Quark Schwinger-Dyson equation in temporal Euclidean space

V. Šauliš and Z. Batíz
1Dept. of Theor. Phys., INP, Řež near Prague, AVČR
2CFTP and Dept. of Phys., IST, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

We present an elementary nonperturbative method to obtain Green’s functions (GFs) for time-like momenta. The observable spectra of hadrons represent clear information about the S-matrix at color singlet channels. The processes $e^-e^+ \rightarrow \text{hadrons}$, τ hadronic decays and so on, are significant sources of experimental information in timelike regime of momenta. The microscopic description of such phenomena at low energy QCD processes are intuitively understood in terms of elementary QCD quanta, although the quantitative description is almost missing. Also other quantum field theoretical nonperturbative problems, e.g. confinement, require computation of Green’s function (GF) for timelike momenta. For understanding QCD the knowledge of GFs is a crucial matter. As QCD is commonly accepted as the ordinary quantum field theory, the amplitudes should be obtainable from elementary QCD GFs which could already encode the information about observables.

Considering the quark propagator in momentum space, the absence of real poles is a tempting idea of quark confinement. The spontaneous dynamical generation of an imaginary part of the quark mass can lead to the absence of real pole. We argue that this is the scenario of confinement by showing the model where the imaginary part of quark wave function has been spontaneously generated even below the standard perturbative threshold. Therefore, our method favors a confinement mechanism based on the lack of real poles.

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I. INTRODUCTION

The observable spectra of hadrons represent clear information about the S-matrix at color singlet channels. The processes $e^-e^+ \rightarrow \text{hadrons}$, τ hadronic decays and so on, are significant sources of experimental information in timelike regime of momenta. The microscopic description of such phenomena at low energy QCD processes are intuitively understood in terms of elementary QCD quanta, although the quantitative description is almost missing. Also other quantum field theoretical nonperturbative problems, e.g. confinement, require computation of Green’s function (GF) for timelike momenta. For understanding QCD the knowledge of GFs is a crucial matter. As QCD is commonly accepted as the ordinary quantum field theory, the amplitudes should be obtainable from elementary QCD GFs which could already encode the information about observables.

Considering the quark propagator in momentum space, the absence of real poles is a tempting idea of quark confinement. The spontaneous dynamical generation of an imaginary part of the quark mass can lead to the absence of real pole. We argue that this is the scenario of confinement by showing the model where the imaginary part of quark wave function has been spontaneously generated even below the standard perturbative threshold. Recall that the threshold value would be otherwise uniquely determined just by the quark pole mass. Using the formalism of Schwinger-Dyson equations, we will exhibit a realistic scenario, in which the quark mass function, as well as the quark renormalization function, become complex for almost all timelike momenta. As usually, the quark propagator remains real for spacelike momenta, where its values correspond to the results performed in the standard Euclidean formalism.

Euclidean space Lattice theory represents the method which in principle provides the information about GFs from the first principles. Minkowski space simulation in QCD is not recently feasible because of oscillating phase factor in the generating functional. To make the method feasible, the continuation to the imaginary time axis is required and the problem is solved in unphysical Euclidean space. Afterward the continuation back is necessary. Such a continuation of lattice data to the timelike momentum axis has been performed only very recently [1] providing thus data for very low momenta only. Avoiding large systematic errors, the continuation can be performed only within imposing of an additional global analytical assumption [2].

In this paper we will solve the quark gap equation which is an alternative way to achieve the non-perturbative solution for QCD Greens functions [3, 4]. The quark gap equation is the part of the Schwinger-Dyson equations (SDEs) which when solved exactly could provide the fully dressed Greens functions as well. As SDEs are an infinite tower of coupled integral equations, they require approximation and/or truncation of the SDEs system. Similarly to lattice formulation of quantum field theory, the most studies of SDEs are performed in the Euclidean space. The trial functions with given analytical properties had been used to make naive continuation to the timelike regime. The results of the paper [3] point towards an analytical structure of the quark propagator with a dominant singularity on the real timelike axis, while the nature of this singularity has not been determined with confidence. In the light of very recent numerical study [1], the singularity can be a branch point and not a real or complex conjugated poles suggested in [3]. The other studies preformed on various assumption also point towards the absence of real pole or they at least challenge that the real pole could be a dominant singularity of the quark propagator [6, 7, 8].

