Solutions of nonlinear PDEs in the completion of uniform convergence spaces

Jan Harm van der Walt
Department of Mathematics and Applied Mathematics
University of Pretoria, South Africa

Abstract

This paper deals with the solution of large classes of systems of nonlinear partial differential equations (PDEs) in spaces of generalized functions that are constructed as the completion of uniform convergence spaces. The existence result for the mentioned systems of equations are obtained as an application of a basic approximation result, which is formulated entirely in terms of usual real valued functions on open subsets of Euclidean $n$-space. The structure and regularity properties of the solutions are explained at the hand of suitable results relating to the structure of the completion of uniform convergence spaces that are defined as initial structures. In this regard, we include also a detailed discussion of the completion of initial uniform convergence spaces in general.

1 Introduction

Uniform spaces, and more generally uniform convergence spaces, appear in many important applications of topology, and in particular analysis. In this regard, the concepts of completeness and completion of a uniform convergence space play a central role. Indeed, Baire’s celebrated Category Theorem asserts that a complete metric space cannot be expressed as the union of a countable family of closed nowhere dense sets. The importance of this result is demonstrated by the fact that the Banach-Steinhauss Theorem, as well as the Closed Graph Theorem in Banach spaces follow from it.

However, in many situations one deals with a space $X$ which is incomplete, and in these cases one may want to construct the completion of $X$. In this regard, the main result, see for instance [6] and [18], is that every Hausdorff uniform convergence space $X$ may be uniformly continuously embedded into a complete, Hausdorff uniform convergence space $X^\sharp$ in such a way that the image of $X$ in $X^\sharp$ is dense. Moreover,
the following *universal property* is satisfied. For every complete, Hausdorff uniform convergence space \( Y \), and any uniformly continuous mapping \( \varphi : X \to Y \)

the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow{\iota_X} && \downarrow{\exists! \varphi^\sharp} \\
X^\sharp & &
\end{array}
\]  

commutes, with \( \varphi^\sharp \) uniformly continuous, and \( \iota_X \) the canonical embedding of \( X \) into its completion \( X^\sharp \).

It is often not only the completion \( X^\sharp \) of a uniform convergence space \( X \) that is of interest, but also the extension \( \varphi^\sharp \) of uniformly continuous mappings from \( X \) to \( X^\sharp \). In this regard, we may recall that one of the major applications of uniform spaces, and recently, see [15], [14] and [16], also uniform convergence spaces, is to the solutions of PDEs. Indeed, let us consider a PDE

\[ Tu = f \]  

with \( T \) a possibly nonlinear partial differential operator which acts on some relatively small space \( X \) of classical functions, \( u \) the unknown function, while the right hand term \( f \) belongs to some space \( Y \). One usually considers some uniformities, or more generally uniform convergence structures, on \( X \) and \( Y \) in such a way that the mapping

\[ T : X \to Y \]  

is uniformly continuous. It is well known that the equation (2) can have solutions of *physical interest* which, however, may fail to be *classical*, in the sense that they do not belong to \( X \). From here, therefore, the particular interest in *generalized solutions* to (2). Such generalized solutions to (2) may be obtained by constructing the completions \( X^\sharp \) and \( Y^\sharp \) of \( X \) and \( Y \), respectively. The mapping (3) extends uniquely to a mapping

\[ T^\sharp : X^\sharp \to Y^\sharp \]
so that the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow{\iota_X} & \quad & \downarrow{\iota_Y} \\
X^\sharp & \xrightarrow{T^\sharp} & Y^\sharp
\end{array} \]

commutes. In view of the diagram (5), one may consider the extended equation

\[ T^\sharp u^\sharp = f \] (6)

as a generalization of the original equation (2). That is, the generalized solutions of (2) are the solutions of (6). Note that the existence of generalized solutions depends on the properties of the mapping \( T^\sharp \) and the uniform convergence structure on \( X^\sharp \) and \( Y^\sharp \), as opposed to the structure and regularity of the generalized solutions, which may be interpreted as the extent to which a generalized solution exhibits characteristics of classical solutions, which depends on the properties of the space \( X^\sharp \) and its elements. It is therefore clear that not only the completion \( X^\sharp \) of a uniform convergence space \( X \) and its elements, but also the the associated extensions of uniformly continuous mappings, defined on \( X \), are of interest.

The example given above indicates a particular point of interest. The uniform convergence structure \( J_X \) on the domain \( X \) of the PDE operator \( T \) is usually defined as the initial uniform convergence structure [3] with respect to some uniform convergence structure \( J_Y \) on \( Y \), and a family of mappings

\[ (\psi_i : X \to Y)_{i \in I} \]

In the case of PDEs, the mappings \( \psi_i \) are typically usual partial differential operators, up to a given order \( m \). A natural question arises as to the connection between the completion of \( X \), and the completion of \( Y \). More generally, consider a set \( X \) and a family of mappings

\[ (\psi_i : X \to X_i)_{i \in I} \]

where each \( X_i \) is a uniform convergence space. If the family \( (\psi_i)_{i \in I} \) separates the points of \( X \), then the initial uniform convergence structure on \( X \) with respect to this family of mappings is also Hausdorff, and we may consider its completion \( X^\sharp \). It appears that the issue of the possible connections between the completions of \( X \) and those of the \( X_i \), respectively, has not yet been fully explored. We aim to clarify the connection between the completion \( X^\sharp \) of \( X \), and the completions \( X_i^\sharp \) of the \( X_i \).
Regarding the above example concerning possibly nonlinear PDEs, we note that the uniform structure on the target space $Y$ is usually induced by some locally convex linear space topology on $Y$, while the initial uniform structure on $X$ is defined in terms of usual linear partial differential operator. Indeed, the Sobolev space $H^1(\Omega)$, for instance, may be defined as the completion of the initial uniform structure on $C^1(\Omega)$ with respect to the family of mappings

$$(D^\alpha : C^1 \rightarrow L_2(\Omega))_{|\alpha| \leq 1}$$

These methods, however, fail to deliver the existence of generalized solutions to any significantly general class of PDEs, particularly in the nonlinear case. This is not due to any conceptual obstacles, and even less so to the limitations of mathematics as such, but rather to the inherent limitations of the linear function analytic methods themselves.

Indeed, the general and type independent theory [10] for the existence of solutions to nonlinear PDEs delivers generalized solutions to very large classes of equations as elements of the Dedekind completion of suitably constructed partially ordered sets. What is more, one obtains a blanket regularity for these generalized solutions, as they may be assimilated with Hausdorff continuous, interval valued functions [1].

As an application of the results on the completion of uniform convergence spaces, we present a significant enrichment of the basic theory of Order Completion [10]. In this regard, we obtain the existence and uniqueness of generalized solutions of $C^k$-smooth systems of nonlinear PDEs, which may be assimilated with functions which are $C^k$-smooth everywhere except on some closed nowhere dense set.

