A simple counterexample for the permanent-on-top conjecture

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Abstract

The permanent-on-top conjecture (POT) was an important conjecture on the largest eigenvalue of the Schur power matrix of a positive semi-definite Hermitian matrix, formulated by Soules. The conjecture claimed that for any positive semi-definite Hermitian matrix $H$, $\text{per}(H)$ is the largest eigenvalue of the Schur power matrix of the matrix $H$. After half a century, the POT conjecture has been proven false by the existence of counterexamples which are checked with the help of computer. It raises concerns about a counterexample that can be checked by hand (without the need of computers). A new simple counterexample for the permanent-on-top conjecture is presented which is a complex matrix of dimension 5 and rank 2.

1 Introduction and notations

The symbol $S_n$ denotes the symmetric group on $n$ objects. The permanent of a square matrix is a vital function in linear algebra that is similar to the determinant. For an $n \times n$ matrix $A = (a_{ij})$ with complex coefficients, its permanent is defined as $\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$. By $\mathcal{H}_n$ we mean the set of all $n \times n$ positive semi-definite Hermitian matrices. The Schur power matrix of a given $n \times n$ matrix $A = (a_{ij})$, denoted by $\pi(A)$, is a $n! \times n!$ matrix with the elements indexed by permutations $\sigma, \tau \in S_n$:

$$\pi_{\sigma \tau}(A) = \prod_{i=1}^{n} a_{\sigma(i),\tau(i)}$$

Conjecture 1. The permanent-on-top conjecture (POT) [9]: Let $H$ be an $n \times n$ positive semi-definite Hermitian matrix, then $\text{per}(H)$ is the largest eigenvalue of $\pi(H)$.

In 2016, Shchesnovich provided a 5-square, rank 2 counterexample to the permanent-on-top conjecture with the help of computer [8].
Definition 1. For an $n \times n$ matrix $A = (a_{ij})$, let $d_A$ be a function $S_n \to \mathbb{C}$ defined by

$$d_A(\sigma) = \prod_{i=1}^{n} a_{\sigma(i)i}$$

This function is also called the "diagonal product" function. Then we can define

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} d_A(\sigma)$$

and

$$\text{per}(A) = \sum_{\sigma \in S_n} d_A(\sigma)$$

For any $n$-square matrix $A$ and $I, J \subset [n]$, $A[I, J]$ denotes the submatrix of $A$ consisting of entries which are the intersections of $i$-th rows and $j$-th columns where $i \in I$, $j \in J$. We define $A(I, J) = A[I^c, J^c]$.

In this paper, we shall study the properties of the spectrum of the Schur power matrix by examining the spectra of the matrices $C_k(A)$ which are defined in the manner:

For any $1 \leq k \leq n$, the matrix $C_k(A)$ is a matrix of size $\binom{n}{k} \times \binom{n}{k}$ with its $(I, J)$ entry defined by $\text{per}(A[I, J]) \cdot \text{per}(A[I^c, J^c])$. There is another conjecture on these matrices $C_k(A)$ which states that:

**Conjecture 2. Pate’s conjecture**

Let $A$ be an $n \times n$ positive semi-definite Hermitian matrix and $k$ be a positive integer number less than $n$, then the largest eigenvalue of $C_k(A)$ is $\text{per}(A)$.

Pate’s conjecture is weaker than the permanent-on-top conjecture (POT) because it is well-known that every eigenvalue of $C_k$ is also an eigenvalue of the Schur power matrix. In the case $k = 1$, in [1], it was conjectured that $\text{per}(A)$ is necessarily the largest eigenvalue of $C_1(A)$ if $A \in \mathcal{H}_n$. Stephen W. Drury has provided an 8-square matrix as a counterexample for this case in the paper [2]. Besides, Bapat and Sunder raise a question as follows:

**Conjecture 3. Bapat & Sunder conjecture:** Let $A$ and $B = (b_{ij})$ be $n \times n$ positive semi-definite Hermitian matrices, then

$$\text{per}(A \circ B) \leq \text{per}(A) \prod_{i=1}^{n} b_{ii}$$

where $A \circ B$ is the entrywise product (Hadamard product).

The Bapat & Sunder conjecture is weaker than the permanent-on-top conjecture and has been proved false by a counterexample which is a positive semi-definite Hermitian matrix of order 7 proposed by Drury [3]. In the present paper, a new simple counterexample for the permanent-on-top conjecture and Pate’s conjecture is presented. It has size $5 \times 5$ and rank 2.

**Conjecture 4. The Lieb permanent dominance conjecture 1966**

Let $H$ be a subgroup of the symmetric group $S_n$ and let $\chi$ be a character of degree $m$ of $H$. Then

$$\frac{1}{m} \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \leq \text{per}(A)$$
holds for all $n \times n$ positive semi-definite Hermitian matrix $A$.

The permanent dominance conjecture is weaker than the permanent-on-top conjecture and still open. The POT conjecture was proposed by Soules in 1966 as a strategy to prove the permanent dominance conjecture.

Definition 2. The elementary symmetric polynomials in $n$ variables $x_1, x_2, \ldots, x_n$ are $e_k$ for $k = 0, 1, \ldots, n$. In this paper, we define $e_k(x_i)$ for $i = 1, 2, \ldots, n$ to be the elementary symmetric polynomial of degree $k$ in $n - 1$ variables obtained by erasing variable $x_i$ from the set $\{x_1, x_2, \ldots, x_n\}$ and, for any subset $I \subset [n]$, the notation $e_k[I]$ denote the elementary symmetric polynomial of degree $k$ in $|I|$ variables $x_i$’s, $i \in I$.

