Kundt spacetimes minimally coupled to scalar field

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Abstract. We derive an exact solutions belonging to Kundt class of spacetimes both with and without a cosmological constant that are minimally coupled to free massless scalar field. We show the algebraic type of these solutions and give interpretation of the results. Subsequently, we look for solutions additionally containing electromagnetic field satisfying nonlinear field equations.

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1. Introduction

Kundt class of spacetimes is among the most important families of exact solutions of Einstein equations. It was derived more than half a century ago [1, 2] and it is defined by the presence of nonexpanding, nonshearing and non-twisting null geodesic congruence [3, 4]. This class contains solutions of various algebraic types and admits, apart from vacuum solutions, cosmological constant, electromagnetic field, gyration source or solutions with supersymmetry. All type D vacuum solutions were classified in classical paper [5]. This family also contains famous pp-wave solution [3, 4] and so called VSI (vanishing scalar invariants) [6] or CSI (constant scalar invariants) [7] spacetimes. Kundt class of spacetimes was also recently generalized to arbitrary dimension [8]. Subclasses of the Kundt family of spacetimes provide the most important examples of so called universal spacetimes—solutions of vacuum field equations of arbitrary gravitational theory whose Lagrangian is a polynomial invariant constructed from metric, Riemann tensor and its derivatives of arbitrary order [9]. Universal metrics also represent classical solutions to string theory because associated quantum corrections vanish in this case [10].

Solutions to Einstein equations containing scalar field serve as a useful tool for understanding General Relativity due to the simplicity of the source. Recently, it becomes evident that fields of this type really do exist (LHC) and play a fundamental role in Standard model of particle physics. In classical General Relativity they were used to study counterexamples to black hole no-hair theorems or cosmic censorship hypothesis and in many other areas. The study of scalar fields in Kundt spacetimes complements the one performed in the closely related Robinson-Trautman family [11].

The Kundt spacetimes with Maxwell electrodynamics were extensively analyzed soon after the appearance of vacuum solutions. One of the most important examples being the conformally flat Bertotti–Robinson solution [12] which contains uniform non-null electromagnetic field. More general solutions possibly containing additional pure radiation and (exact) gravitational waves were found as well (for a review see, e.g., [13]). The pure radiation solutions can be used to support perturbative gravitational waves as well [14].

Nonlinear Electrodynamics (NE) was originally used mainly as a solution to the problem of divergent field of a point charge in the vicinity of its position (see e.g. [15]) also giving satisfactory self-energy of charged particle. The best-known and frequently used form of the theory was introduced already in 1934 by Born and Infeld [16]. Nice overview with a lot of useful information was given in a book by Plebański [17]. Apart from solving the point charge singularity the NE was later used to resolve the spacetime singularity as well.

In this work, we first give explicit solutions of Kundt type with minimally coupled free scalar field. So far the interest was mainly directed towards analyzing pp-waves coupled with scalar field and possibly additional sources (e.g. Yang-Mills fields [18]) which stemmed from the extensive use of pp-waves in string theory. In the next part, we concentrate on solutions containing nonlinear electromagnetic field as an additional source.
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2. Field equations with scalar field

We consider the following action, describing a free massless scalar field minimally coupled to a gravity described by the Einstein-Hilbert action,

\[ S = \int d^4x \sqrt{-g} [R + \nabla_{\mu} \varphi \nabla^{\mu} \varphi - 2\Lambda] \]

where \( R \) is the Ricci scalar for the metric \( g_{\mu\nu} \), and we have included cosmological constant \( \Lambda \). The massless scalar field (SF) \( \varphi \) is considered to be real and we use units in which \( c = \hbar = 8\pi G = 1 \).

By applying variation with respect to the metric we get the following field equations for the action (1),

\[ G_{\mu\nu}^{SF} = \nabla_{[\mu} \varphi \nabla_{\nu]} \varphi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \nabla_{\alpha} \varphi \nabla_{\beta} \varphi \]

We assume the following form of the line element which is suitable for analyzing all the explicit solutions presented below

\[ ds^2 = -Hu^2 - 2dw du - 2W_1 du dx - 2W_2 du dy + \frac{dx^2 + dy^2}{P(u, x, y)^2} \]

with \( H, W_1, W_2 \) being functions of all coordinates.

The Ricci scalar for above metric is

\[ R = 2\Delta (\ln P) + 2P^2 (W_{1,uv} + W_{2,uv}) - \frac{3}{2} P^2 (W_{1,v}^2 + W_{2,v}^2) - 2P^2 (W_{1,uv} + W_{2,uv}) - H_{vv} \]

where

\[ \Delta \equiv P(u, x, y)^2 (\partial_{xx} + \partial_{yy}) \]

By computing optical scalars of the congruence generated by null geodetic vector \( \partial_v \) one indeed obtains vanishing expansion, shear and twist. Such spacetime necessarily belongs to the Kundt family of geometries \[3, 4\] with \( \partial_v \) being the geometrically preferred null direction. The scalar field is assumed to be function of retarded time \( u \) and \( x, y \), i.e. \( \varphi(u, x, y) \). The scalar field must satisfy the corresponding field equation

\[ \Box \varphi(u, x, y) = P^2 (\varphi_{,xx} + \varphi_{,yy} - W_{1,v} \varphi_{,x} - W_{2,v} \varphi_{,y}) = 0 \]

where \( \Box \) is a standard d’Alembert operator for our metric \[4\].

