Gauge-natural parameterized variational problems, vakonomic field theories and relativistic hydrodynamics of a charged fluid

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Abstract

Variational principles for field theories where variations of fields are restricted along a parametrization are considered. In particular, gauge-natural parametrized variational problems are defined as those in which both the Lagrangian and the parametrization are gauge covariant and some further conditions is satisfied in order to formulate a Nöther theorem that links horizontal and gauge symmetries to the relative conservation laws (generalizing what Fernández, García and Rodrigo did in some recent papers). The case of vakonomic constraints in field theory is also studied within the framework of parametrized variational problems, defining and comparing two different concepts of criticality of a section, one arising directly from the vakonomic schema, the other making use of an adapted parametrization. The general theory is then applied to the case of hydrodynamics of a charged fluid coupled with its gravitational and electromagnetic field. A variational formulation including conserved currents and superpotentials is given that turns out to be computationally much easier than the standard one.

1 Introduction

In some recent literature an interest emerged about variational principles in which some specific requirement of physical or mathematical nature force us to restrict the variations of fields according to some a priori given rule.

As an example, let us mention Mechanics with non-integrable constraints on velocities. As it is well known there are at least two different procedures (so called vakonomic and non-holonomic) which lead to different equations of motion, it is now clear [18], at least for affine constraints, that the more realistic trajectories are solutions of the non-holonomic one. However the situation is less clear in more general cases (see [20] and references quoted therein), so that we do know very little about constraints in field theory (see [23, 17]) and we cannot yet decide whether one procedure is better then the other for a
reasonable extension out of Mechanics. In any case we shall not enter this question here, but just mention that both procedures can be implemented as variational problems with constrained variations (see [11]): they differ exactly in the choice of the rule to be imposed to admissible variations.

Another important case which was recently investigated is Euler-Poincaré reduction (see [2, 3, 9]) where a variational problem with constrained variations arises as a Lagrangian reduction of a free variational problem with symmetries.

A more classical example is presented in the well known book by Hawking and Ellis while introducing the equations of motion for relativistic hydrodynamics [13]. There the constraint on variations comes from the physical requirement that variations preserve the continuity equation.

Motivated by all these examples, in this paper we first discuss the formal development of variational problems in which the variations are parametrized by sections of some vector bundle (Section 2). This topic was already addressed in [8] at least for natural constrained field theories arising from the vakonomic technique; we recover here, and sometimes generalize, some of the results of [8] without asking that the parametrization necessarily arises from the vakonomic handling of a constraint, and we develop the case of gauge covariant field theories (much wider then the natural case) for which we develop a full Nöther theory to link horizontal and gauge symmetries to the relative conservation laws (Section 3). In Section 4 we specialize to the case of vakonomic constraints in field theories and define two different concepts of criticality for a section: one arising directly from the vakonomic paradigm, the other following from a compatible parametrization. We also show under what conditions the two concepts coincide. In Section 5, finally, our results are applied to the case of relativistic hydrodynamics of a charged fluid.

An advantage of the framework we shall propose hereafter is to provide a unifying language able to deal with all variational problems with constrained variations, no matter whether they come from the vakonomic scheme, from the non-holonomic one, from a reduction problem or from any other physical or mathematical requirement. In our hope, therefore, this unified language can even allow a better comparison between different schemes.

2 Variational calculus with parametrized variations

Notation follows [7], to which we refer the reader for further details. Let $C \overset{\pi}{\rightarrow} M$ be the configuration bundle of a physical theory the sections $\Gamma(C)$ of which represent the fields (by an abuse of language we will often denote bundles with the same label as their total spaces).

**Definition 1** A Lagrangian of order $k$ on the configuration bundle $C \overset{\pi}{\rightarrow} M$ ($m = \dim M$) is a fibered morphism

$$L : J^k C \longrightarrow \wedge^m T^* M.$$
Definition 2 A parametrization of order $s$ and rank $l$ ($s \geq l$ is required) of the set of constrained variations is a couple $(E, \mathbb{P})$, where $E$ is a vector bundle $E \xrightarrow{\pi_E} C$ while $\mathbb{P}$ is a fibered morphism (section)

$$\mathbb{P} : J^s C \longrightarrow (J^l E)^* \otimes_{\pi_C} VC;$$

here $J$ denotes jet prolongation and $V$ is the vertical functor.

If $(x^\mu, y^a, \sigma^A)$ are local fibered coordinates on $E$ and $\{\partial_a\}$ is the induced fiberwise natural basis of $VC$, a parametrization of order 1 and rank 1 associates to a section $y^a(x)$ of $C$ and a section $\sigma^A(x, y)$ of $E$ the section

$$[p_A^\mu(x^\nu, y^b(x)) \sigma^A(x, y(x)) + p_A^\nu(x^\nu, y^b(x)) \partial_b \sigma^A(x, y(x))] \partial_a$$

of $VC$.

Definition 3 Given a compact submanifold $D \subset M$ with suitably regular boundary $\partial D$, an admissible variation (for a Lagrangian with order $k$ and a parametrization of order $s$ and rank $l$) of a configuration $\sigma \in \Gamma(C)$ on $D$ is a smooth 1-parameter family of sections $\{\sigma_t\}_{t \in [-1, 1]} \subset \Gamma(\pi^{-1} D)$ such that

1. $\sigma_0 = \sigma|_D$
2. $\forall t \in [-1, 1[, \; j^{k-1} \sigma_t|_{\partial D} = j^{k-1} \sigma|_{\partial D}$
3. there exists a section $\varepsilon \in \Gamma(E)$ such that $\frac{d}{dt} \sigma_t|_{t=0} = \mathbb{P} (j^l \varepsilon) \circ j^{s-1} \sigma$ and $j^{k+l-1}(\varepsilon \circ \sigma)|_{\partial D} = 0$.

Definition 4 The set $\{C, L, \mathbb{P}\}$ where $C \xrightarrow{\pi} M$ is a configuration bundle, $L$ is a Lagrangian on $C$ and $\mathbb{P}$ is a parametrization of the set of constrained variations is called a “parametrized variational problem”.

Definition 5 We define critical for the parametrized variational problem $\{C, L, \mathbb{P}\}$ those sections of $C$ such that, for any compact domain $D \subset M$ and for any admissible variation $\{\sigma_t\}$ defined on $D$ one has

$$\frac{d}{dt} \int_D L \circ j_k \sigma_t \bigg|_{t=0} = 0.$$

Accordingly, if we use the trivial parametrization $\mathbb{P} : C \longrightarrow VC^* \otimes_C VC$ that to any $p \in C$ associates the identity of $V_p C$ then the third condition becomes empty and we recover free variational calculus.

For an ordinary variational problem with Lagrangian $L = \mathcal{L} ds$, criticality of a section of $C$ is equivalent in local fibered coordinates $(x^\mu, y^a)$ to the fact that for any compact $D \subset M$ and for any $V \in \Gamma(VC)$ such that $j^{k-1}(V \circ \sigma)|_{\partial D} = 0$ one has

$$\int_D \left[ \frac{\partial \mathcal{L}}{\partial y^a} V^i(x, y(x)) + \frac{\partial \mathcal{L}}{\partial y^a_{\mu_1 \cdots \mu_k}} \frac{\partial y_{\mu_1 \cdots \mu_k}}{\partial y^a} V^i(x, y(x)) + \cdots + \frac{\partial \mathcal{L}}{\partial y^a_{\mu_1 \cdots \mu_k}} d_{\mu_1} \cdots d_{\mu_k} V^i(x, y(x)) \right] ds = 0.$$
Explicit calculations (see [7]) show that the local coordinate expressions given above glue together giving rise to the global expression

\[ \forall D^{\text{cpt}} \subset M, \forall V \in VC \text{ s. t. } j^{k-1}(V \circ \sigma)|_{\partial D} = 0, \int_D <\delta L| j^k V > \circ j^k \sigma = 0 \]  

(1)

where \( \delta L \) is a global fibered (variational) morphism

\[ \delta L : J^k C \longrightarrow (J^k VC)^* \otimes_{J^k C} \Lambda^m T^* M. \]  

(2)

The rigorous definition and a short account on variational morphisms will be given in Appendix A, where the reader can find some non-conventional definitions that we have generalized in order to embrace the case of parametrized variations.

2.1 \( \mathbb{P} \)-first variation formula and \( \mathbb{P} \)-Euler-Lagrange equations

To define criticality for parametrized variational problems with a \( k \)-th order Lagrangian and a parametrization of rank \( l \), we have to restrict variations to those \( V \in VC \) in (1) with \( j^{k-1}(V \circ \sigma)|_{\partial D} = 0 \) that can be obtained through the parametrization from a section \( \varepsilon \) of \( E \) satisfying \( j^{l+k-1}(\varepsilon \circ \sigma)|_{\partial D} = 0 \). Let us make this explicit on a first order Lagrangian and a parametrization with rank 1 and order 1: if \((x^\mu, y^a, \varepsilon^A)\) are local fibered coordinates on \( E \) and \((p_A^a \varepsilon^A + p_A^a \mu d_\mu \varepsilon^A)\partial_a\) is the local representation of \( \langle \mathbb{P} | j^1 \varepsilon \circ j^1 \sigma \rangle \), criticality holds if and only if for any compact \( D \subset M \), for any section \( \varepsilon \) with coordinate expression \( \varepsilon^A(x, y(x)) = 0 \) and \( d_\mu \varepsilon^A(x, y(x)) = 0 \) for all \( x \in \partial D \), we have

\[ \int_D \left[ \frac{\partial \mathcal{L}}{\partial y^a} (p_A^a \varepsilon^A + p_A^a \mu d_\mu \varepsilon^A) + \frac{\partial \mathcal{L}}{\partial y_\mu} d_\mu (p_A^a \varepsilon^A + p_A^a \nu d_\nu \varepsilon^A) \right] ds = 0. \]  

(3)

To set up an intrinsic characterization of critical sections possibly given by a set of differential equations let us introduce the following procedure.

Let us take any parametrization \( \mathbb{P} \) and think of it as a morphism

\[ \mathbb{P} : J^s C \times J^1 E \longrightarrow VC \]

linear in its second argument. Take its \( k \)-th order (holonomic) jet prolongation (see Appendix A, Definition 35)

\[ j^k \mathbb{P} : J^{s+k} C \times J^{l+k} E \longrightarrow J^k VC \]

and read it as a bundle morphism

\[ j^k \mathbb{P} : J^{s+k} C \longrightarrow (J^{l+k} E)^* \otimes_{J^{s+k} C} J^k VC \]

(for simplicity we still denote it by \( j^k \mathbb{P} \), but the same abuse of notation cannot safely be adopted also for \( \mathbb{P} \) and \( \mathbb{P}' \) because the jet prolongations of the two objects give completely different results).

Let us consider the variational morphism \( \delta L : J^k C \longrightarrow (J^k VC)^* \otimes_{J^k C} \Lambda^m T^* M \) introduced in formula (2) and take its formal contraction (see Definition 37) with \( j^k \mathbb{P} \). What we get is the variational \( E \)-morphism
\[ <\delta L \mid j^{k+s}\mathbb{P}^\prime > : J^{k+s}C \to (J^{l+k}E)^* \otimes j^{l+k}C \Lambda^{m}T^*M. \]

Criticality of a section \( \sigma \) can now be recasted in the following global requirement

\[ \forall D \subset M, \forall \varepsilon \in \Gamma(E) \text{ s. t. } j^{l+k-1}(\varepsilon \circ \sigma) |_{\partial D} = 0, \int_{D} \left< \delta L \mid j^{k+s}\mathbb{P}^\prime > \mid j^{l+k}\varepsilon \right> \circ j^{k+s}\sigma = 0 \]

This intrinsic procedure leads in the case of first order Lagrangians and first order first rank parameterizations to the local expression (3).

Now we can try to turn this requirement into a set of differential equations. The first step is to search for a first variation formula for parametrized variational problems.

Thanks to Theorem 45, for any choice of a fibered connection \((\gamma, \Gamma)\) on \( E \to C \to M \) (see Definition 39 and 41), there exists a unique pair \((E, F)\) of variational morphisms

\[ E(L, \mathbb{P}) : J^{2k+l+s}C \to E^* \otimes C \Lambda^{m}T^*M \]
\[ F(L, \mathbb{P}, \gamma) : J^{2k+l+s-1}C \to (J^{l+k-1}E)^* \otimes j^{l+k-1}C \Lambda^{m}T^*M \]

reduced with respect to \((\gamma, \Gamma)\) (see Definition 44) such that \( \forall \varepsilon \in \Gamma(E) \) one has

\[ \left< \delta L \mid j^{k+s}\mathbb{P}^\prime > \mid j^{l+k}\varepsilon \right> = \varepsilon > + \text{Div} < F \mid j^{l+k-1}\varepsilon > . \] (4)

In the previous equation \text{Div} stands for the divergence of a variational morphism (Appendix A, Definition 38) that coincides when computed on a section with the exterior differential of forms.

**Definition 6** We call \( \mathbb{P} \)-Euler-Lagrange morphism the variational morphism \( E(L, S, \mathbb{P}) \) and \( \mathbb{P} \)-Poincaré-Cartan morphism the variational morphism \( F(L, S, \mathbb{P}, \gamma) \).

**Proposition 7** A section \( \sigma \) of \( C \) is critical for the parametrized variational problem \((C, L, \mathbb{P})\) (with \( L \) of order \( k \) and \( \mathbb{P} \) of order \( s \) and rank \( l \)) if and only if \( < E \mid \varepsilon > \circ j^{2k+l+s} \sigma = 0 \).

