A Kähler Structure on the Space of String World-Sheets

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Abstract

Let $(M, g)$ be an oriented Lorentzian 4-manifold, and consider the space $S$ of oriented, unparameterized time-like 2-surfaces in $M$ (string world-sheets) with fixed boundary conditions. Then the infinite-dimensional manifold $S$ carries a natural complex structure and a compatible (positive-definite) Kähler metric $h$ on $S$ determined by the Lorentz metric $g$. Similar results are proved for other dimensions and signatures, thus generalizing results of Brylinski regarding knots in 3-manifolds. Generalizing the framework of Lempert, we also investigate the precise sense in which $S$ is an infinite-dimensional complex manifold.

Running title: The Space of String World-Sheets

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1 Introduction

Given a collection of circles in a 4-dimensional oriented Lorentzian spacetime, one may consider the space $S$ of unparameterized oriented time-like compact 2-surfaces with the given circles as boundary. The main purpose of the present note is to endow $S$ with the structure of an infinite-dimensional Kähler manifold—i.e. with both a complex structure and a Riemannian metric for which this complex structure is covariantly constant. This was motivated by a construction of Brylinski [2], whereby a Kähler structure is given to the space of knots in a Riemannian 3-manifold. In fact, our discussion will be structured so as to apply to codimension 2 submanifolds of a space-time of arbitrary dimension and metrics of arbitrary signature, with the proviso that we only consider those submanifolds for which the normal bundle is orientable and has (positive- or negative-)definite induced metric; thus Brylinski’s construction becomes subsumed as a special case.

As the reader will therefore see, complex manifold theory thus comes naturally into play when one studies codimension 2 submanifolds of a space-time. On the other hand, complex manifold theory makes a quite different kind of appearance when one attempts to study the intrinsic geometry of 2-dimensional manifolds. If some interesting modification of string theory could be found which invoked both of these observations simultaneously, one might hope to thereby explain the puzzling four-dimensionality of the observed world.

Many of the key technical ideas in the present note are straightforward generalizations of arguments due to László Lempert [5], whose lucid study of Brylinski’s complex structure is based on the theory of twistor CR manifolds [4]. One of the most striking features of the complex structures in question is that, while they are formally integrable and may even admit legions of local holomorphic functions, they do not admit enough finite-dimensional complex submanifolds to be locally modeled on any complex topological vector space. This beautifully illustrates the fact, emphasized by Lempert, that the Newlander-Nirenberg Theorem [3] fails in infinite dimensions.
2 The Space of World-Sheets

Let \((M, g)\) be a smooth oriented pseudo-Riemannian n-manifold. We use the term \textit{world-sheet} to refer to a smooth compact oriented codimension-2 submanifold-with-boundary \(\Sigma^{n-2} \subset M^n\) for which the inner product induced by \(g\) on the conormal bundle 

\[ \nu_\Sigma^* := \{ \phi \in T^*M|_{\Sigma} \mid \phi|_{T\Sigma} \equiv 0 \} \]

of \(\Sigma\) is definite at each point. If \(g\) is Riemannian, this just means an oriented submanifold of codimension 2; on the other hand, if \((M, g)\) is a Lorentzian 4-manifold, a world-sheet is exactly an oriented time-like 2-surface.

\textbf{Definition 1} Let \((M, g)\) be a smooth oriented pseudo-Riemannian n-manifold, and let \(B^{n-3} \subset M^n\) be a smooth codimension-3 submanifold which is compact, without boundary. We will then let \(S_{M,B}\) denote the space of smooth oriented world-sheets \(\Sigma^{n-2} \subset M\) such that \(\partial \Sigma = B\).

Of course, this space is sometimes empty— as happens, for example, if \(B\) is a single space-like circle in Minkowski 4-space. This said, \(S_{M,B}\) is automatically a Fréchet manifold, and its tangent space at \(\Sigma\) is

\[ T_\Sigma S_{M,B} = \{ v \in \Gamma(\Sigma, C^\infty(\nu_\Sigma)) \mid v|_{\partial\Sigma} \equiv 0 \} \] .

