On the Optimality of Treating Interference as Noise: A Combinatorial Optimization Perspective

Xinping Yi, Giuseppe Caire
Technical University of Berlin, 10587 Berlin, Germany
{xinping.yi, caire}@tu-berlin.de

Abstract—For single-antenna Gaussian interference channels, we re-formulate the problem of determining the Generalized Degrees of Freedom (GDoF) region achievable by treating interference as noise (TIN) with proper power control from a combinatorial optimization perspective. We show that the TIN power control problem can be cast into an assignment problem, such that the globally optimal power allocation variables can be obtained by well-known polynomial time algorithms. Furthermore, the expression of the TIN-achievable GDoF region can be substantially simplified with the aid of maximum weighted matchings. In addition, we provide conditions under which the TIN-achievable GDoF region is a convex polytope that relax those in [1]. For these new conditions, together with a channel connectivity (i.e., interference topology) condition, we can prove GDoF optimality for a new class of interference networks that is not included, nor includes, the class found in [1].

I. INTRODUCTION

Power control and treating interference as noise (TIN) is one of the most well-known, vastly employed, and yet most attractive, interference management techniques, due to its low complexity and robustness to channel uncertainty. Interestingly, it has also been shown that in some cases TIN is optimal or approximately optimal. For example, we know that TIN achieves the sum-capacity in the noisy regime of the Gaussian 2-user interference channel [2]. In the general $K$-user single-antenna Gaussian interference channel, Geng et al. [3] have shown that, subject to a certain set of conditions on the channel strengths, TIN achieves the optimal Generalized Degrees of Freedom (GDoF) region, and achieves the capacity region to within a constant gap, independent of the channel coefficients and the signal-to-noise ratio (SNR). The TIN optimality condition found in [1] is simply expressed in words as the fact that, for each user (i.e., intended transmitter-receiver pair) the desired signal strength level is no less than the sum of maximum strengths of all interfering signals from the transmitter to the other (unintended) receivers, and to the receiver from the other (unintended) transmitters, when all signal strengths are expressed in log-scale (e.g., in dB). For future reference, we indicate this condition as the “GNAJ” condition, from the initials of the authors of [1]. Under the GNAJ condition, the TIN-Achievable GDoF region (briefly referred to as “TINA”) is a convex polytope defined by the individual GDoF constraints and by the sum GDoF inequalities corresponding to all possible ordered subsets of users (i.e., by $\sum_{m=2}^{K} \binom{K}{m} (m-1)! \approx (K-1)!$ constraints). More recently, it has been also shown by Sun and Jafar in [3] that, by a series of transformations of linear programs, the sum-GDoF characterization can be translated into a minimum weighted matching problem in combinatorial optimization. As such, the sum-GDoF under the GNAJ condition can be characterized as disjoint cycles partition of the interference network.

Such remarkable findings have inspired various related works, such as general X-channels [4], parallel interference networks [3], and compound interference networks [5]. In practice, it is also important to find efficient methods for the TIN power control problem, that is, find the transmit powers that achieve a certain desired GDoF-tuple in the TINA. A recent progress on this problem has been reported by Geng and Jafar in [5], where a simple yet elegant polynomial-time centralized iterative algorithm to find the globally optimal power allocation variables is provided.

Nevertheless, it is worth noting that the GNAJ condition was only proven to be sufficient. An interesting counter-example in [1] showed that there exist partially-connected (in the sense of channel strength levels) interference channels, such that TIN achieves the optimal GDoF region and yet the GNAJ condition is not satisfied. A natural question then arises as to whether there exists a larger class of networks, including partially-connected ones, such that TINA is optimal (i.e., TIN with power control still achieves the optimal GDoF region of the channel).

