Bound on FWER for correlated normal distribution

Nabaneet Das\textsuperscript{1} and Subir K. Bhandari\textsuperscript{2}

\textsuperscript{1}Indian Statistical Institute, Kolkata
\textsuperscript{2}Indian Statistical Institute, Kolkata

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Abstract

In this paper, our main focus is to obtain an asymptotic bound on the family wise error rate (FWER) for Bonferroni-type procedure in the simultaneous hypotheses testing problem when the observations corresponding to individual hypothesis are correlated. In particular, we have considered the sequence of null hypotheses $H_0^i : X_i \sim N(0, 1)$ ($i = 1, 2, ..., n$) and equicorrelated structure of the sequence $(X_1, ..., X_n)$. Distribution free bound on FWER under equicorrelated setup can be found in [19]. But the upper bound provided in [19] is not a bounded quantity as the no. of hypotheses $n$ gets larger and larger and as a result, FWER is highly overestimated for the choice of a particular distribution (e.g., normal). In the equicorrelated normal setup, we have shown that FWER asymptotically is a convex function (as a function of correlation ($\rho$)) and hence an upper bound on the FWER of Bonferroni-$\alpha$ procedure is $\alpha(1 - \rho)$. This implies, Bonferroni’s method actually controls the FWER at a much smaller level than the desired level of significance under the positively correlated case and necessitates a correlation correction.

1 Introduction

Multiple hypothesis testing has been one of the most lively area of research in statistics for the past few decades. The biggest challenge in this area comes from the fact that the models involve an extensive collection of unknown parameters and one has to draw simultaneous inference on a large number of hypotheses mainly with the goal of ensuring a good overall performance (rather than focusing too much on the individual problems). Very often data sets from modern scientific investigations, in the field of Biology, astronomy, economics etc. require such simultaneous testing on thousands of hypotheses.

Various measures of error rate have been proposed over the years. One of the hard-line frequentist approach is to control the family wise error rate (FWER) which is defined as the probability of making at least one false rejection in a family of hypothesis-testing problem.

Bonferroni’s bound provides the classical FWER control method. However, the step-up and step-down algorithms by [11], [17], [7], [12], [10] provides improvement over the Bonferroni’s method in terms of power. While Holm’s procedure provides control over the FWER in general, the other algorithms depend heavily on the independence of the p-values of the individual hypothesis. [3], [4], [7] provides excellent review of the whole theory.

One of the main limitations of these classical methods that control FWER in the strong sense, is their conservative nature which results in lack of power. A substantial improvement in power has been achieved by considering the False discovery rate (FDR) criterion proposed by [1]. See, for example [2], [20], [18], [16], [13] for further details. [14], [7] provides an excellent account on the literature on FDR.

However, most of these works have been done in the context of independent observations. Very
little literature can be found that covered correlated variables. [14] reviews FDR control under dependence set up. [6] clearly shows the effects of correlation on the summary statistics by pointing out that the correlation penalty depends on the root mean square (rms) of correlations. An excellent review of the whole work can be found in [7]. All these works gives immense light in the context that FWER or FDR should be treated more carefully where correlation is present.

Some distribution free bound on FWER can be found in [19] using Chebyshev-type inequalities. But as these inequalities are distribution free, FWER is highly overestimated for choice of particular distributions (e.g.- normal). Also, these inequalities are not of much use for a large number of hypotheses.

In our work, we have considered equicorrelated normal distribution and obtained a sharper bound on the FWER for the Bonferroni type FWER control procedure. We have shown that, asymptotically (For large no. of hypotheses) $FWER(\rho)$ is a convex function in $\rho \in [0,1]$ and hence FWER in Bonferroni-$\alpha$ procedure is bounded by $\alpha(1-\rho)$. This suggests a necessary correlation correction in Bonferroni procedure. While the bound provided in [19] is not a bounded quantity as no. of hypotheses gets larger and larger, the bound provided in this work remains stable even as $n \to \infty$ and shows a clearer picture of the effect of correlation on FWER. This is probably the first attempt in the context of finding most classically used FWER in terms of $\rho$ asymptotically. We, in further communication expect to attempt the same problem in terms of root mean square (rms) of correlations as attempted in [6].

