A NEW CRITERION TO A TWO-CHEMICAL SUBSTANCES CHEMOTAXIS SYSTEM WITH CRITICAL DIMENSION

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ABSTRACT. We mainly investigate the global boundedness of the solution to the following system,
\[
\begin{aligned}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) \quad \text{in } \Omega \times \mathbb{R}^+, \\
    v_t &= \Delta v - v + w \quad \text{in } \Omega \times \mathbb{R}^+, \\
    w_t &= \Delta w - w + u \quad \text{in } \Omega \times \mathbb{R}^+,
\end{aligned}
\]
under homogeneous Neumann boundary conditions with nonnegative smooth initial data in a smooth bounded domain \( \Omega \subset \mathbb{R}^n \) with critical space dimension \( n = 4 \). This problem has been considered by K. Fujie and T. Senba in [5]. They proved that for the symmetric case the condition \( \int_\Omega u_0 < (8\pi)^2 \chi \) yields global boundedness, where \( u_0 \) is the initial data for \( u \). In this paper, inspired by some new techniques established in [3], we give a new criterion for global boundedness of the solution. As a byproduct, we obtain a simplified proof for one of the main results in [5].

1. Introduction. Consider the following chemotaxis system
\[
\begin{aligned}
    u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) \quad \text{in } \Omega \times \mathbb{R}^+, \\
    v_t &= \Delta v - v + w \quad \text{in } \Omega \times \mathbb{R}^+, \\
    w_t &= \Delta w - w + u \quad \text{in } \Omega \times \mathbb{R}^+,
\end{aligned}
\]
in a smooth and bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \leq 4)\), where \( \chi \) is a positive constant. Suppose that the initial data \( (u_0, v_0, w_0) \) satisfies
\[
\begin{aligned}
    u_0 \in C^0(\overline{\Omega}), \quad u_0 \geq 0, \quad \text{in } \Omega, \\
    v_0 \in C^2(\overline{\Omega}), \quad v_0 \geq 0, \quad \text{in } \Omega, \\
    w_0 \in C^2(\overline{\Omega}), \quad w_0 \geq 0, \quad \text{in } \Omega, \\
    \frac{\partial u_0}{\partial \nu} = \frac{\partial v_0}{\partial \nu} = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

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A detailed introduction of this model could be found in K. Fujie and T. Senba in [5] and for more discussion related to chemotaxis models, we refer the reader to survey papers [2, 7, 8]. In this paper, we intend to give an extension as well as a simplified proof for one of the main results in [5] in the critical case \( n = 4 \). In particular, we establish the following result.

**Theorem 1.1.** Assume that \( n = 4 \) and (2) is valid. Then there exists a maximal existence time \( T_{\max} \in (0, \infty] \) and a uniquely determined triple \((u, v, w) \in [C^{2,1}(\bar{\Omega} \times (0, T)) \cap C(\bar{\Omega} \times [0, T))]^3 \) which solves (1) classically in \( \Omega \times (0, T_{\max}) \). Moreover, if \( T_{\max} < \infty \) then

\[
\sup_{t \in [0, T_{\max})} \left( \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{2,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \right) = \infty.
\]

If in addition we assume that either \( \int_{\Omega} u_0 \, dx \) is small enough or \( \{u(\cdot, t)\}_{t \in (0, T_{\max})} \) is uniformly integrable, that is, to each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( E \subset \Omega \) and \( |E| < \delta \) implies \( \sup_{t \in (0, T_{\max})} \int_E u(x, t) \, dx < \varepsilon \), then \( T_{\max} = \infty \) and the solution \((u, v, w)\) is globally bounded in time in the sense that

\[
\sup_{t \in [0, \infty)} \left( \|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{2,\infty}(\Omega)} + \|w(t)\|_{W^{1,\infty}(\Omega)} \right) < \infty.
\]

**Remark 1.** If \( \Omega = B_R(0) \), \( u_0, v_0, w_0 \) are radially symmetric and \( \int_{\Omega} u_0 < \frac{(8\pi)^2}{\chi} \), K. Fujie and T. Senba have shown that (1) admits a unique solution satisfying (see [5, Lemma 7.3])

\[
\sup_{t \in [0, T_{\max})} \int_{\Omega} u \log u \, dx < \infty,
\]

which implies that \( u \) is uniformly integrable. Therefore, our result can be considered as a new and shorter proof of [5, Theorem 1.3].

Without the radially symmetric assumption, we show that if \( \int_{\Omega} u_0 \, dx \leq \varepsilon_0 \) with \( \varepsilon_0 \) small enough, the solution is globally bounded. Here \( \varepsilon \) may be less than \( \frac{(8\pi)^2}{\chi} \). It is still open that whether the condition \( \int_{\Omega} u_0 \, dx < \frac{(8\pi)^2}{\chi} \) can ensure the global boundedness for solution of (1) when \( \Omega \) is general domain (see [5, Remark 1.5]).

