Generalized Pure Lovelock Gravity

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Abstract

We present a generalization of the n-dimensional (pure) Lovelock Gravity theory based on an enlarged Lorentz symmetry. In particular, we propose an alternative way to introduce a cosmological term. Interestingly, we show that the usual pure Lovelock gravity is recovered in a matter-free configuration. The five and six-dimensional cases are explicitly studied.

1 Introduction

The most natural generalization of General Relativity (GR) in \( d \) dimensions satisfying the criteria of general covariance and leading to second order field equations for the metric is given by the Lanczos-Lovelock (LL) gravity theory \cite{1, 2}. In the differential forms language the LL action can be written as the most general \( d \)-form invariant under local Lorentz transformations, constructed out of the spin connection \( \omega^{ab} \), the vielbein \( e^a \) and their exterior derivatives \cite{3, 4},

\[
S_{LL} = \int \sum_{p=0}^{[d/2]} \alpha_p e_{a_1a_2...a_d} R^{a_1a_2...a_{2p-1}} e_{a_{2p+1}...a_d},
\]

where \( R^{ab} = d\omega^{ab} + \omega^a_c \omega^{cb} \) is the Lorentz curvature two-form and the \( \alpha_p \) coefficients are not fixed from first principles.

Different numbers of degrees of freedom emerge depending on the value of the arbitrary coefficients. In particular, the higher curvature terms can produce degenerate sectors with no degrees of freedom. Such degeneracy can be avoided with particular choices of the \( \alpha_p \) constants. In particular, as explained in ref. \cite{5}, there are mainly two ways to avoid degenerate sectors. One of

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them consist in restrict the theory to have a unique degenerate vacuum which leads to a family of gravity theories labeled by the integer $k$ which represents the highest power of curvature. Interestingly, the $\alpha_p$ constants can be fixed requiring that the theory has the maximum possible number of degrees of freedom. Then the LL Lagrangian is a Chern-Simons (CS) form in odd dimensions which is gauge invariant under the $(A)dS$ symmetry. In even dimensions, the LL Lagrangian can be written as a Born-Infeld (BI) gravity Lagrangian which is locally invariant under a Lorentz subalgebra.

Another way of fixing the $\alpha_p$ constants avoiding degeneracy is to demand that there is non-degenerate vacuum. Such requirement leads to the pure Lovelock (PL) theory which consists only in two terms of the full Lovelock Lagrangian,

$$S_{PL} = \int (\alpha_0 L_0 + \alpha_p L_p) ,$$

with

$$L_0 = \epsilon_{a_1 \ldots a_d} e^{a_1} \cdots e^{a_d} ,
\quad
L_p = \epsilon_{a_1 \ldots a_d} R^{a_1 a_2} \cdots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_d} .$$

The coefficients are fixed in terms of the gravitational constant $\kappa$ and the cosmological constant $\Lambda$,

$$\alpha_0 = -\frac{2\Lambda \kappa}{d!} = -\frac{(\mp 1)^p \kappa}{d(d-2p-1)!\ell^{2p}} ,
\quad
\alpha_p = \frac{\kappa}{(d-2p)!} .$$

With this particular choice, the PL theory has a unique nondegenerate $(A)dS$ vacuum in odd dimensions and admits non-degenerate vacua in even dimensions. Additionally, the black holes (BH) solutions of the PL theory behave asymptotically like the $AdS$-Schwarzschild ones. Of particular interest are the BH solutions of the maximal pure Lovelock since the thermodynamical parameters are universal in terms of horizon radius. Recently, a Hamiltonian analysis has shown that the maximum possible number of degrees of freedom of the PL case is the same as in the Einstein-Hilbert (EH) gravity.

The supersymmetric version of the general LL theory is unknown, except in the EH case and in odd dimensions when the LL action can be seen as a CS supergravity action for the $AdS$ superalgebra. The construction of a super LL gravity action, and in particular for a super PL, remains a difficult task. Indeed, there is no clarity in which terms should be considered in the action in order to guarantee the supersymmetric invariance of the theory. A discussion about a five-dimensional supergravity action for the EH term coupled to a Lovelock term can be found in ref. [16]. More recently, the authors of refs. [17, 18] suggested that the supersymmetric version of a PL theory could emerge as a particular limit of a supergravity theory. The procedure that could be used is not new and has already been used to relate GR with different gravity theories [19–23].

