Random Time Change and Related Evolution Equations

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Abstract

In this paper we investigate the long time behavior of solutions to fractional in time evolution equations which appear as results of random time changes in Markov processes. We consider inverse subordinators as random times and use the subordination principle for the solutions to forward Kolmogorov equations. The class of subordinators for which asymptotic analysis may be realized is described.
1 Introduction

We start with a brief description of our framework. Our presentation will be rather informal. For necessary technical conditions and details we refer to the main body of this paper.

Let \( \{X_t, t \geq 0; P_x, x \in E\} \) be a strong Markov process in a phase space \( E \). Denote \( T_t \) its transition semigroup (in a proper Banach space) and \( L \) the generator of this semigroup. Let \( S_t, t \geq 0 \) be a subordinator (i.e., a non-decreasing real-valued Lévy process) with \( S_0 = 0 \) and the Laplace exponent \( \Phi \):

\[
E e^{-\lambda S_t} = e^{-t\Phi(\lambda)} \quad t, \lambda > 0.
\]
We assume that $S_t$ is independent of $X_t$.

Denote by $E_t$, $t > 0$ the inverse subordinator and introduce the time changed process $Y_t = X_{E_t}$. We are interested in the time evolution

$$u(t, x) = E_x[f(Y_t)]$$

for a given initial data $f$. As it was pointed out in several works, see e.g. [To2015], [Chen17], $u(t, x)$ is the unique strong solution (in some proper sense) to the following Cauchy problem

$$\mathbb{D}_t^{(k)} u(t, x) = Lu(t, x) \quad u(0, x) = f(x).$$

Here we have a generalized fractional derivative

$$\mathbb{D}_t^{(k)}\phi(t) = \frac{d}{dt} \int_0^t k(t - s)(\phi(s) - \phi(0))ds$$

with a kernel $k$ uniquely defined by $\Phi$.

Let $u_0(t, x)$ be the solution to a similar Cauchy problem but with ordinary time derivative. In stochastic terminology, it is the solution to the forward Kolmogorov equation corresponding to the process $X_t$. Under quite general assumptions there is a nice and essentially obvious relation between these evolutions:

$$u(t, x) = \int_0^\infty u_0(\tau, x)G_t(\tau)d\tau,$$

where $G_t(\tau)$ is the density of $E_t$. Of course, we may have similar relations for fundamental solutions to the considered equations, for the backward Kolmogorov equations or time evolutions of other related quantities.

Having in mind the analysis of the influence of the random time change on the asymptotic properties of $u(t, x)$, we may hope that the latter formula gives all necessary technical equipments. Unfortunately, the situation is essentially more complicated. The point is about the density $G_t(\tau)$, in general, our knowledge for a generic subordinator is very poor. There are two particular cases in which the asymptotic analysis was already realized. First of all, it is the situation of so-called stable subordinators. Starting with the pioneering works by Meerschaert and his collaborators, this case was studied in details [BM01, MS04].

Another case is related to a scaling property assumed for $\Phi$ [CKKW]. It is, nevertheless, difficult to give an interpretation of this scaling assumption in terms of the subordinator.
The aim of this paper is to describe a class of subordinators for which we may obtain information about the time asymptotic of the fractional dynamics. We propose two methods for the study of this problem. In the first approach we use a modified version of the ratio Tauberian theorem from [LSS07]. This method works under general assumptions about the integrability in time of the solution $u_0(t, x)$. Actually, under this assumption the asymptotics is determined completely by the subordinator characteristics.

There is another side of the problem. In many interesting cases the integrability assumption is not valid. Or, vice versa, we have more detailed information about the behavior of $u_0(t, x)$ which is much stronger than integrability (e.g., exponential decay). We propose an alternative approach to such situations based on the Laplace transform techniques. It gives us a possibility to study solutions without the integrability property and to see the effects of a stronger decay of $u_0(t, x)$.

Finally, we apply our methods to the study of fractional dynamics in several particular models: the heat equation, non-local diffusion, solutions with exponential decays. There we see that the general method is working perfectly in space dimensions $d \geq 3$. But for physically important dimensions $d = 1, 2$ we need our alternative approach.

## 2 General Fractional Derivative

### 2.1 Definitions and Assumptions

In this section we recall the concept of general fractional derivative (GFD) associated to a kernel $k$, see [Koc11] and references therein. The basic ingredient of the theory of evolution equations, [KST06, EIK04] is to consider, instead of the first time derivative, the Caputo-Djrbashian fractional derivative of order $\alpha \in (0, 1)$

$$
(\mathbb{D}_t^{(\alpha)} u)(t) = \frac{d}{dt} \int_0^t k(t - s) (u(s) - u(0)) \, ds, \quad t > 0,
$$

where

$$
k(t) = \frac{t^{-\alpha}}{\Gamma(1 - \alpha)}, \quad t > 0.
$$

More generally, it is natural to consider differential-convolution operators

$$
(\mathbb{D}_t^{(\alpha)} u)(t) = \frac{d}{dt} \int_0^t k(t - s) (u(s) - u(0)) \, ds, \quad t > 0,
$$

and
where \( k \in L_{\text{loc}}^1(\mathbb{R}+) \) \((\mathbb{R}+ := [0, \infty))\) is a non-negative kernel. As an example of such an operator, we consider the distributed order derivative \( D_t^{(\mu)} \) corresponding to

\[
k(t) = \int_0^1 \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \mu(\alpha) \, d\alpha, \quad t > 0,
\]

where \( \mu(\alpha), 0 \leq \alpha \leq 1 \) is a positive weight function on \([0,1]\), see \([\text{APZ09, DGB08, Han07, Koc08b, Koc08a, GU05, MS06}]\).

The class of suitable kernels \( k \) we are interested in is such that the fundamental solution of the corresponding evolution equation \((3.1)\) in Section 3 are probability densities in \( L^\infty(\mathbb{R}+) \cap L^1(\mathbb{R}+) \). Therefore, in this paper we make the following assumptions on the Laplace transform \( K \) of the kernel \( k \in L_{\text{loc}}^1(\mathbb{R}+) \).

\textbf{(H)} Let \( k \in L_{\text{loc}}^1(\mathbb{R}+) \) be a non-negative kernel such that \( \int_0^\infty k(s) \, ds > 0 \) and its Laplace transform

\[
\mathcal{K}(\lambda) := (\mathcal{L}k)(\lambda) := \int_0^\infty e^{-\lambda t} k(t) \, dt \tag{2.5}
\]

exists for all \( \lambda > 0 \) and \( \mathcal{K} \) belongs to the Stieltjes class \( S \) (or equivalently, the function \( \mathcal{L}(\lambda) := \lambda \mathcal{K}(\lambda) \) belongs to the complete Bernstein function class \( CBF \), see Appendix A for definitions), and

\[
\mathcal{K}(\lambda) \to \infty, \text{ as } \lambda \to 0; \quad \mathcal{K}(\lambda) \to 0, \text{ as } \lambda \to \infty; \tag{2.6}
\]

\[
\mathcal{L}(\lambda) \to 0, \text{ as } \lambda \to 0; \quad \mathcal{L}(\lambda) \to \infty, \text{ as } \lambda \to \infty. \tag{2.7}
\]

Under the hypotheses (H), \( \mathcal{L}(\lambda) \) and its analytic continuation admit an integral representation, cf. \((A.6)\) in Appendix A and also \[SSV12\], namely

\[
\mathcal{L}(\lambda) = \int_{(0,\infty)} \frac{\lambda}{\lambda + t} \, d\sigma(t) \tag{2.8}
\]

where \( \sigma \) is a Borel measure on \([0, \infty)\), such that \( \int_{(0,\infty)} (1+t)^{-1} \, d\sigma(t) < \infty \).

Here we give some concrete examples of kernels \( k \) and show that its Laplace transform \( \mathcal{K} \) satisfies \((2.6)\) and \((2.7)\) above.
Example 2.1 ($\alpha$-Stable subordinator). Let $k$ be the kernel \((2.2)\) corresponding to the Caputo-Djrbashian fractional derivative $D_t^{(\alpha)}$ of order $\alpha \in (0, 1)$. Then its Laplace transform is given by

$$K(\lambda) = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty e^{-\lambda t} t^{-\alpha} dt = \lambda^{\alpha-1}.$$  

It is easy to verify that (2.6) and (2.7) are satisfied for $K$ and $L$.

Example 2.2 (Gamma subordinator). Let $k$ be the kernel defined by

$$\mathbb{R}_+ \ni t \mapsto k(t) := a \Gamma(0, bt), \quad a, b > 0,$$

where $\Gamma(\nu, x) := \int_x^\infty t^{\nu-1} e^{-t} dt$ is the upper incomplete Gamma function. The Laplace transform of $k$ is given by

$$K(\lambda) = a \log \left(1 + \frac{\lambda}{b}\right), \quad \lambda > 0.$$  

Again, the properties (2.6) and (2.7) are simple to verify.

