Walks on SPR Neighborhoods

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Abstract

A nearest-neighbor-interchange (NNI) walk is a sequence of unrooted phylogenetic trees, $T_0, T_1, T_2, \ldots$ where each consecutive pair of trees differ by a single NNI move. We give tight bounds on the length of the shortest NNI-walks that visit all trees in a subtree-prune-and-regraft (SPR) neighborhood of a given tree. For any unrooted, binary tree, $T$, on $n$ leaves, the shortest walk takes $\theta(n^2)$ additional steps than the number of trees in the SPR neighborhood. This answers Bryant’s Second Combinatorial Conjecture from the Phylogenetics Challenges List, the Isaac Newton Institute, 2011, and the Penny Ante Problem List, 2009.

1 Introduction

Evolutionary histories, or phylogenies, are essential structures for modern biology [10]. Finding the optimal phylogeny is NP-hard, even when we restrict to tree-like evolution [8, 12]. As such, heuristic searches are used to search the vast set of all trees. There are many search techniques used (see [15] for a survey), but most rely on local search. That is, at each step in the search, the next tree is chosen from the “neighbors” of the current tree. A popular way to define neighbors is in terms of the subtree-prune-and-regraft (SPR) metric (defined in Section [2]). For a given unrooted tree on $n$ leaves, or taxa, the SPR-neighborhood is the number of trees that differ by a single SPR move. The number of trees in the SPR neighborhood is $(2n - 6)(2n - 7)$. The second “Walks on Trees” conjecture of Bryant [5, 14] focuses on efficiently traversing this neighborhood via the nearest-neighbor-interchange (NNI) transformations (defined in Section [2]). Bryant asks:

i. [Question] What is the shortest NNI walk that passes through all binary trees on $n$ leaves?

ii. [Question] Suppose we are given a tree $T$. What is the shortest NNI walk that passes through all the trees that lie at most one SPR (subtree prune and regraft) move from $T$?

Bryant’s conjectures were posed as part of the New Zealand Phylogenetic Meetings’ Penny Ante Problems [8] as well as the Challenges problems from the most recent Phylogenetics Meeting at the Isaac Newton Institute [14].

We answer the second question, proving that the shortest walk takes $\Theta(n^2)$ more steps than the theoretical minimum that visits every tree exactly once (that is, a Hamiltonian path). This builds on past work [6] that showed that a Hamiltonian path was not possible.
2 Background

This section includes definitions and results that we use from Allen & Steel [1]. For a more detailed background on mathematically phylogenetics, see Semple & Steel [13].

**Definition 1.** An unrooted binary phylogenetic tree (or more briefly a binary tree) is a tree whose leaves (degree 1 vertices) are labelled bijectively by a (species) set $S$, and such that each non-leaf vertex is unlabelled and has degree three. We let $UB(n)$ denote the set of such trees for $S = \{1, \ldots, n\}$.

Each internal edge, $e$ of a tree $T \in UB(n)$ yields a natural bipartition, or split of the taxa; We write $A | B$ if there is an edge which partitions the leaf set into the two sets $A$ and $B$. We use the standard notation of $T_A$ to refer to the smallest subtree of $T$ containing leaves only from $A$.

Figure 1 shows several binary trees. Each edge of a tree induces a split of the leaf set. The Nearest Neighbor Interchange (NNI) is a distance metric introduced independently by DasGupta et al. [7] and Li et al. [11]. Roughly, an NNI operation swaps two subtrees that are separated by an internal edge in order to generate a new tree [1].

**Definition 2.** Allen and Steel [1]: Any internal edge of an unrooted binary tree has four subtrees attached to it. A nearest neighbor interchange (NNI) occurs when one subtree on one side of an internal edge is swapped with a subtree on the other side of the edge, as illustrated in Figure 1.

**Definition 3.** The NNI distance, $d_{NNI}(T_1, T_2)$, between two trees $T_1$ and $T_2$ is defined as the minimum number of NNI operations required to change one tree into the other.

The complexity of computing the NNI distance was open for over 25 years, and was proven to be NP-complete by Allen and Steel [1]. For a tree with $n$ uniquely labeled leaves, there are $n - 3$ internal branches. Thus, there are $2(n - 3)$ NNI rearrangements for any tree.

