Macroscopic oscillations locked and synchronized on fixed energy levels by two cooperating drives

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It was unknown whether the energy of a macroscopic object can be confined to a set of discrete values like the energy levels of microscopic systems. Here, through the numerical simulation and theoretical analysis based on an experimentally implementable model, we demonstrate that this phenomenon can occur to a mechanical oscillator, which is simply under the radiation pressure of a cavity field created by two driving lasers. Once the amplitudes and frequencies of the two drives are properly matched, the oscillator will stabilize on one of the fixed trajectories in its phase space. Then both amplitude and phase of its oscillation become frozen on the specific trajectory like an energy level. Above a certain drive amplitude, tiny variation of the oscillator’s initial condition or in the drive amplitudes before reaching stability can affect its proceeding motion but, unlike the aperiodicity in chaotic motion, the oscillator will deterministically end up on one of such energy levels. This finding exemplifies a category of nonlinear dynamical processes, which is dissimilar to any other discovered in the past.

I. INTRODUCTION

Driving nonlinear systems can give rise to interesting phenomena. One category of these phenomena is dynamical synchronization [1,2], which has been studied since the time of C. Huygens [3]. The frequencies and phases of multiple oscillators can be synchronized under weak mutual interaction, to exhibit the behaviors such as the coordinated flashes of fireflies [4] and the injection locking of a laser array to increase output power [5]. Synchronization is accompanied by mode locking. When it is synchronized by a periodic force of constant amplitude, a nonlinear oscillator will be locked to a number of frequencies known as the devil’s staircase. A display of the phenomenon in real physical system is the voltage-current relation called Shapiro steps for a Josephson junction in AC field [7,8]. Accordingly, one may ask the question—whether the amplitude of an oscillation can also be locked to a number of fixed values at the same time? For example, by locking the amplitude A of a mechanical oscillation $X_m(t) = A \sin(\omega_m t)$ with the frequency $\omega_m$, the energy $E_m = \frac{1}{2}(X_m^2 + P_m^2)$ of the mechanical oscillator determined by its displacement $X_m(t)$ and momentum $P_m(t)$ will locate on a number of levels corresponding to the locked discrete values $A_n$ ($n \geq 1$), as if its quantization were realized only by the means of classical physics. For a macroscopic object it is against intuition to conceive the possible existence of its discrete energy levels.

We show that the energy levels like the above mentioned can be created for a macroscopic object through a process of synchronization by two different drives. It is through the system shown in Fig. 1(a), where two coherent fields with their specific frequencies drive a cavity field pressurizing on a mechanical oscillator. The previous researches on similar doubly driven optomechanical systems always concern one strong and one weak field [9], such as in the optomechanically induced transparency [10,11] and the optomechanical chaos [13,14], in addition to the study of mechanical squeezing induced by two drives of different amplitudes [15]. Instead, the phenomena illustrated below emerge under two drives with very close amplitudes $E_1$ and $E_2$. Nonlinear dynamics due to two or more different external drives has not been well explored thus far, except for the stochastic resonance phenomenon involving one noise drive [16,17]. Among the unexplored phenomena of nonlinear dynamics induced by two external drives, we focus on those due to one red detuned drive ($\omega_1 = \omega_c - \omega_m$) and one resonant drive ($\omega_2 = \omega_c$). If acting alone, the former achieves the cooling effect of reducing the mechanical fluctuation in thermal environment [9]. The two drives work together to bring about a previously unknown synchronization to two coupled oscillators that model the system. Such synchronization simultaneously locks the oscillation frequencies and phases for the two oscillators, as well as the amplitude values for one of them to realize the energy levels.

The main results of the current work are divided into four parts. Together with the illustration of a dynamical transition from the linear to the nonlinear regime, the part in Sec. II shows how the first energy level is created under two external drives with their amplitudes and frequency properly matched. The general properties of the mechanical energy levels are described in Sec. III, where the relation between the energy levels and the drive amplitudes, as well as the stabilized cavity oscillation patterns corresponding to the mechanical oscillations on the levels, is explicitly demonstrated. Further understanding of the formation of the mechanical energy levels is provided in Sec. IV, where the functions of the two different drives are clarified. The importance of the evo-
under two external drives. \(a\) The setup of two drives under the condition \(\Delta_1\) of the drive amplitudes, when their frequencies do not match. This system exemplifies a general model of two oscillators with the intrinsical frequencies \(\omega_m\) and \(\omega_m'\) connected by a spring. This system on a cavity with a fixed mirror and a movable mirror (the mechanical oscillator) connected by a spring. This system, as well as the explanation of the notations, the understanding of the supplementary information about the energy levels. In addition to the main text, there are four appendices that provide the energy levels are robust against drive fluctuations, so that it is possible to observe these energy levels. In addition to the main text, there are four appendices that provide the supplementary information about the energy levels.

II. ENERGY LEVEL EMERGING UNDER COOPERATING DRIVES

In terms of two perpendicular quadratures \(X_c\) and \(P_c\) of the cavity field, together with the displacement \(X_m\) and momentum \(P_m\) of the mechanical oscillator, the dynamical equations of the system in Fig. 1(a) read

\[
\begin{align*}
\dot{X}_c &= -\kappa X_c - gX_m P_c + \sqrt{2}\{E_1 \cos(\Delta_1 t) + E_2 \cos(\Delta_2 t)\}, \\
\dot{P}_c &= -\kappa P_c + gX_m X_c + \sqrt{2}\{E_1 \sin(\Delta_1 t) + E_2 \sin(\Delta_2 t)\}, \\
\dot{X}_m &= \omega_m P_m, \\
\dot{P}_m &= -\omega_m X_m - \gamma_m P_m + \frac{\sqrt{2}}{4}g(X_c^2 + P_c^2)
\end{align*}
\]

in the observation system rotating at the cavity frequency \(\omega_c\), where \(\Delta_{1(2)} = \omega_c - \omega_{1(2)}\). A realistic system has a very small coupling constant \(g\) for the quadratic terms in the equations, which, by appearance, simply correct the linear solution at \(g = 0\). However, to a driven system like this, the nonlinear terms can govern the system dynamics. One such example is given in Figs. 1(b1)-1(b2)—with a tiny deviation from \(\Delta_1 = \omega_m\) and \(\Delta_2 = 0\), the displacement \(X_m(t)\) responds linearly to the drive amplitudes, but the drive frequency match locks the amplitude, frequency and phase of \(X_m(t)\) totally, so that the stabilized oscillations become almost the same.

