Derivative Formula and Applications for
Degenerate Diffusion Semigroups *

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Abstract

By using the Malliavin calculus and solving a control problem, Bismut type
derivative formulae are established for a class of degenerate diffusion semigroups
with non-linear drifts. As applications, explicit gradient estimates and Harnack
inequalities are derived.

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tential equation.

1 Introduction

The Bismut derivative formula introduced in [4], also known as Bismut-Elworthy-Li for-
mula due to [6], is a powerful tool to derive regularity estimates on diffusion semigroups.
In the elliptic case this formula can be expressed by using the intrinsic curvature induced
by the generator. But in the degenerate case the required curvature lower bound is no
longer available. Of course, the Malliavin calculus works also for the hypoelliptic case as
shown in e.g. [1] on Riemannian manifolds. In this case the pull-back operator involved

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in the formula is normally less explicit, so that it is hard for one to derive explicit gradient estimates. Nevertheless, as shown in \[11 \text{ [6]}, \text{ in some concrete degenerate cases the derivative formula can be explicitly established by solving certain control problems.} \]

Recently, explicit derivative formulae for damping stochastic Hamiltonian systems have been established in \[16 \] and \[5 \] by using Malliavin calculus and coupling respectively, where the degenerate part is linear. In this case successful couplings with control can be constructed in a very explicit way, so that some known arguments developed in the elliptic setting can be applied. However, when the degenerate part is non-linear, the study becomes much more complicated. The main purpose of this paper is to extend results derived in \[16, 5 \] to the non-linear degenerate case.

Consider the following degenerate stochastic differential equation on \( \mathbb{R}^m \times \mathbb{R}^d \):

\[
\begin{cases}
\mathrm{d}X_t^{(1)} = Z^{(1)}(X_t^{(1)}, X_t^{(2)})\mathrm{d}t, \\
\mathrm{d}X_t^{(2)} = Z^{(2)}(X_t^{(1)}, X_t^{(2)})\mathrm{d}t + \sigma \mathrm{d}B_t,
\end{cases}
\]

where \( X_t^{(1)} \) and \( X_t^{(2)} \) take values in \( \mathbb{R}^m \) and \( \mathbb{R}^d \) respectively, \( \sigma \) is an invertible \( d \times d \)-matrix, \( B_t \) is a \( d \)-dimensional Brownian motion, \( Z^{(1)} \in C^2(\mathbb{R}^{m+d}, \mathbb{R}^m) \) and \( Z^{(2)} \in C^1(\mathbb{R}^{m+d}, \mathbb{R}^d) \). Let \( X_t = (X_t^{(1)}, X_t^{(2)}) \), \( Z = (Z^{(1)}, Z^{(2)}) \). Then the equation can be formulated as

\[
\mathrm{d}X_t = Z(X_t)\mathrm{d}t + (0, \sigma \mathrm{d}B_t).
\]

We assume that the solution is non-explosive, which is ensured by (H1) below. Our purpose is to establish an explicit derivative formula for the associated Markov semigroup \( P_t \):

\[
P_t f(x) = \mathbb{E} f(X_t(x)), \quad t > 0, x \in \mathbb{R}^{m+d}, f \in \mathcal{B}_b(\mathbb{R}^{m+d}),
\]

where \( X_t(x) \) is the solution of (1.2) with \( X_0 = x \), and \( \mathcal{B}_b(\mathbb{R}^{m+d}) \) is the set of all bounded measurable functions on \( \mathbb{R}^{m+d} \).

When \( m = d, \sigma = I_{d \times d} \) and

\[
Z^{(1)}(x, y) = \nabla H(x, \cdot)(y), \quad Z^{(2)}(x, y) = -\nabla H(\cdot, y)(x) - F(x, y)\nabla H(x, \cdot)(y)
\]

for some functions \( H \) and \( F \), (1.1) goes back to the stochastic Hamiltonian system

\[
\begin{cases}
\mathrm{d}X_t = \nabla H(X_t, \cdot)(Y_t)\mathrm{d}t, \\
\mathrm{d}Y_t = -\{\nabla H(\cdot, Y_t)(X_t) + F(X_t, Y_t)\nabla H(X_t, \cdot)(Y_t)\}\mathrm{d}t + \mathrm{d}B_t
\end{cases}
\]

with Hamiltonian function \( H \). See e.g. \[10 \] for the physical background and applications in mechanics of the model, and see \[11 \] for exponential convergence of the system to the invariant probability measure. In particular, if \( H(x, y) = V(x) + \frac{1}{2}|y|^2 \) and \( F \equiv c \) for some constant \( c \), (1.3) is associated to the “kinetic Fokker-Planck equation” in PDE, see e.g. \[12 \] where the hypocoercivity and related regularization estimates w.r.t. the invariant
probability measure are studied; and is known as “stochastic damping Hamiltonian system” in probability theory, see e.g. [8, 15] where some long time behaviors of the system have been investigated.

Following the line of two recent papers [16, 5] where Bismut formula and Harnack inequalities are derived for $P_t$ associated to $L$ with $Z^{(1)}(x, y) = Ay$ for some $m \times d$-matrix $A$, we aim to derive explicit point-wise derivative estimates of $P_t$ for more general settings where $Z^{(1)}(x, y)$ might be non-linear and depend on both variables $x$ and $y$, so that some typical examples for the physical model [13] are covered (see Example 4.1 below).

To compare the present equation with those investigated in [16, 5] where $Z^{(1)}$ is linear, let us recall some simple notations. Firstly, we write the gradient operator on $\mathbb{R}^{m+d}$ as $\nabla = (\nabla^{(1)}, \nabla^{(2)})$, where $\nabla^{(1)}$ and $\nabla^{(2)}$ stand for the gradient operators for the first and the second components respectively, so that $\nabla f : \mathbb{R}^{m+d} \to \mathbb{R}^{m+d}$ for a differentiable function $f$ on $\mathbb{R}^{m+d}$. Next, for a smooth function $\xi = (\xi_1, \ldots, \xi_k) : \mathbb{R}^{m+d} \to \mathbb{R}^k$, let

$$\nabla \xi = \begin{pmatrix} \nabla \xi_1 \\ \vdots \\ \nabla \xi_k \end{pmatrix}, \quad \nabla^{(i)} \xi = \begin{pmatrix} \nabla^{(i)} \xi_1 \\ \vdots \\ \nabla^{(i)} \xi_k \end{pmatrix}, \quad i = 1, 2.$$ 

Then $\nabla \xi, \nabla^{(1)} \xi, \nabla^{(2)} \xi$ are matrix-valued functions of orders $k \times (m + d), k \times m, k \times d$ respectively. Moreover, for an $l \times k$-matrix $M = (M_{ij})_{1 \leq i \leq l, 1 \leq j \leq k}$ and $v = (v_i)_{1 \leq i \leq k} \in \mathbb{R}^k$, let $Mv \in \mathbb{R}^l$ with $(Mv)_i = \sum_{j=1}^k M_{ij}v_j, \ 1 \leq i \leq l$. Finally, we will use $\| \cdot \|$ to denote the operator norm for linear operators, for instance, $\|M\| = \sup_{\|v\| = 1} |Mv|$.

