Classical behaviour in quantum mechanics: a transition probability approach

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Abstract

A formalism is developed for describing approximate classical behaviour in finite (but possibly large) quantum systems. This is done in terms of a structure common to classical and quantum mechanics, viz. a Poisson space with a transition probability. Both the limit where $\hbar \to 0$ in a fixed finite system and the limit where the size of the system goes to infinity are incorporated. In either case, classical behaviour is seen only for certain observables and in a restricted class of states.
1 Dedication

Professor Umezawa viewed physics in a unified way based on quantum field theory. In particular, classical physics emerges through symmetry breaking, which leads to condensation of Goldstone bosons and the accompanying boson transformation [1, 2]. However, symmetry breaking only occurs in infinite systems. Even if these exist, it is desirable to have a formalism that approximates the qualitative features normally associated with infinite systems in their finite approximants. This poses a difficult problem for any approach based on superselection rules (such as Umezawa’s), of which formally no trace is seen in finite systems. Moreover, it is not clear that all classical phenomena in Nature arise in the way described. Indeed, examples related to Bohr’s original correspondence principle, where the classical limit arises when certain quantum numbers become large, do not seem to be covered.

Inspired by Umezawa’s vision, we wish to present the first technical step towards an approach to these problems that avoids some of the difficulties mentioned. It is with sadness that we dedicate these pages to his memory.

2 Observables and states

In quantum mechanics without superselection rules the observables form the self-adjoint (Hermitian) part of \( B(\mathcal{H}) \), the algebra of all bounded operators on some Hilbert space \( \mathcal{H} \). In classical mechanics, the observables consist of real-valued functions on some phase space \( S \). In both cases, they are elements of a real vector space \( \mathfrak{A} \) on which a commutator and an anti-commutator are defined. The former is \( [A, B]_h = i(AB - BA)/h \) in the quantum case, and is the Poisson bracket \( \{f, g\} \) in the classical case, and the latter is \( A \circ B = (AB + BA)/2 \) in quantum mechanics and \( f \circ g = fg \) in classical mechanics.

The commutator satisfies the Jacobi identity, and both operations are intertwined by the Leibniz rule, which says that the commutator is a derivation of the anti-commutator. The only difference between classical and quantum lies in the associativity of \( \circ \): classically we have \( (f \circ g) \circ h - f \circ (g \circ h) = 0 \), whereas quantum-mechanically \( (A \circ B) \circ C - A \circ (B \circ C) \) equals \( h^2[B, [A, C]]/4 \). This discussion can be generalized to systems with superselection rules by replacing \( B(\mathcal{H}) \) with a general \( C^* \)-algebra \( \mathfrak{A} \), and taking a general Poisson manifold \( P \) rather than the symplectic space \( S \).

Quantization is described by a family of maps \( Q_h : C^\infty(P) \to \mathfrak{A} \) such that

\[
\lim_{h \to 0} ([Q_h(f), Q_h(g)]_h - Q_h(\{f, g\})) = 0
\]

(Dirac) and

\[
\lim_{h \to 0} (Q_h(f) \circ Q_h(g) - Q_h(fg)) = 0
\]

(von Neumann), for all reasonable functions \( f, g \) on \( P \). There is a natural equivalence relation between different quantizations, in that \( Q^1_h \) and \( Q^2_h \) are declared equivalent if \( Q^1_h(f) - Q^2_h(f) \to 0 \) for \( h \to 0 \) for all \( f \). This equivalence relation eliminates
operator-ordering ambiguities. The classical limit of quantum mechanics may be described in similar terms, cf. [4, 5].

While the algebraic formulation provides a nice unified description of classical and quantum mechanics, it is more useful for the problem at hand to give a dual description in terms of pure states. Classically, we can look at \( P \) as the space of pure states of the system, which is equipped with a Poisson structure. This amounts to the specification of a Poisson bracket on a suitable set of continuous functions on \( P \), or, equivalently, may be described in terms of a certain geometric structure directly on \( P \) [3].