Example 2.3 (Inverse Gaussian subordinator). Let $a \geq 0$ and $b > 0$ be given and define the kernel $k$ by

$$\mathbb{R}_+ \ni t \mapsto k(t) := \sqrt{\frac{b}{2\pi}} \left(\frac{2}{\sqrt{t}} e^{-\frac{at}{2}} - \sqrt{2\pi} (1 - \text{erf}(z))\right), \quad z := \sqrt{\frac{at}{2}},$$

where $\text{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ is the error function. The Laplace transform of $k$ can be computed and is given by

$$K(\lambda) = \sqrt{\frac{b}{\lambda}} \left(2\sqrt{2\lambda} + a - \sqrt{a}\right), \quad \lambda > 0.$$  

The properties (2.6) and (2.7) follows easily.

2.2 Special Classes of Kernels

Here we collect some classes of kernels $k$ and its Laplace transform asymptotics since they play a major role in this work. Two classes are emphasized, the class corresponding to the distributed order derivative with $k$ given by (2.4) and the class of the general fractional derivative (2.3) for which $K$ is a Stieltjes function.
2.2.1 Distributed order derivatives

The following proposition refers to the special case of distributed order derivative, see [Koc08b] for the proof. We denote the negative real axis by \( \mathbb{R}_- := \{ r \in \mathbb{R}, r \leq 0 \} \).

**Proposition 2.4** (cf. [Koc08b, Prop. 2.2]).

1. Let \( \mu \in C^2([0,1]) \) be given. If \( \lambda \in \mathbb{C} \setminus \mathbb{R}_- \) with \( |\lambda| \to \infty \), then
   \[
   K(\lambda) = \frac{\mu(1)}{\log \lambda} + O \left( (\log |\lambda|)^{-2} \right). 
   \]

   More precisely, if \( \mu \in C^3([0,1]) \), then
   \[
   K(\lambda) = \frac{\mu(1)}{\log \lambda} - \frac{\mu'(1)}{(\log \lambda)^2} + O \left( (\log |\lambda|)^{-3} \right). 
   \]

2. Let \( \mu \in C([0,1]) \) and \( \mu(0) \neq 0 \) be given. If \( \lambda \in \mathbb{C} \setminus \mathbb{R}_- \), then
   \[
   K(\lambda) \sim \frac{1}{\lambda} \log \left( \frac{1}{\lambda} \right)^{-1} \mu(0), \quad \text{as } \lambda \to 0. 
   \]

3. Let \( \mu \in C([0,1]) \) be such that \( \mu(\alpha) \sim a\alpha^s \), \( a > 0, \ s > 0 \). If \( \lambda \in \mathbb{C} \setminus \mathbb{R}_- \), then
   \[
   K(\lambda) \sim a\Gamma(1+s) \frac{1}{\lambda} \log \left( \frac{1}{\lambda} \right)^{-1-s}, \quad \text{as } \lambda \to 0. 
   \]

2.2.2 Classes of Stieltjes functions.

In general if \( k \in L^1_{\text{loc}}(\mathbb{R}_+) \), under the assumption (H), it follows from (2.8) that the Stieltjes function \( K \) admits the integral representation

\[
K(\lambda) = \int_{(0,\infty)} \frac{1}{\lambda + t} d\sigma(t), \quad \lambda > 0. 
\]

In other words, \( K \) is the Stieltjes transform of the Borel measure \( \sigma \). If \( \sigma \) is absolutely continuous with respect to Lebesgue measure with a continuous density \( \varphi \) on \([0,\infty)\), then \( K \) turns out

\[
K(\lambda) = \int_0^\infty \frac{\varphi(t)}{\lambda + t} dt. 
\]
If in addition \( \varphi \) has the asymptotic
\[
\varphi(t) \sim Ct^{-\alpha}, \quad \text{as } t \to \infty, \ 0 < \alpha < 1, \quad (2.14)
\]
\[
\varphi(t) \sim Ct^{\theta-1}, \quad \text{as } t \to 0, \ 0 < \theta < 1, \quad (2.15)
\]
then, \( \varphi \in L^1_{\text{loc}}([0, \infty)) \) and it follows from [Won01, Thm. 1, page 299] (see also [LF02]) that the asymptotic (2.14) implies the asymptotics for \( \mathcal{K} \)
\[
\mathcal{K}(\lambda) \sim C\lambda^{-\alpha}, \quad \text{as } \lambda \to \infty. \quad (2.16)
\]

For the asymptotic of \( \mathcal{K} \) at the origin, we have the following lemma, see [KKdS18, Lem. 7].

**Lemma 2.5.** Suppose that
\[
\varphi(t) = Ct^{\theta-1} + \psi(t), \quad 0 < \theta < 1, \quad (2.17)
\]
where \( |\psi(t)| \leq Ct^{\theta-1+\delta} \), \( 0 < t \leq t_0 \), and \( |\psi(t)| \leq Ct^{-\varepsilon} \), \( t > t_0 \). Here \( 0 < \delta < 1 - \theta \) and \( \varepsilon > 0 \). Then
\[
\mathcal{K}(\lambda) \sim C\lambda^{\theta-1}, \quad \text{as } \lambda \to 0. \]

The function \( [0, \infty) \ni \lambda \mapsto e^{-\tau\mathcal{K}(\lambda)} \), \( \tau > 0 \) is the composition of a complete Bernstein and a completely monotone function, then by Theorem A.8-2 it is a completely monotone function. By Bernstein’s theorem (cf. Theorem A.2), for each \( \tau \geq 0 \), there exists a probability measure \( \nu_\tau \) on \( \mathbb{R}_+ \) such that
\[
e^{-\tau\mathcal{K}(\lambda)} = \int_{(0, \infty)} e^{-\lambda s} d\nu_\tau(s). \quad (2.18)
\]
Define
\[
G_t(\tau) := \int_{(0, t)} k(t - s) d\nu_\tau(s). \quad (2.19)
\]

The function \( G_t(\tau) \) is a central object of this paper, the next lemma collects its fundamental properties.

**Lemma 2.6.** 1. The \( t \)-Laplace transform of \( G_t(\tau) \) is given by
\[
g(\lambda, \tau) := \int_0^\infty e^{-\lambda t} G_t(\tau) dt = \mathcal{K}(\lambda)e^{-\tau\mathcal{K}(\lambda)}. \quad (2.20)
\]
2. The double \((t, \tau)\)-Laplace transform of \(G_t(\tau)\) is equal to
\[
\int_0^\infty \int_0^\infty e^{-\lambda t - p \tau} G_t(\tau) \, dt \, d\tau = \frac{K(\lambda)}{\lambda K(\lambda) + p}.
\]

3. For each fixed \(t \in \mathbb{R}_+\), \(G_t(\tau)\) is a probability density, therefore \(\mathbb{R}_+ \ni \tau \mapsto G_t(\tau) \in L^\infty(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)\).

Proof. 1. It is straightforward to compute the \(t\)-Laplace transform of \(G_t(\tau)\), in fact for any \(\lambda \geq 0\), we have
\[
\int_0^\infty e^{-\lambda t} G_t(\tau) \, dt = \int_0^\infty e^{-\lambda t} \int_{(0, t)} k(t - s) \, d\nu_{\tau}(s) \, dt
= \int_{(0, \infty)} \int_0^\infty e^{-\lambda t} k(t - s) \, dt \, d\nu_{\tau}(s)
= \int_{(0, \infty)} e^{-\lambda s} \int_0^\infty e^{-\lambda t} k(t) \, dt \, d\nu_{\tau}(s)
= K(\lambda) e^{-\tau \lambda K(\lambda)}.
\]

2. It follows immediately from 1.
3. To show that \(G_t(\tau)\) is a probability density for each fixed \(t \in \mathbb{R}_+\), notice that
\[
\int_0^\infty g(\lambda, \tau) \, d\tau = \int_0^\infty e^{-\lambda t} \int_0^\infty G_t(\tau) \, d\tau \, dt = \frac{1}{\lambda}
\]
from which follows that
\[
\int_0^\infty G_t(\tau) \, d\tau = 1. \quad \square
\]

2.3 Probabilistic Interpretation

As the map \([0, \infty) \ni \lambda \mapsto \Phi(\lambda) := \lambda K(\lambda)\) is a complete Bernstein function, then we may define a subordinator \(S\) by its Laplace transform as
\[
\mathbf{E}(e^{-\lambda S(t)}) = e^{-\lambda \phi(\lambda)} = e^{-\lambda \lambda K(\lambda)}, \quad \lambda \geq 0,
\]
and \(\Phi\) is called the \textit{Laplace exponent} or \textit{cumulant} of \(S\). The associated Lévy measure \(\sigma\) has support in \([0, \infty)\), fulfils
\[
\int_{(0, \infty)} (1 \wedge \tau) \, d\sigma(\tau) < \infty, \quad (2.21)
\]
and the Laplace exponent $\Phi$ is represented by
\[ \Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda \tau}) \, d\sigma(\tau). \] (2.22)

The equality (2.22) is known as the Lévy-Khintchine formula for the subordinator $S$. The kernel $k$ is related to the subordinator $S$ via the Lévy measure $\sigma$, namely if we set
\[ k(t) = \sigma((t,\infty)), \quad \forall t \in [0,\infty) \]
it is easy to compute its Laplace transform. In fact, for any $\lambda \geq 0$
\[ \int_0^\infty e^{-\lambda t} \int_0^t d\sigma(s) \, dt = \int_0^\infty \int_0^s e^{-\lambda t} \, dt \, d\sigma(s) = \frac{1}{\lambda} \Phi(\lambda) = K(\lambda). \]

Denote by $E$ the inverse process of the subordinator $S$, that is
\[ E(t) := \inf\{s \geq 0 : S(s) \geq t\} = \sup\{s \geq 0 : S(t) \leq s\}. \] (2.23)

Then the marginal density of $E(t)$ is the function $G_t(\tau), \ t, \tau \geq 0$, more precisely
\[ G_t(\tau) \, d\tau = \partial_\tau P(E(t) \leq \tau) = \partial_\tau P(S(\tau) \geq t) = -\partial_\tau P(S(\tau) < t). \]

## 3 Evolution Equations and the General Method

In this section we develop a general method to study the long time behavior of the subordination by the function $G_t(\tau)$ (introduced in (2.19)) of the solution $u_0(x,t)$ of a Cauchy problem (CP). We choose three of these CPs, namely with exponential time decay, the heat equation and linear non-local diffusions.

