Carter Subgroups, Amalgams, Simple Groups, and the $Z_p^*$-theorem

Geoffrey R. Robinson,
Institute of Mathematics,
University of Aberdeen,
Aberdeen AB24 3UE

February 2, 2015

Abstract

We consider an amalgam of groups constructed from fusion systems for different odd primes $p$ and $q$. This amalgam contains a self-normalizing cyclic subgroup of order $pq$ and isolated elements of order $p$ and $q$.

1 Introduction

In earlier work, ([3],[4]) we used an (iterated) amalgam $X = X_F$ to realise an Alperin fusion system $F$ on a finite $p$-group $P$ via conjugation within $X$, and to obtain explicit linear representations of $X$, thereby relating many finite groups (finite homomorphic images of $X$) to the original fusion system $F$.

To partly motivate what follows, we outline an extension of the example in [4] which will also illustrate some ideas behind the main construction here. If we consider the maximal fusion system on a semidihedral 2-group of order 16, this fusion system is realised by an amalgam $X = GL(2,3) *_D S_4$ where $D$ is a dihedral group of order 8. It is interesting to note that $X$ has a unique conjugacy class of involutions, and that we have $C_X(t) = GL(2,3)$ for each involution $t \in X$. There are only two non-isomorphic finite simple groups $G$ which contain an involution $u$ with $C_G(u) \cong GL(2,3)$ (this is a theorem of R. Brauer).

The amalgam $X$ is certainly not a simple group, but all its proper non-trivial normal subgroups are free (as is implicitly noted in [4]). In [4], we showed that for each odd prime $q$, there is an epimorphism from $X$ to $SL(3,q)$ (when $q \equiv 1, 3 \pmod{8}$) or $SU(3,q)$ (when $q \equiv 5, 7 \pmod{8}$). Hence this fusion system on this single 2-group leads naturally to an infinite number of non-isomorphic finite simple groups (not all of which have a semi-dihedral Sylow 2-subgroup of order 16). Furthermore, each epimorphism constructed has free kernel, so
the finite groups used to build the amalgam embed faithfully in each of these simple epimorphic images, and these epimorphic images are all generated by the images of these finite groups.

In this paper, then, we will construct an amalgam $X$ from two different fusion systems, for two different primes $p$ and $q$. Again, the construction illustrates a general methodology which should be much more widely applicable (and with iterated amalgams). In the case we consider below, each fusion system is of a rather transparent type (and is a fusion system on an extra-special group) and the interaction between the fusion systems is almost minimal.

However, it illustrates the general methodology, and (in our view) it also illustrates that rich structures can arise in this context from uncomplicated building blocks. We were initially led to these considerations by a realisation of a connection between odd analogues of Glauberman’s $Z^*$-theorem, [2], and some troublesome configurations considered in the Feit-Thompson proof of the solvability of finite groups of odd order, [1].

We will construct amalgams $X$ realising related configurations within perfect infinite groups, and show that all proper normal subgroups of the amalgams constructed are free. Each amalgam $X$ constructed is a perfect group which realises a constrained fusion system on an extra-special (finite) $p$-group $P$ of exponent $p$ and a constrained fusion system on an extra-special (finite) $q$-group $Q$ of exponent $q$ where $p$ and $q$ are distinct odd primes. Furthermore, each element of $Z(P)$ and each element of $Z(Q)$ is isolated in $X$. The amalgams $X$ all have free normal subgroups of finite index, so do have non-Abelian finite simple groups as epimorphic images, which are generated by the images of $N_X(P)$ and $N_X(Q)$. We exhibit an explicit pair of generators for each amalgam $X$, one of order $pq$ and one of infinite order. Also, each $X$ is generated by three explicitly identified elements, of respective orders $q, p$ and $pq$. Hence we have explicit generators for any finite simple homomorphic image of $X$, though the order of the image of the element of infinite order is not a priori obvious.

