1. Introduction

The electrical control of magnetization dynamics has been a central issue in the field of spintronics [1, 2], owing to its possible applications in magnetic memory devices with low power consumption. A particularly promising mechanism for the electrical control is to utilize the spin Hall effect [3–6] (SHE) in a normal metal (NM), such as Pt or Ta, to convert an electric current into a spin current, and subsequently to magnetization dynamics in an adjacent magnet via mechanisms such as spin-transfer torque [7, 8] (STT). In reverse, the inverse spin Hall effect [9, 10] (ISHE) can convert the spin current generated by certain means, for instance spin pumping [11, 12], into an electric signal. A particularly intriguing phenomenon that involves both SHE and ISHE is the spin Hall magnetoresistance [13–24] (SMR), in which a charge current in an NM causes a spin accumulation at the edge of the sample due to SHE, yielding a finite spin current at the interface to a ferromagnet. Through ISHE, the spin current in turn gives an electromotive force along the original charge current, effectively changing the magnetoresistance of the NM.

The two major ingredients that determine SMR are the spin diffusion [25] in the NM and the spin current at the NM/ferromagnet interface. The spin diffusion part has been addressed in detail by Chen et al for the NM/ferromagnetic insulator (NM/FMI) bilayer, such as Pt/Y₃Fe₅O₁₂ (Pt/YIG), and FMI/NM/FMI trilayer [23, 24]. This approach solves the spin diffusion equation in the presence of SHE and ISHE in a self-consistent manner, where the spin current at the NM/FMI interface serves as a boundary condition. However, the interface spin current remains an external parameter for which experimental or numerical input is needed [26, 27]. On the other hand, a quantum tunneling formalism has emerged recently as an inexpensive tool to calculate the interface spin current from various material properties such as the insulating gap of the FMI and the interface coupling [28]. The quantum tunneling theory also successfully explains [29] the reduced spin pumping spin current when an additional oxide layer is inserted between NM and FMI [30]. It is then of fundamental importance to combine the spin diffusion approach with the quantum tunneling formalism for the interface spin current to give a
complete theoretical description of the SMR, in particular to quantify how various material properties influence the SMR.

In this article we provide a minimal formalism that bridges the quantum tunneling formalism to the spin diffusion approach. We focus on the SMR in NM/FMI bilayer realized in Pt/YIG, and the NM/ferromagnetic metal (NM/FMM) bilayer realized in Pt/Co and Ta/Co [14]. The spin diffusion in the NM is assumed to be described by the same formalism of Chen et al. [23], whereas the interface spin current is calculated from the quantum tunneling formalism [28, 29]. In the NM/FMM bilayer, we consider an FMM that has long spin diffusion length and a small thickness, such that the spin diffusion effect is negligible and the spin transport is predominately of quantum origin [28]. This is presumably adequate for the case of ultrathin Co films [31], but not for materials with very short spin diffusion length such as permalloy [32, 33]. Within this formalism, the effect of material properties including spin diffusion length of the NM, interface $s - d$ coupling, insulating gap of the FMI, and the thickness of each layer can all be treated on equal footing. In particular, we reveal the signature of quantum interference in SMR in NM/FMM bilayer, and discuss the observability of the predicted signature of quantum interference in SMR. Section 4 gives the complete theoretical description of the SMR, in particular to quantify how various material properties influence the SMR.

The structure of the article is arranged in the following manner. In section 2, we detail the quantum tunneling formalism for the interface spin current in the NM/FMI bilayer, and how it is adopted into the spin diffusion approach that describes the NM. Section 3 generalizes this recipe to the NM/FMM bilayer, and discuss the observability of the predicted signature of quantum interference in SMR. Section 4 gives the concluding remark.

2. NM/FMI bilayer

2.1. Interface spin current

We start with the quantum tunneling formalism that calculates the interface spin current in the NM/FMI bilayer, which later serves as the boundary condition for the spin diffusion equation that determines SMR. The quantum tunneling formalism describes the NM/FMI bilayer shown in figure 1(a) by the Hamiltonian

$$H_N = \frac{p^2}{2m} - \mu_s^e x (-l_N \leq x < 0), \quad (1)$$

$$H_{FI} = \frac{p^2}{2m} + V_0 + \Gamma S \cdot \sigma \quad (0 \leq x \leq l_{FI}), \quad (2)$$

where $\mu_s^e = \pm \mu_s \cdot \xi/2$ is the spin voltage of $\sigma = \{\uparrow, \downarrow\}$ produced by an in-plane charge current $J^x \cdot \hat{S}$. $V_0 - \epsilon_F$ is the insulating gap with $\epsilon_F$ the Fermi energy, and $S = S(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ is the magnetization. We choose $\Gamma < 0$ such that the magnetization has the tendency to align with the conduction electron spin $\sigma$. The wave function near the interface is

$$\psi_N = (A e^{i k_0 x} + B e^{-i k_0 x}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C e^{-i k_0 x} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3)$$

Figure 1. (a) Schematics of the bilayer consists of an NM with thickness $l_N$ and an FMI with thickness $l_{FI}$. (b) The spin mixing conductance $G_{ij}$ versus the FMI thickness $l_{FI}$ at different values of interface $s - d$ coupling strength $-\Gamma S / \ell_F$. The insulating gap strength is fixed at $(V_0 - \epsilon_F) / \ell_F = 1.5$. The absolute units for $G_{ij}$ is $\epsilon^3 h a^2$ which is about $10^{14} - 10^{15} \Omega^{-1} m^{-2}$ depending on the Fermi wave length $a$.

