Convergence over fractals for the Schrödinger equation

Lucà, R. and Ponce-Vanegas, F.

Abstract

We consider a fractal refinement of the Carleson problem for the Schrödinger equation, that is to identify the minimal regularity needed by the solutions to converge pointwise to their initial data almost everywhere with respect to the $\alpha$-Hausdorff measure ($\alpha$-a.e.). We extend to the fractal setting ($\alpha < n$) a recent counterexample of Bourgain [5], which is sharp in the Lebesgue measure setting ($\alpha = n$). In doing so we recover the necessary condition from [23] for pointwise convergence $\alpha$-a.e. and we extend it to the range $n/2 < \alpha \leq (3n + 1)/4$.

1 Introduction

A classic question related to solutions to the linear Schrödinger equation (here $\hbar = 1/(2\pi)$)

$$\begin{cases}
\partial_t u = i\frac{\hbar}{2}\Delta u \\
u(x, 0) = f(x) \in H^s(\mathbb{R}^n),
\end{cases}$$

is: what is the minimal regularity the initial datum must have so that the solution $u$ converge almost everywhere (a.e.) to $f$? More precisely, which is the smallest $s \geq 0$ such that

$$\lim_{t \to 0} u(x, t) = f(x), \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and for all } f \in H^s(\mathbb{R}^n). \quad (1)$$

This problem was introduced by Carleson in [8], where he proved the validity of (1) for $s \geq 1/4$ in dimension $n = 1$. Soon later Dahlberg and Kenig [10] proved this to be sharp. The considerably harder higher dimensional problem
was subsequently studied by many authors [9, 6, 30, 34, 3, 26, 32, 33, 31, 22, 4, 25, 11, 24, 14].

Recently, the problem has been settled, up to the endpoint, thanks to the contributions of Bourgain [5] (see [27] for a nice detailed exposition), who proved the necessity of $s \geq \frac{n}{2(n+1)}$, and of Du–Guth–Li [13] and of Du–Zhang [15], who proved the sufficiency of $s > \frac{n}{2(n+1)}$ in dimensions $n = 2$ and $n \geq 3$, respectively. We mention that, besides Bourgain’s counterexample, the necessity of $s \geq \frac{n}{2(n+1)}$ can be proved also by different counterexamples [23].

In this paper we consider a fractal refinement of the Carleson problem. Given $\alpha \in (0, n]$, the goal is to identify the smallest $0 \leq s \leq n/2$ such that

$$\lim_{t \to 0} u(x, t) = f(x), \quad \text{for } \alpha\text{-a.e. } x \in \mathbb{R}^n \text{ and for all } f \in H^s(\mathbb{R}^n),$$

where $\alpha$-a.e. means almost everywhere with respect to the $\alpha$-dimensional Hausdorff measure.

This fractal refinement of the Carleson problem was introduced in [29]. In [2], the authors gave a complete solutions for $\alpha \in [0, n/2]$, proving that $s > (n - \alpha)/2$ is necessary and sufficient for (2) to hold. The necessity of this condition depends on the Sobolev space framework, since for smaller $s$ there exist initial data in $H^s(\mathbb{R}^n)$ that are not well defined on sets of dimension $\alpha$; see [35]. On the other hand, for $s > (n - \alpha)/2$ one can make sense of the initial data and of the relative solution $\alpha$-a.e.; we refer to the proof of Theorem [9] for details. When $\alpha \in (n/2, n]$, Du and Zhang [15] proved the best known sufficient condition for (2) to hold:

$$s > \frac{n}{2(n+1)} (n + 1 - \alpha).$$

As mentioned, this is optimal (up to the endpoint) when $\alpha = n$, but it is not clear yet whether this is optimal for $\alpha$ strictly smaller. It is worth mentioning that (3) is necessary for the $\alpha$-a.e. pointwise convergence in the periodic setting [16], however in this setting it is still unknown if it is sufficient (not even for $\alpha = n$).

In [23] it was proved that for $(3n + 1)/4 \leq \alpha \leq n$ the condition

$$s > \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n - \alpha),$$

is necessary for (2) to hold. Here we extend this result to the full range $n/2 < \alpha \leq n$ (recall that for smaller $\alpha$ the problem has been solved in [2]); thus
the result is new for $n/2 < \alpha \leq (3n + 1)/4$. To prove this result, we use a modification of the Bourgain counterexample rather than the counterexample in [23]. We consider this fact of independent interest. The possibility of adapting the Bourgain counterexample to the fractal measure setting was also suggested by Lillian Pierce in [27].

**Theorem 1.** Let $n \geq 2$ and $n/2 < \alpha \leq n$. Then for every

\[ s' < s := \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n-\alpha) \]  

(5)

there exists a function $f \in H^{s'}(\mathbb{R}^n)$ such that

\[ \limsup_{t \to 0^+} |e^{it\Delta/2}f(x)| = \infty \]  

(6)

for $x$ in a set of Hausdorff dimension $\geq \alpha$.

For $\alpha \in (n/2, n)$ we can in fact immediately improve the statement, saying that (6) occurs on a set with $\alpha$-Hausdorff measure $= \infty$. This is because in (5) we have a strict inequality. Thus, given $\alpha' > \alpha$ and sufficiently close to $\alpha$ in such a way that

\[ s' < s := \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n-\alpha'), \]

we would in fact prove that (6) occurs on a set of dimension $\geq \alpha'$. When $\alpha = n$ we can not self-improve the statement, however we know by [23] that (6) holds on a set of strictly positive Lebesgue measure.

A consequence of Theorem 1 is the necessity of the condition

\[ s \geq \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)}(n-\alpha) \]

for the validity of the maximal estimate

\[ \int_{B_R} \sup_{t \in (0,1)} |e^{it\Delta}f(x)|^2 d\mu(x) \lesssim C_\mu R^{2s} \|f\|_2^2, \]  

(7)

where $B_R \subset \mathbb{R}^n$ is a ball of radius $R > 1$, and $\mu$ is an $\alpha$-dimensional measure on $B_R \subset \mathbb{R}^n$, i.e. a positive Borel measure that satisfies

\[ \mu(B_r(x)) \lesssim C_\mu r^\alpha. \]
for all balls with center \( x \) and radius \( r > 0 \). One may see (7) as the weighted \( L^2 \) inequality

\[
\int_{B_R} \sup_{t \in (0,1)} |\hat{g}d\sigma(x)|^2 d\mu(x) \lesssim C_\mu R^{2s} \|g\|_{L^2(S)}^2,
\]

where \( S \) is a bounded hypersurface in \( \mathbb{R}^d := \mathbb{R}^{n+1} \) with non zero gaussian curvature (for instance, a portion of the paraboloid in the case of (7)) and \( d\sigma \) is the measure induced on \( S \) by the Lebesgue measure. A closely related family of weighted \( L^2 \) estimates is

\[
\int_{B_1} |\hat{g}d\sigma(Rx)|^2 d\mu(x) \lesssim C_\mu R^{-\gamma} \|g\|_{L^2(S)}^2,
\]

where \( B_1 \) is now a ball in \( \mathbb{R}^d \) of radius 1, and \( \mu \) is an \( \alpha \)-dimensional measure on \( B_1 \subset \mathbb{R}^d \). The problem here is to identify the largest \( \gamma \) such that (9) holds. Interestingly, these problems are very sensitive to the arithmetical structure of the hypersurface \( S \). For instance, the known necessary conditions are different for the sphere and the paraboloid; see [21, 1, 25, 12, 28, 19].

