Hamiltonian dynamics of an exotic action for gravity in three dimensions

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The Hamiltonian dynamics and the canonical covariant formalism for an exotic action in three dimensions are performed. By working with the complete phase space, we report a complete Hamiltonian description of the theory such as the extended action, the extended Hamiltonian, the algebra among the constraints, the Dirac’s brackets and the correct gauge transformations. In addition, we show that in spite of exotic action and tetrad gravity with a cosmological constant give rise to the same equations of motion, they are not equivalent, in fact, we show that their corresponding Dirac’s brackets are quite different. Finally, we construct a gauge invariant symplectic form which in turn represents a complete Hamiltonian description of the covariant phase space.

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I. INTRODUCTION

A dynamical system is characterized by means of its symmetries which constitute an important information in both the classical and the quantum context. It is well-known that the analysis of a dynamical system by means of its equations of motion implies that the phase space is not endowed with a natural or preferred symplectic structure as it has been claimed in [1, 2], and the freedom in the choice of the symplectic structure is an important issue because it could yield different quantum formulations. Hence, in spite we have an infinite way to choose a symplectic structure for any system, the following question arises: are there the same symmetries in two different actions sharing the same equations of motion? The answer in general is not. In fact, it has been showed that two theories sharing the same equations of motion, does not imply that the theories are equivalent even at the classical level [3, 4]. Nonetheless, the study of any theory should be carried out extending the definition of a dynamical system by considering its equations of motion plus an action principle, thus we are in a profitable situation because the action gives us the equations of motion and symmetries; additionally it fixes the symplectic structure of the theory [5, 6]. In this manner, in the study of the symmetries of a dynamical system must be taken into account both, the equations of motion plus an action principle [4]. Nowadays, there exist approaches that can be used for studying the symmetries of any theory, as for instance, Dirac’s canonical formalism and the covariant canonical method, both with their respective advantages. Dirac’s canonical formalism is an elegant approach for obtaining

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relevant physical information of a theory under study, namely, the counting of physical degrees of freedom, the correct gauge transformations, the study of the constraints, the extended Hamiltonian and the extended action [6], all this information is the guideline to make the best progress in the analysis of quantum aspects. On the other hand, in the covariant canonical method, in order to describe all the relevant Hamiltonian description of the covariant phase space [7], we are able to identify a gauge invariant two-form, being an important step to analyze within a complete covariant context the theory under study. Therefore, we think that the complete analysis of any theory should be done by performing a Dirac’s canonical approach and the canonical covariant method, the former because it considers the action to study its symmetries, the latter takes into account the equations of motion in order to construct the covariant phase space. In this respect, usually the way to perform the Dirac formalism is not carried out in a complete form, namely, usually the people prefer to work on a smaller phase space context [8–10]; this means that only those variables that occur in the action with temporal derivative are considered as dynamical, in general in order to obtain a complete study one must perform a pure Dirac’s method, this is, we need to consider the complete set of variables occurring in our theory as dynamical ones. In this respect, we have performed a pure Dirac’s canonical analysis for models as $BF$ theories, the Pontryagin invariant, topological theories, etc., [9, 10] and we have reported the complete structure of the constraints defined on the full phase space, we have commented in those works, that by performing a pure Dirac’s framework we are able to know the symmetries of the theory, as for instance, gauge symmetry and the complete algebra among the constraints defined on the full phase space, fact that usually is not possible to obtain by using a smaller phase space context.

In this manner, the purpose of this paper, is to develop a complete Hamiltonian analysis of an exotic action in three dimensions. It is well-known that Palatini’s gravity with a cosmological constant and exotic action yield the same equations of motion, and there are many works commenting that this fact makes the actions classically equivalent (see [11, 12] and the references therein). However, a complete analysis of an exotic action has not been performed, and therefore the complete symmetries of the theory are not well known. Thus, we show in this paper that the Dirac’s brackets for the dynamical variables that define exotic action and Palatini’s gravity with a cosmological constant are different. In fact, for the former the dynamical variables of the theory are non-commutative and the cosmological constant can not be zero. For the Palatini action with a cosmological constant, the Dirac’s brackets of dynamical variables are commutative and the cosmological constant can be taken as zero, all those ideas will be clarified along the paper. In addition we report the canonical covariant analysis of an exotic action in order to report a complete study of the theory. By constructing a gauge invariant two form on the covariant phase space, we confirm some results obtained by means of Dirac’s framework.
II. HAMILTONIAN DYNAMICS FOR EXOTIC ACTION IN THREE DIMENSIONS

In this section, we will perform a pure Dirac’s analysis for an exotic action given by the following action

\[ S[e, A]_{\text{exotic}} = \frac{1}{2} \int_M A^{IJ} \wedge dA_{IJ} + \frac{2}{3} A^{IK} \wedge A_{KL} \wedge A^{LI} + \int_M \frac{\Lambda}{2} e^I \wedge De^I, \quad (1) \]

where \( A^{IJ} = A^I_{\mu} dx^\mu \) is the Lorentz connection valued in the Lie algebra of \( SO(2,1) \) and \( e^I \) corresponds to the tetrad field or gravitational field. \( \mu, \nu \) are spacetime indices, \( x^\mu \) are the coordinates that label the points for the 3-dimensional spacetime manifold \( M \) and \( I, J = 0, 1, 2 \) are internal indices that can be raised and lowered by internal Lorentzian metric \( \eta_{IJ} = (−1, 1, 1) \).

\[ D_a A^I_{bIJ} = \partial_a A^I_{bIJ} + A^I_{aIK} A^K_{bJ} + A^I_{aJK} A^K_{bI} \]

and

\[ F^I_{abJ} = \partial_a A^I_{bIJ} - \partial_b A^I_{aIJ} + A^I_{aIK} A^K_{bJ} - A^I_{bIK} A^K_{aJ}. \]

It is well-known that this exotic action is the coupling of Chern-Simons theory (the first two terms on the left hand side of (1)) and the Nieh-Yang topological term. In the following lines we will find an analogy among the Nieh-Yang term and Landau’s problem in the Chern-Simons quantization.

