A DYNAMICAL SYSTEM IN THE SPACE OF CONVEX QUADRANGLES

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ABSTRACT. Let us consider a family \( F(\alpha, \beta, \gamma, \delta) \) of convex quadrangles in the plane with given angles \( \{\alpha, \beta, \gamma, \delta\} \) and with the perimeter \( 2\pi \). Such quadrangle \( Q \in F(\alpha, \beta, \gamma, \delta) \) can be considered as a point \((x_1, x_2, x_3, x_4) \in \mathbb{R}^4\), where \( \{x_1, x_2, x_3, x_4\} \) — lengths of edges. Then to \( F \) a finite open segment \( I \subset \mathbb{R}^4 \) is corresponded. A quadrangle in \( F \), that corresponds to the midpoint of \( I \) is called a balanced quadrangle. Let \( M \) be the set of balanced quadrangles. The function \( f : M \to M \) is defined in the following way: angles of the balanced quadrangle \( Q', Q' = f(Q) \), are numerically equal to edges of \( Q \). The map \( f \) defines a dynamical system in the space of balanced quadrangles. In this work we study properties of this system.

1. Introduction

In this work we attempt to construct some kind of a duality in the space of convex quadrangles. Unlike the construction of works [1] and [2] we, as in the work [3], study the "duality" between angles and edges. Precisely, given a convex quadrangle of the perimeter \( 2\pi \) we can consider a new convex quadrangle of the perimeter \( 2\pi \) with angles numerically equal to lengths of edges of the initial one. However, the new quadrangle is not uniquely defined. To achieve the uniqueness we will introduce the notion of a "balanced quadrangle" and in what follows we will work precisely with them.

Let \( ABCD \)

be a convex quadrangle with given angles: \( \angle A = \alpha, \angle B = \beta, \angle C = \gamma \) and \( \angle D = \delta \). Let \( x_1, x_2, x_3 \) and \( x_4, x_1 + x_2 + x_3 + x_4 = 2\pi \), be lengths of edges \( DA, AB, BC \) and \( CD \), respectively. To such quadrangle we correspond a point with coordinates \((x_1, x_2, x_3, x_4) \) in the 4-dimensional space \( \mathbb{R}^4 \) and to the family \( F(\alpha, \beta, \gamma, \delta) \) of such quadrangles — an open interval \( I \subset \mathbb{R}^4 \).

**Definition 1.1.** The quadrangle \( Q \in F(\alpha, \beta, \gamma, \delta) \) to which the midpoint of \( I \) is corresponded is called a balanced quadrangle.

Let \( M \) be the set of balanced quadrangles and let \( f \) be the map \( f : M \to M \), defined in the following way.

**Definition 1.2.** The map \( f \) corresponds to a balanced quadrangle \( Q \) the balanced quadrangle \( Q' \) such, that angles of \( Q' \) (in the clockwise order) are numerically equal to the lengths of edges of \( Q \) (also in the clockwise order).

Iterations of the map \( f \) have the following properties.
2. Explicit Formulas

Let \( \alpha, \beta, \gamma, \delta \) — angles of a convex quadrangle enumerated in the counter-clockwise order. Consider two pairs of angles — \( \{\delta, \alpha\} \) and \( \{\beta, \gamma\} \). In one pair sum of angles is less than \( \pi \), in other — greater, than \( \pi \). The same is true about pairs \( \{\alpha, \beta\} \) and \( \{\gamma, \delta\} \). Let \( \alpha + \delta < \pi \) and \( \gamma + \delta < \pi \). Let us consider the set \( F(\alpha, \beta, \gamma, \delta) \) and the corresponding interval \( I \). Two endpoints of \( I \) correspond to triangles — ”degenerate” quadrangles.

In the left figure above a convex quadrangle is presented with angles \( \alpha, \beta, \gamma, \delta \). Lengths of edges \( DA, AB, BC \) and \( CD \) are \( x_1, x_2, x_3 \) and \( x_4 \), respectively. \( x_1 + x_2 + x_3 + x_4 = 2\pi \). In the middle and right figures triangles \( A'CD \) and \( BCD' \) of the perimeter \( 2\pi \) are presented — two results of the ”degeneration” of \( ABCD \), when \( x_3 = 0 \) in the middle figure and when \( x_2 = 0 \) — in the right.

