THE FACTORIZATION METHOD FOR A PARTIALLY COATED CAVITY IN INVERSE SCATTERING

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Abstract. We consider the interior inverse scattering problem of recovering the shape of an impenetrable partially coated cavity. The scattered fields incited by point source waves are measured on a closed curve inside the cavity. We prove the validity of the factorization method for reconstructing the shape of the cavity. However, we are not able to apply the basic theorem introduced by Kirsch and Grinberg to treat the key operator directly, and some auxiliary operators have to be considered. In this paper, we provide theoretical validation of the factorization method to the problem, and some numerical results are presented to show the viability of our method.

1. Introduction. The inverse scattering problem for acoustic or electromagnetic waves has drawn increased attention in recent years due its importance in various areas, such as medical imaging, ultrasound tomography, nondestructive testing, material science, radar. The typical inverse scattering problems are exterior problems where the measurements are taken outside of the objects (see [5, 6]). But in recent years there has been extensive interest in the interior scattering problem for determining the scattering objects and the structures [7, 13, 21, 23, 25, 26, 29, 30]. The scattering problem with both point sources (incident waves) and measurements (scattered waves) inside a cavity is called the interior scattering problem. In this paper, we consider an inverse scattering problem from an impenetrable partially coated cavity. The goal is to reconstruct the shape of the scattering object from the near field data measured on some curve $C$ inside the cavity. These problems may occur in many industrial applications of non-destructive testing.

Comparing with other sampling methods [5, 6], the factorization method gives an exact characterization of the boundary using the behaviour of the indicator
function. The factorization method developed by authors in [11, 12, 14, 15], Kirsch and Grinberg summarized their works in the monograph [16] in 2008. More related works can be found in [3, 4, 21] and the reference therein.

In this paper, we try to use this method to retrieve the shape and location of a partially coated cavity in $\mathbb{R}^2$. However, in the case of a partially coated cavity, we are not able to apply the basic theorem introduced by Kirsch and Grinberg to treat the key operator. Some auxiliary operators have to be considered. The main challenge is to factorize the auxiliary operators suitably, and show some key properties to the related operators. Then the basic Theorem 2.15 in [16] can be used to retrieve the shape and location of the cavity.

The inverse scattering problem we consider in this paper is to determine the shape of the partially coated cavity from the knowledge of the near field data measured on a known closed curve $C$ inside the cavity.

This paper is organized as follows. In the next section, we will formulate the direct scattering problem, and give the well-posedness of the direct scattering problem by using the variational method. The mathematical basis of the factorization method applied to treat the inverse scattering problem is given in section 3. In section 4, some numerical results are presented to show the viability of our method.

2. The direct scattering problem. We begin with the formulations of the scattering problem. Let $k = \frac{\omega}{c} > 0$ be the wave number, where $\omega > 0$ is the frequency of a time harmonic wave and $c > 0$ is the sound speed. Let $D \subset \mathbb{R}^2$ be a bounded simply connected domain with Lipschitz boundary $\Gamma = \Gamma_D \cup \Pi \cup \Gamma_I$, where $\Gamma_D$ and $\Gamma_I$ are disjoint, relatively open subsets of $\Gamma$, having $\Pi$ as their common boundary in $\Gamma$. Furthermore, the Dirichlet and impedance type of boundary conditions are specified on $\Gamma_D$ and $\Gamma_I$, respectively.

To be precise, we consider the two-dimensional scattering problem of determining the total field $U = u^s + u^i$, from the mixed boundary value problem

\[
\begin{cases}
\Delta U + k^2 U &= 0 \quad \text{in} \quad D, \\
U &= 0 \quad \text{on} \quad \Gamma_D, \\
\frac{\partial U}{\partial \nu} + \lambda U &= 0 \quad \text{on} \quad \Gamma_I,
\end{cases}
\]

where $\nu$ denotes the unit outward normal vector defined almost everywhere on $\Gamma_D \cup \Gamma_I$, and $\lambda$ is a complex-valued impedance coefficient.

Firstly we make some assumptions.

**Assumptions.**

(1). Let $C$ be a Lipschitz closed curve inside $D$, and $D_0$ be the interior domain enclosed by $C$.

(2). The incident wave $u^i$ is a point source in the form

$$u^i(x) = \Phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \quad x \in \mathbb{R}^2, \quad y \in C$$

where $H_0^{(1)}(\cdot)$ is the Hankel function of zero order of the first kind.

(3). The impedance coefficient $\lambda$ satisfies $\Im(\lambda) > 0$.

We introduce some spaces will be used in the following. $H^1(D)$ and $H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ are the usual Sobolev spaces, $H^{1/2}(\Gamma)$ is the trace space, here $\Gamma = \partial D$. 
If \( \Gamma_0 \) is a partial boundary of \( \Gamma \), i.e., \( \Gamma_0 \subset \Gamma \), we define

\[
\begin{align*}
H^{\frac{1}{2}}(\Gamma_0) &= \{ u|_{\Gamma_0} : u \in H^{\frac{1}{2}}(\Gamma) \}, \\
\tilde{H}^{\frac{1}{2}}(\Gamma_0) &= \{ u \in H^{\frac{1}{2}}(\Gamma) : \text{supp}(u) \subseteq \Gamma_0 \}, \\
H^{-\frac{1}{2}}(\Gamma_0) &= (\tilde{H}^{\frac{1}{2}}(\Gamma_0))^\prime \text{ the dual space of } \tilde{H}^{\frac{1}{2}}(\Gamma_0), \\
H_0^1(\Gamma) &= \{ u \in H^1(\Gamma) : u = 0 \text{ on } \Gamma_D \}.
\end{align*}
\]

Here \( H_0^1(D, \Gamma_D) \) is equipped with the norm induced from \( H^1(D) \).

The direct scattering problem is to find scattering wave \( u^s \in H^1(D) \) by the known incident wave \( u^i \) and the boundary \( \Gamma \). Since the incident wave \( u^i = \Phi(x, y) \) satisfies the Helmholtz equation, then the problem (1) is a special case of the following general boundary value problem

\[
\begin{align*}
\Delta u + k^2 u &= 0 \text{ in } D, \\
\frac{\partial u}{\partial n} + \lambda u &= f \text{ on } \Gamma_D, \\
\frac{\partial u}{\partial n} + \lambda u &= h \text{ on } \Gamma_I.
\end{align*}
\]

(2)

**Remark 1.** The problem (1) is a special case of the problem (2) by taking \( u = u^s \), \( f = -u^i \in H^{\frac{1}{2}}(\Gamma_D) \) and \( h = -(\frac{\partial u^i}{\partial n} + \lambda u^i) \in H^{-\frac{1}{2}}(\Gamma_I) \).

In the variational sense, the problem (2) is equivalent to find \( u \in H^1(D) \) satisfying

\[
\int_D (-k^2 u \varphi + \nabla u \nabla \varphi) \, dx + \lambda \int_{\Gamma_I} u \varphi \, ds = \langle h, \varphi \rangle,
\]

for all test function \( \varphi \in H^1_0(D, \Gamma_D) \).

**Theorem 2.1.** Assume that \( \Im(\lambda) > 0 \), then the interior scattering problem (2) has a unique weak solution \( u \in H^1(D) \) (satisfying (3)), and

\[
\|u\|_{H^1(D)} \leq C(\|f\|_{H^{\frac{1}{2}}(\Gamma_D)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma_I)}),
\]

for some positive constant \( C \).

**Proof.** (1) We want to show that \( u = 0 \) if \( f = h = 0 \) in the problem (2). Let \( \varphi = u \) in the variational equation (3), then

\[
\int_D (-k^2 u \varphi + \nabla u \nabla \varphi) \, dx + \lambda \int_{\Gamma_I} u \varphi \, ds = 0.
\]

Taking the imaginary part, we have \( u = 0 \) on \( \Gamma_I \), since \( \Im(\lambda) > 0 \). Thus \( \frac{\partial u}{\partial n} |_{\Gamma_I} = 0 \) by the boundary condition. By Holmgren’s uniqueness theorem in [9], we conclude that \( u = 0 \) in \( D \).

