DEFECT OF EUCLIDEAN DISTANCE DEGREE

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Abstract. Two well studied invariants of a complex projective variety are the unit Euclidean distance degree and the generic Euclidean distance degree. These numbers give a measure of the algebraic complexity for “nearest” point problems of the algebraic variety. It is well known that the latter is an upper bound for the former. While this bound may be tight, many varieties appearing in optimization, engineering, statistics, and data science, have a significant gap between these two numbers. We call this difference the defect of the ED degree of an algebraic variety. In this paper we compute this defect by classical techniques in Singularity Theory, thereby deriving a new method for computing ED degrees of smooth complex projective varieties.

1. Introduction

The unit Euclidean distance degree and the generic Euclidean distance degree are two well-studied invariants which give a measure of the algebraic complexity for “nearest” point problems of an algebraic variety.

Definition 1.1. Let $X$ be an irreducible closed subvariety of $\mathbb{C}^n$. The $w$-weighted Euclidean distance degree of $X$ is the number of complex critical points of

$$d_{u,w}(x) := \sum_{i=1}^{n} w_i(x_i - u_i)^2$$

on the smooth locus $X_{\text{reg}}$ of $X$, for generic data $u = (u_1, \ldots, u_n)$. We write this degree as $\text{EDdeg}_w(X)$. When $w$ is generic (resp., $w = 1$, the all ones vector) we call $\text{EDdeg}_w(X)$ the generic ED degree (resp., unit ED degree) of $X$ and write this as $g\text{EDdeg}(X)$ (resp., $u\text{EDdeg}(X)$).

The Euclidean distance degree was introduced in [8], and has since been extensively studied in areas like computer vision [29, 13, 21], biology [12], chemical reaction networks [1], engineering [7, 33], numerical algebraic geometry [14, 19], and data science [15]. Also of interest are ED-discriminant loci [16, 6], which characterize the meaning of “generic data” in terms of vanishing of polynomials, and the algebraic degree of other optimization problems [5, 17, 26].

For a projective variety $X$ one defines the $w$-weighted Euclidean distance degree of $X$ in terms of affine cones.

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Definition 1.2. If $X$ is an irreducible closed subvariety of $\mathbb{P}^n$, we define the (projective) $w$-weighted Euclidean distance degree of $X$ by $\text{EDdeg}^w(X) := \text{EDdeg}^w(C(X))$, where $C(X)$ is the affine cone of $X$ in $\mathbb{C}^{n+1}$.

It was proved in [22, Theorem 1.3] (see also [2, Theorem 8.1] for the smooth case) that $u\text{EDdeg}$ of a projective variety can be computed as an Euler characteristic weighted by a certain constructible function. More precisely, one has the following result:

Theorem 1.3. Let $X \subset \mathbb{P}^n$ be an irreducible closed subvariety. Then

\begin{equation}
\text{uEDdeg}(X) = (-1)^{\dim(X)} \chi(Eu_X|_{\mathbb{P}^n \setminus (Q \cup H)}),
\end{equation}

where $Eu_X$ is the local Euler obstruction function on $X$, $Q$ is the isotropic quadric $\{(x_0 : \cdots : x_n) \in \mathbb{P}^n | \sum_{i=0}^n x_i^2 = 0\}$, and $H$ is a general hyperplane in $\mathbb{P}^n$. In particular, if $X$ is smooth, then

\begin{equation}
\text{uEDdeg}(X) = (-1)^{\dim(X)} \chi(X \setminus (Q \cup H)).
\end{equation}

Moreover, the above result can be extended to the computation of $\text{EDdeg}^w(X)$, for an arbitrary weight $w$ (see Theorem 2.1 below).

The unit ED degree, $\text{uEDdeg}(X)$, is in general difficult to compute even if $X$ is smooth, since the isotropic quadric $Q$ may intersect $X$ non-transversally. On the other hand, for generic weight $w$, the quadric $Q_w$ intersects $X$ transversally, and the computation of $g\text{EDdeg}(X)$ is more manageable, see e.g., [8], [27], etc.

In this paper, we study the difference

$$\text{EDdefect}(X) := g\text{EDdeg}(X) - \text{uEDdeg}(X)$$

which we refer to as the defect of the Euclidean distance degree. It is known that $\text{EDdefect}(X)$ is non-negative, but for many varieties appearing in optimization, engineering, statistics, and data science, the defect is quite substantial. We give a new topological interpretation of this defect in terms of invariants of singularities of $X \cap Q$ when $X$ is a smooth irreducible complex projective variety in $\mathbb{P}^n$.

Even though both $g\text{EDdeg}(X)$ and $\text{uEDdeg}(X)$ can be computed by topological invariants, as seen in (2) and (8), our new approach provides a direct method for computing Euclidean degree defects. This approach is applied in Section 3 on very concrete examples. In particular, in Example 3.1 we show that the ED defect can be computed much easier than computing $g\text{EDdeg}(X)$ and $\text{uEDdeg}(X)$ individually.

