Better Boosting with Bandits for Online Learning

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Abstract

Probability estimates generated by boosting ensembles are poorly calibrated because of the margin maximization nature of the algorithm. The outputs of the ensemble need to be properly calibrated before they can be used as probability estimates. In this work, we demonstrate that online boosting is also prone to producing distorted probability estimates. In batch learning, calibration is achieved by reserving part of the training data for training the calibrator function. In the online setting, a decision needs to be made on each round: shall the new example(s) be used to update the parameters of the ensemble or those of the calibrator. We proceed to resolve this decision with the aid of bandit optimization algorithms. We demonstrate superior performance to uncalibrated and naively-calibrated on-line boosting ensembles in terms of probability estimation. Our proposed mechanism can be easily adapted to other tasks (e.g. cost-sensitive classification) and is robust to the choice of hyperparameters of both the calibrator and the ensemble.

Keywords: Online learning, Boosting, Bandit optimization, Classifier calibration, Probability estimation, Upper Confidence Bound, Thompson Sampling

1. Introduction

AdaBoost \cite{1} is an extensively studied ensemble learning method, with connections to multiple theoretical frameworks like margin theory \cite{2}, game theory \cite{3}, functional gradient descent \cite{4}, additive logistic regression \cite{5}, probabilistic modelling \cite{6}, to name but a few. AdaBoost classifiers have been very successful as evidenced by extensive experimental comparisons spanning multiple datasets, such as the ones conducted in \cite{7,8}, and applications like face detection in phone cameras \cite{9} and the Yahoo search engine for ranking webpages \cite{10}. The success of AdaBoost is further evidenced in numerous machine learning competitions; indicatively, more than half of the winning Kaggle entries have used gradient boosting\cite{11}.
Yet despite its success in classification, ranking and regression tasks, the probability estimates generated by AdaBoost have been known to be poorly calibrated, i.e. they deviate from empirical class probabilities \[5, 11, 12, 13\]. When the goal is to generate probability estimates rather than just to classify or rank examples, or when it is to solve a cost-sensitive classification problem, the performance of AdaBoost suffers.

Previous work in batch learning –tasks in which all data is available and can be processed at once– has shown that applying some form of calibration to the scores generated by AdaBoost considerably improves performance, in both probability estimation \[11\] and cost-sensitive classification \[14\] tasks. In batch learning scenarios calibration is achieved by reserving part of the training data to train a calibrator function –usually a logistic sigmoid or an isotonic regression– to map uncalibrated raw scores to probability estimates that maximize the likelihood of the model. This is done to avoid overfitting by training both the ensemble and the calibrator on the same datapoints.

Online learning deals with scenarios where data arrive sequentially –either one datapoint at a time or in minibatches– predictions are required as soon as the new datapoints become available and the learner must update its parameters using only the previous datapoint (or minibatch). This way the learner can adapt to changing –even adversarial– environments. Online learning is also a preferable option when dealing with very large amounts of data, when batch learning becomes expensive or even infeasible (due to computational limitations or slow convergence of the generalization error) \[15, 16, 17\]. Despite the increasing relevance of online learning given the growth of streaming and big data applications and despite the success of AdaBoost as a classifier and ranker, the probability estimation quality of online boosting ensembles has not yet been studied in the literature.

In this paper we will demonstrate that online boosting ensembles \[18\] also produce uncalibrated probability estimates under the most common scoring functions, i.e. ways of generating probability estimates. This motivates the need for calibrating the probability estimates of online boosting ensembles.

However, calibration is less straightforward in the online setting. On each round we need to decide whether the new example(s) will be used to update the parameters of the ensemble or those of the calibrator. A naive approach is to use a fixed policy of calibrating on every \(N_c\) rounds. But how do we set this hyperparameter \(N_c\)? Different combinations of problem (data, objective), ensemble (base learner, ensemble size, scoring function used) and calibrator (calibration function, optimization method used) would call for different values of \(N_c\).