(2) Let \( u_0 \in H^1(D) \) be a solution of \( \Delta u_0 = 0 \) in \( D \) with \( u_0 = \tilde{f} \) on \( \Gamma \), when \( \tilde{f} \) is restricted on \( \Gamma_D \), \( \tilde{f}|_{\Gamma_D} = f \), and \( \|\tilde{f}\|_{H^{\frac{1}{2}}(\Gamma)} \leq C \|f\|_{H^{\frac{1}{2}}(\Gamma_D)} \). Then we have

\[
\|u_0\|_{H^1(D)} \leq C_1 \|\tilde{f}\|_{H^{\frac{1}{2}}(\Gamma)} \leq C_2 \|f\|_{H^{\frac{1}{2}}(\Gamma_D)}.
\]

Denote \( w = u - u_0 \). Then \( w \) satisfies

\[
\Delta w + k^2 w = -k^2 u_0 \quad \text{in } D,
\]

and

\[
\frac{\partial w}{\partial n} + \lambda w = h - \left( \frac{\partial u_0}{\partial n} + \lambda u_0 \right) \quad \text{on } \Gamma_I.
\]
We can rewrite the equation (3) as
\[ \int_D (-k^2 w \varphi + \nabla w \nabla \varphi) dx + \lambda \int_{\Gamma_i} w \varphi ds \]
(5) \[ = \langle h, \varphi \rangle - \int_{\Gamma_i} \left( \frac{\partial u_0}{\partial \nu} + \lambda u_0 \right) \varphi ds + \int_D k^2 u_0 \varphi dx. \]
Denote
\[ a(w, \varphi) = \int_D (-k^2 w \varphi + \nabla w \nabla \varphi) dx + \lambda \int_{\Gamma_i} w \varphi ds, \]
\[ l(\varphi) = \langle h, \varphi \rangle - \int_{\Gamma_i} \left( \frac{\partial u_0}{\partial \nu} + \lambda u_0 \right) \varphi ds + \int_D k^2 u_0 \varphi ds, \]
\[ a_1(w, \varphi) = \int_D (w \varphi + \nabla w \nabla \varphi) dx + \lambda \int_{\Gamma_i} w \varphi ds, \]
\[ a_2(w, \varphi) = \int_D (-k^2 - 1) w \varphi dx. \]
Obviously, \( a(w, \varphi) = a_1(w, \varphi) + a_2(w, \varphi) \) is a sesquilinear form and \( l(\varphi) \) is a bounded conjugate linear function. From the Riesz-Fredholm theorem [22], we know that there exists \( w \in H^1_0(D, \Gamma_D) \) satisfying (5), and
\[ \|u\|_{H^1(D)} \leq C_1(\|u_0\|_{H^1(D)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma_i)}) \leq C_2(\|f\|_{H^\frac{1}{2}(\Gamma_D)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma_i)}), \]
with constants \( C_1, C_2 \). Then
\[ \|u\|_{H^1(D)} \leq \|w\|_{H^1(D)} + \|u_0\|_{H^1(D)} \leq C(\|f\|_{H^\frac{1}{2}(\Gamma_D)} + \|h\|_{H^{-\frac{1}{2}}(\Gamma_i)}). \]
The proof of this theorem is completed. \( \square \)

3. The factorization method. The inverse problem we are concerned with is to determine the shape and location of the cavity \( D \) from the knowledge of the near field data \( u^s(x, y), x \in C \) due to point sources \( \Phi(\cdot, y), y \in C \).

Through the scattered fields \( u^s(x, y), x, y \in C \), we define a near field operator \( N : L^2(C) \rightarrow L^2(C) \) given by
\[ (Ng)(x) = \int_C u^s(x, y) g(y) ds(y), \quad x \in C. \]
(6)

Our approach for solving the inverse problem is the factorization method, that is, we try to derive a factorization decomposition to the near field operator \( N \) in the form
\[ N = H^* M H \]
(7)
with different forms of the operators \( H \) and \( M \), and \( H^* \) denotes the adjoint operator of \( H \). The operator \( H \) is compact with dense range and \( M \) is an isomorphism.

Then we try to check that the related operators satisfy the fundamental theorem introduced by Kirsch and Grinberg (see Theorem 2.15 [16]).

Remark 2. In fact, in the following discussion, we find that we are not able to apply the basic theorem introduced by Kirsch and Grinberg to treat the key operator \( N \) directly, and some auxiliary operators have to be considered.

The factorization method is based on the following fundamental theorem (Theorem 2.15 [16]).

**Theorem 3.1.** Let \( X \subset U \subset X^* \) be a Gelfand triple with a Hilbert space \( U \) and a reflexive Banach space \( X \) such that the imbedding is dense. Furthermore, let \( Y \) be a second Hilbert space and let \( N : Y \rightarrow Y, G : X \rightarrow Y \) and \( T : X^* \rightarrow X \) be linear bounded operators such that
\[ N = GTG^*. \]
We have the following assumptions:

(1). $G$ is compact with dense range.

(2). There exists $t \in [0, 2\pi]$ such that $\Re[e^{itT}]$ has the form $\Re[e^{itT}] = C + K$ with some compact operator $K$ and some self-adjoint and coercive operator $C : X^* \to X$, i.e., there is $c_0 > 0$ with

$$\langle \varphi, C\varphi \rangle \geq c_0 \|\varphi\|^2, \quad \text{for all} \quad \varphi \in X^*.$$

(3). $\Im(T)$ is non-negative or non-positive on $X^*$, i.e., $\langle \varphi, \Im(T)\varphi \rangle \geq 0$ or $\langle \varphi, \Im(T)\varphi \rangle \leq 0$, for all $\varphi \in X^*$.

(4). $\Re[e^{itT}]$ is one-to-one or $\Im(T)$ is strictly positive or strictly negative, for $\varphi \in \Re(G^*)$ and $\varphi \neq 0$.

Then the operator $N^\dagger = |\Re[e^{itN}]| + |\Im(N)|$ is positive, and the range of $G$ coincides with the range of $N^\dagger$.

In order to decompose the near field operator $N$, we define a data-to-data operator $G : H^\frac{1}{2}(\Gamma_D) \times H^{-\frac{1}{2}}(\Gamma_I) \to L^2(C)$ which maps the boundary data into the near fields $u^\dagger|_C$, i.e.,

$$G(f, h) = u^\dagger|_C,$$

where $u^\dagger$ is the scattered field of the problem (1). Through a single-layer potential

$$v_\varphi(x) = \int_C \Phi(x, y)\varphi(y)ds(y), \quad \forall \varphi \in L^2(C), \quad x \in \mathbb{R}^2 \setminus \mathcal{C},$$

we define an operator $H = (H_1, H_2) : L^2(C) \to H^\frac{1}{2}(\Gamma_D) \times H^{-\frac{1}{2}}(\Gamma_I)$ given by

$$H_1\varphi = v_\varphi|_{\Gamma_D}, \quad H_2\varphi = (\frac{\partial}{\partial \nu} + \lambda)\varphi|_{\Gamma_I}.$$

To the original problem (1), from the definitions (6)-(10), we have a decomposition

$$N = -GH.$$

Let $(\alpha, \beta) \in \overline{H^{-\frac{1}{2}}(\Gamma_D)} \times \overline{H^{\frac{1}{2}}(\Gamma_I)}$, $g \in L^2(C)$, we have

$$\langle Hg, (\alpha, \beta) \rangle = \int_{\Gamma_D} \overline{\alpha(x)} \int_C \Phi(x, y)g(y)ds(y)ds(x) + \int_{\Gamma_I} \overline{\beta(x)}(\frac{\partial}{\partial \nu} + \lambda)\Phi(x, y)ds(x)ds(y) \tag{12}$$

So the adjoint operator of $H$ is given by

$$(H^* (\alpha, \beta))(x) = \int_{\Gamma_D} \alpha(y)\overline{\Phi(x, y)}ds(y) + \int_{\Gamma_I} \beta(y)(\frac{\partial}{\partial \nu} + \lambda)\overline{\Phi(x, y)}ds(y), \quad x \in C. \tag{13}$$

We use $(\tilde{\alpha}, \tilde{\beta})$ to denote the zero extension of $(\alpha, \beta)$ to $\Gamma$, where $(\tilde{\alpha}, \tilde{\beta}) \in H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, and define a potential

$$W(x) = \int_{\Gamma_D} \tilde{\alpha}(y)\overline{\Phi(x, y)}ds(y) + \int_{\Gamma_I} \tilde{\beta}(y)(\frac{\partial}{\partial \nu} + \lambda)\overline{\Phi(x, y)}ds(y). \tag{14}$$
for \( x \in \mathbb{R}^2 \setminus \Gamma \).

