Time Discrete Approximation of Weak Solutions for Stochastic Equations of Geophysical Fluid Dynamics and Applications

Nathan Glatt-Holtz\textsuperscript{♯}, Roger Temam\textsuperscript{♭}, Chuntian Wang\textsuperscript{♭}

\textsuperscript{♯} Department of Mathematics, Virginia Polytechnic and State University
Blacksburg, VA 24061

\textsuperscript{♭} Department of Mathematics and The Institute for Scientific Computing and Applied Mathematics
Indiana University, Bloomington, IN 47405

emails: negh@vt.edu, temam@indiana.edu, wang211@indiana.edu

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Abstract
As a first step towards the numerical analysis of the stochastic primitive equations of the atmosphere and oceans, we study their time discretization by an implicit Euler scheme. From deterministic viewpoint the 3D Primitive Equations are studied with physically realistic boundary conditions. From probabilistic viewpoint we consider a wide class of nonlinear, state dependent, white noise forcings. The proof of convergence of the Euler scheme covers the equations for the oceans, atmosphere, coupled oceanic-atmospheric system and other geophysical equations. We obtain the existence of solutions weak in PDE and probabilistic sense, a result which is new by itself to the best of our knowledge.

Keywords: Nonlinear Stochastic Partial Differential Equations, Geophysical Fluid Dynamics, Primitive Equations, Discrete Time Approximation, Martingale Solutions, Numerical Analysis of Stochastic PDEs.

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1 Introduction

The primitive equations of the oceans and atmosphere (PEs) are a fundamental model for the large scale fluid flows forming the analytical core of the most advanced general circulation models (GCMs) in use today. In recent years these systems have been a subject of considerable interest in the mathematical community not only because of their wide significance in geophysical applications but also for their delicate nonlinear, nonlocal, anisotropic structure and as a cousin to the other basic equations of mathematical fluid dynamics, namely the incompressible Navier-Stokes and Euler equations.

In this work we study a stochastic version of the PEs and develop techniques which may be viewed as a first step toward their numerical analysis. From the point of view of applications, this work is motivated by a plea from the geophysical community to further develop the theory of nonlinear Stochastic Partial Differential Equations (SPDEs) in a large scale fluid dynamics context and in general, [RTT06]. Indeed, in view of the many sources of uncertainty both physical and numerical which are typically encountered by the modeler, stochastic techniques are playing an increasingly central role in the study of geophysical fluid dynamics. See e.g. [Has76, Ros77, LL79, MT92, PS95, PE08, EP09, BSLP09, ZF10] and also [GTT] for a small sampling of this vast literature.

The primitive equations trace their origins to the beginning of the 20th century with the seminal works of V. Bjerknes and L. F. Richardson ([Bjo94, Ric07]) and have played a central role in the development of climate modeling and weather prediction since that time, [Ped82]. To the best of our knowledge, the development of the mathematical theory for the deterministic PEs began in the early 1990’s with a series of articles by J. L. Lions, R. Temam and S. Wang, [LTW92b, LTW92a, LTW93]. This direction in mathematical geophysics is now a fairly well developed subject with results guaranteeing the global existence of weak solutions which are bounded in $L^2_{x}$, [LTW92b] and the global existence and uniqueness of strong solutions, i.e. solutions evolving continuously in $H^1_{x}$, [Kob06, Kob07, CT07, KZ07]. Of course, these latter developments stand in striking contrast to the current state of the art for the Navier-Stokes equations as proving the global existence and uniqueness of strong solutions is tantamount to solving the famous Clay problem. For further background on the determinstic mathematical theory see the recent surveys [PTZ08, RTT09].

Recently, significant efforts have been made to establish suitable analogues of the above (deterministic) mathematical results in a stochastic setting. In a series of works, [EPT07, GZ09, GH09, GT11a, GT11b, DGHT11, DGHTZ12], the mathematical theory of strong, pathwise\footnote{Here pathwise refers to the fact that solutions are found relative to a prescribed driving noise. In this article we will use the terms ‘pathwise’ and ‘martingale’ as opposed to the alternate terminology of ‘weak’ and ‘strong’ solutions to avoid confusion with the typical PDE terminology for which weak solutions are, roughly speaking, those in $L^2$ ($L^2$) and strong solutions are those in $L^\infty$ ($H^1$).} solutions has been developed. These recent works more or less bring this aspect of the subject to the state of the art, that is they establish, in increasingly physically realistic settings, the global existence and uniqueness of solutions evolving continuously in $H^1_{x}$.

Notwithstanding the above cited body of works, many aspects of the stochastic theory still need further consideration. In this article we develop existence results for weak solutions, that is solutions which remain bounded in time only in $L^2_{x}$. This is a direction which, to the best of our knowledge, remained unaddressed previously. Since such ‘weak solutions’ are not expected to be unique, even in the deterministic setting, it is natural to work within the framework of martingale solutions. In other words we consider below solutions which are weak in both the sense of PDE theory and stochastic analysis.

One particular advantage of this weak-martingale setting is that it allows us to consider physical situations unattainable so far in the above cited works on strong (or strong-pathwise) solutions. From the deterministic point of view we obtain results for the case of inhomogenous, physically realistic boundary conditions. On the other hand, from the stochastic viewpoint our results cover a very general class of state-dependent (multiplicative) noise structures. In particular these noise terms may be interpreted in either the Itô or Stratonovich sense. The later Stratonovich interpretation of noise is important as it may be more realistic in geophysical settings. See e.g. [WR84], [Pen03] for further details. Note that we develop our analysis in a slightly abstract setting which at once allows us to treat the PEs of the oceans, the atmosphere and the...
coupled oceanic/atmospheric system.²

While the results established here take an important further step in the development of the analytical theory for the PEs we believe the main contribution of this article relates to numerical considerations. The approach below centers on an implicit Euler (i.e. time discrete) scheme and we choose this set-up mainly because it may be seen as a mathematical setting suitable for the development of tools needed for the numerical analysis of the stochastic PEs and other nonlinear SPDEs arising in fluid dynamics. Note that while discrete time approximation has been previously employed in [DBD04, DP06], these works treat hyperbolic type systems and only address the case of an additive noise. As such, a number of the techniques developed here, play a crucial role in a work related to the stability and consistency of a class of numerical schemes (both explicit and semi-implicit) for the 2D and 3D stochastic Navier-Stokes equations, [GTW].

Let us now finally turn to sketch some of the main technical challenges and contributions of the article. In fact the first main difficulty is to justify the validity of the implicit scheme on which our analysis centers. While classical arguments involving the Brouwer fixed point theorem can be used to establish the existence of sequences satisfying the implicit scheme, we crucially need that these sequences are adapted to the driving noise. To address this concern we rely on a specifically chosen filtration and a suitable measurable selection theorem from [BT73] (see also [KRN65], [Cas67]).

With suitable solutions to the semi-implicit scheme in hand, basic uniform estimates proceed analogously to the continuous time case with the use of martingale inequalities, etc. In contrast to previous works on Martingale solutions (see e.g. [Ben95, FG95, MS02, DGHT11, GV14]) we circumvent the need for higher moments with suitable stopping time arguments. Another difficulty related to the concern that solutions be adapted appears when we associate continuous time processes with the discrete time schemes in pursuit of compactness and the passage to the limit. In contrast to the deterministic case, [Tem01], [MT98] we must introduce processes which are lagged by a time step. While these processes are indeed adapted, we obtain a time evolution equation with troublesome error terms. In turn these error terms prevent us from addressing compactness directly from the equations and force us to carry out the compactness arguments for a series of interrelated processes.

Organization of the Article

The exposition is organized as follows. In Section 2 we outline an abstract, functional-analytic framework for the stochastic Primitive Equations (and related evolution systems) which may be seen as an “axiomatic”; basis for the rest of the work. The section concludes by recalling the basic notion of Martingale solutions within the context of this framework. In Section 3 we introduce an implicit Euler scheme which discretizes the equations in time. The details of the existence of suitable solutions (adapted to the specific filtration) of this implicit scheme along with associated uniform estimates are given in Propositions 3.1 and 3.2 respectively. In Section 4 we study some continuous time processes associated with the implicit Euler scheme introduced in Section 3. Section 5 then outlines the compactness (tightness) arguments that allow us to pass to the limit and derive the existence of solutions from these approximating continuous time processes. Finally, Section 7 provides extended details connecting the abstract results that we just derived with the concrete example of the primitive equations of the oceans. In this section we also provide a number of examples of possible types of nonlinear state dependent noises covered under the main abstract results. In the interest of making the manuscript as self-contained as possible an Appendix (Section A) collects various technical tools used in the course of our analysis.

2 The Abstract Problem Set-Up

We begin by describing the setting for the abstract evolution equation that we will study below (cf. (2.13) at the end of this Section). As we noted in the introduction, we take this point of view in order to systematically treat the existence of weak solutions for a class of geophysical fluids equations including but not limited to

²We have previously taken such an abstract approach in other work on the stochastic primitive equations, [DGHT11]. There however our focus was on the local existence of strong, pathwise solutions and that framework was, by necessity, more restrictive with respect to domains, noise structures, etc.
the example (7.1)–(7.4) developed below in Section 7. For further details about how to cast other related equations of geophysical fluid dynamics in the following abstract formulation we refer the reader to [PTZ08] and the references therein.

Throughout what follows we fix a Gelfand-Lions inclusion of Hilbert spaces

\[ V(3) \subset V(2) \subset V \subset V' \subset V'(2) \subset V'(3), \]  

(2.1)

Each space is densely, continuously and compactly embedded in the next one. We will denote the norms for \( H \) and \( V \) by \( |\cdot| \) and \( \|\cdot\| \) and the remaining spaces simply by e.g. \( \|\cdot\|_{V'(2)} \). When the context is clear, we will denote the dual pairing between \( V, V', V(2), V(2) \) or \( V(3), V(3) \) by \( \langle \cdot, \cdot \rangle \).

2.1 Basic Operators

We now outline the main elements, a collection of abstract operators, which we use to build the stochastic evolution (2.13) below. We suppose we are given:

- A linear continuous operator \( A : V \mapsto V' \) which defines a bilinear continuous form \( a(U, U') := \langle AU, U' \rangle_{V', V} \) on \( V \). We assume that \( a \) is coercive, i.e.

\[
a(U, U) \geq c_1 \|U\|^2 \quad \text{for all } U \in V. \tag{2.2}
\]

This term will typically capture the diffusive terms in the concrete equations: molecular and eddy viscosity, diffusion of heat, salt, humidity etc.

- A second linear operator \( E \) continuous on both \( H \) and \( V \); \( E \) defines a bilinear continuous form \( e(U, U') := (EU, U') \) on \( H \) (which is also continuous on \( V \)). We suppose furthermore that \( e \) is antisymmetric, that is

\[
e(U, U) = 0 \quad \text{for all } U \in H. \tag{2.3}
\]

This term \( E \) appears in applications to account for the Coriolis (rotational) forces coming from the rotation of the earth.

- A bilinear form \( B \) which continuously maps \( V \times V \) into \( V(2) \); \( B \) gives rise to an associated trilinear form \( b(U, U^\flat, U^\sharp) := \langle B(U, U^\flat), U^\sharp \rangle \) which satisfies the estimates

\[
|b(U, U^\flat, U^\sharp)| \leq c_2 \|U\|^1/2\|U^\flat\|^{1/2}\|U^\sharp\| \quad \text{for all } U, U^\flat, U^\sharp \in V. \tag{2.4}
\]

Moreover we assume the antisymmetry property

\[
b(U, \tilde{U}, \tilde{U}) = 0 \quad \text{for all } U \in V, \tilde{U} \in V(2). \tag{2.5}
\]

Note that, in particular, we may infer from (2.4) that

\[
\|B(U)\|_{V(2)} \leq c_2 \|U\|^3/2 \quad \text{for any } U \in V. \tag{2.6}
\]

Furthermore, we infer from (2.4), (2.5) we may assume that \( B \) is continuous from \( V \times V(2) \) into \( V' \) and satisfies

\[
\|B(U)\|_{V'} \leq c_2 \|U\| \|U\|_{V(2)} \quad \text{for all } U \in V(2). \tag{2.7}
\]

In previous works on the Stochastic PEs, [GT11a, GT11b, DGHT11] we required that this \( a \) be symmetric. In particular such a symmetry was strongly used in these previous works so that we could apply the spectral theorem to the inverse of an associated operator \( A^{-1} \). This is not needed for the arguments presented here and we therefore revert to the more general weak formulation of the PEs given in [PTZ08].
Finally we impose some additional technical convergence conditions on $b$. Firstly we suppose that when $U_k$ converges weakly to $U$ in $V$ then, up to a subsequence $k'$,

$$b(U_k', U_k', U^t) \to b(U, U, U^t) \quad \text{for each } U^t \in V(2).$$

(2.8)

Similarly we assume that if, for some $T > 0$,

$$U_k \to U \quad \text{weakly in } L^2(0, T; V) \text{ and strongly in } L^2(0, T; H),$$

then, again up to a subsequence $k'$,

$$\int_0^T b(U_k', U_k', U^t)dt \to \int_0^T b(U, U, U^t)dt \quad \text{for each } U^t \in L^\infty(0, T; V(3)).$$

(2.9)

$B$ accounts for the main nonlinear (convective) terms in the equations.

- An externally given element $\ell$. We consider $\ell$ to be random in general; it is specified only as a probability distribution on $L^2_{loc}(0, \infty; V')$ subject to the second moment condition (2.17) given below. This term $\ell$ captures various inhomogeneous elements i.e. externally determined body forcings, boundary forcings etc.

In order to define the operators involving the ‘stochastic terms’ in the equations we consider an auxiliary space $\Omega$, on which the underlying driving noise, a cylindrical Brownian motion $W$ evolves (see Section 2.2 below). We suppose $\Omega$ is a separable Hilbert space and use $L_2(\Omega, X)$ to denote the space of Hilbert-Schmidt operators from $\Omega$ into $X$, where, for example $X = H, V$ or $\mathbb{R}$. Sometimes we will abbreviate and write $L_2 := L_2(\Omega, \mathbb{R})$.

Returning to the list of operators we suppose we have defined:

- A (possibly nonlinear) continuous map $\sigma : [0, \infty) \times H \mapsto L_2(\Omega, H)$. We suppose that $\sigma$ is uniformly sublinear, i.e.

$$|\sigma(t, U)|_{L_2(\Omega, H)} \leq c_3(1 + |U|), \quad \text{for every } U \in H \text{ and } t \in \mathbb{R}^+,$$

(2.10)

where the constant $c_3 > 0$ is independent of $t \in [0, \infty)$. For economy of notation we will frequently drop the dependence on $t$ in the exposition below. We define $g : [0, \infty) \times H \times H \mapsto L_2$ according to

$$g(t, U, U^\tau) = (\sigma(t, U), U^\tau) \quad \text{for } U, U^\tau \in H.$$  

(2.9)

The element $\sigma$ determines the structure of the (volumic) stochastic forcing applied to the equations. These stochastic terms typically appear to account for various sources of physical, empirical and numerical uncertainty as we described in the introduction.

- A continuous map $\xi : [0, \infty) \times H \mapsto H$ which is subject to the uniform sublinear condition

$$|\xi(t, U)| \leq c_4(1 + |U|), \quad \text{for every } U \in H \text{ and } t \in \mathbb{R}^+,$$

(2.11)

where $c_4 > 0$ does not depend on $t \geq 0$. We define $s : [0, \infty) \times H \times H \mapsto \mathbb{R}$ by

$$s(t, U, U^\tau) = (\xi(t, U), U^\tau)$$

(2.12)

for $U, U^\tau \in H$. We include $\xi$ in the abstract formulation to allow, in particular, for the treatment of a class of Stratonovich noises; $\xi$ arises when we convert from a Stratonovich into an Itô type noise. This term $S$ therefore allows us to carry out the forthcoming analysis entirely within the Itô framework. See Remarks 2.1, 7.3 below.

With the above abstract framework now in place we may reduce the problem (7.1)–(7.4) below (and related equations) to studying the following abstract stochastic evolution equation in $V(2)$), namely,

$$dU + (AU + B(U) + EU)dt = (\ell + \xi(U))dt + \sigma(U)dW, \quad U(0) = U^0.$$  

(2.13)
This system is to be interpreted in the Itô sense which we recall immediately below in Subsection 2.2.

Note that \( U^0 \) and \( t \) in (2.1) are considered to be random in general. Indeed, since we are studying Martingale Solutions of (2.13) where the underlying stochastic elements in the problem are considered as unknowns, we will specify \( U^0 \) and \( t \) only as probability distributions on \( H \) and \( L^2(0, T; V') \). See Definition 2.1 and the Remark 2.1 following. Note also that, for brevity of notation, we will sometimes write

\[
\mathcal{N}(t, U) := -(AU + B(U) + EU - \xi(t, U)),
\]

in the course of the exposition below. When the context is clear we will sometimes drop the dependence in \( t \) and simply write \( \mathcal{N}(U) \).

2.2 Some Elements of Stochastic Analysis and Abstract Probability Theory

Of course, (2.13) is understood relative to a stochastic basis \( \mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1}) \), that is a filtered probability space with \( \{W^k\}_{k \geq 1} \) a sequence of independent standard 1-d Brownian motions relative to \( \mathcal{F}_t \). Here we may define \( W \) on \( \mathcal{U} \) by considering an associated orthonormal basis \( \{e_k\}_{k \geq 1} \) of \( \mathcal{U} \) and taking \( W = \sum_k W^k e_k \); \( W \) is thus a ‘cylindrical Brownian’ motion evolving over \( \mathcal{U} \).

Actually, this sum \( W = \sum_k W e_k \) is only formal; it does not generally converge in \( \mathcal{U} \). For this reason we will occasionally make use of a larger space \( \mathcal{U}_0 \supset \mathcal{U} \) which we define according to

\[
\mathcal{U}_0 := \left\{ v = \sum_{k \geq 0} \alpha_k e_k : |v|^2_{\mathcal{U}_0} < \infty \right\}, \quad \text{where } |v|^2_{\mathcal{U}_0} := \sum_k \alpha_k^2 \quad \text{and } |v|^2_{\mathcal{U}} := \sum_k \frac{\alpha_k^2}{k^2}.
\]

Note that the embedding of \( \mathcal{U} \subset \mathcal{U}_0 \) is Hilbert-Schmidt. Moreover, using standard martingale arguments with the fact that each \( W_k \) is almost surely continuous we have that, for almost every \( \omega \in \Omega \), \( W(\omega) \in \mathcal{C}([0, T], \mathcal{U}_0) \).

Since, (2.13) is actually short hand for a stochastic integral equation we next briefly recall some elements of the theory of Itô stochastic integration in infinite dimensional spaces. We choose an arbitrary Hilbert space \( X \) and, as above, we use \( L_2(\mathcal{U}, X) \) to denote the collection of Hilbert-Schmidt operators from \( \mathcal{U} \) into \( X \). Given an \( X \)-valued predictable\(^4\) process \( G \in L^2(\Omega; L^2_{\text{loc}}(0, \infty, L_2(\mathcal{U}, X))) \) the (Itô) stochastic integral

\[
M_t := \int_0^t G dW = \sum_k \int_0^t G_k dW_k, \quad \text{where } G_k = G e_k,
\]

is defined as an element in \( \mathcal{M}_X^2 \), the space of all \( X \)-valued square integrable martingales (see [PR07, Section 2.2, 2.3]). For further details on the general theory of infinite-dimensional stochastic integration and stochastic evolution equations we refer the reader to e.g. [DPZ92, PR07].

Since we will be working in the setting of Martingale solutions, where the data in the problem (2.13) is specified only as a probability distribution (over an appropriate function space), it is convenient to introduce some further notations around Borel probability measures. Let \( (\mathcal{H}, \rho) \) be a complete metric space and denote the family of Borel probability measures on \( \mathcal{H} \) by \( Pr(\mathcal{H}) \). Given a Borel measurable function \( f : \mathcal{H} \mapsto \mathbb{R} \) and an element \( \mu \in Pr(\mathcal{H}) \) we will sometime write \( \mu(f) \) for \( \int_\mathcal{H} f(x) d\mu(x) \) when the associated integral makes sense. In particular we will write

\[
\mu(|f|) < \infty \iff \int_\mathcal{H} |f(x)| d\mu(x) < \infty.
\]

We will review some basic properties related to convergence and compactness of subsets of \( Pr(\mathcal{H}) \) in the Appendix, Section A.1, below. We refer the reader to e.g. [Bil99] for an extended treatment of the general theory of probability measures on Polish spaces which include Hilbert spaces such as \( H \) and \( V \).

