GEOMETRY AND TOPOLOGY OF THE SPACE OF
PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. Let Ω be a strongly pseudoconvex domain. We introduce
the Mabuchi space of strongly plurisubharmonic functions in Ω. We
study metric properties of this space using Mabuchi geodesics and es-
establish regularity properties of the latter, especially in the ball. As an
application we study the existence of local Kähler-Einstein metrics.

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Introduction

Let Y be a compact Kähler manifold and \( \alpha_Y \in H^{1,1}(Y, \mathbb{R}) \) a Kähler
class. The space \( \mathcal{H}_{\alpha_Y} \) of Kähler metrics \( \omega_Y \) in \( \alpha_Y \) can be seen as an infinite
dimensional riemannian manifold whose tangent spaces \( T_{\omega_Y} \mathcal{H}_{\alpha_Y} \) can all be
identified with \( \mathcal{C}^\infty(Y, \mathbb{R}) \). Mabuchi has introduced in [Mab87] an \( L^2 \)-metric
on \( \mathcal{H}_{\alpha_Y} \), by setting

\[
\langle f, g \rangle_{\omega_Y} := \int_Y f \, g \, \frac{\omega_Y^n}{V_{\alpha_Y}},
\]

where \( n = \dim \mathbb{C} \ Y \) and \( V_{\alpha_Y} = \int_Y \omega_Y^n = \alpha_Y^n \) denotes the volume of \( \alpha_Y \).
Mabuchi studied the corresponding geometry of \( \mathcal{H}_{\alpha_Y} \), showing in particular that it can formally be seen as a locally symmetric space of non positive
curvature. The (geometry) metric study of the space \( (\mathcal{H}_{\alpha_Y}, \langle \cdot, \cdot \rangle_{\omega_Y}) \)

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has motivated a lot of interesting works in the last decades, see notably [Don99, Chen00, CC02, CT08, Chen09, LV13, DL12, Dar13, Dar14, Dar15].

The purpose of this article is to extend some of these studies to the case when \( Y \) is a smooth strongly pseudoconvex bounded domain of \( \mathbb{C}^n \). We note here that this problem of extension to the local case recently been considered by Rashkovskii [Rash16] and Hosono [Hos16], Rashkovskii studied geodesics for plurisubharmonic functions in the Cegrell class \( F_1 \) on a bounded hyperconvex domain, he also showed that functions with strong singularities generally cannot be connected by (sub)geodesic arcs. Hosono described the behavior of the weak geodesics between toric psh functions with poles at the origin.

Our first interest is the geometry of the space of plurisubharmonic functions, We equipped the space of plurisubharmonic functions with a Levi-Civita connection \( D \) and we describe the tensor curvature and sectional curvature as in a paper of Mabuchi [Mab87]. Our first main result is to establish that the space of plurisubharmonic functions is a locally symmetric space:

**Theorem A.** The Mabuchi space \( \mathcal{H} \) equipped with the Levi-Civita connection \( D \) is a locally symmetric space.

Following the work of Donaldson [Don99] and Semmes [Sem92] in the compact setting, we reinterpret the geodesics as a solution to a homogeneous complex Monge-Ampère equation. Weak geodesics are introduced as an envelope of functions:

\[
\Phi(z, \zeta) = \sup \{ u(z, \zeta) / u \in \mathcal{F}(\Omega \times A, \Psi) \}
\]

Our second main result is to establish regularity properties of geodesics in the ball by adapting the celebrated result of Bedford-Taylor [BT76]:

**Theorem B.** Let \( B \) be the unit ball in \( \mathbb{C}^n \). Let \( \varphi_0 \) and \( \varphi_1 \) be the end geodesic points which are \( C^{1,1} \). Then the Perron-Bremermann envelope

\[
\Phi(z, \zeta) = \sup \{ u(z, \zeta) / u \in \mathcal{F}(\Omega \times A, \Psi) \}
\]

admits second-order partial derivatives almost everywhere with respect to variable \( z \in B \) which locally bounded uniformly with respect to \( \zeta \in A \), i.e. for any compact subset \( K \subset B \) there exists \( C \) which depend on \( K, \varphi_0 \) and \( \varphi_1 \) such that

\[
\| D_z^2 \Phi \|_{L^\infty(K \times A)} \leq C.
\]

The existence of local Kähler-Einstein metrics was studied by Guedj, Kolev and Yeganefar [GKY13] in bounded smooth strongly pseudoconvex domains which are circled. This is equivalent to the resolution of the following Dirichlet problem

\[
(MA)_1 \begin{cases}
(ddc^* \varphi)^n = e^{-\varphi} \mu / \int_\Omega e^{-\varphi} d\mu & \text{in } \Omega \\
\varphi = 0 & \text{on } \Omega
\end{cases}
\]
They treated also the following family of Dirichlet problems

\[(MA)_t \begin{cases} (dd^c \phi_t)^n &= \frac{e^{-t \phi_t} \mu}{\int_{\Omega} e^{-t \phi_t} d\mu}, & \text{in } \Omega \\ \phi_t &= 0, & \text{on } \Omega \end{cases}\]

showing that there is a solution for \( t < (2n)^{1+1/(1+1/n)} \). We apply our study of the geodesics problem and an idea of [DR15, DG16] to prove that the existence of a solution to \((MA)_t\) is equivalent to the coercivity of the Ding functional:

**Theorem C.** Let \( \Omega \subset \mathbb{C}^n \) be a smooth strongly pseudo-convex circled domain. If there exists \( \varepsilon(t), M(t) > 0 \) such that,

\[ F_t(\psi) \leq \varepsilon(t) E(\psi) + M(t) \quad \forall \psi \in \mathcal{H}, \]

then \((MA)_t\) admits a \( S^1 \)-invariant smooth strictly plurisubharmonic function solution. Conversely if \((MA)_t\) admits such a solution \( \phi_t \) and \( \Omega \) is strictly \( \phi_t \)-convex, then there exists \( \varepsilon(t), M(t) > 0 \) such that,

\[ F_t(\psi) \leq \varepsilon(t) E(\psi) + M(t) \quad \forall \psi \in \mathcal{H}. \]

The organization of the paper is as follows.

- Section 1 is devoted to preliminary results and the definition of the space \( \mathcal{H} \) and its geometry.
- In Section 2 we show that the geodesics are continuous (sometimes even Lipschitz) up to the boundary of \( \Omega \times A \).
- In Section 3 we prove Theorem B.
- Finally, we prove Theorem C in Section 4.

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1. **Mabuchi geometry in pseudoconvex domains**

In this section we will study the geometry of the space of plurisubharmonic functions in strongly pseudoconvex domain, based upon works of Mabuchi [Mab87], Semmes [Sem92] and Donaldson [Don99], as it was clarified through lecture notes of Guedj [G14] and Kolev [Kol12].

1.1. **Preliminaries.** In this section we recall some analytic tools which will be used in the sequel. Let \( \Omega \Subset \mathbb{C}^n \) be a smooth pseudoconvex bounded domain. Recall that a bounded domain \( \Omega \Subset \mathbb{C}^n \) is strictly pseudoconvex if there exists a smooth function \( \rho \) defined in neighbourhood \( \Omega' \) of \( \bar{\Omega} \) such that \( \Omega = \{ z \in \Omega' / \rho(z) < 0 \} \) with \( dd^c \rho > 0 \), where

\[ d := \partial + \bar{\partial}, \quad d^c := \frac{i}{2\pi} (\partial - \bar{\partial}) \]
Definition 1.1. We let $PSH(\Omega)$ denote the set of plurisubharmonic functions in $\Omega$. In particular a function $\varphi \in PSH(\Omega)$ is $L^1_{loc}$, upper semi-continuous and such that
\[ dd^c \varphi \geq 0 \]
in the weak sense of positive currents.

The following cone of “test functions” has been introduced by Cegrell [Ceg98]:

Definition 1.2. [Ceg98] We let $E_0(\Omega)$ denote the convex cone of all bounded plurisubharmonic functions $\varphi$ defined in $\Omega$ such that $\lim_{z \to \zeta} \varphi(\zeta) = 0$, for every $\xi \in \partial \Omega$, and $\int_{\Omega} (dd^c \varphi)^n < +\infty$.

Definition 1.3. [Ceg98] The class $E^p(\Omega)$ is a set of functions $u$ for which there exists a sequence of functions $u_j \in E_0(\Omega)$ decreasing towards $u$ in all of $\Omega$, and so that $\sup_j \int_{\Omega} (dd^c u_j)^n < +\infty$.

We will need the following maximum principle:

Proposition 1.4. [BT76] Let $u, v$ be locally bounded plurisubharmonic functions in $\Omega$ such that $\lim inf_{z \to \partial \Omega} (u - v) \geq 0$. Then
\[ (dd^c u)^n \leq (dd^c v)^n \implies u \leq v \text{ in } \Omega. \]

1.2. The space of plurisubharmonic potentials. We begin this section by defining the Mabuchi space of plurisubharmonic functions in $\Omega$.

Definition 1.5. The Mabuchi space of plurisubharmonic functions in $\Omega$ is:
\[ H := \{ \varphi \in C^\infty(\overline{\Omega}, \mathbb{R}) / dd^c \varphi > 0 \text{ in } \overline{\Omega}, \varphi = 0 \text{ on } \partial \Omega \} \]

We now consider the tangent space of $H$ in every $C^\infty(\overline{\Omega}, \mathbb{R})$.

Definition 1.6. The tangent space of $H$ at point $\varphi$, we denote by $T_\varphi H$ is the linearisation of $H$ defined by:
\[ T_\varphi H = \{ \gamma'(0) / \varphi : [-\varepsilon, \varepsilon] \to H \text{ and } \gamma(0) = \varphi \}. \]

The tangent space of $H$ at $\varphi$ can be identified with
\[ T_\varphi H \cong \{ \xi \in C^\infty(\overline{\Omega}, \mathbb{R}) / \xi = 0 \text{ on } \partial \Omega \} \]

Indeed. Let $\xi \in \{ \xi \in C^\infty(\overline{\Omega}, \mathbb{R}) / \xi = 0 \text{ on } \partial \Omega \}$, we put $\gamma(s) := \varphi + s\xi$ for $s$ close enough to 0 we have $\gamma_s \in H$, and
\[ \gamma(0) = \varphi \text{ and } \gamma'(0) = \xi \]

this implies that $\xi \in T_\varphi H$ hence $\{ \xi \in C^\infty(\overline{\Omega}, \mathbb{R}) / \xi|_{\partial \Omega} = 0 \} \subset T_\varphi H$.

Conversely, let $\gamma \in H$ which gives $\gamma|_{\partial \Omega} = 0$ for every $t$. In particular $\dot{\gamma}(0)|_{\partial \Omega} = 0$, therefore
\[ \xi = \dot{\gamma}(0) \in \{ \xi \in C^\infty(\overline{\Omega}, \mathbb{R}) / \xi = 0 \text{ on } \partial \Omega \}. \]

Definition 1.7. [Mab87] The Mabuchi metric is the $L^2$ Riemannian metric. It is defined by
\[ << \psi_1, \psi_2 >>_{\varphi} := \int_\Omega \psi_1 \psi_2 (dd^c \varphi)^n, \]
where $\varphi \in H, \psi_1, \psi_2 \in T_\varphi H$. 

