Combinatorial $t$-designs from quadratic functions

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Received: 15 July 2019 / Revised: 30 October 2019 / Accepted: 8 November 2019 / Published online: 18 November 2019 © Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
Combinatorial $t$-designs have been an interesting topic in combinatorics for decades. It was recently reported that the image sets of a fixed size of certain special polynomials may constitute a $t$-design. Till now only a small amount of work on constructing $t$-designs from special polynomials has been done, and it is in general hard to determine their parameters. In this paper, we investigate this idea further by using quadratic functions over finite fields, thereby obtain infinite families of 2-designs, and explicitly determine their parameters. The obtained designs cover some earlier 2-designs as special cases. Furthermore, we confirm Conjecture 3 in Ding and Tang (ArXiv:1903.07375, 2019).

Keywords Polynomial · Quadratic functions · $t$-Design

Mathematics Subject Classification 51E21 · 05B05 · 12E10

1 Introduction

Let $k$, $t$ and $v$ are positive integers with $1 \leq t \leq k \leq v$. Let $P$ be a set of $v \geq 1$ elements, and let $B$ be a set of $k$-subsets of $P$. The pair $D = (P, B)$ is called an incidence structure, and is said to be a $t$-$(v, k, \lambda)$ design if every $t$-subset of $P$ is contained in exactly $\lambda$ elements of $B$. The elements of $P$ are called points, and those of $B$ are referred to as blocks. We usually

Communicated by L. Teirlinck.

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use $b$ to denote the number of blocks in $B$. A $t$-design is called simple if $B$ has no repeated blocks. A $t$-design is called symmetric if $v = b$ and trivial if $k = t$ or $k = v$. In this paper, we study only simple $t$-designs with $v > k > t$. When $t \geq 2$ and $\lambda = 1$, a $t$-design is called a Steiner system and traditionally denoted by $S(t, k, v)$.

Combinatorial t-designs have found important applications in coding theory, cryptography, communications and statistics. There are two major methods of constructing $t$-designs. One is to construct them from error-correcting codes, and a number of constructions have been presented (see for example [1,4,12,13,19–21]). Recently, Ding and Li [7] obtained infinite families of 2-designs and 3-designs from some special codes and their duals. Afterwards, some $t$-designs were further constructed from some other special codes over finite fields (see [5,6,9–11]). The other method is via group actions of certain permutation groups which are $t$-transitive or $t$-homogeneous on a certain point set. The following theorem, which shows that the incidence structure $(\mathcal{P}, B)$ is always a 2-design by 2-homogeneous group actions (see [2, Proposition 4.6]), was recently employed by Liu and Ding [14] to construct a number of infinite families of 2-designs.

**Theorem 1** [2, Proposition 4.6] Let $\mathcal{P}$ be a set of $v \geq 1$ elements, and let $G$ be a permutation group on $\mathcal{P}$. Let $B \subseteq \mathcal{P}$ be a $k$-subset with $k \geq 2$. Define

$$B = G(B) = \{g(B) : g \in G\},$$

where $g(B) = \{g(b) : b \in B\}$. If $G$ is 2-homogeneous on $\mathcal{P}$ and $|B| \geq 2$, then $(\mathcal{P}, B)$ is a $2$-design with

$$k = |B|, \lambda = b \binom{k}{2} / \binom{v}{2},$$

where $b = \frac{|G|}{|G_B|}$ and $G_B = \{g \in G : g(B) = B\}$ is the stabilizer of $B$ under the group $G$.

Very recently, Ding and Tang [8] presented two constructions of $t$-designs from special polynomials over finite fields, and obtained 2-designs and 3-designs with interesting parameters from their defined $d$-polynomials. However, it is in general hard to determine the parameters of the underlying $t$-designs by their constructions. Motivated by this fact, we obtain infinite families of 2-designs by using quadratic functions over finite fields and determine their parameters explicitly. For other constructions of $t$-designs, see [2,15–18] and the references therein.

The rest of this paper is arranged as follows. Section 2 introduces some basic notation and results of projective planes and affine curves which will be needed in subsequent sections. Based on a generic construction in [8], Sect. 3 gives infinite families of 2-designs with new parameters by quadratic functions over finite fields and the proofs of their parameters are given in Sect. 4. Section 5 summarizes this paper.

**2 Preliminaries**

In this section, we state some notation and basic facts about affine curves and projective planes, which will be used in the following sections.

**2.1 Some notation fixed throughout this paper**

For convenience, we adopt the following notation unless otherwise stated.
– $p$ is a prime number.
– $\gcd(h_1, h_2)$ denotes the greatest common divisor of the two positive integers $h_1$ and $h_2$.
– $q = p^m$, where $m$ and $l$ are positive integers, and $\gcd(m, l) = 1$.
– $\text{GF}(q)$ denotes the finite field with $q$ elements and $\text{GF}(q)^* = \text{GF}(q) \setminus \{0\}$.
– $\text{QR}$ and $\text{NQR}$ denote the set of all nonzero quadratic residues and quadratic non-residues in $\text{GF}(q)$, respectively.