2 Notation, Definitions, Background

A finite $p$-subgroup $S$ of a (possibly infinite) group $G$ is said to be a Sylow $p$-subgroup of $G$ if every finite $p$-subgroup of $G$ is conjugate to a subgroup of $S$. We recall that a Carter subgroup of a finite solvable group $H$ is a self-normalizing nilpotent subgroup of $H$. The fact that such an $H$ always has a Carter subgroup, and that these are all $H$-conjugate, was proved by R. Carter.

We refer to an element $x$ of a group $G$ as isolated if $x$ commutes with none of its other $G$-conjugates. This terminology occurs frequently in existing literature, though usually in the context of finite groups. Glauberman’s $Z^*$-theorem proves that if $G$ is a finite group with no non-trivial normal subgroup of odd order, then any isolated involution $t \in G$ is central in $G$. 

2
For ease of later discussion, we will call a group $H$ torsion simple if all its proper non-trivial normal subgroups are torsion free. In the amalgams we deal with in this paper, torsion free subgroups are always free.

In the Feit-Thompson proof of the solvability of finite groups of odd order,[1], it is shown that a putative minimal finite simple group of odd order must have a self-normalizing cyclic subgroup of order $pq$ for distinct (odd) primes $p$ and $q$ (and that there is a unique conjugacy class of such self-normalizing cyclic subgroups). This self-normalizing cyclic subgroup is the intersection of a pair of maximal subgroups. One of these has an elementary Abelian Sylow $p$-group which is the Frobenius kernel of a Frobenius subgroup of index $q$, and the other has an elementary Abelian normal Sylow $q$-subgroup which is the Frobenius kernel of a Frobenius subgroup of index $p$.

In the Odd Order paper,[1], this possibility is eventually eliminated by a difficult analysis in Chapter VI (the last chapter).

3 The Construction

Let $p$ and $q$ be distinct odd primes. Let $m$ be the smallest positive integer such that $|\text{Sp}(2m, q)|$ is divisible by $p$ and let $n$ be the smallest positive integer such that $|\text{Sp}(2n, p)|$ is divisible by $q$. Let $A$ be the semi-direct product of an extra-special $q$-group $Q$ of order $q^{2m+1}$ and exponent $q$ with a cyclic group of order $p$ acting trivially on $Z(Q)$ and faithfully on $Q/Z(Q)$. Let $B$ be the semi-direct product of an extra-special $p$-group $P$ of order $p^{2n+1}$ and exponent $p$ with a cyclic group of order $q$ acting trivially on $Z(P)$ and faithfully on $P/Z(P)$.

Notice that $A$ has a Carter subgroup which is cyclic of order $pq$ and contains $Z(Q)$, while $B$ has a Carter subgroup which is cyclic of order $pq$ and contains $Z(P)$.

We form the amalgam $X = A *_C B$, where $C$ is a cyclic group of order $pq$ identified (by inclusion) with a Carter subgroup of $A$ and a Carter subgroup of $B$. Then $Q$ is a Sylow $q$-subgroup of $X$ and $P$ is a Sylow $p$-subgroup of $X$.

With the above identification, we take $C = Z(P) \times Z(Q)$. Furthermore, as will become apparent below, the fusion system induced on $Q$ by conjugation within $X$ is just the constrained fusion system induced by $A$ on $Q$ and the fusion system induced on $P$ by conjugation within $X$ is just the constrained fusion system induced on $P$ by $B$. This could be proved from the results of [3], but we will give a self-contained proof below.

We claim that $X$ is torsion simple. Let $N$ be a proper non-trivial normal subgroup of $X$. If $N$ is not free, then (under current hypotheses) it is not torsion free either, and we either have $N \cap A \neq 1$ or $N \cap B \neq 1$, since all elements of finite order in $X$ lie in a conjugate of $A$, or a conjugate of $B$. Hence $N$ contains either an element of order $p$ or an element of order $q$. Suppose that $N$ contains an element of order $p$. If this element lies in $B$, then $N \cap Z(P) \neq 1$
and $[Q, Z(P)] = Q \leq N$. Then also $[P, Z(Q)] = P \leq N$, so $N = X$. A similar argument gives $N = X$ if $N$ contains an element of order $q$. Hence $N$ is free.