Perturbation theory (PT) is the only known method where such continuation is well understood and massively used in practice. In fact in PT at finite order, the analyticity assumptions are more specified: the propagators are analytical functions in the whole complex plane up to a real positive semi axis of $q^2$. When PT is reliable, the particles are revealed in the GFs poles and branch cuts in momentum space. In such circumstances the tree level single pole propagator is dressed within its form constrained by the PT analyticity described above. In general it leads to the
known forms of integral representations and dispersion relations for GFs. This has been used in QCD SDEs formalism long time ago [9], assuming that the spectral representations remain valid for the full nonperturbative solution. More recently, the method of solution based on such spectral representation has been been checked in practice for the number of the toy models [7, 10, 11], providing the correct solution only for rather weak coupling, while it appears to be inefficient when the couplings exceed certain critical values. It is to be noted, that the position of the branch points are uniquely dictated by the mentioned spectral method, which gives us a little freedom for spontaneous generation of complex masses. The above mentioned facts do not disprove the spectral method completely, however the practical failure of the method could be understood as a sign of weakness of the analytical assumptions.

In the next Section we propose new approach based on weaker analytical assumption (compared to the PT or spectral technique discussed above) and 3d Wick rotation is introduced to "rotate" originally space components to imaginary axis. The method is applied to the quark gap equation in the Section 3. The obtained solutions is presented in Section 4.

II. FROM MINKOWSKI SPACE TO TEMPORAL EUCLIDEAN SPACE $E_T$

In lattice theory and in most of the Schwinger-Dyson equations approaches in the literature the so called Wick rotation is used to avoid calculations in Minkowski space, wherein the Green’s functions are singular and the integration is problematic, especially numerically. Besides the singularities problem, there is another aspect impeding momentum integration presented in SDEs, the hyperbolic angle of Minkowskian ”spherical” coordinates:

\[
\begin{align*}
  k_0 &= k \cosh \theta_1 \\
  k_x &= k \sinh \theta_1 \cos \theta_2 \\
  k_y &= k \sin \theta_1 \sin \theta_2 \cos \theta_3 \\
  k_z &= k \sinh \theta_1 \sin \theta_2 \sin \theta_3
\end{align*}
\]  

(2.1)

runs from $-\infty$ to $\infty$ and most (principal valued here) integrals in terms of these hyperbolic angles cannot be found in a closed form. (To that point, there exist a semiperturbative prospect to work directly in Minkowski space, the first iteration of SDE was performed in the paper [12], however single real pole propagator was necessary input to perform some integration analytically)

The mentioned Euclidean space formulation, however, has some drawbacks, since physics involves Minkowski space and not Euclidean one. One needs to ”rotate back” the results to obtain them for timelike arguments. This rotation is basically an analytical continuation on the boundary of perturbative analyticity domain and therefore can be frequently ambiguous, especially if the results are numerically obtained.

In order to circumvent the difficulties stemming from the Minkowski metric or inverted rotation we propose a different procedure: instead of Wick rotating the time variable we rotate the space components. Clearly, this way we maintain the singularity structure, which for a free propagators stays on the exterior boundary of complex contour, but this is a small price to pay for the fact that angular integral are more tractable and especially for the fact that we do not need to rotate the variable twice. The method we will present here is used to obtain results at timelike regime of fourmomenta, while the correlation functions for spacelike arguments can be evaluated in the standard fashion.

We assume there are no singularities in the second and the fourth quadrants of complex planes of the complex variables $k_x, k_y, k_z$. Giving the Lorentz invariance, the singularities in the kernels can be functions only of $p^2$, this assumption is in agreement with the one used in standard Wick rotation, wherein there are no assumed singularities in the first and the third quadrants of the complex $k_0$ plane. This happen for instance when the obtained imaginary part of the square of the mass function is negative, excluding thus any singularities from the I. and III. quadrants and the imaginary $k_i$ axis as well. The afore-mentioned Wick rotation is sketched in Fig. 11 Cauchy theorem gives the following prescription for momentum:

\[
\begin{align*}
  k_{x,y,z} &\to ik_{1,2,3} \, , \\
  i \int d^4k &\to \int d^5k_{E_T} \, ,
\end{align*}
\]  

(2.2)

which in the case of original $3 + 1$ is identical with standard Euclidean $E$ ”spacelike” one. Note only that, the additional $i$ appears when the original Minkowski space is of odd-dimensionality.

