A semigroup $S$ containing a zero element is said to be 0-left cancellative if $st = sr \neq 0$ implies that $t = r$. Given such an $S$ we build an inverse semigroup $\mathcal{H}(S)$, called the inverse hull of $S$. Motivated by the study of certain C*-algebras associated to $\mathcal{H}(S)$ (a task that we will address in a subsequent article) we carry out a detailed analysis of the spectrum of the idempotent semilattice $E(S)$ of $\mathcal{H}(S)$ with a special interest in identifying the ultra-characters. In order to produce examples of characters on $E(S)$, we introduce the notion of strings in a semigroup, attempting to make sense of the infinite paths which are of great importance in the study of graph C*-algebras. Our strongest results are obtained under the assumption that $S$ admits least common multiples, but we also touch upon the notion of finite alignment, motivated by the corresponding notion from the theory of higher rank graphs, and which has also appeared in recent papers by Spielberg and collaborators.

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1. Introduction.

The theory of semigroup C*-algebras has a long history, beginning with Coburn’s work [8] and [9] where the C*-algebra of the additive semigroup of the natural numbers is studied in connection to Toeplitz operators. In [24] Murphy generalized this construction to the positive cone of an ordered group, and later to left cancellative semigroups ([25], [26]). The C*-algebras studied by Murphy turned out to be too wild, even for nice looking semigroups such as $\mathbb{N} \times \mathbb{N}$, and this prompted Li [21] to introduce an alternative C*-algebra for a left cancellative semigroup. By definition a semigroup $S$ is said to be left cancellative provided,

$$st = sr \Rightarrow t = r.$$  \hspace{1cm} (1.1)

Many interesting semigroups possess a zero element, namely an element 0 such that

$$s0 = 0s = 0,$$

for every $s$, and it is obvious that the presence of a zero prevents a semigroup from being left cancellative. In this work we focus on 0-left cancellative semigroups, meaning that (1.1) is required to hold only when the terms in its antecedent are supposed to be nonzero. This dramatically opens up the scope of applications including a wealth of interesting semigroups, such as those arising from subshifts and, more generally, languages over a fixed alphabet. This also allows for the inclusion of the semigroupoids of [11: Section 14] and left cancellative categories, once the multiplication is extended to all pairs of elements by setting it to be zero whenever it is not already defined. See Section (6) for more examples.

Starting with a 0-left cancellative semigroup $S$, the crucial point is first to build an inverse semigroup $\mathcal{H}(S)$, which we call the inverse hull of $S$, in analogy with the notion of the inverse hull of a left cancellative semigroup, cf. [7: Section 1.9] and [6]. Once in possession of the inverse hull, one may invoke any of the now standard constructions of C*-algebras from inverse semigroups, such as the tight C*-algebra [11] or Paterson’s [28] universal C*-algebra. Indeed our initial motivation was to study such C*-algebras, but the present work is instead focused on the passage from the original semigroup to its inverse hull, rather than the much better understood passage from there to the associated C*-algebras. Particularly demanding is the work geared towards understanding the idempotent semilattice of $\mathcal{H}(S)$, which we denote by $\mathcal{E}(S)$, as well as its spectrum. By a standard gadget $\mathcal{E}(S)$ is put in correspondence with a subsemilattice of the power set of $S \setminus \{0\}$, whose members we call the constructible sets, by analogy with a similar concept relevant to Li’s work in [21].

Our proposal, to be further developed in the second part of this article (currently in preparation), is to consider the tight groupoid of $\mathcal{H}(S)$. The unit space of this groupoid is well known to be the tight spectrum of the semilattice $\mathcal{E}(S)$, so it is crucial to understand the tight characters and, in view of [11: 12.9], also the ultra-characters.

One of the main examples motivating our pursuit of the present line of ideas is the path semigroup associated to a graph $E$. See our discussion at the end of section (6) for a definition of this semigroup. According to [18], one may associate to any locally finite
directed graph $E$ (without sinks for simplicity), a C*-algebra $C^*(E)$, which happens to coincide with the C*-algebra of a canonically associated étale groupoid $\mathcal{G}_E$, whose unit space turns out to be the infinite path space of $E$. We therefore set out to look for ways of producing a canonically defined étale groupoid $\mathcal{G}(S)$ from any 0-cancellative semigroup $S$, generalizing the construction of $\mathcal{G}_E$ from the path semigroup of $E$.

In the case of the path semigroup of a graph, the ultra-characters correspond to infinite paths (see also [11:19.11]) so, attempting to make sense of infinite paths on an arbitrary semigroup, we introduce the concept of strings (10.1), an idea already present in [11:19.10], and which is one of our fundamental tools when studying the spectrum of the semilattice of constructible sets. However there are 0-left cancellative semigroups in the wild where the nice relationship between ultra-characters and maximal paths (as observed in the graph case) is all but lost, requiring a much more detailed analysis, which is carried out in section (16). See in particular the example presented after (16.18).

Regarding the problem of fully understanding the spectrum of $\mathfrak{C}(S)$, including the identification of the tight and ultra-characters, we believe the present work represents only a modest beginning in a mammoth task lying ahead. This impression comes from situations in which similar spectra have been more or less understood, such as in [13] and in [10], illustrating the high degree of complexity one should expect.

One of the main working hypothesis adopted in this work is the existence of least common multiples: if $s$ and $t$ are elements of the semigroup $S$, we say that $r$ is a least common multiple of $s$ and $t$, provided $r$ is a common multiple of $s$ and $t$, and $sS \cap tS = rS$. See Definition (5.6) for more details. While the exact form of this notion does not seem to be present in the literature, it is of course motivated by the usual notion from arithmetic as well as similar notions extensively employed in the literature of semigroup C*-algebras, such as in [1], [3], [20], [31] and [32].

Under the assumption that the semigroup admits least common multiples we are able to prove some of our strongest results, beginning with the description in Corollary (7.13) of a normal form for elements of the inverse hull, including of course constructible sets.

Roughly speaking, the difficulty in proving major results in any area of mathematics is inversely proportional to the strength of the chosen set of axioms. The huge generality of semigroups allowed by our limited set of conditions certainly makes that task very difficult but still we believe we have managed to prove a result we think will find interesting applications, namely Theorem (17.11), which essentially characterizes the set of all ultra-characters (and hence the tight spectrum, by taking closures), although it depends on the ad-hoc knowledge of ground ultra-characters (17.8).

Besides semigroups admitting least common multiples we also study finitely aligned semigroups, namely semigroups whose finitely generated right ideals are close under intersection (sometimes called Howson semigroups in analogy with [16]), and which holds true for the semigroup of finite paths on a finitely aligned higher rank graph [19].

In a sense the present work should be thought of as a continuation of the study of semigroupoids started in [11:Section 14] and [12]. In fact, as discussed in (6.4), given a semigroupoid, one may set the undefined products to be zero and thus obtain a semigroup with many of the relevant properties studied here. However, the specific associativity axiom assumed at the beginning of [11:Section 14] is too strong and excludes many interesting
examples which we are now able to treat.

A semigroup with zero $S$ is called categorical at zero (see Definition (4.1)), provided for every $r, s, t \in S$, one has that

$$rs \neq 0, \text{ and } st \neq 0 \Rightarrow rst \neq 0.$$  

Semigroups arising from semigroupoids, as in (6.4), are easily seen to be categorical at zero, but semigroups defined from subshifts, such as in Example (6.2) do not share this property unless the subshift is Markov.

Another well known condition usually considered in the study of semigroups is the existence of right local units. By this we mean that, for every $s$ in $S$, there exists an idempotent element $e$ in $S$ such that $s = se$. In the specific case of 0-left cancellative semigroups this is in fact equivalent to saying that $s \in sS$, for every $s$ in $S$ (see (3.14)). Unital semigroups of course have right local units as do the semigroups arising from categories. However, once more the semigroups coming from subshifts are excluded.

Since one of our main motivations is to be able to apply our theory to subshift semigroups, in many of our general results we have strived to avoid assuming strong hypotheses such as being categorical at zero or the existence of right local units.

Last but not least we should mention the work by Spielberg and collaborators on left cancellative small categories ([29], [30] and [2]), which goes very much in the same direction we are heading, with some significant differences in hypothesis. On the one hand the papers mentioned above only deal with categories, which may be viewed as special cases of the semigroupoids of [11] and [12], but on the other hand they transcend the singly aligned assumption of [11: 20.1] by thoroughly exploring the finitely aligned situation (and to a certain extent the infinitely aligned case as well). However, as already mentioned, semigroups from categories are categorical at zero, and hence they exclude the main examples we have in mind, namely subshift semigroups.

One of the advantages of our theory is that it can work with quotients of left cancellative categories by ideals. If one quotients a left cancellative category by an ideal (e.g., quotient the free monoid or semigroup by the ideal of non-factors of a subshift), the resulting category is no longer left cancellative but the corresponding semigroup is 0-left cancellative.

The results in this paper have already been announced in [15]. In addition, in a forthcoming paper we will apply the results obtained here to study 0-left cancellative semigroups arising from subshifts and their relationship to various C*-algebras that have appeared in the literature motivated by Matsumoto’s original work [22], such as the Carlsen-Matsumoto C*-algebras of [4]. See also [5].
2. Representations of semigroups.

Let \( S \) be a semigroup, namely a nonempty set equipped with an associative operation.

A zero element for \( S \) is a (necessarily unique) element \( 0 \in S \), satisfying

\[
    s0 = 0s = 0, \quad \forall s \in S.
\]

In what follows we will fix a semigroup \( S \) possessing a zero element. Note that one can always adjoin a zero element to a semigroup. The set of idempotent elements of \( S \) will be denoted by \( E(S) \).

2.1. Definition. Let \( \Omega \) be any set. By a representation of \( S \) on \( \Omega \) we shall mean any map

\[
    \pi : S \to \mathcal{I}(\Omega),
\]

where \( \mathcal{I}(\Omega) \) is the symmetric inverse semigroup\(^1\) on \( \Omega \), such that

(i) \( \pi_0 \) is the empty map on \( \Omega \), and

(ii) \( \pi_s \circ \pi_t = \pi_{st} \), for all \( s \) and \( t \) in \( S \).

Given a set \( \Omega \), and any subset \( X \subseteq \Omega \), let \( \text{id}_X \) denote the identity function on \( X \), so that \( \text{id}_X \) is an element of \( E(\mathcal{I}(\Omega)) \), the idempotent semilattice of \( \mathcal{I}(\Omega) \). One in fact has that

\[
    E(\mathcal{I}(\Omega)) = \{ \text{id}_X : X \subseteq \Omega \},
\]

so we may identify \( E(\mathcal{I}(\Omega)) \) with the meet semilattice \( \mathcal{P}(\Omega) \) formed by all subsets of \( \Omega \).

2.2. Definition. Given a representation \( \pi \) of \( S \), for every \( s \) in \( S \) we will denote the domain of \( \pi_s \) by \( F^\pi_s \), and the range of \( \pi_s \) by \( E^\pi_s \), so that \( \pi_s \) is a bijective mapping

\[
    \pi_s : F^\pi_s \to E^\pi_s.
\]

We will moreover let

\[
    f^\pi_s := \pi_s^{-1} \pi_s = \text{id}_{F^\pi_s} \quad \text{and} \quad e^\pi_s := \pi_s \pi_s^{-1} = \text{id}_{E^\pi_s}.
\]

If \( \pi \) is a representation of \( S \) on a set \( \Omega \), and if \( \Omega' \) is a subset of \( \Omega \) such that

\[
    F^\pi_s \subseteq \Omega' \quad \text{and} \quad E^\pi_s \subseteq \Omega', \quad \forall s \in S,
\]

then evidently \( \pi \) may be considered as a representation on \( \Omega' \). Moreover any point of \( \Omega \setminus \Omega' \) will have little relevance for \( \pi \).

An example of a subset of \( \Omega \) satisfying the above is clearly obtained by taking

\[
    \Omega_s^\pi = ( \bigcup_{s \in S} F^\pi_s ) \cup ( \bigcup_{s \in S} E^\pi_s ), \tag{2.3}
\]

which we will henceforth refer to as the essential subset for \( \pi \).

---

\(^1\) The symmetric inverse semigroup on a set \( \Omega \) is the inverse semigroup formed by all partially defined bijections on \( \Omega \).
2.4. Definition. A representation $\pi$ of $S$ is said to be essential provided $\Omega^*_\pi = \Omega$.

2.5. Proposition. Let $\pi$ be a representation of a unital semigroup $S$ on a set $\Omega$, and let $u$ be an invertible element of $s$. Then

$$\Omega^*_\pi = F^*_u \pi = E^*_u \pi.$$

Proof. First note that $F^*_1 \pi = E^*_1 \pi$ as $\pi_1$ is idempotent. Since $\pi_1 \pi_s = \pi_s = \pi_s \pi_1$, it follows that

$$F^*_s \pi, E^*_s \pi \subseteq F^*_1 \pi = E^*_1 \pi$$

for all $s \in S$ and hence $\Omega^*_\pi = F^*_1 \pi = E^*_1 \pi$. If $u$ is invertible with inverse $v$, then $\pi_u \pi_v = \pi_1 = \pi_v \pi_u$ shows that $F^*_1 \pi \subseteq F^*_u \pi$ and $E^*_1 \pi \subseteq E^*_u \pi$. We deduce that $\Omega^*_\pi = F^*_u \pi = E^*_u \pi$ as required. $\square$

Let us fix, for the time being, a representation $\pi$ of $S$ on $\Omega$. Whenever there is only one representation in sight we will drop the superscripts in $F^*_s \pi, E^*_s \pi, f^*_s \pi,$ and $e^*_s \pi$, and adopt the simplified notations $F_s, E_s, f_s,$ and $e_s$.

2.6. Proposition. Given $s$ and $t$ in $S$, one has that

(i) $\pi_s e_t = e_{st} \pi_s$, and
(ii) $f_t \pi_s = \pi_s f_{ts}$.

Proof. We have

$$\pi_s e_t = \pi_s \pi_s^{-1} \pi_s e_t = \pi_s f_s e_t = \pi_s e_t f_s = \pi_s \pi_t \pi_t^{-1} \pi_t \pi_s = \pi_s \pi_t \pi_t^{-1} \pi_t \pi_s = e_{st} \pi_s.$$  

As for (ii), we have

$$f_t \pi_s = f_t \pi_s \pi_s^{-1} \pi_s = f_t e_s \pi_s = e_s f_t \pi_s = \pi_s \pi_s^{-1} \pi_t \pi_t^{-1} \pi_t \pi_s = \pi_s \pi_t \pi_t^{-1} \pi_t \pi_s = \pi_s f_{ts}. \quad \square$$

2.7. Definition.

(i) The inverse subsemigroup of $\mathcal{I}(\Omega)$ generated by the set $\{\pi_s : s \in S\}$ will be denoted by $\mathcal{I}(\Omega, \pi)$.

(ii) Given any $X \in \mathcal{P}(\Omega)$ such that $\text{id}_X$ belongs to $E(\mathcal{I}(\Omega, \pi))$, we will say $X$ is a $\pi$-constructible subset.

(iii) The collection of all $\pi$-constructible subsets of $\Omega$ will be denoted by $\mathcal{P}(\Omega, \pi)$. In symbols

$$\mathcal{P}(\Omega, \pi) = \{X \in \mathcal{P}(\Omega) : \text{id}_X \in E(\mathcal{I}(\Omega, \pi))\}.$$  

Observe that by (2.2), one has that $E_s$ and $F_s$ are $\pi$-constructible sets. For the special case of $s = 0$, we have $E_s = F_s = \emptyset$, so the empty set is $\pi$-constructible as well.

Since $\mathcal{P}(\Omega, \pi)$ corresponds to the idempotent semilattice of $\mathcal{I}(\Omega, \pi)$ by definition, it is clear that $\mathcal{P}(\Omega, \pi)$ is a semilattice, and in particular the intersection of two $\pi$-constructible sets is again $\pi$-constructible. In what follows we would like to characterize the $\pi$-constructible sets.

6
2.8. Lemma. For every $s$ in $S$, and every $X \in \mathcal{P}(\Omega)$, let
\[ s[X] := \pi_s(F_s \cap X), \quad \text{and} \quad s^{-1}[X] := \pi_s^{-1}(E_s \cap X). \]
One then has that
\begin{enumerate}[(i)]
  \item $\pi_s \text{id}_X \pi_s^{-1} = \text{id}_{s[X]}$, and
  \item $\pi_s^{-1} \text{id}_X \pi_s = \text{id}_{s^{-1}[X]}$.
\end{enumerate}
Proof. We have
\[ \pi_s \text{id}_X \pi_s^{-1} = \pi_s f_s \text{id}_X \pi_s^{-1} = \pi_s \text{id}_s \pi_s \pi_s^{-1} = \]
\[ = \pi_s \text{id}_{F_s \cap X} \pi_s^{-1} = \text{id}_{\pi_s(F_s \cap X)} = \text{id}_{s[X]}, \]
proving (i). A similar argument proves (ii). \qed

2.9. Proposition. The family $\mathcal{P}(\Omega, \pi)$ of $\pi$-constructible sets is the smallest subset of $\mathcal{P}(\Omega)$ containing every $E_s$, and which is invariant under the maps
\[ X \mapsto s^{-1}[X], \quad \text{and} \quad X \mapsto s[X], \]
for all $s$ in $S$.

Proof. Given any $s$ in $S$, we have already seen that $E_s \in \mathcal{P}(\Omega, \pi)$. Furthermore, given $X$ in $\mathcal{P}(\Omega, \pi)$, one has that $\pi_s^{-1} \text{id}_X \pi_s$ and $\pi_s \text{id}_X \pi_s^{-1}$ are both idempotent elements of $\mathcal{I}(\Omega, \pi)$, so we may deduce from (2.8) that $s^{-1}[X]$ and $s[X]$ belong to $\mathcal{P}(\Omega, \pi)$.

This proves that $\mathcal{P}(\Omega, \pi)$ satisfies the conditions mentioned in the statement, and it therefore remains to prove that $\mathcal{P}(\Omega, \pi)$ is the smallest such collection. In other words, given any collection $\mathcal{F}$ of subsets of $\Omega$ satisfying the conditions in the statement, we must show that $\mathcal{P}(\Omega, \pi) \subseteq \mathcal{F}$. In order to do this, pick any $X$ in $\mathcal{P}(\Omega, \pi)$, so there exists some $\alpha$ in $\mathcal{I}(\Omega, \pi)$ such that
\[ \text{id}_X = \alpha \alpha^{-1}. \]

By definition of $\mathcal{I}(\Omega, \pi)$, we have that $\alpha$ may be written as a product $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$, where, for each $i$, there is an $s_i \in S$, such that either $\alpha_i = \pi_{s_i}$, or $\alpha_i = \pi_{s_i}^{-1}$.

We will accomplish our goal of showing that $X \in \mathcal{F}$ by induction on $n$. If $n = 1$, and if $\alpha_1 = \pi_{s_1}$, then
\[ \text{id}_X = \alpha_1 \alpha_1^{-1} = \pi_{s_1} \pi_{s_1}^{-1} = \text{id}_{E_{s_1}}, \]
whence $X = E_{s_1}$, so it lies in $\mathcal{F}$, by hypothesis. Still under the assumption that $n = 1$, but supposing now that $\alpha_1 = \pi_{s_1}^{-1}$, we have
\[ \text{id}_X = \alpha_1^{-1} \alpha_1 = \pi_{s_1}^{-1} \pi_{s_1} = \text{id}_{E_{s_1}}, \]
so
\[ X = F_{s_1} = s_1^{-1}[E_{s_1}] \in \mathcal{F}. \]

Next assume that $n > 1$, and let $\beta = \alpha_2 \ldots \alpha_n$, so that $\beta \beta^{-1} = \text{id}_Y$, where $Y$ lies in $\mathcal{F}$ by the induction hypothesis. Moreover
\[ \text{id}_X = \alpha \alpha^{-1} = \alpha_1 \beta \beta^{-1} \alpha_1^{-1} = \alpha_1 \text{id}_Y \alpha_1^{-1}. \]

It therefore follows from (2.8) that $X$ is either equal to $s_1[Y]$ or to $s_1^{-1}[Y]$, according to whether $\alpha_1 = \pi_{s_1}$ or $\alpha_1 = \pi_{s_1}^{-1}$. In any case we conclude that $X \in \mathcal{F}$, completing the proof. \qed
3. Cancellative semigroups.

Beginning with this section we will restrict our attention to semigroups possessing certain special properties regarding cancellation.

3.1. Definition. Let $S$ be a semigroup containing a zero element. We will say that $S$ is 0-left cancellative, or left cancellative away from zero if, for every $r, s, t \in S$,

$$st = sr \neq 0 \Rightarrow t = r,$$

and 0-right cancellative if

$$ts = rs \neq 0 \Rightarrow t = r.$$

If $S$ is both 0-left cancellative and 0-right cancellative, we will say that $S$ is 0-cancellative.

Adjoining a zero to a left cancellative semigroup will result in a 0-left cancellative semigroup and so our study will subsume the classical case.

In what follows we will fix a 0-left cancellative semigroup $S$. Occasionally, we will also assume that $S$ is 0-right cancellative.

If $X \subseteq S$ and $s \in S$, let us write $s^{-1}X$ for the preimage of $X$ under left multiplication by $s$, namely

$$s^{-1}X := \{t \in S : st \in X\}.$$

For any $s$ in $S$ we will let

$$F_s = \{x \in S : sx \neq 0\} = s^{-1}(S \setminus \{0\}),$$

and

$$E_s = \{y \in S : y = sx \neq 0, \text{ for some } x \in S\} = sS \setminus \{0\}.$$

Observe that the correspondence \(x \rightarrow sx\) gives a map from $F_s$ onto $E_s$, which is onto by definition of $E_s$ and one-to-one by virtue of 0-left cancellativity.

3.2. Definition. For every $s$ in $S$ we will denote by $\theta_s$ the bijective mapping given by

$$\theta_s : x \in F_s \mapsto sx \in E_s.$$

Observing that 0 is neither in $F_s$, nor in $E_s$, we see that these are both subsets of

$$S' := S \setminus \{0\},$$

so we may view $\theta_s$ as a partially defined bijection on $S'$, which is to say that $\theta_s \in \mathcal{I}(S')$.

We also notice that when $s = 0$, both $F_s$ and $E_s$ are empty, so $\theta_s$ is the empty map.

3.4. Proposition. The correspondence

$$s \in S \mapsto \theta_s \in \mathcal{I}(S')$$

is a representation of $S$ on $S'$, henceforth called the regular representation of $S$. [8]
Proof. As already seen, $\theta_0$ is the empty map on $I(S')$, so it suffices to check (2.1.ii). Notice that a given element $x$ in $S'$ lies in the domain of $\theta_s \circ \theta_t$ if and only if $tx \neq 0$, and $s(tx) \neq 0$. These two conditions are obviously equivalent to $(st)x \neq 0$, which is to say that $x$ lies in the domain of $\theta_{st}$. Moreover, for any $x$ in this common domain we have

$$\theta_s(\theta_t(x)) = s(tx) = (st)x = \theta_{st}(x),$$

so $\theta_s \circ \theta_t = \theta_{st}$. \hfill \Box

Regarding the notations introduced in (2.2) in relation to the regular representation, notice that

$$F_s = F^\theta_s, \quad \text{and} \quad E_s = E^\theta_s.$$

So far nothing guarantees that the left regular representation is essential, so let us now study its essential subset, beginning with the following trivial fact whose easy proof is left for the reader.

3.5. Lemma. Given an element $s$ in $S$, one has that

$$s \in \bigcup_{t \in S} E^\theta_t \iff s \in S^2, \quad \text{and}$$

$$s \in \bigcup_{t \in S} F^\theta_t \iff Ss \neq \{0\}.$$

We thus see that $s$ fails to be in the essential subset of $\theta$ if and only if $s$ possesses the property defined below:

3.6. Definition. A nonzero element $s$ in $S$ is said to be degenerate if

$$s \notin S^2, \quad \text{and} \quad Ss = \{0\}.$$

The following is thus a simple interpretation of the terms involved:

3.7. Proposition. Denoting the essential subset for $\theta$ by $S'_\#$, one has that

$$S' \setminus S'_\# = \{s \in S' : s \text{ is degenerate}\}.$$

Therefore $\theta$ is essential if and only if $S$ possesses no degenerate elements.

So far nothing guarantees that the left regular representation is injective, but in case injectivity of $\theta$ is desired, let us now discuss the appropriate conditions for this.

3.8. Definition. A semigroup $S$ is called right reductive if it acts faithfully on the left of itself, that is, $sx = tx$ for all $x \in S$ implies $s = t$.

Of course every unital semigroup is right reductive. If $S$ is a right reductive 0-left cancellative semigroup, then it embeds in $I(S')$ via $s \mapsto \theta_s$.

Observe that if $S$ is 0-right cancellative, then a single $x$ for which $sx = rx$, as long as this is nonzero, is enough to imply that $s = t$. So, in a sense, right reductivity is a weaker version of 0-right cancellativity.
3.9. Proposition. Suppose that, besides being 0-left cancellative, $S$ is also right reductive. If $s$ is an element of $S$ such that $sS = S$, then $S$ is a unital semigroup and $s$ is invertible.

Proof. Choosing $u$ in $S$ such that $su = s$, we will prove that $u$ is an identity for $S$. In order to do so, first notice that $s^2S = sS = S$, so, excluding the elementary case in which $S = \{0\}$, we have that $s^2 \neq 0$. Consequently from $sus = ss \neq 0$, we deduce that $us = s$. Since any element $t$ in $S$ may be written in the form $t = sx$, for some $x$ in $S$, we have

$$ut = usx = sx = t,$$

so $u$ is a left identity for $S$. In addition, given $t$ in $S$, we have that

$$tux = tx, \quad \forall x \in S,$$

so $tu = t$ by right reductivity. Therefore $u$ is also a right identity, hence a (two-sided) identity and we see that $S$ is unital.

In order to prove that $s$ is invertible, let $t$ be such that $st = u$. Then

$$sts = us = s = su,$$

and by 0-left cancellativity we get that $ts = u$, so $s$ is invertible and $s^{-1} = t$. □

The following definition introduces one of the main concepts studied in this work.

3.10. Definition. The inverse hull of a 0-left cancellative semigroup $S$, henceforth denoted by $\mathcal{H}(S)$, is the inverse subsemigroup of $\mathcal{I}(S')$ generated by the set $\{\theta_s : s \in S\}$. Thus, in the terminology of (2.7.i) we have

$$\mathcal{H}(S) = \mathcal{I}(S', \theta).$$

The reader should compare the above with the notion of inverse hull considered in [7 : Section 1.9] and [6].

The collection of $\theta$-constructible subsets of $S'$ is of special importance to us, so we would like to give it a special notation:

3.11. Definition. The idempotent semilattice of $\mathcal{H}(S)$, which we will tacitly identify with the semilattice of $\theta$-constructible subsets of $S'$, will be denoted by $\mathcal{E}(S)$. Thus, in the terminology of (2.7.iii) we have

$$\mathcal{E}(S) = \mathcal{P}(S', \theta).$$

Since the $\theta$-constructible sets may be described by (2.9), it is interesting to have a concrete description for the maps mentioned there in the special case of the regular representation.
3.12. Lemma. Regarding the regular representation \( \theta \) of \( S \) on \( S' \), for every \( s \) in \( S \), and every \( X \subseteq S' \), one has that

(i) \( s^{-1}[X] = \{ y \in S' : sy \in X \} \), and
(ii) \( s[X] = \{ y \in S' : y = sx \text{, for some } x \in X \} = sX \setminus \{0\} \).

Proof. Left for the reader. \( \square \)

It will be of importance to identify some properties of 0-left cancellative semigroups that will play a role later.

3.13. Proposition. Let \( S \) be a 0-left cancellative semigroup.

(i) If \( e \in E(S) \) and \( s \in S \setminus \{0\} \), then \( es \neq 0 \), if and only if \( es = s \), that is, \( s \in eS \setminus \{0\} \).
(ii) If \( s \in S \setminus \{0\} \), then \( s \in sS \) if and only if \( se = s \) for a necessarily unique idempotent \( e \).
(iii) If \( ss = S \) and \( S \) is right reductive, then \( S \) is unital and \( s \) is invertible.

Proof. For the non-trivial “only if” direction of the first item, assume that \( es \neq 0 \). Then \( es = ees \neq 0 \) implies \( s = es \) by 0-left cancellativity. For the second item, if \( sx = s \), then \( sx = sx = s \neq 0 \) implies that \( x^2 = x \) by 0-left cancellativity. Also \( sx = s = sy \) implies \( x = y \) by 0-left cancellativity. This establishes the second item. The third item is the content of Proposition (3.9). \( \square \)

A semigroup \( S \) is said to have right local units if \( S = SE(S) \), that is, for all \( s \in S \), there exists \( e \in E(S) \) with \( se = s \). A unital semigroup has right local units for trivial reasons. If \( S \) has right local units, then \( ss = 0 \) implies that \( s = 0 \). From Proposition (3.13) we obtain the following corollary.

3.14. Corollary. Let \( S \) be a 0-left cancellative semigroup. Then \( S \) has right local units if and only if \( s \in sS \) for all \( s \in S \).

3.15. Proposition. Let \( S \) be a right reductive, 0-left cancellative semigroup and suppose that \( e \) and \( f \) are idempotent elements of \( S \) with \( e \neq f \). Then \( ef = 0 \).

Proof. This is obvious if \( e \) or \( f \) is 0. So assume that \( e \neq 0 \neq f \). If \( ef \neq 0 \), then \( ef = f \) by Proposition (3.13.i). But then \( fe = ef = f \neq 0 \) and so \( fe \neq 0 \). Therefore, \( fe = e \) by Proposition (3.13.i). But then \( eS = fS \) and so by Proposition (3.13.i) we obtain that \( ex = x = fx \) for all \( x \in eS = fS \) and \( ex = 0 = fx \) for all \( x \notin eS = fS \). Thus \( e = f \) as \( S \) is right reductive. \( \square \)

3.16. Definition. If \( S \) is a 0-left cancellative, right reductive semigroup with right local units, then for \( s \in S \setminus \{0\} \), we denote by \( s^+ \) the unique idempotent with \( ss^+ = s \). If \( S \) is unital, then \( s^+ = 1 \).

We can associate to a left cancellative category \( C \) (i.e., a category of monics) a semigroup \( S(C) \) by letting \( S(C) \) consist of the arrows of \( C \) together with a zero element 0. Products that are undefined in \( C \) are made zero in \( S(C) \) and the remaining products are as in \( C \). It is straightforward to check that \( S(C) \) is 0-left cancellative, right reductive and has right local units. If \( f : c \to d \) is an arrow of \( C \), then \( f^+ = 1_c \). The case when \( C \) is the category associated to a higher rank graph \([19]\), will be considered later in this paper.
The special case of 0-left cancellative semigroups of the form $S(C)$, with $C$ a left cancellative category, is considered in detail by Spielberg in [30] (which was posted after our announcement of the results of this paper [15], but was done independently of our work). An advantage of our more general framework is that if one has an ideal in a left cancellative category, then the quotient category by this ideal need not be left cancellative, but factoring $S(C)$ by the ideal will result in a 0-left cancellative semigroup. This type of ideal construction, applied to categories of paths, lets one go from shifts of finite type to arbitrary shifts by forbidding patterns.

An ideal in a semigroup $S$ is a non-empty subset $I$ such that $SI \cup IS \subseteq I$. The Rees quotient $S/I$ is the quotient of $S$ by the congruence identifying $I$ to a single element (which will be the zero element of $S/I$). Each element of $S \setminus I$ forms its own equivalence class. The class of 0-left cancellative semigroups is evidently closed under Rees quotients and taking subsemigroups (containing 0).

**3.17. Proposition.** Suppose that $S$ is 0-left cancellative, right reductive and has right local units.

(i) If $s \in S \setminus \{0\}$, then $sx \neq 0$ implies $x \in s^+S$.

(ii) If $s, t \in S \setminus \{0\}$ and $sx \neq 0 \neq tx$, then $s^+ = t^+$.

(iii) If $T$ is a subsemigroup of $S$ containing 0 that is closed under the unary operation $s \mapsto s^+$, then $T$ is 0-left cancellative, right reductive and has right local units.

(iv) If $I$ is a proper ideal of $S$, then the Rees quotient $S/I$ is 0-left cancellative, right reductive and has right local units.

**Proof.** For the first item, $0 \neq sx = ss^+x$ implies $s^+x \neq 0$ and hence $x \in s^+S$ by Proposition (3.13). For the second item, we have that $x \in s^+S \cap t^+S \subseteq s^+t^+S$. It follows that $s^+t^+ \neq 0$ and so $s^+ = t^+$ by Proposition (3.15). Suppose that $T$ is a subsemigroup closed under the unary operation. Then it is obviously 0-left cancellative and has right local units. Suppose that $s \in T \setminus \{0\}$ and $t \in T$ with $sx = tx$ for all $x \in T$. Then $s = ss^+ = ts^+$. We conclude that $t \neq 0$ and so $s^+ = t^+$ by the second item. Therefore, $s = ts^+ = tt^+ = t$. Thus $T$ is right reductive. The final item is proved in the same way as the previous one; we omit the details. \[\square\]

Note that in general a subsemigroup or Rees quotient of a right reductive semigroup need not be right reductive so the right local units play a key role in the last two items.

**4. Categorical at zero semigroups.**

At this point we would like to remark that Definition (2.7.ii), especially when applied to the regular representation $\theta$, is motivated by Li’s use of the term constructible in [21]. However we should notice that, contrary to the situation treated in [21], our $\theta$-constructible subsets are not necessarily related to right ideals.

On the positive side, under special conditions on $S$ we shall soon prove that any $\theta$-constructible set is the nonzero part of a right ideal in $S$.

