A surface with $q = 2$ and canonical map of degree 16

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Abstract

We construct a surface with irregularity $q = 2$, geometric genus $p_g = 3$, self-intersection of the canonical divisor $K^2 = 16$ and canonical map of degree 16.

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1 Introduction

Let $S$ be a smooth minimal surface of general type. Denote by $\phi : S \dashrightarrow \mathbb{P}^{p_g-1}$ the canonical map and let $d := \deg(\phi)$. The following Beauville’s result is well-known.

Theorem 1 (Be). If the canonical image $\Sigma := \phi(S)$ is a surface, then either:

(i) $p_g(\Sigma) = 0$, or

(ii) $\Sigma$ is a canonical surface (in particular $p_g(\Sigma) = p_g(S)$).

Moreover, in case (i) $d \leq 36$ and in case (ii) $d \leq 9$.

Beauville has also constructed families of examples with $\chi(\mathcal{O}_S)$ arbitrarily large for $d = 2, 4, 6, 8$ and $p_g(\Sigma) = 0$. Despite being a classical problem, for $d > 8$ the number of known examples drops drastically: only Tan’s example [Ta, §5], with $d = 9$, the author’s [Ri] example with $d = 12$ and Persson’s example [Pe] with $d = 16$ are known. Du and Gao [DuGa] show that if the canonical map is an abelian cover of $\mathbb{P}^2$, then these examples with $d = 9$ and $d = 16$ are the only possibilities for $d > 8$. These three surfaces are regular, so for irregular surfaces all known examples satisfy $d \leq 8$. We get from Beauville’s proof that lower bounds hold for irregular surfaces. In particular,

$q = 2 \implies d \leq 18$.

In this note we construct an example with $q = 2$ and $d = 16$. The idea of the construction is the following. We start with a double plane with geometric genus $p_g = 3$, irregularity $q = 0$, self-intersection of the canonical divisor $K^2 = 2$ and singular set the union of 10 points of type $A_1$ (nodes) and 8 points of type $A_3$ (standard notation, the resolution of a singularity of type $A_n$ is a chain of $(-2)$-curves $C_1, \ldots, C_n$ such that $C_iC_{i+1} = 1$ and $C_iC_j = 0$ for $j \neq i \pm 1$). Then we take a double covering ramified over the points of type $A_3$ and obtain a surface with $p_g = 3$, $q = 0$ and $K^2 = 4$ with 28 nodes. A double covering ramified over 16 of these 28 nodes gives a surface with $p_g = 3$, $q = 0$ and $K^2 = 8$ with 24
nodes (which is a $\mathbb{Z}_3^2$-covering of $\mathbb{P}^2$). Finally there is a double covering ramified over these 24 nodes which gives a surface with $p_g = 3$, $q = 2$ and $K^2 = 16$ and the canonical map factors through these coverings, thus it is of degree 16.

**Notation**

We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. Linear equivalence of divisors is denoted by $\equiv$. The rest of the notation is standard in Algebraic Geometry.

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**2 $\mathbb{Z}_3^2$-coverings**

The following is taken from [Ca], an alternative reference is [Pa].

**Proposition 2.** A normal finite $G \cong \mathbb{Z}_2^r$-covering $Y \to X$ of a smooth variety $X$ is completely determined by the datum of

1. reduced effective divisors $D_\sigma$, $\forall \sigma \in G$, with no common components;

2. divisor classes $L_1, \ldots, L_r$, for $\chi_1, \ldots, \chi_r$ a basis of the dual group of characters $G^\vee$, such that

$$2L_i \equiv \sum_{\chi_i(\sigma) = -1} D_\sigma.$$

Conversely, given 1. and 2., one obtains a normal scheme $Y$ with a finite $G \cong \mathbb{Z}_2^r$-covering $Y \to X$.

The covering $Y \to X$ is embedded in the total space of the direct sum of the line bundles whose sheaves of sections are the $\mathcal{O}_X(-L_i)$, and is there defined by equations

$$u_{\chi_i} = u_{\chi_i} + x_\sigma \prod_{\chi_i(\sigma) = -1} x_\sigma,$$

where $x_\sigma$ is a section such that $\text{div}(x_\sigma) = D_\sigma$. The scheme $Y$ can be seen as the normalization of the Galois covering given by the equations

$$u_{\chi_i}^2 = \prod_{\chi_i(\sigma) = -1} x_\sigma.$$

The scheme $Y$ is irreducible if $\{ \sigma | D_\sigma > 0 \}$ generates $G$. For the reader’s convenience, we leave here the character table for the group $\mathbb{Z}_3^2$ with generators $x, y, z$. 

