Concentration theorem and relative fixed point formula of Lefschetz type in Arakelov geometry

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Abstract. In this paper we prove a concentration theorem for arithmetic $K_0$-theory, this theorem can be viewed as an analog of R. Thomason’s result (cf. [22]) in the arithmetic case. We will use this arithmetic concentration theorem to prove a relative fixed point formula of Lefschetz type in the context of Arakelov geometry. Such a formula was conjectured of a slightly stronger form by K. Köhler and D. Roessler in [15] and they also gave a correct route of its proof there. Nevertheless our new proof is much simpler since it looks more natural and it doesn’t involve too many complicated computations.

2010 Mathematics Subject Classification: 14C40, 14G40, 14L30, 58J20, 58J52

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1 Introduction

In [14], K. Köhler and D. Roessler proved a Lefschetz type fixed point formula for regular schemes endowed with the action of a diagonalisable group scheme, this formula generalized the
associate an equivariant arithmetic $K$ form classes on reasonable push-forward morphisms. Correspondingly, by defining a suitable element under the action of $\mu$ let $\lambda$ and $\Phi$, let $\lambda = 1$. To every $\mu_n$-equivariant arithmetic variety $X$, we can associate an equivariant arithmetic $K$-group $K_0(X, \mu_n)$ which contains certain set of smooth form classes on $X_{\mu_n}(\mathbb{C})$ as analytic datum where $X_{\mu_n}$ stands for the fixed point subscheme under the action of $\mu_n$. Such an equivariant arithmetic $K_0$-group has a ring structure and moreover it can be made to be an $R(\mu_n)$-algebra. Let $f$ be the structure morphism of $X$ over $D$, and let $N_{X/X_{\mu_n}}$ be the normal bundle with respect to the regular immersion $X_{\mu_n} \hookrightarrow X$, endowed with the quotient metric induced by a chosen Kähler metric of $X_{\mu_n}$. The main theorem in [14] reads: the element $\lambda_{-1}(\overline{N}_X/X_{\mu_n}) := \sum_{j=0}^{rk(N_{X/X_{\mu_n}})} (-1)^j (\overline{N}_X/X_{\mu_n})$ is invertible in $K_0(X_{\mu_n}, \mu_n)^\rho$, the localization of $K_0(X_{\mu_n}, \mu_n)$ with respect to the ideal $\rho$, and we have the following commutative diagram

$$
\begin{array}{ccc}
K_0(X, \mu_n) & \overset{\Lambda_R(f)^{-1}}{\longrightarrow} & K_0(X_{\mu_n}, \mu_n)^\rho \\
\downarrow f_* & & \downarrow f_{\mu_n*} \\
K_0(D, \mu_n) & \overset{\iota}{\longrightarrow} & K_0(D, \mu_n)^\rho
\end{array}
$$

where $\Lambda_R(f) := \lambda_{-1}(\overline{N}_X/X_{\mu_n}) \cdot (1 + R_g(N_{X/X_{\mu_n}}))$, $\tau$ stands for the restriction map and $\iota$ is the natural morphism from a ring or a module to its localization which sends an element $e$ to $\overline{e}$. Here $R_g(\cdot)$ is the equivariant $R$-genus, the definition of the two push-forward morphisms $f_*$ and $f_{\mu_n*}$ involves an important analytic datum which is called the equivariant analytic torsion.

In [16], X. Ma defined the equivariant analytic torsion form which is a higher analog of the equivariant analytic torsion and he also proved the curvature and anomaly formulae for it. Once these preparations are done, one can naturally conjecture a Lefschetz fixed point formula which generalizes K. Köhler and D. Roessler’s results to the relative setting. Precisely speaking, let $X$ and $Y$ be two $\mu_n$-equivariant arithmetic varieties, suppose that we are given a flat $\mu_n$-equivariant morphism $f : X \to Y$ which is smooth on the complex numbers, then we can define reasonable push-forward morphisms $f_*$ and $f_{\mu_n*}$ by using equivariant analytic torsion forms. Correspondingly, by defining a suitable element $M(f)$ in the localization $K_0(X_{\mu_n}, \mu_n)^\rho$, we will get the following commutative diagram

$$
\begin{array}{ccc}
K_0(X, \mu_n) & \overset{M(f)}{\longrightarrow} & K_0(X_{\mu_n}, \mu_n)^\rho \\
\downarrow f_* & & \downarrow f_{\mu_n*} \\
K_0(Y, \mu_n) & \overset{\tau}{\longrightarrow} & K_0(Y_{\mu_n}, \mu_n)^\rho
\end{array}
$$
where \( \tau \) is again the restriction map.

There are two crucial actors in K. Köhler and D. Roessler’s proof of their main theorem. One is an elegant algebro-geometric construction: the deformation to the normal cone, and the other one is a technical result: \( \tilde{K}_0(\cdot, \mu_n) \)-theoretic form of Bismut’s immersion formula. The construction of the deformation to the normal cone also works in the relative setting, so if one can find a general Bismut’s immersion formula in the case of relative setting, the second commutative diagram described above is no longer a conjecture but a theorem. J.-M. Bismut and X. Ma fulfilled this work in [8].

K. Köhler and D. Roessler’s proof is a little hard to follow since it involves too many complicated computations, we will provide a much simpler proof in this paper but on one more condition: the fibre product \( f^{-1}(Y_{\mu_n}) \) is also regular. This limitation will not influence most applications in practice (cf. Remark 6.8 (i)). Especially, the main results in [14] can be totally recovered. Our proof has nothing to do with the construction of the deformation to the normal cone, it relies on an arithmetic concentration theorem in Arakelov geometry. This concentration theorem is an analog of R. Thomason’s result (cf. [22]) at 0-degree level. Consider the closed immersion \( i : X_{\mu_n} \hookrightarrow X \), R. Thomason used the Quillen’s localization sequence for higher equivariant \( K \)-theory to prove that the natural morphisms
\[
i_* : K_*(X_{\mu_n}, \mu_n)_{\rho} \to K_*(X, \mu_n)_{\rho}
\]
are all isomorphisms and their inverse morphisms are of the form \( \lambda_{-1}^{-1}(N^V_{X/X_{\mu_n}}) \cdot i^* \). In the arithmetic case, we will first construct the morphism
\[
i_* : \tilde{K}_0(X_{\mu_n}, \mu_n)_{\rho} \to \tilde{K}_0(X, \mu_n)_{\rho}
\]
and then prove that it is also an isomorphism. The most important result we need to prove the relative Lefschetz fixed point formula in Arakelov geometry is that the inverse morphism of \( i_* \) is given by \( \lambda_{-1}^{-1}(N^V_{X/X_{\mu_n}}) \cdot i^* \).

The structure of this paper is as follows. In Section 2 and Section 3, we recall the algebraic concentration theorem and the equivariant arithmetic \( K_0 \)-theory for the convenience of the reader. In Section 4, we will discuss all analytic results coming from differential geometry which are needed in this paper. In Section 5 and Section 6, we describe and prove the arithmetic concentration theorem and the relative Lefschetz fixed point formula.

Acknowledgements. The author wishes to thank his thesis advisor Damian Roessler for his constant encouragement and for many fruitful discussions between them. Especially, Damian Roessler is the person who made the author realize that it is possible to prove the relative Lefschetz fixed point formula via proving an arithmetic concentration theorem. The author also wishes to thank Xiaonan Ma for his kindly explanation of various purely analytic problems as well as his useful comments concerning a crucial lemma in this paper. Finally, thanks to the referee, whose very detailed comments help to improve the paper a great deal.
2 Algebraic concentration theorem

Let $D$ be a Noetherian integral ring. In this section we fix $S := \text{Spec}(D)$ as the base scheme. Let $n$ be a positive integer, we shall denote by $\mu_n$ the diagonalisable group scheme over $S$ associated to the cyclic group $\mathbb{Z}/n\mathbb{Z}$. By a $\mu_n$-equivariant scheme we understand a separable and of finite type scheme over $S$ which admits a $\mu_n$-action. A $\mu_n$-action on a scheme $X$ is a morphism $m_X : \mu_n \times X \to X$ which satisfies some compatibility properties. Denote by $p_X$ the projection from $\mu_n \times X$ to $X$. For a coherent $\mathcal{O}_X$-module $E$ on $X$, a $\mu_n$-action on $E$ we mean an isomorphism of coherent sheaves $m_E : p_X^*E \to m_X^*E$ which satisfies certain associativity properties. We refer to [13] and [14], Section 2 for the group scheme action theory we are talking about here.

Let $X$ be a $\mu_n$-equivariant scheme, we consider the category of coherent $\mathcal{O}_X$-modules endowed with an action of $\mu_n$ which are compatible with the $\mu_n$-structure of $X$. According to Quillen, to this category we may associate a graded abelian group $G_*(X, \mu_n)$ which is called the higher algebraic equivariant $G$-group. If one replaces the $\mu_n$-equivariant coherent $\mathcal{O}_X$-modules by the $\mu_n$-equivariant vector bundles of finite rank, one gets the higher algebraic equivariant $K$-group $K_*(X, \mu_n)$. It is well known that the tensor product of $\mu_n$-equivariant vector bundles induces a graded ring structure on $K_*(X, \mu_n)$ and a graded $K_*(X, \mu_n)$-module structure on $G_*(X, \mu_n)$. Notice that if $X$ is regular, then the natural morphism from $K_*(X, \mu_n)$ to $G_*(X, \mu_n)$ is an isomorphism.

Denote by $X_{\mu_n}$ the fixed point subscheme of $X$ under the action of $\mu_n$, then the closed immersion $i : X_{\mu_n} \hookrightarrow X$ induces two group homomorphisms $i_* : G_*(X_{\mu_n}, \mu_n) \to G_*(X, \mu_n)$ and $i_* : K_*(X_{\mu_n}, \mu_n) \to K_*(X, \mu_n)$ which satisfy the projection formula. According to [21]. I 4.4, $\mu_n$ is the pull-back of a unique diagonalisable group scheme over $\mathbb{Z}$ associated to the same group, this group scheme will be still denoted by $\mu_n$. Write $R(\mu_n)$ for the group $K_0(\mathbb{Z}, \mu_n)$ which is isomorphic to $\mathbb{Z}[T]/(1 - T^n)$. Let $\rho$ be the prime ideal of $R(\mu_n)$ which is defined to be the kernel of the following canonical morphism

$$\mathbb{Z}[T]/(1 - T^n) \to \mathbb{Z}[T]/(\Phi_n)$$

where $\Phi_n$ stands for the $n$-th cyclotomic polynomial. The prime ideal $\rho$ is chosen to satisfy the condition that the localization $R(\mu_n)_{\rho}$ is a $R(\mu_n)$-algebra in which the elements $1 - T^k$ from $k = 1$ to $n - 1$ are all invertible. This condition plays a crucial role in the proof of the concentration theorem. If the $\mu_n$-equivariant scheme $X$ is regular, then $X_{\mu_n}$ is also regular. We shall write $\lambda_-(N_{X_{\mu_n}/X})$ for the alternating sum $\sum (-1)^j \wedge^j N_{X/X_{\mu_n}}$ where $N_{X/X_{\mu_n}}$ stands for the normal bundle associated to the regular immersion $i$. Then the algebraic concentration theorem in [22] can be described as the following.

**Theorem 2.1.** (Thomason) Let notations and assumptions be as above.

- The $R(\mu_n)_{\rho}$-module morphism $i_* : G_*(X_{\mu_n}, \mu_n)_{\rho} \to G_*(X, \mu_n)_{\rho}$ is actually an isomorphism.
• If $X$ is regular, then $\lambda_1(N_{X/X_{\mu_n}})$ is invertible in $G_*(X_{\mu_n}, \mu_n)_\rho$ and the inverse map of $i_*$ is given by $\lambda_1^{-1}(N_{X/X_{\mu_n}}) \cdot i^*$.