In this work, the optimality of TIN is revisited. The TIN optimality problem was formulated in [1] by first eliminating power allocation variables using the potential theorem [6], to establish the TINA in terms of GDoF variables only, and then by finding the optimal power allocation variables for a given GDoF-tuple in the TINA [5]. In contrast, we re-formulate this problem in a reversed way, from a combinatorial optimization perspective [6]. Interestingly, by first casting power allocation into an assignment problem, the globally optimal power allocation variables corresponding to any feasible GDoF tuple in the TINA can be found by solving the equivalent assignment problem in polynomial time, either in a centralized manner (e.g., Hungarian method [7]) or in a distributed one (e.g., Auction algorithm [8]). Inspired by the duality between the assignment and the maximum weighted matching problems in combinatorial optimization [9], we can express the TINA characterization in terms of a maximum weighted matching problem. In doing so, the TINA region is significantly simplified, requiring only $2^K - 1$ constraints instead of $\approx (K-1)!$. Interestingly, such a representation
also offers an interpretation of the disjoint cycle partition in [3]. By this new formulation, we show that TINA is a convex polytope under a novel channel strength condition that relaxes the GNAJ condition in [1]. This new condition requires that the desired signal strength of each user is no less than the maximum difference between the sum strength of any pair of incoming/outgoing interference signals and the strength of the link between such a pair (all in dB scale). Furthermore, together with a connectivity condition, we are able to establish the optimality of TIN for a new class of networks. Such conditions are not included nor include the GNAJ condition [1].

II. PRELIMINARIES

A. Channel Model

We consider a K-user interference channel where both transmitters (Tx) and receivers (Rx) are equipped with a single antenna each. We shall refer to the j-th Tx-Rx pair as the j-th user. At Rx-j (\forall j \in \mathcal{K} = \{1, \ldots, K\}), the received signal at discrete-time instant t is given by

\[ Y_j(t) = \sum_{i=1}^{K} h_{ij} \tilde{X}_i(t) + Z_j(t) \]  

(1)

where \( \tilde{X}_i(t) \) is the transmitted signal from Tx-i with power constraint \( \mathbb{E}(|\tilde{X}_i(t)|^2) \leq P_i \). \( h_{ij} \) is the channel coefficient between Tx-i and Rx-j, \( Z_j(t) \sim \mathcal{C}\mathcal{N}(0,1) \) is the (normalized) additive white Gaussian noise at Rx-j. Following [1], we translate the signal model in (1) into an equivalent GDoF-friendly form, given by

\[ Y_j(t) = \sum_{i=1}^{K} \sqrt{P_i} e^{\theta_{ij}} X_i(t) + Z_j(t) \]  

(2)

where \( X_i(t) = \frac{\tilde{X}_i(t)}{\sqrt{P_i}} \) is the normalized transmitted signal with power constraint \( \mathbb{E}(|X_i(t)|^2) \leq 1 \). \( \sqrt{P_i} \) and \( \theta_{ij} \) are magnitude and phase of the channel coefficient between Tx-i and Rx-j, respectively, and the exponent \( \alpha_{ij} \) is defined as the corresponding channel strength level

\[ \alpha_{ij} = \log \left( \frac{\max\{1,|h_{ij}|^2P_i\}}{\log P} \right) \]  

(3)

where \( P > 1 \). Given a transmit power \( P^* \) at Tx-i with \( r_i \leq 0 \), the Signal to Interference plus Noise Ratio (SINR) achieved by TIN at Rx-j is given by

\[ \frac{P^*}{\sum_{i \neq j} P^* + P^*r_{ij}} \].

We assume that the transmitters know channel strength levels perfectly for power control, and the receivers have access to both the magnitude and phase of channel coefficients.

B. Definitions

We follow standard definitions for encoding/decoding functions and achievable rates. The individual achievable GDoF of message \( W_k \) is defined as \( d_k = \lim_{P \to \infty} \frac{R_k}{\log P} \) where \( R_k \) is the achievable rate of user \( k \). The (optimal) GDoF region \( \mathcal{P}^* \) is the collection of all achievable GDoF-tuples \((d_1, d_2, \ldots, d_K)\). The TIN-Achievable GDoF region (TINA) defined in [1] is the set of all \( K \)-tuples \((d_1, d_2, \ldots, d_K)\) with components satisfying

\[ d_j \leq \max \left\{ 0, \alpha_{jj} + r_j - \max \{0, \max_{i \neq j} (\alpha_{ij} + r_i)\} \right\} \]  