Our results also recover and give a more conceptual interpretation of a result of Aluffi-Harris [2], which was obtained by characteristic class techniques. Before stating the main result of this paper, let us fix some notations.

Notation 1.4. Let $Q = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n | \sum_{i=0}^n x_i^2 = 0\}$ be the isotropic quadric, and let $X \subset \mathbb{P}^n$ be a smooth irreducible projective variety not contained in $Q$. Let $Z = \text{Sing}(X \cap Q)$ be the singular locus of $X \cap Q$, taken as the schematic intersection. Equivalently, $Z \subset X \cap Q$ is the locus where $X$ intersects $Q$ non-transversally. Let $\mathcal{X}$ be a Whitney stratification of $X \cap Q$, and denote by $\mathcal{X}_0$ the collection of strata contained in $Z$.

In the above notations, our main result can be stated as follows (see Theorem 2.9):
Theorem 1.5 (Main Result). Let $X \subset \mathbb{P}^n$ be a smooth irreducible projective variety not contained in the isotropic quadric $Q$. Then, under Notation 1.4,

\begin{equation}
\text{EDdefect}(X) = \sum_{V \in \mathcal{X}_0} (-1)^{\text{codim}_{X \cap Q} V} \alpha_V \cdot \text{gEDdeg}(V)
\end{equation}

with

\[ \alpha_V = \mu_V - \sum_{\{S \mid V < S\}} \chi_c(L_{V,S}) \cdot \mu_S, \]

where, for any stratum $V \in \mathcal{X}_0$,

\[ \mu_V = \chi(\tilde{H}^*(F_V; \mathbb{Q})) \]

is the Euler characteristic of the reduced cohomology of the Milnor fiber $F_V$ of the hypersurface $X \cap Q \subset X$ at some point in $V$, and $L_{V,S}$ is the complex link of a pair of distinct strata $(V, S)$ with $V \subset S$.

Remark 1.6. For the precise definition of the Milnor fiber $F_V$, see [25], and for the complex link $L_{V,S}$, see [9, Theorem 1.1] and [30, Theorem 2.10].

Remark 1.7. By Thom’s second isotopy lemma, the topological type of Milnor fibers is constant along the strata of a Whitney stratification $\mathcal{X}$ of the hypersurface $X \cap Q$ in $X$.

As an immediate consequence of Theorem 1.5, we get the following result (see Corollary 2.4, and compare also with [2, Corollary 6.3]):

Corollary 1.8 (Isolated Singularities). Under Notation 1.4, assume that $\text{Sing}(X \cap Q)$ consists of isolated points. Then:

\begin{equation}
\text{EDdefect}(X) = \sum_{x \in \text{Sing}(X \cap Q)} \mu_x,
\end{equation}

where $\mu_x$ is the Milnor number of the isolated singularity $x \in \text{Sing}(X \cap Q)$.

Furthermore, if $X \cap Q$ is equisingular along the non-transversal intersection locus $Z$, Theorem 1.5 yields the following:

Corollary 1.9 (Equisingular singular locus). Under Notation 1.4, assume that $Z = \text{Sing}(X \cap Q)$ is connected and $X \cap Q$ is equisingular along $Z$. Then:

\begin{equation}
\text{EDdefect}(X) = \mu \cdot \text{gEDdeg}(Z),
\end{equation}

where $\mu$ is the Milnor number of the isolated transversal singularity at some point of $x$ in $Z$ (i.e., the Milnor number of the isolated hypersurface singularity in a normal slice to $Z$ at $x$).

Theorem 1.5 is motivated by the duality conjecture of [27][(3.5)] in structured low-rank approximation, which predicts a formula for the Euclidean distance degree defect of the restriction of (the dual variety of) $X$ to a linear space $\mathcal{L}$. Since intersecting $X$ with a general linear space $\mathcal{L}$ does not change the multiplicities $\alpha_V$ on the right-hand side of formula (3), we get the following consequence of Theorem 1.5:
Corollary 1.10 (Intersection with linear space). With the notations as in Theorem 1.5, let \( \mathcal{L} \) denote a general linear subspace of \( \mathbb{P}^n \). Then
\[
\text{EDdefect}(X \cap \mathcal{L}) = \sum_{V \in \mathcal{X}_0} (-1)^{\text{codim} X \cap Q} \alpha_V \cdot \text{gEDdeg}(\bar{V} \cap \mathcal{L}).
\]

The proof of our main Theorem 1.5 relies on the theory of vanishing cycles adapted to a pencil of quadrics \( Q_w = \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n \mid w_0 x_0^2 + \cdots + w_n x_n^2 = 0 \} \) on \( X \), see Theorem 2.2. For a quick introduction to hypersurface singularities and vanishing cycles, the interested reader may consult [20][Chapter 10].

