Linear Σ model in the Gaussian wave functional approximation II: Analyticity of the S-matrix and the effective potential/action

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Abstract

We show an explicit connection between the solution to the equations of motion in the Gaussian functional approximation [3] and the minimum of the (Gaussian) effective potential/action of the linear Σ model, as well as with the N/D method in dispersion theory. The resulting equations contain analytic functions with branch cuts in the complex mass squared plane. Therefore the minimum of the effective action may lie in the complex mass squared plane. Many solutions to these equations can be found on the second, third, etc. Riemann sheets of the equation, though their physical interpretation is not clear. Our results and the established properties of the S-matrix in general, and of the N/D solutions in particular, guide us to the correct choice of the Riemann sheet. We count the number of states and find only one in each spin-parity and isospin channel with quantum numbers corresponding to the fields in the Lagrangian, i.e. to Castillejo-Dalitz-Dyson (CDD) poles. We examine the numerical solutions in both the strong and weak coupling regimes and calculate the Källén-Lehmann spectral densities and then use them for physical interpretation.

1 Introduction

In the present paper we extend the study of a chirally invariant, Lorentz invariant, self-consistent mean-field, variational approximation, that goes by the name of Gaussian wave functional approximation [1, 2] to the linear sigma model, that was begun in Ref. [3]. We have shown in Ref. [3] how to ensure chiral symmetry in the Gaussian approximation method, a major improvement over previous treatments. A number of questions have remained open after that paper, however. In particular we have not addressed the connection between the Gaussian approximation to the
canonical equations of motion and the Gaussian effective potential (EP) method, that is rather popular in finite temperature applications [4]. There the meson masses are defined in terms of the curvature (second derivative) of the effective potential evaluated at the minimum. It used to be believed that this definition leads to a violation of the Nambu-Goldstone (NG) theorem for the $\pi$ fields [5], even in the chiral limit. This misunderstanding was cleared up in Ref. [6]; moreover it was shown there that one ought not to minimize the effective potential, which is momentum independent; rather one must minimize the effective action (EA) which leads to momentum-dependent equations. The latter mass definition leads to an equation to be solved for $m_\sigma$. Yet there have been no attempts to solve this equation in the literature, save for Ref. [3]. Thus, most $m_\sigma$ values present in the literature are not acceptable. The equation for $m_\sigma$ is a transcendental one, however, with infinitely many Riemann sheets and an apparently indeterminate number and properties of solutions. Furthermore, the Gaussian method involves certain auxiliary objects, such as the solutions to the gap equations (that are often interpreted as meson masses) whose physical role is also unclear. Similarly, the canonical Gaussian approach involves two-body scattering Bethe-Salpeter (BS) equations that do not seem to appear in the EP approach.

In this paper we shall answer the aforementioned questions and some others not mentioned above: For example we show that exactly the same equations for the meson mass appear in the canonical Gaussian approximation and the Gaussian effective action approach: indeed these equations are the net result of the coupled Bethe-Salpeter and the gap equations. Of course, this fact does not make them any easier to solve, but it offers a useful perspective on the number and nature of the solutions. Inhomogeneous BS equations are scattering equations, and in this particular approximation they will be shown to be equivalent to N/D equations of dispersion theory [7, 8, 9, 10], that ensure manifest unitarity. Some properties of their solutions, such as analyticity, and the physical interpretation of the solutions, follow from unitarity and causality, and have been known since the early 1960’s [7, 8, 9, 10]. Another well known property of N/D equations is the arbitrary number of their solutions: this is the Castillejo-Dalitz-Dyson (CDD) ambiguity [7, 8, 9, 10]. The Gaussian approximation is more restrictive than the N/D approximation, however: all properties, such as the number of CDD poles and values of subtraction constants are determined by the gap and BS equations that are a part of the canonical Gaussian approximation.

In Ref. [3] we have numerically solved the BS equation in the scalar channel on the real $s$ axis and found multiple solutions for certain parameter values, and no solutions at all for others. Yet, in the weak-coupling limit the Gaussian solution is unique and smoothly connected to the perturbative one. We use the Källén-Lehmann spectral representation to show that the “heavy” solutions do not have particle-like properties. In consequence of this, we show that there is only one solution in each spin-parity-isospin channel.

Thus we have established an explicit connection between some previously separate formalisms, such as the effective potential/action (a.k.a. Cornwall-Jackiw-Tomboulis (CJT)) method [11], the Hartree + Random Phase Approximation (RPA), and the N/D method in dispersion theory, as well as shed light on the particle content in this approximation.