(4)

for some assignment of the power allocation variables \((r_1, r_2, \ldots, r_K) \in \mathbb{R}^K_+ \). In the following, we denote the TINA by \( \mathcal{P}^*_S \), where the dependence on the specific network defined by \{\( \alpha_{ij} : i, j \in \mathcal{K} \)\} is clear from the context. From [1] we also know that the polyhedral TINA region is obtained by removing the positive part operator\(^1\) from the right-hand side of (4). Using the potential theorem [6], the authors of [1] are able to find a convex polytope form for the polyhedral TINA region for any subnetwork formed by a subset \( S \subseteq \mathcal{K} \) and its associated desired and interfering links. We shall denote such polytope by \( \mathcal{P}^*_S \). Since removing the positive part in the right-hand side of (4) restricts the GDoF region, then \( \mathcal{P}^*_S \) is achievable by switching off all users in \( S^c = \mathcal{K} \setminus S \) and using TIN for the users in \( S \). We also denote by \( \mathcal{R}^* \) the optimal GDoF region of the interference network, i.e., the region of GDoF-tuples achievable over any possible coding scheme (not restricted to TIN).

The main results in [1] are summarized as below.

**Theorem 1.** [GNAJ [1]] Consider a K-user single-antenna Gaussian interference channel with channel strengths \{\( \alpha_{ij} : i, j \in \mathcal{K} \)\}.

1. For any subnetwork formed by users in \( S \subseteq \mathcal{K} \), \( \mathcal{P}^*_S \) described by\(^2\)

\[ 0 \leq d_k \leq \alpha_{kk}, \forall k \in S, \quad d_i = 0, \forall i \in S^c, \]  

\[ \sum_{k=0}^{m-1} d_k \leq \sum_{k=0}^{m-1} (\alpha_{kk} - \alpha_{[k-1] \mod m,k}), \]  

\[ \forall \text{ ordered subsets } (i_1, \ldots, i_{m-1}) \in S, \quad \forall \ m \in \{2, \ldots, |S|\}. \]  

(5)

2. The TINA of the whole network is given by

\[ \mathcal{R}^*_T = \bigcup_{S \subseteq \mathcal{K}} \mathcal{P}^*_S. \]  

(6)

3. If \( \forall k \in \mathcal{K}, \alpha_{kk} \geq \max_{i \neq k} \{\alpha_{ik} \} + \max_{j \neq k} \{\alpha_{kj}\}, \) then TIN is GDoF-optimal, i.e., \( \mathcal{R}^* = \mathcal{R}^*_T = \mathcal{P}^*_K \) (the whole region is a single convex polytope).

**Remark 1.** It is easy to see that, for \( S = \mathcal{K} \), there are in total \( \sum_{m=2}^{K} \binom{K}{m} (m-1)! = (K-1)! \) constraints in (5). Since the sum-GDoF \( \sum_{k=0}^{m-1} d_k \) does not depend on the order of the indices, for each unordered set of indices \{\( i_1, \ldots, i_{m-1} \)\} there are (\( m-1 \)! inequalities, of which only one is relevant. However, finding which one is relevant involves, in general,

\(^1\)The positive part of \( x \) is \( \max\{0, x\} \).

\(^2\)We use the term ordered subset to indicate that order matters, but elements are not repeated. For example, \((1,2,3)\) and \((1,3,2)\) are two rising such subsets for \( m = 3 \), but \((1,2,2)\) is not valid, because it contains repeated elements.
extensive search, such that finding a general more compact form that eliminates redundant inequalities is non-trivial.

In this work we shall make extensive use of weighted matchings [10]. We recall here some basic definitions. Let \( G = (U, V, E) \) denote a bipartite graph with left vertices \( U \), right vertices \( V \) and edges \( E \subseteq U \times V \). A matching \( M \subseteq E \) is a set of edges, any two of which do not share the same vertex. When weights \( w(u, v) \) are associated to the edges \((u, v) \in E\), we denote by \( w(M) = \sum_{(u, v) \in M} w(u, v) \) the weight of the matching \( M \), and we let \( M^* = \operatorname{arg\,max}_M w(M) \) denote the maximum weighted matching, i.e., the matching with maximum weight. \( M^* \) can be characterized as the solution of the integer program:

\[
\begin{align}
\max & \sum_{(u, v) \in E} w(u, v)x(u, v), \\
\text{s.t.} & \sum_{u \in U} x(u, v) \leq 1, \\
& \sum_{v \in V} x(u, v) \leq 1, \\
& x(u, v) \in \{0, 1\}. 
\end{align}
\] (8)

When equality holds in all constraints, the resulting solution is called a perfect matching, i.e., a matching that covers all vertices. The LP relaxation of (8)-(11), obtained by replacing (11) with \( x(u, v) \in [0, 1] \), is called fractional matching [11].