When $Z^{(1)}(x^{(1)}, x^{(2)})$ depends only on $x^{(2)}$ and $\nabla^{(2)} Z^{(1)}$ is a constant matrix with rank $m$, then equation [1.1] reduces back to the one studied in [5] (and also in [16] for $m = d$). In this case we are able to construct very explicit successful couplings with control, which imply the desired derivative formula and Harnack inequalities as in the elliptic case. But when $Z^{(1)}$ is non-linear, it seems very hard to construct such couplings. The idea of this paper is to split $Z^{(1)}$ into a linear term and a non-linear term, and to derive an explicit derivative formula by controlling the non-linear part using the linear part in a reasonable way. More precisely, let

$$\nabla^{(2)} Z^{(1)} = B_0 + B,$$

where $B_0$ is a constant $m \times d$-matrix. We will be able to establish derivative formulae for $P_t$ provided $B$ is dominated by $B_0$ in the sense that

$$\langle BB_0^* a, a \rangle \geq -\varepsilon |B_0^* a|^2, \ \forall a \in \mathbb{R}^m$$

holds for some constant $\varepsilon \in [0, 1]$.

To state our main result, we first briefly recall the integration by parts formula for the Brownian motion. Let $T > 0$ be fixed. For an Hilbert space $H$, let

$$\mathbb{H}(H) = \left\{ h \in C([0, T]; H) : \ h_0 = 0, \|h\|^2_{\mathbb{H}(H)} := \int_0^T |h_1|^2_H dt < \infty \right\}.$$
be the Cameron-Martin space over $H$. Let $\mathbb{H} = \mathbb{H}(\mathbb{R}^d)$ and, without confusion in the context, simply denote $\| \cdot \|_{\mathbb{H}} = \| \cdot \|_{L^2(\mathbb{H})}$ for any Hilbert space $H$.

Let $\mu$ be the distribution of $\{B_t\}_{t \in [0, T]}$, which is a probability measure (i.e. Wiener measure) on the path space $\Omega = C([0, T]; \mathbb{R}^d)$. The probability space $(\Omega, \mu)$ is endowed with the natural filtration of the coordinate process $B_t(w) := w_t, t \in [0, T]$. A function $F \in L^2(\Omega; \mu)$ is called differentiable if for any $h \in \mathbb{H}$, the directional derivative

$$D_hF := \lim_{\varepsilon \to 0} \frac{F(\cdot + \varepsilon h) - F(\cdot)}{\varepsilon}$$

exists in $L^2(\Omega; \mu)$. If the map $\mathbb{H} \ni h \mapsto D_hF \in L^2(\Omega; \mu)$ is bounded, then there exists a unique $DF \in L^2(\Omega \to \mathbb{H}; \mu)$ such that $\langle DF, h \rangle_{\mathbb{H}} = D_hF$ holds in $L^2(\Omega; \mu)$ for all $h \in \mathbb{H}$. In this case we write $F \in \mathcal{D}(D)$ and call $DF$ the Malliavin gradient of $F$. It is well known that $(D, \mathcal{D}(D))$ is a closed operator in $L^2(\Omega; \mu)$, whose adjoint operator $(\delta, \mathcal{D}(\delta))$ is called the divergence operator. That is,

$$E(D_hF) = \int_\Omega D_hFd\mu = \int_\Omega F\delta(h)d\mu = E(F\delta(h)), \quad F \in \mathcal{D}(D), h \in \mathcal{D}(\delta).$$

For any $s \geq 0$, let $\{K(t, s)\}_{t \geq s}$ solve the following random ODE on $\mathbb{R}^m \otimes \mathbb{R}^m$:

$$\frac{d}{dt}K(t, s) = (\nabla^{(1)}Z^{(1)})(X_t)K(t, s), \quad K(s, s) = I_{m \times m}.$$  

We assume

(H) The matrix $\sigma \in \mathbb{R}^d \otimes \mathbb{R}^d$ is invertible, and there exists $W \in C^2(\mathbb{R}^{m+d})$ with $W \geq 1$ and $\lim_{|x| \to \infty} W(x) = \infty$ such that for some constants $C, l_2 \geq 0$ and $l_1 \in [0, 1]$,

(H1) $LW \leq CW, \quad |\nabla^{(2)}W|^2 \leq CW$, where $L = \frac{1}{2} \text{Tr}(\sigma \sigma^* \nabla^{(2)} \nabla^{(2)}) + Z \cdot \nabla$;

(H2) $\|\nabla Z\| \leq CW^{l_1}, \quad \|\nabla^2 Z\| \leq CW^{l_2}.$

For any $v = (v^{(1)}, v^{(2)}) \in \mathbb{R}^{m+d}$ with $|v| = 1$, we aim to search for $h = h(v) \in \mathcal{D}(\delta)$ such that

$$\nabla_v Pf(x) = E[f(X_T(x))\delta(h)], \quad f \in C^1_0(\mathbb{R}^{m+d})$$

holds. To construct $h$, for an $\mathbb{H}$-valued random variable $\alpha = (\alpha_s)_{s \in [0, T]}$, let

$$g_t = K(t, 0)v^{(1)} + \int_0^t K(t, s)\nabla^{(2)}Z^{(1)}(X_s(x))\alpha_s ds,$$

$$h_t = \int_0^t \sigma^{-1}(\nabla Z^{(2)}(X_s(x))(g_s, \alpha_s) - \dot{\alpha}_s) ds, \quad t \in [0, T].$$
We will show that $h$ satisfies (1.7) provided it is in $\mathcal{D}(\delta)$ and $a_0 = v^{(2)}$, $a_T = 0$, $g_T = 0$, see Theorem 2.1 below for details. In particular, it is the case for $\alpha_s$ given in the following result.

**Theorem 1.1.** Assume (H) and let $\nabla^{(2)} Z^{(1)} = B_0 + B$ for some constant matrix $B_0$ such that (1.4) holds for some constant $\varepsilon \in [0, 1)$. If there exist an increasing function $\xi \in C([0, T])$ and $\phi \in C^1([0, T])$ with $\xi(t) > 0$ for $t \in (0, T]$, $\phi(0) = \phi(T) = 0$ and $\phi(t) > 0$ for $t \in (0, T)$ such that

\[
(1.9) \quad \int_0^t \phi(s)K(T, s)B_0B_0^*K(T, s)^*\,ds \geq \xi(t)I_{m \times m}, \quad t \in (0, T].
\]

Then

1. $Q_t := \int_0^t \phi(s)K(T, s)\nabla^{(2)} Z^{(1)}(X_s)B_0^*K(T, s)^*\,ds$ is invertible for $t \in (0, T]$ with

\[
(1.10) \quad \|Q_t^{-1}\| \leq \frac{1}{(1 - \varepsilon)\xi(t)}, \quad t \in [0, T].
\]

2. Let $h$ be determined by (1.8) for

\[
(1.11) \quad \alpha_t := \frac{T - t}{T}v^{(2)} - \phi(t)B_0^*K(T, t)^*Q_t^{-1}\int_0^T \frac{T - s}{T}K(T, s)\nabla^{(2)} Z^{(1)}(X_s)v^{(2)}\,ds
\]

Then for any $p \geq 2$, there exists a constant $T_p \in (0, \infty)$ if $l_1 = 1$ and $T_p = \infty$ if $l_1 < 1$, such that for any $T \in (0, T_p)$, (1.7) holds with $\mathbb{E}|\delta(h)|^p < \infty$.

3. For any $p > 1$ there exist constants $c_1(p), c_2(p) \geq 0$, where $c_2(p) = 0$ if $l_1 = l_2 = 0$, such that

\[
(1.12) \quad |\nabla P_T f| \leq c_1(p)(P_T|f|^p)^{1/p} \frac{\sqrt{T \wedge 1}\{(T \wedge 1)^2 + \xi(T \wedge 1)\}e^{c_2(p)W}}{\int_0^{T \wedge 1} \xi(s)^2\,ds
\]

holds for all $T > 0$ and $f \in \mathcal{B}_b(\mathbb{R}^{m+d})$.