In this work we propose resolving this decision with the aid of bandit optimization algorithms \[19, 20, 21\]. Bandit algorithms allow us to choose among a set of actions (here: ‘train on new minibatch’ or ‘calibrate on new minibatch’) balancing exploitation and exploration in stochastic or adversarial, stationary or non-stationary settings. They do this by sampling the distribution of rewards of each of the actions (here: the increase in log-likelihood of the model following each action) and maintaining a model of said distribution that is updated upon each feedback. The action to be taken in the next round is then chosen based on
the models of the reward distributions. The different bandit algorithms used –how they model the reward distribution and how they decide which action to take next– are discussed in Section 2.3.

Our bandit-based approach –more specifically UCB1-based policies [22, 23, 24], and Thompson Sampling [25, 26, 27], especially in its discounted-rewards version– shows superior performance to uncalibrated and naively-calibrated (i.e. employing fixed policies of calibrating on every $N_c$ rounds) boosting ensembles in probability estimation. This approach is very easy to adapt to new objectives (e.g. cost-sensitive learning tasks). All we need to do is change the reward function to the appropriate one for the task at hand (e.g. decrease in classification risk after an action). Moreover, the method is very flexible and robust to the ensemble hyperparameter choices, as it will learn an appropriate policy for alternating between the two actions guided by the corresponding rewards of the two actions.

2. Background

This work focusses on binary learning tasks. The examples are considered to be of the form $(x_i, y_i)$, where $x_i$ is the feature vector of the $i$-th example and $y_i \in \{-1, 1\}$ is its class label. Extension to the multiclass case is often handled by breaking down the problem into multiple binary ones, so our analysis and its main results can carry over to the multiclass case. We consider the online setting where examples are presented to the learner in $M$ minibatches\(^2\) of size $b$. On the $n$-th iteration the learner performs the following steps:

1. Receive new examples $x_i, \forall x_i \in \text{minibatch}_n$
2. Predict the label $\hat{y}_i$ and/or the probability estimate $\hat{p}(y_i = 1 | x_i), \forall i \in \text{minibatch}_n$
3. Get true labels $y_i = f(x_i), \forall x_i \in \text{minibatch}_n$, where $f$ is the labelling function
4. Update learner parameters accordingly

The steps above are intentionally left general enough to describe all learning components encountered in the paper. Our goal is to study the quality of the probability estimates generated by online boosting ensembles and strategies for improving it. Online boosting ensembles consist of multiple base learners, themselves also trained in an online fashion and–as we will see– the techniques used for improving the probability estimates (both the calibrator and the reward models of the bandits) are also learners trained in an online fashion. All follow the same general approach defined above: they maintain a model with a fixed number of parameters (i.e. memory and computational complexity are constant w.r.t the number of examples seen so far), which they update every time the labels of a new minibatch become available.

\(^2\)The scenario where the examples are arriving one at a time ($b = 1$) is merely a common special case.
2.1. AdaBoost and Online Boosting

Adaptive Boosting (AdaBoost) is a batch learning algorithm that constructs an ensemble sequentially across multiple rounds. On each round, a new component is added to the model. The principle behind it is to convert a weak learner—a hypothesis whose predictions are marginally more accurate than random guessing—into a strong one—one of error arbitrarily close to the irreducible Bayes error rate. To achieve this, it focuses on each round on correcting the mistakes of the previous model. This can be done either by reweighting or by resampling the dataset on each round, putting more emphasis on examples misclassified in the previous round and less on examples correctly classified in it. Each weak learner is assigned a confidence coefficient based on its predictive performance. Predictions are given by a weighted majority vote among the weak learners, the weight of each learner’s vote being its confidence coefficient.