Fig. 2(a) illustrates how the mode locking phenomenon in Fig. 1(b2) comes into being, using the time evolutions of the mechanical energy \(E_m(t) = \frac{1}{2}(X_m^2(t) + P_m^2(t))\), which is a half of the squared radius of the oscillator’s position in its phase space. The stabilized mechanical energy saturates with the increased drive amplitude \(E\) (this notation stands for \(E_1 = E_2\)). The evolutions courses represented by the black and red curves, which are completely within the nonlinear regime and due to the mechanical energy evolution in Fig. 2(a). The stabilized mechanical energies displayed in Fig. 2(b1) approach the same value of \(\langle E_m \rangle\) (the time average of the mechanical energy) to form something like an energy level. The cooperation of two drives with their proper amplitudes and frequencies is essential for realizing the energy level. First of all, the magnitudes of \(E_1\) and \(E_2\) should be close. For the range of drive amplitude \(E_2\) in Fig. 2(b1), the sole action of the resonant field gives rise to the stabilized \(E_m\) growing continuously with \(E_2\), but the tendency of forming the energy level emerges after adding the cooling field. The stabilized mechanical energies displayed in Fig. 2(b3) approach the same value of \(\langle E_m \rangle\) when \(E_1\) is close to \(E_2\). Secondly, the frequency match is stricter than the closeness of two drive amplitudes. For example, in Fig. 2(c2), the stabilized energy values represented by the black and red curves have a perceivable gap even with their drive frequencies extremely close to the perfect match condition \(\Delta_1 - \Delta_2 = \omega_m\). A phenomenon similar to critical slowing down occurs when the drive amplitude \(E\) is increased from the linear response regime to where the mechanical energy \(\langle E_m \rangle\) is frozen to form the energy level. Fig. 3 shows the processes of evolution of the cavity energy \(E_c(t) = \frac{1}{2}(X_c^2(t) + P_c^2(t))\), with some of them corresponding to those of the mechanical energy evolution in Fig. 2(a). Because the mechanical energy is frozen on the energy level, more energy added into the system by a higher frequency fluctuations and change of initial condition, but such sensitivity to initial condition is not due to chaos. We also specify that the stabilized oscillations on the energy levels are robust against drive fluctuations, so that it is possible to observe these energy levels. In addition to the main text, there are four appendices that provide the supplementary information about the energy levels. In addition to the main text, there are four appendices that provide the supplementary information about the energy levels.
FIG. 2: Formation of the first energy level. The system parameters are the same as those for Figs. 1(b1)-(b2). The real evolution times are connected with the dimensionless time scales $\kappa t$ used here, given the quality factor of a specific cavity. (a) The real time evolutions of the mechanical energy showing that the stabilized values become saturated with the increased $E$. The black, indigo and red curves, which stick together around the energy level, are due to the drive amplitudes indicated in (b1). (b1)-(b3) The process of forming the energy level by gradually increasing the amplitude of the cooling field added to the sole action of the resonant drive. The energy level (for the black and red curves) emerges when the two drive amplitudes get close. (c1)-(c2) The necessity of the frequency match for the used fields in realizing the energy level. Here the two drive amplitudes are equal.

The drive amplitude $E$ has to be in the cavity. For example, the black and red curves, the corresponding mechanical energies of which are completely fixed on the first energy level, have a significant difference for their stabilized cavity energies. The blue and pink curves in the linear response regime also quickly stabilize, and their stabilized values of $\langle E \rangle$ are proportional to the drive amplitude $E$.

The interesting evolution processes exist in the transitional regime, where the green and indigo curves characterized by the temporal dips take much longer time to stabilize. The temporal dip emerges more quickly with increased drive amplitude $E$ until it gradually disappears inside the nonlinear regime. The stabilized oscillations shown in the inset of Fig. 3 allows one to judge if the system has completely reached the energy level—those on the energy level have a pattern close to single mode oscillation (see Fig. 4(c1) below), but those in the transitional regime contain the obvious higher harmonic components.

FIG. 3: Examples of the cavity energy evolution. Here we use six drive amplitudes—$E = 2 \times 10^5 \kappa$ (pink), $E = 2 \times 10^4 \kappa$ (blue), $E = 2 \times 10^3 \kappa$ (green), $E = 4 \times 10^3 \kappa$ (indigo), $E = 2 \times 10^6 \kappa$ (red), and $E_1 = 2 \times 10^7 \kappa$ (black). The inset shows some of their stabilized oscillations.

III. PROPERTIES OF MECHANICAL ENERGY LEVELS

A general model in Fig. 1(a)—two nonlinearly coupled oscillators with one of them doubly driven—encompasses all similar systems. To the abstract model the dimensionless amplitude $E/\kappa$ can be arbitrarily high and, for the real systems, sufficiently high $E/\kappa$ is realizable with a cavity of high quality factor. The stabilized energy $\langle E_m \rangle$ is locked on a series of levels as shown in Fig. 4(a). Around the amplitude $E \approx 5 \times 10^5 \kappa$, our illustrated system undergoes a dynamical transition from the linear response regime to the first energy level. Over the amplitude $E \approx 2.5 \times 10^7 \kappa$, another type of dynamical transition takes place to have the second energy level emerging and seemingly overlapped with the first one. This phenomenon will be discussed later. The “quantized” mechanical energy of the macroscopic oscillator exhibits a power law $\langle E_m \rangle(n) \sim n^{2.2}$.

