FLAT PAIRING AND DIFFERENTIAL HOMOLOGY

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Abstract. Let $h^\bullet$ be a rationally even cohomology theory, $h_\bullet$ its dual homology theory and $\hat{h}^\bullet$ the natural differential refinement, as defined by Hopkins and Singer. We first construct in detail the natural $U(1)$-valued pairing between $h_\bullet$ and the flat part of $\hat{h}^\bullet$, generalizing the holonomy of a flat abelian $p$-gerbe. Then, in order to generalize the holonomy of any abelian $p$-gerbe, we define the differential homology theory $\hat{h}_\bullet$, and the generalized Cheeger-Simons characters. The latter are functions from the differential cycles to $U(1)$, such that the value on a trivial cycle only depends on the curvature.

1. Introduction

Let us consider the ordinary differential cohomology $\hat{H}^\bullet$ on a smooth manifold $X$ [7]. The group $\hat{H}^n(X)$ is canonically isomorphic to the group of the Cheeger-Simons differential characters of degree $n - 1$. An element of the latter is a couple $(\chi, \omega)$, where $\chi$ is an $\mathbb{R}/\mathbb{Z}$-valued group morphism defined on the smooth $(n - 1)$-cycles of $X$ (whose exponential is the holonomy), and $\omega$ is an integral $n$-form on $X$ (the curvature) such that, on a $p$-boundary $\partial D$, one has:

$$\chi(\partial D) = \int_D \omega \mod \mathbb{Z}.$$ 

In particular, when the class is flat the holonomy only depends on the homology class of the cycle, and actually the flat part of $\hat{H}^n(X)$ is canonically isomorphic to $H^{n-1}(X; \mathbb{R}/\mathbb{Z})$.

The aim of the present paper is to generalize this picture to the differential refinement $\hat{h}^\bullet$ of any rationally even cohomology theory $h^\bullet$, as defined by Hopkins and Singer [7, 13]. We start considering the flat case: a flat differential class of degree $n$ provides a morphism from the homology theory $h_\bullet$ to $U(1)$, even if, in general, we cannot consider only classes of degree $n - 1$, as we will clarify in the following. We thus get a pairing between $h_\bullet$ and the flat part of $\hat{h}^\bullet$, with values in $U(1)$. Then, in order to consider even non-flat classes, we define the differential homology groups $\hat{h}_\bullet$. We consider the geometrical definition of the homology theory $h_\bullet$ dual to $h^\bullet$, as described in [8], which provides a good notion of cycles and boundaries. We generalize this definition to the differential case, so that it is possible to define the holonomy of a differential class as a function from the differential cycles to $U(1)$, such that the value on a trivial cycle only depends on the curvature: this construction allows us to define the generalized Cheeger-Simons characters.

The paper is organized as follows. In section 2 we recall the preliminaries about homology and differential cohomology. In section 3 we show the pairing between $h_\bullet$ and the flat part of $\hat{h}^\bullet$. In section 4 we define the differential homology groups and the generalized Cheeger-Simons characters. In section 5 we discuss the case of ordinary differential homology and differential K-homology.

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2. PRELIMINARIES

2.1. COHOMOLOGY AND HOMOLOGY. Let us consider a multiplicative cohomology theory $h^\bullet$ represented by an $\Omega$-spectrum $(E_\bullet, *, \varepsilon_\bullet)$, where $e_n$ is the marked point of $E_n$ and $\varepsilon_n : (\Sigma E_n, \Sigma e_n) \to (E_{n+1}, e_{n+1})$ is the structure map, whose adjoint $\tilde{\varepsilon}_n : E_n \to \Omega e_{n+1} E_{n+1}$ is a homotopy equivalence. Considering the spectrum $(E_\bullet \wedge X, \varepsilon_\bullet \wedge x_0, \varepsilon_\bullet \wedge 1)$, the dual homology theory $h_\bullet$ is defined, on a space with marked point $(X, x_0)$, as

\[ h_n(X, x_0) := \pi_n(E_\bullet \wedge X, e_\bullet \wedge x_0) = \lim_{\rightarrow k} \pi_{n+k}(E_k \wedge X, e_k \wedge x_0). \]

The unreduced groups are defined as $h_n(X) := h_n(X, +\infty)$, for $X_+ = X \uplus \{+\infty\}$. For $\{*\}$ a space with one point and $h^\bullet := h^\bullet(\{*\})$, there is a natural map for every $n \in \mathbb{Z}$ [12]:

\[ \xi^n : h^n(X) \to \text{Hom}_R(h_{n-*}(X), h^\bullet). \]

From (2) we can easily define, for $h^\bullet_R := h^\bullet \otimes_{\mathbb{Z}} \mathbb{R}$:

\[ \xi^n_R : h^n(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \text{Hom}_R(h_{n-*}(X), h^\bullet_R). \]

It follows from the universal coefficient theorem [11] that the map (3) is an isomorphism. We will give a more geometric proof of this fact in the following. Finally, for $h^\bullet_{R/Z} := h^\bullet(\{*\}, \mathbb{R}/\mathbb{Z})$, there is a natural map:

\[ \xi^n_{R/Z} : h^n(X, \mathbb{R}/\mathbb{Z}) \to \text{Hom}_R(h_{n-*}(X), h^\bullet_{R/Z}). \]

For singular cohomology or K-theory [4] is an isomorphism, as a consequence of the universal coefficient theorem formulated via the Ext group [15, 10].

2.2. DUAL HOMOLOGY THEORY. In [8] the author provides a geometric construction of the homology theory dual to a given cohomology theory, and in [4] we provide an equivalent variant of that construction, which we briefly recall in the following, only in the case of a single space $X$.

**Definition 2.1.** Let $h^\bullet$ be a multiplicative cohomology theory. On a space $X$ with the homotopy type of a finite CW-complex, we define:

- the group of $n$-precycles of $h_\bullet$, as the free abelian group generated by the quadruples $(M, u, \alpha, f)$, with:
  - $(M, u)$ a smooth compact $h^\bullet$-manifold (without boundary), whose connected components $\{M_i\}$ have dimension $n + q_i$, with $q_i$ arbitrary;
  - $\alpha \in h^\bullet(M)$, such that $\alpha|_{M_i} \in h^\bullet(M)$;
  - $f : M \to X$ a continuous map;
- the group of $n$-cycles of $h_\bullet$, denoted by $z_n(X)$, as the quotient of the group of $n$-precycles by the free subgroup generated by elements of the form:
  - $(M, u, \alpha + \beta, f) - (M, u, \alpha, f) - (M, u, \beta, f)$;
  - $(M, u, \alpha, f) - (M_1, u|_{M_1}, \alpha|_{M_1}, f|_{M_1}) - (M_2, u|_{M_2}, \alpha|_{M_2}, f|_{M_2})$, for $M = M_1 \sqcup M_2$;
  - $(M, u, \varphi_0, \alpha, f) - (N, v, \alpha, f \circ \varphi)$ for $\varphi : N \to M$ and $\varphi_1 : h^\bullet(N) \to h^\bullet(M)$ the Gysin map;
- the group of $n$-boundaries of $h_\bullet$, denoted by $b_n(X)$, as the subgroup of $z_n(X)$ containing the cycles which are representable by a pre-cycle $(M, u, \alpha, f)$, such that there exits a quadruple $(W, U, A, F)$, where $W$ is a manifold and $M = \partial W$, $U$ is
a $h^\bullet$-orientation of $W$ and $U|_M = u$, $A \in h^\bullet(W)$ and $A|_M = \alpha$, $F : W \to X$ is a continuous map satisfying $F|_M = f$.

We define $h_n(X) := z_n(X) / b_n(X)$.

Describing in this way the dual homology theory, the map (2) corresponds to:

$$\xi^n : h^n(X) \to \Hom_{\mathbb{H}}(h_{n-\bullet}(X), \mathbb{H}^*)$$

$$\alpha \to ([M, u, \beta, f] \to (p_M)_!(\beta \cdot f^* \alpha)),$$

where $p_M : M \to \{\ast\}$. We verify that (5) is well-defined. If we consider two representatives $(M, u, \varphi \beta, f)$ and $(N, v, \beta, f \circ \varphi)$ of the homology class, we have:

$$\xi^n(\alpha)[N, v, \beta, f \circ \varphi] = (p_N)_!(\beta \cdot \varphi^* f^* \alpha) = (p_M)_!(\varphi((\beta \cdot \varphi^* f^* \alpha))$$

$$= (p_M)_!(\varphi(\beta \cdot f^* \alpha) = \xi^n(\alpha)[M, u, \varphi(\beta, f]$$

Let us now suppose that $(M, u, \beta, f) = \partial(W, U, B, F)$. Then we consider a function $\Phi : W \to [0, 1]$, such that $\Phi^{-1}(0) = M$ and $\Phi^{-1}(1) = \emptyset$. Since the Gysin map commutes with the restrictions to the boundaries up to a sign $[4]$, it follows that, for $\Phi$ $\partial W \to \{0, 1\}$ the restriction of $\Phi$, $\Phi'((F^* \alpha \cdot B))|_{\partial W} = (-1)^{\dim X} \Phi_0(F^* \alpha \cdot B)|_{[0, 1]}$. Since the image of $\Phi'$ is only $\{0\}$, it follows that we can identify $\Phi'$ with $p_M$, hence $(p_M)_!(f^* \alpha \cdot \beta) = (-1)^{\dim X} \Phi_0(F^* \alpha \cdot B)|_{[0, 1]}$. This implies that $(\{0, 1\}, \Phi_0(F^* \alpha \cdot B), \id)$ provides an equivalence of homology cycles on the point between $[\{\ast\}, (p_M)_!(f^* \alpha \cdot \beta), \id] \in 0$, hence $(p_M)_!(f^* \alpha \cdot \beta) = 0$ by Poincaré duality. Finally, the image of $\alpha$ is a $\mathbb{H}^\bullet$-module homomorphism, since, for $\gamma \in \mathbb{H}^\bullet$:

$$\xi^n(\alpha)([(M, u, \beta, f)] \cap \gamma) = \xi^n(\alpha)[M, u, \beta \cdot (p_M)^* \gamma, f] = (p_M)_!((\beta \cdot f^* \alpha) \cdot (p_M)^* \gamma)$$

$$= (p_M)_!(\beta \cdot f^* \alpha) \cdot \gamma = \xi^n(\alpha)[M, u, \beta, f] \cdot \gamma$$

We can thus give a more geometric proof of the fact that (3) is an isomorphism.

**Theorem 2.1.** The map (5) induces a natural isomorphism:

$$\xi^\mathbb{R}_n : h^n(X) \otimes \mathbb{R} \xrightarrow{\sim} \Hom_{\mathbb{H}}(h_{n-\bullet}(X), \mathbb{H}_R^\bullet).$$

Proof: It is easy to show the result for $X = \{\ast\}$. In fact, for $\alpha \in \mathbb{H}^n_R$, one has $\xi^n_{\mathbb{R}}(\alpha)(1) = \alpha$, for $1 = [\{\ast\}, 1, \id] \in \mathbb{H}^0$, so surely (5) is injective. It is also surjective, since a homomorphism $\varphi \in \Hom_{\mathbb{H}}(h_{n-\bullet}(X), \mathbb{H}_R^\bullet)$ is completely determined by $\varphi(1)$: in fact, any element of $\mathbb{H}^n_{n-\bullet}$ is of the form $[\{\ast\}, \beta, \id] = 1 \cdot \beta$, and $\varphi(1 \cdot \beta) = \varphi(1) \cdot \beta$.

In general, the Chern character provides an isomorphism:

$$\text{ch}^n : h^n(X) \otimes \mathbb{R} \xrightarrow{\sim} H^n(X, \mathbb{H}^\bullet).$$

A similar isomorphism holds in homology thanks to the homological Chern character:

$$\text{ch}_n : h_n(X) \otimes \mathbb{R} \xrightarrow{\sim} H_n(X, \mathbb{H}^\bullet)$$

$$\text{ch}_n[M, u, \alpha, f] := [M, u, \text{ch}(\alpha) \wedge \hat{A}_hM, f]$$

where $\hat{A}_hM = \int_{N M / M} \text{ch} u$, for $NM$ a representative of the stable normal bundle of $M$. 
Injectivity of (5). We start supposing that $X$ is a smooth manifold. In this case:

$$H^n(X, h^*_p) \simeq \bigoplus_{k \in \mathbb{Z}} H_{dR}^{n-k}(X) \otimes \mathbb{R} h^k_p,$$

for $H^*_R$ the de-Rham cohomology. If $\alpha \in h^n(X) \otimes \mathbb{R}$, $\alpha \neq 0$, there exists a maximum $k \in \mathbb{Z}$ such that $\text{ch}^n(\alpha)(n-k) \neq 0$, where $(n - k)$ denotes the component belonging to $H_{dR}^{n-k}(X) \otimes \mathbb{R} h^k_R$. Let $\omega^{(n-k)}$ be a differential form representing $\text{ch}^n(\alpha)(n-k)$: since it is non-trivial in cohomology, we can find an $(n-k)$-submanifold $Y$ of $X$ such that $\int_Y \omega \neq 0$: for $i : Y \hookrightarrow X$ the embedding, we consider the class $[(Y, 1, i)] \in h_{n-k}(X)$, where $1 \in h^0(Y)$ is defined as $p_Y^*(1)$ for $p_Y : Y \to \{*,\}$. The map (5) associates to $\alpha$ the homomorphism sending $[(Y, 1, i)]$ in $(p_Y)_!(i^* \alpha) \otimes \mathbb{Z} \in h^k_R$. Since the Chern character on a point, i.e. $\text{ch}_1^k : h^k \to H^k(\{\ast\}, h^k_R) \simeq h^k_R$, is given by $\text{ch}_1^k(\alpha) = \alpha \otimes \mathbb{Z} \mathbb{R}$, we get:

$$\text{(8)} \quad (p_Y)_!(i^* \alpha) \otimes \mathbb{Z} \mathbb{R} = \text{ch}^k((p_Y)_!(i^* \alpha)) = \int_Y \hat{A}_h Y \wedge i^* \text{ch}^n(\alpha) = \int_Y \omega^{(n-k)} \neq 0.$$

The only contribution to the integral is the one of $\omega^{(n-k)}$, since the other components of degree $n - k$ can be obtained via lower-degree components of $\text{ch}^n(\alpha)$, but $k$ was the maximum value for which $\text{ch}^n(\alpha)(n-k) \neq 0$. This proves the injectivity of (5) for $X$ a smooth manifold. When $X$ is any space homotopic to a finite CW-complex, the proof is analogous, replacing the submanifold $(Y, i)$ with a couple $(Y, f)$, for $Y$ an $(n-k)$-manifold and $f : Y \to X$ a map such that $f^* \text{ch}^n(\alpha)(n-k) \neq 0$. The latter can therefore be represented by a form $\omega^{(n-k)}$ on $Y$ and equation (8) still holds replacing $i$ with $f$.

Surjectivity of (5). Thanks to the homological Chern character:

$$\text{Hom}_R(h_{n-\ast}(X), h^*_p) \simeq \text{Hom}_R(H_{n-\ast}(X, h^*_p), h^*_p).$$

We can now decompose the singular homology with respect to the coefficient ring, and we get:

$$\text{Hom}_R \left( \bigoplus_{t \in \mathbb{Z}} H_{n-\ast+t}(X, \mathbb{R}) \otimes \mathbb{R} h^*_p, h^*_p \right) = \text{Hom}_R \left( \bigoplus_{p \in \mathbb{Z}} H_{n-p}(X, \mathbb{R}) \otimes \mathbb{R} h^*-p^*_+, h^*_p \right)$$

$$\simeq \bigoplus_{p \in \mathbb{Z}} \text{Hom}_R \left( H_{n-p}(X, \mathbb{R}), \text{Hom}_R \left( (h^*_p)^{p^*_+}, h^*_p \right) \right).$$

On the space $\{\ast\}$ we use the Poincaré duality and the fact that (5) is an isomorphism, as shown above. Hence we get:

$$\bigoplus_{p \in \mathbb{Z}} \text{Hom}_R \left( H_{n-p}(X, \mathbb{R}), \text{Hom}_R \left( (h^*_p)^{p^*_+}, h^*_p \right) \right) \simeq \bigoplus_{p \in \mathbb{Z}} \text{Hom}_R \left( H_{n-p}(X, \mathbb{R}), h^*_p \right)$$

$$\simeq \bigoplus_{p \in \mathbb{Z}} (H_{n-p}(X, \mathbb{R}))^* \otimes h^*_p \simeq \bigoplus_{p \in \mathbb{Z}} H^{n-p}(X, \mathbb{R}) \otimes h^*_p = H^n(X, h^*_p) \simeq h^n(X) \otimes \mathbb{R}.$$

This shows that $h^n(X) \otimes \mathbb{Z} \mathbb{R}$ and $\text{Hom}_R(h_{n-\ast}(X), h^*_p)$ are isomorphic real vector spaces. Since $h^n(X) \otimes \mathbb{Z} \mathbb{R}$ is finite-dimensional, it follows that they are both. Hence, since $\xi^*_p$ is injective, it is also surjective. □
The map (4) corresponds to:

$$\xi^n_{\mathbb{R}/\mathbb{Z}} : h^n(X, \mathbb{R}/\mathbb{Z}) \to \text{Hom}_{\mathbb{R}^{\bullet}}(h_{n-\bullet}(X), h^{\bullet}_{\mathbb{R}/\mathbb{Z}})$$

(9)

$$\alpha \mapsto ([M, u, \beta, f] \mapsto (p_M)(\beta \cdot f^*\alpha)).$$

The product $$\beta \cdot f^*\alpha$$ is provided by the structure of $$h^\bullet$$-module on $$h^\bullet(\cdot, \mathbb{R}/\mathbb{Z})$$. Moreover, because of such a structure the Poincaré duality on $$h^\bullet$$-oriented manifold holds even for $$\mathbb{R}/\mathbb{Z}$$-coefficients, hence we can define $$\langle p_M(\alpha) := \text{PD}_{\{x\}}(p_X) \text{PD}_X(\alpha)$$.

