Stochastic effects on phase-space holes and clumps in kinetic systems near marginal stability

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Abstract
The creation and subsequent evolution of marginally-unstable modes have been observed in a wide range of fusion devices. This behaviour has been successfully explained, for a single frequency shifting mode, in terms of phase-space structures known as a ‘hole’ and ‘clump’. Here, we introduce stochasticity into a 1D kinetic model, affecting the formation and evolution of resonant modes in the system. We find that noise in the fast particle distribution or electric field leads to a shift in the asymptotic behaviour of a chirping resonant mode; this noise heuristically maps onto radial microturbulence via canonical toroidal momentum scattering, affecting hole and clump formation. While the mechanism allowing for the formation of the hole and clump is coherent, the lifetime of a hole and clump is shown to be highly sensitive to initial conditions, affecting the temporal profile of a single bursting event in mode amplitude.

Keywords: stochasticity, turbulence, fast ions, hole and clump, nonlinear dynamics, mode chirping, kinetic instabilities

(Some figures may appear in colour only in the online journal)

1. Introduction

Toroidal Alfvén eigenmodes, (TAEs) are of particular interest to the fusion community. With frequencies on the order of 100kHz [1], they have the ability to become amplified via RF heating and energetic particles. Due to defects in the magnetic field periodicity in real tokamaks, spatially localised modes (gap TAEs) can exist in the frequency gap between TAEs and BAEs (beta-induced Alfvén eigenmodes; frequency <50kHz, close to GAM frequency). Unlike continuum Alfvén modes, gap TAEs exist as coherent waveforms which are resilient to shear damping. Energetic particles undergoing Landau resonance with plasma waves under go radial diffusion, which can lead to large fast particle losses in tokamaks [1–4]; gap TAEs exist for longer timescales than continuum modes, allowing for greater particle loss.

It is well known that hole and clump structures [5–7] can form in the non-linear phase of the evolution of an energetic particle driven mode, such as a TAE. It is understood that mode chirping is directly correlated with hole and clump structures; consequently, chirping modes can allow for greater radial diffusion. As a result, even in the case of continuum Alfvén eigenmodes, rapid mode chirping can also lead to a significant channel for fast ion loss; in such a case, the rate of energy loss via chirping is comparable to the sum of damping rates (e.g. collisional, radiative, continuum damping).

The effects of random, small-scale phenomena have been previously examined in the literature [4–7]: mechanisms such as pitch-angle scattering can destroy holes and clumps. However, the effect of microturbulence on the evolution of a bursting mode in its non-linear phase is relatively unexplored. Recent work by Duarte et al [8, 9] proposes that enhanced...
stochasticity in resonant particle dynamics, in the form of fast ion microturbulence, can be a key mechanism for chirping suppression in several tokamak scenarios. The prediction stimulated dedicated experiments on DIII-D by Van Zeeland et al. [10] with negative triangularity, known for suppressing drift-like instabilities. The experiments have shown a clear correlation between chirping emergence and scenarios with very low turbulent activity.

This motivates the work detailed in this paper: here, we carefully explore the inclusion of stochasticity into 1D kinetic models, allowing us to examine in closer detail the resultant effects on the evolution of resonant modes. In section 4, we explore how small-scale random fluctuations in the mode amplitude can lead to a stochastic lifetime of the hole and clump. In section 5, we explore how random fluctuations in the particle distribution function—heuristically similar to turbulence—can lead to suppression of hole-and-clump formation, and consequently suppress mode chirping.

2. Stochastic model

2.1. Resonant damping

In a tokamak, TAEs resonate with a quasi-2D fast-ion distribution function where the linear stability is determined by competing $df/dv$ and $df/dp_z$ [1]. These correspond to resonance with the poloidal and toroidal transit frequencies $\omega_p$ and $\omega_z$, respectively. [1, 11] given by $\omega + (m + l_p)\omega_p - n\omega_z = 0$. Here, $m$ and $n$ are the poloidal and toroidal modenumbers, and $l_p \in \mathbb{Z} \neq 0$ correspond to poloidal harmonics of the drift velocity (see figure 1).

We model the same key instability physics by examining hole and clump formation on a 1D bump-on-tail distribution function. This allows us to model energetic particle drive via the positive slope of the distribution function between the bulk and the beam, while modelling energetics losses as a damping term [5, 6, 12–14]. The corresponding 1D resonance condition is $\omega - j2\pi v/L_i = 0$ where $j$ is the mode-number, $v$ is the particle velocity, and $L_i$ is the length of the 1D box.

The evolution of the system is determined by coupling the Boltzmann equation to Maxwell’s equations; our model is given here by a multiple species generalisation of kinetic models used by Vann et al. [13] and De-Gol [14]:

\[ \partial_t f_i = C_i[f_i] - v\partial_v f_i - \frac{q_i}{m}E\partial_v f_i/\ell \] (1a)

\[ \partial_t E = -\frac{1}{\epsilon_0} \int_{-\infty}^{\infty} \sum_l (q_l f_l) dv - \alpha E \] (1b)

where $\{C_i\}$ are collision operators, $f_i(x,v,t)$ is the fast particle distribution function for the $i$th species, $E(x,t)$ is the electric field, and $\epsilon_0$ is the permittivity of free space. Damping is effected in the system via $\alpha(x,t)$, and acts as a sink of electromagnetic field energy. Formally, one can show that this augmentation still preserves momentum and energy globally (see derivation in appendix A from classical field theory).

Figure 1. A sketch of a hole and clump on a distribution function peaked near the origin for a chirping, resonant mode interacting with a tokamak plasma. The resonance $\omega + (m + l_p)\omega_p - n\omega_z = 0$ undergoes pitchfork bifurcation as the hole and clump move. Stochastic fluctuations modelled in this paper heuristically map onto fluctuations a function of $p_c$; these effect the formation and evolution of the hole and clump.

2.2. Two species model for turbulence

Here, we model the plasma by using two separate distributions of identical particles: a fast ion distribution $f_{\text{ion}}$ with a deterministic phase-space trajectory in the absence of electric field, and a turbulent distribution $f_{\text{tur}}$ with a stochastic trajectory. By modelling the plasma using two separate distributions, one is able to vary with ease the fraction of particles that are turbulent. Fluctuations in $f_{\text{ion}}$ lead to fluctuations in the electric field via equation (1b). Fluctuations in the electric field interact with $f_{\text{ion}}$ via equation (1a), heuristically mapping via $\alpha E$ onto the energy exchange associated with particle resonance along $df/dv$.