The equations of motion obtained from (1) are given by

\[ \frac{\delta S[e, A]_{\text{exotic}}}{\delta A^I_{\alpha J}} : \epsilon^{\alpha \mu \nu} R_{IJ \mu \nu} [A] - \Lambda \epsilon^{\alpha \mu \nu} e^I_{\mu} e^J_{\nu} = 0, \]

\[ \frac{\delta S[e, A]_{\text{exotic}}}{\delta e^{\alpha I}} : \Lambda \epsilon^{\alpha \mu \nu} D_\mu e^I_{\nu} = 0. \quad (2) \]

The first equations of motion refer to Einstein’s equation written in the first order formalism, and the second refers to the no-torsion condition. By contracting the equations of motion with the inverse \( e^I_{\alpha} \) field, these imply that the spacetime has constant curvature equal to \( 6\Lambda \).

On the other hand, we have commented above with the terminology of a pure Dirac’s method we mean that we will consider in the Hamiltonian framework that all the fields that define our theory are dynamical ones. It is important to remark, that usually the Hamiltonian analysis of any theory is performed by considering as dynamical variables only those that occur in the Lagrangian density with temporal derivative. However, the price to pay for developing the analysis on a smaller phase space is that we cannot know the complete structure of the constraints, their algebra and the gauge transformations defined on the full phase space. Hence, it is mandatory to develop a complete Hamiltonian analysis in order to report all the relevant symmetries of the theory.

By performing the 2+1 decomposition of spacetime, it is assumed that the spacetime manifold is of the form \( M^3 = \Sigma \times R \), where \( \Sigma \) corresponds to Cauchy’s surface and \( R \) represents an evolution parameter. By performing the 2+1 decomposition, we can write the action as

\[ S[e, A]_{\text{exotic}} = \int_M \left[ \frac{1}{2} \epsilon^{0ab} A_0^{IJ} F_{IJab} + \frac{1}{2} \epsilon^{0ab} A_b^{IJ} \dot{A}_a^{IJ} + \frac{\Lambda}{2} \epsilon^{0ab} e_{1b} \dot{e}_a^I + \Lambda \epsilon^{0ab} e^I_a D_b e^I_0 - \frac{\Lambda}{2} \epsilon^{0ab} A_0^{IJ} e_{aI} e_{bJ} \right] dx^3, \quad (3) \]

where we can identify the following Lagrangian density

\[ L = \frac{1}{2} \epsilon^{0ab} A_0^{IJ} F_{IJab} + \frac{1}{2} \epsilon^{0ab} A_b^{IJ} \dot{A}_a^{IJ} + \frac{\Lambda}{2} \epsilon^{0ab} e_{1b} \dot{e}_a^I + \Lambda \epsilon^{0ab} e^I_a D_b e^I_0 - \frac{\Lambda}{2} \epsilon^{0ab} A_0^{IJ} e_{aI} e_{bJ}. \quad (4) \]
Hence, by identifying our set of dynamical variables, a pure Dirac’s method calls for the definition of the momenta \((\Pi^I_\alpha, \Pi^a_IJ)\) canonically conjugate to \((e^I_\alpha, A^a_\alpha)\)

\[
\Pi^I_\alpha = \frac{\delta L}{\delta (e^I_\alpha)} , \quad \Pi^a_\alpha = \frac{\delta L}{\delta e^a_\alpha} .
\]  

(5)

The matrix elements of the Hessian

\[
\frac{\delta^2 L}{\partial (\partial e^I_\alpha) \partial (\partial e^J_\beta)} , \quad \frac{\delta^2 L}{\partial (\partial e^I_\alpha) \partial (\partial A^a_\alpha)} , \quad \frac{\delta^2 L}{\partial (\partial A^a_\alpha) \partial (\partial A^b_\beta)}
\]

are identically zero, the rank is zero, thus, we expect 18 primary constraints. From the definition of the momenta \((\Pi^I_\alpha)\) we identify the following 18 primary constraints

\[
\phi^0_I := \Pi^0_I \approx 0, \\
\phi^a_I := \Pi^a_I - \frac{\Lambda}{2}\epsilon^{ab} e_\text{I}b \approx 0, \\
\phi^0_J := \Pi^0_J \approx 0, \\
\phi^a_J := \Pi^a_J - \frac{\epsilon^{ab}}{2} A_{abIJ} \approx 0.
\]

(7)

The canonical Hamiltonian takes the form

\[
H_c = \int dx^2 \left[ \frac{1}{2} A^a_\alpha e^{ab} F_{abIJ} + \frac{A^a_\alpha}{2} [e_\text{I}a \Pi^a_J - e_\text{J}a \Pi^a_I] - 2 e^a_\text{I}b D_a \Pi^a_J \right],
\]

(8)

and the primary Hamiltonian is given as

\[
H_P = H_c + \int dx^2 \left[ \lambda^a_I \phi^a_I + \lambda^I_\alpha \phi^0_I \right],
\]

(9)

where \(\lambda^a_I, \lambda^I_\alpha\) are Lagrange multipliers enforcing the constraints. For this field theory, the non-vanishing fundamental Poisson brackets are

\[
\{e^I_\alpha(x), \Pi^J_\beta(y)\} = \delta^I_\alpha \delta^J_\beta \delta^2(x - y), \\
\{A^I_\alpha(x), \Pi^J_K(y)\} = \frac{1}{2} \delta^I_\alpha \left( \delta^J_\beta \delta^K_L - \delta^K_\beta \delta^J_L \right) \delta^2(x - y).
\]

(10)

The 18×18 matrix whose entries are the Poisson brackets among the constraints \((\Pi^I_\alpha, \Pi^a_\alpha)\) contains the 18 primary constraints

\[
\phi^0_I = \{ \phi^0_I(x), H_P \} \approx 0 \Rightarrow \psi^0_I = \frac{1}{2} \epsilon^{ab} F_{IJab} + e_\text{I}a \Pi^a_J - e_\text{J}a \Pi^a_I \approx 0,
\]

(11)

and the rank allows us to fix the following values for the Lagrangian multipliers

\[
\phi^a_I = \{ \phi^a_I(x), H_P \} \approx 0 \Rightarrow -\Lambda \epsilon^{ab} \left( \lambda^a_I + D_a I^J \right) \approx 0,
\]

