NEGATIVE LATIN SQUARE TYPE PARTIAL DIFFERENCE SETS IN NONELEMENTARY ABELIAN 2-GROUPS

JAMES A. DAVIS, QING Xiang

Abstract. Combining results on quadrics in projective geometries with an algebraic interplay between finite fields and Galois rings, we construct the first known family of partial difference sets with negative Latin square type parameters in nonelementary abelian groups, the groups $\mathbb{Z}_2^k \times \mathbb{Z}_2^{4\ell-4k}$ for all $k$ when $\ell$ is odd and for all $k < \ell$ when $\ell$ is even. Similarly, we construct partial difference sets with Latin square type parameters in the same groups for all $k$ when $\ell$ is even and for all $k < \ell$ when $\ell$ is odd. These constructions provide the first example that the non-homomorphic bijection approach outlined by Hagita and Schmidt [10] can produce difference sets in groups that previously had no known constructions. Computer computations indicate that the strongly regular graphs associated to the PDSs are not isomorphic to the known graphs, and we conjecture that the family of strongly regular graphs will be new.

1. Introduction

A $k$-element subset $D$ of a finite multiplicative group $G$ of order $v$ is called a $(v, k, \lambda, \mu)$-partial difference set (PDS) in $G$ provided that the multiset of “differences” $\{d_1 d_2^{-1} | d_1, d_2 \in D, d_1 \neq d_2\}$ contains each nonidentity element of $D$ exactly $\lambda$ times and each nonidentity element in $G \setminus D$ exactly $\mu$ times. Partial difference sets are equivalent to strongly regular graphs with a regular automorphism group, and they are connected to projective two-weight codes and two-intersection sets in projective spaces over finite fields. See [19] or [1] for background on these alternative approaches.

Ma’s survey [19] identifies several families of PDSs. Among these are PDSs with parameters $(n^2, r(n - \epsilon), \epsilon n + r^2 - 3\epsilon r, r^2 - \epsilon r)$ for $\epsilon = \pm 1$. When $\epsilon = 1$, the PDS is called a Latin square type PDS, and when $\epsilon = -1$, the PDS is called a negative Latin square type.
PDS. Early results in this area can be found in the paper by Bailey and Jungnickel [11]. We will focus on the case where $n = 4^2\ell$ and $r = 4\ell - 1 + \epsilon$. PDSs with these parameters have been constructed using quadratic forms over $\mathbb{F}_4$, the field with four elements, as we will mention in Section 1.2. This construction, as is true of most of the constructions the authors are aware of, are associated to elementary abelian groups.

Several authors, including those of [6], [7], [13], [14], [18], and [21], have used Galois rings, and more generally finite local rings, to construct PDSs in groups that are not elementary abelian. As far as we know, all PDSs constructed by finite local rings except those in [7] have Latin square parameters. After a presentation on one of these constructions, Mikhail Klin and William Martin [15] asked whether local ring construction can produce PDSs with negative Latin square type parameters. In this paper, we construct the first known family of PDSs with negative Latin square type parameters in nonelementary abelian groups by using Galois rings, and therefore answer the aforementioned question in the affirmative.

Recent work by Mathon [20] on maximal arcs in projective planes motivated the authors to revisit their work [7] on Denniston parameter PDSs. We recognized that one of the Galois ring constructions in that paper could be obtained using a natural map from the finite field construction. The mapping is not an isomorphism, but it is a bijection, and there is a relationship between the character sums in the two group rings. We extend that observation to higher dimensional projective spaces. Hagita and Schmidt [10] recently proposed using bijections between group rings with the property that character sums are preserved, and that is the approach we take in this paper. Wolfmann [23] used a similar approach to constructing Hadamard difference sets in nonelementary abelian groups via bijections between groups: like Hagita and Schmidt’s paper, the examples found in Wolfmann’s paper involved groups which were already known to contain difference sets. This paper uses similar techniques to construct PDSs in groups having no known previous constructions, the first time this approach has led to something new. (We
note that Bruck [3] used a bijection between group rings to construct difference sets in nonabelian groups. His mapping did not preserve character sums in the same way as the Hagita-Schmidt approach.)

We will limit our attention to abelian groups in this paper. In that context, a (complex) character of an abelian group is a homomorphism from the group to the multiplicative group of complex roots of unity. The principal character is the character mapping every element of the group to 1. All other characters are called nonprincipal. Starting with the important work of Turyn [22], character sums have been a powerful tool in the study of difference sets of all types. The following lemma states how character sums can be used to verify that a subset of a group is a PDS.

**Lemma 1.1.** Let $G$ be an abelian group of order $v$ and $D$ be a $k$-subset of $G$ such that \( \{d^{-1} \mid d \in D\} = D \) and \( 1 \notin D \). Then $D$ is a $(v, k, \lambda, \mu)$-PDS in $G$ if and only if, for any complex character $\chi$ of $G$,

\[
\sum_{d \in D} \chi(d) = \begin{cases} 
  k & \text{if } \chi \text{ is principal on } G. \\
  \frac{(\lambda-\mu)\pm\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} & \text{if } \chi \text{ is nonprincipal on } G.
\end{cases}
\]

1.1. **Projective two-intersection sets.** Let $\text{PG}(m-1, q)$ denote the desarguesian $(m-1)$-dimensional projective space over the finite field $\mathbb{F}_q$, where $q$ is a power of a prime $p$, and let $\mathbb{F}_q^m$ be the $m$-dimensional vector space associated with $\text{PG}(m-1, q)$. A projective $(n, m, h_1, h_2)$ set $O = \{\langle y_1 \rangle, \langle y_2 \rangle, \ldots, \langle y_n \rangle\}$ is a proper, non-empty set of $n$ points of the projective space $\text{PG}(m-1, q)$ with the property that every hyperplane meets $O$ in $h_1$ or $h_2$ points. Define $\Omega = \{v \in \mathbb{F}_q^m \setminus \{0\} \mid \langle v \rangle \in O\}$ to be the set of vectors in $\mathbb{F}_q^m$ corresponding to $O$, i.e., $\Omega = \mathbb{F}_q^*O$.

For $(w_1, w_2, \ldots, w_m) \in \mathbb{F}_q^m$, we define a character of the additive group of $\mathbb{F}_q^m$ as follows:

\[
\chi_{(w_1, w_2, \ldots, w_m)} : (v_1, v_2, \ldots, v_m) \mapsto \xi_p^{\sum_{i=1}^m \text{tr}(w_i v_i)}, \quad (v_1, v_2, \ldots, v_m) \in \mathbb{F}_q^m,
\]
where $\xi_p$ is a complex primitive $p$-th root of unity and $\text{tr}$ is the trace from $\mathbb{F}_q$ to $\mathbb{F}_p$. It is easy to see that $\chi_{(w_1, w_2, \ldots, w_m)}$, $(w_1, w_2, \ldots, w_m) \in \mathbb{F}_q^m$, are all the characters of the additive group of $\mathbb{F}_q^m$.

For any nontrivial additive character $\chi_{(w_1, w_2, \ldots, w_m)}$ of $\mathbb{F}_q^m$, we have

$$\chi_{(w_1, w_2, \ldots, w_m)}(\Omega) = (q - 1)|((w_1, w_2, \ldots, w_m)^\perp \cap \{y_1, y_2, \ldots, y_n\})| + (-1)\left(n - |(w_1, w_2, \ldots, w_m)^\perp \cap \{y_1, y_2, \ldots, y_n\}|\right) = q|((w_1, w_2, \ldots, w_m)^\perp \cap \{y_1, y_2, \ldots, y_n\})| - n,$$

where $(w_1, w_2, \ldots, w_m)^\perp = \{((x_1, x_2, \ldots, x_m)) \in \text{PG}(m - 1, q) \mid \sum_{i=1}^m w_i x_i = 0\}$. Using this formula for $\chi_{(w_1, w_2, \ldots, w_m)}(\Omega)$ and Lemma 1.1, one can prove the following lemma.

**Lemma 1.2.** Let $\mathcal{O}$ and $\Omega$ be defined as above. Then $\mathcal{O}$ is a projective $(n, m, h_1, h_2)$ set in $\text{PG}(m - 1, q)$ if and only if $\chi_{(w_1, w_2, \ldots, w_m)}(\Omega) = q h_1 - n$ or $q h_2 - n$, for every nontrivial additive character $\chi_{(w_1, w_2, \ldots, w_m)}$, $(w_1, w_2, \ldots, w_m) \in \mathbb{F}_q^m$. In other words, $\mathcal{O}$ is a projective $(n, m, h_1, h_2)$ set in $\text{PG}(m - 1, q)$ if and only if $\Omega$ is a $(q^m, (q - 1)n, \lambda, \mu)$ partial difference set in the elementary abelian group $(\mathbb{F}_q^m, +)$, where $\lambda = (q - 1)n + (qh_1 - n)(qh_2 - n) + q(h_1 + h_2) - 2n$, and $\mu = (q - 1)n + (qh_1 - n)(qh_2 - n)$.

1.2. **Quadratic forms.** Let $\mathbb{F}_q$ be the field of $q$ elements, where $q$ is a prime power, and let $V$ be an $m$-dimensional vector space over $\mathbb{F}_q$. A function $Q : V \rightarrow \mathbb{F}_q$ is called a quadratic form if

(i). $Q(\alpha v) = \alpha^2 Q(v)$ for all $\alpha \in \mathbb{F}_q$ and $v \in V$;

(ii). the function $B : V \times V \rightarrow \mathbb{F}_q$ defined by $B(v_1, v_2) = Q(v_1 + v_2) - Q(v_1) - Q(v_2)$ is bilinear.

We call $Q$ nonsingular if the subspace $W$ with the property that $Q$ vanishes on $W$ and $B(w, v) = 0$ for all $v \in V$ and $w \in W$ is the zero subspace. If the field $\mathbb{F}_q$ has odd characteristic, then $Q$ is nonsingular if and only if $B$ is nondegenerate; but this may not be
true when \( F_q \) has characteristic 2, because in that case \( Q \) may not be zero on the radical \( \text{Rad}(V) = \{ w \in V \mid B(w,v) = 0 \text{ for all } v \in V \} \). However, if \( V \) is an even-dimensional vector space over an even-characteristic field \( F_q \), then \( Q \) is nonsingular if and only if \( B \) is nondegenerate (cf. [5, p. 14]).