**4.1. Definition.** ([23]) Let $S$ be a semigroup with zero. We will say that $S$ is categorical at zero if, for every $r, s, t \in S$, one has that

\[rs \neq 0, \text{ and } st \neq 0 \Rightarrow rst \neq 0.\]
When applied to unital semigroups, the above concept is not very interesting since it reduces to the absence of zero divisors. In fact, if one is allowed to take $s = 1$, then the above condition would read

$$r \neq 0, \text{ and } t \neq 0 \Rightarrow rt \neq 0. \quad (4.2)$$

The reason for the terminology is that if $C$ is a category, then the semigroup $S(C)$ constructed above is categorical at zero. However not all categorical at zero semigroups have the form $S(C)$, as illustrated by semigroups arising from Markov subshifts to be introduced below.

Recall that a subset $R \subseteq S$ is said to be a right ideal when $Rs \subseteq R$, for every $s$ in $S$. Notice that right ideals always contain the zero element.

Generalizing our notation $S'$ introduced in (3.3), for each $X \subseteq S$, let us put

$$X' = X \setminus \{0\}.$$  

The following result employs the square bracket notation defined in (2.8).

4.3. **Lemma.** Given any $s$ in $S$, and any right ideal $R \subseteq S$, one has that

(i) $s[R'] \cup \{0\}$ is a right ideal in $S$, and

(ii) $s^{-1}[R'] \cup \{0\}$ is a right ideal in $S$, provided $S$ is categorical at zero.

**Proof.** Observing that

$$s[R'] \cup \{0\} \overset{(3.12)}{=} (sR' \setminus \{0\}) \cup \{0\} = sR' \cup \{0\} = sR,$$

the proof of (i) is clear.

Regarding (ii), pick $x$ in $s^{-1}[R'] \cup \{0\}$, and $y$ in $S$. We must then prove that

$$xy \in s^{-1}[R'] \cup \{0\}.$$  

If $xy = 0$, there is nothing to be done, so we suppose that $xy \neq 0$. Consequently $x \neq 0$, and then we see that $x \in s^{-1}[R']$, so $sx \in R'$, by (3.12).

In particular $sx \neq 0$, so $sxy \neq 0$ because $S$ is categorical at zero. Moreover, since $sx \in R$, we also have that $sxy \in R$, and consequently $sxy \in R'$, which implies that $xy \in s^{-1}[R']$, as desired. \qed

To see that being categorical at zero is important in (4.3.ii), let $S$ be a semigroup not possessing this property, and take $s, x, y \in S$ with $sx$ and $xy$ nonzero, but $sxy = 0$.

Considering $S$ as a right ideal in itself, notice that $sx \in S'$, so $x \in s^{-1}[S']$. However $xy$ is not in $s^{-1}[S'] \cup \{0\}$, because neither is $xy = 0$, nor is $sxy$ in $S'$. So $s^{-1}[S'] \cup \{0\}$ is not a right ideal.

4.4. **Proposition.** If $S$ is a 0-left cancellative semigroup which is categorical at zero, then every $\theta$-constructible subset of $S'$ coincides with the set of nonzero elements of some right ideal of $S$.  

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Proof. Letting
\[ \mathfrak{R} = \{ R' : R \text{ is a right ideal of } S \} , \]
we claim that \( \mathcal{P}(S', \theta) \subseteq \mathfrak{R} \). In view of (2.9), in order to prove this claim all we need to do is to show that \( \mathfrak{R} \) contains every \( E^\theta_s \), and that \( \mathfrak{R} \) is invariant under the two maps referred to in (2.9).

Since
\[ E^\theta_s = (sS)' , \]
we see that \( E^\theta_s \) lies in \( \mathfrak{R} \). Moreover, given any \( s \) in \( S \), and any right ideal \( R \subseteq S \), we have by (4.3.ii) that \( T := s^{-1}[R'] \cup \{0\} \) is a right ideal. Observing that \( 0 \not\in s^{-1}[R'] \), we have that
\[ s^{-1}[R'] = T \setminus \{0\} = T' \in \mathfrak{R} , \]
so we see that \( \mathfrak{R} \) is invariant under the first map referred to in (2.9). Regarding the second one, let us again be given a right ideal \( R \subseteq S \). We then have by (4.3.i) that \( T := s[R'] \cup \{0\} \) is a right ideal. Therefore
\[ s[R'] \setminus \{0\} = T \setminus \{0\} = T' \in \mathfrak{R} . \]
This proves that \( \mathfrak{R} \) is invariant under the second map referred to in (2.9), and hence our claim that \( \mathcal{P}(S', \theta) \subseteq \mathfrak{R} \) is verified, from where the statement follows. □

Recalling our discussion right before the statement of (4.4), when we observed that \( s^{-1}[S'] \cup \{0\} \) is not a right ideal, we see that the \( \theta \)-constructible set \( F^\theta_s = s^{-1}[S'] \) is not the nonzero part of a right ideal. This says that the hypothesis that \( S \) is categorical at zero in (4.4) cannot be removed.

Since many of the examples we have in mind involve semigroups which are not categorical at zero, we will unfortunately not be in a position to benefit from Proposition (4.4). It is given above mostly for the purpose of comparing our work with Li’s [21] study of C*-algebras of semigroups. However, in the case of the semigroup associated to a left cancellative category with least common multiples, it will be useful.

4.5. Proposition. Let \( S \) be a categorical at zero, 0-left cancellative, right reductive semigroup with right local units. Let \( s \in S \setminus \{0\} \). Then \( sx \neq 0 \) if and only if \( x \in s^+ S \setminus \{0\} \). In other words, \( F^\theta_s = s^+ S \setminus \{0\} \) and \( E^\theta_s = sS \setminus \{0\} \).

Proof. Note that \( ss^+ = s \neq 0 \) and so \( sx = ss^+ x \) is non-zero if and only \( s^+ x \neq 0 \). □

5. Least common multiples.

We now wish to introduce a class of semigroups possessing a property inspired by the notion of least common multiples from arithmetic. In order to do so we need to consider the question of divisibility.
5.1. Definition. Given \( s \) and \( t \) in a semigroup \( S \), we will say that \( s \) divides \( t \), in symbols \( s \mid t \), or that \( t \) is a multiple of \( s \), when either \( s = t \), or there is some \( u \) in \( S \) such that \( su = t \). In other words, \( s \) divides \( t \) if and only if
\[ t \in \{s\} \cup sS. \]

This should actually be called left-division, since one could alternatively define right-divisibility upon replacing the above expression “\( su = t \)” with “\( us = t \)”. However we will not have any use for right-division, and hence we may safely use the term division to mean left-division.

We observe that division is a reflexive and transitive relation, so it may be seen as a (not necessarily anti-symmetric) order relation upon defining “\( \leq \)” by
\[ s \leq t \iff s \mid t. \] (5.2)

This is the dual of Green’s quasi-order \( \leq_R \), which is usually considered in semigroup theory.

Since any \( s \) in \( S \) divides \( 0 \), one has that \( 0 \) is the maximum element of \( S \). Should \( S \) be a unital semigroup, with unit denoted \( 1_S \), then \( 1_S \) is a minimum element, a property shared by all other invertible elements of \( S \).

Incidentally, when \( S \) is unital, or more generally has right local units, we may define division is a slightly simpler way since
\[ s \mid t \iff t \in sS. \] (5.3)

For the strict purpose of simplifying the description of the division relation, regardless of whether or not \( S \) is unital, we shall sometimes employ the unitized semigroup
\[ \tilde{S} := S \cup \{1\}, \]
where \( 1 \) is any element not belonging to \( S \), made to act like a unit for \( S \). For every \( s \) and \( t \) in \( \tilde{S} \) we therefore have that
\[ s \mid t \iff \exists u \in \tilde{S}, \; su = t. \] (5.4)

Having enlarged our semigroup, we might as well extend the notion of divisibility:

5.4. Definition. Given \( v \) and \( w \) in \( \tilde{S} \), we will say that \( v \mid w \) when there exists some \( u \) in \( \tilde{S} \), such that \( vu = w \), i.e., \( w \in v\tilde{S} \).

Notice that if \( v \) and \( w \) are in \( S \), then the above notion of divisibility coincides with the previous one by (5.3). Analysing the new cases where this extended divisibility may or may not apply, notice that:
\[ \forall w \in \tilde{S}, \; 1 \mid w, \]
\[ \forall v \in \tilde{S}, \; v \mid 1 \iff v = 1. \] (5.5)

The introduction of \( \tilde{S} \) brings with it several pitfalls, not least because \( \tilde{S} \) might not be 0-left cancellative: when \( S \) already has a unit, say \( 1_S \), then in the identity “\( s1_S = s1 \)”, we are not allowed to left cancel \( s \), since \( 1_S \neq 1 \). One should therefore exercise extra care when working with \( \tilde{S} \).

The notion of least common multiples is well studied for unital semigroups. We could not find much in the literature in the non-unital setting and a number of subtleties arise.
5.6. Definition. Let $S$ be a semigroup and let $s,t \in S$. We will say that an element $r \in S$ is a least common multiple for $s$ and $t$ when

(i) $sS \cap tS = rS$,
(ii) both $s$ and $t$ divide $r$.

Observe that when $S$ has right local units then $r \in rS$, by (3.14), and hence condition (5.6.i) trivially implies (5.6.ii), so the former condition alone suffices to define least common multiples. However, in a semigroup without right local units it is not true that condition (5.6.i) implies (5.6.ii). For example, if $S^2 = \{0\}$, then (5.6.i) is always satisfied by any $s,t,r$, but (5.6.ii) is only satisfied for $s \neq t$ when $r = 0$. Notice that if $S^2 = \{0\}$ then $s$ and 0 are least common multiples of $s$ with itself.

The above example indicates some of the strange things that can happen when $S$ lacks local units and is not right reductive. Nevertheless, some of the main examples we have in mind, such as (6.1) below, behave very well with respect to least common multiples even though they do not admit local units.

Regardless of the existence of right local units, observe that when $sS \cap tS = \{0\}$, then 0 is always a least common multiple for $s$ and $t$ because $s$ and $t$ always divide 0. In addition, notice that condition (5.6.ii) holds if and only if $r\bar{S} \subseteq s\bar{S} \cap t\bar{S}$, and therefore one has that $r$ is a least common multiple for $s$ and $t$ if and only if

$$sS \cap tS = rS \subseteq r\bar{S} \subseteq s\bar{S} \cap t\bar{S}. \quad (5.7)$$

5.8. Definition. We shall say that a semigroup $S$ admits least common multiples if there exists a least common multiple for each pair of elements of $S$.

6. Examples.

Even though we believe examples are of fundamental importance in any mathematical work, we have hitherto postponed their presentation to give us time to build the necessary terminology needed to highlight their relevant properties.

Our first class of examples comes from Language Theory. Let $\Lambda$ be any finite or infinite set, henceforth called the alphabet, and let $\Lambda^+$ be the free semigroup generated by $\Lambda$, namely the set of all finite words in $\Lambda$ of positive length (and hence excluding the empty word), equipped with the multiplication operation given by concatenation. Incidentally recall that the free monoid on $\Lambda$ is customarily denoted $\Lambda^*$; it includes the empty string.

Let $L$ be a language on $\Lambda$, namely any nonempty subset of $\Lambda^+$. We will furthermore assume that $L$ is closed under prefixes and suffixes, that is, for every $\alpha$ and $\beta$ in $\Lambda^+$, one has

$$\alpha \beta \in L \Rightarrow \alpha \in L, \text{ and } \beta \in L.$$  

This is equivalent to $L$ being closed under factors: $\alpha \beta \gamma \in L$ implies $\beta \in L$ for all $\beta \in \Lambda^+$ and $\alpha, \gamma \in \Lambda^*$.

Define a multiplication operation on

$$S := L \cup \{0\},$$
where 0 is any element not belonging to $\Lambda^+$, by

$$\alpha \cdot \beta = \begin{cases} 
\alpha \beta, & \text{if } \alpha, \beta \neq 0, \text{ and } \alpha \beta \in L, \\
0, & \text{otherwise.}
\end{cases}$$

The reader will have no difficulty in proving the following:

**6.1. Proposition.** Given any language $L \subseteq \Lambda^+$, closed under prefixes and suffixes, the above multiplication operation is associative thus making $S$ a semigroup with zero. Moreover $S$ is 0-cancellative and admits least common multiples. There are no idempotent elements in $S$, whence $S$ lacks right local units.

One may also see $S$ as the Rees quotient $\Lambda^+/I$, where $I = \Lambda^+ \setminus L$. Because $L$ is closed under factors, $I$ is obviously an ideal of $\Lambda^+$ and hence $S$ is a semigroup. The fact that $S$ is 0-left cancellative may also be deduced from the fact that any Rees quotient of a 0-left cancellative semigroup shares this property.

Notice that $S$ has no nonzero idempotent elements and hence it cannot have right local units. In extreme cases $S$ could also fail to be right reductive such as when all words in $L$ have length one.

We may now easily give an example where $S$ is not categorical at zero: take any nonempty alphabet $\Lambda$, let $L$ be the language consisting of all words of length at most two, and let $S = L \cup \{0\}$, as above. If $a$, $b$ and $c$, are members of $\Lambda$, we have that $abc = 0$, but $ab$ and $bc$ are nonzero, so $S$ is not categorical at zero. Regarding our discussion after the proof of (4.4), observe that $F_\theta^a$ consists of all elements $x$ of $L$ such that $ax \neq 0$, so that $F_\theta^a$ is precisely the set of all words of length 1, which is certainly not the nonzero part of a right ideal in $S$.

One important special case of the above is based on subshifts. Given an alphabet $\Lambda$, as above, consider the left shift, namely the mapping $\sigma: \Lambda^N \rightarrow \Lambda^N$ given by

$$\sigma(x_1x_2x_3\ldots) = x_2x_3x_4\ldots.$$ 

A nonempty subset $\mathcal{X} \subseteq \Lambda^N$ is called a subshift\(^2\) when it is invariant under $\sigma$ in the sense that $\sigma(\mathcal{X}) \subseteq \mathcal{X}$.

Given a subshift $\mathcal{X}$, let $L_\mathcal{X} \subseteq \Lambda^+$ be the language of $\mathcal{X}$, namely the set of all finite words occurring in some infinite word belonging to $\mathcal{X}$. Then $L_\mathcal{X}$ is clearly closed under prefixes and suffixes, and hence we are back in the conditions of example (6.1).

The fact that $\mathcal{X}$ is invariant under the left shift is indeed superfluous, as any nonempty subset $\mathcal{X} \subseteq \Lambda^N$ would lead to the same conclusion. However, languages arising from subshifts have been intensively studied in the literature, hence the motivation for considering this situation.

**6.2. Definition.** Given a subshift $\mathcal{X}$, we will denote by $S_\mathcal{X}$ the semigroup built from $L_\mathcal{X}$ as in (6.1).

\(^2\) The term subshift is often applied to the map obtained by restricting $\sigma$ to $\mathcal{X}$. Moreover, in the field of symbolic dynamics it is also required that $\mathcal{X}$ be closed in the product topology of $\Lambda^N$. 


Given an alphabet Λ, let us be given a matrix

\[ A = \{ A_{x,y} \}_{x,y \in \Lambda} \]

such that \( A_{x,y} \in \{0,1\} \), for all \( x,y \in \Lambda \). Such a matrix is sometimes called a transition matrix. Define \( \mathcal{X}_A \) to be the set of all infinite words

\[ x_1x_2x_3 \ldots \in \Lambda^\mathbb{N}, \]

such that

\[ A_{x_i,x_{i+1}} = 1, \quad \forall i \in \mathbb{N}. \]

It is easy to see that \( \mathcal{X}_A \) is invariant under the left shift hence a subshift. This is usually referred to as the Markov subshift associated to the transition matrix \( A \).

The reader will have no difficulty in checking that \( S_{\mathcal{X}_A} \) is categorical at zero for every transition matrix \( A \), but it is easy to exhibit subshifts \( \mathcal{X} \) for which \( S_{\mathcal{X}} \) does not share this property.

Markov subshifts may be used to exhibit a semigroup which is categorical at zero but is not of the form \( S(C) \), as hinted at in the paragraph following (4.2). In fact a semigroup arising from (6.1) is never isomorphic to some \( S(C) \) since the former has no nonzero idempotent elements while the latter has many, namely the identity morphism of each object of \( C \).

Markov subshifts may indeed be used to produce a semigroup which is categorical at zero and yet is not isomorphic to any subsemigroup of \( S(C) \), no matter which category \( C \) one takes. To see this, consider the alphabet \( \Lambda = \{ x_1, x_2 \} \) and let \( A \) be transition matrix

\[ A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \]

Notice that the words \( x_1x_1, x_1x_2, \) and \( x_2x_1 \), belong to the language of \( \mathcal{X}_A \), but \( x_2x_2 \) is forbidden, precisely because \( A_{x_2,x_2} = 0 \).

Should there exist a category \( C \) such that \( S \) is a subsemigroup of \( S(C) \), the fact that, say, \( x_1x_2 \neq 0 \) would lead one to believe that \( d(x_1) \), namely the domain of \( x_1 \), coincides with \( r(x_2) \), the range of \( x_2 \). But then for similar reasons one would have

\[ d(x_2) = r(x_1) = d(x_1) = r(x_2), \]

which would imply that \( x_2x_2 \neq 0 \), a contradiction.

Another interesting class of examples is obtained from the quasi-lattice ordered groups of [27], which we would now like to briefly describe.

Given a group \( G \) and a unital subsemigroup \( P \subseteq G \), one defines a partial order on \( G \) via

\[ x \leq y \iff x^{-1}y \in P. \]

The quasi-lattice condition says that, whenever elements \( x \) and \( y \) in \( G \) admit a common upper bound, namely an element \( z \) in \( G \) such that \( z \geq x \) and \( z \geq y \), then there exists a least common upper bound, usually denoted \( x \lor y \).
Under this situation, consider the semigroup \( S = P \cup \{0\} \), obtained by adjoining a zero to \( P \). Then, for every nonzero \( s \) in \( S \), i.e., for \( s \) in \( P \), one has that

\[
sS = \{ x \in P : x \geq s \} \cup \{0\},
\]

so that the multiples of \( s \) are precisely the upper bounds of \( s \) in \( P \), including zero.

If \( t \) is another nonzero element in \( S \), one therefore has that \( s \) and \( t \) admit a nonzero common multiple if and only if \( s \) and \( t \) admit a common upper bound in \( P \), in which case \( s \lor t \) is a least common multiple of \( s \) and \( t \).

On the other hand, when \( s \) and \( t \) admit no common upper bound, then obviously \( s \lor t \) does not exist, but still \( s \) and \( t \) admit a least common multiple in \( S \), namely 0.

Summarizing our discussion so far we have the following:

**6.3. Proposition.** Let \((G, P)\) be a quasi-lattice ordered group. Then the 0-left cancellative semigroup \( S := P \cup \{0\} \) admits least common multiples.

We give now an example of a 0-left cancellative semigroup \( S \) where, for all \( s, t \in S \), there exists \( r \in S \) with \( sS \cap tS = rS \), but \( S \) fails to admit least common multiples in our sense. Let \( S = \{a, b, c, ab, ba, c^2, 0\} \) where \( ac = bc = c^2 \) and all other non-obvious products are 0. In particular, any product of three elements of \( S \) is 0 and so \( S \) is associative. Then \( aS \cap bS = aS \cap cS = bS \cap cS = cS = \{c^2, 0\} \) but \( c \) is not a common multiple of \( a \) and \( b \). For all other \( x \in \{ab, ba, c^2, 0\} \), we have \( xS = 0 \).

Another class of examples may be obtained from semigroupoids, as defined in [11: Section 14]. Given a semigroupoid \( \Lambda \), consider the semigroup \( S = \Lambda \cup \{0\} \), where 0 is any element not belonging to \( \Lambda \), with multiplication defined by \( f \cdot 0 = 0 \cdot f = 0 \), for all \( f \) in \( S \), while for \( f \) and \( g \) in \( S \), we put

\[
f \cdot g = \begin{cases} fg, & \text{if } (f, g) \in \Lambda^{(2)}, \\ 0, & \text{otherwise}. \end{cases}
\]

**6.4. Proposition.** Given a semigroupoid \( \Lambda \), let \( S \) be the semigroup constructed above. Then

(i) \( S \) is categorical at zero,

(ii) if every element of \( \Lambda \) is monic [11: Definition 14.5], then \( S \) is 0-left cancellative.

Of course (6.4.i) is a consequence of the choice of the strong associativity property in [11]. Conversely, given a semigroup \( S \) with zero, one could let \( \Lambda = S \setminus \{0\} \), with partial multiplication defined on

\[\Lambda^{(2)} = \{(s, t) \in \Lambda : st \neq 0\} .\]

If \( S \) is categorical at zero one may prove that \( \Lambda \) satisfies the associativity property of [11: Section 14], but one could alternatively generalize the notion of semigroupoid by assuming a less stringent associativity axiom.

We should also mention a few other classes of examples which are in fact special cases of some of the above examples but, given their role in the modern literature, it is perhaps worth singling them out.
Given a small category $C$ in which all arrows are monomorphisms, as already mentioned we may associate to $C$ a semigroup $S(C)$ consisting of the arrows of $C$ together with a zero element, where the product is extended by making it zero whenever not already defined. Then $S(C)$ is a 0-left cancellative semigroup with right local units. This is of course a special case of example (6.4).

Given any directed graph $E$, one may view the collection of all finite paths in $E$ (with or without the vertices which, if included, could be viewed as paths of length zero) as a semigroupoid (also as a category if the vertices are included) in which every element is monic and hence one may again build a semigroup as in (6.4). Should one prefer not to include the vertices, this may also be thought of as a special case of example (6.1), where the alphabet is taken to be the set of all edges, the language consisting of all finite paths in $E$.

The situation in the above paragraph could also be applied to any higher rank graph with similar conclusions.

7. Normal Form.

- Throughout this section we will fix a 0-left cancellative semigroup $S$ admitting least common multiples.

  Given a representation $\pi$ of $S$ on a set $\Omega$, we will now concentrate our attention in giving a concrete description for the elements of $I(\Omega, \pi)$ (see Definition (2.7)), provided $\pi$ satisfies certain special properties, which we will now describe.

  Initially notice that if $s \mid r$, then the range of $\pi_r$ is contained in the range of $\pi_s$ because either $r = s$, or $r = su$, for some $u$ in $S$, in which case $\pi_r = \pi_s \pi_u$. So, using the notation introduced in (2.2),

  \[ E_\pi^r \subseteq E_\pi^s. \]

  When $r$ is a least common multiple of $s$ and $t$, it then follows that

  \[ E_\pi^r \subseteq E_\pi^s \cap E_\pi^t. \]

7.1. Definition. A representation $\pi$ of $S$ is said to respect least common multiples if, whenever $r$ is a least common multiple of elements $s$ and $t$ in $S$, one has that $E_\pi^r = E_\pi^s \cap E_\pi^t$.

  As an example, notice that the regular representation of $S$, defined in (3.4), satisfies the above condition since the fact that $rS = sS \cap tS$ implies that

  \[ E_\theta^r = rS \setminus \{0\} = (sS \cap tS) \setminus \{0\} = (sS \setminus \{0\}) \cap (tS \setminus \{0\}) = E_\theta^s \cap E_\theta^t. \quad (7.2) \]

- From now on we will moreover fix a representation $\pi$ of $S$ on a set $\Omega$, assumed to respect least common multiples.

  Since $\pi$ will be the only representation considered for a while, we will use the simplified notations $F_s$, $E_s$, $f_s$, and $e_s$.

  There is a canonical way to extend $\pi$ to $\tilde{S}$ by setting

  \[ F_1 = E_1 = \Omega, \quad \text{and} \quad \pi_1 = \text{id}_\Omega. \]

  It is evident that $\pi$ remains a multiplicative map after this extension. Whenever we find it convenient we will therefore think of $\pi$ as defined on $\tilde{S}$ as above. We will accordingly extend the notations $f_s$ and $e_s$ to allow for any $s$ in $\tilde{S}$, in the obvious way.
7.3. Proposition. Given $u$ and $v$ in $\tilde{S}$, there exists $w$ in $\tilde{S}$ such that

(i) $uS \cap vS = wS$,

(ii) both $u$ and $v$ divide $w$.

Proof. When $u$ and $v$ lie in $S$, it is enough to take $w$ to be a (usual) least common multiple of $u$ and $v$. On the other hand, if $u = 1$, one takes $w = v$, and if $v = 1$, one takes $w = u$. □

Based on the above we may extend the notion of least common multiples to $\tilde{S}$, as follows:

7.4. Definition. Given $u$ and $v$ in $\tilde{S}$, we will say that an element $w$ in $\tilde{S}$ is a least common multiple of $u$ and $v$, provided (7.3.i-ii) hold. In the exceptional case that $u = v = 1$, only $w = 1$ will be considered to be a least common multiple of $u$ and $v$, even though there might be another $w$ in $S$ satisfying (7.3.i-ii).

It is perhaps interesting to describe the exceptional situation above, where we are arbitrarily prohibiting by hand that an element of $S$ be considered as a least common multiple of 1 and itself, even though it would otherwise satisfy all of the required properties. If $w \in S$ is such an element, then

$$wS = 1S \cap 1S = S,$$

so, in case we throw in the assumption that $S$ is right-reductive, we deduce from (3.9) that $S$ is unital and $w$ is invertible. Thus, in hindsight it might not have been such a good idea to add an external unit to $S$ after all!

On the other hand, when $s$ and $t$ lie in $S$, it is not hard to see that any least common multiple of $s$ and $t$ in the new sense of (7.4) must belong to $S$, and hence it must also be a least common multiple in the old sense of (5.6).

7.5. Proposition. Let $\pi$ be a representation of $S$ on a set $\Omega$. If $\pi$ respects least common multiples then so does its natural extension to $\tilde{S}$. Precisely, if $u$ and $v$ are elements of $\tilde{S}$, and if $w \in \tilde{S}$ is a least common multiple of $u$ and $v$, then $E_w = E_u \cap E_v$.

Proof. If $u$ and $v$ lie in $S$, then $w$ is necessarily a least common multiple of $u$ and $v$ in the old sense of (5.6), so the result follows by hypothesis.

If $u = v = 1$, then $w = 1$ by default\(^3\), and the result follows trivially.

Up to interchanging $u$ and $v$, the last case to be considered is when $u = 1$ and $v \in S$. In this case notice that $v \mid w$, hence $w$ must be in $S$. Moreover,

$$wS = uS \cap vS = S \cap vS = vS.$$ 

---

\(^3\) Should we have allowed in (7.4) that another element $w$ of $S$ be considered a least common multiple of 1 and itself, at this point we would be required to prove that $E_w = \Omega$. This would still be within reach, as long as we loaded up on our hypotheses, requiring $S$ to be right-reductive and $\pi$ to be essential. With all of this we could invoke (3.9) to deduce that $S$ is unital and $w$ is invertible, and then by (2.5) we would obtain the desired equality. In conclusion we believe that adding a little exception to Definition (7.4) is a small price to pay for a result with fewer hypotheses and hence wider applicability.
In an unforeseen twist of fate, if follows from the above that \( w \) is a least common multiple of \( v \) and itself. So, by hypothesis

\[
E_w = E_v \cap E_v = E_v,
\]

whence

\[
E_w = E_v = \Omega \cap E_v = E_1 \cap E_v = E_u \cap E_v,
\]

concluding the proof. \( \square \)

7.6. Lemma. Let \( \pi \) be a representation of \( S \) respecting least common multiples, and let \( w \in \tilde{S} \) be a least common multiple for given elements \( u \) and \( v \) in \( \tilde{S} \). Using (5.3) to write \( w = ux = vy \), with \( x, y \in \tilde{S} \), one has that

(i) \( e_u e_v = e_w \),
(ii) \( \pi_u^{-1} \pi_v = \pi_x f_w \pi_y^{-1} \).

Proof. Since \( \pi \) respects least common multiples, even at the level of \( \tilde{S} \) by (7.5), the hypothesis gives \( E_u \cap E_v = E_w \), from which (i) follows. Regarding (ii) we have

\[
\pi_u^{-1} \pi_v = \pi_u^{-1} \pi_u \pi_x \pi_y^{-1} \pi_v \pi_u \pi_v = \pi_u^{-1} e_u e_v \pi_v = \pi_u^{-1} \pi_w \pi_w^{-1} = \]

\[
= \pi_u^{-1} \pi_u \pi_x \pi_y^{-1} \pi_v = f_u \pi_x \pi_y^{-1} \pi_v = \pi_x f_u \pi_v \pi_y^{-1} = \pi_x f_w \pi_y^{-1}.
\]

As the careful reader may have noticed, we are using (2.6.ii) above for the extended representation, a result that can be proved without any difficulty since no assumption was made regarding faithfulness of the representation. \( \square \)

7.7. Definition. Given a representation \( \pi \) of \( S \), and given any nonempty finite subset \( \Lambda \subseteq \tilde{S} \), we will let

\[
F_\Lambda^\pi = \bigcap_{u \in \Lambda} F_u^\pi, \quad \text{and} \quad f_\Lambda^\pi = \prod_{u \in \Lambda} f_u^\pi.
\]

When there is only one representation of \( S \) in sight, as in the present moment, we will drop the superscripts and use the simplified notations \( F_\Lambda \) and \( f_\Lambda \).

We should remark that, since each \( f_s \) is the identity map on \( F_s \), one has that \( f_\Lambda \) is the identity map on \( F_\Lambda \).

Also notice that, since \( f_1 = \text{id}_\Omega \), the presence of 1 in \( \Lambda \) has no effect in the sense that \( f_\Lambda = f_{\Lambda \cup \{1\}} \), for every \( \Lambda \). Thus, whenever convenient we may assume that \( 1 \in \Lambda \).

As already indicated we are interested in obtaining a description of the inverse semigroup \( \mathcal{I}(\Omega, \pi) \). In that respect it is interesting to observe that most elements of the form \( f_\Lambda \) belong to \( \mathcal{I}(\Omega, \pi) \), but there is one exception, namely when \( \Lambda = \{1\} \). In this case we have

\[
f_{\{1\}} = \text{id}_\Omega,
\]

which may or may not lie in \( \mathcal{I}(\Omega, \pi) \). However, when \( \Lambda \cap S \neq \emptyset \), then surely

\[
f_\Lambda \in \mathcal{I}(\Omega, \pi), \quad (7.8)
\]

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7.9. Lemma. Let $u_1, v_1, u_2, v_2 \in S$, and let $\Lambda_1$ and $\Lambda_2$ be nonempty finite subsets of $\tilde{S}$. Let $w$ be a least common multiple of $v_1$ and $u_2$, and write $w = v_1x = u_2y$, for suitable $x, y \in \tilde{S}$. Then

$$(\pi_{u_1}f_{\Lambda_1}\pi_{v_1}^{-1})(\pi_{u_2}f_{\Lambda_2}\pi_{v_2}^{-1}) = \pi_{u}f_{\Lambda}\pi_{v}^{-1},$$

where $u = u_1x$, $v = v_2y$, and $\Lambda = \Lambda_1x \cup \{w\} \cup \Lambda_2y$.

Proof. We have

$$\pi_{u_1}f_{\Lambda_1}\pi_{v_1}^{-1}\pi_{u_2}f_{\Lambda_2}\pi_{v_2}^{-1} = \pi_{u_1}f_{\Lambda_1}\pi_{x}f_{w}\pi_{y}^{-1}\pi_{v_2}^{-1} = \pi_{u_1}f_{\Lambda_1}\pi_{x}\pi_{y}^{-1}\pi_{v_2}^{-1} = \pi_{u_1}f_{\Lambda_1}\pi_{x}\pi_{y}^{-1} = \pi_{u_1}f_{\Lambda_1}\pi_{x},$$

so $\Lambda$ may be replaced by $\{u\} \cup \Lambda_1 \cup \{v\}$ without altering the above term. Moreover, as in (7.8), observe that if $\Lambda \cap S \neq \emptyset$, then

$$\pi_{u}f_{\Lambda}\pi_{v}^{-1} \in \mathcal{I}(\Omega, \pi).$$

We should remark that, whenever we are looking at a term of the form $\pi_{u}f_{\Lambda}\pi_{v}^{-1}$, we may assume that $u, v \in \Lambda$, because

$$\pi_{u}f_{\Lambda}\pi_{v}^{-1} = \pi_{u}f_{u}\pi_{u}^{-1}\pi_{v}^{-1} = \pi_{u}f_{u}\pi_{v}^{-1} = \pi_{u}f_{u}f_{u}\pi_{v}^{-1} = \pi_{u}f_{u}f_{u}f_{u}\pi_{v}^{-1},$$

so $\Lambda$ may be replaced by $\{u\} \cup \Lambda \cup \{v\}$. In order to prove it, observe first that $\mathcal{J}$ clearly contains the inverse of its elements, so we just need to check that $\mathcal{J}$ is closed under multiplication. Given two elements of $\mathcal{J}$, say

$$\pi_{u_1}f_{\Lambda_1}\pi_{v_1}^{-1} \quad \text{and} \quad \pi_{u_2}f_{\Lambda_2}\pi_{v_2}^{-1},$$

we have by (7.3) that there exists a least common multiple for $v_1$ and $u_2$, say $w$. We may then write $w = v_1x = u_2y$, with $x, y \in \tilde{S}$, and then by (7.9) we have

$$((\pi_{u_1}f_{\Lambda_1}\pi_{v_1}^{-1})(\pi_{u_2}f_{\Lambda_2}\pi_{v_2}^{-1}) = \pi_{u}f_{\Lambda}\pi_{v}^{-1},$$

where $u = u_1x$, $v = v_2y$, and $\Lambda = \Lambda_1x \cup \{w\} \cup \Lambda_2y$. So (7.11.1) indeed represents an element in $\mathcal{J}$, thus proving that $\mathcal{J}$ is an inverse semigroup as claimed.

Given $s$ in $S$, we have

$$\pi_{s} = \pi_{s}f_{s,1}\pi_{1}^{-1} \in \mathcal{J},$$

whence $\mathcal{J}$ contains the inverse semigroup generated by the $\pi_{s}$, namely $\mathcal{I}(\Omega, \pi)$. \qed
With this we may describe the constructible sets in a more concrete way than done in (2.9).

**7.12. Proposition.** Under the assumptions of (7.11), the $\pi$-constructible subsets of $\Omega$ are precisely the sets of the form

$$X = \pi_u(F_\Lambda),$$

where $\Lambda \subseteq \tilde{S}$ is a finite subset, $\Lambda \cap S \neq \emptyset$, and $u \in \Lambda$.

