Systems of germs and theorems of zeros in infinite-dimensional spaces

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Abstract

Systems of germs of sets in infinite-dimensional spaces are introduced and studied. Such a system corresponds to a local zero-set of an ideal of the ring of analytic functions of infinite number of variables. Conversely, this system of germs defines the ideal of germs of analytic functions vanishing on it. A theorem of zeros is proved, stating that this ideal is the radical (in the complex case) or real radical (in the real case) of the initial ideal.

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1 Introduction

In 1952 S. Lang [9] extended Hilbert’s Nullstellensatz to polynomials of infinitely many variables. On the other hand, W. Rückert [16, 17], in 1932, instead of polynomials took germs of complex analytic functions of finitely many variables; his Nullstellensatz involved germs of complex analytic sets. The real case for polynomials of finitely many variables was independently solved by J.-L. Krivine [8], D.W. Dubois [6] and J.-J. Risler [14] in the sixties (of the last century). In their theorem of zeros, the ordinary radical of an ideal, used in the complex version, had to be changed for the real radical. Finally, in 1976 Risler [15, 17] proved finite-dimensional real analytic counterpart of Hilbert’s Nullstellensatz. (See [1] for a more abstract real theorem of zeros.) It is important to notice that the ring of the germs of
analytic functions of finitely many complex or real variables (at some point) is Noetherian. Hence, if $I$ is an ideal of this ring, then the germ of zero-set of $I$ is well defined as $I$ is finitely generated (see e.g. [7]).

In this paper we study the infinite-dimensional analytic (complex and real) case, where an analytic function depends (like Lang’s polynomials) only on a finite number of variables (is \textit{finitely presented}). But, as the number of all variables is infinite, the ring of the germs of such functions is no longer Noetherian and the germs of the zero sets of ideals cannot be defined in the standard way. Moreover, as we have shown in [11], there is no topology in the infinite-dimensional space of all complex or real sequences that would give required properties of the germs of sets. We have been interested there in “local” solutions of infinitely many analytic equations in infinitely many variables. Such equations describe, for instance, indistinguishable states of a (control) system with output and are related to \textit{observability} and \textit{local observability} of the system (see e.g. [3, 4] for the finite-dimensional case and [10, 11, 12] for the infinite-dimensional one). In particular, it is important in local observability whether such equations have locally only one solution (which is the point at which we localize the system and the equations).

Instead of using topology to define the germ of a set, we consider special families of finite dimensional set-germs (\textit{systems of germs}) which approximate in some sense what we want to be an infinite-dimensional set-germ. Systems of germs give rise to a concept of multigerm — the equivalence class of such systems under a natural equivalence relation. We show that it is the right language in this infinite-dimensional world. We can manipulate with multigerms in the same way as we do with finite-dimensional germs. In particular, we can define multigerm of zeros corresponding to an ideal of the ring of germs of finitely presented analytic functions and, conversely, zero ideal of a multigerm. We consider the real and the complex cases. The main result of this paper consists of real and complex theorems of zeros, where we show that the real or ordinary radical of an ideal consists exactly of the germs of finitely presented analytic functions (real or complex, respectively) that vanish on the multigerm of zeros (real or complex again) of the ideal.

We omit here many proofs which are either straightforward or similar to proofs of earlier statements. Instead, we provide several examples which give the flavor of the theory. They concern mostly real functions as the real case is more interesting and closer to applications. Some of them can
be found in D. Mozyrska’s Ph.D. thesis [10], where local observability of infinite-dimensional dynamical systems is studied in detail.

2 Preliminaries and notation

Let $M$ be a topological space, $x_0 \in M$ and $A \subseteq M$. Then by $A_{x_0}$ we shall denote the germ of the set $A$ (or a set-germ) at $x_0$.

Whereas the union and the intersection of finite number of set-germs are well-defined, these operations on an infinite collection of set-germs are not necessarily well-defined.

The germ at $x_0$ of the empty set will be called the empty set-germ and the germ of the whole space — the full set-germ (its representatives are neighborhoods of $x_0$). A germ which is not the empty set-germ will be called a proper set-germ.

**Definition 2.1.** Let $M$ and $N$ be topological spaces. Let $h : M \rightarrow N$ be continuous, $x_0 \in M$, $y_0 = h(x_0)$. Then by the inverse image (at $x_0$) of a germ $A_{y_0}$ with respect to $h$ we will mean the germ at $x_0$ of the inverse image of a representative of $A_{y_0}$, i.e.

$$h_{x_0}^{-1}(A_{y_0}) := h^{-1}(A)_{x_0}. \quad (2.1)$$

**Proposition 2.2.** Let $M, N, P$ be topological spaces, $x_0 \in M$, $y_0 \in N$, $z_0 \in P$. Let $g, f$ be continuous mappings, $g : M \rightarrow N$, $f : N \rightarrow P$, and $y_0 = g(x_0)$, $z_0 = f(y_0)$. If $h = f \circ g$, then $h_{x_0}^{-1}(A_{z_0}) = g_{x_0}^{-1}(f_{y_0}^{-1}(A_{z_0}))$.

From now on let $K = \mathbb{R}$ or $K = \mathbb{C}$. If $x_0 \in M$ and $f$ is a $K$-valued function defined on some neighborhood of $x_0$, then by $f_{x_0}$ we shall denote the germ of the function $f$ (a function-germ) at $x_0$. If $\varphi$ is a function-germ at $x_0$, then a function $f : M \rightarrow K$ such that $f_{x_0} = \varphi$ is called a representative of $\varphi$.