2 Associated matrices

We define the associated matrix of a matrix representation $W : S_n \rightarrow GL_N(\mathbb{C})$ with respect to a $n \times n$ matrix $A$ by:

$$M_W(A) = \sum_{\sigma \in S_n} d_A(\sigma)W(\sigma)$$

Proposition 2.1. The Schur power matrix of a given $n \times n$ Hermitian matrix $A$ is the associated matrix of the left-regular representation with respect to $A$.

Proof. Take a look at the $(\sigma, \tau)$ entry of $M_L(A)$ which is

$$\sum_{\eta \in S_n, \eta \circ \tau = \sigma} d_A(\eta) = d_A(\sigma \circ \tau^{-1}) = \prod_{i=1}^{n} d_{\sigma(i) \tau(i)}$$

the right side is the $(\sigma, \tau)$ entry of $\pi(A)$.

Let us now consider two important matrices $C_1(A)$ and $C_2(A)$ that shall appear frequently from now on.

Definition 3. Let $\mathcal{M}_k : S_n \rightarrow GL_{\binom{n}{k}}(\mathbb{C})$ be the matrix representation given by the permutation action of $S_n$ on $\binom{[n]}{k}$.

Proposition 2.2. For any $n \times n$ Hermitian matrix $A$, the matrix $\mathcal{C}_k(A)$ is the matrix $M_{\mathcal{M}_k}(A)$.

We obtain directly the statement that every eigenvalue of matrix $M_{\mathcal{M}_k}(A)$ is an eigenvalue of the associated matrix of the left-regular representation which is the Schur power matrix. Consequently, Pate’s conjecture is weaker than the permanent-on-top conjecture(POT).
3 Several properties of the Schur power matrix and $\mathcal{C}_1(A)$ in rank 2 case

The main object of this section is $n \times n$ positive semi-definite Hermitian matrices of rank 2. We know that every matrix $A \in \mathcal{H}_n$ of rank 2 can be written as the sum $v_1v_1^* + v_2v_2^*$ where $v_1$ and $v_2$ are two column vectors of order $n$.

**Definition 4.** A matrix $A \in \mathcal{H}_n$ is called ”formalizable” if $A$ can be written in the form $v_1v_1^* + v_2v_2^*$ and every element of $v_1$ vector is non-zero.

**Definition 5.** The formalized matrix $A'$ of a given formalizable matrix $A$ defined in the manner: if $A = v_1v_1^* + v_2v_2^*$ and $v_1 = (a_1, \ldots, a_n)^T$, $a_i \neq 0 \forall i = 1, \ldots, n$; $v_2 = (b_1, \ldots, b_n)^T$ then $A' = v_3v_3^* + v_4v_4^*$ where $v_3 = (1, \ldots, 1)^T$ and $v_4 = (\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n})^T$.

**Proposition 3.1.** Let $A \in \mathcal{H}_n$ be a formalizable matrix, then $\pi(A) = \prod_{i=1}^n |a_i|^2 \pi(A')$.

**Proof.** We compare the $(\sigma, \tau)$-th entries of two matrices.

$$\pi_{\sigma \tau}(A) = \prod_{i=1}^n (a_{\sigma(i)} \overline{a_{\tau(i)}} + b_{\sigma(i)} \overline{b_{\tau(i)}}) = \prod_{i=1}^n |a_i|^2 \prod_{i=1}^n \left(1 + \frac{b_{\sigma(i)} \overline{b_{\tau(i)}}}{a_{\sigma(i)} \overline{a_{\tau(i)}}}\right) = \prod_{i=1}^n |a_i|^2 \pi_{\sigma \tau}(A')$$

**Remark 1.** The same result will be obtained with the matrices $\mathcal{C}_k(A)$ and $\mathcal{C}_k(A')$. It is obvious to see that if the matrix $A$ is a counterexample for the permanent-on-top conjecture and Pate’s conjecture then so is $A'$. Assume that we have an unformalizable matrix $B \in \mathcal{H}$ of rank 2 that is a counterexample for the permanent-on-top conjecture and Pate’s conjecture. That also implies that there is a column vector $x$ such that the following inequality holds

$$\frac{x^* \pi(B)x}{\|x\|^2} > \text{per}(B)$$

By continuity and $B = vv^* + uu^*$, we can change slightly the zero elements of the vector $v$ such the the inequality remains. Therefore, if the permanent-on-top conjecture or Pate’s conjecture is false for some positive semi-definite Hermitian matrix of rank 2 then so is the permanent-on-top conjecture and Pate’s conjecture for some formalizable matrices. That draws our attention to the set of all formalizable matrices.

For any $n \times n$ positive semi-definite Hermitian matrix $A$ of rank 2 there exist two eigenvectors of $v$ and $u$ of $A$ such that $A = vv^* + uu^*$. Let $u_i, v_i$ be the $i$-th row elements of $v$ and $u$ respectively for $i = 1, n$. In the case $A$ has a zero row then $\text{per}(A) = 0$ and the Schur power matrix and matrices $\mathcal{C}_k(A)$ of $A$ are all zero matrices, there is nothing to discuss. Otherwise, every row of $A$ has a non-zero element (so does every column since $A$ is a
The upper bound of rank of the Schur power matrix of rank 2: If A is an Hermitian matrix) which means that for any \( i = 1, n \), the inequalities \( |v_i|^2 + |u_i|^2 > 0 \) hold. Besides, A can be rewritten in the form

\[
(sin(x)v + cos(x)u)(sin(x)v + cos(x)u) + (cos(x)v - sin(x)u)(cos(x)v - sin(x)u) \forall x \in [0, 2\pi]
\]

and the system of n equations \( sin(x)v_i + cos(x)u_i = 0 \), \( i = 1, n \) takes finite solutions in the interval \([0, 2\pi]\). Therefore, there exists \( x \in [0, 2\pi] \) satisfying that \( (sin(x)v + cos(x)u) \) has every element different from 0. Hence, every rank 2 positive semi-definite Hermitian that has no zero-row is formalizable. Several properties about the formalized matrices are presented below.

