Dual -1 Hahn polynomials and perfect state transfer

Luc Vinet ¹ and Alexei Zhedanov²

¹ Centre de recherches mathématiques Université de Montréal, P.O. Box 6128, Centre-ville Station, Montréal (Québec), H3C 3J7
² Institute for Physics and Technology, R.Luxemburg str. 72, 83114 Donetsk, Ukraine

Abstract. We find all the XX spin chains with perfect state transfer (PST) that are connected with the dual -1 Hahn polynomials $R_n(x;\alpha,\beta,N)$. For $N$ odd we recover a model that had already been identified while for $N$ even, we obtain a new system exhibiting PST.

AMS classification scheme numbers: 33C45, 33C90

1. Introduction

The transfer of a quantum state is said to be perfect if there is probability 1 of finding after some time at an end point the state introduced as input at an initial site. It has been realized that perfect state transfer (PST) can be achieved in inhomogeneous XX spin chains in particular, provided the 1-excitation energy eigenvalues satisfy a simple spacing condition. (For reviews, see [1], [2].) It has further been shown recently [11] that a unique (up to trivial rescaling) XX spin chain with nearest neighbor interactions and with PST, corresponds to each such spectrum. This matching proceeds through the association of families of orthogonal polynomials in a discrete variable to XX spin chains with PST. The 1-excitation eigenvalues thus intervene as the orthogonality grid points. We have provided an algorithm to construct the XX Hamiltonians from the spectral data and have shown how to obtain different models with PST from a parent system with that property by performing spectral surgery [11].

The exact solvability of the PST models is intimately related to the characterization of the associated orthogonal polynomials. Among the Dunkl or -1 orthogonal polynomials that we have discovered lately [8], one family, namely that of the dual -1 Hahn polynomials, interestingly relates to PST. We shall here obtain the spin chains with PST to which they are associated and find in the process, a new exactly solvable model.

The outline of the paper is the follows. The dual -1 Hahn polynomials and their relevant properties are recalled in section 1. The relation between perfect state transfer in XX chains and orthogonal polynomials theory is reviewed in section 2. Finally, depending on the parity of $N$ the number of sites minus one - two spin chains with PST are identified and analyzed in section 3. The one for $N$ odd had already been reported.
in [5] and further studied in connection with dual Hahn polynomials in [6]. The one for even \(N\) had escaped attention so far. We conclude the paper by showing how these two models are related.

2. Dual -1 Hahn polynomials

Dual -1 Hahn polynomials \(R_n(x; \alpha, \beta, N)\) were introduced in [9] as \(q = -1\) limits of the dual \(q\)-Hahn polynomials.

These polynomials satisfy the 3-term recurrence relation
\[
R_{n+1}(x) + b_n R_n(x) + u_n R_{n-1}(x) = x R_n(x)
\] (2.1)
and depend on an integer \(N = 1, 2, \ldots\) and two real parameters \(\alpha, \beta\).

The expression of the recurrence coefficients depends on the parity of \(N\). When \(N = 2, 4, 6, \ldots\) is even then
\[
u_n = \begin{cases} 
4n(\alpha - n) & \text{if } n \text{ even} \\
4(N-n+1)(\alpha + \beta - n - 1) & \text{if } n \text{ odd}
\end{cases}
\] (2.2)
and
\[
b_n = \begin{cases} 
2N + 1 - \alpha - \beta & \text{if } n \text{ even} \\
-2N - 3 + \alpha + \beta & \text{if } n \text{ odd}
\end{cases}
\] (2.3)
In compact form we have
\[
u_n = 4[n]_\xi [N-n+1]_\eta, \quad b_n = 2([n]_\xi + [N-n]_\eta) + 1 - \alpha - \beta,
\] (2.4)
where
\[
\xi = \frac{\beta - N - 1}{2}, \quad \eta = \frac{\alpha - N - 1}{2}
\]
and
\[
[n]_\mu = n + \mu(1 - (-1)^n),
\] (2.5)
are the so-called ”\(\mu\)-numbers” which appear naturally in problems connected with the Dunkl operators [4].