2. **preliminaries.** In this part, we give several lemmas which are fatal to our proof, we will use the symbol \( \| \cdot \|_p \) to denote the \( L^p(\Omega) \) norm for simplicity. The first one is the optimal sobolev regularity lemma. Instead of using [6, Theorem 3.1] directly, we adopt [4, Lemma 2.5] (with appropriate modifications) here which is more effective for our purpose.

**Lemma 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be a smooth and bounded domain, \( r, q \in (1, \infty) \). There exists \( C > 0 \) depending on \( q, r, \Omega \) such that for any \( f \in L^r((0, T); L^q(\Omega)) \), \( v_0 \in W^{2,q}(\Omega) \) satisfying \( \frac{\partial v_0}{\partial n} = 0 \) on \( \partial \Omega \) and \( T \in (0, \infty) \), if \( v \in W^{1,r}((0, T); L^q(\Omega)) \cap L^r((0, T); W^{2,q}(\Omega)) \) is the unique strong solution to the following evolution equation

\[
\begin{align*}
\frac{\partial v}{\partial t} &= \Delta v - v + f & \text{in } \Omega \times (0, T), \\
\frac{\partial v}{\partial n} &= 0 & \text{on } \partial \Omega \times (0, T), \\
v(x, 0) &= v_0(x), & \text{in } \Omega,
\end{align*}
\]

then

\[
\int_0^T e^{\frac{s}{T}} \| \Delta v(\cdot, s) \|_{L^r(\Omega)}^r \, ds \leq C \int_0^T e^{\frac{s}{T}} \| f(\cdot, s) \|_{L^r(\Omega)}^r \, ds + C \| v_0 \|_{W^{2,q}(\Omega)}.
\]


Lemma 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $\alpha \in (1, n)$. For all $s \in (0, \infty)$, there exists a constant $C > 0$ such that

$$\|\nabla v\|_{\infty} \leq C \|\Delta v\|_\alpha + C \|v\|_s, \forall v \in W^{2,\alpha}$$ with $\frac{\partial v}{\partial \nu} = 0$ on $\partial \Omega$, \hspace{1cm} (5)

and

$$\|v\|_{\infty} \leq C \|\Delta v\|_\alpha + C \|v\|_s, \forall v \in W^{2,\alpha}$$ with $\frac{\partial v}{\partial \nu} = 0$ on $\partial \Omega$. \hspace{1cm} (6)

Proof. Inequality (5) is just the conclusion of [4, Lemma 4.1]. The inequality (6) can be derived by Gagliardo-Nirenberg interpolation inequality and (5). \hspace{1cm} \Box

Lemma 2.3 (special case of [3, Lemma 2.2]). Let $\Omega \subset \mathbb{R}^4$ be smooth and bounded, $0 < q < 2$, $b = \frac{4-2q}{4-q}$ and $l = \frac{2-q}{2-q+4}$. Assume that $\delta : (0, 1) \to (0, \infty)$ is nondecreasing, then for each $\varepsilon > 0$, we can find $C_\varepsilon > 0$ such that

$$\|\varphi\|_2 \leq \varepsilon \|\nabla \varphi\|_2 \|\varphi\|_{q-1} + C_\varepsilon (b, q, \|\varphi\|_q)$$

for any $\varphi \in F_\delta := \{ \varphi \in W^{1,2}(\Omega) | \text{For all } \varepsilon' \in (0, 1), |E| < \delta(\varepsilon') \text{ implies } \int_E \psi dx < \varepsilon' \}$.

3. Proof of the main result.

Proof of Theorem 1.1. Since the local existence and global boundedness under uniform $L^p$-prior estimate of $u$ has been established in [5] (Proposition 4.1 and Lemma 5.4), we only need to show that there exists $p > 1$ such that $\sup_{0 < t < \tau_{max}} \|u(\cdot, t)\|_p$ is bounded. It is routine to check that $\|u\|_{L^1(\Omega)}$, $\|v\|_{L^1(\Omega)}$ and $\|v\|_{L^1(\Omega)}$ are uniformly bounded in $t$. Hence, throughout the rest of this paper, we always assume that $C$ is a positive constant which may depend on $\sup_{0 < t < \tau_{max}} (\|u\|_{L^1(\Omega)} + \|v\|_{L^1(\Omega)} + \|w\|_{L^1(\Omega)})$ and change from place to place. Multiplying the first equation by $\frac{1}{p} u^{p-1}$, we obtain that

$$\frac{\partial}{\partial t} \left( \int_{\Omega} \frac{1}{p} u^p \right) = \int_{\Omega} u^{p-1} u_t dx$$

$$= - \frac{4(p-1)}{p^2} \int_{\Omega} \nabla u \nabla \frac{x}{u^2} dx + (p-1) \chi \int_{\Omega} u^{p-2} \nabla u \cdot \nabla v dx$$

$$= - \frac{4(p-1)}{p^2} \int_{\Omega} \nabla u \nabla \frac{x}{u^2} dx + \frac{2(p-1)}{p} \chi \int_{\Omega} u^{\frac{2}{p}} \nabla v \nabla \frac{x}{u^2} dx$$

$$\leq - \frac{2(p-1)}{p^2} \int_{\Omega} \nabla u \nabla \frac{x}{u^2} dx + \frac{(p-1)}{2} \chi \int_{\Omega} u^p |\nabla v|^2 dx. \hspace{1cm} (7)$$

Using Gagliardo-Nirenberg interpolation inequality and Young’s inequality, we can estimate $\int_{\Omega} u^p |\nabla v|^2 dx$ as follows:

$$\int_{\Omega} u^p |\nabla v|^2 dx \leq \|u^p\|_{L^p} \|\nabla v\|_{L^{p'}}^2$$

$$= \|u^{\frac{2}{2p}}\|_{L^{2p}} \|\nabla v\|_{2p}^2.$$
derive that

Then we see that

which implies that

θ

where

y

immediately.

