eB|} \right) + \left( \frac{m^{2}}{2|eB|} \right)^{3} \zeta \left( \frac{3}{2} \right) \left( 1 + \frac{m^{2}}{2|eB|} \right).
\]

(63)
Often, in the limit of large or small values of the external field, it is more convenient to work with the asymptotic expansions of the effective action rather than the exact expression as in Eq. (63). First, let us consider the case \( m^2 \ll |eB| \). Then, using the last representation, we easily derive the following asymptotic expansion,

\[
L_{der}^{(2+1)spin}(B) \simeq -\frac{e^2 (\partial_B)^2}{2 \sqrt{4 \pi |eB|} \pi^{3/2}} \sum_{k=0}^{\infty} \frac{5 - 2k}{k!} \Gamma \left( k + \frac{1}{2} \right) \zeta \left( k - \frac{3}{2} \right) \left( \frac{m^2}{2|eB|} \right)^k.
\] (64)

In order to get the asymptotic expansion for \( m^2 \gg |eB| \), we make use of the integral representation in Eq. (62) and obtain

\[
L_{der}^{(2+1)spin}(B) \simeq -\frac{e^2 (\partial_B)^2}{2 \pi^{3/2} m^3} \sum_{k=0}^{\infty} \frac{B_{2k+4}}{(2k+1)!} \Gamma \left( 2k + \frac{3}{2} \right) \left( \frac{2|eB|}{m^2} \right)^{2k},
\] (65)

where \( B_k \) are the Bernoulli numbers.

Now, let us consider the case of the purely electric field background. Without losing the generality, we assume that the field is directed along the first axis of the two-dimensional space. Again the field strength tensor is factored similar to (54),

\[
F^{\mu\nu}(x) = E(x) F^{\mu\nu},
\] (66)

where \( E(x) \) is the magnitude of the electric field. Now the constant matrix \( F^{\mu\nu} \) has nonzero components \( F^{10} = -F^{01} = 1 \), and satisfies the normalization condition: \( F^{\mu\nu} F_{\mu\nu} = -2 \). The general expression (50) simplifies considerably for our choice of the background field. And the derivative part of that expression now reads

\[
tr[U(\tau)|x]_{der}^{(2+1)}(E) = \frac{i \exp(-i \pi/4) e^2 (\partial_B)^2}{(4 \pi |eE|)^{3/2}} \sqrt{\omega} \left( 3 \omega^2 Y^4 - 3 \omega Y^3 - 4 \omega^2 Y^2 + 3 \omega Y + \omega^2 \right)
\]
\[
= -\frac{i \exp(-i \pi/4) e^2 (\partial_B)^2}{2 \sqrt{\omega}} \frac{d^3}{d \omega^3} (\omega \coth \omega),
\] (67)

where now \( \omega = \tau |eE|, \ Y = \coth \omega, \) and \((\partial\| E)^2 \equiv (\partial_0 E \partial_0 E - \partial_1 E \partial_1 E)\). Here we used the Bianchi identity again to show that the electric field does not depend on the second spatial coordinate. Substituting this expression into Eq. (62), we come to the integral representation for the derivative part of the effective Lagrangian,

\[
L_{der}^{(2+1)spin}(E) = \frac{\exp(-i \pi/4) e^2 (\partial_B)^2}{4 (4 \pi |eE|)^{3/2}} \int_0^{\infty} \frac{d\omega}{\sqrt{\omega}} \exp \left(-i \frac{m^2}{|eE|} \omega \right) \frac{d^3}{d \omega^3} (\omega \coth \omega).
\] (68)

As expected in the case of an electric field background, this derivative correction to the effective action contains a nonzero imaginary contribution. A convenient representation of the latter can be obtained in the following way. First, in Eq. (68), we switch to a new variable, \( z = i \omega \), so that the integration runs along the imaginary axis of \( z \) from zero to \( i \infty \). Then, we move the integration contour to the real axis of \( z \). As is easy to check, the integrand has poles at \( z = \pi n \) \((n = 1, 2, \ldots)\). As a result, the real and the imaginary contributions get naturally separated. Indeed, the real part of \( L_{der}^{(2+1)spin} \) is given by the principal value of the integral along the \( Re(z) \) axis, while the imaginary part appears due to the integration along the infinite set of the vanishingly small semi-circles above the poles, \( z = \pi n + \varepsilon \exp(i (\pi - \phi)) \) \((0 < \phi < \pi \) and \( \varepsilon \to 0 \) at the end). In this way, we easily obtain the imaginary part of the right hand side in Eq. (68),

\[
\Im L_{der}^{(2+1)spin}(E) = -\frac{e^2 (\partial_B)^2}{2 \pi^{3/2} |eE|^{3/2}} \sum_{n=1}^{\infty} \frac{1}{n^{5/2}} \exp \left(-\frac{\pi m^2 n}{|eE|} \right)
\]
\[
\times \left[ 15 + 18 \frac{\pi m^2 n}{|eE|} + 12 \left( \frac{\pi m^2 n}{|eE|} \right)^2 + 8 \left( \frac{\pi m^2 n}{|eE|} \right)^3 \right].
\] (69)

We note that the result of the summation in the last expression (as well as in similar formulas later on) can be given in terms of the polylogarithmic function \( Li_\nu(x) \). Equation (69) determines the correction to the probability of the particle-antiparticle pair creation (by definition, the probability density is \( \mathcal{W} = 2 \Im \mathcal{L} \)) in an external electric field.
due to small inhomogeneities in space-time. We emphasize that the correction due to a time derivative of the field
has the “wrong” sign, i.e. it works against the particle creation. The gradient in the space direction parallel to the
field strength, on the other hand, amplifies the process.

As is known, in the case of constant electric field, the imaginary part of the effective Lagrangian is given by
\[
\text{Im} \mathcal{L}^{(2+1)\text{spin}}(E) = \frac{|eE|^{3/2}}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \exp\left( -\frac{\pi m^2}{|eE|} n \right) = \frac{|eE|^{3/2}}{4\pi^2} \text{Li}_{3/2} \left( -\frac{\pi m^2}{|eE|} \right). \tag{70}
\]
This as well as the first correction due to the derivatives remain finite even in the limit of zero fermion mass. Despite
of this fact, we still expect that the derivative expansion (with the electric field background) may fail in the limit of
vanishingly small mass due to higher orders in the number of derivatives. Below we shall see that the same is true in
the spinor QED in 3 + 1 dimensions as well.

VI. SPINOR QED IN 3+1 DIMENSIONS

As was mentioned at the beginning of the paper, the derivative expansion in QED4 was also studied in [2]. The
result of that paper was presented in an explicit form for the special class of the electromagnetic field configurations,
\[
\mathcal{G} = 0, \quad F^{\mu\nu}(x) = \Phi(x)F^{\mu\nu}, \tag{71}
\]
where \(\Phi(x)\) is a slowly varying function that defines the magnitude of the field, and \(F^{\mu\nu}\) is a constant matrix. For
convenience, let us normalize the matrix \(F^{\mu\nu}\) by the condition: \(F^{\mu\nu}F_{\mu\nu} = 2\). Then the scalar function \(\Phi(x)\) is nothing
else but \((-2F)\). As was shown in our previous paper [2], the general result for the diagonal matrix element (50)
in the case of field (71) reduces to the same result as was presented in [2],
\[
\text{tr}(x)U(\tau)|x\rangle_{der}^{(3+1)}(\Phi) = \frac{1}{(4\pi)^2} \frac{\partial_\tau \Phi \partial_\mu \Phi}{\Phi^2} (3\omega^2 Y^4 - 3\omega Y^3 - 4\omega^2 Y^2 + 3\omega Y + \omega^2),
\]
\[
= -\frac{1}{(4\pi)^2} \frac{\partial_\tau \Phi \partial_\mu \Phi}{\Phi^2} \frac{\omega^3}{2} 3 \omega \coth \omega, \tag{72}
\]
where \(\omega = \tau e\Phi, Y = \coth \omega\). As in the (2 + 1)-dimensional theory, here we used the Bianchi identity, which this time
reads
\[
\left( g^{\mu\nu} + (F^2)^{\mu\nu} \right) \partial_\nu \Phi \equiv 0. \tag{73}
\]
In the case of magnetic field along the third axis, for example, this condition means that the specified field cannot
depend on the time and the third spatial coordinates, while in the case of electric field along the first axis, it cannot
depend on the second and third spatial coordinates.

Now, let us consider two particular cases of external field that we studied in 2 + 1 dimensions: purely magnetic and
purely electric field backgrounds. Both of them are just different possibilities of that given in Eq. (71).

Thus, in the case of magnetic field (along the third axis in space) we come to the following integral representation
for the derivative part of the effective Lagrangian,
\[
\mathcal{L}_{der}^{(3+1)\text{spin}}(B) = -\frac{e^2}{(8\pi)^2} \frac{(\partial_\tau B)^2}{|eB|} \int_0^{\infty} \frac{d\omega}{\omega} \exp \left( -\frac{m^2}{|eB|} \omega \right) \frac{d^3}{d\omega^3} (\omega \coth \omega). \tag{74}
\]
Resembling the situation in 2 + 1 dimensions, inhomogeneities of the external magnetic field tend to reduce vacuum
energy density for any value of the ratio \(m^2/|eB|\).

