Theorems of Euclidean Geometry through Calculus

Martin Buysse

Faculté d’architecture, d’ingénierie architecturale, d’urbanisme – LOCI, UCLouvain

We re-derive Thales, Pythagoras, Apollonius, Stewart, Heron, al Kashi, de Gua, Terquem, Ptolemy, Brahmagupta and Euler’s theorems as well as the inscribed angle theorem, the law of sines, the circumradius, inradius and some angle bisector formulae, by assuming the existence of an unknown relation between the geometric quantities at stake, observing how the relation behaves under small deviations of those quantities, and naturally establishing differential equations that we integrate out. Applying the general solution to some specific situation gives a particular solution corresponding to the expected theorem. We also establish an equivalence between a polynomial equation and a set of partial differential equations. We finally comment on a differential equation which arises after a small scale transformation and should concern all relations between metric quantities.

I. THALES OF MILETUS

Imagine that Newton was born before Thales. When considering a triangle with two sides of lengths $x$ and $y$, he could have fantasized about moving the third side parallel to itself and thought: "Well, I am not an ancient Greek geometer but I am rather good in calculus and I feel there might be some connection between the way $x$ and $y$ vary in such circumstances."

He would have materialized his suspicion in a function

$$y = y(x) \quad (1)$$

connecting $x$ and $y$ whatever the position of the third side, as long as it is moved parallel to itself. In particular, after a slight displacement resulting in small deviations $\delta x$ and $\delta y$, he would have had at first order

$$\delta y = y'(x) \delta x \quad (2)$$

But the lengths $\delta x$ and $\delta y$ of the small added segments must themselves obey equation (1), that is

$$\delta y = y(\delta x) \quad (3)$$

To see it, translate those segments to the $(x, y)$ vertex. Developing the right-hand-side member of equation (3) at first order and noticing that $y(0) = 0$, we have

$$\delta y = y'(0) \delta x \quad (4)$$

which, compared to eq. (2), implies that $y'(x)$ is constant. Integrating $y' = k$, $k$ being a positive constant since $y(x)$ is an increasing and smooth function, gives the Thales theorem [1, Book VI, Prop. II]

$$y(x) = kx \quad (5)$$

II. PYTHAGORAS OF SAMOS

If he was born before Thales, Newton was born before Pythagoras too, so that we do not have to make any further unlikely hypothesis. Imagine that driven by his success in suspecting the existence of a Greek theorem, he moved to consider a right triangle of legs of lengths $x$ and $y$ and of hypotenuse of length $z$.

He might have been tempted to speculate about the link, if any, between $x$, $y$ and $z$ in every right triangle. And again, as calculus master, he could have postulated that

$$z = z(x, y) \quad (6)$$

a relation that must be true for any $x$, $y$ and $z$ in a right triangle. In particular, after a slight increase in the length of $x$, leading to a small deviation $\delta x$, while $\delta y = 0$, he would have found, at first order, that

$$\delta z = \partial_x z \delta x \quad (7)$$

Here, $\delta x$ is the length of the hypotenuse of a right triangle with one leg of length $\delta z$. At first order, this small triangle is similar to the initial one. Using the Thales theorem that he has just found out,

$$\delta z = \frac{x}{z} \delta x \quad (8)$$

Substituting this result to $\delta z$ in eq. (7) would have led him to the partial differential equation

$$z \partial_x z = x \quad (9)$$

whose general solution is

$$z^2(x, y) = x^2 + k(y) \quad (10)$$
where \(k(y)\) is an arbitrary function of \(y\). But the function \(z\) has to be symmetric in \(x\) and \(y\) (i.e. he could have made the same reasoning with a non-zero \(\delta y\) while \(\delta x = 0\)) and \(z(x, 0) = x\). Hence

\[
z^2(x, y) = x^2 + y^2
\]

that is, the Pythagorean theorem, which, like Thales’, was probably discovered long before – and published a few centuries later by Euclid \([1\text{, Book I, Prop. XLVII}]\). As will be shown in the last section, the same result can be obtained by considering a small rotation of the hypotenuse around one of its extremities followed by an infinitesimal scale transformation. This proof is known and is published in a slightly different form in \([2, 3]\).

### III. APOLLONIUS OF PERGA

Thales and Pythagoras theorems are not the only ones that are named before famous Greek geometers. Newton could have gone a step further – eastwards, a few centuries later – and assumed that in any triangle of side-lengths \(x\), \(y\) and \(z\), the length \(d\) of the median relative to the \(z\)-length side is a smooth function of \(x\), \(y\) and \(z\), i.e.

\[
d = d(x, y, z)
\]

After an infinitesimal rotation of the \(y\)-length side around the \((y, z)\) vertex resulting in a small deviation \(\delta x\), with \(\delta y = 0\) and \(\delta z = 0\), at first order:

\[
\delta d = \partial_x d \delta x
\]

In order to get an expression for \(\delta x\) and \(\delta d\) and then a differential equation leading to the would-be theorem, consider the infinitesimal arc travelled by the moved vertex, of length \(\delta \ell\).

At first order, it can be seen as the hypotenuse of a small right triangle with one leg of length \(\delta x\), which is similar to a larger right triangle whose corresponding leg is the \(h_y\)-length height relative to the \(y\)-length side, and the hypotenuse is the \(x\)-length side, so that

\[
\delta x = \frac{h_y}{x} \delta \ell
\]

The \(\delta \ell\)-length arc is also the first order hypotenuse of another small right triangle with one leg of length \(\delta d\), which is similar to the triangle with a \(d\)-length hypotenuse and whose corresponding leg is a segment starting from the foot of the median and parallel to – and thus half of the length of – the \(h_y\)-length height, so that

\[
\delta d = \frac{h_y \delta \ell}{2d}
\]

Inserting those deviations in eq. (13) leads to the partial differential equation

\[
d\partial_x d = \frac{x^2}{2}
\]

whose general solution is

\[
d^2(x, y, z) = \frac{x^2}{2} + k(y, z)
\]

with \(k(y, z)\) a function of \(y\) and \(z\). But \(d(x, y, z)\) has to be symmetric in \(x\) and \(y\) (i.e. we can make the same reasoning with a non-zero \(\delta y\) while \(\delta x = 0\)). Hence

\[
d^2(x, y, z) = \frac{x^2 + y^2}{2} + c(z)
\]

with \(c(z)\) a function of \(z\). Furthermore, if \(x = 0\) (or \(y = 0\)), \(y = z\) (or \(x = z\)) and \(d = z/2\). This yields \(c(z) = -z^2/4\), which can alternatively be found by invoking the Pythagorean theorem for \(y = x\). The particular solution reads

\[
d^2(x, y, z) = \frac{x^2 + y^2 - 2(z/2)^2}{2}
\]

that is, Apollonius’s theorem, to be found in a slightly more elaborate form in \([4]\).

### IV. MATTHEW STEWART

Suppose Newton was born before Stewart, an 18th-century Scottish mathematician (and reverend). Well, he was. Perhaps he was not interested, or did not have the time, otherwise he could have used this tool to generalize Apollonius’s theorem to any cevian.
In a triangle of sidelengths \( x, y \) and \( z \), assume that the length \( d \) of a cevian dividing the side of length \( z \) in two segments of lengths \( m \) and \( n \), is a smooth function of \( x, y, m \) and \( n \), that is
\[
d = d(x, y, m, n) \tag{20}
\]
After an infinitesimal rotation of the \( y \)-length side around the \((y, z)\) vertex resulting in a small deviation \( \delta x \), with \( \delta y = 0 \), \( \delta m = 0 \) and \( \delta n = 0 \), at first order:
\[
\delta d = \partial_x d \delta x \tag{21}
\]
Using the same similarities as for the Apollonius’s theorem, with the unique difference that the foot of the cevian is not necessarily the middle of the \((m + n)\)-length side but falls at a distance \( n \) from its right vertex, we find
\[
\delta x = \frac{h_y}{x} \delta \ell \quad \delta d = \frac{nh_y}{(m + n)d} \delta \ell \tag{22}
\]
h\(_y\) being the length of the height relative to \( y \), we have the partial differential equation
\[
d \partial_x d = \frac{n}{m + n} x \tag{23}
\]
whose general solution is
\[
d^2(x, y, m, n) = \frac{n}{m + n} x^2 + k(y, m, n) \tag{24}
\]
where \( k(y, m, n) \) is a function of \( y, m \) and \( n \). But \( d(x, y, m, n) \) must be symmetric in \((x, m)\) and \((y, n)\) (i.e. we can make the same reasoning with a non-zero \( \delta y \) while \( \delta x = 0 \)). Hence
\[
d^2(x, y, m, n) = \frac{nx^2 + my^2}{m + n} + c(m, n) \tag{25}
\]
with \( c(m, n) \) a symmetric function of \( m \) and \( n \). Furthermore, if \( x = 0 \) (or \( y = 0 \)), \( y = m + n \) (or \( x = m + n \)) and \( d = m \) (or \( d = n \)). This yields \( k(m, n) = -mn \). The particular solution reads
\[
d^2(x, y, m, n) = \frac{n(x^2 - m^2) + m(y^2 - n^2)}{m + n} \tag{26}
\]
that is, Stewart’s theorem \( \text{[5]} \).

V. HERON OF ALEXANDRIA

Intoxicated by his findings, Newton could have switched to a more elaborate, though older, challenge – as probably did an Ancient Greek Roman Egyptian mathematician... What if, for any triangle, the area \( A \) could be a smooth function of the sides lengths \( x, y \) and \( z \)? He would have assumed
\[
A = A(x, y, z) \tag{27}
\]
After an infinitesimal rotation of the \( y \)-length side around the \((y, z)\) vertex resulting in a small deviation \( \delta x \), while \( \delta y = 0 \) and \( \delta z = 0 \), at first order:
\[
\delta A = \partial_x A \delta x \tag{28}
\]
First note that the \( h_z \)-length height relative to the \( z \)-length side divides the initial triangle in two right triangles of horizontal legs of lengths \( t \) and \( z - t \) respectively. One can express \( h_z \) as a result of the Pythagorean theorem in both right triangles.