The most popular algorithm for online boosting is the one proposed by Oza—henceforth OnlineBoost. The pseudocode is given in Algorithm 1. The ensemble consists of a fixed number \( T \) of components (weak learners). As in batch AdaBoost, the idea is to increase the weight assigned to examples that have been misclassified by previous models and decrease the weight assigned to examples that have been classified correctly. In OnlineBoost, each example is presented to each weak learner sequentially. If the \( t \)-th weak learner misclassifies an example, the example’s weight for the purpose of updating the parameters of the \((t + 1)\)-th weak learner will increase. Conversely, if the \( t \)-th weak learner classifies an example correctly, the example’s weight for the purpose of updating the \((t + 1)\)-th weak learner will decrease.

The (expected) weight of the current example is captured by the quantity \( \lambda \), which is used as the parameter of the Poisson distribution from which the ‘effective weight’ \( k \) is drawn. A weight of \( \lambda = 1 \) corresponds to no particular emphasis (be it positive or negative) paid to the current example.

The base learner, \( \text{OnlineLearnAlg()} \), is called for updating the parameters of the \( t \)-th weak learner on the \( i \)-th example. Finally, note that \( \lambda_{iw}^t \) is the sum of weights corresponding to correctly classified examples so far by the \( t \)-th weak learner. Conversely, \( \lambda_{iw}^t \) is the sum of weights corresponding to examples misclassified by the \( t \)-th weak learner so far.

In AdaBoost, an example’s weight is adjusted according to the base model’s performance on the entire training set. Instead, in OnlineBoost, the weight adjustment is based on the performance of a base model only on the examples presented so far. This is of course something intrinsic to online learning. It also means that the sequence of parameter updates will depend on the order in which the examples are presented. Oza showed that if a lossless online base learner is used, OnlineBoost converges to the same model as
Algorithm 1 OnlineBoost

Input: Number of weak learners $T$, training examples $\{(x_i, y_i)\mid i = 1, \ldots, N\}$ presented one at a time

For each $i$ do:

Set example weight $\lambda = 1$

For each $t \in \{1, 2, \ldots, T\}$ do:

Set $k$ according to $\text{Poisson}(\lambda)$

Do $k$ times:

$h_t \leftarrow \text{OnlineLearnAlg}(h_t, (x_i, y_i))$

If $h_t(x_i) = y_i$:

$\lambda_t^{sc} \leftarrow \lambda_t^{sc} + \lambda$

$\epsilon_t = \frac{\lambda_t^{sw}}{\lambda_t^{sw} + \lambda_t^{sc}}$

$\lambda \leftarrow \lambda \times \frac{1}{2(1-\epsilon_t)}$

Else:

$\lambda_t^{sw} \leftarrow \lambda_t^{sw} + \lambda$

$\epsilon_t = \frac{\lambda_t^{sc}}{\lambda_t^{sc} + \lambda_t^{sw}}$

$\lambda \leftarrow \lambda \times \frac{1}{2\epsilon_t}$

Prediction: On example $(x, y)$, predict $H(x) = \text{sign} \left[ \sum_{t=1}^{T} h_t(x) \log \frac{1-\epsilon_t}{\epsilon_t} \right]$

an AdaBoost ensemble of the same size trained on the same dataset, as the number of training examples $N \to \infty$.

To simplify the subsequent discussion we denote the confidence weight of the $t$-th weak learner with

$$\beta_t = \log \frac{1-\epsilon_t}{\epsilon_t}$$

(1)

and the ensemble output –the quantity whose sign will equal the final predicted class $H(x)$– with

$$F(x) = \sum_{t=1}^{T} \beta_t h_t(x).$$

(2)

2.2. Probability Calibration

In many applications it is desirable to estimate the probability of a given example belonging to each class. Quantifying the uncertainty about our predictions allows us to capture the reliability of a classification, to combine predictions from different sources or to make cost-sensitive decisions e.g. by using Bayesian Decision Theory principles as done in [14].

training set is identical to that of the corresponding batch learner (e.g. online Naive Bayes).
However, it is not always straightforward to obtain probability estimates from the outputs of a classifier. Most classifiers allow their output to be treated as a score for each test example $x$ that indicates ‘how positive’ $x$ is deemed. The act of converting raw scores to actual probability estimates is called calibration.

