Abstract

We study 5d $\mathcal{N} = 2$ maximally supersymmetric Yang-Mills theory with a gauge group $G$ on $S^2 \times M_3$, where $M_3$ is a 3-manifold. By explicit localization computation we show that the path-integral of the 5d $\mathcal{N} = 2$ theory reduces to that of the 3d $G_C$ Chern-Simons theory on $M_3$, where $G_C$ is the complexification of $G$. This gives a direct derivation of the appearance of the Chern-Simons theory from the compactification of the 6d $(2,0)$ theory, confirms the predictions from the 3d/3d correspondence for $G = SU(N)$, and suggests the generalization of the correspondence to more general gauge groups.
1 Introduction

We have learned over the past a few years that there exists beautiful correspondence (3d/3d correspondence) between supersymmetric 3d $\mathcal{N} = 2$ theories and the geometry of 3-dimensional manifolds [1–4]. (See also related earlier works [5–7].) The correspondence states that for a given 3-manifold $M_3$ there exists a corresponding 3d $\mathcal{N} = 2$ theory $\mathcal{T}[M_3]$ defined from the compactification of the 6d $A_{N-1}$ $(2,0)$ theory on $M_3$, and that the $S^3$ partition function [11–13] or the $S^1 \times S^2$ index [14,15] of the 3d $\mathcal{N} = 2$ theory $\mathcal{T}[M_3]$ coincides with the either full or holomorphic partition function of the $SL(N,\mathbb{C})$ Chern-Simons theory on $M_3$.

While there have already been many works on this subject, the identification of the 3d $\mathcal{N} = 2$ theory tends to rely on indirect physical and mathematical arguments, and there are often restrictions on the choices of $M_3$. It is thus highly desirable to directly verify the claim that the compactification of the 6d $(2,0)$ theory gives rise to the 3d $SL(N,\mathbb{C})$ Chern-Simons theory on $M_3$.

In this paper we study 5d $\mathcal{N} = 2$ maximally supersymmetric Yang-Mills (SYM) theory with a gauge group $G$ on $S^2 \times M_3$. This is the dimensional reduction of the 6d $(2,0)$ theory on $S^1 \times S^2 \times M$. We show directly by supersymmetric localization that the partition function of the 5d $\mathcal{N} = 2$ theory with a gauge group $G$ reduces to the partition function of the 3d $G_{\mathbb{C}}$ Chern-Simons theory, where $G_{\mathbb{C}}$ is the complexification of the gauge group $G$:

$$Z_{5d\, G\, SYM}[S^2 \times M_3] = Z_{3d\, G_{\mathbb{C}}\, CS}[M_3].$$

(1.1)
Let us denote the 5d gauge coupling constant by \( g \), and the radius of \( S^2 \) by \( r \). Both of these parameters are dimension-full, and we find in the correspondence (1.1) that their dimensionless combination \( \frac{8\pi^2 r}{g^2} \) is identified with the complexified level \( t \) of the Chern-Simons theory by

\[
int = \frac{8\pi^2 r}{g^2} .
\]

(1.2)

The right hand side of (1.1) is independent of the size of the manifold \( M_3 \) since Chern-Simons theory is a topological theory. We also find that there is no dependence on the size of \( M_3 \) on the left.

Our computation can be thought of a higher-dimensional lift of the 2d localization on \( S^2 \) \cite{16-18} as well as the localization of the 4d twisted \( \mathcal{N} = 4 \) Yang-Mills on \( I \times M_3 \) \cite{19,20} (see appendix A for the latter). For \( G = SU(N) \) we have \( G_C = SL(N,\mathbb{C}) \) and our results can be regarded as the derivation of the 3d/3d correspondence existing in the literature, when the 3d \( \mathcal{N} = 2 \) theory on \( S^2 \times S^1 \) is dimensionally reduced along the \( S^1 \). While the precise formulation of the 3d/3d correspondence is currently unknown for other gauge groups, our result (1.1) strongly suggests such generalizations.

The rest of this paper is organized as follows. In section 2 we introduce the Lagrangian and the supersymmetry (SUSY) transformations of the 5d \( \mathcal{N} = 2 \) theory on \( S^2 \times M_3 \) partially topologically twisted along the \( M_3 \). In section 3 we add a \( Q \)-exact term to the Lagrangian and show that classical saddle point configurations reproduce the classical action of the complex Chern-Simons theory, while the one-loop determinant is independent of the position of the saddle point. In section 4 we conclude with some supplementary remarks on our results.

We also include three appendices. In appendix A we summarize the 3d \( SL(N,\mathbb{C}) \) Chern-Simons theory and its relation with the 5d \( \mathcal{N} = 2 \) SYM and the 6d \( (2,0) \) theory. In appendix B we study 5d \( \mathcal{N} = 1 \) SYM on \( S^2 \times M_3 \), highlighting the differences from the similar analysis for the 5d \( \mathcal{N} = 2 \) SYM in section 2. In appendix C we review the construction of 2d \( \mathcal{N} = (2,2) \) theory with a twisted vector multiplet and charged twisted chiral multiplets.

**Note Added:** During the preparation of this paper we have been notified that Daniel L. Jafferis and Clay Cordova have been working on closely related projects \cite{25}, and we have coordinated submission of our papers. Towards the completion of this project we received the paper \cite{26}, which also discuss 5d \( \mathcal{N} = 2 \) SYM on \( S^2 \times M_3 \) and obtained 3d TQFT closely related with the 3d Chern-Simons theory. The computational methods, the Lagrangian as well as the resulting expressions, however, seem to be different from ours.

## 2 5d \( \mathcal{N} = 2 \) SYM on \( S^2 \times M_3 \)

In this section we study the Lagrangian and the supersymmetry transformations of the 5d \( \mathcal{N} = 2 \) supersymmetry on \( S^2 \times M_3 \), partially topologically twisted along \( M_3 \). From

\[^3\text{Our work is similar in spirit to the work of } \cite{21,22}, \text{ which discuss 5d SYM on a product of } S^3 \text{ and a 2d Riemann surface and obtained 2d } q\text{-deformed Yang-Mills theory. This is consistent with the results of } \cite{23} \text{ and its reduction } \cite{24}.\]
now on, $\mathcal{N} = 1$ or $\mathcal{N} = 2$ refers to the numbers of supercharges that theories under study preserves in the flat space.

Let us first discuss about the supersymmetry algebra we have to seek for the 5d $\mathcal{N} = 2$ SYM on $S^2 \times M_3$ or its M-theory origin, the 6d $(2,0)$ theory on $S^1 \times S^2 \times M_3$. The supersymmetry algebra of the 6d $(2,0)$ theory in the flat spacetime is $OSp(2,6|4)$ whose bosonic subalgebra $SO(2,6)$ and $Sp(4) \simeq SO(5)$ can be identified as conformal and R-symmetry groups. However, not all of these symmetries can be preserved when the theory is defined on $S^2 \times M_3$. From the 3d/3d correspondence, one expects to have the superconformal index of a certain 3d $\mathcal{N} = 2$ theory upon the compactification along $M_3$. It is therefore natural to require the supersymmetry algebra to be parameterized by Killing spinors on $S^1 \times S^2$ and constant spinors on $M_3$.

Since the Killing spinors depend on the circle $S^1$, it requires twisting the 6d $(2,0)$ theory with $SO(2) \subset Sp(4)$ R-symmetry to obtain the 5d gauge theory on $S^2 \times M_3$. This reduction results in breaking the R-symmetry down to $SO(3)_R \times SO(2)_R$. Since we are interested in the case of a curved 3-manifold $M_3$, one further twists the 5d theory with $SO(3)_R$ to admit constant spinors on $M_3$. Thus the supersymmetry algebra of the 5d theory on $S^2 \times M_3$ should be given by $SU(2|1)$ which contains $SU(2)$ charge generating the isometries of $S^2$ and $U(1)_R$ R-symmetry charge.

In order to place the 5d $\mathcal{N} = 2$ SYM on $S^2 \times M_3$, one might naively imagine the following two-step procedure: first place the physical $\mathcal{N} = 2$ theory on $S^2 \times \mathbb{R}^3$, and then partially topologically twist the theory along $M_3$, when we replace $\mathbb{R}^3$ by a curved 3-manifold $M_3$. In this procedure, the supergroup for the physical 5d $\mathcal{N} = 2$ theory should contain at least $SO(3)_{S^2}$ rotation symmetry along the $S^2$ as well as the $SO(3)_R \times SO(2)_R$ subgroup of the $Sp(4)$ R-symmetry. The $SO(3)_R$-part of the R-symmetry is necessary when we topologically twist the theory along the 3-manifold $M_3$ (cf. [27,28]). The other $SO(2)_R$ R-symmetry should be identified with the $SO(2)$ R-symmetry in the $SU(2|1)$ supersymmetry of the dimensionally-reduced 2d $\mathcal{N} = (2,2)$ theory on $S^2$ [16,17], and plays crucial roles in the 3d/3d correspondence.

We find that it is not possible to place the physical 5d $\mathcal{N} = 2$ SYM on $S^2 \times \mathbb{R}^3$ while preserving the extra $SO(2)_R$ R-symmetry, which should not be involved in the topological twisting. Supergroup classifications also implies that the superalgebra containing the $SO(3)_{S^2} \times SO(2)_R \times SO(3)_R$ symmetries is not consistent either with the 2d supersymmetry algebra on $S^2$ or the topological twisting. We will discuss this subtlety again in appendix E more explicitly at the Lagrangian level.

In this paper we directly construct the Lagrangian and the supersymmetry transformations of the partially topologically twisted 5d $\mathcal{N} = 2$ theory on $S^2 \times M_3$. This theory preserves the 2d $SU(2|1)$ supergroup on $S^2$, fits naturally with the 3d/3d correspondence, and gives rise to our main result (1.1) after localization. Without further ado, we present the explicit Lagrangian for the 5d $\mathcal{N} = 2$ SYM on $S^2 \times M_3$ preserving $SU(2|1)$ supersymmetry.