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2. Computation of ED defect via vanishing cycles

In this section, we compute the defect \( \text{EDdefect}(X) \) by using standard techniques in Singularity theory, such as Milnor fibers, vanishing cycles and the local Euler obstruction function.

We begin with the observation that the proof of Theorem 1.3 in [22] applies without change to the context of the \( w \)-weighted Euclidean distance degree \( \text{EDdeg}_w(X) \) on \( X \), if one uses instead the quadric \( Q_w = \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n \mid w_0 x_0^2 + \cdots + w_n x_n^2 = 0 \} \). More precisely, one has the following result.

Theorem 2.1. Let \( X \subset \mathbb{P}^n \) be an irreducible closed subvariety. Then,
\[
\text{EDdeg}_w(X) = (-1)^{\dim(X)} \chi(\text{Eu}_X|_{\mathbb{P}^n \setminus (Q_w \cup H)}),
\]
with \( H \) a general hyperplane. In particular, if \( X \) is smooth one has:
\[
\text{EDdeg}_w(X) = (-1)^{\dim(X)} \chi(X \setminus (Q_w \cup H)).
\]

From now on we assume in this section that \( X \) is a smooth irreducible complex projective variety in \( \mathbb{P}^n \), which is not contained in the isotropic quadric \( Q \). Then \( X \cap Q_w \) yields a pencil of hypersurfaces \( X_s = \{ f_s = 0 \}_{s \in \mathbb{P}^1} \) on \( X \), where \( s = [s_0 : s_1] \) and
\[
f_s = s_0 (x_0^2 + \cdots + x_n^2) + s_1 (w_0 x_0^2 + \cdots + w_n x_n^2).
\]
The generic member of the pencil is \( X_\infty := X \cap Q_w \) for \( w \) generic, and the singular member is \( X_0 := X \cap Q \) for \( Q = Q_1 \) the isotropic quadric. Moreover, the generic \( w \) can be chosen so that the generic member \( X_\infty = X \cap Q_w \) of the pencil is a smooth hypersurface in \( X \) (since in this case \( X \) and \( Q_w \) intersect transversally), which is transversal to the strata of a Whitney stratification of \( X_0 = X \cap Q \).

Consider the incidence variety of the pencil, that is,
\[
\tilde{X} := \{ (s,x) \in \mathbb{P}^1 \times X \mid x \in X_s \},
\]
which is just the blowup of $X$ along the base locus $X_0 \cap X_\infty$ of the pencil. Let $\pi : \tilde{X} \to \mathbb{P}^1$ be the projection map, hence $X_s = \pi^{-1}(s)$ for any $s \in \mathbb{P}^1$. Let

$$f := \frac{f_0}{f_\infty} : X \setminus X_\infty \subset \tilde{X} \to \mathbb{C}$$

with $f^{-1}(0) = X_0 \setminus X_\infty$.

With the above assumptions and notations, we can now prove the following result:

**Theorem 2.2.** Let $X \subset \mathbb{P}^n$ be a smooth irreducible complex projective variety, and let $w$ be a generic weight. Then:

$$\chi(X) - \chi(X_0) = \chi(\varphi_f(1_{X \setminus X_\infty})), \quad (9)$$

**Proof.** In the above notations and for generic $w$, additivity properties of the Euler characteristic for complex algebraic varieties, together with formulae (2) and (8) yield (here we choose a hyperplane $H$ which is generic in both situations):

$$\chi(X) - \chi(X_0) = \chi(\varphi_f(1_{X \setminus X_\infty})), \quad (10)$$

Furthermore, it follows from [20, Section 10.4] (see also [28, Proposition 5.1] and [23, Proposition 4.1]) that one has the following identity:

$$\chi(X) - \chi(X_0) = \chi(\varphi_f(1_{X \setminus X_\infty})), \quad (11)$$

where

$$\varphi_f : CF(X \setminus X_\infty) \to CF(X_0 \setminus X_\infty)$$

denotes the vanishing cycle functor defined on constructible functions, and $1_{X \setminus X_\infty}$ is the constant function 1 on $X \setminus X_\infty$. Similarly, by restricting to the generic (hence smooth) hyperplane section $X^H := X \cap H$ of $X$, and working with the pencil $X_s^H := X_s \cap H$ on $X^H$ and the restricted function $f^H := f|_H$, one gets that:

$$\chi(X^H_s) - \chi(X^H_0) = \chi(\varphi_f(1_{X^H_s \setminus X^H_\infty})). \quad (12)$$

Using the base change isomorphism of [31, Lemma 4.3.4], we also have that

$$\varphi_f(1_{X^H_s \setminus X^H_\infty}) = \varphi_f(1_{X_s \setminus X_\infty})|_H. \quad (13)$$

Substituting the identities (11), (12), and (13) in (10) we get:

$$\chi(X^H_s) - \chi(X^H_0) = \chi(\varphi_f(1_{X_s \setminus X_\infty})), \quad (14)$$

where the last equality uses the fact that $H$ is generically chosen.