### 2.2 Projective planes and affine curves

Let $\text{GF}(q^{\infty})$ be the algebraic closure of $\text{GF}(q)$. The projective plane $\mathbb{P}^2(\text{GF}(q))$ is defined as

$$
\mathbb{P}^2(\text{GF}(q)) := (\text{GF}(q)^3 \setminus \{(0, 0, 0)\}) / \sim,
$$

where $(X_0, Y_0, Z_0) \sim (X_1, Y_1, Z_1)$ if and only if there is some $\lambda \in \text{GF}(q)^*$ with $X_1 = \lambda X_0$, $Y_1 = \lambda Y_0$ and $Z_1 = \lambda Z_0$. To remind ourselves that points of $\mathbb{P}^2(\text{GF}(q))$ are equivalence classes, we write $(X:Y:Z)$ for the equivalence class of $(X, Y, Z)$ in $\mathbb{P}^2(\text{GF}(q))$. Let $f(x, y) \in \text{GF}(q)[x, y]$ be a polynomial of degree $d$ over $\text{GF}(q)$. Then the affine curve $C_f$ associated to $f$ is defined by

$$
C_f = \{(x, y) \in \text{GF}(q^{\infty})^2 : f(x, y) = 0\}.
$$

The projective closure of the affine curve $C_f$ is

$$
\hat{C}_f = \{(X : Y : Z) \in \mathbb{P}^2(\text{GF}(q^{\infty})) : F(X, Y, Z) = 0\},
$$

where $F(X, Y, Z) = Z^d \cdot f \left( \frac{X}{Z}, \frac{Y}{Z} \right)$ is the homogenization of $f$. For polynomial $F$, $F_X$, $F_Y$ and $F_Z$ denote the formal partial derivatives of $F$ with respect to $X$, $Y$ and $Z$, respectively. A singular point of $\hat{C}_f$ is a point $(X_0 : Y_0 : Z_0) \in \mathbb{P}^2(\text{GF}(q^{\infty}))$ such that

$$
\begin{align*}
F(X_0, Y_0, Z_0) &= 0, \\
F_X(X_0, Y_0, Z_0) &= 0, \\
F_Y(X_0, Y_0, Z_0) &= 0, \\
F_Z(X_0, Y_0, Z_0) &= 0.
\end{align*}
$$

The projective curve $\hat{C}_f$ is nonsingular if it has no singular points. A nonsingular projective plane curve is irreducible.

Let $\mathcal{X}$ be a curve over $\text{GF}(q)$, whose defining equations have coefficients in $\text{GF}(q)$. Then the points on $\mathcal{X}$ with all their coordinates in $\text{GF}(q)$ are called $\text{GF}(q)$-rational points. The set of all $\text{GF}(q)$-rational points of $\mathcal{X}$ is denoted by $\mathcal{X}(\text{GF}(q))$.

The following theorem is the fundamental result in the area of algebraic curves.

**Theorem 2** (Hasse–Weil Theorem) Let $\mathcal{X}$ be a nonsingular projective curve of genus $g$ over the field $\text{GF}(q)$ and set $N = |\mathcal{X}(\text{GF}(q))|$. Then

$$
|N - (q + 1)| \leq 2g\sqrt{q}.
$$

If $q$ is not a perfect square, we can replace the right-hand side of the inequality (1) in Hasse–Weil Theorem with $g\lfloor 2\sqrt{q} \rfloor$.

If $\hat{C}_f$ is a nonsingular projective plane curve corresponding to the polynomial $f(x, y) \in \text{GF}(q)[x, y]$ of degree $d$, then the genus $g$ of $\hat{C}_f$ is given by the Plücker formula

$$
g = \frac{(d - 1)(d - 2)}{2}.
$$

$\square$ Springer
3 $t$-Designs from quadratic functions over $\text{GF}(q)$

Let $f$ be a polynomial over $\text{GF}(q)$, which is always viewed as a function from $\text{GF}(q)$ to $\text{GF}(q)$. For each $(b, c) \in \text{GF}(q)^2$, define

$$B_{(f,b,c)} = \{ f(x) + bx + c : x \in \text{GF}(q) \}.$$ 

Let $k$ be an integer with $2 \leq k \leq q$. Define

$$B_{(f,k)} = \{ B_{(f,b,c)} : |B_{(f,b,c)}| = k, \text{ and } b, c \in \text{GF}(q) \}.$$ 

The incidence structure $\mathbb{D}(f, k) := (\text{GF}(q), B_{(f,k)})$ may be a $t$-$(q, k, \lambda)$ design for some $\lambda$, where $\text{GF}(q)$ is the point set, and the incidence relation is given by the set membership. In such a case, we say that the polynomial $f$ supports a $t$-$(q, k, \lambda)$ design. This construction of $t$-designs with polynomials over finite fields was documented recently in [8].

We define the value spectrum of a polynomial over $\text{GF}(q)$ to be the multiset

$$\text{VS}(f) = \{|B_{(f,b,c)}| : (b, c) \in \text{GF}(q)^2 \}.$$ 

To determine the parameters of $t$-designs supported by a polynomial $f$, we need to know its value spectrum.