2.1 Lagrangian

Let us first summarize our conventions. We use the indices $M,N,\cdots$ ($A,B,\cdots$) for the spacetime (internal space) indices which runs from 1 to 5, while $I,J,\cdots$ for $Sp(4)$ R-
symmetry indices. We use the following representations for the five-dimensional gamma matrices $\Gamma_M$ for the spacetime, and $\hat{\Gamma}^A$ for the internal space:

\begin{align*}
\Gamma^m &= \gamma^m \otimes 1_2 \quad (m = 1, 2), \\
\Gamma^\mu &= \gamma^3 \otimes \gamma^\mu \quad (\mu = 1, 2, 3), \\
\hat{\Gamma}^\mu &= \gamma^\mu \otimes \gamma^3 \quad (\mu = 1, 2, 3), \\
\hat{\Gamma}^i &= 1_2 \otimes \gamma^{1-3} \quad (i = 4, 5),
\end{align*}

(2.1)

where $\gamma^m = (\tau^1, \tau^2)$, $\gamma^\mu = (\tau^1, \tau^2, \tau^3)$ and $\tau_i$ are Pauli matrices. The charge conjugation operator $C$ and the $Sp(4)_R$ invariant tensor $\hat{C}^{IJ}$ are given by

\begin{align*}
C &= (\tau^1)^{ab} \otimes \epsilon^{\dot{a}\dot{b}}, \\
\hat{C} &= \epsilon^{\alpha\beta} \otimes (\tau^1)^{\dot{\alpha}\dot{\beta}},
\end{align*}

(2.2)

where $\mathcal{I} = (a, \dot{a})$ ($I = (\alpha, \dot{\alpha})$) denote the five-dimensional spinor indices ($Sp(4)$ R-symmetry indices). Each of these indices $a$, $\dot{a}$, $\alpha$ and $\dot{\alpha}$ is raised and lowered by the antisymmetric tensor $\epsilon^{ab}$, $\epsilon^{\dot{a}\dot{b}}$, $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot{\alpha}\dot{\beta}}$ with $\epsilon^{12} = -\epsilon_{12} = 1$. Our convention for bilinear of 5-dimensional spinors is

\begin{align*}
\epsilon \lambda &= -\epsilon_{\mathcal{I}} C^{IJ} \lambda_{\mathcal{J}}, \\
\epsilon \Gamma^M \lambda &= -\epsilon_{\mathcal{I}} (C \Gamma^M)^{IJ} \lambda_{\mathcal{J}}, \text{ etc.}
\end{align*}

(2.3)

On Flat $\mathbb{R}^5$  Let us begin with the maximally supersymmetric Yang-Mills theory in flat $\mathbb{R}^5$. The $\mathcal{N} = 2$ vector multiplet contains a gauge field $A_M$, scalar fields $\phi^A$ in $\mathbf{5}$ of $Sp(4)$, and gaugino fields $\lambda_I$. The Lagrangian is given by

\begin{align*}
L_{\mathbb{R}^5} &= \frac{1}{g^2} \text{tr} \left[ \frac{1}{4} F_{MN} F^{MN} + \frac{1}{2} (D_M \phi_A)^2 + \frac{i}{2} \lambda_I \hat{C}^{IJ} \Gamma^M \lambda_J - \frac{1}{4} [\phi_A, \phi_B]^2 \\
&\quad - \frac{i}{2} \lambda_I (\hat{C} \Gamma^A)^{IJ} [\phi^A, \lambda_J] \right].
\end{align*}

(2.4)

This is invariant under the following on-shell SUSY transformation rules$^4$

\begin{align*}
\delta A_M &= i \epsilon_{\mathcal{I}} \hat{C}^{IJ} \Gamma_M \lambda_J, \\
\delta \phi_A &= \epsilon_{\mathcal{I}} (\hat{C} \Gamma_A)^{IJ} \lambda_J, \\
\delta \lambda_I &= -\frac{1}{2} \Gamma^{MN} \epsilon_{\mathcal{I}} F_{MN} + i \Gamma^M (\hat{\Gamma}^A \epsilon)_{IJ} D_M \phi_A + \frac{i}{2} (\hat{\Gamma}^{AB} \epsilon)_{IJ} [\phi_A, \phi_B],
\end{align*}

(2.5)

where we used

\begin{align*}
F_{MN} &= \partial_M A_N - \partial_N A_M - i [A_M, A_N], \\
D_M \sigma &= \partial_M \sigma - i [A_M, \sigma].
\end{align*}

(2.6)

$^4$This can be derived by the dimensional reduction of the on-shell SUSY transformation for the $10d \mathcal{N} = 1$ SYM theory.
On-Shell SUSY on \( S^2 \times M_3 \) From the 3d/3d correspondence we expect that the supersymmetry parameters \( \varepsilon_I \) for the theory on \( S^2 \times M_3 \) should be a Killing spinor on \( S^2 \) and a constant spinor on \( M_3 \) satisfying

\[
\nabla_m \varepsilon_I = -\frac{i}{2r} \Gamma_m \Gamma_{12} \varepsilon_I , \quad \nabla_\mu \varepsilon_I = 0 . \tag{2.7}
\]

In order to have a constant spinor on a curved three-manifold \( M_3 \), one suitably twists the theory with \( SO(3)_R \) subgroup of \( Sp(4) \) (cf. [27, 28]). Let us denote by \( SO(3)_{\text{twist}} \) the diagonal subgroup of the \( SO(3) \) local Lorentz group on \( M_3 \) and the \( SO(3)_R \). The leftover \( SO(2)_R \) is then identified as the \( U(1)_R \) \( R \)-symmetry of the \( SU(2|1) \) supersymmetry algebra on \( S^2 \). Under the symmetry group \( SO(3)_{\text{twist}} \times U(1)_R \), various fields can be decomposed as follows

\[
A_M : \quad 1_0 \oplus 3_0 \equiv A_m \oplus A_\mu ,
\]

\[
\lambda_I : \quad 1_{\pm 1} \oplus 3_{\pm 1} \equiv (\lambda, \bar{\lambda}) \oplus (\psi^\mu, \bar{\psi}^\mu) ,
\tag{2.8}
\]

\[
\phi^A : \quad 1_{\pm 2} \oplus 3_0 \equiv \varphi_\pm \oplus \phi^\mu ,
\]

while the supersymmetry parameters can be decomposed as

\[
\varepsilon_I : \quad 1_{\pm 1} \oplus 3_{\pm 1} . \tag{2.9}
\]

The \( SU(2|1) \) supersymmetry of our interest can be parameterized by the singlets \( (\xi, \bar{\xi}) \) under the \( SO(3)_{\text{twist}} \), which takes the following form

\[
(\varepsilon_I)_{a\dot{a}} = \frac{i}{2} \varepsilon_{a\dot{a}} \left( \xi_a \otimes \varepsilon^+_\dot{a} - (\gamma^3 \bar{\xi})_a \otimes \varepsilon^-_{\dot{a}} \right) , \tag{2.10}
\]

where \( \xi \) and \( \bar{\xi} \) satisfy the Killing spinor equation on the two-sphere

\[
\nabla_m \xi = +\frac{1}{2r} \gamma_m \gamma^3 \xi , \quad \nabla_m \bar{\xi} = -\frac{1}{2r} \gamma_m \gamma^3 \bar{\xi} , \tag{2.11}
\]

and

\[
\varepsilon^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \varepsilon^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{2.12}
\]

For later convenience, let us Define \( \varphi_\pm = \phi^4 \mp i \phi^5 \), \( A_\mu = A_\mu + i \phi_\mu \) and

\[
e^{\dot{a}a} \lambda_{a\dot{a}\alpha\dot{\alpha}} \equiv i \left( \lambda_a \otimes \varepsilon^+_{\dot{\alpha}} - (\gamma^3 \bar{\lambda})_a \otimes \varepsilon^-_{\dot{\alpha}} \right) ,
\]

\[
(e^{\gamma^\mu})^{\dot{a}a} \lambda_{a\dot{a}\alpha\dot{\alpha}} \equiv (\psi^\mu_a \otimes \varepsilon^+_{\dot{\alpha}} - (\gamma^3 \psi^\mu)_a \otimes \varepsilon^-_{\dot{\alpha}}) . \tag{2.13}
\]

In terms of new variables, the on-shell SUSY variation of the fields on \( S^2 \times M_3 \) are given by

\[
\delta A_m = -\frac{1}{2} \left( \xi \gamma^m \bar{\lambda} + \bar{\xi} \gamma^m \lambda \right) ,
\]

\[
\delta \varphi_+ = -\xi \gamma^3 \lambda ,
\]

\[
\delta \varphi_- = +\bar{\xi} \gamma^3 \bar{\lambda} ,
\]

\[
\delta \lambda = (i F_{12} + i D^\mu \phi_\mu) \gamma^3 \xi + i \gamma^m \gamma^3 D_m (\xi \varphi_+) + \frac{i}{2} [\varphi_-, \varphi_+] \xi , \tag{2.14}
\]
and
\[
\begin{align*}
\delta A_\mu &= -\xi_\gamma \bar{\psi}_\mu - \bar{\xi}_\gamma \psi_\mu , \\
\delta \psi_\mu &= -i\gamma^m\gamma^3 F_{m\mu} + \gamma^m \xi D_m \phi_\mu + D_\mu \varphi_+ + \gamma^3 \bar{\xi} [\varphi_+, \phi_\mu] \\
&\quad + \left[ \frac{1}{2} \epsilon_{\mu\nu\rho} \{ F^{\nu\rho} + i(D^\nu \phi^\rho - D^\rho \phi^\nu)\gamma^3 + i[\phi^\nu, \phi^\rho] \} \right] \xi ,
\end{align*}
\]  
(2.15)

where \( \gamma_\pm = \frac{1}{2} (1 \pm \gamma^3) \). Note here that we add \( 1/r \)-correction terms in the SUSY variation rule for the fermion field \( \lambda \). Our convention for bilinear of 2-dimensional spinors is \( \bar{\xi} \lambda = 3 \bar{\xi}^a \lambda_a = -\xi_a \epsilon^{ab} \lambda_b \), \( \bar{\xi} \gamma^3 \lambda = -\xi_a (\epsilon^m)^{ab} \lambda_b \) and so on.

The \( SU(2|1) \) invariant Lagrangian on \( S^2 \times M_3 \) is then given by
\[
\mathcal{L} = \mathcal{L}_{R^5} + \frac{1}{4rg^2} (\mathcal{L}_{CS}(A) - \mathcal{L}_{CS}(\bar{A})) + \frac{i}{2rg^2} \text{tr} [\bar{\lambda} \gamma^3 \lambda] 
\]  
(2.16)

with
\[
\mathcal{L}_{CS}(A) = \epsilon^{\mu\nu\rho} \text{tr} \left[ A_\mu \partial_\nu A_\rho - \frac{2}{3} iA_\mu A_\nu A_\rho \right].
\]  
(2.17)

As commented at the beginning of this section, the Lagrangian of the physical \( \mathcal{N} = 1 \) SYM on \( S^2 \times R^3 \) constructed in appendix B can not be embedded into the above Lagrangian of our interest.

**Off-Shell SUSY on \( S^2 \times M_3 \)** In the next section, we shall use the localization method to reduce the path integral of the 5d \( \mathcal{N} = 2 \) theory on \( S^2 \times M_3 \) down to that of a 3d theory on \( M_3 \). We need the off-shell supersymmetry for this purpose. Introducing an auxiliary field \( D \) whose on-shell value becomes
\[
D = -i D^\mu \phi_\mu ,
\]  
(2.18)

and three complex auxiliary fields \( G_\mu \) whose one-shell values are
\[
G_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho} ,
\]  
(2.19)

the off-shell SUSY transformation rules are given by
\[
\begin{align*}
\delta A_m &= -\frac{1}{2} \left( \xi_\gamma m \bar{\lambda} + \bar{\xi}_\gamma m \lambda \right) , \\
\delta \varphi_+ &= -\xi_\gamma^3 \lambda , \\
\delta \varphi_- &= +\bar{\xi}_\gamma^3 \bar{\lambda} , \\
\delta \lambda &= (iF_{12} - D) \gamma^3 \bar{\xi} + i\gamma^m \gamma^3 D_m (\bar{\xi} \varphi_+) + i[\varphi_-, \varphi_+] \xi , \\
\delta D &= \frac{i}{2} \left( D_m (\xi \gamma^m \gamma^3 \bar{\lambda} - \bar{\xi} \gamma^m \gamma^3 \lambda) + [\varphi_+, \bar{\xi} \lambda ] + [\varphi_-, \xi \lambda] \right) ,
\end{align*}
\]  
(2.20)

and
\[
\begin{align*}
\delta A_\mu &= -\xi_\gamma + \bar{\psi}_\mu - \bar{\xi}_\gamma \psi_\mu , \\
\delta \psi_\mu &= -i\gamma^m \gamma^3 F_{m\mu} + \gamma^m \xi D_m \phi_\mu + D_\mu \varphi_+ \bar{\xi} + \gamma^3 \bar{\xi} [\varphi_+, \phi_\mu] + (G_\mu \gamma_+ + G_\mu \gamma_-) \xi , \\
\delta G_\mu &= +i\bar{\xi}_\gamma m \bar{\psi}_\mu + \xi_\gamma^{-} D_\mu \bar{\lambda} + i\bar{\xi}_\gamma m \gamma_- D_m \psi_\mu + \bar{\xi}_\gamma^3 D_\mu \lambda \\
&\quad + \left[ \phi_\mu, \xi_\gamma^{-} \bar{\lambda} + \bar{\xi}_\gamma^3 + \lambda \right] + i[\varphi_+, \bar{\xi}_\gamma^3 + \psi_\mu] + i[\varphi_-, \xi \gamma \psi_\mu] .
\end{align*}
\]  
(2.21)
The commutator $[\delta_\eta, \delta_\epsilon]$ reads

$$[\delta_1, \delta_2] = -i\mathcal{L}_v + \text{Gauge}(\gamma) + \text{Lorentz}(\Theta) + U(1)_R(\alpha),$$