2
3 The construction

Step 1
Let $T_1, \ldots, T_4 \subset \mathbb{P}^2$ be distinct lines tangent to a smooth conic $H_1$ and

$$\pi : X \rightarrow \mathbb{P}^2$$

be the double cover of the projective plane ramified over $T_1 + \cdots + T_4$. The curve $\pi^*(H_1)$ is of arithmetic genus 3, from the Hurwitz formula, and has 4 nodes, corresponding to the tangencies to $T_1 + \cdots + T_4$. Hence $\pi^*(H_1)$ is reducible,

$$\pi^*(H_1) = A + B$$

with $A, B$ smooth rational curves. From $AB = 4$ and $(A + B)^2 = 8$ we get $A^2 = B^2 = 0$. Now the adjunction formula

$$2g(A) - 2 = AK_X + A^2$$

gives $AK_X = -2$ and then the Riemann-Roch Theorem implies

$$h^0(X, \mathcal{O}_X(A)) \geq 1 + \frac{1}{2}A(A - K_X) = 2.$$ 

Therefore there exists a smooth rational curve $C$ such that $C \neq A, C \equiv A$ and $AC = 0$. The curve

$$H_2 := \pi(C)$$

is smooth rational. The fact $\pi^*(H_2)^2 > C^2$ implies that $\pi^*(H_2)$ is reducible, thus $H_2$ is tangent to the lines $T_1, \ldots, T_4$. As above, there is a smooth rational curve $D$ such that

$$\pi^*(H_2) = C + D$$

and $C^2 = D^2 = 0$. Since $A \equiv C$ and $A + B \equiv C + D$, then $B \equiv D$. 

\[ 
\begin{bmatrix}
  -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & x*y*z \\
  -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & z \\
  -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & y \\
  -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & x \\
  1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & y*z \\
  1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & x*z \\
  1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & x*y \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \text{Id}
\end{bmatrix}
\]
Step 2
Let $x, y, z$ be generators of the group $\mathbb{Z}_3^2$ and

$$\psi : Y \longrightarrow \mathbb{P}^2$$

be the $\mathbb{Z}_2^2$-covering defined by

$$D_1 := D_{xyz} := H_1, D_2 := D_z := H_2, D_3 := D_y := T_1 + T_2, D_4 := D_x := T_3 + T_4,$$

$$D_{yz} := D_{xz} := D_{xy} := 0.$$

Let $d_i$ be the defining equation of $D_i$. According to Section 2, the surface $Y$ is obtained as the normalization of the covering given by equations

$$u_1^2 = d_1 d_2 d_3 d_4, u_2^2 = d_1 d_2, \ldots, u_7^2 = d_3 d_4.$$

Since the branch curve $D_1 + \cdots + D_4$ has only negligible singularities, the invariants of $Y$ can be computed directly. Consider divisors $L_{i...h}$ such that $2L_{i...h} \equiv D_i + \cdots + D_h$ and let $T$ be a general line in $\mathbb{P}^2$. We have

$$L_{1234}(K_{\mathbb{P}^2} + L_{1234}) = 4T \cdot T = 4,$$

$$L_{ij}(K_{\mathbb{P}^2} + L_{ij}) = 2T(-T) = -2,$$

thus

$$\chi(Y) = 8 \chi(\mathbb{P}^2) + \frac{1}{2} (4 + 6 \times (-2)) = 4,$$

$$p_g(Y) = p_g(\mathbb{P}^2) + h^0(\mathbb{P}^2, O_{\mathbb{P}^2}(T)) + 6h^0(\mathbb{P}^2, O_{\mathbb{P}^2}(-T)) = 3.$$

So a canonical curve in $Y$ is the pullback of a line in $\mathbb{P}^2$ and then

$$K_Y^2 = 8.$$

Step 3
Notice that the points where two curves $D_i$ meet transversely give rise to smooth points of $Y$, hence the singularities of $Y$ are:

- 16 points $p_1, \ldots, p_{16}$ corresponding to the tacnodes of $D_1 + \cdots + D_4$;
- 8 nodes $p_{17}, \ldots, p_{24}$ corresponding to the nodes of $D_3$ and $D_4$.

We want to show that $p_1, \ldots, p_{24}$ are nodes with even sum.

The surface $X$ defined in Step 1 is the double plane with equation $u_7^2 = d_3 d_4$, thus the covering $\psi$ factors trough a $\mathbb{Z}_2^2$-covering

$$\varphi : Y \longrightarrow X.$$

The branch locus of $\varphi$ is $A + B + C + D$ plus the 4 nodes given by the points in $D_3 \cap D_4$. The points $p_1, \ldots, p_{16}$ are nodes because they are the pullback of nodes of $A + B + C + D$.