**Proof:** We have

\[ \forall D \subset M, \forall \varepsilon \in \Gamma(E) \text{ s. t. } j^{l+k-1}(\varepsilon \circ \sigma) |_{\partial D} = 0, \int_{D} < \delta L \mid j^{k+s}\mathbb{P}^\prime > \mid j^{l+k}\varepsilon \circ j^{k+s}\sigma = 0 \]

\[ \forall D \subset M, \forall \varepsilon \in \Gamma(E) \text{ s. t. } j^{l+k-1}(\varepsilon \circ \sigma) |_{\partial D} = 0, \int_{D} < E \mid \varepsilon > \circ j^{2k+l+s}\sigma + \int_{D} d( < F \mid j^{l+k-1}\varepsilon > \circ j^{2k+l+s-1}\sigma) = 0 \]

\[ E \circ j^{2k+l+s}\sigma = 0 \]

The first equivalence is ensured by the first variation formula (4) and the Definition 38, while the last holds in force of the fact that \( j^{l+k-1}(\varepsilon \circ \sigma) |_{\partial D} = 0 \) and that \( \varepsilon \) and \( D \) are arbitrary. \( \blacklozenge \)
We stress that the number of derivatives of fields in \( E \) and \( F \) is not sharp: in fact, due to the particular structure of the morphism \( \langle \delta L | j^k \mathcal{P} \rangle \), it is actually always lesser than the one expected from the naive issue. Moreover, the morphism \( \mathcal{E}(L, \mathcal{P}) \) in the splitting is truly unique (i.e. independent on the connection used), while \( \mathcal{F}(L, \mathcal{P}, \gamma) \) is just one representative in the class of variational morphisms fitting into formula (4) chosen so that it is reduced with respect to the fibered connection \( (\gamma, \Gamma) \); more precisely, the final expression of the morphism \( \mathcal{F}(L, \mathcal{P}, \gamma) \) depends at most on the spacetime connection \( \gamma \), while if \( l + k \leq 2 \) (the most common case) also \( \mathcal{F} \) can be defined independently on any part of the fibered connection, so that the splitting algorithm can be performed in coordinates simply by integration by parts the derivatives of the sections of \( E \).

In the case \( k = 1, l = 1 \) and \( s = 1 \), with the same coordinate representation as in formula (3), the morphisms \( \mathcal{E} \) and \( \mathcal{F} \) are the following:

\[
\begin{align*}
\mathcal{E} | \varepsilon \rangle & = \varphi_j^3 \mathcal{E} = \left\{ \left( \frac{\partial L}{\partial y^a} - d_\nu \frac{\partial L}{\partial y^a_\nu} \right) p_A^a \right. - d_\mu \left[ \left( \frac{\partial L}{\partial y^a_\mu} - d_\nu \frac{\partial L}{\partial y^a_\nu} \right) p_A^a \, d_\mu \right] \varepsilon^A \, ds \\
\mathcal{F} | j \varepsilon \rangle & = \varphi_j^2 \mathcal{F} = \left[ \left( \frac{\partial L}{\partial y^a} - d_\nu \frac{\partial L}{\partial y^a_\nu} \right) p_A^a \varepsilon^A + \frac{\partial L}{\partial y^a_\mu} (p_A^a \varepsilon^A + p_A^a d_\nu \varepsilon^A) \right] \, ds
\end{align*}
\]

and they can easily be obtained integrating by parts the integrand of equation (3).

### 2.2 Symmetries

The second natural question arising in our investigations is whether the link between symmetries and conserved currents given in the standard Nöther theory is preserved if we vary configurations only along the parametrized variations. First of all the definition of symmetry for a parametrized variational problem has to be changed with respect to the ordinary one in order to ensure the invariance of the parametrization as well as that of the Lagrangian.

**Definition 8** Let \( C \xrightarrow{\pi_C} M \) be a configuration bundle, \( L \) a Lagrangian of order \( k \) and \( (E, \mathcal{P}) \) a parametrization of order \( r \) and rank \( l \) on the vector bundle \( E \xrightarrow{\pi_E} C \).

Let \( Z : E \rightarrow TE \) be a vector field on the bundle \( E \rightarrow M \) such that \( Y = T\pi_E Z \) is a vector field on \( C \), while \( X = T\pi_E Y \) is a vector field on \( M \).

Calling \( \Psi_{Z,s}^E, \Psi_{Y,s}^C \) and \( \Psi_{Y,s}^{VC} \) the flows of fibered transformations generated by the vector fields \( Z \) and \( Y \) on the bundles \( E, C \) and \( VC \) respectively, we can drag any section \( \varepsilon \in \Gamma(E) \) along \( \Psi_{Z,s}^E \) getting the section \( \varepsilon_s = (\Psi_{Z,s}^E)^* \varepsilon \), any section \( \sigma \in \Gamma(C) \) along \( \Psi_{Y,s}^C \) getting \( \sigma_s = (\Psi_{Y,s}^C)^* \sigma \) and, for any section \( V \in \Gamma(VC) \), we can also drag \( V \circ \sigma \) along \( \Psi_{Y,s}^{VC} \) getting \( (\Psi_{Y,s}^{VC})^* V \circ \sigma \).

The vector field \( Z \) is an *infinitesimal Lagrangian symmetry* of the (parametrized) variational problem \( (C, L, \mathcal{P}) \) if the following covariance identity

\[
\text{Div}(i_X L) + \text{Div} < \alpha | j^k Y >
\]

holds for some morphism \( \alpha : J^{k+h} C \rightarrow (j^k TC)^* \otimes j^k C \Lambda^{m-1} T^* M \), and moreover we have

\[
< \mathcal{P} | j^l \varepsilon_s > \circ j^r \sigma_s = (\Psi_{Y,s}^{VC})^* (< \mathcal{P} | j^l \varepsilon > \circ j^r \sigma >).
\]

6
The definition of infinitesimal symmetry has now been extended to variational problems with parametrized variations. Nevertheless in order to establish a correspondence between symmetries and conserved currents we need some more restrictive hypothesis. Usually (in the case with free variations) one considers the covariance identity \((5)\) and simply splits \(\delta L\) in its Euler-Lagrange part plus the divergence of the Poincaré-Cartan part: being the first vanishing on shell, one gets the conserved current as the argument of the residual divergence. In our framework, on the contrary, field equations arise from the splitting of \(< \delta L \mid j^k \mathbb{P}^k >\) so that we would need a rule to associate to any symmetry \(Y\) and to any configuration \(\sigma\) a section of the bundle of parameters that in turn is mapped by the parametrization into \(\mathcal{L}_Y \sigma\). Without any further hypothesis such a section is not guaranteed to exist. In the next Section we will then state all the necessary assumptions which are needed to get a Nöther theorem in the case of gauge-natural symmetries of gauge-natural variational problems in the next section.

3 Gauge-natural constrained theories

Gauge-natural field theories have been shown to be the most general setting to describe Lagrangian field theories with gauge symmetries, including gauge theories, General Relativity in its different formulations, bosons, spinors and also supersymmetries (see \([5, 7]\)). In the book \([7]\) it was also described in detail how to cope with conservation laws linked to pure gauge (vertical) symmetries and how to implement the conservation of energy and momentum as Nöther currents relative to an horizontal lift of space-time diffeomorphisms. The goal of this Section is to extend the previously known results to the case of constrained variational problems under suitable conditions. The original material on gauge-natural bundles can be found in \([4]\); a standard reference with many theoretical improvements is \([16]\); a more applicative approach can be found in \([7]\), while a friendly introduction to the subject is also provided in Appendix B.

3.1 Morphisms between gauge-natural bundles

In this sub-Section we recall the fundamental definitions about morphisms between gauge-natural bundles, their push-forward along gauge transformations and their Lie derivatives. Gauge-natural morphisms are also defined.

For any principal automorphism \(\Psi = (\phi, \psi)\) of the principal bundle \(P \rightarrow M\) let us denote by \(\mathfrak{B}(\Psi) = (\phi, \tilde{\Psi}_\mathfrak{B})\) the gauge-natural lift of \(\Psi\) to the gauge natural bundle \(\mathfrak{B}(P)\).

**Definition 9** A bundle morphism between gauge-natural bundles \(\mathfrak{M} : \mathfrak{B}(P) \rightarrow \mathfrak{D}(P)\) projecting onto the identity is said to be *gauge-natural* if for any local principal automorphism \(\Psi\) of \(P\) the following diagram commutes

\[
\begin{array}{ccc}
\mathfrak{B}(P) & \xrightarrow{\mathfrak{M}} & \mathfrak{D}(P) \\
\mathfrak{B} & \downarrow \mathfrak{M} & \downarrow \mathfrak{D} \\
\mathfrak{B}(P) & \xrightarrow{\mathfrak{M}} & \mathfrak{D}(P)
\end{array}
\]
Definition 10 Let $\mathcal{M}: \mathcal{B}(P) \to \mathcal{D}(P)$ be a vertical (i.e. projecting onto the identity) morphism between gauge-natural bundles. We define its \textit{push-forward} along the local principal automorphism $\Psi = (\phi, \psi)$ to be the unique vertical morphism $\Psi^*\mathcal{M}: \mathcal{B}(P) \to \mathcal{D}(P)$ such that for all section $\sigma \in \Gamma(\mathcal{B}(P))$ we have

$$\mathcal{D}(\Psi)^*(\mathcal{M} \circ \sigma) = \Psi^*\mathcal{M} \circ \mathcal{B}(\Psi)^*\sigma.$$ 

As a consequence we have the following explicit rule of calculation:

$$\Psi^*\mathcal{M} = \mathcal{D}(\Psi) \circ \mathcal{M} \circ \mathcal{B}(\Psi^{-1}) = \hat{\Psi}_D \circ \mathcal{M} \circ \hat{\Psi}_B^{-1}.$$ 

Definition 11 Let $\mathcal{M}: \mathcal{B}(P) \to \mathcal{D}(P)$ be a vertical morphism between gauge-natural bundles over the same base and $\Psi_s = (\phi_s, \psi_s)$ a 1-parameter family of local principal automorphisms having $\Xi \in \Gamma(\text{IGA}(P))$ as generator. We define the \textit{Lie derivative} of the morphism $\mathcal{M}$ along $\Xi$ to be the morphism $\mathcal{L}_\Xi \mathcal{M}: \mathcal{B}(P) \to \mathcal{V}\mathcal{D}(P)$ that fulfills

$$\mathcal{L}_\Xi \mathcal{M} = \frac{d}{ds}\Psi_s^*\mathcal{M} \bigg|_{s=0}.$$ 

From the definition it follows immediately that the expression of the Lie derivative of a morphism is linked to that of the Lie derivative of sections by means of the following rule: for all $\sigma \in \Gamma(\mathcal{B}(P))$

$$\mathcal{L}_\Xi \mathcal{M} \circ \sigma = \mathcal{L}_\Xi (\mathcal{M} \circ \sigma) - T\mathcal{M} \circ \mathcal{L}_\Xi \sigma.$$ 

If we use coordinates $(x^\mu, y^a)$ on $\mathcal{B}(P)$ and $(x^\mu, z^A(x^\mu, y^a))$ on $\mathcal{D}(P)$, the local expression of $\mathcal{M}$ is $\mathcal{M}: (x^\mu, y^a) \mapsto (x^\mu, z^A(x^\mu, y^a))$. Let us call $\hat{\Xi}_B$ and $\hat{\Xi}_D$ the gauge-natural lifts of an infinitesimal generator of principal automorphisms $\Xi$ respectively to $\mathcal{B}(P)$ and $\mathcal{D}(P)$, and let

$$\hat{\Xi}_B = \Xi^\mu \frac{\partial}{\partial x^\mu} + \hat{\xi}^a_B \frac{\partial}{\partial y^a} \quad \text{and} \quad \hat{\Xi}_D = \Xi^\mu \frac{\partial}{\partial x^\mu} + \hat{\xi}^A_D \frac{\partial}{\partial z^A}$$

be their coordinate expressions.

The Lie derivative of $\mathcal{M}$ has thence local coordinate expression

$$\mathcal{L}_\Xi \mathcal{M} = \left( \hat{\xi}^A_D - \frac{\partial z^A}{\partial y^a} \hat{\xi}^a_B \right) \frac{\partial}{\partial z^A}. $$

Remark 12 Let $\mathcal{B}(P)$ and $\mathcal{D}(P)$ be gauge-natural bundles of order $(r, s)$ and $(j, k)$ respectively. We can think at the Lie derivative as a fibered morphism

$$\mathcal{L}_{\Xi} \mathcal{M} : \mathcal{B}(P) \to \left( J^m \text{IGA}(P) \right)^* \otimes_{\mathcal{D}(P)} \mathcal{V}\mathcal{D}(P)$$

with $m = \max\{r, j\}$ such that if $\Xi$ is vertical (i.e. a section of $\frac{\mathcal{V}\mathcal{P}}{\mathcal{D}(P)} \hookrightarrow \text{IGA}(P)$), $\mathcal{L}_\Xi \mathcal{M}$ depends in fact on the derivatives of $\Xi$ only up to the order $f = \max\{s, k\}$. 

Proposition 13 A necessary and sufficient condition for a vertical morphism between gauge-natural bundles $\mathbb{M} : \mathfrak{B}(P) \to \mathfrak{D}(P)$ to be gauge-natural is that for any 1-parameter family of local automorphisms $\Psi_s$ of $P$ generated by $\Xi$, one has

$$\Psi_s^* \mathbb{M} = \mathbb{M} \quad \text{or} \quad \mathcal{L}_\Xi \mathbb{M} = 0.$$ 

Proposition 14 Any global morphism $\mathbb{M} : \mathfrak{B}(P) \to \mathfrak{D}(P)$ has to be gauge-natural and any local gauge-natural morphism can be extended to a global one.

Proof: (Sketch) A trivialization of $P$ induces both a trivialization of $\mathfrak{B}(P)$ and $\mathfrak{D}(P)$ (see [7]) and gauge-naturality is exactly the same as invariance with respect to changes of trivializations of $P$. ♣

3.2 Variationally gauge-natural morphisms

In this sub-Section we present some technical properties that follow from requiring a local Lagrangian not exactly to be a gauge-natural morphism (that would imply globality) but to satisfy a slightly more relaxed condition that amounts to say that every gauge transformation has to be a Lagrangian symmetry according to definition 8 if a gauge-natural parametrization is provided.

This generalization is needed in order to deal with the case presented as an example in Section 5.1.

The Theorems we shall develop here can be applied to ordinary variational calculus, but they have been introduced with the explicit aim of embracing the case of parametrized variations; their consequences will be analyzed in the next sub-Section.