The proof of Thomason’s algebraic concentration theorem can be split into three steps. The first step is to show that $G_*(U, \mu_n)_\rho \cong 0$ if $U$ has no fixed point, then the claim that $i_* : G_*(X_{\mu_n}, \mu_n)_\rho \rightarrow G_*(X, \mu_n)_\rho$ is an isomorphism follows from Quillen’s localization sequence for higher equivariant $K$-theory, see [22], Théorème 2.1. The second step is to show that $\lambda_1(N_{X/X_{\mu_n}})$ is invertible in $G_*(X_{\mu_n}, \mu_n)_\rho$ if $X$ is regular (cf. [22], Lemme 3.2). The last step is a direct computation using the projection formula for equivariant $K$-theory to show that the inverse map of $i_*$ is exactly $\lambda_1^{-1}(N_{X/X_{\mu_n}}) \cdot i^*$ (cf. [22], Lemme 3.3). The condition that the localization $R(\mu_n)_\rho$ is a $R(\mu_n)$-algebra in which the elements $1 - T^k$ from $k = 1$ to $n - 1$ are all invertible was used in the first and the second step.

3 Equivariant arithmetic $K_0$-theory

By an arithmetic ring $D$ we understand a regular, excellent, Noetherian integral ring, together with a finite set $S$ of embeddings $D \hookrightarrow \mathbb{C}$, which is invariant under a conjugate-linear involution $F_\infty$ (cf. [12], Def. 3.1.1). A $\mu_n$-equivariant arithmetic variety $X$ over $D$ is a regular Noetherian scheme $X$ endowed with a $\mu_n$-projective action over $D$, namely there exists an equivariant closed immersion from $X$ to some $\mu_n$-equivariant projective space $\mathbb{P}^n_D$ (cf. [12], Definition 2.3). Let $X$ be a $\mu_n$-equivariant arithmetic variety, then $X(\mathbb{C})$, the set of complex points of the variety $\prod_{s \in S} X \times_D \mathbb{C}$, is a compact complex manifold. This manifold admits an action of the group of complex $n$-th roots of unity and an anti-holomorphic involution induced by $F_\infty$ which is still denoted by $F_\infty$. It was shown in [22], Prop. 3.1 that the fixed point subscheme $X_{\mu_n}$ is also regular. Fix a primitive $n$-th root of unity $\zeta_n$ and denote its corresponding holomorphic automorphism on $X(\mathbb{C})$ by $g$, by GAGA principle we have a natural isomorphism $X_{\mu_n}(\mathbb{C}) \cong X(\mathbb{C})_g$. Moreover, denote by $A^{p,p}(X(\mathbb{C})_g)$ the set of smooth forms $\omega$ of type $(p,p)$ on $X(\mathbb{C})_g$ which satisfy $F_\infty^{\omega} = (-1)^p \omega$, we shall write $A(X_{\mu_n})$ for the set of form classes

$$\tilde{A}(X(\mathbb{C})_g) := \bigoplus_{p \geq 0} (A^{p,p}(X(\mathbb{C})_g)/(\text{Im}\partial + \text{Im}\bar{\partial})).$$

Similarly, denote by $D^{p,p}(X(\mathbb{C})_g)$ the set of currents $T$ of type $(p,p)$ on $X(\mathbb{C})_g$ which satisfy $F_\infty^T = (-1)^p T$, we shall write $\tilde{U}(X_{\mu_n})$ for the set of current classes

$$\tilde{U}(X(\mathbb{C})_g) := \bigoplus_{p \geq 0} (D^{p,p}(X(\mathbb{C})_g)/(\text{Im}\partial + \text{Im}\bar{\partial})).$$

Definition 3.1. An equivariant hermitian vector bundle $E$ on $X$ is a hermitian vector bundle $E$ on $X$, endowed with a $\mu_n$-action which lifts the action of $\mu_n$ on $X$ such that the hermitian metric on $E_\mathbb{C}$ is invariant under $F_\infty$ and $g$. 
Remark 3.2. Let $\overline{E}$ be an equivariant hermitian vector bundle on $X$, the restriction of $\overline{E}$ to the fixed point subscheme $X_{\mu_n}$ has a natural $\mathbb{Z}/n\mathbb{Z}$-grading structure $\overline{E}|_{X_{\mu_n}} \cong \oplus_{k \in \mathbb{Z}/n\mathbb{Z}} E_k$. $E_k$ is the subbundle of $E|_{X_{\mu_n}}$ such that for open affine subscheme $V$ of $X_{\mu_n}$ the action of $\mu_n(V)$ on $E_k(V)$ is given by $u \cdot e_k = u^k \cdot e_k$ (cf. [21], I, Prop. 4.7.3). We shall often write $\overline{E}_{\mu_n}$ for $\overline{E}_0$.

Following the same notations and definitions as in [14], Section 3, we write $\text{ch}_g(\overline{E})$ for the equivariant Chern character form

$$\text{ch}_g((E_\mathbb{C}, h)) = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \zeta^k \text{ch}(E_{k\mathbb{C}}, h_{E_k})$$

associated to the hermitian holomorphic vector bundle $(E_\mathbb{C}, h)$ on $X(\mathbb{C})$. Moreover, we have the equivariant Todd form

$$\text{Td}_g(\overline{E}) := \frac{c_{\text{ch} E_{\mathbb{C}}}(\overline{E}_{\mu_n \mathbb{C}})}{\text{ch}_g(\sum_{k \in \mathbb{Z}/n\mathbb{Z}} (-1)^j \wedge j E_k^*)}.$$ 

Furthermore, let $\overline{\varphi} : 0 \to \overline{E} \to \overline{E} \to \overline{E} \to 0$ be an exact sequence of equivariant hermitian vector bundles on $X$, we can associate to it an equivariant Bott-Chern secondary characteristic class $\tilde{\text{ch}}_g(\overline{\varphi}) \in \tilde{A}(X_{\mu_n})$ which satisfies the differential equation

$$\text{dd}^c \tilde{\text{ch}}_g(\overline{\varphi}) = \text{ch}_g(\overline{E}^*) - \text{ch}_g(\overline{E}) + \text{ch}_g(\overline{E}^*)$$

where $\text{dd}^c$ is the differential operator $\frac{\partial}{\partial z}$. Similarly, we have the equivariant Todd secondary characteristic class $\tilde{Td}_g(\overline{\varphi}) \in \tilde{A}(X_{\mu_n})$ which satisfies the differential equation

$$\text{dd}^c \tilde{Td}_g(\overline{\varphi}) = \text{Td}_g(\overline{E}^*) - \text{Td}_g(\overline{E}^*) - \text{Td}_g(\overline{E})$$

Let $E$ be an equivariant vector bundle with two different hermitian metrics $h_1$ and $h_2$, we shall write $\tilde{\text{ch}}_g(E, h_1, h_2)$ (resp. $\tilde{Td}_g(E, h_1, h_2)$) for the equivariant Bott-Chern (resp. Todd) secondary characteristic class associated to the exact sequence

$$0 \to (E, h_1) \to (E, h_2) \to 0 \to 0$$

where the map from $(E, h_1)$ to $(E, h_2)$ is the identity map.

Definition 3.3. Let $X$ be a $\mu_n$-equivariant arithmetic variety, we define the equivariant arithmetic Grothendieck group $\tilde{K}_0(X, \mu_n)$ with respect to $X$ as the free abelian group generated by the elements of $\tilde{A}(X_{\mu_n})$ and by the equivariant isometry classes of equivariant hermitian vector bundles on $X$, together with the relations

(i). for every exact sequence $\overline{\varphi}$ as above, $\tilde{\text{ch}}_g(\overline{\varphi}) = \overline{E}^* - \overline{E} + \overline{E}^*$;

(ii). if $\alpha \in \tilde{A}(X_{\mu_n})$ is the sum of two elements $\alpha'$ and $\alpha''$ in $\tilde{A}(X_{\mu_n})$, then the equality $\alpha = \alpha' + \alpha''$ holds in $\tilde{K}_0(X, \mu_n)$.
Remark 3.4. The definition of the arithmetic $K_0$-group implies that there is an exact sequence
\[
\tilde{A}(X_{\mu_n}) \xrightarrow{\alpha} \tilde{K}_0(X, \mu_n) \xrightarrow{\pi} K_0(X, \mu_n) \xrightarrow{} 0
\]
where $\alpha$ is the natural map which sends $\alpha \in \tilde{A}(X_{\mu_n})$ to the class of $\alpha$ in $\tilde{K}_0(X, \mu_n)$ and $\pi$ is the forgetful map.

We now describe the ring structure of $\tilde{K}_0(X, \mu_n)$. We consider the generators of the abelian group $\tilde{K}_0(X, \mu_n)$, for two equivariant hermitian vector bundles $E$, $E'$ on $X$ and two elements $\alpha$, $\alpha'$ in $\tilde{A}(X_{\mu_n})$, we define the rules of the product $\cdot$ as $E \cdot E' := E \otimes E'$, $E \cdot \alpha = \alpha \cdot E := ch_g(E) \wedge \alpha$ and $\alpha \cdot \alpha' := dd_c \alpha \wedge \alpha'$. Note that $\alpha$ and $\alpha'$ are both smooth, so $\alpha \cdot \alpha'$ is well-defined and it is commutative in $\tilde{A}(X_{\mu_n})$. It is easy to verify that our definition is compatible with the two generating relations in Definition 3.3, we leave the verification to the reader.

Furthermore, recall that $R(\mu_n) = \mathbb{Z}[T]/(1 - T^n)$. Let $I$ be the $\mu_n$-equivariant hermitian projective $D$-module whose term of degree 1 is $D$ endowed with the trivial metric and whose other terms are 0, namely $I$ is the structure sheaf $O_D$ of $\text{Spec}(D)$ on which the $\mu_n$-action is given by $u \cdot d = u \cdot d$ where $u \in \mu_n(V)$, $d \in O_D(V)$ and $V$ is an open affine subscheme of $\text{Spec}(D)$. Then we may make $\tilde{K}_0(D, \mu_n)$ an $R(\mu_n)$-algebra under the ring morphism which sends $T$ to $\tilde{T}$. By doing pull-backs, we may endow every arithmetic Grothendieck group we defined before with an $R(\mu_n)$-module structure.

Since the classical arguments of locally free resolutions may not be compatible with the equivariant setting, we summarize some crucial facts we need as follows.

(i). Every equivariant coherent sheaf on an equivariant arithmetic variety is an equivariant quotient of an equivariant locally free coherent sheaf.

(ii). Every equivariant coherent sheaf on an equivariant arithmetic variety admits a finite equivariant locally free resolution.

(iii). An exact sequence of equivariant coherent sheaves on an equivariant arithmetic variety admits an exact sequence of equivariant locally free resolutions.

(iv). Any two equivariant locally free resolutions of an equivariant coherent sheaf on an equivariant arithmetic variety can be dominated by a third one.

All these statements can be found in the proof of [14], Prop. 4.2. For (i) and (ii), one can also see [13], Remark 3.5.

To end this section, we recall the following important lemma which will be used frequently in this paper.

Lemma 3.5. ([14], Lemma 4.5) Let $X$ be a $\mu_n$-equivariant arithmetic variety and let $E$ be an equivariant hermitian vector bundle on $X_{\mu_n}$ such that $E_{\mu_n} = 0$. Then the element $\lambda_{-1}(E)$ is invertible in $\tilde{K}_0(X_{\mu_n}, \mu_n)$. 
4 Analytical preliminaries

4.1 Equivariant analytic torsion forms

In [7], J.-M. Bismut and K. Kähler extended the Ray-Singer analytic torsion to the higher analytic torsion form $T$ for a holomorphic submersion. The differential equation on $dd^c T$ gives a refinement of the Grothendieck-Riemann-Roch theorem. And later, X. Ma generalized J.-M. Bismut and K. Kähler’s results to the equivariant case in his article [10]. In this subsection, we shall recall Ma’s construction of the equivariant analytic torsion form since it will be used to define a reasonable push-forward morphism between equivariant arithmetic $K_0$-groups.