Due to lack of space, we only summarize the results and sketch the proofs in the next sections, and more details and rigorous proofs are given in [12].

III. AN ALTERNATIVE FORMULATION

We re-formulate the TIN problem of [1], [5] from a combinatorial optimization perspective. By casting the power allocation into an assignment problem, we find an alternative form for the TINA via its dual – the maximum weighted matching problem [9].

In what follows, we consider a feasible GDoF tuple in \( \mathcal{P}^{\text{TINA}}_{\mathcal{S}} \) for any user set \( \mathcal{S} \subseteq \mathcal{K} \), where

\[
d_j = \alpha_{jj} + r_j - \max(0, \max_{i \neq j}(\alpha_{ij} + r_i)), \quad j \in \mathcal{S}. \] (12)

By introducing the auxiliary variables

\[
y_u = -r_j \quad \text{and} \quad y_v = \max(0, \max_{i \neq j}(\alpha_{ij} + r_i)), \] (13)

(14)

the individual achievable GDoF can be rewritten as

\[
d_j = \alpha_{jj} - (y_u + y_v). \] (15)

Thus, for \( \mathcal{S} \subseteq \mathcal{K} \), the feasibility of a GDoF-tuple can be guaranteed by the minimization of the auxiliary variables sum:

\[
\min_{j \in \mathcal{S}} (y_u + y_v) \] (16)

\[
\text{s.t.} \quad y_u + y_v \geq \alpha_{jj} - d_j, \forall j \in \mathcal{S}. \] (17)

In general, a given GDoF-tuple in \( \mathcal{P}^{\text{TINA}}_{\mathcal{S}} \) may be achieved by different assignments of the power control variables \( \{r_j : j \in \mathcal{S}\} \). The componentwise minimum configuration is referred to as the globally optimal power control assignment. Using the fact that, for all \( i \neq j \),

\[
y_u + y_v = -r_i + \max(0, \max_{i \neq j}(\alpha_{ij} + r_i)) \] (18)

\[
\geq \alpha_{ij} \] (19)

\[
\geq \alpha_{ij}, \quad \forall i, \quad i \neq j. \] (20)

we have the following theorem that solves the power control problem for a given feasible GDoF-tuple.

Theorem 2. For any \( \{d_j : j \in \mathcal{S}\} \in \mathcal{P}^{\text{TINA}}_{\mathcal{S}} \), the globally optimal power allocation assignment \( \{r_j, j \in \mathcal{S}\} \) can be found by solving the following linear program:

\[
\begin{align}
\min_{\{y_u, y_v\} \in \mathcal{Y}} & \sum_{j \in \mathcal{S}} (y_u + y_v) \\
\text{s.t.} & \sum_{i \in U} y_i \leq 1, \\
& \sum_{j \in V} y_j \leq 1, \\
& y_u + y_v \geq \alpha_{ij} - d_j, \forall j \in \mathcal{S}, \\
& y_u, y_v \geq 0, \forall j \in \mathcal{S}. \] (21a)

(21b)

(21c)

(21d)

where \( r_j = -y_u, \forall j \).

Remark 2. The linear program (21) can be recognized as a dual formulation of an assignment problem [9] which can be solved in polynomial time (e.g., \( O(K^3) \)), by the (centralized) Hungarian method [7] and the (distributed) Auction algorithm [8]. As said, global optimality here means that none of the users can decrease its power to achieve the same GDoF tuple.