**Definition 2.3.** Let $M, N$ be topological spaces and $x_0 \in M$, $y_0 \in N$. Let $g : M \rightarrow N$ be a continuous mapping such that $y_0 = g(x_0)$ and $\varphi$ be the germ of a $K$-valued function at $y_0$. Then we define the pullback of the germ $\varphi$ with respect to the map $g$ in the following way:

$$g_{x_0}^*(\varphi) := (f \circ g|_U)_{x_0}, \quad (2.2)$$

where $f_{y_0} = \varphi$ and $f : V \rightarrow U$, $V \subset N$, $U = g^{-1}(V) \subset M$. 


In applications we will require that $g^*_x$ be injective.

**Proposition 2.4.** Let $g$ be a continuous and open mapping of topological spaces $M$ and $N$. Let $x_0 \in M$ and $y_0 = g(x_0)$. Then $g^*_x$ is a monomorphism from the algebra of germs of all functions at $y_0$ to the algebra of germs of all functions at $x_0$.

Let $M = K^n$ and $x_0 \in M$. By $\mathcal{O}_{x_0}(M)$ we denote the ring of germs of $K$-valued analytic functions at $x_0$. The ring $\mathcal{O}_{x_0}(M)$ is Noetherian and local ([17]). Its only maximal ideal will be denoted by $m^x_{x_0}$. If $I$ is an ideal of $\mathcal{O}_{x_0}(M)$, then $Z(I)$ will denote the zero set-germ at $x_0$ of $I$. Let $J(Z(I))$ be the ideal in $\mathcal{O}_{x_0}(M)$ of all germs of analytic functions that vanish on $Z(I)$.

Let $P$ be a commutative ring with a unit and $I$ be an ideal of $P$ (which will be denoted by $I \triangleleft P$). Then the real radical of $I$, denoted by $\sqrt[\cap]{I}$, is the set of all $a \in P$ for which there are $m \in \mathbb{N}, k \in \mathbb{N} \cup \{0\}$ and $b_1, \ldots, b_k \in P$, such that

$$a^{2m} + b_1^2 + \cdots + b_k^2 \in I.$$  

Then $I \subseteq \sqrt[\cap]{I} \subseteq \sqrt{I}$, where $\sqrt{I}$ denotes the ordinary radical of $I$.

**Proposition 2.5.** Let $f : P \to R$ be a homomorphism of rings. Then $f^{-1}(\sqrt{I}) = \sqrt{f^{-1}(I)}$.

In case of real radicals we have a weaker statement:

**Proposition 2.6.** Let $f : P \to R$ be a homomorphism of rings. Then $f^{-1}(\sqrt[\cap]{I}) \subseteq \sqrt{f^{-1}(I)}$.

**Remark 2.7.** Real and complex analytic theorems of zeros may now be stated as follows, ([15, 7, 17, 16]):

1. Let $x_0 \in \mathbb{R}^n$. If $I$ is an ideal of $\mathcal{O}_{x_0}(\mathbb{R}^n)$ then $J(Z(I)) = \sqrt{I}$.
2. Let $z_0 \in \mathbb{C}^n$. If $I$ is an ideal of $\mathcal{O}_{z_0}(\mathbb{C}^n)$ then $J(Z(I)) = \sqrt{I}$.

### 3 Germs of finitely presented functions

Let $T$ be an arbitrary nonempty set and $K = \mathbb{R}$ or $K = \mathbb{C}$. Consider the product space $\prod_{t \in T} K = K^T = \{x : T \to K\}$ with the product topology. We denote $x_t := x(t)$ for $t \in T$ and $x \in K^T$. 
From now on we assume that $T$ is an infinite set. By $\text{Fin}(T)$ we denote the set of all finite nonempty subsets of $T$. Let $S \in \text{Fin}(T)$. Then $K^S := \{x_S = x|_S : S \to K, x \in K^T\}$. By $\Pi^S_{S_1}$ we denote the projection $\Pi^S_{S_1} : K^{S_2} \to K^{S_1} : x_{S_2} \mapsto x_{S_1}$, where $S_1 \subseteq S_2 \subseteq T$. Let $x_0 \in K^T$. Then $\Pi^S_{S_1}(x_{0S_2}) = x_{0S_1}$ and $\Pi^S_T(x_0) = x_{0S}$. The projection $\Pi^S_{S_1}$ is a continuous and open mapping of topological spaces $K^{S_2}$ and $K^{S_1}$.

The sets of the form $V = (\Pi^S_T)^{-1}(U)$, where $S \in \text{Fin}(T)$ and $U \subseteq K^S$ is open, form the basis of the product topology of $K^T$.

**Definition 3.1.** Let $V$ be an open set in $K^T$ with the product topology. We say that a function $f : V \to K$ is finitely presented on $K^T$ if there are $S \in \text{Fin}(T)$ and a function $\overline{f} : \Pi^S_T(V) \to K$ such that $f = \overline{f} \circ (\Pi^S_T)|_V$. The function $\overline{f}$ is called a representing function of the function $f$. We also say that $f$ depends on a finite number of variables indexed by $S$ or that $S$ is indexing $f$. We say that $f$ is analytic if $\overline{f}$ is analytic. By $\mathcal{F}^T_K$ we denote the family of all analytic finitely presented functions on $K^T$.

Observe that if $S$ is indexing $f$ then any $S'$ such that $S' \supseteq S$ is also indexing $f$ and any $S \in \text{Fin}(T)$ is indexing any constant function.