Let \( Q \) be a nonsingular quadratic form on an \( m \)-dimensional vector space \( V \) over \( F_q \). If \( m \) is odd, then \( Q \) is equivalent to a quadratic form \( x_1x_2 + x_3x_4 + \cdots + x_{m-2}x_{m-1} + cx_m^2 \) for some scalar \( c \in F_q \) and it is called a parabolic quadratic form. If \( m \) is even, then \( Q \) is equivalent to either a quadratic form \( x_1x_2 + x_3x_4 + \cdots + x_{m-1}x_m \) (called hyperbolic, or type +1) or \( x_1x_2 + x_3x_4 + \cdots + x_{m-3}x_{m-2} + p(x_{m-1}, x_m) \), where \( p(x_{m-1}, x_m) \) is an irreducible quadratic form in two indeterminates (called elliptic, or type \(-1\)).

The quadric of the projective space \( \text{PG}(m-1, q) \) corresponding to a quadratic form \( Q \) is the point set \( Q = \{ (v) \in \text{PG}(m-1, q) \mid Q(v) = 0 \} \). The following theorems about the intersections of hyperplanes with quadrics in \( \text{PG}(m-1, q) \) are well-known, see for example [3, 2, p. 151], [4].

**Theorem 1.3.** Let \( Q \) be a nonsingular elliptic quadric in \( \text{PG}(2\ell-1, q) \). Then the hyperplanes of \( \text{PG}(2\ell-1, q) \) intersect \( Q \) in sets of two sizes \( a \) and \( b \), where

\[
a = 1 + \frac{q(q^\ell-1+1)(q^{2\ell-2} - 1)}{q-1}, \quad b = \frac{q^{2\ell-2} - 1}{q-1}.
\]

If we use \( \Omega \) to denote the set of nonzero vectors in \( \mathbb{F}_q^{2\ell} \) corresponding to \( Q \), then for any nontrivial additive character \( \chi \) of \( \mathbb{F}_q^{2\ell} \), we have \( \chi(\Omega) = (q^{\ell-1} - 1) - q^\ell \) or \( (q^{\ell-1} - 1) \) according as the hyperplane of \( \text{PG}(2\ell-1, q) \) corresponding to \( \chi \) meets \( Q \) in \( a \) or \( b \) points.

That is, \( \Omega \) is a \( (q^{2\ell}, q^{\ell} + 1)(q^{\ell-1} - 1), q^{2\ell-2} - q^{\ell-1} - 2, q^{2\ell-2} - q^{\ell-1}) \)-negative Latin square type PDS in the additive group of \( \mathbb{F}_q^{2\ell} \).

**Remarks** (1). Let \( \mathcal{H} \) be a hyperplane of a projective space \( \text{PG}(m-1, q) \), and denote by \( W \) a point outside \( \mathcal{H} \). If \( Q \) is a nonsingular quadric in \( \mathcal{H} \) then the set

\[
\mathcal{C} = \bigcup_{X \in Q}(WX)
\]
is called a cone with vertex $W$ over $Q$. Here $WX$ means the set of points on the line through $W$ and $X$.

(2). The hyperplanes meeting a nonsingular elliptic quadric $Q$ in $\text{PG}(2\ell - 1, q)$ in sets of size $a$ are called tangent hyperplanes; such a hyperplane meets $Q$ in a cone over a nonsingular elliptic quadric in $\text{PG}(2\ell - 3, q)$. Any nontangent hyperplane meets $Q$ in a nonsingular parabolic quadric in that hyperplane.

(3). Similarly, let $\Omega$ be the set of nonzero vectors in $\mathbb{F}_q^{2\ell}$ corresponding to a nonsingular hyperbolic quadric in $\text{PG}(2\ell - 1, q)$. Then for any nontrivial additive character $\chi$ of $\mathbb{F}_q^{2\ell}$, we have $\chi(\Omega) = q^{\ell - (q^{\ell - 1} + 1)}$ or $-(q^{\ell - 1} + 1)$. That is, $\Omega$ is a $(q^{2\ell}, (q^{\ell} - 1)(q^{\ell - 1} + 1), q^{2\ell - 2} + q^{\ell - 1}(q - 1) - 2, q^{2\ell - 2} + q^{\ell - 1})$-Latin square type PDS in the additive group of $\mathbb{F}_q^{2\ell}$.

Even though parabolic quadrics do not give rise to PDSs, we will also need their intersection patterns with hyperplanes.

**Theorem 1.4.** Let $Q'$ be a nonsingular (parabolic) quadric in $\text{PG}(2\ell - 2, q)$ (so the size of $Q'$ is $\frac{q^{2\ell - 2} - 1}{q - 1}$). Then the hyperplanes of $\text{PG}(2\ell - 2, q)$ intersects $Q'$ in sets of three sizes $t$, $h$, and $e$ with respective multiplicities $T$, $H$ and $E$, where

\[
\begin{align*}
t &= \frac{q^{2\ell - 3} - 1}{q - 1}, \\
T &= \frac{q^{2\ell - 2} - 1}{q - 1}, \\
e &= t - q^{\ell - 2}, \\
E &= \frac{q^{2\ell - 2} - q^{\ell - 1}}{2}, \\
h &= t + q^{\ell - 2}, \\
H &= \frac{q^{2\ell - 2} + q^{\ell - 1}}{2}.
\end{align*}
\]

**Remarks** (1). The $T$ hyperplanes with intersection size $t$ are called tangent hyperplanes to $Q'$, such a hyperplane meets $Q'$ in a cone over a parabolic quadric. Each of the $E$ hyperplanes with intersection size $e$ meets $Q'$ in a nonsingular elliptic quadric in that hyperplane, and each of the $H$ hyperplanes with intersection size $h$ meets $Q'$ in a nonsingular hyperbolic quadric in that hyperplane.
(2). If we use \( \Omega' \) to denote the set of nonzero vectors in \( \mathbb{F}_q^{2\ell-1} \) corresponding to \( Q' \), then for any nontrivial additive character \( \chi \) of \( \mathbb{F}_q^{2\ell-1} \), we have \( \chi(\Omega') = -1, -1 - q^{\ell-1}, \) or \(-1 + q^{\ell-1}\). (Here the character values \(-1, -1 - q^{\ell-1}, \) and \(-1 + q^{\ell-1}\) correspond respectively to cone section, elliptic section and hyperbolic section.)

As in the study of all other types of difference sets, one of the central problems in the study of partial difference sets is that for a given parameter set, which groups of the appropriate order contain a partial difference set with these parameters. As far as we know, no examples are known of negative Latin square type PDSs in nonelementary abelian groups, and this paper will construct the first such PDSs. We will focus on the case \( q = 4 \), with \( \mathbb{F}_4 = \{0, 1, \alpha, \alpha^2 = \alpha + 1\} \). We define a quadratic form on \( \mathbb{F}_4^{2\ell} \)

\[
Q_{\ell,j}(x_1, x_2, \ldots, x_{2\ell}) = (\alpha x_1^2 + x_1 x_2 + x_2^2) + (\alpha x_3^2 + x_3 x_4 + x_4^2) + \cdots \\
+ (\alpha x_{2j-1}^2 + x_{2j-1} x_{2j} + x_{2j}^2) + x_{2j+1} x_{2j+2} + x_{2j+3} x_{2j+4} + \cdots + x_{2\ell-1} x_{2\ell}.
\]

**Lemma 1.5.** Let \( Q_{\ell,j}(x_1, x_2, \ldots, x_{2\ell}) \) be the quadratic form on \( \mathbb{F}_4^{2\ell} \) defined above. Then

(a). When \( j \geq 2 \), \( Q_{\ell,j} \) is projectively equivalent to \( Q_{\ell,j-2} \).

(b). When \( j \) is odd, \( Q_{\ell,j} \) is elliptic. When \( j \) is even, \( Q_{\ell,j} \) is hyperbolic.

(c). \( Q_{\ell,j} \) is nonsingular.

(d). \( Q_{\ell,j}(0, x_2, x_3, \ldots, x_{2\ell}) \) is a nonsingular parabolic quadratic form in \( 2\ell - 1 \) indeterminates.

(e). \( Q_{\ell,j}(x_1, x_2, \ldots, x_{2i-1}, x_{2i-1} + x_{2i}, \ldots, x_{2j-1}, x_{2j}, x_{2j+1}, \ldots, x_{2\ell}) = Q_{\ell,j}(x_1, x_2, \ldots, x_{2\ell}) \) for any \( 1 \leq i \leq j \).

**Proof:** (a). The mapping \( x_{2j-3} \mapsto (x'_{2j-3} + x'_{2j-1} + x'_{2j}) \); \( x_{2j-2} \mapsto (\alpha x'_{2j-3} + x'_{2j-2}) \); \( x_{2j-1} \mapsto (x'_{2j-1} + x'_{2j}) \); \( x_{2j} \mapsto (\alpha x'_{2j-3} + x'_{2j-2} + x'_{2j}) \); all other \( x_i \mapsto x'_i \) is an invertible linear transformation from \( Q_{\ell,j} \) to \( Q_{\ell,j-2} \) as required.
(b). If $j$ is even, then $Q_{\ell,j}$ is projectively equivalent to $Q_{\ell,0} = x_1x_2 + x_3x_4 + \cdots + x_{2\ell-1}x_{2\ell}$, which is hyperbolic. Similarly, if $j$ is odd, then $Q_{\ell,j}$ is projectively equivalent to $Q_{\ell,1} = \alpha x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_5x_6 + \cdots + x_{2\ell-1}x_{2\ell}$, which is elliptic.

(c). Let $B(x, x')$ be the bilinear form associated with $Q_{\ell,j}$. Straightforward computations show that

$$B(x, x') = x_1x'_2 + x_2x'_1 + \cdots + x_{2\ell-1}x'_{2\ell-1} + x_{2\ell}x'_{2\ell-1},$$

which is nondegenerate. Hence $Q_{\ell,j}$ is nonsingular.

(d). According to Theorem 22.2.1 in [12], we define the matrix $A = [a_{ik}]$, where $a_{ii} = 2a_i$, $a_{ki} = a_{ik}$ for $i < k$. Here $a_1 = 1$, $a_2 = \alpha$, $a_3 = 1$, $a_4 = \alpha$, $a_5 = 1$, $a_6 = \alpha$, $a_7 = 1$, $a_8 = \alpha$, $a_9 = 1$, $a_{10} = 1$, $a_{11} = 1$, $a_{12} = 1$, $a_{13} = 1$, $a_{14} = \cdots = 0$, $a_{23} = 1$, $a_{45} = 1$, etc.

