Finite axionic electrodynamics from a new non-commutative approach

Patricio Gaete$^1$ and Euro Spallucci$^2$

$^1$ Departamento de Física and Centro Científico—Tecnológico de Valparaíso, Universidad Técnica Federico Santa María, Valparaíso, Chile
$^2$ Dipartimento di Fisica Teorica, Università di Trieste and INFN, Sezione di Trieste, Italy

E-mail: patricio.gaete@usm.cl and spallucci@ts.infn.it

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Abstract

Using the gauge-invariant but path-dependent variable formalism, we compute the static quantum potential for non-commutative axionic electrodynamics (or axionic electrodynamics in the presence of a minimal length). Accordingly, we obtain an ultraviolet finite static potential that is the sum of a Yukawa-type potential and a linear potential, leading to the confinement of static charges. Interestingly, it should be noted that this calculation involves no $\theta$ expansion at all. The present result manifests the key role played by the new quantum of length in our analysis.

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1. Introduction

The formulation and physical consequences of extensions of the standard formalism of field theory to allow non-commuting position operators have been the object of intensive investigations by many authors [1–6]. Let us recall here that these new commutation relations were originally suggested with the goal of avoiding ultraviolet divergences, which appear within the perturbative approach of quantum field theory [7]. In this perspective, it should be recalled that recently considerable attention has been paid to the study of non-commutative field theories due to its natural emergence in string theory [1, 2]. In addition to the string interest, non-commutative quantum field theories have also attracted considerable attention because of some surprising consequences, for example: the ultraviolet–infrared mixing [8], loss of unitarity in models where time does not commute with space coordinates [9] and Lorentz symmetry breaking [10].

It is worth recalling at this point that these studies have been achieved using a star product (Moyal product). More recently, a novel way to formulate non-commutative quantum field theory (or quantum field theory in the presence of a minimal length) has been proposed in [11–13]. The key ingredient of this development is to define the fields as a mean value over coherent states of the non-commutative plane, such that a star product needs not be introduced.
More recently, it has been shown that the coherent state approach can be summarized through the introduction of a new multiplication rule, which is known as the Voros star product [14–18]. Anyway, physics turns out to be independent from the choice of the type of product [19]. An alternative view of these modifications is to consider them as a redefinition of the Fourier transform of the fields. As a consequence, the theory is ultraviolet finite and the cutoff is provided by the non-commutative parameter \( \theta \). It must be clear from this discussion that the existence of a minimal length is determined by the non-commutative parameter \( \theta \). Indeed, since one can incorporate a minimal length \( \sqrt{\theta} \) in spacetime by assuming non-trivial coordinate commutation relations, we then have introduced a non-commutative geometry. Interestingly enough, every point-like structure is smeared out by the presence of the new quantum of length in the manifold.

On the other hand, in previous studies [20, 21], we have considered both axionic electrodynamics and its non-commutative version with a Lorentz invariance violating term, in the presence of a non-trivial constant expectation value for the gauge field strength. This non-commutative version was discussed to leading order in \( \theta \), via the Seiberg–Witten map. We note that these theories experience mass generation due to the breaking of rotational invariance induced by the classical background configuration of the gauge field strength, and in the case of a constant electric field strength expectation value the static potential remains Coulombic for both theories. Nevertheless, this picture drastically changes in the case of a constant magnetic field strength expectation value. In effect, for axionic electrodynamics the potential energy is the sum of a Yukawa and a linear potential, leading to the confinement of static charges. While for non-commutative axionic electrodynamics, the interaction energy is the sum of a Coulomb and a linear potential. Nevertheless, the above models are not ultraviolet finite theories.

Inspired by the above observations, one then naturally asks whether there is a consistent and well-defined axionic electrodynamics in the presence of a minimal length. It appears that this is indeed so as we show in this paper. More specifically, the main goal of this work shall be to examine the effect of the spacetime non-commutativity on a physical observable. To do this, we will work out the static potential for the theory under consideration using the gauge-invariant but path-dependent variable formalism, which is an alternative to the Wilson loop approach [22]. Our treatment is exact for the non-commutative parameter \( \theta \); in other words, there is no \( \theta \) expansion. Interestingly, our analysis leads to a well-defined non-commutative interaction energy. In fact, we obtain an ultraviolet finite static potential that is the sum of a Yukawa-type potential and a linear potential, leading to the confinement of static charges. The key role played by the new quantum of length in triggering a well-defined interaction energy is our main result.

2. Finite electrodynamics

As stated in the introduction, the main focus of this paper is to re-examine the interaction energy between static point-like sources for non-commutative axionic electrodynamics. To carry out such study, we shall compute the expectation value of the energy operator \( H \) in the physical state \( |\Phi_1\rangle \), which we will denote by \( \langle H \rangle_{\Phi_1} \). However, the analysis will be first carried out for a non-commutative version of electrodynamics to more easily grasp the central features and will then be extended to non-commutative axionic electrodynamics.