Performing integration in the right hand side of Eq. (74) by parts (see Eq. (D6) in Appendix D), we find the
following representation (for the representation of the part of the effective action without derivatives in terms of
special functions, see [30]),
\[
\mathcal{L}_{der}^{(3+1)\text{spin}}(B) = -\frac{e^2}{(8\pi)^2} \frac{(\partial_\tau B)^2}{|eB|} \left[ \frac{11}{6} \left( \frac{m^2}{|eB|} \right)^3 + \frac{1}{3} \frac{m^2}{|eB|} - \frac{1}{2} \frac{m^2}{|eB|}^2 \psi \left( 1 + \frac{m^2}{2|eB|} \right) \right.
\]
\[
+ 24\zeta' \left( -2,1 + \frac{m^2}{2|eB|} \right) - 24 \frac{m^2}{|eB|} \zeta' \left( -1,1 + \frac{m^2}{2|eB|} \right)
\]
\[
+ 6 \left( \frac{m^2}{|eB|} \right)^2 \left[ \ln \Gamma \left( 1 + \frac{m^2}{2|eB|} \right) - \ln \sqrt{2\pi} \right] \right]. \tag{75}
\]
As $m^2 \ll |eB|$, this expression allows the following asymptotic expansion,

$$
L_{der}^{(3+1) \text{spin}}(B) \approx -\frac{e^2 (\partial_t B)^2}{(8\pi)^2 |eB|} \left[ 24 \zeta'(-2) + \frac{2m^2}{3|eB|^2} - \frac{m^4}{2|eB|^2} + \frac{m^6}{3|eB|^3} \right.
$$

$$
- \frac{m^8}{2|eB|^4} \sum_{k=0}^\infty \frac{k+1}{k+4} \zeta(k+2) \left( -\frac{m^2}{2|eB|} \right)^k \right],
$$

(76)

where $\zeta'(-2) \approx -0.030$. As $m^2 \gg |eB|$, on the other hand, we obtain

$$
L_{der}^{(3+1) \text{spin}}(B) \approx -\frac{e^2 (\partial_t B)^2}{(2\pi)^2 m^2} \sum_{k=0}^\infty \frac{B_{2k+4}}{2k+1} \left( \frac{2|eB|}{m^2} \right)^{2k}.
$$

(77)

In case of the electric field along the first axis, on the other hand, we obtain the following expression for the derivative part of the effective action,

$$
L_{der}^{(3+1) \text{spin}}(E) = -\frac{ie^2}{(8\pi)^4 |eE|} \int_0^\infty \frac{d\omega}{\omega} \exp \left( -i \frac{m^2}{|eE|} \omega \right) \frac{d^3}{d\omega} (\omega \coth \omega).
$$

(78)

This expression has both real and imaginary part, as always happens in the case of an external electric field. Another representation for it is obtained by analytical continuation of (78) according to the rule $|eB| \to -i|eE|$. The imaginary part though is easily extracted from (78) in a standard way,

$$
\Im L_{der}^{(3+1) \text{spin}}(E) = \frac{e^2}{2^{10} \pi^4 |eE|} \sum_{n=1}^\infty \frac{\left( \frac{\pi m^2 n}{|eE|} \right)^3}{n^3} \exp \left( -\frac{\pi m^2 n}{|eE|} \right)
$$

$$
\times \left[ 6 + 6 \left( \frac{\pi m^2 n}{|eE|} \right)^2 + 3 \left( \frac{\pi m^2 n}{|eE|} \right)^3 \right],
$$

(79)

which determines a correction to the Schwinger result \[10\] for the imaginary part of the effective action in a constant electric field,

$$
\Im L_{der}^{(3+1) \text{spin}}(E) = \frac{(eE)^2}{8\pi^3} \sum_{n=1}^\infty \frac{1}{n^3} \exp \left( -\frac{\pi m^2 n}{|eE|} \right) = \frac{(eE)^2}{8\pi^3} \Li_2 \left[ \exp \left( -\frac{\pi m^2}{|eE|} \right) \right].
$$

(80)

The result in Eq. (74) is in agreement with that of \[10\].

As is easy to establish, both the Schwinger result for a constant field and the first correction due to derivatives are finite in the limit of the vanishing fermion mass. As we argued in the case of the $(2 + 1)$-dimensional spinor QED, this may not be the case in higher orders of the perturbative expansion in the number of derivatives.

**VII. GENERAL RESULT IN SCALAR QED**

Now turning to the calculation of the derivative expansion for the scalar electrodynamics, one does not need to repeat all the calculations similar to those done in Sec. \[14\]. In order to see this, we recall that the effective one-loop Lagrangian in this case reads

$$
L^{(1) \text{scal}}(x) = -i \int_0^\infty \frac{d\tau}{\tau} \langle x | U_{\text{bos}}(\tau) | x \rangle e^{-im^2\tau}.
$$

(81)

The evolution connected with the transition amplitude, $\langle z | U_{\text{bos}}(\tau) | y \rangle$, is described now by the Hamiltonian (compare with Eqs. (2) and (3))

$$
H_{\text{bos}} = -\Pi_\mu \Pi^\mu, \quad \Pi_\mu = -i \partial_\mu + eA_\mu(x).
$$

(82)
Thus, omitting all terms originating from the fermion part in the expression \( \mathcal{L}_{4} \), i.e. putting \( L_{\text{ferm}}^{\text{int}} = 0 \) in Eqs. (10), (12) and \( S_{\text{cl}}^{\text{ferm}} = 0 \) in Eq. (13), we come to the following expression

\[
\langle x | U_{\text{bos}}(\tau) | x \rangle = \langle x | U_{\text{bos}}(\tau) | x \rangle_0
\]

\[
\times \left[ 1 - \frac{i}{8} e F_{\nu \lambda ; \mu} \sum_{j,l} C^{(V)}(f_j, f_l) \left( A^{\nu \lambda \mu \nu \lambda \mu} + 2 A^{\nu \lambda \mu \nu \lambda \mu} A^{\nu \lambda \mu \nu \lambda \mu} + 2 A^{\nu \lambda \mu \nu \lambda \mu} A^{\nu \lambda \mu \nu \lambda \mu} \right) + \frac{i}{18} e^2 F_{\nu \lambda ; \mu} F_{\sigma \rho \nu \lambda ; \mu} \right]
\]

\[
\times \sum_{j,l,k} \left[ C_{1}^{(V)}(f_j, f_l, f_k) \left( A^{\nu \lambda \mu \nu \lambda \mu} A^{\nu \lambda \mu \nu \lambda \mu} + 2 A^{\nu \lambda \mu \nu \lambda \mu} A^{\nu \lambda \mu \nu \lambda \mu} \right) + C_{2}^{(V)}(f_j, f_l, f_k) \left( A^{\nu \lambda \mu \nu \lambda \mu} + 2 A^{\nu \lambda \mu \nu \lambda \mu} A^{\nu \lambda \mu \nu \lambda \mu} + 2 A^{\nu \lambda \mu \nu \lambda \mu} A^{\nu \lambda \mu \nu \lambda \mu} \right) + C_{3}^{(V)}(f_j, f_l, f_k) A^{\nu \lambda \mu \nu \lambda \mu} + C_{4}^{(V)}(f_j, f_l, f_k) A^{\nu \lambda \mu \nu \lambda \mu} A^{\nu \lambda \mu \nu \lambda \mu} \right] \right].
\]

The coefficients used here are the same as in Eq. (10). As for the non-derivative factors, they have the standard form,

\[
\langle x | U_{\text{bos}}(\tau) | x \rangle_0^{(3+1)} = - \frac{i}{(4\pi)^2} \frac{(\tau K^{-})(\tau K^{+})}{\sin(\tau K^{-}) \sinh(\tau K^{+})}
\]

in 3 + 1 dimensions, and

\[
\langle x | U_{\text{bos}}(\tau) | x \rangle_0^{(2+1)} = - \frac{\exp(-i\pi/4)}{(4\pi)^{3/2}} \frac{(\tau \sqrt{2F})}{\sinh(\tau \sqrt{2F})}
\]

in 2 + 1 dimensions, as can be easily checked by using the expressions for the determinants given in Sec. 11 and by taking into account the fact that, because of spin degrees of freedom, we had the additional factor 4 for fermions.

In the case of scalar theory, the renormalization of the electromagnetic field and charge is given by the same formulas \( \mathcal{L}_{4} \) but this time the corresponding constants read

\[
C^{(3+1)} = \frac{1}{48\pi^2} \int \frac{ds}{s} \exp(-sm^2),
\]

\[
C^{(2+1)} = \frac{1}{24\pi^{3/2}} \int \frac{ds}{s} \exp(-sm^2) = \frac{1}{24\pi m}
\]

To get a result of the type as in \( \mathcal{L}_{4} \), one has to expand the coefficient functions in powers of proper time. Thus the expansion for \( \langle x | U_{\text{bos}}(\tau) | x \rangle \) (weak field limit) reads

\[
\langle x | U_{\text{bos}}(\tau) | x \rangle = \langle x | U_{\text{bos}}(\tau) | x \rangle_0
\]

\[
\times \left[ 1 - \frac{ie^2 \tau^3}{30} F_{\nu \lambda ; \mu} \nu \lambda \mu - \frac{ie^2 \tau^3}{180} \left( 4 F_{\nu \lambda ; \mu} \nu \lambda \mu + F_{\nu \lambda ; \mu} \nu \lambda \mu \right) + \ldots \right].
\]

This expansion up to the order \( \tau^5 \) is given in Eq. (B26) in Appendix B. As in the spinor QED, it is useful only in the case of heavy scalar particles (weak fields), when the mass scale is much larger than all other scales in the theory.