Let $f : M \to B$ be a proper holomorphic submersion of complex manifolds, and let $TM, TB$ be the holomorphic tangent bundle on $M, B$. Denote by $J^{Tf}$ the complex structure on the real relative tangent bundle $T_{\mathbb{R}}f$, and assume that $h^{Tf}$ is a hermitian metric on $Tf$ which induces a Riemannian metric $g^{Tf}$. Let $T^H M$ be a vector subbundle of $TM$ such that $TM = T^H M \oplus Tf$, we first give the definition of Kähler fibration as in [5], Def. 1.4.

**Definition 4.1.** The triple $(f, h^{Tf}, T^H M)$ is said to define a Kähler fibration if there exists a smooth real $(1, 1)$-form $\omega$ which satisfies the following three conditions:

(i). $\omega$ is closed;

(ii). $T^H M$ and $T_{\mathbb{R}}f$ are orthogonal with respect to $\omega$;

(iii). if $X, Y \in T_{\mathbb{R}}f$, then $\omega(X, Y) = \langle X, J^{Tf} Y \rangle_{g^{Tf}}$.

It was shown in [5], Thm. 1.5 and 1.7 that for a given Kähler fibration, the form $\omega$ is unique up to addition of a form $f^*\eta$ where $\eta$ is a real, closed $(1, 1)$-form on $B$. Moreover, for any real, closed $(1, 1)$-form $\omega$ on $M$ such that the bilinear map $X, Y \in T_{\mathbb{R}}f \mapsto \omega(J^{Tf} X, Y) \in \mathbb{R}$ defines a Riemannian metric and hence a hermitian product $h^{Tf}$ on $Tf$, we can define a Kähler fibration whose associated $(1, 1)$-form is $\omega$. In particular, for a given $f$, a Kähler metric on $M$ defines a Kähler fibration if we choose $T^H M$ to be the orthogonal complement of $Tf$ in $TM$ and $\omega$ to be the Kähler form associated to this metric.

We now recall the Bismut superconnection of a Kähler fibration. Let $(\xi, h^{\xi})$ be a hermitian holomorphic vector bundle on $M$. Let $\nabla^{Tf}$, $\nabla^{\xi}$ be the holomorphic hermitian connections on $(Tf, h^{Tf})$ and $(\xi, h^{\xi})$. Let $\nabla^{\Lambda(T^*(0,1)f)}_{\xi}$ be the connection induced by $\nabla^{Tf}$ on $\Lambda(T^*(0,1)f)$. Then we may define a connection on $\Lambda(T^*(0,1)f) \otimes \xi$ by setting

$$\nabla^{\Lambda(T^*(0,1)f) \otimes \xi} = \nabla^{\Lambda(T^*(0,1)f)} \otimes 1 + 1 \otimes \nabla^{\xi}.$$ 

Let $E$ be the infinite-dimensional bundle on $B$ whose fibre at each point $b \in B$ consists of the $C^\infty$ sections of $\Lambda(T^*(0,1)f) \otimes \xi |_{f^{-1}b}$. This bundle $E$ is a smooth $\mathbb{Z}$-graded bundle. We define a connection $\nabla^E$ on $E$ as follows. If $U \in T_{\mathbb{R}}B$, let $U^H$ be the lift of $U$ in $T_{\mathbb{R}}^H M$ so that $f_* U^H = U$. Then for every smooth section $s$ of $E$ over $B$, we set

$$\nabla^E_U s = \nabla^{\Lambda(T^*(0,1)f) \otimes \xi}_U s.$$
For $b \in B$, let $\overline{\partial}_b$ be the Dolbeault operator acting on $E_b$, and let $\overline{\partial}_b^*$ be its formal adjoint with respect to the canonical hermitian product on $E_b$ (cf. [10], 1.2). Let $C(T_R f)$ be the Clifford algebra of $(T_R f, h^{T_f})$, then the bundle $\Lambda(T^{*(0,1)} f) \otimes \xi$ has a natural $C(T_R f)$-Clifford module structure. Actually, if $U \in T f$, let $U' \in T^{*(0,1)} f$ correspond to $U$ defined by $U' (\cdot) = h^{T_f}(U, \cdot)$, then for $U, V \in T f$ we set
\[
c(U) = \sqrt{2} U' \wedge, \quad c(\overline{\nabla}) = -\sqrt{2} i \overline{\nabla}
\]
where $i_{(\cdot)}$ is the contraction operator (cf. [9], Definition 1.6). Moreover, if $U, V \in T_b B$, we set $T(U^H, V^H) = -P^{T_f}[U^H, V^H]$ where $P^{T_f}$ stands for the canonical projection from $TM$ to $T f$.

**Definition 4.2.** Let $e_1, \ldots, e_{2m}$ be a basis of $T_b B$, and let $e^1, \ldots, e^{2m}$ be the dual basis of $T_b^* B$. Then the element
\[
c(T) = \frac{1}{2} \sum_{1 \leq \alpha, \beta \leq 2m} e^\alpha \wedge e^\beta \otimes c(T(e^\alpha, e^\beta))
\]
is a section of $(f^*(\Lambda(T^* B) \otimes \Lambda(T^{*(0,1)} f) \otimes \xi))^{odd}$.

**Definition 4.3.** For $u > 0$, the Bismut superconnection on $E$ is the differential operator
\[
B_u = \nabla^E + \sqrt{u(\overline{\partial}^Z + \overline{\partial}^{Z^*})} - \frac{1}{2 \sqrt{2u}} c(T)
\]
on $f^*(\Lambda(T^*_B B) \otimes \Lambda(T^{*(0,1)} f) \otimes \xi)$.

**Definition 4.4.** Let $N_V$ be the number operator on $\Lambda(T^{*(0,1)} f) \otimes \xi$ and on $E$, namely $N_V$ acts as multiplication by $p$ on $\Lambda^p(T^{*(0,1)} f) \otimes \xi$. For $U, V \in T_b B$, set $\omega^{M}(U, V) = \omega^{M}(U^H, V^H)$ where $\omega^{M}$ is the closed form in the definition of Kähler fibration. Furthermore, for $u > 0$, set $N_u = N_V + \frac{i \omega^{M}_u}{u}$. $N_u$ is a section of $(f^*(\Lambda(T^*_B B) \otimes \Lambda(T^{*(0,1)} f) \otimes \xi))^{even}$.

We now turn to the equivariant case. Let $G$ be a compact Lie group, we shall assume that all complex manifolds and holomorphic morphisms considered above are $G$-equivariant and all metrics are $G$-invariant. We will additionally assume that the direct images $R^k f_x \xi$ are all locally free so that the $G$-equivariant coherent sheaf $R^k f_x \xi$ is locally free and hence a $G$-equivariant vector bundle over $B$. [10], 1.2 gives a $G$-invariant hermitian metric (the $L^2$-metric) $h^{R^k f_x \xi}$ on the vector bundle $R^k f_x \xi$.

For $g \in G$, let $M_g = \{ x \in M \mid g \cdot x = x \}$ and $B_g = \{ b \in B \mid g \cdot b = b \}$ be the fixed point submanifolds, then $f$ induces a holomorphic submersions $f_g : M_g \to B_g$. Let $\Phi$ be the homomorphism $\alpha \mapsto (2i\pi)^{-\deg_0/2}$ of $\Lambda^{even}(T^*_R B)$ into itself. We put
\[
\text{ch}_g(R^k f_x \xi, h^{R^k f_x \xi}) = \sum_{k=0}^{\dim M - \dim B} (-1)^k \text{ch}_g(R^k f_x \xi, h^{R^k f_x \xi})
\]
and
\[
\text{ch}'_g(R^k f_x \xi, h^{R^k f_x \xi}) = \sum_{k=0}^{\dim M - \dim B} (-1)^k k \text{ch}_g(R^k f_x \xi, h^{R^k f_x \xi}).
\]
**Definition 4.5.** For $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, let
\[
\zeta_1(s) = -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1}(\Phi \text{Tr}_u[gNu \exp(-B_u^2)]) - \chi'_g(Rf_s \xi, hR_f \xi))du
\]
and similarly for $s \in \mathbb{C}$ with $\text{Re}(s) < \frac{1}{2}$, let
\[
\zeta_2(s) = -\frac{1}{\Gamma(s)} \int_1^\infty u^{s-1}(\Phi \text{Tr}_u[gNu \exp(-B_u^2)]) - \chi'_g(Rf_s \xi, hR_f \xi))du.
\]

X. Ma proves that $\zeta_1(s)$ extends to a holomorphic function of $s \in \mathbb{C}$ near $s = 0$ and $\zeta_2(s)$ is a holomorphic function of $s$.

**Definition 4.6.** The smooth form $T_g(\omega^M, h^\xi) := \frac{\partial}{\partial s}(\zeta_1 + \zeta_2)(0)$ on $B_g$ is called the equivariant analytic torsion form.

**Theorem 4.7.** The form $T_g(\omega^M, h^\xi)$ lies in $\bigoplus_{p \geq 0} A^{p,p}(B_g)$ and satisfies the following differential equation
\[
\text{dd}^c T_g(\omega^M, h^\xi) = \chi_g(Rf_s \xi, hR_f \xi) - \int_{M_g/B_g} \text{Td}_g(Tf, h^{Tf})\chi_g(\xi, h^\xi).
\]

Here $A^{p,p}(B_g)$ stands for the space of smooth forms on $B_g$ of type $(p, p)$.

**Proof.** This is [16], Theorem 2.12. 

We define a secondary characteristic class
\[
\widetilde{\chi}_g(Rf_s \xi, hR_f \xi, h'^R_f \xi) := \sum_{k=0}^{\dim M - \dim B} (-1)^k \widetilde{\chi}_g(R^k f_s \xi, h^R f_s \xi, h'^R f_s \xi)
\]
such that it satisfies the following differential equation
\[
\text{dd}^c \widetilde{\chi}_g(Rf_s \xi, hR_f \xi, h'^R_f \xi) = \chi_g(Rf_s \xi, hR_f \xi) - \chi_g(Rf_s \xi, h'^R_f \xi),
\]
then the anomaly formula can be described as follows.

**Theorem 4.8.** (Anomaly formula) Let $\omega'$ be the form associated to another Kähler fibration for $f : M \to B$. Let $h'^{Tf}$ be the metric on $Tf$ in this new fibration and let $h'^\xi$ be another metric on $\xi$. The following identity holds in $\widetilde{A}(B_g) := \bigoplus_{p \geq 0}(A^{p,p}(B_g)/(\text{Im} \partial + \text{Im} \bar{\partial}))$:
\[
T_g(\omega^M, h^\xi) - T_g(\omega'^M, h'^\xi) = \widetilde{\chi}_g(Rf_s \xi, hR_f \xi, h'^R_f \xi) - \int_{M_g/B_g} \text{Td}_g(Tf, h^{Tf}, h'^{Tf})\chi_g(\xi, h^\xi)
+ \text{Td}_g(Tf, h'^{Tf})\widetilde{\chi}_g(\xi, h'^\xi, h'^\xi).
\]

In particular, the class of $T_g(\omega^M, h^\xi)$ in $\widetilde{A}(B_g)$ only depends on $(h^{Tf}, h^\xi)$.

**Proof.** This is [16], Theorem 2.13. 

\[
\]
4.2 Equivariant Bott-Chern singular currents

The Bott-Chern singular current was defined by J.-M. Bismut, H. Gillet and C. Soulé in [6] in order to generalize the usual Bott-Chern secondary characteristic class to the case where one considers the resolutions of hermitian vector bundles associated to the closed immersions of complex manifolds. In [2], J.-M. Bismut generalized this topic to the equivariant case. We shall recall Bismut’s construction of the equivariant Bott-Chern singular current in this subsection since it plays a crucial role in our later arguments. Bismut’s construction was realized via some current valued zeta function which involves the supertraces of Quillen’s superconnections. This is similar to the non-equivariant case.