Let \( H \in \mathcal{H} \) be a formalizable matrix of the form \( H = vv^* + uu^* \) where \( v = (1, \ldots, 1)^T \) and \( u = (x_1, x_2, \ldots, x_n)^T \). We recall quickly the Kronecker product [10].

**Definition 6.** The Kronecker product (also known as tensor product or direct product) of two matrices \( A \) and \( B \) of sizes \( m \times n \) and \( s \times t \), respectively, is defined to be the \((ms) \times (nt)\) matrix

\[
A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B & \ldots & a_{1n}B \\
a_{21}B & a_{22}B & \ldots & a_{2n}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}B & a_{n2}B & \ldots & a_{nn}B
\end{pmatrix}
\]

**Lemma 1. The upper bound of rank of the Schur power matrix of rank 2:** If A is an \( n \times n \) of rank 2 then rank of \( \pi(A) \) is not larger than \( 2^n - n \).

**Proof.** We observe that rank(A) = 2 implies that dim(Im(A)) = 2 and dim(Ker(A)) = \( n - 2 \). Let \( \langle \{\cdot\} \rangle \) be an orthonormal basis of the orthogonal complement of Ker(A) in \( \mathbb{C}^n \), then denote \( v = Aw, u = At \). Thus, A can be rewritten in the form \( vv^* + uu^* \) where \( v = (a_1, \ldots, a_n)^T, u = (b_1, \ldots, b_n)^T \). It is obvious that \( \text{Im}(A) = \langle v, u \rangle \). Let us denote the Kronecker product of \( n \) copies of the matrix A by \( \otimes^nA \). The mixed-product property of Kronecker product implies that \( \text{Im}(\otimes^nA) = \langle \{\otimes^n_{i=1}t_i, t_i \in \{v, u\} \} \rangle \). Furthermore, the Schur power matrix of A is a diagonal submatrix of \( \otimes^nA \) obtained by deleting all entries of \( \otimes^nA \) that are products of entries of A having two entries in the same row or column. Let define the function \( f \) in the manner that

\[
f: \{\otimes^n_{i=1}t_i, t_i \in \{v, u\}\} \rightarrow \mathcal{V}
\]

and the \( \sigma \)-th element of \( f(\otimes^n_{i=1}t_i) \) vector of order \( n! \) is \( \prod_{i=1}^{n} t_i(\sigma(i)) \) where \( t_i(j) \) is the \( j \)-th row element of the column vector \( t_i \). Let \( \mathcal{B} = \{f(\otimes^n_{i=1}t_i), t_i \in \{v, u\}\} \) then \( \mathcal{B} \) is a generator of \( \text{Im}(\pi(A)) \) since \( \pi(A) \) is a principal matrix of \( \otimes^nA \) and \( \text{Im}(\otimes^nA) = \langle \{\otimes^n_{i=1}t_i, t_i \in \{v, u\}\} \rangle \). We partition \( \mathcal{B} \) into disjoint sets \( S_k \)

\[
k = 0, 1, \ldots, n, S_k = \{f(\otimes^n_{i=1}t_i), t_i \in \{v, u\}, v \text{ appears } k \text{ times in the Kronecker product}\}
\]
Hence, for any \( k = 1, 2, \ldots, n \) the \( \sigma \)-th row element of the sum vector \( \sum_{w \in S_k} w \) is

\[
\sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} a_{\sigma(i_j)} \prod_{t=k+1}^{n} b_{\sigma(i_t)} = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \prod_{j=1}^{k} a_{i_j} \prod_{t=k+1}^{n} b_{i_t}
\]

and \( S_0 = \{(1, 1, \ldots, 1)^T\} \). Therefore, for any \( k = 1, \ldots, n \) then \( S_0 \cup S_k \) is linearly dependent. Hence, by deleting an arbitrary element of each set \( S_k \), \( k = 1, \ldots, n \), then it still remains a generator of \( \text{Im}(\pi(H)) \). Thus

\[
\text{rank}(\pi(A)) = \dim(\text{Im}(\pi(A))) \leq \vert \mathcal{B} \vert - n = 2^n - n
\]

Lemma 2. The permanent of a formalized matrix [5]:

\[
\text{per}(H) = \sum_{k=0}^{n} k!(n-k)! |e_k|^2
\]

Proof. We show that

\[
\text{per}(H) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} (1 + x_{\sigma(i)})
\]

\[
= n! + \sum_{\sigma \in S_n} \sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} x_{i_1} \ldots x_{i_k} \prod_{\sigma \in S_n} x_{\sigma(i_1)} \ldots x_{\sigma(i_k)}
\]

\[
= n! + \sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} k!(n-k)! x_{i_1} \ldots x_{i_k} \prod_{\sigma \in S_n} x_{\sigma(i_1)} \ldots x_{\sigma(i_k)}
\]

\[
= \sum_{k=0}^{n} k!(n-k)! |e_k|^2
\]

We use the elementary symmetric polynomials to examine entries of \( C_1(H) \) with the
Therefore, we have the following proposition.