View $A$ as a matrix over $\mathbb{Z}$, and view $\alpha$ as an indeterminate for the time being. Compute $\Delta = \frac{1}{2}\det(A) = (4\alpha - 1)^j - 1(-1)^{\ell-j}$. Now view $\Delta$ modulo 2, we have $\Delta \neq 0$, by part (i) of Theorem 22.2.1 in [12], this shows that $Q_{\ell,j}(0, x_2, x_3, \ldots, x_{2\ell})$ is nonsingular, and it is necessarily parabolic (note: the associated bilinear form for this quadratic form is degenerate, so we need the more sophisticated argument to demonstrate nonsingularity).

(e). Straightforward computation, which we omit. \hfill \Box

We note that part (e) of the previous lemma will be used in many character sum computations to get a sum of 0 over the pair of elements $(x_1, x_2, \ldots, x_{2\ell-1}, x_{2\ell})$ and $(x_1, x_2, \ldots, x_{2\ell-1}, x_{2\ell-1} + x_{2\ell}, \ldots, x_{2\ell})$.

1.3. **Galois ring preliminaries.** We need to recall the basics of Galois rings. Interested readers are referred to Hammons, et. al [11] for more details. We will only use Galois rings over $\mathbb{Z}_4$. A Galois ring over $\mathbb{Z}_4$ of degree $t$, $t \geq 2$, denoted $\text{GR}(4, t)$, is the quotient ring $\mathbb{Z}_4[x]/\langle \Phi(x) \rangle$, where $\Phi(x)$ is a basic primitive polynomial in $\mathbb{Z}_4[x]$ of degree $t$. Hensel’s lemma implies that such polynomials exist. If $\xi$ is a root of $\Phi(x)$ in $\text{GR}(4, t)$, then
GR(4, t) = \mathbb{Z}_4[\xi] and the multiplicative order of \xi is 2^t - 1. In this paper, we will only need GR(4, 2), and that has the basic primitive polynomial \Phi(x) = x^2 + x + 1.

The ring \( R = \text{GR}(4, t) \) is a finite local ring with unique maximal ideal \( 2R \), and \( R/2R \) is isomorphic to the finite field \( \mathbb{F}_{2^t} \). If we denote the natural epimorphism from \( R \) to \( R/2R \cong \mathbb{F}_{2^t} \) by \( \pi \), then \( \pi(\xi) \) is a primitive element of \( \mathbb{F}_{2^t} \).

The set \( \mathcal{T} = \{0, 1, \xi, \xi^2, \ldots, \xi^{2^t-2}\} \) is a complete set of coset representatives of \( 2R \) in \( R \). This set is usually called a Teichmüller system for \( R \). The restriction of \( \pi \) to \( \mathcal{T} \) is a bijection from \( \mathcal{T} \) to \( \mathbb{F}_{2^t} \), and we refer to this bijection as \( \pi_\mathcal{T} \). An arbitrary element \( \beta \) of \( R \) has a unique \( 2 \)-adic representation

\[
\beta = \beta_1 + 2\beta_2,
\]

where \( \beta_1, \beta_2 \in \mathcal{T} \). Combining \( \pi_\mathcal{T} \) with this \( 2 \)-adic representation and specializing to the case of \( \text{GR}(4, 2) \), we get a bijection \( F_k \) from \( \mathbb{F}_4^{2^t} \) to \( (\text{GR}(4, 2))^k \times \mathbb{F}_4^{2^t - 2k} \) defined by

\[
F_k : (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, \ldots, x_{2\ell}) \mapsto (\pi_\mathcal{T}^{-1}(x_1) + 2\pi_\mathcal{T}^{-1}(x_2), \ldots, \pi_\mathcal{T}^{-1}(x_{2k-1}) + 2\pi_\mathcal{T}^{-1}(x_{2k}), \ldots, x_{2k+1}, x_{2k+2}, \ldots, x_{2\ell}).
\]

The inverse of this map is the map \( F_k^{-1} \) from \( (\text{GR}(4, 2))^k \times \mathbb{F}_4^{2^t - 2k} \) to \( \mathbb{F}_4^{2^t} \),

\[
F_k^{-1} : (\xi_1 + 2\xi_2, \ldots, \xi_{2k-1} + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) \mapsto (\pi_\mathcal{T}(\xi_1), \pi_\mathcal{T}(\xi_2), \ldots, \pi_\mathcal{T}(\xi_{2k-1}), \pi_\mathcal{T}(\xi_{2k}), \ldots, \xi_{2k+1}, \xi_{2k+2}, \ldots, \xi_{2\ell}).
\]

To simplify the notation we will usually omit the subindex in the bijection \( \pi_\mathcal{T}^{-1} : \mathbb{F}_{2^t} \to \mathcal{T} \). (So from now on, \( \pi^{-1} \) means the inverse of the bijection \( \pi_\mathcal{T} : \mathcal{T} \to \mathbb{F}_{2^t} \).) We will show in the next section how we can use \( F_k \) as a character sum preserving bijection to construct a PDS in a nonelementary abelian group, namely the additive group of \( (\text{GR}(4, 2))^k \times \mathbb{F}_4^{2^t - 2k} \).
The Frobenius map $f$ from $R$ to itself is the ring automorphism $f : \beta_1 + 2\beta_2 \mapsto \beta_1^2 + 2\beta_2^2$. This map is used to define the trace $\text{Tr}$ from $R$ to $\mathbb{Z}_4$, namely, $\text{Tr}(\beta) = \beta + \beta f + \cdots + \beta f^{t-1}$, for $\beta \in R$. We note here that the Galois ring trace $\text{Tr} : R \to \mathbb{Z}_4$, is related to the finite field trace $\text{tr} : F_{2^m} \to F_2$ via $\text{tr} \circ \pi = \pi \circ \text{Tr}$.\hspace*{1cm} (1.1)

As a consequence, we have $\sqrt{-1}^{\text{Tr}(2x)} = \sqrt{-1}^{2\text{Tr}(x)} = (-1)^{\text{Tr}(x)} = (-1)^{\text{tr}(\pi(x))}$, for all $x \in R$. The trace of a Galois ring can be used to define all of the additive characters of the ring, as demonstrated in the following well-known lemma.

**Lemma 1.6.** Let $\psi$ be an additive character of $R$. Then there is a $\beta \in R$ so that $\psi(x) = \sqrt{-1}^{\text{Tr}(\beta x)}$ for all $x \in R$.

Since we can write $\beta = \beta_1 + 2\beta_2$ for $\beta \in \text{GR}(4,2)$, where $\beta_k \in \mathcal{T}, k = 1, 2$, we will use the notation $\psi_\beta = \psi_{\beta_1 + 2\beta_2}$ indicating the ring element used to define the character $x \mapsto \sqrt{-1}^{\text{Tr}(\beta x)}$. If $\beta_1 = 0$ but $\beta_2 \neq 0$, then $\psi_{2\beta_2}$ is a character of order 2 and $\psi_{2\beta_2}$ is principal on $2R$. If $\beta_1 \neq 0$, then $\psi_{\beta_1 + 2\beta_2}$ is a character of order 4 and $\psi_{\beta_1 + 2\beta_2}$ is nonprincipal on $2R$. Characters of $(\text{GR}(4,2))^k \times \mathbb{F}_4^{2\ell - 2k}$ will be written

$$\Psi_k = \psi_{(\beta_1 + 2\beta_2, \ldots, \beta_{2k-1} + 2\beta_{2k})} \otimes \chi_{(w_{2k+1}, w_{2k+2}, \ldots, w_{2\ell})}$$ \hspace*{1cm} (1.2)

for $\beta_i \in \mathcal{T}, 1 \leq i \leq 2k$, and $w_i \in \mathbb{F}_4, 2k + 1 \leq i \leq 2\ell$.

2. **Construction of Partial Difference Sets in Nonelementary Abelian 2-Groups**

We are now ready to state the main result of this paper. For $0 \leq k \leq j \leq \ell, \ell \geq 1$, define

$$D_{\ell,j,k} = \{ F_k(x_1, x_2, \ldots, x_{2\ell}) | (x_1, x_2, \ldots, x_{2\ell}) \in \mathbb{F}_4^{2k}\backslash\{0\}, Q_{\ell,j}(x_1, x_2, \ldots, x_{2\ell}) = 0 \}.$$ \hspace*{1cm} (2.1)
That is, 

\[ D_{\ell,j,k} = \{(\xi_1 + 2\xi_2, \ldots, \xi_{2k-1} + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) \in (\text{GR}(4, 2)^k \times \mathbb{F}_4^{2\ell-2k}) \setminus \{0\} | Q_{\ell,j}(F_k^{-1}(\xi_1 + 2\xi_2, \ldots, \xi_{2k-1} + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell})) = 0 \} \]

We can think of \( D_{\ell,j,k} \) as a “lifting” of the set of nonzero vectors \((x_1, x_2, \ldots, x_{2\ell}) \in \mathbb{F}_4^{2\ell} \) satisfying \( Q_{\ell,j}(x_1, x_2, \ldots, x_{2\ell}) = 0 \).

**Theorem 2.1.** For \( j \) odd, \( 1 \leq k \leq j \leq \ell \), the set \( D_{\ell,j,k} \) is a \((4^{2\ell}, (4^\ell + 1)(4^{\ell-1} - 1), 4^{2\ell-2} - 3 \cdot 4^{\ell-1} - 2, 4^{2\ell-2} - 4^{\ell-1})\)-PDS in \( \mathbb{Z}_4^{2k} \times \mathbb{Z}_2^{4\ell-4k} \). For \( j \) even, \( 1 \leq k \leq j \leq \ell \), the set \( D_{\ell,j,k} \) is a \((4^{2\ell}, (4^\ell - 1)(4^{\ell-1} + 1), 4^{2\ell-2} + 3 \cdot 4^{\ell-1} - 2, 4^{2\ell-2} + 4^{\ell-1})\)-PDS in \( \mathbb{Z}_4^{2k} \times \mathbb{Z}_2^{4\ell-4k} \).

This theorem immediately leads to the following corollary, which lists the first known negative Latin square type PDSs in nonelementary abelian groups.

**Corollary 2.2.** There are \((4^{2\ell}, (4^\ell + 1)(4^{\ell-1} - 1), 4^{2\ell-2} - 3 \cdot 4^{\ell-1} - 2, 4^{2\ell-2} - 4^{\ell-1})\)-negative Latin square type PDS in \( \mathbb{Z}_4^{2k} \times \mathbb{Z}_2^{4\ell-4k} \) for every \( k \leq \ell \) except possibly \( \mathbb{Z}_4^{2\ell} \) for \( \ell \) even.