As before, let $g$ be the automorphism corresponding to an element in a compact Lie group $G$. Let $i : Y \to X$ be an equivariant closed immersion of $G$-equivariant Kähler manifolds, and let $\pi$ be an equivariant hermitian vector bundle on $Y$. Assume that $\xi$ is a complex of equivariant hermitian vector bundles on $X$ which provides a resolution of $i_* \eta$. We denote the differential of the complex $\xi$ by $v$. Note that $\xi$ is acyclic outside $Y$ and the homology sheaves of its restriction to $Y$ are locally free and hence they are all vector bundles. We write $H_n = H_n(\xi|_Y)$ and define a $\mathbb{Z}$-graded bundle $H = \bigoplus_n H_n$. For each $y \in Y$ and $u \in TX_y$, we denote by $\partial_u v(y)$ the derivative of $v$ at $y$ in the direction $u$ in any given holomorphic trivialization of $\xi$ near $y$. Then the map $\partial_u v(y)$ acts on $H_y$ as a chain map, and this action only depends on the image $z$ of $u$ in $N_y$ where $N$ stands for the normal bundle of $i(Y)$ in $X$. So we get a chain complex of holomorphic vector bundles $(H, \partial z v)$.

Let $\pi$ be the projection from the normal bundle $N$ to $Y$, then we have a canonical identification

$$(\pi^* H, \partial z) \cong (\pi^*(\wedge N^\vee \otimes \eta), i_z).$$

For this, one can see [3], Section I.b. Moreover, such an identification is an identification of $G$-bundles which induces a family of canonical isomorphisms $\gamma_n : H_n \cong \wedge^n N^\vee \otimes \eta$. Another way to describe these canonical isomorphisms $\gamma_n$ is applying [10], Exp. VII, Lemma 2.4 and Proposition 2.5. These two constructions coincide because they are both locally, on a suitable open covering $\{U_j\}_{j \in J}$, determined by any complex morphism over the identity map of $\eta|_{U_j}$ from $(\xi|_{U_j}, v)$ to the minimal resolution of $\eta|_{U_j}$ (e.g. the Koszul resolution). The advantage of using the construction given in [10] is that it remains valid for arithmetic varieties over any base instead of the complex numbers. Later in [2], for the use of normalization, J.-M. Bismut considered the automorphism of $N^\vee$ defined by multiplying a constant $-\sqrt{-1}$, it induces an isomorphism of chain complexes

$$(\pi^*(\wedge N^\vee \otimes \eta), i_z) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1} i_z)$$

and hence

$$(\pi^* H, \partial z) \cong (\pi^*(\wedge N^\vee \otimes \eta), \sqrt{-1} i_z).$$

This identification induces a family of isomorphisms $\tilde{\gamma}_n : H_n \cong \wedge^n N^\vee \otimes \eta$. By finite dimensional Hodge theory, for each $y \in Y$, there is a canonical isomorphism

$$H_y \cong \{ f \in \xi_y | vf = 0, v^* f = 0 \}.$$
where $v^*$ is the dual of $v$ with respect to the metrics on $\xi$. This means that $H$ can be regarded as a smooth $\mathbb{Z}$-graded $G$-equivariant subbundle of $\xi$ so that it carries an induced $G$-invariant metric. On the other hand, we endow $\wedge N^\vee \otimes \eta$ with the metric induced from $\overline{N}$ and $\overline{\eta}$.

**Definition 4.9.** We say that the metrics on the complex of equivariant hermitian vector bundles $\xi$ satisfy Bismut assumption (A) if the identification $(\pi^*H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), i_z)$ also identifies the metrics, it is equivalent to the condition that the identification $(\pi^*H, \partial_z v) \cong (\pi^*(\wedge N^\vee \otimes \eta), i_z)$ identifies the metrics.

**Remark 4.10.** If the metrics on the complex of equivariant hermitian vector bundles $\xi$ satisfy Bismut assumption (A), then the isomorphisms $\gamma_n$ and $\tilde{\gamma}_n$ are all isometries.

**Proposition 4.11.** There always exist $G$-invariant metrics on $\xi$ which satisfy Bismut assumption (A) with respect to the equivariant hermitian vector bundles $N$ and $\eta$.

**Proof.** This is [2], Proposition 3.5.

From now on we always suppose that the metrics on a resolution satisfy Bismut assumption (A). Let $\nabla^\xi$ be the canonical hermitian holomorphic connection on $\xi$, then for each $u > 0$, we may define a $G$-invariant superconnection

$$C_u := \nabla^\xi + \sqrt{u}(v + v^*)$$

on the $\mathbb{Z}_2$-graded vector bundle $\xi$. Moreover, let $\Phi$ be the map $\alpha \in \wedge (T^*_{\mathbb{R}}X_g) \to (2\pi i)^{-\deg \alpha/2} \alpha \in \wedge (T^*_gX_g)$ and denote

$$\left(\text{Td}^{-1}g\right)(\overline{N}) := \left. \frac{\partial}{\partial b} \right| b=0 \left(\text{Td}_g(b \cdot \text{Id} - \Omega_N^{2\pi i})^{-1}\right)$$

where $\Omega_N^{2\pi i}$ is the curvature form associated to $\overline{N}$. We recall as follows the construction of the equivariant singular current given in [2], Section VI.

**Lemma 4.12.** Let $N_H$ be the number operator on the complex $\xi$, (of homological type), then for $s \in \mathbb{C}$ and $0 < \text{Re}(s) < \frac{1}{2}$, the current valued zeta function

$$Z_g(\xi)(s) := \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \left[ \Phi \text{Tr}_s(N_H g \exp(-C_u^2)) + \left(\text{Td}^{-1}_g(\overline{N}) \text{ch}_g(\overline{\eta}) \delta_{Y_g}\right) \right] du$$

is well-defined on $X_g$ and it has a meromorphic continuation to the complex plane which is holomorphic at $s = 0$.

**Definition 4.13.** The equivariant singular Bott-Chern current on $X_g$ associated to the resolution $\overline{\xi}$, is defined as

$$T_g(\overline{\xi}) := \left. \frac{\partial}{\partial s} \right|_{s=0} Z_g(\overline{\xi})(s).$$
Theorem 4.14. The current $T_g(\xi,.)$ is a sum of $(p,p)$-currents and it satisfies the differential equation
\[
\text{dd}^c T_g(\xi,.) = i_{g*} \text{ch}_g(\eta) \text{Td}_g^{-1}(N) - \sum_k (-1)^k \text{ch}_g(\xi_k).
\]
Moreover, the wave front set of $T_g(\xi,.)$ is contained in $\mathcal{N}_g^{\vee}$.

Proof. This follows from [2, Theorem 6.7 and Remark 6.8].

Finally, we recall a theorem concerning the relationship of the equivariant Bott-Chern singular currents involved in a double complex. This theorem makes sure that our definition for an embedding morphism in arithmetic $K_0$-theory is reasonable.

Theorem 4.15. Let
\[
\overline{\chi} : 0 \to \overline{\eta}_n \to \cdots \to \overline{\eta}_1 \to \overline{\eta}_0 \to 0
\]
be an exact sequence of equivariant hermitian vector bundles on $Y$. Assume that we have the following double complex consisting of resolutions of $i_*\overline{\chi}$ such that all rows are exact sequences.

\[
\begin{array}{cccccccc}
0 & \to & \overline{\xi}_n & \to & \cdots & \to & \overline{\xi}_1 & \to & \overline{\xi}_0 & \to & 0 \\
0 & \to & i_*\overline{\eta}_n & \to & \cdots & \to & i_*\overline{\eta}_1 & \to & i_*\overline{\eta}_0 & \to & 0.
\end{array}
\]

For each $k$, we write $\overline{\epsilon}_k$ for the exact sequence
\[
0 \to \overline{\xi}_{n,k} \to \cdots \to \overline{\xi}_{1,k} \to \overline{\xi}_{0,k} \to 0.
\]

Then we have the following equality in $\overline{U}(X_g) := \bigoplus_{p \geq 0} (D^{p,p}(X_g)/(\text{Im}\partial + \text{Im}\overline{\partial}))$
\[
\sum_{j=0}^{n} (-1)^j T_g(\xi_j,.) = i_{g*} \overline{\text{ch}}_g(\overline{\chi}) \text{Td}_g^{-1}(N) - \sum_k (-1)^k \overline{\text{ch}}_g(\overline{\epsilon}_k).
\]

Here $D^{p,p}(X_g)$ stands for the space of currents on $X_g$ of type $(p,p)$.

Proof. This is [14, Theorem 3.14].

4.3 Bismut-Ma’s immersion formula

In this subsection, we shall recall Bismut-Ma’s immersion formula which reflects the behaviour of the equivariant analytic torsion forms of a Kähler fibration under the composition of an immersion and a submersion. Such a formula can be used to measure, in arithmetic $K_0$-theory, the difference between a push-forward morphism and the composition formed as an embedding morphism followed by a push-forward morphism.
Let $i : Y \rightarrow X$ be an equivariant closed immersion of $G$-equivariant Kähler manifolds. Let $S$ be a complex manifold with trivial $G$-action, and let $f : Y \rightarrow S$, $l : X \rightarrow S$ be two equivariant proper holomorphic submersions such that $f = l \circ i$. Assume that $\mathfrak{n}$ is an equivariant hermitian vector bundle on $Y$ and $\xi$ provides a resolution of $i^* \eta$ on $X$ whose metrics satisfy Bismut assumption (A). Let $\omega^X$, $\omega^X$ be the real, closed and $G$-invariant $(1,1)$-forms on $Y$, $X$ which induce the Kähler fibrations with respect to $f$ and $l$ respectively. We shall assume that $\omega^Y$ is the pull-back of $\omega^X$ so that the Kähler metric on $Y$ is induced by the Kähler metric on $X$. As before, denote by $N$ the normal bundle of $i(Y)$ in $X$. Consider the following exact sequence

$$N : 0 \rightarrow T f \rightarrow T f \mid Y \rightarrow N \rightarrow 0$$

where $N$ is endowed with the quotient metric, we shall write $\tilde{Td}_g(T f, T l \mid Y)$ for $\tilde{Td}_g(N)$ the equivariant Todd secondary characteristic class associated to $N$. It satisfies the following differential equation

$$dd^c \tilde{Td}_g(T f, T l \mid Y) = Td_g(T f, h^T f)Td_g(N) - Td_g(T l \mid Y, h^T l).$$

For simplicity, we shall suppose that in the resolution $\xi$, $\xi_j$ are all $l$-acyclic and moreover $\eta$ is $f$-acyclic. By an easy argument of long exact sequence, we have the following exact sequence

$$\Xi : 0 \rightarrow l_* (\xi_m) \rightarrow l_* (\xi_{m-1}) \rightarrow \ldots \rightarrow l_* (\xi_0) \rightarrow f_* \eta \rightarrow 0.$$

By the semi-continuity theorem, all the elements in the exact sequence above are vector bundles. In this case, we recall the definition of the $L^2$-metrics on direct images precisely as follows. We just take $f_* h^0$ as an example. Note that the semi-continuity theorem implies that the natural map

$$(R^0 f_* \eta)_s \rightarrow H^0(Y_s, \eta \mid Y_s)$$

is an isomorphism for every point $s \in S$ where $Y_s$ stands for the fibre over $s$. We may endow $H^0(Y_s, \eta \mid Y_s)$ with a $L^2$-metric given by the formula

$$< u, v >_{L^2} := \frac{1}{(2\pi)^{d_s}} \int_{Y_s} h^\eta(u, v) \omega^Y d_s d_s!$$

where $d_s$ is the complex dimension of the fibre $Y_s$. It can be shown that these metrics depend on $s$ in a $C^\infty$ manner (cf. [9], p.278) and hence define a hermitian metric on $f_* \eta$. We shall denote it by $f_* h^\eta$.