(2.22)

where

$$\mathcal{L}_v = v^M \partial_M = 2\xi_1 \gamma^m \xi_2 \partial_m,$$

$$\gamma = iv^M A_M + 2\xi_1 \gamma^3 \xi_2 \varphi_+ - 2\xi_1 \gamma^3 \xi_2 \varphi_-,$$

$$\Theta = -\frac{2}{r}\xi_1 \xi_2,$$

$$\alpha = \frac{1}{r}\xi_1 \gamma^3 \xi_2.$$

(2.23)

It is illustrative to rewrite the bosonic part of the on-shell Lagrangian (2.16) in the off-shell SUSY invariant form,

$$\mathcal{L}_b = \mathcal{L}_{tv} + \mathcal{L}_{tc} + \mathcal{L}_W + \mathcal{L}_{\bar{W}},$$

(2.24)

where

$$\mathcal{L}_{tv} = \text{tr} \left[ + \frac{1}{2}(F_{12})^2 + \frac{1}{2}D_m \varphi_+ D_m \varphi_- + \frac{1}{8}[\varphi_+, \varphi_-]^2 + \frac{1}{2}D^2 \right],$$

$$\mathcal{L}_{tc} = \text{tr} \left[ + \frac{1}{2}(F_{m\mu})^2 + \frac{1}{2}(D_m \phi_\mu)^2 + \frac{1}{2}D_\mu \varphi_- D_\mu \varphi_+ - \frac{1}{2}[\phi_\mu, \varphi_+][\phi_\mu, \varphi_-] + \frac{1}{2}\bar{G}_\mu G^\mu + iD(D^\mu \phi_\mu) \right],$$

$$\mathcal{L}_W = \text{tr} \left[ - \frac{i}{2} \partial A^\mu W(A^\mu) G^\mu \right] + i\frac{1}{2r}W(A^\mu),$$

$$\mathcal{L}_{\bar{W}} = \text{tr} \left[ - \frac{i}{2} \partial A^\mu \bar{W}(\bar{A}^\mu) \bar{G}^\mu \right] + i\frac{1}{2r}\bar{W}(\bar{A}^\mu),$$

(2.25)

with

$$W = -\frac{i}{2} \epsilon_{\mu\nu\rho} \text{tr} \left[ A_\mu \partial_\nu A_\rho - \frac{2}{3}iA_\mu A_\nu A_\rho \right],$$

$$\frac{\partial}{\partial A^\mu} W(A^\mu) = -\frac{i}{2} \epsilon_{\mu\nu\rho} F_\nu^\rho.$$

(2.26)

Note that, in the language of $SU(2|1)$ supersymmetry algebra on $S^2$, $(A_m, \varphi_\pm, \lambda, D)$ transforms as a twisted vector multiplet while $(A_\mu, \psi_\mu, G_\mu)$ transform as twisted chiral multiplets. Moreover, the bosonic Lagrangian terms $\mathcal{L}_{tv}$, $\mathcal{L}_{tc}$ and $\mathcal{L}_W$ in (2.25) can be understood as kinetic terms for twisted vector and twisted chiral multiplet, and twisted superpotential terms. Readers are referred to the Appendix C for details.

3 Localization of the Path Integral

We show in this section that the path integral of 5d $\mathcal{N} = 2$ SYM on $S^2 \times M_3$ can be localized to the path integral of 3d $G_C$ Chern-Simons theory on $M_3$ with purely imaginary Chern-Simons level. To this end, we use the supersymmetric localization technique.
3.1 Saddle Point Configurations

Choice of Supercharge  In our localization scheme, we choose a particular supersymmetry generator $Q$ in $SU(2|1)$ which is associated to the following Killing spinors $\xi$ and $\bar{\xi}$ in (2.10)

$$\xi = e^{i\varphi/2} \left( \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right), \quad \bar{\xi} = e^{-i\varphi/2} \left( \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \right). \quad (3.1)$$

Here $(\theta, \varphi)$ denote the polar coordinates on the two-sphere. Given our choice of supercharge, we can show that

$$\delta^2 = -i\mathcal{L}_v + \text{Gauge}(\gamma) + \text{Lorentz}(\Theta) + U(1)_R(\alpha), \quad (3.2)$$

where

$$\mathcal{L}_v = v^M \partial_M = \bar{\xi} \gamma^m \xi \partial_m = -i \partial_{\varphi},$$

$$\gamma = iv^M A_M + \bar{\xi} \gamma^3 \xi \varphi^+ - \xi \gamma^3 \xi \varphi^- = A_\varphi + e^{-i\varphi} \sin \theta \varphi^+ - e^{+i\varphi} \sin \theta \varphi^-, \quad \Theta = -\frac{1}{r} \bar{\xi} \xi = -\frac{1}{r} \cos \theta,$$

$$\alpha = \frac{1}{2r} \bar{\xi} \gamma^3 \xi = \frac{1}{2r}. \quad (3.3)$$

Here we normalize the $U(1)_R$ charge so that $\lambda$ and $\psi^\mu$ carry $+1$ R-charge while $\varphi^\pm$ carry $\pm 2$ R-charge. For instance, the square of the supercharge $Q^2$ generates the following infinitesimal transformation of the fermionic field $\lambda$

$$\delta^2 \lambda = -i\mathcal{L}_v \lambda + i[\gamma, \lambda] + i\Theta \frac{\gamma^3}{2} \lambda + i\alpha \lambda. \quad (3.4)$$

Ghosts and BRST Symmetry  Once introducing the Faddeev-Popov ghost field $c$ to fix a gauge, one can define the BRST transformation as follows

$$Q_B A_M = D_M c, \quad Q_B \phi^A = i[c, \phi^A], \quad Q_B \lambda = i[c, \lambda], \quad Q_B \psi = i[c, \psi], \quad Q_B D = i[c, D], \quad Q_B c = i \frac{1}{2} [c, c] + a_0, \quad (3.5)$$

where $a_0$ is a constant field. With the above rule, one can show that

$$Q_B^2 = \text{Gauge}(a_0). \quad (3.6)$$

The fermionic symmetry we use for the localization is $\hat{Q} = Q + Q_B$ which is required to satisfy

$$\hat{Q}^2 = (Q + Q_B)^2 = -i\mathcal{L}_v + \text{Gauge}(a_0) + \text{Lorentz}(\Theta) + U(1)_R(\alpha). \quad (3.7)$$

It leads that the ghost field $c$ should transform under $Q$ as

$$Qc = -\left( iv^M A_M + \bar{\xi} \gamma^3 \xi \varphi^+ - \xi \gamma^3 \xi \varphi^- \right). \quad (3.8)$$
One can also show that the constant field $a_0$ should satisfy the relations below
\[ Qa_0 = Q_B a_0 = 0 . \] (3.9)

The transformation rules for the anti-ghost fields are given by
\[ Q_B \bar{c} = B , \quad Q_B B = i[a_0, \bar{c}] , \]
\[ Q\bar{c} = 0 , \quad QB = -i\mathcal{L}_c\bar{c} . \] (3.10)

To remove the constant modes of $c$ and $\bar{c}$, denoted by $c_0$ and $\bar{c}_0$, we use the multiplets of constant fields satisfying
\[ \hat{Q}\bar{a}_0 = \bar{c}_0 , \quad \hat{Q}\bar{c}_0 = i[a_0, \bar{a}_0] , \quad \hat{Q}B_0 = c_0 , \quad \hat{Q}c_0 = i[a_0, B_0] . \] (3.11)

**\(\hat{Q}\)-exact Deformations** We now turn into the choice of \(\hat{Q}\)-exact deformation terms. It is convenient to choose them as follows
\[ \mathcal{L}_{\text{def}} \equiv \hat{Q}V = \hat{Q}\left[ \text{tr}\left((\hat{Q}\lambda)^\dagger \lambda + \bar{\lambda}(\hat{Q}\bar{\lambda})^\dagger + (\hat{Q}\psi_\mu)^\dagger \psi_\mu + \bar{\psi}_\mu (\hat{Q}\bar{\psi}_\mu)^\dagger \right) \\ + \text{tr}\left[ \bar{c}f + \bar{c}B_0 + c\bar{a}_0 \right] \right] , \] (3.12)

which provides positive semi-definite bosonic terms once we impose the standard reality conditions on the field variables. Here $f[A_M, \phi^A, \cdots]$ denotes a gauge-fixing condition given by, for instance,
\[ f = i\partial^M A_M + \mathcal{L}_v (\bar{\xi}\gamma^3\bar{\xi}\varphi_+ + i\nu^M A_M) . \] (3.13)

**Saddle Point Configurations** The next step is to find out the saddle point locus where the given path integral can localize on. With the above choice of supercharge $\hat{Q}$ and $\hat{Q}$-exact deformation terms, the supersymmetric saddle point configurations can be described as
\[ \varphi_\pm = F_{12} = D = 0 , \quad A_\mu(x^M) = A_\mu(x^\mu) , \quad G_\mu = 0 , \] (3.14)
and $c = 0$, which solve the conditions $\hat{Q}$(fermions) $= 0$. Note that the saddle point condition for the ghost field, $\hat{Q}c = 0$, leads to
\[ \frac{i}{2}[c, c] + a_0 - \left(i\nu^M \hat{A}_M + \bar{\xi}\gamma^3\bar{\xi}\varphi_+ - \xi\gamma^3\xi\varphi_- \right) = 0 , \] (3.15)
where $\varphi_\pm$ and $\hat{A}_M$ satisfy the condition for the saddle point configurations. That is to say, it turns out that
\[ a_0 = 0 . \] (3.16)
As we will see below, this result (3.16) implies that one-loop determinant is independent of the saddle point configurations.

Plugging these saddle points into the Lagrangian, we learn that the path-integral of the 5d $\mathcal{N} = 2$ theory can be reduced to the path-integral of a $G_C$ Chern-Simons theory on $M_3$ up to a one-loop factor:
\[ Z_{5d} = \int \mathcal{D}A_\mu(x^\nu)\mathcal{D}\bar{A}_\mu(x^\nu) e^{-\frac{2\pi}{3} f_M} \left( \mathcal{L}_{\text{CS}(A)} - \mathcal{L}_{\text{CS}(\bar{A})} \right) Z_{\text{one-loop}}(A, \bar{A}) . \] (3.17)

We will next compute the one-loop determinant $Z_{\text{one-loop}}(A, \bar{A})$. 