Before starting our analysis it may be useful to recall, as effective Lagrangian looks like in the non-commutative coordinates coherent state approach. We mean effective because they are ordinary quantum field theories defined over a smooth, flat [11–13] or curved manifold [23–33], with a ‘memory’ of the non-classical nature of the coordinates below a certain length scale. In a series of papers [11–13], it has been shown how non-commutative geometry effects
are recorded in a long-distance quantum field theory. We can summarize the final results as follows.

(i) Point-like particles have physical meaning. Any physical object has a linear size \( l \geq \sqrt{\theta} \).

(ii) It follows from (i) that there are no point-like sources. Thus, distributions like \( J \propto \delta(\vec{x}) \) have to be replaced by \textit{minimal width Gaussian} functions. From a formal point of view, such a smearing is implemented through the substitution rule

\[
\delta(\vec{x}) \longrightarrow e^{i\theta k^2} \delta(\vec{x}). \tag{1}
\]

(iii) Smeared sources lead to ultraviolet suppressed (Euclidean) Feynman propagators of the form

\[
G(k^2) = e^{-\theta k^2/k^2 + m^2}. \tag{2}
\]

This kind of propagators can be obtained from modified kinetic terms. In the simplest case of a scalar particle, the effective Lagrangian leading to (2) reads

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) e^{\theta \Delta}(\partial^\mu \phi) + \frac{m^2}{2} \phi e^{\theta \Delta} \phi, \tag{3}
\]

where \( \Delta \equiv \partial_\mu \partial^\mu \).

This is a model with derivatives of arbitrary order. In this paper, we are going to investigate the \textit{static} potential between test charges; thus, the d’Alembert operator will be replaced by the Laplace operator and only spatial derivatives will appear. From this point of view, our work is complementary to that of [34], where in a p-adic string model dilaton dynamics is considered and is described by a similar, exponential operator of time derivatives. Both in [34] and this work, the fully covariant case with the exponential of the d’Alembert operator is not addressed. It is well beyond the purpose of this paper, which is focused on the static potential and would deserve an in-depth analysis by itself. Some details on the perturbative treatment of models with higher order derivatives are given in appendices A and B.

It is not difficult to guess that the effective Maxwell–Lagrangian density turns out to be

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} e^{\theta \Delta} F^{\mu\nu}. \tag{4}
\]

We now discuss the interaction energy between static point-like sources for this non-commutative electrodynamics, through two different methods. The first approach is based on the path-integral formalism, whereas the second one makes use of the gauge-invariant but path-dependent variable formalism.

Let us start by writing down the functional generator of Green’s functions, that is,

\[
Z[J] = \exp\left( -\frac{i}{2} \int d^4x \, d^4y J^\mu(x) D_{\mu\nu}(x, y) J^\nu(y) \right), \tag{5}
\]

where \( D_{\mu\nu}(x, y) = \int \frac{dk}{(2\pi)^4} D_{\mu\nu}(k) e^{-iks} \) is the propagator in the Feynman gauge. In this case, the corresponding propagator is given by

\[
D_{\mu\nu}(k) = -\frac{1}{k^2} \left\{ e^{ik^2} \eta_{\mu\nu} + (1 - e^{ik^2}) \frac{k_\mu k_\nu}{k^2} \right\}. \tag{6}
\]

By means of the expression \( Z = e^{iW[J]} \) and employing equation (5), \( W[J] \) takes the form

\[
W[J] = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J^\mu(k) \left[ -\frac{e^{ik^2}}{k^2} \eta^{\mu\nu} - \frac{(1 - e^{ik^2}) k_\mu k_\nu}{k^2} \right] J^\nu(k). \tag{7}
\]
Since the current \( J^\mu (k) \) is conserved, expression (7) then becomes
\[
W [J] = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} F^\mu \left( \frac{\vec{q}^2}{k^2} \right) J^\mu (k).
\] (8)

Next, for \( J_\mu (x) = [Q^\partial (x - x^{(1)}) + \bar{Q}^\partial (x - x^{(2)})] \delta^\mu_0 \), and using standard functional techniques \([35]\), we obtain that the interaction energy of the system is given by
\[
V(r) = Q' \int \frac{d^3k}{(2\pi)^3} \frac{e^{-\vec{q}^2k^2}}{k^2} e^{i\vec{k}\cdot \vec{r}},
\] (9)
where \( r \equiv x^{(1)} - x^{(2)} \). In order to calculate \( V(r) \), we note that the integral over \( k \) can also be written as
\[
\int \frac{d^3k}{(2\pi)^3} \frac{e^{-\vec{q}^2k^2}}{k^2} e^{i\vec{k}\cdot \vec{r}} = \int \frac{d^3k}{(2\pi)^3} \int_0^\infty ds e^{-(\theta/4)k^2} e^{i\vec{k}\cdot \vec{r}}
\]
\[
= \frac{1}{4(\pi)^{3/2}} \gamma (1/2; r^2/4\theta),
\] (10)
with \( r = |\vec{r}| \). Here, \( \gamma (1/2; r^2/4\theta) \) is the lower incomplete Gamma function defined by the following integral representation:
\[
\gamma \left( \frac{1}{2}; \frac{r^2}{4\theta} \right) \equiv \int_0^r du \ e^{-u} u^{1/2}.
\] (11)

