Hegedüs, Gábor and Kasprzyk, Alexander M. (2011) Roots of Ehrhart polynomials of smooth Fano polytopes. Discrete & Computational Geometry, 46 (3). pp. 488-499. ISSN 1432-0444

Access from the University of Nottingham repository: http://eprints.nottingham.ac.uk/30735/1/1004.3817.pdf

Copyright and reuse:

The Nottingham ePrints service makes this work by researchers of the University of Nottingham available open access under the following conditions.

This article is made available under the University of Nottingham End User licence and may be reused according to the conditions of the licence. For more details see: http://eprints.nottingham.ac.uk/end_user_agreement.pdf

A note on versions:

The version presented here may differ from the published version or from the version of record. If you wish to cite this item you are advised to consult the publisher's version. Please see the repository url above for details on accessing the published version and note that access may require a subscription.

For more information, please contact eprints@nottingham.ac.uk
ROOTS OF EHRHART POLYNOMIALS OF SMOOTH FANO POLYTOPES

GÁBOR HEGEDŰS AND ALEXANDER M. KASPRZYK

Abstract. V. Golyshev conjectured that for any smooth polytope $P$ with $\dim(P) \leq 5$ the roots $z \in \mathbb{C}$ of the Ehrhart polynomial for $P$ have real part equal to $-1/2$. An elementary proof is given, and in each dimension the roots are described explicitly. We also present examples which demonstrate that this result cannot be extended to dimension six.

1. Introduction

Let $P$ be a $d$-dimensional convex lattice polytope in $\mathbb{R}^d$. Let $L_P(m) := |mP \cap \mathbb{Z}^d|$ denote the number of lattice points in $P$ dilated by a factor of $m \in \mathbb{Z}_{\geq 0}$. In general the function $L_P$ is a polynomial of degree $d$, called the Ehrhart polynomial \cite{Ehr67}.

The roots of Ehrhart polynomials have recently been the subject of much study (for example \cite{BHWO7, BD08, HHO10, Pfe07}), with a significant portion of this work being based on exhaustive computer calculations using the known classifications of polytopes. It has been conjectured in \cite{BDLD+05} that if $z \in \mathbb{C}$ is a root of $L_P$, then the real part $\Re(z)$ is bounded by $-d \leq \Re(z) \leq d - 1$; Braun has shown \cite{Bra08} that $z$ lies inside the disc centred at $-1/2$ of radius $d(d - 1/2)$.

Definition 1.1. A convex lattice polytope $P$ is called reflexive if the dual polytope $P^\vee := \{ u \in \mathbb{R}^d | \langle u, v \rangle \leq 1 \text{ for all } v \in P \}$ is also a lattice polytope.

There are many interesting and well-known characterisations of reflexive polytopes (for example \cite[Theorem 3.5]{HK10}). They are of particular relevance to toric geometry: reflexive polytopes correspond to Gorenstein toric Fano varieties (see \cite{Bat94}) and have been classified up to dimension four.

Any reflexive polytope $P$ satisfies

\begin{equation}
L_P(m) = L_{\partial P}(m) + L_P(m - 1) \text{ for all } m \in \mathbb{Z}_{>0},
\end{equation}

where $\partial P$ denotes the boundary of $P$. As a consequence, Macdonald’s Reciprocity Theorem \cite{Mac71} tells us that $L_P(-m - 1) = (-1)^d L_P(m)$. In particular we observe that the roots of $L_P$ are symmetrically distributed with respect to the line $\Re(z) = -1/2$.

2010 Mathematics Subject Classification. 52B20 (Primary); 52C07, 11H06 (Secondary).

Research supported in part by OTKA grant K77476.
Theorem 1.2 ([BH07, Proposition 1.8]). Let \( P \) be a \( d \)-dimensional convex lattice polytope such that for all roots \( z \) of \( L_P \), \( \text{Re}(z) = -1/2 \). Then, up to unimodular translation, \( P \) is a reflexive polytope with \( \text{vol}(P) \leq 2^d \).

Theorem 1.3 ([HHO10, Theorem 0.1]). In each dimension \( d \) there exists a reflexive polytope \( P \) such that if \( z \in \mathbb{C} \setminus \mathbb{R} \) is a root of \( L_P \) then \( \text{Re}(z) = -1/2 \).