3.2 One-loop Determinant

**Review**  From our choice of the $\mathcal{Q}$-exact deformation $\mathcal{L}_{\text{def}} = t\mathcal{Q}\mathcal{V}$ (3.12), we can compute the one-loop determinant around the fixed-point configurations. We will follow the standard procedure in the literature, see e.g. [29,30]. In terms of a new set of path-integration variables, one can write

$$V = (\hat{Q}X \ \Psi) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ \hat{Q}\Psi \end{pmatrix},$$  

(3.18)

where $X$ denote collectively bosonic variables while $\Psi$ denote fermionic variables. One can then obtain the kinetic terms as follows

$$\mathcal{L}_{\text{def}} = \mathcal{L}_b + \mathcal{L}_f,$$

(3.19)

where

$$\mathcal{L}_b = (X \ \hat{Q}\Psi) \begin{pmatrix} H \\ 1 \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ \hat{Q}\Psi \end{pmatrix},$$

$$\mathcal{L}_f = (\hat{Q}X \ \Psi) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} 1 \\ H \end{pmatrix} \begin{pmatrix} \hat{Q}X \\ \Psi \end{pmatrix}.$$  

(3.20)

Note that $H = \hat{Q}^2$ commutes with the operators $D_{ij}$. One can show formally the following relation

$$\left( \frac{\det K_f}{\det' K_b} \right)^2 = \frac{\det' H}{\det' X H} = \frac{\det_{\text{Coker}D_{10}} H}{\det_{\text{Ker}D_{10}} H},$$

(3.21)

where $K_f$ and $K_b$ denote the kinetic operators acting on fermionic and bosonic variables. Here $\det'$ indicates that bosonic zero modes should be excluded from the determinant. One can read off from the index below the ratio of determinants

$$\text{ind} \ D_{10} = \text{Tr}_{\text{Ker}D_{10}} [e^{-Ht}] - \text{Tr}_{\text{Coker}D_{10}} [e^{-Ht}].$$

(3.22)

Note that 5d $\mathcal{N} = 2$ SYM on $S^2 \times M_3$ has eighteen bosonic variables ($A_m, \varphi_{\pm}, A_{\mu}, \bar{A}_{\mu}, D, G_{\mu}, \bar{G}_{\mu}, B$) and eighteen fermionic variables ($\lambda, \bar{\lambda}, \psi_{\mu}, \bar{\psi}_{\mu}, c, \bar{c}$). To compute the index of the operator $D_{10}$, the new set of path integration variable is chosen as

$$X_i = (A_m, \varphi_{\pm}, A_{\mu}, \bar{A}_{\mu}; a_0, B_0)$$

$$\Psi_i = (\xi \gamma^3 \bar{\lambda}, c, \bar{c}, \bar{\xi} \gamma^m \psi_{\mu} - \xi \gamma^m \bar{\psi}_{\mu}),$$

(3.23)

and their descendants $\hat{Q}X$ and $\hat{Q}\Psi$. Here $i$ runs from 1 to 9.

**Index Computation**  Recall that the saddle point configurations can be described as

$$A_m = \varphi_{\pm} = 0, \quad A_{\mu} = \hat{A}_{\mu}(x^\mu).$$

(3.24)

For simplicity, let us set $r = 1$ from now on. The index of the operator $D_{10}$ is

$$\text{ind} \ D_{10} = \text{Tr}_X [e^{-tH}] - \text{Tr}_{\Psi} [e^{-tH}],$$

(3.25)
where the symbol $\text{Tr}$ indicates a combined matrix (i.e., over the indices $i$ from 1 to 9 and group indices) and functional trace and

$$H = \hat{Q}^2 = -i\mathcal{L}_v + \text{Lorentz} (\Theta) + \text{Gauge} (a_0) + R_U (1) (\alpha). \quad (3.26)$$

It acts on a field $\mathcal{O}$ in adjoint representation as follows

$$e^{-tH} \mathcal{O}(x^M) = h[\mathcal{O}] \cdot e^{-ia_0(x^M)t} \mathcal{O}(\bar{x}^M) e^{+ia_0(x^M)t}, \quad \bar{x} = e^{+i\theta_\phi x}, \quad (3.27)$$

where the factor $h[\mathcal{O}]$ encodes the action of $H$ on the vector and $U(1)_R$ indices of the field $\mathcal{O}$. As explained in [31], we ignore terms containing constant fields $B_0$ and $\bar{a}_0$ from $\mathcal{V}$ in computing the index. These constant fields are thus regarded as sitting in the kernel of $D_{10}$ leading to a contribution 2 to the index.

From the equations (3.3) and (3.16) one shows that the operator $H$ is independent of the saddle point configurations (3.24). This means that the index of $D_{10}$ should be independent of $A_{\mu}(x^\nu)$ and $\bar{A}_{\mu}(x^\nu)$. It therefore leads to trivial one-loop determinant contributions!

Let us then massage the trace of $e^{-Ht}$ over $X$ fields into

$$\text{Tr}_X \left[ e^{-tH} \right] = \int_{S^2 \times M_3} d^3x \text{tr} \left[ x^M \left| e^{-tH} \right| x^M \right]$$

$$= \int_{M_3} d^3x \text{tr} \left[ x^M \left| \int_{S^2} d^2x \left\langle x^m \left| e^{+it\partial_\phi} \cdot h \cdot e^{-ia_0 t} \right| x^m \right\rangle x^\mu \right]$$

$$= \text{vol}(M_3) \times \int_{S^2} d^2x \text{tr} \left[ x^m \left| e^{+it\partial_\phi} \cdot h \cdot x^m \right\rangle \right] \quad (3.28)$$

where the symbol $\text{tr}$ indicates a matrix trace over $i$ and group indices. We used for the last equality the fact that $a_0 = 0$. It works similarly for the trace of $e^{-Ht}$ over $\Psi$ fields. One can thus reduce the index into the following form

$$\text{ind} \, D_{10} = \text{vol}(M_3) \times \text{ind} \, D_{10}|_{S^2}, \quad (3.29)$$

where $\text{ind} \, D_{10}|_{S^2}$ denotes the index of the operator $D_{10}$ acting only on the two-sphere. Since $D_{10}|_{S^2}$ is transversally elliptic, one can apply the Atiyah-Bott localization formula to compute the reduced index:

$$\text{ind} \, D_{10}|_{S^2} = \sum_{\text{p:fixed points}} \frac{\text{Tr}_X_{s^2(p)}(h) - \text{Tr}_\Psi_{s^2(p)}(h)}{\text{det} \left( 1 - \partial\bar{x}/\partial x \right)|_{S^2}}, \quad (3.30)$$

It is obvious to show that there are two fixed points, one of which is the north pole $\theta = 0$ and another is the south pole $\theta = \pi$. Near the north pole, the operator $e^{Ht}$ acts on the local coordinate $z = \theta e^{i\phi}$ as

$$\tilde{z} = qz, \quad q = e^{+it}. \quad (3.31)$$

It implies that

$$\text{det} \left( 1 - \partial\bar{x}/\partial x \right)|_{S^2} = (1 - q)(1 - q^{-1}). \quad (3.32)$$
One can easily compute the value of $h$ for $X$ and $Y$ fields,
\begin{align}
h[A_2] &= q^{-1}, \quad h[\xi \gamma^3 \bar{\lambda}] = q, \\
h[A_1] &= q, \quad h[c] = 1, \\
h[\varphi_+] &= q^{-1}, \quad h[\bar{c}] = 1,
\end{align}
and
\begin{align}
h[A_\mu] &= 1, \quad h[\xi \gamma^3 \psi_\mu - \xi \gamma^3 \bar{\psi}_\mu] = q, \\
h[\bar{A}_\mu] &= 1, \quad h[\xi \gamma^3 \psi_\mu - \xi \gamma^3 \bar{\psi}_\mu] = 1.
\end{align}
Collecting all the results with the similar contribution from the south pole, one obtains
\begin{align}
\text{ind } D_{10}|_{S^2} &= \left[ -\frac{2}{1-q} + 3 \right]_N + \left[ -\frac{2}{1-q} + 3 \right]_S + 2 \\
&= \left[ -2 - 2 \sum_{n=1} q^n + 3 \right]_N + \left[ 2 \sum_{n=1} (q^{-1})^n + 3 \right]_S + 2 \\
&= -2 \sum_{n=1} q^n + 2 \sum_{n=1} (q^{-1})^n + 2 \times 3,
\end{align}
where the first term in the bracket arises from the twisted vector multiplet while the second from twisted chiral multiplets. Finally the last term in the first line of (3.35) arises from constant modes of $\bar{a}_0$ and $B_0$.

Several remarks are in order. First, one has to be careful to expand the expression in the first line into powers in $q$. This is due to the transversality of elliptic operators. The correct prescription is to expand the terms from the north pole in power of $q$ and the terms from the south pole in power of $q^{-1}$. Second, the term $-2$ in the second line can be understood as fermionic constant mode contributions from the twisted vector multiplet (more precisely, constant modes of $c$ and $\bar{c}$) which are removed by the contribution from constant fields $\bar{a}_0, B_0$. Finally the term $6$ in the finally expression arises from the bosonic zero modes of twisted chiral multiplets, which should be discarded in the one-loop computation. Therefore, one can conclude that
\begin{align}
Z_{\text{one-loop}}(A, \bar{A}) = 1.
\end{align}
Combining this with the previous result (3.17), we obtain the following relation
\begin{align}
Z_{5\text{d SYM}} = \int \mathcal{D}A_\mu(x^\nu) \mathcal{D}\bar{A}_\mu(x^\nu) \, e^{-\frac{2\pi t}{g^2} \int_{\Sigma_3} \left( \mathcal{L}_{\text{cs}}(A) - \mathcal{L}_{\text{cs}}(\bar{A}) \right)},
\end{align}
where we included the volume of $S^2$. Since the right hand side coincides with the partition function of the $G_C$ Chern-Simons theory, (3.37) is precisely the advertised relation (1.1). We moreover obtain (1.2) from the comparison with the Lagrangian of the $SL(N)$ Chern-Simons theory in (A.1) in appendix A. The fact the level $t$ is pure imaginary is consistent with the arguments reviewed in appendix A.

\textsuperscript{5} The right hand side of (3.37) is defined by the path integral over the gauge fields modulo $G$-gauge transformations. However it has turned out that the action on the right is actually invariant the "accidental" $G_C$-gauge transformations. Since $G_C$ is non-compact, the volume of $G_C$ and infinite, making the expression divergent. However since this is only an overall multiplicative constant, we can divide this overall infinity from both sides of (3.37).


4 Concluding Remarks

Let us now conclude with more comments on the relations (1.1), (1.2).

Factorization There is a natural generalization of the relation (1.1).

As discussed in appendix A, it is natural to consider the holomorphic partition function of the analytically continued Chern-Simons theory associated with the classical saddle point $\alpha (A.5)$. This is the basic building block for the full $G_C$ Chern-Simons partition function (A.4).

The relation (1.1) can be thought of as a 5d lift of the relation (A.6). Similarly, we expect to have a 5d lift of the relation (A.7)

$$Z_{\alpha}^{5d \; SYM}[D \times M_3] = Z_{\alpha}^{3d \; CS}[M_3],$$

where $D$ is a 2d cigar (disc); a cigar asymptotes at infinity to a cylinder, with a $U(1)$ symmetry along the extra dimensional direction. Reducing along this $S^1$ one obtains $D/U(1) = \mathbb{R}_{\geq 0}$, with the tip of the cigar corresponding to the endpoint of $\mathbb{R}_{\geq 0}$. The label $\alpha$ represents the boundary condition of the 5d gauge theory at infinity of the cigar.