In order to understand the statement of Bismut-Ma’s immersion formula, we still have to recall an important concept defined by J.-M. Bismut, the equivariant $R$-genus. Let $W$ be a $G$-equivariant complex manifold, and let $\mathfrak{E}$ be an equivariant hermitian vector bundle on $W$. For $\zeta \in S^1$ and $s > 1$ consider the zeta function

$$L(\zeta, s) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s}.$$
and its meromorphic continuation to the whole complex plane. Define the formal power series in $x$

$$\tilde{R}(\zeta, x) := \sum_{n=0}^{\infty} \left( \frac{\partial L}{\partial s} (\zeta, -n) + L(\zeta, -n) \sum_{j=1}^{n} \frac{1}{2j} \right) \frac{x^n}{n!}.$$ 

**Definition 4.16.** The Bismut equivariant $R$-genus of an equivariant hermitian vector bundle $E$ with $E|_{X_\mu} = \sum_\zeta E_\zeta$ is defined as

$$R_g(E) := \sum_{\zeta \in S^1} \left( \text{Tr} \tilde{R}(\zeta, -\Omega E_\zeta) - \Omega E_\zeta(1/2\pi i) \right)$$

where $\Omega E_\zeta$ is the curvature form associated to $E_\zeta$. Actually, the class of $R_g(E)$ in $\tilde{A}(X_g)$ is independent of the metric and we just write $R_g(\cdot)$ for it. Furthermore, the class $R_g(\cdot)$ is additive.

**Theorem 4.17.** (Immersion formula) Let notations and assumptions be as above. Then the equality

$$\sum_{i=0}^{m} (-1)^i T_g(\omega^X, h^S) - T_g(\omega^Y, h^N) + \tilde{c}_g(\Xi, h^{L^2}) = \int_{X_g/S} T_d_g(T_l, h^{T_l}) T_g(\xi)$$

$$+ \int_{Y_g/S} \frac{T_d_g(T_f, h^{T_f})}{T_d_g(\mathcal{N})} \text{ch}_g(\eta) + \int_{X_g/S} T_d_g(T_l) R_g(T_l) \sum_{i=0}^{m} (-1)^i \text{ch}_g(\xi)$$

holds in $\tilde{A}(S)$.

**Proof.** This is the combination of [8], Theorem 0.1 and 0.2, the main theorems in that paper.

---

5 Arithmetic concentration theorem

It is the aim of this section to prove an arithmetic concentration theorem in Arakelov geometry. Let $X$ be a $\mu_n$-equivariant arithmetic variety, we consider a special closed immersion $i : X_{\mu_n} \hookrightarrow X$ where $X_{\mu_n}$ is the fixed point subscheme of $X$. We first claim that the morphism $i$ induces a well-defined group homomorphism $i_* : K_0$-groups as in the algebraic case. To construct $i_*$, some analytic datum, which is the equivariant Bott-Chern singular current, should be involved. Precisely speaking, let $\overline{\eta}$ be a $\mu_n$-equivariant hermitian vector bundle on $X_{\mu_n}$ and let $\overline{\xi}$ be a bounded complex of $\mu_n$-equivariant hermitian vector bundles which provides a resolution of $i_*\overline{\eta}$ on $X$, then we may have an equivariant Bott-Chern singular current $T_g(\overline{\xi}) \in \mathcal{H}(X_{\mu_n})$. Note that the 0-degree part of the normal bundle $N := N_{X/X_{\mu_n}}$ vanishes (cf. [14], Prop. 2.12) so that the wave front set of $T_g(\overline{\xi})$ is the empty set, and hence we know that the current $T_g(\overline{\xi})$ is actually smooth. This fact means that the following definition does make sense.
Definition 5.1. Let notations and assumptions be as above. The embedding morphism

\[ i_* : \hat{K}_0(X_{\mu_n}, \mu_n) \rightarrow \hat{K}_0(X, \mu_n) \]

is defined as follows.

(i). For every \( \mu_n \)-equivariant hermitian vector bundle \( \eta \) on \( X_{\mu_n} \), suppose that \( \xi \) is a resolution of \( i_* \eta \) on \( X \) whose metrics satisfy Bismut assumption (A), 
\[ i_* [\eta] = \sum_k (-1)^k [\xi_k] + T_g(\xi) \] 

(ii). For every \( \alpha \in \hat{A}(X_{\mu_n}) \), \( i_* \alpha = \alpha \text{Td}_g^{-1}(N) \).

Theorem 5.2. The embedding morphism \( i_* \) is a well-defined group homomorphism.

Proof. We have to prove that our definition for \( i_* \) is well-defined and it is compatible with the two generating relations of arithmetic \( K_0 \)-group. Indeed, assume that we are given a short exact sequence

\[ \chi: 0 \rightarrow \eta' \rightarrow \eta \rightarrow \eta'' \rightarrow 0 \]

of equivariant hermitian vector bundles on \( X_{\mu_n} \). As in Theorem 4.15, let \( \xi', \xi \) and \( \xi'' \) be resolutions on \( X \) of \( i_* \eta' \), \( i_* \eta \) and \( i_* \eta'' \) which fit the following double complex

\[
\begin{array}{c}
0 \rightarrow \xi' \rightarrow \xi \rightarrow \xi'' \rightarrow 0 \\
| & | & | \\
0 \rightarrow i_* \eta' \rightarrow i_* \eta \rightarrow i_* \eta'' \rightarrow 0
\end{array}
\]

such that all rows are exact. For each \( k \), we write \( \xi_k \) for the exact sequence

\[ 0 \rightarrow \xi_k \rightarrow \xi_k \rightarrow \xi_k \rightarrow 0. \]

Then Theorem 4.15 implies that the equality

\[ T_g(\xi') - T_g(\xi) + T_g(\xi'') = \frac{\chi_g(\chi)}{\text{Td}_g(N)} - \sum_k (-1)^k \tilde{\chi}_g(\xi_k) \]

holds in \( \hat{A}(X_{\mu_n}) \). This means \( i_* [\eta'] - i_* [\eta] + i_* [\eta''] = 0 \) in the group \( \hat{K}_0(X, \mu_n) \) according to its generating relations. Note that if \( \xi \) is an exact sequence then \( T_g(\xi) \) is equal to \( -\tilde{\chi}_g(\xi) \), so we have \( i_* [0] = 0 \). Moreover, any two resolutions of \( i_* \eta \) are dominated by a third one, then our arguments above also show that \( i_* [\eta] \) is independent of the choice of resolution. Therefore, the embedding morphism \( i_* \) is well-defined and it is compatible with the first generating relation of arithmetic \( K_0 \)-group. On the other hand, the compatibility with the second relation is trivial. So we are done.

Lemma 5.3. (Projection formula) For any elements \( x \in \hat{K}_0(X, \mu_n) \) and \( y \in \hat{K}_0(X_{\mu_n}, \mu_n) \), the equality \( i_*(i^* x \cdot y) = x \cdot i_* y \) holds in \( \hat{K}_0(X, \mu_n) \).
Proof. Assume that $x = \overline{E}$ and $y = \overline{F}$ are equivariant hermitian vector bundles. Let $\xi.$ be a resolution of $i_*\overline{F}$ on $X$, then $\overline{E} \otimes \xi.$ provides a resolution of $i_*(i^*\overline{E} \otimes \overline{F})$. By definition we have

$$i_*(i^*x \cdot y) = \sum (-1)^k [\xi_k \otimes \overline{E}] + \text{ch}_g(\overline{E}) T_g(\xi)$$

which is exactly $x \cdot i_*y$. Assume that $x = \alpha$ is represented by some smooth form and $y = F$ is an equivariant hermitian vector bundle. Again let $\xi.$ be a resolution of $i_*\overline{F}$ on $X$, then

$$i_*(i^*x \cdot y) = \alpha Td_g^{-1}(N)ch_g(\overline{F}) = \alpha[\text{ch}_g(\xi) + \sum (-1)^k \text{ch}_g(\xi_k)]$$

which is exactly $x \cdot i_*y$. Now assume that $x = \overline{E}$ is an equivariant hermitian vector bundle and $y = \alpha$ is represented by some smooth form, then

$$i_*(i^*x \cdot y) = i_*(\text{ch}_g(\overline{E})\alpha) = \text{ch}_g(\overline{E})\alpha Td_g^{-1}(N)$$

which is exactly $x \cdot i_*y$. Finally, if $x$ and $y$ are both represented by smooth forms then $i_*(i^*x \cdot y)$ is trivially equal to $x \cdot i_*y$ by definition. Note that $i_*$ and $i^*$ are group homomorphisms, so we may conclude the projection formula from its correctness on generators. This completes the proof. \qed

Remark 5.4. Lemma 5.3 implies that $i_*$ is even a homomorphism of $R(\mu_n)$-modules so that it induces a homomorphism between arithmetic $K_0$-groups after taking localization, and hence there exists a corresponding projection formula after taking localization.

With Remark 5.4 the arithmetic concentration theorem can be described as follows.

Theorem 5.5. The embedding morphism $i_*: \hat{K}_0(X_{\mu_n}, \mu_n)_\rho \rightarrow \hat{K}_0(X, \mu_n)_\rho$ is an isomorphism and the inverse morphism of $i_*$ is given by $\lambda^{-1}_N(N) \cdot i^*$ where $N$ again stands for the normal bundle $N_{X/X_{\mu_n}}$.

Before giving a complete proof of this concentration theorem, we need to make some purely analytic preliminaries.

Definition 5.6. Let $X$ be a compact complex manifold, and let $\xi.$ be a bounded complex of hermitian vector bundles on $X$ such that the homology sheaves are all locally free i.e. vector bundles. Suppose that the homology sheaves are endowed with some hermitian metrics $h^H$. Such a complex will be called a standard complex. Endow the kernel and the image of every differential with the induced metrics from $\xi.$ We say that a standard complex $(\xi., h^H)$ is homologically split if the following short exact sequences

$$0 \rightarrow \text{Im} \rightarrow \text{Ker} \rightarrow H_\ast \rightarrow 0$$

and

$$0 \rightarrow \text{Ker} \rightarrow \xi_\ast \rightarrow \text{Im} \rightarrow 0$$

of hermitian vector bundles are all orthogonally split.
In [17], X. Ma proved the following uniqueness theorem.

**Theorem 5.7.** Let $X$ be a compact complex manifold, then to each standard complex of hermitian vector bundles $(\xi, h^H)$ on $X$ there is a unique way to associate an element $M(\xi, h^H) \in \tilde{A}(X)$ satisfying the following conditions.

(i). $\dd^c M(\xi, h^H) = \sum (-1)^i \text{ch}(H_i) - \sum (-1)^j \text{ch}(\xi_j)$.

(ii). For any holomorphic morphism $f : X' \to X$, we have $M(f^*\xi, f^*h^H) = f^*M(\xi, h^H)$.

(iii). If $(\xi, h^H)$ is homologically split, then $M(\xi, h^H) = 0$.

The definition of standard complex and Ma’s uniqueness theorem can be easily generalized to the equivariant case. We summarize these generalizations as follows.

**Definition 5.8.** Let $X$ be a compact complex manifold which admits a holomorphic action of a compact Lie group $G$. Fix an element $g \in G$. An equivariant standard complex on $X$ is a bounded complex of $G$-equivariant hermitian vector bundles on $X$ whose restriction to $X_g$ is standard and the metrics on the homology sheaves are $g$-invariant. Again we shall write an equivariant standard complex as $(\xi, h^H)$ to emphasize the choice of the metrics on the homology sheaves.

**Theorem 5.9.** Let $X$ be a compact complex manifold which admits a holomorphic action of a compact Lie group $G$. Fix an element $g \in G$. Then to each equivariant standard complex $(\xi, h^H)$ on $X$, there is a unique additive way to associate an element $M_g(\xi, h^H) \in \tilde{A}(X_g)$ satisfying the following conditions.

(i). $\dd^c M_g(\xi, h^H) = \sum (-1)^i \text{ch}_g(H_i(\xi | X_g)) - \sum (-1)^j \text{ch}_g(\xi_j)$.

(ii). For any equivariant holomorphic morphism $f : X' \to X$, we have $M_g(f^*\xi, f^*h^H) = f_g^*M_g(\xi, h^H)$.

(iii). If $(\xi | X_g, h^H)$ is homologically split, then $M_g(\xi, h^H) = 0$.