The stabilized mechanical oscillations on the energy levels have the invariant patterns in Figs. 4(b1)-4(b4). Corresponding to each mechanical energy level, the cavity oscillations due to different drive amplitudes $E$ also have a fixed frequency spectrum as one of the invariant patterns in Figs. 4(c1)-4(c4), except that their oscillation amplitudes change proportionally to $E$. This feature can be seen from the stabilized $\mathcal{E}_c$ under the condition $E_1 = E_2$, which are illustrated in Fig. 5(d3) or Fig. 6(d3) below. The energy level the oscillator locates can thus be read from the corresponding cavity oscillation pattern. Such synchronization between the cavity and mechanical spectra (the one-to-one correspondence between the pattern in Fig. 4(bi) and the one in Fig. 4(ci), for $1 \leq i \leq 4$), or more generally between the two oscillators of the abstract model in Fig. 1(a), is realized under a pair of specific cooling and resonant fields. Meanwhile, there exists another type of synchronization to be discussed in the next section—the mechanical oscillation...
FIG. 4: Mechanical energy levels and associated oscillation patterns. (a) The beginning energy levels demonstrated with their relations to the dimensionless drive amplitude up to \( E/\kappa = 6.3 \times 10^7 \), in terms of the logarithmic scales. All values of \( \langle E_m \rangle \) displayed are the dynamically stabilized ones. The level \( n = 1 \) is identical to the one in Fig. 2(a) and vanishes with further increased \( E \). These energy levels go up like a quasi parabola as shown in the inset. A higher (lower) nonlinear coupling \( g \) gives the decreased (increased) energies on the levels. (b1)-(b4) and (c1)-(c4) The one-to-one correspondence between the stabilized mechanical oscillations and cavity oscillations. The level, on which the mechanical oscillator is, can be known from the peak number in a half period of the cavity energy oscillation (we use a dash line to make the peak of \( n = 1 \) distinct). The phases of the mechanical spectrum, from the base frequency \( \omega_m \) to the high harmonic components, are completely synchronized with those of the cavity oscillation.

Table I: The oscillation amplitudes \( A_n \) and net displacements of the oscillator, as well as the average mechanical energies, on the beginning five energy levels. These values are obtained with the drive amplitudes used in Fig. D1 of Appendix D. On each level the net displacement \( d_n \) has a small variation, to give rise to the width of the energy level.

| energy level | \( A_n \)  | \( d_n \)  | \( \langle E_m \rangle \) |
|--------------|------------|------------|-----------------|
| \( n = 1 \)  | 5019460.83| 100725.69  | 1.2869 \times 10^{14} |
| \( n = 2 \)  | 16536205.16| 75687.75  | 1.3694 \times 10^{14} |
| \( n = 3 \)  | 27703356.66| 35960.73  | 3.8385 \times 10^{14} |
| \( n = 4 \)  | 38830599.30| 54012.64  | 7.5408 \times 10^{14} |
| \( n = 5 \)  | 49948350.03| 46177.52  | 1.2476 \times 10^{15} |

The numerical simulations based on Eq. (1) show that the mechanical oscillations on the energy levels take the forms

\[
X_m(t) = A_1 \sin(\omega_m t) + d_1, \quad \text{for} \quad n = 1,
\]

\[
X_m(t) = A_n \sin(\omega_m t) + \sum_{k=2}^{n} a_k \sin(k\omega_m t) + d_n, \quad \text{for} \quad n \geq 2
\]

by choosing the proper initial phase for the oscillations. The patterns in Figs. 4(b1)-4(b4) have one more harmonic component after going up one level, as the number of the tiny twists over the curves of \( X_m(t) \) and \( P_m(t) \) (equivalent to the curve of \( E_m(t) \)) increases in this way. The amplitude \( A_n \) of the base frequency component and net displacement \( d_n \) can be read directly from the stable oscillations; see the examples in Tab. I. The amplitudes \( a_k \) (\( 2 \leq k \leq n \)) of the higher harmonic components are
FIG. 5: **Processes of gradually enhancing the resonant field added to the action of a cooling field.** (a1)-(a3) The evolutions of the mechanical energy, the detailed steady states, as well as the associated evolutions of the cavity energy, under the sole action of the cooling field. Here six evenly distributed drive amplitudes—$E_1 = 3.0 \times 10^7 \kappa$ (pink), $E_1 = 3.3 \times 10^7 \kappa$ (green), $E_1 = 3.9 \times 10^7 \kappa$ (red), $E_1 = 4.2 \times 10^7 \kappa$ (indigo), and $E_1 = 4.5 \times 10^7 \kappa$ (blue)—are used for the examples. (b1)-(b3) The corresponding results by adding the resonant field with the ratio $E_2 = 0.5 E_1$. (c1)-(c3) The corresponding results by adding the resonant field with the ratio $E_2 = 0.6 E_1$. (d1)-(d3) The corresponding results by adding the resonant field of $E_2 = E_1$. (d2) shows the stabilized oscillations on the level $n = 1$. Much less than the base frequency component amplitude $A_n$, due to the resonance of the oscillator under the cavity sideband with the frequency $\omega_m$. Their contributions to the mechanical energy can be therefore neglected to have

$$\mathcal{E}_m(t) = \frac{1}{2} (X_m(t))^2 + \frac{1}{2} (P_m(t))^2$$

$$= \frac{1}{2} A_n^2 + \frac{1}{2} d_n^2 + A_n d_n \sin(\omega_m t). \quad (3)$$

The validity of the approximation is demonstrated more clearly by Fig. D1 in Appendix D. The time average $\langle \mathcal{E}_m \rangle(n) = \frac{1}{2} (A_n^2 + d_n^2) \approx \frac{1}{2} A_n^2$ ($d_n \ll A_n$) is the position of an energy level, and $A_n d_n$ is the energy oscillation amplitude on the level as in Figs. 4(b1)-4(b4). The energy $\mathcal{E}_m = \frac{1}{2} (X_m^2 + P_m^2)$ is thus appropriate to illustrate the locking of the mechanical oscillation amplitude to the fixed values $A_n$, though the total energy involving the oscillator also includes the part due to the nonlinear coupling with the cavity field.