2.3. **Differential cohomology.** We fix a multiplicative cohomology theory $$h^\bullet$$ which is rationally even, i.e. $$h^{\text{odd}}_{\mathbb{R}} = 0$$. We suppose that $$h^\bullet$$ is represented by an $$\Omega$$-spectrum $$(E_n, e_n, \varepsilon_n)$$, where $$e_n$$ is the marked point of $$E_n$$ and $$\varepsilon_n : (\Sigma E_n, \Sigma e_n) \to (E_{n+1}, e_{n+1})$$ is the structure map, whose adjoint $$\varepsilon_n : E_n \to \Omega_{e_{n+1}} E_{n+1}$$ is a homeomorphism (not only a homotopy equivalence). We also call $$\mu_{n,m} : E_n \wedge E_m \to E_{n+m}$$ the maps making $$E$$ a ring spectrum (hence the theory $$h^\bullet$$ multiplicative). Moreover, we fix the following data [13]:

- real singular cocycles representing the Chern character $$\iota_n \in C^n(E_n, e_n, h^\bullet_{\mathbb{R}})$$, such that:

$$\iota_{n-1} = \int_{S^1} \varepsilon_{n}^* \iota_n.$$  

- Maps $$\alpha_n : E_n \times E_n \to E_n$$ representing the addition in cohomology, i.e. such that, for $$X$$ a topological space and $$f, g : X \to E_n$$, one has $$[f] + [g] = [\alpha_n \circ (f, g)]$$; we require that, for $$\varphi_n : \Sigma(E_n \times E_n) \to E_{n+1} \times E_{n+1}$$ the structure maps of the spectrum $$E_n \times E_n$$ (defined via the factorization $$\Sigma(E_n \times E_n) \to \Sigma E_n \times \Sigma E_n \to \Sigma E_{n+1} \times \Sigma E_{n+1}$$), one has $$\varepsilon_{n-1} \circ \Sigma \alpha_{n-1} = \alpha_n \circ \varphi_{n-1}$$.

- We call $$\pi_{1,n}, \pi_{2,n} : E_n \times E_n \to E_n$$ the two projections; their homotopy classes correspond to two elements of $$h^n(E_n \times E_n)$$, whose sum is represented by $$\alpha_n \circ (\pi_{1,n}, \pi_{2,n}) = \alpha_n$$, since $$(\pi_{1,n}, \pi_{2,n}) = \text{id}_{E_n \times E_n}$$. Therefore:

$$\pi_{1,n}^* [\iota_n] + \pi_{2,n}^* [\iota_n] = \text{ch}([\pi_{1,n}]) + \text{ch}([\pi_{2,n}]) = \text{ch}([\pi_{1,n} \times \pi_{2,n}]) = \text{ch}[\alpha_n] = \alpha_n^* [\iota_n],$$

hence there exists $$A_{n-1} \in C^{n-1}(E_n \times E_n, e_n \times e_n, h^\bullet_{\mathbb{R}})$$ such that:

$$\pi_{1,n}^* (\iota_n) + \pi_{2,n}^* (\iota_n) - \alpha_n^* (\iota_n) = \delta^{n-1} A_{n-1}.$$  

(11)

Since we are assuming that $$h^{\text{odd}}_{\mathbb{R}} = 0$$, $$A_{n-1}$$ is unique up to a coboundary for $$n$$ even [13]. Then we define $$A_{n-2} := - \int_{S^1} \varphi_{n-1} A_{n-1}$$, where the integration map is defined via the prisma map, and $$\varphi_{n-1}$$ is the structure map of the spectrum $$E_n \times E_n$$ defined above. In this way $$A_{n-1}$$ is unique up to a coboundary for every $$n$$.

- Since, for $$f : X \to E_n$$ and $$g : Y \to E_m$$, one has $$\text{ch}([f] \times [g]) = \text{ch}[f] \times \text{ch}[g]$$, where $$[f] \times [g] = [\mu_{n,m} \circ (f, g)]$$, it follows that, for $$f = \text{id}_{E_n}$$ and $$g = \text{id}_{E_m}$$, one has $$\text{ch}[\mu_{n,m}] = \text{ch}[\text{id}_{E_n}] \times \text{ch}[\text{id}_{E_m}]$$. Hence there exists $$M_{n,m} \in C^{n+m-1}(E_n \wedge E_m, e_n \wedge e_m, h^\bullet_{\mathbb{R}})$$ such that:

$$\delta^{n+m-1} M_{n,m} = \iota_n \times \iota_m - \mu_{n,m}^* \iota_{n+m}.$$  

(12)

Since we are assuming that $$h^{\text{odd}}_{\mathbb{R}} = 0$$, it follows that $$M_{n,m}$$ is unique up to a coboundary for $$n$$ and $$m$$ even [13].
We fix a chain homotopy between the wedge product of two differential forms and the cup product of the two associated singular cochains. In particular, we consider the two maps \( P, Q : \Omega^n(X, V^*) \otimes \Omega^m(X, V^*) \to C^{n+m}(X, V^*) \) defined by \( P(\omega \otimes \rho) := \chi(\omega \wedge \rho) \) and \( Q(\omega \otimes \rho) := \chi(\omega) \cup \chi(\rho) \), for \( \chi : \Omega^*(X, V^*) \to C^*(X, V^*) \) the natural homomorphism. The coboundary of \( \Omega^n(X, V^*) \otimes \Omega^m(X, V^*) \) is \( d(\omega \otimes \rho) := d\omega \otimes \rho + (-1)^{|\omega|}\omega \otimes dp \). There is a chain homotopy \( B : \Omega^n(X, V^*) \otimes \Omega^m(X, V^*) \to C^{n+m-1}(X, V^*) \), which by definition satisfies:

\[
\chi(\omega \wedge \rho) - \chi(\omega) \cup \chi(\rho) = \delta B(\omega \otimes \rho) + Bd(\omega \otimes \rho).
\]

We recall the following definition \([7]\):

**Definition 2.2.** If \( X \) is a smooth manifold (even with boundary), \( Y \) a topological space, \( V^* \) a graded real vector space and \( \kappa_n \in C^n(Y, V^*) \) a real singular cocycle, a differential function from \( X \) to \( (Y, \kappa_n) \) is a triple \((f, h, \omega)\) such that:

- \( f : X \to Y \) is a continuous function;
- \( h \in C^{n-1}(X, V^*) \);
- \( \omega \in \Omega^n(X, V^*) \)

satisfying, for \( \chi : \Omega^*(X, V^*) \to C^*(X, V^*) \) the natural homomorphism:

\[
\delta^{n-1}h = \chi^n(\omega) - f^*\kappa_n.
\]

Moreover, a homotopy between two differential functions \((f_0, h_0, \omega)\) and \((f_1, h_1, \omega)\) is a differential function \((F, H, \pi^*\omega) : X \times I \to (Y, \kappa_n)\), such that \( F \) is a homotopy between \( f_0 \) and \( f_1 \), \( H|_{X \times \{i\}} = h_i \) for \( i = 0, 1 \), and \( \pi : X \times I \to X \) is the natural projection.

A differential function with compact support from \( X \) to \( (Y, y_0, \kappa_n) \) is a differential function \((f, h, \omega) : X \to (Y, \kappa_n)\) such that there exists a compact subset \( K \subset X \) verifying \( f|_{X \setminus K} \equiv y_0 \) and \( h \) and \( \omega \) have support contained in \( K \).

We can now define the natural differential extension of \( h^* \) in the following way, for \( X \) a smooth manifold (even with boundary) \([13, 2]\):

- as a set, \( \tilde{h}^n(X) \) contains the homotopy classes of differential functions \((f, h, \omega) : X \to (E_n, \kappa_n)\).
- The sum is defined as:

\[
[(f, h, \omega)] + [(g, k, \rho)] := [(\alpha_n \circ (f, g), h + k + (f, g)^*A_{n-1}, \omega + \rho)].
\]

- The first Chern class and the curvature are defined as \( I[(f, h, \omega)] := [f] \) and \( R[(f, h, \omega)] := \omega \), and the map \( a : \hat{\Omega}^{*+1}(X, h_0^*)/\text{Im}(d) \to \hat{h}^*(X) \) (v. \([2]\)) is defined as \( a[\rho] := [c_{e_n}, \chi(\rho), dp], \) for \( c_{e_n} \), the constant map whose value is the marked point \( e_n \).

- The \( S^1 \)-integration map is defined in the following way: for \( i : X \times \{1\} \hookrightarrow X \times S^1 \) and \( p : X \times S^1 \to X \) the natural maps, given \([[(f, h, \omega)] \in \tilde{h}^{n+1}(X \times S^1)\) we consider \( [(f, h, \omega)] - p^*i^*[(f, h, \omega)] \in \text{Ker}(i^*) \). The latter can be represented by a triple \((g, k, \omega - p^*i^*\omega)\) such that \( g|_{X \times \{1\}} \) is the constant map with value \( e_{n+1} \) and \( k|_{X \times \{1\}} \equiv 0 \). From \( g : (X \times S^1, X \times \{1\}) \to (E_{n+1}, e_{n+1}) \) we can define \( \tilde{g} : X \to \Omega_{e_{n+1}}E_{n+1} \) and \( \int_{S^1} g := \tilde{\xi}^{-1} \circ \tilde{g} \). Moreover, we define \( \int_{S^1} k \) via the prism map. Hence we consider:

\[
[\left( \int_{S^1} g, \int_{S^1} k, \int_{S^1} \omega \right)]
\]

\(^1\)The map \( a \) is well-defined since \( \delta \chi(\rho) = \chi(dp) \).
The composition:

\[ \delta \]

morphism any more, and the differential Gysin map. We consider the natural embedding

\[ \varphi : X \to Y \]

defines a structure of \( \hat{u} \mapsto \chi(\varphi) \) such that \( \int_{Y} f_{\varphi}^{\ast} \partial \).

For any \( \alpha \in \hat{h}^n(X) \) (without restrictions on \( n \)) there exists \( \alpha' \in \hat{h}^{n+1}(X \times S^1) \) such that \( \int_{S^1} \alpha' = \alpha \) and \( R(\alpha') = dt \wedge \pi^{\ast}R(\alpha) \) [13]. Hence we define, for \( n \) and \( m \) both even:

- for \( \alpha \in \hat{h}^{n-1}(X) \) and \( \beta \in \hat{h}^m(X) \), \( \alpha \cdot \beta := \int_{S^1} \alpha' \cdot \pi^{\ast} \beta' \);
- for \( \alpha \in \hat{h}^{n}(X) \) and \( \beta \in \hat{h}^{m-1}(X) \), \( \alpha \cdot \beta := \int_{S^1} \pi^{\ast} \alpha' \cdot \beta' \);
- for \( \alpha \in \hat{h}^{n-1}(X) \) and \( \beta \in \hat{h}^{m-1}(X) \), \( \alpha \cdot \beta := -\int_{S^1} \pi^{\ast} \alpha' \cdot \beta' \).

2.4. Orientability and Gysin map. A real vector bundle \( E \to X \) of rank \( n \) is orientable with respect to a multiplicative cohomology theory \( \hat{h}^\bullet \) if there exists a Thom class \( u \in \hat{h}^{n}(E^+) \) [11]. If we consider a differential refinement \( \hat{h}^\bullet \) of \( h^\bullet \), in order to define orientability one just has to refine the Thom class \( u \) to a differential Thom class.

**Definition 2.3.** Let \( h^\bullet \) be a multiplicative differential cohomology theory represented by an \( \Omega \)-spectrum \( E = \{ E_k, e_k, e_k \}_{k \in \mathbb{Z}} \), and \( E \to X \) a smooth real vector bundle of rank \( n \). A differential Thom class of \( E \) with respect to the extension \( \hat{h}^\bullet \) defined above is a compactly supported class \( [u, h, \omega] \in \hat{h}^{n}_{cpt}(E) \) such that:

- \([u] \in \hat{h}^{n}(E^+)\) is a Thom class for \( h^\bullet \);
- for every \( x \in X \), \( \int_{E_x} \omega(n) = \pm 1 \), where \( \omega(n) \) is the \( n \)-degree component of \( \omega \).

We can now define the differential Thom morphism, which in general is not an isomorphism any more, and the differential Gysin map. We consider the natural embedding \( \delta : E \to X \times E \), defined by \( (e, e) \to (\pi(e), e) \), and the product \( \hat{h}^\bullet(X) \otimes_{\mathbb{Z}} \hat{h}^\bullet_{cpt}(E) \to \hat{h}^\bullet_{cpt}(X \times E) \).

The composition:

\[
\hat{h}^\bullet(X) \otimes_{\mathbb{Z}} \hat{h}^\bullet_{cpt}(E) \to \hat{h}^\bullet_{cpt}(X \times E) \xrightarrow{\delta^\ast} \hat{h}^\bullet_{cpt}(E)
\]

defines a structure of \( \hat{h}^\bullet(X) \)-module on \( \hat{h}^\bullet_{cpt}(E) \). The differential Thom morphism is the map \( \alpha \to \alpha \cdot \hat{u} \), for \( \hat{u} \) a differential Thom class. For the Gysin map, we start for simplicity from manifolds without boundary. Given an embedding \( i : Y \to X \), we endow the normal bundle \( N_Y X \) with a differential Thom class \( \hat{u} \). Then, we consider a tubular neighborhood \( U \) of \( Y \) in \( X \), a diffeomorphism \( \varphi_U : U \to N_Y X \) and the natural map \( \psi : X \to U^+ \) defined as \( \psi(x) = x \) for \( x \in U \) and \( \psi(x) = \infty \) for \( x \in X \setminus U \). The Gysin map \( i_! : \hat{h}^\bullet(Y) \to \hat{h}^{\bullet+(\dim X - \dim Y)}_{cpt}(X) \) is defined as:

\[
i_!(\alpha) = \psi^\ast(\varphi_U^{\ast})^\ast(\alpha \cdot \hat{u}).
\]

Given a map of compact manifolds \( f : Y \to X \) (not necessarily an embedding), we choose an embedding \( j : Y \to \mathbb{R}^N \), and the embedding \( (f, j) : Y \to X \times \mathbb{R}^N \). Then we consider the Gysin map:

\[
(f, j)_! : \hat{h}^\bullet(Y) \to \hat{h}^{\bullet+(\dim X - \dim Y)}_{cpt}(X \times \mathbb{R}^N)
\]

followed by the integration map:

\[
\int_{\mathbb{R}^N} : \hat{h}^{\bullet+N}_{cpt}(X \times \mathbb{R}^N) \to \hat{h}^\bullet(X)
\]
which we now define. We have defined above the $S^1$-integration map for differential functions. We can generalize it to the $S^N$-integration map. There is a natural projection $\pi^N: (S^1)^N \to S^N$, defined thinking of $S^N$ as $S^1 \land \ldots \land S^1 = (S^1)\land \pi \ldots \land S^1$. For $(f,h,\omega): X \times S^N \to (E_{n+N}, t_{n+N})$, we consider the pull-back $(1 \times \pi^N)^*(f,h,\omega): X \times (S^1)^N \to (E_{n+N}, t_{n+N})$. We define:

$$(20) \quad \int_{S^N} (f,h,\omega) := \int_{S^1} \cdots \int_{S^1} (1 \times \pi^n)^*(f,h,\omega).$$

Given a differential function with compact support $(f,h,\omega): X \times \mathbb{R}^N \to (E_{n+N}, e_{n+N}, t_{n+N})$ (v. def. 2.2), since $S^N$ is the one-point compactification of $\mathbb{R}^N$, we can naturally define $(f,h,\omega)^+: X \times S^N \to (E_{n+N}, t_{n+N})$, and:

$$\int_{\mathbb{R}^N} (f,h,\omega) := \int_{S^N} (f,h,\omega)^+.$$

**Lemma 2.2.** For $(f,h,\omega): X \times \mathbb{R}^N \to (E_{n+N}, e_{n+N}, t_{n+N})$, the homotopy class of $\int_{\mathbb{R}^N} (f,h,\omega)$ only depends on the homotopy class of $(f,h,\omega)$. Moreover, for any $N_1, N_2$ such that $N_1 + N_2 = N$:

$$(21) \quad \int_{\mathbb{R}^N} (f,h,\omega) = \int_{\mathbb{R}^{N_2}} \int_{\mathbb{R}^{N_1}} (f,h,\omega).$$

Proof: If $(F,H,\pi^*\omega): X \times \mathbb{R}^N \times I \to (E_{n+N}, e_{n+N}, t_{n+N})$ is a homotopy between $(f_0, h_0, \omega)$ and $(f_1, h_1, \omega)$, then we can think of the domain of $(F,H,\pi^*\omega)$ as $X \times I \times \mathbb{R}^N$, and $\int_{\mathbb{R}^N} (F,H,\pi^*\omega)$ is a homotopy between $\int_{\mathbb{R}^N} (f_0, h_0, \omega)$ and $\int_{\mathbb{R}^N} (f_1, h_1, \omega)$. Formula (21) easily follows from the factorization $(S^1)^N \simeq (S^1)^{N_2} \times (S^1)^{N_1}$, since, being $(f,h,\omega)$ compactly supported, its extensions commute with any map between compactifications of $\mathbb{R}^N$. □

In the case of manifolds with boundary the definition is similar, remembering that the map must be neat. In particular, when $f: X \to Y$ is not an embedding, instead of considering the embedding $(f,j)$, which is not neat, we apply the following theorem (v. [7] Appendix C) and references therein):

**Theorem 2.3.** Let $f: (Y, \partial Y) \to (X, \partial X)$ be a neat map. Then there exists a neat embedding $\iota: (Y, \partial Y) \to (X \times \mathbb{R}^N, \partial X \times \mathbb{R}^N)$, stably unique up to isotopy, such that $f = \pi_X \circ \iota$ for $\pi_X: X \times \mathbb{R}^N \to X$ the projection.