For instance the free propagator of scalar particle then looks

\[
\frac{1}{p^2 - m^2 + i\varepsilon} \, .
\]  

(2.3)
with positive square
\[ p^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2. \] (2.4)

If necessary, one can use the Euclidean definition of Dirac gamma matrices, \( \gamma_0 \rightarrow \gamma_4; \vec{\gamma} \rightarrow i\vec{\gamma}_E \) and redefined gamma matrices satisfy \( \{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta^{\mu\nu} \).

Since the fixed square Minkowski momentum \( p^2 = \text{const} \) hyperboloid with infinite surface is transformed into the finite four-dimensional sphere in \( E_T \) space, the Cartesian variable are related to the spherical coordinates as usually:
\[
\begin{align*}
k_4 &= k \cos \theta \\
k_1 &= k \sin \theta \cos \beta \\
k_2 &= k \sin \theta \sin \beta \cos \phi \\
k_3 &= k \sin \theta \sin \beta \sin \phi.
\end{align*}
\] (2.5)

III. QUARK SDE

In QCD the quark propagator \( S \) is conventionally characterized by two independent scalars, the mass function \( M \) and renormalization wave function \( Z \) such that
\[ S(p) = \left[ \frac{Z(p)}{\not{p} - M(p) + i\varepsilon} \right]^{-1}, \] (3.1)
or equivalently the functions \( A, B \) (where simply \( M = B/A \), \( A = 1/Z \)) are used when suitable, noting for the bare fermion propagator \( S_0 \) we have \( A = 1 \) and \( B = m_0 \). If the function \( M(p) \) preserves the real pole in the full propagator \( S(p) \) then the insertion of Feynman \( i\varepsilon \) defines the way of loop momentum integration, otherwise it can omitted. For shorthand notation, we will also express the quark propagator in terms of the Dirac vector \( (S_v) \) and the Dirac scalar \( (S_s) \) parts of the propagator:
\[ S(p) = \not{p} S_v(p^2) - S_s(p^2). \] (3.2)

The gap equation for the inverse of \( S \) is
\[
\begin{align*}
S^{-1}(p) &= S_0^{-1}(p) - \Sigma(p), \\
\Sigma(p) &= iC_A g^2 \int \frac{d^4q}{(2\pi)^4} \Gamma_\alpha(q,p) G^{\alpha\beta}(p-q)S(q)\gamma_\beta,
\end{align*}
\] (3.3)
where $\Gamma$ is the quark-gluon vertex, $C_A = T_a T_a = 4/3$ for $SU(3)$ group and $G^{\alpha \beta}$ is the gluon propagator, which in covariant gauges reads

$$G^{\mu \nu}(k) = \left[ -g^{\mu \nu} + \frac{k^\mu k^\nu}{k^2} \right] G(k^2) - \xi \frac{k^\mu k^\nu}{(k^2)^2}, \quad (3.4)$$

where $G$ at tree level reads

$$G(k^2) = k^{-2}, \quad (3.5)$$

The gauge parameter dependent term in Eq. (3.4) remains undressed unless the gauge symmetry is broken, which we assume is the case of QCD.

As QCD is a non-Abelian gauge theory, the GFs are essentially gauge dependent, however in any gauge, the various Greens functions are related through the complicated Slavnov-Taylor identities. These constraints are especially simplified in the Background Field Gauge (BFG), wherein they simplify to the sort of Ward-Takahashi identities [13, 14, 15, 16]. In this case, the symmetry of the system has not been changed by gauge fixing procedure. When solving Schwinger-Dyson equation the BFG can be usefully further exploited [16, 19, 20], even without the exact knowledge of the missing and unknown vertices.

Especially, in BFG the quark-antiquark-gluon-vertex function satisfies QED like Ward identity (WTI):

$$k^\alpha \Gamma_\alpha(p, l) = S^{-1}(p) - S^{-1}(l), \quad (3.6)$$

where $k = p - l$ is the gluon fourmomentum. We use the advantage of BFG and we will consider quark propagator in BFG in this paper.