Definition 15 Let $\mathfrak{C}(P) \to M$ be a gauge-natural bundle and $F \to \mathfrak{C}(P)$ be a vector bundle such that the composition $F \to \mathfrak{C}(P) \to M$ is a gauge natural bundle on $M$ that we call $\mathfrak{F}(P)$. A local variational morphism $\mathbb{M} : \mathfrak{J}^k \mathfrak{C}(P) \to (\mathfrak{J}^h \mathfrak{F}(P))^* \otimes \mathfrak{J}^h \mathfrak{C} \Lambda^{m-n} T^*(M)$ is said to be variationally gauge-natural if for any 1-parameter family of local automorphisms $\Psi_s$ of $P$ generated by $\Xi$ there exists a 1-parameter family of variational morphisms $\{\alpha_s\}$ with

$$\alpha_s : \mathfrak{J}^{k-1} \mathfrak{C}(P) \to (\mathfrak{J}^{h-1} \mathfrak{F}(P))^* \otimes \Lambda^{m-n-1} T^*(M)$$

such that, for every section $X \in \Gamma(\mathfrak{F}(P))$ one has

$$\Psi_s^* < \mathbb{M} \mid j^h X > = < \mathbb{M} \mid \Psi_s^* j^h X > + \text{Div} < \alpha_s \mid \Psi_s^* j^{h-1} X > .$$

Remark 16 In terms of Lie derivatives there are many possible equivalent infinitesimal formulations of the definition of variational gauge-naturality. Let us consider the morphism

$$\mathbb{M} : \mathfrak{J}^k \mathfrak{C}(P) \to (\mathfrak{J}^h \mathfrak{F}(P))^* \otimes \mathfrak{J}^h \mathfrak{C} \Lambda^{m-n} T^*(M)$$.
and let us think of it as a morphism
\[ M' : J^k \mathcal{C}(P) \times J^h \mathcal{F}(P) \rightarrow \Lambda^{m-n} T^*(M) \]
linear in its second argument. Let us suppose that \( \mathcal{C}(P) \) is gauge-natural of order \((r,s)\) and \( \mathcal{F}(P) \) of order \((j,k)\) and that \( m = \min\{r,j\} \), according to Remark 12; then the Lie derivative of \( M' \) with respect to a gauge generator \( \Xi \) is a morphism
\[ \mathcal{L}_\Xi M' : J^k \mathcal{C}(P) \times J^h \mathcal{F}(P) \rightarrow (J^m \text{IGA}(P))^* \otimes_M V \Lambda^{m-n} T^*(M), \]
linear in the second argument, that can also be thought as
\[ \mathcal{L}_\Xi M' : J^k \mathcal{C}(P) \rightarrow (J^h \mathcal{F}(P))^* \otimes (J^m \text{IGA}(P))^* \otimes V \Lambda^{m-n} T^*(M) \]
where the tensor product is on the base \( J^h \mathcal{F}(P) \) (for simplicity we still denote it by \( \mathcal{L}_\Xi M' \)).
Using the map \( \pi_2 \) introduced in Remark 54, we can say that variational gauge-naturality of \( M' \) is equivalent to the existence of a morphism \( \alpha : J^{k-1} \mathcal{C}(P) \rightarrow (J^h \mathcal{F}(P))^* \otimes (J^m \text{IGA}(P))^* \otimes V \Lambda^{m-n} T^*(M) \) such that
\[ \pi_2 \langle \mathcal{L}_\Xi M' | j^h X \rangle = \text{Div} \left( \pi_2 \left( \left< \alpha | j^{m-1} \Xi \right> | j^{h-1} X \right) \right), \]
and the link between \( \alpha \) and the 1-parameter family \( \{\alpha_s\} \) of definition 15 is
\[ -\frac{d}{ds} \alpha_s |_{s=0} = \left< \alpha | j^{m-1} \Xi \right>. \]
Concrete examples in local coordinates of how to check variational gauge-naturality of a Lagrangian morphism as well as that of an Euler-Lagrange morphism will be given in the proof of Proposition 19.

In our exposition, two technical lemmas precede the main basic properties of variationally gauge-natural morphisms and in particular of variationally gauge-natural Lagrangians in order to simplify their proofs. The first lemma is very well known, but we report it here just to display some coordinate expression which will be easily recognized later. To our knowledge the other statements have not yet been proved in the same generality.

**Lemma 17** If \( L = \text{Div} \Lambda \) then for every \( X \in \Gamma(VC) \) there exists a morphism \( \delta \Lambda \) such that \( < \delta L | j^k X > = \text{Div} < \delta \Lambda | j^{k-1} X > \). If \( \Lambda \) is global, so is \( \delta \Lambda \).

**Proof:** Let \( \Lambda \) be a local variational morphism
\[ \Lambda : J^{k-1} C \rightarrow (J^{k-1} VC)^* \otimes_{J^{k-1} C} \Lambda^{m-1} T^*(M) \]
where \( M \) is the base of the configuration bundle \( C \). Let us use coordinates \((x^\mu, y^\alpha_\mu)\) on \( J^k C \) \((\mu \text{ runs from } 1 \text{ to } \text{dim} \ M, \text{ while } \alpha \text{ is a multindex of length } 0 \leq |\alpha| < k)\) and let us express \( \Lambda \) in coordinates as \( \Lambda = \Lambda^\mu(x^\mu, y^\alpha_\mu) \, ds_\mu \) (with \( 0 \leq |\alpha| < k \)) while \( L = d_\mu \Lambda^\mu(x^\mu, y^\alpha_\mu) \, ds \).
The coordinate expression for the morphism \( < \delta L | j^k X > \) is the following
\[ < \delta L | j^k X > = \frac{\partial d_\mu \Lambda^\mu}{\partial y^\alpha_\mu} X^\alpha_\mu \, ds. \]
where the multiindex $\nu$ has length $0 \leq |\nu| \leq k$.

We have

$$
\frac{\partial d_{\mu} \Lambda^\mu}{\partial y^\nu_\alpha} = d_{\mu} \frac{\partial \Lambda^\mu}{\partial y^\nu_\alpha} + \delta_{\nu}^{\alpha} + \hat{\bar{\nu}} \frac{\partial \Lambda^\mu}{\partial y^\alpha_\bar{\nu}},
$$

where the multi-index $\bar{\alpha}$ has length $|\bar{\alpha}| = |\nu| - 1$ and by $\hat{\bar{\nu}}$ we mean a multiindex with 1 entry in position $\mu$ and zero elsewhere, while for $|\nu| = 0$ there is no $\bar{\alpha}$ to fit into the formula so that one has to set $\delta_{\nu}^{\alpha} + \hat{\bar{\nu}} = 0$, while for $|\nu| = k$ we simply have

$$
d_{\mu} \frac{\partial \Lambda^\mu}{\partial y^\nu_\alpha} = 0.
$$

Substituting now into (6) we get

$$
< \delta \mathcal{L} | j^k X > = \left[ d_{\mu} \frac{\partial \Lambda^\mu}{\partial y^\nu_\alpha} + \delta_{\nu}^{\alpha} + \hat{\bar{\nu}} \frac{\partial \Lambda^\mu}{\partial y^\alpha_\bar{\nu}} \right] X^a_\nu \, ds = d_{\mu} \left( \frac{\partial \Lambda^\mu}{\partial y^\alpha_\bar{\nu}} X^a_\alpha \right) \, ds.
$$

We have found the local expression of the morphism $\delta \Lambda$

$$
\delta \Lambda = (\delta \Lambda)^\mu \, ds_{\mu} \quad \text{with} \quad (\delta \Lambda)^\mu = \frac{\partial \Lambda^\mu}{\partial y^\alpha_\bar{\nu}} X^a_\alpha
$$

of which we wanted to prove the existence. It remains to straightforwardly check that if $\Lambda$ is global so is $\delta \Lambda$; and that in this case the previous one is an invariant expression that does not depend on the chosen coordinates.

\[ \clubsuit \]

**Lemma 18** The gauge-natural lift $j^k \hat{\Xi}_{V\mathcal{C}}$ of an infinitesimal generator $\Xi \in \Gamma(\text{IGA}(P))$ of principal automorphisms of $P$ to the $k^{th}$-jet prolongation of the vertical bundle $V\mathcal{C}(P)$ of a gauge-natural bundle $\mathcal{C}(P)$ can be computed from the lift $j^k \hat{\Xi}_{\mathcal{C}}$ to the same jet prolongation of the base bundle $\mathcal{C}(P)$ according to the following rule: let $(x^\mu, y^a_\mu, v^a_\mu)$ be fibered coordinates on $J^k \mathcal{C}(P)$ and $(x^\mu, y^a_\mu, v^a_\mu)$ on $J^k V\mathcal{C}(P)$ ($\nu$ is a multi-index of length $0 \leq |\nu| \leq k$). If

$$
\hat{j}^k \hat{\Xi}_{\mathcal{C}} = \Xi^\mu \frac{\partial}{\partial x^\mu} + (\hat{j}^k \hat{\Xi}_{\mathcal{C}})^b_b \frac{\partial}{\partial y^b_\mu}
$$

is the coordinate expression of $\hat{j}^k \hat{\Xi}_{\mathcal{C}}$ then the coordinate expression of $j^k \hat{\Xi}_{V\mathcal{C}}$ is

$$
\hat{j}^k \hat{\Xi}_{V\mathcal{C}} = \Xi^\mu \frac{\partial}{\partial x^\mu} + (\hat{j}^k \hat{\Xi}_{\mathcal{C}})^b_b \frac{\partial}{\partial y^b_\mu} + (j^k \hat{\Xi}_{V\mathcal{C}})^b_b \frac{\partial}{\partial v^b_\mu}
$$

with

$$
(j^k \hat{\Xi}_{V\mathcal{C}})^b_b = v^a_\mu \frac{\partial}{\partial y^b_\mu} (j^k \hat{\Xi}_{\mathcal{C}})^b_b
$$

where $\bar{\mu}$ is a multi-index of length $0 \leq |\bar{\mu}| \leq k$.

**Proof:** If we have a gauge-natural bundle $\mathfrak{B}(P)$ with a system of local coordinates $(x^\mu, p^A)$, given the gauge-natural lift $\hat{\Xi}_{\mathfrak{B}}$ of any infinitesimal generator $\Xi$ of principal automorphisms of $P$ with local coordinate representation

$$
\hat{\Xi}_{\mathfrak{B}} = \Xi^\mu \frac{\partial}{\partial x^\mu} + \hat{\Xi}_{\mathfrak{B}}^A \frac{\partial}{\partial p^A},
$$

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then the lift $\hat{\mathcal{V}}_{\mathcal{B}}$ on the vertical bundle of $\mathcal{B}$, where we use coordinates $(x^\mu, p^A, \varepsilon^A)$, has the following local representation

$$
\hat{\mathcal{V}}_{\mathcal{B}} = \Xi^\mu \frac{\partial}{\partial x^\mu} + \hat{\mathcal{V}}^A_{\mathcal{B}} \frac{\partial}{\partial p^A} + \varepsilon^B \frac{\partial \hat{\Xi}^A_{\mathcal{B}}}{\partial p^B} \frac{\partial}{\partial \varepsilon^A}.
$$

Now plug in $\mathcal{B}(P) = J^k \mathcal{C}(P)$ and, thanks to the isomorphism $V(J^k \mathcal{C}(P)) \approx J^k(V \mathcal{C}(P))$, one gets the thesis.

\[\clubsuit\]

**Proposition 19** If $L$ is a variationally gauge-natural (local) variational morphism then the gauge-natural morphism $\delta L$ introduced in formula (2) is variationally gauge-natural, too.

**Proof:** Let us introduce a fibered coordinate system $(x^\mu, y^a)$ on the gauge-natural configuration bundle $\mathcal{C}(P)$. On $J^k \mathcal{C}(P)$ we can naturally induce the system of fibered coordinates $(x^\mu, y^{\bar{a}})$ where $\bar{a}$ is a multi-index of length $0 \leq |\bar{a}| \leq k$.

For a $k$-th order Lagrangian and for any section $X$ of the vertical bundle $V \mathcal{C}(P)$ we have

$$
< \delta L \mid j^k X > = \frac{\partial L}{\partial y^a_{\bar{b}}} X^a_{\bar{b}}.
$$

Let us adopt the same notation of the previous Lemma for the lifts of infinitesimal generators of automorphisms of $P$ to the configuration bundle and its vertical bundle.

Furthermore let $(r, s)$ be the order of $\mathcal{C}$, so that the order of $J^k \mathcal{C}$ is at most $(r + k, s + k)$. The Lagrangian $L : J^k \mathcal{C} \rightarrow \Lambda^m T^* M$ is variationally gauge-natural so that a morphism $\alpha : J^{k-1} \mathcal{C}(P) \rightarrow (J^{r+k-1} \text{IGA}(P))^* \otimes V \Lambda^{m-n} T^* (M)$ exists such that

$$
\pi_2 L_{\Xi M^l} = \text{Div} \left( \pi_2 < \alpha \mid j^{m-1} \Xi > \right) = \text{Div} \Lambda.
$$

having called $\Lambda$ the morphism $\Lambda = \pi_2 < \alpha \mid j^{r+k-1} \Xi >$; let $\Lambda^\mu$ be its components with respect to the decomposition $\Lambda = \Lambda^\mu ds^\mu$.

We have in coordinates:

$$
\frac{\partial L}{\partial y^a_{\bar{b}}} d_\alpha L_{\Xi y^a} = d_\mu (L_{\Xi^\mu}) + d_\mu \Lambda^\mu
$$

which is equivalent to

$$
- \frac{\partial L}{\partial x^\alpha} \Xi^\alpha - \frac{\partial L}{\partial y^a_{\bar{b}}} (j^k \hat{\Xi}^a_{\bar{b}})_{\bar{b}} = L d_\mu \Xi^\mu + d_\mu \Lambda^\mu
$$

(7)

Thanks to Lemma 18 we can notice that if the order of $\mathcal{C}(P)$ is $(r, s)$ then the order of $V \mathcal{C}(P)$ is $(r + 1, s + 1)$.

Gauge-naturality of $\delta L : J^k \mathcal{C}(P) \rightarrow (J^k V \mathcal{C}(P))^* \otimes \Lambda^m T^* M$ means that there exists a morphism $\beta : J^{k-1} \mathcal{C}(P) \rightarrow (J^{r+k-1} \text{IGA}(P))^* \otimes (J^{r+k} \text{IGA}(P))^* \otimes V \Lambda^{m-n} T^* (M)$ such that

$$
\pi_2 < \mathcal{L}_{\Xi M^l} \mid j^k X > = \text{Div} \left( \pi_2 \left< \beta \mid j^{r+k} \Xi > \mid j^{k-1} X \right> \right).
$$

Let us then compute $< \mathcal{L}_{\Xi M^l} \mid j^k X >$. We have (both multiindices $\bar{a}$ and $\bar{\mu}$ have length $0 \leq |\bar{a}| = |\bar{\mu}| \leq k$).
the first two addenda cancel, while thanks to Lemma
For any 1-parameter family of local automorphism \( \Psi \) with \( V \in \Gamma(\mathfrak{g}(P)) \) splits into

\[
\pi_2 < \mathcal{L}_\mathcal{M} | j^k X > = \left\{ X_\mu^a \frac{\partial}{\partial y_\mu^a} \frac{\partial L}{\partial y_\mu^a} \mathcal{L} \right\} ds = 
\]

= \left\{ -X_\mu^a \left[ \frac{\partial}{\partial y_\mu^a} \frac{\partial L}{\partial y_\mu^a} \mathcal{L} \right] + \frac{\partial}{\partial y_\mu^a} \sum \left( X_\mu^a \mathcal{L} \right) \right\} ds.