One straightforward choice how to satisfy equation (6) is to consider \( W_1, W_2 \) linear in \( v \) which will make all the terms independent of \( v \). In other words

\[ W_1(u, v, x, y) = v V_1(u, x, y), W_2(u, v, x, y) = v V_2(u, x, y) \]

Since the metric \[4\] is still quite general in the following we will try to simplify its structure using Einstein equations. The form of \( uu, xx \) and \( yy \) components of the Einstein tensor for the metric \[4\] is (with \( G^u_u = G^v_v \))

\[ G^u_u = -\Delta (\ln P) + \frac{P^2}{4} \{ V_1^2 + V_2^2 - 2(V_{1,x} + V_{2,y}) \} \]

\[ G^x_x = \frac{P^2}{4} \{ V_1^2 + 3V_2^2 \} - (PV_{1,y}) P + PV_1 P_{,x} + \frac{H_{vv}}{2} \]

\[ G^y_y = \frac{P^2}{4} \{ 3V_1^2 + V_2^2 \} - (PV_{1,x}) P + PV_2 P_{,y} + \frac{H_{vv}}{2} \]
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From field equations (2) for \(xx\) and \(yy\) components we obtain

\[
xx : \frac{P^2}{4} \{V_1^2 + 3V_2^2\} + \frac{P^2}{2} \{\varphi_x^2 - \varphi_y^2\} - (PV_2)_y P \\
+ PV_1 P_x + \frac{H_{vv}}{2} + \Lambda = 0
\]  
(7)

\[
yy : \frac{P^2}{4} \{3V_1^2 + V_2^2\} - \frac{P^2}{2} \{\varphi_x^2 - \varphi_y^2\} - (PV_1)_x P \\
+ PV_2 P_y + \frac{H_{vv}}{2} + \Lambda = 0
\]  
(8)

and by adding these two equations together we have

\[
P^2 (V_1^2 + V_2^2 - V_{1,x} - V_{2,y}) + H_{vv} + 2 \Lambda = 0 .
\]  
(9)

Since all the terms of equation (9) except \(H_{vv}\) are independent of \(v\) the function \(H\) is necessarily quadratic function in \(v\). The standard form of the Kundt metric possesses quadratic term of \(H\) completely determined by cosmological constant and we will keep this for our \(H\) as well

\[
H(u,v,x,y) = -\Lambda v^2 + M(u,x,y)v + K(u,x,y)
\]  
(10)

So the equation (9) is simplified to

\[
V_1^2 + V_2^2 = V_{1,x} + V_{2,y}
\]  
(11)

Another important equation comes from \(uu\) component of the field equations

\[
\begin{align*}
\frac{P^2}{4} \{V_1^2 + V_2^2 - 2(V_{1,x} + V_{2,y}) - 2(\varphi_x^2 + \varphi_y^2)\} \\
- \Delta (\ln P) + \Lambda = 0
\end{align*}
\]  
(12)

by using (11) in the above equation we can simplify it into

\[
- \Delta (\ln P) + \Lambda = \frac{P^2}{4} \{V_1^2 + V_2^2 + 2(\varphi_x^2 + \varphi_y^2)\}
\]  
(13)

The right-hand side of equation (13) is always positive or zero, i.e.,

\[
\Delta (\ln P) \leq \Lambda
\]  
(14)

The expression \(\Delta (\ln P)\) is the Gaussian curvature of two-spaces of constant \(u,v\) equipped with the induced metric coming from (4). This means that cosmological constant sets its upper bound and if \(\Lambda = 0\) the curvature is necessarily non-positive. In the geometrically simple case of two-surfaces with constant Gaussian curvature which is completely determined by a cosmological constant (therefore we have equality in (14)) the equation (13) implies that the scalar field should not be a function of \(x,y\) and at the same time \(V_1 = V_2 = 0\). We will study this special case in detail in section 4.