By Lemma 1.1, in order to prove Theorem 2.1, we demonstrate that all of the nonprincipal characters of \((\text{GR}(4, 2))^k \times \mathbb{F}_4^{2\ell-2k}\) have a sum over \( D_{\ell,j,k} \) of \(-4^{\ell-1} - 1 \pm 2 \cdot 4^{\ell-1} \) for \( j \) odd and \( 4^{\ell-1} - 1 \pm 2 \cdot 4^{\ell-1} \) for \( j \) even. By Theorem 1.3 and Remark 3 following that theorem, the set \( F_k^{-1}(D_{\ell,j,0}) = D_{\ell,j,0} \) is a PDS in the elementary abelian group of order \( 4^{2\ell} \), so it will have character sums equal to those we are expecting for \( D_{\ell,j,k} \). The following sequence of lemmas will indicate a connection between the character sums over \( D_{\ell,j,k} \) in the additive group of \((\text{GR}(4, 2))^k \times \mathbb{F}_4^{2\ell-2k}\) and the character sums over \( F_k^{-1}(D_{\ell,j,k}) \) in the additive group of \( \mathbb{F}_4^{2\ell} \). We first consider the characters of order two of the additive group of \((\text{GR}(4, 2))^k \times \mathbb{F}_4^{2\ell-2k}\).

**Lemma 2.3.** Let \( \Psi_k \) be a character of \((\text{GR}(4, 2))^k \times \mathbb{F}_4^{2\ell-2j} \) defined by (1.2) with \( \beta_{2l-1} = 0 \) for \( 1 \leq i \leq k \). Then \( \Psi_k(D_{\ell,j,k}) = \chi_{(\pi(\beta_2), \ldots, \pi(\beta_{2k}), 0, \ldots, 0, w_{2k+1}, w_{2k+2}, \ldots, w_{2\ell})}(F_k^{-1}(D_{\ell,j,k})) \).
Proof: Let $(\xi_1 + 2\xi_2, \xi_3 + 2\xi_4, \ldots, \xi_{2k-1} + 2\xi_{2k}, \alpha_{2k+1}, \alpha_{2k+2}, \ldots, \alpha_{2\ell}) \in D_{\ell,j,k}$. The character value of this element is

$$\Psi_k(\xi_1 + 2\xi_2, \xi_3 + 2\xi_4, \ldots, \xi_{2k-1} + 2\xi_{2k}, \alpha_{2k+1}, \alpha_{2k+2}, \ldots, \alpha_{2\ell})$$

$$= \sqrt{-1}^{Tr(\sum_{i=1}^k 2\beta_{2i}\xi_{2i-1})} (-1)^{tr(\sum_{i'=2k+1}^{2\ell} w_{i'}\alpha_{i'})}$$

$$= (-1)^{tr(\sum_{i=1}^k \pi(\beta_{2i})\pi(\xi_{2i-1}) + \sum_{i'=2k+1}^{2\ell} w_{i'}\alpha_{i'})}$$

$$= \chi(\pi(\beta_{2}),\ldots,\pi(\beta_{2k}),\pi(\xi_{2k}),\ldots,\pi(\xi_{2\ell}))(\pi(\xi_1),\pi(\xi_2),\ldots,\pi(\xi_{2k}),\alpha_{2k+1},\ldots,\alpha_{2\ell})$$

The second equality uses the fact that $\sqrt{-1}^{Tr(2\beta_{2i}\xi_{2i-1})} = (-1)^{tr(\pi(\beta_{2i}\xi_{2i-1}))}$ as mentioned earlier in the discussion on trace. (Here Tr is the trace from GR$(4,2)$ to $\mathbb{Z}_4$, and tr is the trace from $\mathbb{F}_4$ to $\mathbb{F}_2$.) This proves the lemma.

Thus, the character sums $\Psi_k(D_{\ell,j,k})$ associated to characters $\Psi_k$ of order two will have the correct sum. In order to prove Theorem 2.1, we only need compute $\Psi_k(D_{\ell,j,k})$, where $\Psi_k$ has order four.

Our strategy for proving Theorem 2.1 goes as follows. We will first prove Theorem 2.1 in the case $k = 1$, then prove the whole theorem by strong induction on $k$. We start by computing the character sum $\Psi_1(D_{\ell,j,1})$, where $\Psi_1 = \psi_{\beta_1 + 2\beta_2} \otimes \chi(w_3,w_4,\ldots,w_{2\ell})$ is a character of order 4, namely $\beta_1 \neq 0$. We will need the following definitions. Let

$$\Omega_0 = \{(2\xi_2,\xi_3,\ldots,\xi_{2\ell}) \in \text{GR}(4,2) \times \mathbb{F}_4^{2\ell-2} \setminus \{0\} \mid Q_{\ell,j}(F_1^{-1}(2\xi_2,\xi_3,\ldots,\xi_{2\ell})) = 0\}, \quad (2.2)$$

and let

$$O_0 = F_1^{-1}(\Omega_0) = \{(0,\pi(\xi_2),\xi_3,\ldots,\xi_{2\ell}) \in \mathbb{F}_4^{2\ell} \setminus \{0\} \mid Q_{\ell,j}(0,\pi(\xi_2),\xi_3,\ldots,\xi_{2\ell}) = 0\}. \quad (2.3)$$

We observe that $\Psi_1(\Omega_0) = \chi(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})(O_0)$. The next lemma shows their common sum when $\beta_1 \neq 0$. 


Lemma 2.4. Suppose $\chi_{(\pi(\beta_2), \pi(\beta_1), w_3, w_4, \ldots, w_{2\ell})}$ is a character of $\mathbb{F}_4^{2\ell}$ with $\beta_1, \beta_2 \in \mathcal{T}$ and $\beta_1 \neq 0$, and let $O_0$ be as above. Then

$$\chi_{(\pi(\beta_2), \pi(\beta_1), w_3, w_4, \ldots, w_{2\ell})}(O_0) = -1 \pm 4^{\ell-1}.$$ 

Proof: For convenience, let

$$O'_0 = \{(x_2, x_3, \ldots, x_{2\ell}) \in \mathbb{F}_4^{2\ell-1} \setminus \{0\} | Q_{\ell,j}(0, x_2, \ldots, x_{2\ell}) = 0\}.$$ 

By the definition of $O_0$, we have

$$\chi_{(\pi(\beta_2), \pi(\beta_1), w_3, w_4, \ldots, w_{2\ell})}(O_0) = \chi'_{(\pi(\beta_1), w_3, \ldots, w_{2\ell})}(O'_0).$$

Since $Q_{\ell,j}(0, x_2, x_3, \ldots, x_{2\ell})$ is a nonsingular parabolic quadratic form in $2\ell - 1$ variables (cf. Lemma 1.5(d)), the corresponding quadric in $\text{PG}(2\ell - 2, 4)$ has three intersection sizes with the hyperplanes, leading to three distinct character values of $-1 \pm 4^{\ell-1}$ or $-1$ (see Theorem 1.4 and the remarks following that theorem). To prove the lemma, we need to show that the condition $\beta_1 \neq 0$ excludes all characters that have a sum of $-1$ over $O'_0$. To do that, we observe that there are $4^{2\ell - 2} - 1$ nonprincipal characters satisfying $\pi(\beta_1) = 0$, which is the same number of characters that have a sum of $-1$ over $O'_0$ since the number of tangent hyperplanes to a nonsingular parabolic quadratic form in $\text{PG}(2\ell - 2, 4)$ is $\frac{4^{2\ell - 2} - 1}{4 - 1}$ (cf. Theorem 1.4). We will show that all of these characters $\chi'_{(0, w_3, \ldots, w_{2\ell})}$ do indeed have a sum of $-1$ over $O'_0$, implying that $\chi'_{(\pi(\beta_1), w_3, \ldots, w_{2\ell})}(O'_0), \pi(\beta_1) \neq 0$, are equal to $-1 \pm 4^{\ell}$.

For any nontrivial character $\chi'_{(0, w_3, \ldots, w_{2\ell})}$, we have

$$\chi'_{(0, w_3, \ldots, w_{2\ell})}(O'_0) = \sum_{x_2^2 + x_3 x_4 + \cdots + x_{2\ell-1} x_{2\ell} = 0} (-1)^{\text{tr}(w_3 x_3 + \cdots + w_{2\ell} x_{2\ell})}$$

$$= \sum_{(0, 0, \ldots, 0) \neq (x_3, x_4, \ldots, x_{2\ell}) \in \mathbb{F}_4^{2\ell-2}} (-1)^{\text{tr}(w_3 x_3 + \cdots + w_{2\ell} x_{2\ell})} \sum_{x_2^2 = x_3 x_4 + \cdots + x_{2\ell-1} x_{2\ell}} 1.$$
Note that the inner sum in the last summation actually has only one term. So
\[
\chi'_j(0, w_3, \ldots, w_{2\ell})(O'_0) = \sum_{(0,0,\ldots,0) \neq (x_3, x_4, \ldots, x_{2\ell}) \in \mathbb{F}_4^{2\ell-2}} (-1)^{\text{tr}(w_3x_3 + \cdots + w_{2\ell}x_{2\ell})} = -1.
\]
As \((w_3, \ldots, w_{2\ell})\) runs through all nonzero \(2\ell - 2\)-tuples, we get \(-1\) as the character sum of \(O'_0\) \((4^{2\ell-2} - 1)\) times, these account for all the tangent hyperplanes. Hence the lemma follows.

The following lemma considers the case \(j\) odd and \(\chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(O_0) = -1 - 4^{\ell-1}\), keeping in mind that the character sum over the entire set \(F_1^{-1}(D_{\ell,j,1})\) is \(-1 - 4^{\ell-1} \pm 2 \cdot 4^{\ell-1}\) since \(F_1^{-1}(D_{\ell,j,1})\) with \(j\) odd is a negative Latin square type PDS in the additive group of \(\mathbb{F}_4^{2\ell}\) (cf. Theorem 1.3).

**Lemma 2.5.** Let \(j\) be odd and \(\chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(O_0) = -1 - 4^{\ell-1}\), then
\[
\chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(F_1^{-1}(D_{\ell,j,1}) \setminus O_0) = \pm 2 \cdot 4^{\ell-1}.
\]

**Proof:** Obvious from the comments before the lemma.

The analogous lemma for the \(j\) even case is given as follows.