The remainder of the paper is organized as follows. In Section 2 we present a general result on the derivative formula by using Malliavin calculus, from which we are able to prove Theorem 3.1 in Section 3. In Section 4 we will verify (1.9) for the following two cases respectively:

(I) $\nabla^{(1)} Z^{(1)}$ is non-constant but $\text{Rank}[B_0] = m$.

(II) $A := \nabla^{(1)} Z^{(1)}$ is constant such that $\text{Rank}[B_0, AB_0, \ldots, A^kB_0] = m$ holds for some $0 \leq k \leq m - 1$. 

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In both cases the $L^p$-gradient estimate (1.12) is derived with specific $\xi$, while in Case (II) the Harnack inequality introduced in [13] is established provided $\nabla Z(1)$ is constant, which extends the corresponding Harnack inequality obtained in [5] for $\nabla Z(1) = 0$ and $\nabla Z(1)$ is constant with rank $m$. This type of Harnack inequality has been applied in the study of heat kernel estimates and contractivity properties of Markov semigroups, see e.g. [5] and references therein.

2 A General Result

In this section we will make use of the following assumption.

($H'$) The function

$$U(x) := \mathbb{E} \exp \left[ 2 \int_0^T \|\nabla Z(X_t(x))\| dt \right], \quad x \in \mathbb{R}^{m+d}$$

is locally bounded.

**Theorem 2.1.** Assume ($H'$) for some $T > 0$. For $v = (v^{(1)}, v^{(2)}) \in \mathbb{H}^{m+d}$, let $(\alpha_s)_{0 \leq s \leq T}$ be an $\mathbb{H}$-valued random variable such that $\alpha_0 = v^{(2)}$ and $\alpha_T = 0$, and let $g_t$ and $h_t$ be given in (1.8). If $g_T = 0$ and $h_t \in \mathcal{D}(\delta)$, then (1.7) holds.

**Proof.** For simplicity, we will drop the initial data of the solution by writing $X_t(x) = X_t$. By ($H'$) and (1.2) we have $X_t \in \mathcal{D}(D)$, and due to the chain rule and the definition of $h_t$,

$$D_hX_t = \int_0^t \nabla Z(X_s)D_hX_s ds + \int_0^t (0, \sigma h_s) ds$$

$$= (0, v^{(2) - \alpha_t}) + \int_0^t \nabla Z(X_s)D_hX_s ds + \int_0^t (0, \nabla Z^{(2)}(X_s)(g_s, \alpha_s)) ds$$

holds for $t \in [0, T]$. Next, it is easy to see that

$$g_t = v^{(1)} + \int_0^t \nabla Z^{(1)}(X_s)(g_s, \alpha_s) ds, \quad t \in [0, T].$$

Combining this with (2.1) we obtain

$$D_hX_t + (g_t, \alpha_t) = v + \int_0^t \nabla Z(X_s)\{D_hX_s + (g_s, \alpha_s)\} ds, \quad t \in [0, T].$$

On the other hand, the directional derivative process

$$\nabla_v X_t := \lim_{\varepsilon \to 0} \frac{X_t(x + \varepsilon v) - X_t(x)}{\varepsilon}$$

$$= \mathbb{E} \exp \left[ 2 \int_0^T \|\nabla Z(X_t(x))\| dt \right], \quad x \in \mathbb{R}^{m+d}$$

is locally bounded.
satisfies the same equation, i.e.

\[(2.2) \quad \nabla v X_t = v + \int_0^t \nabla Z(X_s) \nabla v X_s \, ds, \quad t \in [0, T].\]

Thus, by the uniqueness of the ODE we conclude that

\[D_h X_t + (g_t, \alpha_t) = \nabla v X_t, \quad t \in [0, T].\]

In particular, since \((g_T, \alpha_T) = 0\), we have

\[(2.3) \quad D_h X_T = \nabla v X_T\]

and due to \((H')\) and \((2.2)\),

\[(2.4) \quad \mathbb{E}|D_h X_T|^2 = \mathbb{E}|
abla v X_T|^2 \leq |v|^2 \mathbb{E} \left[ 2 \int_0^T \|\nabla Z\| (X_s) \, ds \right].\]

Combining this with \((1.5)\) and letting \(f \in C^1_0(\mathbb{R}^{m+d})\), we are able to adopt the dominated convergence theorem to obtain

\[\nabla v P_T f = \mathbb{E} \langle \nabla f(X_T), \nabla v X_T \rangle = \mathbb{E} \langle \nabla f(X_T), D_h X_T \rangle = \mathbb{E} D_h f(X_T) = \mathbb{E}[f(X_T) \delta(h)].\]

\[\square\]

**Remark 2.1.** Using the same argument as above, we also have the following derivative formula:

\[(2.5) \quad \mathbb{E} \nabla_v f(X_T) = \mathbb{E} \left( f(X_T) \sum_{i,k} \left[ \delta(h(e_k)) (\nabla X_T)^{-1}_{ki} - D_h(e_k) (\nabla X_T)^{-1}_{ki} \right] v^i \right),\]

where \((e_j)\) is the canonical basis of \(\mathbb{R}^{m+d}\), and \(h(e_j)\) is defined by \((1.8)\) with \(v = e_j\). In fact, since

\[\sum_k (\partial_k X^j_T) (\nabla X_T)^{-1}_{ki} = 1_{i=j}\]

and by \((2.3)\)

\[D_h(e_k) X^j_T = \nabla v X^j_T = \partial_k X^j_T,\]

we have

\[\nabla_v f(X_T) = \sum_i (\partial_v f)(X_T) v^i = \sum_{i,j,k} (\partial_j f)(X_T) (\partial_k X^j_T) (\nabla X_T)^{-1}_{ki} v^i = \sum_{i,j,k} (\partial_j f)(X_T) (D_h(e_k) X^j_T) (\nabla X_T)^{-1}_{ki} v^i = \sum_{i,k} \{D_h(e_k) f(X_T)\} (\nabla X_T)^{-1}_{ki} v^i,\]

which implies \((2.5)\) by the integration by parts formula.
Remark 2.2. For the higher order derivative formula, under further regularity assumptions, for any \( v_1, \cdots, v_j \in \mathbb{R}^{m+d} \) and \( f \in C^4_b(\mathbb{R}^{m+d}) \), we have

\[
\langle \nabla^j \mathbb{E} f(X_T(x)), v_1 \otimes \cdots \otimes v_j \rangle = \mathbb{E} \left[ f(X_T(x)) J_j(T, v_1, \cdots, v_j) \right],
\]

where \( J_1(v) := \delta(h(v)) \) and

\[
J_j(v_1, \cdots, v_j) := J_{j-1}(v_1, \cdots, v_{j-1}) \delta(h(v_j)) + \nabla v_j J_{j-1}(v_1, \cdots, v_{j-1})
\]

- \( D_{\delta(h(v_j))} J_{j-1}(v_1, \cdots, v_{j-1}) \),

where \( h(v) \) is defined by (L8). In fact, as in the proof of Theorem 2.1 we have

\[
\langle \nabla^2 \mathbb{E} f(X_T), v_1 \otimes v_2 \rangle = \nabla v_2 \nabla v_1 \mathbb{E} f(X_T) = \nabla v_2 \mathbb{E} [f(X_T) \delta(h(v_1))] \\
= \mathbb{E} [(\nabla f)(X_T) \cdot \nabla v_2 X_T \cdot \delta(h(v_1))] + \mathbb{E} [f(X_T) \nabla v_2 \delta(h(v_1))]
\]

- \( \mathbb{E} [D_{\delta(h(v_1))} f(X_T)] \delta(h(v_1)) + \mathbb{E} [f(X_T(x)) \nabla v_2 \delta(h(v_1))] \)

The higher derivatives can be obtained by induction.

