ON DELIGNE’S CONJECTURE FOR CENTRAL VALUES OF CERTAIN AUTOMORPHIC L-FUNCTIONS ON GL(3) × GL(2)

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Abstract. We prove Deligne’s conjecture for central critical values of certain automorphic L-functions for GL(3) × GL(2). The proof is based on rationality results for central critical values of triple product L-functions, which follow from establishing explicit Ichino’s formulae for trilinear period integrals for Hilbert cusp forms on totally real étale cubic algebras over Q.

1. Introduction

The purpose of this paper is to establish the explicit Ichino formula for twisted triple product L-functions. As an application of our formula, we establish new cases on the algebraicity of central critical values of certain class of automorphic L-functions for GL(3) × GL(2) divided by the associated Deligne’s periods. To begin with, let f and g be elliptic newforms of weights κ' and κ, level Γ_0(N_1) and Γ_0(N_2), respectively. We let L(s, Sym^2(g) ⊗ f) be the motivic L-function associated with Sym^2(g) ⊗ f. Put

w = 2κ + κ' - 3;  \epsilon = (-1)^{κ'/2-1}.

Denote by Ω_f the Shimura’s periods of f in [Shi77] and define the Deligne’s period Ω_{f,g} ∈ C× by

Ω_{f,g} = \begin{cases} (2π√{-1})^{3-3κ}(√{-1})^{1-κ'}⟨f,f⟩Ω_f & \text{if } 2κ ≤ κ', \\ (2π√{-1})^{2-κ-κ'}⟨g,g⟩Ω_f & \text{if } 2κ > κ'. \end{cases}

Our main result is as follows.

Theorem A. (Cor. 7.1 and 7.2) Suppose that N_1 and N_2 are square-free. We have

\frac{L((w+1)/2, Sym^2(g) ⊗ f)^{σ}}{(2π√{-1})^{3(w+1)/2}Ω_{f,g}} = \frac{L((w+1)/2, Sym^2(g^{σ}) ⊗ f^{σ})^{σ}}{(2π√{-1})^{3(w+1)/2}Ω_{f^{σ},g^{σ}}}.

Remark 1.1. If N_1 = 1 and 2κ > κ', then the above algebraicity result was obtained by Ichino [Ich05 Corollary 2.6] via the explicit pullback formula for Saito-Kurokawa lifts (N_2 = 1 and κ = κ'/2 + 1) and by Xue (N_2 = 1) using a different but closely related approach [Xue]. The first author has generalized Ichino’s pullback formula of Saito-Kurokawa lifts in [Che] if N_2 is further assumed to be odd and cubic-free.

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Our result covers the remaining cases and thus settles down Deligne’s conjecture for the central value of the L-functions for $\text{Sym}^2(g) \otimes f$ at least when the levels of $f$ and $g$ are square-free.

We remark that Raghuram has proved the algebraicity of the central critical values of the Rankin-Selberg L-functions attached to regular algebraic cuspidal automorphic representations on $\text{GL}(n) \times \text{GL}(n-1)$ in a quite general setting [Rag09]. His method is based on a cohomological interpretation of the Rankin-Selberg zeta integral, and specializing the result of Raghuram to $n = 3$, one also obtains the algebraicity of the central critical value of $L(s, \text{Sym}^2(g) \otimes f)$ divided by certain cohomological period for $\text{GL}(3) \times \text{GL}(2)$ in the case $2\kappa > \kappa'$. However, our result in this case is not covered by [Rag09] in the sense that the periods in both results are quite different. More precisely, the periods in our main theorem coincide with Deligne’s period described by Blasius in the appendix to [Orl87] while Raghuram uses the period $p^2(II)$ obtained from the comparison between deRham and Betti cohomology groups for $\text{GL}(3)$ [Rag09 §3.2.1]. It seems a difficult problem to study directly the relation between Deligne’s period and Raghuram’s cohomological period. Our result combined with the non-vanishing hypothesis of central L-values would give a comparison between these two periods.

Our approach also offers the algebraicity of the central critical value of symmetric cube L-functions with the assumption on the non-vanishing of L-values with cubic twist.

**Theorem B** (Corollary 7.3). Suppose that $N_1 > 1$ and there exist a cubic Dirichlet character $\chi$ such that $L\left(\frac{w}{2}, f \otimes \chi\right) \neq 0$. For $\sigma \in \text{Aut}(C)$, we have

$$\left(\frac{L((w+1)/2, \text{Sym}^3(f))}{\pi^{2\kappa-1}(\sqrt{-1})^{\kappa}(f, f)(\Omega_f)^2}\right)^{\sigma} = \frac{L((w+1)/2, \text{Sym}^3(f^\sigma))}{\pi^{2\kappa-1}(\sqrt{-1})^{\kappa}(f^\sigma, f^\sigma)(\Omega_{f^\sigma})^2}.$$

The hypothesis on the non-vanishing of cubic twists of L-values is expected to hold in general but seems unfortunately a far-reaching problem at this moment. So far this hypothesis is only known to be satisfied for cuspidal automorphic representations on $\text{GL}_2(A_K)$ when $Q(\sqrt{-3}) \subset K$ in [BFH05].

Our proof of Theorem A is based on an explicit Ichino’s central value formula for the twisted triple product L-functions. Let $K$ be a real quadratic field and let $g_K$ be the Hilbert modular newform over $K$ associated to $g$ obtained by the base change lift. Let $L(s, g_K \otimes f)$ be the triple product L-function associated to $g_K \otimes f$. Let $\tau_K$ be the quadratic Dirichlet character associated with $K/Q$. From the following factorization of L-functions

$$L(s, g_K \otimes f) = L(s, \text{Sym}^2(g) \otimes f)L(s - \kappa + 1, f \otimes \tau_K),$$

one can deduce easily the algebraicity of $L\left(\frac{w}{2}, \text{Sym}^2(g) \otimes f\right)$ (divided by the associated Deligne’s period) from that of the central value $L\left(\frac{w+1}{2}, g_K \otimes f\right)$ of the twisted triple product and that of the central value $L\left(\frac{w}{2}, f \otimes \tau_K\right)$ of elliptic modular forms whenever $L\left(\frac{w}{2}, f \otimes \tau_K\right)$ does not vanish. The algebraicity of critical L-values of elliptic modular forms with Dirichlet twists is a classical result due to Shimura, so the main task is to choose a nice real quadratic field $K$ with $L\left(\frac{w}{2}, f \otimes \tau_K\right) \neq 0$ and show the algebraicity of the central value $L\left(\frac{w+1}{2}, g_K \otimes f\right)$, for which one appeals to Ichino’s formula in [Ich08]. More precisely, if the global sign in the functional equation of the automorphic L-function for the twisted triple product $g_K \otimes f$ is +1, then Ichino’s formula alluded to above asserts that there exists a quaternion algebra $D$ over $Q$ so that the central critical value $L\left(\frac{w}{2}, g_K \otimes f\right)$ is the ratio between the square of the global trilinear period integral of an automorphic form on $D^\times(A_K) \times D^\times(A)$ and a product of certain local zeta integrals. Taking into account the functional equation and the Galois invariance of the global sign, we may assume the global sign of $\text{Sym}^2(g) \otimes f$ is +1. Then by a result of Friedberg and Hoffstein [EH95], we can choose a real quadratic field $K$ such that (i) $L\left(\frac{w}{2}, f \otimes \tau_K\right) \neq 0$, (ii) the sign of $g_K \otimes f$ is +1 and (iii) the quaternion algebra $D$ in Ichino’s formula is the matrix algebra (resp. a definite quaternion algebra) over $Q$ in the case $2\kappa \leq \kappa'$ (resp. $2\kappa > \kappa'$) if we assume further that $N_1 > 1$.

To obtain the explicit Ichino’s central value formula, we calculate the local period integral at each place (Theorems 6.2 and 6.5) in terms of global period integral, and as a consequence, we obtain the algebraicity of the central value $L\left(\frac{w}{2}, g_K \otimes f\right)$ (Corollaries 6.3 and 6.6) by a standard argument.

The idea of the proof for Theorem B is similar. Assume $\chi$ is a cubic Dirichlet character such that $L\left(\frac{w}{2}, f \otimes \chi\right) \neq 0$. Let $E$ be the totally real cubic Galois extension over $Q$ cut out by $\chi$ and let $f_E$ be the Hilbert modular newform over $E$ associated to $f$ via the base change lift. Consider the degree eight triple
product $L$-function $L(s, f_E)$ associated to $f_E$. Then we have the factorization of $L$-functions:

$$L(s, f_E) = L(s, \text{Sym}^3(f)) L(s - \frac{n}{2} + 1, f \otimes \chi) L(s - \frac{n}{2} + 1, f \otimes \chi^2).$$

Thus the algebraicity of $L \left( \frac{n}{2} + 1, \text{Sym}^3(f) \right)$ is a consequence of the algebraicity of $L \left( \frac{n}{2} + 1, f_E \right)$, which again can be deduced from the explicit Ichino central value formula in this case.

This paper is organized as follows. We first study the local zeta integrals in Ichino’s formula. In §2, we introduce the local zeta integrals and fix the test vectors used in the subsequent local calculation. After recalling basic properties of local matrix coefficients for $GL(2)$ in §3, we carry out the calculations of local zeta integrals in the cases of the matrix algebra and the division algebra in §4 and §5 respectively. In particular, we compute the archimedean zeta integrals explicitly. In §6, we recall Ichino’s formula and in §7, we prove its explicit version in Theorem 6.2 and Theorem 6.5 as well as the main result in §8. In the appendix, we follow a computation of Bhagwat [Bha14] to determine Deligne’s period for $\text{Sym}^3(g) \otimes f$. This is well-known to experts, but we include it here for the sake of completeness.

2. Local zeta integrals

The purpose of this section is to fix the test vectors and to define the local zeta integrals in our local calculation. These local zeta integrals are used to establish explicit Ichino’s formula.

2.1. Notation and assumptions. Let $F$ be a local field of characteristic zero. When $F$ is non-archimedean, denote $\mathcal{O}_F$ the ring of integers of $F$, $\varpi_F$ a prime element, and $\text{ord}_F$ be the valuation on $F$ normalized so that $\text{ord}_F(\varpi_F) = 1$. Let $| \cdot |_F$ be the absolute value on $F$ normalized so that $|\varpi_F|_F^{-1}$ is equal to the cardinality of $\mathcal{O}_F/\varpi_F \mathcal{O}_F$. When $F$ is archimedean, let $| \cdot |_R$ be the usual absolute value on $R$ and $|z|_C = \sqrt{\sigma}$ on $C$.

Let $E$ be an étale cubic algebra over a local field $F$ of characteristic zero. Then $E$ is (i) $F \times F \times F$ three copies of $F$, or (ii) $K \times F$, where $K$ is a quadratic extension of $F$, or (iii) a cubic field extension of $F$. Let $D$ be a quaternion algebra over $F$. If $L$ is a $F$-algebra, let $D^\times(L) := (D \otimes_F L)^\times$. Let $\Pi$ be a unitary irreducible admissible representation of $D^{\times}(E)$ whose central character is trivial on $F^\times$. Let $\Pi'$ be the unitary irreducible admissible representation of $GL_2(E)$ associated to $\Pi$ via the Jacquet-Langlands correspondence. Therefore $\Pi' = \Pi$ if $D = M_2(F)$ is the matrix algebra. Notice that $\Pi' = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3$ (if $E = F \times F \times F$), or $\Pi' = \pi' \boxtimes \pi$ (if $E = K \times F$), where $\pi_j$ $(j = 1, 2, 3)$ and $\pi$ are unitary irreducible admissible representations of $GL_2(F)$, and $\pi'$ is a unitary irreducible admissible representation of $GL_2(K)$. We make the following assumptions on the triplet $(F, E, \Pi)$ in this section and §4, §5.

- If $F$ is archimedean, then $F = R$ and $E = R \times R \times R$.
- When $F = R$, $\Pi'$ is a (limit of) discrete series with the minimal weight $k = (k_1, k_2, k_3)$ and the central character $\text{Sgn}^{k_1} \boxtimes \text{Sgn}^{k_2} \boxtimes \text{Sgn}^{k_3}$ for some positive integers $k_1, k_2, k_3$.
- When $F$ is non-archimedean, we assume $\pi_1, \pi_2, \pi_3$ and $\pi', \pi$ and $\Pi'$ (when $E$ is a field) are unramified (special) representations whose central characters are trivial.
- We assume $\Lambda(\Pi') < 1/2$, where $\Lambda(\Pi')$ is defined in [Ich08] pp. 284-285.
- We assume $\text{Hom}_{D^{\times}(F)}(\Pi, \mathcal{C}) \neq \{0\}$.

Remark 2.1. By the results of Prasad [Pra90] and [Pra92], we have

$$\dim_{\mathcal{C}} \text{Hom}_{D^{\times}(F)}(\Pi, \mathcal{C}) \leq 1.$$

When $F = R$, it follows from [Pra90] Theorem 9.5 that $\text{Hom}_{D^{\times}(R)}(\Pi, \mathcal{C}) \neq \{0\}$ precisely when (i) $D = M_2(F)$ and $2 \cdot \max \{k_1, k_2, k_3\} \geq k_1 + k_2 + k_3$; (ii) $D$ is the division algebra and $2 \cdot \max \{k_1, k_2, k_3\} < k_1 + k_2 + k_3$.

The first case is called the unbalanced case, while the second case is called the balanced case.

2.2. The new line. Denote by $V_{\Pi}$ the representation space of $\Pi$. In what follows, we shall introduce a special one-dimensional subspace in $V_{\Pi}$, which is called the new line $V_{\Pi}^{\text{new}}$ of $V_{\Pi}$. If $F$ is non-archimedean and $a$ is an ideal of $\mathcal{O}_E$, let

$$U_0(a) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_E) \mid c \in a \right\}.$$ 

Suppose that $D = M_2(F)$. If $F$ is non-archimedean, then by [Cas73], there is a unique ideal $c(\Pi)$ of $\mathcal{O}_E$ such that

$$\dim_{\mathcal{C}} V_{\Pi}^{k_0(c(\Pi))} = 1.$$
The ideal $c(\Pi)$ is called the conductor of $\Pi$, and define the new line $V^\text{new}_\Pi := V^{\text{alg}}(c(\Pi))$. If $F = \mathbb{R}$ and $E = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, the new line $V^\text{new}_\Pi$ is the one dimensional subspace of the minimal weight under the $\text{SO}_2(E)$-action.

Suppose that $D$ is division. If $F$ is non-archimedean, and $E \neq K \times F$, then $V_\Pi$ is already one-dimensional according to our assumption. In this case, we put $V^\text{new}_\Pi = V_\Pi$. When $E = K \times F$, we have $\Pi = \pi' \boxtimes \pi$, where $\pi$ (resp. $\pi'$) is a unitary irreducible admissible (resp. generic) representation of $D^\times(F)$ (resp. $\text{GL}_2(K)$). In this case, we have $V_\Pi = V_{\pi'} \otimes V_\pi$ and define the new line $V^\text{new}_\Pi := V^\text{new}_{\pi'} \otimes V_\pi$. Finally, if $F = \mathbb{R}$ and $2 \max \{k_1, k_2, k_3\} < k_1 + k_2 + k_3$, we define the new line $V^\text{new}_\Pi$ to be the one-dimensional subspace $V^{\text{alg}}_\Pi$ of $V_\Pi$ [Pra90, Theorem 9.3].

2.3. Let $\mathfrak{g} = \text{Lie}(\text{GL}_2(\mathbb{R})) \otimes_\mathbb{R} \mathbb{C}$ and $\mathfrak{u}$ be the universal enveloping algebra of $\mathfrak{g}$. We put $\mathfrak{u}_E = \mathfrak{u} \otimes \mathfrak{u} \otimes \mathfrak{u}$. Let

$$V_+ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sqrt{-1} \in \mathfrak{u}$$

be the weight raising operator in [JL70, Lemma 5.6]. Define the normalized operator by

$$\bar{V}_+ := \left( -\frac{1}{8\pi} \right) \cdot V_+.$$

Let $\tau_F \in \text{GL}_2(F)$ be given by

$$(2.1) \quad \tau_F = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } F = \mathbb{R}, \\ 1 & \text{if } F \text{ is nonarchimedean.} \end{cases}$$

Define a special element $t \in \mathfrak{u}_E \times \text{SO}(2, E)$ or $D^\times(E)$ attached to $\Pi$ as follows:

- $F = \mathbb{R}$, $D = \text{M}_2(F)$ and $2 \max \{k_1, k_2, k_3\} \geq k_1 + k_2 + k_3$. Suppose that $k_3 = \max \{k_1, k_2, k_3\}$. Then

$$t = \left( 1 \otimes \bar{V}_+^{k_3-k_1-k_2} \otimes 1, (1, 1, \tau_{\mathbb{R}}) \right) \in \mathfrak{u}_E \times \text{SO}(2, E).$$

- $F$ non-archimedean, $E = F \times F \times F$, $D = \text{M}_2(F)$, $\Pi = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3$ and precisely one of $\pi_j$ is unramified special, say $\pi_1$:

$$t = \left( 1, \left( \overline{\varphi}_F^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \right) \in \text{GL}_2(F) \times \text{GL}_2(F) \times \text{GL}_2(F).$$

- $F$ non-archimedean, $E = K \times F$, $K/F$ is ramified, $D = \text{M}_2(F)$, $\Pi = \pi' \boxtimes \pi$ with $\pi'$ spherical and $\pi$ unramified special:

$$t = \left( \overline{\varphi}_F^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \in \text{GL}_2(K) \times \text{GL}_2(F).$$

- $F$ non-archimedean, $E = K \times F$, $K/F$ is ramified, $D = \text{M}_2(F)$, $\Pi = \pi' \boxtimes \pi$ with $\pi'$ unramified special and $\pi$ spherical:

$$t = \left( \overline{\varphi}_F \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) \in \text{GL}_2(K) \times \text{GL}_2(F).$$

- $F$ non-archimedean, $E$ ramified cubic extension, $D = \text{M}_2(F)$, $\Pi$ unramified special:

$$t = \overline{\varphi}_E^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(E).$$

- For all other cases:

$$t = 1 \in D^\times(E).$$
2.4. **Definition of local zeta integrals.** We are going to define the local zeta integrals in our local computation except for the balanced case, which will be defined by equation (5.6).

Let $\mathcal{J} \in D^\times(E)$ be given as follows:

$$\mathcal{J} = \begin{cases} (\tau_R, \tau_R, \tau_R) \in \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R}) & \text{if } F = \mathbb{R} \text{ and } D = M_2(\mathbb{R}), \\ 1 & \text{otherwise.} \end{cases}$$

Let $\zeta_F(s)$ denote the local zeta function. Therefore,

$$\zeta_F(s) = \begin{cases} 2(2\pi)^{-s} \Gamma(s) & \text{if } K = \mathbb{C}, \\ \pi^{-s/2} \Gamma(s/2) & \text{if } K = \mathbb{R}, \\ (1 - q_F)^{-1} & \text{if } K \text{ is nonarchimedean}, \end{cases}$$

where $q_F$ is the cardinality of the residue field of $F$ when $F$ is non-archimedean.

**Definition 2.2.** Fix a nonzero $D^\times(E)$-invariant pairing $\mathcal{B}_H : V_H \times V_H \to \mathbb{C}$. Let $\phi_H \in V_H^\text{new}$ be a non-zero vector in the new line. The normalized local zeta integral is defined by

$$I(H, t) = \int_{F^\times \backslash D^\times(F)} \frac{\mathcal{B}_H(H(ht)\phi_H, H(t)\phi_H)}{\mathcal{B}_H(H(\mathcal{J})\phi_H, \phi_H)} \, dh,$$

$$I^*(H, t) = \left( \frac{\zeta_F(2)}{\zeta_E(2)} \right) \cdot \frac{L(1, \text{Ad})}{L(1/2, \text{Ad}, r)} \cdot I(H, t).$$

Here the $L$-factors are defined in [Ich08, pp. 282-283].

**Remark 2.3.**

1. Since the central character of $H$ is trivial on $F^\times$, the integrals are well-defined. Moreover, our assumption $\Lambda(H') < 1/2$ implies these integrals converge absolutely [Ich08, lemma 2.1].

2. We note that $\phi_H$ is unique up to a constant as well as $\mathcal{B}_H$. Thus $I(H, t)$ is independence of the choice of $\phi_H$ and $\mathcal{B}_H$. But it does depend on the choice of the measure $dh$.

3. **Matrix coefficients for GL(2)**

Let $F$ be either $\mathbb{R}$ or a non-archimedean local field. Let $\pi$ be a unitary irreducible admissible generic representation of $\text{GL}_2(F)$. Let $\phi_\pi$ be a non-zero vector in the new line $V_\pi^\text{new}$. Fix a non-zero $\text{GL}_2(F)$-invariant bilinear pairing $\mathcal{B}_\pi : \pi \times \pi \to \mathbb{C}$, where $\pi$ is the admissible dual of $\pi$.

**Definition 3.1.** We define the matrix coefficient associate with an element $t \in \mathfrak{U} \times O(2)$ or $t \in \text{GL}_2(F)$ by

$$\Phi_\pi(h; t) = \frac{\mathcal{B}_\pi(\pi(ht)\phi_\pi, \pi(t)\phi_\pi)}{\mathcal{B}_\pi(\pi(\mathcal{J})\phi_\pi, \phi_\pi)}, \quad h \in \text{GL}_2(F).$$

Recall that $\tau_F$ is given by (2.1). When $t = 1$, we simply denote $\Phi_\pi(h)$ for $\Phi_\pi(h; t)$.