2 Description of the problem

Let $X_1, X_2, \ldots$ be a sequence of observations and the null hypotheses are

$$H_{0i} : X_i \sim N(0,1) \, i = 1,2,\ldots$$

Many single hypothesis testing problems focus on the one sided tests which rejects the null hypothesis for large values of the observation. We’ll consider here the one-sided test : - Reject $H_{0i}$ if $X_i > c$. ($c$ is chosen according to the required significance level of the test). The two-sided hypothesis problem can be solved in the similar manner.

Under this setup a classical measure of the type-I error is FWER which is the probability of falsely rejecting at least one null hypothesis (Which happens if $X_i > c$ for some $i$ and the probability is computed under the intersection null hypothesis $\bigcap_{i=1}^{n} H_{0i} = \{X_i \sim N(0,1) \forall i = 1,2,\ldots,n\}$). When we compute under the intersection null hypothesis, FWER and FDR are the same. Hence this study also gives an idea of the behaviour of FDR under this setup.

$$\text{FWER} = P(\text{At least one false rejection}) = P( X_i > c \text{ for some } i \mid H_0 )$$

Suppose, $\text{Corr}(X_i,X_j) = \rho \, \forall i \neq j \, (\rho \geq 0)$.

Our goal is to provide an asymptotic bound on FWER in terms of $\rho$.

Let, $H(\rho) = 1- FWER(\rho) = P(X_i \leq c, \forall i = 1,2,\ldots,n)$

3 Main theorem

Theorem 3.1 Suppose each $H_{0i}$ is being tested at size $\alpha_n$. If $\lim_{n \to \infty} n\alpha_n = \alpha \in (0,1)$ then, as $n \to \infty$, $H'(\rho) \leq 0$ and hence $H(\rho)$ asymptotically is a concave function in $[0,1]$.

Note :-

1. For $\rho = 0$ (Under independence), we must have , $\text{FWER} = 1 - (1-\alpha)^n \approx n\alpha$. 

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2. For $\rho = 1$ (When $X_i = X_j$ a.s. $\forall i \neq j$), we must have FWER = $\alpha$ (Because one rejection would imply rejection of all null hypotheses).

Suppose $y = \mathcal{L}(\rho)$ denotes the line which joins $(1, \alpha)$ and $(0, 1 - (1 - \alpha)^n)$. The following corollary describes the asymptotic behaviour of FWER as a function of $\rho$.

**Corollary 3.1.1** As $n \to \infty$, FWER ($\rho$) is bounded above by the line $\mathcal{L}(\rho)$.

In this section we are going to provide a proof of this theorem.

### 3.1 An alternate form of $H(\rho)$ and it’s derivatives

Under the framework described above, we can say that under $H_0$, the sequence $\{X_n\}_{n \geq 1}$ is exchangeable. (i.e. $(X_1, ..., X_k) \sim N_k(0_k, (1 - \rho)I_k + \rho J_k)$ (Where $J_k$ is a $k \times k$ matrix of ones). Then, $X_k = \theta + Z_k$, $\forall k \geq 1$.

Where $\theta$ is a mean 0, normal random variable, independent of the sequence $\{Z_n\}_{n \geq 1}$ and $Z_i$’s are i.i.d. normal random variables.

Since Cov$(X_i, X_j) = \rho$ this implies, Var$(\theta) = \rho$.

Thus, $\Rightarrow \theta \sim N(0, \rho)$ and $Z_n \sim N(0, 1 - \rho)$ $\forall n \geq 1$

\[ H(\rho) = P(\theta + Z_i \leq c \; \forall i = 1, 2, ..., n) = E[\Phi^n(\frac{c - \theta}{\sqrt{1 - \rho}})] = E[\Phi^n(\frac{c + \sqrt{n}Z}{\sqrt{1 - \rho}})] \]

(Where $Z \sim N(0,1)$ and $\Phi$ is the c.d.f. of N(0,1) distribution)

If we define $d = \frac{c + \sqrt{n}Z}{\sqrt{1 - \rho}}$, then $H(\rho) = E[\Phi^n(d)]$.

