Abstract

A new method for finding first integrals of discrete equations is presented. It can be used for discrete equations which do not possess a variational (Lagrangian or Hamiltonian) formulation. The method is based on a newly established identity which links symmetries of the underlying discrete equations, solutions of the discrete adjoint equations and first integrals. The method is applied to invariant mappings and discretizations of a second order and a third order ODEs. In examples the set of independent first integrals makes it possible to find the general solution of the discrete equations. The method is compared to a direct method of constructing first integrals.
Contents

1 Introduction 3

2 Adjoint equation method for constructing conservation laws for differential equations 4

3 The case of one ordinary differential equation 7
  3.1 The case of an n-th order ODE 7
  3.2 Second order ODEs 10
  3.3 Third order ODEs 11

4 The direct method and integrating factors 19

5 Adjoint equation method for mappings 25

6 Case of mapping involving a single dependent variable 28
  6.1 General theory 28
  6.2 Four-point scalar discrete equation and example 29

7 Integrating factors for mappings 39

8 Discretizations of a scalar ODE 41
  8.1 Theory for difference systems 41
  8.2 Discretization of second order ODEs 44
  8.3 Discretization of third order ODEs 47

9 Conclusion 54
1 Introduction

Considerable progress has been made over the last 25 years in the applications of Lie group theory to difference equations (for reviews see [12, 28, 40] and for original papers [5, 6, 8, 9, 10, 14, 18, 20, 26, 27, 29, 30, 31, 35, 38]). The overall aim of the program is to turn Lie group theory into a tool for solving discrete equations that is as efficient as it is for differential ones.

For ordinary differential equations (ODEs) one of the important applications of Lie group theory is to reduce the order of the equation and ideally to solve it analytically and explicitly. Essentially there are two ways of doing this, once a nontrivial Lie point symmetry group of the equation is found. One is to perform a transformation of the independent and dependent variables that takes the Lie algebra into a convenient form. This also transforms the equation to a form in which the reduction of the order becomes obvious.

An alternative method is to use the Lie point symmetry group to construct first integrals of the equation that are of lower order than the equation (or system of equations) itself. This can be done if a Lagrangian exists and the symmetries are variational ones. If a sufficient number of first integrals can be obtained using the Noether theorem, then the derivatives can be eliminated from the set of first integrals. This provides a solution of the original equation by purely algebraic operations, without any changes of variables or any integration.

If no invariant Lagrangian exists alternative methods of constructing lower order first integrals have been proposed in [2, 4], and in [22, 23, 24]. They make use of the so called adjoint equation solutions of which one uses to construct the required first integrals. We shall call this the ”adjoint equation method”.

The integration methods based on transformations of coordinates have not been adapted to difference equations. The algebraic methods based on invariant Lagrangians and Hamiltonians have been adapted and successfully applied to solve three point difference schemes in [11, 12, 13, 18, 19] and [15, 16, 17], respectively. A research note on adapting the ”adjoint equation method” to difference equations has been published in [41].

The purpose of this article is to present and justify the adjoint equation method for difference systems with an arbitrary number of variables and also to document its usefulness on examples. The paper is organized as follows. In Section 2 we present a brief summary of the adjoint equation method for an arbitrary system of partial or ordinary differential equations (PDEs or ODEs). Section 3 specializes the theory sketched in Section 2 to the case of one scalar ODE. Section 4 describes integrating factors for scalar ODEs. The adjoint equation method for discrete systems is presented in Section 5. This theory is specialized to the case of scalar discrete equations (mappings) and discretizations of scalar ODEs in Sections 6 and 8 respectively. Section 7 presents integrating factors for scalar discrete equations. Finally, Section 9 provides concluding remarks.
2 Adjoint equation method for constructing conservation laws for differential equations

Let us consider a system of \( n \)-th order PDEs

\[
F_\beta(x, u, u_1, u_2, ..., u_n) = 0, \quad \beta = 1, ..., r,
\]

(2.1)

where \( x = (x^1, ..., x^p) \), \( u = (u^1, ..., u^q) \),

\[
u_1 := \{ u_i^k \} = \left\{ \frac{\partial u_i^k}{\partial x^i} \right\}, \quad ..., \quad u_s := \{ u_{i_1...i_s} \} = \left\{ \frac{\partial^s u^k}{\partial x^{i_1}...\partial x^{i_s}} \right\}, \quad ...
\]

\( i = 1, ..., p, \quad k = 1, ..., q. \)

Let \( L_{\alpha\beta} \) be a linear operator

\[
L_{\alpha\beta} = \sum_{k=0}^{\infty} F_{\beta,u_1...u_k} D_{i_1} \cdots D_{i_k}, \quad F_{\beta,u_1...u_k} = \frac{\partial F_\beta}{\partial u_1^{i_1}...u_k^{i_k}}, \quad (2.2)
\]

where

\[
D_i = \frac{\partial}{\partial x^i} + u_i^k \frac{\partial}{\partial u^k} + v_i^k \frac{\partial}{\partial v^k} + u_{ij}^k \frac{\partial}{\partial u_{ij}^k} + v_{ij}^k \frac{\partial}{\partial v_{ij}^k} + v_{ijl}^k \frac{\partial}{\partial v_{ijl}^k} + ...,
\]

then the \textit{adjoint equations} are given by the variational derivatives (or Euler-Lagrange operators):

\[
F^*_\alpha = L^*_{\alpha\beta} v^\beta = \frac{\delta}{\delta u^\alpha} (v^\beta F_\beta) = \sum_{k=0}^{\infty} (-1)^k D_{i_1} \cdots D_{i_k} (v^\beta F_{\beta,u_1...u_k}) = 0, \quad \alpha = 1, ..., q.
\]

(2.3)

We assume summation over repeated indices. Notice that the adjoint equations are always linear equations for \( v = (v^1, ..., v^r) \) with coefficients that in general depend upon \( u \) (solution of (2.1)).

The \textbf{basic operator identity} is the following

\[
v^\beta L_{\alpha\beta} w^\alpha - w^\alpha L^*_{\alpha\beta} v^\beta = D_i C^i,
\]

(2.4)

where \( v^\beta \) and \( w^\alpha \) are some functions of \( x, u \) and derivatives of \( u \). Here

\[
C^i = \sum_{k=0}^{\infty} D_{i_1} \cdots D_{i_k} (w^\alpha) \frac{\delta}{\delta u_{i_1...i_k}^\alpha} (v^\beta F_\beta),
\]

(2.5)

where

\[
\frac{\delta}{\delta u_{i_1...i_k}^\alpha} = \sum_{s=0}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1...i_s}^\alpha}.
\]
are higher order variational operators (or higher order Euler-Lagrange operators). Since the scalar \((q = r = 1)\) relation is probably due to Lagrange (see for example [7], Eq. (2.75) on p. 80), we refer to (2.4) as the Lagrange identity.

We will be interested in Lie symmetries [37, 21, 34]

\[
X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^\infty \zeta^\alpha_{i_1...i_s} \frac{\partial}{\partial u^\alpha_{i_1...i_s}},
\]

(2.6)

where \(\xi^i\) and \(\eta^\alpha\) are some functions of \(x, u\) and a finite number of derivatives of \(u\) and

\[
\zeta^\alpha_{i_1...i_s} = D_{i_1}...D_{i_s}(\eta^\alpha - \xi^i u^\alpha_i) + \xi^i u^\alpha_{i_1...i_s}.
\]

Note that for point symmetries \(\xi^i\) and \(\eta^\alpha\) depend only on \(x\) and \(u\). To each symmetry (2.6) there corresponds the canonical or evolutionary symmetry

\[
\bar{X} = \bar{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s=1}^\infty \bar{\zeta}^\alpha_{i_1...i_s} \frac{\partial}{\partial u^\alpha_{i_1...i_s}},
\]

(2.7)

where

\[
\bar{\eta}^\alpha = \eta^\alpha - \xi^i u^\alpha_i, \quad \bar{\zeta}^\alpha_{i_1...i_s} = D_{i_1}...D_{i_s}(\bar{\eta}^\alpha).
\]

The identity (2.4) can be used to link symmetries of the differential equations (2.1), solutions of the corresponding adjoint equations (2.3) and conservation laws.

Choosing \(w^\alpha = \bar{\eta}^\alpha = \eta^\alpha - \xi^i u^\alpha_i\), we obtain the identity

\[
v^\beta \bar{X} F_\beta = \bar{\eta}^\alpha F^*_\alpha + D_i C^i
\]

(2.8)

for evolutionary operators (2.7). For the Lie symmetry operator (2.6) we obtain

\[
v^\beta X F_\beta = v^\beta \xi^i D_i F_\beta + (\eta^\alpha - \xi^i u^\alpha_i) F^*_\alpha + D_i C^i.
\]

(2.9)

Here the quantities \(C^i\) are

\[
C^i = \sum_{k=0}^\infty D_{i_1}...D_{i_k}(\eta^\alpha - \xi^i u^\alpha_i) \frac{\delta}{\delta u^\alpha_{i_1...i_k}}(v^\beta F_\beta).
\]

(2.10)

The following theorem is based on the Lagrange identity:

**Theorem 2.1** The system of equations (2.1) and its adjoint system (2.3) possess the following conservation law

\[
D_i C^i \bigg|_{2.1, 2.3} = 0
\]

(2.11)

for each Lie symmetry (2.6) of the differential equation (2.1) and for each solution of the adjoint equation (2.3).
Proof. The result follows directly from Eq. (2.9). Indeed, $XF_\beta = 0$ because it is a symmetry criterion for Eq. (2.1), $D_i(F_\beta) = 0$ since it is a differential consequence of Eq. (2.1), and $F_\alpha^* = 0$ on a solution of adjoint equations (2.3). □

Since we are interested in Eqs. (2.1) we need conservation laws for these equations alone, i.e., without using solutions of the adjoint equations (2.3).

One can get rid of the adjoint variables $v_\beta$ figuring in the conservation law (2.11) and subsequent formulas. The identity (2.8) and the idea of solving the adjoint equations in terms of solutions of the original equations were explicitly presented in [2] (see also [4]). These ideas were also suggested and further developed in [22, 23, 24], where numerous examples for ODEs and PDEs were worked out explicitly. The introduction of adjoint variables, of linear equations adjoint to nonlinear ones and the extension variational principles for equations without classical Lagrangians were also considered in [1, 25, 39] and others.

Theorem 2.2 Let the adjoint equations (2.3) be satisfied for all solutions of the differential equations (2.1) upon a substitution

$$v = \varphi(x, u, u_1, u_2, ...), \quad \varphi \neq 0. \tag{2.12}$$

Then, any Lie symmetry (2.6) of the equations (2.1) leads to the conservation law

$$D_iC_i \big|_{2.11} = 0, \tag{2.13}$$

where $v$ and its derivatives should be eliminated via equation (2.12) and its differential consequences.

Remark 2.3 Equations (2.1) and (2.3) can be considered as variational equations for the formal Lagrangian (2.14)

$$\mathcal{L} = v^\beta F_\beta, \tag{2.14}$$

which provides the original and adjoint equations

$$F_\beta = \frac{\delta \mathcal{L}}{\delta v^\beta} = 0, \quad \beta = 1, ..., r, \quad F_\alpha^* = \frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, ..., q. \tag{2.15}$$

Remark 2.4 The same operator identities (2.8) and (2.9) form the basis of the Noether theorem [33] for the Lagrangian systems (see [21] for details). Indeed, consider the case $r = 1$, put $v = 1$ and apply it to a Lagrangian $F = \mathcal{L}(x, u, u_1, u_2, ...).$ Then we get the following identities

$$X\mathcal{L} = \bar{\eta}^\alpha \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i(\bar{N}^i \mathcal{L}),$$

$$\bar{N}^i = \sum_{k=0}^{\infty} D_{i_k} \cdots D_{i_k}(\bar{\eta}^\alpha) \frac{\delta}{\delta u^{i_{i_1} \cdots i_k}}$$
\[ X \mathcal{L} + \mathcal{L} D \xi^i = (\eta^\alpha - \xi^i u_1^\alpha) \frac{\delta \mathcal{L}}{\delta u^\alpha} + D_i (N^i \mathcal{L}), \]
\[ N^i = \xi^i + \sum_{k=0}^{\infty} D_{i_1} \ldots D_{i_k} (\eta^\alpha - \xi^i u_1^\alpha) \frac{\delta}{\delta u_{i_1 \ldots i_k}^\alpha}, \]
which yield a conservation law
\[ D_i \bar{C}^i = 0, \quad \bar{C}^i = \bar{N}^i \mathcal{L} \]
and
\[ D_i C^i = 0, \quad C^i = N^i \mathcal{L}, \]
correspondingly, for the Euler-Lagrange equations
\[ \frac{\delta \mathcal{L}}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, q. \]

Operators \( N^i \) and \( \bar{N}^i \) are called Noether operators.

3 The case of one ordinary differential equation

In this section we restrict ourselves to scalar ordinary differential equations. It is a particular case of the general theory sketched in the previous section. We restrict ourselves to Lie point symmetries because later we will consider the discrete case, to which we wish to adapt the Lie point symmetry approach.

3.1 The case of an \( n \)-th order ODE

Let us consider a scalar ODE of order \( n \)
\[ F(x, u, \dot{u}, \ddot{u}, \ldots, u^{(n)}) = 0. \] (3.1)

We will be interested in Lie point symmetries
\[ X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial \dot{u}} + \zeta_2 \frac{\partial}{\partial \ddot{u}} + \ldots + \zeta_k \frac{\partial}{\partial u^{(k)}} + \ldots, \] (3.2)
where
\[ \zeta_k = D^k (\eta - \xi \dot{u}) + \xi u^{(k+1)} \]
and
\[ D = \frac{\partial}{\partial x} + \dot{u} \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}} + \dddot{u} \frac{\partial}{\partial \ddot{u}} + \ldots + u^{(k+1)} \frac{\partial}{\partial u^{(k)}} + v^{(k+1)} \frac{\partial}{\partial v^{(k)}} + \ldots \]
is the total differentiation.
To each Lie point symmetry (3.2) there corresponds the symmetry in evolutionary form

\[ \bar{X} = \bar{\eta} \frac{\partial}{\partial u} + \bar{\zeta}_1 \frac{\partial}{\partial \dot{u}} + \bar{\zeta}_2 \frac{\partial}{\partial \ddot{u}} + \ldots + \bar{\zeta}_k \frac{\partial}{\partial u^{(k)}} + \ldots, \]  
(3.3)

where

\[ \bar{\eta} = \eta(x, u) - \xi(x, u) \dot{u}, \]

\[ \bar{\zeta}_1 = D(\bar{\eta}), \quad \ldots, \quad \bar{\zeta}_k = D^k(\bar{\eta}). \]

By means of the variational operator

\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D \frac{\partial}{\partial \dot{u}} + D^2 \frac{\partial}{\partial \ddot{u}} + \ldots + (-1)^k D^k \frac{\partial}{\partial u^{(k)}} + \ldots \]  
(3.4)

we introduce the adjoint equation

\[ F^* = \frac{\delta}{\delta u}(vF) = 0. \]  
(3.5)

Thus (2.3) simplifies to

\[ F^* = v \frac{\partial F}{\partial u} - D \left( v \frac{\partial F}{\partial \dot{u}} \right) + D^2 \left( v \frac{\partial F}{\partial \ddot{u}} \right) + \ldots + (-1)^n D^n \left( v \frac{\partial F}{\partial u^{(n)}} \right) = 0. \]

Remark 3.1 Equations (3.7) and (3.5) can be considered as variational equations for the formal Lagrangian [23]

\[ \mathcal{L} = vF, \]  
(3.6)

which provides the original and adjoint equations

\[ F = \frac{\delta L}{\delta v} = 0, \quad F^* = \frac{\delta L}{\delta u} = 0. \]  
(3.7)

Let us define higher order variational (or Euler–Lagrange) operators

\[ \frac{\delta}{\delta u^{(i)}} = \frac{\partial}{\partial u^{(i)}} - D \frac{\partial}{\partial u^{(i+1)}} + D^2 \frac{\partial}{\partial u^{(i+2)}} + \ldots + (-1)^k D^k \frac{\partial}{\partial u^{(i+k)}} + \ldots \]  
(3.8)

Note that Euler–Lagrange operator (3.4) belongs to this set as

\[ \frac{\delta}{\delta u} = \frac{\delta}{\delta u^{(0)}}. \]

Lemma 3.2 (Main identity for scalar ODEs) The following identity holds

\[ vXF = v\xi DF + (\eta - \xi \dot{u})F^* + DI, \]  
(3.9)

where

\[ I = \sum_{i=0}^{n-1} D^i(\eta - \xi \dot{u}) \frac{\delta}{\delta u^{(i+1)}}(vF). \]  
(3.10)
This is a special case of (2.9), (2.10).

We prefer identity (3.9) instead of the corresponding identity for the canonical operator

\[ v\bar{X}F = \bar{\eta}F^* + DI. \] (3.11)

In the discrete case the framework of Lie point symmetries is much better developed in terms of standard vector fields (3.2) than evolutionary ones. The goal of this paper is to develop a discrete analog of the identity (3.9).

Now we examine the identity (3.9) on the solutions of the ODE (3.1). The left hand side turns out to be zero if operator \( X \) is a symmetry of the ODE. The first term on the right hand side contains \( DF \) and drops out as a differential consequence of the ODE. We are left with

\[ (\eta - \xi \dot{u})F^*|_{F=0} + DI|_{F=0} = 0. \] (3.12)

If we can find a substitution for function \( v \) providing \( F^* = 0 \), then we can get rid off the adjoint equation. Thus, we obtain a first integral of the ODE. Let us formulate this as the following theorem.