Integrating by parts the first two addenda and applying (7) we get

\[
\pi_2 < \mathcal{L}_\mathcal{M} | j^k X > = \left\{ X_\mu^a \frac{\partial}{\partial y_\mu^a} \left[ \frac{\partial L}{\partial y_\mu^a} \mathcal{L} \right] + \frac{\partial}{\partial y_\mu^a} \sum \left( X_\mu^a \mathcal{L} \right) \right\} ds = 
\]

Thus we have

\[
\pi_2 < \beta | j^{k+1} \Xi > j^{k-1} X > = < \delta \Lambda | j^{k-1} X >
\]

and \( \delta L \) is variationally gauge-natural.

\[\blacklozenge\]

**Theorem 20** The volume part of a variationally gauge-natural (local) variational morphism is gauge-natural (and thence global).

**Proof:** Let \( P \rightarrow M \) be a principal bundle, \( \mathcal{C}(P) \) a gauge-natural bundle and \( \mathfrak{g}(P) \) a gauge-natural vector bundle. Let \( \mathcal{M} \) be a gauge-natural morphism

\[
\mathcal{M} : J^k \mathcal{C}(P) \rightarrow (J^h \mathfrak{g}(P))^* \otimes J_h \mathfrak{g} \Lambda^m T^* M
\]

that for all \( V \in \Gamma(\mathfrak{g}(P)) \) splits into

\[
\pi_2 < M | j^h V > = < \nabla | V > + \text{Div} < B | j^{h-1} V >
\]

with

\[
\nabla \equiv \nabla(M) : J^{h+k} \mathcal{C}(P) \rightarrow (\mathfrak{g}(P))^* \otimes \mathfrak{g} \Lambda^m T^* M
\]

\[
B \equiv B(M, \gamma) : J^{h+k-1} \mathcal{C}(P) \rightarrow (J^{h-1} \mathfrak{g}(P))^* \otimes J_h \mathfrak{g} \Lambda^{m-1} T^* M.
\]

For any 1-parameter family of local automorphism \( \Psi_s \) of \( P \) and for all \( V \in \Gamma(\mathfrak{g}(P)) \) we have

\[
\Psi^*_s < M | j^h V > = \Psi^*_s < \nabla | V > + \Psi^*_s \left[ \text{Div} < B | j^{h-1} V > \right]
\]

As it happens whenever we consider a natural bundle on \( M \) as gauge-natural on any \( P \rightarrow M \) the gauge-natural lift on \( \Lambda^m T^* M \) of any gauge transformation coincides with
the natural lift of the diffeomorphism onto which the gauge transformation projects, so that if \( \Psi(s) = (\phi_s, \nu_s) \) then \( \Lambda^mT^*(\Psi(s)) = \Lambda^mT^*(\phi_s) \). If computed on a section, the divergence coincides with the exterior differential of forms (see Definition 38). Accordingly, as the push-forward along a diffeomorphism commutes with the exterior differential we have the following splitting

\[
\Psi^* s \langle M | j^h V \rangle = \langle M | \Psi^* j^h V \rangle + \text{Div} \left[ \Psi^* s < B | j^{h-1} V \rangle \right].
\]

By variational gauge-naturality of \( M \) we also have a 1-parameter family of morphisms \( \{ \alpha_s \} \) such that

\[
\Psi^* s \langle M | j^h V \rangle = \langle M | \Psi^* j^h V \rangle + \text{Div} \left[ \alpha_s | \Psi^* (j^{h-1} V) \right]
\]

Thanks to the uniqueness of the volume part of the splitting of a variational morphism we can conclude

\[
\Psi^* s V = V
\]

that ensures gauge-naturality of \( V \).

### 3.3 Gauge-natural variational calculus with parametrized variations

**Definition 21** A gauge-natural variational problem with parametrized variations is defined by the set \((\mathcal{C}(P), L, \mathcal{F}, P, J, \bar{\omega})\) of the following objects:

1. a gauge-natural configuration bundle \( \mathcal{C}(P) \xrightarrow{\pi} M \) of order \((r, s)\) with structure \(G\)-bundle \( P \xrightarrow{\pi} M\);
2. a variationally gauge-natural (local) Lagrangian morphism \( L \) of order \(k\);
3. a vector bundle \( F \xrightarrow{\pi_F} \mathcal{C}(P) \) such that the composite projection \( F \xrightarrow{\pi_F \pi} M \) is a gauge-natural bundle \( \mathfrak{G}(P) \) of order \((p, f)\) called bundle of parameters;
4. a gauge-natural parameterizing morphism \( P : J^1\mathcal{C}(P) \rightarrow (J^q \mathfrak{G}(P))^* \otimes_{J^r\mathcal{C}(P)} V\mathcal{C}(P) \) \((q \in \{0, 1\})\);
5. a gauge-natural morphism \( J : J^k\mathcal{C} \rightarrow (J^{r-q}\text{IGA}(P))^* \otimes_{J^r\mathcal{C}} \mathfrak{G}(P) \) such that we have \( \langle P | j^q < J | j^{r-q} \Xi \rangle \circ j^1\sigma = L \Xi \sigma \);
6. a morphism \( \bar{\omega} : J^k\mathcal{C}(P) \rightarrow \mathcal{C}(M) \times_M \mathcal{C}(P) \) that associates to any configuration a couple \((\gamma, \omega)\) where \( \gamma \) is a linear connection on \( M \) and \( \omega \) is a principal connection on \( P \).

Let us remark that the connection \( \gamma \) is needed to construct the \( \mathbb{P}\)-Poincaré-Cartan morphism \( F(L, S, P, \gamma) \) only if \( k + l > 2 \), otherwise this requirement can be dropped, while the connection \( \omega \) on \( P \) will be needed in every case to distinguish horizontal and vertical gauge symmetries (as we will see in a while).
Theorem 22 A gauge-natural variational problem with parametrized variations \((\mathcal{C}(P), L, \mathfrak{F}, \mathbb{P}, \mathbb{J})\) leads to gauge-natural (global) Euler-Lagrange equations.

**Proof:** According to the splitting
\[
\begin{align*}
\left< \delta L | j^k \mathbb{P}' > \right| j^{l+k} \epsilon = & \left< E | \epsilon \right> + \text{Div} < F | j^{l+k-1} \epsilon >,
\end{align*}
\]
the Euler-Lagrange equations of a gauge-natural variational problem with parametrized variations arise as the boundary part of a variationally gauge-natural variational morphism; in fact \(\delta L\) is variationally gauge-natural by Property 19, and so is \(\mathbb{P}\) by hypothesis (hence also its jet prolongation). Their contraction is hence gauge-natural by definition. Thanks to Property 20 we can conclude that the equations are gauge-natural.

Let us now remark that according to the fact that the Lagrangian is variationally gauge-natural, every gauge-natural lift on \(\mathcal{C}(P)\) of an infinitesimal generator \(\Xi \in \Gamma(\mathcal{A}(P))\) of automorphisms of \(P\) (or, in a more physical language, every gauge transformation) turns out to be a symmetry of our variational problem. With the help of the other requirements of Definition 21 we are able to associate to any of these gauge symmetries a gauge invariant “Nöther” conserved current according to the the rule that follows.

Theorem 23 -Nöther theorem- Let us consider a gauge natural variational problem with parametrized variations \((\mathcal{C}(P), L, \mathfrak{F}, \mathbb{P}, \mathbb{J})\) (orders are as in Definition 21) such that
\[
\pi_2 \mathcal{L}_{\Xi} L = \text{Div} \left( \pi_2 < \alpha | j^{r+k-1} \Xi \right)
\]
The unique fibered morphism (momentum map)
\[
\mathcal{E} : j^{2k+l+s-1} \mathcal{C}(P) \longrightarrow j^{r-q-1} \mathcal{A}(P) \otimes \Lambda^{n-1} T^* M
\]
such that for all \(\sigma \in \Gamma(\mathcal{C}(P))\) and for all \(\Xi \in \Gamma(\mathcal{A}(P))\) one has
\[
< \mathcal{E} | j^{r-q-1} \Xi > \circ j^{2k+l+s-1} \sigma =
\]
\[
= \left( < F | j^{l+k-1} \mathfrak{F} \circ \Xi > - Tp \Xi \circ L - \pi_2 < \alpha | j^{r+k-1} \Xi > \right) \circ j^{2k+l+s-1} \sigma
\]
associates to any couple \((\sigma, \Xi)\) formed by a section \(\sigma \in \Gamma(\mathcal{C}(P))\) and an infinitesimal generator of principal automorphisms \(\Xi \in \Gamma(\mathcal{A}(P))\) a current (i. e., a \((n-1)\)-form) that is conserved (closed) on-shell. Moreover the morphism \(\mathcal{E}\) is gauge-natural (global).

**Proof:** By gauge-naturality of \(L : j^k \mathcal{C} \longrightarrow \Lambda^m T^* M\) one can find a local morphism \(\Lambda : j^{k-1} \mathcal{C} \longrightarrow V \Lambda^{m-1} T^* M\) such that
\[
\pi_2 \mathcal{L}_{\Xi} L = \text{Div} \left( \pi_2 < \alpha | j^{r+k-1} \Xi \right).
\]

\[
\pi_2 \mathcal{L}_{\Xi} L = < \delta L | j^k \mathcal{L}_{\Xi} \sigma > - \text{Div} \left( Tp \Xi \circ L \right) =
\]
\[
= \left< \delta L | j^k \mathbb{P}' > \right| j^{l+k} \epsilon - \text{Div} \left( Tp \Xi \circ L \right) =
\]
\[
= \left< E | \right| j^{r-q} \Xi > + \text{Div} \left< F | j^{l+k-1} < \mathfrak{F} - j^{r-q} \Xi > \right> - \text{Div} \left( Tp \Xi \circ L \right).
\]
Thence
\[-\left\langle \mathcal{E} \right| j^r - q \Xi \right\rangle = \text{Div} \left\langle \mathcal{F} \right| j^{l+k-1} < j^r - q \Xi > \right\rangle - \text{Div} (Tp \Xi \Lambda) - \text{Div} (\pi_2 < \alpha | j^{r+k-1} \Xi >)\]
so
\[-\left\langle \mathcal{E} \right| j^r - q \Xi \right\rangle = \text{Div} < j^r - q - 1 \Xi > . \tag{8}\]

While from the definition of $\mathcal{E}$ one can hardly argue how many derivatives of $\Xi$ will appear in its final expression (cancellations may arise!) from the latter identity (that holds off-shell) we can say that $\Xi$ can enter with its derivatives up to the order $r - q - 1$. If $\sigma$ is a solution of the corresponding Euler-Lagrange equations then we have
\[d \left( \left[ \left( \mathcal{F} \right| j^{l+k-1} < j^r - q \Xi > \right) - Tp \Xi \Lambda - \pi_2 < \alpha | j^{r+k-1} \Xi > \right] \circ j^{2k+l+s-1} \sigma \right) = 0.\]

To prove gauge-naturality of $\mathcal{E}$ let us notice that identity (8) holds; thence, as the push-forward and the exterior differential of forms do commute, it is sufficient to prove gauge-naturality of the left hand side. The morphism $\mathcal{E}$ is gauge-natural by Property 20, and it is contracted on $J$ that is gauge-natural by hypothesis; thus the thesis follows.

In the natural case we were able to construct Noether currents related to diffeomorphisms, and their integrals on $(m-1)$-surfaces were linked to the energy-momentum of the system. In the gauge-natural case we cannot any longer lift diffeomorphisms of $M$, but according to [7], if on $P$ we have a connection gauge-naturally derived from fields, for any infinitesimal generator of principal automorphisms we can define the horizontal part with respect to this connection, and compute the conserved Noether currents relative to it. This should be regarded as the density of energy momentum in a gauge-natural variational principle also in presence of a parametrization relative to a constraint.

## 4 Constrained variational calculus and vakonomic field theories

Let us now apply the previously developed formalism to field theories with constraints within the vakonomic approach. The non-holonomic technique can be studied within the same formalism too, but with a different characterization of admissible variations; this case will be considered in a forthcoming paper.

### 4.1 Vak-criticality

The definitions we will use to implement the principle of constrained least action are a generalization of the ones given by Arnold and others in [1] in the case of Vakonomic Mechanics. We will abandon their line when they introduce Lagrange multipliers to characterize critical sections.
Definition 24 Let $C \xrightarrow{\pi} M$ be the configuration bundle. A constraint of order $p$ and codimension $r$ is a submanifold $S \subset J^p C$ of codimension $r$ that by $\pi^{p-1}$ projects onto the whole $J^{p-1}C$.

Definition 25 A configuration $\sigma \in \Gamma(C)$ is said to be admissible with respect to $S$ if its $p^{th}$-order jet prolongation lies in $S$. The space of admissible configuration with respect to $S$ is

$$\Gamma_S(C) = \{ \sigma \in \Gamma(C) / \Im(j^p \sigma) \in S \}.$$

Definition 26 Given a compact submanifold $D^{cpt} \subset M$, a vak-admissible variation (at order $k$) of an admissible configuration $\sigma \in \Gamma_S(C)$ is a smooth 1-parameter family of sections $\{\sigma_\varepsilon\}_{\varepsilon \in [-1,1]} \subset \Gamma(\pi^{-1}D)$ such that

1. $\sigma_0 = \sigma|_D$
2. $\forall \varepsilon \in [-1,1[, \ j^{k-1}\sigma_\varepsilon|_{\partial D} = j^{k-1}\sigma|_{\partial D}$
3. $\frac{d}{d\varepsilon} j^p \sigma_\varepsilon|_{\varepsilon=0} \in J^p VC \cap TS$.

Definition 27 The set $\{C, L, S\}$ where $C \xrightarrow{\pi} M$ is a configuration bundle, $L$ is a Lagrangian on $C$ and $S$ is a constraint is called a “constrained variational problem”.

Definition 28 We say that an admissible section $\sigma \in \Gamma_S(C)$ is vakonomically critical (or vak-critical) for the variational problem $\{C, L, S\}$ if $\forall D^{cpt} \subset M$ for any vak-admissible variation $\{\sigma_\varepsilon\}_{\varepsilon \in [-1,1]} \in \Gamma(\pi^{-1}D)$ we have

$$\frac{d}{d\varepsilon} \int_D L \circ j^k \sigma_\varepsilon \bigg|_{\varepsilon=0} = 0.$$

An equivalent infinitesimal condition is that

$$\forall D^{cpt} \subset M, \forall V \in V C \text{ s. t. } j^p V \in TS \text{ and } j^{k-1}V|_{\partial D} = 0, \quad \int_D \langle \delta L | j^k V \rangle \circ j^k \sigma = 0$$

where $\delta L$ is the fibered (variational) morphism defined in (2).