\(^4\)For a given stochastic basis \( \mathcal{S} \), let \( \Phi = \Omega \times [0, \infty) \) and take \( \mathcal{G} \) to be the sigma algebra generated by the sets of the form \( (s, t] \times F, \) with \( 0 \leq s < t < \infty \) and \( F \in \mathcal{F}_s; \quad \{0\} \times F; \quad F \in \mathcal{F}_0. \)

Recall that an \( X \) valued process \( U \) is called predictable (with respect to the stochastic basis \( \mathcal{S} \)) if it is measurable from \((\Phi, \mathcal{G})\) into \((X, \mathcal{B}(X))\) where \( \mathcal{B}(X) \) denotes the family of Borelian subsets of \( X \).
2.3 Definition of Martingale Solutions and Statement of the Main Result

We turn now to give a rigorous meaning for the so-called *weak-martingale solutions* of \((2.13)\) which are defined as follows:

**Definition 2.1.** [Weak-Martingale Solutions]** Fix** \(\mu_U^0, \mu_\ell\) **Borel measures respectively on** \(H\) **and** \(L^2_{\text{loc}}(0, \infty; V')\) **with**

\[
\mu_U^0(|| \cdot ||_H^2) < \infty \quad \text{and} \quad \mu_\ell(|| \cdot ||_{L^2_{\text{loc}}(0,T; V')}^2) < \infty, \quad \text{for any} \quad T > 0.
\]  

\((2.17)\)

A *weak-martingale solution* \((\tilde{S}, \tilde{U}, \tilde{\ell})\) of \((2.13)\) consists of a stochastic basis \(\tilde{S} = (\tilde{\Omega}, \tilde{F}, \{\tilde{F}_t\}_{t \geq 0}, \tilde{\mathbb{P}}, \tilde{W})\) and processes \(\tilde{U}\) and \(\tilde{\ell}\) (defined relative to \(\tilde{S}\)) adapted to \(\{\tilde{F}_t\}_{t \geq 0}\). This triple \((\tilde{S}, \tilde{U}, \tilde{\ell})\) will enjoy the following properties

(i) for every \(T > 0\)

\[
\tilde{U} \in L^2(\tilde{\Omega}; L^\infty(0, T; H) \cap L^2(0, T; V)), \quad \tilde{U}\text{ is a.s. weakly continuous in } H,
\]

\[
\tilde{\ell} \in L^2(\tilde{\Omega}; L^2(0, T; V')).
\]  

(ii) For every \(t > 0\) and each test function \(U^\sharp \in V(2)\),

\[
(\tilde{U}(t), U^\sharp) + \int_0^t (a(\tilde{U}, U^\sharp) + b(\tilde{U}, U^\sharp) + e(\tilde{U}, U^\sharp)) ds
\]

\[
= (\tilde{U}(0), U^\sharp) + \int_0^t (\tilde{\ell}(U^\sharp) + s(\tilde{U}, U^\sharp)) dt + \int_0^t g(\tilde{U}, U^\sharp) d\tilde{W},
\]  

almost surely.

(iii) Finally, \(\tilde{U}(0)\) and \(\tilde{\ell}\) have the same laws as \(\mu_U^0, \mu_\ell\), i.e.

\[
\tilde{\mathbb{P}}(\tilde{U}(0) \in \cdot) = \mu_U^0(\cdot) \quad \text{and} \quad \tilde{\mathbb{P}}(\tilde{\ell} \in \cdot) = \mu_\ell(\cdot).
\]  

\((2.20)\)

With this definition in hand we now state one of the main results of the work as follows.

**Theorem 2.1.** Let \(\mu_U^0, \mu_\ell\) be a given pair of Borel measures on respectively \(H\) and \(L^2_{\text{loc}}(0, \infty; V')\) which satisfy the moment conditions \((2.17)\). Then, relative to this data, there exists a *martingale solution* \((\tilde{S}, \tilde{U}, \tilde{\ell})\) of \((2.13)\) in the sense of Definition 2.1.

**Remark 2.1.** Depending on the structure of \(\sigma\) the application of noise leads to a variety of different effects on the behavior of the solutions. In particular \(\sigma\) can be chosen so that the noise either provides a damping or an exciting effect. It is therefore unsurprising that the structure of the stochastic terms in e.g. \((7.1)\) remains a subject of ongoing debate among physicists and applied modelers. In any case, viewed as a proxy for physical and numerical uncertainty, the structure of the noise would be expected to vary by application. With this debate in mind we have therefore sought to treat a very general class of state-dependent noise structures in \(\sigma\) requiring only the sublinear condition \((2.10)\). We have illustrated some interesting examples covered under this condition in Section 7.3 below.

Actually, the Stratonovich interpretation of white noise driven forcing may often be more appropriate for applications in geophysics. See e.g. [WR84], [Pen05] for extended discussions on this connection. Note that although the equations \((2.13)\) are considered in an Itô sense, an additional, state dependent drift term \(\xi\) has been added to the equations which allows us to treat a class of Stratonovich noises with \((2.13)\) via the standard ‘conversion formula’ between Itô and Stratonovich evolutions. See e.g. [Arn74] and also Section 7.3 where we present one such example of Stratonovich forcing in detail.
3 A Discrete Time Approximation Scheme

We now describe in detail the semi-implicit Euler scheme, (3.3), which we use to approximate (2.13). This system is given rigorous meaning in Definition 3.1. We then recall a specific stochastic basis in Section 3.2.1 and establish the existence of solutions of (3.3) in Proposition 3.1 relative to this basis. We conclude this section by providing certain uniform bounds (energy estimates) independent of the time step of the discretization in Proposition 3.2.

3.1 The Implicit Scheme

Fix a stochastic basis $S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1})$ and elements $\ell \in L^2(\Omega; L^2_{loc}(0, \infty; V'))$, $U^0 \in L^2(\Omega; H)$ whose distributions correspond to the externally given $\mu_{\ell}$, $\mu_{U^0}$. For a given $T > 0$ and any integer $N$, let

$$\Delta t = T/N, \quad t^n = t^n_N = n\Delta t,$$

along with the associated stochastic increments

$$\eta^n = \eta^n_N = W(t^n) - W(t^{n-1}), \quad \text{for } n = 1, \ldots, N. \quad (3.1)$$

Using an implicit Euler time discretization scheme we would then like to approximate (2.13) by considering sequences $\{U^n_N\}_{n=1}^N$ satisfying

$$\frac{U^n_N - U^{n-1}_N}{\Delta t} + AU^n_N + B(U^n_N) + EU^n_N = \ell^n_N + \xi(t^n, U^n_N) + \sigma_N(t^n, U^{n-1}_N) \eta^n_N, \quad (3.2)$$

in $V'_{(2)}$ for $n = 1, \ldots, N$. For how to choose $U^n_0$, see Remark 3.1. The terms $\ell^n_N$ are given by

$$\ell^n_N(U^t_\Delta) = \frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} \ell(t, U^t_\Delta) dt \quad \text{for } n = 1, 2, \ldots, N, \quad (3.3)$$

and the operator $\sigma_N : [0, \infty) \times H \to L^2(\Omega, V)$ is any approximation of $\sigma$ which satisfies

$$||\sigma_N(t, U)||^2_{L^2(\Omega; V)} \leq N ||\sigma(t, U)||^2_{L^2(\Omega, H)}; \quad (3.4)$$

$$|\sigma_N(t, U)|^2_{L^2(\Omega, H)} \leq |\sigma(t, U)|^2_{L^2(\Omega, H)}; \quad (3.5)$$

for every $t \geq 0$ and every $U \in H$. Additionally we suppose that, for any $t \geq 0$,

$$\lim_{N \to \infty} \sigma_N(t, U_N) = \sigma(t, U), \quad \text{whenever } U_N \to U \text{ in } H. \quad (3.6)$$

For the existence of such $\sigma_N$, see Remark 3.1. We write $g_N(t, U, U^t) = (\sigma_N(t, U), U^t)^t$.\footnote{The choice of a “time explicit” term in $\sigma_N(t^{n-1}, U^{n-1}_N)$ is needed to obtain the correct (Itô) stochastic integral in the limit as $\Delta t \to 0$. Actually, this adaptivity (measurability) concern also leads us to introduce the approximations of $\sigma$ in (3.3); see Remark 3.1 and (4.6), (4.21) below. Note that, as explained in this Remark approximations of $\sigma$ satisfying (3.5)–(3.7) can always be found via an elementary functional-analytic construction.}

We make the notion of suitable solutions of (3.3) precise in the following definition.

**Definition 3.1.** We consider a stochastic basis $S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1})$. Given $N \geq 1$ and an element $U^0_N \in L^2(\Omega; H)$ which is $(\mathcal{F}_0, \mathcal{B}(H))$ measurable and a process $\ell = (\ell(t) \in L^2(\Omega; L^2(0, T; V')))_{t \geq 0}$ adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, we say that a sequence $\{U^n_N\}_{n=0}^N$ is an admissible solution of the Euler Scheme (3.3), if

(i) For each $n = 1, \ldots, N$, $U^n_0 \in L^2(\Omega; V)$ and $U^n_N$ is $\mathcal{F}_n$ adapted, where $\mathcal{F}_n := \mathcal{F}_n^\infty$, $n = 0, \ldots, N$.\footnote{The choice of a “time explicit” term in $\sigma_N(t^{n-1}, U^{n-1}_N)$ is needed to obtain the correct (Itô) stochastic integral in the limit as $\Delta t \to 0$. Actually, this adaptivity (measurability) concern also leads us to introduce the approximations of $\sigma$ in (3.3); see Remark 3.1 and (4.6), (4.21) below. Note that, as explained in this Remark approximations of $\sigma$ satisfying (3.5)–(3.7) can always be found via an elementary functional-analytic construction.}
(ii) Every pair $U^0_N, U^{n-1}_N$, $n = 1, \ldots, N$, satisfies
\[
(U^0_N - U^{n-1}_N, U^2) + (a(U^0_N, U^2) + b(U^0_N, U^2) + c(U^0_N, U^2)) \Delta t
= (\ell^n(U^2) + s(t^n, U^2_N, U^2)) \Delta t + g_N(t^{n-1}, U^0_N, U^2)\eta^n_N,
\]
almost surely for all $U^2 \in V_{(2)}$.

(iii) For each $n = 1, \ldots, N$, $U^N_n$ and $U^{N-1}_n$ satisfy the ‘energy inequality’, almost surely on $\Omega$:
\[
(U^0_N - U^{N-1}_N, U^2_N) + \Delta t c_1\|U^0_N\|^2 \leq \left(\ell^n(U^0_N) + s(t^n, U^0_N, U^2_N)\right) \Delta t + g_N(U^{N-1}_N, U^2_N)\eta^n_N,
\]
for $n = 1, 2, \ldots, N$ and where $c_1$ is the constant from (2.2).

**Remark 3.1.** At first glance the dependence on $N$ in both the initial condition and in the noise term involving $\sigma$ may seem strange. Indeed, in the deterministic setting, when we approximate (2.13) with (3.3), we would simply take $U^0_N$ to be equal to the initially given $U^0$ for all $N$. Similarly if we were to add deterministic sublinear terms analogous to $\sigma$ to the governing equations no approximation as in (3.5)–(3.7) would be necessary; however, the situation is, in general, more complicated in the stochastic setting as we shall see in detail later on in Section 4, Proposition 4.1. This is essentially because we must construct continuous time processes from the $U^N_n$’s which are adapted to a given filtration. See (4.6), (4.15)–(4.16) (4.17) and (4.21) for specific details.

For now let us describe how we can achieve suitable approximations in the $U^N_n$ and $\sigma^N_n$’s.

- For a given initial probability distributions $\mu_U^0$, on $H$ (with $\mu_U^0(\|\cdot\|_H^2) < \infty$) and having fixed a suitable stochastic basis and an element $U^0 \in L^2(\Omega; H)$, $F_t$-measurable, with distribution $\mu_U^0$. We then pick a sequence $U^0_n \in L^2(\Omega; V_{(2)})$ such that $U^0_n \to U^0$ as $N \to \infty$ in $L^2(\Omega; H)$ but subject to the restriction given in (4.3) below. Such a sequence can be found with a simple density argument. Indeed, since $V_{(2)}$ is dense in $H$, we may initially approximate $U^0$ in $L^2(\Omega, H)$ with a sequence $U^0_{M,N} \in L^2(\Omega; V_{(2)})$. We then define $M(N) = \max\{M \geq 1 : \|U^0_M\|_{L^2(\Omega; V_{(2)})} \leq N^{1/2}\} \wedge N$ and define $U^0_N = U^0_{M(N)}$. Since $M(N) \to \infty$ as $N \to \infty$, $U^0_N$ approximates $U^0$ in $L^2(\Omega; H)$ while maintaining the constraint (4.3).

- We may construct elements $\sigma^N_n$ from $\sigma$ satisfying (3.5)–(3.7) according to the following general functional analytic construction. For any $\psi \in H$, via Lax-Milgram we define $\Psi(U)$ to be the unique solution in $V$ of $((\Psi(U), U^2)) = (U, U^2)$ for all $U^2 \in V$. Classically $\Psi$ is a compact, self-adjoint and injective linear operator on $H$. Thus, by the Spectral Theorem, we may find a complete orthonormal basis for $H$ $\{\Phi_j\}_{j \geq 1}$ which is made up of eigenfunctions of $\Psi$ with a corresponding sequence of eigenvalues $\{\gamma_j\}_{j \geq 1}$ decreasing to zero. For any integer $m$ we let $P_m$ be the projection onto $H_m := \text{span}\{\Phi_1, \ldots, \Phi_m\}$. Now choose a sequence $m_N$ increasing to infinity but so that $\gamma_{m_N}^{-1} \leq N$. It is not hard to see that defined in this way $\sigma^N_n = P_{m_N} \sigma(\cdot)$ satisfies the requirements given in (3.5)–(3.7).

### 3.2 Existence of the $U^N_n$’s

While the existence for a.e. $\omega \in \Omega$ of solutions to (3.3) satisfying (3.9) follows along arguments similar to those found in [PTZ08, Lemma 2.3], some care is required to demonstrate the existence of sequences $\{U^N_n\}_{n=0}^N$ which are adapted to the underlying stochastic basis. For this complication we will make use of a ‘measurable selection theorem’ (Theorem A.2 below in the Appendix Section A.3) from [BT73] (and see also the related earlier works [KRN65], [Cas67]). In order to apply this result we use of a specific stochastic basis defined around the canonical Wiener space whose definition we recall next.

#### 3.2.1 The Wiener measure and its filtration

We recall the canonical Wiener space as follows; see [KS91] for further details. Let
\[
\Omega = C([0, T]; \mathcal{U}_0),
\]
equipped with the Borel σ-algebra denoted as $\mathcal{G}$. We equip $(\Omega, \mathcal{G})$ with the Wiener measure $\mathbb{P}$.\textsuperscript{6} Then the evaluation map $W(\omega, t) := \omega(t)$, $\omega \in \Omega$, $t \in [0, T]$, is a cylindrical Wiener process on $\mathcal{U}_0$. The filtration is given by $\mathcal{G}_t$ defined as

the completion of the sigma algebra generated by the $W(s)$ for $s \in [0, t]$ with respect to $\mathbb{P}$.

Combining these elements $\mathcal{S}_G = (\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P}, W)$ gives a stochastic basis suitable for applying Theorem A.2.

### 3.2.2 Existence of the $U_n^n$’s adapted to $\mathcal{G}_t$\textsuperscript{7}

#### Proposition 3.1

Suppose that

$$N \geq N_0 := 4Tc_4, \quad (\text{or equivalently that } 4\Delta t c_4 < 1),$$

where $c_4$ is the constant arising in (2.11). Consider the stochastic basis $\mathcal{S}_G$ defined as in Section 3.2.1, an $N \geq N_0$, and an element $U_N^n \in L^2(\Omega; H)$ which is $\mathcal{G}_0$-measurable and a process $\ell = \ell(t) \in L^2(\Omega; L^2(0, T; V'))$ measurable with respect to the sigma algebra generated by the $W(s)$ for $s \in [0, t]$. Then there exists a sequence $\{U_N^n\}_{n=0}^N$ which is an admissible solution of the Euler scheme (3.3) in the sense of Definition 3.1.

The rest of this subsection is devoted to the proof of Proposition 3.1. Below we will construct the sequence $\{U_N^n\}_{n=0}^N$ iteratively starting from $U_N^0$ but we first need to take the preliminary step of establishing the existence of a certain Borel measurable map $\Gamma : [0, T] \times V' \to V$ which is used at the heart of this construction.

We define the continuous map $\mathcal{G} : [0, T] \times V \to V_{(2)}$ according to

$$\mathcal{G}(t, U) = U + \Delta t(\lambda U + B(U) + E U - \xi(U, t)),$$

and, for each $t \in [0, T]$ and $F \in V'$ we set:

$$\Lambda(t, F) = \{U \in V : \langle \mathcal{G}(t, U) - F, U' \rangle = 0, \forall U' \in V_{(2)} \text{ and } |U|^2 + \Delta t c_1 ||U||^2 \leq \langle F + \xi(t, U) \Delta t, U \rangle \}.$$  \hspace{1cm} (3.12)

Using this family of sets defined by (3.12) we now establish the following Lemma:

#### Lemma 3.1

There exists a map $\Gamma : (0, T) \times V' \to V$ which is universally Radon measurable (Radon measurable for every Radon measure on $(0, T) \times V'$), such that for every $t \in (0, T) \text{ and every } F \in V'$, $U := \Gamma(t, F) \in \Lambda(t, F)$.

**Proof.** We establish the existence of the desired $\Gamma$ by showing that $\Lambda$ satisfies the conditions of Theorem A.2. More precisely we need to verify that\textsuperscript{7}

(i) for each $t \in [0, T]$, $F \in V'$, the set $\Lambda(t, F)$ is non-empty and that

(ii) $\Lambda(t, F)$ is closed. In other words we need to show that, given any sequences

$$t_n \to t, \quad F_n \to F \text{ in } V', \quad U_n \to U \in V$$

such that, for every $n,$

$$\langle \mathcal{G}(t_n, U_n) - F_n, U' \rangle = 0, \text{ for every } U' \in V_{(2)} \text{ and } |U_n|^2 + \Delta t c_1 ||U_n||^2 \leq \langle F_n + \xi(t_n, U_n) \Delta t, U_n \rangle,$$

we have

$$\langle \mathcal{G}(t, U) - F, U' \rangle = 0, \text{ for every } U' \in V_{(2)} \text{ and } |U|^2 + \Delta t c_1 ||U||^2 \leq \langle F + \xi(t, U) \Delta t, U \rangle.$$

\textsuperscript{6}Using the orthonormal basis $\{e_k\}_{k \geq 1}$ of $\mathcal{U}$, $\mathbb{P}$ is obtained as the product of the independent Wiener measures each one defined on $C([0, T]; \mathbb{R})$.

\textsuperscript{7}To apply Theorem A.2 we actually would like to define $\Lambda$ on the Banach space $\mathbb{R} \times V'$. For this purpose we may simply take $\Lambda(t, F) = \Lambda(T, F)$ when $t > T$ and when $t < 0$ we let $\Lambda(t, F) = \Lambda(0, F)$. 

