Non-Commutative Geometry and the Strong Force

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Abstract
The restrictions imposed on the strong force in the ‘non-commutative standard model’ are examined. It is concluded that given the framework of non-commutative geometry and assuming the electroweak sector of the standard model many details of the strong force can be explained including its vectorial nature.

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1 Introduction

The standard model (SM) is extremely successful at predicting experimental results but from a theoretic/aesthetic point of view it is much less satisfactory. It contains many arbitrary inputs, in particular:

1. **The Higgs Sector.** Whilst the Yang-Mills sector is well motivated geometrically this results in massless particles so the Higgs sector has to be tagged on at the end by hand.

2. **Gauge Group and Group Representation.** The choice of gauge group is arbitrary and given a gauge group the choice of irreducible representation from the infinite number available is also arbitrary.

3. **Parity structure and Spontaneous Symmetry Breaking.** In the SM the SU(2) weak force is assumed to be maximally parity violating and to be spontaneously broken whilst the SU(3) force is assumed to be vectorial and to remain unbroken. In the SM there is no reason for this apparent link between parity structure and spontaneous symmetry breaking.

Non-commutative geometry goes a long way towards answering some of these problems. This paper does not contain the (by now well documented) details of non-commutative geometry. For an extremely thorough mathematical explanation of non-commutative geometry please see [1] or [3], for a clear explanation more suited to the physicist please see [8]. Non-commutative geometry is a generalization of the tools of classical geometry (eg the differential calculus, the notion of a metric space) to a much wider class of manifolds -the collection of ‘non-commutative manifolds’. Since there is a differential calculus on these manifolds it is possible to define a Yang-Mills action over them [4]. This is precisely the area that is of interest to particle physicists.

If a manifold $X = M_4 \times \{0, 1\}$ ie two copies of a Riemannian manifold separated by a finite distance, is considered (a manifold that cannot be treated with classical geometry due to the discreteness of the space) and a pure Yang-Mills theory is constructed over this manifold then something very interesting happens. A ‘gauge boson’ associated with the discreteness of the space occurs. This gauge boson has spin zero and a quartic potential of the form required for spontaneous symmetry breaking. So a pure Yang-Mills theory over such a space automatically has a natural Higgs sector. Furthermore this Higgs sector only arises if the representation of the algebra (associated with the forces -please see below for details) on the Hilbert space of left handed fermions is different to that of the representation on the space of right handed fermion. That is there is a clear explanation for the correlation between spontaneous symmetry breaking and parity structure that is observed in the standard model (as noted in point 3 above). ‘Non-commutative standard models’ also help to solve problem 2. The choice of gauge group is still more or less arbitrary (though the exceptional groups are ruled out -something that is not the
case in the usual formulation of the standard model). However the choice of representations of the gauge group is greatly restricted. The representation of the group comes from the representation of the algebra -this is a very restrictive condition. In general a group has an infinite number of unitary irreducible representations whereas an algebra has typically one or two. See [7] for a detailed analysis of this subject. Non-commutative geometry can therefore be seen to motivate many of the previously arbitrary features of the SM.

The topic of this paper is another constraint imposed on the SM by non-commutative geometry, namely that given the structure of the electroweak sector it follows automatically that the strong force is vectorial (ie non-chiral) and therefore that SU(3) is not broken. This is an explicit proof of an idea of Alain Connes.

2 Basic Framework of Non-Commutative Geometry

Non-commutative Yang-Mills models are constructed via a K-cycle \((\mathcal{A}, \mathcal{H}, D)\) with a real structure \(J\). \(\mathcal{A}\) is an involutive algebra whose group of unitaries is the gauge group of the model. \(\mathcal{H}\) is the Hilbert space of the fermions on which \(\mathcal{A}\) is represented by \(\rho\) and \(D\) is the generalized Dirac operator. \(D\) enables us to define a metric in the non-commutative setting. The real structure \(J\) is the non-commutative generalization of the charge conjugation operator [5]. It satisfies the following conditions:

1. \(JD = DJ\)
2. \(J^2 = \pm 1\)
3. \([\rho(y), J\rho(y')J^{-1}] = 0\) \(y, y' \in \mathcal{A}\)
4. \([[[D, \rho(y)], J\rho(y')J^{-1}] = 0\) \(y, y' \in \mathcal{A}\)

Condition 4, an important condition in the calculations in this paper can also be arrived at by considering Poincaré Duality. Classically all manifolds have an isomorphism known as Poincaré Duality, this is not the case for non-commutative manifolds where Poincaré Duality has to be imposed. The conditions for the existence of such an isomorphism are:

1. \([[[D, y], Jy']J^{-1}] = 0\) \(y, y' \in \mathcal{A}\)
2. \(Tr(\gamma^1[D, y]...[D, y^n]|D|^{-n}) = 0\) \(y^j \in \mathcal{A}\)

Note that condition 1 immediately above is the same as condition 4 on \(J\). For an explanation of condition 2 (not needed for this paper) see [2]. For the non-commutative standard model the K-cycle \((\mathcal{A}, \mathcal{H}, D)\) is taken:

\[
\mathcal{A} = C^\infty(M) \otimes [H \oplus C \oplus M_3(C)]
\]

\[
D = \partial \otimes 1 + \gamma_5 \otimes D_f
\]
where $H$ are the quaternions, $D_f$ is the fermionic mass matrix and $\mathcal{H}$ the Hilbert space of left and right fermions and anti-fermions. Note that in this case it is possible to split the space into a finite part and an infinite part. The finite part of the algebra and the generalized Dirac operator ($\mathcal{A}_f = H \oplus C \oplus M_3(C)$ and $D_f$ respectively) are represented on the finite space. In the calculations that follow just the finite sector is worked with, the full model is then obtained by tensoring with the infinite sector. For details and predictions of the non-commutative standard model see [9].

3 Calculations

Aim: the aim of these calculations is to show that within the framework of non-commutative geometry, if the electroweak sector is assumed the strong force is constrained to be vectorial. The assumptions of this calculation are then

$$[\rho(y), J \rho(y') J^{-1}] = 0 \quad y, y' \in \mathcal{A}$$

$$[[D, \rho(y)], J \rho(y') J^{-1}] = 0 \quad y, y' \in \mathcal{A}$$

these are as discussed a direct consequence of the framework of non-commutative geometry (conditions 3 and 4 in the previous section). Further the form of the electroweak sector is assumed, i.e., the algebra $\mathcal{A}$ is taken to be

$$\mathcal{A} = H \oplus C \oplus X$$

where $X$ (the algebra associated with the strong force) is assumed to be a simple algebra. The representation of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$\rho(y) = \begin{bmatrix} \rho_w(y) \\ \rho_s(y) \end{bmatrix} \quad y \in \mathcal{A}$$

where $y = (a, b, c) \in H \oplus C \oplus X$

$$\rho_w(a, b, c) = \begin{pmatrix} L \\ R \end{pmatrix} \begin{pmatrix} \rho_1(a) \\ \rho_2(b) \end{pmatrix}$$

with $\rho_1(a) = \begin{pmatrix} (u, d)_L \\ a \otimes 1_3 \\ \nu, e \end{pmatrix}$ and $\rho_2(b) = \begin{pmatrix} (u, d)_R \\ B \otimes 1_3 \\ \nu, e \end{pmatrix}$, $B = \begin{pmatrix} b \\ \bar{b} \end{pmatrix}$.

From experimental evidence [6] it is known that quarks exist in ‘threes’ (i.e., what we call colour triplets) of identical mass so the form of the fermionic mass matrix is known. The following
notation is used

\[ D_f = \begin{bmatrix}
0 & M & 0 & 0 \\
M^* & 0 & 0 & 0 \\
0 & 0 & 0 & M \\
0 & 0 & M^* & 0
\end{bmatrix} \text{ where } M = \begin{bmatrix}
M_q \otimes 1_3 & \\
& M_l
\end{bmatrix},
\]

\[ M_q = \begin{bmatrix}
m_u \\
m_d
\end{bmatrix}, \quad M_l = \begin{bmatrix}
m_\nu \\
m_e
\end{bmatrix}.\]

In summary then the only fact that is assumed about the strong force is that its associated algebra is simple.