**Example 3.2.** [2]

Consider the space $\mathbb{R}^N$ with the product topology. A function $f : \mathbb{R}^N \to \mathbb{R}$ is linear and continuous if and only if there are $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathbb{R}$ such that for each $x \in \mathbb{R}^N$, $f(x) = \sum_{i=1}^k a_i x_i$. Hence, linear continuous functionals on $\mathbb{R}^N$ are finitely presented.

Let $f \in \mathcal{F}^T_K$ and $x_0 \in K^T$. Then we may consider the germ of $f$ at $x_0$ in the standard way in the topological space $K^T$ with the product topology. The collection of germs at $x_0$ of functions from $\mathcal{F}^T_K$ forms a commutative ring with a unit (the germ of the constant function equal 1). It is denoted by $\mathcal{O}_{x_0}(K^T)$. We say that $S$ is indexing a germ $\varphi \in \mathcal{O}_{x_0}(K^T)$ if $S$ is indexing some representative of $\varphi$.

Let $x_0 \in K^T$ and $S_1 \subseteq S_2 \subseteq T$. Let us consider the rings $\mathcal{O}_{x_{0S_1}}(K^{S_1})$, $\mathcal{O}_{x_{0S_2}}(K^{S_2})$, of germs of analytic functions, respectively at points $x_{0S_1}, x_{0S_2}$.

Let us consider a monomorphism of the ring $\mathcal{O}_{x_{0S_1}}(K^{S_1})$ into the ring $\mathcal{O}_{x_{0S_2}}(K^{S_2})$:

$$M^{S_2}_{S_1} : \mathcal{O}_{x_{0S_1}}(K^{S_1}) \ni \varphi \mapsto (\Pi^{S_2}_{S_1})^*_{x_{0S_2}}(\varphi) \in \mathcal{O}_{x_{0S_2}}(K^{S_2}). \quad (3.1)$$
It is a particular case of Definition 2.3. Using the monomorphism defined by (3.1) we may identify the ring \( \mathcal{O}_{x_0}(K^{S_1}) \) with the subring of the ring \( \mathcal{O}_{x_0}(K^{S_2}) \) consisting of germs of functions that do not depend on variables with indices from \( S_2 \setminus S_1 \).

**Proposition 3.3.** Let \( x_0 \in K^T \). Then

\[
\mathcal{O}_{x_0}(K^T) = \bigcup_{S \in \text{Fin}(T)} M^T_S \left( \mathcal{O}_{x_0S}(K^{S}) \right) .
\]

*Proof.* Immediately we have that \( \bigcup_{S \in \text{Fin}(T)} M^T_S \left( \mathcal{O}_{x_0S}(K^{S}) \right) \subseteq \mathcal{O}_{x_0}(K^T) \) because of the definition of \( M^T_S \). Now let \( \varphi \in \mathcal{O}_{x_0}(K^T) \) and \( f : V \rightarrow K \) be a representative of \( \varphi \). Then there is \( S \in \text{Fin}(T) \) and \( f : \Pi_T^S(V) \rightarrow K \) such that \( f = f \circ (\Pi_T^S)_{|V} \) and \( \varphi = f_{x_0} = (f \circ (\Pi_T^S)_{|V})_{x_0} = (\Pi_T^S)^* (f_{x_0}) \). Hence \( \varphi \in M^T_S \left( \mathcal{O}_{x_0S}(K^{S}) \right) \). \( \square \)

**Proposition 3.4.** A set \( S \in \text{Fin}(T) \) is indexing \( \varphi \in \mathcal{O}_{x_0}(K^T) \) if and only if \( M^T_S ((M^T_S)^{-1}(\varphi)) = \varphi \).

**Proposition 3.5.** Let \( S_1 \subseteq S_2 \subseteq T \) and \( I \triangleleft \mathcal{O}_{x_0}(K^{S_2}) \). Then \( (M^T_S)^{-1}(I) := \{ \varphi \in \mathcal{O}_{x_0S_1}(K^{S_1}) : M^T_S(\varphi) \in I \} \) is an ideal in the ring \( \mathcal{O}_{x_0S_1}(K^{S_1}) \).

**Definition 3.6.** Let \( I = (\varphi_1, \ldots, \varphi_k) \triangleleft \mathcal{O}_{x_0}(K^T) \) and let \( S_i \) be indexing \( \varphi_i \). Then \( S = \bigcup_{i=1}^k S_i \) is called to be indexing \( I \).

**Proposition 3.7.** Let \( I \triangleleft \mathcal{O}_{x_0}(K^T) \) be finitely generated and let \( S \) be indexing \( I \). Then \( [M^T_S ((M^T_S)^{-1}(I))] \cdot \mathcal{O}_{x_0}(K^T) = I \).

**Proposition 3.8.** Let \( \varphi \in \mathcal{O}_{x_0}(K^T) \). The following conditions are equivalent:

1. \( \varphi \) is invertible in \( \mathcal{O}_{x_0}(K^T) \).
2. \( \exists S \in \text{Fin}(T) : (M^T_S)^{-1}(\varphi) \) is invertible in \( \mathcal{O}_{x_0S}(K^{S}) \).
3. \( \varphi(x_0) \neq 0 \).

**Corollary 3.9.** Consider the ideal \( m_{x_0} = (x_t - x_0)_{t \in T} \) of \( \mathcal{O}_{x_0}(K^T) \) and let \( \varphi \in \mathcal{O}_{x_0}(K^T) \).
1. \( \varphi \) is not invertible \( \iff \varphi \in m_{x_0} \).