1.3. Mabuchi geodesics. Geodesics between two points $\varphi_0, \varphi_1$ in $\mathcal{H}$ are defined as the extremals of the Energy functional

$$\varphi \mapsto H(\varphi) := \frac{1}{2} \int_0^1 \int_{\Omega} (\dot{\varphi}_t)^2 (dd^c \varphi_t)^n.$$

where $\varphi = \varphi_t$ is a path in $\mathcal{H}$ joining $\varphi_0$ to $\varphi_1$. The geodesic equation is obtained by computing the Euler-Lagrange equation of the functional $H$.

**Theorem 1.8.** The geodesic equation is

$$\ddot{\varphi}(t) - |\nabla \varphi(t)|^2_{\varphi(t)} = 0$$

where $\nabla$ is the gradient relative to the metric $\omega_\varphi = dd^c \varphi$.

**Proof.** We need to compute the Euler-Lagrange equation of the Energy functional. Let $(\Phi_{s,t})$ be a variation of $\varphi$ with fixed end points,

$$\phi_{0,t} = \varphi_t, \phi_{s,0} = \varphi_0, \phi_{s,1} = \varphi_1$$

and $\phi_{s,t} = 0$ on $\partial \Omega$

Set $\psi_t := \frac{\partial \phi_t}{\partial t}|_{s=0}$ and observe that $\psi_0 \equiv \psi_1 \equiv 0$ and $\psi_t = 0$ on $\partial \Omega$. Thus

$$\phi_{s,t} = \varphi_t + s\psi_t + \circ(s)$$

and

$$(dd^c \phi_{s,t})^n = (dd^c (\varphi_t + s\psi_t))^n = (dd^c \varphi_t)^n + s.n dd^c \psi_t \wedge (dd^c \varphi_t)^{n-1}.$$ 

A direct computation yields

$$H(\phi_{s,t}) = \frac{1}{2} \int_0^1 \int_{\Omega} (\dot{\phi}_{s,t})^2 (dd^c \phi_{s,t})^n dt$$

$$= H(\varphi_t) + s \int_0^1 \int_{\Omega} \dot{\varphi}_t \psi_t (dd^c \varphi_t)^n dt$$

$$+ \frac{ns}{2} \int_0^1 \int_{\Omega} \dot{\varphi}_t^2 dd^c \psi_t \wedge (dd^c \varphi_t)^{n-1} dt.$$ 

Integration by part, and the fact $\psi_0 \equiv \psi_1 \equiv 0$ yields

$$\int_0^1 \int_{\Omega} \dot{\varphi}_t \psi_t (dd^c \varphi_t)^n dt = - \int_0^1 \int_{\Omega} \psi_t (\dot{\varphi}_t (dd^c \varphi_t)^n + n \dot{\varphi}_t dd^c \psi_t \wedge (dd^c \varphi_t)^{n-1}) dt.$$ 

And we have also by Stokes and the fact $\varphi_t = 0$ on $\partial \Omega$

$$\int_0^1 \int_{\Omega} (\dot{\varphi}_t)^2 dd^c \psi_t \wedge (dd^c \varphi_t)^{n-1} dt = 2 \int_0^1 \int_{\Omega} \psi_t (d\dot{\varphi_t} \wedge (dd^c \varphi_t)^n - d\dot{\varphi_t} \wedge (dd^c \varphi_t)^{n-1}) dt$$

hence

$$H(\phi_{s,t}) = H(\varphi_t) + s \int_0^1 \int_{\Omega} \psi_t \{ - \dot{\varphi}_t (dd^c \varphi_t)^n + nd\dot{\varphi_t} \wedge (dd^c \varphi_t)^n \} dt + \circ(s)$$

which implies

$$0 = d_{\varphi_t}H_\psi$$

$$= \lim_{s \to 0} \frac{H(\phi_{s,t}) - H(\varphi_t)}{s}$$

$$= \int_0^1 \int_{\Omega} \psi_t \{ - \varphi_t (dd^c \varphi_t)^n + nd\dot{\varphi_t} \wedge (dd^c \varphi_t)^n \} dt.$$
Therefore $(\varphi_t)$ is critical point of $H$ if and only if
\[
\dot{\varphi}_t(dd^c\varphi_t)^n = nd\dot{\varphi}_t \wedge d^c\dot{\varphi} \wedge (dd^c\varphi_t)^{n-1}.
\]

\[\square\]

1.4. **Levi-Civita connection.** As for Riemannian manifolds of finite dimension. One can find the local expression of the Levi-Civita connection by polarizing the geodesic equation.

**Definition 1.9.** We define the covariant derivative of the vector field $\psi_t$ along the path $\varphi_t$ in $\mathcal{H}$ by the formula
\[
D\psi := \frac{d\psi}{dt} - \langle \nabla \psi, \nabla \dot{\varphi} \rangle_{\varphi}.
\]

**Theorem 1.10.** $D$ is the Levi-Civita connection.

**Proof.** To show that $D$ is a Levi-Civita connection, we must show that the connection $D$ is metric-compatible and a torsion-free.

i) Metric-compatibility: Let $\psi_1, \psi_2$ be two vector fields
\[
\frac{d}{dt} \langle \psi_1, \psi_2 \rangle_{\varphi} = \frac{d}{dt} \int_\Omega \psi_1 \psi_2 (dd^c\varphi)^n
\]
\[
= \int_\Omega (\psi_1 \psi_2 + \psi_1 \dot{\psi}_2)(dd^c\varphi)^n + n\psi_1 \psi_2 dd^c\dot{\varphi} \wedge (dd^c\varphi)^{n-1}
\]
\[
= \int_\Omega ((\psi_1 \psi_2 - \nabla \psi_1 \nabla \dot{\varphi})\psi_2 (dd^c\varphi)^n
\]
\[
+ \int_\Omega \psi_1 (\dot{\psi}_2 - \nabla \psi_2 \nabla \dot{\varphi})(dd^c\varphi)^n
\]
\[
= \langle \langle D\psi_1, \psi_2 \rangle_{\varphi} + \langle \psi_1, D\psi_2 \rangle_{\varphi}
\]

(The passage from the second line to the third line is a result of the equation
\[
d(\psi_1 \psi_2 dd^c\dot{\varphi} \wedge (dd^c\varphi)^{n-1}) = d(\psi_1 \psi_2) \wedge d^c\dot{\varphi} \wedge (dd^c\varphi)^{n-1} + \psi_1 \psi_2 dd^c\dot{\varphi} \wedge (dd^c\varphi)^{n-1}
\]
and Stokes theorem).

ii) $D$ is a torsion-free, because
\[
D_s \frac{d\varphi}{dt} = D_t \frac{d\varphi}{ds}.
\]

Thus $D$ is a Levi-Civita connection. \[\square\]

1.5. **Curvature tensor.** We will define the curvature tensor and the sectional curvature and we will give those expressions. We will finish by proving that the space of plurisubharmonic functions is locally symmetric. We start by giving some definitions and conventions.

**Definition 1.11.** Let $\psi$ and $\theta$ be two functions in the tangent space of $\mathcal{H}$ at $\varphi$. The Poisson bracket of $\psi$ and $\theta$ compared to the form $\omega_{\varphi} = dd^c\varphi$ is
\[
\{\psi, \theta\}_{\varphi} := i \sum_{\alpha, \beta=1}^{\varphi} \varphi^{\alpha \beta} \left( \frac{\partial \psi}{\partial z_{\beta}} \frac{\partial \theta}{\partial \bar{z}_{\alpha}} - \frac{\partial \psi}{\partial z_{\alpha}} \frac{\partial \theta}{\partial \bar{z}_{\beta}} \right)
\]
where \((\varphi^{a\bar{b}})\) is the inverse matrix of \((\varphi_{a\bar{b}})\).

**Lemma 1.12.** Let \(\psi\), \(\theta\) and \(\eta\) three functions belonging to the tangent space of \(H\) at \(\varphi\). The Poisson bracket satisfies the following properties:

1. \(\{\psi, \theta\} = -\{\theta, \psi\}\)
2. \(\{\psi, \theta\} = \omega_\varphi(X_\psi, X_\theta)\)
3. \(\{\psi, \theta + \eta\} = \{\psi, \theta\} + \{\psi, \eta\}\)
4. \([X_\psi, X_\theta] = X_\psi(X_\theta) - X_\theta(X_\psi) = X_{\{\psi, \theta\}}\)
5. \(\int_H \{\psi, \theta\} \eta(\varphi) n = \int_H \psi \{\theta, \eta\}(\varphi) n\)
6. \(D\{\psi, \theta\} = \{D\psi, \theta\} + \{\psi, D\theta\}\).

where \(X_\psi := i\nabla_\psi\) and \([,\] is the Lie bracket.

Let \(\psi\) be a function in tangent space, the Hessian of \(\psi\) is defined by

\[\text{Hess}_\psi = \nabla_\varphi d\psi\]

where \(\nabla_\varphi\) is the Levi-Civita connection respectively to the form \(\omega_\varphi = d\varphi\). We recall in the next lemma some proprieties of the Hessian well know in the literature.

**Lemma 1.13.** Let \(X\) and \(Y\) be two vector fields. Then the Hessian satisfies the following proprieties:

1. \(\text{Hess}_\psi(X, Y) = \nabla_\varphi (\nabla_\varphi \psi, Y)\)
2. \(\text{Hess}_\psi(X, Y) = X(Y(\psi)) - \nabla_\varphi Y(\psi)\)
3. \(d\varphi(X, iY) = \text{Hess}_\psi(X, Y) + \text{Hess}_\psi(iX, iY)\).

Where \(\nabla_\varphi\) and \(<,>\) are the Levi-Civita connection and the metric respectively associated to the form \(\omega_\varphi = d\varphi\).

In the sequel of this section, we consider a 2-parameters family \(\varphi(t, s) \in \mathcal{H}\) and a vector field \(\psi(t, s) \in T_\varphi \mathcal{H}\) defined along \(\varphi\). We denote by

\[\varphi_t = \frac{d\varphi}{dt}, \quad \varphi_s = \frac{d\varphi}{ds}\]

**Definition 1.14.** The curvature tensor of the Mabuchi metric in \(\mathcal{H}\) is defined by

\[R_\varphi(\varphi_t, \varphi_s)\psi := D_t D_s \psi - D_s D_t \psi\]

where \(\varphi(t, s) \in \mathcal{H}\) is 2-parameters family and vector field \(\psi(t, s) \in T_\varphi \mathcal{H}\).