As shown in refs. [17, 18], the PL Lagrangian can be recovered as a particular limit of CS and BI like Lagrangians constructed out the $C_k$ family. Although the procedure presented in [17, 18] can be reproduced in any spacetime dimension $d$, the obtention of the PL action requires a large amount of extra fields in higher dimensions. Additionally in order to recover the PL dynamics, it is necessary to impose additional restrictions on the fields.
Here we present a generalized pure Lovelock (GPL) gravity theory which leads to the PL action and its dynamics in a matter-free configuration limit without further considerations. In particular, the field content of the theory is spacetime dimension independent. Interestingly, the GPL action allows us to introduce alternatively a generalized cosmological term which generalizes the result obtained in ref. [37] to higher dimensions. Moreover, we show that the GPL gravity action corresponds to a particular case of a more generalized Lovelock (GL) gravity theory. We also show that there is a particular choice of the coefficients appearing in the GL action leading in odd and even dimensions to a CS and BI like gravity, respectively.

2 Generalized Lovelock gravity action

A generalization of the Lanczos-Lovelock gravity action can be performed enlarging the Lorentz symmetry. A Generalized Lovelock (GL) gravity action can be written as the most general d-form invariant under local Lorentz-like transformations, constructed with the spin connection $\omega^{ab}$, the vielbein $e^a$, a Lorentz-like field $k^{ab}$ and their exterior derivatives, without the Hodge dual,

$$S_{GL} = \int \frac{[d/2]}{\sum \alpha_p \binom{p}{p-m} L_{GL}^{(p)},}$$

where

$$L_{GL}^{(p)} = \epsilon_{a_1 a_2 \ldots a_d} R^{a_1 a_2} \ldots R^{a_{2m-1} a_{2m}} F^{a_{2m+1} a_{2m+2}} \ldots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \ldots e^{a_d},$$

with

$$R^{ab} = d\omega^{ab} + \omega^c {\omega^{cb}},$$

$$F^{ab} = dk^{ab} + \omega^c k^{cb} - \omega^b k^{ca} + k^a k^{cb}.$$  

Here the $\alpha_p$ coefficients are arbitrary constants which are not fixed from first principles. The possible permutations of the curvature 2-forms $R^{ab}$ and $F^{ab}$ appearing in the action are reflected in the coefficients $\binom{p}{p-m}$.

Let us note that for $p = 0$ the Lagrangian reproduces the cosmological constant term, while for $p = 1$ the GL Lagrangian is the Einstein-Hilbert term plus the coupling of $F^{ab}$ with the vielbeins. This is an important difference with the usual Lovelock-Cartan gravity theory which contains General Relativity as a particular case. However, a matter-free configuration ($k^{ab} = 0$) allows to recover the usual Lovelock gravity action. In particular, the Einstein-Hilbert term is recovered for $p = 1$ and $k^{ab} = 0$.

The dynamical content is obtained considering the variation of the GL action with respect to $(e^a, \omega^{ab}, k^{ab})$,

$$\delta S_{GL} = \int \delta e^a e_a + \delta \omega^{ab} e_{ab} + \delta k^{ab} k_{ab} = 0, \quad (11)$$
modulo boundary terms. The field equations are given by

\[ \mathcal{E}_a = \sum_{p=0}^{[\frac{d-1}{2}]} \sum_{m=0}^{p} \alpha_p (d - 2p) \mathcal{E}_a^p = 0, \tag{12} \]

\[ \mathcal{E}_{ab} = \sum_{p=1}^{[\frac{d-1}{2}]} \sum_{m=0}^{p} \alpha_p m (d - 2p) \mathcal{E}_{ab}^p = 0, \tag{13} \]

\[ K_{ab} = \sum_{p=1}^{[\frac{d-1}{2}]} \sum_{m=0}^{p} \alpha_p (p - m) (d - 2p) \mathcal{E}_{ab}^p = 0, \tag{14} \]

where

\[ \mathcal{E}_a^p \equiv \left( \frac{p}{p - m} \right) \epsilon_{a b_1 \cdots b_{d-1}} R^{b_1 b_2} \cdots R^{b_{2m-1} b_2m} F^{b_{2m+1} b_{2m+2}} \cdots F^{b_{2p-1} b_{2p}} e_{b_{2p+1}} \cdots e_{b_{d-1}}, \tag{15} \]

\[ \mathcal{E}_{ab}^p \equiv \left( \frac{p}{p - m} \right) \epsilon_{a b a_1 \cdots a_{d-1}} R^{a_1 a_2} \cdots R^{a_{2m-1} a_{2m}} F^{a_{2m+1} a_{2m+2}} \cdots F^{a_{2p-1} a_{2p}} R^{a_{2p+1}} e_{a_{2p+2}} \cdots e_{a_{d-1}}. \tag{16} \]

Here \( R^a = D_\omega e^a + k^a \), with \( D_\omega = d + \omega \) the Lorentz covariant exterior derivative. In particular, the Bianchi identities \( D_\omega R^a = 0 \) and \( D_\omega R^{ab} + k^a R^{cb} - k^b R^{ca} = 0 \) along with \( d^2 = 0 \), assure that the field equations involve only first derivatives of \( e^a \), \( \omega^{ab} \) and \( k^{ab} \).

