Deza graphs with parameters \((n, k, k - 1, a)\) and \(\beta = 1\)

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Abstract
A Deza graph with parameters \((n, k, b, a)\) is a \(k\)-regular graph with \(n\) vertices, in which any two vertices have \(a\) or \(b\) (\(a \leq b\)) common neighbours. A Deza graph is strictly Deza if it has diameter 2, and is not strongly regular. In an earlier paper, the two last authors et al characterised the strictly Deza graphs with \(b = k - 1\) and \(\beta > 1\), where \(\beta\) is the number of vertices with \(b\) common neighbours with a given vertex. Here, we start with a characterisation of Deza graphs (not necessarily strictly Deza graphs) with parameters \((n, k, k - 1, 0)\). Then, we deal with the case \(\beta = 1\) and \(a > 0\), and thus complete the characterisation of Deza graphs with \(b = k - 1\). It follows that all Deza graphs with \(b = k - 1\), \(\beta = 1\) and \(a > 0\) can be made from special strongly regular graphs, and in fact are strictly Deza except for \(K_2\). We present several examples of such strongly regular graphs. A divisible design graph (DDG) is a special Deza graph, and a Deza graph with \(\beta = 1\) is a DDG. The present characterisation reveals an error in a paper on DDGs by the second author et al. We discuss the cause and the consequences of this mistake and give the required errata.

KEYWORDS
Deza graph, divisible design graph, dual Seidel switching, involution, strongly regular graph

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1  |  INTRODUCTION

A $k$-regular graph $\Gamma$ on $n$ vertices is called a Deza graph with parameters $(n, k, b, a)$ if the number of common neighbours of two distinct vertices takes on only two values $a$ or $b$ ($a \leq b$). If the number of common neighbours of two vertices only depends on whether the vertices are adjacent or not, then $\Gamma$ is a strongly regular graph with parameters $(n, k, \lambda, \mu)$, where $\lambda$ (resp. $\mu$) is the number of common neighbours of two adjacent (resp. nonadjacent) vertices; so $\{a, b\} = \{\lambda, \mu\}$. A Deza graph is called a strictly Deza graph if it has diameter 2 and is not strongly regular. Note that the complete graph $K_n$ (which is normally excluded from being strongly regular) is a Deza graph, which is not strictly Deza because it has diameter 1.

Let $\Gamma$ be a Deza graph with parameters $(n, k, b, a)$, and let $v$ be a vertex of $\Gamma$. Denote by $N(v)$ the set of neighbours of a vertex $v$, and let $\beta(v)$ be the number of vertices $u \in V(\Gamma)$ such that $|N(v) \cap N(u)| = b$.

**Lemma 1** (Erickson et al [4] Proposition 1.1). The number $\beta(v)$ does not depend on the choice of $v$ and is given by

$$
\beta(v) = \beta = \begin{cases} 
\frac{k(k-1) - a(n-1)}{b-a} & \text{if } a \neq b, \\
(n-1) & \text{if } a = b.
\end{cases}
$$

Strictly Deza graphs with parameters $(n, k, b, a)$, where $k = b + 1$ and $\beta > 1$ hold, were investigated in [10]. The following theorem was proved.

**Theorem 1** (Kabanov et al [10]). Let $\Gamma$ be a strictly Deza graph with parameters $(n, k, b, a)$ and $\beta > 1$. The parameters $k$ and $b$ of $\Gamma$ satisfy the condition $k = b + 1$ if and only if $\Gamma$ is isomorphic to the strong product of $K_2$ with the complete multipartite graph with parts of size $(n - k + 1)/2$.

Recall that the graph strong product of two graphs $\Gamma_1$ and $\Gamma_2$ has vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and two distinct vertices $(v_1, v_2)$ and $(u_1, u_2)$ are connected iff they are adjacent or equal in each coordinate, that is, for $i \in 1, 2$, either $v_i = u_i$ or $[v_i, u_i] \in E(\Gamma_i)$, where $E(\Gamma_i)$ is the edge set of $\Gamma_i$ (see [2]).

In this paper, we characterise Deza graphs (not necessarily strictly Deza graphs) with parameters $(n, k, k - 1, 0)$ and strictly Deza graphs with parameters $(n, k, k - 1, a)$, $a > 0$ and $\beta = 1$, which completes the characterisation of Deza graphs with parameters $(n, k, k - 1, a)$.

Note that the adverb “strictly” in Theorem 1 cannot be removed, as is shown by the $n$-cycle with $n \geq 5$. However, in the present characterisation, we will see that there are no examples, which are not strictly Deza, except for $K_2$.

2  |  THE CHARACTERISATION

We start this section with a result that characterises Deza graphs (not necessarily strictly Deza graphs) with parameters $(n, k, k - 1, 0)$. 

Proposition 1. Let $\Gamma$ be a Deza graph with parameters $(n, k, k-1, 0)$. Then, one of the following cases holds:

(i) $k = 1$ and $\Gamma$ is a disjoint union of edges;
(ii) $k = 2$ and $\Gamma$ is a disjoint union of cycles, where every cycle has a length different from 4;
(iii) $k \geq 3$ and $\Gamma$ is a disjoint union of a number of complete graphs graphs on $k + 1$ vertices and a number of complete bipartite graphs with removed perfect matchings, where all the parts have size $k + 1$.

Proof. The case $k = 1$ is trivial. If $k = 2$, then $\Gamma$ is a disjoint union of cycles. Therefore, the number of common neighbours of any two vertices is at most 2. A pair of vertices with two common neighbours does arise if and only if the union of cycles has a cycle of length four.

Now, we consider the general case $k \geq 3$. Let $\Gamma_c$ be a connected component of $\Gamma$.

Suppose $\Gamma_c$ has a triangle induced by vertices $x$, $y$, and $z$. Since the vertices $x$ and $y$ have common neighbour $z$, the equality $|N(x, y)| = k - 1$ holds, which means that $y$ is adjacent to all vertices from $N(x) \setminus \{y\}$. Now, any vertex $w$ from $N(x)$ lies in a triangle and, thus, is adjacent to all vertices from $N(x) \setminus \{w\}$. We obtain that the vertices from $\{x\} \cup N(x)$ induce a disjoint clique of size $k + 1$ and $\Gamma_c$ coincides with this clique.