Definition 1.4. A \( d \)-dimensional convex lattice polytope \( P \) is called smooth if the vertices of any facet of \( P \) form a \( \mathbb{Z} \)-basis of the ambient lattice \( \mathbb{Z}^d \).

Clear any smooth polytope is simplicial and reflexive. Smooth polytopes are in bijective correspondence with non-singular toric Fano varieties, and have been classified up to dimension eight [Øbr07].

V. Golyshev conjectured in [Gol09, §5] that, for any smooth polytope \( P \) of dimension \( d \leq 5 \), the roots \( z \in \mathbb{C} \) of \( L_P \) satisfy \( \text{Re}(z) = -1/2 \) (the “canonical line hypothesis”). Notice that it is not required that \( z \notin \mathbb{R} \). We prove Golyshev’s conjecture without resorting to the known classifications – see Sections 2 and 3 below.

Theorem 1.5 (Golyshev). Let \( P \) be a smooth polytope of dimension \( d \leq 5 \). If \( z \in \mathbb{C} \) is a root of \( L_P(m) \) then \( \text{Re}(z) = -1/2 \).

Explicit descriptions of the roots are given in Corollaries 2.6 and 3.8. We summarise them in the following theorem.

Theorem 1.6. Let \( P \) be a smooth \( d \)-dimensional polytope, and suppose that \( z = -1/2 + \beta i \in \mathbb{C} \) is a root of \( L_P \). If \( d = 2 \) then
\[
\beta^2 = -\frac{1}{4} + \frac{2}{f_0}.
\]
If \( d = 3 \) then \( \beta = 0 \) or
\[
\beta^2 = -\frac{1}{4} + \frac{6}{f_0 - 2}.
\]
If \( d = 4 \) then
\[
\beta^2 = -\frac{17}{4} + \frac{3b_2}{b_2 - 2f_0} \pm \sqrt{1 - \frac{12(f_0 + 2)}{b_2 - 2f_0} + \frac{36f_0^2}{(b_2 - 2f_0)^2}}.
\]
If \( d = 5 \) then \( \beta = 0 \) or
\[
\beta^2 = -\frac{5}{4} + \frac{10(f_0 - 2)}{6 + b_2 - 4f_0} \pm \sqrt{1 - \frac{20(f_0 + 4)}{6 + b_2 - 4f_0} + \frac{100(f_0 - 2)^2}{(6 + b_2 - 4f_0)^2}}.
\]

The following example demonstrates that we cannot extend Theorem 1.5 to dimension 6.

Example 1.7. There exist exactly four smooth polytopes in dimension six having roots \( z \) of the Ehrhart polynomial such that \( \text{Re}(z) \neq -1/2 \); in each case \( z \notin \mathbb{R} \). The polytopes have IDs 1895, 1930, 4853, and 5817 in the Graded Ring Database.\[\text{http://grdb.lboro.ac.uk/search/toricsmooth?id_cmp=inkid=1895,1930,4853,5817}\]
are:

\[
\begin{align*}
1 + \frac{31}{10}m + \frac{257}{60}m^2 + \frac{5}{2}m^3 + \frac{19}{12}m^4 + \frac{2}{5}m^5 + \frac{2}{15}m^6, \\
1 + \frac{7}{2}m + \frac{175}{36}m^2 + \frac{35}{12}m^3 + \frac{35}{18}m^4 + \frac{7}{12}m^5 + \frac{7}{36}m^6, \\
1 + \frac{7}{2}m + \frac{21}{4}m^2 + \frac{15}{4}m^3 + \frac{5}{2}m^4 + \frac{3}{4}m^5 + \frac{1}{4}m^6, \\
1 + \frac{31}{10}m + \frac{257}{60}m^2 + \frac{5}{2}m^3 + \frac{19}{12}m^4 + \frac{2}{5}m^5 + \frac{2}{15}m^6.
\end{align*}
\]

The second polytope has roots where Re\((z)\) > 0, and where Re\((z)\) < −1. This demonstrates that the more general “canonical strip hypothesis” does not hold in dimension six.

2. Dimensions Two and Three

One of the fundamental pieces of numerical data associated with a polytope is the \(f\)-vector, which enumerates the number of faces of \(P\). We begin by deriving an expression for the Ehrhart polynomial of a smooth polytope in terms of its \(f\)-vector.

