A JOINT DYNAMIC PRICING AND PRODUCTION MODEL WITH ASYMMETRIC REFERENCE PRICE EFFECT

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(Communicated by Cheng-chew Lim)

Abstract. Reference price plays a significant role in influencing purchase decisions of customers. Due to loss aversion, the asymmetric reference price effect on market demand should be taken into account. This paper develops a joint dynamic pricing and production model with asymmetric reference price effect. In a finite planning horizon, the demand rate is time-varying and depends on price as well as reference price. The decision-making problem with the asymmetric reference price effect turns to be a nonsmooth optimal control problem, which cannot be solved by standard optimal control method. As a special case, we first obtain the joint optimal dynamic pricing and production strategy with symmetric reference price effect by solving the corresponding standard optimal control problem based on Maximum principle. For the case of asymmetric reference price effect, we propose a systematical method on basis of optimality principle to solve the nonsmooth optimal control problem, and obtain the joint strategy. Numerical examples are employed to illustrate the effectiveness of the proposed method. In addition, we assess the sensitivity analysis of system parameters to examine the impacts of asymmetric reference price on optimal pricing and production strategies and total profits.

1. Introduction. Reference price, as a cognitive price benchmark which depends on past product prices and affects purchase decisions of consumers (Kalwani [20]), has received a great deal of attention recently. Numerous studies have indicated that consumer demand will be significantly influenced by reference price (Lattin and Bucklin [24], Popescu and Wu [34]). The reason is that consumers have memory and can form reference price based on past or similar products prices (Taudes and Rudloff [42]). The impact of reference price on demand is mainly due to the differences between the reference price and the shelf price. Specifically, when the current price is lower than the reference price, consumers are likely to sense a gain, which will lead to accelerate demand, and oppositely, when the current price is higher than the reference price, consumers probably feel a loss, which will have a negative effect on demand (Kalyanaram and Little [21]). This demonstrates that the reference price surely influences demand and the whole profit. In addition, the asymmetry in reference price effect should be taken into consideration since that consumers are generally loss-averse according to prospect theory (Kahneman and
Tversky [19]). It means that the effect of losses on their demand is greater than that of gains (Lattin and Bucklin [24], Mazumdar and Papatla [29], Rajendran and Tellis [36], Winer [48]). Due to the asymmetric reference-price effect, the decision-making model to be solved turns to a nonsmooth optimization problem. Fibich et al. [13] proposed a two-stage method to address this problem. Nevertheless, it is hard to apply the method presented in Fibich et al. [13] to solve the optimization problem which contains more than one system state, e.g., the case involving not only reference price but also other variables, like inventory level, being dynamic.

In practice, both the operations strategy and the marketing strategy are important in maximizing the total profits of a firm. Pricing and production, acting as two typical operations and marketing strategies, play key roles in the revenue management for an oligopolistic firm. Based on above considerations, in this paper, we address the joint optimal pricing and production decision-making problem to maximize the profit for a firm in a finite planning horizon where both reference price and inventory level are dynamic, while simultaneously considering the asymmetric reference price effect. The demand function varies in time driven by current price as well as the reference price. The main features contained by our model are as follows: (1) Asymmetric reference price effect; (2) A continuous, dynamic and finite time decision-making environment; (3) Two controls over time: price and production rate. Compared to the existing literature, the joint dynamic pricing and production decision-making model with asymmetric reference price effect presented in this paper is of higher generality and more tally with the actual situations. In this case, the dynamic decision-making problem turns to be a nonsmooth optimal control problem involving two system state variables and two control variables, which is too complex to be directly solved by applying the standard optimal control method or the method presented by Fibich et al. [13]. Thus in this paper we propose a systematical method based on optimality principle and Maximum principle to solve the nonsmooth optimization problem. Numerical examples are employed to illustrate the effectiveness of the solution method. Furthermore, we evaluate the sensitivity analysis of key system parameters to examine their impacts on optimal pricing and production strategies and total profits. To the best of our knowledge, there is no similar method in existing literature to figure out the decision-making model of dynamic pricing and production while simultaneously considering the asymmetric reference price effect.

The remainder of this paper is organized as follows. Section 2 reviews the literature. In Section 3, a joint dynamic pricing and production model with the asymmetric reference-price effect is developed. In Section 4, a systematical method is proposed to solve the corresponding dynamic decision-making problem. In Section 5, numerical examples and sensitivity analysis are presented. Finally, conclusions and future research directions are made in Section 6.

2. Literature review. Literature related to this paper mainly focuses on two aspects. One is about reference price effect, and the other is about joint pricing and production policy.

As an important factor which affects the consumer demand, reference price has been studied since 1980s. Lattin and Bucklin [24] demonstrated that the reference price was consistent with some psychological theories of price perception. Since then, plentiful empirical researches about the effect of reference price on consumer buying behaviors have been discussed. Kalyanaram and Winer [22], Raman and Bass [37], Rust and Zahorik [39] examined the effects of reference price on the
market share of a brand by empirical studies. Kalwani [20], Lattin and Bucklin [24], Pauwels and Siddarth [32] employed empirical researches and considered not only reference price but also other factors, such as inventory level, which influence consumer’s purchase decision. With the development of theoretical studies on reference price, its formation and application in marketing have been studied by an increasing number of researchers. Sorger [41] and Winer [48] indicated that reference price can be assumed as a continuous weight average of past prices with an exponentially decaying weighting function. This formation has been adopted by a large amount of studies, especially in the field of supply chain management, for instance, Zhang et al. [53] and Zhang et al. [55]. Xue et al. [49] also employed the above formation of reference price and obtained optimal dynamic pricing strategy for deteriorating items. Nasiry and Popescu [31] developed a peak-end model in which reference price was a weighted average of the lowest and most recent prices. Mazumdar et al. [30], Arslan and Kachani [3] provided a review of models with reference price. Liu et al. [26] developed a dynamic pricing model of network goods for two firms with bounded rational consumers simultaneously considering the evolving consumers’ preference to the network products. Bi et al. [5] took the stochastic reference effect into account and found that the results under the assumption of stochastic recall memory of consumers differ from findings of certain models. Lin [25] considered the price promotion in a supply chain when taking into account the reference price effects and showed that the reference price effects could mitigate double marginalization effects. Yang et al. [51] studied dynamic pricing strategies with reference effect for a seller facing a classical revenue management problem with stochastic demand, finite horizon, and fixed capacity.

Moreover, a couple of literature associated with reference price researched the optimal pricing strategies in an asymmetric framework. For example, Fibich et al. [13] explored explicit solutions to the problems of how to decide the optimal pricing policy under asymmetric reference-price effects and suggested that the solution structure of optimal pricing strategy was similar to that in the symmetric case, which was in line with the findings of Popescu and Wu [34]. Fibich et al. [14] studied the retailer’s optimal price promotions in the presence of asymmetric reference-price effects.