Let us note that the variation of the action under the spin connection \( \omega^{ab} \) and the Lorentz-like field \( k^{ab} \) imply the same field equation and then the \((d-1)\)-forms \( \mathcal{E}_{ab} \) and \( K_{ab} \) coincide. On the other hand, analogously to the usual Lovelock Cartan gravity, the \((d-1)\)-form \( \mathcal{E}_a \) is independent of the \((d-1)\)-forms \( \mathcal{E}_{ab} \).

Furthermore, using the Bianchi identities for the curvature 2-forms one can show that the following relation holds

\[ D \mathcal{E}_{ab}^p = (d - 1 - 2p) e^b \mathcal{E}_{ba}^{p+1}, \tag{17} \]

for \( 0 \leq p \leq [\frac{d-1}{2}] \). Then, as in refs. \[7, 23\] we have that the previous identity leads to

\[ D \mathcal{E}_a = \sum_{p=1}^{[\frac{d-1}{2}]} \sum_{m=0}^{p} \alpha_p (d + 2 - 2p) (d + 1 - 2p) e^b \mathcal{E}_{ba}^p, \tag{18} \]

which by consistency with \( \mathcal{E}_a = 0 \) must also be zero. Besides, we can see that the following product

\[ e^b \mathcal{E}_{ba} = \sum_{p=1}^{[\frac{d-1}{2}]} \sum_{m=0}^{p} \alpha_p m (d - 2p) e^b \mathcal{E}_{ba}^p \tag{19} \]

must also vanish by consistency with \( \mathcal{E}_{ab} = 0 \). One can easily check that the same result apply for the product \( e^b K_{ba} \).

Thus, fixing the \( \alpha_p \) coefficients could lead to different numbers of degrees of freedom depending on additional constraints of the form \( e^b \mathcal{E}_{ba}^p = 0 \). Following the same arguments of ref. \[7\], there is a particular choice in odd dimensions allowing to avoid additional restrictions and such that \( \mathcal{E}_a \) and \( \mathcal{E}_{ab} \) (or \( K_{ab} \)) are independent.
2.1 Chern-Simons gravity and $\mathcal{C}_4$ algebra

The odd-dimensional GL action (7) reproduces a Chern-Simons action for a particular Lie algebra, known as the $\mathcal{C}_4$ algebra\(^{1,24,27}\), when the $\alpha_p$'s are fixed to the following values:

\[
\begin{align*}
\alpha_0 &= \frac{\kappa}{d\ell^d}, \\
\alpha_p &= \alpha_0 \frac{(2n-1)(2\gamma)^{-p}(n-1)}{(2n-2p-1)} ,
\end{align*}
\]

where $\gamma$ is related to the cosmological constant,

\[
\gamma = -\text{sgn}(\Lambda) \frac{\ell^2}{2},
\]

and $\ell$ is a length parameter related to the cosmological constant.

The $d = 2n - 1$ CS Lagrangian is given by

\[
L_{\text{CS}}^{\mathcal{C}_4} = \kappa \epsilon_{a_1a_2\ldots a_{2n-1}} \sum_{p=0}^{n-1} \sum_{m=0}^{p} \ell^{2(p-n)+1} c_p \left( \frac{p}{p-m} \right) \times R^{a_1a_2} \ldots R^{a_{2m-1}a_{2m}} \Gamma^{a_{2m+1}a_{2m+2}} \ldots \Gamma^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \ldots e^{a_{2n-1}},
\]

with

\[
c_p = \frac{1}{2(n-p) - 1} \left( \frac{n-1}{p} \right).
\]

Let us note that the $d = 3$ CS form reproduces the generalized CS gravity theory presented in\(^{25}\) applying an appropriate change of basis where the $\text{AdS} \oplus \text{Lorentz}$ structure is manifested. The black hole solution of the three-dimensional CS gravity theory based on this symmetry have been recently studied in\(^{28}\).