This paper falls into five sections. After the Introduction, in Sect. II we outline
the Gaussian effective potential/action method and prove its equivalence with the Gaussian approximation to the canonical equations of motion. In Sect. III we demonstrate the latter’s connection with the N/D equations of S-matrix theory. In Sect. IV we show and discuss the numerical solutions to the gap and the Bethe-Salpeter equations, calculate the Källén-Lehmann spectral function and analyze the particle content of the solutions. Finally in Sect. V we summarize and draw conclusions.

2 The Gaussian effective potential/action

We shall use the notation and conventions of Ref. [3]. In Sect. II and III we shall work in the chiral limit \( \varepsilon = 0 \), so as to avoid unnecessary complications. Extension to the non-chiral case is straightforward. Of course, we use the non-chiral equations in the numerical solutions in Sect. IV.

The effective potential, and the effective action methods in quantum field theory (QFT) were popularized in the mid-70’s. These two are objects with certain intriguing theoretical properties: the former is the generating function for the zero-momentum one-particle irreducible (OPI) graphs, the latter is the generating functional for arbitrary momentum OPI graphs [2, 12]. At first only the one-loop perturbative approximation was calculated in the \( \phi^4 \) theory. These two objects need not be perturbative, however, and first attempts at their non-perturbative evaluation were made slightly later.

The Gaussian effective potential is a natural product of a variational calculation based on the Gaussian ground state trial wave functional [1, 2] of a scalar single component \( \phi^4 \) quantum field theory (QFT). The O(N) symmetric effective potential was calculated e.g. in Ref. [5]. In another original approach Cornwall, Jackiw and Tomboulis [CJT], Ref. [11] used certain disconnected (“vacuum”) two-particle irreducible diagrams to define and calculate the ground state (“vacuum”) energy, as per Goldstone’s (many-body) theorem [13]. The resulting vacuum energy defines a (non-perturbative) effective potential [12], which together with the kinetic energy defines the effective action. When one minimizes the CJT vacuum energy using a particular variational Ansatz, the resulting minimization conditions, or the gap equations are equivalent to the ones obtained in the canonical Gaussian variational approximation. It is less obvious that the two-body (Bethe-Salpeter) equation in the latter formalism is equivalent to the mass equation in the former, in the case of spontaneous symmetry breaking. We shall explicate here the proof given in Refs. [14, 6].

Instead of the Cornwall-Jackiw-Tomboulis [CJT] approach we follow Stevenson, Allès and Tarrach’s more direct calculation [5] based on the Gaussian vacuum wave functionals. The latter authors found the “vacuum” (ground state) energy density \( \mathcal{E}(M, \mu; \langle \phi \rangle) \) given by Eq. (3.2) in Ref. [3]. By definition [2, 12], the effective potential \( V_{\text{eff}}(m_i, \langle \phi_i \rangle) \) is

\[
V_{\text{G}}(m_i, \langle \phi_i \rangle) = \mathcal{E}(m_i, \langle \phi_i \rangle) - \mathcal{E}(m_i, \langle \phi_i \rangle = 0). \tag{1}
\]

One may identify the \( \hbar I_1(m_i) \) term in Eq. (3.2) in Ref. [3] with the familiar “zero-point” energy density of a free spinless field of mass \( m_i \). This seems to imply that

\[1\]Indeed, many finite-temperature/density studies [4] have been based on this observation, as the
such one meson states are present in the Gaussian approximation to this theory and that \( m_i \) are their physical masses. That is not the case, however, as we shew in Ref. [14, 15] (see, also later): The \( m_i \) are merely auxiliary quantities (variational parameters) that determine the position of certain particle production thresholds and the corresponding branch cuts, but there are no poles in the propagators at those mass values.

On the other hand it has been shown in two different ways [14, 6, 3] that there are three massless (Nambu-Goldstone) and one massive (\( \sigma \)) state (with a mass different from any of the \( m_i \)) in the Gaussian approximation. Thus the physical content of the Gaussian approximation, though formally well established, remains one of its intuitively most confusing aspects. By working out the connection between various formalisms we shall shed more light on this issue.