$$\psi_{FI} = (D e^{i k_0 x} + E e^{-i k_0 x}) \begin{pmatrix} e^{-i \eta/2} \cos \theta/2 \\ e^{i \eta/2} \sin \theta/2 \end{pmatrix}$$

$$+ (Fe^{i \eta x} + Ge^{-i \eta x}) \begin{pmatrix} -e^{-i \eta/2} \sin \theta/2 \\ e^{i \eta/2} \cos \theta/2 \end{pmatrix}, \quad (4)$$

where $k_{0F} = \sqrt{2m(\epsilon_F + \mu_s^e)}/h$ and $q_{\pm} = \sqrt{2m(\epsilon_0 \pm \Gamma S - \epsilon_F)}/h$. The amplitudes $B \sim E$ are solved in terms of the incident amplitude $A$ by matching wave functions and their first derivative at the interface. The $x < -l_N$ and $x > l_{FI}$ regions are assumed to be vacuum or insulting oxides that correspond to infinite potentials such that the wave functions vanish there for simplicity. We identify the incident flux with $|\alpha|^2 = N_F |\mu_0|/a^3$ where $N_F$ is the density of states per $a^3$ with $a = 2\pi / k_F = \hbar / \sqrt{2m\epsilon_F}$ the Fermi wave length.

The spin current inside the FMI at position $x$ is calculated from the evanescent wave function

$$J_x = \frac{\hbar}{4 \sin \theta} [\psi_{FI}^\dagger (\partial_x \psi_{FI}) - (\partial_x \psi_{FI})^\dagger \sigma \psi_{FI}]. \quad (5)$$

Angular momentum conservation [8, 28] dictates that the interface spin current to be equal to the STT exerts on the magnetization

$$\dot{\mathbf{J}}_0 - \dot{\mathbf{J}}_m = \frac{\tau}{a^2} = \frac{\hbar}{\Gamma_S N_F} \left[ G_0 \hat{S} \times (\hat{S} \times \mu_0) + G_0 \hat{S} \times \mu_0 \right], \quad (6)$$

which defines the field-like $G_0$ and damping-like $G_1$ spin mixing conductance that in turn can be calculated from the interface spin current [28]

$$\Gamma \frac{\hbar}{\Gamma_S} G_0 = \frac{2 j_0 \cos \phi}{|\mu_0| \sin 2\theta} + \frac{j_0 \sin \phi}{|\mu_0| \sin 2\theta} = -\frac{j_0}{|\mu_0| \sin^2 \theta},$$

$$\Gamma \frac{\hbar}{\Gamma_S} G_1 = \frac{j_0 \cos \phi}{|\mu_0| \sin \theta} - \frac{j_0}{|\mu_0| \sin \theta}. \quad (7)$$
A straight forward calculation yields

\[ G_{x,i} = \frac{4}{a^2} \left( q_x \coth q_x l_{FI} - q_x \coth q_x l_{FI}^* \right) \times (\text{Im}, \text{Re}) \left( n_i^+ - n_i^- \right), \]  

(8)

where \( \sigma_{x,y} \) is \( x, y \) component of Pauli matrix, and

\[ n_{\sigma\pm} = \frac{k_{0\sigma}}{k_{0\sigma} + i q_x \coth q_x l_{FI}}, \]

\[ \gamma_0 = \frac{n_{\sigma+} \cos \frac{\theta}{2} + n_{\sigma-} \sin \frac{\theta}{2}}{n_{\sigma+} \sin \frac{\theta}{2} + n_{\sigma-} \cos \frac{\theta}{2}}. \]  

(9)

Equation (8) describes the spin mixing conductance in STT, as well as in spin pumping since the Onsager relation is satisfied in this approach [28]. Both \( G_r \) and \( G_i \) have very weak dependence (at most few percent) on the angle of magnetization \( \theta \) through \( \gamma_0 \), which may be considered as higher order contributions [28]. In the numerical calculation below we set \( \theta = 0.3\pi \) without loss of generality.

Numerical results of the spin mixing conductance \( G_{x,i} \) are shown in figure 1, plotted as a function of the FMI thickness \( l_{FI} \) and at different strength of the interface \( s-d \) coupling \( \Gamma / e_F \). Both \( G_r \) and \( G_i \) increase with \( l_{FI} \) initially and then saturate to a constant as expected, since they originate from the quantum tunneling of conduction electrons that only penetrate into the FMI over a very short distance. At a FMI thickness small compared to Fermi wave length \( l_{FI} \ll a \), we found that \( G_r \propto l_{FI}^3 \) and \( G_i \propto l_{FI}^5 \), therefore the damping-like to field-like ratio is \( |G_i|/G_r| < 1 \) in most of the parameter space, the torque is dominated by field-like component \( |G_i/G_r| < 1 \) throughout the whole range of \( l_{FI} \). Only when the magnitude of \( s-d \) coupling is large compared to the insulating gap \( (V_0 - e_F l_{FI}) \) is the torque dominated by the damping-like component \( |G_i/G_r| > 1 \), consistent with that found previously [28] and also in accordance with the result from first principle calculation [26]. The magnitude of \( G_{x,i} \) generally increases with the \( s-d \) coupling, yet more dramatically for \( G_i \). Note that \( G_r \) and \( G_i \) do not depend on the NM thickness in this quantum tunneling approach.

