Stronger Lower Bounds for Online ORAM*

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Abstract

Oblivious RAM (ORAM), introduced in the context of software protection by Goldreich and Ostrovsky [JACM’96], aims at obfuscating the memory access pattern induced by a RAM computation. Ideally, the memory access pattern of an ORAM should be oblivious of the data being processed. Since the work of Goldreich and Ostrovsky, it was believed that there is an inherent $\Omega(\log n)$ bandwidth overhead in any ORAM working with memory of size $n$. Larsen and Nielsen [CRYPTO’18] were the first to give a general $\Omega(\log n)$ lower bound for any online ORAM, i.e., an ORAM that must process its inputs in an online manner.

In this work, we revisit the lower bound of Nielsen and Larsen, which was proved under an assumption about the format of the memory access pattern of the ORAM. We give an $\Omega(\log n)$ lower bound for the bandwidth overhead of any online ORAM without any such restriction. Our results thus match the model of Boyle and Naor [ITCS’16] who proved that any super-constant lower bound for offline ORAM, i.e., an ORAM that can process its inputs simultaneously, implies super-linear lower bounds on size of sorting circuits – which would constitute a major breakthrough in computational complexity.

As our main technical contribution and to handle the lack of structure, we study the properties of access graph induced naturally by the memory access pattern of an ORAM computation. We identify a particular graph property that can be efficiently tested and that all access graphs of ORAM computation must satisfy with high probability. This property is reminiscent of the Larsen-Nielsen property but it is substantially less structured; that is, it is more generic.

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1 Introduction

Oblivious simulation of RAM machines, initially studied in the context of software protection by Goldreich and Ostrovsky [GO96], aims at protecting the memory access pattern induced by computation of a RAM. In the present day, such oblivious simulation might be needed when performing a computation in the memory of an untrusted server. Despite using encryption for protecting the contents of each memory cell, the memory access pattern might still leak sensitive information. Thus, the memory access pattern should be oblivious of the data being processed and, optimally, depend only on the size of the input.

Constructions. The strong guarantee of obliviousness of the memory access pattern comes at the cost of additional overhead. A trivial solution which scans the whole memory for each memory access induces linear bandwidth overhead, i.e., the multiplicative factor by which the length of a memory access pattern increases in the oblivious simulation of a RAM with \( n \) memory cells. Given the practical applications, an important research direction is to construct an ORAM with as low overhead as possible. The foundational work of Goldreich and Ostrovsky [GO96] already gave a construction with bandwidth overhead \( O(\log^3(n)) \). Subsequent results introduced various improved approaches for building ORAMs (see [Ajt10, CLP14, CP13, DMN11, GGH+13, GO96, GM11, GMOT11, KLO12, PPRY18, RFK+14, SvDS+18, WCS15, WHC+14] and the references therein) leading to the recent construction of Asharov et al. [AKL+18] with bandwidth overhead \( O(\log n) \) for the most natural setting of parameters.

Lower-bounds. It was a folklore belief that an \( \Omega(\log n) \) bandwidth overhead is inherent based on a lower bound presented already in the initial work of Goldreich and Ostrovsky [GO96]. However, the Goldreich-Ostrovsky result was recently revisited in the work of Boyle and Naor [BN16], who pointed out that the lower bound actually holds only in a rather restricted “balls in bins” model where the ORAM is not allowed to read the contents of the data cells it processes. In fact, Boyle and Naor showed that any general lower bound for offline ORAM (i.e., where each memory access of the ORAM can depend on the whole sequence of operations it needs to obliviously simulate) implies non-trivial lower bounds on sizes of sorting circuits which seem to be out of reach of the known techniques in computational complexity.

The first general \( \Omega(\log n) \) lower bound for bandwidth overhead in online ORAM (i.e., where the ORAM must process the operations it has to obliviously simulate in a sequential manner) was subsequently given by Larsen and Nielsen [LN18]. The core of their lower bound was an adaptation of the information transfer technique of Patrascu and Demaine [PD06], originally used for proving lower bounds for data structures in the cell probe model, to the ORAM setting. In fact, the lower bound of Larsen and Nielsen [LN18] for ORAM can be cast as a lower bound for the oblivious Array Maintenance problem and it was recently extended to other oblivious data structures by Jacob et al. [JLN19].

1.1 Our Results

In this work, we further develop the information transfer technique of [PD06] when applied in the context of online ORAMs. We revisit the lower bound of Nielsen and Larsen, which was proved under an assumption about the format of the memory access pattern of the ORAM. Specifically, we prove a matching lower bound in a stronger model without any restriction on the format of the server memory access sequence.

Theorem 1.1 (Informal). Any online ORAM which satisfies the statistical security and has internal memory of size \( m \) must have expected overhead \( \Omega(\log n) \) where \( n \geq m^2 \) and \( n \) is the length of an input sequence of operations. This result holds even when the adversarial server has no information about boundaries between probes corresponding to different operations.

In the computational setting, our techniques give the following.

\footnote{Protecting the memory access of a computation is particularly relevant in the light of the recent Spectre [KGG+18] and Meltdown [LSG+18] attacks.}
Theorem 1.2 (Informal). Any online ORAM which satisfies the computational security and has internal memory of size $m$ must have expected overhead $\omega(1)$. This result holds even when the adversarial server has no information about boundaries between probes corresponding to different operations.