Notations

- \( e(z) = e^{iz} \).
- If \( A \subset \mathbb{R}^n \), then \( |A| \) is its Lebesgue measure, and if \( A \) is a discrete set, then \( |A| \) is the cardinality. For example, if \( I = [a, b] \subset \mathbb{Z} \) denotes the interval of integers \( a \leq k \leq b \), then \( |I| \) is the length of the interval.
- If \( I = [a, b] \subset \mathbb{Z} \), for \( a, b \in \mathbb{R} \), denotes an interval of integers, then we write \( L(I) := \min_{k \in I} k \) and \( R(I) := \max_{k \in I} k \).
- \( B_r(x) \subset \mathbb{R}^n \) is a ball of radius \( r \) and center \( x \)—the center is usually omitted. \( Q(x, l) \subset \mathbb{R}^n \) is a cube with side-length \( l \) and center \( x \).
- If \( x \lesssim y \), then \( x \leq Cy \) for some constant \( C > 0 \), and similarly for \( x \gtrsim y \); if \( x \asymp y \) then \( x \lesssim y \lesssim x \). If \( x \ll y \) then \( x \leq cy \), where \( c \) is a sufficiently small constant, and similarly for \( x \gg y \).
- \( \limsup_{k \to \infty} F_k := \bigcap_{N \geq 1} \bigcup_{k \geq N} F_k \).
• Hausdorff dimension of a set: for $0 < \alpha \leq n$ and $\delta > 0$ we define the outer measure

$$\mathcal{H}_\delta^\alpha(F) := \inf \left\{ \sum_{B_r \in \mathcal{B}} r^\alpha \mid F \subset \bigcup_{B_r \in \mathcal{B}} B_r \text{ and } r < \delta \right\};$$

we do not exclude the case $\delta = \infty$. The $\alpha$-dimensional Hausdorff measure of a set $F$ is $\mathcal{H}^\alpha(F) := \lim_{\delta \to 0} \mathcal{H}_\delta^\alpha(F)$. The Hausdorff dimension of a set $F$ is $\sup \{ \alpha \mid \mathcal{H}^\alpha(F) > 0 \}$.

Acknowledgments

This research is funded by the Basque Government through the BERC 2018-2021 program, and by the Spanish State Research Agency through BCAM Severo Ochoa excellence accreditation SEV-2017-0718 and by the IHAIP project PGC2018-094528-B-I00. Additionally, the first author is supported by Ikerbasque and the second author is supported by the ERCEA Advanced Grant 2014 669689-HADE.

2 Preliminaries

We recall some classic estimates about exponential sums that we will use repeatedly in the rest of the paper.

We recall first a classical result about Gauss quadratic sums, whose proof can be consulted in Lemma 3.1 of \cite{27}.

Lemma 2 (Gauss quadratic sums). If $a, b, q \in \mathbb{Z}$ satisfy the conditions $(a, q) = 1$ and

$$\begin{cases} b \in \mathbb{Z} & \text{when } q \text{ is an odd number}, \\
 b \text{ is even} & \text{when } q \equiv 0 \pmod{4}, \\
 b \text{ is odd} & \text{when } q \equiv 2 \pmod{4}, \end{cases}$$

(10)

then for the quadratic phase

$$f(r) := \frac{a}{q} r^2 + \frac{b}{q} r$$

(11)
it holds that
\[ \sum_{r=0}^{q-1} e(2\pi f(r)) = c_q \sqrt{q}, \quad (12) \]
where \( c_q = 1 \) when \( q \) is odd, and \( c_q = \sqrt{2} \) when \( q \) is even.

The following estimate due to Weyl will be useful to handle incomplete Gauss sums.

**Lemma 3.** Let \( I \) be an integer interval. If \( a, b, q \in \mathbb{Z} \) satisfy the conditions \( (a, q) = 1 \) and \( (10) \), then for the quadratic phase \( f \) in \( (11) \) it holds that
\[ \left| \sum_{k \in I} e(2\pi f(k)) \right| = C \frac{|I|}{\sqrt{q}} + O(\sqrt{q \ln q}), \quad (13) \]
where \( \frac{1}{2} \leq C \leq \sqrt{2} \).

**Proof.** We can assume that \( L(I) := \min_{k \in I} k = 0 \). In fact,
\[ \sum_{k=L(I)}^{R(I)} e(2\pi(a k^2 + b/k/q)) = e(2\pi(a L(I)^2 + b/q L(I))) \sum_{k=0}^{R(I) - 1} e(2\pi(a k^2 + b + 2a L(I)/q k)), \]
and the absolute value at both sides is the same; we observe that the parity of \( b \) and \( b + 2aL(I) \) is preserved.

If \( |I| < q \), then
\[ \left| \sum_{k \in I} e(2\pi f(k)) \right| \leq C \sqrt{q \ln q}; \quad (14) \]
for the proof we refer to Lemma 3.2 of [27].

If \( |I| \geq q \), then we can sum in blocks of length \( q \). Let \( M \) be the largest integer that satisfies \( Mq \leq |I| \), i.e. \( Mq \leq |I| < (M + 1)q \), then
\[ I = [0, Mq - 1] \cup J = \left( \bigcup_{m=0}^{M-1} [mq, mq + q - 1] \right) \cup J, \]
where \( |J| < q \). The sum over each block \( [mq, mq + q - 1] \) is a Gauss quadratic sum, and we arrive to
\[ \sum_{k \in I} e(2\pi f(k)) := M \sum_{r=0}^{q-1} e(2\pi f(r)) + \sum_{k \in J} e(2\pi f(k)). \]
By our election of $M$ we have $M = C|I|/q$, for $\frac{1}{2} < C \leq 1$, and by (14) we have

$$\left| \sum_{k \in I} e(2\pi f(k)) \right| = C\left| \frac{|I|}{q} \sum_{r=0}^{q-1} e(2\pi f(r)) \right| + \mathcal{O}(\sqrt{q \ln q}).$$

Finally, we apply Lemma 2 to get (13).

To deal with perturbations of quadratic sums, we will use the following Lemma, which is consequence of Abel’s summation formula; see Lemma 2.3 of [16].

**Lemma 4.** Let $I$ be an integer interval. Let $a_k \geq 0$ be a sequence of real numbers and $b_k$ be sequences of complex numbers such that

1. $a_{k+1} \leq a_k$,

2. $\left| \sum_{k \in I'} b_k \right| \leq C$, for every interval $I' \subset I$.

Then,

$$\left| \sum_{k \in I'} a_k b_k \right| \leq C a_{L(I')}, \quad \text{for every interval } I' \subset I. \quad (15)$$

If (1) is replaced with $a_{k+1} \geq a_k$, then

$$\left| \sum_{k \in I'} a_k b_k \right| \leq C a_{R(I')}, \quad \text{for every interval } I' \subset I.$$

### 3 The main lower bound

The initial data we consider are modifications of the Bourgain’s counterexample in [5]. Let $\varphi$ be a smooth positive function such that $\text{supp} \hat{\varphi} \subset B_1(0)$ and $\varphi(0) = 1$. We define the function

$$f_D(x) := f_1(x_1)\tilde{f}(\tilde{x}) \quad (16)$$

where

$$f_1(x_1) = e(2\pi R x_1)\varphi(R^{\frac{1}{2}} x_1), \quad \tilde{f}(\tilde{x}) := \prod_{j=2}^{n} \varphi(x_j) \left( \sum_{\frac{R}{2^n} \leq l_j < \frac{R}{2^{n-1}}} e(2\pi D l_j x_j) \right)$$

7
where \( l = (l_2, \ldots, l_n) \in \mathbb{Z}^{n-1} \) and \( x = (x_1, \tilde{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \). For now we set \( D \) as a free parameter, and we will choose its value later as a suitable power of \( R \).

We need the following definition before investigating the divergence set of \( e^{i\hbar t\Delta/2} f_D \); compare with (10).

