Emergent Coulomb forces in reducible Quantum Electrodynamics

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Abstract
This paper discusses an attempt to develop a mathematically rigorous theory of Quantum Electrodynamics (QED). It deviates from the standard version of QED mainly in two aspects: it is assumed that the Coulomb forces are carried by transversely polarized photons, and a reducible representation of the canonical commutation and anti-commutation relations is used. Both interventions together should suffice to eliminate the mathematical inconsistencies of standard QED.

1 Introduction
In recent work [1, 2] Erik Verlinde formulated the claim that gravity is an emergent force. By this is meant that gravitational forces can be derived from other, more fundamental forces. Coulomb forces and gravitational forces have much in common. Both are inversely proportional to the square of the distance. And it is difficult to reconcile them with the aversion of physicists to action at a distance. It is therefore obvious to claim that also the Coulomb forces are emergent forces [3]. While the claim of Verlinde is made in the context of cosmology the emergence of the Coulomb forces is formulated here in the context of quantum field theory.

Two arguments are brought forward. In the common theory of Quantum Electrodynamics (QED), which includes Coulomb forces, it is possible to remove them by a simple transformation of observables [4]. The inverse transformation can be used to reintroduce Coulomb forces given a theory in which they are absent (See Appendix I). The other argument is based on a mathematical proof that electrons bind with long-wavelength photons. This mechanism is discussed further on in the present paper. Evidence for the feasibility of this kind of binding has been reported recently [5] in experiments involving single spins in silicon quantum dots binding with long-wavelength microcavity photons.

A theory of Quantum Electrodynamics which does not include Coulomb forces is appealing because it can avoid some of the technical problems which plague the standard version of the theory. The claim made in [3] is that Coulomb attraction and repulsion are carried by transversely polarized photons. This eliminates the need for longitudinal and scalar photons and allows for building
a rigorous theory of QED. In the present paper the mathematical arguments
supporting this claim are presented in appendix.

2 The Standard Model

The Standard Model of elementary particles summarizes much of our present
day understanding of the fundamental laws of physics. It is a highly effective
theory. It explains almost all phenomena with an amazing degree of precision.
Never the less, it cannot be the final theory of physics, in the first place because
the integration with Einstein’s general relativity theory is missing.

The Standard Model is a quantum field theory, which implies that particles
are described by fields, for instance a photon field or an electron field. Creation
and annihilation operators add or remove one quantum of the corresponding
field. This wording suggests that particles are created or annihilated. However,
it is more careful to say that creation and annihilation operators are a tool to
construct quantum fields and to describe interaction processes.

Two different fields can interact with each other by the exchange of one
or more quanta. Einstein proposed this mechanism to explain the photo-elec-
tric effect. This assumption is a corner stone of all quantum theories. The
interesting question why the exchange of quanta occurs at seemingly random
discrete moments of time is not discussed here.

The prototype of a quantum field theory is Quantum Electrodynamics (QED),
the relativistic quantum theory of electromagnetism. QED was extended to in-
clude all electro-weak interactions, and later also strong interactions. Much
effort goes into the exploration of further extensions. The whole construction
has become an inverted pyramid resting on a few basic principles most of which
were decided on in the early days of quantum mechanics. It is therefore of
utmost importance that existing holes in these foundations are eliminated.

It is indeed worrying that the common formulation of QED is mathematically
inconsistent. A first reason for that is the perturbative approach, which involves
a non-converging series expansion. It is even worse: many of the individual
terms of the expansion are ill-defined. As a consequence, repair techniques are
needed. An alternative is offered by non-perturbative QED. But even then not
all problems disappear.

3 Mathematical problems

The technical difficulties of QED have their consequences. Certain questions
cannot be treated in a reliable manner. Let me mention one.

The physical vacuum differs from the mathematical vacuum. The latter
is defined by a vanishing of all fields. However it is not the state of minimal
energy. The interaction between different fields creates states of negative energy.
A well-known example in the quantum mechanics of particles is the hydrogen
atom. Its eigenstates all have negative energy while the scattering states have
positive energy. By analogy one can expect that in QED there exist states of
negative energy in which the electromagnetic field and the electron/positron
field are bound in some way. If this expectation is correct then the state of
minimal energy is the physical vacuum. No mathematical proof of its existence

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exists. In a non-relativistic context Lieb and coworkers (See for instance [6]) prove the existence of a ground state for models consisting of particles which do not interact among themselves but do interact with the radiation field. An ultra-violet cutoff is applied to the latter.

In Quantum Chromodynamics (QCD) one accepts that a symmetry-breaking phase transition occurs at low energies. Such a phase transition gives a satisfactory explanation for the confinement phenomenon. Also the Englert-Brout-Higgs mechanism, which in the Standard Model is responsible for giving mass to particles, invokes a phase transition. Because these phase transitions are continuous and symmetry-breaking the physical ground state is non-unique and is accompanied by Goldstone bosons. Mathematical proofs for these statements are missing.

4 Reducible QED

One way out to aim at a rigorous theory of Quantum Electrodynamics is to give up one of the axioms of quantum field theory. In the context of constructive quantum field theory [7] evidence exists that in Minkowski space the only models satisfying the Wightman axioms are free-field models. Hence some modification of the underlying assumptions is indicated.

A mild modification of the accepted body of axioms is to allow for reducible representations of the Lie algebra of canonical commutation and anti-commutation relations. Non-commuting observables are at the heart of quantum mechanics, while in classical mechanics all observables mutually commute. The two theories, classical and quantum, can be made as far apart as possible by requiring that in quantum mechanics the only observables which commute with all others are the multiplications with a scalar number. If this is the case then the representation of the algebra of observables is said to be irreducible. In addition, any representation can be decomposed into irreducible representations [20]. However, the argument that it therefore suffices to study the irreducible representations is misleading. This may be so when the decomposition is discrete and involves a small number of irreducible components. The continuous decompositions considered in what follows add a degree of complexity rather than simplify the theory.

Reducible QED is studied in the work of Czachor and collaborators (See [9] [10] [11] [12] and references given in these papers). The main assumption is that the irreducible representations are labeled by a three-dimensional wave vector $k$ and that the decomposition of the reducible representation is then an integral over the wave vector $k$. The version presented here differs from the original version of Czachor by making explicit that for every wave vector $k$ there is a properly normalized wave function $\zeta_k$ and for every observable $\hat{A}$ there is a local copy $A_k$ such that the quantum expectation of $A$ is given by

$$\langle \hat{A} \rangle = \int dk \langle \zeta_k | A_k \zeta_k \rangle.$$ 

For additive quantities such as the total energy this expression is quite obvious: the total energy of the field is obtained by integrating the wave vector-dependent energy density. Details of this formalism are found in Appendix A.


Reducible representations are known in quantum field theory in the context of superselection rules. In the latter case one selects a single component $\zeta_k$ which represents the state of the system. Here the quantum field is represented by all $\zeta_k$ simultaneously. The selected wave vector $k$ is just a filter through which the quantum field is looked at.

5 World view

Historically, the first picture one had of Quantum Electrodynamics was that of ordinary space filled with quantum harmonic oscillators, one pair at each point to cope with the two polarizations of the free electromagnetic field. This picture survives here with the modification that there is one pair of quantum oscillators for each wave vector $k$. Hence, the Euclidean space is replaced by its Fourier space. The 3 classical dimensions appear because the representation of the two-dimensional quantum harmonic oscillator is not the irreducible one but is reducible.

The representation of fermionic fields such as that of electrons and positrons is also reducible. The reducing wave vector is independent of the photonic wave vector. The Hilbert space of the irreducible fermionic representation is finite-dimensional. The electron/positron field has 16 independent states for each wave vector $k$. Its Fock space is obtained from the vacuum state by the action of 4 fermionic creation operators adding an electron or a positron, each with either spin up or spin down.

An important difference between reducible QED and QED of the Standard Model concerns the superposition of photons with unequal wave vector. In the present version of reducible QED this superposition requires another field as intermediary. For instance, expression (13) of [13]

$$\frac{1}{\sqrt{2}} [ | -K\rangle_1 | K\rangle_2 + | -K'\rangle_1 | K'\rangle_2 ]$$

becomes here

$$\zeta_{-k} = | -k, \uparrow\rangle_1, \quad \zeta_k = | k, \uparrow\rangle_2,$$
$$\zeta_{-k'} = | -k', \downarrow\rangle_1, \quad \zeta_k' = | k', \downarrow\rangle_2.$$ 

Here, $| \uparrow\rangle$ and $| \downarrow\rangle$ are for instance the spin up respectively spin down states of an auxiliary electron field. The wave function $\zeta$ describes the simultaneous presence of two combinations. The former describes two photons with wave vector $\pm k$ combined with an electron with spin up, the latter describes two photons with wave vector $\pm k'$ combined with an electron with spin down. In conclusion: with a single field superpositions can be made at a given wave vector. Wave functions at different values of the wave vector are always present and do no require further superposition. Entangled states require two or more distinct fields.

Many expressions in reducible QED look similar to those of common QED except that the integration over wave vectors is missing. It is shifted towards the evaluation of expectation values. Commutation and anti-commutation relations look different because Dirac delta functions are replaced by cosine and sine functions. An immediate advantage is that operator-valued distribution
functions are avoided and that problems caused by ultra-violet divergences are postponed.

In fact, it is not clear what happens with the electromagnetic field at large energy density. High energy fields can be obtained in two ways, either by increasing the energy of individual photons or by increasing the number of photons. Recent experiments are exploring the situation (See for instance [14]).

6 The radiation gauge

In the present version of reducible QED only transversely polarized photons occur (See Appendix B, C). The longitudinal and scalar photons of the traditional theory are absent. This implies that the number of degrees of freedom of the electromagnetic field is 2 rather than 3 or 4. There is no need for the construction of Gupta [15] and Bleuler [16], which intends to remove the nonphysical degrees of freedom. This is an important simplification, which however raises a number of questions.

In a theory containing only transverse photons it is obvious to use the radiation gauge. This is the Coulomb gauge [17], which is often used in Solid State Physics, in absence of Coulomb forces. A drawback of using this gauge is that it is not manifestly Lorentz covariant. What one wins by using this simplifying gauge is lost at the moment one considers a Lorentz boost. Then calculations, needed to restore the radiation gauge, are rather painful. However, this is not a fundamental problem.

If no gauge freedom is left, what is then the role of gauge theories? They are the unifying concept behind the different boson fields appearing in the Standard Model. The reasoning goes that a global gauge symmetry of the free fields becomes a local symmetry of the interacting fields. Total charge $Q$ is a conserved quantity also in reducible QED. The corresponding symmetry group of unitary operators $e^{i\Lambda Q}$ corresponds with the U(1) gauge group of the Standard Model. It multiplies the wave function of an electron with a phase factor $e^{i\Lambda q}$. It is shown in Appendix C that also in reducible QED this global symmetry group can be extended with local symmetries, where 'local' now means local in the space of wave vectors. The constant $\Lambda$ then becomes a function of the wave vectors $k_{\text{ph}}$ of the photon and $k$ of the electron/positron field. It suffices that $\Lambda$ remains constant when $k$ is replace by $\pm k \pm k_{\text{ph}}$. Then $e^{i\Lambda Q}$ is still a symmetry of the Hamiltonian. It reflects that the wave vector of the electron/positron field can only change by emission or absorption of a photon.

7 Emergence of Coulomb forces

The main concern for a theory involving only transverse photons is how to explain the Coulomb forces observed in nature. The explanation given in the present theory is based on an analogy with long range forces which act between polarons. The polaron [18] is a concept of Solid State Physics. A free electron in a dielectric crystal interacts with lattice vibrations, called phonons, and can form a bound state with them. This bound state is the polaron. Two polarons interact with each other because they share the same lattice vibrations. This interaction is long-ranged.
Similarly, an electron of QED interacts with the electromagnetic field. The distinction between a hypothetical bare electron and a dressed electron, surrounded by a cloud of photons, is as old as QED itself. The effect of the electron on itself via its interaction with the electromagnetic field yields a contribution to its energy. This is called the self-energy of the electron. In an electrostatic context the self-energy is the potential energy of the charge of the electron in its own Coulomb field. For a point particle this contribution is infinite. However, in the present paper the Coulomb field is absent by assumption. Hence, the problem of this divergent energy, in its original form, disappears. The interaction between the electron/positron field and the electromagnetic field still yields a static contribution to the energy. As discussed below it decreases the total energy instead of blowing it up to +infinity. In addition, the effects of the dynamic interaction between the two fields are very complicated. For a discussion of the latter see for instance [14].