**Theorem 3.3 (Main theorem for scalar ODEs)** Let the adjoint equation (3.5) be satisfied for all solutions of the original ODE (3.1) upon a substitution

\[ v = \varphi(x,u), \quad \varphi \neq 0. \] (3.13)

Then, any Lie point symmetry (3.2) of the equation (3.1) leads to first integral

\[ I = \sum_{i=0}^{n-1} D^i(\eta - \xi \dot{u})\frac{\delta}{\delta u^{(i+1)}}(vF), \] (3.14)

where \( v \) and its derivatives should be eliminated via equation (3.13) and its differential consequences.

**Remark 3.4** First integrals \( I \), given by (3.14), can depend on \( u^{(n)} \) as well as higher derivatives. We will call such expression higher first integrals. It is reasonable to use the ODE (3.7) and its differential consequences to express these first integrals as functions of the minimal set of variables, i.e., in the form \( \bar{I}(x,u,\dot{u},...,u^{(n-1)}) \). In the examples of this section we will bypass the higher first integrals \( I \) and provide only the final results \( \bar{I} \).

**Remark 3.5** Theorem 3.3 can be extended from point-wise substitutions (3.13) to differential substitutions

\[ v = \varphi(x,u,\dot{u}), \quad ..., \quad v = \varphi(x,u,\dot{u},\ddot{u},...,u^{(n-1)}), \quad \varphi \neq 0. \] (3.15)
3.2 Second order ODEs

The method of the adjoint equation also works for differential equations which possess a Lagrangian. We demonstrate its application for the case of a second order ODE

\[ F(x, u, \dot{u}, \ddot{u}) = 0. \]

Its adjoint equation (3.5) is

\[ F^* = v \frac{\partial F}{\partial u} - D \left( v \frac{\partial F}{\partial \dot{u}} \right) + D^2 \left( v \frac{\partial F}{\partial \ddot{u}} \right) = 0. \]

First integrals (3.14) are given by the formula

\[ I = \left[ (\eta - \dot{u}\xi) \left( \frac{\partial}{\partial \dot{u}} - D \frac{\partial}{\partial \ddot{u}} \right) + D(\eta - \dot{u}\xi) \frac{\partial}{\partial \ddot{u}} \right] (vF). \]

Example: Harmonic oscillator

Let us consider the one-dimensional harmonic oscillator

\[ F = \ddot{u} + u = 0. \tag{3.16} \]

It admits the symmetries [32]

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \sin x \frac{\partial}{\partial u}, \quad X_3 = \cos x \frac{\partial}{\partial u}, \quad X_4 = u \frac{\partial}{\partial u}, \]

\[ X_5 = \sin 2x \frac{\partial}{\partial x} + u \cos 2x \frac{\partial}{\partial u}, \quad X_6 = \cos 2x \frac{\partial}{\partial x} - u \sin 2x \frac{\partial}{\partial u}, \quad X_7 = u \sin x \frac{\partial}{\partial x} + u^2 \cos x \frac{\partial}{\partial u}, \]

\[ X_8 = u \cos x \frac{\partial}{\partial x} - u^2 \sin x \frac{\partial}{\partial u}. \tag{3.17} \]

The adjoint equation (3.5) takes the form

\[ F^* = \ddot{v} + v = 0. \tag{3.18} \]

Note that the equation (3.16) is self-adjoint:

\[ F^* |_{v=u} = F. \]

Let us choose the solution \( v(x, u) = u \), then for symmetries (3.17) we obtain the first integrals

\[ \bar{I}_1 = \dot{u}^2 + u^2, \quad \bar{I}_2 = -\dot{u} \sin x + u \cos x, \quad \bar{I}_3 = -\dot{u} \cos x - u \sin x, \]

\[ \bar{I}_4 \equiv 0, \quad \bar{I}_5 = (\dot{u}^2 - u^2) \sin 2x - 2u\dot{u} \cos 2x, \]

\[ \bar{I}_6 = (\dot{u}^2 - u^2) \cos 2x + 2u\dot{u} \sin 2x, \quad \bar{I}_7 \equiv 0, \quad \bar{I}_8 \equiv 0, \]

respectively. Choosing values of two independent first integrals \( \bar{I}_2 = A \) and \( \bar{I}_3 = B \), we obtain the well-known general solution as

\[ u(x) = A \cos x - B \sin x. \]
3.3 Third order ODEs

To the third order ODE

\[ F(x, u, \dot{u}, \ddot{u}) = 0 \]

there corresponds the adjoint equation

\[ F^* = v \frac{\partial F}{\partial u} - D \left( v \frac{\partial F}{\partial \dot{u}} \right) + D^2 \left( v \frac{\partial F}{\partial \ddot{u}} \right) - D^3 \left( v \frac{\partial F}{\partial \ldots} \right) = 0. \]

First integrals are given as

\[ I = \left( \eta - \dot{u} \xi \right) \left( \frac{\partial}{\partial \dot{u}} - D \frac{\partial}{\partial \ddot{u}} + D^2 \frac{\partial}{\partial \ldots} \right) \]
\[ + D \left( \eta - \dot{u} \xi \right) \left( \frac{\partial}{\partial \ddot{u}} - D \frac{\partial}{\partial \dddot{u}} \right) + D^2 \left( \eta - \dot{u} \xi \right) \frac{\partial}{\partial \ldots} \]
\[ \left( vF \right). \]

Example

Let us investigate the ODE

\[ F = \frac{1}{u^2} \left( \ddot{u} \dot{u} - \frac{3}{2} \dddot{u} \right) - f(x) = 0. \] (3.19)

Its numerical solutions were considered in [5, 6] using a symmetry-preserving discretization.

The first term is the well-known Schwarzian derivative that has many interesting and important applications in mathematics, physics and even (originally) in cartography (for an interesting review see [36]).

In the general case this ODE admits the symmetry group \( SL(2, \mathbb{R}) \). Its Lie algebra is realized as

\[ X_1 = \frac{\partial}{\partial u}, \quad X_2 = u \frac{\partial}{\partial u}, \quad X_3 = u^2 \frac{\partial}{\partial u}, \] (3.20)

for \( f = M = \text{const} \) we get an additional symmetry

\[ X_4 = \frac{\partial}{\partial x}, \] (3.21)

and for \( f = M = 0 \) there are two further symmetries

\[ X_5 = x \frac{\partial}{\partial x}, \quad X_6 = x^2 \frac{\partial}{\partial x}. \] (3.22)

The ODE that we shall consider is

\[ F = \frac{1}{u^2} \left( \dddot{u} \dot{u} - \frac{3}{2} \dddot{u} \right) - M = 0, \quad M = \text{const}. \] (3.23)
The equation (3.23) can be integrated by standard integration techniques. Indeed, it can be rewritten as
\[
\left( \frac{1}{\sqrt{|\dot{u}|}} \right)'' + \frac{M}{2} \frac{1}{\sqrt{|u|}} = 0
\]
and solved for \( \frac{1}{\sqrt{|u|}} \) as a linear (Schrödinger) equation. We find the solution \( u(x) \) for different cases of the parameter \( M \) as follows

\[ M = 0 : \quad u(x) = \frac{1}{C_1 x + C_2} + C_3 \quad \text{or} \quad u(x) = C_1 x + C_2; \quad (3.24) \]

\[ M < 0 : \quad u(x) = C_1 \tanh(\omega x + C_2) + C_3 \]
\[ \quad \text{or} \quad u(x) = C_1 \coth(\omega x + C_2) + C_3 \]
\[ \quad \text{or} \quad u(x) = C_1 e^{\pm 2\omega x} + C_2, \quad \omega = \sqrt{-M/2}; \quad (3.25) \]

\[ M > 0 : \quad u(x) = C_1 \tan(\omega x + C_2) + C_3, \quad \omega = \sqrt{M/2}; \quad (3.26) \]

where \( C_1 \neq 0, C_2 \) and \( C_3 \) are integration constants.

**Remark 3.6** Let us mention that the restriction \( C_1 \neq 0 \) can be removed if instead of the ODE (3.23) we consider the equation
\[
\dot{u}^2 F = \ddot{u} \dddot{u} - \frac{3}{2} \dddot{u}^2 - M \dddot{u}^2 = 0.
\]

Unfortunately, this method will not work in the discrete case, so a different approach is needed and will be based on a discrete version of Theorem 3.3. Let us first solve ODE (3.23) using Theorem 3.3. The idea is to find three independent first integrals of (3.23) and then to eliminate the derivatives \( \dot{u} \) and \( \dddot{u} \) from them (third order ODEs can have at most three first integrals).

The adjoint equation (3.5) takes the form
\[
F^* = -\frac{1}{\dddot{u}} (\dddot{u}^2 + 2M \dddot{u}) = 0. \quad (3.27)
\]

We can consider different cases for the substitutions (3.13) and (3.15).

**Ansatz 1.** Let us look for solutions of the form \( v = v(x) \), which is the simplest Ansatz. It also directly provides the general solution of the adjoint equation. We obtain three independent solutions of the adjoint equation (3.27)

\[ M = 0 : \quad v_a = 1, \quad v_b = x, \quad v_c = x^2; \]
\[ M < 0 : \quad v_a = 1, \quad v_b = \cosh(2\omega x), \quad v_c = \sinh(2\omega x), \quad \omega = \sqrt{-M/2}; \quad (3.28) \]
\[ M > 0 : \quad v_a = 1, \quad v_b = \cos(2\omega x), \quad v_c = \sin(2\omega x), \quad \omega = \sqrt{M/2}. \]
We will use these solutions of the adjoint equation to find first integrals of the ODE (3.23).

Let us use symmetries (3.20), (3.21), (3.22) and solutions of the adjoint equation (3.27) to construct first integrals. The notation $\tilde{I}_{j\alpha}$ means that this integral corresponds to symmetry $X_j$ and solution $v_\alpha$ of the adjoint equation.

For all values of the parameter $M$ there is only one common solution of the adjoint equation, namely

$$v_a(x) = 1.$$  \hspace{1cm} (3.29)

It provides us with the first integrals

$$\tilde{I}_1a = \frac{1}{2} \frac{\dddot{u}}{u^3} + \frac{M}{u}, \quad \tilde{I}_2a = u \left( \frac{1}{2} \frac{\dddot{u}}{u^3} + \frac{M}{u} \right) - \frac{\ddot{u}}{u},$$

$$\tilde{I}_3a = u^2 \left( \frac{1}{2} \frac{\dddot{u}}{u^3} + \frac{M}{u} \right) - 2u \frac{\ddot{u}}{u} + 2\dot{u}, \quad \tilde{I}_4a \equiv -2M.$$

(3.30)

Two additional first integrals for $M = 0$ are trivial:

$$\tilde{I}_{5a} \equiv 0, \quad \tilde{I}_{6a} \equiv -2.$$  \hspace{1cm} (3.31)

The non-trivial first integrals obey the relation

$$\tilde{I}_1a \tilde{I}_3a - \tilde{I}_2a^2 = 2M.$$

Thus we have only two independent first integrals and it is not sufficient for the integration of the third order ODE. To find a sufficient number of first integrals we will consider solutions of the adjoint equation which are specific for particular cases of the parameter $M$. Let us go through different cases of the parameter.

**Case: $M = 0$**

The additional solutions of the adjoint equation are

$$v_b(x) = x \quad \text{and} \quad v_c(x) = x^2.$$  \hspace{1cm}

For $v_b(x) = x$ we obtain first integrals

$$\tilde{I}_{1b} = \frac{x}{2} \frac{\dddot{u}}{u^3} + \frac{\ddot{u}}{u^2}, \quad \tilde{I}_{2b} = u \left( \frac{x}{2} \frac{\dddot{u}}{u^3} + \frac{\ddot{u}}{u^2} \right) - \frac{x}{u} \frac{\ddot{u}}{u} - 1,$$

$$\tilde{I}_{3b} = u^2 \left( \frac{x}{2} \frac{\dddot{u}}{u^3} + \frac{\ddot{u}}{u^2} \right) - 2u \left( \frac{x}{u} + 1 \right) + 2x \ddot{u},$$

$$\tilde{I}_{4b} \equiv 0, \quad \tilde{I}_{5b} \equiv 1, \quad \tilde{I}_{6b} \equiv 0.$$  \hspace{1cm} (3.32)
For $v_c(x) = x^2$ we find

\[ \tilde{I}_{1c} = \frac{x^2 \ddot{u}}{2 \dot{u}^3} + 2x \frac{\dddot{u}}{\dot{u}^2} + \frac{2}{\dot{u}}, \quad \tilde{I}_{2c} = u \left( \frac{x^2 \ddot{u}^2}{2 \dot{u}^3} + 2x \frac{\dddot{u}}{\dot{u}^2} + \frac{2}{\dot{u}} \right) - x^2 \ddot{u} - 2x, \]
\[ \tilde{I}_{3c} = u^2 \left( \frac{x^2 \ddot{u}^2}{2 \dot{u}^3} + 2x \frac{\dddot{u}}{\dot{u}^2} + \frac{2}{\dot{u}} \right) - 2u \left( \frac{x^2 \ddot{u}}{\dot{u}} + 2x \right) + 2x^2 \dot{u}, \]
\[ \tilde{I}_{4c} \equiv -2, \quad \tilde{I}_{5c} \equiv 0, \quad \tilde{I}_{6c} \equiv 0. \quad (3.33) \]

The nontrivial first integrals given in (3.32), (3.33) together with nontrivial first integrals given in (3.30) satisfy the relations

\[ \tilde{I}_{1a} \tilde{I}_{3a} - \tilde{I}_{2a}^2 = 0, \quad \tilde{I}_{1b} \tilde{I}_{3b} - \tilde{I}_{2b}^2 = -1, \quad \tilde{I}_{1c} \tilde{I}_{3c} - \tilde{I}_{2c}^2 = 0, \]
\[ \tilde{I}_{1a} \tilde{I}_{1c} - \tilde{I}_{1b}^2 = 0, \quad \tilde{I}_{2a} \tilde{I}_{2c} - \tilde{I}_{2b}^2 = -1, \quad \tilde{I}_{3a} \tilde{I}_{3c} - \tilde{I}_{3b}^2 = 0. \]

**Integration of the ODE**

We chose three first integrals $\tilde{I}_{1a}$, $\tilde{I}_{2a}$ and $\tilde{I}_{1b}$ and compute the Jacobian

\[ J = \det \left( \frac{\partial (\tilde{I}_{1a}, \tilde{I}_{2a}, \tilde{I}_{1b})}{\partial (u, \dot{u}, \ddot{u})} \right) = -\frac{1}{4} \frac{\dddot{u}^2}{\dot{u}^5}. \]

1. For $J \neq 0$ we can set these first integrals equal to constants

\[ \tilde{I}_{1a} = \frac{1}{2} \frac{\dddot{u}^2}{\dot{u}^3} = A, \quad \tilde{I}_{2a} = \frac{u \dddot{u}^2}{2 \dot{u}^3} - \frac{\dddot{u}}{\dot{u}} = B, \quad \tilde{I}_{1b} = \frac{x \dddot{u}^2}{2 \dot{u}^3} + \frac{\dddot{u}}{\dot{u}^2} = C, \]

where $A \neq 0$, $B$ and $C$ are constants. We rewrite the second and third equations as

\[ Au - \frac{\dddot{u}}{\dot{u}} = B, \quad Ax + \frac{\dddot{u}}{\dot{u}^2} = C \]

and exclude the derivatives

\[ \frac{\dddot{u}}{\dot{u}^2} = (Au - B)(C - Ax) = A. \]

From this equation we can obtain the solution which we present as

\[ u(x) = \frac{1}{C_1 x + C_2} + C_3, \quad (3.34) \]

where $C_1 \neq 0$, $C_2$ and $C_3$ are constants. It is the generic (three-parameter) solution of the ODE.
2. Considering $J = 0$, we solve

$$\ddot{u} = 0, \quad \dot{u} \neq 0$$

and obtain

$$u(x) = C_1 x + C_2, \quad C_1 \neq 0.$$  \hspace{1cm} (3.35)

Direct verification shows that it is a solution of the ODE (3.23). It depends only on two parameters and is thus a degenerate solution.

**Case: $M < 0$**

The specific solutions of the adjoint equation (3.27) are (3.29) and

$$v_b(x) = \cosh(2\omega x) \quad \text{and} \quad v_c(x) = \sinh(2\omega x), \quad \omega = \sqrt{-\frac{M}{2}}.$$  

For these solutions $v_b$ and $v_c$ we find first integrals

$$\tilde{I}_{1b} = \cosh(2\omega x) \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) + 2\omega \sinh(2\omega x) \left( \frac{\ddot{u}}{u^2} \right)$$

$$\tilde{I}_{2b} = \cosh(2\omega x) \left( u \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - \frac{\ddot{u}}{u} \right) + 2\omega \sinh(2\omega x) \left( u \frac{\ddot{u}}{u^2} - 1 \right), \hspace{1cm} (3.36)$$

$$\tilde{I}_{3b} = \cosh(2\omega x) \left( u^2 \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - 2 u \frac{\dddot{u}}{u} - 2 u \right) + 2\omega \sinh(2\omega x) \left( u^2 \frac{\ddot{u}}{u^2} - 2 u \right),$$

$$\tilde{I}_{4b} \equiv 0$$

and

$$\tilde{I}_{1c} = \sinh(2\omega x) \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) + 2\omega \cosh(2\omega x) \left( \frac{\ddot{u}}{u^2} \right),$$

$$\tilde{I}_{2c} = \sinh(2\omega x) \left( u \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - \frac{\ddot{u}}{u} \right) + 2\omega \cosh(2\omega x) \left( u \frac{\ddot{u}}{u^2} - 1 \right), \hspace{1cm} (3.37)$$

$$\tilde{I}_{3c} = \sinh(2\omega x) \left( u^2 \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - 2 u \frac{\dddot{u}}{u} - 2 u \right) + 2\omega \cosh(2\omega x) \left( u^2 \frac{\ddot{u}}{u^2} - 2 u \right),$$

$$\tilde{I}_{4c} \equiv 0,$$

respectively.