It is important to clarify that, unlike the GL Lagrangian, the CS $(2n-1)$-form is invariant not only under Lorentz-like transformation,

\[
\begin{align*}
\delta e^a &= e^b \rho^a_b + e^b \lambda^a_b, \\
\delta \omega^{ab} &= D_o \rho^{ab}, \\
\delta k^{ab} &= D_o \lambda^{ab} + k^c e^b \rho^{cb} - k^b e^c \rho^{ca} + k^c \lambda^{cb} - k^b \lambda^{ca},
\end{align*}
\]

but also under a local $\mathcal{C}_4$ boost

\[
\begin{align*}
\delta e^a &= D_o \rho^a + k^a b^a, \\
\delta \omega^{ab} &= 0, \\
\delta k^{ab} &= \rho^a e^b - \rho^b e^a ,
\end{align*}
\]

where the $\mathcal{C}_4$ gauge parameter is given by

\[
\rho = \frac{1}{2} \rho^{ab} J_{ab} + \frac{1}{2} \lambda^{ab} Z_{ab} + \frac{1}{\ell} \rho^a P_a.
\]

\(^{1}\)Also known as Poincaré semi-simple extended algebra.
In particular, the generators of the $\mathfrak{c}_4$ algebra satisfy the following commutation relations:

\begin{align}
[J_{ab}, J_{cd}] &= \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \\
[Z_{ab}, Z_{cd}] &= \eta_{bc} Z_{ad} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac} + \eta_{ad} Z_{bc}, \\
[J_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b, \\
[Z_{ab}, P_c] &= \eta_{bc} P_a - \eta_{ac} P_b,
\end{align}

Such symmetry can be obtained as a deformation of the Maxwell symmetry and belongs to a generalized family of Lie algebras denoted by $\mathfrak{c}_k$ [27]. At the fermionic level, a recent application has been developed in the three-dimensional CS context where a $(p, q)$ AdS-Lorentz supergravity model was presented [29].

2.2 Born-Infeld gravity and Lorentz-like symmetry

The even-dimensional case requires an alternative approach since eqs. (18) and (19) have not the same number of terms as in the odd-dimensional case.

Following the same procedure introduced in refs. [7, 23], one can note that there is a particular choice of the $\alpha_p$'s

\begin{align}
\alpha_0 &= \frac{\kappa}{d! c}, \\
\alpha_p &= \alpha_0 (2\gamma)^p \left(\begin{array}{c} n \\ p \end{array}\right),
\end{align}

which reproduces a Born-Infeld (BI) like gravity action with $0 \leq p \leq n$. As in the CS case, $\gamma$ is related to the cosmological constant $\gamma = -\text{sgn}(\Lambda) \frac{\ell^2}{2}$. With these coefficients the GL Lagrangian takes the BI-like form

\begin{align}
L_{\mathfrak{c}_4}^{\mathcal{L}_{BI}} &= \kappa \epsilon_{a_1 a_2 \cdots a_{2n}} \sum_{p=0}^{n} \sum_{m=0}^{p} \frac{\ell^{2p-2n}}{2n} \left(\begin{array}{c} n \\ p \end{array}\right) \left(\begin{array}{c} p \\ p-m \end{array}\right) \\
&\quad \times R^{a_1 a_2} \cdots R^{a_{2m-1} a_{2m}} F^{a_{2m+1} a_{2m+2}} \cdots F^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \cdots e^{a_{2n}},
\end{align}

In particular, the Lagrangian can be rewritten in a reduced form,

\begin{align}
L_{\mathfrak{c}_4}^{\mathcal{L}_{BI}} &= \frac{\kappa}{2n} \epsilon_{a_1 a_2 \cdots a_{2n}} \tilde{F}^{a_1 a_2 \cdots a_{2n}},
\end{align}

where

\begin{align}
\tilde{F}^{ab} &= R^{ab} + F^{ab} + \frac{1}{\ell^2} e^a e^b,
\end{align}

is the $\mathfrak{c}_4$ curvature. With this form, the Lagrangian corresponds to the Pfaffian of the 2-form $\tilde{F}^{ab}$ and can be rewritten similarly to the Born-Infeld electrodynamics Lagrangian,

\begin{align}
L_{\mathfrak{c}_4}^{\mathcal{L}_{BI}} &= 2^{n-1} (n-1)! \sqrt{\det \left( R^{ab} + F^{ab} + \frac{1}{\ell^2} e^a e^b \right)}.
\end{align}
It is important to emphasize that the BI-like gravity Lagrangian is only off-shell invariant under a Lorentz-like subalgebra $L_C^4$ of the $C_4$ algebra. This can be clarified through the Levi-Civita symbol $\epsilon_{a_1a_2\cdots a_{2n}}$ in (40) which consists in the only non-vanishing component of the Lorentz-like invariant tensor of rank $n$, namely

$$\langle \tilde{Z}_{a_1a_2}\cdots \tilde{Z}_{a_{2n-1}a_{2n}} \rangle = \frac{2^{n-1}}{n} \epsilon_{a_1a_2\cdots a_{2n}}. \quad (43)$$

Here $\tilde{Z}_{ab} = J_{ab} + Z_{ab}$ satisfy the $C_4$ commutation relations (32)-(34). Such choice of the invariant tensor breaks the full $C_4$ symmetry to its Lorentz-like subgroup $L^4_C$.