This construction is generic in the sense that $t$-designs could be produced by properly selecting the polynomial $f$ over $\text{GF}(q)$. Based on this construction, only a small number of $t$-designs have been constructed. One of the main reasons is that the value spectrum of a polynomial is hard to determine in general. In this paper, we consider constructing $t$-designs from the quadratic function

$$f(x) = x^{p^k + 1}$$

over $\text{GF}(q)$ and determine their parameters.

The following two theorems are the main results of this paper, whose proofs will be postponed to presented in Sect. 4.

**Theorem 3** Let $p = 2, \ell, m$ be two positive integers with $m \geq 3$, $\ell < \frac{m}{4} - 1$ and $\gcd(\ell, m) = 1$. Let $f(x) = x^{2^\ell + 1}$. Then the incidence structure $\mathbb{D}(f(x), k) := (\text{GF}(q), B_{(f(x), k)})$ is a $2$-$(q, k, k(k - 1))$ design, where $k = \frac{2\ell + (-1)^m}{3}$.

**Theorem 4** Let $p$ be an odd prime with $p \equiv 3 \pmod{4}$ and $m \geq 3$ be odd. Let $\ell$ be a positive integer with $\ell < \frac{m-2}{4}$ and $\gcd(\ell, m) = 1$. Let $f(x) = x^{p^k + 1}$. Then the incidence structure $\mathbb{D}(f(x), k) := (\text{GF}(q), B_{(f(x), k)})$ is a $2$-$(q, k, \frac{k(k-1)}{2})$ design, where $k = q - \frac{p - 1}{2(p+1)}$.

As a special case of Theorem 4, we have the following corollary.

**Corollary 1** Let $(p, \ell) = (3, 2)$ and $m \geq 11$ be odd. Then the incidence structure $\mathbb{D}(x^{10}, k) := (\text{GF}(3^m), B_{(x^{10}, k)})$ is a $2$-$(3^m, k, \frac{k(k-1)}{2})$ design, where $k = \frac{5 \cdot 3^m + 1}{8}$.

Note that the conclusion of Corollary 1 also follows if $m \in \{3, 5, 7, 9\}$, which is verified by the Magma program. This means that the conjecture 3 in Ding and Tang [8] is true.

4 Proofs of the main results

Our task of this section is to prove Theorems 3 and 4. To this end, we shall prove a few more auxiliary results before proving the main results of this paper.
4.1 Some auxiliary results

**Lemma 1** Let \( \alpha \in \text{GF}(q)^* \) and \( \beta \in \text{GF}(q)^* \). Let \( \ell \) and \( m \) be integers with \( 1 \leq \ell < m \). Let \( f = f(x, y) = x^{p^\ell+1} + x - \alpha \left( y^{p^\ell+1} + y \right) - \beta \in \text{GF}(q)[x, y] \) and \( N = |C_f(\text{GF}(q))| \). Then

\[
(q + 1 - \delta) - p^\ell (p^\ell - 1) \sqrt{q} \leq N \leq (q + 1) + p^\ell (p^\ell - 1) \sqrt{q},
\]

where \( \delta = \gcd(p^\ell + 1, p^m - 1) \).

**Proof** Let \( \mathcal{X} \) be the projective curve \( \hat{C}_f \). Let

\[
F(X, Y, Z) = X^{p^\ell+1} + XZ^{p^\ell} - \alpha \left( Y^{p^\ell+1} + YZ^{p^\ell} \right) - \beta Z^{p^\ell+1} \in \text{GF}(q)[X, Y, Z]
\]

be the homogenization of \( f(x, y) \) and \((X:Y:Z)\) be a singular point of \( \mathcal{X} \). Then we have

\[
\begin{align*}
F_X &= X^{p^\ell} + Z^{p^\ell} = 0 \\
F_Y &= \alpha \left( Y^{p^\ell} + Z^{p^\ell} \right) = 0 \\
F_Z &= \beta Z^{p^\ell} = 0 \\
F(X, Y, Z) &= 0
\end{align*}
\]

Thus,

\[
\begin{align*}
X^{p^\ell} + Z^{p^\ell} &= 0 \\
Y^{p^\ell} + Z^{p^\ell} &= 0 \\
\beta Z^{p^\ell} &= 0
\end{align*}
\]

From \( \beta \neq 0 \), it follows that \( X = Y = Z = 0 \), a contradiction. Thus, \( \mathcal{X} \) is a nonsingular projective curve. By the Plücker formula (2) and Theorem 2, we have

\[
(q + 1) - p^\ell (p^\ell - 1) \sqrt{q} \leq |\mathcal{X}(\text{GF}(q))| \leq (q + 1) + p^\ell (p^\ell - 1) \sqrt{q}.
\]

(3)

By multiplying through by a nonzero element of \( \text{GF}(q) \), we can assume the right-most nonzero coordinate of a point of \( \mathbb{P}^2(\text{GF}(q)) \) is 1. Therefore, we have

\[
\mathcal{X}(\text{GF}(q)) = \{(x : y : 1) : (x, y) \in C_f \} \cup S,
\]

where \( S = \{(x : 1 : 0) : x \in \text{GF}(q), x^{p^\ell+1} = \alpha \} \). Then

\[
|\mathcal{X}(\text{GF}(q))| = N + |S|.
\]

Since \(|S| \leq \gcd(p^\ell + 1, p^m - 1)\), the desired results follows from Inequality (3).