Now we must have $X = [X, X]$, for otherwise $[X, X]$ is free, so that $A$ and $B$ each embed isomorphically into $X/[X, X]$, a contradiction, as both $A$ and $B$ are non-Abelian. As explained in [4], $X$ has a free (not necessarily normal) subgroup of index $|P||Q|$, so $X$ does have free normal subgroups of finite index. We note in passing that whenever $F$ is a maximal free normal subgroup of finite index in $X$, then $X/F$ acts faithfully on $F/[F, F]$. This is a standard argument, but we reproduce it now: let $N = [F, X] \geq [F, F]$. Then $X/N$ is a central extension of $X/F$ by the finitely generated Abelian group $F/N$. Then $F/N$ must be finite, otherwise there is a perfect central extension of the finite group $X/F$ by $Z/rZ$ for every prime $r$, a contradiction. In particular, $N > [F, F]$, since $F/[F, F]$ is infinite. Hence $C_X(F/[F, F])$ is a proper normal subgroup of $X$, hence free. By the maximal choice of $F$, we must have $F = C_X(F/[F, F])$.

Now we note that $C$ is a self-normalizing cyclic subgroup of $X$. It is a general fact about amalgams that every element of $X \setminus C$ may be written in the form $u_1u_2 \ldots u_n$, where each $u_i$ is either in $A \setminus C$ or in $B \setminus C$, and there is no value of $i$ such that both $u_i$ and $u_{i+1}$ lie in $A$, nor is there any value of $j$ such that both $u_j$ and $u_{j+1}$ lie in $B$. Furthermore, each such product does lie outside $C$. Now let $d$ be a generator of $C$. Consider $u_n^{-1} \ldots u_1^{-1}du_1 \ldots u_n$ and set $v_i = d^{-1}u_id$ for each $i$. Then $u_n^{-1} \ldots u_1^{-1}du_1 \ldots u_n = dv_n^{-1} \ldots (v_1^{-1}u_1) \ldots u_n$. Now $u_i$ does not normalize $C$, since $C$ is self-normalizing in both $A$ and $B$. In particular, $v_1^{-1}u_1 \notin C$, but does lie in the same member of $\{A, B\}$ as $u_1$ does. Hence the given conjugate of $d$ lies outside $C$, since $v_j$ lies outside $C$ for $j > 1$, but does lie in the same member of $\{A, B\}$ as $u_j$ does.

Now we claim that $A \cap A^x$ is a $q'$-group for each $x \in X \setminus A$ (and similarly $B \cap B^y$ is a $p'$-group for each $y \in X \setminus B$). Consider such an element of the form $x = u_1u_2 \ldots u_n$ with each $u_i$ lying outside $C$ but inside $A$ or $B$, such that there is no value of $i$ for which both $u_i$ and $u_{i+1}$ lie in the same member of $\{A, B\}$. If possible, choose an element $a$ of order $q$ in $A \cap A^x$. If $n = 1$, then $x = u_1 \in B \setminus C$, so $x^{-1}ax$ lies outside $A \cup B$ unless $a \in C$. But if $a \in C$, then $\langle a \rangle = Z(Q)$, and $x^{-1}ax$ can only lie in $A$ if it lies in $A \cap B = C$. In that case, $x$ must lie in $N_B(Z(Q)) = C$, a contradiction.

We note that if $n > 1$, then $x$ lies outside $A \cup B$. Suppose now that $n > 1$ and consider $u_n^{-1} \ldots u_1^{-1}au_1 \ldots u_n$. If $a \notin C$, this product lies outside $A \cup B$, for if $u_1 \in A$, then we may bracket it as

$$u_n^{-1} \ldots (u_1^{-1}au_1) \ldots u_n.$$  

Note that $u_1^{-1}au_1$ still lies in $A \setminus C$. On the other hand, if $u_1 \in B$, the expression $u_n^{-1} \ldots u_1^{-1}au_1 \ldots u_n$ is already expressed as a product of elements which lie alternately in $A \setminus C$ or $B \setminus C$.