(13)

\[
\phi^a_J = \{ \phi^a_J(x), H_P \} \approx 0 \Rightarrow e_\text{I}a \left( \lambda^a_J - D_a A^a_\alpha \right) \approx 0.
\]
Consistency requires that their conservation in time vanishes as well. For this theory there are no, third constraints. At this point, we need to identify from the primary and secondary constraints which one corresponds to the first and the second class. For this aim, we need to calculate the rank and the null-vectors of the $24 \times 24$ matrix whose entries will be the Poisson brackets between primary and secondary constraints, the non-zero brackets are given by

\[
\{ \phi_I^a(x), \phi_J^b(y) \} = -\Delta \epsilon^{0ab} \eta_{IJ} \delta^2(x - y),
\]

\[
\{ \phi_I^a(x), \psi_J(y) \} = -\Delta \epsilon^{0ab} [\eta_{IJ} \partial_x \delta^2(x - y) - A_{bIJ} \delta^2(x - y)],
\]

\[
\{ \phi_{IJ}^a(x), \phi_{KL}^b(y) \} = \frac{1}{2} \epsilon^{0ab} [\eta_{IJ} \eta_{KL} - \eta_{IK} \eta_{JL}] \delta^2(x - y),
\]

\[
\{ \phi_{IJ}^a(x), \psi_K(y) \} = [\Pi^a_I \eta_{JK} - \Pi^a_J \eta_{IK}] \delta^2(x - y),
\]

\[
\{ \phi_{IJ}^a(x), \psi_{KL}(y) \} = \frac{1}{2} \epsilon^{0ac} [A_{cIJL} \eta_{JK} - A_{cJIL} \eta_{IK} - A_{cKJL} \eta_{IL} + A_{cKLJ} \eta_{IL} + (\eta_{IK} \eta_{JL} - \eta_{IL} \eta_{JK}) \partial_y] \delta^2(x - y),
\]

\[
\{ \psi_I(x), \psi_{KL}(y) \} = \partial_y \delta^2(x - y) [\eta_{IK} \Pi^a_J - \eta_{IL} \Pi^a_K] + \delta^2(x - y) [A_{aIJL} \Pi^a_K - A_{aKIJ} \Pi^a_L],
\]

\[
\{ \psi_{IJ}(x), \psi_{KL}(y) \} = \frac{1}{4} [\eta_{IK} (\Pi^a_L e_{Ja} - \Pi^a_L e_{La}) + \eta_{JL} (\Pi^a_K e_{Ia} - \Pi^a_K e_{Ka}) + \eta_{JK} (\Pi^a_L e_{La} - \Pi^a_L e_{Ja}) + \eta_{IL} (\Pi^a_K e_{Ka} - \Pi^a_K e_{Ja})] \delta^2(x - y).
\]

This matrix has rank=12 and 12 null vectors, thus, we find that our theory presents a set of 12 first class constraints and 12 second class constraints. By using the contraction of the null vectors with the constraints (7) and (12), we identified the following 12 first class constraints

\[
\gamma^0_I = \Pi^0_I \approx 0,
\]

\[
\gamma^0_{IJ} = \Pi^0_{IJ} \approx 0,
\]

\[
\gamma_I = -2D_a \Pi^a_I + D_a \phi^a_I + \Lambda \epsilon^a_I \phi_I^a,
\]

\[
\gamma_{IJ} = D_a \phi^a_{IJ} + \frac{\epsilon^{0ab}}{2} F_{IJab} + \frac{1}{2} [\Pi^a_I e_{Ja} - \Pi^a_J e_{Ia}],
\]

and the following 12 second class constraints

\[
\chi^a_I = \Pi^a_I - \frac{\Lambda}{2} \epsilon^{0ab} e_{Ib} \approx 0,
\]

\[
\chi^a_{IJ} = \Pi^a_{IJ} - \frac{\epsilon^{0ab}}{2} A_{bIJ} \approx 0,
\]

It is important to remark that these constraints have not been reported in the literature, and its complete structure defined on the full phase space will be relevant in order to know the fundamental gauge transformations. On the other hand, the constraints will play a key role to make progress in the quantization. All this information is only possible by performing a pure Dirac’s analysis.
Now, we will calculate the algebra of the constraints

\[
\{\chi^a_I(x), \chi^b_J(y)\} = -\Lambda \epsilon^{ab} \eta_{IJ} \delta^2(x - y),
\]

\[
\{\chi^a_I(x), \gamma_J(y)\} = \Lambda \chi^a_I \delta^2(x - y) \approx 0,
\]

\[
\{\chi^a_I(x), \gamma_{JN}(y)\} = \frac{1}{2} [\eta_{IJ} \chi^a_N - \eta_{IN} \chi^a_J] \delta^2(x - y) \approx 0,
\]

\[
\{\chi^a_I(x), \gamma_L(y)\} = \frac{1}{2} [\eta_{IL} \chi^a_J - \eta_{IJ} \chi^a_L] \delta^2(x - y) \approx 0,
\]

\[
\{\chi^a_I(x), \gamma_{KL}(y)\} = \frac{1}{2} [\chi^a_{IL} \eta_{JK} - \chi^a_{IJ} \eta_{KL}] \delta^2(x - y) \approx 0,
\]

\[
\{\chi^a_I(x), \chi_{KL}(y)\} = \frac{1}{2} \epsilon^{ab} [\eta_{IL} \eta_{JK} - \eta_{IK} \eta_{JL}] \delta^2(x - y),
\]

where we are able to appreciate that the algebra of the first class constraints is closed and we do not need conditions on the \(\epsilon^{IJK}\) in order to obtain that algebra, this result is different from general relativity expressed by means of Palatinit’s theory, because in Palatinit’s theory in order to obtain a closed algebra it is necessary to use the fact \(\epsilon^{IJK}\) are the structural constants of \(SO(2, 1)\) [15]. Moreover, because of [16] the algebra does not form an \(ISO(2, 1)\) Poincaré algebra, however, it is a Lie algebra. In this respect, we are able to observe that Palatinit’s gravity without a cosmological constant forms an \(ISO(2, 1)\) Poincaré algebra [14, 15]; in the exotic action, the cosmological constant can not be zero, this will be seen in the following lines.

