CONFORMALLY OSSERMAN MANIFOLDS

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Dedicated to the memory of Novica Blažić (1959 – 2005), a remarkable mathematician and a wonderful person.

Abstract. An algebraic curvature tensor is called Osserman if the eigenvalues of the associated Jacobi operator are constant on the unit sphere. A Riemannian manifold is called conformally Osserman if its Weyl conformal curvature tensor at every point is Osserman. We prove that a conformally Osserman manifold of dimension $n \neq 3, 4, 16$ is locally conformally equivalent either to a Euclidean space or to a rank-one symmetric space.

1. Introduction

An algebraic curvature tensor $\mathcal{R}$ on a Euclidean space $\mathbb{R}^n$ is a $(3, 1)$ tensor having the same symmetries as the curvature tensor of a Riemannian manifold. For $X \in \mathbb{R}^n$, the Jacobi operator $\mathcal{R}_X : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $\mathcal{R}_X Y = \mathcal{R}(X, Y) X$. The Jacobi operator is symmetric and $\mathcal{R}_X X = 0$ for all $X \in \mathbb{R}^n$.

Definition 1. An algebraic curvature tensor $\mathcal{R}$ is called Osserman if the eigenvalues of the Jacobi operator $\mathcal{R}_X$ do not depend on the choice of a unit vector $X \in \mathbb{R}^n$.

One of the algebraic curvature tensors naturally associated to a Riemannian manifold (apart from the curvature tensor itself) is the Weyl conformal curvature tensor.

Definition 2. A Riemannian manifold is called (pointwise) Osserman if its curvature tensor at every point is Osserman. A Riemannian manifold is called conformally Osserman if its Weyl tensor at every point is Osserman.

It is well-known (and is easy to check directly) that a Riemannian space locally isometric to a Euclidean space or to a rank-one symmetric space is Osserman. The question of whether the converse is true (“every pointwise Osserman manifold is flat or locally rank-one symmetric”) is known as the Osserman Conjecture [Os]. The first result on the Osserman Conjecture (the affirmative answer for manifolds of dimension not divisible by 4) was published before the conjecture itself [Chi]. In the following almost two decades, the research in the area of Osserman and related classes of manifolds, both in the Riemannian and pseudo-Riemannian settings, was flourishing, with dozens of papers and at least three monographs having been published [G1, G2, GKV].

At present, the Osserman Conjecture is proved almost completely, with the only exception when the dimension of an Osserman manifold is 16 and one of the eigenvalues of the Jacobi operator has multiplicity 7 or 8 [N1, N2, N3, N4]. The main difficulty lies in the fact that the Cayley projective plane (and its hyperbolic dual) are Osserman, with the multiplicities of the eigenvalues of the Jacobi operator being exactly 7 and 8; moreover, the curvature tensor of the Cayley projective plane is essentially different from that of the other rank-one symmetric spaces, as it does not admit a Clifford structure (see Section 2 for details). This is the only known Osserman curvature tensor without a Clifford structure, and to prove the Osserman Conjecture in full it would be largely sufficient to show that there are no other exceptions.

The study of conformally Osserman manifolds was started in [BG1], and then continued in [BG2, BGNSi, G2, BGNSi]. Every Osserman manifold is conformally Osserman (which easily follows from the...
formula for the Weyl tensor and the fact that every Osserman manifold is Einstein), as also is every manifold locally conformally equivalent to an Osserman manifold.

Our main results is the following theorem.

**Theorem 1.** A connected $C^\infty$ Riemannian conformally Osserman manifold of dimension $n \neq 3, 4, 16$ is locally conformally equivalent to a Euclidean space or to a rank-one symmetric space.

Theorem 1 answers, with three exceptions, the conjecture made in [BCNS] (for conformally Osserman manifolds of dimension $n > 6$ not divisible by 4, this conjecture is proved in [BG1, Theorem 1.4]).

Note that the nature of the three excepted dimensions in Theorem 3 is different. In dimension three the Weyl tensor gives no information on a manifold at all. In dimension four, even a “genuine” pointwise Osserman manifold does not have to be locally symmetric (see [GSV, Corollary 2.7], [O1], for the examples of “generalized complex space forms”). As it is proved in [Ch], the Osserman Conjecture is still true in dimension four, but in a more restrictive version: one requires the eigenvalues of the Jacobi operator to be constant on the whole unit tangent bundle (a Riemannian manifold with this property is called *globally Osserman*). One might wonder, whether the conformal counterpart of this result is true. The following elegant characterization in dimension four is obtained in [BG2]: a four-dimensional Riemannian manifold is conformally Osserman if and only if it is either self-dual or anti-self-dual.

In dimension 16, both the conformal and the original Osserman Conjecture remain open (for partial results, see [N3, N4] in the Riemannian case and Theorem 3 in Section 3 in the conformal case).

As a rather particular case of Theorem 1, we obtain the following analogue of the Weyl-Schouten Theorem for rank-one symmetric spaces: a Riemannian manifold of dimension greater than four having the “same” Weyl tensor as that of one of the complex/quaternionic projective spaces or their noncompact duals is locally conformally equivalent to that space. More precisely,

**Theorem 2.** Let $M^n_0$ denote one of the spaces $\mathbb{CP}^{n/2}$, $\mathbb{CH}^{n/2}$, $\mathbb{HP}^{n/4}$, $\mathbb{HH}^{n/4}$, and let $W_0$ be the Weyl tensor of $M^n_0$ at some point $x_0$ in $M^n_0$. Suppose that for every point $x$ of a Riemannian manifold $M^n$, $n > 4$, there exists a linear isometry $i : T_xM^n \to T_{x_0}M^n_0$ which maps the Weyl tensor of $M^n$ at $x$ on a positive multiple of $W_0$. Then $M^n$ is locally conformally equivalent to $M^n_0$.

For $M^n_0 = \mathbb{CP}^{n/2}$, $\mathbb{CH}^{n/2}$ and $n > 6$, the claim follows from [BG1, Theorem 1.4]. The fact that the dimension $n = 16$ is not excluded (compared to Theorem 1) follows from Theorem 3 (see Section 3).

We explicitly require all the object (manifolds, metrics, vector and tensor fields) to be smooth (of class $C^\infty$), although all the results remain valid for class $C^k$, with sufficiently large $k$.

The paper is organized as follows. In Section 2 we give a background on Osserman algebraic curvature tensors and on Clifford structures and prove some technical Lemmas. The proof of Theorem 1 is given in Section 3. Theorem 2 is deduced from a more general Theorem 3. We first prove the local version using the differential Bianchi identity, and then the global version by showing that the “algebraic type” of the Weyl tensor is the same at all the points of a connected conformally Osserman Riemannian manifold (in particular, a nonzero Osserman Weyl tensor cannot degenerate to zero).

## 2. Algebraic curvature tensors with a Clifford structure

### 2.1. Clifford structure.

The property of an algebraic curvature tensor $\mathcal{R}$ to be Osserman is quite algebraically restrictive. In the most cases, such a tensor can be obtained by the following remarkable construction, suggested in [GSV], which generalizes the curvature tensors of the complex and the quaternionic projective spaces.

**Definition 3.** A Clifford structure $\text{Cliff}(\nu; J_1, \ldots, J_\nu; \lambda_0, \eta_1, \ldots, \eta_\nu)$ on a Euclidean space $\mathbb{R}^n$ is a set of $\nu \geq 0$ anticommuting almost Hermitian structures $J_i$ and $\nu + 1$ real numbers $\lambda_0, \eta_1, \ldots, \eta_\nu$, with $\eta_\nu \neq 0$. An algebraic curvature tensor $\mathcal{R}$ on $\mathbb{R}^n$ has a Clifford structure $\text{Cliff}(\nu; J_1, \ldots, J_\nu; \lambda_0, \eta_1, \ldots, \eta_\nu)$ if

$$
\mathcal{R}(X, Y)Z = \lambda_0(\langle X, Z \rangle Y - \langle Y, Z \rangle X) + \sum_{i=1}^\nu \eta_i(2\langle J_iX, Y \rangle J_iZ + \langle J_iZ, Y \rangle J_iX - \langle J_iZ, X \rangle J_iY).
$$

When it does not create ambiguity, we abbreviate $\text{Cliff}(\nu; J_1, \ldots, J_\nu; \lambda_0, \eta_1, \ldots, \eta_\nu)$ to just $\text{Cliff}(\nu)$.
Remark 1. As it follows from Definition\textsuperscript{2}, the operators $J_i$ are skew-symmetric, orthogonal and satisfy the equations $\langle J_i X, J_j X \rangle = \delta_{ij} \|X\|^2$ and $J_i J_j + J_j J_i = -2\delta_{ij} \text{id}$, for all $i, j = 1, \ldots, \nu$, and all $X \in \mathbb{R}^n$. This implies that every algebraic curvature tensor with a Clifford structure is Osserman, as by \textsuperscript{4} the Jacobi operator has the form $\mathcal{R}_X Y = \lambda_0(\|X\|^2 Y - \langle Y, X \rangle X) + \sum_{i=1}^{\nu} 3\eta_i \langle J_i X, Y \rangle J_i X$, so for a unit vector $X$, the eigenvalues of $\mathcal{R}_X$ are $\lambda_0$ (of multiplicity $n-1-\nu$ provided $\nu < n-1$), 0 and $\lambda_0 + 3\eta_i$, $i = 1, \ldots, \nu$.

The converse (“every Osserman algebraic curvature tensor has a Clifford structure”) is true in all the dimensions except for $n = 16$, and also in many cases when $n = 16$, as follows from \textsuperscript{3} (Proposition 1 and the second last paragraph of the proof of Theorem 1 and Theorem 2), \textsuperscript{2} Proposition 1 and \textsuperscript{4} Proposition 2.1. The only known counterexample is the curvature tensor $\mathcal{R} = a R^{10} + b R^1$, where $R^1$ is the curvature tensor of the unit sphere $S^{10}(1)$ and $a \neq 0$.

A Clifford structure $\text{Cliff}(\nu)$ on the Euclidean space $\mathbb{R}^n$ turns it into a Clifford module (we refer to \textsuperscript{1}, Chapter 11, \textsuperscript{2} Chapter 1 for standard facts on Clifford algebras and Clifford modules). Denote $\text{Cl}(\nu)$ a Clifford algebra on $\nu$ generators $x_1, \ldots, x_\nu$, an associative unital algebra over $\mathbb{R}$ defined by the relations $x_i x_j + x_j x_i = -2\delta_{ij}$ (this condition determines $\text{Cl}(\nu)$ uniquely). The map $\sigma : \text{Cl}(\nu) \rightarrow \mathbb{R}^n$ defined on generators by $\sigma(x_i) = J_i$ (and $\sigma(1) = \text{id}$) is a representation of $\text{Cl}(\nu)$ on $\mathbb{R}^n$. As all the $J_i$’s are orthogonal and skew-symmetric, $\sigma$ gives rise to an orthogonal multiplication defined as follows. In the Euclidean space $\mathbb{R}^\nu$, fix an orthonormal basis $e_1, \ldots, e_\nu$. For every $u = \sum_{i=1}^{\nu} u_i e_i \in \mathbb{R}^\nu$ and every $X \in \mathbb{R}^n$, define

$$J_u X = \sum_{i=1}^{\nu} u_i J_i X$$

(when $u = e_i$, we abbreviate $J_{e_i}$ to $J_i$). The map $J : \mathbb{R}^\nu \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by \textsuperscript{2} is an orthogonal multiplication: $\|J_u X\|^2 = \|u\|^2 \|X\|^2$ (similarly, we can define an orthogonal multiplication $J : \mathbb{R}^{\nu+1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $J_u X = \epsilon_0 X + \sum_{i=1}^{\nu} u_i J_i X$, for $u = \sum_{i=0}^{\nu} u_i e_i \in \mathbb{R}^{\nu+1}$, where $\epsilon_0, e_1, \ldots, e_\nu$ is an orthonormal basis for the Euclidean space $\mathbb{R}^{\nu+1}$). For $X \in \mathbb{R}^n$, denote $J_X = \text{Span}(J_1 X, \ldots, J_{\nu} X), \quad I_X = \text{Span}(X, J_1 X, \ldots, J_{\nu} X)$.

Later we will also use the complexified versions of these subspaces which we denote $J_X$ and $I_X$ respectively, for $X \in \mathbb{C}^n$.

If $\mathbb{R}^n$ is a $\text{Cl}(\nu)$-module (equivalently, if there exists an algebraic curvature tensor with a Clifford structure $\text{Cliff}(\nu)$ on $\mathbb{R}^n$), then (see, for instance, \textsuperscript{2} Theorem 11.8.2)

$$\nu \leq 2^h + 8a - 1, \quad \text{where } n = 2^{a+b} c, \quad c \text{ is odd, } 0 \leq b \leq 3.$$ 

From \textsuperscript{3}, we have the following inequalities.

**Lemma 1.** Let $\mathcal{R}$ be an algebraic curvature tensor with a Clifford structure $\text{Cliff}(\nu)$ on $\mathbb{R}^n$. Suppose $n \neq 2, 4, 8, 16$. Then

(i) $n \geq 3\nu + 3$, with the equality only when $n = 6, \nu = 1$, or $n = 12, \nu = 3$, or $n = 24, \nu = 7$.

(ii) $n > 4\nu - 2$, except in the following cases: $n = 24, \nu = 7$ and $n = 32, \nu = 8$.

(iii) there exists an integer $l$ such that $\nu < 2^l < n$.

2.2. Clifford structures on $\mathbb{R}^8$ and the octonions. The proof of Theorem \textsuperscript{2} in the “generic case” will rely upon the fact that $\nu$ is small relative to $n$ (with the required estimates given in Lemma \textsuperscript{1}). However, in the case $n = 8$, the number $\nu$ can be as large as $7$, according to \textsuperscript{3}. Consider this case in more detail. As it is shown in \textsuperscript{2}, not only every Osserman algebraic curvature tensor $\mathcal{R}$ on $\mathbb{R}^8$ has a Clifford structure, but also that Clifford can be taken of one of the two (mutually exclusive) forms: either $\mathcal{R}$ has a $\text{Cliff}(3)$-structure, with $J_1 J_2 = J_3$, or an existing $\text{Cliff}(\nu)$-structure can be “complemented” to a $\text{Cliff}(7)$-structure. More precisely:

**Lemma 2.**

1. Suppose $\mathcal{R}$ is an algebraic curvature tensor on $\mathbb{R}^8$ having a Clifford structure $\text{Cliff}(\nu; J_1, \ldots, J_{\nu}; \lambda_0, \eta_1, \ldots, \eta_{\nu})$. Then exactly one of the following two possibilities may occur: either $\mathcal{R}$ has a Clifford structure $\text{Cliff}(3)$ with $J_1 J_2 = J_3$, or there exist $7 - \nu$ operators $J_{\nu+1}, \ldots, J_7$ such that $J_1, \ldots, J_7$ are
anticommuting almost Hermitian structures with $J_1J_2\ldots J_7 = \text{id}_{\mathbb{R}^8}$ and $\mathcal{R}$ has a Clifford structure 
Cliff$(7; J_1, \ldots, J_7; \lambda_0 - 3\xi, \eta_1 + \xi, \ldots, \eta_7 + \xi, \xi, \ldots, \xi)$, for any $\xi \neq -\eta_i, 0$.

2. Let $\mathbb{O}$ be the octonion algebra with the inner product defined by $\|u\|^2 = uu^*$, where $^*$ is the octonion conjugation, and let $\mathbb{O}' = \mathbb{1}^\perp$, the space of imaginary octonions. Then, in the second case in assertion 1, there exist linear isometries $\iota_1 : \mathbb{R}^8 \to \mathbb{O}$, $\iota_2 : \mathbb{R}^7 \to \mathbb{O}'$ such that the orthogonal multiplication is given by $J_0X = \iota_1(X)\iota_2(u)$.

Proof. 1. This assertion is proved in [N2, Lemma 5]. The proof is based on the fact that every representation $\sigma$ of $\text{Cl}(\nu)$ on $\mathbb{R}^8$, except for the representations of $\text{Cl}(3)$ with $J_1J_2 = \pm J_3$, is a restriction of a representation of $\text{Cl}(7)$ on $\mathbb{R}^8$, to $\text{Cl}(\nu) \subset \text{Cl}(7)$. It follows that the almost Hermitian structures $J_1, \ldots, J_\nu$ defined by $\sigma$ can be complemented by almost Hermitian structures $J_{\nu+1}, \ldots, J_7$ such that $J_1, \ldots, J_7$ anticommute, and so $\mathcal{R}$ can be written in the form (1), with a formal summation up to 7 on the right-hand side (but with $\eta_i = 0$ when $i = \nu + 1, \ldots, 7$). To obtain a Clifford(7)-structure for $\mathcal{R}$, according to Definition [9], we only need to make all the $\eta_i$'s nonzero. This can be done using the identity

$$
(X, Z)Y - (Y, Z)X = \sum_{i=1}^7 \frac{1}{3}(2\langle J_iX, Y \rangle J_iZ + \langle J_iZ, Y \rangle J_iX - \langle J_iX, Z \rangle J_iY)
$$

(which is obtained from the polarized identity $\|X\|^2Y - (X, Y)X = \sum_{i=1}^7 \langle J_iX, Y \rangle J_iX$ which follows from the fact that, for $X \neq 0$, the vectors $\|X\|^{-1}X, \|X\|^{-1}J_1X, \ldots, \|X\|^{-1}J_7X$ form an orthonormal basis for $\mathbb{R}^7$). Then by (1), $\mathcal{R}$ has a Clifford structure \Cliff$(7; J_1, \ldots, J_7; \lambda_0 - 3\xi, \eta_1 + \xi, \ldots, \eta_7 + \xi, \xi, \ldots, \xi)$, for any $\xi \neq -\eta_i, 0$.

2. This assertion is also proved in [N2] (see the beginning of Section 5.1). The proof is based on the following. There are two nonisomorphic representations of $\text{Cl}(7)$ on $\mathbb{R}^8$. Identifying $\mathbb{R}^8$ with the octonion algebra $\mathbb{O}$ via a linear isometry in the two representations these are given by the orthogonal multiplications $J_0X = uX$ and $J_0X = Xu$ respectively [LM § I.8]. As $(uX)^* = X^*u = -X^*u$ for all $u, X \in \mathbb{O}, u \perp 1$, the first representation is orthogonally equivalent to the second one, with the operators $J_i$ replaced by $-J_i$. Since changing the signs of the $J_i$'s does not affect the form of the algebraic curvature tensor (1), we can always assume that a Clifford(7)-structure for an algebraic curvature tensor on $\mathbb{R}^8$ is given by the orthogonal multiplication $J_0X = \iota_1(X)\iota_2(u)$.

In the proof of Theorem [9] for $n = 8$, we will usually identify $\mathbb{R}^8$ with $\mathbb{O}$ and of $\mathbb{R}^7$ with $\mathbb{O}'$ via some fixed linear isometries $\iota_1, \iota_2$ and simply write the orthogonal multiplication in the form

$$
J_0X = Xa,
$$

where $X \in \mathbb{R}^8 = \mathbb{O}$, $u \in \mathbb{O}'$. The proof of Theorem [9] for $n = 8$ extensively uses the computations in the octonion algebra $\mathbb{O}$ (in particular, the standard identities like $a^{*2} = 2\langle a, 1 \rangle 1 - a$, $\langle a, b \rangle = \langle a^*, b^* \rangle = \frac{1}{2}(a^*b + b^*a)$, $\langle ab \rangle = a^2b$, $\langle b, c \rangle = (b^*c)^*$, $\langle ac \rangle = \langle ba, ca \rangle = \|a\|^2\langle b, c \rangle$, for any $a, b, c \in \mathbb{O}$, and the similar ones, see e.g. [HL Section IV]) and the fact that $\mathbb{O}$ is a division algebra (in particular, any nonzero octonion is invertible: $a^{-1} = \|a\|^{-2}a^*$). We will also use the biotetions $\mathbb{O} \otimes \mathbb{C}$, the algebra over the $\mathbb{C}$ with the same multiplication table as that for $\mathbb{O}$. As all the above identities are polynomial, they still hold for biotetions, with the complex inner product on $\mathbb{C}^8$, the underlying linear space of $\mathbb{O} \otimes \mathbb{C}$. However, the biotetion algebra is not a division algebra (and has zero-divisors: $(i1 + e_1)(i1 - e_1) = 0$).