**IV. FORMATION OF FIXED ENERGY LEVELS**

Our concerned processes under two cooperating drives can be better understood by starting with only one of the drives and gradually adding up the amplitude of the other. We numerically simulate the processes using Eq. (1), and provide the qualitative explanations partially with the effective Hamiltonians of Eqs. (C3) and (C4) in Appendix C. These processes provide useful information about how the energy levels are created.

**A. Starting from the cooling field**

In Fig. 5, we start from acting the cooling field alone. In the absence of the resonant field as in Fig. 5(a1), the stabilized mechanical energy increases with the displayed drive amplitude $E_1$ in a quasi linear way, having a continuum spectrum. Now the dominant effect is the exchange of the cavity and mechanical modes as shown in Eq. (C3). In this situation the mechanical oscillator stabilizes quickly under the effective optical damping proportional to the cavity field intensity $\kappa$. The added resonant field brings about an intensified mechanical drive and a gain (squeezing) as seen from Eq. (C4), so the processes from Figs. 5(a1)-5(a2) to Figs. 5(b1)-5(b2) displays a significant increase of the mechanical energy. Meanwhile, under the nonlinear saturation and other damping effects, the evolved mechanical energies $\mathcal{E}_m$ for the different drive amplitudes stabilize in the same range to form something
FIG. 6: Processes of gradually enhancing the cooling field added to the action of a resonant field. (a1)-(a3) The evolutions of the mechanical energy, the detailed view of the oscillations on the first energy level, as well as the associated evolutions of the cavity energy, under the sole action of the resonant field. The drive amplitudes used here are the same as those in Fig. 5—$E_2 = 3.0 \times 10^7 \kappa$ (pink), $E_2 = 3.3 \times 10^7 \kappa$ (green), $E_2 = 3.6 \times 10^7 \kappa$ (red), $E_2 = 3.9 \times 10^7 \kappa$ (black), $E_2 = 4.2 \times 10^7 \kappa$ (indigo), and $E_2 = 4.5 \times 10^7 \kappa$ (blue). (b1)-(b3) The corresponding results by adding the cooling field with the ratio $E_1 = 0.1 E_2$. (c1)-(c3) The corresponding results by adding the cooling field with the ratio $E_1 = 0.9 E_2$. (d1)-(d3) The corresponding results by adding the cooling field of $E_1 = E_2$. (d2) shows the oscillations on the level $n = 3$. (d1)-(d3) The corresponding results by adding the cooling field of $E_1 = E_2$. (d2) shows the stabilized oscillations on the level $n = 3$.

like an energy band in Figs. 5(b1) and 5(b2). The width of the energy band is the difference of the time averages $\langle E_m \rangle$ of the stabilized $E_m$. With a further strengthened resonant field, the one due to the strongest drive (the blue one) jumps up to somewhere like a energy level; see Fig. 5(c1). Accompanying the jump is the change of an associated cavity energy oscillation pattern as in Fig. 5(c3). More energy levels will split out as the resonant field amplitude gets closer to the cooling field amplitude, as shown in Fig. 5(d1). The widths of the energy levels (the difference in $\langle E_m \rangle$) will also be minimized when the two drive amplitudes are equal. Since they are dominated by the base frequency component of $\omega_m$, the stabilized mechanical oscillations in Fig. 5 are in the form

$$X_m(t) = A(E_1, E_2 = \chi E_1) \sin(\omega_m t) + d(E_1, E_2 = \chi E_1)$$

by choosing the proper initial phase, where $0 < \chi \leq 1$. On the energy band in Figs. 5(c1) and 5(c2), for example, the amplitudes $A(E_1, E_2 = 0.6 E_1)$ are almost the same, but the net displacements $d(E_1, E_2 = 0.6 E_1)$ due to different drive amplitudes have difference so that there is a considerable width of the band. The oscillation amplitude $A(E_1, E_2 = 0.6 E_1)$ for the one on the separated energy level is, however, much larger. Such process illustrates a mechanism due to the resonant field—its continuously enhancing amplitude $E_2$ leads to a uniquely nonlinear response of $A(E_1, E_2)$, which increases by discrete steps when $E_2$ is sufficiently high.

B. Starting from the resonant field

On the other hand, one can start from the sole action of the resonant field as in Fig. 6. As seen from the effective Hamiltonian in Eq. (C4), the action of a resonant field carries both intensified cooling and gain (squeezing) effect, as well as the intensified effective mechanical drive that displaces the oscillator. Since it is without the optical damping [9], the stabilization under resonant field is mainly through the more complicated nonlinear saturation. Once the amplitude $E_2$ of the resonant field is up to a sufficiently high value, the mechanism due to the combination of the above mentioned effects leads to the energy levels in Fig. 6(a1). On the energy levels due to a single resonant drive, however, the mechanical oscillations are not synchronized as shown in Fig. 6(a2), where the oscillations

$$X_m(t) = A(E_2) \sin(\omega_m t + \phi(E_2)) + \text{the higher harmonic components} + d(E_2)$$

caused by different $E_2$ have different phases $\phi(E_2)$. An insignificant addition of the cooling field can synchronize
FIG. 7: Functions of the two different drive fields. (a) From the continuum spectrum due to the sole action of the cooling field, the energy bands and energy levels appear after gradually strengthening the resonant field. (b) The energy levels generated by the resonant field alone are lowered by the increased cooling field amplitude. Here the oscillations on an energy level are synchronized after adding the cooling field. In the illustrated example with the level \( n = 1 \), the high harmonic components are also filtered out in the process.

these oscillations, as the phenomenon manifests in Fig. 6(b2). In Fig. 6, the positions of the energy levels are modified by the added cooling field; for an example, compare the average position of the level \( n = 3 \) in Fig. 6(c2) with that of the corresponding level in Fig. 6(d2).

