1. Introduction

Let \( i : X \hookrightarrow \mathbb{P} = \mathbb{P}(V) \) be a smooth connected projective variety embedded into a projective space (we are working over a fixed ground field \( k \)). Set \( \mathcal{O}_X(1) = i^* \mathcal{O}_\mathbb{P}(1) \) and consider the coordinate algebra

\[
A = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(n)).
\]

By construction \( A \) is identified with a quotient algebra \( A = S/I \) where \( S = \text{Sym}(V^*) = k[x_0, \ldots, x_{n-1}] \). The Koszul homology algebra is defined as

\[
H(A) = \bigoplus_{p=0}^{n} \text{Tor}^S_p(A, k).
\]

This is a (bi)graded commutative \( k \)-algebra, finite dimensional as a \( k \)-vector space.

In an inspiring paper \([GKR]\) Gorodentsev, Khoroshkin and Rudakov prove (among others) the following elegant result. Denote by \( K_X \) the canonical class of \( X \).

1.1. Theorem (see \([GKR]\), Sect. 2)). Suppose that

(a) there exists a natural \( N \) such that \( K_X = \mathcal{O}_X(-N) \);
(b) \( H^i(X, \mathcal{O}_X(n)) = 0 \) for all \( n \in \mathbb{Z} \) and \( 0 < i < d := \text{dim} X \).

Then \( H(A) \) is Frobenius.
Here *Frobenius* means that there exists a nondegenerate bilinear pairing
\[ \langle \cdot, \cdot \rangle : H(A) \times H(A) \longrightarrow k, \] suitably compatible with the gradings, such that
\[ \langle ab, c \rangle = \langle a, bc \rangle. \]

The proof in *op. cit.* is very nice; it uses the "sphericity" of certain spectral sequence.

In this note we would like to look at this result from a slightly different perspective. Our point of departure is a fundamental result by Avramov and Golod, [AG]:

1.2. **Theorem.** \( H(A) \) is Frobenius if and only if \( A \) is Gorenstein.

In fact, Avramov and Golod work in the local situation; the passage to our graded context presents no difficulties. Indeed, according to *op. cit.*, \( H(A) \) is Frobenius iff the localisation of \( A \) at 0 is Gorenstein; however, \( A \) is smooth outside this ideal, so this is equivalent to \( A \) being Gorenstein.

So our question reduces to the Gorenstein property of \( A \).

Let us say, following [GKR], that \( X \subset \mathbb{P} \) is subcanonical if the condition (a) of Theorem 1.1 is satisfied. In the present note we prove the following

1.3. **Theorem.** Assume \( \text{char}(k) = 0 \). If \( X \subset \mathbb{P} \) is subcanonical then \( A \) is Gorenstein and has rational singularities.

We establish this using certain Key Lemma from [H] (see Proposition 2.1) giving a sufficient condition for a singularity being Gorenstein and rational. The proof of this lemma uses Grauert-Riemenschneider theorem, and hence the characteristic zero assumption. (On the contrary, although Gorodentsev et al. assume \( k = \mathbb{C} \), their proof of [L] works over an arbitrary field).

1.4. **Corollary.** If \( X \subset \mathbb{P} \) is subcanonical then \( H(A) \) is Frobenius.

So, the condition (b) of Theorem 1.1 is superfluous if \( \text{char} k = 0 \).

The main objects of study in *op cit.* are highest weight orbits of a semisimple algebraic group \( G \). For such \( X \) the authors of [GKR] prove that (b) follows from (a).

In this case we prove that subcanonicity is equivalent to the Gorenstein property of \( A \):

1.5. **Theorem.** Let \( X \subset \mathbb{P}(V) \) be the projectivisation of the highest weight orbit in an irreducible finite dimensional representation \( V \) of a semisimple group \( G \). This embedding is subcanonical if and only if the corresponding coordinate ring \( A \) is Gorenstein (so, iff \( H(A) \) is Frobenius).

1.6. **Acknowledgement.** This note was written during a visit of the first author to the *Institut de Mathématiques de Toulouse*. He thanks this Institute for the hospitality.
2. Proof of Theorem 1.3

We keep the notation of the Introduction. The affine variety \( Z := \text{Spec}(A) \) is the cone over \( X \); therefore it is nonsingular outside 0. It has a very nice desingularization \( Y \) which is the total space of the vector bundle \( E = \mathcal{O}_X(-1) \). Let
\[
(1) \quad p : Y = \text{Spec}(\text{Sym}_{\mathcal{O}_X}(E^*)) \longrightarrow X
\]
be the projection.

The embedding \( \mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow V \) defines an embedding \( Y \longrightarrow X \times V \); the projection to the second factor has image \( Z \subset \text{Spec}(\text{Sym} V^*) = V \) and the map
\[
(2) \quad \pi : Y \longrightarrow Z
\]
is a desingularization.

