Sharp Approximations for the Ramanujan Constant *

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Abstract: In this paper, the authors present sharp approximations in terms of sine function and polynomials for the so-called Ramanujan constant (or the Ramanujan $R$-function) $R(a)$, by showing some monotonicity, concavity and convexity properties of certain combinations defined in terms of $R(a), \sin(\pi a)$ and polynomials. Some properties of the Riemann zeta function and its related special sums are presented, too.

Key Words: The Ramanujan constant; monotonicity; convexity and concavity; approximation; functional inequalities; the Riemann zeta function

Mathematics Subject Classification: 11M06, 33B15, 33C05, 33F05.

1 Introduction

For real numbers $x,y > 0$, the gamma, beta and psi functions are defined as
\[
\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\]
respectively. (For their basic properties, cf. [1, 2, 5, 14].) Let $\gamma = 0.5772156649\cdots$ be the Euler-Mascheroni constant, and for $a \in (0,1)$, let $R(a)$ be defined by
\[
R(a) = -2\gamma - \psi(a) - \psi(1-a)
\]
which is called the Ramanujan constant in literature although it is actually a function of $a$ and probably better to call $R(a)$ the Ramanujan $R$-function (cf. [11]). By the symmetry, we may assume that $a \in (0,1/2]$ in (1.2). It is well known that $R(a)$ is essential in some fields of mathematics such as the zero-balanced Gaussian hypergeometric functions $_2F_1(a,1-a;1;z)$, the theories of Ramanujan’s modular equations and quasiconformal mappings, and the properties of $R(a)$ are indispensable for us to show the properties of $_2F_1(a,1-a;1;z)$ and the functions appearing in generalized Ramanujan’s modular equations. On the other hand, $R(a)$ and the function
\[
B(a) = B(a,1-a) = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)} (0 < a < 1)
\]
are often simultaneously appear in the study of the properties and applications of $R(a)$, and we often need to compare $R(a)$ with $B(a)$. In [11, Section 1], such kind of importance and applications of $R(a)$, and the relation between $R(a)$ and $B(a)$ were described in details. (See also [2, 4–10, 12, 15–17].)

Some authors have obtained some properties, including lower and upper bounds, for $R(a)$. In [11], for instance, power series expansion, integral representation and bounds were obtained for the difference $R(a) - B(a)$. Some related studies showed that $B(a)$ is one of good approximation functions for $R(a)$, and we often require the properties of certain combinations defined in terms of $R(a), B(a), a(1-a)$ and other polynomials. (See [3, 8, 10, 11, 15, 17].)

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The main purpose of this paper is to show some monotonicity, convexity and concavity properties of certain combinations defined in terms of $R(a)$, $B(a)$ and polynomials, by which sharp approximations given by $B(a)$ and polynomials are obtained for $R(a)$. In addition, we shall also show some properties of the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \text{Re } s > 1$$  \hspace{1cm} (1.4)

and its related special sums [1, 23.2]

$$\lambda(n+1) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{n+1}}, \eta(n) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^n}, \beta(n) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)^n}, n \in \mathbb{N}. \hspace{1cm} (1.5)$$

For the later use, we record the following identities and special values

$$\lambda(n + 1) = \left(1 - 2^{-n+1}\right)\zeta(n + 1), \eta(n) = \left(1 - 2^{1-n}\right)\zeta(n), \hspace{1cm} (1.6)$$

$$\zeta(2) = \frac{\pi^2}{6}, \zeta(4) = \frac{\pi^4}{90} = \lambda(2) - \frac{\pi^2}{8} = \lambda(4) - \frac{\pi^4}{96} = \beta(1) - \frac{\pi^2}{4} = \beta(3) - \frac{\pi^4}{32} = 0. \hspace{1cm} (1.7)$$

(See [23.2.19-20, 23.2.24-25 & 23.2.28-31]).

Throughout this paper, the “zero-order derivative” $\varphi^{(0)} \varphi$ will be understood as $\varphi$ itself for real one-variable function $\varphi$, and we always let

$$\begin{aligned}
&\{a_0 = 1, a_1 = -2\eta(2) - 1 = 1 - \pi^2/6 = -2.644934\ldots, a_2 = 2\zeta(3) = 2.404113\ldots, \\
&a_n = [1 + (-1)^n]\zeta(n + 1) - \zeta(n) \in \mathbb{R}, \text{ for } n \geq 3,
\end{aligned} \hspace{1cm} (1.8)$$

$$\begin{aligned}
&b_0 = 5 \log 2 - \pi = 0.324143\ldots, b_1 = -\log 2 + [35\zeta(3) - \pi^3]/8 = 0.690067\ldots,
\end{aligned}\hspace{1cm} (1.9)$$

$$\begin{aligned}
c_n = 1 - \sum_{k=0}^{n} b_k \text{ and } A_n = 2^{n+1} \left(b_0 - \sum_{k=0}^{n} 2^{-k} a_k\right) \text{ for } n \in \mathbb{N} \cup \{0\}. \hspace{1cm} (1.10)
\end{aligned}$$

Clearly, for $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
&\{a_{2n+2} = 2[\zeta(2n+3) - \zeta(2n+1)] = -2 \sum_{k=2}^{\infty} k^{-2n-3}(k^2 - 1) < 0,
\end{aligned} \hspace{1cm} (1.11)$$

$$\begin{aligned}
c_{n+1} = c_n - b_{n+1}, A_{n+1} = 2(A_n - a_{n+1}), n \in \mathbb{N} \cup \{0\}. \hspace{1cm} (1.12)
\end{aligned}$$

Some properties of the constants $a_n$, $b_n$, $c_n$ and $A_n$ will be given in Lemma 2.1 and Corollaries 3.1 and 4.1. We now state the main results of this paper.

**Theorem 1.1.**  (1) The function $f(x) \equiv [1 + x(1 - x)]R(x) - B(x)$ has the following power series expansions

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n (1 - 2x)^{2n}, \hspace{0.5cm} 0 < x \leq \frac{1}{2}. \hspace{1cm} (1.13)$$

(2) $f$ (f') is strictly decreasing and convex (increasing and convex) from (0, 1/2) onto $[b_0, 0]$ ($(a_1, 0]$, respectively), and the function $F(x) \equiv b_0 + b_1(1-2x)^2 + B(x) - [1 + x(1-x)]R(x)$ is strictly completely monotonic on $(0, 1/2)$ with $F(0^+) = b_0 + b_1 - 1 = 0.0142104\ldots$ and $F(1/2) = 0$. Moreover, for each $n \in \mathbb{N} \setminus \{1\}$, the function $F_{1,n}(x) \equiv (-1)^{n+1} f^{(2n)}(x)$ is strictly completely monotonic on $(0, 1/2)$ with $F_{1,n}(0^+) = (-1)^{n+1} n! a_n$, $F_{1,n}(1/2) = 0$ if $n$ is odd, and $F_{1,n}(1/2) = -2^n n! b_{n/2}$ if $n$ is even. In particular, for $x \in (0, 1/2]$,

$$\frac{b_0 + (1 - 2x)P(x) + B(x)}{1 + x(1-x)} \leq R(x) \leq \frac{b_0 + (1 - 2x)Q(x) + B(x)}{1 + x(1-x)}, \hspace{1cm} (1.14)$$

with equality in each instance if and only if $x = 1/2$, where $P(x) = \max\{0, 1 - b_0 - b_1 + b_1(1 - 2x)\}$ and $Q(x) = \min\{1 - b_0, b_1(1 - 2x)\}$.
Theorem 1.2. For each $n \in \mathbb{N} \cup \{0\}$ and $x \in (0, 1/2]$, let $R_n(x) = \sum_{k=0}^{n} a_k x^k$ and $S_n(x) = \sum_{k=0}^{n} b_k (1 - 2x)^{2k}$, $f$ as in Theorem 1.1, and define the functions $f_n$ and $g_n$ on $(0, 1/2)$ by

$$f_n(x) = \frac{f(x) - R_n(x)}{x^{n+1}} \quad \text{and} \quad g_n(x) = \frac{f(x) - S_n(x)}{(1 - 2x)^{2(n+1)}},$$

respectively. Then we have the following conclusions:

1. $f_0$ is strictly increasing and convex from $(0, 1/2)$ onto $(a_1, 2(b_0 - 1)]$. Furthermore, for each $n \in \mathbb{N}$, $f_{2n-1}$ ($f_{2n}$) is strictly increasing and concave (decreasing and convex) from $(0, 1/2)$ onto $(a_{2n}, A_{2n-1})$ ($[A_{2n}, a_{2n+1})$, respectively. In particular, for each $n \in \mathbb{N}$ and all $x \in (0, 1/2]$,

$$\frac{B(x) + R_{2n+2}(x) + A_{2n+2} x^{2n+3}}{1 + x(1 - x)} \leq R(x) \leq \frac{B(x) + R_{2n+1}(x) + A_{2n+1} x^{2n+1}(n+1)}{1 + x(1 - x)}, \quad (1.15)$$

with equality in each instance if and only if $x = 1/2$.

2. For each $n \in \mathbb{N} \cup \{0\}$, $g_n$ is strictly increasing and concave from $(0, 1/2)$ onto $(c_n, b_{n+1})$. Furthermore, for each $m, n \in \mathbb{N} \cup \{0\}$, the function $G_{n,m}(x) \equiv (-1)^{m+1} \delta_n^m(x)$ is strictly completely monotonic on $(0, 1/2)$. In particular, for each $n \in \mathbb{N} \cup \{0\}$ and all $x \in (0, 1/2]$,

$$\frac{B(x) + S_{n+1}(x) + c_{n+1} (1 - 2x)^{2n+3}}{1 + x(1 - x)} \leq R(x) \leq \frac{B(x) + S_n(x)}{1 + x(1 - x)}, \quad (1.17)$$

with equality in each instance if and only if $x = 1/2$.

By Theorem 1.1, it is natural to ask whether the functions

$$F_1(x) \equiv [1 + x(1 - x)] \frac{R(x)}{B(x)}, \quad F_2(x) \equiv \frac{R(x)}{B(x)} - \frac{1}{1 + x(1 - x)} \quad \text{and} \quad F_3(x) \equiv R(x) - \frac{B(x)}{1 + x(1 - x)} \quad (1.19)$$

are monotone on $(0, 1/2)$. Our next result gives the answer to this question.