We treat the electrons as existing within a neutralising background that do not move with the ions, or resonate with the electric field. This allows the model to become a two species plasma:

\[ \partial_t f_{\text{ion}} = C[f_{\text{ion}}] - v\partial_v f_{\text{ion}} - E\partial_v f_{\text{ion}} \] (2a)

\[ \partial_t E = -\int_{-\infty}^{\infty} v(f_{\text{ion}} + f_{\text{tur}}) dv - \alpha E \] (2b)

The normalisation is as follows:

\[ N_{\text{ion}} = 1 ; \quad m_{\text{ion}} = 1 ; \quad \omega_p = 1 ; \quad v_{\text{th}} = 1. \] (3)

where for the non-turbulent fast ions, $N_{\text{ion}}$ is the number of particles, and $\omega_p$ is the ion plasma frequency. The quasithermal quantity $v_{\text{th}}$ normalises the energy of the system, but the equilibrium is not a thermal equilibrium.

For $C = \mathcal{O}(\partial_x^3)$, one can show that the resultant hyperjerk equation $\partial_t f = C[f]$ at fixed $t$ can be represented in the form
\[ \delta g = F(g) \] where \( g \in \mathbb{R}^n \) where \( n \geq 3 \), and \( F(g) \) is a smooth function \([15]\). Via the Poincaré–Bendixson theorem \([16–18]\), this meets the minimum requirements for chaotic behaviour; a simple *reductio ad absurdum* shows that replacing \( n \) with a value less than 3 prevents the formation of chaotic solutions.

It is instructive to note that as two identical species, both the non-turbulent and turbulent fast ions experience the same collision operator. However, while for pitch angle scattering the individual particle trajectories are stochastic, the evolution of the distribution function is deterministic in the absence of electric field; collective diffusion leads to a well defined trajectory for \( f_{\text{ion}} \). In contrast, the turbulent population undergoes processes which lead to a stochastic evolution of \( f_{\text{tur}} \).

Formally, we approximate that for \( f_{\text{tur}} \), one can use
\[ C = O(\varepsilon^2). \]
This is justified mathematically by arguing that the non-turbulent fast ion distribution exists on a subset of phase-space \( \mathbb{R}^p_\text{ion} \subseteq \mathbb{R}^2 \) where its evolution is non-chaotic in the absence of electric field. We utilise a Fokker–Planck diffusive collision operator for \( f_{\text{ion}} \), in alignment with kinetic descriptions of mode chirping based on pitch angle scattering in the literature \([6, 8, 9, 19]\). However, it is important to note that the operator relaxes to the initial distribution function; energy transfer may occur during relaxation.

Similarly to the non-turbulent ions, we state that \( f_{\text{tur}} \) exists on a non-intersecting subset of phase-space \( \mathbb{R}^p_\text{tur} \neq \mathbb{R}^p_\text{ion} \) that is to say it does not exist in non-chaotic regions. We find that a second order collision operator is insufficient to generate \( f_{\text{tur}} \) as we require terms \( O(\varepsilon^2) \). In lieu of an implemented model beyond Fokker–Planck theory, we instead define the trajectory of \( f_{\text{tur}} \) ad hoc to investigate the key resultant physics.

By defining \( f_{\text{ion}} \) and \( f_{\text{tur}} \) on separate subsets of phase-space, we remove the ability to have interspecies collisions.

As a result:

\[
\begin{align*}
\delta f_{\text{ion}} &= \nu \delta \rho_{\text{ion}}(f_{\text{ion}} - F_0) - \nu \delta f_{\text{ion}} - E \delta f_{\text{ion}} \\
\partial_t f_{\text{ion}} &= f_{\text{ion}}(x, v, t) - \nu f_{\text{ion}} - E f_{\text{ion}} \\
\partial_t E &= -\int_{-\infty}^{\infty} v (f_{\text{ion}} + f_{\text{tur}}) \, dv - \alpha E
\end{align*}
\]

where \( F_0(v) \equiv f_{\text{ion}}(x, v, t = 0) \), \( \nu \) is an effective collisional pitch-angle scattering diffusion coefficient, and:

\[
\begin{align*}
f_{\text{ion}} &= 0 \quad \forall \{x, v\} \not\in \mathbb{R}^p_\text{ion} \\
f_{\text{tur}} &= 0 \quad \forall \{x, v\} \not\in \mathbb{R}^p_\text{tur}.
\end{align*}
\]

**2.3. Approximations**

We make a few assumptions to simplify the model computationally. These rely on a Fourier series representation of the distribution function and other quantities:

\[ f(x, v, t) = \frac{1}{2} \sum_j \left[ f_j(v, t) e^{i k x} + \text{c.c.} \right]. \]

where we sum over a set of initial modes, \( \{ j \} \). We define the velocity Fourier transform by symmetric definitions:

\[
\tilde{f}(x, s, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x, v, t) e^{-i s v} \, dv \]
\[
f(x, v, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(x, s, t) e^{i s v} \, ds.
\]

First, we replace \( \alpha(x)E(x) \) with \( \alpha(x) * E(x) \); via the convolution theorem, this allows for a piecewise product of \( \alpha \) and \( E \) in k-space. However, as it does not preserve the canonical form of the Hamiltonian (see appendix A), conservation of energy is violated except for the trivial case where \( \alpha(x) \) is a constant. Here, we examine this trivial case, and therefore \( \forall j : \alpha_j = 2\alpha \).

Secondly, we ignore three-wave coupling; this means that no modes exist except for harmonics of the initial set of modes. We allow this as three-wave coupling of gap TAEs will generate modes which exist in the Alfvén continuum, which are quickly dissipated \([1]\).

Thirdly, we also ignore all harmonics of the initial set of modes except for the fundamental; this is justified by requiring physically that these harmonics are rapidly damped.

Finally, we force \( E_0 \) to be evanescent, and set it to 0 via boundary conditions, removing mean current from the Maxwell–Ampère law. This leads to the caveat:

\[
\left| \alpha_0 E_0 \right| \gg \int_{-\infty}^{\infty} |v (f_{\text{ion},0} + f_{\text{tur},0})| \, dv.
\]

In principle, equation (8) is violated, but we find that the induced mean current is very small. However, it is important to note that there are known physical limitations of this model.