In the effective action of scalar QED, the expansion in Eq. (83) corresponds to the following leading two derivative terms

\[
\mathcal{L}_{1/m^2}^{(3+1)\text{scal}} = \frac{\alpha}{720\pi m^2} \left[ 6 F_{\nu \lambda \mu} \nu \lambda \mu + 4 F_{\nu \lambda \mu} \nu \lambda \mu F_{\nu \lambda \mu} + F_{\nu \lambda \mu} F_{\nu \lambda \mu} \right],
\]

in 3 + 1 dimensions, and

\[
\mathcal{L}_{1/m^2}^{(2+1)\text{scal}} = -\frac{\alpha}{720m^2} \left[ 6 F_{\nu \lambda \mu} \nu \lambda \mu + 4 F_{\nu \lambda \mu} \nu \lambda \mu F_{\nu \lambda \mu} + F_{\nu \lambda \mu} F_{\nu \lambda \mu} \right],
\]

in 2 + 1 dimensions.
VIII. SCALAR QED IN 2+1 DIMENSIONS

Let us start by considering the case of an external magnetic field as in Eq. (82). This time the derivative part of the general expression (82) reduces to

$$\langle x | U_{\text{bos}}(\tau) | x \rangle^{(2+1)}_{\text{der}} = -i e^2 (\partial \cdot B)^2 \frac{1}{4(4 \pi |eB|)^{3/2}} \sqrt{\frac{\omega}{\sinh \omega}} (3\omega Y^3 - 3Y^2 - 2\omega Y + 1)$$

$$= -i e^2 (\partial \cdot B)^2 \frac{\sqrt{\omega}}{2} \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \left( \frac{\omega}{\sinh \omega} \right),$$

(91)

where $\omega = i\tau|eB|$, $Y = \coth \omega$. After substituting the last expression into Eq. (81), we come to the integral representation for the derivative part of the effective Lagrangian (after performing the change of integration variable $\tau \rightarrow \omega = i\tau|eB|$),

$$L^{(2+1)\text{scal}}_{\text{der}}(B) = \frac{e^2 (\partial \cdot B)^2}{(16\pi |eB|)^{3/2}} \int_0^\infty \frac{d\omega}{\sqrt{\omega}} \exp \left( -\frac{m^2}{|eB|} \omega \right) \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \left( \frac{\omega}{\sinh \omega} \right),$$

(92)

which coincides with the result presented in [3]. As in the case of spinor QED, there exists another representation of (82) given in terms of special functions (see Eq. (D11) in Appendix D),

$$L^{(2+1)\text{scal}}_{\text{der}}(B) = \frac{e^2 (\partial \cdot B)^2}{\sqrt{2\pi} (16|eB|)^{3/2}} \left[ 20\zeta \left( \frac{3}{2}, 1 + \frac{m^2}{|eB|} \right) - 18 \frac{m^2}{|eB|} \zeta \left( \frac{1}{2}, \frac{1}{2} + \frac{m^2}{|eB|} \right) \right] + \left( 1 + 3 \left( \frac{m^2}{|eB|} \right)^2 \right) \zeta \left( \frac{1}{2}, \frac{1}{2} + \frac{m^2}{2|eB|} \right) + \frac{1}{2} \left( \frac{m^2}{|eB|} \right)^3 \zeta \left( \frac{3}{2}, \frac{1}{2} + \frac{m^2}{2|eB|} \right).$$

(93)

Numerical study of the integral in (92) shows that inhomogeneities of magnetic field background, in approximation under consideration (one-loop and two derivatives), lead to decreasing the vacuum energy density for $m^2/|eB| \gtrsim 0.927$ and to increasing that density for $m^2/|eB| \lesssim 0.927$, in accordance with Ref. [8].

Analytically, we can obtain only the limiting cases as we did in spinor electrodynamics. In particular, for $m^2 \ll |eB|$, the effective action takes the following asymptotic form,

$$L^{(2+1)\text{scal}}_{\text{der}}(B) \simeq \frac{e^2 (\partial \cdot B)^2}{(16\pi |eB|)^{3/2}} \sum_{k=0}^\infty \frac{1}{k!} \left( 2^k - 2\sqrt{2} (5 - 2k) \zeta \left( k - \frac{3}{2} \right) \right) \left( 2^k - \frac{1}{\sqrt{2}} (1 - 2k) \zeta \left( k + \frac{1}{2} \right) \right) \Gamma \left( k + \frac{3}{2} \right) \left( \frac{|eB|}{m^2} \right)^{2k},$$

(94)

while for $m^2 \gg |eB|$, the expansion reads

$$L^{(2+1)\text{scal}}_{\text{der}}(B) \simeq \frac{e^2 (\partial \cdot B)^2}{32\pi^{3/2} m^3} \sum_{k=0}^\infty \frac{(2^{2k+3} - 1)B_{2k+4} + (2^{2k+1} - 1)B_{2k+2}}{(2k+1)!} \left( \frac{|eB|}{m^2} \right)^{2k} \times \Gamma \left( 2k + \frac{3}{2} \right).$$

(95)

Now, let us consider the case of electric field background. Without losing the generality, we assume that the field is directed along the first axis of space. We obtain

$$\langle x | U_{\text{bos}}(\tau) | x \rangle^{(2+1)}_{\text{der}}(E) = \frac{i \exp(-i\pi/4)}{4(4\pi |eE|)^{3/2}} \frac{e^2 (\partial \cdot E)^2}{\sinh \omega} (3\omega Y^3 - 3Y^2 - 2\omega Y + 1)$$

$$= -\frac{i \exp(-i\pi/4)}{16\pi |eE|)^{3/2}} e^2 (\partial \cdot E)^2 \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \left( \frac{\omega}{\sinh \omega} \right),$$

(96)

where now $\omega = \tau|eE|$ and $Y = \coth \omega$. Substituting this expression into (81), we come to the integral representation for the derivative part of the effective Lagrangian,
In Sec. VI, in the case of scalar QED, the derivative part of the effective Lagrangian is easily obtained by just rewriting it in a different form:

\[
\mathcal{L}_{\text{der}}^{(2+1)\text{scal}} (E) = -\frac{\exp(-i\pi/4)e^2}{(16\pi|eE|)^{3/2}} \int_0^\infty \frac{d\omega}{\omega} \exp \left( -i \frac{m^2}{|eE|} \omega \right) \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \left( \frac{\omega}{\sinh \omega} \right). \tag{97}
\]

And we easily find the imaginary part of this expression,

\[
\Im \mathcal{L}_{\text{der}}^{(2+1)\text{scal}} (E) = \frac{e^2}{8\pi^2} \sum_{n=1}^{\infty} \frac{(1)^n n!}{n^{3/2}} \left[ 15 + 18 \pi m^2 n \langle eE \rangle + 4\pi^2 n^2 \left( 3 \frac{m^4}{|eE|^2} - 1 \right) + 8 m^2 \pi^3 n^3 \left( \frac{m^4}{|eE|^2} - 1 \right) \right], \tag{98}
\]

which determines the correction to the corresponding result for the case of constant electric field,

\[
\Im \mathcal{L}_{\text{der}}^{(2+1)\text{scal}} (E) = \frac{|eE|^{3/2}}{8\pi^2} \sum_{n=1}^{\infty} \frac{(1)^n n!}{n^{3/2}} \left[ - \exp \left( - \frac{\pi m^2 n}{|eE|} \right) \right]. \tag{99}
\]

A simple numerical analysis of the derivative correction in Eq. (98) shows that the sum in the right hand side, being positive for large values of the mass (or small values of the electric field), changes its sign at \(m^2 \approx 0.721|eE|\). Therefore, unlike the case of spinor QED, the time derivative of the field increases (while the gradient in space decreases) the probability of particle-antiparticle pair creation only for \(m^2 \gtrsim 0.721|eE|\).

As in spinor QED, the two-derivative correction to the process of the pair production in scalar QED is convergent even in the limit of the vanishing mass. This observation, of course, is not enough to prove that the derivative expansion is well defined to all orders in the massless theory.

**IX. SCALAR QED IN 3+1 DIMENSIONS**

The derivative expansion for the electromagnetic field of the form (71) was presented in our previous paper \[9\] (we just rewrite it in different form),

\[
\langle x|U_{\text{bos}}(\tau)|x\rangle^{(3+1)}_{\text{der}} = \frac{1}{(8\pi^2)^2} \frac{\partial_{\mu}\Phi \partial^\mu \Phi}{\Phi^2} \frac{\omega}{\sinh \omega} \left( 3\omega Y^3 - 3Y^2 - 2\omega Y + 1 \right)
- \frac{1}{(8\pi^2)^2} \frac{\partial_{\mu}\Phi \partial^\mu \Phi}{\Phi^2} \frac{1}{2} \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \left( \frac{\omega}{\sinh \omega} \right), \tag{100}
\]

with \(\omega = \tau e\Phi, Y = \text{coth} \omega\).

Now, let us consider the two most interesting particular cases as before. As in the case of the fermion theory presented in Sec. VII, in the case of scalar QED, the derivative part of the effective Lagrangian is easily obtained by using (100) with \(\Phi = iB\)

\[
\mathcal{L}_{\text{der}}^{(3+1)\text{scal}} (B) = \frac{e^2}{2(8\pi^2)^2 |eB|} \int_0^\infty \frac{d\omega}{\omega} \exp \left( - m^2 \frac{\omega}{|eB|} \right) \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \left( \frac{\omega}{\sinh \omega} \right). \tag{101}
\]

And again, as is easy to check, the situation with (101) resembles that in the (2+1)-dimensional scalar QED: inhomogeneities of the external magnetic field lead to decreasing the vacuum energy density for large values of the ratio \(m^2/|eB|\) (\(m^2/|eB| \gtrsim 0.41\)) and to increasing for small values (\(m^2/|eB| \lesssim 0.41\)).