**Definition 5 (Admissible fractions).** Let \( p_1, \ldots, p_n, q \in \mathbb{Z} \). A point \( (p_1/q, \ldots, p_n/q) \) is an admissible fraction if \( (p_1, q) = 1 \) and if

\[
\begin{cases}
(p_2, \ldots, p_n) \in \mathbb{Z}^{n-1} & \text{when } q \text{ is an odd number}, \\
p_j \text{ are even} & \text{when } q \equiv 0 \pmod{4}, \\
p_j \text{ are odd} & \text{when } q \equiv 2 \pmod{4}.
\end{cases}
\]  

(17)

**Theorem 6.** Let \( c \ll 1 \) and let \( q > 0 \) be an integer such that \( R_{Dq} \gg \sqrt{\ln q} \). If \( f \) is the initial datum (16), then

\[
\frac{|e^{i\hbar t\Delta/2} f_D(x)|}{\|f_D\|_{L^2}} \gtrsim R^+ \left( \frac{R}{Dq} \right)^{\frac{n-1}{2}}
\]  

(18)

for \((x, t)\) such that \( 0 < t = 2p_1/(D^2 q) \ll 1/R \) and

\[
x \in E_{q,D} \cap [0, c]^n,
\]  

(19)

where \( E_{q,D} \) is the set of points

\[
x_1 \in \frac{p_1 R}{qD^2} + [-cR^{-\frac{1}{2}}, cR^{-\frac{1}{2}}] \quad \text{and} \quad x_j \in \frac{p_j}{Dq} + [-cR^{-1}, cR^{-1}], \quad 2 \leq j \leq n;
\]  

(20)

here \((p_1/q, \ldots, p_n/q)\) is an admissible fraction in the sense of Definition 5; see Fig. 1.

**Proof.** If \( \hat{f} \) is an integrable functions, the solution of the Schrödinger equation with initial datum \( f \) can be represented as

\[
e^{i\hbar t\Delta/2} f(x) = \int \hat{f}(\xi) e(-\pi t|\xi|^2 + 2\pi x \cdot \xi) \, d\xi.
\]

We want to compute the modulus of \( e^{i\hbar t\Delta/2} f_D(x) \) in the region \( |x| < c \) and \( 0 < t < c/R \). We note that

\[
|e^{i\hbar t\Delta/2} f_D(x)| = |e^{i\hbar t\Delta/2} f_1(x_1)| |e^{i\hbar t\Delta/2} \tilde{f}(\tilde{x})|.
\]

8
A direct computation shows that for $|t| \leq c/R$ and
\[
x_1 \in tR + [-cR^{-\frac{3}{2}}, cR^{-\frac{3}{2}}]
\]
we have
\[
|e^{it\Delta}f_1(x_1)| \simeq |\varphi(R^{\frac{1}{2}}(x_1 - tR))| \simeq 1
\] (22)
Again, a direct computation gives ($\tilde{x} \in \mathbb{R}^{n-1}$)
\[
e^{it\Delta/2}\tilde{f}(\tilde{x}) = \prod_{j=2}^{n} \int \varphi(\xi_j)e(-\pi t \xi_j^2 + 2\pi x_j \xi_j)
\]
\[\sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e(-\pi t |Dl_j|^2 + 2\pi Dl_j(x_j - t\xi_j)) \, d\xi_j.\] (23)
To estimate the absolute value of this product, we recall our hypotheses (20):
\[x_j = p_j/(Dq) + \varepsilon_j, \text{ for } |\varepsilon_j| < c/R.\]
We split each factor in (23) into the main term
\[
F_{\text{main}}(t, p_j/q) := e^{it\Delta/2}\varphi(x_j) \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e(-\pi t |Dl_j|^2 + 2\pi l_j \frac{p_j}{q}) \] (24)
and the perturbation

\[ F_{\text{per}}(t, x_j) := \int \hat{\phi}(\xi_j)e(-\pi t\xi_j^2 + 2\pi x_j\xi_j) \]
\[ \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e(-\pi t|Dl|^2 + 2\pi l_j \frac{p_j}{q})(1 - e(2\pi Dl_j(\varepsilon_j - t\xi_j))) d\xi_j. \tag{25} \]

By hypothesis \( t = 2p_1/(D^2q) \), so we can exploit Lemma 3 and the condition \( R/(Dq) \gg \sqrt{\ln q} \) to estimate the main contribution (24) as

\[ |F_{\text{main}}| \simeq | \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e(-2\pi(\frac{p_1}{q}l_j^2 - \frac{p_j}{q}l_j))| \]
\[ \simeq \frac{R}{D\sqrt{q}} \tag{26} \]

we used \(|e^{it\Delta/2}\phi(x_j)| \simeq 1\).

We claim that the perturbation term (25) satisfies \(|F_{\text{per}}| \ll R/(D\sqrt{q})\), which, together with (23) and (26), leads to

\[ |e^{it\Delta/2}\tilde{f}(\tilde{x})| \simeq \left(\frac{R}{D\sqrt{q}}\right)^{n-1}. \tag{27} \]

Then, we multiply by (22) to reach

\[ |e^{it\Delta/2}f_D(x)| \simeq \left(\frac{R}{D\sqrt{q}}\right)^{n-1}. \tag{28} \]

Finally, we divide (28) by \( \|f_D\|_2 \simeq R^{-\frac{1}{2}}(R/D)^{n-1} \) to obtain (18), and so the statement of the Theorem follows up to the claim \(|F_{\text{per}}| \ll R/(D\sqrt{q})\).

To prove the upper bound \(|F_{\text{per}}| \ll R/(D\sqrt{q})\), where \( F_{\text{per}} \) was defined in (25), we begin with

\[ |F_{\text{per}}| \lesssim \sup_{|\xi_j| \leq 1} | \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e(-2\pi \frac{p_1}{q}l_j^2 + 2\pi l_j \frac{p_j}{q})(1 - e(2\pi Dl_j(\varepsilon_j - t\xi_j)))| \]
\[ = \sup_{|\xi_j| \leq 1} | \sum_{\frac{R}{2D} < l_j < \frac{R}{D}} e(-2\pi \frac{p_1}{q}l_j^2 + 2\pi l_j \frac{p_j}{q})\phi_{\varepsilon,t,\xi_j}(l_j) | \]

where \((p_1, q) = 1\), and \( \phi_{\varepsilon,t,\xi_j}(l_j) = 1 - e(2\pi Dl_j(\varepsilon_j - t\xi_j)) \).
By the triangle inequality, it suffices to prove
\[
\left| \sum_{\frac{R}{D q} < l_j < \frac{R}{D q}} e(-2\pi \frac{p_1}{q} l_j^2 + 2\pi \frac{p_j}{q} l_j) \phi_i^j(l_j) \right| \lesssim c \frac{R}{D \sqrt{q}}, \quad c \ll 1 \quad i = 1, 2. \quad (29)
\]
where
\[
\phi^1_i(l_j) := 1 - \cos(2\pi D l_j (\varepsilon_j - t\xi_j)) \quad \text{and} \quad \phi^2_i(l_j) := |\sin(2\pi D l_j (\varepsilon_j - t\xi_j))|.
\]
Again by Lemma 3 and using \( R/(Dq) \gg \sqrt{\ln q} \), we have that
\[
\left| \sum_{l_j \in I} e(-2\pi \frac{p_1}{q} l_j^2 + 2\pi \frac{p_j}{q} l_j) \right| \lesssim |I| + \sqrt{q \ln q} \lesssim \frac{R}{D \sqrt{q}} + \sqrt{q \ln q} \lesssim \frac{R}{D \sqrt{q}}, \quad \forall |I| \leq \frac{R}{D}.
\]
(30)
On the other hand, the functions \( \phi^i_i(l_j) \) are real valued, positive, increasing in \( l_j \), and satisfy
\[
\phi^i_i(l_j) \lesssim |D||l_j|\varepsilon_j - t\xi_j| \lesssim c, \quad c \ll 1;
\]
recall that \( |l_j| \leq \frac{R}{D}, |\varepsilon_j| \leq \frac{c}{R}, |t| \leq \frac{c}{R} \) and \( |\xi_j| \leq 1 \) in the support of \( \hat{\phi}_j \). Thus (29) follows by the second part of Lemma 4 taking
\[
a_{l_j} = \phi^i_i(l_j) \quad \text{and} \quad b_{l_j} = e(-2\pi \frac{p_1}{q} l_j^2 + 2\pi \frac{p_j}{q} l_j),
\]
and the proof is concluded.

\[\square\]

4 Construction of the Examples

According to Theorem 6 the function \( \sup_{0 < t < 1} |e^{it\Delta/2} f_D| \) is large in the set
\[
\bigcup_{1 \leq q \leq Q} (E_{q,D} \cap [0, c]^n) \subset \mathbb{R}^n, \quad 0 < c \ll 1,
\]
as long as \( \frac{R}{DQ} \gg \sqrt{\ln Q} \).
To cover the largest possible area, we should ensure that the collection of sets \( E_{q,D} \), for \( 1 \leq q \leq Q \), is essentially pairwise disjoint. In a unit cell \([0,1/D]^{n-1}\), the number of fractions \((p_2/(Dq), \ldots, p_n/(Dq))\), for \( 1 \leq q \leq Q \), is \( \approx Q^n \), and if we think of the fractions as if they were uniformly distributed, then the average distance between them is \( \approx 1/(Q^{n-1}D) \), so we impose the restriction

\[
R^{-a} := \frac{1}{Q^{n-1}D} \geq R^{-1}.
\] (31)

We remark that \( R/(DQ) = Q^{n-1}R^{-a} \), so the condition \( R/(Dq) \gg \sqrt{\ln q} \), for \( 1 \leq q \leq Q \), is easily satisfied.