Denoting with $N$ the total number of examples, $N_s$ the number of examples with score $s \in S$, were $S$ is finite, and $N_{+,s}$ the number of positives with score $s$, Zadrozny & Elkan [28] give the following definition:

**Definition: 1. Calibrated classifier** A classifier is said to be calibrated if the empirical probability of an example with score $s(x) = s$ belonging to the positive class, $N_{+,s(x)}/N_{s(x)}$, tends to the score value $s$, as $N \to \infty$, $\forall s$.

In practice, most classifiers generate uncalibrated scores, each learner distorts them in its own way according to its inductive bias. We can improve probability estimation by taking measures to correct these distortions, i.e. by mapping the scores generated by the classifier to probability estimates, that maximize the likelihood of the data—or some other measure of quality of the probability estimation. The two most common approaches for doing this are logistic calibration (also known as Platt scaling) [29] and isotonic regression [30].

Logistic calibration finds a sigmoid mapping $s(x) \mapsto \hat{p}(y = 1|x)$. Isotonic regression is non-parametric and more general as it can be used to calibrate scores which exhibit any form of monotonic distortion. It needs more data to avoid overfitting and is less straightforward to adapt to the online setting.

A common measure used for evaluating probabilistic predictions, is the logarithmic loss, which is the negative log-likelihood of the true labels given a probabilistic classifier’s predictions. Denoting positive examples’ labels with $y_i = 1$ and negative examples’ labels with $y_i = 0$, the log-loss over some set of examples is given by

$$L = -\sum_i (y_i \log \hat{p}(y_i = 1|x_i) + (1 - y_i) \log(1 - \hat{p}(y_i = 1|x_i))) \in \mathbb{R}. \quad (3)$$

Another way to assess the calibration of a classifier’s probability estimates is to use reliability curves [33]. These are plots of probability estimates versus empirical probabilities. For perfectly calibrated predictions, the curve should be as close as possible to the diagonal as per the definition above. In this paper we will be mainly using the running average of the log-loss across all predictions so far to capture the progress of the quality of the probability estimates generated by each ensemble. We will only occasionally present reliability curves constructed across all predictions, for illustrative purposes.

Often referred to as ‘cross entropy’, ‘logistic loss’ or simply ‘log-loss’. A common alternative to the log-loss for assessing probability estimates is the Brier score, i.e. the mean squared error of the probability estimates of the sample of size $N'$ in question, $BS = \frac{1}{N'} \sum_i (y_i - \hat{p}(y_i = 1|x_i))^2$, denoting positive and negative labels with $y_i = 1$ and $y_i = 0$, respectively. Both belong to the infinite family of measures known as scoring rules, for more information on which, we direct the reader to [31, 32].
2.3. The multi-armed bandit problem

We will now introduce the basic principles of bandit optimization and the specific techniques we will use in the paper. In Section 5, we will adapt these techniques to automate the process of calibrating probability estimates for OnlineBoost.

The multi-armed bandit problem is a simple model for sequential decision making[19, 20, 21]. The name stems from the one-armed bandit machines found in casinos. When the machine’s lever or ‘arm’ is pulled, a cash reward is received with some probability. In the multi-armed bandit problem we imagine an agent confronted with many such machines, all with differing distributions for the rewards that the agent might receive by playing them. The agent wishes to cumulatively maximise their reward and can do this by pulling the arm with highest reward in expectation. However, the reward distributions are unknown to the agent, and so must be learned. There is an exploration-exploitation tradeoff for the agent in this scenario. The agent must balance exploiting the knowledge they have by pulling what they believe is the best arm and exploring the arms to increase the confidence in this knowledge. There is a large literature on bandits which considers varying assumptions about the reward distributions and the number of arms.