Note that this is still an interesting result. It follows from the work of Boyle and Naor [BN16] that any super constant lower bound for offline ORAM would imply super-linear lower bounds on size of sorting circuits – which would constitute a major breakthrough in computational complexity (for additional discussion, see Section 5).

As an additional contribution, we clarify the ORAM model in which our techniques yield a lower bound. See Definition 2.6 and Section 5 for additional discussions.

Besides online ORAM (i.e., the oblivious Array Maintenance problem), our techniques naturally extend to other oblivious data structures and allow to generalize also the recent lower bounds of Jacob et al. [JLN19] for oblivious stacks, queues, deques, priority queues and search trees.

1.2 Our Techniques

The structure of our proof follows a similar blueprint as the work of Larsen and Nielsen [LN18]. However, we must handle new issues introduced by the more general adversarial model. Most significantly, our proof cannot rely on any formatting of the access pattern, whereas Larsen and Nielsen could have leveraged the fact that the access pattern is split into blocks corresponding to each read/write operation. To handle the lack of structure in the access pattern, we study the properties of the access graph induced naturally by the access pattern of an ORAM computation. We identify a particular graph property that can be efficiently tested and that all access graphs of ORAM computation must satisfy with high probability. This property is reminiscent of the Larsen-Nielsen property but it is substantially less structured; that is, it is more generic.

The access graph is defined as follows: the vertices are timestamps of server probes and there is an edge connecting two vertices if and only if they correspond to two subsequent accesses to the same memory cell. We define a graph property called $\ell$-dense $k$-partition. Graphs with $\ell$-dense $k$-partitions are graphs which may be partitioned into $k$ disjoint subgraphs, each subgraph having at least $\ell$ edges. We show that this property has to be satisfied by access graphs with high probability for any $k$ and an appropriate $\ell$. In Section 3 we prove that if a graph has $\frac{\ell}{k}$-dense $k$-partition for some $\ell$ and $K$ different values of $k$ then the graph must have at least $\Omega(\ell \log K)$ edges.

In Section 4 we prove that access graphs of ORAMs have many dense partitions. Specifically, we show that for $\Omega(n)$ values of $k$, there exist input sequences for which the corresponding graph has $\frac{n}{k}$-dense $k$-partition by a communication-type argument. Applying the indistinguishability of sequences of probes made by ORAM, we get one sequence for which its access graph satisfies $\frac{n}{k}$-dense $k$-partition for $\Omega(n)$ values of $k$ with high probability. Combining the above results from Section 4 with the results from Section 3 we get that the graph of such a sequence has $\Omega(n \log n)$ edges, and thus by definition, $\Omega(n \log n)$ vertices in expectation. This implies that the expected number of probes made by the ORAM on any input sequence of length $n$ is $\Omega(n \log n)$.

2 Preliminaries

In this section, we introduce some basic notation and recall some standard definitions and results. Throughout the rest of the paper, we let $[n]$ for $n \in \mathbb{N}$ to denote the set $\{1, 2, \ldots, n\}$. A function $\text{negl}(n) : \mathbb{N} \to \mathbb{R}$ is negligible if it approaches zero faster than any inverse polynomial.

Definition 2.1 (Statistical Distance). For two probability distributions $X$ and $Y$ on a discrete universe $S$, we define statistical distance of $X$ and $Y$ as

$$\text{SD}(X, Y) = \frac{1}{2} \sum_{s \in S} |\Pr[X = s] - \Pr[Y = s]|.$$
We use the following observation, which characterizes statistical distance as the difference of areas under the curve (see Fact 3.1.9 in Vadhan [Vad99]).

**Proposition 2.2.** Let $X$ and $Y$ be probability distributions on a discrete universe $S$, let $S_X = \{ s \in S : \Pr[X = s] > \Pr[Y = s] \}$, and define $S_Y$ analogously. Then

$$SD(X, Y) = \Pr[X \in S_X] - \Pr[Y \in S_X] = \Pr[Y \in S_Y] - \Pr[X \in S_Y].$$

We also use the following data-processing-type inequality.

**Proposition 2.3.** Let $X$ and $Y$ be probability distributions on a discrete universe $S$. Then for any function $f: S \rightarrow \{0, 1\}$, it holds that $|\Pr[f(X) = 1] - \Pr[f(Y) = 1]| \leq SD(X, Y)$.

**Definition 2.4** (Computational indistinguishability). Two probability ensembles, $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$, are computationally indistinguishable if for every polynomial-time algorithm $D$ there exists a negligible function $\text{negl}(\cdot)$ such that

$$|\Pr[D(X_n, 1^n) = 1] - \Pr[D(Y_n, 1^n) = 1]| \leq \text{negl}(n).$$

### 2.1 Online ORAM

**Definition 2.5** (Array Maintenance Problem [LN13]). The Array Maintenance problem with parameters $(\ell, w)$ is to maintain an array $B$ of $\ell$ $w$-bit entries under the following two operations:

- $(W, a, d)$: Set the contents of $B[a]$ to $d$, where $a \in [\ell]$, $d \in \{0, 1\}^w$. (Write operation)
- $(R, a, d)$: Return the contents of $B[a]$, where $a \in [\ell]$ (note that $d$ is ignored). (Read operation)

We say that a machine implements the Array Maintenance problem with parameters $(\ell, w)$ and probability $p$, if for every input sequence of operations

$$y = (o_1, a_1, d_1), \ldots, (o_n, a_n, d_n),$$

where each $o_i \in \{R, W\}, a_i \in [\ell], d_i \in \{0, 1\}^w$,

and for every read operation in the sequence $y$, the machine returns the correct answer with probability at least $p$.