The 1D electrostatic bump-on-tail model employed here considers a uniform mode structure; particles can move, but radial decoupling \([20, 21]\) does not occur. However, resonance detuning does still occur (as shown later in figure 8). Due to a lack of three-wave coupling, phase locking is also omitted here. Although the model produces eigenmodes that are ion acoustic waves here, one expects a homothetic mapping onto Alfvén waves to exist \([22]\).

Finally, as the problem is solved on a computational grid, we state that a given cell in phase-space will contain some particles that are non-turbulent and some particles that are turbulent. This allows one to freely disregard equation (5).

Overall, for a single mode simulation after spectral decomposition and velocity Fourier transforms:

\[
\begin{align*}
\partial_t \tilde{f}_{\text{ion},j} &= -i \nu s^2 \tilde{f}_{\text{ion},j} + k_j \partial_x \tilde{f}_{\text{ion},j} - i s \tilde{E}_j \tilde{f}_{\text{ion},0} \\
\partial_t f_{\text{ion},0} &= -i \nu s^2 (f_{\text{ion},0} - F_0) - \frac{\nu}{2} \left[ \tilde{E}_j \tilde{f}_{\text{ion},j} + E_j f_{\text{ion},j} \right] \\
\tilde{f}_{\text{tur}} &= \tilde{f}_{\text{tur}}(x, s, t) \\
\partial_t E_j &= -\int_{-\infty}^{\infty} \left[ W(\tilde{f}_{\text{ion},j} + \tilde{f}_{\text{tur},j}) \right] ds - \frac{1}{2} \alpha_j E_j \\
E_0 &= 0
\end{align*}
\]

where \( W(s) \in \mathbb{I} \) is the complex formally divergent integral (which acts a removable singularity in the current):
The standard deviation of $D \cdot 1_L$ is the typical amplitude of stochastic seeding.

2.5. Turbulent distribution

The fast-ions are modelled using a $1D$ bump-on-tail distribution (see figure 2(b)):

$$F_0(v) = \frac{1}{\sqrt{2\pi}} \left[ \eta \exp\left(-\frac{v^2}{\sigma^2}\right) + \frac{1-\eta}{\sqrt{\pi}} \exp\left(-\frac{(v-v_b)^2}{\sigma^2}\right) \right].$$

such that $(1 - \eta)$ is the fraction of particles in the beam, $v_t$ is the beam width, and $v_b$ is the beam velocity. We define the turbulent population as the sum of a top-hat function and a noise term; the noise term is defined to yield a particle population of zero (see figure 3(b)). This allows us to parametrically modify the stochasticity of $f_{\text{tur}}$ via noise without changing the total number of particles. After Fourier transforms:

$$f_{\text{on}}(x, s, t) = \tilde{f}_0(s) + \tilde{f}_{\text{on}}(x, s, t)$$

$$\tilde{f}_{\text{tur},j}(s, t) = \frac{1}{\sqrt{2\pi}} \epsilon_{\text{ej}} \sin\left(\frac{s L_\nu}{2}\right) + \tilde{N}_{\text{ej},j}$$

$$\tilde{f}_{\text{tur},0}(s, t) = 0$$

where $\sin(s) \equiv \sin(s)/s$ is the sinc function. $N_{\text{ej}}(x, s, t)$ is a noise term, and $L_\nu$ is the length of the codomain of $F_0(v)$. Accordingly, the fraction of non-turbulent particles is $1/[1 + \epsilon_{\text{ej}}]$. We assume that broadband noise in $v$-space will still be broadband noise in $s$-space, weighted accordingly via Parseval’s theorem; accordingly we relate amplitudes as:

$$\frac{|\tilde{N}_{\text{ej},j}|}{|\tilde{N}_{\text{ej},j}|} \approx \frac{L_\nu}{\sqrt{2\pi}}.$$

For $N_{\text{ej}}(v, t)$, we use raised cosine noise, with a mean value of 0. The domain is given by:

$$N_{\text{ej}}(v, t) \in \left[-\sigma_{\text{ej}} \sqrt{\frac{3\pi^2}{\pi^2 - 6}}, \sigma_{\text{ej}} \sqrt{\frac{3\pi^2}{\pi^2 - 6}}\right].$$
The typical amplitude is equal to the standard deviation $\sigma_{f, j}$, and accordingly to force positive $f_{\text{tur}}$ everywhere:

$$
\sigma_{f, j} \leq \sigma_{f, j}^{(\text{max})} = \frac{\epsilon_j}{r_{\text{f}}} \sqrt{\frac{\pi^2 - 6}{3r^2}}.
$$

(13)

We also require for conservation of energy and particle number (at constant $\epsilon_j$) that the 0th and 2nd moment of $f_{\text{tur}}$ vanish. To enable this, we enforce that $N_{f, j}(x, s, t)$ has a real part that is odd, and an imaginary part that is even. One finds that the net energy content of the turbulent distribution is given by:

$$
U_{\text{tur}} = \frac{1}{24} r_{\text{f}}^2 L_{\text{v}}. 
$$

(14)

3. Computational method

3.1. DARK

We utilise a new, modular code based on previous work by Arber et al [14, 23]. DARK (D-dimensional Augmented Resonance Kinetic solver) allows for a single framework which can incorporate fundamentally different models and approximations by using a Strang split set of partial flows. Each partial flow is solved separately to yield the full solution across a timestep.

Fourier decomposition reduces $N_v$ grid points in $x$ to $(N_k + 1)$ equations (here the number of modes $N_k = 1$). By using s-space, the code computes collisions and velocity advection at the same time, giving a factor 2 increase in speed, and only requires backwards transforms via FFTW [24] (fastest Fourier transform in the west) on distribution function output, providing a potential $O(N_v \log N_v)$ decrease in computational time on each timestep.

3.2. Global parameters

Our global parameters were selected to be:

$$
v_b = 10 \quad ; \quad v_t = 4 \quad ; \quad \eta = 0.95
$$

$$
\omega_s = 2.0 \quad ; \quad k_j = 0.15
$$

$$
\Delta t = 0.1 \quad ; \quad v \in [-28, 88] \quad ; \quad N_v = 8192.
$$

(15)

Selecting $k_j = 0.15$ means that if we define $k_j$ as the fundamental eigennumber of the system (the length of the 1D box $L_s = 2\pi/k_j$), in turn all of the higher harmonics resonate with the bulk particles, where they undergo strong Landau damping. While some studies have shown that strongly Landau damped modes can be non-linearly unstable [25, 26], here we assume that this is not possible due to small initial mode amplitude; this allows us to justify the lack of three-wave coupling in the model.