We thus define $f_i\alpha := \int_{S^N} \iota \alpha$. This construction of the Gysin map naturally leads to the following definition [7].

**Definition 2.4.** An $\hat{h}$-oriented smooth map is the data of:

- a smooth neat map between compact manifolds $f: Y \to X$;
- a neat embedding $\iota: Y \to X \times \mathbb{R}^N$ for any $N \in \mathbb{N}$, such that $\pi_X \circ \iota = f$;
- a differential Thom class $\hat{u}$ of the normal bundle $N_Y(X \times \mathbb{R}^N)$;
- a neat tubular neighborhood $U$ of $Y$ in $X \times \mathbb{R}^N$ with a diffeomorphism $\varphi: U \to N_Y(X \times \mathbb{R}^N)$.

It follows that $f_i$ is well-defined for an $\hat{h}$-oriented smooth map $f$. 
3. Flat pairing

We are going to define geometrically the Gysin map of \( h^\bullet(\cdot, \mathbb{R}/\mathbb{Z}) \) via the flat part of the differential extension of \( h^\bullet \), supposing that \( h^n \) is finitely generated for every \( n \in \mathbb{Z} \). Then we define the holonomy of a flat differential class over a homology class.

3.1. Flat classes. Given a smooth map \( f : Y \to X \), the Gysin map:

\[
\hat{h}^\bullet(Y) \to \hat{h}^\bullet+(\dim X - \dim Y)(X)
\]

previously defined depends in general on the data involved in the definition \( 2.4 \) of \( \hat{h}^\bullet \)-oriented map, which must be fixed together with the map \( f \) itself. This dependence is clear looking at the curvature: since the latter is a single differential form, and not a cohomology class, changing one of that data also the final curvature will change. We now show that this is the only problem, in the sense that, if we start from a flat class, the result will depend only on the map \( f \) (more precisely on its homotopy class), once that \( Y \) and \( X \) are topologically \( h^\bullet \)-oriented.

There is a natural graded module structure on \( \hat{h}^\bullet(X) \) over \( h^\bullet(X) \), i.e. there exists a product:

\[
\hat{h}^\bullet(X) \otimes \mathbb{Z} h^\bullet(X) \to \hat{h}^\bullet(X).
\]

In fact, the product \( 16 \) restricts to a product \( \hat{h}^\bullet(X) \otimes \mathbb{Z} \hat{h}^\bullet(X) \to \hat{h}^\bullet(X) \), since if one of the two factors has vanishing curvature, also the result has. Actually the product \( \alpha \cdot \beta \), with \( \alpha \) flat, depends only on the first Chern class of \( \beta \). That’s because the equation \( 16 \) can be rewritten, using \( 13 \) for \( (g, k, \rho) \), as:

\[
[(f, h, \omega)] \cdot [(g, k, \rho)] := [(\mu_{n,m} \circ (f, g), h \cup g^* t_m + \chi(\omega) \cup k + B(\omega \otimes \rho) + (f, g)^* M_{n,m}, \omega \wedge \rho)].
\]

Therefore, if \( \omega = 0 \) we get:

\[
[(f, h, 0)] \cdot [(g, k, \rho)] := [(\mu_{n,m} \circ (f, g), h \cup g^* t_m + (f, g)^* M_{n,m}, 0)],
\]

and we see that the result does not depend on \( k \) and \( \rho \). The product \( [(f, h, 0)] \cdot [g] \) is thus well-defined for even degrees. If the degree of \( [(f, h, 0)] \) is odd, by definition we have to consider a class \( \alpha^1 \in h^{*+1}(X \times S^1) \), whose curvature is \( dt \wedge \pi^* R[(f, h, 0)] = 0 \), and apply the definition after formula \( 15 \): since even \( \alpha^1 \) is flat, the product remains well-defined. If the degree of \( [g] \) is odd, we lift it to \( X \times S^1 \), and the product only depends on the Chern class since the first factor remains the flat class \( [(f, h, 0)] \) or its flat lift. Therefore, the product \( 22 \) is well-defined at any degree. The same happens for the exterior product, since the formulas defining it are the same \( 13 \).

This implies that also the module structure \( 17 \) can be refined to:

\[
\hat{h}^\bullet(X) \otimes \mathbb{Z} h^\bullet_{\text{cpt}}(E) \to \hat{h}^\bullet_{\text{cpt}}(E).
\]

Therefore, given a real vector bundle \( E \to X \) of rank \( k \), the Thom morphism:

\[
T_\hat{h} : \hat{h}^\bullet(X) \to \hat{h}^\bullet_{\text{cpt}}(E),
\]

defined as \( \alpha \to \alpha \cdot \hat{u} \) using \( 17 \), actually depends only on the topological Thom class \( u \) of \( E \), not on its differential refinement. From this it easily follows that the Gysin map \( f_! \),

\footnote{Actually the fact of being \( h^n \) finitely generated is not the minimal hypothesis in order to identify \( h^{*+1}(\cdot, \mathbb{R}/\mathbb{Z}) \) with the flat part of \( \hat{h}^\bullet \), we refer to \( 2 \) for the details.}
when applied to a flat class, depends only on the map \( f \). The proof is the same used in [9], pp. 230-233, about the Gysin map in topological K-theory. In fact, if \( f \) is an embedding, the tubular neighborhood is unique up to isotopy (in particular, homotopy) and \( h_n^\bullet \) is a homotopy invariant [2]; if \( f \) is a generic map, from the commutative diagram of p. 233 of [9] we deduce that the result does not depend on the embedding chosen. We also state the following theorem, whose proof is analogous to the one of theorem 5.24 p. 233 of [9].

**Theorem 3.1.** For \( f : Y \to X \) and \( g : Z \to Y \):

- the Gysin map \( f_! : h_n^\bullet(Y) \to h_{n+(\dim X-\dim Y)}^\bullet(X) \) depends only on the homotopy class of \( f \);
- \( (f \circ g)_! = f_! \circ g_! \);
- for \( \hat{\alpha} \in h_n^\bullet(Y) \) and \( \beta \in h^\bullet(X) \), one has:

\[
\hat{f}_!(\hat{\alpha} \cdot f^* \beta) = f_!(\hat{\alpha}) \cdot \beta.
\] (24)

Therefore the map \( [5] \) can be described via the Gysin map for flat differential classes:

\[
\xi_{R/Z}^{n-1} : h_n^\bullet(X) \to \text{Hom}_{h^\bullet}(h_{n-1}, h_n^\bullet(X), h_{n+1}^\bullet) \to \left( [M,u,\beta,f] \to (p_M)!((f^* \hat{\alpha} \cdot \beta)) \right).
\] (25)

In order to show that (25) is well-defined, i.e. that it does not depend on the representative \((M,u,\beta,f)\) of the homology class, we use use an argument similar to the one used for (5). For \( \varphi : N \to M \) and \( \beta \in h^\bullet(N) \), one has \((N,v,\beta,f \circ \varphi) \simeq (M,u,\varphi_! \beta,f)\). Then, thanks to theorem [3], one has:

\[
(p_M)!((f^* \hat{\alpha} \cdot \varphi \beta)) = (p_M)!((\varphi_! f^* \hat{\alpha} \cdot \beta)) = (p_M \circ \varphi)!((f \circ \varphi)^* \hat{\alpha} \cdot \beta) = (p_N)!((f \circ \varphi)^* \hat{\alpha} \cdot \beta).
\]

Let us suppose that \((M,u,\beta,f) = \partial(W,U,B,F)\). Then we consider a function \( \Phi : W \to [0,1] \), such that \( \Phi^{-1}(0) = M \) and \( \Phi^{-1}(1) = \emptyset \). Since the Gysin map commutes with the restrictions to the boundaries up to a sign [1], it follows that \((p_M)!((f^* \hat{\alpha} \cdot \beta) = \Phi_!(F^* \hat{\alpha} \cdot B)|_{\{0\}}\). This implies that \( \Phi_!(F^* \hat{\alpha} \cdot B) \) is a homotopy of differential functions between \((p_M)!((f^* \hat{\alpha} \cdot \beta) \) and 0, hence \([p_M]((f^* \hat{\alpha} \cdot \beta)) = 0 \). Finally, the image of \( \hat{\alpha} \) is a \( h^\bullet \)-module homomorphism, since, for \( \gamma \in h^\bullet \):

\[
\xi_{R/Z}^{n-1}(\hat{\alpha})([M,u,\beta,f] \cap \gamma) = \xi_{R/Z}^{n-1}(\hat{\alpha})[M,u,\beta,(p_M)^*\gamma,f] = (p_M)!((\beta \cdot f^* \hat{\alpha} \cdot (p_M)^*\gamma) = (p_M)!((\beta \cdot f^* \hat{\alpha} \cdot \gamma) = \xi_{R/Z}^{n-1}(\hat{\alpha})[M,u,\beta,f] \cdot \gamma.
\]

**Lemma 3.2.** For \( f : Y \to X \) a map of manifolds:

\[
f_!(0,h,0) = [(0, f_!(h \wedge \hat{A}_h Y) \wedge \hat{A}_h X^{-1}, 0)]
\]

where \( f_! \) in the r.h.s. is the Gysin map in real singular cohomology with coefficients in \( h_R^\bullet \).

This is equivalent to:

\[
f_![0,(ch_R x,0)] = [(0, ch_R(f_! x), 0)]
\] (26)

for any \( x \in h^\bullet(Y) \otimes_Z R \).

\[3\text{In equation (26) we are considering the Chern character as defined on } h^\bullet(X) \otimes_Z R, \text{ in which case it is an isomorphism. If we consider it as defined on } h^\bullet(X), \text{ then } [(0, ch_x,0)] = 0, \text{ and formula (26) implies coherently that } f_!(0, ch_x, 0) = 0.\]
Proof: From (23) we get the differential Thom morphism $[(0, h, 0)] \cdot [u] := [(0, h \cup u^* t_m, 0)]$, and:

$$[h] \cdot u^*[t_m] = [h] \cdot \text{ch}^m[u] = T \left( [h] \cdot \int_{N_Y (X \times \mathbb{R}^N)/Y} \text{ch}^m[u] \right) = T ( [h] \cdot \hat{A}_h (N_Y (X \times \mathbb{R}^N)/Y)^{-1} ).$$

Therefore we get:

$$f_!([h] \cdot \hat{A}_h (N_Y (X \times \mathbb{R}^N)/Y)^{-1}) = f_!([h] \cdot \hat{A}_h ((f^* T (X \times \mathbb{R}^N))/Y)^{-1} \cdot \hat{A}_h (TY/Y))$$

$$= f_!([h] \cdot f^*(\hat{A}_h X)^{-1} \cdot \hat{A}_h Y) = f_!( [h] \cdot \hat{A}_h Y \cdot \hat{A}_h X^{-1} ).$$

Thus:

$$f_!([0, \text{ch}_E x, 0]) = [(0, f_!(\text{ch}_E x \land \hat{A}_h Y) \land \hat{A}_h X^{-1}, 0)] = [(0, \text{ch}_E (f_! x), 0)].$$

□

**Corollary 3.3.** The Gysin map associated to $f : Y \to X$ induces a morphism of exact sequences of $\mathfrak{h}^*$-modules:

$$\cdots \to h^* (Y) \to h^*(Y) \otimes_{\mathbb{Z}} R \to \hat{h}^{*,+1} (Y) \to h^{*+1} (Y) \to \cdots$$

$$\cdots \to h^* (X) \to h^*(X) \otimes_{\mathbb{Z}} R \to \hat{h}^{*,+1} (X) \to h^{*+1} (X) \to \cdots$$

where the map $h^* (X) \otimes_{\mathbb{Z}} R \to \hat{h}^{*,+1} (X)$ is defined by $x \to [(0, \text{ch}_E x, 0)]$. □

**Corollary 3.4.** There is a morphism of complexes of $\mathfrak{h}^*$-modules (the second one not being exact in general):

$$\cdots \to h^n (X) \otimes_{\mathbb{Z}} R \to h^n (X) \otimes_{\mathbb{Z}} R \to \hat{h}^{n+1} (X) \to h^{n+1} (X) \to \cdots$$

where the map $\hat{h}^n (X) \otimes_{\mathbb{Z}} R \to \hat{h}^{n+1} (X)$ is defined by $x \to [(0, \text{ch}_E x, 0)]$. □

Proof: We only have to prove the commutativity of the square under the map $a$. It easily follows from the fact that, for $x \in h^* (X) \otimes_{\mathbb{Z}} R$ and $y \in h^* (X)$:

$$[(0, \text{ch}_E x, 0)] \cdot y = [(0, \text{ch}_E (x y), 0)].$$

This is a direct consequence of formula (23). □

Considering the map (25), we now need a map from $\hat{h}^*_E$ to $\mathbb{R}/\mathbb{Z}$ in order to define the holonomy of a flat class over a cycle.

### 3.2. One-point space.

We use the following notation: for $V^*$ a real vector space and $\omega \in \Omega^n (X, V^*)$, we denote by $\omega^{(m)}$ the $m$-degree component $\omega^{(m)} \in \Omega^m (X, V^{n-m})$; we use the same notation for singular cochains. We construct a natural group homomorphism:

$$\Gamma^{2k+1} : \hat{h}^{2k+1} \to \mathbb{R}/\mathbb{Z}$$

for any cohomology theory $h^*$. Representing $h^*$ via an $\Omega$-spectrum $E = \{ E_k, \varepsilon_k, \varepsilon_k \}_{k \in \mathbb{Z}}$, by definition $\hat{h}^{2k+1}$ is a homotopy class of triples $(f, h, \omega)$ with $f : \{ * \} \to E_{2k+1}$, $h \in C^{2k} (\{ * \}, \hat{h}^*_E)$ and $\omega \in \Omega^{2k+1} (\{ * \}, \hat{h}^*_E)$, such that $\delta^{2k} h = \chi^{2k+1} (\omega) - f^* t_{2k+1}$. Since on a
point there are non-trivial forms only in degree 0, it follows that \( \omega^{(1)} = 0 \). Moreover, also \( f^* \partial_{2k+1} = 0 \), since on a point there are no non-trivial cocycles of odd degree. Hence:

\[
\delta^0 h^{(0)} = \chi^{2k+1}(\omega^{(1)}) - f^* \partial_{2k+1}^{(1)} = 0.
\]

It follows that there is a well-defined map, defined on the single differential functions (not up to homotopy):

\[
(28) \quad (f, h, \omega) \rightarrow h^{(0)}, \quad h^{(0)} \in H^0(\{\star\}, h_2^k) \simeq h_2^k.
\]

From now on we suppose that \( h_2^k \) has a non-trivial free part, otherwise the map (28) vanishes. Actually, we consider the case \( h_2^k \simeq \mathbb{Z} \oplus \text{Tor} h_2^k \), and we suppose such an isomorphism fixed: we will discuss later the case \( h_2^k \simeq \mathbb{Z}^p \oplus \text{Tor} h_2^k \) with \( p > 1 \), which does not actually occur in the most common cases, and the dependence on the isomorphism. We recall that the Chern character on a point:

\[
\text{ch}_n^\infty : h^n \rightarrow H^n(\{\star\}, h_2^*) \simeq H^n(\{\star\}, h_2^k) \simeq h_2^k
\]

is simply defined as \( \text{ch}_n^\infty(\alpha) = \alpha \otimes \mathbb{Z} \mathbb{R} \).

**Lemma 3.5.** If the differential functions \( (f, h_0, \omega) \) and \( (f, h_1, \omega) \) (fixing for the moment the map \( f \)) are homotopic, then \( \hat{h}_1^{(0)} - \hat{h}_0^{(0)} \in \mathbb{Z} \) with respect to the identification \( H^0(\{\star\}, h_2^k) \simeq h_2^k \simeq \mathbb{R} \).

**Proof:** Let us choose a homotopy \((F, H, \pi \omega) : (f, h_0, \omega) \simeq (f, h_1, \omega)\). For \( \pi : I \rightarrow S^1 \) the natural projection, there is a map \( \tilde{F} : S^1 \rightarrow E_{2k+1} \), with \( \tilde{F}(1) = f(\star) \) such that \( F = \tilde{F} \circ \pi \). Moreover:

\[
(29) \quad \delta^0 H^{(0)} = -F^* \partial_{2k+1}^{(1)} = -(\tilde{F} \circ \pi)^* \partial_{2k+1}^{(1)}.
\]

Since \( H^{(0)} \in C^0(I, h_2^k) \), we identify \( \Delta^1 \) with \( I \) and the points of \( I \) with the corresponding 0-simplices, so that, for \( \sigma : I \rightarrow I \) a 1-simplex, one has:

\[
\delta^0 H^{0}(\sigma) = H^{0}(\sigma(1)) - H^{0}(\sigma(0)).
\]

Therefore, from (29) we get \( H^{0}(\sigma(1)) - H^{0}(\sigma(0)) = -\partial_{2k+1}^{(1)}(\tilde{F} \circ \pi \circ \sigma) \). In particular, for \( \sigma = \text{id} \), we get:

\[
(30) \quad \hat{h}_1^{(0)} - \hat{h}_0^{(0)} = -\partial_{2k+1}^{(1)}(\tilde{F} \circ \pi) = -(\tilde{F}^* \partial_{2k+1}^{(1)})(\pi) = - \int_{S^1} \tilde{F}^* \partial_{2k+1}^{(1)}.
\]

By definition \( \tilde{F}^*[2k+1] = \text{ch}^{2k+1}[\tilde{F}] \), for \( [\tilde{F}] \in \hat{h}^{2k+1}(S^1) \). Moreover, \( \int_{S^1} \text{ch}^{2k+1}[\tilde{F}] = \text{ch}^k \int_{S^1} \tilde{F} \), where \( \int_{S^1} \tilde{F} \) can be defined as follows. For \( i : \{1\} \hookrightarrow S^1 \) and \( p : S^1 \rightarrow \{1\} \) the natural maps, we consider \( \tilde{F} - p^* i^* [\tilde{F}] \in \text{Ker}(i^*) \simeq \hat{h}^{2k+1}(S^1) \), and we apply the suspension isomorphism \( \hat{h}^{2k+1}(S^1) \simeq \hat{h}^2(S^0) \simeq h_2^k \). Therefore, from (30) we get:

\[
(31) \quad \hat{h}_1^{(0)} - \hat{h}_0^{(0)} = -\text{ch}^k \int_{S^1} [\tilde{F}] \in \mathbb{Z}.
\]
Theorem 3.7. The choice of the points $S$ natural lift to 1-cycles $\iota$ can define is well-defined even up to homotopies fixing $f$. We now consider a generic homotopy $(F, H, \pi^*\omega) : (f_0, h_0, \omega) \simeq (f_1, h_1, \omega)$. In this case we still get $h_1^{(0)} - h_0^{(0)} = -\iota^{(1)}_{2k+1}(F)$ as in (30), but now $F$ does not necessarily factorize through $S^1$. Since we have the freedom of choosing $\iota_{2k+1}$ within its cohomology class, we argue in the following way:

- for every path-wise connected component $A_\alpha$ of $E_{2k+1}$ (each of them corresponding to a cohomology class in $H^{2k+1}_R$) we choose a marked point $a_\alpha$ and for every $x \in A_\alpha$, a path from $a_\alpha$ to $x$, which we call $F_x$. We require that $F_{a_\alpha}$ is the constant path.