### 2.2. SMR

We adopt the spin diffusion approach of Chen et al [23] to address the effect of the interface spin current in equation (6) on SMR, which is briefly summarized below. The spin diffusion approach is based on the following assumptions for the spin transport in the NM: (1) The spin current in NM consists of two parts, one from the spatial gradient of spin voltage and the other the bare spin current caused directly by SHE,

\[ j_x = -\frac{\sigma_x}{4e} \partial_x \mu_x + \frac{\theta_{SH} \sigma_x E_y}{2} \hat{z}. \]  

(10)

where \( \theta_{SH} \) is spin Hall angle, \( \sigma_x \) is the conductivity of NM, \( E_y \) is applied external electric in \( y \) direction, and \( -e \) is electron charge. (2) The spin voltage obeys the spin diffusion equation \( \nabla^2 \mu_x = \mu_x / \lambda^2 \), where \( \lambda \) is the spin diffusion length. (3) Spin current vanishes at the edge of NM (\( x = -l_b \)), which serves as one boundary condition. (4) The spin current at the NM/FMI interface is described by equation (6), which serves as another boundary condition. The self-consistent solution satisfying (1)-(4) is [23]

\[ \frac{j_x}{j_{SH}} \sin \theta \cos \phi = \beta_1 \sin \theta \sin \phi Re(G) + \sin \theta Im(G), \]

\[ \frac{j_y}{j_{SH}} \sin \theta \sin \phi = \beta_1 \sin \theta \phi Re(G) - \cos \theta Im(G), \]

\[ \frac{j_z}{j_{SH}} = 1 - \frac{\cos \theta}{\sin \theta} - \beta_1 \sin^2 \theta Re(G), \]  

(11)

where

\[ \beta_1 = \sinh \left( \frac{x + l_n}{\lambda} \right) \tan \left( \frac{l_n}{2\lambda} \right), \]

\[ G = \frac{\alpha G_e}{1 - \alpha G_e \coth \left( \frac{x}{\lambda} \right)}, \]

\[ \alpha = \frac{4G \sigma_{y}\epsilon^2\lambda}{\hbar \sigma}. \]  

(12)

and \( j_{SH} = \theta_{SH} \sigma_x E_y / 2e \) is the bare spin current. Here \( \alpha < 0 \) is a negative parameter (because we assume the interface \( s-d \) coupling \( \Gamma < 0 \) that bridges our tunneling formalism to the spin diffusion equation, and \( G_e = G_r + iG_i \) is the complex spin mixing conductance.

Through ISHE, the spin currents in equation (11) is converted back to a charge current in the longitudinal (along \( \hat{y} \)) and transverse (along \( \hat{z} \)) direction

\[ \Delta J_{\text{long}}(x) = -2e \theta_{SH} \left( j_x - \frac{\theta_{SH} \sigma_x E_y}{2e} \hat{z} \right) \cdot \hat{z}, \]  

(13)

\[ \Delta J_{\text{trans}}(x) = 2e \theta_{SH} \left( j_x - \frac{\theta_{SH} \sigma_x E_y}{2e} \hat{z} \right) \cdot \hat{y}. \]  

(14)

The conductivity averaged over the NM layer then follows

\[ \sigma_{\text{long}} = \sigma + \frac{1}{l_b E_F} \int_{-l_b}^{0} dx \Delta J_{\text{long}}(x), \]  

(15)

\[ \sigma_{\text{trans}} = \frac{1}{l_b E_F} \int_{-l_b}^{0} dx \Delta J_{\text{trans}}(x). \]  

(16)

Using \( \theta_{SH}^2 \sim 0.01 \ll 1 \), the longitudinal and transverse component of SMR read

\[ \rho_{\text{long}} = \sigma_{\text{long}}^{-1} \approx \rho + \Delta \rho_0 + \sin^2 \theta \Delta \rho_1, \]

\[ \rho_{\text{trans}} = -\sigma_{\text{trans}}/\sigma_{\text{long}}^2 \approx \cos \theta \sin \theta \sin \phi \Delta \rho_1 - \sin \theta \cos \phi \Delta \rho_2, \]  

(17)
As a function of the FMI thickness $l_{FI}$, which is expected since conduction electrons only tunnel into the FMI over a short depth, so the interface spin current saturates once the FMI is thicker than this tunneling depth. The insulating gap $\Delta$ affects the tunneling depth, and is particularly influential on the magnitude of $\Delta \rho_1 / \rho$ as can be seen by comparing plots with different $(V_0 - \epsilon_f) / \epsilon_f$ in figure 2. The magnitude of $\Delta \rho_1 / \rho$ also generally increases with the $s - d$ coupling $\Gamma S / \epsilon_f$, while $\Delta \rho_2 / \rho$ at large $\Gamma S / \epsilon_f$ displays a nonmonotonic dependence on the FMI thickness. On the other hand, as a function of NM thickness $l_N$, both SMR components increase and peak at around $l_N / \lambda \sim 2$ and then decrease monotonically for large $l_N$. This can be understood because both $\Delta \rho_1$ and $\Delta \rho_2$ are interface effects that become less significant compared to bulk resistivity $\rho$ when NM thickness increases, and the spin voltage is known to be maximal when the NM thickness is comparable to the spin diffusion length $l_{SD} \sim l_{FI}$.