The first item, (i) may be established with a Galerkin scheme and the Brouwer fixed point theorem along standard arguments typically used to prove the existence of solutions for nonlinear elliptic equations of the type of Navier-Stokes and primitive equations (see Lemma 2.3, Page 26 in [PTZ08]). Since some specifics are different here we briefly sketch some details of this argument. Fix any \( t \in [0, T] \) and any \( F \in V' \) and consider a family \( \{ \Psi_k \}_{k \geq 1} \subset V(2) \) which is free and total in \( V \). For each \( m \geq 1 \) we seek an element \( U_m = \sum_{j=1}^{m} \beta_j \Psi_j \) such that

\[
\langle \Theta(t, U_m) - F, \Psi_k \rangle = 0 \quad \text{for every} \quad k = 1, \ldots, m.
\] (3.13)

Observe that, for any \( U_m \) of this form, using (2.2), (2.3), (2.5), and (2.11) we estimate

\[
\langle \Theta(t, U_m) - F, U_m \rangle = |U_m|^2 + \Delta t (a(U_m, U_m) - \langle \xi(t, U_m), U_m \rangle) - \langle F, U_m \rangle
\]

\[
\geq |U_m|^2 + \Delta t (c_1 |U_m|^2 - 2c_4 (1 + |U_m|^2)) - |F|_{V'} |U_m|\]

\[
\geq \frac{\Delta t c_1}{2} |U_m|^2 - \frac{1}{2} \left( 1 + \frac{1}{\Delta t c_1} |F|^2_{V'} \right).
\]

The last inequality follows from the assumption (3.10) which implies that \( 2c_4 \Delta t \leq 1 \). The existence of solutions for (3.13) for any given \( t, F \) of the form \( U_m = \sum_{j=1}^{m} \beta_j \Psi_j \) thus follows for each \( m \) from the Brouwer fixed point theorem.

We next seek bounds on the resulting sequence of \( U_m \)'s in \( V \) independent of \( m \). Starting from (3.13) we find that

\[
|U_m|^2 + c_1 \Delta t |U_m|^2 \leq \Delta t (\xi(t, U_m), U_m) + \langle F, U_m \rangle
\]

\[
\leq 2c_4 \Delta t (1 + |U_m|^2) + \frac{1}{2 \Delta t c_1} |F|^2_{V'} + \frac{\Delta t c_1}{2} |U_m|^2.
\] (3.14)

Using once again the standing assumption (3.10) we have that \( U_m \) is bounded in \( V \) independently of \( m \). Passing to a subsequence as needed and using that \( V \) is compactly embedded in \( H \) we infer the existence of an element \( U \) such that \( U_m \to U \) weakly in \( V \) and strongly in \( H \).

Returning to (3.14) and using the lower semicontinuity of weakly convergent sequences we obtain that

\[
|U|^2 + c_1 \Delta t |U|^2 \leq \langle \xi(t, U) + F, U \rangle.
\]

To show that \( U \) satisfies \( \langle \Theta(t, U) - F, U \rangle = 0 \) for every \( U \in V(2) \) we simply invoke (2.8) for \( B \) and the other continuity assumptions on \( A, E \) and \( \xi \) and obtain this identity for \( U = \Psi_k \) for each \( k \geq 1 \). By linearity and density we therefore infer the identity for arbitrary \( U \in V(2) \).

With this we now have established (i). The second item, (ii), to show that \( \Lambda \) is closed, follows immediately from the continuity of \( \Theta \) from \( [0, T] \times V \) into \( V' \) and the continuity of \( \xi \) from \( [0, T] \times H \) into \( H \). The proof of Lemma 3.1 is therefore complete.

\[
\square
\]

Construction of an Adapted Solution

**Step 1.** We will build the desired sequence \( \{U^n_N\}_{n=0}^{N} \) inductively as follows:

\[
U^n_N = f^n_N(W^n_{[0,t_n]}),
\] (3.15)

with \( f^n_N : \mathcal{C}([0, t_n]; \mathcal{U}_0) \to V \) measurable for \( V \) equipped with \( \mathcal{B}(V) \) and \( \mathcal{C}([0, t_n]; \mathcal{U}_0) \) equipped with \( \mathcal{G}_n := \mathcal{G}_{t_n} \) (defined as in Section 3.2.1).

Suppose that we have obtained \( U^{n-1}_N \) for some \( n \geq 2 \). Since \( \mathcal{G}_{n-1} \) is the completion of \( \mathcal{B}(\mathcal{C}([0, t_{n-1}]; \mathcal{U})) \) with respect to the Wiener measure \( \mathbb{P}_\beta \), \( f^{n-1}_N \) is \( \mathbb{P} \)-measurable. Now we define \( \mathcal{D}^n_N : V \times V' \times \mathcal{C}([0, t_n]; \mathcal{U}_0) \to V' \) by setting

\[
\mathcal{D}^n_N(x, y, z) = x + y \Delta t + \sigma_N(t^{n-1}, U)z.
\] (3.16)

\footnote{We observe that the sigma algebra generated by the \( W(s) \) for \( s \in (0, t) \) is just \( \phi_t^{-1}(\mathcal{B}(\mathcal{C}([0, T]; \mathcal{U}_0)), \text{ where } \phi_t : \mathcal{C}([0, T]; \mathcal{U}_0) \to \mathcal{C}([0, T]; \mathcal{U}_0) \text{ is the mapping } (\phi_t^{-1} \omega)(s) = \omega(t \wedge s); 0 \leq s \leq T \text{ (see [KS91]).} \)
Then we can define
\[ U_N^n = \Gamma \left( t^n, \mathcal{D}^n_N \left( U_N^{n-1}, \ell_N^n, \eta_N^n \right) \right) \]
\[ := \chi \left( t^n, U_N^{n-1}, \ell_N^n, \eta_N^n \right). \tag{3.17} \]
Since \( \sigma_n \) is a continuous map, clearly \( \mathcal{D}_n \) is a continuous map. Moreover \( \Gamma \) is universally Radon measurable thanks to Lemma 3.1, hence Corollary A.1 applies and we infer that \( \chi \) is universally Radon measurable from the Borel sigma algebra on \( V \times V' \times \mathcal{C}([0, t_n]; \mu_0) \) to the Borel sigma algebra on \( V \).

Since \( \ell = \ell(t) \) is a process assumed to be measurable with respect to the sigma algebra generated by the \( W(s) \) for \( s \in [0, t] \), \( \ell_N^n \) is measurable with respect to the sigma algebra generated by the \( W(s) \) for \( s \in [0, t_n] \) thanks to (3.4). Hence by Theorem A.1 in the Appendix with \( X \) as \( \Omega \), \( (Y, \mathcal{M}) \) as \( (\mathcal{C}([0, t_n]; \mu_0), \mathcal{B}(\mathcal{C}([0, t_n]; \mu_0))) \), \( \psi \) as \( W|_{[0, t_n]} \), \( \mathcal{H} \) as \( V \), we see that there exists a function \( L_N^n : \mathcal{C}([0, t_n]; \mu_0) \to V \) which is Borel measurable, such that
\[ \ell_N^n = L_N^n(W|_{[0, t_n]}). \tag{3.18} \]
From (3.17) and (3.18) we infer
\[ U_N^n = \kappa(t^n, f_N^{n-1}(W|_{[0, t_n-1]}), L_N^n(W|_{[0, t_n]}), \eta_N^n) \]
\[ := f_N^n(W|_{[0, t_n]}). \tag{3.19} \]
Since \( L_N^n \) and \( f_N^{n-1} \) are \( \mathbb{P} \)-measurable and \( \kappa \) is universally Radon measurable, Theorem A.3 applies and we infer that \( f_N^n \) is \( \mathbb{P} \)-measurable, that is, \( f_N^n \) is measurable with respect to \( \mathcal{G}_n \).

**Step 2.** We infer that \( U_N^n : \Omega \to V \) is measurable with respect to \( \mathcal{G}_n \) as desired.

Observe moreover that, according to Lemma 3.1 (cf. (3.12)), \( \langle \Theta(t^n, U_N^n), U_N^n \rangle = \langle \mathcal{D}_N^n \left( U_N^{n-1}, \ell_N^n, \eta_N^n \right), U_N^n \rangle \), for every \( U^t \in V_2 \) and \( |U_N^n|^2 + \delta t c_1 \| U_N^n \|^2 \leq \langle \mathcal{D}_N^n \left( U_N^{n-1}, \ell_N^n, \eta_N^n \right), U_N^n \rangle \) which is to say that \( U_N^{n-1} \) and \( U_N^n \) satisfy (3.3) and (3.9).

It remains to show that \( U_N^n \in L^2(\Omega; V) \). We start from (3.9), now established for \( U_N^n \) and \( U_N^{n-1} \), and use the elementary identity \( 2(U - U^t, U) = |U|^2 - |U^t|^2 + |U - U^t|^2 \) and obtain,
\[ |U_N^n|^2 - |U_N^{n-1}|^2 + |U_N^n - U_N^{n-1}|^2 + 2\delta t c_1 \| U_N^n \|^2 \]
\[ \leq 2\Delta t \left( \ell_N^n(U_N^n) + s(t^n, U_N^n, U_N^n) \right) + 2g_N(U_N^{n-1}, U_N^n) \eta_N^n, \tag{3.20} \]
almost surely. To address the terms involving \( \ell \) we have that (cf. (3.4))
\[ |2\Delta t \ell_N^n(U_N^n)| \leq 2 \int_{(n-1)\Delta t}^{n\Delta t} \| \ell(t) \|_V \| U_N^n \|_V dt \leq c_1 \Delta t \| U_N^n \|^2 + c_1^{-1} \zeta_N^n \]
where we define \( \zeta_N^n \) according to
\[ \zeta_N^n = \int_{(n-1)\Delta t}^{n\Delta t} \| \ell(t) \|_V^2 dt. \tag{3.21} \]

For the terms involving \( s \) defined as in (2.12) we simply infer from (2.11)
\[ 2\Delta t |s(t^n, U_N^n, U_N^n)| \leq 4\Delta t c_4 (1 + |U_N^n|^2). \tag{3.22} \]

With Hölder’s inequality we find
\[ |2g_N(U_N^{n-1}, U_N^n - U_N^{n-1}) \eta_N^n| \leq \frac{1}{2} |U_N^n - U_N^{n-1}|^2 + 2|\sigma_N(U_N^{n-1}) \eta_N^n|^2. \tag{3.23} \]
Then using that $g_N$ is linear in its second argument,
\[
g_N(U_N^{n-1}, U_N^n) = g_N(U_N^{n-1}, U_N^n) = g_N(U_N^{n-1}, U_N^n - U_N^{n-1})\eta_N^n + g_N(U_N^{n-1}, U_N^n - U_N^{n-1})\eta_N^n \\
\leq g_N(U_N^{n-1}, U_N^n - U_N^{n-1})\eta_N^n + |g_N(U_N^{n-1}, U_N^n - U_N^{n-1})\eta_N^n| \\
\leq (\text{thanks to (3.23)}) \\
\leq g_N(U_N^{n-1}, U_N^n - U_N^{n-1})\eta_N^n + \frac{1}{2}|U_N^n - U_N^n|^2 + 2|\sigma_N(U_N^{n-1})\eta_N^n|^2. \tag{3.24}
\]

Using these observations for $g_N$, $\ell_N^n$ and $s$ we rearrange and infer that, up to a set of measure zero,
\[
|U_N|^2 - |U_N^{n-1}|^2 + \frac{1}{2}|U_N^n - U_N^{n-1}|^2 + \Delta t c_1\|U_N^n\|^2 \\
\leq c_1^2 \zeta_N^n + 4\Delta t c_4 (1 + |U_N^n|^2) + 2g_N(U_N^{n-1}, U_N^n - U_N^{n-1})\eta_N^n + 2|\sigma_N(U_N^{n-1})\eta_N^n|^2. \tag{3.25}
\]

Using (2.10), (3.6) and that $U_N^{n-1}$ is $G_{n-1}$-measurable and in $L^2(\Omega; H)$ we have that
\[
\mathbb{E}g_N(U_N^{n-1}, U_N^n - U_N^{n-1})\eta_N^n = 0, \quad \mathbb{E}|\sigma_N(U_N^{n-1})\eta_N^n|^2 = \Delta t \mathbb{E}|\sigma_N(U_N^{n-1})|^2_{L_2(\Omega, H)} \leq 2\Delta t c_3^2 \mathbb{E}(1 + |U_N^{n-1}|^2).
\]

From this observation, (3.25) and (3.10) we infer
\[
\mathbb{E}\Delta t c_1\|U_N^n\|^2 \leq \mathbb{E} \left((4\Delta t c_4 - 1)|U_N^n|^2 + c(|U_N^{n-1}|^2 + \zeta_N^n + 1)\right) \leq c\mathbb{E}(|U_N^{n-1}|^2 + \zeta_N^n + 1),
\]
which implies that $U_N^n \in L^2(\Omega; H)$, as needed.

We have thus established the iterative step in the construction of $\{U_N^n\}_{n=0}^N$. The base case, $n = 1$, is established in an identical fashion to the iterative steps. The proof of Proposition 3.1 is now complete.

**Remark 3.2.** Although necessary for the establishment of the existence of the $U_N^n$’s in Proposition 3.1, it is not necessary to assume the underlying stochastic basis to be $\mathcal{S}_{G}$ (defined in subsection 3.2.1) in the results throughout Section 3.3 to Section 5.1. The reason is that these results are true whenever such $U_N^n$’s defined as in Definition 3.1 exist; in other words they are independent of the choice of the underlying stochastic basis. Similarly, it is not necessary at this point to assume that $U^0$ and $\ell$ have laws which coincide with those of the externally given $\mu_{U^0}$ and $\mu_\ell$ for these results.

However, it is necessary that we resume these assumptions of $\mathcal{S}_{G}$, $\mu_{U^0}$ and $\mu_{\ell}$ starting in Section 5.2.

### 3.3 Uniform ‘Energy’ Estimates for the $U_N^n$

Starting from (3.9) we next determine certain uniform bounds, independent of $N$, for (suitable) sequences $\{U_N^n\}_{n=1}^N$ satisfying (3.3) as follows:

**Proposition 3.2.** Let
\[
N_1 := 12Tc_5, \quad \text{with } c_5 := 8c_4 + 80c_3^2, \tag{3.26}
\]
where $c_3$ and $c_4$ are from (2.10) and (2.11). Let $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1})$ be the given stochastic basis and assume that $\ell = \ell(t) \in L^2(\Omega; L^2(0, T; V'))$ is measurable with respect to $\mathcal{F}_t$. For each $N \geq N_1$ we assume that $U_N^0 \in L^2(\Omega, H)$, is $\mathcal{F}_0$ measurable and such that
\[
\sup_{N \geq N_1} \mathbb{E}|U_N^0|^2 < \infty. \tag{3.27}
\]
Then for each $N \geq N_1$, consider the sequences $\{U_N^n\}_{n=1}^N \subset L^2(\Omega; V)$ which satisfy (3.3) starting from $U_N^0$ and relative to $\ell$ in the sense of Definition 3.1. Then
\[
\sup_{N \geq N_1} \mathbb{E} \left(\max_{0 \leq t \leq N} |U_N^n|^2 + \sum_{k=1}^N (|U_N^k - U_N^{k-1}|^2 + \Delta t |U_N^k|^2)\right) < \infty. \tag{3.28}
\]
Lemma 3.2. Assume that \( \{M^n\}_{n \geq 0} \) is a (discrete) martingale on a Hilbert space \( \mathcal{H} \) (with norm \( |\cdot| \)), relative to a given filtration \( \{\mathcal{F}_n\}_{n \geq 0} \). We assume, additionally that \( M_0 = 0 \) and that \( \mathbb{E}|M_n|^q < \infty \), for all \( n \geq 0 \). Then, for any \( q \geq 1 \) and any \( n \geq 1 \)

\[
\mathbb{E} \max_{1 \leq m \leq n} |M^n|^q \leq c_q \mathbb{E}(A^n)^{q/2} , \tag{3.31}
\]

where \( c_q \) is a universal positive constant depending only on \( q^9 \) (which is independent of \( n \) and \( \{M^m\}_{m \geq 0} \)) and \( A^n \) is the quadratic variation defined by

\[
A^n = \sum_{m=1}^{n} \mathbb{E}(|M^m - M^{m-1}|^2 |\mathcal{F}_{m-1}) . \tag{3.32}
\]

Hence with the observation that \( \mathbb{1}_{\tilde{n}_K \geq k} \) is \( \mathcal{F}_{k-1} \)-measurable we compute the quadratic variation of \( \{M^{m,n \wedge \tilde{n}_K}_{n=m} \}_{n=m}^N \) in view of (3.32) as follows

\[
A^{m,m,n}_{n} = \sum_{k=m}^{n} \mathbb{E}(|M^{m,m,k \wedge \tilde{n}_K}_{n} - M^{m,m,(k-1) \wedge \tilde{n}_K}_{n} |^2 |\mathcal{F}_{k-1}) = \sum_{k=m}^{n} \mathbb{E}(\mathbb{1}_{\tilde{n}_K \geq k}|g_N(U^{k-1}_N, t^{k-1}_N)|^2 |\mathcal{F}_{k-1})

= \sum_{k=m}^{\tilde{n}_K \wedge n} |g_N(U^{k-1}_N, t^{k-1}_N)|^2 \Delta t; \tag{3.29}
\]

\( \text{Proof.} \) The starting point for the estimates leading to (3.28) is of course (3.9) and from this inequality we can use the same proof as in Proposition 3.1 to obtain (3.25). In order to make suitable estimates for the final two terms in (3.25) we need to take advantage of some martingale structure in the terms involving \( \sigma_N \). For any \( 1 \leq m \leq n \leq N \) we define the stochastic processes

\[
M^{m,m}_N := \sum_{k=m}^{n} g_N(U^{k-1}_N, t^{k-1}_N)^{k}_N, \quad Q^{m,m}_N := \sum_{k=m}^{n} |\sigma_N(U^{k-1}_N)|^{k}_N . \tag{3.29}
\]

Summing (3.25) for \( 1 \leq m \leq n = k \leq l \leq N \) we find,

\[
|U^k_N|^2 + \sum_{k=m}^{l} \left( \frac{1}{2}|U^k_N - U^{k-1}_N|^2 + \Delta t \right) \|U^k_N\|^2 \leq |U^m_N|^2 + \sum_{k=m}^{l} \left( c_1^{-1} \delta_N + 4 \Delta t c_4 (1 + |U^k_N|^2) \right) + 2 M^{m,l}_N + 2 Q^{m,l}_N . \tag{3.30}
\]

Since \( \{U_N^n\}_{n=0}^N \subset L^2(\Omega; H) \) and is adapted to \( \mathcal{F}_n := \mathcal{F}_{t^n} \), it is easy to see that \( \{M^{m,m}_N\}_{n=m}^N \) is a martingale relative to \( \{\mathcal{F}_n\}_{n=m}^N \) with \( M^{m,m}_m = 0 \). We would like to apply a discrete version of the Burkholder-Davis-Gundy inequality, recalled here as in Lemma 3.2 to obtain estimates for \( \mathbb{E} \max_{m \leq l \leq n} |M^{m,m}_N| \). Unfortunately it is not clear that \( \{M^{m,m}_N\}_{n=m}^N \) is square integrable so we have to apply a localization argument to make proper use of this inequality. For any \( K > 0 \) we define the stopping times

\[
\tilde{n}_K = \min_{l \geq m} \{ |U^{l-1}_N| \geq K \} \wedge N .
\]

Since \( \{U_N^n\}_{n=0}^N \subset L^2(\Omega; H) \) we have that \( \tilde{n}_K \uparrow N \) almost surely as \( K \uparrow \infty \). Clearly \( \{M^{m,n \wedge \tilde{n}_K}_N\}_{n=m}^N \) is a square-integrable martingale. For the moment let us recall a discrete analogue of the Burkholder-Davis-Gundy Inequality. This result and other related martingale inequalities can be found in e.g. [Dur10].
Thus, by Lemma 3.2, (2.10) and (3.6) we infer
\[ E \max_{m \leq t \leq n} |M_N^{m,t/\tau_N}| \leq 3E \left( \sum_{k=m}^{n} |g_N(U_N^{k-1}, U_N^{k-1})|_{L^2}^2 \Delta t \right)^{1/2} \leq 3E \left( \sum_{k=m}^{n} |\sigma_N(U_N^{k-1})|_{L^2(U,H)}^2 |U_N^{k-1}|^2 \Delta t \right)^{1/2} \allowbreak \leq 3c_3E \left( \sum_{k=m}^{n} 2(1 + |U_N^{k-1}|^2)|U_N^{k-1}|^2 \Delta t \right)^{1/2} \leq 4 \frac{1}{4}E \max_{m \leq k \leq n} |U_N^{k-1}|^2 + 18c_3^2E \sum_{k=m}^{n} (1 + |U_N^{k-1}|^2) \Delta t. \]