By means of expression (10) together with \( Q' = -Q \), the interaction energy reduces to
\[
V(r) = - \frac{Q^2}{4(\pi)^{3/2}} \frac{1}{r} \gamma (1/2; r^2/4\theta).
\] (12)

From this expression, it should be clear that the interaction energy is regular at the origin, in contrast to the usual Maxwell theory. In this respect, the above result clearly shows the key role played by the ‘smeared propagator’ in equation (8).

Next, we compute the interaction energy from the viewpoint of the gauge-invariant but path-dependent variable formalism, along the lines of \([20–22]\). Within this framework, we shall compute the expectation value of the energy operator \( H \) in the physical state \( |\Phi\rangle \), which we will denote by \( \langle H | \Phi \rangle \). Nevertheless, to obtain the corresponding Hamiltonian we must carry out the quantization of the theory. At this point, special care has to be exercised since expression (4) contains higher time derivatives. However, as was mentioned before, this paper is aimed at studying the static potential of the above theory, so that \( \Delta \) can be replaced by \(-\nabla^2\). At the moment for notational convenience we will maintain \( \Delta \), but it should be borne in mind that this paper essentially deals with the static case. In addition, it is interesting to note that if we expand expression (4) up to first order in \( \theta \), we obtain the Lagrangian density of the Abelian Lee–Wick model \([36, 37]\). We shall come back to these points in appendix A.

We now turn our attention to the calculation of the interaction energy. In order to obtain the corresponding Hamiltonian, the canonical quantization of this theory from the Hamiltonian point of view is straightforward. The canonical momenta are found to be \( \Pi^\mu = -\phi^{\partial \Delta} F^{0\mu} \), and one immediately identifies the usual primary constraint \( \Gamma_0 = 0 \) and \( \Pi^i = \phi^{\partial \Delta} F^{i0} \). The canonical Hamiltonian is now obtained in the usual way by a Legendre transform, that is,
\[
H_C = \int d^3x \left\{ -A_0 \partial_0 \Pi^i - \frac{1}{2} \Pi_\mu \Pi^\mu_e - \frac{1}{4} F_{ij} F^{ij} \right\}.
\] (13)

Time conservation of the primary constraint, \( \Pi_0 = 0 \), leads to the usual Gauss constraint \( \Gamma_1 (x) = \partial_0 \Pi^0 = 0 \). The extended Hamiltonian that generates translations in time then reads \( H = H_C + \int d^3x \left( c_0 (x) \Pi_0 (x) + c_1 (x) \Gamma_1 (x) \right) \), where \( c_0 (x) \) and \( c_1 (x) \) are the Lagrange
multipliers. Since $\Pi^0 = 0$ for all time and $A_0(x) = [A_0(x), H] = c_0(x)$, which is completely arbitrary, we discard $A^0$ and $\Pi^0$ because they add nothing to the description of the system. Thus, the extended Hamiltonian is now given as

$$H = \int d^3x \left\{ c(x) \partial_\mu \Pi^\mu - \frac{1}{2} \Pi^\mu \Pi_\mu - \frac{1}{4} F_{ij} F^{ij} \right\},$$

where $c(x) = c_1(x) - A_0(x)$ is an arbitrary parameter reflecting the gauge invariance of the theory. As is well known, to avoid this trouble we must fix the gauge. A particularly convenient choice is found to be

$$\Gamma_2(x) = \int_{C_i} dz^\nu A_\nu(z) \equiv \int_0^1 d\lambda \lambda^i A_\lambda(\lambda x) = 0,$$

where $\lambda(0 \leq \lambda \leq 1)$ is the parameter describing the space-like straight path $x^i = \xi^i + \lambda (x - \xi)^i$ and $\xi$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^i = 0$. The choice (15) leads to the Poincaré gauge [38]. As was explained in [38], we can now write down the only non-vanishing Dirac bracket for the canonical variables. This is a fairly long calculation which will not repeat here:

$$[A_i(x), \Pi_j(y)]^\ast = \delta^i_j \delta^{(3)}(x - y) - \partial_j^i \int_0^1 d\lambda \lambda^i \delta^{(3)}(\lambda x - y).$$