**Remark 3.2.** Note that $\Phi_\pi(h; t)$ is independent of the choice of elements $\phi_\pi$ and $\phi_\pi$ in the one dimensional subspaces of $\pi$ and $\pi$ which consisting of either weight $k$ elements or newforms, respectively. Moreover, it is also independent of $\mathcal{B}_\pi$ and the models for which we used to realize $\pi$ and $\pi$.

3.1. **A formulas of $\Phi_\pi(h, t)$: the archimedean case.** Let $\pi$ be a (limit of) discrete series representation of $\text{GL}_2(\mathbb{R})$ with minimal weight $k \geq 1$ and the central character $\text{sgn}^k$. Note that $\pi \cong \pi$. Let $\psi$ be the additive character of $\mathbb{R}$ defined by $\psi(x) = e^{2\pi \sqrt{-1} x}$. Let $W(\pi, \psi)$ be the Whittaker model of $\pi$ with respect to $\psi$. Let $\mathcal{B}_\pi : \mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi, \psi) \to \mathbb{C}$ be the $\text{GL}_2(\mathbb{R})$-invariant bilinear pairing given by

$$\mathcal{B}_\pi(W, W') = \int_{\mathbb{R}^\times} W \left( \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right) W' \left( \begin{array}{cc} -t & 0 \\ 0 & 1 \end{array} \right) d^\times t,$$

for $W, W' \in \mathcal{W}(\pi, \psi)$. Here $d^\times t = |t|_{\mathbb{R}}^{-1} \, dt$, and $dt$ is the usual Lebesgue measure on $\mathbb{R}$.

Let $W_\pi \in \mathcal{W}(\pi, \psi)$ be the weight $k$ element characterized by

$$W_\pi \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) = a^k \psi \cdot I_{\mathbb{R}_+}(a), \quad a \in \mathbb{R}^\times.$$
For each $m \in \mathbb{Z}_{\geq 0}$, we put $W^m = \rho(V^m)W$. Here $\rho$ denotes the right translation. In particular, we have $W^0 = W$. We note that $W^m$ has weight $k + 2m$.

The following recursive formula can be deduced from the proof of [Lj70] Lemma 5.6

\[(3.3) \quad W^{m+1} = 2a \cdot \frac{d}{da} W^m + (k + 2m - 4\pi a) \cdot W^m.
\]

**Lemma 3.3.** We have

\[
W^m = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}
\]

Proof. This is [Ike98, Lemma 2.1].

For each $\rho \in \rho^m$ we have $\varphi(\rho) = \varphi^m$. In particular, we have $W^m$. We note that $W^m$ has weight $k + 2m$.

**Lemma 3.4.** Let $a \in \mathbb{R}^x$ and $x \in \mathbb{R}$. Then

\[
\mathfrak{B}_\pi \left( \rho \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) W^m
\]

is equal to

\[
2^{-k + 2m} \pi^{-k} \mathbb{B}_\pi \mathbb{B}_\pi (a) \sum_{i,j=0}^m (-2)^{i+j} \begin{pmatrix} m \\ i \end{pmatrix} \begin{pmatrix} m \\ j \end{pmatrix} \pi k \frac{(1-a) \Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)} \frac{(a)^{k+i}}{\Gamma(k+i)\Gamma(k+j)} [\ Psi x ]^{k+i+j}.
\]

Let $N$ be a nonnegative integer. We have the following identity

\[
\sum_{i=0}^N (-1)^i \binom{N}{i} \frac{\Gamma(z+i)}{\Gamma(w+i)} = \frac{\Gamma(z)}{\Gamma(w)} \cdot \frac{\Gamma(w-z+N)}{\Gamma(w+N)}
\]

for every $z, w \in \mathbb{C}$.

**Lemma 3.5.** Let $m \geq 0$. We have the following identity

\[
\sum_{i=0}^m (-1)^i \binom{m}{i} \frac{\Gamma(z+i)}{\Gamma(w+i)} = \frac{\Gamma(z)}{\Gamma(w)} \cdot \frac{\Gamma(w-z+m)}{\Gamma(w+m)}
\]

for every $z, w \in \mathbb{C}$.

**Proof.** This is [Ike98, Lemma 2.1].

**Lemma 3.6.** We have

\[
\mathfrak{B}_\pi \left( \rho \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \right) W^m = 4^{-k+m} \pi^{-k} \Gamma(k+m+1).
\]
Using \([\text{Sch02}, \text{Prop. 3.1.2}]\), one can deduce that
\[
\mathcal{B}_\pi \left( \rho \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) W^m, W^m_\pi \right) = 4^{-k+m} \pi^{-k} \Gamma(k+m)^2 \sum_{i,j=0}^m (-1)^{i+j} \binom{m}{i} \binom{m}{j} \frac{\Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)}.
\]

Applying Lemma 3.5, we find that
\[
\sum_{i,j=0}^m (-1)^{i+j} \binom{m}{i} \binom{m}{j} \frac{\Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)} = \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{\Gamma(m-i)}{\Gamma(-i)\Gamma(k+m)} = (-1)^m \frac{\Gamma(0)}{\Gamma(-m)\Gamma(k+m)} = \frac{\Gamma(m+1)}{\Gamma(k+m)}.
\]

This proves the lemma.

Combining the above results, we obtain the following corollary.

**Corollary 3.7.** Let \(m \in \mathbb{Z}_{\geq 0}\), \(x \in \mathbb{R}\) and \(a \in \mathbb{R}^\times\). We have

\[
\Phi_\pi \left( \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} ; V^m \right) = 2k + 2m \frac{\Gamma(k+m)^2}{\Gamma(k)} \sum_{i,j=0}^m (-2)^{i+j} \binom{m}{i} \binom{m}{j} \frac{\Gamma(k+i+j)}{\Gamma(k+i)\Gamma(k+j)} \frac{(-a)^{k+i}}{(1-a) + \sqrt{-1}x}.
\]

\[(1)\]

**3.2. A formula of \(\Phi_\pi(h,t)\): the non-archimedean case.** Let \(F\) be a non-archimedean local field. Let \(B(F)\) be the subgroup of upper triangular matrices in \(GL_2(F)\). Denote by \(St_F\) the Steinberg representation of \(GL_2(F)\). Namely, \(St_F\) is the unique irreducible subrepresentation in the induced representation

\[\text{Ind}_{B(F)}^{GL_2(F)}(\lambda_1 \otimes \lambda_2).\]

**Lemma 3.8.** Suppose that \(\pi = \text{Ind}_{B(F)}^{GL_2(F)} (\lambda_1 \otimes \lambda_2)\) is spherical. Let \(\alpha = \lambda_1^{1/2}.\) Then for \(n \in \mathbb{Z}\), we have

\[\Phi_\pi \left( \begin{pmatrix} \alpha^n & 0 \\ 0 & 1 \end{pmatrix} ; \gamma \right) = \frac{(-a)^{k/2}}{\sqrt{-1}x} \Phi_\pi \left( \begin{pmatrix} \alpha^n & 0 \\ 0 & 1 \end{pmatrix} \right)\]

**Proof.** This is Macdonald’s formula. For example, see [Bump98 Theorem 4.6.6].

**Lemma 3.9.** Suppose \(\pi = St_F \otimes \chi\), where \(\chi\) is a unramified quadratic character of \(F^\times\). Then for \(n \in \mathbb{Z}\), we have

\[\Phi_\pi \left( \begin{pmatrix} \alpha^n & 0 \\ 0 & 1 \end{pmatrix} \right) = \chi(\alpha^n) q_F^{-\frac{1}{2}} \left( \alpha \right)^{1+\frac{1}{2}} \left( \frac{1}{1+\alpha} \right)^{1/2} \left( \frac{1}{1-\alpha} \right)^{1/2} \]

**Proof.** For the ease of notation, we put \(\alpha = \alpha_F\) and \(q = q_F\). Let \(\psi\) be an additive character of \(F\) of order zero. Let \(W(\pi, \psi)\) be the Whittaker model of \(\pi\) with respect to \(\psi\). Since \(\pi\) is self-dual, we have \(W(\pi, \psi) = W(\pi, \psi)\) by the uniqueness of the Whittaker model. Let \(W_\pi \in W(\pi, \psi)\) be the newform with \(W_\pi(1) = 1\). By \([\text{Sch02}, \text{Summary}]\) we have

\[W_\pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \chi(a)a \cdot \Phi_\pi(a), \quad a \in F^\times.
\]

Using \([\text{Sch02}, \text{Prop. 3.1.2}]\), one can deduce that

\[(3.4) \quad W_\pi \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = -q^{-1} \chi(a)a \cdot \Phi_\pi(a).
\]
Let \( B_\pi : W(\pi, \psi) \times W(\pi, \psi) \to \mathbf{C} \) be the \( GL_2(F) \)-invariant bilinear pairing given by (3.1) with \( R^\times \) replaced by \( F^\times \). The Haar measure \( d^\times t \) on \( F^\times \) is determined by \( \text{Vol}(\mathcal{O}_F^\times, d^\times t) = 1 \). Let \( n \in \mathbb{Z} \).

\[
B_\pi \left( \rho \left( \begin{array}{cc} \varpi^n & 0 \\ 0 & 1 \end{array} \right) \right) W_\pi, W_\pi = \int_{F^\times} W_\pi \left( \begin{array}{cc} \varpi^n t & 0 \\ 0 & 1 \end{array} \right) W_\pi \left( \begin{array}{cc} -t & 0 \\ 0 & 1 \end{array} \right) d^\times t
\]

\[
= \chi(\varpi^n) q^{-n} \int_{\mathcal{O}_F^\times} |t|^2 d^\times t
\]

\[
= \chi(\varpi^n) q^{-n-2\delta(n)} \cdot \zeta_F(2),
\]

where

\[
\delta(n) = \begin{cases} 0 & \text{if } n \geq 0, \\ -n & \text{if } n < 0. \end{cases}
\]

In particular, we have

(3.5) \[ B_\pi(W_\pi, W_\pi) = \zeta_F(2), \]

and hence

\[
\Phi_\pi \left( \begin{array}{cc} \varpi^n & 0 \\ 0 & 1 \end{array} \right) = \chi(\varpi^n) q^{-n-2\delta(n)} = \chi(\varpi^n) q^{|n|}.
\]

Using (3.4) and by similar calculations, we find that

\[
B_\pi \left( \rho \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \begin{array}{cc} \varpi^n & 0 \\ 0 & 1 \end{array} \right) W_\pi, W_\pi = -\chi(\varpi^n) q^{-n-1-2\delta'(n)} \cdot \zeta_F(2),
\]

where

\[
\delta'(n) = \begin{cases} -1 & \text{if } n \geq 1, \\ -n & \text{if } n < 1. \end{cases}
\]

Since \( B_\pi(W_\pi, W_\pi) = \zeta_F(2) \) by (3.5), we obtain

\[
\Phi_\pi \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \begin{array}{cc} \varpi^n & 0 \\ 0 & 1 \end{array} \right) = -\chi(\varpi^n) q^{-n-1-2\delta'(n)} = -\chi(\varpi^n) q^{-|n-1|}.
\]

This finishes the computation. \qed

4. The calculation of local zeta integral (I)

In this section, let \( D = M_2(F) \). We compute the normalized local zeta integral \( I^\ast(H, t) \) in Definition 2.2

4.1. Haar measures. If \( F = \mathbf{R} \), let \( dx \) be the usual Lebesgue measure on \( \mathbf{R} \), and the Haar measure \( d^\times x \) on \( \mathbf{R}^\times \) is given by \( |x|_{\mathbf{R}}^{-1} d^\times x \). The Haar measure \( dh \) on \( GL_2(\mathbf{R}) \) is given by

\[
dh = \frac{dz}{|z|_{\mathbf{R}}} \frac{dxdy}{|y|_{\mathbf{R}}} \frac{dk}{dk}
\]

for \( h = z \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k \) with \( x, y \in \mathbf{R}^\times, z \in \mathbf{R}_+^\times, k \in SO(2) \), where \( dx, dy, dz \) are the usual Lebesgue measures and \( dk \) is the Haar measure on \( SO(2) \) such that \( \text{Vol}(SO(2), dk) = 1 \).

If \( F \) is non-archimedean, let \( dx \) be the Haar measure on \( F \) so that the total volume of \( \mathcal{O}_F \) is equal to 1 and let \( d^\times x \) on \( F^\times \) be the Haar measure on \( F^\times \) so that \( \mathcal{O}_F^\times \) also has volume 1. On \( GL_2(F) \), we let \( dh \) be the Haar measure determined by \( \text{Vol}(GL_2(\mathcal{O}_F), dh) = 1 \).

The measure on the quotient space \( F^\times \backslash GL_2(F) \) is the unique quotient measure induced from the measure \( dh \) on \( GL_2(F) \) and the measure \( d^\times x \) on \( F^\times \).
4.2. The archimedean case. Let \( \pi_j (j = 1, 2, 3) \) be a (limit of) discrete series representation of \( \text{GL}_2(\mathbb{R}) \) with minimal weight \( k_j \geq 1 \) and central character \( \text{sgn}^{k_j} \) such that

\[
2 \max \{k_1, k_2, k_3\} \geq k_1 + k_2 + k_3.
\]

We may assume \( k_3 = \max \{k_1, k_2, k_3\} \) and let \( 2m = k_3 - k_1 - k_2 \) for some integer \( m \geq 0 \).

**Proposition 4.1.** We have

\[
I^* (\Pi, t) = 2^{k_1 + k_2 - k_3 + 1}.
\]

**Proof.** Note that the \( L \)-factor given by

\[
L(s, \Pi, r) = \zeta_C (s + (k_3 + k_2 + k_1 - 3)/2) \zeta_C (s + (k_3 - k_2 - k_1 + 1)/2) \\
\times \zeta_C (s + (k_3 - k_2 + k_1 - 1)/2) \zeta_C (s + (k_3 + k_2 - k_1 - 1)/2).
\]

We proceed to compute \( I(\Pi, t) \). By definition

\[
I(\Pi, t) = \int_{\mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R})} \Phi_{\pi_1} (h) \Phi_{\pi_2} \left( h; V^m_+ \right) \Phi_{\pi_3} (h; \tau_\mathbb{R}) dh
\]

\[
= \left( \frac{1}{8\pi} \right)^{2m} \int_{\mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R})} \Phi_{\pi_1} (h) \Phi_{\pi_2} \left( h; V^m_+ \right) \Phi_{\pi_3} (h; \tau_\mathbb{R}) dh.
\]

Put

\[
\Phi(h) = \Phi_{\pi_1} (h) \Phi_{\pi_2} \left( h; V^m_+ \right) \Phi_{\pi_3} (h; \tau_\mathbb{R}), \quad h \in \text{GL}_2(\mathbb{R}).
\]

We now focus our attention to compute the following integral:

\[
I := \int_{\mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R})} \Phi(h) dh.
\]

Note that \( \Phi(h) \) is right \( \text{SO}(2) \)-invariant. Since the total volume of \( \text{SO}(2) \) is 1, it follows that

\[
\int_{\mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R})} \Phi(h) dh = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \Phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) + \Phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \right) d^x a dx,
\]

by the Iwasawa decomposition. Since \( \Phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \) vanishes when \( a \in \mathbb{R}_+ \), we find that

\[
I = \int_{\mathbb{R}^\times \setminus \text{GL}_2(\mathbb{R})} \Phi(h) dh = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \Phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a & 0 \\ 0 & 1 \end{pmatrix} \right) d^x a dx.
\]

By Corollary 3.7 we have \( \Phi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) \) is equal to \( 2^{2k_3} \frac{\Gamma(k_2 + m)^2}{\Gamma(k_3^2)} \) times

\[
(4.1) \quad \mathbb{I}_R - (a) \sum_{i,j=0}^m (-2)^{i+j} \binom{m}{i} \binom{m}{j} \frac{\Gamma(k_2 + i + j)}{\Gamma(k_2 + i) \Gamma(k_2 + j)} \frac{(a)_{k_3 - m + i}}{(1 - a) - \sqrt{-1}x} |k_3| |1 - a| |\sqrt{-1}x|^{k_3 - 2m + i + j}.
\]

By (4.1) we have

\[
I = 2^{2k_3} (k_2 + m)^2 \sum_{i,j=0}^m (-2)^{i+j} \binom{m}{i} \binom{m}{j} \frac{\Gamma(k_2 + i + j)}{\Gamma(k_2 + i) \Gamma(k_2 + j)} \cdot I_{i,j},
\]

where for \( 0 \leq i, j \leq m \),

\[
I_{i,j} := \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left( \frac{a_{k_3 - m + i}}{(a + 1)^2 - \sqrt{-1}x} \right) \frac{\Gamma(k_3 - m + i)}{(1 + \sqrt{-1}x)^{k_3 - 2m + i + j - 1}} d^x a dx
\]

\[
= \left( \int_{\mathbb{R}^+} \frac{a_{k_3 - m + i}}{(a + 1)^2 - \sqrt{-1}x} d^x a \right) \int_{\mathbb{R}} \frac{dx}{(1 + \sqrt{-1}x)^{k_3 - 2m + i + j - 1}}
\]

\[
= 2^{-(2k_3 - 2m + i + j)} \pi \frac{\Gamma(k_3 - m + i - 1) \Gamma(k_3 - m + j)}{\Gamma(k_3 - 2m + i + j) \Gamma(k_3)}.
\]

The last equality follows from the following lemma.
Lemma 4.2. For $|\arg z| < \pi$, $0 < \text{Re}(\beta) < \text{Re}(\alpha)$, we have
\[
\int_{\mathbb{R}^+} \frac{t^\beta}{(t+z)^\alpha} dt = z^{\beta-\alpha} \cdot \frac{\Gamma(\alpha-\beta)\Gamma(\beta)}{\Gamma(\alpha)}.
\]
For $\text{Re}(\alpha + \beta) > 1$, we have
\[
\int_{\mathbb{R}} \frac{dx}{(1 + \sqrt{-1}x)^\alpha(1 - \sqrt{-1}x)^\beta} = 2^{2-\alpha-\beta} \cdot \pi \cdot \frac{\Gamma(\alpha + \beta - 1)}{\Gamma(\alpha)\Gamma(\beta)}.
\]

PROOF. These are [Ike98, Lemma 2.4 and 2.5] \qed

Thus we obtain
\[
I = 2^{2+2m} \pi \frac{\Gamma(k_2 + m)^2}{\Gamma(k_2)\Gamma(k_3)} \sum_{i,j=0}^m (-1)^{i+j} \binom{m}{i} \binom{m}{j} \frac{\Gamma(k_2 + i + j)}{\Gamma(k_2 + i)} \frac{\Gamma(k_3 - m + i + j)}{\Gamma(k_3 - 2m + i + j)}.
\]

To simply the above expression of $I$, we need one more combinatorial identity from [Orl87, Lemma 3].

Lemma 4.3. Let $N \in \mathbb{Z}_{\geq 0}$ and $t, \alpha, \beta \in \mathbb{C}$. Then
\[
\Gamma(\alpha + N) \sum_{i=0}^N (-1)^i \binom{N}{i} \frac{\Gamma(t + i)}{\Gamma(\alpha + i)} = (-1)^N \frac{\Gamma(t + \beta + \alpha + N - 1 + i)}{\Gamma(2t + \beta + i)} \frac{\Gamma(t + \beta + N)}{\Gamma(t + \beta)} \frac{\Gamma(t - \alpha + 1)}{\Gamma(t - \alpha + 1)}.
\]

Now we write
\[
I = 2^{2+2m} \pi \frac{\Gamma(k_2 + m)^2}{\Gamma(k_1)\Gamma(k_2)} \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{\Gamma(k_2 + j + i)}{\Gamma(k_2 + i)} \frac{\Gamma(k_3 - 2m + j + i)}{\Gamma(k_3 - 2m + j + i)}.
\]

Applying Lemma 4.3 to $I'$ with $t = k_2 + j$, $\alpha = k_2$ and $\beta = k_3 - 2m - 2k_2 - j$, we find that
\[
I' = (-1)^m \cdot \frac{\Gamma(k_2 + j)\Gamma(k_3 - m - 1)}{\Gamma(k_3 - m + j)} \cdot \frac{\Gamma(k_3 - 2m - 2k_2 - j)}{\Gamma(k_3 - 2m) \cdot \Gamma(j + 1)}.
\]

It follows that
\[
I = (-1)^m 2^{2+2m} \pi \frac{\Gamma(k_3 - m - 1)\Gamma(k_2 + m)\Gamma(k_1 + m)}{\Gamma(k_3)\Gamma(k_2)\Gamma(k_1)} \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{\Gamma(1 + j)}{\Gamma(1 - m + j)}.
\]

Applying Lemma 4.5 we obtain
\[
I = 2^{2+2m} \pi \frac{\Gamma(k_3 - m - 1)\Gamma(k_2 + m)\Gamma(k_1 + m)\Gamma(m + 1)}{\Gamma(k_1)\Gamma(k_2)\Gamma(k_3)}.
\]

Therefore we find that
\[
I(\Pi, \mathbf{t}) = \left( \frac{1}{8\pi} \right)^{2m} I = 2^{2-4m} \pi^{1-2m} \frac{\Gamma(k_3 - m - 1)\Gamma(k_2 + m)\Gamma(k_1 + m)\Gamma(m + 1)}{\Gamma(k_1)\Gamma(k_2)\Gamma(k_3)}.
\]

and the proposition follows. \qed

We deduce a consequence from Proposition 4.1. Let $m_1, m_2$ be two non-negative integers such that $m_1 + m_2 = m$. Put
\[
\mathbf{t}_{m_1, m_2} = \left( \mathcal{Y}^{m_1}_{+} \otimes \mathcal{Y}^{m_2}_{+} \otimes 1, (1, 1, \tau_R) \right) \in \mathcal{U}_E \times O(2, E).
\]

Then our original element $\mathbf{t}$ is $\mathbf{t}_{0, m}$. Put
\[
I^* (\Pi; \mathbf{t}_{m_1, m_2}) = \frac{L(1, \Pi, \text{Ad})}{\zeta_R(2)^2 L(1/2, \Pi, r)} \cdot I(\Pi, \mathbf{t}_{m_1, m_2}),
\]
where
\[
I (II, t_{m_1, m_2}) = \int_{\mathbb{R}^\times \setminus GL_2(\mathbb{R})} \Phi_{\pi_1} \left( h; \tilde{V}_+^{m_1} \right) \Phi_{\pi_2} \left( h; \tilde{V}_+^{m_2} \right) \Phi_{\pi_3} (h; \tau_R) dh
\]

Then \( I^* (II, t) \) in Definition 2.2 is nothing but \( I^* (II, t_{0, m}) \).