The timestep was picked to be small enough to allow for a reasonable frequency analysis without becoming too computationally expensive. At $\Delta t = 0.1$, using a window size of 2000 timesteps, we obtain a frequency resolution $\Delta \omega = \pi/100$ provided that $\delta \omega < \Delta \omega$ across the timeframe of the bin. Furthermore, the Fourier spectrum of $E_{\text{N}}$ should be dominated by structures in the region $\omega \gg \omega_p$.

It is also a requirement for the code to adhere to the Courant–Friedrichs–Lewy (CFL) [27] limit for spatial advection in s-space via the piecewise parabolic method (PPM) [28] routine; the number of $v$-points $N_v$ and the domain of $v$ adheres to the CFL limit.

Noise in the system is provided by using a pseudorandom number generator (PRNG) with a given seed value; by using such a method of noise generation, the results are reproducible.

3.3. Benchmarking

Energy is not conserved in these simulations; the Fokker–Planck collisions heat the fast ion distribution, always aiming to restore the energy content in the distribution function to $U_0$:

$$
U_0 \equiv \frac{1}{2} \int_{-\infty}^{\infty} F_0 v^2 dv = \frac{1}{2} \left[ \eta + (1 - \eta) (v_b^2 + v_t^2) \right].
$$

(16)

For the purpose of benchmarking, we use a very simple model for symmetric mode flattening with a local population transfer $(f_0 - F_0) \sim -(\nu - \nu_0) \exp[-(\nu - \nu_0)^2]$ near resonance. This yields the corresponding energy flux from collisions in the weakly non-linear regime:

$$
\hat{U}_{\text{coll}} \equiv \frac{1}{2} \int \hat{\sigma}_s^2 (f_0 - F_0) v^2 dv \sim -\nu_0 \nu. 
$$

(17)
We test the energy conservation by examining a simulation with the same parameters employed in section 4, at $\alpha_j = 0.6$. As is shown in figure 4, the total energy content in the system is roughly constant, however we observe a small discrepancy in the Figure 5. We believe that this discrepancy is due to approximations made regarding higher harmonics of $f_{\text{ion}}$ and $E_j$, the total energy injected into the system via $\partial_i \alpha_j$ increases sharply. One expects this energy loss to be equal to that lost to the mode $E_j$, and damping $\alpha_j$; the total energy injected into the system via $\nu$ at this time is very small.

Once the distribution function has suitably relaxed, the deficit in the energy content should asymptote to that lost via damping $\alpha_j$, however we observe a small discrepancy in the energy content (roughly 0.45% of the total energy content). We believe that this discrepancy is due to approximations made regarding higher harmonics of $f_{\text{ion}}$ and $E_j$, and deem this discrepancy to be negligible for the single burst simulations examined here.

4. Stochastic lifetime of hole and clump

Here, we consider a system with noise only in the electric field; that is $N_j = 0$. We consider no particles in the turbulent population $f_{\text{tur}}$, such that the simulations reduce to single species.

The electric field is only seeded by noise; $D_k = 10^{-7}$ was used, with $\epsilon_{E_j} = 0$. A set of 2500 simulations were employed, allowing for 50 varying values of $t_{\text{NL}} = (\gamma_j \Omega_j^2) / (|\omega_j^0|)^2 \nu$, each tested for 50 different initial seeds of the PRNG. $k$ and $\nu$ were fixed to $0.150$ and $10^{-7}$ respectively, with 50 values of $\alpha_j$ on the interval $[0.06,0.158]$. The low value of $\nu$ justifies a relatively large timestep of $\Delta t = 0.1$.

4.1. Burst characterisation

To characterise the behaviour of a single bursting event, a set of simulations were used to produce data for the length of four temporal regions: lag, growth, plateau, and decay. Each of these regions are labelled in figure 5 for a sketch burst.

For an overall burst time $t_b = t_l + t_g + t_p + t_d$, the constituent times can fluctuate (functional dependences determined from simulation). The theoretical maximum and minimum amplitudes were used to create a fit routine, allowing one to acquire from the mode amplitude $|E_j|$ the constituent times as a function of the parameters $k_j$, $\alpha_j$, and $\nu$. We find that these times are not deterministic, but instead are stochastic:

\[
\begin{align*}
\text{lag} : & \quad t_l = t_l^{(0)} + \delta t_l \\
\text{growth} : & \quad t_g = t_g^{(0)} \\
\text{plateau} : & \quad t_p = t_p^{(0)} + \delta t_p \\
\text{decay} : & \quad t_d = t_d^{(0)} + \delta t_d.
\end{align*}
\]

where $\{\delta t_x\}$ denote stochastic terms. The fluctuation of each of the times has a well defined mean and standard deviation, and are analysed such that the mean is 0; here, we find that the lag, plateau, and decay times are highly stochastic. We find that at small $t_{\text{NL}}$ the mean plateau time $t_p^{(0)}$ is roughly constant, while the mean decay time $t_d^{(0)}$ is a linear function of $t_{\text{NL}}$, while the mean lag time $t_l^{(0)}$ is a non-linear function of $t_{\text{NL}}$.

4.2. Linear phase

When a single mode bursts, the electric field grows linearly via the bump-on-tail instability, provided that the mode lies on the positive slope of the ‘bump’. For $\nu = E_S = E_N = 0$, the frequency and growth rate in the linear phase are then determined by [29]:

\[
p + \alpha_j \frac{\Omega_j}{2} = \int \frac{v \partial_v F_0}{p + i k_j \Omega_j} dv,
\]

where $\Omega$ is the suitable Landau contour for the problem, and $p \equiv \gamma_j - i \omega_j^{(0)}$, such that the overall linear growth rate for the $j$th mode is given by:

\[
\gamma_j(k_j, \alpha_j) = \gamma_j(k_j, \alpha_j) - \frac{\alpha_j}{2}.
\]
where the unperturbed linear growth rate $\gamma_{jL}$ is equivalent to $\gamma_j$ in the absence of dissipation.

The frequency $\omega_j^{(0)} = \omega_j(t = 0)$ is the initial eigenfrequency of the $j$th mode. If one solves equation (9d) for negligible current, on average:

$$|\delta E_j|_{\text{min}} \approx \frac{D_j}{\alpha_j/2} \left[ 1 - e^{-\alpha_j \Delta t/2} \right] + O \left( e^{-\left(\alpha_j \Delta t\right)^2} \right). \quad (21)$$

The simulated noise is static over a timestep, leading to an error which manifests as the term $\sim \exp(-\alpha_j \Delta t/2)$.