**IV. METRIC TENSOR TRUNCATION OF QUARK SDE**

In this section we transform the Minkowski quark SDE into the two-dimensional equation in Euclidean temporal space. To proceed this we first specify the approximation of the SDE. In order to go step further beyond the simplest ladder approximation we will use the exact WTI of BFG to treat product $k \cdot \Gamma$ in the kernel of SDE (3.3). This allows to entirely evaluate the contribution which stem from this term.

The remaining what needs to be specified is the product of the full vertex $\Gamma$ with the metric tensor part $g$ of the gluon propagator. As an introductory approximation made in the temporal Euclidean space, we simply take for the product $g^{\mu \nu} \Gamma^\mu \approx \gamma^\nu$. The approximation is improvable by making a loop expansion with dressed internal propagators (i.e. skeleton expansion). This is a future program which, in addition, will check the reliability of approximation used here.

For convenience we will denote

$$\Sigma = \Sigma_T + \Sigma_{L \xi} = \Sigma_g + \Sigma_L + \Sigma_{L \xi} \quad (4.1)$$

$$\Sigma_i(p) = \delta B_i(p) - \delta A_i(p) \not p \quad (4.2)$$

for $i = g, L, L \xi$, where $\Sigma_T$ stems from the dressed transverse part of the gluon propagator and $L, (L \xi)$ labels the selfenergy contribution which follows from the dressed (undressed gauge) longitudinal term in gluon propagator, clearly $T = g + L$ in our notation.

In this notation the appropriate terms explicitly read

$$\Sigma_g(p) = -i Z_1 g^2 C_A \int_k \Gamma_\mu(k, p) g^{\mu \nu} G(q) S(k) \gamma^\nu, \quad (4.3)$$

$$\Sigma_L(p) = i Z_1 g^2 C_A \int_k \Gamma_\mu(k, p) \frac{q^\mu q^\nu}{q^2} G(q) S(k) \gamma^\nu, \quad (4.4)$$

$$\Sigma_{L \xi} = -i Z_1 g^2 C_A \xi \int_k \Gamma_\mu(k, p) \frac{q^\mu q^\nu}{(q^2)^2} S(k) \gamma^\nu, \quad (4.5)$$

where we have used the shorthand notation $\int_k$ for the fourdimensional integral $\int \frac{d^4 k}{(2\pi)^4}$. 

Using the WTI we can get for $\Sigma_L\xi$

$$\Sigma_L\xi(p) = -iZ_l g^2 C_A \int k \frac{k}{(k^2)^2}$$

$$+ iZ_l g^2 C_A \int S^{-1}(p) S(k) \frac{k}{(q^2)^2}.$$  \hspace{1cm} \text{(4.6)}

The first term (4.6) is zero since it is odd in the variable $k$.

Performing 3d Wick rotation and integrating over the Euclidean angles we get the following contribution to the renormalization function:

$$\delta A_{L\xi}(x) = \frac{\xi g^2 C_A}{(4\pi)^2} \left[ B(x) \int_0^x dy \frac{y}{x} S_s(y) + A(x) \int_x^\infty dy S_s(y) \right],$$  \hspace{1cm} \text{(4.7)}

where $x = p^2_\xi$ and $y = q^2_\xi$.

For the contribution to the function $B$ we can obtain

$$\delta B_{L\xi}(x) = \frac{\xi g^2 C_A}{2(4\pi)^2} \left[ A(x) \int_0^x dy \frac{y}{x} S_s(y) + B(x) \int_x^\infty dy S_s(y) \right].$$  \hspace{1cm} \text{(4.8)}

In order to calculate $\Sigma_T$ the transverse part of the full gluon propagator needs to be specified. At low $Q^2$ the BFM gluon propagator is unknown function of momenta and gauge parameter, the only known is the undressed longitudinal part. To that point we will consider Landau gauge, assuming that various recent studies SDEs and lattice calculations performed in this gauge, offer already reasonable estimate. The most ambiguous is the deep infrared behaviour $q^2 \Lambda^2_{QCD}$, depending on the details, most of the recent studies shows up that tree level $q^2 = 0$ pole singularity is softened $1/(q^2)^a, a < 1$, with possible infrared finite solution [17], [18].