Nontrivial first integral of this case, which are given in (3.30), (3.36) and (3.37), obey the relations

$$\tilde{I}_{1a} \tilde{I}_{3a} - \tilde{I}_{2a}^2 = 2M, \quad \tilde{I}_{1b} \tilde{I}_{3b} - \tilde{I}_{2b}^2 = -2M, \quad \tilde{I}_{1c} \tilde{I}_{3c} - \tilde{I}_{2c}^2 = 2M,$$

$$\tilde{I}_{1a}^2 + \tilde{I}_{1c}^2 = \tilde{I}_{1b}^2, \quad \tilde{I}_{2a}^2 + \tilde{I}_{2c}^2 = \tilde{I}_{2b}^2 - 2M, \quad \tilde{I}_{3a}^2 + \tilde{I}_{3c}^2 = \tilde{I}_{3b}^2.$$
Integration of the ODE

We select $\tilde{I}_{1a}$, $\tilde{I}_{2a}$ and $\tilde{I}_{1b}$ and compute the Jacobian

$$J = \det \left( \frac{\partial (\tilde{I}_{1a}, \tilde{I}_{2a}, \tilde{I}_{1b})}{\partial (u, \dot{u}, \ddot{u})} \right)$$

$$= -\frac{\omega \ddot{u}^2 - 4\omega^2 \dot{u}^2}{\dot{u}^2} \left( \sinh(2\omega x)\ddot{u}^2 + 4\omega \cosh(2\omega x)\dot{u}\ddot{u} + 4\omega^2 \sinh(2\omega x)\dot{u}^2 \right).$$

1. In the case $J \neq 0$ we can set these first integrals equal to constants and obtain solutions

$$u(x) = C_1 \tanh(\omega x+C_2)+C_3 \quad \text{and} \quad u(x) = C_1 \coth(\omega x+C_2)+C_3, \quad (3.38)$$

where $C_1 \neq 0$, $C_2 \neq 0$ and $C_3$ are constants.

2. The case $J = 0$ gets split into two subcases.

(a) The system

$$\ddot{u}^2 - 4\omega^2 \dot{u}^2 = 0, \quad \dot{u} \neq 0$$

leads to

$$\ddot{u} = \pm 2\omega \dot{u}, \quad \dot{u} \neq 0.$$  

We obtain two degenerate solutions of the ODE

$$u(x) = C_1 e^{\pm 2\omega x} + C_2, \quad C_1 \neq 0. \quad (3.39)$$

(b) The system

$$\sinh(2\omega x)\ddot{u}^2 + 4\omega \cosh(2\omega x)\dot{u}\ddot{u} + 4\omega^2 \sinh(2\omega x)\dot{u}^2 = 0, \quad \dot{u} \neq 0$$

can be solved as quadratic for $\ddot{u}$:

$$\ddot{u} = -2\omega \coth(\omega x)\dot{u} \quad \text{or} \quad \ddot{u} = -2\omega \tanh(\omega x)\dot{u}, \quad \dot{u} \neq 0,$$

which can be solved as

$$u(x) = C_1 \tanh(\omega x) + C_2 \quad \text{or} \quad u(x) = C_1 \coth(\omega x) + C_2, \quad C_1 \neq 0. \quad (3.40)$$

Finally solutions (3.38), (3.39) and (3.40) can be written together as given in (3.25).
Case: $M > 0$

The specific solutions of the adjoint equation are (3.29) and

$$v_b(x) = \cos(2\omega x) \quad \text{and} \quad v_c(x) = \sin(2\omega x), \quad \omega = \sqrt{\frac{M}{2}}.$$  

For $v_b$ we compute the first integrals

$$\tilde{I}_{1b} = \cos(2\omega x) \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - 2\omega \sin(2\omega x) \left( \frac{\dddot{u}}{u^2} \right),$$

$$\tilde{I}_{2b} = \cos(2\omega x) \left( u \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - \frac{\dddot{u}}{u} \right) - 2\omega \sin(2\omega x) \left( u \frac{\dddot{u}}{u^2} - 1 \right), \quad (3.41)$$

$$\tilde{I}_{3b} = \cos(2\omega x) \left( u^2 \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - 2u \frac{\dddot{u}}{u} + 2\ddot{u} \right) - 2\omega \sin(2\omega x) \left( u^2 \frac{\dddot{u}}{u^2} - 2u \right),$$

$$\tilde{I}_{4b} \equiv 0.$$

For $v_c$ we get

$$\tilde{I}_{1c} = \sin(2\omega x) \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) + 2\omega \cos(2\omega x) \left( \frac{\dddot{u}}{u^2} \right),$$

$$\tilde{I}_{2c} = \sin(2\omega x) \left( u \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - \frac{\dddot{u}}{u} \right) + 2\omega \cos(2\omega x) \left( u \frac{\dddot{u}}{u^2} - 1 \right), \quad (3.42)$$

$$\tilde{I}_{3c} = \sin(2\omega x) \left( u^2 \left( \frac{1}{2} \frac{\dddot{u}}{u^3} - \frac{M}{u} \right) - 2u \frac{\dddot{u}}{u} + 2\ddot{u} \right) + 2\omega \cos(2\omega x) \left( u^2 \frac{\dddot{u}}{u^2} - 2u \right),$$

$$\tilde{I}_{4c} \equiv 0.$$

The nontrivial first integrals (3.30), (3.41), (3.42) satisfy the relations

$$\tilde{I}_{1a} \tilde{I}_{3a} - \tilde{I}_{2a}^2 = 2M, \quad \tilde{I}_{1b} \tilde{I}_{3b} - \tilde{I}_{2b}^2 = -2M, \quad \tilde{I}_{1c} \tilde{I}_{3c} - \tilde{I}_{2c}^2 = -2M,$$

$$\tilde{I}_{1b}^2 + \tilde{I}_{1c}^2 = \tilde{I}_{1a}^2, \quad \tilde{I}_{2b}^2 + \tilde{I}_{2c}^2 = \tilde{I}_{2a}^2 + 2M, \quad \tilde{I}_{3b}^2 + \tilde{I}_{3c}^2 = \tilde{I}_{3a}^2.$$

Integration of the ODE

Similarly to the previous cases we chose $\tilde{I}_{1a}$, $\tilde{I}_{2a}$ and $\tilde{I}_{1b}$ as three first integrals. The Jacobian is

$$J = \det \left( \frac{\partial(\tilde{I}_{1a}, \tilde{I}_{2a}, \tilde{I}_{1b})}{\partial(u, \dot{u}, \ddot{u})} \right) = \frac{\omega \dddot{u}^2 + 4\omega^2 \dddot{u}^2}{\dddot{u}^9} \left( \sin(2\omega x) \dddot{u}^2 + 4\omega \cos(2\omega x) \dddot{u} \dddot{u} - 4\omega^2 \sin(2\omega x) \dddot{u}^2 \right).$$
1. For $J \neq 0$ we obtain the solutions

$$u(x) = C_1 \tan(\omega x + C_2) + C_3,$$  \hspace{1cm} (3.43)

where $C_1 \neq 0$, $C_2 \neq \frac{\pi}{2} n$, $n \in \mathbb{Z}$ and $C_3$ are constants.

2. The equality $J = 0$ can happen in two cases

(a) First system

$$\ddot{u}^2 + 4\omega^2 \dot{u}^2 = 0, \quad \dot{u} \neq 0$$

has no solutions.

(b) Second system

$$\sin(2\omega x)\ddot{u}^2 + 4\omega \cos(2\omega x)\dot{u}\ddot{u} - 4\omega^2 \sin(2\omega x)\dot{u}^2 = 0, \quad \dot{u} \neq 0$$

can be solved for $\ddot{u}$: leads to

$$\ddot{u} = 2\omega \tan(\omega x)\dot{u} \quad \text{or} \quad \ddot{u} = -2\omega \cot(\omega x)\dot{u}, \quad \dot{u} \neq 0.$$

These equations are solved as

$$u(x) = C_1 \tan(\omega x) + C_2 \quad \text{or} \quad u(x) = C_1 \cot(\omega x) + C_2,$$  \hspace{1cm} (3.44)

where $C_1 \neq 0$ and $C_2$ are integration constants.

Finally, we unite solutions (3.43) and (3.44) into the generic solution of the ODE given in (3.26).

**Remark 3.7** It is possible to use a different Ansatz for solutions of the adjoint equation.

**Ansatz 2.** If we look for solutions of the form $v = v(x, u, \dot{u})$, we find

$$v(x, u, \dot{u}) = \frac{A(x, u)}{\sqrt{\dot{u}}} + \frac{B(u)}{\dot{u}} + C(x),$$

where

$$A(x, u) = \alpha(x)u + \beta(x), \quad \ddot{\alpha} + \frac{M}{2} \alpha = 0, \quad \dddot{\beta} + \frac{M}{2} \beta = 0,$$

$$B(u) = \gamma_2 u^2 + \gamma_1 u + \gamma_0, \quad \dddot{C} + 2M\dot{C} = 0.$$
Here we have to consider cases $M = 0$, $M < 0$ and $M > 0$ separately. We obtain the following independent solutions for $\alpha(x)$ and $\beta(x)$:

$M = 0$ : $\alpha_1 = 1, \quad \alpha_2 = x$, 
$\beta_1 = 1, \quad \beta_2 = x$;

$M < 0$ : $\alpha_1 = \cosh(\omega x), \quad \alpha_2 = \sinh(\omega x)$,
$\beta_1 = \cosh(\omega x), \quad \beta_2 = \sinh(\omega x), \quad \omega = \sqrt{-M/2}$;

$M > 0$ : $\alpha_1 = \cos(\omega x), \quad \alpha_2 = \sin(\omega x)$,
$\beta_1 = \cos(\omega x), \quad \beta_2 = \sin(\omega x), \quad \omega = \sqrt{M/2}$.

Solutions for $C(x)$ are the same as solutions for $v(x)$ given in (3.28). In each case we will obtain 10 independent solutions of the adjoint equation. The solutions of the original ODE (3.23) are of course the same as obtained with Ansatz 1.

4 The direct method and integrating factors

In this section we compare the direct method [3] with the method presented in the previous section. We consider a scalar ODE

$$F(x, u, \dot{u}, \ddot{u}, ..., u^{(n)}) = 0 \tag{4.1}$$

and assume that this ODE is solved with respect to the highest derivative $u^{(n)}$:

$$F = u^{(n)} - f(x, u, \dot{u}, \ddot{u}, ..., u^{(n-1)}) = 0. \tag{4.2}$$

We are interested in first integrals

$$I = I(x, u, \dot{u}, ..., u^{(n-1)}) \tag{4.3}$$

of this ODE such that

$$D(I) = \Lambda F, \tag{4.4}$$

holds identically in the whole space for some nonsingular function

$$\Lambda = \Lambda(x, u, \dot{u}, ..., u^{(n-1)}), \tag{4.5}$$

called an integrating factor. Since the left hand side of relation (4.4) is a total derivative it is annihilated by the action of the variational operator. Therefore we obtain the equation

$$\frac{\delta}{\delta u}(\Lambda F) = 0. \tag{4.6}$$
Remark 4.1 The assumption that the ODE is solved with respect to the highest derivative \(u^{(n)}\) allows us to restrict the form of the integrating factor to \(\Lambda = \Lambda(x, u, \dot{u}, ..., u^{(n-1)})\) (otherwise \(\Lambda\) could depend on the highest derivative). It provides us with the possibility of splitting equation (4.6) into \(1 + \lfloor n/2 \rfloor\) equations [3].

Let us provide comparison with the adjoint equation

\[
\frac{\delta}{\delta u}(vF) \bigg|_{F=0} = 0, \quad v = \varphi(x, u, \dot{u}, ..., u^{(n-1)}). \tag{4.7}
\]

If we consider \(\Lambda\) and \(\varphi\) which depend on the same variables, we obtain the following result.

**Proposition 4.2** An integrating factor is always a solution of adjoint equation independently of whether a symmetry of the underlying equation exists. The inverse statement is not true.

Using integrating factors, we have to solve the equation (4.4) in order to find a first integral \(I\). Explicit line integral formulas which provide first integrals were given in [3]. The approach based on the solution of the adjoint equation yields the first integral by formula (3.14), which does not require any integration.

It was observed on the examples of the previous section that the pair consisting of a symmetry \(X\) and a solution of the adjoint equation \(v\) can generate a trivial first integral. To the contrary, an integrating factor \(\Lambda\) provides a non-trivial first integral. On the other hand, it will be observed on the example given below that the approach based on Theorem 3.3 has the advantage that we can use a simpler Ansatz for \(\varphi\) then for \(\Lambda\).

**Example: application of the direct method**

We demonstrate the direct method using the example of ODE (3.23). We multiply the equation by the unknown factor \(\Lambda(x, u, \dot{u}, \ddot{u})\) and apply a variational operator (3.4):

\[
\frac{\delta}{\delta u} \left( \Lambda(x, u, \dot{u}, \ddot{u}) \left( \frac{\ddot{u}}{\dot{u}} - \frac{3}{2} \left( \frac{\ddot{u}}{\dot{u}} \right)^2 - M \right) \right) = 0. \tag{4.8}
\]

To simplify the problem we are looking for an integrating factor of the simplest form. It can be shown that there are no integrating factors of the form \(\Lambda(x, u)\). Therefore we chose the form

\[
\Lambda = \Lambda(x, u, \dot{u}). \tag{4.9}
\]
Equation (4.8) appears as

\[- \left( \frac{2}{\dot{u}} \frac{\partial \Lambda(x, u, \dot{u})}{\partial \dot{u}} + \frac{\Lambda(x, u, \dot{u})}{\dot{u}^2} \right) u^{(4)} + \ldots, \quad (4.10)\]

where the rest of equation does not contain \( u^{(4)} \). Thus we get the equation

\[\frac{2}{\dot{u}} \frac{\partial \Lambda(x, u, \dot{u})}{\partial \dot{u}} + \frac{\Lambda(x, u, \dot{u})}{\dot{u}^2} = 0. \quad (4.11)\]

Integration for variable \( \dot{u} \) yields

\[\Lambda = \frac{f(x, u)}{\sqrt{|\dot{u}|}}. \quad (4.12)\]

where \( f(x, u) \) is some function.

Substitution of \( \Lambda \) into (4.8) yields

\[\frac{3}{4 \sqrt{|\dot{u}|}} \left( \frac{2f_{xx} + Mf_{uu}}{\dot{u}^2} - 2f_{uu} \right) \ddot{u} - |\dot{u}|^\frac{3}{2} f_{wuu} - 3\text{sgn}(\dot{u}) \sqrt{|\dot{u}|} f_{xuu} - \frac{3}{2} \sqrt{|\dot{u}|} (2f_{xxx} + Mf_x) = 0, \quad (4.13)\]

where

\[\text{sgn}(x) = \begin{cases} 
1, & x > 0; \\
0, & x = 0; \\
-1, & x < 0.
\end{cases}\]

Equating to zero the coefficients of different powers of \( \ddot{u} \) and then those of different powers of \( \dot{u} \), we obtain the system

\[f_{uu} = 0, \quad f_{xx} + \frac{M}{2} f = 0. \quad (4.14)\]

This system has the following solutions for different cases of \( M \):

\[M = 0 : \quad f(x, u) = (C_1 x + C_2) u + C_3 x + C_4; \quad (4.15)\]

\[M < 0 : \quad f(x, u) = (C_1 e^{\omega x} + C_2 e^{-\omega x}) u + C_3 e^{\omega x} + C_4 e^{-\omega x}, \quad \omega = \sqrt{-M/2}; \quad (4.16)\]

\[M > 0 : \quad f(x, u) = (C_1 \cos(\omega x) + C_2 \sin(\omega x)) u + C_3 \cos(\omega x) + C_4 \sin(\omega x), \quad \omega = \sqrt{M/2}. \quad (4.17)\]
Thus we can present a number of independent solutions $\Lambda$ and find the corresponding integrals $I(x, u, \dot{u}, \ddot{u})$ using the relation

$$D(I) = \frac{\partial I}{\partial x} + \dot{u} \frac{\partial I}{\partial u} + \ddot{u} \frac{\partial I}{\partial \dot{u}} + \frac{\partial I}{\partial \ddot{u}} = \Lambda F.$$ (4.18)

This equation is solved in the same manner as (4.13), starting from the highest derivative $\ddot{u}$.