Suppose $\Gamma_c$ has no triangles. Let $x$ be a vertex from $\Gamma_c$. Then, any vertex from $N(x)$ is adjacent to $k - 1$ vertices from $N_2(x)$, where $N_2(x)$ is the second neighbourhood of $x$. Any vertex from $N_2(x)$ has at least one common neighbour with $x$ by definition of $N_2(x)$. Thus, any vertex from $N_2(x)$ has precisely $k - 1$ neighbours in $N(x)$ (the common neighbours with $x$). The double counting edges between $N(x)$ and $N_2(x)$ gives $k(k - 1) = |N_2(x)|(k - 1)$ and, therefore, $|N_2(x)| = k$. Any two distinct vertices $y_1, y_2$ from $N(x)$ have $k - 2$ common neighbours in $N_2(x)$ and, in particular, distinct $(k - 1)$-sets of neighbours in $N_2(x)$. This means that for any partition $Y \cup \{z\}$ of $N_2(x)$, where $|Y| = k - 1$, there is a vertex $y$ in $N(x)$ such that $y$ is adjacent to all vertices from $Y$ and is not adjacent to $z$. This implies that for any two distinct vertices $z_1, z_2$ from $N_2(x)$ there is a vertex in $N(x)$, which is adjacent to $z_1$ and is not adjacent to $z_2$. Thus, we have $N(x, z_1) \neq N(x, z_2)$, and the vertices $z_1, z_2$ have $k - 2$ neighbours in $N(x)$. Since $k \geq 3$ and $\Gamma_c$ has no triangles, the vertices $z_1, z_2$ are nonadjacent. Since $z_1, z_2$ is an arbitrary pair of vertices from $N_2(x)$, the subgraph induced by $N_2(x)$ is a coclique. We obtain that any vertex from $N_2(x)$ has precisely one neighbour in $N_3(x)$ and any two vertices from $N_2(x)$ have precisely one common neighbour in $N_3(x)$, where $N_3(x)$ is the third neighbourhood of $x$. This implies that $N_3(x)$ consists of a single vertex $x_0$, which is adjacent to all vertices from $N_2(x)$, and $\Gamma_c$ is a complete bipartite graph with removed perfect matching, where the parts are $\{x\} \cup N_2(x)$ and $N(x) \cup \{x_0\}$.

As we see from Proposition 1, there are no strictly Deza graphs with parameters $(n, k, k - 1, 0)$. We present two constructions of Deza graphs with parameters $(n, k, b, a)$, where $k = b + 1$, $a > 0$ and $\beta = 1$. Both constructions use a strongly regular graph $\Delta$ with parameters $(m, \ell, \lambda, \mu)$, where $\lambda = \mu - 1$.

Construction 1. Let $\Gamma_1$ be the strong product of $K_2$ and $\Delta$. The graph $\Gamma_1$ is a strictly Deza graph with parameters $(n, k, k - 1, a)$ and $\beta = 1$, where $n = 2m$, $k = 2\ell + 1$, $a = 2\mu$. 
So, if $B$ and $A_1$ are the adjacency matrices of $\Delta$ and $\Gamma_1$, respectively, then $A_1 = (B \otimes J_2) - I_n$ (where $J_m$ is the $m \times m$ all-ones matrix, and $I_m$ is the identity matrix of order $m$).

Suppose that $\Delta$ has an involution that interchanges only nonadjacent vertices. Let $P$ be the corresponding permutation matrix, then $B' = PB$ is a symmetric $(0, 1)$-matrix (because $P = P^T$ and $PBP = B$) with zero diagonal (because $P$ interchanges only nonadjacent vertices). So $B'$ is the adjacency matrix of a graph $\Delta'$ (say), which is a Deza graph because $B' = 2PBP = B^2$. This construction was given in [4] and the method has been called dual Seidel switching (see [5]).

Next, let $\Gamma'_2$ be the strong product of $K_2$ and $\Delta'$. Modify $\Gamma'_2$ as follows: For any transposition $(xy)$ of the involution, take the corresponding two pairs of vertices $xx', xx''$ and $yy', yy''$, delete the edges $xx', xx''$ and $yy', yy''$, and insert the edges $xy', xy''$. Define $\Gamma_2$ to be the resulting graph. If $A_2$ is the adjacency matrix of $\Gamma_2$, then we can also construct $\Gamma_2$ from $\Gamma_1$ using dual Seidel switching in the following way: $A_2 = P_1A_1$, where $P_1 = P \otimes I_2$. We easily have that $A_2^2 = A_1^2$, which shows that $\Gamma_2$ is a Deza graph with the same parameters as $\Gamma_1$.

**Construction 2.** The graph $\Gamma_2$ is a strictly Deza graph with parameters $(n, k, k - 1, a)$ and $\beta = 1$, where $n = 2m$, $k = 2\ell + 1$, $a = 2\mu$.

Note that in $\Gamma_1$ any two vertices with $b$ common neighbours are adjacent. For $\Gamma_2$, this is not true; therefore, $\Gamma_1$ and $\Gamma_2$ are nonisomorphic.

**Theorem 2.** If $\Gamma$ is a strictly Deza graph with parameters $(n, k, k - 1, a)$, $k > 1$ and $\beta = 1$, then $\Gamma$ can be obtained either from Construction 1 or from Construction 2.

In case $k = 1$, $\Gamma$ consists of $n/2$ disjoint edges and $\beta = 1$ implies $\Gamma = K_2$.

### 3 | PROOF OF THEOREM 2

In view of Proposition 1, we assume that $\Gamma$ is a strictly Deza graph with parameters $(n, k, k - 1, a)$ with $k > 2$, $a > 0$ and $\beta = 1$. For a vertex $x$ of $\Gamma$, denote by $x_b$, the unique vertex of $\Gamma$ that has $b$ common neighbours with $x$. Note that $(x_b)_b = x$ holds. A vertex $x$ in $\Gamma$ is said to be an $A$-vertex ($NA$-vertex) if $x$ is adjacent (not adjacent) to the vertex $x_b$.

**Lemma 2.** Let $x$ be an $A$-vertex. Then, the equality $N(x) \setminus \{x_b\} = N(x_b) \setminus \{x\}$ holds.

**Proof.** It follows from Lemma 2. \qed

**Lemma 3.** Let $x, y$ be two $A$-vertices, $\{x, x_b\} \neq \{y, y_b\}$. Then, there are either all possible edges between $\{x, x_b\}$ and $\{y, y_b\}$ or no such edges.