One of the most popular moves used to search treespace is the Subtree-Prune-and-Regraft (SPR). Roughly, an SPR move prunes a selected subtree and then reattaches it on an edge selected from the remaining tree (see Figure 1).

**Definition 4.** Allen and Steel [1]: A subtree prune and regraft (SPR) on a phylogenetic tree $T$ is defined as cutting any edge and thereby pruning a subtree, $t$, and then regrafting the subtree by the same cut edge to a new vertex obtained by subdividing a pre-existing edge in $T - t$. We also apply a forced contraction to maintain the binary property of the resulting tree (see Figure 1).

**Definition 5.** The SPR distance, $d_{SPR}(T_1, T_2)$, between two trees is the minimal number of SPR moves needed to transform the first tree into the second tree.
The calculation of SPR distances has been proven NP-complete for both rooted and unrooted trees \[4, 9\]. Approximation algorithms for SPR on rooted trees exist \[2, 3\].

**Definition 6.** Let \( T_0 \) be an unrooted, binary tree. Define \( N_{SPR}(T_0) \) to be the **SPR-neighborhood** of \( T_0 \), namely,

\[
N_{SPR}(T_0) = \{ T \mid d_{SPR}(T_0, T) = 1 \}
\]

When the tree in question is obvious, we will drop the argument and call the neighborhood \( N_{SPR} \).

**Definition 7.** Let \( T_0 \) be an unrooted, binary tree and let \( S \) be a set of trees that are 1 SPR move from \( T_0 \). Define \( N_{NNI}(S, T_0) \) to be the **NNI-neighbors** of \( S \), namely,

\[
N_{NNI}(S, T_0) = \{ T \mid \exists T' \in S, d_{NNI}(T, T') = 1 \text{ and } d_{SPR}(T_0, T') = 1 \}
\]

We note that for every subset \( S \) of the SPR neighborhood of \( T_0 \), \( S \subseteq N(S) \).

A “sibling pair” or “cherry” in a tree are two leaves that have the same parent. A “caterpillar tree” refers to the unrooted tree with exactly 2 sibling pairs.

### 3 Results

Theoretically, the shortest walk of the SPR neighborhood would be if each tree could be visited exactly once (that is, a “Hamiltonian path”). In \[6\], this was shown to be impossible for \( n \geq 6 \). This was done by showing that in the SPR neighborhood of caterpillar trees, there are at least 4 ‘isolated triangles’ on the outer edge of the neighborhood (see Figure 2) that force at least two trees to be visited twice.

To bound the number of steps needed to visit the SPR neighborhood, we first introduce a new concept of an orbit of an edge. The orbit is all trees created by breaking that edge (in either direction) in an SPR move. More formally:

**Definition 8.** Define for each edge \( e \) of the tree \( T_0 \), the **orbit** of \( e \), \( O_e \), to be all the trees that are one SPR move from \( T_0 \) where the edge broken by the SPR move is \( e \).

In Figure 2 the SPR-neighborhood, as well as orbits, of the 7-taxon caterpillar tree are shown.

To calculate the size of the SPR neighborhood of a tree, Allen and Steel (proof of Theorem 2.1, \[1\]) characterized the relationship between the trees in the neighborhood.
Theorem 9. Allen and Steel [11]: Let $T_0$ be an unrooted phylogenetic tree on $n$ leaves and let $N_{SPR} = N_{SPR}(T_0)$ be all trees that are a single SPR move from $T_0$.

1. The size of the SPR neighborhood is $|N_{SPR}| = 2(n - 3)(2n - 7)$.
2. The number of trees in $N_{SPR}$ that can be obtained by more than one SPR move from $T_0$ are exactly those from the NNI transformations. Thus, there are $2n - 6$ of them.
3. The number of trees in $N_{SPR}$ that can be obtained by only one SPR move from $T_0$ is $4(n - 3)(n - 4)$.

From this, we make the following simple observations about orbits:

Observation 10. Let $T_0$ be an unrooted phylogenetic tree on $n$ leaves:

1. Every tree $T \in N_{SPR}$ belongs to some orbit $O_e$ for $e \in E(T_0)$.
2. Each orbit contains $T_0$.
3. Excluding $T_0$, there are exactly $2n - 6$ trees that are included in two orbits.
4. The number of orbits is the number of edges in the tree, $2n - 3$.
5. The size of each orbit is $2n - 5$.