\[(i, j)\text{-th entry defined by } (1 + x_i x_j), \text{ per}(H(i|j))\] and

\[
\text{per}(H(i|j)) = \sum_{\sigma \in S_n; \; \sigma(i) = j} \prod_{i \neq l} (1 + x_l x_{\sigma(l)})
\]

\[
= \sum_{\sigma \in S_n; \; \sigma(i) = j} \prod_{k=0}^{n-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n; \; i_m \neq i} x_{i_1} \ldots x_{i_k} x_{\sigma(i_1)} \ldots x_{\sigma(i_k)}
\]

\[
= \sum_{k=0}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n; \; i_m \neq i} k!(n-1-k)! x_{i_1} \ldots x_{i_k} e_k(x_j)
\]

\[
= \sum_{k=0}^{n-1} k!(n-1-k)! e_k(x_i) e_k(x_j)
\]

And notice that

\[
e_k = x_i e_{k-1}(x_i) + e_k(x_i) \forall k = 1, \ldots, n
\]

Then

\[
\frac{\text{per}(H)}{n} = \frac{1}{n} \sum_{k=0}^{n} k!(n-k)! |e_k|^2
\]

\[
= (n-1)! (|e_0|^2 + |e_n|^2) + \sum_{k=1}^{n-1} \frac{k!(n-k)!}{n} (x_i e_{k-1}(x_i) + e_k(x_i))(x_j e_{k-1}(x_j) + e_k(x_j))
\]

Hence

\[
(1 + x_i x_j), \text{ per}(H(i|j)) - \frac{\text{per}(H)}{n}
\]

\[
= \sum_{k=1}^{n-1} \left( k!(n-1-k)! - \frac{k!(n-k)!}{n} \right) e_k(x_i) e_k(x_j)
\]

\[
+ \left( (k-1)!(n-k)! - \frac{k!(n-k)!}{n} \right) x_i e_{k-1}(x_i) x_j e_{k-1}(x_j)
\]

\[
- \frac{k!(n-k)!}{n} (x_i e_{k-1}(x_i) e_k(x_j) + x_j e_{k-1}(x_j) e_k(x_i))
\]

\[
= \sum_{k=1}^{n-1} \frac{(k-1)!(n-1-k)!}{n} (ke_k(x_i) - (n-k)x_i e_{k-1}(x_i))(ke_k(x_j) - (n-k)x_j e_{k-1}(x_j))
\]

\[
= \sum_{k=1}^{n-1} \frac{(k-1)!(n-1-k)!}{n} (ne_k(x_i) - (n-k)e_k)(ne_k(x_j) - (n-k)e_k)
\]

Therefore, we have the following proposition.
Proposition 3.2. The matrix $\mathcal{C}_1(H)$ can be rewritten in the form

$$\mathcal{C}_1(H) = \frac{\operatorname{per}(H)}{n} v v^* + \sum_{k=1}^{n-1} \frac{(k-1)!(n-1-k)!}{n} v_k v_k^*$$

where $v = (1, \ldots, 1)^T$ of order $n$, for $k = 1, \ldots, n-1$, $v_k = (\ldots, ne_k(x_i) - (n-k)e_k, \ldots)^T$

Proposition 3.3. For any $k = 1, \ldots, n-1$, $\langle v, v_k \rangle = 0$

Proof. 

$$\langle v, v_k \rangle = \sum_{i=1}^{n} (ne_k(x_i) - (n-k)e_k)$$

$$= n \sum_{i=1}^{n} e_k(x_i) - n(n-k)e_k$$

$$= 0$$

Proposition 3.4. The rank of $\mathcal{C}_1(H)$ is the cardinality of the set $\{x_i, i = \overline{1,n}\}$. In formula, $\operatorname{rank}(\mathcal{C}_1(H)) = |\{x_i, i = \overline{1,n}\}|$.

Proof. For the $i$-th element of $v_k$, we have

$$ne_k(x_i) - (n-k)e_k$$

$$= ke_k - nx_i e_{k-1}(x_i)$$

$$= ke_k + n \sum_{j=1}^{k} (-1)^j e_{k-j} x_i^j$$

which leads us to a conclusion that $\langle v, v_1, \ldots, v_{n-1} \rangle = \langle p_0, \ldots, p_{n-1} \rangle$ where $p_j = (\ldots, x_i^j, \ldots)^T$

which is equal to $|\{x_i, i = \overline{1,n}\}|$ by the determinantal formula of Vandermonde matrices.

Proposition 3.5. The determinant of $\mathcal{C}_1(H)$ is given by

$$\det(\mathcal{C}_1(H)) = \frac{\operatorname{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)!(n-1-k)! \cdot \prod_{i<j} |x_i - x_j|^2$$

Proof. Case 1: There are indices $i$ and $j$ such that $x_i = x_j$ then rank$(\mathcal{C}_1(H)) < n$ that is equivalent to det$(\mathcal{C}_1(H)) = 0$.

Case 2: $x_i$’s are distinct then $\{v, v_1, \ldots, v_{n-1}\}$ makes a basis of $\mathbb{C}^n$. Therefore, $\mathcal{C}_1(H)$ is
similar to the Gramian matrix of \( n \) vectors

\[
\begin{pmatrix}
\frac{\text{per}(H)}{n} v_1; \sqrt{\frac{(k-1)!(n-1-k)!}{n}} v_k, & k = 1, n-1
\end{pmatrix}.
\]

Thus

\[
\det(\mathcal{G}_1(H)) = \det\left( G\left( \frac{\text{per}(H)}{n} v_1; \sqrt{\frac{(k-1)!(n-1-k)!}{n}} v_k, \ k = 1, n-1 \right) \right)
\]

\[
= \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} \frac{(k-1)!(n-1-k)!}{n} \cdot \det(G(v, v_1, \ldots, v_{n-1}))
\]

And from the proof of proposition 3.4, we obtain that

\[
\begin{pmatrix}
1 & \ldots & ke_k & \ldots & (n-1)e_{n-1} \\
0 & \ldots & (-1)^2 ne_{k-1} & \ldots & (-1)^2 ne_{n-2} \\
& \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & (-1)^i ne_{j-i} & \ldots \\
& \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & (-1)^{n-1}n
\end{pmatrix}
\]