In reducible QED one can prove that an electron can form a bound state with a transversely polarized photon field (See Appendix H). The binding energy is minus twice the kinetic energy of the photon field. This result is typical for a linear interaction of the electron field with a photon field described by a quadratic Hamiltonian. In some cases the self-energy is negative (See Appendix H.3). The bound state formed in this way is similar to the polaron. In addition, Appendix H.4 shows that the binding also exists for photons with long wavelength and with a wave vector almost parallel to that of the electron field. This is important because it makes it plausible that different regions of the electron field develop a long-range interaction, which is then the Coulomb interaction. However, a mathematical proof that the Coulomb attraction and repulsion are reproduced by this mechanism is still missing.

8 Discussion

The assumption that the Coulomb forces are emergent instead of being fundamental forces has far reaching consequences, some of which have been discussed above. A number of problems of the standard theory are eliminated or can at least be avoided. No superfluous degrees of freedom appear because only transversely polarized photons are taken into account. The problem of the divergence of the self-energy of the electron is absent.

The combination of reducible QED with the assumption of emergent Coulomb forces allows for the development of a mathematically consistent theory of QED. In particular, ultraviolet divergences are not hindering because the integration over wave vectors is postponed to the moment of evaluation of quantum expectations. In this context rigorous proofs can be given of the existence of bound states due to the interaction of the electron field with long wavelength photons. An analogy with the polarons of Solid State Physics makes it then plausible that Coulomb forces are carried by these transversely polarized long wavelength photons.

Some experimental evidence for the present version of reducible QED is found in Solid State Physics. It is generally accepted that free electrons in metals do not experience any long range Coulomb repulsion. In the present context this is an immediate consequence of the characteristic property of metals that long wavelength photons cannot propagate inside the material. Evidence for the
binding of the electron spin with long-range photons is given in [5]. A prediction of the present version of reducible QED is that entanglement of photons with distinct wave vectors requires an ancillary field. It is not easy to test this property because in any experiment entanglement with the environment is hard to avoid.

The claim that the Coulomb forces are emergent, if correct, requires significant modifications to the Standard Model. The concept of gauge theories survives in a modified form, as indicated. The gauge freedom is not any longer due to the presence of superfluous degrees of freedom but is the consequence of working in a reducible representation. What this means for the weak and strong interactions has still to be investigated.

It is tempting to extrapolate the present work in the direction of quantum gravity. Technical difficulties seem treatable. The existence of long wavelength gravitational waves has been established recently [19]. However, the existence of the graviton as the quantum of the gravitational field is still an open question. It is also hard to believe that the gravitational forces, which we experience all the time, would be carried by low energy quantum particles about which we do not know anything.

Appendices

A Reducible quantum fields

A.1 Definition

Let $\mathcal{H}$ is a given Hilbert space, either finite dimensional or separable, and $K$ an open subset of $\mathbb{R}^n$. In the sequel $K$ will be either $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{0\}$. Normalized elements of $\mathcal{H}$ are called wave functions, elements of $K$ are called wave vectors. Maps of $K$ into $\mathcal{H}$ are called quantum fields. Let $\Gamma$ denote the linear space of continuous fields $\zeta : k \in K \mapsto \zeta_k \in \mathcal{H}$.

In the terminology of [20] $\Gamma$ is a continuous field of Hilbert spaces. A family of sesquilinear forms $(\cdot, \cdot)_k$, $k \in K$, is defined on $\Gamma$ by

$$(\phi, \zeta)_k = (\phi_k, \zeta_k).$$

The corresponding semi-norms $||\zeta||_k \equiv ||\zeta_k||$ turn $\Gamma$ into a locally convex Hausdorff space.

A subspace $\Gamma_{\text{norm}}$ of $\Gamma$ is formed by the $\zeta \in \Gamma$ for which the map $k \mapsto ||\zeta_k||$ is bounded continuous. A norm is defined on this subspace by

$$||\zeta|| = \sup_{k \in K} ||\zeta_k||.$$ 

It turns $\Gamma_{\text{norm}}$ into a Banach space. Fields belonging to this subspace are said to be bounded in norm.

In standard quantum mechanics the normalization of wave functions is important. In the present context this leads to the axiom that states of the quantum field theory are represented by elements $\zeta$ of $\Gamma$ which satisfy the normalization condition

$$||\zeta_k|| = 1 \quad \text{for all} \quad k \in K.$$
If this condition is satisfied then $\zeta \in \Gamma$ is said to be properly normalized.

A.2 Transposed fields

The dual $\Gamma^*$ of $\Gamma$ consists of all continuous conjugate-linear functions of $\Gamma$. Introduce

**Definition A.1** A dual field $\theta$ is a map $k \in K \mapsto \theta_k \in \Gamma^*$ which is point-wise continuous.

The space of dual fields is denoted $\Gamma^\dagger$. Introduce also the notation

$$ (\theta, \zeta)_k = \overline{\theta_k(\zeta)}, \quad \zeta \in \Gamma, \theta \in \Gamma^\dagger. $$

Because $\zeta \mapsto \theta_k(\zeta)$ is conjugate-linear the form $(\cdot, \cdot)$ is sesquilinear. Following the Physics convention it is linear in the second argument. The requirement of point-wise continuity in the definition means that the map $k \mapsto (\theta, \zeta)_k$ is continuous for any field $\zeta$ in $\Gamma$.

Given $\theta \in \Gamma$ let $\theta_k^T$ be defined by

$$ \theta_k^T : \zeta \in \Gamma \mapsto (\zeta_k | \theta_k). $$

It belongs to $\Gamma^*$ and the map $\theta^T : k \mapsto \theta_k^T$ is a dual field. This shows that $\Gamma$ is embedded in the set of dual fields by the injection $\theta \mapsto \theta^T$. One has

$$ (\theta^T, \zeta)_k = \overline{(\zeta_k | \theta_k)} = \langle \theta_k | \zeta_k \rangle $$

and

$$ (\theta^T, \zeta)_k = \overline{(\zeta^T, \theta)_k}. $$

The space of transposed fields $\theta^T, \theta \in \Gamma^*$ is denoted $\Gamma^\dagger$ and is a subspace of $\Gamma^\dagger$. The inverse transposition is the map $\theta^T \mapsto \theta$. It is tradition to call this inverse map also a transposition and to convene that $(\theta^T)^T = \theta$.

A.3 Diagonal operators

A linear operator $\hat{A}$ in $\Gamma$ is a diagonal operator if there exists a map $k \in K \mapsto A_k$, where $A_k$ is a linear operator on $\mathcal{H}$, and a subspace $\mathcal{D}$ of $\Gamma$, called the domain of $\hat{A}$, such that for all $\zeta$ in $\mathcal{D}$

- $\zeta_k$ is in the domain of $A_k$ for all $k$;
- $k \mapsto A_k \zeta_k$ is continuous;
- $\hat{A} \zeta$ equals the map $k \mapsto A_k \zeta_k$.

The diagonal operators generalize the concept of block-diagonal matrices for which all blocks have the same size. In fact, if $K$ is a finite set and $\mathcal{H}$ is finite-dimensional then any diagonal operator is represented by a block-diagonal matrix.
Any operator $A$ on $\mathcal{H}$ defines a diagonal operator $\hat{A}$ on $\Gamma$ by

$$[\hat{A}\zeta]_k = A\zeta_k \quad \text{for all } k \in K.$$ 

The domain of this operator is the set

$$\mathcal{D} = \{\zeta \in \Gamma : \zeta_k \text{ is in the domain of } A \text{ for all } k \in K\}.$$ 

In particular, the identity operator $I$ is a diagonal operator which satisfies $\hat{I}\zeta = \zeta$ for all $\zeta \in \Gamma$.

**Proposition A.2** If $A$ is a bounded operator on $\mathcal{H}$ then

1) $\hat{A}$ is a continuous operator defined on all of $\Gamma$;

2) If $\zeta \in \Gamma$ is bounded in norm then also $\hat{A}\zeta$ is bounded in norm and $||\hat{A}|| = ||A||$.

**Proof**

1) For any $\zeta \in \Gamma$ is

$$||A\zeta_k - A\zeta_k'|| \leq ||A|| ||\zeta_k - \zeta_k'||.$$ 

Hence, continuity of $k \mapsto A\zeta_k$ follows from the continuity of $k \mapsto \zeta_k$. This shows that any $\zeta$ in $\Gamma$ belongs to the domain of $\hat{A}$. Finally, continuity of $\hat{A}$ follows because it suffices that for each $k$ the seminorm $||A\zeta||_k$ is bounded above by $||A|| ||\zeta||_k$.

2) If $\zeta \in \Gamma$ is bounded in norm then

$$||\hat{A}\zeta|| = \sup_k ||[\hat{A}\zeta]_k|| = \sup_k ||A\zeta_k|| \leq ||A|| \sup_k ||\zeta_k|| = ||A|| ||\zeta||.$$ 

Hence $k \mapsto A\zeta_k$ is bounded in norm and $||\hat{A}|| \leq ||A||$. Equality $||\hat{A}|| = ||A||$ follows from the action of $\hat{A}$ on constant fields.

**Proposition A.3** If $A$ is a closed operator on $\mathcal{H}$ then $\hat{A}$ is a closed operator on the Banach space $\Gamma_{\text{norm}}$.

**Proof**

Assume $\zeta^{(n)} \equiv \hat{A}\zeta^{(n)}$ converge to $\zeta$ and $\eta^{(n)} \equiv \hat{A}\eta^{(n)}$ converge to $\eta$. Then for each $k \in K$ converges $\zeta^{(n)}_k$ to $\zeta_k$ and $\eta^{(n)}_k = A\zeta^{(n)}_k$ converge to $\eta_k$. Because $A$ is closed with given domain $\mathcal{D}_A \subset \mathcal{H}$ the vector $\zeta_k$ belongs to $\mathcal{D}_A$ and $A\zeta_k = \eta_k$. 


Because $\eta \in \Gamma_{\text{norm}}$ the map $k \mapsto \eta_k$ is continuous. Hence, $\zeta$ belongs to $\mathcal{D}$ and $\hat{A}\zeta = \eta$.

An example of diagonal operators is found in the book of Dixmier [20]. Given two fields $\zeta, \eta$ in $\Gamma$ introduce the bounded operators $A_k$ defined by

$$A_k \theta_k = \langle \eta_k | \theta_k \rangle \zeta_k.$$

Then the diagonal operator $\hat{A}$ is defined on all of $\Gamma$. To prove this use that the map $k \mapsto \langle \eta_k | \theta_k \rangle$ is continuous.

### A.4 Integral operators

The diagonal operators generalize a certain type of diagonal block matrices. The analogue of non-diagonal block matrices are then integral operators of the type defined below.

The integral operator $\hat{J}$ with measurable kernel $J(k, k')$ is defined by

$$[\hat{J} \zeta]_k = \int dk' J(k, k') \zeta_{k'}.$$

The domain of definition of $\hat{J}$ is the subspace of $\Gamma$ consisting of all $\zeta$ for which

- $\zeta_{k'}$ is in the domain of $J(k, k')$ for all $k$ and almost all $k'$;
- the map $k' \mapsto J(k, k') \zeta_{k'}$ is integrable for all $k$;
- the map $k \mapsto \int dk' J(k, k') \zeta_{k'}$ is continuous.

Formally, a diagonal operator $\hat{A}$ is an integral operator $\hat{J}$ with kernel $J(k, k') = \delta(k - k')A_k$. However, this kernel does not satisfy the condition of integrability.

Given kernels $J(k, k')$ and $L(k, k')$ the product of the operators $\hat{J}$ and $\hat{L}$ involves a convolution of their kernels and can be written as $\hat{J}\hat{L} = (J \ast L)$. This follows from

$$[\hat{J}\hat{L} \zeta]_k = \int dk' J(k, k')[\hat{L} \zeta]_{k'} = \int dk' J(k, k') \int dk'' L(k', k'') \zeta_{k''} = \int dk'' \left( \int dk' J(k, k')L(k', k'') \right) \zeta_{k''} = [(J \ast L) \zeta]_k$$

with the convolution of kernels $J$ and $L$ defined by

$$(J \ast L)(k, k'') = \int dk' J(k, k')L(k', k'').$$
A.5 Adjoint operators

The adjoint $\hat{A}^\dagger$ of an operator $\hat{A}$ on $\Gamma$ is an operator on $\Gamma^\dagger$ satisfying

$$(\hat{A}^\dagger\theta,\zeta)_k = (\theta,\hat{A}\zeta)_k$$

for all $k \in K, \theta \in \Gamma^\dagger, \zeta \in \Gamma$.