ε

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where θ ∈ (1, 2) is some constant to be determined, θ′ = \( \frac{θ}{θ-1} \) and α = \( \frac{2p-2/θ}{2p-1} \) ∈ (0, 1). By Gagliardo-Nirenberg inequality, we have

\[
\frac{1}{(1-a)p} \int_Ω u^p dx \leq \frac{p-1}{2p^2} \int_Ω |∇u|^2 dx + C \left( \int_Ω u dx \right)^θ.
\]

(9)

Substituting (8) and (9) into (7) and denoting y(t) := \( \int_Ω u^p dx \), r := \( \frac{2}{1-θ} \), we derive that

\[
y'(t) + \frac{r}{2} y(t) \leq C\|∇v\|_α^5 \int_Ω u^p dx + C, \tag{10}
\]

which implies that

\[
y(t) \leq e^{-\frac{r}{2}t} y(0) + C \int_0^t e^{\frac{r}{2}(s-t)} (\|∇v\|_α^5 - C^* \|∇u\|_2^2) ds + C. \tag{11}
\]

To estimate \( \int_0^t e^{\frac{r}{2}(s-t)} \|∇v\|_α^5 ds \), we utilize (5), (6) with s = 1 and Lemma 3 to derive that

\[
\int_0^t e^{\frac{r}{2}(s-t)} \|∇v\|_α^5 ds \leq C \int_0^t e^{\frac{r}{2}(s-t)} \||∇v|^α\|_α^5 ds + C
\]

\[
\leq C \int_0^t e^{\frac{r}{2}(s-t)} \|w\|_α^5 ds + C
\]

\[
\leq C \int_0^t e^{\frac{r}{2}(s-t)} \|\Delta v\|_α^5 ds + C
\]

\[
\leq C \int_0^t e^{\frac{r}{2}(s-t)} \|u\|_p^5 ds + C, \tag{12}
\]

where α, β satisfy α = \( \frac{4θ}{2+θ} \) and β = \( \frac{2α}{2+3θ} = \frac{4θ}{5θ-2} \) respectively.

Choosing p = β ∈ (1, \( \frac{2}{3} \)) and substituting (12) into (11), we get

\[
y(t) \leq e^{-\frac{r}{2}t} y(0) + C \int_0^t e^{\frac{r}{2}(s-t)} (\|u\|_p^5 - C^* \|∇u\|_2^2) ds + C. \tag{13}
\]

For the case that u(\cdot, t) is uniformly integrable, it is routine to check that \( u^p \) belongs to \( F_\beta \) with \( q = \frac{p}{2} \). Therefore, we can derive the following result by choosing \( q = \frac{2}{p} \) in Lemma 2.3:

\[
\|u\|_p^r = \|u^\frac{2}{p}\|_2^2 \leq ε \|∇u\|_2^2 \|u^\frac{2}{p}\|_2^{2(1-l)} + C_ε, \tag{14}
\]

where b = \( \frac{2p-2}{2p-1} \) and l ∈ (0, 1). Since p = β = \( \frac{4θ}{5θ-2} \) implies θ = \( \frac{2p}{5p-3} \), we have

\[
r = \frac{2}{p(1-α)} = \frac{2}{p(1-\frac{2p-2/θ}{2p-1})} = \frac{2p-2}{p(\frac{2}{3} - 1)} = \frac{2p-1}{2p-2}.
\]

Then we see that \( \frac{2p}{p} b = \frac{2(2p-2)}{4p-3} \cdot \frac{4-2θ}{4-θ} = 2 \). Choosing ε small enough such that \( ε(\int_Ω u_0 dx)^{(1-l)} < C^* \) in (14), and substituting it into (13), we get boundedness of y immediately.
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For the case that \( \int_{\Omega} u_0 dx \leq \varepsilon_0 \), by Gagliardo-Nirenberg inequality, we have
\[
\|u\|_p = \|u^\frac{b}{p}\|_2 \leq \tilde{C}\|\nabla u^\frac{b}{p}\|_2 (1-b)\|u^\frac{b}{p}\|_p^{(1-b)} + C\|u^\frac{b}{p}\|_p.
\]
(15)

Setting \( \varepsilon_0 = \left( \frac{C^*}{\tilde{C}} \right)^{\frac{1}{1-b}} \) and substituting (15) into (13), we get the boundedness of \( y \).

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