Now, an application of dominated convergence theorem would yield,

\[ H'(\rho) = E[n\Phi^{n-1}(d)\phi(d)G(\rho, Z)] \quad (1) \]

(where $G(\rho, Z) = \frac{\partial^2}{\partial \rho^2} = \frac{1}{2(1-\rho)^2}[c + \frac{Z}{\sqrt{\rho}}]$ and $\phi(.)$ is the N(0,1) p.d.f.)

And again by D.C.T.,

\[ H''(\rho) = E[\frac{n}{2}\Phi^{n-2}(d)\phi(d)[aG^2(\rho, Z) + bG(\rho, Z) + \frac{c\Phi(d)}{4\rho(1-\rho)^{\frac{3}{2}}}]] \quad (2) \]

Where, $a = (n - 1)\phi(d) - d\Phi(-d)$ and $b = \frac{(4\rho - 1)\Phi(d)}{2\rho(1-\rho)}$.

Let’s define, $\alpha_1 = \Phi(-d)$.

Then, note that,

\[ H''(\rho) = E[\frac{n}{2}(1 - \alpha_1)^{n-2}\phi(d)[aG^2(\rho, Z) + bG(\rho, Z) + \frac{c(1-\alpha_1)}{4\rho(1-\rho)^{\frac{3}{2}}}] = \int_{-\infty}^{\infty} \frac{n}{2}(1 - \alpha_1)^{n-2}\phi(d)[aG^2(\rho, z) + bG(\rho, z) + \frac{c(1-\alpha_1)}{4\rho(1-\rho)^{\frac{3}{2}}}]\phi(z)dz \]
4 Proof of the main theorem

The proof of the main theorem involves three steps.

- **Step 1:** The second and third term in \( H''(\rho) \to 0 \) as \( n \to \infty \). (Proof is given in appendix (lemma I)).

- **Step 2:** Suppose at \( z = z_0, \alpha_1(z_0) = \Phi(-d(z_0)) = \frac{1}{n} \). If \( G(\rho, z_0) > 0 \) then, the first term is asymptotically \( \leq 0 \).

- **Step 3:** If \( G(\rho, z_0) \leq 0 \) then the first term \( \to 0 \) as \( n \to \infty \).

Proof of step 1 is given in appendix (lemma I). We shall proceed with the proof of step 2.

4.1 Proof of step 2

We know that, \( x\Phi(x) \sim \phi(x) \) for large enough \( x \).

\[ d = \frac{c+\sqrt{\pi}z}{\sqrt{1-\rho}} \to^{s.s.} \infty \text{ as } n \to \infty \] (Because \( c \to^{s.s.} \infty \text{ as } n \to \infty \)).

So, \( a = (n-1)(\phi(d) - d\Phi(d) \sim d(n\alpha_1 - 1) \).

By lemma II in appendix, we can say that,

\[ E[\frac{n}{2}(1-\alpha_1)^{n-2}\phi(d)G^2(\rho, Z)] \sim E[\frac{n}{2}(1-\alpha_1)^{n-2}\phi(d)(n\alpha_1 - 1)G^2(\rho, Z)]. \]

\[ E[\frac{n}{2}(1-\alpha_1)^{n-2}\phi(d)(n\alpha_1 - 1)G^2(\rho, Z)] = \int n(n\alpha_1 - 1)(1-\alpha_1)^{n-2}\phi(d)G^2(\rho, Z)dz \]

\[ = \int n(n\alpha_1 - 1)(1-\alpha_1)^{n-2}\phi(d)G^2(\rho, z)\phi(z)dz \]

\[ \approx \int_0^1 n(n\alpha_1 - 1)(1-\alpha_1)^{n-2}K(\alpha_1)d\alpha_1 \]

(Where \( K(\alpha_1) = dG^2(\rho, z)\phi(z) \))

Here we have assumed that \( G(\rho, z_0) > 0 \).