Indeed, if we choose a tubular neighborhood of \(\Sigma\) which is identified with the normal bundle of an open extension \(\Sigma_\epsilon\) of \(\Sigma\) beyond its boundary, every section of \(\nu_\Sigma \to \Sigma\) which vanishes on \(\partial\Sigma\) is thereby identified with an imbedded submanifold of \(M\), and this submanifold is still a world-sheet provided the \(C^1\) norm of the section is sufficiently small. This provides \(S_{M,B}\) with charts which take values in Fréchet spaces, thus giving it the desired manifold structure.

Since the normal bundle \(\nu_\Sigma = (\nu_\Sigma^*)^* = TM/T\Sigma = (T\Sigma)\perp\) of our world-sheet is of rank 2 and comes equipped with an orientation as well as a metric induced by \(g\), we may identify \(\nu_\Sigma\) with a complex line bundle by taking \(J : \nu_\Sigma \to \nu_\Sigma, J^2 = -1\) to be rotation by \(+90^\circ\). This then defines an endomorphism \(J\) of \(TS\) by

\[ J : T_\Sigma S_{M,B} \to T_\Sigma S_{M,B} : v \to J \circ v . \]
Clearly $J^2 = -1$, so that $J$ gives $S$ the structure of an almost-complex Fréchet manifold—i.e. every tangent space of $S$ can now be thought of as a complex Fréchet space by defining $J$ to be multiplication by $\sqrt{-1}$. In the next sections, we shall investigate the integrability properties of this almost-complex structure.

3 Integrability of the Complex Structure

Let $(M, g)$ denote, as before, an oriented pseudo-Riemannian manifold. Let $Gr^+_2(M)$ denote the bundle of oriented 2-planes in $T^*M$ on which the inner product induced by $g$ is definite. This smooth $(3n-4)$-dimensional manifold then has a natural CR structure \cite{4, 7} of codimension $n-2$. Let us review how this comes about.

Let $\tilde{N} \subset \left( C \otimes T^*M - T^*M \right)$ denote the set of non-real null covectors of $g$, and let $N \subset P(C \otimes T^*M)$ be its image in the fiber-wise projectivization of the complexified cotangent bundle. There is then a natural identification of $N$ with $Gr^+_2(M)$. Namely, using pairs $u, v \in T^*_x M$ of real covectors satisfying $\langle u, v \rangle = 0$ and $\langle u, u \rangle = \langle v, v \rangle$, we define a bijection between these two spaces by

$$Gr^+_2(M) \ni \text{oriented span}(u, v) \leftrightarrow [u + iv] \in N \subset P(C \otimes T^*_x M)$$

which is independent of the representatives $u$ and $v$. But, letting $\vartheta = \sum p_j dx^j$ denote the canonical complex-valued 1-form on the total space of $C \otimes T^*M \to M$, and letting $\omega$ be the restriction of $d\vartheta$ to $\tilde{N}$, the distribution

$$\hat{D} = \ker(\omega : C \otimes T\tilde{N} \to C \otimes T^*\tilde{N})$$

is involutive by virtue of the fact that $\omega$ is closed; since $\hat{D}$ also contains no non-zero real vectors as a consequence of the fact that $\tilde{N} \cap T^*M = \emptyset$, $\hat{D}$ is a CR structure on $\tilde{N}$, the codimension of which can be checked to be $n-2$. This CR structure is invariant under the natural action of $C^\times$ on $\tilde{N}$ by scalar multiplication, and thus descends to a CR structure $D$ on $N = Gr^+_2(M)$, again of codimension $n-2$. Moreover, $\vartheta|_N$ descends to $N$ as a CR line-bundle-valued 1-form

$$\theta \in \Gamma(N, \mathcal{E}^{1,0}(L)) \ , \quad \bar{\partial}_b \theta = 0 \ ,$$
where, letting $T^{1,0}N := (C \otimes TN)/D$, $L\otimes(2n-3) = \wedge^{2n-3}T^{1,0}N$, $\mathcal{E}^{1,0}(L) := C^\infty(L \otimes (T^{1,0}N)^*)$, and $\partial_b$ is naturally induced by $d|_D$.