**Lemma 2.6.** Let \(j\) be even and \(\chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(O_0) = -1 + 4^{\ell-1}\), then
\[
\chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(F_1^{-1}(D_{\ell,j,1}) \setminus O_0) = \pm 2 \cdot 4^{\ell-1}.
\]

**Proof:** Note that when \(j\) is even, the character sum over the entire set \(F_1^{-1}(D_{\ell,j,1})\) is \(-1 + 4^{\ell-1} \pm 2 \cdot 4^{\ell-1}\) since \(F_1^{-1}(D_{\ell,j,1})\) with \(j\) even is a Latin square type PDS in the additive group of \(\mathbb{F}_4^{2\ell}\). The conclusion of the lemma is obvious from this observation.

We now consider the other case from Lemma 2.5, namely that \(\chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(O_0) = -1 + 4^{\ell-1}\) in the case \(j\) odd; and \(\chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(O_0) = -1 - 4^{\ell-1}\) in the case \(j\) even.
Lemma 2.7. Let $j$ be odd and let $\chi(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})$ be a character of $\mathbb{F}_2^{2\ell}$ with $\beta_1, \beta_2 \in T$, and $\beta_1 \neq 0$. If $\chi(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})(O_0) = -1 + 4^\ell - 1$, then

$$\chi(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})(F_1^{-1}(D_{\ell,j,1})\setminus O_0) = 0.$$ 

**Proof:** Since $j$ is odd, $F_1^{-1}(D_{\ell,j,1})$ corresponds to a nonsingular elliptic quadric in $\text{PG}(2\ell-1,4)$, hence its character values $\chi(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})(F_1^{-1}(D_{\ell,j,1}))$ are $-1 - 4^\ell - 1 \pm 2 \cdot 4^\ell - 1$. Assume to the contrary that $\chi(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})(F_1^{-1}(D_{\ell,j,1})\setminus O_0) \neq 0$. By the assumption that $\chi(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})(O_0) = -1 + 4^\ell - 1$, we have

$$\chi(\pi(\beta_2),\pi(\beta_1),w_3,\ldots,w_{2\ell})(F_1^{-1}(D_{\ell,j,1})) = -1 - 4^\ell - 1 - 2 \cdot 4^\ell - 1,$n that is, the hyperplane $\mathcal{H} : \pi(\beta_2)x_1 + \pi(\beta_1)x_2 + w_3x_3 + \cdots + w_{2\ell}x_{2\ell} = 0$ meets $Q_{\ell,j}$ in a cone $\mathcal{C}$ with vertex $W$ over a nondegenerate elliptic quadric in $\text{PG}(2\ell-3,4)$. (Here $Q_{\ell,j}$ is the elliptic quadric defined by $Q_{\ell,j}$ in $\text{PG}(2\ell-1,4)$.) We write $\mathcal{H} \cap Q_{\ell,j} = \mathcal{C}.$

Let $\mathcal{H}_1$ denote the hyperplane $x_1 = 0$. We know that $\mathcal{H}_1 \cap Q_{\ell,j} = Q'_{\ell,j}$ is a nonsingular parabolic quadric with equation $x_2^2 + \alpha x_3^2 + x_4^2 + \cdots + x_{2\ell}x_{2\ell} = 0$, which will determine the character value $\chi(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})(O_0).$ We have

$$\mathcal{H} \cap Q'_{\ell,j} = \mathcal{H}_1 \cap (\mathcal{H} \cap Q_{\ell,j}) = \mathcal{H}_1 \cap \mathcal{C}.$$ 

Note that $\mathcal{C}$ is a cone over an elliptic quadric, so any hyperplane not through the vertex $W$ meets $\mathcal{C}$ in an elliptic quadric [2 p. 177]. Hence if we can show that $\mathcal{H}_1$ does not go through the vertex $W$, then we know that $\mathcal{H}_1 \cap \mathcal{C} = \mathcal{H} \cap Q'_{\ell,j}$ is not hyperbolic, thus $\chi(\pi(\beta_2),\pi(\beta_1),w_3,w_4,\ldots,w_{2\ell})(O_0) \neq -1 + 4^\ell - 1$, which is a contradiction.

The cone $\mathcal{C}$ is a degenerate quadric with a 1-dimensional radical being its vertex $W$. So we compute the radical of $\mathcal{C} = \mathcal{H} \cap Q_{\ell,j}$. Let $B(X,X') = Q_{\ell,j}(X + X') - Q_{\ell,j}(X) - Q_{\ell,j}(X')$ be the bilinear form associated with $Q_{\ell,j}$, where $X = (x_1, x_2, \ldots, x_{2\ell})$ and $X' = (x'_1, x'_2, \ldots, x'_{2\ell})$. Then...
\[
\text{Rad}(\mathcal{H} \cap \mathcal{Q}_{\ell,j}) = \{ X \mid B(X, X') = 0 \text{ for all } X' \in \mathcal{H} \}
\]
\[
= \{(x_1, x_2, \ldots, x_{2\ell}) \mid x_1x_2' + x_2x_1' + \cdots + x_{2\ell-1}x_{2\ell}' + x_{2\ell}x_{2\ell-1}' = 0, \text{ for all } (x_1', x_2', \ldots, x_{2\ell}') \in \mathcal{H} \}
\]
\[
= \{\epsilon((\pi(\beta_1), \pi(\beta_2), w_4, w_3, \ldots, w_{2\ell}, w_{2\ell-1}) \mid \epsilon \in \mathbb{F}_4^* \}
\]

Therefore the vertex of \( C \) is \( W = (\pi(\beta_1), \pi(\beta_2), w_4, w_3, \ldots, w_{2\ell}, w_{2\ell-1}) \). Since \( \beta_1 \neq 0 \), we see that \( \mathcal{H}_1 : x_1 = 0 \) does not go through \( W \). The proof is now complete.

The analogous lemma for the \( j \) even case is given below.

**Lemma 2.8.** Let \( j \) be even and let \( \chi_{(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})} \) be a character of \( \mathbb{F}_4^{2\ell} \) with \( \beta_1, \beta_2 \in \mathcal{T} \), and \( \beta_1 \neq 0 \). If \( \chi_{(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})}(O_0) = -1 - 4^{\ell-1} \), then

\[
\chi_{(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})}(F_1^{-1}(D_{\ell,j,1})\backslash O_0) = 0.
\]

The proof of this lemma is completely parallel to that of Lemma 2.7. We omit the details.

The above five lemmas are enough to prove Theorem 2.1 in the case \( k = 1 \) for both \( j \) odd and even (see the proof below). For the purpose of doing induction on \( k \), we define,
for any integers $j, k$, $2 \leq k < j \leq \ell$, the following sets

\begin{align*}
\Upsilon_{k-1,k} &= \{(\xi_1 + 2\xi_2, \ldots, 0 + 2\xi_{2k-2}, 0 + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k}\}; \\
\Upsilon_{k-1} &= \{(\xi_1 + 2\xi_2, \ldots, 0 + 2\xi_{2k-2}, \xi_{2k-1} + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k}, \xi_{2k-1} \neq 0\}; \\
\Upsilon_{k} &= \{(\xi_1 + 2\xi_2, \ldots, \xi_{2k-3} + 2\xi_{2k-2}, 0 + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k}, \xi_{2k-3} \neq 0\}; \\
\Upsilon &= \{(\xi_1 + 2\xi_2, \ldots, \xi_{2k-3} + 2\xi_{2k-2}, \xi_{2k-1} + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k}, \xi_{2k-1} \neq 0, \xi_{2k-3} \neq 0\}; \\
U_{k-1,k} &= \{(\xi_1 + 2\xi_2, \ldots, 0, \pi(\xi_{2k-2}), 0, \pi(\xi_{2k}), \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k-2}\}; \\
U_{k-1} &= \{(\xi_1 + 2\xi_2, \ldots, 0, \pi(\xi_{2k-2}), \pi(\xi_{2k-1}), 0, \pi(\xi_{2k}), \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k-2}, \pi(\xi_{2k-1}) \neq 0\}; \\
U_{k} &= \{(\xi_1 + 2\xi_2, \ldots, \pi(\xi_{2k-3}), \pi(\xi_{2k-2}), 0, \pi(\xi_{2k}), \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k-2}, \pi(\xi_{2k-3}) \neq 0\}; \\
U &= \{(\xi_1 + 2\xi_2, \ldots, \pi(\xi_{2k-3}), \pi(\xi_{2k-2}), \pi(\xi_{2k-1}), 0, \pi(\xi_{2k}), \xi_{2k+1}, \ldots, \xi_{2\ell}) \in D_{\ell,j,k-2}, \\
&\quad \pi(\xi_{2k-1}) \neq 0, \pi(\xi_{2k-3}) \neq 0\};
\end{align*}

We observe that $D_{\ell,j,k} = \Upsilon_{k-1,k} \cup \Upsilon_{k-1} \cup \Upsilon_{k} \cup \Upsilon$ and $D_{\ell,j,k-2} = U_{k-1,k} \cup U_{k-1} \cup U_{k} \cup U$. The following lemma connects the character sum over $D_{\ell,j,k}$ with the character sum over $D_{\ell,j,k-2}$.

**Lemma 2.9.** For $k \geq 2$, let $\Psi_k$ be defined as in (1, 3), and let $\Psi_{k-2} = \psi(\beta_1 + 2\beta_2, \ldots, \beta_{2k-5} + 2\beta_{2k-4}) \otimes \chi(\pi(\beta_{2k-2}), \pi(\beta_{2k-3}), \pi(\beta_{2k}), \pi(\beta_{2k-1}), w_{2k+1}, \ldots, w_{2\ell})$. If $\beta_{2k-3}$ and $\beta_{2k-1}$ are both nonzero, then $\Psi_k(\Upsilon_{k-1,k}) = \Psi_{k-2}(U_{k-1,k}); \Psi_k(\Upsilon_{k-1}) = -\Psi_{k-2}(U_{k-1}); \Psi_k(\Upsilon_{k}) = -\Psi_{k-2}(U_{k});$ and $\Psi_k(\Upsilon) = \Psi_{k-2}(U)$.