The operator $\hat{A}$ on $\Gamma$ is said to be symmetric if $\hat{A}^\dagger\zeta = (\hat{A}\zeta)^\dagger$ for all $\zeta \in \Gamma$.

**Proposition A.4** Consider an operator $\hat{A}$ on $\Gamma$, which is everywhere defined and continuous. Then there exists a unique adjoint $\hat{A}^\dagger$ with domain all of $\Gamma^\dagger$.

**Proof**

Fix $\theta$ in $\Gamma^\dagger$. Let $\eta^\dagger_k(\zeta) = \theta^\dagger_k(\hat{A}\zeta)$. Then the map $\zeta \mapsto \eta^\dagger_k(\zeta)$ is continuous because $\zeta \mapsto \hat{A}\zeta$ is continuous by assumption and $\theta^\dagger_k$ belongs to $\Gamma^*$. In addition is $k \mapsto \eta^\dagger_k(\zeta)$ continuous for any $\zeta \in \Gamma$ because $k \mapsto \theta^\dagger_k$ is pointwise continuous.

Hence, $k \mapsto \eta^\dagger_k$ belongs to $\Gamma^\dagger$. Define the linear operator $\hat{A}^\dagger$ by $\hat{A}^\dagger\theta = \eta$. One verifies that

$$(\hat{A}^\dagger\theta,\zeta)_k = (\eta,\zeta)_k = \overline{(\eta_k(\zeta))} = \overline{(\zeta,\hat{A}\zeta)} = (\zeta,\hat{A}\zeta)_k.$$

This shows that $\hat{A}^\dagger$ is an adjoint of $\hat{A}$.

Assume now that $(\zeta,\theta)_k = (\zeta,\eta)_k$ for all $\zeta$ and $k$. This means $\theta_k(\zeta) = \eta_k(\zeta)$ so that the functions $\theta_k$ and $\eta_k$ coincide for all $k$. This implies $\theta = \eta$ and hence uniqueness of the adjoint $\hat{A}^\dagger$.

Consider a diagonal operator $\hat{A}$ defined by bounded operators $A_k$. Then $\hat{A}$ is continuous and everywhere defined. Hence the proposition applies and the adjoint $\hat{A}^\dagger$ is well-defined. In addition one has for all $\theta \in \Gamma$ that $\hat{A}^\dagger\theta^\dagger = \eta^\dagger$ with the field $\eta$ defined by $\eta_k = A_k^\dagger\theta_k$. This implies that $\hat{A}^\dagger$ maps the subspace $\Gamma^\dagger$ of $\Gamma^\dagger$ into itself.

On the other hand, if $\hat{J}$ is an integral operator with kernel $J_{k,k'}$, then one cannot expect that $\hat{J}^\dagger$ maps the subspace $\Gamma^\dagger$ of $\Gamma^\dagger$ into itself. Indeed, one calculates

$$
\left(\hat{J}^\dagger\theta^\dagger,\zeta\right)_k = \left(\theta^\dagger,\hat{J}\zeta\right)_k \\
= \left\langle \theta_k|\hat{J}\zeta|k\right\rangle_k \\
= \int dk' \left\langle \theta_k|J_{k,k'}\zeta_{k'}\right\rangle \\
= \int dk' \left\langle J^\dagger_{k,k'}\theta_k|\zeta_{k'}\right\rangle \\
= \int dk' \left\langle \eta^\dagger(k),\zeta_{k'}\right\rangle_{k'}
$$

with $\eta_{k'}(k) = J^\dagger_{k,k'}\theta_k$. This result is not of the form $(\eta^\dagger,\zeta)_k$. 

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A.6 Isometries

Consider an operator $\hat{U}$ on $\Gamma$ which conserves field normalization. Continuity of the map $\hat{U}$ follows from

$$|||\hat{U}\zeta_k||| = ||\zeta_k||$$

for all $k \in K$.

Hence, by Proposition A.4 $\hat{U}^\dagger$ is defined on all of $\Gamma^\dagger$. In addition, if $\zeta$ and $\theta$ belong to $\Gamma$ then one has

$$\left(\hat{U}^\dagger(\hat{U}\theta)^\tau, \zeta\right)_k = \left(\hat{U}\theta, \hat{U}\zeta\right)_k = \langle \theta | \zeta \rangle_k = \langle \theta^\tau, \zeta \rangle_k.$$ 

This implies $\hat{U}^\dagger(\hat{U}\theta)^\tau = \theta^\tau$ for all $\theta \in \Gamma$.

**Proposition A.5** Any strongly continuous map $k \mapsto U_k$ into the isometries of $H$ defines a diagonal operator $\hat{U}$ which conserves field normalization.

**Proof**

One has

$$||U_k\zeta_k - U_{k'}\zeta_{k'}|| \leq ||(U_k - U_{k'})\zeta_k|| + ||U_{k'}(\zeta_k - \zeta_{k'})||$$

$$= ||(U_k - U_{k'})\zeta_k|| + ||\zeta_k - \zeta_{k'}||.$$

Hence continuity of the map $k \mapsto ||U_k\zeta_k||$ follows from the strong continuity of $k \mapsto U_k$ and continuity of $k \mapsto \zeta_k$. This shows that $\hat{U}\zeta$ belongs to Gamma for all $\zeta$ and therefore that $\hat{U}$ is defined on all of $\Gamma$. That it conserves field normalization follows immediately. 

□

A.7 Quantum expectations

The quantum expectation of an operator $\hat{A}$ on $\Gamma$, given a properly normalized field $\zeta$ belonging to its domain, equals

$$\langle \hat{A} \rangle = \ell^3 \int dk \left( \zeta^\tau, \hat{A}\zeta \right)_k,$$

whenever this integral converges. The constant length $\ell$ has been added to make the field $\zeta$ dimensionless. The quantity $\left( \zeta^\tau, \hat{A}\zeta \right)_k$ is interpreted as being the quantum expectation of $\hat{A}$ conditioned on the knowledge of the value of the wave vector $k$. This conditioning is meaningful because the reduction of the representation over the wave vector is a classical, i.e. non-quantum aspect of the theory. A weighing of the integration over $k$ may be added if there is physical evidence for it.

B The scalar boson field

The standard representation of quantum mechanics is said to be irreducible because the only operators commuting with momentum and position operators $P$ and $Q$ are the multiples of the identity operator. The reducible representation,
used in the present work, is built by integrating irreducible representations. Following the original work of Marek Czachor and coworkers (See [9, 10, 11, 12] and papers cited in these works) integration over the wave vector \( k \) is used to decompose the reducible representation into irreducible ones. This means that for a given wave vector \( \mathbf{k} \) the standard representation of quantum mechanics in a Hilbert space \( \mathcal{H} \) is used. The dependence of the wave vector involves a field of Hilbert spaces \( \Gamma \). It is the linear space which consists of all continuous fields \( \zeta: \mathbb{R}^3 \ni \mathbf{k} \rightarrow \zeta_k \in \mathcal{H} \). Note the exclusion of \( \mathbf{k} = 0 \). It is assumed that the wave vector of a massless boson field cannot vanish.

**B.1 The irreducible components**

A scalar boson at a given wave vector \( \mathbf{k} \) in \( \mathbb{R}^3 \) is described by a quantum harmonic oscillator.

The Hilbert space \( \mathcal{H} \) equals the space \( \mathcal{L}^2(\mathbb{R}, \mathbb{C}) \) of quadratically integrable complex functions over the real line. The momentum operator \( P \) and the position operator \( Q \) are self-adjoint operators defined in the usual manner. The annihilation operator \( a \), and its adjoint \( a^\dagger \), are defined by

\[
a = \frac{1}{r\sqrt{2}}Q + i\frac{r}{\hbar\sqrt{2}}P.
\]

The positive constant length \( r \) is introduced to make the operators \( a \), and \( a^\dagger \) dimensionless. The Hamiltonian \( H \) of the harmonic oscillator can then be written as

\[
H = \hbar \omega a^\dagger a,
\]

with \( \omega > 0 \) the frequency of the oscillator. Note that the so-called ground state energy is omitted. In what follows the frequency \( \omega \) will depend on a 3-dimensional wave vector \( \mathbf{k} \), with a so-called linear dispersion relation

\[
\omega(\mathbf{k}) = c|\mathbf{k}|.
\]

Here, \( c \) is the speed of light.

The eigenstates of the Hamiltonian \( H \) are denoted \( \ket{n} \), \( n = 0, 1, 2 \cdots \). They can be constructed starting from the ground state \( \ket{0} \) by the action of the creation operator \( a^\dagger \).

**B.2 Coherent states**

Fix a complex number \( z \). The following wave function determines a coherent state

\[
|z\rangle^c = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}z^n |n\rangle
\]

The sum is convergent and the wave function is normalized to one:

\[
\langle |z\rangle^c | |z\rangle^c \rangle = \sqrt{\langle z|z\rangle^c} = 1.
\]
All coherent states belong to the domain of the annihilation operator $a$ and satisfy

$$a|z⟩^c = z|z⟩^c.$$  

They also belong to the domain of the creation operator $a^\dagger$. The maps $w \rightarrow |w⟩^c$ and $|w⟩^c \rightarrow w$ are one-to-one and continuous.

### B.3 Coherent fields

Let be given a continuous complex function $F(k)$. Use it to define the wave function $|F⟩^c$ of a coherent field by $[|F⟩^c]_k = |F(k)⟩^c$. This coherent field is a properly normalized element of $Γ$. Clearly is

$$a[|F⟩^c]_k = F(k)[|F⟩^c]_k$$  

for all $k \in \mathbb{R}^3$.

Coherent fields play an important role further on in the development of the free field theory.

With some abuse of notation the constant field $k \mapsto |0⟩^c$ will be denoted $|0⟩^c$ as well as $|0⟩$. It is the vacuum state of the free field theory.

The extension $\hat{a}$ of the annihilation operator $a$ to a diagonal operator on $Γ$ satisfies the following property, which can be proved easily.

**Proposition B.1** Let be given a continuous complex function $F(k)$. The coherent field $|F⟩^c$ belongs to the domain of the diagonal operator $\hat{a}$ and satisfies $[\hat{a}|F⟩^c]_k = F(k)[|F⟩^c]_k$ for all $k \in \mathbb{R}^3$. If $F(k)$ is bounded then $\hat{a}|F⟩^c$ is bounded in norm.

A similar result holds for the creation operator $\hat{a}^\dagger$.

**Proposition B.2** Let be given a continuous complex function $F(k)$. The coherent field $|F⟩^c$ belongs to the domain of the diagonal operator $\hat{a}^\dagger$. If $F(k)$ is bounded then $\hat{a}^\dagger|F⟩^c$ is bounded in norm.

### B.4 The free field Hamiltonian

The free-field Hamiltonian $\hat{H}$ is an unbounded symmetric operator on $Γ$. It is the diagonal operator defined by

$$[\hat{H}ζ]_k = H_kζ_k$$  

with $H_k = \hbar c|k|a^\dagger a$, \hspace{1cm} (1)

where $a$ is the annihilation operator introduced before. Its domain of definition is the subspace of $Γ$ consisting of all $ζ$ in $Γ$ such that

- $ζ_k$ is in the domain of the self-adjoint operator $a^\dagger a$ for all $k$;
- $k \in \mathbb{R}^3 \mapsto |k|a^\dagger aζ_k$ is continuous.

Physically acceptable free fields necessarily are superpositions with the vacuum field. Only then it is feasible to obtain a finite value for the expectation value of the energy operator $H$. 

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B.5 The classical wave equation

A large class of solutions of the free wave equation $\Box_x \phi = 0$ consists of functions $\phi(x)$ of the form

$$\phi(x) = 2R \int dk \frac{f^{3/2}}{N_0(k)} f(k) e^{-ik \cdot x},$$

(2)

where $f$ is a continuous function of $\mathbb{R}^3$.

The so-called normalization factor $N_0(k)$ is the usual one

$$N_0(k) = \sqrt{(2\pi)^3 2 |k|},$$

(3)

except that the constant $\ell$ is inserted also here to make it dimensionless. The insertion of this normalization factor leads further on to a satisfactory physical interpretation of the profile function $f(k)$.