Theorem 1.3. (1) There exists a unique number $x_1 \in (1/4, 1/2)$ such that the function $F_1$ defined by (1.19) is strictly increasing on $(0, x_1]$, and decreasing on $[x_1, 1/2]$, with $F_1(0^+) = 1$ and $F_1(1/2) = (5 \log 2)/\pi = 1.103178 \cdots$. However, $F_1$ is neither convex nor concave on $(0, 1/2)$. In particular, for $x \in (0, 1/2]$,

$$\frac{\alpha B(x)}{1 + x(1 - x)} < R(x) < \frac{\delta B(x)}{1 + x(1 - x)}, \quad (1.20)$$

with the best possible coefficients $\alpha = 1$ and $\delta = F_1(x_1)$. Moreover,

$$1.111592 \cdots = \frac{19 \sqrt{2} \log 8}{16\pi} < \delta < 1.112146. \quad (1.21)$$

(2) There exists a unique number $x_2 \in (0, 1/2)$ such that the function $F_2$ defined by (1.19) is strictly increasing on $(0, x_2]$, and decreasing on $[x_2, 1/2]$ with $F_2(0^+) = 0$ and $F_2(1/2) = 4b_0/(5\pi) = 0.082542 \cdots$. However, $F_2$ is neither convex nor concave on $(0, 1/2)$.

(3) The function $F_3$ defined by (1.19) is strictly decreasing and convex from $(0, 1/2)$ onto $[\rho, 1)$, where $\rho = 4b_0/5 = \log 16 - 4\pi/5 = 0.259314 \cdots$. In particular, for $x \in (0, 1/2]$,

$$\rho + \frac{B(x)}{1 + x(1 - x)} \leq R(x) \leq \rho + (1 - \rho)(1 - 2x) + \frac{B(x)}{1 + x(1 - x)}, \quad (1.22)$$

with equality in each instance if and only if $x = 1/2$. 

3
2 Preliminaries

In this section, we prove two technical lemmas needed in the proofs of our main results stated in Section 1. Our first lemma shows some properties of \( b_n, \lambda(n) \) and \( \beta(n) \).

**Lemma 2.1.** (1) The functions \( \lambda(x) = \sum_{k=0}^{\infty} (2k + 1)^{-x} \) and \( \varphi_1(x) = \lambda(x) - \lambda(x + 1) \) are both strictly decreasing and convex on \((1, \infty)\). Also, with \( \lambda(1, \infty) = (1, \infty) \) and \( \varphi_1(1, \infty) = (0, 0) \). Moreover, for each \( c \in [C_1, \infty) \), the function \( \varphi_2(x) \equiv \lambda(x + c)/\lambda(x) \) is strictly increasing from \([2, \infty)\) onto \([C_2, 1]\), where \( C_1 = \log(\pi^2/8) \)/\( \log 3 = 0.191166 \cdots \) \( \cdots \) \( \beta \).

(2) The function \( \beta(x) = \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{-x} \) is strictly increasing from \([1, \infty)\) onto \([\pi/4, 1]\), and concave on \([2/\log 3, \infty)\).

(3) For \( n \in \mathbb{N} \), the sequences \( \{b_{n+1}\} \) and \( \{(n + 1)b_{n+1}\} \) are both strictly increasing, while the sequence \( \{x \} \) is strictly decreasing, with \( \lim_{n \rightarrow \infty} nb_n = \lim_{n \rightarrow \infty} [(n + 1)b_{n+1} - nb_n] = 0 \). In particular, for \( n \in \mathbb{N} \setminus \{1\} \),

\[
-0.027624 \cdots = 2b_2 \leq nb_n < (n + 1)b_{n+1} \leq \min[0, nb_n + 3b_3 - 2b_2],
\]

with equality in each instance if and only if \( n = 2 \).

**Proof.** (1) The monotonicity of \( \lambda \) is clear. Since the derivative

\[
\lambda'(x) = -\sum_{k=1}^{\infty} \frac{\log(2k + 1)}{(2k + 1)^x}
\]

is clearly increasing on \((1, \infty)\), the function \( \lambda \) is convex on \((1, \infty)\).

It is easy to see that \( \varphi_1(1) = \infty \) and \( \varphi_1(\infty) = 0 \). By (2.2), we have

\[
\varphi_1'(x) = -2\sum_{k=1}^{\infty} \frac{k}{(2k + 1)^{x+1}} \log(2k + 1),
\]

which is negative and strictly increasing on \((1, \infty)\), and hence the result for \( \varphi_1 \) follows.

Clearly, \( \varphi_2(2) = \lambda(2 + c)/\lambda(2) = 8\lambda(2)/\pi^2 = C_2 \) and \( \varphi_2(\infty) = 1 \). By differentiation and (2.2),

\[
\frac{\lambda(x)^2 \varphi_2'(x)}{\varphi_2(x)} = \sum_{k=1}^{\infty} \frac{\log(2k + 1)}{(2k + 1)^{x+c}} \left[(2k + 1)^c \lambda(x + c) - \lambda(x)\right] > \varphi_1(x) \sum_{k=1}^{\infty} \frac{\log(2k + 1)}{(2k + 1)^{x+c}},
\]

where \( \varphi_1(x) = 3^c \lambda(x + c) - \lambda(x) \), and by (1.5),

\[
\varphi_1(x) = \sum_{k=0}^{\infty} \frac{3^c - (2k + 1)^c}{(2k + 1)^{x+c}} = 3^c - 1 - \sum_{k=0}^{\infty} \frac{(2k + 1)^c - 3^c}{(2k + 1)^{x+c}}
\]

which is clearly strictly increasing on \((1, \infty)\). Since \( c \geq [\log(\pi^2/8)/\log 3 = [\log \lambda(2)/\log 3, \]

\[
\varphi_1(2) = 3^c \lambda(2 + c) - \lambda(2) \geq \lambda(2)[\lambda(2 + c) - 1] > 0.
\]

Hence the result for \( \varphi_2 \) follows from (2.4).

(2) Clearly, \( \lim_{x \rightarrow \infty} \beta(x) = 1 \) and \( \beta(1) = \pi/4 \). Set \( \beta_1(t) = (\log t)/t \) for \( t \in [3, \infty) \). Then by differentiation,

\[
\beta'(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\log(2k + 1)}{(2k + 1)^x} = \sum_{k=1}^{\infty} \frac{\log[2(2k - 1) + 1]}{[2(2k - 1) + 1]^x} - \sum_{k=1}^{\infty} \frac{\log[2(2k + 1)]}{[2(2k + 1)]^x} = \frac{1}{\pi} \sum_{k=1}^{\infty} [\beta_1((4k - 1)^{y}) - \beta_1((4k + 1)^{y})].
\]

(2.5).
It is easy to verify that the function $\beta_1$ is strictly decreasing on $[e, \infty)$. Since $(4k - 1)^x \geq 3^x \geq 3 > e$ for $k \geq 1$ and $x \geq 1$, $\beta'(x) > 0$ for $x \in [1, \infty)$ by (2.5), so that the monotonicity of $\beta(x)$ follows.

By the first equality in (2.5) and by differentiation,

$$\beta''(x) = \frac{1}{x^2} \sum_{k=1}^{\infty} \left[ \beta_2((4k + 1)^x) - \beta_2((4k - 1)^x) \right],$$

where $\beta_2(t) = (\log t)^2/t$ for $t \geq 3$. It is easy to show that $\beta_2$ is strictly decreasing on $[e^2, \infty)$. Hence if $(4k - 1)^x \geq 3^x \geq e^2$, that is, $x \geq 2/\log 3$, then $\beta''(x) < 0$. This shows that $\beta$ is concave on $[2/\log 3, \infty)$.

(3) It follows from (1.7) and [1, Table 23.3] that

$$\phi_3(3) = \mu_1 = 5\lambda(4) - \lambda(2) - 4\beta(4) = \frac{5\pi^4}{96} - \frac{\pi^2}{8} - 4\beta(4) = -0.116088 \cdots.$$  

Clearly, $\phi_3(\infty) = 0$, and $k(k + 1) - 1 + (-1)^k \geq 6$ for all $k \geq 2$. By (1.5), we have

$$\phi_3(x) = -4 \sum_{k=2}^{\infty} \frac{k(k + 1) - 1 + (-1)^k}{(2k + 1)^{x+1}},$$  

(2.6)

$$\phi_3(x) = 4 \sum_{k=2}^{\infty} \frac{k(k + 1) - 1 + (-1)^k}{(2k + 1)^{x+1}} \log(2k + 1).$$  

(2.7)

Hence the result for $\phi_3$ follow from (2.6) and (2.7).

Next, for each $k \in \mathbb{N} \setminus \{1\}$ and for $x \in [3, \infty)$, let $\phi_{8,k}(x) = (x - 3 + 1/\log 5)(2k + 1)^x$. Then $\phi_{8,k}(3) = 1/[(2k + 1)^5 \log 5]$, $\phi_{8,k}(\infty) = 0$, and by (2.6),

$$\phi_4(x) = -4 \sum_{k=2}^{\infty} \frac{k(k + 1) - 1 + (-1)^k}{2k + 1} \phi_{8,k}(x)$$  

(2.8)

Clearly, $\phi_4(3) = \mu_2$, and $\phi_4(\infty) = 0$ by (2.8). Since

$$\phi_{8,k}'(x) = (2k + 1)^{-x}[1 - (x - 3 + 1/\log 5)\log(2k + 1)]$$

$$\leq -x(3/2k + 1)^{-x}\log 5 < 0 \quad \text{for} \quad x > 3,$$

$\phi_{8,k}$ is strictly decreasing from $[3, \infty)$ onto $(0, \phi_{8,k}(3))$, so that the monotonicity of $\phi_4$ follows from (2.8).

Since $3 - 1/\log 5 = 2.378656 \cdots > 0$, and since

$$-\phi_5(x) = [-\phi_4(x)][1 + (3 - 1/\log 5)/(x - 3 + 1/\log 5)]$$

which is a product of two positive and strictly decreasing functions on $[3, \infty)$, the monotonicity of $\phi_5$ follows.

Clearly, $\phi_6(3) = \mu_4 = \phi_5(5) - \phi_3(3) = 5[\lambda(6) - \lambda(4) - 4\beta(6)] - \mu_3 = 0.337348 \cdots$ and $\phi_6(\infty) = 0$. It follows from (2.6) that

$$\phi_6(x) = (x + 2)\phi_3(x + 2) - x\phi_3(x) = 8 \sum_{k=2}^{\infty} \frac{k(k + 1) - 1 + (-1)^k}{(2k + 1)^{x+3}}[2k(2k + 1)x - 1].$$  

(2.9)

Differentiation gives

$$\frac{d}{dx} \left[ \frac{2(k^2 + k)x - 1}{(2k + 1)^{x+3}} \right] = \frac{2k(k + 1) \log(2k + 1)}{(2k + 1)^{x+3}} \left[ \frac{1}{\log(2k + 1)} + \frac{1}{2k(k + 1)} - x \right]$$

$$\leq 2k(k + 1) \log(2k + 1) \left[ \frac{1}{\log 5} + \frac{1}{12} - 3 \right] < 0$$

for $x \in [3, \infty)$, since $(1/\log 5) + (1/12) = 0.704668 \cdots$. Hence the monotonicity of $\phi_6$ follows from (2.9).