- We require that $\iota^{(1)}_{2k+1}(F_x) = 0$ for every $x \in E_{2k+1}$. This is possible, since a 0-cochain corresponds to a function from $E_{2k+1}$ to $\mathbb{R}^{2k+1}$. Therefore, given a representative of the Chern character $\iota^{(1)}_{2k+1} \in C^1(E_{2k+1}, H^{2k+1}_R)$, we consider the 0-cochain $j_{2k+1}$ corresponding to the function $x \to \iota^{(1)}_{2k+1}(F_x)$, and we define $\iota^{(1)}_{2k+1} := \iota^{(1)}_{2k+1} - \delta^0 j_{2k+1}$. Then $\iota^{(1)}_{2k+1}(F_x) = \iota^{(1)}_{2k+1}(F_x) - (j_{2k+1}(x) - j_{2k+1}(a_\alpha)) = \iota^{(1)}_{2k+1}(F_x) - (\iota^{(1)}_{2k+1}(F_x) - 0) = 0$.

Lemma 3.6. The choice of $\iota^{(1)}_{2k+1}$ described above can be extended to the choice of cocycles $\iota^{(t)}_{2k+t}$, for every $t > 1$, compatible with the relation (10).

Proof: Let us call $\iota^{(t)}_{2k+t}$ the representatives previously chosen. Then $\iota^{(1)}_{2k+1} := \iota^{(1)}_{2k+1} - \delta^0 j_{2k+1}$, for $j_{2k+1}$ defined above. We define $j^{(t-1)}_{2k+t}$ such that the cocycles $\iota^{(t)}_{2k+t} := \iota^{(t)}_{2k+t} - \delta^{t-1} j^{(t-1)}_{2k+t}$ are compatible with (10). For $t = 2$, let us consider the 0-cycles $\{x\}_{x \in E_{2k+1}}$, and their natural lift to 1-cycles $S_x$ on $\Sigma E_{2k+1}$. The push-forward $(\varepsilon_{2k+1})_*$ is injective on the family $\{S_x\}$, since $\varepsilon_{2k+1} : E_{2k+1} \to \Omega_{E_{2k+2}} E_{2k+2}$ is a homeomorphism by hypothesis. Hence, we can define $j^{(1)}_{2k+2}(\varepsilon_{2k+1} S_x) := (2k+1) \varepsilon_{2k+1}(x)$, so that $\int_{S_x} \varepsilon_{2k+1} j^{(1)}_{2k+2} = j_{2k+1}$.

For $t = 2$, we argue in the same way considering the 1-cycles $\{(\varepsilon_{2k+1})_x S_x\}_{x \in E_{2k+2}}$ previously defined, and their natural lifts to 2-cycles $S_x'$ on $\Sigma E_{2k+2}$. The construction can be repeated inductively for any $t$.

Theorem 3.7. Choosing $\iota^{(1)}_{2k+1}$ as above, the map $[(f, h, \omega)] \to [h^{(0)}]$ is well-defined in $H^2_R / \text{Im} \text{ch}^{2k}_\ast \simeq \mathbb{R}/\mathbb{Z}$ up to homotopies of differential functions, and does not depend on the choice of the points $a_\alpha$ and of the paths $F_x$.

Proof: Let $(F, H, \pi^*\omega) : (f_0, h_0, \omega) \simeq (f_1, h_1, \omega)$ be a homotopy. Then $h_1^{(0)} - h_0^{(0)} = -\iota^{(1)}_{2k+1}(F)$, and, if $f_0(*) = x$ and $f_1(*) = y$, one has\footnote{The last equality in the equation is due to the fact that, for $\varphi, \psi : I \to X$ 1-simplices such that $\varphi(1) = \psi(0)$, the chain $\varphi * \psi - \varphi - \psi$ is a boundary, as one can show constructing a 2-simplex which, one the three sides of $\Delta_2$, restricts to $-\varphi, -\psi, \varphi * \psi$.}

$$\iota^{(1)}_{2k+1}(F) = -\iota^{(1)}_{2k+1}(F_x) + \iota^{(1)}_{2k+1}(F_y) + \iota^{(1)}_{2k+1}(F_x) - \iota^{(1)}_{2k+1}(F_y) + \iota^{(1)}_{2k+1}(F_x) = \iota^{(1)}_{2k+1}(F_x - F_y * F_x).$$
The latter belongs to $\text{Im} \, \text{ch}^2_{\{s\}}$, since, being $F_y^{-1} \ast F \ast F_x$ a cycle, using the notation of the proof of lemma 3.5, it follows that:

$$\iota_{2k+1}^{(1)}(F \ast F_y^{-1} \ast F_x) = \text{ch}^0 \int_{S^1} [F \ast F_y^{-1} \ast F_x].$$

If we choose other points $a_\alpha$ or paths $F_x$, we simply get a different cycle $F \ast F_y^{-1} \ast F_x$, but the value of $\iota_{2k+1}^{(1)}$ belongs anyway to the image of the Chern character. □

Summarizing, thanks to lemma 3.7 if $\mathfrak{h}^{2k} \simeq \mathbb{Z}$ we get a well-defined map:

$$(33) \quad \Gamma^{2k+1} : \mathfrak{h}^{2k+1} \to \mathfrak{h}^2_{\mathbb{R}} / \text{Im} \, \text{ch}^{2k}_{\{s\}} \simeq \mathbb{R} / \mathbb{Z}.$$  

In order to prove that it is a group homomorphism, we must consider the term involving $A_{2k}^{(0)}$ in the sum of differential functions (14).

**Theorem 3.8.** Choosing $\iota_{2k+1}^{(1)}$ as above (v. comments before lemma 3.6), it is possible to choose $\iota_{2k+1}^{(2)}$ in such a way that $A_{2k}^{(0)}$ is an integral 0-cochain (so that it takes values in $\text{Im} \, \text{ch}^{2k}_{\{s\}} \simeq \mathbb{Z}$ on every point of $E_{2k+1} \times E_{2k+1}$).

**Proof:** From equation (11) we get:

$$(34) \quad \pi^*_{1,2k+1}(\iota_{2k+1}^{(1)}) + \pi^*_{2,2k+1}(\iota_{2k+1}^{(1)}) - \alpha^*_{2k+1}(\iota_{2k+1}^{(1)}) = \delta^0 A^{(0)}_{2k}.$$  

Let us consider a 1-chain $\varphi : I \to E_{2k+1} \times E_{2k+1}$, which provides three chains from $I$ to $E_{2k+1}$: $\varphi_1 := \pi_{1,2k+1} \circ \varphi$, $\varphi_2 := \pi_{2,2k+1} \circ \varphi$, and $\varphi_{1+2} := \alpha_{2k+1} \circ \varphi$. Evaluating both sides of the equation (34) on $\varphi$ we get:

$$(35) \quad \iota_{2k+1}^{(1)}(\varphi_1 + \varphi_2 - \varphi_{1+2}) = A^{(0)}_{2k}(\varphi(1)) - A^{(0)}_{2k}(\varphi(0)).$$

Thanks to the choices above, we can construct three cycles:

$$\psi_1 := F_{\varphi_1(0)} \ast \varphi_1 \ast F^{-1}_{\varphi_1(1)}; \quad \psi_2 := F_{\varphi_2(0)} \ast \varphi_2 \ast F^{-1}_{\varphi_2(1)}; \quad \psi_{1+2} := F_{\varphi_{1+2}(0)} \ast \varphi_{1+2} \ast F^{-1}_{\varphi_{1+2}(1)}$$

and (35) becomes:

$$(36) \quad \iota_{2k+1}^{(1)}(\psi_1 + \psi_2 - \psi_{1+2}) = A^{(0)}_{2k}(\varphi(1)) - A^{(0)}_{2k}(\varphi(0)).$$

If we think of $\psi_i : S^1 \to E_{2k+1}$ for $i = 1, 2, 1 + 2$, we get that:

$$\iota_{2k+1}^{(1)}(\psi_i) = [\iota_{2k+1}^{(1)}(\psi_i)] = \int_{S^1} \psi_i^* [\iota_{2k+1}^{(1)}] = \int_{S^1} \text{ch}_{2n+1}[\psi_i] = \text{ch}_n^2 \int_{S^1} [\psi_i].$$

Hence, from (36) we get:

$$A^{(0)}_{2k}(\varphi(1)) - A^{(0)}_{2k}(\varphi(0)) \in \mathbb{Z}.$$  

Since $\varphi$ was generic, we have shown that $\delta^0 A^{(0)}_{2k}$ is an integral cochain, thus $A^{(0)}_{2k}$ is integral up to a real 0-cocycle, i.e. up to a locally constant function on $E_{2k+1} \times E_{2k+1}$. Therefore, it remains to show that we can find one point for each path-connected component of $E_{2k+1} \times E_{2k+1}$, on which the value of $A^{(0)}_{2k}$ is integral. We recall that $A^{(0)}_{2k}$, being 2k even, is defined as $A^{(0)}_{2k} := -\int_{S^1} \varphi^*_{2k+1} A^{(1)}_{2k+1}$ for $\varphi_{2k+1} : \Sigma(E_{2k+1} \times E_{2k+1}) \to E_{2k+2} \times E_{2k+2}$ the structure maps of the spectrum $E_n \times E_n$. We fix a point $(p_1, p_2)$ for each connected component of $E_{2k+1} \times E_{2k+1}$, and we consider the corresponding 1-cycle $\varphi : I \to \Sigma(E_{2k+1} \times E_{2k+1})$. Then:

$$A^{(0)}_{2k}(p_1, p_2) = \int_I (\varphi_{2k+1} \circ \varphi)^* A^{(1)}_{2k+1} = A^{(1)}_{2k+1}(\varphi_{2k+1} \circ \varphi).$$
Since $\partial^\bullet$ is rationally even, it follows that $H_1(E_{2k+2} \times E_{2k+2}, \mathbb{R}) = 0$ (v. [13], comments after Assumption 2.10), hence there exists a 2-cycle $\Psi$ such that $\partial\Psi = \varphi_{2k+1} \circ \varphi$, hence:

$$A_{2k+1}^{(1)}(\varphi_{2k+1} \circ \varphi) = (\pi_{1,2k+2}^*(f_{2k+2}^{(2)}) + \pi_{2,2k+2}^*(f_{2k+2}^{(2)}) - \alpha_{2k+2}^*(f_{2k+2}^{(2)}))\Psi.$$  

Now it is enough to add to $f_{2k+2}^{(2)}$ a coboundary $\delta\xi$ on each connected component such that:

$$\xi(\pi_{1,2k+2} \circ \varphi_{2k+1} \circ \varphi + \pi_{2,2k+2} \circ \varphi_{2k+1} \circ \varphi + \alpha_{2k+2} \circ \varphi_{2k+1} \circ \varphi) = x$$

with $x + A_{2k}^{(0)}(p_1, p_2) \in \mathbb{Z}$. We the same technique of lemma 3.6, we can extend the choice to cocycles $f_{2k+t}^{(t+1)}$, for every $t > 1$, compatible with the relation (10).

We now show the behavior of the map $\Gamma^{2k+1}$ under the action of $h^\bullet$. This fact will be important for the definition of the pairing between homology and flat differential cohomology.

**Theorem 3.9.** It is possible to choose the cocycles $f_{2n}^{(2)}$, compatibly with the choices involved in lemma 3.6, such that, for $\hat{\alpha} \in \hat{h}^{2k+1}$ and $\beta \in h^{2k}$ we get:

$$\Gamma^{2k+2n+1}(\hat{\alpha} \cdot \beta) = \Gamma^{2k+1}(\hat{\alpha}) \cdot \text{ch}(\beta).$$

Proof: For $\hat{\alpha} = [(f, h, 0)]$ and $\beta = [g]$, we lift $\hat{\alpha}$ to $\hat{\alpha}^1 = [(f^1, h^1, 0)] \in \hat{h}^{2k+2}(S^1)$. In particular we define $f^1(t) = \varepsilon_{2k+1}([t, f(*))]$, i.e. $f^1 = \varepsilon_{2k+1}(f(*))$, and we suppose $h^1|_{X \times \{1\}} = 0$, so that we can apply formula (15) and the product is:

$$\hat{\alpha} \cdot \beta = (\int_{S^1} M_{2k+2,2h} \circ (f^1, g^1), \int_{S^1} h^1 \cup g^1 + (f^1, g^1) M_{2k+2,2h}), 0).$$

We have to show that $\int_{S^1} (f^1, g^1)^* M_{2k+2,2h}$ can be turned integral making the proper choices, because in this case we get (mod $\mathbb{Z}$ when necessary):

$$\Gamma^{2k+2n+1}(\hat{\alpha} \cdot \beta) = [\int_{S^1}(h^1 \cup g^1 + ^0_{2h})]^{(0)} = h^{(0)} \cdot (f^1, g^1) = \Gamma^{2k+1}(\hat{\alpha}) \cdot \text{ch}(\beta)).$$

We call $1_{S^1} : I \to S^1$ the natural 1-simplex representing a generator in homology. One has that $\gamma_{2h}^{(1)} \in C^1(E_{2h}, \mathbb{R}_{2h-1}) = 0$, and, since $H^1(E_{2k+2}, \mathbb{R}) = 0$ (v. [13], comments after Assumption 2.8), we get from equation (12):

$$\int_{S^1} (f^1, g^1) M_{2k+2,2h}^{(1)} = (f^1, g^1) M_{2k+2,2h}^{(1)}(1_{S^1}) = M_{2k+2,2h}^{(1)}(\partial \Sigma) = \delta^1 M_{2k+2,2h}^{(1)}(\Sigma)$$

$$= (\partial_{2k+2} \times \gamma_{2h}^{(0)}(\Sigma)) + (\gamma_{2k+2}^{(2)} \times \gamma_{2h}^{(2)}(\Sigma) + (\gamma_{2k+2}^{(2)} \times \gamma_{2h}^{(2)}(\Sigma)$$

We start with the term $\gamma_{2k+2,2h}(\mu_{2k+2,2h} \circ \Sigma)$, then we analyze the remaining ones. Because of the properties of the multiplication $\mu$:

$$\partial(\mu_{2k+2,2h} \circ \Sigma) = \mu_{2k+2,2h} \circ \partial \Sigma = \mu_{2k+2,2h} \circ (f^1, g^1) \circ 1_{S^1} = (\mu_{2k+1,2h} \circ (f, g))^1 \circ 1_{S^1}.$$

We now use the notation of the comments before the lemma 3.6, in particular we refer to the paths $F_x$ and the points $a_0$. The path $F_{\mu_{2k+1,2h} \circ (f,g)(*)}$ in $E_{2k+2h+1}$ lifts in $\Sigma E_{2k+2h+1}$.
to a cylinder (projected to the suspension) between the loop lifting \(a_\alpha\) and the loop lifting \(\mu_{2k+1,2h} \circ (f,g)(*)\). We call such a cylinder \(F^1_{\mu_{2k+1,2h} \circ (f,g)(*)}\). From formula (10) we get:

\[
\mu_{2k+2,2h}((\varepsilon_{2k+2h+1})_* F^1_{\mu_{2k+1,2h} \circ (f,g)(*)}) = \mu_{2k+2,2h} F^1_{\mu_{2k+1,2h} \circ (f,g)(*)} = 0,
\]

moreover:

\[
\partial((\varepsilon_{2k+2h+1})_* F^1_{\mu_{2k+1,2h} \circ (f,g)(*)}) = (\mu_{2k+1,2h} \circ (f,g))^1 \circ 1S_1 - a^1_\alpha \circ 1S_1,
\]

i.e. the two boundaries are (39) and the lift of \(a_\alpha\). Since \(H_1(E_{2k+2h+2}, \mathbb{R}) = 0\) (v. [13], comments after Assumption 2.8), we consider \(\Xi\) such that \(\partial \Xi = a^1_\alpha \circ 1S_1\), i.e. we put a plug on the common boundary of the push-forwards of the cylinders \(F^1_{\mu_{2k+1,2h} \circ (f,g)(*)}\) (one plug for each connected component). Because of (39) the other plug is \(\mu_{2k+2,2h} \circ \Sigma\), therefore we try to modify \(i^{(2)}_{2k+2h+2}\) in such a way that the value it takes on the second plug, i.e. the last term of (39), coincides with the value on the push-forward of the entire cylinder with two plugs, which is integral since the latter is a cycle. If \(i^{(2)}_{2k+2h+2}\) is the old representative, we define:

\[
i^{(2)}_{2k+2h+2} := i^{(2)}_{2k+2h+2} - \delta \alpha,
\]

where \(\alpha\) has the value \(i^{(2)}_{2k+2h+2}(\Xi)\) on the image via \((\varepsilon_{2k+2h+2})_*\) of the lift of each point pathwise connected to \(a_\alpha\). In this way the value of \(\delta \alpha\) on the image of the lift of a path \(F_x\) is 0, since \(\alpha\) takes the same value on the two boundaries of the cylinder, and it does not change the choice of lemma [3,7] (in particular, \(\int_{S_1}(\varepsilon_{2k+2h+2})_* \alpha\) is constant on the pathwise-connected components, therefore it is a 0-coboundary and has no effect on \(i^{(1)}_{2k+2h+1}\)). Then:

\[
i^{(2)}_{2k+2h+2}((\varepsilon_{2k+2h+2})_* F^1_{\mu_{2k+1,2h} \circ (f,g)(*)} + \Xi) = (i^{(2)}_{2k+2h+2} - \delta \alpha)((\varepsilon_{2k+2h+2})_* F^1_{\mu_{2k+1,2h} \circ (f,g)(*)} + \Xi)
\]

Hence:

\[
i^{(2)}_{2k+2h+2}(\mu_{2k+2,2h} \circ \Sigma) = i^{(2)}_{2k+2h+2}(\mu_{2k+2,2h} \circ \Sigma + (\varepsilon_{2k+2h+2})_* F^1_{\mu_{2k+1,2h} \circ (f,g)(*)} + \Xi)
\]

and \(\partial(\mu_{2k+2,2h} \circ \Sigma + (\varepsilon_{2k+2h+2})_* F^1_{\mu_{2k+1,2h} \circ (f,g)(*)} + \Xi) = 0\), hence the value of \(i^{(2)}_{2k+2h+2}\) is integral.