**Proof:** It is straightforward to see that $\Psi_k(\Upsilon_{k-1,k}) = \Psi_{k-2}(U_{k-1,k})$. We will show that $\Psi_k(\Upsilon_{k-1}) = -\Psi_{k-2}(U_{k-1})$: the other arguments are similar. We use part (e) of Lemma 1.5 to organize our sum to take advantage of pairs of the form $\Psi_k(\xi_1 + 2\xi_2, \ldots, \xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell}) + \Psi_k(\xi_1 + 2\xi_2, \ldots, \xi_{2k} + 2(\xi_{2k-1} + \xi_{2k}), \xi_{2k+1}, \ldots, \xi_{2\ell}) = \Psi_k(\xi_1 + 2\xi_2, \ldots, \xi_{2k-1} + 2\xi_{2k}, \xi_{2k+1}, \ldots, \xi_{2\ell})(1 + (-1)^{\text{tr}(\pi(\beta_{2k-1})\pi(\xi_{2k-1}))}).$ This last term will be 0 unless $\pi(\xi_{2k-1}) = \pi(\beta_{2k-1}^{-1})$. We note that we will have the same term $(1 +
\((-1)^{tr({\pi(\beta_{2k-1})\pi(\xi_{2k-1})})}\) in the \(\Psi_{k-2}(U_{k-1})\) sum. Thus, this character sum, while originally over all elements of \(\Upsilon_{k-1}\) or \(U_{k-1}\), will only be over elements with \(\pi(\xi_{2k-1}) = \pi(\beta_{2k-1}^{-1})\). We can factor out the term \(i^{Tr(\beta_{2k-1}\xi_{2k-1})} = i^{Tr(\beta_{2k-1}\beta_{2k-1}^{-1})} = i^{Tr(1)} = -1\) from the \(\Psi_k(\Upsilon_{k-1})\) sum, and what is left will be \(\Psi_{k-2}(U_{k-1})\). Thus, \(\Psi_k(\Upsilon_{k-1}) = -\Psi_{k-2}(U_{k-1})\). \(\Box\)

We will use strong induction, and hence we will assume that the character sum \(\Psi_{k-2}(D_{\ell,j,k-2})\) will have the correct values. The following lemma shows how we can get a correct sum for \(\Psi_k(\Upsilon_{k-1})\) and \(\Psi_k(\Upsilon_k)\) when \(\beta_{2k-3} = \beta_{2k-1} \neq 0\).

**Lemma 2.10.** If \(\beta_{2k-3} = \beta_{2k-1} \neq 0\), then \(\Psi_k(\Upsilon_{k-1}) = \Psi_k(\Upsilon_k) = 0\).

**Proof:** Suppose \(\beta_{2k-3} = \beta_{2k-1} \neq 0\). We will show \(\Psi_k(\Upsilon_{k-1}) = 0\): the other case is similar. Using the fact that we only have to sum over elements of \(\Upsilon_{k-1}\) with \(\xi_{2k-1} = \beta_{2k-1}^{-1}\), and the fact that \(\pi(\xi_{2k-2}) = \alpha^2\pi(\xi_1) + \pi(\xi_1)^2\pi(\xi_2) + \pi(\xi_2) + \cdots + \pi(\xi_{2k-4}) + \alpha^2\pi(\xi_{2k-1}) + \pi(\xi_{2k-2})^2\pi(\xi_{2k}) + \cdots + \alpha^2\xi_{2j-1} + \xi_{2j}^2 + \xi_{2j+1}^2 + \cdots + \xi_{2\ell-1}^2\xi_{2\ell}\) from the quadratic form, we get

\[
\Psi_k(\Upsilon_{k-1}) = \sum_{\xi_{2k-3}=0} i^{Tr(\sum_{i=1}^k \beta_{2i-1}\xi_{2i-1})} (-1)^{tr(\sum_{i=1}^k (\pi(\beta_{2i-1}\xi_{2i}) + \pi(\beta_{2i}\xi_{2i-1}))) + w_{2k+1}\xi_{2k+1} + \cdots + w_{2\ell}\xi_{2\ell}}
\]

\[
= \sum_{\xi_{2k-3}=0} i^{Tr(\sum_{i=1}^{k-2} \beta_{2i-1}\xi_{2i-1} + \beta_{2k-1}\xi_{2k-1})} (-1)^{tr(\sum_{i=1}^{k-2} (\pi(\beta_{2i-1}\xi_{2i}) + \pi(\beta_{2i}\xi_{2i-1})))}
\]

\[
(-1)^{tr(\pi(\beta_{2k-3})(\alpha^2\pi(\xi_1) + \pi(\xi_1)^2\pi(\xi_2)) + \pi(\xi_2) + \cdots + \pi(\xi_{2k-4}) + \alpha^2\pi(\xi_{2k-1}) + (\pi(\beta_{2k-1})^{-1})^2\pi(\xi_{2k}) + \pi(\xi_{2k}) + \cdots + \xi_{2\ell-1}^2\xi_{2\ell})}
\]

\[
(-1)^{tr(\pi(\beta_{2k-2})(0) + \pi(\beta_{2k-1}\xi_{2k}) + \pi(\beta_{2k}\beta_{2k-1}^{-1}) + w_{2k+1}\xi_{2k+1} + \cdots + w_{2\ell}\xi_{2\ell})}
\]

This last sum ranges over arbitrary values for \(\xi_i\) except for \(i = 2k - 3\) and \(i = 2k - 1\), where the values are fixed. We can rearrange the sum so there is an inner sum over all possible values of \(\xi_{2k}\). If we do that, this inner sum will be \(\sum_{\xi_{2k}} (-1)^{tr((\pi(\beta_{2k-1}) + \pi(\beta_{2k-3}) + \pi(\beta_{2k}\beta_{2k-1}^{-1}))\pi(\xi_{2k}))}\).
Since $\beta_{2k-3} = \beta_{2k-1}$, this sum reduces to $\Sigma_{\xi_{2k}}(-1)^{tr(\pi(\beta_{2k-1})\pi(\xi_{2k}))}$, and this sum is 0, proving the lemma.

From Lemmas 2.9 and 2.10 we conclude that $\Psi_k(D_{\ell,j,k}) = \Psi_{k-2}(D_{\ell,j,k-2})$. Induction will tell us the value of $\Psi_{k-2}(D_{\ell,j,k-2})$, taking care of this case.

The final piece we need to prove the main theorem is to compute the value of $\Psi_k(\Upsilon_{k-1})$ when $\beta_{2k-3} \neq \beta_{2k-1}$, where both $\beta_{2k-3}$ and $\beta_{2k-1}$ are nonzero. We have the same inner sum as in the proof of the last lemma, but the inner sum in this case, namely $\Sigma_{\xi_{2k}}(-1)^{tr((\pi(\beta_{2k-1})+\pi(\beta_{2k-3})+\pi(\beta_{2k-3}^2\beta_{2k-1}^{-1}))\pi(\xi_{2k}))$ will be 0 since $\pi(\beta_{2k-1})+\pi(\beta_{2k-3})+\pi(\beta_{2k-3}^2\beta_{2k-1}^{-1}) = 0$ for all possible choices. We can extend this idea to each pair $(\xi_{2i-1},\xi_{2i})$, $i \leq k-2$ to get “inner sums” of $\Sigma_{\xi_{2i-1},\xi_{2i}}^k tr((\beta_{2i-1}\xi_{2i-1})(-1)^{tr((\pi(\beta_{2i-1})+\pi(\beta_{2i}+\pi(\beta_{2i-1})+\pi(\beta_{2i-1})+\pi(\beta_{2i-1})+\pi(\beta_{2i-1})))))\). We are assuming $\beta_{2i-1} \neq 0$, since otherwise we could combine $\Psi_k(D_{\ell,j,k}) = \Psi_{k-1}(D_{\ell,j,k-1})$ with the inductive hypothesis to get the correct character sum. (Note here that $\Psi_{k-1} = \psi(\beta_1+2\beta_2,...,2\beta_{2k-3}+2\beta_{2k-2}) \otimes \chi(\pi(\beta_{2k}),\pi(\beta_{2k-1}),\pi(\beta_{2k-2}),...)$.) We can use Lemma 1.5 (e) to restrict the possible values for $\xi_{2i-1}$, and then consider the sums over $\xi_{2i-1} = 0$ and $\xi_{2i-1} = \beta_{2i-1}^{-1}$ separately. These sums are $\Sigma_{\xi_{2i-1}=0,\xi_{2i}}(-1)^{tr((\pi(\beta_{2i-1})+\pi(\beta_{2i}))\pi(\xi_{2i}))$ and $(-1)^{tr((\pi(\beta_{2i})+\pi(\beta_{2i}))\pi(\xi_{2i}))}\Sigma_{\xi_{2i-1}=\beta_{2i-1},\xi_{2i}}(-1)^{tr((\pi(\beta_{2i-1})+\pi(\beta_{2i}-\pi(\beta_{2i}))\pi(\xi_{2i}))$. If $\beta_{2i-1} = \beta_{2k-3}$, then the first sum is 4 and the second sum is 0; if $\beta_{2i-1} \neq \beta_{2k-3}$, then the first sum is 0 and the second sum is $\pm 4$. Thus, in either case, the total sum over the pair $(\xi_{2i-1},\xi_{2i})$ is $\pm 4$. When $k < j$, the “inner sum” for pairs $(\xi_{2i-1},\xi_{2i})$, $k+1 \leq i \leq j$, will be $\Sigma_{\xi_{2i-1},\xi_{2i}}(-1)^{tr((w_{2i-1}\xi_{2i}+w_{2i}\xi_{2i})+\pi(\beta_{2k-3})(\alpha^2\xi_{2i}+\xi_{2i}))}$. This is the character sum of a quadratic form, which is $\pm 4$ (see [16] p. 341, Exercise 6.30).

For $i > j$, we have “inner sums” of the form $\Sigma_{\xi_{2i-1},\xi_{2i}}(-1)^{tr((\pi(\beta_{2k-3})^2\xi_{2i-1}\xi_{2i}+w_{2i-1}\xi_{2i})+w_{2i}\xi_{2i})}. Again this is the character sum of a quadratic form, which is $\pm 4$. Thus, we have $\ell - 1$ “inner sums”, each of which are $\pm 4$. This proves the following lemma.
Lemma 2.11. Suppose $β_{2i-1} \neq 0$ for $i \leq k$ and $β_{2k-3} \neq β_{2k-1}$. Then $Ψ_k(Υ_{k-1}) = ±4^{ℓ-1}$ and $Ψ_k(Υ_k) = ±4^{ℓ-1}$.