Equating those expressions yields \( x^2 - t^2 = y^2 - (z - t)^2 \) and hence
\[
t = \frac{x^2 - y^2 + z^2}{2z} \quad z - t = \frac{y^2 - x^2 + z^2}{2z} \tag{29}
\]
Again, \( \delta \ell \) is the length of the infinitesimal arc travelled by the moved vertex. Like in the two last sections, it can be considered as the first-order hypotenuse of a small triangle whose similarity with a larger one allows to find \( \delta x \). But it is also, at first order, the hypotenuse of another small triangle with one leg of length \( \delta h_z \), which is similar to the large right triangle whose corresponding leg is the \((z - t)\)-length segment, and the hypotenuse the \( y \)-length side. Since \( h_y = \frac{2A}{y} \) and \( \delta h_z = 2\delta A/z \), we have
\[
\delta x = \frac{2A}{xy} \delta \ell \quad \delta A = \frac{y^2 - x^2 + z^2}{4y} \delta \ell \tag{30}
\]
Plugging in results \( \text{[30]} \) into equation \( \text{[28]} \), gives the partial differential equation
\[
A \partial_x A = \frac{1}{8} [x(y^2 + z^2) - x^3] \tag{31}
\]
which can be integrated out to give the general solution

\[ A^2(x, y, z) = \frac{1}{16}[2x^2(y^2+z^2) - x^4 + k(y, z)] \] (32)

where \( k(y, z) \) is an homogeneous function of \( y \) and \( z \). Since \( A(x, y, z) \) must be symmetric in \( x, y \) and \( z \) (i.e. we can make the same reasoning with a non-zero \( \delta y \) or \( \delta z \)), 

\[ k(y, z) = 2y^2z^2 - y^4 - z^4. \]

Hence

\[ A(x, y, z) = \frac{1}{4}\sqrt{2(x^2y^2 + x^2z^2 + y^2z^2) - (x^4 + y^4 + z^4)} \] (33)

which can be factorized into the Heron theorem [6]

\[ A(x, y, z) = \sqrt{x + y + z - x + y + z - x - y - z} \]

whose discovery could actually be Archimedes’ [7].

VI. JAMSHID AL-KASHI

Newton could have chosen to deal with angles – besides calculus, he knew a bit about trigonometry. Let us send him to Persia, a few centuries before his birth, and wonder wether in any triangle of sidelengths \( x, y \) and \( z \), the angle \( \gamma = (x, y) \) could be a smooth function of \( x, y \) and \( z \), that is

\[ \gamma = \gamma(x, y, z) \] (35)

After an infinitesimal rotation of the \( y \)-length side around the \((x, y)\) vertex resulting in a small deviation \( \delta z \), with \( \delta x = 0 \) and \( \delta y = 0 \), at first order:

\[ \delta \gamma = \partial_z \gamma \delta z \] (36)

While \( \delta \gamma \) is easy to connect to \( \delta \ell \), the length of the arc travelled by the moved vertex (in the illustrative figures, \( x \) and \( z \) have been swapped for aesthetic reasons), \( \delta z \) can be determined thanks to the same similiarity as in the three previous sections. We have thus

\[ \delta \gamma = \frac{\delta \ell}{y} \quad \delta z = \frac{h_y}{z} \delta \ell \quad \text{with} \quad h_y = x \sin \gamma \] (37)

Inserting those deviations in eq. (36) yields the partial differential equation

\[ \sin \gamma \partial_z \gamma = \frac{z}{xy} \] (38)

whose general solution is

\[ \cos[\gamma(x, y, z)] = \frac{z^2 + k(x, y)}{2xy} \] (39)

where \( k(x, y) \) is a symmetric, homogeneous function of \( x \) and \( y \). According to Pythagoras, when \( \gamma = \pi/2 \), \( z^2 = x^2 + y^2 \), i.e. \( k(x, y) = -x^2 - y^2 \). Hence

\[ \cos[\gamma(x, y, z)] = \frac{-z^2 + x^2 + y^2}{2xy} \] (40)

that is, al-Kashi’s theorem [8] – also known as the law of cosines or generalized Pythagorean theorem, and already familiar to Euclid [1] Book II, Prop. XII & XIII].

VII. OLRY TERQUEM

Completely exhilarated, Newton could have taken on a bigger piece and assumed that in any triangle of sidelengths \( x, y \) and \( z \), the length \( d \) of the \( \gamma = (x, y) \) angle bisector is a smooth function of \( x, y \) and \( z \), i.e.

\[ d = d(x, y, z) \] (41)

After an infinitesimal rotation of the \( y \)-length side around the \((x, y)\) vertex resulting in a small deviation \( \delta z \), with \( \delta x = 0 \) and \( \delta y = 0 \), at first order:

\[ \delta d = \partial_z d \delta z \] (42)

Again, thanks to the same similarity as in the four previous sections, \( \delta z \) can easily be linked to \( \delta \ell \), the length of the arc travelled by the moved vertex.
It is a little more complicated for $\delta d$. First note that in the illustrative figure, $\delta d < 0$, so that we will consider the positive length $-\delta d$. Then observe that when the $y$-length side infinitesimally rotates around the $(x, y)$ vertex, the foot of the $\gamma = (x, y)$ angle bisector moves along a perpendicular to the $y$-length side, just like the $(z, y)$ vertex. But the angle between this perpendicular and the angle bisector is the complementary of $\gamma/2$. Thus in the small right triangle of legs of lengths $-\delta d$ and $d \delta \gamma/2$, the opposite angle to the $-\delta d$-length leg is, at first order, equal to $\gamma/2$, implying that $\tan(\gamma/2) = -\delta d / (d \delta \gamma/2)$. Hence

$$\delta z = \frac{h_y}{z} \delta \ell$$

$$\delta d = -\tan \frac{\gamma}{2} \frac{d \delta \gamma}{2} \delta \ell = y \delta \gamma$$ \hspace{1cm} (43)

$h_y$ being the length of the height relative to $y$. Using

$$\tan \frac{\gamma}{2} = \frac{\sin \gamma}{1 + \cos \gamma}$$ \hspace{1cm} (44)

with

$$\sin \gamma = \frac{h_y}{x} \quad \text{and} \quad \cos \gamma = -\frac{z^2 + x^2 + y^2}{2xy}$$ \hspace{1cm} (45)

we have the partial differential equation

$$\frac{\partial \ell}{d} = -\frac{z}{-z^2 + (x + y)^2}$$ \hspace{1cm} (46)

whose general solution is

$$d(x, y, z) = k(x, y) \sqrt{(x + y)^2 - z^2}$$ \hspace{1cm} (47)

where $k(x, y)$ is a symmetric function of $x$ and $y$. To determine it, note that in the particular case of a right triangle with hypotenuse of length $z$, the angle bisector is the diagonal of the inscribed square of sidelength $xy/(x + y)$ as can be deduced from similarities between the right triangles generated by the square in the initial triangle. We find $k(x, y) = \sqrt{xy/(x + y)}$. Hence

$$d(x, y, z) = \sqrt{xy \left(1 - \frac{z^2}{(x + y)^2}\right)}$$ \hspace{1cm} (48)

that is, the length of the angle bisector, as Terquem computed in the 19th century [9].

VIII. JEAN-PAUL DE GUÁ DE MALVES

Armed with this powerful theorem-finding tool, Newton could have moved on to even bolder challenges, like leaving the plane for the real space, and imagining, say, a generalization of the Pythagorean theorem in three dimensions! Let him consider a trirectangular tetrahedron, that is a tetrahedron with a right angle corner, like the corner of a cube: what if, for any of them, the area of the face opposite to the right angle was a function of the areas of the other faces?

A convenient way to parametrize the problem is to give arbitrary lengths to the three edges from the right angle vertex, say $x$, $y$ and $z$. The areas of the three right triangle faces are $xy/2$, $xz/2$ and $yz/2$. For the area of the last face, opposite to the right angle, say $A$, we can have an expression by choosing a base, say the edge of length $\sqrt{y^2 + z^2}$ (thanks Pythagoras) and the relative height of length $h$. We have

$$A = \frac{1}{2} \sqrt{y^2 + z^2} \ h$$ \hspace{1cm} (49)

Let us go back to Newton and his obsession. He could have stated that $A$ is a smooth function of $x$ and $y$:

$$A = A(x, y, z)$$ \hspace{1cm} (50)

Choosing to slightly increase $x$, while leaving $y$ and $z$ invariants, that is, an infinitesimal deviation $\delta x$, with $\delta y = 0$ and $\delta z = 0$, we find

$$\delta A = \partial_x A \delta x$$ \hspace{1cm} (51)

Eq. (49) implies that

$$\delta A = \frac{1}{2} \sqrt{y^2 + z^2} \ h$$ \hspace{1cm} (52)

But what do we know of $\delta h$? First note that the foot of the $h$-length height is not affected by the deviation $\delta x$ since this $h$-length height and the $x$-length edge are in a plane orthogonal to the base of the $A$-area face. In this plane, we can check that at first order, the right triangle with $h$-length hypotenuse and $x$-length leg is similar to the one with $\delta x$-length hypotenuse and $\delta h$-length leg, so that

$$\delta h = \frac{x}{h} \delta x$$ \hspace{1cm} (53)

Combining this equation with result (52), itself plugged into eq. (51) with $\delta y = 0$, we have

$$\frac{1}{2} \sqrt{y^2 + z^2} \ x \delta x = \partial_x A \delta x$$ \hspace{1cm} (54)

Simplifying by $\delta x$ and using eq. (49) to get rid of $h$, we find a partial differential equation

$$A \partial_x A = \frac{1}{4} (y^2 + z^2) x$$ \hspace{1cm} (55)
It can be integrated out to give the general solution

\[ A^2(x, y, z) = \frac{1}{4}[(y^2 + z^2)x^2 + k(y, z)] \]  

with \( k(y, z) \) an homogeneous and symmetric function of \( y \) and \( z \). Since \( A(x, y, z) \) must itself be symmetric in \( x \), \( y \) and \( z \) (i.e. we can make the same reasoning with a non-zero \( \delta y \) or \( \delta z \)), \( k(y, z) = y^2z^2 \). Hence

\[ A^2(x, y, z) = \left(\frac{xy}{2}\right)^2 + \left(\frac{xz}{2}\right)^2 + \left(\frac{yz}{2}\right)^2 \]  

known as de Gua’s theorem [11], first formulated by Descartes [11], which states that in any trirectangular tetrahedron, the square of the area of the face opposite to the right corner is equal to the sum of the squares of the areas of the other faces – a three-dimensional generalization of the Pythagorean theorem.