3. Singular model

First we consider solution which does not have constant Gaussian curvature of two-surfaces spanned by \(x,y\). Further simplification of our problem can be obtained by considering the following components of Einstein equations

\[
xu : \left[ P^2 \left( V_2 V_{2,x} - V_2 V_{1,y} + \frac{V_{1,yy} - V_{2,xy}}{2} \right) \right]
\]
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\[ + P P_y (V_1 y - V_2 x) + V_1 \Lambda \] v + \left( \frac{P_u}{P} \right)_{,x} + \frac{M_x}{2} \\
+ V_1 \frac{P_u}{P} - \frac{V_1 u}{2} + \varphi, u \varphi, x = 0 \quad (15) \\
y_u : \left[ P^2 \left( V_1 V_{1,y} - V_1 V_{2,x} + \frac{V_{2,xx} - V_{1,xy}}{2} \right) \right. \\
+ P P_x (V_{2,x} - V_{1,y}) + V_2 \Lambda \] v + \left( \frac{P_u}{P} \right)_{,y} + \frac{M_y}{2} \\
+ V_2 \frac{P_u}{P} - \frac{V_2 u}{2} + \varphi, u \varphi, y = 0 \quad (16)

Here again the terms with different powers of \( v \) need to vanish separately. Looking for completely general explicit solution for the metric and scalar field seems too complicated so we are going to make specific choices leading to solvable case. For simplification and in analogy with the Robinson-Trautman geometry in standard form \[3, 4\] one can assume the function \( M \) in the following form

\[ M(u, x, y) = -2 \frac{P, u(u, x, y)}{P(u, x, y)} \quad (17) \]

and consider separation of variables for \( \varphi \) and \( P \) of this form

\[ \varphi(u, x, y) = \phi(u) + \psi(x, y) \]
\[ P(u, x, y) = \frac{P(x, y)}{U(u)} \quad (18) \]

By substituting these assumptions into (6) we obtain factorization of dependence on coordinate \( u \) for functions \( V_1 = f(u) V_1(x, y) + V_3(x, y), V_2 = f(u) V_2(x, y) + V_1(x, y) \) with \( V_3 = -\frac{\psi, x}{\psi, y} \). Further using equation (11) concludes that \( f = const \). So from now on \( V_1, V_2 \) are functions of \( x, y \) only. After substituting the above definitions (18) in (15) we obtain the following relations when considering the zero order terms in \( v \)

\[ \frac{V_1}{\psi, x} = \frac{\phi, u}{(\ln U), u} = \frac{V_2}{\psi, y} \quad (19) \]

which leads to these forms of functions \( V_1, V_2 \) (the possible arbitrary constant coming from the above relations is fixed to one using (6))

\[ V_1(x, y) = \frac{\partial \psi(x, y)}{\partial x}, \quad V_2(x, y) = \frac{\partial \psi(x, y)}{\partial y} \quad (20) \]
\[ \phi(u) = \ln(U(u)) \quad (21) \]

Using the first order term in \( v \) from equation (15) we see that \( \Lambda \) has to vanish. From (6), one can find the scalar field explicitly now

\[ \varphi(u, x, y) = \ln(U(u)) - \ln(a + \ln(x^2 + y^2)) \quad (22) \]

where \( a \) is a constant. The equation (13) simplifies into

\[ \frac{\Delta(\ln P)}{P^2} = -\frac{3}{4}(\psi_{xx} + \psi_{yy}) \quad (23) \]

which can be now solved for \( P \)

\[ P(x, y) = \sqrt{x^2 + y^2} \left[ a + \ln(x^2 + y^2) \right]^{3/4} \quad (24) \]
The final nontrivial equation which was not mentioned yet is the $vu$ component of Einstein equations (2) which now simplifies into the following form

$$vu : \Delta K + K \Delta \psi + P^2 [K_{,x} \psi_{,x} + K_{,y} \psi_{,y}]$$

$$- 4U U_{,uu} - 2U_{,u}^2 = 0 \quad (25)$$

We will use the separation of variables once more in this case

$$K(u, x, y) = k(u) K(x, y)$$

and the equation (25) splits into a condition for function $k(u)$

$$vu : k(u) = C_0 \left( 4U U_{,uu} + 2U_{,u}^2 \right) \quad (26)$$

where $C_0$ is a constant and linear second order elliptic PDE with known coefficients whose solution always exists.

Now, let us determine the algebraic types of important tensors characterizing our solution. We would like our solution to be of reasonable generality which can be checked based on Petrov or Segre classification. We will use the following tetrad ($i$ is an imaginary unit)

$$l = \partial_u - \left[ \frac{H}{2} + \frac{v^2 P^2}{2U^2} (V_1^2 + V_2^2) \right] \partial_v + \frac{vP^2}{U^2} \left( V_1 \partial_x + V_2 \partial_y \right) \quad (27)$$

$$m = \frac{P}{\sqrt{2U}} (\partial_x + i \partial_y) \quad (28)$$

There are three nonvanishing Weyl scalars in this frame

$$\Psi_0 = \frac{P}{4U(u)^2} \left\{ K_{,yy} - K_{,xx} + 2i K_{,xy} \right\}$$

$$+ K \left( V_{2,y} - V_{1,x} + i (V_{1,y} + V_{2,x}) \right)$$

$$+ \left( \frac{2 \zeta + 1}{4U^2 \zeta^2} \right) \frac{K_{,y} + i K_{,x}}{\sqrt{x^2 + y^2}} (y + ix)$$