**Remark 2.3.** We further note that for generic weight $w$, the constructible function $\varphi_f(1_{X \setminus Q_w})|_{X \setminus (Q_w \cup H)}$ is in fact supported on the (complement of $H$ in the) singular locus of the zero-fiber of $f$, i.e., on $\text{Sing}(X \cap Q) \setminus (Q_w \cup H)$.

As an immediate consequence of Theorem 2.2, we get the following result (also proved in [2, Corollary 6.3] by using characteristic classes):
Corollary 2.4. Under Notation 1.4, assume that \( \text{Sing}(X \cap Q) \) consists of isolated points. Then:

\[
\text{EDdefect}(X) = \sum_{x \in \text{Sing}(X \cap Q)} \mu_x,
\]

where \( \mu_x \) is the Milnor number of the isolated singularity \( x \in \text{Sing}(X \cap Q) \).

Remark 2.5. In the statement of Corollary 2.4 we use the fact that the Milnor fibration of a hypersurface singularity germ does not depend on the choice of a local equation for the germ. In particular, at points \( x \not\in X_0 \cap X_\infty \) one can use freely \( f \) in place of \( f_0 \) (and vice versa) when considering Milnor fibers of such points in \( X_0 = X \cap Q \).

As another important special case, assume that \( Z = \text{Sing}(X \cap Q) \) is a closed (smooth and connected) stratum in a Whitney stratification of \( X_0 = X \cap Q \), that is, \( X_0 \) is equisingular along \( Z \). Then the Milnor fiber of \( f_0 \) at any point \( x \in Z \) has the homotopy type of a bouquet of spheres of dimension \( \dim(X_0) - \dim(Z) \), and let us denote by \( \mu \) the number of these spheres (this is the transversal Milnor number at \( x \in Z \), i.e., the Milnor number of the isolated singularity at \( x \) in a normal slice to the stratum \( Z \)). In particular, using Remark 2.5, we get in this case that:

\[
\chi(\varphi_f(1_{X \setminus X_\infty})|_{X \setminus (X_\infty \cup H)}) = (-1)^{\dim(X_0) - \dim(Z)} \mu \cdot \chi(Z \setminus (Q_w \cup H))
\]

\[
= (-1)^{\dim(X_0)} \mu \cdot g\text{EDdeg}(Z),
\]

where the second equality follows from (8). Then Theorem 2.2 yields the following:

Corollary 2.6. Under Notation 1.4, assume that \( Z = \text{Sing}(X \cap Q) \) is connected and \( X \cap Q \) is equisingular along \( Z \). Then:

\[
\text{EDdefect}(X) = \mu \cdot g\text{EDdeg}(Z),
\]

where \( \mu \) is the Milnor number of the isolated transversal singularity at some point of \( Z \).

For the remaining of this section, we deal with the case when \( Z = \text{Sing}(X \cap Q) \) is itself Whitney stratified by arbitrary singularities. Choose as before a Whitney stratification \( \mathcal{X} \) of the hypersurface \( X_0 = X \cap Q \) in \( X \) so that \( Z \) is a union of strata. Recall that any strata \( W, V \in \mathcal{X} \) satisfy the frontier condition: \( W \cap \bar{V} \neq \emptyset \) implies that \( W \subset \bar{V} \). In particular, \( \mathcal{X} \) is partially ordered by:

\[
W \leq V \iff W \subset \bar{V}.
\]

We write \( W < V \) if \( W \leq V \) and \( W \neq V \). By the genericity assumption, the strata of \( \mathcal{X} \) are intersected transversally by \( H \) and \( Q_w \) (for \( w \) generic). Let \( \mathcal{X}_0 \) (with the induced partial order \( \leq \)) denote the collection of singular strata of \( \mathcal{X} \), i.e., strata of \( X_0 \) which are contained in \( Z \). Recall that the constructible function

\[
\alpha := \varphi_f(1_{X \setminus Q_w})|_{X \setminus (Q_w \cup H)}
\]

of Theorem 2.2 is supported on \( Z \setminus (Q_w \cup H) \). Our goal is to express \( \alpha \) in terms of the constructible functions \( \text{Eu}_V|_{\mathbb{P}^n \setminus (Q_w \cup H)} \), with \( V \in \mathcal{X}_0 \).

We first recall some well-known facts about constructible functions. Let \( CF_{\mathcal{X}_0}(Z) \) denote the abelian group of \( \mathcal{X} \)-constructible functions on \( X \cap Q \) which are supported on \( Z \). Then we have the following:
Lemma 2.7. The collection \( \{ \overline{E}_V \mid V \in \mathcal{X}_0 \} \) is a basis of \( CF_{\mathcal{X}_0}(Z) \).

Proof. This is well-known. We sketch a proof to set the notations for further use.