The paper is organized as follows. In Section 2 we discuss the structure of the completion $Y^\sharp$ of a subspace $Y$ of a uniform convergence structure $X$ relative to that of the completion $X^\sharp$ of $X$. Section 3 addresses the structure of the completion of the Cartesian product of a family of uniform convergence spaces. In particular, we show that the Wyler completion preserves Cartesian products. As an application of the results on subspaces and products of uniform convergence spaces, we investigate the structure of the completion of the initial uniform convergence structure on a set $X$, with respect to uniform convergence structures $J_{X_i}$ on sets $X_i$, and a family of mappings

$$(\psi_i : X \rightarrow X_i)_{i \in I}$$

in Section 4. In the context of nonlinear PDEs, as explained above, the results we obtain in this regard may be considered as a regularity result. In Section 5 we apply the results of the preceding sections to systems of nonlinear PDEs.

## 2 Subspaces and Embeddings

It can easily be shown that the Bourbaki completion of a uniform space $X$ preserves subspaces. In particular, the completion $Y^\sharp$ of any subspace of $X$ is isomorphic to a
subspace of the completion $X^♯$ of $X$. For uniform convergence spaces in general, and the associated Wyler completion [13], this is not the case. In this regard, consider the following\(^1\).

**Example 1** Consider the real line $\mathbb{R}$ equipped with the uniform convergence structure associated with the usual uniformity. Also consider the set $\mathbb{Q}$ of rational numbers equipped with the subspace uniform convergence structure induced from $\mathbb{R}$. The Wyler completion $\mathbb{Q}^♯$ of $\mathbb{Q}$ is the set $\mathbb{R}$ equipped with a suitable uniform convergence structure. As such, the inclusion mapping $i : \mathbb{Q} \to \mathbb{R}$ extends to a uniformly continuous bijection

$$i^♯ : \mathbb{Q}^♯ \to \mathbb{R}$$

(7)

Furthermore, a filter $\mathcal{F}$ on $\mathbb{Q}^♯$ converges to $x^♯$ if and only if

$$\mathcal{V}(x^♯)|_{\mathbb{Q}} \cap [x^♯] \subseteq \mathcal{F}$$

(8)

where $\mathcal{V}(x^♯)$ denotes the neighborhood filter at $x^♯$ in $\mathbb{R}$. As such, it is clear that the neighborhood filter at $x^♯$ does not converge in $\mathbb{Q}^♯$. Therefore the mapping (7) does not have a continuous inverse, so that it is not an embedding.

In view of Example[1] it is clear that Wyler completion does not preserve subspaces. Indeed, even in case the uniform convergence structure is induced by a uniformity, the completion of a subspace of a uniform convergence space $X$ will in general not be a subspace of the completion $X^♯$. Before we proceed to establish with our investigation of the completion of a subspace $Y$ of a uniform convergence structure $X$, we note that the Wyler completion is minimal among complete uniform convergence spaces containing a given uniform convergence space $X$ as a dense subspace. Indeed, this is an easy consequence of the universal property [11].

**Proposition 2** Consider a Hausdorff uniform convergence space $X$. For any complete, Hausdorff uniform convergence space $X^♯_0$ that contains $X$ a dense subspace, there is a bijective and uniformly continuous mapping

$$i^♯_{X,0} : X^♯ \to X^♯_0.$$  

(9)

**Proof.** Consider the inclusion mapping

$$i_{X,0} : X \to X^♯_0,$$

(10)

which is clearly a uniformly continuous embedding. As such, there is a unique uniformly continuous mapping

$$i^♯_{X,0} : X^♯ \to X^♯_0$$

(11)

\(^1\)This example was communicated to the author by Prof. H. P. Butzmann
which extends the mapping (10). We show that (11) is a bijection. In this regard, consider any point $x^\sharp_0 \in X^\sharp_0$. Since $X$ is dense in $X^\sharp_0$, there is some Cauchy filter $\mathcal{F}$ on $X$ such that $[\mathcal{F}]_{X^\sharp_0}$ converges to $x^\sharp_0$ in $X^\sharp_0$. Furthermore, there is some $x^\sharp \in X^\sharp$ so that $[\mathcal{F}]_{X^\sharp}$ converges to $x^\sharp$ in $X^\sharp$. Since

$$i_{X,0}^\sharp ([\mathcal{F}]_{X^\sharp}) = i_{X,0} (\mathcal{F}) = [\mathcal{F}]_{X^\sharp_0}$$

(12)

it follows that $i_{X,0}^\sharp (x^\sharp) = x^\sharp_0$. Hence (11) is surjective.

To see that (11) is injective, consider any $x^\sharp, y^\sharp \in X^\sharp$ and Cauchy filters $\mathcal{F}$ and $\mathcal{G}$ on $X$ which converge to $x^\sharp$ and $y^\sharp$, respectively, in $X^\sharp$. Suppose that

$$i_{X,0}^\sharp (x^\sharp) = i_{X,0}^\sharp (y^\sharp) = z^\sharp_0$$

(13)

for some $z^\sharp_0 \in X^\sharp_0$. It now follows by (12) that $[\mathcal{F}]_{X^\sharp_0}$ and $[\mathcal{G}]_{X^\sharp_0}$ both converge to $z^\sharp_0$ in $X^\sharp_0$. As such, $[\mathcal{F}]_{X^\sharp_0} \cap [\mathcal{G}]_{X^\sharp_0}$ converges to $z^\sharp_0$ so that $\mathcal{F} \cap \mathcal{G}$ is a Cauchy filter on $X$. This shows that $x^\sharp_0 = y^\sharp_0$ so that (11) is injective. This completes the proof. ■

The main result of this section is the following.

**Proposition 3** Let $Y$ be a subspace of the uniform convergence space $X$. Then there is an injective, uniformly continuous mapping

$$i^\sharp : Y^\sharp \rightarrow X^\sharp$$

(14)

which extends the inclusion mapping $i : Y \rightarrow X$. In particular,

$$i^\sharp (Y^\sharp) = a_{X^\sharp} (i_X (Y)) .$$

(15)

Furthermore, the uniform convergence structure on $Y^\sharp$ is the smallest complete, Hausdorff uniform convergence structure on $a_{X^\sharp} (Y)$, with respect to inclusion, so that $Y$ is contained as a dense subspace.

**Proof.** In view of the fact that the inclusion mapping $i : Y \rightarrow X$ is a uniformly continuous embedding, we obtain a uniformly continuous mapping

$$i^\sharp : Y^\sharp \rightarrow X^\sharp$$

(16)

so that the diagram

$$\begin{array}{ccc}
Y & \xrightarrow{i} & X \\
\downarrow{i_Y} & & \downarrow{i_X} \\
Y^\sharp & \xrightarrow{i^\sharp} & X^\sharp
\end{array}$$

(17)
commutes. To see that the mapping (16) is injective, consider any \(y^\sharp_0, y^\sharp_1 \in Y^\sharp\) and suppose that

\[
i^\sharp \left(y^\sharp_0\right) = i^\sharp \left(y^\sharp_1\right) = x^\sharp\]