Recently, diverse BI-like gravity theories have been constructed with different purposes [18, 21, 22]. At the supersymmetric level, similar constructions have been done based on the MacDowell-Mansouri formalism [30–36].

3 Generalized Pure Lovelock gravity action

An alternative way of fixing the $\alpha_p$’s can be implemented such that a generalized pure Lovelock (GPL) gravity action can be constructed. The new coefficients are fixed in terms of the gravitational constant $\kappa$ and the cosmological constant $\Lambda$ in the same way as in refs. [5, 13],

$$\alpha_0 = -\frac{2\Lambda\kappa}{d!} = -\frac{(\mp 1)^p \kappa}{d(d-2p-1)!\ell^{2p}}, \quad \text{(44)}$$

$$\alpha_p = \frac{\kappa}{(d-2p)!}. \quad \text{(45)}$$

Such generalization contains only two term of the full GL series (given by eq. (7)),

$$S_{GPL} = \int (\alpha_0 L_0 + \alpha_p L_p), \quad \text{(46)}$$

where

$$L_0 = \epsilon_{a_1a_2\cdots a_d} e^{a_1} e^{a_2} \cdots e^{a_d}, \quad \text{(47)}$$

$$L_p = \sum_{m=0}^{p} \binom{p}{p-m} \epsilon_{a_1a_2\cdots a_d} R^{a_1a_2} \cdots R^{a_{2m-1}a_{2m}} F^{a_{2m+1}a_{2m+2}} \cdots F^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \cdots e^{a_d}. \quad \text{(48)}$$

Interestingly, the $p$-order term in the curvature 2-forms reproduces a generalized cosmological term in any spacetime dimension $d$. This particular case of the generalized Lovelock series has been first introduced in four dimensions in refs. [33, 37]. Subsequently in ref. [18], a generalized cosmological constant term has been obtained in even dimensions as a particular configuration limit of a BI-like gravity theory. Thus, the generalized pure Lovelock action presented here, through the Lorentz-like gauge field $k^{ab}$, allows us to introduce alternatively a cosmological term in arbitrary dimensions.

Let us note that the Einstein-Hilbert term appears only in the case $p = 1$ along with the following term:

$$\epsilon_{a_1a_2a_3\cdots a_d} F^{a_1a_2} e^{a_3} \cdots e^{a_d}. \quad \text{(49)}$$
On the other hand, topological densities are obtained in even dimensions for \( p = d/2 \),
\[
L_{d/2} = \sum_{m=0}^{p} \left( \frac{d/2}{d/2 - m} \right) \epsilon_{a_1a_2 \ldots a_d} R^{a_1a_2} \ldots R^{a_{2m-1}a_{2m}} F^{a_{2m+1}a_{2m+2}} \ldots F^{a_{d-1}a_d} .
\] (50)

These terms are related to Euler type characteristic classes. Then, we have that for \( 1 \leq p \leq \frac{d-2}{2} \) the even-dimensional GPL gravity action reproduces truly dynamical actions.

Interestingly, let us note that in a matter-free configuration \( k^{ab} = 0 \) the GPL action (46) reduces to the pure Lovelock action,
\[
S_{PL} = \int \left( \alpha_0 L_0 + \alpha_p L_p \right) ,
\] (51)
with
\[
L_0 = \epsilon_{a_1a_2 \ldots a_d} e^{a_1} e^{a_2} \ldots e^{a_d} ,
\] (52)
\[
L_p = \epsilon_{a_1a_2 \ldots a_d} R^{a_1a_2} \ldots R^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \ldots e^{a_d} .
\] (53)

Obtaining the PL gravity action in a particular matter-free configuration limit is not new and has already been presented in refs. \([17, 18]\). Nevertheless, the techniques considered in \([17, 18]\) require an excessive amount of new extra fields as the spacetime dimension grows. In our case, the field content of the theory does not depend of the spacetime dimension.