**Corollary 4.4.** Notation is as above. We have
\[
I^* (II, t_{m_1, m_2}) = I^* (II, t)
\]
for every non-negative integers \( m_1, m_2 \) such that \( m_1 + m_2 = m \).

**Proof.** This is in fact an easy consequence form the multiplicity one result of local trilinear forms, Proposition 4.1 together with the local Rankin-Selberg integral. More precisely, let \( \mu_2 = | \cdot |_{(k_2-1)/2} \) and \( \nu_2 = | \cdot |_{(1-k_2)/2} \text{sgn}^{k_2} \) be two characters of \( \mathbb{R}^\times \). Then \( \pi_2 \) can be realized as the unique irreducible subrepresentation of \( \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})} (\mu_2 \boxtimes \nu_2) \) which we denote by \( \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})} (\mu_2 \boxtimes \nu_2) \). For every non-negative integer \( n \), we let \( f_{\pi_2}^n \in \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})} (\mu_2 \boxtimes \nu_2) \) be the element characterized by requiring
\[
f_{\pi_2}^n \left( \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right) = e^{i(k_2+2n)\theta}.
\]

We have the following relation, which can be found in [JL70, Lemma 5.6 (iii)]
\[
\rho (\tilde{V}_+) f_{\pi_2}^n = 2(k_2 + n) f_{\pi_2}^{n+1}.
\]

Inductively we find that
\[
\rho (\tilde{V}_+) f_{\pi_2}^n = c(\pi_2, n, \ell) f_{\pi_2}^{n+\ell},
\]
where
\[
c(\pi_2, n, \ell) = 2 \frac{\Gamma(k_2 + n + \ell)}{\Gamma(k_2 + n)},
\]
for every \( \ell \geq 0 \).

Let \( \Psi : \mathcal{W}(\pi_1, \psi) \boxtimes \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})} (\mu_2 \boxtimes \nu_2) \boxtimes \mathcal{W}(\pi_3, \psi) \to \mathbb{C} \) be the local Rankin-Selberg integral defined by
\[
\Psi (W_1 \otimes f_2 \otimes W_3) = \int_{\mathbb{R}^\times N(\mathbb{R}) \setminus GL_2(\mathbb{R})} W_1 (\tau_R g) W_3 (g) f_2 (g) dg,
\]
for \( W_1 \in \mathcal{W}(\pi_1, \psi) \), \( f_2 \in \text{Ind}_{B(\mathbb{R})}^{GL_2(\mathbb{R})} (\mu_2 \boxtimes \nu_2) \) and \( W_3 \in \mathcal{W}(\pi_3, \psi) \). Here
\[
N = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in GL_2 \right\}.
\]

One check easily that this integral converges absolutely and certainly it defines a \( GL_2(\mathbb{R}) \)-invariant trilinear form. From the multiplicity one result of such trilinear form and the fact that \( I^* (II, t) \neq 0 \), one can deduce that following equality easily
\[
I^* (II, t_{m_1, m_2}) = \left( \frac{c(\pi_2, m_1, m_2)}{c(\pi_2, 0, m_2)} \right)^2 \left( \frac{\Psi (W_{\pi_1}^{m_1} \otimes f_{\pi_2}^{m_2} \otimes \rho(\tau_R) W_{\pi_3})}{\Psi (W_{\pi_1} \otimes f_{\pi_2}^m \otimes \rho(\tau_R) W_{\pi_3})} \right)^2.
\]

Recall that \( W_{\pi_1}^n = \rho (\tilde{V}_+^n) W_{\pi_1} \) for every \( n \geq 0 \). Our task now is to compute the ratio of these two Rankin-Selberg integrals. Since we can let \( m_1, m_2 \) vary, it suffices to compute the numerator. Applying Lemma 3.3,
By letting $m$ and Lemma 3.5, we find that the numerator is
\[
\Psi(W_{\pi_1}^{m_1} \otimes f_{\pi_2}^{m_2} \otimes \rho(\tau_R)W_{\chi}) = \int_{SO(2)} \int_{R^x} W_{\pi_1}((a, 0)k)W_{\pi_3}((-a, 0)k) f_{\pi_2}((0, 1)k) \frac{d^x a}{|a|_R} dk
\]
\[
= \int_{R^x} W_{\pi_1}((-a, 0)) W_{\pi_3}((-a, 0)) |a|_R^{-1} d^x a
\]
\[
= 2^{k_1+k_3+m_1} \sum_{j=0}^{m_1} (-4\pi)^j \Gamma(k_1+m_1) \left( m_1 \right) \int_{0}^{\infty} a^{k_1+k_3+k_j} e^{-4\pi a d^x a}
\]
\[
= 2^{k_1+k_3+m_1} (4\pi)^{1-k_1-k_3-k_j} \Gamma(k_1+m_1) \sum_{j=0}^{m_1} (-1)^j \left( m_1 \right) \Gamma(k_2+m_1+m_2)
\]
\[
= (-1)^m 2^{k_1+k_3+m_1} (4\pi)^{1-k_1-k_3-\frac{1}{2}} \Gamma(k_1) \left( m_1 \right) \Gamma(k_2+m_1+m_2)
\]

By letting $m_1 = 0$ and $m_2 = m$, we obtain the value of denominator. Combining with equation (4.2), we find that the right hand side of the equation (4.3) is equal to 1. The corollary follows. □

4.3. The non-archimedean case. Let $F$ be a non-archimedean local field. Write $\varpi = \varpi_F$ and $q = q_F$ for simplicity. Recall that we have assumed
\[
\text{Hom}_{GL_2(F)}(II, C) \neq \{0\}.
\]

According to the results of Prasad [Pra90] and [Pra92] and our assumption on II, (4.4) holds for following cases. (i) Suppose $E = F \times F \times F$ so that $II = \pi_1 \boxtimes \pi_2 \boxtimes \pi_3$. Then (i-a) one of $\pi_1, \pi_2, \pi_3$ is spherical; (i-b) $\pi_j = St_F \otimes \chi_j$ are unramified special representations for $j = 1, 2, 3$ with $\chi_1^2 \chi_2 \chi_3(\varpi) = -1$. (ii) Suppose $E = K \times F$ so that $II = \pi^{\prime} \boxtimes \pi$. Then (ii-a) $\pi$ is spherical; (ii-b) $\pi^{\prime}$ is spherical, $\pi = St_F \otimes \chi$ is a unramified special representation, $K/F$ is ramified and $\chi(\varpi) = -1$; (ii-c) $\pi^{\prime} = St_K \otimes \chi^{\prime}$, $\pi = St_F \otimes \chi$ are unramified special representations, $K/F$ is ramified or $K/F$ is unramified and $\chi^{\prime}(\varpi) = 1$. (iii) Suppose $E$ is a field. Then (iii-a) $II$ is spherical; (iii-b) $II = St_E \otimes \chi$ is a unramified special representation with $\chi(\varpi) = -1$.

We say that $E$ is unramified over $F$ if either $E = F \times F \times F$, or $E = K \times F$, where $K$ is the unramified extension over $F$, or $E$ is the unramified cubic extension over $F$. The evaluation of $I^*(II, t)$ has been carried out in the following cases.

**Proposition 4.5.**

1. Suppose $E$ is unramified over $F$ and $II$ is spherical. Then we have
   \[
   I^*(II, t) = 1.
   \]

2. Suppose $E = F \times F \times F$ and $\pi_j = St_F \otimes \chi_j$, where $\chi_j$ are unramified quadratic characters of $F^{\times}$ for $j = 1, 2, 3$. Then we have
   \[
   I^*(II, t) = 2q^{-1}(1 + q^{-1}).
   \]

3. Suppose $E = F \times F \times F$ and one of $\pi_j (j = 1, 2, 3)$ is spherical and the other two are unramified special. Then we have
   \[
   I^*(II, t) = q^{-1}.
   \]

Here $I^*(II, t)$ is defined in (2.4).

**Proof.** Part (1) is [Ichi08] Lemma 2.2, (2) is in [Ill10] Section 7 and (3) is a result of [Ne11] Lemma 4.4. □

We proceed to compute $I^*(II, t)$ in the remaining cases. For $\Phi \in L^1(F^{\times}\backslash GL_2(F))$ such that $\Phi(\varpi^{k}h) = \Phi(h)$ for every $h \in GL_2(F)$ and $k, k^{\prime} \in K_0(\varpi)$, where
\[
K_0(\varpi) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(O_F) \mid c \in \varpi O_F, \right\}
\]
we have the integration formula

\[
\int_{F^\times \setminus GL_2(F)} \Phi(h) dh = (1 + q)^{-1} \left\{ \sum_{n \in \mathbb{Z}} \Phi \left( \begin{pmatrix} \infty^n & 0 \\ 0 & 1 \end{pmatrix} \right) q^{\vert n \vert} + \sum_{n \in \mathbb{Z}} \Phi \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \infty^n & 0 \\ 0 & 1 \end{pmatrix} \right) q^{\vert n-1 \vert} \right\}
\]

(cf. [110] Section 7).

**Proposition 4.6.** Let \( E = F \times F \times F \). Suppose one of \( \pi_j \) is unramified special and the other two are spherical. Then we have

\[ I^* (\Pi, t) = q^{-1} (1 + q^{-1})^{-1}. \]

**Proof.** In this case, the \( L \)-factor is given by

\[
L(s, \Pi, r) = (1 - \chi(\varepsilon) \alpha \beta q^{-s-1/2})^{-1} (1 - \chi(\varepsilon) \alpha^{-1} \beta^{-1} q^{-s-1/2})^{-1} \times (1 - \chi(\varepsilon) \alpha \beta^{-1} q^{-s-1/2})^{-1} \times (1 - \chi(\varepsilon) \alpha^{-1} \beta^{-1} q^{-s-1/2})^{-1}.
\]

We continue to compute \( I(\Pi, t) \). Assume \( \pi_1 = \text{St}_F \otimes \chi \) for some unramified quadratic character \( \chi \) of \( F^\times \), and

\[ \pi_j = \text{Ind}_{B(F)}^{GL_2(F)} \left( \cdot \mid |_{F^\times} \otimes \cdot \mid |_{F^\times} \right) \]

for \( j = 2, 3 \). Let \( \alpha = |\varepsilon|_{F}^{\lambda_2} \) and \( \beta = |\varepsilon|_{F}^{\lambda_3} \). Then we have

\[ I (\Pi, t) = \int_{F^\times \setminus GL_2(F)} \Phi(h) dh, \]

where

\[ \Phi(h) = \Phi_{\pi_1}(h) \Phi_{\pi_2}(h) \Phi_{\pi_3} \left( h: \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right), \quad h \in GL_2(F). \]

By [135], Lemma 5.8 and Lemma 3.9 we find that

\[
I(\Pi, t) = (1 + q)^{-1} \left\{ \sum_{n = -\infty}^{\infty} \Phi \left( \begin{pmatrix} \infty^n & 0 \\ 0 & 1 \end{pmatrix} \right) q^{\vert n \vert} + \sum_{n = -\infty}^{\infty} \Phi \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \infty^n & 0 \\ 0 & 1 \end{pmatrix} \right) q^{\vert n-1 \vert} \right\}
\]

\[
= (1 + q)^{-1} \frac{(1 - q^{-1}) (1 - \alpha^2 q^{-1}) (1 - \alpha^{-2} q^{-1}) (1 - \beta^2 q^{-1}) (1 - \beta^{-2} q^{-1})}{(1 - \chi(\varepsilon) \alpha \beta q^{-1}) (1 - \chi(\varepsilon) \alpha^{-1} \beta q^{-1}) (1 - \chi(\varepsilon) \alpha^{-1} \beta^{-1} q^{-1}) (1 - \chi(\varepsilon) \alpha \beta^{-1} q^{-1})}.
\]

This completes the proof. \( \square \)

**Proposition 4.7.** Let \( E = K \times F \) and \( \pi \) be a spherical representation of \( GL_2(F) \).

(1) If \( K \) is ramified over \( F \) and \( \pi' \) is spherical, then we have

\[ I^* (\Pi, t) = 1. \]

(2) If \( \pi' \) is unramified special, then we have

\[
I^* (\Pi, t) = \begin{cases} q^{-1} (1 + q^{-1})^{-2} (1 + q^{-2}) & \text{if } K \text{ is unramified over } F, \\ q^{-1} (1 + q^{-1})^{-1} & \text{if } K \text{ is ramified over } F. \end{cases}
\]

**Proof.** Let

\[ \pi = \text{Ind}_{B(F)}^{GL_2(F)} \left( \cdot \mid |_{F}^{\lambda} \otimes \cdot \mid |_{F}^{-\lambda} \right), \quad \beta = |\varepsilon|_{F}^{\lambda}. \]

We begin with (1). Let

\[ \pi' = \text{Ind}_{B(K)}^{GL_2(K)} \left( \cdot \mid |_{K}^{\lambda'} \otimes \cdot \mid |_{K}^{-\lambda'} \right), \quad \alpha = |\varepsilon|_{K}^{\lambda'}. \]

For a non-negative integer \( n \), let \( X_n \) be the image of

\[ GL_2(\mathcal{O}_F) \begin{pmatrix} \varepsilon^n & 0 \\ 0 & 1 \end{pmatrix} \]

in \( F^\times \setminus GL_2(F) \). Note that

\[ \text{vol}(X_n, dh) = \begin{cases} 1 & \text{if } n = 0, \\ q^n (1 + q^{-1}) & \text{if } n \geq 1. \end{cases} \]
By Lemma 3.8, we have
\[
I(\Pi, t) = \sum_{n=0}^{\infty} \Phi_{\pi'} \left( \left( \begin{array}{c} \overline{w}^n \\ 0 \\ 1 \end{array} \right) \right) \Phi_{\pi} \left( \left( \begin{array}{c} \overline{w}^n \\ 0 \\ 1 \end{array} \right) \right) \text{vol}(X_n, dh)
\]
\[
= \frac{(1 - \alpha^2 q^{-1})(1 - \alpha^2 q^{-1})(1 + \beta q^{-1/2})(1 + \beta^{-1} q^{-1/2})}{(1 - \alpha^2 q^{-1/2})(1 - \alpha^2 q^{-1/2})(1 - \alpha^2 \beta^{-1} q^{-1/2})(1 - \alpha^2 \beta q^{-1/2})}
\]
Recall that the L-factor is given by
\[
L(s, \Pi, r) = (1 - \alpha \beta p^{-s})^{-1}(1 - \alpha \beta^{-1} p^{-s})^{-1}(1 - \beta p^{-s})^{-1}
\]
\[
\times (1 - \beta^{-1} p^{-s})^{-1}(1 - \alpha^{-1} \beta p^{-s})^{-1}(1 - \alpha^{-1} \beta^{-1} p^{-s})^{-1}.
\]
This shows (1).

Now we consider (2). Let \( \pi' = \text{St}_K \otimes \chi' \) for some unramified quadratic character \( \chi' \) of \( K^\times \). Suppose \( K \) is unramified over \( F \). By definition
\[
I(\Pi, t) = \int_{F^\times \setminus \text{GL}_2(F)} \Phi_{\pi'}(h) \Phi_{\pi}(h) dh.
\]
Applying Lemma 3.8, Lemma 3.9 and Lemma 5.4
\[
I(\Pi, t) = (1 + q)^{-1} \sum_{n=-\infty}^{\infty} \chi'(\overline{w})^n \Phi_{\pi} \left( \left( \begin{array}{c} \overline{w}^n \\ 0 \\ 1 \end{array} \right) \right) \left( q^{-|n|} - q^{-|n-1|} \right)
\]
\[
= \frac{(1 - q^{-1})(1 + q^{-2})}{(1 + q^{-1})} \frac{(1 - \chi'(\overline{w})\alpha q^{-1/2})(1 - \chi'(\overline{w})\alpha^{-1} q^{-1/2})}{(1 - \chi'(\overline{w})\alpha^{-3/2})(1 - \chi'(\overline{w})\alpha^{-1} q^{-3/2})}.
\]
Suppose \( K \) is ramified over \( F \). Similar calculations shows
\[
I(\Pi, t) = q^{-1} \frac{(1 - q^{-1})}{(1 + q^{-1})} \frac{(1 - \alpha^2 q^{-1})(1 - \alpha^{-2} q^{-1})}{(1 - \alpha^2 q^{-3/2})(1 - \alpha^{-2} q^{-3/2})}.
\]
Finally, if \( K/F \) is unramified, we have
\[
L(s, \Pi, r) = (1 + \chi'(\overline{w})\alpha q^{-s})^{-1}(1 - \chi'(\overline{w})\alpha q^{-s-1})^{-1}
\]
\[
\times (1 + \chi'(\overline{w})\alpha^{-1} q^{-s})^{-1}(1 - \chi'(\overline{w})\alpha^{-1} q^{-s-1})^{-1},
\]
while if \( K/F \) is ramified,
\[
L(s, \Pi, r) = (1 - \alpha q^{-s-1})^{-1}(1 - \alpha^{-1} q^{-s-1})^{-1}.
\]
This shows (2) and our proof is complete. \( \square \)

**Proposition 4.8.** Let \( E = K \times F \) and \( \pi = \text{St}_F \otimes \chi \), where \( \chi \) is an unramified quadratic character of \( F^\times \).

(1) If \( \pi' \) is spherical, \( \chi(\overline{w}) = -1 \) and \( K \) is ramified over \( F \), then we have
\[
I^*(\Pi, t) = 2q^{-1}(1 + q^{-1})^{-1}.
\]

(2) If \( \pi' = \text{St}_K \otimes \chi' \), where \( \chi' \) is a unramified quadratic character of \( K^\times \), then we have
\[
I^*(\Pi, t) = \begin{cases} 
2q^{-1}(1 + q^{-1})^{-1}(1 + q^{-2}) & \text{if } K \text{ is unramified over } F \text{ and } \chi' \chi(\overline{w}) = 1, \\
q^{-1} & \text{if } K \text{ is ramified over } F.
\end{cases}
\]

**Proof.** We first consider (1). By definition,
\[
I(\Pi, t) = \int_{F^\times \setminus \text{GL}_2(F)} \Phi(h) dh,
\]
where
\[
\Phi(h) = \Phi_{\pi'} \left( h; \left( \begin{array}{c} \overline{w}^K \\ 0 \\ 1 \end{array} \right) \right) \Phi_{\pi}(h), \quad h \in \text{GL}_2(F).
\]
By \cite{Le}, Lemma 3.8 and Lemma 3.9 we have
\[ I(II, t) = (1 + q)^{-1}(1 - \chi(\varpi)) \sum_{n=-\infty}^{\infty} \chi(\varpi)^n \phi_n \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} = 2q^{-1} \frac{(1 - q^{-1})}{(1 + q^{-1})^2} \frac{(1 + \alpha^2 q^{-1})(1 + \alpha^{-2} q^{-1})}{(1 - \alpha^2 q^{-1})(1 - \alpha^{-2} q^{-1})}. \]

Notice that
\[ L(s, II, r) = (1 - \chi(\varpi)\alpha^2 q^{-s-1/2})^{-1}(1 - \chi(\varpi)\alpha^{-2} q^{-s-1/2})^{-1}(1 - \chi(\varpi)q^{-s-1/2})^{-1}. \]

This shows (1).