**Lemma 2** Let \( \alpha \in \text{GF}(q) \setminus \{0, 1\} \), \( \ell \) and \( m \) be integers with \( 1 \leq \ell < m \). Let \( f(x, y) = x^{p^\ell+1} + x - \alpha \left( y^{p^\ell+1} + y \right) \in \text{GF}(q)[x, y] \) and \( N = |C_f(\text{GF}(q))| \). Then

\[
q + 1 - \delta \leq N \leq q + 1,
\]

where \( \delta = \gcd(p^\ell + 1, p^m - 1) \).
Proof Let $\mathcal{X}$ be the projective curve $\hat{C}_f$ and $F(X, Y, Z) = X^{p^\ell+1} + XZ^{p^\ell} - \alpha \left( Y^{p^\ell+1} + YZ^{p^\ell} \right) \in \text{GF}(q)[X, Y, Z]$ be the homogenization of $f(x, y)$.

Let $(X : Y : Z) \in \mathcal{X}(\text{GF}(q))$. Then we have

$$(X - \alpha Y) Z^{p^\ell} = \alpha Y^{p^\ell+1} - X^{p^\ell+1}.$$  

If $X = \alpha Y$, then $0 = \alpha Y^{p^\ell+1} - \alpha^{p^\ell+1}Y^{p^\ell+1} = \alpha \left( 1 - \alpha^{p^\ell} \right) Y^{p^\ell+1}$. By $\alpha \neq 0$ and $\alpha \neq 1$, we know that $(X : Y : Z)$ must be the point $(0 : 0 : 1)$.

If $X \neq \alpha Y$ and $Y = 0$, then $XZ^{p^\ell} = -X^{p^\ell+1}$. Thus, $(X : Y : Z)$ must be the point $(1 : 0 : -1)$.

If $X \neq \alpha Y$ and $Y \neq 0$, then

$$Z = \left( \frac{\alpha Y^{p^\ell+1} - X^{p^\ell+1}}{X - \alpha Y} \right)^{p^m - \ell}.$$  

Thus, $(X : Y : Z)$ must be the point

$$(x : 1 : \left( \frac{\alpha - x^{p^\ell+1}}{x - \alpha} \right)^{p^m - \ell})$$  

with $x \in \text{GF}(q) \setminus \{ \alpha \}$. Hence,

$$|\mathcal{X}(\text{GF}(q))| = q + 1. \quad (4)$$

Note that

$$\mathcal{X}(\text{GF}(q)) = \{(x : y : 1) : (x, y) \in C_f \} \cup S,$$

where $S = \{(x : 1 : 0) : x \in \text{GF}(q), x^{p^\ell+1} = \alpha \}$. It then follows that

$$|\mathcal{X}(\text{GF}(q))| = N + |S|.$$  

Since $|S| \leq \gcd(p^\ell + 1, p^m - 1)$, the proof is then completed by Eq. (4). \hfill \square

By Lemmas 1 and 2, we have the following corollary.

Corollary 2 Let $(\alpha, \beta) \in \text{GF}(q)^* \times \text{GF}(q)$ with $(\alpha, \beta) \neq (1, 0)$. Let $\ell$ and $m$ be integers with $1 \leq \ell < m$. Let $N = |C_f(\text{GF}(q))|$, where $f(x, y) = x^{p^\ell+1} + x - \alpha \left( y^{p^\ell+1} + y \right) - \beta \in \text{GF}(q)[x, y]$. Then

$$(q - p^\ell) - p^\ell(p^\ell - 1) \sqrt{q} \leq N \leq (q + 1) + p^\ell(p^\ell - 1) \sqrt{q}.$$  

Lemma 3 Let $(\alpha, \beta) \in \text{GF}(q)^* \times \text{GF}(q)$ with $(\alpha, \beta) \neq (1, 0)$. Let $\ell$ and $m$ be integers with $1 \leq \ell < m$. Let $N = |C_f(\text{GF}(q))|$, where $f(x, y) = x^{p^\ell+1} + x - \alpha \left( y^{p^\ell+1} + y \right) - \beta \in \text{GF}(q)[x, y]$. Define

$$B_\ell = \left\{ x^{p^\ell+1} + x : x \in \text{GF}(q) \right\}. \quad (5)$$

If $B_\ell = \alpha B_\ell + \beta$, then

$$N \geq 2q - |B_\ell|.$$  

Proof Let $h_0(x) = x^{p^\ell+1} + x$ and $h_1(x) = \alpha \left( x^{p^\ell+1} + x \right) + \beta$. Let $k = |B_\ell|$ and $B_\ell = \{z_1, \cdots, z_k\}$. For any $z \in \text{GF}(q)$, let $h_0^{-1}(z) = \{x \in \text{GF}(q) : h_0(x) = z\}$. Then we have

$$N = \sum_{y \in \text{GF}(q)} |h_0^{-1}(h_1(y))|.$$
Since $B_\ell = \alpha B_1 + \beta$, we have $|h_0^{-1}(h_1(y))| \geq 1$ for any $y \in \text{GF}(q)$. Then

$$N = q + \sum_{y \in \text{GF}(q)} \left( |h_0^{-1}(h_1(y))| - 1 \right)$$

$$\geq q + \sum_{i=1}^{k} \left( |h_0^{-1}(z_i)| - 1 \right)$$

$$= q - k + \sum_{i=1}^{k} |h_0^{-1}(z_i)|$$

$$= 2q - k.$$

This then completes the proof. □

In order to obtain Corollary 3, we need the following result which was proved in [3, Theorem 5.6].