In the following, starting from the power allocation solution of Theorem 2 and exploiting the duality between assignment and maximum weighted matching problems, we shall re-formulate the TINA in a more useful and compact form. First, we introduce the following matrix associated with the assignment problem (21):

\[
A_{ij} = \begin{cases} 
\alpha_{ij}, & i \neq j \\
\alpha_{ij} - d_j, & i = j.
\end{cases} \] (22)

Then, we observe that the dual problem of (21) is given by

\[
\begin{align}
\max_{(i,j) \in E} & \sum_{(i,j) \in E} A_{ij}x(i,j), \\
\text{s.t.} & \sum_{(i,j) \in E} x(i,j) \leq 1, \\
& \sum_{(i,j) \in E} x(i,j) \leq 1, \\
& x(i,j) \in [0, 1]. \] (23a)

(23b)

(23c)

(23d)

It is known that this linear program has integer-valued optimal solutions for bipartite graphs [10]. Since in our case the graph associated to the Tx’s and Rx’s in \( \mathcal{S} \) and corresponding intended and interfering links is bipartite by construction, then (23) coincides with a maximum weighted matching problem, obtained by replacing (23d) with \( x(i,j) \in \{0, 1\} \).

Next, we note that an edge \( (i,j) \) belongs to the maximum-weight matching if and only if \( y_u + y_v = A_{ij} \) [9]. For a given \( \mathcal{S} \subseteq \mathcal{K} \), due to (15) we have that a feasible GDoF-tuple implies
For any user subset \( S \subseteq K \), we define the subgraph \( \mathcal{G}[S] = (S, S, S \times S) \) with weights \( \{ \alpha_{ij}^*: i, j \in S \} \). By the observation above, the sum of \( \{ A_{jj} : j \in S \} \) must be no less than \( w_0(M) \) for any matching \( M \) of \( \mathcal{G}[S] \). Hence, we can write
\[
\sum_{j \in S} (\alpha_{jj} - d_j) \geq \max_w w(M_S) = w(M_S^*),
\]
where \( M_S^* \) is the matching of \( \mathcal{G}[S] \) with the maximum weight. This yields the following results, where the proof of the equivalence to Theorem 2 of [1] is relegated to Appendix.

**Theorem 3.** Consider a \( K \)-user single-antenna Gaussian interference channel with channel strengths \( \{ \alpha_{ij} : i, j \in K \} \). For any user subset \( S \subseteq K \), \( \mathcal{P}_{S}^{TINA} \) is given by:
\[
0 \leq d_k \leq \alpha_{kk}, \quad \forall k \in S, d_i = 0, \forall i \in K \setminus S \quad \text{(26)}
\]
\[
\sum_{k \in S'} d_k \leq \sum_{k \in S'} \alpha_{kk} - w(M_S^*), \quad \forall S' \subseteq S \quad \text{(27)}
\]

**Remark 3.** Using Th. 3 into (6) we find that we need only \( 2^K - 1 \) inequalities, one for each non-trivial subset of \( K \), to describe the whole TINA region \( \mathcal{R}^{TINA} \).

**Example 1.** We consider the example in [5, Fig. 8] to show the efficiency of our formulation, as shown in Fig. 1(a). According to Theorem 3, the TINA GDoF region is immediately given as
\[
\mathcal{P}_{\{1,2,3\}}^{TINA} = \{ (d_1, d_2, d_3) : 0 \leq d_1 \leq 2, 0 \leq d_2 \leq 1, 0 \leq d_3 \leq 1.5, d_1 + d_2 \leq 2.3, d_2 + d_3 \leq 1.5, d_1 + d_3 \leq 2.4, d_1 + d_2 + d_3 \leq 2.5 \},
\]
which is identical to the expression found in [5]. In order to solve the power allocation for a given GDoF-tuple (say \( (0.5, 0.6, 0.7) \) in this case), we take the weight matrix in Fig. 1(b) as the input of Hungarian method and we obtain:
\[
y_{u_1} = 1.2, y_{u_2} = 0.4, y_{u_3} = 0.7, y_{v_1} = 0.3, y_{v_2} = 0, y_{v_3} = 0.1.
\]
Thus, the globally optimal power allocation assignment is:
\[
r_1 = -1.2, r_2 = -0.4, r_3 = -0.7,
\]
which coincide with what found in [5].

To start the Hungarian method, \( y_{u_j} \) and \( y_{v_j} \) are usually initialized respectively with the maximum value of the \( j \)-th row of \( A \) and 0. Following the procedure of the Hungarian method in [7], we gradually decrease \( y_{u_j} \) and increase \( y_{v_j} \) to make sure the constraints in (21) satisfied. Note that \( r_j = -y_{u_j} \) is increasing during this procedure. Once we find one solution, it will be the global optimum assignment, because it is impossible to decrease \( r_j \) (correspondingly increase \( y_{u_j} \)) and find another solution in the region that we have already explored.