Another stream of research focuses on the joint pricing and production policy. This literature has been substantially growing since Whitin [47] first studied the problem with the demand dependent on price. Bitran and Caldentey [6], Elmaghraby and Keskinocak [12] and Ramasesh [38] gave overviews of recent developments of pricing models and current practices. Pekelman [33], Adida and Perakis [1] proposed a continuous time optimal control model without shortage and used Pontryagin’s maximum principle to solve the joint pricing and production strategy problem. Bakal et al. [4] analyzed the problems involving the selection of markets or customers and inventory models with price-sensitive demand. Transchel and Minner [43] provided both analytical and numerical results on the impact of joint optimization on the pricing and replenishment decisions. Kabirian [18] considered some linear and non-linear functions of demand and different costs of production and obtained the optimal production quantity and sale price. Tsao and Sheen [44] and Zhang et al. [54] studied joint pricing and replenishment decision for deteriorating items considering lot-size and preservation technology investment, respectively. Yang et al. [50] obtained optimal pricing and replenishment strategies
for a multi-market deteriorating product with time-varying and price-sensitive demand. Ghoreishi et al. [15] studied joint pricing and replenishment decisions for non-instantaneous deteriorating items with partial backlogging and customer returns. Lu et al. [28] focused on the optimal dynamic pricing and replenishment policy for perishable items by introducing limited replenishment capacity. Rabbani et al. [35] proposed a model about joint optimization of dynamic pricing and inventory replenishment policies for items with simultaneous deterioration of quality and physical quantity. Based on dynamic programming methodology, Zhang et al. [52], and Caccetta and Mardaneh [8] developed finite-horizon joint decision-making models to obtain pricing and production policies under the assumption that the unsatisfied demand was backlogged. Under supply chain environment, Arcelus et al. [2] compared the pricing and ordering policies when retailers were risk-neutral, risk-averse and risk-seeking, respectively. Guajardo et al. [16] proposed an optimization model involving pricing and production decisions and approached a coordination mechanism in both single-period and multi-period models. Chen and Xiao [9] studied pricing and replenishment policies for a supply chain with multiple competing retailers with both retail-competition and retail-cooperation models. Chen [10] examined the optimizing pricing, replenishment and rework decisions for imperfect and deteriorating items in a manufacturer-retailer channel.

Several researches not only considered reference price effects, but also took joint pricing and replenishment strategy into account. Urban [45] analyzed a single-period joint inventory and pricing model with reference price effects, demand uncertainty and price elasticity. Wang et al. [46] studied how an electronic product supply chain performs and what are the optimal production and pricing decisions under reference price effects. Dye and Yang [11] proposed a joint dynamic pricing and preservation technology investment model for a deteriorating inventory system with reference price effects in a discrete environment. Hsieh and Dye [17] characterized the optimal selling pricing and inventory strategies for deteriorating items with a demand rate that depends on displayed stock level and selling price simultaneously by incorporating reference price effects. Liu et al. [27] obtained a newsvendor’s optimal pricing and ordering quantity decisions to discuss the role of reference effect on newsvendor’s decision behavior in a market with strategic customers. However, as far as we know, extant research is silent on studying the problem of joint continuous and dynamic pricing and production while simultaneously considering the asymmetric reference price effects. In addition, we apply the optimal control theory to solve the corresponding optimization problem, distinguishing our work with others which mainly deal with static or discrete optimization models. Moreover, the asymmetric reference effects in a continuous and dynamic environment leads to a nonsmooth optimization problem, which cannot be solved by the standard optimal control method, and we propose a systematic method on basis of optimality principle to solve it. This innovative method serves as a complement to the existing literature with nonsmooth optimal control problem.

3. Model formulation. In this section, we consider the joint dynamic pricing and production policy for products. The influence of reference price on product demand has been taken into account. When consumers decide whether to buy a product or not, they will compare the current price of the product with the reference price, which can be affected by the consumers’ memory of past prices. Denote by \( r(t) \) and \( p(t) \) the consumers’ reference price and current price of the product at time \( t \), respectively. According to Sorger [41] and Winer [48], the reference price \( r(t) \) can...
be assumed as a continuous weight average of past prices \( p(t) \) with an exponentially decaying weighting function, which follows
\[
\dot{r}(t) = \delta (p(t) - r(t)), \quad r(0) = r_0,
\]
where \( r_0 \) is the initial reference price, and a constant \( \delta \) means the memory effect.

According to Fibich et al. [13], the demand function can be specified by
\[
D(t) = \alpha - \beta p(t) - \eta (p(t) - r(t)),
\]
where \( \alpha, \beta, \eta \) are positive constants. \( \alpha \) represents the basic market potential, \( \beta \) denotes the sensitivity of the demand with respect to price, and \( \eta \) implies the sensitivity of consumers to the gap between the reference price and the sales price.

According to prospect theory (Kahneman and Tversky [19]) and empirical studies (Briesch et al. [7], Kalwani [20], Krishnamurthi et al. [23], Lattin and Bucklin [24]), the effect of losses on demand is larger than that of gains, which is the well-known loss aversion. Thus, \( \eta \) is given by
\[
\eta = \begin{cases} 
\eta_1, & r \leq p, \\
\eta_2, & r > p,
\end{cases}
\]
where \( \eta_1 \geq \eta_2 \geq 0 \). The loss effect of the reference price on demand is greater than the gain effect when \( \eta_1 > \eta_2 \). The asymmetry in reference price effect disappears when \( \eta_1 = \eta_2 \).

During a fixed selling cycle \([0, T]\), a firm selling durable products should decide the price \( p(t) \) and the production rate \( u(t) \). The inventory level dynamics can be described by
\[
\dot{I}(t) = -D(t) + u(t), \quad I(0) = I_0, \quad I(T) = 0,
\]
where \( I_0 \) is the initial inventory level, \( I(T) = 0 \) implies that there is no backlog and no salvage in the end of the selling period.

Note that \( I(t) \geq 0 \) means inventory level and that \( I(t) < 0 \) means backlog level.

The inventory (or backlog) cost function and the production cost function are assumed to be quadratic in the inventory and production, respectively, i.e.,
\[
C_I(I) = \frac{h}{2} I^2, \quad C_u(u) = \frac{g}{2} u^2,
\]
where \( h > 0, g > 0 \) are the measure values of inventory (or backlog) cost and production cost, respectively.

Hence, we can get the objective function of the firm as
\[
J = \int_0^T (p(t)D(t) - C_I(I) - C_u(u)) \, dt.
\]

The goal of this paper is to obtain the joint optimal pricing and production policy maximizing the profit \( J \). Taking Equations (1)-(7) together, the optimization
4. Solution method. In this section, we first solve the optimization problem with symmetric effect of reference price based on Maximum principle, then propose a systematical method to solve the optimization problem in the case of asymmetric effect of reference price. To facilitate understanding, we use the subscript “s”, “a” to signify “symmetric case”, “asymmetric case”, respectively.

4.1. The symmetric case. Consider the optimal control problem (8) with $\eta = \eta_1 = \eta_2$. In this case, the joint optimal pricing and production problem turns to be a standard linear quadratic optimal control problem involving two states and two controls.