One can interpret this physically as a finite bandwidth for the noise; we expect ITG (ion temperature gradient) turbulence spectral frequencies to typically be much slower than the plasma frequency \cite{30}, however here we examine noise with a frequency spectrum that is typically above the plasma frequency, corresponding to high frequency turbulence. The peak value for the electric field is the non-linear saturation point, which can be approximated by the following value \cite{5}:

$$|E_j|_{\text{max}} \approx (\gamma_{jL})^2. \quad (22)$$

Accordingly, as we expect exponential growth in the linear phase, the total time spent in the linear phase is given by:

$$(t_g^{(0)})_{\text{theory}} = \frac{2}{\gamma_j} \log \left[ \frac{\alpha_j \gamma_{jL}}{2D_j \left[ 1 - \exp(-\alpha_j \Delta t/2) \right]} \right]. \quad (23)$$

Simulations were found to strongly agree with this value; we find $t_g^{(0)} = (-91.0 \pm 8.1) + (1.15 \pm 0.01) \cdot (t_g^{(0)})_{\text{theory}}.$ The quantity $t_g$ does not appear to be stochastic; fluctuations in the value of $t_g$ as a function of PRNG seed are typically around $2$ or $3$ orders of magnitude lower than the mean value $\bar{t}_g^{(0)}.$ One finds that the accuracy improves at low $\alpha_j$; we find that this is in accordance with theory, as our value for $(t_g^{(0)})_{\text{theory}}$ assumes slow damping (small $\alpha_j \Delta t$).

At high $\alpha_j$, the expected growth time grows logarithmically until $\gamma_{jL}$ dominates:

$$\lim_{\alpha_j \to \infty} (t_g^{(0)})_{\text{theory}} = \frac{2}{\gamma_j} \left( \log \left[ \frac{\gamma_{jL}}{2D_j} \right] + \log \alpha_j \right).$$

Interestingly, one finds that even if the linear growth rate is non-zero, the seed electric field can prevent the mode from growing. This can be shown by setting the growth time to zero and solving for $\gamma_{jL}$:

$$(\gamma_j)_{\text{min}} \approx \frac{2D_j}{\alpha_j} - \frac{\alpha_j}{2}. \quad (24)$$

If the linear growth rate is below this minimum, the seed electric field quenches the mode before it has a chance to burst; in order to preserve the true meaning of the linear growth rate, one should impose a limit on $D_j$ when using high $\alpha_j$:

$$D_j \leq \frac{\alpha_j^2}{4}. \quad (25)$$

This hard limit on the noise level allows one to properly investigate simulations close to the linear stability boundary.

4.3. Non-linear phase

Once the mode reaches the non-linear saturation point, resonance broadening occurs, flattening the background distribution function $f_{0,0}$ in the close vicinity of the resonant phase velocity $\omega_j^{(0)}/k_j$. If the mode is marginally unstable, to the extent that $\var{5}$:

$$\frac{\alpha_j}{2} > 0.2\gamma_{jL}. \quad (26)$$

a phase-space bifurcation in the form of a hole and clump form $f_{0,0}.$

For diffusive Fokker–Planck collisions, the time spent in the plateau and decay regions has previously been shown to be a function of the timescale $t_{\text{NL}}.$

We take first order Taylor expansions in $t_{\text{NL}}$ as follows:

$$t_x^{(0)} = a_x + b_x t_{\text{NL}} + O(t_{\text{NL}}^2) = \frac{1}{50} \sum_{\text{seed}} t_x. \quad (27)$$

where we sum over $50$ PRNG seeds. From the simulations, $t_x^{(0)}$ appears to be a non-linear function of $t_{\text{NL}},$ while $t_{d}^{(0)}$ appears to be linear.

The mean plateau time $t_p^{(0)}$ appears to be constant at low $t_{\text{NL}}$; however at high $t_{\text{NL}},$ large error in the linear fit reduces our ability to determine the mean time.

We find that $a_d = (2.56 \pm 0.01) \times 10^2$, $b_d = (3.62 \pm 0.35) \times 10^{-2},$ and $a_p = (1.41 \pm 0.04) \times 10^5.$ One should note that the errors here are errors in the linear fit to mean values; they represent confidence in the functional dependence on $t_{\text{NL}},$ not the stochasticity. We find that $O(b_p) = 10^{-3},$ allowing us to state $t_p^{(0)} \approx a_p$ for $t_{\text{NL}} \ll 10^6.$

4.4. Burst stochasticity

We once again take first order Taylor expansions in $t_{\text{NL}},$ but now examining the standard deviations in $t_x$:

$$\sigma_X = c_X + \sigma_X t_{\text{NL}} + O(t_{\text{NL}}^2) = \frac{1}{50} \sum_{\text{seed}} \delta t_x. \quad (28)$$

where we once again sum over $50$ PRNG seeds. We find that generally, $\sigma_X$ does not appear to be a function of $t_{\text{NL}}.$ The lag time $t_l$ is strongly stochastic, with $\sigma_l/t_l^{(0)} \sim 10^0.$ This is in accordance with theory, as at very low amplitude, $\delta_t |E|$ is strongly dependent on the noise term, which is stochastic.

The plateau exists while the hole and clump have a static population of particles, and therefore, once the phase-space structures dissipate, the mode drops significantly in amplitude. The time spent in this region, $t_p,$ is stochastic; as is shown in figure 6(a), the relative fluctuation $\sigma_p/t_p^{(0)} \sim 10^{-2}.$ We conclude that the point at which this occurs is stochastic, leading to a stochastic life time of the hole and clump.

The growth and decay times $t_g$ and $t_d$ are defined by the minimum and maximum mode amplitude. Therefore, any stochastic behaviour reflects fluctuation in the growth rate and decay rate of the mode. We find that $\sigma_g/t_g$ is negligible,
implying that mode growth is not stochastic, as one might expect. However, we find that $\sigma_d/t_d^{(0)} \sim 10^{-1}$, implying a large fluctuation in the decay rate of the mode (see figure 6(b)).

5. Stochastic suppression of hole and clump

Here, we consider a case with a wholly non-stochastic electric field ($\epsilon_N = 0$), and examined the effect of a stochastic distribution function $f_{\text{stoch}}$ on mode chirping.