2.3. Technical lemma. In the proof of Theorem [9] we will use the following lemma.

Lemma 3. Suppose that $n > 4$, and additionally, if $n = 8$, then $\nu \leq 3$, and if $n = 16$, then $\nu \leq 7$.

1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a homogeneous polynomial map of degree $m$ such that for all $X \in \mathbb{R}^n$, $F(X) \in \mathcal{F}X$ (respectively $F(X) \in \mathcal{I}X$). Then there exist homogeneous polynomials $c_i$, $i = 1, \ldots, \nu$ (respectively $i = 0, 1, \ldots, \nu$), of degree $m - 1$ such that $F(X) = \sum_{i=1}^\nu c_i(X)J_iX$ (respectively $F(X) = c_0(X)X + \sum_{i=1}^\nu c_i(X)J_iX$).

2. Let $1 \leq k \leq \nu$ and let $a_j$, $1 \leq j \leq \nu$, $j \neq k$, be $\nu - 1$ vectors in $\mathbb{R}^n$ such that for all $Y \in \mathbb{R}^n$,$$
(6)
\sum_{j \neq k}(a_j, J_kY)J_jY + \langle a_j, Y \rangle J_kJ_jY = 0.
$$
Then either $a_j = 0$ for all $j \neq k$, or $\nu = 1$, or $\nu = 3$, $J_1 J_2 = \varepsilon J_3$, $\varepsilon = \pm 1$, and $a_j = J_j v$ for all $j \neq k$, where $v \neq 0$.

3. Suppose $n$ and $\nu$ are arbitrary numbers satisfying (3). Let $N^n$ be a smooth Riemannian manifold and let $J_1, \ldots, J_\nu$ be anticommuting almost Hermitian structures on $N^n$. Suppose that for every nowhere vanishing smooth vector field $X$ on $N^n$, the distribution $\mathcal{J}X = \text{Span}(J_1 X, \ldots, J_\nu X)$ is smooth (that is, the $\nu$-form $J_1 X \wedge \cdots \wedge J_\nu X$ is smooth). Then for every $x \in N^n$, there exists a neighbourhood $U = U(x)$ and smooth anticommuting almost Hermitian structures $\tilde{J}_1, \ldots, \tilde{J}_\nu$ on $U$ such that $\text{Span}(\tilde{J}_1 X, \ldots, \tilde{J}_\nu X) = \text{Span}(J_1 X, \ldots, J_\nu X)$, for any vector field $X$ on $U$.

Proof. 1. It is sufficient to prove the assertion for the case $F(X) \in \mathfrak{X}$. 

As for every $X \neq 0$, the vectors $X, J_1 X, \ldots, J_\nu X$ are orthogonal and have the same length $\|X\|$, we have $\|X\|^2 F(X) = f_0(X) X + \sum_{\nu=1}^\nu f_\nu(X) J_\nu X$, where $f_0(X) = \langle F(X), X \rangle$, $f_\nu(X) = \langle F(X), J_\nu X \rangle$ are homogeneous polynomials of degree $m + 1$ of $X$ (or possibly zeros). Taking the squared lengths of the both sides we get $\|X\|^2 \|F(X)\|^2 = f_0^2(X) + \sum_{\nu=1}^\nu f_\nu^2(X)$, so the sum of squares of $\nu + 1$ polynomials $f_0(X), f_\nu(X), \ldots, f_\nu(X)$ is divisible by $\|X\|^2$. Let for $X = (x_1, \ldots, x_n)$, $\|X\|^2$ be the ideal of $\mathbb{R}[X]$ generated by $\|X\|^2 = \sum_{j=1}^n x_j^2$, and let $\mathcal{R} = \mathbb{R}[X]/\langle\|X\|^2\rangle$. We have $\sum_{\nu=0}^\nu f_\nu^2 = 0$, where $f_\nu$ is the image of $f_\nu$ under the natural projection $\pi : \mathbb{R}[X] \to \mathcal{R}$. If at least one of the $f_\nu$‘s is nonzero (say the $\nu$-th one), then $\sum_{\nu=0}^\nu (f_\nu/f_0)^2 = -1$ in $\mathcal{R}$, the field of fractions of the ring $\mathcal{R}$. The field $\mathcal{R}$ is isomorphic to the field $\mathbb{L}_{n-1} = \mathbb{R}(x_1, \ldots, x_{n-1}, \sqrt{d})$, where $d = x_1^2 + \cdots + x_{n-1}^2$ (an isomorphism from $\mathbb{L}_{n-1}$ to $\mathcal{R}$ is induced by the map $(a+b/c)/c \to (a + bx/c, a, b, c \in \mathbb{R}[x_1, \ldots, x_{n-1}], c \neq 0)$). By Theorem 3.13, the level of the field $\mathbb{L}_{n-1}$, the minimal number of elements whose sum of squares is $-1$, is $2^l$, where $2^l < n \leq 2^{l+1}$. It follows that in all the cases when $\nu < 2^l < n$ we arrive at a contradiction. This means that $f_\nu = 0$, for all $i = 0, \ldots, \nu$, so each of the $f_\nu$‘s is divisible by $\|X\|^2$ in $\mathbb{R}[X]$, so $F(X) = (\|X\|^2 f_0(X)) X + \sum_{\nu=1}^\nu (\|X\|^2 f_\nu(X)) J_\nu X$, with all the nonzero coefficients on the right-hand side being homogeneous polynomials of degree $m - 1$. The claim now follows from assertion (iii) of Lemma 4.

If $\nu = 1$, equation (6) is trivially satisfied. If $\nu = 2$, the claim immediately follows by taking the inner product of (6) with $J_1 J_2 Y$. If $\nu = 3$, let $k = 3$ (without loss of generality). Taking the inner product of (6) with $J_1 Y$ we obtain $\langle a_1, J_1 Y \rangle \|Y\|^2 = \langle a_1, J_1 J_2 J_3 Y, Y \rangle$. It follows that the polynomial $\langle J_1 J_2 J_3 Y, Y \rangle$ is divisible by $\|Y\|^2$. As the operator $J_1 J_2 J_3$ is symmetric and orthogonal, it equals $\pm i d$. Hence $J_1 J_2 = \varepsilon J_3$, $\varepsilon = \pm 1$. Then (6) takes the form $\langle a_1, J_3 J_2 Y + \varepsilon (a_1, J_2 J_3 Y + \varepsilon J_3 Y) J_2 Y = 0$, which is equivalent to $a_1 = -\varepsilon J_3 a_2$. Acting by $J_1$ on the both sides we obtain $J_1 a_1 = J_2 a_2$, so $a_2 = J_1 v$, with $v = -J_1 a_2 = -J_2 a_2$ (we can assume $v \neq 0$, as otherwise $a_2 = 0$).

Now assume $\nu > 3$ and denote $L = \text{Span}(a_1)$. As it follows from (6), if $Y \perp L$, then $J_1 J_2 Y \perp L$, so $L$ is $J_3$-invariant. Polarizing (6) we obtain $\sum_{\nu \neq k} \langle a_\nu, J_\nu J_k Y \rangle + \langle a_\nu, J_\nu J_k Y \rangle J_\nu J_k Y + \langle a_\nu, J_\nu J_k Y \rangle J_\nu J_k Y = 0$.

$\sum_{\nu \neq k} \langle a_\nu, J_\nu J_k Y \rangle + \langle a_\nu, J_\nu J_k Y \rangle J_\nu J_k Y + \langle a_\nu, J_\nu J_k Y \rangle J_\nu J_k Y = 0$.

It follows that for all $\nu = 1, \ldots, \nu$, $Y \in \mathbb{R}^n$, $\sum_{\nu \neq k} \langle a_\nu, J_\nu J_k Y \rangle + \langle a_\nu, J_\nu J_k Y \rangle J_\nu J_k Y = 0$, that is, $J_{a_\nu} = \frac{\langle a_\nu, J_\nu J_k Y \rangle}{\langle a_\nu, J_\nu J_k Y \rangle} e_j$, $\langle a_\nu, J_\nu J_k Y \rangle e_j$.

Note that $u(Y), v(Y)$ $\perp e_k$. Now, fix an arbitrary $Y \in \mathbb{R}^n$ and choose a unit vector $v \perp u(Y), v(Y), e_k$ (this is possible, as $\nu > 3$). Then $J_{a_\nu} = \frac{\langle a_\nu, J_\nu J_k Y \rangle}{\langle a_\nu, J_\nu J_k Y \rangle} e_j$, $\langle a_\nu, J_\nu J_k Y \rangle e_j$.

Note that $v(Y) \neq 0$, then the operator $\langle v(Y) \rangle^{-1} J_{a_\nu} = \frac{\langle a_\nu, J_\nu J_k Y \rangle}{\langle a_\nu, J_\nu J_k Y \rangle} e_j$, $\langle a_\nu, J_\nu J_k Y \rangle e_j$.

This is a contradiction. Hence $v(Y) = 0$ for all $Y \in \mathbb{R}^n$, so all the $\alpha_j$‘s are zeros.

3. We first prove the lemma assuming $2\nu \leq n$. In this case, the proof closely follows the arguments of the proof of [N1] Lemma 3.1.

Let $Y_0 \in T_x N^n$ be a unit vector. As $2\nu \leq n$, there exists a vector $E \in T_x N^n$ which is not in the range of the map $\Phi : S^{n-1} \times S^{n-1} \to S^{n-1}$ defined by $\Phi(u, v) = J_u J_v Y_0$. Then $\mathcal{J} E \cap \mathcal{J} Y_0 = 0$. It follows that on some neighbourhood $U'$ of $x$ there exist smooth unit vector fields $Y$ and $E_n$ such that $E_n(x) = E$, $Y(x) = Y_0$ and $\mathcal{J} E_n \cap \mathcal{J} Y_0 = 0$ at every point $y \in U'$. By the assumption, the $\nu$-dimensional distribution $\mathcal{J} E_n$ is smooth, so we can choose $\nu$ smooth orthonormal sections $E_1, \ldots, E_\nu$ of it, and then define anticommuting almost Hermitian structures $\tilde{J}_\nu$ on $U'$ by $J_{\tilde{J}_\nu} = E_n$ (so that $J_{\tilde{J}_\nu} = \sum_{\beta=1}^\nu a_{\alpha\beta} J_{\beta}$, where $(a_{\alpha\beta})$ is the $\nu \times \nu$ orthogonal matrix given by $a_{\alpha\beta} = \langle E_\alpha, J_\beta E_\nu \rangle$).
Let $E_{n+1}, \ldots, E_{n-1}$ be orthonormal vector fields on $U'$ such that $E_1, \ldots, E_n$ is an orthonormal frame, and let, for a vector field $X$ on $U'$, $\hat{J}X$ denote the $n \times \nu$ matrix whose column vectors are $\hat{J}_1X, \ldots, \hat{J}_\nu X$ relative to the frame $E_1, \ldots, E_n$. Then $(\hat{J}X)^t \hat{J}X = ||X||^2 I_\nu$ and all the $\nu \times \nu$ minors of the matrix $\hat{J}X$ are smooth functions on $U'$. Moreover, the entries of the matrices $\hat{J}E_i$, $i = 1, \ldots, n$, are the rearranged entries of the matrices $\hat{J}_\alpha$, $\alpha = 1, \ldots, \nu$, relative to the basis $\{E_i\}$, so to prove that the $\hat{J}_\alpha$'s are smooth it suffices to show that all the entries of the matrices $\hat{J}E_i$ are smooth (on a possibly smaller neighbourhood). Denote $\hat{J}E_i = (K_i-P_i)$, where $K_i$ and $P_i$ are $\nu \times \nu$ and $(n-\nu) \times \nu$ matrices-functions on $U'$ respectively (note that $\hat{J}E_n = (I_0)$). For an arbitrary $t \in \mathbb{R}$, all the $\nu \times \nu$ minors of the matrix $\hat{J}(E_i+tE_n) = (K_i+\nu)$ are smooth. For every entry $(P_i)_{k\alpha}$, $k = \nu + 1, \ldots, n$, $\alpha = 1, \ldots, \nu$, the coefficient of $\nu^{-1}$ in the $\nu \times \nu$ minor of $\hat{J}(E_i+tE_n)$ consisting of $\nu-1$ out of the first $\nu$ rows (omitting the $\alpha$-th row) and the $k$-th row is $\pm(P_i)_{k\alpha}$, so all the entries of all the $P_i$'s are smooth.

For the vector field $Y$, constructed at the beginning of the proof, denote $\hat{J}Y = (K_i-P_i)$. As $P = \sum_{i=1}^\nu (Y,E_i)P_i$, all the entries of $P$ are smooth on $U'$. Moreover, as $TY \cap IE_n = 0$, the spans of the vector columns of the matrices $\hat{J}Y$ and $\hat{J}E_n = (I_0)$ have trivial intersection, so $rk P = \nu$, at every point $y \in U'$. Therefore we can choose the rows $\nu+1 \leq b_1 < \cdots < b_\nu \leq n$ of the matrix $P$ at the point $x$ such that the corresponding minor $P_{(b)} = P_{b_1 \ldots b_\nu}$ is nonzero. Then the same minor $P_{(b)}$ is nonzero on a (possibly smaller) neighbourhood $U \subset U'$ of $x$. Taking all the $\nu \times \nu$ minors of $\hat{J}Y$ consisting of $\nu-1$ out of $\nu$ rows of $P_{(b)}$ and one row of $K$ we obtain that all the entries of $K$ are smooth on $U$. Moreover, for an arbitrary $t \in \mathbb{R}$, all the $\nu \times \nu$ minors of the matrix $\hat{J}(tE_1+Y) = (tK_1+K)$ are smooth. Computing the coefficient of $t$ in all the $\nu \times \nu$ minors of $\hat{J}(tE_1+Y)$ consisting of $\nu-1$ out of $\nu$ rows of $(tP_1+P)_{(b)}$ and one row of $tK_1+K$ and using the fact that all the entries of $K, P$ and $P_i$ are smooth on $U$ we obtain that all the entries of $K_1$ are also smooth on $U$. Therefore all the entries of all the matrices $\hat{J}E_i$ are smooth on $U$, hence the anticommuting almost Hermitian structures $\hat{J}_\alpha$ are also smooth on $U$.

As $\nu$ and $n$ must satisfy inequality $\mathbf{3}$ (hence the inequalities of Lemma $\mathbf{1}$), the above proof works in all the cases except for the following: $n = 4$, $\nu = 3$ and $n = 8$, $\nu = 5, 6, 7$. The case $n = 4$, $\nu = 3$ is easy: taking any smooth orthonormal frame $E_1$ on a neighbourhood of $x$ and defining $\hat{J}_\alpha = \sum_{\beta=1}^3 a_{\alpha\beta}J_\beta$ (with the orthogonal $3 \times 3$ matrix $(a_{\alpha\beta})$ given by $a_{\alpha\beta} = \langle E_\alpha, J_\beta E_4 \rangle$) we obtain that all the entries of the $\hat{J}_\alpha$ relative to the basis $E_i$ are $\pm 1$ and 0.

The proof in the cases $n = 8, \nu = 5, 6, 7$ is based on the fact that any set of anticommuting almost Hermitian structures $J_1, \ldots, J_6$ on $\mathbb{R}^8$, except when $\nu = 3$ and $J_1J_2 = \pm J_3$, can be complemented by almost Hermitian structures $J_{\nu+1}, \ldots, J_7$ to a set $J_1, \ldots, J_7$ of anticommuting almost Hermitian structures on $\mathbb{R}^8$ (assertion 1 of Lemma $\mathbf{2}$).

If $n = 8$, $\nu = 7$, choose an arbitrary smooth almost Hermitian structure $J_7$ on some neighbourhood $U$ of $x$ and complement it by anticommuting almost Hermitian structures $J_1, \ldots, J_6$ at every point of $U$. Then $\text{Span}(J_1X, \ldots, J_6X) = (\text{Span}(X,J_7X))^\perp$ is a smooth distribution, for every smooth nowhere vanishing vector field $X$ on $U$. This reduces the case $n = 8, \nu = 7$ to the case $n = 8, \nu = 6$.

Let $n = 8, \nu = 6$, and let $J_7$ be an almost Hermitian structure complementing $J_1, \ldots, J_6$ at every point $x \in N^n$. Using the first part of the proof (or the fact that $J_7X$ spans the one-dimensional smooth distribution $(\text{Span}(J_1X, \ldots, J_6X) \oplus \mathbb{R}X)^\perp$, for every nonvanishing smooth vector field $X$) we can assume that $J_7$ is smooth on a neighbourhood $U$ of $x \in N^n$. Choose a smooth orthonormal frame $E_1, \ldots, E_8$ on (a possibly smaller neighbourhood) $U$ such that the matrix of $J_7$ relative to $E_i$ is $\left( \begin{array}{cc} 0 & I_2 \\ 0 & 0 \end{array} \right)$ and define the almost Hermitian structure $\hat{J}_6$ on $U$ by $\hat{J}_6E_2 = E_1$, $\hat{J}_6E_4 = E_3$, $\hat{J}_6E_6 = -E_5$, $\hat{J}_6E_8 = -E_7$. Then $J_7$ and $\hat{J}_6$ anticommute, hence we can complement them by almost Hermitian structures $J'_1, \ldots, J'_6$ on $U$ in such a way that $J'_1, \ldots, J'_5$ are anticommuting almost Hermitian structures. Moreover, as both $J_7$ and $\hat{J}_6$ are smooth on $U$, the five-dimensional distribution $\text{Span}(J'_1X, \ldots, J'_6X) = (\text{Span}(X,J_7X,\hat{J}_6X))^\perp$ is smooth, for every smooth nowhere vanishing vector field $X$ on $U$. This reduces the case $n = 8, \nu = 6$ to the case $n = 8, \nu = 5$. Indeed, if $J_1, \ldots, J_5$ are smooth anticommuting almost Hermitian structures on $U$ such that $\text{Span}(\hat{J}_1X, \ldots, \hat{J}_5X) = \text{Span}(J'_1X, \ldots, J'_5X)$, for every vector field $X$, then $\hat{J}_1, \ldots, \hat{J}_5, \hat{J}_6$ are the required almost Hermitian structures, as $\text{Span}(\hat{J}_1X, \ldots, \hat{J}_6X) = \text{Span}(J'_1X, \ldots, J'_6X, \hat{J}_6X)$ =
(\text{Span}(X, J_7X))^\perp = \text{Span}(J_1X, \ldots, J_6X)$, for every vector field $X$ on $\mathcal{U}$, and $J_6$ anticommutes with every $J_\alpha$, $\alpha = 1, \ldots, 5$, since it anticommutes with every $J'_{\alpha}$, $\alpha = 1, \ldots, 5$.