**Definition 2.6** (Online Oblivious RAM). For $m,w \in \mathbb{N}$, let $\text{RAM}^*(m, w)$ denote a probabilistic random access machine $M$ with $m$ cells of internal memory, each of size $w$ bits, which has access to a data structure, called server, implementing the Array Maintenance problem with parameters $(2^w, w)$ and probability 1. In other words, in each step of computation $M$ may probe the server on a triple $(o, a, d) \in \{R, W\} \times [2^w] \times \{0, 1\}^w$ and on input $(R, a, d)$, where $a \in [2^w], d \in \{0, 1\}^w$, the server returns to $M$ the data last written in $B[a]$. We say that $\text{RAM}^*$ probes the server whenever it makes an Array Maintenance operation to the server.

Let $m, w$ be any natural numbers such that $M \leq 2^w$. An online Oblivious RAM $M$ with address range $M$, cell size $w$ bits and $m$ cells of internal memory is a $\text{RAM}^*(m, w)$ satisfying online access sequence, correctness, and statistical (resp. computational) security as defined below.

**Online Access Sequence:** For any input sequence $y = y_1, \ldots, y_n$ the $\text{RAM}^*$ machine $M$ gets $y_i$ one by one, where each $y_i \in \{R, W\} \times [M] \times \{0, 1\}^w$. Upon the receipt of each operation $y_i$, the machine $M$ generates a possibly empty sequence of server probes $(o_1, a_1, d_1), \ldots, (o_{\ell_i}, a_{\ell_i}, d_{\ell_i})$, where each $(o_i, a_i, d_i) \in \{R, W\} \times [2^w] \times \{0, 1\}^w$, and updates its internal memory state in order to correctly implement the request $y_i$. We define the access sequence corresponding to $y_i$ as $A(M, y_i) = a_1, a_2, \ldots, a_{\ell_i}$. For the input sequence $y$, the access sequence $A(M, y)$ is defined as

$$A(M, y) = A(M, y_1), A(M, y_2), A(M, y_3), \ldots, A(M, y_n).$$

Note that the definition of the machine $M$ is online, and thus for each input sequence $y = y_1, \ldots, y_n$ and each $i \in [n - 1]$, the access sequence $A(M, y_i)$ does not depend on $y_{i+1}, \ldots, y_n$. 

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Correctness: $M$ implements the Array Maintenance problem with parameters $(M, w)$ with probability $1 - p_{\text{fail}}$.

Statistical Security: For any two input sequences $y, y'$ of the same length, the statistical distance of the distributions of access sequences $A(M, y)$ and $A(M, y')$ is at most $\frac{1}{4}$.

Computational Security: For any infinite families of input sequences $\{y_n\}_{n \in \mathbb{N}}$ and $\{y'_n\}_{n \in \mathbb{N}}$ such that $|y_n| = |y'_n| \geq n$ for all $n \in \mathbb{N}$, the probability ensembles $\{A(M, y_n)\}_{n \in \mathbb{N}}$ and $\{A(M, y'_n)\}_{n \in \mathbb{N}}$ are computationally indistinguishable.

As customary, in the computational security definition we consider infinite families of ORAM where we allow $m, M, w$ to be functions of $n$, the length of the input sequence. So the computational security is defined not for a single fixed choice of an ORAM machine but for an infinite family.

For ease of exposition, we assume perfect correctness of the ORAM (i.e., $p_{\text{fail}} = 0$). However, our lower bounds can be extended also to ORAM with imperfect correctness (see discussion after Lemma 4.2).

The parameters of the ORAM model from Definition 2.6 are depicted in Figure 2.1. We use different sizes of arrows on server and RAM side to denote the asymmetry of the communication ($\text{RAM}^*$ sends type of operation, address and data, and server returns just data in case of read operation and nothing in case of write). Note that the input sequence $y$ of ORAM consists of a sequence of all operations, whereas the access sequence $A(M, y)$ consists of a sequence of addresses of all probes.

3 Dense Graphs

In this section, we define an efficiently testable property of graphs that we show to be satisfied by graphs induced by the access pattern of any statistically secure ORAM. This property implies that the overhead of such ORAM must be logarithmic.

We say a directed graph $G = (V, E)$ is ordered if $V$ is a subset of integers and for each edge $(u, v) \in E$, $u < v$. For a graph $G = (V, E)$ and $S, T \subseteq V$, we let $E(S, T) \subseteq E$ be the set of edges that start in $S$ and end in $T$, and for integers $a \leq m \leq b \in V$ we let $E(a, m, b) = E\{a, a + 1, \ldots, m - 1\}, \{m, m + 1, \ldots, b - 1\}$.

Definition 3.1. A $k$-partition of an ordered graph $G = (V = \{0, 1, 2, \ldots, N - 1\}, E)$ is a sequence $0 = b_0 \leq m_0 \leq b_1 \leq m_1 \leq \cdots \leq b_k = N$. We say that the $k$-partition is $\ell$-dense if for each $i \in \{0, \ldots, k - 1\}$, $E(b_i, m_i, b_{i+1})$ is of size at least $\ell$. 