Applying the optimal control theory and introducing two adjoint variables $\lambda_1, \lambda_2$ associated with the state variables $I, r$, respectively, we can get the Hamiltonian function as follows

$$H(I, r, p, u, \lambda_1, \lambda_2, t) = p(\alpha - \beta p - \eta(p - r)) - \frac{h}{2}I^2 - \frac{g}{2}u^2 + \lambda_1(-\alpha - \beta p - \eta(p - r)) + \lambda_2(p - r).$$

(9)

According to the Maximum principle (Sethi and Thompson [40]), the optimal price $p^*_s$ and production rate $u^*_s$ of the symmetric case should maximize the Hamiltonian function $H$ at each instant $t$, namely,

$$p^*_s = \text{arg max}_{p(t)} H(I, r, p, u, \lambda_1, \lambda_2, t),$$

(10)

$$u^*_s = \text{arg max}_{u(t)} H(I, r, p, u, \lambda_1, \lambda_2, t).$$

(11)

Considering the adjoint variables $\lambda_1, \lambda_2$ which satisfy $\dot{\lambda}_1 = -\frac{\partial H}{\partial I}$ and $\dot{\lambda}_2 = -\frac{\partial H}{\partial r}$, respectively, we can get

$$\dot{\lambda}_1 = hI,$$

(12)

$$\dot{\lambda}_2 = \eta\lambda_1 + \delta \lambda_2 - \eta p, \quad \lambda_2(T) = 0,$$

(13)

where the transversality condition $\lambda_2(T) = 0$ comes from the freedom of the state variable $r(t)$ in the terminal time $T$.

The following proposition characterizes the optimal pricing policy $p^*_s$, production policy $u^*_s$, reference price $r^*_s$ and inventory level $I^*_s$ for the symmetric problem.

**Proposition 1.** In the symmetric scenario, the optimal pricing policy $p^*_s$, production policy $u^*_s$, inventory level $I^*_s$ and reference price $r^*_s$ during the interval $[0, T]$, 

...
respectively, are given by
\[
p(s)(t) = \frac{1}{2c_0} \left( k_1(c_0 + c_1 - c_2)e^{r_1t} + k_2(c_0 + c_1 + c_2)e^{-r_1t} + k_3(c_0 + c_1 + c_3)e^{r_3t} + k_4(c_0 + c_1 - c_3)e^{-r_3t} + c_4 \right),
\]
\[
u(s)(t) = \frac{1}{g} \left( k_1 e^{r_1t} + k_2 e^{-r_1t} + k_3 e^{r_3t} + k_4 e^{-r_3t} \right) - \frac{\alpha}{\beta g + 2},
\]
\[
I(s)(t) = \frac{k_1 m_1}{2h g(\eta + \beta)} e^{r_1t} - \frac{k_2 m_1}{2h g(\eta + \beta)} e^{-r_1t} + \frac{k_3 m_2}{2h g(\eta + \beta)} e^{r_3t} - \frac{k_4 m_2}{2h g(\eta + \beta)} e^{-r_3t},
\]
\[
\lambda(s)(t) = \frac{\eta + \beta}{c_0 \delta} \left( k_1(c_1 - c_2)e^{r_1t} + k_2(c_1 + c_2)e^{-r_1t} + k_3(c_1 + c_3)e^{r_3t} + k_4(c_1 - c_3)e^{-r_3t} + k_4(c_1 - c_3)e^{-r_3t} - \frac{(g\beta + 1)\eta}{(g\beta + 2)\beta} \right),
\]
and the corresponding adjoint variables are
\[
\lambda_1 = k_1 e^{r_1t} + k_2 e^{-r_1t} + k_3 e^{r_3t} + k_4 e^{-r_3t} - \frac{g\alpha}{\beta g + 2},
\]
\[
\lambda_2 = \frac{\eta + \beta}{c_0 \delta} \left( k_1(c_1 - c_2)e^{r_1t} + k_2(c_1 + c_2)e^{-r_1t} + k_3(c_1 + c_3)e^{r_3t} + k_4(c_1 - c_3)e^{-r_3t} - \frac{\eta \alpha}{\delta \beta (g\beta + 2)} \right),
\]
where \(m_1, m_2, r_1, r_3, c_i, i = 0, ..., 4\) are characterized in the Appendix, and the constants \(k_1, k_2, k_3\) and \(k_4\) simultaneously satisfy the following equations
\[
\begin{align*}
I(s)(0) &= I_0, \\
r(s)(0) &= r_0, \\
I(s)(T) &= 0, \\
\lambda_2(T) &= 0.
\end{align*}
\]

**Proof.** See the Appendix. \(\square\)

It should be mentioned that Proposition 1 gives the analytical solutions for the optimal controls and system states except the parameters \(k_1, k_3, k_3\) and \(k_4\) which can be numerically calculated from (18). With the results presented in Proposition 1, one can compute the maximum profit.

4.2. The asymmetric case. Consider the optimal control problem (8) with \(\eta_2 < \eta_1\). Note that the asymmetric reference effect does not work in solving (8) if there is no crossing in \(p(t)\) and \(r(t)\) over the whole horizon \([0, T]\). In this case, the optimal solutions are the same as those presented in Proposition 1. Specifically, one can get the optimal solutions by setting \(\eta = \eta_1\) in Proposition 1 if \(r(t) \leq p(t)\), for any \(t \in [0, T]\); Otherwise, one can get the optimal solutions by setting \(\eta = \eta_2\).

In the following, we consider a general case that \(p(t)\) and \(r(t)\) have at least one intersection, which incurs a nonsmooth optimization problem. Specifically, the asymmetry of the reference price effect on demand stemming from the loss aversion leads to the nonsmooth inventory dynamics and the non-continuity of objective functional with respect to the reference price, and brings about a nonsmooth optimal control problem (8) which cannot be solved by using the standard optimal control method. To solve the nonsmooth optimization problem, we first consider that there
only exits one intersection of \( p(t) \) and \( r(t) \) with the intersection time denoted as \( \tau \) which satisfies \( p(\tau) = r(\tau) \). Consider a low initial reference price \( r_0 \) such that \( r \leq p \) for \( t \in [0, \tau] \). Then we have

\[
\eta = \begin{cases} 
\eta_1, & t \in [0, \tau], \\
\eta_2, & t \in (\tau, T],
\end{cases}
\]

where \( \eta_1 > \eta_2 \). Under this assumption, we can rewrite the profit function as

\[
J_a = \int_0^\tau \left( p(t)(\alpha - \beta p(t) - \eta_1 (p(t) - r(t)) - \frac{h}{2} I(t)^2 - \frac{g}{2} u(t)^2) \right) dt \\
+ \int_\tau^T \left( p(t)(\alpha - \beta p(t) - \eta_2 (p(t) - r(t)) - \frac{h}{2} I(t)^2 - \frac{g}{2} u(t)^2) \right) dt.
\]

According to the optimality principle, the optimality in the second interval \( (\tau, T] \) should be guaranteed no matter what the result in the first interval \( [0, \tau] \) would be.