$$- \left( \frac{2 \zeta + 3}{2U^2 \zeta^3} \right) \frac{K_{,y} + i K_{,x}}{\sqrt{x^2 + y^2}} \left( \frac{10 v^2}{U^4 \zeta^2} \right) \frac{(y + ix)^2}{(x^2 + y^2)^2} \quad (30)$$

$$\Psi_1 = \frac{5}{\sqrt{2} P U^3} \frac{v(x - iy)}{U^2} \quad (31)$$

$$\Psi_2 = - \frac{5}{6 \sqrt{U} U^2} \quad (32)$$

where

$$\zeta = a + \ln (x^2 + y^2) \quad (33)$$

We use the classification process described in [19] which is based on [20] and can be used for arbitrary tetrad. Computing the invariants

$$I = \Psi_0 \Psi_4 - 4\Psi_1 \Psi_3 + 3\Psi_2^2, \quad J = \det \begin{pmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{pmatrix}$$

one can immediately confirm that $I^3 = 27J^2$ is satisfied so that we are dealing with type II or more special. At the same time we have generally $IJ \neq 0$ so it cannot be just
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Additionally, the spinor covariant $R_{ABCDEF}$ has nonzero components

$$R_{000000} = \Psi_1 (3\Psi_0 \Psi_2 - 2\Psi_1^2)$$

$$R_{000001} = \frac{1}{2} \Psi_2 (3\Psi_0 \Psi_2 - 2\Psi_1^2)$$

which means that generally the spacetime cannot be just of type D. So indeed our scalar field solution is of algebraic type II which is the most general one in case of vacuum Kundt subclass.

Next, we will consider the Ricci tensor whose nonzero frame components are the following

$$\Phi_{11} = -\frac{1}{2 \sqrt{\zeta} U^2}$$

$$\Phi_{00} = 128 v^2 \Phi_{11}^2 + 8 v \Phi_{11} \frac{U'}{U} - \frac{U'^2}{2U^2}$$

$$\Phi_{01} = \frac{(x - iy)}{\sqrt{2} P U^3} \left( \frac{U'}{2U \Phi_{11}} + 4v \right)$$

$$\Phi_{02} = 2 \Phi_{11} \frac{(y + ix)^2}{x^2 + y^2}$$

hence the Plebański spinor has three nonzero components $\chi_0, \chi_1, \chi_2$ and so the Petrov–Plebański type is II according to the classification process described in [21]. In this case the Segre type is necessarily $[2,11]$. This means that the Ricci tensor is non-degenerate and there is no invariance group associated with it.

From the Weyl scalars and the metric functions one can observe that point $\zeta = 0$ (see (33)) looks like a curvature singularity of our solution. We can confirm this by computing Ricci and Kretschman scalars

$$R = -\frac{4}{\sqrt{\zeta} U^2}, \quad K = \frac{15}{4} R^2$$

The singularity is located on the cylinder with nonzero radius $\rho = \sqrt{x^2 + y^2} = \exp(-\frac{a^2}{2})$ which lies along the direction of propagation of the possible gravitational waves. This singularity is evidently sourced by the scalar field, namely its spatial part $\psi$ which influences the geometrically important function $P$ through (23). Considering the metric (4) and function $P$ (24) the induced metric on this cylinder is diverging. Naturally, one considers only range of coordinates $x, y$ covering the exterior of this singular cylinder.

4. Scalar waves

Now we will study the case of constant Gaussian curvature given only by a cosmological constant

$$\Delta(\ln P) = \Lambda$$

Using the results obtained at the end of section 2 and the separation of variables for function $P$ (18) the line element (4) reduces to the following form

$$ds^2 = -H(u, v, x, y) du^2 - 2 dudv + \frac{U(u)^2}{P(x, y)^2}(dx^2 + dy^2)$$
The Ricci scalar is now given by

$$R = -\frac{2\Delta \ln P(x,y)}{U(u)^2} - \frac{\partial^2 H(u,v,x,y)}{\partial u^2}$$  \hfill (42)$$

In this case the scalar field is necessarily only function of $u$ as mentioned at the end of section 2 and the only nonzero component of the energy momentum tensor for such a scalar field is

$$S^v T^w_u = \left(\frac{\partial \varphi(u)}{\partial u}\right)^2$$  \hfill (43)$$