The numerical result of SMR is shown in figure 2, plotted as a function of the FMI thickness $l_{FI}$ and NM thickness $l_N$ at several values of insulating gap $(V_0 - \epsilon_f) / \epsilon_f$ and $s - d$ coupling $\Gamma S / \epsilon_f$. As a function of the FMI thickness $l_{FI}$, both $\Delta \rho_1 / \rho$ and $\Delta \rho_2 / \rho$ initially increase and then saturate at around $l_{FI} / \lambda \sim 2$, which is expected since conduction electrons only tunnel into the FMI over a short depth, so the interface spin current saturates once the FMI is thicker than this tunneling depth. The insulating gap $(V_0 - \epsilon_f) / \epsilon_f$ obviously affects the tunneling depth, and is particularly influential on the magnitude of $\Delta \rho_1 / \rho$, as can be seen by comparing plots with different $(V_0 - \epsilon_f) / \epsilon_f$ in figure 2.

The magnitude of $\Delta \rho_1 / \rho$ also generally increases with the $s - d$ coupling $\Gamma S / \epsilon_f$, while $\Delta \rho_2 / \rho$ at large $\Gamma S / \epsilon_f$ displays a nonmonotonic dependence on the FMI thickness. On the other hand, as a function of NM thickness $l_N$, both SMR components increase and peak at around $l_N / \lambda \sim 2$ and then decrease monotonically for large $l_N$. This can be understood because both $\Delta \rho_1$ and $\Delta \rho_2$ are interface effects that become less significant compared to bulk resistivity $\rho$ when NM thickness increases, and the spin voltage is known to be maximal when the NM thickness is comparable to the spin diffusion length $l_{SD} \sim l_{FI}$.

Clearly the FMI thickness $l_{FI}$ affects $\Delta \rho_1$ and $\Delta \rho_2$ only through the $\bar{G} = \bar{G}(l_{FI})$ in equation (12).

To perform numerical calculation of equation (18), we make the following assumption on the parameter $\alpha$ in equation (12) that connects the quantum tunneling formalism with the spin diffusion equation. Firstly, $\alpha$ contains the density of state per $a^2$ at the Fermi surface, which is assumed to be the inverse of Fermi energy $N_F = 1/\epsilon_F$. The combined parameter $\Gamma S N_F = \Gamma S / \epsilon_f$ therefore represents the strength of $s - d$ coupling. Other parameters that influence $\alpha$ are the spin diffusion length assumed to be $\lambda \approx 10$ nm, the conductivity of the NM film taken to be $\sigma_{C} \approx 5 \times 10^{8} \text{ } \Omega^{-1} \text{ } \text{m}^{-1}$, and Fermi wave length assumed to be roughly equal to the lattice constant $a \approx 0.4$ nm, all of which are the typical values for commonly used materials such as Pt. These lead to the dimensionless parameter $\alpha G_{0} \approx 10 \times (\Gamma S / \epsilon_f) \times (G_{0} \epsilon^{2} / \hbar a^{2})$ in equation (12) being expressed in terms of the relative strength of $s - d$ coupling and the spin mixing conductance divided by its unit. In what follows, we examine the effect of FMI thickness, NM thickness, insulating gap, and interface $s - d$ coupling on SMR. On the contrary, the spin Hall angle, spin diffusion length, and conductivity are treated as constants, although in reality they may also depend on the layer thickness or on each other in such thin films [35].

The numerical result of SMR is shown in figure 2, plotted as a function of the FMI thickness $l_{FI}$ and NM thickness $l_N$ at several values of insulating gap $(V_0 - \epsilon_f) / \epsilon_f$ and $s - d$ coupling $\Gamma S / \epsilon_f$. As a function of the FMI thickness $l_{FI}$, both $\Delta \rho_1 / \rho$ and $\Delta \rho_2 / \rho$ initially increase and then saturate at around $l_{FI} / \lambda \sim 2$, which is expected since conduction electrons only tunnel into the FMI over a short depth, so the interface spin current saturates once the FMI is thicker than this tunneling depth. The insulating gap $(V_0 - \epsilon_f) / \epsilon_f$ obviously affects the tunneling depth, and is particularly influential on the magnitude of $\Delta \rho_1 / \rho$, as can be seen by comparing plots with different $(V_0 - \epsilon_f) / \epsilon_f$ in figure 2. The magnitude of $\Delta \rho_1 / \rho$ also generally increases with the $s - d$ coupling $\Gamma S / \epsilon_f$, while $\Delta \rho_2 / \rho$ at large $\Gamma S / \epsilon_f$ displays a nonmonotonic dependence on the FMI thickness. On the other hand, as a function of NM thickness $l_N$, both SMR components increase and peak at around $l_N / \lambda \sim 2$ and then decrease monotonically for large $l_N$. This can be understood because both $\Delta \rho_1$ and $\Delta \rho_2$ are interface effects that become less significant compared to bulk resistivity $\rho$ when NM thickness increases, and the spin voltage is known to be maximal when the NM thickness is comparable to the spin diffusion length $l_{SD} \sim l_{FI}$.