Furthermore, one can show that the PL dynamics in \([5, 13, 14]\) can also be reproduced in a matter-free configuration leading appropriately to the pure Lovelock gravity theory. In fact considering \( k^{ab} = 0 \), we have
\[
\alpha_0 \epsilon_{a_1a_2 \ldots a_d} e^{a_1} e^{a_2} \ldots e^{a_{d-1}} + \alpha_p \epsilon_{a_1a_2 \ldots a_d} R^{a_1a_2} \ldots R^{a_{2p-1}a_{2p}} e^{a_{2p+1}} \ldots e^{a_{d-1}} = 0 ,
\] (54)
\[
\alpha_p \epsilon_{a_1a_2 \ldots a_d} R^{a_1a_2} \ldots R^{a_{2p-1}a_{2p}} T^{a_{2p+1}} e^{a_{2p+2}} \ldots e^{a_d} = 0 ,
\] (55)
where \( T^a = D_a e^a \). Unlike the procedure presented in previous works, the truly PL dynamics is recovered here without imposing any identification on the fields.

One could obtain the same result considering \( k^{ab} \) as the true spin-connection and \( \omega^{ab} \) as the new extra-field, but it is straightforward to see that the curvatures (9)-(10) reproduces the usual Lorentz curvature only when \( \omega^{ab} \) is identified as the true spin-connection one-form.

Thus, we have obtained the PL theory considering not only an appropriate limit in the GPL action (46) but also a right dynamical limit without any identification of the fields.

### 3.1 The five-dimensional case

The five-dimensional generalized pure Lovelock gravity reproduces two diverse actions depending on the value of \( p \). Indeed, for \( p = 1 \) the GPL action reduces to
\[
S_{GPL}^{p=1} = \int_{M_5} \epsilon_{ab f e} \left[ \alpha_0 e^a e^b e^c e^d e^e + \alpha_1 \left( R^{ab} e^d e^f + F^{ab} e^c e^d e^e \right) \right] ,
\] (56)
Here \( \alpha_0 \) and \( \alpha_1 \) are given by eqs. (44)-(45). The \( p = 1 \) GPL action can be seen as the coupling of a generalized cosmological term \( \mathcal{L}_\Lambda \) to the Einstein-Hilbert term,
\[
S_{GPL}^{p=1} = \int_{M_5} \mathcal{L}_\Lambda + \mathcal{L}_{EH} ,
\] (57)
where
\[ L_\Lambda = \alpha_0 \epsilon_{abcde} e^a e^b e^c e^d e^e + \alpha_1 \epsilon_{abcde} \left( D_\omega k^{ab} e^c e^d e^e + k^a j^b e^c e^d e^e \right) . \] (58)

One can see that the matter-free configuration limit \( (k^{ab} = 0) \) leads to the
\[ S_{PL}^{p=1} = \int_{M_5} \epsilon_{abcde} \left( \alpha_0 \epsilon^{a} e^b e^c e^d e^e + \alpha_1 R^{ab} e^c e^d e^e \right) , \] (59)

where the \( \alpha_p \) coefficients are identical to the PL ones. Let us note that the
\[ p = 1 \] PL action corresponds to the standard General Relativity action in presence of a cosmological constant term. The obtention of GR in a matter-free configuration limit of the GPL theory is a desirable feature in order to generalize gravity since it should satisfy the correspondence principle. Furthermore, in a matter-free configuration, the field equations read
\[ \epsilon_{abcde} \left( \alpha_0 \epsilon^{a} e^b e^c e^d e^e + \alpha_1 R^{ab} e^c e^d e^e \right) \delta e^e = 0 , \] (60)
\[ \epsilon_{abcde} \left( \alpha_1 T^e e^c e^d e^e \right) \delta \omega^{ab} = 0 , \] (61)
\[ \epsilon_{abcde} \left( \alpha_1 T^e e^d e^e \right) \delta k^{ab} = 0 , \] (62)

which correspond to the appropriate \( p = 1 \) PL dynamics described in refs. [5, 13, 14].

On the other hand, the \( p = 2 \) case does not contain the EH term and the GPL action is given by
\[ S_{GPL}^{p=2} = \int_{M_5} \epsilon_{abcde} \left[ \alpha_0 \epsilon^{a} e^b e^c e^d e^e + \alpha_2 \left( R^{ab} R^{cd} e^e + R^{ab} R^{cd} e^e + R^{ab} R^{cd} e^e \right) \right] , \] (63)

which can be seen as the coupling of a generalized cosmological term \( L_\Lambda \) to an Lanczos-Lovelock term \( L_{LL} \),
\[ S_{GPL}^{p=2} = \int_{M_5} L_\Lambda + L_{LL} . \] (64)

The \( L_\Lambda \) term includes the usual cosmological term plus additional terms depending on the Lorentz-like field \( k^{ab} \),
\[ L_\Lambda = \alpha_0 \epsilon_{abcde} e^a e^b e^c e^d e^e + \alpha_1 \epsilon_{abcde} \left( R^{ab} D k^{cd} e^e + D k^{ab} D k^{cd} e^e \right) , \] (65)

with \( D = d + \omega + k \).