**IX. THE INSCRIBED ANGLE**

Let us move on to the circle, and confront Newton to a simple problem.

Assume that an arbitrary inscribed angle \( \alpha \) is a smooth function of the central angle \( \theta \) that intercepts the same arc on the circle, i.e.

\[ \alpha = \alpha(\theta) \]  

After an infinitesimal deviation \( \delta \theta \), at first order,

\[ \delta \alpha = \alpha' \delta \theta \]  

At first order again, the small right triangle with one leg of length \( y \delta \alpha \) and hypotenuse of length \( \delta \ell \) is similar to the triangle of corresponding sidelengths \( y/2 \) and \( R \) respectively.

Thus we have

\[ y \delta \alpha = \frac{y}{2} \delta \ell \]  

Since \( \delta \theta = \delta \ell / R \), this leads to the differential equation

\[ \alpha' = \frac{1}{2} \]  

whose general solution is

\[ \alpha(\theta) = \frac{\theta}{2} + k \]  

where \( k \) is a constant that vanishes since \( \alpha(0) = 0 \). Hence

\[ \alpha(\theta) = \frac{\theta}{2} \]  

that is, the inscribed angle theorem.

**X. THE CIRCUMRADIUS**

We could keep Newton in the circle and think to another question: of course, for any triangle, the circumradius length \( R \) should be determined by the sidelengths \( x \), \( y \) and \( z \). It was known to Euclid [1, Book IV, Prop. V] and it is actually simple to prove it with the help of Heron’s theorem. But let him play the game of finding it from scratch, that is, by postulating that

\[ R = R(x, y, z) \]  

If the \((x, y)\) vertex is slightly moved along the circumcircle, it generates infinitesimal deviations \( \delta x \) and \( \delta y \), while \( \delta z = 0 \) and \( \delta R = 0 \). At first order, we have:

\[ \partial_x R \delta x + \partial_y R \delta y = 0 \]  

We need to go through some geometric considerations before proceeding: any angle in the triangle has the same magnitude as the ones between its opposite side and the tangent lines to the circumcircle from the two other vertices, since they intercept the same arc.

After the infinitesimal displacement of our vertex along the circumcircle, we consider the right triangles with \( \delta \ell \)-length hypotenuse and \( \delta x \)-length leg, and with \( \delta \ell \)-length hypotenuse and \( -\delta y \)-length leg, respectively, and derive

\[ \delta x = \cos \alpha \delta \ell \quad -\delta y = \cos \beta \delta \ell \]
Applying the al-Kashi theorem yields

\[ \delta x = \frac{y^2 - x^2 + z^2}{2yz} \delta \ell \quad \delta y = \frac{x^2 - y^2 + z^2}{2xz} \delta \ell \]  

(67)

Plugging in results (67) into eq. (65), simplifying by \( \delta \ell / 2z \) and isolating \( \partial_y R \) gives the partial differential equation

\[ \partial_y R = \frac{x}{y} \frac{y^2 - x^2 + z^2}{x^2 - y^2 + z^2} \partial_x R \]  

(68)

A similar expression arises for \( \partial_z R \) when the \((x, z)\) vertex is slightly moved along the circumcircle:

\[ \partial_z R = \frac{z^2 - x^2 + y^2}{x z^2 - x^2 + y^2} \partial_x R \]  

(69)

Finally, we consider the infinitesimal scale transformation \( R \rightarrow R + \delta R \). Since each of the sidelengths increases in line with itself and \( \delta R \), that is \( \delta x = x/R \delta R \), etc., we have the partial differential equation

\[ R = x \partial_x R + y \partial_y R + z \partial_z R \]  

(70)

which guarantees that each quantity at stake is taken into account with its appropriate dimension. Plugging in eq. (68) and (69) into the last one (70) yields the partial differential equation

\[ \frac{\partial_x R}{R} = \frac{x^4 - y^4 - z^4 + 2(y^2 z^2 + x^2 z^2 + x^2 y^2)}{x^4 - y^4 - z^4 + 2(y^2 z^2 + x^2 z^2 + x^2 y^2)} \]  

(71)

which can be integrated out to give the general solution

\[ R(x, y, z) = \frac{x k(y, z)}{\sqrt{x^4 - y^4 - z^4 + 2(y^2 z^2 + x^2 z^2 + x^2 y^2)}} \]  

(72)

with \( k(y, z) \) a symmetric function of \( y \) and \( z \). Since \( R(x, y, z) \) must itself be symmetric in \( x, y \) and \( z \), \( k(y, z) = c y z \) where \( c \) is a constant. We just have to compute the circumradius length of, say, an equilateral triangle of side length 1, that is \( R(1, 1, 1) = \sqrt{3}/3 \), to find out that \( c = 1 \). Furthermore, like for Heron’s theorem, the argument of the square root can be factorized to give

\[ R(x, y, z) = \frac{xyz}{\sqrt{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)}} \]  

(73)

which is indeed the expression of the circumradius as a function of the sidelengths.

XI. THE LAW OF SINES

Let us go back to the Thales theorem that Newton could have discovered by the grace of calculus if only he was born two millenia before: \( y(x) = k x \) is already a nice result, that we have used all along this paper, but what about \( k \)? If he knew the angle \( \gamma = (x, y) \) between the two sides, and one of the angles, say \( \beta = (z, x) \), between the \( x \)-length side and the third side, of length \( z \), he could have speculated that \( y \) is a smooth function of \( x \), \( \beta \) and \( \gamma \), that is

\[ y = y(x, \beta, \gamma) \]  

(74)

It is possible to find the function \( y \) with a simple rotation of one side around one of the adjacent vertices, like in sections III to VII, but the differential equations that arise are a bit complicated to solve: one is linear and the other one is the Bernoulli type. Even if it is not straightforward, it happens to be much simpler to work within the circumcircle. After an infinitesimal and clockwise displacement of the \( \gamma \) vertex along the circumcircle, resulting in a small deviations \( \delta y, \delta x \) and \( \delta \beta \) while \( \delta \gamma = 0 \), at first order,

\[ \delta y = \partial_x y \delta x + \partial_y y \delta \beta \]  

(75)

The deviations \( \delta x \) and \( \delta y \) are the same as in section X, eq. (66). \(-\delta \beta \) can be computed by considering \(-x \delta \beta \) as the length of the leg opposite to \( \alpha \) in the small right triangle of other leg of length \( \delta x \) and hypotenuse of length \( \delta \ell \). Using \( \alpha = \pi - (\beta + \gamma) \) to get rid of \( \alpha \), we have at first order

\[ -\delta y = \cos \beta \delta \ell \quad \delta x = -\cos(\beta + \gamma) \delta \ell \quad -\delta \beta = \frac{\sin(\beta + \gamma)}{x} \delta \ell \]  

(76)

Combining eq. (75) and (76) leads to the differential equation

\[ x \cos \beta = x \cos(\beta + \gamma) \partial_x y + \sin(\beta + \gamma) \partial_y y \]  

(77)
If we now consider an infinitesimal and clockwise displacement of the \( \beta \) vertex along the circumcircle, resulting in deviations \( \delta x \) and \( \delta y \) while \( \delta y = 0 \) and \( \delta \gamma = 0 \), at first order
\[
\partial_x \gamma \delta x + \partial_y \gamma \delta y = 0 \tag{81}
\]

Using the analogous first-order similarities and the fact that \( \alpha = \pi - (\beta + \gamma) \), we find
\[
- \delta x = - \cos(\beta + \gamma) \delta \ell \quad \delta \gamma = \frac{\sin(\beta + \gamma)}{x} \delta \ell \tag{79}
\]
Combining eq. (78) and (79) leads to the differential equation
\[
x \cos(\beta + \gamma) \partial_x y + \sin(\beta + \gamma) \partial_x y = 0 \tag{80}
\]

Finally, we consider the infinitesimal scale transformation \( R \rightarrow R + \delta R \), where \( R \) is the circumradius. Since each of the sidelines increases in line with itself and \( \delta R/R \), that is \( \delta x = x/R \delta R \), but \( \delta \beta = 0 \) and \( \delta \gamma = 0 \), we have the partial differential equation
\[
y = x \partial_x y \tag{81}
\]

Using eq. (81) allows to simplify eq. (77) and (80) to yield
\[
\begin{align*}
\partial_y \beta y + \frac{y}{\tan(\beta + \gamma)} &= \frac{x \cos \beta}{\sin(\beta + \gamma)} \tag{82} \\
\partial_y \gamma y + \frac{y}{\tan(\beta + \gamma)} &= 0 \tag{83}
\end{align*}
\]

They are both linear. Moreover, the second one is separable and admits the general solution
\[
y(x, \beta, \gamma) = \frac{K(x, \beta)}{\sin(\beta + \gamma)} \tag{84}
\]
where \( K(x, \beta) \) is a function of \( x \) and \( \beta \). But this solution must satisfy eq. (81), that is \( K(x, \beta) = k(\beta) x \) where \( k(\beta) \) is a function of \( \beta \). The solution must also satisfy eq. (82), thus we find \( k(\beta) = \sin \beta + C \) with \( C \) a real constant. Finally, since for \( \beta = 0 \), \( y = 0 \), we have
\[
y(x, \beta, \gamma) = \frac{x \sin \beta}{\sin(\beta + \gamma)} \tag{85}
\]
which, using \( \sin(\beta + \gamma) = \sin \alpha \), is the law of sines, proved by Nasir al-Din al-Tusi, Persian mathematician – and architect – regarded as a founder of trigonometry [12].