Hence, letting \( K \to \infty \), we have, by the monotone convergence theorem,
\[ E \max_{m \leq t \leq n} |M_N^{m,t}| \leq \frac{1}{4} E \max_{m \leq k \leq n} |U_N^{k-1}|^2 + 18c_3^2E \sum_{k=m}^{n} (1 + |U_N^{k-1}|^2). \tag{3.33} \]

On the other hand since \( U_N^0 \) is adapted to \( \mathcal{F}_n \), given the condition (2.10) on \( \sigma \) and (3.6) we infer that
\[ E|Q_N^{m,n}| = \sum_{k=m}^{n} E|\sigma_N(U_N^{k-1})|_{L^2(U,H)}^2 \Delta t \leq 2c_3^2E \sum_{k=m}^{n} (1 + |U_N^{k-1}|^2). \tag{3.34} \]

We now use (3.33), (3.34) with (3.30) and infer that
\[ E \max_{m \leq t \leq n} |U_N^t|^2 \leq \left( 2|U_N^{m-1}|^2 + \sum_{k=m}^{n} (c_1^{-1}\zeta_k^k + 4c_4\Delta t(1 + |U_N^{k-1}|^2)) + 2 \max_{m \leq t \leq n} |M_N^{m,t}| + 2Q_N^{m,n} \right) \leq \left( 2|U_N^{m-1}|^2 + \sum_{k=m}^{n} (c_1^{-1}\zeta_k^k + 4c_4\Delta t(1 + |U_N^{k-1}|^2)) + 40c_3^2\Delta t \sum_{k=m}^{n} (1 + |U_N^{k-1}|^2) \right) + \frac{1}{2} E \max_{m \leq k \leq n} |U_N^{k-1}|^2. \]

Rearranging we find that
\[ E \max_{m \leq t \leq n} |U_N^t|^2 \leq \left( 2|U_N^{m-1}|^2 + 2c_1^{-1} \sum_{k=m}^{n} \zeta_k^k + c_5 \Delta tE \sum_{k=m}^{n+1} (1 + |U_N^{k-1}|^2) \right) \leq \left( 2|U_N^{m-1}|^2 + 2c_1^{-1} \sum_{k=m}^{n} \zeta_k^k + c_5 \Delta t(n - m + 2)(1 + E \max_{m \leq k \leq n+1} |U_N^{k-1}|^2) \right). \tag{3.35} \]

for the constant \( c_5 = 8c_4 + 80c_3^2 \) which in particular depends only on \( c_3, c_4 \). Thus, subject to the condition:
\[ c_5 \Delta t(n - m + 2) \leq \frac{1}{2}, \quad \text{i.e.} \quad \frac{n - m + 2}{N} \leq \frac{1}{2c_5T}, \tag{3.36} \]
we have
\[ E \max_{m \leq t \leq n} |U_N^t|^2 \leq c_6E \left( |U_N^{m-1}|^2 + \sum_{k=m}^{n} \zeta_k^k + 1 \right), \tag{3.37} \]

where \( c_6 = \max\{4c_1^{-1}, 7\} \). Thus, by iterating this inequality and noting, cf. (3.21), that \( \sum_{k=1}^{N} \zeta_n^k = \|\|_{L^2(0,T;V')} \), we finally conclude that,
\[ E \max_{1 \leq t \leq N} |U_N^t|^2 \leq c_7E \left( |U_N^0|^2 + \|\|_{L^2(0,T;V')}^2 + 1 \right), \quad \text{for all } N \geq N_1. \tag{3.38} \]
Note carefully that, in view of (3.36), we need not iterate (3.37) more than, say, \( r 16c_5 T \) times to obtain (3.38).\(^{10}\) As such we may take \( c_7 = (1 + c_6)^{16c_5 T} = (1 + \max\{4c_1^{-1}, 7\})^{16T(8c_4 + 80c_5^2)} \) which, crucially, is independent of \( N \).

We now return to (3.30). With (3.34) we infer,
\[
\mathbb{E} \sum_{k=1}^{N} (|U_N^k - U_N^{k-1}|^2 + 2c_1 \Delta t \|U_N^k\|^2) \leq c_8 \mathbb{E} \left( |U_0^N|^2 + \sum_{k=1}^{N} (c_1^{-1} c_k + 4c_4 \Delta t(1 + |U_N^k|^2)) + 4c_3^2 \Delta t \sum_{k=1}^{N} (1 + |U_N^{k-1}|^2) \right) \\
\leq c_8 \mathbb{E} \left( |U_0^N|^2 + \max_{1 \leq t \leq N} |U_N^t|^2 + \left\| \ell(t) \right\|_{L^2(0,T;V')}^2 + 1 \right),
\]
where we can take, \( c_8 = \max\{1, c_7^{-1}, 4T(c_3^2 + c_4)\} \). As such, (3.38) and (3.39) with (3.27) imply (3.28), completing the proof of Proposition 3.2.

\[\Box\]

4 Continuous Time Approximations and Uniform Bounds

In this section we detail how the sequences \( \{U_n^N\}_{n=0}^{N} \) defined in the sense of Definition 3.1 may be used to define continuous time processes that approximate (2.13). The details of establishing the compactness of the associated sequences of probability laws and of the passage to the limit are given further on in Section 5.

We now fix sequences \( \{U_n^N\}_{n=0}^{N} \) satisfying (3.3) in the sense of Definition 3.1. For \( N \geq N_1 \), with \( N_1 \) as in (3.26), let:
\[
U_N(t) = \begin{cases} 
U_0^N & \text{for } t \in [0, t^1], \\
U_N & \text{for } t \in (t^n, t^{n+1}], \ n = 1, \ldots, N - 1.
\end{cases}
\]

Of course we do not have any time derivatives of the \( U_N \)'s (even fractional in time) as are typically needed for compactness. Furthermore we would like to be able to associate an approximate stochastic equation for (2.13) with these \( \{U_n^N\}_{n=0}^{N} \)'s. For these dual concerns we introduce further stochastic processes and consider:
\[
\bar{U}_N(t) = \begin{cases} 
U_0^N & \text{for } t \in [0, t^1], \\
U_N^t - U_N^{t^n} & \text{for } t \in (t^n, t^{n+1}], \ n = 1, \ldots, N - 1.
\end{cases}
\]

**Remark 4.1.** The processes \( U_N \) and \( \bar{U}_N \) are slightly different than those typically used in the deterministic case. See, e.g. [Tem01]. Actually, these processes are essentially their deterministic analogues evaluated at time \( t \) by their value at time \( t - \Delta t \). With this choice we crucially obtain processes which are adapted to \( \mathcal{F}_t \) for \( t \geq 0 \). Not surprisingly however the present definitions of \( U_N, \bar{U}_N \) leads to bothersome error terms in (4.6) below. In turn these error terms dictate the additional convergences in \( \sigma \) and \( U^T \) when we initially defined the discrete scheme (3.3); cf. (3.5)-(3.7) and Remark 3.1 above. These error terms also complicate compactness arguments further on in Section 5 and see Remark 4.2.

The rest of this section is now devoted to proving the following desirable properties of \( U_N \) and \( \bar{U}_N \):

\[^{10}\text{Indeed, for } N \geq N_1, \text{ let } \mathfrak{N}(N) \text{ be the minimum number of iterations of (3.37), subject to the constraint (3.36), which are needed to establish (3.38). Take } \mathfrak{F}(N) \text{ to be the ‘fraction of the time interval that can be covered at each step’, namely,}
\]
\[
\mathfrak{F}(N) := \max_{n \in \mathbb{N}} \left\{ \frac{n}{N} : n + 2 \leq N \right\} > \frac{1}{2c_5 T} - \frac{3}{N} \geq \frac{1}{4c_5 T},
\]
where the last inequality follows from the standing assumption (3.26). Since \( \mathfrak{N}(N) \mathfrak{F}(N) \leq 2 \) we finally estimate:
\[
\mathfrak{N}(N) \mathfrak{F}(N) \leq 16c_5 T.
\]
Here \(^{p} \text{p}^{\text{p}} \) is the smallest integer that is larger than or equal to \( p \).
Proposition 4.1. Let $S = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \{W^k\}_{k \geq 1})$ be a stochastic basis, and let $N_1$ be as in (3.26) in Proposition 3.2. Consider a sequence $\{U^N_n\}_{N \geq N_1}$ bounded in $L^2(\Omega, H)$ independently of $N$, with $U^N_0 \mathbb{F}_0$-measurable for each $N$ and such that
\[
\mathbb{E} \left( (1 + \|U^N_0\|^2)(1 + \|U^N_0\|^2_{L^2(2)}) \right) \leq c \Delta t^{-1} = cN,
\]for a constant $c > 0$, independent of $N$.\(^{11}\) Suppose we also have defined a process $\ell = \ell(t) \in L^2(\Omega; L^2(0, T; V'))$ adapted to $\{\mathcal{F}_t\}_{t \geq 0}$.

For each $N \geq N_1$, we consider sequences $\{U^N_N\}_{N=1}^N$ which satisfy (3.3) starting from $U^N_0$ in the sense of Definition 3.1. Once these sequences $\{U^N_N\}_{N=0}^N$ exists, then we define the continuous time processes $\{U_N\}_{N \geq 1}$ and $\{\tilde{U}_N\}_{N \geq 1}$ according to (4.1) and (4.2) respectively. Then,

(i) for each $N \geq N_1$, $U_N$ and $\tilde{U}_N$ are $\{\mathcal{F}_t\}_{t \geq 0}$-adapted and
\[
\{U_N\}_{N \geq N_1} \text{ and } \{\tilde{U}_N\}_{N \geq N_1} \text{ are bounded in } L^2(\Omega; L^2(0, T; V) \cap L^\infty(0, T; H)).
\]

Moreover we have that
\[
\lim_{N \uparrow \infty} \mathbb{E} \int_0^T |U_N - \tilde{U}_N|^2 dt = 0.
\]

(ii) $U_N$ and $\tilde{U}_N$ satisfy a.s. and for every $t \geq 0$,
\[
\tilde{U}_N(t) = U^N_0 + \int_0^t (N(U_N) + \ell_N) dt + \int_0^t \sigma_N(U_N) dW + \mathcal{E}^D_N(t) + \mathcal{E}^S_N(t),
\]
subject to error terms $\mathcal{E}^D_N(t) \in L^2(\Omega; L^2(0, T; V'))$, $\mathcal{E}^S_N(t) \in L^2(\Omega; L^2(0, T; H))$ which are defined explicitly in (4.15), (4.16) below.

(iii) These error terms $\mathcal{E}^D_N(t)$, $\mathcal{E}^S_N(t)$ satisfy
\[
\lim_{N \uparrow \infty} \mathbb{E}\|\mathcal{E}^D_N\|^2_{L^2(0, T; V')} = 0,
\]
\[
\lim_{N \uparrow \infty} \mathbb{E}\|\mathcal{E}^S_N\|^2_{L^2(0, T; H)} = 0,
\]
and moreover
\[
\sup_{N \geq N_1} \mathbb{E}\|\mathcal{E}^S_N\|^2_{L^\infty(0, T; H) \cap L^2(0, T; V')} < \infty.
\]

We proceed to prove Proposition 4.1 in a series of subsections below. The proof of (i) is essentially a direct application of Proposition 3.2 and we provide the details in the subsection immediately following. In Subsection 4.2 we provide the details of the derivation of (4.6) and in particular explain the origin of the error terms $\mathcal{E}^D_N$, $\mathcal{E}^S_N$. The final Subsection 4.3 provides details of the estimates for these error terms which lead to (4.7)–(4.9).

Remark 4.2. It is not straightforward to obtain fractional in time estimates for $\tilde{U}_N$ from (4.6) in view of the error terms which have a rather complicated structure (see (4.15), (4.16) below). As such, we can not establish sufficient compactness for the sequence $\tilde{U}_N$ directly to facilitate the passage to the limit. For this reason we choose to introduce additional continuous time processes in Section 5 below. An alternate approach will be presented later on in the related work [GTW].

\(^{11}\)The constraint (4.3) is necessary for (4.4),(4.7). This is not a serious restriction when we pass to the limit in Section 5; as we described above in Remark 3.1, for any given $U^0 \in L^2(\Omega; H)$ we may obtain a sequence $U^0_N$ approximating $U^0$ which maintains (4.3).
4.1 Uniform Bounds and Clustering

It is clear from (4.1) that $U_N$ is $\{F_t\}_{t \geq 0}$-adapted and that

$$
\mathbb{E} \left( \sup_{t \in [0,T]} |U_N|^2 + \int_0^T \|U_N\|^2 dt \right) = \mathbb{E} \left( \max_{0 \leq m \leq N-1} |U_N^m|^2 + \sum_{m=0}^{N-1} \Delta t \|U_N^m\|^2 \right).
$$

Thus, since (3.27) holds we have the uniform bound (3.28) from Proposition 3.2 and we immediately infer that

$$
\sup_{N \geq N_1} \mathbb{E} \left( \sup_{t \in [0,T]} |U_N|^2 + \int_0^T \|U_N\|^2 dt \right) < \infty, \quad (4.10)
$$

with $N_1$ the integer appearing in (3.26).

As with the $U_N$ above, it is easy to see from (4.2) that $\bar{U}_N$ is adapted to $\{F_t\}_{t \geq 0}$ and that $\{U_N^n\}_{n=1}^N$ is adapted to $F_n$ ($= F_{t^n}$). Furthermore, direct calculations show that:

$$
U_N - \bar{U}_N(t) = \begin{cases} 
0 & \text{for } t \in [0,t^1], \\
\frac{U_N^n - U_N^{n-1}}{\Delta t} (t^{n+1} - t) & \text{for } t \in (t^n, t^{n+1}], \ n = 1, \ldots, N - 1.
\end{cases} \quad (4.11)
$$

Using (4.11) we compute, similarly to e.g. [Tem01], that

$$
\mathbb{E} \int_0^T |U_N - \bar{U}_N|^2 dt = \sum_{n=1}^{N-1} \mathbb{E} |U_N^n - U_N^{n-1}|^2 \int_{t^n}^{t^{n+1}} \left( \frac{t^{n+1} - t}{\Delta t} \right)^2 dt = \frac{\Delta t}{3} \mathbb{E} \sum_{n=1}^N |U_N^n - U_N^{n-1}|^2.
$$

We thus infer (4.5) directly from this observation and (3.28). Based on similar considerations we also have

$$
\mathbb{E} \int_0^T \|\bar{U}_N\|^2 \leq c \Delta t \mathbb{E} \sum_{n=0}^N \|U_N^n\|^2 = c \Delta t \mathbb{E} \|U_N^0\|^2 + \Delta t \mathbb{E} \sum_{n=1}^N \|U_N^n\|^2.
$$

Thus, once again due to (4.3) and (3.28), we finally have

$$
\sup_{N \geq N_1} \mathbb{E} \left( \sup_{t \in [0,T]} \|\bar{U}_N\|^2 + \int_0^T \|\bar{U}_N\|^2 dt \right) < \infty. \quad (4.12)
$$

With (4.10) and (4.12) we have now established the first item in Proposition 4.1.

4.2 The Approximate Stochastic Evolution Systems

We next derive the equation (4.6) relating $U_N$ and $\bar{U}_N$ giving explicit expressions for $E_N^0$, $E_N^S$. We observe that, almost surely and for almost every $t \geq 0$ (in fact for every $t \notin \{t_0, t_1, \ldots, t_N\}$)

$$
\frac{d}{dt} \bar{U}_N(t) = \sum_{n=1}^{N-1} \frac{U_N^n - U_N^{n-1}}{\Delta t} \chi_{[t^n, t^{n+1})}(t), \quad (4.13)
$$

where $\chi(t_1, t_2)$ denotes the indicator function of $(t_1, t_2)$. Recall that $\eta_N^n = W(t^n) - W(t^{n-1})$ and let $N^*_t := \min\{n : t^n \geq t\}$ in other words we take $N^*_t$ such that

$$
N^*_t \Delta t \leq t < (N^*_t + 1) \Delta t.
$$
Working from (4.13) and (3.3) we therefore compute

\[ \tilde{U}_N(t) = U_N^0 + \int_0^t \sum_{n=1}^{N-1} \frac{U_N^n - U_N^{n-1}}{\Delta t} \chi(t^n, t^{n+1})(s) ds \]

\[ = U_N^0 + \int_0^t \sum_{n=1}^{N-1} (\mathcal{N}(U_N^n) + \ell_N^n) \chi(t^n, t^{n+1})(s) ds + \int_0^t \sum_{n=1}^{N-1} \sigma_N(U_N^{n-1}) \frac{\eta_N^0}{\Delta t} \chi(t^n, t^{n+1})(s) ds \]

\[ = U_N^0 + \int_0^t (\mathcal{N}(U_N) + \ell_N) ds + \int_0^t \sigma_N(U_N) dW + \mathcal{E}_N^D(t) + \mathcal{E}_N^S(t), \quad (4.14) \]

where the ‘error terms’, \( \mathcal{E}_N^D(t) \) and \( \mathcal{E}_N^S(t) \), are defined as:

\[ \mathcal{E}_N^D(t) := -\mathcal{N}(U_N^0) \Delta t \wedge t - \left( \int_0^t \ell_N ds + \ell_N^{N-1}(t^{N-1} - t) \chi_{t>t^1} \right) = \mathcal{E}_N^{D,1}(t) + \mathcal{E}_N^{D,2}(t), \quad (4.15) \]

and

\[ \mathcal{E}_N^S(t) := -\sigma_N(U_N^{N-2}) \frac{\eta_N^{N-1}}{\Delta t} (t^{N-1} - t) \chi_{t>t^1} - \int_0^t \sigma_N(U_N) dW := \mathcal{E}_N^{S,1}(t) + \mathcal{E}_N^{S,2}(t). \quad (4.16) \]

To understand the origin of these error terms we observe that

\[ \int_0^t \sum_{n=1}^{N-1} \mathcal{N}(U_N^n) \chi(t^n, t^{n+1})(s) ds = \int_0^t \sum_{n=0}^{N-1} \mathcal{N}(U_N^n) \chi(t^n, t^{n+1})(s) ds - \mathcal{N}(U_N^0) \Delta t \wedge t \]

\[ = \int_0^t \mathcal{N}(U_N) ds + \mathcal{E}_N^{D,1}(t). \]

Moreover, using the definition of the \( \ell_N^n \)’s in (3.4), we have

\[ \int_0^t \sum_{n=1}^{N-1} \ell_N^n \chi(t^n, t^{n+1})(s) ds = \int_0^t \sum_{n=1}^{N-1} \ell_N^n \chi(t^n, t^{n+1})(s) ds + \left( \int_0^t \ell_N^{N-1} ds \right) \chi_{t>t^1} \]

\[ = \sum_{n=1}^{N-1} \ell_N^n \Delta t + \ell_N^{N-1}(t - t^{N-1}) \chi_{t>t^1} = \int_0^t \ell_N^{N-1} ds + \ell_N^{N-1}(t - t^{N-1}) \chi_{t>t^1} \]

\[ = \int_0^t \ell_N ds - \int_{t^{N-1}}^t \ell_N ds + \ell_N^{N-1}(t - t^{N-1}) \chi_{t>t^1}. \]

On the other hand for the error terms \( \mathcal{E}_N^S(t) \) involving \( \sigma_N \) in (4.16), we compute,

\[ \int_0^t \sum_{n=1}^{N-1} \sigma_N(U_N^{n-1}) \frac{\eta_N^n}{\Delta t} \chi(t^n, t^{n+1})(s) ds = \int_0^t \sum_{n=1}^{N-1} \sigma_N(U_N^{n-1}) \frac{\eta_N^n}{\Delta t} \chi(t^n, t^{n+1})(s) ds - \int_t^t \sigma_N(U_N^{N-2}) \frac{\eta_N^{N-1}}{\Delta t} ds \chi_{t>t^1} \]

\[ = \sum_{n=1}^{N-1} \sigma_N(U_N^{n-1}) \eta_N^n - \sigma_N(U_N^{N-2}) \frac{\eta_N^{N-1}}{\Delta t} (t^{N-1} - t) \chi_{t>t^1} \]

\[ = \int_0^t \sigma_N(U_N) dW - \sigma_N(U_N^{N-2}) \frac{\eta_N^{N-1}}{\Delta t} (t^{N-1} - t) \chi_{t>t^1} \]

\[ = \int_0^t \sigma_N(U_N) dW + \mathcal{E}_N^S(t). \]
4.3 The Estimates for the Error Terms

We next proceed to make estimates on the error terms $\mathcal{E}_N^D$ and $\mathcal{E}_N^S$ as desired in (4.7), (4.9). Perusing (4.15) we begin with estimates for $\mathcal{E}_N^{D,1}$. Invoking the bounds provided by (2.7) along with the continuity properties of the other operators making up $\mathcal{N}$ in (2.14) defined in Section 2.1 we have:

$$E \sup_{t \in [0,T]} \|\mathcal{E}_N^{D,1}(t)\|_{V'}^2 \leq \Delta t^2 E \|\mathcal{N}(U_N^0)\|_{V'}^2 \leq c \Delta t^2 E \left(1 + \|Y_N^0\| \right)^2 (1 + \|U_N^0\|_{V'}^2).$$

As such, in the view of the standing condition (4.3) (cf. Remark 3.1) we conclude that

$$\lim_{N \to \infty} E\|\mathcal{E}_N^{D,1}\|_{L^2(0,T;V')}^2 = \lim_{N \to \infty} E\|\mathcal{E}_N^{D,1}\|_{L^\infty(0,T;V')}^2 = 0. \quad (4.17)$$

For $\mathcal{E}_N^{D,2}$ we estimate in $L^2(0,T;V')$

$$\int_0^T \left\| \int_{t^{k-1}}^t \ell dt \right\|_{V'}^2 dt \leq \int_0^T \int_{t^{k-1}}^t \|\ell\|_{V'}^2 ds(t-t^{k-1}) dt = \sum_{k=1}^{N-1} \int_{t^{k-1}}^{t^k} \|\ell\|_{V'}^2 ds(t-t^{k-1}) dt \leq c \Delta t^2 \int_0^T \|\ell\|_{V'}^2 dt,$$

and

$$\int_0^T \left\| \mathcal{E}_N^{D,2}(t^{k-1}) \chi_{t>t^k} \right\|_{V'}^2 dt = \sum_{k=1}^{N-1} \|\ell_k\|_{V'}^2 \int_{t^k}^{t^{k+1}} (t^{k+1}-t)^2 dt \leq \frac{\Delta t^2}{3} \sum_{k=1}^{N-1} \left\| \int_{t^k}^{t^{k+1}} \ell ds \right\|_{V'}^2 \leq \frac{\Delta t^2}{3} \int_0^T \|\ell\|_{V'}^2 dt.$$

In summary we have

$$\lim_{N \to \infty} E\|\mathcal{E}_N^{D,2}\|_{L^2(0,T;V')}^2 = 0 \quad (4.18)$$

and so we conclude (4.7) from (4.17) and (4.18).