It is seen that \(u_0 = u_{N+1} = 0\) as required for finite orthogonal polynomials. The positivity condition \(u_n > 0, n = 1, 2, \ldots, N\) is equivalent to the conditions
\[
\alpha > N, \quad \beta > N.
\] (2.6)

Let us define the Bannai-Ito (BI) grid
\[
y_s = \begin{cases} 
-\alpha - \beta + 2s + 1 & \text{if } s \text{ even,} \\
\alpha + \beta - 2s - 1 & \text{if } s \text{ odd.}
\end{cases}
\] (2.7)
The polynomials \(R_n(x)\) are orthogonal on the \(N + 1\) points \(y_0, y_1, \ldots, y_N\) of the BI grid
\[
\sum_{s=0}^{N} w_s y_s R_n(y_s) R_m(y_s) = \kappa_0 u_1 u_2 \ldots u_n \delta_{nm},
\] (2.8)
where the discrete weights are defined as
\[
w_{2s} = (-1)^s (\frac{N/2}{s!})^s \frac{(1-\alpha/2)_s (1-\alpha/2-\beta/2)_s}{(1-\beta/2)_s (N/2+1-\alpha/2-\beta/2)_s}
\] (2.9)
and

\[ w_{2s+1} = (-1)^s \frac{(N/2)_s}{s!} \frac{(1 - \alpha/2)_s(1 - \alpha/2 - \beta/2)_s}{(1 - \beta/2)_s(N/2 + 1 - \alpha/2 - \beta/2)_{s+1}}. \]  

(2.10)

The normalization coefficient is

\[ \kappa_0 = \frac{(1 - \frac{\alpha+\beta}{2})_{N/2}}{(1 - \frac{\beta}{2})_{N/2}}. \]  

(2.11)

Assume that \( \alpha = N + \epsilon_1, \beta = N + \epsilon_2, \) where \( \epsilon_{1,2} \) are arbitrary positive parameters. (This parametrization corresponds to the positivity condition for the dual -1 Hahn polynomials.) Then it is easily verified that all the weights are positive \( w_s > 0, \ s = 0, 1, \ldots, N. \)

Moreover, the spectral points \( y_s \) are divided into two non-overlapping discrete sets of the real line:

\[ \{1 - \delta, -3 - \delta, -7 - \delta, \ldots, -2N + 1 - \delta\} \]

and

\[ \{1 + \delta, 5 + \delta, 9 + \delta, \ldots, 2N - 3 + \delta\}, \]

where \( \delta = \epsilon_1 + \epsilon_2 > 0. \) The first set corresponds to \( y_s \) with even \( s \) and contains \( 1 + N/2 \) points; the second set corresponds to \( y_s \) with odd \( s \) and contains \( N/2 \) points.

When \( N = 1, 3, 5, \ldots \) is odd we have the BI grid \( y_s \)

\[ y_s = \begin{cases} \alpha + \beta + 2s + 1 & \text{if } s \ \text{even} \\ -\alpha - \beta - 2s - 1 & \text{if } s \ \text{odd} \end{cases} \]  

(2.12)

and the recurrence coefficients

\[ u_n = \begin{cases} 4n(N + 1 - n) & \text{if } n \ \text{even} \\ 4(\alpha + n)(\beta + N + 1 - n) & \text{if } n \ \text{odd} \end{cases} \]  

(2.13)

and

\[ b_n = \begin{cases} -1 + \alpha + \beta & \text{if } n \ \text{even} \\ -1 + \alpha - \beta & \text{if } n \ \text{odd} \end{cases}. \]  

(2.14)

In compact form we have

\[ u_n = 4[n]\xi[N - n + 1]_\eta, \quad b_n = 2([n]_\xi + [N - n]_\eta) - 2N - 1 - \alpha - \beta, \]  

(2.15)

with \( \xi = \alpha/2, \eta = \beta/2. \)

Note that the parameters \( \xi \) and \( \eta \) have different expressions in (2.4) or (2.15).