2. \( \mathcal{O}_{x_0}(K^T) \) is a local ring with the maximal ideal \( m_{x_0} = (x_t - x_{0t})_{t \in T} \).

By \( \text{Fin}(I) \) we denote the set of all finite nonempty subsets of \( I \subset \mathcal{O}_{x_0}(K^T) \).

Let \( \alpha \in \text{Fin}(I) \), where \( I \subset \mathcal{O}_{x_0}(K^T) \). Then \( (\alpha) \) denotes the ideal generated by all the elements of \( \alpha \).

**Proposition 3.10.** Let \( I \) be an ideal of \( \mathcal{O}_{x_0}(K^T) \). Then

\[
\sqrt{I} = m_{x_0} \iff \forall S \in \text{Fin}(T) \exists \alpha \in \text{Fin}(I) : \sqrt{(\alpha)} = (x_t - x_{0t})_{t \in S}.
\]

Observe that for the real radical Proposition 3.10 does not hold.

**Example 3.11.** Let \( x_0 = 0 \), \( T = \mathbb{N} \) and the ideal \( I \) of \( \mathcal{O}_0(\mathbb{R}^N) \) be generated by germs \( x_1^2 + (x_2 - x_3)^2, x_2^2 + (x_4 - x_5)^2, \ldots, x_{2k+1}^2 + (x_{2k+2} - x_{2k+3})^2, \ldots \). Then \( \sqrt{I} = m_{x_0} \) but for each \( \alpha \in \text{Fin}(I) \) and \( S \in \text{Fin}(T) \): \( \sqrt{(\alpha)} \neq (x_t - x_{0t})_{t \in S} \).

As the ring \( \mathcal{O}_{x_0}(K^T) \) is not Noetherian, the zero set-germ of an ideal may not be well defined. In Section 3 we will define the zero-system of an ideal \( I \subset \mathcal{O}_{x_0}(K^T) \) using zero set-germs of finite subsets of \( I \).

Let \( \alpha = \{ \varphi_1, \ldots, \varphi_k \} \subset \mathcal{O}_{x_0}(K^T) \). Then we define the zero set-germ of \( \alpha \) in a standard way: \( Z(\alpha) = Z(\varphi_1) \cap \ldots \cap Z(\varphi_k) \), where \( Z(\varphi) = Z(\tilde{\varphi})_{x_0} \) for some representative of \( \varphi \). Let \( S \in \text{Fin}(T) \) be indexing \( \alpha \). Then \( (M_{\mathcal{S}}^T)^{-1}(\alpha) = \{(M_{\mathcal{S}}^T)^{-1}(\varphi_1), \ldots, (M_{\mathcal{S}}^T)^{-1}(\varphi_k)\} \subset \mathcal{O}_{x_0s}(K^S) \) and \( Z((M_{\mathcal{S}}^T)^{-1}(\alpha)) \) is the set-germ in \( K^S \).

**Proposition 3.12.** Let \( \alpha = \{ \varphi_1, \ldots, \varphi_k \} \subset \mathcal{O}_{x_0}(K^T) \) and let \( S \in \text{Fin}(T) \) be indexing \( \alpha \). Then \( (\Pi_{\mathcal{S}}^T)^{-1}(Z((M_{\mathcal{S}}^T)^{-1}(\alpha))) = Z(\alpha) \).

**Proof.** Let for each \( i = 1, \ldots, k : (f_i)_{x_0} = \varphi_i \) and \( f_i : V_i \rightarrow K \). Then there are functions \( \tilde{f}_i : \Pi_{\mathcal{S}}^T(V_i) \rightarrow K \) such that \( f_i = \tilde{f}_i \circ (\Pi_{\mathcal{S}}^T)_{|V_i} \). Let \( y \in K^T \). Then:

\[
y \in (\Pi_{\mathcal{S}}^T)^{-1}(Z(\tilde{f}_1, \ldots, \tilde{f}_k)) \iff \Pi_{\mathcal{S}}^T(y) \in Z(\tilde{f}_1, \ldots, \tilde{f}_k) \iff \forall i = 1, \ldots, k : \tilde{f}_i(\Pi_{\mathcal{S}}^T(y)) = 0 \iff y \in (f_1, \ldots, f_k).
\]

Hence \( (\Pi_{\mathcal{S}}^T)^{-1}(Z(\tilde{f}_1, \ldots, \tilde{f}_k))_{x_0} = Z(f_1, \ldots, f_k) \).

**Corollary 3.13.** Assume that \( \alpha, \beta \in \text{Fin}(\mathcal{O}_{x_0}(K^T)) \). If \( Z(\alpha) \subseteq Z(\beta) \) then there is \( S \in \text{Fin}(T) \) such that \( Z((M_{\mathcal{S}}^T)^{-1}(\alpha)) \subseteq Z((M_{\mathcal{S}}^T)^{-1}(\beta)) \).