The total energy of the classical field $\phi$ is given by

$$E^{cl} = \hbar c^2 \ell^2 \int_{\mathbb{R}^3} dx \left[ \left( \frac{\partial \phi}{\partial x^0} \right)^2 + \sum a \left( \frac{\partial \phi}{\partial x^a} \right)^2 \right].$$

(4)

From (2) one obtains

$$E^{cl} = \int_{\mathbb{R}^3} dk \hbar c |k||f(k)|^2.$$  

(5)

The interpretation in the context of quantum mechanics is standard. The factor $|f(k)|^2$ is the density of particles with wave vector $k$ and corresponding energy $\hbar c |k|$.

The particle density $|f(k)|^2$ has the dimension of the inverse of a volume in $k$-space. Introduce therefore the dimensionless function

$$F(k) = \ell^{-3/2} f(k)$$

and use it to construct the coherent field $|F\rangle^c$ (See Appendix B.3). Consider now the Hamiltonian $\hat{H}$ of the free boson field as given by (1). Its irreducible components satisfy

$$\langle F(k)|H_k|F(k)\rangle^c = \hbar c |k|^2 \langle F(k)|a^+ a|F(k)\rangle^c$$

$$= \hbar c |k|^2 |F(k)|^2$$

$$= \ell^{-3} \hbar c |k||f(k)|^2.$$  

(6)

This result allows us to write the classical energy (5) in terms of the free-field Hamiltonian $\hat{H}$ and the coherent field $|F\rangle^c$

$$E^{cl} = \ell^3 \int_{\mathbb{R}^3} dk \langle F(k)|H_k|F(k)\rangle^c \leq +\infty.$$  

B.6 Correspondence principle

Introduce field operators $\hat{\phi}(x)$, with $x$ in Minkowski space $\mathbb{R}^4$, defined by

$$[\hat{\phi}(x) \zeta]_k = \phi_k(x) \zeta_k$$
with
\[ \phi_k(x) = \frac{1}{N_0(k)} \left( e^{-ik_\mu x^\mu} a + e^{ik_\mu x^\mu} a^\dagger \right). \]  
(7)

The eigenstates \(|n\rangle\), \(n = 0, 1, \cdots\) of the harmonic oscillator belong to the domain of the r.h.s. of (7), as well as all coherent states \(|z\rangle\), \(z \in \mathbb{C}\). It is obvious to define \(\phi_k\) as the self-adjoint extension of the r.h.s. of (7). The map \(k \mapsto \phi_k(x)\) defines a diagonal operator \(\hat{\Phi}(x)\) of \(\Gamma\). It is called the free field operator.

The free field operators satisfy the commutation relations
\[ [\hat{\Phi}(x), \hat{\Phi}(y)]^- = \left( k \mapsto \frac{1}{(2\pi)^{3/2}|k|} \sin(k_\mu(y^\mu - x^\mu)) \right). \]

The r.h.s. of this expression is a bounded diagonal operator which commutes with all other diagonal operators of \(\Gamma\).

Derivatives to all orders of \(\hat{\Phi}(x)\) with respect to \(x^\mu\) are again diagonal operators. In particular the free field operators satisfy the operator-valued wave equation
\[ \Box_x \hat{\Phi}(x) = 0. \]

**Proposition B.3** Given any continuous complex function \(f\) of \(\mathbb{R}^3\) and the corresponding function \(F(k) = \ell^{-3/2}f(k)\), the coherent field \(|F\rangle\) belongs to the domain of the free field operator \(\hat{\Phi}(x)\) for any \(x\) in Minkowski space \(\mathbb{R}^4\).

### C Electromagnetic fields

The vector potential \(A_\mu(x)\) of classical electromagnetism has a so-called gauge freedom. This means that it is not fully determined by the physical quantities which are the electric and magnetic forces. It is tradition to fix this freedom by use of the Lorentz gauge. It has the advantage of leading to a theory which is manifestly Lorentz covariant. However, it does not eliminate all freedom of choice. The description of free electromagnetic fields is most convenient in the so-called transverse gauge. It limits the number of degrees of freedom to two transversely propagating electromagnetic waves.

#### C.1 The classical vector potential

An electromagnetic wave traveling in direction 3 with electric component in direction 1 can be described by the vector potential
\[ A^i(x) \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cos(k_{\text{ph}}(x^3 - x^0)). \]

Here \(k_{\text{ph}}\) is the wave vector. The index ‘ph’ is used to label wave vectors of the electromagnetic field. The electric and magnetic fields can be derived from the vector potential \(A^i(x)\) by
\[ E^a_{cl} = -\frac{\partial A^i_{cl}}{\partial t} - c\frac{\partial A^i_{cl}}{\partial x^a}; \]

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One then finds

\[ E_{1}^{c} \approx c k \sin(k h(x^3 - x^0)) \]

and \( B_{2}^{c} = -\frac{1}{c} E_{1}^{c} \) and \( E_{3}^{c} = E_{1}^{c} = c B_{1}^{c} = c B_{2}^{c} = 0 \).

Now let \( \Xi(k h) \) be a rotation matrix which rotates the arbitrary wave vector \( k h \in \mathbb{R}^{3} \) into the positive \( z \)-direction. Then an electromagnetic wave with wave vector \( k h \) is described by the vector potential with components \( A_{cl}^{0}(x) = 0 \) and

\[ A_{cl}^{\alpha}(x) = \Re \int d k h \frac{\lambda^{\beta/2}}{N_{0}(k h)} f(k h) \Xi_{1,\alpha}(k h) e^{-ik h_{x} x^{\mu}}. \]

After smearing out with a complex weight function \( f(k h) \), and inserting a normalization factor as before (See (3)), this becomes

\[ A_{cl}^{\alpha}(x) = \Re \int d k h \frac{\lambda^{\beta/2}}{N_{0}(k h)} \sum_{\beta=1,2} f_{\beta}(k h) \Xi_{\beta,\alpha}(k h) e^{-ik h_{x} x^{\mu}}. \]  

The parameter \( \lambda \) could be absorbed into the weight function \( f(k h) \). However, it is kept for dimensional reasons.

The free electromagnetic wave has two possible polarizations. The second linear polarization is obtained by replacing \( \Xi_{1,\alpha}(k h) \) by \( \Xi_{2,\alpha}(k h) \) in the previous expression. In addition, the two polarizations can be combined by adding up the corresponding vector potentials. The general expression is of the form

\[ A_{cl}^{\alpha}(x) = \Re \int d k h \frac{\lambda^{\beta/2}}{N_{0}(k h)} \sum_{\beta=1,2} f_{\beta}(k h) \Xi_{\beta,\alpha}(k h) e^{-ik h_{x} x^{\mu}}. \]  

C.2 Field operators

Because the electromagnetic wave has two polarizations it is obvious to consider a 2-dimensional quantum harmonic oscillator instead of the single oscillator used in Appendix B on scalar bosons.

Let \( a_{h} \) and \( a_{v} \) be the annihilation operators for a photon with horizontal respectively vertical polarization. The free-field Hamiltonian of the quantized electromagnetic field \( \hat{H}^{h} \) is the diagonal operator defined by

\[ H_{k h}^{h} = \hbar c|k h| (a_{h}^{\dagger} a_{h} + a_{h}^{\dagger} a_{h}) . \]

Field operators \( \hat{A}_{\alpha}(x) \) are defined by

\[ A_{\alpha,k h}(x) = \frac{\lambda}{2 N_{0}(k h)} \hat{\varepsilon}_{\alpha}^{(H)}(k h) \left[ e^{-ik h_{x} x^{\mu}} a_{h} + e^{-ik h_{x} x^{\mu}} a_{h}^{\dagger} \right] + \frac{\lambda}{2 N_{0}(k h)} \hat{\varepsilon}_{\alpha}^{(V)}(k h) \left[ e^{-ik h_{x} x^{\mu}} a_{v} + e^{-ik h_{x} x^{\mu}} a_{v}^{\dagger} \right], \]

with polarization vectors \( \hat{\varepsilon}_{\alpha}^{(H)}(k h) \) and \( \hat{\varepsilon}_{\alpha}^{(V)}(k h) \) given by two rows of the rotation matrix \( \Xi \)

\[ \hat{\varepsilon}_{\alpha}^{(H)}(k h) = \Xi_{1,\alpha}(k h) \quad \text{and} \quad \hat{\varepsilon}_{\alpha}^{(V)}(k h) = \Xi_{2,\alpha}(k h) . \]
Note that $a_H$ and $a_V$ commute and that $[a_H, a_H^\dagger] = I$ and $[a_V, a_V^\dagger] = I$. This can be used to verify that the field operators $A_\alpha(x)$ satisfy Heisenberg’s equation of motion

$$i\hbar\partial_0 A_\alpha(x) = \left[\hat{A}_\alpha(x), \hat{H}_{ph}\right].$$

Fix a properly normalized field $\zeta$ in $\Gamma$. The quantum expectation of the field operators becomes

$$A_{\alpha}(x) = \frac{\ell^3}{2} \int d\mathbf{k} \langle \zeta | A_{\alpha}(x) | \zeta \rangle_{k_{ph}}$$

$$= \frac{\ell^3}{2} \int \mathbf{k} \frac{\lambda}{N_0(k_{ph})} e^{-i k_{\mu} x^\mu} \times \left[ \varepsilon^{(H)}_{\alpha}(k_{ph}) \langle \zeta_{k_{ph}} | a_H \zeta_{k_{ph}} \rangle + \varepsilon^{(V)}_{\alpha}(k_{ph}) \langle \zeta_{k_{ph}} | a_V \zeta_{k_{ph}} \rangle \right].$$

This is of the form (8) with

$$f_1(k_{ph}) = \frac{\ell^3}{2} \langle \zeta_{k_{ph}} | a_H \zeta_{k_{ph}} \rangle$$

and

$$f_2(k_{ph}) = \frac{\ell^3}{2} \langle \zeta_{k_{ph}} | a_V \zeta_{k_{ph}} \rangle.$$  

Operator-valued electric and magnetic fields are defined by

$$\hat{E}_{\alpha} = -c\partial_0 \hat{A}_{\alpha},$$

$$\hat{B}_{\alpha} = \sum_{\beta,\gamma} \varepsilon_{\alpha,\beta,\gamma} \frac{\partial}{\partial x^\beta} \hat{A}_{\gamma},$$

Gauss’s law in absence of charges is satisfied. Indeed, the divergence of the electric field operators vanishes, as follows from

$$\sum_{\alpha} \partial_\alpha E_{\alpha,k_{ph}} = \frac{1}{2N_0(k_{ph})} \lambda c |k_{ph}| \left( \sum_{\alpha} k_{\alpha}^{ph} \varepsilon^{(H)}_{\alpha}(k_{ph}) \right) \left[ e^{-i k_{\mu} x^\mu} a_H + e^{i k_{\mu} x^\mu} a_H^\dagger \right]$$

$$+ \frac{1}{2N_0(k_{ph})} \lambda c |k_{ph}| \left( \sum_{\alpha} k_{\alpha}^{ph} \varepsilon^{(V)}_{\alpha}(k_{ph}) \right) \left[ e^{-i k_{\mu} x^\mu} a_V + e^{i k_{\mu} x^\mu} a_V^\dagger \right]$$

$$= 0,$$

because

$$\sum_{\alpha} k_{\alpha}^{ph} \varepsilon^{(H)}_{\alpha}(k_{ph}) = (\Xi(k_{ph}) | k_{ph}^{ab} \rangle)_{1} = |k_{ph}^{ab}|(e_3)_1$$

vanishes, as well as a similar expression for the vertical polarization.