The sum $Ψ_k(Υ_{k-1}) + Ψ_k(Υ_k)$ is either 0 or $±2 \cdot 4^{ℓ-1}$. If the sum is 0, then we can use the same idea as in the comments after Lemma 2.10 to show that $Ψ_k(D_{ℓ,j,k}) = Ψ_{k-2}(D_{ℓ,j,k-2})$, and induction will tell us that this has the correct sum. The following lemma finishes the computations we need to prove the main theorem.

Lemma 2.12. Let $Ψ_k$ be defined as in (1.2) and $Ψ_{k-2}$ as in Lemma 2.9 with $β_{2k-3} \neq 0$ and $β_{2k-1} \neq 0$, and suppose $D_{ℓ,j,k-2}$ is a PDS (with parameters depending on $j$ odd or even).

If $j$ is odd and $Ψ_k(Υ_{k-1}) + Ψ_k(Υ_k) = ±2 \cdot 4^{ℓ-1}$, then $Ψ_k(Υ_{k-1,k}) + Ψ_k(Υ) = −1 − 4^{ℓ-1}$. If $j$ is even and $Ψ_k(Υ_{k-1}) + Ψ_k(Υ_k) = ±2 \cdot 4^{ℓ-1}$, then $Ψ_k(Υ_{k-1,k}) + Ψ_k(Υ) = −1 + 4^{ℓ-1}$.

Proof: By Lemma 2.9 $Ψ_{k-2}(U_{k-1,k}) + Ψ_{k-2}(U) = Ψ_{k-2}(D_{ℓ,j,k-2}) − Ψ_{k-2}(U_{k-1}) − Ψ_{k-2}(U_k) = Ψ_{k-2}(D_{ℓ,j,k-2}) + Ψ_k(Υ_{k-1}) + Ψ_k(Υ_k)$. If $j$ is odd and $Ψ_k(Υ_{k-1}) + Ψ_k(Υ_k) = 2 \cdot 4^{ℓ-1}$, then $Ψ_{k-2}(U_{k-1,k}) + Ψ_{k-2}(U) = (−1 − 4^{ℓ-1} ± 2 \cdot 4^{ℓ-1}) + 2 \cdot 4^{ℓ-1}$ (the case $Ψ_k(Υ_{k-1}) + Ψ_k(Υ_k) = −2 \cdot 4^{ℓ-1}$ is similar, as is the $j$ even case). We will show that $Ψ_{k-2}(U_{k-1,k}) + Ψ_{k-2}(U) = −1 + 3 \cdot 4^{ℓ-1}$ leads to a contradiction. Define a character $Ψ'_{k-2}$ to have $β'_i = β_i$ except

(i): $β'_{2k-2}$ chosen to satisfy $tr(π(β'_{2k-2}^{-1}β_{2k-3}^{-1})) = 1 + tr(π(β_{2k-2}^{-1}β_{2k-3}^{-1}))$

(ii): $β'_{2k}$ chosen to satisfy $tr(π(β'_{2k}β_{2k-1}^{-1})) = 1 + tr(π(β_{2k}β_{2k-1}^{-1}))$.

Neither $π(β_{2k-2})$ or $π(β_{2k})$ appears in $Ψ_{k-2}(U_{k-1,k})$, so $Ψ'_{k-2}(U_{k-1,k}) = Ψ_{k-2}(U_{k-1,k})$. Both $π(β_{2k-2})$ and $π(β_{2k})$ appear in $Ψ_{k-2}(U)$ in the term $(-1)^tr(π(β_{2k-2}β_{2k-3}^{-1}) + π(β_{2k}β_{2k-1}^{-1}))$.

(Here again we use the fact that if $π(ξ_{2i-1}) = π(β_{2i-1}^{-1})$ then Lemma 1.3 (e) will allow us to pair elements to sum to 0.) The conditions on $β'_{2k-2}$ and $β'_{2k}$ imply that

$(-1)^tr(π(β'_{2k-2}β_{2k-3}^{-1}) + π(β_{2k}β'_{2k-1}^{-1})) = (-1)^tr(π(β_{2k-2}β_{2k-3}^{-1}) + π(β_{2k}β_{2k-1}^{-1}))$, so $Ψ'_{k-2}(U) = Ψ_{k-2}(U)$.

The element $β_{2k}$ appears in $Ψ_{k-2}(U_{k-1})$ in the term $(-1)^tr(π(β_{2k}β_{2k-1}^{-1}))$, but $β_{2k-2}$ does not appear in this sum, implying that $Ψ'_{k-2}(U_{k-1}) = −Ψ_{k-2}(U_{k-1})$. Similarly, $Ψ'_{k-2}(U_k) = −Ψ_{k-2}(U_k)$. Hence,
\[ \Psi'_{k-2}(D_{\ell,j,k-2}) = \Psi'_{k-2}(U_{k-1,k}) + \Psi'_{k-2}(U_{k-1}) + \Psi'_{k-2}(U_k) \]
\[ = \Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U_{k-1}) - \Psi_{k-2}(U_k) \]
\[ = \Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U_{k-1}) + \Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_k) \]

If \( \Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) = -1 + 3 \cdot 4^{\ell-1} \), then \( \Psi'_{k-2}(D_{\ell,j,k-2}) = -1 + 5 \cdot 4^{\ell-1} \). We assumed that \( D_{\ell,j,k-2} \) was a PDS, but \( -1 + 5 \cdot 4^{\ell-1} \) is not a correct character sum for this set, so that contradiction proves the lemma.

**Proof of Theorem 2.1** We proceed by induction on \( k \). For the \( k = 1 \) case, Lemma 2.3 shows that the characters \( \Psi_1 \) of order 2 have the correct character sums. So we only need to consider characters \( \Psi_1 = \psi_{\beta_1 + 2\beta_2} \otimes \chi(w_1,w_2,...,w_{2\ell}) \) of order 4 (i.e., \( \beta_1 \neq 0 \)). We will only give the detailed arguments in the case \( j \) odd using Lemma 2.5 and 2.7. The case \( j \) even is similar (using Lemma 2.6 and 2.8).

We break \( \Psi_1(D_{\ell,j,1}) \) into three sums,

\[ \Psi_1(D_{\ell,j,1}) = \sum_{(\xi_1+2\xi_2,\xi_3,...,\xi_{2\ell}) \in D_{\ell,j,1}} (\sqrt{-1})^{\text{Tr}(\beta_1+2\beta_2)(\xi_1+2\xi_2)}(-1)^{\text{tr}(\sum_{i=3}^{2\ell} w_i \xi_i)} \]
\[ = \sum_{(\xi_1+2\xi_2,\xi_3,...,\xi_{2\ell}) \in D_{\ell,j,1}, \text{Tr}(\beta_1 \xi_1) = 0} (-1)^{\text{tr}(\beta_2 \xi_1+\beta_1 \xi_2)}(-1)^{\text{tr}(\sum_{i=3}^{2\ell} w_i \xi_i)} \]
\[ - \sum_{(\xi_1+2\xi_2,\xi_3,...,\xi_{2\ell}) \in D_{\ell,j,1}, \text{Tr}(\beta_1 \xi_1) = 2} (-1)^{\text{tr}(\beta_2 \xi_1+\beta_1 \xi_2)}(-1)^{\text{tr}(\sum_{i=3}^{2\ell} w_i \xi_i)} \]
\[ + \sum_{(\xi_1+2\xi_2,\xi_3,...,\xi_{2\ell}) \in D_{\ell,j,1}, \text{Tr}(\beta_1 \xi_1) = \text{odd}} (\sqrt{-1})^{\text{Tr}(\beta_1 \xi_1)}(-1)^{\text{tr}(\beta_2 \xi_1+\beta_1 \xi_2)}(-1)^{\text{tr}(\sum_{i=3}^{2\ell} w_i \xi_i)} \]

The third sum above is over elements of \( D_{\ell,j,1} \) with the property that \( \text{Tr}(\beta_1 \xi_1) \) is odd. In that case, the observation following Lemma 1.5 implies that each such element has a matching element so that the pair will combine for a character sum of 0, so the sum over
all these elements is 0. We note that this is also true for the sum of \( \chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell}) \) over the elements of \( F_{1}^{-1}(D_{\ell,j,1}) \) with \( \text{tr}(\pi(\beta_1)\pi(\xi_1)) = 1 \).

For the first sum above, note that \( \text{Tr}(\beta_1 \xi_1) = 0 \) implies that \( \xi_1 = 0 \), we see that the first sum is over the set of elements of \( D_{\ell,j,1} \) with \( \xi_1 = 0 \). Hence

\[
\sum_{(\xi_1 + 2\xi_2, \xi_3, \ldots, \xi_{2\ell}) \in D_{\ell,j,1}} (\sqrt{-1})^{\text{Tr}((\beta_1 + 2\beta_2)(2\xi_2))} (-1)^{\text{tr}(\sum_{i=3}^{2\ell} w_i \xi_i)} = \Psi_1(\Omega_0)
\]

\[
= \chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(O_0),
\]

where \( \Omega_0 \) and \( O_0 \) are defined in (2.2) and (2.3), respectively. By Lemma 2.4, the first sum is equal to \(-1 \pm 4^{\ell-1}\).

The second sum above is over elements of \( D_{\ell,j,1} \) satisfying \( \xi_1 \neq 0 \) and \( \text{Tr}(\beta_1 \xi_1) \) is even. Since both \( \beta_1 \) and \( \xi_1 \) are in the Teichm"uller set \( \mathcal{T} \), their product is in \( \mathcal{T} \) as well. The only nonzero element of \( \mathcal{T} \) with an even trace is 1, and \( \text{Tr}(1) = 2 \). This implies that the second sum is

\[
S = \sum_{(\beta_1^{-1} + 2\xi_2, \xi_3, \ldots, \xi_{2\ell}) \in D_{\ell,j,1}} \chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(\pi(\xi_1), \pi(\xi_2), \xi_3, \ldots, \xi_{2\ell})
\]

Combining this with the observation after our analysis of the third sum, we see that

\[
S = \chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(F_{1}^{-1}(D_{\ell,j,1}) \setminus O_0).
\]

Hence we finally have

\[
\Psi_1(D_{\ell,j,1}) = \chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(O_0) - \chi(\pi(\beta_2), \pi(\beta_1), w_3, \ldots, w_{2\ell})(F_{1}^{-1}(D_{\ell,j,1}) \setminus O_0). \quad (2.4)
\]

Thus, using Lemmas 2.4, 2.5 and 2.7 we get \( \Psi_1(D_{\ell,j,1}) = -1 - 4^{\ell-1} \pm 2 \cdot 4^{\ell-1} \) as required. This completes the proof of the theorem in the case where \( k = 1 \).