(18)

for some \(x^\sharp \in X^\sharp\). Since \(\iota_Y\left(Y\right)\) is dense in \(Y^\sharp\) there exists Cauchy filters \(\mathcal{F}\) and \(\mathcal{G}\) on \(Y\) such that \(\iota_Y\left(\mathcal{F}\right)\) converges to \(y^\sharp_0\) and \(\iota_Y\left(\mathcal{G}\right)\) converges to \(y^\sharp_1\). From the diagram above it follows that \(\iota_X\left(i\left(\mathcal{F}\right)\right)\) and \(\iota_X\left(i\left(\mathcal{G}\right)\right)\) converges to \(x^\sharp\). Therefore the filter

\[
\mathcal{H} = \iota_X\left(i\left(\mathcal{F}\right)\right) \cap \iota_X\left(i\left(\mathcal{G}\right)\right)
\]

converges to \(x^\sharp\) in \(X^\sharp\). Note that the filter

\[
i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)
\]

is a Cauchy filter on \(Y\) so that \(\iota_Y\left(i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)\right)\) must converge in \(Y^\sharp\) to some \(y^\sharp\). But \(\iota_Y\left(i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)\right) \subseteq \iota_Y\left(\mathcal{F}\right)\) and \(\iota_Y\left(i^{-1}\left(\iota_X^{-1}\left(\mathcal{H}\right)\right)\right) \subseteq \iota_Y\left(\mathcal{G}\right)\) so that \(\iota_Y\left(\mathcal{F}\right)\) and \(\iota_Y\left(\mathcal{G}\right)\) must converge to \(y^\sharp\) as well. Since \(Y^\sharp\) is Hausdorff it follows by (18) that \(y^\sharp_0 = y^\sharp_1 = y^\sharp\). Therefore \(i^\sharp\) is injective.

Clearly \(i^\sharp\left(Y^\sharp\right) \subseteq \alpha_{X^\sharp}\left(\iota_X\left(Y\right)\right)\). To verify the reverse inclusion, consider any \(x^\sharp \in \alpha_{X^\sharp}\left(\iota_X\left(Y\right)\right)\). Then

\[
\exists \mathcal{F} \text{ a filter on } \iota_X\left(Y\right) : \\
[\mathcal{F}]_{X^\sharp} \text{ converges to } x^\sharp \text{ in } X^\sharp.
\]

(19)

Then there is a Cauchy filter \(\mathcal{G}\) on \(X\) so that

\[
\iota_X\left(\mathcal{G}\right) \cap [x^\sharp] \subseteq [\mathcal{F}]_{X^\sharp}
\]

(20)

This implies that the Cauchy filter \(\mathcal{G}\) has a trace \(\mathcal{H} = \mathcal{G}_{\iota_Y}\) on \(Y\), which is a Cauchy filter on \(Y\). The result now follows by the commutative diagram (17).

The last statement follows immediately from Proposition 2. ■

The following is an immediate consequence of Proposition 3.

**Corollary 4** Let \(X\) and \(Y\) be uniform convergence spaces, and \(\varphi : X \to Y\) a uniformly continuous embedding. Then there exists an injective uniformly continuous mapping \(\varphi^\sharp : X^\sharp \to Y^\sharp\), where \(X^\sharp\) and \(Y^\sharp\) are the completions of \(X\) and \(Y\) respectively, which extends \(\varphi\).

It should be noted that there are many different ‘completions’ which one may associate with a given Hausdorff uniform convergence space, each designed so as to preserve a specific property, or properties, of a uniform convergence space. E. Reed [11] made a definitive study of several such completions. Furthermore, the completion of a convergence vector space [7], the completion of a convergence group [5], and the Wyler completion [18] of a uniform convergence space are in general all different. Indeed, the Wyler completion is typically not compatible with the algebraic structure of a convergence group or convergence vector space [3], while the convergence group completion of a convergence vector space does in general not induce a vector space convergence structure [3]. Among all possible completions, the Wyler completion is the only one that satisfies the universal property [11]. None of the mentioned completions, however, will, in general, preserve subspaces.


3 Products of Uniform Convergence Spaces

Besides subspaces, the product uniform convergence structure on the Cartesian product of a family of uniform convergence spaces is the simplest example of an initial uniform convergence structure, and in this section we consider the completions of these initial uniform convergence spaces. As is the case for subspaces of Hausdorff uniform convergence spaces, the Wyler completion of the product of a family of Hausdorff uniform convergence spaces is in general different from the product of the completions of the components. Indeed, Kent and Ruiz de Eguino [8] gave the following example.

**Example 5** Consider on $\mathbb{Q}$ the uniform convergence structure induced by the usual metric, and let $\mathbb{Q} \times \mathbb{Q}$ carry the product uniform convergence structure. The completion $\mathbb{Q}^\sharp$ of $\mathbb{Q}$ is $\mathbb{R}$, equipped with a suitable uniform convergence structure. In particular, a filter $\mathcal{F}$ converges to $x$ in $\mathbb{Q}^\sharp$ if and only if

$$\left\{ \left\{ (x, y) \cup \{x\} \cap \{x\} : \epsilon > 0 \right\} \right\} \subseteq \mathcal{F}.$$  

On the other hand, the completion $(\mathbb{Q} \times \mathbb{Q})^\sharp$ of $\mathbb{Q} \times \mathbb{Q}$ is $\mathbb{R} \times \mathbb{R}$, equipped with the appropriate uniform convergence structure. A filter $\mathcal{G}$ converges to $(x, y)$ in $(\mathbb{Q} \times \mathbb{Q})^\sharp$ if and only if

$$\left\{ \left\{ (x - \epsilon, x + \epsilon) \times (y - \epsilon, y + \epsilon) \cap \mathbb{Q} \times \mathbb{Q} \} \cup \{ (x, y) \} : \epsilon > 0 \right\} \subseteq \mathcal{G}.$$  

Consider now the filter

$$\mathcal{H} = \left\{ \left\{ \left( \frac{1}{n}, \pi \right) : n \geq k \right\} : k \in \mathbb{N} \right\}.$$  

Clearly the filter $\mathcal{H}$ converges to $(0, \pi)$ in $\mathbb{Q}^\sharp \times \mathbb{Q}^\sharp$. On the other hand, $\mathcal{H}$ cannot converge to $(0, \pi)$ in $(\mathbb{Q} \times \mathbb{Q})^\sharp$.

The above example shows that the completion of the product of a family of Hausdorff uniform convergence spaces may differ from the product of the completions of the components. However, applying the results from Section 2 we obtain the following result concerning the structure of the completion of the product of a family of uniform convergence spaces.

**Theorem 6** Let $(X_i)_{i \in I}$ be a family of Hausdorff uniform convergence spaces, and let $X$ denote their Cartesian product, equipped with the product uniform convergence structure. Then there exists an bijective uniformly continuous mapping

$$\iota_X : X^\sharp \to \prod_{i \in I} X_i,$$

where $X^\sharp$ and $X_i^\sharp$ denote the completions of $X$ and the $X_i$, respectively.
Proof. First note that $\prod_{i \in I} X_i^\sharp$ is complete [18]. For every $i$, let $\iota_{X_i} : X_i \to X_i^\sharp$ be the uniformly continuous embedding associated with the completion $X_i^\sharp$ of $X_i$. Define the mapping $\iota_X : X \to \prod_{i \in I} X_i^\sharp$ through

$$\iota_X : x = (x_i)_{i \in I} \mapsto (\iota_{X_i}(x_i))_{i \in I}$$

For each $i$, let $\pi_i : X \to X_i$ be the projection. Since each $\iota_{X_i}$ is injective, so is $\iota_X$. Moreover, we have

$$U \in J_X \Rightarrow (\pi_i \times \pi_i)(U) \in J_{X_i} \Rightarrow (\iota_{X_i} \times \iota_{X_i})(((\pi_i \times \pi_i)(U)) \in J_{X_i}$$
$$\Rightarrow \prod_{i \in I} (\iota_{X_i} \times \iota_{X_i})((\pi_i \times \pi_i)(U)) \in J_{\prod_i}$$
$$\Rightarrow (\iota_X \times \iota_X)(U) \in J_{\prod_i}$$