Lengths of edges \( x'_1, x'_2, x'_3, x'_4 \) of the ”degenerated quadrangle” \( A'CD \) are:

\[
x'_1 = \frac{2\pi \cdot \sin(\alpha + \delta)}{\sin(\alpha) + \sin(\delta) + \sin(\alpha + \delta)} , \quad x'_2 = \frac{2\pi \cdot \sin(\delta)}{\sin(\alpha) + \sin(\delta) + \sin(\alpha + \delta)} , \quad x'_3 = 0 , \quad x'_4 = \frac{2\pi \cdot \sin(\alpha)}{\sin(\alpha) + \sin(\delta) + \sin(\alpha + \delta)} .
\]

Lengths of edges \( x''_1, x''_2, x''_3, x''_4 \) of the ”degenerated quadrangle” \( BCD' \) are:

\[
x''_1 = \frac{2\pi \cdot \sin(\gamma)}{\sin(\gamma) + \sin(\delta) + \sin(\gamma + \delta)} , \quad x''_2 = 0 , \quad x''_3 = \frac{2\pi \cdot \sin(\delta)}{\sin(\gamma) + \sin(\delta) + \sin(\gamma + \delta)} , \quad x''_4 = \frac{2\pi \cdot \sin(\gamma + \delta)}{\sin(\gamma) + \sin(\delta) + \sin(\gamma + \delta)} .
\]

Now lengths of edges of the balanced quadrangle with angles \( \alpha, \beta, \gamma, \delta \) are:

\[
x_1 = \frac{x'_1 + x''_1}{2} , \quad x_2 = \frac{x'_2 + x''_2}{2} , \quad x_3 = \frac{x'_3 + x''_3}{2} , \quad x_4 = \frac{x'_4 + x''_4}{2} .
\]

(1)

Proposition 1. Let \( \varphi > 0, \psi > 0 \) \( \varphi + \psi < \pi \). Then

\[
\frac{\sin(\varphi)}{\sin(\varphi) + \sin(\psi) + \sin(\varphi + \psi)} < \frac{1}{2} \quad \text{and} \quad \frac{\sin(\varphi + \psi)}{\sin(\varphi) + \sin(\psi) + \sin(\varphi + \psi)} < \frac{1}{2} .
\]

Proof. We have that

\[
\frac{\sin(\varphi + \psi)}{\sin(\varphi) + \sin(\psi) + \sin(\varphi + \psi)} = \frac{\cos \frac{\varphi + \psi}{2}}{\cos \frac{\varphi - \psi}{2} + \cos \frac{\varphi + \psi}{2}} .
\]

As \( \frac{\varphi - \psi}{2} < \frac{\varphi + \psi}{2} \) then

\[
\cos \frac{\varphi - \psi}{2} > \cos \frac{\varphi + \psi}{2}
\]

and the second inequality is proved.
Now let us consider the fraction
\[ \frac{\sin(\varphi)}{\sin(\varphi) + \sin(\psi) + \sin(\varphi + \psi)} \]
The derivative of the denominator
\[ (\sin(\varphi) + \sin(\psi) + \sin(\varphi + \psi))' = 2 \cdot \cos\frac{\varphi}{2} \cdot \cos\frac{2\psi + \varphi}{2} \]
is positive in the interval \(0 < \psi < \frac{\pi - \varphi}{2}\) and is negative in the interval \(\frac{\pi - \varphi}{2} < \psi < \pi - \varphi\). Hence, the fraction is maximal at points 0 and \(\pi - \varphi\). But here the value of the fraction is \(\frac{1}{2}\). \(\square\)

**Consequence 2.1.** A balanced quadrangle has two adjacent edges of the length \(\leq \frac{\pi}{2}\).

*Proof.* Indeed, these are edges \(AB\) and \(BC\). \(\square\)

**Consequence 2.2.** A convex quadrangle of the perimeter \(2\pi\) with three edges with the lengths \(> \frac{\pi}{2}\) cannot be balanced.