In addition to the representation (101), we find the following one (see Eq. (D12) in Appendix D)

\[
\mathcal{L}_{\text{der}}^{(3+1)\text{scal}} (B) = \frac{e^2}{2(8\pi^2)^2 |eB|} \left[ \frac{11}{6} \left( \frac{m^2}{|eB|} \right)^3 - \frac{m^2}{|eB|} \left( 1 + \left( \frac{m^2}{|eB|} \right)^2 \right) \psi \left( \frac{1}{2} + \frac{m^2}{2|eB|} \right) \right.
+ 7 \frac{m^2}{6 |eB|} + 2 \left[ 1 + 3 \left( \frac{m^2}{|eB|} \right)^2 \right] \ln \Gamma \left( \frac{1}{2} + \frac{m^2}{2|eB|} \right) - \ln \sqrt{2\pi}
+ 24 \zeta' \left( -2, \frac{1}{2} + \frac{m^2}{2|eB|} \right) - 24 m^2 |eB| \zeta \left( -1, \frac{1}{2} + \frac{m^2}{2|eB|} \right) \right]. \tag{102}
\]
In the limit $m^2 \ll |eB|$, this expression allows the following asymptotic expansion,

$$\mathcal{L}_{der}^{(3+1)\text{scal}}(B) \simeq \frac{e^2(\partial_i B)^2}{2(8\pi)^2|eB|} \left[ -18\epsilon'(-2) - \ln 2 + \frac{2m^2}{3|eB|} + \frac{m^6}{3|eB|^3} 
- \frac{m^4}{2|eB|^2} \sum_{k=0}^{\infty} \frac{k+1}{k+2}(2k+1)(k+2) \left( -\frac{m^2}{2|eB|} \right) \right].$$

(103)

In the limit $m^2 \gg |eB|$, on the other hand, we obtain

$$\mathcal{L}_{der}^{(3+1)\text{scal}}(B) \simeq -\frac{e^2(\partial_i B)^2}{(8\pi)^2m^2} \sum_{k=0}^{\infty} \frac{(2k+3-1)B_{2k+4} + (2k+1-1)B_{2k+2}}{2k+1} \left( \frac{|eB|}{m^2} \right)^{2k}. \quad (104)$$

In the case of the electric field directed along the first axis, we obtain

$$\mathcal{L}_{der}^{(3+1)\text{scal}}(E) = \frac{ie^2(\partial_i E)^2}{2(8\pi)^2|eE|} \int_0^\infty \frac{d\omega}{\omega} \exp \left( -i \frac{m^2}{|eE|} \omega \right) \left( \frac{d^3}{d\omega^2} + \frac{d}{d\omega} \right) \left( \frac{\omega}{\sinh \omega} \right). \quad (105)$$

Thus, the imaginary part of derivative part of the Lagrangian reads

$$\mathcal{I}m\mathcal{L}_{der}^{(3+1)\text{scal}}(E) = \frac{e^2(\partial_i E)^2}{27\pi^4|eE|} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \exp \left( -\frac{\pi m^2}{|eE|} n \right) \times \left[ 6 + 6 \frac{\pi m^2}{|eE|} + \pi^2 n^2 \left( 3 \frac{m^4}{|eE|^2} - 1 \right) + \frac{\pi^2 m^4 n^4}{|eE|^2} \left( \frac{m^4}{|eE|^2} - 1 \right) \right],$$

(106)

which determines the correction to the probability of particle-antiparticle creation in a constant electric field expressed through

$$\mathcal{I}m\mathcal{L}_{der}^{(3+1)\text{scal}}(E) = \frac{(eE)^2}{16\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp \left( -\frac{\pi m^2}{|eE|} n \right)$$

$$= -\frac{(eE)^2}{16\pi^3} \text{Li}_2 \left( -\exp \left( -\frac{\pi m^2}{|eE|} \right) \right). \quad (107)$$

As in $(2 + 1)$-dimensional case, we observe that the sum in the right hand side of Eq. (106) is positive only for the large enough values of the mass ($m^2 > 0.388|eE|$).

The expression (106) concludes the list of our results describing the influence of slowly varying external electromagnetic fields on the spinor and scalar QED vacuum in two-derivative approximation.

**X. HOW TO GET HIGHER DERIVATIVE TERMS?**

Obviously, the method of the present paper can be applied for calculating the higher derivative terms (with their total number equal to four or higher) of the low energy effective action in QED. However, the computational work with increasing the total number of derivatives is getting so hard that obtaining already all the four derivative terms seems to be impossible without use of a computer. Just to get feeling how difficult this problem is, let us consider the classification of all the relevant Feynman diagrams in four derivative approximation.

To facilitate the calculation of the perturbative expansion in number of derivatives in the problem at hand, it is appropriate to develop the Feynman diagram technique. Our starting point will be the system of equations (14) and (19). Then, as is seen, the derivative expansion results from all (connected as well as disconnected) vacuum diagrams produced by (19). A somewhat disappointing feature of our Lagrangian is an infinite number of local interactions. Nevertheless, as will become clear in a moment, while working at any finite order of the perturbative theory, one requires only a finite number of those interactions.
We observe that there are two different types of local interactions in (14). The first (bosonic) type contains only the bosonic fields, \( x_\mu(t) \). The corresponding vertices are shown in Figure 1. The other interactions involve both the boson, \( x_\mu(t) \), and the spinor fields, \( \psi_\mu(t) \). These latter produce the vertices given in Figure 2. The integers in the vertices denote the number of derivatives (later called the weights of vertices) of the electromagnetic field with respect to space-time. Some legs in the diagrams are marked by circles and bullets. The circles correspond to legs related to the first Lorentz index (\( \nu \)) of the tensor weight, \( F_{\nu\lambda,\mu_1...\mu_n} \), assigned to the vertex, while the bullets, on the other hand, mark legs which contain the derivatives with respect to the proper time. The latter act on the (bosonic) propagators attached to the marked legs.

The Feynman rules for writing expressions corresponding to Feynman diagrams are more or less standard. One has to use the propagators given in (11) and (12) for connecting the bosonic (solid) and the fermion (dashed) legs, respectively. The combinatoric factors can be straightforwardly derived. Two simplest diagrams giving a nonzero contribution to two-derivative terms in the effective action are represented in Figure 3.

Let us mention the most general rules. To start with, we classify all diagrams leading to terms with a given finite number (called the weight of the corresponding diagrams from now on) of derivatives in the expansion. First of all, we see that diagrams with weight \( \mathcal{N} \) may contain different number of vertices. We denote by \( \text{Der}(\mathcal{N}) \) the set of all diagrams of a given weight \( \mathcal{N} \). By marking the bosonic (Fig. 1) and the fermion (Fig. 2) vertices with \( n \) derivatives just by \([n]\) and \([\bar{n}]\), respectively, we see that the set \( \text{Der}(\mathcal{N}) \) contains a finite number of elements: \( \text{Der}(\mathcal{N}) = \{ [\mathcal{N}], [\mathcal{N}], [\mathcal{N} - 1] \oplus [\mathcal{N}], [\mathcal{N} - 1] \oplus [1], [\mathcal{N} - 1] \oplus [\mathcal{N}], [\mathcal{N} - 1] \oplus [1],... \} \). Each of the elements in \( \text{Der}(\mathcal{N}) \) produces in its turn a (finite) number of Feynman diagrams differing from one another by all possible connections (by means of propagators) between all legs of the vertices. Thus, the diagrams of weight two in Figure 3, related to \( C^{W} \) and \( C^{I} \) in the general expression (50), correspond to elements \([2]\) and \([2]\) in the set \( \text{Der}(2) \), respectively.

Any element of \( \text{Der}(\mathcal{N}) \) specifies the number of different vertices as well as their separate weights. If the number of different vertices in a diagram is given by integers \( \{ V_1, V_2, ..., V_k \} \) then the overall factor in front of the corresponding expression is \( 1/(V_1!V_2!...V_k!) \). Next, let the total number of bosonic and the fermion vertices are \( k_B \) and \( k_F \), respectively. Then the total number of bosonic legs of all the vertices in such a diagram is \( 2k_B + \mathcal{N} \), while the number of the fermion legs is \( 2k_F \). Since, we are interested in vacuum diagrams (with all legs being connected) only, the diagrams of an odd weight \( \mathcal{N} \) are not relevant for our derivative expansion. So, we put \( \mathcal{N} = 2n \). As is easy to count, the total number of all possible connections (by means of \( k_B + n \) bosonic and \( k_F \) fermion propagators) between these vertices is \( (N + 2k_B - 1)!!(2k_F - 1)!! \), where we assume that \((1)!! \equiv 1 \). This is an upper bound for the number of different diagrams with the given vertex set corresponding to the given element \( \overline{[V_1] \oplus ... \oplus [V_{k_F}]} \oplus \overline{[V_{k_F+1}] \oplus ... \oplus [V_k]} \in \text{Der}(\mathcal{N}) \). However, due to the symmetry of the vertices with respect to permutations of their non-marked legs as well as with respect to permutations of identical vertices, some of the diagrams are in fact equivalent. For example, the naive number of all relevant diagrams for the two-derivative terms in the effective action is 25. On the other hand, as is seen from our general result (13), the actual number of non-equivalent terms is 11.

Now let us say several words about the sign factors of diagrams. First, all diagrams of weight \( \mathcal{N} = 2n \) have an overall factor \((-i)^n\). To get the right sign resulting from the the fermion loops, one preliminary has to assign the direction of the the fermion flow in the diagram by adding arrows on the the fermion (dashed) lines. Then the overall sign factor is obtained by multiplying sign factors for each the fermion loop of the diagram. Each of the loop factors is defined by the formula: \((-1)^{N_0+1}\) where \( N_0 \) is the number of arrows running into circles of loop vertices. This rule takes into account the fact that the fermion propagators are antisymmetric with respect to the simultaneous permutation of their Lorentz indices and proper time coordinates as well as the fact that tensor weight at the fermion vertices feels the order of first two indices.

Concluding this section, we would like to express a hope that the brief description of the Feynman technique given here would be enough for writing a code in some of the languages used for analytical computations if such a need appears.

**XI. CONCLUSIONS**

In conclusion, here we further develop the method of our previous paper [3] and generalize it to quantum electrodynamics in 2 + 1 dimensions. The distinctive feature of our approach is the use of a special matrix basis (in Lorentz indices) in order to deal with functions of antisymmetric tensors such as the (background) field strength tensor in QED. In Sec. [11], we give the explicit representation for these matrices as well as demonstrate how they facilitate the calculation. It is also the use of these matrices that allowed us to obtain the derivative expansion in the fully covariant form.
Then, in this paper, we derived explicit expression of the two-derivative term in the derivative expansion of the effective action in QED in both fermion and scalar QED in 2+1 and 3+1 dimensions. In addition, we also calculated the leading order corrections to the probability of the particle-antiparticle creation rate produced by space-time gradients of the electric field background. The latter gives a non-trivial generalization of the famous Schwinger result in a constant electric field \[3\].