First of all note that, \( \int_0^1 n(n\alpha_1 - 1)(1-\alpha_1)^{n-2}d\alpha_1 = 0 \)

Suppose, \( f_n(\alpha_1) = n(n\alpha_1 - 1)(1-\alpha_1)^{n-2} \)

We have, \( f_n(\alpha_1) > (\alpha_1 < \frac{1}{2}) \) and \( \int_0^1 f_n(\alpha_1)d\alpha_1 = 0 \).

Since \( \Phi(-d(z_0)) = \frac{1}{n} \), \( \frac{1}{n} < \frac{1}{2} \) for large enough \( n \), this means, \( d(z_0) > 0 \) for large \( n \).

Also, observe that, \( \alpha_1 \) is a decreasing function of \( z \). Since \( G(\rho, z_0) > 0 \), we can say that,

\( K(\alpha_1(z)) > K(\alpha_1(z')) \forall z' < z_0, z > z_0 \)

This means, \( \int_0^\frac{1}{2} |f_n(\alpha_1)|K(\alpha_1)d\alpha_1 > \frac{1}{2} \int_0^\frac{1}{2} |f_n(\alpha_1)|K(\alpha_1)d\alpha_1 \).

Since \( f_n(\alpha_1) < 0, \forall \alpha_1 \in (0, \frac{1}{2}) \), this means \( \int_0^1 f_n(\alpha_1)K(\alpha_1)d\alpha_1 \leq 0 \)

And thus, \( E[\frac{n}{2}(1-\alpha_1)^{n-2}\phi(d)(n\alpha_1 - 1)G^2(\rho, Z)] \leq 0 \) which gives us the desired result.

This completes the proof of step 2.

4.2 Proof of step 3

In this part we have assumed that \( G(\rho, z_0) \leq 0 \). The whole real line is broken in three disjoint regions \( \{ \alpha_1 > \frac{6\log n}{n} \}, \{ \alpha_1 < \frac{1}{n\log n} \} \) and \( \{ \frac{1}{n\log n} < \alpha_1 \leq \frac{6\log n}{n} \} \). We’ll show that the integral in these three regions separately \( \to 0 \) as \( n \to \infty \).
• **Case 1 :** \(\{\alpha_1 > \frac{6 \log n}{n}\} \)

If \(\alpha_1 > \frac{6 \log n}{n}\), then \((1 - \alpha_1)^{n-2} < \left(1 - \frac{6 \log n}{n}\right)^{n-2}\).

Using the fact that, \((1 - \frac{x}{n})^n \leq e^{-x} \forall x > 0, \forall n \in \mathbb{N}\), we can say that, \((1 - \alpha_1)^{n-2} < \frac{1}{n^4}\).

So, \(E[\mathcal{A}^2(1 - \alpha_1)^{n-2}\phi(d) aG^2(\rho, Z) I_{(\alpha_1 > \frac{6 \log n}{n})}] \leq \frac{1}{n} E[\frac{aG^2(\rho, Z) I_{(\alpha_1 > \frac{6 \log n}{n})}}{n^4}]\).

Each \(H_{0i}\) is being tested at size \(\alpha_n\) and we reject \(H_{0i}\) if \(X_i > c\).

So, \(\alpha_n = \Phi(-c)\).

As \(n \to \infty, c \to \infty\) by the condition \(\lim_{n \to \infty} n \alpha_n = \alpha \in (0, 1)\).

For large \(c\), we have \(\Phi(-c) \sim \frac{\phi(c)}{c} = \frac{1}{\sqrt{2\pi}c} \ldots \ldots \ldots \) (i)

Now, observe that, \(a = (n - 1)\phi(d) - d\Phi(d) \leq n - 1 + |d| \leq n - 1 + \frac{c + \sqrt{n}Z}{\sqrt{1 - \rho}}\)

and \(G^2(\rho, Z) \leq \frac{c^2 + \frac{\sigma^2}{n}}{2(1 - \rho)}\).