Let \( x \) approach the boundary \( \Gamma \) from inside of the domain \( D \), and using the jump relationships of the single- and double-layer potentials across the boundary, we have

\[
W(x)\big|_{\Gamma} = \int_{\Gamma} \tilde{\alpha}(y) \Phi(x, y) ds(y) + \int_{\Gamma} \tilde{\beta}(y) \left( \frac{\partial}{\partial \nu_y} + \lambda \right) \Phi(x, y) ds(y) - \frac{1}{2} \tilde{\beta}(x),
\]

and

\[
\frac{\partial}{\partial \nu} + \lambda)W(x)\big|_{\Gamma} = \frac{\partial}{\partial \nu_x} \left[ \int_{\Gamma} \tilde{\alpha}(y) \Phi(x, y) ds(y) + \int_{\Gamma} \tilde{\beta}(y) \left( \frac{\partial}{\partial \nu_y} + \lambda \right) \Phi(x, y) ds(y) \right]
\]

\[
+ \lambda \int_{\Gamma} \tilde{\alpha}(y) \Phi(x, y) ds(y) + \int_{\Gamma} \tilde{\beta}(y) \left( \frac{\partial}{\partial \nu_y} + \lambda \right) \Phi(x, y) ds(y)
\]

\[
+ \tilde{\alpha} + \frac{(\lambda - \gamma) \tilde{\beta}}{2}.
\]

**Remark 3.** For simplicity, in our discussion, we use "(\( \cdot \))_+" or "(\( \cdot \))_−" to denote the limit approaching the boundary from outside and inside of the corresponding domain, respectively.

Analogous to [9], \( S, K, K' \) and \( T \) are boundary integral operators defined by

\[
S_\varphi(x) = \int_{\Gamma} \varphi(y) \Phi(x, y) dy, \quad x \in \Gamma,
\]

\[
K_\varphi(x) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu_y} dy, \quad x \in \Gamma,
\]

\[
K'_\varphi(x) = \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu_x} dy, \quad x \in \Gamma,
\]

\[
T_\varphi(x) = \frac{\partial}{\partial \nu_x} \int_{\Gamma} \varphi(y) \frac{\partial \Phi(x, y)}{\partial \nu_y} dy, \quad x \in \Gamma.
\]

We use \( S_{\Gamma_D} \) and \( S_{\Gamma_I} \) to denote the restriction of \( S \) on the partial boundary \( \Gamma_D \) and \( \Gamma_I \), respectively, and \( K_{\Gamma_D}, K'_{\Gamma_D}, T_{\Gamma_D}, K_{\Gamma_I}, K'_{\Gamma_I}, T_{\Gamma_I} \) have the similar meaning.

**Remark 4.** (1). It is well-known (see [22]) that the operators

\( S : H^{-\frac{1}{2}+s}(\Gamma) \rightarrow H^{\frac{1}{2}+s}(\Gamma) \), \( K : H^{\frac{1}{2}+s}(\Gamma) \rightarrow H^{\frac{1}{2}+s}(\Gamma) \),

\( K' : H^{-\frac{1}{2}+s}(\Gamma) \rightarrow H^{-\frac{1}{2}+s}(\Gamma) \), \( T : H^{\frac{1}{2}+s}(\Gamma) \rightarrow H^{-\frac{1}{2}+s}(\Gamma) \)

are bounded for \( s \in (-1, 1) \).

(2). The restriction operators

\( S_{\Gamma_D} |_{\Gamma_J} : H^{-\frac{1}{2}+s}(\Gamma_J) \rightarrow H^{\frac{1}{2}+s}(\Gamma_J) \), \( K_{\Gamma_D} |_{\Gamma_J} : H^{\frac{1}{2}+s}(\Gamma_J) \rightarrow H^{\frac{1}{2}+s}(\Gamma_J) \),

\( K'_{\Gamma_D} |_{\Gamma_J} : H^{-\frac{1}{2}+s}(\Gamma_J) \rightarrow H^{-\frac{1}{2}+s}(\Gamma_J) \), \( T_{\Gamma_D} |_{\Gamma_J} : H^{\frac{1}{2}+s}(\Gamma_J) \rightarrow H^{-\frac{1}{2}+s}(\Gamma_J) \)

are bounded for \( s \in (-1, 1) \), for \( \Gamma_J, \Gamma_J = \Gamma_D \) or \( \Gamma_I \).

Restrict (15) and (16) on \( \Gamma_D \) and \( \Gamma_I \), respectively, and remember that \( \tilde{\alpha}|_{\Gamma_I} = 0 \) and \( \tilde{\beta}|_{\Gamma_D} = 0 \), we have

\[
\left( \frac{\partial}{\partial \nu} + \lambda \right)W(x)\big|_{\Gamma_D} = M \left( \begin{array}{c} W_{\Gamma_D} \\ \alpha \end{array} \right),
\]

where

\[
M = \left( \begin{array}{ccc} S_{\Gamma_D} |_{\Gamma_J} & (K_{\Gamma_I} + \lambda S_{\Gamma_D}) |_{\Gamma_J} \\ (K'_{\Gamma_D} + \lambda S_{\Gamma_D}) |_{\Gamma_J} & (T_{\Gamma_I} + \lambda K_{\Gamma_D}) |_{\Gamma_J} \end{array} \right).
\]
The factorization method for a partially coated cavity

By the definitions of the operators \( \mathcal{H}^* \) in (13) and \( G \) in (8), combining with the equation (18), we have a decomposition

(19) \[ \mathcal{H}^*(\alpha, \beta) = GM(\alpha, \beta). \]

**Theorem 3.2.** Assume that \( k^2 \) is not an eigenvalue of \(-\triangle\) in \( D_0 \), we have

(1) The imaginary part of \( M \) is strictly negative, i.e.,

\[ \langle \Im(M)(\alpha, \beta), (\alpha, \beta) \rangle < 0, \]

for all \((\alpha, \beta) \in \tilde{H}^{-\frac{1}{2}}(\Gamma_D) \times \tilde{H}^{\frac{1}{2}}(\Gamma_I)\) and \((\alpha, \beta) \neq (0, 0)\).

(2) \( M \) is invertible.

(3) The near field operator \( N \) can be decomposed into

(20) \[ N = -\mathcal{H}^*M^{-1}\mathcal{H}. \]

(4) The operator \( \mathcal{H}^* \) is compact and has a dense range in \( L^2(C) \).