3 Proof of Theorem 1.1

The idea of the proof is to apply Theorem 2.1 for the given process \( \alpha_s \). Obviously, (H1) implies that for any \( l \geq 1 \), there exists a constant \( C_l \) such that \( LW_t^l \leq C_l W_t^l \), so that \( \mathbb{E} W(X_t(x))^l \leq e^{C_l t} W(x)^l \) and thus, the process is non-explosive; while (H2) imply that \( \|\nabla Z\| + \|\nabla^2 Z\| \leq CW_t^{l_1} v_2 \) holds for some \( C > 0 \), so that

\[
\mathbb{E} \left( (\|\nabla Z\|^p + \|\nabla^2 Z\|^p)(X_t) \right) \leq e^{c(p) l_1} W^{p(l_1+1)} , \ t \geq 0
\]

holds for any \( p \geq 1 \) with some constant \( c(p) > 0 \). The following lemma ensures that (H) implies (H') for all \( T > 0 \) if \( l_1 < 1 \) and for small \( T > 0 \) if \( l_1 = 1 \).

Lemma 3.1. If (H1) holds, then for any \( T > 0 \),

\[
\mathbb{E} \exp \left[ \frac{2}{T^2 C^4 + 2 C^2 T} \int_0^T W(X_t) \sigma^2 \, dt \right] \leq \exp \left[ \frac{2W}{TC^4 + 2 C^2 T} \right].
\]

Consequently, (H2) imply that \( U := \mathbb{E} \exp[2 \int_0^T \|\nabla Z\|(X_t) \sigma^2 \, dt] \) is locally bounded on \( \mathbb{R}^{m+d} \) if either \( l_1 < 1 \) or \( l_1 = 1 \) but \( T^2 C^2 \|\sigma^2\|^2 e^{4+2 C T} \leq 1 \).

Proof. It suffices to prove the first assertion. By the Itô formula and (H1), we have

\[
dW(X_t) = \langle \nabla^{(2)} W(X_t), \sigma dB_t \rangle + LW(X_t) \sigma dB_t \leq \langle \nabla^{(2)} W(X_t), \sigma dB_t \rangle + CW(X_t)dt.
\]
So, for \( t \in [0, T] \),
\[
d\{e^{-(C+2/T)t}W(X_t)\} \leq e^{-(C+2/T)t} \langle \nabla^{(2)} W(X_t), \sigma dB_t \rangle - \frac{2}{T} e^{-CT-2} W(X_t)dt.
\]
Thus, letting \( \tau_n = \inf\{t \geq 0 : W(X_t) \geq n\} \), for any \( n \geq 1 \) and \( \lambda > 0 \) we have
\[
\mathbb{E}\exp\left[\frac{2\lambda}{Te^{CT+2}} \int_0^{T \land \tau_n} W(X_t)dt\right] \\
\leq e^{AW} \mathbb{E}\exp\left[\lambda \int_0^{T \land \tau_n} e^{-(C+2/T)t} \langle \nabla^{(2)} W(X_t), \sigma dB_t \rangle\right] \\
\leq e^{AW} \left(\mathbb{E}\exp\left[2\lambda^2 C \|\sigma\|^2 \int_0^{T \land \tau_n} W(X_t)dt\right]\right)^{1/2},
\]
where the second inequality is due to the exponential martingale and (H1). By taking
\[
\lambda = \frac{1}{TC\|\sigma\|^2 e^{CT+2}},
\]
we arrive at
\[
\mathbb{E}\exp\left[\frac{2}{T^2 C \|\sigma\|^2 e^{4+2CT}} \int_0^{T \land \tau_n} W(X_t)dt\right] \leq \exp\left[\frac{2W}{TC\|\sigma\|^2 e^{2+CT}}\right].
\]
This completes the proof by letting \( n \to \infty \).

To ensure that \( \mathbb{E}|\delta(h)|^p < \infty \), we need the following two lemmas.

**Lemma 3.2.** Assume (H). Then there exists a constant \( c > 0 \) such that
\[
\|DX_t\|_H \leq \sqrt{t}\|\sigma\|e^{C\int_0^t W^{(1)}(X_s)ds}, \ t \geq 0.
\]
Consequently, if \( l_1 < 1 \), then for any \( p \geq 1 \),
\[
\mathbb{E}\left(\sup_{t \in [0, T]} \|DX_t\|_H^p\right) < \infty, \ T \geq 0;
\]
and if \( l_1 = 1 \), then for any \( p \geq 1 \) there exists a constant \( T_p > 0 \) such that
\[
\mathbb{E}\left(\sup_{t \in [0, T]} \|DX_t\|_H^p\right) < \infty, \ T \in (0, T_p).
\]

**Proof.** Due to Lemma 3.1 it suffices to prove (3.2). From (1.2) we see that for any \( h \in H \),
\( D_hX_t \) solves the following random ODE:
\[
D_hX_t = \int_0^t \langle \nabla Z(X_s)D_hX_sds + (0, \sigma h(t)).
\]
Combining this with (H2) and $|h(t)| \leq \sqrt{t} \|h\|_{\mathbb{H}}$, we obtain

$$|D_hX_t| \leq C \int_0^t W^{l_1}(X_s)|D_hX_s|ds + \sqrt{t} \|\sigma\| \cdot \|h\|_{\mathbb{H}}, \quad h \in \mathbb{H}.$$ 

Therefore,

$$\|DX_t\|_{\mathbb{H}} \leq C \int_0^t W^{l_1}(X_s)\|DX_s\|_{\mathbb{H}}ds + \sqrt{t} \|\sigma\|.$$ 

This implies (3.2) by Gronwall’s inequality. \hfill \Box

**Lemma 3.3.** Assume (H). Then for any $s \in [0, T]$,

(3.3) \quad $\|K(T, s)\| \leq Ce^{C \int_s^T W^{l_1}(X_r)dr} \quad \|\partial_s K(T, s)\| \leq CW^{l_1}(X_s)e^{C \int_s^T W^{l_1}(X_r)dr},$

and

(3.4) \quad $\|DK(T, s)\|_{\mathbb{H}} \leq Ce^{C \int_s^T W^{l_1}(X_r)dr} \int_s^T W^{l_2}(X_r)\|DX_r\|_{\mathbb{H}}dr.$

Consequently, for any $p > 1$ there exists $T_p \in (0, \infty)$ if $l_1 = 1$ and $T_p = \infty$ if $l_1 < 1$ such that

$$\mathbb{E} \left( \sup_{t \in [0, T]} \|DK(T, t)\|_{\mathbb{H}}^p \right) < \infty, \quad T \in (0, T_p).$$

**Proof.** By Lemma 3.2 and $\sup_{t \in [0, T]} \mathbb{E} W^{l}(X_t) < \infty$ for any $l > 0$ as observed in the beginning of this section, it suffices to prove (3.3) and (3.4). First of all, by (1.6) and (H2), we have

$$\|K(t, s)\| \leq 1 + \int_s^t \|\nabla^{(1)} Z^{(1)}(X_r)\| \|K(r, s)\|dr \leq 1 + C \int_s^t W^{l_1}(X_r)\|K(r, s)\|dr,$$

which yields the first estimate in (3.3) by Gronwall’s inequality. Moreover, noticing that

$$\partial_s K(t, s) = \int_s^t (\nabla^{(1)} Z^{(1)})(X_r)\partial_s K(r, s)dr - (\nabla^{(1)} Z^{(1)})(X_s),$$

by (H2) we have

$$\|\partial_s K(t, s)\| \leq C \int_s^t W^{l_1}(X_r)\|\partial_s K(r, s)\|dr + CW^{l_1}(X_s).$$