Note that gradient of the scalar field is now aligned with the null congruence defining the properties of spacetime ($\nabla^\mu \varphi \propto \partial \mu$) which is not possible in the case of Robinson-Trautman family \cite{11} where the nonzero expansion of the congruence disallows completely aligned scalar field. Such alignment means that scalar field propagates along this null direction and can be interpreted as a scalar wave. Einstein tensor then simplifies into the following form for metric (41)

$$G^u u = G^v v = -\frac{\Delta \ln P(x,y)}{U(u)^2}$$  \hfill (44)$$

$$G^v u = -\frac{1}{2} \frac{\Delta H}{U(u)^2} + \frac{H_v U'(u) + 2U''(u)}{U(u)}$$  \hfill (45)$$

$$G^x u = \frac{P(x,y)^2}{2U(u)^2} H_{xv}, \quad G^y u = \frac{P(x,y)^2}{2U(u)^2} H_{yv}$$  \hfill (46)$$

$$G^v x = -\frac{1}{2} H_{xv}, \quad G^v y = -\frac{1}{2} H_{yv}$$  \hfill (47)$$

$$G^x x = G^y y = \frac{1}{2} \frac{\partial^2 H}{\partial v^2}$$  \hfill (48)$$

From (2) we obtain three equations

$$\frac{\Delta \ln P(x,y)}{U(u)^2} = \Lambda$$  \hfill (49)$$

$$\frac{1}{2} \frac{\partial^2 H}{\partial u^2} = -\Lambda$$  \hfill (50)$$

$$-\frac{1}{2} \frac{\Delta H}{U(u)^2} + \frac{H_v U'(u) + 2U''(u)}{U(u)} = \varphi^2$$  \hfill (51)$$

From equation (50) it follows that function $H$ should be quadratic in $v$ with the quadratic term given by a cosmological constant and the linear term independent of $x,y$ because of equation (46) and the form of energy momentum tensor (43)

$$H(u,v,x,y) = -\Lambda v^2 + h(u)v + K(u,x,y)$$  \hfill (52)$$

From equation (51) and using the above form of $H$ (52) we necessarily have

$$\Delta H(u,v,x,y) = C_0 k(u)$$  \hfill (53)$$

with some unknown function $k(u)$.

We will split the investigation into two cases: in the first one we assume $\Lambda = 0$ and in the second one we demand $U(u) = \text{constant}$. First, we will study $\Lambda = 0$ case.
In this case it is obvious from (49) that
\[ \Delta (\ln P(x,y)) = 0 \]
so the transversal two-spaces spanned by \( x, y \) are flat since the above expression is their Gaussian curvature. The function \( H(u,v,x,y) \) reduces to
\[ H(u,v,x,y) = h(u)v + K(u,x,y). \]

The relation (51) is the only remaining equation that needs to be solved. Using equation (53) the \( vu \) component of Einstein equations (51) now reduces into
\[ k(u)C_0 - \frac{h(u)U'(u) + 2U''(u)}{U(u)} \left( \frac{\partial \varphi(u)}{\partial u} \right)^2 = 0 \]  
(54)
and one can generate explicit solutions \( U(u) \) by arbitrarily specifying functions \( k(u) \), \( h(u) \) and \( \varphi(u) \), and constant \( C_0 \). We can make the problem even more explicit by assuming the following form of the arbitrary functions
\[ h(u) = \frac{D(u)}{U(u)} \]  
(55)
\[ k(u) = q(u)U(u) \]  
(56)
\[ \left( \frac{\partial \varphi(u)}{\partial u} \right)^2 = \frac{\Phi(u)^2}{U(u)} \]  
(57)
and we reduce the solution to just double integration with given functions \( D, q \) and \( \Phi \).

\[ U(u) = \frac{1}{2} \int \int \left( \frac{1}{2} C_0 q + \Phi^2 - D \right) du + C_1 u + C_2. \]  
(58)

Note, that from (57) and (43) the function \( \Phi^2 \) is proportional to the energy of the scalar field.

Now we would like to see how general is this family of solutions. Therefore we will determine its algebraic type. The only non zero Weyl scalar is
\[ \Psi_0 = \frac{P}{4U(u)^2} \{ 2i(PK_{xy} + P_{yx}K_y + P_{yx}K_x) + PK_{yy} \}

- PK_{xx} + 2P_{xy}K_x - 2P_{yx}K_x \} \]  
(59)
Now, we can again determine the type irrespective of possible non-optimal choice of tetrad by using explicit methods in [19]. We immediately see that \( IJ = 0 \) so it is a geometry of algebraic type III or more special. Additionally, the spinor covariant \( Q_{ABCD} \) whose components are quadratic expressions in Weyl scalars is identically zero which means that the algebraic type is N. This is usually interpreted as a geometry representing exact gravitational wave. The scalar field produces an energy momentum tensor which can be classified according to its Petrov-Plebański and Segre types (for a review of classification strategies see [21]). In our subclass we have the only nonzero component of the tracefree Ricci spinor \( \phi_{00} = -1/2\varphi^2 \) hence the Plebański tensor is vanishing and we have Petrov-Plebański type O. Since \( \phi_{11} - \phi_{01}\phi_{21} = 0 \) the Segre type is \([2,11]\). This Segre type corresponds to radiative sources (pure radiation field or null Maxwell field [3]) so we have aligned gravitational and scalar waves.