We next turn to make estimates for $\mathcal{E}_N^S$. We begin with estimates in $L^2(0,T;H)$. For $\mathcal{E}_N^{S,1}$ we observe with (2.10) and (3.6) (cf. (3.34)) that

$$E \int_0^T \|\mathcal{E}_N^{S,1}\|_{\mathcal{L}^2(0,T;H)}^2 dt = \sum_{k=1}^{N-1} E \left|\sigma_N(U_N^{k-1}) \frac{\eta_N}{\Delta t}\right|^2 \int_{t^k}^{t^{k+1}} (t^{k+1}-t)^2 dt = \Delta t \sum_{k=1}^{N-1} E \left|\sigma_N(U_N^{k-1}) \eta_N\right|^2 \leq c \Delta t \sum_{k=1}^{N-1} E \left((1 + |U_N^{k-1}|)^2\right) \Delta t,$$

and infer from (3.28) in Proposition 3.2 that

$$\lim_{N \to \infty} E\|\mathcal{E}_N^{S,1}\|_{L^2(0,T;H)}^2 = 0. \quad (4.19)$$

On the other hand, with the Itô isometry and another application of (2.10) and (3.6) we have

$$E \int_0^T \|\mathcal{E}_N^{S,2}\|_{\mathcal{L}^2(0,T;H)}^2 dt = \sum_{k=1}^{N-1} E \int_{t^k}^{t^{k+1}} \left|\int_{t^k}^{t} \sigma_N(U_N)dW\right|^2 dt = \sum_{k=1}^{N-1} E \int_{t^k}^{t^{k+1}} \int_{t^k}^{t} |\sigma_N(U_N)|^2 dW dt \leq c \Delta t \sum_{k=1}^{N-1} E \left((1 + |U_N^{k-1}|)^2\right) \Delta t,$$
so that
\[ \lim_{N \to \infty} \mathbb{E}\|\mathcal{E}^{S,2}_N\|_{L^2(0,T;H)}^2 = 0. \]  
(4.20)

By combining now (4.19) and (4.20) we obtain (4.8).

We turn now to establishing the uniform bounds announced in (4.9). Estimates similar to those leading to (4.19), (4.20) but which instead make use of the condition (3.5) yield bounds in \( L^2(0,T;V) \) namely,
\[ \mathbb{E} \int_0^T \|\mathcal{E}^{S,1}_N\|^2 dt = \frac{\Delta t}{3} \sum_{k=1}^{N-1} \mathbb{E} \|\sigma_N(U^{k-1}_N)\|_{L^2(I\cup V)}^2 \Delta t \leq \frac{T}{3} \sum_{k=1}^{N-1} \mathbb{E} \|\sigma(U^{k-1}_N)\|_{L^2(I\cup V)}^2 \Delta t \leq c \sum_{k=1}^{N-1} \mathbb{E}(1 + |U^{k-1}_N|^2) \Delta t, \]
and similarly
\[ \mathbb{E} \int_0^T \|\mathcal{E}^{S,2}_N\|^2 dt = \sum_{k=1}^{N-1} \mathbb{E} \|\sigma_N(U^{k-1}_N)\|_{L^2(I\cup V)}^2 \int_{t_k}^{t_{k+1}} (t - t_k) dt \leq c \sum_{k=1}^{N-1} \mathbb{E}(1 + |U^{k-1}_N|^2) \Delta t, \]
so that, taken together we infer that:
\[ \sup_{N \geq N_1} \mathbb{E}\|\mathcal{E}^S_N\|_{L^2(0,T;V)} < \infty. \]  
(4.21)

Finally we supply a bound for \( \mathcal{E}^S_N \) in \( L^\infty(0,T;H) \). For \( \mathcal{E}^{S,3}_N \) we observe with (2.10), (3.6) that
\[ \mathbb{E} \sup_{t \in [0,T]} |\mathcal{E}^{S,1}_N|^2 \leq \sum_{k=1}^{N-1} \mathbb{E} \sup_{t \in [t^k,t^{k+1}]} |\mathcal{E}^{S,1}_N|^2 \leq \sum_{k=1}^{N-1} \mathbb{E} |\sigma_N(U^{k-1}_N)\eta_k|^2 \leq c \sum_{k=1}^{N-1} \mathbb{E}(1 + |U^{k-1}_N|^2) \Delta t. \]

To estimate \( \mathcal{E}^{S,2}_N \) we use Doob’s inequality and (2.10) to infer
\[ \mathbb{E} \sup_{t \in [0,T]} |\mathcal{E}^{S,2}_N|^2 \leq \sum_{k=1}^{N-1} \mathbb{E} \sup_{t \in [t^k,t^{k+1}]} |\mathcal{E}^{S,2}_N|^2 \leq \sum_{k=1}^{N-1} \mathbb{E} \sup_{t \in [t^k,t^{k+1}]} \int_{t_k}^t |\sigma_N(U_N) dW|^2 ds \leq \sum_{k=1}^{N-1} \mathbb{E} \sup_{t \in [0,T]} |\sigma_N(U_N)|_{L^2(I\cup H)}^2 \Delta t \leq c \sum_{k=1}^{N-1} \mathbb{E}(1 + |U^{k-1}_N|^2) \Delta t. \]

With these bounds and (3.28) we conclude that
\[ \sup_{N \geq N_1} \mathbb{E}\|\mathcal{E}^S_N\|^2_{L^\infty(0,T;H)} < \infty. \]  
(4.22)

In turn, (4.21), (4.22) directly imply (4.9) and so the proof of Proposition 4.1 is now complete.

## 5 Compactness and The Passage to the Limit

In this section we detail the compactness arguments that we use to prove the existence of Martingale solutions of (2.13) using the processes \( U_N \) and \( \tilde{U}_N \) defined in the previous section. As it is not clear how to obtain compactness directly from \( \tilde{U}_N \), (cf. Remark 4.2) we must introduce further processes to achieve this end.

Recalling (4.1), (4.2), (4.15), (4.16) we define
\[ U^*_N = U_N - \mathcal{E}^S_N, \quad U^*_N = U_N - \mathcal{E}^D_N, \]  
(5.1)
and then consider the associated probability measures
\[ \mu_N(\cdot) := \mathbb{P}(U_N \in \cdot), \quad \mu^*_N(\cdot) := \mathbb{P}(U^*_N \in \cdot), \quad \mu^{*\prime}_N(\cdot) := \mathbb{P}(U^{*\prime}_N \in \cdot). \]  
(5.2)
Notice that, due to Proposition 4.1, $\mu_N, \mu_N^*$ are defined on the space $\mathcal{X} := L^2(0, T; H)$. Regarding the elements $\mu_N^*$ we observe that, as a consequence of (4.6)

$$U_N^*(t) = U_N^0 + \int_0^t (\mathcal{N}(U_N) + \ell) dt + \int_0^t \sigma_N(U_N) dW.$$  \hspace{1cm} (5.3)

As a result of this identity and Proposition 4.1, the elements $\mu_N^*$ may be regarded as measures on the space $\mathcal{Y} := L^2(0, T; V') \cap C([0, T]; V_1'(3))$.

We will show below that $\mu_N$ and $\mu_N^*$ converge weakly to a common measure $\mu$ and then make careful usage of the Skorohod embedding theorem to pass to the limit in (5.3) on a new stochastic basis. The former compactness arguments, which rely on the intermediate measures $\mu_N$, will be carried out in the next subsection and the details of the Skorohod embedding will be discussed in Subsection 5.2 further on.

### 5.1 Tightness Arguments

In this section we will establish the following compactness properties of the $\{\mu_N\}_{N \geq N_1}$ and $\{\mu_N^*\}_{N \geq N_1}$.

**Proposition 5.1.** The assumptions are precisely those in Proposition 4.1. Define $\{U_N\}_{N \geq N_1}$ and $\{U_N^*\}_{N \geq N_1}$ according to (4.1) and (5.1) and where $N_1$ is as in (3.26). Let $\{\mu_N\}_{N \geq N_1}$, $\{\mu_N^*\}_{N \geq N_1}$ be the associated Borel measures on

$$\mathcal{X} := L^2(0, T; H), \quad \mathcal{Y} := L^2(0, T; V') \cap C([0, T]; V_1'(3)),$$

defined according to (5.2). Then, there exists a Borel measure $\mu$ on $L^2(0, T; H) \cap C([0, T]; V_1'(3))$ such that, up to a subsequence$^{12}$

$$\mu_N \rightharpoonup \mu, \quad \text{(weakly) on } \mathcal{X},$$  \hspace{1cm} (5.4)

and

$$\mu_N^* \rightharpoonup \mu, \quad \text{(weakly) on } \mathcal{Y}.$$  \hspace{1cm} (5.5)

The rest of this subsection is devoted to the proof of Proposition 5.1. We proceed as follows: First we show that $\{\mu_N\}_{N \geq N_1}$ is tight (cf. Appendix A.1) in $L^2(0, T; H)$ by employing a suitable variant of the Aubin-Lions compactness theorem which we establish in Proposition A.4 below. We next show that $\{\mu_N^*\}_{N \geq N_1}$ is tight in $C([0, T]; V_1'(3))$ via an Arzelà-Ascoli type compact embedding from [FG95] and [Tem95]. We finally employ the estimates (4.5), (4.7) along with the general convergence results recalled in Lemma A.1 to finally infer (5.4) and (5.5).

#### 5.1.1 Tightness for $\mu_N^*$ in $L^2(0, T; H)$

With the aid of Proposition A.4 we identify some compact subsets of $\mathcal{X} = L^2(0, T; H)$ that, in conjunction with suitable estimates (see (5.10)–(5.13) immediately below) are used to establish the tightness of $\{\mu_N^*\}_{N \geq N_1}$ in $\mathcal{X}$. For $U \in \mathcal{X}$, $n > 0$, define

$$[U]_j := \left( j \sup_{0 \leq \theta \leq j^{-6}} \int_0^{T - \theta} \|U(t + \theta) - U(t)\|_{V_1'(3)}^{4/3} dt \right)^{3/4},$$  \hspace{1cm} (5.6)

and, for each $R > 0$, consider

$$B_R := \left\{ U \in \mathcal{X} : \|U\|_{L^2(0, T; V)} + \|U\|_{L^\infty(0, T; H)} + \sup_{j \geq 1} [U]_j \leq R \right\}. $$  \hspace{1cm} (5.7)

$^{12}$We recall the notion of weak compactness of probability measures along with the equivalent notion of tightness in the Appendix, Section A.1 below.
It is not hard to show that each set $B_R$ is a closed subset of $\mathcal{X}$. Perusing (5.6) it is clear that the condition (A.4) holds uniformly for elements in $B_R$. Thus, as a consequence of Proposition A.4, (ii) these sets $B_R$ are compact in $\mathcal{X} = L^2(0, T; H)$ for each $R > 0$.

Now, for each $R > 0$, we have:

$$\mu^*_N(B_R^c) \leq P\left(\|U^*_n\|_{L^2(0,T,V)} + \|U^*_n\|_{L^\infty(0,T,H)} > R/2\right) + P\left(\sup_{j \geq 1}[U^*_n]_j > R/2\right) \tag{5.8}$$

As a consequence of (4.4), (4.9) and (5.1) we have

$$P\left(\|U^*_n\|_{L^2(0,T,V)} + \|U^*_n\|_{L^\infty(0,T,H)} > R/2\right) \leq \frac{c}{R^2}, \tag{5.9}$$

for some constant $c$ independent of $N$.

Next we need to establish suitable uniform estimates for $\sup_{j \geq 1}[U^*_n]_j$ (cf. (5.6)). To this end we observe with (4.14) and (5.1) that for any $\theta > 0$,

$$\int_0^{T-\theta} \|U^*_n(t + \theta) - U^*_n(t)\|_{V^\prime_n(2)}^{4/3} dt \leq I^D_N(\theta) + I^S_N(\theta), \tag{5.10}$$

with

$$I^D_N(\theta) = c \int_0^{T-\theta} \left\| \int_t^{t+\theta} \sum_{n=1}^{N-1} (\mathcal{N}(U^*_n) + \ell^e_n) \chi_{(\tau^n, \tau^{n+1})}(s) ds \right\|_{V^\prime_n(2)}^{4/3} dt,$$

$$I^S_N(\theta) = c \int_0^{T-\theta} \left\| \int_t^{t+\theta} \sigma_N(U^*_n) dW \right\|_{V^\prime_n(2)}^{4/3} dt.$$

To address $I^D_N(\theta)$ we observe, with (2.6) and the standing assumptions on the operators that make up $\mathcal{N}$ in (2.14), that for any $U \in V$,

$$\|\mathcal{N}(U)\|_{V^\prime_n(2)}^{4/3} \leq c(|U|^{2/3} + 1)(\|U\|^2 + 1). \tag{5.11}$$

Furthermore it is clear from (3.4) and Hölder’s inequality that, a.s.

$$\int_0^T \sum_{n=1}^{N-1} \sum_{l=1}^{n-1} \left( \int_t^{t+\theta} \|\ell^e_n\|_{V^\prime_n(2)}^{4/3} \chi_{(\tau^n, \tau^{n+1})}(s) ds \right) \chi_{(\tau^n, \tau^{n+1})}(s) dt \leq \int_0^T \|\ell\|_{V^\prime_n(2)}^{4/3} dt.$$

Combining these observations we infer that, a.s.

$$I^D_N(\theta) \leq c\theta^{1/3} \int_0^T \sum_{n=1}^{N-1} \sum_{l=1}^{n-1} \left( \int_t^{t+\theta} \|\mathcal{N}(U^*_n) + \ell^e_n\|_{V^\prime_n(2)}^{4/3} \chi_{(\tau^n, \tau^{n+1})}(s) ds \right) dt$$

$$\leq c\theta^{1/3} \int_0^T \sum_{n=1}^{N-1} \sum_{l=1}^{n-1} \left( (|U^*_n|^{2/3} + 1)(\|U^*_n\|^2 + 1) + \|\ell\|_{V^\prime_n(2)}^{4/3} \right) \chi_{(\tau^n, \tau^{n+1})}(s) ds$$

$$\leq c\theta^{1/3} \left( \max_{0 \leq l \leq N} \left( 1 + |U^*_n|^{2/3} \right) \sum_{j=1}^{N} \Delta t(\|U^*_n\|^2 + 1) + \int_0^T \|\ell\|_{V^\prime_n(2)}^{4/3} dt \right)$$

$$\leq c\theta^{1/3} \left( \max_{0 \leq l \leq N} \left( 1 + |U^*_n|^{2/3} \right) \sum_{j=1}^{N} \Delta t(\|U^*_n\|^2 + 1) + \int_0^T (1 + \|\ell\|^2_{V^\prime_n}) dt \right). \tag{5.12}$$
For the term \( I_N^S \) we estimate, for \( 0 \leq \theta \leq \delta \),

\[
\mathbb{E} \left( \sup_{0 \leq \theta \leq \delta} I_N^S(\theta) \right) \leq c \int_0^T \left( \mathbb{E} \sup_{0 \leq \theta \leq \delta} \left\| \int_t^{(t+\theta) \wedge T} \sigma_N(U_N) \, dW \right\|_{V'_2(\omega)}^2 \right)^{2/3} \, dt \\
\leq c\delta^{2/3} \mathbb{E} \sup_{t \in [0, T]} (1 + |U_N|^2),
\]

where the second line follows from Doob’s inequality and the standing assumptions (2.10) on \( \sigma \) and (3.6) on \( \sigma_N \):

\[
\mathbb{E} \sup_{0 \leq \theta \leq \delta} \left\| \int_t^{(t+\theta) \wedge T} \sigma_N(U_N) \, dW \right\|_{V'_2(\omega)}^2 \leq c \mathbb{E} \int_t^{(t+\theta) \wedge T} \|\sigma_N(U_N)\|_{V'_2(\omega)}^2 \, ds \leq c\delta \mathbb{E} \sup_{t \in [0, T]} (1 + |U_N|^2).
\]

The estimates (5.12), (5.13) allow the second term in (5.8) to be treated as follows. Observe that according to (5.6), (5.10) we have

\[
\sup_{j \geq 1} |U_N|^{4/3} \leq \sup_{j \geq 1} \left( j \sup_{|\theta| \leq j^{-6}} I_N^S(\theta) \right) + \sup_{j \geq 1} \left( j \sup_{|\theta| \leq j^{-6}} I_N^S(\theta) \right).
\]

For the first term we observe with (5.12) that

\[
\sup_{j \geq 1} \left( j \sup_{|\theta| \leq j^{-6}} I_N^S(\theta) \right) \leq c \left( \max_{0 \leq \ell \leq 1} (1 + |U_N|) \sum_{r=1}^N \Delta t (\|U_N\|^{2} + 1) + \int_0^T (\|A\|_{V'_2 \ast}^{2} + 1) \, dt \right)
\leq c(T_1^N T_2^N + T_3^N).
\]

Regarding the second term we simply bound

\[
\sup_{j \geq 1} \left( j \sup_{|\theta| \leq j^{-6}} I_N^S(\theta) \right) \leq \sum_{j \geq 1} j \sup_{|\theta| \leq j^{-6}} I_N^S(\theta) := T_4^N,
\]

so that for \( \rho > 0 \), sufficiently large,

\[
\mathbb{P} \left( \sup_{j \geq 1} |U_N|^{4/3} > \rho \right) \leq c(T_1^N T_2^N + T_3^N + T_4^N) \leq c(T_1^N T_2^N + T_3^N + T_4^N) \leq c \left( \max_{0 \leq \ell \leq 1} (1 + |U_N|) \sum_{r=1}^N \Delta t (\|U_N\|^{2} + 1) + \int_0^T (\|A\|_{V'_2 \ast}^{2} + 1) \, dt \right)
\leq c \frac{\mathbb{E}(T_1^N + T_2^N + T_3^N + T_4^N)}{\sqrt{c}}.
\]

In view of the uniform bound (3.28) established in Proposition 3.2, \( \sup_N \mathbb{E} T_1^N \) and \( \sup_N \mathbb{E} T_2^N \) are both finite. The term \( \sup_N \mathbb{E} T_3^N \), which is independent of \( N \), is finite due to the standing assumption on \( \ell \) (cf. (2.17)).