**Proof.** Suppose $x$ is adjacent to $y$. Since $x$ is an $A$-vertex, $x_b$ is adjacent to $y$. Since the vertices $y, y_b$ have $k - 1$ common neighbours and $y$ is adjacent to $x$ and $x_b$, the vertex $y_b$ is adjacent to at least one of the vertices $x$ and $x_b$. The fact that $x, x_b$ are $A$-vertices implies that $y_b$ is adjacent to both the vertices $x$ and $x_b$. \qed
Let \( x \) be an NA-vertex. Then, \( N(x) \) contains precisely one vertex, which is not adjacent to \( x_b \). Denote this vertex by \( x' \).

**Lemma 5.** Let \( x \) be an NA-vertex. Then, the vertex \( (x_b)' \) belongs to \( N_2(x) \).

**Proof.** By definition, \( (x_b)' \) is the neighbour of \( x_b \), which is not adjacent to \( (x_b)_b = x \). □

**Lemma 6.** Let \( x \) be an NA-vertex. The following statements hold:

1. The vertex \( (x_b)' \) is the unique neighbour of \( x_b \) in \( N_2(x) \).
2. The common neighbours of \( x_b \) and \( (x_b)' \) lie in \( N(x) \).
3. The vertex \( (x_b)' \) has precisely \( a \) neighbours in \( N(x) \).
4. Any vertex \( u \in N(x, x') \) is adjacent to both \( x_b \) and \( (x_b)' \).
5. The equality \( N(x, x') = N(x_b, x') = N(x, (x_b)') = N(x', (x_b)') \) holds.
6. If a vertex is adjacent to any three ones from the set \( \{x, x', x_b, (x_b)\} \), then it is adjacent to all of them.

**Proof.**

1. It follows from the fact that the vertex \( x_b \) has precisely \( b = k - 1 \) neighbours in \( N(x) \).
2. It follows from item (1).
3. As \( \beta = 1 \), one has \( |N(x, (x_b)')| \neq |N(x, x_b)| = b \), hence \( |N(x, (x_b)')| = a \).
4. The vertex \( u \) has precisely \( a \) neighbours in \( N(x) \); one of them is \( x' \). The vertex \( x_b \) is adjacent to all vertices in \( N(x) \) but \( x' \), in particular it is adjacent to \( u \). Moreover, \( u \) and \( x_b \) have precisely \( a - 1 \) common neighbours in \( N(x) \). The only neighbour of \( x_b \) in \( N_2(x) \) is the vertex \( (x_b)' \). Thus, \( (x_b)' \) is a common neighbour of \( u \) and \( x_b \), and, in particular, \( u \) is adjacent to \( (x_b)' \).
5. It follows from item (4).
6. It follows from item (5). □

For an NA-vertex \( x \), put \( W(x) = N(x, x') \).

**Lemma 7.** For any NA-vertex \( x \) and for any vertex \( y \in W(x) \), the vertex \( y_b \) belongs to \( W(x) \).

**Proof.** Since the vertices \( x, x', x_b, (x_b)' \) are neighbours of \( y \), and \( |N(y, y_b)| = k - 1 \), the vertex \( y_b \) is adjacent to at least three of them. By Lemma 6(6), the vertex \( y_b \) is adjacent to all of them. □

**Lemma 8.** The parameter \( a \) is even.

**Proof.** It follows from Lemma 7 and the fact that \( |W(x)| = a \). □

**Lemma 9.** For any NA-vertex \( x \), the vertex \( x' \) is an NA-vertex.
**Proof.** Suppose that \((x')_b\) and \(x'\) are adjacent. Since \(x\) is a neighbour of \(x'\), then, in view of Lemma 4, the vertices \(x, x_b\) belong to \(N(x', (x')_b)\). In particular, we obtain that \(x'\) is adjacent to \(x_b\), which is a contradiction because \(x'\) is not adjacent to \(x_b\) by definition. □

**Lemma 10.** For an NA-vertex \(x\), the vertex \((x')_b\) is not adjacent to \(x\).

**Proof.** Suppose that \((x')_b\) belongs to \(N(x)\). Since \((x')_b \neq x'\), the vertex \((x')_b\) is adjacent to \(x_b\). This means that \(x_b\) is the unique vertex, which is adjacent to \((x')_b\) and is not adjacent to \(x'\). Thus, every vertex \(y\) from \(N(x, (x')_b)\) is adjacent to \(x'\), and, consequently, belongs to \(N(x, x') = W(x)\). Since \(|W(x)| = |N(x, (x')_b)| = a\), we have \(W(x) = N(x, (x')_b)\). In particular, this gives the inclusion \(W(x) \subseteq N((x')_b)\). On the other hand, it follows from Lemma 6(5) that \(W(x) \subseteq N((x')_b)\).

Let us count the number of common neighbours of the vertices \((x')_b\) and \((x')_b'\). Clearly, \((x')_b \neq (x')_b'\) since \((x')_b \in N(x)\) by assumption, and \((x')_b' \notin N(x)\) by definition. Thus, we have \(a = |N((x')_b, (x')_b')| \geq |W(x) \cup [x_b]| = a + 1\), which is a contradiction. □

**Lemma 11.** For an NA-vertex \(x\), the equality \(x'' = x\) holds.

**Proof.** The vertex \(x'\) is adjacent to \(x\) by definition. By Lemma 10, the vertex \((x')_b\) is not adjacent to \(x\). This proves the lemma. □

**Lemma 12.** For an NA-vertex \(x\), the equality \((x')_b = (x_b)'\) holds.

**Proof.** Suppose \((x')_b \neq (x_b)'\) and let \(y \in N(x', (x')_b) \cap N_2(x)\). Observe, by Lemma 6(5), that \(N(x', (x_b)') \subseteq N(x)\), thus it follows that \(y \notin N((x_b)')\). Now, let \(w \in N(x_b, y)\) be arbitrary. Since \(w \neq (x_b)'\) from above, and since \((x_b)'\) is the only neighbour of \(x_b\) not in \(N(x)\), we conclude that \(w \in N(x)\). This establishes the inclusion \(N(x_b, y) \subseteq N(x, y)\) in which case \(N(x_b, y) = N(x, y)\) since both sets have equal size \(a\). But this is a contradiction since \(x' \in N(x, y)\) while \(x' \notin N(x, y)\). □

Further, in view of Lemma 12, we use the simplified notation \(x'_b\) for the vertex \((x')_b = (x_b)'\). For an NA-vertex \(x\), put \(C(x) = \{x, x', x_b, x'_b\}\).