Using the distinguished basis \( \{ 1_V \mid V \in \mathcal{X}_0 \} \) of \( CF_{\mathcal{X}_0}(Z) \), we write
\[
\overline{E}_V = \sum_{W \leq V} a_{W,V} \cdot 1_W,
\]
with transition matrix \( A = (a_{W,V}) \) given by:
\[
a_{W,V} = \overline{E}_V(w), \quad \text{for } w \in W.
\]

By the properties of the local Euler obstruction function, we have that
\[
\overline{E}_V|_V = 1_V,
\]
and, for \( w \in W, \overline{E}_V(w) \neq 0 \) only if \( W \leq V \). So the transition matrix \( A = (a_{W,V}) \) is upper-triangular with respect to the partial order \( \leq \), with all diagonal entries equal to 1. In particular, \( A \) is invertible, so \( \{ \overline{E}_V \mid V \in \mathcal{X}_0 \} \) is indeed a basis of \( CF_{\mathcal{X}_0}(Z) \). \( \square \)

For any stratum \( V \in \mathcal{X}_0 \), we can now write \( 1_V \) in the basis of Lemma 2.7 as:
\[
1_V = \sum_{W \leq V} b_{W,V} \cdot \overline{E}_W,
\]
where the matrix \( B = (b_{W,V}) \) is the inverse of the matrix \( A = (a_{W,V}) \) from the proof of the above lemma. In particular, \( B \) is upper triangular with all diagonal entries equal to 1 and with non-zero off-diagonal entries computed inductively by the inversion formula of [32, Proposition 3.6.2]. We also note that
\[
b_{W,V} = (-1)^{\dim W} e_{W,V},
\]
where \( e_{W,V} \) is the Euler obstruction of the pair of strata \( (W, V) \), e.g., see [9, Section 1.1]. With this interpretation, a result of Kashiwara [18] (see also [10, 8.2], or [9, Theorem 1.1]), states that the non-zero off-diagonal entries of \( B \) can be given a topological interpretation in terms of complex links of pairs of strata. Specifically, for strata \( W < V \), one has:
\[
b_{W,V} = -\chi_c(L_{W,V}),
\]
where \( \chi_c \) denotes the Euler characteristic of compactly supported cohomology, and \( L_{W,V} \) is the complex link of the pair of strata \( W, V \) (that is, the intersection of \( V \) with a nearby hyperplane near \( W \) and normal to \( W \); see, e.g., [9, Theorem 1.1] for a precise definition).

In the above notations, we have the following:

Lemma 2.8. Let \( \delta \in CF_{\mathcal{X}_0}(Z) \) be a constructible function, written in terms in the above distinguished bases as:
\[
\delta = \sum_{V \in \mathcal{X}_0} \mu_V \cdot 1_V = \sum_{V \in \mathcal{X}_0} \alpha_V \cdot \overline{E}_V
\]
for some integers \( \mu_V, \alpha_V \). Then for any \( W \in \mathcal{X}_0 \) one has:
\[
\alpha_W = \sum_{\{V \mid W \leq V\}} b_{W,V} \cdot \mu_V.
\]
Proof. Evaluate \((21)\) at \(w \in W\) to get:
\[
\mu_W = \sum_{\{V \mid W \leq V\}} \alpha_V \cdot \text{Eu}_V (w) = \sum_{\{V \mid W \leq V\}} \alpha_V \cdot a_{W,V}.
\]
Then \((22)\) follows since \(B = (b_{W,V})\) is the inverse of \(A = (a_{W,V})\). \(\square\)

In order to deal with the function \(\alpha = \varphi_f (1_{X\setminus Q \cap X \setminus (Q_w \cup H)})\) of Theorem 2.2, we need to restrict the statements of Lemma 2.7 and Lemma 2.8 to \(Z \setminus (Q_w \cup H)\). Using Remark 2.5, the coefficients \(\mu_V\) of \(\alpha\) in the basis \(\{1_{V \setminus (Q_w \cup H)} \mid V \in \mathcal{X}_\emptyset\}\) of constructible functions supported on \(Z \setminus (Q_w \cup H)\) are given by:
\[
(23) \quad \mu_V = \chi(\tilde{H}^*(F_V; \mathbb{Q})),
\]
\(\text{i.e., the Euler characteristic of the reduced cohomology of the Milnor fiber } F_V \text{ of } f_0 \text{ at some point in } V. \)

Plugging \((23)\) and \((20)\) into \((22)\), and expressing \(\alpha\) in terms of the basis \(\{\text{Eu}_V | P_n \setminus (Q_w \cup H) \mid V \in \mathcal{X}_\emptyset\}\) of constructible functions with support on \(Z \setminus (Q_w \cup H)\), we get by Theorem 2.2 and Remark 2.5 the following generalization of \((16)\) to arbitrary singularities:

**Theorem 2.9.** Under Notation 1.4,
\[
(24) \quad \text{EDdefect}(X) = \sum_{V \in \mathcal{X}_\emptyset} (-1)^{\text{codim}_{X \cap Q} V} \alpha_V \cdot \text{gEDdeg}(\overline{V})
\]
with
\[
\alpha_V = \sum_{\{S \mid V \leq S\}} b_{V,S} \cdot \mu_S = \mu_V - \sum_{\{S \mid V < S\}} \chi_c(L_{V,S}) \cdot \mu_S.
\]
Here, for any stratum \(V \in \mathcal{X}_\emptyset\), \(\mu_V\) is the Euler characteristic of the reduced cohomology of the Milnor fiber \(F_V\) of the hypersurface \(X \cap Q \subset X\) at some point in \(V\), and \(L_{V,S}\) denotes the complex link of a pair of distinct strata \((V,S)\) with \(V \subset S\).