**Proof.** The complex $\xi$ splits on $X_g$ orthogonally into a series of standard complexes $\xi_\zeta$, for all $\zeta \in S^1$. Using the non-equivariant Bott-Chern-Ma classes on $X_g$, we define

$$M_g(\xi, h^H) = \sum_{\zeta \in S^1} \zeta M(\xi_\zeta, h^{H_\zeta}).$$

Then the axiomatic characterization follows from the non-equivariant one in Theorem 5.7 and the definition of $\text{ch}_g$. For the uniqueness, first note that by the condition (ii), the relation $M_g(\xi, h^H) = M_g(\xi | X_g, h^H)$ should be satisfied, then we may reduce our proof to the case where $X$ is equal to $X_g$. Since $M_g$ is required to be additive, we only have to show that for every $\zeta \in S^1$, $M_g(\xi_\zeta, h^{H_\zeta}) = \zeta M(\xi_\zeta, h^{H_\zeta})$. This follows from Theorem 5.7 since every compact complex manifold can be regarded as an equivariant complex manifold (with the trivial action), on which any standard complex can be endowed with a $g$-structure as multiplication by $\zeta$. Such an approach is similar to the proof of [13], Theorem 3.4.
Remark 5.10. (i). The condition of compactness in Definition 5.6 and Theorem 5.7 is not necessary; it was only used in the proof of Theorem 5.9 given above.

(ii). If one directly generalizes the proof of Theorem 5.7 to the equivariant case (by trivially adding the subscript $g$ to every notation), then the condition of additivity in Theorem 5.9 can be removed. Actually the additivity is a byproduct of such a proof.

Now let $\xi$ be an equivariant standard complex on $X$. Then we can always split $\xi|_{X_g}$ into a series of short exact sequences

$$0 \to \text{Im} \to \text{Ker} \to \mathcal{H}_s(\xi|_{X_g}) \to 0$$

and

$$0 \to \text{Ker} \to \xi^*|_{X_g} \to \text{Im} \to 0$$

of equivariant hermitian vector bundles. Denote the alternating sum of the equivariant Bott-Chern secondary characteristic classes of the short exact sequences above by $C(\xi, h^H)$ such that it satisfies the following differential equation

$$dd^c C(\xi, h^H) = \sum (-1)^i \text{ch}_g(\mathcal{H}_i(\xi|_{X_g})) - \sum (-1)^j \text{ch}_g(\xi^j),$$

then it is clear that the class $C(\xi, h^H)$ is an element in $\tilde{A}(X_g)$ and it satisfies the three conditions in Theorem 5.9. To every equivariant standard complex $\xi$ on $X$, we may associated a new canonical equivariant standard complex in which the metrics on homology bundles $H_s(\xi|_{X_g})$ are induced by the metrics on $\xi$. (see Section 3.2). This special choice of metrics will be denoted by $h^H_{\text{ind}}$. It is easy to compute the difference of $C(\xi, h^H)$ and $C(\xi, h^H_{\text{ind}})$. It is the alternating sum of secondary characteristic classes

$$\sum (-1)^i \text{ch}_g(H_i(\xi|_{X_g}), h^H, h^H_{\text{ind}}).$$

Now we define another equivariant secondary class associated to $(\xi, h^H_{\text{ind}})$ by using the supertraces of Quillen’s superconnections as follows.

For $s \in \mathbb{C}$ with $\text{Re}(s) > 1$, let

$$\zeta_1(s) = -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1} \{ \Phi \text{Tr}_s[N \text{exp}(-A^2_0)] - \Phi \text{Tr}_s[N \text{exp}(-\nabla^H(\xi), 2)] \} du$$

and similarly for $s \in \mathbb{C}$ with $\text{Re}(s) < \frac{1}{2}$, let

$$\zeta_2(s) = -\frac{1}{\Gamma(s)} \int_1^\infty u^{s-1} \{ \Phi \text{Tr}_s[N \text{exp}(-A^2_0)] - \Phi \text{Tr}_s[N \text{exp}(-\nabla^H(\xi), 2)] \} du.$$
Lemma 5.11. Define $\zeta(\xi, h^H) := \zeta(\xi, h^H_{\text{ind}}) + \sum (-1)^k \tilde{\chi}_{g}(H_i(\xi_{\mid X_g}), h^H, h^H_{\text{ind}})$. Then $\zeta(\xi, h^H)$ determines an element in $\tilde{A}(X_g)$ which satisfies the three conditions in Theorem 5.9 and hence we have $\zeta(\xi, h^H) = C(\xi, h^H)$.

Proof. Actually, according to [17], Proposition 10.4, one just need to add the subscript $g$ to every step in the argument given in [4], Cor. 1.30 and nothing else should be changed. Here, we roughly describe that why $\zeta(\xi, h^H_{\text{ind}}) = 0$ when $(\xi_{\mid X_g}, h^H)$ is homologically split (note that in this case $h^H$ should be equal to $h^H_{\text{ind}}$). This can be seen from the following argument. If $(\xi_{\mid X_g}, h^H)$ is homologically split, then up to isometries we may write $F_k := \xi_k \mid_{X_g} \cong F_k \oplus \overline{F}_k \oplus \overline{F}_{k-1}$ where $\{F_k\}$ is a family of hermitian vector bundles on $X_g$. Moreover, the differential $v$ is given by $(v_1, v_2, v_3) \mapsto (v_3, 0, 0)$. So we compute directly that $A^2_{v} \mid \overline{F}_k = \nabla^2 + u(\text{Id} \overline{F}_k \oplus \text{Id} \overline{F}_{k-1})$. This equality implies that

$$
\text{Tr}_s \left[ N \text{exp}(-A^2_{v}) \right] = \sum_k (-1)^k \{ \text{Tr} \mid \overline{F}_k \left[ k \text{exp}(-\nabla^2 - u \text{Id}) \right] + \text{Tr} \mid \overline{F}_k \left[ k \text{exp}(-\nabla^2) \right] + \text{Tr} \mid \overline{F}_{k-1} \left[ k \text{exp}(-\nabla^2 - u \text{Id}) \right] \}
$$

and hence

$$
\zeta_1(s) + \zeta_2(s) = -\frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} e^{-u} \{ \Phi \text{Tr}_s \left[ N \text{exp}(-\nabla^2) \right] \} du = -\Phi \text{Tr}_s \left[ N \text{exp}(-\nabla^2) \right]
$$

which has nothing to do with $s$. So we get $\zeta(\xi, h^H) = \frac{\partial}{\partial s}(\zeta_1 + \zeta_2)(0) = 0$. \hfill $\square$

Corollary 5.12. We have $\zeta(\xi, h^H_{\text{ind}}) = C(\xi, h^H_{\text{ind}})$ in $\tilde{A}(X_g)$.

Now we go back to the arithmetic case. We consider the closed immersion $i : X_{\mu_n} \hookrightarrow X$ with hermitian normal bundle $\overline{N}$, and as before let $\overline{\eta}$ be an equivariant hermitian vector bundle on $X_{\mu_n}$. Assume that the complex $\overline{\xi}$ provides a resolution of $i_* \overline{\eta}$ on $X$ by equivariant hermitian vector bundles whose metrics satisfy Bismut assumption (A). Then the restriction of $\overline{\xi}$ to $X_g$ is naturally a standard complex such that $h^H$ is equal to $h^H_{\text{ind}}$.

Lemma 5.13. Let notations and assumptions be as above. Then the equality

$$
\lambda_{-1}(\overline{N}^\vee) \cdot \overline{\eta} - \sum_j (-1)^j i^*(\overline{\xi}_j) = T_g(\overline{\xi})
$$

holds in $\tilde{K}_0(X_{\mu_n}, \mu_n)$.

Proof. According to Remark 4.10, we can split $\overline{\xi} \mid_{X_{\mu_n}}$ into the following series of exact sequences of equivariant hermitian vector bundles

$$
0 \to \text{Im} \to \text{Ker} \to \wedge^s \overline{N}^\vee \otimes \overline{\eta} \to 0
$$

and

$$
0 \to \text{Ker} \to \overline{\xi} \mid_{X_{\mu_n}} \to \text{Im} \to 0.
$$
Then by the definition of arithmetic $K_0$-theory, $\lambda_1(\overline{N}^\vee) \cdot \eta - \sum_j (-1)^j i^*(\xi_j)$ is nothing but $C(\xi, h^\mathbb{H})$ or $C(\tilde{\xi}, h_{\text{ind}}^\mathbb{H})$ in $\tilde{K}_0(X_{\mu_n}, \mu_n)$.

Comparing with the construction of the equivariant Bott-Chern singular current recalled in Section 3.2 or with more details in [2], Section VI, we claim that in our special situation $T_g(\xi)$ is equal to $\zeta(\xi, h_{\text{ind}}^\mathbb{H})$ defined before this lemma. Actually since $\xi$ is supposed to admit the metrics satisfying Bismut assumption (A), the superconnection $A_u$ in the definition of $\zeta(\xi, h_{\text{ind}}^\mathbb{H})$ is exactly the superconnection $C_u$ in the definition of $T_g(\xi)$. Moreover, since $(\hat{H}_u(\xi, |x_{\mu_n}), h_{\text{ind}}^\mathbb{H})$ are isometric to $\wedge^* N_C \otimes \eta_C$ the supertrace $T_g[N \exp(-\nabla H(\xi), 2)]$ in the definition of $\zeta(\xi, h_{\text{ind}}^\mathbb{H})$ is equal to $-(\text{Td}_g^{-1})'(\nabla)\text{ch}_g(\eta)$ in the definition of $R_g(\xi)$, this can be seen directly from the computation [2], (6.26). So according to Corollary [5, 12] we have $C(\xi, h_{\text{ind}}^\mathbb{H}) = \zeta(\xi, h_{\text{ind}}^\mathbb{H}) = T_g(\xi)$ in $\tilde{A}(X_{\mu_n})$ and hence they are equal in the group $\tilde{K}_0(X_{\mu_n}, \mu_n)$. This implies the equality in the statement of this lemma.

**Remark 5.14.** The kernel of the proof of Lemma 5.13 is the equality $C(\xi, h_{\text{ind}}^\mathbb{H}) = T_g(\xi)$ which is in the purely analytic setting. We would like to indicate that it is possible to prove this equality by using the construction of the deformation to the normal cone. It is a little complicated but it is independently interesting. We shall fulfill this work in a coming paper.

We are now ready to give a complete proof of our arithmetic concentration theorem.