For \( T_4^N \) we refer back to (5.13) and apply the monotone convergence theorem to infer:

\[
\mathbb{E} T_4^N \leq \sum_{n \geq 1} n^{-3} \mathbb{E} \sup_{t \in [0, T]} (1 + |U_N|^2) < \infty.
\]

We finally conclude that

\[
\mathbb{P} \left( \sup_{j \geq 1} |U_N| \geq R/2 \right) = \mathbb{P} \left( \sup_{j \geq 1} |U_N|^{4/3} > (R/2)^{4/3} \right) \leq \frac{c}{R^{4/3}}.
\]

Combining (5.8), (5.9) and (5.16) we now conclude that (cf. Appendix A.1)

\[ \{\mu_N^r\}_{N \geq 1} \text{ is tight in } \mathcal{X} = L^2(0; T; H). \]
5.1.2 Tightness for $\mu_N^{**}$ in $\mathcal{C}([0,T]; V'_3)$

We next show that $\mu_N^{**}$ is tight in $\mathcal{C}([0,T], V'_3)$. For this purpose we make appropriate usage of a compact embedding from [FG95] (see also [Tem95]). Let us fix any $p \in (2, \infty), \alpha \in (0, 1/2)$ such that $\alpha p > 1$. According to [FG95]:

$$W^{1,4/3}(0,T; V'_2) \subset \subset \mathcal{C}([0,T]; V'_3), \quad W^{\alpha,p}(0,T; V'_2) \subset \subset \mathcal{C}([0,T]; V'_3), \quad (5.18)$$

that is, the embeddings are continuous and compact. We now define

$$B_R := \left\{ X \in \mathcal{C}([0,T]; V'_3) : \|X\|_{W^{1,4/3}(0,T; V'_2)} \leq R \right\} + \left\{ Y \in \mathcal{C}([0,T]; V'_3) : \|Y\|_{W^{\alpha,p}(0,T; V'_2)} \leq R \right\}$$

for any $R > 0$. With (5.18), it is clear that $B_R$ is compact in $\mathcal{C}([0,T]; V'_3)$ for every $R > 0$. Observe moreover that, in view of (5.3)

$$\{U_N^{**} \in B_R\} \supset \left\{ U_N^0 + \int_0^T (N(U_N) + \ell) ds \in B_R^0 \right\} \cap \left\{ \int_0^T \sigma_N(U_N) dW \in B_R^S \right\},$$

and thus that

$$\mu_N^{**}(B_R^0) \leq \mathbb{P} \left( \left\| U_N^0 + \int_0^T (N(U_N) + \ell) ds \right\|_{W^{1,4/3}(0,T; V'_2)} > R \right) + \mathbb{P} \left( \left\| \int_0^T \sigma_N(U_N) dW \right\|_{W^{\alpha,p}(0,T; V'_2)} > R \right)$$

$$:= S_R^N + T_R^N. \quad (5.19)$$

Hence we will infer that $\{\mu_N^{**}\}$ is tight in $\mathcal{C}([0,T], V'_3)$ if we can show that $T_N^R, S_N^R$ converge uniformly in $N$ to zero as $R \uparrow \infty$.

For $T_N^R$ we estimate, with (5.11)

$$\left\| U_N^0 + \int_0^T (N(U_N) + \ell) ds \right\|^{4/3}_{W^{1,4/3}(0,T; V'_2)}$$

$$\leq c (1 + |U_N^0|^2) + c \int_0^T \|N(U_N) + \ell\|_{V'_2}^{4/3} dt$$

$$\leq c (1 + |U_N^0|^2) + c \int_0^T \left( (|U_N|^{2/3} + 1)(|U_N|^2 + 1) + \|\ell\|_{V'_2}^2 \right) dt,$$

$$\leq c \sup_{t \in [0,T]} (1 + |U_N|^2) \cdot \left( \int_0^T (1 + |U_N|^2 + \|\ell\|_{V'_2}^2) dt + 1 \right).$$

Thus we find, cf. (5.15):

$$T_N^R \leq \mathbb{P} \left( c \sup_{t \in [0,T]} (1 + |U_N|^2) \cdot \left( \int_0^T (1 + |U_N|^2 + \|\ell\|_{V'_2}^2) dt + 1 \right) > R \right)$$

$$\leq \mathbb{P} \left( c \sup_{t \in [0,T]} (1 + |U_N|^2) > R^{1/2} \right) + \mathbb{P} \left( \int_0^T (1 + |U_N|^2 + \|\ell\|_{V'_2}^2) dt + 1 > R^{1/2} \right)$$

$$\leq \frac{c}{R^{1/2}} \mathbb{E} \sup_{t \in [0,T]} (1 + |U_N|^2) + \frac{1}{R^{1/2}} \mathbb{E} \left( \int_0^T (1 + |U_N|^2 + \|\ell\|_{V'_2}^2) dt + 1 \right). \quad (5.20)$$
We turn to $S^R_N$. For this purpose let us define for any $R > 0$ the stopping times
\[ \tau_R := \inf_{t \geq 0} \left\{ \sup_{s \in [0,t]} |U_N| \geq R \right\} \] \[ \wedge T = \sup_{t \geq 0} \left\{ \sup_{s \in [0,t]} |U_N| < R \right\} \wedge T. \]

Using $\tau_R$ we now estimate with the Chebyshev inequality that
\[ S^R_N \leq \mathbb{P}\left( \left\| \int_0^{\wedge \tau_R} \sigma_N(U_N) dW \right\|_{W^p(0,T;V^{(2)})} > R, \tau_R \geq T \right) + \mathbb{P}(\tau_R < T) \]
\[ \leq \mathbb{P}\left( \left\| \int_0^{\wedge \tau_R} \sigma_N(U_N) dW \right\|_{W^p(0,T;V^{(2)})} > R \right) + \mathbb{P}\left( \sup_{s \in [0,T]} |U_N| \geq R \right) \]
\[ \leq \frac{1}{R^p} \mathbb{E}\left( \left\| \int_0^{\wedge \tau_R} \sigma_N(U_N) dW \right\|_{W^p(0,T;V^{(2)})}^p \right) + \frac{1}{R^2} \mathbb{E}\sup_{s \in [0,T]} |U_N|^2. \quad (5.21) \]

Now in order to treat this final stochastic integral term we recall the following generalization of the Burkholder-Davis-Gundy inequality from e.g. [FG95]: for a given Hilbert space $X$, $p \geq 2$ and $\alpha \in [0,1/2)$ we have for all $X$-valued predictable $G \in L^p(\Omega; L^p_{\text{loc}}(0, \infty, L_2(\mathcal{U}, X)))$
\[ \mathbb{E}\left( \left\| \int_0^T G dW \right\|_{W^p(0,T;X)}^p \right) \leq c \mathbb{E}\left( \int_0^T |G|_{L_2(\mathcal{U}, X)}^p dt \right), \]
which holds with a constant $c$ depending only on $\alpha$ and $p$. Continuing now from (5.21) we have
\[ S^R_N \leq \frac{c}{R^p} \mathbb{E}\int_0^{\wedge \tau_R} |\sigma_N(U_N)|_{L_2(\mathcal{U}, H)}^p dt + \frac{1}{R^2} \mathbb{E}\sup_{s \in [0,T]} |U_N|^2 \leq \frac{c}{R^p} \mathbb{E}\sup_{s \in [0,T \wedge \tau_R]} (1 + |U_N|^p) + \frac{1}{R^2} \mathbb{E}\sup_{s \in [0,T]} |U_N|^2 \]
\[ \leq \frac{c(1 + R^{p-2})}{R^p} \mathbb{E}\sup_{s \in [0,T]} (1 + |U_N|^2) + \frac{1}{R^2} \mathbb{E}\sup_{s \in [0,T]} |U_N|^2 \leq \frac{c}{R^2} \mathbb{E}\sup_{s \in [0,T]} (1 + |U_N|^2). \quad (5.22) \]

Combining the estimates (5.20), (5.22) with (4.4) we finally conclude
\[ \sup_{N \geq N_1} \mu^*_N(B_R) \geq 1 - \frac{c}{R^{1/2}} \]
and hence infer
\[ \{\mu^*_N\}_{N \geq N_1} \text{ is tight in } C([0,T]; V^{(2)}_H). \quad (5.23) \]

**Remark 5.1.** Let us observe that the tightness bounds for $\mu^*_N$ and $\mu^*_N$ could be carried out differently if we had available, for example, the uniform bounds on 'higher moments' like
\[ \sup_{N \geq 1} \mathbb{E}\left( \max_{0 \leq k \leq N} |U_N|^4 + \left( \sum_{k=1}^N \Delta t\|U_N\|^2 \right)^2 \right) < \infty \quad (5.24) \]
or equivalently that
\[ \sup_{N \geq 1} \mathbb{E}\left( \sup_{t \in [0,T]} |U_N|^4 + \left( \int_0^T \|U_N\|^2 dt \right)^2 \right) < \infty. \quad (5.25) \]
Indeed, in numerous other previous works related to stochastic fluids equations (see e.g. [Ben95, FG95, MS02, DGHT11, GV14]) estimates analogous to (5.25) are established essentially via Itô’s lemma in order to achieve tightness in the probability laws associated to a regularization scheme.

In the current situation, instead due to the way we carry out the estimates in (5.15), (5.20) and (5.21)–(5.22), we have adopted a different approach, namely, we establish tightness (compactness) estimates without recourse to such higher moment estimates.

A different method using higher moments will be shown in the related work [GTW].

5.1.3 Cauchy Arguments and Conclusions

With (5.17) and (5.23) now in hand it is then simply a matter of collecting the various convergences above to complete the proof of Proposition 5.1.

By making use of Prohorov’s theorem (cf. Section A.1 in the Appendix) with (5.17) we infer the existence of a probability measure $\mu$ such that, up to a subsequence,

$$
\mu_N^* \rightarrow \mu \quad \text{on } X = L^2(0, T; H) \quad \text{and also on } L^2(0, T; V').
$$

Due to (5.1) with (4.5) and (4.8) it is clear that $U_N' - U_N$ converges to zero in $X = L^2(0, T; H)$ and hence in $L^2(0, T; V')$ a.s. Hence, by now invoking (4.7) and referring back once more to (5.1), we have that $U_N' - U_N^*$ converges to zero in $L^2(0, T; V')$ a.s. Thus, invoking Lemma A.1, we conclude, again up to a subsequence, that:

$$
\mu_N^* \rightarrow \mu \quad \text{on } L^2(0, T; V') \quad \text{and } \mu_N \rightarrow \mu \quad \text{on } X = L^2(0, T; H).
$$

(5.26)

In particular this is the first desired convergence for $\{\mu_N\}_{N \geq N_1}$, (5.4). On the other hand invoking Prohorov’s theorem with (5.23) and the convergence just established for $\{\mu_N^*\}_{N \geq N_1}$ in $L^2(0, T; V')$ we see that $\mu_N^*$ is tight in $Y = L^2(0, T; V') \cap C([0, T]; V'_{(3)})$. By Prohorov’s theorem in the other direction and passing to a further subsequence as needed we have

$$
\mu_N^* \rightarrow \bar{\mu} \quad \text{on } Y = L^2(0, T; V') \cap C([0, T], V'_{(3)}).
$$

Since, clearly, $\bar{\mu} = \mu$ this yields the second desired item (5.5). The proof of Proposition 5.1 is therefore complete.

5.2 Proof of Theorem 2.1

Conclusion: Almost Sure convergence and the Passage to the Limit on the Skorokhod Basis

We now have all of the ingredients to finally prove one the main results of this article, namely Theorem 2.1. Suppose that we are given $\mu_{U_0} \in Pr(H)$ and $\mu_{\ell} \in Pr(L^2_{loc}(0, \infty; V'))$ according to the conditions specified in Definition 2.1. As mentioned in Remark 3.2 now it is necessary to introduce the stochastic basis $\mathcal{F}_T$ (defined as in subsection 3.2.1), an element $U^0$ which is $\mathcal{F}_0$ measurable and a process $\ell = \ell(t)$ measurable with respect to the sigma algebra generated by the $W(s)$ for $s \in [0, t]$\textsuperscript{13}, whose laws coincide with those of $\mu_{U_0}, \mu_{\ell}$. Thus Proposition 3.1 applies and we obtain the existence of the $U^*_N$’s adapted to $\mathcal{F}_T$.

We then approximate $U^0 \in L^2(\Omega; H)$ with a sequences of elements $\{U^0_N\}_{N \geq 1} \subseteq L^2(\Omega, V(2))$, which maintains the bound (4.3) as described in Remark 3.1 above. Proposition 4.1 applies and hence we can use this sequence $\{U^0_N\}_{N \geq N_1}$, the process $\ell$, and the sequence $U_N^*$ to define processes $\{U_N\}_{N \geq N_1}, \{U_N^*\}_{N \geq N_1}$ according to (4.1) and (5.1) respectively ($N_1$ is given by (3.26)). In order to pass to the limit in the associated evolution equation (5.3), we consider the product measures:

$$
\nu_N(\cdot) := \mathbb{P}(\{U^*_N, U_N, \ell, W\} \in \cdot)
$$

\textsuperscript{13}Note that since the sigma algebra generated by the $W(s)$ for $s \in [0, t]$ is the smallest respect to which $W(t)$ is measurable, so $\ell(t)$ is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, and hence all the previous results applies.
which are defined on the space
\[ \mathcal{Z} = \mathcal{Y} \times \mathcal{X} \times L^2(0; V') \times \mathcal{C}([0, T]; \mathcal{U}_0). \] (5.27)

where, as above, \( \mathcal{Y} = L^2(0; V') \cap \mathcal{C}([0, T], V'_{(3)}) \), \( \mathcal{X} = L^2(0; T; H) \), and \( \mathcal{U}_0 \) is defined as in Section 2.2, (2.15). By invoking Proposition 5.1 we have that (passing to a subsequence as needed) \( \mu_N \rightarrow \mu \) on \( \mathcal{X} \) and \( \mu_N^{**} \rightarrow \mu \) on \( \mathcal{Y} \), where \( \mu_N \) and \( \mu_N^{**} \) are defined as in (5.2). It follows, again up to passing to a subsequence, that \( \nu_N \) converges weakly to a measure \( \nu \) on \( \mathcal{Z} \) (defined in (5.27)). Furthermore, recalling (5.1) and making use of (4.5), (4.7), (4.8) it is not hard to see that:
\[ \nu(\{(U^{**}, U, \ell, W) \in \mathcal{Z} : U \neq U^{**}\}) = 0. \]

Thus, by making use of the Skorokhod embedding theorem (see Section A.1) we obtain, relative to a new probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), a sequence of random variables
\[ (\hat{U}_N^{**}, \hat{U}_N, \hat{\ell}_N, \hat{W}_N) \rightarrow (\bar{U}, \bar{\ell}, \bar{W}) \quad \hat{\Omega} \text{ a.s. in } \mathcal{Z}. \] (5.28)

Moreover, the uniform bounds for \( \{U_N\}_{N \geq N_1} \) in \( L^2(\Omega; L^2(0, T; V) \cap L^\infty(0, T; H)) \) from Proposition 4.1, (4.4) imply that in addition to (5.28) we also have
\[ \tilde{U}_N \rightarrow \bar{U} \quad \text{weakly in } L^2(\Omega; L^2(0, T; V)) \text{ and weakly-star in } L^2(\Omega; L^\infty(0, T; H)). \] (5.29)

Following a procedure very similar to [Ben95] we may now show that \( \bar{W}_N \) is a cylindrical Brownian motion relative to the filtration \( \hat{\mathcal{F}}_t^N \) defined as the sigma algebra generated by the \((\tilde{U}_N^{**}(s), \tilde{U}_N(s), \tilde{\ell}_N(s), \tilde{W}_N(s))\) for \( s \leq t \) and that \((\tilde{U}_N^{**}, \tilde{U}_N, \tilde{\ell}_N, \bar{W}_N)\) satisfies (5.3) on the ‘Skorokhod space’ \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) viz.
\[ \tilde{U}_N^{**}(t) = \tilde{U}_N^{**}(0) + \int_0^t (\tilde{N}(\tilde{U}_N) + \tilde{\ell}_N)ds + \int_0^t \sigma_N(\tilde{U}_N)d\bar{W}_N. \] (5.30)

Using the convergences in (5.28)–(5.29) with (5.30) it is standard\(^{14}\) to show that \( \bar{U} \) satisfies (2.1)–(2.20) relative to the stochastic basis \( \hat{\mathcal{S}} := (\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{\mathbb{P}}, \{\hat{W}_k\}_{k \geq 1}) \) where \( \{\hat{\mathcal{F}}_t\}_{t \geq 0} \) is defined as the sigma algebra generated by the \((\tilde{U}(s), \tilde{\ell}(s), \bar{W}(s))\) for \( s \leq t \) and \( W_k = (W, e_k)_{\mathcal{U}} \). Therefore \((\hat{S}, U, \hat{\ell})\) is a Martingale solution of (2.13) relative to \( \mu_{U_0}, \mu_\ell \) in the sense of Definition 2.1 and the proof of Theorem 2.1 is complete.

### 6 Convergence of the Euler Scheme

We conclude by reinterpreting from the point of view of numerical analysis, the study above as a result of convergence for the Euler scheme (3.3).

**Theorem 6.1.** We assume given \( \mu_{U_0} \in \text{Pr}(H) \) and \( \mu_\ell \in \text{Pr}(L^2_{loc}(0, \infty; V')) \) according to Definition 2.1. We also assume given the stochastic basis \( \mathcal{S}_G \) (defined as in subsection 3.2.1), an element \( U^0 \) which is \( \mathcal{G}_0 \) measurable and a process \( \ell = \ell(t) \) measurable with respect to the sigma algebra generated by the \( W(s) \) for \( s \in [0, t] \), whose laws coincide with those of \( \mu_{U_0}, \mu_\ell \). Let a sequences of elements \( \{U^0_N\}_{N \geq 1} \subseteq L^2(\Omega, V(2)) \) approximate \( U^0 \) in \( L^2(\Omega; H) \) as described in Remark 3.1. Then the processes \( \{U_N\}_{N \geq N_1} \), defined according to (4.1) \((N_1 \text{ is given by (3.26)}\) adapted to \( \{\mathcal{G}_t\}_{t \geq 0} \) exist.

Moreover the family \( \{\mu_N\} \) of probability laws of \( \{U_N\} \), is weakly compact over the phase space \( L^2(0, T; H) \cap \mathcal{C}([0, T], V'_{(3)}) \) and hence converges weakly to a probability measure \( \mu \) on the same phase space up to a subsequence. Furthermore, there exists a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\) and a subsequence of random vectors \((\tilde{U}_N, \tilde{\ell}_N, \tilde{W}_N)\) with values in \( \mathcal{Z}_1 := L^2(0, T; H) \cap \mathcal{C}([0, T], V'_{(3)}) \times L^2(0, T; V') \times \mathcal{C}([0, T]; \mathcal{U}_0) \) such that

(i) \((\tilde{U}_N, \tilde{\ell}_N, \tilde{W}_N)\) have the same probability distribution as \((U_N, \ell, W)\).