The sectional curvature is given by

\[K_\varphi(\varphi_t, \varphi_s) := < R_\varphi(\varphi_t, \varphi_s)\varphi_t, \varphi_s >_\varphi\]

**Theorem 1.15.** The curvature tensor of the Mabuchi metric in \(\mathcal{H}\) can be expressed as

\[R_\varphi(\varphi_t, \varphi_s)\psi = -\{\varphi_t, \varphi_s\}, \psi\}\]

The sectional curvature is the following

\[K_\varphi(\varphi_t, \varphi_s) = -\|\{\varphi_t, \varphi_s\}\|_\varphi^2 \leq 0,\]

where \(\{,\}\) is the Poisson bracket associate to the form \(\omega_\varphi = d\varphi\).

**Proof.** To compute the curvature tensor of \(D\), we compute the first term in the definition of the curvature tensor. Indeed, let \(\psi\) be the vector field, its derivative along the path \(\varphi_s\)

\[D_s\psi = \psi_s - < \nabla_\psi, \nabla_\varphi >_\varphi \psi_s = \psi_s + \Gamma_\varphi(\psi, \varphi_s),\]
where\[\Gamma_\varphi(\psi, \varphi_s) = -\nabla_\psi \cdot \nabla \varphi_s > \varphi,\]
we derive the \(D_t D_s \psi\) along the path \(\varphi_t\) as follows
\[
D_t D_s \psi = D_t (\psi_s + \Gamma_\varphi(\psi, \varphi_s))
= \frac{d}{dt}(\psi_s + \Gamma_\varphi(\psi, \varphi_s)) + \Gamma_\varphi(\psi_s + \Gamma_\varphi(\psi, \varphi_s), \varphi_t)
= \psi_{st} + \frac{d}{dt}(\Gamma_\varphi(\psi, \varphi_s)) + \Gamma_\varphi(\psi_s, \varphi_t) + \Gamma_\varphi(\Gamma_\varphi(\psi, \varphi_s), \varphi_t).
\]
We express the second term in RHS of the last equation as:
\[
\frac{d}{dt}(\Gamma_\varphi(\psi, \varphi_s)) = \Gamma_\varphi(\psi_s, \varphi_t) + dd^c \varphi_t(\nabla \varphi_s, i \nabla \psi).
\]
By applying of the three properties of lemma 1.13 by taken \(X = \nabla \varphi_s\) and \(Y = \nabla \psi\), then we express the last term in the last equation as follows:
\[
 \frac{d}{dt}(\nabla \varphi_s, i \nabla \psi) = \operatorname{Hess}(\varphi_t)(\nabla \varphi_s, \nabla \psi) + \operatorname{Hess}(\varphi_t)(i \nabla \varphi_s, i \nabla \psi).
\]
Which gives
\[
\frac{d}{dt} \Gamma_\varphi(\varphi_t, \psi) = \Gamma_\varphi(\psi, \varphi_{ts}) + \Gamma_\varphi(\psi_s, \varphi_t) + \operatorname{Hess}(\varphi_t)(\nabla \varphi_s, \nabla \psi) + \operatorname{Hess}(\varphi_t)(i \nabla \varphi_s, i \nabla \psi).
\]
We develop the fourth term in the RHS in the last equation by applying the second properties of the lemma 1.13, taken \(X = \nabla \varphi_s\) and \(Y = \nabla \psi\):
\[
\operatorname{Hess}(\varphi_t)(\nabla \varphi_s, \nabla \psi) = \nabla \varphi_s(\nabla_\psi(\nabla \varphi_t)) - (\nabla^c_\varphi \nabla \psi)(\varphi_t)
= \nabla \varphi_s(\nabla_\psi(\nabla \varphi_t)) + \nabla \varphi_s(\nabla_\psi(\nabla \varphi_t)) - \nabla \varphi_s(\nabla_\psi(\nabla \varphi_t)).
\]
We have also by applying the first properties of lemma 1.13:
\[
\operatorname{Hess}(\varphi_t)(i \nabla \varphi_s, i \nabla \psi) = \nabla^c_\varphi i \nabla \varphi_s(\nabla_\psi(\nabla \varphi_t)) + \nabla^c_\varphi i \nabla \varphi_s(\nabla_\psi(\nabla \varphi_t)).
\]
Where \(X_h = i \nabla h\). Then we have:
\[
\frac{d}{dt} \Gamma_\varphi(\varphi_s, \psi) = \Gamma_\varphi(\psi, \varphi_{ts}) + \Gamma_\varphi(\psi_s, \varphi_t) + \Gamma_\varphi(\Gamma_\varphi(\varphi_s, \psi), \varphi_t)
- \operatorname{Hess}(\psi)(\nabla \varphi_t, \nabla \varphi_s) + \omega_c(\nabla^c_\varphi X_{\varphi_t}, X_s).
\]
After all previous equations, we get the expression of \(D_t D_s \psi\) as follows:
\[
D_t D_s \psi = \psi_{st} + \Gamma_\varphi(\psi, \varphi_{ts}) + \Gamma_\varphi(\psi_s, \varphi_t) + \Gamma_\varphi(\Gamma_\varphi(\varphi_s, \psi), \varphi_t) - \operatorname{Hess}(\psi)(\nabla \varphi_t, \nabla \varphi_s)
+ \omega_c(\nabla^c_\varphi X_{\varphi_t}, X_s) + \Gamma_\varphi(\psi_t, \varphi_s) + \Gamma_\varphi(\Gamma_\varphi(\psi, \varphi_t), \varphi_s).
\]
We get the expression of \(D_s D_t \psi\) by reversing the roles of \(t\) and \(s\) as follows:
\[
D_s D_t \psi = \psi_{st} + \Gamma_\varphi(\psi, \varphi_{ts}) + \Gamma_\varphi(\psi_s, \varphi_t) + \Gamma_\varphi(\Gamma_\varphi(\varphi_s, \psi), \varphi_t) - \operatorname{Hess}(\psi)(\nabla \varphi_s, \nabla \varphi_t)
+ \omega_c(\nabla^c_\varphi X_{\varphi_t}, X_s) + \Gamma_\varphi(\psi_t, \varphi_s) + \Gamma_\varphi(\Gamma_\varphi(\psi, \varphi_t), \varphi_s).
\]
Therefore we get
\[
R_\varphi(\varphi_t, \varphi_s)\psi = D_t D_s \psi - D_s D_t \psi \\
= \omega_\varphi(\nabla_{X_{\varphi_t}} X_{\varphi_s}, X_{\psi}) - \omega_\varphi(\nabla_{X_{\varphi_s}} X_{\varphi_t}, X_{\psi}) \\
= \omega_\varphi([X_{\varphi_t}, X_{\varphi_s}], X_{\psi}) \\
= \omega_\varphi(\{\varphi_t, \varphi_s\}, X_{\psi}) \\
= -\{\{\varphi_t, \varphi_s\}, \psi\}
\]

In the line three we use the fact that the Levi-Civita connection is torsion free. In the line four we use the fourth property in lemma 1.12, in the last line we use the second property in lemma 1.12. We calculate the sectional curvature as follow:
\[
K_\varphi(\varphi_t, \varphi_s) = \langle \langle R_\varphi(\varphi_t, \varphi_s)\varphi_t, \varphi_s \rangle \rangle_\varphi \\
= \int_\Omega R_\varphi(\varphi_t, \varphi_s)\varphi_t \varphi_s (dd^c \varphi)^n \\
= -\int_\Omega \{\{\varphi_t, \varphi_s\}, \varphi_t\} \varphi_s (dd^c \varphi)^n \\
= -\int_\Omega \{\varphi_t, \varphi_s\} \varphi_t \varphi_s (dd^c \varphi)^n \\
= -||\{\varphi_t, \varphi_s\}||_{dd^c \varphi}^2
\]

We use in line three the expression of the curvature tensor, in the line four we use fifth property in lemma 1.12.

□

**Definition 1.16.** We say a connection \(D\) in \(\mathcal{H}\) is locally symmetric if its curvature tensor is parallel i.e \(DR = 0\).

**Theorem 1.17.** The Mabuchi space \(\mathcal{H}\) provided by the Levi-Civita connection \(D\) is a locally symmetric space.

**Proof.** Let \(\varphi(t, s, r)\) be 3-parameters family in \(\mathcal{H}\).
\[
D_r(R_\varphi(\varphi_t, \varphi_s)\psi) = D_r(-\{\{\varphi_t, \varphi_s\}, \psi\}) \\
= -\{D_r\{\varphi_t, \varphi_s\}, \psi\} - \{\{\varphi_t, \varphi_s\}, D_r\psi\} \\
= -\{\{D_r \varphi_t, \varphi_s\} + \{\varphi_t, D_r \varphi_s\}, \psi\} - \{\{\varphi_t, \varphi_s\}, D_r\psi\} \\
= \{\{D_r \varphi_t, \varphi_s\} - \{\varphi_t, D_r \varphi_s\}, \psi\} - \{\{\varphi_t, \varphi_s\}, D_r\psi\} \\
= R_\varphi(D_r \varphi_t, \varphi_s)\psi + R_\varphi(\varphi_t, D_r \varphi_s)\psi + R_\varphi(\varphi_t, D_r \varphi_s)(D_r \psi)
\]

We use the expression of the curvature tensor and the sixth property in the lemma 1.12 of the Poisson bracket. Therefore
\[
(D_r R_\varphi)(\varphi_t, \varphi_s)\psi = D_r(R_\varphi(\varphi_t, \varphi_s)\psi) - R_\varphi(D_r \varphi_t, \varphi_s)\psi - R_\varphi(\varphi_t, D_r \varphi_s)\psi - R_\varphi(\varphi_t, \varphi_s)(D_r \psi) = 0,
\]
hence \(\mathcal{H}\) is locally symmetric.

□

2. The Dirichlet problem

We now study the regularity of geodesics using pluripotential theory, the tools using are developed by Bedford and Taylor [BT76, BT82].
2.1. Semmes trick. We are interested in the boundary value problem for the geodesic equation: given \( \varphi_0, \varphi_1 \) two distinct points in \( \mathcal{H} \), can one find a path \((\varphi(t))_{0 \leq t \leq 1}\) in \( \mathcal{H} \) which is a solution of (1) with end points \( \varphi(0) = \varphi_0 \) and \( \varphi(1) = \varphi_1 \)? For each path \((\varphi_t)_{t \in [0,1]}\) in \( \mathcal{H} \), we set

\[
\Phi(z, \zeta) = \varphi_t(z) , \ z \in \Omega \text{ and } \zeta = e^{t+i}s \in A = \{ \zeta \in \mathbb{C} : 1 < |\zeta| < e \}
\]

We will show in this section that the geodesic equation in \( \mathcal{H} \) is equivalent to Monge-Ampère equation on \( \Omega \times A \) as in Semmes [Sem92].