For the inductive step, suppose that \( D_{\ell,j,k'} \) is a PDS for all \( k - 1 \geq k' \geq 0, \ell \geq j \geq k > 1 \), with the appropriate parameters depending on \( j \) odd or even. To show that \( D_{\ell,j,k} \) is also
a PDS, we compute the character sums $\Psi_k(D_{\ell,j,k})$, where

$$\Psi_k = \psi(\beta_1 + 2\beta_2, ..., \beta_{2k-1} + 2\beta_{2k}) \otimes \chi(w_{2k+1}, ..., w_{2\ell}).$$

As in the $k = 1$ case, Lemma 2.3 shows that the characters $\Psi_k$ of order 2 have the correct character sum. For characters $\Psi_k$ of order 4, we consider three cases.

The first case is when there is a $\beta_{2i-1} = 0$ for some $1 \leq i \leq k$. We can assume without loss of generality that $i = k$, permuting the coordinates if necessary. In this case, $\Psi_k(D_{\ell,j,k}) = \Psi_{k-1}(D_{\ell,j,k-1})$, and the inductive hypothesis implies that this sum is correct (note here that $\Psi_{k-1} = \psi(\beta_1 + 2\beta_2, ..., \beta_{2k-3} + 2\beta_{2k-2}) \otimes \chi(\pi(\beta_{2k}), 0, w_{2k+1}, ..., w_{2\ell})$ since $\pi(\beta_{2k-1}) = 0$).

The second case is when none of the $\beta_{2i-1}$ are 0 for $i \leq k$ but there is a pair $\beta_{2i-1} = \beta_{2i'}-1$ for $i, i' \leq k, i \neq i'$. Again, without loss of generality we can assume that $i = k - 1$ and $i' = k$. By applying Lemmas 2.9 and 2.10 we get

$$\Psi_k(D_{\ell,j,k}) = \Psi_k(\Upsilon_{k-1,k}) + \Psi_k(\Upsilon_{k-1,k-1}) + \Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon) = \Psi_{k-2}(U_{k-1,k}) + \Psi_{k-2}(U) = \Psi_{k-2}(D_{\ell,j,k-2})$$

Induction tells us that $\Psi_k(D_{\ell,j,k})$ has the correct value.

The final case is when none of the $\beta_{2i-1}$ are 0 for $i \leq k$, and all of the $\beta_{2i-1}$ are distinct. According to Lemma 2.11 there are 3 possibilities for $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_{k})$: 0 or $\pm 2 \cdot 4^{\ell-1}$. If the sum is 0, then the remarks after Lemma 2.10 indicate that $\Psi_k(D_{\ell,j,k}) = \Psi_{k-2}(D_{\ell,j,k-2})$, so induction implies that the character sum has the correct value. When $\Psi_k(\Upsilon_{k-1}) + \Psi_k(\Upsilon_{k}) = \pm 2 \cdot 4^{\ell-1}$, Lemma 2.12 shows that $\Psi_k(\Upsilon_{k-1,k}) + \Psi_k(\Upsilon) = -1 - 4^{\ell-1}$ when $j$ is odd, so $\Psi_k(D_{\ell,j,k}) = -1 - 4^{\ell-1} \pm 2 \cdot 4^{\ell-1}$ as required (the $j$ even case is similar).

Thus, in all cases the character sum $\Psi_k(D_{\ell,j,k})$ is as required, proving the theorem. \qed
Remarks (1). We have checked that lifting a nonsingular elliptic quadric in PG(3, q), where q ≠ 4, in the same way as in Theorem 2.1 (namely using a straightforward bijection from multiple copies of the field to the Teichmüller set representation of the Galois ring elements), will not produce a PDS in nonelementary abelian 2-group.

(2). Since $D_{2,1,0}$ is a PDS in $G_0 = (\mathbb{F}_4^4, +)$ (here $D_{2,1,0}$ corresponds to an elliptic quadric in PG(3, 4)), the Cayley graph $(G_0, D_{2,1,0})$ is a strongly regular graph with vertex set $G_0$. Using GAP, it is found that the full automorphism group of this graph has order $2^{12} \cdot 3^2 \cdot 5 \cdot 17$. Similarly, since $D_{2,1,1}$ is a PDS in $G_1 = Z_4^2 \times \mathbb{F}_4^2$, the Cayley graph $(G_1, D_{2,1,1})$ is a strongly regular graph with vertex set $G_1$. Again by using GAP, it is found that the full automorphism group of this graph has order $2^{12} \cdot 3^2 \cdot 5$. Even though the two strongly regular Cayley graphs $(G_0, D_{2,1,0})$ and $(G_1, D_{2,1,1})$ have the same parameters, they have different full automorphism groups, hence they are nonisomorphic. Based on further computations of automorphism groups, we conjecture that the strongly regular Cayley graphs arising from $D_{t,j,k}$, $k > 0$, are never isomorphic to the classical strongly regular Cayley graphs arising from quadratic forms over $\mathbb{F}_4$.

(3) The PDSs $D_{t,j,k}$ and $D_{t,j',k}$ are equivalent if $j - j'$ is even. The mapping defined in the proof of Lemma 1.5 (a) induces a group automorphism $\phi$ on $(\text{GR}(4, 2))^k \times \mathbb{F}_4^{2t-2k}$ satisfying $\phi(D_{t,j,k}) = D_{t,j-2,k}$ for $0 \leq k \leq j - 2$ ($\phi$ will fix all elements of the group $(\text{GR}(4, 2))^k \times \mathbb{F}_4^{2t-2k}$ except $x_{2j-3}$ through $x_{2j}$, and these are mapped as directed in the proof). We chose to do the proof in the general form since that made the induction easier to read, but we could have stated the result in terms of $D_{t,j,j}$ and $D_{t,j+1,j}$ and gotten a PDS in each equivalence class.

3. Future Directions

The following list describes possible implications of the results in this paper.

(1). Are there other groups of the appropriate order that support PDSs with the same parameters in this paper? In particular, are there other combinations of $\mathbb{Z}_4$ and $\mathbb{Z}_2$ that
contain PDSs? A good candidate might be to see if $\mathbb{Z}_4^4$ contains a PDS with negative Latin square parameters, finishing off the one case we can’t do in Corollary 2.2.

(2). Can the “bijection idea” produce any more sets where the character sums are so well preserved?

(3). Can we do anything in other contexts to get groups such as $\mathbb{Z}_8$ involved in a PDS group?

**Acknowledgement:** We thank Frank Fiedler for helping us compute the automorphism groups of some strongly regular graphs. The research of Qing Xiang was supported in part by NSA grant MDA 904-99-1-0012. We would also like to thank the anonymous referee for suggestions that improved the exposition of this paper.

**References**

[1] R.A.Bailey and D. Jungnickel, Translation nets and fixed-point-free group automorphisms, *J. Combin. Theory Ser. A*, 55 (1990), 1–13.

[2] A. Beutelspacher and U. Rosenbaum, *Projective Geometry: From Foundations to Applications*, Cambridge University Press, Cambridge, 1998.

[3] R. H. Bruck, Difference sets in a finite group, *Trans. Amer. Math. Soc.* 78 (1955), 311–317.

[4] R. A. Calderbank, W. M. Kantor, The geometry of two-weight codes, *Bull. London Math. Soc.* 18 (1986), 97–122.

[5] P. J. Cameron, Finite geometry and coding theory, Lecture Notes for Socrates Intensive Programme “Finite Geometries and Their Automorphisms”, Potenza, Italy, June 1999.

[6] Y. Q. Chen, D. K. Ray-Chaudhuri, Q. Xiang, Constructions of partial difference sets and relative difference sets using Galois rings. II, *J. Combin. Theory Ser. A*, 76 (1996), 179–196.

[7] J. A. Davis, Q. Xiang, A family of partial difference sets with Denniston parameters in nonelementary abelian 2-groups, *European J. Combin.* 21 (2000), 981–988.

[8] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Research Report*, Suppl. No. 10.

[9] R. A. Games, The geometry of quadrics and correlations of sequences, *IEEE Transactions on Information Theory*, IT-32 (1986), 423–426.
[10] M. Hagita, B. Schmidt, Bijections between group rings preserving character sums, *Designs, Codes and Cryptography*, **24** (2001), 243–254.

[11] A. R. Hammons, P. V. Kumar, A. R. Calderbank, N. J. A. Sloane, and P. Sole, The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes. *IEEE Transactions on Information Theory*, **40** (1994), 301–319.

[12] J. W. P. Hirschfeld, J. Thas, *General Galois Geometries*, The Clarendon Press, Oxford University Press, New York, 1991.

[13] Xiang-dong, Hou, Bent functions, partial difference sets, and quasi-Frobenius local rings. *Des. Codes Cryptogr.*, **20** (2000), no. 3, 251–268.

[14] Xiang-dong, Hou, Ka Hin Leung and Qing Xiang, New partial difference sets in $\mathbb{Z}_{p^2}^t$ and a related problem about Galois rings, *Finite Fields and Appl.*, **7** (2001), 165–188.

[15] M. Klin and W. J. Martin, personal conversation.

[16] R. Lidl and H. Niederreiter, *Finite Fields*, Cambridge University Press, Cambridge, 1997.

[17] K. H. Leung, S. L. Ma, Constructions of partial difference sets and relative difference sets on $p$-groups, *Bull. London Math. Soc.*, **22** (1990), 533–539.

[18] K. H. Leung, S. L. Ma, Partial difference sets with Paley parameters, *Bull. London Math. Soc.*, **27** (1995), 553–564.

[19] S. L. Ma, A survey of partial difference sets, *Designs, Codes and Cryptography*, **4** (1994), 221–261.

[20] R. Mathon, New maximal arcs in Desarguesian planes, *J. Combin. Theory (A)* **97** (2002), 353–368.

[21] D. K. Ray-Chaudhuri, Q. Xiang, Constructions of partial difference sets and relative difference sets using Galois rings, Special issue dedicated to Hanfried Lenz. *Des. Codes Cryptogr.*, **8** (1996), 215–227.

[22] R. J. Turyn, Character sums and difference sets. *Pacific J. Math.*, **15** (1965), 319–346.

[23] J. Wolfmann, Difference sets in $\mathbb{Z}_4^m$ and $\mathbb{F}_2^{2m}$, *Designs, Codes, and Cryptography*, **20** (2000), 73–88.