4.2 Parametrized variations and vakonomic criticality

In order to turn the vak-criticality condition into a differential equation, the classical strategy (at least in Mechanics) is to introduce Lagrange multipliers (see [1]). Here we try to profitably use a suitable parametrization of the set of constrained variations. This idea was first formally developed by Fernandez, Garcia and Rodrigo in the very interesting paper [8].
Definition 29 Let $C \xrightarrow{\pi} M$ be the configuration bundle. A parametrization
\[ P_S : J^sC \rightarrow (J^lE)^* \otimes_{J^sC} VC \]
of order $s$ and rank $l$ ($s \geq l$ is required) of the set of constrained variations is said to be vakonomically adapted to the constraint $S \subset J^pC$ if for all $\varepsilon \in \Gamma(E)$ and $\sigma \in \Gamma_S(C)$ the vertical vector field $j^p(<P|j^l\varepsilon > \circ j^s\sigma) \in TS$.

Definition 30 A parametrization $P_S : J^sC \rightarrow (J^lE)^* \otimes_{J^sC} VC$ vak-adapted to a constraint $S$ is said to be faithful on $\sigma \in \Gamma_S(C)$ to $S$ if for all $V \in VC$ such that both $j^p(V \circ \sigma) \in TS$ and $j^{k-1}(V \circ \sigma)|_{\partial D} = 0$ hold, there exist a section $\varepsilon \in \Gamma(E)$ such that $<P|j^l\varepsilon > \circ j^s\sigma = V$ and $j^{k-1+1}(<\varepsilon \circ \sigma)|_{\partial D} = 0$.

The fundamental problem of the existence of a (possibly faithful) parametrization vakonomically adapted to a constraint $S$ has not yet been studied in general and will not be faced here. For the moment we limit ourselves to consider a given specific parametrization as a part of the variational problem, deferring the study of the general case to further investigations. The same was done in all the previous papers on this topic (see e.g. [8]).

Once we have a vak-adapted parametrization $P_S$ we can study $P_S$-criticality, constructing, as we have shown in Section 2.1, the relevant $P_S$-Euler-Lagrange equations.

It is important to remark that even having defined the Lagrangian on the whole configuration bundle, the first variation formula arising from it does not depend on the value of the Lagrangian outside the constraint. In fact if one has a Lagrangian $\tilde{L} : J^kC \rightarrow \wedge^m T^*M$ such that $\tilde{L} = L + N$ with $N$ proportional to the equation of the constraint $S$, then on the constraint $S$ it automatically holds $<\delta N | j^kV > \circ j^s\sigma = 0$ for all $V \in VC$ s.t. $j^pV \in TS$ and for all $\sigma \in \Gamma_S(C)$ therefore the addendum $N$ does not contribute to the first variation formula, neither in the field equation nor in the boundary part.

The link between vak-criticality and $P_S$-criticality is given by the following Proposition.

Proposition 31 Let $\sigma \in \Gamma_S(C)$ be a vakonomically critical section for the constrained variational problem $\{C, L, S\}$; then for any adapted parametrization $P_S \sigma$ is $P_S$-critical.
Proof: We have

\[ \sigma \in \Gamma_S(C) \text{ is critical} \]

\[ \Downarrow \]

\[ \forall D \subset M, \forall \text{ adm. var. } \{\sigma_{\epsilon}\}, \quad \frac{d}{d\epsilon} \int_D L \circ j^k \sigma_{\epsilon} \bigg|_{\epsilon=0} = 0 \]

\[ \Downarrow \]

\[ \forall D \subset M, \forall V \in VC \text{ s. t. } j^p V \in TS \text{ and } j^{k-1}\sigma|_{\partial D} = 0, \quad \int_D \langle \delta L | j^k V \rangle \circ j^k \sigma = 0 \]

\[ \Downarrow \text{(a)} \]

\[ \forall D \subset M, \forall \epsilon \in \Gamma(E) \text{ s. t. } j^{l+k-1}\epsilon|_{\partial D} = 0, \quad \int_D \langle \delta L | j^{l+k} \epsilon \rangle \circ j^{k+l} \sigma = 0 \]

\[ \Downarrow \]

\[ \mathbb{E} \circ j^{2k+l+k} \sigma = 0. \]

Step (a) is not an equivalence because there can be admissible infinitesimal variations vanishing at the boundary with their derivatives up to the desired order that do not come from sections of the bundle of parameters that do vanish on the boundary. The last equivalence holds in force of Stokes’s theorem, the vanishing of \( j^{l+k-1}\epsilon \) on the boundary and the independence of the generators of \( E \).

**Corollary 32** Let \( \sigma \in \Gamma_S(C) \) be an admissible \( P_S \)-critical section for the constrained variational problem \( \{C, L, S\} \) and let also \( P_S \) be faithful to \( S \) on \( \sigma \); then \( \sigma \) is vakonomically critical for the constrained variational problem \( \{C, L, S\} \).

5 Examples:

To our knowledge, parametrized variational problems were first formally investigated by Fernández, García and Rodrigo [8] in relation with their application to Lagrangian reduction and in particular to Euler-Poincaré reduction [2, 9, 3]. They were already implicitly used in the literature to deal with Vakonomic Mechanics [1, 11] and Relativistic Hydrodynamics [13]. What we want to present here is a gauge-natural example that also helps to clarify the distinction between a vak-critical section of a constrained variational problem and a \( P_S \)-critical section, of which we study the equations of motions and the conserved Noether currents.

5.1 A gauge-natural example: charged general relativistic fluids.

A charged general relativistic fluid can be thought as a congruence of curves filling a causal domain \( D \subset M \), each of them carrying some rest mass and charge, being source of gravity and of electromagnetic interaction.

Our kinematical description is inspired by [13], where the gravitational field is represented by a Lorentzian metric with components \( g_{\mu\nu} \), the electromagnetic potential by a connection \( A_{\mu} \) on a principal \( U(1) \)-bundle \( P \) and the fluid degrees of freedom are represented by
a nowhere vanishing 3-form $J = J^\mu ds_\mu$ such that the unit timelike vector $u^\mu = J^\mu/|J|$ is
tangent to the flow lines and that the volume form $\sqrt{|g|} J = |J|$ represents the matter density in $D$. For what concerns the charge density we simply assume that at each spacetime point it is proportional to the matter density by an “elementary charge” $q$.

An alternative (more common) approach to describe the fluid degrees of freedom (see [22, 14, 21]) is to associate to any space-time point $x^\mu$ three scalar fields $R^a(x^\mu)$ ($a = 1, 2, 3$) identifying the “abstract fluid particles”. We remark that using the tangent vector to flow lines as fluid variable, we loose the direct kinematical identification between different points of the same flow line. In fact, having a configuration $J^\mu(x)$, in order to know whether two points of spacetime are connected by the same flow line, we have to solve the ODE that defines the integral curves of $u^\mu$ with initial condition in one of them.

In order to implement conservation of matter we only allow for closed $J$, so that for any closed 3-surface $\Sigma$ in $D$ the flow of $J$ through $\Sigma$ is zero. The bundle of configurations is thence

$$\mathcal{C}(P) = \text{Lor}(M) \times_M \mathcal{C}(P) \times_M \Lambda^3 T^* M$$

that is gauge-natural having $P \rightarrow M$ with fiber $U(1)$ as structure bundle. The field $J$, moreover, is subject to the constraint $S \subset \mathcal{F} \Lambda^3 T^* M$ represented by the equation $dJ = 0$, or locally $d\mu J^\mu = 0$ (we also require $J$ to be time-like with respect to the unknown metric, but this just selects an open set in the configuration bundle, so that the condition is automatically preserved by any infinitesimal variation).

The dynamics of the system is governed by the Lagrangian

$$L = L_H + L_{EM} + L_F + L_{int}$$  \hspace{1cm} (9)

with

$$L_H(g_{\alpha\beta}, \partial_\lambda g_{\alpha\beta}, \partial_\lambda \omega g_{\alpha\beta}) = \frac{\sqrt{|g|}}{2 \kappa} R ds$$

$$L_{EM}(g_{\alpha\beta}, A_\nu, \partial_\sigma A_\nu) = - \frac{1}{4} \sqrt{|g|} F_{\rho\sigma} F_{\rho\sigma} ds$$

$$L_F(J^\mu, g_{\alpha\beta}) = - \sqrt{|g|} \left[ \rho (1 + e(\rho)) \right] ds$$

$$L_{int} = q J^\mu A_\mu ds$$

where $R$ is the scalar curvature of the metric, $F_{\rho\sigma}$ are the curvature coefficients (field strength) of the connection $A$ (gauge potential) and $e(\rho)$ is a generic function of the scalar $\rho = \sqrt{\frac{\rho_0 J^\mu J_\mu}{|g|}}$, physically interpreted as the internal energy of the fluid. It gives rise to the pressure $P = \rho^2 \frac{de}{d\rho}$.

The gauge-natural bundle of parameters we introduce is then

$$\mathfrak{f}(P) = (\pi \mathcal{C}(P))^* \text{Lor}(M) \times \mathcal{C}(P) \times \mathcal{C}(P) \times \mathcal{C}(P) \times \mathcal{C}(P) \times \mathcal{C}(P) \times TM$$

and the parametrization is such that for any section $\mathcal{S} = \delta g_{\mu\nu}, \delta A_\mu, X^\mu \in \Gamma(E)$ and for any configuration $\sigma \in \Gamma(C)$ we have

$$\langle \mathcal{P}_\sigma | j^1 \mathcal{S} \rangle = \langle \delta g_{\mu\nu}(x^\sigma, g_{\alpha\beta}), \delta A_\mu(x, A_\alpha), \mathcal{L}_X J^\mu \rangle$$
where $\mathcal{L}_X J^\mu$ denotes the formal Lie derivative of $J$, i.e. $\mathcal{L}_X J^\mu = \nabla_\nu J^\mu X^\nu - J^\nu \nabla_\nu X^\mu + J^\mu \nabla_\nu X^\nu$. In this way we are freely varying the gravitational and electromagnetic degrees of freedom, while variations of the fluid variables are taken to be tangent to the constraint $S$; in fact for every vector field $X$ and for every admissible configuration $J$ (i.e. such that $\partial_\mu J^\mu = 0$ holds) we have $\partial_\mu \mathcal{L}_X J^\mu = 0$.

Let us remark that this parametrization is trivially gauge-natural; in fact it partially coincides with the identity and partially with the Lie derivative.

Remark 33 We want to spend some words to stress that our parametrization is adapted to the constraint $S$, but in general not faithful to it; nevertheless this is exactly what Physics requires. Varying $J$ according to our prescription is equivalent to drag $J$ along a 1-parameter family of diffeomorphisms generated by $X$ and this amounts to drag the integral curves of $u^\mu$ and to adjust the density $\rho$ to keep $J$ conserved (see [13]). If we consider a compact subset $D \subset \mathcal{M}$ and find a tangent vector $X$ whose flow moves the integral curves of $u^\mu$ also on the boundary of $D$ ($X|_{\partial D} \neq 0$) but leaves their tangent vectors fixed ($\mathcal{L}_X J^\mu|_{\partial D} = 0$, as it is possible in some cases), we have that it produces a family of deformed $J$ that are fixed on the boundary and that fulfill conservation of matter, but do not come from a vector field that vanishes on the boundary as it would be requested by faithfulness. Nevertheless, this is exactly what we want to do from the physical viewpoint: when we think of the congruence of curves that represent our fluid and we imagine to vary them without moving the boundary we do not want to move particles, not only tangent vectors. To support our choice to vary fields along our parametrization, we stress that also in the alternative approach of “abstract fluid particles” variations leave unchanged the particle identification on the boundary. Solutions of the Euler-Lagrange equations that we are going to find will then be $\mathbb{P}_S$-critical sections of the parametrized constrained variational problem, without being necessarily vak-critical solutions of the variational problem with the constraint $S$ given by $dJ = 0$.

The first variation formulae arising from the freely varied Lagrangians $L_H$ and $L_{EM}$ are the well known

$$
\langle \delta L_H | j^2 < \mathbb{P}_S | j^1 \varepsilon > \rangle = \frac{\sqrt{|g|}}{2\kappa} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) \delta g^{\mu\nu} \, ds + \nabla_\alpha \langle \mathbb{P}_S^\alpha \rangle | j^1 \varepsilon > \rangle \tag{10}
$$

with

$$
\langle \mathbb{P}_S^\alpha \rangle | j^1 \varepsilon > \rangle = \frac{\sqrt{|g|}}{2\kappa} (g^{\alpha\lambda} g_{\rho\sigma} - \delta^\alpha_\rho \delta^\lambda_\sigma) \nabla_\lambda \delta g^{\rho\sigma} \, ds
$$

where the covariant derivative is relative to the Levi-Civita connection and

$$
\langle \delta L_{EM} | j^1 \varepsilon > \rangle = -\sqrt{|g|} (\nabla_\mu F^{\mu\nu}) \delta A_\nu \, ds - \sqrt{|g|} \nabla_\mu (F^{\mu\nu} \delta A_\nu) \, ds
$$

$$
- \frac{\sqrt{|g|}}{2} H_{\mu\nu}^{(EM)} \delta g^{\mu\nu} \, ds
$$

with

$$
\sqrt{|g|} H_{\mu\nu}^{(EM)} \, ds = -2 \frac{\partial L_{EM}}{\partial g^{\mu\nu}} = \sqrt{|g|} (F_{\mu\rho} F^{\rho\nu} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu}) \, ds.
$$
From the parametrized variation of $L_F$, by taking into account the definitions

$$\mu = \rho (1 + e(\rho)) \quad \text{and} \quad P = \rho^2 \frac{\partial e}{\partial \rho}$$

together with their consequences

$$\rho \frac{\partial \mu}{\partial \rho} = \mu + P \quad \text{and} \quad \rho \nabla_\nu \left( \frac{\partial \mu}{\partial \rho} \right) = \nabla_\nu P$$

we have

$$\langle \delta L_F \mid < P S \mid j^1 \varepsilon > \rangle = - \frac{\partial \mu}{\partial \rho} u_\mu (\nabla_\nu J^\mu X^\nu - J^\nu \nabla_\nu X^\mu + J^\mu \nabla_\nu X^\nu) \, ds.$$ 