Lemma 2.1. The Monge-Ampère measure of the function \( \Phi \) in \( \Omega \times A \) is:

\[
(dd^c_z \Phi(z, \zeta))^{n+1} = (dd^c_z \Phi(z, \zeta))^{n+1} + (n+1)(dd^c_z \Phi(z, \zeta))^n \wedge R
\]

with

\[
R = R(z, \zeta) = d_z d_c^z \Phi + d_\zeta d_c^z \Phi + d_\overline{\zeta} d_c^z \Phi,
\]

Proof. We write \( d_{z, \zeta} \Phi = d_z \Phi + d_\zeta \Phi \) and \( d_{z, \overline{\zeta}} \Phi = d_z \Phi + d_{\overline{\zeta}} \Phi \), and we give also the expression of \( dd^c_{z, \zeta} \Phi(z, \zeta) \) in \( \Omega \times A \). Indeed

\[
\begin{align*}
dd^c_{z, \zeta} \Phi &= (d_z + d_\zeta)(d_z^c \Phi + d_{\overline{\zeta}}^c \Phi) \\
&= d_z d_c^z \Phi + d_\zeta d_c^z \Phi + d_{\overline{\zeta}} d_c^z \Phi \\
&\quad + d_\bar{\zeta} d_c^z \Phi + R(z, \zeta)
\end{align*}
\]

with \( R = d_z d_c^z \Phi + d_\zeta d_c^z \Phi + d_{\overline{\zeta}} d_c^z \Phi \) such that \( R^4 = 0 \). Then we can find the expression of \( (dd^c_{z, \zeta} \Phi)^{n+1} \) in \( \Omega \times A \). Indeed

\[
(dd^c_{z, \zeta} \Phi)^{n+1} = (dd^c_z \Phi + R)^{n+1}
\]

\[
= \sum_{j=0}^{n+1} C_j^{n+1}(dd^c_z \Phi)^j \wedge (R)^{n+1-j}
\]

\[
= (dd^c_z \Phi)^{n+1} + (n+1)(dd^c_z \Phi)^n \wedge R
\]

\[
+ \frac{n(n+1)}{2}(dd^c_z \Phi)^{n-1} \wedge R^2
\]

On the second line we use Leibniz formula and the fact that \( R^3 = R \wedge R \wedge R = 0 \) on the third line.

\[
\square
\]

Theorem 2.2. \((\varphi_t)_{0 \leq t \leq 1}\) is a geodesic if and only if \( (dd^c_{z, \zeta} \Phi(z, \zeta))^{n+1} = 0 \).

Proof. From the previous lemma, we have

\[
(dd^c_{z, \zeta} \Phi(z, \zeta))^{n+1} = (dd^c_z \Phi(z, \zeta))^{n+1} + (n+1)(dd^c_z \Phi(z, \zeta))^n \wedge R
\]

\[
+ \frac{n(n+1)}{2}(dd^c_z \Phi(z, \zeta))^{n-1} \wedge R^2
\]

The first term in RHS of the last equation equal to 0 a cause of bi-degree. We have

\[
d_\zeta \Phi = \partial_\zeta \Phi + \overline{\partial}_\zeta \Phi = \frac{\partial \Phi}{\partial \zeta} d_\zeta + \frac{\partial \Phi}{\partial \overline{\zeta}} d_{\overline{\zeta}} = \varphi_t(z)(d_\zeta + d_{\overline{\zeta}})
\]

and

\[
d_c^z \Phi = i \frac{1}{2}(\overline{\partial} \Phi - \partial \Phi) = i \frac{1}{2} \frac{\partial \Phi}{\partial \zeta} d_\zeta - \frac{\partial \Phi}{\partial \overline{\zeta}} d_{\overline{\zeta}} = i \frac{1}{2} \varphi_t(z)(d_\zeta - d_{\overline{\zeta}})
\]
and we have also $d_{\bar{z}}d_{\bar{z}}^c \Phi = i \bar{\partial} \Phi(z) d\zeta \wedge d\bar{\zeta}$, which gives

$$R = i \bar{\partial} \Phi(z) d\zeta \wedge d\bar{\zeta} + \frac{i}{2} dz \bar{\partial} \Phi(z) + \frac{i}{2} d\bar{z} \partial \Phi(z) + d_{\bar{z}} \bar{\partial} \Phi(z) \wedge d\bar{\zeta}$$

and

$$R^2 = 2id_{\bar{z}} \bar{\partial} \Phi(z) \wedge d_{\bar{z}} \bar{\partial} \Phi(z) \wedge d\zeta \wedge d\bar{\zeta}$$

Now we can explain the second term also. Indeed

$$(dd^c_z \Phi)^n \wedge R = (dd^c_z \Phi(z))^n \wedge (i \bar{\partial} \Phi(z) d\zeta \wedge d\bar{\zeta} + \frac{i}{2} dz \bar{\partial} \Phi(z) + \frac{i}{2} d\bar{z} \partial \Phi(z) + d_{\bar{z}} \bar{\partial} \Phi(z) \wedge d\bar{\zeta})$$

And also for third term, we have

$$(dd^c_z \Phi)^{n-1} \wedge R^2 = (dd^c_z \Phi(z))^{n-1} \wedge R \wedge R = (dd^c_z \Phi(z))^{n-1} \wedge 2id_{\bar{z}} \bar{\partial} \Phi(z) \wedge d\zeta \wedge d\bar{\zeta} = -2id_{\bar{z}} \bar{\partial} \Phi(z) \wedge (dd^c_z \Phi(z))^{n-1} \wedge d\zeta \wedge d\bar{\zeta}$$

After the previous equations we have,

$$(dd^c_{z,\zeta} \Phi)^{n+1} = (n+1)(dd^c_{z,\zeta} \Phi(z,\zeta))^n \wedge R + \frac{n(n+1)}{2} (dd^c_{z,\zeta} \Phi(z,\zeta))^{n-1} \wedge R^2$$

$$= i(n+1)(\bar{\partial} (dd^c_{z,\zeta} \Phi)^n - nd_{\bar{z}} \bar{\partial} \Phi(z) \wedge d_{\bar{z}} \partial \Phi(z) \wedge (dd^c \Phi(z))^n-1 \wedge d\zeta \wedge d\bar{\zeta})$$

$$= i(n+1) \left( \Phi_1 - \frac{nd_{\bar{z}} \bar{\partial} \Phi(z) \wedge d_{\bar{z}} \partial \Phi(z) \wedge (dd^c \Phi(z))^{n-1}}{(dd^c \Phi(z))^n} \right) (dd^c \Phi(z))^n \wedge d\zeta \wedge d\bar{\zeta}$$

From the fact that $nd(\Phi_1) \wedge d^c(\Phi_1) \wedge (dd^c \Phi(z))^{n-1} = \bar{\partial} \Phi_1 (dd^c \Phi(z))^n$, we infer that $\Phi_1$ is geodesic if and only if

$$(dd^c_{z,\zeta} \Phi(z,\zeta))^{n+1} = 0.$$

$$\Box$$

After the previous theorem we deduce that the geodesic problem in Mabuchi space is equivalent to the following Dirichlet problem:

$$\begin{cases}
(dd^c_{z,\zeta} \Phi(z,\zeta))^{n+1} = 0 & \Omega \times A \\
\Phi(z,\zeta) = \varphi_0(z) & \Omega \times \{|\zeta| = 1\} \\
\Phi(z,\zeta) = \varphi_1(z) & \Omega \times \{|\zeta| = e\} \quad (3) \\
\Phi(z,\zeta) = 0 & \partial \Omega \times A
\end{cases}$$

2.2. Continuous envelopes. We have that $\varphi_0$ and $\varphi_1$ are smooth, in the sequel we can assume that $\varphi_0$ and $\varphi_1$ are only $C^{1,1}$.

Definition 2.3. The Perron-Bremermann envelope is defined by

$$\Phi(z,\zeta) = \sup\{u(z,\zeta) \in \mathcal{F}(\Psi,\Omega \times A)\}$$

with

$$\mathcal{F}(\Psi,\Omega \times A) = \{u \in PSH(\Omega \times A) \cap C^0(\bar{\Omega} \times \bar{A}) \mid u^* \leq \Psi \text{ on } \partial(\Omega \times A)\}$$

Where $\Psi|_{\partial \Omega \times \bar{A}} = 0$ and $\Psi_{\partial A \times \Omega} = \begin{cases}
\varphi_0(z) & \{|\zeta| = 1\}; \\
\varphi_1(z) & \{|\zeta| = e\}
\end{cases}$. 

Theorem 2.4. If $\Psi \in C^0(\partial(\Omega \times A))$. Then the Perron-Bremermann envelope $\Phi$ satisfies the following conditions:

i) $\Phi \in PSH(\Omega \times A) \cap C^0(\Omega \times A)$.

ii) $\Phi|_{\partial(\Omega \times A)} = \Psi$.

iii) $(dd^c_{z,\zeta} \Phi(z,\zeta))^{n+1} = 0$ in $\Omega \times A$.

Proof. Let $\rho$ be a strictly plurisubharmonic defining of $\Omega = \{\rho < 0\}$. Observe that the family $F(\Psi, \Omega \times A)$ is not empty.

i) We start by proving the plurisubharmonicity of $\Phi$ in $\Omega \times A$. We can write the Dirichlet problem on the following way:

$$
\begin{cases}
(dd^c_{z,\zeta} \Phi(z,\zeta))^{n+1} = 0 & \Omega \times A \\
\Phi(z,\zeta) = \Psi(z,\zeta) & \partial(\Omega \times A)
\end{cases}
$$

with $\Psi(z,\zeta) = \frac{1}{e} \left(\varphi_1(\zeta)(|\zeta|^2 - 1) - \varphi_0(z)(|\zeta|^2 - e^2)\right)$. Let $h \in Har(\Omega \times A) \cap C^0(\Omega \times A)$ be a harmonic function in $\Omega \times A$, continuous up to the boundary of $\Omega \times A$, the solution of the following Dirichlet problem

$$
\begin{cases}
\Delta_{z,\zeta} h(z,\zeta) = 0 & \Omega \times A \\
h = \Psi, & \partial(\Omega \times A)
\end{cases}
$$

Exists, since $\Omega \times A$ is a regular domain.