**Lemma 13.** For an NA-vertex \(x\), the equalities \(C(x) = C(x') = C(x_b) = C(x'_b)\) hold.

**Proof.** It follows from the equality \((x_b)_b = x,\) Lemma 11 and Lemma 12. □

**Lemma 14.** The set of NA-vertices in \(\Gamma\) can be partitioned into quadruples of the form \(\{x, x', x_b, x'_b\}\).

**Proof.** It follows from Lemma 13. □

**Lemma 15.** Let \(x\) be an NA-vertex, and let \(y \notin C(x)\). Then, there are either all possible edges between \(\{x, x_b\}\) and \(\{y, y_b\}\) or no such edges.
Proof. If \( y \) is an \( A \)-vertex, then the result follows from Lemma 4. Thus, we assume that \( y \) is an NA-vertex. We show that if \( x \) and \( y \) are adjacent, then \( x \) is adjacent to \( y_b \), \( x_b \) is adjacent to \( y \), and \( x_b \) is adjacent to \( y_b \).

Suppose \( x \) is not adjacent to \( y_b \). Then, \( x = y' \) holds, which is a contradiction since, by Lemma 14, the condition \( C(x) \cap C(y) = \emptyset \) holds. Using similar arguments, we can show that \( x_b \) is adjacent to \( y \) and \( x_b \) is adjacent to \( y_b \). The lemma is proved. □

Lemma 16. For any two NA-vertices \( x, y \) such that \( C(x) \neq C(y) \), the following statements hold.

(1) The vertices \( x' \) and \( y \) are adjacent if and only if the vertices \( x \) and \( y' \) are adjacent.

(2) The vertices \( x \) and \( y \) are adjacent if and only if the vertices \( x' \) and \( y' \) are adjacent.

Proof.

(1) Suppose there exist two NA-vertices \( x, y \), \( C(x) \cap C(y) = \emptyset \) such that \( x' \) is adjacent to \( y \) and \( x \) is not adjacent \( y' \). For any vertex \( z \in N(x, y) \), \( z \neq x', y' \), in view of Lemma 15, there are all possible edges between \( \{z, z_b\} \) and \( \{x, x_b\} \), as well as between \( \{z, z_b\} \) and \( \{y, y_b\} \). In particular, \( z_b \) belongs to \( N(x, y) \). Moreover, \( x' \) belongs to \( N(x, y) \) and \( y' \) does not belong to \( N(x, y) \). Finally, \( x_b, y_b \) do not belong to \( N(x, y) \) since \( x, y \) are NA-vertices, and \( x_b, y_b \) do not belong to \( N(x, y) \) by Lemma 5. This means that \( a = |N(x, y)| \) is odd, which contradicts to Lemma 8.

(2) It follows from item (1) by replacing \( x' \) by \( x \). □

Lemma 17. Let \( x, y \) be two NA-vertices, \( C(x) \neq C(y) \). Then, the following statements hold.

(1) If a vertex from \( \{x, x_b\} \) is adjacent to a vertex from \( \{y, y_b\} \), then there are all possible edges between \( \{x, x_b\} \) and \( \{y, y_b\} \), and there are all possible edges between \( \{x', x_b'\} \) and \( \{y', y_b'\} \).

(2) If a vertex from \( \{x', x_b'\} \) is adjacent to a vertex from \( \{y', y_b'\} \), then there are all possible edges between \( \{x, x_b\} \) and \( \{y, y_b\} \), and there are all possible edges between \( \{x', x_b'\} \) and \( \{y', y_b'\} \).

(3) If a vertex from \( \{x, x_b\} \) is adjacent to a vertex from \( \{y', y_b'\} \), then there are all possible edges between \( \{x, x_b\} \) and \( \{y', y_b'\} \), and there are all possible edges between \( \{x', x_b'\} \) and \( \{y, y_b\} \).

(4) If a vertex from \( \{x', x_b'\} \) is adjacent to a vertex from \( \{y, y_b\} \), then there are all possible edges between \( \{x, x_b\} \) and \( \{y', y_b'\} \), and there are all possible edges between \( \{x', x_b'\} \) and \( \{y, y_b\} \).

(5) There are either 0, or 8, or 16 edges between the sets \( C(x) \) and \( C(y) \).
Proof.

(1) Without loss of generality, assume that $x$ and $y$ are adjacent. By Lemma 15, there are all possible edges between $\{x, x_b\}$ and $\{y, y_b\}$. On the other hand, by Lemma 16 (2), the vertices $x'$ and $y'$ are adjacent. Then, by Lemma 15 again, there are all possible edges between $\{x', x'_b\}$ and $\{y', y'_b\}$.

(2)-(4) They follow from item (1) by suitable replacement of vertices.

(5) It follows from items (1) to (4). □

Lemma 18. Let $x$ be an NA-vertex and $y$ be an A-vertex. The following statements hold.

(1) If $x$ and $y$ are adjacent, then the vertices $y$ and $y'_b$ lie in $W(x)$.

(2) There are either all possible edges between the sets $\{x, x'_b, x_b\}$ and $\{y, y_b\}$, or no such edges.

Proof.

(1) Let $z$ be a vertex in $N(x, y)$, $z \neq x', y'_b$. We prove that $z_b$ belongs to $N(x, y)$. If $z$ is an A-vertex, then it follows from Lemma 4. Suppose $z$ is an NA-vertex. Then, by Lemma 4, $z_b$ belongs to $N(y)$, and by Lemma 15, $z_b$ belongs to $N(x)$, which implies that $z_b$ belongs to $N(x, y)$. Moreover, $y'_b$ belongs to $N(x, y)$. In view of Lemma 8, $|N(x, y)|$ is even, which implies that $x'$ belongs to $N(x, y)$, and, in particular, $y$ is adjacent to $x'$. This means that $y$ belongs to $N(x, x')$, and, consequently, $y'_b$ belongs to $N(x, x')$.

(2) It follows from item (1). □

Let $\Gamma'$ be the graph obtained from $\Gamma$ by removing all edges $\{x, x_b\}$, where $x$ is an A-vertex, and all edges $\{x, x'_b\}$, where $x$ is an NA-vertex.

Lemma 19. The following statements hold.