**Proof.** Let \( S \) be indexing \( \alpha \cup \beta \). Then by Proposition 3.12 we have that:

\[
(\Pi_{\mathcal{S}}^T)^{-1}(Z((M_{\mathcal{S}}^T)^{-1}(\alpha))) \subseteq (\Pi_{\mathcal{S}}^T)^{-1}(Z((M_{\mathcal{S}}^T)^{-1}(\beta))) \quad \text{As } \Pi_{\mathcal{S}}^T \text{ is surjective we get } Z((M_{\mathcal{S}}^T)^{-1}(\alpha)) \subseteq Z((M_{\mathcal{S}}^T)^{-1}(\beta)) \quad \blacksquare
\]

7
4 Systems of germs

By a directed set of indeces we mean an ordered pair \((\Lambda, \ll)\), where \(\Lambda\) is an arbitrary set and \(\ll\) is a transitive relation in \(\Lambda\) that satisfies the Moore-Smith’s condition:[13]:

\[
\forall \alpha \in \Lambda \ \forall \beta \in \Lambda \ \exists \gamma \in \Lambda : (\alpha \ll \gamma \land \beta \ll \gamma) .
\] (4.1)

Definition 4.1. Let \(x_0 \in K^T\) and \((\Lambda, \ll)\) be a directed set of indeces. Then we define a **system of germs at** \(x_0\) as a set of set-germs at \(x_0\):

\[
\{A^\alpha, \alpha \in \Lambda\}
\]

such that

\[
\forall \alpha \in \Lambda \ \exists S \in \text{Fin}(T) \ \exists A \subseteq K^S : A^\alpha = (\Pi_S^T)^{-1}(A_{x_0S}) \text{ and}
\]

\[
\forall (\alpha, \beta \in \Lambda) : \beta \ll \alpha \implies A^\alpha \subseteq A^\beta .
\]

Definition 4.2.

1. A system \(\{A^\alpha, \alpha \in \Lambda\}\) such that for every \(\alpha, \beta \in \Lambda\), \(A^\alpha = A^\beta\) will be called a **constant system**.

2. The system of germs \(\{A^S, S \in \text{Fin}(T)\}\) such that for each \(S\):

\[
A^S = (\Pi_S^T)^{-1}(x_{0S})
\]

will be called the **system of point-germs at** \(x_0\) and will be denoted by \(\{x_0\}_{\text{Fin}(T)}\).

3. The system \(\{A_S, S \in \text{Fin}(T)\}\) such that for each \(S\):

\[
A_S \text{ is the germ of the space } K^T,
\]

will be called the **system of full germs at** \(x_0\). (It is a constant system).

4. We say that a system \(\{A^\alpha, \alpha \in \Lambda\}\) is **trivial** if there exists \(\alpha \in \Lambda\) such that \(A^\alpha = \emptyset\). Then \(\alpha \ll \beta \implies A^\beta = \emptyset\).

5. The system of germs at \(x_0\): \(\{A^\alpha, \alpha \in \Lambda\}\) will be called **proper** if for each \(\alpha \in \Lambda\) the germ \(A^\alpha\) is a proper germ (\(x_0 \in A^\alpha\)).

Let \((\Lambda, \ll_\Lambda), (\Gamma, \ll_\Gamma)\) be directed sets. Let us consider \(\Lambda \times \Gamma = \{(\alpha, \beta) : \alpha \in \Lambda, \beta \in \Gamma\}\). In the product we define the relation \(\ll_{\Lambda \times \Gamma}\) in the following way:

\[(\alpha_1, \beta_1) \ll (\alpha_2, \beta_2) \text{ iff } \alpha_1 \ll_\Lambda \alpha_2 \text{ and } \beta_1 \ll_\Gamma \beta_2.\]

**Proposition 4.3.** \((\Lambda \times \Gamma, \ll_{\Lambda \times \Gamma})\) is a directed set.

**Definition 4.4.** Let \(\mathcal{A} = \{A^\alpha, \alpha \in \Lambda\}\), \(\mathcal{B} = \{B^\beta, \beta \in \Gamma\}\) be systems of germs at \(x_0 \in K^T\). Then we define
1. \( A \cap B := \{ A^\alpha \cap B^\beta, (\alpha, \beta) \in \Lambda \times \Gamma \} \).

2. \( A \cup B := \{ A^\alpha \cup B^\beta, (\alpha, \beta) \in \Lambda \times \Gamma \} \).

**Proposition 4.5.** The union and the intersection of systems of germs at \( x_0 \) are systems of germs.

**Proposition 4.6.** The union and the intersection of proper systems of germs are proper.

Observe that, in general, \( A \cup B \neq B \cup A \) and \( A \cap B \neq B \cap A \). We shall introduce a relation (between two systems) that will allow to compare two systems in some way. In particular we will be allowed to compare a system of germs at \( x_0 \) with the point-germs system.

**Definition 4.7.** Let us consider two systems of germs at \( x_0 \): \( A = \{ A^\alpha, \alpha \in \Lambda \} \) and \( B = \{ B^\beta, \beta \in \Gamma \} \). Then

\[
A \preceq B \iff \forall \beta \in \Gamma \exists \alpha \in \Lambda : A^\alpha \subseteq B^\beta.
\]

It is easy to notice that the relation \( \preceq \) from Definition 4.7 is reflexive and transitive.

**Remark 4.8.** A similar relation defined for flags of finite-dimensional algebraic varieties was used in [18].

**Proposition 4.9.** Let \( \{ x_0 \}_{\text{Fin}(T)} \) be the system of point-germs at \( x_0 \) and \( A = \{ A^\alpha, \alpha \in \Lambda \} \) be a proper system of germs at \( x_0 \). Then \( \{ x_0 \}_{\text{Fin}(T)} \preceq A \).

The relation \( \preceq \) between two systems of germs has similar properties as the relation of inclusion of sets.