Finally let us calculate the commutation relations

$$[A_{\alpha,k_{ph}}(x), A_{\beta,k_{ph}}(y)] = -\frac{i}{(2\pi)^{3/2} |k_{ph}| \ell} \lambda^2 \sin(k_{ph}^\mu (x - y)^\mu)$$

$$\times \left( \varepsilon^{(H)}_{\alpha}(k_{ph}) \varepsilon^{(H)}_{\beta}(k_{ph}) + \varepsilon^{(V)}_{\alpha}(k_{ph}) \varepsilon^{(V)}_{\beta}(k_{ph}) \right).$$

These commutation relations differ from the standard ones in the first place because the integration over the $k_{ph}$ vector, found in the standard theory, is missing.
C.3 Coherent fields

A pair of complex numbers $z, w$ determines a coherent state of the two-dimensional quantum harmonic oscillator. It satisfies $a^{|z, w\rangle} = z^{|z, w\rangle}$ and $a^{|z, w\rangle} = w^{|z, w\rangle}$. Given two complex continuous functions $f_1(k\text{ph})$, $f_2(k\text{ph})$ a properly normalized field $\zeta$ in $\Gamma$ is defined by

$$\zeta_{k\text{ph}} = |F_1(k\text{ph}), F_2(k\text{ph})\rangle,$$

with $F_i(k\text{ph}) = \ell^{-3/2} f_i(k\text{ph})$, $i = 1, 2$. This field $\zeta$ describes a coherent electromagnetic field. It belongs to the domain of the free Hamiltonian $H\text{ph}$. The quantum expectation of the latter equals

$$E_{\text{qu}} = \ell^3 \int dk (\zeta, H\zeta)_k$$

$$= \int dk h c |k\text{ph}| (|F_1(k\text{ph})|^2 + |F_2(k\text{ph})|^2) \leq +\infty.$$  

The interpretation is obvious: $|f_1(k\text{ph})|^2$ and $|f_2(k\text{ph})|^2$ are the expected densities for horizontally, respectively vertically polarized photons with kinetic energy $h c |k\text{ph}|$.

C.4 Single photon states

An important example of an incoherent field is the electromagnetic field produced by a single photon. In the present formalism this field requires a wave vector-dependent superposition of the single photon wave function, say $|1, 0\rangle$ for a horizontally polarized photon, with the ground state $|0, 0\rangle$ of the two-dimensional harmonic oscillator. This superposition can be written as

$$\zeta_{k\text{ph}} = \sqrt{\rho(k\text{ph}) e^{i\phi(k\text{ph})}} |1, 0\rangle + \sqrt{1 - \rho(k\text{ph})} |0, 0\rangle.$$

The energy of the electromagnetic wave equals

$$E_{\text{qu}} = h c \int dk |k\text{ph}| \rho(k\text{ph}).$$

The wave vector distribution $|k\text{ph}| \rho(k\text{ph})$ must be integrable to keep the total energy finite. In particular, $\rho(k\text{ph})$ cannot be taken constant. Therefore, the superposition of the one-photon wave function and the ground state wave function is a necessity.

The quantum expectation of the vector potential evaluates to

$$A_{\text{cl}}(x) = \lambda \int d^3k \sqrt{\rho(k\text{ph}) (1 - \rho(k\text{ph}))} e^{ik\text{ph} \cdot x} \text{Re} e^{i\phi(k\text{ph})} e^{-ik_{\text{ph}}^\mu x^\mu}.$$

Note that the contribution to the classical electromagnetic field comes from the region where the overlap with the ground state is neither 0 nor 1.

The one-photon field discussed above is linearly polarized. An example of circularly polarized one-photon field is obtained by choosing

$$\zeta_{k\text{ph}} = \sqrt{\rho(k\text{ph}) e^{i\phi(k\text{ph})}} \frac{1}{\sqrt{2}} (|1, 0\rangle \pm i|0, 1\rangle) + \sqrt{1 - \rho(k\text{ph})} |0, 0\rangle.$$
The spin is given by $S_2 = \pm \ell^3 \int \mathrm{d}k \, \rho(k)$. Note that $\frac{1}{\sqrt{2}} (|1,0\rangle \pm i|0,1\rangle)$ is a wave function of the 2-dimensional harmonic oscillator, just like $|1,0\rangle$ or $|0,1\rangle$. Both linearly and circularly polarized one-photon fields exist in the present theory.

D Scalar fermions

D.1 The Klein-Gordon equation

This section concerns the quantum field description of fermions with a rest mass $m > 0$. The appropriate wave equation is the Klein-Gordon equation

$$ (\square + \kappa^2) \phi(x) = 0 \quad \text{with} \quad \kappa = \frac{mc}{\hbar}. \quad (14) $$

For $m = 0$ it reduces to the d’Alembert equation $\square \phi = 0$, discussed in Appendix B.5. Propagating wave solutions are of the same form as in Appendix B.5

$$ \phi(x) = 2\Re \int_{\mathbb{R}^3} \mathrm{d}k \frac{\ell^{3/2}}{N_\kappa(k)} f(k)e^{-ik_\mu x^\mu}, \quad (15) $$

but with a dispersion relation given by the positive square root

$$ k^0 = \omega(k)/c \quad \text{with} \quad \omega(k) = c\sqrt{\kappa^2 + |k|^2}, $$

and a corresponding normalization

$$ N_\kappa(k) = \sqrt{(2\pi)^3 2\ell \omega(k)/c}. $$

The constant $\ell$ is inserted in (15) for dimensional reasons. It makes $|f(k)|^2$ into a density.

D.2 Larmor precession

We use the harmonic oscillator in the description of bosons because it exhibits periodic motion. An alternative model exhibiting periodic motion is that of Larmor precession. It involves the Pauli matrices $\sigma_\alpha$, $\alpha = 1,2,3$. The time evolution is

$$ \sigma_1(t) = \sigma_1 \cos(\omega t) + \sigma_2 \sin(\omega t), \quad (16) $$

$$ \sigma_2(t) = \sigma_2 \cos(\omega t) - \sigma_1 \sin(\omega t), \quad (17) $$

$$ \sigma_3(t) = \sigma_3. $$

The Hamiltonian reads

$$ H = -\frac{1}{2} \hbar \omega \sigma_3. \quad (18) $$

D.3 Fermionic state space

Let $\Gamma_2$ denote the linear space of continuous fields $\zeta: \mathbb{R}^3 \to \zeta_k \in \mathbb{C}^2$. Like in the case of bosonic fields it is a locally convex Hausdorff space. However,
because the Hilbert space $\mathbb{C}^2$ is finite-dimensional it is also a Banach space. An element $\zeta$ of $\Gamma_2$ is said to be properly normalized if $||\zeta_k|| = 1$ for all $k$. States of the fermionic quantum field theory are represented by properly normalized fields.

Note that any properly normalize field $\zeta$ of $\Gamma_2$ can be written into the form

$$\zeta_k = \left( \frac{\sqrt{1 - \rho(k)} e^{i\chi(k)}}{\sqrt{\rho(k)} e^{i\xi(k)}} \right),$$

(19)

where $\rho$, $\chi$ and $\xi$ are real-valued functions of $k \in \mathbb{R}^3$. By adopting this way of writing one tacitly assumes that $(1,0)^T$ means absence of the fermion, while $(0,1)^T$ means presence of the fermion. Note the analogy with single photon states as described in Appendix C.4. With this interpretation $\rho(k)$ becomes the density of the fermion field with wave vector $k$.

The Hamiltonian (18) of Larmor precession defines a diagonal operator $\hat{H}_{\text{el}}$ by

$$\hat{H}_{\text{el}} = \frac{1}{2} \hbar \omega(k) (I - \sigma_3).$$

(20)

A constant matrix has been added to make the Hamiltonian non-negative. This does not change the dynamics of the Larmor precession. The domain of definition of $\hat{H}_{\text{el}}$ is all of $\Gamma_2$.

With the help of (19) the quantum expectation of the Hamiltonian becomes

$$\langle \hat{H}_{\text{el}} \rangle = \ell^3 \int \frac{d^3k}{(2\pi)^3} \langle \zeta, \hat{H}_{\text{el}} \zeta \rangle_k = \ell^3 \int \frac{d^3k}{(2\pi)^3} \hbar \omega(k) \rho(k).$$

(21)

This reveals that $\rho(k)$ is a distribution of quantum particles with dispersion relation $\omega(k)$. It is restricted by the condition that $0 \leq \rho(k) \leq 1$ for all $k$. Because the energy must remain finite the distribution $\rho(k)$ should go to 0 fast enough for large values of the wave vector $|k|$.

### D.4 Field operator

Introduce now the field operator $\hat{\phi}(x)$ defined by $[\hat{\phi}(x)\zeta]_k = \phi_k \zeta_k$ with

$$\phi_k(x) = \frac{1}{N_k(k)} \left[ \sigma_+ (t) e^{ik \cdot x} + \sigma_- (t) e^{-ik \cdot x} \right].$$

(22)

It is tradition to decompose this field operator into so-called positive-frequency and negative-frequency operators

$$\hat{\phi}(x) = \hat{\phi}^+(x) + \hat{\phi}^-(x), \quad \text{with}$$

$$\phi_k^+(x) = \frac{1}{N_k(k)} \sigma_+ (t) e^{ik \cdot x}, \quad \text{and} \quad \phi_k^-(x) = \frac{1}{N_k(k)} \sigma_- (t) e^{-ik \cdot x}.$$  

They satisfy the anti-commutation relations

$$\hat{\phi}^+(x) \hat{\phi}^+(y) = 0,$$
\[
\{ \hat{\phi}^+(x), \hat{\phi}^-(y) \}_+ = \left( k \mapsto \frac{c}{2(2\pi)^3\ell \omega(k)} e^{-ik\cdot(x-y)} \right).
\]

These anti-commutation relations are non-canonical. Note that

\[
\hat{\phi}^-(x) = (\hat{\phi}^+(x))^\dagger.
\]

The classical field \( \phi^c(x) \) corresponding with the field operator \( \hat{\phi}(x) \) equals

\[
\phi^c(x) = \ell^3 \int \frac{dk}{N(k)} \frac{1}{\sqrt{\rho(k)(1-\rho(k))}} e^{-i(x(k)-\xi(k))} e^{-ik\cdot x}.
\]

The free Dirac equation

\textbf{E.1 The algebra of creation and annihilation operators}

The electron wave is fermionic. It has two polarizations, which are related to the spin of the electron. In addition, the electron has an anti-particle, which is the positron. This means that the electron field has 4 internal degrees of freedom and that we need 4 copies of the spin matrices \( \sigma_\pm \) instead of the single copy introduced in Appendix D.2. The corresponding matrices are denoted \( \sigma_s^\pm \), with the index \( s \) running from 1 to 4. They satisfy the anti-commutation relations

\[
\{ \sigma_s^+, \sigma_t^+ \}_+ = 0,
\]

\[
\{ \sigma_s^+, \sigma_t^- \}_+ = \delta_{s,t}.
\]

The hermitian conjugate of \( \sigma_s^+ \) is \( \sigma_s^- \). Together they generate an algebra known as a Clifford algebra. An explicit representation of the operators as 16-by-16 matrices is easily constructed (See for instance Section 3-9 of [22]). However, it is not needed in the sequel.

Basis vectors of the 16-dimensional Hilbert space \( \mathcal{H}_{16} \) are specified by subsets \( \Lambda \subset \{1,2,3,4\} \) and are given by

\[
|\Lambda\rangle = [\sigma_i^1]_{i \in \Lambda} [\sigma_3^1]_{i \in \Lambda} [\sigma_2^1]_{i \in \Lambda} [\sigma_1^1]_{i \in \Lambda} |\emptyset\rangle.
\]

For instance, if \( \Lambda = \{1,3\} \) then \(|\{1,3\}\rangle = \sigma_3^- |\emptyset\rangle\).

The field operators \( \hat{\phi}_s(x) \) are diagonal operators on \( \Gamma_2 \) defined by matrices \( \phi_{s,k}(x) \). The latter can be written in the form \([22]\). They satisfy the anti-commutation relations

\[
\{ \phi_{s,k}^+(x), \phi_{t,k'}^+(y) \}_+ = 0 \quad \text{and}
\]
\[
\{\phi^{(+)}_{s,k}(x), \phi^{(-)}_{t,k'}(y)\} = \frac{c}{2(2\pi)^3 i\ell \omega(k)} \delta_{s,t} e^{-ik_\mu x^\mu} e^{i k'_\mu y^\mu}.
\]

(25)

Note that (22) implies that

\[
\phi^{(+)}_{s,k}(x) = \frac{1}{N_\kappa(k)} e^{-ik_\mu x^\mu} \sigma^{(+)}_s.
\]

A familiar notation for these operators, evaluated at \(x = 0\), is

\[
b^{\uparrow} = \sigma^{(+)}_1, \quad b^{\downarrow} = \sigma^{(+)}_2, \quad d^{\downarrow} = \sigma^{(+)}_3, \quad d^{\uparrow} = \sigma^{(+)}_4.
\]

This alternative notation is not used here.