The second estimate in (3.3) follows. As for (3.4), since

$$\frac{d}{dt} DK(t, s) = (\nabla DX_t, \nabla^{(1)} Z^{(1)}(X_t)K(t, s) + (\nabla^{(1)} Z^{(1)}(X_t)DK(t, s),$$
with $DK(s, s) = 0$, it follows from (H2) and (3.3) that

$$\|DK(t, s)\|_{H} \leq \int_{s}^{t} \|\nabla \nabla^{(1)} Z^{(1)}(X_{r})\| \|DX_{r}\|_{H} \|K(r, s)\|dr$$

$$+ \int_{s}^{t} \nabla X^{(1)}(X_{r}) \|DK(r, s)\|_{H} dr$$

$$\leq Ce^{C} \int_{s}^{t} W_{t}(X_{r}) \|DX_{r}\|_{H} dr$$

$$+ C \int_{s}^{t} W_{t}(X_{r}) \|DK(r, s)\|_{H} dr.$$ 

This implies (3.4). \hfill \Box

Proof of Theorem 1.1 (1) Let $a \in \mathbb{R}^{m}$. By (1.4), (1.9) and $\nabla^{(2)} Z^{(1)} = B_{0} + B$ we have

$$\langle Q_{t}a, a \rangle = \int_{0}^{t} \phi(s) \left(\langle K(T, s)B_{0}B_{s}K(T, s)^{*}a, a \rangle + \langle K(T, s)B(X_{s})B_{s}K(T, s)^{*}a, a \rangle \right)ds$$

$$\geq (1 - \varepsilon) \int_{0}^{t} \phi(s)|B_{s}K(T, s)^{*}a|^2 ds \geq (1 - \varepsilon)\xi(t)|a|^2.$$

This implies that $Q_{t}$ is invertible and (1.10) holds.

(2) According to Lemma 3.1 (H) implies (H') for all $T > 0$ if $l_{1} < 1$ and for small $T > 0$ if $l_{1} = 1$. Next, we intend to prove that $h \in D(\delta)$ and $E|\delta(h)|^{p} < \infty$ for small $T > 0$ if $l_{1} = 1$ and for all $T > 0$ if $l_{1} < 1$. Indeed, by Lemmas 3.2, 3.3, 3.1, and the fact that

$$DQ_{t}^{-1} = -Q_{t}^{-1}(DQ_{t})Q_{t}^{-1},$$

there exists $T_{p} > 0$ if $l_{1} = 1$ and $T_{p} = \infty$ if $l_{1} < 1$ such that

$$\sup_{t \in [0, T]} E|DQ_{t}|^{p} < +\infty, \ T \in (0, T_{p}),$$

and by (1.10),

$$\left(\mathbb{E}\|DQ_{t}^{-1}\|_{H}^{p}\right)^{1/p} \leq \left(\mathbb{E}|DQ_{t}|^{p}\right)^{1/p} \left(1 - \varepsilon\xi(t)\right)^{1/p}, \ t \in (0, T),$$

$$\sup_{t \in [0, T]} \left(\mathbb{E}\|D\alpha_{t}\|_{H}^{p} + \mathbb{E}\|Dg_{t}\|_{H}^{p}\right)^{1/p} < \infty, \ T \in (0, T_{p}).$$

Since

$$\dot{h}_{t} = \sigma^{-1}\{\nabla Z^{(2)}(X_{t})(g_{t}, \alpha_{t}) - \dot{\alpha}_{t}\},$$

$$\|D\dot{h}_{t}\|_{H} \leq \|\sigma^{-1}\{\nabla^{2} Z^{(2)}(X_{t})\|DX_{t}\|_{H} \|(g_{t}, \alpha_{t})\|_{H} + \|\nabla Z^{(2)}(X_{t})\| \|Dg_{t}, D\alpha_{t}\|_{H} + \|D\dot{\alpha}_{t}\|_{H}\},$$
we conclude from (H2), (3.11) and (3.6) that

\[ \mathbb{E} \left( \int_0^T \| D\tilde{h}_t \|_{H}^2 dt \right)^{p/2} + \mathbb{E} \| h \|_H^p < \infty, \quad T \in (0, T_p). \]

Therefore, according to e.g. [8, Proposition 1.5.8], we have \( h \in \mathcal{D}(\delta) \) and \( \mathbb{E} |\delta(h)|^p < \infty \) provided \( T \in (0, T_p) \).

Now, to prove (1.7), it remains to verify the required conditions of Theorem 2.1 for \( \alpha_t \) given by (1.11). Since \( \phi(0) = \phi(T) = 0 \), we have \( \alpha_0 = v(2) \) and \( \alpha_T = 0 \). Moreover, noting that

\[
I_1 := \frac{1}{\int_0^T \xi(t)^2 dt} \int_0^T \phi(t)K(T, t)\nabla^{(2)}Z^{(1)}(X_t)B_0^sK(T, t)^* dt \int_t^T \xi(s)^2Q_s^{-1}K(T, 0)v^{(1)} ds
\]

\[
= \frac{1}{\int_0^T \xi(t)^2 dt} \int_0^T \hat{Q}_t dt \int_t^T \xi(s)^2Q_s^{-1}K(T, 0)v^{(1)} ds
\]

\[
= \frac{1}{\int_0^T \xi(t)^2 dt} \int_0^T \xi(t)^2Q_tQ_t^{-1}K(T, 0)v^{(1)} dt = K(T, 0)v^{(1)}
\]

and

\[
I_2 := \left( \int_0^T \phi(t)K(T, t)\nabla^{(2)}Z^{(1)}(X_t)B_0^sK(T, t)^* dt \right) Q_T^{-1} \int_0^T \frac{T - s}{T}K(T, s)\nabla^{(2)}Z^{(1)}(X_s)v^{(2)} ds
\]

\[
= Q_T Q_T^{-1} \int_0^T \frac{T - s}{T}K(T, s)\nabla^{(2)}Z^{(1)}(X_s)v^{(2)} ds = \int_0^T \frac{T - s}{T}K(T, s)\nabla^{(2)}Z^{(1)}(X_s)v^{(2)} ds,
\]

we obtain by (1.11)

\[
g_T = K(T, 0)v^{(1)} + \int_0^T K(T, t)\nabla^{(2)}Z^{(1)}(X_t)\alpha_t dt
\]

\[
= K(T, 0)v^{(1)} - I_1 + \int_0^T \frac{T - t}{T}K(T, t)\nabla^{(2)}Z^{(1)}(X_t)v^{(2)} dt - I_2 = 0.
\]

(3) By an approximation argument, it suffices to prove the desired gradient estimate for \( f \in C^1_b(\mathbb{R}^{m+d}) \). Moreover, by the semigroup property and the Jensen inequality, we only have to prove for \( p \in (1, 2) \) and \( T \in (0, T_p \wedge 1) \). In this case we obtain from (1.7) that

\[
|\nabla P_T f| \leq (P_T |f|^p)^{1/p}(\mathbb{E} |\delta(h)|^q)^{1/q},
\]

where \( q := \frac{p}{p-1} \geq 2 \). Therefore, it remains to find constants \( c_1, c_2 \geq 0 \), where \( c_2 = 0 \) if \( l_1 = l_2 = 0 \), such that

\[
(3.8) \quad (\mathbb{E} |\delta(h)|^q)^{1/q} \leq \frac{c_1 \sqrt{T(T^2 + \xi(T))e^{cW}}}{\int_0^T \xi(s)^2 ds}.
\]
To this end, we take $\phi(t) = \frac{t(T-t)}{T^2}$ such that $0 \leq \phi \leq 1$ and $|\dot{\phi}(t)| \leq \frac{1}{T}$ for $t \in [0, T]$. Since $\xi$ is increasing, by (3.3) and (1.9), we have for some constant $C > 0$,