### XII. CLAUDIUS PTOLEMY

We now take Newton back to Antiquity, in Roman Egypt, and make him look at an ancient problem: given a cyclic quadrilateral, determine the length of its diagonals as functions of the sidelengths \( x, y, u \) and \( v \), that is, for the \( z \)-length segments joining the \((v, x)\) and \((y, u)\) vertices:
\[
z = z(x, y, u, v) \tag{86}
\]

After a small displacement of the \((x, y)\) vertex along the circumcircle, resulting in deviations \( \delta x \) and \( \delta y \) while \( \delta y = 0 \) as well as \( \delta z = 0 \), we have at first order
\[
\partial_x \gamma \delta x + \partial_y \gamma \delta y = 0 \tag{87}
\]

The deviations \( \delta x, \delta y \) are those of section X, eq. (66):
\[
\delta x = \cos \alpha \delta \ell \quad -\delta y = \cos \beta \delta \ell \tag{88}
\]

where \( \alpha \) and \( \beta \) are the angles opposite to the \( x \) and \( y \)-length sides of the \((x, y, z)\) triangle.

Plugging in those results into eq. (87) and considering the equivalent after a small displacement of the \((u, v)\) vertex, we find the two partial differential equations
\[
\begin{align*}
\cos \alpha \partial_x z - \cos \beta \partial_y z &= 0 \tag{89} \\
\cos \rho \partial_u z - \cos \phi \partial_v z &= 0 \tag{90}
\end{align*}
\]

where \( \rho \) and \( \phi \) are the angles opposite to the \( u \) and \( v \)-length sides of the \((u, v, z)\) triangle.

If we now slightly move the \((y, u)\) vertex along the circumcircle, with small deviations \( \delta y, \delta u \) and \( \delta z \), while \( \delta x = \delta v = 0 \), we have at first order
\[
\delta z = \partial_y z \delta y + \partial_z \delta y \tag{91}
\]

The deviations \( \delta y, \delta u \) and \( \delta z \) are given by
\[
\begin{align*}
\delta y &= \cos \beta \delta \ell \\
-\delta u &= \cos \rho \delta \ell \\
-\delta z &= \cos \gamma \delta \ell 
\end{align*} \tag{92}
\]

where \( \delta y \) and \( -\delta u \) are calculated analogously to \( \delta x \) and \( -\delta y \) above, while \( -\delta z \) is to be seen as the length of one of the legs in a small right triangle of hypotenuse of length \( \delta \ell \) and adjacent angle of same magnitude \( \gamma \) as the one between the \( x \) and \( y \)-length sides.
Inserting those results into eq. (91) yields
\[ \cos \gamma = \cos \rho \partial_x z - \cos \beta \partial_y z \] (93)
Moreover, from the infinitesimal scale transformation
\[ R \to R + \delta R, \] a fourth differential equation arises:
\[ z = x \partial_x z + y \partial_y z + u \partial_u z + v \partial_v z \] (94)
Combining equations (89), (90), (93) and (94), isolating \( \partial_x z \) and using \( x \cos \beta + y \cos \alpha = z \) and \( u \cos \rho + v \cos \phi = z \) to simplify the expression, we have
\[ \partial_x z = \frac{(x \cos \beta - \cos \gamma) \cos \beta}{\cos \rho \cos \phi + \cos \alpha \cos \beta} \] (95)
Looking at the intercepted arcs, we observe that \((u, v)\) and \(\gamma\) are supplementary. We then exploit \( \gamma = \pi - (\alpha + \beta) = \rho + \phi \) and the angle sum identities to find
\[ \partial_x z = \frac{\sin \rho \sin \phi \cos \beta}{\sin \rho \sin \phi + \sin \alpha \sin \beta} \] (96)
Thanks to the law of sines applied to both \((x, y, z)\) and \((u, v, z)\) triangles, and the al-Kashi theorem applied to \(\cos \beta\), this equation finally becomes
\[ 2z \partial_x z = \frac{uv(x^2 - y^2 + z^2)}{(uv + xy)x} \] (97)
and turns out to be linear in \(z^2\). Its general solution reads
\[ z^2(x, y, u, v) = \frac{uv(x^2 + y^2) + c(y, u, v)x}{uv + xy} \] (98)
where the function \(c(y, u, v)\) must be symmetric in \(u\) and \(v\), while \(z^2(x, y, u, v)\) must notably be symmetric in \(x\) and \(y\). Thus \(c(y, u, v) = y(u^2 + v^2)\). Hence, after factorization,
\[ z^2(x, y, u, v) = \frac{(ux + vy)(vx + uy)}{uv + xy} \] (99)
which, if multiplied by the square of the other diagonal length, yields the theorem of Ptolemy [15]. Eq. (99) might have been first stated by Brahmagupta [16] §28.

### XIII. BRAHMAGUPTA

A few centuries later in India, was solved the problem of expressing the area \(A\) of any cyclic quadrilateral as a function of its sidelengths \(x, y, u\) and \(v\). Well it could have been solved through calculus too. Let us start from
\[ A = A(x, y, u, v) \] (100)
Like in the last section, we draw the \(z\)-length diagonal, which divides the quadrilateral in two triangles of sides of length \(x, y, z, u, v, z\) respectively. Selecting the \(y\)-length side as the base of the \((x, y, z)\) triangle, we compute its area by multiplying \(y\) and \(x \sin \gamma\) divided by two, where \(\gamma\) is the \((x, y)\) angle magnitude. But the intercepted arcs of the circumcircle tell us that in the other triangle, \((u, v)\) is the supplementary of \((x, y)\). Thus we have for their respective areas \(A_{xyz} = xy \sin \gamma/2\) and \(A_{uvz} = uv \sin \gamma/2\). This yields
\[ \sin \gamma = \frac{2A}{xy + uv} \] (101)
where \(A = A_{xyz} + A_{uvz}\) is the total area of the quadrilateral.

Let us go back to the project. Rather than moving the \((x, y)\) vertex along the circumcircle, which leads to complicated equations, we invite Newton to follow the reasoning developed in section V for the Heron theorem and make an infinitesimal rotation of the \(y\)-length side around the \((y, z)\) vertex resulting in a small deviation \(\delta x\), while \(\delta y = 0\) and \(\delta z = 0\). We have at first order
\[ \delta A = \partial_x A \delta x \] (102)
This slight departure from the cyclical nature of the quadrilateral will be of no consequence. The reasoning is valid in the triangle and as soon as the needed result is secured, we will get back into the circumcircle to proceed. Here \(h\) will stand for the length of the height relative to the \(z\)-length side in the \((x, y, z)\) triangle, such that \(A_{xyz} = zh/2\) and thus \(\delta A = z\delta h/2\) since \(\delta A_{uvz} = 0\). We know a little more about trigonometry than in section V so that we can express, at first order, the deviations \(\delta x\) and \(\delta h\) in terms of \(\gamma\) and \(\alpha\), the \((y, z)\) angle magnitude:
\[ \delta x = \sin \gamma \delta \ell \quad \delta h = \cos \alpha \delta \ell \] (103)
where \(\delta \ell\) is the distance travelled by the moved vertex. Plugging in these into eq. (102), using eq. (101) for the sine and al-Kashi theorem for the cosine, we find
\[ \partial_x A = \frac{xy + uv}{8y} [-x^2 + y^2 + z^2] \] (104)
that is, applying eq. (99) to express \(z^2\) in terms of the sidelengths,
\[ \partial_x A = \frac{1}{8}[-x^3 + (y^2 + u^2 + v^2)x + 2uyv] \] (105)
which can be integrated out to give the general solution

\[ A^2(x, y, u, v) = \frac{1}{16} \left[ -x^4 + 2(y^2 + u^2 + v^2)x^2 + 8xuv + k(y, u, v) \right] \]

where \( k(y, u, v) \) is a homogeneous function of \( y, u \) and \( v \).

A\((x, y, u, v)\) must be symmetric in \( x, y, u \) and \( v \). Consequently \( k(y, u, v) = -y^4 - u^4 - v^4 + 2(y^2u^2 + y^2v^2 + u^2v^2) \).

After factorization, we come across the Brahmagupta theorem \[16, \S 21\]

\[ A(x, y, u, v) = \sqrt{-x + y + u + v} \]

**XIV. LEONHARD EULER**

This one has a simple calculus part, but is a bit more trickier for the geometric principles. Imagine that Newton, or even Euler, would have been interested in determining the distance \( d \) between the circumcenter and incenter of any triangle. Let him assume that \( d \) depends on the inradius \( r \) and the circumradius \( R \), i.e.

\[ d = d(r, R) \]

After a slight displacement of one of the vertices, say \( O \), along the circumcircle, resulting in a second triangle with small deviations \( \delta d \) and \( \delta r \), while \( \delta R = 0 \), we at first order

\[ \delta d = \partial d \delta r \]

Newton could not ignore that the circumcenter is the intersection of the perpendicular side bisectors and that the incenter is the intersection of the internal angle bisectors. Some geometrical considerations need to be taken into account in order to proceed.