It would be interesting to directly derive the expression (4.1) from the 5d localization of the $\mathcal{N} = 2$ theory. If we can show this, it follows automatically from (1.1), (4.1) and (A.4) that

$$Z_{\alpha}^{5d \; SYM}[S^2 \times M_3] = \sum_{\alpha, \bar{\alpha}} n_{\alpha, \bar{\alpha}} Z_{\alpha}^{5d \; SYM}[D \times M_3] Z_{\bar{\alpha}}^{5d \; SYM}[\bar{D} \times M_3],$$

where $n_{\alpha, \bar{\alpha}} \in \mathbb{Z}$ and $\bar{D}$ is a cigar with orientation reversed.

6d Lift We expect for consistency that there are further lifts of the relations (1.1), (4.1) to 6d, which is natural from the discussion in appendix A (cf. [32]):

$$Z_{6d \; (2, 0)}[(S^1 \times S^2)_q \times M_3] = Z_{3d \; CS}[M_3],$$

and

$$Z_{6d \; (2, 0)}[(S^1 \times D)_q \times M_3] = Z_{3d \; CS}[M_3],$$

where $(S^1 \times S^2)_q$ and $(S^1 \times D)_q$ represents twist by the $U(1)$ isometry of the $S^2$ ($D$) along the $S^1$ direction. This twist is the 6d lift of the similar twist for the 3d superconformal index on $S^1 \times S^2$. This 6d lift in practice involves a suitable resummation of the perturbative series expansion in $G_C$ Chern-Simons theory.

General Gauge Groups As already stated in introduction, our result (1.1) strongly suggests that the 3d/3d duality should be generalize to a general gauge group $G$ and their complexifications. Most of the works in the literature deals with the case $G_C = SL(2, \mathbb{C})$. The case of $G_C = SL(N, \mathbb{C})$ with $N > 2$ can be deal with from the mathematical work of [33], see also [34] for recent discussion. For ADE gauge groups, we expect 3d $\mathcal{N} = 2$ theory $T_G[M]$ should arise from the boundaries of the 4d $\mathcal{N} = 2$ theories of class $\mathcal{S}$, similar to the cases discussed in [1, 5, 7].
**Strong Coupling Limit**  In the relation (1.2), the 6d limit (i.e. the strong coupling limit) \( g \to \infty \) corresponds to the limit \( t \to 0 \), which represents a highly quantum regime of the Chern-Simons theory (A.1). We can trade this with the classical limit \( t \to \infty \) under the S-duality of the Chern-Simons theory \( t \to t' \sim t^{-1} \), which originates from the S-duality of the twisted 4d \( \mathcal{N} = 4 \) SYM.

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\[^6\text{In the notation of appendix A this is } \Psi \to \Psi' \sim \Psi^{-1}, \text{ where the parameter } \Psi \text{ is the parameter for the twisted } \mathcal{N} = 4 \text{ theory and is identified with the Chern-Simons level } t/2.\]
Appendix

A 3d Complex Chern-Simons

In this appendix we provide minimal enlightening review of the 3d Chern-Simons theory with a non-compact gauge group $G_C$ \cite{19,35,37}, and their relation with the 5d $\mathcal{N} = 2$ SYM. We mostly focus on the $G_C = SL(N,\mathbb{C})$ case.

Let us consider a 3-manifold $M_3$, and an $SL(N,\mathbb{C})$ flat connection $A$ on $M_3$. We denote the complex conjugate of $A$ by $\bar{A}$. The Lagrangian of the theory is given by

$$L_{CS}[A,\bar{A}] = \frac{t}{8\pi} Tr(A \wedge dA - \frac{2i}{3} A \wedge A \wedge A) + \bar{t} \frac{8\pi}{\pi} Tr(\bar{A} \wedge d\bar{A} - \frac{2i}{3} \bar{A} \wedge \bar{A} \wedge \bar{A}),$$

(A.1)

where $t = k + i\sigma$ ($k,\sigma \in \mathbb{R}$) is the complexified level. The parameter $k$, the ordinary level, is quantized as usual $k \in \mathbb{Z}$\[^7\] whereas the imaginary level $\sigma$ can be chosen to be a continuous parameter.

The partition function of the theory is defined by

$$Z_{SL(N)}_{CS}[M_3] = \int DAD\bar{A} e^{i \int_{M_3} L_{CS}[A,\bar{A}]}.$$

(A.2)

The fact that the gauge group is non-compact means that the trace $Tr$ in (A.1) is not positive definite. This causes a problem for the Yang-Mills kinetic terms, making the energy unbounded from below. However this is not problem for the Chern-Simons theory which has a trivial Hamiltonian.

The level $t$ plays the role of the inverse Planck constant, and we can choose to do the perturbative expansion. The classical saddle points are given by the solutions of the equations of motion $F_A = \bar{F}_{\bar{A}} = 0$, which is locally trivial but global can be non-trivial due to the presence of Wilson lines. The moduli space of the classical solutions are the moduli space of flat connections:

$$\mathcal{M}_{\text{flat}} = \text{Hom}(\pi_1(M), SL(N,\mathbb{C})).$$

(A.3)

Let us label the holomorphic (anti-holomorphic) flat connections by $\alpha$ ($\bar{\alpha}$).

Let us expand the path-integral around the classical flat connections $\alpha,\bar{\alpha}$. Since the Lagrangian (A.1) is written as a sum of the holomorphic and the anti-holomorphic part, the expansion around the classical solution also factorizes into the holomorphic part $Z_\alpha(t)$ and the anti-holomorphic part $Z_{\bar{\alpha}}(\bar{t})$. However this factorization breaks down when we choose to sum over the flat connections\[^8\]

$$Z_{CS}(t,\bar{t}) = \sum_{\alpha,\bar{\alpha}} n_{\alpha,\bar{\alpha}} Z_\alpha(t) Z_{\bar{\alpha}}(\bar{t}),$$

(A.4)

\[^7\] The Lagrangian (A.1) relies on the choice of the trivialization of the gauge bundle, and the partition function is independent of this choice only when $k \in \mathbb{Z}$.

\[^8\] The equation (A.4) follows from the fact that the real integration cycle of the $(G_C)_C = G_C \times G_C$ theory decomposes into the linear combination of $C_\alpha \times C_{\bar{\alpha}}$.  

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for some integer coefficient $c_{\alpha,\bar{\alpha}}$. The holomorphic (anti-holomorphic) partition function $Z_\alpha (\bar{Z}_{\bar{\alpha}})$ can be obtained by evaluating the path integral \[ (A.2) \) over a middle-dimensional integration cycle $C_\alpha (C_{\bar{\alpha}})$, defined by the downward Morse flow from the saddle point $\alpha$ ($\bar{\alpha}$) with respect to the real part of the action:

\[
Z_\alpha(t) = \int_{C_\alpha} \mathcal{D}A \ e^{i t \int_{M_3} L_{CS}[A]}, \tag{A.5}
\]

**Relation with the 6d (2, 0) Theory**  Let us next quickly explain why this theory is related to the 5d $\mathcal{N} = 2$ SYM discussed in this paper, and how it is related to 3d $\mathcal{N} = 2$ theories (cf. \[ [4] \]).

Let us begin with the 6d (2, 0) theory on $S^1 \times S^2 \times M$. The 2-sphere $S^2$ can be thought of as a $\tilde{S}^1$-fibration over an interval $I$, where the $\tilde{S}^1$ fiber shrinks at the endpoint of $I$. By compactifying the 6d (2, 0) theory on $S^1 \times \tilde{S}^1$, we have the the 4d $\mathcal{N} = 4$ theory on $I \times M$ discussed in \[ [19, 20] \].

Since $M_3$ is a curved manifold we need to topologically twist the 4d $\mathcal{N} = 4$ theory. The natural twist here is the one in \[ [27, 28] \], which is discussed in the context of geometric Langlands correspondence \[ [38] \]; this mixes the $SO(3)_R$ part of the $SO(6)_R$ R-symmetry with the rotational $SO(3)$ of the tangential directions of $M_3$. As explained in \[ [38] \], there is a $\mathbb{CP}^1$-family of such topological twist, and the resulting theory depends only on a single parameter $\Psi$, which is a some combination of the 4d theta-angle and the point $p$ on $\mathbb{CP}^1$. This topological twist complexifies the gauge field $A_\mu$ along $M_3$ into $A_\mu = A_\mu + i \phi_\mu$, where $\phi_\mu$’s are the three scalars out of the six scalars in 4d $\mathcal{N} = 4$ theory. The Lagrangian of the topologically twisted theory is $Q$-exact, except that there are boundary contributions which are given by complex Chern-Simons terms for $A_\mu$, with level $t = \Psi/2$ \[ [38] \]. In our case, since we have two boundaries we have two Chern-Simons terms, with opposite Chern-Simons levels due to orientation reversal, leading to

\[
Z_{4d\ G\mathcal{N}=4}[I \times M_3] = Z_{3d\ G_C\ CS}[M_3]. \tag{A.6}
\]

We can instead consider the 4d $\mathcal{N} = 4$ SYM on a half-line $\mathbb{R}_{\geq 0} \times M_3$. The discussion is similar with a complex CS term induced on the boundary, except that we not have to specify that boundary condition at the infinity of $\mathbb{R}_{\geq 0}$. This is specified by a flat $SL(N)$ connection, which we again label by $\alpha$.

\[
Z_{4d\ G\mathcal{N}=4}[\mathbb{R}_{\geq 0} \times M_3] = Z_{3d\ G_C\ CS}[M_3](t). \tag{A.7}
\]

Lifting the story back to 5d, the interval is lifted to $S^2$ and we have the 5d $\mathcal{N} = 2$ SYM on $S^2 \times M_3$. This means that the result derived in the main text \[ (1.1) \) is a 5d lift of the relation \[ (A.6) \). In particular the fact that the two Chern-Simons terms have opposite levels matches nicely with the result from the localization computation \[ (3.37) \), provided $t$ is imaginary.  

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9 This partition function is sometimes loosely referred to as that of the $G_C$ Chern-Simons theory. However in our terminology the $G_C$ Chern-Simons theory refers to the full partition function \[ (A.4) \), and the holomorphic partition function \[ (A.5) \) should rather be thought of as a analytic continuation of the $SU(N)$ Chern-Simons theory.

10 This happens when the parameter $\Psi$ is imaginary, for example when the 4d theta-angle is trivial and the point $p$ on $\mathbb{CP}^1$ lies on the real axis.
Similarly, we can lift the half-line $\mathbb{R}_{\geq 0}$ in (A.7) to a two-dimensional cigar (disc) $D$ with a boundary condition labeled by $\alpha$, leading to the natural relation (4.1). In this setup, the factorized form (A.4) of the 3d Chern-Simons theory is now translated into the geometrical factorization of the $S^2$ into two cigars $D$.

Finally, we can go back to the 6d $(2,0)$ theory on $S^2 \times S^1 \times M$, and choose to first compactify on the 3-manifold $M_3$ to obtain the theory $T[M_3]$ on $S^1 \times S^2$. The partition function on $S^1 \times S^2$, or the 3d superconformal index, is known to take a factorized form, which is the counterpart of (A.4) [39, 40]. In 3d $\mathcal{N} = 2$ theory the label $\alpha$ represents the vacuum of the theory compactified on $S^1$, and the level $t$ is related with the fugacity $q$ of the 3d index, where the latter parameter is identified with the same parameter in the 6d $(2,0)$ theory in (4.3), (4.4).