$$\int_0^t \xi(s)^2 ds \leq \xi(t)^2 \leq Ct^2, \quad t \in [0, 1].$$

Thus, by Lemmas 3.1, 3.2, 3.3 and (1.1), it is easy to see that for any $\theta \geq 2$ there exist constants $c_1, c_2 \geq 0$, where $c_2 = 0$ if $l_1 = l_2 = 0$, such that for all $0 < t \leq T \leq T_p \wedge 1$,

$$\left(\mathbb{E}\|DX_t\|_H^\theta\right)^{1/\theta} \leq c_1 \sqrt{T} t^2 e^{c_2 W}, \quad \left(\mathbb{E}\|DK(T, t)\|_H^\theta\right)^{1/\theta} \leq c_1 T^{3/2} e^{c_2 W},$$

$$\left(\mathbb{E}\|DQ_t^{-1}\|_H^\theta\right)^{1/\theta} \leq \left\{\mathbb{E}(\|DQ_t^{-1}\|_H^\theta\|Q_t^{-1}\|_H^\theta\right\}^{1/\theta} \leq c_1 T^{7/2} e^{c_2 W},$$

$$\left(\mathbb{E}\|D\alpha_t\|_H^\theta\right)^{1/\theta} \leq c_1 T^{3/2} e^{c_2 W}, \quad \left(\mathbb{E}\|\dot{h}_t\|_H^\theta\right)^{1/\theta} \leq c_1 T^{7/2} e^{c_2 W},$$

Combining these with (3.7), (H2) and (3.1), we obtain

$$\|h\|_{D^{1,q}} := \left(\mathbb{E}\|Dh\|_{L^\infty}^q\right)^{1/q} + \|h\|_H \leq \sqrt{T} \left\{\mathbb{E}\left(\frac{1}{T} \int_0^T \|D\dot{h}_t\|_H^2 dt\right)^{q/2}\right\}^{1/q} + \|h\|_H$$

$$\leq \sqrt{T} \left(\frac{1}{T} \int_0^T \mathbb{E}\|Dh_t\|_H^q dt\right)^{1/q} + \left(\mathbb{E} \int_0^T \|\dot{h}_t\|_H^2 dt\right)^{1/2}$$

$$\leq \frac{c_1 \sqrt{T(T^{3/2} + \xi(T)) e^{c_2 W}}}{\int_0^T \xi(s)^2 ds}.$$

This implies (3.8) since $\delta : D^{1,q} \to L^q$ is bounded, see e.g. Proposition 1.5.8 in [9].

$$\square$$

4 Two Specific Cases

As indicated in the end of Section 1, we intend to apply Theorem 1.1 to Case (I) and Case (II) respectively with concrete functions $\xi$ satisfying (1.9).

4.1 Case (I): $\text{Rank}[B_0] = m$

**Theorem 4.1.** Assume (H) and (1.3) for some $\varepsilon \in [0, 1)$. If $\text{Rank}[B_0] = m$, then there exist constants $c_1, c_2 > 0$ such that (1.9) holds for

$$\xi(t) = c_1 \int_0^t \phi(s)e^{-c_2(T-s)}ds, \quad t \in [0, T].$$
Consequently, for any \( p > 1 \) there exist two constants \( c_1(p), c_2(p) \geq 0 \), where \( c_2(p) = 0 \) if \( l_1 = l_2 = 0 \), such that

\[
|\nabla P_T f| \leq \frac{c_1(p)(P_T |f|^p)^{1/p}}{(T \wedge 1)^{3/2}} e^{c_2(p)W}, \quad T > 0.
\]

**Proof.** It is easy to see that the desired gradient estimate follows from (1.12) for the claimed \( \xi \) with \( \phi(t) = \frac{t(T-t)}{2} \), we only prove the first assertion. Since \( \nabla^{(1)} Z^{(1)} \) is bounded, there exists a constant \( C > 0 \) such that

\[
|K(T,s)^* a| \geq e^{-C(T-s)}|a|, \quad a \in \mathbb{R}^m.
\]

If \( \text{Rank}[B_0] = m \), then \( |B_0^* a| \geq c'|a| \) holds for some constant \( c' > 0 \) and all \( a \in \mathbb{R}^m \). Therefore,

\[
M_t := \int_0^t \phi(s) K(T,s) B_0 B_0^* K(T,s)^* ds
\]

satisfies

\[
\langle M_t a, a \rangle = \int_0^t \phi(s) |B_0^* K(T,s)^* a|^2 ds \geq c'^2 \int_0^t \phi(s) e^{-2C(T-s)} |a|^2 ds.
\]

This completes the proof. \( \square \)

**Example 4.1.** Consider the stochastic Hamilton system (1.3), where \( m = d \) and \( \nabla^{(2)} Z^{(1)} = \text{Hess}_{H(x,\cdot)}(y) \) is symmetric. If for some \( C > 0 \),

\[
(4.1) \quad CI_{d \times d} \leq \nabla^{(2)} Z^{(1)}, \quad \text{or} \quad \nabla^{(2)} Z^{(1)} \leq -CI_{d \times d}.
\]

Then we take \( B_0 = CI_{d \times d} \) if \( \nabla^{(2)} Z^{(1)} \geq CI_{d \times d} \), while \( B_0 = -CI_{d \times d} \) if \( \nabla^{(2)} Z^{(1)} \leq -CI_{d \times d} \). It is trivial to see that \( \text{Rank}[B_0] = d = m \) and (1.4) holds for \( \varepsilon = 0 \).

A typical choice of \( H \) in the physical model such that (4.1) holds is that (cf. [10, Chapter XIII])

\[
H(x,y) = V(x) + \frac{1}{2} \langle M(x)y, y \rangle,
\]

where \( M(x) \), called mass matrix of the system, is a \( d \times d \)-real symmetric, smooth and positive definite matrix; and \( V(x) \), called potential energy, is a smooth function. Assume that

\[
M(x) \geq CI_{d \times d},
\]

then (4.1), and hence (1.4) with \( \varepsilon = 0 \), holds. If moreover \( F \in C_b^2, M \in C_b^3, \) and \( V \geq 0 \) (equivalently, bounded from below since one may add a constant to \( H \)) such that

\[
\|\nabla^k V(x)\| \leq C(V(x) + 1), \quad k = 2, 3,
\]

then Assumption (H) holds with \( W(x,y) = H(x,y) + 1 \) and \( l_1 = l_2 = 1 \). Therefore, Theorem 4.1 applies.
4.2 Case (II): $A := \nabla^{(1)}Z^{(1)}$ is constant

Throughout this subsection we assume that

(A) (Kalman condition) $A := \nabla^{(1)}Z^{(1)}$ is constant and there exists an integer number $0 \leq k \leq m - 1$ such that

(4.2) $\text{Rank}[B_0, AB_0, \cdots, A^k B_0] = m.$

When $k = 0$, (4.2) means $\text{Rank}[B_0] = m$ which has been considered in Theorem 4.1.

Theorem 4.2. Assume (H), (A) and (1.4) for some $\varepsilon \in (0, 1)$. Let $\phi(t) = \frac{t(T-t)}{T^2}$. Then:

1. There exist constants $c_1, c_2 > 0$ such that (11.9) holds for $\xi(t) = c_1(t \wedge 1)^{2(k+1)} T e^{c_2 T}, t \in [0, T]$.