The SPR neighborhood is the union of the orbits, but surprisingly, these orbits are mostly disjoint. Roughly, the overlap of orbits is very small and they have very few neighbors in common. Formally:

Lemma 11. Let $T_0$ be an unrooted phylogenetic tree on $n$ taxa. Let $T_i, T_j \in N_{SPR}(T_0)$ and there exists $e \in E(T_0)$, $T_i, T_j \in O_e$. Let $e_i$ be the target edge of the move that created $T_i$ for $i = 1, 2$ (that is, $T_1$ is formed by grafting some pruned subtree of $T_0$ to $e_1$ and $T_2$ is the result of grafting a pruned subtree to $e_2$).

Then, $T_1$ and $T_2$ differ by a single NNI move if and only if $e_1$ and $e_2$ have a common endpoint in $T_0$.

Proof. $\iff$: Assume that $e_1$ and $e_2$ have a common endpoint in $T_0$. Let $M$ be the subtree pruned by the SPR move that creates $T_1$. Without loss of generality, let the split induced by $e_1$ in $T_0$ be $ABC | DEM$ and the split induced by $e_2$ in $T_0$ be $AB | CDEM$, where $A, B, C, D, E$, and $M$ are the leaves of subtrees of $T_0$. Let $T_X$ refer to the subtree with leaves only from the set $X$.

If $T_M$ is pruned to create $T_1$, then we have that $T_1$ contains the splits: $ABM | CDE$ and $AB | MCDE$. If $T_M$ is also pruned to create $T_2$, then we have that $T_2$ contains the splits: $ABCM | DE$ and $ABC | MDE$. Thus, $T_1$ and $T_2$ differ by a single NNI move (swapping $T_C$ and $T_M$), and the hypothesis holds.

So, assume that $T_M$ is not pruned to create $T_2$, but instead that $e$ is pruned in the other direction. Let $N = S - M$, where $S$ is the set of leaves of $T$. Since $e_1$ and $e_2$ share an endpoint, at least one of them must be the edge pruned, $e$. If both are $e$, then $T_1 = T_2 = T$, and the hypothesis is trivially true. If only one, say $e_2$, is $e$, then $e_1$ must be a neighbor of $e$ in $T$ which implies $T_2 = T_1 = T$, and again the hypothesis is trivially true.

$\implies$: By assumption $T_1$ and $T_2$ differ by a single NNI move. By definition of the NNI move, there exists an edge $e' \in E(T_i)$ that when removed, breaks $T_i$ into 4 distinct subtrees, $T_A, T_B, T_C, T_D$ with leaf sets, $A, B, C, D$, and the split $AB | CD$ belongs to $T_1$ while $BC | AD$ belongs to $T_2$. Since both $T_1$ and $T_2$ are in the same orbit, the same edge $e$ is pruned to create both. We note that since they differ by only the NNI move, that, by the argument above, the pruning of $e$ must occur in the same direction for both to be result in non-trivial trees. Further, $e$ must prune one of the subtrees: $A, B, C, D$, since only one move is allowed and $T_1$ and $T_2$ contain exactly the same trees. Without loss of generality, assume that $A$ is pruned. We note the trees induced by the leaves of $B, C, D$ are identical for these subtrees: $T_0|_{L(BUCD)} = T_1|_{L(BUCD)} = T_2|_{L(BUCD)}$. It follows that $e_1$ and $e_2$ share a common endpoint,
namely the intersection point of $B, C, D$. \hfill \Box

We can immediately give an upper bound on the NNI-walk of the SPR neighborhood as $O(n^2)$ steps. The underlying idea is to traverse each orbit separately, and then link these paths to form a traversal of the entire SPR neighborhood:

**Lemma 12.** The SPR neighborhood has an NNI-walk of length $O(n^2)$.

**Proof.** We will break the NNI-walk of the SPR neighborhood into a NNI-walk of the orbit of each edge in $T_0$. Since each orbit contains the initial tree $T_0$, we can glue together the walks of the orbit to make a walk of the entire space. We note that since each orbit contains at most $2n - 5$, walking the $2n - 3$ orbits in this fashion yields a walk with the number of steps is bounded by $2(2n - 5)(2n - 3) = O(n^2)$.