\((v, v_1, \ldots, v_{n-1}) = (p_0, p_1, \ldots, p_{n-1})\)

The matrix in the right side is the transition matrix given by

\[
\text{The } (i, j)\text{-th entry } = \begin{cases} 
(-1)^i ne_{j-i} & \text{ if } i > 1 \\
(j-1)e_{j-1} & \text{ if } i = 1 \text{ and } j > 1 \\
1 & \text{ if } (i, j) = (1, 1) 
\end{cases}
\]

with convention that \( e_0 = 1; \ e_t = 0 \text{ if } t < 0 \). Moreover, we observe that the transition matrix is an upper triangular matrix with the absolute value of diagonal entries equal to \( n \) except the \((1,1)\)-th entry equal to 1 and \((p_0, p_1, \ldots, p_{n-1})\) is a Vandermonde matrix. Hence

\[
\det(\mathcal{G}_1(H)) = \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)! (n-1-k)! \cdot \det(G(p_0, p_1, \ldots, p_{n-1}))
\]

\[
= \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)! (n-1-k)! \cdot |\det(p_0, p_1, \ldots, p_{n-1})|^2
\]

\[
= \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)! (n-1-k)! \cdot \prod_{i < j} |x_i - x_j|^2
\]

The right side is also equal to 0 if there are indices \( i \neq j \) such that \( x_i = x_j \). Hence the equality holds in both cases. \( \blacksquare \)
Remark 2. From the proposition 3.5, we are able to calculate the determinant of $C_1(H)$ of any positive semi-definite Hermitian matrix $H$ of rank 2 in the way:

Let $A$ be an $n \times n$ positive semi-definite Hermitian matrix of rank 2 then $A$ can be written in the form $vv^* + uu^*$ with $v_i, u_i$ are the $i$-th elements of $v$ and $u$ respectively. Then the following formula for the determinant of $C_1(H)$ is achieved.

Theorem 1. Let $H = vv^* + uu^*$ be an $n \times n$ positive semi-definite Hermitian matrix then:

$$\det(C_1(H)) = \frac{\text{per}(H)}{n} \prod_{k=1}^{n-1} n(k-1)!(n-1-k)! \cdot \prod_{i<j} |v_i u_j - v_j u_i|^2$$

where $v_i$ and $u_i$ are $i$-th elements of the vector $v$ and $u$ respectively.

4 A counterexample for the conjectures 1 and 2 in the case $n = 5$

Let us take the values of $u_i$’s and $v_i$’s, $a \in \mathbb{R}$

$$u_1 = ai, u_2 = -a, u_3 = -ai, u_4 = a, u_5 = 0, v_i = 1 \forall i = 1, \ldots, 5$$

then $e_1 = e_2 = e_3 = e_5 = 0, e_4 = -a^4$

For any matrix of the form, the spectrum of $C_1(H)$ is determined clearly by the mentioned above properties and theorems.

By lemma 3.1, $\text{rank}(\pi(H)) \leq 2^5 - 5 = 27$ which means that there are at most 27 positive engenvalues.

By lemma 3.2,

$$\text{per}(H) = 120 + 24|e_1|^2 + 12|e_2|^2 + 12|e_3|^2 + 24|e_4|^2 + 120|e_5|^2 = 120 + 24a^8$$

and the proposition 3.2 implies that

$$C_1(H) = \frac{\text{per}(H)}{5} vv^* + \frac{6}{5} v_1 v_1^* + \frac{2}{5} v_2 v_2^* + \frac{2}{5} v_3 v_3^* + \frac{6}{5} v_4 v_4^*$$

where

$$v_1 = \begin{pmatrix} -5ai \\ 5a \\ 5ai \\ -5a \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -5a^2 \\ 5a^2 \\ -5a^2 \\ 5a^2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 5a^3 i \\ 5a^3 \\ -5a^3 i \\ -5a^3 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} a^4 \\ a^4 \\ a^4 \\ a^4 \\ -4a^4 \end{pmatrix}$$
Notice that \( \{v, v_1, v_2, v_3, v_4\} \) is orthogonal, thus those vectors are eigenvectors of \( \mathcal{C}_1(H) \) corresponding to the eigenvalues

\[
\text{per}(H) = 120 + 24a^8, \quad \frac{6}{5} ||v_1||^2 = 120a^2, \quad \frac{2}{5} ||v_2||^2 = 40a^4, \quad \frac{2}{5} ||v_3||^2 = 40a^6, \quad \frac{6}{5} ||v_4||^2 = 24a^8
\]

We replace \( a^2 = c \), then \( \text{tr}(\pi(H)) = 120(1 + c)^4 \). The spectrum of \( \mathcal{C}_1(H) \) is

\[
\{120 + 24c^4, 120c, 40c^2, 40c^3, 24c^4\}
\]

Moreover, every eigenvalue of \( \mathcal{C}_1(H) \) except \( \text{per}(H) \) is an eigenvalue of \( \pi(H) \) with multiplicity at least 4 and, every eigenvalue of \( \mathcal{C}_2(H) \) except eigenvalues of \( \mathcal{C}_1(H) \) is an eigenvalue of \( \pi(H) \) with multiplicity at least 5. Therefore, if we can calculate the sum and the sum of squares of at most 2 unknown positive eigenvalues of \( \pi(H) \), then the spectrum is determined. We compute the trace of \( \mathcal{C}_2(H) \). The \((i, j)(i, j)\)-th diagonal entry of \( \mathcal{C}_2(H) \) is given by

\[
\text{per}(H[i, j], \{i, j\}) + \text{per}(H\{i, j\}, \{i, j\}) = (2 + |e_1[i, j]|^2 + 2|e_2[i, j]|^2 + 6|e_1[i, j]|^2 + 2|e_3[i, j]|^2) + 6|e_3[i, j]|^2
\]