where $J_{\prod_i}$ denotes the product uniform convergence structure on $\prod_{i \in I} X_i^\sharp$. Hence $\iota_X$ is uniformly continuous. Similarly, if the filter $\mathcal{V}$ on $\iota_X(X) \times \iota_X(X)$ belongs to the subspace uniform convergence structure, then

$$(\pi_i \times \pi_i)(\mathcal{V}) \in J_{X_i} \Rightarrow (\iota_{X_i}^{-1} \times \iota_{X_i}^{-1})((\pi_i \times \pi_i)(\mathcal{V})) \in J_{X_i}$$
$$\Rightarrow \prod_{i \in I} (\iota_{X_i}^{-1} \times \iota_{X_i}^{-1})((\pi_i \times \pi_i)(\mathcal{V})) \in J_X$$
$$\Rightarrow (\iota_X^{-1} \times \iota_X^{-1})(\mathcal{V}) \in J_X$$

so that $\iota_X^{-1}$ is uniformly continuous. Hence $\iota_X$ is a uniformly continuous embedding. It now follows by Corollary 4 that the mapping $\iota_X$ extends to an injective uniformly continuous mapping

$$\iota_X^\sharp : X^\sharp \to \prod_{i \in I} X_i.$$

To see that the mapping (21) is surjective, we note that $\iota_{X_i}(X_i)$ is dense in $X_i^\sharp$ for each $i \in I$. That is,

$$\forall i \in I : \forall x_i^\sharp \in X_i^\sharp : \exists \mathcal{F}_i \text{ a Cauchy filter on } X_i : \iota_{X_i}(\mathcal{F}_i) \text{ converges to } x_i^\sharp \text{ in } X_i^\sharp$$

The filter $\mathcal{F} = \prod_{i \in I} \mathcal{F}_i$ is a Cauchy filter on $X$. As such, there is some $x^\sharp \in X^\sharp$ so that $\mathcal{F}$ converges to $x^\sharp$ in $X^\sharp$. Furthermore,

$$\iota_X(\mathcal{F}) = \prod_{i \in I} \iota_i(\mathcal{F}_i)$$

so that $\iota_X^\sharp([\mathcal{F}]_{X^\sharp})$ converges to $\left(x_i^\sharp\right)_{i \in I}$ in $\prod_{i \in I} X_i^\sharp$. As such, it follows by the uniform continuity of $\iota_X^\sharp$ that $\iota_X^\sharp(x^\sharp) = \left(x_i^\sharp\right)_{i \in I}$. This completes the proof. ■
We note here that, similar to the case of the completion of subspaces of Hausdorff uniform convergence spaces, the Wyler completion $X^\#$ of $X = \prod_{i \in I} X_i$ is simply the set $\prod_{i \in I} X_i^\#$ equipped with the finest complete, Hausdorff uniform convergence structure such that $X$ is contained as a dense subspace. In particular, we have the following result, which follows by Theorem 6 and Proposition 3.

**Proposition 7** The uniform convergence structure on $X^\#$ is the smallest complete, Hausdorff uniform convergence structure on the set $\prod_{i \in I} X_i^\#$ such that $\prod_{i \in I} X_i$ is contained in $\prod_{i \in I} X_i^\#$ as a dense subspace.

## 4 Completion of Initial Uniform Convergence Spaces

In view of the fact that the Wyler completion of uniform convergence spaces does not, in general, preserve subspaces or Cartesian products, initial structures are not invariant under the formation of completions. That is, if $X$ carries the initial uniform convergence structure with respect to a family of mappings\(^{(24)}\)

$$(\psi_i : X \to X_i)_{i \in I},$$

into Hausdorff uniform convergence spaces $X_i$, then the completion $X^\#$ of $X$ does not necessarily carry the initial uniform convergence structure with respect to

$$(\psi_i^\# : X^\# \to X_i^\#)_{i \in I},$$

where $\psi_i^\#$ denotes the uniformly continuous extension of $\psi_i$ to $X^\#$. In this regard, one can at best obtain a generalization of Proposition 3 and Theorem 6. The first, and in fact quite obvious, result in this regard is the following.

**Proposition 8** Suppose that $X$ is equipped with the initial uniform convergence structure with respect to a family of mappings

$$(\varphi_i : X \to X_i)_{i \in I}, \quad (24)$$

where each uniform convergence space $X_i$ is Hausdorff, and the family of mappings\(^{(24)}\) separates the points on $X$. Then each mapping $\varphi_i$ extends uniquely to a uniformly continuous mapping

$$\varphi_i^\# : X^\# \to X_i^\# \quad (25)$$

and the uniform convergence structure on $X^\#$ is finer than the initial uniform convergence structure with respect to the mappings\(^{(24)}\).

Concerning the uniform convergence structure on $X^\#$, Proposition 8 is, in the general case, the sharpest result. However, this result does not give any information concerning the structure of the set $X^\#$ and its elements, which is the main interest of this paper. In this regard, we have the following interesting results.
Lemma 9 For each $i \in I$, let $X_i$ be a Hausdorff uniform convergence space, with uniform convergence structure $\mathcal{J}_{X_i}$. Let the uniform convergence space $X$ carry the initial uniform convergence structure $\mathcal{J}_X$ with respect to the family of mappings

$$(\psi_i : X \mapsto X_i)_{i \in I}$$

Assume that $(\psi_i)_{i \in I}$ separates the points of $X$. Then the mapping

$$\Psi : X \ni x \mapsto (\psi_i(x))_{i \in I} \in \prod_{i \in I} X_i$$

is a uniformly continuous embedding. In particular, the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\psi_i} & X_i \\
\downarrow{\Psi} & & \downarrow{\pi_i} \\
\prod X_i & & \\
\end{array}$$

commutes for every $i \in I$.

Proof. Since the family $(\varphi_i)_{i \in I}$ separates the points of $X$, the mapping (26) is injective. Furthermore, the diagram (27) clearly commutes. Note that

$$\forall i \in I : (\psi_i \times \psi_i)(U) \in \mathcal{J}_{X_i}.$$ 

Hence we have

$$\forall i \in I : (\pi_i \times \pi_i)(\Psi \times \Psi)(U) \in \mathcal{J}_{X_i}.$$ 

Therefore $$(\Psi \times \Psi)(U) \in \mathcal{J}_{\Pi},$$ which is the product uniform convergence structure, so that $\Psi$ is uniformly continuous.

Let $\mathcal{V} \in \mathcal{J}_\Pi$ be a filter on $\prod_{i \in I} X_i \times \prod_{i \in I} X_i$ with a trace on $\Psi(X) \times \Psi(X)$. Then

$$\forall i \in I :$$

a) $$(\pi_i \times \pi_i)(\mathcal{V}) \in \mathcal{J}_{X_i}$$

b) $W \in (\pi_i \times \pi_i)(\mathcal{V}) \Rightarrow W \cap (\psi_i(X) \times \psi_i(X)) \neq \emptyset$$

so that

$$\forall i \in I : (\psi_i \times \psi_i)((\Psi^{-1} \times \Psi^{-1})(\mathcal{V})) \supseteq (\pi_i \times \pi_i)(\mathcal{V})$$
Form the definition of an initial uniform convergence structure, and in particular the product uniform convergence structure, it follows that \((\Psi^{-1} \times \Psi^{-1}) (\mathcal{V}) \in \mathcal{J}_X\). Hence \(\Psi\) is a uniformly continuous embedding. ■ The following is now a straightforward application of Lemma 9, Theorem 6 and Proposition 3.