At first, we solve the second (last) stage of the problem for a given intersection time \( \tau \) where the objective functional is denoted as \( J_2 \), which introduces the following optimization problem

\[
\max_{p(.), u(.)} J_2 = \int_\tau^T \left( p(t)(\alpha - \beta p(t) - \eta_2 (p(t) - r(t)) - \frac{h}{2} I(t)^2 - \frac{g}{2} u(t)^2) \right) dt,
\]

s.t.

\[
\dot{I}(t) = -\alpha + \beta p(t) + \eta_2 (p(t) - r(t)) + u(t), \\
\dot{r}(t) = \delta (p(t) - r(t)), \\
p(\tau) = r(\tau), \quad I(T) = 0.
\]

By solving the optimization problem (21), we can get the following proposition.

\textbf{Proposition 2.} In the asymmetric scenario, the optimal pricing policy \( p^*_a \), production policy \( u^*_a \), inventory level \( I^*_a \) and reference price \( r^*_a \) during the interval \( (\tau, T] \), respectively, are given by

\[
p^*_a(t) = \frac{1}{2c_0} \left( k'_1(c_0 + c_1 - c_2)e^{r_1 t} + k'_2(c_0 + c_1 + c_2)e^{-r_1 t} + k'_3(c_0 + c_1 + c_3)e^{r_3 t} \\
+ k'_4(c_0 + c_1 - c_3)e^{-r_3 t} + c_4 \right),
\]

\[
u^*_a(t) = \frac{1}{g} \left( k'_1 e^{r_1 t} + k'_2 e^{-r_1 t} + k'_3 e^{r_3 t} + k'_4 e^{-r_3 t} \right) - \frac{\alpha}{\beta g + 2},
\]

\[
I^*_a(t) = \frac{k'_1 m_1}{2h g (\eta + \beta)} e^{r_1 t} - \frac{k'_2 m_1}{2h g (\eta + \beta)} e^{-r_1 t} + \frac{k'_3 m_2}{2h g (\eta + \beta)} e^{r_3 t} - \frac{k'_4 m_2}{2h g (\eta + \beta)} e^{-r_3 t},
\]

\[
r^*_a(t) = \frac{\eta + \beta}{c_0 \delta} \left( k'_1 (1 - c_1 - c_2)e^{r_1 t} + k'_2 (1 + c_1 + c_2)e^{-r_1 t} + k'_3 (1 + c_1 + c_3)e^{r_3 t} \\
+ k'_4 (-c_1 - c_3)e^{-r_3 t} - \frac{(g \beta + 1) \alpha}{(g \beta + 2) \beta} \right).
\]
and the corresponding adjoint variables are

\[
\begin{align*}
\lambda_1 &= k'_1 e^{rt} + k'_2 e^{rt} + k'_3 e^{rt} + k'_4 e^{rt} - \frac{g\alpha}{\beta g + 2}, \\
\lambda_2 &= \frac{\eta + \beta}{c_0 \beta} \left( k'_1 (c_1 - c_2) e^{rt} + k'_2 (c_1 + c_2) e^{rt} + k'_3 (c_1 + c_3) e^{rt} + k'_4 (c_1 - c_3) e^{rt} \right) \\
&\quad - \frac{\eta \alpha}{\delta \beta (g\beta + 2)},
\end{align*}
\]

where \(m_1, m_2, r_1, r_3, c_i, i = 0, \ldots, 4\) are characterized in the Appendix, and the constants \(k'_1, k'_2, k'_3, k'_4, \bar{r}_a\) and \(z\) simultaneously satisfy the following equations

\[
\begin{align*}
I^*_a(\tau) &= z, \\
p^*_a(\tau) &= r^*_a(\tau) = \bar{r}_a, \\
I^*_a(T) &= 0, \\
\lambda_2(T) &= 0.
\end{align*}
\]

Proof. The proof is similar to that of the Proposition 1, thus it is omitted here. \(\square\)

Next, we consider the sub-problem in the first stage \([0, \tau]\). Here the objective functional is

\[
J_1 = \int_0^\tau \left( p(t) (\alpha - \beta p(t) - \eta_1 (p(t) - r(t))) - \frac{h}{2} I(t)^2 - \frac{g}{2} u(t)^2 \right) dt.
\]

(27)

According to the optimality principle, the optimality in the first interval \([0, \tau]\) should also be guaranteed. Thus we formulate the following optimization problem

\[
\max_{p(.), u(.)} J_1 = \int_0^\tau \left( p(t) (\alpha - \beta p(t) - \eta_1 (p(t) - r(t))) - \frac{h}{2} I(t)^2 - \frac{g}{2} u(t)^2 \right) dt,
\]

s.t.

\[
\begin{align*}
\dot{I}(t) &= -\alpha + \beta p(t) + \eta_1 (p(t) - r(t)) + u(t), \\
\dot{r}(t) &= \delta (p(t) - r(t)), \\
I(0) &= I_0, \quad I(\tau) = z, \quad r(0) = r_0, \quad r(\tau) = p(\tau) = \bar{r}_a.
\end{align*}
\]

(28)

By solving the optimization problem (28), we can get the following proposition.

**Proposition 3.** In the asymmetric scenario, the optimal pricing policy \(p^*_a\), production policy \(u^*_a\), inventory level \(I^*_a\) and reference price \(r^*_a\) during the interval \([0, \tau]\), respectively, are given by

\[
p^*_a(t) = \frac{1}{2c_0} \left( k''_1 (c_0 + c_1 - c_2) e^{rt} + k''_2 (c_0 + c_1 + c_2) e^{rt} + k''_3 (c_0 + c_1 + c_3) e^{rt} + k'_4 (c_0 + c_1 - c_3) e^{rt} + c_0 \right),
\]

\[
u^*_a(t) = \frac{1}{g} \left( k''_1 e^{rt} + k''_2 e^{-rt} + k''_3 e^{rt} + k''_4 e^{-rt} \right) - \frac{\alpha}{\beta g + 2},
\]

\[
I^*_a(t) = \frac{k'_1 m_1}{2h g (\eta + \beta)} e^{rt} - \frac{k'_2 m_1}{2h g (\eta + \beta)} e^{-rt} + \frac{k'_3 m_2}{2h g (\eta + \beta)} e^{rt} - \frac{k'_4 m_2}{2h g (\eta + \beta)} e^{-rt},
\]

\[
r^*_a(t) = \frac{\eta + \beta}{c_0 \delta} \left( k''_1 (c_1 - c_2) e^{rt} + k''_2 (c_1 + c_2) e^{rt} + k''_3 (c_1 + c_3) e^{rt} \right) + k'_4 (c_1 - c_3) e^{rt} - \frac{(g\beta + 1)\alpha}{(g\beta + 2)\beta},
\]

\[
(29)
\]

\[
(30)
\]

\[
(31)
\]