Hence, we use the table to represent all the diagonal entries of \( \mathcal{C}_2(H) \).

| Coordinates | Values |
|-------------|--------|
| (1, 2)\( (1, 2) \) | \( (2 + 2c + 2c^2)(6 + 4c + 2c^2) \) |
| (1, 3)\( (1, 3) \) | \( (2 + 2c^2)(6 + 2c^2) \) |
| (1, 4)\( (1, 4) \) | \( (2 + 2c + 2c^2)(6 + 4c + 2c^2) \) |
| (1, 5)\( (1, 5) \) | \( (2 + c)(6 + 2c + 2c^2 + 6c^3) \) |
| (2, 3)\( (2, 3) \) | \( (2 + 2c + 2c^2)(6 + 4c + 2c^2) \) |
| (2, 4)\( (2, 4) \) | \( (2 + 2c^2)(6 + 2c^2) \) |
| (2, 5)\( (2, 5) \) | \( (2 + c)(6 + 2c + 2c^2 + 6c^3) \) |
| (3, 4)\( (3, 4) \) | \( (2 + 2c + 2c^2)(6 + 4c + 2c^4) \) |
| (4, 5)\( (4, 5) \) | \( (2 + c)(6 + 2c + 2c^2 + 6c^3) \) |
| \( \text{tr}(\mathcal{C}_2(H)) \) | \( 120 + 48c^8 + 104c^3 + 152c^2 + 120c \) |

Furthermore, we use the symmetric polynomials to calculate the sum of all squares of eigenvalues.

\[
\text{tr}(\pi(H))^2 = \sum_{\sigma \in S_5} \sum_{\tau \in S_5} \left| \prod_{i=1}^{5} (1 + u_{\sigma(i)}u_{\tau(i)}) \right|^2
\]

\[
= 120 \sum_{\sigma \in S_5} \left| \prod_{i=1}^{5} (1 + u_{\sigma(i)}) \right|^2
\]

\[11\]
We know that $u_5 = 0$, and for $k = 1, \ldots, 4$ we have $u_k = a_i^k$ with $a^2 = c$ then

$$\text{tr}(\pi(H)^2) = 120 \sum_{\sigma \in S_5} \left| \prod_{i=1}^{5} (1 + u_i \mu_{\sigma(i)}) \right|^2$$

$$= 120 \left( \sum_{k=1}^{4} \sum_{\sigma \in S_5, \sigma(k)=5} \left| \prod_{j \neq k} (1 + u_j \mu_{\sigma(j)}) \right|^2 + \sum_{\sigma \in S_5, \sigma(5)=5} \left| \prod_{j=1}^{4} (1 + u_j \mu_{\sigma(i)}) \right|^2 \right)$$

$$= 120 \left( \sum_{k=1}^{4} \sum_{\sigma \in S_5, \sigma(k)=5} \left| \prod_{j \neq k} (1 + c \cdot i^{1-\sigma(j)} \right|^2 + \sum_{\sigma \in S_5, \sigma(5)=5} \left| \prod_{j=1}^{4} (1 + c \cdot i^{1-\sigma(j)}) \right|^2 \right)$$

**Lemma 3.** By the fundamental theorem of symmetric polynomials and $e_1 = e_2 = e_3 = e_5 = 0$ then every monomial symmetric polynomial in 5 variables of degree non-divisible by 4 takes $(u_1, u_2, u_3, u_4, u_5)$ as a root.

The lemma 4.1 reduces the sums

$$\sum_{k=1}^{4} \sum_{\sigma \in S_5, \sigma(k)=5} \left| \prod_{j \neq k} (1 + c \cdot i^{1-\sigma(j)}) \right|^2$$

$$= \sum_{k=1}^{4} \sum_{\sigma \in S_5, \sigma(k)=5} (1 + c^2)^3 + (1 + c^2) \sum_{j \neq k} 2 \text{Re}(i^{1-\sigma(j)})$$

$$+ (1 + c^2)^2 \sum_{i_1 < i_2 \neq k, 5} (i^{i_1-\sigma(i_1)} + i^{\sigma(i_1)-i_1})(i^{i_2-\sigma(i_2)} + i^{\sigma(i_2)-i_2}) + c^3 \prod_{j \neq k, 5} (i^{1-\sigma(j) + i^{\sigma(j)-j}})$$

$$= 96(1 + c^2)^3 + \sum_{k=1}^{4} \sum_{\sigma \in S_5, \sigma(k)=5} c^2(1 + c^2)2 \text{Re} \left( \sum_{i_1 < i_2 \neq k, 5} i^{i_1-\sigma(i_1)} - i^{i_2-\sigma(i_2)+\sigma(i_1)} \right)$$

$$= 96(1 + c^2)^3 + \sum_{k=1}^{4} \sum_{\sigma \in S_5, \sigma(k)=5} c^2(1 + c^2) \text{Re} \left( \sum_{i_1 \neq i_2 \neq k, 5} e^{i_1-i_2} \sum_{\sigma \in S_5, \sigma(k)=5} i^{\sigma(i_2)-\sigma(i_1)} \right)$$

combine with

$$\sum_{\sigma \in S_5, \sigma(k)=5} i^{\sigma(i_2)-\sigma(i_1)} = 2 \sum_{\alpha=1}^{4} i^\alpha \sum_{\beta \neq \alpha} i^\beta = -2.4 = -8$$

We attain

$$\sum_{k=1}^{4} \sum_{\sigma \in S_5, \sigma(k)=5} \left| \prod_{j \neq k, 5} (1 + c \cdot i^{1-\sigma(j)}) \right|^2 = 96(1 + c^2)^3 - 8c^2(1 + c^2) \sum_{k=1}^{4} \text{Re} \left( \sum_{i_1 \neq i_2 \neq k, 5} e^{i_1-i_2} \right)$$

$$= 96(1 + c^2)^3 + 64c^2(1 + c^2)$$

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Hence, the spectrum of the matrix $H$ ample to the permanent-on-top conjecture (POT).