In quantum mechanics (without superselection rules to start with) we may start from the pure state space \( \mathbb{P}H \) (the projective space of the Hilbert space \( H \), obtained from the latter by imposing \( \langle \psi | \phi \rangle = 1 \) and identifying \( |\psi\rangle \) with \( \exp(i\alpha)|\psi\rangle \); we denote the image of \( |\psi\rangle \in H \) in \( \mathbb{P}H \) by \( \psi \)). There is a natural Poisson structure on \( \mathbb{P}H \) (which derives from its Fubini-Study Kähler structure, cf. [3] or section 5 below). If we associate a function \( f_A \) on \( \mathbb{P}H \) to each Hermitian operator \( A \) on \( H \), defined by
\[
 f_A(\psi) = \langle \psi | A | \psi \rangle,
\]
then this Poisson structure is specified by the rule
\[
 \{f_A, f_B\} = f_{[A,B]}(\psi)\hbar.
\]

This time, however, we cannot reconstruct the system from the space \( \mathbb{P}H \) with its Poisson structure, since we have not incorporated the fact that in quantum mechanics not all functions on \( \mathbb{P}H \), but only those of the form \( f_A \), are observables. Also, we do not yet know how to compute the anti-commutator \( f_A \circ f_B \). This additional information turns out to be encoded in the transition probabilities on \( \mathbb{P}H \).

3 Transition probability spaces

A transition probability space is a set \( \mathcal{P} \) with a function \( p \) from \( \mathcal{P} \times \mathcal{P} \) to the interval \([0, 1]\), such that \( p(\psi, \phi) = 1 \) implies \( \psi = \phi \). In standard quantum mechanics, \( \mathcal{P} = \mathbb{P}H \) and \( p(\psi, \phi) = |\langle \psi | \phi \rangle|^2 \). If there are superselection rules, the pure state space is the union of all sectors \( \mathbb{P}H_S \); the transition probabilities between two different sectors identically vanish. In classical mechanics, \( \mathcal{P} = P \) and \( p^cl(\psi, \phi) \neq 0 \) only if \( \psi = \phi \) (in which case it equals 1). Hence each point forms its own little superselection sector, cf. [6]. See [7] for general information on transition probability spaces.

To capture classical and quantum mechanics in one picture, the transition probability space \( \mathcal{P} \) should carry a unitary Poisson structure. This means the following: given a fixed point \( \psi \in \mathcal{P} \), we define a function \( p_\psi \) on \( \mathcal{P} \) by \( p_\psi(\phi) = p(\psi, \phi) \). (In standard quantum mechanics, this function is represented by the operator \( |\psi\rangle\langle \psi| \).) The Poisson structure then leads to a vector field \( \xi_\psi \) by the usual rule \( \xi_\psi f = \{p_\psi, f\} \). This vector field defines a flow \( \phi \rightarrow \phi(t) \) for each \( \phi \in \mathcal{P} \), such that \( d\phi(t)/dt = \xi_\phi(\phi(t)) \).

The unitarity condition is a compatibility requirement between the transition probabilities and the Poisson structure, viz. that for every \( \psi \) this flow must leave \( p \) invariant, in the sense that \( p(\phi_1(t), \phi_2(t)) = p(\phi_1, \phi_2) \) for all \( t \) and all \( \phi_1, \phi_2 \in \mathcal{P} \). See [8] for details.

Given the transition probabilities, in standard quantum mechanics the Poisson
structure on $\mathbb{P}\mathcal{H}$ is actually determined by unitarity, up to a multiplicative constant, which may be identified with $1/\hbar$. (If there are superselection rules, each sector could in principle have its own $\hbar$, but this possibility does not seem to be realized in Nature.) In classical mechanics any Poisson structure is unitary.

It is possible to characterize quantum mechanics (with superselection rules) in terms of certain axioms on a Poisson space with a transition probability, and the algebra of observables $\mathfrak{A}$ may then be reconstructed from $\mathcal{P}$ [3]. An observable is here regarded as a function on $\mathbb{P}$ (rather than e.g. an operator on a Hilbert space), and the basic point is that every observable is a linear combination of functions of the type $p_\psi$. Moreover, every observable $f = \sum_i \mu_i p_{\psi_i}$ (where all $\mu_i \in \mathbb{R}$), has a spectral representation $f = \sum \lambda_\alpha p_\alpha$, where $p(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. This spectral theorem holds for more general transition probability spaces than those describing quantum mechanics; even in the latter case one is led to a new proof of the usual spectral theorem (which is a restatement of the one used here), which does not use Hilbert space theory [3].