Recall the following

2.1. Proposition (see [H]). Let \( \pi : Y \longrightarrow Z \) be a proper birational map with \( Y \) smooth and \( Z \) normal. Let \( \omega_Y \) be the sheaf of higher differentials on \( Y \). Assume there exists a morphism \( \phi : \mathcal{O}_Y \rightarrow \omega_Y \) such that \( \pi_*\phi : \pi_*\mathcal{O}_Y \rightarrow \pi_*\omega_Y \) is an isomorphism. Then \( Z \) is Gorenstein and has rational singularities.

We wish to apply this to our desingularization \( \pi : Y \rightarrow Z \). Note that \( Z = \text{Spec}(A) \) is normal.

The short exact sequence of vector bundles on \( Y \)
\[
(3) \quad 0 \longrightarrow p^*E \longrightarrow T_Y \longrightarrow p^*T_X \longrightarrow 0
\]
yields an isomorphism
\[
(4) \quad \omega_Y = p^*(\omega_X \otimes E^*).
\]
We wish to calculate the global sections of \( \omega_Y \). First of all, we have
\[
(5) \quad p_*\omega_Y = p_*p^*(\omega_X \otimes E^*) = \omega_X \otimes E^* \otimes \text{Sym}_{\mathcal{O}_X}E^* = \bigoplus_{n \geq 1} \omega_X \otimes \mathcal{O}_X(n)
\]
since \( p \) is an affine morphism.

2.2. Proof of Theorem 1.3 Let \( \omega_X = \mathcal{O}_X(-N) \). One has an obvious map
\[
\mathcal{O}_X = \omega_X \otimes \mathcal{O}_X(N) \overset{\cong}{\longrightarrow} \bigoplus_{n \geq 1} \omega_X \otimes \mathcal{O}_X(n) = p_*\omega_Y
\]
which gives by adjunction a map \( \phi : \mathcal{O}_Y \longrightarrow \omega_Y \).

We will check now that \( \phi \) induces an isomorphism of the global sections. Applying to \( \phi \) the direct image functor \( p_* \) we get a morphism
\[
(6) \quad p_*(\phi) : \bigoplus_{n \geq 0} \mathcal{O}_X(n) \longrightarrow \bigoplus_{n \geq 1} \omega_X \otimes \mathcal{O}_X(n)
\]
which is obviously a map of $p_*(\mathcal{O}_Y)$-modules. By definition it carries $1 \in p_*(\mathcal{O}_Y)$ to a generator of $\omega_X(N) = \mathcal{O}_X$, so the map $p_*(\phi)$ carries isomorphically the summand $\mathcal{O}_X(n)$ of the left-hand side to the summand $\omega_X \otimes \mathcal{O}_X(N + n)$ of the right-hand side. For $n < N$ one has on the right-hand side

$$\Gamma(X, \omega_X \otimes \mathcal{O}_X(n)) = \Gamma(X, \mathcal{O}_X(n - N)) = 0,$$

so $p_*(\phi)$ induces an isomorphism of the global sections.

3. Homogeneous case

Let now $G$ be a semisimple Lie group, $V$ a simple finite dimensional highest weight $G$-module, $v \in V$ be a highest weight vector. Let $P$ be the stabilizer of $Cv$ in $P(V)$. This is a parabolic subgroup of $G$. A $G$-equivariant embedding $i : X := G/P \rightarrow \mathbb{P}(V)$ is induced.

The closure $Z$ of $Gv$ is a cone in $V$. We have $Z = \text{Spec}(A)$ where $A$ is the homogeneous coordinate ring of $X = G/P$ with respect to $i$.

In this case the converse of the theorem 1.3 is valid. One has

3.1. Theorem. The space $Z$ is Gorenstein iff $\omega_X = \mathcal{O}_X(-N)$ for some $N$.

Note that the conclusion of the Theorem is not true for an arbitrary (nonhomogeneous) $X$ (for example it follows easily from the results of Mukai [M] that a generic curve of genus 7 embedded canonically in $\mathbb{P}^6$ has a Gorenstein coordinate ring).

Proof. The dualizing complex of $Z$ can be calculated as

$$\omega_Z = R\text{Hom}_{SV^*}(A, SV^*)[\dim V - \dim Z]$$

(the shift is chosen so that $\omega_Z$ is concentrated in degree 0 when $A$ is Cohen-Macaulay).

Its cohomology keeps the grading of $SV^*$ and $A$; therefore, if $A$ is Gorenstein so that $\omega_Z$ is an invertible $A$-module, it has to be isomorphic to $A$.

Choose an isomorphism $\theta : A \rightarrow \omega_Z$.