**IV. A NEW TINA OPTIMALITY CONDITION**

Besides the reduction of the number of inequalities, this new formulation enables us to identify a relaxed channel strength condition such that the TINA region is a convex polytope. The proof is relegated to Appendix.

**Theorem 4.** Consider a \( K \)-user single-antenna Gaussian interference channel with channel strengths \( \{ \alpha_{ij} : i, j \in K \} \). If
\[
\alpha_{kk} \geq \max_{i,j : i,j \neq k} \{ \alpha_{ik} + \alpha_{kj} - \alpha_{ij} \}, \quad \forall k \in K, \quad \text{(28)}
\]
then \( \mathcal{P}_{S}^{TINA} \) is monotonically non-decreasing, i.e., if \( S_1 \subseteq S_2 \subseteq K \) then \( \mathcal{P}_{S_1}^{TINA} \subseteq \mathcal{P}_{S_2}^{TINA} \). Also, \( \mathcal{R}^{TINA} = \mathcal{P}_{K}^{TINA} \) is a convex polytope.

It is immediate to see that the condition of Th. 4 is a relaxation of the GNAJ condition. Next, we add to this new condition a topology condition that defines a special class of partially connected interference channels. These two conditions together enable us to establish the TINA optimality for a class of networks that is not included in the class of network defined by the GNAJ condition. The converse proof follows the approach in [1] and is presented in Appendix.

**Theorem 5.** Consider a \( K \)-user single-antenna Gaussian interference channel with channel strengths \( \{ \alpha_{ij} : i, j \in K \} \). Assume that (28) holds, and, in addition, that for every \( S \subseteq K \) with \( |S| > 2 \), the corresponding fully connected weighted subgraph \( \mathcal{G}[S] = (S, S, S \times S) \) with weights \( \{ \alpha_{ij}^*: i, j \in S \} \),
\[
\exists (i, j) \in M^*_S, \text{ s.t. } \alpha_{ij}^* = 0. \quad \text{(29)}
\]
Then, \( \mathcal{R}^* = \mathcal{R}^{TINA} = \mathcal{P}_{K}^{TINA} \).

**Remark 4.** When condition (28) holds but GNAJ does not hold, the topology condition (29) allows us to establish the optimality of TINA since, under this condition, we can prove that the converse is tight. This, however, is only a sufficient condition and there might be a larger class of networks, including both the subclass defined by Th. 1 and the one defined by Th. 5, for which TIN is GDoF-optimal.

**Example 2.** We illustrate the new TINA optimality conditions by the example in Fig. 2. It is easy to verify that (28) holds for the entire network, while the original GNAJ TIN optimality condition does not hold for user 2. Note that \( M^* = \{ (1, 3), (2, 1), (3, 2) \} \) is a (non-unique) maximum weighted matching and contains \( \alpha_{13}^* = \alpha_{13} = 0 \), such that also condition (29) holds. Thus, from Theorems 5, 4 and 3, the
optimal GDoF region of this network is the polytope defined by:

$$
P^\text{TINA}_{(1,2,3)} = \{(d_1, d_2, d_3): \begin{align*}
0 &\leq d_i \leq 1, \forall i \in \{1, 2, 3\} \\
d_1 + d_2 &\leq 1.1, d_2 + d_3 \leq 1.3 \\
d_1 + d_3 &\leq 1.2, d_1 + d_2 + d_3 \leq 1.8
\end{align*} \}.
$$

Remark 5. A subclass of network topologies for which (29) holds is the class of networks that have no perfect matchings in any unweighted subgraph of $G$ with zero-weight edges removed. A bipartite graph has no perfect matchings if Hall’s condition does not hold [10]. The so-called triangular networks in [13] belong to this category.

V. CONCLUSION

The GDoF optimality problem of treating interference as noise for Gaussian interference channels has been re-formulated in this work from a combinatorial optimization perspective. Thanks to this new formulation, we cast power allocation into an assignment problem, which can be solved in polynomial time. A new expression for the TIN-Achievable GDoF region is provided, which is more compact and useful than what known before since it eliminates many redundant inequalities. A relaxed version of the condition in [1] on the channel coefficients is given, for which the TIN-Achievable GDoF region is convex polytope. Finally, a new TIN optimality condition is also revealed, by which TIN still achieves the optimal GDoF region for a class of networks different from the one identified in [1]. It is also worth noting that our new TIN optimality condition does not violate the conjecture in [1] that GNAJ condition is also necessary “except for a set of channel gain values with measure zero.”