The spectral representation is used to define the anti-commutator via $f^2 \equiv f \circ f = \sum_\alpha \lambda_\alpha^2 p_\alpha$, and then $f \circ g = ((f + g)^2 - (f - g)^2)/4$. Unitarity then implies that the commutator (defined as the Poisson bracket) and the anti-commutator are related by the Leibniz identity. The reader may immediately check that in classical mechanics (where the sums extend over an uncountable number of terms) every observable is automatically in spectral form, so that $f \circ g = fg$ in the usual sense of pointwise multiplication. In quantum mechanics, however, one recovers the usual anti-commutator in the above way. Finally, the norm of an observable is simply given by $\|f\| = \sup_{\psi \in \mathbb{P}} |f(\psi)|$.

4 Classical germs

From the point of view of Poisson spaces with a transition probability, quantization theory and the classical limit of quantum mechanics are described in one and the same way. One starts from a Poisson manifold $P$ (the pure state space of the classical system) and a quantum pure state space $\mathbb{P}$ (e.g., $\mathbb{P} = \mathbb{P}\mathcal{H}$ for some Hilbert space $\mathcal{H}$). The basic ingredient is a family of injections $q_\hbar : P \rightarrow \mathbb{P}$ (defined for $\hbar$ in a certain interval $(0, \hbar_0)$) which satisfy

$$\lim_{\hbar \rightarrow 0} p(q_\hbar(x), q_\hbar(y)) = \delta_{xy}$$

for all $x, y \in P$ (recall that $\delta_{xy}$ is just the classical transition probability $p^c(x, y)$). Thus each $q_\hbar$ embeds the classical state space into its quantum counterpart, in such a way that for small $\hbar$ the classical points all become almost mutually orthogonal. Such a family $q_\hbar$ generalizes the notion of a coherent state, and may be referred to as a classical germ [4] (also cf. [2]). Two germs $q^1_\hbar$ and $q^2_\hbar$ may be declared equivalent if $\lim_{\hbar \rightarrow 0} p(q^1_\hbar(x), q^2_\hbar(x)) = 1$ for all $x \in P$.

Consider a function $f$ on $P$ which is nonzero only at a finite number of points; its spectral representation is $f = \sum_x f(x) p^c_x$. Using a classical germ, we can define a quantization of $f$ by $Q_\hbar(f) = \sum_x f(x) p_{q_\hbar(x)}$. For small $\hbar$ the r.h.s. will approximate...
the spectral representation of $Q_{\hbar}(f)$, so that $Q_{\hbar}(f)^2$ is approximately $\sum_x f(x)^2p_{\hbar}(x)$, which equals $Q_{\hbar}(f^2)$. Hence $Q_{\hbar}(f)^2 \to Q_{\hbar}(f^2)$ for $\hbar \to 0$, which if true for all $f$ is equivalent to von Neumann’s condition. In practice, $f$ will have support in an uncountable set, and the sum will be replaced by an integral. The above prescription then suggests quantizations of the type $Q_{\hbar}(f) = \int_{\rho} d\mu(x) f(x)p_{\hbar}(x)$, where the measure $\mu$ is normalized by the requirement $Q_{\hbar}(1) = 1$, and, in case that $P$ is symplectic, is usually proportional to the Liouville measure. Since in usual notation $p_{\hbar}(x) = |q_{\hbar}(x)\rangle\langle q_{\hbar}(x)|$, we see that coherent state quantization schemes (cf. [9]) are a special case of this.

Given an equivalence class of classical germs, one can consider a family $A_{\hbar}$ of observables on $\mathcal{P}$ (we regard an observable as a function on $\mathcal{P}$, and will not distinguish between an operator $A$ and its associated function $f_A$) which depend on $\hbar$ in such a way that $\lim_{\hbar \to 0} A_{\hbar}(q_{\hbar}(x)) \equiv A_0(x)$ exists for all $x$, and defines a continuous function $A_0$ on $\mathcal{P}$. (In that case, the family $A_{\hbar}$ may be seen as a quantization of $A_0$.) We will refer to such a family as a classical funnel.

For any function $F$ on $\mathcal{P}$, denote by $q_{\hbar}^*F$ the function on $\mathcal{P}$ defined by $q_{\hbar}^*F(x) = F(q_{\hbar}(x))$. We then impose the requirement on the classical germ that for those classical funnels $A_{\hbar}$ and $B_{\hbar}$ for which the limit functions $A_0$ and $B_0$ are differentiable, $q_{\hbar}^*[A_{\hbar}, B_{\hbar}]_\hbar$ approaches the Poisson bracket $\{q_{\hbar}^*A_{\hbar}, q_{\hbar}^*B_{\hbar}\}$ when $\hbar \to 0$. Cf. [10] for the coherent state analogue, and [3] for necessary conditions on $q_{\hbar}$. This requirement is the state space analogue of Dirac’s condition.