C. Specific roles of the different drives

Despite the complexity in forming the energy levels, their general tendencies changing with the two different fields are clear. As summarized in Fig. 7, the positions of the energy levels go up with strengthened resonant field, but go down under intensified cooling field. These energy levels will stabilize on the fixed positions when the amplitudes of the two fields are equal to each other. A strong resonant field, with its amplitude considerably higher than the scale of saturating the continuum spectrum on the first level as in Fig. 2, gives rise to significant mechanical displacement and gain effect. Together with the nonlinear saturation, these effects make the stabilized \( \langle \mathcal{E}_m \rangle \) jump by discrete steps while the resonant drive is continuously enhanced, to split the continuum spectrum due to a sole cooling field into energy bands and energy levels. The action of a single resonant field with sufficiently high amplitude thus brings about nonlinear dynamical behavior, in contrast to the regime of weaker drives where the nonlinearity should appear under well matched frequencies of two drives as in Fig. 1(b2). On the other hand, the effects of the cooling field are to lower the discrete energy levels caused by the resonant field and synchronize all oscillations on a certain level to the same phase. The top of the continuum part of the spectrum due to a single resonant field is lowered by the cooling field so that the first level can be realized with \( E \sim 10^5 \kappa \). In terms of the effective cooling intensity \( J = gE/\omega_m \) [18], the level \( n = 1 \) in our example exists at \( J \approx 0.1 \), which is experimentally achievable by the current optomechanical systems.

FIG. 8: Distribution of the stabilized average mechanical energy in the regime of high drive amplitudes. Here the distribution of the stabilized average mechanical energy along the horizontal axis is irregular, but the values on the vertical axis are completely fixed. (a) One section along the horizontal axis of Fig. 4(a), viewed with the scale in the order of \( 10^3 \). (b) The view of another range starting from \( E/\kappa = 10^8 \) with the scale of \( 10^{-4} \). In (b) a level transition takes place with \( \delta E = 10^{-12} E \), and the logarithmic scale on the vertical axis appears uneven.
V. EVOLUTION TOWARD ENERGY LEVELS

Next we look at the transient periods of the oscillator before the system reaches the stability. Above the drive amplitude leading to the higher levels \( n \geq 2 \), the evolution process to the energy levels can become complicated. In what follows, we reveal the fact by showing how the external drive amplitude fluctuations and the system’s initial condition affect the evolution courses.

A. Sensitivity to the drive amplitudes

One phenomenon due to a strong resonant field is that, when its amplitude is sufficiently high, the evolution of the system becomes sensitive to the change of the drive amplitudes. This regime is characterized by the simultaneously intensified cooling effect (the tendency to lower the mechanical energy) and squeezing effect (the tendency to increase the mechanical energy) as indicated in Eq. (C4). They join with the nonlinearity to make the机械能量) and squeezing effect (the tendency to lower the mechanical energy) as indicated in Eq. (C4). They join with the nonlinearity to make the

\[ \lim_{t \to \infty} (E_m(E + \delta E, t) = (E_m(E + \delta E, n + 1) \neq (E_m(E, n) + \frac{d}{dE} (E_m(E, n))\delta E + \cdots, (6) \]

in which the Taylor expansion with respect to \( E \) for the function \( (E_m)(E) \) becomes invalid. In Figs. 8(a)-8(b) the energy \( (E_m) \) distributes irregularly along the horizontal axis, but its values on the vertical axis are nonetheless fixed to those of the energy levels.

B. Effects of the drive amplitude fluctuations

The sensitivity of evolution processes to drive amplitudes provides an unusual example that the straightforward perturbation treatment for the dynamics related to drives breaks down. It is therefore imaginable that a tiny fluctuation in either of two high amplitudes \( E_1 \) and \( E_2 \) would make the system evolve to a different energy level. However, it actually depends on when the fluctuation takes effect. Fluctuations will influence the evolution of the system only if they exist before the system has stabilized. Upon evolving to the stability, the oscillations on the energy levels are rather robust against drive fluctuations.

Here we give the following example of adding a small fluctuation

\[ \delta H_d = i\eta(H(\kappa t - \kappa t_d) - H(\kappa t - \kappa t_s - \kappa t_d)) \times (\hat{a}^\dagger e^{i\omega_m t} - \hat{a} e^{-i\omega_m t}) \] (7)

in the cooling drive, where \( H(t) \) is the Heaviside function, to the Hamiltonian

\[ H_d = iE(\hat{a}^\dagger e^{i\omega_m t} - \hat{a} e^{-i\omega_m t}) + iE(\hat{a}^\dagger - \hat{a}) \] (8)

of the two drives without fluctuation (the notation here follows Eq. (B3)). This fluctuation is in the form of

FIG. 9: Influence of the square shaped fluctuations. The amplitudes are \( E = 10^8 \kappa \) and \( \eta = 200\kappa \), and the evolution courses are compared with those without fluctuations. (a) The fluctuations start at \( t = 0 \). (b) The fluctuations act at the different delay times \( t = t_d \).
square pulse with the amplitude $|\eta| \ll E$ and an action period of $t_s$. First, we let the fluctuation start from $t = 0$ and end at $t = t_s$. The simulation in Fig. 9(a) indicates that, even if the fluctuation acts for a very short period of time, the evolution of the system will be changed to another energy level. However, by postponing the action of the fluctuation, a big difference will arise as in Fig. 9(b). If its action is delayed to a time $t = t_d$ that is close to the time of approaching the stability under two unperturbed drives, the ending state of the evolution will never be changed. After it has completely evolved to one of the energy levels, the system will stably remain on the level even under fluctuations with much large amplitudes. These scenarios illustrate the fact that a small term like $\delta H_d$ does not simply play the role of perturbation to system dynamics.