We have developed a pure Dirac’s analysis and there are second class constraints, however, they can be eliminated through Dirac’s bracket for the theory. In fact, by observing that the matrix whose elements are only the Poisson brackets among second class constraints is given by

\[
C_{\alpha\beta} = \begin{pmatrix}
0 & -\Lambda \eta_{IJ} & 0 & 0 \\
\Lambda \eta_{IJ} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} [\eta_{IL} \eta_{JK} - \eta_{IK} \eta_{JL}] \\
0 & 0 & -\frac{1}{2} [\eta_{IL} \eta_{JK} - \eta_{IK} \eta_{JL}] & 0
\end{pmatrix} \epsilon^{ab} \delta^2(x - y), \tag{19}
\]

its inverse will be

\[
C^{-1}_{\alpha\beta} = \begin{pmatrix}
0 & \frac{1}{2} \eta^{IJ} & 0 & 0 \\
-\frac{1}{2} \eta^{IJ} & 0 & 0 & 0 \\
0 & 0 & 0 & -2 [\eta^{IL} \eta^{JK} - \eta^{IK} \eta^{JL}] \\
0 & 0 & 2 [\eta^{IL} \eta^{JK} - \eta^{IK} \eta^{JL}] & 0
\end{pmatrix} \epsilon^{ab} \delta^2(x - y). \tag{20}
\]

The Dirac’s brackets among two functionals \(A, B\) are expressed by

\[
\{A(x), B(y)\}_D = \{A(x), B(y)\}_P + \int dudv \{A(x), \zeta^\alpha(u)\} C^{-1}_{\alpha\beta}(u, v) \{\zeta^\beta(v), B(y)\}, \tag{21}
\]

where \(\{A(x), B(y)\}_P\) is the usual Poisson brackets between the functionals \(A, B\) and \(\zeta^\alpha(u) = \)
\((\chi^a, \chi_{IJ}^a)\). Hence, we obtain the following Dirac’s brackets of the theory

\[\{e'_a(x), \Pi^b J(y)\}_D = \{e'_a(x), \Pi^b J(y)\}_P + \int dudv\{e'_a(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), \Pi^b J(y)\}, \]

\[= \frac{1}{2} \delta^b_a \delta^I_J \delta^2(x-y), \tag{22}\]

\[\{e'_a(x), e'_b(y)\}_D = \{e'_a(x), e'_b(y)\}_P + \int dudv\{e'_a(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), e'_b(y)\}, \]

\[= \frac{1}{A} \eta^I_J \varepsilon_{a0b}\delta^2(x-y), \tag{23}\]

\[\{\Pi^a I(x), \Pi^b J(y)\}_D = \{\Pi^a I(x), \Pi^b J(y)\}_P + \int dudv\{\Pi^a I(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), \Pi^b J(y)\}, \]

\[= \frac{A}{4} \eta_{IJ} \varepsilon_{0ab}\delta^2(x-y), \tag{24}\]

\[\{A^{IJ}_a(x), \Pi^b LN(y)\}_D = \{A^{IJ}_a(x), \Pi^b LN(y)\}_P + \int dudv\{A^{IJ}_a(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), \Pi^b LN(y)\}, \]

\[= \frac{1}{4} \delta^b_a \left[ \delta^I_J \delta^N L - \delta^J N \delta^I L \right] \delta^2(x-y), \tag{25}\]

\[\{A^{IJ}_a(x), A^{LN}_b(y)\}_D = \{A^{IJ}_a(x), A^{LN}_b(y)\}_P + \int dudv\{A^{IJ}_a(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), A^{LN}_b(y)\}, \]

\[= \frac{1}{2} \left[ \eta^{IJ} \eta^J N - \eta^J N \eta^{IJ} \right] \varepsilon_{a0b}\delta^2(x-y), \tag{26}\]

\[\{\Pi^a IJ(x), \Pi^b LN(y)\}_D = \{\Pi^a IJ(x), \Pi^b LN(y)\}_P + \int dudv\{\Pi^a IJ(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), \Pi^b LN(y)\}, \]

\[= \frac{1}{8} [\eta_{IJ} \eta_{LN} - \eta_{IN} \eta_{JL}] \varepsilon_{0ab}\delta^2(x-y). \tag{27}\]

\[\{e'_a(x), A^{LN}_b(y)\}_D = \{e'_a(x), A^{LN}_b(y)\}_P + \int dudv\{e'_a(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), A^{LN}_b(y)\}, \]

\[= 0, \tag{28}\]

\[\{e'_a(x), \Pi^b LN(y)\}_D = \{e'_a(x), \Pi^b LN(y)\}_P + \int dudv\{e'_a(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), \Pi^b LN(y)\}, \]

\[= 0, \tag{29}\]

\[\{A^{IJ}_a(x), \Pi^b L(y)\}_D = \{A^{IJ}_a(x), \Pi^b L(y)\}_P + \int dudv\{A^{IJ}_a(x), \zeta^a(u)\}C^{-1}\alpha\beta(u, v)\{\zeta^\beta(v), \Pi^b L(y)\}, \]

\[= 0. \tag{30}\]
It is important to remark that the fields \( e, A \) and their canonical momenta have become non-commutative, and the cosmological constant can not be fixed to zero. On the other hand, in Palatini’s gravity, by performing a pure Hamiltonian analysis, Dirac’s brackets among the fields \( e \) and \( A \) become commutative and the cosmological constant can be taken as zero (see [15]). This result marks a difference at the classical level among exotic and Palatini actions. Furthermore, we notice that the term of Nieh-Yang becomes a magnetic like term, just as is present in Landau’s problem. In fact, for a charged particle of mass \( m \) confined by a quadratic potential that moves in a uniform magnetic field, the Lagrangian is given by [16]

\[
L = \frac{m}{2} \dot{x}_i^2 + \frac{B}{2} \epsilon_{ij} \dot{x}^i x^j - \frac{K}{2} x_i^2, \tag{31}
\]

where \( B \) is the magnetic field and \( K \) is a constant. In general the action (31) is not singular and the Hamiltonian analysis is easy to carry out. Because of the action is not singular, we can take \( B \) or \( K \) as zero without problem. However, by taking the limit \( m \to 0 \) the system becomes singular and after a Dirac’s analysis of (31) in that limit, there are second class constraints, and the coordinates are non-commutative; Dirac’s brackets of the theory are given by