The CR structure $D$ of $N$ may be expressed in the form

$$D = \{v - iJv \mid v \in H\}$$

for a unique rank $2n-2$ sub-bundle $H$ of the real tangent bundle $TN$ and a unique endomorphism $J$ of $H$ satisfying $J^2 = -1$. In these terms the geometric meaning of the CR structure of $N$ is fairly easy to describe. Indeed, if $\pi : Gr^+(M) \rightarrow M$ is the tautological projection, then $H_P = (\pi_*P)^{-1}(P)$ for every oriented definite 2-plane $P \subset TM$. On vertical vectors, $J$ acts by the standard complex structure on the quadric fibers of $N \rightarrow M$; whereas $J$ acts on horizontal vectors by $90^\circ$ rotation in the 2-plane $P \subset TM$. This point of view, however, obscures the fact that $D$ is both involutive and unaltered by conformal changes $g \mapsto e^f g$.

A compact $(n-2)$-dimensional submanifold-with-boundary $S \subset N$, will be called a transverse sheet if its tangent space is everywhere transverse to the CR tangent space of $N$:

$$TN|_S = TY \oplus H|_Y.$$

As before, let $B^{n-3} \subset M$ denote a compact codimension-3 submanifold, and let $\varpi : N \rightarrow M$ be the canonical projection. We will then let $\hat{S}_{N,B}$ denote the set of transverse sheets $S \subset N$ such that $\varpi$ maps $\partial S$ diffeomorphically onto $B$. Thus $\hat{S}_{N,B}$ is a Fréchet manifold whose tangent space at $S$ is given by

$$T\hat{S}_{N,B}|_S = \{v \in \Gamma(S, C^\infty(H|_S)) \mid \varpi_*(v|_{\partial S}) \equiv 0\},$$

and hence $J : H \rightarrow H$ induces an almost-complex structure $\hat{J}$ on $\hat{S}_{N,B}$ by $\hat{J}(v) := J \circ v$.

**Proposition 3.1** The almost-complex structure $\hat{J}$ on the space $\hat{S}_{N,B}$ of transverse sheets is formally integrable— i.e.

$$\tau(v, w) := \hat{J}[v, w] - [v, \hat{J}w] - [\hat{J}v, w] - [\hat{J}\hat{J}v, \hat{J}w] = 0$$

for all smooth vector fields $v, w$ on $\hat{S}_{N,B}$.
Proof. The Fröhlicher-Nijenhuis torsion $\tau(v, w)$ is tensorial in the sense that its value at $S$ only depends on the values of $v$ and $w$ at $S$. Given $v_S, w_S \in \{ v \in \Gamma(S, C^\infty(H|_S)) \mid \varpi(v|_{\partial S}) \equiv 0 \}$, we will now define preferred extensions of them as vector fields near $S \in \hat{S}_{N,B}$ in such a manner as to simplify the computation of $\tau(v, w) = \tau(v_S, w_S)$. To do this, we may first use a partition of unity to extend $v_S$ and $w_S$ as sections $\hat{v}, \hat{w} \in \Gamma(N, C^\infty(H))$ defined on all of of $N$ in such a manner that $\hat{v}$ and $\hat{w}$ are tangent to the fibers of $\varpi$ along all of $\varpi^{-1}(B)$. Now let $U \subset N$ be a tubular neighborhood of $S$ which is identified with the normal bundle $H$ of some open extension $S_0$ of $S$, and let $\hat{U} \subset \hat{S}_{N,B}$ be the set of transverse sheets $S' \subset U$. We may now define our preferred extensions of $v$ and $w$ of $v_S$ and $w_S$ on the domain $\hat{U}$ by letting the values of $\hat{v}$ and $\hat{w}$ at $S' \subset U$ be the restrictions of $\hat{v}$ and $\hat{w}$ to $S'$. Notice that $[\hat{v}, \hat{w}]$ is then precisely the vector field on $\hat{U}$ induced by $[\hat{v}, \hat{w}]$, whereas $\hat{J}v$ is the vector field induced by $J\hat{v}$. Since the integrability condition for $(N, D)$ says that $J([\hat{v}, \hat{w}] - [\hat{J}\hat{v}, \hat{J}\hat{w}]) = [\hat{v}, \hat{J}\hat{w}] + [\hat{J}\hat{v}, \hat{w}]$, it therefore follows that $\hat{J}[v, w] - \hat{J}[\hat{J}v, \hat{J}w] = [v, \hat{J}w] + [\hat{J}v, w]$, so that $\tau(v, w) = 0$, as claimed.