\[
(32)
\]
and the corresponding adjoint variables are
\[ \lambda_1 = k_1'' e^{r_1 t} + k_2'' e^{-r_1 t} + k_3'' e^{r_3 t} + k_4'' e^{-r_3 t} - \frac{\alpha}{\beta g + 2}, \]
\[ \lambda_2 = \frac{\eta + \beta}{c_0 \lambda} \left( k_1'' (c_1 - c_2) e^{r_1 t} + k_2'' (c_1 + c_2) e^{-r_1 t} + k_3'' (c_1 + c_3) e^{r_3 t} + k_4'' (c_1 - c_3) e^{-r_3 t} \right. \]
\[ \left. - \frac{\eta \alpha}{\delta \beta (g \beta + 2)} \right), \]
where \( m_1, m_2, r_1, r_3, c_i, i = 0, ..., 4 \) are characterized in the Appendix, and the constants \( k_1'', k_2'', k_3'' \) and \( k_4'' \) simultaneously satisfy the following equations
\[
\begin{cases}
I_a^*(0) = I_0, \\
r_a^*(0) = r_0, \\
I_a^*(\tau) = z, \\
p_a^*(\tau) = r_a^*(\tau) = \bar{r}_a.
\end{cases}
\] (33)

Proof. The proof is similar to that of the Proposition 1, thus it is omitted here. \( \Box \)

When solving the two sub-problems, there actually exist ten parameters, namely, \( k_1', k_2', k_3', \bar{r}_a, z, k_1'', k_2'', k_3'' \) and \( k_4'' \), which need to be determined. These ten parameters can be calculated by solving Equations (26) and (33) simultaneously, which involves ten equations totally. Note that for a given intersection time \( \tau \), the corresponding optimal solutions in each substage can be obtained from Propositions 2 and 3 respectively, and the corresponding profit in each substage can be described as a function of \( \tau \). To obtain the optimal solutions on the whole horizon \([0, T]\), the intersection time \( \tau \in (0, T) \) should also be optimized to maximize the total profit \( J_a = J_1 + J_2 \), which formulates the following optimization problem
\[
\max_{\tau \in (0, T)} J_1(\tau) + J_2(\tau). \] (34)

The optimization problem (34) is a one-dimensional parametric optimization problem. Although the analytical solution for the optimal \( \tau^* \) cannot be obtained due to the complicate objective function, the optimal numerical solution can be easily calculated by using one-dimensional search algorithm performed in the bounded interval \([0, T]\).

Note that the above solution procedure applies to the case with a low initial reference price \( r_0 \). Similarly, we can also obtain the result in the case with a high initial reference price \( r_0 \) such that \( r(t) > p(t) \) for \( t \in [0, \tau'] \). Thus, by comparing the profits under the two cases, we can obtain the optimal results in the asymmetric case of a single intersection time instant for any initial reference price. In addition, although we only focus the case with one intersection time, the method proposed above can be generalized to the situations involving more than one intersection between \( p(t) \) and \( r(t) \) during the selling cycle \([0, T]\). Specifically, assuming that there exist \( n \) intersection times \( \tau_1, \tau_2, ..., \) and \( \tau_n \) in the interval \([0, T]\), we only need to first solve \( n + 1 \) optimization sub-problems with \( \eta \) taking \( \eta_1 \) or \( \eta_2 \) depending on the comparisons of \( r \) and \( p \), and then find the optimal intersection times \( \tau_1, \tau_2, ..., \tau_n \) through solving the \( n \)-dimensional parametric optimization problem. On the other hand, numerical studies show that in the optimal solutions there only exists an intersection time at most, though we cannot prove it analytically due to the complexity.
5. Numerical study. In this section, we present numerical examples in both symmetric and asymmetric cases to illustrate the effectiveness of the above method and examine the impacts of key parameters on the optimal outcomes under both scenarios.

5.1. The symmetric case. Consider the parameters \( \alpha = 25, \beta = 0.5, \eta = 0.55, \delta = 1.2, h = 0.05, g = 0.25, T = 2 \). The initial inventory level and reference price are given as \( I_0 = 20, r_0 = 20 \). These demand parameters and cost parameters are similarly set according to the previous studies in marketing and operations management, for example, Xue et al. [49].

According to Proposition 1, the optimal pricing policy \( p^*_s \), production policy \( u^*_s \), inventory level \( I^*_s \) and reference price \( r^*_s \) during the interval \([0, T] \), respectively, are given by

\[
p^*_s(t) = -0.7351 e^{-0.8399 t} - 1.980 e^{-0.4545 t} + 0.002525 e^{0.4545 t} - 1.419 e^{0.8399 t} + 26.47,
\]

\[
u^*_s(t) = 0.2277 e^{-0.8399 t} - 9.926 e^{-0.4545 t} + 0.05004 e^{0.4545 t} - 0.4081 e^{0.8399 t} + 11.76,
\]

\[
I^*_s(t) = -0.9562 e^{-0.8399 t} + 22.56 e^{-0.4545 t} + 0.1137 e^{0.4545 t} - 1.714 e^{0.8399 t},
\]

\[
r^*_s(t) = -2.449 e^{-0.8399 t} - 3.188 e^{-0.4545 t} + 0.001832 e^{0.4545 t} - 0.8350 e^{0.8399 t} + 26.47.
\]

The corresponding maximum total profit can be calculated from Equation (7) as \( J^*_s = 592.3300 \).

The relationship between the optimal price and reference price is shown in Figure 1. It indicates that the optimal price \( p^*_s \) always decreases. However, the reference price \( r^*_s \) increases at the beginning of sales cycle and then reduces. It is notable that the optimal price \( p^*_s \) and reference price \( r^*_s \) has only one intersection. The whole dynamics can be described as follows. At the beginning of the selling cycle, the firm adopts the strategy with a higher price, contributing to enhancing the reference price of the consumers. At the end of the planning horizon, a selling price lower than the reference price will be employed to expand demand. During this stage, both selling price and reference price decrease.

![Figure 1. Optimal price \( p^*_s \) and reference price \( r^*_s \).](image)

Table 1 reports the sensitivity analysis in the symmetric case, in which the plus sign ‘+’ means that the value of outcome increases when the parameter increases from one value to another. Accordingly, a minus sign ‘−’ means a negative influence on the corresponding value with the parameter increasing. Additionally, there exists a sign ‘−, +’ in Table 1, which means a decreasing trend at the beginning and an
Table 1. Variations in optimal outcomes in the symmetric case.

|        | \( p_t^* \) | \( u_t^* \) | \( I_t^* \) | \( r_t^* \) | \( J_t^* \) |
|--------|--------------|--------------|--------------|--------------|--------------|
| \( \delta(0.8; 1.0; 1.2; 1.4) \) | +            | +            | +            | +            | +            |
| \( \beta(0.25; 0.5; 0.75; 1.0) \) | -            | -            | -            | -            | -            |
| \( \eta(0.35; 0.55; 0.75; 0.95) \) | -            | +            | -            | -            | +            |

increasing trend at the end. Specifically, for the cell \((J_t^*, \eta)\), \(J_t^*\) reduces when \(\eta\) ranges from 0.35 to 0.75, however, it increases when \(\eta\) varies from 0.75 to 0.95. In addition, when a parameter is varying, all the others remain at their benchmark values as given at the beginning of this subsection.