**Proof.** We leave it for the reader to check that all sets of the above form are $\pi$-constructible, and let us instead show that every $\pi$-constructible set $X$ is of the above form.

Given any such $X$, we have that $\text{id}_X$ is an idempotent element of $I(\Omega, \pi)$, so there exists some $\alpha$ in $I(\Omega, \pi)$ such that $\text{id}_X = \alpha \alpha^{-1}$. By (7.11) we may write $\alpha = \pi_u f_\Lambda \pi_v^{-1}$, where $\Lambda$ is a finite subset of $\tilde{S}$, with $\Lambda \cap S \neq \emptyset$, and $u, v \in \Lambda$. It follows that

$$\text{id}_X = \alpha \alpha^{-1} = (\pi_u f_\Lambda \pi_v^{-1}) (\pi_v f_\Lambda \pi_u^{-1}) = \pi_u f_\Lambda f_v f_\Lambda \pi_u^{-1} =$$

$$= \pi_u f_\Lambda \pi_u^{-1} = \pi_u \text{id}_{F_\Lambda} \pi_u^{-1} = \text{id}_{\pi_u(F_\Lambda)},$$

so $X = \pi_u(F_\Lambda)$, as desired. $\square$

Recalling that the regular representation of $S$ respects least common multiples, our last two results apply to give:

**7.13. Corollary.** Let $S$ be a 0-left cancellative semigroup admitting least common multiples. Then

$$\mathcal{F}(S) = \{\theta_u f_\Lambda \theta_v^{-1} : \Lambda \subseteq \tilde{S} \text{ is finite, } \Lambda \cap S \neq \emptyset, \text{ and } u, v \in \Lambda\},$$

and

$$\mathcal{E}(S) = \{uF_\Lambda, \Lambda \subseteq \tilde{S} \text{ is finite, } \Lambda \cap S \neq \emptyset, \text{ and } u \in \Lambda\}.$$
7.15. Proposition. Given a finite subset $\Lambda \subseteq \tilde{S}$, with $\Lambda \cap S \neq \emptyset$, and given $u$ in $\Lambda$, suppose that $X$ is a $\theta$-constructible set such that

$$X \subseteq uF_{\Lambda}.$$  

Then there is some $x$ in $\tilde{S}$, and a finite subset $\Delta \subseteq \tilde{S}$, with $\Lambda x \subseteq \Delta$, such that

$$X = uxF_{\Delta}.$$  

Proof. Using (7.13), write $X = vF_{\Gamma}$, with $v \in \Gamma \subseteq \tilde{S}$, and $\Gamma \cap S \neq \emptyset$. Observe that the sets $vF_{\Gamma}$ and $uF_{\Lambda}$ are respectively the ranges of the idempotent elements $\theta_v f_{\Gamma} \theta_v^{-1}$ and $\theta_u f_{\Lambda} \theta_u^{-1}$. By hypothesis we then have that

$$\text{id}_X = \text{id}_{uF_{\Lambda} \cap X} = \text{id}_{uF_{\Lambda}} \text{id}_X = (\theta_u f_{\Lambda} \theta_u^{-1})(\theta_v f_{\Gamma} \theta_v^{-1}).$$

Let $w$ be a least common multiple of $u$ and $v$, and write $w = ux = vy$, for suitable elements $x$ and $y$ in $\tilde{S}$. By (7.9) the above product turns out to be

$$\theta_{ux} f_{\Lambda x \cup \{w\} \cup \Gamma y} \theta_{vy}^{-1} = \theta_{w} f_{\Lambda x \cup \{w\} \cup \Gamma y} \theta_{w}^{-1} = \theta_{w} f_{\Delta} \theta_{w}^{-1},$$

where $\Delta = \Lambda x \cup \{w\} \cup \Gamma y$. We then conclude that

$$X = wF_{\Delta} = uxF_{\Delta}. \quad \square$$

In the case of semigroups having right local units we may give a slightly more precise description for $I(\Omega, \pi)$.

7.16. Proposition. Let $S$ be a 0-left cancellative semigroup admitting least common multiples and right local units. Also let $\pi$ be a representation of $S$ on a set $\Omega$, assumed to respect least common multiples. Then any nonzero element in $I(\Omega, \pi)$ may be written as

$$\pi_s f_{\Lambda} \pi_t^{-1},$$

where $\Lambda$ is a nonempty finite subset of $S$, and $s, t \in \Lambda$. If moreover $S$ is right-reductive, one may also assume that $s^+ = t^+$, and that $\Lambda \subseteq Ss^+$.

Proof. Given any nonzero element $g \in \mathcal{S}(S)$, use (7.11) to write

$$g = \pi_u f_{\Lambda} \pi_v^{-1},$$

where $\Lambda \subseteq \tilde{S}$ is finite, $\Lambda \cap S \neq \emptyset$, and $u, v \in \Lambda$.

We then claim that we may assume that $u$ lies in $S$. In order to see this, pick any $s$ in $\Lambda \cap S$, and recall that there is an idempotent element $s^+$ of $S$ such that $s = ss^+$. Then $\pi_{s^+}$ is an idempotent element in $I(\Omega)$, and in particular $\pi(s^+) = \pi(s^+)^{-1}$. So

$$f_s = \pi_s^{-1} \pi_s = \pi_{s^+}^{-1} \pi_s = \pi_{s^+}^{-1} \pi_{s^+}^{-1} \pi_s = \pi_{s^+} f_s,$$  

(7.16.1)
and consequently

\[ g = \pi_u f_\Lambda \pi_v^{-1} = \pi_u f_s f_\Lambda \pi_v^{-1} = \pi_u \pi_s + f_s f_\Lambda \pi_v^{-1} = \pi_{us} + f_\Lambda \pi_v^{-1}. \]

Noticing that \( us^+ \in S \), and that the argument presented in (7.10) allows us to assume that \( us^+ \in \Lambda \), the claim is proven. In an entirely similar way one checks that \( v \) may also be taken in \( S \).

We will therefore assume that \( u, v \in S \), so that

\[ u, v \in \Lambda \cap S =: \Lambda'. \]

The only difference between \( \Lambda \) and \( \Lambda' \), if any, is that 1 might be in the former but not in the latter. In any case it is clear that \( f_\Lambda = f'_\Lambda \), so

\[ g = \pi_u f_\Lambda \pi_v^{-1} = \pi_u f'_\Lambda \pi_v^{-1}, \]

proving the first part of the statement. To address the last part let us suppose from now on that \( S \) is right-reductive.

Observing that \( u = uu^+ \), and that \( \pi_u + \) is and idempotent element, and hence commutes with \( f_\Lambda \), we have

\[ g = \pi_u f_\Lambda \pi_v^{-1} = \pi_u \pi_u + f_\Lambda \pi_v + \pi_v^{-1} = \pi_u f_\Lambda \pi_u + \pi_v + \pi_v^{-1} = \pi_u f_\Lambda (u^+ + \pi_v^{-1}). \]

We then must have that \( u^+ = v^+ \) since otherwise (3.15) gives \( u^+ v^+ = 0 \), and we would deduce from the above that \( g = 0 \).

Given any \( t \in \Lambda \), notice that

\[ \pi_u f_\Lambda = \pi_u \pi_u + f_t f_\Lambda \overset{(7.16.1)}{=} 1 \pi_u \pi_t + f_t f_\Lambda = \pi_u (u + t^+) f_t f_\Lambda. \]

Should \( t^+ \) not coincide with \( u^+ \), the above would again imply that \( g = 0 \), so necessarily \( t^+ = u^+ \), and hence \( t = tt^+ = tu^+ \in Su^+ \), thus proving that \( \Lambda \subseteq Su^+ \). This concludes the proof.

As before we may also describe \( \pi \)-constructible sets based on (7.16).

**7.17. Corollary.** Under the conditions of (7.16), any \( \pi \) constructible subset of \( \Omega \) may be written as

\[ \pi_s (F_\Lambda), \]

where \( \Lambda \) is a finite subset of \( S \), and \( s \in \Lambda \). If moreover \( S \) is right-reductive, one may also assume that that \( \Lambda \subseteq S s^+ \).

**Proof.** Left for the reader.

Let us say that \( g \) dominates \( f \) in an inverse semigroup if \( g \geq f \). In the special case in which \( S \) is also 0-right cancellative we have:
7.18. Proposition. Let $S$ be a 0-cancellative\(^4\) semigroup admitting least common multiples. Then $\mathcal{S}(S)$ is a 0-$E$-unitary\(^5\) inverse semigroup. Conversely, if $S$ is 0-left cancellative and has right local units, then $\mathcal{S}(S)$ 0-$E$-unitary implies that $S$ is 0-cancellative.

Proof. Pick any $g \in \mathcal{S}(S)$, and use (7.13) to write

$$g = \theta_u f_\Lambda \theta_v^{-1},$$

where $\Lambda \subseteq \tilde{S}$ is finite, $\Lambda \cap S \neq \emptyset$, and $u, v \in \Lambda$, and suppose that $g$ dominates a nonzero idempotent. Since the idempotent elements in $\mathcal{I}(S')$ are identity functions on their domains, it follows that $g$ admits a fixed point, say $p \in S'$.

Notice that $p$ lies in the domain of $g$, which is a subset of $E_v$, so we may write $p = vs$, for some $s$ in $S$. We then have

$$vs = p = g(p) = us \neq 0.$$

We now wish to conclude from the above that $u = v$, which will in turn imply that $g$ is idempotent and the proof will be finished. In case both $u$ and $v$ lie in $S$, the desired conclusion that $u = v$ clearly follows from the fact that $S$ is 0-right cancellative. If both $u$ and $v$ lie in $\tilde{S} \setminus S = \{1\}$, then $u = v$ for obvious reasons. We must therefore deal with the remaining situation in which one of $u$ and $v$ lie in $S$, while the other coincides 1. By symmetry we will suppose, without loss of generality, that $u$ is in $S$ and $v = 1$. It then follows that $s = us$, whence also $us = u^2s$, so 0-right cancellativity implies that $u^2 = u$. Therefore $\theta_u$ is idempotent, and hence so is $\theta_u f_\Lambda = g$.

Suppose that $S$ is 0-left cancellative and has right local units. Assume that $\mathcal{S}(S)$ is 0-$E$-unitary and that $su = tu \neq 0$ with $s, t, u \in S$. We compute that

$$\theta_s \theta_t^{-1} \theta_{tu} \theta_{tu}^{-1} = \theta_s \theta_t^{-1} \theta_t \theta_u \theta_u^{-1} \theta_t^{-1} \theta_{su} \theta_{tu}^{-1},$$

which is idempotent and non-zero as $su = tu \neq 0$. Thus $\theta_s \theta_t^{-1}$ is idempotent. Let $e$ be an idempotent with $te = t$. From $0 \neq tu = teu$, we obtain $u = eu$ and hence $0 \neq su = seu$ implies that $se \neq 0$ and so $se = se^2$ implies $s = se$. Thus $\theta_s \theta_t^{-1}(te) = se = s$. But by idempotence, $\theta_s \theta_t^{-1}(te) = te = t$. Thus $s = t$ and so $S$ is right 0-cancellative. \(\square\)

The description of a given element of $\mathcal{S}(S)$ in the form $\theta_u f_\Lambda \theta_v^{-1}$, as in (7.13), is far from unique and, in fact, it might not be easy to find a unique representation. However, should an element of $\mathcal{S}(S)$ posses two distinct representations of the above form, certain relations between these may be identified. In order to carry out this analysis, we will first develop a few technical tools.

The first such tool is intended to point out a situation in which uniqueness does fail.

---

\(^4\) Recall that a semigroup is 0-cancellative when it is both 0-left cancellative and 0-right cancellative.

\(^5\) An inverse semigroup is called 0-$E$-unitary, or $E^*$-unitary, if whenever an element $g$ dominates a nonzero idempotent, then $g$ itself is idempotent.
7.19. Lemma. Let \( S \) be a 0-left cancellative semigroup admitting least common multiples. Let \( \Lambda \) be a finite subset of \( \tilde{S} \), with \( \Lambda \cap S \neq \emptyset \), and let \( u, v \in \Lambda \). Given \( w \in S \), suppose that the range of \( \theta_u f_\Lambda \theta_v^{-1} \) is contained in \( E_{uw} \). Then

(i) \( F_\Lambda \subseteq E_w \),
(ii) \( \theta_u f_\Lambda \theta_v^{-1} = \theta_{uw} f_{\Lambda w} \theta_{vw}^{-1} \).

Proof. The hypothesis about the range of \( \theta_u f_\Lambda \theta_v^{-1} \) implies that

\[
\theta_u f_\Lambda \theta_v^{-1} = e_{uw} \theta_u f_\Lambda \theta_v^{-1} = \theta_{uw} \theta_{uw}^{-1} \theta_u f_\Lambda \theta_v^{-1} = \theta_{uw} \theta_{uw}^{-1} \theta_{uw} \theta_{uw}^{-1} \theta_u f_\Lambda \theta_v^{-1} = \\
= \theta_{uw} \theta_{uw}^{-1} f_u f_\Lambda \theta_v^{-1} = \theta_{uw} \theta_{uw}^{-1} f_\Lambda \theta_v^{-1} = \theta_{uw} f_{\Lambda w} \theta_{vw}^{-1} = \theta_{uw} f_{\Lambda w} \theta_{vw}^{-1},
\]

proving (ii). Given that \( u, v \in \Lambda \), we have

\[
f_\Lambda = f_u f_\Lambda f_v = \theta_u^{-1} \theta_u f_\Lambda \theta_v^{-1} \theta_v = \theta_u^{-1} e_{uw} \theta_u f_\Lambda \theta_v^{-1} \theta_v = \theta_u^{-1} \theta_u e_w f_\Lambda \theta_v^{-1} \theta_v = f_u e_w f_\Lambda f_v = e_w f_\Lambda.
\]

Therefore

\[
\text{id}_{F_\Lambda} = \text{id}_{E_w} \text{id}_{F_\Lambda} = \text{id}_{E_w \cap F_\Lambda},
\]

so \( F_\Lambda = E_w \cap F_\Lambda \), and then \( F_\Lambda \subseteq E_w \). \( \square \)

We next prove our first uniqueness result, assuming two given representations of the same element of \( S(S) \) already share some ingredients. From now on we will have to rely on 0-right cancellativity.

7.20. Lemma. Let \( S \) be a 0-cancellative semigroup admitting least common multiples. For each \( i = 1, 2 \), let \( \Lambda_i \) be a finite subset of \( \tilde{S} \) having a nonempty intersection with \( S \), and let \( u_i, v_i \in \Lambda_i \) be such that

\[
\theta_{u_i} f_{\Lambda_1} \theta_{v_1}^{-1} = \theta_{u_2} f_{\Lambda_2} \theta_{v_2}^{-1} \neq 0, \quad \text{and} \quad u_1 = u_2.
\]

Then \( f_{\Lambda_1} = f_{\Lambda_2} \). In addition,

(i) if either both \( v_1 \) and \( v_2 \) lie in \( S \), or both \( v_1 \) and \( v_2 \) lie in \( \tilde{S} \setminus S = \{1\} \), then \( v_1 = v_2 \).
(ii) if \( v_1 \in S \) and \( v_2 = 1 \), then \( v_1 \) is an idempotent element of \( S \), and \( \theta_{v_1} \geq f_{\Lambda_1} \).
(iii) same as in (ii) with subscripts “1” and “2” interchanged.

Proof. Based on (i) we will simply write \( u \) for \( u_1 \) or \( u_2 \).

Let \( z \) be any nonzero element in the common domain of \( \theta_u f_{\Lambda_1} \theta_{v_1}^{-1} \) and \( \theta_u f_{\Lambda_2} \theta_{v_2}^{-1} \), so that \( z \in E_{v_1} \cap E_{v_2} \).

By definition we have that \( E_{v_i} = v_i S \setminus \{0\} \) (in fact when \( v_i = 1 \), this is not quite the definition of \( E_{v_i} \), although it is still obviously true) so we may write \( z = v_1 x_1 = v_2 x_2 \), with \( x_1, x_2 \in S \). It is perhaps worth insisting that \( x_1 \) and \( x_2 \) indeed lie in \( S \), as opposed to \( \tilde{S} \).

We then have

\[
x_1 x_2 = \theta_u f_{\Lambda_1} \theta_{v_1}^{-1}(z) = \theta_u f_{\Lambda_2} \theta_{v_2}^{-1}(z) = u x_2 \neq 0,
\]

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so \( x_1 = x_2 \), by 0-left cancellativity (regardless of the fact that \( \tilde{S} \) might not satisfy this property), and hence
\[
v_1 x_1 = v_2 x_2 = v_2 x_1 \neq 0. \tag{7.20.1}
\]

In order to prove (i) we must check that \( v_1 = v_2 \). Under the first alternative of (i), this follows from 0-right cancellativity, while it is plain obvious under the second alternative. Still under the conditions of (i) we then have that
\[
f_{\Lambda_1} = f_u f_{\Lambda_1} f_{v_1} = \theta_{u}^{-1} \theta_u f_{\Lambda_1} \theta_{v_1}^{-1} \theta_{v_1} =
\]
\[
= \theta_{u}^{-1} \theta_u f_{\Lambda_2} \theta_{v_2}^{-1} \theta_{v_2} = f_u f_{\Lambda_2} f_{v_2} = f_{\Lambda_2},
\]
completing the proof under (i). So now let us assume that \( v_1 \in S \) and \( v_2 = 1 \). We then have from (7.20.1) that \( v_1 x_1 = x_1 \), so also \( v_1^2 x_1 = v_1 x_1 \), hence by 0-right cancellativity we deduce that \( v_1 \) is idempotent. Therefore \( \theta_{v_1} = \theta_{v_1}^{-1} = f_{v_1} \), and then
\[
f_{\Lambda_1} = f_u f_{\Lambda_1} f_{v_1} = \theta_{u}^{-1} \theta_u f_{\Lambda_1} \theta_{v_1}^{-1} = \theta_{u}^{-1} \theta_u f_{\Lambda_2} \theta_{v_2}^{-1} = f_{\Lambda_2}.
\]

We finally have
\[
f_{\Lambda_1} \theta_{v_1} = f_{\Lambda_1} f_{v_1} = f_{\Lambda_1},
\]
so \( \theta_{v_1} \geq f_{\Lambda_1} \), proving (ii), while (iii) is proved in a similar way. \( \square \)

The following result is our best shot at identifying relations between two descriptions of a single element of \( \mathcal{S}(S) \) when uniqueness fails.

**7.21. Theorem.** Let \( S \) be a 0-cancellative semigroup admitting least common multiples. For \( i = 1, 2 \), let \( \Lambda_i \) be a finite subset of \( \tilde{S} \) intersecting \( S \), and let \( u_i, v_i \in \Lambda_i \) be such that
\[
\theta_{u_i} f_{\Lambda_1} \theta_{v_1}^{-1} = \theta_{u_2} f_{\Lambda_2} \theta_{v_2}^{-1} \neq 0.
\]

Then there are \( x_1, x_2 \in \tilde{S} \), such that
\begin{enumerate}
\item \( \theta_{u_i} f_{\Lambda_1} \theta_{v_1}^{-1} = \theta_{u_i} x_i f_{\Lambda_i} x_i \theta_{v_1}^{-1} x_i \), for \( i = 1, 2 \),
\item \( u_1 x_1 = u_2 x_2 \), and \( F_{\Lambda_1 x_1} = F_{\Lambda_2 x_2} \).
\end{enumerate}

Moreover at least one of the following three properties hold:
\begin{enumerate}
\item \( v_1 x_1 = v_2 x_2 \), or
\item \( v_1 x_1 \) is an idempotent element in \( S \), and \( \theta_{v_1} x_1 \geq f_{\Lambda_1 x_1} \), and \( v_2 = x_2 = 1 \), or
\item same as in (b) with subscripts “1” and “2” interchanged.
\end{enumerate}

**Proof.** Notice that the range of the nonempty map mentioned in the hypothesis is contained in \( E_{u_1} \cap E_{u_2} \). Letting \( w \) be a least common multiple of \( u_1 \) and \( u_2 \), write \( w = u_1 x_1 = u_2 x_2 \), with \( x_1, x_2 \in \tilde{S} \).

Since \( u_1 S \cap u_2 S = w S \), we have that \( E_{u_1} \cap E_{u_2} = E_w \), so for every \( i = 1, 2 \), the range of \( \theta_{u_i} f_{\Lambda_1} \theta_{v_1}^{-1} \) is contained in \( E_w = E_{u_i x_i} \). By (7.19) we then conclude that
\[
\theta_{u_i} f_{\Lambda_1} \theta_{v_1}^{-1} = \theta_{u_i x_i} f_{\Lambda_i x_i} \theta_{v_1}^{-1 x_i},
\]
thus proving (i) and the first part of (ii). Having already seen that \( u_1 x_1 = u_2 x_2 = w \), notice that
\[
\theta_w f_{\Lambda_1 x_1} \theta_{v_1}^{-1 x_1} = \theta_w f_{\Lambda_2 x_2} \theta_{v_2}^{-1 x_2} \neq 0,
\]
so the conclusion follows from (7.20). \( \square \)
Finally, we handle the case of a categorical at zero semigroup, generalizing several known results in the literature. This theorem applies, in particular, to the inverse hull of a left cancellative category.

**7.22. Theorem.** Let $S$ be a categorical at zero semigroup that is 0-left cancellative, right reductive, has right local units and least common multiples. Then the non-zero elements of the inverse hull $\mathcal{H}(S)$ are precisely those elements of the form $\theta_s\theta_t^{-1}$ with $s^+ = t^+$. Moreover, if $s_1^+ = t_1^+$ and $s_2^+ = t_2^+$, then $\theta_{s_1}\theta_{t_1}^{-1} = \theta_{s_2}\theta_{t_2}^{-1}$ if and only if there exist $x, y$ with $xy = s_1^+$, $yx = s_2^+$, $s_1x = s_2$, $t_1x = t_2$, $s_2y = s_1$ and $t_2y = t_1$.

**Proof.** Since $S$ is categorical at zero, we have that $f_s = \theta_{s^+}$ for all $s \in S$ by Proposition (4.5). Thus if $s^+ = t^+$ and $\Lambda \subseteq S\theta^+$, then

$$\theta_s f_{\Lambda} \theta_t^{-1} = \theta_s \theta_{s^+} \theta_t^{-1} = \theta_s \theta_t^{-1}.$$  

Also note that

$$F_s = s^+S \setminus \{0\} = t^+S \setminus \{0\} \supseteq ts \setminus \{0\} = E_t,$$

and so $\theta_s \theta_t^{-1} \neq 0$ if $s^+ = t^+$.

Assume that $s_1, s_2, t_1, t_2, x, y$ are as above. Then $xS = s_1^+S = t_1^S$ since $xy = s_1^+$ and $s_1x = s_2 \neq 0$ implies $s_2^+x \neq 0$ and so $x \in s_1^+S$. Thus $\theta_x x^{-1} = \theta_{s_1}^+$ and hence

$$\theta_{s_2} \theta_{t_1}^{-1} = \theta_{s_2} \theta_{s_1}^{-1} \theta_{t_1} = \theta_{s_1} \theta_{t_1}^{-1}.$$  

Conversely, if $\theta_{s_1} \theta_{t_1}^{-1} = \theta_{s_2} \theta_{t_2}^{-1}$, then $s_1S = s_2S$ and so $s_1x = s_2$ and $s_2y = s_1$ for some $x, y \in S$. Then $s_1xy = s_1$ and $s_2xy = s_2$. Therefore, $xy = s_1^+$ and $yx = s_2^+$ by 0-left cancellativity. Now $s_1 = \theta_{s_1} \theta_{t_1}^{-1}(t_1) = \theta_{s_2} \theta_{t_2}^{-1}(t_1)$ and so $t_1 = t_2z$ with $s_2z = s_1 = s_2y$, whence $y = z$. Therefore, $t_1 = t_2y$. Similarly, $t_2 = t_1w$ with $s_1w = s_2 = s_1x$ and hence $w = x$, whence $t_2 = t_1x$. This completes the proof.  

Notice that in Theorem (7.22) if $S = S(C)$ where $C$ is a left cancellative category with least common multiples, then the elements $x, y$ above will be isomorphisms in $C$.

**8. Finitely aligned semigroups.**

We consider here a generalization of the lcm property.

**8.1. Definition.** A 0-left cancellative semigroup $S$ is said to be **finitely aligned**, or to have the (right) **Howson property**, provided that, for every $s$ and $t$ in $S$, there is a finite sequence $\{r_j\}_{j=1}^n$ of elements of $S$ such that

(i) $sS \cap tS = \bigcup_{j=1}^n r_j S$,

(ii) both $s$ and $t$ divide $r_j$, for every $j = 1, \ldots, n$.

Notice that when $S$ has right local units one has that $r \in rS$, for all $r \in S$, so condition (8.1.ii) above follows from (8.1.i).

For simplicity, we will stick to the case that $S$ has right local units so for this section, let us assume that $S$ is a 0-left cancellative semigroup admitting right local units.

Under the present hypothesis we then have that the intersection of finitely generated right ideals is finitely generated, hence the motivation for the terminology adopted above [16].
8.2. Definition. We shall say that $S$ is strongly finitely aligned if, for all $s, t \in S$, there exists a finite set $B \subseteq S \setminus \{0\}$ (possibly empty) such that $sS \cap tS = BS$ and $bS \cap b'S = \{0\}$ for $b \neq b' \in B$. Here we interpret $\emptyset S = 0$. We call $B$ a basis for $sS \cap tS$.

For example, any 0-left cancellative right lcm semigroup with right local units is strongly finitely aligned. We shall get more examples from higher rank graphs.

Our main example of a strongly finitely aligned semigroup comes from a finitely aligned higher rank graph [19]; the reader is referred to [19] for all undefined notions. Let $\Lambda$ be a $k$-graph with degree functor $d$. We write use $\vee$ for the pointwise maximum on $\mathbb{N}^k$. Put $S = \Lambda \cup \{0\}$ where all undefined products in $\Lambda$ are made 0. Then $S$ is a 0-cancellative, categorical at zero, right and left reductive semigroup with local units. We claim that $S$ is strongly finitely aligned in our sense if and only if $\Lambda$ is finitely aligned in the usual sense; note that it is singly aligned when $S$ has lcms.

First assume that $S$ is strongly finitely aligned in our sense and let $\lambda, \mu \in \Lambda$. Let $B$ be a basis for $\lambda S \cap \mu S$ and assume that $\lambda S \cap \mu S \neq \{0\}$. We claim that if $\gamma \in B$, then $d(\gamma) = d(\lambda) \vee d(\mu)$. Indeed, by the unique factorization property, we must have $\gamma = \rho \eta$ with $d(\rho) = d(\lambda) \vee d(\mu)$ and $\rho \in \lambda S \cap \mu S$ by the unique factorization property. Thus $\rho = \gamma' \tau$ with $\gamma' \in B$. From $\gamma = \gamma' \tau \eta$, we deduce that $\gamma = \gamma'$ and $d(\tau \eta) = 0$, i.e., $\tau$ and $\eta$ are identities. Thus $d(\gamma) = d(\rho) = d(\lambda) \vee d(\mu)$. Next observe that if $\rho \in \lambda S \cap \mu S$ and $d(\rho) = d(\lambda) \vee d(\mu)$, then $\rho \in B$. Indeed, $\rho = \gamma \eta$ with $\gamma \in B$. Hence $d(\rho) = d(\gamma)$ by what we just observed and so $\rho = \gamma$. It now follows that there is a bijection $\Lambda^{\min}(\lambda, \mu) \to B$ given by $(\alpha, \beta) \mapsto \lambda \alpha = \mu \beta$ and so $\Lambda$ is finitely aligned.

Next assume that $\Lambda$ is finitely aligned in the usual sense and suppose that $\lambda S \cap \mu S \neq \{0\}$. Let $B = \{\lambda \alpha \mid (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)\}$. We claim that $B$ is a basis for $\lambda S \cap \mu S$. Clearly, $B \subseteq \lambda S \cap \mu S$. If $0 \neq \gamma \neq \lambda S \cap \mu S$, then by the unique factorization property we must have that $\gamma = \rho \eta$ with $d(\rho) = d(\lambda) \vee d(\mu)$. Then if $\rho = \lambda \alpha = \mu \beta$, we have that $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ and so $\rho = \lambda \alpha \in B$. Also if $\gamma, \gamma' \in B$ and $0 \neq \tau \in \gamma S \cap \gamma' S$, then from $d(\gamma) = d(\gamma')$ we must have $\gamma = \gamma'$ by the unique factorization property. Thus $B$ is a basis. We conclude that $S$ is strongly finitely aligned. We shall show in our sequel paper that the tight $C^*$-algebra of the strongly finitely aligned 0-cancellative semigroup $S(\Lambda)$ associated to a finitely aligned $\Lambda$ is the higher rank graph $C^*$-algebra.

For the remainder of this work, we will focus on the case of right lcm semigroups, but future work will consider further the finitely aligned case.

9. Free product.

In this section, we study free products of 0-left cancellative monoids in order to produce new examples lcm monoids and finitely aligned monoids.

If $M$ and $N$ are monoids, their free product $M \ast N$ is their coproduct in the category of monoids. It is a standard fact that $M$ and $N$ embed in their free product and each element of $M \ast N$ can be uniquely expressed as a product of the form $m_1 n_1 m_2 n_2 \cdots m_k n_k$ with $m_i \in M \setminus \{1\}$, for $2 \leq i \leq k$, $n_i \in N \setminus \{1\}$ for $1 \leq i \leq k - 1$ and $m_1 \in M$, $n_k \in N$.

Assume now that $M$ and $N$ are monoids with zero. Let us denote by $M \ast_0 N$ their coproduct in the category of monoids with zero and call it the 0-free product of $M$ and $N$. In other words $M \ast_0 N$ is a monoid with zero equipped zero-preserving homomorphisms.
$M \rightarrow M *_0 N$ and $N \rightarrow M *_0 N$ such that any zero-preserving homomorphisms $M \rightarrow T$ and $N \rightarrow T$ to a monoid with zero $T$ ‘extend’ uniquely to $M *_0 N$.

9.1. Proposition. Let $M$ and $N$ be monoids with zero. Let $I$ be the ideal of $M * N$ generated by the respective zeroes $0_M, 0_N$ of $M$ and $N$. Then $M *_0 N \cong (M * N)/I$.

Proof. It is clear that if $M \rightarrow T$ and $N \rightarrow T$ are zero-preserving maps, then their extension to $M * N$ maps $I$ to 0 and hence factors uniquely through $(M * N)/I$. It follows that $(M * N)/I$ (equipped with the canonical maps $M \rightarrow (M * N)/I$ and $N \rightarrow (M * N)/I$) has the correct universal property to be $M *_0 N$. □

9.2. Corollary. Suppose that $M$ and $N$ are non-trivial monoids with zero. Then $M$ and $N$ embed into $M *_0 N$ and each non-zero element of $M *_0 N$ can be uniquely written in the form $m_1 n_1 m_2 n_2 \cdots m_k n_k$ with $m_i \in M \setminus \{0, 1\}$, for $2 \leq i \leq k$, $n_i \in N \setminus \{0, 1\}$ for $1 \leq i \leq k - 1$ and $m_1 \in M \setminus \{0\}$, $n_k \in N \setminus \{0\}$.

Proof. This is immediate from the normal form theorem for free products of monoids and the observation that if $M$ and $N$ are non-trivial, then an element of $M * N$ belongs to the ideal generated by the zeroes of $M$ and $N$ if and only if its normal form contains a 0 syllable. □

For a non-zero, non-identity element $u$ of $M *_0 N$, we say that the normal form of $u = m_1 n_1 m_2 n_2 \cdots m_k n_k$ ends in an $M$-syllable if $n_k = 1$, and that $m_k$ is the last syllable of $u$, and otherwise we say that it ends in an $N$-syllable and $n_k$ is the last syllable of $u$. The total number of non-identity syllables in the normal form of $u$ is called the syllable length of $u$. We take 0 and 1 to have syllable length 0.

As a consequence, one can show that the 0-free product of non-trivial 0-left (0-right) cancellative monoids is 0-left (0-right) cancellative.

9.3. Theorem. Let $M$ and $N$ be non-trivial 0-left cancellative monoids. Then $M *_0 N$ is 0-left cancellative. The dual result holds for 0-right cancellative monoids.

Proof. By induction on syllable length, it is enough to show that if $x \in M \cup N$ and $xu = xv \neq 0$, then $u = v$. By symmetry, we may assume that $x \in M \setminus \{0\}$. Let $u = m_1 n_1 \cdots m_k n_k$ and $v = m'_1 n'_1 \cdots m'_r n'_r$ be the normal forms as per (9.2). Then $xu$ has normal form $(xm_1)n_1 \cdots m_k n_k$ and $xv$ has normal form $(xm'_1)n'_1 \cdots m'_r n'_r$ and so $xm_1 = xm'_1 \neq 0$, $k = r$ and $m_i = m'_i$, $n_j = n'_j$ for $2 \leq i \leq k$ and $1 \leq j \leq k$. As $M$ is 0-left cancellative $m_1 = m'_1$ and so $u = v$. □

Next we want to prove that being an lcm monoid or a (strongly) finitely aligned monoid is closed under 0-free product. We begin by describing a set of representatives of the principal right ideals of a 0-free product. Recall that the $R$-class of an element of a monoid is the set of all elements which generate its principal right ideal.

9.4. Proposition. Let $M$ and $N$ be non-trivial monoids with 0. Let $T_M$ and $T_N$ be a complete set of representatives of the non-zero $R$-classes of $M$ and $N$, respectively, with $1 \in T_M$ and $1 \in T_N$. Then a complete set of representatives of the $R$-classes of $M *_0 N$ consists of 0, 1 and all elements whose normal forms end in a syllable from $(T_M \cup T_N) \setminus \{1\}$.
Proof. We prove that the $\mathcal{R}$-class of $w \in M \ast_0 N$ appears in the list above by induction on the syllable length of $w$. If the syllable length of $w$ is zero, there is nothing to prove. Suppose that $w \in M \ast_0 N \setminus \{0, 1\}$ has normal form $ux$ with $x \in M \setminus \{0, 1\} \cup N \setminus \{0, 1\}$ and $u$ has syllable length one less than $x$. Without loss of generality, assume that $x \in M$. If $x$ is a right invertible element of $M$, then $ux$ generates the same right ideal as $u$ and the result follows by induction. If $x$ is not right invertible, then $xM = x'M$ for a unique $x' \in T_M \setminus \{1\}$. Then $ux'$ generates the same right ideal as $w = ux$ and $ux'$ belongs to our list of representatives.