The lemma 4.1 also reduces the sum

$$\sum_{\sigma \in S_4, \sigma(5)=5} \prod_{i=1}^4 (1 + c \cdot i^{j-\sigma(j)})^2 = \sum_{\sigma \in S_4} \prod_{i=1}^4 (1 + c \cdot i^{j-\sigma(j)})^2$$

$$= \sum_{\sigma \in S_4} \left| 1 + c^4 + c^3 \sum_{i=1}^4 i^{\sigma(j)-j} + c^2 \sum_{j_1<j_2} i^{j_1+j_2-\sigma(j_1)-\sigma(j_2)} \right|^2$$

$$= 24(1 + c^4)^2 + (c^6 + c^2) \sum_{\sigma \in S_4} \prod_{i=1}^4 i^{j-\sigma(j)}^2 + c^4 \sum_{\sigma \in S_4} \sum_{j_1<j_2} i^{j_1+j_2-\sigma(j_1)-\sigma(j_2)}^2$$

We compute each part separately by the lemma 4.1

$$\sum_{\sigma \in S_4} \prod_{i=1}^4 i^{j-\sigma(j)}^2 = 24 \cdot 4 - 8 \sum_{j_1 \neq j_2} i^{j_1-j_2} = 96 + 32 = 128$$

$$\sum_{\sigma \in S_4} \sum_{j_1<j_2} i^{j_1+j_2-\sigma(j_1)-\sigma(j_2)}^2 = \sum_{\sigma \in S_4} \left( \binom{4}{2} + \frac{1}{4} \sum_{\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}} i^{\sigma(i_1)+\sigma(i_2)-\sigma(i_3)-\sigma(i_4)} + \sum_{j_1<j_2} i^{j_1-j_2+\sigma(j_2)-\sigma(j_1)} \right)$$

$$= 144 + 2 \sum_{\{i_1, i_2, i_3, i_4\}} i^{i_1+i_4-i_2-i_3} - 16 \sum_{j_1<j_2} i^{j_1-j_2} = 208 - 4 \sum_{j_1<j_2} i^{2j_1+2j_2} = 224$$

Thus, we obtain $\text{tr}(\pi(H)^2) = 120(24(1 + c^4)^2 + 128(c^6 + c^2) + 224c^4 + 96(1 + c^2)^3 + 64c^2(1 + c^2))$.

Hence, the spectrum of $\pi(H)$ is

- $\text{per}(H) = 120 + 24c^4$ of multiplicity 1
- $120c, 40c^2, 40c^3, 24c^4$ of multiplicity 4
- $64c^3, 112c^2$ of multiplicity 5
- 0 of multiplicity 93

We observe that $c = 2$ is a solution of the inequality $120 + 24c^4 - 64c^3 < 0$. Therefore, the matrix $H = vv^* + uu^*$ where $v = (1, \ldots, 1)^T$, $u = \sqrt{2}(i, -1, -i, 1, 0)^T$ is a counterexample to the permanent-on-top conjecture (POT).

$$H = \begin{pmatrix} 3 & 1 - 2i & -1 & 1 + 2i & 1 \\ 1 + 2i & 3 & 1 - 2i & -1 & 1 \\ -1 & 1 + 2i & 3 & 1 - 2i & 1 \\ 1 - 2i & -1 & 1 + 2i & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$
The spectrum of this counterexample is also given by above calculations:

- \( \text{per}(H) = 504 \) of multiplicity 1
- \( 240, 160, 320, 384 \) of multiplicity 4
- \( 512 \) and \( 448 \) of multiplicity 5
- \( 0 \) of multiplicity 93

Once, I have the counterexample, a shorter way to prove the matrix \( H \) is a counterexample for Pate’s conjecture in the case \( n = 5 \) and \( k = 2 \) is available by Tensor product. For the purposes of this paper let us describe the tensor product of vector spaces in terms of bases:

**Definition 7.** Let \( V \) and \( W \) be vector spaces over \( \mathbb{C} \) with bases \( \{v_i\} \) and \( \{w_j\} \), respectively. Then \( V \otimes W \) is the vector space spanned by \( \{v_i \otimes w_j\} \) subject to the rules:

\[
(\alpha v + \alpha' v') \otimes w = \alpha (v \otimes w) + \alpha' (v' \otimes w)
\]

\[
v \otimes (\alpha w + \alpha' w') = \alpha (v \otimes w) + \alpha' (v \otimes w')
\]

for all \( v, v' \in V \) and \( w, w' \in W \) and all scalars \( \alpha, \alpha' \).

If \( \langle \cdot, \cdot \rangle \) is an inner product on \( V \) then we can define an inner product \( \langle \cdot, \cdot \rangle \) on \( V \otimes V \) in the manner:

\[
\langle v_{i_1} \otimes v_{i_2}, v_{i_3} \otimes v_{i_4} \rangle = \langle v_{i_1}, v_{i_3} \rangle \langle v_{i_2}, v_{i_4} \rangle
\]

for any \( v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4} \) vectors.

On \( \mathbb{C}[x,y] \), we consider the inner product, and the resulting Euclidean norm \( |\cdot| \), such that monomials are orthogonal and \( |x^n y^k|^2 = n!k! \).