**Definition 4.10.** Let \( A, B \) be systems of germs at \( x_0 \). We define \( A \approx B \iff A \preceq B \) and \( B \preceq A \).

**Proposition 4.11.** The relation \( \approx \) is an equivalence relation in the collection of systems of germs at \( x_0 \).

**Definition 4.12.** The equivalence class of the system \( A \) will be denoted by \([A]\) and called the multigerm at \( x_0 \) determined by the system \( A \). The multigerm determined by the point-germ system \( \{ x_0 \}_{\text{Fin}(T)} \) will be denoted by \([\{ x_0 \}]\) and called the point-multigerm. A trivial system determines the empty multigerm denoted by \([\emptyset]\).
In the collection of multigerms at $x_0$ the operations like union and intersection are well-defined. Namely let $\mathcal{A}, \mathcal{B}$ be systems of germs at $x_0$. Then $[\mathcal{A}] \cup [\mathcal{B}] := [\mathcal{A} \cup \mathcal{B}], [\mathcal{A}] \cap [\mathcal{B}] := [\mathcal{A} \cap \mathcal{B}]$. We say that $[\mathcal{A}] \subseteq [\mathcal{B}]$ if there are $\tilde{\mathcal{A}} \in [\mathcal{A}]$ and $\tilde{\mathcal{B}} \in [\mathcal{B}]$ such that $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$.

The following proposition describes a particular case of a system of germs that determines the point-multigerm.

**Proposition 4.15.** Let $\mathcal{A} = \{A^\alpha, \alpha \in \Lambda\}$ be a system of germs at $x_0$. If for each $S \in \text{Fin}(T)$ there exists $\alpha \in \Lambda$ such that $A^\alpha = (\Pi_S^{-1})(x_0S)$ then $[\mathcal{A}] = \{\{x_0\}_{\text{Fin}(T)}\}$.

The converse to the implication given in Proposition 4.13 does not hold. We illustrate this situation in the following example.

**Example 4.14.** Let $T = \mathbb{N}, x_0 = 0 \in \mathbb{R}^N$ and $\mathcal{A} = \{A^n, n \in \mathbb{N}\}$, where $A^n = \{x \in \mathbb{R}^N : x_1 = 0, \ldots, x_{n-1} = 0, x_n = x_{n+1}\}x_0$. Of course the condition from the definition of the system of germs is satisfied. We have that $\mathcal{A} \approx \{x_0\}_{\text{Fin}(\mathbb{N})}$, because for $S \in \text{Fin}(\mathbb{N})$ it is enough to take $n = 1 + \max\{k \in S\} \in \mathbb{N}$. Then $A^n \subset (\Pi_S^N)^{-1}(x_0S)$, but no $A^n$ is equal to $(\Pi_S^N)^{-1}(x_0S)$.

**Proposition 4.15.** Let $[\mathcal{A}], [\mathcal{B}], [\mathcal{C}]$ be multigerms at $x_0$. Then

1. $[\mathcal{A}] \subseteq [\mathcal{A}][B] \subseteq [\mathcal{A}]$.
2. $[\mathcal{A}] \cap [\mathcal{B}] = [\mathcal{B}] \cap [\mathcal{A}]$ and $[\mathcal{A}] \cup [\mathcal{B}] = [\mathcal{B}] \cup [\mathcal{A}]$.
3. $[\mathcal{A}] \cap ([\mathcal{B}] \cap [\mathcal{C}]) = ([\mathcal{A}] \cap [\mathcal{B}]) \cap [\mathcal{C}]$ and $[\mathcal{A}] \cup ([\mathcal{B}] \cup [\mathcal{C}]) = ([\mathcal{A}] \cup [\mathcal{B}]) \cup [\mathcal{C}]$.
4. $[\emptyset] \subseteq [\mathcal{A}], [\mathcal{A}] \cap [\mathcal{A}] = [\mathcal{A}]$ and $[\mathcal{A}] \cup [\mathcal{A}] = [\mathcal{A}]$.
5. $[\mathcal{A}] \cap [\emptyset] = [\emptyset], [\mathcal{A}] \cup [\emptyset] = [\mathcal{A}]$.
6. for the proper multigerm $[\mathcal{A}] : [\mathcal{A}] \cap \{x_0\} = \{x_0\}, [\mathcal{A}] \cup \{x_0\} = [\mathcal{A}]$ and $\{\{x_0\}\} \subseteq [\mathcal{A}]$.

## 5 Theorems of zeros

**Definition 5.1.** Let $I$ be an ideal of the ring $\mathcal{O}_{x_0}(K^T)$. The **zero-system of an ideal $I$** is defined to be the system of germs at $x_0 : \mathcal{Z}(I) = \{Z(\alpha), \alpha \in$
Proof. Hence \( Z(\alpha) \) is the zero set-germ of \( \alpha \). We consider \( \text{Fin}(I) \) with the inclusion relation. An equivalence class of \( Z(I) \) (with respect to the relation \( \approx \)) is called the zero-multigerm of the ideal \( I \) and it is denoted by \([Z(I)]\).

**Proposition 5.2.** Let \( I, J \triangleleft \mathcal{O}_x(K^T) \). If \( J \subseteq I \) then \([Z(I)] \subseteq [Z(J)]\).