The positivity condition \( u_n > 0, \ n = 1, 2, \ldots, N \) is equivalent either to

\[ \alpha > -1, \beta > -1 \]  

(2.16)

or to \( \alpha < -N, \beta < -N. \) It is sufficient to consider (2.16).

The polynomials \( R_n(x) \) are orthogonal on the set of \( N + 1 \) points \( y_s \)

\[ \sum_{s=0}^{N} w_s R_n(y_s) R_m(y_s) = \kappa_0 u_1 u_2 \ldots u_n \delta_{nm}, \]  

(2.17)
where the discrete weights are defined as
\[ w_{2s} = (-1)^s \frac{(-N - 1)/2)_s}{s!} \frac{(1/2 + \alpha/2)_s(1 + \alpha/2 + \beta/2)_s}{(1/2 + \beta/2)_s(N/2 + 3/2 + \alpha/2 + \beta/2)_s}, \tag{2.18} \]
and
\[ w_{2s+1} = (-1)^s \frac{-(N - 1)/2)_s}{s!} \frac{(1/2 + \alpha/2)_{s+1}(1 + \alpha/2 + \beta/2)_s}{(1/2 + \beta/2)_{s+1}(N/2 + 3/2 + \alpha/2 + \beta/2)_s}. \tag{2.19} \]

The normalization coefficient is
\[ \kappa_0 = \frac{(1 + \frac{\alpha+\beta}{2})_{(N+1)/2}}{(\frac{\beta+1}{2})_{(N+1)/2}}. \tag{2.20} \]

Assume that \( \alpha = -1 + \epsilon_1, \ \beta = -1 + \epsilon_2 \), where \( \epsilon_{1,2} \) are arbitrary positive parameters. (This parametrization corresponds to the positivity condition for the dual -1 Hahn polynomials for \( N \) odd.) Then it is easily verified that the weights are positive \( w_s > 0 \), \( s = 0, 1, \ldots, N \).

Again, the spectral points \( y_s \) are divided into two non-overlapping discrete sets of the real line:
\[ \{-1 - \delta, -5 - \delta, -9 - \delta, \ldots, -2N + 1 - \delta\} \]
and
\[ \{-1 + \delta, 3 + \delta, 7 + \delta, \ldots, 2N - 3 + \delta\}, \]
where \( \delta = \epsilon_1 + \epsilon_2 > 0 \). Both sets contain \( (N - 1)/2 \) points.

The dual -1 Hahn polynomials can be explicitly expressed in terms of the ordinary dual Hahn polynomials; depending on whether \( N \) and \( n \) are even or odd, the formulas take different forms (see [9] for details).

In [10] it was shown that the -1 dual Hahn polynomials appear as Clebsch-Gordan coefficients for the irreducible representations of the algebra \( sl_{-1}(2) \) which is the \( q = -1 \) limit of the quantum algebra \( sl_q(2) \). The algebra \( sl_{-1}(2) \) is generated by 4 operators \( J_0, J_+, J_-, R \) satisfying the commutation relations
\[ [J_0, J_{\pm}] = \pm J_{\pm}, \quad [R, J_0] = 0, \quad \{J_+, J_-\} = 2J_0, \quad \{R, J_{\pm}\} = 0, \] \tag{2.21} 
where \( [A, B] = AB - BA \) and \( \{A, B\} = AB + BA \). The operator \( R \) is an involution operator, i.e. it satisfies the property
\[ R^2 = I. \tag{2.22} \]