**Proposition 4.1.** The permanent of the Gram matrix of any 1-forms \( f_j \in \mathbb{C}x \oplus \mathbb{C}y \) is \( |\prod f_j|^2 \).

**Proof.** We prove the generalization of the statement which states that if \( f_1, f_2, \ldots, f_n; g_1, g_2, \ldots, g_n \) be \( 2n \) 1-forms and \( A \) be an \( n \times n \) matrix with \( (i,j) \)-th entry \( \langle f_i, g_j \rangle \), then

\[
\text{per}(A) = \begin{vmatrix}
\prod_{i=1}^{n} f_i, \prod_{i=1}^{n} g_i
\end{vmatrix}
\]

Let \( f_i = \alpha_i x + \beta_i y, g_i = \alpha'_i x + \beta'_i y \) for any \( i \in \{1,2,\ldots,n\} \).

We compute each side of the equality:
The left side is

$$\text{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \langle f_i, g_{\sigma(i)} \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} \langle \alpha_i x + \beta_i y, \alpha'_{\sigma(i)} x + \beta'_{\sigma(i)} y \rangle = \sum_{\sigma \in S_n} \prod_{i=1}^{n} (\alpha_i \cdot \alpha'_{\sigma(i)} + \beta_i \cdot \beta'_{\sigma(i)})$$

$$= \sum_{\sigma \in S_n} \sum_{k=0}^{n} \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n \atop 1 \leq k+1 < \cdots < i_n \leq n} \alpha_{i_1} \cdots \alpha_{i_k} \beta_{i_k+1} \cdots \beta_{i_n} \cdot \prod_{1 \leq i_1 < \cdots < i_k \leq n \atop 1 \leq k+1 < \cdots < i_n \leq n} \alpha'_{i_1} \cdots \alpha'_{i_k} \beta'_{i_k+1} \cdots \beta'_{i_n} \right)$$

$$= \sum_{k=0}^{n} k! (n-k)! \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n \atop 1 \leq k+1 < \cdots < i_n \leq n} \alpha_{i_1} \cdots \alpha_{i_k} \beta_{i_k+1} \cdots \beta_{i_n} \right) \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n \atop 1 \leq k+1 < \cdots < i_n \leq n} \alpha'_{i_1} \cdots \alpha'_{i_k} \beta'_{i_k+1} \cdots \beta'_{i_n} \right)$$

and the right side is

$$\left\langle \prod_{i=1}^{n} f_i, \prod_{i=1}^{n} g_i \right\rangle$$

$$= \left\langle \prod_{k=0}^{n} x^k y^{n-k} \sum_{1 \leq i_1 < \cdots < i_k \leq n \atop 1 \leq k+1 < \cdots < i_n \leq n} \alpha_{i_1} \cdots \alpha_{i_k} \beta_{i_k+1} \cdots \beta_{i_n} \right\rangle$$

$$= \sum_{k=0}^{n} k! (n-k)! \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n \atop 1 \leq k+1 < \cdots < i_n \leq n} \alpha_{i_1} \cdots \alpha_{i_k} \beta_{i_k+1} \cdots \beta_{i_n} \right) \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n \atop 1 \leq k+1 < \cdots < i_n \leq n} \alpha'_{i_1} \cdots \alpha'_{i_k} \beta'_{i_k+1} \cdots \beta'_{i_n} \right)$$

Let $f_j = x + yi \sqrt{2}$ ($j = 1, 2, 3, 4$) and $f_5 = x$. Their Gram matrix is the given matrix $H$ with $\text{per}H = |f_1 f_2 f_3 f_4 f_5|^2 = |x^5 - 4xy^4|^2 = 512 \cdot 24 = 504$ (according to the proposition 4.1). When $\{p, q, r, s, t\} = \{1, 2, 3, 4, 5\}$, define $F_{p,q} = f_p f_q \otimes f_r f_s f_t$ and an inner product on $\mathbb{C}[x,y] \otimes \mathbb{C}[x,y]$ as the definition 4.1. It is obvious that $\mathcal{G}_2(H)$ of $H$ is the Gram matrix of the ten tensors $F_{p,q}$ with $\{p, q, r, s, t\} = \{1, 2, 3, 4, 5\}$, $p < q$, and $r < s < t$. We observe that

$$(1 + i)F_{41} + (-1 + i)F_{12} + (-1 - i)F_{23} + (1 - i)F_{34} - 2iF_{51} + 2F_{52} + 2iF_{53} - 2F_{54}$$

$$= 16\sqrt{2}x^2 \otimes y^3 - 32\sqrt{2}xy \otimes xy^2 + 16\sqrt{2}y^2 \otimes x^2y,$$

whose norm squared is

$$2^9 \cdot 2!3! + 2^{11} \cdot 2! + 2^9 \cdot 2! \cdot 2! = 512 \cdot 24,$$

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while the norm squared of the coefficient vector is
\[ |1 + i|^2 + |-1 + i|^2 + |1 - i|^2 + |1 - i|^2 + |2i|^2 + 2^2 + |2i|^2 + |-2|^2 = 24. \]

Therefore, a linear operator mapping eight orthonormal vectors to \( F_{12}, F_{23}, F_{34}, F_{41}, F_{51}, F_{52}, F_{53}, F_{54} \) has norm at least \( \sqrt{512} \), so the Gram matrix of these eight tensors, which is an 8-square diagonal submatrix of \( \mathcal{C}_2(H) \), has norm (=largest eigenvalue) at least 512, whence so does \( \mathcal{C}_2(H) \) itself. In fact, the norm of \( \mathcal{C}_2(H) \) is 512.

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