\[
\{ x^i, x^j \}_D = -\epsilon^{ij},
\]

thus, for this singular theory \( B \) can not be zero; in fact, the spectra of energy depend on a factor \( \frac{1}{B} \) [13]. In this manner, in analogy with the action (31), in the exotic action the Nieh-Yang term is a “magnetic field” like term (see the term \( \Lambda^2 \epsilon^{0a} \epsilon^{b} \dot{e}^I a \epsilon^{Ib} \) of (3)), namely, the cosmological constant becomes to be the magnetic field \( B \) and the field \( e \) the non-commutative coordinates (see eq. (23)). Of course, the Chern-Simons term can be treated in the same form; however, Chern-Simons gives non-commutative connections \( A \) (see eq (26)), and the Nieh-Yang term gives non-commutative fields \( e \). Thus, the Nieh-Yang term becomes a non-commutative gauge theory for the triad field. Therefore, we realize that for a singular theory it is not a correct step to fix the parameters that occur in the theory before developing a detailed analysis. In order to study a singular system with arbitrary parameters, first, it is mandatory to perform a detailed Dirac’s analysis, then, we could study the behavior of the action by taking the limit \( m \to 0 \) of the parameters. The exotic action is a singular system and our detailed analysis indicates that the cosmological constant cannot be fixed to zero. Moreover, the identification of the constraints will allow us to identify the extended action. By using the first class constraints (17), the second class constraints (18), and the Lagrangian multipliers (15) we find that the extended action takes the form

\[
S_E[e^I \alpha, A^{IJ} \alpha, \Pi^\alpha I, \Pi^\alpha J, v_0^I, a_0^{IJ}, u_0^I, \dot{u}_0^I, u^I, \dot{u}^J, v_a^I, \dot{v}_a^I] = \int_M \left[ \dot{e}^I \alpha \Pi^\alpha I + \dot{\dot{A}}^{IJ} \alpha \Pi^\alpha J - H' - u_0^I \gamma_0^I - u_0^{IJ} \gamma_{IJ}^0 - u^I \gamma_I - u^{IJ} \gamma_{IJ} - v_a^I \chi^I - v_a^{IJ} \chi_{IJ}^a \right] dx^3, \tag{32}
\]

where \( H' \) is the linear combination of first class constraints

\[
H' = \int \left[ e_0^I \gamma_0^I - A_0^{IJ} \gamma_{IJ}^0 \right] dx^2, \tag{33}
\]

and \( u_0^I, u_0^{IJ}, u^I, u^{IJ}, v_a^I, v_a^{IJ} \) are Lagrange multipliers enforcing the first and second class constraints. From the extended action we can identify the extended Hamiltonian given by

\[
H_E = H' + \int \left[ v_0^I \gamma_0^I + u_0^{IJ} \gamma_{IJ}^0 + u^I \gamma_I + u^{IJ} \gamma_{IJ} \right] dx^2. \tag{34}
\]
It is important to remark that the theory under study has an extended Hamiltonian which is a linear combination of first class constraints reflecting the general covariance of the theory, just as General Relativity, thus, in order to perform a quantization of the theory, it is not possible to construct the Schrödinger equation because the action of the Hamiltonian on physical states is annihilation.

In Dirac’s quantization of systems with general covariance, the restriction of our physical state is archived by demanding that the first class constraints in their quantum form must be satisfied and the Dirac’s brackets must be taken into account as well, thus in this paper we have all the tools for studying the quantization of the theory by means of a canonical framework.

One of the most important symmetries that can be studied by using the Hamiltonian method, are the gauge transformations. Gauge transformations are fundamental in the identification of physical observables [6]. In this respect, we have commented above that a detailed analysis will give us the correct gauge symmetry. In fact, the correct gauge symmetry is obtained according to Dirac’s conjecture by constructing a gauge generator using the first class constraints, and the structure of the constraints defined on the full phase space will give us the fundamental gauge transformations.

For this aim, we will apply the Castellani’s algorithm to construct the gauge generator. We define the generator of gauge transformations as

$$G = \int \left[ D_0 \varepsilon_0^I \gamma^0_I + D_0 \varepsilon_0^{IJ} \gamma^0_{IJ} + \varepsilon^I \gamma_I + \varepsilon^{IJ} \gamma_{IJ} \right].$$

(35)

Therefore, we find that the gauge transformations on the phase space are

$$\delta_0 e^I_0 = D_0 \varepsilon_0^I,$$

$$\delta_0 e^I_a = D_a \varepsilon^I + \varepsilon^{IJ} e_a J,$$

$$\delta_0 A_0^{IJ} = D_0 \varepsilon_0^{IJ},$$

$$\delta_0 A_a^{IJ} = \frac{\Lambda}{2} [e^j_a \varepsilon^I - e^I_a \varepsilon^j] - D_a \varepsilon^{IJ},$$

$$\delta_0 \Pi_0^I = 0,$$

$$\delta_0 \Pi_0^{IJ} = 0,$$

$$\delta_0 \Pi_a^I = \frac{\Lambda}{2} \varepsilon^{ab} \partial_b \varepsilon^I + \Lambda \varepsilon^{IJ} \Pi^a_{IJ} - \varepsilon^I \Pi^a J,$$

$$\delta_0 \Pi_a^{IJ} = -\varepsilon^K \varepsilon^a L - \varepsilon^L \varepsilon^a K + \varepsilon^K \varepsilon^L \Pi_0^{aIJ} + \frac{1}{2} \varepsilon^a b \partial_b \varepsilon_{IJ}.$$  

(36)