Integrating by parts, we get the first variation formula

$$\langle \delta L_F \mid < P S \mid j^1 \varepsilon > \rangle = - \sqrt{|g|} \left[ (u_\mu u^\nu + \delta^\nu_\mu) \nabla_\nu P + (\mu + P) u^\nu \nabla_\nu u_\mu \right] X^\mu \, ds +$$

$$- \frac{\mu + P}{\rho} u^\alpha X^\alpha (\nabla_\mu J^\mu) + \sqrt{|g|} \nabla_\nu \left[ (\mu + P) (\delta^\nu_\mu + u_\mu u^\nu)X^\mu \right] ds +$$

$$- \frac{\sqrt{|g|}}{2} H^F_{\mu\nu} \delta g^{\mu\nu} \, ds$$

with

$$\sqrt{|g|} H^F_{\mu\nu} \, ds = 2 \frac{\partial L_F}{\partial g^{\mu\nu}} = \sqrt{|g|} \left[ P g^{\mu\nu} - (\mu + P) u_\mu u_\nu \right] ds.$$ 

Let us compute the first variation formula for the interaction term $L_{(int)}$:

$$\langle \delta L_{int} \mid < P S \mid j^1 \varepsilon > \rangle = \frac{\partial L_{int}}{\partial J^\mu} J^\mu \nabla_\nu P + \frac{\partial L_{int}}{\partial A_\nu} \delta A_\nu =$$

$$= q A_\mu (\nabla_\nu J^\mu X^\nu - J^{\nu\mu} \nabla_\mu X^\nu + J^\mu \nabla_\nu X^\nu) \, ds + q J^\nu \delta A_\nu \, ds =$$

$$= q J^\nu \delta A_\nu + q J^\mu F^\mu_{\nu\mu} X^\nu \, ds + q \nabla_\nu (A_\mu J^\mu X^\nu - A_\mu J^\nu X^\nu) \, ds$$

The complete set of the equations of motion for the Lagrangian (9) is thence the following

$$\begin{cases}
\nabla_\mu J^\mu = 0 & \text{(constraint)} \\
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa (H_{\mu\nu}^{(EM)} + H^{(F)}_{\mu\nu}) \\
\sqrt{|g|} \nabla_\mu F^{\mu\nu} = q J^\nu \\
(u_\mu u^\nu + \delta^\nu_\mu) \nabla_\nu P + (\mu + P) u^\nu \nabla_\nu u_\mu - J^\mu F_{\mu\nu} = 0.
\end{cases}$$

We are now going to prove that the Lagrangian (9) is variationally gauge-natural when restricted to the prolongation of the constraint submanifold $S \subset J^1 \mathfrak{g}(P)$ identified by equation $\nabla_\mu J^\mu = 0$.

If $(x^\mu, \theta)$ are coordinates on the structure bundle $P$, an infinitesimal generator of principal automorphisms of $P$ is a vector field $Y = Y^\mu (x^\alpha) \partial_\mu + y(x^\alpha) \frac{\partial}{\partial \theta} \in \Gamma(TP)$ projecting onto
X = TπY = Y^μ∂_μ. The Lie derivative of a section σ of the gauge-natural configuration bundle \(Lor(M) \times C(P) \times \Lambda^3 T^* M\) with respect to Y is

\[\mathcal{L}_Y \sigma = (\mathcal{L}_X g_{αβ}) \frac{∂}{∂y_{αβ}} + (\mathcal{L}_Y A_ν) \frac{∂}{∂A_ν} + (\mathcal{L}_X J^μ) \frac{∂}{∂J^μ}\]

with coefficients

\[\mathcal{L}_X g_{αβ} = \nabla_α Y_β + \nabla_β Y_α\]
\[\mathcal{L}_Y A_σ = Y^ν F_νσ + \nabla_σ (y + A_ρ Y^ρ)\]
\[\mathcal{L}_Y F_{αβ} = Y^ρ \nabla_ρ F_{αβ} + \nabla_α Y^ρ F_ρβ + \nabla_β Y^ρ F_αρ\]
\[\mathcal{L}_X J^μ = \nabla_ν J^μ Y^ν - J^ν \nabla_ν Y^μ + J^μ \nabla_υ Y^ν\].

Thus the configuration bundle is gauge-natural with order (1, 1).

A gauge-natural morphism \(\mathbb{J} : J^1 \mathfrak{E} \rightarrow (J^1 \text{IGA}(P))^* \otimes J^1 \mathfrak{E}(P)\) such that we have

\[\langle \mathbb{P}_S | j^1 \mathbb{J} j^1 Y > \circ j^1 \mathbb{J} = \mathcal{L}_Y \sigma\]

(14)

can be defined by \(< \mathbb{J} j^1 Y > \circ j^1 \mathbb{J} = (\mathcal{L}_X g_{αβ}, \mathcal{L}_Y A_σ, X)\).

Let us now consider the Lie derivative of the morphism L with respect to Y. We find:

\[\pi_2 \mathcal{L}_Y L = \frac{∂L}{∂y_{αβ}} \mathcal{L}_X g_{αβ} + \frac{∂L}{∂y_{αβ}} \nabla_α L + \frac{∂L}{∂y_{αβ}} \nabla_β L + \frac{∂L}{∂y_{αβ}} \nabla_σ L + \frac{∂L}{∂y_{αβ}} \nabla_υ L + \frac{∂L}{∂y_{αβ}} \nabla_μ L\]

\[+ \frac{∂L}{∂y_{αβ}} \partial L + \frac{∂L}{∂y_{αβ}} \partial L + \frac{∂L}{∂y_{αβ}} \partial L + \frac{∂L}{∂y_{αβ}} \partial L + \frac{∂L}{∂y_{αβ}} \partial L + \frac{∂L}{∂y_{αβ}} \partial L\]

Analyzing this term by term we have that the identities \(\mathcal{L}_Y L_H = 0, \mathcal{L}_Y L_{EM} = 0\) are well known to be satisfied (see [7]) and they can be recasted using the first variation formulae (10) and (11) together with the identity (14) into

\[\frac{\sqrt{|g|}}{2κ} (R_{µν} - \frac{1}{2} R g_{µν}) \mathcal{L}_Y g^{µν} = \nabla_α \mathcal{E}^α_H\]

(15)

with

\[\mathcal{E}^α_H ds_α = \frac{\sqrt{|g|}}{2κ} \left[ \frac{3}{2} R^α_λ - R δ^α_λ \right] Y^λ + \left( g^{βγ} δ^α_λ - g^{αβ} δ^λ_γ \right) \nabla_β Y^γ \right] ds_α\]

and into

\[-\sqrt{|g|} (\nabla_µ F^{µν}) \mathcal{L}_Y A_ν - \frac{\sqrt{|g|}}{2} H^{EM}_{µν} \mathcal{L}_Y g^{µν} = \nabla_α \mathcal{E}^α_{EM}\]

(16)

with

\[\mathcal{E}^α_{EM} ds_α = -\sqrt{|g|} \left[ g^{µσ} H^{EM}_{σµ} Y^µ + F^{µν} \nabla_µ (y + A_ρ Y^ρ) \right] ds_ν\].

For the Lagrangian \(L_F\) a straightforward calculation enables us to verify that the identity \(\mathcal{L}_Y L_F = 0\) holds too, and that it can be split (thanks to the first variation formula (12) together with identity (14)) into the following:

\[-\sqrt{|g|} \left( (u_μ^ν u^{µν} + δ^{µν}_μ) \nabla_ν P + (µ + P_μ + P) u^{ν} \nabla_ν u_μ^α \right) X^μ +\]
\[-\frac{µ + P}{ρ} u_μ^α \nabla_µ J^μ \right] - \frac{\sqrt{|g|}}{2} H^{EM}_{µν} \mathcal{L}_X g^{µν} = \nabla_α \mathcal{E}^α_F\]

(17)
with
\[ \mathcal{E}_F^\alpha ds_\alpha = -\sqrt{|g|} (g^{\alpha \mu} H^{(F)}_{\mu \nu} Y^\nu) ds_\alpha. \]

What remains is just to show that the Lagrangian \( L_{\text{int}} \) is variationally gauge-natural when restricted to the constraint submanifold. In fact we have
\[
\pi_2 \mathcal{L}_Y L_{\text{int}} = \left[ \frac{\partial L_{\text{int}}}{\partial J^\mu} \mathcal{L}_X J^\mu + \frac{\partial L_{\text{int}}}{\partial A_\nu} \mathcal{L}_Y A_\nu - \nabla_\alpha (Y^\alpha L_{\text{int}}) \right] ds =
\]
\[ = q \left[ A_\mu (\nabla_\nu J^\nu Y^\mu - J^\nu \nabla_\nu Y^\mu + J^\mu \nabla_\nu Y^\nu) + J^\nu [Y^\rho F_{\rho \nu} \nabla_\nu (y + A_\rho Y^\rho)] + A_\nu J^\nu Y^\alpha - Y^\alpha J^\nu \nabla_\alpha A_\nu - Y^\alpha A_\rho \nabla_\alpha J^\nu \right] ds =
\]
\[ = q \left[ J^\nu \nabla_\nu y \right] ds = q \left[ \nabla_\nu (y J^\nu) - y \nabla_\nu J^\nu \right] ds
\]
where the first addendum is a (local) divergence, while the second one vanishes on the constraint. Thanks to the first variation (13) together with identity (14), the previous formula can be recasted into the identity
\[
q J^\nu \mathcal{L}_Y A_\nu + q J^\nu F_{\mu \nu} X^\nu = \nabla_\alpha \mathcal{E}_\text{int}^\alpha \tag{18}
\]
with
\[ \mathcal{E}_\text{int}^\alpha = -q (y + A_\mu Y^\mu) J^\alpha \]

Summing up identities (15), (16), (17) and (18) we find that the sum of the left hand sides vanishes if composed with a \( \mathbb{P}_S \)-critical section, while the current
\[ \mathcal{E}^\alpha ds_\alpha = (\mathcal{E}^\alpha_H + \mathcal{E}^\alpha_{EM} + \mathcal{E}^\alpha_F + \mathcal{E}^\alpha_{\text{int}}) ds_\alpha \]
is conserved.

If in particular we compute the current \( \mathcal{E}^\alpha ds_\alpha \) relative to the horizontal part \( Y_{(\text{hor})} = Y^\mu \partial_\mu - A_\rho Y^\rho \frac{\partial}{\partial \theta^\rho} \) of an infinitesimal generator of automorphisms of \( P \), what we get is
\[ \mathcal{E}^\alpha_{(\text{hor})} ds_\alpha = \mathcal{E}^\alpha_H ds_\alpha - \sqrt{|g|} \left[ g^{\alpha \mu} (H^{(F)}_{\mu \nu} + H^{(F)}_{\nu \mu}) Y^\nu \right] ds_\alpha. \]

and this is the conserved current related with the energy-momentum of the system. Moreover, the Nöther current relative to the vertical part \( Y_{(V)} = (y + A_\mu Y^\mu) \frac{\partial}{\partial \theta^\mu} \) is
\[ \mathcal{E}^\alpha_{(V)} = -\sqrt{|g|} F^{\alpha \mu} \nabla_\mu (y + A_\rho Y^\rho) - (y + A_\mu Y^\mu) q J^\alpha. \]

Integrating by part the covariant derivative of \((y + A_\mu Y^\mu)\) we can split this vertical current according to
\[ \mathcal{E}^\alpha_{(V)} = (\sqrt{|g|} \nabla_\mu F^{\alpha \mu} - q J^\alpha)(y + A_\mu Y^\mu) - \nabla_\mu \left[ \sqrt{|g|} F^{\alpha \mu} (y + A_\mu Y^\rho) \right] \]
where the first summand vanishes on-shell while the argument of the divergence is called the superpotential \( \mathcal{U} = -\frac{q}{2} \sqrt{|g|} F^{\alpha \mu} (y + A_\mu Y^\rho) ds_\alpha \mu \) and it is a 2-form that is closed even off-shell. We stress that the interaction Lagrangian does not contribute to the electromagnetic superpotential.
Conclusions

Motivated by many physical and mathematical examples, we have studied variational problems with parametrized variations. In particular, starting from the results of [8] we have worked on a twofold generalization: by one side we studied parametrized variational problems, no matter from where the parametrization comes from, and we have recovered and ever generalized known results. In a second moment we also focused on the application to vakonomic constraints, for which we have defined and compared the two different concepts of vak-criticality and $P$-criticality. A detailed study of nonholonomic field theory could also be based on the same framework and it shall certainly be performed in the future developments. Hopefully our unified language will also help to understand how constraints on the derivatives of fields have to be handled in the case of field theories.

For what concerns the second direction of generalization, we have formulated a variational theory for gauge-natural parametrized field theories including conserved currents that provides, in the case of relativistic hydrodynamics of a charged fluid, a non-conventional procedure to define field equations, Nöther currents and superpotentials and that turns out to be computationally much easier then the standard one. A last observation that deserves to be further investigated is that a $P$-Euler Lagrange equations arising from the parametrized variational problems is usually not variational in the standard sense. Nevertheless we have a very efficient machinery to study symmetries and conservation laws. Do this apply to any known non-variational equations?

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A Variational morphisms and their splittings

In this Appendix we report the theory of variational morphisms. The theory is modeled on [7] and [10], where complete proofs are given. To adapt them to the constrained case we have to modify a bit the definitions. In particular, the modified definitions of formal connection and variational morphism seem to appear here for the first time.

A.1 Variational morphisms

Variational morphisms are an abstract model of the integrands appearing in global variational calculus. They were introduced (see [6] for a discussion and some bibliography on alternative approaches) to provide a general framework in which we can implement two specific algorithmic procedures based on integration by parts. The first one explicitly construct a splitting of such integrands into a volume and a boundary covariant part.
Definition 34 Let $C \xrightarrow{\pi} M$ be a configuration bundle over a $m$ dimensional base and $E \xrightarrow{\pi_E} C$ a vector bundle over $C$. Let $k \geq h$ and $n$ be natural numbers (zero included). A variational $E$-morphism is a vertical (projecting onto the identity) bundle morphism

$$\mathbb{M} : J^k C \rightarrow (J^h E)^* \otimes J^n C \pi^* \Lambda^{m-n} T^*(M).$$

The minimal $k$ is called the order of $\mathbb{M}$, $h$ is the rank, while $m - n$ is the degree and $n$ the codegree.