One finds that in cases of high collisionality, we must enforce $\nu \lesssim \Delta \tau^2 / \Delta t$ to avoid numerical inaccuracies where collisions dissipate structures much faster than the timestep. We fix the linear growth rate to $\gamma_j = 0.0534$, to reduce the parameter space while still allowing hole and clump formation (see equation (24)). We fix $D$ at $10^{-7}$ and $1/(1 + \epsilon_{\text{eff}}) = 98\%$ to investigate a small electric field drive and a small turbulent population. We define the relative stochasticity as $R_j \equiv \log_{10}(\sigma_{f_j}/\sigma_{f_j}^{\text{max}})$.

In theory, $f_{\text{stoch}}$ can affect $\delta E$ via equation (9d). As $(W \cdot f_{\text{stoch}}) \in 1$, one can see that real stochastic noise will produce an imaginary stochastic term in equation (9d), which will lead to a scrambling of the phase of coherent structures with wavenumber $k_j$. Consequently, the coherence of hole and clump structures can be destroyed, allowing them to rapidly dissipate.

In contrast with $E_N$, as $f_{\text{stoch}}$ exchanges no energy on pseudorandom time average with $E$, it can instantaneously create perturbations in $E$ which cannot be induced by finite $E_N$.

First, we show 3 simulations with varying collisionality $\log_{10} \nu = \{-2, -5, -6\}$, and no noise ($R_j = 0$). A timestep of $\Delta t = 0.01$ to allow us safe exploration of $\nu \sim 10^{-2}$. As is shown in figure 7, high collisionality ($\nu = 10^{-2}$) suppresses mode chirping, medium collisionality ($\nu = 10^{-5}$) allows for repeated bursting, and low collisionality ($\nu = 10^{-6}$) allows for only a single event.

Next, we highlight the effect of $\sigma_{f_j}$ on the asymptotic behaviour of the mode in figures 7 and 8; we show results from 3 simulations with $\nu = 10^{-5}$ using a coarser timestep of $\Delta t = 0.1$, and $R_j = \{-1, -2, -4\}$. For low stochasticity ($R_j = -4$), the effective collisionality increases; we still observe repeated bursting, however the period between repeated bursts is characteristic of simulations with $\nu \sim 10^{-4}$. Here, $f_{\text{stoch}}$ affects the stability of $f_{\text{sat}}$. If not make the fast-ion population stable; as is shown in figure 8(a), the initial hole and clump undisturbed, and repeated bursts still occur.

For medium stochasticity ($R_j = -2$), we find that the repeated bursting are suppressed. We give two equivalent explanations: the electric field produced by $f_{\text{stoch}}$ approaches the maximum amplitude of the repeated bursts, saturating them. As is shown in figure 8(b), the initial hole and clump still exists, but repeated bursts do not occur.

As we increase to high stochasticity ($R_j = -1$), we find that the initial burst is suppressed. We find that this is when the electric field produced by $f_{\text{stoch}}$ has an amplitude close to the non-linear saturation point; at this point, the mode does not resonate, even though it is unstable. Alternatively, $f_{\text{stoch}}$ prevents the mode from bursting by increasing the effective collisionality; as is shown in figure 8(c), the electric field produced by $f_{\text{stoch}}$ creates features on the spatially averaged distribution $f_0(\tau) = f_{\text{ion}}(\tau, t = 1200)$ which affect the hole and clump.

6. Conclusions and future work

In the case of weak stochasticity, we conclude that mode chirping is not wholly deterministic; the shape of the burst in mode amplitude can be determined on average analytically, but varies depending on the noise seed employed. We hope that this theory will lead to further analytical work, allowing one to freely predict the shape of a burst as a function of the whole parameter space: it is reasonable to assume that the empirical coefficients $\{a_j\}$ and $\{b_j\}$ are dependent on the shape of $F_0$.

Further work expanding this theory to include the time between bursts could lead to predictions of the burst frequency between Alfvén mode chirping events in tokamaks, which would allow for a greater understanding of fast ion loss. We
plan to couple predictions about the shape of a given burst to analytic theory by Berk et al [5] to create an analytical theory of hole-and-clump destabilisation in future work.

We conclude that the lifetime of a hole and clump is stochastic, and that the decay rate of the mode is also stochastic. Again, we hope that the accuracy of analysis for $t_p$ and $t_d$ improves with further work; in reality, the plateau region has a slight negative slope. It is our belief that an upgraded model with a negative gradient for $t_p$ would yield greater accuracy on the non-linear dependence of $t_p$ on $t_{NL}$.

We find that increasing the stochasticity in the system is initially equivalent to increasing the effective diffusive

![Figure 7](image1.png)

Figure 7. Plots of log $|E_j|(t)$ from linearly unstable simulations in section 4. Asymptotic behaviours for a single mode are shown for deterministic cases with varying collisionality $\nu$ (top plots), and stochastic cases with fixed collisionality $\nu = 10^{-5}$ (bottom plots). (a): Low effective collisionality: the mode amplitude $|E_j|$ undergoes a single bursting event at $t \approx 0$, corresponding to mode chirping. Top plot shows $\nu = 10^{-6}$, $R_j = 0$; bottom plot shows $\nu = 10^{-5}$, $R_j = -2$. (b): Medium effective collisionality: $|E_j|$ undergoes repeated bursting events ($t \approx \{0, 22000, 24500, \ldots\}$), each corresponding to mode chirping. Top plot shows $\nu = 10^{-5}$, $R_j = 0$; bottom plot shows $\nu = 10^{-5}$, $R_j = -4$. (c): High effective collisionality: the system does not undergo mode chirping. Top plot shows $\nu = 10^{-2}$, $R_j = 0$; bottom plot shows $\nu = 10^{-5}$, $R_j = -1$.

![Figure 8](image2.png)

Figure 8. Plots of $\omega_j(t)$ and $f_0(v) = f_{\text{sonic}}(v, t = 1200)$ (top plots and bottom plots respectively) from simulations in section 5 with $\nu = 10^{-5}$; the relative stochasticity $R_j$ is varied between simulations. Black dotted lines in the bottom plots show the existence (or lack of) hole and clump at $t = 1200$. (a): $R_j = -4$; noise in $f_{\text{fus}}$ produces an electric field, but repeated bursting still occurs. (b): $R_j = -2$; noise in $f_{\text{fus}}$ produces an electric field which prevents repeated bursts from occurring, but does not prevent the mode from initially chirping. (c): $R_j = -1$; noise in $f_{\text{fus}}$ produces an electric field which prevents a hole and clump from forming; the system is non-linearly stable.
collisionality. This is as one might expect from theory; stochasticity in the turbulent distribution function or electric field affects the damping term $\alpha E$, which is analogous to the energetic particle drive $df/d\phi_\nu$. Accordingly, simulations with increased stochasticity produce similar overall results to theory based on stochastic perturbations to momentum scattering via microturbulence-induced radial diffusion [31]. However, an important nuance appears when considering repeated bursting; low amplitude repeated bursts are saturated by the electric field produced by $f_{\text{sat}}$, leading to an asymptotic behaviour characteristic of a decrease in the effective collisionality.