The divisor $\varphi^*(A + C)$ is even ($A + C \equiv 2A$), double ($A + C$ in the branch locus of $\varphi$), with smooth support ($A + C$ smooth) and $p_1, \ldots, p_{16} \in \varphi^*(A + C)$,
Consider the minimal resolution of the singularities of $Y$ 

$$\rho : Y' \rightarrow Y$$

and let $A_1, \ldots, A_{24} \subset Y'$ be the $(-2)$-curves corresponding to the nodes $p_1, \ldots, p_{24}$. The divisor $(\varphi \circ \rho)^*(A + C)$ is even and there exists a divisor $E$ such that

$$(\varphi \circ \rho)^*(A + C) = 2E + \sum_{i=1}^{16} A_i.$$ 

Thus there exists a divisor $L_1$ such that $\sum_{i=1}^{16} A_i \equiv 2L_1$.

Analogously one shows that the nodes $p_{17}, \ldots, p_{24}$ have even sum, i.e. there exists a divisor $L_2$ such that $\sum_{i=1}^{24} A_i \equiv 2L_2$. This follows from $\psi^*(T_1 + T_3)$ even, double, and with support of multiplicity 1 at $p_{17}, \ldots, p_{24}$ and of multiplicity 2 at 8 of the nodes $p_1, \ldots, p_{16}$.

**Step 4**

So there is a divisor $L := L_1 + L_2$ such that

$$\sum_{i=1}^{24} A_i \equiv 2L.$$ 

Consider the double covering $S \rightarrow Y$ ramified over $p_1, \ldots, p_{24}$ and determined by $L$. More precisely, given the double covering $\eta : S' \rightarrow Y'$ with branch locus $\sum_{i=1}^{24} A_i$, determined by $L$, $S$ is the minimal model of $S'$. We have

$$\chi(S') = 2\chi(Y') + \frac{1}{2} L(K_{Y'} + L) = 8 - 6 = 2.$$ 

Since the canonical system of $Y$ is given by the pullback of the system of lines in $\mathbb{P}^2$, the canonical map of $Y$ is of degree 8 onto $\mathbb{P}^2$. We want to show that the canonical map of $S'$ factors through $\eta$.

One has

$$p_g(S') = p_g(Y') + h^0(Y', \mathcal{O}_{Y'}(K_{Y'} + L)),$$ 

so the canonical map factors if

$$h^0(Y', \mathcal{O}_{Y'}(K_{Y'} + L)) = 0.$$ 

Let us suppose the opposite. Hence the linear system $|K_{Y'} + L|$ is not empty and then $A_i(K_{Y'} + L) = -1, i = 1, \ldots, 24$, implies that $\sum_{i=1}^{24} A_i \equiv 2L$ is a fixed component of $|K_{Y'} + L|$. Therefore

$$h^0(Y', \mathcal{O}_{Y'}(K_{Y'} + L - 2L)) = h^0(Y', \mathcal{O}_{Y'}(K_{Y'} - L)) > 0$$ 

and then

$$h^0(Y', \mathcal{O}_{Y'}(2K_{Y'} - 2L)) = h^0\left(Y', \mathcal{O}_{Y'}\left(2K_{Y'} - \sum_{i=1}^{24} A_i\right)\right) > 0.$$ 

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This means that there is a bicanonical curve $B$ through the 24 nodes of $Y$. We claim that there is exactly one such curve. In fact, the strict transform in $Y'$ of the line $T_1$ is the union of two double curves $2T_a, 2T_b$ such that

$$T_a \sum_{i=1}^{24} A_i = T_b \sum_{i=1}^{24} A_i = 6$$

and $T_a \rho^*(B) = T_b \rho^*(B) = 4$. This implies that $\rho^*(B)$ contains $T_a$ and $T_b$. Analogously $\rho^*(B)$ contains the reduced strict transform of $T_2, T_3$ and $T_4$. There is only one bicanonical curve with this property, with equation $u_7 = 0$ (the bicanonical system of $Y$ is induced by $O_{\mathbb{P}^2}(2)$ and $u_2, \ldots, u_7$).

As

$$h^0(Y', O_{Y'}(2K_{Y'} - 2L)) = 1 \implies h^0(Y', O_{Y'}(K_{Y'} - L)) = 1,$$

then such bicanonical curve is double. This is a contradiction because the curve given by $u_7 = 0$ is not double.

So $h^0(Y', O_{Y'}(K_{Y'} + L)) = 0$ and we conclude that the surface $S$ has invariants $p_g = 3$, $q = 2$, $K^2 = 16$ and the canonical map of $S$ is of degree 16 onto $\mathbb{P}^2$.

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