Now we consider (2). By definition,
\[ I(II, t) = \int_{F^* \setminus GL_2(F)} \Phi(h) dh, \]
where
\[ \Phi(h) = \Phi_{\varpi}(h) \phi_{\varpi}(h), \quad h \in GL_2(F). \]

Suppose \( K \) is unramified over \( F \). Applying \cite{Le}, Lemma 3.9 we find that
\[ I(II, t) = (1 + q)^{-1} \left\{ \sum_{n=-\infty}^{\infty} \phi \left( \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \right) q^{n|} + \sum_{n=-\infty}^{\infty} \phi \left( \begin{pmatrix} 0 & 1 \\ \varpi^n & 0 \end{pmatrix} \right) q^{n-1} \right\} = (1 + q)^{-1}(1 + \chi'(\varpi)) \frac{(1 + \chi'(\varpi)q^{-2})}{(1 - \chi'(\varpi)q^{-2})}. \]

When \( K \) is ramified over \( F \), a similar calculation shows that
\[ I^*(II, t) = q^{-1} \frac{(1 + \chi'(\varpi)q^{-1})}{(1 - \chi'(\varpi)q^{-2})}. \]

Note that the L-factors are
\[ L(s, II, r) = \begin{cases} (1 - \chi'(\varpi)q^{-s-3/2})^{-1}(1 - q^{-2s-1})^{-1} & \text{if } K/F \text{ is unramified}, \\ (1 - \chi(\varpi)q^{-s-3/2})^{-1}(1 - \chi(\varpi)q^{-s-1/2})^{-1} & \text{if } K/F \text{ is ramified}. \end{cases} \]

This proves the proposition. \( \square \)

**Proposition 4.9.** Let \( E \) be a field.
\( (1) \) If \( E \) is ramified over \( F \) and \( \Pi \) is spherical, then we have
\[ I^*(II, t) = 1 \]
\( (2) \) If \( \Pi = St_E \otimes \chi, \) where \( \chi \) is the non-trivial unramified quadratic character of \( E^* \), then we have
\[ I(II, t) = \begin{cases} 2q^{-1}(1 + q^{-1})^{-1}(1 - q^{-1} + q^{-2}) & \text{if } E/F \text{ is unramified}, \\ 2q^{-1}(1 + q^{-1})^{-1} & \text{if } E/F \text{ is ramified}. \end{cases} \]

**Proof.** Suppose \( \Pi \) is spherical and \( E/F \) is ramified. Let
\[ \Pi = Ind_{B(E)}^{GL_2(E)} \left( \begin{pmatrix} \alpha & 0 \\ 0 & \varpi \end{pmatrix} \right), \quad \alpha = |\varpi E|_E. \]

For a non-negative integer \( n \), let \( X_n \) be the image of
\[ GL_2(O_F) \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} GL_2(O_F) \]
in \( F^* \setminus GL_2(F) \). Note that
\[ \text{Vol}(X_n, dh) = \begin{cases} 1 & \text{if } n = 0, \\ q^n(1 + q^{-1}) & \text{if } n \geq 1. \end{cases} \]

Applying Lemma 3.8
\[ I(II, t) = \sum_{n=0}^{\infty} \phi_n \begin{pmatrix} \varpi^n & 0 \\ 0 & 1 \end{pmatrix} \text{Vol}(X_n, dh) = \frac{(1 - q^{-1})(1 + \alpha q^{-1/2})(1 + \alpha^{-1} q^{-1/2})}{(1 - \alpha^2 q^{-1/2})(1 - \alpha^{-2} q^{-1/2})}. \]
Notice that
\[ L(s, \Pi, r) = (1 - \alpha^3 p^{-s})^{-1}(1 - \alpha p^{-s})^{-1}(1 - \alpha^{-1} p^{-s})^{-1}(1 - \alpha^{-3} p^{-s})^{-1}. \]
This proves (2).
Suppose \( \Pi = \text{St}_E \otimes \chi \), where \( \chi \) is the non-trivial unramified quadratic character of \( E^\times \). If \( E/F \) is unramified,
\[ I(\Pi, t) = \int_{F^\times \backslash \text{GL}_2(F)} \Phi_\Pi(h) \, dh. \]
By (4.5) and Lemma 3.9 we obtain
\[ 1, \]
where
\[ \iota \rightarrow x \rightarrow dh \]
its maximal compact subring. Then
\[ \text{unramified extension over } F \]
\[ \text{non-archimedean and } \]
\[ \text{is ramified, similar calculations show} \]
\[ = (1 + q)^{-1} \left( 1 - \chi(\varpi) \frac{1 + \chi(\varpi) q^{-2}}{1 - \chi(\varpi) q^{-2}} \right). \]
When \( E/F \) is ramified, similar calculations show
\[ I^*(\Pi, t) = \frac{2q^{-1}}{1 + q^{-1}}. \]
On the other hand, the \( L \)-factors are
\[ L(s, \Pi, r) = \begin{cases} (1 - \chi(\varpi) q^{-s-3/2})^{-1} (1 + \chi(\varpi) q^{-s-1/2} + q^{-2s-1})^{-1} & \text{if } E/F \text{ is unramified}, \\ (1 - \chi(\varpi) q^{-s-3/2})^{-1} & \text{if } E/F \text{ is ramified}. \end{cases} \]
This completes the proof. \( \square \)

5. The Calculation of Local Zeta Integral: (II)

The purpose of this section is to compute the normalized zeta integral \( I^*(\Pi, t) \) in Definition 2.2. when \( D \) is a division algebra over \( F \).

5.1. Haar Measures. Haar measures on \( F \) and \( F^\times \) are the same as in \[4.1\]. We describe the choice of Haar measures on \( D^\times (F) \). When \( F = \mathbb{R} \), let \( dh \) be the Haar measure on \( D^\times (\mathbb{R}) \) such that \( \text{Vol}(D^\times (\mathbb{R})/\text{GL}_2(\mathbb{R}), dh/d^x t) = 1 \), where \( d^x t = |t|_{\mathbb{R}}^{-1} dt \) and \( dt \) is the usual Lebesgue measure on \( \mathbb{R} \). When \( F \) is non-archimedean, let \( \mathcal{O}_D \) be its maximal compact subring. Then \( dh \) is chosen so that \( \text{Vol}(\mathcal{O}_D, dh) = 1 \).

In any cases, the measure on the quotient space \( F^\times \backslash D^\times (F) \) is the unique quotient measure induced from the measure \( dh \) on \( D^\times (F) \) and the measure \( d^x x \) on \( F^\times \).

5.2. Embeddings. We fix various embeddings in this section. Following results depend on these embeddings. When \( F = \mathbb{R} \), we embedded \( D(\mathbb{R}) \) in \( M_2(\mathbb{C}) \) in the usual way. More precisely, we let
\[ D(\mathbb{R}) = \mathcal{H} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}) \right\}. \]
When \( F \) is non-archimedean and \( E = K \times F \), we have \( D(E) = M_2(K) \times D(F) \) and we fix an embedding:
\[ \iota : D(F) \rightarrow M_2(K), \]
so that
\[ \iota(D(F)) = \left\{ \frac{\alpha}{\omega \beta} \frac{\beta}{\bar{\alpha}} \mid \alpha, \beta \in K \right\}, \]
where \( x \mapsto \bar{x} \) is the non-trivial Galois action on \( x \in K \), and \( \omega \) is either \( \varpi \) or a unit \( u \) such that \( F(\sqrt{\omega}) \) is the unramified extension over \( F \), according to \( K \) is unramified or ramified over \( F \). We then identify \( D(F) \) with its image under the embedding \( \iota \). The maximal order \( \mathcal{O}_D \) in \( D(F) \) is then
\[ \left\{ \frac{\alpha}{\omega \beta} \frac{\beta}{\bar{\alpha}} \mid \alpha, \beta \in \mathcal{O}_K \right\}. \]
Let
\[ \varpi_D = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix} \quad \text{or} \quad \varpi_D = \begin{pmatrix} \varpi_K & 0 \\ 0 & -\varpi_K \end{pmatrix}, \]
according to $K$ is unramified or ramified over $F$. We have

$$F^\times \setminus D^\times (F) = \left( \mathcal{O}_F^\times \setminus \mathcal{O}_D^\times \right) \sqcup \varpi_D \left( \mathcal{O}_F^\times \setminus \mathcal{O}_D^\times \right).$$

Note that

$$\text{Vol}(\mathcal{O}_F^\times \setminus \mathcal{O}_D^\times, dh) = 1,$$

according to our choice of measures.

5.3. **The archimedean case.** In this case, we have following realizations

$$(\pi_j, V_{\pi_j}) = (\rho_{k_j}, \mathcal{L}_{k_j}(C))$$

for $j = 1, 2, 3$, where

$$\mathcal{L}_{k_j}(C) = \bigoplus_{n_j=0}^{k_j-2} C \cdot X_j^{n_j} \cdot Y_j^{k_j-2-n_j}$$

with $\rho_{k_j}(g) P(X_j, Y_j) = P((X_j, Y_j) g) \det(g)^{-k_j/2-1},$

for $g \in D^\times (\mathbb{R})$ and $P(X_j, Y_j) \in \mathcal{L}_{k_j-2}(C)$. The representation space of $\Pi$ is given by

$$\mathcal{V}_\Pi = \mathcal{L}_{k_1}(C) \otimes \mathcal{L}_{k_2}(C) \otimes \mathcal{L}_{k_3}(C).$$

The new line $V_{\Pi}^{\text{new}}$ in this case is the one dimensional subspace fixed by $D^\times (\mathbb{R})$. Let $P_{\underline{k}}$ be the distinguished vector in $V_{\Pi}^{\text{new}}$ defined by

$$P_{\underline{k}} = \text{det} \begin{pmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{pmatrix}^{k_1} \otimes \text{det} \begin{pmatrix} X_2 & X_3 \\ Y_2 & Y_3 \end{pmatrix}^{k_2} \otimes \text{det} \begin{pmatrix} X_3 & X_1 \\ Y_3 & Y_1 \end{pmatrix}^{k_3}$$

where $k_1 = (k_1 + k_2 - k_3 - 2)/2$, $k_2 = (k_2 + k_3 - k_1 - 2)/2$ and $k_3 = (k_1 + k_3 - k_2 - 2)/2$. Its clear that $P_{\underline{k}}$ is non-zero and invariant by $D^\times (\mathbb{R})$. Therefore, we have

$$V_{\Pi}^{\text{new}} = C \cdot P_{\underline{k}}.$$

Let $\langle , \rangle_{k_j}$ be the $D^\times (\mathbb{R})$-invariant bilinear pairing on $\mathcal{L}_{k_j-2}(C)$ defined by

$$\langle X_j^{n_j} Y_j^{k_j-2-n_j}, X_j^{m_j} Y_j^{k_j-2-m_j} \rangle_{k_j} = \begin{cases} (-1)^{n_j} \binom{k_j - 2}{n_j}^{-1} & \text{if } n_j + m_j = k_j - 2, \\
0 & \text{if } n_j + m_j \neq k_j - 2, \end{cases}$$

for $0 \leq n_j, m_j \leq k_j - 2$. Let $\langle , \rangle_{\underline{k}}$ be the $D^\times (\mathbb{R})$-invariant pairing on $V_{\Pi}$ given by

$$\langle , \rangle_{\underline{k}} = \langle , \rangle_{k_1} \otimes \langle , \rangle_{k_2} \otimes \langle , \rangle_{k_3}.$$ 

In this case, the normalized local zeta integral $I^\star(\Pi, t)$ in Definition 2.12 is equal to

$$I^\star(\Pi, t) = \frac{\zeta_F(2)}{\zeta_E(2)} \cdot \frac{L(1, \Pi', \text{Ad})}{L(1/2, \Pi'', r)} \cdot \langle P_{\underline{k}}, P_{\underline{k}} \rangle_{\underline{k}}.$$

where $\Pi'$ is the Jacquet-Langlands lift of $\Pi$ to $GL_2(\mathbb{R})$. We proceed to compute the value $\langle P_{\underline{k}}, P_{\underline{k}} \rangle_{\underline{k}}$. Let $\ell$ be the linear map

$$\ell : V_{\Pi} \to V_{\Pi}^{\text{new}}(\mathbb{R}) = V_{\Pi}^{\text{new}}, \quad v \mapsto \ell(v) = \int_{R^\times \setminus D^\times (\mathbb{R})} \Pi(h) v \, dh.$$

Since $\ell(P_{\underline{k}}) = P_{\underline{k}} \neq 0$, we have $\ell \neq 0$ and hence surjective. We have the following equality

$$\langle P_{\underline{k}}, P_{\underline{k}} \rangle_{\underline{k}} \cdot \langle \ell(v_1), \ell(v_2) \rangle_{\underline{k}} = \langle v_1, P_{\underline{k}} \rangle_{\underline{k}} \cdot \langle v_2, P_{\underline{k}} \rangle_{\underline{k}}$$

for every $v_1, v_2 \in \mathcal{L}(C)$.

**Proposition 5.1.** We have

$$I^\star(\Pi, t) = \frac{(k_1 - 1)(k_2 - 1)(k_3 - 1)}{4 \pi^2}.$$
Proof. Note that the $L$-factor is given by
\[
L(s, \Pi, r) = \zeta_C(s + (k_1 + k_2 + k_3 - 3)/2))\zeta_C(s + (-k_1 + k_2 + k_3 - 1)/2) \\
\times \zeta_C(s + (k_1 - k_2 - k_3 + 1)/2)\zeta_C(s + (k_1 + k_2 - k_3 - 1)/2).
\]
In view of (5.6), it suffices to show that
\[
(\mathbf{P}_k \cdot \mathbf{P}_k)_k = \frac{\Gamma(k_1^* + k_2^* + k_3^* + 2)\Gamma(k_1^* + 1)\Gamma(k_2^* + 1)\Gamma(k_3^* + 1)}{\Gamma(k_1^* + k_2^* + 1)\Gamma(k_1^* + k_3^* + 1)\Gamma(k_2^* + k_3^* + 1)}.
\]
By direct computation, we have
\[
\mathbf{P}_k = \sum_{n_1=0}^{k_1^*} \sum_{n_2=0}^{k_2^*} \sum_{n_3=0}^{k_3^*} \binom{k_1^*}{n_1} \binom{k_2^*}{n_2} \binom{k_3^*}{n_3} (-1)^{(k_1^*+k_2^*+k_3^*)-(n_1+n_2+n_3)} X_1^{k_2^*-n_2+n_3} Y_1^{k_3^*-n_2+n_3} \otimes X_2^{k_1^*-n_1-n_3} Y_2^{k_1^*-n_1-n_3} \otimes X_3^{k_1^*-n_1+n_2} Y_3^{k_2^*-n_1+n_2}.
\]
The coefficient in front of the vector $v_1 := X_1^{k_1^*} \otimes Y_2^{k_2^*} \otimes X_3^{k_3^*}$ in the expression of $\mathbf{P}_k$ is equal to $(-1)^{k_1^*+k_2^*}$. On the other hand, the coefficient in front of the vector $v_2 := Y_1^{k_1^*} \otimes X_2^{k_2^*} \otimes X_3^{k_3^*}$ in $\mathbf{P}_k$ is $(-1)^{k_1^*}$. It follows that
\[
(\langle v_1, \mathbf{P}_k \rangle_k \cdot \langle v_2, \mathbf{P}_k \rangle_k = (-1)^{k_1^*+k_2^*+k_3^*} \langle v_1, v_2 \rangle_k = (-1)^{k_1^*+k_2^*+k_3^*} \left( k_1^* + k_2^* \right)^{-2}.
\]
On the other hand, we have
\[
\langle \ell(v_1), \ell(v_2) \rangle_k = \int_{\mathbf{R}^3 \setminus D^\times} \langle \Pi(h)v_1, v_2 \rangle_k dh.
\]
Note that
\[
\mathbf{R}^3 \setminus D^\times \cong \{ \pm 1 \} \setminus \text{SU}(2).
\]
We parametrize $u = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \text{SU}(2)$ by setting $\alpha = \cos \theta \cdot e^{i\phi}$ and $\beta = \sin \theta \cdot e^{i\chi}$ with $0 \leq \theta \leq \pi/2$ and $0 \leq \varphi, \chi \leq 2\pi$. For $\Phi \in L^1(\text{SU}(2))$, we have
\[
\int_{\text{SU}(2)} \Phi(u) du = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \Phi(\theta, \varphi, \chi) \cdot \sin 2\theta d\theta d\varphi d\chi.
\]
Our choice of the Haar measure on $\mathbf{R}^3 \setminus D^\times$ implies the total volume of $\text{SU}(2)$ is equal to 2.
Let $u = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \text{SU}(2)$. By (5.9), we have
\[
\int_{\mathbf{R}^3 \setminus D^\times} \langle \Pi(h)v_1, v_2 \rangle_k dh = \frac{1}{2} \int_{\text{SU}(2)} \langle \Pi(u)v_1, v_2 \rangle_k du
\]
\[
= (-1)^{k_1^*+k_2^*+k_3^*} \left( k_1^* + k_2^* \right)^{-1} \sum_{j=0}^{k_1^*} \binom{k_1^*}{j} \binom{k_2^*}{j} \frac{(-1)^j}{2j} \int_{\text{SU}(2)} \left| \alpha \right|^{k_1^*+k_2^*+k_3^*} \left| \beta \right|^j C \, du
\]
\[
= (-1)^{k_1^*+k_2^*+k_3^*} \frac{(-1)^j}{j} \binom{k_1^*}{j} \binom{k_2^*}{j} \binom{k_3^*}{j}.
\]
Its clear that the coefficient of the term $X^{5.4}$. cases being considered in the following proposition.

algebra over $F$. According to the results of Prasad [Pra90], [Pra92] and our assumption on $\mathcal{H}$, the last equality follows from Lemma 5.2 below. This completes the proof of Proposition 5.1.

Lemma 5.2. Let $a, b$ and $n$ be non-negative integers. Suppose $a \geq n$. Then we have

(5.10) \[ \sum_{j=0}^{n} (-1)^{j} \binom{a}{j} \left(\frac{a+b+n-j}{a+b}\right) = \binom{b+n}{b}. \]

Proof. Consider the function

\[ f(X) = \sum_{j=0}^{a} (-1)^{j} \binom{a}{j} \left(1 + X\right)^{a+b+n-j} \]

where $X$ is a variable. Since we have assumed $a \geq n$, the coefficient of the term $X^{a+b}$ in $f(X)$ is the right hand side of (5.10). On the other hand, we have

\[ f(X) = \sum_{j=0}^{a} (-1)^{j} \binom{a}{j} \left(1 + X\right)^{a+b+n-j} \]

\[ = (1 + X)^{a+b+n} \sum_{j=0}^{a} (-1)^{j} \binom{a}{j} \left(1 + X\right)^{-j} \]

\[ = (1 + X)^{a+b+n} (1 - (1 + X)^{-1})^a = X^a (1 + X)^{b+n}. \]

It is clear that the coefficient of the term $X^{a+b}$ in $X^a (1 + X)^{b+n}$ is equal to the left hand side of (5.10). This finishes the proof of lemma.

5.4. The non-archimedean case. Let $F$ be a non-archimedean local field and $D$ is the quaternion division algebra over $F$. Recall that we have assumed

$\text{Hom}_{D^+}(\mathcal{H}, \mathbb{C}) \neq \{0\}$.

According to the results of Prasad [Pra90], [Pra92] and our assumption on $\mathcal{H}$, this happens precisely for the cases being considered in the following proposition.

Proposition 5.3. Let $\nu_D : D^+ \to \mathbb{G}_m$ be the reduced norm of $D$.

1. Let $E = F \times F \times F$. If $\pi_j = \chi_j \circ \nu_D$, where $\chi_j$ is a unramified quadratic character of $F^\times$ with $\chi_1 \chi_2 \chi_3(\varpi) = 1$. Then we have

$\mathcal{I}^*(\mathcal{H}, \mathfrak{t}) = 2(1 - q^{-1})^2$

2. Let $E = K \times F$ and $\pi = \chi \circ \nu_D$, where $\chi$ is a unramified quadratic character of $F^\times$. Then we have

$\mathcal{I}^*(\mathcal{H}, \mathfrak{t}) = \begin{cases} 
2 & \text{if } \pi' \text{ is spherical and } K/F \text{ is unramified}, \\
2(1 + q^{-2}) & \text{if } \pi' \text{ is spherical, } \chi(\varpi) = 1 \text{ and } K/F \text{ is ramified}.
\end{cases}$

Here $\chi'$ is a unramified quadratic character of $K^\times$. 
(3) Let $E$ be a field. If $\Pi$ is the trivial character of $D^\times(E)$, then we have

$$I^*(\Pi, t) = \begin{cases} 2(1+q^{-1}+q^{-2}) & \text{if } E/F \text{ is unramified}, \\ 2 & \text{if } E/F \text{ is ramified}. \end{cases}$$

Here $I^*(\Pi, t)$ is the local zeta integral in Definition 6.2.

**Proof.** We first treat (1). Since $\chi_1\chi_2\chi_3(\varpi) = 1$, we have

$$I(\Pi, t) = \int_{F^\times \backslash D^\times(F)} \chi_1\chi_2\chi_3(\nu_D(h))dh = \Vol(F^\times \backslash D^\times(F), dh) = 2.$$ 

The $L$-factor is

$$L(s, \Pi') = (1 - \chi_1\chi_2\chi_3(\varpi)q^{-s-1/2} - 2(1 - \chi_1\chi_2\chi_3(\varpi)q^{-s-3/2} - 1,$$

where $\Pi'$ is the Jacquet-Langlands lift of $\Pi$ to $GL_2(F)$. This shows (1).

We proceed to show (2). Suppose $\pi'$ is spherical. Then by Lemma 6.8 and (5.1), we find that

$$I(\Pi, t) = \int_{F^\times \backslash D^\times(F)} \Phi_{\pi'}(h)\pi(h)dh = 1 + \Phi_{\pi'}(\varpi_D)\chi(\varpi)$$

$$= \begin{cases} (1+q^{-2})^{-1}(1+\chi(\varpi)aq^{-1})(1+\chi(\varpi)\alpha^{-1}q^{-1}) & \text{if } K/F \text{ is unramified}, \\ 2 & \text{if } K/F \text{ is ramified}. \end{cases}$$

Suppose $\pi' = \St_K \otimes \chi'$. In this case,

$$I(\Pi, t) = \int_{F^\times \backslash D^\times(F)} \Phi_{\pi'}(h)\pi(h)dh = 1 + \Phi_{\pi'}(\varpi_D)\chi(\varpi) = (1 - \chi'(\varpi)).$$

Here we use Lemma 6.8 and the observation that $O_D^\times$ is contained in the Iwahori subgroup of $GL_2(K)$. The $L$-factors are given in Proposition 6.7 and Proposition 6.8. This shows (2).

For the case (3), we have

$$I(\Pi, t) = \Vol(F^\times \backslash D^\times(F), dh) = 2.$$ 

The $L$-factors are given in Proposition 6.9. This completes the proof. □

6. Explicit central value formulae and algebraicity for triple product

The purpose of this section is to give explicit central value formulae for the triple product $L$-functions by combining Ichino’s formula [Ich08 Theorem 1.1 and Remark 1.3] with the local calculations in the previous sections. We use these formulae to prove the algebraicity of the central values.

Since the work of Garrett [Gar79], special values of triple product $L$-functions have been studied extensively by many people such as Orloff [Orl87], Satoh [Sat87], Harris and Kudla [HK91], Garrett and Harris [GH93], Gross and Kudla [GK92], Bocherer and Schulze-Pillot [BSP96], Furusawa and Morimoto [FM04], [FM06].