Every proper ideal of the ring \( \mathcal{O}_x(K^T) \) is contained in \( m_{x_0} = (x_t - x_0)_{t \in T} \). Therefore:

**Corollary 5.3.** For a proper ideal \( I \) of \( \mathcal{O}_x(K^T) \): \([Z(m_{x_0})] \subseteq [Z(I)]\).

**Proposition 5.4.** \([Z(m_{x_0})] = [\{x_0\}]\).

*Proof.* The system \( Z(m_{x_0}) \) is proper and from Corollary 4.15 \([\{x_0\}] \subseteq [Z(m_{x_0})]\). Now let \( S \in \text{Fin}(T) \) and we take \( \alpha = \{x_t - x_0\}_{t \in S} \in \text{Fin}(m_{x_0}) \).

Then \( Z(\alpha) = (\Pi_S)^{-1}(x_0S) \). Therefore \([Z(m_{x_0})] = [\{x_0\}]\). \( \square \)

**Corollary 5.5.** If \( I \) is a proper ideal of \( \mathcal{O}_x(K^T) \) then \([Z(I)] \neq [\emptyset]\).

**Proposition 5.6.** Let \( I, J \triangleleft \mathcal{O}_x(K^T) \). Then \([Z(I + J)] = [Z(I)] \cap [Z(J)]\).

*Proof.* From Proposition 5.2 we have that: \([Z(I + J)] \subseteq [Z(I)] \) and \([Z(I + J)] \subseteq [Z(J)]\). Hence \([Z(I + J)] \subseteq [Z(I)] \cap [Z(J)]\).

Now let \( \alpha \in \text{Fin}(I + J) \), \( \alpha = \{\varphi_1, \ldots, \varphi_k\} \), where \( \varphi_i = \psi_{i,1} + \psi_{i,2} \) and \( \psi_{i,1} \in I, \psi_{i,2} \in J \). Then let \( \beta = (\beta_1, \beta_2) \in \text{Fin}(I) \times \text{Fin}(J) \) be such that \( \beta_1 = \{\psi_{1,1}, \ldots, \psi_{k,1}\}, \beta_2 = \{\psi_{1,2}, \ldots, \psi_{k,2}\} \). Then \( Z(\beta_1) \cap Z(\beta_2) \subseteq Z(\alpha) \).

Hence \( Z(I) \cap Z(J) \subseteq Z(I + J) \). \( \square \)

**Proposition 5.7.** Let \( I \) be an ideal of \( \mathcal{O}_x(\mathbb{R}^T) \). Then \([Z(I)] = [Z(\sqrt[\mathbb{R}]{T})]\).

*Proof.* Because \( I \subseteq \sqrt[\mathbb{R}]{T} \), from Proposition 5.2 we get: \([Z(\sqrt[\mathbb{R}]{T})] \subseteq [Z(I)]\). To prove that \( Z(I) \subseteq Z(\sqrt[\mathbb{R}]{T}) \) we must show that for any \( \beta \in \text{Fin}(\sqrt[\mathbb{R}]{T}) \) there is \( \alpha \in \text{Fin}(I) \), such that \( Z(\alpha) \subseteq Z(\beta) \). Let \( \beta \in \text{Fin}(\sqrt[\mathbb{R}]{T}) \), hence \( \beta = \{\varphi_1, \ldots, \varphi_p\} \) and for \( i = 1, \ldots, p \); \( \varphi_i \in \sqrt[\mathbb{R}]{T} \). Then for each \( i = 1, \ldots, p \) there are \( m_i \in \mathbb{N}, k_i \in \mathbb{N} \cup \{0\} \) and \( \psi_{i,1}, \ldots, \psi_{i,k_i} \in \mathcal{O}_x(K^T) \), such that:

\[
\varphi_i^{2m_i} + \sum_{j=1}^{k_i} \psi_{i,j}^2 \in I. \quad \text{Let } \alpha = \{\varphi_1^{2m_1} + \sum_{j=1}^{k_1} \psi_{1,j}^2, \ldots, \varphi_p^{2m_p} + \sum_{j=1}^{k_p} \psi_{p,j}^2\}. \quad \text{Then } Z(\alpha) \subseteq Z(\beta). \quad \square
\]

**Proposition 5.8.** Let \( I \) be an ideal of \( \mathcal{O}_x(\mathbb{C}^T) \). Then \([Z(I)] = [Z(\sqrt{T})]\).
Let \([\mathcal{A}]\) be a multigerm at \(x_0\). Then we define the set \(J([\mathcal{A}]) := \{\varphi \in O_{x_0}(K^T) : [\mathcal{A}] \subseteq [\mathcal{Z}(\varphi)]\}\), where \([\mathcal{Z}(\varphi)]\) is the multigerm generated by the zero-system of the ideal \((\varphi)\).

**Proposition 5.9.** Let \([\mathcal{A}]\) be a multigerm at \(x_0\) Then \(J([\mathcal{A}])\) is an ideal of the ring \(O_{x_0}(K^T)\).

**Proof.** Let \(\varphi \in J([\mathcal{A}])\) and \(\psi \in J([\mathcal{A}])\). Then \([\mathcal{A}] \subseteq [\mathcal{Z}(\varphi)]\) and \([\mathcal{A}] \subseteq [\mathcal{Z}(\psi)]\). Hence \([\mathcal{A}] \subseteq [\mathcal{Z}(\varphi)] \cap [\mathcal{Z}(\psi)] = [\mathcal{Z}((\varphi) + (\psi))] \subseteq [\mathcal{Z}((\varphi + \psi))]\), (from Propositions 5.6 and 5.2).