The Casimir operator \( Q \), commuting with \( J_0 \) and \( J_{\pm} \) is [10]
\[ Q = J_+ J_- q^{-J_0} - \frac{2}{(q^2 - 1)(q - 1)}(q^{J_0} - 1 + q^{-J_0}). \tag{2.23} \]

The algebra \( sl_{-1}(2) \) admits a nontrivial addition rule (i.e. coproduct)
\[ \tilde{J}_0 = J_0 \otimes I + I \otimes J_0, \quad \tilde{J}_{\pm} = J_{\pm} \otimes R + I \otimes J_{\pm}, \quad \tilde{R} = R \otimes R. \tag{2.24} \]
such that the operators \( \tilde{J}_0, \tilde{J}_{\pm}, \tilde{R} \) again satisfy relations (2.21).

The Clebsch-Gordan coefficients (CGC) of \( sl_{-1}(2) \) arise as overlap coefficients between two canonical bases in the coproduct representation space. For details on the identification of the CGC with -1 dual Hahn polynomials see [10].
3. Perfect state transfer and orthogonal polynomials

Consider the spin chain with Hamiltonian

\[ H = \frac{1}{2} \sum_{l=0}^{N-1} J_{l+1} (\sigma^x_l \sigma^x_{l+1} + \sigma^y_l \sigma^y_{l+1}) + \frac{1}{2} \sum_{l=0}^{N} b_l (\sigma^z_l + 1), \]  

(3.1)

where \( J_l > 0 \) are the constants coupling the sites \( l - 1 \) and \( l \) and \( b_l \) are the strengths of the magnetic field at the sites \( l \) (\( l = 0, 1, \ldots, N \)). The symbols \( \sigma^x_l, \sigma^y_l, \sigma^z_l \) stand for the Pauli matrices.

Introduce the basis vectors

\[ |e_n\rangle = (0, 0, \ldots, 1, \ldots, 0), \quad n = 0, 1, 2, \ldots, N, \]

where the only 1 (spin up) occupies the \( n \)-th position. In that basis, the restriction \( J \) of \( H \) to the one-excitation subspace is given by the following \((N + 1) \times (N + 1)\) Jacobi matrix

\[
J = \begin{pmatrix}
    b_0 & J_1 & 0 & \ldots & 0 \\
    J_1 & b_1 & J_2 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & J_N & b_N
\end{pmatrix}
\]

Equivalently, we have.

\[ J|e_n\rangle = J_{n+1}|e_{n+1}\rangle + b_n|e_n\rangle + J_n|e_{n-1}\rangle. \]  

(3.2)

The boundary conditions

\[ J_0 = J_{N+1} = 0 \]  

(3.3)

are assumed.

Let \( x_s, \ s = 0, 1, \ldots, N \) denote the eigenvalues of the matrix \( J \). They are all real and nondegenerate. They are labeled in increasing order, i.e. \( x_0 < x_1 < x_2 < \ldots x_N \).

To the Jacobi matrix \( J \) one can associate the monic orthogonal polynomials \( P_n(x) \) defined by the 3-term recurrence relation

\[ P_{n+1}(x) + b_n P_n(x) + u_n P_{n-1}(x) = xP_n(x), \]

\[ n = 0, 1, \ldots, N, \quad P_{-1} = 0, \ P_0(x) = 1, \]  

(3.4)

where \( u_n = J_n^2 > 0 \).

\( P_{N+1}(x) \) is the characteristic polynomial

\[ P_{N+1}(x) = (x - x_0)(x - x_1) \ldots (x - x_N). \]  

(3.5)

The orthogonality relation reads

\[ \sum_{s=0}^{N} P_n(x_s)P_m(x_s)w_s = h_n \delta_{nm}, \]  

(3.6)

where

\[ h_n = u_1 u_2 \ldots u_n \]
and the discrete weights \( w_s > 0 \) are uniquely determined by the recurrence coefficients \( b_n, u_n \).