We now proceed with the calculation of the interaction energy between point-like sources for the model under consideration. As we have noted before, we will calculate the expectation value of the energy operator $H$ in the physical state $|\Phi\rangle$. At this point, we also recall that the physical state $|\Phi\rangle$ can be written as

$$|\Phi\rangle = |\overline{\Psi}(y)\Psi(y')\rangle = \overline{\Psi}(y) \exp \left( -i q \int_{C_y} dz^\nu A_\nu(z) \right) \Psi(y')(0),$$

where the line integral is along a space-like path on a fixed time slice, $q$ is the fermionic charge and $|0\rangle$ is the physical vacuum state. Note that the charged matter field together with the electromagnetic cloud (dressing) which surrounds it is given by $\Psi(y) = \exp(-i q \int_{C_y} dz^\nu A_\nu(z)) \psi(y)$. With the help of our path choice, this physical fermion then becomes $\Psi(y) = \exp(-i q \int_{C_y} dz^\nu A_\nu(z)) \psi(y)$. In other words, each of the states $|\Phi\rangle$ represents a fermion–antifermion pair surrounded by a cloud of gauge fields to maintain gauge invariance.

Next, by taking into account the above Hamiltonian structure, we observe that

$$\Pi_i(x)|\overline{\Psi}(y)\Psi(y')\rangle = \overline{\Psi}(y) \Psi(y') \Pi_i(x)|0\rangle + q \int_y^{y'} dz^\nu \delta^{(3)}(z - x)|\Phi\rangle.$$

Having made this observation and since the fermions are taken to be infinitely massive (static) we can substitute $\Delta$ by $-\nabla^2$ in equation (14). Therefore, the expectation value $\langle H \rangle_{\Phi}$ becomes

$$\langle H \rangle_{\Phi} = \langle H \rangle_0 + \langle H \rangle_{\Phi}^{(1)}.$$

where $\langle H \rangle_0 = \langle 0 | H | 0 \rangle$. The $\langle H \rangle_{\Phi}^{(1)}$ term is given by

$$\langle H \rangle_{\Phi}^{(1)} = -\frac{q^2}{2} \int d^3x \int_y^{y'} dz^\nu \delta^{(3)}(x - z') e^{\theta(z')} \int_y^{y'} dz^\nu \delta^{(3)}(x - z),$$

which can also be expressed solely in terms of the new Green function

$$\langle H \rangle_{\Phi}^{(1)} = -\frac{q^2}{2} \int y^{y'} dz^\nu \int_y^{y'} dz^\nu \nabla^2 \tilde{G}(z, z').$$
Figure 1. Shape of the potential, equation (23). Note that $V(L) = \frac{q^2}{8\pi}\frac{1}{\sqrt{\theta}}V(y)$, with $y = L/2\sqrt{\theta}$. The dashed line represents the Coulomb potential.

In this case, $\tilde{G}$ is the new Green function

$$\tilde{G}(z, z') = \frac{1}{4(\pi)^{3/2}}\frac{1}{r}y(1/2; r^2/4\theta),$$  \hspace{1cm} (22)

where $r = |z - z'|$.

Employing equation (21) and remembering that the integrals over $z'\dagger$ and $\zeta'\theta$ are zero except on the contour of integration, the potential for two opposite charges, located at $y$ and $y'$, reduces to the Coulomb-type potential. In other words,

$$V(L) = -\frac{q^2}{4(\pi)^{3/2}}\frac{1}{L}y(1/2; L^2/4\theta),$$  \hspace{1cm} (23)

with $|y - y'| \equiv L$. One immediately sees that both approaches, despite being completely different, lead to the same result which seems to indicate that they are equivalent term by term. Incidentally, it is of interest to note that the above result comes from the constraint structure of the theory under consideration. Furthermore, in contrast to our previous analysis [20, 21] via the Seiberg–Witten map, unexpected features are found. Interestingly, it should be noted that the above calculation of $V$ involves no $\theta$ expansion at all. Note also that setting the non-commutative parameter to zero reproduces the standard expression for Maxwell theory. In fact, it is observed that the introduction of the non-commutative space induces a finite Coulombic potential for $L \rightarrow 0$ (see figure 1). This then implies that the self-energy and the electromagnetic mass of a point-like particle are finite in this version non-commutative of electrodynamics. Thus, the present theory is extremely simple but still rich in content and its static potential is remarkably similar to the one found in the Abelian Lee–Wick model. In addition, it is worthwhile to note that the potential (23) is spherically symmetric, although the external fields break the isotropy of the problem in a manifest way. Also, we stress that the choice of the gauge in this approach is really arbitrary. Being the formalism completely gauge invariant, we would obtain exactly the same result in any gauge.

We shall next discuss two alternative derivations of the result (23). Instead of working with the Lagrangian density (4), we might as well formulate the discussion in terms of smeared sources, as was first demonstrated in [11–13]. It also permits us to check the internal consistency of our methodology.
As a first step, we shall begin by considering the usual Lagrangian density for the Maxwell field, namely

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$  \hfill (24)

Following the same steps employed for obtaining (19), the above term is expressed as

$$\langle H \rangle_{(1)} = -\frac{q^2}{2} \int d^3x \int_0^y d\zeta \delta^{(3)}(x - z') \int_y^y d\zeta' \delta^{(3)}(x - z).$$  \hfill (25)

We now consider the formulation of this theory in the presence of a minimal length. To do this, the source \( \delta(3)(x - y) \) is replaced by the smeared source \( e^{2\theta \nabla^2} \delta(3)(x - y) \). Hence, expression (18) reduces to

$$\Pi_i(x) \equiv \Psi(y) \Psi(y') \Pi_i(0) + q \int_y^y d\zeta e^{2\theta \nabla^2} \delta^{(3)}(z - x)(x) |\Phi \rangle.$$  \hfill (26)

Thus, by employing relation (26) into equation (25), we immediately recover the result (23).