First note that if $u$ ends in an $N$-syllable and $x \in T_M \setminus \{1\}$, then $1 \notin xM$ and so $u$ is a prefix of the normal form of any element of $ux(M \ast_0 N)$. A similar observation holds for $ux$ if $u$ ends in an $M$-syllable and $x \in T_N \setminus \{1\}$. It now follows that the elements on our list generate distinct principal right ideals.

□

9.5. Theorem. Let $M$ and $N$ be non-trivial 0-left cancellative lcm ((strongly) finitely aligned) monoids. Then $M \ast_0 N$ is also a 0-left cancellative lcm ((strongly) finitely aligned) monoid.

Proof. We know $M \ast_0 N$ is 0-left cancellative by (9.3). Let $T_M$ and $T_N$ be a complete set of representatives of the non-zero $\mathcal{R}$-classes of $M$ and $N$, respectively, with $1 \in T_M$ and $1 \in T_N$. Then a complete set of representatives of the $\mathcal{R}$-classes of $M \ast_0 N$ consists of 0, 1 and all elements whose normal forms end in a syllable from $(T_M \cup T_N) \setminus \{1\}$ by (9.4). If $ux$ is a normal form with $x \in T_M \setminus \{1\}$ and $u$ empty or ending in an $N$-syllable, then the non-zero right multiples of $ux$ have normal form $uyz$ where $y$ is a right multiple of $x$ in $M$ and $z$ is empty or has first syllable from $N$. The situation is dual if $x \in T_N \setminus \{1\}$ and $u$ is empty or ends in an $M$-syllable. It follows that if $R$ and $R'$ are principal right ideals with $R \cap R' \neq \emptyset$, then either $R \subseteq R'$, $R' \subseteq R$ or $R$ and $R'$ are generated by $ux$ and $ux'$ (written in normal form) with $x, x' \in (T_M \cup T_N) \setminus \{1\}$. If $M$ and $N$ are lcm monoids, then the least common multiple of $ux, ux'$ is $uy$ where $y$ is a least common multiple of $x, x'$ in the respective factor $M$ or $N$. If $M$ and $N$ are (strongly) finitely aligned and $B$ is a finite generating set (basis) for $xM \cap x'M$, if $x, x' \in T_M$, or for $xN \cap x'N$, if $x, x' \in T_N$, then $uB$ is a generating set (basis) for $ux(M \ast_0 N) \cap ux'(M \ast_0 N)$.

Note that if $G$ is a group, then $G \cup \{0\}$ is always a 0-left cancellative lcm monoid. So we can build lots of strongly finitely aligned 0-cancellative monoids by taking free products of groups with adjoined zeroes and higher rank $k$-graph on one vertex.

10. Strings.

Regarding the semigroup $S_\mathcal{X}$ constructed from a subshift $\mathcal{X}$, as in example (6.2), suppose we want to recover $\mathcal{X}$ from the algebraic structure of $S_\mathcal{X}$. Given a generic element of $\mathcal{X}$, namely an infinite word

$$x = x_1x_2x_3 \ldots,$$

we may approximate $x$ by members of $S_\mathcal{X}$ by considering the sequence of finite words $\{s_n\}_{n \in \mathbb{N}}$, given by

$$s_n = x_1x_2 \ldots x_n, \quad \forall n \in \mathbb{N}.$$
We will not attempt to give a precise definition for the meaning of the word *approximate* in this context, but we will instead introduce a general concept which is expected to play the role of the above heuristic method for an arbitrary semigroup.

Throughout this section $S$ will be a fixed $0$-left cancellative semigroup.

**10.1. Definition.** A nonempty subset $\sigma \subseteq S$ is said to be a *string* in $S$, if

(i) $0 \not\in \sigma$,
(ii) for every $s$ and $t$ in $S$, if $s \mid t$, and $t \in \sigma$, then $s \in \sigma$,
(iii) for every $s_1$ and $s_2$ in $\sigma$, there is some $s$ in $\sigma$ such that $s_1 \mid s$, and $s_2 \mid s$.

An elementary example of a string is the set of divisors of any nonzero element $s$ in $S$, namely,

$$\delta_s = \{ t \in S : t \mid s \}.$$  \hspace{1cm} (10.2)

Considering the partially ordered set $\mathcal{P}$ formed by all principal right ideals $s \tilde{S}$ of $S$, whose smallest element is $\{0\}$, notice that for every string $\sigma$, one has that

$$\mathcal{F} := \{ s \tilde{S} : s \in \sigma \}$$

is a proper filter [11:12.1] on $\mathcal{P}$. Conversely, if $\mathcal{F}$ is a proper filter on $\mathcal{P}$, then

$$\sigma := \{ s \in S : s \tilde{S} \in \mathcal{F} \}$$

is a string. Therefore, strings are essentially the same as proper filters on $\mathcal{P}$.

Strings often contain many elements, but there are some exceptional strings consisting of a single semigroup element. To better study these it is useful to introduce some terminology.

**10.3. Definition.** Given a nonzero $s$ in $S$ we will say that $s$ is:

(i) *prime*, if the only divisor of $s$ is $s$, itself, or, equivalently, if $\delta_s = \{s\}$,
(ii) *irreducible*, if there are no two elements $x$ and $y$ in $S$ such that $s = xy$, or, equivalently, if $s \not\in S^2$.

It is evident that any irreducible element is prime, but there might be prime elements which are not irreducible. For example, in the semigroup $S = \{0, s, e\}$, with multiplication table given by

|   | 0 | e | s |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| e | 0 | e | 0 |
| s | 0 | s | 0 |

one has that $s$ is prime but not irreducible because $s = se \in S^2$. 

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10.4. Proposition. A singleton \{s\} is a string if and only if \(s\) is prime.

Proof. If \(s\) is prime then the singleton \{\(s\)\} coincides with \(\delta_s\), and hence it is a string. Conversely, supposing that \{\(s\)\} is a string, we have by (10.1.ii) that \(\delta_s \subseteq \{s\}\), from where it follows that \(s\) is prime. \(\Box\)

10.5. Definition. The set of all strings in \(S\) will be denoted by \(S^*\).

From now on our goal will be to define an action of \(S\) on \(S^*\).

10.6. Proposition. Let \(\sigma\) be a string in \(S\), and let \(r \in S\). Then

(i) if \(0\) is not in \(r\sigma\), one has that

\(r \ast \sigma := \{t \in S : t | rs, \text{ for some } s \in \sigma\}\)

is a string whose intersection with \(rS\) is nonempty.

(ii) If \(\sigma\) is a string whose intersection with \(rS\) is nonempty, then

\(r^{-1} \ast \sigma := \{t \in S : rt \in \sigma\}\)

is a string, and \(0\) is not in \(r(r^{-1} \ast \sigma)\).

Proof. In order to prove that \(r \ast \sigma\) satisfies (10.1.i) we argue by contradiction: if \(0\) is in \(r \ast \sigma\), then there exists some \(s\) in \(\sigma\) such that \(0 | rs\), whence \(rs = 0\), which is ruled out by hypotheses.

Since division is a transitive relation, it is clear that \(r \ast \sigma\) satisfies (10.1.ii).

Regarding (10.1.iii), for each \(i = 1, 2\), let \(t_i \in r \ast \sigma\), and pick \(s_i \in \sigma\), such that \(t_i \mid rs_i\). We may then choose \(u_i \in \tilde{S}\) so that

\(t_iu_i = rs_i\).

Since \(s_1\) and \(s_2\) lie in the \(\sigma\), we may furthermore choose \(v_1, v_2 \in \tilde{S}\), such that

\(s_1v_1 = s_2v_2 \in \sigma\).

Setting \(w_i = u_iv_i\), we then have that

\(t_1w_1 = t_1u_1v_1 = rs_1v_1 = rs_2v_2 = t_2u_2v_2 = t_2w_2\).

Given that \(s_1v_1 \in \sigma\), it is clear that \(rs_1v_1 \in r \ast \sigma\), so we may complement our findings above by writing

\(t_1w_1 = t_2w_2 \in r \ast \sigma\),

hence proving (10.1.iii) for \(r \ast \sigma\). The final requirement of (i) is easily checked by noticing that

\(\emptyset \neq r\sigma \subseteq (r \ast \sigma) \cap rS\).

This also proves the required condition that \(r \ast \sigma\) be nonempty.
Addressing (ii), notice that since $\sigma \cap rS \neq \emptyset$, we may pick some $t$ in $S$ such that $rt \in \sigma$, so we see that $t$ lies in $r^{-1} \ast \sigma$, proving the latter to be a nonempty set. Also, $0$ is not in $r^{-1} \ast \sigma$, since otherwise $r \cdot 0$ would be in $\sigma$, hence $r^{-1} \ast \sigma$ satisfies (10.1.i). Assuming that

$$s \mid t \in r^{-1} \ast \sigma,$$

we have that

$$rs \mid rt \in \sigma,$$

and we deduce that $rs \in \sigma$, hence $s \in r^{-1} \ast \sigma$, thus proving that $r^{-1} \ast \sigma$ satisfies (10.1.ii).

Let us now prove that $r^{-1} \ast \sigma$ satisfies (10.1.iii). For this let us pick $t_1$ and $t_2$ in $r^{-1} \ast \sigma$, so that $rt_1, rt_2 \in \sigma$, and then we may find $u_1, u_2 \in \tilde{S}$, such that

$$rt_1u_1 = rt_2u_2 \in \sigma.$$

By 0-left cancellativity we deduce that $t_1u_1 = t_2u_2$, and it is clear that the element represented by either side of this equality lies in $r^{-1} \ast \sigma$.

To finish we observe that if $t \in r^{-1} \ast \sigma$, then $rt$ lies in $\sigma$, so $rt \neq 0$. □

It should be noted that, under the assumptions of (10.6.i), one has that $r \sigma \subseteq r \ast \sigma$, (10.7)

and in fact $r \ast \sigma$ is the hereditary closure of $r \sigma$ relative to the order relation (5.2). In addition we have:

**10.8. Proposition.** If $\sigma$ and $\tau$ are strings with $r \sigma \subseteq \tau$, then $\sigma$ satisfies the assumption of (10.6.i), and $r \ast \sigma \subseteq \tau$.

*Proof.** If $r \sigma \subseteq \tau$, then $0$ is clearly not in $r \sigma$, while the inclusion $r \ast \sigma \subseteq \tau$ follows from the fact that $\tau$ is hereditary, and the observation already made that $r \ast \sigma$ is the hereditary closure of $r \sigma$. □

We will now define a representation of $S$ on the set $S^*$ of all strings in $S$, as follows.

**10.9. Proposition.** For each $r$ in $S$, put

$$F_r^* = \{ \sigma \in S^* : r \sigma \not\ni 0 \}, \quad \text{and} \quad E_r^* = \{ \sigma \in S^* : \sigma \cap rS \neq \emptyset \}.$$

Also let

$$\theta_r^* : F_r^* \to E_r^*$$

be defined by $\theta_r^*(\sigma) = r \ast \sigma$, for every $\sigma \in F_r^*$. Then:

(i) $\theta_r^*$ is bijective, and its inverse is the mapping defined by

$$\sigma \in E_r^* \mapsto r^{-1} \ast \sigma \in F_r^*.$$

(ii) Viewing $\theta^*$ as a map from $S$ to $\mathcal{I}(S^*)$, one has that $\theta^*$ is a representation of $S$ on $S^*$.
Proof. (i) For $\sigma \in F^*_r$, and $t \in S$, one has that
\[
t \in r^{-1} \ast (r \ast \sigma) \iff rt \in r \ast \sigma \iff \exists s \in \sigma, \ rt \mid rs \iff
\]
\[
\iff \exists s \in \sigma, \ \exists u \in \tilde{S}, \ rtu = rs
\]
Observing that $rs \neq 0$, by hypothesis, we may use 0-left cancellativity to conclude that the above is equivalent to
\[
\exists s \in \sigma, \ \exists u \in \tilde{S}, \ tu = s \iff \exists s \in \tilde{S}, \ tu \in \sigma \iff t \in \sigma.
\]
This shows that $r^{-1} \ast (r \ast \sigma) = \sigma$, and we will next prove that $r \ast (r^{-1} \ast \sigma) = \sigma$. For this, pick $\sigma$ in $E^*_r$, and notice that for any given $t \in S$, one has that
\[
t \in r \ast (r^{-1} \ast \sigma) \iff \exists s \in r^{-1} \ast \sigma, \ t \mid rs \iff \exists s \in S, \ (rs \in \sigma) \land (t \mid rs).
\]
The last sentence above implies that $t$ divides an element of $\sigma$, and hence that $t \in \sigma$, therefore showing that $r \ast (r^{-1} \ast \sigma) \subseteq \sigma$.

To prove the reverse inclusion, pick $t$ in $\sigma$. Observing that $\sigma$ is in $E^*_r$, we may find $x$ in $S$ such that $rx \in \sigma$. Using (10.1.iii), pick $u$ and $v$ in $\tilde{S}$, such that $tu = rxv \in \sigma$, so $t \mid rxv$ and, upon choosing $s = xv$, we see that (10.9.1) holds. So $t \in r \ast (r^{-1} \ast \sigma)$, concluding the proof of (i).

(ii) It is clear that $F^*_0 = E^*_0 = \emptyset$, so $\theta^*_0$ is the empty map. Given $r_1$ and $r_2$ in $S$, we must now prove that the domain of $\theta^*_{r_1} \circ \theta^*_{r_2}$ coincides with the domain of $\theta^*_{r_1 r_2}$, namely $F^*_{r_1 r_2}$, and that
\[
r_1 \ast (r_2 \ast \sigma) = (r_1 r_2) \ast \sigma,
\]
for every $\sigma$ in the above common domain.

For this, notice that a given string $\sigma$ lies in the domain of $\theta^*_{r_1} \circ \theta^*_{r_2}$ if and only if
\[
(\sigma \in F^*_r) \land (r_2 \ast \sigma \in F^*_r) \iff (0 \notin r_2 \sigma) \land (0 \notin r_1 (r_2 \ast \sigma)).
\]
Suppose by way of contradiction that a string $\sigma$ satisfying the above equivalent condition fails to be in $F^*_{r_1 r_2}$. Then
\[
0 \in r_1 r_2 \sigma \overset{(10.7)}{\subseteq} r_1 (r_2 \ast \sigma),
\]
which contradicts (10.9.3). Therefore we see that the domain of $\theta^*_{r_1} \circ \theta^*_{r_2}$ is contained in $F^*_{r_1 r_2}$.

In order to prove the reverse inclusion, pick $\sigma$ in $F^*_{r_1 r_2}$. Then $0 \notin r_1 r_2 \sigma$, from where one deduces that $0 \notin r_2 \sigma$, thus verifying the first condition in the right hand side of (10.9.3), and we claim that the second condition also holds. Arguing by contradiction,
suppose that \(0 \in r_1(r_2 * \sigma)\), which is to say that \(r_1 t = 0\), for some \(t \in r_2 * \sigma\). It follows that there are elements \(u \in \tilde{S}\), and \(s \in \sigma\), such that \(tu = r_2 s\), and then

\[
0 = r_1 tu = r_1 r_2 s \in r_1 r_2 \sigma,
\]
a contradiction. This proves that the whole right hand side of (10.9.3) holds, and hence that \(\sigma\) lies in the domain of \(\theta^*_r\), as needed.

In order to prove (10.9.2), let \(\sigma\) be in \(F^*_r\), and notice that

\[
r_1 r_2 \sigma \subseteq r_1 (r_2 * \sigma) \subseteq r_1 (r_2 * \sigma)
\]
so the hereditary closure of \(r_1 r_2 \sigma\), namely \((r_1 r_2) * \sigma\), is contained in \(r_1 (r_2 * \sigma)\). On the other hand, if \(t \in r_1 (r_2 * \sigma)\), then \(tu = r_1 y\), for suitable \(u \in \tilde{S}\), and \(y \in r_2 * \sigma\). We may moreover write \(yv = r_2 s\), with \(v \in \tilde{S}\), and \(s \in \sigma\), so

\[
tuv = r_1 yv = r_1 r_2 s \in r_1 r_2 \sigma \subseteq (r_1 r_2) * \sigma,
\]
whence \(t \in (r_1 r_2) * \sigma\). This shows that \(r_1 (r_2 * \sigma) \subseteq (r_1 r_2) * \sigma\), and hence verifies (10.9.2), concluding the proof of (ii). \(\square\)

Useful alternative characterizations of \(F^*_r\) and \(E^*_r\) are as follows:

**10.10. Proposition.** Given \(r\) in \(S\), and given any string \(\sigma\) in \(S^*\), one has that:

(i) \(\sigma \in F^*_r \iff \sigma \subseteq F^\theta_r\),

(ii) \(\sigma \in E^*_r \iff \sigma \cap E^\theta_r \neq \emptyset\),

(iii) \(\sigma \in E^*_r \Rightarrow r \in \sigma\). In addition the converse holds provided \(r \in rS\) (e.g. if \(S\) has right local units).

**Proof.** (i) A given string \(\sigma\) lies in \(F^*_r\), if and only if \(rs \neq 0\), for every \(s \in \sigma\), which is to say that \(\sigma \subseteq F^\theta_r\).

(ii) A string \(\sigma\) belongs to \(E^*_r\), if and only if \(\sigma \cap rS \neq \emptyset\), but since \(0\) is not in \(\sigma\), this is equivalent to

\[
\emptyset \neq \sigma \cap (rS \setminus \{0\}) = \sigma \cap E^\theta_r.
\]

(iii) If \(\sigma \in E^*_r\), there exists some \(s \in S\) such that \(rs \in \sigma\), hence \(r \in \sigma\), by (10.1.ii). Conversely, if \(r \in rS\), and \(r \in \sigma\), then \(r \in rS \cap \sigma\), whence \(rS \cap \sigma\) is nonempty and we see that \(\sigma \in E^*_r\). \(\square\)

**10.11. Remark.** If \(rS\) is generated by \(X \subseteq S\) as a right ideal, that is, \(rS = \bigcup_{s \in X} s \tilde{S}\), then \(\sigma \in E^*_r\) if and only if \(\sigma \cap X \neq \emptyset\).

Recall from (7.13) that, when \(S\) has least common multiples, every \(\theta\)-constructible subset of \(S'\) has the form \(\theta_u(F^\theta_{\Lambda})\), where \(\Lambda \subseteq \tilde{S}\) is finite, \(\Lambda \cap S \neq \emptyset\), and \(u \in \Lambda\). By analogy this suggests that it might also be useful to have a characterization of \(\theta^*_u(F^*_\Lambda)\) along the lines of (10.10).
10.12. Proposition. Let $\Lambda$ be a finite subset of $\tilde{S}$ having a nonempty intersection with $S$, and let $u \in \Lambda$. Then $\theta_u(F_\Lambda^*)$ consists precisely of the strings $\sigma$ such that

$$\emptyset \neq \sigma \cap E_\Lambda^0 \subseteq \theta_u(F_\Lambda^0).$$

Proof. Noticing that $\theta_u(F_\Lambda^*)$ is a subset of $E_\Lambda^*$, one has that any given string $\sigma$ lies in $\theta_u(F_\Lambda^*)$ if and only if

$$\left(\sigma \in E_\Lambda^* \right) \land \left(\theta_u^{-1}(\sigma) \in F_\Lambda^*\right) \iff \left(\sigma \cap E_\Lambda^0 \neq \emptyset \right) \land \left(\theta_u^{-1}(u) \subseteq F_\Lambda^0\right),$$

where we observe that our application of (10.10.i-ii) for $u$, though $\neq 1$, one has that (10.10.i-ii) is trivially true.

Note that, since $\sigma$ does not contain $0$, the definition of $u^{-1} \ast \sigma$ is equivalent to

$$u^{-1} \ast \sigma = \{t \in F_\Lambda^0 : \theta_u(t) \in \sigma \cap E_\Lambda^0\} = \theta_u^{-1}(\sigma \cap \emptyset).$$

So (10.12.1) is further equivalent to

$$\left(\sigma \cap E_\Lambda^0 \neq \emptyset \right) \land \left(\theta_u^{-1}(\sigma \cap E_\Lambda^0) \subseteq F_\Lambda^0\right) \iff \emptyset \neq \sigma \cap E_\Lambda^0 \subseteq \theta_u(F_\Lambda^0).$$

□

After (10.9) we now have two natural representations of $S$, namely the regular representation $\theta$ acting on $S^*$, and $\theta^*$ acting on $S^*$. We shall next prove that the correspondence $r \mapsto \delta_r$ is covariant (i.e., $S$-equivariant) for these representation. We begin by proving a technical result designed, among other things, to show that the domains and ranges are matched accordingly.

10.13. Lemma. Let $\Lambda$ be a finite subset of $\tilde{S}$, with $\Lambda \cap S \neq \emptyset$, and let $u \in \Lambda$. Then for every $r \in S$, one has that

$$r \in \theta_u(F_\Lambda^0) \iff \delta_r \in \theta_u(F_\Lambda^0).$$

Proof. If $r \in \theta_u(F_\Lambda^0)$, then $\delta_r \cap E_\Lambda^0$ is nonempty because $r$ belongs to this set. In order to show that $\delta_r \in \theta_u(F_\Lambda^*)$ it therefore suffices to show that

$$\delta_r \cap E_\Lambda^0 \subseteq \theta_u(F_\Lambda^0),$$

by (10.12). Given $x$ in $\delta_r \cap E_\Lambda^0$, we may write $x = us$, for some $s \in S$ and, observing that $x \mid r$, we have that $xv = r$, for some $v$ in $\tilde{S}$. Then $r = xv = usv$, whence $sv = \delta_u^{-1}(r) \in F_\Lambda^0$, and it easily follows that $s \in F_\Lambda^0$, as well, as $F_\Lambda^0$ is hereditary. Thus $x = us \in \theta_u(F_\Lambda^0)$. Having checked the inclusion displayed above, we have proven that $\delta_r \in \theta_u(F_\Lambda^0)$. Conversely, assuming that $\delta_r \in \theta_u(F_\Lambda^0)$, we have by (10.12) that

$$\emptyset \neq \delta_r \cap E_\Lambda^0 \subseteq \theta_u(F_\Lambda^0).$$

We may thus pick $x$ in $\delta_r \cap E_\Lambda^0$, so $x = us$, and $xv = r$, where $s \in S$, and $v \in \tilde{S}$, as above. It follows that

$$r = xv = usv \in \delta_r \cap E_\Lambda^0 \subseteq \theta_u(F_\Lambda^0),$$

concluding the proof. □
10.14. Proposition. The map

\[ \delta : s \in S' \mapsto \delta_s \in S^*, \]

where \( \delta_s \) is defined in (10.2), is covariant relative to \( \theta \) and \( \theta^* \).

Proof. For every \( r \) in \( S \), we need to prove that, \( \delta(F_r^\theta) \subseteq F_r^\theta \), \( \delta(E_r^\theta) \subseteq E_r^\theta \), and that the diagram

\[
\begin{array}{ccc}
F_r^\theta & \xrightarrow{\theta_r^\delta} & E_r^\theta \\
\delta \downarrow & & \downarrow \delta \\
F_r^* & \xrightarrow{\theta_r^*} & E_r^*
\end{array}
\]

commutes.

The first two facts follow immediately from (10.13), so we need only check the commutativity of the diagram, which boils down to proving that

\[ r \ast \delta_s = \delta_{rs}, \quad \forall s \in F_r^\theta. \]

Given \( s \) in \( F_r^\theta \), for obvious reasons we have that \( r \delta_s \subseteq \delta_{rs} \), so by (10.8) we obtain \( r \ast \delta_s \subseteq \delta_{rs} \).

On the other hand, if \( t \in \delta_{rs} \), then \( t \mid rs \), so there exists \( u \in \tilde{S} \), such that \( tu = rs \) (10.7)

Therefore \( t \) is in \( r \ast \delta_s \), showing the reverse inclusion \( \delta_{rs} \subseteq r \ast \delta_s \), and hence proving our diagram to be commutative. \( \square \)

Let us now consider the question of whether \( \theta^* \) is an essential representation. Recall from (2.3) that the essential subset for \( \theta^* \) is

\[ S^*_\theta = ( \bigcup_{s \in S} F_s^* ) \cup ( \bigcup_{s \in S} E_s^* ). \]

10.15. Proposition. Let \( \sigma \) be a string in \( S^* \). Then:

(i) \( \sigma \) does not belong to \( \bigcup_{t \in S} E_t^* \), if and only if \( \sigma = \{s\} \), for some irreducible \( s \in S \),

(ii) \( \sigma \) does not belong to \( S^*_\theta \), if and only if \( \sigma = \{s\} \), for some degenerate element \( s \) in \( S \).

Proof. (i) Observe that, for every \( t \) in \( S \), one has that

\[ \sigma \notin E_t^* \iff \sigma \cap E_t^\theta = \emptyset. \] (10.15.1)

Assuming that \( \sigma \notin E_t^* \), for every \( t \) in \( S \), we will now show that \( \sigma \) is a singleton, so we suppose that \( s,t \in \sigma \). Then, by (10.1.iii) there are \( u,v \in \tilde{S} \), such that \( su = tv \in \sigma \). If \( v \neq 1 \), then \( v \in S \), whence

\[ tv \in tS \setminus \{0\} = E_t^\theta, \]

contradicting (10.15.1). Thus \( v = 1 \), and the same reason shows that \( u = 1 \), whence \( s = t \), and we see that \( \sigma \) contains precisely one element.

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Letting \( s \) denote the single element in \( \sigma \), one has that \( s \) is necessarily irreducible since otherwise we could write \( s = tr \), with \( t, r \in S \), and then \( s \in \sigma \cap E_{t}^{\theta} \), again contradicting (10.15.1).

Conversely, assuming that \( s \) is irreducible then \( \sigma := \{ s \} \) is a string by (10.4). Observing that for every \( t \) in \( S \), one has \( E_{t}^{\theta} \subseteq S^2 \), it is clear that \( \sigma \cap E_{t}^{\theta} = \emptyset \), whence \( \sigma \notin E_{t}^{\theta} \), thanks to (10.15.1).

(ii) In order to prove the “only if” part, let \( \sigma \in S^{*} \setminus S_i^{*} \). Then clearly \( \sigma \) does not belong to \( \bigcup_{t \in S} E_{t}^{\theta} \), so \( \sigma = \{ s \} \), for some irreducible \( s \in S \), by (i), and all we need to do is show that \( SS = 0 \). Arguing by contradiction, suppose that \( rs \neq 0 \), for some \( r \) in \( S \). Then \( \sigma \subseteq F_{r}^{\theta} \), hence \( \sigma \in F_{r}^{\theta} \), by (10.10.i) contradicting the hypothesis.

Conversely, suppose that \( \sigma = \{ s \} \), where \( s \) is degenerate, hence in particular irreducible. By (i) we then have that \( \sigma \) is not in any \( E_{t}^{\theta} \), so it suffices to prove that \( \sigma \) is not in any \( F_{t}^{\theta} \), either. But since \( SS = \{ 0 \} \), we see that \( s \) is not in any \( F_{t}^{\theta} \), whence \( \sigma \notin F_{t}^{\theta} \), so \( \sigma \notin F_{t}^{\theta} \), by (10.10.i).

We then obtain a result similar to (3.7):

**10.16. Proposition.** Denoting the essential subset for \( \theta^{*} \) by \( S_{i}^{*} \), one has that
\[
S^{*} \setminus S_{i}^{*} = \{ \{ s \} : s \text{ is degenerate} \}.
\]
Therefore \( \theta^{*} \) is essential if and only if \( S \) possesses no degenerate elements.

Observe that when \( S \) has right local units, then it has no irreducible elements, much less degenerate ones. We therefore obtain the following consequence of the above result and of (3.7):

**10.17. Corollary.** If \( S \) has right local units, then both \( \theta \) and \( \theta^{*} \) are essential representations.

Observe that the union of an increasing family of strings is a string, so any string is contained in a maximal one by Zorn’s Lemma.

**10.18. Definition.** The subset of \( S^{*} \) formed by all maximal strings will be denoted by \( S^{\infty} \).

Our next result says that \( S^{\infty} \) is invariant under \( \theta^{*} \).

**10.19. Proposition.** For every \( r \) in \( S \), and for every maximal string \( \sigma \) in \( F_{r}^{*} \), one has that \( \theta_{r}^{*}(\sigma) \) is maximal.

*Proof.* Suppose that \( r \ast \sigma \) is contained in another string \( \mu \). Since \( r \ast \sigma \) is in \( F_{r}^{*} \), we have that
\[
\emptyset \neq (r \ast \sigma) \cap rS \subseteq \mu \cap rS,
\]
so \( \mu \) is in \( E_{r}^{\theta} \). It is then easy to see that
\[
\sigma = r^{-1} \ast (r \ast \sigma) \subseteq r^{-1} \ast \mu,
\]
so \( \sigma = r^{-1} \ast \mu \), by maximality, whence,
\[
r \ast \sigma = r \ast (r^{-1} \ast \mu) = \mu,
\]
proving that \( r \ast \sigma = \theta_{r}^{*}(\sigma) \) is maximal. \( \square \)
Observe that the above result says that $S^\infty$ is invariant under each $\theta^*_r$, but not necessarily under $\theta^{*-1}_r$.

An example to show that $S^\infty$ may indeed not be invariant under $\theta^{*-1}_r$ is as follows. Consider the language $L$ on the alphabet $\Sigma = \{a, b\}$ given by

$$L = \{a, b, aa, ba\}.$$ 

Then $\sigma = \{b, ba\}$ is a maximal string, while $\theta^{*-1}_b(\sigma) = \{a\}$ is not maximal.

However, when $S$ is categorical at zero, the situation is much better, as we shall now see.

10.20. Proposition. Suppose that $S$ is categorical at zero. Then, for every $r$ in $S$, and for every maximal string $\sigma$ in $E^*_r$, one has that $\theta^{*-1}_r(\sigma)$ is maximal.

Proof. Suppose that $\mu$ is a string with

$$\theta^{*-1}_r(\sigma) \subseteq \mu. \quad (10.20.1)$$

Notice that $\theta^{*-1}_r(\sigma)$ lies in $F^*_r$ for obvious reasons, but it is far from obvious that the same applies to $\mu$. However, under the present hypothesis we shall show that indeed $\mu \in F^*_r$.

Given $t$ in $\mu$, choose any $s$ in $\theta^{*-1}_r(\sigma)$. Since both $t$ and $s$ lie in $\mu$, we may pick $u, v \in \tilde{S}$, such that $su = tv \in \mu$. Observing that $\theta^{*-1}_r(\sigma) \in F^*_r$, we see that

$$s \in \theta^{*-1}_r(\sigma) \subseteq F^\theta_r,$$

so $rs \neq 0$. Clearly also $su \neq 0$, so we may use the fact that $S$ is categorical at zero (even though $u$ is not necessarily an element of $S$) to conclude that

$$0 \neq rsu = rtv,$$

whence $rt \neq 0$, and we see that $t \in F^\theta_r$, thus proving that $\mu \subseteq F^\theta_r$. Again by (10.10) we then get that $\mu \in F^*_r$, as desired.

Therefore from (10.20.1) it follows that

$$\sigma = \theta^*_r(\theta^{*-1}_r(\sigma)) \subseteq \theta^*_r(\mu),$$

so $\sigma = \theta^*_r(\mu)$, by maximality, and

$$\theta^{*-1}_r(\sigma) = \theta^{*-1}_r(\theta^*_r(\mu)) = \mu,$$

thus proving that $\theta^{*-1}_r(\sigma)$ is maximal.

As an immediate consequence we have the following:

10.21. Corollary. Let $S$ be a 0-left cancellative, categorical at zero semigroup, and let $r \in \tilde{S}$. Then:

(i) $\theta^*_r(F^*_r \cap S^\infty) \subseteq S^\infty$,

(ii) $\theta^{*-1}_r(E^*_r \cap S^\infty) \subseteq S^\infty$, and

(iii) $S^\infty$ is an invariant subset of $S^*$ under the natural action of $\mathcal{I}(S^*, \theta^*)$.

Proof. Points (i) and (ii) are restatements of (10.19) and (10.20), respectively, while (iii) is a consequence of (i–ii), as well as the fact that $\mathcal{I}(S^*, \theta^*)$ is generated by all of the $\theta^*_r$, together with their inverses. \qed
11. Open strings.

Throughout this section $S$ will be a fixed 0-left cancellative semigroup, as before.

Given a string $\sigma$ in $S^*$, and given $s, p \in S$, one has by the very definition of strings that

$$sp \in \sigma \Rightarrow s \in \sigma.$$ 

Clearly the converse of the above implication is not true and in fact, knowing that $s$ lies in $\sigma$, does not even guarantee the existence of some $p$ in $S$ such that $sp \in \sigma$. To clarify this issue we introduce the following concept:

11.1. Definition. Given any string $\sigma$ in $S^*$, the interior of $\sigma$ is the subset of $S$ given by

$$\hat{\sigma} = \{s \in S : \exists p \in S, \ sp \in \sigma\}.$$ 

In addition, we shall say that $\sigma$ is an open string when $\sigma = \hat{\sigma}$.

Observe that when $S$ has right local units, every string is automatically open.

11.2. Proposition. Given any string $\sigma$, one has that

(i) $\hat{\sigma} \subseteq \sigma$,
(ii) $\sigma$ fails to be open if and only if $\sigma = \delta_r$ for some $r$ in $S$ such that $r \notin rS$; in this case such an $r$ is unique,
(iii) if $s_1, s_2 \in \hat{\sigma}$, and if $r$ is a least common multiple for $s_1$ and $s_2$, then $r \in \hat{\sigma}$,
(iv) if $\hat{\sigma}$ is nonempty, and if $S$ admits least common multiples, then $\hat{\sigma}$ is a string.