C. Sensitivity to the initial condition

Among the previously known phenomena of nonlinear dynamics, chaotic motion is typical to have a tiny change of initial condition leading to huge difference in proceeding evolution. This character also exists to the model in Fig. 1(a), when the drives are strong enough to create the higher energy levels. In Fig. 10 the evolution trajectories of the energy $E_m(t)$ are compared for two situations, in one of which the oscillator is slightly touched at $t = 0$ so that there is a tiny difference in the initial conditions. Even though the initial difference is as small as $\delta E_m(0) = 10^{-14}$, the system will evolve to two different levels with a huge gap in the order of $10^{14}$. On the other hand, this type of evolutions is without the other character of chaos—the motion never repeats itself during all time. Once the system parameters are determined, the system will evolve to one of the fixed energy levels, no matter how the initial condition is modified.

VI. SUMMARY

From the perspective of the general model illustrated in Fig. 1(a), we find a type of synchronization for two nonlinearly coupled oscillators, which must be realized under two external drives having their amplitudes and frequencies properly matched. Upon reaching such synchronization, the two oscillators oscillate with a fixed spectrum of entrained frequencies, respectively. Phase dynamics [19] is the primary concern in synchronization problems, including those in chaotic systems [20–22] and systems operating in quantum regime [23–25]. The uniqueness in the current problem is a simultaneous phase locking on all entrained frequency components of the two oscillators, rather than on a couple of frequency components only.

Accompanying the synchronization is the amplitude locking for one of the oscillators, which exhibits a behavior of oscillating on its discrete energy levels. When the two drive amplitudes are equal, the positions of these energy level determined by the system parameters are completely fixed. The oscillations due to different drive amplitudes but locked on the same energy level also have their oscillation phases synchronized.

The evolution processes to the synchronization are intriguing too. As the external drives become stronger, an evolution process will become more sensitive to slight variations in the drive amplitudes, to have the evolution course be changed to another final state with huge difference. However, the stabilized oscillations on the energy levels are robust against drive fluctuations so that the energy levels can be observed. Similar to chaotic motion, the evolution of the oscillator can sensitively respond to the change of initial condition. But it is different from chaos because, given any pair of cooperating drives, the oscillator will be always stabilized on one of the fixed energy levels in the end. The system will be nonetheless synchronized to a state having the one-to-one correspondence between the oscillation patterns of the two oscillators as in Fig. 4, for whatever initial condition and drive fluctuation in the beginning period of evolution. As the real physical systems used for example, these phenomena associated with the energy levels of a macroscopic object are expected to be observable with the suitable optomechanical systems.

Appendix A: Method and notations

We consider two external drives, one cooling field and one resonant field, which act on a cavity coupled to a me-
channic oscillator. Strong nonlinearity can arise due to the existence of the resonant drive, so, unlike in the cooling of the mechanical oscillator \[1\], the system dynamics cannot be linearized. The classical nonlinear dynamical equations, i.e., Eq. (1) in the main text, are numerically integrable with high precision. For the different parameter regimes, sufficiently high precision for numerics can be chosen by considering the proper balance with the calculation efficiency (speed). Another example of numerically approachable nonlinearity in optomechanics is self-induced oscillation induced by one external drive; see, e.g., \[26–28\].

The dimensionless variables, \(X_c\), \(P_c\), \(X_m\) and \(P_m\), are adopted for the model described by Eq. (1). The conversions of these variables to the real ones are simply by the multiplications of the respective constant factors. By the use of these dimensionless variables, the cavity energy \(E_c(t) = \frac{1}{2}(X^2_c(t) + P^2_c(t))\) is the same as the photon number in the cavity, and the mechanical energy \(E_m(t) = \frac{1}{2}(X^2_m(t) + P^2_m(t))\) is equivalent to the phonon number in the mechanical oscillation. All system parameters in the equations are with the unit \(s^{-1}\) or \(Hz\). The drive amplitude is related to the drive power \(P_{1(2)}\) as 
\[
E_{1(2)} = \sqrt{\frac{\kappa P_{1(2)}}{\hbar \omega_{1(2)}}}.
\]
For the convenience in the numerical calculations, we use the relative parameters with respect to the cavity damping rate \(\kappa\), so that the calculations only involve the dimensionless quantities. For example, the drive amplitudes \(E_1\) and \(E_2\) are taken as how many times of the parameter \(\kappa\).

### Appendix B: Hamiltonian approach to the system dynamics

As in the study of the macroscopic quantumness of optomechanical systems \[2\], the cavity field and mechanical oscillator are modeled by two oscillation modes with 
\[
\frac{1}{2}(X^2_c + P^2_c) = \hat{a} \hat{a} + \frac{\kappa}{2} \quad \text{and} \quad \frac{1}{2}(X^2_m + P^2_m) = \hat{b} \hat{b} + \frac{\kappa}{2},
\]
respectively. The system Hamiltonian of two such coupled quantum mechanical oscillators consists of two parts. One is the following (\(h = 1\))
\[
H_s = \omega_c \hat{a} \hat{a} + \omega_m \hat{b} \hat{b} - G_m \hat{a} \hat{a} (\hat{b} + \hat{b})
\]  
(B1)