In the present paper we assume $q^2$ is a branch point of gluon propagator, which does not have a purely real pole in its transverse part. More specifically, we will assume that the product of the coupling with gluon propagator can be expressed through the following integral representation:

$$\frac{g^2}{4\pi} G(q^2, \Lambda_{QCD}) = \int_0^\infty d\nu \frac{\rho_2(\nu, \Lambda_{QCD})}{q^2 - \nu + i\varepsilon}. $$  \hspace{1cm} \text{(4.9)}

Thus, contrary to studied quark propagator, the standard analyticity for gluon propagator is still assumed. As already mentioned, such a representation has been already used in SDE context [9], however that the exact gluon propagator has not the assumed analytical properties is quite possible which would complicate our analysis in this case.

To do our best we will use the reasonable model of the gluon propagator at all scales. Below, we discuss several basic requirements which should be satisfied.

Firstly, the prescription (4.9) will respect asymptotic freedom thus for sufficiently large $q^2$ the leading power behaviour must be softened by standard perturbative log corrections such that

$$\frac{g^2}{4\pi} G(q^2, \Lambda_{QCD}) \simeq \frac{1}{q^2 \log(q^2/\Lambda_{QCD}^2)} + ...$$  \hspace{1cm} \text{(4.10)}

where the dots represents higher order scheme dependent contribution.

It will have no unphysical singularity (known from naive use of perturbative theory at strong coupling). At last but not at least, the $\rho_2$ in the gluon propagator may involve confinement. The last two requirements listed above are automatically satisfied for any regular function $\rho_2$. To comply with this we will not assume that $\rho_2$ includes Dirac delta as it would be when free particle mode is expected.

To satisfy all the requirements simultaneously the propagator function $G$ can be constructed by considering the function

$$\rho_2(x) = 2\frac{\alpha(x) \rho_0(x)}{x},$$  \hspace{1cm} \text{(4.11)}

where the function $\alpha(x)$ is calculated through

$$\rho_0(x) = \frac{4\pi/\beta}{\pi^2 - \ln^2(x/\Lambda_{QCD}^2)},$$

$$\alpha(x) = P_\nu \int_0^\infty \frac{d\nu \rho_0(\nu)}{x - \nu}.$$  \hspace{1cm} \text{(4.12)}
where symbol $\mathcal{P}$ stands for Cauchy principal value integration and $\beta$ in (1.12) represents the beta function coefficient, for which we take $4\pi/\beta = 1$ (recall, $4\pi/\beta = 1.396$ for three active quarks in perturbative QCD).

Recall also, the auxiliary functions $\rho_g$, $\alpha(x)$ correspond to the imaginary and real parts of the analyticized 1-loop effective charge $\alpha_{QCD}(x)$ constructed in [21, 22, 23], however the original meaning of $\alpha(x)$ is lost here. In our approach it is the gluon propagator, and not the running charge, which satisfies dispersion relation (4.9). The full expression for $\alpha$ can be found in the original paper.

Substituting IR (4.9) into $\Sigma$ we can write for $\delta A_g$

$$\delta A_g(p^2) = -\frac{Tr(\hat{\rho}\Sigma_g(p))}{4p^2},$$

where $q = p - k$. Performing the 3d Wick rotation and integrating over the Euclidean angles we get

$$\delta A_g(p^2) = \frac{-C_A}{\pi^2} \int_0^\infty dy y \sqrt{y/x} S_v(y) \int_0^\infty d\nu \rho_g(\nu) I_2(x, y, \nu),$$

(4.14)

where the function $I_2$ is defined below (4.17). Similarly we can easily derive the contribution from $g$ to the function $B$

$$\delta B_g(x) = \frac{Tr}{4} \Sigma_g = \frac{2C_A}{\pi^2} \int_0^\infty dy y S_v(y) \int_0^\infty d\nu \rho_g(\nu) I(x, y, \nu).$$

(4.15)

The functions $I, I_2$ in (4.15) and (4.14) are the complex non-holomorphic functions defined through the angular integral in the following way

$$I(x, y, \nu) = \int_0^{2\pi} d\theta \sin^2 \theta \frac{\sin^2 \theta}{x + y - \nu - 2\sqrt{xy} \cos \theta + i\epsilon},$$

(4.16)

$$I_2(x, y, \nu) = \int_0^{2\pi} d\theta \sin^2 \theta \cos \theta \frac{\sin^2 \theta}{x + y - \nu - 2\sqrt{xy} \cos \theta + i\epsilon}.$$  

(4.17)

Both integrals above can be evaluated in a closed form and we list the results in the Appendix A.