It remains to analyze the other terms of the last step of formula (38). The term \((i^{(1)}_{2k+2} \times i^{(1)}_{2h})(\Sigma)\) vanishes since the push-forward of any subsimplex of \(\Sigma\) to \(E_{2h}\) is the point \(g(*)\), hence it is a boundary in degree 1. From \((i^{(2)}_{2k+2} \times i^{(0)}_{2h})(\Sigma)\) we get \(i^{(2)}_{2k+2}(\Sigma) \cdot i^{(0)}_{2h}(g(*))\), where we still call \(\Sigma\) its projection to \(E_{2h+2}\). Both terms are integral: the first one because of the same choices we made above, with the same proof replacing \(\mu_{2k+2,2h} \circ \Sigma\) with \(\Sigma\); the second one because \(i^{(0)}_{2h}\) represents an integral 0-class and \(g(*)\) is a cycle. It remains to analyze \((i^{(0)}_{2k+2} \times i^{(2)}_{2h})(\Sigma)\). In this case we get products between \(i^{(0)}_{2k+2}\) evaluated on some vertices, which is integral since \(i^{(0)}_{2k+2}\) is, and \(i^{(2)}_{2h}\) evaluated on some 2-cycles collapsing to the point \(g(*)\). Thus, we have to turn \(i^{(2)}_{2h}\) integral on points. If \(p_n\) indicates an n-simplex collapsing to the point p, we have that \(\partial p_n = p_n\), hence we consider the 1-simplex \(\beta\) on \(E_{2h}\) such that \(\beta(p_1) = i^{(2)}_{2h}(p_2)\) and we replace \(i^{(2)}_{2h}(p_2) - \delta \beta\). This has no effects on (40), because \(\alpha\) takes non-trivial values on some 1-cycles: between those 1-cycles, the
only one collapsing to a point is the marked point $e_{2h}$, but by definition $\iota^{(2)}_{2n} \simeq 0$. There is a natural map:

\begin{equation}
\eta^* : h^* \to \check{h}^*.
\end{equation}

For odd degrees, given $\eta : a > 0$ an odd degree, we define $\eta[f] := [(f, 0, 0)]$. It is well-defined since $f^* \iota_{2k+1} \in C^{2k+1}(\{\ast\}, h^*)$ can be non-trivial only in odd degrees, but any cocycle of odd degree on a point is vanishing. For even degrees, given $\eta : \{\ast\} \to E_{2k}$, the pull-back $f^* \iota_{2k}$ is made by a real number for each even degree: in degree 0 it is also a 0-form, while in degree $2a > 0$ there is a unique $\check{h}^{(2a-1)}$ such that $\delta^{2a-1} \check{h}^{(2a-1)} = f^* \iota_{2k}$, thus we can define $\eta[f] := [(f, \check{h}, f^* \iota_{2k})]$. The map $\eta^*$ induces a splitting $\check{h}^* \simeq h^* \oplus (h^*_{\mathbb{R}} / \text{Im} \ ch)$. In particular:

\begin{equation}
\check{h}^{2k+1} \simeq h^{2k+1} \oplus (h^*_{\mathbb{R}} / \text{Im} \ ch)
\end{equation}

The map $\Gamma^{2k+1}$ is the projection $\check{h}^{2k+1} \to h^*_{\mathbb{R}} / \text{Im} \ ch$, followed by the isomorphism $h^*_{\mathbb{R}} / \text{Im} \ ch \simeq \mathbb{R} / \mathbb{Z}$ when $h^*_{\mathbb{R}} \simeq \mathbb{R}$. In particular, it catches all the possible non-integral information.

With a generic choice of $\iota^{(2)}_{2n}$, for $[f] \in h^{2k+1}$ and $[g] \in h^{2h}$, the product is:

\begin{equation}
[(f, 0, 0)] \cdot [(g, 0, g^* \iota_{2k})] = \int_{S^1} [(\mu_{2k+2, 2h} \circ (f^1, g), (f^1, g)^* M_{2k+2, 2h}, 0)]
\end{equation}

\begin{equation}
= [(\mu_{2k+1, 2h} \circ (f, g), J(f, g), 0)] = [(fg, 0, 0)] \cdot [(0, J(f, g), 0)].
\end{equation}

The class of $J(f, g)$, up to $\text{Im} \ ch$, only depends on the first Chern classes of $f$ and $g$. The choice of theorem 3.4 is the one for which $J = 0$, i.e. the one for which $\eta(\alpha \beta) = \eta(\alpha) \eta(\beta)$. Thus, the diagram of the corollary 3.4 becomes:

\begin{equation}
\begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\check{h}^n(X) \otimes \mathbb{R} & \overset{\pi}{\longrightarrow} & \check{h}_{2k+1}^n(X) & \overset{\iota}{\longrightarrow} & h^{n+1}(X) & \overset{\iota}{\longrightarrow} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\overset{\text{Hom}_{\mathbb{R}}(\check{h}^n(X), h^*_{\mathbb{R}})}{\longrightarrow} & \overset{\text{Hom}_{\mathbb{R}}(\check{h}^n_{\mathbb{R}}(X), h^*_{\mathbb{R}} / \text{Im} \ ch)}{\longrightarrow} & \overset{\text{Hom}_{\mathbb{R}}(\check{h}_{2k+1}^n(X), h^*_{\mathbb{R}})}{\longrightarrow} & \overset{\text{Hom}_{\mathbb{R}}(h^{n+1}(X), h^*_{\mathbb{R}})}{\longrightarrow} & \cdots
\end{array}
\end{equation}

where $\alpha'(\varphi) = (|\varphi|, 0)$ and $\iota'(\varphi, \psi) = \psi$.

### 3.3. Flat pairing

We can now define a natural $\mathbb{R} / \mathbb{Z}$-valued pairing on a manifold $X$ between $\check{h}^*_{\mathbb{R}}$ and $h^*$, that, in case of singular differential cohomology, reduces to the holonomy of a flat abelian $p$-gerbe. In the following definition, the cohomology of the point is involved. We could use the Poincaré duality on the point, in order to involve only the homology in the definition of the pairing (in the topological side), but we avoid it in order to maintain the previous notation.

**Definition 3.1.** For $X$ a differential manifold, there is a natural pairing:

\begin{equation}
\Xi^{n+1} : \check{h}^{n+1}(X) \to \text{Hom}_{\mathbb{R}}\left( \bigoplus_{k \in \mathbb{Z}, h^{2k}_{\mathbb{R}} \neq 0} h^{n-2k}(X), \mathbb{R} / \mathbb{Z} \right).
\end{equation}

\[\footnote{Choosing another representative for $\iota_{2n}^{(2)}$, there will be an isomorphism of differential extensions making $J$ appear.}\]
For \( \hat{\alpha} \in \hat{h}_n^{n+1}(X) \) and \([M, u, \beta, f] \in h_{n-2k}(X)\), we define (44) as:

\[
\Xi^{n+1}(\hat{\alpha})[M, u, \beta, f] = \Gamma^{2k+1} \circ (p_M)(f^* \hat{\alpha} \cdot \beta).
\]

Here \( p_M : M \to \{*\} \) and the product \( f^* \hat{\alpha} \cdot \beta \) is defined by (22). The invariance by \( h^* \) is defined by:

\[
\Xi^{n+1}(\hat{\alpha})([M, u, \beta, f] \cdot \gamma) = (\Xi^{n+1}(\hat{\alpha})[M, u, \beta, f]) \cdot \text{ch}(\gamma).
\]

Formula (46) follows directly from formula (37), since:

\[
\Xi^{n+1}(\hat{\alpha})([M, u, \beta, f] \cdot \gamma) = \Gamma^{2k+1} \circ (p_M)(f^* \hat{\alpha} \cdot \beta \cdot (p_M)^* \gamma)
\]

\[
= \Gamma^{2k+1}((p_M)(f^* \hat{\alpha} \cdot \beta) \cdot \gamma) = (\Gamma^{2k+1} \circ (p_M)(f^* \hat{\alpha} \cdot \beta)) \cdot \text{ch}(\gamma)
\]

\[
= \Xi^{n+1}(\hat{\alpha})[M, u, \beta, f] \cdot \text{ch}(\gamma).
\]

If we take the free part of \( h^* \), which we are supposing to be 0 or \( \mathbb{Z} \), we get from (13):

\[
\cdots \xrightarrow{r} h^n(X) \otimes_{\mathbb{R}} \mathbb{R} \xrightarrow{\text{inv}} h_{n-1}^{n+1}(X) \xrightarrow{f} h_{n-1}^{n+1}(X) \xrightarrow{r} \cdots
\]

\[
\cdots \xrightarrow{r} \text{Hom}_e\left( (\bigoplus_{p=0}^{2k} h_{n-2k}(X)) \otimes_{\mathbb{R}} \mathbb{R} \right) \xrightarrow{\text{inv}} \text{Hom}_e\left( (\bigoplus_{p=0}^{2k} h_{n-2k}(X)) \otimes_{\mathbb{R}/\mathbb{Z}} \mathbb{R} \right) \xrightarrow{f} \text{Hom}_e\left( (\bigoplus_{p=0}^{2k} h_{n-1-k}(X)) \otimes_{\mathbb{R}/\mathbb{Z}} \mathbb{R} \right) \xrightarrow{r} \cdots
\]

We define the holonomy of a flat differential class over a cycle as the exponential of \( \Xi^{n+1} \).

In particular:

**Definition 3.2.** For \( \hat{\alpha} \in \hat{h}_n^{n+1}(X) \) and \([M, u, \beta, f] \in z_{n-1-2k}(X)\), we define the holonomy of \( \hat{\alpha} \) over \([M, u, \beta, f]\) as:

\[
\text{Hol}_\alpha[M, u, \beta, f] := \exp \circ \Gamma^{2k+1} \circ (p_M)(\beta \cdot f^* \hat{\alpha}).
\]

One could expect that (44) is an isomorphism, but it seems not to be the case in general. In fact, in order to show that it is for ordinary cohomology and, as we show in the following, for K-theory (10), it is necessary to use the universal coefficient theorem in its formulation via the Ext group (6) (13), which does not hold for a generic cohomology theory (11). Therefore, it seems that in general there are non-trivial kernel and cokernel, and we do not know if it is possible to find an explicit characterization. We can just state the following partial result, which at least shows that the pairing \( \Xi^* \) is far from being trivial for any cohomology theory.

**Theorem 3.10.** \( \Xi^n \) does not vanish on the non-trivial non-torsion differential classes with trivial first Chern class. Hence, for any fixed first Chern class, it can vanish at most on a subset of differential classes differing by a torsion one. Moreover, its image contains the homomorphisms which are liftable to an \( \mathbb{R} \)-valued one.

Proof: Let us consider \( \hat{\alpha} = [(0, \text{ch}_{\mathbb{R}}(\alpha), 0)] \) non-torsion. Using the notation of corollary (3.4) one has that \( \Xi^n(\hat{\alpha}) = 0 \) if and only if \( \xi_{\mathbb{R}/\mathbb{Z}}(\hat{\alpha}) = 0 \). Moreover \( \hat{\alpha} = a(\alpha) \), and \( \xi_{\mathbb{R}/\mathbb{Z}}(\alpha) = 0 \) if and only if \( \xi_{\mathbb{R}/\mathbb{Z}} \circ a(\alpha) = 0 \), hence \( \alpha \circ \xi_{\mathbb{R}}(\alpha) = 0 \). Hence \( \xi_{\hat{\mathbb{R}}}(\alpha) \) takes values in the integral cohomology of the point, in particular in the rational one. Thus, considering the isomorphism \( \xi Q \) given by theorem (2.1) replacing \( \mathbb{R} \) with \( \mathbb{Q} \), we get that \( \alpha \) is a rational class, so \( n\alpha \) is integral for \( n \in \mathbb{N} \). Hence \( n\alpha = a(n\alpha) = 0 \), thus \( \hat{\alpha} = 0 \) since it was non-torsion. Finally, if a homomorphism is liftable to an \( \mathbb{R} \)-valued one, it trivially follows from corollary (3.4) and theorem (2.1) that it belongs to the image of \( \Xi^n \). \( \square \)
Corollary 3.11. For \( \hat{\alpha} \in \hat{h}_n^N(X) \) non-torsion, if there exits \( n \in \mathbb{N} \) such that \( nI(\hat{\alpha}) = 0 \), then \( \Xi^n(\hat{\alpha}) \neq 0 \). Similarly, for \( \varphi \) an element of the codomain of \( \Xi^n \), if \( n\varphi \) is liftable to an \( \mathbb{R} \)-valued morphism, then \( \varphi \) belongs to the image of \( \Xi^n \). \( \square \)

Up to now we considered the cases \( h^{2k} \simeq \mathbb{Z} \oplus \text{Tor} h^{2k} \) or \( h^{2k} \simeq \text{Tor} h^{2k} \). If \( h^{2k} \simeq \mathbb{Z}^p \oplus \text{Tor} h^{2k} \) with \( p > 1 \), one possibility is to define the holonomy as taking values in \((\mathbb{R}/\mathbb{Z})^p\) or \((U(1))^p\), in which case all the previous results hold. Otherwise, we can compose such a holonomy with the morphism \((\mathbb{R}/\mathbb{Z})^p \rightarrow \mathbb{R}/\mathbb{Z}\) given by the sum of the components. This definition seems more natural, even if it can vanish even when the projection \( h^{2k+1} \rightarrow h_{\mathbb{R}}^{2k}/\text{Im} \text{ch}\) does not, therefore it is not true any more that the map \( \Gamma^{2k+1} \) catches all the possible non-integral information; in particular, theorem \( 3.10 \) does not hold any more. Anyway this is not a problem, it is just a property lost.

Finally, we have to consider the dependence on the isomorphism \( h^{2k} \simeq \mathbb{Z} \oplus \text{Tor} h^{2k} \). Actually we used it only to get \( h_{\mathbb{R}}^{2k} \simeq \mathbb{R} \), in such a way that the real images of the classes in \( h^{2k} \) correspond to \( \mathbb{Z} \subset \mathbb{R} \). Therefore, we are only interested in the induced surjective morphism from the free part of \( h^{2k} \) to \( \mathbb{Z} \), which is unique up to a sign. For \( k = 0 \) the sign is fixed since \( h^0 \) is a ring and we require the isomorphism to be a ring isomorphism (in particular, in the case of singular cohomology there is no ambiguity any more, as expected). For \( k \neq 0 \) the sign actually depends on the isomorphism chosen. When \( h^{2k} \simeq \mathbb{Z}^p \oplus \text{Tor} h^{2k} \) with \( p > 1 \) the situation is different, since the automorphisms of the free part are of course more complicated. For example, if \( h^{2k} \simeq \mathbb{Z}^2 \), we can consider the automorphism \((n, m) \rightarrow (n + km, m)\) for any \( k \neq 0 \) (which extends to \( h_{\mathbb{R}}^{2k} \simeq \mathbb{R}^2 \)), and in this case the holonomy changes not only up to a sign, whether it takes values in \((\mathbb{R}/\mathbb{Z})^2\) or in \( \mathbb{R}/\mathbb{Z} \) via the sum. Anyway, in the most common applications this situation does not occur.

4. Differential homology

We now define the differential homology groups dual to a differential cohomology theory \( \hat{h}^\bullet \).

**Definition 4.1.** An \( \hat{h}^\bullet \)-orientation on a smooth connected manifold without boundary \( X \) is the data of:

- en embedding \( i : X \hookrightarrow \mathbb{R}^N \) for any \( N \in \mathbb{N} \);
- a differential Thom class \( \hat{u} \) of the normal bundle \( N_X\mathbb{R}^N \);
- a tubular neighborhood \( U \) of \( X \) in \( \mathbb{R}^N \) with a diffeomorphism \( \varphi : U \rightarrow N_X\mathbb{R}^N \).

When the manifold is not connected, we choose an \( \hat{h}^\bullet \)-orientation on each connected component. A manifold with an \( \hat{h}^\bullet \)-orientation is called \( \hat{h}^\bullet \)-manifold.

It follows that an \( \hat{h}^\bullet \)-orientation on \( X \) is an \( \hat{h}^\bullet \)-orientation on the map \( p_X : X \rightarrow \{\ast\} \). The following theorem is well-known, but we prove it anyway for completeness.