From now on $L$ denotes always a slowly varying function (SVF), see Appendix A, and $C, C'$ are constants which change from line to line.

### 3.1 The General Method

Let $L$ be a generic heuristic Markov generator defined on functions $u_0(x,t), \ t > 0, \ x \in \mathbb{R}^d$. In Subsection 3.2 we present concrete examples of such
Markov generators. Consider the evolution equations of the following type

\[
\begin{cases}
\frac{\partial u_0(x,t)}{\partial t} = (\mathcal{L}u_0)(x,t) \\
u_0(x,0) = \xi(x),
\end{cases}
\tag{3.1}
\]

which we assume a solution \(u_0(x, \cdot) \in L^1(\mathbb{R}_+)\) is known. We are interested in studying the subordination of the solution \(u_0(x, t)\) by the density \(G_t(\tau)\), that is the function \(u(x, t)\) defined by

\[
u(x, t) := \int_0^\infty u_0(x, \tau)G_t(\tau)\,d\tau, \quad x \in \mathbb{R}^d, \quad t \geq 0.
\tag{3.2}
\]

The subordination principle, see \[\text{Baz00}\], tells that \(u(x, t)\) is the solution of the general fractional differential equation

\[
\begin{cases}
(D_t^{(k)}u)(x, t) = (\mathcal{L}u)(x, t) \\
u(x, 0) = \xi(x),
\end{cases}
\tag{3.3}
\]

with the same operator \(\mathcal{L}\) acting in the spatial variables \(x\) and the same initial condition \(\xi\).

Remark 3.1. 1. The appropriate notions of the solutions of (3.1) and (3.3) depend on the specific setting, they were explained

(a) in \[\text{Koc11}\] for the case where \(L\) is the Laplace operator on \(\mathbb{R}^n\),

(b) in \[\text{Baz00, Baz01, Baz15}\] with abstract semigroup generators for special classes of kernels \(k\),

(c) in \[\text{Prü93}\] for abstract Volterra equations.

2. There is also a probabilistic interpretation of the subordination identities (see, for example, \[\text{Kol11}\]). In the models of statistical dynamics we deal with a subordination of measure flows that will give a weak solution to the corresponding general fractional equation.

In order to study the time evolution of \(u(x, t)\) one possibility is to define its Cesaro mean

\[
M_t(u(x, t)) := \frac{1}{t} \int_0^t u(x, s)\,ds
\]
and investigate its long time behavior. Notice that the Cesaro mean of $u(t, x)$ may be written as

$$M_t(u(x,t)) = \int_0^\infty u_0(x, \tau) \left( \frac{1}{t} \int_0^t G_s(\tau) \, ds \right) \, d\tau$$

$$= \int_0^\infty u_0(x, \tau) M_t(G_t(\tau)) \, d\tau. \quad (3.4)$$

Therefore, we are led to investigate the Cesaro mean of the density $G_t(\tau)$ which determine the long time behavior of $u(x, t)$ once the integral in (3.4) exists. To this end, first we introduce a suitable class of admissible $k(t)$, then we show a theorem which, for each fixed $\tau \in [0, \infty)$, gives a connection between the Cesaro mean of $G_t(\tau)$ and Cesaro mean of $k(t)$. We assume $u_0(x, \cdot) \in L^1(\mathbb{R}_+)$, then the asymptotic of the integral in (3.4) is a consequence of the pointwise convergence in $\tau$ and a uniform bound that give a possibility to apply Lebesgue’s dominated convergence theorem.

**Definition 3.2 (Admissible kernels - $\mathcal{K}(\mathbb{R}_+)$).** The subset $\mathcal{K}(\mathbb{R}_+) \subset L^1_{\text{loc}}(\mathbb{R}_+)$ of admissible kernels $k$ is defined by those elements in $L^1_{\text{loc}}(\mathbb{R}_+)$ satisfying (H) such that for some $s_0 > 0$

$$\liminf_{\lambda \to 0+} \frac{1}{\mathcal{K}(\lambda)} \int_0^{s_0/\lambda} k(t) \, dt > 0 \quad \text{(A1)}$$

and

$$\lim_{t, r \to \infty} \left( \int_0^t k(s) \, ds \right) \left( \int_0^r k(s) \, ds \right)^{-1} = 1. \quad \text{(A2)}$$

The assumptions (A1) and (A2) are easy to check for the classes we introduced in Section 2.

The following theorem establishes an asymptotic relation between the Cesaro means of the density $G_t(\tau)$ and Cesaro mean of $k(t) \in \mathcal{K}(\mathbb{R}_+)$, for each fixed $\tau \in [0, \infty)$.

**Theorem 3.3.** Let $\tau \in [0, \infty)$ be fixed and $k \in \mathcal{K}(\mathbb{R}_+)$ a given admissible kernel. Define the map $G.: [0, \infty) \to \mathbb{R}_+$, $t \mapsto G_t(\tau)$ such that $\int_0^\infty e^{-\lambda t} G_t(\tau) \, dt$ exists for all $\lambda > 0$. Then

$$\lim_{\lambda \to 0+} e^{-\tau \mathcal{K}(\lambda)} = 1$$
\[ \lim_{t \to \infty} \left( \int_0^t G_s(\tau) \, ds \right) \left( \int_0^t k(s) \, ds \right)^{-1} = 1 \]

or

\[ M_t(G_t(\tau)) = \frac{1}{t} \int_0^t G_s(\tau) \, ds \sim \frac{1}{t} \int_0^t k(s) \, ds = M_t(k(t)), \quad t \to \infty \]

and \( M_t(G_t(\tau)) \) is uniformly bounded in \( \tau \in \mathbb{R}_+ \).

**Proof.** The \( t \)-Laplace transform of \( G_t(\tau) \) exists for any \( \lambda > 0 \), cf. Lemma 2.6-1. Then the result of the theorem for each \( \tau > 0 \) follows from Corollary 3.3. in [LSS07] with \( X_+ = \mathbb{R}_+ \), \( G_t = u(t) \), \( k = g \) and \( x = 1 \). The uniform bound in \( \tau \) follows from the obvious uniform bound

\[ e^{-\tau \lambda \mathcal{K}(\lambda)} \leq 1. \]

\( \square \)

We have now all the necessary tools to investigate the Cesaro mean of the density \( G_t(\tau) \) for all the classes of admissible kernels. The following three classes of admissible kernels \( k \in \mathcal{K}(\mathbb{R}_+) \) are studied, and they are given in terms of their Laplace transform \( \mathcal{K}(\lambda) \) as \( \lambda \to 0 \)

\[ \mathcal{K}(\lambda) = \lambda^{\theta - 1}, \quad 0 < \theta < 1. \quad \text{(C1)} \]

\[ \mathcal{K}(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad L(x) := \mu(0) \log(x)^{-1}. \quad \text{(C2)} \]

\[ \mathcal{K}(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad L(x) := C \log(x)^{-1-s}, \quad s > 0, \quad C > 0. \quad \text{(C3)} \]

**(C1).** We have in this case

\[ \mathcal{K}(\lambda) = \lambda^{\theta - 1} = \lambda^{-\rho} L \left( \frac{1}{\lambda} \right), \]

where \( \rho := 1 - \theta \geq 0 \) and \( L(x) := 1 \) is a ‘trivial’ SVF. Applying the Karamata-Tauberian theorem, see Theorem B.4, we obtain

\[ \int_0^t k(s) \, ds \sim Ct^\rho L(t) \iff M_t(k(t)) = \frac{1}{t} \int_0^t k(s) \, ds \sim Ct^{-\theta}, \quad \text{as} \ t \to \infty. \]
We have, as $\lambda \to 0$

$$K(\lambda) \sim \lambda^{-1} \log \left( \frac{1}{\lambda} \right)^{-1} \mu(0) = \lambda^{-1} L \left( \frac{1}{\lambda} \right), \text{ as } \lambda \to 0,$$

where $L(x) := \mu(0) \log(x)^{-1}$ is a SVF. Hence, by the Karamata-Tauberian theorem we obtain

$$M_t(k(t)) = \frac{1}{t} \int_0^t k(s) \, ds \sim C \log(t)^{-1}, \quad \text{as } t \to \infty.$$  

(C3). The Laplace transform for each $s > 0$

$$K(\lambda) \sim C \lambda^{-1} \log \left( \frac{1}{\lambda} \right)^{-1-s} = \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad \text{as } \lambda \to 0,$$

where $L(x) := C \log(x)^{-1-s}$ is a SVF. Then, by the Karamata-Tauberian theorem we obtain

$$M_t(k(t)) = \frac{1}{t} \int_0^t k(s) \, ds \sim C \log(t)^{-1-s}, \quad \text{as } t \to \infty.$$  