(1) The graph $\Gamma'$ is $(k - 1)$-regular.

(2) For any vertices $x, y$ in $\Gamma'$ such that $y \neq x_b$, there are either all possible edges in $\Gamma'$ between $\{x, x_b\}$ and $\{y, y_b\}$ or no such edges.

(3) For any vertices $x, y$ in $\Gamma'$ such that $y \neq x_b$, the equalities $N_{\Gamma'}(x, y) = N_{\Gamma'}(x, y) = N_{\Gamma'}(x_b, y) = N_{\Gamma'}(x, y'_b)$ hold.

Proof.

(1) It follows from the fact that precisely one edge was removed for each vertex.

(2) If $x, y$ are A-vertices, then it follows from Lemma 3. If one of the vertices $x, y$ is an A-vertex and the other is an NA-vertex, then it follows from Lemma 18. If $x, y$ are NA-vertices, then it follows from Lemma 17.

(3) It follows from the fact that in the graph $\Gamma'$, any vertex $x$ has the same neighbourhood as the vertex $x_b$. □
Let $\Gamma''$ be the graph whose vertex set is the set of all pairs of vertices $[x, x_b]$ in $\Gamma'$, and two vertices $[x, x_b]$ and $[y, y_b]$ are adjacent in $\Gamma''$ whenever there are all possible edges between then sets of vertices $\{x, x_b\}$ and $\{y, y_b\}$ in $\Gamma'$.

**Lemma 20.**

1. The graph $\Gamma''$ is a Deza graph with parameters $(n/2, (k - 1)/2, a/2, (a - 2)/2)$.
2. Let $[x, x_b]$ and $[y, y_b]$ be vertices of $\Gamma''$ such that $x$ and $y$ are both NA-vertices and there are precisely eight edges between the sets $\{x, x_b\}$ and $\{y, y_b\}$ in graph $\Gamma$. Then, $[x, x_b]$ and $[y, y_b]$ have $a/2$ common neighbours iff $[x, x_b]$ and $[y, y_b]$ are adjacent in $\Gamma''$.
3. In all remaining cases, $[x, x_b]$ and $[y, y_b]$ have $a/2$ common neighbours iff $[x, x_b]$ and $[y, y_b]$ are nonadjacent in $\Gamma''$.

**Proof.**

1. It follows from Lemma 19(1) that the graph $\Gamma''$ is $(k - 1)/2$-regular. For arbitrary two distinct vertices $[x, x_b]$ and $[y, y_b]$ in $\Gamma''$, we consider all possible cases and prove that the number of their common neighbours is either $a/2$ or $(a - 2)/2$.

**Case 1.** The vertices $x$ and $y$ are $A$-vertices. By Lemma 19(2), there are either all possible edges between $\{x, x_b\}$ and $\{y, y_b\}$ or no such edges.

Suppose there are all possible edges between $\{x, x_b\}$ and $\{y, y_b\}$. Then, $N_{\Gamma'}(x, y) = \{x_b, y_b\} \cup N_{\Gamma'}(x, y)$, which, in view of Lemma 19(3), implies that the adjacent vertices $[x, x_b]$ and $[y, y_b]$ have $|N_{\Gamma'}(x, y)|/2 = (a - 2)/2$ common neighbours in $\Gamma''$.

Suppose there are no edges between $\{x, x_b\}$ and $\{y, y_b\}$. Then, $N_{\Gamma'}(x, y) = N_{\Gamma'}(x, y)$, which, in view of Lemma 19(3), implies that the nonadjacent vertices $[x, x_b]$ and $[y, y_b]$ have $|N_{\Gamma'}(x, y)|/2 = a/2$ common neighbours in $\Gamma''$.

**Case 2.** The vertex $x$ is an NA-vertex and the vertex $y$ is an $A$-vertex. By Lemma 18(2), there are either all possible edges between $\{x, x_b\}$ and $\{y, y_b\}$ or no such edges.

Suppose there are all possible edges between $\{x, x_b\}$ and $\{y, y_b\}$. Then, $N_{\Gamma'}(x, y) = \{x', y_b\} \cup N_{\Gamma'}(x, y)$, which, in view of Lemma 19(3), implies that the adjacent vertices $[x, x_b]$ and $[y, y_b]$ have $|N_{\Gamma'}(x, y)|/2 = (a - 2)/2$ common neighbours in $\Gamma''$.

Suppose there are no edges between $\{x, x_b\}$ and $\{y, y_b\}$. Then, $N_{\Gamma'}(x, y) = N_{\Gamma'}(x, y)$, which, in view of Lemma 19(3), implies that the nonadjacent vertices $[x, x_b]$ and $[y, y_b]$ have $|N_{\Gamma'}(x, y)|/2 = a/2$ common neighbours in $\Gamma''$.

**Case 3.** The vertices $x$ and $y$ are NA-vertices. If $C(x) = C(y)$ (in other words, $[y, y_b] = \{x', x_b'\}$), then $N_{\Gamma'}(x, y) = N_{\Gamma'}(x, y)$, which, in view of Lemma 19(3), implies that the nonadjacent vertices $[x, x_b]$ and $[y, y_b]$ have $|N_{\Gamma'}(x, y)|/2 = a/2$ common neighbours in $\Gamma''$. If $C(x) \neq C(y)$ (in other words, $[y, y_b] \neq \{x', x_b'\}$), then, by Lemma 17, there are either 0, or 8, or 16 edges between the sets.
Suppose there are no edges between \( \{x, x_b, x_b'\} \) and \( \{y, y_b, y_b'\} \). Then, \( N_\Gamma(x, y) = N_\Gamma(y, x) \), which, in view of Lemma 19(3), implies that the nonadjacent vertices \([x, x_b] \) and \([y, y_b] \) have \( |N_\Gamma(x, y)|/2 = a/2 \) common neighbours in \( \Gamma'' \).

Suppose there are all 16 possible edges between \( \{x, x_b, x_b'\} \) and \( \{y, y_b, y_b'\} \). Then, \( N_\Gamma(x, y) = \{x', y'\} \cup N_\Gamma(x, y) \), which, in view of Lemma 19(3), implies that the adjacent vertices \([x, x_b] \) and \([y, y_b] \) have \( |N_\Gamma(x, y)|/2 = (a-2)/2 \) common neighbours in \( \Gamma'' \).