Let $n = 8$, $\nu = 5$, and let $J_6, J_7$ be anticommuting almost Hermitian structures complementing $J_1, \ldots, J_6$ at every point $x \in \mathbb{N}^n$. As $\text{Span}(J_6X, J_7X) = (\text{Span}(J_1X, \ldots, J_6X))^\perp$, by the first part of the proof, we can choose such $J_6$ and $J_7$ to be smooth on a neighbourhood $\mathcal{U}$ of $x \in \mathbb{N}^n$. Choose a smooth orthonormal frame $E_1, \ldots, E_8$ on (a possibly smaller neighbourhood) $\mathcal{U}$ as follows. First choose an arbitrary smooth unit vector field $E_1$ on $\mathcal{U}$. The vector fields $J_6E_1$ and $J_7E_1$ are orthonormal; set $E_2 = -J_6J_7E_1$. The unit vector field $J_6J_7E_1$ is orthogonal to $E_1, J_6E_1, J_7E_1$; set $E_4 = -J_6J_7E_1$.

Choose an arbitrary smooth unit vector field $E_2$ of the smooth distribution $(\text{Span}(E_1, E_2, E_3, E_4))^\perp$ on $\mathcal{U}$. That distribution is both $J_6$- and $J_7$-invariant, so we can set, similar to above, $E_6 = J_6E_5$, $E_7 = J_7E_5$, $E_8 = -J_6J_7E_5$. Now define the almost Hermitian structure $J_5$ on $\mathcal{U}$ whose matrix relative to the frame $E_i$ is $(-I_4 I_4)$. Then $J_5, J_6, J_7$ are anticommuting almost Hermitian structures on $\mathcal{U}$, with $J_5J_6 \neq \pm J_7$, hence we can complement them by almost Hermitian structures $J'_1, \ldots, J'_4$ on $\mathcal{U}$ in such a way that $J'_1, J'_4, J'_5, J'_6, J'_7$ are anticommuting almost Hermitian structures. Moreover, as $J_5, J_6, J_7$ are smooth on $\mathcal{U}$, the four-dimensional distribution $\text{Span}(J'_1X, \ldots, J'_4X) = (\text{Span}(X, J_5X, J_6X, J_7X))^\perp$ is smooth, for every smooth nowhere vanishing vector field $X$ on $\mathcal{U}$. By the first part of the proof, we can find smooth anticommuting almost Hermitian structures $J_1, \ldots, J_4$ on (a possibly smaller) neighbourhood $\mathcal{U}$ such that $\text{Span}(J_1X, \ldots, J_4X) = \text{Span}(J'_1X, \ldots, J'_4X)$, for every vector field $X$. Then $J_1, \ldots, J_4, J_5$ are the required almost Hermitian structures, as $\text{Span}(J_1X, \ldots, J_5X) = \text{Span}(J'_1X, \ldots, J'_4X, J_5X) = (\text{Span}(X, J_5X, J_6X, J_7X))^\perp = \text{Span}(J_1X, \ldots, J_5X)$, for every vector field $X$ on $\mathcal{U}$, and $J_5$ anticommutes with every $J_\alpha$, $\alpha = 1, 2, 3, 4$, since it anticommutes with every $J'_\alpha$, $\alpha = 1, 2, 3, 4$.

3. Conformally Osserman Manifolds. Proof of Theorem 1

Let $M^n$, $n \neq 3, 4$, be a smooth conformally Osserman Riemannian manifold. If $n = 2$, the manifold is locally conformally flat, so we can assume that $n > 4$. Combining the results of [3] (Proposition 1 and the second last paragraph of the proof of Theorem 1 and Theorem 2), [2] Proposition 1 and [4] Proposition 2) we obtain that the Weyl tensor of $M^n$ has a Clifford structure, for all $n \neq 16$, and also for $n = 16$ provided the Jacobi operator $W_X$ has an eigenvalue of multiplicity at least 9 (note that the Jacobi operator of any Osserman algebraic curvature tensor on $\mathbb{R}^{16}$ has an eigenvalue of multiplicity at least 7, by topological reasons). In the latter case, $W$ has a Clifford structure $\text{Cliff}(\nu)$, with $\nu \leq 6$, at every point on $M^n$.

To prove Theorem 1 it therefore suffices to prove the following theorem.

**Theorem 3.** Let $M^n$ be a connected smooth Riemannian manifold whose Weyl tensor at every point $x \in M^n$ has a Clifford structure $\text{Cliff}(\nu(x))$. Suppose that $n > 4$, and additionally that if $n = 16$, then $\nu(x) \leq 4$. Then there exists a space $M^n_0$ from the list $\mathbb{R}^n, \mathbb{CP}^{n/2}$, $\mathbb{CH}^{n/2}$, $\mathbb{RP}^{n/4}$, $\mathbb{HP}^{n/4}$ (the Euclidean space and the rank-one symmetric spaces with their standard metrics) such that $M^n$ is locally conformally equivalent to $M^n_0$.

Note that by Theorem 1 every point of $M^n$ has a neighbourhood conformally equivalent to a domain of the same “model space”. Also note that Theorem 1 in comparison to Theorem 1 says something also in the case $n = 16$.

We start with a brief informal sketch of the proof of Theorem 1. First of all, we show that the Clifford structure for the Weyl tensor can be chosen locally smooth on an open, dense subset $M' \subset M^n$ (see Lemma 4 for the precise statement). To simplify the form of the curvature tensor $R$ of $M^n$, we combine the $J_0$-part of $W$ (from 1) with the difference $R - W$, so that $R$ has the form 7 for some smooth symmetric operator field $\rho$, at every point of $M'$. The technical core of the proof is Lemma 6 and Lemma 7 which establish various identities for the covariant derivatives of $\rho$, the $J'_i$'s and the $\eta_i$'s, using the differential Bianchi identity for the curvature tensor of the form 7. Lemma 6 treats the case $(n, \nu) = (8, 7)$ and uses the octonion arithmetic, and Lemma 7 all the other cases (and uses the fact that $\nu$ is small compared to $n$, see Lemma 1). It follows from the identities of Lemma 6 and Lemma 7 that, unless the Weyl tensor vanishes, the metric on $M'$ can be locally changed to a conformal one whose curvature tensor again has the form 7, but with the two additional features: firstly, all the $\eta_i$'s
are locally constant, and secondly, $\rho$ is a Codazzi tensor, that is, $(\nabla_X \rho)Y = (\nabla_Y \rho)X$. By the result of [DS], the exterior products of the eigenspaces of a symmetric Codazzi tensor are invariant under the curvature operator on the two-forms. Using that, we prove in Lemma 7 that $\rho$ must be a multiple of the identity, so, by (7), $M'$ is locally conformally equivalent to an Osserman manifold. The affirmative answer to the Osserman Conjecture in the cases for $n$ and $\nu$ considered in Theorem 3 [N1, Theorem 1.2] implies that $M'$ is locally conformally equivalent to one of the spaces listed in Theorem 3. This proves Theorem 3 at the “generic” points. To prove Theorem 3 globally, we first show (using Lemma 8) that $M'$ splits into a disjoint union of a closed subset $M_0$, on which the Weyl tensor vanishes, and nonempty open connected subsets $M_\alpha$, each of which is locally conformal to one of the rank-one symmetric spaces $\mathbb{CP}^{n/2}$, $\mathbb{CH}^{n/2}$, $\mathbb{HP}^{n/4}$, $\mathbb{HH}^{n/4}$. On every $M_\alpha$, the conformal factor $f$ is a well-defined positive smooth function. Assuming that there exists at least one $M_\alpha$ and that $M_0 \neq \emptyset$ we show that there exists a point $x_0 \in M_0$ on the boundary of a geodesic ball $B \subset M_\alpha$ such that both $f(x)$ and $\nabla f(x)$ tend to zero when $x \to x_0$, $x \in B$ (Lemma 9). Then the positive function $u = f^{(n-2)/4}$ satisfies elliptic equation (37) in $B$, with $\lim_{x \to x_0, x \in B} u(x) = 0$, hence by the boundary point theorem, the limiting value of the inner derivative of $u$ at $x_0$ must be positive. This contradiction implies that either $M = M_0$ or $M = M_\alpha$.

**Proof of Theorem 3** Let $M^n$, $n > 4$, be a connected smooth Riemannian manifold whose Weyl tensor at every point has a Clifford structure. Define the function $N : M^n \to \mathbb{N}$ as follows: for $x \in M^n$, $N(x)$ is the number of distinct eigenvalues of the Jacobi operator $W_X$ associated to the Weyl tensor, where $X$ is an arbitrary nonzero vector from $T_x M^n$. As the Weyl tensor is Osserman, the function $N(x)$ is well-defined. Moreover, as the set of symmetric operators having no more than $N_0$ distinct eigenvalues is closed in the linear space of symmetric operators on $\mathbb{R}^n$, the function $N(x)$ is lower semi-continuous (every subset $\{x : N(x) \leq N_0\}$ is closed in $M^n$). Let $M'$ be the set of points where the function $N(x)$ is continuous. It is easy to see that $M'$ is an open and dense (but possibly disconnected) subset of $M^n$. The following lemma shows that the Clifford structure for the Weyl tensor is locally smooth on every connected component of $M'$.

**Lemma 4.** Let $M^n$, $n > 4$, be a smooth Riemannian manifold whose Weyl tensor has a Clifford structure at every point. If $n = 16$, we additionally require that at every point $x \in M^{16}$, the Weyl tensor has a Clifford structure $\text{Cliff}(\nu(x))$ with $\nu(x) \neq \emptyset$.

Let $M'$ be the (open, dense) subset of $M^n$ at the points of which the number of distinct eigenvalues of the Jacobi operator associated to the Weyl tensor of $M^n$ is locally constant. Then for every $x \in M'$, there exists a neighbourhood $U = U(x)$, a number $\nu \geq 0$, smooth functions $\eta_1, \ldots, \eta_\nu : U \to \mathbb{R} \setminus \{0\}$, a smooth symmetric linear operator field $\rho$ and smooth anticommuting almost Hermitian structures $J_i$, $i = 1, \ldots, \nu$, on $U$ such that the curvature tensor of $M^n$ has the form

$$R(X, Y)Z = \langle X, Z \rangle \rho Y + \langle \rho X, Z \rangle Y - \langle Y, Z \rangle \rho X - \langle \rho Y, Z \rangle X$$

$$+ \sum_{i=1}^{\nu} \eta_i \langle 2\langle J_i X, Y \rangle J_i Z + \langle J_i Z, Y \rangle J_i X - \langle J_i Z, X \rangle J_i Y \rangle, \quad \text{for all } y \in U \text{ and } X, Y, Z \in T_x M^n.$$  

Moreover, if $n = 8$, then the curvature tensor has the form (11) either with $\nu = 3$ and $J_1 J_2 = \pm J_3$, or with $\nu = 7$, for all $y \in U$.

**Proof.** Let $X$ be a smooth unit vector field on $M^n$. As the Weyl tensor $W$ is a smooth Osserman algebraic curvature tensor, the characteristic polynomial of $W_X X^\perp$ (of the restriction of the Jacobi operator $W_X$ to the subspace $X^\perp$) does not depend on $X$ and is a well-defined smooth map $p : M^n \to \mathbb{R}_{n-1}[t]$, $y \to p_y(t)$, where $\mathbb{R}_{n-1}[t]$ is the $(n-1)$-dimensional affine space of polynomials of degree $n-1$ with the leading term $(-t)^{n-1}$. As all the roots of $p_y(t)$ are real and the number of different roots is constant on every connected component of $M'$, the eigenvalues $\mu_0, \mu_1, \ldots, \mu_{\nu}$ of $W_X X^\perp$ are smooth functions and their multiplicities $m_0, m_1, \ldots, m_{\nu}$ are constant, on every connected component of $M'$ (we chose the labelling in such a way that $m_0 = \max(m_0, m_1, \ldots, m_{\nu})$).

First consider the case $n \neq 8$. The Weyl tensor has a Clifford structure given by (11) at every point of $M'$. By Lemma 11 for $n > 4$, $n \neq 8, 16$, $n - 1 - \nu > \nu$, for any Clifford structure on $\mathbb{R}^n$. By (3), for $n = 16$, $\nu \leq 8$, so by the assumption, the inequality $n - 1 - \nu > \nu$ also holds for $n = 16$. Then the biggest multiplicity of an eigenvalue of $W_X X^\perp$ is $n - 1 - \nu$ (see Remark 1). So
the number $\nu = n - 1 - m_0$ is constant and the function $\lambda_0 = \mu_0$ is smooth on every connected component of $M'$. Moreover, for every smooth unit vector field $X$ on $M'$ and every $i = 1, \ldots, l$, the $\mu_i$-eigendistribution of $W_{X|X^+}$ is $\text{Span}_{\lambda_0+3\eta_j=\mu_i}(J_iX)$. As $\lambda_0$ and $\mu_i$ are smooth functions on every connected component of $M'$, $\eta_j$ also is. Moreover, on every connected component of $M'$, every distribution $\text{Span}_{\lambda_0+3\eta_j=\mu_i}(J_iX)$ is smooth and has a constant dimension $m_i$, for any nowhere vanishing smooth vector field $X$. By assertion 3 of Lemma 3 there exists a neighbourhood $U(x)$ and smooth anticommuting almost Hermitian structures $\tilde{J}_j$ (for the $j$'s such that $\lambda_0 + 3\eta_j = \mu_i$) on $U(x)$ such that $\text{Span}_{\lambda_0+3\eta_j=\mu_i}(J_iX) = \text{Span}_{\lambda_0+3\eta_j=\mu_i}(\tilde{J}_iX)$. Let $W$ be the algebraic curvature tensor on $U = \cap_{x \in U(x)} W(x)$ with the Clifford structure $\text{Cliff}(\nu; \tilde{J}_1, \ldots, \tilde{J}_n; \lambda_0, \eta_1, \ldots, \eta_n)$. Then $\nu = n - 1 - m_0$ is constant and all the $\tilde{J}_i, \eta_i$ and $\lambda_0$ are smooth on $U$. Moreover, for every unit vector field $X$ on $U$, the Jacobi operators $\tilde{W}_X$ and $W_X$ have the same eigenvalues and eigenvectors by construction, hence $W_X = W_X$, which implies $W = W$.

Now consider the case $n = 8$. By Lemma 2 at every point $x \in M'$, the Weyl tensor either has a Cliff(3)-structure, with $J_1 J_2 = J_3$, or a Cliff(7)-structure (but not both). As on every connected component $M_\alpha$ of $M'$, the number and the multiplicities of the eigenvalues of the operator $W_{X|X^+}$, $X \neq 0$, are constant, it follows from Remark 1 that when $M_\alpha$ may potentially contain the points of the both kinds is when one of the eigenvalues of $W_{X|X^+}$, $X \neq 0$, on $M_\alpha$ has multiplicity 4 and the Clifford structure at every point $x \in M_\alpha$ is either $\text{Cliff}(3; J_1, J_2, J_3; \lambda_0, \eta_1, \eta_2, \eta_3)$ with $J_1 J_2 = J_3$, or $\text{Cliff}(7; J_1, \ldots, J_l; \lambda_0 - 3\eta_i = \xi_0, \eta_0 + \xi, \eta_0 + \xi, \eta_0 + \xi, \eta_0 + \xi, \eta_0 + \xi, \xi, \xi)$, where $\eta_1, \eta_2, \eta_3 \neq 0$ (some of them can be equal) and $\xi \neq -\eta_i, 0$. The eigenvalues of $W_{X|X^+}$, $\|X\| = 1$, at every point $x \in M_\alpha$ are $\lambda_0$, of multiplicity 4, and $\lambda_0 + 3\eta_i$. Let $X$ be an arbitrary nowhere vanishing smooth vector field on a neighbourhood $U \subset M_\alpha$ of a point $x \in M_\alpha$. Then the four-dimensional eigendistribution of the operator $W_{X|X^+}$ corresponding to the eigenvalue of multiplicity 4 is smooth, therefore its orthogonal complement, the distribution $\text{Span}(J_1 X, J_2 X, J_3 X)$ is also smooth. By assertion 3 of Lemma 3 there exist smooth anticommuting almost Hermitian structures $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$ on (a possibly smaller) neighbourhood $U$ such that $\text{Span}(\tilde{J}_1 X, \tilde{J}_2 X, \tilde{J}_3 X) = \text{Span}(J_1 X, J_2 X, J_3 X)$. By assertion 1 of Lemma 3 every $\tilde{J}_i$ is a linear combination of the $J_j$'s: $\tilde{J}_i = \sum_{j=1}^3 a_{ij}J_j$, and moreover, the matrix $(a_{ij})$ must be orthogonal, as the $\tilde{J}_i$'s are anticommuting almost Hermitian structures. It follows that $\tilde{J}_1 \tilde{J}_2 \tilde{J}_3 = \pm J_1 J_2 J_3$. The operator on the left-hand side is smooth on $U$, the one on the right-hand side is $\pm id_{R^8}$, at the points where the Clifford structure is Cliff(3) with $J_1 J_2 = J_3$, and is symmetric with trace zero, at the points where the Clifford structure is Cliff(7) (which follows from the identity $J_4(J_1 J_2 J_3) J_4 = J_1 J_2 J_3$). Therefore all the point of $U$ either have a Cliff(3)-structure with $J_1 J_2 = J_3$, or a Cliff(7)-structure. In the both cases, the Clifford structure for $W$ can be taken smooth: in the first case, we follow the arguments as in the first part of the proof, as $\nu < n - 1 - \nu$; in the second one, we apply assertion 3 of Lemma 3 to every eigendistribution of $W_{X|X^+}$.

Thus for any $x \in M'$, the Weyl tensor on a neighbourhood $U = U(x)$ has the form (11), with a constant $\nu$ and smooth $\lambda_0, \eta_i$ and $J_i$. Then the curvature tensor has the form (7), with the operator $\rho$ given by $\rho = \frac{1}{n-2} \text{Ric} + \left( \frac{\nu}{n-2} \lambda_0 - \frac{n+2}{n-2} \eta_i J_i \right) id$, where $\text{Ric}$ is the Ricci operator and $\text{scal}$ is the scalar curvature. As $\lambda_0$ is a smooth function, the operator field $\rho$ is also smooth. 

Remark 2. In effect, the proof shows that if an algebraic curvature tensor $R$ field has a Clifford structure at every point of a Riemannian manifold, (and $\nu \neq 8$ when $n = 16$) then it has a Clifford structure of the same class of differentiability as $R$ on a neighbourhood of every generic point of the manifold.

Remark 3. As it follows from assertion 1 of Lemma 2 (in fact, from equation (4)), in the case $n = 8$, $\nu = 7$ we can replace in (7) $\rho$ by $\rho - \frac{3}{4} f id$ and $\eta_i$ by $\eta_i + f$, without changing $R$, where $f$ is an arbitrary smooth function on $U$ (if we want the resulting Clifford structure to be Cliff(7), we additionally require that $\eta_i + f$ is nowhere zero).