**Lemma 4** [3, Theorem 5.6] Let $F$ be an arbitrary finite field of characteristic $p$, $s$ be a power of $p$ and $F \cap \text{GF}(s) = \text{GF}(t)$. Let $0 \neq b \in F$ and $N_0$ denote the number of $b$ such that the polynomial $x^{s+1} - bx + b$ has no rational root in $F$. Then

$$N_0 = \begin{cases} 
\frac{m+1-t}{2(p+1)} & \text{if } \hat{m} \text{ is even}, \\
\frac{p^{m+1} - 1}{2(p+1)} & \text{if } \hat{m} \text{ is odd and } s \text{ is odd}, \\
\frac{p^{m+1} + 1}{2(p+1)} & \text{if } \hat{m} \text{ is odd and } s \text{ is even}, 
\end{cases}$$

(6)

where $\hat{m} = [F : \text{GF}(t)]$.

**Corollary 3** Let $\ell$ be a positive integer with $\gcd(\ell, m) = 1$. Let $\hat{N}$ denote the number of $c \in \text{GF}(q)^*$ such that the polynomial $x^{p\ell+1} + x + c$ has no rational root in $\text{GF}(q)$. Then

$$\hat{N} = \begin{cases} 
\frac{p^{m+1} - p^{\ell}}{2(p+1)} & \text{if } m \text{ is even}, \\
\frac{p^{m+1} - 1}{2(p+1)} & \text{if } m \text{ is odd and } p^\ell \text{ is odd}, \\
\frac{p^{m+1} + 1}{2(p+1)} & \text{if } m \text{ is odd and } p^\ell \text{ is even}. 
\end{cases}$$

(7)

**Proof** In Lemma 4, we let $F = \text{GF}(p^m)$ and $s = p^\ell$ with $\gcd(\ell, m) = 1$. Then

$$F \cap \text{GF}(s) = \text{GF}(p^m) \cap \text{GF}(p^\ell) = \text{GF}(p)$$

and

$$\hat{m} = [F : \text{GF}(t)] = [\text{GF}(p^m) : \text{GF}(p)] = m.$$

Further, since $x^{p\ell}$ is a permutation of $\text{GF}(p^m)$, we have

$$x^{s+1} - bx + b = x^{p\ell+1} - bx + b$$

$$= x^{p\ell+1} - b^{p\ell} x + b^{p\ell}$$

$$= (-by)^{p\ell+1} - b^{p\ell} (-by) + b^{p\ell}$$

$$= b^{p\ell+1} y^{p\ell+1} + b^{p\ell+1} y + b^{p\ell}$$

$$= b^{p\ell+1} (y^{p\ell+1} + y + b^{-1}).$$
Since \( b \in \mathbb{GF}(p^m)^* \), \( b^{p^\ell+1}(y^{p^\ell+1} + y + b^{-1}) = 0 \) is equivalent to
\[
x^{p^\ell+1} + x + c = 0,
\]
where \( c \in \mathbb{GF}(p^m)^* \). The desired conclusion then follows from Lemma 4. \( \square \)

**Lemma 5** Let \( m \) and \( \ell \) be a positive integer with \( \gcd(\ell, m) = 1 \). Then
\[
|B_\ell| = \begin{cases} 
q - \frac{p^{m+1} - p}{2(p+1)} & \text{if } m \text{ is even}, \\
q - \frac{p^{m+1} - 1}{2(p+1)} & \text{if } m \text{ is odd and } p^\ell \text{ is odd}, \\
q - \frac{p^{m+1} + p}{2(p+1)} & \text{if } m \text{ is odd and } p^\ell \text{ is even},
\end{cases}
\]
where \( B_\ell \) was defined by (5).

**Proof** By definition, we have
\[
|B_\ell| = q - \hat{N}
\]
where \( \hat{N} \) was defined by Corollary 3. This means that Eq. (8) follows. This completes the proof. \( \square \)

**Lemma 6** Let \( m \) and \( \ell \) be a positive integer with \( \gcd(\ell, m) = 1 \). Define
\[
\text{Stab}(B_\ell) = \{ux + v : (u, v) \in \mathbb{GF}(q)^* \times \mathbb{GF}(q), \ uB_\ell + v = B_\ell\}
\]
and \( \mu = |\text{Stab}(B_\ell)| \), where \( B_\ell \) was defined by (5). Then we have the following.

(I) If \( p = 2, m \geq 4 \) is even and \( 2\ell + 2 < m/2 \), then \( \mu = 1 \).

(II) If \( p = 2, m \geq 3 \) is odd and \( 2\ell + 2 < m/2 \), then \( \mu = 1 \).

(III) If \( p \equiv 3 \mod 4, m \geq 3 \) is odd and \( 2\ell + 1 < m/2 \), then \( \mu = 1 \).