**Theorem 10** For each \(i \in I\), let \(X_i\) be a Hausdorff uniform convergence space, with uniform convergence structure \(\mathcal{J}_{X_i}\). Let the uniform convergence space \(X\) carry the initial uniform convergence structure \(\mathcal{J}_X\) with respect to the family of mappings

\[
(\psi_i : X \mapsto X_i)_{i \in I}
\]

Assume that \((\psi_i)_{i \in I}\) separates the points of \(X\). Then there exists a unique injective, uniformly continuous mapping

\[
\Psi^\sharp : X^\sharp \to \prod_{i \in I} X_i^\sharp
\]

such that, for each \(i \in I\), the diagram

\[
\begin{array}{ccc}
X^\sharp & \xrightarrow{\psi_i^\sharp} & X_i^\sharp \\
\downarrow{\Psi^\sharp} & & \downarrow{\pi_i} \\
\prod X_i^\sharp & \xrightarrow{\psi_i} & X_i^\sharp
\end{array}
\]

commutes, with \(\pi_i\) the projection, and \(\psi_i^\sharp\) the unique extension of \(\psi_i\) to \(X^\sharp\).

Within the context of nonlinear PDEs, as explained in Section 1, Theorem 10 may be interpreted as a regularity result. Indeed, consider some space \(X \subseteq \mathcal{C}^\infty(\Omega)\) of classical, smooth functions on an open, nonempty subset \(\Omega\) of \(\mathbb{R}^n\). Equip \(X\) with the initial uniform convergence structure \(\mathcal{J}_X\) with respect to the family of mappings

\[
D^\alpha : X \to Y, \ \alpha \in \mathbb{N}^n
\]

where \(Y\) is some space of functions on \(\Omega\) that contains \(D^\alpha(X)\) for each \(\alpha \in \mathbb{N}^n\). In view of Lemma 9 the mapping

\[
D : X \ni u \to (D^\alpha u) \in Y^\mathbb{N},
\]

while Theorem 10 guarantees that the extension

\[
D^\sharp : X^\sharp \ni u \to (D^\alpha u) \in Y^{\sharp \mathbb{N}}
\]
of (32) is injective and that the diagram

\[ \begin{array}{ccc}
X^z & \xrightarrow{D^{\alpha z}} & Y^z \\
\downarrow & & \downarrow \\
D^z & \xrightarrow{\pi_\alpha} & \mathbb{Y}_D^{2\mathbb{N}}
\end{array} \]

commutes. Here

\[ D^{\alpha z} : X^z \to Y^z, \ \alpha \in \mathbb{N}^n \]  \hspace{1cm} (35)

are the uniformly continuous extension of the mappings (31). As such, each generalized function \( u^z \in X^z \) may be identified with \( D^z u^z \in \mathbb{Y}_D^{2\mathbb{N}} \).

## 5 An Application to Nonlinear PDEs

The Order Completion Method [10] for nonlinear partial differential equations is a general and type independent theory for the existence and regularity of generalized solutions of nonlinear PDEs. The generalized solutions obtained through this method are constructed as elements of the Dedekind completion of suitable spaces of piecewise smooth functions. Recently, see [14] through [17], this method was significantly enriched by reformulating it in terms of suitable uniform convergence structures, notably the uniform order convergence structure [14].

We now present, as an application of the results obtained in Sections 2, 3 and 4, a further enrichment of the basic theory. In particular, we prove existence and basic regularity results for generalized solutions of arbitrary \( C^k \)-smooth systems of nonlinear PDEs. In this regard, consider a system of \( K \) nonlinear PDEs, each of order at most \( m \), of the form

\[ T(x, D)u(x) = f(x), \ x \in \Omega, \]  \hspace{1cm} (36)

where \( \Omega \subseteq \mathbb{R}^n \) is some nonempty open subset of \( \mathbb{R}^n \). The right hand term \( f : \Omega \to \mathbb{R}^K \) is assumed to be a \( C^k \)-smooth function on \( \Omega \), with components \( f_1, ..., f_K \), while the partial differential operator \( T(x, D) \) is supposed to be defined through a \( C^k \)-smooth function

\[ F : \Omega \times \mathbb{R}^M \to \mathbb{R}^K \]  \hspace{1cm} (37)
by
\[
\forall \quad u \in \mathcal{C}^k(\Omega)^K : \\
\forall \quad x \in \Omega : \\
T(x,D)u(x) = F(x,u(x),...,D^\alpha u(x),...) , \ |\alpha| \leq m
\]
with \( k \in \mathbb{N} \cup \{\infty\} \). We also make the following technical assumption:
\[
\forall \quad x \in \Omega : \\
f(x) \in \text{int}\{F(x,\xi) : \xi \in \mathbb{R}^M\}
\]
Note that (39) is merely a necessary condition for the existence of a classical solution to (36) on a neighborhood of \( x \in \Omega \).

We construct generalized solutions to (36) which may be assimilated with functions which are \( \mathcal{C}^k \)-smooth everywhere on \( \Omega \), except possible on a closed nowhere dense set \( \Gamma \subset \Omega \). In this regard, we consider the space \( \mathcal{N}\mathcal{L}(\Omega)^K \) of nearly finite, normal lower semi-continuous functions on \( \Omega \). Recall [4], see also [14], that an extended real valued function
\[
u : \Omega \to \mathbb{R}
\]
is normal lower semi-continuous at \( x \in \Omega \) whenever
\[
(I \circ S)(u)(x) = u(x),
\]
and \( u \) is normal lower semi-continuous on \( \Omega \) whenever it is normal lower semi-continuous at every \( x \in \Omega \). Here \( I \) and \( S \) are the Lower and Upper Baire Operators, respectively, defined through
\[
I(u)(x) = \sup\{\inf\{u(y) : \|x - y\| < \delta\} : \delta > 0\}, \quad x \in \Omega
\]
and
\[
S(u)(x) = \inf\{\sup\{u(y) : \|x - y\| < \delta\} : \delta > 0\}, \quad x \in \Omega.
\]
It is clear that an extended real valued mapping on \( \Omega \) is normal lower semi-continuous at \( x \in \Omega \) whenever \( u \) is real valued and continuous at \( x \). A normal lower semi-continuous function is called nearly finite whenever the set
\[
\{x \in \Omega : u(x) \in \mathbb{R}\}
\]
is open and dense in \( \Omega \). The set \( \mathcal{N}\mathcal{L}(\Omega) \) is a fully distributive and Dedekind complete lattice with respect to the pointwise order
\[
u \leq \mu \iff \left( \forall \quad x \in X : \nu(x) \leq \mu(x) \right).
\]
We consider the following subspaces of \( \mathcal{N}\mathcal{L}(\Omega) \). Namely, for \( l \in \mathbb{N} \cup \{\infty\} \), we consider the set
\[
\mathcal{M}\mathcal{L}^l(\Omega) = \left\{ u \in \mathcal{N}\mathcal{L}(\Omega) \left| \exists \Gamma \subset \Omega \text{ closed nowhere dense : } u \in \mathcal{C}^l(\Omega \setminus \Gamma) \right. \right\}.
\]
Theorem 11 For each $l \geq 0$, the space $\mathcal{ML}^l(\Omega)$ is a fully distributive lattice with respect to the pointwise order (44).