Proof. (i) Obvious in view of (10.1.ii).

(ii) Assuming that $\hat{\sigma} \neq \sigma$ we may choose $r \in \sigma \setminus \hat{\sigma}$, so that $r \in \sigma$, but $rp \notin \sigma$, for all $p$ in $S$. Observing that $\delta_r \subseteq \sigma$, we will show that in fact $\delta_r = \sigma$. For this, let $s \in \sigma$, and use (10.1.iii) to find $u, v \in \hat{S}$ such that

$$su = rv \in \sigma.$$ 

Notice that $v = 1$, since otherwise $r \in \hat{\sigma}$. Therefore $su = r$, so $s$ divides $r$, whence $s \in \delta_r$. To see that $r \notin rS$, notice that otherwise there would be some $t$ in $S$ such that $r = rt$, and again this will conflict with the choice of $r$ outside $\hat{\sigma}$.

To show that $r$ is unique let us assume that $\delta_r = \delta_{r'}$. Then $r$ and $r'$ divide each other, so there exist $u$ and $v$ in $\hat{S}$, such that $ru = r'$, and $r'v = r$. From this we get that $r = ruv$, and then necessarily $u = v = 1$, or otherwise $r \in rS$. Thus $r = r'$.

Conversely, if $\sigma = \delta_r$, with $r \notin rS$, we claim that $r \in \delta_r \setminus \hat{\delta_r}$. On the one hand it is evident that $r \in \delta_r$. On the other hand, supposing by contradiction that $r \in \hat{\delta_r}$, there exists $p$ in $S$ such that $rp \in \delta_r$, whence $rp \mid r$, so we may find $u$ in $\hat{S}$ with $rpu = r$. This implies that $r \in rS$, contradicting the assumptions. Therefore $\delta_r \neq \hat{\delta_r}$, as desired.
(iii) Given that \( s_1 \) and \( s_2 \) are in \( \bar{\sigma} \), choose \( p_1 \) and \( p_2 \) in \( S \), such that \( s_1p_1 \) and \( s_2p_2 \) belong to \( \sigma \). Using (10.1.iii), we furthermore choose \( u_1 \) and \( u_2 \) in \( \bar{S} \), such that

\[
s := s_1p_1u_1 = s_2p_2u_2 \in \sigma.
\]

It then follows that

\[
s \in s_1S \cap s_2S = rS,
\]

so there is some \( t \) in \( S \) such that \( s = rt \), so \( r \) divides \( s \) and we deduce that \( r \in \sigma \).

(iv) It is easy to see that \( \bar{\sigma} \) satisfies (10.1.i-ii), while (10.1.iii) follows immediately from (iii) and the existence of least common multiples. \( \square \)

11.3. Proposition. Let \( \sigma \) be a maximal string. Then either \( \sigma \) is open, or \( \sigma = \delta_r \) for some \( r \) in \( S \), such that \( rS = \{0\} \).

Proof. Supposing that the maximal string \( \sigma \) is not open, we have that \( \sigma = \delta_r \), with \( r \notin rS \), by (11.2.ii), so it suffices to prove that \( rS = \{0\} \). Supposing otherwise, let \( s \in S \) be such that

\[
t := rs \neq 0.
\]

Since \( r \mid t \), we have that \( \delta_r \subseteq \delta_t \), so \( \delta_r = \delta_t \) by maximality. It follows that \( t \mid r \), so we may find \( u \) in \( \bar{S} \) such that \( tu = r \). Therefore

\[
r = tu = rsu \in rS,
\]

contradicting the fact that \( r \) is not in \( rS \). This concludes the proof. \( \square \)

Let us now study open strings in relation to the representation \( \theta^* \).

11.4. Proposition. Let \( \sigma \) be an open string in \( S \), and let \( r \in S \).

(i) If \( \sigma \in F^*_r \), then \( \theta^*_r(\sigma) \) is open.

(ii) If \( \sigma \in E^*_r \), then \( \theta^*_r^{-1}(\sigma) \) is open.

Proof. (i) Let \( t \in \theta^*_r(\sigma) \), so that \( t \mid rs \), for some \( s \) in \( \sigma \), and hence we may write \( tx = rs \), for a suitable \( x \) in \( \bar{S} \). Since \( \sigma \) is open, we may pick some \( p \) in \( S \) such that \( sp \in \sigma \), whence

\[
txp = rsp \in r\sigma \subseteq r * \sigma = \theta^*_r(\sigma).
\]

Since \( xp \in S \), this proves (i).

(ii) Let \( t \in \theta^*_r^{-1}(\sigma) \), so that \( rt \in \sigma \). Since \( \sigma \) is open, we may pick some \( p \) in \( S \) such that \( rtp \in \sigma \). Therefore

\[
 tp \in r^{-1} * \sigma = \theta^*_r^{-1}(\sigma),
\]

proving (ii). \( \square \)

We will return to the study of open strings in future sections.
12. Representing the inverse hull on strings.

In (2.1) we introduced the notion of semigroup representations on a set $\Omega$. The semigroups we had in mind there were simply associative semigroups with zero, but from now on we will also consider representations of inverse semigroups, such as $\mathcal{H}(S)$.

There is no need to amend Definition (2.1) when the semigroup considered is an inverse semigroup but it is worth noticing that if $\rho$ is a representation of a given inverse semigroup $S$ on a set $\Omega$, then

$$\rho(s^{-1}) = \rho(s)^{-1}, \quad \forall s \in S,$$

a fact that follows easily from the uniqueness of inverses.

Our goal now is to show that the action of a 0-left cancellative semigroup $S$ on strings extends to the inverse hull $\mathcal{H}(S)$. We do this by first proving a more general result.

Let $X$ be a set equipped with a preorder, that is, a transitive and reflexive relation $\leq$. Thus, $X$ is a partially ordered set except for the fact that the anti-symmetric property is not required to hold.

The goal is to later allow the example in which $X$ is a semigroup and $\leq$ is the order given by division.

By abuse of language we will refer to $\leq$ as the order on $X$, even though, strictly speaking, this is not an order relation.

If $x$ and $y$ are in $X$, the interval from $x$ to $y$ is the set

$$[x, y] = \{z \in X : x \leq z \leq y\}.$$

We will also work with the unbounded intervals

$$(-\infty, y] = \{z \in X : z \leq y\}, \quad \text{and} \quad [x, +\infty) = \{z \in X : x \leq z\}.$$

12.1. Definition. A subset $A \subseteq X$ is said to be:

(i) convex, if whenever $x, y \in A$, one has that $[x, y] \subseteq A$,

(ii) hereditary, if whenever $x \in A$, one has that $(-\infty, x] \subseteq A$,

(iii) directed, if $X$ is nonempty and whenever $x, y \in A$, there exists some $z$ in $A$ such that $x \leq z$ and $y \leq z$,

(iv) a string, if $A$ is hereditary and directed.

The set of all directed subsets of $X$ will be denoted by $\Delta(X)$, and the set of all strings in $X$ will be written $\Sigma(X)$. Note that a set $D$ is directed if and only if each of its finite subsets (including the empty set) has an upper bound. What we call a string is usually called an ideal in order theory but since the term “ideal” has other meanings in this paper we shall use the alternative terminology. Convex sets are precisely the sets which are the intersection of a hereditary set and the complement of a hereditary set.

For the sake of symmetry, observe that the notion of directed sets may also be expressed using intervals: $A \neq \emptyset$ is directed if and only if, whenever $x, y \in A$, one has that

$$[x, +\infty) \cap [y, +\infty) \cap A \neq \emptyset.$$

Observe that the empty set is convex and hereditary by vacuity. It is however not directed since it has been explicitly ruled out in (12.1.iii)
12.2. Definition. If $A$ is any subset of $X$, we will denote by $h(A)$ the hereditary closure of $A$, namely

$$h(A) = \{x \in X : \exists y \in A, \ x \leq y\} = \bigcup_{y \in A} (-\infty, y].$$

It is easy to see that $h(A)$ is the smallest hereditary subset of $X$ containing $A$. Consequently $h(h(A)) = h(A)$, and $A$ is hereditary if and only if $h(A) = A$.

If $D$ is a directed subset of $X$, it is easy to see that $h(D)$ is also directed, and hence $h(D)$ is a string. We may therefore view $h$ as a map

$$h: \Delta(X) \to \Sigma(X).$$

(12.3)

If $A$ and $B$ are nonempty subsets of $X$, with $A \subseteq B$, recall that $A$ is said to be cofinal in $B$, provided for every $x$ in $B$, there exists some $y$ in $A$, such that $x \leq y$. Notice that if $B$ is nonempty, then $A$ must be nonempty as well.

12.4. Proposition. Let $A$ and $D$ be subsets of $X$, with $A \subseteq D$. If $A$ is cofinal in $D$, and $D$ is directed, then $A$ is also directed.

Proof. As observed above, $A \neq \emptyset$. Given $x$ and $y$ in $A$, use that $D$ is directed to produce some $z$ in $D$, such that $x, y \leq z$. By cofinality, there is $w$ in $A$, with $z \leq w$, whence $x, y \leq w$. \hfill \Box

The following is a useful condition, similar to cofinality, that applies even when $A$ is not a subset of $B$.

12.5. Definition. Given subsets $D$ and $E$ of $X$, with $D$ directed, we will say that $D$ is asymptotically contained in $E$, in symbols

$$D \sqsubseteq E,$$

provided $E \cap D$ is cofinal in $D$.

The above notion could easily be applied to a not necessarily directed set $D$, but we will have no use for it outside the situation outlined above. Moreover, in most applications of that notion, $E$ will be a convex set.

The following will be useful later.

12.6. Lemma. Let $D$ be a directed subset of $X$, and let $E \subseteq X$.

(i) If $D \subseteq E$, then $h(D) \subseteq E$.

(ii) If $D \subseteq E$, then $E \cap D$ is directed.

Proof. (i) We need to show that $E \cap h(D)$ is cofinal in $h(D)$, so let $x \in h(D)$. Then there exists $y$ in $D$ such that $x \leq y$. By hypothesis, there exists $z$ in $E \cap D$, such that $y \leq z$. Therefore

$$x \leq y \leq z \in E \cap D \subseteq E \cap h(D),$$

as desired.

(ii) follows immediately from (12.4). \hfill \Box

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The relevance of asymptotical containment for strings is emphasized next.

12.7. Lemma. Let \( \sigma \) be a string in \( X \), and let \( E \) be any subset of \( X \). Then the following are equivalent:

(i) \( \sigma \sqsubseteq E \),
(ii) \( \sigma = h(E \cap \sigma) \),
(iii) \( \sigma = h(A) \), for some subset \( A \subseteq E \).

Proof. (i)\( \Rightarrow \) (ii) Since \( \sigma \) is hereditary and \( E \cap \sigma \subseteq \sigma \), it is clear that \( h(E \cap \sigma) \subseteq \sigma \). To prove the reverse inclusion, pick any \( x \) in \( \sigma \). Then by (i) there exists \( y \) such that \( x \leq y \in E \cap \sigma \), so we see that \( x \in h(E \cap \sigma) \).

(ii)\( \Rightarrow \) (iii) Obvious.

(iii)\( \Rightarrow \) (i) Since \( A \) is evidently cofinal in \( h(A) \), the hypothesis implies that \( A \) is cofinal in \( \sigma \). Since \( A \subseteq E \cap \sigma \), we have that \( E \cap \sigma \) is cofinal in \( \sigma \), whence \( \sigma \sqsubseteq E \) \( \square \).

Recall that \( \mathcal{I}(X) \) denotes the collection of all partial bijections on \( X \). It is well known that \( \mathcal{I}(X) \) is an inverse semigroup, known as the symmetric inverse semigroup of \( X \).

12.8. Definition. Let \( \varphi \in \mathcal{I}(X) \).

(i) The domain of \( \varphi \) will be denoted by \( F_\varphi \) and its range will be written \( E_\varphi \).

(ii) We will say that \( \varphi \) is order preserving if, for every \( x \) and \( y \) in \( F_\varphi \), one has that

\[
x \leq y \iff \varphi(x) \leq \varphi(y).
\]

(iii) We shall denote by \( \mathcal{I}^+(X) \), the collection of all order preserving \( \varphi \) in \( \mathcal{I}(X) \), such that \( F_\varphi \) and \( E_\varphi \) are convex.

12.9. Proposition. \( \mathcal{I}^+(X) \) is an inverse subsemigroup of \( \mathcal{I}(X) \).

Proof. It is easy to see that if \( \varphi \) is order preserving then so is \( \varphi^{-1} \), so \( \mathcal{I}^+(X) \) is seen to be invariant under taking inverses. Given \( \varphi \) and \( \psi \) in \( \mathcal{I}^+(X) \), recall that the domain and range of the composition \( \varphi \psi \) are given respectively by

\[
F_{\varphi \psi} = \psi^{-1}(F_\varphi \cap E_\psi), \quad \text{and} \quad E_{\varphi \psi} = \varphi(F_\varphi \cap E_\psi).
\]

We will next prove that \( F_{\varphi \psi} \) is convex. For this, suppose that \( x, y \in F_{\varphi \psi} \), and \( z \in [x, y] \). Since \( F_{\varphi \psi} \subseteq F_\psi \), and the latter is convex, we have that \( z \in F_\psi \). So

\[
\psi(x) \leq \psi(z) \leq \psi(y),
\]

and since \( \psi(x) \) and \( \psi(y) \) lie in the convex set \( F_\varphi \), it follows that \( \psi(z) \in F_\varphi \cap E_\psi \), whence \( z \in F_{\varphi \psi} \).

This proves that \( F_{\varphi \psi} \) is convex and a similar argument shows that \( E_{\varphi \psi} \) is also convex. Trivially, \( \varphi \psi \) is order preserving. \( \square \)
Our main goal will be to describe a canonical action of \( I_+(X) \) on \( \Sigma(X) \). As an intermediate step, for each \( \varphi \) in \( I_+(X) \), we will build a partial mapping \( \varphi^\Delta \) on \( \Delta(X) \).

**12.10. Definition.** Given any subset \( E \subseteq X \), we will denote by \( E^\Delta \) the subset of \( \Delta(X) \) consisting of all directed subsets \( D \) of \( X \), such that \( D \subseteq E \).

Since every nonempty directed subset of \( X \) is clearly asymptotically contained in \( X \), we have that

\[
X^\Delta = \Delta(X).
\]

Given \( \varphi \) in \( I_+(X) \), and given \( D \) in \( F_\varphi^\Delta \), we have that \( F_\varphi \cap D \) is directed by (12.6.ii). Since \( \varphi \) is an order-isomorphism from \( F_\varphi \) to \( E_\varphi \), it follows that \( \varphi(F_\varphi \cap D) \) is also directed. In addition, since \( \varphi(F_\varphi \cap D) \) is contained in \( E_\varphi \), it is obvious that \( \varphi(F_\varphi \cap D) \subseteq E_\varphi \), meaning that \( \varphi(F_\varphi \cap D) \in E_\varphi^\Delta \). We therefore have a well defined mapping

\[
D \in F_\varphi^\Delta \mapsto \varphi(F_\varphi \cap D) \in E_\varphi^\Delta.
\]

**12.11. Definition.** Given any \( \varphi \) in \( I_+(X) \), we will denote by \( \varphi^\Delta \) the above map from \( F_\varphi^\Delta \) to \( E_\varphi^\Delta \), namely

\[
\varphi^\Delta(D) = \varphi(F_\varphi \cap D), \quad \forall D \in F_\varphi^\Delta.
\]

We view \( \varphi^\Delta \) as a partial mapping on \( \Delta(X) \).

If \( \psi = \varphi^{-1} \), we may ask ourselves what is the relationship between \( \psi^\Delta \) and \( \varphi^\Delta \), but one should not expect these maps to be the inverse of each other. The reason is that

\[
\psi^\Delta(\varphi^\Delta(D)) = F_\varphi \cap D, \quad \forall D \in F_\varphi^\Delta,
\]

and so when \( D \) is asymptotically contained in \( F_\varphi \), without being a subset of \( F_\varphi \), one would have that \( \psi^\Delta(\varphi^\Delta(D)) \neq D \). By turning our attention to strings, rather than directed sets, we will soon fix this anomaly. Nonetheless, the assignment \( \varphi \mapsto \varphi^\Delta \) is a homomorphism.

**12.12. Proposition.** For \( \varphi, \psi \in I_+(X) \), one has \( \varphi^\Delta \psi^\Delta = (\varphi \psi)^\Delta \).

*Proof.* Suppose that \( D \in F_\varphi^\Delta \psi_\psi \). Then \( D \cap F_\psi \psi \) is cofinal in \( D \). So if \( x \in D \), then there is \( y \in D \cap F_\psi \psi \subseteq F_\psi \) with \( x \leq y \). Thus \( D \in F_\psi^\Delta \) and \( \psi^\Delta(D) = \psi(D \cap F_\psi) \). If \( z \in \psi(D \cap F_\psi) \) and \( z = \psi(d) \) with \( d \in D \), then by assumption, there is \( d' \in D \cap F_\psi \psi \) with \( d \leq d' \). Then \( \psi(d') \geq \psi(d) = z \) and \( \psi(d') \in \psi(D \cap F_\psi) \cap F_\varphi \). Thus \( \psi^\Delta(D) \in F_\varphi^\Delta \) and \( \varphi^\Delta \psi^\Delta(D) = \varphi(\psi(D \cap F_\psi) \cap F_\varphi) = \psi^\Delta(D \cap F_\varphi \psi) \).

It remains to show that the domain of \( \varphi^\Delta \psi^\Delta \) is contained \( F_\varphi^\psi \). If \( D \in F_\varphi^\Delta \) and \( \psi^\Delta(D) \in F_\varphi^\Delta \), then \( D \cap F_\psi \psi \) is cofinal in \( D \) and \( \psi(D \cap F_\psi) \cap F_\varphi \) is cofinal in \( \psi(D \cap F_\psi) \). So if \( x \in D \), then there exists \( y \in D \cap F_\psi \psi \) with \( x \leq y \). Then \( \psi(y) \leq z \) with \( z \in \psi(D \cap F_\psi) \cap F_\varphi \). So \( z = \psi(d) \) with \( d \in D \cap F_\psi \). Then \( d \in D \cap F_\psi \psi \) and \( x \leq y \leq d \) (as \( \psi(y) \leq \psi(d) \) implies \( y \leq d \)). This completes the proof. \( \square \)

**12.13. Definition.** Given any subset \( E \) of \( X \), we will denote by \( E^\Sigma \) the collection of all strings \( \sigma \) in \( X \) such that \( \sigma \subseteq E \). Equivalently

\[
E^\Sigma = \Sigma(X) \cap E^\Delta.
\]

Since every string in \( X \) is clearly asymptotically contained in \( X \), we have that \( X^\Sigma = \Sigma(X) \).

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12.14. Proposition. For every subset $E$ of $X$, one has that $h$ maps $E^\Delta$ into $E^\Sigma$.

Proof. Given $D$ in $E^\Delta$, we have that $h(D) \subseteq E$, by (12.6.i). Since $h(D)$ is a string by (12.3), we conclude that $h(D) \in E^\Sigma$. \hfill $\square$

12.15. Definition. Given any $\varphi$ in $\mathcal{I}_+(X)$, consider the map $\varphi^\Sigma$ from $F^\Sigma_\varphi$ to $E^\Sigma_\varphi$ given by $\varphi^\Sigma = h \circ \varphi^\Delta |_{F^\Sigma_\varphi}$. Diagramatically:

$$
\begin{array}{ccc}
F^\Sigma_\varphi & \longrightarrow & E^\Sigma_\varphi \\
\cap & \uparrow h & \\
F^\Delta_\varphi & \longrightarrow & E^\Delta_\varphi
\end{array}
$$

Our main goal is to prove that the correspondence $\varphi \to \varphi^\Sigma$ is an action of $\mathcal{I}_+(X)$ on $\Sigma(X)$ by partial bijections.

The slightly complex definition of $\varphi^\Sigma$ tends to cause formulas to quickly grow in size. The following technical fact is designed to contain the buildup of formulas by knocking off an “$h$”.

12.16. Lemma. If $\varphi \in \mathcal{I}_+(X)$, then $h\varphi^\Delta = h\varphi^\Delta h$ as partial mappings.

Proof. Suppose that $D \subseteq F_\varphi$. Then $h(D) \subseteq F^\Delta_\varphi$ by (12.6.i) and so the domain of $h\varphi^\Delta$ is contained in the domain of $h\varphi^\Delta h$. Suppose that $D$ is in the domain of $h\varphi^\Delta h$. Then $h(D) \in F^\Delta_\varphi$ and so $h(D) \cap F_\varphi$ is cofinal in $h(D)$. Let $x \in D$. Then since $x \in h(D)$, there is $y \in h(D) \cap F_\varphi$ with $x \leq y$. Then $y \leq z$ with $z \in D$ by definition of $h(D)$. Then $z \leq w$ with $w \in h(D) \cap F_\varphi$, again by cofinality of $h(D) \cap F_\varphi$ in $h(D)$. As $y \leq z \leq w$ and $y, w \in F_\varphi$, we have $z \in F_\varphi$ by convexity of $F_\varphi$. We conclude that $x \leq z \in D \cap F_\varphi$ and so $D \in F^\Delta_\varphi$. Thus the partial mappings $h\varphi^\Delta$ and $h\varphi^\Delta h$ have the same domain.

Now we prove if $D \in F^\Delta_\varphi$, then

$$h(\varphi(F_\varphi \cap h(D))) = h(\varphi(F_\varphi \cap D)).$$

Since $h(D)$ contains $D$, the inclusion “$\supseteq$” is evident. In order to prove the reverse inclusion, let $x$ be any element of $h(\varphi(F_\varphi \cap h(D)))$. Then, there exists some $y$ in $F_\varphi \cap h(D)$, such that $x \leq \varphi(y)$. We may then pick $z$ in $D$, with $y \leq z$.

Since $D \subseteq F_\varphi$, we have that $F_\varphi \cap D$ is cofinal in $D$, so there exists $w$ such that $z \leq w \in F_\varphi \cap D$. It follows that $y \leq w$, and since both $y$ and $w$ lie in $F_\varphi$, we have

$$x \leq \varphi(y) \leq \varphi(w) \in \varphi(F_\varphi \cap D),$$

whence $x \in h(\varphi(F_\varphi \cap D))$. \hfill $\square$

12.17. Proposition. For every $\varphi$ in $\mathcal{I}_+(X)$, one has that $\varphi^\Sigma$ is a bijective mapping from $F^\Sigma_\varphi$ to $E^\Sigma_\varphi$, and its inverse is given by $(\varphi^{-1})^\Sigma$.
Proof. Initially notice that \( F_{\varphi^{-1}} = E_\varphi \), and \( E_{\varphi^{-1}} = F_\varphi \), whence \( (\varphi^{-1})^\Sigma \) is a map from \( E_\varphi^\Sigma \) to \( F_\varphi^\Sigma \), as expected. Next observe that \( h((\varphi^{-1})^\Delta h_\varphi^\Delta = h(\varphi^{-1})^\Delta \varphi^\Delta = h(\varphi^{-1})^\Delta = h(1_{F_\varphi})^\Delta \) by (12.16) and (12.12). But if \( \sigma \) is a string in \( F_\varphi^\Sigma \), then \( h((\varphi^{-1})^\Delta \varphi^\Delta = h((\varphi^{-1})^\Delta) \varphi^\Delta = h(\varphi^{-1})^\Delta \varphi^\Delta = (\varphi^{-1})^\Delta = 1_{F_\varphi} \). A dual argument completes the proof. □

The result above shows that \( \varphi^\Sigma \) is indeed an element of the inverse semigroup \( I(\Sigma(X)) \), and we next plan to prove that the correspondence

\[ \varphi \in I_+(X) \mapsto \varphi^\Sigma \in I(\Sigma(X)), \]

is an inverse semigroup homomorphism.

12.18. Proposition. For every \( \varphi \) and \( \psi \) in \( I_+(X) \), one has that \( (\varphi \psi)^\Sigma = \varphi^\Sigma \psi^\Sigma \). That is, the assignment \( I_+(X) \rightarrow I(\Sigma(X)) \) given by \( \varphi \mapsto \varphi^\Sigma \) is a homomorphism of inverse semigroups.

Proof. We compute

\[ \varphi^\Sigma \psi^\Sigma = (h_\varphi^\Delta)|_{\Sigma(X)}(h_\psi^\Delta)|_{\Sigma(X)} = (h_\varphi^\Delta h_\psi^\Delta)|_{\Sigma(X)} = h_\varphi^\Delta h_\psi^\Delta \psi^\Sigma \]

by (12.16) and (12.12). This completes the proof. □

Our main result is now an easy consequence of (12.17) and (12.18):

12.19. Theorem. Let \( X \) be a set equipped with a transitive and reflexive relation “\( \leq \)”. Then there exists a semigroup homomorphism

\[ \varphi \in I_+(X) \mapsto \varphi^\Sigma \in I(\Sigma(X)) \]

such that, for all \( \varphi \) in \( I_+(X) \), and for every string \( \sigma \) in \( F_\varphi^\Sigma \), one has that

\[ \varphi^\Sigma(\sigma) = h(\varphi(F_\varphi \cap \sigma)). \]

As an application, let \( S \) be a 0-left cancellative semigroup, and let \( S' = S \setminus \{0\} \). Given \( s \) and \( t \) in \( S' \), recall that \( s \) is said to divide \( t \), in symbols \( s \mid t \), if there exists \( u \) in \( \tilde{S} = S \cup \{1\} \) such that \( su = t \). Setting

\[ s \leq t \iff s \mid t, \]

we have that \( S' \) becomes a (possibly not anti-symmetric) ordered set. For each \( s \) in \( S \), recall that

\[ F_s = \{ x \in S' : sx \neq 0 \}, \quad \text{and} \quad E_s = sS \setminus \{0\}, \]

and that

\[ \theta_s : x \in F_s \mapsto sx \in E_s. \]

12.20. Proposition. For every \( s \) in \( S \), one has that \( F_s \) and \( E_s \) are convex in \( S' \), and \( \theta_s \) is order preserving.
Proof. Given \( x, y, z \in S' \), with \( x \leq y \leq z \), and \( x, z \in F_s \), pick \( u \) in \( \tilde{S} \) such that \( yu = z \). Then

\[ syu = sz \neq 0, \]

so \( sy \neq 0 \), whence \( y \in F_s \). This shows that \( F_s \) is convex. Notice that \( x \) did not play any role above, which means that \( F_s \) is in fact a hereditary set.

Now let \( x, y, z \in S' \), with \( x \leq y \leq z \), and \( x, z \in E_s \). We may then pick \( u \) in \( \tilde{S} \) such that \( xu = y \). We may also write \( x = st \), for some \( t \) in \( S \), so

\[ y = xu = stu, \]

whence \( y \in E_s \). This shows that \( E_s \) is convex. Notice that \( z \) did not play any role above, which means that \( E_s \) is in fact a hereditary set for the reverse order.

Given \( x \) and \( y \) in \( F_s \), with \( x \leq y \), pick \( u \) in \( \tilde{S} \) such that \( xu = y \). Then

\[ \theta_s(x)u = sxu = sy = \theta_s(y), \]

whence \( \theta_s(x) \leq \theta_s(y) \).

Instead of assuming that \( x \leq y \), suppose that \( \theta_s(x) \leq \theta_s(y) \), so we may find \( u \) in \( \tilde{S} \), such that \( \theta_s(x)u = \theta_s(y) \), which is to say that \( sxu = sy \), whence \( xu = y \) by virtue of 0-left cancellativity, proving that \( x \leq y \). This concludes the proof that \( \theta_s \) is order preserving. \( \square \)

The result above implies that each \( \theta_s \) lies in \( \mathcal{I}_+(S') \), and hence the inverse semigroup generated by the \( \theta_s \), namely \( \mathcal{H}(S) \), is a subset of \( \mathcal{I}_+(S') \).

We may then view the composition

\[ \rho : \mathcal{H}(S) \hookrightarrow \mathcal{I}_+(S') \rightarrow \mathcal{I}(\Sigma(S')), \]

where the rightmost arrow is the representation defined by (12.19), as a representation of \( \mathcal{H}(S) \) on \( \Sigma(S') \).

In the context of 0-left cancellative semigroups notice that the notion of strings, as introduced in (10.1), coincides with the concept defined in (12.1.iv) for the above order relation on \( S' \). In other words,

\[ S^* = \Sigma(S'), \]

so \( \rho \) is seen to be a representation of \( \mathcal{H}(S) \) on \( S^* \).

12.21. Corollary. Regarding the representation \( \rho \) above, one has that

\[ \rho(\theta_s) = \theta_s^*, \]

for every \( s \) in \( S \).

Proof. We must first prove that \( \theta_s^* \) and \( \rho(\theta_s) \) share domains an ranges. Recall that the domain of \( \theta_s^* \) is given by

\[ F_s^* = \{ \sigma \in S^* : s\sigma \not\geq 0 \}, \]

and its range is the set

\[ E_s^* = \{ \sigma \in S^* : \sigma \cap sS \neq \emptyset \}. \]
On the other hand, the domain and range of \( \rho(\theta_s) = \theta_s^\Sigma \) are respectively given by \( F_s^\Sigma \) and \( E_s^\Sigma \).

In order to prove that \( F_s^* = F_s^\Sigma \), let \( \sigma \) be a string in \( F_s^* \). Then clearly \( \sigma \subseteq F_s \), whence \( \sigma \subseteq F_s^\Sigma \). Conversely, if \( \sigma \in F_s^\Sigma \), suppose by contradiction that there exists \( x \) in \( \sigma \) such that \( sx = 0 \). By assumption there exists some \( y \) in \( F_s \cap \sigma \), such that \( x \leq y \), so we may find \( u \) in \( \tilde{S} \), such that \( xu = y \). Therefore

\[
sy = sxu = 0,
\]

contradicting the fact that \( y \in F_s \). This proves that \( \sigma \) lies in \( F_s^* \), and hence that \( F_s^* = F_s^\Sigma \).

We will next prove that \( E_s^* = E_s^\Sigma \), so pick any \( \sigma \) in \( E_s^* \). Then there exists some \( x \) in \( \sigma \) of the form \( x = st \), with \( t \in S \). Given any \( y \) in \( \sigma \), we may use the fact that \( \sigma \) is directed to find some \( z \) in \( \sigma \), such that \( x, y \leq z \). Therefore there exists \( u \) in \( \tilde{S} \), such that \( xu = z \), so

\[
z = xu = stu \in E_s \cap \sigma.
\]

Since \( y \leq z \), this proves that \( E_s \cap \sigma \) is cofinal in \( \sigma \), whence \( \sigma \in E_s^\Sigma \). This proves that \( E_s^* \subseteq E_s^\Sigma \). Conversely, given \( \sigma \) in \( E_s^\Sigma \), we have that \( E_s \cap \sigma \) is cofinal in \( \sigma \), so in particular \( E_s \cap \sigma \) is nonempty, and this clearly implies that \( \sigma \in E_s^* \). This concludes the proof that \( E_s^* = E_s^\Sigma \).

Given any \( \sigma \in F_s^\Sigma = F_s^* \), we have seen that \( \sigma \subseteq F_s \), whence

\[
\rho(\theta_s)(\sigma) = \theta_s^\Sigma(\sigma) = h(\theta_s(F_s \cap \sigma)) = h(\theta_s(\sigma)) = \{ t \in S' : t \leq x, \text{ for some } x \in s\sigma \} =
\]

\[
\{ t \in S' : t \mid sr, \text{ for some } r \in \sigma \} = \theta_s^*(\sigma).
\]

This completes the proof. \( \square \)

The big conclusion of all this is as follows:

**12.22. Proposition.** Let \( S \) be a 0-left cancellative semigroup. Then there exists a unique representation \( \rho \) of \( \mathcal{S}(S) \) on \( S^* \), such that the following diagram commutes.

\[
\begin{array}{ccc}
S & \xrightarrow{\theta^*} & \mathcal{I}(S^*) \\
\theta \downarrow & & \downarrow \rho \\
\mathcal{S}(S) & &
\end{array}
\]

Observing that a homomorphism of inverse semigroups must restrict to the corresponding idempotent semilattices, we obtain the following:

**12.23. Corollary.** Let \( S \) be a 0-left cancellative semigroup. Then there exists a semilattice representation

\[
\varepsilon: \mathcal{E}(S) \rightarrow \mathcal{P}(S^*),
\]

such that

\[
\varepsilon(\theta_u(F_\Lambda^\theta)) = \theta_u^*(F_\Lambda^*),
\]

whenever \( \Lambda \) is a finite subset of \( \tilde{S} \) intersecting \( S \) nontrivially, and \( u \in \Lambda \).
Proof. Identifying the idempotent semilattices of $\mathcal{H}(S)$ and $\mathcal{I}(S^\ast)$ with $\mathcal{E}(S)$ and $\mathcal{P}(S^\ast)$, respectively, it is enough to take $\varepsilon$ to be the restriction of the representation $\rho$ of (12.22) to $\mathcal{E}(S)$.

Observing that $E^\theta_r = \theta_r(F^\theta_r)$, notice that

$$
\varepsilon(E^\theta_r) = \theta^*_r(F^*_r) = E^*_r.
$$

The subset of $S^\ast$ consisting of all open strings was shown in (11.4) to be invariant under both $\theta^*_r$ and $\theta^*_r^{-1}$, for every $r$ in $S$. As an immediate consequence we thus obtain the following:

12.24. Proposition. The set of all open strings in $S$ is invariant under the representation of $\mathcal{H}(S)$ on $S^\ast$ described in (12.22).

We now prove that the mapping $r \mapsto \delta_r$ from $S' = S \setminus \{0\}$ to $\Sigma(S)$ is covariant with respect to $\rho$. More precisely, we prove the following.

12.25. Theorem. Let $\varphi \in \mathcal{H}(S)$ and $r \in S' = S \setminus \{0\}$. Then $r \in F_{\varphi}$ if and only if $\delta_r \in F^\varphi$. Moreover, if $r \in F_{\varphi}$, then $\delta_{\varphi(r)} = \varphi^\Sigma(\delta_r)$.