Let $x \in M'$ and let $U = U(x)$ be the neighbourhood of $x$ defined in Lemma 3. By the second Bianchi identity, $\left( \nabla_U R \right)(X, Y)Y + \left( \nabla_Y R \right)(U, X)Y + \left( \nabla_X R \right)(Y, U)Y = 0$. Substituting $R$ from (7) and using the fact that the operators $J_i$'s and their covariant derivatives are skew-symmetric and the operator $\rho$ and
its covariant derivatives are symmetric we get:
\[
\langle X, Y \rangle (\langle \nabla_U \rho \rangle Y - (\nabla_Y \rho) U) + \| Y \|^2 (\langle \nabla_X \rho \rangle U - (\nabla_U \rho) X) + \langle U, Y \rangle (\langle \nabla_Y \rho \rangle X - (\nabla_X \rho) Y) + \langle (\nabla_Y \rho) U - (\nabla_U \rho) Y, Y \rangle X + \langle (\nabla_X \rho) Y - (\nabla_Y \rho) X, Y \rangle U + \langle (\nabla_U \rho) X - (\nabla_X \rho) U, Y \rangle Y + \sum_{i=1}^\nu 3 \langle X(\eta_i) \rangle (J_i, Y, U) - U(\eta_i) (J_i, J_i, Y) Y \\
(8) + \sum_{i=1}^\nu Y(\eta_i) (2 \langle J_i, U, X \rangle J_i Y + \langle J_i, Y, X \rangle J_i U - \langle J_i, Y, U \rangle J_i X) + \sum_{i=1}^\nu \eta_i (3 \langle \langle \nabla_U J_i \rangle, X, Y \rangle + 3 \langle \langle \nabla_X J_i \rangle, Y, U \rangle + 2 \langle \langle \nabla_Y J_i \rangle U, X \rangle) J_i Y + 3 \langle J_i, X, Y \rangle (\nabla_U J_i) Y + 3 \langle J_i, Y, U \rangle (\nabla_X J_i) Y + 2 \langle J_i, U, X \rangle (\nabla_Y J_i) Y + \langle (\nabla_Y J_i) Y, X \rangle J_i U + (J_i, Y, X) (\nabla_Y J_i) U - \langle (\nabla_Y J_i) Y, U \rangle J_i X - (J_i, Y, U) (\nabla_Y J_i) X) = 0.
\]
Taking the inner product of \( \Box \) with \( X \) and assuming \( X, Y \) and \( U \) to be orthogonal we obtain
\[
\| X \|^2 \langle Q(Y), U \rangle + \| Y \|^2 \langle Q(X), U \rangle + \sum_{i=1}^\nu 3 \langle X(\eta_i) \rangle (J_i, Y, U) - U(\eta_i) (J_i, J_i, Y) Y \\
(9) + \sum_{i=1}^\nu \eta_i (2 \langle \langle \nabla_U J_i \rangle, X, Y \rangle + \langle (\nabla_X J_i) Y, U \rangle + \langle (\nabla_Y J_i) U, X \rangle) J_i Y + \langle (\nabla_X J_i) Y, X \rangle J_i U - \langle J_i, Y, U \rangle (\nabla_Y J_i) Y) = 0,
\]
where \( Q : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the quadratic map defined by
\[
\langle Q(X), X \rangle = 0.
\]
Note that \( \langle Q(X), X \rangle = 0 \).

**Lemma 5.** In the assumptions of Lemma 4, let \( x \in M' \) and let \( \mathcal{U} \) be the corresponding neighbourhood of \( x \). Suppose that if \( n = 8 \), then \( \nu = 3 \) and \( J_1 J_2 = J_3 \) on \( \mathcal{U} \), and if \( n = 16 \), then \( \nu \leq 4 \). For every point \( y \in \mathcal{U} \), identify \( T_y M' \) with the Euclidean space \( \mathbb{R}^n \) via a linear isometry. Then

(i) there exist vectors \( m_i, b_{ij} \in \mathbb{R}^n \), \( i, j = 1, \ldots, \nu \), such that for all \( X, Y, U \in \mathbb{R}^n \), and all \( i = 1, \ldots, \nu \),

\[
(11a) \quad Q(Y) = 3 \sum_{k=1}^\nu \langle m_k, Y \rangle J_k Y, \\
(11b) \quad \langle \nabla_X J_i \rangle X = \eta_i^{-1} (\| X \|^2 m_i - \langle m_i, X \rangle X) + \sum_{j=1}^\nu \langle b_{ij}, X \rangle J_j X, \\
(11c) \quad b_{ij} + b_{ji} = \eta_i^{-1} J_j m_i + \eta_j^{-1} J_i m_j, \\
(11d) \quad \nabla \eta_i = 2 J_i m_i, \\
(11e) \quad \sum_{j \neq i} \langle \eta_i b_{ij} + \eta_j b_{ji}, J_i J_j Y \rangle J_i J_j Y + \langle \eta_i b_{ij} + \eta_j b_{ji}, Y \rangle J_i J_j Y = 0.
\]

(ii) the following equations hold:

\[
(12a) \quad (\nabla_Y \rho) U - (\nabla_U \rho) Y = \sum_{i=1}^\nu \langle 2 \langle J_i, Y, U \rangle m_i - \langle m_i, Y \rangle J_i U + \langle m_i, U \rangle J_i Y, \\
(12b) \quad b_{ij} (3 - \eta_i \eta_j^{-1}) + b_{ji} (3 - \eta_j \eta_i^{-1}) = 0, \quad i \neq j, \\
(12c) \quad J_i m_i = \eta_i p, \quad i = 1, \ldots, \nu, \quad \text{for some } p \in \mathbb{R}^n.
\]

**Proof.** (i) We split the proof of this assertions into the two cases: the **exceptional case**, when either \( n = 6, \nu = 1 \), or \( n = 12, \nu = 3 \), \( J_1 J_2 = \pm J_3 \), or \( n = 8, \nu = 3 \), \( J_1 J_2 = J_3 \), and the **generic case**: all the other Clifford structures considered in the lemma.

**Generic case.** From (9) we obtain
\[
(13) \quad \| X \|^2 \langle Q(Y), U \rangle + \| Y \|^2 \langle Q(Y), U \rangle = 0, \quad \text{for all } X \perp T Y, Y \perp T U, X, Y, U \neq 0.
\]
We want to show that \( \langle Q(X), U \rangle = 0 \), for all \( X \perp T U \). This is immediate when \( n > 3 \nu + 3 \). Indeed, for any \( U \neq 0 \) and any unit \( X \perp T U \), \( \text{codim}(T U + T X) > \nu + 1 \), so we can choose unit vectors
Y_1, Y_2 \perp TU+IX such that Y_1 \perp TY_2. Then (13) implies that \langle Q(X), U \rangle = -\langle Q(Y_1), U \rangle = \langle Q(Y_2), U \rangle = -\langle Q(X), U \rangle.

Consider the case \( n \leq 3\nu + 3 \). By assertion (i) of Lemma 11 this could only happen when \( n = 12 \), \( \nu = 3 \) or \( n = 24 \), \( \nu = 7 \) (for the pairs \((n, \nu)\) belonging to the generic case), and in both the cases \( n = 3\nu + 3 \). Choose and fix an arbitrary \( U \neq 0 \) and consider the quadratic form \( q(X) = \langle Q(X), U \rangle \) defined on the \((2\nu+2)\)-dimensional space \( L = (TU)^\perp \). Assume \( q \neq 0 \). By (13), the restriction of \( q \) to the unit sphere of \( L \) is not a constant, so it attains its maximum (respectively minimum) on a great sphere \( S_1 \) (respectively \( S_2 \)). The subspaces \( L_1 \) and \( L_2 \) defined by \( S_1 \) and \( S_2 \) are orthogonal. Moreover, by (13), \( L_2 \subseteq (IX)^\perp \cap L \), for any nonzero \( X \in L_1 \), which implies that \( \dim L_2 \geq \nu + 1 \). Similarly \( \dim L_1 \geq \nu + 1 \), so, let \( L_1 \subseteq L_2, \dim L_1 = \dim L_2 = \nu + 1 \), and \( L = L_1 \oplus L_2 \). It follows that for some \( c > 0 \), \( q(X) = c(\|\pi_1 X\|^2 - \|\pi_2 X\|^2) \), where \( \pi_1 : L \to L_1 \) is the orthogonal projection. Moreover, \( L_2 = (IX)^\perp \cap L \), for all nonzero \( X \in L_1 \), which means that the subspace \( L_1 = L_2^\perp \cap L \) and similarly \( L_2 \) is \( \pi T \)-invariant, where \( \pi : \mathbb{R}^n \to L \) is the orthogonal projection, and even more: \( \pi T X = L_{\alpha}, \) for every nonzero \( X \in L_{\alpha}, \alpha = 1, 2, \) by the dimension count.

Let \( X = X_1 + X_2, Y = Y_1 + Y_2 \in L, \) where \( X_\alpha = \pi_\alpha X, Y_\alpha = \pi_\alpha Y. \) The condition \( Y \perp IX \) is equivalent to \( (X_1, Y_1) + (X_2, Y_2) = (\pi J, X_1, Y_1) + (\pi J, X_2, Y_2) = 0, \) for all \( i = 1, \ldots, \nu. \) Take arbitrary orthonormal bases for \( L_1 \) and \( L_2 \) and denote \( M_{\alpha}(X_\alpha), \alpha = 1, 2, \) the \((\nu + 1) \times (\nu + 1)-\)matrix whose columns relative to the chosen basis for \( L_{\alpha} \) are \( X_{\alpha}, \pi_1 J X_1, \ldots, \pi_1 J X_{\nu}. \) Then \( Y \perp IX \) if and only if \( M_1(X_1) Y_1 = -M_2(X_2) Y_2. \) Since for \( \alpha = 1, 2, \) and any nonzero \( X_\alpha \in L_{\alpha}, \) the columns of \( M_\alpha(X_\alpha) \) span \( L_{\alpha}, \) we obtain \( Y_2 = -(M_2(X_2))^{-1} M_1(X_1) Y_1, \) for any \( X_2 \neq 0. \) Then, as \( q(X) = c(\|X_1\|^2 - \|X_2\|^2), \) \( q(Y) = c(\|Y_1\|^2 - \|Y_2\|^2), \) equation (13) implies \( \|Y_1\|^2 \|X_2\|^2 - \|X_1\|^2 \|Y_2\|^2 = 0, \) so \( \|Y_1\|^2 \|X_2\|^2 = \|X_1\|^2 \|Y_2\|^2 = 0, \) for any \( X_1, Y_1 \in L_1 \) and any nonzero \( X_2 \in L_2. \) It follows that \( \|X_1\|^2 = (M_1(X_1))^{-1} = \|X_2\|^2 (M_2(X_2))^{-1}, \) for any nonzero \( X_\alpha \in L_\alpha. \) Thus for some positive definite symmetric \((\nu + 1) \times (\nu + 1)-\)matrix \( T \), we have \( M_{\alpha}(X_\alpha)^T = T_{\alpha} = \|X_\alpha\|^2 T, \) for all \( X_\alpha \in L_{\alpha}, \) \( \alpha = 1, 2. \) Then for any \( X = X_1 + X_2 \in L, X_\alpha \in L_{\alpha}, \) and any \( i = 1, \ldots, \nu, \) \( \|X_i^\| \|X\|^2 = \|\pi_1 J X_i \|^2 \|\pi_1 J X_i \|^2 = \|\pi_1 J X_i \|^2 \|\pi_1 J X_i \|^2 = \|\pi_1 J X_i \|^2 \|\pi_1 J X_i \|^2 = T_{\alpha} = \|X_{\alpha} \|^2 T_{\alpha} = \|X_\alpha \|^2. \) On the other hand, for any \( X \in L, \pi J X = X_i = X_i - \|U\|^{-2} \sum_{j=1}^\nu (J_i X_i J_i X_i J_i X_i) X_i = T_{\alpha} - \|X_\alpha \|^2 T_{\alpha}. \) This implies \( T_{\alpha} = 1, \) so \( X \perp J_i X, \) for all \( i = 1, \ldots, \nu \) and all \( X \in L = (TU)^\perp \). Therefore \( J_i X, J_i U \in TU, \) for all \( i = 1, \ldots, \nu \) and \( X \in \mathbb{R}^n \) for which the quadratic form \( q(X) = \langle Q(X), U \rangle \) defined on \((TU)^\perp \) is nonzero. If this is true for at least one \( U \), then this is true for a dense subset of \( \mathbb{R}^n \), which implies that \( J_i X, J_i U \in TU, \) for all \( i = 1, \ldots, \nu \) and all \( U \in \mathbb{R}^n \).

Then by assertion 1 of Lemma 3 for \( i \neq j \), \( J_i J_i U = \sum_{k=1}^\nu a_{ijk} J_k U \), for some constants \( a_{ijk} \), which implies that \( J_k J_k U = a_{ijk} \|U\|^2 \), so for all the triples of pairwise distinct \( i, j, k \), the symmetric operator \( J_k J_k \) on \( \mathbb{R}^n \) is a multiple of the identity. This is impossible when \( \nu > 3 \) (as for \( i \neq j, k \), the operator \( J_k J_k \) must be orthogonal and symmetric). The only remaining cases are \( n = 12, \nu = 3, \) with \( J_2 J_3 J_2 = \pm id, \) and \( n = 6, \nu = 1, \) which are considered under the exceptional case below.

Thus \( Q(X), U \neq 0, \) for \( X \perp TU, \) so \( Q(X) \in TU, \) for all \( X \in \mathbb{R}^n. \) By assertion 1 of Lemma 3 (and the fact that \( \langle Q(X), X \rangle = 0 \)), this implies equation (11a), with some vectors \( m_i \in \mathbb{R}^n. \)

To prove (11b) and (11c), we first show that for an arbitrary \( X \neq 0 \), there is a dense subset of the \( Y \)'s in \((IX)^\perp \) such that \( JF \cap JY = 0 \). This follows from the dimension count (compare to [11, Lemma 3.2 (1)]). For \( X \neq 0 \), define the cone \( CX = \{ J_i X, u : u, v \in \mathbb{R}^n \} \) (see (2)). As \( \dim CX \leq 2\nu - 1 - n - (\nu + 1) = \dim (IX)^\perp \) (the inequality in the middle follows from assertion (i) of Lemma 1), the complement to \( CX \) is dense in \((IX)^\perp \). This complement is the required subset, as the condition \( Y \notin CX \) is equivalent to \( JF \cap JY = 0 \). Substituting such \( X, Y \) into (9) we obtain by (11a):

\[ \sum_{i=1}^\nu \|X\|^2(m_i, X) - \eta_i \langle (\nabla X J_i X, Y) \rangle J_i Y + \sum_{i=1}^\nu \|Y\|^2(m_i, X) - \eta_i \langle (\nabla Y J_i Y, X) \rangle J_i X = 0. \]

As \( JF \cap JY = 0 \), all the coefficients vanish, so \( \|X\|^2(m_i, X) - \eta_i \langle (\nabla X J_i X, Y) \rangle = 0, \) for all \( X \in \mathbb{R}^n, \) all \( i = 1, \ldots, \nu, \) and all \( Y \) from a dense subset of \((IX)^\perp \), which implies that \( \langle (\nabla X J_i X - \eta_i \|X\|^2 m_i) X, Y \rangle \) for all \( X \in \mathbb{R}^n. \) Equation (11b) then follows from assertion 1 of Lemma 3. Equation (11c) follows from (11b) and the fact that \( \langle (\nabla X J_i X, J_i X), J_i X, Y \rangle + \langle (\nabla Y J_i Y, J_i Y), X, J_i X \rangle = 0. \)
To prove (11d) and (11e), substitute \( X = J_k Y, U \perp X, Y \) into (9). Consider the first term in the second summation. As \( \langle J_i Y, X \rangle = \|Y\|^2 \delta_{i,k} \), that term equals \( 3 \eta_k (2 \langle (\nabla U J_k) X, Y \rangle + \langle (\nabla X J_k) Y, U \rangle + \langle (\nabla Y J_k) U, X \rangle) \). As \( J_k \) is orthogonal and skew-symmetric, \( (\nabla U J_k) X, Y \rangle = (\nabla U J_k Y, J_k Y) = 0 \). Next, \( (\nabla Y J_k) U, X \rangle = -(\nabla Y J_k) J_k Y, U \rangle = (J_k (\nabla Y J_k) Y, U \rangle = \langle \eta_k^2 \|Y\|^2 J_k m_k + \sum_{j=1}^n b_{kj} J_k Y, J_k Y, U \rangle \) by (11b). Similarly, as \( Y = -J_k X \), it follows from (11b) that \( (\nabla X J_k) Y, U \rangle = (J_k (\nabla X J_k) Y, U \rangle = (J_k (\nabla X J_k) X, U \rangle = (J_k (\eta_k^2 \|X\|^2 - m_k, X) + \sum_{j=1}^n b_{kj} X, J_k Y, U \rangle = \langle \eta_k^2 \|Y\|^2 J_k m_k + \sum_{j \neq k} b_{kj} J_k J_k Y - b_{kj}, J_k Y, J_k Y \rangle \). Substituting this into (9) and using (11a) and (11b) we obtain after simplification:

\[
(14) \quad \|Y\|^2 (2 J_k m_k - J_k U(\eta_k)) + \sum_{j=1}^n \langle \eta_k b_{kj} + \eta_j b_{jk}, (J_k Y, J_k Y) J_k Y + (J_k Y, J_k Y) U \rangle = 0.
\]

By Lemma 3.2(3), for all \( U \in \mathbb{R}^n \), we can find a nonzero \( Y \) such that \( U \perp J Y + J J Y \). Substituting such a \( Y \) into (14) proves (11d). Then (14) simplifies to (11e).

Exceptional case (either \( n = 6, \nu = 1 \), or \( n = 12, \nu = 3, J_1 J_2 = \pm J_3, \) or \( n = 8, \nu = 3, J_1 J_2 = J_3 \)).

In all these cases, the Clifford structure has the following "\( J^2 \)-property"; for every \( X \in \mathbb{R}^n \), \( J J X = I J X = I X \). In particular, if \( Y \perp I X \), then \( J Y \perp I X \).

Substitute \( X = J_k U \) and \( Y \perp I X = I U \) to (8) and take the inner product of the resulting equation with \( J_k Y \). Using the fact that \( (\nabla Y J_k) U, J_k U \rangle = (\nabla Y J_k) Y, J_k Y \rangle = 0 \) and the \( J^2 \)-property we get

\[
-J_k ((\nabla U J_k) U - (\nabla U J_k) J_k U) + 2 \|U\|^2 \eta_k + 3 \eta_k (\nabla U J_k) J_k U - (\nabla U J_k) J_k U \in I U.
\]

The expression \( F(U) \) on the left-hand side is a quadratic map on \( \mathbb{R}^n \) to itself. By assertion 1 of Lemma 3, \( F(U) \) is a linear combination of \( U, J_1 U, \ldots, J_n U \) whose coefficients are linear forms of \( U \).

In particular, the cubic polynomial \( F(U) \), \( J_k U \) must be divisible by \( \|U\|^2 \). As \( J_k \) is orthogonal and skew-symmetric, \( (\nabla U J_k) U - (\nabla U J_k) J_k U, J_k U \rangle = 0 \), so there exists a vector \( m_k \in \mathbb{R}^n \) such that \( (\nabla U J_k) U - (\nabla U J_k) J_k U, J_k U \rangle = -3 \|U\|^2 (m_k, U) \). It follows that the quadratic map \( Q \) defined by (10) satisfies \( Q(U), J_k U \rangle = 3 \|U\|^2 (m_k, U) \), for all \( U \in \mathbb{R}^n \) and all \( k = 1, \ldots, \nu \). As \( \langle Q(U), U \rangle = 0 \), we can define a quadratic map \( T : \mathbb{R}^n \to \mathbb{R}^n \) such that for all \( U \in \mathbb{R}^n \),

\[
(15) \quad Q(U) = T(U) + 3 \sum_{k=1}^\nu \langle m_k, U \rangle J_k U, \quad T(U) \perp I U.
\]

Taking \( U = J_k X, \ X, U \perp I Y \) in (9) and using (11) we obtain

\[
-J_k ((\nabla U J_k) U - (\nabla U J_k) J_k U) + 2 \|U\|^2 \eta_k + 3 \eta_k (\nabla U J_k) J_k U - (\nabla U J_k) J_k U \in I U.
\]

From assertion 1 of Lemma 3, it follows that the expression on the left-hand side is a linear combination of \( Y, J_1 Y, \ldots, J_n Y \) whose coefficients are linear forms of \( Y \), so for some vectors \( b_{ij} \in \mathbb{R}^n \),

\[
(16) \quad (\nabla Y J_i J_j) Y = \eta_k^{-1} \|m_i\|^2 - \langle m_i, Y \rangle \rangle - (3 \eta_k^{-1} J_i T(Y) + \sum_{j=1}^n \langle b_{ij}, J_j Y \rangle J_j Y.
\]

As \( \langle \nabla Y J_i J_j Y, J_j Y \rangle \) is antisymmetric in \( i \) and \( j \) and \( J_i T(Y) \perp I Y \) by (11) and the \( J^2 \)-property, the \( b_{ij} \)'s satisfy (11c).