for the two external drives with (amplitude, frequency) = \((E_1, \omega_1)\) and \((E_2, \omega_2)\), respectively. In the following discussions we use an observation system rotating at the cavity frequency \(\omega_c\), so that the system dynamics is determined by
\[
H^I_s(t) = e^{i\omega_c t} \hat{a} \hat{a} (H_s + H_d - \omega_c \hat{a} \hat{a}) e^{-i\omega_c t}\]
\[
= \omega_m \hat{b} \hat{b} - G_m \hat{a} \hat{a} (\hat{b} + \hat{b}) + iE_1 \hat{a} e^{i\Delta_1 t} - \hat{a} e^{-i\Delta_1 t}
\]
\[
+ iE_2 \hat{a} e^{i\Delta_2 t} - \hat{a} e^{-i\Delta_2 t},
\]
(B3)

where \(\Delta_{1(2)} = \omega_c - \omega_{1(2)}\). The dampings of the two oscillators can be described in terms of their couplings to the environments \[29\]:
\[
H_{sr}(t) = i\sqrt{2\kappa} \{ \hat{a} \hat{c}_c(t) - \hat{a} \hat{c}_c(t) \}
\]
\[
+ i\sqrt{2\gamma_m} \{ \hat{b} \hat{e}_m(t) - \hat{b} \hat{e}_m(t) \},
\]
(B4)

where we adopt a form conforming to those in \[9\], and a difference up to a phase factor for the mechanical noise operator \(\hat{e}_m\) may arise due to the rotating wave approximation in deriving the above stochastic Hamiltonian \[30\].

The overall action \(U(t) = T \exp \{- i \int_0^t d\tau (H^I_s(\tau) + H_{sr}(\tau))\}\) as a time-ordered exponential leads to the following nonlinear dynamical equations:
\[
\dot{\hat{a}} = -\kappa \hat{a} + iG_m (\hat{b} + \hat{b}) \hat{a} + E_1 e^{i\Delta_1 t} + E_2 e^{i\Delta_2 t}
\]
\[
\dot{\hat{b}} = -\gamma_m \hat{b} - i\omega_m \hat{b} + iG_m \hat{a} \hat{a} + i\sqrt{2\gamma_m} \hat{e}_m(t)\]
(B5)

Eq. (1) in the main text is equivalent to the mean-field approximation, i.e., \(\hat{a}(\hat{b}) \rightarrow \langle \hat{a} \rangle (\langle \hat{b} \rangle)\) and \(\langle \hat{a} \hat{b} \rangle = \langle \hat{a} \rangle \langle \hat{b} \rangle\), for the above equations, except for a more general damping term proportional to the oscillator’s velocity. Moreover, the notation \(g = \sqrt{2G_m}\) is used there for simplifying the coefficients in the equations. An interpretation of purely classical physics for Eq. (1) is with the Hamiltonian \(\hat{H}^I_s\), which has the operators in Eq. (B3) replaced by their expectation values in addition to the cavity oscillation Hamiltonian. Then Eq. (1) is from the Hamiltonian’s equations of motion, \(\dot{q}_i = \partial \hat{H}^I_s / \partial \dot{p}_i\) and \(\dot{p}_i = -\partial \hat{H}^I_s / \partial q_i\) (\(i = 1, 2\)), plus the damping terms, where \((q_1, q_2) = (X_c, X_m)\) and \((p_1, p_2) = (P_c, P_m)\). The phenomena described by Eq. (1) are classical since \(q_i\) and \(p_i\) commute.

The effects of the quantum noises \(\hat{c}_c\) and \(\hat{e}_m\), which are important in the situations such as the cooling of the mechanical oscillator to its ground state, are averaged out for the classical dynamics. To our concerned processes and systems, they are negligible because of two facts: (1) the intensity \(\langle \sqrt{2\kappa} \hat{c}_c(t) \sqrt{2\kappa} \hat{c}_c(t') \rangle = 2\kappa \delta(t - t')\) of \(\hat{c}_c\) is extremely small as compared to \(E_{1(2)}\) and its fluctuations; (2) the mechanical noise intensity \(\langle \sqrt{2\gamma_m} \hat{e}_m(t) \sqrt{2\gamma_m} \hat{e}_m(t') \rangle = 2\gamma_m n_{th} \delta(t - t')\) is also tiny due to the low damping rate \(\gamma_m\) and in an environment of not too high temperature (the thermal occupation \(n_{th}\) is not very high). In Sec. V-B we consider the fluctuations of \(E_{1(2)}\), whose averages are non-zero. From the effects of such regular fluctuations, one can understand how a random distribution of these fluctuations (a classical noise of the drives with its vanishing average) affects the system dynamics.
FIG. D1: Comparison of the stabilized mechanical energy and associated cavity quadratures predicted with the exact nonlinear dynamical equations and the linearized equations. The solid curves are obtained with the numerical simulations based on the nonlinear dynamical equations (Eq. (1)), and the dashed curves are from the linearized Eq. (D1). Here, five different drive amplitude values respectively leading to five different energy levels are used for the illustrations from (a1)-(a3) to (e1)-(e3). The drive amplitude $E$ leading to the level $n = 2$ is lower than the amplitude that realizes the level $n = 3$, as a demonstration of the phenomenon in Fig. 8.

Appendix C: Actions of the drives and nonlinearity together

The system Hamiltonian in Eq. (B3) has the nonlinear term and drive terms. The expectation value of any system operator $\hat{O}$ after the system has evolved for a time $t$ is given by

$$
\langle \hat{O}(t) \rangle = \text{Tr}(\hat{O}\rho(t)) = \text{Tr}(U^{\dagger}(t)\hat{O}U(t)\rho_0), \quad (C1)
$$

where $U(t) = e^{-i\int_0^t d\tau (H_I^s(\tau) + H_{sr}(\tau))}$. The initial state $\rho_0$ has the cavity in a vacuum and the mechanical oscillator in a thermal state (its thermal equilibrium with the environment), and also includes the reservoir states. The interplay of the drives and the nonlinearity can be seen from the factorizations of the evolution operator $U(t)$ \footnote{\cite{18, 31}}.