Proof. Consider any $u, v \in \mathcal{ML}^l(\Omega)$. Then there is a closed and nowhere dense subset $\Gamma$ of $\Omega$ such that $u, v \in \mathcal{C}^l(\Omega \setminus \Gamma)$. Define the open subsets $U$, $V$ and $W$ of $\Omega \setminus \Gamma$ through

$$U = \{ x \in \Omega \setminus \Gamma : u(x) < v(x) \},$$

$$V = \{ x \in \Omega \setminus \Gamma : v(x) < u(x) \}$$

and

$$W = \text{int}\{ x \in \Omega \setminus \Gamma : u(x) = v(x) \}.$$

It is clear that the function

$$\varphi : \Omega \ni x \mapsto \sup\{ u(x), v(x) \} \in \mathbb{R}$$

is $\mathcal{C}^l$-smooth on $U \cup V \cup W$. Furthermore, the set $U \cup V \cup W$ is dense in $\Omega \setminus \Gamma$. As such, it follows by [14, Theorem 1] that $\sup\{ u, v \}$ belongs to $\mathcal{ML}^l(\Omega)$.

The existence of the infimum of $u, v \in \mathcal{ML}^l(\Omega)$ in $\mathcal{ML}^l(\Omega)$ follows in the same way. The distributivity of $\mathcal{ML}^l(\Omega)$ now follows by the corresponding property for $\mathcal{NL}(\Omega)$.

With the nonlinear partial differential operator (38) we may associate a mapping

$$T : \mathcal{ML}^{m+k}(\Omega)^K \rightarrow \mathcal{ML}^k(\Omega)^K. \quad (46)$$

In particular, the components of the mapping (46) may be defined through

$$T_j : \mathcal{ML}^{m+k}(\Omega)^K \ni u \mapsto (I \circ S)(F_j(\cdot, ..., D^a u, ...)) \in \mathcal{ML}^k(\Omega) \quad (47)$$

where, for $j = 1, ..., K$, the mappings $F_j : \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}$ are the components of (37).

On the space $\mathcal{ML}^{m+k}(\Omega)^K$ we define the equivalence relation

$$\forall \ u, v \in \mathcal{ML}^{m+k}(\Omega)^K : u \sim_T v \iff Tu = Tv.$$ 

(48)

and we denote the quotient space $\mathcal{ML}^{m+k}(\Omega)^K / \sim_T$ by $\mathcal{ML}^k_T(\Omega)$. We may associate with the mapping $T$ an injective mapping

$$\hat{T} : \mathcal{ML}^{m+k}_T(\Omega) \rightarrow \mathcal{ML}^k(\Omega)^K.$$ 

(49)
so that the diagram

\[
\begin{array}{ccc}
\mathcal{ML}^{m+k}(\Omega)^K & \xrightarrow{T} & \mathcal{ML}^k(\Omega)^K \\
\downarrow q_T & & \downarrow i \\
\mathcal{ML}_T^{m+k}(\Omega) & \xrightarrow{\hat{T}} & \mathcal{ML}^k(\Omega)^K
\end{array}
\]

commutes, with \(q_T\) the canonical quotient mapping associated with the equivalence relation (48), and \(i\) the identity. In view of the commutative diagram (50), the equation

\[ Tu = f, \]  

which is an extension of the system of PDEs (36), is equivalent to

\[ \hat{T}U = f \]  

in the sense that

\[ \forall \ u \in \mathcal{ML}^{m+k}(\Omega)^K : \]

\[ Tu = f \iff \hat{T}(q_T(u)) = f \]  

and

\[ \forall \ U \in \mathcal{ML}_T^{m+k}(\Omega) : \]

\[ \hat{T}U = f \iff T(q_T^{-1}(U)) = \{f\} \]  

Let us now introduce suitable uniform convergence structures on \(\mathcal{ML}^{m+k}(\Omega)^K\) and \(\mathcal{ML}_T^{m+k}(\Omega)\). In this regard, recall [9] that a sequence \((x_n)\) on a partially ordered set \(L\) order converges to \(x \in L\) if and only if

\[ \exists \ (\lambda_n, \mu_n) \subset L : \]

\[ 1) \ n \in \mathbb{N} \Rightarrow \begin{pmatrix} 1.1 \\ 1.2 \end{pmatrix} : \]

\[ \lambda_n \leq \lambda_{n+1} \leq \mu_{n+1} \leq \mu_n \]

\[ 2) \ \sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\} \]  

In general, the order convergence of sequences cannot be induced through a topology. That is, for a partially ordered set \(L\), there is in general no topology \(\tau\) on \(L\) so that a sequence \((x_n)\) on \(L\) converges to \(x \in L\) with respect to \(\tau\) if and only if \((x_n)\) order converges to \(x\).
However, in view of Theorem 11, the order convergence of sequences on $\mathcal{M} \mathcal{L}^k (\Omega)$ may be induced by a convergence structure [2]. In particular, the order convergence structure $\lambda_o$, which is defined through

$$\forall \ n \in \mathbb{N} : \exists [\lambda_n, \mu_n] \subset \mathcal{M} \mathcal{L}^k (\Omega) :$$

1) $n \in \mathbb{N} \Rightarrow [\lambda_{n+1}, \mu_{n+1}] \subseteq [\lambda_n, \mu_n]$

2) $\sup\{\lambda_n : n \in \mathbb{N}\} = u = \inf\{\mu_n : n \in \mathbb{N}\}$

3) $\{[\lambda_n, \mu_n] : n \in \mathbb{N}\} \subseteq \mathcal{F}$

induces the order convergence of sequences. Furthermore, $\lambda_o$ is Hausdorff and first countable, see [2, Theorem 17]. The Cartesian product $\mathcal{M} \mathcal{L}^k (\Omega)^K$ is equipped with the product convergence structure $\lambda_o^K$, see [3], which is defined through

$$\forall U \text{ a filter on } \mathcal{M} \mathcal{L}^k (\Omega)^K \times \mathcal{M} \mathcal{L}^k (\Omega)^K :$$

$$\exists F_1, \ldots, F_k \text{ filters on } \mathcal{M} \mathcal{L}^k (\Omega)^K :$$

1) $F_i \in \lambda_o (u_i), \ i = 1, \ldots, K$

2) $U \supseteq (F_1 \times F_1) \cap \ldots \cap (F_k \times F_k)$

(58)

is uniformly Hausdorff and complete. Furthermore, the uniform convergence structure (58) induces the convergence structure $\lambda_o^K$ on $\mathcal{M} \mathcal{L}^k (\Omega)^K$.

The space $\mathcal{M} \mathcal{L}^m + k (\Omega)$ will carry the initial uniform convergence structure $\mathcal{J}_T$ with respect to the injective mapping

$$\hat{T} : \mathcal{M} \mathcal{L}^m + k (\Omega) \rightarrow \mathcal{M} \mathcal{L}^k (\Omega)^K.$$ 

That is,

$$U \in \mathcal{J}_T \Leftrightarrow (\hat{T} \times \hat{T}) (U) \in \mathcal{J}_K.$$ 

(59)

The following is now immediate.