Among other results, here we derived the Feynman rules for generating the perturbative expansion of the effective action in the number of derivatives. This means that, in principle, an arbitrary finite order of the derivative expansion is calculable in our approach. For obvious reasons, the complexity of calculation explodes at higher orders and, in the case of the four-derivative approximation, the computational work already becomes so hard that it is almost impossible to get a result in the closed form without using a computer. By making use of the Feynman rules, derived in this paper, one can write a computer code in order to calculate higher order approximations.

At the end, let us also make a few remarks about possible tests and applications of derivative expansion obtained in this paper.

As in the case of the Euler-Heisenberg action, the derivative corrections will affect, among other things, the photon-photon scattering amplitude. For a vanishing background field, the latter is discussed in detail in \[1\]. Obviously, when the background field is non-zero the corresponding amplitude and the energy dependence of the cross section are going to change. As for the explicit form of the result, it will be given elsewhere.

Besides that, it is likely that the explicit dependence of the photon-photon cross section would be of great interest in studies of some real systems which exist under extremely large magnetic fields. The vicinity of the neutron stars and the early Universe \[31\] are the most natural candidates of such systems.

The formal derivative expansion might also be useful in other problems, such as the generalization of the theory of magnetic catalysis of chiral symmetry breaking in QED\(_4\) \[32\] and QED\(_5\) \[33\] to the case of inhomogeneous external fields.

**XII. ACKNOWLEDGMENTS**

We would like to thank Theodore Hall for pointing out mistakes in the expressions for the imaginary part of the photon-photon scattering amplitude. For a vanishing background field, the latter is discussed in detail in \[1\]. Obviously, when the background field is non-zero the corresponding amplitude and the energy dependence of the cross section are going to change. As for the explicit form of the result, it will be given elsewhere.

Besides that, it is likely that the explicit dependence of the photon-photon cross section would be of great interest in studies of some real systems which exist under extremely large magnetic fields. The vicinity of the neutron stars and the early Universe \[31\] are the most natural candidates of such systems.

The formal derivative expansion might also be useful in other problems, such as the generalization of the theory of magnetic catalysis of chiral symmetry breaking in QED\(_4\) \[32\] and QED\(_5\) \[33\] to the case of inhomogeneous external fields.

**APPENDIX A: COEFFICIENT FUNCTIONS WHICH APPEAR IN THE DERIVATIVE EXPANSION**

Here we give the functions\(^1\) used in \[30\] and \[33\]:

\[
C^W(\bar{\alpha}, \bar{\beta}) = \tau^2 \tanh(\alpha \tau) H(\beta \tau),
\]

\[
C^V(\bar{\alpha}, \bar{\beta}) = \alpha \tau^3 H(\alpha \tau) H(\beta \tau) - \frac{\alpha \tau}{\beta^2 - \alpha^2} [H(\beta \tau) - H(\alpha \tau)],
\]

\[
C^W_1(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \frac{\tau^3}{8} \tanh(\alpha \tau) \tanh(\beta \tau) H(\gamma \tau),
\]

\[
C^W_2(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \frac{\tau^2}{4} \left[ \tanh(\alpha \tau) + \tanh(\beta \tau) \right] \frac{H(\alpha \tau + \beta \tau) - H(\gamma \tau)}{\alpha + \beta - \gamma} - \frac{H(\gamma \tau)}{\alpha + \beta},
\]

\[
C^V_1(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = -\frac{\tau^3}{4} \tanh(\alpha \tau) \left( \beta \tau H(\beta \tau) H(\gamma \tau) - \frac{H(\beta \tau) - H(\gamma \tau)}{\tau(\beta + \gamma)} \right),
\]

\(^1\)Here we corrected the typos which appeared in \[30\], namely we (i) omitted an extra term in the expression for \(C^V_1\) that was mistakenly present; (ii) replaced the wrong factor \(H(\tau \beta)\) in the last term of \(C^V_4\) by \(H(\tau \gamma)\), and (iii) added the third term in \(C^V_4\) which was originally missing. In addition, we rewrote \(C^V_4\) in a slightly different form.
that in [9] we ignored this difference.

Here we used the following notation

\[ H(x) = \frac{x \coth x - 1}{x^2}, \]  

and the letters with bars differ from the letters without those only in a factor of the electric charge: \( \alpha = e\bar{\alpha}. \) Note that in [9] we ignored this difference.

As \( \tau \to 0, \) these coefficient functions have the following asymptotic behavior

\[ C_W^W(\bar{\alpha}, \bar{\beta}) \simeq \frac{\alpha \tau^3}{3} - \frac{\alpha \tau^5}{45} (5\alpha^2 + \beta^2) + O(\tau^7), \]  

\[ C_V^W(\bar{\alpha}, \bar{\beta}) \simeq \frac{2\alpha \tau^3}{15} - \frac{\alpha \tau^5}{105} (\alpha^2 + \beta^2) + O(\tau^7), \]
\[
C^1_{\bar{W}W}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq \frac{\alpha \beta \tau^5}{24} + O(\tau^7), \tag{A15}
\]
\[
C^2_{\bar{W}W}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq -\frac{\tau^3}{12} + \frac{\tau^5}{180} (4\alpha^2 + 4\beta^2 + \gamma^2 - 7\alpha \beta - \alpha \gamma - \beta \gamma) + O(\tau^7), \tag{A16}
\]
\[
C^1_{VV}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq \frac{\alpha \tau^5}{180} (\gamma - 6\beta) + O(\tau^7), \tag{A17}
\]
\[
C^2_{VV}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq -\frac{\alpha \beta \tau^5}{90} + O(\tau^7), \tag{A18}
\]
\[
C^4_{VV}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq \frac{\tau^3}{90} - \frac{\tau^5}{945} (2\alpha^2 + 2\beta^2 + 2\gamma^2 - 51\alpha \beta - 9\alpha \gamma + 9\beta \gamma) + O(\tau^7), \tag{A19}
\]
\[
C^2_{VV}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq -\frac{4\tau^3}{45} + \frac{\tau^5}{1890} (10\alpha^2 + 10\beta^2 + 13\gamma^2 + 15\alpha \beta + 3\alpha \gamma - 3\beta \gamma) + O(\tau^7), \tag{A20}
\]
\[
C^3_{VV}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq -\frac{\tau^3}{45} + \frac{\tau^5}{945} (2\alpha^2 + 2\beta^2 + 2\gamma^2 + 9\alpha \beta) + O(\tau^7), \tag{A21}
\]
\[
C^4_{VV}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq \frac{2\tau^3}{45} - \frac{4\tau^5}{945} (\alpha^2 + \beta^2 + \gamma^2) + O(\tau^7), \tag{A22}
\]
\[
C^5_{VV}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \simeq \frac{\tau^3}{1890} - \frac{\tau^5}{1890} (20\alpha^2 + 23\beta^2 + 23\gamma^2 - 12\alpha \beta - 12\alpha \gamma + 6\beta \gamma) + O(\tau^7). \tag{A23}
\]

APPENDIX B: EXPANSION OF THE DERIVATIVE TERMS IN POWERS OF THE PROPER TIME

In this appendix we give the proper time expansion of the derivative terms, as in Eqs. (56) and (88), up to the order \(\tau^5\). In case of spinor QED, from Eq. (50), we derive the expansion

\[
\text{tr}(\langle x|U(\tau)|x\rangle \simeq \text{tr}(\langle x|U(\tau)|x\rangle_0 + \frac{ie^2 \tau^3}{20} F^{\nu\lambda} F_{\nu\lambda,\mu}^\mu \\
+ \frac{ie^2 \tau^3}{180} (\frac{7}{2} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} - F^{\nu\lambda} F_{\nu\mu,\mu}) - \frac{ie^4 \tau^5}{315} F_{\nu\lambda,\mu,\nu} (16 F^{\mu\nu\lambda} F^{\nu\lambda} + F^{\nu\lambda}(F^2)^{\mu\nu}) \\
+ \frac{ie^4 \tau^5}{1890} (2 F_{\nu\lambda,\mu} F^{\nu\rho} (F^2)^\lambda_{\mu} - 2 F_{\nu\lambda,\mu} F^{\nu\lambda,\rho} (F^2)^\mu_{\rho} - 37 F_{\nu\lambda,\mu} F^{\nu\sigma,\mu} (F^2)^{\lambda}_{\sigma} + F_{\nu,\mu} F^{\sigma,\mu} (F^2)^{\nu\sigma}) \\
- \frac{ie^4 \tau^5}{2520} F_{\nu\lambda,\mu} F_{\nu\sigma,\mu} (38 \eta^{\mu\nu \lambda,\rho} F^{\nu\lambda} - 12 \eta^{\nu\lambda,\rho} F^{\lambda,\rho} F^{\mu\sigma} + 47 \eta^{\mu\nu \lambda,\rho} F^{\nu\lambda} F^{\sigma,\rho} + 16 \eta^{\mu\nu \lambda,\rho} F^{\nu\lambda} F^{\sigma,\rho}) \bigg). \tag{B1}
\]