For $L$ contribution we first use the WTI (3.6) and the integral representation (4.9), then the appropriate contribution can be written like

$$\Sigma_L(p^2) = i4\pi C_A \int_k \frac{h}{q^2} S(k) S^{-1}(p) \int_0^\infty d\nu \rho_g(\nu) \int_0^\infty \frac{d\nu \rho_g(\nu)}{q^2 - \nu + i\epsilon}.$$  

(4.18)

Making the appropriate trace projections, performing the 3d Wick rotation and after some trivial manipulations we get

$$\delta A_L(x) = \frac{c_p}{x} \int_0^\infty d\nu \rho_g(\nu) \int_0^\infty dy y \left[ -S_v(y)A(x) \int_0^\infty \frac{d\theta q \cdot k \sin^2 \theta}{q^2(q^2 - \nu + i\epsilon)} + S_v(y)B(x) \int_0^\infty \frac{d\theta q \cdot p \sin^2 \theta}{q^2(q^2 - \nu + i\epsilon)} \right],$$

$$\delta B_L(x) = c_p \int_0^\infty d\nu \rho_g(\nu) \int_0^\infty dy y \left[ -S_v(y)B(x) \int_0^\infty \frac{d\theta q \cdot k \sin^2 \theta}{q^2(q^2 - \nu + i\epsilon)} + S_v(y)A(x) \int_0^\infty \frac{d\theta q \cdot p \sin^2 \theta}{q^2(q^2 - \nu + i\epsilon)} \right],$$  

(4.19)

where

$$c_p = \frac{4\pi C_A}{(2\pi)^3}.$$  

(4.20)

In the above formula we do not state explicitly the fact that the all scalar products are in $E_T$ space. The scalar products $k \cdot q = k^2 - k \cdot p$ and $q \cdot p = k \cdot p - p^2$ in the numerators lead finally to the result that can be expressed by the integrals $I$ and $I_2$ explicitly. We get:

$$\delta A_L(x) = -c_p \int_0^\infty dy y S_v(y) A(x) \int_0^\infty \frac{d\nu \rho_g(\nu)}{\nu} \left[ yI(x, y, \nu) - \sqrt{xy} I_2(x, y, \nu) - yI(x, y, 0) + \sqrt{xy} I_2(x, y, 0) \right]$$

$$- c_p \int_0^\infty dy y B(x) S_v(y) \int_0^\infty \frac{d\nu \rho_g(\nu)}{\nu} \left[ I(x, y, \nu) - \sqrt{xy} I_2(x, y, \nu) - I(x, y, 0) + \sqrt{xy} I_2(x, y, 0) \right].$$  

(4.21)
Similarly the function $B_L$ can be written in the following form:

$$
\delta B_L(x) = c_p \int_0^\infty dy dy \left( S_v(y) B(x) + A(x) S_s(y) \right) \int_0^\infty du \rho_p(\nu) \frac{I(x, y, \nu)}{2} 
- c_p \int_0^\infty dy dy \left( S_v(y) B(x) - A(x) S_s(y) \right) \int_0^\infty du \rho_s(\nu) (y - x) \frac{I(x, y, \nu) - I(x, y, 0)}{2\nu}.
$$

(4.22)

V. SOLUTION OF SDE IN $E_T$

Assuming regularity of functions $S_s, S_v$ on the real axis the quark SDE is transformed into two coupled complex integral equations which are free of non-integrable singularities and so they are prepared for suitable numerical treatment. Beside, assuming a perturbative asymptotic ultraviolet solution, the SDE requires renormalization. For this purpose we use the momentum subtraction renormalization scheme, so the SDE for unrenormalized functions $A, B$ which formally reads

$$
B = m_o + \sum_i \delta B_i ; A = 1 + \sum_i \delta A_i ; i = T, L, \zeta ;
$$

(5.1)

are rewritten into the SDE for renormalized ones. The renormalization constant $Z_1$ is absorbed defining the renormalized propagator, however here we are working in $E_T$ space and a certain care is needed. First, we avoid the mixture of different computational approaches by choosing a timelike renormalization scale. Further, we keep the renormalization constant real, thus only the real parts of the functions $\delta A, \delta B$ can be subtracted. Hence the renormalization is performed as the follows:

$$
\delta A_R(p, \mu) = Re\delta A(p) - Re\delta A(\mu) + i\mu \delta A(p),
\delta B_R(p, \mu) = Re\delta B(p) - Re\delta B(\mu) + i\mu \delta B(p),
$$