APPENDIX

A. Proof of Theorem 3

For the sake of this proof, we denote by $P_S$ the region defined by (5), and by $P^{\text{TINA}}_S$ the region defined by (26). Our goal is to show that $P_S = P^{\text{TINA}}_S$ for any $S \subseteq K$.

$\mathcal{P}_S \subseteq \mathcal{P}^{\text{TINA}}_S$: To prove this, we show that for any inequality presented in $\mathcal{P}^{\text{TINA}}_S$, we can always find the same one in $\mathcal{P}_S$.

Given a subnetwork $G[S]$, the matching with the maximum weight is a degree-1 subgraph. 3 Connecting the direct links will lead to single or multiple disjoint cycles, as all the nodes has degree-2.

For the single-cycle case, this cycle corresponds to a sum GDoF constraint in $\mathcal{P}_S$. For the multiple-cycle case, each cycle corresponds to a sum GDoF constraint in $\mathcal{P}_S$ of the users involved in this cycle. Thus, the sum GDoF constraint with the maximum weighted matching in $\mathcal{P}^{\text{TINA}}_S$ corresponds to the combination of these sum GDoF constraints in $\mathcal{P}_S$.

As such, $\mathcal{P}_S$ contains or implies all the constraints in $\mathcal{P}^{\text{TINA}}_S$. As $\mathcal{P}_S$ has more constraints, it follows that $\mathcal{P}_S \subseteq \mathcal{P}^{\text{TINA}}_S$.

$\mathcal{P}^{\text{TINA}}_S \subseteq \mathcal{P}_S$: To prove this, we show that for any subset of users $S$, the TINA GDoF region confined by $\mathcal{P}_S$ is no larger that that by $\mathcal{P}^{\text{TINA}}_S$. It is clear that $\mathcal{P}_S$ is determined by individual GDoF and sum GDoF constraints of any subset of users in $S$. The individual GDoF constraints of two regions are identical. Thus, our focus will be on the sum GDoF constraints for users in $S$ with $|S| \geq 2$.

For the user set $S$, the sum GDoF constraints in $\mathcal{P}_S$ only come from (1) the sum GDoF constraints with all possible permutations of $S$, and (2) the combination of a number of individual and/or sum GDoF constraints of subsets of $S$. For the first case, the sum GDoF constraint in $\mathcal{P}_S$ is dominated by the maximum weight of any possible matchings (associated with cyclic sequences). For the second case, suppose the combination involves a number of subnetworks, where the subnetworks may have intersections. This combination of constraints involves every user with equal times (say $b$ times), otherwise, the combination will not lead to a sum GDoF constraint, because it is a weighted sum GDoF constraint and can be implied by the combination of other sum GDoF constraints. Each sum GDoF constraint for a subnetwork involves a cyclic sequence and hence forms a matching. Thus, the combination of sum GDoF constraints corresponds to a fractional perfect matching by assigning $x(u, v)$ in (8)-(11) with $1/b$. In bipartite graphs, the weight of any fractional perfect matching equals the weight of a perfect matching [10], [11].

Thus, neither the weight of any matching nor of any fractional matching is greater than the maximum weighted matching, such that the sum GDoF constraints in $\mathcal{P}^{\text{TINA}}_S$ will be more restrictive than those or any combinations in $\mathcal{P}_S$, i.e., $\mathcal{P}^{\text{TINA}}_S \subseteq \mathcal{P}_S$. This completes the proof.

B. Proof of Theorem 4

In what follows, we prove that under condition (28), $\mathcal{P}^{\text{TINA}}_S$ is monotonicly increasing. Hence, from (6) this immediately implies that $\mathcal{R}^{\text{TINA}}_S = \mathcal{P}^{\text{TINA}}_S$ which, by inspection, is a convex polytope.