(4) Clearly, $b_n+1 = \phi_5(2n + 2)$, $(n + 1)b_n+1 = \phi_5(2n + 2)/2$ and $(n + 2)b_{n+2} - (n + 1)b_{n+1} = \phi_6(2n + 2)/2$.

Hence part (4) follows from part (3). □
Lemma 2.2. For \( x \in (0, 1/2] \),

\[
R(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(1 + (-1)^n\right) \zeta(n + 1) x^n = \log 16 + 4 \sum_{n=1}^{\infty} \lambda(2n + 1)(1 - 2x)^{2n}, \tag{2.10}
\]

\[
B(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(1 + (-1)^{n+1}\right) \eta(n + 1) x^n = 4 \sum_{n=0}^{\infty} \beta(2n + 1)(1 - 2x)^{2n}, \tag{2.11}
\]

\[
R\left(\frac{1}{4}\right) = -2\gamma - \psi\left(\frac{1}{4}\right) - \psi\left(\frac{3}{4}\right) = 6 \log 2 = 4.158883 \ldots, \tag{2.12}
\]

\[
R'\left(\frac{1}{4}\right) = \psi'\left(\frac{3}{4}\right) - \psi'\left(\frac{1}{4}\right) = -16\beta(2) = -14.655449 \ldots. \tag{2.13}
\]

Proof. It is well known that

\[
\psi(1 + x) = \psi(x) + \frac{1}{x}, \tag{2.14}
\]

\[
\psi(1 + x) + \gamma = \sum_{n=1}^{\infty} (-1)^n \zeta(n + 1) x^n, \quad |x| < 1, \tag{2.15}
\]

\[
\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k + x)^{n+1}}, \tag{2.16}
\]

(Cf. \[1, 6.3.5, 6.3.14 & 6.4.10\].) By (2.14) and (2.15),

\[
R(x) = \frac{1}{x} - \left[\gamma + \psi(1 - x)\right] - \left[\gamma + \psi(1 + x)\right]
= \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^n \zeta(n + 1)(-x)^n + \sum_{n=1}^{\infty} (-1)^n \zeta(n + 1)x^n,
\]

yielding the first equality in (2.10). By differentiation,

\[
R^{(n)}(x) = (-1)^{n+1} \psi^{(n)}(1 - x) - \psi^{(n)}(x). \tag{2.17}
\]

Hence by (2.16) and (2.17), \( R^{(2n-1)}(1/2) = 0 \) and

\[
R^{(2n)}\left(\frac{1}{2}\right) = -2\psi^{(2n)}\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{2(2n)!}{(k + 1/2)^{2n+1}} = 4^{n+1}(2n)!\lambda(2n + 1),
\]

so that \( R(x) \) has the following power series expansion

\[
R(x) = R\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{R^{(n)}(1/2)}{n!} \left(x - \frac{1}{2}\right)^n = \log 16 + \sum_{n=1}^{\infty} \frac{R^{(2n)}(1/2)}{(2n)!} \left(x - \frac{1}{2}\right)^{2n}
= \log 16 + 4 \sum_{n=1}^{\infty} \lambda(2n + 1)(1 - 2x)^{2n}
\]

at \( x = 1/2 \). This yields the second equality in (2.10).

By \[1, 4.3.68 & 23.1.18\],

\[
\frac{1}{\sin z} = \frac{1}{z} + 2 \sum_{n=1}^{\infty} \frac{\eta(2n)}{\pi^{2n}} z^{2n-1} = \frac{1}{z} + \sum_{n=1}^{\infty} \left[1 + (-1)^{n+1}\right] \frac{\eta(n + 1)}{\pi^{n+1}} z^n, \tag{2.18}
\]

for \( |z| < \pi \). Hence the first equality in (2.11) follows from (1.3) and (2.18). By \[1, 4.3.69 & 23.1.18\], we have

\[
\frac{1}{\cos z} = 2 \sum_{n=0}^{\infty} \left(\frac{2}{\pi}\right)^2 \frac{\beta(2n + 1)}{\pi^{2n+1}} z^{2n}, \quad |z| < \frac{\pi}{2}, \tag{2.19}
\]
from which it follows that

\[ B(x) = \frac{\pi}{\sin(\pi x)} = \frac{\pi}{\cos[\pi(1 - 2x)/2]} = 4 \sum_{n=0}^{\infty} \beta(2n + 1)(1 - 2x)^{2n}, \]

yielding the second equality in (2.11).

Next, by \[1], 6.3.3 & 6.3.8,\]

\[ \psi\left(\frac{1}{2}\right) = -\gamma - \log 4, \quad \psi(2x) = \frac{1}{2} \psi(x) + \frac{1}{2} \psi\left(x + \frac{1}{2}\right) + \log 2, \quad (2.20) \]

from which we obtain

\[ R\left(\frac{1}{4}\right) = -2\gamma - \left[\psi\left(\frac{1}{4}\right) + \psi\left(\frac{3}{4}\right)\right] = -2\gamma - 2 \left[\psi\left(\frac{1}{2}\right) - \log 2\right] = 6 \log 2. \]

Finally, by (1.15), it is easy to see that

\[ \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)^2} = \frac{1}{2(2n) + 1} - \sum_{n=0}^{\infty} \frac{1}{(2n + 1)^2}. \quad (2.21) \]

It follows from (2.16), (2.21) and \[1\] Table 23.3 that

\[ R'\left(\frac{1}{4}\right) = \psi'\left(\frac{3}{4}\right) - \psi'\left(\frac{1}{4}\right) = 16 \left[\sum_{n=0}^{\infty} \frac{1}{(4n + 3)^2} - \sum_{n=0}^{\infty} \frac{1}{(4n + 1)^2} \right] = -16 \beta(2). \]

**Remark 2.3.** In \[16\] Theorem 2.2, it was proved that \( R(x) = (1/x) + 2 \sum_{n=1}^{\infty} \zeta(2n + 1)x^{2n} \) for \( x \in (0, 1/2] \), which is consistent with the first equality in (2.10). However, its proof given in \[16\] is quite complicated.

**3 Proof of Theorem 1.1**

(1) Since \( f(x) = R(x) - B(x) + x(1 - x)R(x) \), it follows from Lemma 2.2 \[1.6\] and \[1.8\] that

\[
\begin{align*}
f(x) &= \sum_{n=1}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n + 1) - \left[ 1 + (-1)^{n+1} \right] \eta(n + 1) \right] x^n \\
&\quad + (1 - x) \left\{ 1 + \sum_{n=1}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n + 1) x^{n+1} \right\} \\
&= 1 - x + \sum_{n=1}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n + 1) - \left[ 1 + (-1)^{n+1} \right] \eta(n + 1) \right] x^n \\
&\quad + \sum_{n=1}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n + 1) x^{n+1} - \sum_{n=1}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n + 1) x^{n+2} \\
&= 1 - \left[ 1 + 2 \eta(2) \right] x + 2 \zeta(3) x^2 + \sum_{n=3}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n + 1) - \left[ 1 + (-1)^{n+1} \right] \eta(n + 1) \right] x^n \\
&\quad + \sum_{n=2}^{\infty} \left[ 1 + (-1)^{n+1} \right] \zeta(n) x^n - \sum_{n=3}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n - 1) x^n \\
&= 1 - \left[ 1 + 2 \eta(2) \right] x + 2 \zeta(3) x^2 + \sum_{n=3}^{\infty} \left[ 1 + (-1)^n \right] \zeta(n + 1) + \left[ 1 + (-1)^{n+1} \right] \zeta(n) \\
&\quad - \left[ 1 + (-1)^{n+1} \right] \eta(n + 1) - \left[ 1 + (-1)^n \right] \zeta(n - 1) \right] x^n \\
&= 1 - \left[ 1 + \zeta(2) \right] x + 2 \zeta(3) x^2 + \sum_{n=3}^{\infty} \zeta(3) x^n = \sum_{n=0}^{\infty} a_n x^n,
\end{align*}
\]

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which yields the first power series expansion in (1.13).

Clearly, $1 + x(1 - x) = [5 - (1 - 2x^2)]/4$. It follows from (2.10), (2.11), (1.6), (1.7) and (1.9) that
\[
f(x) = \left[\frac{5}{4} - \frac{(1 - 2x)^2}{4}\right] \log 16 + 4 \sum_{n=1}^{\infty} \alpha(2n + 1)(1 - 2x)^{2n} - 4 \sum_{n=0}^{\infty} \beta(2n + 1)(1 - 2x)^{2n}
\]
\[
= b_0 - (1 - 2x)^2 \log 2 - \sum_{n=1}^{\infty} \alpha(2n + 1)(1 - 2x)^{2n+2} + \sum_{n=1}^{\infty} [5\alpha(2n + 1) - 4\beta(2n + 1)](1 - 2x)^{2n}
\]
\[
= b_0 + [5\alpha(3) - 4\beta(3) - \log 2](1 - 2x)^2 + \sum_{n=2}^{\infty} [5\alpha(2n + 1) - \alpha(2n - 1) - 4\beta(2n + 1)](1 - 2x)^{2n}
\]
\[
= b_0 + b_1(1 - 2x)^2 + \sum_{n=2}^{\infty} b_n(1 - 2x)^{2n} = \sum_{n=0}^{\infty} b_n(1 - 2x)^{2n},
\]
yielding the second series expansion in (1.13).