### E.2 The Hamiltonian

The Hamiltonian of the electron field \(\hat{H}_e\) is the sum of 4 copies of the scalar Hamiltonian (20). It is defined by

\[
H^{\sigma}_k = \frac{1}{2} \hbar \omega(k) \sum_{s=1}^{4} (1 - \sigma^3_s).
\]

(26)

Note that the Hamiltonian is positive. It is tradition to assign negative energies to positrons and positive energies to electrons. This tradition is not followed here because it does not make sense. It is a remainder of Dirac’s interpretation of positrons as holes in a sea of electrons. The alternative treatment assigns the vacuum state to one of the eigenstates of \(\sigma_3\) instead of assigning a particle/anti-particle pair to the two eigenstates. The dimension of the Hilbert space goes up from 4 (the number of components of a Dirac spinor) to 16. This is meaningful because the Dirac equation considered here is an equation for field operators and not the original one which holds for classical field spinors (See (33) below).

Number operators \(N_s\) are defined by

\[
N_s = \sigma^{-}_s \sigma^{(+)}_s \quad s = 1, 2, 3, 4.
\]

(27)

They appear in the Hamiltonian

\[
H^{\sigma}_k = \hbar \omega(k) \sum_{s=1}^{4} N_s.
\]

(28)

The field operators \(\hat{\phi}^{(+)}_s(x)\) satisfy Heisenberg’s equations of motion

\[
i\hbar c \partial_0 \hat{\phi}^{(+)}_s(x) = \left[ \hat{\phi}^{(+)}_s(x), \hat{H}^\sigma \right]_{-}.
\]

In principle, this is all that is needed for a description of electron/positron fields. However, the notion of electric current is needed for a description of the interactions between the electromagnetic field and the electron/positron field. The derivation below is far from trivial and follows the approach initiated by Dirac. Let us start by introducing Dirac’s equation for quantum field operators.
E.3 Dirac’s equation

Introduce the gamma matrices. In the standard representation they read
\[
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \gamma_\alpha = \begin{pmatrix} 0 & -\sigma_\alpha \\ \sigma_\alpha & 0 \end{pmatrix}.
\]

Next introduce auxiliary field operators \( \hat{\psi}_r, r = 1, 2, 3, 4 \). They are called the Dirac field operators and are defined by
\[
\psi_r, k(x) = \sqrt{2} \ell_k \sum_{s=1,2} u^{(s)}_r(k) \phi^{(+)}_{s,k}(x) + \sum_{t=3,4} v^{(t)}_r(k) \phi^{(-)}_{t,k}(x)
\]
\[
= \frac{1}{\sqrt{(2\pi)^3}} \sum_{s=1,2} u^{(s)}_r(k) e^{-i\kappa_x x^\mu \sigma^{(+)}_s} + \sum_{t=3,4} v^{(t)}_r(k) e^{i\kappa_x x^\mu \sigma^{(-)}_t}.
\]  

(29)

The vectors \( u^{(1)}, u^{(2)}, v^{(3)}, v^{(4)} \) are the analogues of the polarization vectors of the photon. They are partly fixed by the requirement that the vector with components \( \hat{\psi}_r \) satisfies Dirac’s equation
\[
i\gamma^\mu \partial_\mu \hat{\psi}(x) = \kappa \hat{\psi}(x).
\]  

(30)

Indeed, using
\[
\partial_\mu \phi^{(\pm)}_{s,k} = \mp ik_\mu \phi^{(\pm)}_{s,k}
\]

one finds that a sufficient condition for (30) to hold is
\[
\gamma^\mu k_\mu u^{(s)} = \kappa u^{(s)} \quad \text{and} \quad \gamma^\mu k_\mu v^{(t)} = -\kappa v^{(t)}.
\]

Each of these two equations has two independent solutions. They can be chosen to satisfy the orthogonality relations
\[
\sum_{r} u^{(s)}_r(k) u^{(s')}_{r'}(k) = \delta_{s,s'},
\]
\[
\sum_{r} v^{(t)}_r(k) v^{(t')}_{r'}(k) = \delta_{t,t'},
\]
\[
\sum_{r} u^{(s)}_r(k) v^{(t)}_{r'}(-k) = 0.
\]  

(31)

An electron/positron field is now determined by a properly normalized field \( \zeta \) of the form
\[
\zeta_k = \sum_{\Lambda \subset \{1,2,3,4\}} z^\Lambda_k |\Lambda\rangle,
\]

with complex coefficients \( z^\Lambda_k \) satisfying
\[
|z^\Lambda_k|^2 = 1, \quad \text{for all } k.
\]
It defines a Dirac spinor containing classical fields by

$$\phi^\alpha_r(x) = \ell^3 \int d\mathbf{k} \langle \zeta_\mathbf{k} | \psi_{r,k} \zeta_\mathbf{k} \rangle,$$

whenever the integral converges. This Dirac spinor $\phi^\alpha_r(x)$ with 4 components satisfies the Dirac equation

$$i\gamma^\mu \partial_\mu \phi^\alpha_r(x) = \kappa \phi^\alpha_r(x).$$

Finally note that each of the Dirac field operators, as constructed here, is not only a solution of the Klein-Gordon equation

$$[\Box + \kappa] \psi_{r,k} = 0,$$

but also of the partial equations

$$[c^2 \partial_0^2 + |\hbar \omega(k)|^2] \psi_{r,k}(x) = 0,$$
$$[\Delta + |\mathbf{k}|^2] \psi_{r,k}(x) = 0.$$  

A Lorentz transformation can mix up these two equations.

### E.4 Charge conjugation

The charge conjugation matrix $C$ is defined by

$$C \gamma^\mu C^{-1} = - (\gamma^\mu)^T.$$

Using the standard representation of the gamma matrices it equals $C = i\gamma^2 \gamma^0$ (See for instance Section 10.3.2 of [23]). In explicit form is

$$C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$  

The main properties of the matrix $C$ are

- $C^{-1} = C^\dagger = CT = -C$;
- $\sum_{r'} C_{r,r'} u_{r'}^{(1)}(\mathbf{k}) = v_r^{(4)}(-\mathbf{k})$;
- $\sum_{r'} C_{r,r'} u_{r'}^{(2)}(\mathbf{k}) = v_r^{(3)}(-\mathbf{k})$.

The charge conjugation operator $C_c$ is a linear operator on $\mathcal{H}_{16}$ with the properties that $C_{c}^{-1} = C_c^\dagger = -C_c$ and

$$C_c \psi_{r,k}(x) C_{c}^{-1} = - \sum_{r'} C_{r,r'} \psi_{r',k}(x),$$
$$C_c \psi_{r,k}^\alpha(x) C_{c}^{-1} = \sum_{r'} C_{r,r'} \psi_{r',k}(x).$$

25
F  The Dirac current

F.1 Two-point correlations

Fix a properly normalized electron field \( \zeta \). A two-point correlation function for the Dirac field operators \( \hat{\psi}_r(x) \) is defined by

\[
G_{r',r}(x,x') = \ell^3 \int \frac{d \mathbf{k}}{2 \pi} \int \frac{d \mathbf{k}'}{2 \pi} \langle \zeta_\mathbf{k} | \hat{\psi}_{r',\mathbf{k}}(x) \hat{\psi}_{r',\mathbf{k}'}(x') \zeta_{\mathbf{k}'} \rangle,
\]

(36)

whenever the integrals converge. Note the order of the indices \( r, r' \). A short calculation using Dirac’s equation shows that the vector \( r(x) \) with 4 components

\[
r_\mu(x) = \text{Tr} \gamma_\mu G(x,x)
\]

satisfies the continuity equation

\[
0 = \frac{\partial}{\partial x^\mu} \text{Tr} \gamma^\mu G(x,x).
\]

(37)

The vector \( r(x) \), introduced above, describes a current, which however is not yet the electric current. The components of \( r(x) \) are real numbers. Indeed, using \( (\gamma_\mu)\gamma^0 = \gamma^0 \gamma_\mu \) one verifies that

\[
r_\mu(x) = \text{Tr} \gamma^\mu G(x,x) = \text{Tr} G^\dagger(x,x) \gamma^0 \gamma_\mu \gamma^0 = \sum_{r,r'} \langle \zeta_{\mathbf{k}} | [\hat{\psi}_{r',\mathbf{k}}(x)]^\dagger \hat{\psi}_{r,\mathbf{k}}(x) \zeta_{\mathbf{k}} \rangle \gamma^0 \gamma^\mu \gamma^0 \gamma_{r',r}.
\]

\[
= \sum_{r,r'} \ell^3 \int \frac{d \mathbf{k}}{2 \pi} \int \frac{d \mathbf{k}'}{2 \pi} \langle \zeta_{\mathbf{k}} | [\hat{\psi}_{r',\mathbf{k}}(x)]^\dagger \hat{\psi}_{r,\mathbf{k}}(x) \zeta_{\mathbf{k}} \rangle \gamma^0 \gamma^\mu \gamma_{r',r} = \text{Tr} G^\dagger(x,x) \gamma^\mu = r_\mu(x).
\]

F.2 The electric current

The electric current operators \( \hat{J}_\mu(x) \) are integral operators defined by the symmetric kernels

\[
J_{\mu,k,k'}^\mu(x) = \frac{1}{2} q_\mu c \left( R_{\mu,k,k'}^\mu(x) - C R_{\mu,k,k'}^\mu(x) C^{-1} \right).
\]

(38)

Here, \( q_\mu \) is a unit of charge. The domain of definition of \( \hat{J}_\mu(x) \) consists of the fields \( \zeta \in \Gamma_2 \) for which the integrals

\[
\int d \mathbf{k}' J_{\mu,k,k'}^\mu(x) \zeta_{\mathbf{k}'}
\]

are absolutely convergent. Because \( \hat{R} \) satisfies the continuity equation also \( \hat{J} \) does. One can show that

\[
J_{\mu,k,k'}^\mu(x) = \frac{1}{2} q_\mu c \sum_{r,r'} \gamma_{r',r,k}^\mu \psi_{r',k}(x) \psi_{r,k} (x) - \frac{1}{2} q_\mu c \sum_{r,r'} \gamma_{r',r,k}^\mu \psi_{r,k}(x) \psi_{r',k} (x).
\]
This is a well-known expression for the Dirac current, adapted to the present context.

The total charge $\hat{Q}$ is the diagonal operator satisfying

$$\frac{1}{c} \int \mathrm{d}x \, J^0_{k,k'}(x) = \delta(k - k') Q_k.$$  

(40)

One finds

$$\frac{1}{c} \int \mathrm{d}x \, J^0_{k,k'}(x) = q_{el} \delta(k - k') (N_1 + N_2 - N_3 - N_4).$$

This implies

$$\hat{Q} = q_{el} (N_1 + N_2 - N_3 - N_4).$$

The obvious interpretation is that the components 1 and 2 of the field describe an electron with charge $q_{el}$, and that components 3 and 4 describe a positron with charge $-q_{el}$.

G Interaction of Photons and Electrons

G.1 The state space

The reducible representations of the free electromagnetic field and the free electron/positron field each have their own wave vector used to label the irreducible components. They are denoted $k^{ph}$, respectively $k$. A field $\zeta$ of the interacting system associates with each pair $k^{ph} \neq 0$, $k$ of wave vectors a wave function $\zeta_{k^{ph}, k}$ in the product Hilbert space $\mathcal{H}_{em} \times \mathcal{H}_{16}$, where $\mathcal{H}_{em}$ is the Hilbert space of a two-dimensional harmonic oscillator, and $\mathcal{H}_{16}$ is the 16-dimensional Hilbert space representing the possible states of an electron/positron field. Basis vectors in the product Hilbert space are denoted $|m, n\rangle \times |\Lambda\rangle \equiv |m, n, \Lambda\rangle$, where $m, n$ count photons and $\Lambda$ is a subset of $\{1, 2, 3, 4\}$. The space of continuous fields of the form

$$(k^{ph}, k) \in \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \zeta_{k^{ph}, k} \in \mathcal{H}_{em} \times \mathcal{H}_{16},$$

is denoted $\Gamma^{ph, el}$.