Note that all type N Kundt spacetimes with null radiation source are known [3]. However, the above explicit solution represents the source in the form of scalar field which obeys the corresponding field equations.

We give several explicit solutions based on equation (54) for some simple choices of \( k(u) \), \( h(u) \) and \( \varphi(u) \) in Table 1.
curvature based on the value of a cosmological constant.

Now, we can rewrite the equation (51) while considering the separation of variables (18) and (53) in the following way:

\[ \gamma \ln \left( \frac{\alpha + u}{\beta + u} \right) \frac{h_0}{u} \frac{1}{e^{n(u + \omega)}} \left\{ C_1 (u + \beta) \frac{1}{\alpha - \beta} \text{HeunC} \left( \frac{\alpha - \beta}{2} \omega, \epsilon, 0, \frac{\alpha^2}{\beta - \alpha}, \frac{\beta + u}{\alpha - \beta} \right) + C_2 (u + \beta) \frac{1}{\alpha - \beta} \text{HeunC} \left( \frac{\alpha - \beta}{2} \omega, -\epsilon, 0, \frac{\alpha^2}{\beta - \alpha}, \frac{\beta + u}{\alpha - \beta} \right) \right\} \]

\[ \gamma \ln \left( \alpha + \frac{\beta}{u} \right) \frac{h_0}{u} \frac{1}{e^{n(u + \omega)}} \left\{ C_1 u - \frac{\beta}{\alpha} u \frac{1}{\alpha - \beta} \text{HeunC} \left( \frac{\beta}{\alpha}, \omega, 0, \frac{\alpha^2}{\beta - \alpha}, \frac{\beta^2 + \beta u}{\alpha - \beta} \right) + C_2 u \frac{1}{\alpha - \beta} \text{HeunC} \left( \frac{\beta}{\alpha}, -\omega, 0, \frac{\alpha^2}{\beta - \alpha}, \frac{\beta^2 + \beta u}{\alpha - \beta} \right) \right\} \]

\[ \alpha u + \beta \frac{h_0 u}{u} \right\{ C_1 \text{KummerM} \left( \frac{2 \alpha^2 + 4 \beta u + C_0}{4 \beta}, \frac{\beta}{2}, \frac{\beta}{4} u^2 \right) + C_2 \text{KummerW} \left( \frac{2 \alpha^2 + 4 \beta u + C_0}{4 \beta}, \frac{\beta}{2}, \frac{\beta}{4} u^2 \right) \right\} \]

\[ e^{-\alpha u + \beta} \frac{h_0}{u} \right\{ C_1 \text{BesselJ} \left( -\frac{\sqrt{4 \alpha^2 + 4 \beta}}{4 \alpha}, \frac{1}{4} \sqrt{\alpha^2 + \beta} \right) + C_2 \text{BesselY} \left( -\frac{\sqrt{4 \alpha^2 + 4 \beta}}{4 \alpha}, \frac{1}{4} \sqrt{\alpha^2 + \beta} \right) \right\} \]

\[ \sin \alpha u \frac{h_0}{u} \right\{ C_1 \text{MathieuC} \left( \frac{4 \alpha^2 + \frac{1}{4} \beta}{16 \alpha^2}, \frac{1}{8}, \alpha u \right) + C_2 \text{MathieuS} \left( \frac{4 \alpha^2 + \frac{1}{4} \beta}{16 \alpha^2}, \frac{1}{8}, \alpha u \right) \right\} \]

**Table 1.** Explicit solutions for \( k(u) = U^2(u) \), where we have defined \( \Omega = \frac{1}{2} \sqrt{8 \gamma^2 + h_0^2 - 4 \beta_0^2}, \omega = \sqrt{h_0^2 + 4 \beta_0}, \epsilon = \sqrt{2} \gamma + 1 \)

### 4.2. Case \( U(u) = \text{constant} \) and \( \Lambda \neq 0 \)

In this case we choose \( U(u) = 1 \) for simplicity. One can immediately see from (49) that

\[ \Delta(\ln P(x, y)) = \Lambda \] (60)

so the transversal two-spaces spanned by \( x, y \) have constant positive or negative curvature based on the value of a cosmological constant.

For the function \( H \) in the form (52) the equation (50) is identically satisfied. Now, we can rewrite the equation (51) while considering the separation of variables (18) and (53) in the following way:

\[ \frac{k(u)C_0}{2} + \left( \frac{\partial \varphi(u)}{\partial u} \right)^2 = 0 \] (61)

We again compute Weyl scalars for a natural tetrad of this solution and obtain the following nonzero components

\[ \Psi_0 = \frac{P}{4} \left\{ 2i(P K_{x,y} + P_{x} K_{,y} + P_{y} K_{,x}) + P K_{,yy} \right\} - P K_{,xx} + 2 P_{,y} K_{,y} - 2 P_{,x} K_{,x} \] (62)

\[ \Psi_2 = \frac{\Lambda}{6} - \frac{1}{12} H_{,uv} \] (63)