### 2.1 The Gaussian effective potential

We shall use the fact that the effective potential is the generating function of OPI zero-momentum Feynman diagrams. In other words, the \( n \)-th derivative of the effective potential is the OPI Green’s function evaluated at zero external momenta. Thus the curvature (the second derivative) of the effective potential with respect to the corresponding fields evaluated at the minimum yield the inverse of the two-point Green functions \( \Gamma^{-1}_{ii} \) evaluated at zero momentum squared \( p^2 = 0 \),

\[
-\Gamma^{-1}_{ii}(p = 0) = \left( \frac{d^2 V_G(\langle \phi_j \rangle)}{d \langle \phi_i \rangle^2} \right)_{\text{min}}
\]

which, in turn, are interpreted as squared masses of the corresponding states by way of

\[
-\Gamma^{-1}_{ii}(p = 0) = m_i^2,
\]

thus equating the meson masses with Eqs. (2). It had been tacitly assumed that such a definition of \( \sigma, \pi \) masses

\[
\begin{align*}
m_{\sigma}^2 &= \left( \frac{d^2 V_G(\langle \phi_j \rangle)}{d \langle \phi_{0} \rangle^2} \right)_{\text{min}} \\
m_{\pi}^2 &= \left( \frac{d^2 V_G(\langle \phi_{i} \rangle)}{d \langle \phi_{i} \rangle^2} \right)_{\text{min}}, \quad i = 1, 2, 3,
\end{align*}
\]

is equivalent to the “single-particle”, or “gap equation” masses \( M, \mu \), i.e. that \( m_{\sigma} = M \) and \( m_{\pi} = \mu \), but that assumption leads to an apparent violation of the Nambu-Goldstone [NG] theorem, since \( \mu > 0 \), even in the chiral limit \( (\varepsilon = 0) \). This conclusion is incorrect, albeit very common in the literature. Many studies were devoted to attempts at its rectification: for example it was shown that \( \mu \to 0 \) in the \( 1/N \to 0 \) limit, and that was supposed to restore the NG theorem, see Ref. [5]. A zero-point energy and the effective potential point of view offer an “obvious” extension of the zero-temperature/density formalism.
straightforward evaluation of the derivatives in Eq. (4) yields [6], however,

\[ m^2_\sigma = M^2 \left( \frac{1 + 2\lambda_0 [3I_{MM}(0) - I_{\mu\mu}(0)] - 24\lambda_0^2 I_{MM}(0)I_{\mu\mu}(0)}{1 - \lambda_0 [3I_{MM}(0) + 3I_{\mu\mu}(0)] + 12\lambda_0^2 I_{MM}(0)I_{\mu\mu}(0)} \right) \neq M^2 \]  

\[ m^2_\pi = 0 \neq \mu^2, \]  

where \( I_{\mu\mu}(p^2), I_{MM}(p^2) \) are logarithmically multivalued functions of \( p^2 \) defined by

\[ I_{mm}(s) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m^2 + i\epsilon][(k - p)^2 - m^2 + i\epsilon]}, \]  

where \( m = M, \mu \), explicitly evaluated in Eqs. (3.21), (3.22) and (3.23) in Ref. [3].

The result Eq. (6), of course, restores the NG theorem, but it does so almost per fiat: it gives one no insight into the mechanism that brought it about, e.g. it tells us nothing about the sum of Feynman diagrams that leads to it. After all, each effective potential generates a certain class of loop diagrams (at zero external momenta) that one may wish to identify. It is very difficult to see which class of diagrams corresponds to Eqs. (5),(6), and which Schwinger-Dyson [SD] equations sum up that class. The first, brief answer to that question was given in Ref. [15]; now we shall elaborate on it. Once we have identified the Feynman diagrams one immediately sees that Eq. (5) is not quite right. That fact was also recognized earlier, Ref. [6], but without reference to Feynman diagrams, and the correction was also given there, but only with a formal mathematical justification. Here we shall give an explicit Feynman diagrammatic interpretation of that formal definition.

### 2.2 The Gaussian effective action

We shall use the fact that the effective action is the generating functional of one-particle irreducible (OPI) Green functions [2], i.e.,

\[ \Gamma_{ij}^{-1}(x, y) = \left( \frac{\delta^2 S[\langle \phi_k \rangle]}{\delta \langle \phi_i(x) \rangle \delta \langle \phi_j(y) \rangle} \right)_{\text{min}}, \]  

where \( \delta \) is the functional derivative. Then the correct definition of the particle mass is the position of the pole in the two-point Green function, i.e.