**Proof.** (1). Firstly from the expression of \( W \) (in (14)), we have

(21) \[ \alpha = -\left( \frac{\partial W^+}{\partial \nu} - \frac{\partial W^-}{\partial \nu} \right)|_{\Gamma_D}, \quad (W^+ - W^-)|_{\Gamma_D} = 0, \]

and

(22) \[ \beta = (W^+ - W^-)|_{\Gamma_I}, \quad \lambda \beta = -\left( \frac{\partial W^+}{\partial \nu} - \frac{\partial W^-}{\partial \nu} \right)|_{\Gamma_I}. \]

From (17) we have

\[ \langle M(\alpha, \beta), (\alpha, \beta) \rangle = \langle (f, h), (\alpha, \beta) \rangle \]

\[ = \int_{\Gamma_D} W^- \left( \frac{\partial W^-}{\partial \nu} - \frac{\partial W^+}{\partial \nu} \right) ds + \int_{\Gamma_I} \left( \frac{\partial W^-}{\partial \nu} + \lambda W^- \right)(W^+ - W^-) ds \]

\[ = \int_{\Gamma_D} (W^- \frac{\partial W^-}{\partial \nu} - W^+ \frac{\partial W^+}{\partial \nu}) ds + \int_{\Gamma_I} \left( \frac{\partial W^-}{\partial \nu} (W^+ - W^-) + \lambda W^- (W^+ - W^-) \right) ds \]

\[ = \int_{\Gamma_D} (W^- \frac{\partial W^-}{\partial \nu} - W^+ \frac{\partial W^+}{\partial \nu}) ds + \int_{\Gamma_I} \left( \left[ \frac{\partial W^+}{\partial \nu} + \lambda W^+ \right] (W^+ - W^-) \right) W^+ \]

\[ - \frac{\partial W^-}{\partial \nu} W^- + \lambda W^- (W^+ - W^-) \} ds \]

\[ = \int_{\Gamma_D} (W^- \frac{\partial W^-}{\partial \nu} - W^+ \frac{\partial W^+}{\partial \nu}) ds + \int_{\Gamma_I} \left( \frac{\partial W^+}{\partial \nu} W^+ - \frac{\partial W^-}{\partial \nu} W^- \right) ds \]

\[ + \int_{\Gamma_I} \left[ \lambda (W^+ - W^-) \right] W^+ + \lambda W^- (W^+ - W^-) \} ds. \]
Taking the imaginary part in (23), we have
\[ \Im\langle M(\alpha, \beta), (\alpha, \beta) \rangle = \Im\left\{ \int_{\Gamma_D} \left( W - \frac{\partial W^+}{\partial \nu} - W - \frac{\partial W^-}{\partial \nu} \right) ds + \int_{\Gamma_I} \left( \frac{\partial W^+}{\partial \nu} W^+ - \frac{\partial W^-}{\partial \nu} W^- \right) ds \right\} \]

+ \Im\left\{ \int_{\Gamma_I} \left[ \lambda(W^+ - W^-) W^+ + \lambda W(W^+ - W^-) \right] ds \right\}

\[ = \Im\left\{ \int_{\Gamma_D} \left( W - \frac{\partial W^+}{\partial \nu} - W + \frac{\partial W^-}{\partial \nu} \right) ds - \int_{\Gamma_I} \left( \frac{\partial W^+}{\partial \nu} W^+ - \frac{\partial W^-}{\partial \nu} W^- \right) ds \right\} \]

\[ - \Im(\lambda) \int_{\Gamma_I} |W^+ - W^-|^2 ds \]

From (22), we have \( M(\alpha, \beta) \) is injective. Letting \( W \to \infty \) in \( D \), we have
\[ \Im\langle M(\alpha, \beta), (\alpha, \beta) \rangle = 0. \]

From Helmholtz’s Lemma and unique continuation, we have \( W = 0 \) in \( \mathbb{R}^2 \setminus \overline{D} \), and \( W|_{x=0} = 0 \). From (22), we have \( \beta = (W^+ - W^-)|_{\Gamma_I} = 0 \), and \( \frac{\partial W^+}{\partial \nu} - \frac{\partial W^-}{\partial \nu} |_{\Gamma_I} = -\lambda \beta = 0 \). Then we have \( W|_{\Gamma_I} = \frac{\partial W^-}{\partial \nu} |_{\Gamma_I} = 0 \). Since \( W \) satisfies the Helmholtz equation in \( D \), by the Holmgren’s uniqueness theorem [9], we have \( W = 0 \) in \( D \). Then \( \alpha = -\frac{\partial W^+}{\partial \nu} - \frac{\partial W^-}{\partial \nu} |_{\Gamma_D} = 0 \). This deduces that the imaginary part of \( M \) is strictly negative.

(2). If \( M(\alpha, \beta) = 0 \), obviously \( \Im\langle M(\alpha, \beta), (\alpha, \beta) \rangle = 0 \). From the above discussion in (1), we have \( \alpha = \beta = 0 \), so \( M \) is injective.

The operator \( M \) can be decomposed into
\[ M \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \tilde{M} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right). \]
This implies that the invertibility of $M$ is equivalent to the invertibility of $\tilde{M}$, and
\[
\tilde{M} = \begin{pmatrix}
S_{\Gamma_D}|_{\Gamma_D} & (K_{\Gamma_I} + \lambda S_{\Gamma_D})|_{\Gamma_I} \\
-(K_{\Gamma_D} + \lambda S_{\Gamma_D})|_{\Gamma_I} & (-T_{\Gamma_I} - \lambda T_{\Gamma_I} - \lambda K_{\Gamma_I} - |\lambda|^2 S_{\Gamma_I} + i\Delta(\lambda) I)|_{\Gamma_I}
\end{pmatrix}.
\]
Because $M$ is injective, we know that $\tilde{M}$ is injective too.

In order to use the Fredholm theory, it is sufficient to show that $\tilde{M}$ is a Fredholm operator with zero index. From those we can obtain that the operator $\tilde{M}$ has bounded inverse, and the operator $M$ also is bounded invertible. With the help of the following Lemma 3.3, we will prove $\tilde{M}$ is a Fredholm operator with index zero.

Lemma 3.3. (see Lemma 1.14 and Theorem 1.26 in \cite{16}). Let $S_0, K_0, K_0', T_0$ be the operators corresponding to the operators $S, K, K', T$ with kernel $\tilde{\Phi}(x, y)$ replaced by the kernel $\Phi_0(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|}$, respectively. Denote $L_S = S - S_0$, $L_K = K - K_0$, $L_{K'} = K' - K_0'$ and $L_T = T - T_0$. We have
(i). The operators $L_S, L_K, L_{K'}, L_T$ are compact.
(ii). $K_0$ and $K_0'$ are adjoint operators, and $S_0$ is self-adjoint.
(iii). $S_0$ and $-T_0$ are coercive, i.e., there exists $c_0$ such that
\[\langle S_0 \varphi, \varphi \rangle \geq c_0 \|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)}^2 \quad \text{and} \quad \langle -T_0 \varphi, \varphi \rangle \geq c_0 \|\varphi\|_{H^{\frac{1}{2}}(\Gamma)}^2.
\]

Denote
\[
A_0 = \begin{pmatrix}
S_0|_{\Gamma_D} & (K_0 + \lambda S_0)|_{\Gamma_D} \\
(-K_0' - \lambda S_0)|_{\Gamma_D} & (-T_0 - i\Delta(\lambda) I)|_{\Gamma_I}
\end{pmatrix},
\]
and
\[
A_e = \begin{pmatrix}
L_S|_{\Gamma_D} & (L_K + \lambda L_S)|_{\Gamma_D} \\
(-L_{K'} - \lambda L_S)|_{\Gamma_I} & (-L_T + \lambda K + \lambda K' + |\lambda|^2 S)|_{\Gamma_I}
\end{pmatrix},
\]
then $\tilde{M} = A_0 + A_e$.