(2) Clearly, $f(0^+) = a_0 = 1$ and $f(1/2) = b_0$ by (1.13). By (1.13) and differentiation, we obtain
\[
f'(x) = -2 \sum_{k=1}^{\infty} (2k)b_k(1 - 2x)^{2k-1},
\]
(3.1)
\[
f''(x) = (-2)^2 \sum_{k=0}^{\infty} (2k + 2)(2k + 1)b_{k+1}(1 - 2x)^{2k},
\]
(3.2)
\[
f'''(x) = (-2)^3 \sum_{k=1}^{\infty} (2k + 2)(2k + 1)(2k)b_{k+1}(1 - 2x)^{2k-1}.
\]
(3.3)

Generally, by the mathematical induction, it is not difficult to obtain the following two expressions
\[
f^{(2n-1)}(x) = -2^{2n-1} \sum_{k=1}^{\infty} \frac{(2n + 2k - 2)!}{(2k - 1)!} b_{k+n-1}(1 - 2x)^{2k-1},
\]
(3.4)
\[
f^{(2n)}(x) = 2^n \sum_{k=0}^{\infty} \frac{(2n + 2k)!}{(2k)!} b_{n+k}(1 - 2x)^{2k},
\]
(3.5)
for $n \in \mathbb{N}$. By (3.3) and (3.5), $f^{(2n-1)}(1/2) = 0$ and $f^{(2n)}(1/2) = 4^n(2n)!b_n$ for $n \in \mathbb{N}$. The limiting value $f^{(n)}(0^+) = n!a_n$ follows from the first equality in (1.13). Hence it follows from (2.11), (3.4) and (3.5) that for all $n \geq 2$, $f^{(2n-1)}(x)$ is strictly decreasing and convex (increasing and concave) from $(0, 1/2]$ onto $[0, (2n - 1)!a_{2n-1})$ ((2n)!a_{2n}, 4^n(2n)!b_n), respectively.

It follows from (2.1) and (3.2) that $f''$ is strictly increasing and concave on $(0, 1/2]$, with $f''(0^+) = 2a_2 > 0$ by (1.13). Hence $f'$ is strictly increasing and convex on $(0, 1/2]$ with $f'(1/2) = 0$ by (3.1) and $f'(0^+) = a_1$ by (1.13), and $f$ is strictly decreasing and convex from $(0, 1/2)$ onto $[b_0, 1)$. Clearly, $F(0^+) = b_0 + b_1 - f(0^+) = b_0 + b_1 - 1 = 0.0142104 \cdots$ and $F(1/2) = b_0 - f(1/2) = 0$. It follows from (1.13) that $F$ has the following power series expansion
\[
F(x) = -\sum_{k=2}^{\infty} b_k(1 - 2x)^{2k}.
\]
(3.6)

Applying Lemma (2.14) and (3.6), one can easily see that $F$ is strictly completely monotonic on $(0, 1/2]$. Next, for $m, n, k \in \mathbb{N}$ and $x \in (0, 1/2)$,
\[
(-1)^m f_{1,n}^{(m)}(x) = (-1)^{m+n+1} f_{n+m}^{(m)}(x) = \begin{cases} -f^{(2k)}(x), & \text{if } m + n = 2k, \\ f^{(2k-1)}(x), & \text{if } m + n = 2k - 1. \end{cases}
\]
(3.7)

If $n \geq 2$, then $m + n \geq 3$ and $k \geq 2$. It follows from Lemma (2.14) and the monotonicity properties of $f^{(2k-1)}$ and $f^{(2k)}$, $k \geq 2$, that for $x \in (0, 1/2)$,
\[
-f^{(2k)}(x) > -f^{(2k)}(1/2) = -4^k(2k)!b_k > 0,
\]
\[
f^{(2k-1)}(x) > f^{(2k-1)}(1/2) = 0.
\]
Hence by (3.7), \((-1)^mF_{1,n}^{(m)}(x) > 0\) for each \(m \in \mathbb{N}\), \(n \in \mathbb{N} \setminus \{1\}\) and for all \(x \in (0, 1/2)\). This yields the strictly complete monotonicity of \(F_{1,n}\).

Clearly, \(F_{1,n}(0^+) = (-1)^{n+1} f^{(n)}(0^+) = (-1)^{n+1} n!a_n\). By (3.4) and (3.5), \(F_{1,n}(1/2) = 0\) if \(n\) is odd, and \(F_{1,n}(1/2) = 2^n n! b_{n/2}\) if \(n\) is even.

Finally, it follows from the monotonicity and convexity properties of \(f\) and \(F\) that

\[
\begin{align*}
\frac{b_0 + B(x) + (1 - b_0)(1 - 2x)}{1 + x(1 - x)} & \geq R(x) \geq \frac{b_0 + B(x)}{1 + x(1 - x)}, & (3.8) \\
\frac{b_0 + b_1(1 - 2x)^2 + B(x)}{1 + x(1 - x)} & \geq R(x) \geq \frac{b_0 + (1 - b_0 - b_1)(1 - 2x) + b_1(1 - 2x)^2 + B(x)}{1 + x(1 - x)}, & (3.9)
\end{align*}
\]

with equality in each instance if and only if \(x = 1/2\). Hence (1.14) and its equality case follow. \(\square\)

**Corollary 3.1.** Let \(a_n\), \(b_n\) and \(c_n\) be as in (1.8)-(1.10), and set \(d_n = \sum_{k=0}^n b_k \lambda (2n - 2k + 2) - (n + 1)b_{n+1}\) for \(n \in \mathbb{N} \cup \{0\}\). \(D_0 = 5d_0 - b_0\) and \(D_n = 5d_n - d_{n-1} - b_n\) for \(n \in \mathbb{N}\). Then we have the following conclusions:

1. The following identities hold

\[
\sum_{k=0}^\infty a_k = \sum_{k=0}^\infty b_k = 1, \quad (3.10)
\]

\[
\sum_{k=0}^\infty 2^{-k}d_k = b_0 = 5 \log 2 - \pi. \quad (3.11)
\]

2. For \(n \in \mathbb{N}\), the sequence \(\{c_n\}\) is strictly increasing. In particular, for \(n \in \mathbb{N} \setminus \{1\}\),

\[
-0.014210\cdots = c_1 < -0.000398\cdots = c_2 \leq c_n < \lim_{n \to \infty} c_n = 0, \quad (3.12)
\]

\[
1 < \sum_{k=0}^n b_k = 1 - c_n \leq 1 - c_2. \quad (3.13)
\]

3. \(d_0 = -0.290171\cdots, d_1 = 1.207861\cdots, d_2 = 1.08920\cdots, d_3 = 1.00824\cdots, \lim_{n \to \infty} d_n = 1, \) and

\[
d < d_n < \bar{d}_n < d_2 < d_3 \quad \text{for} \quad n \geq 3, \quad (3.14)
\]

where \(d = (1 - \pi^2/8)(b_0 + b_1) + \pi^2/8 = 0.996679\cdots\) and \(\bar{d} = d_3 - [(\pi^4/96) - 1]b_2 - [(\pi^2/8) - 1]b_3 = 1.001117\cdots\).

4. \(D_0 = [5\pi^2/8 - 1]b_0 - 5b_1 = -1.7750006\cdots, D_1 = 5d_1 - d_0 - b_1 = 5.639413\cdots, D_2 = 5d_2 - d_1 - b_2 = 3.850551\cdots, D_3 = 5d_3 - d_2 - b_3 = 3.995587\cdots, \) and

\[
D_n > 5d - \bar{d}_n = 3.982277\cdots, \quad n \geq 4. \quad (3.15)
\]

**Proof.** (1) Let \(f\) be as in Theorem 1.1. Then \(f(1-x) = f(x)\) by the symmetry. Hence \(f(1^-) = f(0^+) = a_0 = 1\), so that \(\sum_{k=0}^\infty a_k = \sum_{k=0}^\infty b_k = 1\). We obtain (3.11) by taking \(x = 1/2\) in (1.13).

By (1.10), \(\lim_{n \to \infty} c_n = 0\). Since \(c_{n+1} - c_n = -b_{n+1} > 0\) for \(n \in \mathbb{N}\) by Lemma 2.14 and (1.12), the monotonicity of \(c_n\) and (1.12) follow. (3.13) holds by (3.12).

(3) By (1.6), (1.7), (1.9) and [1], Table 23.3], we obtain

\[
\begin{align*}
d_0 = b_0 \lambda (2) - b_1 = (\pi^2/2b_0/8) - b_1 & = -0.290171\cdots, \\
d_1 = b_0 \lambda (4) + b_1 \lambda (2) - 2[\lambda (5) - \lambda (3) - 4\beta (5)] & = 1.207861\cdots, \\
d_2 = b_0 \lambda (6) + b_1 \lambda (4) + b_2 \lambda (2) - 3b_3 & = 1.08920\cdots, \\
d_3 = b_0 \lambda (8) + b_1 \lambda (6) + b_2 \lambda (4) + b_3 \lambda (2) - 4b_4 & = 1.00824\cdots.
\end{align*}
\]

Applying Lemma 2.11) and (4), one can easily show that \(\lim_{n \to \infty} \sum_{k=2}^n b_k \lambda (2n - 2k + 2) - 1 = 0\) by the definition of limit. Hence by Lemma 2.11) and (4),

\[
\lim_{n \to \infty} d_n = \lim_{n \to \infty} \left( b_0 \lambda (2n + 2) + b_1 \lambda (2n) + \sum_{k=2}^n b_k - (n + 1)b_{n+1} + \sum_{k=2}^n b_k \lambda (2n - 2k + 2) - 1 \right) = \sum_{k=0}^\infty b_k = 1.
\]
Since \( \lambda(m) > 1 \) and \( b_m < 0 \) for \( m \geq 2 \) by Lemma 2.1(1) and (4), it follows that for \( n \geq 3 \),

\[
d_n = b_0 \lambda(2n + 2) + b_1 \lambda(2n) + \sum_{k=2}^{n} b_k \lambda(2n - 2k + 2) - (n + 1)b_{n+1}
\]

\[
< \tilde{d}_n \equiv b_0 \lambda(2n + 2) + b_1 \lambda(2n) + \sum_{k=2}^{n} b_k - (n + 1)b_{n+1}
\]

\[
= b_0[\lambda(2n + 2) - 1] + b_1[\lambda(2n) - 1] + 1 - c_n - (n + 1)b_{n+1},
\]

which strictly decreasing in \( n \) by part (2) and Lemma 2.1(1) and (4). Hence for \( n \geq 3 \),

\[
\tilde{d}_n < \bar{d}_3 = \bar{d} = b_0[\lambda(8) - 1] + b_1[\lambda(6) - 1] + 1 - c_3 - 4b_4
\]

\[
= d_3 - [\lambda(4) - 1]b_2 - [\lambda(2) - 1]b_3 = 1.001117 \cdots < d_2.
\]

Hence the second and third inequalities in (3.14) hold.