**Definition 2.1.** Let \(P\) be a \(d\)-dimensional convex polytope. Define \(f_{-1} := 1\), \(f_d := 1\), and \(f_i\) equal to the number of \(i\)-dimensional faces of \(P\), for any \(0 \leq i \leq d - 1\). The \(f\)-vector of \(P\) is the sequence \((f_{-1}, f_0, \ldots, f_d)\).

**Lemma 2.2.** Let \(P\) be a \(d\)-dimensional smooth polytope. Then

\[
L_P(m) = \sum_{i=-1}^{d-1} f_i \binom{m}{i+1} \quad \text{and} \quad L_{\partial P}(m) = \sum_{i=0}^{d-1} f_i \binom{m-1}{i}.
\]

**Proof.** Clearly

\[
L_{\partial P}(m) = f_0 + \sum_F \left| (mF)^\circ \cap \mathbb{Z}^d \right|,
\]

where the sum is taken over all \(i\)-dimensional faces \(F\) of \(P\), \(i > 0\), and \(Q^\circ\) denotes the (relative) interior of \(Q\). Since \(P\) is smooth, \(F \cap \mathbb{Z}^d\) forms part of a basis for the underlying lattice \(\mathbb{Z}^d\) for any face \(F\). Hence

\[
L_{\partial P}(m) = \sum_{i=0}^{d-1} f_i \binom{m-1}{i}.
\]
To calculate $L_P(m)$ we make use of (1.1):

$$L_P(m) = 1 + \sum_{k=1}^{m} \frac{m}{i} \binom{k-1}{i} = 1 + \sum_{i=0}^{d-1} f_i \binom{m}{i+1}.$$
Proof. $d = 2$: By Corollary 2.3 we know that

$$L_P(m) = 1 + \frac{1}{2}f_0m + \frac{1}{2}f_0m^2.$$ 

Let $\alpha + \beta i \in \mathbb{C}$ be a root of $L_P$, where $\alpha, \beta \in \mathbb{R}$. Assume that $\beta \neq 0$. By considering the imaginary part we obtain

$$\beta(1 + 2\alpha) = 0,$$

hence $\alpha = -1/2$ as required. The real part simplifies to

$$\beta^2 = \frac{2}{f_0} - \frac{1}{4}.$$ 

Theorem 2.4 tells us that this is always positive, thus we obtain both roots of $L_P$.

$d = 3$: In this case Corollary 2.3 tells us that

$$L_P(m) = 1 + \frac{1}{6}(f_0 + 10)m + \frac{1}{2}(f_0 - 2)m^2 + \frac{1}{3}(f_0 - 2)m^3,$$

giving real and imaginary parts:

(2.1) \quad $$1 + \frac{1}{6}(f_0 + 10)\alpha + \frac{1}{2}(f_0 - 2)(\alpha^2 - \beta^2) + \frac{1}{3}(f_0 - 2)(\alpha^2 - 3\beta^2)\alpha = 0,$$

(2.2) \quad $$\frac{1}{6}(f_0 + 10)\beta + (f_0 - 2)\alpha\beta + \frac{1}{3}(f_0 - 2)(3\alpha^2 - \beta^2)\beta = 0.$$

Assume that $\beta \neq 0$. Equation (2.2) gives us

(2.3) \quad $$(f_0 - 2)\beta^2 = \frac{1}{2}f_0 + 5 + 3(f_0 - 2)\alpha + 3(f_0 - 2)\alpha^2.$$ 

Substituting (2.3) into (2.1) gives

$$\frac{1}{12}(2\alpha + 1) \left(4(f_0 - 2)(2\alpha + 1)^2 + 26 - f_0\right).$$

Clearly $\alpha = -1/2$ is one possible solution. The discriminant of $4(f_0 - 2)(2\alpha + 1)^2 + 26 - f_0$, regarded as a quadratic in $2\alpha + 1$, is $16(f_0 - 2)(f_0 - 26)$. This is negative when $2 \leq f_0 \leq 26$, and by Theorem 2.4 this covers all possible values of $f_0$. Hence $\alpha = -1/2$ is the only solution.