Suppose there are eight edges between \( \{x, x_b, x_b'\} \) and \( \{y, y_b, y_b'\} \). Then, there are two subcases with respect to Theorem 17. If there are all possible edges between \( \{x, x_b\} \) and \( \{y, y_b\} \) and there are all possible edges between \( \{x', x_b'\} \) and \( \{y', y_b'\} \), then \( N_\Gamma(x, y) = N_\Gamma(x, y) \), which, in view of Lemma 19(3), implies that the adjacent vertices \([x, x_b] \) and \([y, y_b] \) have \( |N_\Gamma(x, y)|/2 = a/2 \) common neighbours in \( \Gamma'' \). If there are all possible edges between \( \{x, x_b\} \) and \( \{y', y_b'\} \) and there are all possible edges between \( \{x', x_b'\} \) and \( \{y, y_b\} \), then \( N_\Gamma(x, y) = \{x', y'\} \cup N_\Gamma(x, y) \), which, in view of Lemma 19(3), implies that the nonadjacent vertices \([x, x_b] \) and \([y, y_b] \) have \( |N_\Gamma(x, y)|/2 = (a-2)/2 \) common neighbours in \( \Gamma'' \).

(2) It follows from the proof of item (1).

(3) It follows from the proof of item (1).

By Lemma 17(3), for any two NA-vertices \( x, y \) in \( \Gamma \) such that \( C(x) \neq C(y) \), there are either 0, or 8, or 16 edges between the sets \( C(x) = \{x, x_b, x_b'\} \) and \( C(y) = \{y, y_b, y_b'\} \). This implies that there are either 0, or 2, or 4 edges between the sets of vertices \([x, x_b, x_b']\) and \([y, y_b, y_b']\). Note that, in the case when there are two edges, the switching edges between \([x, x_b, x_b']\) and \([y, y_b, y_b']\) preserves regularity of \( \Gamma'' \). Let us make all such switchings in \( \Gamma'' \). Denote by \( \Gamma''' \) the resulting graph.

**Lemma 21.** The following statements hold.

(1) The adjacency matrix of \( \Gamma''' \) can be obtained from the adjacency matrix of \( \Gamma'' \) by permuting rows in pairs corresponding to the vertices \([x, x_b, x_b']\) and \([y, y_b, y_b']\) for all quadruples \([x, x_b, x_b']\) of NA-vertices. In particular, the adjacency matrices of \( \Gamma''' \) and \( \Gamma'' \) coincide in the case when \( \Gamma \) has no NA-vertices.

(2) The graph \( \Gamma''' \) is strongly regular with parameters \((v/2, (k-1)/2, (a-2)/2, a/2)\).

**Proof.**

(1) It follows from the definition of \( \Gamma''' \) and Lemmas 17 and 18(2).

(2) It follows from Lemma 20 and item (1).

In view of Lemma 21, if \( \Gamma \) has no NA-vertices, then it comes from Construction 1. Let us prove that, if \( \Gamma \) has NA-vertices, then it comes from Construction 2. Lemmas 20 and 21 imply that the adjacency matrix of the Deza graph \( \Gamma'' \) can be obtained from the adjacency matrix of the strongly regular \( \Gamma''' \) by swapping rows in pairs corresponding to the vertices \([x, x_b] \) and \([x', x_b']\).
for all quadruples \( \{x, x', x_b, x'_b\} \) of \( \overline{\text{NA}} \)-vertices. It follows from [4] that this permutation is an order 2 automorphism of \( \Gamma'' \) that interchanges only nonadjacent vertices. Thus, \( \Gamma'' \) can be obtained from \( \Gamma''' \) by dual Seidel switching, which implies that \( \Gamma \) comes from Construction 2 by definition. The theorem is proved.

4 | STRONGLY REGULAR GRAPHS WITH \( \lambda = \mu - 1 \)

Note that, if \( \Gamma \) is a strongly regular graph with \( \lambda = \mu - 1 \), then so is its complement. So both \( \Gamma \) and its complement satisfy the condition for Construction 1. For Construction 2, \( \Gamma \) needs an involution that interchanges only nonadjacent vertices. In this section, we survey strongly regular graphs with \( \lambda = \mu - 1 \), and look for the desired involutive automorphism.

4.1 | Paley graphs of square order

The Paley graph \( P(r) \) is a graph with vertex set \( \mathbb{F}_r \), where \( r \) is a prime power such that \( r \equiv 1 \mod 4 \). Two vertices \( x \) and \( y \) of \( P(r) \) are adjacent whenever \( x - y \) is a nonzero square in \( \mathbb{F}_r \). See [9] for an excellent survey of Paley graphs. The Paley graph is a strongly regular graph with parameters \( (r, (r - 1)/2, (r - 5)/4, (r - 1)/4) \), so it satisfies the conditions of Construction 1, which leads to strictly Deza graphs with parameters \( (2r, r, r - 1, (r - 1)/2) \). The complement of \( P(r) \) is isomorphic to \( P(r) \), since for any nonsquare \( a \) in \( \mathbb{F}_r \) the map \( x \to ax \) interchanges edges and nonedges in \( P(r) \).

If \( r = q^2 \) is a square, then the Paley graph \( P(q^2) \) satisfies the conditions of Construction 2. To explain this, we need some properties of the field \( \mathbb{F}_{q^2} \). Let \( d \) be a nonsquare in \( \mathbb{F}_q \). The elements of the finite field of order \( q^2 \) can be considered as

\[
\mathbb{F}_{q^2} = \{x + y\alpha \mid x, y \in \mathbb{F}_q\},
\]

where \( \alpha \) is a root of the polynomial \( f(t) = t^2 - d \).

Let \( \beta \) be a primitive element of the finite field \( \mathbb{F}_{q^2} \). Then, we have \( \mathbb{F}_{q^2}^* = \{\beta^{(q+1)} \mid i \in \{0, \ldots, q - 2\}\} \). Since \( q + 1 \) is even, each element of \( \mathbb{F}_{q^2}^* \) is a square in \( \mathbb{F}_q^* \). It also follows that \( x^{q-1} = \beta^{q^2-1} = 1 \) for every \( x \in \mathbb{F}_q^* \).

Lemma 22. For any \( \gamma = x + y\alpha \) from \( \mathbb{F}_{q^2} \), the following equalities hold:

(1) \( \gamma^q = x - y\alpha \);
(2) \( \gamma - \gamma^q = 2y\alpha \).