As in the \( p = 1 \) case, the \( p = 2 \) PL theory is recovered in a matter-free configuration limit,
\[ S_{PL}^{p=2} = \int_{M_5} \epsilon_{abcde} \left[ \alpha_0 \epsilon^{a} e^b e^c e^d e^e + \alpha_2 \epsilon^{a} e^b e^c e^d e^e \right] , \] (66)

while the field equations considering \( k^{ab} = 0 \) reproduce the \( p = 2 \) PL dynamics,
\[ \epsilon_{abcde} \left( \alpha_0 \epsilon^{a} e^b e^c e^d + \alpha_1 R^{ab} R^{cd} e^e \right) \delta e^e = 0 , \] (67)
\[ \epsilon_{abcde} \left( \alpha_1 R^{cd} e^e \right) \delta \omega^{ab} = 0 , \] (68)
\[ \epsilon_{abcde} \left( \alpha_1 R^{cd} e^e \right) \delta k^{ab} = 0 . \] (69)
Let us note that the value and the sign of $\alpha_0$ is different for any value of $p$ and thus $\alpha_0$ is distinct for $p = 1$ and $p = 2$. In particular, for even value of $p$ the $\alpha_0$ coefficient has a negative sign which makes the pure Lovelock theory to have a unique nondegenerate $ds$ and $AdS$ vacuum \textsuperscript{15}. Interestingly, no further considerations on the $\alpha_p$ constants or in the fields must be imposed in order to obtain appropriately the pure Lovelock theory. A similar procedure has been considered in ref. \textsuperscript{17} in order to recover the five-dimensional PL theory. However, the obtention of the PL action and dynamics in \textsuperscript{17} required the introduction not only of four new extra-fields but also appropriate identifications on the extra-fields.

### 3.2 The six-dimensional case

The six-dimensional generalized pure Lovelock action also reproduces two diverse gravity actions depending on the value of $p$. Each case describes an alternative way to introduce a cosmological term. Let us note that only $p = 1$ and $p = 2$ reproduce non-trivial actions meanwhile the $p = 3$ case does not correspond to a GPL action since it leads to topological terms. For $d \geq 7$, a $p = 3$ GPL action can be constructed.

The $p = 1$ GPL action consists in the Einstein-Hilbert term plus a six-dimensional generalized cosmological term,

$$S_{GPL}^{p=1} = \int_{M_6} \epsilon_{abcdef} \left[ \alpha_0 e^a e^b e^c e^d e^e e^f + \alpha_1 \left( R^{ab} e^c e^d e^e e^f + F^{ab} e^c e^d e^e e^f \right) \right].$$

The GPL action can be rewritten explicitly as

$$S_{GPL}^{p=1} = \int_{M_6} \mathcal{L}_\Lambda + \mathcal{L}_{EH},$$

where

$$\mathcal{L}_\Lambda = \alpha_0 \epsilon_{abcdef} e^a e^b e^c e^d e^e e^f + \alpha_1 \epsilon_{abcdef} \left( D_\omega k^{ab} e^c e^d e^e e^f + k^a k^b e^c e^d e^e e^f \right).$$

The $p = 1$ GPL corresponds to one of the simplest generalizations of the GR theory. Interestingly GR in presence of the cosmological constant, which corresponds to the $p = 1$ pure Lovelock action, emerges considering a matter-free configuration limit ($k^{ab} = 0$),

$$S_{PL}^{p=1} = \int_{M_6} \epsilon_{abcdef} \left( \alpha_0 e^a e^b e^c e^d e^e e^f + \alpha_1 R^{ab} e^c e^d e^e e^f \right).$$

The $p = 1$ PL action always corresponds to the standard General Relativity action in presence of a cosmological constant. Moreover, in a matter-free configuration, the field equations read

$$\epsilon_{abcdef} \left( \alpha_0 e^a e^b e^c e^d e^e + \alpha_1 R^{ab} e^c e^d e^e \right) \delta e^f = 0,$$

$$\epsilon_{abcdef} \left( \alpha_1 T^{cde} e^e e^f \right) \delta \omega^{ab} = 0,$$

$$\epsilon_{abcdef} \left( \alpha_1 T^{cde} e^e e^f \right) \delta k^{ab} = 0,$$

which describe appropriately the $p = 1$ PL dynamics.
On the other hand, the \( p = 2 \) GPL action describes an alternative way to introduce a cosmological term,

\[
S_{\text{GPL}}^{p=2} = \int_{M_6} \epsilon_{abcdef} \left[ \alpha_0 e^a b^c d^e f^e + \alpha_2 \left( R^{ab} R^{cd} e^f + R^{ab} F^{cd} e^f + F^{ab} F^{cd} e^f \right) \right],
\]

which can be seen as the coupling of a generalized cosmological term \( \mathcal{L}_\Lambda \) to a Gauss-Bonnet term \( \mathcal{L}_{GB} \),

\[
S_{\text{GPL}}^{p=2} = \int_{M_6} \mathcal{L}_\Lambda + \mathcal{L}_{GB}.
\]