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Figure 1: Schema of online ORAM from Definition 2.6
There is a simple greedy algorithm running in time $O(|V|^2 \cdot |E|)$ which tests for given integers $k, \ell$ whether a given ordered graph $G = (V, E)$ has an $\ell$-dense $k$-partition. (The algorithm looks for the $k$ parts one by one greedily from left to right.)

**Lemma 3.3.** Let $K \subseteq \mathbb{N}$ be a subset of powers of 4. Let $\ell \in \mathbb{N}$ be given. Let $G = (\{0, \ldots, N-1\}, E)$ be an ordered graph which for each $k \in K$ has an $(\ell/k)$-dense $k$-partition. Then $G$ has at least $\frac{\ell}{2} \cdot |K|$ edges.

**Proof.** We use the following claim to bound the number of edges.

**Claim 3.3.** Let $k > k' > 0$ be integers. Let $0 = b_0 \leq m_0 \leq b_1 \leq \cdots \leq b_k = N$ be a $k$-partition of $G$, and $0 = b'_0 \leq m'_0 \leq b'_1 \leq \cdots \leq b'_k = N$ be a $k'$-partition of $G$. Then for at least $k-k'$ distinct $i \in \{0, \ldots, k-1\}$

$$E(b_i, m_i, b_{i+1}) \cap \bigcup_{j \in \{0, \ldots, k'-1\}} E(b'_j, m'_j, b'_{j+1}) = \emptyset.$$  \hspace{1cm} (1)

**Proof.** For any $j \in \{0, \ldots, k'-1\}$ and $(u, v) \in E(b'_j, m'_j, b'_{j+1})$, if $(u, v) \in E(b_i, m_i, b_{i+1})$ for some $i$ then $b_i < m'_j < b_{i+1}$ (as $b_i \leq v < m'_j < b_{i+1}$). Thus, $i$ is uniquely determined by $j$. Hence, $E(b_i, m_i, b_{i+1})$ may intersect $\bigcup_{j \in \{0, \ldots, k'-1\}} E(b'_j, m'_j, b'_{j+1})$ only if $b_i < m'_j < b_{i+1}$, for some $j \in \{0, \ldots, k'-1\}$. Thus, such an intersection occurs only for at most $k'$ different $i$. The claim follows.

Now we are ready to prove Lemma 3.2. For each $k \in K$, pick an $(\ell/k)$-dense $k$-partition $0 = b_0 \leq m_0 \leq b_1 \leq m_1 \leq \cdots \leq b_k = N$ of $G$ and define the set of edges $E_k$:

$$E_k = \bigcup_{i \in \{0, \ldots, k-1\}} E(b_i, m_i, b_{i+1}).$$

For each $k \in K$, we lower-bound $|E_k \setminus \bigcup_{k' \in K, k' \not\leq k} E_{k'}|$ by $\ell/2$. Since $K$ contains powers of 4, $\sum_{k' \in K, k' \not\leq k} k' \leq \ell/2$. By the above claim, for at least $k - \sum_{k' \in K, k' \not\leq k} k' \geq k'/2$ different $i \in \{0, \ldots, k-1\}$, $E(b_i, m_i, b_{i+1}) \cap \bigcup_{k' \in K, k' \not\leq k} E_{k'} = \emptyset$. By density, $|E(b_i, m_i, b_{i+1})| \geq \ell/k$, so $|E_k \setminus \bigcup_{k' \in K, k' \not\leq k} E_{k'}| \geq \frac{\ell}{2} \cdot \frac{k}{2} = \ell/2$. Hence, $|\bigcup_{k \in K} E_k| = \sum_{k \in K} |E_k \setminus \bigcup_{k' \in K, k' \not\leq k} E_{k'}| \geq |K| \cdot \frac{\ell}{2}$.

In the following corollary, we show that the property of having many dense partitions with some probability implies proportionally many edges. (Note that the $\lfloor \log_4 t \rfloor - \lceil \log_4 s \rceil$ term corresponds exactly to the number of powers of four between $t$ and $s$.)

**Corollary 3.4.** Let $\ell, s, t$ be natural numbers, where $s \leq t$. Let $p \in [0, 1]$ be a real. Let $G$ be an ordered graph picked at random from a distribution such that for each integer $k$, $s \leq k \leq t$, the randomly chosen ordered graph $G$ has an $(\ell/k)$-dense $k$-partition with probability at least $p$. Then the expected number of edges in $G$ is at least $\frac{\ell t}{2} \cdot (\lfloor \log_4 t \rfloor - \lceil \log_4 s \rceil)$.

**Proof.** Let $K$ be the set of integers such that $k \in K$ if and only if $k$ is a power of 4 and $G$ has an $(\ell/k)$-dense $k$-partition. $K$ is a random variable. The expected size of $K$ is at least $p(\lfloor \log_4 t \rfloor - \lceil \log_4 s \rceil)$. By Lemma 3.2 the expected number of edges in $G$ is at least $\frac{\ell}{2} \cdot \left(\lfloor \log_4 t \rfloor - \lceil \log_4 s \rceil \right)$.