**Proof.** (of Theorem 5.5) Denote by $U$ the complement of $X_{\mu_n}$ in $X$, then $j : U \hookrightarrow X$ is an $\mu_n$-equivariant open subscheme of $X$ whose fixed point set is empty. We consider the following double complex

\[
\begin{array}{cccccc}
\tilde{A}(X_{\mu_n})_p & \xrightarrow{i_*} & \tilde{A}(X_{\mu_n})_p & \xrightarrow{j^*} & \tilde{A}(U_{\mu_n})_p & \longrightarrow 0 \\
\downarrow a & & \downarrow a & & \downarrow a & \\
\tilde{K}_0(X_{\mu_n}, \mu_n)_p & \xrightarrow{i_*} & \tilde{K}_0(X, \mu_n)_p & \xrightarrow{j^*} & \tilde{K}_0(U, \mu_n)_p & \longrightarrow 0 \\
\downarrow x & & \downarrow x & & \downarrow x & \\
K_0(X_{\mu_n}, \mu_n)_p & \xrightarrow{i_*} & K_0(X, \mu_n)_p & \xrightarrow{j^*} & K_0(U, \mu_n)_p & \longrightarrow 0 \\
& \downarrow 0 & & \downarrow 0 & & \downarrow 0 & \\
& 0 & & 0 & & 0 & \\
\end{array}
\]

whose columns are all exact according to Remark 3.4. The third row of this complex is exact by Quillen’s localization sequence for higher equivariant $K$-theory. We next claim that the other two rows are also exact. Actually, this follows from an easy argument of diagram chasing. Note that the algebraic concentration theorem implies that $K_0(U_{\mu_n})_p$ all vanish, then together with the fact that $U_{\mu_n}$ is the empty set we conclude that the elements of the third column of this double complex are all equal to 0. So our claim is equivalent to say $i_+ : \tilde{K}_0(X_{\mu_n}, \mu_n)_p \rightarrow \tilde{K}_0(X, \mu_n)_p$ is surjective. Indeed, for any element $x \in \tilde{K}_0(X, \mu_n)_p$ we may find an element
y ∈ \overline{K}_0(X_{\mu_n}, \mu_n)_\rho such that \( i_* \pi(y) = \pi(x) \). This means \( x - i_* (y) \) is in the kernel of \( \pi \), so there exists an element \( \alpha \in \tilde{A}(X_{\mu_n}) \) such that \( \alpha = x - i_* (y) \) in \( \overline{K}_0(X_{\mu_n})_\rho \). Set \( \beta = \alpha Td_g(\overline{N}) \), we get \( i_* (y + \beta) = i_* (y) + \alpha = x \) in \( \overline{K}_0(X_{\mu_n})_\rho \). Hence, \( i_* \) is surjective. By constructing its inverse morphism, we conclude that the embedding morphism \( i_* : \overline{K}_0(X_{\mu_n}, \mu_n)_\rho \to \overline{K}_0(X, \mu_n)_\rho \) is really an isomorphism. Concerning such inverse morphism, let \( \eta \) be an equivariant hermitian vector bundle on \( X_{\mu_n} \), then we have

\[
\lambda^{-1}_- (\overline{N}') \cdot i^* i_* (\eta) = \lambda^{-1}_- (\overline{N}') \cdot i^* (\sum_k (-1)^k [\xi_k] + T_d (\xi)) = [\eta].
\]

The last equality follows from Lemma [5,13]. Moreover, let \( \alpha \) be an element in \( \tilde{A}(X_{\mu_n}) \), then we have the equalities

\[
\lambda^{-1}_- (\overline{N}') \cdot i^* i_* (\alpha) = \lambda^{-1}_- (\overline{N}') \cdot [\alpha Td_g (\overline{N})] = [\chi_g (\lambda^{-1}_- (\overline{N}'))] = [\alpha].
\]

Therefore, the inverse morphism of the embedding morphism \( i_* \) is of the form \( \lambda^{-1}_- (\overline{N}') \cdot i^* \cdot [\alpha] \). This completes the proof of the arithmetic concentration theorem.

### 6 Relative fixed point formula of Lefschetz type in Arakelov geometry

Let \( X, Y \) be two \( \mu_n \)-equivariant arithmetic varieties over some fixed arithmetic ring \( D \) and let \( f : X \to Y \) be a \( \mu_n \)-equivariant morphism over \( D \) which is smooth over the complex numbers. By definition \( X \) admits a \( \mu_n \)-projective action over \( D \), then the morphism \( f \) is automatically \( \mu_n \)-projective. Fix a \( \mu_n(\mathbb{C}) \)-invariant Kähler metric on \( X(\mathbb{C}) \) so that we get a Kähler fibration with respect to the holomorphic submersion \( f_\mathbb{C} : X(\mathbb{C}) \to Y(\mathbb{C}) \). Let \( E \) be an \( f \)-acyclic \( \mu_n \)-equivariant hermitian vector bundle on \( X \), we know that the direct image \( f_* E \) is a coherent sheaf which is locally free on \( Y(\mathbb{C}) \) hence it can be endowed with a natural equivariant structure and the \( L^2 \)-metric. Moreover, \( (f_* E, f_* h^E) \) always admits a resolution by equivariant hermitian vector bundles on \( Y \). Assume that \( \xi_{\mathbb{C}}^{\xi_{\mathbb{C}}} \) is such a resolution and we denote by \( \chi_g (\xi_{\mathbb{C}}^{\xi_{\mathbb{C}}}) \) the secondary Bott-Chern characteristic class of the following exact sequence

\[
0 \to \xi_{\mathbb{C}}^{\xi_{\mathbb{C}}} \to \cdots \to \xi_{\mathbb{C}}^{\xi_{\mathbb{C}}} \to \xi_{\mathbb{C}}^{\xi_{\mathbb{C}}} \to (f_* E_{\mathbb{C}}, f_* h^E) \to 0.
\]

Let \( \overline{K}_{0_{ac}} (X, \mu_n) \) be the group generated by \( f \)-acyclic equivariant vector bundles on \( X \) and the elements of \( \tilde{A}(X_{\mu_n}) \), with the same relations as in Definition [3,3]. A theorem of Quillen (cf. [19], Cor.3 p. 111) for the algebraic analogs of this group implies that the natural map \( \overline{K}_{0_{ac}} (X, \mu_n) \to \overline{K}_{00} (X, \mu_n) \) is an isomorphism. So the following definition does make sense.

**Definition 6.1.** Let notations and assumptions be as above. The push-forward morphism

\[
f_* : \overline{K}_0 (X, \mu_n) \to \overline{K}_0 (Y, \mu_n)
\]
is defined as follows.

(i). For every $f$-acyclic $\mu_n$-equivariant hermitian vector bundle $\overline{E}$ on $X$, the direct image of $\overline{E}$ is given by $f_*(-\overline{E}) = \sum_k (-1)^k \overline{E}_k + \tilde{\text{ch}}_g(\xi, \overline{E}) - T_g(\omega_X, h_E)$ where $\xi, \overline{E}$ is a resolution of $(f_* E, f_* h_E)$ on the base variety $Y$.

(ii). For every element $\alpha \in \tilde{A}(X, \mu_n)$, $f_* \alpha = \int_{X g / Y g} T\text{d}g(T f, h_{T f}) \alpha \in \tilde{A}(Y, \mu_n)$.

**Theorem 6.2.** The push-forward morphism $f_*$ is a well-defined group homomorphism.

**Proof.** Again we have to prove that our definition for $f_*$ is independent of the choice of resolution and it is compatible with the two generating relations of arithmetic $K_0$-group. Indeed, assume that we are given a short exact sequence

$$0 \to \overline{E}' \to \overline{E} \to \overline{E}'' \to 0$$

of $f$-acyclic equivariant hermitian vector bundles on $X$. Then we may construct the following short exact sequence of resolution

$$0 \to \xi_m' \to \cdots \to \xi_1' \to \xi_0' \to (f_* E', f_* h_{E''}) \to 0$$

According to the generating relations of $\tilde{K}_0(Y, \mu_n)$, our definition of the push-forward morphism $f_*$ and the double complex formula of equivariant secondary Bott-Chern classes, we have an equation

$$f_* \overline{E}' - f_* \overline{E} + f_* \overline{E}'' = -T_g(\omega_X, h_{E'}) + T_g(\omega_X, h_E) - T_g(\omega_X, h_{E''}) + \tilde{\text{ch}}_g(f_* \overline{E})$$

in $\tilde{K}_0(Y, \mu_n)$. On the other hand, we apply Bismut-Ma’s immersion formula to the case where the fibration is with respect to $f_C : f_C^{-1}(Y_g) \to Y_g$ and the closed immersion is the identity map,
then the equality
\[ T_y(\omega^X, h^{E'}) - T_y(\omega^X, h^E) + T_y(\omega^X, h^{E''}) - \hat{c}_g(f_\ast \bar{\tau}) = - \int_{X_g/Y_g} \operatorname{Td}_g(T f, h^{T f}) \tilde{c}_g(\bar{\tau}) \]
holds in \( \tilde{A}(Y_{\mu_n}) \). Combing these two computations above, we finally get
\[ f_\ast \mathcal{E}' - f_\ast \mathcal{E} + f_\ast \mathcal{E}'' = \int_{X_g/Y_g} \operatorname{Td}_g(T f, h^{T f}) \tilde{c}_g(\bar{\tau}). \]
This final expression means that the push-forward morphism \( f_\ast \) is compatible with the first generating relation of arithmetic \( K_0 \)-theory. For the second one, it is rather clear from definition. It is easily seen from the generating relation of \( K_0(Y, \mu_n) \) that \( f_\ast 0 \) is equal to 0. Moreover, since any two resolutions of \( f_\ast E \) are dominated by a third one, our arguments above also show that \( f_\ast \mathcal{E} \) is independent of the choice of resolution. So we are done. \( \square \)

**Lemma 6.3.** (Projection formula) For any elements \( y \in \tilde{K}_0(Y, \mu_n) \) and \( x \in \tilde{K}_0(X, \mu_n) \), the equality \( f_\ast (f^\ast y \cdot x) = y \cdot f_\ast x \) holds in \( \tilde{K}_0(Y, \mu_n) \).

**Proof.** Assume that \( y = \mathcal{E} \) is an equivariant hermitian vector bundle and \( x = \mathcal{F} \) is an \( f \)-acyclic equivariant hermitian vector bundle, then \( f^\ast y \cdot x = f^\ast \mathcal{E} \otimes \mathcal{F} \). By projection formula for direct images and the definition of \( L^2 \)-metric, we know that \( (f_\ast (f^\ast \mathcal{E} \otimes \mathcal{F}), f_\ast h^{f^\ast \mathcal{E} \otimes \mathcal{F}}) \) is isometric to \( \mathcal{E} \otimes (f_\ast \mathcal{F}, f_\ast h^\mathcal{F}) \). Moreover, concerning the analytic torsion form, we have \( T_y(\omega^X, h^{f^\ast \mathcal{E} \otimes \mathcal{F}}) = \hat{c}_g(\mathcal{E})T_y(\omega^X, h^\mathcal{F}) \). So the projection formula \( f_\ast (f^\ast y \cdot x) = y \cdot f_\ast x \) holds in this case.

Assume that \( y = \mathcal{E} \) is an equivariant hermitian vector bundle and \( x = \alpha \) is represented by some smooth form. We write \( f_\ast^y \) and \( f_\ast \) for the pull-back and push-forward of smooth forms respectively, then
\[ f_\ast (f^\ast y \cdot x) = f_\ast (f_\ast^y x) = f_\ast (f_\ast^y \hat{c}_g(\mathcal{E}) \alpha) \]
\[ = f_\ast (f_\ast^y \hat{c}_g(\mathcal{E}) \alpha \operatorname{Td}_g(T f)) = \hat{c}_g(\mathcal{E}) f_\ast (\alpha \operatorname{Td}_g(T f)) \]
\[ = \hat{c}_g(\mathcal{E}) \int_{X_g/Y_g} \alpha \operatorname{Td}_g(T f) = y \cdot f_\ast x. \]
Here we have used the projection formula of smooth forms \( p_\ast (p^\ast \alpha_1 \wedge \alpha_2) = \alpha_1 \wedge p_\ast \alpha_2 \) (cf. [11], Prop. IX p. 303).

Assume that \( y = \beta \) is represented by some smooth form and \( x = \mathcal{F} \) is an \( f \)-acyclic hermitian vector bundle, then
\[ f_\ast (f^\ast y \cdot x) = f_\ast (f_\ast^y \beta \hat{c}_g(\mathcal{F})) = f_\ast (f_\ast^y \beta \hat{c}_g(\mathcal{F}) \operatorname{Td}_g(T f)) \]
\[ = \beta f_\ast (\hat{c}_g(\mathcal{F}) \operatorname{Td}_g(T f)) = \beta \int_{X_g/Y_g} \hat{c}_g(\mathcal{F}) \operatorname{Td}_g(T f) \]
\[ = \beta \hat{c}_g(f_\ast^y \mathcal{F}) - \beta dd' \hat{c}_g(\omega^X, h^\mathcal{F}) \]
which is exactly \( y \cdot f_\ast x \).
Finally, assume that $y = \beta$ and $x = \alpha$ are both represented by smooth forms, then
\[
       f_* (f^* y \cdot x) = f_* (f_g^* \beta (\dd \alpha)) = f_g_* (f_g^* \beta (\dd \alpha) \Td_g (Tf)) = \beta \dd \alpha f_g_* (\alpha \Td_g (Tf))
\]
which is also equal to $y \cdot f_* x$.

Since $f_*$ and $f^*$ are both group homomorphisms, we may conclude the projection formula by linear extension.