3.2 Applications to Concrete Examples

3.2.1 Exponential decay

Let us assume that the solution $u_0(x,t)$ of the Cauchy problem (3.1) is such that

$$\sup_{x \in \mathbb{R}^d} |u_0(x,t)| \leq C e^{-\gamma t}, \quad \gamma > 0.$$  

(3.5)

This behavior of $u_0$ may be justified in a number of cases of PDEs. We derive the long time behavior of the subordination $u(x,t)$ defined in (3.2) using the general method above. As the function $\mathbb{R}_+ \ni t \mapsto u_0(x,t) \in \mathbb{R}_+$ is integrable, then the long time behavior of the Cesaro mean of

$$u(x,t) = \int_0^\infty u_0(x,\tau) G_t(\tau) \, d\tau$$

reduces to the study of the Cesaro mean of the admissible kernel $k(t)$. We derive the long time behavior of the Cesaro mean of $k(t)$ through its Laplace transform $K(\lambda)$ by an application of the Karamata-Tauberian theorem.
For the first class of kernels (C1) it is easy to see that the Cesaro mean of $k$ is given, as before, by

$$M_t(u(x,\cdot)) \sim C t^{-\theta}, \; t \to \infty. \quad (3.6)$$

(C2) For the class (C2), an application of the Karamata-Tauberian theorem gives

$$M_t(u(x,\cdot)) \sim C \log(t)^{-1}, \; t \to \infty. \quad (3.7)$$

(C3) Now we look at class (C3) and again by the Karamata-Tauberian theorem we obtain

$$M_t(u(x,\cdot)) \sim C \log(t)^{-1-s}, \; t \to \infty. \quad (3.8)$$

3.2.2 The Heat Equation

We consider the Cauchy problem given by

$$\begin{cases}
\frac{\partial u_0(x,t)}{\partial t} = \Delta u_0(x,t) \\
u_0(x,0) = \varphi(x),
\end{cases} \quad (3.9)$$

where $\varphi \in L^1(\mathbb{R}^d)$. If $G_t(x)$ denotes the fundamental solution (also known as Green function) of the Cauchy problem (3.9), then the solution $u_0(x,t)$ is written as a convolution between the initial condition $\varphi$ and $G_t$, that is

$$u_0(x,t) = (\varphi \ast G_t)(x).$$

Using the Young convolution inequality $\|u_0(\cdot,t)\|_{\infty} \leq \|\varphi\|_{L^1} \|G_t\|_{\infty}$, the solution $u_0(x,t)$ is time continuous and bounded in $x$ in the supremum norm. In addition, it is not difficult to see that $u_0(x,t)$ satisfies

$$\sup_{x \in \mathbb{R}^d} |u_0(x,\tau)| \leq C, \; \tau \in [0,1] \quad (3.10)$$

and

$$\sup_{x \in \mathbb{R}^d} |u_0(x,\tau)| \leq \frac{C}{\tau^{d/2}}, \; \tau \in ]1, \infty). \quad (3.11)$$

The function $u(x,t)$ is defined as the subordination of $u_0(x,t)$ by the density $G_t(\tau)$, that is

$$u(x,t) = \int_0^\infty u_0(x,\tau)G_t(\tau) \, d\tau.$$
As \( u_0(x,t) \) is bounded in a neighbourhood of \( \tau = 0^+ \), then the only important contribution for the long time behavior of \( u(x,t) \) comes from \( \tau > 1 \). On the other hand, the map \([1, \infty) \ni \tau \to \frac{1}{\sqrt[2]{\tau}} \in \mathbb{R}^+ \) belongs to \( L^1(\mathbb{R}^+) \) for \( d \geq 3 \). Therefore using the results from Subsection 3.1 we may derive the long time behavior of the Cesaro mean of \( u(x,t) \) as in the previous example for each classes (C1), (C2), and (C3). See (3.6), (3.7) and (3.8). Notice that for \( d = 1 \) and \( d = 2 \) this method does not allow us to take any conclusion on the long time behavior of the Cesaro mean of \( u(x,t) \) since \( \frac{1}{\sqrt[2]{\tau}} \notin L^1(\mathbb{R}^+) \).

On Section 4 we use an alternative method which allow us to do so.

### 3.2.3 Linear Non-local Diffusion

We consider the linear non-local diffusion, see for instance [AVMR10, Ch. 1]

\[
\begin{aligned}
\frac{\partial u_0(x,t)}{\partial t} &= a \ast u_0(x,t) - u_0(x,t) = \int_{\mathbb{R}^d} a(x-y)(u_0(y,t) - u_0(x,t)) dy - u_0(x,t) \\
u_0(x,0) &= \varphi(x),
\end{aligned}
\]

for \( x \in \mathbb{R}^d, t > 0 \), and \( a \in C(\mathbb{R}^d, \mathbb{R}) \) is radial density function, that is a nonnegative radial function with \( a(0) > 0 \) and \( \langle a \rangle := \int_{\mathbb{R}^d} a(x) dx = 1 \). The notion of a solution of (3.12) is a function \( u_0 \in C(\mathbb{R}^+, L^1(\mathbb{R}^d)) \) such that (3.12) is satisfied in the integral sense

\[
u_0(x,t) = \varphi(x) + \int_0^t \int_{\mathbb{R}^d} a(x-y)u_0(y,s) dy - u_0(x,s) ds.
\]

The existence and uniqueness of solutions of the CP (3.12) may be shown using the Fourier transform technic. In the sequel, \( \hat{f} \) denotes de Fourier transform of \( f \in L^1(\mathbb{R}^d) \) defined by

\[
\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i(x,\xi)} f(x) dx,
\]

where \( \langle \cdot, \cdot \rangle \) denotes de scalar product in \( \mathbb{R}^d \). The following theorem states under which conditions on \( a \) and \( \varphi \) the CP (3.12) has a unique solution, see Theorem 1.3 in [AVMR10, Ch. 1] for more details and other properties of the solution \( u_0(x,t) \). Here we emphasize the uniform bound of \( u_0(x,t) \) in \( x \) as the most relevant for our considerations below.
Theorem 3.4. Assume that there exist \( A > 0 \) and \( 0 < r \leq 2 \) such that
\[
\hat{a}(\xi) = 1 - A|\xi|^r + o(|\xi|^r) \quad \text{as } \xi \to 0.
\]
For any nonnegative \( \varphi \) such that \( \varphi, \hat{\varphi} \in L^1(\mathbb{R}^d) \), there exits a unique solution \( u(x, t) \) of the CP (3.12) such that
\[
\|u_0(\cdot, t)\|_{L^\infty(\mathbb{R}^d)} \leq Ct^{-d/r}.
\]

Remark 3.5. As the solution \( u_0(x, t) \) is time continuous and uniformly bounded in \( x \), then it is easy to derive the following properties of \( u_0(x, t) \)
\[
\sup_{x \in \mathbb{R}^d} |u_0(x, \tau)| \leq C, \quad \tau \in [0, 1], \quad (3.13)
\]
\[
\sup_{x \in \mathbb{R}^d} |u_0(x, \tau)| \leq C\tau^{-d/r}, \quad \tau \in ]1, \infty[. \quad (3.14)
\]

Our aim now is to study the function \( u(x, t) \) given by the subordination of \( u_0(x, t) \) by the density \( G_t(\tau) \), namely
\[
u(x, t) = \int_0^\infty u_0(x, \tau)G_t(\tau) d\tau,
\]
that is determine the long time behavior of \( u(x, t) \) for all the classes of admissible kernels \( k \in \mathbb{K}(\mathbb{R}_+) \).

For \( d \geq 3 \) the function \( \mathbb{R}_+ \ni \tau \mapsto \tau^{-d/r} \in \mathbb{R}_+ \) is integrable, therefore the long time behavior of \( M_t(u(x, t)) \) reduces to that of \( M_t(G_t(\tau)) \). For the three classes of admissible kernels \( k \in \mathbb{K}(\mathbb{R}_+) \), they are given by (3.6), (3.7), (3.8).

### 4 Alternative Method for Subordinated Dynamics

In this section we investigate the long time behavior of the subordination dynamics \( u(x, t) \) for the three CP problems from Subsection 3.2 using an alternative method, the Laplace transform. The possibility to apply this alternative method is related to the a priori information of the initial solution \( u_0(x, t) \). Here we would like to emphasize the results obtained for the heat equation and the linear non-local diffusion. More precisely, the general method from Section 3 does not allow us to obtain the long time behavior of \( M_t(u(x, t)) \) for these examples if the dimension \( d = 1 \) and \( d = 2 \), while the Laplace transform method does for any dimension \( d \geq 1 \).
4.1 Exponential decay

We have the exponential decay of the initial solution \( u_0(x,t) \), see (3.5). Computing the \( t \)-Laplace transform of \( u(x,t) \) and using (2.20) to obtain

\[
(\mathcal{L} u(x,\cdot))(\lambda) = C \int_0^\infty e^{-\tau \gamma} (\mathcal{L} G_\tau)(\lambda) \, d\tau
\]

\[
= C K(\lambda) \int_0^\infty e^{-\tau \gamma} e^{-\tau \lambda K(\lambda)} \, d\tau
\]

\[
= C \frac{K(\lambda)}{\lambda K(\lambda) + \gamma}.
\]

We investigate each class of admissible kernels \( k \in \mathbb{K}(\mathbb{R}^+) \), that is (C1), (C2) and (C3).