## 4 ORAM Lower Bound

In this section, we fix integers $n, m, M, w \geq 1$ such that $m \leq \sqrt{n}$, $n \leq M \leq 2^w$, and an ORAM $\mathcal{M}$ with address range $M$, cell size $w$ and $m$ cells of internal memory (see Definition 2.6). We argue that any statistically secure ORAM $\mathcal{M}$ must make $\Omega(n \log n)$ server probes in expectation in order to implement a sequence of $n$ input operations. We also show that any computationally secure ORAM $\mathcal{M}$ must make $\omega(n)$ server probes in expectation on any input sequence of length $n$. 

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Definition 4.1. Let \( A(\mathcal{M}, y) = a_0, \ldots, a_{N-1} \) be an access sequence of \( \mathcal{M} \) for some input sequence \( y \). We define a directed graph \( G(A(\mathcal{M}, y)) = (V, E) \) called access graph as follows: \( V = \{0, \ldots, N-1\} \) and \( (i, j) \in E \) iff \( i < j \) and \( a_i = a_j \) and for each \( k \in \{i+1, \ldots, j-1\} \), \( a_k \neq a_i \).

Notice that every vertex of an access graph has an outdegree as well as indegree at most one.

In the following we consider input sequences of length \( n \in \mathbb{N} \) where \( n \) is even. First, we define \( Y_{n,0} = [(W,0,0^w), (R,0,0^w)]^{n/2} \), i.e., a sequence of alternating writes and reads at address 0 with data \( d = 0^w \).

Second, for each \( k \in \{1, 2, \ldots, \frac{n}{2}\} \), let \( \ell = \left\lfloor \frac{n}{2\ell} \right\rfloor \), we define a distribution \( Y_{n,k} \) of input sequences as

\[
Y_{n,k} = (W, 1, b_{1,1}), (W, 2, b_{1,2}), \ldots, (W, \ell, b_{1,\ell}), (R, 1, 0^w), (R, 2, 0^w), \ldots, (R, \ell, 0^w),
\]

\[
(W, 1, b_{2,1}), (W, 2, b_{2,2}), \ldots, (W, \ell, b_{2,\ell}), (R, 1, 0^w), (R, 2, 0^w), \ldots, (R, \ell, 0^w),
\]

\[
\ldots,
\]

\[
(W, 1, b_{k,1}), (W, 2, b_{k,2}), \ldots, (W, \ell, b_{k,\ell}), (R, 1, 0^w), (R, 2, 0^w), \ldots, (R, \ell, 0^w),
\]

\[
(W, 0, 0^w), (R, 0, 0^w), (W, 0, 0^w), \ldots, (R, 0, 0^w),
\]

where each \( b_{i,j} \in \{0,1\}^w \) is an independently uniformly chosen bit string. We define the \( i \)-th block of writes \( W_i = (W, 1, b_{1,1}), (W, 2, b_{1,2}), \ldots, (W, \ell, b_{1,\ell}) \) and the \( i \)-th block of reads \( R_i \) to be the sequence of operations \( (R, 1, 0^w), (R, 2, 0^w), \ldots, (R, \ell, 0^w) \) following right after \( W_i \). Note that after the \( k \)-th block of reads the sequence is padded by a sequence of alternating writes and reads to length \( n \). We use the notation \( G_{n,k} = G(A(\mathcal{M}, Y_{n,k})) \) and \( G_{n,0} = G(A(\mathcal{M}, Y_{n,0})) \), where \( \mathcal{M} \) is an ORAM as defined in the beginning of this section.

The following lemma uses only correctness of ORAM and does not depend on its security. The proof of the lemma uses the information transfer technique similarly to Lemma 2 in [LN18].

Lemma 4.2. Let \( n, m, M, w, \mathcal{M} \) be as in the beginning of this section, moreover suppose \( n \geq 10 \) is an even integer. Let \( k \geq 1 \) be an integer such that \( k \leq \frac{2\log n + 11}{10(m+2\log n+11)} \). Let \( A(\mathcal{M}, Y_{n,k}) \) be the access sequence of \( \mathcal{M} \) and \( G_{n,k} \) be the corresponding access graph. \( G_{n,k} \) is a random variable that depends on \( Y_{n,k} \) and the internal randomness of \( \mathcal{M} \). With probability at least \( 1 - \frac{1}{n^2} \), \( G_{n,k} \) has \((n/5k)\)-dense \( k \)-partition.

Proof. By our assumption from the beginning of this section, \( n \leq M \), and thus for any \( k \in \{1, 2, \ldots, \frac{n}{2}\} \) all sequences \( Y_{n,k} \) have all addresses in the correct range. Fix any \( k \) satisfying the assumptions of this lemma and set \( \ell = \left\lfloor \frac{n}{2\ell} \right\rfloor \). Let \( W_i \) and \( R_i \) be the \( i \)-th block of writes and reads in \( Y_{n,k} \), respectively. Let \( U_i \) be the vertices of \( G_{n,k} \) corresponding to \( W_i \), and \( V_i \) be the vertices corresponding to \( R_i \). It suffices to prove that for each \( i \in \{1, \ldots, k\} \), the probability that there are fewer than \( n/5k \) edges between \( U_i \) and \( V_i \) is less than \( 1/n^2 \). If this is true then by the union bound the lemma follows.