**Theorem 4.1.** For \( X \) an \( \hat{h}^\bullet \)-manifold without boundary, there exists a differential form \( \hat{A}_h(X) \) on \( X \), such that for every \( \hat{\alpha} \in \hat{h}^\bullet(X) \) (we recall that \( R \) denotes the curvature):

\[
R((\pi_X)_{\hat{\alpha}}) = \int_X R(\hat{\alpha}) \wedge \hat{A}_h(X).
\]

The form \( \hat{A}_h(X) \) is a representative of the cohomology class \( \hat{A}_h(X) = \int_{N_X/X} \text{ch} u \).
Lemma 4.2. Let $\varphi : Y \to X$ be an $\hat{h}^\ast$-oriented smooth map. Then an $\hat{h}^\ast$-orientation on $X$ naturally induces an $\hat{h}^\ast$-orientation on $Y$ via $\varphi$.

Proof: Let the orientations of $\varphi$ and $X$ be given by the embeddings $\iota: Y \hookrightarrow X \times \mathbb{R}^L$, with $\pi_X \circ \iota = \varphi$, and $\jmath : X \hookrightarrow \mathbb{R}^N$. Let the differential Thom classes be $\hat{v}$ on $N_Y(X \times \mathbb{R}^L)$ and $\hat{u}$ on $N_X \mathbb{R}^N$, and the tubular neighborhoods be $V$ for $Y$ in $X \times \mathbb{R}^N$ and $U$ for $X$ in $\mathbb{R}^N$. The orientation induced on $Y$ is defined by the following data:

Proof: For $\varphi_U^+ \circ \psi : \mathbb{R}^N \to (N_X \mathbb{R}^N)^+$ the map appearing in (18) and $\omega$ a compactly-supported form on $N_X \mathbb{R}^N$, one has:

$$\int_{\mathbb{R}^n} (\varphi_U^+ \circ \psi)^* \omega = \int_{N_X \mathbb{R}^N} \omega.$$  

It follows that, if $\hat{u} = [(u, h_u, \omega_u)]$ is the differential Thom class of $N_X \mathbb{R}^N$, and $\pi : N_X \mathbb{R}^N \to X$ the projection, we get:

$$R((\pi_X) \hat{\alpha}) = \int_{N_X \mathbb{R}^N} \pi^* R(\hat{\alpha}) \wedge \omega_u = \int_X R(\hat{\alpha}) \wedge \left( \int_{(N_X \mathbb{R}^N)/X} \omega_u \right).$$  

Therefore we get the thesis for:

(47) $\hat{A}_h(X) = \int_{(N_X \mathbb{R}^N)/X} \omega_u.$

Due to the normalization condition, the 0-degree component of $\hat{A}_h(X)$ is 1. In particular, in the case of the ordinary differential cohomology we get $\hat{A}_H(X) = 1$ as expected. □

When defining the topological Gysin map, we do not need the definition of orientation for the map itself, as in the differential case. We can show that this depends on the fact that, given two vector bundles $E, F \to X$, a topological orientation of two elements between $E, F$ and $E \oplus F$ determines an orientation of the third. In fact, the definition of differential orientation for a smooth map $f : Y \to X$ reduces, in the topological case, to a Thom class of the normal bundle $N_Y(X \times \mathbb{R}^N)$, which can be defined independently on $N$ since the normal bundle is stably unique. An orientation of $X$ determines an orientation of $X \times \mathbb{R}^N$ canonically. Therefore, an orientation of two elements between $X, Y$ and $f$ determines an orientation of the third one. Given a smooth map $f : Y \to X$ between two topologically oriented manifolds, we compute the Gysin map endowing $f$ with the orientation determined by the ones of $X$ and $Y$, that’s why we do not have to define it explicitly.

For what concerns differential orientations, given two oriented bundles $E, F \to X$, with orientation $\hat{u}$ and $\hat{v}$ respectively, we can find a canonical orientation on $E \oplus F$, defined in the following way: for $p_E : E \oplus F \to E$ and $p_F : E \oplus F \to F$ the projection, we consider the orientation $p_E^* \hat{u} \cdot p_F^* \hat{v}$. Instead, it is not possible in general to orient $F$ from $E$ and $E \oplus F$, or vice versa. Hence, the fact that an orientation of two elements between $X, Y$ and $f$ determines an orientation of the third one cannot be generalized to the differential case: what still holds is the fact that, given an orientation on $X$ and one on $f$, there is a natural induced orientation on $Y$, as we prove in the following lemma. Therefore, when we consider a map $f : Y \to X$ between differential oriented manifolds (v. def. 1.2), we can require that the orientation of $f$ is compatible with the ones of $X$ and $Y$, in the sense that the orientation of $Y$ is the one induced by $X$ and $f$.

Lemma 4.2. Let $\varphi : Y \to X$ be an $\hat{h}^\ast$-oriented smooth map. Then an $\hat{h}^\ast$-orientation on $X$ naturally induces an $\hat{h}^\ast$-orientation on $Y$ via $\varphi$. 

Proof: Let the orientations of $\varphi$ and $X$ be given by the embeddings $\iota : Y \hookrightarrow X \times \mathbb{R}^L$, with $\pi_X \circ \iota = \varphi$, and $\jmath : X \hookrightarrow \mathbb{R}^N$. Let the differential Thom classes be $\hat{v}$ on $N_Y(X \times \mathbb{R}^L)$ and $\hat{u}$ on $N_X \mathbb{R}^N$, and the tubular neighborhoods be $V$ for $Y$ in $X \times \mathbb{R}^N$ and $U$ for $X$ in $\mathbb{R}^N$. The orientation induced on $Y$ is defined by the following data:
the embedding \( \xi = (j, 1) \circ t : Y \hookrightarrow \mathbb{R}^{N+L} \);

- on the normal bundle \( N_Y \mathbb{R}^{N+L} \simeq N_Y (X \times \mathbb{R}^L) \oplus N_{X \times \mathbb{R}^L} \mathbb{R}^{N+L} |_Y \simeq N_Y (X \times \mathbb{R}^L) \oplus (\pi_L^* N_X \mathbb{R}^N) |_Y \), for \( \pi_L : \mathbb{R}^{N+L} \to \mathbb{R}^N \), we put the differential orientation induced from the ones on \( N_Y (X \times \mathbb{R}^L) \) and \( N_X \mathbb{R}^N \);

- for the tubular neighborhood, we consider the tubular neighborhood of \( Y \) in \( X \times \mathbb{R}^L \), which is the image under \( \varphi_V^{-1} \) of \( N_Y (X \times \mathbb{R}^L) \), and, for each of its points, we consider the image under \((\varphi_U^{-1}, 1_{\mathbb{R}^L})\) of the corresponding fiber of \( N_{X \times \mathbb{R}^L} \mathbb{R}^{N+L} \simeq \pi_L^* N_X \mathbb{R}^N \simeq N_X \mathbb{R}^N \times \mathbb{R}^L \). The diffeomorphism is defined in the following way: given a vector \((V, W)_y \in N_Y (X \times \mathbb{R}^L) \oplus (\pi_L^* N_X \mathbb{R}^N) |_Y \), we apply \( \varphi_V^{-1} \) to \( V \) in \( y \) getting a point in \( p \in X \times \mathbb{R}^L \), then we apply \((\varphi_U^{-1}, 1_{\mathbb{R}^L})\) to \( W \), the latter taken as orthogonal to \( X \times \mathbb{R}^L \) in \( y \) and translated from \( y \) to \( p \).

\( \square \)

**Remark:** When translating \( W \) from \( y \) to \( p \), we are actually assuming that a vector of \( \mathbb{R}^{N+L} \), which is orthogonal to \( X \times \mathbb{R}^L \) in a point \( y \in Y \), remains transversal to \( X \times \mathbb{R}^L \) when translated to any point of the tubular neighborhood of \( Y \). This always happens if the tubular neighborhood of \( Y \) is small enough; otherwise, we have to approximate it with a sequence having this property and use a limit argument. Moreover, the translation is possible since the ambient space is \( \mathbb{R}^{N+L} \). We could define in a similar way the composition of orientations of two maps \( \varphi : Y \to X \) and \( \psi : X \to W \), and lemma 4.2 would represent the particular case \( W = \{ \ast \} \). The problem in this case is that, since the ambient space would be \( W \times \mathbb{R}^{N+L} \), instead of \( \mathbb{R}^{N+L} \), we could not translate a vector without more information. In particular, we should fix a metric on \( W \) and use the Levi-Civita connection on \( W \times \mathbb{R}^{N+L} \). Thus, the composition of orientations is well-defined for maps between manifolds with metric. Since we do not need to fix a metric in the following, lemma 4.2 is enough.

The immediate generalization of theorem 3.1 does not hold at all for non-flat classes. Surely the homotopy-invariance is lost, and formula (21) does not hold. In fact, such a formula is due to the fact that, thinking for simplicity of an embedding, the multiplication by \( \beta \) in the r.h.s. is equivalent to the multiplication by \( \beta |_U \), where \( U \) is the tubular neighborhood of \( Y \) in \( X \), since \( f_1(\alpha) \) is vanishing outside \( U \). But, being \( U \) a deformation retract of \( Y \), \( \beta |_U \) is equivalent to \( \varphi_U^* \pi_N^* \beta |_Y \), for \( \pi_N : N_Y X \to Y \) the normal bundle. Since the Thom isomorphism is a \( h_* (Y) \)-module morphism, the result follows. In the differential case, the fact of being \( U \) a deformation retract of \( Y \) is not enough, since the curvature depends on \( U \). Hence, the formula in general fails, and we must only consider a weaker statement involving \( \varphi_U^* \pi_N^* \beta |_Y \). For the composition, the same argument holds, since in the construction of lemma 4.2 we translate a vector from the normal bundle in \( Y \) to the whole tubular neighborhood, hence the differential Thom class must be invariant by such a translation. That’s why we must impose suitable conditions in order to recover the properties analogous to the ones of theorem 3.1.

We use the notations of the proof of lemma 4.2 and \( V \) is the tubular neighborhood of \( Y \) in \( X \times \mathbb{R}^L \), \( \varphi_V : V \to N_Y (X \times \mathbb{R}^L) \) the diffeomorphism, and \( \dot{u} \) the differential orientation of \( N_X \mathbb{R}^N \). We briefly give the idea of the condition we are going to introduce. The parallel transport in \( \mathbb{R}^{N+L} \) allows us to identify the bundle \((N_{X \times \mathbb{R}^L} \mathbb{R}^{N+L}) |_V \) with the “propagation” along \( V \) of \((N_{X \times \mathbb{R}^L} \mathbb{R}^{N+L}) |_Y \): we require that the orientation on \((N_{X \times \mathbb{R}^L} \mathbb{R}^{N+L}) |_V \)
We can endow (49) with two differential orientations:

\[ N_Y \mathbb{R}^{N+L} \to N_Y (X \times \mathbb{R}^L), \]

canonically isomorphic to \( \pi^*_N (N_Y \mathbb{R}^{N+L})_Y \). We consider the pull-back of (48):

\[ \varphi^*_V (N_Y \mathbb{R}^{N+L}) \to V. \]

We can endow (49) with two differential orientations:

- the parallel translation in \( \mathbb{R}^{N+L} \) provides an isomorphism of bundles:

\[
P : \varphi^*_V (N_Y \mathbb{R}^{N+L}) \xrightarrow{\cong} (N_X \times \mathbb{R}^L)_{|Y}.
\]

Hence, we can put on (49) the orientation \( P^* (\pi^*_L | V) \).

- Since (48) is canonically isomorphic to \( \pi^*_N (N_X \times \mathbb{R}^L)_Y \), we can put-back to it the orientation \( (\pi^*_L | Y) \). There is a natural map \( \varphi^*_V : \varphi^*_V (N_Y \mathbb{R}^{N+L}) \to N_Y \mathbb{R}^{N+L} \) (the projection on the first factor of the fiber product), hence we can put on (49) the orientation \( \varphi^*_V (\pi^*_N (N_X \times \mathbb{R}^L)_Y) \).

We will need to require that the two orientations coincide:

\[
P^* (\pi^*_L | V) = \varphi^*_V (\pi^*_N (N_X \times \mathbb{R}^L)_Y).
\]

**Definition 4.2.** Let \( Y \) and \( X \) be \( \hat{\mathbb{h}} \)-manifolds. A map between \( \hat{\mathbb{h}} \)-manifolds compatible with the orientations is a smooth \( \hat{\mathbb{h}} \)-oriented map \( \varphi : Y \to X \) such that:

- the orientation on \( Y \) coincides with the one induced from \( X \) and \( \varphi \) as stated in lemma 4.2;
- equation (51) holds.

We still use the notations of the proof of lemma 4.2. For \((x, v) \in V \subset X \times \mathbb{R}^L \) there exists \( y \in Y \) such that \( \varphi_V (x, v) = w_y \), with \( w_y \) belonging to the fiber of \( y \in Y \) in \( N_Y (X \times \mathbb{R}^L) \). Given a smooth manifold \( M \) and a map \( f : X \to M \), we can require that \( f(x) = f(y) \) in this case. In other words we require that:

\[
(f \circ \pi_L)|_V = (f \circ \pi_L)|_{\varphi_V} \circ \pi_{N_Y (X \times \mathbb{R}^L)} \circ \varphi_V.
\]

**Definition 4.3.** Let \( \varphi : Y \to X \) a differential oriented map, \( M \) a manifold and \( f : X \to M \) a smooth map. Then \( f \) respects the orientation of \( \varphi \) if formula (52) holds.

**Theorem 4.3.** For \( \varphi : Y \to X \) a differential oriented map compatible with the orientations of \( X \) and \( Y \), and \( f : X \to M \) a smooth map respecting the orientation of \( \varphi \), the following properties hold:

- \( (p_Y)_! = (p_X)_! \circ \varphi_Y \), for \( p_X : X \to \{*\} \) and \( p_Y : Y \to \{*\} \);
- for \( \alpha \in \mathbb{h}^*(Y) \) and \( \beta \in \mathbb{h}^*(M) \), one has:

\[
\varphi_! (\alpha \cdot \varphi^* f^* \beta) = \varphi_! (\alpha) \cdot f^* \beta.
\]
Proof: Let us consider the following diagram:

\[
\begin{array}{c}
\xymatrix{ h^\bullet(Y) \ar[r]^{\alpha} & h^\bullet_{cpt}(X \times \mathbb{R}^L) \ar[r]^{(j_1)_L} & h^\bullet_{cpt}(\mathbb{R}^{N+L}) \\
\downarrow f_{\mathbb{R}L} & \downarrow f_{\mathbb{R}L} & \downarrow f_{\mathbb{R}N+L} \\
\xymatrix{ h^\bullet(X) \ar[r]^{j_1} & h^\bullet_{cpt}(\mathbb{R}^N) \ar[r]^{f_{\mathbb{R}N}} & \mathfrak{h}^\bullet.}
\end{array}
\]

We first prove that the upper line until \(\mathfrak{h}^\bullet\) is \((p_Y)_!\). In particular, we have to prove that \((j_1)_! \circ \iota_! = ((j, 1) \circ \iota)_!\). Let us fix \(\alpha \in h^\bullet(Y)\). If we calculate \(((j, 1) \circ \iota)_!\), the application of the Thom isomorphism of \(N_Y \mathbb{R}^{N+L}\) gives the class:

\[
(54) \quad \alpha \cdot (p_{p_Y(X \times \mathbb{R}^N)}^\bullet((\pi_L^* \hat{u})|_Y) \cdot p_{\pi_L^*(N_X \mathbb{R}^N)}^\bullet),
\]

for:

\[
p_{p_Y(X \times \mathbb{R}^N)} : N_Y \mathbb{R}^{N+L} \to N_Y(X \times \mathbb{R}^N), \quad p_{\pi_L^*(N_X \mathbb{R}^N)} : N_Y \mathbb{R}^{N+L} \to \pi_L^*(N_X \mathbb{R}^N),
\]

the product by \(\alpha\) being given by \((17)\). The diffeomorphism from \(N_Y \mathbb{R}^{N+L}\) to the tubular neighborhood splits by definition in the action of \(\varphi_V\) followed by the one of \(\varphi_U\), the last acting via parallel translation. One has:

\[
((\varphi_V^\dagger)^* (\alpha \cdot (p_{p_Y(X \times \mathbb{R}^N)}^\bullet((\pi_L^* \hat{u})|_Y) \cdot p_{\pi_L^*(N_X \mathbb{R}^N)}^\bullet)) = ((\varphi_V^\dagger)^* (\alpha \cdot p_{\pi_L^*(N_X \mathbb{R}^N)}^\bullet)) \cdot (P^{-1})^* \varphi_V((\pi_L^* \hat{u})|_Y)
\]

\[
= ((\varphi_V^\dagger)^* (\alpha \cdot p_{\pi_L^*(N_X \mathbb{R}^N)}^\bullet)) \cdot (\pi^\bullet_L \hat{u})|_Y.
\]

The last equality follows from \((51)\).

The lower line is \((p_X)_!\), and the path from \(h^\bullet(Y)\) to \(\mathfrak{h}^\bullet\) passing through the lower line is \((p_X)_! \circ f_!\). Therefore, we have to show the commutativity of the diagram. The commutativity of the right triangle follows from lemma \((2,2)\) while for the central square it follows from the fact that the Thom class on \(N_X \mathbb{R}^{N+L} \simeq \pi_L^* N_X \mathbb{R}^N\) is by definition \(\pi_L^* \hat{u}\).

