ON A CONVEXITY PROPERTY

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Abstract. In this article we proved an interesting property of the class of continuous convex functions. This leads to the form of pre-Hermite-Hadamard inequality which in turn admits a generalization of the famous Hermite-Hadamard inequality. Some further discussion is also given.

1. Introduction

Most general class of convex functions is defined by the inequality

\[
\frac{\phi(x) + \phi(y)}{2} \geq \phi\left(\frac{x+y}{2}\right).
\]

A function which satisfies this inequality in a certain closed interval \(I\) is called convex in that interval. Geometrically it means that the midpoint of any chord of the curve \(y = \phi(x)\) lies above or on the curve.

Denote now by \(Q\) the family of weights i.e., non-negative real numbers summing to 1. If \(\phi\) is continuous, then the inequality

\[
p\phi(x) + q\phi(y) \geq \phi(px + qy)
\]

holds for any \(p, q \in Q\). Moreover, the equality sign takes place only if \(x = y\) or \(\phi\) is linear (cf. [HL]).

The same is valid for so-called Jensen functional, defined as

\[
J_\phi(p, x) := \sum p_i \phi(x_i) - \phi\left(\sum p_i x_i\right),
\]

where \(p = \{p_i\}_1^n \in Q, x = \{x_i\}_1^n \in I, n \geq 2\).

Geometrically, the inequality (1.2) asserts that each chord of the curve \(y = \phi(x)\) lies above or on the curve.

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2. Results and proofs

Main contribution of this paper is the following

**Proposition X** Let \( f(\cdot) \) be a continuous convex function defined on a closed interval \([a, b] := I\). Denote

\[
F(s, t) := f(s) + f(t) - 2f\left( \frac{s + t}{2} \right).
\]

Prove that

\[
\max_{s, t \in I} F(s, t) = F(a, b).
\] (1)

**Proof.** It suffices to prove that the inequality

\[
F(s, t) \leq F(a, b)
\]

holds for \( a < s < t < b \).

In the sequel we need the following assertion (which is of independent interest).

**Lemma 2.1.** Let \( f(\cdot) \) be a continuous convex function on some interval \( I \subseteq \mathbb{R} \). If \( x_1, x_2, x_3 \in I \) and \( x_1 < x_2 < x_3 \), then

\[
\begin{align*}
(i) & \quad \frac{f(x_2) - f(x_1)}{2} \leq f\left( \frac{x_2 + x_3}{2} \right) - f\left( \frac{x_1 + x_3}{2} \right); \\
(ii) & \quad \frac{f(x_3) - f(x_2)}{2} \geq f\left( \frac{x_1 + x_3}{2} \right) - f\left( \frac{x_1 + x_2}{2} \right). \\
\end{align*}
\]

**Proof**

We shall prove the first part of the lemma; proof of the second part goes along the same lines.

Since \( x_1 < x_2 < \frac{x_2 + x_3}{2} < x_3 \), there exist \( p, q; 0 \leq p, q \leq 1, p + q = 1 \) such that \( x_2 = px_1 + q\frac{x_2 + x_3}{2} \).

Hence,

\[
\begin{align*}
& \frac{f(x_1) - f(x_2)}{2} + f\left( \frac{x_2 + x_3}{2} \right) = \frac{1}{2}[f(x_1) - f(px_1 + q\frac{x_2 + x_3}{2})] + f\left( \frac{x_2 + x_3}{2} \right) \\
& \geq \frac{1}{2}[f(x_1) - (pf(x_1) + qf\left( \frac{x_2 + x_3}{2} \right))] + f\left( \frac{x_2 + x_3}{2} \right) = \frac{q}{2}f(x_1) + \frac{2 - q}{2}f\left( \frac{x_2 + x_3}{2} \right) \\
& \geq f\left( \frac{q}{2}x_1 + \frac{2 - q}{2}\left( \frac{x_2 + x_3}{2} \right) \right) = f\left( \frac{q}{2}x_1 + \left( \frac{x_2 + x_3}{2} \right) - \frac{1}{2}(x_2 - px_1) \right) = f\left( \frac{x_1 + x_3}{2} \right). \\
\end{align*}
\]

For the proof of second part we can take \( x_2 = p\left( \frac{x_2 + x_3}{2} \right) + qx_3 \) and proceed as above.
Now, applying the part (i) with $x_1 = a, x_2 = s, x_3 = b$ and the part (ii) with $x_1 = s, x_2 = t, x_3 = b$, we get

\[
\frac{f(s) - f(a)}{2} \leq f\left(\frac{s + b}{2}\right) - f\left(\frac{a + b}{2}\right),
\]

(2)

\[
\frac{f(b) - f(t)}{2} \geq f\left(\frac{s + b}{2}\right) - f\left(\frac{s + t}{2}\right),
\]

(3)

respectively.

Subtracting (2) from (3), the desired inequality follows.

\[\square\]

**Remark 2.2.** A challenging task is to find a geometric proof of the property (1).

We shall quote now a couple of important consequences. The first one is used in a number of articles although we never saw a proof of it.

**Corollary 2.3.** Let $f$ be defined as above. If $x, y \in [a, b]$ and $x + y = a + b$, then

\[f(x) + f(y) \leq f(a) + f(b).\]

**Proof.** Obvious, as a simple application of Proposition X.

\[\square\]

**Corollary 2.4.** Under the conditions of Proposition X, the double inequality

\[2f\left(\frac{a + b}{2}\right) \leq f(pa + qb) + f(pb + qa) \leq f(a) + f(b)\]

(4)

holds for arbitrary weights $p, q \in Q$.