The results in Table 1 allow for the following comments. The optimal price, reference price, and inventory level all raise as the ‘memory parameter’ \(\delta\) increases, which means a bigger \(\delta\) will lead to higher selling price, reference price, and inventory level. The ‘memory parameter’ \(\delta\) reflects consumers’ ability to remember past price. A larger \(\delta\) is, a weaker memory is. Therefore, the firm has incentives to set higher selling price and reference price to obtain a higher margin when facing the consumers with their reference price influenced by the selling price rapidly. In this case, although the demand and the corresponding optimal production drop a little, the total profit increases with \(\delta\) increasing, which shows that the consumers with ‘poor memory’ would benefit the firm. In addition, note that a larger \(\beta\) means that the consumers are more sensitive about selling price. Facing the consumers with more sensitivities about selling price, the firm will incur a great loss in demand when attempting to induce a high reference price by setting a high selling price. Thus, as shown in Table 1, all of the optimal solutions, including \(p_t^*, u_t^*, I_t^*, r_t^*\) and the total profit \(J_t^*\) decrease as \(\beta\) increases. Finally, note that the parameter \(\eta\) reflects the sensitivity of reference price on the demand. It is shown that a larger parameter \(\eta\) leads to lower price and reference price. Meanwhile, it gives rise to higher production and inventory. However, the total profit drops first and then increases with \(\eta\) increasing. This interesting result may stem from the two fold effect on demand of the consumers’ sensitivity to the gap between the reference price and the sales price. Specifically, if \(p > r\), a higher \(\eta\) leads to a lower demand, while generates a higher demand if \(p < r\). As a result, with \(\eta\) increasing, whether the total profit decreases or increases is both of possibility.

5.2. The asymmetric case. Set \(\eta_1 = 0.6\), \(\eta_2 = 0.5\), and all the other parameters remain the same as those in the symmetric case.

Consider the case involving only one intersection between \(p(t)\) and \(r(t)\) during the selling cycle \([0, T]\). On basis of Propositions 2 and 3, we can get the relationship between the total profit \(J_a\) and intersection time \(\tau\), depicted in Figure 2, which shows that there exists a unique maximizer, i.e., the optimal intersection time \(\tau^* = 1.14\) maximizing the total profit \(J_a\).

Then according to Propositions 2 and 3, the optimal pricing policy \(p_t^a\), production policy \(u_t^a\), inventory level \(I_t^a\) and reference price \(r_t^a\) during the interval \([0, T]\),
respectively, are given by

\[
\begin{align*}
    p_\alpha^*(t) &= \begin{cases} 
    0.250 e^{-0.82 t} - 9.996 e^{-0.45 t} + 0.032 e^{0.45 t} - 0.416 e^{0.82 t} + 11.76, & t \in [0, \tau^*], \\
    0.220 e^{-0.86 t} - 10.18 e^{-0.46 t} - 0.171 e^{0.46 t} - 0.342 e^{0.86 t} + 11.76, & t \in (\tau^*, T], 
    \end{cases} \\
    u_\alpha^*(t) &= \begin{cases} 
    -0.738 e^{-0.82 t} - 2.076 e^{-0.45 t} + 0.0014 e^{0.45 t} - 1.331 e^{0.82 t} + 26.47, & t \in [0, \tau^*], \\
    -0.778 e^{-0.86 t} - 1.950 e^{-0.46 t} - 0.01 e^{0.46 t} - 1.297 e^{0.86 t} + 26.47, & t \in (\tau^*, T], 
    \end{cases} \\
    I_\alpha^*(t) &= \begin{cases} 
    -1.028 e^{-0.82 t} + 22.66 e^{-0.45 t} + 0.0724 e^{0.45 t} - 1.790 e^{0.82 t}, & t \in [0, \tau^*], \\
    -0.942 e^{-0.86 t} + 23.17 e^{-0.46 t} - 0.389 e^{0.46 t} - 1.469 e^{0.86 t}, & t \in (\tau^*, T], 
    \end{cases} \\
    r_\alpha^*(t) &= \begin{cases} 
    -2.345 e^{-0.82 t} - 3.337 e^{-0.45 t} + 0.001 e^{0.45 t} - 0.790 e^{0.82 t} + 26.47, & t \in [0, \tau^*], \\
    -2.735 e^{-0.86 t} - 3.143 e^{-0.46 t} - 0.007 e^{0.46 t} - 0.756 e^{0.86 t} + 26.47, & t \in (\tau^*, T].
    \end{cases}
\]

The corresponding maximum total profit can be calculated from Equation (7) as \( J_\alpha^* = 590.0597 \). In addition, it can be verified by numerical simulations that the case with more than one intersection between \( p(t) \) and \( r(t) \) during the selling cycle \([0, T]\) is not the optimal.

The optimal pricing policy \( p_\alpha^* \) and reference price \( r_\alpha^* \) are depicted in Figure 3. In the interval \([0, \tau^*]\), the price \( p_\alpha^* \) is higher than the reference price \( r_\alpha^* \) and the outcome is opposite after time \( \tau^* \). This result is similar to that in the symmetric case.
In the following, we assess the sensitivity analysis of some important parameters under the asymmetric case.

Table 2. The optimal intersection time $\tau^*$ with different $\theta$.

| $\theta$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 |
|----------|------|------|------|------|------|------|
| $\tau^*$ | 1.14 | 1.21 | 1.29 | 1.36 | 1.45 | 1.53 |

At first, we examine the impact of changes of asymmetry on the optimal pricing strategy $p^*_a$, production policy $u^*_a$ and profit $J^*_a$. Set $\eta_1 = \eta + \theta$ and $\eta_2 = \eta - \theta$. The positive parameter $\theta$ reflects the asymmetric property. The greater $\theta$ is, the greater the asymmetry is. When $\theta$ equals 0, the situation reduces to the case of symmetric reference effect. The 'asymmetric parameter' $\theta$ is chosen from $\{0, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30\}$ and $\eta = 0.55$.

It can be seen from Table 2 that the optimal intersection time $\tau^*$ increases with $\theta$. This result shows that a greater asymmetry leads to a longer optimal intersection time. Figure 4 illustrates that the optimal price increases while the production reduces as $\theta$ increases. It means that a stronger asymmetry pushes an increase of price while causes a lower production rate. Moreover, different from that of symmetric reference price effect, there will be a down jump in the time path of
the production rate in the case of asymmetric reference price effect, and the jump amplitude raises as $\theta$ increases. Finally, we examine the impact of $\theta$ on the total profit $J^*$. Figure 5 indicates that the total profit decreases with $\theta$ increasing. Thus it is concluded that the asymmetry has a negative influence on the profit. The reason behind the above results is as follows. Note that a higher $\theta$ reflects that the consumers are more loss-averse. Loss-averse consumers weigh losses more heavily than equally sized gains, therefore, the firm will increase the selling price to control the reference price at a high level. This increased price generates a lower demand, which enables the firm to reduce the production. Consequently, the firm suffers a loss from the consumers’ loss-averse behavior. Our study about the impacts of $\theta$ gives some hints to the firms when making the joint pricing and production decision. Facing the loss-averse consumers, the firm should increase price while decrease production, compared to the case that the consumers are loss-neutral.