A.2 Operations with (variational) morphisms

Definition 35 Given a vertical fibered morphism $\mathbb{F} = (id, \Phi)$ between two bundles $B$ and $D$ over the same base $M$ we call $s$-order jet prolongation (or $s$-order formal derivative) of $\Phi$ the unique fibered morphism $j^s \mathbb{F} = (id, j^s \Phi)$ that makes commutative the following diagram

![Diagram](image)

When one prolongs a morphism projecting onto the identity from a jet bundle to another bundle, e.g. $\mathbb{F} : J^1 p^* B \rightarrow D$, one gets a morphism $j^1 \mathbb{F} : J^1 (J^p B) \rightarrow J^1 D$. The bundle $J^{p+1} B$ is naturally embedded in $J^1 (J^p B)$ by a canonical inclusion $i$. We are often more interested into the map $j^1 \mathbb{F} \circ i$ rather then in the prolongation itself. We call $j^1 \mathbb{F} \circ i$ the first order holonomic prolongation of $\mathbb{F}$. In the sequel, when we mention the prolongation of a morphism starting from a jet bundle we will always mean the holonomic one.

Remark 36 We warn the reader that by $j^k$ we denote three different concepts of prolongation: prolongation of a section, of a vertical morphism and of a vector field. They are strongly linked, though not identical concepts. For a discussion of this link we refer the reader to [7], but we just remark that every vector field $X$ on a bundle $B$ can be seen as a vertical morphism $X : B \rightarrow TB$, and if we compute the first jet prolongation of this morphism we get another
morphism $j^h X : J^h B \to J^h TB$ that is not a vector field on $J^h B$ as it should be if thought as the prolongation of a vector field. If $(x^\mu, y^i_\nu)$ are coordinates on $J^h B$ ( $\nu$ is a multiindex of length $0 \leq |\nu| \leq h$), and $(x^\mu, y^i_\nu, v^\mu_\nu, v^0_\nu)$ coordinates on $J^1 TB$, the coordinate expression of the prolonged morphism $j^h X$ is $j^h X(x^\mu, y^i_\nu) = (x^\mu, y^i_\nu, d_\nu v^\mu, d_\nu v^0)$.

Luckily enough, for all integers $h$, a global canonical map $r_h : J^h TB \to T J^h B$ can be defined (see [19, 12]), whose coordinate expression is $r_h(x^\mu, y^i_\nu, v^\mu_\nu) = (x^\mu, y^i_\nu, v^\mu_\nu, d_\nu v^\mu - \sum_\alpha v^\alpha_\nu d_\nu - \alpha^\nu v^\mu_\nu)$ where $\tilde{\alpha}$ is any possible multiindex of length $0 \leq |\tilde{\alpha}| < |\nu| = h$ and $1_\mu$ is a multiindex with 1 entry in position $\mu$ and zero elsewhere. The ordinary prolongation $j^h X$ of the vector field $X$ as a vector field on $J^h B$ can be recovered just by composing $r_h$ with the prolonged morphism $j^X$.

**Definition 37** Let $B \to M$ be a bundle and $E$ a vector bundle over the same base, let $\mathcal{M}$ be a fibered morphism $\mathcal{M} : B \to (E)^*$ and $\mathcal{P}$ another bundle morphism $\mathcal{P} : B \to E$.

We define formal contraction of $\mathcal{M}$ on $\mathcal{P}$ the unique variational morphism

$$< \mathcal{M} | \mathcal{P} > : B \to M \times \mathbb{R}$$

such that

$$\forall \rho \in \Gamma(B), \quad < \mathcal{M} | \mathcal{P} > \circ \rho = < \mathcal{M} \circ \rho | \mathcal{P} \circ \rho >$$

where $< | >$ in the right-hand side is the usual fiberwise contraction between a section of a vector bundle and a section of its dual.

**Definition 38** Let $\mathcal{M} : J^k C \to \Lambda^{m-n}T^*(M)$ be a variational morphism. We define the divergence of $M$ to be the unique variational morphism $\text{Div}(\mathcal{M}) : j^{k+1} C \to \Lambda^{m-n+1}T^*(M)$ such that for any section $\sigma \in \Gamma(C)$ the following holds

$$\text{Div}(\mathcal{M}) \circ j^{k+1} \sigma := d(\mathcal{M} \circ j^k \sigma).$$

### A.3 Formal connections and fibered connections

The following definitions generalize the ones given in [10] to the settings of constrained variational calculus. Let us consider a composite projection $E \xrightarrow{\pi_E} B \xrightarrow{\pi} M$ and the fibered morphism (section)

\[ B \xrightarrow{\pi} E \xrightarrow{\pi} M \]

with $E$ vector bundle on $B$.

Let $(x^\mu, y^i)$ be a coordinate system on $B$ and $(x^\mu, y^i, v^A)$ on $E$ (with respect to a fiberwise local basis $\{e_A\}$) a local representation of $\mathcal{F}$ is $v^A(x^\mu, y^i)e_A$, while a local representation
of its first jet prolongation (as a fibered morphism) \( j^1F \) is \( v^A(x^\mu, y^i) e_A + d_\mu v^A e^\mu_A \), where \( \{e^\mu_A\} \) is a base of the fiber of \( J^1E \to E \) and \( d_\mu v^A = \partial_\mu v^A + \partial_i v^A y^i_\mu \) is called the formal derivative of \( F \).

The transformation rules of the formal derivatives under a fibered change of coordinates are not tensorial.

Consider a set \( \left\{ \Gamma^A_{B\mu}(x^\mu, y^i, y^i_\mu) \right\}_{A,B\in\{1,\ldots,\text{rank}E\}, \mu\in\{1,\ldots,\text{dim}M\}} \) of local coefficients that fit into the expression

\[
\nabla_\mu v^A = d_\mu v^A + \Gamma^A_{B\mu} v^B
\]

(19)

giving rise to an object at the left hand side of the equalities that under a change of fibered coordinates

\[
\begin{align*}
x'^\mu &= x'^\mu(x^\mu) \\
y'^i &= y'^i(x^\mu, y^i) \\
y'^i_\mu &= (J^{-1})^\mu_i d_\mu y^i \quad (J^{-1})^\mu_i (J^i_\mu + J^i_\mu y^i_\mu) \\
v'^A &= M_A^i (x^\mu, y^i) v^A
\end{align*}
\]

transforms according to the rule

\[
\nabla_\mu' v'^A = (J^{-1})^\mu_i \nabla_\mu v^A M_A^i
\]

(20)

that are nothing but the transition functions of the vector bundle \( E \otimes_B T^*M \to B \).

The resulting transformation law for the coefficients is

\[
\Gamma^A_{B'\mu'} = J^\mu_\mu' (\Gamma^A_{B\mu} M_A^i M_B^j - d_\mu M_A^i M_B^j)
\]

(21)

It’s easy to verify that relations (21) form a cocycle over the manifold \( J^1B \), thus they define the transition functions of an affine bundle \( FC(E) \to J^1B \) modeled on the vector bundle \( E^* \otimes_B E \otimes_B T^*M \).

**Definition 39** A formal connection on the composite fiber bundle \( E \to B \to M \) is a global section of the affine bundle \( FC(E) \).

Let us consider the bundle \( \mathbb{L}(E) \to B \) of linear frames of the vector bundle \( E \xrightarrow{\pi_E} B \).

Let \( r = \text{rank}E \). \( \mathbb{L}(E) \) is a principal \( GL(r, \mathbb{R}) \) bundle and the free natural right action \( r : \mathbb{L}(E) \times GL(r, \mathbb{R}) \to \mathbb{L}(E) \) can be defined on it.

Let us call \( \mathbb{L}_M(E) \) the bundle \( \mathbb{L}(E) \to M \) obtained by the obvious composition of projections. It is no longer principal, nevertheless the action \( a \) induces on it a free right action \( a_M \), that can be lifted to a right action \( j^1a_M \) on its first jet extension \( J^1\mathbb{L}_M(E) \). The action \( j^1a_M \) is still free and admits a quotient manifold \( \frac{J^1\mathbb{L}_M(E)}{GL(r, \mathbb{R})} \) that has an affine bundle structure over the base \( J^1B \). The two affine bundles \( FC(E) \) and \( \frac{J^1\mathbb{L}_M(E)}{GL(r, \mathbb{R})} \) are isomorphic and this can be proved directly by showing that they share the same transition functions, providing a more intrinsic characterization of the bundle of formal connections.

Consider now the bundle \( J^1E \xrightarrow{j^1\pi_E} j^1B \); the presence of a formal connection on \( E \), allows us to define the global vector bundle morphism \( \nabla \) by the following commutative diagram
whose local coordinate expression is (19).

**Definition 40** Let $\sigma$ be a section of $J^1E_{J^1B}$. We call $\nabla \circ \sigma$ the **formal covariant derivative** of $\sigma$.

**Definition 41** A **fibered connection** on $E \xrightarrow{\pi E} B$ is a couple $(\gamma, \Gamma)$, where $\gamma$ is a linear connection on $M$ and $\Gamma$ is a formal connection on $E$.

Given a fibered connection on $E$ we can extend the morphism $\nabla$ to $J^1(E_{J^1B} \otimes T^p_q(M))$ where $T^p_q(M)$ is the algebra of $p$-times covariant and $q$-times contravariant tensors on $M$ with the obvious tensorization procedure, getting a fibered morphism

$$J^1(E_{J^1B} \otimes T^p_q(M)) \xrightarrow{\nabla} E \otimes_B T^p_{q+1}(M)$$

$$J^1B \xrightarrow{\pi B_0^1} B$$

that we denote by the same symbol $\nabla$.

**Definition 42** Let $\sigma$ be a section of $J^1(E_{J^1B} \otimes T^p_q(M))$, let us call $\nabla \circ \sigma$ the **formal covariant derivative** of $\sigma$.

**Definition 43** The **formal covariant derivative** of a morphism $F : B \to E$ is the unique fibered morphism $\nabla F : J^1B \to E \otimes_B T^*M$ such that

$$\forall \rho \in \Gamma(B), \nabla F \circ j^1\rho = \nabla \circ j^1(F \circ \rho).$$

A formal connection on a composite fiber bundle $E \to B \to M$ can be constructed from a linear connection $A$ on the vector bundle $E \to B$ and the first jet prolongation $j^1\sigma$ of a section of the bundle $B \to M$. Let $(x^\mu, y^a, v^B)$ be a fibered coordinate system on $E$; the connection $A$ has the coordinate representation

$$A = dx^\mu \otimes (\frac{\partial}{\partial x^\mu} - A^B_{D\mu}v^D \frac{\partial}{\partial y^B}) + dy^i \otimes (\frac{\partial}{\partial y^i} - A^B_{Di}v^D \frac{\partial}{\partial v^B}).$$

The coefficients $\Gamma^B_{D\mu} = A^B_{D\mu} + A^B_{D\mu}y^i$ transform according to (21) and thus they glue together into a global formal connection on $E \to B \to M$. 29
A.4 Local expression and reduction of a variational morphism with respect to a fibered connection

Fixed a fiberwise basis \( \{ e_A(x), e_{A1}(x), \ldots, e_{A^{\lambda_1\cdots\lambda_h}}(x) \} \) of \( J^h_x E \) and its dual \( \{ e^A(x), e_{A1}(x), \ldots, e_{A^{\lambda_1\cdots\lambda_h}}(x) \} \) in \( (J^h_x E)^* \), the local expression of a variational morphism \( M : J^k C \rightarrow (J^h E)^* \otimes \Lambda^{m-n} T^*(M) \) is the following

\[
M = \frac{1}{n!} \left( v_{A}^{\mu_1\cdots\mu_n} e^A + v_{A}^{\mu_1\cdots\mu_n\lambda_1} e_{A\lambda_1} + \cdots + v_{A}^{\mu_1\cdots\mu_n\lambda_1\cdots\lambda_h} e_{A^{\lambda_1\cdots\lambda_h}} \right) \otimes ds_{\mu_1\cdots\mu_n}, \tag{22}
\]

where the indices \( \mu_1 \cdots \mu_n \) are skew-symmetric while the \( \lambda_1 \cdots \lambda_h \) (if any) are symmetric and where the coefficients are functions of the point \( j^k_x \rho \in J^k C \).

Given a fibered connection \( (\gamma^{\alpha}_{\beta\mu}, \Gamma^A_{\beta\mu}) \) we can change basis on \( J^h_x E \) in a way that a holonomic section \( j^h X \in \Gamma(J^h E) \) in the new basis (called \textit{basis of symmetrized covariant derivatives}) has components

\[
\begin{align*}
\hat{v}^A &= v^A(x) \\
\hat{v}^A_{\lambda_1} &= \nabla_{\lambda_1} v^A(x) = d_{\lambda_1} v^A + \Gamma^A_{B\lambda_1} v^B \\
&\vdots \\
\hat{v}^A_{\lambda_1\cdots\lambda_h} &= \nabla(\lambda_1 \cdots \lambda_h) v^A(x).
\end{align*}
\]

Accordingly, using the dual basis on \( (J^h E)^* \) we can define a new coordinate system \( (x^\alpha, \hat{v}_{A}^{\mu_1\cdots\mu_n}, \hat{v}_{A}^{\mu_1\cdots\mu_n\lambda_1}, \hat{v}_{A}^{\mu_1\cdots\mu_n\lambda_1\cdots\lambda_h}) \) on \( (J^h E)^* \otimes j^h C \Lambda^{m-n} T^*(M) \) such that the local expression of \( M \) is

\[
\hat{M} = \frac{1}{n!} \left( \hat{v}_{A}^{\mu_1\cdots\mu_n} \hat{e}^A + \hat{v}_{A}^{\mu_1\cdots\mu_n\lambda_1} \hat{e}_{A\lambda_1} + \cdots + \hat{v}_{A}^{\mu_1\cdots\mu_n\lambda_1\cdots\lambda_h} \hat{e}_{A^{\lambda_1\cdots\lambda_h}} \right) \otimes ds_{\mu_1\cdots\mu_n}.
\]

We remark that in the previous expression, as in formula (22), the coefficients have to be meant as functions of the point \( j^k_x \rho \in J^k C \), and the indices \( \mu_1 \cdots \mu_n \) are skew-symmetric while the \( \lambda_1 \cdots \lambda_h \) (if any) are symmetric.

The explicit change of basis and coordinates in the case \( h = 1 \) is

\[
\begin{align*}
\hat{e}^A &= e^A \\
\hat{e}_{A}^{\lambda_1} &= e_{A\lambda_1} + \Gamma^A_{B\lambda_1} e^B \\
v_A &= \hat{v}^A + \hat{v}_{A}^{\lambda_1} \Gamma^A_{B\lambda_1},
\end{align*}
\]

The introduction of a fibered connection can select a “preferred” kind of variational morphism having nice properties when written in the symmetrized covariant derivatives coordinate system.