Perfect state transfer (PST) occurs \([2]\) if there exists a time \( T \) such that
\[
e^{iTJ}|e_0\rangle = e^{i\phi}|e_N\rangle,
\]
where \( \phi \) is a real number. This means that the initial state \(|e_0\rangle\) evolves into the state \(|e_N\rangle\) (up to an inessential phase factor \( e^{i\phi} \)).

It is known that the PST property is equivalent to the two conditions \([2]\):

(i) the eigenvalues \( x_s \) satisfy
\[
x_{s+1} - x_s = \frac{\pi}{T} M_s,
\]
where \( M_s \) are positive odd numbers.

(ii) the matrix \( J \) is mirror-symmetric \( RJR = J \), where the matrix \( R \) (reflection matrix) is
\[
R = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & \ldots & 0 & 0
\end{pmatrix}
\]

Property (ii) is equivalent to the property \([11]\)

(ii') the weights \( w_s \) (up to a normalization) are given by the expression
\[
w_s = \frac{1}{|P'_{N+1}(x_s)|} > 0.
\]

4. **XX spin chains with PST and and dual -1 Hahn OPs**

We now wish to determine the circumstances for which the Jacobi matrix \( J \) corresponding to the -1 dual Hahn polynomials will possess the PST property.

Crucial is the mirror symmetry condition (ii). In terms of the recurrence coefficients it means
\[
u_{N-n+1} = u_n, \quad b_{N-n} = b_n
\]
for all \( n = 0, 1, \ldots N \).

Consider first the case of odd \( N \). In this case formulas (2.15) immediately imply that conditions (4.1) hold iff \( \xi = \eta \), or, equivalently, iff \( \alpha = \beta \). Under this condition we have
\[
u_n = 4[n]_{\alpha/2}[N - n + 1]_{\alpha/2}, \quad b_n = -1.
\]
The positivity condition for the measure is \( \alpha < -1 \).

The spectrum \( x_s \) of the Jacobi matrix \( J \) coincides with the BI grid (2.12) and consists of two uniform subgrids \( G_- \) and \( G_+ \) (with a step of 4 between the neighbor points).

The subgrid \( G_- \) consists of the \((N + 1)/2\) points \( x_0 < x_1 < \ldots < x_{(N-1)/2} \):
\[
G_- = \{-2N + 1 + \delta, -2N + 5 + \delta, \ldots, -1 - \delta\}.
\]
The subgrid $G_+$ consists of the $(N + 1)/2$ points $x_{(N+1)/2} < x_{(N+3)/2} < \ldots < x_N$:

$$G_+ = \{-1 + \delta, 4 + \delta, \ldots, 2N - 3 + \delta\}$$

(4.4)

where $\delta = 2(\alpha + 1) > 0$. There is a gap of length $2\delta = 4(\alpha + 1) > 0$ between these 2 subgrids.

Condition (i) in this case is equivalent to the restriction

$$\alpha = \frac{M_2}{M_1}, \quad M_2 > M_1,$$

(4.5)

where $M_2$ is even and $M_1$ is odd ($M_2$ and $M_1$ are assumed to be coprime).

The Jacobi matrix with coefficients (4.2) and with the PST property, was first introduced in [5]. In [6] the corresponding orthogonal polynomials $P_n(x)$ were related to the dual Hahn polynomials. We see, that in fact these polynomials coincide with the dual -1 Hahn polynomials for $N$ odd. Note that in [5], [6] the diagonal coefficients vanish $b_n = 0$. This corresponds to a simple shift of the argument of the polynomials $P_n(x) \rightarrow P_n(x - 1)$ as is seen from (4.2).

Consider now the case of $N$ even. The recurrence coefficients are given by formulas (2.15)

The mirror symmetry condition (4.1) is again equivalent to the condition $\alpha = \beta$. We then have

$$u_n = 4[n]_\xi [N - n + 1]_\xi, \quad b_n = 4\xi (-1)^{n+1} - 1,$$

(4.6)

where $2\xi = \alpha - N - 1$. The positivity condition implies $\xi > -1/2$.