As a second derivation of our previous result, it may be recalled that [38]

$$V = q(A_0(y) - A_0(y')),$$  \hfill (27)

where the physical scalar potential is given by

$$A_0(x^0, x) = \int_0^1 d\xi x^i E_i(\xi x),$$  \hfill (28)

with \( i = 1, 2, 3 \). This follows from the vector gauge-invariant field expression

$$A_\mu(x) \equiv A_\mu(x) + \partial_\mu \left( -\int_\xi^x d\eta A_\mu(\eta) \right),$$  \hfill (29)

where, as in equation (17), the line integral is along a space-like path from the point \( \xi \) to \( x \), on a fixed slice time. The gauge-invariant variables (29) commute with the sole first constraint (Gauss’ law), corroborating that these fields are physical variables [39]. In passing, we note that Gauss’ law for the present theory reads \( \partial_i \Pi^i = J^0 \), where we have included the external current \( J^0 \) to represent the presence of two opposite charges. For, \( J^0(x) = q e^{2\theta \nabla^2} \delta^{(3)}(x) \), we then have that the electric field may be written as

$$E_i = q \partial_i G(x).$$  \hfill (30)

Finally, replacing this result in (28) and using (27), the potential for a pair of point-like opposite charges \( q \), located at \( 0 \) and \( L \), takes the form

$$V(L) = -\frac{q^2}{4(\pi)^{3/2}} \frac{1}{L} \gamma(1/2; L/4\theta),$$  \hfill (31)

where \( |L| \equiv L \). It must be clear from this discussion that a correct identification of physical degrees of freedom is a key feature for understanding the physics hidden in gauge theories. According to this viewpoint, once that identification is made, the computation of the potential is carried out by means of Gauss’ law.
3. Finite axionic electrodynamics

We now extend our analysis for considering axionic electrodynamics [20], which is the main thrust of this work. The gauge theory we are considering is defined by

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4} \varphi \tilde{F}^{\mu\nu} F_{\mu\nu}. \]  
(32)

However, in order to put our discussion into context it is useful to summarize the relevant aspects of the analysis described previously [20]. Thus, our first undertaking is to carry out the integration over the \( \varphi \)-field. We also recall that in [20] we have considered static scalar fields, as a consequence we may replace \( \Delta \varphi = -\nabla^2 \varphi \), with \( \Delta \equiv \partial^\mu \partial_\mu \). Once this is done, we arrive at the following effective theory:

\[ \mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\lambda^2}{32} (\tilde{F}_{\mu\nu} F^{\mu\nu}) \left( \frac{1}{\nabla^2 - m^2} \tilde{F}_{\alpha\beta} F^{\alpha\beta} \right). \]  
(33)

Next, after splitting \( F_{\mu\nu} \) in the sum of a classical background, \( \langle F_{\mu\nu} \rangle \), and a small fluctuation, \( f_{\mu\nu} \), the corresponding Lagrangian density up to quadratic terms in the fluctuations becomes

\[ \mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\lambda^2}{32} \left( \frac{v^0 f_{\mu\nu} \varepsilon_{\mu\nu} v^0}{\nabla^2 - m^2} \right) f_{\mu\nu}, \]  
(34)

where \( f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \) \( a_\mu \) stands for the fluctuation, and \( \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \equiv v^{\mu
u} \) and \( \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \equiv v^{\mu
u} \).

Having made this observation, we now turn our attention to the calculation of the interaction energy in the \( v^0 \neq 0 \) and \( v^i = 0 \) case (referred to as the electric one in what follows). In such a case, the Lagrangian (34) reads

\[ \mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{\lambda^2}{32} \left( \frac{v^0 f_{\mu\nu} \varepsilon_{\mu\nu} v^0}{\nabla^2 - m^2} \right) f_{\mu\nu}. \]  
(35)

It is now again straightforward to apply the formalism discussed in the preceding section. For this purpose, we start by observing that the canonical Hamiltonian can be worked as usual and is given by

\[ H_C = \int d^3 x \left\{ \Pi_\mu \partial^\mu \varphi + \frac{1}{2} \frac{\Pi^2}{(\nabla^2 - M^2)} + \frac{1}{2} B^2 \right\}, \]  
(36)

where \( B \) is the magnetic field, and \( M^2 = m^2 + \frac{\lambda^2}{4} \nabla^2 \).