We realize that the fundamental gauge transformations of the exotic action are given by (36) and do not correspond to diffeomorphisms, but they are \(\Lambda\)-deformed \(ISO(2,1)\) Poincaré transformations. However, any theory with a dynamical background metric is diffeomorphisms covariant, and this symmetry must be obtained from the fundamental gauge transformation. Hence, the diffeomorphisms can be found by redefining the gauge parameters as \(\varepsilon_0^I = \varepsilon^I = \xi^\rho \varepsilon^I_\rho, \varepsilon_0^{IJ} = \varepsilon^{IJ} = -\xi^\rho A_\rho^{IJ},\)

and the gauge transformation (36) takes the following form

$$e^I_\alpha \rightarrow e^I_\alpha + \xi_\delta e^I_\alpha + \xi^\rho \left[ D_\alpha e^I_\rho - D_\rho e^I_\alpha \right],$$

$$A_\alpha^{IJ} \rightarrow A_\alpha^{IJ} + \xi_\delta A_\alpha^{IJ} + \xi^\rho \left[ R^{IJ}_\alpha \rho - \frac{\Lambda}{2} (e^I_\alpha e^J_\rho - e^J_\alpha e^I_\rho) \right].$$

(37)

Therefore, diffeomorphisms are obtained (on shell) from the fundamental gauge transformations as an internal symmetry of the theory. With the correct identification of the constraints, we can
carry out the counting of degrees of freedom in the following form: there are 36 canonical variables \((e^I\alpha, A^I\alpha, \Pi^I\alpha, \Pi^{IJ}\alpha)\), 12 first class constraints \((\gamma^0_I, \gamma^0_{IJ}, \gamma_I, \gamma_{IJ})\) and 12 second class constraints \((\chi^Ia, \chi_{IJ}a)\) and one concludes that the exotic action for gravity in three dimensions is devoid of degrees of freedom, therefore, the theory is topological.

As a conclusion of this part, we have performed a pure Hamiltonian analysis for the exotic action by working with the complete configuration space. With the present analysis, we have obtained the extended action, the extended Hamiltonian, the complete structure of the constraints on the full phase space, and the algebra among them. The price to pay for working on the complete phase space, is that the theory presents a set of first and second class constraints; by using the second class constraints we have constructed Dirac’s brackets and they will be useful in the quantization of the theory.

### III. THE SYMPLECTIC METHOD FOR EXOTIC ACTION

In order to develop a complete analysis, in this section we shall carry out the covariant canonical formalism for the theory, and we shall confirm some results obtained in the above section.

Let us start by calculating the variation of the exotic action

\[
\delta S[A, e]_{\text{exotic}} = \int_M \left[ \frac{1}{2} (\epsilon^{\alpha\mu\nu} F_{IJ\mu\nu}[A] - \Lambda \epsilon^{\alpha\mu\nu} e_{I\mu} e_{J\nu}) \delta A^I_J + (\Lambda \epsilon^{\alpha\mu\nu} D_{\mu} e^I_{\nu}) \delta e^I_{\alpha} \right] - \int_M \partial_{\mu} \left( \Lambda \epsilon^{\mu\alpha\nu} e_{I\alpha} \delta e^I_{\nu} + \epsilon^{\mu\alpha\nu} A^{IJ}_{\alpha} \delta A_{\nu I J} \right),
\]

where we can identify the equations of motion (2) and the integral kernel for the symplectic structure from the boundary term

\[
\Psi^\mu_{\text{exotic}} = \epsilon^{\mu\alpha\nu} \Lambda e_{I\alpha} \delta e^I_{\nu} + \epsilon^{\mu\alpha\nu} A^{IJ}_{\alpha} \delta A_{\nu I J}.
\]

which does not contribute locally to the dynamics, but generates the symplectic form on the phase space \(\mathcal{Z}\).

Now, we define the fundamental concept in the studio of the symplectic formalism of the theory: the covariant phase space for the theory described by (5) is the space of solutions of (2), and we shall call it \(Z\). In this manner, we can obtain the fundamental two-form of the geometric structure for the theory by means of the variation (exterior derivative on \(\mathcal{Z}\)) of the symplectic potential (39)

\[
\varpi = \int_{\Sigma} J^\mu d\Sigma_{\mu} = \int_{\Sigma} \delta \Psi^\mu d\Sigma_{\mu} = \int_{\Sigma} (\epsilon^{\mu\alpha\nu} \Lambda \delta e_{I\alpha} \wedge \delta e^I_{\nu} + \epsilon^{\mu\alpha\nu} \delta A^{IJ}_{\alpha} \wedge \delta A_{\nu I J}) d\Sigma_{\mu},
\]

where \(\Sigma\) is a Cauchy hypersurface. We are able to observe in the geometric structure (40) the non-commutative character of the dynamical variables.

So, we will prove that our symplectic form is closed and gauge invariant. Moreover, the integral kernel of the geometric form \(J^\mu\) is conserved, which guarantees that \(\varpi\) is independent of \(\Sigma\). We need to remember that the closeness of \(\varpi\) is equivalent to the Jacobi identity that the Poisson brackets satisfy in the usual Hamiltonian scheme.
In order to prove the closeness of $\varpi$, we can observe that $\delta^2 e^I_\mu = 0$ and $\delta^2 A_\alpha^{IJ} = 0$, because $e^I_\mu$ and $A_\alpha^{IJ}$ are independent 0-forms on the covariant phase space $Z$ and $\delta$ is nilpotent, thus
\[
\delta \varpi = \int_\Sigma \left\{ e^{\mu \nu} \Lambda^2 e_{I\alpha} \wedge \delta e^I_\nu - e^{\mu \nu} \Lambda \delta e_{I\alpha} \wedge \delta^2 e^I_\nu + e^{\mu \rho \nu} \delta^2 A_\alpha^{IJ} \wedge \delta A_{\nu LJ} \right\} d\Sigma_\mu = 0,
\]
therefore the geometric form is closed.