5 Classical germs from coherent states

An interesting class of examples of classical germs comes from a particular type of coherent states, which we will now describe in a geometric way. The material in the next paragraph may be found in [3], and a heuristic presentation is in [10].

Any Hilbert space $\mathcal{H}^x$ carries a canonical symplectic form $\omega_{\hbar}$. If we identify the tangent space $T\mathcal{H}^x$ with $\mathcal{H}^x$, this is defined by $\omega_{\hbar}(|\psi\rangle, |\varphi\rangle) = -2\hbar \Im \langle \varphi | \psi \rangle$ (we assume $\hbar \neq 0$). It quotients to the projective space $\mathbb{P}\mathcal{H}^x$, on which it gives the Fubini-Study form. Now let $U^x$ be an irreducible unitary representation of a connected Lie group $G$ on $\mathcal{H}^x$. This naturally defines an action of $G$ (which we denote by the same symbol $U^x$) on $\mathbb{P}\mathcal{H}^x$, which turns out to be strongly Hamiltonian. Thus we find an equivariant momentum map $J^x_{\hbar} : \mathbb{P}\mathcal{H}^x \to \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of $G$, and $\mathfrak{g}^*$ its dual. This just means that for any generator $T_a$ of $\mathfrak{g}$ we have a function $< J^x_{\hbar}(T_a) > = J_a$ on $\mathbb{P}\mathcal{H}^x$ which generates the action of $G$ on $\mathbb{P}\mathcal{H}^x$ as a canonical transformation. The Poisson brackets of the $J_a$ reproduce the Lie algebra , i.e., $\{J_a, J_b\} = f_{ab}^c J_c$. This is equivalent to global equivariance, that is, $J^x_{\hbar} \circ U^x = Co \circ J^x_{\hbar}$, where $Co$ is the co-adjoint action of $G$ on $\mathfrak{g}^*$. Explicitly, $J_a(\psi) = -i\hbar \langle \psi | dU^x(T_a) | \psi \rangle$, where $\langle \psi | \psi \rangle = 1$, and $dU^x(T_a)$ is the anti-Hermitian representative of $T_a$ (so that $[dU^x(T_a), dU^x(T_b)] = f_{ab}^c dU^x(T_c)$).

Take a fixed $\psi_0 \in \mathbb{P}\mathcal{H}^x$. It may happen that its orbit $U^x(G)\psi_0$ is a symplectic subspace of $\mathbb{P}\mathcal{H}^x$. In that case, $U^x(G)\psi_0$ is a covering space of the co-adjoint orbit $\mathcal{O}^x_{\hbar}$ through $J(\psi_0)$ in $\mathfrak{g}^*$ (see Thm. 14.6.5 in [3]; $\mathcal{O}^x_{\hbar}$ is here assumed to be equipped
with its canonical symplectic form). In most examples, $U^\chi(G)\psi_0$ is actually homeomorphic to $O^\chi_h$; the momentum map then provides an identification of the two as symplectic spaces. The dependence of $O^\chi_h$ on $\hbar$ comes from the fact that the symplectic form on $O^\chi_h$ is proportional to $\hbar$. In cases of interest to the classical limit of quantum mechanics, the label $\chi$ is of the form $\chi = L\chi_0$, where $L$ is some positive number (which is quantized if $G$ is compact, as is $\chi_0$). In that case, the orbit $O^\chi_h$ coincides with $O^\chi_0 = O^\chi$ as a manifold, and has the symplectic form $hL\omega_{\chi_0}$, where $\omega_{\chi_0}$ is the symplectic form on $O^\chi_0$. In the classical regime in the sense of Bohr, the quantum number $L$ is very large. With $\hbar$ a fixed constant of Nature, this means that $O^\chi_h$ will blow up as $L \to \infty$. To avoid this, in practice one keeps the classical scale $hL$ fixed (and equal to 1), and stipulates that this fixed scale is large compared to $\hbar$.

This means that one lets $\hbar \to 0$ and therefore (still) $L \to \infty$. If $L$ is quantized then clearly $\hbar$ can no longer assume arbitrary values, and the classical limit is achieved along a sequence $\{h = 1/L\}_{L \in \mathbb{N}}$. This latter procedure is the one we will follow. In particular, $\chi$ blows up in the classical limit. If necessary, the classical scale may be varied by changing $\chi_0$.