Proof. Suppose first that $r \in F_{\varphi}$. Thence since $r$ is the maximum element of $\delta_r$, clearly $\delta_r \subseteq F_{\varphi}$ and so $\delta_r \in F^\varphi$. Conversely, if $\delta_r \in F^\varphi$, then $\delta_r \cap F_{\varphi}$ is cofinal in $\delta_r$. But then there exists $s \in \delta_r \cap F_{\varphi}$ such that $r \leq s$. But also, $s \leq r$. It follows since $F_{\varphi}$ is convex that $r \in F_{\varphi}$.

Assume that $r \in F_{\varphi}$. By definition $\varphi^\Sigma(\delta_r) = h(\varphi(F_{\varphi} \cap \delta_r))$. So $\varphi(r) \in \varphi^\Sigma(\delta_r)$, whence $\delta_{\varphi(r)} \subseteq \varphi^\Sigma(\delta_r)$. If $s \in \varphi^\Sigma(\delta_r)$, then $s \leq \varphi(t)$ with $t \in \delta_r \cap F_{\varphi}$. Since $\varphi$ is order preserving, $s \leq \varphi(t) \leq \varphi(r)$ and so $s \in \delta_{\varphi(r)}$. This completes the proof.

13. Unbounded strings and backward invariance.

Recall from (10.21) that, for semigroups which are categorical at zero, the space $S^\infty$ of maximal strings is invariant under every $\theta^*_r$. However, even though semigroups obtained from subshifts (see (6.2)) are not necessarily categorical at zero, the above invariance may be shown to hold.

Here we would like to present a general result about invariance of $S^\infty$ under $\theta^*_r$, which does not rely on the property of being categorical at zero and hence applies to subshift semigroups.

Let $S$ be a 0-left cancellative semigroup, fixed throughout this section.

13.1. Definition. Let $N$ be a totally ordered set. An $N$-valued length function for $S$ is a function

$$
\ell : S' = S \setminus \{0\} \to N,
$$

such that for every $r, s, t$ in $S'$, one has that

(i) if $s \mid t$, then $\ell(s) \leq \ell(t)$,
(ii) if $r \mid st \neq 0$, and $\ell(r) \leq \ell(s)$, then $r \mid s$.
For an example consider the semigroup $S = L \cup \{0\}$, given by a language $L$ invariant under prefixes and suffixes, as in (6.1). Letting

$$\ell: S' = L \to \mathbb{N}$$

be defined by setting $\ell(s)$ equal to the usual word-length of $s$, it is easy to see that $\ell$ is a length function in the sense of (13.1).

**13.2. Definition.** Let $\ell$ be an $\mathbb{N}$-valued length function on $S$.

(i) We will say that a subset $X \subseteq S'$ is bounded (relative to $\ell$), provided there exists $n_0 \in \mathbb{N}$, such that $\ell(s) \leq n_0$, for all $s$ in $X$.

(ii) We will say that $\ell$ is homogeneous if, for every $r$ in $S$, and for every bounded subset $X \subseteq F^\theta_r$, one has that $rX$ is bounded.

If $\ell$ is the word-length mentioned above, one has that

$$\ell(rs) = \ell(r) + \ell(s),$$

whenever $rs \neq 0$, and from this it easily follows that $\ell$ is homogeneous.

**13.3. Lemma.** Let $S$ be a 0-left cancellative semigroup equipped with a homogeneous length function $\ell: S' \to \mathbb{N}$. Given $r$ in $S$, let $\sigma$ be an unbounded string in $E^*_r$. Then

(i) $\theta^*_r^{-1}(\sigma)$ is unbounded,

(ii) if $\mu$ is a string with $\theta^*_r^{-1}(\sigma) \subseteq \mu$, then $\mu \in F^*_r$,

(iii) if $\sigma$ is maximal, then $\theta^*_r^{-1}(\sigma)$ is also maximal.

**Proof.** (i) Supposing by contradiction that

$$\tau := \theta^*_r^{-1}(\sigma) = r^{-1} \ast \sigma$$

is bounded, then $r\tau$ is also bounded by homogeneity. Noticing that $\sigma = r \ast \tau$, by (10.9), and hence that $\sigma$ is is the hereditary closure of $r\tau$, it would follow that $\sigma$ is also bounded, a contradiction. This shows that $\theta^*_r^{-1}(\sigma)$ is unbounded.

(ii) Arguing again by contradiction, suppose that

$$\theta^*_r^{-1}(\sigma) = r^{-1} \ast \sigma \subseteq \mu,$$

and that $\mu$ is not in $F^*_r$, so there exists some $x \in \mu$, with $rx = 0$.

By (i) we have that $r^{-1} \ast \sigma$ is unbounded, so in particular $\ell(x)$ cannot be a bound for $r^{-1} \ast \sigma$. Therefore there exists some $y \in r^{-1} \ast \sigma$, such that $\ell(y) \not\leq \ell(x)$, and because $\mathbb{N}$ is totally ordered this means that $\ell(y) > \ell(x)$.

Since $x, y \in \mu$, there are $u, v \in \tilde{S}$, such that $xu = yv \in \mu$. It follows that $x \mid yv$, and hence by (13.1.ii) we deduce that $x \mid y$. So $y = xw$, for some $w$ in $\tilde{S}$, whence

$$0 = rxw = ry \in \sigma,$$

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a contradiction. This proves that \( \mu \in F_r^* \).

(iii) Suppose that \( \mu \) is a string with

\[ \theta_r^{*-1}(\sigma) \subseteq \mu. \]

We then have by (ii) that \( \mu \in F_r^* \), so

\[ \sigma = \theta_r^{*}(\theta_r^{*-1}(\sigma)) \subseteq \theta_r^{*}(\mu), \]

and then \( \sigma = \theta_r^{*}(\mu) \), by maximality, whence

\[ \theta_r^{*-1}(\sigma) = \theta_r^{*-1}(\theta_r^{*}(\mu)) = \mu, \]

thus proving that \( \theta_r^{*-1}(\sigma) \) is maximal. \( \square \)

13.4. Corollary. Let \( S \) be a 0-left cancellative semigroup, admitting a homogeneous length function relative to which every maximal string is unbounded. Then \( S^\infty \) is an invariant subset of \( S^* \) under the natural action of \( I(S^*, \theta^*) \).

Proof. Follows immediately from the above result as in (10.21). \( \square \)

If \( \mathcal{X} \) is a subshift, it is easy to see that the maximal strings in \( S^*_\mathcal{X} \) are unbounded, so the result above applies even though such semigroups are not always categorical at zero.
PART TWO

Semilattices

Given a 0-left cancellative semigroup $S$, we will now concentrate on studying the semilattice $\mathfrak{C}(S)$ of constructible sets, focusing in particular on the question of determining its spectrum.

We begin by discussing some general aspects of semilattices.

14. Preliminaries on semilattices.

Let $\mathcal{E}$ be a semilattice with zero. By a character on $\mathcal{E}$ [11: 12.4] we mean any nonzero map

$$\varphi : \mathcal{E} \to \{0, 1\},$$

such that $\varphi(0) = 0$, and $\varphi(xy) = \varphi(x)\varphi(y)$, for all $x$ and $y$ in $\mathcal{E}$. The set of all characters on $\mathcal{E}$, usually denoted$^6$ by $\hat{\mathcal{E}}$, is called the spectrum of $\mathcal{E}$. It is well known that $\hat{\mathcal{E}}$ is a locally compact, Hausdorff topological space when equipped with the product topology induced from $\{0, 1\}^\mathcal{E}$.

The topology on $\hat{\mathcal{E}}$ therefore admits a basis of open sets of the form

$$D(f_1) \cap \cdots \cap D(f_m) \cap D(e_1)^c \cap \cdots \cap D(e_n)^c,$$

with $f_1, \ldots, f_m, e_1, \ldots, e_n \in \mathcal{E}$, where

$$D(e) := \{\varphi \in \mathcal{E} : \varphi(e) = 1\},$$

and $X^c$ is the complement of $X$ in $\hat{\mathcal{E}}$. Setting $e = f_1 \ldots f_m$, and observing that

$$\varphi(f_1) = \cdots = \varphi(f_m) = 1 \iff \varphi(e) = 1,$$

we see that the set in (14.1) coincides with

$$D(e) \cap D(e_1)^c \cap \cdots \cap D(e_n)^c,$$

so we may take the above to be the general form of a basic open set. Note that one may assume without loss of generality that $e_i \leq e$, for $i = 1, \ldots, n$.

A filter on $\mathcal{E}$ is by definition [11: 12.1 & 12.2] a nonempty subset $\xi \subseteq \mathcal{E}$, such that

- $0 \notin \xi$,
- $x, y \in \xi \Rightarrow xy \in \xi$

$^6$ The spectrum of $\mathcal{E}$ is however denoted by $\hat{\mathcal{E}}_0$ in [11], which incidentally is our basic reference for the theory of semilattices.
• $x \geq y \in \xi \Rightarrow x \in \xi$,
for every $x$ and $y$ in $\mathcal{E}$.

Recall from [11] that there is a one-to-one correspondence between characters and filters, as follows: given a character $\varphi$, the corresponding filter is given by

$$\xi = \{ x \in \mathcal{E} : \varphi(x) = 1 \}.$$ 

On the other hand, given a filter $\xi$, the corresponding character is given by

$$\varphi(x) = [x \in \xi], \quad \forall x \in \mathcal{E},$$

where brackets stand for Boolean value.

By definition an *ultrafilter* is a filter which is not properly contained in any other filter. If a character $\varphi$ corresponds to an ultrafilter, then $\varphi$ is called an *ultracharacter*. The set of all ultracharacters is denoted $\hat{\mathcal{E}}_\infty$.

It is well known [11: 12.3] that a filter $\xi$ is an ultrafilter if and only if $e \notin \xi$ implies that $ef = 0$ for some $f \in \xi$. Also, every filter is contained in an ultrafilter.

The topology on the set of filters, inherited from the product topology via the above correspondence of filters and characters, still has basic sets of the form (14.3), except that now we should interpret (14.2) as

$$D(e) := \{ \xi : e \in \xi \}.$$ 

Given any subset $F \subseteq \mathcal{E}$, we shall say that a subset $Z \subseteq F$ is a *cover for $F$* [11: 11.5] if, for every nonzero $x \in F$, there exists $z \in Z$ such that $zx \neq 0$. If $y \in \mathcal{E}$ and $Z$ is a cover for $F := \{ x \in \mathcal{E} : x \leq y \}$, we will say that $Z$ is a *cover for $y$*.

Given finite subsets $X, Y \subseteq \mathcal{E}$, we shall denote by $\mathcal{E}^{X,Y}$ the subset of $\mathcal{E}$ given by

$$\mathcal{E}^{X,Y} = \{ z \in \mathcal{E} : z \leq x, \forall x \in X, \text{ and } zy = 0, \forall y \in Y \}.$$ 

A character $\varphi$ of $\mathcal{E}$ is said to be *tight* [11: 11.6] if, for all finite subsets $X, Y \subseteq \mathcal{E}$, and for every finite cover $Z$ for $\mathcal{E}^{X,Y}$, one has that

$$\bigvee_{z \in Z} \varphi(z) = \bigwedge_{x \in X} \varphi(x) \wedge \bigwedge_{y \in Y} \neg \varphi(y).$$
In view of the fact that characters are nonzero by definition, and hence satisfy [11: 11.7.(i)], one has by [11: 11.8] that a character \( \varphi \) is tight if and only if, for every \( x \in \mathcal{E} \) and for every finite cover \( Z \) for \( x \), one has that

\[
\bigvee_{z \in Z} \varphi(z) = \varphi(x).
\]

Every ultracharacter is necessarily tight [11: 12.7], and in fact the set \( \hat{\mathcal{E}}_{\text{tight}} \) formed by the tight characters coincides with the closure of \( \hat{\mathcal{E}}_{\infty} \) in \( \hat{\mathcal{E}} \) [11: 12.9].

If \( \xi_1 \) and \( \xi_2 \) are two filters in \( \mathcal{E} \), and if \( \varphi_1 \) and \( \varphi_2 \) are the corresponding characters, observe that

\[
\xi_1 \subseteq \xi_2 \iff \varphi_1 \leq \varphi_2. \tag{14.4}
\]

We note that characters are functions taking values in the ordered set \( \{0, 1\} \), and that the order among characters mentioned above is supposed to mean pointwise order.

Based on this one may give a characterization of ultracharacters which does not explicitly mention the associated filters:

**14.5. Proposition.** Let \( \mathcal{E} \) be a semilattice and let \( \varphi \in \hat{\mathcal{E}} \). Then \( \varphi \) is an ultracharacter if and only if,

\[
\forall \psi \in \hat{\mathcal{E}}, \ \varphi \leq \psi \Rightarrow \varphi = \psi.
\]

*Proof.* Follows immediately from (14.4). \( \square \)

Let \( \mathcal{E} \) be a semilattice with zero and let \( J \) be an ideal in \( \mathcal{E} \) in the sense that \( J \mathcal{E} \subseteq \mathcal{E} \). For every filter \( \eta \) in \( J \), consider the filter in \( \mathcal{E} \) generated by \( \eta \), namely

\[
\langle \eta \rangle_{\mathcal{E}} = \{ x \in \mathcal{E} : \exists y \in \eta, \ y \leq x \}.
\]

On the other hand, for every filter \( \xi \) in \( \mathcal{E} \), such that \( \xi \cap J \neq \emptyset \), it is evident that \( \xi \cap J \) is a filter in \( J \).

**14.6. Proposition.** The correspondence

\[
i : \eta \in \hat{J} \mapsto \langle \eta \rangle_{\mathcal{E}} \in \hat{\mathcal{E}},
\]

is a homeomorphism from \( \hat{J} \) onto the open subset of \( \hat{\mathcal{E}} \) given by

\[
U = \{ \xi \in \hat{\mathcal{E}} : \xi \cap J \neq \emptyset \}.
\]

Moreover, the inverse of the above correspondence is given by

\[
p : \xi \in U \mapsto \xi \cap J \in \hat{J}.
\]

Furthermore, \( i \) and \( p \) are order isomorphisms where we order filters by inclusion.
Proof. Given any filter $\eta$ in $\hat{J}$, it is evident that

$$\eta \subseteq \langle \eta \rangle_E \cap J,$$  \hspace{1cm} (14.6.1)

so $\langle \eta \rangle_E \cap J$ is nonempty and hence we see that $\langle \eta \rangle_E \in U$.

Observe that (14.6.1) is in fact an equality, since for every $x \in \langle \eta \rangle_E \cap J$, there exists $y \in \eta$, with $y \leq x$, so obviously $x \in \eta$. The fact that $\eta = \langle \eta \rangle_E \cap J$ may then be expressed as $\eta = p(i(\eta))$, so the composition $p \circ i$ is the identity on $J$.

Given $\xi$ in $U$, it is easy to see that

$$\langle \xi \cap J \rangle_E \subseteq \xi,$$  \hspace{1cm} (14.6.2)

and we again claim that this inclusion is an equality. In fact, given any $x$ in $\xi$, choose any $y$ in the nonempty set $\xi \cap J$, and notice that that

$$x \geq xy \in \xi \cap J,$$

so $x \in \langle \xi \cap J \rangle_E$. This proves that $\langle \xi \cap J \rangle_E = \xi$ or, equivalently, that $i \circ p$ is the identity on $U$. Obviously, $i$ and $p$ are order preserving.

To see that $U$ is open, notice that we may write it as

$$U = \bigcup_{x \in J} \{ \xi \in \hat{E} : x \in \xi \},$$

which is a union of basic open subsets of $\hat{E}$.

We check now that $i$ and $p$ are continuous. We will subscript $D$ by the space we are working in. If $\eta \in \hat{J}$, then a basic neighborhood of $i(\eta)$ is of the form

$$D_{\hat{E}}(e) \cap D_{\hat{E}}(e_1)^c \cap \cdots \cap D_{\hat{E}}(e_n)^c$$

with $e, e_1, \ldots, e_n \in \mathcal{E}$. Since $e \in i(\eta)$, there exists $f \in \eta$ with $f \leq e$. Let $f_i = fe_i$; note that $f_i \in J$ and $f_i \notin \eta$. Then $\eta \in D_j(f) \cap D_j(f_1)^c \cap \cdots \cap D_j(f_n)^c$. Moreover, if $\xi \in D_j(f) \cap D_j(f_1)^c \cap \cdots \cap D_j(f_n)^c$, then $e \in i(\xi)$, but $e_1, \ldots, e_n \notin i(\xi)$. This shows that $i$ is continuous.

Next, if $\xi \in U$ and

$$V = D_j(e) \cap D_j(e_1)^c \cap \cdots \cap D_j(e_n)^c$$

with $e, e_1, \ldots, e_n \in J$ is a basic neighborhood of $p(\xi)$ in $\hat{J}$, then $\xi \in D_{\hat{E}}(e) \cap D_{\hat{E}}(e_1)^c \cap \cdots \cap D_{\hat{E}}(e_n)^c$ and this neighborhood clearly maps into $V$ under $p$. This completes the proof that $i$ and $p$ are homeomorphisms. \hfill \Box

As mentioned earlier, filters are in one-to-one correspondence with characters. Seeing things from the latter point of view, one has:
14.7. Proposition. Identifying \( \hat{J} \) with its image in \( \hat{E} \) under the map \( i \) of (14.6), one has that

\[
\hat{J} = \{ \varphi \in \hat{E} : \varphi|_J \neq 0 \}.
\]

Proof. Given \( \varphi \) in \( \hat{E} \), let \( \xi \) be the associated filter, namely

\[
\xi = \{ x \in E : \varphi(x) = 1 \}.
\]

Thus, a necessary and sufficient for \( \varphi|_J \) to be nonzero is that \( \xi \cap J \neq \emptyset \), which is to say that \( \xi \) lies in the set \( U \) of (14.6), namely the range of \( i \). This completes the proof. \( \square \)

14.8. Proposition. Let \( \eta \) be a filter in \( J \). Then \( \eta \) is an ultrafilter if and only if \( \langle \eta \rangle_E \) is an ultrafilter in \( \hat{E} \).

Proof. Observe that the open set \( U \) in (14.6) is an upper set. Hence a filter in \( U \) is maximal in \( U \) if and only if it is an ultrafilter. The proposition now follows because \( i \) and \( p \) in (14.6) are order isomorphisms. \( \square \)

15. Representations of semilattices.

Given a semilattice \( E \), and and given any representation

\[
\pi : E \to \mathcal{I}(\Omega),
\]

on some set \( \Omega \), it is easy to see that the range of \( \rho \) must in fact be contained in the semilattice \( \mathcal{P}(\Omega) \). This motivates the following:

15.1. Definition. Let \( E \) be a semilattice and let \( B \) be a Boolean algebra\(^7\). A map \( \pi : E \to B \) will be called a representation of \( E \) in \( B \), provided \( \pi(0) = 0 \), and \( \pi(xy) = \pi(x) \wedge \pi(y) \), for every \( x \) and \( y \) in \( E \).

In case \( B \) coincides with the Boolean algebra \( \mathcal{P}(\Omega) \), for a given set \( \Omega \), the above concept therefore reduces to the notion of a representation of \( E \) on \( \Omega \), as defined in (2.1).

If \( E \) is a subsemilattice of \( \mathcal{P}(\Omega) \) (always supposed to include the zero element of \( \mathcal{P}(\Omega) \), namely the empty set), then the inclusion map

\[
E \hookrightarrow \mathcal{P}(\Omega)
\]

is evidently a representation of \( E \) on \( \Omega \).

Suppose we are given a semilattice \( E \), and a representation \( \pi \) of \( E \) on a set \( \Omega \). For each \( \omega \) in \( \Omega \), consider the mapping \( \varphi^\pi_\omega : E \to \{0, 1\} \), defined by

\[
\varphi^\pi_\omega (x) = [\omega \in \pi(x)], \quad \forall x \in E,
\]

(15.2)

where brackets stand for Boolean value.

\(^7\) We do not require Boolean algebras to have a top element: for us they have meets, joins, a bottom element and relative complements.
It is clear that \( \varphi^\pi_\omega \) is a multiplicative map, so it is a character as long as it is nonzero. It is moreover clear that \( \varphi^\pi_\omega \) is nonzero if and only if \( \omega \) is in the essential subset \( \Omega^\# \) for \( \pi \). Therefore

\[
\hat{E}_\Omega := \{ \varphi^\pi_\omega : \omega \in \Omega \setminus \{0\} \} = \{ \varphi^\pi_\omega : \omega \in \Omega^\# \}
\] (15.3)

is a subset of the spectrum \( \hat{E} \) of \( E \).

We would now like to discuss the question of how big is \( \hat{E}_\Omega \) within \( \hat{E} \). The answer will of course depend on \( \pi \) since, when \( \pi \) is the identically zero map, for instance, one should not expect \( \hat{E}_\Omega \) to be very big at all. Based on similar results pertaining to other representation theories, such as of C*-algebras, one might expect that \( \hat{E}_\Omega \) is dense in \( \hat{E} \) when \( \pi \) is injective, but this is unfortunately not true.

For example, if \( \Omega = \{1, 2, 3\} \), \( E = \mathcal{P}(\Omega) \), and \( \pi \) is the identity map from \( E \) to \( \mathcal{P}(\Omega) \), then \( \hat{E} \) is finite and the character \( \varphi \) defined by

\[
\varphi(X) = [\{1, 2\} \subseteq X], \quad \forall X \in E,
\]

is neither in \( \hat{E}_\Omega \), nor in its closure.

The next result will give a measure of the size of \( \hat{E}_\Omega \), provided \( \pi \) does not send a nonzero element to the empty set. Its proof is deceptively simple, yet the result is significant.

**15.4. Proposition.** Let \( \pi \) be a representation of the semilattice \( E \) on a set \( \Omega \). If \( \pi(x) \) is nonempty for every nonzero \( x \) in \( E \), then the closure of \( \hat{E}_\Omega \) in \( \hat{E} \) contains all tight characters of \( E \).

**Proof.** Since the ultracharacters are dense in the space of tight characters, it suffices to show that each neighborhood of an ultracharacter \( \psi \) intersects \( \hat{E}_\Omega \). As already mentioned, a basic neighborhood of \( \psi \) is given by \( D(e) \), with \( 0 \neq e \in E \). Let \( \omega \in \pi(e) \). Then \( \varphi^\pi_\omega(e) = 1 \) and so \( \varphi^\pi_\omega \in D(e) \). This completes the proof. \( \square \)

As an immediate consequence we:

**15.5. Corollary.** Let \( E \) be a subsemilattice of \( \mathcal{P}(\Omega) \). Then every tight character of \( E \) lies in the closure of the set \( \{ \varphi_\omega : \omega \in \Omega \} \setminus \{0\} \), where each \( \varphi_\omega \) is defined by

\[
\varphi_\omega(X) = [\omega \in X], \quad \forall X \in E.
\]

**Proof.** Follows by applying (15.4) to the identity representation \( \text{id} : E \hookrightarrow \mathcal{P}(\Omega) \). \( \square \)

Proposition (15.4) speaks about the abundance of characters of \( E \) obtained by composing the representation \( \pi \) with *principal* characters of \( \mathcal{P}(\Omega) \), namely characters of the form

\[
X \mapsto [\omega \in X].
\]

Given a representation \( \pi \) of \( E \) in a Boolean algebra \( B \), it is therefore interesting to determine which characters \( \varphi \) of \( E \) factor as

\[
\varphi = \chi \circ \pi,
\] (15.6)
for some character $\chi$ of $B$, preserving meets and joins. An obvious necessary condition is that if $x, y_1, \ldots, y_n$ are elements of $E$ with $\pi(x) = \bigvee_{i=1}^n \pi(y_i)$, one must have that $\varphi(x) = \bigvee_{i=1}^n \varphi(y_i)$.

There are some useful equivalent forms of the above condition which will be important in the sequel. First recall that the category of Boolean algebras is equivalent to the category of Boolean rings. (Recall that a Boolean ring is a ring, necessarily commutative, in which every element is idempotent.) If $R$ is a Boolean ring, then the Boolean algebra structure on $R$ is given by $x \land y = xy$, $x \lor y = x + y - xy$ and $x \setminus y = x - y$ where $x \setminus y$ is the relative complement for $y \leq x$. Conversely, if $B$ is a Boolean algebra, the ring structure takes $\land$ as the multiplication and defines addition by $x + y = (x \setminus y) \lor (y \setminus x)$. Boolean algebra characters on $B$ correspond to surjective ring homomorphisms to the two-element field $\mathbb{F}_2$.

15.7. Proposition. Let $\pi$ be a representation of the semilattice $E$ in a Boolean algebra $B$, and let $\varphi$ be a character on $E$. Then the following are equivalent:

(i) for every $x, y_1, \ldots, y_n \in E$, one has that

$$\pi(x) = \bigvee_{i=1}^n \pi(y_i) \Rightarrow \varphi(x) = \bigvee_{i=1}^n \varphi(y_i),$$

(ii) for every $x, y_1, \ldots, y_n \in E$, one has that

$$\pi(x) \leq \bigvee_{i=1}^n \pi(y_i) \Rightarrow \varphi(x) \leq \bigvee_{i=1}^n \varphi(y_i).$$

(iii) for every $x, y_1, \ldots, y_n \in E$ with $y_i \leq x$, for $i = 1, \ldots, n$, one has that

$$\prod_{i=1}^n (\pi(x) - \pi(y_i)) = 0 \Rightarrow \prod_{i=1}^n (\varphi(x) - \varphi(y_i)) = 0$$

for the Boolean ring structures on $B$ and $\{0, 1\}$.

Proof. (i) $\Rightarrow$ (ii): Assuming that $\pi(x) \leq \bigvee_{i=1}^n \pi(y_i)$, we have that

$$\pi(x) = \pi(x) \land \left( \bigvee_{i=1}^n \pi(y_i) \right) = \bigvee_{i=1}^n \pi(x) \land \pi(y_i) = \bigvee_{i=1}^n \pi(xy_i),$$

so we may use (i) to deduce that

$$\varphi(x) = \bigvee_{i=1}^n \varphi(xy_i) = \bigvee_{i=1}^n \varphi(x) \land \varphi(y_i) \leq \bigvee_{i=1}^n \varphi(y_i).$$

8 In the theory of Boolean algebras, in fact also in the theory of lattices, characters are usually assumed to preserve meets and joins, meaning that $\varphi(x \land y) = \varphi(x) \land \varphi(y)$, and $\varphi(x \lor y) = \varphi(x) \lor \varphi(y)$, for all $x$ and $y$. Virtually all characters of Boolean algebras in this work will be supposed to preserve meets and joins but, since we are simultaneously dealing with semilattices and Boolean algebras, we will try to be explicit every time these properties are required of a character. Note that a character of a boolean algebra preserves joins and meets if and only if it is an ultracharacter.
proving (ii).

(ii) ⇒ (i): Notice that if \( \pi(x) = \bigvee_{i=1}^{n} \pi(y_i) \), then for each \( i \), one has that \( \pi(y_i) \leq \pi(x) \), so (ii) implies that \( \varphi(y_i) \leq \varphi(x) \), whence

\[
\bigvee_{i=1}^{n} \varphi(y_i) \leq \varphi(x).
\]

Since the reverse inequality also follows from (ii), the proof is concluded.

(ii) ⇒ (iii): Note that

\[
0 = \prod_{i=1}^{n} (\pi(x) - \pi(y_i)) = \pi(x) \setminus \bigvee_{i=1}^{n} \pi(y_i)
\]

and so \( \pi(x) \leq \bigvee_{i=1}^{n} \pi(y_i) \). Therefore, by (ii), we have that \( \varphi(x) \leq \bigvee_{i=1}^{n} \varphi(y_i) \) and so

\[
\prod_{i=1}^{n} (\varphi(x) - \varphi(y_i)) = \varphi(x) \setminus \bigvee_{i=1}^{n} \varphi(y_i) = 0
\]

establishing (iii).

(iii) ⇒ (ii): Notice that \( \pi(x) \leq \bigvee_{i=1}^{n} \pi(y_i) \) implies that \( \pi(x) \leq \bigvee_{i=1}^{n} \pi(y_i) \). Therefore,

\[
\prod_{i=1}^{n} (\pi(x) - \pi(y_i)) = \pi(x) \setminus \bigvee_{i=1}^{n} \pi(y_i) = 0.
\]

So (iii) implies

\[
0 = \prod_{i=1}^{n} (\varphi(x) - \varphi(y_i)) = \varphi(x) \setminus \bigvee_{i=1}^{n} \varphi(y_i)
\]

and hence

\[
\varphi(x) \leq \bigvee_{i=1}^{n} \varphi(y_i) \leq \bigvee_{i=1}^{n} \varphi(y_i)
\]

as required. \(\square\)

The frequency with which we will use this condition largely justifies giving it a name:

15.8. Definition. Let \( \pi \) be a representation of the semilattice \( \mathcal{E} \) in a Boolean algebra \( \mathcal{B} \), and let \( \varphi \) be a character on \( \mathcal{E} \). We will say that \( \varphi \) is tight relative to \( \pi \), or simply \( \pi \)-tight, provided it satisfies the equivalent conditions of (15.7). The set of all \( \pi \)-tight characters on \( \mathcal{E} \) will be written as \( \hat{\mathcal{E}}_{\pi} \).

Observing that a character is \( \pi \)-tight if and only if it satisfies a set of equations, namely those in (15.7.i), it is clear that the set of all \( \pi \)-tight characters is closed in \( \hat{\mathcal{E}} \). For future reference we record this fact below.

15.9. Proposition. Let \( \pi \) be a representation of the semilattice \( \mathcal{E} \) in a Boolean algebra \( \mathcal{B} \). Then \( \hat{\mathcal{E}}_{\pi} \) is a closed subset of \( \hat{\mathcal{E}} \).
As the reader might have already suspected, the necessary condition for the question raised in (15.6) to have a positive answer is also sufficient, as we shall prove next.

It will be convenient to use some rudiments of Stone duality between Boolean rings (algebras) and locally compact Hausdorff spaces with a basis of compact open sets (generalized Stone spaces). Most of this is well known in the unital case and is folklore in the non-unital case but one must take care with morphisms. If \( X \) is a generalized Stone space, then the ring \( C_c(X, \mathbb{F}_2) \) of continuous functions with compact support from \( X \) to \( \mathbb{F}_2 \) (with the discrete topology) is a Boolean ring (note that every Boolean ring has characteristic 2 and so is an \( \mathbb{F}_2 \)-algebra). Conversely, if \( R \) is a Boolean ring, then \( \text{Spec}(R) = \text{hom}(R, \mathbb{F}_2) \) is a generalized Stone space with the topology of pointwise convergence. These two constructions are inverse to each other up to isomorphism. If \( \pi: R \to R' \) is a surjective homomorphism of Boolean rings, then it is obvious that \( \text{Spec}(R') \) embeds as a closed subspace of \( \text{Spec}(R) \). Suppose that \( R' \subseteq R \) is a subring. To show that restriction induces a surjective continuous map \( \text{Spec}(R) \to \text{Spec}(R') \), we need that every character of \( R' \) extends to \( R \). This is well known in the unital case and here is the proof in the non-unital case.

15.10. **Proposition.** Let \( \mathcal{B} \) be a Boolean algebra and \( \mathcal{B}' \) a non-zero subBoolean algebra. Then every ultracharacter of \( \mathcal{B}' \) extends to \( \mathcal{B} \).

**Proof.** We prove this in the language of ultrafilters. Let \( \xi \) be an ultrafilter on \( \mathcal{B}' \). Then \( \eta = \{ x \in \mathcal{B} : \exists y \in \xi, x \geq y \} \) is a filter on \( \mathcal{B} \) with \( \eta \cap \mathcal{B}' = \xi \). It follows by Zorn’s lemma that there is an ultrafilter \( \zeta \) on \( \mathcal{B} \) containing \( \eta \). Then \( \zeta \cap \mathcal{B}' \) is a filter on \( \mathcal{B}' \) containing \( \xi \). Since \( \xi \) is an ultrafilter, we must have \( \zeta \cap \mathcal{B}' = \xi \). This completes the proof. \( \square \)

We remark that if \( \mathcal{E} \) is a semilattice, the semigroup algebra \( \mathbb{F}_2 \mathcal{E} \) is a Boolean ring and each character of \( \mathcal{E} \) extends uniquely to a ring homomorphism \( \mathbb{F}_2 \mathcal{E} \to \mathbb{F}_2 \). Thus \( \text{Spec}(\mathbb{F}_2 \mathcal{E}) \) can be identified with \( \hat{\mathcal{E}} \). We now show that if \( \pi: \mathcal{E} \to \mathcal{B} \) is a representation into a Boolean algebra, then the \( \pi \)-tight characters are precisely those that factor through \( \pi \).

15.11. **Theorem.** Let \( \pi \) be a representation of the semilattice \( \mathcal{E} \) into the Boolean algebra \( \mathcal{B} \) and let \( \varphi \) be a character on \( \mathcal{E} \). Then there exists a character \( \chi \) of \( \mathcal{B} \), preserving meets and joins, such that \( \varphi = \chi \circ \pi \) if and only if \( \varphi \) is \( \pi \)-tight.