It follows from the definition of the Thom isomorphism that:

\[
T(\hat{\alpha} \cdot \varphi^* \hat{\beta}) = T(\hat{\alpha} \cdot (\varphi, j)^* (f \circ \pi_L)^* \hat{\beta}) = T(\hat{\alpha}) \cdot \pi_L^* N_Y(X \times \mathbb{R}^N)(f \circ \pi_L)^* \hat{\beta}.
\]

hence, using \((52)\):

\[
\varphi(\hat{\alpha} \cdot \varphi^* \hat{\beta}) = \int_{\mathbb{R}L} (\varphi, j)_!(\hat{\alpha}) \cdot \varphi_V^* \pi_{N_Y(X \times \mathbb{R}^N)}^\bullet(f \circ \pi_L)^* \hat{\beta}
\]

\[
= \int_{\mathbb{R}L} (\varphi, j)_!(\hat{\alpha}) \cdot (f \circ \pi_L)^* \hat{\beta} = \int_{\mathbb{R}L} (\varphi, j)_!(\hat{\alpha}) \cdot f^* \hat{\beta} = \varphi(\hat{\alpha}) \cdot f^* \hat{\beta},
\]

the product by \((f \circ \pi_L)^* \hat{\beta}\) being well-defined since \(\varphi(\hat{\alpha})\) has compact support. \(\square\)

**Lemma 4.4.** Let \(Y\) and \(X\) be \(h^\bullet\)-manifolds and \(M\) a manifold. Then:

- any smooth map \(\varphi : Y \to X\) admits an \(h^\bullet\)-orientation compatible with a differential refinement of the orientations of \(Y\) and \(X\);
- if \(\varphi\) is an embedding, any smooth map \(f : X \to M\) is homotopic to a map \(f'\) which respects the differential orientation of \(\varphi\).
Proof: Let us put on \( \varphi \) the topological orientation induced by the ones of \( Y \) and \( X \), i.e. for any embedding \( \iota : Y \to X \times \mathbb{R}^L \) we choose the orientation of the normal bundle induced from the ones of \( Y \) and \( X \). Let us choose a differential refinement of this orientation of \( \varphi \), and a differential refinement of the orientation of \( X \). Then, by construction, the differential orientation induced on \( Y \) is a refinement of the topological one. If \( \iota : Y \hookrightarrow X \times \mathbb{R}^N \) is the embedding chosen, we can consider the homotopy \( \Psi : X \times \mathbb{R}^L \to X \times \mathbb{R}^L \) contracting the closure of the tubular neighborhood of \( Y \) to \( Y \) itself (in particular, \( Y \) is fixed). The pull-back of the orientation of \( \varphi \) via \( \Psi \) will be a differential orientation which satisfies (51) and refines the same topological orientation.

If \( \varphi \) is an embedding, the map \( \Psi \) previously considered can be defined for \( L = 0 \), i.e. \( \Psi : X \to X \). Then, given \( f : X \to M \), we consider \( f' = f \circ \Psi \), which is homotopic to \( f \) via \( f \circ \Psi \). Then formula (52) holds by construction. □

Remark: In the second part of theorem 4.4, requiring that \( \varphi \) is an embedding is not the minimal hypothesis. The statement can be generalized supposing that the image of \( \varphi \) is a sufficiently regular subspace of \( X \). We do not develop the details since they are not needed in the following, but we just remark this fact in order to underline that the definitions 4.2 and 4.3 are not too restrictive. □

We now introduce some tools about manifolds with boundary. We use the notation \( \mathbb{R}_+^N = \{ (x_1, \ldots, x_N) \in \mathbb{R}^N | x_N \geq 0 \} \).

**Definition 4.4.** An \( \hat{h}^\bullet \)-orientation on a smooth connected manifold with boundary \( X \) is the data of:

- a neat embedding \( i : X \hookrightarrow \mathbb{R}_+^N \) for any \( N \in \mathbb{N} \);
- a differential Thom class \( \hat{u} \) of the normal bundle \( N_X \mathbb{R}^N \);
- a neat tubular neighborhood \( U \) of \( X \) in \( \mathbb{R}_+^N \) with a diffeomorphism \( \varphi : U \to N_X \mathbb{R}^N \).

When the manifold is not connected, we choose an \( \hat{h}^\bullet \)-orientation on each connected component.

**Lemma 4.5.** An \( \hat{h}^\bullet \)-orientation on a manifold with boundary \( X \) is equivalent to the data of:

- a neat map \( \Phi : X \to I \) such that \( \partial X = \Phi^{-1}\{0\} \);
- an \( \hat{h}^\bullet \)-orientation of \( \Phi \).

Proof: It trivially follows from the homeomorphism \( \mathbb{R}_+^N \simeq \mathbb{R}^{N-1} \times [0, 1) \). □

**Theorem 4.6.** For \( X \) an \( \hat{h}^\bullet \)-manifold with boundary and \( \Phi : X \to I \) defined as in lemma 4.5, there exists a differential form \( \hat{A}_{\hat{h}}(X) \) on \( X \), such that for every \( \hat{\alpha} \in \hat{h}^\bullet(X) \):

\[
\int_0^1 R(\Phi_! \hat{\alpha}) = \int_X R(\hat{\alpha}) \wedge \hat{A}_{\hat{h}}(X).
\]

\(^9\)In order to construct such a homotopy, we must suppose that the closure of the tubular neighborhood chosen is contained in a bigger tubular neighborhood.
Proof: For $\varphi^+ \circ \psi : I \times \mathbb{R}^N \to U^+$ the map appearing in (18) and $\omega$ a compactly-supported form on $N_X(I \times \mathbb{R}^N)$, one has:

$$\int_0^1 \int_{\mathbb{R}^n} (\varphi^+ \circ \psi)^* \omega = \int_{N_X(I \times \mathbb{R}^N)} \omega.$$ 

It follows that, if $\hat{u} = [(u, h, \omega_u)]$ is the differential Thom class of $N_X(I \times \mathbb{R}^N)$, and $\pi : N_X(I \times \mathbb{R}^N) \to X$ the projection, we get:

$$\int_0^1 R(\Phi! \hat{\alpha}) = \int_{N_X(I \times \mathbb{R}^N)} \pi^* R(\hat{\alpha}) \wedge \omega_u = \int_X R(\hat{\alpha}) \wedge \left( \int_{N_X(I \times \mathbb{R}^N)}/X \omega_u \right).$$

Therefore we get the thesis for:

(55) $\hat{A}_h(X) = \int_{(N_X(I \times \mathbb{R}^N))/X} \omega_u.$

Due to the normalization condition, the 0-degree component of $\hat{A}_h(X)$ is 1. In particular, in the case of the ordinary differential cohomology we get $\hat{A}_H(X) = 1$ as expected. □

At the topological level, an orientation on a manifold with boundary canonically induces an orientation on the boundary. This fact can be directly generalized to the differential case.

**Lemma 4.7.** Let $X$ be a smooth connected compact manifold with boundary. Then an $\hat{h}^\bullet$-orientation on $X$ naturally induces an $\hat{h}^\bullet$-orientation on $\partial X$.

Proof: Let the orientation on $X$ be given by the embedding $j : X \hookrightarrow \mathbb{R}^N$, the differential Thom class $\hat{u}$ on $N_X \mathbb{R}^N$, and the neat tubular neighborhood $U$ of $X$ in $\mathbb{R}^N$. The orientation induced on $\partial X$ is defined by the following data:

- the embedding $\xi = j \circ i_{\partial X} : X \hookrightarrow \mathbb{R}^{N-1}$;
- on the normal bundle $N_{\partial X} \mathbb{R}^{N-1} \simeq N_X \mathbb{R}^N |_{\partial X}$ we just consider the restriction of $\hat{u}$;
- since $U$ is a neat tubular neighborhood, by definition $U \cap \mathbb{R}^{N-1}$ is a tubular neighborhood of $\partial X$, and the diffeomorphism $\varphi_U$ restricts to a suitable diffeomorphism for $\partial X$.

□

**Lemma 4.8.** For $X$ an $\hat{h}^\bullet$-manifold with boundary and $\Phi : X \to I$ defined as in lemma 4.5, endowing $\partial X$ with the orientation induced by $X$ as stated in lemma 4.7, one has for every $\hat{\alpha} \in \hat{h}^\bullet(\partial X)$:

(56) $(p_{\partial X})_!(\hat{\alpha}|_{\partial X}) = (\Phi_! \hat{\alpha})|_{\{0\}}.$

Proof: It directly follows from lemma 4.5 since all the structures involved in the definition of the Gysin map for $p_{\partial X}$ are the restrictions to the boundary of the corresponding structures for $\Phi$. □

We are now ready to define differential homology groups.

**Definition 4.5.** On a smooth compact manifold $X$, we define:

- the group of $n$-precycles of $\hat{h}^\bullet$ as the free abelian group generated by the quadruples $(M, \hat{u}, \hat{\alpha}, f)$, with:
We define $h^\bullet$-character of degree $n - 1$ on $X$ is a couple $(\chi_{n-1}, \omega_n)$, where:

$$
\chi_{n-1} \in \text{Hom}_{\hat{h}^\bullet} \left( \bigoplus_{k \in \mathbb{Z}, b^n_k \neq 0} \hat{z}_{n-1-2k}(X), \mathbb{R}/\mathbb{Z} \right)
$$

and $\omega_n \in \Omega^n(X, \hat{h}^\bullet_{\mathbb{R}})$, such that, if $(M, \hat{u}, \hat{\beta}, f) = \partial(W, \hat{U}, \hat{B}, F)$, then:

$$
\chi_{n-1}[(M, \hat{u}, \hat{\beta}, f)] = \int_W F^*\omega_n \wedge R(\hat{B}) \wedge \hat{A}_i(W) \mod \mathbb{Z}.
$$

The invariance by $h^\bullet$ is defined (using the map $\eta$ defined in (11)) by:

$$
\chi_{n-1}(\hat{\alpha})([M, \hat{u}, \hat{\beta}, f] \cdot \eta(\gamma)) = \chi_{n-1}(\hat{\alpha})[M, \hat{u}, \hat{\beta}, f] \cdot \text{ch}(\gamma).
$$

We denote by $\hat{h}^{n-1}(X)$ the group of characters of degree $n$.

As for the flat pairing, we could use the Poincaré duality on the point in order to involve only the homology in the definition of the Cheeger-Simons characters, but we avoid it in order to maintain the notation used up to now. In formula (59) we can consider a differential class of the form $\eta(\gamma)$, since $\gamma$ is an even-degree class, therefore the map $\eta$ is an isomorphism (v. formula (12)).
Theorem 4.9. There is a natural group morphism:

\[
CS^n_h : \hat{h}^n(X) \to \hat{h}^{n-1}(X)
\]

where \( \chi \) is defined, for \( [(M, \hat{\alpha}, \hat{\beta}, f)] \in \hat{c}_{n-1-k}(X) \), by:

\[
\chi([(M, \hat{\alpha}, \hat{\beta}, f)]) := \Gamma^{2k+1} \circ (p_M)!(\hat{\beta} \cdot f^*\hat{\alpha}).
\]

Proof: If we consider two representatives \( (M, u, \varphi_1\beta, f) \) and \( (N, \nu, \beta, f \circ \varphi) \) of the homology class, we have, thanks to theorem 4.3:

\[
\chi([N, \nu, \beta, f \circ \varphi]) = \Gamma^{2k+1} \circ (p_N)!(\hat{\beta} \cdot \varphi^*f^*\hat{\alpha}) = \Gamma^{2k+1} \circ (p_M)!(\varphi_1(\hat{\beta} \cdot \varphi^*f^*\hat{\alpha})
\]

\[
= \Gamma^{2k+1} \circ (p_M)!(\varphi_1(\hat{\beta} \cdot f^*\hat{\alpha})) = \chi([(M, \hat{\alpha}, \varphi_1\beta, f)]).
\]

Let us now suppose that \( (M, \hat{\alpha}, \hat{\beta}, f) = \partial(W, \hat{U}, \hat{B}, F) \). Then, for \( \Phi \) defined as in lemma 4.5, thanks to lemma 4.8 one has:

\[
(p_M)!(\beta \cdot f^*\hat{\alpha}) = (\Phi_!(\hat{B} \cdot F^*\hat{\alpha}))|\{0\}.
\]

Let \( \Phi_!(\hat{B} \cdot F^*\hat{\alpha}) = [(f, h, \omega)] \in \hat{c}^{-k+1}_{k+1}(1) \). From (27) and the previous equation we get:

\[
\gamma^{-k+1}(p_M)!(\beta \cdot f^*\hat{\alpha}) = [h(0)]|\{0\}.
\]

One has \( \delta^0h(0) = \chi(\omega(1)) - f^\ast\iota_{k+1}(1) \). Since \( h(0) \) is a 0-cocycle, it follows that \( h(0) = \chi(h(0)), \)

therefore we get:

\[
\chi(dh^0 - \omega(1)) = -f^\ast\iota_{k+1}^{-1}. \]

Because of the previous choice of \( \iota_{k+1}^{-1} \), for \( 1 : I \to I \) the identity 1-simplex, \( f^\ast\iota_{k+1}^{-1}(1) \in \text{Im} \chi^{(0)}_{\iota_{k+1}^{-1}} \). Therefore:

\[
0 \equiv \chi(dh^0 - \omega(1))(1) = \int_I (dh^0 - \omega(1)) = h^0(0) - \int_I \omega(1).
\]

Moreover, from theorem 4.6:

\[
\int_I \omega(1) = \int_I \Phi_!(\hat{B} \cdot F^*\hat{\alpha})) = \int_W R(\hat{B} \cdot F^*\hat{\alpha})) \wedge \hat{A}_h(W).
\]

Hence:

\[
\chi([(M, \hat{\alpha}, \hat{\beta}, f)]) = \int_W F^*\Phi_!(\hat{B} \cdot F^*\hat{\alpha})) \wedge \hat{A}_h(W).
\]

This is exactly formula (58) for \( \omega_n = R(\hat{\alpha}). \)

Formula (59) follows from formula (37) and the fact that the map \( p_M : M \to \{\ast\} \)

respects the orientations of \( M \) and \( \{\ast\} \) by definition, and the identity \( \text{id} : \{\ast\} \to \{\ast\} \)

respects the orientation of \( p_M \) (being constant). Therefore, formula (53) holds with \( \varphi^* = p_M^* \) and \( f^* = \text{id} \), hence:

\[
\chi(\hat{\alpha})([M, \hat{\alpha}, \hat{\beta}, f] \cdot \eta(\gamma)) = \Gamma^{2k+1} \circ (p_M)!(f^*\hat{\alpha} \cdot \hat{\beta} \cdot (p_M)^*\eta(\gamma))
\]

\[
= \Gamma^{2k+1}((p_M)!(f^*\hat{\alpha} \cdot \hat{\beta}) \cdot \gamma) = (\Gamma^{2k+1} \circ (p_M)!(f^*\hat{\alpha} \cdot \beta)) \cdot \text{ch}(\gamma)
\]

\[
= \chi_{n-1}(\hat{\alpha})[M, \hat{\alpha}, \hat{\beta}, f] \cdot \text{ch}(\gamma).
\]

\[ \square \]
The proof of the following theorem is straightforward from the previous definition.

**Theorem 4.10.** When $\alpha$ is flat, the value of the associated Cheeger-Simons character over $[M, \hat{\mu}, \hat{\beta}, f]$ coincides with the value of $(\ref{44})$, computed with respect to the homology class represented by the underlying topological cycle $[M, I(\hat{\mu}), I(\hat{\beta}), f]$. \(\Box\)

We define the holonomy of a differential class over a differential cycle as the exponential of $\chi_{n-1}$. In particular:

**Definition 4.7.** For $\hat{\alpha} \in \hat{h}^n(X)$ and $[M, \hat{\mu}, \hat{\beta}, f] \in \hat{z}_{n-1-2k}(X)$, we define the holonomy of $\hat{\alpha}$ over $[M, \hat{\mu}, \hat{\beta}, f]$ as:

$$\text{Hol}_{\hat{\alpha}}[M, \hat{\mu}, \hat{\beta}, f] := \exp \circ \Gamma^{2k+1} \circ (p_M)(\hat{\beta} \cdot f^* \hat{\alpha}).$$

A remark is now in order. It follows from formula $(\ref{58})$ that, when $\hat{\alpha}$ is flat, the value of $\chi_{n-1}$ only depends on the differential homology class, not on the single cocycle. Moreover, because of theorem $(\ref{4.10})$ its value corresponds to the value of the pairing $(\ref{44})$, which actually depends only on the underlying homology class, not on the differential refinement. Therefore, one is led to suppose that the differential homology groups are isomorphic to the topological ones. We now prove that this is the case: in other words, the definition of differential homology does not provide new homology groups, but it provides another way to define the cycles and the boundaries of the topological homology groups, in such a way that it is possible to integrate a differential class over a cycle. Cycles are really important at a differential level, as for ordinary cohomology, and the topological ones are not enough. When the class is flat, the integration depends only on the homology class, which is the same in the two cases.

**Theorem 4.11.** The natural group morphism:

$$\Phi : \hat{h}_\bullet(X) \rightarrow h_\bullet(X)$$

$$[(M, \hat{\mu}, \hat{\alpha}, f)] \rightarrow [(M, I(\hat{\mu}), I(\hat{\alpha}), f)]$$

is an isomorphism.

Proof: We divide the proof in three steps.