On the other hand, by part (2), and by Lemma 2.1(1) and (4), we have

\[
d_n \geq b_0 \lambda(2n + 2) + b_1 \lambda(2n) + \lambda(2) \sum_{k=2}^{n} b_k - (n + 1)b_{n+1} = \bar{d}_n
\]

\[
\equiv b_0[\lambda(2n + 2) - \lambda(2)] + b_1[\lambda(2n) - \lambda(2)] + \lambda(2)(1 - c_n) - (n + 1)b_{n+1},
\]

which strictly decreasing in \( n \) by part (2) and Lemma 2.1(1) and (4), and hence

\[
d_n \geq \bar{d}_n > \lim_{n \to \infty} \bar{d}_n = \bar{d} \equiv b_0 \left(1 - \frac{\pi^2}{8}\right) + b_1 \left(1 - \frac{\pi^2}{8}\right) + \frac{\pi^2}{8} = 0.996679 \cdots.
\]

This yields the first inequality in (3.14).

(4) By computation, one can obtain the values of \( D_n \) for \( 0 \leq n \leq 3 \). It follows from (2.1) and (3.14) that

\[
D_n = 5d_n - d_{n-1} - b_n > 5d - \bar{d} = 3.982277 \cdots
\]

for \( n \geq 4 \), which yields the inequality (3.15). \( \Box \)

4 Proof of Theorem 1.2

(1) Clearly, \( f_0(0^+) = f'(0^+) = a_1 \) and \( f_0(1/2) = 2(b_0 - 1) \). Let \( h_0(x) = xf'(x) - [f(x) - a_0] \). Then

\[
h_0(0^+) = 0, \quad f'_0(x) = \frac{h_0(x)}{x^2}, \quad h'_0(x) \left[ \frac{d}{dx}(x^2) \right]^{-1} = \frac{1}{2} f''(x).
\]

By Theorem 1.1(2), \( f'' \) is strictly increasing on \((0, 1/2)\), and hence so is \( f'_0 \) by [2, Theorem 1.25]. Since \( f'_0(0^+) = f''(0^+)/2 = a_2 > 0 \) by l’Hôpital’s Rule, the monotonicity and convexity properties of \( f_0 \) follow.

Let \( h_1(x) = f(x) - R_0(x) \) and \( h_2(x) = x^{n+1} \). Then by (1.13),

\[
h_1(x) = \sum_{k=n+1}^{\infty} a_k x^k = x^{n+1} \sum_{k=0}^{\infty} a_{n+k+1} x^k, \quad h_2(x) = \frac{h_1(x)}{h_2(x)} = \frac{\sum_{k=0}^{\infty} a_{n+k+1} x^k}{\sum_{k=0}^{\infty} a_{n+k+1} x^k}, \quad (4.1)
\]

\[
h^{(m)}(0^+) = h^{(m)}(0) = 0 \text{ for } m \in \mathbb{N} \cup \{0\} \text{ with } 0 \leq m \leq n, \text{ and by differentiation,}
\]

\[
\frac{h^{(n+1)}_{1}(x)}{h^{(n+1)}_{2}(x)} = \frac{f^{(n+1)}(x)}{(n+1)!},
\]

so that \( f_n \) has the same monotonicity property as that of \( f^{(n+1)} \) by [2, Theorem 1.25]. Hence the monotonicity properties of \( f_{2n} \) and \( f_{2n-1} \) follow from Theorem 1.1(2).
Next, differentiation gives

\[ f_n'(x) = \frac{1}{x^{n+2}} \left\{ x \left[ f'(x) - \sum_{k=0}^{n} k a_k x^{k-1} \right] - (n + 1) \left[ f(x) - \sum_{k=0}^{n} a_k x^k \right] \right\} \]

\[ = \frac{1}{x^{n+2}} \left\{ x f'(x) - (n + 1) f(x) + \sum_{k=0}^{n} (n + 1 - k) a_k x^k \right\} = \frac{h(x)}{h_4(x)} \]  \hspace{1cm} (4.2)

where \( h_4(x) = x^{n+2} \) and

\[ h_3(x) = x f'(x) - (n + 1) f(x) + \sum_{k=0}^{n} (n + 1 - k) a_k x^k \]

\[ = \sum_{k=n+2}^{\infty} (k - n - 1) a_k x^k = x^{n+2} \sum_{k=0}^{\infty} (k + 1) a_{n+k+2} x^k \]

by (1.13). It is easy to verify that \( h_3^{(n+1)}(x) = x f^{(n+2)}(x) \). Hence \( h_3^{(m)}(0^+) = h_4^{(m)}(0) = 0 \) for \( m \in \mathbb{N} \cup \{0\} \) with \( 0 \leq m \leq n + 1 \), and

\[ \frac{h_3^{(n+1)}(x)}{h_4^{(n+1)}(x)} = \frac{f^{(n+2)}(x)}{(n + 2)!} \]  \hspace{1cm} (4.3)

which shows that \( f_n' \) has the same monotonicity property as that of \( f^{(n+2)} \) by [3, Theorem 1.25] and (4.2). Consequently, the convexity (concavity) of \( f_2 \) ( \( f_{2n+1} \), respectively) follows from Theorem (1.12).

The limiting value \( f_0(0^+) = d_{n+1} \) follows from (4.1). By the definition of \( f_n \), \( f_n(1/2) = A_n \). Hence the double inequality (1.15) and its equality case follow from the monotonicity and concavity properties of \( f_{2n-1} \) and (1.12). Taking \( n = 1 \) in (1.15), we obtain (1.16) and its equality case.

(2) Let \( h_5(x) = f(x) - S_n(x) = (1 - 2x)^{(2n+1)} \sum_{k=0}^{\infty} b_{n+k+1}(1 - 2x)^{2k} \) and \( h_6(x) = (1 - 2x)^{2(n+1)} \). Then \( h_5^{(1/2)} = h_6^{(m)}(1/2) = 0 \) for \( m \in \mathbb{N} \cup \{0\} \) with \( 0 \leq m \leq 2n + 1 \), and

\[ g_n(x) = \frac{h_5(x)}{h_6(x)} = \sum_{k=0}^{\infty} b_{n+k+1}(1 - 2x)^{2k}. \]  \hspace{1cm} (4.4)

By the definition of \( g_n \), \( g_n(0^+) = h_5(0^+) = c_n \), and by (4.2), \( g_n((1/2)^-) = b_{n+1} \). Hence it follows from (4.4) and Lemma (4.14) that \( g_n \) is strictly increasing and concave from \( (0, 1/2) \) onto \( (c_n, b_{n+1}) \).

Next, applying the method used to prove (4.4) and (4.5), we can obtain the following derivatives

\[ g_n^{(2m+1)}(x) = -2^{2m+1} \sum_{k=0}^{\infty} \frac{(2m + 2k)!}{(2k - 1)!} b_{n+m+k+1}(1 - 2x)^{2k-1}, \]  \hspace{1cm} (4.5)

\[ g_n^{(2m)}(x) = 2^{2m} \sum_{k=0}^{\infty} \frac{(2m + 2k)!}{(2k)!} b_{n+m+k+1}(1 - 2x)^{2k}. \]  \hspace{1cm} (4.6)

Hence by Lemma (4.14), \( g_n^{(2m)} \) ( \( g_n^{(2m+1)} \) ) is strictly increasing (decreasing, respectively) on \( (0, 1/2) \) for \( m \in \mathbb{N} \cup \{0\} \). This also yields the concavity (convexity) property of \( g_n^{(2m)} \) ( \( g_n^{(2m+1)} \), respectively).

The proof of the complete monotonicity property for \( G_{n,m} \) is similar to that for \( F_{1,n} \) in Theorem (1.12), and we omit the details. By (4.5) and (4.6), we obtain the following limiting values

\[ G_{n,2m+1}(0^+) = -2^{2m+1} \sum_{k=0}^{\infty} \frac{(2m + 2k)!}{(2k - 1)!} b_{n+m+k+1}, \] \hspace{0.5cm} \( G_{n,2m+1}((1/2)^-) = 0 \),

\[ G_{n,2m}(1/2^-) = -2^{2m} \sum_{k=0}^{\infty} \frac{(2m + 2k)!}{(2k)!} b_{n+m+k+1}, \] \hspace{0.5cm} \( G_{n,2m+1}(0^+) = -4^m(2m)!b_{n+m+1} \).

The double inequality (1.17) and its equality case follow from the monotonicity and concavity properties of \( g_n \). Taking \( n = 1 \) in (1.17), we obtain (1.18) and its equality case. \( \Box \)
Corollary 4.1. Let $a_n, b_n, c_n$ and $A_n$ are as in (1.8)–(1.10). Then we have the following conclusions:

1. For all $n \in \mathbb{N} \cup \{0\}$, $c_n < b_{n+1}$.
2. $A_{2n+1}$ ($A_{2n}$) is strictly increasing (decreasing, respectively) in $n \in \mathbb{N}$, and for all $n \in \mathbb{N}$,
   \[ a_{2n+2} < A_{2n+1} < \lim_{n \to \infty} A_n = 0 < a_{2n+1} + \frac{1}{2}a_{2n+2} < A_{2n} < a_{2n+1}. \]  
   \(4.7\)

3. $|a_n|$ is strictly decreasing in $n \in \mathbb{N} \setminus \{1\}$. In particular, for $n \in \mathbb{N} \setminus \{1\}$ and $\mu = 65/108 = 0.601851 \cdots$,
   \[ \lim_{n \to \infty} a_n = 0 < -a_{2n+2} < a_{2n+1} < -a_{2n} < \cdots < a_3 < a_2 < -a_1, \]  
   \(4.8\)

\[ \left(1 - \frac{1 - \mu}{2^{n-1} - \mu}\right)\zeta(n) < \zeta(n+1) < \frac{\zeta(n)+1}{2} < \zeta(n). \]  
   \(4.9\)

Proof. (1) The inequality in part (1) holds by the result for $g_n$ in Theorem 1.2.

(2) It follows from the monotonicity properties of $f_{2n+1}$ and $f_{2n}$ stated in Theorem 1.2 that
   \[ A_{2n} < a_{2n+1} \quad \text{and} \quad a_{2n+2} < A_{2n+1}, \quad n \in \mathbb{N}. \]  
   \(4.10\)

By (3.10), $\lim_{n \to \infty} a_n = 0$. From (1.12) and (4.10), we obtain
   \[ a_{2n+2} < A_{2n+1} = 2(A_{2n} - a_{2n+1}) < 0, \]  
   \(4.11\)

and hence $\lim_{n \to \infty} A_n = 0$ by Pinching Theorem for limits.