So, \(E[\frac{aG^2(\rho, Z)}{n^4} I_{(\alpha_1 > \frac{6 \log n}{n})}] \leq E[\frac{n(1 + \frac{c + \sqrt{n}Z}{\sqrt{1 - \rho}})(c^2 + \frac{\sigma^2}{n})}{(2(1 - \rho)n^4)}] \to 0\) as \(n \to \infty\).

• **Case 2 :** \(\{\alpha_1 < \frac{1}{n(\log n)^\gamma}\} \)

For all \(d > 0\), we must have \(\phi(d) \leq ad_1\).

This implies, \(|a| = |(n - 1)\phi(d) - d\Phi(d)| \leq n|d|\alpha_1 + |d|\).

\(n\alpha_1 < \frac{1}{(\log n)^\gamma}\). So, for large \(n, n\alpha_1 < 1\).

This implies, in this case, \(|a| \leq 2|d|\) for large \(n\).

Using the fact \(\phi(d) \leq ad_1\) we note that,

\(E[\mathcal{A}^2(1 - \alpha_1)^{n-2}\phi(d) aG^2(\rho, Z) I_{(\alpha_1 < \frac{1}{n(\log n)^\gamma})}] \leq E[\frac{aG^2(\rho, Z)}{(\log n)^\gamma} I_{(\alpha_1 < \frac{1}{n(\log n)^\gamma})}]\).

From the previous observation (in case 1) we can say that, \(d^2G^2(\rho, Z) \leq \frac{(c + \sqrt{n}Z)^\gamma(\frac{c^2 + \frac{\sigma^2}{n}}{2})}{2\rho(1 - \rho)^\gamma}\)

**Notation :** \(x_n = \Theta(y_n)\) if \(\exists c_1, c_2 > 0\) and \(M \in \mathbb{N}\) such that, \(c_1y_n \leq x_n \leq c_2y_n \forall n \geq M\)

Note that, \(\log n = \Theta(c^2).\) (By (i))

This implies, \(E[\frac{aG^2(\rho, Z)}{(\log n)^\gamma} I_{(\alpha_1 < \frac{1}{n(\log n)^\gamma})}] \to 0\).

Thus, \(E[\mathcal{A}^2(1 - \alpha_1)^{n-2}\phi(d) aG^2(\rho, Z) I_{(\alpha_1 < \frac{1}{n(\log n)^\gamma})}] \to 0\)

• **Case 3 :** \(\{\frac{1}{n(\log n)^\gamma} \leq \alpha_1 \leq \frac{6 \log n}{n}\} \)

If \(\{G(\rho, z_0) \leq 0\}\) then \(z_0 \leq -\sqrt{pc} \Rightarrow |z_0| \geq \sqrt{pc}\). (This means \(z_0\) takes a very large negative value.

Note that, \(d^2G^2(\rho, Z) = p(Z, c)\) (A polynomial in \(Z\) and \(c\)).

Since \(\lim_{|z| \to \infty} p(x, c)e^{-\frac{2}{|z|^2}} = 0\), this implies, \(\lim_{n \to \infty} p(z, c) \phi(z) = 0\) if \(z z_0\)

Also, note that, \(\int f_n(\alpha_1)d\alpha_1 = \int (k - 1)(1 - \frac{k}{n})^{n-2}dk \leq \int (x - 1)e^{-\frac{x}{n}}dx < \infty\)

\(= \int f_n(\alpha_1)d\alpha_1\) is bounded.

And hence, \(E[\mathcal{A}(1 - \alpha_1)^{n-2}\phi(d) d(n\alpha_1 - 1)G^2(\rho, Z) I_{(\alpha_1 > \frac{1}{n})}] = \int f_n(\alpha_1)K(\alpha_1)d\alpha_1 \to 0\)

Now we need to consider the region \(\{\frac{1}{n(\log n)^\gamma} \leq \alpha_1 \leq \frac{1}{n}\}\).