The slabs that form \( E_{q,D} \) have dimensions \( cR^{-\frac{1}{2}} \times cR^{-1} \times \cdots \times cR^{-1} \), for \( c \ll 1 \), and they do not overlap in the \( \tilde{x} \)-space because \( R/(DQ) \gg 1 \), however they may overlap in the \( x_1 \) direction. To exploit the whole area of the slabs, we impose the new restriction

\[
R^{-b} := \frac{R}{QD^2} \geq R^{-\frac{1}{2}};
\] (32)

see Figure 1.

The conditions (31) and (32) allow us to solve for \( Q \) and \( D \) as

\[
D = R^{(n-(n-1)a+nb)/(n+1)} \quad \text{and} \quad Q = R^{n-1/(2a-b-1)}.
\] (33)

Since \( Q \geq 1 \), then we have to be sure that \( 2a \geq 1 + b \), so we can write our conditions as

\[
0 < a \leq 1, \quad 0 < b \leq \frac{1}{2} \quad \text{and} \quad 2a \geq 1 + b;
\] (34)
in particular, \( a \geq \frac{1}{2} \).
Definition 7 (Divergence Sets). Let $a$ and $b$ satisfy the conditions (34), and let $A_k$, for $k \geq k_0 \gg 1$, be the collection of slabs $s$ such that:

1. $s$ has dimensions $cR_k^{-\frac{3}{2}} \times cR_k^{-1} \times \cdots \times cR_k^{-1}$, for $R_k = 2^k$ and $c \ll 1$.
2. $s$ has center at 
   $$
   \left(\frac{2p_1 R_k}{(qD_k^2)}, \frac{p_2}{(D_k q)}, \ldots, \frac{p_n}{(D_k q)}\right),
   $$
   where $(p_1/q, \ldots, p_n/q)$ is an admissible fraction (Definition 5) with $1 \leq q \leq Q_k$, and $D_k$ and $Q_k$ are given by (33).

A $(a, b)$-set of divergence $F$ is defined as

$$
F := \limsup_{k \to \infty} F_k, \quad F_k := \bigcup_{s \in A_k} s.
$$

For fixed $a$ and $b$, we define the initial datum

$$
g_{a, b} = \sum_{k \geq k_0} R_k^s \frac{k}{\|f_{D_k}\|^2} f_{D_k},
$$

where $R_k = 2^k$ and $k_0 \gg 1$. Inequality (18) dictates the value of $s$, and in terms of $a$ and $b$ we have

$$
s := \frac{1}{4} + \frac{n - 1}{2(n + 1)}(n - (n - 1)a - b).
$$

Since

$$
\|f\|^2_{H^{s'}(\mathbb{R}^n)} = \sum_{k \geq k_0} k R_k^{2(s' - s)} < \infty, \quad \text{for } s' < s,
$$

we have that $f \in H^{s'}(\mathbb{R}^n)$ for every $s' < s$.

We have to prove that the different terms in the sum (36) do not interfere with each other. We need the following Lemma.

Lemma 8. If the Fourier transform of $\varphi \in S(\mathbb{R})$ is supported in $(-1, 1)$, then for every $N \geq 1$ it holds

$$
|e^{ith\Delta/2} \varphi(x)| \leq C_N \frac{1}{|x|^N}, \quad \text{for } |x| > 2t.
$$
Proof. We use the principle of non-stationary phase. We assume that \( x > 2t \); the other case is similar. The solution is

\[
e^{i \theta \Delta / 2} \varphi(x) = \int \hat{\varphi}(\xi) e(-\pi t |\xi|^2 + 2\pi x \cdot \xi) \, d\xi.
\]

Since \( \partial_\xi e(-\pi t |\xi|^2 + 2\pi x \cdot \xi) = -2\pi i (t \xi - x) e(-\pi t |\xi|^2 + 2\pi x \cdot \xi) \), then by repeated integration by parts we obtain

\[
|e^{i \theta \Delta / 2} \varphi(x)| \leq C_N \frac{1}{|x - t|^N},
\]

which is the statement of the Lemma.

Before proving the main result of this section, we need to make an observation on the way we define solutions. For \( f \in H^s \), we define solutions for Sobolev functions, in such a way that they are well defined on sets with large Hausdorff dimension. Recall that \( Q(N) \) is the cube of side \( N \) centered at zero. We set

\[
e^{i \theta \Delta / 2} f(x) = \lim_{N \to \infty} S_N(t) f(x), \tag{40}
\]

where

\[
S_N(t) f(x) = \int_{Q(N)} \hat{f}(\xi) e(-\pi t |\xi|^2 + 2\pi x \cdot \xi) \, d\xi. \tag{41}
\]

The limit (40) is usually taken with respect to the \( L^2 \) norm, but here we take all the limits pointwise at each point \( x \) where they exist. When \( f \in L^2(\mathbb{R}) \), it is known that the limit exists pointwise for almost every \( x \in \mathbb{R} \) and that it coincides with the \( L^2 \)-limit. When \( n = 1 \), this result is due to Carleson [7], whose proof extends to higher dimensions as proved, for instance, in [18]. Moreover, we can show that this limit exists \( \gamma \)-almost everywhere for every \( f \in H^s \) with \( s \in (0, n/2] \), as long as \( \gamma > n - 2s \); see the appendix of [16]. This can be regarded as a refinement of Carleson’s result, although it does not recover it.

Theorem 9. If \( g_{a,b} \) is the initial datum defined in (36), then

\[
\limsup_{t \to 0^+} |e^{i \theta \Delta / 2} g_{a,b}(x)| = \infty \tag{42}
\]

for every \( x \in (F \cap ([c_0, c_1] \times [0, c_1]^{n-1})) \setminus \Omega \), where

- \( c_0 := \frac{1}{10} c_1 \), \quad c_1 \ll c \ll 1 \);
Figure 2: The gray lines represent the regions where the functions $e^{it\Delta/2}h_k$ concentrate.

- $F$ is a $(a, b)$-set of divergence;
- $\mathcal{H}^*(\Omega) = 0$ for $\gamma > n - 2s$.