First of all, the perpendicular side bisectors and the internal angle bisectors intersect on the circumcircle. To be convinced, just notice that the angles of the triangle are inscribed angles in the circumcircle. According to the inscribed angle theorem, their magnitude is determined by the intercepted arc. Hence, any angle bisector bisects the intercepted arc. So does the corresponding perpendicular side biserctor. Let us call \( Q \) the intersection of the angle bisector from \( O \) and the perpendicular bisector of its opposite side. Obviously, this angle bisector still passes through \( Q \) after the \( O \) vertex has been moved along the circle.

Secondly, when a vertex is moved along the circumcircle, the incenter moves along another circle whose center is precisely the intersection of the angle bisector from the moved vertex, and the perpendicular bisector of the opposite side, that is, \( Q \).

It is simple to show that the trajectory of the incenter is circular: since the angle of the moved vertex remains constant throughout the movement, so does the angle of the moved vertex of a new triangle formed by the two remaining fixed vertices and the incenter; the inscribed angle theorem again tells us that the incenter moves along a circle passing through the two fixed vertices.

To see that the center of this circle is on the circumcircle – and corresponds to \( Q \) – just move the incenter along its circular trajectory until it lays on the perpendicular bisector; in this symmetric configuration, a simple angle chase teaches us that the distance between the incenter and \( Q \) is the same as between each one of the fixed vertices and \( Q \), so that this distance must be the radius of the circle, and \( Q \) its center.

As a consequence, since it moves along a circle whose center is on the angle bisector from the moved vertex, at first order the incenter necessarily moves perpendicularly to this angle bisector.

We can now go back to equation (109) and try to find an expression for \( \delta d \) and \( \delta r \) by observing the way the initial triangle slightly moves to the second one.
Note that $\delta d$ (positive in the illustrative figure) is one of the legs of a right triangle whose hypotenuse length is the distance, say $\delta p$, between the incenters of the two triangles. Since at first order, the $\delta p$-length segment is perpendicular to the angle bisector from the moved vertex, and that the other leg is perpendicular to $d$-length segment, this right triangle is similar to another one of $d$-length hypotenuse and useful leg of length, say, $q$. Hence
\[ \delta d = \frac{q}{d} \delta p \]  
(110)

For $\delta r$ (negative in the illustrative figure), we need to draw the inradius perpendicularly to the side opposite to the moved vertex, and observe the way it evolves. Here, it is $-\delta r$ that can be seen as the length of one of the legs of a right triangle with $\delta p$-length hypotenuse, which is similar to another one of hypotenuse of length $R$ and corresponding leg of length $q$. We have
\[ \delta r = -\frac{q}{R} \delta p \]  
(111)

Dividing equations (110) and (111) member to member and using equation (109) yields the differential equation
\[ d\partial r_d = -R \]  
(112)

which we integrate to obtain the general solution
\[ d^2(r, R) = -2Rr + k(R) \]  
(113)

with $k(R) \geq 2Rr$ a real function of $R$. Since in an equilateral triangle, $d^2(R/2, R) = 0$, we find $k(R) = R^2$. Hence
\[ d^2(r, R) = R(R - 2r) \]  
(114)

that is, Euler’s theorem [12], first proved by Chapple [13].

**XV. THE ANGLE BISECTOR**

We already have a section [VII] dedicated to the angle bisector. But it was about its full length: from the vertex to the foot. What if Newton got now interested to the part between the vertex and the incenter? It should also be a smooth function of the sidelenath $x$, $y$ and $z$, i.e.
\[ c = c(x, y, z) \]  
(115)

Since the position of the incenter matters, and that we know how it moves in the circuncircle, we will let Newton continue his considerations in this very circuncircle. After a small clockwise displacement of the $(z, y)$ vertex along the circuncircle, we have infinitesimal deviations $\delta y$ and $\delta z$, while $\delta x = 0$ (in the illustrative figures, $x$ and $z$ have been swapped for aesthetic reasons). At first order,
\[ \delta c = \partial_op \delta y + \partial_yc \delta z \]  
(116)

In the small right triangle with legs of length $-\delta c$ and $c\delta\gamma/2$ – where $\gamma = (x, y)$ – the angle opposite to the $c\delta\gamma/2$-length side has the same magnitude as the one between the $x$-length side and the $\beta = (x, z)$ angle bisector, that is $\beta/2$, since they both are inscribed angles intercepting the same arc of the circle along which the incenter travels. Thus we have
\[ \tan \frac{\beta}{2} = \frac{c\delta\gamma/2}{-\delta c} \]  
(117)

\[ -\delta y \]
\[ \delta c \]
\[ \delta \gamma \]
\[ \beta/2 \]
\[ \gamma/2 \]
\[ \alpha/2 \]
\[ \beta/2 \]
\[ \gamma/2 \]
\[ \alpha/2 \]
\[ \beta/2 \]
\[ \gamma/2 \]
\[ \alpha/2 \]
\[ \beta/2 \]
\[ \gamma/2 \]
\[ \alpha/2 \]
\[ \beta/2 \]
\[ \gamma/2 \]
Furthermore, based on the same reasoning as in the last section, we can connect \(-\delta y\) and \(\delta z\) to \(\delta \ell\), as well as \(y \delta \gamma = \sin \beta \delta \ell\), so that using eq. (44) for the tangent, we find
\[
- \delta y = \cos \beta \delta \ell \quad \delta z = \cos \gamma \delta \ell \quad \delta c = -\frac{c(1 + \cos \beta)}{2y} \delta \ell
\] (118)

Plugging in results (48) into equation (44) gives a partial differential equation whose analogue can be obtained by considering a clockwise rotation of the \((x, z)\) vertex instead of the \((z, y)\) one. Both equations read
\[
\frac{c}{2y} (1 + \cos \beta) = \cos \beta \partial_y c - \cos \gamma \partial_z c \quad (119)
\]
\[
\frac{c}{2x} (1 + \cos \alpha) = \cos \alpha \partial_x c - \cos \gamma \partial_z c \quad (120)
\]

Combining those with the differential equation arising from the infinitesimal scale transformation \(R \rightarrow R + \delta R\), that is
\[
c = x \partial_x c + y \partial_y c + z \partial_z c
\] (121)
and isolating \(\partial_z c\), yields
\[
\frac{\partial_z c}{c} = \frac{\cos \alpha + \cos \beta}{x \cos \beta \cos \gamma + y \cos \alpha \cos \gamma + z \cos \alpha \cos \beta}
\] (122)
which, using al-Kashi’s theorem for the cosines and working things out a little bit, can be reexpressed as
\[
\frac{\partial_z c}{c} = \frac{x + y}{(x + y + z)(x + y - z)}
\] (123)
and integrated out to give the general solution
\[
c(x, y, z) = k(x, y) \sqrt{\frac{x + y - z}{x + y + z}}
\] (124)
where \(k(x, y)\) is a symmetric function of \(x\) and \(y\). In the particular case of a right triangle with hypotenuse of length \(z\), \(c = \sqrt{2}r\) where \(r\) is the inradius length. Computing the area of this right triangle yields \(r(x + y + z) = xy\). Substituting \(r\) in the expression of \(c\) and inserting the latter in eq. (124), we find \(k(x, y) = \sqrt{xy}\). Hence
\[
c(x, y, z) = \sqrt{xy} \frac{x + y - z}{x + y + z}
\] (125)
that is, the distance between the incenter and the foot of the \(\gamma = (x, y)\) angle bisector. Dividing \(c(x, y, z)\) in this equation by the full length of the bisector \(d(x, y, z)\) in eq. (48) gives the ratio \((x + y)/(x + y + z)\), meaning that the relative position/height of the incenter on the angle bisector (measured from its foot) is equal to the ratio of its corresponding sidelength and the perimeter of the triangle, that is \(z/(x + y + z)\); a result that can easily be verified in barycentric coordinates.

**XVI. THE INRADIUS**

If we were able to do it for the circumradius, section X, we should not doubt of Newton’s appetancy to do it for the inradius \(r!\) It would also be a smooth function of the sidelength \(x, y\) and \(z\), that is
\[
r = r(x, y, z)
\] (126)

Strange as it may seem, since the position of the incenter matters, the circumcircle looks again to be the best place to do the job for the inradius. After a small displacement of the \((x, y)\) vertex along the circumcircle, we have infinitesimal deviations \(\delta x\) and \(\delta y\), while \(\delta z = 0\). At first order,
\[
\delta r = \partial_x r \delta x + \partial_y r \delta y
\] (127)
\(\delta x\) and \(\delta y\) are determined in the same way as in the last sections – see for instance eq. (60) for the case of the circumradius:
\[
\delta x = \cos \alpha \delta \ell \quad - \delta y = \cos \beta \delta \ell
\] (128)
For \(\delta r\) we will refer to the geometrical considerations of section XIV and consider the small right triangle of legs of lengths \(-\delta z\) and \(\delta s\) where \(\delta s\) is the distance travelled by the foot of the inradius perpendicular to the z-length
side. First of all, we check that this inradius foot divides the $z$-length side in two segments of lengths $(x - y + z)/2$ and $(-x + y + z)/2$ respectively, so that after the infinitesimal displacement it is moved of $\delta s = (\delta x - \delta y)/2$. Secondly, in this small right triangle, the angle opposite to the $-\delta r$-length leg is equal to the angle between the internal $\gamma$ angle bisector and the height from the same vertex (or any line perpendicular to the opposite $z$-length side), that is $\gamma/2 - (\pi/2 - \alpha) = (\alpha - \beta)/2$ since $\gamma = \pi - (\alpha + \beta)$. We have

$$-\delta r = \tan \left(\frac{\alpha - \beta}{2}\right) \frac{\delta x - \delta y}{2}$$

$$= \frac{\sin(\alpha - \sin \beta)}{\cos \alpha + \cos \beta} \delta \ell$$

(129)

thanks to eq. (128) and the development of the tangent

$$\tan \frac{\alpha - \beta}{2} = \frac{\sin \alpha - \sin \beta}{\cos \alpha + \cos \beta}$$