where

$$
\Delta \rho_1 / \rho = -\theta_{S H}^{2} \frac{2 \lambda}{l_{N}} \tan \left(\frac{l_{N}}{2 \lambda}\right),
$$

$$
\Delta \rho_2 / \rho = -\theta_{S H}^{2} \frac{\lambda}{l_{N}} \tan \left(\frac{l_{N}}{2 \lambda}\right) \Re(\bar{G}),
$$

$$
\Delta \rho_2 / \rho = \theta_{S H}^{2} \frac{\lambda}{l_{N}} \tan \left(\frac{l_{N}}{2 \lambda}\right) \Im(\bar{G}).
$$

Figure 2. The $\Delta \rho_1 / \rho$ and $-\Delta \rho_2 / \rho$ component of SMR in the NM/FMI bilayer described by equations (17) and (18), plotted against the FMI thickness in units of Fermi wave length $l_{FI}/a$ and NM thickness in units of the spin diffusion length $l_{SD}/\lambda$, at various strength of $s - d$ coupling $-\Gamma S / \epsilon_f$ and the insulating gap $(V_0 - \epsilon_f) / \epsilon_f$. Note that the color scale of each plot is different.
3. NM/FMM bilayer

3.1. Interface spin current and SMR

We proceed to address the SMR in the NM/FMM bilayer, with the assumption that the FMM film itself has negligible SHE and is much thinner than its spin diffusion length $l_{\text{FM}} \ll \lambda$, such that quantum tunneling is the dominant mechanism for spin transport in the FMM, while the spin diffusion inside the FMM can be ignored. The calculation of the spin current at the NM/FMM interface starts with the model schematically shown in figure 3(a). The NM and FMM occupy $-l_N \leq x < 0$ and $0 \leq x \leq l_{\text{FM}}$, respectively. The NM region is described by equations (1) and (3), while the FMM layer is described by $H_{\text{FM}} = \mu/2m + \mathbf{1}\Sigma \cdot \sigma$ and the wave function

$$\psi_{\text{FM}} = (D e^{i k x} + F^{-i k x})(e^{i \pi/2 \cos \theta/2} e^{i \pi/2 \sin \theta/2}) + (E e^{i k x} + G e^{-i k x})(-e^{-i \pi/2 \sin \theta/2} e^{i \pi/2 \cos \theta/2}), \quad (19)$$

where $k_\pm = \sqrt{2m(eF \mp \Gamma S)/\hbar}$. The wave functions outside of the bilayer in $x > l_{\text{FM}}$ and $x < -l_N$ are assumed to vanish for simplicity. The coefficients $A \sim I$ are again determined by matching wave functions and their first derivative at the interface. The interface spin current and the spin mixing conductance are calculated from equations (5) to (7), with replacing $\psi_I$ by $\psi_{\text{FM}}$ and $l_I$ to $l_{\text{FM}}$, resulting in

$$G_{o\beta} = \frac{1}{a \gamma f^2 (\text{Im}, \text{Re})} \left( Z_{i-\beta} \delta_{i+\alpha} \right) \times (u_{+-} - u_{++} - u_{-+} + u_{++}), \quad (20)$$

where

$$u_{o\beta} = \frac{e^{i(\alpha k + \beta k_{\text{FM}})}}{\alpha k + \beta k_{\text{FM}}}, \quad W_{o\beta \gamma} = \frac{k_{o\alpha} + \beta k_{o\alpha}}{2k_{o\alpha}},$$

$$Z_{o\alpha \beta} = W_{o\alpha \gamma} e^{-i k_{\text{FM}}}, \quad Z_{o\alpha \beta} = W_{o\beta \gamma} e^{i k_{\text{FM}}},$$

$$\gamma_{\beta} = \cos \theta/2 + Z_{\beta-} Z_{\beta+} \sin \theta/2. \quad (21)$$

To get SMR, we use equations (17) and (18) while taking the $G = G_e + i G_i$ obtained from equation (20). The results for the $\Delta \rho_1/\rho$ and $\Delta \rho_2/\rho$ components of SMR as functions of FMM thickness $l_{\text{FM}}$ and NM thickness $l_N$ are shown in figure 4, for several values of $s - d$ coupling $\Gamma S/eF$. As a function of NM thickness, both components reach a maximum at around the spin diffusion length $l_N/\lambda \sim 1$ and then decrease monotonically, similar to that reported in figure 2 for NM/FMI bilayer and is due to the spin diffusion effect explained in section 2.2. On the other hand, as a function of FMM thickness, both components show clear modulations with an average periodicity that decreases with increasing $s - d$ coupling, a trend similar to that of $G_e$ and $G_i$ shown in figure 3 and is attributed to the quantum interference of spin transport. Intuitively, a larger $s - d$ coupling renders a faster precession of conduction electron spin when it travels inside the FMM, hence more modulations appear for a given FMM thickness. The $\Delta \rho_1$ component of SMR is found to be generally one order of magnitude smaller than the $\Delta \rho_2$ component.