It follows from (1.4), (1.5) and (1.11) that
   \[ a_{2n+1} + \frac{1}{2}a_{2n+2} = \zeta(2n+3) + \zeta(2n+1) - 2\eta(2n+2) = \sum_{k=2}^{\infty} \frac{[k+(-1)^k]k}{2^{2n+3}} > 0. \]  
   \(4.13\)

Hence (4.7) holds by (4.10)–(4.13).

Next, by (1.12) and (4.11), we obtain
   \[ A_{2n+1} - A_{2n+3} = 2[(A_{2n} - a_{2n+1}) - (A_{2n+2} - a_{2n+3})] \]  
   \[ = 2[A_{2n} - 2(A_{2n+1} - a_{2n+2}) + a_{2n+3} - a_{2n+1}] \]  
   \[ = 2[2a_{2n+2} + a_{2n+3} - 3(A_{2n} - a_{2n+1})] \]  
   \[ < 2\left(2a_{2n+2} + a_{2n+3} - \frac{3}{2}a_{2n+2}\right) = a_{2n+2} + 2a_{2n+3}, \]  

and hence by (1.11), (1.4) and by (1.5),
   \[ A_{2n+1} - A_{2n+3} < 2[\zeta(2n+3) - \zeta(2n+1)] + 4[\zeta(2n+3) - \eta(2n+4)] \]  
   \[ = 2[3\zeta(2n+3) - \zeta(2n+1) - 2\eta(2n+4)] \]  
   \[ = -2\sum_{k=3}^{\infty} \frac{k^3 - 3k - 2(-1)^k}{k^{2n+4}} < -8\sum_{k=3}^{\infty} \frac{k+1}{k^{2n+4}} < 0 \]  
   \(4.14\)

since $k^3 - 3k \geq 4k + 6 \geq 4(k+1) + 2(-1)^k$ for $k \geq 3$. This yields the monotonicity of $A_{2n+1}$.

By (4.10) and (4.12), we have
   \[ A_{2n+2} - A_{2n} < a_{2n+3} - a_{2n+1} - \frac{1}{2}a_{2n+2}, \]  

and hence it follows from (1.4), (1.5) and (1.11) that
   \[ A_{2n+2} - A_{2n} < 2[\zeta(2n+3) - \eta(2n+4) - \zeta(2n+1) + \eta(2n+2)] - \zeta(2n+3) + \zeta(2n+1) \]  
   \[ = \zeta(2n+3) - \zeta(2n+1) + 2[\eta(2n+2) - \eta(2n+4)] \]  
   \[ = -\sum_{k=2}^{\infty} \frac{k^2 - 1}{k^{2n+4}} \left[k + 2(-1)^k\right] < 0. \]
This yields the monotonicity of $A_{2n}$.

(3) The limiting value of $a_n$ follows from (3.10). Computation gives: $a_4 = -0.395066 \cdots$, $a_3 = 0.510048 \cdots$, $a_2 = 2.404113 \cdots$ and $a_1 = -2.644934 \cdots$ Hence $-a_4 < a_3 < a_2 < -a_1$.

By (1.11), in order to prove the monotonicity of $|a_n|$, we need only to prove (4.8). By applying the Monotone l’Hôpital’s Rule [2, Theorem 1.25], one can easily verify that the function $P_1(x) \equiv [(x + 1)^8 - 1]/[x(x + 1)^7]$ is strictly decreasing from (0, 1/2) onto [6305/2187, 8), by which we see that

$$P_2(m) \equiv 2m + 1 - 2m \left(\frac{2m}{2m + 1}\right)^7 = \frac{[1 + (2m)^{-1}]^8 - 1}{(2m)^{-1}[1 + (2m)^{-1}]^7} = P_1\left(\frac{1}{2m}\right)$$

is strictly increasing in $m \in \mathbb{N}$ with $P_2(1) = \omega \equiv 6305/2187 = 2.882944 \cdots$ and $P_2(\infty) = 8$. Hence it follows from (1.11), (4.14) and (1.5) that for $n \in \mathbb{N} \setminus \{1\}$,

$$\frac{a_{2n+1} + a_{2n+2}}{2} = \zeta(2n + 3) - \eta(2n + 2) = \sum_{k=2}^{\infty} \frac{(-1)^k k + 1}{k^{2n+3}} = \frac{1}{2^{2n+3}} \left[3 - 2 \left(\frac{2n+3}{3}\right) + \frac{1}{4^{2n+3}} \left[5 - 4 \left(\frac{4}{5}\right)^{2n+3}\right] + \cdots\right] + \frac{1}{(2m)^{2n+3}} \left[(2m + 1) - 2m \left(\frac{2m}{2m + 1}\right)^{2n+3}\right] + \cdots$$

$$\geq \sum_{m=1}^{\infty} \frac{P_2(m)}{(2m)^{2n+3}} > \frac{\omega}{2^{2n+3}} \sum_{m=1}^{\infty} \frac{1}{m^{2n+3}} = \frac{\omega}{2^{2n+3}} \zeta(2n + 3), \quad (4.15)$$

and

$$\frac{a_{2n+1} + a_{2n}}{2} = 2\zeta(2n + 1) - \eta(2n + 2) - \zeta(2n - 1) = \sum_{k=1}^{\infty} \frac{2k - k^3 + (-1)^k}{k^{2n+2}}$$

$$< - \sum_{k=2}^{\infty} k^{-2n-2} \left(\frac{k^2}{2} + k\right) = 1 - \frac{\zeta(2n) + \zeta(2n+1)}{2} < 0 \quad (4.16)$$

since $k^3 - k^2/2 - 5k/2 \geq 1 \geq (-1)^k$ for $k \geq 2$. Hence it follows from (1.11), (4.15) and (4.16) that $0 < -a_{2n+2} < a_{2n+1} < -a_{2n}$, so that (4.8) holds.

Similarly to (4.15), we can easily obtain

$$\zeta(n + 1) - \eta(n) = \sum_{k=2}^{\infty} \frac{(-1)^k k + 1}{k^{n+1}} > \frac{1}{2n+1} \sum_{m=1}^{\infty} \frac{P_3(m)}{m^{n+1}}$$

$$> \frac{P_3(1)}{2n+1} \sum_{m=1}^{\infty} \frac{1}{m^{n+1}} = 2^{1-n} \mu \zeta(n + 1),$$

for $n \in \mathbb{N}$, where $P_3(m) = 2m + 1 - 2m[2m/(2m + 1)]^3$ with $P_3(1) = 65/27 = 4\mu$. Hence by (1.6),

$$\zeta(n + 1) > \frac{\eta(n)}{1 - 2^{1-n} \mu} = \frac{1 - 2^{1-n} \mu}{1 - 2^{1-n} \mu} \zeta(n) = \left(1 - \frac{1 - \mu}{2^{n-1} - \mu}\right) \zeta(n)$$

for $n \in \mathbb{N} \setminus \{1\}$. This yields the first inequality in (4.9). On the other hand, we have

$$\zeta(n + 1) - \zeta(n) = - \sum_{k=2}^{\infty} \frac{k - 1}{k^{n+1}} < - \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k^n} = 1 - \frac{\zeta(n)}{2}$$

for $n \in \mathbb{N} \setminus \{1\}$, which yields the second inequality in (4.9). The third inequality in (4.9) is clear.
5 Proof of Theorem 1.3

(1) Clearly, \( F_1(1/2) = (5 \log 2)/\pi \). By (2.10) and (2.11), it is easy to see that \( F_1(0^+) = 1 \).

Let \( f \) be as in Theorem 1.1. Then \( F_1(x) = [f(x)/B(x)] + 1 \). Differentiation gives

\[
\frac{B(x)F_1'(x)}{4(1 - 2x)} = H(x) \equiv \frac{f'(x) + f(x)H_1(x)}{4(1 - 2x)},
\]

where \( H_1(x) = \psi(1 - x) - \psi(x) \). Clearly, \( H_1(1/2) = 0 \) and \( H_1(0^+) = \infty \).

By [1, 6.3.7, 4.3.67 & 23.2.16] and (1.6), we have

\[
H_1(x) = \psi(1 - x) - \psi(x) = \pi \cot(\pi x) = \pi \tan\left(\pi \frac{1}{2} - x\right),
\]

\[
\tan z = 2 \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\right) 2^n \lambda(2n)z^{2n-1}, \quad |z| < \frac{\pi}{2}.
\]

It follows from (5.2) and (5.3) that

\[
H_1(x) = \pi \tan\left(\pi \frac{1}{2} - x\right) = -\sum_{k=1}^{\infty} 2^{2k+1} \lambda(2k) \left(x - \frac{1}{2}\right)^{2k-1} = 4 \sum_{k=1}^{\infty} \lambda(2k)(1 - 2x)^{2k-1}
\]

by which we obtain the following limiting values

\[
H_1^{(2n+1)}(1/2) = -2^{2n+3}(2n + 1)! \lambda(2n + 2) \quad \text{and} \quad H_1^{(2n)}(1/2) = 0, \quad n \in \mathbb{N} \cup \{0\}.
\]

Let \( H_2(x) = f(x)H_1(x) \). Then by differentiation and the Leibniz formula,

\[
H_2^{(n)}(x) = n! \sum_{k=0}^{n} C_n^k f^{(k)}(x) H_1^{(n-k)}(x),
\]

where \( C_n^k = n!/(n - k)!k! \). By Theorem 1.11, \( f^{(2n+1)}(1/2) = 0 \) and \( f^{(2n)}(1/2) = 4^n(2n)!b_n \) for \( n \in \mathbb{N} \cup \{0\} \).