It suffices to show that there is a 2-walk of each orbit $O_e$ for $e \in E(T_0)$. Each tree, $T \in O_e$, is created by pruning the edge $e$ in $T_0$ and re-grafting the pruned subtree to another edge in $T_0$ (see Figure 2). Every tree in the orbit corresponds to an edge in $T_0$ (namely, the target edge), and the trees in the orbit are connected exactly when their target edges share an endpoint in $T_0$ by Lemma 11. Thus, the orbit can be traversed by at most $2(2n - 3)$ steps by starting at $T_0$ and following a depth-first-search of the tree (each tree in the orbit is visited at most once on the way “down” the search and once on the way “up” the search). \hfill \Box

To show the lower bound takes more work. It follows from this lemma that every orbit has very small overlap with the other orbits:

**Lemma 13.** Let $T_1, T_2 \in N_{SPR}(T_0)$ such that $T_1$ and $T_2$ are a single NNI move apart. Then $d_{NNI}(T_0, T_1), d_{NNI}(T_0, T_2) \leq 2$.

**Proof.** We note that if there exists an $e \in E(T_0)$ such that $T_1, T_2 \in O_e$, then by Lemma 11 the lemma holds.

So, let us assume that there exists $e_1, e_2 \in E(T_0)$, $T_1 \in O_{e_1}$, $T_1 \notin O_{e_2}$, $T_1 \in O_{e_1}$ and $T_1 \notin O_{e_2}$. Let $M_i$ be the leaves of the subtree pruned with $e_i$ from $T_0$ to create tree $T_i$, $i = 1, 2$. Since $T_1$ and $T_2$ are a single NNI move apart. By definition, there exists a split in $T_1$, $AB \mid CD$ that is rearranged in $T_2$: $BC \mid AD$.

We will argue, by cases, that both $T_1$ and $T_2$ are within 2 NNI moves of $T_0$. Without loss of generality, we will assume that $M_1 \cap T_A \neq \emptyset$

**Case 1:** $M_1 \subseteq T_A$. Then, let $A' = A - M_1$. So, we have that $T_1$ contains the split $A'M_1B|CD$ and $T_2$ contains the split $BC \mid A'M_1D$. Since $T_1$ is only one SPR move from $T_0$, the structure of the 2 trees is identical without $M_1$, that is, $T_1|A'\cup UBUCD = T_0|A'\cup UBUCD$, and $T_0$ includes and edge that splits $A'$ and $B$ from $C$ and $D$. Since $T_2$ does not contain such an edge, the move that creates it must prune one of $M_1$, $T_A'$, $T_B$, $T_C$, or $T_D$. Pruning $T_{M_1}$ is not possible since $T_1$ and $T_2$ are in different orbits. Pruning $T_A'$ is only possible if $T_{M_1}$ and $T_B$ have the same parent, in which case the $d_{NNI}(T_0, T_1), d_{NNI}(T_0, T_2) = 2$ and the lemma holds. Pruning $T_B$ to create $T_2$ means that the subtree $T_{M_1}$ is on the edge separating $A'$ and $B$ from $C$ and $D$ in $T_0$ and $d_{NNI}(T_0, T_1), d_{NNI}(T_0, T_2) = 2$. Lastly, pruning $T_B$, $T_C$, or $T_D$ is only possible if $T_0 = T_1$, in which case, $d_{NNI}(T_0, T_1) = 0, d_{NNI}(T_0, T_2) = 1$.

**Case 2:** $M_1 = A$. So, we have that $T_1$ contains the split $M_1B|CD$ and $T_2$ contains the split $BC \mid M_1D$. We have three possibilities for $T_0$, namely, it could contain one of the following three splits: $M_1B \mid CD, BC \mid M_1D$, or $BD \mid M_1C$. In each of these cases, we have $d_{NNI}(T_0, T_1), d_{NNI}(T_0, T_2) \leq 1$ and the lemma holds.