Figure 6. Impact of $\delta$ on the optimal price $p^*_a$ and production $u^*_a$.

Figure 7. Impact of $\delta$ on the total profit $J^*_a$.

The ‘memory parameter’ $\delta$ is chosen from $\{0.8, 1.0, 1.2, 1.4\}$, which is the same as that in symmetric case. Figures 6 and 7 respectively show the optimal price, production rate and the total profit for each level of $\delta$. We can get that the optimal price and the total profit raise while the optimal production drops as the ‘memory parameter’ $\delta$ increases, which are in accordance with the symmetric scenario. However, different from that of symmetric reference price effect, there exists a down jump in production rate after the intersection time $\tau$ in the case of asymmetric reference price effect.
As that in the symmetric case, the parameter $\beta$ is chosen from \{0.25, 0.5, 0.75, 1.0\} as well. Figures 8 and 9 respectively show the optimal price, production rate and the total profit for each level of $\beta$ in the asymmetric case. From Figure 8, we know that a larger price sensitivity $\beta$ leads to a higher sales price. This result is in line with that under the symmetric case. Figure 8 shows the relationship between the optimal production and parameter $\beta$ is sophisticated under the asymmetric case due to the existence of the jump. Specifically, a smaller $\beta$ implies a longer inter-
section time $\tau$, and incurs a larger production before the intersection time $\tau$ while a less production after $\tau$. Overall, the whole production in the planning horizon shrinks with the increasing of $\beta$. From Figure 9, the total profit also reduces with $\beta$ increasing which is similar to that of the symmetric case.

Furthermore, we examine the impact of the parameter $\eta$ on the optimal solutions. We set $\eta_1 = \eta + 0.05$ and $\eta_2 = \eta - 0.05$. The parameter $\eta$ is chosen from \{0.35, 0.55, 0.75, 0.95\}. Figures 10 and 11 respectively show the optimal price, production rate and the total profit via different $\eta$ in the asymmetric case. As shown from Figures 10 and 11, the optimal price drops while the total profit drops first and then increases with $\eta$ increasing, which is in line with the results under the symmetric case. Additionally, Figure 10 shows that a smaller $\eta$ implies a longer intersection time $\tau$, and incurs a less production.

6. Conclusions. Asymmetric reference price effect, as an important factor to be considered in pricing models, will lead to a nonsmooth optimization problem which
is hard to solve. In this study, based on Maximum principle and optimality principle, we introduce a systematical method to solve the nonsmooth optimization problem in which a decision-maker determines both pricing and production policy while simultaneously considers. Numerical examples show the effectiveness of the proposed method. Sensitivity analysis of key parameters is conducted to render some managerial insights. It is shown that the asymmetry has a negative influence on the optimal production and total profit. Additionally, the intensity of reference price effect brings a subtle impact on profit. As the reference price effect increases, the profit reduces first and then increases.

Although our paper has identified the effect of both symmetric and asymmetric reference price on the joint pricing and production model, it has a few limitations, and some valuable extensions should be noted. First, our inventory model is restrained to durable products, which may not be applicable to the deteriorating products. Also, it is assumed for mathematical tractability that the inventory cost coefficient and the shortage cost coefficient take the same value. This assumption could be relaxed to better match the practical situation.

Acknowledgments. This work was supported by National Natural Foundation of China No. 61473204, Humanity and Social Science Youth Foundation of Ministry of Education of China No. 14YJCZH204.
Appendix. Proof of Proposition 1 Note that the Hamiltonian function $H$ is concave in $p$ and $u$. Thus, according to Equations (9)-(11), the first-order conditions for optimality are

$$\frac{\partial H}{\partial p} = \alpha - 2\beta p - 2\eta p + \eta r + \beta \lambda_1 + \eta \lambda_1 + \delta \lambda_2 = 0,$$

$$\frac{\partial H}{\partial u} = -gu + \lambda_1 = 0,$$  

which yield the optimal pricing and production policy respectively as

$$p^* = \frac{\alpha + \eta r + \beta \lambda_1 + \eta \lambda_1 + \delta \lambda_2}{2\beta + 2\eta}, \quad (A3)$$

$$u^* = \frac{\lambda_1}{g}. \quad (A4)$$

Substituting Equations (A3)-(A4) into Equations (1), (2), (4), (12) and (13) respectively, we can get

$$\dot{I} = -\frac{1}{2}\eta r + \left(\frac{1}{2}(\eta + \beta) + \frac{1}{g}\right)\lambda_1 + \frac{1}{2}\delta \lambda_2 - \frac{1}{2}a, \quad (A5)$$

$$\dot{r} = -\frac{2\beta \delta - \eta \delta}{2\beta + 2\eta} r + \frac{\delta}{2} \lambda_1 + \frac{\delta^2}{2\beta + 2\eta} \lambda_2 + \frac{\alpha \delta}{2\beta + 2\eta}, \quad (A6)$$

$$\dot{\lambda}_1 = hI, \quad (A7)$$

$$\dot{\lambda}_2 = -\frac{\eta^2}{2\beta + 2\eta} r + \frac{\eta}{2} \lambda_1 + \frac{2\beta \delta + \eta \delta}{2\beta + 2\eta} \lambda_2 - \frac{\alpha \delta}{2\beta + 2\eta}. \quad (A8)$$

Rewriting them, we have

$$\begin{bmatrix} \dot{I} \\ \dot{r} \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = A \begin{bmatrix} I \\ r \\ \lambda_1 \\ \lambda_2 \end{bmatrix} + b, \quad (A9)$$

where

$$A = \begin{bmatrix} 0 & -\frac{1}{2}\eta & \frac{1}{2}(\eta + \beta) + \frac{1}{g} & \frac{1}{2}\delta \\ 0 & \frac{2\beta \delta - \eta \delta}{2\beta + 2\eta} & \delta & \frac{\delta^2}{2\beta + 2\eta} \\ h & 0 & 0 & 0 \\ 0 & \frac{-\eta^2}{2\beta + 2\eta} & \frac{\eta}{2} & \frac{2\beta \delta + \eta \delta}{2\beta + 2\eta} \end{bmatrix}, \quad (A10)$$

and

$$b = \begin{bmatrix} -\frac{1}{2}a \\ \frac{-\frac{1}{2}a}{2\beta + 2\eta} \\ 0 \\ -\frac{\alpha \delta}{2\beta + 2\eta} \end{bmatrix}. \quad (A11)$$
The eigenvalues of $A$ and the corresponding matrix of eigenvectors of $A$ are given by

$$
\begin{bmatrix}
  r_1 \\
r_2 \\
r_3 \\
r_4 \\
\end{bmatrix} = \begin{bmatrix}
  \sqrt{\frac{a+b}{2g(\eta+\beta)}} \\
  -\sqrt{\frac{a-b}{2g(\eta+\beta)}} \\
  \sqrt{\frac{a-\sqrt{b}}{2g(\eta+\beta)}} \\
  -\sqrt{\frac{a+\sqrt{b}}{2g(\eta+\beta)}}
\end{bmatrix},
$$