It is easy to check that $A_e : H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma_D) \times H^{\frac{1}{2}}(\Gamma_I)$ is compact, and $A_0 : H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma_D) \times H^{-\frac{1}{2}}(\Gamma_I)$ defines a sesquilinear form
\[
\langle A_0 \tilde{\psi}, \psi \rangle = \langle S_0 \tilde{\alpha}, \tilde{\alpha} \rangle + \langle (K_0 + \lambda S_0) \tilde{\beta}, \tilde{\alpha} \rangle - \langle (K_0' + \lambda S_0) \tilde{\alpha}, \tilde{\beta} \rangle
\]
\[
-\langle (T_0 - i\Delta(\lambda) I) \tilde{\beta}, \tilde{\beta} \rangle
\]
\[
= \langle S_0 \tilde{\alpha}, \tilde{\alpha} \rangle - \langle T_0 \tilde{\beta}, \tilde{\beta} \rangle + \langle (K_0 + \lambda S_0) \tilde{\beta}, \tilde{\alpha} \rangle
\]
\[
-\langle \tilde{\alpha}, (K_0 + \lambda S_0) \tilde{\beta} + i\Delta(\lambda) \tilde{\beta}, \tilde{\beta} \rangle,
\]
where $\tilde{\psi} = (\tilde{\alpha}, \tilde{\beta})$ denotes the zero extension of $\psi = (\alpha, \beta)$ on $\Gamma$. Taking the real part of (28), we have
\[
\Re(A_0 \tilde{\psi}, \tilde{\psi}) = \Re((S_0 \tilde{\alpha}, \tilde{\alpha}) - (T_0 \tilde{\beta}, \tilde{\beta})) \geq c_0(\|\tilde{\alpha}\|_{H^{-\frac{1}{2}}(\Gamma)}^2 + \|\tilde{\beta}\|_{H^{\frac{1}{2}}(\Gamma)}^2).
\]
Furthermore
\[
\Re(A_0 \psi, \psi) \geq c_0(\|\alpha\|_{H^{-\frac{1}{2}}(\Gamma_D)}^2 + \|\beta\|_{H^{\frac{1}{2}}(\Gamma_I)}^2) = c_0 \|\psi\|^2.
\]

Now we obtain that $\tilde{M}$ is the Fredholm operator with zero index, combining $\tilde{M}$ is injective, the operators $\tilde{M}$ and $M$ are invertible.

(3). Combining (11), (19) and $M^{-1}$ is bounded, we have
\[
N = -\mathcal{H}^* M^{-1} \mathcal{H}.
\]
(4) From the regularity for elliptic differential equations (see [22]), we know \( u|_{D_0} \in H^1(D_0) \) and \( u|_C \in H^2(C) \). So we obtain that \( G : H^2(D) \times H^{-2}(I) \to L^2(C) \) is compact, since the embedding operator \( I : H^2(C) \to L^2(C) \) is compact.

From the continuation of kernels, the compactness of \( H^* \) is immediately obtained.

The denseness of \( \text{Range}(H^*) \) is equivalent to the injectivity of its adjoint operator \( H^* \).

Let \( H\phi = 0 \), then \( v\phi|_{D} = 0 \) and \( (\frac{\partial}{\partial \nu} + \lambda)v\phi|_{I} = 0 \). This implies that \( v\phi \) is the solution of the exterior boundary condition problem

\[
\begin{cases}
\Delta v + k^2 v &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
v &= 0 \quad \text{on } \Gamma_D, \\
\frac{\partial v}{\partial \nu} + \lambda v &= 0 \quad \text{on } \Gamma_I, \\
\lim_{r \to \infty} \sqrt{r} (\frac{\partial v}{\partial r} - ikv) &= 0.
\end{cases}
\]

By using the uniqueness result to the problem (31) (see [5]), we have \( v\phi = 0 \) in \( \mathbb{R}^2 \setminus \overline{D} \).

Then from the continuation of \( v\phi \), we have \( v\phi = 0 \) in \( \mathbb{R}^2 \setminus \overline{D_0} \). Since \( v\phi \) is a single layer potential, we have \( v\phi|_{C} = 0 \). Hence \( v\phi = 0 \) in \( D_0 \) since \( k^2 \) is not the eigenvalue of \( -\Delta \) in \( D_0 \).

Then \( \phi = \frac{\partial v\phi}{\partial \nu} - \frac{\partial v\phi}{\partial r} = 0 \) on \( C \). So we obtain \( H \) is injective, this implies that \( H^* \) has dense range.

The proof of this theorem has been completed.

But the real part of the operator \( M \) fails to be the sum of a coercive operator and a compact operator, which is in contrast to the second part of Theorem 3.1. In what follows, we will introduce a modified operator \( N_D \) to replace \( N \).

To this end, we select that \( \Omega \) is a priori known open and bounded domain with \( C^2 \) boundary \( \partial \Omega \), such that

\[
\overline{D_0} \subset \Omega \quad \text{and} \quad \overline{\Omega} \subset D,
\]

and the boundary \( \partial \Omega \) of \( \Omega \) is disjoint with \( C \).

For some complex parameters \( \rho_1 \) and \( \rho_2 \), we define

\[
N_D = N - \rho_1 \tilde{H}^* \tilde{H}, \quad \text{and} \quad N_I = N + \rho_2 \tilde{H}^* \tilde{H},
\]

where \( \tilde{H} : L^2(C) \to L^2(\partial \Omega) \) is given by

\[
(\tilde{H} \varphi)(x) = \int_C \Phi(x,y) \varphi(y) ds(y), \quad x \in \partial \Omega,
\]

for \( \varphi(x) \in L^2(C) \).

**Theorem 3.4.** Assume that \( k^2 \) is neither a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \), nor a Dirichlet eigenvalue of \( -\Delta \) in \( D_0 \). Then

1. we have a decomposition

\[
N_D = - \left( \begin{array}{c} \mathcal{H}_1 \\ \tilde{H} \end{array} \right)^* M_1 \left( \begin{array}{c} \mathcal{H}_1 \\ \tilde{H} \end{array} \right),
\]

with

\[
M_1 = M_{\text{ID}} + M_{\text{IC}}
\]

and

\[
N_I = - \left( \begin{array}{c} \tilde{H} \\ \mathcal{H}_2 \end{array} \right)^* M_2 \left( \begin{array}{c} \tilde{H} \\ \mathcal{H}_2 \end{array} \right),
\]
with
\begin{equation}
M_2 = M_{20} + M_{2c},
\end{equation}
where the operators $M_{10}, M_{20}$ are coercive and $M_{1c}, M_{2c}$ are compact.

(2). The operator $\Im(M_1)$ is strictly positive, and $\Im(M_2)$ is strictly negative.

(3). The operators $(\mathcal{H}_1^*, \mathcal{H}_2^*) = \left( \frac{\mathcal{H}_1}{\mathcal{H}} \right)^*$ and $(\mathcal{H}_1^*, \mathcal{H}_2^*) = \left( \frac{\mathcal{H}}{\mathcal{H}_2} \right)^*$ are compact
and have dense ranges in $L^2(C)$.

Proof. (1). Define $R = (R_1, R_2) : L^2(\partial \Omega) \rightarrow H^{\frac{1}{2}}(\Gamma_D) \times H^{-\frac{1}{2}}(\Gamma_I)$ by $R_1 f_1 = w|_{\Gamma_D}$ and $R_2 f_1 = (\frac{\partial}{\partial \nu} + \lambda) w|_{\Gamma_I}$, respectively, where $w$ solves the exterior Dirichlet boundary value problem with $f_1 \in L^2(\partial \Omega)$, that is
\begin{equation}
\begin{cases}
\Delta w + k^2 w &= 0 \quad \text{in } x \in \mathbb{R}^2 \setminus \Omega, \\
w|_{\Gamma_D} &= f_1 \quad \text{on } \partial \Omega, \\
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ik w \right) &= 0.
\end{cases}
\end{equation}
The existence and regularity of the solutions to the problem (38) with $L^2$–boundary data implies the operator $R$ is compact (refer to [1, 2]).