As we pointed before, by employing relation (26), the expectation value reads

\[ \langle H \rangle_\varphi = \langle H \rangle_0 + \langle H \rangle_\varphi^{(1)} + \langle H \rangle_\varphi^{(2)}, \]  
(37)

with \( \langle H \rangle_0 = \langle 0 | H | 0 \rangle \), while the terms \( \langle H \rangle_\varphi^{(1)} \) and \( \langle H \rangle_\varphi^{(2)} \) are given by

\[ \langle H \rangle_\varphi^{(1)} = -\frac{q^2}{2} \int d^3 x \int^\gamma_\beta d\gamma' e^2 \nabla^2 \delta^{(3)}(x - z') \left( 1 - \frac{M^2}{\nabla^2} \right)^{-1} \int^\gamma_\chi d\gamma'' e^2 \nabla^2 \delta^{(3)}(x - z), \]  
(38)

and

\[ \langle H \rangle_\varphi^{(2)} = \frac{m^2 q^2}{2} \int d^3 x \int^\gamma_\beta d\gamma' e^2 \nabla^2 \delta^{(3)}(x - z') \left( 1 - \frac{M^2}{\nabla^2} \right) \int^\gamma_\chi d\gamma'' e^2 \nabla^2 \delta^{(3)}(x - z). \]  
(39)

Following our earlier discussion, these expressions become

\[ \langle H \rangle_\varphi^{(1)} = -\frac{q^2}{2} \int^\gamma d\gamma' \int^\gamma d\gamma'' \nabla^2 \tilde{G}(z, z'), \]  
(40)

and

\[ \langle H \rangle_\varphi^{(2)} = \frac{q^2 m^2}{2} \int^\gamma d\gamma' \int^\gamma d\gamma'' \tilde{G}(z, z'). \]  
(41)
Accordingly, the new Green function takes the form
\[
\tilde{G} = \frac{e^{M^2\theta}}{4\pi^{3/2}} \sqrt{\frac{2M}{r}} \left[ K_{1/2}(r) - \frac{1}{2} \int_{r/(2M\theta)}^{\infty} dy \, y^{-1/2} e^{-\frac{\pi}{2}(y+1/y)} \right].
\] (42)

By using \(\sqrt{\frac{\pi}{2\pi}}K_{1/2}(x) = \frac{\pi}{2\pi} e^{-x}\), it follows that this new Green function can be rewritten as
\[
\tilde{G} = \frac{e^{M^2\theta}}{4\pi} \left[ e^{-Mr} - \frac{1}{\sqrt{\pi}} \int_{r/(2\theta)}^{\infty} du \, u^{-1/2} e^{-u-(M^2r^2/4u)} \right],
\] (43)

which is finite in the limit \(r \to 0\), that is, \(\tilde{G}(0) = -\frac{M^2}{4\pi} e^{M^2\theta}\).

Finally, following our earlier procedure, the potential for a pair of point-like opposite charges \(q\) located at \(0\) and \(L\) takes the form
\[
V(L) = -\frac{q^2}{4\pi} \frac{e^{M^2\theta}}{L} \left[ e^{-ML} - \frac{1}{\sqrt{\pi}} \int_{L^2/(4\theta)}^{\infty} dy \, y^{-1/2} e^{-y-(M^2L^2/4y)} \right] + \frac{q^2}{8\pi} m^2 e^{M^2\theta} E_1(M^2\theta) L,
\] (44)

where \(E_1(M^2\theta)\) is the exponential integral. Once again, this result explicitly shows the effect of including a smeared source in the form of an ultraviolet finite static potential which is the sum of a Yukawa and a linear potential, leading to the confinement of static charges (see figure 2). Another crucial feature of this result is that the entire effect of non-commutativity is properly captured in the string tension. This improves the situation as compared to our previous studies \([20, 21]\), where an ultraviolet cutoff has been introduced.

Note that in figure 2, we defined \(V(L) = \frac{q^2 e^{M^2\theta}}{4\pi} V(x)\). The plot represents the potential energy for the case \(4M^2\theta = 0.2\) and \(\frac{m^2}{M^2} E_1(M^2\theta) = 2\).

4. Final remarks

In summary, within the gauge-invariant but path-dependent variable formalism, we have considered the confinement versus screening issue for axionic electrodynamics in the presence of a minimal length. Once again, a correct identification of physical degrees of freedom has
been a key tool for understanding the physics hidden in gauge theories. Interestingly, we have obtained an ultraviolet finite static potential that is the sum of a Yukawa-type potential and a linear potential, leading to the confinement of static charges. This may be contrasted with our previous studies [20, 21], where a cutoff has been introduced in order to avoid ultraviolet divergences. As already expressed, this calculation involves no $\theta$ expansion at all. The above analysis reveals the key role played by the new quantum of length in our analysis. Finally, it seems a challenging work to extend the above analysis to the non-Abelian case as well as to three-dimensional gauge theories. We expect to report on progress along these lines soon.