**Case: $M = 0$**

There are four integrating factors

$$\Lambda_1 = \frac{1}{\sqrt{|\dot{u}|}}, \quad \Lambda_2 = \frac{x}{\sqrt{|\dot{u}|}}, \quad \Lambda_3 = \frac{u}{\sqrt{|\dot{u}|}}, \quad \Lambda_4 = \frac{ux}{\sqrt{|\dot{u}|}},$$

which provide us with the corresponding first integrals

$$I_1 = \frac{1}{\sqrt{|\dot{u}|}} \ddot{u}, \quad I_2 = \frac{1}{\sqrt{|\dot{u}|}} \left(x \frac{\ddot{u}}{\dot{u}} + 2\right), \quad I_3 = \frac{1}{\sqrt{|\dot{u}|}} \left(u \frac{\ddot{u}}{\dot{u}} - 2\ddot{u}\right), \quad I_4 = \frac{1}{\sqrt{|\dot{u}|}} \left(xu \frac{\ddot{u}}{\dot{u}} - 2x\ddot{u} + 2u\right),$$ (4.19)

which obey the relation

$$I_1I_4 - I_2I_3 = 4\text{sgn}(\dot{u}).$$

**Integration of the ODE**

We select first integrals $I_1, I_2$ and $I_3$ and compute the Jacobian

$$J = \det \left( \frac{\partial(I_1, I_2, I_3)}{\partial(u, \dot{u}, \ddot{u})} \right) = \text{sgn}(\dot{u}) \frac{\ddot{u}}{|\dot{u}|^3}.$$

1. In the case $J \neq 0$ we can use the first integrals. Setting them to be equal to constants

$$I_1 = \frac{1}{\sqrt{|\dot{u}|}} \ddot{u} = \text{const}, \quad I_2 = \frac{1}{\sqrt{|\dot{u}|}} \left(x \frac{\ddot{u}}{\dot{u}} + 2\right) = \text{const},$$

$$I_3 = \frac{1}{\sqrt{|\dot{u}|}} \left(u \frac{\ddot{u}}{\dot{u}} - 2\ddot{u}\right) = \text{const},$$

we can find the the generic solution

$$u(x) = \frac{1}{C_1 x + C_2} + C_3,$$ (4.20)

where $C_1 \neq 0, C_2$, and $C_3$ are constants.
2. The case $J = 0$ leads to the system

$$\ddot{u} = 0, \quad \dot{u} \neq 0.$$  

Thus, we obtain the special solution using the ODE

$$u(x) = C_1 x + C_2, \quad C_1 \neq 0. \quad (4.21)$$

**Case: $M < 0$**

There are four independent integrating factors

$$\Lambda_1 = \frac{e^{\omega x}}{\sqrt{|u|}}, \quad \Lambda_2 = \frac{e^{-\omega x}}{\sqrt{|u|}}, \quad \Lambda_3 = \frac{e^{\omega x} u}{\sqrt{|u|}}, \quad \Lambda_4 = \frac{e^{-\omega x} u}{\sqrt{|u|}}.$$  

They generate the first integrals

$$I_1 = \frac{e^{\omega x}}{\sqrt{|u|}} \left( \frac{\ddot{u}}{u} + 2\omega \right), \quad I_2 = \frac{e^{-\omega x}}{\sqrt{|u|}} \left( \frac{\ddot{u}}{u} - 2\omega \right),$$

$$I_3 = \frac{e^{\omega x}}{\sqrt{|u|}} \left( \frac{\ddot{u}}{u} - 2\ddot{u} + 2\omega u \right), \quad I_4 = \frac{e^{-\omega x}}{\sqrt{|u|}} \left( \frac{\ddot{u}}{u} - 2\ddot{u} - 2\omega u \right). \quad (4.22)$$

These four first integrals obey the relation

$$I_1 I_4 - I_2 I_3 = -8\omega \text{sgn}(\dot{u}).$$

**Integration of the ODE**

We chose the three first integrals $I_1$, $I_2$ and $I_3$, compute the Jacobian

$$\text{det} \left( \frac{\partial (I_1, I_2, I_3)}{\partial (u, \dot{u}, \ddot{u})} \right) = -2\omega e^{\omega x} \text{sgn}(\dot{u}) \frac{\ddot{u}}{|\dot{u}|^2}$$

and consider two cases.

1. In the case $J \neq 0$ we can use the values of the first integrals to obtain the generic solutions of the ODE

$$u(x) = C_1 \tanh(\omega x + C_2) + C_3 \quad \text{and} \quad u(x) = C_1 \coth(\omega x + C_2) + C_3,$$

where $C_1 \neq 0$, $C_2$ and $C_3$ are constants, as well as one special solution

$$u(x) = C_1 e^{2\omega x} + C_2, \quad C_1 \neq 0. \quad (4.23)$$

2. In the case $J = 0$ we can use the values of the first integrals to obtain the generic solutions of the ODE

$$u(x) = C_1 x + C_2, \quad C_1 \neq 0.$$  

$\quad (4.24)$
2. If $J = 0$, we obtain the system

$$\ddot{u} + 2\omega \dot{u} = 0, \quad \dot{u} \neq 0.$$ 

and find the special solution

$$u(x) = C_1 e^{-2\omega x} + C_2, \quad C_1 \neq 0. \quad (4.25)$$

**Case: $M > 0$**

As in the previous case there are four independent integrating factors

$$\Lambda_1 = \frac{\cos(\omega x)}{\sqrt{|\dot{u}|}}, \quad \Lambda_2 = \frac{\sin(\omega x)}{\sqrt{|\dot{u}|}}, \quad \Lambda_3 = \frac{\cos(\omega x)u}{\sqrt{|\dot{u}|}}, \quad \Lambda_4 = \frac{\sin(\omega x)u}{\sqrt{|\dot{u}|}},$$

which let us find the corresponding first integrals

$$I_1 = \frac{1}{\sqrt{|\dot{u}|}} \left( \cos(\omega x) \frac{\ddot{u}}{\dot{u}} - 2\omega \sin(\omega x) \right), \quad I_2 = \frac{1}{\sqrt{|\dot{u}|}} \left( \sin(\omega x) \frac{\ddot{u}}{\dot{u}} + 2\omega \cos(\omega x) \right),$$

$$I_3 = \frac{1}{\sqrt{|\dot{u}|}} \left( \cos(\omega x) \left( \frac{\dddot{u}}{\dot{u}} - 2\ddot{u} \right) - 2\omega \sin(\omega x) u \right),$$

$$I_4 = \frac{1}{\sqrt{|\dot{u}|}} \left( \sin(\omega x) \left( \frac{\dddot{u}}{\dot{u}} - 2\ddot{u} \right) + 2\omega \cos(\omega x) u \right). \quad (4.26)$$

The first integrals satisfy the relation

$$I_1 I_4 - I_2 I_3 = 4\omega \text{sgn}(\dot{u}).$$

**Integration of the ODE**

We find the Jacobian

$$\det \left( \frac{\partial(I_1, I_2, I_3)}{\partial(u, \dot{u}, \ddot{u})} \right) = \omega \text{sgn}(\dot{u}) \frac{\cos(\omega x) \dddot{u} - 2\omega \sin(\omega x) \ddot{u}}{|\dot{u}|^2}$$

and consider two cases.

1. For $J \neq 0$ we obtain the solution in the form

$$u(x) = C_1 \tan(\omega x + C_2) + C_3, \quad (4.27)$$

where $C_1 \neq 0$, $C_2 \neq \pi n$, $n \in \mathbb{Z}$ and $C_3$ are constants.
2. The determinant of the Jacobian equals to zero if
\[ \cos(\omega x) \ddot{u} - 2\omega \sin(\omega x) \dot{u} = 0, \quad \dot{u} \neq 0. \]

Thus, we obtain the special solution
\[ u(x) = C_1 \tan(\omega x) + C_2, \quad C_1 \neq 0. \] (4.28)

Let us sum up the comparison of the two methods for the example (3.23).

1. Both methods allow to find the general solution;
2. Integration factors as opposed to the adjoint equation method always provide non-trivial first integrals;
3. We had to use a more complicated Ansatz for \( \Lambda \) than for \( v \). Searching for integrating factors, we needed \( \Lambda = \Lambda(x, u, \dot{u}) \). It was sufficient to consider the simple Ansatz \( v = v(x) \) for the solution of the adjoint equation;
4. Using \( \Lambda \) in the direct method, we have to integrate for obtaining first integrals. Using \( v \) and symmetry operator \( X \) in the adjoint equation method, we apply a formula which does not require integration. This advantage may be crucial in the case of discrete equations.

## 5 Adjoint equation method for mappings

In this section we will consider systems of discrete equations and develop a theory analogous to the continuous case results reviewed in Paragraph 3.1. It should be noted that discrete equations might not possess continuous limits. Such discrete equations have no relation to discretizations of ODEs. As in the previous section we will assume summation over repeated indices.

Let us consider discrete equations with the dependent variable
\[ u_m = (u^1_m, ..., u^q_m), \quad m \in \mathbb{Z}. \]

Discrete systems of order \( n \) can be presented as equations involving \( n + 1 \) points
\[ F_\beta(m, u_m, u_{m+1}, u_{m+2}, ..., u_{m+n}) = 0, \quad \beta = 1, 2, ..., r. \] (5.1)

We will assume that these equations can be resolved for \( u_m \) and \( u_{m+n} \). This assumption is necessary to solve the Cauchy problem to the left and to the right from the lattice points containing initial values.

We consider Lie point symmetries
\[ X = \eta^\alpha(u) \frac{\partial}{\partial u^\alpha}. \] (5.2)
For application to functions on lattice points we consider symmetry operators which are prolonged to all points involved in equations (5.1)

\[ X = \eta^\alpha_m \frac{\partial}{\partial u^\alpha_m} + \eta^\alpha_{m+1} \frac{\partial}{\partial u^\alpha_{m+1}} + \ldots + \eta^\alpha_{m+n} \frac{\partial}{\partial u^\alpha_{m+n}}, \quad \eta^\alpha_l = \eta^\alpha(u_l). \]  

(5.3)

It is helpful to introduce forward and backwards shift operators \( S_+ \) and \( S_- \):

\[ S_+ m = m + 1, \quad S_+ u_m = u_{m+1}, \]

\[ S_- m = m - 1, \quad S_- u_m = u_{m-1}. \]

Discrete variational operators are defined by the relation

\[ \delta \sum_m \mathcal{F}(m, u_m, u_{m+1}, \ldots, u_{m+n}) = \sum_m \delta u^\alpha_m \sum_{k=0}^\infty S_k^- \frac{\partial}{\partial u^\alpha_{m+k}} \mathcal{F}(m, u_m, u_{m+1}, \ldots, u_{m+n}). \]

We suppose \( \mathcal{F} \to 0 \) sufficiently fast when \( m \to \pm \infty \) so that the difference functional is well defined. The relation provides us with operators

\[ \frac{\delta}{\delta u^\alpha_m} = \sum_{k=0}^\infty S_k^- \frac{\partial}{\partial u^\alpha_{m+k}} = \frac{\partial}{\partial u^\alpha_m} + S_- \frac{\partial}{\partial u^\alpha_{m+1}} + \ldots + S_k^- \frac{\partial}{\partial u^\alpha_{m+k}} + \ldots \]  

(5.4)

Note that these operators are given for the system (5.1) of arbitrary order \( n \).

We will make use of adjoint variables \( v_m = (v^1_m, \ldots, v^r_m) \) and adjoint equations

\[ F^\ast_\alpha = \frac{\delta}{\delta u^\alpha_m} (v^\beta_m F_\beta) = 0, \quad \alpha = 1, \ldots, q, \]  

(5.5)

which are always linear for the adjoint variables \( v_m \). These equations can be presented as

\[ F^\ast_\alpha = v^\beta_m \frac{\partial F_\beta}{\partial u^\alpha_m} + v^\beta_{m-1} S_- \left( \frac{\partial F_\beta}{\partial u^\alpha_{m+1}} \right) + \ldots + v^\beta_{m-k} S_k^- \left( \frac{\partial F_\beta}{\partial u^\alpha_{m+k}} \right) + v^\beta_{m-n} S_n^- \left( \frac{\partial F_\beta}{\partial u^\alpha_{m+n}} \right) = 0. \]

**Remark 5.1** The system which consists of the original equations (5.1) together with the adjoint equations (5.5) can be considered as variational equations for the formal Lagrangian

\[ L = v^\beta_m F_\beta. \]  

(5.6)
Now we will obtain the main identity which will be used to find first integrals.

Let us fix the value of index \( m \), which corresponds to the left point in the equations (5.1), and define higher order discrete Euler–Lagrange operators

\[
\frac{\delta}{\delta u^\alpha_m(j)} = \sum_{k=0}^{\infty} S_k \frac{\partial}{\partial u^\alpha_{m+j+k}} = \frac{\partial}{\partial u^\alpha_{m+j}} + S_1 \frac{\partial}{\partial u^\alpha_{m+j+1}} + \ldots + S_k \frac{\partial}{\partial u^\alpha_{m+j+k}} + \ldots \quad (5.7)
\]

We note that variational operators (5.4) belong to this family:

\[
\frac{\delta}{\delta u^\alpha_m} = \frac{\delta}{\delta u^\alpha_{m(0)}}.
\]

Lemma 5.2 (Main identity for discrete equations) The following identity holds

\[
v^\beta_m X F_\beta = \eta^\alpha_m F^*_\alpha + (1 - S_-)J, \quad (5.8)
\]

where

\[
J = \sum_{j=1}^{n} \eta^\alpha_{m+j} \frac{\delta}{\delta u^\alpha_{m(j)}} (v^\beta_m F_\beta) \quad (5.9)
\]

Proof. The identity can be shown by a direct calculation. □

An alternative derivation of the main identity (5.8) can be based on the following operator identity.

Lemma 5.3 The following operator identity (no summation for \( \alpha \)) holds

\[
\sum_{k=0}^{\infty} \eta^\alpha_{m+k} \frac{\partial}{\partial u^\alpha_{m+k}} = \eta^\alpha_m \sum_{k=0}^{\infty} S_k \frac{\partial}{\partial u^\alpha_{m+k}} + (1 - S_-) \sum_{j=1}^{\infty} \eta^\alpha_{m+j} \frac{\delta}{\delta u^\alpha_{m(j)}} \quad (5.10)
\]

If we take summation of the identities (5.10) for all \( \alpha = 1,\ldots,q \) and apply the resulting operator identity to the formal Lagrangian (5.6), we get the identity (5.8).

Let us adapt the results of the continuous case, given in Paragraph 3.1, to the discrete case.

Theorem 5.4 (Main theorem for discrete equations) Let the adjoint equations (5.5) be satisfied for all solutions of the original equations (5.1) upon a substitution

\[
v_m = \varphi(m, u_m), \quad \varphi \neq 0. \quad (5.11)
\]

Then, any Lie point symmetry (5.2) of the equations (5.1) leads to first integral

\[
J = \sum_{j=1}^{n} \eta^\alpha_{m+j} \frac{\delta}{\delta u^\alpha_{m(j)}} (v^\beta_m F_\beta), \quad (5.12)
\]

where values \( v_m, \ldots, v_{m-n} \) should be eliminated by means of the equations (5.11) and their shifts to the left.
Proof. The result follows from the identity (5.8).

Remark 5.5 Generally first integrals $J$, given by (5.12), can depend on more than $n$ points. We will call such expression higher first integrals. Using the discrete equations (5.1), we can always reduce this number of points to minimal set, for example, to points $m, m+1, \ldots, m+n-1$, i.e., $J(m, u_m, u_{m+1}, u_{m+2}, \ldots, u_{m+n-1})$.

Remark 5.6 Instead of point substitutions (5.11) we can use generalized substitutions which involve neighbouring points. For systems (5.1), we can consider substitutions like

$$v_m = \varphi(m, u_m, u_{m+1}), \quad \ldots, \quad v_m = \varphi(m, u_m, u_{m+1}, \ldots, u_{m+n-1}).$$

(5.13)

Remark 5.7 The requirement that the substitution (5.11) annihilates the adjoint equations on the solutions of the original equations, which is used in Theorem 5.4, can be replaced by a weaker condition

$$\eta^\alpha_m F^*_\alpha = 0,$$

which should hold for a given symmetry $X$ of the system (5.1) on the solutions of this system. This is a weaker condition than the requirement of the theorem that all equations $F^*_\alpha = 0$ hold individually.

In the following sections we will consider applications of these results.

6 Case of mapping involving a single dependent variable

6.1 General theory

In this section we will consider scalar discrete equations of order $n$

$$F(m, u_m, u_{m+1}, u_{m+2}, \ldots, u_{m+n}) = 0$$

(6.1)

admitting symmetries of the form

$$X = \eta(u) \frac{\partial}{\partial u}.$$ (6.2)

Such symmetries are prolonged as

$$X = \eta_m \frac{\partial}{\partial u_m} + \eta_{m+1} \frac{\partial}{\partial u_{m+1}} + \ldots + \eta_{m+n} \frac{\partial}{\partial u_{m+n}}, \quad \eta_l = \eta(u_l).$$ (6.3)

to all points involved in the equation (6.1).
The corresponding adjoint equation (5.5) has the form

\[ F^* = \frac{\delta}{\delta u_m} (v_m F) = 0, \quad (6.4) \]

where

\[ \frac{\delta}{\delta u_m} = \sum_{k=0}^{\infty} S^k \frac{\partial}{\partial u_{m+k}} = \frac{\partial}{\partial u_m} + S_k \frac{\partial}{\partial u_{m+1}} + \ldots + \frac{\partial}{\partial u_{m+k}} + \ldots \quad (6.5) \]

is the discrete variational operator. Explicitly we have

\[ F^* = v_m \frac{\partial F}{\partial u_m} + v_{m-1} S_{-} \left( \frac{\partial F}{\partial u_{m+1}} \right) + \ldots + v_{m-k} S_{-} \left( \frac{\partial F}{\partial u_{m+k}} \right) + \ldots + v_{m-n} S_{-} \left( \frac{\partial F}{\partial u_{m+n}} \right) = 0. \]

Theorem 5.4 restricted to the case of this section states the following.

**Theorem 6.1 (Main theorem for scalar discrete equations)** Let the adjoint equation (6.4) be satisfied for all solutions of the original equation (6.1) upon a substitution

\[ v_m = \varphi(m, u_m), \quad \varphi \neq 0. \quad (6.6) \]

Then, any Lie point symmetry (6.2) of the equation (6.1) leads to a first integral

\[ J = \sum_{j=1}^{n} \eta_{m+j} \frac{\delta}{\delta u_{m(j)}} (v_m F), \quad (6.7) \]

where

\[ \frac{\delta}{\delta u_{m(j)}} = \sum_{k=0}^{\infty} S^k \frac{\partial}{\partial u_{m+j+k}} = \frac{\partial}{\partial u_{m+j}} + S_k \frac{\partial}{\partial u_{m+j+1}} + \ldots + \frac{\partial}{\partial u_{m+j+k}} + \ldots \]

and values \( v_m, \ldots, v_{m-n} \) should be eliminated by means of the Eq. (6.6) and its shifts to the left.