We remark that an oscillating dependence on FMM layer thickness due to quantum effects has been reported for other spin transport related quantities, such as magnetoresistance [36], anisotropic magnetoresistance (AMR) [37], and interlayer exchange coupling [38]. Recent SMR experiments in NiFe/Pt also show hints for a nonmonotonic dependence on the NiFe thickness [39], although in reality other effects not taken into account by our minimal model, such as surface roughness, may also be the origin. We also mention that the longitudinal resistance $\rho_{\text{long}}$ in equation (17) is invariant under inversion of the magnetization $S \rightarrow -S$, in accordance with the first harmonic resistance $R_\omega$ measured in Ta/Co and Pt/Co bilayers, where $\omega$ is the frequency of the applied a.c. current [14]. On the other hand, a unidirectional spin Hall magnetoresistance (USMR) in the second harmonic $R_\omega$ that is odd under inversion of magnetization has also been observed [14], and has been related to a spin-dependent interface resistance due to SHE induced spin accumulation. This USMR is anticipated to be outside of the scope of our minimal model, and may require a certain Boltzmann equation type of approach to capture.

Notice that since the FMM itself is conducting and sandwiched between two different materials, an in-plane charge current is expected to induce a spin accumulation in the FMM due to Rashba spin–orbit coupling (SOC) [40–42], which may subsequently modify the boundary conditions discussed below equation (10) that we used to solve the spin diffusion equation. Given that the quantification of the relative weight between the spin accumulation induced by Rashba SOC and that induced by SHE requires a separate treatment, we ignore this SOC-induced spin accumulation and examine solely the influence of SHE. Despite this simplification, the calculated SMR shown in figure 4 has a magnitude similar to that reported experimentally, meaning that the SHE alone already contributes significantly to the SMR. Thus we also anticipate that the predicted quantum interference effect should still be manifest even if the Rashba SOC is included.
3.2. To observe the predicted oscillation in SMR

The experimental detection of the oscillation of SMR with respect to FMM thickness $h_{FM}$ shown in figure 4 would be a direct proof of our approach. In a typical NM/FMM setup, however, the resistance of both NM and FMM contribute to the total resistance measured in experiments, therefore it is important to investigate whether there is a situation in which the predicted oscillation of SMR can manifest. To explore this possibility, we use a three-resistor model to characterize the total longitudinal resistance [14], which contains the resistor that represents the NM layer ($R_N$), the FMM layer ($R_{FM}$), and the interface layer ($R_I$) connected in parallel, each denoted by $R_{i0}\delta R_i$ with $i = \{N, I, F\}$. Here $R_{i0}$ is the contribution to the longitudinal resistance in layer $i$ that does not depend on the angle of the magnetization, and $\delta R_i$ is the part that depends on the angle which is generally much smaller $\delta R_i \ll R_{i0}$. Expanding the total longitudinal resistance to leading order in $\delta R_i$ yields

$$R_{tot} \approx R_{tot}^0 + \left( \frac{R_{i0}^0 R_i^0}{B} \right)^2 \delta R_N + \left( \frac{R_{i0}^0 R_i^0}{B} \right)^2 \delta R_I + \left( \frac{R_{N0}^0 R_{I0}^0}{B} \right)^2 \delta R_{FM},$$

$$R_{tot}^0 = \frac{R_{N0}^0 R_{I0}^0 R_{F0}^0}{B},$$

$$B = R_{N0}^0 R_{I0}^0 + R_{I0}^0 R_{F0}^0 + R_{N0}^0 R_{F0}^0.$$ (22)

Each resistance is assumed to satisfy the usual relation to the sample size $R_{i0} l_i \rho_i = \rho_{i0} l_i$, where $L$ and $h$ are the length and the width of the sample, respectively, $\rho_{i0}$ and $\delta \rho_i$ are the corresponding resistivity, and $l_i$ is the thickness of layer $i$.

The thickness of the interface layer $I$ does not depend on the angle of the magnetization, and $\delta \rho_{IF}$ is the corresponding resistivity, and $l_i$ is the thickness of layer $i$. The thickness of the interface layer $I$ is assumed to be intrinsically constant, in contrast to $l_N$ and $l_F$ that can be varied experimentally [14]. The percentage change of the total resistance due to the angle of the magnetization is

$$R_{tot} - R_{tot}^0 \approx \left( \frac{\rho_{IF}^0}{l_{IF} \rho_{F0}} \right) \delta \rho_{IF} \left( \frac{\rho_{N0}}{l_N \rho_{N0}} \right) \delta \rho_N + \left( \frac{\rho_{N0} \rho_{F0}}{l_N l_{IF} \rho_{F0}} \right) \delta \rho_{IF} \left( \frac{\rho_{I0}}{l_I \rho_{I0}} \right) \delta \rho_I + \left( \frac{\rho_{F0} \rho_{I0}}{l_{IF} l_I \rho_{I0}} \right) \delta \rho_{IF},$$

$$C = \frac{\rho_{N0} \rho_{F0}}{l_N l_{IF} \rho_{F0}} + \frac{\rho_{N0} \rho_{F0}}{l_N l_{IF} \rho_{F0}} + \frac{\rho_{F0} \rho_{I0}}{l_{IF} l_I \rho_{I0}}.$$ (23)

Note that the $\rho_{i0} l_i \rho_{IF}^0 l_{IF}$ factors are monotonic functions of the layer thicknesses $l_{N, I, F}$, and are independent from the angle of the magnetization.