For all $v \in F(\Psi, \Omega \times A)$, we have $v^* \leq \Psi$ on $\partial(\Omega \times A)$, which implies

$$(v - h)^* \leq 0 \text{ on } \partial(\Omega \times A)$$

Furthermore we have

$$\Delta_{z,\zeta} (v - h)(z,\zeta) = \Delta_{z,\zeta} v(z,\zeta) \geq 0 \text{ in } \Omega \times A$$

Then by maximum principle

$$v(z,\zeta) \leq h(z,\zeta) \text{ in } \Omega \times A$$

the last inequality holds for every function in $F(\Psi, \Omega \times A)$, hence it holds for upper envelope of subsolution

$$\Phi(z,\zeta) \leq h(z,\zeta) \text{ in } \Omega \times A$$

It also holds for its upper semi-continuous regularization on the boundary $(\Omega \times A)$, we get

$$\Phi(z,\zeta))^* \leq \Psi(z,\zeta) \text{ on } \partial(\Omega \times A)$$

and consequently

$$\Phi^* \in F(\Psi, \Omega \times A)$$

Since the function $\Phi^*$ is plurisubharmonic in $\Omega \times A$ and

$$\Phi(z,\zeta) \leq \Phi(z,\zeta))^* \text{ in } \Omega \times A$$

we infer that

$$\Phi(z,\zeta))^* = \Phi(z,\zeta) \text{ in } \Omega \times A.$$
Fix $a = (a_1, a_2) \in \mathbb{C}^n \times \mathbb{C}$ with $|a| \leq \beta$. So, we have the following inequality
\[ \Phi(\xi + a_1, \eta + a_2) \leq \Psi(\xi, \eta) + \varepsilon \text{ if } (\xi, \eta) \in (\Omega \times A \setminus \{a\}) \cup \partial(\Omega \times A) \]
and
\[ \Phi^*(z + a_1, \zeta + a_2) \leq \Psi(z + \alpha, \zeta + a_2) + \varepsilon \leq \Phi(z, \zeta) + \varepsilon \text{ if } \Omega \times A \cap \partial((\Omega \times A) \setminus \{a\}) \]
It follows that the function
\[ W(z, \zeta) = \begin{cases} \max(\Phi(z, \zeta), \Phi(z + a_1, \zeta + a_2) - 2\varepsilon & (z, \zeta) \in (\Omega \times A) \cap (\Omega \times A) \cap \{a\}; \\ \Phi(z, \zeta) & (z, \zeta) \in (\Omega \times A) \setminus (\Omega \times A) \setminus \{a\}. \end{cases} \]
is plurisubharmonic in $\Omega \times A$ because
1) If $(z, \zeta) \in (\Omega \times A) \cap (\Omega \times A) \setminus \{a\}$ it coincides with $\Phi$ which is plurisubharmonic.
2) If $(z, \zeta) \in (\Omega \times A) \setminus (\Omega \times A) \setminus \{a\}$, it is the maximum of two plurisubharmonic functions.
3) After the two previous inequalities, we infer that the function $W$ coincides on the boundary, furthermore
\[ W \leq \Psi \text{ on } \partial(\Omega \times A) \]
Which implies $W \in \mathcal{F}(\Omega \times A, \Psi)$, finally we get
\[ \Phi(z + a_1, \zeta + a_2) - 2\varepsilon \leq \Phi(z, \zeta) \text{ for } (z, \zeta) \in \Omega \times A \text{ and } a \in \mathbb{C}^{n+1}, |a| \leq \beta \]
Thus $\Phi$ is lower semi-continuous, therefore it is continuous.
i) We are going to prove that
\[ \lim_{\Omega \times A \ni (z, \zeta) \to (\xi_0, \eta_0) \in \partial(\Omega \times A)} \Phi(z, \zeta) = \Psi(\xi_0, \eta_0) \]
Firstly, since $\Phi \in \mathcal{F}(\Psi, \Omega \times A)$ we have
\[ \limsup_{(z, \zeta) \to (\xi_0, \eta_0)} \Phi(z, \zeta) \leq \Psi(\xi_0, \eta_0) \forall (\xi_0, \eta_0) \in \partial(\Omega \times A) \]
To prove the reverse of inequality, we construct a plurisubharmonic barrier function at each point $(\xi_0, \eta_0) = \gamma_0 \in \partial(\Omega \times A)$. Since $\rho$ is strictly plurisubharmonic function, we can choose $B$ large enough so that the function
\[ b(z, \xi) := B\rho(z) - |z - \xi|^2 - |\zeta - \eta_0|^2 \]
is plurisubharmonic in $\Omega \times A$ and continuous up to the boundary such that $b(\xi_0, \eta_0) \leq 0$ with $b < 0$ for all $(z, \zeta) \in \Omega \times A \setminus \gamma_0$. Fix $\varepsilon > 0$ and take $\eta > 0$ such that $\Psi(\gamma_0) - \varepsilon \leq \Psi(\gamma) \forall \gamma \in \partial(\Omega \times A)$ and $|\gamma - \gamma_0| \leq \eta$. We choose a big constant $C$ so that
\[ Cb + \Psi(\gamma_0) - \varepsilon \leq \Psi \text{ on } \partial(\Omega \times A) \]
This implies that the function $V(z, \zeta) = Cb(z, \zeta) + \Psi(\gamma_0) - \varepsilon \in PSH(\Omega \times A)$ is
\[ V \leq \Psi \text{ on } \partial(\Omega \times A) \]
Thus we have $V \in \mathcal{F}(\Psi, \Omega \times A)$ which implies $V(z, \zeta) \leq \Phi(z, \zeta)$ in $\Omega \times A$. We get
\[ \Psi(\xi_0, \eta_0) - \varepsilon \leq \liminf_{(z, \zeta) \to (\xi_0, \eta_0)} \Phi(z, \zeta) \]
therefore
\[ \lim_{(z, \zeta) \to (\xi_0, \eta_0)} \Phi(z, \zeta) = \Psi(\xi_0, \eta_0) \forall (\xi_0, \eta_0) \in \partial(\Omega \times A) \]
iii) The Perron Bremermann envelope

\[ \Phi(z, \zeta) = \sup \{ u(z, \zeta) \in \mathcal{F}(\Omega \times A, \Psi) \} \]

is plurisubharmonic continuous up the boundary of \( \Omega \times A \) and \( \Phi|_{\partial(\Omega \times A)} = \Psi \).

By a lemma due to Choquet, this envelope can be realised by a countable family

\[ \Phi(z, \zeta) = \sup \{ u(z, \zeta) \in \mathcal{F}(\Omega \times A, \Psi) \} = \sup \{ u_j(z, \zeta) \in \mathcal{F}(\Omega \times A, \Psi) \} \]

We put

\[ \Phi_j(z, \zeta) = \max(u_1(z, \zeta), u_2(z, \zeta), \ldots, u_j(z, \zeta)) \]

the function \( \Phi_j \) is increasing and

\[ (\Phi(z, \zeta))^* = (\sup_j \{ \Phi_j(z, \zeta) \})^* \]

Let \( B \subset \subset \Omega \times A \) be any ball, we consider the following Dirichlet problem

\[ \begin{cases} 
(\dd^c u_j(z, \zeta))^{n+1} = 0, & B; \\
\partial^B. 
\end{cases} \]

since

\[ (\dd^c z, \zeta u_j(z, \zeta))^{n+1} \leq (\dd^c z, \zeta \Phi_j(z, \zeta))^{n+1} \]

and

\[ u_j = \Phi_j \text{ on } \partial B \]

we have

\[ \Phi_j(z, \zeta) \leq u_j(z, \zeta) \text{ in } B. \]

We consider the following function

\[ \Theta(z, \zeta) = \begin{cases} 
u_j(z, \zeta), & (z, \zeta) \in B; \\
\Phi_j(z, \zeta), & (z, \zeta) \in \Omega \times A \setminus B. 
\end{cases} \]

The function \( \Theta_j \) belongs to \( \mathcal{F}(\Omega \times A, \Psi) \}. \) This implies

\[ \Theta_j(z, \zeta) \leq \Phi_j(z, \zeta) \text{ in } \Omega \times A \]

furthermore

\[ \Theta_j = \Phi_j = \Psi \text{ on } \partial(\Omega \times A) \]

then

\[ u_j(z, \zeta) = \Phi_j(z, \zeta) \text{ in } B \]

therefore

\[ (\dd^c z, \zeta (\Phi_j(z, \zeta)))^{n+1} = (\dd^c z, \zeta (u_j(z, \zeta)))^{n+1} = 0 \text{ in } B \]

since \( B \) is arbitrary we give

\[ (\dd^c z, \zeta \Phi_j(z, \zeta))^{n+1} = 0 \text{ in } \Omega \times A \]

By the continuity property of Monge-Amèpre operators of Bedford and Taylor along monotone sequences, we have

\[ (\dd^c z, \zeta (\Phi_j(z, \zeta)))^{n+1} \rightarrow (\dd^c z, \zeta (\Phi(z, \zeta)))^{n+1} = 0 \]

i.e

\[ (\dd^c z, \zeta (\Phi(z, \zeta)))^{n+1} = 0 \text{ in } \Omega \times A. \]
2.3. Lipschitz regularity. In this subsection we will give the geodesic regularity Lipschitz in time and in space. We begin by regularity Lipschitz in time. We use a barrier argument as noted by Berndtsson [Ber15]

**Proposition 2.5.** The Perron Bremermann envelope \( \Phi(z, \zeta) = \sup \{ u(z, \zeta)/u \in \mathcal{F}(\Omega \times A, \Psi) \} \) is Lipschitz function with respect to \( t = \log |\zeta| \).

**Proof.** The proof follows from a classical balayage technique. Indeed, we consider the following function

\[
\chi(z, \zeta) = \max(\varphi_0(z) - A \log |\zeta|, \varphi_1(z) + A(\log |\zeta| - 1))
\]

where \( A > 0 \) is a big constant. Furthermore

\[
\begin{align*}
\chi(z, \zeta)|_{\Omega \times \{|\zeta| = 1\}} &= \max(\varphi_0(z), \varphi_1(z) - A) = \varphi_0(z) \\
\chi(z, \zeta)|_{\Omega \times \{|\zeta| = e\}} &= \max(\varphi_0(z) - A, \varphi_1(z)) = \varphi_1(z) \\
\chi(z, \zeta)|_{\partial \Omega \times A} &= \max(-A \log |\zeta|, A(\log |\zeta| - 1) - 1) \leq 0
\end{align*}
\]

The last line follows by \( \varphi_0 = \varphi_1 = 0 \) on \( \partial \Omega \) and \( 1 < |\zeta| < e \). Then \( \chi \) it belongs to \( \mathcal{F}(\Omega \times A, \Psi) \) and

\[
\chi(z, \zeta) \leq \Phi(z, \zeta) \text{ in } \Omega \times A
\]

since \( \Phi(z, \zeta) = \varphi(z, \log |\zeta|) \) and \( \chi(z, \zeta) = \chi(z, \log |\zeta|) \), which implies

\[
\frac{\varphi(z, \log |\zeta|) - \varphi(z, 1)}{\log |\zeta|} \geq \frac{\chi(z, \zeta) - \varphi(z, 1)}{\log |\zeta|} = \frac{\chi(z, \zeta) - \chi(z, 1)}{\log |\zeta|}
\]

\[
\lim_{|\zeta| \to 1} \frac{\chi(z, \zeta) - \chi(z, 1)}{\log |\zeta|} = \lim_{|\zeta| \to 1} \frac{\varphi_0(z) - A \log(|\zeta|) - \varphi_0(z)}{\log |\zeta|} = -A
\]

which gives \( \dot{\varphi}(z, 0) \geq -A \), similarly for other case \( \dot{\varphi}(z, 1) \leq A \). Since the function \( \varphi_t \) is convex along \( t \) (by subharmonicity in \( \zeta \)), we infer that for almost everywhere \( z, t \),

\[
-A \leq \dot{\varphi}(z, 0) \leq \dot{\varphi}(z, t) \leq \dot{\varphi}(z, 1) \leq A
\]

then \( \varphi_t \) is uniformly Lipschitz at \( t = \log |\zeta| \).

We will prove the regularity Lipschitz in space by adapting the method of Bedford and Taylor [BT76] (see also [GZ17]).

**Theorem 2.6.** The Perron Bremermann envelope \( \Phi(z, \zeta) = \sup \{ u(z, \zeta)/u \in \mathcal{F}(\Omega \times A, \Psi) \} \) is Lipschitz function up to the boundary with respectively to space variable.