Take \( X = J_k Y, U \perp I Y \in I X \) in (9). As \( (\nabla U J_k) J_k Y, Y \rangle = 0, \ (\nabla Y J_k) U, X \rangle = -(\nabla Y J_k) J_k Y, U \rangle = (J_k (\nabla Y J_k) Y, U \rangle, \ and similarly, \ (\nabla X J_k) Y, U \rangle = -(\nabla X J_k) J_k X, U \rangle = (J_k (\nabla X J_k) X, U \rangle, \ we obtain from (15) after simplification that

\[
(17) \quad 2 T(Y) + 2T(J_k Y - 3 \|Y\|^2 (\nabla \eta_k - 2 J k m_k) \in I Y.
\]

In the case \( n = 6, \nu = 1 \), we can prove the remaining identities (11a) (11b) (11d) (11e) of assertion (i) as follows. Taking in (9) nonzero \( X, Y, U \) such that the subspaces \( I X, I Y \) and \( I U \) are mutually orthogonal we obtain by (13) \( \|X\|^2 (T(X), U) + \|Y\|^2 (T(Y), U) = 0 \) (which is, essentially, (13)). Replacing \( Y \) by \( J_1 Y \) and using (17) we get \( 2T(X) + 3 \|X\|^2 (\nabla \eta_1 - 2 J_1 m_1) \in I X \). The same is true with \( X \) replaced by \( J_1 X \). Then by (17), \( \nabla \eta_1 - 2 J_1 m_1 \in I X \), for all \( X \in \mathbb{R}^n \), so \( \nabla \eta_1 - 2 J_1 m_1 = 0 \) (which is (11d)). Then \( T(X) \in I X \), hence \( T(X) = 0 \), as \( T(X) \perp I X \) by (15). Now (11a) follows from (15), (11b) follows from (16), and (11c) is trivially satisfied, as \( \nu = 1 \).

In the cases \( n = 8, 12, \nu = 3, J_1 J_2 = J_3 \) (if \( J_1 J_2 = -J_3 \), we replace \( J_3 \) by \( -J_3 \), without changing the curvature tensor \( \tau \)), we argue as follows. Adding equation (17) with \( k = 1 \) and with \( k = 2 \) and then subtracting (17) with \( k = 3 \) and \( Y \) replaced by \( J_1 Y \) we get \( 4T(Y) \in 3 \|Y\|^2 (\nabla \eta_1 - 2 J_1 m_1) + (\nabla \eta_1 - 2 J_2 m_2) - (\nabla \eta_1 - 2 J_3 m_3) \in I Y \). This remains true under a cyclic permutation of the indices \( 1, 2, 3 \), which implies \( (\nabla \eta_k - 2 J_k m_k) \in I Y \), for all \( i, k = 1, 2, 3 \) and all \( Y \in \mathbb{R}^n \). Then
∇η_k - 2J_km_k = ∇η_i - 2J_i m_i = \frac{4}{3} V, for some vector V ∈ ℝ^n, and T(Y) - ||Y||^2 V ∈ IY from the above. As T(Y) ∥ IY by (15), we obtain T(Y) = ||Y||^2 V - ⟨Y, V⟩Y - \sum_{j=1}^{3}(J_j Y, V)J_j Y, so

\[
\nabla \eta_i = 2J_i m_i + \frac{4}{3} V, \quad Q(Y) = ||Y||^2 V - ⟨Y, V⟩Y + \sum_{j=1}^{3}(3m_j + J_j Y, V)J_j Y,
\]

(18)

(∇Y · Jj)Y = (3ηj)^{-1} [(||Y||^2)(3m_i - J_i V) - (3m_i - J_i V, Y)Y + \sum_{j=1}^{3}(3ηjb_{ij} - J_i J_j Y, V)J_j Y]

(the second equation follows from (15); the third one, from (16) and the fact that J1J2 = J3).

Substitute X = JkY into (5) again, with an arbitrary U ∥ X, Y. Using (18) and the fact that the J_i’s are skew-symmetric, orthogonal and anticommuting, we obtain after simplification:

\[
\sum_{i=1}^{3}(3a_{ik} - 2J_i J_k V, J_i J_k Y)J_i J_k Y + \sum_{i=1}^{3}(3a_{ik} - 2J_i J_k V, Y)J_k J_i Y ∈ \text{Span}(Y, J_i Y),
\]

where a_{ik} = η_i b_{ki} + η_k b_{ik}. Taking k = 1 and using the fact that J_1J_2 = J_3 we get from the coefficient of J_2J : 3J_1a_{12} - 4J_2V + 3a_{12} = 0, so 4V = -3J_2a_{12} + 3J_2a_{12}. Cyclically permuting the indices 1, 2, 3 and using the fact that a_{ik} = a_{ki} we obtain V = 0, which implies (11c). As V = 0, equations (11a) (11b) (11b) follow from (18).

(ii) By (10) and (11a), \((∇X ρ)U - (∇U ρ)X, X) = 3 \sum_{i=1}^{n}(m_i, X)(J_i X, U), for all X, U ∈ ℝ^n. Polaring this equation and using the fact that the covariant derivative of ρ is symmetric we obtain \(⟨(∇X ρ)U, Y⟩ + (⟨∇U ρ)(Y, X), X) - 2(⟨(∇U ρ)(Y, X), Y⟩ = 3 \sum_{i=1}^{n}(m_i, Y)(J_i X, U) + (⟨m_i, X⟩)(J_i X, U)⟩ + (⟨m_i, X⟩)(J_i X, U)⟩ + (⟨m_i, X⟩)(J_i X, U)⟩), which proves (12b).

To establish (12c), substitute X \perp IY, U = JkY into (8). Using the equations of assertion (i) and (12a) we obtain after simplification:

\[
3(∇X · Jk)Y - (∇Y · Jk)X = -3η_k^{-1}(m_k, Y)X + \sum_{i=1}^{n}(η_i b_{ki} + 2δ_{ik} J_k m_k, Y)J_i X \mod(IY).
\]

Subtracting three times polarized equation (11b) (with i = k) and solving for \((∇Y · Jk)X\) we get

\[
(∇Y · Jk)X = \sum_{i=1}^{n}\frac{1}{2} η_k^{-1}(3η_i b_{ki} - η_i b_{ki} - 2δ_{ik} J_k m_k, Y)J_i X \mod(IY),
\]

for all X ∥ IY. Choose s ≠ k and define the subset S_{ks} ⊂ ℝ^n ∥ ℝ^n by S_{ks} = \{(X, Y) : X, Y ≠ 0, X, J_k X, J_i X ∥ J_i Y\}. It is easy to see that \((X, Y) ∈ S_{ks} ↔ (Y, X) ∈ S_{ks}\) and that replacing J_i Y by IY in the definition of S_{ks} gives the same set S_{ks}. Moreover, the set \{X : (X, Y) ∈ S_{ks}\} (and hence the set \{Y : (X, Y) ∈ S_{ks}\}) spans ℝ^n. If n = 8, ν = 3, J_1J_2 = J_3, this easily follows from the J^2-property; in all the other cases, from (N1) Lemma 2.4 (4)]. For (X, Y) ∈ S_{ks}, take the inner product of (10) with J_k Y. Since \((∇Y · Jk)X, J_i X\) is antisymmetric in k and s, we get \((3 - η_k η_s^{-1})b_{ks} + (3 - η_s η_k^{-1})b_{sk}, Y) = 0\), for a set of the Y’s spanning ℝ^n. This proves (12b).

To prove (12c), we apply assertion 2 of Lemma (6) to equation (11c). If ν = 1, there is nothing to prove (in fact, if ν = 1 and n ≥ 8, the claim of Theorem 1 follows from (BG1) Theorem 1.1). If η_i b_{ij} + η_j b_{ij} = 0 for all i ≠ j, then (12b), b_{ij} + b_{ji} = 0 for all i ≠ j, so by (11c), η_i^{-1} J_i m_i = -η_j^{-1} J_j m_j. Acting by J_i J_j we obtain that the vector η_i^{-1} J_i m_i is the same, for all i = 1, …, ν.

The only remaining possibility is ν = 3, J_1J_2 = J_3 (if J_1J_2 = J_3 we can replace J_3 by J_3 without changing the curvature tensor (7)), and η_k b_{ki} + η_k b_{ki} = J_3 v, for all the triples \{i, j, k\} = \{1, 2, 3\}, where v ≠ 0. We will show that this leads to a contradiction. Note that by (3), the existence of a Cliff(3)-structure implies that n is divisible by 4, so by the assumption of the lemma, n ≥ 8.

If η_i = η_k for some i ≠ k, then from (12c) and η_k b_{ki} + η_k b_{ki} = J_3 v it follows that v = 0, a contradiction. Otherwise, if the η_i’s are pairwise distinct, we get b_{ik} = (3η_i - η_k)/(4η_i(η_i - η_k) J_3 v for \{i, j, k\} = \{1, 2, 3\}. Substituting this to (11c) and acting by J_1 on the both sides we get η_i^{-1} J_i m_i = η_i^{-1} J_k m_k = \frac{1}{4} ε_{ik}(η_i^{-1} + η_k^{-1}) v, for \{i, j, k\} = \{1, 2, 3\}, where for i ≠ k we define ε_{ik} = ±1 by J_1J_k = ε_{ik} J_j. It is easy to see that ε_{ik} = -ε_{jk} and ε_{jk} = ε_{ij}, where \{i, j, k\} = \{1, 2, 3\}. Then \sum_{i=1}^{3} η_i^{-1} = 0 and η_i^{-1} J_i m_i = \frac{1}{4} ε_{ik}(η_i^{-1} + η_k^{-1}) v + w, for some w ∈ ℝ^n. It then follows from (11c) that \nabla η_i = \frac{1}{4} ε_{jk} η_k(η_i^{-1} - η_k^{-1}) v + 2η_i w, which implies \nabla ln|η_i η_j| = 6w and \nabla ln|η_i^{-1}| = -\frac{1}{4} ε_{ik} η_i^{-1} v. Let U ⊂ U be a neighbourhood of x on which \nabla ln|η_i η_j| ≠ 0. Then v is a nowhere vanishing smooth vector field on U. Multiplying
the metric on $\mathcal{U}$ by a function $e^f$ changes neither the Weil tensor, nor the $J_i$'s, and multiplies every $\eta_i$ by $e^{-f}$ and $\nabla$ acting on functions by $e^{-f}$. Taking $f = \frac{1}{4} \ln|\eta_1\eta_2\eta_3|$ we can assume that $w = 0$ on $\mathcal{U}'$, so that $C = \eta_1\eta_2\eta_3$ is a constant. Then, as $\sum_{i=1}^3 \eta_i = 0$, we get $\nabla \eta_i = \pm \frac{1}{2} \sqrt{1 - 4C^{-1}\eta_i^2}$.

It follows that $v = \nabla t$ for some smooth function $t: \mathcal{U}' \to \mathbb{R}$ such that $\eta_i = -36C\rho(t + c_i)$, where $\rho$ is the Weierstrass function satisfying $(\frac{d}{dz} \rho(t))^2 = 4\rho(t)^3 + 6 - 4C^{-2}$ and $c_i \in \mathbb{R}$. Summarizing the identities of this paragraph, we have pointwise pairwise nonequal functions $\eta_i: \mathcal{U}' \to \mathbb{R} \setminus \{0\}$ satisfying

$$ m_i = -\frac{1}{12} \eta_j \eta_k (\eta_i^{-1} - \eta_j^{-1}) J_i J_j J_k, \quad b_{ij} = \frac{1}{12} \eta_j (\eta_i^{-1} - \eta_k^{-1}) v, \quad b_{ij} = (3\eta_i - \eta_j)(4\eta_i - \eta_j)(1 - 3\eta_i) \mathbf{J}_i J_j $$

for $i, j, k = \{1, 2, 3\}$, where we used (11c) to compute $b_{ij}$. Then equation (10) simplifies to $\left(\nabla Y J_k \right) X = \sum_{i \neq k} \frac{1}{2} (\eta_k - \eta_i)^{-1} (\langle J_i v, Y \rangle J_i X \mod (\mathcal{I}))$, for all $X \perp \mathcal{I} Y$. By the $J^2$-property, $\mathcal{I} Y \perp \mathcal{I} X$, so to find the “mod($\mathcal{I}$)”-part, we have to compute the inner products of $\left(\nabla Y J_k \right) X$ with $Y, J_i Y, J_i J_j X$, $J_i J_j J_k$. Since $\langle \nabla Y J_k X, Y \rangle = \langle \nabla Y J_k J_i Y, X \rangle = \langle \nabla Y J_i J_k X, Y \rangle = \langle \varepsilon(a_3 J_i J_k) Y, J_k \rangle J_i J_j Y, f \rangle$, for $j, k = \{1, 2, 3\}$, these products can be found using (11b). Simplifying by (20) and using the above equation we get after some calculations:

$$ (\nabla Y J_k X) = \frac{1}{12} \varepsilon_{ijk} (\eta_i^{-1} - \eta_j^{-1}) (\langle J_i v, X \rangle J_i Y + \langle v, X \rangle J_i Y) + \frac{1}{\eta_k} \sum_{i \neq k} (\langle J_i v, X \rangle J_i Y - \langle J_i Y, X \rangle J_i v - \langle X, Y \rangle J_i v) + \sum_{i \neq k} \frac{1}{4} (\eta_i - \eta_k)^{-1} (\langle J_i v, Y \rangle J_i X) $$

for all $X \perp \mathcal{I} Y$, where $i, j, k = \{1, 2, 3\}$. To compute $\left(\nabla Y J_k \right) X$ when $X \in \mathcal{I} Y$ we again use (11b) and the fact that $\left(\nabla Y J_k \right) J_i = -f_i (\nabla Y J_k)$ and $\left(\nabla Y J_k \right) J_i = \varepsilon_k \left(\nabla Y J_k \right) J_i - f_i (\nabla Y J_k J_i)$. Simplifying by (20) and using the above equation we get after some calculations:

$$ (\nabla Y J_k X) = \frac{1}{12} \varepsilon_{ijk} (\eta_i^{-1} - \eta_j^{-1}) (\langle J_i v, X \rangle Y + \langle v, X \rangle J_i Y - \langle X, J_i Y \rangle J_i v + \langle X, J_i Y \rangle J_i v) + \sum_{i \neq k} \frac{1}{4} (\eta_i - \eta_k)^{-1} (\langle J_i v, Y \rangle J_i X) $$

for all $X, Y \in \mathbb{R}^n$, where $i, j, k = \{1, 2, 3\}$. Let for $a, b \in \mathbb{R}^n$, $a \wedge b$ be the skew-symmetric operator defined by $(a \wedge b) X = \langle a, X \rangle b - \langle b, X \rangle a$. Then the above equation can be written in the form $\nabla Y J_k = \frac{1}{12} \varepsilon_{ijk} (\eta_i^{-1} - \eta_j^{-1}) (J_i v, Y + v, J_i v) + \frac{1}{4} \eta_k^{-1} \sum_{i \neq k} (J_i v \wedge J_i Y + J_i Y \wedge Y) + \sum_{i \neq k} \frac{1}{4} (\eta_k - \eta_i)^{-1} (J_i v, Y) J_i Y$, that is,

$$ \nabla Y J_k = \left( J_k, AY \right) $$

for $i, j, k = \{1, 2, 3\}$, where we used the fact that $\left[ J_k, a \wedge b \right] = J_k a \wedge b + a \wedge J_k b$ and $\left[ J_k, J_i \right] = 2\varepsilon_k J_i J_j$, for $i, j, k = \{1, 2, 3\}$. By the Ricci formula, $\nabla Y J_k = \nabla Z J_k - \nabla Z J_k X$, where the tensor field $\nabla Z$ is defined by $\nabla Y J_k = \nabla Z (\nabla Y J_k) - \nabla Z (J_k X)$ for vector fields $Y, Z$ on $\mathcal{U}'$. As $\nabla Y J_k = \left[ J_k, AY \right]$ by (21), this is equivalent to the fact that the operator $F(Y, Z) = (\nabla Y A) Y - (\nabla Z A) Z - [AY, AZ] - R(Y, Z)$ commutes with all the $J_i$'s, for all $Y, Z \in \mathbb{R}^n$ and all $i = 1, 2, 3$. As by (7), $R(Y, Z) = Y \wedge Z + Z \wedge Y + \sum_{i=1}^3 \eta_i (J_i Y \wedge J_i Z + 2 J_i J_i Y)$, we obtain using (21) and the identities $[a \wedge b, c \wedge d] = (a, d) c \wedge b - (a, c) d \wedge b$ and $[a, b] c \wedge a + [b, c] d \wedge a$, $[J_k, a \wedge b] = J_k a \wedge b + a \wedge J_k b$:
where \( \{i,j,k\} = \{1,2,3\} \), and \( H \) is the symmetric operator associated to the Hessian of the function \( t \) (that is, \( (HY,Z) = Y'(Zt) - (\nabla_Y)Zt \), for vector fields \( Y,Z \) on \( U' \)).

As \( [F(Y,Z),J_s] = 0 \) and the subspace \( IY + IZ \) is \( J \)-invariant (hence \( (IY + IZ) \wedge \mathbb{R}^n \) is ad\( J \)-invariant), it follows from (22) that for all \( Y, Z \in \mathbb{R}^n \) and all \( s = 1,2,3 \),

\[
[V(Y,Z),J_s] + \sum_{i=1}^{3} (K_iY,Z)[J_i,J_s] \in (IY + IZ) \wedge \mathbb{R}^n.
\]

(24)

Take \( Y, Z \in IY \) in (24). Then by the \( J \)-property, \( IY + IZ = IY \) and \( [V(Y,Z),J_s] \in IY \wedge IY \), so (24) simplifies to \( \sum_{i,j,s} \varepsilon_{ijs}(K_iY,Z)J_j \in IY \wedge \mathbb{R}^n \), where \( \{i,j,s\} = \{1,2,3\} \). Projecting this to the subspace \( (IY)^{\perp} \wedge (IY)^{\perp} \subset \mathfrak{o}(n) \) (with respect to the standard inner product on \( \mathfrak{o}(n) \)) and using the fact that \( (IY)^{\perp} \) is \( J \)-invariant and \( n \geq 8 \), we get \( (K_iY,Z) = 0 \), for all \( i = 1,2,3 \) and all \( Y,Z \in IY \). Introduce the operators \( \hat{J}_i = \pi_{IY} J_i \pi_{IY} \). As \( IY \) is \( J \)-invariant, the \( \hat{J}_i \)'s are anticommuting almost Hermitian octonion structures on \( IY \). Then the condition \( \langle K_iY,Z \rangle = 0 \), \( Y \in IY \), and (23) imply

\[
(4\omega_i + \lambda_i)v \wedge \hat{J}_i v + 4\varepsilon_{jk}(\omega_j + \omega_k)\hat{J}_j v \wedge \hat{J}_k v + \lambda_i(48C + \|v\|^2)\hat{J}_i + \hat{J}_i \hat{H} + \hat{H} \hat{J}_i = 0.
\]

Multiplying by \( \hat{J}_i \) and taking the trace we obtain \( 4\|v\|^2(\omega_i + \omega_j + \omega_k) + \lambda_i(96C + 3\|v\|^2) + \text{Tr} \hat{H} = 0 \), where \( \{i,j,k\} = \{1,2,3\} \), so \( \lambda_i(96C + 3\|v\|^2) \) does not depend on \( i = 1,2,3 \). As the \( \lambda_i \)'s are pairwise distinct (otherwise the condition \( \sum_{i=1}^{3} \eta_i = 0 \) from (20) is violated), we get \( \|v\|^2 = -32C \).