Proof.

(1) As \( \alpha^{2(q-1)} = d^{q-1} = 1 \), and as \( -1 \) is the unique involution in \( \mathbb{F}_q^* \), it follows that \( \alpha^{q-1} = -1 \). Thus, we have \( (x + y\alpha)^q = x^q + y^q\alpha^q = x - y\alpha \).

(2) It follows from item (1). \( \square \)
For any $\gamma = x + y\alpha \in \mathbb{F}_q^*$, define the norm mapping $N$ by $N(\gamma) = \gamma^{q+1} = \gamma\gamma^q = (x + y\alpha)(x - y\alpha) = x^2 - y^2d$. The norm mapping is a homomorphism from $\mathbb{F}_q^*$ to $\mathbb{F}_q^*$ with $\text{Im}(N) = \mathbb{F}_q^*$. Thus, the kernel $\text{Ker}(N)$ is the subgroup of order $q + 1$ in $\mathbb{F}_q^*$.

Now, we make some remarks on squares in finite fields.

**Lemma 23.**

1. The element $-1$ is a square in $\mathbb{F}_q^*$ iff $q \equiv 1 \mod 4$;
2. For any nonsquare $e$ in $\mathbb{F}_q^*$, the element $-e$ is a square in $\mathbb{F}_q^*$ iff $q \equiv 3 \mod 4$.

The following lemma can be used to test whether an element $\gamma = x + y\alpha \in \mathbb{F}_q^*$ is a square.

**Lemma 24** (Baker [1], Lemma 2). An element $\gamma = x + y\alpha \in \mathbb{F}_q^*$ is a square iff $N(\gamma) = x^2 - y^2d$ is a square in $\mathbb{F}_q^*$.

Lemma 25 immediately follows by Lemmas 23 and 24, and the fact that $N(\alpha) = -d$.

**Lemma 25.** The element $\alpha$ is a square in $\mathbb{F}_q^*$ iff $q \equiv 3 \mod 4$.

Denote by $\varphi$ the automorphism of $P(q^2)$ that sends $\gamma$ to $\gamma^q$. Note that $\varphi$ fixes the elements from $\mathbb{F}_q^*$.

Lemma 26 follows from Lemmas 25 and 22(2) and the fact all elements from $\mathbb{F}_q^*$ are squares in $\mathbb{F}_q^*$.

**Lemma 26.** The following statements hold.

1. If $q \equiv 1 \mod 4$, then $\varphi$ interchanges only nonadjacent vertices.
2. If $q \equiv 3 \mod 4$, then $\varphi$ interchanges only adjacent vertices.

In view of Lemma 26, and because the Paley graph $P(q^2)$ is isomorphic to its complement, we can conclude that $P(q^2)$ satisfies the condition of Construction 2 for every odd prime power $q$. Thus, if the parameters of $P(q^2)$ are $(4\mu + 1, 2\mu, \mu - 1, \mu)$, where $\mu = (q^2 - 1)/4$, we can obtain the following three strictly Deza graphs from $P(q^2)$:

1. a strictly Deza graph with parameters $(4\mu + 1, 2\mu, \mu - 1, \mu)$ obtained using dual Seidel switching in $P(q^2)$,
2. a strictly Deza graph with parameters $(8\mu + 2, 4\mu + 1, 4\mu, 2\mu)$ obtained from Construction 1,
3. a strictly Deza graph with parameters $(8\mu + 2, 4\mu + 1, 4\mu, 2\mu)$ obtained from Construction 2.

### 4.2 Symmetric conference matrices

An $m \times m$ matrix $C$ with zeros on the diagonal, and $\pm 1$ elsewhere, is a conference matrix if $CC^T = (m - 1)I$. If a conference matrix $C$ is symmetric with constant row (and column) sum $r$, then $r = \pm \sqrt{m - 1}$, and $B = 1/2(I_m - I_m - C)$ is the adjacency matrix of a strongly regular graph with parameter set $$(\lambda, \mu, 0, \mu, -1)$$.
Note that $\mathcal{P}(-r)$ is the complementary parameter set of $\mathcal{P}(r)$. Symmetric conference matrices with constant row sum have been constructed by Seidel (see [11], Theorem 13.9). If $q$ is an odd prime power and $r = \pm q$, then such a conference matrix can be obtained from the Paley graph of order $q^2$. Let $B'$ be the adjacency matrix of $P(q^2)$, and put $S = I_{q^2} - I_{q^2} - 2B'$ ($S$ is the so-called Seidel matrix of $P(q^2)$). Define

$$C' = \begin{bmatrix} 0 & 1^T \\ 1 & S \end{bmatrix}$$

(1 is the all-ones vector). Then, $C'$ is a symmetric conference matrix of order $m = q^2 + 1$. However, $C'$ does not have constant row sum.

Next, we shall make the row and column sums constant by multiplying some rows and the corresponding columns of $C'$ by $-1$. This operation is called Seidel switching, and it is easily seen that Seidel switching does not change the conference matrix property. To describe the required rows and columns, we use the notation and description of $P(q^2)$ given in Section 4.1. If $q \equiv 3 \mod 4$, we take the complement of the described Paley graph. Then, the involution $\varphi$ given in Lemma 26 interchanges only nonadjacent vertices in all cases. For $x \in \mathbb{F}_q$, define $V_x = \{x + y\alpha \mid y \in \mathbb{F}_q\}$. Then, the sets $V_x$ partition the vertex set of $P(q^2)$, and each class is a coclique. Moreover, each partition class is fixed by the involution $\varphi$. Let $V$ be the union of $1/2(q - 1)$ classes $V_x$. Then, $V$ induces a regular subgraph of $P(q^2)$ of degree $1/4(q - 1)^2 - 1$ with $1/2q(q - 1)$ vertices. Now, make the matrix $C$ by Seidel switching in $C'$ with respect to the rows and columns that correspond with $V$. Then, $C$ is a regular symmetric conference matrix, and $B'JIC = 1/2(-JIC)$ is the adjacency matrix of a strongly regular graph $\Gamma$ with parameter set $\mathcal{P}(q)$, and $\varphi$ remains an involution that interchanges only nonadjacent vertices. (It is worthwhile to remark that any union of $1/2(q - 1)$ partition classes leads to a strongly regular graph with parameter set $\mathcal{P}(q)$, but different choices may lead to nonisomorphic graphs.) So $\Gamma$ satisfies the conditions for Construction 1 and 2. For the complement of $\Gamma$, we found no involutions that interchanges only nonadjacent vertices. Thus, we find Deza graphs with parameters $(q^2 + 1, 1/2(q^2 - q), 1/4(q - 1)^2, 1/4(q - 1)^2 - 1)$ (by dual Seidel switching in $\Gamma$), $(2q^2 + 2, q^2 - q + 1, q^2 - q, 1/2(q - 1)^2)$ (by Construction 1 and 2 applied to $\Gamma$), and $(2q^2 + 2, q^2 + q + 1, q^2 + q, 1/2(q + 1)^2)$ (by Construction 1 applied to the complement of $\Gamma$). If $q = 3$, $\Gamma$ is the Petersen graph. It has the unique involutive automorphism that interchanges only nonadjacent vertices. This automorphism has four fixed and six moved vertices. The Deza graph obtained from the Petersen graph with dual Seidel switching has diameter 3. However, in all other cases, the obtained Deza graphs are strictly Deza. The complement of the Petersen graph has no involutive automorphisms that interchanges only nonadjacent vertices. We expect that this is the case for all strongly regular graphs with parameter set $\mathcal{P}(r)$ and $r < 0$.