Notice that despite the difference between the two sets of matrices \(A_{\nu\mu}^{\alpha\beta} \) in \(2 + 1 \) and \(3 + 1 \) dimensions, the expression in square brackets is independent of the dimension up to this order in the expansion. In calculation, we took into account the Bianchi identity to show that many seemingly different terms appearing in the expansion reduce to the same structures. In particular, the following relations are the identities that we needed

\[
F_{\nu\lambda,\mu} \eta^{\nu\lambda,\rho} F^{\nu\rho} = \frac{1}{2} F_{\nu\lambda,\mu} \eta^{\nu\lambda,\rho} F^{\nu\rho}, \tag{B2}
\]
\[
F_{\nu\lambda,\mu} F^{\nu\rho} (F^2)^{\lambda}_{\mu} = \frac{1}{2} F_{\nu\lambda,\mu} F^{\nu\lambda} (F^2)^{\mu\lambda}, \tag{B3}
\]
\[
F_{\nu\lambda,\mu} F^{\nu\rho} F^{\sigma,\mu} = \frac{1}{2} F_{\nu\lambda,\mu} F^{\nu\lambda} F^{\sigma,\mu}, \tag{B4}
\]
\[
F_{\nu\lambda,\mu} F^{\nu\sigma} F^{\mu\rho} = \frac{1}{2} F_{\nu\lambda,\mu} F^{\nu\lambda} F^{\mu\sigma} + \frac{1}{2} F_{\nu\lambda,\mu} F^{\nu\lambda} F^{\mu\sigma}, \tag{B5}
\]
\[
F_{\nu\lambda,\mu} F^{\nu\rho} F^{\lambda,\rho} = -\frac{1}{2} F_{\nu\lambda,\mu} F^{\nu\lambda} F^{\sigma,\rho}, \tag{B6}
\]
\[
F_{\nu\lambda,\mu} F^{\nu\sigma} F^{\mu\rho} = -2 F_{\nu\lambda,\mu} F^{\nu\lambda} F^{\sigma,\rho}, \tag{B7}
\]
\[
F_{\nu\lambda,\mu} F^{\nu\lambda} F^{\sigma,\rho} = \frac{1}{4} F_{\nu\lambda,\mu} F^{\nu\lambda} F^{\sigma,\rho}. \tag{B8}
\]
After expanding $Tr(x|U(\tau)|x)$ in Eq. (33) in powers of $\tau$ up to the terms of order $\tau^3$ and substituting the obtained expression in the definition of the effective action, we arrive at the following two-derivative correction of the order $1/m^6$,

$$
\mathcal{L}^{(3+1)\text{spin}}_{1/m^6} = -\frac{11\alpha^2}{630m^6} F^{\beta\gamma} F_{\beta\gamma} F^{\nu\lambda} F_{\nu\lambda,\mu} + \frac{4\alpha^2}{315m^6} (F^2)_{\mu\nu}^\lambda F^{\nu\lambda} F_{\nu\lambda,\mu} \\
+ \frac{\alpha^2}{270m^6} F^{\beta\gamma} F_{\beta\gamma} \left( \frac{7}{2} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} - F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} \right) \\
- \frac{2\alpha^2}{945m^6} \left( 2 F^{\nu\lambda,\mu} F^{\nu\lambda,\rho} (F^2)_{\mu\rho} - 2 F^{\nu\lambda,\mu} F^{\nu\lambda,\rho} (F^2)_{\mu\rho} - 3 F^{\nu\lambda,\mu} F^{\nu\lambda,\rho} (F^2)_{\mu\rho} + F_{\nu\mu,\rho} F_{\nu\mu,\rho} (F^2)_{\mu\rho} \right) \\
+ \frac{\alpha^2}{630m^6} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} \left( 38\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} - 12\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} + 47\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} + 16\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} \right),
$$

(B15)

to the one-loop effective action in spinor QED in $3 + 1$ dimensions, and the correction of the order $1/m^7$,

$$
\mathcal{L}^{(2+1)\text{spin}}_{1/m^7} = -\frac{11\alpha^2}{336m^7} F^{\beta\gamma} F_{\beta\gamma} F^{\nu\lambda} F_{\nu\lambda,\mu} + \frac{\alpha^2}{42m^7} (F^2)_{\mu\nu}^\lambda F^{\nu\lambda} F_{\nu\lambda,\mu} \\
+ \frac{\alpha^2}{144m^7} F^{\beta\gamma} F_{\beta\gamma} \left( \frac{7}{2} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} - F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} \right) \\
- \frac{\alpha^2}{252m^7} \left( 2 F^{\nu\lambda,\mu} F^{\nu\lambda,\rho} (F^2)_{\mu\rho} - 2 F^{\nu\lambda,\mu} F^{\nu\lambda,\rho} (F^2)_{\mu\rho} - 3 F^{\nu\lambda,\mu} F^{\nu\lambda,\rho} (F^2)_{\mu\rho} + F_{\nu\mu,\rho} F_{\nu\mu,\rho} (F^2)_{\mu\rho} \right) \\
+ \frac{\alpha^2}{336m^7} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} \left( 38\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} - 12\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} + 47\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} + 16\eta^{\mu\rho} F^{\lambda\sigma} F^{\nu\kappa} \right),
$$

(B16)

to the effective action in $2 + 1$ dimensions. It turns out that these latter can be further simplified. Indeed, after integrating by parts, the results can be expressed through the following seven Lorentz scalars,

$$
L_1 = F^{\beta\gamma} F_{\beta\gamma} F^{\nu\lambda} F_{\nu\lambda,\mu},
$$

(B17)

$$
L_2 = F^{\beta\gamma} F_{\beta\gamma} F^{\nu\lambda} F_{\nu\lambda,\mu},
$$

(B18)

$$
L_3 = F^{\beta\gamma} F_{\beta\gamma} F^{\nu\lambda} F_{\nu\lambda,\mu},
$$

(B19)

$$
L_4 = F_{\nu\lambda,\mu} F^{\nu\lambda} (F^2)_{\mu\nu},
$$

(B20)

$$
L_5 = F^{\nu\lambda} F_{\nu\lambda,\mu} F^{\lambda\sigma} F_{\sigma,\mu},
$$

(B21)

$$
L_6 = F_{\nu\lambda,\mu} F^{\nu\lambda} (F^3)_{\mu\nu},
$$

(B22)

$$
L_7 = F_{\nu\lambda,\mu} F^{\nu\lambda} (F^2)_{\mu\nu}.
$$

(B23)

Thus, the final results in $3 + 1$ and in $2 + 1$ dimensions read

$$
\mathcal{L}^{(3+1)\text{spin}}_{1/m^6} = -\frac{16\alpha^2}{315m^6} L_1 - \frac{8\alpha^2}{315m^6} L_2 + \frac{2\alpha^2}{315m^6} L_3 - \frac{\alpha^2}{945m^6} L_4 \\
- \frac{11\alpha^2}{945m^6} L_5 - \frac{26\alpha^2}{945m^6} L_6 + \frac{4\alpha^2}{189m^6} L_7,
$$

(B24)
\[ L_{1/m^6}^{2+1} = -\frac{2\alpha^2 \pi}{21m^6} L_1 - \frac{\alpha^2 \pi}{21m^6} L_2 + \frac{\alpha^2 \pi}{84m^6} L_3 - \frac{\alpha^2 \pi}{504m^6} L_4 \]
\[ - \frac{11\alpha^2 \pi}{504m^6} L_5 - \frac{13\alpha^2 \pi}{252m^6} L_6 + \frac{5\alpha^2 \pi}{126m^6} L_7, \]  
\( \text{B25} \)

respectively. This should be compared with the result of Ref. \( \text{[3]} \) (see Eq. (14) there). Notice that the photon field in Ref. \( \text{[3]} \) describes on-shell quanta, and, as a result, the terms containing \( L_1, L_3, L_6 \) and \( L_7 \) do not appear (they are proportional to \( k^2 = 0 \)).

In a similar way, in the case of scalar QED we obtain the following expression for the expansion of Eq. \( \text{[3]} \),

\[ \langle x|\bar{U}_{\text{bos}}(\tau)|x \rangle \simeq \langle x|\bar{U}_{\text{bos}}(\tau)|x \rangle_0 \left[ 1 - \frac{\alpha^2 \pi^3}{30} F^{\nu\lambda} F_{\nu\lambda,\mu} \right. \]
\[ \left. - \frac{\alpha^2 \pi^3}{180} \left( 4 F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} + F^{\nu\lambda,\mu} F_{\nu,\mu,\lambda} \right) + \frac{\alpha^2 \pi^5}{480} F_{\nu\lambda,\mu,\kappa} \right] \]
\[ + \frac{\alpha^2 \pi^7}{7560} \left( 8 F_{\nu\lambda,\mu,\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} + 13 F_{\nu\lambda,\mu,\rho} F^{\nu\lambda,\rho} (F^2)^{\mu\lambda} + 20 F_{\nu\lambda,\mu,\rho} F^{\nu,\sigma,\mu} (F^2)^{\lambda\sigma} + 4 F_{\nu\lambda,\mu,\rho} F_{\sigma,\rho} (F^2)^{\lambda\sigma} \right) \]
\[ + \frac{\alpha^2 \pi^5}{5040} F_{\nu\lambda,\mu,\sigma,\rho} \left( 8 \eta^{\mu\rho} F^{\nu\lambda,\sigma\rho} (F^2)^{\lambda\rho} - 4 \eta^{\mu\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} - 17 \eta^{\mu\rho} F^{\nu\lambda,\sigma} (F^2)^{\lambda\sigma} + 24 \eta^{\mu\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} \right), \]  
\( \text{B26} \)