\(^{14}\)Note that, in particular, the stochastic terms involving \( \sigma_N(U_N) \) converge due to (3.7).
7 Applications for Equations in Geophysical Fluid Dynamics

In this section we apply the above framework culminating in Theorem 2.1 and Theorem 6.1 to a stochastic version of the Primitive Equations. Our presentation here will focus on the case of the equations of the oceans. Note however that the abstract setting introduced above is equally well suited to derive results for analogous systems for the atmosphere or for the coupled oceanic-atmospheric system (COA).15 We refer the interested reader to [PTZ08] for further details on these other interesting situations.

7.1 The Oceans Equations

The stochastic primitive equations of the Oceans take the form:

\[ \partial_t \mathbf{v} + \nabla \cdot \mathbf{v} + w \partial_z \mathbf{v} + \frac{1}{\rho_0} \nabla p + f \times \mathbf{v} - \mu \nabla \mathbf{v} - \nu \nabla \mathbf{v} = F_{\mathbf{v}} + \sigma_{\mathbf{v}}(\mathbf{v}, T, S) \tilde{W}_1, \tag{7.1a} \]

\[ \partial_t p = -\rho g, \tag{7.1b} \]

\[ \nabla \cdot \mathbf{v} + \partial_z w = 0 \tag{7.1c} \]

\[ \partial_t T + \nabla \cdot \mathbf{v} T + w \partial_z T - \mu_T \Delta T - \nu_T \partial_z T = F_T + \sigma_T(\mathbf{v}, T, S) \tilde{W}_2, \tag{7.1d} \]

\[ \partial_t S + \nabla \cdot \mathbf{v} S + w \partial_z S - \mu_S \Delta S - \nu_S \partial_z S = F_S + \sigma_S(\mathbf{v}, T, S) \tilde{W}_3, \tag{7.1e} \]

\[ \rho = \rho_0 (1 + \beta_T (T - T_r) + \beta_S (S - S_r)). \tag{7.1f} \]

Here, \( U := (\mathbf{v}, T, S) = (u, v, T, S) \), \( p, \rho \) represent the horizontal velocity, temperature, salinity, pressure and density of the fluid under consideration; \( \mu, \nu, \mu_T, \nu_T, \mu_S, \nu_S \) are positive coefficients which account for the eddy and molecular diffusivities (viscosity) in the equations for \( \mathbf{v}, T \) and \( S \). The terms \( F_{\mathbf{v}}, F_T, F_S \) are volumetric sources of momentum, heat and salt which are zero in idealized situations but which we consider to be random in general.

The state dependent stochastic terms are driven by independent Gaussian white noise processes \( \tilde{W}_j \), \( j = 1, 2, 3 \) which are formally delta correlated in time. The stochastic terms may be written in the expansion

\[ \sigma_U(\mathbf{v}) \tilde{W} = \begin{pmatrix} \sigma_{\mathbf{v}}(\mathbf{v}) \tilde{W}_1(t, \mathbf{x}) \\ \sigma_T(\mathbf{v}) \tilde{W}_2(t, \mathbf{x}) \\ \sigma_S(\mathbf{v}) \tilde{W}_3(t, \mathbf{x}) \end{pmatrix} = \sum_{k \geq 1} \begin{pmatrix} \sigma_{\mathbf{v}}^k(\mathbf{v})(t, \mathbf{x}) \tilde{W}_1^k(t) \\ \sigma_T^k(\mathbf{v})(t, \mathbf{x}) \tilde{W}_2^k(t) \\ \sigma_S^k(\mathbf{v})(t, \mathbf{x}) \tilde{W}_3^k(t) \end{pmatrix}, \tag{7.2} \]

where the elements \( \tilde{W}_j^k \) are independent 1-D white (in time) noise processes. We may interpret the multiplication in (7.2) in either the Itô or the Stratonovich sense; as we detail in one example below the classical correspondence between the Itô and Stratonovich systems allows us to treat both situations within the framework of the Itô evolution (2.13). We will describe some physically interesting configurations of these ‘stochastic terms’ in detail below in Subsection 7.3.

The operators \( \Delta = \partial_{xx} + \partial_{yy} \) and \( \nabla = (\partial_x, \partial_y) \) are the horizontal laplacian and gradient operator. Here the operator \( \nabla \mathbf{v} \) captures part of the convective (material) derivative and is defined according to

\[ \nabla \mathbf{v} := \mathbf{v} \cdot \nabla = u \partial_x + v \partial_y. \tag{7.3} \]

\[ \text{15Via a suitable change of variables, the dynamical equations for the compressible gases which constitute the earth’s atmosphere may be shown to take a mathematical form essentially similar to the incompressible equations for the oceans.} \]
**Remark 7.1.** As given, the model \((7.1)\), expresses the equations for Oceanic flows in the ‘beta-plane approximation’, that is to say we make use of the fact that the earth is locally flat. This setting is suitable for regional studies and we will focus on this case for the simplicity of presentation. With suitable adjustments to the definition of the operators \(\Delta, \nabla, \nabla_v\) and to the domain introduced below we could consider the evolutions in the full spherical geometry of the earth. We refer to [LTW92b] (and also to [PTZ08]) for further details on how to cast a global circulation model in the form of e.g. \((2.13)\).

### 7.1.1 Domain and Boundary Conditions

The evolution \((7.1)\) takes place on a bounded domain \(\mathcal{M} \subset \mathbb{R}^3\) which we define as follows. Fix a bounded, open domain \(\Gamma_i \subset \mathbb{R}^2\) with sufficiently smooth boundary \((C^3,\text{say})\); \(\Gamma_i\) represents the surface of the ocean in the region under consideration. We suppose we have defined a ‘depth’ function \(h = h(x, y) : \Gamma_i \to \mathbb{R}\) which is at least \(C^2\) and is subject to the restriction \(0 < h \leq h(x, y) \leq \bar{h}\). With these ingredients we then let
\[
\mathcal{M} := \{x := (x, y, z) \in \mathbb{R}^3 : (x, y) \in \Gamma_i, z \in (-h(x, y), 0)\}.
\]

The boundary \(\partial \mathcal{M}\) of \(\mathcal{M}\), is divided into its top \(\Gamma_i\) lateral \(\Gamma_l\) and bottom \(\Gamma_b\) boundaries. We denote the outward unit normal to \(\partial \mathcal{M}\) by \(n\) and the normal to \(\Gamma_l\) in \(\mathbb{R}^2\) by \(n_H\).

We next prescribe the following, physically realistic boundary conditions for equation \((7.1)\) considered in \(\mathcal{M}\). See e.g. [PTZ08] for further details. On \(\Gamma_i\) we suppose
\[
\partial_z \mathbf{v} + \alpha_v (\mathbf{v} - \mathbf{v}^a) = \tau_v, \quad w = 0, \quad \partial_z T + \alpha_T (T - T^a) = 0, \quad \partial_z S = 0, \tag{7.4}
\]
where \(\alpha_v, \alpha_T\) are fixed positive constants and \(\tau_v, \mathbf{v}^a, T^a\) are in general random and non-constant in space and time. Physically speaking, the first two equations in \((7.4)\) account for a boundary layer model where \(\mathbf{v}^a, T^a\) represent the values for velocity and temperature of the atmosphere at the surface of the oceans; \(\tau_v\) accounts for the shear of the wind.

At the bottom of the ocean \(\Gamma_b\) we take
\[
\mathbf{v} = 0, \quad w = 0, \quad \partial_n T = 0, \quad \partial_n S = 0. \tag{7.5}
\]
Finally for the lateral boundary \(\Gamma_l\)
\[
\mathbf{v} = 0, \quad \partial_n T = 0, \quad \partial_n S = 0. \tag{7.6}
\]
Note that, in view of the Neumann (no-flux) boundary conditions imposed on \(S\) in \((7.4)\)–\((7.6)\), there is no loss in generality in assuming
\[
\int_{\mathcal{M}} Sd\mathcal{M} = 0 = \int_{\mathcal{M}} F_Sd\mathcal{M}. \tag{7.7}
\]
See [PTZ08] for further details. Finally \((7.1)\)–\((7.7)\) are supplemented with initial conditions for \(\mathbf{v}, T\) and \(S\), that is
\[
\mathbf{v} = \mathbf{v}_0, \quad T = T_0, \quad S = S_0, \quad \text{at } t = 0. \tag{7.8}
\]
7.1.2 A Reformulation of the Equations

Starting from the incompressibility condition, (7.1c) and the hydrostatic equation (7.1b) we may derive an equivalent form for (7.1) as follows.

\[ \begin{align*}
\partial_t \mathbf{v} + \nabla \cdot \mathbf{v} + w(\mathbf{v})\partial_z \mathbf{v} + \frac{1}{\rho_0} \nabla p + f k \times \mathbf{v} - \mu \Delta \mathbf{v} - \nu \partial_z^2 \mathbf{v} &= F \mathbf{v} - \nabla P + \sigma(\mathbf{v}, T, S) \hat{W}_1, \\
\partial_t T + \nabla \cdot T + w(\mathbf{v})\partial_z T - \mu T \Delta T - \nu T \partial_z^2 T &= F_T + \sigma(T, \mathbf{v}, T, S) \hat{W}_2, \\
\partial_t S + \nabla \cdot S + w(\mathbf{v})\partial_z S - \mu S \Delta S - \nu S \partial_z^2 S &= F_S + \sigma_S(\mathbf{v}, T, S) \hat{W}_3, \\
\rho &= \rho_0 (1 - \beta_T (T - T_r) + \beta_S (S - S_r)), \\
w(\mathbf{v})(\cdot, z) &= \int_0^z \nabla \cdot \mathbf{v} dz, \quad \nabla \cdot \int_{-h}^0 \mathbf{v} dz = 0.
\end{align*} \]  

(7.9a) – (7.9e)

This reformulation is desirable as, in particular, it is more suitable for the typical functional setting of the equations which we describe next. The unknowns and parameters in the equations are precisely those given above immediately after (7.1). Of course (7.9) is subject to the same initial and boundary conditions as in (7.1), namely (7.4)–(7.8). For further details concerning the equivalence of (7.9) and (7.1) see [PTZ08].

7.2 The Functional Setting and Connections with the Abstract Framework

We now proceed to introduce the basic function spaces associated with the Primitive equations (7.9) (equivalently (7.1)) and then introduce and explain the variational formulation of the various terms in equation connecting them with the abstract assumptions laid out above in Section 2.

7.2.1 Basic Function Spaces

To begin we define the smooth test functions

\[ \mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2 = \left\{ \mathbf{v} \in C^\infty(\mathcal{M})^2 : \nabla \cdot \int_{-h}^0 \mathbf{v} dz = 0, \mathbf{v} |_{\Gamma_i \cap \Gamma_b} = 0 \right\} \times \left\{ (T, S) \in C^\infty(\mathcal{M})^2 : \int_\mathcal{M} S d\mathcal{M} = 0 \right\}. \]

We now take \( H \) to be the closure of \( \mathcal{V} \) in \( L^2(\mathcal{M})^4 \) or, equivalently, \( H := H_1 \times H_2 \) where

\[ \begin{align*}
\left\{ \mathbf{v} \in L^2(\mathcal{M})^2 : \nabla \cdot \int_{-h}^0 \mathbf{v} dz = 0, n_H \cdot \int_{-h}^0 \mathbf{v} dz = 0 \quad \text{on} \quad \partial \Gamma_i \right\} &\times \left\{ (T, S) \in L^2(\mathcal{M})^2 : \int_\mathcal{M} S d\mathcal{M} = 0 \right\}. \quad (7.10)
\end{align*} \]

On \( H \) it is convenient to define the inner product and norm according to:

\[ (U, \tilde{U})_H := \int_\mathcal{M} (\mathbf{v} \cdot \tilde{\mathbf{v}} + K_T \mathbf{T} \tilde{T} + K_S \mathbf{S} \tilde{S}) d\mathcal{M}, \quad |U| := (U, U)^{1/2}_H. \]

The constants \( K_T, K_S > 0 \), which are introduced for coercivity in the principal linear terms in the equations, are chosen in order to fulfill (2.2) for (7.14) below. We define \( \Pi \) to be the orthogonal (Leray-type) projection from \( L^2(\mathcal{M})^4 \) onto \( H \).

We shall next define the \( H^1 \) type space \( V = V_1 \times V_2 \) where

\[ \left\{ \mathbf{v} \in H^1(\mathcal{M})^2 : \int_{-h}^0 \nabla \cdot \mathbf{v} dz = 0, \mathbf{v} = 0 \text{ on} \Gamma_i \cup \Gamma_b \right\} \times \left\{ (T, S) \in H^1(\mathcal{M})^2 : \int_\mathcal{M} S d\mathcal{M} = 0 \right\}, \]

(7.11)

We endow \( V \) with the inner product and norm

\[ ((U, \tilde{U}))_V := ((U, \tilde{U}))_V + K_T ((U, \tilde{U}))_T + K_S ((U, \tilde{U}))_S, \quad ||U|| := ((U, U))^{1/2}. \]  

(7.12)
where
\[
((U, \tilde{U}))_v := \int_{\mathcal{M}} (\mu_v \nabla v \cdot \nabla \tilde{v} + \nu_v \partial_z v \cdot \partial_z \tilde{v}) d\mathcal{M} + \alpha_v \int_{\Gamma_i} v \cdot \tilde{v} d\Gamma_i,
\]
\[
((U, \tilde{U}))_T := \int_{\mathcal{M}} (\mu_T \nabla T \cdot \nabla \tilde{T} + \nu_T \partial_z T \cdot \partial_z \tilde{T}) d\mathcal{M} + \alpha_T \int_{\Gamma_i} T \tilde{T} d\Gamma_i,
\]
\[
((U, \tilde{U}))_S := \int_{\mathcal{M}} (\mu_S \nabla S \cdot \nabla \tilde{S} + \nu_S \partial_z S \cdot \partial_z \tilde{S}) d\mathcal{M}.
\]

From (7.11)–(7.12) we may deduce the Poincaré type inequality \( |U| \leq c\|U\| \), for every \( U \in V \). This justifies taking \( \| \cdot \| \) as the norm for \( V \) (which is equivalent to the \( H^1 \) norm). Finally we define:

\[
V_{(2)}, V_{(3)} \text{ are the closures of } V \text{ in } H^2(\mathcal{M})^4, H^3(\mathcal{M})^4 \text{ norms respectively}
\]

and simply endow \( V_{(2)} \) and \( V_{(3)} \) with, respectively, the \( H^2(\mathcal{M}) \) and \( H^3(\mathcal{M}) \) norms. Let \( V' \) (resp. \( V'_{(2)}, V'_{(3)} \)) be the dual of \( V \) (resp. \( V_{(2)}, V_{(3)} \)) relative to the \( H \) inner product.

It is clear with the Rellich-Kondrachov theorem and standard facts about Hilbert spaces that the spaces introduced in (7.10)–(7.13) provide a suitable Gelfand-Lions inclusion as desired for (2.1). On this functional basis we now turn to describe the variational form of (7.9).

### 7.2.2 The Variational Form of the Equations

To capture most of the linear structure in (7.9) we define the operator \( A \) as a continuous linear map from \( V \) to \( V' \) via the bilinear form:

\[
a(U, \tilde{U}) := ((U, \tilde{U}))_v - \int_{\mathcal{M}} \left( g \int_0^1 (\beta_S S - \beta_T T) d\tilde{z} \right) \nabla \cdot \tilde{v} d\mathcal{M}. \tag{7.14}
\]

We observe that if \( K_T, K_S \) in (7.12) are chosen sufficiently large then, \( a \) is coercive, namely it satisfies the condition required by (2.2).

We next define the main nonlinear portion of (7.9). Motivated by (7.9e) we take \( w = w(U) := \int_0^1 \nabla \cdot v d\tilde{z} \) and then define a bilinear form \( B : V \times V \to V_{(2)} \) via the trilinear form

\[
b(U, \tilde{U}, U^*) := b_v(U, \tilde{U}, U^*) + K_T \cdot b_T(U, \tilde{U}, U^*) + K_S \cdot b_S(U, \tilde{U}, U^*), \tag{7.15}
\]

where

\[
b_v(U, \tilde{U}, U^*) := \int_{\mathcal{M}} ((v \cdot \nabla) \tilde{v} + w(U) \partial_z \tilde{v}) \cdot v^* d\mathcal{M},
\]
\[
b_T(U, \tilde{U}, U^*) := \int_{\mathcal{M}} ((v \cdot \nabla) \tilde{T} + w(U) \partial_z \tilde{T}) T^* d\mathcal{M},
\]
\[
b_S(U, \tilde{U}, U^*) := \int_{\mathcal{M}} ((v \cdot \nabla) \tilde{S} + w(U) \partial_z \tilde{S}) S^* d\mathcal{M}.
\]

To capture the rotation (Coriolis) term in (7.9a) we define \( E : H \to H \) via:

\[
c(U, \tilde{U}) = \int_{\mathcal{M}} (2f + \nabla \times v) \cdot \tilde{v} d\mathcal{M}. \tag{7.16}
\]

Note carefully that \( a, e \) and \( b \) satisfy the conditions imposed in Section 2.1 which we used in the abstract result Theorem 2.1. The inhomogenous terms in (7.9) are given by the element \( \ell \) defined according to

\[
\ell(\tilde{U}) = \int_{\mathcal{M}} (F_v \tilde{v} + K_T F_T \tilde{T} + K_S F_S \tilde{S}) d\mathcal{M} + \int_{\mathcal{M}} \left( g \int_0^1 (1 + \beta_T T - \beta_S S) d\tilde{z} \right) \nabla \cdot \tilde{v} d\mathcal{M}
\]
\[
+ \int_{\Gamma_i} (\tau_v + \alpha_v v^0) \cdot \tilde{v} + \alpha_T T^0 \tilde{T} |d\Gamma_i. \tag{7.17}
\]
Note that $v^a, \tau_v, T^a$, which represent the velocity, shear force of the wind and the temperature at the surface of ocean are have significant uncertainties and should thus be considered to have a random component in practice.

### 7.3 Some Stochastic Forcing Regimes

It remains to complete the connection between (7.1) and (2.13) by describing various physically interesting scenarios for $\sigma(U) \dot{W}$. We connect these ‘concrete descriptions’ with the terms $\sigma$ and $\xi$ in the abstract equation (2.13) (or equivalently to $g, s$ in (2.19)). We consider three situations in detail below. In each case we describe how to define $\sigma_U$ appearing in (7.9) and we then take $\sigma(\cdot) = \Pi\sigma_U(\cdot)$.

#### 7.3.1 Additive Noise

The most classical case is to consider an additive noise where we suppose that $\sigma_U$ is independent of $U = (v, T, S)$. In other words $\sigma_U : [0, \infty) \times \mathcal{M} \to (L_2(U, L^2(M)))^4$. In order to satisfy (2.10) we would require that

$$\sup_{t \geq 0} \sum_{k \geq 1} |\sigma^k_U(t)|^2 = \sup_{t \geq 0} |\sigma_U(t)|^2_{L_2(U, H)} < \infty. \quad (7.18)$$

Note that since, the Itô and Stratonovich interpretations of (7.2) coincide in the additive case we may take $\xi \equiv 0$ so that (2.11) is automatically satisfied.

We also observe that in this case we may give an explicit (if formal) characterization of the space-time correlation structure of the noise

$$\mathbb{E} \left[ \sigma_U(t, x) \dot{W}(t, x) \cdot \sigma_U(s, y) \dot{W}(s, y) \right] = K(t, s, x, y) \delta_{t-s} \quad (7.19)$$

where the correlation kernel $K$ is given by

$$K(t, s, x, y) = \sum_{k \geq 1} \sigma^k_U(t, x) \cdot \sigma^k_U(s, y).$$

**Remark 7.2.** Given the condition (7.18) the case of space-time white noise is ruled out under our framework. Of course such a space-time white noise is very degenerate in space (not even defined in $L^2_x$) and so such a situation is far from reach due to the highly nonlinear character of the PEs. Similar remarks apply to the 3D stochastic Navier-Stokes equations but see [DPD02] for the 2-D case.