We now observe that there is a canonical imbedding

$$S_{M,B} \xrightarrow{\varpi} \hat{S}_{N,B}$$

obtained by sending a world-sheet to its normal-bundle, thought of as the image of a section of $Gr^+_2(M)|_{\Sigma} = N|_{\Sigma}$; thought of in this way, it is easy to see that $\nu_{\Sigma} \subset N$ is a transverse submanifold.

Theorem 3.2 The imbedding $\varpi$ realizes $(S_{M,B}, J)$ as a complex submanifold of $(\hat{S}_{N,B}, \hat{J})$. In particular, the almost-complex structure $J$ of $S_{M,B}$ is formally integrable.

Proof. The projection $\varpi : N \to M$ induces a map $\hat{\varpi} : \hat{S}_{N,B} \to S_{M,B}$ which is a left inverse of $\varpi$ and satisfies $\hat{\varpi}_*\hat{J} = J\varpi_*$. It therefore suffices to show
that the tangent space of the image of $\Psi$ is $\hat{J}$-invariant. Now the condition for a transverse sheet $S \subset N$ to be the $\Psi$-lifting of the world-sheet $\varpi(S) \subset M$ is exactly that $\theta|_S \equiv 0$. When $S$ satisfies this condition, a connecting field $v \in \Gamma(S, \mathcal{E}(H))$ then represents a vector $\hat{v} \in T\hat{S}$ which is tangent to the image of $\Psi$ iff

$$v \lrcorner d\theta|_{TS} + d(v \lrcorner \theta)|_{TS} \equiv 0 ;$$

the exterior derivative of $\theta$ may here be calculated in any local trivialization for the line bundle $L$, since the left-hand side rescales properly under changes of trivialization so as define an $L$-valued 1-form on $S$. But since $\theta \in \Gamma(N, \mathcal{E}^{1,0}(L))$ satisfies $\overline{\partial}_b \theta = 0$, it follows that

$$(Jv \lrcorner d\theta)|_{TS} + d(Jv \lrcorner \theta)|_{TS} = i(Jv \lrcorner d\theta)|_{TS} + id(Jv \lrcorner \theta)|_{TS}$$

because $\theta$ and $d\theta$ are of types $(1,0)$ and $(2,0)$, respectively. The tangent space of the image of $\Psi$ is therefore $\hat{J}$-invariant, and the claim follows.

**Definition 2** Let $(\mathfrak{X}, \mathfrak{J})$ be an almost-complex Fréchet manifold, and let $f : U \to \mathbb{C}$ be a differentiable function defined on an open subset of $\mathfrak{X}$. We will say that $f$ is $\mathfrak{J}$-holomorphic if

$$(\mathfrak{J}v)f = ivf \quad \forall v \in TU .$$

**Definition 3** An almost-complex Fréchet manifold $(\mathfrak{X}, \mathfrak{J})$ is called weakly integrable if for each real tangent vector $w \in T\mathfrak{X}$ there is a $\mathfrak{J}$-holomorphic function $f$ defined on a neighborhood of the base-point of $w$ such that $wf \neq 0$.

**Theorem 3.3** Suppose that $(M, g)$ is real-analytic. Then $(\hat{S}_{N,B}, \hat{J})$ is weakly integrable.

**Proof.** If $(M, g)$ is real-analytic, so is the CR manifold $(N, D)$, and we can therefore realize $(N, D)$ as a real submanifold of a complex manifold $(2n-3)$-manifold $\mathcal{N}$. This can even be done explicitly by taking $\mathcal{N}$ to be a space of complex null geodesics for a suitable complexification of $(M, g)$.

Now let $S \subset N \subset \mathcal{N}$ be any transverse sheet. Then there is a neighborhood $V \subset \mathcal{N}$ of $S$ which can be holomorphically imbedded in some $\mathbb{C}^\ell$. 

7
Indeed, let \( Y \subset \mathcal{N} \) be a totally real \((2n - 3)\)-manifold containing \( S \), let \( f : Y \to \mathbb{R}^\ell \) be a smooth imbedding, and let \( Y_0 \) be a precompact neighborhood of \( S \subset Y \) with smooth boundary. By [3], the component functions \( f^j|_{Y_0} \) are limits in the \( C^1 \) topology of the restrictions of holomorphic functions. Using such an approximation of \( f \), we may therefore imbed \( Y_0 \) as a totally real submanifold of \( \mathbb{C}^\ell \) by a map which extends holomorphically to a neighborhood of \( Y_0 \), and this holomorphic extension then automatically yields a holomorphic imbedding of some open neighborhood \( V \supset S \) in \( \mathbb{C}^\ell \).