**Proposition 12** The mapping $\hat{T}$ is a uniformly continuous embedding of the uniform convergence space $\mathcal{M}_T^k (\Omega)$ into the uniform convergence space $\mathcal{M} \mathcal{L}^k (\Omega)^K$. 

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As in the rest of the paper, we denote by $\mathcal{ML}^{m+k}_T (\Omega)^{\sharp}$ the uniform convergence space completion of $\mathcal{ML}^{m+k}_T (\Omega)$. The extension of the uniformly continuous embedding 

$$\hat{T} : \mathcal{ML}^{m+k}_T (\Omega) \to \mathcal{ML}^k (\Omega)^K$$

is denoted $\hat{T}^{\sharp}$. The generalized equation, corresponding to (36), now takes the form

$$\hat{T}^{\sharp} U^{\sharp} = f$$

(60)

In view of the equivalence of the equations (51) and (52), a solution $U^{\sharp}$ of (60) is interpreted as a generalized solution of (36).

The existence of solutions of (60) follows as an application of the following basic approximation result [15].

**Theorem 13** Consider a system of nonlinear PDEs of the form (36) through (38) that also satisfies (39). For every $\epsilon > 0$ there exists a closed nowhere dense set $\Gamma_\epsilon \subset \Omega$, and a function $u_\epsilon \in C^\infty (\Omega \setminus \Gamma_\epsilon)^K$ such that

$$f_j (x) - \epsilon \leq T_j (x, D) u_\epsilon (x) \leq f_j (x), \ x \in \Omega \setminus \Gamma_\epsilon$$

(61)

for every $j = 1, \ldots, K$.

The main result of this section is the following.

**Theorem 14** Consider a system of nonlinear PDEs of the form (36) through (38). For every $f \in C^k (\Omega)^K$ that satisfies (39), there exists a unique $U^{\sharp} \in \mathcal{ML}^{m+k}_T (\Omega)^{\sharp}$ such that

$$\hat{T}^{\sharp} U^{\sharp} = f.$$

**Proof.** First let us show existence. For every $n \in \mathbb{N}$, Theorem 13 yields a closed nowhere dense set $\Gamma_n \subset \Omega$ and a function $u_n \in C^\infty (\Omega \setminus \Gamma_n)^K$ such that

$$x \in \Omega \setminus \Gamma_n \Rightarrow f_j (x) - \frac{1}{n} \leq T_j (x, D) u_n (x) \leq f_j (x)$$

(62)

for every $j = 1, \ldots, K$. Since $\Gamma_n$ is closed nowhere dense we associate $u_n$ with the function $v_n \in \mathcal{ML}^{m+k}_T (\Omega)^K$, the components of which are defined as

$$v_{n,i} = (I \circ S) (u_{n,i}).$$

Clearly, we now have, for each $n \in \mathbb{N}$ and $j = 1, \ldots, K$, the inequalities

$$f_j - \frac{1}{n} \leq T_j v_n \leq f_j$$

(63)

Consider now, for each $n \in \mathbb{N}$, the equivalence class $U_n \in \mathcal{ML}^{m+k}_T (\Omega)$ associated with the function $v_n$ through (18).
Clearly, the sequence \( \left( \widehat{T}U_n \right) \) converges to \( f \) in \( \mathcal{ML}^k(\Omega)^K \). It now follows that \((U_n)\) is a Cauchy sequence in \( \mathcal{ML}_T^{m+k}(\Omega) \) so that there exists \( U^\sharp \in \mathcal{ML}_T^{m+k}(\Omega) \) that satisfies (60).

Since the mapping \( \widehat{T} : \mathcal{ML}_T^{m+k}(\Omega) \rightarrow \mathcal{ML}^k(\Omega)^K \) is a uniformly continuous embedding, the uniqueness of the solution \( U^\sharp \) found above now follows by Corollary 4.

Note that the uniqueness of the generalized solution \( U^\sharp \) to (60) should not be misinterpreted as implying that any, possibly classical, solutions are disregarded. In fact, quite the contrary. Recall that the completion of a uniform convergence space \( X \) may be obtained constructively. In particular, it consists of all equivalence classes of Cauchy filters on \( X \) so that the members of an equivalence class \([\mathcal{F}]\) all converge to the same element of the completion \( X^\sharp \). That is, if we denote by \( X_C \) the set of Cauchy filters on \( X \), then

\[
X^\sharp = X_C / \sim_C \tag{64}
\]

where \( \sim_C \) is the equivalence relation on \( X_C \) defined as

\[
\mathcal{F} \sim_C \mathcal{G} \iff \mathcal{F} \cap \mathcal{G} \in X_C. \tag{65}
\]

In view of this, the unique generalized solution is in fact the totality of all approximate solutions in \( \mathcal{ML}_T^{m+k}(\Omega)^K \). In particular, every classical solution \( u \in C^{m+k}(\Omega)^K \) of (36), and every solution \( u \in \mathcal{ML}^{m+k}(\Omega)^K \) of (51) generates a Cauchy filter on \( \mathcal{ML}_T^{m+k}(\Omega) \) which converges to \( U^\sharp \) in \( \mathcal{ML}_T^{m+k}(\Omega)^\sharp \). As such, the unique generalized solution of (36) contains also all of the mentioned usual solutions, should such solutions exist.

Notice also that the mapping

\[
\widehat{T}^\sharp : \mathcal{ML}_T^{m+k}(\Omega)^\sharp \rightarrow \mathcal{ML}^k(\Omega)^K
\]

is injective. As such, we may consider the completion \( \mathcal{ML}_T^{m+k}(\Omega)^\sharp \) of \( \mathcal{ML}_T^{m+k}(\Omega) \) as a subset of \( \mathcal{ML}^k(\Omega)^K \), equipped with a suitable uniform convergence structure. Hence, as a bonus, we also have a blanket regularity in the sense that every element \( U^\sharp \) of \( \mathcal{ML}_T^{m+k}(\Omega)^\sharp \) may be assimilated with elements of \( \mathcal{ML}^k(\Omega)^K \).

The results on existence, uniqueness and regularity of generalized solutions to (36) obtained in this section are, to a certain extent, maximal with respect to the regularity of solutions within the framework of the so called pullback spaces of generalized functions considered here. In this regard, let us now formulate the construction of a generalized solution in an abstract framework. Consider spaces \( X \) and \( Y \) of functions \( g : \Omega \rightarrow \mathbb{R}^K \) such that \( f \in Y \), and the nonlinear partial differential operator \( T \) associated with (36) acts as

\[
T : X \rightarrow Y. \tag{66}
\]

Also suppose that \( Y \) is equipped with a complete and Hausdorff uniform convergence structure \( J_Y \) which is first countable. Proceeding in the same way as is done in this
section, we introduce an equivalence relation on $X$ through

$$u \sim_T v \Leftrightarrow Tu = Tv,$$

and associate with the mapping (66) the injective mapping

$$\hat{T}_X : X_T \rightarrow Y,$$  \hspace{1cm} (67)

where $X_T$ is the quotient space $X/\sim_T$. In particular, the mapping (67) is supposed to satisfy

$$\forall \ U \in X_T :$$

$$\forall \ u \in U :$$

$$Tu = \hat{T}_X U = f.$$

If we equip $X_T$ with the initial uniform convergence structure $J_T$ with respect to the mapping (67), then $J_T$ is Hausdorff and first countable. In particular, the mapping (67) is a uniformly continuous embedding, and extends uniquely to a injective uniformly continuous mapping

$$\hat{T}_X^\sharp : X_T^\sharp \rightarrow Y,$$  \hspace{1cm} (68)

where $X_T^\sharp$ is the completion of $X_T$. A generalized solution of the systems of nonlinear PDEs

$$Tu = f$$

in this context is any solution $U^\sharp \in X_T^\sharp$ of the equation

$$\hat{T}_X^\sharp U^\sharp = f.$$  \hspace{1cm} (69)

Note that, in view of the fact that the mapping (67) is a uniformly continuous embedding, and (68) therefore an injection, the equation (69) can have at most one solution.