By the definition of $v_\varphi$ in (9), the single-layer potential $v_\varphi$ solves the problem (38) with boundary value $f_1|_{\partial \Omega} = v_\varphi|_{\partial \Omega} = \mathcal{H}_\varphi$. By the definition of $\mathcal{H}$ in (10) and the definition of $R$, we have
\begin{equation}
(39) \quad \mathcal{H}_1 = R_1 \mathcal{H} \quad \text{and} \quad \mathcal{H}_2 = R_2 \mathcal{H}.
\end{equation}

Now we have
\begin{align}
N_D &= N - \rho_1 \mathcal{H}^* \mathcal{H} \\
&= -\mathcal{H}^* M^{-1} \mathcal{H} - \rho_1 \mathcal{H}^* \mathcal{H} \\
&= -\left( \frac{\mathcal{H}_1}{\mathcal{H}} \right)^* \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) M^{-1} \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) \left( \frac{\mathcal{H}_1}{\mathcal{H}} \right) - \rho_1 \mathcal{H}^* \mathcal{H} \\
&= -\left( \frac{\mathcal{H}_1}{\mathcal{H}} \right)^* \left[ \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) M^{-1} \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & \rho_1 I \end{array} \right) \right] \left( \frac{\mathcal{H}_1}{\mathcal{H}} \right).
\end{align}
\begin{equation}
(40)
\end{equation}

Denote
\begin{equation}
(41) \quad M_1 = \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) M^{-1} \left( \begin{array}{cc} I & 0 \\ 0 & R_2 \end{array} \right) + \left( \begin{array}{cc} 0 & 0 \\ 0 & \rho_1 I \end{array} \right),
\end{equation}
then (34) is proved.

For simplicity, we rewrite the operator $M$ as
\begin{equation}
M = \left( \begin{array}{cc} S_0 & 0 \\ 0 & T_0 \end{array} \right) + M^c.
\end{equation}
By Lemma 3.3, we know $M^{-1}$ is bounded, and $M^{-1}$ can be expressed by
\begin{equation}
(42) \quad M^{-1} = \left( \begin{array}{cc} S_0^{-1} & 0 \\ 0 & T_0^{-1} \end{array} \right) - M^{-1} M^c \left( \begin{array}{cc} S_0^{-1} & 0 \\ 0 & T_0^{-1} \end{array} \right).
\end{equation}
Substituting (42) into (41), we have
\[
M_1 = \begin{pmatrix} S_0^{-1} & 0 & 0 \\ 0 & \rho_1 I & 0 \\ 0 & 0 & R_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ R_1 T_0 R_2 & 0 \\ 0 & 0 \\ R_2 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ R_2 & 0 \end{pmatrix} M^{-1} M^c \begin{pmatrix} S_0^{-1} & 0 \\ 0 & T_0^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \\ R_2 & 0 \end{pmatrix}.
\]

Denote
\[
M_{10} = \begin{pmatrix} S_0^{-1} & 0 & 0 \\ 0 & \rho_1 I & 0 \\ 0 & 0 & R_1 \end{pmatrix},
\]
and
\[
M_{1c} = \begin{pmatrix} 0 & 0 \\ R_2 T_0^{-1} R_2 & 0 \\ 0 & 0 \\ R_2 & 0 \\ 0 & 0 \\ I & 0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & R_1 \end{pmatrix} M^{-1} M^c \begin{pmatrix} S_0^{-1} & 0 \\ 0 & T_0^{-1} \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \\ R_2 & 0 \end{pmatrix}.
\]

Set $\Re(\rho_1) > 0$, then $M_{10}$ and $M_{1c}$ are coercive and compact, respectively, since $R_2$ and $M^c$ are compact.

Similarly, we have
\[
N_I = -\begin{pmatrix} \widehat{H} \\ \widehat{H}_2 \end{pmatrix}^* M_2 \begin{pmatrix} \widehat{H} \\ \widehat{H}_2 \end{pmatrix},
\]
with
\[
M_2 = \begin{pmatrix} R_1^* & 0 \\ 0 & I \end{pmatrix} M^{-1} \begin{pmatrix} R_1 & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} \rho_2 I & 0 \\ 0 & 0 \end{pmatrix}.
\]

Then we rewrite $M_2$ as
\[
M_2 = \begin{pmatrix} -\rho_2 I & 0 \\ 0 & T_0^{-1} \end{pmatrix} + \begin{pmatrix} R_1^* S_0^{-1} R_1 & 0 \\ 0 & 0 \end{pmatrix}
- \begin{pmatrix} R_1^* & 0 \\ 0 & I \end{pmatrix} M^{-1} M^c \begin{pmatrix} S_0^{-1} & 0 \\ 0 & T_0^{-1} \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ 0 & I \end{pmatrix}.
\]

Denote
\[
M_{20} = \begin{pmatrix} -\rho_2 I & 0 \\ 0 & T_0^{-1} \end{pmatrix},
\]
and
\[
M_{2c} = \begin{pmatrix} R_1^* S_0^{-1} R_1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} R_1^* & 0 \\ 0 & I \end{pmatrix} M^{-1} M^c \begin{pmatrix} S_0^{-1} & 0 \\ 0 & T_0^{-1} \end{pmatrix} \begin{pmatrix} R_1 & 0 \\ 0 & I \end{pmatrix},
\]
let $\Re(\rho_2) > 0$, then $M_{20}$ is coercive. By the compactness of $R_1$ and $M^c$, we have that $M_{2c}$ is compact.

(2). For any $\phi = (\phi_1, \phi_2) \in H^2(\Gamma_D) \times L^2(\partial \Omega)$, we have
\[
\Re \langle M_1 \phi, \phi \rangle = \Re \langle \begin{pmatrix} I & 0 \\ 0 & R_2 \end{pmatrix} M^{-1} \begin{pmatrix} I & 0 \\ 0 & R_2 \end{pmatrix} \phi, \phi \rangle + \Re \langle \rho_1 \phi_2, \phi_2 \rangle
= \Re \langle \phi, \tilde{M} \phi \rangle + \Re \langle \rho_1 \rangle \| \phi_2 \|^2
= -\Re \langle \tilde{M} \phi, \phi \rangle + \Re \langle \rho_1 \rangle \| \phi_2 \|^2.
\]

Let $\Re(\rho_1) > 0$, then we have $\Re \langle M_1 \phi, \phi \rangle \geq 0$, since $\Re \langle \tilde{M} \phi, \phi \rangle \leq 0$. Where $\phi = M^{-1} \begin{pmatrix} I & 0 \\ 0 & R_2 \end{pmatrix} \phi = M^{-1} \begin{pmatrix} \phi_1 \\ R_2 \phi_2 \end{pmatrix}$.

If $\Re \langle M_1 \phi, \phi \rangle = 0$, this implies $\Re \langle \tilde{M} \phi, \phi \rangle = \Re \langle \rho_1 \rangle \| \phi_2 \|^2 = 0$, i.e.,
\[
\phi_2 = 0 \quad \text{and} \quad \begin{pmatrix} \phi_1 \\ R_2 \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
So $\phi = \hat{0}$, thus $\Im(M_1)$ is strictly positive.

Analogously, we can deduce that $\Im(M_2\phi, \phi) = \Im(M\hat{\phi}, \phi) - \Im(\rho_2)\|\phi_1\|^2$, for $\phi = (\phi_1, \phi_2) \in L^2(\partial\Omega) \times H^{-\frac{1}{2}}(\Gamma_I)$, and $\hat{\phi} = M^{-1}\left(\begin{array}{cc} R_1 & 0 \\ 0 & I \end{array}\right)\phi$.

(3). The compactness of the operator $\tilde{\mathcal{H}}^*$ can be seen immediately, since

$$(\tilde{\mathcal{H}}^*\psi)(x) = \int_{\partial\Omega} \overline{\Phi(x, y)}\psi(y)dy, \quad x \in C$$

has a continuous kernel.

Next, we show $\tilde{\mathcal{H}}$ is injective.