Proof. We define $h_k := kR_k^{-s}f_{D_k}/\|f_{D_k}\|_2$, where $R_k := 2^k$. From the proof of Theorem [4] we know that for $t = 2p_1/(D_k^2q) \ll 1/R_k$ the value of the solution at $x \in F_k \cap [0, c_1]^n = \bigcup_{1 \leq q \leq Q_k} E_{q,D_k} \cap [0, c_1]^n$ is

$$|e^{it\Delta/2}h_k(x)| \gtrsim k. \quad (43)$$

We fix $k^* \geq k_0 \gg 1$ and $x \in F_{k^*}$, and we know that

$$x_1 = tR_{k^*} + \mathcal{O}(R_{k^*}^{-1}), \quad c_2R_{k^*}^{-1} < t < c_1R_{k^*}^{-1}. \quad (44)$$

It suffices to prove

$$|e^{it\Delta/2}h_k(x)| \lesssim R_k^{-1}, \quad \text{for} \quad k \neq k^*. \quad (45)$$

because then for $t = 2p_1/(D_k^2q) \ll 1/R_{k^*}$ we would have, for all $k_1 \geq k^*$, the following (recall (11))

$$|S_{2k_1}(t)g_{a,b}(x)| > |e^{it\Delta/2}h_{k^*}(x)| - \sum_{k_0 \leq k \neq k^* \leq k_1} |e^{it\Delta/2}h_k(x)| \gtrsim k^*. \quad (46)$$

15
as long as \( k_0 \gg 1 \); then in order to deduce (42) we note that for all \( x \in F \cap ([c_0,c_1] \times [0,c_1])^{n-1} \) we can choose any \( k^* \geq k_0 \gg 1 \), and we have a lower bound as (40) and the sequence of times \( t = 2p_1/(D_k^2, q) \ll 1/R_{k^*} \) goes to zero as \( k^* \to \infty \). More precisely, since we have

\[
e^{i\theta \Delta/2} f(x) = \lim_{N \to \infty} S_N(t)g_{a,b}(x) = \lim_{k_1 \to \infty} S_{2k_1}(t)g_{a,b}(x) \quad (47)
\]

except possibly on sets \( \Omega_t, t = 2p_1/(D_k^2, q) \) with \( \mathcal{H}^\gamma(\Omega_t) = 0 \) and these sets are countably many, then (42) would follow by (46)-(47), taking

\[
\Omega := \bigcup_{t=2p_1/(D_k^2, q)} \Omega_t.
\]

It remains to prove (45). From (23), we see that we can bound \( e^{i\theta \Delta/2} \tilde{h}_k(x) \) with the crude estimate

\[
|e^{i\theta \Delta/2} \tilde{f}_{D_k}(x)| \lesssim \left( \frac{R_k}{D_k} \right)^{n-1},
\]

so we can control each term \( e^{i\theta \Delta/2} h_k(x) \) as

\[
|e^{i\theta \Delta/2} h_k(x)| \leq |[e^{i\theta R_k \Delta/2} \varphi(R_k^2(x - tR_k))]| kR_k^2 \left( \frac{R_k}{D_k} \right)^{\frac{n-1}{2}} R_k^{-s}
\]

\[
\lesssim |e^{i\theta R_k \Delta/2} \varphi(R_k^2(x - tR_k))]| kQ_k \frac{n-1}{s},
\]

and we can apply Lemma 8 to \( \varphi \).

We verify the hypotheses of Lemma 8 when \( R_k < R_{k^*} \). By (44) we get

\[
\frac{R_k^2(x - tR_k)}{tR_k} = \frac{R_k^2(t(R_{k^*} - R_k) + O(R_k^{-3}))}{tR_k} \gtrsim R_k^{-\frac{1}{2}} R_{k^*} > 2,
\]

for \( k, k^* \geq k_0 \gg 1 \); hence, \( |e^{i\theta \Delta/2} h_k(x)| \lesssim_N kQ_k^{\frac{n-1}{s}} R_k^{-\frac{N}{s}} \lesssim R_k^{-1} \), for \( N \gg 1 \).

We verify now the hypotheses of Lemma 8 when \( R_k > R_{k^*} \):

\[
\frac{R_k^2(tR_k - x_1)}{tR_k^3} = \frac{R_k^2(t(R_k - R_{k^*}) + O(R_k^{-3}))}{tR_k^3} \gtrsim R_k^{\frac{1}{2}} > 2,
\]

for \( k, k^* \geq k_0 \gg 1 \); hence, \( |e^{i\theta \Delta/2} h_k(x)| \lesssim_N kQ_k^{\frac{n-1}{s}} (R_{k^*} R_k^{-\frac{3}{s}})^N \lesssim R_k^{-1} \), for \( N \gg 1 \).

\[
\square
\]

16
5 Dimension of the Divergence Set

In the previous section we constructed initial data parameterized by $a$ and $b$. To simplify matters, we choose those values of $a$ and $b$ for which computations are easier and exhaust all possible outcomes. Our choices are:

(I) $\frac{1}{2} < a \leq \frac{3}{4}$ and $b = 2a - 1$ \hspace{1cm} (48)

(II) $\frac{3}{4} < a \leq 1$ and $b = \frac{1}{2}$. \hspace{1cm} (49)

We refer to these $(a, b)$-sets of divergence (Definition 7) as of type I and type II. We remark that for I we have $Q = 1$, and that $a = 1$ and $b = \frac{1}{2}$ is Bourgain’s example.

**Theorem 10.** Let $0 < c_0 \leq 1$. If $F = \limsup_{k \to \infty} F_k$ is a $(a, b)$-set of divergence (Definition 7), then $\dim(F \cap [0, c_0]^n) \leq \alpha := \frac{1}{2} + (n - 1)a + b$.

**Proof.** Fix a scale $0 < \lambda \ll 1$ and choose $k'$ such that $R^{-1}_{k'} < \lambda$. Since $F_k$ is union of $\approx R^{(n-1)a+b}_k$ slabs with dimensions $R^{-1}_k \times R^{-1}_k \times \cdots \times R^{-1}_k$, and each slab can be covered by $R^2_k$ balls $B_r$, for $r = R^{-1}_k$, then we can find a collection $B_k$ with $|B_k| = R^{\alpha}_k$ of balls with radius $R^{-1}_k$ covering $F_k$, so that

$$\mathcal{H}_\lambda^\beta(F) := \inf \left\{ \sum_{B_{\rho} \in B} \rho^\beta \mid F \subset \bigcup_{B_{\rho} \in B} B_{\rho} \text{ and } \rho < \lambda \right\} \leq \sum_{k > k'} \sum_{B_{\rho} \in B_k} R^{-\beta}_k,$$

and the last sum is smaller than $\sum_{k > k'} R^{\alpha-\beta}_k$, which tends to zero as $k' \to \infty$ whenever $\beta > \alpha$. \qed

To prove the corresponding lower bound of $\dim F$, we employ the techniques in Section 4 of [24]. We recall a result of Falconer, which is consequence of Theorem 3.2 and Corollary 4.2 in [17].

**Lemma 11.** Let $0 < c \leq 1$. Suppose that there exists a constant $C > 0$ such that, for all $\delta > 0$ and all cubes $Q(x, \delta) \subset [0, c]^n$, we have the density condition

$$\liminf_{k \to \infty} \mathcal{H}^\beta_{\infty}(F_k \cap Q(x, \delta)) \geq C\delta^\beta,$$

where $\{F_k\}_{k \geq 0}$ is a sequence of open subsets of $B(0, 1)$. Then, for all $\beta' < \beta$,

$$\mathcal{H}^{\beta'}_{\infty}(\limsup_{k \to \infty} F_k) > 0.$$
We prove now the lower bound of $\dim F$ in the easier case, in the case of sets of type I.

**Theorem 12.** If $F = \limsup_{k \to \infty} F_k$ is a set of type I, that is, $\frac{1}{2} < a \leq \frac{3}{4}$ and $b = 2a - 1$, then $\dim F \cap [0, c_0]^n \geq \alpha$ where

$$\alpha := \frac{1}{2} + (n - 1)a + b.$$  \hfill (50)

**Proof.** From Lemma 11 it will be sufficient to show that

$$\mathcal{H}_\infty^\beta(F_k \cap Q(x, \delta)) \geq C\delta^\beta, \quad \forall Q(x, \delta) \subseteq [0, c_0]^n,$$  \hfill (51)

holds for all $k$ sufficiently large, where $\beta = \alpha - \varepsilon$ for $0 < \varepsilon \ll 1$. The size of $k$ for which (51) holds will depend on $\delta$. To prove (51) we define an auxiliary measure which is a uniform mass measure over $F_k \cap Q(x, \delta)$, namely

$$\mu_k(A) := \frac{|A \cap F_k \cap Q(x, \delta)|}{|F_k \cap Q(x, \delta)|}.$$  

Note that $\mu_k$ depends on the set $F_k \cap Q(x, \delta)$, but we will only stress the dependence on $k$ in the notation.