\[ \Gamma_{ii}^{-1}(p^2 = m_i^2) = 0 \]  

where

\[ \Gamma_{ij}^{-1}(p) = \left( \frac{\delta^2 \tilde{S}_G[\langle \phi_k \rangle]}{\delta \langle \phi_i(p) \rangle \delta \langle \phi_j(0) \rangle} \right)_{\text{min}}, \]  

and \( \tilde{S} \) is the Fourier transform of the (Gaussian) effective action,

\[ S_G[\langle \phi_i \rangle] = \int d^4x (T - V_G), \]

\[ T = \frac{1}{2} (\partial_\mu \langle \phi_i \rangle)^2 \] is the kinetic energy density. Eq. (9) shows the distinction between the effective action and the effective potential methods: the effective potential is
momentum independent and thus cannot correctly describe the pole in the propagator, except when the pole happens to be at zero momentum/mass. Thus the mass obtained from the effective potential method agrees with the one obtained from the effective action only when the mass vanishes. Therefore we must solve only Eq. (9) for the mass, as the latter appears on both sides of the equation. This distinction is insignificant for massless (NG) states and its importance increases with the mass: once the mass crosses the lowest (particle pair) production threshold, it acquires an imaginary part that cannot be neglected. Thus the pion mass calculated via the effective potential might be OK, because it lies below all hadronic production thresholds, but the scalar meson mass is definitely not OK.

Equations resulting from Eq. (10) in the linear \( \sigma \) model have been written down, but not solved (except in the trivial NG pion case) in Ref. [6]: Eqs. (10) yield \((N-1)=3\) massless states (pions in the chiral limit) and one massive state (\(\sigma\) meson) whose mass \( p^2 = m^2_{\sigma} \) is determined by the roots of the following equation

\[
p^2 = M^2 \left( 1 + \frac{3\lambda_0 \left[ 3I_{MM}(p^2) + I_{\mu\mu}(p^2) \right]}{1 - \lambda_0 \left[ 3I_{MM}(p^2) + 5I_{\mu\mu}(p^2) \right]} - 12\lambda_0^2 I_{MM}(p^2)I_{\mu\mu}(p^2) \right). \tag{12}
\]

After making the replacement \( p^2 = m^2_{\sigma} \), and a slight rearrangement Eq. (12) turns into

\[
m^2_{\sigma} = M^2 \left( 1 + 2\lambda_0 \left[ 3I_{MM}(m^2_{\sigma}) - I_{\mu\mu}(m^2_{\sigma}) \right] - 12\lambda_0^2 I_{MM}(m^2_{\sigma})I_{\mu\mu}(m^2_{\sigma}) \right). \tag{13}
\]

To show equivalence of these results to those of the canonical Gaussian approximation, we must first remember that in Ref. [3] we showed the scalar \((\sigma)\) sector coupled Bethe-Salpeter (BS) equations that sum infinite classes of connected, though not necessarily OPI Feynman diagrams. The mass of the scalar meson is determined by the “pole condition” in the scalar channel BS solution, Eq. (3.26) in Ref. [3], that reads

\[
(s - M^2)\mathcal{D}(s) = 0. \tag{14}
\]

where

\[
\mathcal{D}(s) = 1 - \lambda_0 \left[ 3 \left( 1 + 3\frac{M^2}{s-M^2} \right) I_{MM}(s) + \left( 5 + 3\frac{M^2}{s-M^2} \right) I_{\mu\mu}(s) \right] + 2\lambda_0^2 I_{MM}(s)I_{\mu\mu}(s) \left( 1 + 3\frac{M^2}{s-M^2} \right), \tag{15}
\]

is the discriminant of the coupled BS equations (see Ref. [3]). Collecting terms we find

\[
s = M^2 \left( 1 + 2\lambda_0 \left[ 3I_{MM}(s) - I_{\mu\mu}(s) \right] - 12\lambda_0^2 I_{MM}(s)I_{\mu\mu}(s) \right) \left( 1 - \lambda_0 \left[ 3I_{MM}(s) + 5I_{\mu\mu}(s) \right] + 12\lambda_0^2 I_{MM}(s)I_{\mu\mu}(s) \right). \tag{16}
\]

Upon replacing \( s = m^2_{\sigma} \), this equation becomes identical to Eq. (13) for the \( \sigma \) mass. In other words, the results of the Gaussian effective action approach are exactly identical to those of the Gaussian BS equation (or, in the many-body theory language, to the mean-field theory + RPA), proving which was one of our goals.
As stated above, in the Gaussian approximation, it is unclear which (classes of) OPI diagrams are generated by the effective action. We have readily calculated the analytic form of the effective potential curvature at the minimum, Eq. (13), but it would be a major challenge to identify the corresponding class of OPI diagrams without the benefit of the above insights obtained from the Gaussian Bethe-Salpeter equation in Ref. [3]. Furthermore, the factor \((s - M^2)\) in the equation (14) explicitly shows that the \(M\)-particle pole has been “amputated” from the amplitude, i.e., that the summed diagrams are one-\(M\)-particle irreducible. Some confusion arose due to the fact that \(\Gamma_{ij}(p^2 = s)\) is a two-point OPI Green function, whereas the BS equation defines a connected four-point Green function. There is no contradiction, however, as one can see after “amputating” the external “legs” of the BS amplitude: the result is just an \(s\)-channel propagator, i.e., a two-point Green function. We have thus given an explicit proof of a formal property of the effective action, but this does not begin to tell us what branch of Eq. (13) to solve.