(C1). It follows that

\[
(\mathcal{L} u(x,\cdot))(\lambda) = C \frac{\lambda^{\theta-1}}{\lambda^{\theta} + \gamma} = \lambda^{-(1-\theta)} L \left( \frac{1}{\lambda} \right), \quad L(x) := \frac{C}{x^{-\theta} + \gamma}.
\]

Then the Karamata-Tauberian theorem gives

\[
M_t(u(x,t)) \sim C t^{-\theta} \frac{1}{t^{-\theta} + \gamma} \sim C t^{-\theta}, \ t \to \infty.
\]

(C2). We have, as \( \lambda \to 0 \)

\[
(\mathcal{L} u(x,\cdot))(\lambda) \sim C \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad L(x) := C \frac{(\log(x))^{-s}}{(\log(x))^{-s} + \gamma}.
\]

And again, an application of the Karamata-Tauberian theorem yields

\[
M_t(u(x,t)) \sim C \log(t)^{-1} \frac{1}{(\log(t))^{-1} + \gamma} \sim C \log(t)^{-1}, \ t \to \infty.
\]

(C3). For that class one obtains

\[
(\mathcal{L} u(x,\cdot))(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad L(x) := C \frac{(\log(x))^{-1-s}}{(\log(x))^{-1-s} + \gamma}.
\]

By the Karamata-Tauberian theorem we have

\[
M_t(u(x,t)) \sim C \log(t)^{-1-s} \frac{1}{(\log(t))^{-1-s} + \gamma} \sim C \log(t)^{-1-s}, \ t \to \infty.
\]

In conclusion, this alternative method reproduces the same type of decay of the Cesaro mean of \( u(x,t) \) as the general method from Section 3 for this example.
4.2 The Heat Equation

We compute the $t$-Laplace transform of $u(x, t)$ and then apply the Karamata-Tauberian theorem. We have, again using (2.20), that

$$(Lu(x, \cdot))(\lambda) = K(\lambda) \int_0^\infty u_0(x, \tau)e^{-\tau\lambda\mathcal{K}(\lambda)} \, d\tau.$$  

It follows from (3.10) and (3.11) that the solution $u_0(x, \tau)$ is bounded in a neighborhood of $\tau = 0+$, hence the long time behavior of $M_t(u(x, t))$ is only influenced as $\tau > 1$, that is the factor

$$CK(\lambda) \int_1^\infty \tau^{-d/2}e^{-\tau\lambda\mathcal{K}(\lambda)} \, d\tau.$$  

The integral on the right-hand side is computed using the upper incomplete Gamma function

$$\int_b^\infty \tau^\nu e^{-\tau x} \, d\tau = x^{-\nu-1}\Gamma(\nu + 1, bx), \quad \Re(x) > 0.$$  

Hence, neglecting the constant for $\tau \in [0, 1]$, the $t$-Laplace transform of $u(x, t)$ has the form

$$(Lu(x, \cdot))(\lambda) = CK(\lambda)(\lambda\mathcal{K}(\lambda))^{d/2-1}\Gamma(1 - d/2, \lambda\mathcal{K}(\lambda)).$$  

Now we study each class of admissible kernels with the behavior described by (C1), (C2) and (C3).

(C1). We have $\mathcal{K}(\lambda) = \lambda^{\theta-1}$ and we distinguish the following cases:

1. For $d = 1$, as $\lambda \to 0$

$$(Lu(x, \cdot))(\lambda) = C\lambda^{-(1-\theta/2)}\Gamma(1/2, \lambda^\theta) = \lambda^{-\rho}L\left(\frac{1}{\lambda}\right),$$  

where $\rho = 1 - \theta/2$ and $L(x) := C\Gamma(1/2, x^{-\theta})$ is a SVF. In fact, to see that $L(x)$ is a SVF first we use the relation

$$\Gamma(s, x) = \Gamma(s) - \gamma(s, x), \quad s \neq 0, -1, -2, \ldots,$$  

where $\gamma(s, x)$ is the lower incomplete Gamma function, the fact that $x^{-\theta} \to 0$ when $x \to \infty$ together with

$$\gamma(s, x) \sim \frac{x^s}{s}, \quad x \to 0.$$  

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Hence, by the Karamata-Tauberian theorem the Cesaro mean of $u(x,t)$ behaves as
\[ M_t(u(x,t)) \sim Ct^{-\theta/2}L(t) \sim Ct^{-\theta/2}, \quad t \to \infty. \]

2. For $d = 2$, as $\lambda \to 0$

\[ \left( \mathcal{L}u(x, \cdot) \right)(\lambda) \sim \lambda^{-(1-\theta)}L \left( \frac{1}{\lambda} \right), \]

where $L(x) := CT(0,x^{-\theta}) = CE_1(x^{-\theta})$ and $E_1(x)$, $x > 0$ is the exponential integral, see [AS92, Eq. (5.1.1)]. For $x \to 0$ we have, cf. [AS92, Eq. (5.1.11)]

\[ E_1(x) \sim -\gamma - \ln(x), \quad (4.4) \]

where $\gamma$ is the Euler-Mascheroni constant. Then it is simple to show that $L(x) = CE_1(x^{-\theta})$ is a SVF. And again, by the Karamata-Tauberian theorem we obtain

\[ M_t(u(x,t)) \sim Ct^{-\theta}L(t) \sim Ct^{-\theta} \left( \gamma + \log(t^{-\theta}) \right), \quad t \to \infty. \quad (4.5) \]

3. For $d \geq 3$, as $\lambda \to 0$

\[ \left( \mathcal{L}u(x, \cdot) \right)(\lambda) \sim \lambda^{-(1-\theta)}L \left( \frac{1}{\lambda} \right), \]

where $L(x) := x^{\theta(1-d/2)}\Gamma(1-d/2,x^{-\theta})$. To show that $L(x)$ is a SVF use the relation

\[ \Gamma(s, x) \sim -\frac{x^s}{s}, \quad \Re(s) < 0, \quad x \to 0. \quad (4.6) \]

Once more, the Karamata-Tauberian theorem gives

\[ M_t(u(x,t)) \sim Ct^{-\theta}L(t) \sim Ct^{-\theta}, \quad t \to \infty. \]

(C2). The Laplace transform $\mathcal{K}(\lambda)$ behaves as $\lambda \to 0$

\[ \mathcal{K}(\lambda) \sim \lambda^{-1}L \left( \frac{1}{\lambda} \right), \quad L(x) := \mu(0)\log(x)^{-1}. \]

We distinguish the cases $d = 1$, $d = 2$ and $d \geq 3$. 

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1. For \( d = 1 \) as \( \lambda \to 0 \)
\[
(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1}L\left(\frac{1}{\lambda}\right),
\]
where
\[
L(x) := C\log(x)^{-1/2}\Gamma(1/2, \mu(0)\log(x)^{-1}).
\]
To verify that \( L(x) \) is a SVF notice that \( L(x) \) is the product of two SVF, then \( L(x) \) is SVF by Proposition \( B.3-3 \). Hence, by the Karamata-Tauberian theorem and (4.2) the Cesaro mean of \( u(x, t) \) is
\[
M_t(u(x, t)) \sim C\log(t)^{-1}\left(\log(t)^{1/2}\Gamma(1/2, \mu(0)\log(t)^{-1})\right)
\]
\[
\sim C\log(t)^{-1} + C'\log(t)^{-1/2}, \quad t \to \infty.
\]
2. For \( d = 2 \) as \( \lambda \to 0 \)
\[
(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1}L\left(\frac{1}{\lambda}\right),
\]
where \( L(x) := \mu(0)\log(x)^{-1}E_1(\mu(0)\log(x)^{-1}) \). Again, \( L(x) \) is a SVF because it is the product of two SVF. Then an application of the Karamata-Tauberian theorem and (4.4) yields
\[
M_t(u(x, t)) \sim C \log(t)^{-1}E_1(\mu(0)\log(t)^{-1})
\]
\[
\sim C \log(t)^{-1}\left[\gamma + \log \left(\mu(0)\log(t)^{-1}\right)\right], \quad t \to \infty.
\]
3. In general, for any \( d \geq 3 \) as \( \lambda \to 0 \) we have
\[
(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1}L\left(\frac{1}{\lambda}\right),
\]
where
\[
L(x) := (\mu(0)\log(x)^{-1})^{d/2}\Gamma(1-d/2, \mu(0)\log(x)^{-1}).
\]
It is clear that \( L(x) \) is a SVF, hence the Karamata-Tauberian theorem together with (4.6) implies the long time behavior for \( M_t(u(x, \cdot)) \), namely
\[
M_t(u(x, t)) \sim C \log(t)^{-1}\left(\log(t)^{1-d/2}\Gamma(1-d/2, \mu(0)\log(t)^{-1})\right)
\]
\[
\sim C \log(t)^{-1}.
\]
(C3). Finally, let us investigate the Cesaro mean of $u(x, t)$ for the class (C3), that is where $K(\lambda)$ behaves as $\lambda \to 0$

$$K(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right), \quad L(x) := C (\log(x))^{-1-s}, \quad s > 0, \quad C > 0.$$ 