Hence from (5.5) and (5.6), we obtain the following values

\[
H_2^{(2n)}\left(\frac{1}{2}\right) = \sum_{k=0}^{2n} C_{2n}^k f^{(k)}\left(\frac{1}{2}\right) H_1^{(2n-k)}\left(\frac{1}{2}\right)
\]

\[
= \sum_{m=1}^{n} C_{2m}^{2m-1} f^{(2m-1)}\left(\frac{1}{2}\right) H_1^{(2m-2m+1)}\left(\frac{1}{2}\right) + \sum_{m=0}^{n} C_{2m}^{2m} f^{(2m)}\left(\frac{1}{2}\right) H_1^{(2m-2m)}\left(\frac{1}{2}\right) = 0,
\]

\[
H_2^{(2n+1)}\left(\frac{1}{2}\right) = \sum_{k=0}^{2n+1} C_{2n+1}^k f^{(k)}\left(\frac{1}{2}\right) H_1^{(2n-k+1)}\left(\frac{1}{2}\right)
\]

\[
= \sum_{m=1}^{n+1} C_{2m+1}^{2m-1} f^{(2m-1)}\left(\frac{1}{2}\right) H_1^{(2m-2m+2)}\left(\frac{1}{2}\right) + \sum_{m=0}^{n} C_{2m+1}^{2m} f^{(2m)}\left(\frac{1}{2}\right) H_1^{(2m-2m+1)}\left(\frac{1}{2}\right)
\]

\[
= \sum_{m=0}^{n} C_{2m+1}^{2m+1} f^{(2m+1)}\left(\frac{1}{2}\right) H_1^{(2m-2m+1)}\left(\frac{1}{2}\right) = -(2n + 1)!2^{2n+3} \sum_{m=0}^{n} b_m \lambda(2n - 2m + 2)
\]

for \( n \in \mathbb{N} \cup \{0\} \). Therefore, \( H_2(x) \) has the following power series expansion

\[
H_2(x) = \sum_{n=0}^{\infty} \frac{H_2^{(2n)}(1/2)}{(2n + 1)!} \left(x - \frac{1}{2}\right)^{2n+1} = 4 \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} b_k \lambda(2n - 2k + 2) \right] (1 - 2x)^{2n+1}.
\]

Let \( d_n \) be as in Corollary 3.1. Then it follows from (1.13), (5.1) and (5.7) that

\[
H(x) = \frac{4 \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} b_k \lambda(2n - 2k + 2) \right] (1 - 2x)^{2n+1} - 4 \sum_{n=0}^{\infty} n b_n (1 - 2x)^{2n}}{4(1 - 2x)}
\]

\[
= \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} b_k \lambda(2n - 2k + 2) - (n + 1)b_{n+1} \right] (1 - 2x)^{2n} = \sum_{n=0}^{\infty} d_n (1 - 2x)^{2n}.
\]
Clearly, $H((1/2)^-) = d_0 = -0.290171 \cdots$, and $H(0^+) = [a_1 + a_0 H((0^+))]/4 = \infty$ by (5.1) and Thm. 1.1. It follows from Corollary 3.1 and (5.3) that $H$ is strictly decreasing and convex from $(0, 1/2)$ onto $(d_0, \infty)$. Hence $H$ has a unique zero $x_1 \in (0, 1/2)$ such that $H(x) > 0$ for $0 < x < x_1$, and $H(x) < 0$ for $x_1 < x < 1/2$, so that the piecewise monotonicity property of $F_1$ follows from (5.1).

By (1.3), (5.1) and (5.2), we have

$$4(1 - 2x)H(x) = (1 - 2x)R(x) + [1 + x(1 - x)]R'(x) + \frac{\pi^2 \cos(\pi x)}{\sin^2(\pi x)} \left( [1 + x(1 - x)]R(x) - \frac{\pi}{\sin(\pi x)} \right) H_1(x),$$

and hence

$$H\left( \frac{1}{4} \right) = \frac{1}{2} \left[ 3 \log 2 + \frac{19}{16} R'\left( \frac{1}{4} \right) + \pi^2 \sqrt{2} + \pi \left[ \frac{19}{16} R\left( \frac{1}{4} \right) - \pi \sqrt{2} \right] \right] = 0.095698 \cdots$$

by (2.12) and (2.13). This shows that $x_1 \in (1/4, 1/2)$.

Next, it follows from (5.2) that

$$B'(x) = -B(x)H_1(x) \quad \text{and} \quad H'_1(x) = -\frac{\pi^2}{\sin^2(\pi x)} = -B(x)^2. \quad (5.9)$$

By (5.1) and (5.8),

$$F'_1(x) = \frac{4(1 - 2x)H(x)}{B(x)} = \frac{4}{B(x)} \sum_{n=0}^{\infty} d_n(1 - 2x)^{2n+1},$$

and hence by (5.9) and differentiation,

$$F''_1(x) = \frac{4}{B(x)} \left[ H_1(x) \sum_{n=0}^{\infty} d_n(1 - 2x)^{2n+1} - 2 \sum_{n=0}^{\infty} (2n + 1)d_n(1 - 2x)^{2n} \right]. \quad (5.10)$$

It follows from (5.3), (5.10) and Corollary 3.1(3) that

$$F''_1(1/2) = -8d_0/\pi > 0. \quad (5.11)$$

On the other hand, by (5.1) and (5.2),

$$F'_1(x) = \frac{f'(x) + f(x)H_1(x)}{B(x)} = \frac{f'(x) \sin(\pi x) + \pi f(x) \cos(\pi x)}{\pi},$$

and by differentiation and (5.2), we obtain

$$F''_1(x) = \frac{1}{\pi} \left[ f''(x) \sin(\pi x) + 2\pi f'(x) \cos(\pi x) - \pi^2 f(x) \sin(\pi x) \right], \quad (5.12)$$

and hence we have

$$F''_1(0^+) = 2f'(0^+) = 2a_1 = -(2 + \pi^2/3) < 0. \quad (5.13)$$

By (5.11) and (5.13), we see that $F_1$ is neither convex nor concave on $(0, 1/2)$.

The double inequality (1.20) is clear. Since $F_1(0^+) = 1$, the coefficient $a = 1$ is best possible.

It follows from (2.12) and the piecewise monotonicity property of $F_1$ that

$$\delta = F_1(x_1) > F_1\left( \frac{1}{4} \right) = \frac{19R(1/4)}{16\pi \sqrt{2}} = \frac{19 \log 8}{8\pi \sqrt{2}} = 1.111592 \cdots,$$

and hence the first inequality in (1.21) holds. On the other hand, it follows from (1.18) and (1.3) that

$$R(x) \leq \frac{B(x)}{1 + x(1 - x)}[1 + h_8(x)] \quad (5.14)$$
for all $x \in (0, 1/2]$, where $h_8(x) = \frac{1}{\pi} S_2(x) \sin(\pi x)$ and $S_2(x) = b_0 + b_1(1 - 2x)^2 + b_2(1 - 2x)^4$. Let $h_{10}(x) = b_1 + 2b_2(1 - 2x)^2$ and $h_{11}(x) = (1 - 2x)\tan(\pi x)$. Then by differentiation,

$$h_8'(x)/\cos(\pi x) = h_9(x) \equiv S_2(x) - 4h_{10}(x)h_{11}(x)/\pi.$$  

(5.15)

Clearly, $h_9(0) = b_1 + b_2 > 0$ and $h_9(1/2^-) = b_0 - 8b_1/\pi^2 = -0.235204 \cdots$. Since $b_2 < 0$ and since $b_1 + 2b_2 = 0.662442 \cdots$, $h_{10}$ is strictly increasing from $[0, 1/2]$ onto $[b_1 + 2b_2, b_1]$ and

$$S_2'(x)/[4(1 - 2x)] = -h_{10}(x) < -(b_1 + 2b_2) < 0, 0 \leq x \leq 1/2,$$

so that $S_2$ is strictly decreasing from $[0, 1/2]$ onto $[b_0, b_0 + b_1 + b_2]$. It is easy to show that the function $h_{11}$ is strictly increasing from $[0, 1/2)$ onto $(0, 2/\pi)$. Hence by (5.15), $h_9$ is strictly decreasing from $[0, 1/2]$ onto $(b_0 - 8b_1/\pi^2, b_0 + b_1 + b_2)$, and $h_9$ has a unique zero $x_0 \in (0, 1/2)$ such that $h_9'(x) > 0$ for $x \in (0, x_0)$, and $h_9'(x) < 0$ for $x \in (x_0, 1/2]$. Computation gives:

$$h_9(0.276937) = 0.00000007895 \cdots, h_9(0.276938) = -0.00000137425 \cdots$$

showing that $x_0 \in (0.276937, 0.276938)$. Therefore

$$h_8(x) \leq h_8(x_0) < \frac{1}{\pi} S_2(0.276937) \sin(0.276938\pi) = 0.11214596 \cdots < 0.112146,$$

which yields the second inequality in (1.21) by (5.14).

(2) Clearly, $F_2(x) = f(x)/[1 + x(1 - x)]B(x)$. By differentiation,

$$\frac{[1 + x(1 - x)]^2}{1 - 2x} B(x)F_2'(x) = H_3(x) \equiv 4[1 + x(1 - x)]H(x) - f(x),$$  

(5.16)

where $H$ is as in (5.1). It follows from (1.12) and (5.8) that

$$H_3(x) = \left[5 - (1 - 2x)^2\right]H(x) - f(x)$$

(5.17)

$$= \sum_{n=0}^{\infty} 5d_n(1 - 2x)^{2n} - \sum_{n=0}^{\infty} d_n(1 - 2x)^{2(n+1)} = \sum_{n=0}^{\infty} b_n(1 - 2x)^{2n}$$

$$= \sum_{n=0}^{\infty} (5d_n - b_n)(1 - 2x)^{2n} - \sum_{n=1}^{\infty} d_{n-1}(1 - 2x)^{2n} = \sum_{n=0}^{\infty} D_n(1 - 2x)^{2n},$$  

(5.18)

where $D_n$ is as in Corollary 3.1. Clearly, $H_3(0^+) = 4H(0^+) - a_0 = \infty$ and $H_3(1/2) = D_0 < 0$. Hence by (5.18) and Corollary 3.1(4), it is clear that $H_3$ is strictly completely monotonic from $(0, 1/2)$ onto $(D_0, \infty)$. This shows that $H_3$ has a unique zero $x_3 \in (0, 1/2)$ such that $H_3(x) > 0$ for $x \in (0, x_3)$, and $H_3(x) < 0$ for $x \in (x_3, 1/2]$, and hence the piecewise monotonicity property of $F_2$ follows from (5.16).