(130)

Plugging in results (128) and (129) into equation (127) gives a partial differential equation whose analogue can be obtained by considering a rotation of the $(x, y)$ vertex instead of the $(x, y)$ one. Both equations read

$$\frac{\sin \beta - \sin \alpha}{2} = \cos \alpha \partial x r - \cos \beta \partial y r$$

(131)

$$\frac{\sin \alpha - \sin \gamma}{2} = \cos \gamma \partial z r - \cos \alpha \partial x r$$

(132)

Combining those with the differential equation arising from the infinitesimal scale transformation $R \mapsto R + \delta R$, that is

$$r = x \partial x r + y \partial y r + z \partial z r$$

(133)

and isolating $\partial x r$, yields

$$\partial x r \frac{\cos \beta \cos \gamma}{x \cos \beta \cos \gamma + y \cos \alpha \cos \gamma + z \cos \alpha \cos \beta}$$

$$= \frac{y(\sin \beta - \sin \alpha) \cos \gamma + z(\sin \gamma - \sin \alpha) \cos \beta}{2(x \cos \beta \cos \gamma + y \cos \alpha \cos \gamma + z \cos \alpha \cos \beta)}$$

(134)

which, using the fundamental law for the sines, al-Kashi’s theorem for the cosines and working things out a little bit, can be reexpressed as

$$\partial x r \frac{(x^2 - y^2 - z^2)(x^2 + y^2 - z^2)}{x(2y^2z^2 + x^2z^2 + x^2y^2) - x^4 - y^4 - z^4}$$

$$= \frac{(y - x)(x^2 + y^2 - z^2) + (x - z)(x^2 - y^2 + z^2)}{2x \sqrt{(y^2)^2 + x^2z^2 + x^2y^2) - x^4 - y^4 - z^4}}$$

(135)

This equation is linear. Its general solution reads

$$r(x, y, z) = \frac{-x^3 + (y + z)x^2 - (y + z)(y - z) + x c(y, z)}{2 \sqrt{(y^2)^2 + x^2z^2 + x^2y^2) - x^4 - y^4 - z^4}}$$

where $c(y, z)$ is a homogeneous function of $y$ and $z$. Since $r(x, y, z)$ must be symmetric in $x$, $y$ and $z$, we have $c(y, z) = y^2 + z^2 + mz$ where $m$ is a real constant that can easily be determined by considering, for instance, an equilateral triangle with $x = y = z = 2\sqrt{3} r$. We find $m = -2$. Hence, after factorization and simplification,

$$r(x, y, z) = \frac{4(x + y + z)(x - y - z)}{4(x + y + z)}$$

(136)

that is, the expected expression of the inradius as the area (see section V) divided by twice the perimeter.

**XVII. THE ANGLE BISECTORS PROBLEM**

There is a well-known problem about the internal angle bisectors of a triangle: given three real and positive numbers $a$, $b$, $c$, find the sidelength $x$, $y$, $z$ of the triangle that would admit those numbers as angle bisector lengths. A lot of things have been said about it. We would not be far from the truth if we say that there is a solution, but that it is the solution of a polynomial equation of too high a degree to be solved [9]. See [17–19] for a historical perspective and more recent attempt. We will just propulse Newton as a new challenger for a similar – but not equivalent – problem, that is identifying $a$, $b$, $c$ not as the full lengths of the angle bisectors, but of their part going from the vertex to the incenter. This problem has a solution, but as it is the solution of a third degree polynomial equation, we will not try to write it down. Let Newton state it this way: each of the sidelengths must be a smooth function of $a$, $b$, $c$, the length of the segments which bisect the angles $\alpha$, $\beta$ and $\gamma$ respectively, opposite to the sides of lengths $x$, $y$, $z$ respectively. For $z$, we have

$$z = z(a, b, c)$$

(137)

Again, since the position and the movement of the incen-ter matter, the circumcircle is the best place to operate. After a small displacement of the $(x, y)$ vertex along the circumcircle, we have the infinitesimal deviations $\delta a$, $\delta b$ and $\delta c$, while $\delta z = 0$. At first order,

$$\partial_a z \delta a + \partial_b z \delta b + \partial_c z \delta c = 0$$

(138)

If $\delta p$ is the distance between the incenters before and after the small displacement, the angle between this $\delta p$-length segment and the $-\delta a$-length is an inscribed angle of the circle along which the incenter travels, and has magnitude $\beta/2$. Analogously, the angle between the $\delta p$-length segment and the $\delta b$-length has magnitude $\alpha/2$. 
Thus

\[ \delta a = -\cos \frac{\beta}{2} \delta p \quad \delta b = \cos \frac{\alpha}{2} \delta p \]  

(139)

It is a bit more complicated for \( \delta c \). We first observe that 

\[ -\delta c \] is the length of one of the legs of a small right triangle

with hypotenuse of length \( \delta \ell \), the distance travelled by 

the moved vertex along the circumcircle. The magnitude 

of the angle between those two small sides is given by 

\[ \gamma/2 + \beta \]. Since \( \gamma = \pi - (\alpha + \beta) \), the magnitude of the 

complementary of this angle is \( (\alpha - \beta)/2 \). Hence

\[ \delta c = -\sin \frac{\alpha - \beta}{2} \delta \ell \]  

(140)

To connect \( \delta \ell \) to \( \delta p \), just consider \( \delta \ell \) as the product of 

the circumradius length \( R \) by the corresponding central 

angle, that is twice the inscribed angle \( \delta \alpha = -\delta \beta \) (since 

\( \delta \gamma = 0 \)). This inscribed angle \( \delta \alpha \) is itself connected to 

\( \delta p \) by trigonometry: \( a \delta \alpha/2 \) is the second leglength of the 

small right triangle of first leglength \(-\delta a\) and hypotenuse \( \delta p \). But the circumradius length is not supposed to appear in our equations; we can get rid of it in favour of \( z \) by reminding that \( \sin \gamma = z/(2R) \), as can be seen by moving the \((x, y)\) vertex along the circumcircle, thus preserving \( \gamma \), until the \( x \) or \( y \)-length side coincides with a diameter. These considerations can be summarized as follows:

\[ \delta \ell = 2R \delta \alpha \quad a \frac{\delta \alpha}{2} = \sin \frac{\beta}{2} \delta p \quad 2R = \frac{z}{\sin \gamma} \]  

(141)

Inserting those equations in eq. (140) yields

\[ \delta c = -\sin \frac{\alpha - \beta}{2} \frac{z}{a} \sin \frac{\beta}{2} \delta p \]  

(142)

We need a little trigonometry to move forward. Applying the law of sines in at least two of the three triangles generated by the \( a, b \) and \( c \)-length segments in the initial triangle, we find

\[ a \sin \frac{\alpha}{2} = b \sin \frac{\beta}{2} = c \sin \frac{\gamma}{2} \]  

(143)

Furthermore, if we apply the al-Kashi theorem to the angle \( (a, b) = \pi - (\alpha + \beta)/2 = \pi/2 + \gamma/2 \), we have

\[ \sin \frac{\gamma}{2} = \frac{z^2 - a^2 - b^2}{2ab} \]  

(144)

Inserting this in eq. (143) gives

\[ \sin \frac{\alpha}{2} = \frac{c}{a} \frac{z^2 - a^2 - b^2}{2ab} \quad \sin \frac{\beta}{2} = \frac{c}{b} \frac{z^2 - a^2 - b^2}{2ab} \]  

(145)

Applying the al-Kashi theorem to the angles \( (a, z) = \alpha/2 \) and \( (b, z) = \beta/2 \) respectively, we find

\[ \cos \frac{\alpha}{2} = \frac{z^2 + a^2 - b^2}{2az} \quad \cos \frac{\beta}{2} = \frac{z^2 - a^2 + b^2}{2bz} \]  

(146)

Inserting those results in the development of \( \cos(\gamma/2) = \sin(\alpha/2 + \beta/2) = \sin(\alpha/2) \cos(\beta/2) + \cos(\alpha/2) \sin(\beta/2) \) yields

\[ \cos \frac{\gamma}{2} = \frac{cz}{ab} \frac{z^2 - a^2 - b^2}{2ab} \]  

(147)

Note that submitting \( \sin(\gamma/2) \) in eq. (144) and \( \cos(\gamma/2) \) in eq. (147) to the fundamental law leads to a polynomial equation of degree 3 in \( z^2 \), that is, the complicated but existing solution to our problem.