Using (19) (20) we can immediately confirm that \( I^3 = 27 J^2 \) is satisfied so that we are dealing with type II or more special. At the same time we have generally \( IJ \neq 0 \) so
it cannot be just type III. Additionally, the spinor covariant $R_{ABCDEF}$ has nonzero component
\[ R_{000001} = \frac{1}{2} \Psi_2 (3\Psi_0 \Psi_2 - 2\Psi_1^2) \] (64)
which means that generally the spacetime cannot be of type D. So indeed our scalar field solution is of algebraic type II which is the most general one in case of vacuum Kundt subclass. We have one nonzero component of the tracefree Ricci spinor $\phi_{00} = \frac{1}{4} \Delta H$ hence the Petrov-Plebański type is O and the Segre type is [(2, 11)] again so the source corresponds to null radiation given by a scalar wave.

The above described case is generalization of a specific subcase of Kundt solutions with cosmological constant described in [22]. Namely, our solution additionally supports the scalar field.

If one would want to restrict the algebraic type solely to D (keeping $\Psi_2 \neq 0$ but setting $\Psi_0 = 0$) one can derive from (62) that necessarily $K$ is independent of $x, y$ which by using (63) means $k = 0$. But that means (see (61)) the vanishing of the contribution of scalar field energy momentum tensor. In accordance with [4] we might interpret our type II solution as gravitational wave on type D background. So as we have just seen the scalar wave is necessarily accompanied by a gravitational wave in a kind of nontrivial interaction where the scalar wave generates the gravitational one.

Of particular interest might be a question whether our subclasses admit pp-wave solutions that still retain the scalar field. Due to the construction of Kundt class the geometry reduces to pp-wave if the principal null direction $l = \partial v$ is covariantly constant
\[ l_{\alpha;\beta} dx^\alpha dx^\beta = -\frac{1}{2} \frac{\partial H(u, v, x, y)}{\partial v} = 0. \] (65)
For the case of model described in subsection 4.1 this translates into $h = 0$ and $D = 0$. In the case of model described in subsection 4.2 we have $h = 0$ and $\Lambda = 0$. In both cases the scalar field is generally nonvanishing so we have pp-wave solutions with scalar field.

5. Nonlinear Electrodynamic

So far we considered only scalar field with possibly a nonvanishing cosmological constant. It is interesting to investigate the possibility of having some form of the nonlinear electrodynamics (NE) as an additional source. We assume general Lagrangian of the nonlinear electromagnetic field $L(F)$ to be an arbitrary function of the invariant $F = F_{\mu\nu}F^{\mu\nu}$ constructed from closed Maxwell 2-form $F_{\mu\nu}$ with the following energy momentum tensor
\[ NE T_{\mu\nu} = \frac{1}{2} \left\{ \delta^{\mu}_{\nu} L - (F_{\nu\lambda} F^{\mu\lambda}) L_F \right\} \] (66)
which contributes to Einstein equations
\[ G_{\mu\nu} = SP T_{\mu\nu} + NE T_{\mu\nu} - \Lambda \delta_{\mu\nu}, \] (67)
and the modified Maxwell (nonlinear electrodynamics) field equations are given in the following form
\[ \partial_{\mu}(\sqrt{-g} L_{F} F^{\mu\nu}) = 0 \] (68)
in which $\mathcal{L}_F = \frac{d\mathcal{L}(F)}{dF}$. If we assume a specific Maxwell 2-form

$$\mathbf{F} = E(u, v)du \wedge dv$$

(69)

then from (68) and the metric (41) one can find

$$\mathcal{L}_F F_{uv} = F_0$$

(70)

where $F_0$ is a constant. The energy momentum tensor given in (66) can be expressed in the form

$$^{\text{NE}}T_{\mu\nu} = \text{diag}\left\{ \frac{\mathcal{L}}{2} - F\mathcal{L}_F, \frac{\mathcal{L}}{2} - F\mathcal{L}_F, \frac{\mathcal{L}}{2}, \frac{\mathcal{L}}{2} \right\}$$

(71)

As one can see from (70), it is possible to find the form of nonlinear electrodynamics Lagrangian explicitly

$$\mathcal{L}(F) = -\alpha \sqrt{-F}$$

(72)

where $\alpha = \sqrt{8}F_0$. Although, it can be found from $uu$ component of (67) as well (when considering $U(u) = \text{const.}$)

$$-\Delta (\ln P(x, y)) + \Lambda = \frac{\mathcal{L}}{2} - F\mathcal{L}_F.$$  

(73)

It is clear that $\frac{\mathcal{L}}{2} - F\mathcal{L}_F$ should be constant. For simplicity one may choose this constant as zero. So the form of NE Lagrangian would be the same as (72) and we can have (60).