### 6.2 Four-point scalar discrete equation and example

Let us specify the general results of the previous paragraph to one scalar four-point discrete equation

\[ F(m, u_m, u_{m+1}, u_{m+2}, u_{m+3}) = 0. \]

The invariance condition reads as follows

\[ X(F) \big|_{F=0} = \left[ \eta_m \frac{\partial F}{\partial u_m} + \eta_{m+1} \frac{\partial F}{\partial u_{m+1}} + \eta_{m+2} \frac{\partial F}{\partial u_{m+2}} + \eta_{m+3} \frac{\partial F}{\partial u_{m+3}} \right]_{F=0} = 0. \]
The adjoint equation takes the form
\[ F^* = v_m \frac{\partial F}{\partial u_m} + v_{m-1} S_{-} \left( \frac{\partial F}{\partial u_{m+1}} \right) + v_{m-2} S_{-}^2 \left( \frac{\partial F}{\partial u_{m+2}} \right) + v_{m-3} S_{-}^3 \left( \frac{\partial F}{\partial u_{m+3}} \right) = 0. \]

The first integral gets presented as
\[ J = \eta_{m+1} \left[ v_m \frac{\partial F}{\partial u_{m+1}} + S_{-} \left( v_m \frac{\partial F}{\partial u_{m+2}} \right) + S_{-}^2 \left( v_m \frac{\partial F}{\partial u_{m+3}} \right) \right] + \eta_{m+2} \left[ v_m \frac{\partial F}{\partial u_{m+2}} + S_{-} \left( v_m \frac{\partial F}{\partial u_{m+3}} \right) \right] + \eta_{m+3} \left[ v_m \frac{\partial F}{\partial u_{m+3}} \right]. \]

**Example**

Let us consider the discrete equation
\[ F = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - K = 0, \quad K \neq 0. \quad (6.8) \]

This equation was considered in [5, 6] as a part of the system (8.31), which will be examined below.

We note that for \( K = 0 \) this equation is equivalent to the system
\[ u_{m+2} - u_m = 0, \]
\[ u_{m+1} - u_m \neq 0, \]
which can be easily solved as
\[ u_m = A(-1)^m + B, \quad A \neq 0. \]

The equation (6.8) admits symmetries
\[ X_1 = \frac{\partial}{\partial u}, \quad X_2 = u \frac{\partial}{\partial u}, \quad X_3 = u^2 \frac{\partial}{\partial u}. \quad (6.9) \]

The adjoint equation (6.4) (after use of the original equation \( F = 0 \)) is
\[ F^* = \frac{K(u_{m+2} - u_{m+1})}{(u_{m+2} - u_m)(u_{m+1} - u_{m+1})}(v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3}) = 0. \quad (6.10) \]

It gets simplified to a linear mapping
\[ v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3} = 0. \quad (6.11) \]
It is easy to find solutions \( v_m = v_m(m) \). We obtain three independent solutions of the adjoint equation

\[
K = 4 : \quad v^a_m = 1, \quad v^b_m = m, \quad v^c_m = m^2; \\
0 < K \text{ or } K > 4 : \quad v^a_m = 1, \quad v^b_m = \mu_1^m, \quad v^c_m = \mu_2^m; \\
0 < K < 4 : \quad v^a_m = 1, \quad v^b_m = \cos(2\phi m), \quad v^c_m = \sin(2\phi m); 
\]

where

\[
\mu_{1,2} = \frac{(K - 2) \pm \sqrt{K^2 - 4K}}{2} \quad \text{and} \quad \phi = \arccos\left(\frac{\sqrt{K}}{2}\right). 
\]

First of all we consider the solution of the adjoint equation \( v^a_m = 1 \), which is common for all values \( K \neq 0 \). Applying Theorem 5.4 with this solution and symmetries \( X_1, X_2 \) and \( X_3 \) and simplifying the obtained first integrals as described in Remark 5.5 we get the first integrals

\[
\tilde{J}_{1a} = 2 \left( \frac{K}{u_{m+2} - u_m} - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right), \\
\tilde{J}_{2a} = \frac{K(u_{m+2} + u_m)}{u_{m+2} - u_m} - \frac{2u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{2u_{m+1}}{u_{m+1} - u_m}, \\
\tilde{J}_{3a} = 2 \left( \frac{Ku_{m+2}u_m}{u_{m+2} - u_m} - \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}^2}{u_{m+1} - u_m} \right),
\]

respectively.

These three first integrals, which hold for all \( K \neq 0 \), are not independent. They satisfy the relation

\[
\tilde{J}_{1a}\tilde{J}_{3a} - (\tilde{J}_{2a})^2 = 4K - K^2. \tag{6.14}
\]

To integrate the discrete equation \((6.8)\) we need one more independent first integral (it should be a first integral which involves \( m \)). As in the continuous case we need to consider different cases of the parameter \( K \) separately.

**Case: \( K = 4 \)**

For \( K = 4 \) the other solutions of the adjoint equation are

\[
v^b_m = m \quad \text{and} \quad v^c_m = m^2.
\]

For \( v^b_m = m \) and symmetries \((6.9)\) we get first integrals

\[
\tilde{J}_{1b} = 2m \left[ \frac{K}{u_{m+2} - u_m} - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right] \\
- K \left( \frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m} \right) + \frac{3}{u_{m+2} - u_{m+1}} + \frac{3}{u_{m+1} - u_m},
\]

\[31\]
In total we obtained nine nontrivial first integrals. They satisfy 6 relations

\[ J_{2b} = m \left[ \frac{K(u_{m+2} + u_m)}{u_{m+2} - u_m} - \frac{2u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{2u_{m+1}}{u_{m+1} - u_m} \right] \]

\[ J_{3b} = 2m \left[ \frac{K u_{m+2} u_m}{u_{m+2} - u_m} - \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}^2}{u_{m+1} - u_m} \right] \]

\[ J_{1c} = 2m^2 \left[ \frac{1}{u_{m+2} - u_m} - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right] \]

\[ J_{2c} = m^2 \left[ \frac{K (u_m + u_{m+2})}{u_{m+2} - u_m} - \frac{2u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{2u_{m+1}}{u_{m+1} - u_m} \right] \]

\[ J_{3c} = 2m^2 \left[ \frac{K u_{m+2} u_m}{u_{m+2} - u_m} - \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}^2}{u_{m+1} - u_m} \right] \]

For \( v_m^c = m^2 \) we get first integrals

\[ J_{1c} = \frac{1}{2m} \left[ \frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m} \right] \]

\[ J_{2c} = \frac{6}{m} \left[ \frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m} \right] \]

\[ J_{3c} = \frac{6}{m} \left[ \frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m} \right] \]

In total we obtained nine nontrivial first integrals. They satisfy 6 relations

\[ J_{1a} J_{3a} - J_{2a}^2 = 0, \quad J_{1b} J_{3b} - J_{2b}^2 = -4, \quad J_{1c} J_{3c} - J_{2c}^2 = 4, \]

\[ J_{1a} J_{1c} - J_{1b}^2 - \frac{1}{4} J_{1a}^2 = 0, \quad J_{2a} J_{2c} - J_{2b}^2 - \frac{1}{4} J_{2a}^2 = -4, \quad J_{3a} J_{3c} - J_{3b}^2 - \frac{1}{4} J_{3a}^2 = 0. \]
As we know, for a four-point equation we have at most three independent first integrals

Integration of the mapping

Let us choose three first integrals $\tilde{J}_{1a}$, $\tilde{J}_{2a}$ and $\tilde{J}_{1b}$. The Jacobian is

$$J = \det \left( \frac{\partial (\tilde{J}_{1a}, \tilde{J}_{2a}, \tilde{J}_{1b})}{\partial (u_m, u_{m+1}, u_{m+2})} \right) = \frac{16(u_{m+2} - 2u_{m+1} + u_m)^4}{(u_{m+1} - u_m)(u_{m+2} - u_m)^3(u_{m+2} - u_{m+1})^3}$$

1. For $J \neq 0$ we can set these first integrals equal to constants, eliminate $u_{m+1}$ and $u_{m+2}$ from them and express $u_m$ in terms of $m$ and the constants. We obtain

$$u_m = \frac{1}{C_1 m + C_2} + C_3,$$

where $C_1 \neq 0$, $C_2$ and $C_3$ are constants.

2. For $J = 0$ we solve the system

$$u_{m+2} - 2u_{m+1} + u_m = 0,$$

$$u_{m+1} \neq u_m, \quad u_{m+2} \neq u_m$$

and obtain its solution

$$u_m = C_1 m + C_2, \quad C_1 \neq 0.$$  \hspace{1cm} (6.16)

Using substitution into the discrete equation (6.8), we confirm that (6.16) is a solution. It is a degenerate solution since it depends only on two constants.

Case: $K < 0$ or $K > 4$

In this case two specific solutions of the adjoint equation (6.10) are

$$v^b_m = \mu_1^m \quad \text{and} \quad v^c_m = \mu_2^m, \quad \mu_{1,2} = \frac{(K-2) \pm \sqrt{K^2 - 4K}}{2}.$$  \hspace{1cm} (6.17)

For $v^b,c_m = \mu_i^m$ ($i = 1$ for $b$ and $i = 2$ for $c$) we obtain first integrals

$$\tilde{J}_{1b,1c} = \mu_i^m \left[ \frac{K}{u_{m+2} - u_m} - \frac{K}{u_{m+1} - u_m} \right] + \mu_i^{m-1} \left[ K \left( \frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m} \right) - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right] - \mu_i^{m-2} \left[ \frac{1}{u_{m+2} - u_{m+1}} + \frac{1}{u_{m+1} - u_m} \right],$$

33
\[ J_{2b,3c} = \mu_i^m \left[ \frac{K u_m}{u_{m+2} - u_m} - \frac{K u_m}{u_m + u_{m+1} - u_m} \right] + \mu_i^{m-1} \left[ \frac{K}{2} \left( \frac{u_{m+2} + u_m}{u_{m+2} - u_m} + \frac{u_{m+1} + u_m}{u_{m+1} - u_m} \right) - \frac{u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}}{u_{m+1} - u_m} \right] - \mu_i^{m-2} \left[ \frac{u_{m+1}}{u_{m+2} - u_{m+1}} + \frac{u_{m+1}}{u_{m+1} - u_m} \right], \]

\[ J_{3b,3c} = \mu_i^m \left[ \frac{K u_m^2}{u_{m+2} - u_m} - \frac{K u_m^2}{u_m + u_{m+1} - u_m} \right] + \mu_i^{m-1} \left[ K \left( \frac{u_{m+2} u_m}{u_{m+2} - u_m} + \frac{u_{m+1} u_m}{u_{m+1} - u_m} \right) - \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}^2}{u_{m+1} - u_m} \right] - \mu_i^{m-2} \left[ \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} + \frac{u_{m+1}^2}{u_{m+1} - u_m} \right]. \]

These first integrals correspond to three symmetries (6.9), respectively.

The first integrals of this case obey the relations

\[ \tilde{J}_{1a} \tilde{J}_{3a} - \tilde{J}_{2a}^2 = K(4 - K), \quad \tilde{J}_{1b} \tilde{J}_{3b} - \tilde{J}_{2b}^2 = 0, \quad \tilde{J}_{1c} \tilde{J}_{3c} - \tilde{J}_{2c}^2 = 0, \]

\[ \tilde{J}_{1b} \tilde{J}_{1c} - \frac{K}{4} \tilde{J}_{1a}^2 = 0, \quad \tilde{J}_{2b} \tilde{J}_{2c} - \frac{K}{4} \tilde{J}_{2a}^2 = \frac{K^2(4 - K)}{4}, \quad \tilde{J}_{3b} \tilde{J}_{3c} - \frac{K}{4} \tilde{J}_{3a}^2 = 0. \]

Integration of the mapping

Let us chose \( \tilde{J}_{1a}, \tilde{J}_{2a} \) and \( \tilde{J}_{1b} \) as three first integrals and find the Jacobian

\[ J = \text{det} \left( \frac{\partial(\tilde{J}_{1a}, \tilde{J}_{2a}, \tilde{J}_{1b})}{\partial(u_m, u_{m+1}, u_{m+2})} \right) = K(\mu_1 - 1)\mu_1^{m-2} \]

\[ \times \frac{K(u_{m+2} - 2u_{m+1} + u_m)^2 + (4 - K)(u_{m+2} - u_m)^2}{(u_{m+1} - u_m)^3(u_{m+2} - u_m)^3(u_{m+2} - u_{m+1})^3} (\mu_1 u_{m+2} - (\mu_1 + 1)u_{m+1} + u_m)^2. \]

1. For \( J \neq 0 \) we set first integrals equal to constants and obtain

\[ u_m = C_1 \frac{(4 - K)(\mu_2 - \mu_1) - C_2(1 - \mu_1)^2\mu_m^n + \frac{K}{C_2}(1 - \mu_2)^2\mu_1^n}{K(\mu_2 - \mu_1) - C_2(1 - \mu_1)^2\mu_2^n + \frac{K}{C_2}(1 - \mu_2)^2\mu_1^n} + C_3, \]  \( (6.17) \)

where \( C_1 \neq 0, C_2 \neq 0 \) and \( C_3 \) are constants.

2. The case \( J = 0 \) splits into two subcases.
(a) The system
\[ K(u_{m+2} - 2u_{m+1} + u_m)^2 + (4 - K)(u_{m+2} - u_m)^2 = 0, \]
leads to
\[ u_{m+2} - 2u_{m+1} + u_m = \pm \sqrt{\frac{K - 4}{K}(u_{m+2} - u_m)}, \]
and provides us with the degenerate solutions
\[ u_m = C_1\mu_1^m + C_2 \quad \text{and} \quad u_m = C_1\mu_2^m + C_2, \quad C_1 \neq 0. \quad (6.18) \]

(b) The system
\[ \mu_1 u_{m+2} - (\mu_1 + 1)u_{m+1} + u_m = 0, \]
leads to
\[ u_m = C_1\mu_1^{-m} + C_2 = C_1\mu_2^m + C_2, \quad C_1 \neq 0, \]
which is the second of the two degenerate solutions (6.18) already obtained.

The generic solution (6.17) can be conveniently rewritten as follows

- \( K > 4: \)
  \[ u_m = C_1 \tanh(\psi m + C_2) + C_3 \quad (6.19) \]
  or
  \[ u_m = C_1 \coth(\psi m + C_2) + C_3 \quad (6.20) \]

- \( K < 0: \)
  \[ u_m = \begin{cases} 
  C_1 \tanh(\psi m + C_2) + C_3 & \text{if } m \text{ is even} \\
  C_1 \coth(\psi m + C_2) + C_3 & \text{if } m \text{ is odd}
  \end{cases} \quad (6.21) \]
  or
  \[ u_m = \begin{cases} 
  C_1 \coth(\psi m + C_2) + C_3 & \text{if } m \text{ is even} \\
  C_1 \tanh(\psi m + C_2) + C_3 & \text{if } m \text{ is odd}
  \end{cases} \quad (6.22) \]
Here
\[ \psi = \frac{1}{2} \ln |\mu_1| = \frac{1}{2} \ln \left| \frac{K - 2 + \sqrt{K^2 - 4K}}{2} \right| \]
and \( C_1 \neq 0, C_2 \) and \( C_3 \) are constants.

In addition to the generic solutions there are the degenerate solutions
\[ u_m = C_1 \mu_1^m + C_2 \quad \text{and} \quad u_m = C_1 \mu_2^m + C_2, \quad C_1 \neq 0, \quad (6.23) \]
which can be rewritten as
\[ u_m = C_1 (\text{sgn} K)^m e^{\pm 2\psi m} + C_2. \quad (6.24) \]

**Case: \( 0 < K < 4 \)**

In this case we obtain two specific solutions of the adjoint equation (6.10)
\[ v_m^b = \cos(2\phi m) \quad \text{and} \quad v_m^c = \sin(2\phi m), \quad \phi = \arccos \left( \frac{\sqrt{K}}{2} \right). \]