G.2 The interaction Hamiltonian

The Hamiltonian $\hat{H}$ is of the usual form

$$\hat{H} = \hat{H}^{ph} + \hat{H}^{el} + \hat{H}^{i}.$$  

(41)

The kinetic energy of the photon field is given by [9]

$$H^{ph}_{k^{ph}} = \hbar c |k^{ph}| \left( a^+_n a_n + a^+_v a_v \right),$$

that of the electron/positron field by [20, 22]

$$H^{el}_k = \frac{1}{2} \hbar \omega(k) \sum_{s=1}^{4} N_s,$$
where $N_s$ is the number operator indicating the presence of a particle of type $s$ (electron or positron, spin up or down). The interaction term involves the Dirac current $\tilde{J}^\mu(x)$ and the electromagnetic potential operators $\tilde{A}_\mu(x)$. The obvious definition is

$$\tilde{H}(x^0) = \int_{\mathbb{R}^3} dx \tilde{A}_\mu(x) \tilde{J}^\mu(x).$$

For $x^0 = 0$ this defines the interaction Hamiltonian in the Schrödinger picture.

### G.3 Gauge transformations

The charge operator $\hat{Q}$ commutes with $\tilde{H}^{ph}$, $\tilde{H}^{el}$ and hence with the full Hamiltonian $\tilde{H}$. The one-parameter group $\Lambda \in \mathbb{R} \mapsto \exp(i\Lambda \hat{Q})$ is a global symmetry of electrodynamics and corresponds with the U(1) gauge group of the Standard Model. This raises the question whether this symmetry group can be extended to include local symmetries. Local means here local in the space of wave vectors.

Given a smooth function $\Lambda(k^{ph}, k)$ introduce the diagonal operator $\hat{U}_\Lambda$ defined by

$$[\hat{U}_\Lambda \zeta|_{k^{ph}, k} = e^{i\Lambda(k^{ph}, k)Q} \zeta|_{k^{ph}, k}.$$ 

**Proposition G.1** Assume that any of the 4 possibilities $k' = \pm k \pm k^{ph}$ implies that $\Lambda(k^{ph}, k') = \Lambda(k^{ph}, k)$. Then $\hat{U}_\Lambda$ commutes with the Hamiltonian $\tilde{H}$.

**Proof**

It clearly commutes with $\tilde{H}^{ph}$ and $\tilde{H}^{el}$. Because $\hat{Q}$ commutes with $\tilde{J}^\alpha$ one has

$$[\hat{U}_\Lambda^{-1} \tilde{H} \hat{U}_\Lambda \zeta|_{k^{ph}, k} = \int_{\mathbb{R}^3} dx \sum_\alpha A_{\alpha, k^{ph}}(0, x)$$

$$\times e^{i\int dk' e^{[\Lambda(k^{ph}, k') - \Lambda(k^{ph}, k)]Q} J_{k', k}^\alpha(0, x) \zeta|_{k^{ph}, k'}}.$$ 

The $x$-dependence of $A_{\alpha, k^{ph}}(0, x)$ involves factors $e^{ik^{ph} \cdot x}$. The $x$-dependence of $J_{k', k}^\alpha(0, x)$ involves factors $e^{[\pm k \pm k'] \cdot x}$. Hence the integration over $x$ produces Dirac delta functions $\delta(\pm k^{ph} \pm k \pm k')$. By assumption these restrictions on the wave vectors imply that $\Lambda(k^{ph}, k') = \Lambda(k^{ph}, k)$. Therefore the factor $e^{[\Lambda(k^{ph}, k') - \Lambda(k^{ph}, k)]Q}$ may be omitted in the above expression. The result is that

$$[\hat{U}_\Lambda^{-1} \tilde{H} \hat{U}_\Lambda \zeta|_{k^{ph}, k} = [\tilde{H} \zeta|_{k^{ph}, k}.$$ 

This implies that $\tilde{H}$ commutes with $\hat{U}_\Lambda$.

Let $k = k_\| + k_\perp$ be the decomposition of $k$ into a part parallel to $k^{ph}$ and an orthogonal part. With this notation, the condition of the proposition is satisfied when $\Lambda(k^{ph}, k)$ is a function of $k^{ph}$ and $|k_\perp|$ only. It is easy to
construct functions $\Lambda$ which depend on $k^0$ and $|k_\perp|$ in a non-trivial manner. Hence, the Hamiltonian $H$ is invariant under non-trivial gauge transformations. On the other hand, it is also easy to construct functions $\Lambda$ such that $\hat{U}_\Lambda$ does not commute with $H$.

H Bound states

H.1 The unperturbed vacuum

The ground state of the non-interacting system is given by $\zeta_{k^0, k} = |0, 0, \emptyset\rangle$. The two zeroes indicate the absence of horizontally, respectively vertically polarized photons. The empty set indicates the absence of electrons and positrons. This state is not an eigenstate of the interacting Hamiltonian $\hat{H}$. One has

$$[\hat{H} \zeta_{k^0, k}] = [\hat{H}^2 \zeta_{k^0, k}]$$

$$= \int_{\mathbb{R}^3} dx A_\mu(x) J^\mu(x) \zeta_{k^0, k}$$

$$= \int_{\mathbb{R}^3} dx A_\mu k^0, k(x)|0, 0\rangle \langle 0, 0|$$

$$= \frac{\lambda}{2N_0(k^0)} \sum_{\alpha} \sqrt{q_\alpha} \left[ \langle \delta^{(0)}(k^0)|0, 0\rangle + \langle \delta^{(V)}(k^0)|0, 1\rangle \right]$$

$$\times \sum_{s=1, 2} \sum_{t=3, 4} \left[ \langle u^{(s)}(k)|0\rangle \gamma^{0} \gamma^{\alpha} v^{(t)}(k^0) \right]$$

$$\times \sigma_3^{(-)} \sigma_3^{(-)} |\emptyset\rangle$$

$$- \frac{\lambda}{2N_0(k^0)} \sum_{\alpha} \sqrt{q_\alpha} \left[ \langle \delta^{(0)}(k^0)|0, 0\rangle + \langle \delta^{(V)}(k^0)|0, 1\rangle \right]$$

$$\times \sum_{s=1, 2} \sum_{t=3, 4} \langle u^{(s)}(k)|0\rangle \gamma^{0} \gamma^{\alpha} v^{(t)}(-k - k^0) \langle [s, t] |$$

The action of the Hamiltonian $H$ on the free vacuum creates an electron/positron pair together with a single photon.

The quantum expectation $\langle \zeta | \hat{H} \zeta \rangle$ of the energy vanishes. An important question is whether there exist fields $\zeta$ in $\Gamma^{k_0, k}$ for which $\langle \zeta | \hat{H} \zeta \rangle$ is negative, or even diverges to minus infinity. This question is related to the problem of stability of matter and has been studied extensively (See for instance the work of Lieb and coworkers [3]). A systematic study in the present context is postponed.

H.2 Trial wave function

Let us try to find a wave function $\zeta_{k^0, k}$ for which the expectation value $\langle \zeta | \hat{H} \zeta \rangle$ is strictly negative.

For all $\zeta$ of the form $\zeta_{k^0, k} = \sum_{m, n} a_{m, n}(k^0, k)|m, n, 0\rangle$ is $\langle \zeta | \hat{H} \zeta \rangle \geq 0$. As a next step consider wave functions of the form

$$\zeta_{k^0, k} = \sqrt{\rho(k^0, k)}|0, 0, \emptyset\rangle$$

$$+ a_{0, 0}(k^0, k)|0, 0, \{1\} + a_{1, 0}(k^0, k)|1, 0, \{1\} + a_{0, 1}(k^0, k)|0, 1, \{1\}$$
They describe a single electron with spin up, eventually accompanied by an electromagnetic wave which is a superposition of a horizontally and a vertically polarized photon. The superposition with a wave function with vanishing electron/positron field is needed in order to satisfy the conflicting requirements of proper normalization and of a finite quantum expectation of the energy. Proper normalization requires that for all \( k^h, k \)

\[
1 = \rho^{\text{vac}}(k^h, k) + |a_{0,0}(k^h, k)|^2 + |a_{1,0}(k^h, k)|^2 + |a_{0,1}(k^h, k)|^2.
\]

Finiteness of the total energy requires that the density of the vacuum \( \rho^{\text{vac}}(k^h, k) \) tends to 1 fast enough when \( |k^h| \) and \( |k| \) become large.

The kinetic energy of the fields equals

\[
\mathcal{E}^{\text{kin}} = \ell^3 \int \mathcal{E}^{\text{kin}}(k) \rho^{\text{kin}}(k),
\]

with

\[
\rho^{\text{kin}}(k^h) = \ell^3 \int \mathcal{E}^{\text{kin}}(k) \rho^{\text{kin}}(k),
\]

\[
\rho^{\text{kin}}(k) = \ell^3 \int \mathcal{E}^{\text{kin}}(k) \rho^{\text{kin}}(k),
\]

Before evaluating the interaction energy first consider

\[
[H\chi]\rho_{k^h,k} = \int dx \sum_a A_{a,k^h,k}(x)[0,0)
\]

\[
\times \int dk' J_{a,k,k'}(x)a_{0,0}(k^h,k')\{1\}]_{x^4=0}
\]

\[
+ \int dx \sum_a A_{a,k^h,k}(x)[1,0)
\]

\[
\times \int dx' J_{a,k,k'}(x)a_{1,0}(k^h,k')\{1\}]_{x^4=0}
\]

\[
+ \int dx \sum_a A_{a,k^h,k}(x)[0,1)
\]

\[
\times \int dx' J_{a,k,k'}(x)a_{0,1}(k^h,k')\{1\}]_{x^4=0}
\]

\[
\frac{\lambda e}{4N_0(k^h)(2\pi)^3} \int dx \sum_a \epsilon^H_{\alpha}(k^h) e^{-ik^h \cdot x} a^\dagger_{\alpha} + \epsilon^V_{\alpha}(k^h) e^{-i\vec{k}^h \cdot \vec{x}} a^\dagger_{\alpha} [0,0)
\]

\[
\times \int dk' \left[ e^{-i(k-k') \cdot x} \{u^{(1)}(k)\} \gamma^\alpha \gamma^\alpha u^{(1)}(k') \right] + e^{i(k-k') \cdot x} \{u^{(1)}(k)\} \gamma^\alpha \gamma^\alpha u^{(1)}(k') \right]_{x^4=0}
\]

\[
+ \frac{\lambda e}{4N_0(k^h)(2\pi)^3} \int dx \sum_a \epsilon^H_{\alpha}(k^h) e^{ik^h \cdot x} a_{\alpha} [1,0)
\]

\[
\times \int dk' \left[ e^{-i(k-k') \cdot x} \{u^{(1)}(k)\} \gamma^\alpha \gamma^\alpha u^{(1)}(k') \right] + e^{i(k-k') \cdot x} \{u^{(1)}(k)\} \gamma^\alpha \gamma^\alpha u^{(1)}(k') \right]_{x^4=0}
\]

30
The omitted terms are orthogonal to $|0,0,\emptyset\rangle$ and $|m,n,\{1\}\rangle$. Integration over $x$ gives

\[
\begin{aligned}
&= \frac{\lambda_{q_c} \eta_c}{4N_0(k^{ph})(2\pi)^3} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_\alpha \left[ \varepsilon^{(H)}_\alpha(k^{ph})|1,0\rangle + \varepsilon^{(V)}_\alpha(k^{ph})|0,1\rangle \right] \\
&\times \left[ \langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}') \rangle + \langle u^{(1)}(\mathbf{k}')|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}) \rangle \right] a_{0,0}(k^{ph},k')|\{1\}\rangle \\
&\quad + \ldots
\end{aligned}
\]

\[
\begin{aligned}
&= \frac{\lambda_{q_c} \eta_c}{4N_0(k^{ph})(2\pi)^3} \int \frac{d\mathbf{k}}{(2\pi)^3} \sum_\alpha \left[ \varepsilon^{(H)}_\alpha(k^{ph})|1,0\rangle + \varepsilon^{(V)}_\alpha(k^{ph})|0,1\rangle \right] \\
&\times \left[ \langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k} + k^{ph}) \rangle a_{0,0}(k^{ph},k + k^{ph}) \\
&\quad + \langle u^{(1)}(\mathbf{k} - k^{ph})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}) \rangle a_{0,0}(k^{ph},k - k^{ph}) \right] \\
&\quad + \frac{\lambda_{q_c} \eta_c}{4N_0(k^{ph})(2\pi)^3} \sum_\alpha \left[ \varepsilon^{(H)}_\alpha(k^{ph})|1,0\rangle + \varepsilon^{(V)}_\alpha(k^{ph})|0,1\rangle \right] \\
&\times \left[ \langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k} - k^{ph}) \rangle a_{1,0}(k^{ph},k - k^{ph}) \\
&\quad + \langle u^{(1)}(\mathbf{k} + k^{ph})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}) \rangle a_{1,0}(k^{ph},k + k^{ph}) \right] \\
&\quad + \frac{\lambda_{q_c} \eta_c}{4N_0(k^{ph})(2\pi)^3} \sum_\alpha \left[ \varepsilon^{(V)}_\alpha(k^{ph})|0,0\rangle + \varepsilon^{(V)}_\alpha(k^{ph})|0,1\rangle \right] \\
&\times \left[ \langle u^{(1)}(\mathbf{k})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k} - k^{ph}) \rangle a_{0,1}(k^{ph},k - k^{ph}) \\
&\quad + \langle u^{(1)}(\mathbf{k} + k^{ph})|\gamma^0\gamma^\alpha u^{(1)}(\mathbf{k}) \rangle a_{0,1}(k^{ph},k + k^{ph}) \right]
\end{aligned}
\]

\[\cdots\]
The functions $w$ with

$$+ \langle u^{(1)}(k + k^b)|\gamma^0\gamma^a u^{(1)}(k)\rangle a_{0,1}(k^b, k + k^b)$$