3. Examples

**Example 3.1** \((2 \times 2 \text{ Determinant})\). Let \(X\) denote the smooth irreducible subvariety of \(\mathbb{P}^3\) defined by \(x_0x_3 - x_1x_2 = 0\), and let \(Q\) denote the isotropic quadric \(\{(x_0 : \cdots : x_3) \in \mathbb{P}^n \mid \sum_{i=0}^3 x_i^2 = 0\}\). The variety \(X \cap Q\) consists of four lines and has precisely four isolated singularities. This is illustrated in Figure 3.1 where we restrict \(X\) to an affine chart by setting \(x_0 = 1\) and make a change of coordinates to plot the figures effectively. By Corollary 2.4, we have
\[
\text{EDdefect}(X) = \sum_{x \in \text{Sing}(X \cap Q)} \mu_x = 1 + 1 + 1 + 1
\]
where \(\mu_x = 1\) is the Milnor number of the isolated singularity \(x \in \text{Sing}(X \cap Q)\). This agrees with computations from [8], as \(\text{gEDdeg}(X) = 6\) (cf. [8][Example 7.11]) and \(\text{uEDdeg}(X) = 2\) (cf. [8][Example 2.4]). Furthermore, it is much easier to compute \(\text{EDdefect}(X)\) directly rather than computing the two Euler characteristics in \((2)\) and \((8)\) separately.
Example 3.2 (Kinetic Proofreading Networks: McKeithan Model). The following example is motivated by chemical reaction networks and was initially proposed by McKeithan [24]. We follow the formulation from [1][Section 3.3].

The affine $N$-site McKeithan Variety is an affine toric variety given by the image of the map
\[ \mathbb{C}^2 \to \mathbb{C}^{N+2}, \quad (r, s) \mapsto (rs, rs, \ldots, rs, r, s) = (x_1, \ldots, x_N, a, b). \]

The set of implicit equations of the projective closure $X_N$ that is obtained by homogenizing with respect to $x_0$ is
\[ \{ x_1 = x_2 = \ldots = x_N, \quad x_0x_N = ab \}. \]

When $N = 1$, this specializes to the $2 \times 2$ determinant in our previous example. The results of [1], imply $gE\text{D}\text{deg}(X_N) = 6$ and $E\text{D}\text{defect}(X_N) = 0$ for $N > 1$. On the other hand, the set of implicit equations of the projective closure $Y_N$ that is obtained by homogenizing with respect to $\sqrt{N}x_0$ is
\[ \{ x_1 = x_2 = \ldots = x_N, \quad \sqrt{N}x_0x_N = ab \}. \]

While the generic Euclidean distance degrees of $X_N$ and $Y_N$ coincide for all $N$, the unit Euclidean distance degrees can be different. In fact, it follows as in the previous example that $E\text{D}\text{defect}(Y_N) = 4$ for all $N \geq 1$. To see this, note the intersection of $Y_n$ with the isotropic quadric consists of four line intersecting at four points as in Figure 3.1 but in a higher dimensional ambient space.

Example 3.3 (Rank one matrices). The variety $X$ in Example 3.1 consists of $2 \times 2$ matrices with rank equal to one. More generally, let $X = X_{s,t}$ denote the subvariety of
The variety \( X \) is smooth and irreducible. In fact, \( X \) is the image of the Segre embedding \( \sigma : \mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \to \mathbb{P}^{st-1} \). Instead of studying the intersection \( X \cap Q \) in \( \mathbb{P}^{st-1} \), we study the isomorphic variety \( \sigma^{-1}(Q) \) in \( \mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \). Let \( y_1, \ldots, y_s \) and \( z_1, \ldots, z_t \) be the homogeneous coordinates of \( \mathbb{P}^{s-1} \) and \( \mathbb{P}^{t-1} \) respectively. Since the isotropic quadric \( Q \subset \mathbb{P}^{st-1} \) is defined by \( \sum_{i,j} x_{ij}^2 = 0 \), the preimage \( \sigma^{-1}(Q) \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \) is defined by \( \sum_{i,j} (y_i z_j)^2 = 0 \). Notice that \( \sum_{i,j} (y_i z_j)^2 = (\sum_i y_i^2) \cdot (\sum_j z_j^2) \). Thus, \( \sigma^{-1}(Q) \) consists of two smooth irreducible components,

\[
Z_1 := \{ [y_i] \in \mathbb{P}^{s-1} | \sum_i y_i^2 = 0 \} \times \mathbb{P}^{t-1}
\]

and

\[
Z_2 := \mathbb{P}^{s-1} \times \{ [z_j] \in \mathbb{P}^{t-1} | \sum_j z_j^2 = 0 \}. 
\]