For fixed $\hbar > 0$, we define $q_h(x) = (J^h_k)^{-1}(x)$. If $x = Co(g)J(\psi_0)$ then $q_h(x)$ corresponds to the coherent state $U^\chi(g)|\psi_0\rangle$ in the usual formalism [10]. In view of the above, we see that the states $q_h(x)$ lie in different (projective) Hilbert spaces as $\hbar$ varies. This is no problem in the context of our formalism, as we may take the space of pure states $\mathcal{P}$ to be the union of all projective unitary representation spaces of $G$ (this is nothing but the pure state space of the group algebra $C^*(G)$, and therefore a perfectly natural object). We then look at the $q_h$ as a collection of injective maps from $O^\chi_0$ into $\mathcal{P}$. It may then be verified that the $q_h$ indeed define a classical germ. For a given value of $\hbar$, the lack of classical behaviour is measured by the non-zero-ness of the transition probabilities $p(q_h(x), q_h(y))$ for $x \neq y$.

Classical funnels $A_h$ are functions of the operators $-i\hbar U^\chi(T_h)$; that is, $A_h$ is nonzero only on $\mathbb{P}\mathcal{H}^{\chi_0}/\hbar$. If $A_h = a(-i\hbar U^\chi(T_1), \ldots, -i\hbar U^\chi(T_n))$ for some function $a$, then $A_0 = a(T_1, \ldots, T_n)$, regarded as a function on $O^\chi_0$, cf. [10].

6 Coherent state examples of classical germs

The following examples are well known (e.g., [10, 11, 12]), but it is useful to see them reformulated in the language described above.

Consider the Heisenberg group $G = H_n$ in $n$ dimensions. The generators of $g \simeq G \simeq \mathbb{R}^{2n+1}$ are $\{P_i, Q_j, Z\} \ (i, j = 1 \ldots n)$, with Lie brackets $[P_i, Q_j] = \delta_{ij}Z$ and $[P_i, Z] = [Q_j, Z] = 0$. We denote the dual basis in $g^*$ by $\{\dot{P}_i, \dot{Q}_j, \dot{Z}\}$. The co-adjoint orbit $O^\chi (\chi \neq 0)$ through $\lambda \dot{Z}$ may be identified with $T^*\mathbb{R}^n$; under this identification, a point $(p, q) \in T^*\mathbb{R}^n$ is identified with $p_i\dot{P}_i + q_j\dot{Q}_j + \lambda \dot{Z}$ (summation convention). One finds

$$Co(p_jQ_j - q_iP_i)\lambda \dot{Z} = p_i\dot{P}_i + q_j\dot{Q}_j + \lambda \dot{Z}.$$ 

The symplectic form on $O^\chi$ is given by $\lambda dp_i \wedge dq_i$.

The irreducible representations $U^L (L \neq 0)$ are all realized on the same Hilbert space $\mathcal{H}^L \equiv \mathcal{H} = L^2(\mathbb{R}^n)$, and may be specified by $dU^L(P_i) = \partial/\partial x_i$, $dU^L(Q_j) =$
$iLx_j$, and $dU^L(Z) = iL$. If we now take $\psi_0 \in \mathbb{P}\mathcal{H}$ defined by
\[
\langle x|\psi_0 \rangle = (L/\pi)^{n/4} \exp(-Lx^2/2)
\]
then $J^L_h(\psi_0) = hL\hat{Z}$. The classical germ $q_h$ is a family of maps from $\mathcal{O}^1 \simeq T^*\mathbb{R}^n$ into $\mathbb{P}\mathcal{H}$. Hence we put $L = 1/h$. By construction, $q_h(p, q)$ is then given by $U^{1/h}(p_jQ_j - q_iP_i)\psi_0$, which is represented in $\mathcal{H}$ by the wave function
\[
\langle x|q_h(p, q)\rangle = (\pi h)^{-n/4} \exp(-\frac{1}{2}ip_jq_j/h) \exp(ip_jx_j/h) \exp(-(x-q)^2/2h).
\]
One checks without difficulty that all requirements on a classical germ are indeed satisfied. Classical operators are functions of $-i\hbar dU^{1/h}(T_a)$, where $T_a$ is $P_i$, $Q_j$, or $Z$. Clearly,
\[
-i\hbar dU^{1/h}(P_i) = -i\hbar \partial/\partial x_i; \quad -i\hbar dU^{1/h}(Q_j) = x_j; \quad -i\hbar dU^{1/h}(Z) = 1.
\]