The contribution to the angular dependent part of $R_{IF}$ comes from the AMR which takes the form [43, 44] $\delta \rho_{IF} \propto (j \cdot \hat{m})^2 \propto (m^2)$ since the in-plane charge current $j^i$ runs along $\hat{y}$ as shown in figure 3(a), and we denote $\hat{m} = \hat{S}/S = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$ as the unit vector along the direction of the magnetization. In addition, Zhang et al [44] showed that the interface resistance has a quadratic dependence on $\delta \rho_{IF}$.
dependence on both $m^r$ and $m^t$, a result of surface spin–orbit scattering. On the other hand, the SMR in the NM has the angular dependence [23] described by equation (17). These considerations lead to the parameterization of resistivity by

$$\begin{align}
\rho_F^0 + b\rho_F^0 &= \rho_F^0 + \Delta \rho_F^b (m^t)^2, \\
\rho_1^0 + b\rho_1^0 &= \rho_1^0 + \Delta \rho_1^b (m^t)^2 + \Delta \rho_1^c (m^t)^2, \\
\rho_N^0 + b\rho_N^0 &= (\rho + \Delta \rho_0^0) + \Delta \rho_1 [(m^t)^2 + (m^t)^2].
\end{align}$$

Combining this with equation (23) motivates us to propose the following experiment that should isolate the effect of longitudinal SMR represented by $b\rho_N$. From equation (24), we see that $b\rho_F$ and $b\rho_1$ vanish if the magnetization does not have an in-plane component, i.e. $m^r = m^t = 0$, while $b\rho_N$ remains finite as long as the out-of-plane component is nonzero $m^t \neq 0$. Thus we propose to fix the magnetization of the FMM film to be out-of-plane $m^t = 0$, in which case the percentage change of total longitudinal resistance as a function of FMM thickness takes the form

$$\frac{R_{\text{tot}} - R_{\text{tot}}^0}{R_{\text{tot}}^0} \approx \frac{l_1}{l_F + l_1 + l_2} \times \frac{\Delta \rho_1}{\rho + \Delta \rho_0} (m^t)^2,$$

(for $m^r = m^t = 0$) (25)

where $\rho, \Delta \rho_0$, and $\Delta \rho_1$ are those in equations (17) and (18), $l_1 = l_N^0 / \rho_1^0$ and $l_2 = l_N^1 / \rho_1^1$ are two length scales that can be treated as fitting parameters in experiments. Equation (25) indicates that, for the case of only out-of-plane magnetization, the percentage change of magnetoresistance decays with the FMM thickness $l_F$ due to the $l_1(l_F + l_1 + l_2)$ factor, but also oscillates with $l_F$ due to the $\Delta \rho_1/(\rho + \Delta \rho_0) \approx \Delta \rho_1/\rho$ factor as quantified in equation (18) and shown in figure 4. Thus varying FMM thickness while keeping its magnetization out-of-plane may be a proper set up to observe the predicted oscillation of longitudinal SMR, provided the FMM thickness remains thinner than its spin relaxation length $l_F \ll \lambda$.

Note that the convention of labeling coordinate in SMR or STT experiments is that the charge current flows by definition along $\hat{x}$ and the direction normal to the film is along $\hat{z}$. Therefore the coordinate in our tunneling formalism $(x, y, z)$ corresponds to $(z, x, y)$ in the experimental convention. Finally, we remark that the surface roughness and surface absorption in a realistic NM/FMM bilayer may cause difficulty in experimentally observing the predicted quantum size effect, whose precise effects are however beyond the scope of our minimal model.

4. Conclusion

In summary, a quantum tunneling formalism is incorporated into the spin diffusion approach to study the effect of various material properties on SMR, in particular the effect of layer thickness, insulating gap, and interface $s-d$ coupling. In contrast to first principle calculations that target one particular combination of materials at a time, our minimal model is a very efficient tool to quantify how these material properties affect SMR, and treat them on equal footing. For the NM/FMI case, we reveal an SMR that saturates at large FMI thickness since the conduction electrons only tunnels into the FMI over a short distance, whereas the $\Delta \rho_1$ and $\Delta \rho_2$ components of SMR display different dependence on the insulating gap and interface $s-d$ coupling. For the NM/FMM case, we predict that SMR may display a pattern of oscillation as increasing FMM thickness due to quantum interference, and propose an experiment to observe it by using fixed out-of-plane magnetization to isolate SMR from other contributions. We anticipate that our minimal model that combines the quantum and diffusive approach may be used to guide the search for suitable materials that optimize the SMR, and help to predict novel spin transport effects in ultrathin heterostructures in which quantum effects shall not be overlooked.
Acknowledgments

The authors acknowledge the fruitful discussions with P. Gambardella, F. Casanova, and J. Mendil. WC and MS are grateful for the financial support through a research grant of the Swiss National Science Foundation.