**Proof.** We work with Boolean rings. Consider the extension \( \pi: \mathbb{F}_2 \mathcal{E} \to \mathcal{B} \). By (15.10), we may replace \( \mathcal{B} \) by \( \pi(\mathbb{F}_2 \mathcal{E}) \) and so we assume without loss of generality that \( \pi \) is onto. Thus we want to show that \( \varphi: \mathbb{F}_2 \mathcal{E} \to \mathbb{F}_2 \) factors through \( \pi \) if and only if the corresponding character is \( \pi \)-tight. The surjective homomorphism \( \pi \) embeds \( \text{Spec}(\mathcal{B}) \) into \( \hat{\mathcal{E}} \) and so if \( X \) is the set of characters \( \varphi \) with a factorization as \( \chi \circ \pi \), then we can identify \( \mathbb{F}_2 \mathcal{E} \) with \( C_c(\hat{\mathcal{E}}, \mathbb{F}_2) \) and \( \mathcal{B} \) with \( C_c(X, \mathbb{F}_2) \) and, moreover, we can identify \( \pi \) with the restriction map \( f \mapsto f|_X \). The kernel of the restriction map was determined in [33: Proposition 5.2] in a more general setting (one must take \( k = \mathbb{F}_2 \) and \( S = \mathcal{E} \) in that theorem). Namely, ker \( \pi \) is the ideal generated by all products \( \prod_{i=1}^n (e - e_i) \) such that \( e, e_1, \ldots, e_n \in E, e_i \leq e \) and

\[
D(e) \cap D(e_1)^c \cap \cdots \cap D(e_n)^c \cap X = \emptyset.
\]  

(15.11.1)

We claim that (15.11.1) holds if and only if \( \prod_{i=1}^n (\pi(e) - \pi(e_i)) = 0 \). Indeed, if we have \( \prod_{i=1}^n (\pi(e) - \pi(e_i)) = 0 \), then trivially, for any character \( \chi \) of \( \mathcal{B} \), \( \prod_{i=1}^n (\chi(\pi(e)) - \chi(\pi(e_i))) = 0 \).
0 and hence either \( \chi(\pi(e)) = 0 \) or \( \chi(\pi(e)) = 1 = \chi(\pi(e_i)) \) for some \( i \). Thus \( \chi \circ \pi \notin D(e) \cap D(e_1^c) \cap \cdots \cap D(e_n^c) \) and so (15.11.1) holds. Conversely, if \( x = \prod_{i=1}^n (\pi(e) - \pi(e_i)) \neq 0 \), then there is a character \( \chi \) of \( B \) with \( \chi(x) = 1 \) (choose an ultrafilter containing the principal filter generated by \( x \)). Then \( 1 = \chi(x) = \prod_{i=1}^n (\chi(\pi(e)) - \chi(\pi(e_i))) \) and so \( \chi \circ \pi \in D(e) \cap D(e_1^c) \cap \cdots \cap D(e_n^c) \cap X \) and hence (15.11.1) fails.

It now follows from (15.7) and [33: Proposition 5.2] that \( \varphi \) factors through \( \pi \) if and only if it is \( \pi \)-tight. \qed

Notice that the above proof shows that if \( \pi: E \to B \) is a homomorphism such that \( \pi(E) \) generates \( B \) as a Boolean algebra, then \( \text{Spec}(B) \) is homeomorphic to \( \hat{E}_\pi \). The following special case will be used repeatedly.

15.12. Theorem. Let \( \pi \) be a representation of the semilattice \( E \) on a set \( \Omega \), and let \( \varphi \) be a \( \pi \)-tight character on \( E \). Then there exists a character \( \chi \) of \( P(\Omega) \), preserving meets and joins, such that \( \varphi = \chi \circ \pi \).

We shall now be interested in representing semilattices in associative algebras.

15.13. Definition. Let \( E \) be a semilattice and let \( A \) be an associative algebra over a field \( K \). A mapping \( \pi: E \to A \) is said to be a representation of \( E \) in \( A \), if \( \pi(0) = 0 \), and \( \pi(xy) = \pi(x)\pi(y) \), for all \( x \) and \( y \) in \( A \).

Given a semilattice \( E \), and a representation \( \pi \) of \( E \) in an algebra \( A \), one may often assume that \( A \) is abelian by replacing \( A \) with the subalgebra of \( A \) generated by the range of \( \pi \). If \( A \) is indeed abelian, we may view \( \pi \) as a representation in a canonically defined Boolean algebra as follows:

\[
B_A := \{ e \in A : e^2 = e \}.
\]

Under the operations
\[
e \land f = ef, \quad \text{and} \quad e \lor f = e + f - ef, \quad \forall e, f \in B_A
\]
it is easy to see that \( B_A \) is a Boolean algebra and clearly \( \pi \) takes values in \( B_A \). All concepts relating to Boolean algebra representations, such as \( \pi \)-tightness, therefore immediately apply to representations in associative algebras by considering the associated representation in \( B_A \). If \( A \) is a commutative \( K \)-algebra, then its spectrum \( \hat{A} \) is the space of non-zero \( K \)-algebra homomorphisms \( f: A \to K \) equipped with the topology of pointwise convergence (where \( K \) is viewed as a discrete space).

15.14. Proposition. Let \( \pi \) be a representation of a semilattice \( E \) in an associative \( K \)-algebra \( A \), such that \( A \) is generated by the range of \( \pi \). Then:

(i) The spectrum of \( A \), which we will denote by \( \hat{A} \), is homeomorphic to \( \hat{E}_\pi \).

(ii) \( A \) is isomorphic to the algebra \( C_c(\hat{E}_\pi, K) \), consisting of all locally constant, compactly supported, \( K \)-valued functions on \( \hat{E}_\pi \).

Proof. Since \( A \) is an abelian algebra generated by idempotents, it follows from [17: Corollaire 1] that \( A \) is naturally isomorphic to \( C_c(\text{Spec}(B_A), K) \). Since \( \pi: E \to B_A \) extends to a surjective homomorphism \( \pi: \mathbb{F}_2 E \to B_A \), (15.11) and its proof shows that we can identify
Spec(\(\mathcal{B}_A\)) with \(\hat{\pi}\). It remains to show that \(\hat{\pi}\) is homeomorphic to \(Spec(\mathcal{B}_A)\). The restriction map \(\hat{\pi} : Spec(\mathcal{B}_A) \to Spec(\mathcal{B}_A)\) is clearly continuous and injective (the latter since \(\pi(\mathcal{E})\) generates \(\mathcal{A}\)). It is onto because we can identify \(\mathcal{B}_A\) with the characteristic functions of compact open subsets of \(Spec(\mathcal{B}_A)\) and then if \(\xi \in Spec(\mathcal{B}_A)\), the corresponding character is evaluation at \(\xi\). But this extends to \(C_c(Spec(\mathcal{B}_A), \mathbb{K})\) as evaluation at \(\xi\). We show that the restriction map is open. A basic neighborhood \(U\) in \(\hat{\pi}\) specifies the value of a character at finitely many elements of \(\mathcal{B}_A\) (since each element can be expressed as a finite linear combination of idempotents and hence the value of a character at any element is determined by its value at finitely many idempotents). Since a character can only take on values \(0\) and \(1\) on an idempotent (because \(\mathbb{K}\) is a field) it follows that the image of \(U\) is the open subset of \(Spec(\mathcal{B}_A)\) of characters taking the same values on those specified idempotents.

We now begin to apply some of this machinery in the strongly finitely aligned case. This will be used in our sequel paper to show that the tight \(C^*\)-algebra of the inverse hull of the semigroup associated to a finitely aligned higher rank graph is the corresponding higher rank graph \(C^*\)-algebra. Again, we shall assume our strongly finitely aligned semigroup has right local units.

15.15. Lemma. Let \(S\) be strongly finitely aligned and let \(s, t \in S\). Suppose that \(sS \cap tS\) has basis \(B = \{w_1, \ldots, w_n\}\). Write \(w_i = sx_i = ty_i\) with \(x_i, y_i \in S\).

(i) \(\theta_s^{-1}\theta_i \theta_{y_i} \theta_{y_i}^{-1} = \theta_x \theta_{y_i} \theta_{y_i}^{-1}\).

(ii) The set \(\{\theta_y \theta_{y_i}^{-1} \theta_t \theta_{t}^{-1} \theta_t \mid 1 \leq i \leq n\}\) is a cover of \(\theta_i^{-1} \theta_s \theta_i^{-1} \theta_t\).

Proof. (i) If \(u \in \mathcal{B}\) is in the domain of the left hand side, then \(u = y_i z\) with \(z \in S, tu \neq 0\) and \(\theta_s^{-1} \theta_t \theta_{y_i} \theta_{y_i}^{-1}(u) = \theta_s^{-1}(tu)y_i = \theta_s^{-1}(sx_i z) = x_i z\). On the other hand, \(w_i z = ty_i z = tu \neq 0\) and so \(\theta_{x_i} \theta_{y_i} \theta_{y_i}^{-1}(u) = \theta_{x_i}(z) = x_i z\). Similarly, if \(u \in \mathcal{B}\) is in the domain of \(\theta_{x_i} \theta_{y_i} \theta_{y_i}^{-1}\), then \(u = y_i z\) with \(w_i z \neq 0\) and \(\theta_{x_i} \theta_{y_i} \theta_{y_i}^{-1}(u) = \theta_{x_i}(z) = x_i z\). But \(tu = ty_i z = w_i z \neq 0\) implies that \(\theta_s^{-1} \theta_t \theta_{y_i} \theta_{y_i}^{-1}(u) = \theta_s^{-1}(tu) = \theta_s^{-1}(sx_i z) = x_i z\) as \(tu = ty_i z = sx_i z\).

(ii) First note that if \(\theta_y \theta_{y_i}^{-1} \theta_t \theta_{t}^{-1} \theta_t(x) = x,\) then \(x = y_i z\) with \(tx = ty_i z = sx_i z \neq 0\). Then \(\theta_t^{-1} \theta_s \theta_i^{-1} \theta_t(x) = \theta_t^{-1} \theta_s \theta_i^{-1}(tx) = \theta_t^{-1} \theta_s \theta_i^{-1}(sx_i z) = \theta_t^{-1}(tx) = x\) and so we have \(\theta_y \theta_{y_i}^{-1} \theta_t \theta_{t}^{-1} \theta_t \leq \theta_t^{-1} \theta_s \theta_i^{-1} \theta_t\).

Suppose now that \(0 \neq f \leq \theta_t^{-1} \theta_s \theta_i^{-1} \theta_t\) and that \(f(x) \neq 0\). Then \(\theta_t^{-1} \theta_s \theta_i^{-1} \theta_t(x) \neq 0\) and so \(tx \neq 0\) and \(tx = sy\) for some \(y \in S\). Therefore, \(tx \in sS \cap tS\) and so \(tx = w_i z = ty_i z = sx_i z\) for some \(z \in S\) and \(1 \leq i \leq n\). Then \(x = y_i z\) by \(0\)-left cancellativity and so \(\theta_y \theta_{y_i}^{-1} \theta_t \theta_{t}^{-1} \theta_t(x) = x\). Thus \(f \theta_y \theta_{y_i}^{-1} \theta_t \theta_{t}^{-1} \theta_t \neq 0\).

Let us say that a representation \(\pi : T \to A\) of \(T\) in an associative algebra \(A\) is cover-to-join if whenever \(\{f_1, \ldots, f_n\}\) is a cover of an idempotent \(e\) of \(E(T)\) with \(f_i \leq e\) for all \(i = 1, \ldots, n\), then \(\pi(e) = \bigvee_{i=1}^n \pi(f_i)\) (where the join is taken in the commutative algebra generated by \(\pi(E(T))\)), i.e., \(\bigcap_{i=1}^n (\pi(e) - \pi(f_i)) = 0\).

15.16. Corollary. Let \(S\) be a strongly finitely aligned \(0\)-left cancellative semigroup and let \(s, t \in S\). Suppose that \(sS \cap tS\) has basis \(B = \{w_1, \ldots, w_n\} S\). Write \(w_i = sx_i = ty_i\) with \(x_i, y_i \in S\). If \(\pi : \mathcal{S}(S) \to A\) is a cover-to-join \(*\)-representation into an associative \(*\)-algebra
\( A \) and if we put \( \pi_s = \pi(\theta_s) \), then
\[
\pi_s^* \pi_t = \sum_{i=1}^{m} \pi_{x_i} \pi(f_{w_i}) \pi_{y_i}^* .
\]
holds. In particular, if \( S \) is categorical at zero and right reductive, then the formula
\[
\pi_s^* \pi_t = \sum_{i=1}^{m} \pi_{x_i} \pi_{y_i}^*
\]
holds.

**Proof.** Since \( w_i S \cap w_i S = \{0\} \) for \( i \neq j \), we have by (15.15) that
\[
\pi_t^* \pi_s \pi_t^* = \sum_{i=1}^{n} \pi_t^* \pi_{y_i} \pi_{y_i}^* ,
\]
and so
\[
\pi_s^* \pi_t = \sum_{i=1}^{n} \pi_s^* \pi_{y_i} \pi_{y_i}^* = \sum_{i=1}^{n} \pi_{x_i} \pi(f_{w_i}) \pi_{y_i}^* ,
\]
where the last equality follows from (15.15). The final statement follows because \( w_i = sx_i \) implies \( w_i^+ = x_i^+ \) and hence \( \theta_{x_i} f_{w_i} = \theta_{x_i} \theta_{w_i^+} = \theta_{x_i} \).

As a corollary, we deduce that the image of \( \mathcal{S}(S) \) of a strongly finitely aligned semigroup under a cover-to-join \(*\)-representation \( \pi \) is spanned by elements of the form \( \pi_s \pi(f_{\Lambda}) \pi_t^{-1} \) where \( \Lambda \) is a finite subset of \( S \), which is similar to what happens in the lcm case.

**15.17. Theorem.** Let \( S \) be a strongly finitely aligned \( 0\)-left cancellative semigroup. Let \( \pi : \mathcal{S}(S) \to A \) be a cover-to-join \(*\)-representation to an associative \(*\)-algebra \( A \) such that \( \pi(\mathcal{S}(S)) \) spans \( A \) and write \( \pi_s \) for \( \pi(\theta_s) \) for \( s \in S \). Then \( A \) is spanned by elements of the form \( \pi_s \pi(f_{\Lambda}) \pi_t^* \) with \( \Lambda \subseteq S \) finite and \( s, t \in \Lambda \). If \( S \) is right reductive and categorical at \( 0 \), then \( A \) is spanned by elements of the form \( \pi_s \pi_t^* \) with \( s, t \in S \).

**Proof.** Let \( T \) be the set of elements of the form \( \pi_s \pi(f_{\Lambda}) \pi_t^* \) with \( \Lambda \subseteq S \) finite and \( s, t \in \Lambda \). Trivially, \( T \subseteq \pi(\mathcal{S}(S)) \). Also, if \( e \in E(S) \) with \( se = s \), then \( \pi_s = \pi_s \pi(f_s) \pi_e^* \in T \). So it suffices to prove that \( T \) is an inverse semigroup. Since \( (\pi_s \pi(f_{\Lambda}) \pi_t^*)^* = \pi_t \pi(f_{\Lambda}) \pi_s^* \), it suffices to show that \( T \) is a semigroup. So let \( s_i, t_i \in \Lambda_i \subseteq S \), for \( i = 1, 2 \), with \( \Lambda_i \) finite. Let \( \{w_1, \ldots, w_n\} \) be a basis for \( t_1 S \cap t_2 S \) and write \( w_i = t_1 x_i = s_2 y_i \).

First observe that
\[
\pi_{s_1} \pi(f_{\Lambda_1}) \pi_{x_1} \pi(f_{w_1}) \pi_{y_1}^* \pi(f_{\Lambda_2}) \pi_{t_2}^* = \pi_{s_1} \pi(f_{\Lambda_1 x_i \cup \{w_i\} \cup \Lambda_2 y_i}) \pi_{t_2}^* ,
\]
and \( s_1 x_i, t_2 y_i \in \Lambda_1 x_i \cup \{w_i\} \cup \Lambda_2 y_i \). Then applying this and (15.16), yields
\[
\pi_{s_1} \pi(f_{\Lambda_1}) \pi_{t_1}^* \cdot \pi_{s_2} \pi(f_{\Lambda_2}) \pi_{t_2}^* = \sum_{i=1}^{n} \pi_{s_1} \pi(f_{\Lambda_1}) \pi_{x_1} \pi(f_{w_1}) \pi_{y_1}^* \pi(f_{\Lambda_2}) \pi_{t_2}^* =
\]
\[
= \sum_{i=1}^{n} \pi_{s_1} \pi(f_{\Lambda_1 x_i \cup \{w_i\} \cup \Lambda_2 y_i}) \pi_{t_2}^* y_i ,
\]
as required.

The statement in the categorical at zero case is proved similarly, using the final formula in (15.16) .
Actually, if one follows the above proofs carefully one can weaken the assumption that \( \pi \) is cover-to-join to just ask that if a constructible set \( X \) can be written as a finite union of constructible sets \( X = \bigcup_{i=1}^{n} X_i \), then \( \pi(1_X) = \bigvee_{i=1}^{n} \pi(1_{X_i}) \). That is, Theorem (15.17) is true for \( \theta \)-tight \( * \)-representations \( \pi \) where \( \theta: E(S(S)) \to \mathcal{P}(S \setminus \{0\}) \) is the restriction of the regular representation of \( S(S) \).

16. The spectrum of the semilattice of constructible sets.

Throughout this section we will fix a 0-left cancellative semigroup \( S \) admitting least common multiples.

As mentioned in the preamble to Part (2), we are interested in the study of the semilattice \( E(S) \) of constructible sets, with a special emphasis on its spectrum. In order to exhibit examples of characters on \( E(S) \) we shall make use of the idea behind (15.2) so, considering the representation

\[
\varepsilon: E(S) \to \mathcal{P}(S^*),
\]

introduced in (12.23), and given \( \sigma \) in the essential subset for \( \varepsilon \), we may define

\[
\varphi^\varepsilon_\sigma: X \in E(S) \mapsto [\sigma \in \varepsilon(X)] \in \{0, 1\}.
\]

which is a character on \( E(S) \), as discussed near (15.3).

Observe that, for trivial reasons, the essential subset for \( \varepsilon \) coincides with the essential subset \( S^* \) for \( \theta^* \), which we have seen in (10.16) to consist of all strings except for the singletons \( \{s\} \), where \( s \) is a degenerate\(^9\) element of \( S \).

16.1. Definition.

(i) A string \( \sigma \) will be called degenerate if \( \sigma = \{s\} \), where \( s \) is a degenerate element. The set of all non-degenerate strings is therefore \( S^*_\sigma \).

(ii) For every non-degenerate string \( \sigma \), we shall denote by \( \varphi_\sigma \) the character of \( E(S) \) given by

\[
\varphi_\sigma(X) = [\sigma \in \varepsilon(X)], \quad \forall X \in E(S).
\]

Employing the terminology introduced in (15.3) we have that

\[
\hat{E}(S^*_\sigma) = \{\varphi_\sigma : \sigma \in S^*_\sigma\},
\]

which is thus a subset of \( \hat{E}(S) \), allowing for a first glimpse of our main object of study.

16.2. Proposition. One has that \( \hat{E}(S^*_\sigma) \) is dense in the tight spectrum of \( E(S) \).

Proof. We will prove this as an application of (15.4), and hence our task consists in showing that \( \varepsilon(X) \) is nonempty for every nonempty \( X \) in \( E(S) \). But this is immediate from Theorem (12.25), which implies that \( r \in X \) if and only if \( \delta_r \in \varepsilon(X) \). \qed

\(^9\) Recall from (3.6) that \( s \) is degenerate if \( s \) is irreducible and \( Ss = \{0\} \).
We should note that, since $E(S)$ is a subset of $P(S')$, one may use (15.5) to produce another set of characters which is also dense in the $E_{\text{tight}}(S)$. However, characters arising from the identity representation of $E(S)$ on $S'$ have a very small chance of being tight, so we shall not be able to benefit much from (15.5) in the present context. As we shall see later (see (16.18)), characters arising from strings are much more likely to be ultracharacters, and hence tight.

Suppose we are given $\varphi$ and we want to recover $\sigma$ from $\varphi$. In the special case in which $S$ has right local units, we have that

$$\varphi_{\sigma}(E^\theta_s) = 1 \iff \sigma \in \varepsilon(E^\theta_s) = E^*_s \iff s \in \sigma, \quad (16.3)$$

so $\sigma$ is recovered as the set $\{s \in S : \varphi_{\sigma}(E^\theta_s) = 1\}$. Without assuming right local units, the last part of (16.3) cannot be trusted, but it may be replaced with

$$\cdots \iff \sigma \cap E^\theta_s \neq \emptyset, \quad (16.4)$$

so we at least know which $E^\theta_s$ have a nonempty intersection with $\sigma$.

**16.5. Proposition.** Given any string $\sigma$, one has that the set

$$\{s \in S : \varphi_{\sigma}(E^\theta_s) = 1\},$$

coincides with $\hat{\sigma}$, namely the interior of $\sigma$, as defined in (11.1).

*Proof.** Observe that since $0 \notin \sigma$, one has for every $s$ in $S$, that

$$\sigma \cap sS = \sigma \cap (sS \setminus \{0\}) = \sigma \cap E^\theta_s.$$ 

By (16.4) one then has that $s$ lies in the set displayed in the statement if and only if $\sigma \cap sS \neq \emptyset$, so the statement follows. \qed

Given any character $\varphi$ of $E(S)$, regardless of whether or not it is of the form $\varphi_{\sigma}$ as above, we may still consider the set

$$\sigma_\varphi := \{s \in S : \varphi(E^\theta_s) = 1\}, \quad (16.6)$$

so that, when $\varphi = \varphi_{\sigma}$, we get $\sigma_\varphi = \hat{\sigma}$, by (16.5).

**16.7. Proposition.** If $\varphi$ is any character of $E(S)$, and $\sigma_\varphi$ is nonempty, then $\sigma_\varphi$ is a string which is moreover closed under least common multiples.

*Proof.** Since $E^\theta_0 = \emptyset$, we have that $\varphi(E^\theta_0) = 0$, whence $0 \notin \sigma_\varphi$. If $t \in \sigma_\varphi$, and if $s$ divides $t$, then $tS \subseteq sS$, whence also $E^\theta_t \subseteq E^\theta_s$, and then $1 = \varphi(E^\theta_t) \leq \varphi(E^\theta_s)$, so $s \in \sigma_\varphi$.

To prove that $\sigma_\varphi$ is closed under least common multiples, let $s, t \in \sigma$, and let $r$ be a least common multiple of $s$ and $t$. Then $E^\theta_r = E^\theta_s \cap E^\theta_t$, whence

$$\varphi_{\sigma}(E^\theta_r) = \varphi_{\sigma}(E^\theta_s)\varphi_{\sigma}(E^\theta_t) = 1,$$

so $r \in \sigma_\varphi$. This also implies that $\sigma_\varphi$ satisfies (10.1.iii). \qed
Based on (16.1.ii) we may define a map from the set of all non-degenerate strings to \( \widehat{E}(S) \), the spectrum of \( E(S) \), by

\[
\Phi : \sigma \in S^* \mapsto \varphi_\sigma \in \widehat{E}(S),
\]

but if we want the dual correspondence suggested by (16.6), namely

\[
\varphi \mapsto \sigma_\varphi,
\]

to give a well defined map from \( \widehat{E}(S) \) to \( S^* \), we need to worry about its domain because we have not checked that \( \sigma_\varphi \) is always nonempty, and hence \( \sigma_\varphi \) may fail to be a string. The appropriate domain is evidently given by the set of all characters \( \varphi \) such that \( \sigma_\varphi \) is nonempty but, before we formalize this map, it is interesting to introduce a relevant subsemilattice of \( E(S) \).

16.10. Proposition. The subset of \( E(S) \) given by\(^{10}\)

\[ E_1(S) = \{ sF^\theta_\Lambda, \ \Lambda \subseteq S \text{ is finite, and } s \in \Lambda \}, \]

is an ideal of \( E(S) \). Moreover, for every \( X \) in \( E(S) \), one has that \( X \) lies in \( E_1(S) \) if and only if \( X \subseteq E^\theta_s \), for some \( s \) in \( S \).

Proof. Let us first prove the last sentence of the statement. The “only if” part is evident since

\[ sF^\theta_\Lambda \subseteq E^\theta_s, \]

so let us focus on the “if” part. We thus suppose that \( X \in E(S) \) is such that \( X \subseteq E^\theta_s \), for some \( s \) in \( S \). Observing that

\[ X \subseteq E^\theta_s = sF^\theta_s, \]

we see that \( X = sxF^\theta_\Delta \), where \( x \) and \( \Delta \) are as in (7.15), whence \( X \in E_1(S) \). The first part of the statement, namely that \( E_1(S) \) is an ideal, now follows easily. \( \Box \)

Observe that if \( S \) admits right local units, then \( E_1(S) = E(S) \), thanks to (7.17).

By (14.6) we may then view \( \widehat{E}_1(S) \) as an open subset of \( \widehat{E}(S) \). The next result is intended to distinguish the elements of \( \widehat{E}_1(S) \) within \( \widehat{E}(S) \).

16.11. Proposition. Let \( \varphi \) be a character on \( E(S) \). Then the following are equivalent:

(i) \( \varphi \in \widehat{E}_1(S) \),

(ii) \( \varphi(E^\theta_s) = 1 \), for some \( s \) in \( S \),

(iii) \( \sigma_\varphi \) is nonempty, and hence it is a string by (16.7).

Proof. The equivalence between (i) and (ii) follows from (14.7) and (16.10), while (ii) and (iii) are obviously equivalent. \( \Box \)

\(^{10}\) This should be contrasted with (7.13), where the general form of an element of \( E(S) \) is shown to be \( uF^\theta_\Lambda \), where \( u \) is in \( \tilde{S} \), rather than \( S \).
If $S$ admits right local units, we have seen that $\mathfrak{E}_1(S) = \mathfrak{E}(S)$, so $\sigma_\varphi$ is a string for every character $\varphi \in \mathfrak{E}(S)$.

The vast majority of non-degenerate strings $\sigma$ lead to a character $\varphi_\sigma$ belonging to $\mathfrak{E}_1(S)$, but there are exceptions.

16.12. **Proposition.** If $\sigma$ is a non-degenerate string in $S_1^*$ then $\varphi_\sigma$ does not belong to $\mathfrak{E}_1(S)$ if and only if $\sigma = \{s\}$, where $s$ is an irreducible element of $S$.

**Proof.** By (16.11) to say that $\varphi_\sigma$ is not in $\mathfrak{E}_1(S)$, is to say that for all $t$ in $S$ we have that $\varphi_\sigma(E^\theta_t) = 0$, or, equivalently that $\sigma \notin E_t^*$, by (10.10.ii). The conclusion then follows from (10.15.i). □

By (16.11) we have that the largest set of characters on which the correspondence described in (16.9) produces a bona fide string is precisely $\mathfrak{E}_1(S)$, so we may now formally introduce the map suggested by that correspondence.

16.13. **Definition.** We shall let

$$\Sigma : \mathfrak{E}_1(S) \mapsto S^*,$$

be the map given by

$$\Sigma(\varphi) = \sigma_\varphi = \{s \in S : \varphi(E^\theta_s) = 1\}, \quad \forall \varphi \in \mathfrak{E}_1(S).$$

For every string $\sigma$, excluding the exceptional ones discussed in (16.12), we then have that

$$\Phi(\sigma) = \varphi_\sigma \in \mathfrak{E}_1(S),$$

and

$$\Sigma(\Phi(\sigma)) = \tilde{\sigma}, \quad (16.14)$$

by (16.5). The nicest situation is for open strings:

16.15. **Proposition.** If $\sigma$ is an open string, then

(i) $\sigma$ is non-degenerate,

(ii) $\Phi(\sigma) \in \mathfrak{E}_1(S)$, and

(iii) $\Sigma(\Phi(\sigma)) = \tilde{\sigma}$.

**Proof.** If $s$ is an irreducible element in $S$, then the string $\{s\}$ is certainly not open, so an open string cannot be any of the exceptional strings discussed in (16.12), much less a degenerate string. Therefore $\Phi(\sigma) \in \mathfrak{E}_1(S)$. The third point follows from (16.14) and the fact that $\sigma = \tilde{\sigma}$. □

Given that the composition $\Sigma \circ \Phi$ is so well behaved for open strings, we will now study the reverse composition $\Phi \circ \Sigma$ on a set of characters related to open strings.

16.16. **Definition.** A character $\varphi$ in $\mathfrak{E}(S)$ will be called an open character if $\sigma_\varphi$ is a (nonempty) open string. 71
We remark that every open character belongs to $\widehat{E}_1(S)$ by (16.11), although not all characters in $\widehat{E}_1(S)$ are open.

By (16.15) it is clear that $\varphi_\sigma$ is an open character for every open string $\sigma$.

If $S$ admits right local units, we have seen that every string in $S^*$ is open, and also that $\sigma_\varphi$ is a string for every character. Therefore every character in $\widehat{E}(S)$ is open.

The composition $\Phi \circ \Sigma$ is not as well behaved as the one discussed in (16.15), but there is at least some relationship between a character $\varphi$ and its image under $\Phi \circ \Sigma$, as we shall now see.

16.17. Proposition. Given any open character $\varphi$, one has that

$$\varphi \leq \Phi(\Sigma(\varphi)).$$

Proof. In an effort to decongest notation, throughout this proof we will write $\sigma$ for $\sigma_\varphi$, so the inequality in the statement reads $\varphi \leq \varphi_\sigma$. In order to prove it, it is clearly enough to argue that there is no $X$ in $E(S)$ such that $\varphi(X) = 1$, and $\varphi_\sigma(X) = 0$.

Arguing by contradiction, we assume that such an $X$ exists. Observing that $\varphi$ is in $\widehat{E}_1(S)$, we may choose $s$ in $S$ such that $\varphi(E_s^\theta) = 1$. Setting $X' = X \cap E_s^\theta$, we have that

$$\varphi(X') = \varphi(X) \varphi(E_s^\theta) = 1, \quad \text{and} \quad \varphi_\sigma(X') = \varphi_\sigma(X) \varphi_\sigma(E_s^\theta) = 0,$$

which means that we may suppose without loss of generality that the originally chosen $X$ is a subset of $E_s^\theta$. Using (16.10) we have that $X \in \widehat{E}_1(S)$, so we may write $X = rF_\Lambda^\theta$, where $\Lambda$ is a finite subset of $S$, and $r \in \Lambda$. Noticing that $X \subseteq E_r^\theta$, we have

$$1 = \varphi(X) \leq \varphi(E_r^\theta),$$

so we see that $r \in \sigma = \sigma_\varphi$. Since $\varphi$ is an open character, $\sigma$ is an open string, so there exists some $t$ in $S$, such that $rt \in \sigma$. Therefore $\sigma \cap E_r^\theta \neq \emptyset$, and we deduce from (10.10.ii) that $\sigma$ is in $E_r^*$. Notice that to say that $\varphi_\sigma(X) = 0$ is the same as saying that

$$\sigma \notin \varepsilon(X) = \theta_r^*(F_\Lambda^*),$$

which implies that

$$r^{-1} * \sigma = \theta_r^{-1}(\sigma) \notin F_\Lambda^*,$$

or, equivalently, that

$$r^{-1} * \sigma \not\subset F_\Lambda^\theta,$$

by (10.10.i). This said, we may pick $y \in r^{-1} * \sigma$, such that $ty = 0$, for some $t \in \Lambda$. In particular $ry \in \sigma$, so

$$\varphi(E_r^\theta) = 1.$$
We next claim that $X$ and $E_{ry}^\theta$ are disjoint. In fact, should this not be the case, we could find some
\[ s \in X \cap E_{ry}^\theta = rF_\Lambda^\theta \cap E_{ry}^\theta, \]
which may therefore be written as
\[ s = rp = r y q, \]
for suitable $p$ in $F_\Lambda^\theta$, and $q$ in $S$. Therefore $p = y q$, by 0-left cancellativity, and then
\[ 0 \neq tp = tyq = 0, \]
a contradiction. This proves that $X \cap E_{ry}^\theta = \emptyset$, so
\[ 0 = \varphi(X \cap E_{ry}^\theta) = \varphi(X) \varphi(E_{ry}^\theta) \stackrel{\text{(16.17.2)}}{=} \varphi(X), \]
contradicting (16.17.1), and thus concluding the proof. \qed

We have thus arrived at an important result.

16.18. Theorem. Let $S$ be a 0-left cancellative semigroup admitting least common multiples. Then, for every open, maximal string $\sigma$ over $S$, one has that $\varphi_\sigma$ is an ultracharacter.

Proof. Let $\psi$ be a character such that $\varphi_\sigma \leq \psi$. For every $s$ in $S$ we then have that
\[ s \in \sigma \stackrel{\text{(16.15.iii)}}{\Rightarrow} \varphi_\sigma(E_s^\theta) = 1 \Rightarrow \psi(E_s^\theta) = 1 \Rightarrow s \in \sigma_\psi, \]
which means that $\sigma \subseteq \sigma_\psi$, and hence that $\sigma = \sigma_\psi$, by maximality. It follows that $\psi$ is an open character, so (16.17) applies for $\psi$, and we deduce that
\[ \varphi_\sigma \leq \psi \leq \Phi(S(\psi)) = \Phi(\sigma) = \varphi_\sigma, \]
so $\varphi_\sigma = \psi$, proving that $\varphi_\sigma$ is maximal. \qed

The previous result raises the question as to whether $\sigma_\varphi$ is a maximal string for every ultracharacter $\varphi$, but this is not true in general. Consider for example the unital semigroup
\[ S = \{1, a, aa, 0\}, \]
in which $a^3 = 0$. The $\theta$-constructible subsets of $S$ are precisely

\begin{center}
\begin{tabular}{|c|c|}
\hline
$E_1^\theta = F_1^\theta = \{1, a, aa\}$ & \\
\hline
$F_a^\theta = \{1, a\}$ & $E_a^\theta = aF_a^\theta = \{a, aa\}$ \\
\hline
$F_{aa}^\theta = \{1\}$ & $aF_{aa}^\theta = \{a\}$ \\
\hline
$E_{aa}^\theta = aaF_{aa}^\theta = \{aa\}$ & \\
\hline
\end{tabular}
\end{center}

List of $\theta$-constructible sets
and there are three strings over \( S \), namely

\[
\begin{array}{|c|c|c|}
\hline
\delta_1 &=& \{1\} \\
\delta_a &=& \{1, a\} \\
\delta_{aa} &=& \{1, a, aa\} \\
\hline
\end{array}
\]

Since the correspondence \( s \mapsto \delta_s \) is a bijection from \( S' \) to \( S^* \), we see that \( \theta^* \) is isomorphic to \( \theta \), and in particular the \( \theta^* \)-constructible subsets of \( S^* \), listed below, mirror the \( \theta \)-constructible ones.

\[
\begin{array}{|c|c|c|}
\hline
E_1^* &=& F_1^* = \{\delta_1, \delta_a, \delta_{aa}\} \\
F_a^* &=& \{\delta_1, \delta_a\} \\
E_a^* &=& aF_a^* = \{\delta_a, \delta_{aa}\} \\
F_{aa}^\theta &=& \{\delta_1\} \\
aF_{aa}^\theta &=& \{\delta_a\} \\
E_{aa}^\theta &=& aaF_{aa}^\theta = \{\delta_{aa}\} \\
\hline
\end{array}
\]

List of \( \theta^* \)-constructible sets

Observe that the string \( \sigma := \delta_a = \{1, a\} \) is a proper subset of the string \( \{1, a, aa\} \), and hence \( \sigma \) is not maximal. But yet notice that \( \varphi_\sigma \) is an ultracharacter, since \( \{\delta_a\} \) is a minimal\(^{11}\) member of \( \mathcal{P}(S^*, \theta^*) \). We thus get an example of

“A string \( \sigma \) which is not maximal but such that \( \varphi_\sigma \) is an ultracharacter.”