(A12)

and

$$
H = \begin{bmatrix}
  \frac{m_1}{c_1-c_2} & -\frac{m_1}{c_1+c_2} & \frac{m_1}{c_1-c_2} & -\frac{m_1}{c_1+c_2} \\
  \frac{m_2}{c_1-c_2} & \frac{m_2}{c_1+c_2} & -\frac{m_2}{c_1-c_2} & \frac{m_2}{c_1+c_2} \\
  \frac{m_3}{c_1-c_2} & \frac{m_3}{c_1+c_2} & -\frac{m_3}{c_1-c_2} & \frac{m_3}{c_1+c_2} \\
  \frac{m_4}{c_1-c_2} & \frac{m_4}{c_1+c_2} & -\frac{m_4}{c_1-c_2} & \frac{m_4}{c_1+c_2}
\end{bmatrix},
$$

(A13)

where

$$a = 2h\eta g\beta + 2hn + \beta^2 g + 2h\beta + 2g\delta^2 \beta + h\eta g^2,$$

$$b = 6h^2 \eta^2 g^2 \beta^2 + 12h^2 \eta^2 g \beta^2 + 4h^2 \eta g^2 \beta^2 + 12h^2 \eta \gamma \beta - 4h \beta \gamma g \delta^2 - 8h \delta^2 g \beta^2 + 8h^2 \eta \beta g + h \beta^2 g^2 + 4h^2 \beta^3 g + 4g \beta^4 \delta^2 + h^2 g^2 \eta + 4h^2 \eta^2 + 4h^2 \beta^2 - 8h \eta g \beta \delta^2 + 4g \beta^3 \delta \eta h \gamma^2,$$

$$m_1 = \sqrt{g(\eta+\beta)(a+\sqrt{b})},$$

$$m_2 = \sqrt{-g(\eta+\beta)(-a+\sqrt{b})},$$

$$c_1 = -4\eta^3 h g^3 \delta^2 - 8\eta^2 \delta \beta h g^2 - 12\eta^2 h \delta g^3 \beta^2 + 2\eta \delta g^2 a + 16\eta \delta g^2 \sqrt{b} - 12\eta^2 h g^3 \beta^3 + 8\beta g^3 \sqrt{b} - 8\delta^2 g h g^2,$$

$$c_2 = 2m_1 \eta^3 h g^2 + 6\eta^2 m_1 \eta h g \beta + 4m_1 \eta h g^2 + 6m_1 \beta g h^2 + 8\eta m_1 \beta h g - m_1^{3/2} + 4m_1 \beta g^2 + 2m_1 \beta^3 h g^2,$$

$$c_3 = 2m_2 \eta^3 h g^2 + 6\eta^2 m_2 \beta h g^2 + 4m_2 \eta h g \beta + 6m_2 \eta^2 h g^2 + 8\eta m_2 \beta h g - m_2^{3/2} + 4m_2 h g^2 + 2m_2 \beta^3 h g^2.$$

Hence,

$$
\begin{bmatrix}
  I \\
r \\
  \lambda_1 \\
  \lambda_2
\end{bmatrix} = H \begin{bmatrix}
  e^{rt} & 0 & 0 & 0 \\
  0 & e^{rt} & 0 & 0 \\
  0 & 0 & e^{rt} & 0 \\
  0 & 0 & 0 & e^{rt}
\end{bmatrix} \begin{bmatrix}
  k_1 \\
  k_2 \\
  k_3 \\
  k_4
\end{bmatrix} - A^{-1} b.
$$

(A14)

According to Equation (18), that is

$$
\begin{align*}
I(0) &= I_0, \\
r(0) &= r_0, \\
I(T) &= 0, \\
\lambda_2(T) &= 0,
\end{align*}
$$

the parameters $k_1, k_2, k_3, k_4$ can be determined. Then we obtain the optimal inventory level $I^*_0(t)$, reference price $r^*_0(t)$ and their corresponding adjoint variables.
\( \lambda_1(t) \) and \( \lambda_2(t) \) as follows

\[
I^*_\lambda(t) = \frac{k_1m_1}{2h_g(\eta+\beta)} e^{r_1t} - \frac{k_2m_1}{2h_g(\eta+\beta)} e^{-r_1t} + \frac{k_3m_2}{2h_g(\eta+\beta)} e^{r_3t} - \frac{k_4m_2}{2h_g(\eta+\beta)} e^{-r_3t}, \tag{A15}
\]

\[
r^*_\lambda(t) = \frac{\eta+\beta}{c_0\delta} \left( k_1(-c_1-c_2)e^{r_1t} + k_2(-c_1+c_2)e^{-r_1t} + k_3(-c_1+c_3)e^{r_3t} + k_4(-c_1-c_3)e^{-r_3t} - \frac{(g\beta+1)\alpha}{(g\beta+2)\beta} \right), \tag{A16}
\]

\[
\lambda_1 = k_1 e^{r_1t} + k_2 e^{-r_1t} + k_3 e^{r_3t} + k_4 e^{-r_3t} - \frac{g\alpha}{\beta g+2}, \tag{A17}
\]

\[
\lambda_2 = \frac{\eta+\beta}{c_0\delta} \left( k_1(c_1-c_2)e^{r_1t} + k_2(c_1+c_2)e^{-r_1t} + k_3(c_1+c_3)e^{r_3t} + k_4(c_1-c_3)e^{-r_3t} - \frac{\eta \alpha}{\delta \beta (g\beta+2)} \right), \tag{A18}
\]

where

\[ c_0 = 8g^3\beta \eta(\eta+\beta)^3. \]

Substituting (A16)-(A18) into (A3) and (A4), we can obtain the optimal price \( p^*_\lambda(t) \) and production \( u^*_\lambda(t) \) as

\[
p^*_\lambda(t) = \frac{1}{2c_0} \left( k_1(c_0+c_1-c_2)e^{r_1t} + k_2(c_0+c_1+c_2)e^{-r_1t} + k_3(c_0+c_1+c_3)e^{r_3t} + k_4(c_0+c_1-c_3)e^{-r_3t} + c_4 \right), \tag{A19}
\]

\[
u^*_\lambda(t) = \frac{1}{g} \left( k_1 e^{r_1t} + k_2 e^{-r_1t} + k_3 e^{r_3t} + k_4 e^{-r_3t} - \frac{\alpha}{\beta g+2} \right) - \frac{\eta}{2\beta^2+2\eta} - \frac{g\alpha}{2\beta g+4} - \frac{\eta \alpha(\beta+\eta)c_0}{\delta^2 (g\beta+2)\beta}. \tag{A20}
\]

The proof is complete.

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Received March 2016; revised March 2018.

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