In the next round of examples, $G$ is a connected compact semi-simple Lie group. What follows is merely a reformulation of some of the results in [3, 13, 14, 15]. The label $\chi$ stands for a highest weight (relative to a choice of a maximal torus $T \subset G$ and of a fundamental Weyl chamber), and we assume that $\chi = L\chi_0$ for $L \in \mathbb{N}$ and some highest weight $\chi_0$. Each such $\chi_0$ defines a co-adjoint orbit $\mathcal{O}^{\chi_0}$: this is the orbit through $\chi_0$, which originally was an element of $\mathfrak{t}^*$ but is now regarded as an element of $\mathfrak{g}^*$ by putting it equal to zero on the orthocomplement of $\mathfrak{t}$ in $\mathfrak{g}$ with respect to the Killing metric. If $|\Psi_0\rangle$ is the highest weight vector in $\mathcal{H}^\chi$, then $J^h_0\Psi_0$ lies in the orbit through $hL\chi_0$, and $J^h_0$ is a symplectomorphism between $U^\chi(G)\Psi_0$ and $\mathcal{O}^{h\chi}$. As explained above, we now take $\mathcal{O}^{\chi_0}$ as the fixed classical phase space, and put $h = 1/L$. The map $q_h$ then injects $\mathcal{O}^{\chi_0}$ into $\mathbb{P}\mathcal{H}^\chi$, and defines a classical germ.

For example, for $G = SU(2)$ the co-adjoint orbit $\mathcal{O}^{\chi_0}$ is a sphere $S^2$ in $\mathfrak{g}^* = \mathbb{R}^3$ with radius $j_0$; the symplectic form is $j_0$ times the Fubini-Study form on $S^2 \simeq \mathbb{CP}^2$. With $j_0 = 1$ and $h = 1/j$, one finds that $p(q_h(z), q_h(w)) = (\cos \frac{1}{2} \Theta)^{j/h}$, where $\Theta$ is the angle between $z$ and $w$. Clearly, $\lim_{h \to 0} p(q_h(z), q_h(w)) = \delta_{zw}$.

Our last example of this sort (cf. [2]) provides a bridge towards the systems studied in the next section. We take $G = U(M)$; its Lie algebra comprises the set of observables of an $M$-level system. We realize $\mathfrak{g}^*$ as the space of all Hermitian $M \times M$ matrices $\rho$, with pairing $<\rho, T_a> = -i\text{Tr} \rho T_a$, where $T_a \in \mathfrak{g}$ is realized in its defining representation (i.e., as a skew-Hermitian $M \times M$ matrix). A co-adjoint orbit is labeled by an $M$-tuple of real numbers, and consists of all Hermitian matrices having these numbers as eigenvalues. The co-adjoint orbit $\mathcal{O}^1$ of interest equals the set of matrices $\rho$ with eigenvalues $(1, 0, \ldots, 0)$, and corresponds to the highest weight $\chi_0 = (1, 0, \ldots, 0)$. Any such $\rho$ can be written as $\rho = |\psi\rangle \langle \psi|$ for some $|\psi\rangle \in \mathbb{C}^M$ with $\langle \psi|\psi\rangle = 1$. It follows that $\mathcal{O}^1 \simeq \mathbb{CP}^M$, equipped with the Fubini-Study symplectic form. When convenient, we label its points simply by $\psi$, which of course stands for the matrix $|\psi\rangle \langle \psi|$. This is our fixed classical phase space. The highest weight $\chi_0$, in turn, corresponds to the defining representation $U^{\chi_0} \equiv U^1$ of $U(M)$ on $\mathcal{H}^{\chi_0} \equiv \mathcal{H}^1 = \mathbb{C}^M$. 