Now take \( Y, Z \perp IY \) in (24). Projecting to \( IY \wedge IY \) and using the fact that \( IY \wedge IY \) is ad\( J \)-invariant we obtain that the operator \( V(Y,Z) + \sum_{i=1}^{3} (K_iY,Z)\hat{J}_i \) on \( IY \) commutes with every \( \hat{J}_s \). The centralizer of the set \( \{\hat{J}_1,\hat{J}_2,\hat{J}_3\} \) in the Lie algebra \( \mathfrak{o}(4) = \mathfrak{o}(2\mathbb{R}) \) is the three-dimensional subalgebra spanned by \( v \wedge \hat{J}_i v - \varepsilon_{jk} \hat{J}_j v \wedge \hat{J}_k v \), \( (i,j,k) = \{1,2,3\} \) (“the right multiplication by the imaginary quaternions”). Substituting \( V(Y,Z) \) from (22) and using the fact that \( \hat{J}_i = \|v\|^2(v \wedge \hat{J}_i v + \varepsilon_{jk} \hat{J}_j v \wedge \hat{J}_k v) \) we obtain that the operator \( V(Y,Z) + \sum_{i=1}^{3} (K_iY,Z)\hat{J}_i \) commutes with all the \( \hat{J}_i \)'s, for \( Y, Z \perp IY \), if and only if \( -\frac{1}{2}(J_iY,Z)(\lambda_i^2 + \lambda_i^k - \lambda_i^k) - \lambda_i^k \lambda_i - \lambda_i^k \lambda_i + 2\|v\|^2(K_iY,Z) = 0 \), for all \( i = 1,2,3 \). Substituting the \( \lambda_i \)'s from (21) and \( (K_iY,Z) \) from (24) and taking into account that \( \|v\|^2 = -32C \), which is shown above, we obtain \( \langle J_iH + HJ_i - 32C\lambda_i\pi_{IY} J_iY,Z \rangle = 0 \), for all \( Y, Z \perp IY \) and all \( i = 1,2,3 \). Then \( \pi(J_iH + HJ_i)\pi = 32C\lambda_i\pi J_i\pi \), where \( \pi = \pi_{IY} \). Multiplying both sides by \( \pi J_i\pi \) from the right and using the fact that \( \{\pi,J_i\} = 0 \) (as \( IY \) is \( J \)-invariant) we get \( \pi(J_iHJ_i - H)\pi = -32C\lambda_i\pi \). Taking the traces of the both sides we obtain

\[
-2\text{Tr}(\pi H \pi) = -32C\lambda_i(n - 4),
\]

which is a contradiction, as \( n > 4 \) and the \( \lambda_i \)'s are pairwise distinct (which follows from the equation \( \sum_{i=1}^{3} \eta_i = 0 \) of (20)). 

The next lemma shows that the relations similar to (14) (22) of Lemma 5 also hold in all the remaining cases when \( n = 8 \) (that is, when \( \nu \neq 3 \) and when \( \nu = 3 \) and \( J_1J_2 \neq \pm J_3 \)). As it is shown in Lemma 4 in all these cases the Weyl tensor has a smooth \( \text{Cliff}(7) \)-structure in a neighbourhood \( U \) of every point \( x \in M' \). Moreover, by assertion 2 of Lemma 2 that \( \text{Cliff}(7) \)-structure is an “almost Hermitian octonion structure”, in the following sense. For every \( y \in U \), we can identify \( \mathbb{R}^8 = T_yM^8 \) with \( \mathbb{O} \) and of \( \mathbb{R}^7 \) with \( \mathbb{O}' = \mathbb{O}^1 \) via linear isometries \( t_1, t_2 \) respectively in such a way that the orthogonal multiplication \( \mathbb{O} \) defined by \( \text{Cliff}(7) \) has the form \( \{J_iX = Xu \mid X \in \mathbb{R}^8 = \mathbb{O} \}, \nu \in O' \).

**Lemma 6.** Let \( x \in M' \subset M^8 \) and let \( U \) be the neighbourhood of \( x \) defined in Lemma 4. For every point \( y \in U \), identify \( \mathbb{R}^8 = T_yM^8 \) with \( \mathbb{O} \) via a linear isometry in such a way that the Clifford structure \( \text{Cliff}(7) \) on \( \mathbb{R}^8 \) is given by (4). Then there exist \( m,t,b_{ij} \in \mathbb{R}^8 = \mathbb{O} \), \( i,j = 1,\ldots,7 \), such that for all \( X,U \in \mathbb{R}^8 = \mathbb{O} \),

\[
(25a) \quad (\nabla_U J_t)X = \sum_{j=1}^{7} (b_{ij},U)X e_j + (X(U^*m) - \langle m,U \rangle X)e_i + \langle m,U e_i \rangle X,
\]

\[
(25b) \quad b_{ij} + b_{ji} = 0,
\]

\[
(25c) \quad (\nabla_X \rho)U - (\nabla_U \rho)X = \frac{1}{4}(X \wedge U)t + 2 \sum_{j=1}^{7} \eta_j(\langle me_i, U \rangle X e_i - \langle me_i, X \rangle U e_i) + 2(X e_i, U)me_i,
\]

\[
(25d) \quad \nabla \eta_i = -4\eta_i m - \frac{1}{4}t.
\]

**Proof.** In the proof we use standard identities of the octonion arithmetic (some of them are given in Subsection 2.2).
By [N2, Lemma 7], for the Clifford structure \(\text{Cliff}(7)\) given by (5), there exist \(b_{ij} \in \mathbb{R}^8\), \(i, j = 1, \ldots, 7\), satisfying (25a) and an \((\mathbb{R}\text{-})\)linear operator \(A: \mathbb{O} \to \mathbb{O}'\) such that for all \(X, U \in \mathbb{R}^8 = \mathbb{O},\)

\[
(\nabla_U J_i)X = \sum_{j=1}^{7} (b_{ij}, U) X e_j + (X \cdot AU)e_i + \langle AU, e_i \rangle X.
\]

Equation (5) is a polynomial equation in 24 real variables, the coordinates of the vectors \(X, Y, U \in \mathbb{R}^8\). It still holds, if we allow \(X, Y, U\) to be complex and extend the tensors \(J_i, \nabla J_i\) and \(\langle \cdot, \cdot \rangle\) to \(\mathbb{C}^8\) by the complex linearity. The complexified inner product \(\langle \cdot, \cdot \rangle\) takes values in \(\mathbb{C}\) and is a nonsingular quadratic form on \(\mathbb{C}^8\). Moreover, equation (5) is still true, if we identify \(\mathbb{C}^8\) with the bioctonion algebra \(\mathbb{O} \oplus \mathbb{O}\), and \(\mathbb{C}^7\) with \(1^\perp = \mathbb{O}' \oplus \mathbb{O}\), the orthogonal complement to \(1\) in \(\mathbb{O} \oplus \mathbb{O}\).

Let \(Y \in \mathbb{O} \oplus \mathbb{O}\) be a nonzero isotropic vector (that is, \(Y^* Y = 0\)) and let \(J \mathbb{O} Y = \text{Span}_{\mathbb{C}}(J_i Y_1, \ldots, J_i Y_7)\). Then \(Y \in J \mathbb{O} Y\) and the space \(J \mathbb{O} Y\) is isotropic: the inner product of any two vectors from \(J \mathbb{O} Y\) vanishes. Choose \(X, U \in J \mathbb{O} Y\) and take the inner product of the complexified equation (5) with \(X\). As \(X, Y\) and \(U\) are mutually orthogonal, we get (4), which further simplifies to \(\sum_{i=1}^{7} \eta_i \langle J_i X, U \rangle \langle \nabla Y J_i Y, X \rangle = 0\), as \(\|X\|^2 = \|Y\|^2 = \langle J_i Y, X \rangle = \langle J_i Y, U \rangle = 0\). Using (20) we obtain \(\sum_{i=1}^{7} \eta_i \langle J_i X, U \rangle \langle Y \cdot AY e_i, X \rangle = 0\), for all isotropic vectors \(Y\) and for all \(X, U \in J \mathbb{O} Y\). It follows that \(Y \cdot AY = \sum_{i=1}^{7} \eta_i \langle J_i X, U \rangle X e_i\), for all \(X, U \in J \mathbb{O} Y\). As \(Y \cdot AY = J AY Y \in J \mathbb{O} Y\) and \(J \mathbb{O} Y\) is isotropic, we get \(Y \cdot AY \perp J \mathbb{O} Y\), so \(Y \cdot AY = \text{Span}_{\mathbb{C}}(\sum_{i=1}^{7} \eta_i \langle J_i X, U \rangle J_i X | X \in J \mathbb{O} Y\rangle).\) Following the arguments in the proof of [N2, Lemma 8] starting with formula (29), we obtain \(AU = U^* m - (U, m) 1\), for some \(m \in \mathbb{O}\).

Then equation (25a) follows from (26).

To prove (25c) and (25d), introduce the vectors \(f_{ij} \in \mathbb{R}^8\), \(i, j = 1, \ldots, 8\), and the quadratic map \(T: \mathbb{R}^8 \to \mathbb{R}^8\) (similar to the map \(Q\) of (10)) by

\[
f_{ij} = (\eta_i - \eta_j) b_{ij} + \delta_{ij}(\nabla \eta_i - 2\eta_i m),
\]

\[
\langle T(X), U \rangle = \frac{1}{2} \langle (\nabla X \rho) U - (\nabla U \rho) X, X \rangle - \sum_{i=1}^{7} \eta_i \langle me_i, X \rangle \langle X e_i, U \rangle.
\]

Note that \(f_{ij} = f_{ji}\) and \(\langle T(X), X \rangle = 0\). Take \(X, Y, U\) to be mutually orthogonal vectors in \(\mathbb{R}^8\). By (25a) and (25b), \(\langle (\nabla_U J_i)X, Y \rangle = \sum_{j=1}^{7} \langle b_{ij}, U \rangle \langle X e_j, Y \rangle - \langle m, U \rangle \langle X e_i, Y \rangle + \langle (X^* m) e_i, Y \rangle = \sum_{j=1}^{7} \langle b_{ij} - \delta_{ij} m, U \rangle \langle X e_j, Y \rangle + \langle m((e_i Y^*) X), U \rangle\), so every term on the left-hand side of (4) can be written as the inner product of a vector depending on \(X\) and \(Y\) by \(U\). As \(U \perp X, Y\) is arbitrary, we find after substituting (4) and (25a) into (4) and rearranging the terms:

\[
\|X\|^2 T(Y) + \|Y\|^2 T(X) + 2 \sum_{i=1}^{7} \eta_i \langle Y e_i, X \rangle \langle m((e_i Y^*) X) + (Y (X^* m)) e_i, e_i \rangle
\]

\[
+ \sum_{i,j=1}^{7} \langle Y e_j, X \rangle \langle f_{ij}, X e_i \rangle - \langle f_{ij}, Y \rangle X e_i \rangle - \sum_{i,j=1}^{7} \langle Y e_i, X \rangle \langle Y e_j, Y \rangle f_{ij} \in \text{Span}(X, Y),
\]

for all \(X \perp Y\) (where we used the fact that \(X(Y^* m)) e_i = -(Y(X^* m)) e_i\), as \(X \perp Y\). Taking the inner products with \(X\) and with \(Y\) we obtain

\[
\|X\|^2 T(Y) + \|Y\|^2 T(X) + 2 \sum_{i=1}^{7} \eta_i \langle Y e_i, X \rangle \langle m((e_i Y^*) X) + (Y (X^* m)) e_i, e_i \rangle
\]

\[
+ \sum_{i,j=1}^{7} \langle Y e_j, X \rangle \langle f_{ij}, X e_i \rangle - \langle f_{ij}, Y \rangle X e_i \rangle - \sum_{i,j=1}^{7} \langle Y e_i, X \rangle \langle Y e_j, Y \rangle f_{ij} = \langle T(Y), X \rangle + \langle T(X), Y \rangle,
\]

for all \(X \perp Y\). Taking \(X = Yu, \ u = \sum_{i=1}^{7} u_i e_i \in \mathbb{O}'\) and re-grouping the terms we obtain

\[
\|u\|^2 T(Y) + T(Y u) + 2 \sum_{i=1}^{7} \eta_i u_i (2 \langle Y, me_i \rangle Y u - 2 \langle Y u, me_i \rangle Y) + 2 \|Y\|^2 (mu) e_i
\]

\[
+ \sum_{i,j=1}^{7} u_{ij} \langle f_{ij} + 8 \delta_{ij} \eta_i m, Y \rangle Y e_i - \langle f_{ij}, Y \rangle Y e_i \rangle - \sum_{i,j=1}^{7} \|Y\|^2 u_{ij} f_{ij}
\]

\[= \|Y\|^2 \langle T(Y), Y \rangle + \|Y\|^2 \langle T(Y), Y \rangle Y,
\]

where we used \(m((e_i Y^*) X) + (Y (X^* m)) e_i = 2 \langle Y, me_i \rangle Y u - 2 \langle Y u, me_i \rangle Y + 4 \langle Y u, m \rangle Y e_i - 4 \langle Y, m \rangle \langle Y u, e_i \rangle + 2 \|Y\|^2 (mu) e_i\), which follows from \(m((e_i Y^*) X) = (Y (X^* m)) e_i - 2 \langle m, Y e_i \rangle X - 2 \langle X, me_i \rangle Y\), for all \(X, Y,\)
and \( (Y(X^*m))e_i = -2(Y,m)(Yu)e_i - 2(Y,m)Ye_i + \|Y\|^2(mu)e_i \), for \( X = Yu, \ u \perp 1 \). By assertion 1 of Lemma 3 (with \( \nu = 1 \) and \( \overline{Y} = \text{Span}(Y,Yu) \)) we obtain that both coefficients on the right-hand side of (29), \( \|Y\|^2(T(Y),Y)u \) and \( \|Y\|^2(T(Yu),Y) \), are linear forms of \( Y \in \mathbb{R}^8 \), for every \( u \in \Omega' \). As \( (T(Y),Y) = 0 \), this implies that there exists an \((\mathbb{R})\)-linear operator \( C : \Omega \to \Omega' \) such that \( \|Y\|^2Y^*T(Y) = CY \), so \( T(Y) = Y \cdot CY \), for all \( Y \in \Omega \). Substituting this to (29) and rearranging the terms we obtain

\[
(Yu) \left( C(Yu) - \sum_{i,j=1}^7 u_j(f_{ij} + 8\delta_{ij}m,Y)e_i \right)
+ Y \left( \left\|u\right\|^2CY + 4\sum_{i,j=1}^7 \eta_{u\iota}(Y,me_i)u - (Yu,me_i)1 + Y^*(mu)e_i \right)
+ \sum_{i,j=1}^7 u_j(f_{ij} + 8\delta_{ij}m,Y)e_i - \sum_{i,j=1}^7 u_iu_jY^*f_{ij} - \langle CY,Yu \rangle + \langle C(Yu),u \rangle 1 = 0,
\]

The left-hand side of (30) has the form \( YuL(Y,u) + YF(Y,u) \), where \( L(Y,u) \) and \( F(Y,u) \) are \((\mathbb{R})\)-linear operator on \( \Omega \), for every \( u \in \Omega' \). By [22] Lemma 6, for every unit octonion \( u \in \Omega' \), \( L(Y,u) = \langle a(u),Y \rangle 1 + \langle t(u),Y \rangle u + Y^*p(u) \), for some functions \( a,t,p : S^0 \subset \Omega' \to \Omega \). Extending \( a,t,p \) by homogeneity (of degree 1,0,1 respectively) to \( \Omega' \) we obtain \( C(Yu) - \sum_{i,j=1}^7 u_j(f_{ij} + 8\delta_{ij}m,Y)e_i = \langle a(u),Y \rangle 1 + \langle t(u),Y \rangle u + Y^*p(u) \), for all \( u \in \Omega' \). Moreover, \( p(u) = -a(u) \), as \( C(Y) \perp 1 \). By the linearity of the left-hand side by \( u \), we get \( \langle t(u_1 + u_2) - a(u_1) - a(u_2),Y \rangle 1 + \langle t(u_1 + u_2) - t(u_1),Y \rangle u_1 + \langle t(u_1 + u_2) - t(u_2),Y \rangle u_2 + Y^*(a_1 + a_2) = 0 \), for all \( u_1, u_2 \in \Omega' \). Then \( Y^*(a(u_1 + u_2) - a(u_1) - a(u_2)) \in \text{Span}(1,u_1,u_2), \) for all \( Y \in \Omega \), which is only possible when \( a(u) \) is linear, that is \( a(u) = Bu \), for some \((\mathbb{R})\)-linear operator \( B : \Omega' \to \Omega \). It follows that \( t(u_1 + u_2) = t(u_1) = t(u_2) \), that is, \( t \in \Omega \) is a constant. So \( C(Yu) = \sum_{i,j=1}^7 u_j(f_{ij} + 8\delta_{ij}m,Y)e_i + (Bu,Y)1 + \langle t(Y),u \rangle Y - Y^*Bu \). Taking the inner product of the both sides with \( v \in \Omega' \) and subtracting from the resulting equation the same equation with \( u \) and \( v \) interchanged we obtain \( C(Yu,v) - \langle C(Yu),v \rangle \langle C(Yu,u) \rangle - Bu,Yv \rangle \), where \( C^t \) is the operator adjoint to \( C \). Now taking \( u \perp v \) and \( v = uv \) we get \( \|u\|^2(C^t - Bu,Yu) = -\|v\|^2(C^t - Bu,u) \), which implies \( C = B^t \). Then from the above, \( \langle C(Yu),e_i \rangle = \sum_{j=1}^7 u_j(f_{ij} + 8\delta_{ij}m,Y) + \langle t(Y),u_i - Bu,Ye_i \rangle = \langle Be_i,Yu \rangle, \) so \( \sum_{j=1}^7 u_j(f_{ij} + 8\delta_{ij}m,Y) + (Bu)e_i + (Be_i)u = 0 \). Therefore

\[
T(Y) = Y \cdot CY = Y \cdot B^tY, \quad f_{ij} = -\delta_{ij}(8\eta_{m} + t) - (Be_i)e_j - (Be_j)e_i.
\]

Substituting (31) to (30) and simplifying we obtain \( -\langle Lu,Y \rangle Y - \langle Lu,Y \rangle Y + \|Y\|^2Lu \cdot u = 0 \), where \( Lu = 4Bu - tu - 4 \sum_{i=1}^7 \eta_{u\iota}(me_i) \). Taking \( Y \perp Lu, Lu \perp u \) we get \( Lu = 0 \), so

\[
Bu = \frac{1}{4}Lu + \sum_{i=1}^7 \eta_{u\iota}(me_i).
\]

Substituting (32) to the first equation of (31) and then to (28) and simplifying we obtain that for arbitrary \( X, U \in \Omega \), \( \langle (\nabla_X \rho)U - (\nabla_U \rho)X, X \rangle = \frac{1}{4}\langle (t,X)(Y,X) - \|X\|^2(t,U) \rangle + 6 \sum_{i=1}^7 \eta(Xe_i,U)(me_i, X) \). Polarizing this equation we get

\[
\langle (\nabla_Y \rho)U - (\nabla_U \rho)Y, X \rangle + \langle (\nabla_X \rho)U - (\nabla_U \rho)X, Y \rangle = \frac{1}{4}\langle (t,Y)(X,Y) - (t,Y)(X,Y) - 2\langle X,Y \rangle(t,U) \rangle
+ 6 \sum_{i=1}^7 \eta(Xe_i,U)(me_i, Y) + \langle Ye_i,U \rangle(me_i, X).
\]

Subtracting the same equation, with \( X \) and \( U \) interchanged and using the fact that \( \rho \) is symmetric we get (25a). The second equation of (31) and (32) give \( f_{ii} = -6\eta_{m} - \frac{1}{2}t \), which by (27) implies (25a). \( \square \)

**Lemma 7.** In the assumptions of Theorem 3 let \( x \in M' \), where \( M' \subset M^n \) is defined in Lemma 4. Then there exists a neighbourhood \( \Omega = \Omega(x) \) and a smooth metric on \( \Omega \) conformally equivalent to the original metric whose curvature tensor has the form (7), with \( \rho \) a multiple of the identity.