2. For any $p > 1$, there exist two constants $c_1(p), c_2(p) \geq 0$, where $c_2(p) = 0$ if $l_1 = l_2 = 0$, such that

$$|\nabla P_T f| \leq \frac{c_1(p)(P_T|f|^p)^{1/p}}{(T \wedge 1)^{(4k-1)\sqrt{0+3/2}}} e^{c_2(p)W}, \quad T > 0.$$  

3. If $\nabla^{(2)}Z^{(1)} = B_0$ is constant and $l_1 < \frac{1}{2}$, then there exists a constant $c > 0$ such that

$$|\nabla P_T f| \leq \lambda \left\{ P_T f \log f - (P_T f) \log P_T f \right\}$$

$$+ \frac{c}{\lambda} \left\{ \frac{l_1 W}{(1 + \lambda^{-1})^2} + \frac{(1 + \lambda^{-1})^4 l_1/(1-2l_1)}{(T \wedge 1)^{4k+2-2l_1}(1-2l_1)} + \frac{1}{(1 \wedge T)^{4k+3}} \right\} P_T f, \quad \lambda > 0, T > 0$$

holds for all $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$, the set of positive functions in $\mathcal{B}_b(\mathbb{R}^{m+d})$.

4. If $\nabla^{(2)}Z^{(1)} = B_0$ is constant and $l_1 = \frac{1}{2}$, then there exist constants $c, c' > 0$ such that for any $T > 0, \lambda \geq \frac{c}{(T \wedge 1)^{2\pi}}$ and $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d}),$

$$|\nabla P_T f| \leq \lambda \left\{ P_T f \log f - (P_T f) \log P_T f \right\} + \frac{c'((1 \wedge T)^2 W + 1)}{\lambda(T \wedge 1)^{4k+3}} P_T f.$$ 

Proof. Since (2) is a direct consequence of (1.12) and (1), we only prove (1), (3) and (4).

Let

$$M_t = \int_0^t \frac{s(T-s)}{T^2} e^{(T-s)A} B_0 B_0^* e^{(T-s)A^*} ds, \quad U_t = \int_0^t e^{sA} B_0 B_0^* e^{sA^*} ds, \quad t \in [0, T].$$
According to [9, §3], the limit

$$Q := \lim_{t \to 0} t^{-(2k+1)\Gamma_t U_t \Gamma_t}$$

exists and is an invertible matrix, where $\Gamma_t$ is a family of projection matrices. Thus, $U_t \geq c(t \wedge 1)^{2k+1}$ holds for some constant $c > 0$ and all $t > 0$. Then there exist constants $c_1, c_2 > 0$ such that for any $t \in (0, T)$,

$$M_t \geq \frac{t}{4T} \int_{t/2}^{t} e^{(T-s)A} B_0 B_0^* e^{(T-s)A^*} ds \geq \frac{t e^{-|A| T}}{4T} \int_0^{t/2} e^{sA} B_0 B_0^* e^{sA^*} ds \geq \frac{c_1 t^{2(k+1)}}{4T e^{c_2 T}} I_{m \times m}$$

holds. This proves the first assertion.

(3) By the semigroup property and the Jensen inequality, we assume that $T \in (0, 1]$. Let $\nabla^{(2)} Z^{(1)} = B_0$ be constant. Then $h$ given in Theorem 1.1 is adapted such that

$$\delta(h) = \int_0^T \langle \dot{h}_t, dB_t \rangle.$$

Moreover, it is easy to see that for $\xi(t)$ given in (1) and $T \in (0, 1]$,

$$|\dot{h}_t| \leq \frac{c_1 (T W^{l_1}(X_t) + 1)}{T^{2(k+1)}}, \quad t \in [0, T]$$

holds for some constant $c_1 > 0$ independent of $T$. Thus, for any $\lambda > 0$,

$$\mathbb{E} e^{\delta(h)/\lambda} = \mathbb{E} \exp \left[ \frac{1}{\lambda} \int_0^T \langle \dot{h}_t, dB_t \rangle \right] \leq \left( \mathbb{E} \exp \left[ \frac{2}{\lambda^2} \int_0^T |\dot{h}_t|^2 dt \right] \right)^{1/2}$$

$$\leq \left( \mathbb{E} \exp \left[ \frac{c_2}{\lambda^2} \left( \frac{\int_0^T W^{2l_1}(X_t) dt}{T^{4k+2}} + \frac{1}{T^{4k+3}} \right) \right] \right)^{1/2}.$$  \hfill (4.3)

On the other hand, since $l_1 \in [0, 1]$, by Lemma 3.1 and the Jensen inequality, there exist two constants $c_3, c_4 > 0$ such that

$$\mathbb{E} \exp \left[ \frac{c_3 l_1}{T} \int_0^T W(X_t) dt \right] \leq e^{c_4 l_1 W}, \quad T \in (0, 1].$$ \hfill (4.4)

Moreover, since $2l_1 < 1$, there exists a constant $c_5 > 0$ such that

$$\frac{c_2 W^{2l_1}}{\lambda^2 T^{4k+2}} \leq \frac{c_3 l_1 W}{(1 + \lambda)^2 T} + \frac{c_5 (1 + \lambda^{-1})^{4l_1/(1-2l_1)}}{\lambda^2 T^{(4k+2-2l_1)/(1-2l_1)}}, \quad \lambda, T > 0.$$

Combining this with (4.3) and (4.4), we conclude that

$$\log \mathbb{E} e^{\delta(h)/\lambda} \leq \frac{c l_1 W}{(1 + \lambda)^2} + \frac{c (1 + \lambda^{-1})^{4l_1/(1-2l_1)}}{\lambda^2 T^{(4k+2-2l_1)/(1-2l_1)}} + \frac{c}{\lambda^2 T^{4k+3}}, \quad T \in (0, 1], \lambda > 0.$$
holds for some constant $c > 0$. This completes the proof of (3) by (1.7) and the Young inequality (see [2, Lemma 2.4])

$$|\nabla P_T f| = |\mathbb{E}[f(X_T)\delta(h)]| \leq \lambda \{ P_T f \log f - (P_T f) \log P_T f \} + \lambda (P_T f) \log \mathbb{E}e^{\delta(h)/\lambda}.$$  

(4) Again, we only consider $T \in (0,1]$. Let $c_2$ and $C$ be in (4.3) and Lemma 3.1 respectively. Then there exists a constant $c > 0$ such that for any $T \in (0,1]$, $\lambda \geq \frac{c}{T^2}$ implies

$$\frac{c_2}{\lambda^2 T^{4k+2}} \leq \frac{2}{T^2 C \|\sigma\|^{2e^{4+2CT}}}.$$  

Thus, by (4.3) and Lemma 3.1, if $\lambda \geq \frac{c}{T^2}$ then

$$\log \mathbb{E}e^{\delta(h)/\lambda} \leq \frac{c_2 T^2 C \|\sigma\|^{2e^{4+2CT}}}{4\lambda^2 T^{4k+2}} \log \mathbb{E} \exp \left[ \frac{2 \int_0^T W(X_t) dt}{T^2 C \|\sigma\|^{2e^{4+2CT}}} \right] + \frac{c_2}{\lambda^2 T^{4k+2}} \leq \frac{c'(T^2 W + 1)}{\lambda^2 T^{4k+2}}$$

holds for some constant $c' > 0$ independent of $T$. Combining this with (4.5) we finish the proof.

To derive the Harnack inequality of $P_T$ from Theorem 4.2 (3) and (4), let us recall a result of [5]. If there exist a constant $\lambda_0 > 0$ and a positive measurable function $\gamma : [\lambda_0, \infty) \times \mathbb{R}^{m+d} \to [0, \infty)$ such that

$$|\nabla \gamma P_T f| \leq \lambda \{ P_T f \log f - (P_T f) \log P_T f \} + \gamma(\lambda, \cdot) P_T f, \quad \lambda \geq \lambda_0$$

holds for some constant $\lambda_0 \in (0, \infty]$ and all $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$, then by [5, Proposition 4.1],

$$P_T f(x) \leq (P_T f^p)^{1/p}(x + v) \exp \left[ \int_0^1 \gamma \left( \frac{p-1}{1+(p-1)s}, x + sv \right) \frac{1}{1+ps} ds \right]$$

holds for all $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$ and $p \geq 1 + \lambda_0$. Then we have the following consequence of Theorem 4.2 (3) and (4).