**Proof** We now prove the three cases as follows.

(I) By definition, it is clear that \((1, 0) \in \text{Stab}(B_\ell)\). Suppose that \( \mu = |\text{Stab}(B_\ell)| \neq 1 \), then there must exist \((\alpha, \beta) \in \text{Stab}(B_\ell)\) with \((\alpha, \beta) \neq (1, 0)\). From Corollary 2, it follows that
\[
N \leq (q + 1) + p^\ell(p^\ell - 1)\sqrt{q},
\]
where \( N \) was defined by Corollary 2.

Meanwhile, by Lemmas 3 and 5, we have
\[
N \geq 2q - k = q + \frac{p^{m+1} - p}{2(p+1)} = (q + 1) + \frac{p^{m+1} - 3p - 2}{2(p+1)}.
\]

Since \( p = 2 \) and \( m \geq 4 \) is even, we have
\[
\frac{p^{m+1} - 3p - 2}{2(p+1)} - 2^{m-2} = \frac{2^{m+1} - 8 - 2^{m-2}}{6} = \frac{1}{3}(2^{m-2} - 4) \geq 0.
\]
This means that
\[
\frac{p^{m+1} - 3p - 2}{2(p+1)} \geq 2^{m-2}.
\]
Therefore, from Eqs. (10) and (11), we have
\[
N \geq (q + 1) + 2^{m-2}.
\]
Furthermore, by $2\ell + 2 < m/2$ and Eq. (9) we have

$$N \leq (q + 1) + 2^{(\ell + 1)/2} < (q + 1) + 2^{(\ell + m/2)}/(q + 1) + 2^{m-2},$$

which contradicts to Eq. (12). This means that there does not exist $(\alpha, \beta) \in \text{Stab}(B_\ell)$ with $(\alpha, \beta) \neq (1, 0)$. Hence, $\mu = |\text{Stab}(B_\ell)| = |\{(1, 0)\}| = 1$.

(II) The proof is similar to case (I) and we omit it here. The desired conclusion then follows from Lemma 3 and Corollary 2.

(III) By definition, it is clear that $(1, 0) \in \text{Stab}(B_\ell)$. Suppose that $\mu = |\text{Stab}(B_\ell)| \neq 1$, then there must exist $(\alpha, \beta) \in \text{Stab}(B_\ell)$ with $(\alpha, \beta) \neq (1, 0)$. From Corollary 2, we have

$$N \leq (q + 1) + p^{(\ell + 1)/2}(p^\ell - 1)\sqrt{q},$$

where $N$ is defined by Corollary 2. Meanwhile, by Lemmas 3 and 5, we have

$$N \geq 2q - k = q + \frac{p^{m+1} - 1}{2(p + 1)} = (q + 1) + \frac{p^{m+1} - 2p^3 - 3}{2(p + 1)}.$$

Since $\rho \geq 3$ and $m \geq 3$ is odd, we have

$$(p^{m+1} - 2p - 3) - 2(p + 1)p^{m-1} = p(p^{m-2}(p^2 - 2p - 2) - 2) - 3 \geq 0.$$

This means that

$$\frac{p^{m+1} - 2p - 3}{2(p + 1)} \geq p^{m-1}.$$

Therefore, from Eqs. (14) and (15), we have

$$N \geq (q + 1) + p^{m-1}.$$  

Further, by $2\ell + 1 < m/2$ and Eq. (13) we have

$$N \leq (q + 1) + p^{(\ell + 1)/2}p^{m/2} < (q + 1) + p^{2\ell + m/2} < (q + 1) + p^{m-1},$$

which is a contradiction to Eq. (16). This means that there does not exist $(\alpha, \beta) \in \text{Stab}(B_\ell)$ with $(\alpha, \beta) \neq (1, 0)$. Hence, $\mu = |\text{Stab}(B_\ell)| = |\{(1, 0)\}| = 1$.

This completes the proof.

\[ \square \]

**Lemma 7** Let $p \geq 3$ and $p \equiv 3 \text{ mod } 4$. Let $m$ be odd and $\ell$ be a positive integer with $\gcd(\ell, m) = 1$. Define the group

$$\text{GA}_1(\text{GF}(q)) = \{ux + v : (u, v) \in \text{GF}(q)^* \times \text{GF}(q), u \in \text{QR}\}.$$

Then the group $\text{GA}_1(\text{GF}(q))$ is 2-homogeneous on $\text{GF}(q)$.