Let $\tilde{\mathcal{H}}\varphi = 0$, i.e., $\int_C \Phi(x, y)\varphi(y)dy = 0$ for any $x \in \partial\Omega$. Then

$$v_{\varphi}(x) = \int_C \Phi(x, y)\varphi(y)dy, \quad x \in \mathbb{R}^2 \setminus \bar{C}$$

satisfies the Helmholtz equation in the domain $\Omega$ with $v_{\varphi}|_{\partial\Omega} = 0$. Because $k^2$ is not the Dirichlet eigenvalue of $-\Delta$ in $\Omega$, so $v_{\varphi} = 0$ in $\Omega$, furthermore $v_{\varphi} = 0$ in $\mathbb{R}^2$, thus $\varphi = 0$. □

**Remark 5.** From Theorem 3.4, we can see that

(1). The operator $N_D$ satisfies all the conditions of Theorem 3.1, so $\text{Range}((N_D^2)^{\frac{1}{2}}) = \text{Range}(\tilde{\mathcal{H}}_1^*, \tilde{\mathcal{H}}^*)$, where $N_D^2 = |\Re(N_D)| + |\Im(N_D)|$. For $\phi = (\phi_1, \phi_2) \in \tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_D) \times L^2(\partial\Omega)$, we have

$$\begin{equation}
(\tilde{\mathcal{H}}_1^*, \tilde{\mathcal{H}}^*)\phi(x) = \int_{\Gamma_D} \phi_1(y)\overline{\Phi(x, y)}dy + \int_{\partial\Omega} \phi_2(y)\overline{\Phi(x, y)}dy, \quad x \in C.
\end{equation}$$

(2). The operator $N_I$ satisfies all the conditions of Theorem 3.1, so $\text{Range}((N_I^2)^{\frac{1}{2}}) = \text{Range}(\tilde{\mathcal{H}}^*, \tilde{\mathcal{H}}_2^*)$, where $N_I^2 = |\Re(N_I)| + |\Im(N_I)|$. For $\phi = (\phi_1, \phi_2) \in L^2(\partial\Omega) \times \tilde{\mathcal{H}}^{\frac{1}{2}}(\Gamma_I)$, we have

$$\begin{equation}
(\tilde{\mathcal{H}}^*, \tilde{\mathcal{H}}_2^*)\phi(x) = \int_{\partial\Omega} \phi_1(y)\overline{\Phi(x, y)}dy + \int_{\Gamma_I} \phi_2(y)(\frac{\partial}{\partial\nu} + \lambda)\overline{\Phi(x, y)}dy, \quad x \in C.
\end{equation}$$

**Lemma 3.5.** Assume that $k^2$ is neither a Dirichlet eigenvalue of $-\Delta$ in $\Omega$, nor a Dirichlet eigenvalue of $-\Delta$ in $D_0$. For any piecewise smooth nonintersecting arc $L$ without caps, defining

$$\begin{equation}
\Phi_L^1(x) = \int_L \varphi_1(y)\overline{\Phi(x, y)}dy + \int_{\partial\Omega} \varphi_2(y)\overline{\Phi(x, y)}dy,
\end{equation}$$

for $\varphi = (\varphi_1, \varphi_2) \in \tilde{\mathcal{H}}^{-\frac{1}{2}}(L) \times L^2(\partial\Omega)$ and $\varphi_1(x) \neq 0$ for all $x \in L$. Then

$$L \subseteq \Gamma_D \iff \Phi_L^1 \in \text{Range}(\tilde{\mathcal{H}}_1^*, \tilde{\mathcal{H}}^*).$$

**Proof.** Assume $L \subseteq \Gamma_D$. From $\tilde{\mathcal{H}}^{-\frac{1}{2}}(L) \subset \tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_D)$, $\Phi_L^1 \in \text{Range}(\tilde{\mathcal{H}}_1^*, \tilde{\mathcal{H}}^*)$ is obtained immediately.

On the contrary, if we have $L \not\subseteq \Gamma_D$, but $\Phi_L^1 \in \text{Range}(\tilde{\mathcal{H}}_1^*, \tilde{\mathcal{H}}^*)$. This implies that there exists $\hat{\varphi} = (\hat{\varphi}_1, \hat{\varphi}_2) \in \tilde{\mathcal{H}}^{-\frac{1}{2}}(\Gamma_D) \times L^2(\partial\Omega)$, such that

$$\begin{equation}
\Phi_L^1(x) = \int_{\Gamma_D} \hat{\varphi}_1(y)\overline{\Phi(x, y)}dy + \int_{\partial\Omega} \hat{\varphi}_2(y)\overline{\Phi(x, y)}dy.
\end{equation}$$

We select a point $z \in L$, but $z \not\in \Gamma_D$, then $\Phi_L^1(x)$ has a singular point at $x = z$, but the right term in (46) is analytic. This is a contradiction. □
From Theorems 3.1, 3.4 and Lemma 3.5, we have

**Theorem 3.6.** Assume that \( k^2 \) is neither a Dirichlet eigenvalue of \(-\Delta\) in \( \Omega \), nor a Dirichlet eigenvalue of \(-\Delta\) in \( D_0 \). Then

(1)
\[
L \subseteq \Gamma_D \iff \Phi_L^1 \in \text{Range}(\mathcal{H}_1^*, \mathcal{H}_1^*),
\]

and

(2)
\[
L \subseteq \Gamma_D \iff \sum_{j=1}^{\infty} \frac{|\langle \Phi_L^1, \psi_j \rangle_{L^2(\Sigma)}|^2}{|\lambda_j|} < \infty.
\]

Here \((\lambda_j, \psi_j)\) is an eigensystem of \( N^4_{D_0} \), recall \( N^4_{D_0} = |\Re(N_D)| + |\Im(N_D)| \).

Analogously, we can reshape the boundary of \( \Gamma_I \) from the following conclusions.

**Lemma 3.7.** Assume that \( k^2 \) is neither a Dirichlet eigenvalue of \(-\Delta\) in \( \Omega \), nor a Dirichlet eigenvalue of \(-\Delta\) in \( D_0 \). For any piecewise smooth nonintersecting arc \( L \) without cups, defining

(49)
\[
\Phi_L^2(x) = \int_{\partial\Omega} \varphi_1(y)\overline{\Phi(x, y)}ds(y) + \int_L \varphi_2(y)(\frac{\partial}{\partial \nu} + \lambda)\overline{\Phi(x, y)}ds(y),
\]

for \( \varphi = (\varphi_1, \varphi_2) \in L^2(\partial\Omega) \times \tilde{H}^1(L) \) and \( \varphi_2(x) \neq 0 \) for all \( x \in L \). Then

(50)
\[
L \subseteq \Gamma_I \iff \Phi_L^2 \in \text{Range}(\mathcal{H}_2^*, \mathcal{H}_2^*),
\]

and

(51)
\[
L \subseteq \Gamma_I \iff \sum_{j=1}^{\infty} \frac{|\langle \Phi_L^2, \psi_j \rangle_{L^2(\Sigma)}|^2}{|\lambda_j|} < \infty.
\]

Here \((\lambda_j, \psi_j)\) is an eigensystem of \( N^4_I \), recall \( N^4_I = |\Re(N_I)| + |\Im(N_I)| \).

4. **The numerical examples.** In this section, we study the applicability of our method through some numerical examples in \( \mathbb{R}^2 \). In every example, for the forward numerical solution, we use the fully discrete collocation method on the boundary integral equation method as suggested in [19, 20, 24].