More formally, a general class of bandit problems is described by a set of arms \( A \) with an associated set of reward distributions \( \nu_{a,n}(\theta_{a,n}) \) for \( a \in A \), denoting by \( \theta_{a,n} \) the corresponding distribution parameters. The agent interacts with the bandit in a series of rounds. At each round \( n \), the agent chooses an action \( a_n \in A \) and then receives the reward \( X_n(a) \sim \nu_{a,n}(\theta_{a,n}) \). We denote the expected reward at round \( n \) of an arm as \( \mu_n(a) = \mathbb{E}[\nu_{a,n}(\theta_{a,n})] \) and so the largest expected reward at time \( n \) is given by \( \mu_n(*) = \max_a \mu_n(a) \). The regret of a decision \( a_n \) is given by \( r_n = \mu_n(*) - \mu_n(a_n) \). The cumulative regret after \( M \) rounds will be, \( R_M = \sum_{n=1}^{M} r_n \). The goal of the agent is to minimise the expected cumulative regret, \( \mathbb{E}[R_M] \).

In the standard problem, the reward distributions are assumed to be stationary and Bernoulli such that \( \nu_{a,n}(\theta_{a,n}) = \nu_{a,n'}(\theta_{a,n'}) = \text{Bernoulli}(\theta_a) \), \( \forall n,n' \) and a finite number of arms [19, 34]. There are many variations of the problem, ranging from bounded rewards [22], adversarial rewards [35], non-stationary rewards[36, 37, 38, 39], and infinite number of arms [40, 41, 42]. The bandit problem can be extended in many other ways; for example there is much work on contextual bandits [43, 44, 45], i.e. policies that take a context, e.g. an example’s feature vector \( x \) into account when deciding the next arm to pull.

In this work, we will assume that the rewards are stochastic (i.e. non-adversarial) but the reward distributions are not necessarily stationary. The non-stationarity will be handled by using discounted rewards as will be explained in Section 5. We will now discuss the specific bandit strategies used in this paper.

2.3.1. Policies

Thompson Sampling. [25] is a popular strategy due to having both theoretical justification[26, 46, 34, 47] and strong empirical performance[27]. The strategy assumes a given family of arm reward distributions \( \nu \). A distribution is used to model \( \mathbb{P}(\theta_a|H_n) \) where \( H_n \) is the history of past actions and associated rewards
up to round $n$. After receiving a reward for a given action $a$ the posteriors can be updated via Bayes rule. The agent chooses an arm by first drawing a sample $\hat{\theta}_a(n) \sim P(\theta_a|H_n)$ from each arm $a$. The agent chooses to pull arm $a_n = \max_a E \left[ \nu(\hat{\theta}_a(n)) \right]$. This is the arm with the highest mean reward conditioned on the sampled parameters $\hat{\theta}_a(n)$.

For the case of Gaussian rewards with known variance $\sigma^2$, since $\nu$ is Gaussian the unknown parameter $\theta_a = \mu_a$ is the mean reward of the arm. Due to the self-conjugacy of the Gaussian distribution, $P(\theta_a|H_n)$ is also modelled as a Gaussian distribution with parameters $\hat{\mu}_a(n)$ and $\hat{\sigma}_a^2(n)$, derived in closed-form. The policy, henceforth ‘Gaussian Thompson Sampling’, is shown in Algorithm 2.

Algorithm 2 Gaussian Thompson Sampling (with known variance $\sigma^2$)

| Let $\hat{\mu}_a(1) = 0$, $\hat{\sigma}_a^2(1) = 1$ for $a \in A$. |
| for $n = 1, \ldots, M$ do |
| Sample $\hat{\theta}_a(n) \sim \mathcal{N}(\hat{\mu}_a(n), \hat{\sigma}_a^2(n))$, for $a \in A$. |
| Pull arm $a_n = \arg\max_a \hat{\theta}_a(n)$ |
| Let $\hat{\mu}_a(n + 1) = \frac{\hat{\mu}_a(n) + \sum_{j=1}^{n} X_j(a_n)\hat{\sigma}_a^2(n)}{\hat{\sigma}_a^2(n)}$ |
| $\hat{\mu}_a(n + 1) = \frac{\hat{\mu}_a(n) + \sum_{j=1}^{n} X_j(a_n)\hat{\sigma}_a^2(n)}{\hat{\sigma}_a^2(n) + \sigma^2}$ |
| Let $\hat{\mu}_a(n + 1) = \hat{\mu}_a(n)$ |
| $\hat{\sigma}_a^2(n + 1) = \hat{\sigma}_a^2(n)$ for $a \in A \setminus \{a_n\}$. |