Now suppose that \( v \) is a smooth section of \( H \) along \( S \). We may express \( v \) uniquely as \( u + Jw \), where \( u \) and \( w \) are tangent to the \( Y \). By changing \( Y \) if necessary, we can furthermore assume that \( u \neq 0 \). Let \( F \) be a smooth function on \( Y \) which vanishes on \( S \) and such that the derivative \( vF \) is non-negative and supported near some interior point of \( S \); and let \( \varphi \) be a real-valued smooth \((n - 2)\)-form on \( Y \) whose restriction to \( S \) is positive on the support of \( vF \). Set \( \psi = F\varphi \). Using our imbedding of \( Y \) in \( \mathbb{C}^\ell \), we can express \( \psi \) as a family of component functions—e.g. by arbitrarily declaring that all contractions of \( \psi \) with elements of the normal bundle \((TY)^{\perp} \subset TC^\ell \) shall vanish. But, again by [3], these component functions are \( C^1 \)-limits on \( Y_0 \) of restrictions of holomorphic functions from a neighborhood of \( Y_0 \subset \mathbb{C}^\ell \). Thus, by perhaps replacing \( V \) with a smaller neighborhood, there is a holomorphic \((n - 2)\)-form \( \beta \) on \( V \) which approximates \( \psi \) well enough that

\[
\Re \int_S v \lrcorner d\beta > \frac{1}{2} \int_S u \lrcorner d\psi > 0
\]

and

\[
\Re \int_{\partial S} v \lrcorner \beta > -\frac{1}{2} \int_S u \lrcorner d\psi .
\]

Let \( \hat{V} := \{ S' \in \mathcal{S}_{N,B} \mid S' \subset V \} \), and define \( f_\beta : \hat{V} \to \mathbb{C} \) by \( f_\beta(S') = \int_{S'} \beta \). Then \( f_\beta \) is a holomorphic function on the open set \( \hat{V} \subset \mathcal{S}_{N,B} \). Indeed, if \( \gamma \) is any smooth \((n - 2)\)-form on \( V \), and if we set \( f_\gamma(S') = \int_{S'} \gamma \), then, for \( S' \subset V \), the derivative of \( f_\gamma \) in the direction of \( w \in \Gamma(S',C^\infty(H)) \), \( \varpi_*(w)|_{\partial S'} \equiv 0 \), is given by

\[
w f_\gamma|_S = \int_S w \lrcorner d\gamma + \int_{\partial S'} w \lrcorner \gamma ;
\]

for if \( w \) is extended to \( V \) as a smooth vector field \( \hat{w} \) tangent to the fibers of
and \( S_t \) is obtained by pushing \( S' \) along the flow of the vector field \( \dot{w} \), then
\[
w f \gamma |_{S'} = \frac{d}{dt} \left[ \int_{S_t} \gamma \right]_{t=0} = \int_{S'} \mathcal{L} \dot{w} \gamma
\]
\[
= \int_{S'} [\dot{w} \lrcorner d\gamma + d(\dot{w} \lrcorner \gamma)]
\]
\[
= \int_{S'} w \lrcorner d\gamma + \int_{\partial S'} w \lrcorner \gamma.
\]

But since \( \beta \) is the restriction of a holomorphic \((n-2)\)-form from a region of \( \mathcal{N} \), it therefore follows that
\[
(\mathcal{J} w) f |_{S'} = \int_{S'} (Jw) \lrcorner d\beta + \int_{\partial S'} (Jw) \lrcorner \beta
\]
\[
= i \int_{S'} w \lrcorner d\beta + i \int_{\partial S'} w \lrcorner \beta
\]
\[
= i w f |_{S'},
\]
showing that the function \( f_\beta \) induced by \( \beta \) is \( \mathcal{J} \)-holomorphic, as claimed.