Next, it follows from (1.3), (5.1), (5.16) and (5.18) that

$$F_2'(x) = \frac{[1 + x(1 - x)][f'(x) + f(x)H_3(x)] - (1 - 2x)f(x)}{[1 + x(1 - x)]^2 B(x)}$$

$$= \frac{[1 + x(1 - x)][f'(x) \sin(\pi x) + \pi f(x) \cos(\pi x)] - (1 - 2x)f(x) \sin(\pi x)}{\pi[1 + x(1 - x)]^2}$$

$$= \sum_{n=0}^{\infty} D_n(1 - 2x)^{2n+1}$$

$$= \frac{1}{[1 + x(1 - x)]^2 B(x)} \left(2 \sum_{n=0}^{\infty} (2n + 1)D_n(1 - 2x)^{2n} + \left[2 - \frac{1 - 2x}{1 + x(1 - x)} - H_1(x)\right] \sum_{n=0}^{\infty} D_n(1 - 2x)^{2n+1}\right)$$  

(5.19)

By (5.9) and the third equality in (5.19), and by differentiation,
from which we obtain

\[ F_2'(1/2) = -32D_0/(25\pi) = 0.723202 \cdots > 0. \] (5.20)

On the other hand, by the second equality in (5.19) and by differentiation,

\[ \pi F_2''(x) = \frac{f''(x) \sin(\pi x) + 2\pi f'(x) \cos(\pi x) - \pi^2 f(x) \sin(\pi x)}{1 + x(1 - x)} + \frac{2f(x) \sin(\pi x)}{[1 + x(1 - x)]^2} \]

\[ - \frac{2(1 - 2x)}{[1 + x(1 - x)]^2} \left\{ f'(x) \sin(\pi x) + \pi f(x) \cos(\pi x) - \frac{(1 - 2x)f(x) \sin(\pi x)}{1 + x(1 - x)} \right\}, \]

by which we have

\[ F_2''(0^+) = 2[f'(0^+) - f(0^+)] = 2(a_1 - a_0) = -(4 + \pi^2/3) < 0. \] (5.21)

The assertion on the convexity and concavity property of \( F_2 \) now follows from (5.20) and (5.21).

(3) Since \( F_3(x) = f(x)/[1 + x(1 - x)] \), the monotonicity property of \( F_3 \) follows from Theorem 1.1(2).

Clearly, \( F_3(1/2) = 4f(1/2)/5 = \log 16 - 4\pi/5 = \rho \), and \( F_3(0^+) = f(0^+) = a_0 = 1 \). For \( x \in (0, 1/2] \), let

\[ F_4(x) = [1 + x(1 - x)]^2f''(x) - 2(1 - 2x)[1 + x(1 - x)]f'(x) + 2[2 - 3x(1 - x)]f(x). \]

Clearly, the function \( x \mapsto 2 - 3x(1 - x) \) is strictly decreasing from \([0, 1/2] \) onto \([5/4, 2] \). By Theorem 1.1(2), \( f(x) > 0, f'(x) < 0 \) and \( f''(x) > 0 \) for all \( x \in (0, 1/2] \). Hence \( F_4(x) > 0 \) for \( x \in (0, 1/2] \). Differentiation gives

\[ [1 + x(1 - x)]^3 F_3''(x) = F_4(x) > 0 \]

for all \( x \in (0, 1/2] \), which yields the convexity of \( F_3 \).

The double inequality (1.22) and its equality case are clear. \( \square \)

**Remark 5.1.** We present the comparison of the bounds of \( R(x) \) given in Theorems 1.4 and 1.3 as follows.

1. The double inequality (1.14) is better than (1.22). In fact, it is clear that

\[ \frac{b_0 + (1 - 2x)P(x)}{1 + x(1 - x)} \geq \frac{b_0}{1 + x(1 - x)} \geq \frac{4}{5}b_0 = \rho \]

for \( x \in (0, 1/2] \), with equality if and only if \( x = 1/2 \). On the other hand, we have

\[ \frac{b_0 + (1 - 2x)Q(x) + B(x)}{1 + x(1 - x)} \leq \frac{b_0 + (1 - b_0)(1 - 2x) + B(x)}{1 + x(1 - x)}, \quad x \in (0, 1/2], \]

with equality if and only if \( x = 1/2 \). Put \( y = 1 - 2x \). Then \( y \in [0, 1], 1 + x(1 - x) = (5 - y^2)/4 \), and

\[ \frac{b_0 + B(x) + (1 - b_0)(1 - 2x)}{1 + x(1 - x)} \leq \rho + (1 - \rho)(1 - 2x) + \frac{B(x)}{1 + x(1 - x)} \leftrightarrow \]

\[ 4[b_0 + (1 - b_0)y] \leq (5 - y^2) [\rho + (1 - \rho)y] \leftrightarrow (1 - \rho)y^2 + \rho y - 1 \leq 0, \]

which is true since the function \( y \mapsto (1 - \rho)y^2 + \rho y - 1 \) is strictly increasing from \([0, 1] \) onto \([-1, 0] \).

2. It is clear that the lower bound of \( R(x) \) given in (1.14) is better than that given in (1.20).

3. Let \( F_5(x) = [b_0 + (1 - b_0)(1 - 2x)] \sin(\pi x) \) for \( x \in (0, 1/2] \). Then \( F_5(0) = 0, F_5(1/2) = b_0 \), and

\[ b_0 + B(x) + (1 - b_0)(1 - 2x) \leq \delta B(x) \quad \text{\( \Rightarrow \) \( F_5(x) \leq \pi(\delta - 1) \)}, \] (5.22)

\[ b_0 + B(x) + (1 - b_0)(1 - 2x) \geq \delta B(x) \quad \text{\( \Rightarrow \) \( F_5(x) \geq \pi(\delta - 1) \)}. \] (5.23)

Differentiation gives

\[ \frac{F_5'(x)}{\cos(\pi x)} = F_6(x) \equiv \pi[b_0 + (1 - b_0)(1 - 2x)] - 2(1 - b_0)\tan(\pi x), \]

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which is clearly strictly decreasing from \((0, 1/2)\) onto \((−\infty, \pi]\). Hence \(F_6\) has a unique zero \(x_5 \in (0, 1/2)\) such that \(F_5\) is strictly increasing on \((0, x_5]\), and decreasing on \([x_5, 1/2]\). By (1.21), \(b_0 < 0.35057 < \pi(\delta - 1) < 0.112146\pi = 0.35231704 \cdots < 0.3523171\). Since \(F_3(1/4) = (1 + b_0)/(2\sqrt{2}) = 0.468155 \cdots > 0.112146\pi > \pi(\delta - 1)\), there exist two numbers \(x_6, x_7 \in (0, 1/2)\) with \(x_6 < x_7\) such that

\[
F_5(x) = \begin{cases} < \pi(\delta - 1), & \text{if } x \in (0, x_6) \cup (x_7, 1/2], \\ = \pi(\delta - 1), & \text{if } x = x_6 \text{ or } x_7, \\ > \pi(\delta - 1), & \text{if } x \in (x_6, x_7). \end{cases}
\] (5.24)

Consequently, by (5.22)–(5.24), the first upper bound of \(R(x)\) given in (1.14) is better (worse) than that given in (1.20) for \(x \in (0, x_6) \cup [x_7, 1/2]\) (for \(x \in [x_6, x_7]\), respectively).

Next, let \(F_7(x) = \left[b_0 + b_1(1 - 2x)^2\right] \sin(\pi x)\). Then \(F_7(0) = 0, F_7(1/2) = b_0\), and

\[
b_0 + b_1(1 - 2x)^2 + B(x) \leq \delta B(x) \Leftrightarrow F_7(x) \leq \pi(\delta - 1),
\]

\[
b_0 + b_1(1 - 2x)^2 + B(x) \geq \delta B(x) \Leftrightarrow F_7(x) \geq \pi(\delta - 1).
\] (5.25) (5.26)

By differentiation,

\[
F_7'(x) = \frac{F_8(x) - \pi v_1}{\cos(\pi x)} = \pi \left[b_0 + b_1(1 - 2x)^2\right] - 4b_1h_1(x),
\]

where \(h_1\) is as in (5.15). Clearly, \(F_8\) is strictly decreasing on \((0, 1/2)\) with \(F_8(0) = \pi(b_0 + b_1) > 0\) and \(F_8(1/2)^-\) = \(\pi b_0 - 8b_1/\pi = -0.738915 \cdots\). Hence \(F_8\) has a unique zero \(x_8 \in (0, 1/2)\) such that \(F_7\) is strictly increasing on \((0, x_8]\), and decreasing on \([x_8, 1/2]\). Since \(F_7(0.28) = 0.352694 \cdots > 0.112146\pi > \pi(\delta - 1)\), there exist two numbers \(x_9, x_{10} \in (0, 1/2)\) with \(x_9 < x_{10}\) such that

\[
F_7(x) = \begin{cases} < \pi(\delta - 1), & \text{if } x \in (0, x_9) \cup (x_{10}, 1/2], \\ = \pi(\delta - 1), & \text{if } x = x_9 \text{ or } x_{10}, \\ > \pi(\delta - 1), & \text{if } x \in (x_9, x_{10}). \end{cases}
\] (5.27)

Hence by (5.25)–(5.27), the second upper bound of \(R(x)\) given in (1.14) is better (worse) than that given in (1.20) for \(x \in (0, x_9) \cup [x_{10}, 1/2]\) (for \(x \in [x_9, x_{10}]\), respectively).

(4) Let \(R_n\) and \(S_n\) be as in Theorem 1.2 and \(\delta\) as in Theorem 1.3. Then it follows from (1.14), (1.15), (1.17) and (1.20) that for all \(n \in \mathbb{N}\) and \(x \in (0, 1/2)\),

\[
\frac{B(x) + D_1(x)}{1 + x(1 - x)} \leq R(x) \leq \frac{B(x) + D_2(x)}{1 + x(1 - x)},
\] (5.28)

with equality in each instance if and only if \(x = 1/2\), where

\[
D_1(x) = \max \left\{ b_0 + (1 - 2x)P(x), R_{2n+2}(x) + A_{2n+2}x^{2n+3}, S_{n+1}(x) + c_{n+1}(1 - 2x)^{2n+3} \right\},
\]

\[
D_2(x) = \min \left\{ b_0 + (1 - 2x)Q(x), R_{2n+1}(x) + A_{2n+1}x^{2n+1}, S_{n+1}(x), (\delta - 1)B(x) \right\}.
\]

(5) Applying [2, Theorem 1.25] and Theorem 1.3, one can easily show that the function \(F_9(x) = [F_3(x) - 1]/x\) is strictly increasing from \((0, 1/2)\) onto \((-2 - \pi^2)/6, -2(1 - \rho))\), while \(F_{10}(x) = [F_3(x) - \rho]/(1 - 2x)\) is strictly decreasing from \((0, 1/2)\) onto \((0, 1 - \rho)\), where \(F_3\) is as in Theorem 1.3

Remark 5.2. Theorems 1.1–1.3 show that the function \(B(x)/\sin(\pi x) = \pi/[1 + x(1 - x)]\sin(\pi x)\) is a very good approximation of \(R(x)\).

Conjecture 5.3. Let \(F_3\) be as in Theorem 1.3. Our computation supports the following conjecture: \(F_3\) is strictly completely monotonic on \((0, 1/2)\).

If this conjecture is true, then we can obtain sharp lower and upper bounds for \(R(x)\), which are expressed in terms of \(B(x)/[1 + x(1 - x)]\) and polynomials.
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