B 5d $\mathcal{N} = 1$ SYM on $S^2 \times \mathbb{R}^3$

This section is for those who are interested in 5d $\mathcal{N} = 1$ SYM theory instead of the $\mathcal{N} = 2$ theory. In contrast with the $\mathcal{N} = 2$ case, it turns out that one can place the physical five-dimensional gauge theory with eight supercharges on $S^2 \times \mathbb{R}^3$ while preserving the full $SU(2)$ $R$-symmetry.

We also discuss the supersymmetric localization of the $\mathcal{N} = 1$ pure SYM theory on $S^2 \times \mathbb{R}^3$ [11] Due to the absence of an extra $U(1)$ R-symmetry, the choice of supercharge for the localization cannot be compatible with the topological twist along $\mathbb{R}^3$. Consequently, the result will not be directly related with the 3d/3d correspondence nor 2d $\mathcal{N} = (2,2)$ theory on $S^2$ after KK reduction. Nevertheless the localization of the theory itself could be of interest from other perspectives and we report on our computations in this appendix.

B.1 $\mathcal{N} = 1$ Supersymmetry on $S^2 \times \mathbb{R}^3$ and Lagrangian

Vector Multiplet Let us begin with the 5d $\mathcal{N} = 1$ pure SYM on $S^2 \times \mathbb{R}^3$. An $\mathcal{N} = 1$ vector multiplet consists of a gauge field $A_M$, a real scalar field $\sigma$, and gaugino field $\lambda_a$ where $a$ denotes $SU(2)_R$ indices. The SUSY variation rules for the theory on the flat space $\mathbb{R}^5$ are given by

\[
\begin{align*}
\delta^{(0)} A_M &= i\epsilon^{ab} \varepsilon_a \Gamma_M \lambda_b , \\
\delta^{(0)} \sigma &= \epsilon^{ab} \varepsilon_a \lambda_b , \\
\delta^{(0)} \lambda_a &= -\frac{1}{2} \Gamma^{MN} \varepsilon_a F_{MN} + i \Gamma^M \varepsilon_a D_M \sigma + \varepsilon_b D_a \epsilon^{bc} , \\
\delta^{(0)} D_{ab} &= -i \left( \varepsilon_a \Gamma^M D_M \lambda_b + \varepsilon_b \Gamma^M D_M \lambda_a \right) + i \left[ \sigma, \varepsilon_a \lambda_b + \varepsilon_b \lambda_a \right] ,
\end{align*}
\]

where $D_{ab} = D_{ba}$.

In order to put the 5d $\mathcal{N} = 1$ pure SYM on $S^2 \times \mathbb{R}^3$, one needs to introduce additional terms to the variation rules above, coupled to the curvature of $S^2$. The supersymmetry

\[\text{\footnote{Since 5d $\mathcal{N} = 1$ theories are non-renormalizable and UV-incomplete, localization computation could break down in the UV, when the gauge coupling is large. However there are examples where we can discuss the UV fixed point within the validity of effective 5d gauge theory (41,42), see [43,44] for related computation for $S^5$ localization.}}\]
transformation parameters $\varepsilon_I$ also have to satisfy a Killing spinor equation on the two-sphere $S^2$

$$\nabla_m \varepsilon_a = \Gamma_m \bar{\varepsilon}_a, \quad \partial_\mu \varepsilon_a = 0 \quad (B.2)$$

with

$$\bar{\varepsilon}_a = -\frac{i}{2r} \Gamma_{12} \varepsilon_a. \quad (B.3)$$

The SUSY variation rules consistent to those in the two-sphere $S^2$ takes the following form

$$\delta \lambda_a = \delta^{(0)} \lambda_a + 2i\sigma \bar{\varepsilon}_a, \quad (B.4)$$

while all other variations are unmodified. One can show that this modified supersymmetry algebra closes off-shell. The commutator $[\delta_\eta, \delta_\varepsilon]$ on the vector multiplet reads

$$[\delta_\eta, \delta_\varepsilon] = -i \mathcal{L}_v + \text{Lorentz}(\Theta) + \text{Gauge}(\gamma) + SU(2)_R(\alpha), \quad (B.5)$$

where

$$\mathcal{L}_v = v^M \partial_M = (2\epsilon_{ab} \eta_a \Gamma^M \varepsilon_b) \partial_M, \quad \gamma = i\nu^M A_M + \epsilon_{ab} \eta_a \varepsilon_b \sigma, \quad \Theta = \frac{2i}{r} \epsilon^{ab} \eta_a \varepsilon_b, \quad \alpha^b_a = \frac{1}{r} (\eta_a \Gamma_{12} \varepsilon^b + \eta^b \Gamma_{12} \varepsilon_a). \quad (B.6)$$

The supersymmetry algebra on $S^2 \times \mathbb{R}^3$ therefore becomes a hybrid of $OSp(3|2)$ Lie superalgebra and the super Poincaré algebra in $\mathbb{R}^3$.

**Supersymmetric Lagrangian** The $\mathcal{N} = 1$ pure SYM Lagrangian is given by

$$\mathcal{L} = \frac{1}{g^2} \text{tr} \left[ \frac{1}{4} F_{MN}^2 + \frac{1}{2} (D_M \sigma)^2 + i \frac{e}{2} \epsilon^{ab} \lambda_a \Gamma^M \lambda_b - i \frac{e}{2} \epsilon^{ab} \lambda_a [\sigma, \lambda_b] - \frac{1}{2} D_{ab} D^{ab} + \frac{\sigma^2}{2r^2} - \frac{\sigma}{2r} \epsilon^{mn} F_{mn} \right] - \frac{i}{2rg^2} \epsilon^{\mu\nu\rho} \text{tr} \left[ A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right], \quad (B.7)$$

where $r$ denotes the radius of two-sphere. Here the dimensionless 3d Chern-Simons level $k = \frac{8\pi^2 r^2}{g^2}$ is quantized. Note the presence of the the term $i \epsilon^{12}$, which leads to an obstruction for constructing the physical $\mathcal{N} = 2$ SYM theory on $S^2 \times \mathbb{R}^3$ respecting either $Sp(4)_R$ or $SO(3)_R \times SO(2)_R$ R-symmetry.

**Comment on the Reduction to $S^2$** In the language of 2d $\mathcal{N} = (2, 2)$ supersymmetry on $S^2$, i.e. $SU(2|1)$, the 5d $\mathcal{N} = 1$ pure SYM Lagrangian can be understood as a theory involving a vector multiplet and a chiral multiplet in adjoint representation. In order to see this, one first needs to identify $U(1) \subset SU(2)_R$ as the R-symmetry group of $SU(2|1)$. However, this identification prevents us from performing the topological twist along $\mathbb{R}^3$. 
The supersymmetry parameter \( \varepsilon_a \) can be then decomposed as

\[
\varepsilon_a = \frac{1}{\sqrt{2}} \left( \xi_{a\dot{a}} \otimes \varepsilon^+_{\dot{a}} - \left( \gamma^3 \bar{\xi} \right)_{a\dot{a}} \otimes \varepsilon^-_{\dot{a}} \right),
\]

where

\[
D_m \xi_{a\dot{a}} = + \frac{1}{2r} \gamma_m \gamma^3 \xi_{a\dot{a}} , \quad D_m \bar{\xi}_{\dot{a}a} = - \frac{1}{2r} \gamma_m \gamma^3 \bar{\xi}_{\dot{a}a} .
\]

and

\[
\varepsilon^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \varepsilon^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .
\]

Here \( a \) denote the spinor indices on \( S^2 \) while \( \dot{a} \) denotes spinor indices on \( \mathbb{R}^3 \). Performing the KK reduction, one can reduce the 5d \( \mathcal{N} = 1 \) SYM down to 2d \( \mathcal{N} = (4,4) \) supersymmetric theory on the two-sphere. The rotation symmetry \( SO(3)_{\mathbb{R}^3} \) can be now identified as \( SU(2) \) of the \( \mathcal{N} = (4,4) \) supersymmetry algebra \( SU(2|2) \). Choosing

\[
\xi_{a\dot{a}} = \xi_a \otimes \varepsilon^+_{\dot{a}} , \quad \bar{\xi}_{\dot{a}a} = \bar{\xi}_{\dot{a}} \otimes \varepsilon^-_{a} ,
\]

further breaks the supersymmetry down to \( \mathcal{N} = (2,2) \). For later convenience, let us decompose the gaugino field \( \lambda_a \) into the following form

\[
\lambda_a = - \frac{1}{\sqrt{2}} \left( \lambda_a \otimes \varepsilon^+_{\dot{a}} + i (\gamma^3 \psi)_a \otimes \varepsilon^+_{\dot{a}} \right) \otimes \varepsilon^+_{\dot{a}} + \frac{1}{\sqrt{2}} \left( (\gamma^3 \bar{\lambda})_a \otimes \varepsilon^-_{\dot{a}} - i \bar{\psi}_a \otimes \varepsilon^+_{\dot{a}} \right) \otimes \varepsilon^-_{\dot{a}}
\]

Given the above reduction to 2d \( \mathcal{N} = (2,2) \), one can show that the above 5d \( \mathcal{N} = 1 \) SUSY transformation rules and the supersymmetric Lagrangian can be reduced to those of a vector multiplet \( (A_m, \sigma, A_5, \lambda) \) and an adjoint chiral multiplet \( (A_3, A_4, \psi) \) with R-charge \( q = 0 \) \([16,17]\). Upon KK reduction, it is illustrative to rewrite the bosonic part of the Lagrangian (B.7)

\[
\mathcal{L}_b = \mathcal{L}_v + \mathcal{L}_c
\]

with

\[
\mathcal{L}_v = \frac{1}{g^2} \mathrm{tr} \left[ \frac{1}{2} (F_{12} - \frac{\sigma}{r})^2 + (D_m \sigma)^2 + (D_m A_5)^2 - [\sigma, A_5]^2 + D^2 \right],
\]

\[
\mathcal{L}_c = \frac{1}{2g^2} \mathrm{tr} \left[ D_m \bar{\phi} D_m \phi - [\sigma, \bar{\phi}] [\sigma, \phi] - [A_5, \bar{\phi}] [A_5, \phi] + i \bar{\phi} [D, \phi] - i \bar{\phi} A_5, \phi] + \bar{F} F \right],
\]

where the former is the \( SU(2|1) \) invariant kinetic Lagrangian for the vector multiplet while the latter is the kinetic terms for the adjoint chiral multiplet. It is obvious that this theory should not be embedded into the model we studied in the main text where the latter can be reduced to \( \mathcal{N} = (2,2) \) twisted gauge theory on \( S^2 \).
B.2 Localization of Path Integral

Saddle Points  We choose the supercharge for the localization as follows

\[ \xi_a = e^{i\varphi/2} \left( \cos \frac{\theta}{2}, \sin \frac{\theta}{2} \right), \quad \bar{\xi}_a = e^{-i\varphi/2} \left( \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right), \]  

where \( \theta \) and \( \varphi \) denote the polar coordinates of the two-sphere. Given the above choice of supercharge, one can show that

\[ \delta^2 = -i L_v + \text{Lorentz}(\Theta) + \text{Gauge}(\gamma) + U(1)_R(\alpha) \]  

with

\[ L_v = v^M \nabla_M = -i \partial_\varphi + \partial_5, \]

\[ \Theta = \cos \theta \frac{r}{r}, \]

\[ \gamma = \cos \theta \sigma + iv^M A_M, \]

\[ \alpha = \frac{1}{2r}, \]  

(B.17)

Here we normalize the \( U(1)_R \) charge such that \( \lambda \) and \( \psi \) carry +1 R-charge. For instance, the square of the supercharge \( Q \) acts on the gaugino \( \lambda_a \) as follows

\[ Q^2 \lambda = -i L_v \lambda + \frac{i}{2} \gamma^3 \Theta \lambda + i [\hat{\sigma}, \lambda] + i \alpha \lambda. \]  

(B.18)

Note that the vector field \( \partial_5 \) acts freely on \( x^5 \).