Now let \(\varphi \in J([\mathcal{A}]), \psi \in O_{x_0}(K^T)\). As \((\varphi \cdot \psi) \subseteq (\varphi)\), then from the Proposition 5.2 \([\mathcal{A}] \subseteq [\mathcal{Z}(\varphi)] \subseteq [\mathcal{Z}(\varphi \cdot \psi)]\). Thus \(\varphi \cdot \psi \in J([\mathcal{A}])\).

The ideal \(J([\mathcal{A}])\) has similar properties to the corresponding ideal in the finite-dimensional case.

**Proposition 5.10.** Let \([\mathcal{A}], [\mathcal{B}]\) be the multigerms at \(x_0\). Then

1. if \([\mathcal{A}] \subseteq [\mathcal{B}]\) then \(J([\mathcal{B}]) \subseteq J([\mathcal{A}])\).
2. \(J([\mathcal{A} \cup \mathcal{B}]) = J([\mathcal{A}]) \cap J([\mathcal{B}])\).

**Corollary 5.11.** \(J([\emptyset]) = O_{x_0}(K^T)\).

Let \(I \triangleleft O_{x_0}(K^T)\). Let us consider its zero-multigerm \([\mathcal{Z}(I)]\). Then in \(O_{x_0}(K^T)\) the ideal \(J([\mathcal{Z}(I)]) = \{\varphi \in O_{x_0}(K^T) : [\mathcal{Z}(I)] \subseteq [\mathcal{Z}(\varphi)]\}\) is well-defined. The condition \([\mathcal{Z}(I)] \subseteq [\mathcal{Z}(\varphi)]\) means that there is \(\alpha \in \text{Fin}(I)\) such that \(Z(\alpha) \subseteq Z(\varphi)\).

Now we are ready to state the main results of this paper.

**Theorem 5.12.** Let \(I\) be an ideal of \(O_{x_0}(\mathbb{R}^T)\). Then \(J([\mathcal{Z}(I)]) = \sqrt[\varphi]{I}\).

**Proof.** First we show that \(J([\mathcal{Z}(I)]) \subseteq \sqrt[\varphi]{I}\).

Let \(\varphi \in J([\mathcal{Z}(I)])\). Then \(Z(I) \triangleleft Z(\varphi)\). Hence there is \(\alpha \in \text{Fin}(I)\) such that \(Z(\alpha) \subseteq Z(\varphi)\). From Corollary 3.13 there is \(S \in \text{Fin}(T)\) such that: \(Z((M^T_S)^{-1}(\alpha)) \subseteq Z((M^T_S)^{-1}(\varphi))\). Then \(J(Z((M^T_S)^{-1}(\varphi))) \subseteq J(Z((M^T_S)^{-1}(\alpha)))\) in \(O_{x_0S}(\mathbb{R}^S)\). Now from Risler’s theorem for the ring \(O_{x_0S}(\mathbb{R}^S)\) we get that \(\sqrt[\varphi]{(M^T_S)^{-1}(\alpha)} \subseteq \sqrt[\varphi]{(M^T_S)^{-1}(\varphi)}\). Observe now that \((M^T_S)^{-1}(\varphi) \in \sqrt[\varphi]{(M^T_S)^{-1}(\varphi)}\). Hence \((M^T_S)^{-1}(\varphi) \in \sqrt[\varphi]{(M^T_S)^{-1}(\alpha)}\). Then

\[
\varphi = M^T_S ((M^T_S)^{-1}(\varphi)) \in M^T_S \left(\sqrt[\varphi]{(M^T_S)^{-1}(\varphi)}\right) \subseteq M^T_S \left(\sqrt[\varphi]{(M^T_S)^{-1}(\alpha)}\right).
\]
Now from Proposition 2.6 we obtain
\[ \varphi \in M^T \left( (M^T)^{-1} \left( \sqrt[\alpha]{(\alpha)} \right) \right) \subseteq \sqrt[\alpha]{(\alpha)} \subseteq \sqrt[\beta]{I}. \]

If \( \varphi \in \sqrt[\beta]{I} \) then \( (\varphi) \subseteq \sqrt[\beta]{I} \). Hence from Proposition 5.2 \( [Z(\sqrt[\beta]{I})] \subseteq [Z(\varphi)]. \) In Proposition 5.7 we showed that \( [Z(\sqrt[\beta]{I})] = [Z(I)]. \) Then also \( [Z(I)] \subseteq [Z(\varphi)]. \) Therefore \( \varphi \in J([Z(I)]). \)

**Theorem 5.13.** Let \( I \) be an ideal of \( O_{x_0}(C^T) \). Then \( J([Z(I)]) = \sqrt{I}. \)

**Proof.** The proof is similar to the proof for the real case. The inclusion \( J([Z(I)]) \subseteq \sqrt{I} \) follows from the Rückert theorem [17, 16] in the finite-dimensional case.

**Corollary 5.14.** Let \( I \) be an ideal of \( O_{x_0}(R^T) \). Then \( [Z(I)] = [\{x_0\}] \iff \sqrt[\alpha]{I} = m_{x_0}. \)

**Remark 5.15.** The condition \( [Z(I)] = [\{x_0\}] \) in the real case describes local observability of an infinite-dimensional dynamical system. The ideal \( I \) is generated by the germs at \( x_0 \) of the functions from the observation algebra of the system that vanish at \( x_0 \). See [10, 12] for details.

**Corollary 5.16.** Let \( I \) be an ideal of \( O_{x_0}(C^T) \). Then \( [Z(I)] = [\{x_0\}] \iff \sqrt{I} = m_{x_0}. \)

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