On the other hand, since \( \sigma = \sigma_\varphi \), this also provides an example of

“An ultracharacter \( \varphi \) such that \( \sigma_\varphi \) is not maximal.”

This suggests the need to single out the strings which give rise to ultracharacters:

**16.19. Definition.** We will say that a string \( \sigma \) is *quasi-maximal* whenever \( \varphi_\sigma \) is an ultracharacter. The set of all quasi-maximal strings will be denoted by \( S^\infty \).

Adopting this terminology, the conclusion of (16.18) states that every open, maximal string is quasi-maximal.

**16.20. Theorem.** Let \( S \) be a 0-left cancellative semigroup admitting least common multiples. Then, every open ultracharacter on \( \mathfrak{E}(S) \) is of the form \( \varphi_\sigma \) for some open, quasi-maximal string \( \sigma \).

*Proof.* Let \( \varphi \) be an open ultracharacter on \( \mathfrak{E}(S) \). Letting \( \sigma = \sigma_\varphi \), we have that \( \sigma \) is open by definition, and by (16.17) it follows that \( \varphi \leq \varphi_\sigma \), and hence \( \varphi = \varphi_\sigma \), by maximality. That \( \sigma \) is a quasi-maximal string is due to the fact that \( \varphi_\sigma \) is an ultracharacter. \( \square \)

\(^{11}\) Whenever \( e_0 \) is a nonzero minimal element of a semilattice \( E \), the character \( \varphi(e) = [e_0 \leq e] \) is an ultracharacter.
The importance of quasi-maximal strings evidenced by the last result begs for a better understanding of such strings. While we are unable to provide a complete characterization, we can at least exhibit some further examples beyond the maximal ones.

To explain what we mean, recall from (10.10.i) that a string \( \sigma \) belongs to some \( F_\Lambda^\theta \) if and only if \( \sigma \) is contained in \( F_\Lambda^\theta \). It is therefore possible that \( \sigma \) is maximal among all strings contained in \( F_\Lambda^\theta \), and still not a maximal string. An example is the string \{1, a\} mentioned above, which is maximal within \( F_a^\theta \), but not maximal in the strict sense of the word.

16.21. Proposition. Let \( \Lambda \) be a nonempty finite subset of \( S \) and suppose that \( \sigma \) is an open string such that \( \sigma \subseteq F_\Lambda^\theta \). Suppose moreover that \( \sigma \) is maximal among the strings contained in \( F_\Lambda^\theta \), in the sense that for every string \( \mu \), one has that \( \sigma \subseteq \mu \subseteq F_\Lambda^\theta \Rightarrow \sigma = \mu. \)

Then \( \varphi_\sigma \) is an ultracharacter, and hence \( \sigma \) is a quasi-maximal string.

Proof. We begin by following the first steps of the proof of (16.18): let \( \psi \) be a character such that \( \varphi_\sigma \leq \psi \). For every \( s \) in \( S \) we then have that

\[
s \in \sigma \Rightarrow \varphi_\sigma(E_s^\theta) = 1 \Rightarrow \psi(E_s^\theta) = 1 \Rightarrow s \in \sigma_\psi,
\]

which means that \( \sigma \subseteq \sigma_\psi \). By hypothesis we have that \( \sigma \subseteq F_\Lambda^\theta \), hence \( \sigma \in F_\Lambda^\star \), by (10.10.i), so

\[
1 = [\sigma \in F_\Lambda^\star] = [\sigma \in \varepsilon(F_\Lambda^\theta)] = \varphi_\sigma(F_\Lambda^\theta) \leq \psi(F_\Lambda^\theta),
\]

and we conclude that \( \psi(F_\Lambda^\theta) = 1 \). We claim that this entails that \( \sigma_\psi \subseteq F_\Lambda^\theta \). In fact, given any \( s \in \sigma_\psi \), we have

\[
1 = \psi(E_s^\theta) \psi(F_\Lambda^\theta) = \psi(E_s^\theta \cap F_\Lambda^\theta),
\]

so \( E_s^\theta \cap F_\Lambda^\theta \) cannot possibly be the empty set. Picking any \( r \) in \( E_s^\theta \cap F_\Lambda^\theta \), we have that \( r = sx \), for some \( x \) in \( S \), and for every \( t \) in \( \Lambda \), one has

\[
0 \neq tr = tsx,
\]

so in particular \( ts \neq 0 \), showing that \( s \in F_\Lambda^\theta \). We have thus proved that \( \sigma \subseteq \sigma_\psi \subseteq F_\Lambda^\theta \), and we deduce from the relative maximality of \( \sigma \) that \( \sigma = \sigma_\psi \).

It follows that \( \psi \) is an open character, so (16.17) applies for \( \psi \), and we deduce that

\[
\varphi_\sigma \leq \psi \leq \Phi(\Sigma(\psi)) = \Phi(\sigma) = \varphi_\sigma,
\]

so \( \varphi_\sigma = \psi \), proving that \( \varphi_\sigma \) is maximal. \( \square \)

Here are some further questions we have come across and which are still to be answered:

16.22. Questions.

(i) Is there an intrinsic characterization of quasi-maximal strings?

(ii) Under which assumptions on \( S \) is every quasi-maximal string maximal?

(iii) Is it possible to characterize the strings \( \sigma \) for which \( \varphi_\sigma \) is a tight character? These should be called tight strings.
17. Ground characters.

In the last section we were able to fruitfully study open characters using strings, culminating with Theorem (16.20), stating that every open ultracharacter is given in terms of a string. However nothing of interest was said about an ultracharacter when it is not open. The main purpose of this section is thus to obtain some useful information about non-open ultracharacters. The main result in this direction is Theorem (17.11), below.

Throughout this section we fix a 0-left cancellative semigroup $S$ admitting least common multiples. For each $s$ in $S$ let

$$\hat{F}_s = \{ \varphi \in \hat{E}(S) : \varphi(F_s^0) = 1 \}, \quad \text{and} \quad \hat{E}_s = \{ \varphi \in \hat{E}(S) : \varphi(E_s^0) = 1 \},$$

and for every $\varphi$ in $\hat{F}_s$, consider the character $\hat{\theta}_s(\varphi)$ given by

$$\hat{\theta}_s(\varphi)(X) = \varphi(\theta_s^{-1}(E_s^0 \cap X)), \quad \forall X \in \mathcal{E}(S).$$

Observing that

$$\hat{\theta}_s(\varphi)(E_s^0) = \varphi(\theta_s^{-1}(E_s^0)) = \varphi(F_s^0) = 1,$$

we see that $\hat{\theta}_s(\varphi)$ is indeed a (nonzero) character, and that $\hat{\theta}_s(\varphi)$ belongs to $\hat{E}_s$. As a consequence we get a map

$$\hat{\theta}_s : \hat{F}_s \to \hat{E}_s,$$

which is easily seen to be bijective, with inverse given by

$$\hat{\theta}_s^{-1}(\varphi)(X) = \varphi(\theta_s(F_s^0 \cap X)), \quad \forall \varphi \in \hat{E}_s, \quad \forall X \in \mathcal{E}(S).$$

We may then see each $\hat{\theta}_s$ as an element of $\mathcal{I}(\hat{E}(S))$, and it is not hard to see that the correspondence

$$\hat{\theta} : s \in S \mapsto \hat{\theta}_s \in \mathcal{I}(\hat{E}(S))$$

is a representation of $S$ on $\hat{E}(S)$.

All of this may also be deduced from the fact that any inverse semigroup, such as $\mathcal{H}(S)$, admits a canonical representation on the spectrum of its idempotent semilattice (see [11: Section 10]), and that $\hat{\theta}$ may be obtained as the composition

$$S \xrightarrow{\theta} \mathcal{H}(S) \to \mathcal{I}(\hat{E}(S)),$$

where the arrow in the right-hand-side is the canonical representation mentioned above.

17.2. Definition. We shall refer to $\hat{\theta}$ as the dual representation of $S$.

In order to study the relationship between the dual representation and the representation $\rho$ of $\mathcal{H}(S)$ described in (12.22), let us prove the following technical result.
17.3. Lemma. Given \( s \) in \( S \), and \( \sigma \) in \( S^*_s \), one has that

(i) \( \varphi_\sigma \in \hat{F}_s \iff \sigma \in F^*_s \),

(ii) if the equivalent conditions in (i) are satisfied, then \( \hat{\theta}_s(\varphi_\sigma) = \varphi_{\theta^*_s(\sigma)} \),

(iii) \( \varphi_\sigma \in \hat{E}_s \iff \sigma \in E^*_s \),

(iv) if the equivalent conditions in (iii) are satisfied, then \( \hat{\theta}^{-1}_s(\varphi_\sigma) = \varphi_{\theta^*_s^{-1}(\sigma)} \).

Proof. (i) We have

\[
\varphi_\sigma \in \hat{F}_s \iff \varphi_\sigma(F^*_s) = 1 \iff \sigma \in \varepsilon(F^*_s) \quad (12.23)
\]

(iii) Follows as above by replacing the letter “\( F \)” by the letter “\( E \)”.

(ii) Assuming (i), one has for every \( X \in \mathcal{E}(S) \), that

\[
\hat{\theta}_s(\varphi_\sigma)(X) = \varphi_\sigma(\theta^{-1}_s(E^*_s \cap X)) = [\sigma \in \varepsilon(\theta^{-1}_s(E^*_s \cap X))] \quad (12.22)
\]

\[
= [\sigma \in \theta^*_s^{-1}(\varepsilon(E^*_s \cap X))] = [\theta^*_s(\sigma) \in \varepsilon(E^*_s \cap X)] = \cdots \quad (17.3.1)
\]

Observe that

\[
\varepsilon(E^*_s \cap X) = \varepsilon(E^*_s) \cap \varepsilon(X) = E^*_s \cap \varepsilon(X),
\]

and since \( \theta^*_s(\sigma) \) is evidently in \( E^*_s \), one has that (17.3.1) coincides with

\[
[\theta^*_s(\sigma) \in \varepsilon(X)] = \varphi_{\theta^*_s(\sigma)}(X),
\]

thus proving (ii).

(iv) Assuming (iii), one has for every \( X \in \mathcal{E}(S) \), that

\[
\hat{\theta}^{-1}_s(\varphi_\sigma)(X) = \varphi_\sigma(\theta_s(F^*_s \cap X)) = [\sigma \in \varepsilon(\theta_s(F^*_s \cap X))] = \]

\[
= [\sigma \in \theta^*_s(\varepsilon(F^*_s \cap X))] = [\theta^*_s^{-1}(\sigma) \in \varepsilon(F^*_s \cap X)],
\]

and conclusion follows as in the proof of (ii). \( \square \)

Considering the representation \( \theta^*_s \) of \( S \) on \( S^*_s \), observe that \( S^*_s \) is an invariant\(^{12}\) subset of \( S^* \), and it is easy to see that it is also invariant under the representation \( \rho \) of \( \mathfrak{H}(S) \) described in (12.22). Together with the dual representation of \( \mathfrak{H}(S) \) on \( \hat{\mathcal{E}}(S) \) mentioned above, we thus have two natural representations of \( \mathfrak{H}(S) \), which are closely related, as the following immediate consequence of the above result asserts:

17.4. Proposition. The mapping

\[
\Phi: S^*_s \to \hat{\mathcal{E}}(S)
\]

of (16.8) is covariant relative to the natural representations of \( \mathfrak{H}(S) \) referred to above.

\(^{12}\) The essential subset for a representation is evidently invariant!

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Proof. Follows immediately from (17.3), and the fact that $\mathcal{H}(S)$ is generated by the $\theta_s$ and their inverses. \hfill \square

The fact that the correspondence between strings and characters (see e.g. (16.14) and (16.17)) is not a perfect one is partly responsible for the fact that expressing the covariance properties of the map $\Sigma$ of (16.13) cannot be done in the same straightforward way as we did for $\Phi$ in (17.4).

Let us first treat the question of covariance regarding $\hat{\theta}_s^{-1}(\varphi)$. Of course, for this to be a well defined character we need $\varphi$ to be in $\hat{E}_s$, meaning that $\varphi(E_\theta^s) = 1$, which is also equivalent to saying that $s \in \sigma_\varphi$. In particular characters with empty strings are immediately ruled out.

17.5. Lemma. For every $s$ in $S$, and every character $\varphi$ in $\hat{E}_s$, one has that

$$\sigma_{\hat{\theta}_s^{-1}(\varphi)} = \{p \in S : sp \in \sigma_\varphi\}.$$

Proof. For $p$ in $S$, notice that $p \in \sigma_{\hat{\theta}_s^{-1}(\varphi)}$, iff

$$1 = \hat{\theta}_s^{-1}(\varphi)(E_\theta^p) = \varphi(\theta_s(F_\theta^s \cap E_\theta^p)) = \varphi(E_{sp}),$$

which in turn is equivalent to saying that $sp \in \sigma_\varphi$. \hfill \square

The set appearing in the right hand side of the equation displayed in (17.5) is precisely the same set mentioned in definition (10.6.ii) of $s^{-1} \ast \sigma_\varphi$, except that this notation is reserved for the situation in which the intersection of $\sigma$ with $sS$ is nonempty, which precisely means that $s \in \tilde{\sigma}_\varphi$.

17.6. Proposition. Pick $s$ in $S$ and let $\varphi$ be any character in $\hat{E}_s$. Then $s \in \sigma_\varphi$, and moreover

(i) if $s$ is in $\tilde{\sigma}_\varphi$, then $\sigma_\varphi \in E_\theta^s$, and $\sigma_{\hat{\theta}_s^{-1}(\varphi)} = \theta_s^{-1}(\sigma_\varphi)$,

(ii) if $s$ is not in $\tilde{\sigma}_\varphi$, then $\sigma_{\hat{\theta}_s^{-1}(\varphi)} = \emptyset$.

Proof. If $s$ is in the interior of $\sigma_\varphi$, there is some $p$ in $S$ such that $sp \in \sigma_\varphi$, whence also $sp \in \sigma_\varphi \cap E_\theta^s$. It follows that $\sigma_\varphi \cap E_\theta^s$ is nonempty, so (10.10.ii) implies that $\sigma_\varphi$ lies in $E_\theta^s$. The last statement of (i) then follows at once from (17.5).

On the other hand, if $s$ is not in the interior of $\sigma_\varphi$, the conclusion again follows from (17.5). \hfill \square

Regarding the behavior of strings associated to characters of the form $\hat{\theta}_s(\varphi)$, we have:

17.7. Lemma. For every $s$ in $S$, and every character $\varphi$ in $\hat{F}_s$, one has that $\hat{\theta}_s(\varphi)$ belongs to $\hat{E}_1(S)$ (and hence (16.11) implies that $\sigma_{\hat{\theta}_s(\varphi)}$ is a string), and moreover

(i) if $\sigma_\varphi$ is nonempty, then $\sigma_\varphi \in F_\theta^s$, and $\sigma_{\hat{\theta}_s(\varphi)} = \theta_s^*(\sigma_\varphi)$,

(ii) if $\sigma_\varphi$ is empty, then $\sigma_{\hat{\theta}_s(\varphi)} = \delta_s$.
Proof. The fact that $\hat{\sigma}(\varphi)$ belongs to $\hat{E}_1(S)$ follows from (17.1) and (16.11).

Given any $t$ in $\sigma$, we have that $\varphi(E^0_t) = 1$, so
\[ \varphi(E^0_t \cap F^0_s) = \varphi(E^0_t) \varphi(F^0_s) = 1, \]
whence $E^0_t \cap F^0_s \neq \emptyset$. Analyzing any element in this nonempty intersection one quickly realizes that $t \in F^0_s$, from whence it follows that $\sigma \varphi \subseteq F^0_s$. Under the assumption of (i) one has that $\sigma \varphi$ is a string by (16.7), and the first conclusion of (i) then follows from (10.10.i).

Regardless of any assumption on $\sigma \varphi$, pick any $t$ in $S$, and let $r$ be a least common multiple of $s$ and $t$, so that $E^0_r = E^0_s \cap E^0_t$, and $r = su = tv$, for suitable $u$ and $v$ in $\bar{S}$. We then have that
\[ t \in \sigma_{\hat{\theta}_s}(\varphi) \iff \hat{\theta}_s(\varphi)(E^0_t) = 1 \iff \varphi(\theta^{-1}_s(E^0_s \cap E^0_t)) = 1 \iff \varphi(\theta^{-1}_s(E^0_r)) = 1 \iff \ldots \]
Notice that
\[ E^0_r = E^0_{su} = \theta_s(E^0_u \cap F^0_s), \]
so,
\[ \varphi(\theta^{-1}_s(E^0_r)) = \varphi(\theta^{-1}_s(E^0_u \cap F^0_s)) = \varphi(E^0_u \cap F^0_s) \]
We then deduce that
\[ t \in \sigma_{\hat{\theta}_s}(\varphi) \iff \varphi(E^0_u \cap F^0_s) = 1. \quad (17.7.1) \]

In case $t \mid s$, in which case one may take $r = s$, and $u = 1$, above, one obviously has that $\varphi(E^0_u \cap F^0_s) = 1$, so the above argument shows that $t \in \sigma_{\hat{\theta}_s}(\varphi)$, thus proving that
\[ \delta_s \subseteq \sigma_{\hat{\theta}_s}(\varphi). \]

Now suppose that $t \nmid s$, and that we are under the conditions of (ii). Then the element $u$ above is necessarily different from 1, so it must lie in $S$. Moreover $\varphi(E^0_u) = 0$, since otherwise $u \in \sigma$, whence
\[ \varphi(E^0_u \cap F^0_s) = \varphi(E^0_u) \varphi(F^0_s) = 0, \]
and then $t \notin \sigma_{\hat{\theta}_s}(\varphi)$, by (17.7.1). This concludes the proof of (ii), and it now remains to prove the last assertion of (ii). In order to do this we first observe that $\theta^*_s(\sigma) \in E^*_s$, so $s \in \theta^*_s(\sigma)$ by (10.10.iii), and hence also
\[ \delta_s \subseteq \theta^*_s(\sigma). \quad (17.7.2) \]
Choosing any $t \in \sigma_{\hat{\theta}_s}(\varphi)$, let $r$, $u$, and $v$ be as above. If $u = 1$, then $t \mid s$, so $t \in \theta^*_s(\sigma)$, by (17.7.2). Henceforth assuming that $u \in S$, we have by (17.7.1) that $\varphi(E^0_u) = 1$, so $u \in \varphi$, and since $t \mid su$, it follows that $t \in \theta^*_s(\sigma)$, thus showing that $\sigma_{\hat{\theta}_s(\varphi)} \subseteq \theta^*_s(\sigma)$.

In order to prove the reverse inclusion, pick any $t \in \theta^*_s(\sigma)$. Then $t \mid sp$, for some $p$ in $\varphi$, so we may write $tx = sp$, for some $x$ in $\bar{S}$. A moment’s reflection will convince the reader that
\[ \theta_s(E^0_p \cap F^0_s) \subseteq E^0_s \cap E^0_t, \]
so
\[ \hat{\theta}_s(\varphi)(E^0_t) = \varphi(\theta^{-1}_s(E^0_s \cap E^0_t)) \geq \varphi(E^0_p \cap F^0_s) = \varphi(E^0_p) \varphi(F^0_s) = \varphi(E^0_p) = 1. \]
This shows that $t \in \sigma_{\hat{\theta}_s(\varphi)}$, concluding the proof of (ii).
We may interpret the above result, and more specifically the identity

\[ \sigma_{\hat{\theta}_s(\varphi)} = \theta_1^*(\sigma_{\varphi}), \]

as saying that the correspondence \( \varphi \mapsto \sigma_{\varphi} \) is covariant with respect to the actions \( \hat{\theta} \) and \( \theta_1^* \), on \( \hat{E}(S) \) and \( S^* \), respectively, except that the term “\( \sigma_{\varphi} \)” appearing is the right-hand-side above is not a well defined string since it may be empty, even though the left-hand-side is always well defined. In the problematic case of an empty string, (17.7.ii) then gives the undefined right-hand-side the default value of \( \delta_s \).

17.8. Definition. A character \( \varphi \) in \( \hat{E}(S) \) will be called a ground character if \( \sigma_{\varphi} \) is empty.

By (16.11), the ground characters are precisely the members of \( \hat{E}(S) \setminus \hat{E}_1(S) \).

Besides the ground characters, a character \( \varphi \) may fail to be open because \( \sigma_{\varphi} \), while being a bona fide string, is not an open string. In this case we have by (11.2.ii) that \( \sigma_{\varphi} = \delta_s \), for some \( s \) in \( S \) such that \( s \notin sS \).

17.9. Lemma. Let \( \varphi \) be a character such that \( \sigma_{\varphi} = \delta_s \), where \( s \) is such that \( s \notin sS \). Then \( \varphi \in \hat{E}_s \), and \( \hat{\theta}_s^{-1}(\varphi) \) is a ground character.

Proof. Since \( s \in \sigma_{\varphi} \), we have that \( \varphi(E_\theta^s) = 1 \), so \( \varphi \in \hat{E}_s \), whence

\[ \psi := \hat{\theta}_s^{-1}(\varphi) \]

is a well defined character. In order to prove that \( \psi \) is a ground character, we argue by contradiction and suppose instead that there exists some \( t \in \sigma_\psi \). Observing that \( \psi \in \hat{F}_s \), we then have that

\[ st \in s\sigma_\psi \subseteq s*\sigma_\psi = \theta_1^*(\psi) \overset{(17.7.1)}{=} \sigma_{\hat{\theta}_s(\psi)} = \sigma_{\varphi} = \delta_s. \]

It follows that \( st | s \), whence \( stu = s \), for some \( u \) in \( \tilde{S} \), so

\[ s = stu \in sS, \]

contradicting the hypothesis. This shows that \( \psi \) is indeed a ground character, concluding the proof. \( \square \)

We may now give a precise characterization of non-open characters in terms of the ground characters:

17.10. Proposition. Denote by \( \hat{E}_{op}(S) \) the set of all open characters on \( E(S) \). Then

\[ \hat{E}(S) \setminus \hat{E}_{op}(S) = \{ \hat{\theta}_u(\varphi) : u \in \tilde{S}, \varphi \text{ is a ground character in } \hat{F}_u \}. \]

Moreover for each \( \psi \) in the above set, there is a unique pair \( (u, \varphi) \), with \( u \) in \( \tilde{S} \), and \( \varphi \) a ground character, such that \( \psi = \hat{\theta}_u(\varphi) \).
Proof. If \( \varphi \) is a non-open character, then \( \sigma_\varphi \) is either empty, in which case \( \varphi \) is a ground character, or \( \sigma_\varphi \) is a (nonempty) non-open string. In the latter case we have by (11.2.ii) that \( \sigma_\varphi = \delta_s \), for some \( s \in S \), such that \( s \notin sS \), so it follows from (17.9) that

\[
\psi := \hat{\theta}^{-1}(\varphi)
\]

is a ground character, necessarily in \( \hat{F}_s \). Observing that \( \varphi = \hat{\theta}_s(\psi) \), we see that \( \varphi \) lies in the set appearing in the right-hand-side in the statement.

Conversely, if \( \varphi \) is a ground character in \( \hat{F}_u \), we must show that \( \hat{\theta}_u(\varphi) \) is not open. In case \( u = 1 \), there is nothing to do since ground characters are obviously not open, so we henceforth suppose that \( u \in S \). In keeping with our tradition of naming elements in \( S \) by \( s, t, \) and \( r \), while reserving \( u \) and \( v \) for elements which, in principle, are allowed to range in all of \( \tilde{S} \), we will write \( s \) for \( u \), so that \( \varphi \in \hat{F}_s \), and our task consists in showing that \( \hat{\theta}_s(\varphi) \) is not open.

Since \( \varphi \) is a ground character, we have by (17.7.ii) that

\[
\sigma_{\hat{\theta}_s(\varphi)} = \delta_s.
\]

Arguing by contradiction, suppose that \( \hat{\theta}_s(\varphi) \) is open, whence \( \delta_s \) is an open string. Observing that \( s \) lies in \( \delta_s \), there exists some \( p \) in \( S \) such that \( sp \in \delta_s \), and then we may find \( x \) in \( \tilde{S} \) such that \( spx = s \). Letting \( e = px \), we then claim that \( F_s^\vartheta \subseteq E_e^\vartheta \). To see this, it is enough to notice that if \( t \in F_s^\vartheta \), then

\[
0 \neq st = set,
\]

so 0-left cancellativity applies giving \( t = et \in E_e^\vartheta \), and proving our claim. Recalling that \( \varphi \in \hat{F}_s \), we then have

\[
1 = \varphi(F_s^\vartheta) \leq \varphi(E_e^\vartheta),
\]

whence \( e \in \sigma_\varphi = \emptyset \), a contradiction. This shows that \( \hat{\theta}_s(\varphi) \) is not open, as desired.

To prove the last assertion in the statement, suppose that

\[
\hat{\theta}_{u_1}(\varphi_1) = \hat{\theta}_{u_2}(\varphi_2),
\]

where \( u_1, u_2 \in \tilde{S} \), and \( \varphi_1 \) and \( \varphi_2 \) are ground characters. We first observe that one cannot have \( u_1 \in S \), and \( u_2 = 1 \) (or vice-versa), since otherwise \( \hat{\theta}_{u_1}(\varphi_1) \) is not a ground character by (17.7), so it cannot possibly coincide with the ground character \( \varphi_2 \). If both \( u_1 \) and \( u_2 \) coincide with 1, there is nothing to do, so we suppose from now on that

\[
s_i := u_i \in S, \quad \forall i = 1, 2,
\]

hence our hypothesis reads:

\[
\hat{\theta}_{s_1}(\varphi_1) = \hat{\theta}_{s_2}(\varphi_2).
\]

Using (17.7.ii) we have that

\[
\delta_{s_1} = \sigma_{\hat{\theta}_{s_1}(\varphi_1)} = \sigma_{\hat{\theta}_{s_2}(\varphi_2)} = \delta_{s_2}.
\]

By the first part of the proof we have that \( \hat{\theta}_{s_1}(\varphi_1) \) is not open, so the string displayed above is likewise not open, and we deduce from the uniqueness in (11.2.ii) that \( s_1 = s_2 \). The fact that \( \varphi_1 = \varphi_2 \) now follows easily. \( \square \)
We may now combine several of our earlier results to give a description of all ultracharacters on $\mathcal{E}(S)$:

17.11. Theorem. Let $S$ be a 0-left cancellative semigroup admitting least common multiples. Denote by $\mathcal{E}_\infty(S)$ the set of all ultracharacters on $\mathcal{E}(S)$, and by $\mathcal{E}^\text{op}_\infty(S) = \mathcal{E}_\text{op}(S) \cap \mathcal{E}_\infty(S)$, namely the subset formed by all open ultracharacters. Then

(i) $\mathcal{E}^\text{op}_\infty(S) = \{ \varphi_\sigma : \sigma \text{ is an open, quasi-maximal string in } S \}$, and
(ii) $\mathcal{E}_\infty(S) \setminus \mathcal{E}^\text{op}_\infty(S) = \{ \hat{\theta}_u(\varphi) : u \in \tilde{S}, \varphi \text{ is a ground, ultracharacter in } \hat{\mathcal{F}}_u \}$.

Proof. In order to prove (i) observe that if $\varphi$ is an open ultracharacter, then $\varphi = \varphi_\sigma$ for some open, quasi-maximal string $\sigma$ by (16.20). On the other hand, if $\sigma$ is an open, quasi-maximal string, then $\varphi_\sigma$ is an ultracharacter by definition, while

$$\sigma = \Sigma(\Phi(\sigma)),$$

by (16.15.iii), so $\varphi_\sigma = \Phi(\sigma)$ is an open character.

By [14: Proposition 3.2], one has that the dual representation of $\mathcal{H}(S)$ on $\mathcal{E}(S)$ leaves the set of ultracharacters invariant, so a character $\varphi$ in $\hat{\mathcal{F}}_u$ is an ultracharacter if and only if the same holds for $\hat{\theta}_u(\varphi)$. This said, point (ii) follows immediately from (17.10). □

The upshot of all this is that, in order to fully understand the ultracharacters of $\mathcal{E}(S)$, one first needs to describe the open, quasi-maximal strings. The remaining ultracharacters are therefore obtained as the orbit under $\hat{\theta}$ of the ground, ultracharacters.

Should the above program be brought to completion, one would therefore be able to describe all tight characters, since $\mathcal{E}^\text{tight}(S)$ is well known to be the closure of the set of ultracharacters.

List of symbols

| Symbol | Description                                      | Page |
|--------|-------------------------------------------------|------|
| $\mathcal{I}(\Omega)$ | Symmetric inverse semigroup on the set $\Omega$ | 5    |
| $\pi$  | Representation of a semigroup                    | 5    |
| id$_X$ | Identity function on the set $X$                 | 5    |
| $\mathcal{P}(\Omega)$ | The set of all subsets of $\Omega$              | 5    |
| $F^\pi_s$ | Domain of $\pi_s$                              | 5    |
| $E^\pi_s$ | Range of $\pi_s$                               | 5    |
| $F_s$  | Simplified notation for domain of $\pi_s$       | 5    |
| $E_s$  | Simplified notation for range of $\pi_s$        | 5    |
| $f^\pi_s$ | Identity function on $F^\pi_s$                  | 5    |
| $e^\pi_s$ | Identity function on $E^\pi_s$                  | 5    |
| $\Omega^\#_s$ | Essential subset                                | 5    |
\(\mathcal{I}(\Omega, \pi)\) The inverse semigroup generated by the range of a representation of \(\pi\) \text{ ... 6}

\(\mathcal{P}(\Omega, \pi)\) The collection of all \(\pi\)-constructible sets \text{ ...................... 6}

\(F_s\) Domains for the regular representation \text{ ........................................ 8}

\(E_s\) Ranges for the regular representation \text{ ........................................ 8}

\(\theta\) Regular representation \text{ .......................................... 8}

\(S'\) Set of all nonzero elements of \(S\) \text{ ...................................... 8}

\(\mathcal{H}(S)\) Inverse hull of \(S\) \text{ ........................................... 10}

\(\mathcal{E}(S)\) Set of all \(\theta\)-constructible subsets of \(S'\) \text{ ........................................ 10}

\(s^+\) Right local unit for \(s\) \text{ ............................................. 11}

\(s | t\) \(s\) divides \(t\) \text{ .................................................. 15}

\(\widetilde{S}\) Unitized semigroup \(S \cup \{1\}\) \text{ ........................................ 15}

\(F^\pi_{\Lambda}\) Intersection of \(F^\pi_u\), for \(u\) in \(\Lambda\) \text{ ........................................ 22}

\(f^\pi_{\Lambda}\) Product of \(f^\pi_u\), for \(u\) in \(\Lambda\) \text{ ........................................ 22}

\(\sigma\) A string \text{ ........................................... 34}

\(\delta_s\) The string of divisors of \(s\) \text{ ........................................ 34}

\(S^*\) The set of all strings in \(S\) \text{ ........................................ 35}

\(r * \sigma\) Product of the semigroup element \(r\) by the string \(\sigma\) \text{ ........... 35}

\(r^{-1} * \sigma\) Inverse product of the semigroup element \(r\) by the string \(\sigma\) \text{ .......... 35}

\(\theta^*\) Representation of \(S\) on \(S^*\) \text{ ............................................. 36}

\(F^*_r\) Domain of \(\theta^*_r\) \text{ ............................................. 36}

\(E^*_r\) Range of \(\theta^*_r\) \text{ ............................................. 36}

\(S^\infty\) The set of all maximal strings \text{ ....................................... 41}

\(\hat{\sigma}\) The interior of the string \(\sigma\) \text{ ......................................... 43}

\(\varepsilon\) Representation of \(\mathcal{E}(S)\) on \(S^*\) \text{ ........................................... 52}

\(\ell\) Length function \text{ ................................................... 53}

\(\hat{\mathcal{E}}_\pi\) Set of all \(\pi\)-tight characters on the semilattice \(\mathcal{E}\) \text{ .............. 63}

\(\varphi_\sigma\) Character associated with the string \(\sigma\) \text{ ..................................... 68}

\(\sigma_\varphi\) String associated with the character \(\varphi\) (when nonempty) \text{ .......... 69}

\(\hat{\mathcal{E}}(S)\) The spectrum of \(\mathcal{E}(S)\) \text{ ........................................... 70}

\(\Phi\) Map sending non-degenerate strings to their associated characters \text{ .......... 70}

\(\mathcal{E}_1(S)\) A subsemilattice of \(\mathcal{E}(S)\) \text{ ........................................... 70}

\(\Sigma\) Map sending characters to their associated strings \text{ .................................. 71}

\(S^\infty\) The set of all quasi-maximal strings \text{ ....................................... 74}

\(\hat{\theta}\) Dual representation of \(S\) on \(\hat{\mathcal{E}}(S)\) \text{ ........................................... 76}

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\[
\begin{align*}
\hat{F}_s & \quad \text{Domain of } \hat{\theta}_s & \quad 76 \\
\hat{E}_s & \quad \text{Range of } \hat{\theta}_s & \quad 76 \\
\mathcal{E}_{\text{op}}(S) & \quad \text{Set of all open characters on } \mathcal{E}(S) & \quad 80
\end{align*}
\]

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