Proceeding as before we distinguish the following cases:

1. For $d = 1$, as $\lambda \to 0$ we have

$$\left( \mathcal{L} u(x, \cdot) \right)(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where

$$L(x) := C \log(t)^{-1-s} \Gamma(1/2, C \log(x)^{-1-s})$$

$$= C \log(t)^{-1-s} \left( \sqrt{\pi} - \gamma(1/2, \log(x)^{-1-s}) \right)$$

is a SVF since it is the product of two SVF. Then, the Karamata-Tauberian theorem yields

$$M_t(u(x,t)) \sim C \log(t)^{-1-s} \left( \sqrt{\pi} - 2 \log(t)^{(1-s)/2} \right), \quad t \to \infty.$$ 

2. For $d = 2$, as $\lambda \to 0$ we have

$$\left( \mathcal{L} u(x, \cdot) \right)(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where $L(x) := C \log(t)^{-1-s} E_1(C \log(x)^{-1-s})$. Then it follows from Karamata-Tauberian theorem and (4.4) that

$$M_t(u(x,t)) \sim C \log(t)^{-1-s} \left[ \gamma + \log(C \log(t)^{-1-s}) \right], \quad t \to \infty.$$ 

3. For $d \geq 3$, as $\lambda \to 0$

$$\left( \mathcal{L} u(x, \cdot) \right)(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where $L(x) := C \log(x)^{-2-s+d/2} \Gamma(1 - d/2, C \log(x)^{-1-s})$. Again, $L(x)$ is a SVF as it is a product of two SVF. Then by the Karamata-Tauberian theorem and (4.6) we obtain

$$M_t(u(x,t)) \sim C \log(t)^{-1-s} \left[ \log(t)^{d/2-1} \Gamma(1 - d/2, C \log(t)^{-1-s}) \right]$$

$$\sim C \log(t)^{-1-s}.$$
Remark 4.1. As a conclusion, the alternative method produces the same long time decay of the Cesaro mean of \( u(x, t) \) compared to the general method from Section 3 for \( d \geq 3 \). In addition, with the Laplace transform method we can handle the dimensions \( d = 1 \) and \( d = 2 \) which was not possible with the general method.

4.3 Linear Non-local Diffusion

It follows from (3.13) and (3.14) that the solution \( u_0(x, \tau) \) is bounded in a neighbourhood of \( \tau = 0^+ \), therefore the long time behavior of \( M_t(u(x, t)) \) depends only on \( \tau > 1 \), that is the factor

\[
CK(\lambda) \int_1^{\infty} \tau^{-d/r} e^{-\tau \lambda K(\lambda)} d\tau.
\]

The integral on the right hand side above is computed using (4.1) such that (neglecting a constant)

\[
(\mathcal{L} u(x, \cdot))(\lambda) = CK(\lambda)(\lambda K(\lambda))^{d/r} \Gamma(1 - d/r, \lambda K(\lambda)).
\]

We investigate the long time behavior of \( M_t(u(x, t)) \) for the three classes of admissible kernels (C1), (C2) and (C3). The analysis below is similar to the analysis of the heat equation assuming \( 1 < r \leq 2 \).

(C1). We have \( K(\lambda) = \lambda^{\theta - 1}, 0 < \theta < 1 \) and

\[
(\mathcal{L} u(x, \cdot))(\lambda) = \lambda^{-(1-\theta d/r)} L \left( \frac{1}{\lambda} \right),
\]

where \( L(x) = C \Gamma(1 - d/r, x^{-\theta}) \) is a SVF.

1. For \( d = 1 \) it follows that

\[
(\mathcal{L} u(x, \cdot))(\lambda) = \lambda^{-(1-\theta/r)} \Gamma(1 - 1/r, \lambda^{-\theta})
\]

with \( 1 - \theta/r > 0 \) and \( 1 - 1/r \in (0, 1/2] \). As \( \Gamma(1 - 1/r, \lambda^{-\theta}) \) is a SVF, then the Karamata-Tauberian theorem gives

\[
M_t(u(x, t)) \sim C \lambda^{-\theta/r} \Gamma(1 - 1/r, \lambda^{-\theta})
\]

and using the equality (4.2) we obtain

\[
M_t(u(x, t)) \sim Ct^{-\theta/r} L(t) \sim Ct^{-\theta/r}.
\]
2. For $d = 2$ we have

$$(\mathcal{L}u(x, \cdot))(\lambda) \geq \lambda^{-(1-2\theta/r)} \Gamma(1 - 2/r, \lambda^{-\theta})$$

such that to have $1 - 2\theta/r > 0$ implies that $r = 2$. This case is similar to the heat equation, see (4.5). Thus, we have

$$M_t(u(x,t)) \sim Ct^{-\theta} (\gamma + \log(t^{-\theta})), \quad t \to \infty.$$

3. For $d \in [3, r/\theta \vee 3)$, we have

$$(\mathcal{L}u(x, \cdot))(\lambda) \geq \lambda^{-(1-\theta)} L \left( \frac{1}{\lambda} \right),$$

where $L(x) = x^{\theta(1-d/r)} \Gamma(1 - d/r, x^{-\theta})$ is a SVF using (4.6). Therefore, we derive the long time behavior of $M_t(u(x,t))$ as a consequence of the Karamata-Tauberian theorem, namely

$$M_t(u(x,t)) \sim Ct^{-\theta} L(t) \sim Ct^{-\theta}, \quad t \to \infty.$$

(C2). That is the case when $\mathcal{K}(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right)$, $L(x) := \mu(0) \log(x)^{-1}$ which implies, as $\lambda \to 0$

$$(\mathcal{L}u(x, \cdot))(\lambda) \sim C \lambda^{-1} L \left( \frac{1}{\lambda} \right)^{d/r} \Gamma \left( 1 - d/r, L \left( \frac{1}{\lambda} \right) \right).$$

1. For $d = 1$ as $\lambda \to 0$, we have

$$(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where $L(x) = C \log(x)^{-1/r} \Gamma(1 - 1/r, \mu(0) \log(x)^{-1})$ is a SVF. Then the Karamata-Tauberian theorem and (4.2) yields

$$M_t(u(x,t)) \sim C \log(t)^{-1} \log(t)^{-1-1/r} \Gamma(1 - 1/r, \mu(0) \log(x)^{-1})$$

$$\sim C \log(t)^{-1} \left( \Gamma(1 - 1/r) \log(t)^{-1-1/r} - C' \right), \quad t \to \infty.$$

2. Now for $d = 2$ we have, as $\lambda \to 0$

$${\mathcal L}(u(x, \cdot))(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where $L(x) = C \log(x)^{-2/r} \Gamma(1 - 2/r, \mu(0) \log(x)^{-1})$ is a SVF.
(a) For the special case $r = 2$ it reduces to

$$L(x) = C \log(x)^{-1} E_1(\mu(0) \log(x))^{-1}.$$  

Then an application of the Karamata-Tauberian theorem and (4.4) yields

$$M_t(u(x, t)) \sim C \log(t)^{-1} E_1(\mu(0) \log(t))^{-1}$$

$$\sim C \log(t)^{-1} \left[ \gamma + \log(\mu(0) \log(t))^{-1} \right], \quad t \to \infty.$$  

(b) For $1 < r < 2$, then $-1 < 1 - 2/r < 0$ and by (4.6)

$$M_t(u(x, t)) \sim C \log(t)^{-1} \log(x)^{1-2/r} \Gamma(1 - 2/r, \mu(0) \log(x))^{-1}$$

$$\sim C \log(t)^{-1}, \quad t \to \infty.$$  

3. For $d \geq 3$, we obtain as $\lambda \to 0$

$$(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

where $L(x) = C \log(x)^{-d/r} \Gamma(1 - d/r, \mu(0) \log(x))$ is a SVF. As $1 - d/r < 0$, then by the Karamata-Tauberian theorem and (4.6) follows

$$M_t(u(x, t)) \sim C \log(t)^{-1} \log(x)^{1-d/r} \Gamma(1 - d/r, \mu(0) \log(x))^{-1}$$

$$\sim C \log(t)^{-1}, \quad t \to \infty.$$  

(C3). The third class of admissible kernels has Laplace transform

$$\mathcal{K}(\lambda) \sim \lambda^{-1} L \left[ \frac{1}{\lambda} \right], \quad L(x) := C(\log(x))^{-1-s}, \quad s > 0, \quad C > 0$$

such that

$$(\mathcal{L}u(x, \cdot))(\lambda) \sim C \lambda^{-1} L \left( \frac{1}{\lambda} \right)^{d/r} \Gamma(1 - d/r, L \left( \frac{1}{\lambda} \right)),$$

1. First we take $d = 1$ and obtain

$$(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1} L \left( \frac{1}{\lambda} \right),$$

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where $L(x) = C \log(x)^{-1-s}/r(1 - 1/r, C \log(x)^{-1-s})$ is a SVG. Then by the Karamata-Tauberian theorem and (4.2) follows

$$M_t(u(x, t)) \sim C \log(t)^{-1-s} \log(t)^{1+s-(1+s)/r} \Gamma(1 - 1/r, C \log(t)^{-1-s})$$

$$\sim C \log(t)^{-1-s} \left(\log(t)^{1+s-(1+s)/r} \Gamma(1 - 1/r) + C'\right), \quad t \to \infty.$$ 

2. For $d = 2$ we have

$$(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1} L \left(\frac{1}{\lambda}\right),$$

where $L(x) = C \log(x)^{-2(1+s)/r} \Gamma(1 - 2/r, C \log(x)^{-1-s})$ is a SVF.