The spectrum $x_s$ of the corresponding Jacobi matrix $J$ coincides with the BI grid (2.7) and consists of two uniform grids (with a step of 4 between the neighbor points) containing $N + 1$ points and $N$ points. There is a gap of length $4(\alpha - N) > 0$ between these sets.

Condition (i) in this case is equivalent to the restriction

$$\alpha = N + \frac{M_1}{M_2},$$

(4.7)

where $M_1, M_2$ are positive coprime odd integers.

This example seems to be have been overlooked. Note that in contrast to the case with $N$ odd, the diagonal recurrence coefficients $b_n$ in (4.6) are not constant. The only case when $b_n = -1$ occurs when $\xi = 0$ which corresponds to the Krawtchouk polynomials.

It is interesting to point out that there is a connection based on the Christoffel transform between these two spin models with PST. To see this, suppose we have two models $A$ and $B$. Assume that the spin chain $A$ with PST corresponds to the Jacobi matrix $J_A$ with $N + 1$ spectral points $x_0, x_1, \ldots, x_N$ that satisfy condition (i). Similarly, take spin chain $B$ with PST to correspond to the Jacobi matrix $J_B$ with $N$ eigenvalues $x_0, x_1, x_2, \ldots, x_{N-1}$ also satisfying (i). The matrix $J_A$ has dimension $N + 1 \times N + 1$ while the matrix $J_B$ has dimension $N \times N$ and the spectrum of $J_B$ differs from the spectrum of $J_A$ by the elimination of the level $x_N$. It is then easy to show [11] that the matrix $J_B$ is
obtained from the matrix $J_A$ by a Christoffel transform. Equivalently, this means that the monic orthogonal polynomials $\tilde{P}_n(x)$ corresponding to the matrix $J_B$ are obtained from the polynomials $P_n(x)$ corresponding to matrix $J_A$ by the formula

$$\tilde{P}_n(x) = \frac{P_{n+1}(X) - K_n P_n(x)}{x - x_N},$$  \hspace{1cm} (4.8)

where

$$K_n = \frac{P_{n+1}(x_N)}{P_n(x_N)}. \hspace{1cm} (4.9)$$

Formula (4.8) is equivalent to the well known Christoffel transform for the orthogonal polynomials [7]. The corresponding recurrence coefficients can be obtained by the formulas [11]

$$\tilde{u}_n = u_n \frac{K_n}{K_{n-1}}, \quad \tilde{b}_n = b_{n+1} + K_{n+1} - K_n.$$

Returning to our systems, assume that the Jacobi matrix $J_A$ has size $N + 1 \times N + 1$ with odd $N$ and corresponds to the spin chain with coefficients (4.2). The spectrum of this matrix consists of $N + 1$ points $x_s$ given by (4.3), (4.4). This matrix corresponds to the PST model proposed in [5]. The Jacobi matrix $J_B$ has size $N \times N$ and its spectrum consists of the two subgrids:

$$\tilde{G}_- = \{x_0, x_1, \ldots, x_{(N-1)/2}\} = G_- \hspace{1cm} (4.11)$$

and

$$\tilde{G}_+ = \{x_{(N+1)/2}, x_{(N+3)/2}, \ldots x_{N-1}\}, \hspace{1cm} (4.12)$$

i.e. the subgrid $\tilde{G}_-$ coincides with $G_-$ and contains $(N + 1)/2$ points. The subgrid $\tilde{G}_+$ contains $(N - 1)/2$ points.