**Corollary 4.3.** Let (H) and (A) hold such that $\nabla^{(2)} Z^{(1)} = B_0$ is constant.

1. If $l_1 \in [0, 1/2)$, then there exists a constant $c > 0$ such that

$$P_T f(x) \leq (P_T f^p)^{1/p}(x + v) \times \exp \left[ \frac{c|v|^2}{p-1} \left( \frac{p-1}{p-1} \frac{l_1}{p-1} W(x + sv) ds + \frac{(1 + p|v|/p-1)l_1/(1-2l_1)}{(T \land 1)^{(4k+2-2l_1)/(1-2l_1)} + \frac{1}{T^{4k+3}}} \right) \right]$$

holds for all $x, v \in \mathbb{R}^{m+d}, T > 0, p > 1$ and $f \in \mathcal{B}_b^+(\mathbb{R}^{m+d})$.  

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(2) If \( l_1 = 1 \) then there exist two constants \( c, c' > 0 \) such that for any \( T > 0, f \in \mathbb{R}^{m+d} \) and \( x, v \in \mathbb{R}^{m+d} \),

\[
P_T f(x) \leq (P_T f p)^{1/p} (x + v) \exp \left[ \frac{c'|v|^2 \left\{ 1 + (T \wedge 1)^2 \int_0^1 W(x + sv) ds \right\}}{(p - 1)(T \wedge 1)^{4k+3}} \right]
\]

holds for \( p \geq 1 + \frac{|v|}{(T \wedge 1)^2} \).

Proof. (1) Let \( v \in \mathbb{R}^{m+d} \) with \(|v| > 0\). By Theorem 4.2 (3), we have

\[
|\nabla v P_T f| \leq \lambda |v| \left\{ P_T f \log f - (P_T f) \log P_T f \right\}
\]

\[\]

\[
+ \frac{c|v|}{\lambda} \left\{ \frac{l_1 W}{(1 + |v|^{-1})^2} + \frac{1}{(T \wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(T \wedge 1)^{4k+3}} \right\} P_T f, \lambda > 0.
\]

Replacing \( \lambda \) by \( \frac{\lambda}{|v|} \), we see that (4.6) holds for any \( \lambda_0 > 0 \) and

\[
\gamma(\lambda, \cdot) = \frac{c|v|^2}{\lambda} \left\{ \frac{l_1 W}{(1 + |v|^{-1})^2} + \frac{1}{(T \wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(T \wedge 1)^{4k+3}} \right\}, \lambda > 0.
\]

Then the desired Harnack inequality follows from (4.7) since

\[
\int_0^1 \gamma(\frac{p-1}{1+(p-1)s}; x + sv) \frac{ds}{1+(p-1)s}
\]

\[
= \frac{c|v|^2}{p-1} \int_0^1 \left\{ \frac{l_1 W(x + sv)}{1 + \frac{|v|(1+(p-1)s)}{p-1}} + \frac{1}{(T \wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(T \wedge 1)^{4k+3}} \right\} ds
\]

\[
\leq \frac{c|v|^2}{p-1} \left( \frac{l_1(p-1)}{p-1 + |v|} \int_0^1 W(x + sv) ds + \frac{1}{(T \wedge 1)^{(4k+2-2l_1)/(1-2l_1)}} + \frac{1}{(T \wedge 1)^{4k+3}} \right).
\]

(2) Let \( v \in \mathbb{R}^{m+d} \) with \(|v| > 0\). By Theorem 4.2 (4),

\[
|\nabla v P_T f| \leq |v| \lambda \left\{ P_T f \log f - (P_T f) \log P_T f \right\} + \frac{c'|v|((1 \wedge T)^2 W + 1)}{\lambda(T \wedge 1)^{4k+3}} P_T f
\]

holds for \( \lambda \geq \frac{c|v|}{(T \wedge 1)^2} \). Using \( \frac{\lambda}{|v|} \) to replace \( \lambda \), we see that (4.6) holds for \( \lambda_0 = \frac{c|v|}{(T \wedge 1)^2} \) and

\[
\gamma(\lambda, \cdot) = \frac{c|v|^2((1 \wedge T)^2 W + 1)}{\lambda(T \wedge 1)^{4k+3}}.
\]

Then the proof is completed by (4.7). \( \square \)

Finally, according to e.g. [14, §4.2], the Harnack inequalities presented above imply explicit heat kernel estimates and entropy-cost inequalities for the invariant probability measure (if exists).

Since there exist many non-trivial examples of \( A \) and \( B_0 \) such that (A) holds (see [7]), it is easy to construct corresponding examples to illustrate Theorem 4.2. For instance, for Theorem 4.2 (3) and (4) only simply consider (1.3) with \( H(x, y) = \langle Ax, y \rangle + W(y) \) such that \( \nabla W = B_0 \), and for assertion (2) a small perturbation of \( W \) is allowed.
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References

[1] M. Arnaudon, A. Thalmaier, *The differentiation of hypoelliptic diffusion semigroups*, Arxiv preprint [arXiv:1004.2174].

[2] M. Arnaudon, A. Thalmaier, F.-Y. Wang, *Gradient estimates and Harnack inequalities on non-compact Riemannian manifolds*, Stoch. Proc. Appl. 119(2009), 3653–3670.

[3] D. Bakry, P. Cattiaux, A. Guillin, *Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincare*, J. Func. Anal. 254 (2008), 727–759.

[4] J. M. Bismut, *Large Deviations and the Malliavin Calculus*, Boston: Birkhäuser, MA, 1984.

[5] A. Guillin, F.-Y. Wang, *Degenerate Fokker-Planck equations: Bismut formula, gradient estimate and Harnack inequality*, Arxiv preprint [arXiv:1103.2817].

[6] K.D. Elworthy and X.-M. Li, *Formulae for the derivatives of heat semigroups*, J. Funct. Anal. 125(1994), 252–286.

[7] R. E. Kalman, P. L. Falb, M. A. Arbib, *Topics in Mathematical Control Theory*, McGraw-Hill Book Co. New York, 1969.

[8] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, Berlin, 2006.

[9] T. Seidman, *How violent are fast controls?* Mathematics of Control Signals Systems, 1(1988), 89-95.

[10] C. Soize, *The Fokker-Planck Equation for Stochastic Dynamical Systems and Its Explicit Steady State Solutions*, Series on Advances in Mathematics for Applied Sciences 17, World Scientific, Singapore, 1994.

[11] D. Talay, *Stochastic Hamiltonian systems: exponential convergence to the invariant measure and discretization by the implicit Euler scheme*, Markov Processes Related Fields 8(2002), 1–36.

[12] C. Villani, *Hypocoercivity*. Mem. Amer. Math. Soc. 202 (2009), no. 950.

[13] F.-Y. Wang, *On estimation of the logarithmic Sobolev constant and gradient estimates of heat semigroups*, Probab. Theory Relat. Fields 108(1997), 87–101.

[14] F.-Y. Wang, *Derivative formula and Harnack inequality for jump processes*, [arXiv:1104.5531](https://arxiv.org/abs/1104.5531)
[15] L. Wu, *Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems*, Stoch. Proc. Appl., 91 (2001), 205–238.

[16] X.-C. Zhang, *Stochastic flows and Bismut formulas for stochastic Hamiltonian systems*, Stoch. Proc. Appl. 120 (2010), 1929–1949.