As a result, we posit that in a given regime, an increase in micro-turbulence leads to an anomalous decrease in the effective collisionality; it is implied that in this regime, micro-turbulence reduces the ability for the distribution function to reconstitute via pitch-angle scattering.

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**Appendix A. Resonant damping**

The classical Lagrangian and Hamiltonian densities of the electromagnetic field are given by [32]:

$$\mathcal{L} = -\frac{F^\alpha{}^\beta F_{\alpha}{}^\beta}{4\mu_0} - A_{\alpha}J^\alpha$$

$$\mathcal{H}_0 = \Pi^{\alpha\beta} \partial_\beta A_\alpha - \mathcal{L}_0$$  \hspace{1cm} (A.1)

where $F^{\alpha\beta}$ is the electromagnetic force tensor, $\mu_0$ is the permeability of free space, $J^\alpha$ is the four-current. $A_{\alpha}$ and $\Pi^{\alpha\beta}$ are the four-potential and conjugate tensor:

$$A_{\alpha} = (\phi/c, \mathbf{A}) \hspace{1cm} \Pi^{\alpha\beta} = \frac{\partial \mathcal{E}}{\partial (\partial^\beta A_\alpha)}$$  \hspace{1cm} (A.2)

where $\phi$ is the electric scalar potential, and $\mathbf{A}$ is the magnetic vector potential.

**A.1. Augmentation tensor, $G^{\alpha\beta}$**

Let us define $\mathcal{L}(A_\alpha, \partial_\beta A_\alpha) = \mathcal{L}_0 + \delta \mathcal{L}$. Then, $\mathcal{H}(A_\alpha, \Pi^{\alpha\beta})$ is given by the appropriate Legendre transformation:

$$\mathcal{H} = \Pi^{\alpha\beta} \partial_\beta A_\alpha - \mathcal{L}$$

$$= \left[ \frac{\partial \mathcal{L}_0}{\partial (\partial_\beta A_\alpha)} \right] \partial_\beta A_\alpha - L_0 + \left[ \frac{\partial (\delta \mathcal{L})}{\partial (\partial_\beta A_\alpha)} \right] \partial_\beta A_\alpha - \delta \mathcal{L}.$$

where we define $\mathcal{H}_0 : \delta \mathcal{L} = 0$. We seek the perturbation $\delta \mathcal{H} = 0$ so as to preserve the canonical form of the Hamiltonian. Therefore:

$$\delta \mathcal{L} = \frac{\partial (\delta \mathcal{L})}{\partial (\partial_\beta A_\alpha)} \partial_\beta A_\alpha.$$

This trivial partial differential equation solves to give:

$$\delta \mathcal{L} = G^{\alpha\beta} (A_\mu) \cdot \partial_\beta A_\alpha,$$

where the augmentation tensor $G^{\alpha\beta} (A_\mu)$ preserves the Lorentz invariance of the Lagrangian density, but is only a function of the four-potential.

**A.2. Canonical form**

From $\mathcal{L} = \mathcal{L}_0 + \delta \mathcal{L}$, the augmented Maxwell’s equations for the system are given by the generalized Euler-Lagrange equations:

$$\partial_\beta \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\beta A_\alpha)} \right] - \frac{\partial \mathcal{L}}{\partial A_\alpha} = 0.$$

(A.3)

By examining a 1D Cartesian space with no $B$-field, we seek an augmentation tensor that satisfies:

$$-\alpha E = \frac{1}{\epsilon_0} \left[ A_\mu \frac{\partial G^{0\mu}}{\partial A_x} + \partial_\mu A_\alpha \frac{\partial G^{1\alpha}}{\partial A_x} - (G^{01} + G^{11}) \right].$$

(A.4)

Doing so allows the augmentation to the Gauss–Ampere law *a posteriori* to manifest as a Berk–Breizman sink of energy via a global dissipation channel [5, 13]. This is non-trivially satisfied, but the simplest case is when $\partial G^{0\mu}/\partial A = 0$, and:

$$G^{01} + G^{11} = -\alpha \epsilon_0 (\partial_\phi \mathcal{E} + \mathcal{A}).$$  \hspace{1cm} (A.5)

One can also show that further constraints on the augmentation tensor allow Gauss’ law to retain the exact same form as before. Therefore, this family of augmentation tensors produce the modified Maxwell–Ampere law and preserve the canonical form of the energy density:

$$\partial_\mu E = -\frac{1}{\epsilon_0} - \alpha E \hspace{1cm} U = \frac{1}{\epsilon_0} \int_{-\infty}^{\infty} E^2 \, dx.$$  \hspace{1cm} (A.6)

The Lorentz force on charged particles due to the electromagnetic field is given by the particle Lagrangian [32]. If we examine a single particle:

$$L_p = \frac{1}{2} m u_{\mu} u^\mu + q u^\mu A_\mu.$$

(A.7)

where $m$ and $q$ are the particle mass and charge, and $u^\mu$ is the four-velocity. One can show that if we constrain the definition for the four-potential to be invariant under the augmentation, then:
\[ m \frac{d^2 \alpha}{dt^2} = -q \mu_0 \Pi_{\alpha \beta} \frac{d\alpha}{dt}. \]  

(A.8)

The augmentation does perturb \( \Pi_{\alpha \beta} \), however the force does not work; it is in fact a fictitious force, and therefore can be omitted. Finally, we omit the spatially averaged current to avoid a build up of loop voltage; one can show that if we take the spatially averaged part of the Maxwell–Ampere law:

\[- \int \left( f_{\text{ion},0} + f_{\text{ae},0} \right) dv = \partial_t E_0 + \alpha_0 E_0.\]

We require that the mean current is very small, and is dominated by exponential decay. In such a case, we find that the spatially averaged electric field \( E_0(t) \) must be temporally evanescent:

\[ E_0(t) \approx E_0(t=0)e^{-\alpha_0 t}. \]

This in turn allows us to remove the spatially averaged electric field by setting \( E_0(t=0) = 0 \) as a boundary condition.