Assume we have proved

$$\mu_k(B_r) \leq Cr^\beta \delta^{-\beta}$$  \hfill (52)

for all sufficiently large $k$ (the size of $k$ will depend on $\delta$). Using (52) we can prove (51) easily, noting that if $\mathcal{B}$ is a collection of balls $B_r$ that covers $F_k$, then

$$1 = \mu_k(F_k \cap Q(x, \delta)) \leq \sum_{B_r \in \mathcal{B}} \mu_k(B_r) \leq C\delta^{-\beta} \sum_{B_r \in \mathcal{B}} r^\beta.$$  

Thus we have reduced to prove (52). To do so we have to work at several scales. It will be useful to keep in mind that if $k \gg 1$ then

$$|F_k \cap Q(x, \delta)| \simeq R_k^{\alpha - n} \delta^n$$  \hfill (53)

and that $R_k \to \infty$ as $k \to \infty$. Many estimates below will be indeed justified taking $k$ large enough, depending on $\delta$.  

18
1. Scale $r < R_k^{-1}$. In the worst case a ball is entirely contained in a slab from $F_k$, so

$$\mu_k(B_r) \lesssim r^n R_k^{-\alpha+n} \delta^{-n} \leq r^\alpha \delta^{-n} = r^{\beta} \delta^{-\beta} r^{\alpha-\beta} \delta^{\beta-n} < r^{\beta} \delta^{-\beta} R_k^{-(\alpha-\beta)} \delta^{\beta-n},$$

since $\alpha - \beta > 0$ and $r < R_k^{-1}$ we have $R_k^{-(\alpha-\beta)} \delta^{\beta-n} < 1$ for $k \gg \delta$ 1 thus \(^{[72]}\) holds at this scale.

2. Scale $R_k^{-1} < r < R_k^{-\alpha}$. Recall that $R_k^{-\alpha} < R_k^{-\frac{1}{2}}$, so a ball $B_r$ cannot contain a slab. On the other hand, since $r < R_k^{-\alpha}$ a ball $B_r$ intersects at most one slab, so

$$\mu_k(B_r) \lesssim r^n R_k^{-(n-1)} R_k^{-\alpha+n} \delta^{-n} = r^{\alpha} R_k^{(n-1)-\alpha} \delta^{-n} < r^\alpha \delta^{-n},$$

using $R_k^{-1} < r$ and $\alpha > 1$. Using also $r < R_k^{-\alpha}$ we see that

$$\mu_k(B_r) \lesssim r^{\beta} R_k^{(\beta-\alpha)} \delta^{-n} < r^{\beta} \delta^{-\beta}, \quad k \gg \delta 1.$$ 

3. Scale $R_k^{-\alpha} < r < R_k^{-\frac{1}{2}}$. A ball $B_r$ intersects $\lesssim R_k^{(n-1)a} r^{n-1}$ slabs, so

$$\mu_k(B_r) \lesssim r^n R_k^{(n-1)a-n+1} R_k^{-\alpha+n} \delta^{-n} \leq r^n R_k^{\frac{1}{2} \alpha-n+b+1} \delta^{-n},$$

where we used \(^{[50]}\). Since $r < R_k^{-\frac{1}{2}}$ we have that

$$\mu_k(B_r) \lesssim r^{\beta} R_k^{\frac{1}{2} \beta-n-b+\frac{1}{2}} \delta^{-n} < r^{\beta} R_k^{\frac{1}{2}(\beta-\alpha)} \delta^{-n}$$

where we used

$$\alpha := (n-1)a + b + \frac{1}{2} = \frac{n-3}{2} b + \frac{n}{2} + 1 + 2b - 1 < n + 2b - 1. \quad (54)$$

Thus

$$\mu_k(B_r) < r^{\beta} \delta^{-\beta}, \quad k \gg \delta 1.$$ 

4. Scale $R_k^{-\frac{1}{2}} < r < R_k^{-b}$. A ball $B_r$ contains $\lesssim R_k^{(n-1)a} r^{n-1}$ slabs, so recalling again \(^{[50]}\) we get

$$\mu_k(B_r) \lesssim r^n R_k^{(n-1)a-n+1} R_k^{-\alpha+n} \delta^{-n} = r^{n-1} R_k^{-b} \delta^{-n} < r^{n-1+2b} \delta^{-n},$$

where we used $R_k^{-\frac{1}{2}} < r$. From $r < R_k^{-b}$ and \(^{[54]}\) we have that

$$\mu_k(B_r) \lesssim r^{\beta} R_k^{-b(n+2b-1-\beta)} \delta^{-n} < r^{\beta} R_k^{-b(\alpha-\beta)} \delta^{-n} < r^{\beta} \delta^{-\beta}, \quad k \gg \delta 1.$$
a) \[ \geq \frac{1}{qq'} > \frac{1}{Q^2} \]

\[ \frac{p_2}{p_2} \frac{q_2}{q'} \]

x_2

\[ \frac{p_2}{p_2} \frac{q_2}{q'} \]

x_2

b) \[ x_3 \]

fractions are very crowded over this line

\[ x_2 \]

Figure 3: (a) When \( n = 2 \) the fractions are already well separated; Lemma 13 is unnecessary. (b) When \( n \geq 3 \) the fractions might concentrate around some regions, which prohibits the Frostman measure technique we used in Lemma 12.

5. Scale \( R_k^{-b} < r < \delta \). A ball intersects \( \lesssim R_k^{(n-1)a+b-n} \) slabs, so

\[ \mu_k(B_r) \lesssim r^n R_k^{(n-1)a+b-n+\frac{1}{2}} R_k^{-\alpha+n} \delta^{-n} = r^n \delta^{-n} < r^\beta \delta^{-\beta}. \]

The inequality (52) thus holds, and so the statement of the Theorem.

The lower bound for type II sets is harder to prove, and we need a Lemma that assures us that for all \( F_k \) we can find a large sub-collection of slabs uniformly distributed. Similar arguments were used in Lemma 4.3 of [16] and in Sections 5.6–5.8 of [27].

Lemma 13. Let \( F = \lim \sup_{k \to \infty} F_k \) be a set of type II, that is, \( \frac{3}{4} < a \leq 1 \) and \( b = \frac{1}{2} \). If \( A_k \) is the collection of slabs in \( F_k \cap Q(x, \delta) \), for \( \delta < 1 \), then, for every \( \varepsilon > 0 \) and \( k \gg \varepsilon \), we can extract a sub-collection of slabs \( A_k' \subset A_k \) such that
\[(i) \ |A_k'| \gtrsim R_k^{-\varepsilon} |A_k|.
\]

\[(ii) \text{If } x = (x_1, \bar{x}) \text{ and } y = (y_1, \bar{y}) \text{ are the centers of two slabs in } A_k' \text{ and } \bar{x} \neq \bar{y}, \text{ then } |\bar{x} - \bar{y}| \gtrsim 1/(Q_k^{n-1} D_k).
\]

**Proof.** The sets \(F_k := \bigcup_{s \in A_k} s\) have a periodic structure. In fact, recall that the centers of the slabs are

\[
(2p_1 R_k/(qD_k^2), p_2/(D_k q), \ldots, p_n/(D_k q)),
\]

where \((p_1/q, \ldots, p_n/q)\) is an admissible fraction (Definition 5); hence, \(F_k\) is made up of translation of the slabs in the unit cell \([0, 2R_k/D_k^2] \times [0, 1/D_k]^{n-1}\). We assume that \(k\) is so large that the number of unit cells not entirely contained in \(Q(x, \delta)\) is negligible. Therefore, the number of slabs in \(Q(x, \delta)\) is \(|A_k| \simeq D_k^{n+1} R_k^{-1} \delta^{-n} |\{\text{slabs per unit cell}\}|\), and the Lemma reduces to extract a large number of admissible fractions in \([0, 1]^n\) with denominator \(\leq Q_k\).

We drop the subscript \(k \gg 1\). Let \(A^0\) be the set of admissible fractions, and let \(A^1 \subset A^0\) be the collection of fractions \((p_1/q, \ldots, p_n/q)\) with \(q \equiv 0 \pmod{4}\) and \(p_j\) even for \(2 \leq j \leq n\), so that \(|A^1| \approx |A^0|\).