This particular form of Lagrangian, namely a square root of invariant $F$, is not new and it has been investigated previously (see e.g. [23, 24]). Namely, it was shown that the spherically symmetric purely electric solution is absent in this model, or, in other words, the electric monopole is vanishing by definition. On the other hand, gauge theory with such a Lagrangian contains interesting string-like solutions [25] leading to possible confinement. Also, radiation modes (or null field solutions satisfying $F = 0$) do not appear naturally but can be recovered using a magnetic condensation in effective four-dimensional theory coming from 6D compactification. This square root Lagrangian is also a special case of the so called Power Maxwell model [26] $((F^s)^s)$ when $s = \frac{1}{2}$ which was subject of intensive study. One problem in this type of models is usually connected with the energy conditions, however, we can easily avoid this by properly selecting the constant $\alpha$.

The next equation is the $xx$ component of (67)

$$\frac{1}{2} \frac{\partial^2 H(u, v, x, y)}{\partial v^2} + \Lambda = \frac{\mathcal{L}}{2}$$

(74)

If we define

$$H(u, v, x, y) = \chi(u, v) - \Lambda v^2 + h(u)v + K(u, x, y),$$

(75)

the above equation reduces to a simpler form

$$\mathcal{L} = \frac{\partial^2 \chi(u, v)}{\partial v^2}$$

(76)

Since in NE the (mixed component) energy momentum tensor does not have any non-diagonal terms the $vu$ component of (67) has the scalar field as the only source. So we can retrieve the equation which is the same as (54).

The electromagnetic field considered above is evidently non-null ($F_{\mu\nu}F^{\mu\nu} \neq 0$) and since $F_{\mu\nu}F^{\mu\nu} = 0$ it is purely electric in preferred frame. The Petrov type of the Weyl tensor is easily determined since the nonzero Weyl scalars are still given by (62).
and so the type is D. The complete energy momentum tensor (both \((71)\) and \((43)\)) is of Petrov–Plenański type D since we have nonzero tracefree Ricci spinor components
\[
\phi_{00} = \frac{1}{4} \Delta H, \quad \phi_{11} = \frac{1}{8} \chi_{vv}
\]
and the Segre type is then [2, (11)] according to classification scheme in [21].

One may be surprised that we have non-null electromagnetic field but there is no source in the nonlinear Maxwell equation \((68)\). However, using Leibniz rule one can split the left-hand side of this equation and rearrange the equation into the form
\[
\partial_\mu (\sqrt{-g} F^{\mu \nu}) = -\sqrt{-g} F^{\mu \nu} \partial_\mu \ln \mathcal{L}_F
\]
Now, the left-hand side represents standard Maxwell equation and the right-hand side represents source generated by nontrivial \(\partial_\mu \ln \mathcal{L}_F\). So from the point of view of standard Maxwell theory the solution of vacuum nonlinear electrodynamics has a source. This interpretation was already noted in the original paper [16].

Finally we give the explicit form of \(E(u, v)\). From \((76)\) and the relation \(F = -2E^2\) we immediately get
\[
|E(u, v)| = -\frac{1}{4F_0} \frac{\partial^2 \chi(u, v)}{\partial u^2}
\]
(77)

Here we can again search for a pp-wave geometry in our results. In this case, from equations \((75)\) and \((65)\) we see that the Lagrangian \((76)\) and the electric field \((77)\) itself necessarily vanishes. However, if we do not make a definition \((75)\) and analyze equation \((74)\) directly we can see that the Lagrangian and therefore the electric field is constant. So the pp-wave spacetime in our class of solutions admits only constant electric field together with scalar field.

6. Conclusion and Final Remarks

We have investigated scalar field coupled to Kundt spacetime. The explicit solutions were given based on the behavior of Gaussian curvature of transversal two-spaces. The first explicit solution corresponds to a spacetime with a singular cylinder with the singularity sourced by the scalar field. This solution can be thought of as an analog of the scalar field solution in the Robinson–Trautman class given in [11].

We have also given two explicit (up to one integration) subclasses of Kundt family containing scalar field wave. Their respective algebraic types of the Weyl tensor are N and II. The notable absence of type D confirms the results of [24] where it was shown that Kundt type D subclass does not admit null massless scalar field. One can easily see that our assumption about the functional dependence of the scalar field necessarily means that it is null. This scalar field is naturally radiative. Since our type II Kundt geometry is interpreted as an exact gravitational wave on type D background we can see that the presence of scalar wave necessarily generates accompanying gravitational wave. Both classes of spacetimes admit pp-wave solutions with scalar field as a special case.

In the final section we have considered general electrodynamic field with nonlinear Lagrangian as an additional source and obtained solutions for geometry, electromagnetic field and the specific form of Lagrangian. The physical significance of the derived Lagrangian was investigated in several previous works. In this case the pp-wave condition results in trivial solution containing only constant electric field. The algebraic type of the Weyl tensor was II in this case. The electromagnetic field is non-null and sourceless. However, it has a source generated by nonlinear Lagrangian when interpreted in the scope of Maxwell theory.
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