As the solutions to Eq. (13) are expected to lie in the complex \(p^2\) plane, one must specify the sheet (“branch”) of the Riemann surface that one is working in. That is, in the effective action/potential approach at least, \textit{a priori} impossible: there is no reason why one branch should be preferred to another. Moreover, equations, similar to Eq. (13), in models with fermions [16] have been found to contain roots on the real axis of the second-, as well as of infinitely many other lower Riemann sheets. In the present case there are bound to be even more roots as there are two thresholds and two sets of infinitely many Riemann sheets. This (“sheet”) ambiguity in the effective potential formalism can be resolved by referring to its connection to the BS equation. So we turn to the study of analytic properties of the Gaussian BS equation.

### 3 Analyticity of the Gaussian BS equation

We shall show the exact equivalence of the solutions to the Gaussian BS equation and the so-called N/D equations in S-matrix theory. N/D equations are one way of implementing the (two-body) unitarity and causality conditions in a relativistic setting, which, in turn translate into analytic properties of the scattering amplitude. These analytic properties are important as they tell us what branch of the equation to solve and the solution’s physical interpretation (bound state, resonance, “anti-bound state”) [7, 8, 9, 10].

The broken-symmetry connected four-point Green function, Eq. (3.12) in Ref. [3], for the scattering of two non-identical (“pion-sigma” scattering) scalar particles has the same generic form of a geometric series as in the symmetric phase, see Ref. [1], but with an additional pole term in the “potential” due to the “elementary” (massive) “pion” excitation, see Eq. (3.11) and Fig. 4 in Ref. [3]. Such a pole term is known in the S-matrix literature as the Castillejo-Dalitz-Dyson (CDD) pole. It has been known for some time [17] that such a geometric progression of Feynman diagrams corresponds to the solution of the (S-wave) N/D equations in the \(s\)-channel

\[
D_\pi(s) = \frac{N_\pi(s)}{1 + \frac{1}{\pi} \int \frac{d t}{t - s - i\epsilon} N_\pi(t) \text{Im} \Pi_\pi(t)}
\]  

(17)

The solutions to the N/D equations are not unique, however, the arbitrariness show-
ing up in the form of so called Castillejo-Dalitz-Dyson (CDD) poles. The position of a CDD pole, and the coefficient multiplying it are arbitrary in the usual S-matrix, or “bootstrap” approach, but in our approach they are completely determined by the Gaussian approximation to the underlying σ model Lagrangian. The physical interpretation of CDD poles used to be controversial, but the present-day consensus is that they correspond to elementary particles/fields in the theory, which conjecture is confirmed by our results in the Gaussian approximation.

We may rewrite the pion-channel kernel (“polarization function”) \( \Pi_\sigma(s) \) of the Bethe-Salpeter equation

\[
\Pi_\sigma(s) = I_{M\mu}(s) ,
\]

in the “dispersive” form (see Eqs. (20))

\[
I_{M\mu}(s) = i \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - M^2 + i\epsilon][(-P)^2 - \mu^2 + i\epsilon]}
= I_{M\mu}(0) - \frac{s}{(4\pi)^2} K_{M\mu}(s) = \frac{1}{2\lambda_0} \left( \frac{\mu^2}{\mu^2 - M^2} \right) - \frac{s}{(4\pi)^2} K_{M\mu}(s)
= \frac{1}{2\lambda_0} \left( \frac{\mu^2}{\mu^2 - M^2} \right) - \frac{s}{16\pi^2} \int \frac{dt}{t - s - i\epsilon} \text{Im} K_{M\mu}(t) ,
\]

where \( s = P^2 \) and the real and imaginary parts are

\[
\text{Im} K_{M\mu}(s) = \frac{1}{s} \text{Im} I_{M\mu}(s)
= \frac{\pi}{s} \sqrt{\left( 1 - \frac{(M - \mu)^2}{s} \right) \left( 1 - \frac{(M + \mu)^2}{s} \right) \theta(s - (M + \mu)^2)}
\]

\[
\text{Re} K_{M\mu}(s) = \frac{2}{s} \left[ \frac{(M^2 - \mu^2)}{2s} \right] \log M \mu + \frac{1}{2} \left( 1 + \frac{(M^2 + \mu^2)}{(M^2 - \mu^2)} \right) \log M \mu
- \sqrt{\left( 1 - \frac{(M - \mu)^2}{s} \right) \left( 1 - \frac{(M + \mu)^2}{s} \right) \tanh^{-1} \sqrt{\frac{s - (M + \mu)^2}{s - (M - \mu)^2}}} .
\]