**Step 1.** If $I(\hat{\mu}) = I(\hat{\mu}')$, $I(\hat{\alpha}) = I(\hat{\alpha}')$ and $f$ is homotopic to $f'$, then $[(M, \hat{\mu}, \hat{\alpha}, f)] = [(M, \hat{\mu}', \hat{\alpha}', f')]$ in $\hat{h}_\bullet(X)$. In fact, since $\hat{\alpha}' = \hat{\alpha} + a(\rho)$, we can consider on $M \times I$ the class $A = \pi_I^*\hat{\alpha} + a(t \cdot \pi_I^*\rho)$, which links $\hat{\alpha}$ to $\hat{\alpha}'$. The same construction for $\hat{\mu}$ and $\hat{\mu}'$ leads an orientation $\hat{U}$ on $M \times I$ in the following way: we consider the projection $M \times I \rightarrow I$ and we orient it via the orientation of $M$, considering the embedding $M \hookrightarrow \mathbb{R}^N$, which determines the embedding $M \times I \hookrightarrow \mathbb{R}^N \times I$, with normal bundle $\pi_I^*(N_M \mathbb{R}^N)$. On such a bundle we put the orientation $\hat{U} = \pi_I^*\hat{\mu} + a(t \cdot \pi_I^*\eta)$. It is clear that $\hat{U}$ is a differential Thom class, since on each fiber the first Chern class is the same of $\hat{\mu}$ and the curvature differs by an exact form, whose integral is vanishing. Hence, for $F : M \times I \rightarrow X$ a homotopy between $f$ and $f'$, one has $\partial(M \times I, \hat{U}, \hat{A}, F) = (M, \hat{\mu}, \hat{\alpha}, f) - (M, \hat{\mu}', \hat{\alpha}', f')$.\[\text{\textsuperscript{10}}\]

\[\text{\textsuperscript{10}}\text{The class } A \text{ is not a homotopy of differential functions, since its curvature can have a leg } dt \text{ on } I, \text{ and this is forbidden by definition } [\text{ III}]. \text{ That’s why } A \text{ is defined even if } \hat{\alpha}' \neq \hat{\alpha}.\]
Step 2. Given two equivalent topological precycles \((M, u, \varphi, f) \simeq (N, v, \alpha, f \circ \varphi)\), for \(\varphi : N \hookrightarrow M\) an embedding, any two differential refinements \((M, u, \alpha', f)\) (with \(I(\alpha') = \varphi(\alpha)\)) and \((N, \dot{v}, \dot{\alpha}, f \circ \varphi)\) are equivalent as differential homology classes (not as differential cycles). In fact, thanks to lemma 4.4, we can find a differential orientation of \(\varphi\) compatible with differential refinements of \(\dot{u}'\) of \(u\) and \(\dot{v}'\) of \(v\), and a map \(f'\), homotopic to \(f\), which respects the orientation of \(\varphi\). Then by definition \([(M, \dot{u}', \varphi(\alpha'), f')] = [(N, \dot{v}', \dot{\alpha} f \circ \varphi)]\). By the first step, this implies that \([(M, \dot{u}, \alpha', f)] = [(N, \dot{v}, \dot{\alpha}, f \circ \varphi)]\).

Step 3. The morphism \(\Phi\) is clearly well-defined and surjective. Therefore, we only have to prove the injectivity. Let us suppose that \(\Phi[(M, \dot{u}, \dot{\alpha}, f)] = 0\). Then \([(M, I(\dot{u}), I(\dot{\alpha}), f)]\) is equivalent, as a cocycle, to \([(N, v, \beta, g)]\) such that \((N, v, \beta, g) = \partial(W, V, B, G)\). This means that there exists a sequence of pre-cycles \((M_i, u_i, \alpha_i, f_i)\), for \(i = 0, \ldots, n\), such that \((M_0, u_0, \alpha_0, f_0) = (M, I(\dot{u}), I(\dot{\alpha}), f)\), \((M_n, u_n, \alpha_n, f_n) = (N, v, \beta, g)\) and such that there exists a map \(\varphi_i : M_i \to M_{i+1}\) or \(\psi_i : M_{i+1} \to M_i\) such that \(f_i = f_{i+1} \circ \varphi_i\) and \(\alpha_{i+1} = \varphi_i(\alpha_i)\), or the analogue for \(\psi_i\). We choose a differential refinement \((M_i, u_i, \alpha_i, f_i)\) for each \(i\), such that for \(i = 0\) it coincides with \((M, \dot{u}, \dot{\alpha}, f)\), for \(i = n\) it is a refinement \((N, \dot{v}, \dot{\beta}, g)\) of \((N, v, \beta, g)\). Thanks to the construction in [3], we can suppose that each map \(\varphi_i\) or \(\psi_i\) is a section of a sphere bundle or a diffeomorphism, in any case an embedding. Hence, by the second step, we get that \([(M, \dot{u}_i, \dot{\alpha}_i, f_i)] = [(M_{i+1}, \dot{u}_{i+1}, \dot{\alpha}_{i+1}, f_{i+1})]\), hence \([(M, \dot{u}, \dot{\alpha}, f)] = [(N, \dot{v}, \dot{\beta}, g)]\). We now consider a differential refinement \((W, V, B, G)\) of \((W, V, B, G)\) by the step 1, \([(N, \dot{v}, \dot{\beta}, g)] = [(N, \dot{V}|_N, B|_N, g)] = 0\).□

We call \(\text{Codom}(\Xi)\) the codomain of \(\Xi^n\), defined by formula (14). There is an embedding \(i : \text{Codom}(\Xi^n) \to \hat{h}^{n-1}(X)\), since a morphism \(\varphi : h_{n-1-2k}(X) \to U(1)\) determines a unique morphism \(\chi : \hat{z}_{n-1-2k}(X) \to \mathbb{R}/\mathbb{Z}\) such that \(\exp_\chi[M, \dot{\beta}, f] = \varphi[M, I(\dot{u}), I(\dot{\beta}), f]\), and we define \(i(\varphi) = (\chi, 0)\). It follows from theorem 4.11 that the image of \(i\) is the subgroup of generalized Cheeger-Simons characters with vanishing curvature, which we call \(\hat{h}^{n-1}(X)\). In fact, given a character \((\chi, 0)\), it follows from formula (58) that \(\chi_{n-1}\) only depends on the homology class, hence, thanks to theorem 4.11, it depends on the topological homology class, thus it belongs to the image of \(i\). It follows from theorem 4.11 that \(i\) restricts to an embedding \(i' : \ker(\Xi^n) \hookrightarrow \ker(CS_h^n)\), and that there is an embedding \(j : \text{Im}(\Xi^n) \hookrightarrow \text{Im}(CS_h^n)\). Because of \(i\) and \(j\) we can construct a morphism \(a : \text{Coker}(\Xi^n) \to \text{Coker}(CS_h^n)\). We can now show that actually \(i'\) and \(a\) are isomorphisms.

In particular, in the case of K-theory and singular cohomology, \(CS_h^n\) is an isomorphism, as we analyze in more detail in the following theorem.

Theorem 4.12. The following canonical isomorphisms hold:

\[
\text{Ker}(\Xi^n) \cong \text{Ker}(CS_h^n), \quad \text{Coker}(\Xi^n) \cong \text{Coker}(CS_h^n).
\]

Proof: If \(\hat{\alpha} \in \hat{h}^{n}(X)\) is not flat, then \(CS_h^n(\hat{\alpha}) \neq 0\), since \(CS_h^n(\hat{\alpha}) = (\chi, R(\hat{\alpha}))\) and \(R(\hat{\alpha}) \neq 0\). Hence \(\text{Ker}(CS_h^n) \subset \text{Ker}(\Xi^n)\) and the equality follows. Moreover, \(\hat{h}^{n-1}(X) \cap \text{Im}(CS_h^n) = \text{Im}(\Xi^n)\), hence \(a : \text{Coker}(\Xi^n) \to \text{Coker}(CS_h^n)\) is an embedding. Moreover, if \((\chi_{n-1}, \omega_n) \in \hat{h}^{n-1}(X)\), we consider a class \(\hat{\alpha} \in \hat{h}^{n}(X)\) such that \(R(\hat{\alpha}) = \omega_n\), and we call \((\chi', \omega_n) := CS_h(\hat{\alpha})\). Then \((\chi'_n - \chi_{n-1}, 0) \in \hat{h}^{n-1}(X)\), and, in \(\text{Coker}(CS_h^n)\), one has \([(\chi_{n-1}, \omega_n)] = [(\chi'_n - \chi_{n-1}, 0)] \in \text{Im} a\). Therefore \(a\) is also surjective. □

Remark: Given \((\chi_{n-1}, \omega_n) \in \hat{h}^{n-1}(X)\), it can happen that \(\omega_n \neq 0\) but \(\chi_{n-1} = 0\): for example, in the case of singular cohomology, if \(n = 0\) this is always the case. Actually,
5. Singular cohomology and K-theory

In the case of singular cohomology, a flat differential character $\hat{\alpha} \in \hat{H}_n^1(X)$ defines a differential character $\chi_{n-1} : H_{n-1}(X) \to U(1)$. Such a character corresponds to the holonomy of the flat abelian $(n - 2)$-gerbe classified by $\hat{\alpha}$. If we think of $\hat{\alpha}$ as a smooth Deligne cohomology class, there is an explicit formula for the holonomy [5]. When a homology class is representable by a submanifold $M \subset X$, the holonomy can be also defined in the following way: we consider $\hat{\alpha}|_M$, which, for dimensional reasons, has trivial first Chern class. Therefore, it can be represented as a class $h \in H^{n-1}(M, \mathbb{R})/H^{n-1}(M, \mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$.

The isomorphism is defined via the evaluation $h([M])$, for $[M]$ the fundamental class of $M$, and the holonomy is its exponential. If we represent a generic class in the form $[(M, u, \beta, f)]$, then we argue in the same way considering $f^*\hat{\alpha} \cdot \beta$ on $M$: we now show that this is exactly the pairing (44) in the case of singular cohomology. In fact, the map (27) is non-trivial only for $k = 0$, and since $\hat{H}^1([\ast]) \simeq H^0([\ast], \mathbb{R}/\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$, the map $\Gamma^1 : \hat{H}^1([\ast]) \to \mathbb{R}/\mathbb{Z}$ is the identity up to canonical isomorphism. More precisely, $\Gamma^1([0, h, 0]) = [h] \in H^0([\ast], \mathbb{R}/\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$. Therefore, the pairing (44) reduces to:

$$
\Xi^{n+1} : \hat{H}_n^{n+1}(X) \to \text{Hom}(H_n(X, \mathbb{Z}), \mathbb{R}/\mathbb{Z})
$$

$$
\Xi^{n+1}(\hat{\alpha})[M, u, \beta, f] = (p_M)(f^*\hat{\alpha} \cdot \beta).
$$

We suppose for simplicity $M$ connected. Then $\dim M = n+q$ and $\beta \in H^q(M)$. Since $f^*\hat{\alpha} \cdot \beta \in \hat{H}_n^{n+q+1}(M)$, and $H^{n+q+1}(M) = 0$ for dimensional reasons, then $f^*\hat{\alpha} \cdot \beta = [(0, h, 0)]$ for $h \in C^{n+q}(M, \mathbb{R})$. Because of lemma 13.2, we get $(p_M)!([(0, h, 0)]) = [(0, (p_M)h, 0)]$, and $(p_M)!h = h([M])$, as claimed. In particular, it follows that (44) is an isomorphism (as a consequence of the universal coefficient theorem [3], as is well-known).

If we consider also non-flat classes, their holonomy is defined on singular cycles, with no need of differential refinement. This is not applicable to cycles represented in the form $[M, u, \beta, f]$, since the product $f^*\hat{\alpha} \cdot \beta$ is not well-defined if $\hat{\alpha}$ is not flat. In this case we need to consider the differential refinement of the cycles, as in def. 13.6. Anyway, it follows from formula (61) that (60) is an isomorphism. Therefore, there is a canonical isomorphism between the group of Cheeger-Simons characters defined on singular cycles (in the usual sense), and the group of Cheeger-Simons characters defined on differential cycles (as defined in the present paper), since both groups are canonically isomorphic to $\hat{H}^n(X)$. We can explicitly describe this isomorphism for a large class of cycles. In fact, let us consider a submanifold $M \subset X$. It determines a cycle in the usual sense, therefore we can compute the holonomy. Otherwise, we consider the differential $(n-1)$-cycle $[(M, \hat{\alpha}, 1, i)]$, for $\hat{\alpha}$ any differential orientation refining the topological one, and $i : M \hookrightarrow X$ the embedding. Then, for $\alpha \in \hat{H}^n(X)$, $\hat{\alpha}|_M$ is flat for dimensional reasons, therefore the
push-forward only depends on the topological orientation of \( M \). Then we argue as above. Thus, \( \exp(\rho_M)(\hat{\alpha}|_M) \) coincides with the holonomy in the usual sense. The same holds if we consider a cycle \([ (M, \hat{u}, 1, f) ]\), for \( f \) not necessarily an embedding, considering the class \( f^* \hat{\alpha} \). Moreover, once that we compute the value of a character over a differential cycle, we compute it over all the homologous cycles via formula (58). Hence, the values of a Cheeger-Simons character in the usual sense or in the sense of the present paper, can be directly computed one from the other on cycles whose homology class belongs to the image of the natural map from the \((n - 1)\)-bordism group of \( X \) to the singular homology (for low-dimensional manifolds this map is surjective). Otherwise, we have to pass through the canonical isomorphisms with \( h^n(X) \).

Finally, we consider the case of complex K-theory. Topologically, it satisfies the Bott periodicity, i.e. \( K^n(X) \cong K^{n-2}(X) \). One way to express this isomorphism is to consider a generator \( \gamma_0 \in K^{-2}(\{\ast\}) \) and the map \( \alpha \to \alpha \cdot \gamma_0 \). The inverse is \( \alpha \to \alpha \cdot \gamma_0^{-1} \), for \( \gamma_0 \in K^{2}(\{\ast\}) \) a generator. Such a periodicity can be extended to differential K-theory, considering the map \( \hat{\alpha} \to \hat{\alpha} \cdot \eta(\gamma_0) \), for \( \eta \) defined by formula (41). It is still an isomorphism, since \( \eta(\gamma_0)\eta(\gamma_0^{-1}) = \eta(\gamma_0\gamma_0^{-1}) = \eta(1) = 1 \). The periodicity can be extended to differential cycles, via the isomorphism:

\[
B : \hat{\Omega}^n(X) \to \hat{\Omega}^{n-2}(X)

\[
[M, \hat{u}, \hat{\beta}, f] \to [M, \hat{u}, \hat{\beta} \cdot \eta(\gamma_0), f].
\]

Therefore, a generalized Cheeger-Simons character, as defined by formula (57), is uniquely determined by its restriction to \( \hat{\Omega}^{n-1} \), because of formula (59). The same holds for the pairing (44) (v. [10] for an analytic description of the pairing). It follows from the universal coefficient theorem for K-theory, formulated via the Ext group [15, formula 3.1], that (44) is an isomorphism, therefore also (60) thanks to the isomorphisms (51). Therefore, in the case of K-theory, the pairing (44) and theorem 4.9 can be summarized an enriched as follows.

**Definition 5.1.** A Cheeger-Simons differential \( \hat{K}^\bullet \)-character of degree \( n - 1 \) on \( X \) is a couple \((\chi_{n-1}, \omega_n)\), where:

\[
\chi_{n-1} : \hat{\Omega}^{n-1}(X) \to \mathbb{R}/\mathbb{Z}
\]

and \( \omega_n \in \Omega^n(X, K^\bullet) \), such that, if \( (M, \hat{u}, \hat{\beta}, f) = \partial(W, \hat{U}, \hat{B}, F) \), then \( \chi_{n-1}[ (M, \hat{u}, \hat{\beta}, f) ] = \int_W F^* \omega_n \wedge R(\hat{B}) \wedge \hat{A}_K(W) \mod \mathbb{Z} \). We denote by \( \hat{K}^{n-1}(X) \) the group of characters of degree \( n \).

**Theorem 5.1.** There is a natural group isomorphism:

\[
CS^n_K : \hat{K}^n(X) \to \hat{K}^{n-1}(X)
\]

\[
\hat{\alpha} \to (\chi, R(\hat{\alpha})),
\]

---

11Actually, one can prove that, in the case of ordinary cohomology, the push-forward with respect to a fibration (in this case over a point) never depends on the differential refinement of the orientation, that’s why it is possible to integrate in smooth Deligne cohomology without specifying the differential orientation chosen.
where \( \chi \) is defined, for \([ (M, \hat{u}, \hat{\beta}, f) ] \in \hat{z}_{n-1}(X) \), by \( \chi[(M, \hat{u}, \hat{\beta}, f)] := \Gamma^{2k+1} \circ (p_M)^!(\hat{\beta} \cdot f^* \hat{\alpha}) \).

Restricting to flat classes and exponentiating, we get an isomorphism:

\[
\Xi^n : \hat{K}^n(X) \to \text{Hom}(K_{n-1}(X), \mathbb{R}/\mathbb{Z}).
\]

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**References**

[1] J. F. Adams, *Stable homotopy and generalized cohomology*, Chicago Lectures in Mathematics
[2] U. Bunke and T. Schick, *Uniqueness of smooth extensions of generalized cohomology theories*, J Topology (2010) 3 (1): 110-156, arXiv:0901.4423
[3] E. Dyer, *Cohomology theories*, W. A. Benjamin (1969)
[4] F. Ferrari Ruffino, *Geometric homology revisited*, preprint, arXiv:1301.5882
[5] K. Gomi and Y. Terashima, *Higher dimensional parallel transport*, Mathematical Research Letters 8, 2533 (2001)
[6] A. Hatcher, *Algebraic topology*, Cambridge university press, 2002
[7] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, J.Diff.Geom. 70 (2005) 329-452, arXiv:math/0211216
[8] A. Jakob, *A bordism-type description of homology*, manuscripta math. 96, 67-80 (1998)
[9] M. Karoubi, *K-theory: an Introduction*, Springer-Verlag, 1978
[10] J. Lott, \( \mathbb{R}/\mathbb{Z} \) *Index Theory*, Comm. Anal. Geom. 2 (1994), no. 2, 279-311
[11] Y. B. Rudyak, *On Thom spectra, orientability and cobordism*, Springer monographs in mathematics
[12] R. M. Switzer, *Algebraic topology*, Springer-Verlag
[13] M. Upmeier, *Products in Generalized Differential Cohomology*, arXiv:1112.4173
[14] G. W. Whitehead, *Generalized homology theories*, Transactions of the American Mathematical Society, Vol. 102, No. 2 (Feb., 1962), pp. 227-283
[15] Z. Yosimura, *Universal coefficient sequences for cohomology theories of CW-spectra*, Osaka J. Math. 12 (1975), 305-323