leading to the \( 1/m^6 \) correction to the effective Lagrangian density,

\[ L_{1/m^6}^{(3+1)\text{scal}} = -\frac{\alpha^2}{210m^6} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} + \frac{\alpha^2}{210m^6} (F_2)^{\mu\kappa} F^{\nu\lambda} F_{\nu\lambda,\mu} \kappa \]
\[ - \frac{\alpha^2}{1080m^6} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} + F^{\nu\lambda,\mu} F_{\nu,\mu,\lambda} \]
\[ + \frac{\alpha^2}{3780m^6} \left( 8 F_{\nu\lambda,\mu,\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} + 13 F_{\nu\lambda,\mu,\rho} F^{\nu\lambda,\rho} (F^2)^{\mu\lambda} + 20 F_{\nu\lambda,\mu,\rho} F^{\nu,\sigma,\mu} (F^2)^{\lambda\sigma} + 4 F_{\nu\lambda,\mu,\rho} F_{\sigma,\rho} (F^2)^{\lambda\sigma} \right) \]
\[ + \frac{\alpha^2}{2520m^6} F_{\nu\lambda,\mu,\sigma,\rho} \left( 8 \eta^{\mu\rho} F^{\nu\lambda,\sigma\rho} (F^2)^{\lambda\rho} - 4 \eta^{\mu\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} - 17 \eta^{\mu\rho} F^{\nu\lambda,\sigma} (F^2)^{\lambda\sigma} + 24 \eta^{\mu\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} \right), \]  
\( \text{B27} \)

in \( 3 + 1 \) dimensions, and the \( 1/m^7 \) correction,

\[ L_{1/m^7}^{(2+1)\text{scal}} = -\frac{5\alpha^2 \pi}{330m^6} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} - \frac{\alpha^2 \pi}{112m^7} (F_2)^{\mu\kappa} F^{\nu\lambda} F_{\nu\lambda,\mu} \kappa \]
\[ + \frac{\alpha^2 \pi}{576m^6} F^{\nu\lambda,\mu} F_{\nu\lambda,\mu} + F^{\nu\lambda,\mu} F_{\nu,\mu,\lambda} \]
\[ - \frac{\alpha^2 \pi}{2016m^7} \left( 8 F_{\nu\lambda,\mu,\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} + 13 F_{\nu\lambda,\mu,\rho} F^{\nu\lambda,\rho} (F^2)^{\mu\lambda} + 20 F_{\nu\lambda,\mu,\rho} F^{\nu,\sigma,\mu} (F^2)^{\lambda\sigma} + 4 F_{\nu\lambda,\mu,\rho} F_{\sigma,\rho} (F^2)^{\lambda\sigma} \right) \]
\[ - \frac{\alpha^2 \pi}{1344m^7} F_{\nu\lambda,\mu,\sigma,\rho} \left( 8 \eta^{\mu\rho} F^{\nu\lambda,\sigma\rho} (F^2)^{\lambda\rho} - 4 \eta^{\mu\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} - 17 \eta^{\mu\rho} F^{\nu\lambda,\sigma} (F^2)^{\lambda\sigma} + 24 \eta^{\mu\rho} F^{\nu\lambda,\rho} (F^2)^{\lambda\mu} \right), \]  
\( \text{B28} \)

in \( 2 + 1 \) dimensions.

Up to a divergence, the derived corrections to the effective action are equivalent to

\[ L_{1/m^6}^{(3+1)\text{scal}} = -\frac{13\alpha^2}{2520m^6} L_1 - \frac{\alpha^2}{840m^6} L_2 - \frac{\alpha^2}{2520m^6} L_3 + \frac{\alpha^2}{1890m^6} L_4 \]
\[ + \frac{\alpha^2}{3780m^6} L_5 - \frac{\alpha^2}{3780m^6} L_6 + \frac{\alpha^2}{1890m^6} L_7, \]  
\( \text{B29} \)

\[ L_{1/m^7}^{(2+1)\text{scal}} = -\frac{13\alpha^2}{1344m^7} L_1 + \frac{\alpha^2}{48m^7} L_2 + \frac{\alpha^2}{1344m^7} L_3 - \frac{\alpha^2}{1008m^7} L_4 \]
\[ - \frac{\alpha^2}{2016m^7} L_5 + \frac{\alpha^2}{2016m^7} L_6 - \frac{\alpha^2}{1008m^7} L_7, \]  
\( \text{B30} \)

in \( 3 + 1 \) and \( 2 + 1 \) dimensions, respectively. Here we used the same seven scalars as in the case of spinor QED above.
APPENDIX C: COEFFICIENT FUNCTIONS WHICH APPEAR IN PURELY ELECTRIC AND PURELY MAGNETIC CASES

In this appendix we list the formulas, similar to that in Eq. (60), which appear in the course of reduction the general expression for the derivative contribution to the case of a pure magnetic (electric) field background. These are

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} A^{(\rho)}_{(l)} = -2i C^{(W)} (\alpha, \alpha) \sum_{i=1}^{2} \partial_{i} \partial_{i} B,
\]

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) \left( A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} A^{(\rho)}_{(l)} + 2 A^{(\rho)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} \right) = 4i C^{(V)} (\alpha, \alpha) \sum_{i=1}^{2} \partial_{i} \partial_{i} B,
\]

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) C^{(W)}_{1} (f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} = 4 C^{(W)}_{1} (\alpha, \alpha, \alpha) \sum_{i=1}^{2} (\partial_{i} B)^{2},
\]

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) C^{(W)}_{2} (f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} = -2 C^{(W)}_{2} (\alpha, \alpha, \alpha, \alpha) \sum_{i=1}^{2} (\partial_{i} B)^{2},
\]

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) C^{(W)}_{1} (f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} = 4 C^{(W)}_{1} (\alpha, \alpha, \alpha) \sum_{i=1}^{2} (\partial_{i} B)^{2},
\]

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) C^{(W)}_{2} (f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} = -2 C^{(W)}_{2} (\alpha, \alpha, \alpha, \alpha) \sum_{i=1}^{2} (\partial_{i} B)^{2},
\]

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) C^{(W)}_{3} (f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} = C^{(W)}_{3} (\alpha, \alpha, \alpha) \sum_{i=1}^{2} (\partial_{i} B)^{2},
\]

\[
C^{(V)}_{4} (f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} = C^{(V)}_{4} (\alpha, \alpha, \alpha) \sum_{i=1}^{2} (\partial_{i} B)^{2},
\]

\[
F_{\nu,\lambda,\mu,\rho}(f, j, i, k) C^{(W)}_{5} (f, j, i, k) A^{(\nu)}_{(j)} A^{(\lambda)}_{(i)} A^{(\mu)}_{(k)} = C^{(W)}_{5} (\alpha, \alpha, \alpha) \sum_{i=1}^{2} (\partial_{i} B)^{2},
\]

where \( \alpha = \sqrt{2F} \).

The latter expressions contain the coefficient functions from Appendix [A] calculated for a particular values of their arguments. The convenient representation for them reads
where, by definition, 

\[ C^V(\bar{a}, \bar{a}) = \frac{\tau^2}{2\omega}(3\omega^2H^2 + 3H - 1), \]  

\[ C^W(\bar{a}, \bar{a}) = \tau^2 \tanh(\omega)H, \]  

\[ C_1^{WW}(\bar{a}, \bar{a}, \bar{a}) = \frac{\tau^3}{8} \tanh^2(\omega)H, \]  

\[ C_2^{WW}(\bar{a}, -\bar{a}, \bar{a}) = -\frac{\tau^3}{4} (1 - \tanh^2(\omega))H, \]  

\[ C_1^{VV}(\bar{a}, \bar{a}, -\bar{a}) = -\frac{\tau^3}{4\omega} \tanh(\omega)(2\omega^2H^2 + 3H - 1), \]  

\[ C_2^{VV}(\bar{a}, \bar{a}, \bar{a}) = -\frac{\tau^3}{4\omega} \tanh(\omega)(\omega^2H^2 + 3H - 1), \]  

\[ C_1^{VV}(\bar{a}, \bar{a}, -\bar{a}) = \frac{\tau^3}{4\omega^2} (4\omega^4H^3 + 7\omega^2H^2 - 2\omega^4H + 3H - 1), \]  

\[ C_1^{VV}(\bar{a}, -\bar{a}, \bar{a}) = -\frac{\tau^3}{2\omega^2} (4\omega^4H^3 + 10\omega^2H^2 - 3\omega^4H + 6H - 2), \]  

\[ C_1^{VV}(\bar{a}, -\bar{a}, \bar{a}) = -\frac{\tau^3}{4\omega^2} (2\omega^4H^3 - \omega^2H^2 - 3H + 1), \]  

\[ C_2^{VV}(\bar{a}, \bar{a}, \bar{a}) = \frac{\tau^3}{2\omega^2} (2\omega^4H^3 + 5\omega^2H^2 - 2\omega^2H + 3H - 1), \]  

\[ C_2^{VV}(\bar{a}, -\bar{a}, \bar{a}) = -\frac{\tau^3}{2\omega^2} (2\omega^4H^3 + 11\omega^2H^2 - 2\omega^2H + 9H - 3), \]  

\[ C_3^{VV}(\bar{a}, \bar{a}, -\bar{a}) = \frac{\tau^3}{4\omega^2} (4\omega^4H^3 + 13\omega^2H^2 - 4\omega^2H + 9H - 3), \]  

\[ C_3^{VV}(\bar{a}, -\bar{a}, \bar{a}) = -\frac{\tau^3}{2\omega^2} (\omega^4H^3 + 4\omega^2H^2 - \omega^2H + 3H - 1), \]  

\[ C_4^{VV}(\bar{a}, \bar{a}, \bar{a}) = -\frac{\tau^3}{4\omega^2} (2\omega^4H^3 + 5\omega^2H^2 - 2\omega^2H + 3H - 1), \]  

\[ C_5^{VV}(\bar{a}, -\bar{a}, \bar{a}) = -\frac{\tau^3}{4\omega^2} (2\omega^4H^3 - \omega^2H^2 - 2\omega^2H - 3H + 1), \]  

\[ C_5^{VV}(\bar{a}, -\bar{a}, \bar{a}) = -\frac{\tau^3}{\omega^2} (2\omega^4H^3 + 5\omega^2H^2 - 2\omega^2H + 3H - 1), \]  

where, by definition, \( \omega = e\bar{a}\tau \) and \( H = H(\omega) \).