Upper Confidence Bound (UCB) policies. refer to a particular class of bandit policy [22, 23, 24]. As the name suggests such policies manage exploration through the use of upper confidence bounds on the estimates of mean arm rewards. In this way UCB policies follow a principle of optimism in the face of uncertainty. A UCB policy starts by pulling each arm once. After this, for each arm the number of pulls of the arm $k_a(n)$ and an estimate of the mean $\hat{\mu}_a(n) = \frac{1}{k_a(n)} \sum_{j=1}^{n} X_j(a)I(a_j = a)$ is maintained. This is then combined with a padding function $c(k_a(n), t)$ to give a upper confidence bound for the arm of $U_a(n) = \hat{\mu}_a(n) + c(k_a(n), n)$. The agent chooses to pull arm $a_n = \max_a U_a(n)$. As an example of a padding function, the one used by UCB1 [22] is

$$c(k_a(n), n) = \sqrt{\frac{2\ln n}{k_a(n)}}.$$

Padding functions have been further improved with policies such as KL-UCB [23]. In the same paper an improved version of UCB1 was also introduced (see Proposition 4 of [23]), to which we will henceforth refer as ‘UCB1-Improved’. 

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3. Probability Estimates under Online Boosting Ensembles

It is straightforward to adapt the two most common scoring functions, i.e. ways of producing probability estimates for batch AdaBoost ensembles, to the OnlineBoost case. The first choice, is to use the \textit{weighed fraction of base learners voting for the positive class} \cite{11}.

\[ s(x) = \frac{\sum_{t: h_t(x) = 1} \beta_t h_t(x)}{\sum_{t=1}^T \beta_t h_t(x)} \in [0, 1]. \tag{4} \]

Another choice, motivated by the view of Boosting as an additive logistic regression procedure \cite{5} is

\[ s'(x) = \frac{1}{1 + e^{-F(x)}} \in [0, 1]. \tag{5} \]

Both scores of the form of Eq.(4) and of the form of Eq.(5) tend to values close to 0 and 1. For the case of the former, this behaviour is connected to the \textit{margin maximization} properties of boosting. The (normalized) \textit{hypothesis (a.k.a. voting) margin} of a training example \((x, y)\) under the ensemble \(F\) is defined as

\[ \text{margin}(x, y) = \frac{yF(x)}{\sum_{t=1}^T \beta_t} \in [-1, 1]. \tag{6} \]

It is a combined measure of confidence and correctness of the classification of the example under \(F\). Its sign encodes whether the example was correctly classified (positive) or misclassified (negative), while the magnitude of the margin measures the \textit{confidence of the final hypothesis}. AdaBoost greedily maximizes the \textit{margins of the training examples} \cite{5}, promoting correct classifications for which the ensemble is highly confident. In fact, this margin maximizing behaviour of boosting algorithms has been connected to their nice generalization properties as classifiers \cite{2}. Theorem 1, given below, shows that OnlineBoost –like AdaBoost– also greedily maximizes the margins of the training examples. It allows much of the theory behind AdaBoost, including the general form of the scoring functions and their properties, to carry over to OnlineBoost.

**Theorem 1.** OnlineBoost greedily minimizes the exponential loss of the margin \(L(y, F(x)) = e^{-yF(x)}\) via stochastic gradient descent steps in the space of functions \(F(x)\).