**Proof.** Let \( x \in M' \) and let \( \Omega \) be the neighbourhood of \( x \) on which the Weyl tensor has the smooth Clifford structure defined in Lemma 4. We can assume that \( \nu > 0 \), as in the case of a Clifford(0)-structure, the curvature tensor given by (7) has the form \( R(X,Y)Z = (X,Z)\rho Y + \langle \rho X, Z \rangle Y - (Y,Z)\rho X - \langle \rho Y, Z \rangle X \), so the Weyl tensor vanishes. Then the metric on \( \Omega \) is locally conformally flat, that is, is conformally equivalent to a one with \( \rho = 0 \).
If \( n = 8, \nu = 7, \) and all the \( \eta_i \)'s at \( x \) are equal, then they are equal at some neighbourhood of \( x \) (by definition of \( M' \)). By Remark 3 we can replace \( \rho \) by \( \rho + 3/4 \eta_1 \) id and \( \eta_1 \) by \( 0 = \eta_1 - \eta_1 \) in (i) arriving at the case \( \nu = 0 \) considered above.

For the remaining part of the proof, we will assume that in the case \( n = 8, \nu = 7, \) at least two of the \( \eta_i \)'s at \( x \) are different; up to relabelling, let \( \eta_1 \neq \eta_2 \) at \( x \), and also on a neighbourhood of \( x \) (replace \( \mathcal{U} \) by a smaller neighbourhood, if necessary). Let \( f \) be a smooth function on \( \mathcal{U} \) and let \( \langle \cdot, \cdot \rangle' = e^f \langle \cdot, \cdot \rangle \). Then \( W' = W, J_1 = J_1, \eta'_1 = e^{-f} \eta_1 \) and, on functions, \( \nabla' = e^{-f} \nabla \), where we use the dash for the objects associated to metric \( \langle \cdot, \cdot \rangle' \). Moreover, the curvature tensor \( R' \) still has the form (7), and all the identities of Lemma 4 and of Lemma 5 remain valid.

In the cases considered in Lemma 5 the ratios \( \eta_i/\eta_1 \) are constant, as it follows from (11d) and (12a). In particular, taking \( f = \ln |\eta_1| \) we obtain that \( \eta_i' \) is a constant, so all the \( \eta' \)'s are constant, \( m'_\alpha = 0 \) by (11d), so \((\nabla\rho')y - (\nabla\rho')y = 0 \) by (12a). In the case \( n = 8, \nu = 7 \) (Lemma 5), take \( f = \ln |\eta_1 - \eta_2| \). Then by (25c), \( \nabla f = -4m \) and \( \nabla \eta_1' = -2e^{-2f} \) which implies \( m' = -\frac{1}{4} \nabla \ln |\eta_1 - \eta_2| = 0 \), \( t' = e^{-2f} \), again by (25c) for the metric \( \langle \cdot, \cdot \rangle' \). Then by (25c), \((\nabla\rho')y - (\nabla\rho')y = \frac{1}{4}(X'\rho') \). By Remark 3 we can replace \( \rho' \) by \( \tilde{\rho}' = \rho' + \frac{3}{2} (\eta_1 + C) \) and \( \eta_i' \) by \( \tilde{\eta}_i = (\eta_i' + C) \) without changing the curvature tensor \( R' \) given by (7) (\( C \) is a constant chosen in such a way that \( \tilde{\eta}_i \neq 0 \) anywhere on \( \mathcal{U} \)). Then by (25c) and (25a) for the metric \( \langle \cdot, \cdot \rangle', (\nabla\rho')y - (\nabla\rho')y = 0 \).

Dropping the dashes and the tildes, we obtain that, up to a conformal smooth change of the metric on \( \mathcal{U} \), the curvature tensor has the form (7), with \( \rho \) satisfying the identity

\[
(\nabla\rho')y = (\nabla X)'y,
\]

for all \( X, Y \), that is, with \( \rho \) being a symmetric Codazzi tensor.

Then by [DS Theorem 1], at every point of \( \mathcal{U} \), for any three eigenspaces \( E_\beta, E_\gamma, E_{a,0} \) of \( \rho \), with \( \alpha \notin \{ \beta, \gamma \} \), the curvature tensor satisfies \( R(X, Y)Z = 0 \), for all \( X \in E_\beta, Y \in E_\gamma, Z \in E_{a,0} \). It then follows from (7) that

\[
\sum_{i=1}^\nu \eta_i (2\langle J_i X, Y \rangle J_i Z + \langle J_i X, Y \rangle J_i X - \langle J_i Z, Y \rangle J_i Y) = 0,
\]

for all \( X \in E_\beta, Y \in E_\gamma, Z \in E_{a,0}, \alpha \notin \{ \beta, \gamma \} \).

Suppose \( \rho \) is not a multiple of the identity. Let \( E_1, \ldots, E_p, p \geq 2 \), be the eigenspaces of \( \rho \). If \( p > 2 \), denote \( E'_1 = E_1, E'_2 = E_2 \), \( \vdots \), \( E'_p = E_p \). Then by linearity, (33) holds for any \( X, Y \in E'_a, Z \in E'_b \), such that \( \{ \alpha, \beta \} = \{ 1, 2 \} \). Hence to prove the lemma it suffices to show that (33) leads to a contradiction, in the assumption \( p = 2 \). For the rest of the proof, suppose that \( p = 2 \). Denote \( \dim E_{a,0} = d_a, d_1 \leq d_2 \).

Choose \( Z \in E_{a,0}, X, Y \in E_\beta, \alpha \neq \beta \), and take the inner product of (33) with \( X \). We get

\[
\sum_{i=1}^\nu \eta_i (J_i X, Y, J_i X, Z) = 0.
\]

It follows that for every \( X \in E_{a,0} \), the subspaces \( E_1 \) and \( E_2 \) are invariant subspaces of the symmetric operator \( R_X \in \text{End}(\mathbb{R}^n) \) defined by \( R_X Y = \sum_{i=1}^\nu \eta_i (J_i X, Y, J_i X) \).

So \( R_X \) commutes with the orthogonal projections \( \pi_\beta : \mathbb{R}^n \to E_\beta, \beta = 1, 2 \). Then for all \( \alpha, \beta = 1, 2 \) (\( \alpha \) and \( \beta \) can be equal), all \( X \in E_{a,0} \) and all \( Y \in \mathbb{R}^n \),

\[
\sum_{i=1}^\nu \eta_i (J_i X, \pi_\beta Y, J_i X, \pi_\gamma) = \sum_{i=1}^\nu \eta_i (J_i X, \pi_\beta Y, \pi_\beta J_i X).
\]

Taking \( Y = J_j X \) we get that \( \pi_\beta J_j X \subset J_j X \), that is \( \pi_\beta J_j X \subset J_j X \), for all \( X \in E_{a,0}, \alpha, \beta = 1, 2 \). As \( \alpha + \beta = 2 \), we obtain \( J_j X \subset \pi_\beta J_j X \subset J_j X \), hence \( J_j X = \pi_\beta J_j X \). As every function \( f_{a,b} : E_{a,0} \to Z, \alpha, \beta = 1, 2 \), defined by \( f_{a,b}(X) = \dim \pi_\beta J_j X, X \in E_{a,0} \), is lower semi-continuous, and \( f_{a,1} + f_{a,2} = \nu \) for all nonzero \( X \in E_{a,0} \), there exist constants \( c_{a,b} \), with \( c_{a,1} + c_{a,2} = \nu \), such that \( \dim \pi_\beta J_j X = c_{a,b} \), for all \( \alpha, \beta = 1, 2 \) and all nonzero \( X \in E_{a,0} \).

Let \( X, Y \in E_{a,0}, Z \in E_\beta, \beta \neq \alpha \). Taking the inner product of (33) with \( J_j Z, j = 1, \ldots, \nu \), we get

\[
2\eta_j \langle J_j Z, Y \rangle \|Z\|^2 = \sum_{i \neq j} \eta_i \langle (J_i Z, Y) (J_i J_j Z) - \langle J_j Z, Y, J_i Z \rangle \rangle.
\]

As \( \langle J_j Z, X \rangle = \langle J_j \pi_\beta Z, X \rangle = -\langle Z, \pi_\beta J_j X \rangle \) (and similarly for \( \langle J_i Z, Y \rangle \)), the right-hand side, viewed as a quadratic form of \( Z \) in \( E_\beta \), vanishes for all \( Z \in (\pi_\beta J_j X)^\perp \cap (\pi_\beta J_j X)^\perp \), that is, on a subspace of dimension at least \( d_j - 2c_{a,b} \). So for \( \alpha \neq \beta \), either \( 2c_{a,b} \geq d_j \), or \( J_\beta E_{a,0} \perp E_{a,0} \), that is, \( \pi_\beta J_j X = J_j X \), for all \( X \in E_{a,0} \), so \( c_{a,b} = \nu \).

Similarly, if \( Z \in E_{a,0}, X, Y \in E_\beta, \beta \neq \alpha \), the inner product of (33) with \( J_j X, j = 1, \ldots, \nu \), gives

\[
\eta_j \langle J_j Z, Y \rangle \|X\|^2 = \sum_{i=1}^\nu \eta_i \langle -2(J_i X, Y) (J_i J_j X) + \langle J_j X, Z \rangle (J_i J_j X) \rangle.
\]
As \(\langle J, X, Y \rangle = -\langle X, \pi_3 J Y \rangle, \langle J, Z, X \rangle = -\langle X, \pi_3 J Z \rangle\), the right-hand side, viewed as a quadratic form of \(X \in E_\beta\), vanishes on the subspace \((\pi_3 J Y)^\perp \cap (\pi_3 J Z)^\perp\) whose dimension is at least \(d_\beta - c_\beta - c_\beta\). We obtain that for \(\alpha \neq \beta\), either \(c_\alpha + c_\beta \geq d_\beta\), or \(J E_\alpha \perp E_\beta\), that is, \(\pi_3 J Z = 0\), for all \(Z \in E_\alpha\), so \(c_\alpha = 0\). As the equation \(c_\alpha = 0\) contradicts both \(2c_\alpha \geq d_\beta\) and \(c_\alpha = \nu\) (as \(\nu > 0\)), we must have \(c_\alpha + c_\beta \geq d_\beta\). Then \(2\nu = \sum c_\alpha c_\beta \geq d_1 + d_2 = n\).

This proves the lemma in all the cases when \(\nu < n\), that is, in all the cases except for \(n = 8, \nu \geq 4\) (as it follows from Lemma 1).

Consider the case \(n = 8\). We identify \(\mathbb{R}^8\) with \(\mathbb{O}\) and assume that the \(J_i\)’s act as in \([8]\). Let \(D : \mathbb{O} \to \mathbb{O}\) be the symmetric operator defined by \(D 1 = 0\), \(D e_i = \eta_i e_i\). By \([1]\), condition \((32)\) still holds if we replace \(D\) by \(D + e\text{Im}\), where \(\text{Im}\) is the operator of taking the imaginary part of an octonion. So we can assume that the eigenvalue of the maximal multiplicity of \(D_{\mathbb{O}'}\) is zero (one of them, if there are more than one).

Then in \((33)\), \(\nu = \text{rk } D\). By construction, \(\nu \leq 6\), and we only need to consider the cases when \(\nu \geq 4\), as it is shown above.

By \((34)\), \((J, X, Y) J, Z = \langle X e_i, Y \rangle Z e_i = \langle e_i, X^* Y \rangle Z e_i\), so \(\sum \eta \langle J, X, Y \rangle J, Z = \sum \eta \langle e_i, X^* Y \rangle Z e_i = \sum (D e_i, X^* Y) Z e_i = Z D (X^* Y)\), as \(D\) is symmetric and \(D 1 = 0\). Then \((33)\) can be rewritten as

\[
2 Z D (X^* Y) + X D (Z^* Y) - Y D (Z^* X) = 0,
\]

for all \(X, Y \in E_\beta, Z \in E_\alpha, \alpha \neq \beta\).

Taking the inner product of \((34)\) with \(X\) (and using the fact that \(D\) is symmetric, \(D 1 = 0\) and \(Y^* X = 2 \langle X, Y \rangle - X^* Y\) we obtain \((D (X^* Y), X^* Z) = 0\). It follows that for every \(X \in E_\beta\), the subspaces \(E_\alpha\) and \(E_\beta\) are invariant subspaces of the symmetric operator \(L_X D L_X^\perp\), where \(L_X : \mathbb{O} \to \mathbb{O}\) is the left multiplication by \(X\) (note that \(L_X^\perp = L_X^\perp\) and that \(L_X D L_X^\perp\) coincides with the operator \(R_X\) introduced above).

So \(L_X D L_X^\perp\) commutes with the both orthogonal projections \(\pi_\alpha : \mathbb{R}^8 = E_\alpha, \alpha = 1, 2\). It follows that for every \(\alpha, \beta\) (not necessarily distinct) and every \(X \in E_\beta\), the operator \(D\) commutes with \(L_X^\perp \pi_\alpha L_X = \|X\|^2 \pi_X e_\alpha\), that is,

\[
\text{the space } X^* E_\alpha \text{ is an invariant subspace of } D, \text{ for all } \alpha, \beta, \text{ and all } X \in E_\beta.
\]

Consider all the possible cases for the dimensions \(d_\alpha\) of the subspaces \(E_\alpha\).

Let \((d_1, d_2) = (1, 7)\), and let \(u\) be a nonzero vector in \(E_1\). Then by \((35)\), every line spanned by \(X^* u, X \perp u\) (that is, every line in \(\mathbb{O}'\)) is an invariant subspace of \(D\). It follows that \(D_{\mathbb{O}'}\) is a multiple of the identity, which is a contradiction, as \(\text{rk } D = \nu, 4 \leq \nu \leq 6\).

Let \((d_1, d_2) = (2, 6)\), and let \(E_1 = \text{Span}(u, u e), e \in \mathbb{O}'\), \(\|e\| = \|u\| = 1\). Then \(E_2 = u L\), where \(L = \text{Span}(1, e)^\perp\). By \((35)\), with \(E_\alpha = E_1\) and \(X = u^* u = -u^* U \in E_2, U \in L\), every two-plane \(\text{Span}(U, (U^* u)/e), U \in L\), is an invariant subspace of \(D\). Note that \((U u^*)(u)e) = U^* e\), for all \(U \in L\), and moreover, the operator \(J\) defined by \(J U = (U u^*)/e\) (and \(u^*\)) is an almost Hermitian structure on \(L\).

Then \(L\) is an invariant subspace of \(D\) (as the sum of the invariant subspaces \(\text{Span}(U, J U), U \in L\) and \(J D_{\perp} U \in \text{Span}(U, J U)\), for all \(U \in L\) (as \(\text{Span}(U, J U)\)) is both \(J\)- and \(D_{\perp}\)-invariant). From assertion 1 of Lemma 3 it follows that the operator \(J D_{\perp}\) is a linear combination of \(i d_{\perp}\) and \(J\). As \(D\) is symmetric and its eigenvalue of the maximal multiplicity is zero, \(D_{\perp} = 0\). Then \(\nu = \text{rk } D < 1\), which is a contradiction.

For the cases \((d_1, d_2) = (3, 5), (4, 4)\), we use the notion of the Cayley plane. A four-dimensional subspace \(C \subset \mathbb{O}\) is called a Cayley plane, if for orthonormal octonions \(X, Y, Z \in C, X(Y^* Z) \in C\). This definition coincides with [HL] Definition IV.1.23], if we disregard the orientation. We will need the following properties of the Cayley plane (they can be found in [HL] Section IV) or proved directly:

(i) A Cayley plane is well-defined; moreover, if \(X(Y^* Z) \in C\) for some triple \(X, Y, Z\) of orthonormal octonions in \(C\), then the same is true for any triple \(X, Y, Z \in C\) (possibly, non-orthonormal).

(ii) If \(C\) is a Cayley plane, then the subspace \(X^* C\) is the same for all nonzero \(X \in C\); we call this subspace \(C^* C\).

(iii) If \(C\) is a Cayley plane, then \(C^\perp\) is also a Cayley plane and \(C^* = C^\perp\). Moreover, for all nonzero \(X \in C^\perp\), the subspace \(X^* C\) is the same and is equal to \(C^\perp\).

(iv) For every nonzero \(e \in \mathbb{O}\) and every pair of orthonormal imaginary octonions \(u, v\), the subspace \(C = \text{Span}(e, u, e, v, (e u) v)\) is a Cayley plane; every Cayley plane can be obtained in this way.

Let \((d_1, d_2) = (3, 5)\). Then \(E_1\) is contained in a Cayley plane \(C\) (spanned by \(E_1\) and \(X(Y^* Z)\), for some orthonormal vectors \(X, Y, Z \in E_1\)), so \(C^\perp \subset E_2\). Let \(U\) be a unit vector in the orthogonal plane.
complement to \( C^\perp \) in \( E_2 \). Then for every nonzero \( X \in C^\perp \), \( X^*E_2 = C^\perp \mathbb{C} \oplus \mathbb{R}(X^*U) \), by properties (iii) (iii). As for any two invariant subspaces of a symmetric operator, their intersection and the orthogonal complements to it in each of them are also invariant, it follows from (iii) that both \( C^\perp \mathbb{C} \) and every line \( \mathbb{R}(X^*U) \), \( X \in C^\perp \), are invariant subspaces of \( D \). Then the restriction of \( D \) to the four-dimensional space \( (C^\perp)^*U \) is a multiple of the identity on that space. As the eigenvalue of the maximal multiplicity of \( D \) is zero, \( \mathbb{R}1 \oplus (C^\perp)^*U \subset \text{Ker} D \). Then \( \nu = \text{rk} D \leq 3 \), which is again a contradiction.

Let now \( d_1 = d_2 = 4 \). First assume that \( E_1 \) is not a Cayley plane. Let \( X_1, X_2 \) be orthonormal vectors in \( E_1 \). Then \( X_1^*E_1 \cap X_2^*E_1 \subset \text{Span}(1, X_1^*X_2) \), as \( X_2^*X_1 = -X_1^*X_2 \). Moreover, for any unit vector \( Y \in X_1^*E_1 \cap X_2^*E_1 \) orthogonal to \( \text{Span}(1, X_1^*X_2) \) we have \( Y = X_3^*X_3 = X_2^*X_4 \) for some \( X_3, X_4 \in E_1 \). \( X_3, X_4 \perp X_1, X_2 \), which implies \( X_2(X_1^*X_3) = X_4 \in E_1 \), so \( X_1 \) is a Cayley plane by property (i). It follows that \( X_1^*E_1 \cap X_2^*E_1 = \text{Span}(1, X_1^*X_2) \). As by (iii) both subspaces on the left-hand side are invariant under \( D \) and as \( \mathbb{R}1 \) is an invariant subspace of \( D \), we obtain that every line \( \mathbb{R}(X_1^*X_2) \), \( X_1, X_2 \in E_1 \) is an invariant subspace of \( D \) (that is, \( X_1^*X_2 \) is an eigenvector of \( D \)). Then the space \( L = \text{Span}(E_1^*E_1) \) lies in an eigenspace of \( D \), so \( D_{L^*} \) is a multiple of \( \text{id}_{L^*} \). If \( X_1, X_2, X_3 \in E_1 \) are orthonormal, then \( X_2^*X_3 \notin X_1^*E_1 \), as \( E_1 \) is not a Cayley plane. So dim \( L \geq 5 \). As the eigenvalue of the maximal multiplicity of \( D \) is zero, \( \nu = \text{rk} D \leq 3 \), a contradiction.