However, we have also carefully chosen \( \beta \) so that the real part of the expression \( \int_S v \lrcorner d\beta + \int_{\partial S} v \lrcorner \beta = v f_\beta \) is positive. For every real tangent vector \( v \) on \( S_{N,B} \), one can thus find a locally-defined \( \mathcal{J} \)-holomorphic function whose derivative is non-trivial in the direction \( v \). Hence \( \mathcal{J} \) is weakly integrable, as claimed.

**Corollary 3.4** If \((M,g)\) is real-analytic, then \((S_{M,B}, \mathcal{J})\) is weakly integrable.

**Proof.** By Theorem 3.2 and 3.3, \((S_{M,B}, \mathcal{J})\) can be imbedded in the weakly integrable almost-complex manifold \((\mathcal{N}_{N,B}, \mathcal{J})\). Since the restriction of a holomorphic function to an almost-complex submanifold is holomorphic, it follows that \((S_{M,B}, \mathcal{J})\) is weakly integrable.

One might instead ask whether \((S_{M,B}, \mathcal{J})\) is strongly integrable— i.e. locally biholomorphic to a ball in some complex vector space. The answer is no; in contrast to any strongly integrable almost-complex manifold, \((S_{M,B}, \mathcal{J})\) contains very few finite-dimensional complex submanifolds:
Proposition 3.5 Suppose that $(M, g)$ is real-analytic. At a generic point $S \in S_{M,B}$, a generic $(n-1)$-plane is not tangent to any $(n-1)$-dimensional $\mathcal{J}$-complex submanifold.

Proof. Let $S \subset M$ be a world-sheet which is not real-analytic near $p \in S$. Let $q \in N = Gr^+_2(M)$ be given by $q = T_p^p S^\perp$, and let $v_1, \ldots, v_{n-1} \in H_q$ be a set of real vectors such that the $v_j + iJv_j$ form a basis for $D_q$. Extend these vectors as sections $\hat{v}_j$ of $H|\Psi(S)$ which satisfy equation (1) along the sheet; this may be done, for example, by first extending each $v_j$ to just a 1-jet at $p$ satisfying (1) at $p$, projecting this to a 1-jet via $\varpi$ to yield a 1-jet of a normal vector field on $S \subset M$, extending this 1-jet as a section of the normal bundle of $S$, and finally lifting this section using $\Psi_s$. Let $u_1, \ldots, u_n$ be the elements of $T_S S_{M,B}$ represented by $\hat{v}_1, \ldots, \hat{v}_{n-1}$, and let $P \subset T^{1,0}_S S_{M,B}$ be spanned by $u_1 - iJu_1, \ldots, u_n - iJu_n$.

Now suppose there were an $\mathcal{J}$-holomorphic submanifold $X \subset S_{M,B}$ through $S$ with $(1,0)$-tangent plane equal to $P$. Then $X$ represents a family of transverse sheets in $N$ which foliates a neighborhood $U \subset N$ of $q$; moreover, because $X$ represents a holomorphic family, the leaf-space projection $\ell : U \to X$ is CR in the sense that $\ell_u(D) \subset T^{0,1}X$. Since we have assumed that $(M, g)$ is real-analytic, we may also assume that $U$ has a real-analytic CR imbedding $U \hookrightarrow \mathbb{C}^{2n-3}$. Moreover the twistor CR manifold $N$ is automatically “anticlastic,” by which I mean that the Levi form $\mathcal{L} : D \to TN/H : v \mapsto [v, \bar{v}] \mod H$ is surjective at each point of $N$. This gives rise to a Bochner-Hartogs extension phenomenon: every CR function on $U$ extends to a holomorphic function on some neighborhood of $U \subset \mathbb{C}^{2n-3}$. In particular, every CR map defined on $U$ must be real-analytic, and this applies in particular to the leaf-space projection $\ell$. Thus $\Psi(S)$ is real-analytic near $q$, and $S$ is therefore real-analytic near $p$. This proves the result by contradiction.  