The (momentum) \( s \) dependent part of this integral is an analytic function in the cut complex \( s \) plane. There are in general two logarithmic branch cuts (one stretching from \( M + \mu \) to \(+\infty\), another from \( M - \mu \) to \(-\infty\), though on the first (“physical”) sheet only the right-hand-side cut appears) determining a Riemann surface with infinitely many sheets. This is rather different from the corresponding nonrelativistic case which has only one (square root) cut with two sheets.

Comparing Eq. (3.12) in Ref. [3] with Eq. (17) above, it becomes clear that the form of the Gaussian approximation \( \sigma \) propagator demands that the numerator function \( N_\pi(s) \) equal the pion channel “potential” \( V_\pi(s) \), Eq. (3.11) in Ref. [3] and the form of the denominator in Eq. (17):

\[
N_\pi(s) = V_\pi(s) = 2\lambda_0 \left( 1 + \frac{M^2}{s - \mu^2} \right) \tag{21}
\]

\[
\frac{1}{\pi} \int \frac{dt}{t - s - i\epsilon} \text{Im} I_{M\mu}(t) = N_\pi(s) \left[ I_{M\mu}(0) + \frac{s}{\pi} \int \frac{dt}{t(t - s - i\epsilon)} \text{Im} I_{M\mu}(t) \right] . \tag{22}
\]
Equation (21) tells us that $s = \mu^2$ is the position, and $2\lambda_0 M^2$ is the coupling strength of the CDD pole, whereas Eq. (22) dictates the value of the “subtraction constant”

$$\Pi_\pi(0) = I_{M\mu}(0) = \frac{I_0(M) - I_0(\mu)}{M^2 - \mu^2} = \frac{1}{2\lambda_0} \left( 1 - \frac{M^2 - \varepsilon/v}{M^2 - \mu^2} \right) ,$$

(23)

that is, in turn, fixed by the gap equations.

Since the N/D approximation is unitary by construction we conclude that the $s$ channel Gaussian BS scattering amplitude is also unitary. By the same token, one can show that the $\sigma$ channel propagator can be written as a solution to the matrix N/D equations, see Ref. [8]. The subtraction constants are fixed as in Ref. [3].

Now that we have established analytic properties of the Gaussian BS equation, we may look for its solutions. As noted above, physically interesting solutions are to be found as follows: (1) bound states on the real axis of the physical sheet, below all thresholds; (2) resonances in the fourth quadrant of the “second” sheet; (3) “antibound states” on the real axis of the second sheet, below both thresholds. There are also new kinds of S-matrix singularities in the relativistic quantum field theory (QFT) that do not appear in nonrelativistic quantum mechanics. One such possible new relativistic singularity is the CDD “pole” (or “zero” in the denominator). CDD poles are associated with “elementary” particles/fields in the theory, see p. 400 in Ref. [7]. In our case this means a field in the $\sigma$ model Lagrangian. If we remember that the linear $\sigma$ model has been shown to be the low-energy limit of the Nambu–Jona-Lasinio (NJL) chiral quark model, we may say that the “elementary” particles here are just the bare NJL quark-antiquark states. These states lead to real poles only in the weak meson-meson coupling limit, as we shall show below. Thus the $\sigma$ resonance may become a stable state only in the limit of weak interactions.

It should be noted, however, that $I_{MM}(s), I_{\mu\mu}(s)$ are analytic functions with imaginary parts above the corresponding thresholds, so that Eq. (14) actually implies two equations: one for the real and one for the imaginary part. Usually only the real part is considered, however. That is all right if the (real part of the) root lies below all thresholds. If it does not, as in our case, one must look at the equation for the imaginary part, as well. As one moves away from the real $s$ axis, each equation yields a line of roots in the complex $s$ plane. The intersection of the two lines (the real and imaginary roots) then yields the position of the pole. Only the pole on the second lower sheet $^2$, if it exists at all, determines the mass and width of the resonance.