(a) For $r = 2$ the SVF $L(x)$ reduces to

$$L(x) = C \log(x)^{-(1+s)} E_1(C \log(x)^{-1-s})$$

and then using (4.4) we obtain

$$M_t(u(x, t)) \sim C \log(t)^{-1-s}(\gamma + \log(C \log(t)^{-1-s})), \quad t \to \infty.$$ 

(b) For $1 < r < 2$ we have $-1 < 1 - 2/r < 0$ and

$$M_t(u(x, t)) \sim C \log(t)^{-1-s} \log(t)^{(1+s)(1-2/r)} \Gamma(1 - 2/r, C \log(t)^{-1-s})$$

$$\sim C \log(t)^{-1-s}, \quad t \to \infty.$$ 

3. Finally for $d \geq 3$ we have

$$(\mathcal{L}u(x, \cdot))(\lambda) \sim \lambda^{-1} L \left(\frac{1}{\lambda}\right),$$

where $L(x) = C \log(x)^{-d(1+s)/r} \Gamma(1 - d/r, C \log(x)^{-1-s})$ is a SVF. As before, we obtain

$$M_t(u(x, t)) \sim C \log(t)^{-1-s} \log(t)^{(1+s)(1-d/r)} \Gamma(1 - d/r, C \log(t)^{-1-s})$$

$$\sim C \log(t)^{-1-s}, \quad t \to \infty.$$ 

In conclusion, both methods produces the same type of long time behavior for $d \geq 3$, in addition for $d = 1$ and $d = 2$ we are also able to obtain a decay using this alternative Laplace transform method.
A  Bernstein, Complete Bernstein and Stieltjes Functions

In this appendix we collect certain notions of functions theory needed throughout the paper. Namely, the classes of completely monotone, Stieltjes, Bernstein functions and complete Bernstein functions. They are used in connection with the properties of the Laplace transform (LT). More details on these classes may be found in [SSV12].

Completely monotone functions. The LT (one-sided) of a function \( f : [0, \infty) \to [0, \infty) \) or a measure \( \mu \) on \( B([0, \infty)) \) is defined by

\[
\tilde{f}(p) := (L f)(p) := \int_0^{\infty} e^{-p\tau} f(\tau) \, d\tau \quad \text{or} \quad (L \mu)(p) := \int_{[0,\infty)} e^{-p\tau} \, d\mu(\tau),
\]

respectively, whenever these integrals converge. It is clear that \( L u = L \mu_u \) if \( d\mu_u(\tau) = u(\tau) \, d\tau \). Finite measures on \([0, \infty)\) are uniquely determined by their LT.

**Definition A.1.** A \( C^\infty \)-function \( \varphi : [0, \infty) \to \mathbb{R} \) is called completely monotone if

\[
(-1)^n \varphi^{(n)}(\tau) \geq 0, \quad \forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \tau > 0.
\]

The family of all completely monotone functions will be denoted by \( \mathcal{CM} \).

The function \([0, \infty) \ni \tau \mapsto e^{-\tau t}, 0 \leq t < \infty\) is a prime example of a completely monotone function. In fact, any element \( \varphi \in \mathcal{CM} \) can be written as an integral mixture of this family. This is precisely the contents of the next theorem, due to Bernstein, on the characterization of the class \( \mathcal{CM} \) in terms the LT of positive measures supported on \([0, \infty)\). For the proof we refer to [SSV12, Thm. 1.4].

**Theorem A.2** (Bernstein). Let \( \varphi : (0, \infty) \to \mathbb{R} \) be a completely monotone function.

1. Then there exists a unique measure \( \mu \) on \([0, \infty)\) such that

\[
\tilde{\varphi}(p) = \int_{[0,\infty)} e^{-p\tau} \, d\mu(\tau), \quad p > 0.
\]
2. Conversely, whenever \( \tilde{\varphi}(p) < \infty, \forall p > 0 \), the function \([0, \infty) \ni p \mapsto \tilde{\varphi}(p)\) is completely monotone, that is \(\varphi\) belongs to the class \(\mathcal{CM}\).

**Remark A.3.** The class \(\mathcal{CM}\) of completely monotone functions is easily seen to be closed under pointwise addition, multiplication and convergence. However, the composition of elements of the class \(\mathcal{CM}\) is, in general, not completely monotone.

**Stieltjes functions.** A subclass of the completely monotone functions is the, so called Stieltjes functions, and they play a central role in the study of complete Bernstein functions, defined below.

**Definition A.4.** A non-negative function \(\varphi : (0, \infty) \longrightarrow [0, \infty)\) is a Stieltjes function if it can be written in the form

\[
\varphi(\tau) = \frac{a}{\tau} + b + \int_{(0, \infty)} \frac{1}{\tau + t} d\sigma(t),
\]

where \(a, b \geq 0\) and \(\sigma\) is a Borel measure on \((0, \infty)\) such that

\[
\int_{(0, \infty)} (1 + t)^{-1} d\sigma(t) < \infty.
\]

The family of all Stieltjes functions we denote by \(\mathcal{S}\).

**Remark A.5.** 1. The integral in (A.1) is called the Stieltjes transform of the measure \(\sigma\).

2. Using the elementary identity

\[
(\tau + t)^{-1} = \int_0^\infty e^{-s(\tau + t)} \, ds
\]

and the Fubini theorem we see that the integral appearing in (A.1) is also a double Laplace transform and \(\varphi\) may be written as

\[
\varphi(\tau) = \frac{a}{\tau} + b + \int_0^\infty e^{-\tau s} h(s) \, ds,
\]

where

\[
h(s) = \int_{(0, \infty)} e^{-st} d\sigma(t)
\]
is a completely monotone function whose LT \( \tilde{h}(p) \) exists for any \( p > 0 \). In particular, we see that \( S \subset \mathcal{C}M \) and \( S \) consists of all \( \varphi \in \mathcal{C}M \) such that its representation measure (from Theorem A.2) has a completely monotone density on \( (0, \infty) \), for \( \varphi \in S \) is of the form

\[
\varphi(p) = (\mathcal{L} a \cdot dt)(p) + (\mathcal{L} b \cdot \delta_0(dt))(p) + (\mathcal{L}(\mathcal{L} \sigma)(t)dt)(p).
\]

**Example A.6.** The following are examples of Stieltjes functions for any \( \tau, t > 0 \)

\[
\begin{align*}
\varphi_1(\tau) &= 1, &\varphi_2(\tau) &= \frac{1}{\tau}, &\varphi_3(\tau) &= (\tau + t)^{-1}, &\varphi_4(\tau) &= \frac{1 + t}{\tau + t}, \\
\varphi_5(\tau) &= \tau^{a-1}, &\varphi_6(\tau) &= \frac{1}{\sqrt{\tau}} \arctan \frac{1}{\sqrt{\tau}}, &\varphi_7(\tau) &= \frac{1}{\tau} \log(1 + \tau).
\end{align*}
\]

**Bernstein functions.** Now we introduce the class of Bernstein functions which are closely related to completely monotone function. Bernstein functions are also known in probabilistic terms as Laplace exponents.

**Definition A.7.** 1. A \( C^\infty \)-function \( \varphi : (0, \infty) \rightarrow \mathbb{R} \) is called a Bernstein function if \( \varphi(\tau) \geq 0 \) for all \( \tau > 0 \) and

\[
(-1)^{n-1} \varphi^{(n)}(\tau) \geq 0, \quad \forall n \in \mathbb{N}, \tau > 0.
\]

2. Equivalently, a function \( \varphi : (0, \infty) \rightarrow \mathbb{R} \) is a Bernstein function, if, and only if, it admits the representation

\[
\varphi(\tau) = a + b\tau + \int_{(0,\infty)} (1 - e^{-\tau t})d\mu(t), \quad (A.3)
\]

where \( a, b \geq 0 \) and \( \mu \) is a Borel measure on \( (0, \infty) \), called the Lévy measure, satisfying

\[
\int_{(0,\infty)} (1 \wedge t) d\mu(t) < \infty. \quad (A.4)
\]

The Lévy triplet \((a, b, \mu)\) determines \( \varphi \) uniquely and vice versa. In particular,

\[
a = \varphi(0+), \quad b = \lim_{\tau \to \infty} \frac{\varphi(\tau)}{\tau}.
\]
3. The class of Bernstein function will be denoted by $\mathcal{BF}$.

The following structural characterization theorem of Bernstein functions is due to Bochner, see [SSV12, Thm 3.7] for the proof.

**Theorem A.8.** Let $\varphi : (0, \infty) \rightarrow \mathbb{R}$ be a positive function. The following assertions are equivalent.

1. $\varphi \in \mathcal{BF}$.
2. $f \circ \varphi \in \mathcal{CM}$, for every $f \in \mathcal{CM}$.
3. $e^{-\tau \varphi} \in \mathcal{CM}$ for every $\tau > 0$.