The values for $\beta$ are determined by (2.3):

$$\beta^2 = \frac{26 - f_0}{4f_0 - 8}.$$ 

If we allow $\beta = 0$ then (2.1) becomes

$$\frac{1}{24}(2\alpha + 1) \left((f_0 - 2)(2\alpha + 1)^2 + 26 - f_0\right).$$

Once more the discriminant of the quadratic component tells us that the only solution is when $\alpha = -1/2$. \qed

The proof of Proposition 2.5 gives us explicit equations for the roots of $L_P$. 

ROUTES OF EHRHART POLYNOMIALS OF SMOOTH FANO POLYTOPES 5
Corollary 2.6. Let $P$ be a smooth $d$-dimensional polytope, and suppose that $z = -1/2 + \beta i \in \mathbb{C}$ is a root of $L_P$. If $d = 2$ then
$$\beta^2 = -\frac{1}{4} + \frac{2}{f_0}.$$ If $d = 3$ then $\beta = 0$ or
$$\beta^2 = -\frac{1}{4} + \frac{6}{f_0 - 2}.$$

3. Dimensions Four and Five

In order to prove Theorem 1.5 in dimension 4 we require a some additional results. Throughout we write $b_2 := |\partial(2P) \cap \mathbb{Z}^d|$, where $d$ is the dimension of $P$.

Lemma 3.1 ([HK10, Corollary 4.4]). Let $P$ be a four-dimensional smooth polytope. Then
$$5f_0 - 10 \leq b_2 \leq 5f_0.$$

Lemma 3.2. Let $P$ be a four-dimensional smooth polytope. Then
$$(b_2 - 8f_0)^2 > 24(b_2 - 2f_0).$$

Proof. From Lemma 3.1 we have that
$$(b_2 - 8f_0)^2 = (b_2 - 16f_0)b_2 + 64f_0^2 \geq (10 - 5f_0)(10 + 11f_0) + 64f_0^2 = 9f_0^2 + 60f_0 + 100 = (3f_0 + 10)^2.$$ Clearly $72f_0 < (3f_0 + 10)^2$, and since $24(b_2 - 2f_0) \leq 72f_0$ (by Lemma 3.1) we obtain the result. □

We shall also make use of the following trivial observation:

Lemma 3.3. Let $g(x) := ax^4 + bx^2 + c \in \mathbb{R}[x]$ be a polynomial such that $a > 0$, $b < 0$, $c > 0$ and $b^2 - 4ac > 0$. Then $g$ has four distinct real roots.

Proposition 3.4. Let $P$ be a four-dimensional smooth polytope. If $z \in \mathbb{C}$ is a root of $L_P(m)$ then $\text{Re}(z) = -1/2$.

Proof. In four dimensions the Ehrhart polynomial simplifies to
$$L_P(m) = 1 + \frac{1}{12}(8f_0 - b_2)m(m + 1) - \frac{1}{24}(2f_0 - b_2)m^2(m + 1)^2.$$ If $z = \alpha + i\beta$ is a root of $L_P$ then, by considering the real and imaginary parts, we obtain
$$(3.1) \quad 24 + 12f_0((\alpha + 1)\alpha - \beta^2) - (2f_0 - b_2)\alpha(\alpha + 1)(\alpha(\alpha + 1) - 2 - 6\beta^2) - (2f_0 - b_2)\beta^2(\beta^2 + 1) = 0,$$ and
$$(3.2) \quad (6f_0 - (2f_0 - b_2) ((\alpha + 1)\alpha - \beta^2 - 1)) (2\alpha + 1)\beta = 0.$$ Clearly $\alpha = -1/2$ is a possible solution to equation (3.2), in which case $\beta$ satisfies (by (3.1))
$$(3.3) \quad 16(b_2 - 2f_0)\beta^4 + 8(5b_2 - 34f_0)\beta^2 + 3(128 + 3b_2 - 22f_0) = 0.$$
This quadratic in $\beta^2$ has distinct real solutions if and only if

$$(b_2 - 8f_0)^2 - 24(b_2 - 2f_0) > 0.$$ 

By Lemma 3.2 we know that this is always true.

Now we consider the signs of the coefficients of $(3.3)$. The leading coefficient is equal to $1/2f_2$, and so is positive. The coefficient of $\beta^2$ is always negative by Lemma 3.1, and the constant term is positive by Lemma 3.2. Hence, by Lemma 3.3 there are four distinct real solutions to equation $(3.1)$.