4 Kähler Structure

The complex structure $\mathcal{J}$ on $S$ depends only on the conformal class $[g] = \{e^Jg\}$ of our metric, but we will now specialize by fixing a specific pseudo-Riemannian metric $g$. Our reason for doing so is that we thereby induce an
$L^2$-metric on $\mathcal{S}$. Indeed, each tangent space

$$T_{\Sigma}S_{M,B} = \{ v \in \Gamma(\Sigma, C^\infty(\nu_\Sigma)) \mid v|_{\partial \Sigma} \equiv 0 \}$$

may be equipped with a positive-definite inner product by setting

$$h(v, w) := \int_{\Sigma} g(v, w) \ d\text{vol}_{g_{|\Sigma}}.$$ 

We shall now see that this metric has some quite remarkable properties.

**Theorem 4.1** The Riemannian metric $h$ on $\mathcal{S}$ is Kähler with respect to the previously-defined complex structure $\mathcal{J}$.

**Proof.** Let $\Omega$ denote the volume n-form of $g$, and define a 2-form $\omega$ on $\mathcal{S}$ by

$$\omega(v, w) := \int_{\Sigma} (v \wedge w) \wedge \Omega.$$ 

Obviously, $\omega$ is $\mathcal{J}$-invariant and

$$h(v, w) = \omega(\mathcal{J}v, w).$$

We therefore just need to show that $\omega$ is closed.\footnote{The reader may ask whether it is actually legitimate to call a Riemannian manifold Kähler when the almost-complex structure in question is at best weakly integrable. However, formal integrability and the closure of the Kähler form are the only conditions necessary to insure that the almost-complex structure tensor is parallel, even in infinite dimensions.} To check this, let us introduce the universal family

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\pi} & \mathcal{S} \\
& \searrow & \downarrow \pi \\
& & M
\end{array}
\]
where the fiber of $\pi$ over $\Sigma \in \mathcal{S}$ is defined to be $\Sigma \subset M$. We can then pull $\Omega$ back to $\mathcal{F}$ to obtain a closed $n$-form $\alpha = p^*\Omega$ which vanishes on the boundary $B = \partial \Sigma$ of every fiber of $\pi$. But $\omega$ is just obtained from $\alpha$ by integrating on the fibers of $\pi$:

$$\omega = \pi_*\alpha .$$

Since $\pi_*$ commutes with $d$ on forms which vanish along the fiber-wise boundary (cf. [1], Prop. 6.14.1), it follows that

$$d\omega = d(\pi_*\alpha) = \pi_*d\alpha = \pi_*d(p^*\Omega) = \pi_*p^*d\Omega = \pi_*p^*0 = 0 .$$

Thus $h$ is a Kähler metric, with Kähler form $\omega$.

To conclude this note, we now observe that $(\mathcal{S}, h)$ is formally of Hodge type— but non-compact, of course!

**Proposition 4.2** Modulo a multiplicative constant, the Kähler form $\omega$ of $h$ represents an integer class in cohomology. If, moreover, $M$ is non-compact, $\omega$ is actually an exact form, and its cohomology class thus vanishes.

**Proof.** If $M$ is compact, we may assume that $g$ has total volume 1, so that its volume form $\Omega$ then represents an element of integer cohomology. Since $\omega = \pi_*p^*\Omega$, its cohomology class $[\omega] = \pi_*p^*[\Omega]$ is therefore integral. If, on the other hand, $M$ is non-compact, $\Omega = d\Upsilon$ for some $(n - 1)$-form $\Upsilon$, and hence $\omega = d(\pi_*p^*\Upsilon)$.

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**References**

[1] R. Bott and L.T. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1982.

[2] J.-L. Brylinski, *The Kähler Geometry of the Space of Knots in a Smooth Threefold*, preprint, Penn State, 1990.
[3] R. Harvey and R.O. Wells, Jr. *Holomorphic Approximation and Hyper-function Theory on a $C^1$ Totally Real Submanifold of a Complex Manifold*, Math. Ann. 197 (1972) 287–318.

[4] C. LeBrun, *Twistor CR Manifolds and Three-Dimensional Conformal Geometry*, Trans. Am. Math. Soc. 284 (1984) 601–616.

[5] L. Lempert, *Loop Spaces as Complex Manifolds*, J. Diff. Geom, to appear.

[6] A. Newlander and L. Nirenberg, *Complex Analytic Coordinates in Almost Complex Manifolds*, Ann. Math. 65 (1957) 391–404.

[7] H. Rossi, *LeBrun's Non-Imbeddibility Theorem in Higher Dimensions*, Duke Math. J. 52 (1985) 457–474.