However, if $a \in C$, then we might as well suppose that $u_1 \in B \setminus C$, since
\[O_q(C) = Z(A).\] But then, as before, we can express the product as
\[a[a^{-1}u_n^{-1}a] \ldots [a^{-1}u_1^{-1}au_1]u_2 \ldots u_n.\]

Now notice that \([a, u_1] \in B.\] If \([a, u_1] \in C,\) then as above, we obtain \(u_1 \in N_B(Z(Q)) = C,\) contrary to assumption. Thus \([a, u_1] \in B\setminus C,\) which shows that the stated product lies outside \(A \cup B.\)

Now we have proved that \(A \cap A^z\) is a \(q'-\)group for each \(x \in X\setminus A,\) and by symmetry, \(B \cap B^y\) is a \(p'-\)group for each \(y \in X\setminus B.\) We now note that \(z\) is isolated whenever \(z \in Z(Q)^#,\) (and an analogous statement holds for any \(w \in Z(P)^#\)).

For suppose that \(z^y \neq z\) commutes with \(z.\) Then \(\langle z, z^y \rangle\) is finite of order \(q^2\) so is conjugate to a subgroup of \(Q.\) Then \(\langle z^y, z^{y^2} \rangle \leq Q\) for some \(y \in X.\) Then \(y \in A\) since \(z \in A \cap yAy^{-1}.\) Similarly \(xy \in A,\) so that \(x \in A = C_X(z)\) and \(z = z^y.\)

Notice that if \(N\) is any proper normal subgroup of \(X,\) then \(z \notin N\) as \(N\) is free, but \(zN\) is not central in \(X/N.\) For \(X/N\) is not Abelian as \(X = [X, X],\) so if \(zN\) is central in \(X/N,\) then \(z\) lies in a proper normal subgroup of \(X,\) which is free, a contradiction.

**Remark:** We note that \(X\) (and hence every homomorphic image of \(X\)) is generated by two elements. For any \(u \in Q\setminus Z(Q),\) and \(v \in P\setminus Z(P),\) we have \(X = Y = \langle uv, C, \rangle.\) For let \(z\) be a generator of \(Z(Q)\) and \(w\) be a generator of \(Z(P).\) Then \(uvz \in Y,\) so \(v^{-1}v^2 = (uv)^{-1}(uv)^2 \in Y.\) Now \(v^{-1}v^2 \in P\setminus Z(P),\) so that \(P = \langle v^{-1}v^2, z \rangle\) and \(P \leq Y.\) Similarly, \(Q \leq Y,\) so \(Y = X.\) Notice, however, that \(uv\) has infinite order. We can certainly generate \(X\) by three elements of finite order, as \(X = \langle u, v, zuv \rangle.\)

In fact, we claim that \(T = \langle z^b, uv \rangle = X\) for any \(b \in B.\) For \(T\) contains \(\langle u, v \rangle = A,\) from the above argument. Hence \(T\) contains \(\langle z^b, v \rangle\) which is a \(B\)-conjugate of \(\langle z, bvb^{-1} \rangle = B\) as above. Thus \(T = X.\)

We also remark that every element lying outside all conjugates of \(A\) and outside all conjugates of \(B\) has a centralizer which is free of rank 1. For if \(x \in X\) and no conjugate of \(x\) lies in \(A \cup B\) then \(x\) has infinite order. Hence \(x \notin C_X(a)\) for any \(a \in A^#,\) and similarly for any conjugate of \(A.\) A similar argument holds for conjugates of \(B.\) Thus \(C_X(x)\) is torsion free, and hence free. But a free group with non-trivial centre is free of rank 1.