**Proof.** Let \( \rho \) be a smooth defining of \( \Omega \) which is strictly psh in a neighbourhood \( \Omega' \) of \( \Omega \), and also \( \alpha \) be a smooth defining of \( A \) which is strictly psh in a neighbourhood \( A' \) of \( A \). We will construct an extension of function defined on the boundary of \( \Omega \times A \) by

\[
\Psi(z, \zeta) = \left\{ \begin{array}{ll}
\varphi_0(z) & \Omega \times \{|\zeta| = 1\} \\
\varphi_1(z) & \Omega \times \{|\zeta| = e\} \\
0 & \partial \Omega \times A
\end{array} \right.
\]

Let \( \chi \) be a smooth function with compact support defined in \([0, 1]\) by \( \chi(t) = 1 \) near of \( 0 \) and by \( \chi(t) = 0 \) near of \( 1 \). We put

\[
\tilde{\chi}(\zeta) = \chi(\log |\zeta|)
\]
is a smooth function in $\bar{A}$. We have $\tilde{\chi}(\zeta) = 1$ near of $|\zeta| = 1$ and $\tilde{\chi}(\zeta) = 0$ near of $|\zeta| = \epsilon$.

We consider the following function:

$$F(z, \zeta) = \tilde{\chi}(\zeta)\bar{\varphi}(z, \zeta) + (1 - \tilde{\chi}(\zeta))\bar{\varphi}_1(z, \zeta) + B\alpha(\zeta),$$

where $\bar{\varphi}_0(z, \zeta) = \varphi_0(z)$, $\bar{\varphi}_1(z, \zeta) = \varphi_1(z)$. The function $F$ satisfies

$$F|_{\Omega \times \partial A} = \begin{cases} \varphi_0(z), & \Omega \times \{|\zeta| = 1\} \\ \varphi_1(z), & \Omega \times \{|\zeta| = \epsilon\} \\ 0, & \partial\Omega \times A \end{cases}$$

The function $F$ is extension plurisubharmonic of the function $\Psi$ defined in $\Omega \times \partial A$ to $\Omega \times A$. We can also extend the function $\Psi$ defined in $\partial\Omega \times A$ by putting

$$F(z, \zeta) = D\rho(z),$$

where $D$ is a big constant.

On two cases the function $F$ satisfies the following properties

$$F \leq \Phi \text{ on } \partial(\Omega \times A) \text{ and } (dd^c_{\bar{\zeta}}F)^{n+1} \geq (dd^c_{\bar{\zeta}}\Phi)^{n+1} \text{ in } \Omega \times A$$

By maximum Principle we get

$$F(z, \zeta) \leq \Phi(z, \zeta) \text{ in } \Omega \times A$$

Applying the same process to the boundary data $-\Psi$ we choose $C^{1,1}$ function defined in $\Omega \times A$ such that $G = -\Psi$ on $\partial(\Omega \times A)$, the maximum Principle implies

$$\Phi(z, \zeta) \leq -G(z, \zeta) \text{ in } \Omega \times A$$

After the two previous inequalities we have

$$F(z, \zeta) \leq \Phi(z, \zeta) \leq -G(z, \zeta) \text{ in } \Omega \times A$$

Since $F(\cdot, \zeta) \leq \Phi(\cdot, \zeta)$ in $\Omega$, the envelope $\Phi(\cdot, \zeta)$ can be extended respectively to variable $z$ as a plurisubharmonic function in $\Omega'$ by setting $\Phi(\cdot, \zeta) = F(\cdot, \zeta)$ in $\Omega' \setminus \Omega$ with $\zeta$ fixed in $A$. Fix $\delta > 0$ so small that $z + h \in \Omega$ whenever $z \in \bar{\Omega}$ and $||h|| < \delta$, this set noted in sequel by $\Omega_h$. Fix $h \in \mathbb{C}^n$ such that $||h|| < \delta$. Recall that $F$ and $G$ are Lipschitz in each variable, thus

$$|F(z + h, \zeta) - F(z, \zeta)| \leq C||h|| \text{ and } |G(z + h, \zeta) - G(z, \zeta)| \leq C||h||$$

for any $z \in \Omega$ and $\zeta \in \bar{A}$.

Observe that the function $v(z, \zeta) := \Phi(z + h, \zeta) - C||h||$ is well defined psh in the open set $\Omega \times A$. If $z \in \partial\Omega \cap \Omega_h$ and $\zeta \in A$, then

$$v(z, \zeta) = \Phi(z + h, \zeta) - C||h|| \leq -G(z + h, \zeta) - C||h|| \leq -G(z, \zeta) = \Psi(z, \zeta).$$

If $z \in \Omega \cap \partial\Omega_h$ and $\zeta \in A$, then

$$v(z, \zeta) = \Phi(z + h, \zeta) - C||h|| \leq F(z + h, \zeta) - C||h|| \leq F(z, \zeta) \leq \Phi(z, \zeta).$$

This shows that the function $w$ defined by

$$w(z, \zeta) := \begin{cases} \max(v(z, \zeta), \Phi(z, \zeta)) & \text{if } (z, \zeta) \in \Omega \cap \Omega_h \times A \\ \Phi(z, \zeta) & \text{if } (z, \zeta) \in \Omega \setminus \Omega_h \times A \end{cases}$$

is plurisubharmonic in $\Omega \times A$. Since $w \leq \Psi$ on $\partial(\Omega \times A)$ we get $w \leq \Phi$ in $\Omega \times A$. We have shown that

$$\Phi(z + h, \zeta) - \Phi(z, \zeta) \leq C||h||$$
which proves that $\Phi(\cdot, \zeta)$ is Lipschitz in every $z \in \Omega$. \hfill \Box

### 3. Case of the unit ball

In this section we shall show how to use the proof of Bedford and Taylor [BT76], which is simplified by Demailly [Dem93] in the unit ball for giving the regularity in space variable for our geodesics problem. We need some preparation for prove this regularity. The open subset $\mathbb{B} := \{ z \in \mathbb{C}^n / |z_1|^2 + |z_2|^2 + \ldots + |z_n|^2 < 1 \}$ of $\mathbb{C}^n$ is called the unit ball. First we shall define the Mobius transformation of the unit ball. Let $a \in \mathbb{B} \setminus \{0\} \subset \mathbb{C}^n$. Denote the orthogonal projection onto the subspace of in $\mathbb{C}^n$ generated by the vector $a$ by,

$$P_a(z) := \frac{<z, a>}{||a||^2}.$$ 

The Mobius transformation associated with $a$ is the mapping

$$T_a(z) := \frac{P_a(z) - a + \sqrt{1 - ||a||^2}(z - P_a(z))}{1 - <z, a>}.$$ 

With $<z, a> = \sum_{i=1}^n z_i \bar{a}_i$ denote the hermitian scalar product of $z$ and $a$. For every $a \in B$, the Mobius transformation has the following properties

i) $T_a(0) = -a$ and $T_a(a) = 0$.

ii) An elementary computation yields

\begin{equation}
T_a(z) = z - a + <z, a> a + O(||a||^2) = z - h + O(||a||^2),
\end{equation}

with $h = h(a, z) := a - <z, a> z$ and $O(||a||^2)$ is uniformly with respect of $z \in \mathbb{B}$. We need in the sequel the following useful lemma for giving the regularity in unit ball.

**Lemma 3.1.** Let $u$ be a plurisubharmonic function in domain $\Omega \subset \subset \mathbb{C}^n$, assume that there exists $B, \delta > 0$ such that

$$u(z + h) + u(z - h) - 2u(z) \leq B||h||^2, \quad \forall 0 < ||h|| < \delta$$

and for all $z \in \Omega$ and $\text{dist}(z, \partial \Omega) > \delta$. Then $u$ is $C^{1,1}$-smooth, ant its second derivative, which exists almost everywhere, satisfies

$$||D^2u||_{L_{\infty}(\Omega)} \leq B.$$ 

**Proof.** Let $u_\varepsilon = u * \chi_\varepsilon$ denote the standard regularization of $u$ defined in $\Omega_\varepsilon = \{ z \in \Omega / \text{dist}(\partial \Omega, z) > \varepsilon \}$ for $0 < \varepsilon << 1$. Fix $\delta > 0$ small enough and $0 < \varepsilon < \frac{\delta}{2}$. Then for $0 < ||h|| < \frac{\delta}{2}$ we have

\begin{equation}
(3) \quad u_\varepsilon(z + h) + u_\varepsilon(z - h) - 2u_\varepsilon(z) \leq B||h||^2
\end{equation}

It follows from Taylors formula that if $z \in \Omega_\varepsilon$,

$$\frac{d^2}{dt^2} u_\varepsilon(z + th)|_{t=0} := \lim_{t \to 0^+} \frac{u_\varepsilon(z - th) + u_\varepsilon(z + th) - 2u_\varepsilon(z)}{t^2}$$
therefore by having $D^2u_\varepsilon(z).h^2 \leq B||h||^2$ for all $z \in \Omega_\varepsilon$ and $h \in \mathbb{C}^n$. Now for $z \in \Omega_\varepsilon$

$$D^2u_\varepsilon(z).h^2 = \sum_{i,j=1}^{n} \left( \frac{\partial^2 u_\varepsilon}{\partial z_i \partial \bar{z}_j} h_i h_j + 2 \frac{\partial^2 u_\varepsilon}{\partial z_i \partial z_j} h_i \bar{h}_j + \frac{\partial^2 u_\varepsilon}{\partial z_i \partial \bar{z}_j} \bar{h}_i \bar{h}_j \right)$$

Recall that $u_\varepsilon$ is plurisubharmonic in $\Omega_\varepsilon$ hence

$$D^2u_\varepsilon(z).h^2 + D^2u_\varepsilon(z).(ih)^2 = 4 \sum_{i,j=1}^{n} \frac{\partial^2 u_\varepsilon}{\partial z_i \partial \bar{z}_j} h_i \bar{h}_j \geq 0.$$ 

The above upper-bound also yields a lower-bound of $D^2u_\varepsilon$

$$D^2u_\varepsilon.h^2 \geq -D^2u_\varepsilon.(ih)^2 \geq -B||h||^2$$

For any $z \in \Omega$ and $h \in \mathbb{C}^n$. This implies that

$$||D^2u_\varepsilon||_{L^\infty(\Omega)} \leq B$$

Thus, we have shown that $Du_\varepsilon$ is uniformly Lipschitz in $\Omega_\varepsilon$. We infer that $Du$ is Lipschitz in $\Omega$ and $Du_\varepsilon \rightarrow Du$ uniformly in compact subsets of $\Omega$. Since the dual of $L^1$ is $L^\infty$, it follows from the Alaoglu-Banach theorem that, up to extracting a subsequence, there exists a bounded function $V$ such that

$$D^2u_\varepsilon \rightarrow V$$

hence $V = D^2u$. Therefore $u$ is $C^{1,1}$ in $\Omega$ and its second-order derivative exists almost everywhere with $||D^2u||_{L^\infty(\Omega)} \leq B$. 

\[\square\]

**Theorem 3.2.** Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$. Let $\varphi_0$ and $\varphi_1$ be the end geodesic points which are $C^{1,1}$. Then the Perron-Bremermann envelope

$$\Phi(z, \zeta) = \sup \{ u(z, \zeta) / u \in \mathcal{F}(\Omega \times A, \Psi) \}$$

admits second-order partial derivates almost everywhere with respect to variable $z \in \mathbb{B}$ which locally bounded uniformly with respect to $\zeta \in A$, i.e for any compact subset $K \subset \mathbb{B}$ there exists $C$ which depend on $K, \varphi_0$ and $\varphi_1$ such that

$$||D^2\Phi||_{L^\infty(K \times A)} \leq C.$$

**Proof.** For proving the theorem, we need to prove the following inequality

$$\Phi(z + h, \zeta) + \Phi(z - h, \zeta) - 2\Phi(z, \zeta) \leq A||h||^2,$$

for any $||h|| << 1$, $z \in \mathbb{B}$ and $\zeta \in A$.