Application of Theorem 5.4 with symmetries (6.9) and solution \( v_m^b \) gives us the first integrals
\[
\begin{align*}
\tilde{J}_{1b} &= \cos(2\phi m) \left[ \frac{K}{u_{m+2} - u_m} - \frac{K}{u_{m+1} - u_m} \right] \\
+ & \cos(2\phi (m-1)) \left[ K \left( \frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m} \right) - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right] \\
- & \cos(2\phi (m-2)) \left[ \frac{1}{u_{m+2} - u_{m+1}} + \frac{1}{u_{m+1} - u_m} \right];
\end{align*}
\]
\[
\begin{align*}
\tilde{J}_{2b} &= \cos(2\phi m) \left[ \frac{K u_m}{u_{m+2} - u_m} - \frac{K u_m}{u_{m+1} - u_m} \right] \\
+ & \cos(2\phi (m-1)) \left[ \frac{K}{2} \left( \frac{u_{m+2} + u_m}{u_{m+2} - u_m} + \frac{u_{m+1} + u_m}{u_{m+1} - u_m} \right) - \frac{u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}}{u_{m+1} - u_m} \right] \\
- & \cos(2\phi (m-2)) \left[ \frac{u_{m+1}}{u_{m+2} - u_{m+1}} + \frac{u_{m+1}}{u_{m+1} - u_m} \right];
\end{align*}
\]
\[
\begin{align*}
\tilde{J}_{3b} &= \cos(2\phi m) \left[ \frac{K u_m^2}{u_{m+2} - u_m} - \frac{K u_m^2}{u_{m+1} - u_m} \right] \\
+ & \cos(2\phi (m-1)) \left[ K \left( \frac{u_{m+2} u_m}{u_{m+2} - u_m} + \frac{u_{m+1} u_m}{u_{m+1} - u_m} \right) - \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}^2}{u_{m+1} - u_m} \right] \\
- & \cos(2\phi (m-2)) \left[ \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} + \frac{u_{m+1}^2}{u_{m+1} - u_m} \right];
\end{align*}
\]
For \( v_m^c \) we get the first integrals

\[
\begin{align*}
\tilde{J}_{1c} &= \sin(2\phi m) \left[ \frac{K}{u_{m+2} - u_m} - \frac{K}{u_{m+1} - u_m} \right] \\
&+ \sin(2\phi(m-1)) \left[ K \left( \frac{1}{u_{m+2} - u_m} + \frac{1}{u_{m+1} - u_m} \right) - \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+1} - u_m} \right] \\
&- \sin(2\phi(m-2)) \left[ \frac{1}{u_{m+2} - u_{m+1}} + \frac{1}{u_{m+1} - u_m} \right],
\end{align*}
\]

\[
\begin{align*}
\tilde{J}_{2c} &= \sin(2\phi m) \left[ \frac{K u_m}{u_{m+2} - u_m} - \frac{K u_m}{u_{m+1} - u_m} \right] \\
&+ \sin(2\phi(m-1)) \left[ K \left( \frac{u_{m+2} + u_m}{u_{m+2} - u_m} + \frac{u_{m+1} + u_m}{u_{m+1} - u_m} \right) - \frac{u_{m+1}}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}}{u_{m+1} - u_m} \right] \\
&- \sin(2\phi(m-2)) \left[ \frac{u_{m+1}}{u_{m+2} - u_{m+1}} + \frac{u_{m+1}}{u_{m+1} - u_m} \right],
\end{align*}
\]

\[
\begin{align*}
\tilde{J}_{3c} &= \sin(2\phi m) \left[ \frac{K u_m^2}{u_{m+2} - u_m} - \frac{K u_m^2}{u_{m+1} - u_m} \right] \\
&+ \sin(2\phi(m-1)) \left[ K \left( \frac{u_{m+2} u_m}{u_{m+2} - u_m} + \frac{u_{m+1} u_m}{u_{m+1} - u_m} \right) - \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} - \frac{u_{m+1}^2}{u_{m+1} - u_m} \right] \\
&- \sin(2\phi(m-2)) \left[ \frac{u_{m+1}^2}{u_{m+2} - u_{m+1}} + \frac{u_{m+1}^2}{u_{m+1} - u_m} \right].
\end{align*}
\]

The first integrals of this case together with first integrals \( \tilde{J}_{1a}, \tilde{J}_{2a} \) and \( \tilde{J}_{3a} \) satisfy the relations

\[
\begin{align*}
\tilde{J}_{1a} \tilde{J}_{3a} - \tilde{J}_{2a}^2 &= K(4-K), \\
\tilde{J}_{1b} \tilde{J}_{3b} - \tilde{J}_{2b}^2 &= \frac{K^2(K - 4)}{4}, \\
\tilde{J}_{1c} \tilde{J}_{3c} - \tilde{J}_{2c}^2 &= \frac{K^2(K - 4)}{4}, \\
\tilde{J}_{1b} \tilde{J}_{1c} - \tilde{J}_{2a} \tilde{J}_{2b} &= 0, \\
\tilde{J}_{2b}^2 + \tilde{J}_{2c}^2 - \frac{K}{4} \tilde{J}_{1a}^2 &= 0, \\
\tilde{J}_{3c}^2 + \tilde{J}_{3b}^2 - \frac{K}{4} \tilde{J}_{3a}^2 &= 0.
\end{align*}
\]

**Integration of the mapping**

As in the previous case we chose three first integrals \( \tilde{J}_{1a}, \tilde{J}_{2a} \) and \( \tilde{J}_{1b} \). The Jacobian is

\[
J = \det \left( \frac{\partial (\tilde{J}_{1a}, \tilde{J}_{2a}, \tilde{J}_{1b})}{\partial (u_m, u_{m+1}, u_{m+2})} \right)
= \frac{K(u_{m+2} - 2u_{m+1} + u_m)^2 + (4 - K)(u_{m+2} - u_m)^2}{(u_{m+1} - u_m)^2(u_{m+2} - u_m)^2(u_{m+2} - u_{m+1})^2} \cdot \frac{KR_1 R_2}{\cos(2\phi(m+1)) - \cos(2\phi m)},
\]

\[37\]
where
\[ R_1 = \alpha (u_{m+2} - 2u_{m+1} + u_m) + \beta (u_{m+2} - u_m), \]
\[ \alpha = \sin 2\phi (\sin(2\phi m) + \sin \phi), \quad \beta = (1 - \cos 2\phi) (\cos(2\phi m) - \cos \phi) \]
and
\[ R_2 = \gamma (u_{m+2} - 2u_{m+1} + u_m) + \delta (u_{m+2} - u_m), \]
\[ \gamma = \sin 2\phi (\sin(2\phi m) - \sin \phi), \quad \delta = (1 - \cos 2\phi) (\cos(2\phi m) + \cos \phi). \]

1. In the case \( J \neq 0 \) we set these first integrals equal to constants and obtain the generic solution
\[ u_m = C_1 \tan(\phi m + C_2) + C_3, \quad (6.25) \]
where \( C_1 \neq 0, C_2 \neq -\frac{3}{2}\phi + \frac{\pi}{2} k, k \in \mathbb{Z} \) and \( C_3 \) are constants.

2. Analysis of the case \( J = 0 \) splits into three subcases.
   (a) The system
\[ K(u_{m+2} - 2u_{m+1} + u_m)^2 + (4 - K)(u_{m+2} - u_m)^2 = 0, \]
\[ u_{m+1} \neq u_m, \quad u_{m+2} \neq u_m \]
has no solutions.
(b) The system
\[ R_1 = 0, \]
\[ u_{m+1} \neq u_m, \quad u_{m+2} \neq u_m \]
has solutions
\[ u_m = C_1 \tan(\phi m + C_2) + C_3, \quad C_1 \neq 0, \quad C_2 = -\frac{3}{2}\phi + \frac{\pi}{2} k. \quad (6.26) \]
Verification shows that these functions are solutions of the discrete equations (6.8).
(c) The system
\[ R_2 = 0, \]
\[ u_{m+1} \neq u_m, \quad u_{m+2} \neq u_m \]
has solutions
\[ u_m = C_1 \tan(\phi m + C_2) + C_3, \quad C_1 \neq 0, \quad C_2 = -\frac{3}{2}\phi + \frac{\pi}{2} + \pi k. \quad (6.27) \]
which are solutions of the discrete equation.

Finally, we unite the obtained solutions into the generic solution of the form
\[ u_m = C_1 \tan(\phi m + C_2) + C_3, \quad (6.28) \]
where \( C_1 \neq 0, C_2 \) and \( C_3 \) are constants.
7 Integrating factors for mappings

In this section we consider the direct method for discrete equations. It is an
adaptation of the continuous case method. We would like to find first integrals

\[ I(m, u_m, u_{m+1}, ..., u_{m+n-1}) = \text{const} \quad (7.1) \]

which hold on the solutions of the scalar discrete equation

\[ F(m, u_m, u_{m+1}, ..., u_{m+n}) = 0. \quad (7.2) \]

we require

\[ (S_+ - 1)I = \Lambda F, \quad \Lambda = \Lambda(m, u_m, u_{m+1}, ..., u_{m+n}). \quad (7.3) \]

**Remark 7.1** In the continuous case it was useful to assume that the equation
is linear with respect to the highest derivative (see Remark 4.1). This lead to
integrating factors independent of the highest derivative. In the discrete case
there is no analog of this property and a discrete integrating factor can depend on
the same variables as the discrete equation. This makes application of the direct
method for discrete equations more difficult. One needs to come up with a good
Ansatz for the integration factor \( \Lambda \) in order to determine it.

Relation \((7.3)\) should hold identically, i.e., not only on the solutions of the
discrete equation. Since the left hand side is a total difference we can apply the
discrete variational operator \((6.5)\) to it and obtain the equation

\[ \frac{\delta}{\delta u_m}(\Lambda F) = 0 \quad (7.4) \]

for \( \Lambda \).

Finally we note that all solutions of equation \((7.4)\) always satisfy the discrete
adjoint equation

\[ \left. \frac{\delta}{\delta u_m}(v_m F) \right|_{F=0} = 0 : \quad (7.5) \]

For integrating factors equation \((7.4)\) holds identically, while solutions of the
adjoint equation hold on the solutions of the original discrete equation.

**Example**

We demonstrate the direct method on the example of the discrete equation

\[ F(u_m, u_{m+1}, u_{m+2}, u_{m+3}) = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - K = 0, \quad K = \text{const} \quad (7.6) \]
In general, the integrating factors for \((7.6)\) have the following form
\[
\Lambda(m, u_m, u_{m+1}, u_{m+2}, u_{m+3}).
\] (7.7)

Equation \((7.4)\) becomes
\[
\left( \frac{\partial}{\partial u_m} + S_+ \frac{\partial}{\partial u_{m+1}} + S_2 \frac{\partial}{\partial u_{m+2}} + S_3 \frac{\partial}{\partial u_{m+3}} \right) (\Lambda F) = 0
\]
or in detail
\[
\frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} \frac{\partial \Lambda}{\partial u_m} + \frac{(u_{m+2} - u_m)(u_{m+1} - u_{m-1})}{(u_{m+2} - u_{m+1})(u_m - u_{m-1})} \frac{\partial \Lambda^-}{\partial u_m} + \frac{(u_m - u_{m-2})(u_{m-1} - u_{m-3})}{(u_m - u_{m-1})(u_{m-2} - u_{m-3})} \frac{\partial \Lambda^{--}}{\partial u_m}
\]
\[
+ \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} \frac{\partial \Lambda}{\partial u_m} + \frac{(u_{m+2} - u_m)(u_{m+1} - u_{m-1})}{(u_{m+2} - u_{m+1})(u_m - u_{m-1})} \frac{\partial \Lambda^-}{\partial u_m} + \frac{(u_m - u_{m-2})(u_{m-1} - u_{m-3})}{(u_m - u_{m-1})(u_{m-2} - u_{m-3})} \frac{\partial \Lambda^{--}}{\partial u_m}
\]
\[
- K \frac{\partial}{\partial u_m} (\Lambda + \Lambda^- + \Lambda^{--} + \Lambda^{--}) = 0, \quad (7.8)
\]

where \(\Lambda^- = S_+ \Lambda, \Lambda^{--} = S_2 \Lambda\) and \(\Lambda^{--^-} = S_3 \Lambda\).

Even to find a special solution of the last equation is a very difficult task. Practically, we have to introduce some kind of Ansatz to find a solution. If some integrating factor is obtained, we can use the equation
\[
(S_+ - 1) I(m, u_m, u_{m+1}, u_{m+2}) = \Lambda F
\] (7.9)
to find the corresponding first integral. We will not pursue this issue here. Instead we will show how the direct method can provide the same first integrals which were given in the previous section, where equation \((7.9)\) was solved.

Let us try
\[
\Lambda = f_1(m, u_m, u_{m+1}, u_{m+2}) + f_2(m, u_{m+1}, u_{m+2}, u_{m+3}).
\] (7.10)
as an Ansatz. Among such functions there are integration factors
\[
\Lambda = \left( \frac{1}{u_{m+2} - u_{m+1}} - \frac{1}{u_{m+2} - u_m} \right) (a_2(m) u_{m+2}^2 + a_1(m) u_{m+2} + a_0(m))
\]
\[
+ \left( \frac{1}{u_{m+3} - u_{m+1}} - \frac{1}{u_{m+2} - u_{m+1}} \right) (a_2(m+1) u_{m+1}^2 + a_1(m+1) u_{m+1} + a_0(m+1)),
\] (7.11)
where \( a_2(m), a_1(m) \) and \( a_0(m) \) are solutions of the equation

\[
a_i(m) + (1-K)a_i(m-1) + (K-1)a_i(m-2) - a_i(m-3) = 0, \quad i = 0, 1, 2. \quad (7.12)
\]

(Substitution of \( \Lambda \) in the form \((7.11)\) into the equation \((7.8)\) gives equations \((7.12)\) for functions \( a_2(m), a_1(m) \) and \( a_0(m) \).) Note that this is the same equation as equation \((6.11)\), which was studied before. So, we have the same solution.

Using integrating factor Ansatz \((7.11)\), we can obtain the same first integrals as given in Paragraph \(6.2\). Specification of the functions \( a_0(m), a_1(m) \) and \( a_2(m) \) which provides these first integrals is given in Table 1. Here we use three linearly independent solutions of the equation \((7.12)\): a constant solution and functions

\[
\alpha(m) = \begin{cases} 
-(m + (m-1)), & \text{if } K = 4, \\
-(\mu_1^m + \mu_1^{m-1}) & \text{if } K < 0 \text{ or } K > 4, \\
-(\cos(2\phi m) + \cos(2\phi(m-1))) & \text{if } 0 < K < 4,
\end{cases} \quad (7.13)
\]

and

\[
\beta(m) = \begin{cases} 
-(m^2 + (m-1)^2) & \text{if } K = 4, \\
-(\mu_2^m + \mu_2^{m-1}) & \text{if } K < 0 \text{ or } K > 4, \\
-(\sin(2\phi m) + \sin(2\phi(m-1))) & \text{if } 0 < K < 4,
\end{cases} \quad (7.14)
\]

where

\[
\mu_{1,2} = \frac{(K - 2) \pm \sqrt{K^2 - 4K}}{2} \quad \text{and} \quad \phi = \arccos \left( \frac{\sqrt{K}}{2} \right).
\]

8 Discretizations of a scalar ODE

8.1 Theory for difference systems

In this section we are interested in discretizations of a scalar ODE. For the discretization of an ODE of order \( n \) we need a difference stencil with at least \( n+1 \) points. We will use precisely \( n+1 \) points, namely, points \( x_m, ..., x_{m+n} \). These points are not specified in advance and will be defined by an additional mesh equation \([12]\).

As a discretization we will consider a discrete equation on \( n+1 \) points

\[
F(x_m, u_m, x_{m+1}, u_{m+1}, ..., x_{m+n}, u_{m+n}) = 0, \quad (8.1)
\]

on a mesh

\[
\Omega(x_m, u_m, x_{m+1}, u_{m+1}, ..., x_{m+n}, u_{m+n}) = 0. \quad (8.2)
\]

These two equations form the difference system to be used. In the continuous limit the first equation goes into the original ODE and the second equation turns into an identity (for example, \( 0 = 0 \)).

41
Table 1. First integrals corresponding to integrating factors (7.11) with specified functions $a_0(m)$, $a_1(m)$ and $a_2(m)$. Functions $\alpha(m)$ and $\beta(m)$ are given in (7.13) and (7.14), respectively.

| $a_0(m)$ | $a_1(m)$ | $a_2(m)$ | first integral |
|----------|----------|----------|----------------|
| $-2$     | 0        | 0        | $\tilde{J}_{1a}$ |
| 0        | $-2$     | 0        | $\tilde{J}_{2a}$ |
| 0        | 0        | $-2$     | $\tilde{J}_{3a}$ |
| $\alpha(m)$ | 0        | 0        | $\tilde{J}_{1b}$ |
| 0        | $\alpha(m)$ | 0        | $\tilde{J}_{2b}$ |
| 0        | 0        | $\alpha(m)$ | $\tilde{J}_{3b}$ |
| $\beta(m)$ | 0        | 0        | $\tilde{J}_{1c}$ |
| 0        | $\beta(m)$ | 0        | $\tilde{J}_{2c}$ |
| 0        | 0        | $\beta(m)$ | $\tilde{J}_{3c}$ |
The Lie point symmetry
\[ X = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} \tag{8.3} \]
gets prolonged to the points of the difference stencil as
\[ X = \xi_m \frac{\partial}{\partial x_m} + \eta_m \frac{\partial}{\partial u_m} + \ldots + \xi_{m+n} \frac{\partial}{\partial x_{m+n}} + \eta_{m+n} \frac{\partial}{\partial u_{m+n}}, \tag{8.4} \]
\[ \xi_l = \xi(x_l, u_l), \quad \eta_l = \eta(x_l, u_l). \]

The discrete variational operators (5.4) take the form
\[ \frac{\delta}{\delta u_m} = \sum_{k=0}^{\infty} S_k \frac{\partial}{\partial u_{m+k}} + \ldots + S_{m-k} \frac{\partial}{\partial u_{m+k}} + \ldots, \tag{8.5} \]
\[ \frac{\delta}{\delta x_m} = \sum_{k=0}^{\infty} S_k \frac{\partial}{\partial x_{m+k}} + \ldots + S_{m-k} \frac{\partial}{\partial x_{m+k}} + \ldots \tag{8.6} \]

To the system of difference equations (8.1),(8.2) there correspond the adjoint equations
\[ F^* = \frac{\delta}{\delta u_m} (v_m F + w_m \Omega) = 0 \tag{8.7} \]
and
\[ \Omega^* = \frac{\delta}{\delta x_m} (v_m F + w_m \Omega) = 0, \tag{8.8} \]
where \( v_m \) and \( w_m \) are adjoint variables. In detail they are
\[ F^* = v_m \frac{\partial F}{\partial u_m} + v_{m-1} S_- \left( \frac{\partial F}{\partial u_{m+1}} \right) + \ldots + v_{m-k} S_k \left( \frac{\partial F}{\partial u_{m+k}} \right) + \ldots + v_{m-n} S_n \left( \frac{\partial F}{\partial u_{m+n}} \right) \]
\[ + w_m \frac{\partial \Omega}{\partial u_m} + w_{m-1} S_- \left( \frac{\partial \Omega}{\partial u_{m+1}} \right) + \ldots + w_{m-k} S_k \left( \frac{\partial \Omega}{\partial u_{m+k}} \right) + \ldots + w_{m-n} S_n \left( \frac{\partial \Omega}{\partial u_{m+n}} \right) = 0 \]
and
\[ \Omega^* = v_m \frac{\partial F}{\partial x_m} + v_{m-1} S_- \left( \frac{\partial F}{\partial x_{m+1}} \right) + \ldots + v_{m-k} S_k \left( \frac{\partial F}{\partial x_{m+k}} \right) + \ldots + v_{m-n} S_n \left( \frac{\partial F}{\partial x_{m+n}} \right) \]
\[ + w_m \frac{\partial \Omega}{\partial x_m} + w_{m-1} S_- \left( \frac{\partial \Omega}{\partial x_{m+1}} \right) + \ldots + w_{m-k} S_k \left( \frac{\partial \Omega}{\partial x_{m+k}} \right) + \ldots + w_{m-n} S_n \left( \frac{\partial \Omega}{\partial x_{m+n}} \right) = 0. \]

In this setting Theorem 5.4 takes the following form
Theorem 8.1 (Main result for discretized ODE) Let the adjoint equations \((8.7), (8.8)\) be satisfied for all solutions of the original equations \((8.1), (8.2)\) upon a substitution
\[
v_m = \varphi_1(m, x_m, u_m), \quad \varphi_1 \neq 0 \text{ or } \varphi_2 \neq 0.
\]
\[
w_m = \varphi_2(m, x_m, u_m),
\]
\[\varphi_1 \neq 0 \text{ or } \varphi_2 \neq 0. \tag{8.9}\]

Then, any Lie point symmetry \((8.3)\) of the equations \((8.1), (8.2)\) leads to first integral
\[
J = \sum_{j=1}^{n} \left( \xi_{m+j} \frac{\delta}{\delta x_{m+j}} + \eta_{m+j} \frac{\delta}{\delta u_{m+j}} \right) (v_m F + w_m \Omega), \tag{8.10}
\]
where
\[
\frac{\delta}{\delta u_{m(j)}} = \sum_{k=0}^{\infty} S_k \frac{\partial}{\partial u_{m+j+k}} = \frac{\partial}{\partial u_{m+j}} + S_1 \frac{\partial}{\partial u_{m+j+1}} + \cdots + S_k \frac{\partial}{\partial u_{m+j+k}} + \cdots \tag{8.11}
\]
and
\[
\frac{\delta}{\delta x_{m(j)}} = \sum_{k=0}^{\infty} S_k \frac{\partial}{\partial x_{m+j+k}} = \frac{\partial}{\partial x_{m+j}} + S_1 \frac{\partial}{\partial x_{m+j+1}} + \cdots + S_k \frac{\partial}{\partial x_{m+j+k}} + \cdots \tag{8.12}
\]
are higher order discrete Euler–Lagrange operators and \(v_m, w_m, \ldots, v_{m-n}, w_{m-n}\) should be eliminated by means of Eqs. \((8.9)\) and their shifts to the left.