For contradiction, assume there exists \( i \in \{1, \ldots, k\} \) such that the probability that there are fewer than \( n/5k \) edges between \( U_i \) and \( V_i \) is \( \geq 1/n^2 \). Here, the randomness is taken over the choice of an input sequence \( y \leftarrow Y_{n,k} \) and the internal randomness of \( \mathcal{M} \). Fix such an \( i \). Fix all the randomness except for the choice of \( b_{i,1}, \ldots, b_{i,\ell} \) in \( Y_{n,k} \) so that \( G_{n,k} \) obtained from this restricted distribution has fewer than \( n/5k \) edges between \( U_i \) and \( V_i \) with probability \( \geq 1/n^2 \) over the choice of \( b_{i,1}, \ldots, b_{i,\ell} \). (This is possible by an averaging argument.) Let \( B \subseteq \{0,1\}^{w\times\ell} \) be the set of choices for \( b_{i,1}, \ldots, b_{i,\ell} \) which give fewer than \( n/5k \) edges between \( U_i \) and \( V_i \) in \( G_{n,k} \). Clearly, \( |B| \geq 2^{w\ell}/n^2 \).

We use \( \mathcal{M} \) to construct a deterministic protocol that transmits any string from \( B \) from Alice to Bob, two communicating parties, using at most \( \log |B| - 10 \) bits. That gives a contradiction as such an efficient transmission violates the pigeon-hole principle.

On input \( b \in B \) to Alice, Alice sends a single message to Bob who can determine \( b \) from the message. They proceed as follows. Both Alice and Bob simulate \( \mathcal{M} \) on \( Y_{n,k} \) up until reaching \( W_i \). All the randomness used before the \( i \)-th block of writes \( W_i \) is fixed and known both to Alice and Bob. Then Alice continues with the simulation of \( \mathcal{M} \) on \( W_i \) with data \( b_{i,1}, b_{i,2}, \ldots, b_{i,\ell} \) set to \( b \). Once she finishes it, she sends the content of the internal memory of \( \mathcal{M} \) to Bob using \( w\ell \) bits. Then Alice continues with the simulation of \( \mathcal{M} \) on \( R_i \) and whenever \( \mathcal{M} \) makes a server probe to read from a location that was written last time during the simulation of \( W_i \), Alice sends over the address and the content of that cell to Bob. Overall, Alice sends at...
most $mw + 2wn/5k$ bits of communication to Bob that can be concatenated into a single message of this size.

On receiving side, Bob uses the internal state of $\mathcal{M}$ communicated by Alice to continue with the computation on $R_i$, while he uses the state of the server he obtained initially before reaching $W_i$. He simulates all server probes by himself, except for read operations that match the list sent by Alice, where he initially uses the content provided by Alice. Clearly, Bob can determine $b$ from the simulation.

As $k \leq \frac{n}{10(m + 2 \log n + 11)}$, $mw + 2wn/5k \leq (n/2k - 2 \log n - 11)w$, so $mw + 2wn/5k \leq (\ell - 2 \log n - 10)w$, hence, the number of communicated bits is $mw + 2wn/5k \leq \log |B| - (2w - 2) \log n - 10w$, which is a contradiction.

Using good error-correcting codes [MS77], this lemma could be generalized to the case when $\mathcal{M}$ implements Array Maintenance problem with probability $1 - p_{\text{fail}} < 1$, i.e., $\mathcal{M}$ is allowed to return a wrong value for each of its input read operations with a small constant probability $p_{\text{fail}}$. The graph $G_{n,k}$ would still have $(cn/k)$-dense $k$-partition with $1 - 1/n$ probability for some $\epsilon > 0$ which depends only on the allowed failure probability $p_{\text{fail}}$.

Lemma 4.2 shows that there is an input sequence such that the corresponding access graph has $\frac{n}{5k}$-dense $k$-partition for some $k$. We show that by statistical security of $\mathcal{M}$, this property holds for a single input sequence and many different values of $k$.

**Lemma 4.3.** Let $n, m, M, w, \mathcal{M}$ be as in the beginning of this section, and assume $n$ is even and $n \geq 10$. Let $y$ be an input sequence to $\mathcal{M}$ of length $n$. If $\mathcal{M}$ is a statistically secure online ORAM then for every $k \in \{1, 2, \ldots, \frac{n}{10(m + 2 \log n + 11)}\}$

$$\Pr[G(A(\mathcal{M}, y)) \text{ has an (n/5k)-dense k-partition}] \geq \frac{3}{5}.$$  

**Proof.** For contradiction, suppose that for some $k$ the probability is less than $3/5$. From the statistical security of $\mathcal{M}$ we know that the statistical distance $\text{SD}(A(\mathcal{M}, y), A(\mathcal{M}, Y_{n,k})) \leq \frac{1}{3}$. By Lemma 4.2 $G_{n,k}$ has an $(n/5k)$-dense $k$-partition with probability at least $1 - 1/n \geq 9/10$. Define a function $f_{\ell,k}$ on ordered graphs that is an indicator of having an $\ell$-dense $k$-partition. Applying Proposition 2.3 with $X \leftarrow G(A(\mathcal{M}, y)), Y \leftarrow G_{n,k}$, and $f = f_{n/5k,k}$, we can conclude that $G(A(\mathcal{M}, y))$ has an $(n/5k)$-dense $k$-partition with probability at least $3/4 - 1/10 \geq 3/5$.  

We are ready to prove our main theorem for statistically secure ORAM.