One choice is to factorize the actions of the drive term with the detuning $\Delta_1 = \omega_m$ and the mechanical oscillation term, so that

$$
U(t) = V_1(t) \times T \exp \left\{ -i \int_0^t d\tau V_1^\dagger(\tau) (H_1^s(\tau) + H_{sr}(\tau) - H_1(\tau)) V_1(\tau) \right\}, \quad (C2)
$$

where $V_1(t) = \exp\{-iH_1t\}$ and $H_1(t) = iE_1(\hat{a}\dagger e^{i\omega_m t} - \hat{a}e^{-i\omega_m t}) + \omega_m\hat{b}\dagger\hat{b}$. The nonlinear term in the second evolution operator is transformed to
in which there is an intensified coupling term proportional to \( \hat{a}^\dagger \hat{b} \) (without an oscillating phase factor) by the drive amplitude \( E_1 \) for realizing the exchange of the modes \( \hat{a} \) and \( \hat{b} \) as in cooling the mechanical oscillator. Similarly, by factorizing out the actions of the resonant field term and the mechanical oscillation term, we have

\[
V_2^1(\tau)(-G_m \hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}) V_2(\tau) = -G_m E_2 \tau e^{-i\omega_m \tau} \hat{b} + \hat{b}^\dagger e^{i\omega_m \tau} + \text{h.c.}
\]

\[
+ G_m E_2 \tau^2 (\hat{b} + \hat{b}^\dagger)
\]

\[
- G_m \hat{a}^\dagger \hat{a} (\hat{b} e^{-i\omega_m \tau} + \hat{b}^\dagger e^{i\omega_m \tau}), \quad \text{(C3)}
\]

where \( V_2(t) = \exp(-iH_2 t) \) and \( H_2(t) = iE_2 (\hat{a}^\dagger - \hat{a}) + \omega_m \hat{b} \hat{b}^\dagger \). The terms growing with time (those proportional to \( G_m E_2 \tau e^{-i\omega_m \tau} \) and \( G_m E_2 \tau^2 \) ) are characteristic of a resonant drive, indicating the simultaneously intensified cooling and squeezing (gain for both cavity and mechanical modes) effects, as well as a greatly intensified effective mechanical driving.

The effective Hamiltonians in Eqs. (C3) and (C4) have the terms carrying the product of a weak nonlinearity \((G_m \ll \kappa)\) but the strong drives \((E_{1(2)} \gg \kappa)\), explicitly demonstrating their joint actions. The flexibility of the quantum Hamiltonian approach allows one to see these effects. Through the mean-field approach the quantum dynamics is reduced to the classical dynamics, for which the effects of these terms also exist.

**Appendix D: Validity of the single mode approximation for stabilized mechanical oscillations**

By plugging the stabilized mechanical oscillation Eq. (2), with the higher harmonic components neglected, into Eq. (1), one has the linearized dynamical equations

\[
\dot{X}_c = -\kappa X_c - g(A_n \sin(\omega_m t) + d_n) P_c + \sqrt{2E \cos(\omega_m t)} + \sqrt{2E} F_X(t),
\]

\[
\dot{P}_c = -\kappa P_c + g(A_n \sin(\omega_m t) + d_n) X_c + \sqrt{2E \sin(\omega_m t)} F_P(t) \quad \text{(D1)}
\]

for the stabilized cavity field quadratures, where \( \Delta_1 = \omega_m, \Delta_2 = 0 \) and \( E_1 = E_2 = E \). The solution to this linear differential equations takes the form:

\[
\begin{pmatrix}
X_c(t) \\
P_c(t)
\end{pmatrix} = \int_0^t d\tau \exp\left\{\int_\tau^t dt' \begin{pmatrix}
-\kappa & -g(A_n \sin(\omega_m t') + d_n) \\
g(A_n \sin(\omega_m t') + d_n) & -\kappa
\end{pmatrix} \right\} \begin{pmatrix}
F_X(\tau) \\
F_P(\tau)
\end{pmatrix}
\]

\[
= \int_0^t d\tau e^{-\kappa(t-\tau)} \begin{pmatrix}
0 & 0 \\
0 & -\kappa
\end{pmatrix} \exp\left\{\int_\tau^t dt' \begin{pmatrix}
0 & -g(A_n \sin(\omega_m t') + d_n) \\
g(A_n \sin(\omega_m t') + d_n) & 0
\end{pmatrix} \right\} D(t, \tau)
\]

\[
= \int_0^t d\tau D(t, \tau) \tilde{\lambda}(\tau) + \int_0^t d\tau \tilde{D}(t, \tau) \int_\tau^t dt' \tilde{M}(t') \tilde{\lambda}(\tau) + \frac{1}{2!} \int_0^t d\tau \tilde{D}(t, \tau) \left( \int_\tau^t dt' \tilde{M}(t') \right)^2 \tilde{\lambda}(\tau) + \cdots, \quad \text{(D2)}
\]
where the time-ordered exponential function in the solution is factorized into the product of two ordinary exponential functions of matrix.

The integrals involving the trigonometry functions in the above equation can be straightforwardly performed to find all Fourier components of the cavity field. The amplitude $A_n$ fixed to a set of discrete values divides the amplitudes of the Fourier components into the groups corresponding to the energy levels. The Fourier components corresponding to a fixed $A_n$ comprise an invariant oscillation pattern, except for their uniformly changed oscillation amplitudes according to the drive amplitude $E$. In Fig. D1, together with the stabilized mechanical energy, we compare the quadratures $X_e$ and $P_e$ obtained from Eq. (D1) (numerically integrating the equations without resorting to the above formal expansion) with those evolved according to the nonlinear dynamical equations. A good consistency for the results found in the two different ways provides the evidence for the validity of the approximation with the base frequency component for Eq. (2).

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