**Proof Sketch.** See Appendix A.

The scores of the form of Eq.(4) assigned to the training examples can be expressed \cite{13} in terms of their corresponding margins as follows:

\[ s(x) = \begin{cases} \frac{1}{2} (1 + \text{margin}(x, y)), & \text{if } y = 1 \\ \frac{1}{2} (1 - \text{margin}(x, y)), & \text{if } y = -1. \end{cases} \tag{7} \]

\[ \text{Eq. (5) differs from the one given in } \cite{5} \text{ by a factor of 2 that multiplies } F \text{ in the latter. This is simply because the formulation of OnlineBoost of Algorithm 1 uses } \beta_t = \log \frac{1 + \epsilon_t}{\epsilon_t} \text{ as the confidence weight of the } t\text{-th weak learner, while } \cite{5} \text{ uses } \beta_t = \frac{1}{2} \log \frac{1 + \epsilon_t}{\epsilon_t}. \text{ Both forms result in the same weight updates after normalizing the latter and to equivalent predictions } H(x). \]
We see that as margin\((x, y) \to 1\), the scores assume values \(s(x) \to 0\) for negative examples and \(s(x) \to 1\) for positive examples. In other words, maximizing the margins—something not only AdaBoost and OnlineBoost, but all boosting algorithms do, by virtue of minimizing monotonically decreasing loss functions of the margin—forces the ensemble to learn to assign scores that tend to 0 and 1.

As for scores of the form of Eq. (5), they can be expressed \([6]\) as a Product of Experts (PoE) \([9]\)

\[
s'(x) = \frac{\prod_{t=1}^{T} \hat{p}_t(y = 1|x)}{\prod_{t=1}^{T} \hat{p}_t(y = 1|x) + \prod_{t=1}^{T} \hat{p}_t(y = -1|x)},
\]

with experts’ probability estimates of the form

\[
\hat{p}_t(y = 1|x) = \begin{cases} 
\epsilon_t, & \text{if } h_t(x) = -1 \\
1 - \epsilon_t, & \text{if } h_t(x) = 1,
\end{cases}
\]

\[
\hat{p}_t(y = -1|x) = \begin{cases} 
1 - \epsilon_t, & \text{if } h_t(x) = -1 \\
\epsilon_t, & \text{if } h_t(x) = 1,
\end{cases}
\]

where \(\epsilon_t = \frac{\lambda_s^w}{\lambda_s^w + \lambda_l^w}\) is the weighted error of the \(t\)-th weak learner on the examples seen so far, and \(h_t(x) \in \{-1, 1\}\) its prediction on example \(x\).

In larger ensembles, the outputs tend to be more and more distorted. One reason for this is that the PoE assumes that the experts produce independent estimates and the more experts we add to the ensemble, the more likely we are to deviate from such an assumption. Another reason is that a single expert producing a score of \(\hat{p}_t(y = 1|x) = 0\) or \(\hat{p}_t(y = 1|x) = 1\) to a given example \(x\) suffices to dominate the ensemble’s score \(s'(x)\) on that example. This was discussed in the case of batch AdaBoost in \([14]\) and holds for the OnlineBoost ensembles as well.

In our experiments we use scores of the form of Eq. (4), motivated by previous work in batch boosting \([11, 10, 14]\). Indeed, as we see in Figure 3, the scores tend to be skewed towards 0 or 1, and OnlineBoost ensembles tend to be very poorly calibrated. Note that poor probability estimation does not necessarily lead to poor classification. In fact, as we discussed, the very reason that makes boosting a successful classifier, namely its margin maximization property, is also responsible for its poor performance as a probability estimator, since it forces the ensemble to produce probability estimates skewed towards 0 or 1.

4. Naive Calibration of Online Boosting

In the previous section, we saw theoretical and empirical evidence that suggests that the probability estimates generated under OnlineBoost ensembles are distorted and need to be properly calibrated. To our knowledge, the calibration of online boosting ensembles has not been studied before in the literature.
Figure 1: [TOP] Reliability diagrams for OnlineBoost ensembles on two sample datasets. The main diagonal corresponds to the ideal (perfectly calibrated) probability estimator. We can see that the probability estimates generated by OnlineBoost are far from ideal. [BOTTOM] Histogram of the probability estimates (scores) assigned by the boosting ensemble. We see the scores tend to values close to 0 and 1. The results shown here are averages and 95% confidence intervals calculated across 10 runs of training an ensemble of $T = 10$ online Naive Bayes classifiers using minibatch sizes of $b = 50$. 
In this section we present a simple calibration policy, which directly draws from previous work on batch learning \[11, 14\]. In the next section we will further refine this approach.