The idea is to study the boundary behavior of the plurisubharmonic function $(z, \zeta) \rightarrow \frac{1}{2}(\Phi(z + h, \zeta) + \Phi(z - h, \zeta))$ in order to compare it with the function $\Phi$ in $\mathbb{B} \times A$. This does not make sense since the translations do not preserve the boundary. We are instead going to move point $z$ by automorphisms of the unit ball: the group of holomorphic automorphisms of the latter acts transitively on it and this is the main reason why we prove this result for the unit ball rather than for a general strictly pseudoconvex domain (which has generically few such automorphisms).

By the fact $\Phi$ is Lipschitz with respectively to $z$ variable (theorem 2.6) and expansion (2) we have

$$|\Phi(T_a(z), \zeta) - \Phi(z - h, \zeta)| \leq C||T_a(z) - (z - h)|| \leq C||a||^2$$
and
\[ |\Phi(T_{-a}(z), \zeta) - \Phi(z + h, \zeta)| \leq C||T_{-a}(z) - (z + h)|| \leq C||a||^2 \]
which implies
\[ \Phi(z + h, \zeta) + \Phi(z - h, \zeta) \leq \Phi(T_a(z), \zeta) + \Phi(T_{-a}(z), \zeta) + 2C||a||^2. \]
We consider the following functions:
\[ F(z, \zeta) := \Phi(T_a(z), \zeta) + \Phi(T_{-a}(z), \zeta) + 2C||a||^2, \]
and \[ G(z, \zeta) = 2\Phi(z, \zeta) + D||a||^2, \]
we observe that the functions \( F \) and \( G \) are well defined in \( \mathbb{B} \times A \) and plurisubharmonic in \( \mathbb{B} \times A \). We need to show that
\[ F(z, \zeta) \leq G(z, \zeta) \quad \text{in} \quad \mathbb{B} \times A. \]
For showing the last inequality we will apply the maximum Principle, then we need to prove
\[ F(z, \zeta) \leq G(z, \zeta) \quad \text{on} \quad \partial(\mathbb{B} \times A) \]
and
\[ (dd^c z, \zeta) F(z, \zeta))^{n+1} \geq (dd^c z, \zeta) G(z, \zeta))^{n+1} \quad \text{in} \quad \mathbb{B} \times A \]
The last inequality is easy follows from the fact that \( F \) is a plurisubharmonic and \( (dd^c z, \zeta) \Phi)^{n+1} = 0 \) in \( \mathbb{B} \times A \) by (Theorem 2.4).
We need to compare \( F \) and \( G \) in the boundary of \( \mathbb{B} \times A \). Indeed, since \( \partial(\mathbb{B} \times A) = (\partial \mathbb{B} \times A) \cup (\mathbb{B} \times \partial A) \), then we will compare in two parts, we begin by the part \( \partial \mathbb{B} \times A \), in this part we get
\[ F|_{\partial \mathbb{B} \times A} = 2C||a||^2 \quad \text{and} \quad G|_{\partial \mathbb{B} \times A} = D||a||^2. \]
For shows that \( F|_{\partial \mathbb{B} \times A} \leq G|_{\partial \mathbb{B} \times A} \), we take just \( D = 2C \).
For the second part \( \mathbb{B} \times \partial A \), We compare in \( \mathbb{B} \times A \) only, because \( \partial \mathbb{B} \times A \) belongs to the previous part, since \( \partial A = \{|\zeta| = 1\} \cup \{|\zeta| = e\} \), we begin this part by comparing in case \( \mathbb{B} \times \{|\zeta| = e\} \), we have
\[ F|_{\mathbb{B} \times \{|\zeta| = 1\}} = \varphi_0(T_a(z)) + \varphi_0(T_{-a}(z)) + 2C||a||^2 \]
and
\[ G|_{\mathbb{B} \times \{|\zeta| = 1\}} = 2\varphi_0(z) + D||a||^2 \]
We apply Taylor expansion and we get
\[ \varphi_0(T_a(z)) = \varphi_0(z - h + O(|a|^2) = \varphi_0(z) - d\varphi(z).h + O(|a|^2) \]
and
\[ \varphi_0(T_{-a}(z)) = \varphi_0(z + h + O(|a|^2) = \varphi_0(z) + d\varphi(z).h + O(|a|^2) \]
Which is implies
\[ \varphi_0(T_a(z)) + \varphi_0(T_{-a}(z)) \leq 2\varphi_0(z) + 2C_0|a|^2 \]
where \( C_0 \) depend only on the \( \varphi_0 \) then
\[ F(z, \zeta) \leq 2\varphi_0(z) + 2C_1|a|^2 + 2C|a|^2 \]
If we take \( D = 2(C_0 + C) \), we get \( F(z, \zeta) \leq G(z, \zeta) \) on \( \mathbb{B} \times \{|\zeta| = 1\} \). By same methods we get \( F(z, \zeta) \leq G(z, \zeta) \) on \( \mathbb{B} \times \{|\zeta| = 1\} \) for \( D = 2(C_1 + C) \),
where $C_1$ depend only on the $\varphi_1$ which concludes the second part. By part one and two we infer
\[ F(z, \zeta) \leq G(z, \zeta) \quad \text{in} \quad \partial(\mathbb{B} \times A) \]
From the maximum Principle we get
\[ F(z, \zeta) \leq G(z, \zeta) \quad \text{in} \quad \mathbb{B} \times A \]
Which is implies
\[ \Phi(z + h, \zeta) + \Phi(z - h, \zeta) - 2\Phi(z, \zeta) \leq \Phi(T_a(z), \zeta) + \Phi(T_{-a}(z), \zeta) + 2C||a||^2 - 2\Phi(z, \zeta) \leq D||a||^2 \]
Observe that the mapping $a \mapsto h(a, z) = a - \langle z, a \rangle z$ is a local diffeomorphism in neighborhood of the origin as long as $||z|| < 1$, which depend on $z \in \mathbb{B}$ smoothly and its inverse $h \mapsto a(h, z)$ is linear with a norm less than or equal to $\frac{1}{1-||z||^2}$ since
\[ ||h|| \geq ||a|| - ||a||||z||^2 = ||a||(1 - ||z||^2) \]
which gives
\[ \Phi(z + h, \zeta) + \Phi(z - h, \zeta) - 2\Phi(z, \zeta) \leq \frac{D||h||^2}{(1 - ||z||^2)^2} \]
Fix a compact set $K \subset \mathbb{B}$ compact, there exists $\delta > 0$ such that $\forall z \in K$ and $\forall 0 < ||h|| < \delta$ we have
\[ \Phi(z + h, \zeta) + \Phi(z - h, \zeta) - 2\Phi(z, \zeta) \leq \frac{D||h||^2}{\text{dist}^2(K, \partial \mathbb{B})} \]
after the previous lemma we get
\[ ||2\Phi||_{L^\infty(K \times A)} \leq D \]
with $C = \frac{D}{\text{dist}^2(K, \partial \mathbb{B})}$.

4. Moser-Trudinger Inequalities

In this section we assume $\Omega$ is pseudoconvex circled domain. We try to solve the complex Monge-Ampère equation
\[ (dd^c \varphi_t)^n = \frac{\mu}{\int_{\Omega} e^{-t \varphi_t} d\mu} \]
with $\varphi_t$ smooth and plurisubharmonic, $\varphi_t|_{\partial \Omega} = 0$ and $\mu$ just the Lebesgue normalised so that $\mu(\Omega) = 1$. We know that
- We can solve this equation if $t$ is not too large ($t = 1$ is treated in [GKY13] and even $t < (2n)^{1+1/n}(1 + 1/n)^{(1+1/n)}$).
- One can not solve the equation if $t$ is too large, cf [GKY13, section 6.2] and [BB11].
We denote by
\[ E(\varphi) := \frac{1}{n+1} \int_{\Omega} \varphi(dd^c \varphi)^n \]
the Monge-Ampère energy functional of a plurisubharmonic function \( \varphi \), which is defined as the primitive of Monge-Ampère operator. The expression
\[ F_t(\varphi) := E(\varphi) + \frac{1}{t} \log \left[ \int_{\Omega} e^{-t\varphi} d\mu \right] \]
defines the Ding functional.

**Definition 4.1.** We say the functional \( F_t \) is coercive, if there exist \( \varepsilon > 0 \) and \( B > 0 \) such that:
\[ F_t(\varphi) \leq \varepsilon E(\varphi) + B \quad \forall \varphi \in \mathcal{H} \]

**Definition 4.2.** Set \( \Phi_s(z) = \Phi(z, e^s) \). The continuous family \( (\Phi_s)_{0 \leq s \leq 1} \) is called the geodesic joining \( \varphi_0 \) and \( \varphi_1 \).

We show that \( E \) is linear along of geodesics, this result is in [GKY13, lemma 22], and was proven by Rashkovskii [Rash16] in the Cegrell class, we reprove it for continuous geodesics for convenience of the reader.