**Proof** Let $\{x_1, x_2\} \subseteq \text{GF}(q)$ and $\{y_1, y_2\} \subseteq \text{GF}(q)$ be any two 2-subsets of $\text{GF}(q)$. Let

\[
\begin{cases}
    u_1x_1 + v_1 = y_1 \\
    u_1x_2 + v_1 = y_2
\end{cases}
\quad \text{or} \quad
\begin{cases}
    u_1x_1 + v_1 = y_2 \\
    u_1x_2 + v_1 = y_1
\end{cases}.
\]

Then we have

\[
\begin{cases}
    u_1 = (x_1 - x_2)^{-1}(y_1 - y_2) \\
    v_1 = y_1 - (x_1 - x_2)^{-1}(y_1 - y_2)x_1
\end{cases}
\quad \text{or} \quad
\begin{cases}
    u_1 = (x_1 - x_2)^{-1}(y_1 - y_2)(-1) \\
    v_1 = y_2 - (x_1 - x_2)^{-1}(y_1 - y_2)(-1)x_1.
\end{cases}
\]

By assumption, we have $-1 \in \text{NQR}$. We then deduce that one is a quadratic residue and the other is a quadratic non-residue in the two values $(x_1 - x_2)^{-1}(y_1 - y_2)$ and $(x_1 - x_2)^{-1}(y_1 - y_2)(-1)$ of Eq. (17). This means that there exists $\sigma(x) = (ux + v) \in \text{GA}_1(\text{GF}(q))$ such that
σ(x) sends \{x_1, x_2\} to \{y_1, y_2\}, where \( u \in \text{QR} \) is a quadratic residue and \( v \in \text{GF}(q) \). The desired conclusion then follows from the definition of 2-homogeneity.

\[ \square \]

**Lemma 8** With the symbols and notation above, let

\[
A_1 = \{ B_{f,b,c} : (b, c) \in \text{GF}(q)^* \times \text{GF}(q) \}, \\
A_2 = \{ uB_\ell + v : (u, v) \in \text{GF}(q)^* \times \text{GF}(q) \}
\]

and

\[
A_3 = \{ uB_\ell + v : (u, v) \in \text{GF}(q)^* \times \text{GF}(q) \text{ and } u \in \text{QR} \},
\]

where \( f(x) = x^{p^\ell+1} \) and \( B_\ell \) was defined by eqnspsJe11). Then we have the following results.

(I) If \( p = 2 \) and \( \gcd(\ell, m) = 1 \), then \( A_1 = A_2 \).

(II) If \( p \geq 3 \), \( p \equiv 3 \mod 4 \), \( m \) is odd with \( \gcd(\ell, m) = 1 \), then \( A_1 = A_3 \).

**Proof** For each \( (b, c) \in \text{GF}(q)^* \times \text{GF}(q) \), we have

\[
f(x) + bx + c = x^{p^\ell+1} + bx + c = x^{p^\ell+1} + b^{p^\ell}x + c \quad (b \text{ is replaced with } b^{p^\ell})
\]

\[
= (bx)^{p^\ell+1} + b^{p^\ell}(bx) + c \quad (x \text{ is replaced with } bx)
\]

\[
= b^{p^\ell+1}(x^{p^\ell+1} + x) + c. \quad (18)
\]

(I) For any \( B_{f,b,c} = \{ f(x) + bx + c : x \in \text{GF}(q) \} \in A_1 \), from Eq. (18) we have

\[
B_{f,b,c} = \{ f(x) + bx + c : x \in \text{GF}(q) \} = \{ b^{p^\ell+1}B_\ell + c : x \in \text{GF}(q) \} \in A_2,
\]

which means that \( A_1 \subseteq A_2 \). Next we prove \( A_2 \subseteq A_1 \).

For each \( (u, v) \in \text{GF}(q)^* \times \text{GF}(q) \), we have

\[
u(x^{p^\ell+1} + x) + v = (u^{(p^\ell+1)^{-1}}x)^{p^\ell+1} + u^{1-(p^\ell+1)^{-1}}(u^{(p^\ell+1)^{-1}}x) + v = (ux)^{p^\ell+1} + ux + v \quad (\text{since } p = 2, (p^\ell + 1)^{-1} = 1)
\]

\[
= x^{p^\ell+1} + 1 \cdot x + v \quad (ux \text{ is replaced with } x) \quad (19)
\]

Hence, for any \( uB_\ell + v \in A_2 \), by Eq. (19) we have \( uB_\ell + v = B_{f,1,v} \in A_1 \). This means that \( A_2 \subseteq A_1 \). The desired conclusion then follows.

(II) By definition, \( \gcd(p^\ell + 1, q - 1) = 2 \). Thus, \( b^{p^\ell+1} = (b^{(p^\ell+1)/2})^2 \in \text{QR} \) in Eq. (18), which means that \( A_1 \subseteq A_3 \). The proof of \( A_3 \subseteq A_1 \) is similar to the proof of \( A_2 \subseteq A_1 \) of case (I) and we omit it here.

\[ \square \]

### 4.2 The proofs of Theorems 3 and 4

It is now time to show the results as stated in Theorems 3 and 4.