Suppose \(\Phi(-d(z_1)) = \frac{1}{\frac{1}{n(\log n)^\gamma}}\) and \(z_1 = z_0 + \delta_n\).

\(\Rightarrow \frac{\Phi(-d(z_1))}{\Phi(-d(z_1))} = (\log n)^3\)
For large $n$, $d(z_i)\Phi(-d(z_i)) \sim \phi(-d(z_i))$ for $i = 0, 1$.

\[ \Rightarrow \frac{d(z_i)e^{d(z_i)/2}}{d(z_0)e^{d(z_0)/2}} \sim (\log n)^3 \]

\[ \Rightarrow \log \left( \frac{d(z_i)}{d(z_0)} \right) + \frac{d^2(z_i) - d^2(z_0)}{2} \sim 3 \log(\log n) \]

Since $d(z_1) > d(z_0)$, we have

\[ \Rightarrow d(z_1) - d(z_0) = \sqrt{\left( \frac{\delta_n}{d(z_1) + d(z_0)} \right)} \leq \frac{6 \log(\log n)}{d(z_1) + d(z_0)} \]

Since $d(z_1) > d(z_0) = -\Phi^{-1}(\frac{1}{n}) = \Theta(\sqrt{\log n})$ this implies, $\frac{6 \log(\log n)}{d(z_1) + d(z_0)} \rightarrow 0$ as $n \rightarrow \infty$.

So, $E\left[ \left( \frac{n}{2} \right) (1 - \alpha_1)^{n-2}\phi(d)(d(n\alpha_1 - 1))G^2(\rho, Z)I_{\left( \frac{1}{n(\log n)^3} \alpha_1 \leq \frac{1}{n} \right)} \right]$

\[ \leq \int_{\frac{1}{n(\log n)^3}} n(1 - \alpha_1)^{n-2}K(\alpha_1) d\alpha_1 \]

Note that, $|K(\alpha_1)| \leq \max_{z \in [2, 0, z_0 + \delta_n]} p(z, c)\phi(z)$ (as $K(\alpha_1) = p(z, c)\phi(z)$)

Since $\delta_n \rightarrow 0$ and $p(z_0, c)\phi(z_0) \rightarrow 0$ as $n \rightarrow \infty$, by continuity of $p(z, c)\phi(z)$ we can say that, $\max_{z \in [2, 0, z_0 + \delta_n]} p(z, c)\phi(z) \rightarrow 0$ as $n \rightarrow \infty$.

\[ \int_{\frac{1}{n(\log n)^3}} n(1 - \alpha_1)^{n-2}d\alpha_1 = \int_{\frac{1}{(\log n)^3}} (1 - \frac{k}{n})^{n-2}dk \leq \int_{0}^{1} e^{-\frac{k}{n}}dk \leq 1. \]

So, $\int_{\frac{1}{n(\log n)^3}} n(1 - \alpha_1)^{n-2}K(\alpha_1) d\alpha_1 \rightarrow 0$

and hence $E\left[ \left( \frac{n}{2} \right) (1 - \alpha_1)^{n-2}\phi(d)(d(n\alpha_1 - 1))G^2(\rho, Z)I_{\left( \frac{1}{n(\log n)^3} \alpha_1 \leq \frac{1}{n} \right)} \right] \rightarrow 0$.

This completes the proof of main theorem.

### 5 Conclusion

From the theorem in the previous section, we have $H''(\rho) \leq 0$ asymptotically.

\[ \Rightarrow \text{FWER}''(\rho) \geq 0 \text{ as } n \rightarrow \infty. \]

This means for large $n$, FWER $(\rho)$ is bounded by $\mathcal{L}(\rho)$ in $[0, 1]$.

From this, we can conclude that,

**Theorem 5.1** For large $n$, FWER $(\rho) \leq \alpha_n + (1 - \rho)[1 - \alpha_n - (1 - \alpha_n)^n]$.

For large $n$, $1 - (1 - \alpha_n)^n \approx n\alpha_n$ and this implies, FWER $(\rho) \leq \alpha_n[n - (n - 1)\rho]$.