(5.2)

which leaves us with the renormalized SDE

$$
A_R(p, \mu) = 1 + \int dy \left( [ReK_A(x, y) - ReK_A(\mu, y)] + i\mu K_A(x, y) \right),
B_R(p, \mu) = m(\mu) + \int dy \left( [ReK_B(x, y) - ReK_B(\mu, y)] + i\mu K_B(x, y) \right),
$$

(5.3)

where, for clarity we have explicitly indicated

$$
\int dy K_A(x, y) = \sum_i \delta A_i,
$$

(5.4)

in order to show how the subtraction procedure works for the integral kernels. The same is performed for similarly for the kernel $K_B$.

As in the case of perturbation theory, the imaginary part is expected to be finite and untouched by renormalization. Clearly, this procedure maintains the hermicity of the Lagrangian.

For very low momenta the quark masses should approximately correspond to the known values of various constituent quark models, where $M(0) \approx \Lambda_{QCD}$ for up and down quarks. Assuming that the real part of the mass function is continuous when crossing zero, this value is actually available from Euclidean (spacelike) SDE studies: a typical estimate of the infrared mass lays in the range $250 - 600 \text{MeV}$, while the renormalized mass at few $\text{GeV}$ $m_{u,d}(2\text{GeV}) = 2 - 8 \text{MeV}$ is the standard input. Here, working in $E_T$ space instead of large, we rather choose low renormalization scale $\mu$, concretely

$$
\mu = \Lambda_{QCD}/4
$$

(5.5)

adjusting the renormalized function is $B(\mu) = \Lambda_{QCD}$ and $A(\mu) = 1$.

In practice the integrals are replaced by the discrete sums on suitable grid. Setting large upper bound $\epsilon^{16} \Lambda_{QCD}$ and taking the large number $N = 300 - 1000$ of Gaussian mesh points shows up reasonable stability of the numeric. The functions $A$ and $B$ are separated to their real and imaginary parts and we solve resulting four coupled integral equations simultaneously by the method of iterations. Comparing to the Euclidean spacelike case, the resulting kernel of Euclidean timelike SDE (5.3) is not a completely smooth function a more careful analysis is required. To speed up numeric significantly we first integrate over the integral variable $\nu$ before running the iterations.
Recently we have obtained the results for Landau gauge $\xi = 0$, where we have achieved a good stability of our numerical solution. In Fig. 2 we present the resulting functions obtained for 600 points and $e^{12}\Lambda_{QCD}$ cutoff, enlarging cutoff or decreasing the number of points makes the infrared behaviour more chaotic (leaving the smooth average approximately constant).

In PT the propagator is purely real under the threshold scale. Here, this is the main result of our presented study, the imaginary parts of the functions $B$ and $A$ are generated below the expected perturbative threshold. The resulting mass function becomes complex and a real pole is not present on the real axis of square of momenta.

More interestingly, we plot the absolute values of the mass function and the inverse of renormalization function in Fig. 2. The function $|M|$ shows up the maximum at $2.3\Lambda_{QCD}$ where it also cuts the linear function of $p$. The phase $\phi_M$ of the mass function extracted from $M = |M|e^{2\phi_M}$ where we got $\phi_M \approx -25^\circ$ at $q = 2.3\Lambda_{QCD}$. The mass phase is a slowly varying function in the full momentum regime and it monotonously goes to small negative value in UV.

The function $A = |A|e^{2\phi_A}$ is predominantly real, slowly varying, affecting quantitative behaviour of the function $M$ far from the renormalization point. The results for its absolute value and phase are added to the Fig. 2 and Fig. 3 respectively. Keeping the low scale renormalization point, it has a minimum at few $\Lambda_{QCD}$ and it logarithmically increases in UV (the same is also true also for spacelike regime).