**Proof.** Applying Proposition X with $s = pa + qb, t = pb + qa; s, t \in I$ we get the right-hand side of (4). The left-hand side inequality is obvious since, by definition,

\[
\frac{f(pa + qb) + f(pb + qa)}{2} \geq f\left[\frac{(pa + qb) + (pb + qa)}{2}\right] = f\left(\frac{a + b}{2}\right).
\]

\[\square\]

**Remark 2.5.** The relation (4) represents a kind of pre-Hermite-Hadamard inequalities. Indeed, integrating both sides of (4) over $p \in [0, 1]$, we obtain the form of Hermite-Hadamard inequality (cf. [NP]),

\[f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}.\]
Moreover, the inequality (4) admits a generalization of the Hermite-Hadamard inequality.

**Proposition Y** Let \( g \) be an arbitrary non-negative and integrable function on \( I \). Then, with \( f \) defined as above, we get

\[
2f\left(\frac{a + b}{2}\right) \int_a^b g(t)dt \leq \int_a^b (g(t) + g(a + b - t))f(t)dt \leq (f(a) + f(b)) \int_a^b g(t)dt. \tag{5}
\]

**Proof.** Multiplying both sides of (4) with \( g(pa + qb) \) and integrating over \( p \in [0,1] \), we obtain

\[
2f\left(\frac{a + b}{2}\right) \int_a^b g(t)dt \leq \int_a^b (f(t) + f(a + b - t))g(t)dt \leq (f(a) + f(b)) \int_a^b f(t)dt, \tag{5}
\]

and, because

\[
\int_a^b (f(t) + f(a + b - t))g(t)dt = \int_a^b (g(t) + g(a + b - t))f(t)dt,
\]

the inequality (5) follows. \( \square \)

We shall give in the sequel some illustrations of this proposition.

**Corollary 2.6.** For any \( f \) that is convex and continuous on \( I := [a, b], 0 < a < b \) and \( \alpha \in \mathbb{R}/\{0\} \), we have

\[
2f\left(\frac{a + b}{2}\right) \leq \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b [t^{\alpha-1} + (a + b - t)^{\alpha-1}]f(t)dt \leq f(a) + f(b).
\]

Also, for \( \alpha \to 0 \), we get

**Corollary 2.7.**

\[
2f\left(\frac{a + b}{2}\right) \frac{\log(b/a)}{a + b} \leq \int_a^b \frac{f(t)}{t(a + b - t)} dt \leq [f(a) + f(b)] \frac{\log(b/a)}{a + b}.
\]

Similarly,

**Corollary 2.8.**

\[
2f\left(\frac{\pi}{2}\right) \leq \int_0^\pi f(t)\sin t dt \leq f(0) + f(\pi);
\]

\[
2f\left(\frac{\pi}{4}\right) \leq \int_0^{\pi/2} |\sin t + \cos t| f(t)dt \leq f(0) + f(\pi/2).
\]

Estimations of the convolution of symmetric kernel on a symmetric interval are also of interest.
Corollary 2.9. Let $f$ and $g$ be defined as above on a symmetric interval $[-a, a], a > 0$. Then we have that
\[
2f(0) \int_{-a}^{a} g(t)dt \leq \int_{-a}^{a} [g(-t) + g(t)]f(t)dt \leq [f(-a) + f(a)] \int_{-a}^{a} g(t)dt.
\]

Remark 2.10. There remains the question of possible extensions of the relation (1). In this sense one can try to prove, along the lines of the proof of (1), that
\[
\max_{p,q \in Q; x,y \in [a,b]} F^*(p, q; x, y) = F^*(p, q; a, b),
\]
where
\[
F^*(p, q; x, y) := pf(x) + qf(y) - f(px + qy).
\]

Anyway the result will be wrong, as simple examples show (except from the case $f(x) = x^2$).

On the other hand, it was proved in [S] that for $p_i \in Q$ and $x_i \in [a, b]$ there exist $p, q \in Q$ such that
\[
J_f(p, x) = \sum p_i f(x_i) - f(\sum p_i x_i) \leq pf(a) + qf(b) - f(pa + qb),
\]
for any continuous function $f$ which is convex on $[a, b]$.

Therefore, an important conclusion follows.

Corollary 2.11. For arbitrary $p_i \in Q$ and $x_i \in [a, b]$, we have that
\[
\sum p_i f(x_i) - f(\sum p_i x_i) \leq \max_p [pf(a) + qf(b) - f(pa + qb)]:= T_f(a, b),
\]
where $T_f(a, b)$ is an optimal upper global bound, depending only on $a$ and $b$ (cf. [S]).

An answer to the above remark is given by the next

Proposition Z If $f$ is continuous and convex on $[a, b]$, then
\[
\max_{p,q \in Q; x,y \in [a,b]} F^*(p, q; x, y) \leq F(a, b).
\]

Proof. We shall prove just that
\[
F^*(p, q; x, y) \leq F(x, y),
\]
for all $p, q \in Q$ and $x, y \in [a, b]$.

Indeed,
\[
F(x, y) - F^*(p, q; x, y) = qf(x) + pf(y) + f(px + qy) - 2f\left(\frac{x+y}{2}\right).
\]
\[ \geq f(qx + py) + f(px + qy) - 2f\left(\frac{x + y}{2}\right) \geq 2f\left(\frac{qx + py + px + qy}{2}\right) - 2f\left(\frac{x + y}{2}\right) = 0. \]

The rest of the proof is an application of Proposition X. □

Putting there \( x = a, y = b \) and combining with (6), we obtain another global bound for Jensen functional.

**Corollary 2.12.** We have that
\[
\mathcal{J}_f(p, x) \leq f(a) + f(b) - 2f\left(\frac{a + b}{2}\right) := T'_f(a, b).
\]

The bound \( T'_f(a, b) \) is not so precise as \( T_f(a, b) \) but is much easier to calculate.

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