Remark 8.2 As in the general case the condition that the adjoint equations are satisfied, i.e., \(F^\star = \Omega^\star = 0\), can be substituted by a weaker condition
\[
\xi_m \Omega^\star + \eta_m F^\star = 0,
\]
which should hold for a given symmetry \(X\) of the system \((8.1), (8.2)\) on the solutions of this system.

8.2 Discretization of second order ODEs

Let us specify the formulas of the previous section for the three-point case. We get the difference equation
\[
F(x_m, u_m, x_{m+1}, u_{m+1}, x_{m+2}, u_{m+2}) = 0
\]
on the mesh
\[
\Omega(x_m, u_m, x_{m+1}, u_{m+1}, x_{m+2}, u_{m+2}) = 0.
\]
The adjoint equations take the form

\[ F^* = v_m \frac{\partial F}{\partial u_m} + v_{m-1} S_- \left( \frac{\partial F}{\partial u_{m+1}} \right) + v_{m-2} S_-^2 \left( \frac{\partial F}{\partial u_{m+2}} \right) \]

\[ + w_m \frac{\partial \Omega}{\partial u_m} + w_{m-1} S_- \left( \frac{\partial \Omega}{\partial u_{m+1}} \right) + w_{m-2} S_-^2 \left( \frac{\partial \Omega}{\partial u_{m+2}} \right) = 0 \]

and

\[ \Omega^* = v_m \frac{\partial F}{\partial x_m} + v_{m-1} S_- \left( \frac{\partial F}{\partial x_{m+1}} \right) + v_{m-2} S_-^2 \left( \frac{\partial F}{\partial x_{m+2}} \right) \]

\[ + w_m \frac{\partial \Omega}{\partial x_m} + w_{m-1} S_- \left( \frac{\partial \Omega}{\partial x_{m+1}} \right) + w_{m-2} S_-^2 \left( \frac{\partial \Omega}{\partial x_{m+2}} \right) = 0. \]

First integrals are given as

\[ J = \left[ \xi_{m+1} \left( \frac{\partial}{\partial x_{m+1}} + S_- \frac{\partial}{\partial x_{m+2}} \right) + \eta_{m+1} \left( \frac{\partial}{\partial u_{m+1}} + S_- \frac{\partial}{\partial u_{m+2}} \right) \right. \]

\[ + \xi_{m+2} \frac{\partial}{\partial x_{m+2}} + \eta_{m+2} \frac{\partial}{\partial u_{m+2}} \left] \left( v_m F + w_m \Omega \right) \right. \]

Example: Harmonic oscillator

Let us consider the one-dimensional harmonic oscillator

\[ \ddot{u} + u = 0. \] (8.13)

As a discretization we consider the scheme

\[ \frac{2}{x_{m+2} - x_m} \left( \frac{u_{m+2} - u_{m+1}}{x_{m+2} - x_{m+1}} - \frac{u_{m+1} - u_m}{x_{m+1} - x_m} \right) + \frac{u_{m+2} + 2u_{m+1} + u_m}{4} = 0 \] (8.14)

on the uniform mesh

\[ x_{m+2} - x_{m+1} = x_{m+1} - x_m. \] (8.15)

This discretization of the harmonic oscillator was considered in \[17\].

Let us rewrite the scheme in an equivalent form

\[ F = \frac{u_{m+2} - u_{m+1}}{x_{m+2} - x_{m+1}} - \frac{u_{m+1} - u_m}{x_{m+1} - x_m} + \frac{x_{m+2} - x_m u_{m+2} + 2u_{m+1} + u_m}{2} = 0, \]

\[ \Omega = (x_{m+2} - x_{m+1}) - (x_{m+1} - x_m) = 0. \] (8.16)

It is not difficult to verify that the difference system (8.16) admits the symmetries generated by the operators

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \sin(\omega x) \frac{\partial}{\partial u}, \quad X_3 = \cos(\omega x) \frac{\partial}{\partial u}, \quad X_4 = u \frac{\partial}{\partial u}, \] (8.17)
where

\[ \omega = \arctan(\frac{h}{2}) \quad \Rightarrow \quad h = x_{m+2} - x_{m+1} = x_{m+1} - x_m. \]

The adjoint equations are

\[
\begin{align*}
F^* &= v_m \left( \frac{1}{x_{m+1} - x_m} + \frac{x_{m+2} - x_m}{8} \right) \\
& \quad + v_{m-1} \left( -\frac{1}{x_{m+1} - x_m} - \frac{1}{x_{m-1} - x_{m-2}} + \frac{x_{m+1} - x_{m-1}}{4} \right) \\
& \quad + v_{m-2} \left( \frac{1}{x_{m-1} - x_{m-2}} + \frac{x_{m} - x_{m-2}}{8} \right) = 0 \quad (8.18)
\end{align*}
\]

and

\[
\begin{align*}
\Omega^* &= v_m \left( -\frac{u_{m+1} - u_m}{(x_{m+1} - x_m)^2} - \frac{u_{m+2} + 2u_{m+1} + u_m}{8} \right) \\
& \quad + v_{m-1} \left( \frac{u_{m+1} - u_m}{(x_{m+1} - x_m)^2} + \frac{u_m - u_{m-1}}{(x_{m} - x_{m-1})^2} \right) \\
& \quad + v_{m-2} \left( -\frac{u_m - u_{m-1}}{(x_{m} - x_{m-1})^2} + \frac{u_m + 2u_{m-1} + u_{m-2}}{8} \right) \\
& \quad + w_m - 2w_{m-1} + w_{m-2} = 0 \quad (8.19)
\end{align*}
\]

considered on the solutions of the equations (8.16).

It is easy to check that on the solutions of the equations (8.16) the adjoint equations (8.18), (8.19) have the particular solution

\[ v^a_m = 0, \quad w^a_m = x_m. \quad (8.20) \]

For symmetries (8.3) with \( \xi = 0 \) we can consider the equation (8.18) instead of the system (8.18), (8.19) (see Remark 8.2). In this case we find the special solution

\[ v^b_m = u_m, \quad w^b_m = 0. \quad (8.21) \]

Let us use these solutions to find first integrals with the help of Theorem 8.1 and symmetries (8.17). We will bypass the higher first integrals and provide only the final results for both pairs (8.20) and (8.21).

- \( v^a_m = 0, \ w^a_m = x_m \)

Applying the theorem with symmetry \( X_1 \) gives first integral

\[ \tilde{J}_1^a = x_m - x_{m+1} = -h. \quad (8.22) \]

The other symmetries provide trivial first integrals.
\[ v^b_m = u_m, \; w^b_m = 0 \]

For symmetries \( X_2, \; X_3 \) and \( X_4 \) we obtain first integrals

\[ \tilde{J}^b_2 = \left(\frac{1}{h} + \frac{h}{4}\right) (-u_{m+1} \sin(\omega x_{m+1}) + u_m \sin(\omega x_m)), \quad (8.23) \]

\[ \tilde{J}^b_3 = \left(\frac{1}{h} + \frac{h}{4}\right) (-u_{m+1} \cos(\omega x_{m+1}) + u_m \cos(\omega x_m)), \quad (8.24) \]

\[ \tilde{J}^b_4 = -h \left[ \left(\frac{u_{m+1} - u_m}{h}\right)^2 + \left(\frac{u_{m+1} + u_m}{2}\right)^2 \right], \quad (8.25) \]

where we used \( h = x_{m+1} - x_m \) and \( x_{m+2} = x_{m+1} + h \).

Using values of the first integrals \( \tilde{J}^a_1, \; \tilde{J}^b_2 \) and \( \tilde{J}^b_3 \), we can express the solution of the difference system in the form

\[ u_m = A \cos(\omega x_m) + B \sin(\omega x_m). \quad (8.26) \]

The mesh for this solution

\[ x_m = x_0 + mh, \quad m = 0, \pm 1, \pm 2, \ldots \quad (8.27) \]

can be obtained by integration of the linear equation (8.22). Here \( A, \; B, \; h > 0 \) and \( x_0 \) are constants. Note that \( x_0 \) appears from the integration of the linear equation (8.22).

### 8.3 Discretization of third order ODEs

Here we specify the general formulas for the four-point case. We get the difference equation

\[ F(x_m, u_m, x_{m+1}, u_{m+1}, x_{m+2}, u_{m+2}, x_{m+3}, u_{m+3}) = 0 \]

on the mesh

\[ \Omega(x_m, u_m, x_{m+1}, u_{m+1}, x_{m+2}, u_{m+2}, x_{m+3}, u_{m+3}) = 0. \]

The adjoint equations are

\[ F^* = v_m \frac{\partial F}{\partial u_m} + v_{m-1} S_- \left( \frac{\partial F}{\partial u_{m+1}} \right) + v_{m-2} S_- \left( \frac{\partial F}{\partial u_{m+2}} \right) + v_{m-3} S_- \left( \frac{\partial F}{\partial u_{m+3}} \right) \]

\[ + w_m \frac{\partial \Omega}{\partial u_m} + w_{m-1} S_- \left( \frac{\partial \Omega}{\partial u_{m+1}} \right) + w_{m-2} S_- \left( \frac{\partial \Omega}{\partial u_{m+2}} \right) + w_{m-3} S_- \left( \frac{\partial \Omega}{\partial u_{m+3}} \right) = 0 \]
and
\[
\Omega^* = v_m \frac{\partial F}{\partial x_m} + v_{m-1} S_-(\frac{\partial F}{\partial x_{m+1}}) + v_{m-2} S_2^- \left( \frac{\partial F}{\partial x_{m+2}} \right) + v_{m-3} S_3^- \left( \frac{\partial F}{\partial x_{m+3}} \right) \\
+ w_m \frac{\partial \Omega}{\partial x_m} + w_{m-1} S_-(\frac{\partial \Omega}{\partial x_{m+1}}) + w_{m-2} S_2^- \left( \frac{\partial \Omega}{\partial x_{m+2}} \right) + w_{m-3} S_3^- \left( \frac{\partial \Omega}{\partial x_{m+3}} \right) = 0.
\]

First integrals take the form
\[
J = \left[ \xi_{m+1} \left( \frac{\partial}{\partial x_{m+1}} + S_2 \frac{\partial}{\partial x_{m+3}} ight) + \eta_{m+1} \left( \frac{\partial}{\partial u_{m+1}} + S_2 \frac{\partial}{\partial u_{m+3}} \right) + \xi_{m+2} \left( \frac{\partial}{\partial x_{m+2}} + S_2 \frac{\partial}{\partial x_{m+3}} \right) + \eta_{m+2} \left( \frac{\partial}{\partial u_{m+2}} + S_2 \frac{\partial}{\partial u_{m+3}} \right) + \xi_{m+3} \frac{\partial}{\partial x_{m+3}} + \eta_{m+3} \frac{\partial}{\partial u_{m+3}} \right] (v_m F + w_m \Omega).
\]

Example

Let us return to the ODE
\[
F = \frac{1}{u^2} \left( \dddot{u} - \frac{3}{2} \ddot{u}^2 \right) - M = 0,
\]
which we examined in the Paragraph 3.3. We recall that in the general case it admits symmetries
\[
X_1 = \frac{\partial}{\partial u}, \quad X_2 = u \frac{\partial}{\partial u}, \quad X_3 = u^2 \frac{\partial}{\partial u}, \quad X_4 = \frac{\partial}{\partial x}.
\]
For \(M = 0\) there are additional symmetries
\[
X_5 = x \frac{\partial}{\partial x}, \quad X_6 = x^2 \frac{\partial}{\partial x}.
\]
We will consider these two cases separately.

Case: \(M = 0\)

As a discretization we consider the invariant scheme
\[
F = \frac{u_{m+3} - u_{m+1} u_{m+2} - u_m}{x_{m+3} - x_{m+1} x_{m+2} - x_m} - \frac{u_{m+3} - u_{m+2} u_{m+1} - u_m}{x_{m+3} - x_{m+2} x_{m+1} - x_m} = 0,
\]
\[
\Omega = \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)} - K = 0, \quad K \neq 0,
\]
which was introduced in [5, 6]. It admits all six symmetries (8.29), (8.30).

The adjoint system for the presented scheme is

\[ F^* = \frac{\alpha(u_{m+2} - u_{m+1})}{(u_{m+2} - u_m)(u_{m+1} - u_m)} (v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3}) = 0 \]

and

\[ \Omega^* = -\frac{\alpha(x_{m+2} - x_{m+1})}{(x_{m+2} - x_m)(x_{m+1} - x_m)} (v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3}) \]

\[ + \frac{K(x_{m+2} - x_{m+1})}{(x_{m+2} - x_m)(x_{m+1} - x_m)} (w_m + (1 - K)w_{m-1} + (K - 1)w_{m-2} - w_{m-3}) = 0, \]

where

\[ \alpha = \frac{u_{m+3} - u_{m+1} u_{m+2} - u_m}{x_{m+3} - x_{m+1} x_{m+2} - x_m} \]

\[ = \frac{u_{m+3} - u_{m+2} u_{m+1} - u_m}{x_{m+3} - x_{m+2} x_{m+1} - x_m}. \]

Variables \( u_{m+3} \) and \( x_{m+3} \) in the coefficient \( \alpha \) should be expressed in term of the other variables involved in the scheme.

The adjoint equations lead to the system of linear mappings

\[ v_m + (1 - K)v_{m-1} + (K - 1)v_{m-2} - v_{m-3} = 0, \]

\[ w_m + (1 - K)w_{m-1} + (K - 1)w_{m-2} - w_{m-3} = 0. \]

One can use pairs \((v_m, w_m)\) which solve this system to find first integrals.

However, it is more convenient to rewrite the scheme (8.31) in the equivalent form

\[ \tilde{F} = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - K = 0, \]

\[ \Omega = \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)} - K = 0. \]

Note that the system is symmetric under the interchange of \( u \) and \( x \). We can use the results obtained for discrete equation (6.8) to integrate this scheme. We need to consider different subcases for different values of \( K \).