### 3. The Exceptional Behavior

If a balanced quadrangle \(Q\) has two equal opposite edges, then from (1) it follows, that \(Q'\) has edges \(\{a, a, b, b\}\), \(a < b, b = \pi - a\). It means, that \(Q''\) is a trapezoid with equal side edges. The edges of \(Q''\) have lengths
\[ \left(\frac{\pi}{2 + 2 \cos(a)}, \frac{\pi}{2}, \frac{\pi}{2 + 2 \cos(a)}, \frac{\pi}{2} + \frac{\pi \cdot \cos(a)}{1 + \cos(a)}\right). \]
Thus, \(Q''\) is of the same type, as \(Q\). So, \(f\) maps trapezoids to quadrangles with equal opposite angles and them to trapezoids:

A trapezoid with angles \(\{a, a, b, b\}\) after double iteration of \(f\) becomes a trapezoid with angles \(\{c, c, d, d\}\), \(c < d\), where
\[ c = \frac{\pi}{1 + \sin\frac{\pi}{2 + 2 \cos(a)} + \cos\frac{\pi}{2 + 2 \cos(a)}}. \]
The convex function \(c(a), 0 < a < \frac{\pi}{2}\), increases from the value \(c(0) = \frac{\pi}{\sqrt{2^2 + 1}} \approx 1.3\) to the value \(c\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\). Its plot in the square \([1.4, \frac{\pi}{2}] \times [1.4, \frac{\pi}{2}]\) is presented below (the scaling is not preserved):
The curve $c(a)$ and the line $c = a$ intersects in two points: $a \approx 1.48342158769377952440379165224$ and $a = \frac{\pi}{2}$. The first point defines an attracting 2-cycle (the derivative $c'(a)$ at this point $\approx 0.8$). The point $a = \frac{\pi}{2}$ is a repelling stationary point — the square. Below are presented angles of quadrangles in the attracting 2-cycle: 
\[(1.48342\ldots, 1.48342\ldots, 1.65817\ldots, 1.65817\ldots) \leftrightarrow (1.44472\ldots, \frac{\pi}{2}, 1.44472\ldots, \frac{\pi}{2} + 0.25214\ldots)\]

4. **The general case**

Computations demonstrate that iterations of a generic balanced quadrangle converge to an attracting 2-cycle $Q_1 \leftrightarrow Q_2$, where balanced quadrangles $Q_1$ and $Q_2$ are congruent up to mirror symmetry. Angles $\alpha, \gamma$ and $\delta$ of $Q_1$ are defined by relations
\[
\begin{align*}
\alpha &= \frac{\pi \sin(\alpha + \delta)}{\sin(\alpha) + \sin(\delta) + \sin(\alpha + \delta)} + \frac{\pi \sin(\gamma)}{\sin(\gamma) + \sin(\delta) + \sin(\gamma + \delta)} \\
\delta &= \frac{\pi \sin(\alpha + \delta)}{\sin(\alpha) + \sin(\delta) + \sin(\alpha + \delta)} \\
\gamma &= \frac{\pi \sin(\gamma)}{\sin(\gamma) + \sin(\delta) + \sin(\gamma + \delta)}
\end{align*}
\]
\[(2)\]

and
\[
\begin{align*}
\alpha &\approx 1.54819305248669225152933985324 \\
\beta &\approx 1.82405188512759300508614890573 \\
\gamma &\approx 1.41515953031350909799654144250 \\
\delta &\approx 1.49578083925179212231325656509
\end{align*}
\]
\[(3)\]

It must be noted that modules of some derivatives of righthand parts of relations (2) at the point (3) are greater, than 1. Thus, the attracting property of this 2-cycle does not have a simple explanation.

**Remark 4.1.** Let generic quadrangles $R$ and $S$ be congruent up to mirror symmetry. As expected, if $f^{(n)}(R)$ is close to $Q_1$, $f^{(n)}(S)$ is close to $Q_2$, and vice versa.

**References**

[1] J. Cantarella, T. Needham, C. Shonkwiler, *Random triangles and polygons in the plane*, The American Mathematical Monthly, 126(2), 2019, 113-134.

[2] I. Busjatskaja, Y. Kochetkov, *Dual quadrangles in the plane*, arXiv: 1911.09321.

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