\section*{Remark 6.4.}
Lemma 6.3 implies that $f_*$ is a homomorphism of $R(\mu_n)$-modules, and hence it induces a push-forward morphism after taking localization.

We next translate Bismut-Ma’s immersion formula to a $\hat{K}_0$-theoretic version. Let notations and assumptions be as before. Assume that the morphism $f : X \to Y$ has a factorization $f = l \circ i$ such that $i : X \to P$ is a closed immersion of equivariant arithmetic varieties and $l : P \to Y$ is still equivariant and smooth on the complex numbers. We additionally suppose that the $\mu_n$-action on $Y$ is trivial. Let
\[
       \Xi : 0 \to \xi_m \to \xi_{m-1} \to \cdots \xi_0 \to i_* \eta \to 0
\]
be a resolution by $l$-acyclic equivariant hermitian vector bundles on $P$ of an $f$-acyclic equivariant hermitian vector bundle $\eta$ on $X$. We suppose that the Kähler metric on $X$ is induced by a $\mu_n(C)$-invariant Kähler metric on $P$ and the normal bundle $N$ of $i_*(X)$ in $P$ carries the quotient metric. As usual, suppose that the metrics on $\xi_\cdot$ satisfy Bismut assumption (A).

\textbf{Theorem 6.5.} Let notations and assumptions be as above. Then the equality
\[
       f_* (\eta) - \sum_{j=0}^m (-1)^j l_* (\xi_j) = \int_{X/y} \ch_g (\eta) R_g (N) \Td_g (Tf) + \int_{P/y} T_g (\xi_j) \Td_g (Tl, h^Tl)
\]
\[
       + \int_{X/y} \ch_g (\eta) \Td_g (Tf, Tl |_X) \Td_g^{-1} (N)
\]
holds in $\hat{K}_0 (Y, \mu_n)$.

\textbf{Proof.} The verification follows rather directly from the generating relations of arithmetic $K_0$-theory. In fact
\[
       f_* (\eta) - \sum_{j=0}^m (-1)^j l_* (\xi_j) = (f_* \eta, f_* h^\eta) - T_g (\omega^X, h^\eta) - \sum_{j=0}^m (-1)^j ((l_* \xi_j, l_* h^\xi_j) - T_g (\omega^P, h^\xi_j))
\]
\[
       = \ch_g (l_* \Xi_C) - T_g (\omega^X, h^\eta) + \sum_{j=0}^m (-1)^j T_g (\omega^P, h^\xi_j).
\]
And the right-hand side of the last equality is exactly the left-hand side of Bismut-Ma’s immersion formula. One just need to note that to simplify the right-hand-side of Bismut-Ma’s immersion formula, we have used an Atiyah-Segal-Singer type formula for immersion
\[
       i_g_* (\Td_g^{-1} (N) \ch_g (x)) = \ch_g (i_* (x)).
\]
This formula is the content of [14], Theorem 6.16.
Remark 6.6. The $\hat{K}_0$-theoretic version of Bismut-Ma’s immersion formula even holds without acyclicity conditions on the bundles $\eta$ and $\xi_j$. One can carry out the same principle of the proof of [20], Theorem 6.7 to show this fact because every equivariant arithmetic variety is supposed to admit a $\mu_n$-projective action.

We now turn to the description of the relative fixed point formula of Lefschetz type and its proof. Let $f : X \to Y$ be an equivariant morphism of $\mu_n$-equivariant arithmetic varieties, which is flat and smooth over the complex numbers. We additionally suppose that the fibre product $f^{-1}(Y_{\mu_n})$ is regular. We shall naturally endow $X_{\mu_n}(\mathbb{C})$ and $f^{-1}(Y_{\mu_n})(\mathbb{C})$ with the Kähler metric induced by the Kähler metric of $X(\mathbb{C})$.

We have the following Cartesian square

$$
\begin{array}{ccc}
X_{\mu_n} & \overset{f_{\mu_n}}{\longrightarrow} & Y_{\mu_n} \\
\downarrow & & \downarrow \\
X & \overset{f}{\longrightarrow} & Y
\end{array}
$$

whose rows are both closed immersions of $\mu_n$-equivariant arithmetic varieties and whose columns are both flat. Then the normal bundle $N_{X/f^{-1}(Y_{\mu_n})}$ is isomorphic to the pull-back of normal bundle $f^*N_{Y/Y_{\mu_n}}$. Therefore by restricting to $X_{\mu_n}$ we get a short exact sequence

$$
\mathcal{N} : 0 \to N_{f^{-1}(Y_{\mu_n})/X_{\mu_n}} \to N_{X/X_{\mu_n}} \to f^*N_{Y/Y_{\mu_n}} \to 0
$$

of equivariant hermitian vector bundles. Here all metrics on these normal bundles are the quotient metrics. Define

$$
M(f) := (\lambda_{-1}(N_{X/X_{\mu_n}}')\lambda_{-1}(f^*N_{Y/Y_{\mu_n}}) + \hat{T}d_g(N)Td_g^{-1}(f^*N_{Y/Y_{\mu_n}}))
$$

$$
\cdot (1 - R_g(N_{X/X_{\mu_n}}) + R_g(f^*N_{Y/Y_{\mu_n}})).
$$

Note that $f_{\mu_n}$ is still smooth over the complex numbers because $f_{\mu_n\mathbb{C}} = f_g$ is still a submersion. Then we can define the push-forward morphism $f_{\mu_n*}$. 

Theorem 6.7. (Relative fixed point formula) Let notations and assumptions be as above. Then the following diagram

$$
\begin{array}{ccc}
\hat{K}_0(X, \mu_n) & \overset{M(f)\cdot \tau}{\longrightarrow} & \hat{K}_0(X_{\mu_n}, \mu_n) \\
\downarrow f_* & & \downarrow f_{\mu_n*} \\
\hat{K}_0(Y, \mu_n) & \overset{\tau}{\longrightarrow} & \hat{K}_0(Y_{\mu_n}, \mu_n)
\end{array}
$$

commutes, where $\tau$ stands for the restriction map.
Proof. Since \( f \) is flat, the Cartesian square introduced before this theorem induces a commutative diagram in arithmetic \( K_0 \)-theory

\[
\begin{array}{ccc}
\hat{K}_0(X, \mu_n) & \xrightarrow{\tau} & \hat{K}_0(f^{-1}(Y_{\mu_n}), \mu_n) \\
\downarrow f_* & & \downarrow f_* \\
\hat{K}_0(Y, \mu_n) & \xrightarrow{\tau} & \hat{K}_0(Y_{\mu_n}, \mu_n).
\end{array}
\]

From the exact sequence

\[
\overline{N} : 0 \to \overline{N}_f^{-1}(Y_{\mu_n}/X_{\mu_n} \to \overline{N}_X/X_{\mu_n} \to f^* \overline{N}_Y/Y_{\mu_n} \to 0,
\]

we know that

\[
\lambda_1^{-1}(\overline{N}_f^{-1}(Y_{\mu_n}/X_{\mu_n})) \lambda_1^{-1}(f^* \overline{N}_Y/Y_{\mu_n}) - \lambda_1^{-1}(\overline{N}_X/X_{\mu_n}) = Td_g(\overline{N})
\]

according to \([14]\), Lemma 7.1 and that

\[
R_g(N_f^{-1}(Y_{\mu_n}/X_{\mu_n} + R_g(f^* N_Y/Y_{\mu_n}) = R_g(N_X/X_{\mu_n})
\]

since the equivariant \( R \)-genus is additive. Therefore, we may reduce our proof to the case where the \( \mu_n \)-action on the base variety \( Y \) is trivial. In this case, denote by \( i \) the canonical closed immersion \( X_{\mu_n} \to X \). Then for any element \( x \in \hat{K}_0(X, \mu_n) \), we have

\[
f_*(x) = f_\mu n i_* ^{-1}(x) = f_\mu n i_*(\lambda_1^{-1}(\overline{N}_X/X_{\mu_n}) \cdot (x)).
\]

Consider the factorization \( f_\mu n = f \circ i \), we have to compute the difference \( f_\mu n - f_\mu n i_* \) in arithmetic \( K_0 \)-theory. Indeed, this difference can be measured by Bismut-Ma’s immersion formula. By applying Theorem \([6,3]\) and Remark \([6,6]\) to the closed immersion \( i \), for any equivariant hermitian vector bundle \( \overline{\eta} \) on \( X_{\mu_n} \) we have

\[
f_\mu n (\overline{\eta}) - f_\mu n i_* (\overline{\eta}) = \int_{X_{\mu_n}/X} \eta \cdot Td_g(\overline{N}_X/X_{\mu_n}) Td_g(Tf, h_{Tf}) = f_\mu n (\eta \cdot R_g(N_X/X_{\mu_n})).
\]

The first equality holds because the exact sequence

\[
0 \to T_{f_\mu n} \to Tf \mid_{X_{\mu_n}} \to \overline{N}_X \to 0
\]

is orthogonally split on \( X_g \) so that \( Td_g(Tf, h_{Tf}) = 0 \). The second equality follows from \([14]\), Lemma 7.3 and the fact that \( \eta \cdot R_g(N_X/X_{\mu_n}) \) is dd*-closed. On the other hand, let \( \alpha \) be an element in \( \overline{A}(X_{\mu_n}) \), we have

\[
f_\mu n (\alpha) - f_\mu n i_* (\alpha) = \int_{X_{\mu_n}/X} \alpha Td_g(Tf, h_{Tf}) = \int_{X_{\mu_n}/X} \alpha Td_g^{-1}(\overline{N}_X/X_{\mu_n}) Td_g(Tf, h_{Tf})
\]

\[
= \int_{X_{\mu_n}/X} \alpha Td_g^{-1}(\overline{N}_X/X_{\mu_n}) dd^* Td_g(Tf, h_{Tf}) \mid_{X_g} = 0.
\]
Combing these two computations, by the ring structure of $\hat{K}_0(\cdot)$, we know that the equality
\[
f_{\mu_n\ast}(y) - f_{\mu_n\ast}(y \cdot R_g(N_{X/X_{\mu_n}})) = f_{\mu_n\ast}(y) - f_{\mu_n\ast}(y) = 0
\]
holds for any element $y \in \hat{K}_0(X_{\mu_n}, \mu_n)$, since both two sides are additive. Now continue our computation for $f_{\ast}(x)$, we obtain that
\[
f_{\ast}(x) = f_{\mu_n\ast}(\lambda_{\ast}^{-1}(N_{X/X_{\mu_n}}) \cdot \tau(x)) - f_{\mu_n\ast}(\lambda_{\ast}^{-1}(N_{X/X_{\mu_n}}) \cdot \tau(x) \cdot R_g(N_{X/X_{\mu_n}}))
\]
\[
= f_{\mu_n\ast}(\lambda_{\ast}^{-1}(N_{X/X_{\mu_n}}) \cdot (1 - R_g(N_{X/X_{\mu_n}})) \cdot \tau(x)).
\]
The last thing should be indicated is that the following square
\[
\begin{array}{c}
\hat{K}_0(X, \mu_n) \xrightarrow{i} \hat{K}_0(Y, \mu_n) \\
\downarrow f_{\ast} \downarrow f_{\ast}
\end{array}
\]
is naturally commutative. So the relative fixed point formula holds in the case where the $\mu_n$-action on the base variety $Y$ is trivial. By the observation given at the beginning of this proof, it is enough to conclude the statement in our theorem. 

\[\Box\]

**Remark 6.8.** (i). The condition $f^{-1}(Y_{\mu_n})$ is regular can be satisfied if the $\mu_n$-action on the base variety is trivial or the morphism $f$ is not only smooth over the complex numbers but also smooth everywhere. This is already enough for the applications in practice. For instance, our main result implies various formulae stated as conjectural in [18], in particular Proposition 2.3 in that article.

(ii). The condition of flatness on the morphism $f$ is only used in the reduction of general case to the case where the $\mu_n$-action on the base variety is trivial. If the $\mu_n$-action on $Y$ is trivial, one can certainly remove the condition of flatness.

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