Bonferroni’s method suggests us to take $\alpha_n = \frac{\alpha}{n}$ if we want to maintain $\alpha$-FWER level. This satisfies the criterion of the main theorem of section 2.

When $\alpha_n = \frac{\alpha}{n}$, then $\alpha_n[n - (n - 1)\rho] \sim \alpha(1 - \rho)$.

This implies, the FWER of the Bonferroni’s procedure is bounded by $\alpha(1 - \rho)$.

### 6 Appendix

#### 6.1 Lemma I

**Lemma 6.1** The second and third term in $H''(\rho) \rightarrow 0$ as $n \rightarrow \infty$.

**Proof**: We shall do it only for the third term. The other one follows similarly. This proof is similar to the proof of case 2 $(G(\rho, z_0) < 0)$ of step 3 of the main theorem.
Third term in $H''(\rho)$ is $\propto E[n(c(1-\alpha_1)^{n-1}\phi(d))] = \int_{-\infty}^{\infty} nc(1-\alpha_1)^{n-1}\phi(d)\phi(z)dz$.
Let, $\phi(z) = M(\alpha_1)$.
Then, $\int_{-\infty}^{\infty} nc(1-\alpha_1)^{n-1}\phi(d)\phi(z)dz = c \int_{0}^{\frac{1}{n}}(1-\frac{k}{n})^{n-1}M(\frac{k}{n})dk$
Since $\phi(d) \leq d^2\alpha_1$ for large enough $d$, so
$$E[nc(1-\alpha_1)^{n-1}\phi(d)I_{(\alpha_1<\frac{1}{n(logn)^2})}] \to 0$$
Also, if $\alpha_1 > \frac{6log n}{n}$ then $(1-\alpha_1)^{n-2} \leq \frac{1}{n}$ and hence
$$E[nc(1-\alpha_1)^{n-1}\phi(d)I_{(\alpha_1>\frac{6log n}{n})}] \to 0$$
Let, $\Phi(-d(z_0-\delta''_n)) = \frac{1}{n(logn)^2}$ and $\Phi(-d(z_0+\delta''_n)) = \frac{6log n}{n}$.
Similar to the case 2 ($G(\rho, z_0) < 0$) of step 3 of the main theorem we can say that, $\max{\{\delta'_n, \delta''_n\}} \to 0$ as $n \to \infty$.
\Rightarrow If $\frac{1}{n(logn)^2} < \alpha_1 < \frac{6log n}{n}$ then $cM(\alpha_1) \leq \max_{z\in[z_0-\delta'_n, z_0+\delta''_n]} c\phi(z) \to 0$ as $n \to \infty$.
Thus, $E[nc(1-\alpha_1)^{n-1}\phi(d)I_{(\frac{1}{n(logn)^2}<\alpha_1<\frac{6log n}{n})}] \to 0$ and the third term in $H''(\rho) \to 0$.

6.2 Lemma II

Lemma 6.2 $E[\frac{E}{2}(1-\alpha_1)^{n-2}\phi(d)G^2(\rho, Z)|a-d(n\alpha_1-1)|] \to 0$ as $n \to \infty$.

Proof :-

$|a-d(n\alpha_1-1)| = (n-1)|\phi(d) - d\Phi(d)|$.
If $d \leq 0$ then $\alpha_1 \geq \frac{1}{2}$ and hence, $(1-\alpha_1)^{n-2} \leq \left(\frac{1}{2}\right)^{n-2}$. 
Since $n = \Theta(c \frac{\rho^2}{\epsilon})$, this immediately implies that,
$$E[\frac{E}{2}(1-\alpha_1)^{n-2}\phi(d)G^2(\rho, Z)|a-d(n\alpha_1-1)|I_{(d<0)}] \to 0$$

When $d > 0$, $|\phi(d) - d\Phi(d)| \leq K\alpha_1$ for a constant $K > 0$.
This implies, $E[\frac{E}{2}(1-\alpha_1)^{n-2}\phi(d)G^2(\rho, Z)|a-d(n\alpha_1-1)|I_{(d>0)}] \leq KE[n^2\alpha_1(1-\alpha_1)^{n-2}\phi(d)G^2(\rho, Z)I_{(d>0)}]$.
A similar idea to the proof of lemma-I will tell us that, we need to consider the region corresponding to $\frac{1}{n(logn)^2} < \alpha_1 < \frac{8log n}{n}$ and the integral corresponding to this region $\to 0$ follows exactly in the similar way as lemma - I.