Clearly, \( Z_1 \) intersects \( Z_2 \) transversally and \( \sigma^{-1}(Q) \) is equisingular along \( Z_1 \cap Z_2 \). Therefore, \( \sigma^{-1}(Z) = Z_1 \cap Z_2 \) or equivalently \( Z = \sigma(Z_1 \cap Z_2) \). Take a point \( P \in Z_1 \cap Z_2 \) and take a two-dimensional general slice \( V \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \) passing through \( P \). Near \( P \), \( V \cap Z_1 \) and \( V \cap Z_2 \) are two smooth curves intersecting transversally at \( P \). It is well-known that the Milnor number of a nodal curve singularity is 1. Therefore, by Corollary 2.6,

\[
\text{EDdefect}(X) = \mu \cdot \text{gEDdeg}(Z) = 1 \cdot \text{gEDdeg}(Z) = (-1)^{\dim Z} \chi(Z \setminus (Q_w \cup H))
\]

where \( w \) is a generic weight and \( H \) a general hyperplane. The last term of the above formula can be computed as follows. Since the Euler characteristic is additive on sub-varieties, we have

\[
\chi(Z \setminus (Q_w \cup H)) = \chi(Z) - \chi(Z \cap Q_w) - \chi(Z \cap H) + \chi(Z \cap Q_w \cap H).
\]

Furthermore, since \( Z \cong \sigma^{-1}(Z) = Z_1 \cap Z_2 \), we have

\[
\chi(Z) = \chi(Z_1 \cap Z_2)
\]

\[
\chi(Z \cap Q_w) = \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_w))
\]

\[
\chi(Z \cap H) = \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(H))
\]

\[
\chi(Z \cap Q_w \cap H) = \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_w) \cap \sigma^{-1}(H)).
\]

All the intersections in the above equations are smooth, hence each right hand side can be computed using Chern classes, see e.g. [21, Page 15].

Notice that \( \sigma^{-1}(Q_w) \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \) is a hypersurface of bidegree \((2, 2)\) and \( \sigma^{-1}(H) \subset \mathbb{P}^{s-1} \times \mathbb{P}^{t-1} \) is a hypersurface of bidegree \((1, 1)\). Thus, the values of \( \chi(Z_1 \cap Z_2) \), \( \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_w)) \), \( \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(H)) \) and \( \chi(Z_1 \cap Z_2 \cap \sigma^{-1}(Q_w) \cap \sigma^{-1}(H)) \) is equal to the
coefficient of \([H_1]^{s-1}[H_2]^{t-1}\) in the following power series, respectively,

\[
\begin{align*}
(25) \quad & 2[H_1] \cdot 2[H_2] \cdot \frac{(1 + [H_1])^s(1 + [H_2])^t}{(1 + 2[H_1])(1 + 2[H_2])}, \\
(26) \quad & 2[H_1] \cdot 2[H_2] \cdot \frac{(1 + [H_1])^s(1 + [H_2])^t}{(1 + 2[H_1])(1 + 2[H_2])}, \\
(27) \quad & 2[H_1] \cdot 2[H_2] \cdot \frac{(1 + [H_1])^s(1 + [H_2])^t}{(1 + 2[H_1])(1 + 2[H_2])}, \\
(28) \quad & 2[H_1] \cdot 2[H_2] \cdot \frac{(1 + [H_1])^s(1 + [H_2])^t}{(1 + 2[H_1])(1 + 2[H_2])}, \\
\end{align*}
\]

where \([H_1]\) and \([H_2]\) are considered as formal variables. Thus, we have \(\text{EDdefect}(X)\) is equal to the coefficient of \([H_1]^{s-1}[H_2]^{t-1}\) in

\[
\begin{align*}
(29) \quad & 2[H_1] \cdot 2[H_2] \cdot \frac{(1 + [H_1])^s(1 + [H_2])^t}{(1 + 2[H_1])(1 + 2[H_2])}, \\
& \frac{1}{(1 + 2[H_1] + 2[H_2])(1 + [H_1] + [H_2])}.
\end{align*}
\]

After expanding the factors that do not depend on \(s, t\), we see (29) equals

\[
(1 + [H_1])^s(1 + [H_2])^t \cdot \sum_{0 \leq i,j} c_{i,j} [H_1]^i[H_2]^j
\]

where the \(c_{i,j}\) are integers that do not depend on \(s, t\). Thus, for any \(s, t\),

\[
\text{EDdefect}(X) = \sum_{k=0}^{s-1} \sum_{t=0}^{t-1} \binom{s}{k} \binom{t}{\ell} c_{s-1-k,t-1-\ell}.
\]

**Remark 3.4.** A generalization of the above setup to rank-one tensors is considered in [2, Example 9.6], where the \(\mu\text{EDdeg}\) is computed using a similar method.