We denote by \(PA^1\) the projection of \(A^1\) into the plane \((x_2, \ldots, x_n)\), so \(PA^1\) is the set of fractions \((p_2/q, \ldots, p_n/q)\) with \(q \equiv 0 \pmod{4}\) and even \(p_j\). The Dirichlet’s approximation Theorem asserts that for \(2y \in \mathbb{R}^{n-1}\) there
exists \((p'_2, \ldots, p'_n) \in \mathbb{Z}^{n-1}\) such that

\[
|2y - \frac{p'_j}{q'}| \leq \frac{1}{q'(Q/4)^{\frac{1}{n-1}}}, \quad \text{for some } 1 \leq q' \leq Q/4, \quad (55)
\]

so if we write \(q = 4q'\) and \(p_j = 2p'_j\), then we can assert that for every \(y \in \mathbb{R}^{n-1}\) there exists a fraction \((p_2/q, \ldots, p_n/q)\), for \(q \equiv 0 \pmod{4}\) and \(p_j\) even, such that

\[
|y - \frac{p_j}{q}| \leq 2^{\frac{n+1}{n-1}} \frac{1}{qQ^{\frac{n}{n-1}}}, \quad \text{for some } 1 \leq q \leq Q.
\]

In general, a point \(y \in [0, 1]^{n-1}\) cannot be sufficiently well approximated by fractions if it satisfies (55) with a fraction \((p'_2/q', \ldots, p'_n/q')\) with small \(q'\), so it is convenient to ignore those points. The volume in \([0, 1]^{n-1}\) occupied by those undesirable points is less than

\[
\sum_{1 \leq q' \leq Q/2^{n+2}} \left( \frac{1}{q'(Q/4)^{\frac{1}{n-1}}} \right)^{n-1} (2q')^{n-1} = \frac{1}{2}.
\]

Let \(G := \{ y \in [0, 1]^{n-1} \mid y \text{ satisfies (55) for some } Q/2^{n+2} < q' \leq Q/4 \}\), then by (56) the volume of \(G\) is \(> \frac{1}{2}\). Cover \(G\) with cubes \(Q(y, l)\), where \(y \in G\) and \(l := 2^{n+2+\frac{n^2}{n-1}}/Q^{\frac{n}{n-1}}\). By Vitali’s covering Theorem we can find a disjoint collection of cubes \(\{Q(y_j, l)\}_{1 \leq j \leq N}\) such that

\[
G \subset \bigcup_{j=1}^{N} Q(y_j, 3l);
\]

hence, \(N \geq c_n Q^n\). We pick from within each \(Q(y_j, l)\) a fraction and construct so a collection of fractions \(\mathcal{C} \subset P \mathcal{A}^1\); we define \(\mathcal{A}^2 \subset \mathcal{A}^1\) as the set of fractions such that \(P \mathcal{A}^2 = \mathcal{C}\). By construction, \(|P \mathcal{A}^2| \gtrsim Q^n\) and any two points in \(P \mathcal{A}^2\) lie at distance \(\gtrsim 1/Q^{\frac{n}{n-1}}\); the latter, after dilation by \(1/D\), implies the condition (ii).

The fractions in \(\mathcal{A}^2\) that lie over \((p_2/q, \ldots, p_n/q) \in P \mathcal{A}^2\) is in number at least \(\varphi(q)\), where \(\varphi\) is the Euler’s totient function. Since \(\varphi(q) \geq q^{1-\varepsilon}\) for every \(\varepsilon > 0\) and \(q \gg \varepsilon\)—see Theorem 327 in [20]—then the number of fractions in \(\mathcal{A}^2\) is \(\geq Q^{1-\varepsilon} |P \mathcal{A}^2| \gtrsim Q^{n+1-\varepsilon} \approx Q^{-\varepsilon} |\mathcal{A}^0|\), where \(\mathcal{A}^0\) is the set of admissible fractions; this concludes the verification of condition (i).

\(\square\)
Theorem 14. Let $0 < c_0 \leq 1$. If $F = \limsup_{k \to \infty} F_k$ is a set of type II, that is, $\frac{3}{4} < a \leq 1$ and $b = \frac{1}{2}$, then $\dim F \cap [0, c_0]^n \geq \alpha$ where

$$\alpha := 1 + (n - 1)\alpha. \quad (57)$$

Proof. We use the same method as in Theorem 12. For fixed $\varepsilon > 0$, let $A_k'$ be the collection of slabs provided by Lemma 13, and let $F_k'$ be the corresponding set. Given $Q(x, \delta) \subseteq [0, c_0]^n$, we define again a measure $\mu_k$ on $F_k \cap Q(x, \delta)$ that will be useful in the proof; the measure is

$$\mu_k(A) := \frac{|A \cap F_k' \cap Q(x, \delta)|}{|F_k' \cap Q(x, \delta)|}.$$

If $k \gg \varepsilon$ then

$$|F_k' \cap Q(x, \delta)| \gtrsim R_k^{\alpha - n - \varepsilon} \delta^n.$$

We take $\beta := \alpha - 2n \varepsilon < \alpha - \varepsilon$. The goal is again to prove (52), from which we deduce Theorem 14 proceeding as we did in the proof of Theorem 12.

Since $b = \frac{1}{2}$, we can think of the slabs over $(p_2/(qD_k), \ldots, p_n/(qD_k))$ as a single tube of length 1.

1. Scale $r < R^{-1}$. In the worst case a ball is entirely contained in a slab from $F_k$, so

$$\mu_k(B_r) \lesssim r^n R_k^{-(\alpha + \varepsilon)} \delta^{-n} \lesssim r^{\alpha - \varepsilon} \delta^{-n} = r^\beta \delta^{-\beta} (r^{\alpha - \varepsilon} \delta^{-n});$$

since $\alpha - \beta > \varepsilon$ and $r < R_k^{-1}$, then $\mu_k(B_r) < r^\beta \delta^{-\beta}$ whenever $k \gg \delta 1$.

2. Scale $R_k^{-1} < r < R_k^{-a}$. By the properties of separation of the slabs in $A_k'$, a ball $B_r$ intersects at most one slab—recall Lemma 13(ii) and (31)—so

$$\mu_k(B_r) \lesssim r R_k^{-(n-1) - \alpha + \varepsilon} \delta^{-n} = r R_k^{-(\alpha + 1 + \varepsilon)} \delta^{-n} < r^{\alpha - \varepsilon} \delta^{-n},$$

where we used $\alpha > 1$. Since $r < R_k^{-a}$ we see that

$$\mu_k(B_r) \lesssim r^\beta R_k^{(\alpha - \varepsilon) \delta^{-n}}, \quad k \gg \delta 1.$$
since \( r < R_k/D_k^2 \leq R_k^{-\frac{1}{n+1}} \), we see that
\[
\mu_k(B_r) \lesssim r^\beta \delta^{-\beta}, \quad k \gg \delta. 
\]

4. Scale \( R_k/D_k^2 < r < \delta \). A ball \( B_r \) contains \( \simeq D_k^{n+1} R_k^{-1} r^n \) translations of the unit cell \([0, 2R_k/D_k^2] \times [0, 1/D_k]^{n-1}\). If \( V \) is the volume of \( F_k' \) per unit cell, then \(|B_r \cap F_k'| \simeq V D_k^{n+1} R_k^{-1} r^n\)
and
\[
|Q(x, \delta) \cap F_k'| \simeq V D_k^{n+1} R_k^{-1} \delta^n;
\]
thus
\[
\mu_k(B_r) \lesssim r^n \delta^{-n} < r^\beta \delta^{-\beta}.
\]
The inequality \( \mu_k(B_r) \leq Cr^\beta \delta^{-\beta} \) holds for \( k \) sufficiently large (depending on \( \delta \)), so the proof is complete.

6 Conclusion of the proof

We are now ready to prove our statement combining the results from the previous section. First we take \( a, b \) as in (48)-(49) and recall that we have defined
\[
\alpha := \frac{1}{2} + (n - 1)a + b. \tag{58}
\]
Note that we have a bijection between \( a \in (1/2, 1] \) (which predicts also the value of \( b \) by (48)-(49)) and \( \alpha \in (n/2, n] \), which is the range we are interested in (the case \( \alpha = n \) was handled in [5]).