In order to fix a gauge, we introduce the Faddeev-Popov ghost field \( c \), and define the BRST transformation as follows

\[ Q_B A_M = D_M c, \quad Q_B \sigma = i [c, \sigma], \quad Q_B \lambda = i \{c, \lambda\}, \]

\[ Q_B \psi = i \{c, \psi\}, \quad Q_B D = i [c, D], \quad Q_B c = i \{c, c\} + a_0, \]  

(B.19)

and

\[ Qc = -\gamma = -\cos \theta \sigma - iv^M A_M. \]  

(B.20)

Here \( a_0 \) is a constant field. The transformation rules for the anti-ghost fields are given by

\[ Q_B \bar{c} = B, \quad Q_B B = i [a_0, \bar{c}], \quad Q \bar{c} = 0, \quad Q_B = -i L_v \bar{c}. \]  

(B.21)

The fermionic symmetry we use for the localization is \( \hat{Q} = Q + Q_B \) whose square becomes

\[ \hat{Q}^2 = (Q + Q_B)^2 = -i L_v + \text{Gauge}(a_0) + i \alpha R, \]  

(B.22)

To remove the constant modes of \( c \) and \( \bar{c} \), we also introduce the multiplets of constant fields which transform under \( \hat{Q} \) as follows

\[ \hat{Q} a_0 = \bar{c}_0, \quad \hat{Q} \bar{c}_0 = i [a_0, \bar{a}_0], \quad \hat{Q} B_0 = c_0, \quad \hat{Q} c_0 = i [a_0, B_0]. \]  

(B.23)
We choose $\hat{Q}$-exact deformations whose bosonic terms are positive definite as follows

$$\mathcal{L}_{\text{def}} = \hat{Q} \text{Tr} \left[ \mathcal{V} + \bar{c} f + c B_0 + c \bar{a}_0 \right]$$ \hspace{1cm} (B.24)

with

$$\mathcal{V} = \frac{1}{2} \left( (Q \bar{\lambda}) \gamma^3 \lambda - (Q \bar{\psi}) \gamma^3 \psi + \text{c.c} \right),$$ \hspace{1cm} (B.25)

and the gauge condition

$$f = i \partial^M A_M - i \cos \theta \partial \sigma - i \partial A \varphi.$$ \hspace{1cm} (B.26)

One can then show that the path-integral of our interest can localize onto saddle point configurations

$$A_m = \hat{A}_m(x^m), \quad A_\mu = \hat{A}_\mu(x^\mu), \quad \sigma = \hat{\sigma} = +\frac{rB}{2}. \hspace{1cm} (B.27)$$

with

$$\hat{A}_m dx^m = \frac{B}{2} (\kappa - \cos \theta) d\varphi, \quad \hat{F}_{\mu\nu} = 0, \quad [\sigma, \hat{F}_{MN}] = 0.$$ \hspace{1cm} (B.28)

Here the flux $B$ over the two-sphere is GNO-quantized. For simplicity, let us set $r = 1$ from now on. Note that the saddle point condition for the ghost field is

$$\frac{i}{2} [c, c] + a_0 - \cos \theta \hat{\sigma} - iv^M \hat{A}_M = 0,$$ \hspace{1cm} (B.29)

where $\hat{\sigma}$ and $\hat{A}_M$ satisfy the condition for the saddle point configurations. In other words,

$$a_0 = \cos \theta \hat{\sigma} + iv^M \hat{A}_M.$$ \hspace{1cm} (B.30)

It implies that the path-integral of the 5d $\mathcal{N} = 1$ SYM on $S^2 \times \mathbb{R}^3$ can be reduced to

$$Z_{5d} = \sum_B \int \mathcal{D} \hat{A}_\mu(x^\mu) \delta(\hat{F}_{\mu\nu}) e^{\frac{i 2\pi}{g^2} \mathcal{L}_{\text{CS}}(\hat{A})} Z_{\text{one-loop}}(\hat{A}_\mu, B). \hspace{1cm} (B.31)$$

We will show that the one-loop determinant is trivial.

**Cohomological Basis** Note that there are ten bosonic variables $(A_M, \sigma, D, F, \bar{F}, B)$ and ten fermionic variables $(\lambda, \psi, c, \bar{c})$. For later convenience, let us first define

$$\Xi_1 = \frac{1}{2} \left( \xi \gamma^3 \psi + \bar{\xi} \gamma^3 \bar{\psi} \right),$$

$$\Xi_2 = \frac{1}{2i} \left( \xi \gamma^3 \psi - \bar{\xi} \gamma^3 \bar{\psi} \right),$$

$$\Xi_3 = \frac{1}{2} \left( \bar{\xi} \gamma^3 \bar{\lambda} - \xi \gamma^3 \lambda \right).$$ \hspace{1cm} (B.32)
and
\[ \Lambda = \hat{Q}\sigma = \frac{1}{2} (\bar{\xi} \lambda - \xi \bar{\lambda}) + i[c, \sigma] , \]
\[ \Lambda_m = \hat{Q} A_m = -i \frac{1}{2} (\bar{\xi} \rho_m \lambda + \xi \rho_m \bar{\lambda}) + D_m c , \]
\[ \Lambda_3 = \hat{Q} A_3 = \frac{1}{2} (\bar{\xi} \psi + \xi \bar{\psi}) + D_3 c , \]
\[ \Lambda_4 = \hat{Q} A_4 = -i \frac{1}{2} (\bar{\xi} \psi - \xi \bar{\psi}) + D_4 c . \] (B.33)

The cohomological basis for the fields are organized as follows,
\[ X_a = (A_1, A_2, A_3, A_4, \sigma; \bar{a}_0, B_0) : 5_B + \text{two constant fields} , \]
\[ \Psi_a = (\Xi_1, \Xi_2, \Xi_3, c, \bar{c}) : 5_F , \] (B.34)
and their descendants \( \hat{Q} X_a \) and \( \hat{Q} \Psi_a \) where \( a \) runs from one to five.

Index Computation As reviewed in section 3.2, one can read off the one-loop determinant contributions from the index of the operator \( D_{10} \) defined as
\[ \text{ind} \ D_{10} = \text{Tr}_X \left[ e^{-tH} \right] - \text{Tr}_\Psi \left[ e^{-tH} \right] , \] (B.35)
where the symbol \( \text{Tr} \) indicates a combined matrix (i.e. over \( a \) and group indices) and functional trace and
\[ H = \hat{Q}^2 = -iL_v + \text{Lorentz}(\Theta) + \text{Gauge}(a_0) + R_{U(1)}(\alpha) . \] (B.36)

It acts on a field \( O \) in adjoint representation as follows
\[ e^{-tH} O(x^M) = h_{[O]} \cdot e^{-ia_0(x^M)t} O(x^M)e^{+ia_0(x^M)t} , \quad \tilde{x} = e^{t(\partial_5 + i \partial_5)} x , \] (B.37)
where the factor \( h_{[O]} \) encodes the action of \( H \) on the vector and \( U(1)_R \) indices of the field \( O \) and
\[ a_0 = \cos \theta \hat{\sigma} + \hat{A}_\mu(x^m) + i\hat{A}_5(x^\mu) . \] (B.38)

From the fact that the saddle point configurations except \( \hat{A}_\mu(x^\mu) \) are constant in \( \mathbb{R}^3 \), one can massage the index into the following form
\[ \text{ind} \ D_{10} = \int_{\mathbb{R}^3} d^3 x \ \text{tr} \left[ \langle x^\mu | e^{+it\partial_5} e^{-ia_0 t} | x^\mu \rangle \right] \times \text{ind} \ D'_{10} \big|_{S^2} , \] (B.39)
where \( \text{tr} \) is performed over gauge group indices and \( D'_{10} \big|_{S^2} \) denotes the operator \( D_{10} \) for the Abelian SYM theory reduced on the two-sphere. Since \( D_{10} \big|_{S^2} \) is transversally elliptic, one can apply the Atiyah-Bott localization formula to compute the reduced index. As computed in [16], we obtain
\[ \text{ind} D'_{10} \big|_{S^2} = 0 \rightarrow \text{ind} \ D_{10} = 0 . \] (B.40)

This means immediately that the one-loop determinant from the vector multiplet is trivial
\[ Z_{\text{one-loop}}(\hat{A}_\mu, B) = 1 . \] (B.41)
\[ \mathcal{N} = (2, 2) \] Twisted Multiplets on \( S^2 \)

In this appendix we review the construction of Euclidean two-dimensional \( \mathcal{N} = (2, 2) \) gauge theories involving a twisted vector multiplet and charged twisted chiral multiplets studied in [18, 45].

Twisted Vector Multiplet

An \( \mathcal{N} = (2, 2) \) twisted vector multiplet contains a gauge field \( A_m \), two real scalar fields \( \varphi, \bar{\varphi} \) and gaugino field \( \eta \). The \( SU(2|1) \) representation on this multiplet is given by

\[
\delta \varphi = -\xi \gamma^3 \eta ,
\]
\[
\delta \bar{\varphi} = -\bar{\xi} \gamma^3 \bar{\eta} ,
\]
\[
\delta A_m = \frac{1}{2} (\xi \gamma^m \bar{\eta} + \bar{\xi} \gamma^m \eta) ,
\]
\[
\delta \eta = i \gamma^m \gamma^3 D_m (\bar{\xi} \varphi) + \frac{i}{2} \xi [\bar{\varphi}, \varphi] + (iF_{12} - D) \gamma^3 \xi ,
\]
\[
\delta \bar{\eta} = i \gamma^m \gamma^3 D_m (\xi \bar{\varphi}) + \frac{i}{2} \bar{\xi} [\bar{\varphi}, \varphi] - (iF_{12} + D) \gamma^3 \bar{\xi} ,
\]
\[
\delta D = -\frac{i}{2} \left( D_m (\xi \gamma^m \gamma^3 \bar{\eta} + \bar{\xi} \gamma^m \gamma^3 \eta) + [\varphi, \bar{\xi} \bar{\eta}] - [\bar{\varphi}, \xi \eta] \right) .
\]

The \( U(1)_R \) charges of the component fields are summarized in the table below

| \( U(1)_R \) | \( A_m \) | \( \varphi \) | \( \bar{\varphi} \) | \( \chi \) | \( \bar{\chi} \) | \( D \) |
|-------------|-----------|--------|--------|-------|--------|------|
| 0           | +2        | -2     | +1     | -1    | 0      |      |