7 Simulation results

Theorem 5.1 tells us for large $n$, FWER for Bonferroni’s method (with level of significance $\alpha$) is asymptotically bounded above by $\alpha(1-\rho)$. In order to verify this result empirically, some simulation results have been provided in table 1. In our simulation experiments, we have considered $\rho = 0, 0.1, 0.3, 0.5, 0.7, 0.9$ and $\alpha = 0.01, 0.05, 0.1, 0.4, 0.6, 0.7$. In each combination of $(\rho, \alpha)$, 10000 replications have been made to estimate the FWER (the estimate obtained is denoted by $FWER$). In each replication, we have generated 10000 equicorrelated normal random variables each with mean 0 and variance 1. Bonferroni’s method suggests us to reject $H_0$, at level $\alpha$ if $Z_i > (1 - \frac{\alpha}{10000})$-th quantile of $N(0,1)$ distribution. In each replication we have to note whether or not any of the 10000 $Z_i$’s exceeds that cut-off and then $FWER$ is obtained accordingly from the 10000 replications. Each $FWER$ obtained at the combination $(\rho, \alpha)$ is compared with $\alpha(1-\rho)$ (the upper bound
mentioned in section 5). It is impressive that in all the cases FWER is substantially smaller than \( \alpha(1 - \rho) \) (except at \((\rho, \alpha) = (0.1, 0.01)\) although the difference is not noteworthy). All these observations suggest that in positively correlated setup, Bonferroni’s method actually controls the FWER at a much smaller level than the desired level of significance which makes this method more conservative in this case.

| \( \rho \) | \( \alpha \) | 0.01  | 0.05  | 0.1   | 0.4   | 0.6   | 0.7   |
|----------|----------|-------|-------|-------|-------|-------|-------|
| 0.9      | FWER     | 9.00E-05 | 0.00046 | 0.00053 | 0.00221 | 0.00324 | 0.0031 |
|          | \( \alpha(1 - \rho) \) | 1.00E-03 | 0.005  | 0.01   | 0.04   | 0.06   | 0.07   |
| 0.7      | FWER     | 0.00101 | 0.00363 | 0.00588 | 0.01617 | 0.02149 | 0.023  |
|          | \( \alpha(1 - \rho) \) | 0.003  | 0.015  | 0.03   | 0.12   | 0.18   | 0.21   |
| 0.5      | FWER     | 0.00347 | 0.01156 | 0.01918 | 0.04909 | 0.06414 | 0.07042 |
|          | \( \alpha(1 - \rho) \) | 0.005  | 0.025  | 0.05   | 0.2    | 0.3    | 0.35   |
| 0.3      | FWER     | 0.00683 | 0.02523 | 0.04363 | 0.11495 | 0.15013 | 0.16494 |
|          | \( \alpha(1 - \rho) \) | 0.007  | 0.035  | 0.07   | 0.28   | 0.42   | 0.49   |
| 0.1      | FWER     | 0.00996 | 0.04367 | 0.07978 | 0.23801 | 0.31105 | 0.34295 |
|          | \( \alpha(1 - \rho) \) | 0.009  | 0.045  | 0.09   | 0.36   | 0.54   | 0.63   |
| 0        | FWER     | 0.01018 | 0.0486  | 0.09424 | 0.32914 | 0.45065 | 0.50499 |
|          | \( \alpha(1 - \rho) \) | 0.01   | 0.05   | 0.1    | 0.4    | 0.6    | 0.7    |

Table 1: Simulation results
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