In the examples, the near field operator \( N \) is represented by a \( C^{64 \times 64} \) matrix, where each entry is the near field pattern \( u^s(x_j, y_l), \ j, l = 1, \ldots, 64 \), with \( x_j \in C \) and \( y_l \in C \) are the equidistant points on the curve \( C \). Then the operator \( N_D = N - \rho_1\tilde{\mathcal{H}}^*\mathcal{H} \) can be computed explicitly, and notice that

\[
(\tilde{\mathcal{H}}^*\tilde{\mathcal{H}}\psi)(\hat{x}) = 2\pi R \int_{|\hat{y}|=1} J_0(kR|\hat{x} - \hat{y}|)\psi(\hat{y})ds(\hat{y}), \quad |\hat{x}| = 1.
\]

**Remark 6.** Refer to [16] on page 85, we have an example to calculate \((\tilde{\mathcal{H}}^*\tilde{\mathcal{H}}\psi)(\hat{x})\) to the case that \( \Omega \) is a disk with center 0 and radius \( R \).
The real and imaginary part of the matrix $N_1$ is given by
\[ \mathcal{R}(N_1) = \frac{N_1 + N_1^*}{2} \quad \text{and} \quad \mathcal{I}(N_1) = \frac{N_1 - N_1^*}{2i}. \]

We define the absolute value of a matrix $A \in \mathbb{C}^{N \times N}$ with a singular value decomposition $A = U \Lambda V^*$ as $|A| = U|\Lambda|V^*$ with $|\Lambda| = \text{diag} |\lambda_j|$, $j = 1, \ldots, N$.

For our reconstructions, we used a grid of $121 \times 121$ equally spaced sampling points in $[-6, 6] \times [-6, 6]$ and $[-4, 4] \times [-4, 4]$. Let $\{\lambda_j, \psi_j\}_{j=1}^{64}$ be the eigensystem of $N_1^t = |\mathcal{R}(N_1)| + |\mathcal{I}(N_1)|$. Then we compute
\[ g(z) = \sum_{j=1}^{64} \frac{|\langle \Phi(\cdot, z), \psi_j \rangle_{L^2(C)}|^2}{|\lambda_j|}. \]

In what follows, we consider two different geometries as the cavity, one is a kite and the other is a peanut. The wave number $k = 1$ and the noise is 3%. We plot the contour of $\ln g(z)$ to reconstruct the shapes of the cavities.

In figure 1, the shape of the cavity is a kite which is denoted by
\[ \partial D = \{ x = (2.5 \cos(z) + 1.25 \cos(2z), 2.5 \sin(z)) : z \in [0, 2\pi] \}. \]

We choose the curve $C$ to be a circle with radius 0.8 center at the origin, and set a small square with side length 2 between $C$ and $\partial D$. We specify the Dirichlet boundary condition for $z \in [0, \pi]$ and the impedance boundary condition for $z \in [\pi, 2\pi]$ with $\lambda = 50(1 + 3i)$.

In figure 2, the shape of the cavity is a peanut which is denoted by
\[ \partial D = \{ x = (2 \cos(z) + 0.4 \cos(3z), 2 \sin(z) + 0.4 \sin(3 \cdot z)) : z \in [0, 2\pi] \}. \]

We choose the curve $C$ to be a circle with radius 1.2 center at the origin, and set a small rectangle $[-1, 1] \times [-0.5, 0.5]$ between $C$ and $\partial D$. We specify the Dirichlet boundary condition for $z \in [0, \pi]$ and the impedance boundary condition for $z \in [\pi, 2\pi]$ with $\lambda = 2 + 6i$.

**Figure 1.** Reconstruction with $\lambda = 50(1 + 3i)$, $k = 1$ and noise 3% (left). One reconstruction contour is denoted by blue colour (right).
Figure 2. Reconstruction with $\lambda = 2 + 6i$, $k = 1$ and noise 3% (left). One reconstruction contour is denoted by blue colour (right).

REFERENCES

[1] T. Angell and R. Kleinmann, The Helmholtz equation with $L^2$–boundary values, SIAM J. Math. Anal., 16 (1985), 259–278.
[2] T. Angell and A. Kirsch, Optimization Methods in Electromagnetic Radiation, Springer-Verlag, New York, 2004.
[3] O. Bondarenko and X. Liu, The factorization method for inverse obstacle scattering with conductive boundary condition, Inverse Problems, 29 (2013), 095021, 25pp.
[4] Y. Boukari and H. Haddar, The factorization method applied to cracks with impedance boundary conditions, Inverse Problems and Imaging, 7 (2013), 1123–1138.
[5] F. Cakoni, D. Colton and P. Monk, The direct and inverse scattering problems for partially coated obstacles, Inverse Problems, 17 (2001), 1997–2015.
[6] F. Cakoni and D. Colton, The linear sampling method for cracks, Inverse Problems, 19 (2003), 279–295.
[7] F. Cakoni, D. Colton and S. Meng, The inverse scattering problem for a penetrable cavity with internal measurements, AMS Contemp. Math., 615 (2014), 71–88.
[8] M. Chamaillard, N. Chaulet and H. Haddar, Analysis of the factorization method for a general class of boundary conditions, J. of Inverse and Ill-posed Problems, 22 (2013), 643–670.
[9] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Springer-Verlag, Berlin, 1998.
[10] K. Daisuke, Error estimates of the DtN finite element method for the exterior Helmholtz Problem, J. Comp. Appl. Math., 200 (2007), 21–31.
[11] N. I. Grinberg and A. Kirsch, The factorization method for obstacles with a priori separated sound-soft and sound-hard parts, Math. Comput. Simulation, 66 (2004), 267–279.
[12] N. I. Grinberg, The operator factorization method in inverse obstacle scattering, Integral Equations and Operator Theory, 54 (2006), 333–348.
[13] Y. Hu, F. Cakoni and J. Liu, The inverse problem for a partially coated cavity with interior measurements, Appl. Anal., 93 (2014), 936–956.
[14] A. Kirsch, Characterization of the shape of a scattering obstacle using the spectral data of the far field operator, Inverse Problems, 14 (1998), 1489–1512.
[15] A. Kirsch, Factorization of the far field operator for the inhomogeneous medium case and an application in inverse scattering theory, Inverse Problems, 15 (1999), 413–429.
[16] A. Kirsch and N. I. Grinberg, The Factorization Method for Inverse Problems, Oxford University Press, New York, 2008.
[17] A. Kirsch and X. Liu, Direct and inverse acoustic scattering by a mixed-type scatterer, Inverse Problems, 29 (2013), 065005, 19pp.
[18] A. Kirsch and X. Liu, A modification of the factorization method for the classical acoustic inverse scattering problem, Inverse Problems, 30 (2014), 035013, 14pp.
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[19] R. Kress, On the numerical solution of a hypersingular integral equation in scattering theory, *J. Comput. Appl. Math.*, 61 (1995), 345–360.

[20] R. Kress and K. M. Lee, Integral equation method for scattering from an impedance crack, *J. Comp. Appl. Math.*, 161 (2003), 161–177.

[21] X. D. Liu, The factorization method for cavities, *Inverse problems*, 30 (2014), 015006, 18pp.

[22] W. Mclean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge University Press, Cambridge, 2000.

[23] S. X. Meng, H. Haddar and F. Cakoni, The factorization method for a cavity in an inhomogeneous medium, *Inverse Problems*, 30 (2014), 045008, 20pp.

[24] L. Mönch, On the inverse acoustic scattering problem by an open arc: The sound-hard case, *Inverse Problems*, 13 (1997), 1379–1392.

[25] H. Qin and F. Cakoni, Nonlinear integral equations for shape reconstruction in inverse interior scattering problem, *Inverse Problems*, 27 (2011), 035005, 17pp.

[26] H. Qin and D. Colton, The inverse scattering problem for cavities, *Appl. Numer. Math.*, 62 (2012), 699–708.

[27] J. Yang, B. Zhang and H. Zhang, The factorization method for reconstructing a penetrable obstacle with unknown buried objects, *SIAM. J. Appl. Math.*, 73 (2013), 617–635.

[28] J. Yang, B. Zhang and H. Zhang, Reconstruction of complex obstacles with generalized impedance boundary conditions from far-field data, *SIAM J. Appl. Math.*, 74 (2014), 106–124.

[29] F. Zeng, F. Cakoni and J. Sun, An inverse electromagnetic scattering problem for a cavity, *Inverse Problems*, 27 (2011), 125002.

[30] F. Zeng, P. Suarez and J. Sun, A decomposition method for an interior scattering problem, *Inverse Problems and Imaging*, 7 (2013), 291–303.

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