## Appendix B. Seed electric field

If one chooses the following form for the seed contribution:

\[ S_j \equiv -\frac{1}{2} \sum \{ A_{js} e^{-i\omega_s t} + \text{c.c.} \}, \]

where \( \{ A_{js} \} \in \mathbb{C} \), one finds the following partial differential equation:

\[ \partial_t E_{S_j} + \frac{1}{2} \alpha_j E_{S_j} = \frac{1}{2} \sum \{ A_{js} e^{-i\omega_s t} + \text{c.c.} \}. \]

We define Laplace forwards and backwards transforms by:

\[ \hat{f}(p) \equiv \int_0^\infty f(t) e^{-pt} dt ; f(t) \equiv \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{\sigma-i\tau}^{\sigma+i\tau} \hat{f}(p) e^{pt} dp. \]

This allows one to find that under forward Laplace transforms:

\[ \hat{E}_{S_j} = \frac{1}{p + \alpha_j/2} \left\{ E_{S_j}(t=0) + \sum \{ \frac{A_{js}}{p + \omega_s} + \frac{A_{js}'}{p - \omega_s} \} \right\}. \]

Formally, for convergence of the forward transform:

\[ \exists \sigma < \text{Re}(p) : \lim_{t \to \infty} |f(t)| = e^{\sigma t}. \]

Therefore, if the traditional Bromwich contour is shifted to examine a line integral along \( \text{Re}(p) < \sigma \), we can examine singularities in \( \hat{E}_{S_j} \); residues of these singularities allow us to recover the solution for the electric field via the residue theorem. By examining \( \text{Re}(p) \to -\infty \), we find that the only remaining contribution to the backwards transform is the residues, as the rest of the integral becomes exponentially small:

\[ E_{S_j} \approx 2\pi i \cdot \sum \text{Res}(\hat{E}_{S_j}, e^{p_i}, p_i) \]

where \( p_i \) are the locations of the singularities for \( \text{Re}(p) < \sigma \). When calculated, this yields:

\[ E_{S_j}(x,t) = E_S^{(\text{ev})}(x)e^{-\alpha_j t/2} + \frac{1}{2} \sum \{ E_{S_{j\mu}}(x)e^{-i\omega_{j\mu} t} + \text{c.c.} \}. \]

where \( E_S^{(\text{ev})} \) refers to non-propagating evanescent modes given by the simple pole at \( p = -\alpha_j/2 \). We can deny these modes from existing by setting \( E_S^{(\text{ev})} = 0 \) as a boundary condition.

\[ E_{S_{j\mu}}(x) = \frac{A_{js}}{\alpha_j/2 - i\omega_s}. \]

If we select \( A_s \in \mathbb{R} \), we find that this reduces to the form:

\[ E_S(x,t) = \sum \frac{A_s}{\alpha_j^2/4 + \omega_s^2} \cos(\omega_s t). \]

If one solves the modified Maxwell–Ampere law when there is very little change to the distribution function (negligible instability drive) and no noise, we find the solution:

\[ \delta E(t) = -\epsilon E_S(x,t). \]

As can be seen, for these modes there is no net electric field overall; the seed mode and the perturbation are counterpropagating.

We then consider the addition of a noise term \( E_N \). By similar analysis, one can show that the noise is representable as a distinct set of frequencies, \( \{ \omega_b \} \). However, in the limit that \( \{ \omega_b \} \) form a continuum, we find that we can represent the noise term contribution in the form:

\[ \lim_{\delta \omega_b \to 0} \frac{1}{2} \sum \{ A_{bs} e^{-i\omega_b t} + \text{c.c.} \} \equiv -N_E \]

where \( N_E \) is a pseudorandom noise term that seeds instabilities. We can once again ignore evanescent effects via boundary conditions, leaving only the propagating contribution. Again, one finds from a similar analysis that the noise term leads to no net perturbation of \( E \), in accordance with the conservation of energy.

However, one can show that by solving equation (9d) for negligible current:

\[ \delta E_j(t + \Delta t) \approx \frac{D_j}{\alpha_j/2} \left[ 1 - \exp \left( -\frac{1}{2} \alpha_j \Delta t \right) \right] + O(\Delta t^2). \]

The numerical flaw associated with timestep size is what gives the initial drive; clearly, the limit as \( \Delta t \to 0 \) yields \( \delta E_j = 0 \).

## Appendix C. Computational method

### C.1. Strang splitting

Formally, the system’s equations of motion are defined by a rank two tensor of differential operators \( \tilde{Z}_p^\mu \), acting in the phase-space \( (\mathbf{x}, \mathbf{v}) \) on the state vector \( F^\nu \):

\[ \partial_t F^\nu = \tilde{Z}_p^\mu F^\nu. \]

(C.1)

The flow operator tensor \( \Phi_p^\mu \) yields the trajectory of the system:
\[ F^\mu(x, v, t + \Delta t) = \Phi_\mu^\nu(\Delta t) \circ F^\nu(x, v, t). \]

We then suppose that \( Z^n_\mu \) is representable as a linear sum of \( n \) operators. In such a case:

\[ Z^n_\mu = \sum_{j=1}^{n} (Z_j)_\mu. \]

Therefore, via the Baker–Campbell–Hausdorff formula [33], one finds that as for two matrices \( A \) and \( B \):

\[ e^{(A+B)\Delta t} = e^{A\Delta t} e^{B\Delta t} + O(\Delta t) \]

the overall flow can be split into \( n \) partial flows via the symmetric Strang splitting method [34, 35]:

\[ F^\mu(x, v, t + \Delta t) = \left[ \prod_{j=1}^{n} (\Phi_j)_\mu^\nu(\Delta t/2) \circ \right] F^\nu(x, v, t) + O(\Delta t^2). \]

Accordingly, we first split the problem by evolving one of the functions in the state vector while keeping the rest constant. Secondly, we split the flow for this state function into partial flows. Then each partial flow is solved numerically in forward flow order and then reverse flow order, for a half a timestep each.

For example, in the \( x - v \) code, we utilise a state vector of \( (f, E) \), keeping the electric field constant while the distribution function is evolved, and vice versa. The partial flows examine spatial advection, velocity advection, collisions, and electric field evolution.

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