When combining the above example with Corollary 1.10, we get the following:

**Corollary 3.5.** If \(X\) is the variety of rank one matrices as in Example 3.1 and \(\mathcal{L}\) is a general linear space, then \(\text{EDdefect}(X \cap \mathcal{L}) = g\text{EDdeg}(Z \cap \mathcal{L})\) where \(Z\) is the singular locus of \(X \cap Q\).

**Example 3.6 (Quadric surface).** Consider the hypersurface \(X\) defined by

\[
f = (x_1 - \sqrt{-1}x_0)^2 + 2(x_3 - \sqrt{-1}x_2)^2 + q
\]

where \(q = x_0^2 + x_1^2 + x_2^2 + x_3^2\). The variety \(X \cap Q\) consists of three lines. Two of the lines \(L_1, L_2\) are generically reduced, but one of the lines \(L_3\) is with multiplicity two. This explains why the degree of the ideal \(\langle f, q \rangle\) is four. The radical ideal of \(L_3\) is \(\langle x_2 + \sqrt{-1}x_3, x_1 - \sqrt{-1}x_0 \rangle\) and defines \(Z\). The union of lines \(L_1 \cup L_2\) intersect \(L_3\) at two distinct points: \(\{P_1, P_2\}\).

We stratify \(Z\) by \(S_0 = Z \setminus \{P_1, P_2\}\) and \(S_i = \{P_i\}\) for \(i = 1, 2\). For \(i = 1, 2\), the complex link \(L_{S_i, S_0}\) consists of a point. According to Equation (22):

\[
\begin{bmatrix}
\alpha_{P_2} \\
\alpha_{P_1} \\
\alpha_{S_0}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu_{P_2} \\
\mu_{P_1} \\
\mu_{S_0}
\end{bmatrix}.
\]
The Milnor fiber $F_{S_0}$ is homotopy equivalent to \( \{ x^2 = 1 \} \subset \mathbb{C}^2 \), and for \( i = 1, 2 \) the Milnor fiber $F_{S_i}$ is homotopy equivalent to \( \{ x^2 y = 1 \} \simeq \mathbb{C}^* \). Therefore, $\chi(F_{S_0}) = 2$ and $\chi(F_{S_i}) = 0$ for $i = 1, 2$. According to Theorem 2.9,
\[
\text{EDdefect}(X) = \alpha_{S_0} \cdot \text{gEDdeg}(S_0) - \alpha_{P_1} \cdot \text{gEDdeg}(P_1) - \alpha_{P_2} \cdot \text{gEDdeg}(P_2) \\
= \alpha_{S_0} \cdot 1 - \alpha_{P_1} \cdot 1 - \alpha_{P_2} \cdot 1 \\
= \mu_{S_0} \cdot 1 - (\mu_{P_1} - \mu_{S_0}) \cdot 1 - (\mu_{P_2} - \mu_{S_0}) \cdot 1 \\
= 1 \cdot 1 - (-1 - 1) - (-1 - 1) \\
= 5.
\]

**Remark 3.7.** These topological computations agree with the fact that $\text{gEDdeg}(X) = 6$ and $u \text{EDdeg}(X) = 1$. We computed these numbers using our Macaulay2 [11] package EuclideanDistanceDegree, which is available at https://github.com/JoseMath/EuclideanDistanceDegree/

This package implements Grobner basis methods and continuation methods (specifically, we used Bertini [3, 4]).

```plaintext
- Macaulay2 code to compute EDdefect(V(F)) *

    i1 : loadPackage"EuclideanDistanceDegree";
    i2 : kk=QQ[I]/ideal(I^2+1);
    i3 : T=kk[x0,x1,x2,x3];
    i4 : q=x0^2+x1^2+x2^2+x3^2;
    i5 : F={(x1-1*x0)^2+2*(x3-1*x2)^2+q};
    --Symbolic computation (Grobner bases method):
    i6 : EDDefect=(determinantalGenericEuclideanDistanceDegree F-
        determinantalUnitEuclideanDistanceDegree F)/(degree kk)
    o6 = 5
    --Numerical computation (Continuation method):
    ----Note: Bertini needs to be installed for this to work.
    -- (i7-i10) Create directories and write Bertini files
    -- (i11) Run Bertini and computes EDdefect(V(F))
    i7 : (dir1,dir2)=(temporaryFileName(),temporaryFileName());
    i8 : {dir1,dir2}/mkdir;
    i9 : leftKernelGenericEDDegree(dir1,F);
    i10 : leftKernelUnitEDDegree(dir2,F);
    i11 : EDDefect=runBertiniEDDegree(dir1)-runBertiniEDDegree(dir2)
    o11 = 5
```

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