**Theorem 4.4.** There are constants $c_0, c_1 > 0$ such that for any integers $m, w \geq 1$ and $M \geq n \geq c_0$ where $m \leq \sqrt{n}$ and $M \leq 2^n$, any statistically secure online ORAM $\mathcal{M}$ with address range $M$, cell size $w$ bits and $m$ cells of internal memory must perform at least $c_1 n \log n$ server probes in expectation (the expectation is over the randomness of $\mathcal{M}$) on any input sequence of length $n$.

**Proof.** Fix an ORAM machine $\mathcal{M}$. Consider any input sequence $y$ to $\mathcal{M}$ of length $n$. By Lemma 4.3 for every $k$, such that $1 \leq k \leq \frac{n}{10(m + 2 \log n + 11)}$, we get that

$$\Pr[G(A(\mathcal{M}, y)) \text{ has an (n/5k)-dense k-partition}] \geq \frac{3}{5}.$$  

Applying Corollary 3.4 with $s = 1$, $t = \frac{n}{10(m + 2 \log n + 11)}$, $\ell = \left\lfloor \frac{n}{5} \right\rfloor$, and $p = 3/5$, we can lower bound the expected number of edges in $G(A(\mathcal{M}, y))$ by

$$\frac{3n}{50} \left\lfloor \frac{n}{10(m + 2 \log n + 11)} \right\rfloor.$$
For $n \geq 1000$, \( \left\lfloor \frac{n}{10(n+2\log n+10)} \right\rfloor \geq \frac{\sqrt{n}}{100} \). Hence, the expected number of edges in $G(A(\mathcal{M}, y))$ is at least $\frac{3}{1000} \cdot n \log \frac{\sqrt{n}}{100} \geq \frac{1}{100} \cdot n \log n$, provided $c_0$ is large enough. Since the indegree of each vertex of an access graph is at most one, the expected number of vertices in $G(A(\mathcal{M}, y))$, which is the same as the expected number of probes in $A(\mathcal{M}, y)$, is at least $\frac{1}{100} \cdot n \log n$.

Next we prove $\omega(1)$ lower bound for computationally secure ORAM.

**Theorem 4.5.** Let $m, M, w : \mathbb{N} \to \mathbb{N}$ be non-decreasing functions such that for all $n$ large enough: $m(n) \leq \sqrt{n}$ and $n \leq M(n) \leq 2^w(n)$. Let $\{\mathcal{M}(n)\}_{n \in \mathbb{N}}$ be a sequence of online ORAM’s with address range $M(n)$, cell size $w(n)$ bits and $m(n)$ cells of internal memory that is computationally secure. Let $\{y_n\}_{n \in \mathbb{N}}$ be a sequence of input sequences where $|y_n| = n$, for each $n \in \mathbb{N}$.

For any constant $c_1 > 0$ there is a constant $c_0 > 0$, such that for any $n \geq c_0$, $\mathcal{M}(n)$ must perform in expectation at least $c_1 n$ server probes on the input sequence $y_n$.

In particular there is no computationally secure online ORAM with constant overhead $O(1)$.

**Proof.** For each $n \in \mathbb{N}$, define $k(n)$ to be the smallest $k$ such that $\Pr[G(A(\mathcal{M}(n), y_n)) \text{ has } (n/5k)\text{-dense } k\text{-partition}] < 1/2$. Using Corollary 3.4 we get for each $n$ large enough that the expected number of edges in $G(A(\mathcal{M}(n), y_n))$ is at least $c \cdot n \log k(n)$, for some absolute constant $c > 0$. It suffices to show that $k(n) \to \infty$ as $n \to \infty$. There cannot exist a constant $k$ such that $Y_n$ has $(n/5k)$-dense $k$-partition with probability less than $\frac{1}{2}$ for infinitely many $n$. Otherwise $\{y_n\}_n$ would be computationally distinguishable from $\{Y_{n,k}\}_n$ (by the greedy algorithm which has $k$ hard-wired). So, $k(n) \to \infty$ as $n \to \infty$.

## 5 Alternative Definitions for Oblivious RAM

In this section, we recall some alternative definitions for ORAM which appeared in the literature and explain the relation of our lower bound to those models.

**The definition of Larsen and Nielsen.** Larsen and Nielsen [LN18] required that for any two input sequences of equal length, the corresponding distributions of access sequences cannot be distinguished with probability greater than $1/4$ by any algorithm running in polynomial time in the sum of the following terms: the length of the input sequence, logarithm of the number of memory cells (i.e., $\log n$), and the size of a memory cell (i.e., $\log n$ for the most natural parameters). We show that their definition implies statistical closeness as considered in our work (see the statistical security property in Definition 2.6). Therefore, any lower bound on the bandwidth overhead of ORAM satisfying our definition implies a matching lower bound w.r.t. the definition of Larsen and Nielsen [LN18].