We have found four distinct roots when $\text{Re}(z) = -1/2$. Since $LP$ is of degree four, we are done. $\square$

Finally we consider dimension five.

**Lemma 3.5 ([HK10, Corollary 4.4]).** Let $P$ be a five-dimensional smooth polytope. Then

$$42f_0 - 105 \leq 7b_2 \leq 52f_0 - 90.$$ 

**Lemma 3.6.** Let $P$ be a five-dimensional smooth polytope. Then

$$100(f_0 - 2)^2 + (6 + b_2 - 4f_0)^2 > 20(6 + b_2 - 4f_0)(f_0 + 4).$$

**Proof.** We begin by observing that the statement is equivalent to

$$(10(f_0 - 2) - (6 + b_2 - 4f_0))^2 > 120(6 + b_2 - 4f_0),$$

which in turn is equivalent to

$$(13(f_0 - 2) - (b_2 - f_0))(13(f_0 - 2) - (b_2 - f_0) + 120) > 1200(f_0 - 2).$$

From Lemma 3.5 we have that

$$13(f_0 - 2) - (b_2 - f_0) \geq \frac{46}{7}f_0 - \frac{92}{7},$$

which is always positive since $f_0 \geq 6$. Hence

$$-(13(f_0 - 2) - (b_2 - f_0))(13(f_0 - 2) - (b_2 - f_0) + 120) - 1200(f_0 - 2) \geq \left(\frac{46}{7}f_0 - \frac{92}{7}\right)\left(\frac{46}{7}f_0 - \frac{92}{7} + 120\right) - 1200(f_0 - 2)$$

$$= \frac{4}{49}(f_0 - 2)(529f_0 - 6098).$$

This is positive for all $f_0 \geq 12$.

To prove the inequality when $f_0 \leq 11$ we consider

$$-(13(f_0 - 2) - (b_2 - f_0))(13(f_0 - 2) - (b_2 - f_0) + 120) - 1200(f_0 - 2) \geq (13(f_0 - 2) - (b_2 - f_0))\left(\frac{46}{7}f_0 - \frac{92}{7} + 120\right) - 1200(f_0 - 2)$$

$$= -\frac{2}{7}(23f_0 + 374)b_2 + \frac{4}{7}(161f_0^2 + 219f_0 - 662).$$

We wish to show that

$$-\frac{2}{7}(23f_0 + 374)b_2 + \frac{4}{7}(161f_0^2 + 219f_0 - 662) > 0$$
whenever $6 \leq f_0 \leq 11$. It is enough to prove that, in the given range,

\begin{equation}
\label{3.4}
b_2 < \frac{2(161f_0^2 + 219f_0 - 662)}{23f_0 + 374}.
\end{equation}

Now

\[ b_2 - f_0 = f_1 \leq \frac{f_0}{2}, \]
and so

\[ b_2 \leq \frac{f_0(f_0 + 1)}{2}. \]

We shall show that

\[ \frac{f_0(f_0 + 1)}{2} < \frac{2(161f_0^2 + 219f_0 - 662)}{23f_0 + 374}. \]

But this is trivial; the cubic

\[ f_0(f_0 + 1)(23f_0 + 374) - 4(161f_0^2 + 219f_0 - 662) \]

\[ = 23f_0^3 - 247f_0^2 - 502f_0 + 2648 \]

is negative when $6 \leq f_0 \leq 11$, hence equation \eqref{3.4} holds. \hfill \Box

**Proposition 3.7.** Let $P$ be a five-dimensional smooth polytope. If $z \in \mathbb{C}$ is a root of $L_P(m)$ then $\text{Re}(z) = -1/2$.