Appendix A. Higher time derivatives and Lee–Wick model

In this appendix, we wish to further elaborate on our previous observation after expression (4), that is, by dealing with static configurations we may use the standard Legendre transformation to construct the Hamiltonian.

The initial point of our analysis is the Lagrangian density (4),

$$L = -\frac{1}{4} F_{\mu\nu} e^{\theta/L} F_{\mu\nu} - J_\mu A^\mu, \quad (A.1)$$

where $J_\mu$ is an external source. In order to illustrate the role played by higher time derivatives, for simplicity, we consider expression (A.1) to the lowest order in $\theta$. In this case,

$$L = -\frac{1}{4} F_{\mu\nu} (1 + \theta/L) F_{\mu\nu} - J_\mu A^\mu, \quad (A.2)$$

which is similar to Lee–Wick electrodynamics [40]. However, expression (A.2) can also be written as

$$L = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{\theta}{2} \partial_\lambda F^{\lambda\alpha} \partial_\rho F_{\rho\alpha} - J_\mu A^\mu, \quad (A.3)$$

which is known as Podolsky’s electrodynamics [41]. In passing, we note that this Lagrangian is the simplest system with second time derivatives.

One immediately sees that the Euler–Lagrange equations are

$$(1 + \theta/L) \partial_\mu F^{\mu\lambda} = J_\lambda, \quad (A.4)$$

where $\Delta \equiv \partial_\mu \partial^\mu$. Expressed in terms of the electric and magnetic fields, $E^i = F^{i0}$ and $B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}$, the equations of motion take the form

$$(1 + \theta/L) \nabla \cdot E = J^0, \quad (A.5)$$

and

$$(1 + \theta/L) (\dot{E} - \nabla \times B) = J. \quad (A.6)$$

Interestingly, it is observed that in the electrostatic case ($\dot{E} = 0$ and $B = 0$), and $J = 0$, these equations reduce to

$$(\nabla^2 - 1/\theta) (\nabla \cdot E) = J^0. \quad (A.7)$$

For $J^0(t, \mathbf{x}) = q \delta^{(3)}(\mathbf{x})$, the electric field is given by

$$E^i = q \delta^i_j (G(\mathbf{x}) + \tilde{G}(\mathbf{x})), \quad (A.8)$$

where $G(\mathbf{x}) = \frac{1}{4\pi} \frac{1}{|\mathbf{x}|}$ and $\tilde{G}(\mathbf{x}) = \frac{\nabla}{4\pi |\mathbf{x}|}$ are the Green functions in three space dimensions. Next, replacing this result in equation (28), the potential for a pair of point-like opposite charges $q$ located at $\mathbf{0}$ and $\mathbf{L}$ becomes

$$V = -\frac{q^2}{4\pi L} \left(1 - e^{-L/|\mathbf{x}|}\right), \quad (A.9)$$
with \( L \equiv |L| \). Incidentally, the above static potential is identical to the one encountered in [37] in the Hamiltonian approach, using the standard Legendre transformation. But we do not think that the agreement is an accidental coincidence.

In order to understand more precisely this agreement, we will re-examine our previous Hamiltonian analysis. In this case, it is well known that the Hamiltonian approach to higher derivatives theories was first developed by Ostrogradsky [42] for non-singular systems, and his method consists in defining one more pair of canonical variables and so doubling the dimension of the phase space. For singular higher derivatives systems (our case), one can generalize Dirac’s theory for constrained systems to include the Ostrogradsky approach. Accordingly, in Lagrangian (A.3) the velocities have to be taken as independent canonical variables. Hence, the phase-space coordinate for the theory under consideration is given by

\[
(A_\mu, \Pi^\nu) \oplus (\dot{A}_\mu, \Pi^{(1)\nu}) \tag{A.10}
\]

where \( \Pi^{(1)\nu} \) is the canonical momentum conjugate to \( \dot{A}_\mu \). This implies that the canonical Hamiltonian \( H_C \) takes the form

\[
H_C = \int d^3x \left( p_\mu \dot{A}_\mu + \Pi^{(1)}_{\mu} \ddot{A}_\mu - \mathcal{L} \right). \tag{A.11}
\]

According to usual procedure, the momenta are defined as [43]

\[
\Pi_{\mu}^{(1)} \equiv \frac{\partial \mathcal{L}}{\partial \ddot{A}_\mu} \tag{A.12}
\]

and

\[
p_{\mu} \equiv \frac{\partial \mathcal{L}}{\partial A_\mu} - 2\partial_k \left[ \frac{\partial \mathcal{L}}{\partial \partial_0 \partial_k A_\mu} \right] - \partial_0 \left( \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \right). \tag{A.13}
\]

Using these definitions, we obtain the following expressions for the momenta:

\[
\Pi_{\mu}^{(1)} = \theta \left[ \partial_0 F_{\mu 0} \delta_\mu^0 + \partial^k F_{k\mu} \right] \tag{A.14}
\]

and

\[
p_{\mu} = -F_{\mu 0} - \theta \left[ 2\partial_k \partial_0 F_{\mu 0} \delta_\mu^0 - \partial_0 \partial_k F_{k\mu} \right]. \tag{A.15}
\]