6.1. Notation. We fix some notations here. If $F$ is a number field, let $O_F$ be its ring of integers, $\mathcal{D}_F$ be its absolute discriminant, and $h_F$ be its class number. Let $A$ be the ring of adeles of $Q$ and $\hat{Z} = \prod_p Z_p$ be the profinite completion of $Z$. We will denote by $v$ a place of $Q$ and by $p$ a finite prime of $Q$. If $R$ is a $Q$-algebra, let $A_R = A \otimes_Q R$ and $R_\psi = R \otimes Q \hat{Q}_\psi$. For an abelian group $M$, let $\hat{M} = M \otimes_{\hat{Z}} \hat{Z}$.

We fix an additive character $\psi = \prod_p \psi_p : \hat{Q} \setminus A \to \mathbb{C}^\times$ defined by $\psi_\infty(x) = e^{2\pi \sqrt{-1}x}$ for $x \in \mathbb{R}$, and $\psi_p(x) = e^{-2\pi \sqrt{-p}x}$ for $x \in Z[p^{-1}]$.

6.2. Modular forms and Automorphic forms. We briefly review the definitions of modular forms and automorphic forms on certain quaternion algebras, and we write down an explicit correspondence between them. We follow the exposition of [Shi81 section 1], but with some modifications, so that it will be suitable for our application here.

We first introduce some notations. Let $d \geq 1$ be an integer and $\mathfrak{h}^d$ be the $d$-fold product of the upper half complex plane $\mathfrak{h}$. Let $GL_2^+(\mathbb{R})$ be the identity connect component of $GL_2(\mathbb{R})$. If $d = 1$, we let $h \in GL_2^+(\mathbb{R})$
acting on \( z \in \mathcal{O} \) and we define the factor \( J(h, z) \) by

\[
h \cdot z = \frac{az + b}{cz + d},
\]

\[
J(h, z) = \det(g)^{-\frac{1}{2}}(cz + d) \quad h = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).
\]

In general, we let \( \text{GL}_d^+(\mathbb{R})^d \) acting on \( \mathcal{O} \) component-wise. If \( \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d \), we put

\[
J(h, z)^{\mathbf{k}} = \prod_{j=1}^{d} j(h(j_z), z)^{k_j} \quad \text{for} \quad h = (h_1, \ldots, h_d) \in \text{GL}_d^+ (\mathbb{R})^d \quad \text{and} \quad z = (z_1, \ldots, z_d) \in \mathcal{O}.
\]

Let \( C^\infty(\mathcal{O}) \) be the space of \( C \)-valued smooth functions on \( \mathcal{O} \). Let \( k \) be an integer. Recall the Maass-Shimura differential operators \( \delta_k \) and \( \varepsilon \) on \( C^\infty(\mathcal{O}) \) are given by

\[
\delta_k = \frac{1}{2\pi \sqrt{-1}} \left( \frac{\partial}{\partial z} + \frac{k}{2\sqrt{-1}y} \right) \quad \text{and} \quad \varepsilon = -\frac{1}{2\pi \sqrt{-1}} y \frac{\partial}{\partial z} \quad y = \text{Im}(z)
\]

(cf. [Hi93] page 310). If \( m \geq 0 \) is an integer, we put \( \delta^m_k = \delta_{k + 2m - 2} \cdots \delta_{k + 2} \delta_k \). In general, if \( \mathbf{k} = (k_1, \ldots, k_d) \), \( \mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d \) with \( m_j \geq 0 \) for \( 1 \leq j \leq d \), we let \( \delta^{\mathbf{m}}_{\mathbf{k}} \) and \( \varepsilon^\mathbf{m} \) be given by

\[
\delta^{\mathbf{m}}_{\mathbf{k}} = (\delta_{k_1}^{m_1}, \ldots, \delta_{k_d}^{m_d}) \quad \text{and} \quad \varepsilon^\mathbf{m} = (\varepsilon^{m_1}, \ldots, \varepsilon^{m_d}),
\]

and acting on \( f \in C^\infty(\mathcal{O}) \) coordinate-wise.

Let \( F \) be a totally real number field over \( \mathbb{Q} \) with degree \( d = [F : \mathbb{Q}] \). Let \( \Sigma_F := \text{Hom}_\mathbb{Q}(F, \mathbb{C}) \) and \( \mathcal{O}_{F} \) be the \( d \)-fold product of \( \mathcal{O} \). Let \( D \) be a quaternion algebra over \( F \). Let \( G = D^{\times} \) viewed as an algebraic group defined over \( F \). For any \( F \)-algebra \( L \), \( G(L) = (D \otimes_F L)^{\times} \). We assume \( D \) is either totally definite or totally definite. In other words, we assume either \( G(F_\infty) \cong \text{GL}_2(\mathbb{R})^{\Sigma_F} \) or \( G(F_{\infty}) \cong (\mathbb{H}^\times)^{\Sigma_F} \), where \( \mathbb{H} \) is the Hamiltonian quaternion algebra.

6.2.1. The totally indefinite case. Let \( \mathbf{k} = (k_\sigma)_{\sigma \in \Sigma_F}, \mathbf{m} = (m_\sigma)_{\sigma \in \Sigma_F} \in \mathbb{Z}^{\Sigma_F} \) with \( k_\sigma > 0 \) and \( m_\sigma \geq 0 \) for all \( \sigma \in \Sigma_F \). The zero and the identity element \( \mathbb{Z}^{\Sigma_F} \) will be denoted by \( \mathbf{0} \) and \( \mathbf{1} \), respectively. Let \( U \subset G(\mathbb{F}) \) be an open compact subgroup. We assume \( \nu_D(U) = \mathcal{O}_{F}^\times \), where \( \nu_D \) is the reduced norm of \( D \) and we extend it to a map on \( D \otimes_F \mathbb{F} \) in an obvious way.

Denote by \( \mathcal{N}_k^{(m)}(D, F; U) \) the space of functions \( f : \mathcal{O}_{F} \times G(\mathbb{F}) \to \mathbb{C} \) such that \( f(z, ahu) = f(z, h) \) for \( z \in \mathcal{O}_{F} \) and \( (a, h, u) \in \mathbb{F}^{\times} \times G(\mathbb{F}) \times U \). Also, for each \( h \in G(\mathbb{F}) \), the function \( f_h(z) := f(z, h) \in C^\infty(\mathcal{O}_{F}) \) is slowly increasing and \( \varepsilon^\mathbf{m} + \frac{1}{2} \text{f}_h = 0 \). Finally, it satisfies the following automorphism condition:

\[
(6.1) \quad f_h(\gamma \cdot z)J(\gamma, z)^{-\mathbf{k}} = f_h(z), \quad \gamma \in G(\mathbb{F}) \cap \left( G^{+}(F_{\infty}) \times hUh^{-1} \right),
\]

where \( G^{+}(F_{\infty}) \) is the identity connect component of \( G(F_{\infty}) \). We put \( \mathcal{N}_k(D, F; U) = \bigcup_m \mathcal{N}_k^{(m)}(D, F; U) \). Notice that if \( f \in \mathcal{N}_k(D, F; U) \), then \( \delta^m f \in \mathcal{N}_{k+2m}(D, F; U) \) (cf. [Hi93] page 312). Assume \( D = M_2 \) is the matrix algebra. Let \( \mathfrak{n} \subset \mathcal{O}_{F} \) be an ideal. Put

\[
K_0(\mathfrak{n}) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}_2(\mathcal{O}_{F}) \mid c \in \mathfrak{n} \right\}.
\]

Then \( \mathcal{N}_k^{(0)}(M_2, F; K_0(\mathfrak{n})) = \mathcal{A}_k(M_2, F; K_0(\mathfrak{n})) \) is the space of holomorphic Hilbert modular forms of \( F \) of weight \( k \) and level \( \mathfrak{n} \). Let \( \mathcal{S}_k(M_2, F; K_0(\mathfrak{n})) \) be the subspace of holomorphic cusp forms in \( \mathcal{M}_k(M_2, F; K_0(\mathfrak{n})) \).

We also define a subspace of automorphic forms on \( G(\mathcal{A}_{F}) \) as follows. Let \( \mathbf{k} \) and \( U \) be as above. We identify \( U \) and \( G(F_{\infty}) \) with subgroups of \( G(\mathcal{A}_{F}) \) in an obvious way. Let \( \mathcal{A}_k(D, F; U) \) be the space of automorphic forms \( f : G(\mathcal{A}_{F}) \to \mathbb{C} \) (cf. [Bj79] section 4) such that

\[
f(\alpha \gamma h k(\theta) u) = f(h)e^{\sqrt{-1}k\theta}, \quad k \cdot \theta = \sum_{\sigma \in \Sigma_F} k_\sigma \theta_\sigma,
\]

\[
(a \in A_{\mathcal{F}}, \quad \gamma \in G(F), \quad \theta = (\theta_\sigma)_{\sigma \in \Sigma_F}, \quad k(\theta) = (k(\theta_\sigma))_{\sigma \in \Sigma_F}, \quad k(\theta) = \left( \begin{array}{cc} \cos \theta_\sigma & \sin \theta_\sigma \\ -\sin \theta_\sigma & \cos \theta_\sigma \end{array} \right), \quad u \in U).
\]
Denote by $A^0_k(D, F; U)$ the subspace of cusp forms in $A_k(D, F; U)$ (if there is no such, we let $A^0_k(D, F; U)$ be the whole space). Suppose $F = \mathbb{Q}$. Let $\tilde{V}_\pm : A_k(D, F; U) \rightarrow A_{k \pm 2}(D, F; U)$ be the normalized weight raising/lowering elements ([BL70] page 155) given by

$$
\tilde{V}_\pm = -\frac{1}{8\pi} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes 1 \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \sqrt{-1} \in \text{Lie}(\text{GL}_2(\mathbb{R})) \otimes \mathbb{R} \mathbb{C}.
$$

In general, we have $\tilde{V}_\pm^{\sigma} : A_k(D, F; U) \rightarrow A_{k \pm 2\sigma}(D, F; U)$, where $\tilde{V}_\pm^{\sigma}(\cdot)_\sigma \in \Sigma_F$ acts on the archimedean component of $f \in A_k(D, F; U)$ coordinate-wisely.

We write down an explicit correspondence between the spaces $N_k(D, F; U)$ and $A_k(D, F; U)$. Fix a set of representatives $\{x_1, \cdots, x_h\}$ for the double cosets $G(F) \backslash G(\mathbb{A}_F)/G^+(F) \backslash U$. Then

$$
G(\mathbb{A}_F) = \Pi_{j=1}^h G(F)x_j G^+(F) \backslash U
$$

is a disjoint union. We may assume every archimedean component of $x_j$ is one for $1 \leq j \leq r$, and we regard $x_j$ as elements in $G(\hat{F})$. For each $f \in N_k(D, F; U)$, we define $\Phi(f) \in A_k(D, F; U)$ the adelic lift of $f$ by the formulae

$$
\Phi(f)(\gamma x_j h u) = f(x_j(h \cdot 1), j h u, 1)_{\pm}^k, \quad i = (\sqrt{-1}, \cdots, \sqrt{-1}) \in \mathbb{I}^{\Sigma_F},
$$

$$(\gamma \in G(F), h_u \in G^+(F) \backslash U, u \in U, 1 \leq j \leq h).$$

Conversely, we can recover $f$ form $\Phi(f)$ by setting

$$
f(z, h) = \Phi(f)(h_u h, j h u, 1)_{\pm}^k, \quad h_u \in G^+(F) \backslash U, h \cdot 1 = z.
$$

The weight raising/lowering operators are the adelic version of the Maass-Shimura differential operators $\delta^m_k$ and $\varepsilon^m_k$ on the space of automorphic forms. More precisely, one check that

$$
\tilde{V}_\pm^k \Phi(f) = \Phi(\delta^m_k f) \quad \text{and} \quad \tilde{V}_\pm^m \Phi(f) = \Phi(\varepsilon^m_k f).
$$

In particular, $f$ is holomorphic if and only if $\tilde{V}_\pm^k \Phi(f) = 0$.

6.2.2. $D$ is totally definite. Let $k = (k_\sigma)_{\sigma \in \Sigma_F}$ and $U$ be as above. We assume $k_\sigma \geq 2$ for all $\sigma \in \Sigma_F$. We identify $G(F_\infty)$ with $(H^\times)^{\Sigma_F} \subset GL_2(\mathbb{C})^{\Sigma_F}$. Let $(\rho_{k_\sigma}, L_{k_\sigma}(\mathbb{C}))$ be the $(k_\sigma - 1)$-dimensional irreducible representation of $H^\times$, and $\langle \cdot, \cdot \rangle_{k_\sigma}$ be the bilinear pairing on $L_{k_\sigma}(\mathbb{C})$ defined in [BL73] respectively. We form an irreducible representation $(\rho_{k}, L_{k}(\mathbb{C}))$ of $G(F_\infty)$ by setting

$$
\rho_{k} = \otimes_{\sigma \in \Sigma_F} \rho_{k_\sigma}, \quad L_k(\mathbb{C}) = \otimes_{\sigma \in \Sigma_F} L_{k_\sigma}(\mathbb{C}).
$$

Then $\langle \cdot, \cdot \rangle_k = \otimes_{\sigma \in \Sigma_F} \langle \cdot, \cdot \rangle_{k_\sigma}$ defines a bilinear pairing on $L_k(\mathbb{C})$.

Let $M_k(D, F; U)$ be the space of $L_k(\mathbb{C})$-valued automorphic forms of type $\rho_k$, which consists of functions $f : G(\mathbb{A}_F) \rightarrow L_k(\mathbb{C})$ such that

$$
f(a \gamma h u) = \rho_k(h_u)^{-1} f(h),
$$

$$(a \in A^\times_F, \gamma \in G(F), h_u \in G(F_\infty), u \in U).
$$

Let $A(G(\mathbb{A}_F))$ be the space of $\mathbb{C}$-valued automorphic forms on $G(\mathbb{A}_F)$ (Cf. [BL79] section 4). For $v \in L_k(\mathbb{C})$ and $f \in M_k(D, F; U)$, we define a function $\Phi(v \otimes f) : G(F) \backslash G(\mathbb{A}_F) \rightarrow \mathbb{C}$ by

$$
\Phi(v \otimes f)(h) = \langle v, f(h) \rangle_k.
$$

Then the map $v \otimes f \mapsto \Phi(v \otimes f)$ gives rise to a $G(F_\infty)$-equivariant morphism $L_k(\mathbb{C}) \rightarrow A(G(\mathbb{A}_F))$ for every $f \in M_k(D, F; U)$. Let $A_k(D, F; U)$ the subspace of $A(\mathbb{A}_F)$, consisting of functions $\Phi(v \otimes f) : G(\mathbb{A}_F) \rightarrow \mathbb{C}$ for $v \in L_k(\mathbb{C})$ and $f \in M_k(D, F; U)$.

More generally, suppose $F = F_1 \times \cdots \times F_r$, where $F_j$ are totally real number fields. Let $D$ be a quaternion $\mathbb{Q}$-algebra and put $D_{F_j} = D \otimes \mathbb{Q} F_j$, $D_F = D \otimes \mathbb{Q} F$. Let $U_j \subset G_j(\hat{F}_j)$ be open compact subgroups, where $G_j := D_{F_j}^\times$ viewed as an algebraic group defined over $F_j$. Let $\mathbb{Z}^{\Sigma_F}$ be sets of positive integers. Put $U = (U_1, \ldots, U_r)$ and $\mathbb{k} = (k_1, \ldots, k_r)$. If $D$ is definite, we define

$$
M_k(D, F; U) = \otimes_{j=1}^r M_k(D_{F_j}, F_j; U_j) \quad \text{and} \quad A_k(D, F; U) = \otimes_{j=1}^r A_k(D_{F_j}, F_j; U_j).
$$

If $D$ is indefinite, similar definitions apply to the spaces $N_k(D, F; U)$ and $A_k(D, F; U)$.
6.3. **Global settings.** Let $E$ be an étale cubic $\mathbb{Q}$-algebra. Then $E$ is (i) $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ three copies of $\mathbb{Q}$, or (ii) $F \times \mathbb{Q}$, where $F$ is a quadratic extension of $\mathbb{Q}$, or (iii) $E$ is a field. Let $\mathcal{O}_E$ be the maximal order in $E$ and let $D_E$ be the absolute discriminant of $E$. Put

$$
(6.3) 
\epsilon = \begin{cases} 
3 & \text{if } E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}, \\
2 & \text{if } E = F \times \mathbb{Q}, \\
1 & \text{if } E \text{ is a cubic extension of } \mathbb{Q}.
\end{cases}
$$

Here $F$ is a quadratic extension over $\mathbb{Q}$. We assume

$$
(6.4) 
E_\infty = E \otimes_\mathbb{Q} \mathbb{R} \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}.
$$

In particular, $F$ is a real quadratic extension over $\mathbb{Q}$, and $E$ is a real cubic extension over $\mathbb{Q}$ if it is a field.

Let $\mathfrak{n} \subset \mathcal{O}_E$ be an ideal. We have $\mathfrak{n} = (N_1 \mathbb{Z}, N_2 \mathbb{Z}, N_3 \mathbb{Z})$ or $\mathfrak{n} = (n_\mathbb{F}, N \mathbb{Z})$ according to $E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ or $E = F \times \mathbb{Q}$, respectively. Here $N_j, N (j = 1, 2, 3)$ are positive integers and $n_\mathbb{F}$ is an ideal of $\mathcal{O}_F$. Let $\mathbb{k} = (k_1, k_2, k_3)$ be a triple of positive even integers with $k_j \geq 2$ for $j = 1, 2, 3$. We put

$$
(6.5) 
w = k_1 + k_2 + k_3 - 3.
$$

Let $f_E \in \mathcal{M}_k(M_2, E; K_0(\mathfrak{n}))$ be a normalized Hilbert newform of weight $\mathbb{k}$ and level $K_0(\mathfrak{n})$ (cf. [Shi78] page 652). More precisely, if $E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, then $f_E = f_1 \otimes f_2 \otimes f_3$, where $f_j \in \mathcal{S}_{k_j}(M_1, \mathbb{Q}; K_0(N_j \mathbb{Z}))$ is a normalized newform of weight $k_j$ and level $K_0(N_j \mathbb{Z})$. On the other hand, if $E = F \times \mathbb{Q}$, then $f_E = g_F \otimes f$, where $g_F \in \mathcal{S}_{k_1}(M_2, F; K_0(\mathfrak{n}_F))$ is a normalized Hilbert newform of weight $(k_1, k_2)$ and level $K_0(\mathfrak{n}_F)$, and $f \in \mathcal{S}_{k_3}(M_2, Q; K_0(N \mathbb{Z}))$ is a normalized newform of weight $k_3$ and level $K_0(N \mathbb{Z})$. Let $f_E = \Phi(f_E)$ be the adelic lift to $\mathcal{A}_k(M_2, E; K_0(\mathfrak{n}))$. Let $\Pi$ be the unitary irreducible cuspidal representation of $GL_2(A_E)$ generated by $f_E$. By the tensor product theorem [Fal79], $\Pi \cong \otimes'_v \Pi_v$, where $\Pi_v$ are irreducible admissible representations of $GL_2(E_v)$. We define the $L$-function and $\epsilon$-factor associated to $\Pi$ and $r$ as product of local $L$-factors and $\epsilon$-factors. That is, we put

$$
L(s, \Pi, r) = \prod_v L(s, \Pi_v, r_v) \quad \text{and} \quad \epsilon(s, \Pi, r, \psi) = \prod_v \epsilon(s, \Pi_v, r_v, \psi_v).
$$

Note that $L(s, \Pi, r)$ is holomorphic at $s = 1/2$.

Ichino’s formula relates the period integrals of triple products of certain automorphic forms on quaternion algebras along the diagonal cycles and the central values of triple $L$-functions. To describe the choice of the quaternion algebra, we define the local root number $\epsilon(\Pi_v) \in \{\pm\}$ associated to $\Pi_v$ for each place $v$ by the following condition

$$
\epsilon(\Pi_v) = 1 \leftrightarrow \text{Hom}_*(\Pi_v, C) \neq \{0\},
$$

where $* = GL_2(Q_p)$ or $(g, K)$ according to $v = p$ or $v = \infty$, respectively.