To this end, let us show that if two distributions of access sequences are not statistically close, then they are distinguishable in the sense of Larsen and Nielsen. Assume there exist two input sequences $y$ and $y'$ of equal lengths, for which the access sequences $A(\mathcal{M}, y)$ and $A(\mathcal{M}, y')$ have statistical distance greater than $1/4$. We define a distinguisher algorithm $D$ that on access sequence $x$ outputs 1 whenever $\Pr[A(\mathcal{M}, y) = x] > \Pr[A(\mathcal{M}, y') = x]$, outputs 0 whenever $\Pr[A(\mathcal{M}, y) = x] < \Pr[A(\mathcal{M}, y') = x]$, and outputs a uniformly random bit whenever $\Pr[A(\mathcal{M}, y) = x] = \Pr[A(\mathcal{M}, y') = x]$. It follows from definition of $D$, basic properties of statistical distance (see Proposition 2.2), and our assumption about the statistical distance of $A(\mathcal{M}, y)$ and $A(\mathcal{M}, y')$ that

\[
|\Pr[D(A(\mathcal{M}, y)) = 1] - \Pr[D(A(\mathcal{M}, y')) = 1]| = \text{SD} (A(\mathcal{M}, y), A(\mathcal{M}, y')) > \frac{1}{4}.
\]

Note that $D$ is allowed to have all information about the distributions $A(\mathcal{M}, y)$ and $A(\mathcal{M}, y')$ hardwired. Thus, $D$ can run in linear time in the length of the access sequence (which is polynomial in the length of the input sequence) and distinguishes the two access sequences with probability greater than $1/4$. 


The definition of Goldreich and Ostrovsky. The definition of Goldreich and Ostrovsky (Definition 2.9 in [GO96]) suffers from an issue which, to the best of our knowledge, was not pointed out in the literature. In particular, the definition can be satisfied almost trivially without any indistinguishability of the access sequences. Recall that the Goldreich-Ostrovsky definition requires that for any two input sequences $y$ and $y'$, if the length distributions $|A(M, y)|$ and $|A(M, y')|$ are identical, then $A(M, y)$ and $A(M, y')$ are identical as well. As we show, this requirement can be satisfied by creating an ORAM that makes sure that on any two distinct sequences $y, y'$, the length distributions $|A(M, y)|$ and $|A(M, y')|$ are different (note that no indistinguishability is required in that case).

In particular, we create an ORAM with a constant overhead so that $|A(M, y)| \in \{2|y|, 2|y| + 1\}$ and the distribution $|A(M, y)|$ encodes the sequence $y$. The ORAM proceeds by performing every operation $y_i$ directly on the server followed by a read operation from address 0. After the last instruction in $y$, the ORAM selects a random sequence of operations $r$ of length $|y|$ and if $r$ is lexicographically smaller than $y$ then the ORAM performs an extra read from address 0. Note that this ORAM can be efficiently implemented using constant amount of internal memory by comparing the input sequence to the randomly selected one online. Also, the machine does not need to know the length of the sequence in advance. Finally, the length distribution $|A(M, y)|$ is clearly different for each input sequence $y$. Given that the above definition allows an almost trivial ORAM with constant overhead, we do not hope to extend our lower bound towards this definition.

Simulation-based definitions. The recent work of Asharov et al. [AKL18] employs a simulation-based definition. Specifically, their definition is more general than the one employed in this work and we currently do not know how to transfer the information transfer technique into their setting.

Our definition of computational ORAM. Larsen and Nielsen claimed that their lower bound for statistically secure online ORAM implies a matching lower bound for computationally secure online ORAM. We see two caveats in this claim.

First, a straightforward extension of the definition of Larsen and Nielsen that considers only pairs of sequences and introduces a security parameter $\lambda$ would allow solutions that can have bandwidth overhead $O(\lambda)$ which might eventually be $\Omega(n)$. In other words, the growing security parameter eventually allows achieving identical memory access by scanning the whole memory. Therefore, in our Definition 2.6 we extend the definition of statistically secure online ORAM into the computational setting by considering infinite families of input sequences.

Second, we currently do not know how to extend our techniques from the statistical regime into the computational setting with a matching bound. Our technique gives an $\omega(1)$ lower bound for the bandwidth overhead of computationally secure online ORAM. We leave it as an intriguing open problem whether it is possible to prove an $\Omega(\log n)$ lower bound for online ORAM w.r.t. our definition of computational security.

However, note that the $\omega(1)$ lower bound is still an interesting result. It follows from the work of Boyle and Naor [BN16] that any super-constant lower bound for offline ORAM would imply super-linear lower bounds on size of sorting circuits – which would constitute a major breakthrough in computational complexity. Boyle and Naor [BN16] proved the following theorem (rephrased using our notation).

**Theorem 5.1** (Theorem 3.1 [BN16]). Suppose there exists a Boolean circuit ensemble $C = \{C(n, w)\}_{n,w}$ of size $s(n, w)$, such that each $C(n, w)$ takes as input $n$ words each of size $w$ bits, and outputs the words in sorted order. Then for word size $w \in \Omega(\log n) \cap n^{O(1)}$ and constant internal memory $m \in O(1)$, there exists a secure offline ORAM (as per Definition 2.8 [BN16]) with total bandwidth and computation $O(n \log w + s(2n/w, w))$.

Moreover, the additive factor of $O(n \log w)$ follows from the transpose part of the algorithm of [BN16] (see Figures 1 and 2 in [BN16]). As Boyle and Naor showed in their appendix (Remark B.3 [BN16]) this additive factor in total bandwidth may be reduced to $O(n)$ if the size of internal memory is $m \geq w$. Thus, sorting circuit of size $O(nw)$ implies offline ORAM with total bandwidth $O(n + 2\frac{n}{w}w) = O(n)$. Or the other way around lower bound $\omega(n)$ for total bandwidth of offline ORAM implies $\omega(nw)$ lower bound for circuits sorting $n$ words of size $w$ bits, each.
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