**Example A.9.** The following are Bernstein functions

$$\varphi_1(\tau) = \tau^\alpha, \ 0 < \alpha < 1, \ \text{or} \ \varphi_2(\tau) = \frac{\tau}{1 + \tau}, \ \text{or} \ \varphi_3(\tau) = \log(1 + \tau)$$

which are obtained as an integral mixture of the extremal Bernstein functions

$$e_0(\tau) = \tau, \quad e_t(\tau) = \frac{1 + t}{t} (1 - e^{-\tau t}), \ 0 < t < \infty, \ \text{and} \ e_\infty(\tau) = 1$$

by the measures $d\mu(t) = \frac{\alpha}{\Gamma(1-\alpha)} t^{-1-\alpha}, \ 0 < \alpha < 1$, $d\mu(t) = e^{-t}$ and $d\mu(t) = t^{-1} e^{-t}$, respectively.

**Complete Bernstein functions.** Finally, we introduce the fourth class of functions, so called complete Bernstein functions, which are Bernstein functions where the Lévy measure $\mu$ in the representation (A.3) has a nice density.

**Definition A.10.** A Bernstein function $\varphi$ is said to be a complete Bernstein function if its Lévy measure $\mu$ in (A.3) has a density $\rho$ with respect to the Lebesgue measure with $\rho \in \mathcal{CM}$. Thus, (A.3) takes the form

$$\varphi(\tau) = a + b\tau + \int_0^\infty (1 - e^{-\tau t}) \rho(t) \, dt, \quad (A.5)$$

such that by (A.4) we have

$$\int_0^\infty (1 \wedge t) \rho(t) \, dt < \infty.$$

The class of complete Bernstein functions we denote by $\mathcal{CBF}$. 

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The following theorem gives the characterization of complete Bernstein functions, cf. [SSV12, Thm 6.2]

**Theorem A.11.** Let $\varphi : (0, \infty) \rightarrow \mathbb{R}$ be a given non-negative function, then the following expression are equivalent.

1. $\varphi \in \text{CBF}$.

2. The function $(0, \infty) \ni \tau \mapsto \tau^{-1}\varphi(\tau)$ belongs to $\mathcal{S}$.

3. There exists a Bernstein function $\psi$ such that
   
   \[ \varphi(\tau) = \tau^2(\mathcal{L}\psi)(\tau), \quad \tau > 0. \]

4. $\varphi$ has an analytic continuation to the upper plane $\mathbb{C}_{>0} := \{z \in \mathbb{C} \mid \Im z > 0\}$ such that $\Im \varphi(z) \geq 0$ for all $z \in \mathbb{C}_{>0}$ and the limit $\varphi(0+) = \lim_{\tau \downarrow 0} \varphi(\tau)$ exists and is real.

5. $\varphi$ has an analytic continuation to the cut complex plane $\mathbb{C} \setminus (\infty, 0]$ such that $\Im z \cdot \Im \varphi(z) \geq 0$ for all $z \in \mathbb{C} \setminus (\infty, 0]$ and the limit $\varphi(0+) = \lim_{\tau \downarrow 0} \varphi(\tau)$ exists and is real.

6. $\varphi$ has an analytic continuation to $\mathbb{C}_{>0}$ which is given by
   
   \[ \varphi(z) = a + bz + \int_{(0, \infty)} \frac{z}{z + t} d\sigma(t), \quad (A.6) \]
   
   where $a, b \geq 0$ and $\sigma$ is a Borel measure on $(0, \infty)$ satisfying (A.2).

**Remark A.12.** The constants $a, b$ appearing in both representations (A.6) and (A.5) are the same. The relation between the density $\rho$ appearing in (A.5) of the function $\varphi \in \text{CBF}$ and the measure $\sigma$ corresponding to the Stieltjes function $\psi(\tau) = \tau^{-1}\varphi(\tau)$ is given by

\[ \rho(\tau) = \int_{(0, \infty)} e^{-\tau t} d\sigma(t). \]

The next theorem shows certain nonlinear properties of the class $\text{CBF}$ which gives rise to many applications of this class. Below we use the shorthand notation $\text{CBF} \circ \mathcal{S} \subset \mathcal{S}$ to indicate that the composition of any $\varphi \in \text{CBF}$ and $f \in \mathcal{S}$ is an element of $\mathcal{S}$, etc.
Theorem A.13.  1. \( \varphi \in \text{CBF}\{0\} \) if, and only if, \( \varphi^*(\tau) := \tau/\varphi(\tau) \) belongs to \( \text{CBF} \). The call \((\varphi, \varphi^*)\) the conjugate pair of complete Bernstein functions.

2. A function \( \varphi \neq 0 \) is a complete Bernstein function if, and only if, \( 1/\varphi \) is a non-trivial Stieltjes function.

3. \( \varphi \in \text{CBF} \) if, and only if, \((\tau + \varphi)^{-1} \in S\) for every \( \tau > 0 \).

4. \( \text{CBF} \circ S \subset S \).

5. \( S \circ \text{CBF} \subset \text{CBF} \).

6. \( \text{CBF} \circ \text{CBF} \subset \text{CBF} \).

7. \( S \circ S \subset \text{CBF} \).

We conclude this subsection with some examples of elements in the class \( \text{CBF} \).

Example A.14. The following are typical examples of complete Bernstein functions

\[ \varphi_1(\tau) = 1, \quad \varphi_2(\tau) = \tau, \quad \text{and} \quad \varphi_3(\tau) = \frac{\tau}{\tau + t}, \quad 0 < t < \infty. \]

Using the representation (A.6) with Stieltjes measures \( \sigma \) of the forms

\[ \frac{1}{\pi} \sin(\alpha \pi) t^{\alpha - 1} dt, \quad \mathbb{1}_{(0,1)}(t) \frac{dt}{2\sqrt{t}} \quad \text{and} \quad \frac{1}{t} \mathbb{1}_{(1,\infty)}(t) dt, \]

we see that the functions

\[ \varphi_4(\tau) = \tau^\alpha, \quad 0 < \alpha < 1, \quad \varphi_5(\tau) = \sqrt{\tau} \arctan \frac{1}{\sqrt{\tau}}, \quad \text{and} \quad \varphi_6(\tau) = \log(1 + \tau) \]

are also complete Bernstein functions.
The Karamata Tauberian Theorem

Tauberian theorems deals with the deduction of the asymptotic behavior of functions from a certain class (regular varying in the original of Karamata [Kar33]) from the asymptotic behavior of their transforms (e.g., their Laplace-Stieltjes transforms). We refer to [Sen76, Sec. 2.2] and [BGT87] for more details and proofs.

Let $A > 0$ be given and denote by $\mathcal{F}_+(A)$ the class of positive measurable functions defined on $[A, \infty)$.

**Definition B.1** (Regular and slowly varying functions). Let $f \in \mathcal{F}_+(A)$ be given. We say that $f$ is

1. **regular varying function** (RVF) at infinity in the sense of Karamata if the limit
   \[ K_f(\lambda) := \lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} \]
   exists and is finite for all $\lambda > 0$.

2. **slowly varying function** (SVF) if
   \[ K_f(\lambda) = 1, \quad \forall \lambda > 0. \]

**Proposition B.2.** Let $f \in \mathcal{F}_+(A)$ be a RVF.

1. Then there is a real number $\rho$ (called index of the function $f$) such that
   \[ K_f(\lambda) = \lambda^\rho, \quad \lambda > 0. \]
   The index $\rho = 0$ characterizes the SVF, that is $K_f(\lambda) = \lambda^0 = 1$.

2. Any RVF $f$ of index $\rho$ is represented as
   \[ f(x) = x^\rho l(x), \quad \forall x \geq A, \]
   where $l$ is a corresponding SVF.

**Proposition B.3** (See [BGT87, Prop. 1.3.6]).

1. If $L$ is a SVF, then \( \frac{\log(L(x))}{\log(x)} \to 0, \ x \to \infty. \)

2. If $L$ is a SVF, so does $(L(x))^\alpha$ for any $\alpha \in \mathbb{R}$. 

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3. If $L_1, L_2$ vary slowly, so do $L_1(x)L_2(x)$, $L_1(x)+L_2(x)$, and (if $L_2(x) \to \infty$ as $x \to \infty$) $L_1(L_2(x))$.

We say that the functions $f$ and $g$ are asymptotically equivalent at infinity, and denote $f \sim g$ as $x \to \infty$, meaning that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$ 

**Theorem B.4** (Karamata Tauberian Theorem). Let $U : [0, \infty) \to \mathbb{R}$ be a monotone non-decreasing function such that

$$w(p) := \int_0^\infty e^{-pt} dU(t) < \infty, \quad \forall p > 0.$$ 

Then, if $C, \rho \geq 0$ and $L$ is a slowly varying function, we have

1. $w(p) = p^{-\rho}L\left(\frac{1}{p}\right)$ as $p \to 0^+$ $\implies U(t) \sim t^\rho L(t)/\Gamma(\rho + 1)$ as $t \to \infty$;

2. $w(p) = p^{-\rho}L(p)$ as $p \to \infty$ $\implies U(t) \sim t^\rho L\left(\frac{1}{t}\right)/\Gamma(\rho + 1)$ as $t \to 0^+$.

**Founding**

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