Now, in order to show the existence of a solution of (69), we must construct a sequence $(u_n)$ in $X$ so that $(Tu_n)$ converges to $f$ in $Y$. In this regard, the most general such result is given by Theorem 13. As such, within such a general context as considered here, it follows that, if the mapping (37) is $C^k$-smooth, we have

$$X \supseteq \mathcal{M}C^{n+k}(\Omega)^K.$$  \hspace{1cm} (70)

It now follows by (66) and (70) that

$$Y \supseteq \mathcal{M}C^k(\Omega)^K.$$  \hspace{1cm} (71)
This may be summarized in the following commutative diagram.

\[
\begin{array}{c}
\mathcal{M}L^{m+k}(\Omega)^K \ni \mathcal{T} \quad \mathcal{M}L^k(\Omega)^K \\
\downarrow \quad \uparrow \\
X \quad Y
\end{array}
\]

Combining the diagram (72) with (50) and

\[
\begin{array}{c}
X \ni \mathcal{T} \quad Y \\
\downarrow \quad \uparrow \\
\hat{T}_X \quad X_
\end{array}
\]

we obtain an injective mapping

\[
\iota_T : \mathcal{M}L^{m+k}_T(\Omega) \rightarrow X_T
\]

so that the diagram

\[
\begin{array}{c}
\mathcal{M}L^{m+k}_T(\Omega) \ni \hat{T} \quad \mathcal{M}L^k(\Omega)^K \\
\downarrow \quad \uparrow \\
X_T \quad Y
\end{array}
\]

(75)
commutes. In particular, if the subspace convergence structure induced on $\mathcal{ML}_{m+k}^T(\Omega)$ from $Y$ is coarser than the order convergence structure, then the mapping $i^\sharp_T$ is uniformly continuous. Furthermore, in this case the mapping $i^\sharp_T$ extends to an injective uniformly continuous mapping

$$i^\sharp_T : \mathcal{ML}_{m+k}^T(\Omega)^\sharp \to X^\sharp_T$$

so that the extended diagram

$$\mathcal{ML}_{T}^{m+k}(\Omega)^\sharp \xrightarrow{i^\sharp_T} \mathcal{ML}_{k}(\Omega)^K$$

$$\xrightarrow{\hat{T}^\sharp} Y$$

The existence of the injective mapping $i^\sharp_T$ may be interpreted as follows. Any pullback type space of generalized functions $X^\sharp_T$ which is constructed as above, and subject to the condition of generality of the nonlinear partial differential operator $T$ must contain the space $\mathcal{ML}_{m+k}^T(\Omega)^\sharp$. As such, within the context of general, continuous systems of nonlinear PDEs, the generalized functions in $\mathcal{ML}_{m+k}^T(\Omega)^\sharp$ may be considered to be ‘more regular’ than those in any other space of generalized functions constructed in this way.

6 Conclusion

In this paper we have shown that initial uniform convergence structures are, in general, not preserved by the Wyler completion. In particular, we discuss the completion of subspaces and products of uniform convergence spaces in some detail. Nevertheless, some insight into the structure of the completion of an initial uniform convergence space is obtained.

As an application of these results, we obtain the existence of generalized solutions of arbitrary $C^k$-smooth systems of nonlinear PDEs. In addition, a blanket regularity result is obtained, in the sense that every generalized solution may be assimilated with functions which are $C^k$-smooth everywhere except on a closed nowhere dense set. These results are shown to be maximal, with respect to regularity, within the setting of the so called pullback spaces of generalized functions used here and in [10, 14] and [15].
References

[1] R. Anguelov and E. E. Rosinger, Solving large classes of nonlinear systems of PDE’s, *Computers and Mathematics with Applications* **53** (2007), 491-507.

[2] R. Anguelov and J. H. van der Walt, Order convergence structure on $C(X)$, *Quaestiones Mathematicae* **28** (2005), 425-457.

[3] R. Beattie and H. P. Butzmann, *Convergence structures and applications to functional analysis* Kluwer Academic Publishers, Dordrecht, Boston, London (2002).

[4] R. P. Dilworth, The normal completion of the lattice of continuous functions, *Transactions of the AMS* **68** (1950), 427-438.

[5] R. Fric and D. C. Kent, Completion of pseudotopological groups, *Mathematische Nachrichten* blf 99 (1980), 99-103.

[6] W. Gähler W, *Grundstrukturen der anlysis II*, Birkhäuser Verlag, Basel (1978).

[7] S. Gähler, W. Gähler and G. Kneis, Completion of pseudo-topological vector spaces *Mathematische Nachrichten* blf 75 (1976), 185-206.

[8] D. C. Kent and R. Ruiz de Eguino, On products of Cauchy completions, *Mathematische Nachrichten* **155** (1992), 47-55.

[9] W. A. Luxemburg and A. C. Zaanen, *Riesz Spaces I* North-Holland, Amsterdam, London (1971).

[10] M. Oberguggenberger and E. E. Rosinger, *Solution of continuous nonlinear PDEs through order completion*, North-Holland, Amsterdam, London, New York, Tokyo (1994).

[11] E. E. Reed, Completions of uniform convergence spaces, *Mathematische Annalen* **194** (1971), 83-108.

[12] E. E. Rosinger, *Nonlinear partial differential equations, an algebraic view of generalized solutions*, North Holland Mathematics Studies, vol. 164 (1990).

[13] J. H. van der Walt, Order convergence on Archimedean vector lattices with applications, MSc Thesis, University of Pretoria, 2006.

[14] J. H. van der Walt, The uniform order convergence structure on $\mathcal{ML}(X)$, *Quaestiones Mathematicae* **31** (2008), 55-77.

[15] J. H. van der Walt, The order completion method for systems of nonlinear PDEs: Pseudotopological perspectives, *Acta Applicandae Mathematicae* **103** (2008), 1-17.

[16] J. H. van der Walt, The order completion method for systems of nonlinear PDEs revisited, To Appear in *Acta Applicandae Mathematicae*.
[17] J. H. van der Walt, The order completion method for systems of nonlinear PDEs: Regularity of generalized solutions, Technical Report UPWT 2008/??, University of Pretoria, 2008.

[18] O. Wyler, Ein komplettierungsfunktor für uniforme limesräume, *Mathematische Nachrichten* 40 (1970), 1-12.