**Case 3:** $M_1 \supseteq A$. So, $M_1 \cap B \neq \emptyset$. Since $T_{M_1}$ is a subtree of $T_1$ and of $T_2$, it must contain all of $B$. If $M_1 = A \cup B$, then the target edges in $T_1$ and $T_2$ must separate $C$ and $D$, and are identical. Similarly, if $M_1 \subseteq A \cup B$, $M_1$ must contain all of $C$ or all of $D$, and the taget edges in $T_1$ and $T_2$ must preserve the rooting of the remaining subtree, and thus, are identical. Thus, $d_{NNI}(T_0, T_1), d_{NNI}(T_0, T_2) = 0$. \hfill \Box
Lemma 14. Let $U \subseteq O_e$ be connected consist of trees more than 2 NNI moves from $T_0$. Let $n = |U|$. Then any NNI-circuit of $U$ takes at least $\frac{3}{2}(n-1)$ steps.

Proof. By induction on the size of $|U|$.

For $|U| = 1$: This is trivially true.

For $|U| > 1$, choose $x \in U$ closest to $T_0$. Since $x$ is closest to $T_0$, not all of the neighbors of $x$ are in $U$ (if so, then there is an element in $U$ closer to $T_0$). Since $T_0$ is binary, it has at most 4 neighbors in $N_{SPR}$. If $x$ has one neighbor in $U$, then, a circuit of $U$ must traverse the same edge from $x$ to its neighbor twice, and the number of steps needed is at least two more than the number of steps needed for the smaller set $|U| - \{x\}$. By inductive hypothesis, this smaller set takes at least $\frac{3}{2}(|U| - \{x\} - 1)$ steps. So, the number of steps for $U$ is:

$$\frac{3}{2}(|U| - \{x\} - 1) + 2 \geq \frac{3}{2}(|U| - 1)$$

If $x$ has two neighbors in $U$, then call the sub-trees rooted at $x$’s neighbors, $U_1$ and $U_2$. If the neighbors of $x$ are connected, then it takes 3 steps to visit $x$ in a circuit of $x$, $U_1$, and $U_2$. If they are not connected, it takes 4 steps. Thus, by inductive hypothesis, the number of steps needed is:

$$\frac{3}{2}(|U_1| - 1) + \frac{3}{2}(|U_2| - 1) + 3 \geq \frac{3}{2}(|U| - 1)$$

If $x$ has 3 neighbors in $U$, then by similar argument, we have the lower bound.

If $x$ has 4 neighbors in $U$, then it is not the closest element of $U$ to $T_0$, giving a contradiction.

From the last two lemmas, we have that the orbits are mostly isolated – the only trees having neighbors from outside the orbit are within 2 steps of $T_0$. Each of these isolated arms of the orbit must be visited in an NNI walk of the SPR neighborhood, and the walks of the isolated arms take many extra steps. This yields our lower bound:

Lemma 15. It takes $\Omega(n)$ extra steps to make a circuit of an orbit.

Proof. Let $e \in E(T_0)$ and $O_e$ its orbit. Since each orbit has $2n - 5$ trees (Observation 10) and by Lemma 11 at most 8 have neighbors from $N_{SPR} - O_e$.

It follows from Lemma 11, the $2n - 13$ remaining trees are in two connected sets. By the Pigeonhole Principal, one set has at least $n - 7$ trees. By Lemma 14, it takes $\Omega((n-7)-(1)/2) = \Omega(n)$ extra steps to visit the larger connected set. Thus, it takes $\Omega(n)$ extra steps to traverse the orbit.

The above lemmas can be combined to show that $\theta(n^2)$ extra steps are needed to traverse the neighborhood, since there are $2n - 3$ orbits, and each has minimal overlap with other orbits.

Theorem 16. Every SPR neighborhood takes $(2n - 6)(2n - 7) + \Theta(n^2)$ steps to traverse.

Proof. The upper bound follows by Lemma 12.

For the lower bound: by Lemma 13 every orbit, $O_e$ has $\Omega(n)$ trees that have no neighbors in other orbits. By Lemma 15, it takes $\Omega(n)$ extra steps to traverse these regions of $O_e$. Since, by Theorem 9 there are $2n - 3$ orbits, we have that any path must take $\geq (2n-3)\Omega(n) = \Omega(n^2)$ extra steps.

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