We note that (by essentially the same proof as given in [4]), all normal subgroups of finite index of \(X\) are free of rank greater than 1. For let \(N\) be a normal subgroup of \(X\) of finite index. We know already that \(N\) is free, so that \(A\) and \(B\) embed isomorphically into \(X/N.\) Hence \([X : N]\) id divisible by \(p^{2n+1}q^{2m+1}.\) If \(N\) has \(r\) generators, then \(\chi(N) = \frac{1}{[X : N]}\). We also have
\[\chi(X) = \frac{1}{q^{2n+1}} + \frac{1}{pq^{2m+1}} - \frac{1}{pq},\]
which yields
\[r = 1 + \frac{(p^2q^{2m} - p^{2n}q^{2m})[X : N]}{p^{2n}q^{2m}} > 1.\]
In particular, this implies that every non-identity element of $X$ has an infinite number of conjugates. Hence over any field, the group algebra $FX$ (consisting of finite $F$-linear combinations of elements of $X$) has one-dimensional centre.

4 Remarks on the finite case

Although much more sophisticated counting arguments and character-theoretic arguments are used in Chapter V of ([1]), to reach the configuration which is left to be eliminated in Chapter VI of that work, as a matter of interest, we give a direct proof here of an easier result. We prove that the $p$-local and $q$-local structure which we have shown to be present in $X$ can’t all be present in a finite group.

Theorem: There is no finite group $G$ with the following properties: i) There are distinct odd prime divisors $p$ and $q$ of $|G|$ such that $G$ has a self-normalizing cyclic subgroup $C$ of order $pq$ with $C = Z(P) \times Z(Q)$ for $P$ a Sylow $p$-subgroup and $Q$ a Sylow $q$-subgroup of $G$.

ii) $N_G(P) = PZ(Q) = C_G(Z(P))$ and $N_G(Q) = QZ(P) = C_G(Z(Q))$.

iii) $P \cap P^g = 1$ for all $g \in G \setminus N_G(P)$ and $Q \cap Q^h = 1$ for all $h \in G \setminus N_G(Q)$.

Proof: Suppose otherwise, and set $A = N_G(P)$, $B = N_G(Q)$. We may, and do, suppose that $G = \langle P, Q \rangle$. Let $N$ be a normal subgroup of $G$. Then $A \cap N \neq 1$ or $B \cap N = 1$ gives $Z(P) \leq N$ or $Z(Q) \leq N$, so either $Q = [Q, Z(P)] \leq N$ or $P = [P, Z(Q)] \leq N$. Suppose that that $N$ is a $\{p, q\}$-group. Now $P$ is not cyclic, otherwise $G$ has a normal $p$-complement, whereas $P = [P, Z(Q)] \leq G'$, and likewise $Q$ is not cyclic. Now $N \leq (C_G(x) : x \in P^#) \leq A$, so that $N = 1$. Hence $G$ is a simple group. Now $G$ has $[G : A](|P| - 1)$ non-identity $p$-elements and $[G : B](|Q| - 1)$ non-identity $q$-elements. Also $G$ has (at least) $|G| \left\lceil \frac{p-1}{p} \right\rceil \left\lceil \frac{q-1}{q} \right\rceil$ elements of order $pq$. This accounts for $\left\lceil \frac{1}{p} + \frac{1}{q} - \frac{1}{p|P|} - \frac{1}{q|Q|} + \frac{1}{p} - \frac{1}{q} \right\rceil$ elements. However, this is greater than $|G|$ as $p$ and $q$ are both odd, a contradiction.

Bibliography

[1] Feit, Walter; Thompson, John G., Solvability of groups of odd order, Pacific J. Math., 13, (1963), 775-1029.

[2] Glauberman, George, Central elements in core-free groups, J. Algebra, 4, (1966), 403-420.

[3] Robinson, Geoffrey R., Amalgams, blocks, weights, fusion systems and finite simple groups, J. Algebra, 314,2, (2007), 912-923.

[4] Robinson, Geoffrey R., Reduction mod q of fusion system amalgams, Trans. Amer. Math. Soc. 363, 2, (2011), 1023-1040.