Let again \( d_1 = d_2 = 4 \), and let \( E_2 \) be a Cayley plane. Then \( E_2 = (E_1^\perp)^\perp \) is also a Cayley plane by property (iii). Moreover, by the same property, \( E_1^*E_1 = E_2^*E_2 = V_1 \) and \( E_1^*E_2 = E_1^\perp E_2 = V_2 \), where \( V_1, V_2 \) are mutually orthogonal four-dimensional subspaces of \( \mathbb{O} \), and \( 1 \in V_1 \). From (iii), each of the two spaces \( V_1, V_2 \) is invariant under \( D \). Let \( X, Y \in E_1 \), \( Z, W \in E_2 \), with \( X, Z \neq 0 \), and let \( u = X^\perp Y, v = Z^\perp W \). As \( X^* \perp X \), \( Z^* \perp Z \), \( X \perp E_1 \) is nonsingular, \( L_X(V_1 \cap 1^\perp) \) is a three-dimensional subspace of \( E_1 \). The same is true with \( X \) replaced by \( X' \). Therefore, for some \( u, v \in V_1 \cap 1^\perp \), \( X u = X'v \), hence \( X' = -\|v\|^2(Xu)v \). As \( X' \perp X \), we get \( (X, Xu)v = 0 \), so \( u \perp v \). Thus \( \langle D(Z^*X), Z^*X \rangle = 0 \) for any \( Z \in E_2 \) and any orthogonal \( X, X' \in E_1 \). As \( Z^*E_1 \neq 0 \), for any nonzero \( Z \in E_2 \), by properties (ii) (iii), and the operator \( L_{Z^*} \) is orthogonal when \( \|Z\| = 1 \) we get \( \langle Du_1, v_2 \rangle = 0 \), for any two orthogonal vectors \( u_1, v_2 \in V_2 \). It follows that the restriction of \( D \) to its invariant subspace \( V_2 \) is a multiple of the identity. As \( V_2 \subset \mathbb{O} \) and the eigenvalue of \( D_{V_2} \) of the maximal multiplicity is zero we obtain that \( \mathbb{R}1 \oplus V_2 \subset \text{Ker} D \). Then \( \nu = \text{rk} D \leq 3 \) which is a contradiction. \( \square \)

Remark 4. As it follows from the proof of Lemma 7 the algebraic statement “a symmetric operator satisfying (iii) is a multiple of the identity” is valid when \( 2\nu < n \). In particular, when \( n = 16 \), it remains true, if we relax the restrictions \( \nu \leq 4 \) of Theorem 5 to \( \nu \neq 8 \) (as for \( n = 16 \), \( \nu \leq 8 \) by (iii)).

Lemma 7 implies Theorem 5 at the generic points. Indeed, by Lemma 7 every \( x \in M' \) has a neighbourhood \( \mathcal{U} \) which is either conformally flat or is conformally equivalent to a Riemannian manifold whose curvature tensor has the form (7), with \( \rho \) being a multiple of the identity, that is, whose curvature tensor has a Clifford structure. It follows from [1, Theorem 1.2], [2] Proposition 2] that \( \mathcal{U} \) is conformally equivalent to an open subset of one of the five model spaces: the rank-one symmetric spaces \( \mathbb{C}P^{n/2}, \mathbb{CH}^{n/2}, \mathbb{H}P^{n/4}, \mathbb{H}H^{n/4}, \) or the Euclidean space.

To prove Theorem 5 in full, we show that, firstly, the same is true for any \( x \in M' \), and secondly, that the model space to which \( \mathcal{U} \) is conformally equivalent is the same, for all \( x \in M' \).

We normalize the standard metric \( g \) on each of the spaces \( \mathbb{C}P^{n/2}, \mathbb{CH}^{n/2}, \mathbb{H}P^{n/4}, \mathbb{H}H^{n/4} \) in such a way that the sectional curvature \( K_{x} \) satisfies \( |K_{x}| \in [1, 4] \). Then the curvature tensor of each of them has a Clifford structure \( \text{Cliff}(\nu; J_1, \ldots, J_{\nu+1}, \varepsilon, \varepsilon, \ldots, \varepsilon) \) (\( \nu + 1 \varepsilon \)'s), where \( \nu = 1, 3, \varepsilon = \pm 1 \) and the \( J_i \)'s are smooth anticommuting almost Hermitian structures, with \( J_1 J_2 = \pm J_1 \) when \( \nu = 3 \) and with \( \nabla_{Z}J_{i} = \sum_{j=1}^{m} \omega_{i}^{j}(Z)J_{j} \), where \( \omega_{i}^{j} \) are smooth 1-forms with \( \omega_{i}^{j} + \omega_{i}^{j} = 0 \), and \( \nabla \) is the Levi-Civita
connection for $\tilde{g}$. Denote the corresponding spaces by $M_{\nu,\varepsilon}$ (and their Weyl tensors, by $W_{\nu,\varepsilon}$), so that
\[M_{1,1} = (\mathbb{C}P^n/2, \tilde{g}), \quad M_{1,-1} = (\mathbb{C}H^n/2, \tilde{g}), \quad M_{3,1} = (\mathbb{H}P^n/4, \tilde{g}), \quad M_{3,-1} = (\mathbb{H}H^n/4, \tilde{g}).\]

We start with the following technical lemma:

**Lemma 8.** Let $(N^n, \langle \cdot, \cdot \rangle)$ be a smooth Riemannian space locally conformally equivalent to one of the $M_{\nu,\varepsilon}$, so that $\tilde{g} = f^2 \circ \nu^* \langle \cdot, \cdot \rangle$, for a positive smooth function $f = e^{2\phi} : N^n \to \mathbb{R}$. Then the curvature tensor $R$ and the Weyl tensor $W$ of $(N^n, \langle \cdot, \cdot \rangle)$ satisfy
\[
R(X, Y, Z, W) = \varepsilon f(X \wedge Y, J_1 Y - J_2 Y, J_2 X + J_1 X, Z), \quad K = H(\phi) - \nabla \phi \otimes \nabla \phi + \frac{1}{2} \| \nabla \phi \|^2 \text{id},
\]
\[
W(X, Y, Z, W) = W_{\nu,\varepsilon}(X, Y) = \varepsilon f(-\frac{3}{n-1} X \wedge Y + T(X, Y)),
\]
\[
\|W\|^2 = C_{\nu n} f^2, \quad C_{\nu n} = 6 \nu n(n + 2)(n - \nu - 1)(n - 1)^{-1},
\]
\[
(\nabla W)(X, Y, Z, W) = \varepsilon f(-\frac{3}{n-1} X \wedge Y + T(X, Y)) + \frac{1}{2} \varepsilon (\langle T(X, Y), \nabla f \wedge Z \rangle + T((\nabla f \wedge Z)X, Y) + T(X, (\nabla f \wedge Z)Y),
\]

where $X \wedge Y$ is the linear operator defined by $(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$, $H(\phi)$ is the symmetric operator associated to the Hessian of $\phi$, and both $\nabla$ and the norm are computed with respect to $\langle \cdot, \cdot \rangle$.

**Proof.** The curvature tensor of $M_{\nu,\varepsilon}$ has the form $R(X, Y, Z, W) = \varepsilon f(X \wedge Y, J_1 Y - J_2 Y, J_2 X + J_1 X, Z)$, where $(X \wedge Y)Z = \tilde{g}(X, Z)Y - \tilde{g}(Y, Z)X$. Under the conformal change of metric $\tilde{g} = f^2 \circ \nu^* \langle \cdot, \cdot \rangle$, the curvature tensor transforms as $R(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W}) = f^{-1} R(X, Y, Z, W)$.

For every point $x \in M'$, there exists a neighbourhood $U$ of $x$ and a positive smooth function $f : U \to \mathbb{R}$ such that the Riemannian space $(U, f^2 \nu^* \langle \cdot, \cdot \rangle)$ is isometric to an open subset of one of the five model spaces $(M_{\nu,\varepsilon}$ or $\mathbb{R}^n)$, so at every point $x \in M'$, the Weyl tensor $W$ of $M^n$ either vanishes, or has the form given in (36b). The Jacobi operators associated to the different Weyl tensors $W_{\nu,\varepsilon}$ in (36b) differ by the multiplicities and the signs of the eigenvalues, so every point $x \in M'$ has a neighbourhood conformally equivalent to a domain of exactly one of the model spaces. Moreover, the function $f > 0$ is well-defined at all the points where $W \neq 0$, as $\|W\|^2 = C_{\nu n} f^2$ by (36c).

By continuity, the Weyl tensor $W$ of $M^n$ either has the form $W_{\nu,\varepsilon}$ or vanishes, at every point $x \in M^n$ (as $M'$ is open and dense in $M^n$, see Lemma 4). Moreover, every point $x \in M^n$, at which the Weyl tensor has the form $W_{\nu,\varepsilon}$, has a neighbourhood, at which the Weyl tensor has the same form. Hence $M^n = M_0 \cup \bigcup_{\alpha} M_{\alpha}$, where $M_0 = \{ x : W(x) = 0 \}$ is closed, and every $M_{\alpha}$ is a nonempty open connected subset, with $\partial M_{\alpha} \subset M_0$, such that the Weyl tensor has the same form $W_{\nu,\varepsilon} = W_{\nu(\alpha),\varepsilon(\alpha)}$ at every point $x \in M_{\alpha}$. In particular, $M_{\alpha} \subset M'$, for every $\alpha$, so that each $M_{\alpha}$ is locally conformally equivalent to one of the model spaces $M_{\nu,\varepsilon}$.

If $M = M_0$ or if $M_0 = \emptyset$, the theorem is proved. Otherwise, suppose that $M_0 \neq \emptyset$ and that there exists at least one component $M_{\alpha}$. Let $y \in \partial M_{\alpha} \subset M_0$ and let $B_{\delta}(y)$ be a small geodesic ball of $M$ centered at $y$ which is strictly geodesically convex (any two points from $B(y)$ can be connected by a unique geodesic segment lying in $B_{\delta}(y)$ and that segment realizes the distance between them). Let $x \in B_{\delta/3}(y) \cap M_{\alpha}$ and let $r = \text{dist}(x, M_0)$. Then the geodesic ball $B = B_r(x)$ lies in $M_\alpha$ and is strictly
convex. Moreover, \( \partial B \) contains a point \( x_0 \in M_0 \). Replacing \( x \) by the midpoint of the segment \([x_0x]\) and \( r \) by \( r/2 \), if necessary, we can assume that all the points of \( \partial B \), except for \( x_0 \), lie in \( M_\alpha \).

The function \( f \) is positive and smooth on \( \overline{B} \setminus \{x_0\} \) (that is, on an open subset containing \( \overline{B} \setminus \{x_0\} \), but not containing \( x_0 \)). We are interested in the behavior of \( f(x) \), when \( x \in B \) approaches \( x_0 \).

**Lemma 9.** When \( x \to x_0 \), \( x \in B \), both \( f \) and \( \nabla f \) have a finite limit. Moreover, \( \lim_{x \to x_0, x \in B} f(x) = 0 \).

**Proof.** The fact that \( \lim_{x \to x_0, x \in B} f(x) = 0 \) follows from Lemma 9 and the fact that \( W_{[\xi]} = 0 \) (as \( x_0 \in M_0 \)).

As the Riemannian space \((B, f(\langle \cdot, \cdot \rangle))\) is locally isometric to a rank-one symmetric space \( M_\nu, \varepsilon \) and is simply connected, there exists a smooth isometric immersion \( \iota : (B, f(\langle \cdot, \cdot \rangle)) \to M_\nu, \varepsilon \). Since \( f \) is smooth on \( B \setminus \{x_0\} \) and \( \lim_{x \to x_0, x \in B} f(x) = 0 \), the range of \( \iota \) is a bounded domain in \( M_\nu, \varepsilon \). Moreover, as \( \lim_{x \to x_0, x \in B} f(x) = 0 \), every sequence of points in \( B \) converging to \( x_0 \) in the metric \( \langle \cdot, \cdot \rangle \) is a Cauchy sequence for the metric \( f(\langle \cdot, \cdot \rangle) \). It follows that there exists a limit \( \lim_{x \to x_0, x \in B} \iota(x) \in M_\nu, \varepsilon \). Defining for every \( x \in B \) the point \( J_{[\alpha]} = \text{Span}(J_1, \ldots, J_\nu) \) in the Grassmanian \( G(\nu, \Lambda^2 T_x M^n) \), we find that there exists a limit \( \lim_{x \to x_0, x \in B} J_{[\alpha]} = J_{[\alpha]} = J_{[x_0]} \in G(\nu, \Lambda^2 T_{x_0} M^n) \). In particular, if \( Z \) is a continuous vector field on \( B \), then there exists a unit continuous vector field \( Y \) on \( B \) such that \( Y \perp Z, J \) on \( B \). For such two vector fields, the function \( \theta(Y, Z) = \langle \sum_{j=1}^\nu (\nabla_{E_j} W)(E_j, Y)Y, Z \rangle \) (where \( E_j \) is an orthonormal frame on \( B \)) is well-defined and continuous on \( B \). Using (36d) we obtain by a direct computation at the points of \( B \), \( \theta(Y, Z) = \frac{\nu(n-3)}{2(n-1)}(3\nabla f \wedge Y - (n-1)T(\nabla f, Y)Y, Z) \). This fact, and hence \( \nabla f \), is continuous on \( B \), and since \( Z \) is an arbitrary continuous vector field on \( B \), \( \nabla f \) has a finite limit when \( x \to x_0, x \in B \). □

As \( \lim_{x \to x_0, x \in B} f(x) = 0 \) and the \( J_i \)'s are orthogonal, the second term on the right-hand side of equation (36a) tends to \( 0 \) when \( x \to x_0 \) in \( B \). Therefore the (3,1) tensor field defined by \( (X, Y) \to (X \wedge KY + KX \wedge Y) \) has a finite limit (namely \( \nu_{[\alpha]} \)) when \( x \to x_0 \) in \( B \). It follows that the symmetric operator \( K \) has a finite limit at \( x_0 \). Computing the trace of \( K \) and using the fact that \( \phi = \frac{1}{4} \ln f \) we get

\[
\Delta u = Fu, \quad \text{where } u = f^{(n-2)/4}, \quad F = \frac{1}{4}(n-2)\text{Tr}K
\]
on \( B \). Both functions \( F \) and \( u \) are smooth on \( B \setminus \{x_0\} \) and have a finite limit at \( x_0 \). Moreover, \( \lim_{x \to x_0, x \in B} u(x) = 0 \) by Lemma 9 and \( u(x) > 0 \) for \( x \in B \setminus \{x_0\} \). The domain \( B \) is a small geodesic ball, so it satisfies the inner sphere condition (the radii of curvature of the sphere \( \partial B \) are uniformly bounded). By the boundary point theorem [1] Section 2.3, the inner directional derivative of \( u \) at \( x_0 \) (which exists by Lemma 9 if we define \( u(x_0) = 0 \) by continuity) is positive.

As \( \nabla u = \frac{1}{(n-2)}f^{(n-6)/4}\nabla f \) in \( B \), we arrive at a contradiction with Lemma 9 in all the cases, except for \( n = 6 \). To finish the proof in that case, we will show that the limit \( \lim_{x \to x_0, x \in B} \nabla f(x) \), which exists by Lemma 9 is zero. When \( n = 6 \), we have \( \nu = 1 \) by (3), so \( T(X, Y) = JX \wedge JY + 2(JX, Y)J \). Taking \( J = J(x) \) smooth on \( B \setminus \{x_0\} \), we have a finite limit when \( x \to x_0, x \in B \) (see the proof of Lemma 9). Using the covariant derivative of \( T \) computed in Lemma 9 and (36d), we obtain that on \( B \),

\[
\langle \nabla_U \nabla_Z W(X, Y) \rangle = \varepsilon(\langle H(f)U, Z \rangle(\frac{1}{2}X \wedge Y + T(X, Y)) + \langle H(f)U \wedge Z, T(X, Y) \rangle + \langle (H(f)U \wedge Z)X, Y \rangle + T(X, (H(f)U \wedge Z)Y)) + \varepsilon f^{-1}Zf\langle [T(X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, Y) + T(X, (\nabla f \wedge U)Y) \rangle \\
+ \varepsilon f^{-1}Zf\langle [T(X, Y), \nabla f \wedge U] + T((\nabla f \wedge U)X, Y) + T(X, (\nabla f \wedge U)Y) \rangle + \varepsilon f^{-1}(T((\nabla f \wedge X)Y), \nabla f \wedge U) + T((\nabla f \wedge U)(\nabla f \wedge X)Y) + T((\nabla f \wedge X)(\nabla f \wedge U)Y) + \varepsilon f^{-1}(T(X, (\nabla f \wedge Z)Y), \nabla f \wedge U) + T((\nabla f \wedge U)(\nabla f \wedge Z)Y) + T(X, (\nabla f \wedge U)(\nabla f \wedge Z)Y),
\]

where \( H(f) \) is the symmetric operator associated to the Hessian of \( f \). Taking \( U = Z = E_j \), where \{\( E_j \)\} is an orthonormal basis, and summing up by \( j \) we find after some computations:

\[
\sum_{j=1}^6 (\nabla_{E_j} \nabla_{E_j} W(X, Y)) = \varepsilon \Delta f(-\frac{1}{2}X \wedge Y + T(X, Y)) - \varepsilon f^{-1}\|\nabla f\|^2T(X, Y) + \varepsilon f^{-1}(T(X, Y)\nabla f \wedge \nabla f + T((X \wedge Y)\nabla f, \nabla f)) + \frac{3}{2} \varepsilon f^{-1}(\nabla f \wedge (X \wedge Y)\nabla f + J\nabla f \wedge (X \wedge Y)J\nabla f).
\]
As both \( \nabla f \) and \( J \) are smooth on \( \overline{B} \setminus \{x_0\} \) and have limits when \( x \to x_0, \ x \in B \), there exist unit vector fields \( X, Y \), continuous on \( B \) and satisfying \( IX, JY \perp \nabla f, IX \perp JY \). For such \( X \) and \( Y \),

\[
\sum_{j=1}^{6} (\nabla_{E_j} \nabla_{E_j} W)(X, Y) = \varepsilon \triangle f \left(-\frac{3}{8} X \wedge Y + JX \wedge JY\right) - \varepsilon f^{-1} \|\nabla f\|^2 JX \wedge JY.
\]

As the left-hand side is continuous on \( B \) and \( \lim_{x \to x_0, x \in B} \triangle f = 0 \) by (37) and Lemma 9, we obtain that the field \( f^{-1} \|\nabla f\|^2 JX \wedge JY \) of skew-symmetric operators has a limit at \( x_0 \). Taking the trace of its square we find that there exists a limit \( \lim_{x \to x_0, x \in B} f^{-1} \|\nabla f\|^4 \) which implies \( \lim_{x \to x_0, x \in B} \nabla f = 0 \) by Lemma 9. We again arrive at a contradiction with the boundary point theorem for the function \( u = f \) satisfying (37). □

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