References

[1] Wolf S A, Awschalom D D, Buhrman R A, Daughton J M, von Molnár S, Roukes M L, Chtchelkanova A Y and Treger D M 2001 Science 294 1488
[2] Žužić I, Fabian J and Das Sarma S 2004 Rev. Mod. Phys. 76 323
[3] Dyakonov M I and Perel V I 1971 Phys. Lett. A 35 459
[4] Hirsch J E 1999 Phys. Rev. Lett. 83 1834
[5] Sinova J, Culcer D, Niu Q, Sinitsyn N A, Jungwirth T and MacDonald A H 2004 Phys. Rev. Lett. 92 126603
[6] Sinova J, Valenzuela S O, Wunderlich J, Back C H and Jungwirth T 2015 Rev. Mod. Phys. 87 1213
[7] Berger L 1996 Phys. Rev. B 54 9353
[8] Slonczewski J C 1996 J. Magn. Magn. Mater. 159 L1–7
[9] Saitoh E, Ueda M, Miyajima H and Tatara G 2006 Appl. Phys. Lett. 88 182509
[10] Kimura T, Otani Y, Sato T, Takahashi S and Maekawa S 2007 Phys. Rev. Lett. 98 156601
[11] Tserkovnyak Y, Brataas A and Bauer G E W 2002 Phys. Rev. Lett. 88 117601
[12] Zhang S and Li Z 2004 Phys. Rev. Lett. 93 127204
[13] Nakayama H et al 2013 Phys. Rev. Lett. 110 206601
[14] Avci C O, Garello K, Ghosh A, Gabureac M, Alvarado S F and Gambardella P 2015 Nat. Phys. 11 570
[15] Althammer M et al 2013 Phys. Rev. B 87 224401
[16] Vlietstra N, Shan J, Castel V, van Wees B J and Ben Youssef J 2013 Phys. Rev. B 87 184421
[17] Hahn C, de Loubens G, Klein O, Viret M, Naleto V V and Ben Youssef J 2013 Phys. Rev. B 87 174417
[18] Vlietstra N, Shan J, Castel V, Ben Youssef J, Bauer G E W and van Wees B J 2013 Appl. Phys. Lett. 103 032401
[19] Isasa M, Bedoya-Pinto A, Vélez S, Golmar F, Sánchez F, Hueso L E, Fontcuberta J and Casanova F 2014 Appl. Phys. Lett. 105 142402
[20] Vlietstra N, Shan J, van Wees B J, Isasa M, Casanova F and Ben Youssef J 2014 Phys. Rev. B 90 174436
[21] Marmion S R, Ali M, McLaren M, Williams D A and Hickey B J 2014 Phys. Rev. B 89 224404
[22] Kim J, Sheng P, Takahashi S, Mitani S and Hayashi M 2016 Phys. Rev. Lett. 116 097201
[23] Chen Y-T, Takahashi S, Nakayama H, Althammer M, Goennenwein S T B, Saitoh E and Bauer G E W 2013 Phys. Rev. B 87 144411
[24] Chen Y-T, Takahashi S, Nakayama H, Althammer M, Goennenwein S T B, Saitoh E and Bauer G E W 2016 J. Phys.: Condens. Matter 28 103004
[25] Zhang S 2000 Phys. Rev. Lett. 85 393
[26] Jia X, Liu K, Xia K and Bauer G E W 2011 Europhys. Lett. 96 17005
[27] Burrowes C, Heinrich B, Kardasz B, Montoya E A, Girt E, Sun Y, Song Y-Y and Wu M 2012 Appl. Phys. Lett. 100 092403
[28] Chen W, Sigrist M, Sinova J and Manske D 2015 Phys. Rev. Lett. 115 217203
[29] Chen W, Sigrist M and Manske D 2016 Phys. Rev. B 94 104412
[30] Du C H, Wang H L, Pu Y, Meyer T L, Woodward P M, Yang F Y and Hammel P C 2013 Phys. Rev. Lett. 111 247202
[31] Bass J and Pratt W P Jr 2007 J. Phys.: Condens. Matter 19 183201
[32] Steenwyk S D, Hsu S Y, Loloee R, Bass J and Pratt W P Jr 1997 J. Magn. Magn. Mater. 170 L1
[33] Dubois S, Piriaux L, George J M, Ounadjela K, Duval J L and Fert A 1999 Phys. Rev. B 60 477
[34] Maekawa S, Valenzuela S O, Saitoh E and Kimura T 2012 Spin Current 1st edn (Oxford: Oxford University Press) ch 8
[35] Sagasta E, Omori Y, Isasa M, Gradhand M, Hueso L E, Niimi Y, Otani Y and Casanova F 2016 Phys. Rev. B 94 060412(R)
[36] Okuno S N and Inomata K 1994 Phys. Rev. Lett. 72 1553
[37] Li J X, Jia M W, Sun L, Ding Z, Chen B L and Wu Y Z 2015 IEEE Magn. Lett. 6 2500304
[38] Bloemen P J H, van de Vorst M T H, Johnson M T, Coehoorn R and de Jonge W J M 1994 J. Appl. Phys. 76 7081
[39] Yang X, Xu Y, Yao K and Wu Y 2016 AIP Adv. 6 065203
[40] Manchon A and Zhang S 2008 Phys. Rev. B 78 214205
[41] Manchon A and Zhang S 2009 Phys. Rev. B 79 094422
[42] Gambardella P and Miron I M 2011 Phil. Trans. R. Soc. A 369 3175
[43] McGuire T and Potter R 1975 IEEE Trans. Magn. 11 1018
[44] Zhang S S-L, Vignale G and Zhang S 2015 Phys. Rev. B 92 024412