For future useful calculations we shall obtain the linearized equations of motion of the theory. For this purpose, we replace $A_{\nu LJ} \to \delta A_{\nu LJ}$ and $e^I_\mu \to \delta e^I_\mu$ in (2) and keep only the first-order terms, we obtain
\[
e^{\mu \nu} \left[ D_\rho \delta A_{\nu LJ} \right] - \Lambda \left[ e_{\alpha I} \delta e_{\nu J} + \delta e_{\alpha l} e_{\nu J} \right] = 0,
\]
\[
\Lambda e^{\mu \nu} \left[ D_\alpha \delta e_{\nu J} + e^I_\nu \delta A_\alpha^{IJ} \right] = 0.
\]
Furthermore, we can see that under fundamental gauge transformations given in (36) and for some infinitesimal variation we have
\[
\delta A_\alpha^{IJ} = \delta A_\alpha^{IJ} - \delta A_\alpha^I K^J - \delta A_\alpha^K J^I,
\]
\[
\delta e^I_\alpha = \delta e^I_\alpha + \delta e_{\alpha J} e^{IJ},
\]
thus, by using (43), we find that $\varpi$ transforms
\[
\varpi' = \int_\Sigma \left( e^{\mu \nu} \Lambda \delta e^I_\alpha \wedge \delta e^I_\nu + e^{\mu \nu} \delta A_\alpha^{IJ} \wedge \delta A_{\nu LJ} \right) d\Sigma_\mu
= \varpi + \int_\Sigma \mathcal{O}(\epsilon^2) d\sigma.
\]
Therefore, $\varpi$ is a $SO(2, 1)$ singlet. This result allows us to prove that
\[
\partial_{\mu} J^\mu = D_\mu J^\mu = \Lambda e^{\mu \nu} \left[ D_\nu \delta e_{\alpha I} \wedge \delta e^I_\nu + \delta e_{\alpha I} \wedge D_\nu \delta e^I_\nu + D_\mu \delta A_\alpha^{IJ} \wedge \delta A_{\nu LJ} \right]
= -\Lambda e^{\mu \nu} e^I_\nu \delta A_{\nu LJ} - \Lambda e^{\mu \nu} e^J_\nu \delta e_{\alpha I} \wedge \delta A_{\mu LJ} + \frac{\Lambda}{2} e^{\mu \nu} e^I_\mu \delta e_{\alpha I} \wedge \delta A_{\nu LJ}
+ \frac{\Lambda}{2} e^{\mu \nu} e^J_\mu \delta A_{\nu LJ} \wedge \delta e_{\alpha I} + \frac{\Lambda}{2} e^{\mu \nu} e^J_\nu \delta A_{\nu LJ} \wedge \delta e_{\alpha I} = 0,
\]
where we have used the linearized equations given in (12), and the antisymmetry of 1-forms $\delta e^I_\nu$ and $\delta A_\alpha^{IJ}$. Therefore, $\varpi$ is independent of $\Sigma$, thus performing a Lorentz transformation $\Sigma_t \to \Sigma'_t$ and $\varpi \to \varpi'$
\[
\varpi' = \int_\Sigma \delta \Psi^\alpha d\Sigma_\alpha = \int_\Sigma \delta \Psi^\alpha d\Sigma_\alpha = \varpi.
\]
In this manner, with these results we have constructed a Lorentz and gauge invariant symplectic structure on the phase space and it is possible to formulate the Hamiltonian theory in a manifestly covariant way.

In order to reproduce the gauge transformations found in the Hamiltonian formalism by using now the symplectic method, let us consider that upon picking $\Sigma$ to be the standard initial value surface $t = 0$, hence equation (40) takes the standard form
\[
\varpi = \int_\Sigma \left[ \delta \Pi_{IJ} \wedge \delta e^I_\alpha + \delta \Pi_{IJ} \delta A_{\alpha^{IJ}} \right],
\]
where $\Pi^a_I \equiv \Lambda \epsilon^{ab} e_{Ib}$ and $\Pi^{a}_{IJ} \equiv \epsilon^{ab} A_{bIJ}$.

For two 0-forms $f, g$ defined on $Z$, the Hamiltonian vector field defined by the symplectic form \cite{17}

$$X_f \equiv \int \delta f \frac{\delta}{\delta e_a^I} - \frac{\delta f}{\delta \Pi_{IJ}} \delta A_{ab}^J - \frac{\delta f}{\delta \Pi_{IJ}} \delta A_{ab}^J, \tag{48}$$

and the Poisson bracket $\{f, g\}$ is given by

$$\{f, g\} \equiv \int \frac{\delta f}{\delta e_a^I} \delta \Pi_{IJ} - \frac{\delta f}{\delta \Pi_{IJ}} \delta e_a^I + \frac{\delta f}{\delta A_{ab}^J} \delta \Pi_{IJ} - \frac{\delta f}{\delta \Pi_{IJ}} \delta A_{ab}^J. \tag{49}$$

Furthermore, smearing the constraints \cite{14} with test fields we obtain

$$\gamma_I \left[ C^I \right] = \int \left[ -2 D_a \Pi_{Ia}^a + D_a \phi^a_I + \Lambda \epsilon^{J} \phi_{IJ} \right], \tag{50}$$

$$\gamma_{IJ} \left[ C^{IJ} \right] = \int \left[ 2 \Lambda \epsilon^{J} \phi_{IJ} + \frac{1}{2} \epsilon^{ab} F_{IJab} + \frac{1}{2} \left[ \Pi_{I}^a e_{Ja} - \Pi_{J}^a e_{Ia} \right] \right]. \tag{51}$$

By inspection, the functional derivatives different from zero are given by

$$\frac{\delta \gamma_I \left[ C^I \right]}{\delta e_a^I} = -\Lambda \epsilon^{ab} D_b C_I - \Lambda C^J \phi_{IJ}^a, \quad \frac{\delta \gamma_I \left[ C^I \right]}{\delta \Pi_{IJ}^a} = D_a C_I, \quad \frac{\delta \gamma_{IJ} \left[ C^{IJ} \right]}{\delta e_a^I} = -\Pi_{IJ}^a, \quad \frac{\delta \gamma_{IJ} \left[ C^{IJ} \right]}{\delta \Pi_{IJ}^a} = C^{IJ} e_{aJ},$$

$$\frac{\delta \gamma_{IJ} \left[ C^{IJ} \right]}{\delta A_{ab}^J} = \frac{1}{2} \left[ C_I \Pi_J^a - C_J \Pi_I^a \right], \quad \frac{\delta \gamma_I \left[ C^I \right]}{\delta A_{ab}^J} = \frac{1}{2} \left[ C_I^F \Pi_F^a - C_J^F \Pi_I^a \right], \quad \frac{\delta \gamma_I \left[ C^I \right]}{\delta A_{ab}^J} = \frac{1}{2} \epsilon^{ab} F_{IJ} + \frac{1}{2} \left[ \Pi_{I}^a e_{Ja} - \Pi_{J}^a e_{Ia} \right]. \tag{52}$$