The representation $U^{L\chi_0} \equiv U^L_\mathfrak{g}$ is realized on $\mathcal{H}^L_S = \otimes^L_\mathfrak{g}\mathbb{C}^M$, the symmetrized tensor product of $L$ copies of $\mathbb{C}^M$. This is the state space of a system of $L$ identical bosons. The momentum map in this representation is given by linear extension of

$$J^L_\hbar(\psi_1 \otimes_S \ldots \otimes_S \psi_L) = \hbar \sum_{i=1}^L |\psi_i\rangle \langle \psi_i|.$$ 

The highest weight vector $|\Psi_0\rangle$ in $\mathcal{H}^L_S$ is the $L$-fold tensor product $\otimes^L |e_1\rangle$ of $L$ copies of the first basis vector $|e_1\rangle$. For the coherent states we get $J^L_\hbar(\otimes^L \psi) = \hbar L |\psi\rangle \langle \psi|$. (Since $U^L_\mathfrak{g}(g)\Psi_0 = \otimes^L U(g)e_1$, any point in the orbit $U^L_\mathfrak{g}(G)\Psi_0$ has the form $\otimes^L \psi$.) Thus $J^L_\hbar$ provides a symplectomorphism between $U^L_\mathfrak{g}(G)\Psi_0$ and the co-adjoint orbit consisting of all matrices with eigenvalues $(\hbar L, 0, \ldots, 0)$. As before, we now put $\hbar = 1/L$. By the general theory, this leads to a classical germ, whose member $q_\hbar$ injects $\mathcal{O}^1$ into $\mathcal{H}^L_S$. Explicitly, $q_\hbar(\psi) = \otimes^L \psi$. Hence

$$p(q_\hbar(\psi), q_\hbar(\varphi)) = |\langle \varphi |\psi\rangle|^2,$$

and it immediately follows that $\lim_{\hbar \to 0} p(q_\hbar(\psi), q_\hbar(\varphi)) = \delta_{\varphi\psi}$.

Finally, we note that classical funnels must be functions of $-i\hbar dU^L_\mathfrak{g}(T_a) = (-i/L) \sum_{j=1}^L T_a^{(j)}$, where $T_a^{(j)}$ is defined by linear extension of the operator

$$T_a^{(j)} |\psi_1\rangle \otimes_S \ldots \otimes_S |\psi_L\rangle = |\psi_1\rangle \otimes_S \ldots T_a |\psi_j\rangle \ldots \otimes_S |\psi_L\rangle.$$ 

The corresponding function $f_{T_a^{(j)}}$ on $\mathbb{P}\mathcal{H}^L_S$ is simply given by

$$f_{T_a^{(j)}}(\psi_1 \otimes_S \ldots \otimes_S \psi_L) = f_{T_a}(\psi_j).$$

Thus a classical funnel must be a function of single-particle operators averaged over all bosons in the system. This is a consequence of the irreducibility of $U^L_\mathfrak{g}$, which would not hold if the particles were distinguishable (so that the tensor product $\otimes^L \mathbb{C}^M$ is not symmetrized).

### 7 Mean-field systems

A spectacular occurrence of classical behaviour in a quantum system is encountered in mean-field systems. These include certain formulations of the BCS model of superconductivity, the Dicke laser model, Josephson junctions, etc. The current theoretical understanding of these models has emerged from the papers [16, 17, 11, 18, 19, 20, 21, 22], and others. Below we only study the so-called homogeneous case.

As in the previous section, we look at $L$ copies of an $M$-level system, but this time the particles are distinguishable, and may be thought of as sitting at the points of a lattice containing $L$ sites. The algebra of observables $\mathfrak{A}_\infty$ of the infinite system ($L = \infty$) is defined as the $(C^*\text{-inductive})$ limit of the algebra generated by operators of the type $A_1 \otimes A_2 \ldots \otimes A_N \otimes 1 \ldots$, where $N$ is finite (but varies), and the tail only consists of unit operators 1. Classical behaviour is found by focusing on the subset of the pure state space consisting of the permutation invariant states (such states
are invariant under permutations of the $A_i$ in the string above). It can be shown that permutation invariant pure states are of the form $\otimes^N \psi \equiv \Psi$, where $\psi$ is a pure state on $\mathfrak{A}_1$, i.e., $\psi \in \mathcal{O}^1 = \mathbb{PC}^M$. Hence the pure permutation invariant states themselves form the space $\mathbb{PC}^M$. The transition probability between different $\Psi$'s vanishes - this is intuitively obvious from the previous section, since the permutation invariant states act on the observables as if the particles were indistinguishable bosons. Moreover, the transition probability between arbitrary local perturbations of $\Psi$ and $\Psi' \neq \Psi$ vanishes. (In fact, the subset $S^P$ of all permutation invariant states of the total state space of $\mathfrak{A}_\infty$ is a so-called Bauer simplex, whose boundary consists of the primary, or ‘macroscopically pure’, permutation invariant states. Hence all states in $S^P$ have a unique decomposition into states which describe pure phases of the system. These properties make $S^P$ a classical object.)