Summary of calculations:
From \([y, Jy'J^{-1}] = 0\) it follows that \([\rho_w, \rho_s] = 0\) so \(\rho_s\) is block diagonal. Also since \(\rho_s\) is an algebra representation it cannot depend on \(a\) if it is to commute with \(\rho_w\). So

\[ \rho_s(b, c) = \begin{bmatrix}
\rho_3(b, c) \\
\rho_4(b, c)
\end{bmatrix} \]

the condition \([\rho_w, \rho_s] = 0\) is then equivalent to

\[ [\rho_1, \rho_3] = [\rho_2, \rho_4] = 0 \quad (i).\]

Now consider the second condition \([[D, y], Jy'J^{-1}] = 0\). Given (i) it is trivial matrix multiplication to show that this second condition is equivalent to

\[ M\rho_2\rho_4 - \rho_1M\rho_4 - \rho_3M\rho_2 + \rho_3\rho_1M = 0. \]

Now only \(\rho_1\) depends on \(a\) so this equation is actually two equations which must be satisfied separately

\[ M\rho_2\rho_4 - \rho_3M\rho_2 = 0 \]

and

\[ -\rho_1M\rho_4 + \rho_3\rho_1M = 0. \]

These equations contain essentially the same information so consider only the latter. If \(a\) is taken to be 1 then since \(\rho_1\) is a representation we have \(\rho_1(1) = 1\)

\[ \rho_3M = M\rho_4 \quad (ii).\]

Given (i) and Schur’s lemma it follows that as a matrix

\[ \rho_3 = 1_2 \otimes R \oplus 1_2K \quad R \in M_3(C), K \in C \]

furthermore it is known that \(\rho_3\) is a representation of \(C\) and \(X\) therefore \(R = c\) or \(\bar{c}\) and \(K = b\) or \(\bar{b}\). Similarly it can be seen that

\[ \rho_4(b, c) = 1_2 \otimes T \oplus \Delta \]
where $T = c$ or $\bar{c}$ and $\Delta = \begin{bmatrix} b & \bar{b} \\ b & b \end{bmatrix}, \begin{bmatrix} b & \bar{b} \\ b & \bar{b} \end{bmatrix}$ or $\begin{bmatrix} b & \bar{b} \\ \bar{b} & b \end{bmatrix}$.

Now imposing condition (ii) forces $R = T$ and $K_{12} = \Delta$\footnote[1]{ this latter equality only holds for $m_\nu \neq 0$ please see below for details on massless neutrinos} ie

$$\rho_s(b, c) = \begin{bmatrix} 1_2 \otimes c & b_{12} \\ b_{12} & 1_2 \otimes c \end{bmatrix}$$ or

$$\rho_s(b, c) = \begin{bmatrix} 1_2 \otimes c & \bar{b}_{12} \\ \bar{b}_{12} & 1_2 \otimes c \end{bmatrix}$$

Note then that $\rho_3 = \rho_4$ and that $\rho_s$ commutes with the fermionic mass matrix so it has been shown that the strong force is constrained to be vectorial. Also the quark doublet $\begin{bmatrix} u \\ d \end{bmatrix}$ is acted upon by $1_2 \otimes c$ where $c$ must be a $3 \times 3$ matrix. So it follows that the strong force doesn’t see flavour and that $X = M_3(C)$ or $M_3(R)$ ie that it’s gauge group is either $U(3)$ or a subgroup of $U(3)$.

It can be seen in the calculation above that I have assumed the presence of a massive right handed neutrino. This is not a necessary assumption, the right handed neutrino can be projected out at any stage without affecting the calculations. However if a massless right handed neutrino is assumed then it cannot be shown that the force on the right handed neutrino is the same as that on the left handed neutrino. See [10] for work on right handed neutrinos in the framework of non-commutative geometry.

Extra generations: The calculations above have been performed for $N_G = 1$, the generalization to $N_G = 3$ follows the same pattern. There is a slight complication that, due to the increased size of the Hilbert space a larger number of representations $\rho_3$ and $\rho_4$ satisfy (i), for instance the undesirable

$$\rho_3 = (1_2 \otimes c \otimes 1_3) \oplus (1_2 \otimes b_{13})$$

in which the strong force mixes the three generations of quarks rather then the three colours but these are ruled out by (ii). So it is found that on the quarks the strong force is again vectorial and that it is the same on all three generations. The force (associated to $C$) on the leptons is again vectorial but is not necessarily the same on every generation. As for $N_G = 1$ if an f-neutrino ($\nu_f, f = e, \mu, \tau$) is assumed to be massless then it cannot be shown that the force on the left f-neutrino is the same as that on the right f-neutrino.
4 Conclusions

So, given the framework of non-commutative geometry and details of the weak interaction it is possible to predict almost the exact form of the strong force. In particular that:

1. the strong force is vectorial

2. the strong force is the same for up as for down quarks

Number (1) is in my opinion the strongest and most surprising prediction and represents yet another instance in which non-commutative geometry helps to reduce the number of arbitrary inputs into the standard model. Point (2) follows more trivially from the fact that $\rho_s$ must be independent of $a \in H$.

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