We now apply the Duality isomorphism, see [H1], VII.3.4, to the proper morphism $\pi : Y \rightarrow Z$. It gives, in particular, an isomorphism

$$\text{Hom}_{D(Y)}(F, \pi^!G) \sim \text{Hom}_{D(Z)}(R\pi_*F, G)$$

for any $F \in D^-(Y)$, $G \in D^+(Z)$.

We apply this to $F = \mathcal{O}_Y$ and $G = \omega_Z$. By a general result of Kempf [K] $Z$ has rational singularities, so $R\Gamma(Y, \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y) = A$. Moreover, $\pi^!(\omega_Z) = \omega_Y$. Thus, Duality isomorphism gives us

$$\text{Hom}_{D(Y)}(\mathcal{O}_Y, \omega_Y) \sim \text{Hom}_{D(Z)}(\mathcal{O}_Z, \omega_Z).$$
We see that the map $\theta : A \to \omega_Z$ is adjoint to a map $\theta_Y : \mathcal{O}_Y \to \omega_Y$ which in turn can be rewritten as a morphism

$$\theta_X : \mathcal{O}_X \to p_*(\omega_Y) = \bigoplus_{n \geq 1} \omega_X(n).$$

We intend to prove now that each direct component $\theta_{X,n} : \mathcal{O}_X \to \omega_X(n)$ is either isomorphism or vanishes. This will immediately imply the theorem.

Note that the formula (7) shows that the group $G$ naturally acts on $\omega_Z$. We claim that $\theta : A \to \omega_Z$ is necessarily $G$-equivariant.

In fact, the $G$-action on $A$-module $\omega_Z$ is compatible with $G$-action on $A$:

$$g(ax) = g(a)g(x), \quad g \in G, \quad a \in A, \quad x \in \omega_Z.$$  

Another $G$-module structure on $\omega_Z$ compatible with the $G$-action on $A$ is given by $\theta$. These two actions define two group homomorphisms

$$\rho_1, \rho_2 : G \to \text{Aut}_\mathbb{C}(\omega_Z).$$

The “difference” between the two defined by the formula

$$\rho_{12} : g \mapsto \rho_1(g^{-1}) \circ \rho_2(g)$$

gives rise to a crossed homomorphism $\rho_{12} : G \to \text{Aut}_A(\omega_Z) = \mathbb{C}^*$. Since the action of $G$ on $\mathbb{C}^*$ is trivial and $G$ is semisimple, $\rho_{12}$ is trivial, which means that $\theta$ is $G$-equivariant.

Let us show that the maps $\theta_Y$ and $\theta_X$ obtained from $\theta$ via Duality isomorphism, are also $G$-equivariant.

Choose $g \in G$ and let $g_X : X \to X$, $g_Y : Y \to Y$, $g_Z : Z \to Z$ denote the corresponding automorphisms of the varieties.

An action of $g \in G$ on $\mathcal{O}_Z$ and $\omega_Z$ are expressed as isomorphisms $g_Z^*(\mathcal{O}_Z) \to \mathcal{O}_Z$ and $g_Z^*(\omega_Z) \to \omega_Z$. Since $\theta$ is equivariant, it gives rise to a commutative diagram

$$
\begin{array}{ccc}
g_X^*(\mathcal{O}_Z) & \xrightarrow{g_Z^*} & g_X^*(\mathcal{O}_Z) \\
\downarrow \quad & & \downarrow \\
\mathcal{O}_Z & \xrightarrow{\theta} & \omega_Z
\end{array}
$$

The map $\theta_Y$ can be described as the composition

$$\mathcal{O}_Y \xrightarrow{\pi^! R\pi_*(\mathcal{O}_Y)} \pi^! \mathcal{O}_Z \xrightarrow{\pi^! \omega_Z}.$$
so that it suffices to check that the first morphism is $G$-equivariant. The latter can be expressed as the commutativity of the diagram

$$
g_Y^*(\mathcal{O}_Y) \longrightarrow g_Y^*(\pi^! R\pi_*(\mathcal{O}_Y))$

(12)

$$\mathcal{O}_Y \longrightarrow \pi^! R\pi_*(\mathcal{O}_Y)$$

for each $g \in G$, and this follows from the relations

$$g_Y^* \pi^! = \pi^! g_Y^*, \quad g_Y^* R\pi_* = R\pi_* g_Y^*.$$

All this proves that $\theta_Y$ is $G$-equivariant; the similar fact for $\theta_X$ is even more transparent.

We have already understood that the components $\theta_{X,n}$ of the map $\theta_X : \mathcal{O}_X \longrightarrow \bigoplus \omega_X(n)$ are $G$-equivariant. This implies that the map of fibers at $1P \in G/P$ is $P$-equivariant. The fibers are one-dimensional representations of $P$; any $P$-morphism is either zero or an isomorphism. This proves the theorem. □

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