We now return to the finite system ($L < \infty$), and take the pure state space $\mathcal{P}$ to be the union of all $\mathcal{P}_L$, where $\mathcal{P}_L$ is the pure state space of $\mathfrak{A}_L$ (the $L$-particle system). Clearly, $\mathcal{P}_L = \mathbb{PH}^L$, with $\mathbb{H}^L = \otimes^L \mathbb{C}^M$. The group $G = U(M)$ acts on $\mathbb{H}^L$ by the reducible unitary representation $U^L = \otimes^L U^1$. The corresponding momentum map is given by essentially the same formula as in the symmetrized case. With $\hbar = 1/L$ in what follows, we are therefore led to define the classical germ as a family of maps from $\mathcal{O}^1$ (as in the previous example) into $\mathcal{P}$. In view of the above, we take $q_h(\psi) = \otimes^L \psi \in \mathcal{P}_L \subset \mathcal{P}$. The transition probability is $p(q_h(\psi), q_h(\varphi)) = |\langle \varphi | \psi \rangle|^2L$, exactly as before.

The preceding paragraphs relate to any lattice model. What characterizes homogeneous mean-field models is their time-evolution. Namely, the Hamiltonian $H_L$ of the $L$-particle system is assumed to be of the form $H_L = LH$ where $H$ is a function of the scaled generators $-i\hbar dU^L(T_a)$ of $U(M)$ (cf. [22] for a wider class of Hamiltonians). Here $-i\hbar dU^L(T_a) = (-i/L) \sum_{j=1}^L T_a^{(j)}$, where $T_a^{(j)}$ is given by essentially the same expression as in the symmetrized case, i.e., it acts as $T_a$ on the $j$’th copy of $\mathbb{C}^M$. If $H$ is non-linear, a particle interacts with all other particles. Note that there are many more classical funnels than those of the type $H$ alone: for example, all operators in $\mathfrak{A}_\infty$ are included.

In view of the long-range nature of the Hamiltonian, a time-evolution on the infinite system $\mathfrak{A}_\infty$ does not exist in the usual sense (that is, as a one-parameter automorphism group of the algebra of observables). Instead, the limit $L \to \infty$ of the evolution described by $H_L$ only exists in certain representations. These include those induced by permutation invariant states. For those states (and their local perturbations) one can define a limiting Schrödinger picture time-evolution.

Recall that the permutation invariant pure states $\Psi$ of the infinite system form the manifold $\mathbb{PC}^M$, which we identify with the co-adjoint orbit $\mathcal{O}^1$ in $\frak{g}^*$, equipped with its canonical symplectic structure. The spectacular fact is now that the full quantum Schrödinger time-evolution of the states $\Psi$ in $\mathcal{O}^1$ coincides with the classical time-evolution generated by $\tilde{H}$, now regarded as a function on $\mathcal{O}^1$ through the replacement of $-i\hbar dU^L(T_a)$ by $T_a$ in its arguments (here $T_a \in \frak{g}$ is seen as a function on $\frak{g}^*$, and hence on its subspace $\mathcal{O}^1$, by linear evaluation). In particular, quantum ground states simply correspond to stationary points of the classical dynamics.
For the finite systems this means, from our point of view, that

$$\lim_{\hbar \to 0} p(q_{\hbar}(\psi)(t), q_{\hbar}(\psi(t))) = 1,$$

where $\psi \to \psi(t)$ is the classical time-evolution in $\mathcal{O}_1^1$ generated by $\hat{H}$, and $q_{\hbar}(\psi) \to q_{\hbar}(\psi)(t)$ is the time-evolution in $\mathcal{P}_L$ (with $L = 1/\hbar$, as always) generated by the Hamiltonian $H_L$. For each fixed finite size $L$ we can monitor the departure from classical behaviour by computing the static transition probabilities $p(q_{\hbar}(\psi), q_{\hbar}(\varphi))$ and the dynamical ones $p(q_{\hbar}(\psi)(t), q_{\hbar}(\psi(t)))$. The hypothetical infinite system only enters through its classical shadow, the finite phase space $\mathcal{O}_1^1$.

In our opinion, these models provide strong support for the belief that the existence of the classical world is compatible with quantum mechanics.

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