**Proof.** Let $z = \alpha + i\beta \in \mathbb{C}$ be a root of $L_P$, where $P$ is a five-dimensional smooth polytope. By Corollary \ref{2.3} we see that $\alpha$ and $\beta$ must satisfy

\begin{equation}
\label{3.5}
(2\alpha + 1)\left( (6 + b_2 - 4f_0)((\alpha - 1)\alpha(\alpha + 1)(\alpha + 2) - 10(\alpha + 1)\alpha\beta^2 + 5(\beta^2 + 1)\beta^2) + 20(f_0 - 2)((\alpha + 1)\alpha - 3\beta^2) + 120 \right) = 0,
\end{equation}

\begin{equation}
\label{3.6}
(14f_0 - b_2 + 94)\beta + 5(16f_0 - b_2 - 30)\alpha\beta + 20(f_0 - 2)(3\alpha^2 - \beta^2)\beta - 10(4f_0 - b_2 - 6)(\alpha^2 - \beta^2)\alpha\beta - (4f_0 - b_2 - 6)(5\alpha^4 - 10\alpha^2\beta^2 + \beta^4)\beta = 0.
\end{equation}

Clearly $\alpha = -1/2, \beta = 0$ is always a solution. Suppose that $\alpha = -1/2$ and $\beta \neq 0$. Equation \eqref{3.5} holds, and from \eqref{3.6} we obtain

\begin{equation}
\label{3.7}
16(6 + b_2 - 4f_0)\beta^4 + 40(22 + b_2 - 12f_0)\beta^2 + 2134 + 9b_2 - 116f_0 = 0.
\end{equation}

This quadratic in $\beta^2$ has distinct real solutions if and only if

\[ 100(f_0 - 2)^2 + (6 + b_2 - 4f_0)^2 > 20(6 + b_2 - 4f_0)(f_0 + 4), \]

which holds by Lemma \ref{3.6}.

As in the four-dimensional case we consider the signs of the coefficients of \eqref{3.7}. The leading coefficient equals $1/2f_4$ and so is positive. The coefficient of $\beta^2$ is negative by Lemma \ref{3.5} and the fact that $f_0 \geq 6$, and the constant term is positive (again by Lemma \ref{3.5}). Thus, by Lemma \ref{3.3} equation \eqref{3.7} has four distinct real solutions.

Hence we have found all five roots of $L_P$, and in each case $\text{Re}(z) = -1/2$ as required. \hfill \Box

From equations \eqref{3.3} and \eqref{3.7} we have
Corollary 3.8. Let $P$ be a smooth $d$-dimensional polytope, and suppose that $z = -1/2 + \beta i \in \mathbb{C}$ is a root of $L_P$. If $d = 4$ then
\[
\beta^2 = -\frac{17}{4} + \frac{3b_2}{b_2 - 2f_0} \pm \sqrt{1 - \frac{12(f_0 + 2)}{b_2 - 2f_0} + \frac{36f_0^2}{(b_2 - 2f_0)^2}}.
\]
If $d = 5$ then $\beta = 0$ or
\[
\beta^2 = -\frac{5}{4} + \frac{10(f_0 - 2)}{6 + b_2 - 4f_0} \pm \sqrt{1 - \frac{20(f_0 + 4)}{6 + b_2 - 4f_0} + \frac{100(f_0 - 2)^2}{(6 + b_2 - 4f_0)^2}}.
\]

4. Concluding Remarks

In four dimensions one can prove Theorem 1.5 without knowing the explicit equation for the Ehrhart polynomial. We require the following result.

Proposition 4.1 ([BHW07, Proposition 1.9]). Let $P$ be a four-dimensional reflexive polytope. Every root $z \in \mathbb{C}$ of $L_P(m)$ has $\text{Re}(z) = -1/2$ if and only if
\begin{enumerate}[(i)]
    \item $2 |\partial P \cap \mathbb{Z}^4| \leq 9 \text{vol}(P) + 16$, and
    \item $(|\partial P \cap \mathbb{Z}^4| - 4 \text{vol}(P))^2 \geq 16 \text{vol}(P)$.
\end{enumerate}

Alternative proof in dimension four. First we show that condition (i) of Proposition 1.1 is satisfied. Since $P$ is smooth, $f_0 = |\partial P \cap \mathbb{Z}^4|$. It follows from Lemma 3.1 that $15f_0 \leq 3b_2 + 30$. Hence $9f_0 \leq 3(b_2 - 2f_0) + 30$. By Theorem 2.4 we have that $f_0 \leq 12$, giving us the (very crude) inequality
\begin{equation}
16f_0 < 3(b_2 - 2f_0) + 128.
\end{equation}
In four dimensions we have that $f_3 = b_2 - 2f_0$ ([HK10, Theorem 4.2]) and, since $P$ is smooth, $f_3 = 24 \text{vol}(P)$. Substituting into equation (4.1) gives condition (i).