**APPENDIX D: SPECIAL FUNCTION REPRESENTATION FOR THE INTEGRALS WHICH APPEAR IN THE PURELY ELECTRIC AND PURELY MAGNETIC CASES**

In the main text, we saw that the calculation of the effective action for spinor QED in an external magnetic field reduces to evaluating the following integral (with \( \mu = 1/2 \) in 2 + 1 dimensions, and \( \mu = 0 \) in 3 + 1 dimensions)

\[
I^{(\text{spin})}(\sigma; \mu) = \int_0^\infty d\omega \omega^{\mu - 1} e^{-\sigma \omega} \frac{d^3}{d\omega^3} (\omega \coth(\omega)) = \int_0^\infty d\omega \omega^{\mu - 1} e^{-\sigma \omega} \left( \omega \coth(\omega) - 1 - \frac{\omega^2}{3} \right)
\]

\[
= -\int_0^\infty d\omega \left( \omega \coth(\omega) - 1 - \frac{\omega^2}{3} \right) \frac{d^3}{d\omega^3} (\omega^{\mu - 1} e^{-\sigma \omega})
\]

\[
= \int_0^\infty d\omega \omega^{\mu - 1} e^{-\sigma \omega} \left( \coth(\omega) - \frac{1}{\omega} \right) \frac{\omega^2 (2 - 3\mu) (1 - \mu)}{\omega^2} + 3\sigma (2 - \mu) (1 - \mu) + 3\sigma^2 (1 - \mu) + \sigma^3 \omega \right),
\]  

(D1)
where we integrated by parts (to avoid divergences as \( \omega \to 0 \) we subtracted the first two terms of the hyperbolic cotangent asymptotes). For large enough values of the parameter \( \mu \), one can apply the following table integrals \[27\]

\[
\int_0^\infty d\omega \omega^{\mu-1} e^{-\sigma \omega} \coth \omega = \Gamma(\mu) \left[ 2^{1-\mu} \zeta \left( \mu, 1 + \frac{\sigma}{2} \right) + \sigma^{-\mu} \right], \tag{D2}
\]

\[
\int_0^\infty d\omega \omega^{\mu-1} e^{-\sigma \omega} = \sigma^{-\mu} \Gamma(\mu). \tag{D3}
\]

Thus, the integral in Eq. \((D1)\) yields

\[
I^{(\text{spin})}(\sigma; \mu) = 2^{-\mu} \Gamma(\mu + 1) \left[ \sigma^3 \zeta \left( \mu + 1, 1 + \frac{\sigma}{2} \right) + 6\sigma^2 \frac{1-\mu}{\mu} \zeta \left( \mu, 1 + \frac{\sigma}{2} \right) - 12\sigma \frac{2-\mu}{\mu} \zeta \left( \mu - 1, 1 + \frac{\sigma}{2} \right) + 8\sigma \frac{3-\mu}{\mu} \zeta \left( \mu - 2, 1 + \frac{\sigma}{2} \right) \right]. \tag{D4}
\]

As one can easily check, the original integral in Eq. \((D1)\) is well defined for \( \mu > -1 \). Therefore, the last expression should allow a well defined analytical continuation to the whole that range of values of \( \mu \). Notice that this should be true even despite the fact that the intermediate integrals, as in Eqs. \((D2)\) and \((D3)\), may not be well defined for all values \( \mu > -1 \). In particular, by an analytical continuation, we obtain the results for the values of \( \mu \) which are of interest,

\[
I^{(\text{spin})}(\sigma; \mu) = 2 \sqrt{2\pi} \left[ 5 \zeta \left( \frac{-3}{2}, 1 + \frac{\sigma}{2} \right) - 9 \frac{\sigma}{2} \zeta \left( \frac{-1}{2}, 1 + \frac{\sigma}{2} \right) + 3 \left( \frac{\sigma}{2} \right)^2 \zeta \left( \frac{1}{2}, 1 + \frac{\sigma}{2} \right) + \left( \frac{\sigma}{2} \right)^3 \zeta \left( \frac{3}{2}, 1 + \frac{\sigma}{2} \right) \right], \tag{D5}
\]

\[
I^{(\text{spin})}(\sigma; 0) = \frac{11}{6} \sigma^3 + \sigma^2 - \frac{1}{3} \sigma - \frac{3}{2} \sigma^3 \psi \left( 1 + \frac{\sigma}{2} \right) + 6\sigma^2 \left[ \ln \Gamma \left( 1 + \frac{\sigma}{2} \right) - \ln \sqrt{2\pi} \right] - 24\sigma \zeta' \left( -1, 1 + \frac{\sigma}{2} \right) + 24\zeta' \left( -2, 1 + \frac{\sigma}{2} \right). \tag{D6}
\]

where the prime denotes the derivative of the zeta function with respect to its first argument. In derivation of the second expression we used the following identities \[27\]

\[
\zeta(-1, q) = -\frac{q^2}{2} + \frac{q}{2} - \frac{1}{12}, \quad \zeta(0, q) = \frac{1}{2} - q, \quad \zeta(-2, q) = -\frac{q^3}{3} + \frac{q^2}{2} - \frac{q}{6}, \quad \zeta'(0, q) = \frac{\partial \zeta(z, q)}{\partial z} \bigg|_{z=0} = \ln \Gamma(q) - \ln \sqrt{2\pi},
\]

\[
\lim_{z\to 1} \left( \zeta(z, q) - \frac{1}{z-1} \right) = -\psi(q). \tag{D7}
\]

In the case of scalar QED, we come to the integral (again, with \( \mu = 1/2 \) in \( 2+1 \) dimensions, and \( \mu = 0 \) in \( 3+1 \) dimensions)

\[
I^{(\text{scal})}(\sigma; \mu) = \int_0^\infty d\omega \omega^{\mu-1} e^{-\sigma \omega} \left( \frac{d^3}{d\omega^3} + \frac{d}{d\omega} \right) \frac{\omega}{\sinh \omega}
\]

\[
= \int_0^\infty d\omega \omega^{\mu-1} e^{-\sigma \omega} \left[ \frac{d^3}{d\omega^3} \left( \frac{\omega}{\sinh \omega} - 1 + \frac{\omega^2}{6} \right) + \frac{d}{d\omega} \left( \frac{\omega}{\sinh \omega} - 1 \right) \right]
\]

\[
= -\int d\omega \left[ \left( \frac{\omega}{\sinh \omega} - 1 + \frac{\omega^2}{6} \right) \frac{d^3}{d\omega^3} \right] \left( \omega^{\mu-1} e^{-\sigma \omega} \right)
\]

\[
= \int_0^\infty d\omega \omega^{\mu-1} e^{-\sigma \omega} \left[ \left( \frac{1}{\sinh \omega} - \frac{1}{\omega} + \frac{\omega}{6} \right) \frac{(3-\mu)(2-\mu)(1-\mu)}{\omega^2} \right]
\]

26
\[ \frac{3\sigma(2-\mu)(1-\mu)}{\omega} + 3\sigma^2(1-\mu) + \sigma^3\omega + \left(\frac{1}{\sinh\omega} - \frac{1}{\omega}\right)(1-\mu + \sigma\omega) \],

(D8)

where we integrated by parts as in the spinor case. In addition to the table integral in (D3), we need also the following one,

\[ \int_0^\infty \frac{d\omega\omega^{\mu-1}e^{-\sigma\omega}}{\sinh\omega} = 2^{1-\mu}\Gamma(\mu)\zeta\left(\mu, \frac{1+\sigma}{2}\right). \]

(D9)

Thus, we obtain

\[ I^{(scal)}(\sigma; \mu) = 2^{-\mu}\Gamma(\mu+1) \left[ \sigma(1+\sigma^2)\zeta\left(\mu + 1, \frac{1+\sigma}{2}\right) + 2(1+3\sigma^2)\frac{1-\mu}{\mu}\zeta\left(\mu, \frac{1+\sigma}{2}\right) \right. \]

\[ -12\sigma\frac{2-\mu}{\mu}\zeta\left(\mu - 1, \frac{1+\sigma}{2}\right) + 8\frac{3-\mu}{\mu}\zeta\left(\mu - 2, \frac{1+\sigma}{2}\right) \left. \right]. \]

(D10)

And, finally, by analytical continuation, we obtain the results for two values of \( \mu \) that are of interest,

\[ I^{(scal)}\left(\sigma; \frac{1}{2}\right) = \sqrt{\frac{\pi}{2}} \left[ 20\zeta\left(3, \frac{1+\sigma}{2}\right) - 18\zeta\left(-\frac{1}{2}, \frac{1+\sigma}{2}\right) \right. \]

\[ + (1+3\sigma^2)\zeta\left(2, \frac{1+\sigma}{2}\right) \left. \right] \]

\[ + (1+3\sigma^2)\zeta\left(2, \frac{1+\sigma}{2}\right) \]

\[ = \frac{11}{6}\sigma^3 + \frac{7}{6}\sigma - \sigma(1+\sigma^2)\psi\left(\frac{1+\sigma}{2}\right) + 2(1+3\sigma^2) \left[ \ln\Gamma\left(\frac{1+\sigma}{2}\right) - \ln\sqrt{2\pi} \right] \]

\[ - 24\sigma\zeta'\left(-1, \frac{1+\sigma}{2}\right) + 24\zeta'\left(-2, \frac{1+\sigma}{2}\right). \]

(D11)

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FIG. 1. Diagrammatic notations for the boson interaction vertices. The curly brackets denote symmetrization of the type:

$$F_{\nu\mu_1\ldots\mu}\{\lambda,\mu_1\ldots\mu\} = F_{\nu\lambda,\mu_1\ldots\mu\mu} + F_{\nu\mu_1,\lambda\ldots\mu\mu} + \ldots + F_{\nu\mu_n,\lambda\ldots\mu\mu}.$$
FIG. 2. Diagrammatic notations for the fermion-boson interaction vertices.
FIG. 3. Two simplest examples of diagrams related to the two-derivative terms $C^W$ and $C^V$ in our general expression for spinor QED.