We have numerically solved the real part of the scalar meson mass/Gaussian BS equation on the real axis of the physical sheet, looking for bound states. The results are shown in Fig. 9 of Ref. [3]. Note the double-valuedness of the solutions. The question arises: can one ascribe particle-like properties to the heavy branch of the solution. Short of a pole search in the second sheet, that question can only be answered by calculating the Källén-Lehmann spectral function.

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$^2$I_{MM}(s), I_{\mu\mu}(s)$ have logarithmic branch points and therefore infinitely many sheets, in contrast with the nonrelativistic case where the branch points are of the square root type, with only two sheets.
4 Källén-Lehmann Spectral function

As argued in Sect. III, the Gaussian pion propagator $D_\pi(s)$ is an analytic function in the cut $s$-plane and, as such, it allows a dispersive, or Källén-Lehmann representation

$$D_\pi(s) = -\int dt \frac{\rho_\pi(t)}{t - s - i\epsilon},$$

(24)

where

$$\rho_\pi(s) = -\frac{1}{\pi} \text{Im} D_\pi(s),$$

is the spectral density function. The latter represents the mass distribution of physical excitations in this channel. In Ref. [15] we have explicitly shown that in the pion ($J^P = 0^-$) channel and in the chiral limit, the spectral function

$$\rho_\pi(s) = a \delta(s) + c(s) \theta(s - (M + \mu)^2),$$

(25)

contains only one Dirac delta function, instead of two, as naively expected. In Fig. 1 we show the same spectral density in the nonchiral case (explicitly broken O(4) symmetry) and again find only one Dirac delta function, this time at $\sqrt{s} = 140$ MeV. Another way of saying this is that the strength with which the state at $s = \mu^2$ appears in the spectrum is zero, i.e., the state decouples from the single-particle spectrum. The heavier excitations $\phi_i, i = 1, 2, 3$ correspond to unstable quasi-particles [18] that decay into an odd number of lighter Goldstone bosons. Thus there is no particle doubling in the Gaussian approximation, contrary to suggestions e.g. by Törnqvist [19].

Similar comments hold for the $\sigma$ sector, the analysis being more complicated due to two different kinds of intermediate state being possible there. The sigma channel is phenomenologically more interesting than the pion one because that is where many experimental “supernumerary” states have been observed. In Figs. 2, 3, and 4 we show the scalar Källén-Lehmann spectral density

$$\rho_\sigma(s) = -\frac{1}{\pi} \text{Im}(D_\sigma(s)),$$

at various free parameter values. Here the scalar propagator is defined as

$$D_\sigma(s) = \frac{-2\lambda_0}{(s - M^2)D(s)},$$

(26)

the remaining terms in the $D_{ij}$ matrix elements contribute to the effective $\sigma\phi_i\phi_i$, $i = 0, 1, 2, 3$ vertex form factors. Once again we find only one bump, or Dirac delta function in the density of states, depending on the coupling strength $\lambda_0$ and other free parameters.

We showed in Fig. 8 of Ref. [3] that the scalar channel (dressed $\sigma$ meson) pole position is always shifted downward from the “elementary” sigma field’s ($\phi^0$) CDD pole at $s = M^2$, in accord with the variational nature of the Gaussian approximation. As the coupling constant $\lambda_0$ drops below some critical value $\lambda_c$, which is a function of the masses $\mu, M$ and the cutoff $\Lambda$, the $\sigma$ meson mass becomes purely real as its position drops below the $4\mu^2$ threshold, see Fig. 2. Thus, within the
Gaussian approximation, the $\sigma$ meson can be (quasi-)stable at weak couplings, but at moderate and strong couplings it is always a broad resonance. The behaviour of the critical value of the $\sigma$ meson mass $m_\sigma^c$ as a function of the cutoff $\Lambda$ can be gleaned from Fig. 8 in Ref. [3] (and similar graphs for values of $\Lambda$ between 0.4 and 1 GeV): the crossing point of the $m_\sigma(M)$ and $2\mu(M)$ curves lies around $M \simeq 300$ - 400 MeV until it disappears altogether for $\Lambda > 0.5$ GeV.

In Ref. [3] (see Fig. 9.) we showed that at intermediate values of the coupling $\lambda_0$ and low cutoff $\Lambda$ there is a second, much heavier solution to the real part of the BS equation besides the usual light one. Thus there is once again the possibility of a second pole in the S matrix. Such a conclusion would be premature, as can be seen in Fig. 3, and 4: there is no enhancement of the scalar spectral function at high energy, whereas at low $s$ the standard solution pole can be seen as a Dirac delta function turning first into a narrow then a wider peak as one moves up in energy. Thus we see that there is no enhancement in the density of states at the corresponding energy/mass and thus there is no second resonance peak. We conclude that the heavy solution exists only in the real part of the BS equation, whereas the imaginary part does not have a root in the vicinity.