That Proposition 1.1 (ii) holds is immediate from Lemma 3.2 and the fact that $b_2 - 2f_0 = 24 \text{vol}(P)$.

Theorem 1.6 tells us that in order to compute the roots of the Ehrhart polynomial we need only know $f_0$ and, in dimensions four and five, $b_2 := |\partial(2P) \cap \mathbb{Z}^d|$. Clearly $f_0 \geq d + 1$, and Theorem 2.4 provides a sharp upper bound. The values of $b_2$ can be calculated from Øbro’s classification [Øbr07]. The possible pairs $(f_0, b_2)$ are reproduced in Tables 1 and 2.

Acknowledgments. The authors wish to express their gratitude to Alessio Corti for alerting them to [Gol09].
Table 2. The possible pairs \((f_0, b_2)\) for the 866 five-dimensional smooth polytopes.

| \(f_0\) | 6 | 7 | 7 | 8 | 8 | 8 | 8 | 9 | 9 | 9 | 9 | 9 | 10 | 10 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(b_2\) | 21 | 27 | 28 | 33 | 34 | 35 | 36 | 40 | 41 | 42 | 43 | 44 | 46 | 49 | 50 |

| \(f_0\) | 10 | 10 | 10 | 11 | 11 | 11 | 11 | 12 | 12 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(b_2\) | 51 | 52 | 53 | 56 | 58 | 59 | 60 | 61 | 62 | 66 | 67 | 72 | 76 | 86 |

References

[Bat94] Victor V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. 3 (1994), no. 3, 493–535.

[BD08] Benjamin Braun and Mike Develin, *Ehrhart polynomial roots and Stanley’s non-negativity theorem*, Integer points in polyhedra—geometry, number theory, representation theory, algebra, optimization, statistics, Contemp. Math., vol. 452, Amer. Math. Soc., Providence, RI, 2008, pp. 67–78.

[BDLD+05] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle, and R. P. Stanley, *Coefficients and roots of Ehrhart polynomials*, Integer points in polyhedra—geometry, number theory, algebra, optimization, Contemp. Math., vol. 374, Amer. Math. Soc., Providence, RI, 2005, pp. 15–36.

[BHW07] Christian Bey, Martin Henk, and Jörö M. Wills, *Notes on the roots of Ehrhart polynomials*, Discrete Comput. Geom. 38 (2007), no. 1, 81–98.

[Bra08] Benjamin Braun, *Norm bounds for Ehrhart polynomial roots*, Discrete Comput. Geom. 39 (2008), no. 1-3, 191–193.

[Cas06] Cinzia Casagrande, *The number of vertices of a Fano polytope*, Ann. Inst. Fourier (Grenoble) 56 (2006), no. 1, 121–130.

[Ehr67] Eugène Ehrhart, *Sur un problème de géométrie diophantienne linéaire. II. Systèmes diophantiens linéaires*, J. Reine Angew. Math. 227 (1967), 25–49.

[Gol09] V. V. Golyshev, *On the canonical strip*, Uspekhi Mat. Nauk 64 (2009), no. 1(385), 139–140.

[HHO10] Takayuki Hibi, Akihiro Higashitani, and Hidefumi Ohsugi, *Roots of Ehrhart polynomials of Gorenstein Fano polytopes*, arXiv:1001.4165v1 [math.CO]

[HK10] Gábor Hegedüs and Alexander M. Kasprzyk, *The boundary volume of lattice polytopes*, arXiv:1002.2815v2 [math.CO].

[Mac71] I. G. Macdonald, *Polynomials associated with finite cell-complexes*, J. London Math. Soc. (2) 4 (1971), 181–192.

[Obro7] Mikkel Öbro, *An algorithm for the classification of smooth Fano polytopes*, arXiv:0704.0049v1 [math.CO], classifications available from http://grdb.lboro.ac.uk/

[Pfe07] Julian Pfeifle, *Gale duality bounds for roots of polynomials with nonnegative coefficients*, arXiv:0707.3010v2 [math.CO].

Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040 Linz, Austria

E-mail address: gabor.hegedues@oeaw.ac.at

School of Mathematics and Statistics, University of Sydney, Sydney NSW 2006, Australia

E-mail address: a.m.kasprzyk@usyd.edu.au