Here

\[
\mathcal{L}_\Lambda = \alpha_0 \epsilon_{abcdef} e^a b^c d^e f^e + \alpha_1 \epsilon_{abcdef} \left( R^{ab} D_k c^d e^f + D_k a b D_k c^d e^f \right),
\]

with \( D = d + \omega + k \).

Considering \( k^{ab} = 0 \) we recover the six-dimensional \( p = 2 \) PL theory,

\[
S_{\text{PL}}^{p=2} = \int_{M_6} \epsilon_{abcdef} \left[ \alpha_0 e^a b^c d^e f^e + \alpha_2 R^{ab} R^{cd} e^f \right],
\]

while the field equations read in a matter-free configuration limit

\[
\epsilon_{abcdef} \left( \alpha_0 e^a b^c d^e f^e + \alpha_1 R^{ab} R^{cd} e^f \right) \delta e^f = 0, \quad (81)
\]

\[
\epsilon_{abcdef} \left( \alpha_1 R^{cd} T^e f^f \right) \delta \omega^{ab} = 0, \quad (82)
\]

\[
\epsilon_{abcdef} \left( \alpha_1 R^{cd} T^e f^f \right) \delta k^{ab} = 0, \quad (83)
\]

which correspond to the appropriate \( p = 1 \) PL dynamics \([5, 13, 14]\).

As in odd-dimensions, no further considerations have to be imposed in order to obtain appropriately the pure Lovelock theory. A similar procedure has been considered in ref. \([18]\) in order to recover the even-dimensional PL theory from a Born-Infeld like gravity theory. Nevertheless, the obtention of the PL theory in ref. \([18]\) requires much more conditions and an excessive amount of extra fields.

4 Discussion

In the present work, we have presented a generalized Lovelock gravity theory introducing an additional field which enlarge the symmetry to a Lorentz-like algebra. Interestingly, a generalized pure Lovelock theory is obtained fixing the \( \alpha_p \) coefficients which consists only in two terms of full generalized Lovelock action. The generalized PL action considered here allows us to present an alternative way of introducing a generalized cosmological term. Our result generalizes the four-dimensional case presented in ref. \([37]\) to arbitrary dimensions.

In addition, the usual pure Lovelock theory is recovered in a matter-free configuration of the GPL theory. Unlike refs. \([17, 18]\), not only the PL action is recovered but also the right PL dynamics is directly obtained in the matter-free configuration limit. Such limit is considered without imposing any identifications on the fields. Besides the field content of the GL gravity theory is independent of the spacetime dimension avoiding excessive number of terms in higher dimensions.
The results obtained here, along with the ones presented in \cite{17, 18}, could be useful in order to construct a supersymmetric extension of the pure Lovelock theory. Furthermore, the same procedure could be applied in other (super)gravities in order to establish explicit relations between non-trivial (super)gravity actions. In particular it would be interesting to explore the existence of a configuration limit in order to derive the CJS supergravity. In refs. \cite{38, 39}, it has been suggested that the Maxwell like superalgebras could be useful for such task.

Additionally, there are interesting features of the Lovelock formalism which deserve to be explored in our generalized Lovelock theory. Of particular relevance in the AdS/CFT context are the black hole solutions of the Lovelock gravity \cite{40–42}. On the other hand, various problems in the Lovelock gravity can be solved exactly \cite{43}, leading to a particular interest in the effect of higher-curvature terms in the holography context \cite{44, 45}. Moreover, matter conformally-coupled to gravity can be seen as an extension of the Lovelock gravity theory \cite{46}. In this model, interesting problems related to the black hole geometry can be solved exactly \cite{47}, allowing to study Hawking-Page phase transitions \cite{48, 49}. Further interesting studies about the Lovelock gravity theory can be found in refs. \cite{50–54}.

Finally, it would be worth exploring our generalization to the quasi-topological gravity which consist in higher curvature gravity \cite{55–63}. The field equations of such theory reduce intriguingly to second order differential equations for spherically symmetric spacetimes and have exact solutions similar to the Lovelock ones. Such interesting behavior is not unique but appears in a bigger family of theories that contains the Lovelock and the quasi-topological theories, as well as the recent Einsteinian cubic gravity theory \cite{64–66} as particular examples \cite{67–69}.

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