1. \( K = 4 \)

We obtain the solution

\[ u_m = \frac{1}{C_1 m + C_2} + C_3 \quad \text{or} \quad u_m = C_1 m + C_2 \]

(8.33)

on the mesh

\[ x_m = \frac{1}{C_4 m + C_5} + C_6 \quad \text{or} \quad x_m = C_4 m + C_5, \]

(8.34)

where \( C_1 \neq 0, C_2, C_3, C_4 \neq 0, C_5 \) and \( C_6 \) are constants.
2. \( K > 4 \)

We obtain the solution

\[ u_m = C_1 \tanh(\psi m + C_2) + C_3 \]  \hspace{1cm} (8.35)

or

\[ u_m = C_1 \coth(\psi m + C_2) + C_3 \]  \hspace{1cm} (8.36)

or

\[ u_m = C_1 \mu_{1,2}^m + C_2 = C_1 e^{\pm 2\psi m} + C_2 \]  \hspace{1cm} (8.37)

on the mesh

\[ x_m = C_4 \tanh(\psi m + C_5) + C_6 \]  \hspace{1cm} (8.38)

or

\[ x_m = C_4 \coth(\psi m + C_5) + C_6 \]  \hspace{1cm} (8.39)

or

\[ x_m = C_4 \mu_{1,2}^m + C_5 = C_4 e^{\pm 2\psi m} + C_5, \]  \hspace{1cm} (8.40)

where \( C_1 \neq 0, C_2, C_3, C_4 \neq 0, C_5 \) and \( C_6 \) are constants. Here

\[ \mu_{1,2} = \frac{K - 2 \pm \sqrt{K^2 - 4K}}{2} \]

and

\[ \psi = \frac{1}{2} \ln \mu_1 = \frac{1}{2} \ln \left( \frac{K - 2 + \sqrt{K^2 - 4K}}{2} \right). \]

3. \( K < 0 \)

We obtain the solution

\[ u_m = \begin{cases} 
  C_1 \tanh(\psi m + C_2) + C_3 & \text{if } m \text{ is even} \\
  C_1 \coth(\psi m + C_2) + C_3 & \text{if } m \text{ is odd}
\end{cases} \]  \hspace{1cm} (8.41)

or

\[ u_m = \begin{cases} 
  C_1 \coth(\psi m + C_2) + C_3 & \text{if } m \text{ is even} \\
  C_1 \tanh(\psi m + C_2) + C_3 & \text{if } m \text{ is odd}
\end{cases} \]  \hspace{1cm} (8.42)

or

\[ u_m = C_1 \mu_{1,2}^m + C_2 = C_1 (-1)^m e^{\pm 2\psi m} + C_2 \]  \hspace{1cm} (8.43)

on the mesh

\[ x_m = \begin{cases} 
  C_4 \tanh(\psi m + C_5) + C_6 & \text{if } m \text{ is even} \\
  C_4 \coth(\psi m + C_5) + C_6 & \text{if } m \text{ is odd}
\end{cases} \]  \hspace{1cm} (8.44)
or
\[ x_m = \begin{cases} 
  C_1 \coth(\psi m + C_2) + C_3 & \text{if } m \text{ is even} \\
  C_1 \tanh(\psi m + C_2) + C_3 & \text{if } m \text{ is odd}
\end{cases} \tag{8.45} \]

or
\[ x_m = C_4 \mu_{1,2}^m + C_5 = C_4 (-1)^m e^\pm 2\psi m + C_5, \tag{8.46} \]

where \( C_1 \neq 0, C_2, C_3, C_4 \neq 0, C_5 \) and \( C_6 \) are constants. Here
\[ \mu_{1,2} = \frac{K - 2 \pm \sqrt{K^2 - 4K}}{2} \]

and
\[ \psi = \frac{1}{2} \ln(-\mu_1) = \frac{1}{2} \ln \left( \frac{K - 2 + \sqrt{K^2 - 4K}}{2} \right). \]

4. \( 0 < K < 4 \)

We obtain the solution
\[ u_m = C_1 \tan(\phi m + C_2) + C_3 \tag{8.47} \]
on the mesh
\[ x_m = C_4 \tan(\phi m + C_5) + C_6 \tag{8.48} \]

where \( C_1 \neq 0, C_2, C_3, C_4 \neq 0, C_5 \) and \( C_6 \) are constants. Here
\[ \phi = \arccos \left( \frac{\sqrt{K}}{2} \right). \]

**Remark 8.3** Let us note that all these solutions can be presented in the unified form
\[ u_m = \frac{1}{\alpha x_m + \beta} + \gamma \quad \text{or} \quad u_m = \alpha x_m + \beta, \tag{8.49} \]

where \( \alpha \neq 0, \beta \) and \( \gamma \) are constants. They should be considered on the corresponding meshes, which are different for different values of the parameter \( K \). Thus, the discretization (8.31) provides the exact solution of the ODE (8.28) for \( M = 0 \). For the case \( K = 4 \) this was observed in [5, 6].

It should be noted that for \( K < 0 \) we do not get monotonicity for mesh points \( x_m \). This is clearly seen in mesh equations (8.44), (8.45) and (8.46). Though we obtain the exact solution of the ODE in these points, we can not speak about a mesh on which we have a discretization of the ODE.
Case: $M \neq 0$

As a discretization we consider the invariant scheme

$$F = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - \frac{(x_{m+3} - x_{m+1})(x_{m+2} - x_m)}{(x_{m+3} - x_{m+2})(x_{m+1} - x_m)}$$

$$\times \left(1 - \frac{M}{6}(x_{m+3} - x_m)(x_{m+2} - x_{m+1})\right) = 0,$$

(8.50)

$$\Omega(x_{m+3} - x_{m+2}, x_{m+2} - x_{m+1}, x_{m+1} - x_m) = 0.$$

It admits the four symmetries (8.29). To find solutions we specify the mesh as a regular one

$$\Omega = x_{m+1} - x_m - h = 0,$$

(8.51)

where $h > 0$ is a constant. The first equation will take the form

$$F = \frac{(u_{m+3} - u_{m+1})(u_{m+2} - u_m)}{(u_{m+3} - u_{m+2})(u_{m+1} - u_m)} - \bar{K} = 0,$$

(8.52)

where

$$\bar{K} = 4 \left(1 - \frac{M}{2} h^2\right).$$

For the equation (8.52) we can use results obtained for the mapping (6.8) in Paragraph 6.2. Since $h \neq 0$ we have $\bar{K} \neq 4$. For nontrivial cases $\bar{K} \neq 0$ there can be three possibilities.

1. $0 < \bar{K} < 4$ ($M > 0$, $0 < h < \sqrt{2/M}$)

   We obtain the solution

   $$u_m = C_1 \tan(\bar{\phi}m + C_2) + C_3,$$

   (8.53)

   where $C_1 \neq 0$, $C_2$ and $C_3$ are constants, on the mesh

   $$x_m = x_0 + hm.$$

   (8.54)

   Here

   $$\bar{\phi} = \arccos\left(\frac{\sqrt{\bar{K}}}{2}\right).$$

2. $\bar{K} > 4$ ($M < 0$)

   We obtain the solution

   $$u_m = C_1 \tanh(\bar{\psi}m + C_2) + C_3$$

   (8.55)
or

\[ u_m = C_1 \coth(\bar{\psi}m + C_2) + C_3 \]  
(8.56)

or

\[ u_m = C_1 \mu_{1,2}^m + C_2 = C_1 e^{\pm 2\bar{\psi}m} + C_2, \]  
(8.57)

where \( C_1 \neq 0, C_2 \) and \( C_3 \) are constants, on the regular mesh (8.54). Here

\[ \bar{\mu}_{1,2} = \frac{K - 2 \pm \sqrt{K^2 - 4K}}{2} \]

and

\[ \bar{\psi} = \frac{1}{2} \ln \bar{\mu}_1 = \frac{1}{2} \ln \left( \frac{K - 2 + \sqrt{K^2 - 4K}}{2} \right). \]

3. \( \bar{K} < 0 \) \((M > 0, h > \sqrt{2/M})\)

We obtain the solution

\[ u_m = \begin{cases} 
C_1 \tanh(\bar{\psi}m + C_2) + C_3 & \text{if } m \text{ is even} \\
C_1 \coth(\bar{\psi}m + C_2) + C_3 & \text{if } m \text{ is odd}
\end{cases} \]  
(8.58)

or

\[ u_m = \begin{cases} 
C_1 \coth(\bar{\psi}m + C_2) + C_3 & \text{if } m \text{ is even} \\
C_1 \tanh(\bar{\psi}m + C_2) + C_3 & \text{if } m \text{ is odd}
\end{cases} \]  
(8.59)

or

\[ u_m = C_1 \bar{\mu}_{1,2}^m + C_2 = C_1 (-1)^m e^{\pm 2\bar{\psi}m} + C_2, \]  
(8.60)

where \( C_1 \neq 0, C_2 \) and \( C_3 \) are constants, on the regular mesh (8.54). Here

\[ \bar{\mu}_{1,2} = \frac{K - 2 \pm \sqrt{K^2 - 4K}}{2}. \]

and

\[ \bar{\psi} = \frac{1}{2} \ln(-\bar{\mu}_1) = \frac{1}{2} \ln \left( -\frac{K - 2 + \sqrt{K^2 - 4K}}{2} \right). \]

Note that because of the steplength restriction \( h > \sqrt{2/M} \) we can not speak about consistent discretization of the ODE in this case.

It should be noted that for sufficiently small steplengths \( h \ll 1 \) we will always have \( K > 0 \) and, thus, avoid jumping solutions of the last case.
Remark 8.4 We recall that in the case \( M = 0 \) the scheme (8.31) provided us with the exact solution of the ODE (8.28). In the present case \( M \neq 0 \) the scheme (8.50) with regular mesh specification (8.51) provides the exact solutions of the ODE (8.28) if we apply the scheme to the modified equation

\[
F_{\text{mod}} = \frac{1}{\bar{u}} \left( \bar{u} \ddot{u} - \frac{3}{2} \bar{u}^2 \right) - M_{\text{mod}} = 0.
\] (8.61)

The original equation parameter \( M > 0 \) should be changed to the modified value

\[
M_{\text{mod}} = \frac{2}{h^2} \sin^2 \left( \sqrt{\frac{M}{2} h} \right), \quad 0 < h < \sqrt{2/M}
\]

and the parameter \( M < 0 \) should be changed to the modified value

\[
M_{\text{mod}} = -\frac{2}{h^2} \sinh^2 \left( \sqrt{-\frac{M}{2} h} \right).
\]

Application of the scheme (8.50) to the modified equation (8.61) with the modified constant \( M_{\text{mod}} \) gives exact solution of the ODE (8.28) with constant \( M \). Note that in both cases \( M_{\text{mod}} \to M \) as \( h \to 0 \).

We note that modification of the constant \( M \) can be interpreted as scaling of the independent variable \( x \).

9 Conclusion

This paper consists of two parts. The first is a brief review of two known methods of obtaining first integrals of differential equations with nontrivial Lie symmetries: the "adjoint equation method" (Sections 2 and 3) and the "direct method" (Section 4). Both of them are particularly useful either when no Lagrangian exists, or when the symmetries of the equation are not Lagrangian ones and the Noether theorem can not be applied. The methods are valid both for ordinary and partial differential equations. We apply them to obtain first integrals and general solutions of a third order nonlinear ODE (the Schwarzian equation (3.23)).

The second part is an adaptation of the adjoint equation method first to mappings, then to difference systems. The mappings are equations involving several discrete points. The second are difference equations on lattices that arise e.g. when differential equations are solved numerically. In both cases (see Sections 6 and 8, respectively) we apply the discretized adjoint equation method to a specific four-point equation, respectively four-point difference systems. These systems have the Schwarzian ODE (3.23) as a continuous limit and and share its Lie point symmetry group. We have also treated a simpler example, namely a discrete linear harmonic oscillator. The results for both examples can be summed up as follows:
1. The adjoint equation method makes it possible to obtain complete sets of functionally independent first integrals of the differential equations and difference systems. These in turn provide the general solutions of these equations.

2. The invariant discretization of continuous ODE (3.23) with \( M = 0 \) considered here is exact. The solutions of the difference system coincide with the solutions of the original ODE. The invariant discretizations of the other continuous ODEs considered here, namely of the harmonic oscillator and ODE (3.23) with \( M \neq 0 \), can be made exact if we allow a parameter modification.

In the paper we restricted ourselves to ordinary difference equations. However, the presented approach can be extended to differential–difference equations as well as to partial difference equations.

References

[1] R. W. Atherton and G. M. Homsy (1975) On the existence and formulation of variational principles for nonlinear differential equations Stud. Appl. Math. 54 (1) 31–60.

[2] S. Anco and G. Bluman (1997) Direct construction of conservation laws from field equations, Phys. Rev. Lett. 78 2869–2873.

[3] G. Bluman and S. Anco (2002) Symmetry and Integration Methods for Differential Equations (New York: Springer)

[4] G. Bluman, A. Cheviakov and S. Anco (2010) Applications of Symmetry Methods to Partial Differential Equations, Vo. 168, Appl. Math. Sci. (New York: Springer)

[5] A. Bourlioux, C. Cyr-Gagnon and P. Winternitz (2006) Difference schemes with point symmetries and their numerical tests J. Phys A: Math. Gen. 39 (22) 6877–6896.

[6] A. Bourlioux, R. Rebelo and P. Winternitz (2008) Symmetry preserving discretization of \( SL(2,\mathbb{R}) \) invariant equations J. Nonlinear Math. Phys 15 362–372.

[7] R. Dennemeyer (1968) Introduction to partial differential equations and boundary value problems (New York: McGraw-Hill)

[8] V. A. Dorodnitsyn (1991) Transformation groups in mesh spaces J. Sov. Math. 55 N1 1490, Plenum Publishing Corporation.
[9] V. A. Dorodnitsyn (1993) Finite–difference models entirely inheriting symmetry of original differential equations Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics 191, Kluwer Academic Publishers.

[10] V. A. Dorodnitsyn (1993) The finite-difference analogy of Noether’s theorem Doklady RAN 328 (6) 678–682 (in Russian). Translation in Phys. Dokl. 38 (2) 66–68 (1993).

[11] V. Dorodnitsyn (2001) Noether–type theorems for difference equations, Applied Numerical Mathematics 39 307–321.

[12] V. Dorodnitsyn (2011) Applications of Lie Groups to Difference Equations Chapman & Hall/CRC differential and integral equations series.

[13] V. Dorodnitsyn and E. Kaptsov (2013) Discretizing of second order ODEs possessing symmetries Journ. of computational mathematics and mathematical physics 53 (8) 1329–1355.

[14] V. Dorodnitsyn and R. Kozlov (2003) A heat transfer with a source: the complete set of invariant difference schemes J. Nonlinear Math. Phys. 10 (1) 16–50.

[15] V. Dorodnitsyn and R. Kozlov (2009) First integrals of difference Hamiltonian equations J. Phys. A: Math. Theor. 42 454007.

[16] V. Dorodnitsyn and R. Kozlov (2010) Invariance and first integrals of continuous and discrete Hamiltonian equations J. Engrg. Math. 66 253–270.

[17] V. Dorodnitsyn and R. Kozlov (2011) Lagrangian and Hamiltonian formalism for discrete equations: symmetries and first integrals, SMS Lecture Notes, SMS Lecture Notes, Cambridge University Press, 7–49.

[18] V. Dorodnitsyn, R. Kozlov and P. Winternitz (2000) Lie group classification of second-order ordinary difference equations, J. Math. Phys. 41 (1) 480–504.

[19] V. Dorodnitsyn, R. Kozlov and P. Winternitz (2004) Continuous symmetries of Lagrangians and exact solutions of discrete equations J. Math. Phys. 45 (1) 336–359.

[20] P. E. Hydon (2000) Symmetries and first integrals of ordinary difference equations R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456 (2004) 2835–2855.

[21] N. H. Ibragimov (1985) Transformation Groups Applied to Mathematical Physics (Dordrecht: Reidel)
[22] N. Ibragimov (2007) A new conservation theorem J. Math. Anal. Appl. 333, 311–328.

[23] N. H. Ibragimov (2011) Nonlinear self-adjointness and conservation laws J. Phys A: Math. Gen. 44 432002.

[24] N. Ibragimov (2010–2011) Nonlinear self-adjointness in constructing conservation laws, Archives of ALGA 7/8, ALGA Publications, Karlskrona, Sweden.

[25] A. H. Kara and F. M. Mahomed (2000) Relationship between symmetries and conservation laws Internat. J. Theoret. Phys. 39 (1) 23–40.

[26] D. Levi and P. Winternitz (1991) Continuous symmetries of discrete equations Phys. Lett. A 152 (7) 335–338.

[27] D. Levi and P. Winternitz (1996) Symmetries of discrete dynamical systems J. Math. Phys. 37 5551–5576.

[28] D. Levi and P. Winternitz (2006) Continuous symmetries of difference equations J. Phys. A 39 (2) R1–R63.

[29] D. Levi, P. Winternitz and R. I. Yamilov. (2010) Lie point symmetries of differential–difference equations J. Phys. A Math. Theor. 43 (29) 292002.

[30] D. Levi, Z. Thomova and P. Winternitz, (2011) Are there contact transformations for discrete equations? J. Phys. A: Math. Theor. 44 265201.

[31] D. Levi, C. Scimiterna, Z. Thomova and P. Winternitz (2012) Contact transformations for difference schemes J. Phys. A: Math. Theor. 45 022001.

[32] M. Lutzky (1978) Symmetry groups and conserved quantities for the harmonic oscillator J. Phys. A 11 (2) 249–258.

[33] E. Noether (1918) Invariante Variationsprobleme, Nachr. Konig. Gesell. Wissen., Gottingen, Math.-Phys. Kl. 2 235–257.

[34] P. J. Olver (1993) Applications of Lie groups to differential equations Second edition (New York: Springer–Verlag)

[35] P. J. Olver (2001) Geometric foundations of numerical algorithms and symmetry Appl. Alg. Engin. Comp. Commun. 11 417–436.

[36] V. Ovsienko and S. Tabachnikov (2009) What is the Schwarzian derivative? Notices of the AMS 56 (1) 34–36.

[37] L. V. Ovsyannikov (1982) Group analysis of differential equations (New York: Academic)
[38] G. R. W. Quispel, H. W. Capel and R. Sahadevan (1992) Continuous sym-
metries of differential–difference equations *Phys. Lett.* 170A 379–383.

[39] V. Rosenhaus and G. H. Katzin (1994) On symmetries, conservation laws,
and variational problems for partial differential equations *J. Math. Phys.* 35
(4) 1998–2012.

[40] P. Winternitz (2011) Symmetry preserving discretization of differential equa-
tions and Lie point symmetries of differential-difference equations. In D. Levi,
P. J. Olver, Z. Thomova and P. Winternitz, editors, Symmetries and Inte-
grability of Difference Equations, 292–341. Cambridge University Press.

[41] P. Winternitz, V. Dorodnitsyn, E. Kaptsov and R. Kozlov, First integrals
of difference equations which do not possess a variational formulation (in
Russian), *Doklady Mathematics* (To appear); [arXiv:1307.7585] (10 pages).