But in this quest, we were following Newton on another path: it is now time to exploit the expressions of the small deviations in eq. (139) and (142) - developing \( \sin(\alpha/2 - \beta/2) \) and using \( \sin \gamma = 2 \sin(\gamma/2) \cos(\gamma/2) \) in the latter - and plug them into eq. (138), to finally obtain our first partial differential equation

\[ \frac{z^2 - a^2 + b^2}{2b} \partial_a z - \frac{z^2 + a^2 - b^2}{2a} \partial_b z + c \frac{b^2 - a^2}{ab} \partial_c z = 0 \]  

(148)

Combining it with the differential equation arising from the infinitesimal scale transformation \( R \rightarrow R + \delta R \), that is

\[ z = a \partial_a z + b \partial_b z + c \partial_c z \]  

(149)
Following a similar reasoning as previously we find the differential equation

\[ \frac{\partial y}{\partial z} = \frac{z^2 + a^2 - b^2}{2a} - \frac{z^2 - a^2 - b^2}{2b} \partial_z z \] \hspace{1cm} (150)

\[ \frac{\partial z}{\partial z} = \frac{z^2 - a^2 + b^2}{2b} - \frac{z^2 - a^2 - b^2}{2a} \partial_z z \] \hspace{1cm} (151)

That was the easy part. We need a third partial differential equation to complete the system and be able to solve it. The analogue of eq. (148) can be obtained by considering a small displacement of the \((y, z)\) vertex along the circumcircle, instead of the \((x, y)\) one, but it is more complicated since now \(z \neq 0\). At first order

\[ \delta z = \partial_z z \delta a + \partial_b z \delta b + \partial_c z \delta c \] \hspace{1cm} (152)

Following a similar reasoning as previously we find

\[ \delta a = -\sin \frac{\beta - \gamma}{2} z \frac{\sin \gamma}{b} \frac{2}{\sin \gamma} \delta p \] \hspace{1cm} (153)

\[ \delta b = -\cos \frac{\gamma}{2} \delta p \] \hspace{1cm} (154)

\[ \delta c = \cos \frac{\beta}{2} \delta p \] \hspace{1cm} (155)

\[ \delta z = -\cos \gamma \frac{z}{\sin \gamma} \frac{2}{b} \sin \gamma \frac{\delta p}{2} \] \hspace{1cm} (156)

where the latter derives from \(\delta z = -\cos \gamma \delta \ell\), by analogy with what was done, for instance, to determine \(\delta y\) in eq. (60). Using the same trigonometry as for the previous case, plus \(\cos \gamma = \cos^2(\gamma/2) - \sin^2(\gamma/2)\), and plugging in the results into eq. (152), we have

\[ c^2z^2 - a^2b^2 \left[ z^2 - a^2 - b^2 \right] / \left( \frac{ab^2c}{ab} \frac{z^2 - a^2 - b^2}{2b} \right) \partial_z z + \frac{c^2z^2 - a^2 - b^2}{2a} \partial_b z - \frac{z^2 - a^2 - b^2}{2z} \partial_c z \] \hspace{1cm} (157)

All that remains is to replace, in this equation, \(\partial_z z\) and \(\partial_b z\) by their expression in eq. (150) and (151) respectively. After a tedious calculation we finally obtain the differential equation

\[ c \partial_z z = \frac{z(z^2 - a^2 - b^2)(-z^4 + 4(a^2 + b^2)z^2 - (a^2 - b^2)^2)}{z^6 - 3(a^2 + b^2)z^4 + 3(a^2 - b^2)^2z^2 - (a^2 + b^2)(a^2 - b^2)^2} \] \hspace{1cm} (158)

which can be integrated out to give the general solution

\[ \sqrt{(a + b)^2 - z^2}(z^2 - (a - b)^2) / z(z^2 - a^2 - b^2) \] \hspace{1cm} (159)

where \(k(a, b)\) is a symmetric function of \(a\) and \(b\). Here \(z(a, b, c)\) is implicitly given as a root of a polynomial equation of degree 3 in \(z^2\). And this equation happens to be the same as the one we have mentioned earlier if we take \(k(a, b) = \pm 1/(ab)\).

XVIII. THE POLYNOMIALS

One of the lessons that we can draw from this last attempt is that the method cannot evade the sometimes complicated question of finding the roots of a polynomial. Newton surely was interested in polynomials. Could he have tried to find their roots through calculus? Let us first look at a polynomial \(P(x)\) of degree 2 with real coefficients \(a, b\) and \(c\). Each of its roots, if any, satisfies the equation

\[ ax^2 + bx + c = 0 \] \hspace{1cm} (160)

Assume that the root \(x\) of our polynomial is a smooth function of the coefficients:

\[ x = x(a, b, c) \] \hspace{1cm} (161)

After an infinitesimal deviation of those coefficients, we have, at first order:

\[ \delta x = \partial_x x \delta a + \partial_b x \delta b + \partial_c x \delta c \] \hspace{1cm} (162)

But eq. (160) must also remain valid, so that, at first order,

\[ x^2 \delta a + x \delta b + \delta c = -(2ax + b) \delta x \] \hspace{1cm} (163)

If we consider first \(\delta c \neq 0\) with \(\delta a = \delta b = 0\), then \(\delta b \neq 0\) with \(\delta a = \delta c = 0\), and finally \(\delta a \neq 0\) with \(\delta b = \delta c = 0\), we have respectively

\[ \delta c = -(2ax + b) \delta x \] \hspace{1cm} (164)

\[ x^2 \delta a + x \delta b + \delta c = 0 \] \hspace{1cm} (165)

\[ (2ax + b)^2 - b^2 = k_0(a, c) \] \hspace{1cm} (166)

\[ (2ax + b)^2 - b^2 = k_0(b, a) \] \hspace{1cm} (167)

where \(k_0(a, b), k_0(b, a),\) and \(k_0(a, c)\) are arbitrary functions. Using the two last solutions leads to \(k_0(a, c) = k_0(b, c)a = k(c)a\) where \(k(c)\) is an arbitrary function; comparing this to the first solution yields \(4k_0(a, b) = K\) and \(k(c) = K - 4c\), with \(K\) a constant that must vanish since \(x\) is a root of the polynomial; which brings us back to eq. (160).

For a polynomial of degree \(n\) with real coefficients, the roots satisfy the polynomial equation

\[ \sum_{k=0}^{n} a_k x^k = 0 \] \hspace{1cm} (168)

Assume that the root \(x\) is a smooth function of the coefficients:

\[ x = x(a_0, \ldots, a_n) \] \hspace{1cm} (169)
After an infinitesimal deviation of those coefficients, we have, at first order:
\[ \delta x = \sum_{k=0}^{n} \partial_{a_k} x \delta a_k \]  
(170)

If we replace \( a_k \) by \( a_k + \delta a_k \) and \( x \) by \( x + \delta x \) in eq. (168), we have at first order again
\[ \sum_{k=0}^{n} x^k \delta a_k = - \sum_{i=0}^{n} i a_i x^{i-1} \delta x \]  
(171)

Now, by successively keeping only one non-zero coefficient deviation \( \delta a_k \) for \( k = 0, \ldots, n \), we find the \((n + 1)\) relations
\[ x^k \delta a_k = - \sum_{i=0}^{n} i a_i x^{i-1} \delta x \]  
(172)

that can be plugged in into eq. (170) to yield the \((n + 1)\) partial differential equations
\[ \left( \sum_{i=0}^{n} i a_i x^{i-1} \right) \partial_{a_k} x + x^k = 0 \]  
(173)

for \( k = 0, \ldots, n \). The general solution of this system should provide the roots of the polynomial! Each of these equations can be expressed as an ordinary first-order differential equation that turns out to be exact. For each \( k \), we can indeed see our polynomial as a function \( u_k \) of \( x \) and \( a_k \), i.e.
\[ u_k(x, a_k) = \sum_{i=0}^{n} a_i x^i \]  
(174)

Its partial derivatives
\[ \partial_x u_k = \sum_{i=0}^{n} i a_i x^{i-1} \]  
and \[ \partial_{a_k} u_k = x^k \]  
(175)

are naturally the coefficients of eq. (173) so that for each \( k \), we have the total differential \( du_k = 0 \), and \( u_k(x, a_k) \) must be equal to an arbitrary function of all the \( a_i \) for \( i \neq k \). Equating those \((n + 1)\) arbitrary functions forces them to be constant and eventually to vanish since \( x \) is a root of the polynomial. We end up with eq. (168).

In other words, we have found a set of partial differential equations equivalent to the polynomial equation, but the solution of this set is naturally given in the implicit form of the polynomial equation itself, as we could have expected! Newton, at least, would have...

XIX. ARCHIMEDES OF SYRACUSE

We don’t need to speculate on Newton’s aspirations, to pretend to discover that the method can be used to compute the area of a circle and the volume of a sphere as functions of their radius length \( r \) by observing the way they behave under the transformation \( r \mapsto r + \delta r \).

For the circle area \( A(r) \), we have \( \delta A = A' \delta r \), and it is easy to calculate \( \delta A = 2\pi r \delta r \), so that \( A' = 2\pi r \) and \( A(r) = \pi r^2 + k \) where \( k \) is a constant equal to 0 since \( A(0) = 0 \).

We cannot do it for the perimeter \( P(r) \) of the circle since the intermediate result \( \delta P = 2\pi \delta r \) that leads to \( P' = 2\pi \) and \( P(2\pi) = 2\pi r \), precisely relies on the fact that \( P = 2\pi r \), that is, the definition of \( \pi \).

If \( V_s(r) \) is the volume of the sphere, then \( \delta V_s = V_s' \delta r \). By imparting a small thickness \( \delta r \) to the spherical envelope of area \( 4\pi r^2 \), we can calculate \( V_s = 4\pi r^2 \delta r \), so that \( V_s' = 4\pi r^2 \) and \( V_s(r) = 4\pi r^3 / 3 + k \) where \( k \) is a constant equal to 0 since \( V_s(0) = 0 \).

We could do it for the area of the sphere \( A_s(r) \), but we did not find a way to compute \( \delta A_s = 8\pi r \delta r \) without using integral calculus and/or, at least, the cylindrical projection that Archimedes used to directly compute \( A_s(r) = 4\pi r^2 \) [20].