Let us start with $|S| = 2$. Suppose without loss of generality $S = \{k, j\}$. Due to the condition (28), $\min\{\alpha_{kk}, \alpha_{jj}\} \geq \alpha_{kj} + \alpha_{jk}$, then $\mathcal{P}^{\text{TINA}}_k \subseteq \mathcal{P}^{\text{TINA}}_{\{k, j\}}$ and $\mathcal{P}^{\text{TINA}}_j \subseteq \mathcal{P}^{\text{TINA}}_{\{k, j\}}$.

If direct links are in the matching, we can eliminate them from $S$, which does not affect our proof.
Then, we prove the general cases with the following lemma.

Lemma 1. Given a subgraph $G[S]$ with weights \( \{\alpha_{ij}, i, j \in S\} \), the difference of maximum weighted matching with and without the user $k$ is bounded by

\[
w(M^*_S) - w(M^*_{S\setminus\{k\}}) \leq \max_{i,j \in S, i,j \neq k} \{\alpha_{ik} + \alpha_{kj} - \alpha_{ij}\}
\]

Proof: Suppose without loss of generality that the maximum weighted matching of $G[S]$ ($k \in S$) includes links $(i, k)$ and $(k, j)$ with weights $\alpha_{ik}$ and $\alpha_{kj}$ respectively. After removing user $k$ and edges $(i, k)$, $(k, j)$ from the matching, and adding the link $(i, j)$ with weight $\alpha_{ij}$, we have a matching for $S \setminus \{k\}$. Thus, for all $\{(i, k), (k, j)\} \in M^*_S$, we have

\[
w(M^*_S) - w(M^*_{S\setminus\{k\}}) \leq \max_{i,j \in S, i,j \neq k} \{\alpha_{ik} + \alpha_{kj} - \alpha_{ij}\} \geq w(M^*_S) - w(M^*_{S\setminus\{k\}})
\]

Together with the condition (28), we have

\[
\alpha_{kk} \geq w(M^*_S) - w(M^*_{S\setminus\{k\}})
\]

for any user $k \in S$. It follows immediately that $P^TINA_{S(k)} \subseteq P^TINA_S$ for all $S \subseteq K$. More generally, if $S_1 \subseteq S_2$, then $P^TINA_{S_1} \subseteq P^TINA_{S_2}$. This completes the proof.

C. Proof of Theorem 5

Due to the fact that $R^* \supseteq R^TINA$ and that, under condition (28), $R^TINA = P^TINA$, achievability trivially follows.

For the converse, we employ the cyclic outer bounds first revealed in [14] and later used to prove the optimality of TIN condition in [1].

The individual GDoF bounds can be simply obtained as in [1]. For sum GDoF bounds, given $m \geq 2$ and any ordered subset $\pi = \{0, 1, \ldots, m-1\} \subseteq K$, we follow the same footsteps as in [1] and have the cyclic sum GDoF bound

\[
\sum_{j=0}^{m-1} d_j \leq \min_{\pi \in \pi(S)} \{f_\pi, g_\pi, 0, \ldots, g_\pi, m-1\},
\]

where we define

\[
f_\pi = \sum_{j=0}^{m-1} \max\{0, \alpha_{ij_{j+1}}, \alpha_{ij_{j+1}} - \alpha_{ij_{j+1}}\}
\]

\[
g_{\pi, k} = \sum_{j=0}^{m-1} (\alpha_{ij_{j+1}} - \alpha_{ij_{j+1}}) + \alpha_{ik_{k+1}}, \quad k = 0, \ldots, m-1,
\]

and where the index subscript arithmetic is modulo $m$.

When $m = 2$, then condition (28) is equivalent to the GNJ condition and the bound is known to be tight. When $m > 2$, let us first consider the bound formed by the “g” terms in (33). Notice that the left-hand side of (33) depends only on the indices in $\pi$ but not on its order. Hence, letting $S$ denote a given unordered subset of size $m$ of $K$ and using the short-cut notation $\pi \in \pi(S)$ to indicate the ordered sets formed with the elements of $S$, i.e., the permutations of $S$, we can write

\[
\min_{\pi \in \pi(S)} \{f_\pi, g_{\pi, 0}, \ldots, g_{\pi, m-1}\}
\]

which coincides with $P^TINA_S$ for every $S \subseteq K$. Under the condition (28), the TINA is the largest polyedral region, so the converse bound is tight.

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