**Definition 44** Let \( M : J^k C \rightarrow (J^h E)^* \otimes j^h C \Lambda^{m-n} T^*(M) \) be a variational morphism and \( M_l \) its term of rank \( 0 \leq l \leq h \). The term \( M_l \) is said to be \textit{reduced} with respect to the fibered connection \( (\gamma^{\alpha}_{\beta\mu}, \Gamma^A_{\beta\mu}) \) if, written in the symmetrized covariant derivatives coordinate system, it has the coefficients symmetric in the first \( n+1 \) upper indices, i.e. if \( \hat{v}_{A}^{[\mu_1\cdots\mu_n\lambda_1] \cdots \lambda_l} = 0 \). The variational morphism \( M \) is reduced if each one of its terms is reduced.
Notice that the term of rank zero is always reduced with respect to any connection, having always antisymmetric upper indices and that in the case of Mechanics \((m = 1)\) all variational morphisms are reduced with respect to any connection.

Now we are ready to state the most important results of the theory, that justify the name “variational” for the morphism we have introduced, providing a sort of “first variation formula” from any variational morphism. The proofs of the next two Theorems can be found in [7], but as a consequence of the minor modifications we introduced here one has to reinterpret the symbols according to the previous definitions.

**Theorem 45 - “Splitting lemma”**- Let \(\mathbb{M} : J^k C \to (J^h E)^* \otimes \rho_C \Lambda^m T^* M\) be a global variational morphism with codegree \(n = 0\) and let \(h, k \in \mathbb{N}\) (zero included). Chosen a fibered connection \((\gamma, \Gamma)\) on \(E\) there exist a unique pair \(\mathbb{V}, \mathbb{B}\) of variational morphisms

\[
\mathbb{V} \equiv \mathbb{V}(\mathbb{M}) : J^{h+k} C \to E^* \otimes C \Lambda^m T^* M
\]

\[
\mathbb{B} \equiv \mathbb{B}(\mathbb{M}, \gamma) : J^{h+k-1} C \to (J^{h-1} E)^* \otimes j^{h-1} C \Lambda^{m-1} T^* M
\]

reduced with respect to \((\gamma, \Gamma)\) such that \(\forall V \in \Gamma(E)\) the following holds true

\[
< \mathbb{M} | j^h V > = < \mathbb{V} | V > + \text{Div} < \mathbb{B} | j^{h-1} V > . \tag{23}
\]

The variational morphism \(\mathbb{V}\) is called the volume part of \(\mathbb{M}\) and \(\mathbb{B}\) its boundary part.

**Theorem 46 - “Reduction lemma”**- Let \(\mathbb{M} : J^k C \to (J^h E)^* \otimes \rho_C \Lambda^{m-n} T^* M\) be a global variational morphism of codegree \(n \geq 1\) and let \(h, k \in \mathbb{N}\) (zero included). Chosen a fibered connection \((\gamma, \Gamma)\) on \(E\) there exist a unique pair \(\mathbb{V}, \mathbb{B}\) of variational morphisms

\[
\mathbb{V} \equiv \mathbb{V}(\mathbb{M}) : J^{h+k} C \to (J^h E)^* \otimes \rho_C \Lambda^{m-n} T^* M
\]

\[
\mathbb{B} \equiv \mathbb{B}(\mathbb{M}, \gamma) : J^{h+k-1} C \to (J^{h-1} E)^* \otimes j^{h-1} \Lambda^{m-n-1} T^* M
\]

reduced with respect to \((\gamma, \Gamma)\) such that \(\forall V \in \Gamma(E)\) the following holds true

\[
< \mathbb{M} | j^h V > = < \mathbb{V} | j^h V > + \text{Div} < \mathbb{B} | j^{h-1} V > . \tag{24}
\]

The variational morphism \(\mathbb{V}\) is called the volume part of \(\mathbb{M}\) and \(\mathbb{B}\) its boundary part.

Constructive proofs of these Theorems are given in [7] where we can find a collection of older results; they are carried on by induction on the rank of the morphism. If the rank is 1 then one just needs to integrate by parts and no fibered connection is needed. At any rank they provide an algorithm out of which we can perform explicitly the splitting or the reduction.

In both the Theorems the volume part is uniquely defined even if we do not require it to be reduced, and finally it does not depend on the fibered connection. The boundary parts, on the contrary, are defined modulo a divergenceless term, but the condition of being reduced with respect to any specific fixed fibered connection determines them uniquely (if the rank is 1 the boundary part is still unique and reduced with respect to any fibered connection). Moreover it was proved [15] that the boundary parts depend in fact only on the connection on the base, while the formal connection on \(E\) reveals itself to be just an intermediate object useful to explicitly construct the splitting, but in the end unessential.
B Gauge natural bundles

Definition 47 Let $G$ be a Lie group. $\mathcal{P}^n(G)$ will denote the category of principal $G$-bundles whose base manifold is $n$-dimensional and whose maps are $G$-bundle maps projecting onto local diffeomorphism between the two bases.

Definition 48 A covariant functor $\mathcal{B}$ from the category $\mathcal{P}^n(G)$ to the category of bundles and bundle morphisms is a gauge-natural functor if

a) for all principal $G$-bundle $P \xrightarrow{p} M$, $\mathcal{B}(P)$ is a bundle with structure group $G$ having the same base $M$ as $P$;

b) any principal morphism $\Phi : P \rightarrow P'$ projecting onto $\phi : M \rightarrow M$ induces a fibered morphism $\mathcal{B}(\Phi) : \mathcal{B}(P) \rightarrow \mathcal{B}(P')$ also projecting over $\phi$;

c) for any open $U \subseteq M$ the inclusion morphism $i : p^{-1}(U) \rightarrow P$ is mapped into the inclusion morphism $\mathcal{B}(i) : \mathcal{B}(p^{-1}(U)) \rightarrow \mathcal{B}(P)$.

Definition 49 A gauge-natural bundle $\mathcal{G} \xrightarrow{\gamma} M$ having the principal bundle $P \xrightarrow{p} M$ as structure bundle is the image of $P$ through a gauge-natural functor $\mathcal{B}$.

Examples of gauge-natural bundles are all natural bundles (trivial case), the bundle of principal connections $\mathcal{C}(P)$ on a principle bundle $P$, the bundle $\text{IGA}(P)$ of its Infinitesimal Generators of principal Automorphisms, the bundle of spin frames and others (see [7]). Notice that jet prolongations of a gauge-natural bundle are gauge-natural with the same structure bundle, and that every composite fibration $Z \xrightarrow{\pi_Z} Y \xrightarrow{\pi_Y} X$ where $Y$ is a gauge-natural bundle with structure bundle $P$ and $\pi_Z$ is natural makes $\pi_Y \circ \pi_Z$ a gauge-natural bundle with structure bundle $P$.

We want to stress that for our purposes, the most relevant property of a gauge-natural bundle $\mathcal{B}(P)$ is the possibility of lifting local principal automorphisms of the structure $G$-bundle $P$ to local fibered morphisms of $\mathcal{B}(P)$. We will call “gauge transformation” on $\mathcal{B}(P)$ the gauge-natural lift of a local principal automorphism of $P$ and “gauge group” the structure group $G$.

The same, via the tangent map, can be done for infinitesimal generators: an infinitesimal generator of principal automorphisms can be lifted to a vector field on $\mathcal{B}(P)$ that will be called its “gauge-natural lift”.

Let us consider a 1-parameter family of infinitesimal generators of principal automorphisms $\{\Psi_s = (\phi_s, \psi_s)\}$. It can be proved (see [7]) that any infinitesimal generator $\Xi \in \Gamma(TG) = \frac{d}{ds}\Psi_s|_{s=0}$ is a right invariant projectable vector field on $P$ and that we can explicitly construct the bundle $\text{IGA}(P) \rightarrow M$ whose sections are in one to one correspondence with the infinitesimal generators of such families and with right invariant sections of $TP$. Infinitesimal generators of vertical principal automorphisms $\{\Psi_s = (id, \psi_s)\}$ can be identified with sections of a subbundle $V_P \rightarrow \text{IGA}(P)$ that is associated to $P \xrightarrow{p} M$ by means of the adjoint action of the Lie algebra $\mathfrak{g}$ of the gauge group $G$. 
By means of a trivialization of $P$ we can locally define a set \( \{ \rho_A \}_{A \in \{ 1, \ldots, \dim G \}} \) of right invariant vector fields on $P$ that at every point form a basis of $VP$. Let \((x^\mu, y^a)\) be a fibered system of coordinates on $P$; every infinitesimal generator of principal automorphisms $\Xi$ can be locally written as a right invariant vector field

\[
\Xi(x^\mu, y^a) = \xi^\mu(x^\mu) \frac{\partial}{\partial x^\mu} + \xi^A(x^\mu) \rho_A(x^\mu, y^a)
\]
on $P$.

**Definition 50** We say that a gauge-natural bundle $\mathcal{B}(P)$ is of \((r, s)\)-order \(s \leq r\) if the gauge-natural lift is a morphism

\[ L : \mathcal{B}(P) \rightarrow (J^r \text{IGA}(P))^* \otimes T\mathcal{B}(P) \]
such that if $\Xi$ is vertical (i.e. it is a section of $\frac{VP}{G} \hookrightarrow \text{IGA}(P)$), the vector field $\hat{\Xi}_{\mathcal{B}} = < L \mid j^r \Xi >$ on $\mathcal{B}(P)$ is vertical, too, and depends in fact on the derivatives of $\Xi$ only up to the order $s$.

**Definition 51** Let $\Psi = (\phi, \psi)$ be a local principal automorphism of $P$ and $\mathcal{B}(\Psi) = (\phi, \hat{\Psi}_{\mathcal{B}})$ be its gauge-natural lift to the gauge-natural bundle $\mathcal{B}(P)$. Let $\sigma$ be a section of $\mathcal{B}(P)$; we define its push-forward along $\Psi$ to be the section $\Psi^* \sigma = \hat{\psi}_{\mathcal{B}} \circ \sigma \circ \phi^{-1}$.

**Definition 52** Let $\Xi$ be the infinitesimal generator of the 1-parameter family of local principal automorphisms $\{ \Psi_s = (\phi_s, \psi_s) \}$; we define the Lie derivative of a section $\sigma$ of $\mathcal{B}(P)$ along $\Xi$ to be the unique section of the pull back bundle $\mathcal{L}_\Xi \sigma \in \Gamma(\sigma^* V\mathcal{B}(P))$ such that

\[ \mathcal{L}_\Xi \sigma = -\frac{d}{ds} \Psi_s^* \sigma. \]

**Remark 53** We can easily prove that if $\hat{\Xi}_{\mathcal{B}}$ is the gauge-natural lift of $\Xi$ on $\mathcal{B}(P)$, and $\Xi$ is the projection of $\Xi$ on $M$, the Lie derivative of a section can be computed through the following formula

\[ \mathcal{L}_\Xi \sigma = T\sigma \circ \pi \Xi - \hat{\Xi}_{\mathcal{B}}. \]

In coordinates, if $(x^\mu, y^i)$ are fibered coordinates on $\mathcal{B}(P)$ while the lift can be written as $\hat{\Xi}_{\mathcal{B}} = \xi^\mu \frac{\partial}{\partial x^\mu} + \hat{\Xi}_{\mathcal{B}}^i \frac{\partial}{\partial y^i}$, one has $\mathcal{L}_\Xi \sigma = \mathcal{L}_y y^i \frac{\partial}{\partial y^i}$, with

\[ \mathcal{L}_y y^i = d_\mu y^i(x) \xi^\mu(x) - \hat{\Xi}_{\mathcal{B}}^i(x, y(x)) \]

It also turns out that the Lie derivative operator on sections of the gauge-natural bundle $\mathcal{B}(P)$ of order \((r, s)\) (where $r \geq s$) can be interpreted as a fibered morphism

\[ \mathcal{D}_\mathcal{L} : J^1 \mathcal{B}(P) \rightarrow (J^r \text{IGA}(P))^* \otimes V\mathcal{B}(P) \]
such that for all sections $\sigma \in \Gamma(\mathcal{B}(P))$ and for all infinitesimal generator of principal automorphism $\Xi \in \Gamma(\text{IGA}(P))$ one has $< \mathcal{D}_\mathcal{L} \mid J^r \Xi > \circ j^1 \sigma = \mathcal{L}_\Xi \sigma$. 

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Notice that if Ξ is vertical (i.e. it is a section of $\nabla_P \rightarrow \text{IGA}(P)$) then $\mathcal{L}_\Xi \sigma$ depends in fact on the derivatives of Ξ only up to the order $s$.

**Remark 54** When in differential geometry one computes the Lie derivative of a vector field on a manifold $M$ one gets another vector field and so happens for differential forms and tensors. According to our definition, on the contrary, the Lie derivative of a vector field is a section of the vertical bundle $VT_{TM}$. To reconcile this apparent contradiction we have to remark that whenever $B$ is a vector bundle with base $M$ then its vertical $V_B$ is isomorphic (see [16]) to the fibered product $B \times_M B$ and one has a global vertical fibered morphism $\pi_2 : V_B \rightarrow B$ that realizes the projection onto the second fibered factor. If $(x^\mu, y^a, v^a)$ is a local coordinate system on $V_B$ then the map $\pi_2$ acts as follows: $\pi_2 : (x^\mu, y^a, v^a) \mapsto (x^\mu, v^a)$. If $\sigma$ is a section of a gauge-natural vector bundle then $\pi_2 \mathcal{L}_\Xi \sigma$ would be what traditionally is meant as its Lie derivative, the coordinate computation being identical, but with a slightly different interpretation.

Let $P \rightarrow M$ be a principal bundle; let moreover $U \subset M$ be open, $(x^\mu, g^a)$ be a coordinate system on $p^{-1}(U) \subset P$ and $\{\rho_A(x^\mu, g^a)\}_{A=1}^{\dim G}$ a fiberwise right invariant basis of vertical vector fields. An infinitesimal generator of principal automorphisms has the local form $\Xi(x^\mu, g^a) = \xi^\mu(\partial_\mu + \xi^A(x^\mu, g^a)) \rho_A(x^\mu, g^a)$ (see [7]). Given a principal connection $\omega(x^\mu, g^a) = dx^\mu \otimes (\partial_\mu - \omega^A_\mu(x^\mu, g^a)) \rho_A(x^\mu, g^a)$ we can also split $\Xi$ into a vertical and a horizontal part according to

$$\Xi = \xi^\mu(\partial_\mu - \omega^A_\mu \rho_A) \oplus (\omega^A_\mu \xi^\mu + \xi^A) \rho_A = \Xi(H) \oplus \Xi(V).$$

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