#### 7.3.2 Nemytskii Type Operators

We next consider stochastic forcings of transformations of the unknown $U$ as follows. Let $\Psi = (\Psi_v, \Psi_T, \Psi_S) : \mathbb{R}^4 \to \mathbb{R}^4$ and suppose, for simplicity, that $\Psi$ is smooth. We denote the partial derivatives of $\Psi$ with respect to the $v, T, S$ variables by $\partial_v \Psi, \partial_T \Psi, \partial_S \Psi$ and the gradient by $\nabla_U \Psi$. Take a sequence of smooth functions $\alpha^k = \alpha^k(x) : \mathcal{M} \to \mathbb{R}$ and define

$$\sigma_U^k(U, t, x) = \Psi(U) \alpha^k(x). \quad (7.20)$$

We may formally interpret $\sigma_U(U) \dot{W} = \Psi(U) \dot{\eta}$ where:

- $\dot{\eta}$ is a white in time Gaussian process with the spatial-temporal correction structure $\mathbb{E}(\dot{\eta}(t, x) \dot{\eta}(s, y)) = K(x, y) \delta_{t-s}$ where $K(x, y) = \sum_{k \geq 1} \alpha^k(x) \cdot \alpha^k(y)$.

- The ‘multiplication’ $\Psi(U)$ and $\dot{\eta}$ may be taken in either the Itô or the Stratonovich sense.
We now connect (7.20) to (2.13) in the Itô or the Stratonovich situations in turn illustrating conditions on $\Psi$ and the $\alpha_k$’s that guarantee that (2.10) holds and in the Stratonovich case that (2.11) holds.

**The Itô Case:** Suppose that
\[ |\Psi(U)|^2 \leq c_\phi(1 + |U|^2) \quad \text{for all } U \in \mathbb{R}^4, \tag{7.21} \]
and for the elements $\alpha^k$ we suppose that
\[ \sum_{k \geq 1} \|\alpha^k\|_{V(2)}^2 < \infty. \tag{7.22} \]
Under (7.21)–(7.22) we have
\[ |\sigma(U)|_{L^2(\mathcal{U},H)}^2 \leq \sum_{k \geq 1} |\Psi(U)\alpha^k|_{L^2}^2 \leq c\phi \sum_{k \geq 1} \|\alpha^k\|_{L^\infty(M)} (1 + |U|_H^2), \leq c \sum_{k \geq 1} \|\alpha^k\|_{V(2)}^2 (1 + |U|_H^2), \]
so that (2.10) holds for constant $c_3$ that depends on $c_\phi, \sum_{k \geq 1} \|\alpha^k\|_{V(2)}^2$, and the constant in Agmon’s inequality. Note that, since we are considering the case of an Itô noise, $\xi \equiv 0$.

**The Stratonovich Case:** If we understand the multiplication $\Psi(U)\dot{\eta}$ in the Stratonovich sense then we may convert back to an Itô type evolution according to:
\[ \Psi(U)\dot{\eta} = \sum_{k \geq 1} \Psi(U)\alpha^k \circ dW^k = \xi(U)(U) + \sum_{k \geq 1} \Psi(U)\alpha^k dW^k \tag{7.23} \]
where
\[ \xi(U)(U,x) = \Psi(U) \cdot \nabla_U \Psi(U) \sum_{k=1}^{\infty} \alpha^k(x)^2 \]
See e.g. [Arn74, KP92] for further details on this conversion formula. Under the additional assumption
\[ |\nabla_U \Psi(U)| \leq c < \infty \quad \text{for all } U \in \mathbb{R}^4, \tag{7.24} \]
we define $\xi(U) := \Pi \xi(U)$ for any $U \in H$. It is clear that $\xi$ satisfies (2.11).

**Remark 7.3.** We note here that the relationship (7.23) is, for now, only formal; we prove the existence of martingale solutions for the system that results from a formal application of this conversion formula (see e.g. [Arn74, KP92]). We leave the rigorous justification of (7.23) and the related issues of an approximation of Wong-Zakai type ([WZ65]) of (2.13) for future work. Note however that (7.23) has already been explored in [GS96, Twa96, CM11] in an infinite dimensional fluids context for pathwise solutions and in [TZ06] for martingale solutions of a class of abstract, nonlinear, stochastic PDEs.

### 7.3.3 Stochastic Forcing of Functionals

Finally we examine the case when we stochastically force **functionals of the unknown** i.e. terms which have a non-local dependence on the solution $U$. For example consider, for $k \geq 1$ continuous (not necessarily linear) $\phi^k := \phi^k(U) : H \rightarrow \mathbb{R}$, and sufficiently smooth $\alpha^k = \alpha^k(t,x) : [0,\infty) \times \mathcal{M} \rightarrow \mathbb{R}^4$. We define
\[ \sigma^U_k(U,t,x) = \dot{\phi}^k(U)\alpha^k(t,x). \tag{7.25} \]
Here, we interpret $\sigma^U(U)\dot{W}$ in the Itô sense. Subject to, for example,
\[ \sup_k |\dot{\phi}^k(U)|^2 \leq c(1 + |U|^2), \quad \sup_{t \geq 0} \sum_{k \geq 1} \|\alpha^k(t)\|^2 < \infty \tag{7.26} \]
we obtain a $\sigma$ from (7.25) which satisfies (2.10). For a ‘concrete example’ of a $\sigma$ of the form (7.25) which satisfies (7.26) let $\{\psi^k\}_{k \geq 1}$ be a sequence of elements in $L^2(\mathcal{M})^2$ with $\sup_k |\psi^k|_{L^2(M)} < \infty$ and let $\alpha^k \in V$ satisfying the sumability condition in (7.26). We take $\phi^k(U) = \int_M \dot{v}(x) \cdot \dot{\psi}^k(x) d\mathcal{M}$ and obtain
\[ \sigma(U)\dot{W} = \sum_{k \geq 1} \left( \int_M \dot{v}(x) \cdot \dot{\psi}^k(x) d\mathcal{M} \right) \alpha^k(t,x) dW^k(t). \tag{7.27} \]
A Appendix: Technical Complements

We collect here, for the convenience of the reader, various technical results which have been used in the course of the analysis above. While some of the material may be considered to be somewhat ‘classical’ by specialists we believe that the stochastic type results will be useful to the non-probabilists and that the deterministic results will be helpful for the probabilists.

A.1 Some Convergence Properties of Measures

We next briefly review some basic notations of convergence for collections of Borel probability measures. In particular we highlight a certain abstract convergence lemma that has been used in a crucial way in the passage to the limit several times above. For further details concerning the general theory of convergence in spaces of probability measures see e.g. [Bil99] and [RY99].

Let $(\mathcal{H}, \rho)$ be a complete metric space and denote by $Pr(\mathcal{H})$ the collection of Borel probability measures on $\mathcal{H}$. We recall that a sequence $\{\mu_n\}_{n \geq 1} \subset Pr(\mathcal{H})$ is said to converge weakly to a measure $\mu$ on $\mathcal{H}$ if $\mu_n \Rightarrow \mu$ if and only if

$$\lim_{n \to \infty} \int f(x)d\mu_n(x) = \int f(x)d\mu(x) \text{ for every bounded continuous function } f : \mathcal{H} \to \mathbb{R}. \quad (A.1)$$

We recall that a collection $\Lambda \subset Pr(\mathcal{H})$ is said to be weakly relatively compact if every sequence $\{\mu_n\}_{n \geq 1} \subset \Lambda$ possesses a weakly convergent subsequence. On the other hand we say that $\Lambda \subset Pr(\mathcal{H})$ is tight if, for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset \mathcal{H}$ such that $\mu(K_\epsilon) \geq 1 - \epsilon$, for each $\mu \in \Lambda$. The Prokhorov theorem asserts that these two notions, namely tightness and weak compactness of probability measures are equivalent.

We also make use of the Skorokhod embedding theorem which states that, whenever $\mu_n \Rightarrow \mu$ on $\mathcal{H}$, then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of random variables $X_n : \tilde{\Omega} \to \mathcal{H}$ such that $\tilde{\mathbb{P}}(X_n \in \cdot) = \mu_n(\cdot)$ and which converges a.s. to a random variable $X : \tilde{\Omega} \to \mathcal{H}$ with $\mathbb{P}(X \in \cdot) = \mu(\cdot)$.

The following convergence result, found in e.g. [Bil99], relates roughly speaking weak convergence and clustering in probability, and was used to facilitate the proof of (5.26) in Section 5.1.3:

Lemma A.1. Let $(\mathcal{H}, \rho)$ be an arbitrary metric space. Suppose $X_n$ and $Y_n$ are $\mathcal{H}$-valued random variables and let $\mu_n(\cdot) = \mathbb{P}(X_n \in \cdot)$ and $\nu_n(\cdot) = \mathbb{P}(Y_n \in \cdot)$ be the associated sequences of the probability laws. If the sequence $\{\mu_n\}_{n \geq 0}$ converges weakly to a probability measure $\mu$ and if, for all $\epsilon > 0$

$$\lim_{n \to \infty} \mathbb{P}(\rho(X_n, Y_n) \geq \epsilon) = 0.$$

Then $\nu_n$ also converges weakly to $\mu$.

A.2 An extension of the Doob-Dynkin Lemma

We extend the Doob-Dynkin Lemma (see e.g. [Oks03]) to the case where the image space of the measurable functions are complete separable metric spaces. In order to achieve this goal, let us recall the following notions and results from [Dud02].

If $(\Omega, \mathcal{F})$ is a measure space and $E \subset \Omega$, let $\mathcal{F}_E := \{B \cap E : B \in \mathcal{F}\}$. Then $\mathcal{F}_E$ is a sigma algebra of subsets of $E$, and $\mathcal{F}_E$ will be called the relative sigma algebra (of $\mathcal{F}$ on $E$).

Proposition A.1. Let $(\Omega, \mathcal{F})$ be any measurable space and $E$ any subset of $\Omega$ (not necessarily in $\mathcal{F}$). Let $f$ be a function on $E$ with values in a Polish space $\mathcal{H}$ and measurable with respect to $\mathcal{F}_E$. Then $f$ can be extended to a function on all of $\Omega$, measurable with respect to $\mathcal{F}$.

Proof. The proof is direct combining Theorem 4.2.5 and Proposition 4.2.6 in [Dud02].
Now let $(\mathcal{Y}, \mathcal{M})$ be a measure space, $\mathcal{X}$ any set, and $\psi$ a function from $\mathcal{X}$ into $\mathcal{Y}$. Let $\psi^{-1}[\mathcal{M}] := \{\psi^{-1}(M) : M \in \mathcal{M}\}$. Then $\psi^{-1}[\mathcal{M}]$ is a sigma algebra of subsets of $\mathcal{X}$.

**Theorem A.1.** We are given a set $\mathcal{X}$, a measure space $(\mathcal{Y}, \mathcal{M})$, and a function $\psi$ from $\mathcal{X}$ into $\mathcal{Y}$. If a function $\ell$ on $\mathcal{X}$ with values in a Polish space $\mathcal{H}$ is $\psi^{-1}[\mathcal{M}]$ measurable, then there exists an $\mathcal{M}$-measurable function $L$ on $\mathcal{Y}$ such that $\ell = L \circ \psi$.

**Proof.** Whenever $\psi(u) = \psi(v)$, we have $\ell(u) = \ell(v)$, for if not, let $B$ be a Borel set in $\mathcal{H}$ with $\ell(u) \in B$ but $\ell(v) \notin B$. Then $\ell^{-1}(B) = \psi^{-1}(C)$ for some $C \in \mathcal{M}$, with $\psi(u) \in C$ but $\psi(v) \notin C$, a contradiction. Thus, $\ell = L \circ \psi$ for some function $L$ from $D := \text{range } \psi$ into $\mathcal{H}$. For any Borel set $E \subset \mathcal{H}$, $\psi^{-1}(L^{-1}(E)) = \ell^{-1}(E) = \psi^{-1}(F)$ for some $F \in \mathcal{M}$, so $F \cap D = L^{-1}(E)$ and $L$ is $\mathcal{M}_D$ measurable. By Proposition A.1, $L$ has a $\mathcal{M}$-measurable extension to all of $\mathcal{Y}$.

**A.3 A Measurable Selection Theorem**

We turn now to restate the measurable selection theorem which was proven in [BT73] and is based on the earlier works [KRN65], [Cas67]. We employed this result above to establish the existence of adapted solutions of (3.8) in Proposition A.1.

First we recall the definition of a Radon measure. Let $X$ be a locally compact Hausdorff spaces and $B(X)$ be the Borel sigma algebra on $X$. A Radon measure on $X$ is a measure defined on $B(X)$ that is finite on all compact sets, outer regular on all Borel sets, and inner regular on all open sets (Page 212, [Fol99]).

**Theorem A.2.** Let $X$ and $Y$ be separable Banach spaces and suppose that $\Lambda$ is a ‘multivalued map’ from $X$ into $Y$ i.e. a map from $X$ into the subsets of $Y$. We assume that $\Lambda$ takes values in closed, non-empty subsets of $Y$ and that its graph is closed viz.

$$\text{if } x_n \to x \text{ in } X, \text{ and } y_n \to y \text{ in } Y, \text{ with } y_n \in \Lambda x_n, \text{ then } y \in \Lambda x.$$

Then, $\Lambda$ admits a universal Radon measurable section, $\Gamma$, that is there exists a map $\Gamma : X \to Y$ such that $\Gamma x \in \Lambda x$ for every $x$, and such that $\Gamma$ is Radon measurable for every Radon measure on $X$.

**Remark A.1.** Note that since $X$ is a separable Banach space, any probability measure on $X$ is Radon; this is because any separable Banach space is a Polish space (separable and complete metric space) and that every Polish space is a Radon space (A Hausdorff space $X$ is called a Radon space if every finite Borel measure on $X$ is a Radon measure, i.e. is inner regular (see [Sch73])).

The following results are from [Sch73] and [DS88]. The final goal is to establish Corollary A.1 below, which we have employed in the article to prove that the map $\gamma$ defined in (3.17) (Section 3.2.2) is universally Radon measurable. For that purpose, we need the to introduce the following results (Proposition A.2 to Theorem A.3).

**Definition A.1.** (Lusin $\mu$-measurable) Let $X$ be a topological space. Let $\mu$ be a Radon measure on $X$ and let $h$ maps $X$ into $Y$ where $Y$ is a Hausdorff topological space. Then the mapping $h$ is said to be Lusin $\mu$-measurable if, for every compact set $K \subset X$ and every $\delta > 0$, there exists a compact set $K_\delta \subset K$ with $\mu(K - K_\delta) \leq \delta$ such that $h$ restricted to $K_\delta$ is continuous.

**Proposition A.2.** A function whose restriction to every compact set is continuous, is Lusin measurable for every Radon measure (Page 25, [Sch73]).

**Proposition A.3.** The assumptions are the same as in Definition A.1. If $h : X \to Y$ is Lusin $\mu$-measurable, then $h$ is $\mu$-measurable, and conversely, if $Y$ is metrizable and separable, then every $\mu$-measurable function is also Lusin $\mu$-measurable (Page 26, [Sch73]).

**Theorem A.3.** Let $X$, $Y$ and $Z$ to be separable Banach spaces and $\mu$ be a Radon measure on $X$. Let $\varphi : X \to Y$ be a $\mu$-measurable mapping. Let $\Gamma : Y \to Z$ be universally Radon measurable. Then $G := \Gamma \circ \varphi$ is $\mu$-measurable on $X$. 

Proof. From Proposition A.3, $\varphi$ is Lusin $\mu$-measurable. Then Theorem A.3 follows from the proof of Theorem 3.2 in [BT73].

**Corollary A.1.** Let $X$, $Y$ and $Z$ to be separable Banach spaces and $\varphi : X \to Y$ be a continuous mapping. Let $\Gamma : Y \to Z$ be universally Radon measurable. Then $G := \Gamma \circ \varphi$ is universally Radon measurable.

Proof. This can be directly deduced from Proposition A.2 and Theorem A.3.

### A.4 Compact Embedding Results

In order to establish the compactness of a sequence of probability measures associated with the solutions of (3.3) we made use of the following compact embedding theorem which is close to that found in [Tem83] and of course generalizes the classical Aubin-Lions Compactness theorem (see [Aub72])

**Proposition A.4.** Let $Z \subset X \subset Y$ be a collection of three Banach spaces with $Z$ compactly embedded in $Y$ and $Y$ continuously embedded in $X$.

(i) Suppose that $\mathcal{G}$ is a bounded subset of $L^p(\mathbb{R}, Z) \cap L^\infty(\mathbb{R}, Y)$, where $1 < p \leq \infty$, and assume that for some $1 < q < \infty$

$$\int_{-\infty}^{\infty} |g(t + s) - g(s)|_X^q ds \to 0 \quad \text{as } t \to 0,$$

(A.2)

uniformly for $g \in \mathcal{G}$ and that there exists $L > 0$ such that

$$\text{supp}\{g\} \subset [-L, L], \text{ for every } g \in \mathcal{G}.$$  (A.3)

Then, the set $\mathcal{G}$ is relatively compact in $L^p(\mathbb{R}, Y)$.

(ii) For $T > 0$ if $\mathcal{G}$ is a bounded subset of $L^p(0, T, Z) \cap L^\infty(0, T, Y)$ and

$$\int_{0}^{T-a} |g(t + s) - g(s)|_X^q ds \to 0 \quad \text{as } t \to 0,$$

(A.4)

uniformly for elements in $\mathcal{G}$, then $\mathcal{G}$ is relatively compact in $L^p(0, T, Y)$.

Proof. The proof is a fairly straightforward generalization of [Tem95, Theorem 13.2]. Observe that if $q > p$ then (A.2) and (A.3) taken together imply that

$$\int_{-\infty}^{\infty} |g(t + s) - g(s)|_X^p ds \to 0 \quad \text{as } t \to 0,$$

uniformly for $g \in \mathcal{G}$. Therefore there is no loss of generality in supposing that $q \leq p$ in what follows.

For $a > 0$, define the *averaging operator* $J_a$ according to

$$(J_a f)(s) = \frac{1}{2a} \int_{s-a}^{s+a} f(t)dt = \frac{1}{2a} \int_{-a}^{a} f(s+t)dt.$$  

We take $\mathcal{G}_a = \{J_a g : g \in \mathcal{G}\}$. Arguing exactly as in [Tem83] we have, for $a > 0$, that $\mathcal{G}_a$ is relatively compact in $L^p(\mathbb{R}; Y)$.

To show that $\mathcal{G}$ is itself relatively compact in $L^p(\mathbb{R}; Y)$, we prove that it is a *totally bounded* subset of $L^p(\mathbb{R}; Y)$; in other words we prove that, for every $\epsilon > 0$, there exists finitely many elements $g_1, \ldots, g_N$ in $L^p(\mathbb{R}, Y)$ such that $\mathcal{G}$ is contained in the union of the $\epsilon$ balls centered at these points.

Again, arguing exactly as in [Tem83] we have that, as a consequence of (A.2), for every $\delta > 0$ there exists $a = a(\delta) > 0$ such that

$$|J_a g - g|_{L^p(\mathbb{R}, X)} \leq \delta, \quad \text{for every } g \in \mathcal{G}.$$  (A.5)
On the other hand, from [Tem01, Chapter 3, Lemma 2.1] we infer that, for every \( \eta > 0 \), there exists \( C_\eta > 0 \) such that, for every \( g \in L^p(\mathbb{R}, Z) \)

\[
|J_a g - g|_{L^p(\mathbb{R}, Z)} \leq C_\eta |J_a g - g|_{L^p(\mathbb{R}, X)} + \eta |J_a g - g|_{L^p(\mathbb{R}, Z)} \leq C_\eta |J_a g - g|_{L^p(\mathbb{R}, X)} + 2\eta |g|_{L^p(\mathbb{R}, Z)}. \tag{A.6}
\]

The last inequality follows from the fact that, \((\text{Arn74})\) L. Arnold, 

\begin{align*}
\text{Acknowledgments} \\
\text{References}
\end{align*}

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Nathan Glatt-Holtz
Department of Mathematics
Virginia Polytechnic Institute and State University
Web: http://www.math.vt.edu/people/negh/
Email: negh@vt.edu

Roger Temam
Department of Mathematics
Indiana University
Web: http://mypage.iu.edu/~temam/
Email: temam@indiana.edu

Chuntian Wang
Department of Mathematics
Indiana University
Web: http://mypage.iu.edu/~wang211/
Email: wang211@email.iu.edu