Thus, by using \cite{50} and \cite{51} (the motion on $Z$ generated by $\gamma_I \left[ C^I \right]$ and $\gamma_{IJ} \left[ C^{IJ} \right]$), is given by

$$e'^{IJ}_I \rightarrow e'^{IJ}_I + \frac{\xi}{2} D_a C_I + \xi C^{IJ} e_{aJ} + O(\xi^2),$$

$$A'^{IJ}_a \rightarrow A^{IJ}_a - \frac{\xi}{2} \left[ e'^{aI}_a C^J - e'^{aJ}_a C^I \right] + \xi D_a C^{IJ} + O(\xi^2),$$

$$\Pi'^{a}_{IJ} \rightarrow \Pi^{a}_{IJ} + \frac{\xi}{2} \epsilon^{ab} D_b C_I + \xi C^J \Pi_{IJ}^a - \xi C^J \Pi_{IJ}^a + O(\xi^2),$$

where $\xi$ is an infinitesimal parameter \cite{17}. We are able to observe that the gauge transformations \cite{55} are those found using Dirac’s method (see \cite{36}), and correspond to $\Lambda$-deformed Poincaré transformations. Furthermore, it is well-known that any background independent theory is diffeomorphisms covariant and this symmetry should be manifest in our geometric structure, in order to prove that $\varpi$ is diffeomorphisms covariant we observe that \cite{37} for some infinitesimal variation takes the form

$$\delta e'^{IJ} \rightarrow \delta e'^{IJ} + \xi^\mu \partial_\mu \delta e'^{IJ} + \delta e'^{IJ} \partial_\alpha \xi^\mu,$$

$$\delta A'^{IJ}_a \rightarrow \delta A^{IJ}_a + \xi^\mu \partial_\mu \delta A^{IJ}_a + \delta A^{IJ}_a \partial_\alpha \xi^\mu,$$  \tag{53}

thus by using \cite{53}, $\varpi$ will undergo the transformation as

$$\varpi' = \varpi + \int_{\Sigma} \left( \epsilon^{\mu\nu} \Lambda \delta e'^{\mu} + \epsilon^{\mu} \delta A'^{\mu}_a \Lambda \delta A'^{\nu}_a \right) d\Sigma^\mu, \tag{54}$$
Moreover, $\mathcal{L}_\xi \varpi = \xi \cdot d\varpi + d(\xi \cdot \varpi)$, but $\varpi$ is closed ($d\varpi = 0$), hence the term on the right hand side is a surface term. Therefore, we have showed that $\varpi$ is invariant under infinitesimal diffeomorphisms. As a conclusion of this section, we have constructed a gauge invariant symplectic form on $Z$ which in turn represents a complete Hamiltonian description of the covariant phase space for the theory, and it will allow us to analyze the quantum treatment in forthcoming works.

IV. CONCLUSIONS AND PROSPECTS

In this paper, a detailed Hamilton analysis for an exotic action has been performed; in our analysis, the price to pay by working on the complete phase space is that the theory presents a set of first and second class constraints and we have identified their full structure. By identifying the complete structure of the constraints, we found the fundamental gauge transformations of the theory corresponding to deformed Poincaré transformations and by defining the gauge parameters, diffeomorphisms can be obtained from the fundamental gauge symmetry. It is important to comment, that only by using a pure Dirac’s analysis it is possible to identify the complete gauge symmetry of the theory. On the other hand, we constructed the fundamental Dirac’s brackets and we showed that the exotic action is non-commutative and presents problems when the cosmological constant takes the value $\Lambda = 0$, because there is a singularity at the level of Dirac’s brackets. In this respect, we observed an analogy with the case of Landau’s problem identifying the cosmological constant with the magnetic field and the field $e$ with the non-commutative coordinates. Additionally we have showed that the exotic action is different from Palatini’s theory even at the classical level; in Palatini’s theory by performing a complete analysis, their Dirac’s brackets among the dynamical variables are commutative and the cosmological constant can take the zero value, and there are no singularities. On the other hand, we developed the canonical covariant formalism, we constructed a gauge invariant symplectic form and we confirmed the results obtained by means of Dirac’s framework. In this manner, we have developed all tools to analyze the quantization aspects of the exotic action by using Dirac’s canonical method or canonical covariant formalism.

Finally, our analysis can be extended to others actions sharing the same equations of motion with three dimensional gravity, namely

$$S[A,e] = S'[A,e] + \frac{1}{\gamma} \tilde{S}[A,e],$$

where $S'[A,e]$ is the Palatini action, and

$$\tilde{S}[A,e] = \frac{1}{\sqrt{|\Lambda|}} \left[ \frac{1}{2} \int_M A^{IJ} \wedge dA_{IJ} + \frac{2}{3} A^{IK} \wedge A^{KL} \wedge A^{LI} \right] + s \sqrt{|\Lambda|} \int_M e_I \wedge De^I,$$

where $s$ is a constant, $\Lambda$ is the cosmological constant and $\gamma$ is an Immirzi-like parameter. In fact, the action gives rise to the same equations of motion of the Palatini action, however, from our analysis we can observe that Dirac’s brackets of the canonical variables $A$ and $e$ will be non-commutative. In was performed a canonical analysis on a smaller phase space context of
the action \([55]\), however, we have observed that it is mandatory to perform a detailed canonical analysis in order to know the complete symmetries. In fact, in \([16]\), it was not discussed the fundamental gauge symmetry of \([55]\), and the case of \(\Lambda = 0\) was studied on a smaller phase space, obtaining that \([55]\) is reduced to gravity without a cosmological constant; however, already there exists non-commutativity among the dynamical variables, thus, this is not a complete study because we have commented that Palatini’s gravity is commutative among their dynamical variables. In this manner, it is necessary to perform a complete Hamiltonian analysis in order to obtain a complete description of the theory \([18]\). Furthermore, it is important to comment that there exist formulations of 3D gravity where has been introduced correctly the Immirzi parameter \([19, 21]\). In fact, the parameter introduced in these papers, vanishes on half-shell, this is, when the torsion-free condition holds, which is also how the four-dimensional Immirzi parameter disappears from the Holst action. Hence, it will be useful to compare the difference among the results given in \([16]\) and those reported in \([19, 21]\).

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