**Lemma 4.3.** Let \( (\Phi_s)_{0 \leq s \leq 1} \) be a continuous geodesic. Then \( s \mapsto E(\Phi_s) \) is affine.

**Proof.** After the proof of theorem 2.2 we have
\[
(dd^c_\zeta \Phi(z, \zeta))^{n+1} = (n+1)(dd^c_\zeta \Phi(z, \zeta))^n \wedge R + \frac{n(n+1)}{2}(dd^c_\zeta \Phi(z, \zeta))^{n-1} \wedge R^2
\]
\[
= (n+1)(d_\zeta d_\zeta \Phi \wedge (d_\zeta d_\zeta \Phi)^n - nd_\zeta d_\zeta \Phi \wedge d_\zeta d_\zeta \Phi \wedge (d_\zeta d_\zeta \Phi)^{n-1})
\]
We have by definition of \( E \)
\[
E(\Phi(., \zeta)) = \frac{1}{n+1} \int_{\Omega} \Phi(z, \zeta)(d_\zeta d_\zeta \Phi(z, \zeta))^n
\]
Which implies
\[
d_\zeta E = \frac{1}{n+1} \int_{\Omega} d_\zeta \Phi \wedge (d_\zeta d_\zeta \Phi)^n
\]
\[
d_\zeta d_\zeta E(\Phi) = \frac{1}{n+1} \left( \int_{\Omega} d_\zeta d_\zeta \Phi \wedge (d_\zeta d_\zeta \Phi)^{n-1} + n \int_{\Omega} d_\zeta \Phi \wedge d_\zeta d_\zeta \Phi \wedge (d_\zeta d_\zeta \Phi)^n \right)
\]
\[
= \frac{1}{n+1} \left( \int_{\Omega} d_\zeta d_\zeta \Phi \wedge (d_\zeta d_\zeta \Phi)^{n-1} - n \int_{\Omega} d_\zeta \Phi \wedge d_\zeta d_\zeta \Phi \wedge (d_\zeta d_\zeta \Phi)^{n-1} \right)
\]
\[
= \frac{1}{(n+1)^2} \int_{\Omega} (dd^c_\zeta \Phi)^{n+1}
\]
where the second equality follows from Stokes theorem because \( d_\zeta \Phi = 0 \) on \( \partial \Omega \), and the last one be above calculation. Thus, it follows from theorem 2.4 that \( \zeta \in A \mapsto E(\Phi(., \zeta)) \in \mathbb{R} \) is harmonic in \( \zeta \). Since \( \Phi \) is invariant by rotation with respect to \( \zeta \), hence it is affine in \( t = \log |\zeta| \).

We recall here [GKY13, proposition 23].
Proposition 4.4. Assume that $\Omega$ is circled, let $\varphi_t$ be an $S^1$-invariant solution of $(MA)_t$. Then

\[ F_t(\varphi_t) = \sup_{\psi \in I(\Omega)} F_t(\psi), \]

where $I(\Omega)$ denotes all $S^1$-invariant plurisubharmonic functions $\psi$ in $\Omega$ which are continuous up to the boundary, with zero boundary value.

Proof. Let $(\Phi)_0 \leq s \leq 1$ be a geodesic joining $\Phi_0 := \varphi_t$ to $\Phi_1 = \psi$. It follows from work of Berndtsson [Ber06] that

\[ s \mapsto -\frac{1}{t} \log \left( \int_{\Omega} e^{-t\Phi_s} d\mu \right) \]

is convex, since $s \mapsto E(\Phi_s)$ is affine from lemma 4.3. Then $s \mapsto F(\Phi_s)$ is concave.

therefore it is sufficient to show that the derivative of $F_t(\Phi_s)$ at $s = 0$ is non-negative to conclude $F_t(\varphi_t) = F(\Phi_0) \geq F_t(\Phi_s)$ for all $s$, in particular at $s = 1$ where it yields $F_t(\varphi_t) \geq F_t(\psi)$. When $s \mapsto \Phi_s$ is smooth, a direct computation yields,

\[ \frac{d}{ds} F_t(\Phi_s) = \int_{\Omega} \Phi_s \left[ (d\phi_s)^n - \frac{e^{-t\Phi_s} d\mu}{\int_{\Omega} e^{-t\Phi_s} d\mu} \right] = 0 \]

For the general case, the same method as in the proof of [BBGZ13, theorem 6.6] applies.

Lemma 4.5. The Functional $F_t$ is upper semi-continuous in $E^1_\omega(\Omega) = \{ \psi \in E^1(\Omega)/\psi = 0 \text{ on } \partial \Omega \text{ and } E(\psi) \geq -C \}$.

Proof. Recall $F_t(\psi) = E(\psi) + \frac{1}{4} \log \left( \int_{\Omega} e^{-t\psi} d\mu \right)$. The first term is upper semi-continuous in $E^1_t(\Omega)$. For the second term we apply Skoda uniform integrability theorem[Zer01].

Assume without loss of generality that $t = 1$. We need to check that $\psi \in E^1_t(\Omega) \mapsto \int_{\Omega} e^{-\psi} d\mu$ is upper semi-continuous.

Let $\psi_j$ be a sequence in $E^1_t(\Omega)$ converging to $\psi$ these functions have zero Lelong number. The following extension: $g_j = \psi_j + \psi$ to $\Omega \subset K \subset \Omega'$ as $\tilde{g}_j = g_j \text{ in } \Omega, \tilde{g}_j = 0 \text{ in } \Omega' \setminus \Omega$. We apply Skoda’s uniform integrability estimates:

\[ \int_{\Omega} e^{-2(\psi + \psi_j)} d\mu \leq \int_{K} e^{-2(\psi + \psi_j)} d\mu \leq C \]

\[ |\int_{\Omega} e^{-\psi_j} d\mu - \int_{\Omega} e^{-\psi} d\mu| \leq \int_{\Omega} |\psi - \psi_j| e^{-(\psi_j + \psi)} d\mu \leq C ||\psi_j - \psi||_{L^2(\mu)}. \]

as follows from the Cauchy-Schwarz inequality and the elementary inequality

\[ |e^a - e^b| \leq |a - b| e^{a+b}, \text{ for all } a, b \geq 0 \]

The conclusion follows since $(\psi_j)$ converges to $\psi \text{ in } L^2(\mu)$. \hfill \Box

We recall that the Dirichlet problem $(MA)_t$ has a solution for $t = 1$ by [GKY13], we moreover have uniqueness if $\Omega \text{ is strictly } \varphi \text{-convex}( \Omega \text{ is strictly convex for the metric } dd^c \varphi)$. We recall here the main result of [GKY13].
Theorem 4.6. Let $\Omega \subset \mathbb{C}^n$ be a bounded smooth strongly pseudoconvex domain which is circled. Let $\varphi$ be a smooth $S^1$-invariant strictly plurisubharmonic solution of the complex Monge-Ampère problem $(MA)_1$. If $\Omega$ is strictly $\varphi$-convex, then $\varphi$ is the unique $S^1$-invariant solution of $(MA)_1$.

Inspired by Dinezze-Guedj [DG16, theorem 5.5], we now prove the following theorem

Theorem 4.7. Let $\Omega \subset \mathbb{C}^n$ be a smooth strongly pseudo-convex circled domain. If there exists $\varepsilon(t), M(t) > 0$ such that,

$$F_t(\psi) \leq \varepsilon(t)E(\psi) + M(t) \quad \forall \psi \in \mathcal{H},$$

then $(MA)_t$ admits a $S^1$-invariant smooth strictly plurisubharmonic function solution.

Conversely, if $(MA)_t$ admits such a solution $\varphi_t$ and $\Omega$ is strictly $\varphi_t$-convex, then there exists $\varepsilon(t), M(t) > 0$ such that,

$$F_t(\psi) \leq \varepsilon(t)E(\psi) + M(t) \quad \forall \psi \in \mathcal{H}.$$

Proof. If we assume the following inequality holds,

$$F_t(\psi) \leq \varepsilon(t)E(\psi) + M(t)$$

then the same method of [GKY13] applies, if only we change $\varphi$ by $t\varphi$.

Conversely, as $\varphi_t$ is a solution of $(MA)_t$ then from the (proposition 4.4) we have

$$F_t(\varphi_t) := \sup\{F_t(\psi)/\psi \in \mathcal{H} \cap I(\Omega)\}$$

assume for contradiction that there is no $\varepsilon > 0$ such that

$$F_t(\psi) \leq \varepsilon E(\psi) + M$$

for all $\psi \in \mathcal{H}$. Put $\varepsilon_j = \frac{1}{j}$ and $M = F_t(\varphi_t) + 1$. Then we can find a sequence $(\varphi_j) \subset \mathcal{H}$ such that

$$F_t(\varphi_j) > \frac{E(\varphi_j)}{j} + F_t(\varphi_t) + 1$$

We discuss here two cases, the first case if $E(\varphi_j)$ does not blow up to $-\infty$, we reach a contradiction, by letting $j$ go to $+\infty$. Indeed we can assume that $E(\varphi_j)$ bounded and $\varphi_j$ converges to some $\psi \in E^1(\Omega)$ which is $S^1$-invariant. Since $F_t$ is upper semi-continuous by lemma 4.5, we infer $F_t(\psi) \geq F_t(\varphi_t) + 1 > F_t(\varphi_t)$ contradiction because $\varphi_t$ is the solution of $(MA)_t$.

The second case if $E(\varphi_j) \rightarrow -\infty$. It follows that $d_j = -E(\varphi_j) \rightarrow +\infty$.

We let $(\phi_{s,j})_{0 \leq s \leq d_j}$ denote the weak geodesic joining $\varphi_t$ to $\varphi_j$ and set $\psi_j := \phi_{1,j}$. We know that is $s \mapsto E(\phi_{s,j})$ is affine along of the Mabuchi geodesic. Thus $E(\phi_{s,j}) = a_j s + b_j$, where $a_j$ and $b_j$ are real numbers. For $s = 0$ we have

$$E(\phi_{0,j}) = b_j = E(\varphi_t)$$

and for $s = d_j$ we have

$$E(\varphi_j) = E(\phi_{d_j,j}) = a_j d_j + E(\varphi_t)$$
therefore \( a_j = \frac{E(\varphi_j) - E(\varphi_t)}{d_j} \). Then

\[ E(\varphi_{s,j}) = \frac{E(\varphi_j) - E(\varphi_t)}{d_j} s + E(\varphi_t) \]

Since \( s \mapsto E(\varphi_{s,j}) \) is affine along the Mabuchi geodesic and by Berndtsson [Bern06] convexity result, we infer that the map \( s \mapsto \mathcal{F}_t(\varphi_{s,j}) \) is concave, which implies with (5) that

\[ 0 \geq \mathcal{F}_t(\phi_{1,j}) - \mathcal{F}_t(\phi_{0,j}) \geq \frac{\mathcal{F}_t(\phi_{d,j}) - \mathcal{F}_t(\phi_{0,j})}{d_j} > -\frac{1}{j} + \frac{1}{d_j} \]

thus \( \mathcal{F}_t(\psi_j) \to \mathcal{F}_t(\varphi_t) \). This shows that \( (\psi_j) \) is a maximizing sequence for \( \mathcal{F}_t \). If we take \( t = 1 \) on equation (6) we get

\[ E(\psi_j) = \frac{E(\varphi_j) - E(\varphi_t)}{d_j} + E(\varphi_t) = -1 - \frac{E(\varphi_t)}{d_j} + E(\varphi_t) \geq -1 + E(\varphi_t) \]

Passing to subsequence, we can assume that \( \psi_j \) converge to \( \psi \in E^1(\Omega) \) which is \( S^1 \)-invariant. Since \( \mathcal{F}_t \) is upper semi-continuous and \( \psi_j \) is a maximizing sequence for \( \mathcal{F}_t \) then we have \( \mathcal{F}_t(\psi) = \mathcal{F}_t(\varphi_t) \) and so \( \psi = \varphi_t \) thanks to the uniqueness. Letting \( j \) to infinity in (7) we get

\[ E(\psi) = -1 + E(\varphi_t) \]

This yields a contradiction. \( \square \)

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