In the following, we assume the global root number $\epsilon(\Pi)$ associated to $\Pi$ is equal to 1. Namely, we assume

$$
\epsilon(\Pi) := \prod_v \epsilon(\Pi_v) = 1.
$$

Notice that $\epsilon(\Pi_v) = 1$ for almost all $v$ by the results of [Pra90] Theorem 1.2 and [Pra92] Theorem B. By this assumption, there is a unique quaternion $\mathbb{Q}$-algebra $D$ such that $D_v$ is the division $\mathbb{Q}_v$-algebra if and only if $\epsilon(\Pi_v) = -1$. Applying [Pra90] Theorem 1.2 and [Pra92] Theorem B, we see that the Jacquet-Langlands lift $\Pi^D = \otimes'_v R_v^D$ of $\Pi$ to $D^\times(A_E)$ exists, where $R_v^D$ is a unitary irreducible admissible representation of $D^\times(E_v)$. Moreover, by the way we chose $D$, the following local root number condition is satisfied:

$$
\epsilon(\Pi_v) = \begin{cases} 
1 & \text{if } D_v \text{ is the matrix algebra}, \\
-1 & \text{if } D_v \text{ is the division algebra}.
\end{cases}
$$

Let $\Sigma_D$ be the ramification set of $D$ and $\Sigma_D^{(\infty)} \subset \Sigma_D$ be the subset without the infinite place. For each $v \notin \Sigma_D$, we fix an isomorphism $\iota_v : M_2(Q_v) \cong D \otimes_\mathbb{Q} Q_v$ once and for all. Let $O_D$ be the maximal order of $D$ such that $D \otimes \mathbb{Z} p = \mathfrak{t}_p(M_2(Z_p))$ for all $p \notin \Sigma_D$. If $R$ is a $\mathbb{Q}$-algebra, we put $D(R) := D \otimes_\mathbb{Q} R$. We introduce following
three sets of places of $\mathbb{Q}$:

\[
\begin{align*}
\Sigma_3 &= \{ v \mid E_v \cong \mathbb{Q}_v \times \mathbb{Q}_v \times \mathbb{Q}_v \}, \\
\Sigma_2 &= \{ v \mid E_v \cong K_v \times \mathbb{Q}_v \text{ for some quadratic extension } K_v \text{ of } \mathbb{Q}_v \}, \\
\Sigma_1 &= \{ v \mid E_v \text{ is a cubic extension of } \mathbb{Q}_v \}.
\end{align*}
\]

(6.8) Note that by our assumption, we have $\infty \in \Sigma_3$. Also for every $p \notin \Sigma_D$, the map $\iota_p$ induces isomorphisms $D(E_p) \cong M_2(E_p)$ and $\mathcal{O}_D \otimes_{\mathbb{Z}} \mathcal{O}_{E_p} \cong M_2(\mathcal{O}_{E_p})$, where $\mathcal{O}_{E_p}$ is the maximal order of $E_p$. For $v \notin \Sigma_2 \cap \Sigma_D$, the canonical diagonal embedding $\mathbb{Q}_v \hookrightarrow E_v$ induces a diagonal embedding $D_v \hookrightarrow D(E_v)$. On the other hand, for each $p \in \Sigma_2 \cap \Sigma_D$, we choose an isomorphism $D(K_p) \cong M_2(K_p)$ so that the embedding $D_p \hookrightarrow D(E_p) \cong M_2(K_p) \times D_p$ is the identity map in the second coordinate, and is given by the one in (5.2) for the first coordinate. In any case, we identify $D_v$ as subalgebras of $D(E_v)$ via these embeddings. Suppose $E$ is a field, we note that the finite ramification sets $\Sigma_{D(F)}$ and $\Sigma_{D(E)}$ of $D(F)$ and $D(E)$ are given by

\[
\begin{align*}
\Sigma_{D(F)} &= \{ p \in \mathcal{O}_F \text{ prime ideal } \mid p \text{ divides } p \text{ for some } p \in \Sigma_3 \cap \Sigma_D \}, \\
\Sigma_{D(E)} &= \{ p \in \mathcal{O}_E \text{ prime ideal } \mid p \text{ divides } p \text{ for some } p \in (\Sigma_1 \cup \Sigma_3) \cap \Sigma_D \}.
\end{align*}
\]

We put

\[
(6.9) \quad N^- = \prod_{p \in \Sigma_{D(F)}} p \quad \text{and} \quad \mathfrak{N}_F^- = \prod_{p \in \Sigma_{D(F)}} p \quad \text{and} \quad \mathfrak{N}_E^- = \prod_{p \in \Sigma_{D(E)}} p.
\]

Recall that $\mathfrak{n}$ is an ideal in $\mathcal{O}_E$ and $\hat{\mathfrak{n}} = \prod_p \mathfrak{n}_p$ is the closure of $\mathfrak{n}$ in $\hat{E}$. In the following, we further assume that

\[
(6.10) \quad \mathfrak{n} \text{ is square-free.}
\]

More precisely, we assume $N_1, N_2$ and $N_3$ are square-free integers if $E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ and $\mathfrak{n}_F \subset \mathcal{O}_F$, $N \in \mathbb{Z}$ are square-free if $E = F \times \mathbb{Q}$. Let

\[
(6.11) \quad M = \prod_{p \mid N_F^-(\mathfrak{n})} p.
\]

If $L > 0$ is an integer coprime to $N^-$, we denote by $R'_L$ the standard Eichler order of level $L$ contained in $\mathcal{O}_D$. Similar notation is used to indicate the standard Eichler orders of $D(F)$ and $D(E)$. We define the order $R_{II,D}$ of $D(E)$ by

\[
R_{II,D} = \left\{ \begin{array}{ll}
R'_{N_1/N^-} \times R'_{N_2/N^-} \times R'_{N_3/N^-} & \text{if } E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}, \\
R'_{\mathfrak{n}/\mathfrak{n}^-} \times R'_{N/N^-} & \text{if } E = F \times \mathbb{Q}, \\
R'_{\mathfrak{n}/\mathfrak{n}^-} & \text{if } E \text{ is a field.}
\end{array} \right.
\]

We mention that the divisibility of each ideals appeared in the definition of $R_{II,D}$ follows from the results of [Pra90] and [Pra92]. We also define an order $R_{M/N^-}$ of $D$, which is a twist of the standard Eichler order $R'_{M/N^-}$. More precisely, for $p$ such that $p \in \Sigma_{E,2}$ with $p \mid D_E$ and $\mathfrak{n} \mathcal{O}_{E_p} = \omega_K \mathcal{O}_{E_p} \times \mathbb{Z}_p$, we require

\[
R_{M/N^-} \otimes \mathbb{Z} \mathbb{Z}_p = \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) K_0(p) \left( \begin{array}{cc} p^{-1} & 0 \\ 0 & 1 \end{array} \right).
\]

Notice that these are precisely the places $p$ so that $E_p = K_p \times \mathbb{Q}_p$ with $K_p/\mathbb{Q}_p$ is unramified, and $II_D = II_p = \pi'_p \otimes \pi_p$ where $\pi'_p$ (resp. $\pi_p$) is a special representation of $GL_2(K_p)$ (resp. $GL_2(\mathbb{Q}_p)$).

To describe our formula, we need a notation. Let $\nu(II)$ be the number of prime $p$ such that

- $p \in \Sigma_3$, $II_p = \pi_1,p \boxtimes \pi_2,p \boxtimes \pi_3,p$ and $\pi_j,p$ are special representations of $GL_2(\mathbb{Q}_p)$ for $j = 1, 2, 3$.
- $p \in \Sigma_2$, $II_p = \pi'_p \boxtimes \pi_p$ and $\pi'_p$ (resp. $\pi_p$) is a special representation of $GL_2(K_p)$ (resp. $GL_2(\mathbb{Q}_p)$).
- $p \in \Sigma_2$, $K_p/\mathbb{Q}_p$ is ramified, $II_p = \pi'_p \boxtimes \pi_p$ and $\pi'_p$ (resp. $\pi_p$) is a unramified representation (resp. special representation) of $GL_2(K_p)$ (resp. $GL_2(\mathbb{Q}_p)$).
- $p \in \Sigma_1$ and $II_p$ is a special representation of $GL_2(E_p)$.
6.4. Unbalanced case. Assume $\epsilon(\Pi_\infty) = 1$ in this section. We assume without loss of generality that $k_3 = \max\{k_1, k_2, k_3\}$. Then $\epsilon(\Pi_\infty) = 1$ implies $k_3 \geq k_1 + k_2$. In this case, we have

$$D^\times(E_\infty) = \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R}) \times \text{GL}_2(\mathbb{R}) \quad \text{and} \quad \Pi_\infty^D = \Pi_\infty,$$

is the discrete series representation of $D^\times(E_\infty)$ of minimal weight $k$ and trivial central character. Let $\mathcal{A}(D^\times(A_E))$ be the space of $C$-valued automorphic forms on $D^\times(A_E)$ and let $\mathcal{A}(D^\times(A_E))_{\Pi^D}$ be the underlying space of $\Pi^D$ in $\mathcal{A}(D^\times(A_E))$. Put

$$\mathcal{A}_\mathcal{A}(D, E; \hat{\Pi}_D^{\infty})[\Pi^D] = \mathcal{A}_\mathcal{A}(D, E; \hat{\Pi}_D^{\infty}) \cap \mathcal{A}(D^\times(A_E))_{\Pi^D}.$$ 

By the multiplicity one theorem and the theory of newform, we have

$$\mathcal{A}_\mathcal{A}(D, E; \hat{\Pi}_D^{\infty})[\Pi^D] = C f_E^D,$$

for some non-zero element $f_E^D \in \Pi^D$. We normalize $f_E^D$ in the following way. Consider $(f_E^D)^*(h) = f_E^D(h\tau_\infty)$ for $h \in D^\times(A_E)$, where

$$\tau_\infty = \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \in D^\times(E_\infty).$$

Since $\Pi^D$ is unitary and self-contragredient, we have $\Pi^D \cong \Pi^D \cong \Pi^D$, where $\Pi^D$ is the conjugate representation of $\Pi^D$. The multiplicity then one theorem implies $(f_E^D)^* \in \Pi^D$. By the theory of newform, there exists a non-zero constant $\alpha$ such that $f_E^D(h) = \alpha \cdot f_E^D(h\tau_\infty)$ for all $h \in D^\times(A_E)$. Since $((f_E^D)^*)^* = f_E^D$, we see that $\alpha = 1$. As $\alpha$ can be written as $\beta/\bar{\beta}$ for some $\beta \in C^\times$, we may assume

$$f_E^D(h\tau_\infty) = f_E^D(h) \quad \text{for all} \quad h \in D^\times(A_E).$$

Of course this normalization is only up to a non-zero real number, but its enough for us.

Let $f_E^D \in \mathcal{A}_\mathcal{A}(D, E; \hat{\Pi}_D^{\infty})$ so that $\Phi(f_E^D) = f_E^D$. We define the norm $\langle f_E^D, f_E^D \rangle_{\hat{\Pi}_D^{\infty}}$ of $f_E^D$ as follows. Fix a set of representatives $\{x_1, \cdots, x_r\}$ for the double cosets $D^\times(E) \backslash D^\times(A_E)/D^\times(E_\infty)^+ \hat{\Pi}_D^{\infty}$, where $D^\times(E_\infty)^+$ is the three-fold product of $\text{GL}_2^+(\mathbb{R})$. We may assume every archimedean component of $x_j$ is one for $1 \leq j \leq r$. Let

$$\Gamma_j = D^\times(E) \cap \left( D^\times(E_\infty)^+ \times x_j \hat{\Pi}_D^{\infty} \right), \quad 1 \leq j \leq r.$$

The functions $f_{E, x_j} : \mathcal{H}^3 \to \mathbb{C}$ satisfy the automorphy condition (6.1) for $\gamma \in \Gamma_j$. We define

$$\langle f_E^D, f_E^D \rangle_{\hat{\Pi}_D^{\infty}} = \sum_{j=1}^{r} \int_{x_j \backslash \mathcal{H}^3} |f_{E, x_j}^D(z)|^2 \text{Im}(z)^k d\mu(z), \quad z = (z_1, z_2, z_3) \in \mathcal{H}^3, \quad \text{Im}(z)^k = \prod_{\ell=1}^{3} \text{Im}(z_\ell)^{k_\ell}.$$ 

The measure $d\mu(z)$ on $\mathcal{H}^3$ is given by

$$d\mu(z) = \prod_{\ell=1}^{3} y_{\ell}^{-2} dx_\ell dy_\ell \quad (z_\ell = x_\ell + iy_\ell, \quad 1 \leq \ell \leq 3),$$

where $dx_\ell$ and $dy_\ell$ are the usual Lebesgue measures on $\mathbb{R}$. Clearly, $\langle f_E^D, f_E^D \rangle_{\hat{\Pi}_D^{\infty}}$ is independent of the choice of the set $\{x_1, \cdots, x_r\}$. Similarly, we can define the norm $\langle f_E, f_E \rangle_{K_0(h)}$ of $f_E$.

On the other hand, the Petersson norms of $f_E$ and $f_E^D$ are given by

$$\int_{\mathcal{A}_\mathcal{A}(D^\times(E)) \backslash \text{GL}_2(E) \times \text{GL}_2(A_E)} |f_E(h)|^2 dh \quad \text{and} \quad \int_{\mathcal{A}_\mathcal{A}(D^\times(E)) \backslash D^\times(A_E)} |f_E^D(h)|^2 dh,$$
where \( dh \) are the Tamagawa measures on \( A^\times_E \setminus \GL_2(A_E) \) and \( A^\times_E \setminus D^\times(A_E) \), respectively. By [Wal85b, Lemma 6.1 and Lemma 6.3], we have

\[
\langle f_E, f_E \rangle_{K_0(\hat{n})} = h_E \left[ \GL_2(\hat{O}_E) : K_0(\hat{n}) \right] D^3/2_E \zeta_E(2) \int_{A^\times_E \setminus \GL_2^1(E) \setminus \GL_2^2(A_E)} |f_E(h)|^2 dh,
\]

(6.13)

\[
\langle f^D_E, f^D_E \rangle_{\hat{R}^\times_E} = h_E \left[ \hat{O}^\times_{D(E)} : \hat{R}^\times_{D(E)} \right] \prod_{p \mid N^-} (p-1) \prod_{p \in \Sigma_1 \cap \Sigma_D} (p-1)^2 \prod_{p \in \Sigma_1 \cap \Sigma_D, p^3 \parallel M} (p^2 + p + 1) \times D^3/2_E \zeta_E(2) \int_{A^\times_E \setminus D^\times_E(A_E)} |f^D_E(h)|^2 dh.
\]

Here \( h_E := \sharp \left( E^\times \setminus A^\times_E \setminus E^\times \hat{O}_E \right) \) is the class number of \( E \). We mention that

\[
dh = 8 \prod_{p \mid N^-} (p-1)^{-1} \prod_{p \in \Sigma_1 \cap \Sigma_D} (p-1)^{-2} \prod_{p \in \Sigma_1 \cap \Sigma_D, p^3 \parallel M} (p^2 + p + 1)^{-1} D^{-3/2}_E \zeta_E(2)^{-1} \prod_v dh_v,
\]

where \( dh \) is the Tamagawa measures on \( A^\times_E \setminus D^\times(A_E) \) and \( dh_v \) is the Haar measure on \( E_v^\times \setminus D^\times(E_v) \) defined in [4.1] and [5.1] for each place \( v \) of \( Q \).

**Lemma 6.1.** We have

\[
\langle f_E, f_E \rangle_{K_0(\hat{n})} = 2^{-k_1-k_2+k_3} h_E D_N E Q(\hat{n}) \cdot L(1, II, \text{Ad}),
\]

where \( c \) is given by (6.19).

**Proof.** By specializing the formula in [Wal85b, Proposition 6.1], we have

\[
\int_{A^\times_E \setminus \GL_2(E) \setminus \GL_2(A_E)} |f_E(h)|^2 dh = 2^{-k_1-k_2+k_3} \zeta_E(2)^{-1} D^{-1/2}_E N(\hat{n}) \left[ \GL_2(\hat{O}_E) : K_0(\hat{n}) \right]^{-1} L(1, II, \text{Ad}).
\]

The lemma follows form combining this with the equation (6.13). \( \square \)

For each place \( v \), let \( t_v \in D^\times(E_v) \) be the element defined in [2.3] for \( \Pi^D_v \) and put \( t = \otimes_v t_v, \hat{t} = \otimes_p t_p \). Recall \( N^- = \prod_{p \in \Sigma_D} p \) and \( M = \prod_{p \mid N^-(\hat{n})} p \). Let

\[
\Gamma^D_M = \Gamma^D_M(\hat{N}^-) \cap \left( D^\times(\hat{R}) \times \hat{R}^\times_{M/N^-} \right) \subset \SL_2(\hat{R}),
\]

which is a Fuchsian group of the first kind. Remember that \( k_3 \geq k_1 + k_2 \). Set

\[
2m = k_3 - k_1 - k_2.
\]

Recall that \( \nu(II) \) is the non-negative integer defined in the last paragraph of [6.3].

**Theorem 6.2.** Suppose \( f^D_E \) is normalized as [6.12].

1. We have

\[
\left( \int_{\hat{R}^\times_D} (1 \otimes \delta_{k_2} \otimes 1) f^D_E ((z, z, -z), \hat{t}) g^{k_3-2} dxdy \right)^2 = 2^{-2k_3-1+\nu(II)} M^D^{-1/2} \langle f^D_E, f^D_E \rangle_{\hat{R}^\times_D} L \left( \frac{1}{2}, II, r \right).
\]

2. The central value is non-negative, that is

\[
L \left( \frac{1}{2}, II, r \right) \geq 0.
\]

**Proof.** By the normalization [6.12], we have

\[
\int_{A^\times_E \setminus D^\times(E) \setminus D^\times(A_E)} f^D_E(h) f^D_E(h \tau_\infty) dh = \int_{A^\times_E \setminus D^\times(E) \setminus D^\times(A_E)} |f^D_E(h)|^2 dh.
\]
On the other hand, since $\text{Ad}_R(V_+)=V_+ - 2\sqrt{-1}I_2$ and $H^D$ has trivial central character, we also have

$$
\int_{A^*D^* \times A^* (\AA)} \Pi^D_\infty (t_{\infty}) f^D_E (h t) dh = \int_{A^*D^* \times A^* (\AA)} \Pi^D_\infty (t_{\infty} \tau_{\infty}) \overline{\Pi^D_\infty (h t)} dh
$$

$$
= \int_{A^*D^* \times A^* (\AA)} \Pi^D_\infty (t_{\infty}) f^D_E (h t) dh.
$$

By Ichino’s formula ([Ich08, Theorem 1.1 and Remark 1.3]) and the choices of Haar measures in [4.11 and 5.1] we find that

$$
\frac{|\int_{A^*D^* \times A^* (\AA)} \Pi^D t f^D_E (h) dh|}{|\int_{A^*E^* \times A^*E (\AA)} f^D_E (h) dh|}^2 = \left( \frac{\int_{A^*D^* \times A^* (\AA)} \Pi^D t f^D_E (h) dh}{\int_{A^*E^* \times A^*E (\AA)} f^D_E (h) dh} \right)^2
$$

$$
= 2^{1-c} \prod_{p \mid N-} (p - 1)^{-1} \cdot \frac{\zeta_E (2)}{\zeta_Q (2)^2} \cdot \frac{L (1/2, \Pi, r)}{L (1, \Pi, \text{Ad})} \cdot \prod_v I^* (\Pi^D_v, t_v),
$$

where $c$ is given by [6.3]. Since $L (1, \Pi, \text{Ad}) > 0$ by Lemma [6.1] and $L^* (\Pi_v, t_v) > 0$ for all $v$ by our results in the previous sections, we see immediately that assertion (2) holds.

To drive our formula, we note that from (6.2) and the definition of $t_{\infty}$, the function $\Pi^D t f^D_E$ is the adelic lift of the automorphic function

$$
((z_1, z_2, z_3), h) \mapsto (1 \otimes \delta_k^m \otimes 1) f^D_E ((z_1, z_2, -z_3), h t), \quad ((z_1, z_2, z_3), h) \in \hat{\Gamma}_3 \times \text{GL}_2 (\hat{E}).
$$

Applying lemmas 6.1 and 6.3 in [IP], we obtain

$$
\left( \int_{A^*D^* \times A^* (\AA)} \Pi^D t f^D_E (h) dh \right)^2 = \zeta_Q (2)^{-2} \prod_{p \mid M/N-} (1 + p)^{-2} \prod_{p \mid N-} (p - 1)^{-2}
$$

$$
\times \left( \int_{P_{M/N-} \times A^* (\AA)} (1 \otimes \delta_k^m \otimes 1) f^D_E ((z, z, -z), h) y^{k_3 - 2} dxdy \right)^2.
$$

The theorem follows from combining this with Lemma [6.1] and our results for $L^* (\Pi^D_v, t_v)$ in [IP] and [6].

Now we apply Theorem [6.2] to prove the algebraicity of the central critical values of the triple product $L$-functions in certain cases. We keep the notations and assumptions in [6.3]. We further assume that $E$ is not a field. To unify our statements, let $K = Q \times Q$ or $K = F$. Let $D_K$ be its absolute discriminant. We have $E = K \times Q$. Let $n_K = (N_1 Z, N_2 Z)$, $N_3 = N$ and $g_K = f_1 \otimes f_2$, $f_3 = f$ when $K = Q \times Q$. In any case, we have

$$
f_E = g_K \otimes f \quad \text{and} \quad n = (n_K, N Z).
$$

We define the motivic $L$-function and its associated completed $L$-function for $g_K \otimes f$ by

$$
L(s, g_K \otimes f) = \prod_p L \left( s - \frac{w}{2}, \Pi_p, r_p \right) \quad \text{and} \quad L(s, g_K \otimes f) = L \left( s - \frac{w}{2}, \Pi, r \right).
$$

Recall that $w$ is given by [6.5]. Define the Petersson norm of $f$ by

$$
\langle f, f \rangle_{\Gamma_0 (N)} = \int_{\Gamma_0 (N) \backslash \hat{\Gamma}_3} |f(z)|^2 y^{k_3 - 2} dxdy, \quad (z = x + iy).
$$

Here $dx, dy$ are the usual Lebesgue measures on $R$.