4.3 | The Hoffman-Singleton graph

We consider Robertson's pentagons and pentagrams construction (see [8]) of the Hoffman-Singleton graph, which is strongly regular with parameters $(50, 7, 0, 1)$. 

$$\mathcal{P}(r) = \left(r^2 + 1, \frac{1}{2}(r^2 - r), \frac{1}{4}(r - 1)^2 - 1, \frac{1}{4}(r - 1)^2 \right).$$
The 50 vertices of the Hoffman-Singleton graph are grouped into five pentagons \(P_0, \ldots, P_4\) and five pentagrams \(Q_0, \ldots, Q_4\) labelled in such a way that the pentagrams are the complements of the pentagons; there are no edges between any two distinct pentagons, nor between any two distinct pentagrams. Edges between pentagon and pentagram vertices are defined by the rule: each vertex \(i \in \{0, 1, 2, 3, 4\}\) of a pentagon \(P_j, j \in \{0, 1, 2, 3, 4\}\) is adjacent to the vertex \((i + jk) \mod 5\) of a pentagram \(Q_k, k \in \{0, 1, 2, 3, 4\}\).

Let \(\varphi\) be the permutation of vertices of the Hoffman-Singleton graph that fixes each vertex of \(P_0, Q_0\), and interchanges \(P_1, Q_1\) with \(P_4, Q_4\) and \(P_2, Q_2\) with \(P_3, Q_3\). The permutation \(\varphi\) is the unique involutive automorphism of the Hoffman-Singleton graph that interchanges only nonadjacent vertices. The Deza graph obtained from the Hoffman-Singleton graph with dual Seidel switching has diameter 3. However, each of Constructions 1 and 2 produces a strictly Deza graph with parameters (100, 15, 14, 2). Construction 1 applied to the complement gives a strictly Deza graph with parameters (100, 85, 84, 72).

5 | DIVISIBLE DESIGN GRAPHS (DDGs)

For a Deza graph \(\Gamma\) with parameters \((n, k, b, a)\) and \(a \neq b\), we define two graphs \(\Gamma_a\) and \(\Gamma_b\) on the vertex set of \(\Gamma\), where two vertices \(x\) and \(y\) are adjacent in \(\Gamma_a\) (resp. \(\Gamma_b\)) if \(x\) and \(y\) have \(a\) (resp. \(b\)) common neighbours. Clearly, \(\Gamma_a\) and \(\Gamma_b\) are each other’s complement, and regular of degree \(\alpha\) and \(\beta\), respectively \(\Gamma_a\) and \(\Gamma_b\) have been called the children of \(\Gamma\).

If \(\Gamma_a\) or \(\Gamma_b\) is the disjoint union of complete graphs, then \(\Gamma\) is called a DDG. DDGs are interesting structures on their own, and have been studied in \([3,7]\). If \(A\) is the adjacency matrix of a DDG, then \(A\) also satisfies the conditions for the incidence matrix of a divisible design, which explains the name.

If \(\Gamma\) is a Deza graph with \(\alpha = 1\) (resp. \(\beta = 1\)), then \(\Gamma_a\) (resp. \(\Gamma_b\)) consist of \(n/2\) disjoint edges, so \(\Gamma\) is a DDG. Such a DDG has been called thin (see [3]). Thus, Theorem 2 also characterises thin DDGs with \(b = k - 1\) (for DDGs, one uses \(\lambda_1\) and \(\lambda_2\) instead of \(b\) and \(a\)). The characterisation of DDGs with \(\lambda_1 = k - 1\) is also claimed in Theorem 4.11 of [7]. However, this claim is not correct, because it only uses Construction 1 (which corresponds to Construction 4.10 in [7]), and Construction 2 is not mentioned. The proof in [7] is based on the characterisation of divisible designs with \(k - \lambda_1 = 1\) from [6]. But Haemers et al [7] overlooked the fact that isomorphic divisible designs may correspond to nonisomorphic DDGs. Indeed, \(A\) and \(A’\) are adjacency matrices of isomorphic DDGs whenever there exists a permutation matrix \(P\) such that \(PAP^T = A’\), but for \(A\) and \(A’\) to be incidence matrices of isomorphic divisible designs it is only required that \(PAQ = A’\) for permutation matrices \(P\) and \(Q\). This is precisely what happens if one DDG can be obtained from the other by dual Seidel switching. So Theorem 4.11 of [7] can be repaired with basically the same proof by inserting Construction 2 into the statement.

A graph is walk-regular if the number of closed walks of any given length at a vertex \(x\) is independent of the choice of \(x\). From the results in [3], it follows that the graphs made by Construction 1 are walk-regular, and those from Construction 2 are not. It also follows that the matrices of Constructions 1 and 2 have different eigenvalues. Because of this, the discovered mistake in [7] has some consequences for Table 1 in [7] and [3], and the second author would like to take the opportunity to give the necessary corrections. For the parameter sets (18, 9, 8, 4) and (20, 7, 6, 2), there can occur an additional eigenvalue 1, and the multiplicities are not determined by the parameters. Moreover, for these parameter sets, the “no” occurring in the column headed by “notWR” in Table 1 of [3] should be replaced by “yes.”
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