5 Summary and Conclusions

The mean-field method was initially fraught with problems when applied to the linear $\sigma$ model with spontaneously broken internal symmetry - the Goldstone theorem did not seem to “work”. This problem was solved, at the price of opening new questions [14]: The Goldstone boson found in the Gaussian approximation [14] turned out to be a composite state that apparently co-exists with the massive “elementary” state with identical quantum numbers. A similar situation occurs in the scalar sector. At first this looks like a “doubling” of the number of scalar states, sometimes invoked in the phenomenological literature on the $\sigma$ meson [19]. We have showed here that this doubling is only apparent: we investigated the question of the particle content of the $\phi^4$ theory in the Gaussian approximation by employing the Källén-Lehmann representation [2]. Thus we found that the massive “elementary” pion does not appear in the Källén-Lehmann spectral function as an excitation in the pseudoscalar channel. This is the same as saying that the Castillejo-Dalitz-Dyson (CDD) “pole” in the $\pi$ channel has a vanishing residue in the MFA to the linear sigma model. Similarly, only one state has been found to exist in the $\sigma$ channel of the MFA. Moreover, we showed that the mass of the composite $\sigma$ state agrees exactly with that calculated in the CJT formalism. Another interpretation, (that might be only semantically different from the above one) was derived by using operator many-body (“quasi-particle RPA”) methods [18, 13]. In that theoretical framework the “lighter” states are particles and the “heavier” ones are so-called quasi-particles that appear due to the interactions of the particles.

In summary, we have: 1) established equivalence of the standard Gaussian approximation with the effective potential method; 2) shown equivalence to the N/D equations and thus proven unitarity. 3) solved the resulting equations and investigated the theories particle content via the Källén-Lehmann representation.

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Figure 1: The isovector pseudoscalar (pion channel) spectral density $\rho_\pi(s)$ in the Gaussian approximation to the O(4) linear sigma model as a function of the CM energy $\sqrt{s}$ for various values of the free parameters. The vertical line at $\sqrt{s} = 140$ MeV represents the Dirac delta function. Note the absence of other delta functions. The threshold of the continuum is at $\sqrt{s} = M + \mu$.

Figure 2: The isoscalar scalar (sigma channel) spectral density $\rho_\sigma(s)$ in the Gaussian approximation to the O(4) linear sigma model as a function of the CM energy $\sqrt{s}$ for $\Lambda = 1$ GeV and various values of the variational parameter (mass) $M$, i.e. of the coupling constant $\lambda_0$. The vertical line represents the Dirac delta function. The first threshold of the continuum is at $\sqrt{s} = 2\mu$, the second threshold (2$\sigma$) is at $\sqrt{s} = 2M$ where a cusp in the spectral density can be seen. In both cases the physical $\sigma$ mass lies below the lower threshold at $\sqrt{s} = 2\mu$, so the $\sigma$ meson is stable.

Figure 3: As in Fig. 2, but for cutoff $\Lambda = 0.4$ GeV and three values of the variational parameter $M = 330$, 370 and 500 MeV (with $\mu = 146$, 156, and 165 MeV, respectively). In one case the physical $\sigma$ mass lies below the lower threshold at $\sqrt{s} = 2\mu$, in the other two it lies above their respective thresholds, so accordingly the $\sigma$ meson is either stable or a resonance. Note the widening of the $\sigma$ resonance peak as its mass increases, and the coupling constant grows $\lambda_0$ with it.

Figure 4: Same as in Fig. 3, but rescaled to show higher values of $\sqrt{s}$: note the absence of any enhancement in the region of the “second solution” (the cusp is due to the second threshold).
$\Lambda_4 = 1$ GeV, $\mu = 143$ MeV, $M = 204$ MeV

$\Lambda_4 = 1$ GeV, $\mu = 200$ MeV, $M = 406$ MeV
\[ \rho_\sigma = 0.4 \text{ GeV}, M = 330 \text{ MeV} \]
\[ \Lambda_4 = 0.4 \text{ GeV}, M = 370 \text{ MeV} \]
\[ \Lambda_4 = 0.4 \text{ GeV}, M = 501 \text{ MeV} \]
$\Lambda_4 = 0.4 \text{ GeV}, M = 500 \text{ MeV}$