XX. A SIMPLE EQUATION

The equations (70), (81), (94), (121), (133) and (149) that were used to determine the circumradius, the law of sines, etc. could be formulated in a more general way – still specific to Euclidean Geometry:

\[ f(x_1, \ldots, x_p) = \frac{1}{n} \sum_{i=1}^{n} n_i x_i \partial_{x_i} f \]  
(176)

where \( f \) is a metric quantity (length, area, volume, ...) and \( n \) its dimension \((1, 2, 3, \ldots)\), while \( x_i \) are the metric quantities on which \( f \) depends, and \( n_i \) their own dimensions. Every theorem of Euclidean Geometry expressed as a smooth function, every length, surface, volume, hypersurface formula should satisfy equation (176) which insures that all metric quantities involved in it are dimensionally correctly mixed together. It is naturally true for all the theorems re-derived in this article.

Besides the six cases mentionned above, we show how we could have used equation (176) to derive Pythagoras and Heron’s theorems. In the first subsection we observe how this equation specifically acts by restricting a large class of solutions. The second subsection is devoted to the study of a \( n = 2 \) case, since Heron deals with the area of triangles.

A. Pythagoras

Imagine that instead of linearly increasing \( x \) (and then \( y \)) of an infinitesimal \( \delta x \), we would have chosen to perform an infinitesimal rotation of the hypotenuse around, say, the \((y, z)\) vertex. Here \( \delta z = 0 \), we would then have
\[ \partial_x z \delta x + \partial_y z \delta y = 0 \]  
(177)

where the small deviations \( \delta x \) and \( -\delta y \) happen to be the leglengths of a right triangle similar to the initial one – since its hypotenuse is, at first order, perpendicular to...
the hypotenuse of the initial triangle $x$, yielding

$$\delta y = \frac{x}{y} \delta x$$  \hspace{1cm} (178)$$

Plugging in this equation into equation \([177]\) gives

$$y \partial_x z - x \partial_y z = 0$$  \hspace{1cm} (179)$$

which admits a large class of solutions of the form $z(x, y) = f(x^2 + y^2)$. To complete the derivation of the Pythagorean theorem, we need to consider the way $z$ behaves, at first order, under a scale transformation, that is, $R \rightarrow R + \delta R$, where $R$ is the circumradius (here $R = z/2$) or any other length to be used as a scale. We have

$$z = x \partial_x z + y \partial_y z$$  \hspace{1cm} (180)$$

which corresponds to equation \([176]\). Joining equations \([179]\) and \([180]\) in a system and isolating $\partial_x z$ yields

$$\frac{\partial_x z}{z} = \frac{x}{x^2 + y^2}$$  \hspace{1cm} (181)$$

and its general solution

$$z(x, y) = k(y) \sqrt{x^2 + y^2}$$  \hspace{1cm} (182)$$

with $k(y)$ a function of $y$. But the equation \([181]\) has its analogous for $\partial_y z$, generating by integration a function $k(x)$ which must be equal to $k(y)$, that is, to a real and positive constant $k$. Considering a triangle degenerated in a segment implies that $k = 1$.

B. Heron

We still postulate that

$$A = A(x, y, z)$$  \hspace{1cm} (183)$$

but we now consider the triangle in its circumcircle. If the $(x, y)$ vertex is slightly moved along the circumcircle, it generates the infinitesimal deviations $\delta x$ and $\delta y$, while $\delta z = 0$. At first order, we have:

$$\delta A = \partial_x A \delta x + \partial_y A \delta y$$  \hspace{1cm} (184)$$

The deviations $\delta x$ and $\delta y$ are the same as in the circumcircle section, equation \([67]\) while $\delta A = z \delta h/2$. To determine $\delta h$, we observe that it is one of the leglengths of a small right triangle of hypotenuse of length $\delta \ell$, and that this right triangle is similar to another one of hypotenuse of length $R$ and leg with corresponding length $d_z$, the distance between the circumcenter and the height of length $h$, that is, the distance between the middle of the $z$-length side and the foot of the $h$-length height, or else

$$d_z = \frac{y^2 - x^2}{2z}$$ \hspace{1cm} (185)$$

This can be seen by subtracting $t$, in eq. \([29]\) to $z/2$. Hence

$$\delta h = \frac{d_z \delta \ell}{R} = \frac{y^2 - x^2}{2zR} \delta \ell$$ \hspace{1cm} (186)$$

But in the circumcircle, the angle between the $h$-length height and the $x$-length side is the same as the one between the $y$-length side and the circumradius from de $(x, y)$ vertex, thus $h/x = y/(2R)$.

Isolating $R$ and substituting $2A/z$ to $h$ yields

$$R = \frac{xyz}{4A}$$ \hspace{1cm} (187)$$

Thus

$$\delta A = A \frac{y^2 - x^2}{xy} \delta \ell$$ \hspace{1cm} (188)$$

Plugging in equations \([67]\) and \([188]\) into equation \([184]\), simplifying by $\delta \ell/z$ and re-arranging the terms leads to

$$2A = x \frac{y^2 - x^2 + z^2}{y^2 - x^2} \partial_x A - y \frac{x^2 - y^2 + z^2}{y^2 - x^2} \partial_y A$$ \hspace{1cm} (189)$$

This equation can be joined to its symmetric counterpart corresponding to the infinitesimal displacement of the $(z, x)$ vertex along the circumsircle, while $\delta y = 0$, that is

$$2A = z \frac{x^2 - z^2 + y^2}{x^2 - z^2} \partial_x A - x \frac{z^2 - x^2 + y^2}{x^2 - z^2} \partial_y A$$ \hspace{1cm} (190)$$

and the equation originating from the infinitesimal scale transformation $R \leftrightarrow R + \delta R$

$$2A = x \partial_x A + y \partial_y A + z \partial_z A$$ \hspace{1cm} (191)$$

in a system of three equations that we can solve in $\partial_x A$ to obtain the partial differential equation

$$\partial_x A \frac{1}{A} = \frac{1}{2} \left[ -4x^2 - 4x(y^2 + z^2) - 2(2y^2 z^2 + 2z^2 y^2 + 2x^2 y^2 + x^2 z^2 + y^2 z^2) \right]$$ \hspace{1cm} (192)$$

which can be integrated out to give the general solution

$$A(x, y, z) = k(y, z) \sqrt{-x^4 - y^4 - z^4 + 2(y^2 z^2 + x^2 z^2 + x^2 y^2 + \ldots}$$ \hspace{1cm} (193)$$

where $k(y, z)$ is a function of $y$ and $z$. Since the function $A$ must be symmetric in $x$, $y$ and $z$, $k(y, z) = c$ where $c$ is a constant. We just have to compute the area of, say, an isocèle right triangle of leglength 1, that is $A(1, 1, \sqrt{2}) = 1/2$, to find out that $c = 1/4$. After factorization, the result is equation \([34]\), that is, Heron's theorem.
CONCLUSION

With this paper, we present an alternative way to derive classical theorems in Euclidean geometry. Not all theorems, of course. It does not work for theorems in discrete geometry or involving number theory, for theorems stating that this or that line cuts another at this or that point, is perpendicular or tangent to this or that circle, etc. It has to be a theorem involving an equation that defines a function, which will be seen as a particular solution of a (system of) differential equation(s). The proofs that we propose are not necessarily simpler than others. They do not evade the geometric difficulties at stake. We displace the argument of the proof into the game of infinitesimals, but it remains as geometric.

The main advantage of this method is that the theorem does not need to be known. We start with a function, any function, and observe the way it behaves, at first order, under small deviations of some quantities. In the best case, it gives us a (system of) differential equation(s) that we can solve and, therefore, discover the theorem. We chose to use it to rediscover about 20 theorems or identities in the long history of Euclidean geometry, but we hope it may be used to discover new theorems, perhaps in other fields of mathematics.

ACKNOWLEDGMENTS

I thank Raphaël Lefevere for a critical reading of the manuscript and a fruitful collaboration.

The initial idea comes from a previous work in particle physics, looking for natural relations between the mixing angles and the fermion mass ratios [21], supervised by Jean-Marc Gérard, whom I also thank.

This paper is dedicated to Jacques Weyers, who disappeared last Fall.

[1] Euclid of Alexandria, Elements (c. 300 BC).
[2] M. Staring, Mathematics Magazine 69, 45 (1996).
[3] B. C. Berndt, The Mathematical Intelligencer 10, 24 (1988).
[4] Apollonius of Perga, De Loci Planis, Book II, Prop. Ib (c. 200 BC).
[5] S. Stewart, Some General Theorems of Considerable Use in the Higher Parts of Mathematics, Prop. II (1746).
[6] Heron of Alexandria, Metrica, Book I, Prop. VIII (c. 70 AD).
[7] T. Heath, History of Greek Mathematics, Vol. II, 321 (1921).
[8] J. al-Kashi, Mifta al-isab, Book IV, Chap. I (1427).
[9] O. Terquem, Nouv. Ann. Math. (1) 1, 79 (1842).
[10] J.-P. de Guia de Malves, Mémoires de l’Académie royale des sciences – ann. 1783, 374 (1786).
[11] R. Descartes, Opuscules, Cogitationes privatae (1619).
[12] L. Euler, Novi Commentarii academiae scientiarum Petropolitanae 11, 103 (1767).
[13] W. Chapple, Misc. Curios. Math. 4, 117 (1746).
[14] N. al-Din al-Tusi, Kitab al-Shakl al-qatta, Book III, Chap. II (1427).
[15] C. Ptolemy, Almagest, Book I, Chap. IX (c. 150 AD).
[16] Brahmagupta, Brahmasphutasiddhanta, Chap. XII, Sec. IV (628).
[17] N. Altshiller Court, Scripta Math. 19, 218-219 (1953).
[18] P. Mironescu and L. Panaitopol, Amer. Math. Monthly 101, 58-60 (1994).
[19] G. Dinca and J. Mawhin, Bull. Belg. Math. Soc. Simon Stevin 17, 2, 333-341 (2010).
[20] Archimedes of Syracuse, On the Sphere and Cylinder, Book I, Prop. XXXIII (c. 225 BC).
[21] M. Buysse, arXiv (2002) hep-ph/0205213