In order to obtain the recurrence coefficients $\tilde{u}_n, \tilde{b}_n$ corresponding to the matrix $J_B$ we need the expression for the coefficients $K_n$ resulting from (4.9). A simple calculation gives

$$K_n = \begin{cases} 2(N + \alpha - n) & \text{if } n \text{ even} \\ 2(N - n) & \text{if } n \text{ odd} \end{cases}.$$  \hspace{1cm} (4.13)

Equivalently, in terms of $\mu$-numbers (2.5), we have

$$K_n = 2[N - n]_{\alpha/2}. \hspace{1cm} (4.14)$$

Substituting this expression into formulas (4.10) we obtain

$$\tilde{u}_n = 4[n]_{\alpha/2}[N - n]_{\alpha/2}, \quad \tilde{b}_n = -3 - 2(-1)^n \alpha.$$  \hspace{1cm} (4.15)

These recurrence coefficients correspond to (4.6) (replacing $\alpha \rightarrow 2\xi$ and performing a shift of the coefficient $b_n$).

We have thus demonstrated how the spin chain with the PST property for even $N$ can be obtained from the corresponding spin chain for odd $N$.

It was shown in [11] that provided the eigenvalues $x_s$, $s = 0, 1, \ldots$ of the Jacobi matrix $J$ satisfy the condition (i), the weight function $w_s$ constructed from (3.9)
determines uniquely the mirror-symmetric matrix $J$ and hence, the spin chain with PST.

Thus, the only spin chain with the PST property, corresponding to the eigenvalues $x_s$ of the BI type (2.7), (2.12) are those whose Jacobi matrices $J$ generate the dual $-1$ Hahn polynomials with $\beta = \alpha$. We should stress, that the positivity conditions (2.6) and (2.16) are crucial. They imply, in particular that the BI grid consists of two uniform subgrids separated by a positive gap. Only for BI grids of such kind can one associate the dual $-1$ Hahn polynomials uniquely. More general BI grids with an overlap of the two subgrids lead to more complicated orthogonal polynomials. This problem will be considered elsewhere.

**Acknowledgments**

AZ thanks Centre de Recherches Mathématiques (Université de Montréal) for hospitality. The authors would like to thank Mathias Christandl for stimulating discussions. The research of LV is supported in part by a research grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada.
[1] C. Albanese, M. Christandl, N. Datta, A. Ekert, *Mirror inversion of quantum states in linear registers*, Phys. Rev. Lett. 93 (2004), 230502.

[2] A. Kay, *A Review of Perfect State Transfer and its Application as a Constructive Tool*, arXiv:0903.4274.

[3] R. Koekoek, R. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report no. 98-17, Delft University of Technology, 1998.

[4] M. Rosenblum, *Generalized Hermite Polynomials and the Bose-like Oscillator Calculus*, in: Oper. Theory Adv. Appl., vol. 73, Birkhauser, Basel, 1994, pp. 369–396. ArXiv:math/9307224.

[5] T. Shi, Y. Li, A. Song, C. P. Sun, *Quantum-state transfer via the ferromagnetic chain in a spatially modulated field*, Phys. Rev. A 71 (2005), 032309, 5 pages, quant-ph/0408152.

[6] N. Stoilova, J. Van der Jeugt, *An exactly solvable spin chain related to Hahn polynomials*, SIGMA 7 (2011), 033, 13 pages.

[7] G. Szegő, *Orthogonal Polynomials*, fourth edition, AMS, 1975.

[8] S. Tsujimoto, L. Vinet and A. Zhedanov, *Dunkl shift operators and Bannai-Ito polynomials*, arXiv:1106.3512.

[9] S. Tsujimoto, L. Vinet and A. Zhedanov, *Dual -1 Hahn polynomials: "classical" polynomials beyond the Leonard duality*, arXiv:1108.0132.

[10] S. Tsujimoto, L. Vinet and A. Zhedanov, *From sl_q(2) to a Parabosonic Hopf Algebra*, SIGMA 7 (2011), 093, 13 pages. arXiv:1108.1603.

[11] L. Vinet and A. Zhedanov, *How to construct spin chains with perfect state transfer*, arXiv:1110.6474.