**Proof of Theorem 3** Recall that \( p = 2 \) and \( f(x) = x^{p^\ell+1} = x^{2^\ell+1} \). By definition, from Lemma 5, it follows that

\[
k = |B_\ell| = |\{x^{p^\ell+1} + x : x \in \text{GF}(q)\}| = \frac{2q + (-1)^m}{3}. \quad (20)
\]

Define the group

\[
\text{GA}(\text{GF}(q)) = \{ ux + v : (u, v) \in \text{GF}(q)^* \times \text{GF}(q) \}.
\]
It is clear that $GA(GF(q))$ is the general affine group and its size is $|GA(GF(q))| = q(q - 1)$. The stabilizer of $B_\ell$ under $AG(GF(q))$ is defined by

$$Stab(B_\ell) = \{ux + v : (u, v) \in GF(q)^* \times GF(q), uB_\ell + v = B_\ell\},$$

where $B_\ell$ is defined by (5). We then deduce that

$$|B_{(f,k)}| = \frac{|GA(GF(q))|}{|Stab(B_\ell)|} = \frac{(q - 1)q}{|Stab(B_\ell)|} = q(q - 1)$$

by Lemma 6. This means that all blocks $B_{(f,b,c)}$ with $(b, c) \in GF(q)^* \times GF(q)$ are pairwise distinct. Note that $GA(GF(q))$ is 2-homogeneous on $GF(q)$. By definitions and the result (I) of Lemma 8, the incidence structure $(GF(q), B_{(f(x), k)})$ can be seen as $(GF(q), B)$, which is constructed by the base block $B_\ell$ under the action of $GA(GF(q))$, where

$$B = \{gB_\ell : g \in GA(GF(q))\}.$$

Further, from Theorem 1, it then follows that the incidence structure $\mathbb{D}(f(x), k) := (GF(q), B_{(f(x), k)})$ is a 2-$(q, k, \lambda)$ design, where $k$ was defined by Eq. (20) and

$$\lambda = |B_{(f,k)}| \left(\begin{array}{c} k \\ 2 \end{array}\right)/\left(\begin{array}{c} q \\ 2 \end{array}\right) = k(k - 1),$$

The proof is then completed. \qed

**Proof of Theorem 4** The proof is similar to that of Theorem 4. By definition, from Lemma 5 we have

$$k = |B_\ell| = \left|\left\{x^{p^2+1} + x : x \in GF(q)\right\}\right| = q - \frac{pq - 1}{2(p + 1)}.$$  \hspace{1cm} (21)

Define the group

$$GA_1(GF(q)) = \{ux + v : (u, v) \in GF(q)^* \times GF(q), u \in QR\}.$$

It is clear that the size of the group $GA_1(GF(q))$ is $|GA_1(GF(q))| = q(q - 1)/2$. The stabilizer of $B_\ell$ under $AG_1(GF(q))$ is defined by

$$Stab(B_\ell) = \{ux + v : (u, v) \in GF(q)^* \times GF(q), u \in QR, uB_\ell + v = B_\ell\},$$

where $B_\ell$ was defined by (5). We then deduce that

$$|B_{(f,k)}| = \frac{|GA_1(GF(q))|}{|Stab(B_\ell)|} = q(q - 1)/2$$

by Lemma 6. By definitions and the result (II) of Lemma 8, $(GF(q), B_{(f(x), k)})$ can be seen as $(GF(q), B)$ constructed by the base block $B_\ell$ under the action of $GA_1(GF(q))$, where

$$B = \{gB_\ell : g \in GA_1(GF(q))\}.$$

From Theorem 1 and Lemma 7, it then follows that the incidence structure $(GF(q), B_{(f(x), k)})$ is a 2-$(q, k, \lambda)$ design, where $k$ was defined by Eq. (21) and

$$\lambda = |B_{(f,k)}| \left(\begin{array}{c} k \\ 2 \end{array}\right)/\left(\begin{array}{c} q \\ 2 \end{array}\right) = k(k - 1)/2,$$

The desired conclusion then follows. \qed
5 Summary and concluding remarks

In this paper, based on the general constructions of t-designs from polynomials over GF\((q)\) in [8], quadratic functions were used to construct t-designs. It was shown that infinite families of 2-designs were produced and their parameters were also explicitly determined. Furthermore, the results in this paper gave an affirmative answer to Conjecture 3 in Ding and Tang [8] and generalized the result. We remark that this paper does not consider the case that \(q\) is an odd prime power with \(q \equiv 1 \pmod{4}\), since Magma program shows that the corresponding incidence structures are not 2-designs. To conclude this paper, we further presents the following two conjectures, which are the complements of the main results of this paper.

**Conjecture 1** Let \(p = 2\), \(\ell\), \(m\) be two positive integers with \(m \geq 3\), \(m > \ell \geq m - 1\) and \(\gcd(\ell, m) = 1\). Let \(f(x) = x^{2^\ell+1}\). Then the incidence structure \(D(f(x), k) := (\text{GF}(q), B_{f(x), k})\) is a \(2-(q, k, k(k - 1))\) design, where \(k = \frac{2q(-1)^m}{3}\).

**Conjecture 2** Let \(p\) be an odd prime with \(p \equiv 3 \pmod{4}\) and \(m \geq 3\) be odd. Let \(\ell\) be a positive integer with \(m^2 - 4 \leq \ell < m - 1\) and \(\gcd(\ell, m) = 1\). Let \(f(x) = x^{p^\ell+1}\). Then the incidence structure \(D(f(x), k) := (\text{GF}(q), B_{f(x), k})\) is a \(2-(q, k, \frac{(k-1)}{2})\) design, where \(k = q - \frac{pq-1}{2(p+1)}\).

**Acknowledgements** The authors are very grateful to the reviewers and the Editor, for their comments and suggestions that improved the presentation and quality of this paper. The research of C. Xiang was supported by the National Natural Science Foundation of China under Grant No. 61672015.

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