Hence, we find

\[
\Pi^{(1)} = \theta [\dot{\mathbf{E}} - (\nabla \times \mathbf{B})] \tag{A.16}
\]

and

\[
p = -\mathbf{E} + \theta [\dot{\mathbf{E}} - \partial_0 (\nabla \times \mathbf{B}) - 2 \nabla (\nabla \cdot \mathbf{E})]. \tag{A.17}
\]

Again, it is easy to see that in the electrostatic case (\( \dot{\mathbf{E}} = 0 \) and \( \mathbf{B} = 0 \)), these equations reduce to

\[
\Pi^{(1)} = 0 \tag{A.18}
\]

and

\[
p = -\mathbf{E} + 2 \theta \nabla (\nabla \cdot \mathbf{E}). \tag{A.19}
\]

Also, it should be noted that \( p^0 = 0 \) and \( \Pi^{(1)}_0 = 0 \).

Effectively, therefore, in the electrostatic case our canonical Hamiltonian takes the form

\[
H_C = \int d^3x (p_\mu \dot{A}_\mu - \mathcal{L}). \tag{A.20}
\]
which is the standard Legendre transformation. Thus, the agreement between the result (A.9) and our previous Hamiltonian treatment [37] has been clarified. It follows from this that although the Lagrangian density (A.3) contains higher time derivatives, in the electrostatic case the canonical momentum conjugate to velocities disappears. Therefore, the new Legendre transformation reduces to the standard Legendre transformation.

Evidently, the next step would be to verify, order by order, that the above conclusion is preserved. The above result suggests that this is the case. On the other hand, and in order to support this observation, we also call our attention to the fact that our physical state is defined solely in terms of \( A_i \)-fields, that is,

\[
|\Phi\rangle = |\overline{\Psi}(y)\Psi(y')\rangle = \overline{\Psi}(y) \exp \left( i q \int_y^{y'} dx' A_i(z) \right) \psi(y') |0\rangle.
\]  

(A.21)

One can now observe that, although in principle there may be canonical momenta conjugate to velocities, they will contribute to the calculation of the energy only terms proportional to the canonical momentum conjugated to \( A_i \). This is consistent because our physical state is a gauge-invariant state, so this state cannot contain terms proportional to the velocities in order to preserve gauge invariance. In other terms, our physical state is essentially a static state.

**Appendix B. Non-perturbative versus perturbative results**

In this appendix, we would like to show the difference between exact calculations and perturbative expansion in \( \theta \). We shall consider the simple case of the classical Coulomb potential in a non-commutative (Euclidean) background geometry. The Poisson equation reads

\[
\nabla^2 \phi(\vec{x}) = -4 \pi e \exp(\theta \nabla^2) \delta(\vec{x}),
\]  

(B.1)

where the rhs of equation (B.1) is the smeared source obtained through the rule (ii). The Poisson equation can be exactly solved through the standard Fourier method. The resulting Coulomb potential is

\[
\phi(\vec{x}) = -\frac{e}{\sqrt{\pi |\vec{x}|}} \gamma(1/2; \vec{x}^2/4\theta).
\]  

(B.2)

\( \phi(\vec{x}) \) is regular in the origin

\[
\phi(0) = -\frac{e}{\sqrt{\pi \theta}}.
\]  

(B.3)

The divergence of the classical Coulomb potential has been cured by the short-distance fluctuations of the coordinates.

Suppose we ignore the possibility to obtain an exact solution of equation (B.1) and proceed through a ‘perturbative’ approach, i.e. we expand the exponential operators on the rhs up to the first order in \( \theta \). This is the standard procedure adopted in hundreds of papers implementing non-commutative effects through the Wigner–Weyl–Moyal \( \ast \)-product. In our toy model, this is equivalent to approximate the Poisson equation with

\[
\nabla^2 \phi(\vec{x}) = -4 \pi e (1 + \theta \nabla^2 + O(\theta^2)) \delta(\vec{x}).
\]  

(B.4)

It is immediate to find

\[
\phi(\vec{x}) = -\frac{e}{|\vec{x}|} - 4 \pi e \theta \delta(\vec{x}) + O(\theta^2).
\]  

(B.5)

Thus, the order-\( \theta \) result is the standard Coulomb potential plus a divergent contact interaction in \( \vec{x} = 0 \). It is easy to see that also the successive corrections do not improve
the short-distance behavior of $\phi$. This follows from the fact that at any finite order in $\theta$ the charge keeps its point-like nature. Removing the divergence in $\vec{x} = 0$ is a non-perturbative effect which cannot be seen at any finite order in a $\theta$-expansion. This is a key feature of non-commutative field theories which does not seem to have been fully understood by people working in this research field.

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