+ \cdots .$$

The quantum expectation of the interaction energy becomes

$$\mathcal{E}_{\text{int}} = -\ell^3 \int \text{d}k \frac{\lambda q_a c}{4N_0(k^b)} \sum_{\alpha}$$

$$\times \left[ \varepsilon^{(H)}_\alpha(k^b)w^{(H)}_\alpha(k^b) + \varepsilon^{(V)}_\alpha(k^b)w^{(V)}_\alpha(k^b) \right]$$

with

$$w^{(H)}_\alpha(k^b) = -2\Re \ell^3 \int \text{d}k a_{1,0}(k^b, k)a_{0,0}(k^b, k + k^b)$$

$$\times \langle u^{(1)}(k)|\gamma^0\gamma^a u^{(1)}(k + k^b) \rangle$$

and

$$w^{(V)}_\alpha(k^b) = -2\Re \ell^3 \int \text{d}k a_{0,1}(k^b, k)a_{0,0}(k^b, k + k^b)$$

$$\times \langle u^{(1)}(k)|\gamma^0\gamma^a u^{(1)}(k + k^b) \rangle .$$

The terms which contribute describe the interaction of the photon with the spin of the electron.

**H.3 Variational approach**

Consider now the problem of minimizing the total energy given a fixed value for the density of the vacuum $\rho^{\text{vac}}(k^b, k)$. Variation of $a_{1,0}(k^b, k)$ gives

$$a_{1,0}(k^b, k) = -U^{(H)}(k^b, k)a_{0,0}(k^b, k + k^b)$$

with

$$U^{(H/V)}(k^b, k) = \frac{\lambda q_a c}{4N_0(k^b)\hbar|k^b|} \sum_{\alpha}$$

$$\times \varepsilon^{(H/V)}_\alpha(k^b)\langle u^{(1)}(k)|\gamma^0\gamma^a u^{(1)}(k + k^b) \rangle .$$

Similarly,

$$a_{0,1}(k^b, k) = -U^{(V)}(k^b, k)a_{0,0}(k^b, k + k^b).$$

The normalization condition becomes

$$1 = \rho^{\text{vac}}(k^b, k) + |a_{0,0}(k^b, k)|^2 + U_+^2(k^b, k) |a_{0,0}(k^b, k + k^b)|^2, \quad (42)$$

with

$$U_+^2(k^b, k) = |U^{(H)}(k^b, k)|^2 + |U^{(V)}(k^b, k)|^2 .$$

The functions $w^{(H)}_\alpha(k^b)$ and $w^{(V)}_\alpha(k^b)$ are of the form

$$w^{(H/V)}_\alpha(k^b) = 2\Re \ell^3 \int \text{d}k U^{(H/V)}(k^b, k)a_{0,0}(k^b, k + k^b)$$

$$\times \langle u^{(1)}(k)|\gamma^0\gamma^a u^{(1)}(k + k^b) \rangle .$$
Above it is shown that for a wave function of the form H.4 Long-wavelength analysis

The density of the electron field equals

\[ \rho_{\text{el}}(k) = \int \frac{\lambda_q c}{4N_0(k^*b)} |a_{0,0}(k^*b, k)|^2 \]

The interaction energy becomes

\[ \mathcal{E}_{\text{int}} = -2\ell^6 \int dk^*b \int dk \frac{\lambda_q c}{4N_0(k^*b)} |a_{0,0}(k^*b, k + k^*b)|^2 \]

\[ \times \sum_{\alpha} \left[ \varepsilon^{(H)}_{\alpha}(k^*b)U^{(H)}(k^*b, k) + \varepsilon^{(V)}_{\alpha}(k^*b)U^{(V)}(k^*b, k) \right] \]

\[ \times \langle u^{(1)}(k)|\gamma_0^\alpha u^{(1)}(k + k^*b) \rangle \]

\[ = -2\ell^6 \int dk^*b \int dk |a_{0,0}(k^*b, k)|^2 U^2_{\perp}(k^*b, k) \]

\[ = -2\mathcal{E}_{\text{ph}}. \]

The interaction energy is minus twice the kinetic energy of the photon field. This result is typical for a quadratic minimization problem. During the minimization the energy of the electron field is kept constant. Hence one can conclude that for an electron field with a given kinetic energy and no photons present there always exists an interacting system where the energy spectrum of the electron field is unmodified but the total energy is lowered by adding the photon field.

For further use, note that the kinetic energy of the photon field can be written as

\[ \mathcal{E}_{\text{ph}} = \ell^6 \int dk^*b \ h c |k^*b| \int dk |a_{0,0}(k^*b, k)|^2 U^2_{\perp}(k^*b, k - k^*b). \]  

(43)

The density of the electron field equals

\[ \rho^a(k) = \ell^3 \int dk^*b \left[ |a_{0,0}(k^*b, k)|^2 + |a_{1,0}(k^*b, k)|^2 + |a_{0,1}(k^*b, k)|^2 \right] \]

\[ = \ell^3 \int dk^*b \left[ |a_{0,0}(k^*b, k)|^2 + U^2_{\perp}(k^*b, k)|a_{0,0}(k^*b, k + k^*b)|^2 \right]. \]  

(44)

H.4 Long-wavelength analysis

Above it is shown that for a wave function of the form

\[ \xi_{k^*b, k} = \sqrt{\rho_{\text{ph}}(k^*b, k)}|0, 0, \emptyset| + a_{0,0}(k^*b, k)|0, 0, \{1\} \]

\[ - U^{(H)}(k^*b, k)a_{0,0}(k^*b, k + k^*b)|1, 0, \{1\} \]

\[ - U^{(V)}(k^*b, k)a_{0,0}(k^*b, k + k^*b)|0, 1, \{1\} \]

the interaction energy is minus twice the kinetic energy of the photon field. By explicit calculation one shows that coefficients \(a_{0,0}(k^*b, k)\) exist such that the wave function is physically acceptable. These calculations lead to

\[ U^{(H/V)}(k^*b, k) = -\frac{\lambda_q c}{4N_0(k^*b)h|k^*b|} \sum_{\alpha} \varepsilon^{(H/V)}_{\alpha}(k^*b)k_{\alpha} \]

\[ \times \frac{\omega(k) + \omega(k + k^*b) + 2ck}{\sqrt{\omega(k + k^*b)[\omega(k + k^*b) + ck]\sqrt{\omega(k)[\omega(k) + ck]}}} \]

(45)

This gives

\[ U^2_{\perp}(k^*b, k) = \left( \frac{\lambda_q c}{4N_0(k^*b)h|k^*b|} \right)^2 \left( |k|^2 - \frac{(k \cdot k^*b)^2}{|k^*b|^2} \right) \]

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\[ \begin{align*}
&\times \frac{[\omega(k) + \omega(k + k^b) + 2c\kappa]^2}{\omega(k + k^b)[\omega(k + k^b) + c\kappa][\omega(k) + c\kappa]} \\
&\geq \left( \frac{\lambda q_c c}{4N_0(k^b)\hbar|k^b|} \right)^2 \left( \frac{|k|^2 - (k \cdot k^b)^2}{|k^b|^2} \right) \frac{4}{\omega(k)\omega(k + k^b)}. \\
\end{align*} \]

A long wavelength approximation is

\[ U_2^\perp(k^b, k) = \left( \frac{\lambda q_c c}{4N_0(k^b)\hbar|k^b|} \right)^2 \left( \frac{|k|^2 - (k \cdot k^b)^2}{|k^b|^2} \right) \frac{4}{\omega^2(k)} \left[ 1 - \frac{k \cdot k^b}{k^2 + |k^b|^2} + O(|k^b|^2) \right]. \]

If \( k^b \) and \( k \) are not parallel then this expression diverges as \( |k^b|^{-3} \). From the normalization condition (42) then follows that \( a_{0,0}(k^b, k) \) should vanish in the long-wavelength limit as \( |k^b|^3 \) or it should vanish in all regions away from the longitudinal direction. This observation leaves room for two types of solutions.

I Emergent Coulomb forces

In standard QED the electromagnetic field is described by 4 independent operators \( \hat{A}_\mu(x) \). In the present approach one component, namely \( \hat{A}_0(x) \) is identically zero and the 3 remaining components \( \hat{A}_\alpha(x), \alpha = 1, 2, 3, \) satisfy the orthogonality relation

\[ \sum_{\alpha} k_{\alpha}^b A_{\alpha, k^b}(x) = 0. \]

Hence, the electromagnetic quantum field, as treated in the present work, has only two degrees of freedom. In particular, the electric field operators \( \hat{E}_\alpha(x) \) satisfy Gauss’ law in absence of charges (See (12)). This can be justified with the argument that the full law, including a source term in the r.h.s., will emerge after the interaction with the electron/positron field is taken into account. This argument is supported by the existence [3] of a transformation of the field operators \( \hat{E}_\alpha(x) \) which maps the homogeneous law of Gauss onto the full version of the law.

Introduce new electric field operators

\[ \hat{E}'_\alpha(x) = \hat{E}_\alpha(x) + \frac{\mu_0 c}{4\pi} \frac{\partial}{\partial x^\alpha} \int dy \frac{1}{|x - y|} \times \hat{U}(-x^0, y^0) \hat{j}^0(y, 0) \hat{U}(x^0). \]

Here, \( \hat{U}(x^0) = \exp(-ix^0\hat{H}/\hbar c) \) is the time evolution of the interacting system. The new operators are marked with a double prime to distinguish them from the operators of the non-interacting system and those of the interacting system. The latter are denoted with a single prime. One verifies immediately that Gauss’ law is satisfied

\[ \sum_{\alpha} \frac{\partial}{\partial x^\alpha} \hat{E}'_\alpha(x) = -\mu_0 c \hat{j}^0(x). \]
The second term of \(46\) is the Coulomb contribution to the electric field. The curl of this term vanishes. Hence it is obvious to take
\[
\hat{B}''_{\alpha}(x) \equiv \hat{B}'_{\alpha}(x).
\]
This implies the second of the four equations of Maxwell, stating that the divergence of \(\hat{B}''_{\alpha}(x)\) vanishes. In addition, the fourth equation, absence of magnetic charges, follows immediately because \(\hat{E}''(x)\) and \(\hat{E}'(x)\) have the same curl. Remains Faraday’s law to be written as
\[
(\nabla \times \hat{B}''(x))_{\alpha} - \frac{1}{c} \frac{\partial}{\partial x^0} \hat{E}''_{\alpha}(x) = -\mu_0 \hat{j}''_{\alpha}(x)
\]
with
\[
\hat{j}''_{\alpha}(x) = -\frac{1}{\mu_0 c} \frac{\partial}{\partial x^0} \left( \hat{E}''_{\alpha}(x) - \hat{E}'_{\alpha}(x) \right).
\]
Finally, take \(\hat{j}''_0(x) = \hat{j}'_0(x)\). A short calculation shows that the newly defined current operators \(\hat{j}_\mu''(x)\) satisfy the continuity equation.

One concludes that a formalism of QED is possible which does not postulate the existence of longitudinal or scalar photons. Two pictures coexist: the original Heisenberg picture and what is called here the emergent picture. In both pictures the time evolution of all operators is the same, but the definition of the electromagnetic field operators differs. In the original description only transversely polarized photons exist. On the other hand, the field operators of the emergent picture satisfy the full Maxwell equations, including Coulomb forces.

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