**Corollary 6.3.** Assume either $\epsilon (\Pi) = -1$, or $\epsilon (\Pi_v) = 1$ for all $v$. Then for every $\sigma \in \text{Aut} (C)$, we have

$$
\left( \frac{L((w + 1)/2, g_K \otimes f)}{D_K^{1/2} \pi^{2k_3} (f, f)_{\Gamma_0 (N)}} \right)^{\sigma} = \frac{L((w + 1)/2, g_K^\sigma \otimes f^\sigma)}{D_K^{1/2} \pi^{2k_3} (f^\sigma, f^\sigma)_{\Gamma_0 (N)}}.
$$
Proof. First note that we have $\epsilon(\Pi_\sigma) = \epsilon(\Pi)$ for all $\sigma \in \text{Aut}(\mathbb{C})$. In fact, this follows from $\Pi_\sigma^\infty \cong \Pi_\infty$ and Lemma \[\text{[HK04]}\]. Also, if $\epsilon(\Pi) = -1$, then by the results of \[\text{[HK04]}\] and \[\text{[PSP08]}\], we have

$$L \left( \frac{w+1}{2}, g_\mathcal{K} \otimes f \right) = 0,$$

(see also the remark after this corollary). As $\epsilon(\Pi_\sigma) = \epsilon(\Pi) = -1$, we conclude that

$$L \left( \frac{w+1}{2}, g_\mathcal{K} \otimes f^\sigma \right) = L \left( \frac{w+1}{2}, g_\mathcal{K} \otimes f \right) = 0.$$

Now we assume $\epsilon(\Pi_v) = 1$ for all $v$. Then $D = M_2$. Let $\iota : \mathcal{H} \to \mathcal{H}^2$ be the diagonal embedding $z \mapsto (z, z)$. The $GL_2(\mathbb{Q}_p)$ component of $t_\rho \in GL_2(K_v) \times GL_2(\mathbb{Q}_p)$ is equal to $1$ for all $p$. Thus we may view $t$ as an element in $GL_2(K)$. Note that $(1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}$ is a nearly holomorphic Hilbert modular form over $K$ of weight $(k_1, k_2 + 2m)$, where $\rho$ denote the right translation of $GL_2(K)$. Let $\iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K})(z) = (1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}((z, z), 1)$ be its pullback along $\iota$ at the identity cusp. Then it is a nearly holomorphic modular form of weight $k_3$ and level $\Gamma_0(M)$. We consider the period integral $\langle \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}), f \rangle$ defined by

$$\langle \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}), f \rangle = \int_{\Gamma_0(M) \backslash \mathcal{H}} \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K})(z) \overline{f(z)} y_k^{-2} \, dx \, dy,$$

where $z = x + iy$ and $dx, dy$ are the usual Lebesgue measures on $\mathbb{R}$. Let $\sigma \in \text{Aut}(\mathbb{C})$. By our normalization of $g_\mathcal{K}$, we have

$$(\iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}))^\sigma = \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}).$$

Since $\iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K})$ is nearly holomorphic and $f$ is a newform, by \[\text{[Stu80]}\] Theorem 4 and \[\text{[Shi76]}\], we have

$$\frac{\langle \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}), f \rangle}{\langle f, f \rangle_{\Gamma_0(N)}} = \frac{\langle \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}), f^\sigma \rangle}{\langle f^\sigma, f^\sigma \rangle_{\Gamma_0(N)}}.$$

In particular, we have $\langle \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}), f \rangle \in \mathbb{R}$. Note that

$$(1 \otimes \delta_{k_2}^m \otimes 1)\rho(\hat{t})f_E((z, z, -\mathcal{F}), 1) = \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K})(\mathcal{F})f(z).$$

By Theorem 6.2 we have

$$\left\langle \iota^*((1 \otimes \delta_{k_2}^m)\rho(\hat{t})g_\mathcal{K}), f \right\rangle^2 = 2^{-2k_3 - 1 + \nu(\Pi)} M_{\hat{\chi}} L_{\hat{\chi}} \left( \frac{w+1}{2}, g_\mathcal{K} \otimes f \right).$$

Applying $\sigma$ on both sides and note that $\epsilon(\Pi_\sigma) = \epsilon(\Pi_v)$, $\iota^*(\Pi_v, t_\sigma) = \iota^*(\Pi_v, t_v)$ for all $v$, as well as $\nu(\Pi) = \nu(\Pi_\sigma)$. The corollary follows from applying our central value formula to the left hand side again. \hfill \Box

Remark 6.4. We mention that there are three ways to define the triple $L$-function: (1) from the Galois side, as we did in this paper; (2) by the Langlands-Shahidi method; (3) by the theory of local zeta integrals \[\text{[PSR87]}, \text{[Ike92]}\]. However, for questions of vanishing or nonvanishing at 1/2, it makes no difference which definition of the $L$-function we choose, since any two of them are only different from a finite number of local $L$-factors, which we know are all non-vanishing at 1/2.

6.5. Balanced case. Assume $\epsilon(\Pi_\infty) = -1$ in this section. We have

$$D^\times(E_{\infty}) = \mathcal{H}^\times \times \mathcal{H}^\times \times \mathcal{H}^\times$$

and $(\Pi_\infty^\infty, V_H) = (\rho_E, L_k(\mathbb{C}))$. Let $\mathcal{A}(D^\times(A_E))_{\Pi_D}$ be the underlying space of $\Pi_D$ in $\mathcal{A}(D^\times(A_E))$ and put

$$\mathcal{A}_k(D, E; \hat{R}_D)[[\Pi_D]] = \mathcal{A}_k(D, E; \hat{R}_D^\infty, V_H) \cap \mathcal{A}(D^\times(A_E))_{\Pi_D}.$$
To state our central value formula for the balanced case, we need some notations. Let $\Cl(R_{\Pi^D})$ and $\Cl(R_{M/N^-})$ be sets of representatives of $\hat{E}^x D^x(E)\backslash D^x(\hat{E})/\hat{R}_{\Pi^D}^{x}$ and $\hat{Q}^x D^x(Q)\backslash D^x(Q)/\hat{R}_{M/N^-}^{x}$, respectively. Let $\Gamma$ be finite sets defined by

\[
(D^x(E) \cap \hat{E}^x \alpha \hat{R}_{\Pi^D}^{x} \alpha^{-1})/E^x \quad \text{or} \quad (D^x(Q) \cap \hat{Q}^x \alpha \hat{R}_{M/N^-}^{x} \alpha^{-1})/Q^x,
\]

according to $\alpha \in \Cl(R_{\Pi^D})$ or $\alpha \in \Cl(R_{M/N^-})$, respectively. We put

\[
\langle f_E^D, f_E^D \rangle_{\hat{R}_{\Pi^D}^{x}} = \sum_{\alpha \in \Cl(R_{\Pi^D})} \frac{1}{t_\Gamma(\alpha)} \langle f_E^D(\alpha), f_E^D(\alpha) \rangle_{E^x}.
\]

For each place $v$, let $t_v \in D^x(E_v)$ be the element defined in [163] for $\Pi_v^D$ and put $t = \otimes_v t_v$. Recall that $M = \prod_{p \mid N^\circ(a)} P$ and that $v(\Pi)$ is the non-negative integer defined in the last paragraph of [163].

**Theorem 6.5.** (1) We have

\[
\left( \sum_{\alpha \in \Cl(R_{M/N^-})} \frac{1}{t_\Gamma(\alpha)} \langle f_E^D(\alpha t), P_{E^x}^{\alpha} \rangle_{E^x} \right)^2 = 2^{-(k_1+k_2+k_3+1)+\nu(\Pi)} M \frac{1}{\Gamma(1/2, \Pi, r)} \int_{A^x \backslash D^x(A)} \frac{f_E^D(ht)dh}{\Gamma(1/2, \Pi, r)}.
\]

(2) The central value is non-negative, that is

\[
L \left( \frac{1}{2}, \Pi, r \right) \geq 0.
\]

**Proof.** (1) By Lemmas 6.1 and 6.3 in [IP], we have

\[
\left( \sum_{\alpha \in \Cl(R_{M/N^-})} \frac{1}{t_\Gamma(\alpha)} \langle f_E^D(\alpha t), P_{E^x}^{\alpha} \rangle_{E^x} \right)^2 = \frac{1}{24} \prod_{p \mid M/N^-} \left( 1 + p \prod_{p \mid N^-} (p-1) \int_{A^x \backslash D^x(A)} f_E^D(ht)dh \right).
\]

where $dh$ is the Tamagawa measure on $A^x \backslash D^x(A)$. On the other hand, applying some lemmas, we obtain

\[
\langle f_E^D, f_E^D \rangle_{\hat{R}_{\Pi^D}^{x}} = 2^{-6-3} h_E^2 \frac{\pi^{2\nu(\Pi)}}{\prod_{p \mid N^-} (p-1)} \int_{A^x \backslash D^x(A)} f_E^D(ht)dh,
\]

where $dh$ is the Tamagawa measure on $A^x \backslash D^x(A)$. Schur’s orthogonal relation implies

\[
\int_{A^x \backslash D^x(A)} f_E^D(ht)fh_E^D(h)dh = \frac{\langle P_{E^x}, P_{E^x} \rangle_{E^x}}{(k_1-1)(k_2-1)(k_3-1)} \int_{A^x \backslash D^x(A)} f_E^D(ht)fh_E^D(h)dh.
\]

The measure $dh$ on the RHS of the equation above is also the Tamagawa measure on $A^x \backslash D^x(A)$. By Ichino’s formula [Ich08 Theorem 1.1 and Remark 1.3] and the choices of Haar measures in [111] and [151], we find that

\[
\frac{\left( \int_{A^x \backslash D^x(A)} f_E^D(ht)dh \right)^2}{\int_{A^x \backslash D^x(A)} f_E^D(ht)fh_E^D(h)dh} = 2^{3-c-3} \prod_{p \mid N^-} (p-1)^{-1} \cdot \frac{\zeta_E(2)}{\zeta_E(2)} \cdot \frac{L(1/2, \Pi, r)}{L(1, \Pi, Ad)} \cdot \prod_v \Gamma_v(\Pi_v^D, t_v).
\]

Here the constant $c$ is given by [163]. The central value formula follows from the equations above together with Lemma 4.1 and the results for $\Gamma_v(\Pi_v^D, t_v)$ in [141] and [151].

To prove (2), it suffices to show that the ratio

\[
\frac{\left( \int_{A^x \backslash D^x(A)} f_E^D(ht)dh \right)^2}{\int_{A^x \backslash D^x(A)} f_E^D(ht)fh_E^D(h)dh}
\]
is non-negative. To do this we consider \((f^D_E)^*(h) = \overline{f^D_E(h\tau_\infty)}\) for \(h \in D^\times(A_E)\), where
\[
\tau_\infty = \left(\begin{array}{ccc} 0 & 1 \\ -1 & 0 \end{array}\right), \left(\begin{array}{ccc} 0 & 1 \\ -1 & 0 \end{array}\right), \left(\begin{array}{ccc} 0 & 1 \\ -1 & 0 \end{array}\right) \in D^\times(E_\infty).
\]
The function \((f^D_E)^*\) satisfy the same conditions as \(f^D_E\). By the uniqueness, there exists a non-zero constant \(\alpha\) such that
\[
f^D_E(h) = \alpha \cdot \overline{f^D_E(h\tau_\infty)} \text{ for all } h \in D^\times(A_E).
\]
On one hand, we have
\[
\int_{A^* \cdot D^\times(Q) \setminus D^\times(A)} \langle f^D_E(ht), P_\xi \rangle_k dh = \alpha \int_{A^* \cdot D^\times(Q) \setminus D^\times(A)} \langle f^D_E(h\tau_\infty), P_\xi \rangle_k dh
\]
\[
= \alpha \cdot \int_{A^* \cdot D^\times(Q) \setminus D^\times(A)} \langle f^D_E(ht), P_\xi \rangle_k dh.
\]
On the other hand, recall that
\[
\mathcal{H}_L(v, w) = \langle v, \Pi^D_\infty(\tau_\infty) \tilde{w} \rangle_{\xi_k} \quad v, w \in L(C),
\]
defines an \(D^\times(E_\infty)\)-invariant Hermitian pairing on \(V_{H_L}\). We have
\[
\int_{A^* \cdot D^\times(E) \setminus D^\times(A_E)} \langle f^D_E(h), f^D_E(h) \rangle_k dh = \int_{A^* \cdot D^\times(E) \setminus D^\times(A_E)} \langle f^D_E(h), f^D_E(h) \rangle_k dh
\]
\[
= \alpha \cdot \int_{A^* \cdot D^\times(E) \setminus D^\times(A_E)} \mathcal{H}_L(f^D_E(h), f^D_E(h)) dh.
\]
This finishes the proof. \(\square\)

Define the motivic \(L\)-function for \(f_E\) by
\[
L(s, f_E, r) = \prod_p L\left(\frac{s}{2}, \Pi_{pr}, r_p\right).
\]
Recall that \(w\) is given by (6.3). We have the following corollary, which proves the Deligne’s conjecture for the central critical value of \(L(s, f_E, r)\) in the balanced range.

**Corollary 6.6.** Let \(\sigma \in \text{Aut}(C)\). We have
\[
\left(\frac{L\left((w+1)/2, f_E, r\right)}{D_E^{1/2} \pi^{w+2} \langle f_E, f_E \rangle_{K_0(\tilde{a})}}\right)^\sigma = \frac{L\left((w+1)/2, f_E^\sigma, r\right)}{D_E^{1/2} \pi^{w+2} \langle f_E^\sigma, f_E^\sigma \rangle_{K_0(\tilde{a})}}.
\]

**Proof.** If \(\epsilon(\Pi) = -1\), then by the same argument in Corollary 6.3, we have
\[
L\left(\frac{w+1}{2}, f_E^\sigma, r\right) = L\left(\frac{w+1}{2}, f_E, r\right) = 0,
\]
for every \(\sigma \in \text{Aut}(C)\).

Assume \(\epsilon(\Pi) = 1\) and put
\[
\langle f^D_E, P_\xi \rangle_k = \left(\sum_{\alpha \in \text{Cl}(R_{M/N})} \frac{1}{n_\alpha} \langle f^D_E(\alpha t), P_\xi \rangle_k\right)^2.
\]

By [Wal85 Lemma II.1.1], we have
\[
\left(\frac{\langle f^D_E, P_\xi \rangle_k}{(f^D_E, f^D_E)_{R^\sigma_D}}\right)^\sigma = \left(\frac{\langle f^D_E^\sigma, P_\xi \rangle_k}{(f^D_E^\sigma, f^D_E^\sigma)_{R^\sigma_D}}\right)^\sigma
\]
for all \(\sigma \in \text{Aut}(C)\). The rest of the proof is similar to that in the last paragraph of Corollary 6.3. \(\square\)
In this section, we prove our main results of this paper. Let $N_1, N_2$ be positive square free integers, and $\kappa', \kappa$ be positive even integers. Put $w = 2\kappa + \kappa' - 3$. Let $N = \gcd(N_1, N_2)$ and $M = \lcm(N_1, N_2)$. Let $f \in S_{\nu'}(\Gamma_0(N_1))$ and $g \in S_\nu(\Gamma_0(N_2))$ be normalized elliptic newforms and $f$ and $g$ be the adelic lifts of $f$ and $g$, respectively. Let $\tau = \otimes_v \tau_v$ and $\pi = \otimes_v \pi_v$ be the irreducible unitary cuspidal automorphic representations of $\GL_2(\A)$ generated by $f$ and $g$, respectively.

If $F'$ is a cyclic extension of $\Q$ with prime degree, we let $\pi_{F'}$ be the base change lift of $\tau$ to $\GL_2(\A_{F'})$. Since $N_2$ is assumed to be square-free, $\pi_{F'}$ is a unitary irreducible cuspidal automorphic representation of $\GL_2(\A_{F'})$ with trivial central character $\left[\text{AC89}\right]$.

We define the motivic $L$-function and its associated completed $L$-function for $\Sym^2(g) \otimes f$ by
\[
L(s, \Sym^2(g) \otimes f) = \prod_p L\left(s - \frac{w}{2}, \Sym^2(\pi_p) \otimes \tau_p\right), \quad \Lambda(s, \Sym^2(g) \otimes f) = \prod_v L\left(s - \frac{w}{2}, \Sym^2(\pi_v) \otimes \tau_v\right).
\]

Note that $L(s, \Sym^2(g) \otimes f)$ is holomorphic at $s = (w + 1)/2$.

**Corollary 7.1.** Assume $\kappa' \geq 2\kappa$. Let $\epsilon = (-1)^{\kappa'/2 - 1}$. For $\sigma \in \Aut(\C)$, we have
\[
\left( \frac{L((w + 1)/2, \Sym^2(\sigma) \otimes f)}{\pi^{3\kappa'/2}(\sqrt{-1})^{\kappa'/2 - 1}} \right)_{\sigma} = \frac{L((w + 1)/2, \Sym^2(\sigma) \otimes f^\sigma)}{\pi^{3\kappa'/2}(\sqrt{-1})^{\kappa'/2 - 1}} \cdot \Omega_f^\sigma.
\]

Here $\Omega_f^\sigma$ are the periods of $f$ defined by Shimura in $\left[\text{Shi77}\right]$.

**Proof.** Define $\Xi$ to be the set of real quadratic extensions $K/\Q$ such that
- $D_K$ is prime to $M$.
- $\left(\frac{D_K}{p}\right) = -1$ for $p \mid N$.
- $\left(\frac{D_K}{p}\right) = 1$ for $p \mid M/N_2$.

Here $D_K$ is the discriminant of $K/\Q$. Let $K \in \Xi$ and $\chi_K = \otimes_v \chi_{K,v} : K^\times \backslash A_K^\times \rightarrow \C$ be the idele class character associated to $K$ by class field theory. Put $\Pi = \pi_K \otimes \tau$. By the properties of $K$, we have $\epsilon(\Pi_v) = 1$ for all $v$. On the other hand, by the results of $\left[\text{Pra90}\right]$ and $\left[\text{Pra92}\right]$, we have
\[
\epsilon(\Pi_v) = \epsilon\left(\frac{1}{2}, \Pi_v, r_v, \psi_v\right) \chi_{K,v}(-1),
\]
for all place $v$. In particular, $\epsilon(1/2, \Pi, r, \psi) = 1$ and the matrix algebra $M_2$ is the unique quaternion algebra over $\Q$ satisfying $\left[\text{6.7}\right]$. We see from the factorization $\epsilon(s, \Pi, r, \psi) = \epsilon(s, \Sym^2(\pi) \otimes \tau, \psi) \epsilon(s, \tau \otimes \chi_K, \psi)$ that
\[
\epsilon(\frac{1}{2}, \Sym^2(\pi) \otimes \tau, \psi) = \epsilon\left(\frac{1}{2}, \tau \otimes \chi_K, \psi\right).
\]

If $\epsilon(1/2, \tau \otimes \chi_K, \psi) = -1$, then $\epsilon(1/2, \Sym^2(\pi) \otimes \tau, \psi) = -1$. On the other hand, by Corollary $\left[\text{7}\right]$ we also have $\epsilon(1/2, \Sym^2(\pi^\sigma) \otimes \tau^\sigma, \psi) = -1$. Therefore
\[
L\left(\frac{w + 1}{2}, \Sym^2(\sigma) \otimes f^\sigma\right) = L\left(\frac{w + 1}{2}, \Sym^2(g) \otimes f\right) = 0,
\]
for all $\sigma \in \Aut(\C)$ by functional equation. Otherwise, by the nonvanishing theorem of $\left[\text{FH95}\right]$, there exists $K' \in \Xi$ such that $L(\kappa'/2, f \otimes \chi_{K'}) \neq 0$. Let $\sigma \in \Aut(\C)$. By $\left[\text{Shi77}\right]$, we have
\[
\left(\frac{L(\kappa'/2, f \otimes \chi_{K'})}{D_{K'}^{1/2}\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_{f,K'}^\sigma}\right)_{\sigma} = \frac{L(\kappa'/2, f^\sigma \otimes \chi_{K'})}{D_{K'}^{1/2}\pi^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_{f,K'}^\sigma},
\]
\[
\left(\frac{\langle f, f \rangle}{(\sqrt{-1})^{\kappa'-1}\Omega_f^\sigma}\right)_{\sigma} = \frac{\langle f^\sigma, f^\sigma \rangle}{(\sqrt{-1})^{\kappa'-1}\Omega_{f,K'}^\sigma}.
\]
Corollary 7.3. Assume $g_{K'}$ be the normalized Hilbert modular newform associated to $π_{K'}$, the base change lift of $π$ to $GL_2(\mathbb{A}_{K'})$. By Corollary 6.3, we have

$$\left( \frac{L((w+1)/2, g_{K'} \otimes f)}{D_{K'}^{1/2} \pi^{2\zeta c}(f, f')^2} \right)^{\sigma} = \frac{L((w+1)/2, g_{K'}^\sigma \otimes f^\sigma)}{D_{K'}^{1/2} \pi^{2\zeta c}(f^\sigma, f'^\sigma)^2}. $$

Note that $g_{K'}^\sigma = (g^\sigma)_{K'}$. Now the corollary follows from combining these equations with the following factorization

$$L \left( \frac{w+1}{2}, g_{K'} \otimes f \right) = L \left( \frac{w+1}{2}, \text{Sym}^2(g) \otimes f \right) L \left( \frac{\kappa'}{2}, f \otimes \chi_{K'} \right).$$

This completes the proof. □

Define the Petersson norm of $g$ by

$$\langle g, g \rangle = \int_{\Gamma_0(N_2) \backslash \mathcal{H}} |g(\tau)|^2 y^{\kappa - 2} \, d\tau.$$  

**Corollary 7.2.** Assume $2\kappa > \kappa'$ and $N_1 > 1$. Let $\epsilon = (-1)^{\kappa'/2 - 1}$. For $\sigma \in \text{Aut}(\mathcal{C})$, we have

$$\left( \frac{L((w+1)/2, \text{Sym}^2(g) \otimes f)}{\pi^{2\kappa + \kappa'/2 - 1}(\sqrt{-1})^{\kappa/2 - 1}(g, g)^2 \Omega_f} \right)^{\sigma} = \frac{L((w+1)/2, \text{Sym}^2(g^\sigma) \otimes f^\sigma)}{\pi^{2\kappa + \kappa'/2 - 1}(\sqrt{-1})^{\kappa'/2 - 1}(g^\sigma, g^\sigma)^2 \Omega_f^{\sigma}}.$$  

Here $\Omega_f^{\sigma}$ are the periods of $f$ defined by Shimura in [Shi77].