In finite temperature and density QCD it is sometimes suggested, that confinement/deconfinement phenomena goes hand by hand with chiral symmetry breaking/chiral symmetric phases. In our formalism, although there is no space for temperature definition, but the description of chiral symmetry breaking could be a QCD must. Nowadays, the lack of a reasonably precise description of chiral symmetry breaking is a basic weakness of our presented $E_T$ formalism. Actually, within our setting, taking the Lagrangian mass to zero (also avoiding forbidden mass subtraction) we got the zero dynamical mass everywhere. A bit vaguely pronounced: the kernel of the SDE is not strong enough to produce this nonperturbative effect. Without going into technical details, the phenomena of dynamical mass generation could be available by further modeling of SDE kernel, (e.g. most naively, by further enhancing of the gluon propagator in the infrared). However, as we have found, the price we would pay is an unpleasant (and sometimes drastic) loss of numerical stability.

VI. SUMMARY AND CONCLUSIONS

We have presented a first analysis of the quark gap equation in the temporal Euclidean space. Given fact that 3d Wick rotated kernel is non-analytical function at timelike axis of momenta, we do not have at hand the powerful method as in the case of standard (spacelike) Euclidean formalism. Nevertheless, at this level the method really works and allows us to solve quark gap equation with a good accuracy.

We obtain the solution with spontaneous infrared complexification of the quark mass function, as opposed to
the perturbation theory, the quark mass function becomes complex from the beginning of the momentum axis. In Landau gauge, $B$ is the main source of the absorptive part of the quark propagator in the infrared region, while the renormalization function appears to be marginal for the confinement due to its small generated imaginary part. The absolute value of the complex mass function is enhanced at few $\Lambda_{QCD}$, with the nonzero quark mass function phase $\phi_M \approx 25^\circ$ responsible for the absence of the quark propagator pole.

The method provides not only a qualitative but even a quantitative description of propagator of confined quarks. Following the fact that our kernel approximation is only too weak in order to produce correct chiral symmetry breaking, we can expect that the observed complexification phenomena will persist for more realistic kernels of the quark SDE. Using the advantage of BFG, the contribution from the longitudinal gluons to quarks selfenergy has been already fully taken into account. The product of metric tensor with the improved quark-gluon vertex could provide the known slope of the mass function (already known form spacelike studies). To justify our estimate explicitly, an improved study of the quark propagator with a stable numeric is required.
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APPENDIX A: INTEGRAL $I$

Consider the integral $I$:

$$I(x, y, \nu) = \int_0^{2\pi} d\theta \frac{\sin^2 \theta}{a - b \cos \theta + i\varepsilon}$$  \hspace{1cm} (A1)

where $a = x + y - \nu$ is a real number and $b = 2\sqrt{xy}$ is a positive real number. Since $\nu$ is positive the integrand has a singularity in the integration range, so we keep the $i\varepsilon$ prescription of the propagator.

Making the standard substitution $t = \tan \frac{\theta}{2}$ we arrive at the following formula:

$$I(x, y, \nu) = \frac{8}{a + b} \int_0^\infty dt \frac{t^2}{(1 + t^2)^2(t^2 - c + i\varepsilon)}$$  \hspace{1cm} (A2)

where

$$c = \frac{b - a}{b + a}$$

Performing the principal value integration one can arrive to the following result

$$\frac{1}{\pi} I(x, y, \nu) = \frac{a}{b^2} + \frac{b - a}{b^2} \theta(-c) - i2\frac{\sqrt{c}}{b} \theta(c)$$  \hspace{1cm} (A3)

with an integrable singularity in $\sqrt{c}, (\sqrt{c})^{-1}$.

The integral

$$I_2(x, y, \nu) = \int_0^{2\pi} d\theta \frac{\sin^2 \theta \cos \theta}{a - b \cos \theta + i\varepsilon}$$  \hspace{1cm} (A4)

can be evaluated in a similar fashion.

The result is irregular at $c = 0$ and regular for positive or negative $c$. For $c > 0$ it reads

$$I_2(x, y, \nu) = \frac{\pi}{a + b} \left[ \frac{1}{1 + c} - \frac{8c}{(1 + c)^3} - i4 \frac{(1 - c) \sqrt{c}}{(1 + c)^3} \right]$$  \hspace{1cm} (A5)

For $c < 0$ we can get

$$I_2(x, y, \nu) = \frac{\pi}{a + b} \frac{1 - \sqrt{-c}}{(1 + \sqrt{-c})^3},$$  \hspace{1cm} (A6)

which is again a finite function.

The special cases $\nu = 0$ simplify, the functions $I(x, y, 0), I_2(x, y, 0)$ can be obtained by considering the appropriate limits.

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