First we claim that given any\( s' < s := \frac{n}{2(n+1)} + \frac{n-1}{2(n+1)} (n - \alpha) \tag{59} \)
we can find a solution \( u(x, t) \) with initial datum \( u_0 \in H^{s'}(\mathbb{R}^n) \) such that
\[
\lim_{t \to 0^+} \sup |u(x, t)| = \infty
\]
for \( x \in (F \cap ([c_0, c_1] \times [0, c_1]^{n-1})) \setminus \Omega \), where \( F \) is an \((a, b)\)-set of divergence, \( 0 < c_0 := \frac{1}{10} c_1 \ll 1 \) and \( \Omega \) has dimension \( \leq n - 2s \). Indeed, it suffices to choose \( u_0 := g_{a,b} \) defined in (36) so that \( u_0 \in H^{s'}(\mathbb{R}^n) \) for
\[
s' < s := \frac{1}{4} + \frac{n-1}{2(n+1)} (n - (n - 1)a - b); \tag{60}
\]
see (38)-(37). Since under (38) the inequality (60) becomes (59), then the claim follows invoking Theorem 9.

Thus, to conclude the proof, we need to show that

$$\dim \left( \left( F \cap ([c_0, c_1] \times [0, c_1]^{n-1}) \right) \setminus \Omega \right) \geq \alpha. \tag{61}$$

First, covering \((F \cap ([c_0, c_1] \times [0, c_1]^{n-1}))\) with \(\simeq (c_1/c_0)^{n-1}\) cubes of side \(c_0\), we see as consequence of Theorems 12 and 14 that

$$\dim(F \cap ([c_0, c_1] \times [0, c_1]^{n-1})) \geq \alpha.$$  

On the other hand, we know that \(\dim \Omega \leq n - 2s\) (see Theorem 9). Thus, since for our choice (59) of \(s\) we have \(\alpha > n - 2s\) when \(\alpha > n/2\), then (61) follows and the proof is concluded.

References

[1] J.A. Barceló, J.M. Bennett, A. Carbery, A. Ruiz, and M.C. Vilela. Some special solutions of the Schrödinger equation. Indiana University Mathematics Journal, 56(4):1581–1593, 2007.

[2] Juan Antonio Barceló, Jonathan Bennett, Anthony Carbery, and Keith M. Rogers. On the dimension of divergence sets of dispersive equations. Math. Ann., 349(3):599–622, 2011.

[3] J. Bourgain. A remark on Schrödinger operators. Israel J. Math., 77(1-2):1–16, 1992.

[4] J. Bourgain. On the Schrödinger maximal function in higher dimension. Tr. Mat. Inst. Steklova, 280:53–66, 2013.

[5] J. Bourgain. A note on the Schrödinger maximal function. J. Anal. Math., 130:393–396, 2016.

[6] Anthony Carbery. Radial Fourier multipliers and associated maximal functions. In Recent progress in Fourier analysis (El Escorial, 1983), volume 111 of North-Holland Math. Stud., pages 49–56. North-Holland, Amsterdam, 1985.

[7] Lennart Carleson. On convergence and growth of partial sums of Fourier series. Acta Math., 116:135–157, 1966.
[8] Lennart Carleson. Some analytic problems related to statistical mechanics. In *Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979)*, volume 779 of *Lecture Notes in Math.*, pages 5–45. Springer, Berlin, 1980.

[9] Michael G. Cowling. Pointwise behavior of solutions to Schrödinger equations. In *Harmonic analysis (Cortona, 1982)*, volume 992 of *Lecture Notes in Math.*, pages 83–90. Springer, Berlin, 1983.

[10] Björn E. J. Dahlberg and Carlos E. Kenig. A note on the almost everywhere behavior of solutions to the Schrödinger equation. In *Harmonic analysis (Minneapolis, Minn., 1981)*, volume 908 of *Lecture Notes in Math.*, pages 205–209. Springer, Berlin-New York, 1982.

[11] Ciprian Demeter and Shaoming Guo. Schrödinger maximal function estimates via the pseudoconformal transformation. Preprint. arXiv:1608.07640.

[12] Xiumin Du. Upper bounds for Fourier decay rates of fractal measures. *Journal of the London Mathematical Society*, 102(3):1318–1336, 2020.

[13] Xiumin Du, Larry Guth, and Xiaochun Li. A sharp Schrödinger maximal estimate in $\mathbb{R}^2$. *Ann. of Math. (2)*, 186(2):607–640, 2017.

[14] Xiumin Du, Larry Guth, Xiaochun Li, and Ruixiang Zhang. Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates. *Forum Math. Sigma*, 6:e14, 18, 2018.

[15] Xiumin Du and Ruixiang Zhang. Sharp $L^2$ estimates of the Schrödinger maximal function in higher dimensions. *Ann. of Math. (2)*, 189(3):837–861, 2019.

[16] D. Eceizabarrena and R. Lucà. Convergence over fractals for the periodic Schrödinger equation. Preprint. arXiv:2005.07581.

[17] K. J. Falconer. Classes of sets with large intersection. *Mathematika*, 32:191–205, 1985.

[18] Charles Fefferman. On the convergence of multiple Fourier series. *Bull. Amer. Math. Soc.*, 77:744–745, 1971.
[19] L. Guth, A. Iosevich, Y. Ou, and H. Wang. On Falconer’s distance set problem in the plane. *Invent. Math.*, 219(3):779–830, 2020.

[20] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers.* Edited and revised by D. R. Heath-Brown and J. H. Silverman. With a foreword by Andrew Wiles. 6th ed. Oxford: Oxford University Press, 6th ed. edition, 2008.

[21] Alex Iosevich and Michael Rudnev. Distance measures for well-distributed sets. *Discrete & Computational Geometry*, 38(1):61–80, July 2007.

[22] Sanghyuk Lee. On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$. *Int. Math. Res. Not.*, pages Art. ID 32597, 21, 2006.

[23] R. Lucà and K. M. Rogers. A note on pointwise convergence for the Schrödinger equation. *Math. Proc. Camb. Philos. Soc.*, 166(2):209–218, 2019.

[24] Renato Lucà and Keith M. Rogers. Coherence on fractals versus pointwise convergence for the Schrödinger equation. *Comm. Math. Phys.*, 351(1):341–359, 2017.

[25] Renato Lucà and Keith M. Rogers. Average decay of the Fourier transform of measures with applications. *J. Eur. Math. Soc. (JEMS)*, 21(2):465–506, 2019.

[26] A. Moyua, A. Vargas, and L. Vega. Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$. *Duke Math. J.*, 96(3):547–574, 1999.

[27] Lillian B. Pierce. On Bourgain’s counterexample for the Schrödinger maximal function. Preprint. arXiv:1912.10574v4.

[28] Felipe Ponce-Vanegas. Examples of measures with slow decay of the spherical means of the Fourier transform. *Proc. Am. Math. Soc.*, 146(6):2617–2621, 2018.

[29] Peter Sjögren and Per Sjölin. Convergence properties for the time-dependent Schrödinger equation. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 14(1):13–25, 1989.
[30] Per Sjölin. Regularity of solutions to the Schrödinger equation. *Duke Math. J.*, 55(3):699–715, 1987.

[31] T. Tao. A sharp bilinear restrictions estimate for paraboloids. *Geom. Funct. Anal.*, 13(6):1359–1384, 2003.

[32] T. Tao and A. Vargas. A bilinear approach to cone multipliers. I. Restriction estimates. *Geom. Funct. Anal.*, 10(1):185–215, 2000.

[33] T. Tao and A. Vargas. A bilinear approach to cone multipliers. II. Applications. *Geom. Funct. Anal.*, 10(1):216–258, 2000.

[34] Luis Vega. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc.*, 102(4):874–878, 1988.

[35] Darko Žubrinić. Singular sets of Sobolev functions. *C. R. Math. Acad. Sci. Paris*, 334(7):539–544, 2002.