**Proof.** Since $N_1 > 1$, by the non-vanishing results of [FH95], we can choose a real quadratic field $K$ with fundamental discriminant $\mathcal{D} > 0$ such that $L(\kappa'/2, f \otimes \chi_{\mathcal{D}}) \neq 0$, where $\chi_{\mathcal{D}}$ is the Dirichlet character associated to $K/\mathbb{Q}$ by class field theory. Let $g_K$ be the normalized Hilbert modular newform associated to $π_K$ and $g_{K'} \in π_{K'}$ be its adelic lift. By equation (6.13), the Petersson norm of $g_K$ is given by

$$\langle g_K, g_K \rangle = h_K \left[ \text{GL}_2(\hat{\mathcal{O}}_K) : K_0(N_2O_K) \right] D_K^{3/2} \zeta_K(2) \int_{\mathcal{A}_K \backslash \text{GL}_2^\varphi(K) \backslash \text{GL}_2^\varphi(\mathbb{A}_K)} |g_K(h)|^2 \, dh,$$

where $h_K$ is the class number of $K$ and $dh$ is the Tamagawa measure on $\mathcal{A}_K \backslash \text{GL}_2(\mathbb{A}_K)$. We have

$$\left( \frac{\langle g_K, g_K \rangle}{\langle g, g \rangle^2} \right)^{\sigma} = \frac{\langle g^\sigma K, g^\sigma K \rangle_{K'}}{(g^\sigma, g^\sigma)^2}.$$  

This equality follows from combining the factorization

$$L(1, π_K, \text{Ad}) = L(1, π, \text{Ad})L(1, π, \text{Ad}, \chi),$$

and a result of Sturm [Stu89]. The rest of proof is similar to that of Corollary 7.1 except we use Corollary 6.6 here instead. □

We consider the case when $E$ is a cubic Galois extension over $\mathbb{Q}$. Under some assumptions, we prove Deligne’s conjecture for the central critical value of $L(s, \text{Sym}^3(f))$, where $L(s, \text{Sym}^3(f))$ is the motivic $L$-function for $\text{Sym}^3(f)$ defined by

$$L(s, \text{Sym}^3(f)) = \prod_p L \left( s - \frac{w}{2}, \text{Sym}^3(\tau_p) \right).$$

Here $w = 3\kappa' - 3$.

**Corollary 7.3.** Assume $N_1 > 1$ and there exist a cubic Dirichlet character $\chi$ such that $L \left( \frac{\kappa'}{2}, f \otimes \chi \right) \neq 0$. For $\sigma \in \text{Aut}(\mathcal{C})$, we have

$$\left( \frac{L((w+1)/2, \text{Sym}^3(f))}{\pi^{2\kappa - 1}(\sqrt{-1})^{\kappa}(f, f')\Omega_f^2} \right)^{\sigma} = \frac{L((w+1)/2, \text{Sym}^3(f^\sigma))}{\pi^{2\kappa - 1}(\sqrt{-1})^{\kappa}(f^\sigma, f'^\sigma)\Omega_f^{\sigma^2}}.$$
Using again [Shi77], we have

\[ L(1, \pi_E, \text{Ad}) = L(1, \pi, \text{Ad})L(1, \pi, \chi_E) \]

together with Sturm’s result [Stu80] yield

\[ \left( \frac{\langle f, f \rangle^3}{\langle f, f \rangle} \right) = \left( \frac{\langle f^\sigma, f^\sigma \rangle^3}{\langle f^\sigma, f^\sigma \rangle} \right) \]

Using again [Shi77], we have

\[ \left( \frac{L(\kappa'/2, f \otimes \chi)}{G(\chi)^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_f^2} \right)^\sigma = \frac{L(\kappa'/2, f^\sigma \otimes \chi)}{G(\chi)^{\kappa'/2}(\sqrt{-1})^{\kappa'/2}\Omega_f^2} \]
\[ \left( \frac{L(\kappa'/2, f \otimes \bar{\chi})}{G(\bar{\chi})^{\kappa'/2}(\sqrt{-1})^{\bar{\kappa}'/2}\Omega_f^2} \right)^\sigma = \frac{L(\kappa'/2, f^\sigma \otimes \bar{\chi})}{G(\bar{\chi})^{\kappa'/2}(\sqrt{-1})^{\bar{\kappa}'/2}\Omega_f^2} \]
\[ \left( \frac{\langle f, f \rangle}{(\sqrt{-1})^{\kappa'-\kappa}\Omega_f^2} \right)^\sigma = \left( \frac{\langle f^\sigma, f^\sigma \rangle}{(\sqrt{-1})^{\kappa'-\kappa}\Omega_f^2} \right) \]

Here \( G(\chi) \) (resp. \( G(\bar{\chi}) \)) is the Gauss sum associated to \( \chi \) (resp. \( \bar{\chi} \)) defined in [Shi77]. Notice that since the Hecke field of \( f \) is totally real, we have

\[ L \left( \frac{\kappa'}{2}, f \otimes \bar{\chi} \right) = L \left( \frac{\kappa'}{2}, f \otimes \chi \right) \neq 0. \]

Also, as \( E/Q \) is Galois, \( D_E \) is a square. The corollary then follows from these equations together with Corollary 6.6 and the factorization

\[ L \left( \frac{w+1}{2}, f_E, r \right) = L \left( \frac{w+1}{2}, \text{Sym}^3(f) \right) L \left( \frac{\kappa'}{2}, f \otimes \chi \right) L \left( \frac{\kappa'}{2}, f \otimes \bar{\chi} \right). \]

This finishes the proof.

\[ \square \]

**Appendix : Root numbers and Deligne’s periods**

The appendix consists of two parts. In the first part, we explain that the various local root numbers are invariant under the Galois action. In the second part, we compute the Deligne’s period of the motive associated to \( \text{Sym}^3(g) \otimes f \).

**Root numbers.** Let \( F \) be a non-archimedean local field of characteristic zero. Let \( E \) be an étale cubic algebra over \( F \). Let \( D \) be the quaternion division algebra over \( F \). The definition of the local root numbers in [6.3] is valid in more general settings. More precisely, let \( \Pi \) be an irreducible admissible generic representation of \( \text{GL}_2(E) \) whose central character is trivial on \( F^\times \). Define \( \epsilon(\Pi) \in \{ \pm 1 \} \) by the following condition

\[ \epsilon(\Pi) = 1 \Leftrightarrow \text{Hom}_{\text{GL}_2(E)}(\Pi, C) \neq \{0\}. \]

We call \( \epsilon(\Pi) \) the (local) root number associated to \( \Pi \). We can also define the local root number for the archimedean case as we did in the same section, but in terms of the category of \( (g, K) \)-modules. The results of Prasad [Pra90], [Pra92] imply that if \( \epsilon(\Pi) = -1 \), then the Jacquet-Langlands lift \( \Pi' \) of \( \Pi \) to \( D^\times(E) \) is non-zero, and \( \text{Hom}_{D^\times}(\Pi', C) \neq \{0\} \).

Let \( \sigma \in \text{Aut}(C) \) and \( (\pi, V) \) be a representation of a group \( G \). Following [Wal85a, section 1], we define a representation \( \pi^\sigma \) of \( G \) as follows. Let \( V' \) be another \( C \)-linear space with a \( \sigma \)-linear isomorphism \( t' : V \to V' \). We define

\[ \pi^\sigma(g) = t' \circ \pi(g) \circ t'^{-1}, \quad g \in G. \]

If \( \pi = \chi \) is a character, then \( \chi^\sigma = \sigma(\chi) \).

Notice that \( \Pi^\sigma \) is an irreducible admissible generic representation of \( \text{GL}_2(E) \) with central character \( \omega_\Pi^\sigma \), where \( \omega_\Pi \) is the central character of \( \Pi \). In particular, \( \omega_\Pi \) is trivial on \( F^\times \) if and only if \( \omega_\Pi^\sigma \) is.
**Lemma A.** For every $\sigma \in \text{Aut}(C)$, we have
\[ \epsilon (\Pi^\sigma) = \epsilon (\Pi). \]

**Proof.** This follows immediately from the definition. Indeed, we have a $\sigma$-linear isomorphism,
\[ \text{Hom}_{\text{GL}_2(F)}(\Pi, C) \to \text{Hom}_{\text{GL}_2(F)}(\Pi^\sigma, C), \]
defined by $\ell \mapsto \ell' := \sigma \circ \ell \circ t'^{-1}$. This finishes the proof. \(\square\)

We have a corollary.

**Corollary A.** Let $\psi$ be a non-trivial additive character of $F$. Let $\pi$ and $\tau$ be two irreducible admissible generic representations of $\text{GL}_2(F)$ with central character $\omega_\pi$ and $\omega_\tau$, respectively. Let $\sigma \in \text{Aut}(C)$.

1. Suppose $\omega_\pi^2 \cdot \omega_\tau = 1$. Then $\epsilon \left( \frac{1}{2}, \text{Sym}^2(\pi) \otimes \tau, \psi \right) \in \{ \pm 1 \}$ is independent of $\psi$, and we have
\[ \epsilon \left( \frac{1}{2}, \text{Sym}^2(\pi^\sigma) \otimes \tau^\sigma \right) = \epsilon \left( \frac{1}{2}, \text{Sym}^2(\pi) \otimes \tau \right). \]

2. Suppose $\omega_\pi^3 = 1$. Then $\epsilon \left( \frac{1}{2}, \text{Sym}^3(\pi), \psi \right) \in \{ \pm 1 \}$ is independent of $\psi$, and we have
\[ \epsilon \left( \frac{1}{2}, \text{Sym}^3(\pi^\sigma) \right) = \epsilon \left( \frac{1}{2}, \text{Sym}^3(\pi) \right). \]

**Proof.** We only prove (1) since the proof of (2) is similar. Let $\Pi = \pi \boxtimes \pi \boxtimes \tau$. By the results of [Pra90], [Gan08] Theorem 1.2 and [Ram00] Theorem 4.4.1, $\epsilon \left( \frac{1}{2}, \Pi, r, \psi \right) \in \{ \pm 1 \}$ is independent of $\psi$ and
\[ \epsilon(\Pi) = \epsilon \left( \frac{1}{2}, \Pi, r \right). \]
Since $\tau \otimes \chi_\pi$ is self-dual, we have $\epsilon \left( \frac{1}{2}, \tau \otimes \omega_\pi, \psi \right) \in \{ \pm 1 \}$ is independent of $\psi$. By the factorization
\[ \epsilon \left( s, \Pi, r, \psi \right) = \epsilon \left( s, \tau \otimes \chi_\omega_\tau, \psi \right) \epsilon \left( s, \text{Sym}^2(\pi) \otimes \tau, \psi \right), \]
we see that $\epsilon \left( \frac{1}{2}, \text{Sym}^2(\pi) \otimes \tau, \psi \right) \in \{ \pm 1 \}$ is also independent of $\psi$.

By the lemma A, we only need to show
\[ \epsilon \left( \frac{1}{2}, \tau^\sigma \otimes \omega_\pi^\sigma \right) = \epsilon \left( \frac{1}{2}, \tau \otimes \omega_\pi \right). \]
But this is a result of [Wal85a] Proposition I.2.5, which said
\[ \epsilon \left( \frac{1}{2}, \tau^\sigma \otimes \omega_\pi^\sigma \right) = \epsilon \left( \frac{1}{2}, \tau^\sigma \otimes \omega_\pi^\sigma, \psi^\sigma \right) = \sigma \epsilon \left( \frac{1}{2}, \tau \otimes \omega_\pi, \psi \right) = \epsilon \left( \frac{1}{2}, \tau^\sigma \otimes \omega_\pi^\sigma \right), \]
where $\psi^\sigma = \sigma \circ \psi$. This completes the proof. \(\square\)

**Deligne’s periods.** Notations being the same as in the previous section. In [Yos01], H. Yoshida define fundamental periods of a pure motive over $Q$ whose construction including Deligne’s periods. In particular, Yoshida give a formula for Deligne’s periods of the tensor product of two pure motives over $Q$ in terms of the fundamental periods of the two motives. Specializing the formula of Yoshida, C. Bhagwat give a more explicit formula in [Bha14] for pure motive whose nonzero Hodge numbers are one. In this section we use formula in [Bha14] to compute Deligne’s periods of the motive associated to $\text{Sym}^2(g) \otimes f$. It turns out that there are no fundamental periods other than Deligne’s periods in our case. Let $M(f)$ and $M(g)$ be the motives over $Q$ with coefficients in $Q(f)$ and $Q(g)$, respectively. For their construction, see [Sch00]. We consider the symmetric square $\text{Sym}^2(M(g))$ (resp. the symmetric cube $\text{Sym}^3(M(f))$) of the motive $M(g)$ (resp. $M(f)$). We follow [Del79] and [Yos01] for the conventions and notations. All motives below have coefficients in $Q(f, g)$, and we write $\sim$ for the equivalence relation defined by $Q(f, g)^\times$.

In [Bha14], the exponent of $(e^+(M')e^-(M'))$ in Theorem 3.2 should be $a_k - k$ in stead of $a_k - k - 1$.

**Proposition A.** We have
\[
e^\pm(\text{Sym}^2(M(g) \otimes M(f)) = \begin{cases} 
(2\pi\sqrt{-1})^{3-3\kappa}(\sqrt{-1})^{1-\kappa'}(f, f)\Omega^\pm_j & \text{if } \kappa' \geq 2\kappa, \\
(2\pi\sqrt{-1})^{2-2\kappa-\kappa'}(g, g)2\Omega^\pm_j & \text{if } 2\kappa > \kappa'. 
\end{cases}
\]
\[
e^\pm(\text{Sym}^3(M(f)) = (2\pi\sqrt{-1})^{1-\kappa}(\sqrt{-1})^{1-\kappa'}(f, f)(\Omega^\pm_j)^2.
\]
The Hodge decomposition and the Hodge filtration are given by
\[
\begin{align*}
\kappa &> \kappa, \\
\omega &> \omega(M), \\
\kappa &< \kappa(M).
\end{align*}
\]
Since \( L(M,s) = \zeta(s) \zeta_{\mathbb{R}}(s - \kappa + 2) \). We have
\[
H_B(M) \otimes \mathbb{Q} C = H^{0,2k-2}(M) \oplus H^{k-1,\kappa-1}(M) \oplus H^{2k-2,0}(M),
\]
\[
H_{DR}(M) = F^0(M) \supsetneq F^{k-1}(M) \supsetneq F^{2k-2}(M) \supsetneq \{0\}.
\]
It is well known that
\[
\begin{align*}
w(M) = \kappa' - 1, \\
d^+(M) = 1, \\
c^+(M) = \Omega^+_f, \\
d^-(M) = 1, \\
\delta(M) = (2\pi \sqrt{-1})^{1-\kappa'}, \\
c^+(M)c^-(M) = (\sqrt{-1})^{1-\kappa'}(f, f).
\end{align*}
\]
The Hodge decomposition and the Hodge filtration are given by
\[
H_B(M) \otimes \mathbb{Q} C = H^{0,\kappa-1}(M) \oplus H^{k-1,\kappa',1}(M),
\]
\[
H_{DR}(M) = F^0(M) \supsetneq F^{k-1}(M) \supsetneq \{0\}.
\]
For the motive \( N \), we have \( w(N) = 2\kappa + \kappa' - 3 = w \), and \( d^{-}(N) = 3 \). (Since \( d^{+}(N) = d^{+}(M)d^{+}(M') + d^{-}(M)d^{-}(M') = 3 \).) Following the notation in §3 of [Bha14], we have
\[
\begin{align*}
p_1 &= 0, \\
p_2 &= \kappa' - 1, \\
q_1 &= 0, \\
q_2 &= \kappa - 1, \\
k &= 1, \\
k' &= 1, \\
n &= 2\kappa - 2, \\
\epsilon(M') &= 1.
\end{align*}
\]
Note that \( \mathcal{P} = \mathcal{P}' = \emptyset \).

Assume \( \kappa' \geq 2\kappa \). Then we have
\[
L_{\infty}(N, s) = \zeta_{\mathbb{C}}(s) \zeta_{\mathbb{C}}(s - (\kappa - 1)) \zeta_{\mathbb{C}}(s - (2\kappa - 2)).
\]
Therefore,
\[
H_B(N) \otimes \mathbb{Q} C = H^{0,\omega'}(N) \oplus H^{k-1,2\kappa'-2}(N) \oplus H^{2k-2,\kappa'-1}(N) \oplus H^{\kappa'-1,2\kappa-2}(N) \oplus H^{2\kappa'-2,\kappa-1}(N) \oplus H^{\omega,0}(N),
\]
\[
H_{DR}(N) = F^0(N) \supsetneq F^{k-1}(N) \supsetneq F^{2k-2}(N) \supsetneq F^{\kappa'-1}(N) \supsetneq F^{2\kappa'-2}(N) \supsetneq F^\omega(N) \supsetneq \{0\}.
\]
In the notation of §§2.2 in [Bha14], we have \( k_0 = 3 \) and \( a_3 = 2\kappa - 2 \). Thus \( a_1 = 3 \), \( a_2 = 0 \), \( a_1^3 = 1 \), and \( a_3^3 = 1 \).

Note that \( M' = M'(2\kappa - 2) \). Therefore, by equation (5.1.7) in §5 of [Del79], we have
\[
\begin{align*}
\delta(M') &\sim c^+(M')c^-(M')^{-1} \\
&= c^+(M')c^-(M')^{-1}(2\pi \sqrt{-1})^{-(2\kappa - 2)d^-(M')} \\
&= \delta(M)(2\pi \sqrt{-1})^{-2(\kappa - 2)d^-(M')} \\
&= (2\pi \sqrt{-1})^{3-3\kappa}.
\end{align*}
\]
By Theorem 3.2 in [Bha14], we have
\[
\begin{align*}
c^\pm(N) &= c^\pm(M)\delta(M')(c^+(M)c^-(M)) \\
&= c^\pm(M)(2\pi \sqrt{-1})^{3-3\kappa}(c^+(M)c^-(M)) \\
&= (2\pi \sqrt{-1})^{3-3\kappa}(\sqrt{-1})^{1-\kappa'}(f, f)\Omega^f_1.
\end{align*}
\]
Assume \( 2\kappa > \kappa' \). Then we have
\[
L_{\infty}(N, s) = \zeta_{\mathbb{C}}(s) \zeta_{\mathbb{C}}(s - (\kappa - 1)) \zeta_{\mathbb{C}}(s - (\kappa' - 1)).
\]
If $\kappa < \kappa'$, then
\[
H_B(N) \otimes_{Q} \mathbf{C} = H^{0,\omega}(N) \oplus H^{k-1,\kappa + \kappa'-2}(N) \oplus H^{k-1,2\kappa-2}(N) \oplus H^{k+\kappa'-2,\kappa-1}(N) \oplus H^{\kappa+\kappa'-2,\kappa-1}(N) \oplus H^{0,\omega}(N),
\]
\[
H_{DR}(N) = F^0(N) \supseteq F^{k-1}(N) \supseteq F^{\kappa'-1}(N) \supseteq F^{2\kappa-2}(N) \supseteq F^{\kappa+\kappa'-2}(N) \supseteq F^{\omega}(N) \supseteq \{0\}.
\]
In this case, we have $k_0 = 3$ and $r_3 = \kappa' - 1$.

If $\kappa > \kappa'$, then
\[
H_B(N) \otimes_{Q} \mathbf{C} = H^{0,\omega}(N) \oplus H^{k-1,2\kappa-2}(N) \oplus H^{k-1,\kappa + \kappa'-2}(N) \oplus H^{k+\kappa'-2,\kappa-1}(N) \oplus H^{2\kappa-2,\kappa'-1}(N) \oplus H^{\omega,0}(N),
\]
\[
H_{DR}(N) = F^0(N) \supseteq F^{k-1}(N) \supseteq F^{\kappa'-1}(N) \supseteq F^{\kappa+\kappa'-2}(N) \supseteq F^{2\kappa-2}(N) \supseteq F^{\omega}(N) \supseteq \{0\}.
\]
In this case, we have $k_0 = 3$ and $r_3 = \kappa - 1$.

If $\kappa = \kappa'$, then
\[
H_B(N) \otimes_{Q} \mathbf{C} = H^{0,\omega}(N) \oplus H^{k-1,2\kappa-2}(N) \oplus H^{2\kappa-2,\kappa-1}(N) \oplus H^{\omega,0}(N),
\]
\[
H_{DR}(N) = F^0(N) \supseteq F^{k-1}(N) \supseteq F^{2\kappa-2}(N) \supseteq F^{\omega}(N) \supseteq \{0\}.
\]
In this case, we have $k_0 = 2$ and $r_2 = \kappa - 1$.

In all cases, we have $a_1 = 2$, $a_2 = 1$, $a_1' = 2$, and $a_3' = 0$. By Theorem 3.2 in [Bha14], we have
\[
c^\pm(N) = c^\pm(M) \delta(M)(c^+(M)c^-(M))
\]
\[
\sim \Omega_f^2(2\pi\sqrt{-1})^{1-\kappa}c^+(M)(M(g))c^-(M(g))\delta(M(g))
\]
\[
\sim \Omega_f^2(2\pi\sqrt{-1})^{2-2\kappa}(\sqrt{-1})^{2\kappa}g(2\pi\sqrt{-1})^{1-\kappa}
\]
\[
\sim (2\pi\sqrt{-1})^{2-\kappa}\kappa'(g, g)^2\Omega_f^2.
\]

The formula for $c^\pm(\text{Sym}^3M(f))$ follows from Prop. 7.7 in [Del79], This completes the proof. \(\square\)

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