Twisted Chiral Multiplet

Let us then consider a twisted chiral multiplet in representation \( \mathbf{R} \) under the gauge group \( G \). It contains a complex scalar field \( Y \), a complex Dirac spinor \( \chi \), and an auxiliary field \( G \). On the two-sphere, the \( SU(2|1) \) supersymmetry transformation rules for the component fields are given by

\[
\delta Y = \tilde{\xi} \gamma_- \chi - \xi \gamma_+ \chi ,
\]
\[
\delta \bar{Y} = \tilde{\xi} \gamma_+ \bar{\chi} - \xi \gamma_- \bar{\chi} ,
\]
\[
\delta \chi = i \gamma^m \gamma_+ \xi D_m Y - i \gamma^m \gamma_- \bar{\xi} D_m Y - i \gamma_- \bar{\xi} \varphi Y + i \gamma_+ \xi \bar{\varphi} Y - \gamma_- \xi G + \gamma_+ \bar{\xi} \bar{G} ,
\]
\[
\delta \bar{\chi} = i \gamma^m \gamma_- \bar{D}_m \bar{Y} - i \gamma^m \gamma_+ D_m \bar{Y} + i \gamma_+ \bar{\xi} \bar{\varphi} - i \gamma_- \bar{\xi} \bar{Y} \varphi - \gamma_+ \xi \bar{G} + \gamma_- \tilde{\xi} \tilde{G} ,
\]
\[
\delta G = i \xi \gamma_- \left( \gamma^m D_m \chi - \bar{\eta} Y - \varphi \chi \right) - i \tilde{\xi} \gamma_+ \left( \gamma^m D_m \bar{\chi} - \eta Y - \varphi \bar{\chi} \right) ,
\]
\[
\delta \bar{G} = i \xi \gamma_+ \left( \gamma^m D_m \bar{\chi} + \bar{\eta} \bar{Y} + \bar{\chi} \varphi \right) - i \tilde{\xi} \gamma_- \left( \gamma^m D_m \bar{\chi} + \bar{\eta} \bar{Y} + \bar{\chi} \varphi \right) .
\]

The \( U(1)_R \) charges of the component fields are summarized in the table below

| \( U(1)_R \) | \( Y \) | \( \bar{Y} \) | \( \chi_- \) | \( \chi_+ \) | \( \bar{\chi}_- \) | \( \bar{\chi}_+ \) | \( G \) | \( \bar{G} \) |
|-------------|-------|--------|--------|--------|--------|--------|-------|-------|
| 0           | 0     | +1     | -1     | -1     | +1     | 0      |       |      |
Supersymmetric Lagrangian  The kinetic Lagrangian for an $\mathcal{N} = (2, 2)$ twisted vector multiplet takes the following form

$$L_{tv} = \frac{1}{2}\text{tr} \left[ F_{12}^2 + D_m \bar{\phi} D_m \phi + \frac{1}{4} [\phi, \bar{\phi}]^2 - i \bar{\eta} \gamma^m D_m \eta - i \bar{\eta} \gamma^3 [\phi, \bar{\eta}] + \frac{i}{2} \eta \gamma^3 [\bar{\phi}, \eta] ight] + D^2 - \frac{i}{r} \bar{\eta} \gamma^3 \eta . \quad (C.5)$$

The $\mathcal{N} = (2, 2)$ twisted chiral multiplets, minimally coupled to the twisted vector multiplet, have the kinetic Lagrangian

$$L_{tc} = D_m \bar{Y} D_m Y + \frac{1}{2} \bar{Y} [\phi, \bar{\phi}] Y + i \bar{Y} D Y + i \bar{\chi} \gamma^m D_m \chi + i \bar{\chi} (\phi \gamma_+ + \bar{\phi} \gamma_-) \chi$$

$$+ i \bar{Y} (\gamma_- \eta + \gamma_+ \bar{\eta}) \chi - i \bar{\chi} (\gamma_+ \eta + \gamma_- \bar{\eta}) Y + \bar{G} G , \quad (C.6)$$

which is invariant under the $SU(2|1)$ SUSY transformation. Twisted superpotential couplings for the twisted chiral multiplet can be written in terms of a holomorphic function $W(Y)$. The interaction terms

$$L_W = - i W'(Y) G - W''(Y) \chi \gamma_- \chi + \frac{i}{r} W(Y) ,$$

$$L_{\bar{W}} = - i \bar{W}'(\bar{Y}) \bar{G} - \bar{W}''(\bar{Y}) \bar{\chi} \gamma_+ \bar{\chi} + \frac{i}{r} \bar{W}(\bar{Y}) , \quad (C.7)$$

are invariant under the above $SU(2|1)$ SUSY transformation rules.

Remark  The Lagrangian and the supersymmetry transformations discussed in this appendix coincide with those discussed in section 2 when the latter are compactified along $M_3$. The identification of various component fields are given by

$$\phi_+ = \phi , \quad \phi_- = \bar{\phi} , \quad \mathcal{A} = Y , \quad \bar{\mathcal{A}} = \bar{Y} ,$$

$$\lambda = \eta , \quad \bar{\lambda} = - \bar{\eta} , \quad \psi_+ = - \bar{\chi}_+ , \quad \psi_- = - \chi_- , \quad \bar{\psi}_+ = - \bar{\chi}_- , \quad \bar{\psi}_- = - \chi_- . \quad (C.8)$$

References

[1] Y. Terashima and M. Yamazaki, *SL(2, $\mathbb{R}$) Chern-Simons, Liouville, and Gauge Theory on Duality Walls*, JHEP 1108 (2011) 135, [arXiv:1103.5748](http://arxiv.org/abs/1103.5748).

[2] T. Dimofte, D. Gaiotto, and S. Gukov, *Gauge Theories Labelled by Three-Manifolds*, [arXiv:1108.4389](http://arxiv.org/abs/1108.4389).

[3] S. Cecotti, C. Cordova, and C. Vafa, *Braids, Walls, and Mirrors*, [arXiv:1110.2115](http://arxiv.org/abs/1110.2115).

[4] T. Dimofte, D. Gaiotto, and S. Gukov, *3-Manifolds and 3d Indices*, [arXiv:1112.5179](http://arxiv.org/abs/1112.5179).

[5] N. Drukker, D. Gaiotto, and J. Gomis, *The Virtue of Defects in 4D Gauge Theories and 2D CFTs*, [arXiv:1003.1112](http://arxiv.org/abs/1003.1112).
[6] T. Dimofte, S. Gukov, and L. Hollands, Vortex Counting and Lagrangian 3-manifolds, arXiv:1006.0977.

[7] K. Hosomichi, S. Lee, and J. Park, AGT on the S-duality Wall, arXiv:1009.0340.

[8] Y. Terashima and M. Yamazaki, 3d N=2 Theories from Cluster Algebras, arXiv:1301.5902.

[9] Y. Terashima and M. Yamazaki, Emergent 3-manifolds from 4d Superconformal Indices, Phys.Rev.Lett. 109 (2012) 091602, arXiv:1203.5792.

[10] M. Yamazaki, Quivers, YBE and 3-manifolds, JHEP 1205 (2012) 147, arXiv:1203.5784.

[11] A. Kapustin, B. Willett, and I. Yaakov, Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter, JHEP 03 (2010) 089, arXiv:0909.4559.

[12] D. L. Jafferis, The Exact Superconformal R-Symmetry Extremizes Z, arXiv:1012.3210.

[13] N. Hama, K. Hosomichi, and S. Lee, Notes on SUSY Gauge Theories on Three-Sphere, arXiv:1012.3512.

[14] S. Kim, The Complete superconformal index for N=6 Chern-Simons theory, Nucl.Phys. B821 (2009) 241–284, arXiv:0903.4172.

[15] Y. Imamura and S. Yokoyama, Index for three dimensional superconformal field theories with general R-charge assignments, JHEP 1104 (2011) 007, arXiv:1101.0557.

[16] F. Benini and S. Cremonesi, Partition functions of N=(2,2) gauge theories on S² and vortices, arXiv:1206.2356.

[17] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, Exact Results in D=2 Supersymmetric Gauge Theories, arXiv:1206.2606.

[18] J. Gomis and S. Lee, Exact Kahler Potential from Gauge Theory and Mirror Symmetry, JHEP 1304 (2013) 019, arXiv:1210.6022.

[19] E. Witten, Analytic Continuation Of Chern-Simons Theory, arXiv:1001.2933.

[20] E. Witten, A New Look At The Path Integral Of Quantum Mechanics, arXiv:1009.6032.

[21] Y. Fukuda, T. Kawano, and N. Matsumiya, 5D SYM and 2D q-Deformed YM, Nucl.Phys. B869 (2013) 493–522, arXiv:1210.2855.

[22] T. Kawano and N. Matsumiya, 5D SYM on 3D Sphere and 2D YM, Phys.Lett. B716 (2012) 450–453, arXiv:1206.5966.
[23] A. Gadde, L. Rastelli, S. S. Razamat, and W. Yan, The 4d Superconformal Index from q-deformed 2d Yang-Mills, arXiv:1104.3850.

[24] T. Nishioka, Y. Tachikawa, and M. Yamazaki, 3d Partition Function as Overlap of Wavefunctions, arXiv:1105.4390.

[25] C. Cordova and D. L. Jafferis, Complex Chern-Simons from M5-branes on the Squashed Three-Sphere, arXiv:1305.2891.

[26] J. Yagi, 3d TQFT from 6d SCFT, arXiv:1305.0291.

[27] N. Marcus, The Other topological twisting of N=4 Yang-Mills, Nucl.Phys. B452 (1995) 331–345, hep-th/9506002.

[28] M. Blau and G. Thompson, Aspects of NT ≥ 2 topological gauge theories and D-branes, Nucl.Phys. B492 (1997) 545–590, hep-th/9612143.

[29] V. Pestun, Localization of gauge theory on a four-sphere and supersymmetric Wilson loops, arXiv:0712.2824.

[30] M. F. Atiyah, Elliptic operators and compact groups. Lecture Notes in Mathematics, Vol. 401. Springer-Verlag, Berlin, 1974.

[31] N. Hama and K. Hosomichi, Seiberg-Witten Theories on Ellipsoids, JHEP 1209 (2012) 033, arXiv:1206.6359.

[32] E. Witten, Fivebranes and Knots, arXiv:1101.3216.

[33] V. Fock and A. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, Publ. Math. Inst. Hautes Études Sci. (2006), no. 103 1–211.

[34] T. Dimofte, M. Gabella, and A. B. Goncharov, K-Decompositions and 3d Gauge Theories, arXiv:1301.0192.

[35] E. Witten, Quantization of chern-simons gauge theory with complex gauge group, Commun. Math. Phys. 137 (1991) 29–66.

[36] S. Gukov, Three-dimensional quantum gravity, Chern-Simons theory, and the A polynomial, Commun. Math. Phys. 255 (2005) 577–627, hep-th/0306165.

[37] T. Dimofte, S. Gukov, J. Lenells, and D. Zagier, Exact Results for Perturbative Chern-Simons Theory with Complex Gauge Group, Commun. Num. Theor. Phys. 3 (2009) 363–443, arXiv:0903.2472.

[38] A. Kapustin and E. Witten, Electric-Magnetic Duality And The Geometric Langlands Program, Commun.Num.Theor.Phys. 1 (2007) 1–236, hep-th/0604151.

[39] S. Pasquetti, Factorisation of N = 2 Theories on the Squashed 3-Sphere, JHEP 1204 (2012) 120, arXiv:1111.6905.

[40] C. Beem, T. Dimofte, and S. Pasquetti, Holomorphic Blocks in Three Dimensions, arXiv:1211.1986.
[41] N. Seiberg, *Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics*, Phys.Lett. B388 (1996) 753–760, [hep-th/9608111].

[42] K. A. Intriligator and N. Seiberg, *Mirror symmetry in three-dimensional gauge theories*, Phys.Lett. B387 (1996) 513–519, [hep-th/9607207].

[43] D. L. Jafferis and S. S. Pufu, *Exact results for five-dimensional superconformal field theories with gravity duals*, arXiv:1207.4359.

[44] B. Assel, J. Estes, and M. Yamazaki, *Wilson Loops in 5d N=1 SCFTs and AdS/CFT*, arXiv:1212.1202.

[45] J. Gomis and N. Doroud, *to appear*, .