4.1. Online Platt Scaling

As a calibration method we choose Platt-scaling (logistic calibration). This was done in part because OnlineBoost will tend to generate probability estimates that tend towards 0 or 1 as discussed above - i.e. amenable to sigmoid correction- and in part because the adaptation of the method in the online scenario is easy and efficient. Platt-scaling consists of finding a logistic sigmoid mapping of scores to probability estimates. The probability estimates are thus given by:

\[
\hat{p}(y = 1|x) = \frac{1}{1 + e^{w_1 s(x) + w_0}},
\]

(8)

where \(s(x)\) are the uncalibrated scores of the form of Eq. (4) and \(w_0\) and \(w_1\) are the parameters to be fitted.

We update the parameters of the sigmoid on one minibatch at a time (provided said minibatch is used for calibration –see next subsection), such that the log-loss of Eq. (3) is minimized\(^7\). To account for class imbalance, the Bayesian prior correction proposed by Platt \[29\] was applied. Rather than using \(y_i = 1\) for positive labels and \(y_i = 0\) for negative labels, in Eq. (3), we use respectively

\[
y_i = \frac{N_+ + 1}{N_+ + 2} \text{ and } y_i = \frac{1}{N_- + 2}.
\]

(9)

On every minibatch (be it used for calibration or for training), we update \(N_+\) and \(N_-\), the current numbers of positive and negative examples respectively encountered in the dataset so far. If the data distribution is non-stationary, this also allows the predictions to adapt to prior probability shift.

Hence the calibrator always keeps track of 4 quantities: \(N_+\) and \(N_-\) mentioned above and \(w_0\) and \(w_1\), the current sigmoid parameters to be updated on the next iteration.

4.2. A Naive Calibration Policy

A simple strategy is to use every \(N_c\)-th example to calibrate (update \(w_0\) and \(w_1\)) and the remaining ones to train the ensemble\(^8\). As the calibrator function has only two parameters, it is reasonable to expect that in the long term using most minibatches to train the ensemble would yield better results. We experiment with values \(N_c \in \{2, 4, 6, 8, 10, 12, 14\}\), corresponding to fractions of \{50%, 25%, 16.7%, 12.5%, 10%, 8.3%, 7.1%\} of the data, respectively. Figure 2 shows reliability diagrams for uncalibrated OnlineBoost and naively-calibrated OnlineBoost with \(N_c = 2\) on two sample datasets. We see that even this most naive calibration policy considerably improves the probability estimation behaviour of online boosting. In the next section we will refine the naive calibration proposed here.

\(^7\)In the paper we use log-loss to assess probabilistic predictions. Had we been using some other scoring rule (e.g. Brier score), it would be sensible to minimize the same loss (e.g. squared loss) to train the parameters of the sigmoid.

\(^8\)The first round is always used to train the ensemble.
Figure 2: Reliability diagrams for uncalibrated OnlineBoost (green) and naively-calibrated OnlineBoost with $N_c = 2$ (blue) on two sample datasets. The main diagonal corresponds to the ideal (perfectly calibrated) probability estimator. We can see that some calibration—even of the most naive form—considerably improves the probability estimation behaviour of online boosting. The results shown here are averages and 95% confidence intervals calculated across 10 runs of training an ensemble of $T = 10$ online Naive Bayes classifiers using minibatch sizes of $b = 50$.

For completeness we should mention another obvious candidate naive policy of calibration: On each minibatch, we can construct two models, one by performing each of the two actions (train, calibrate). We then retain the model that leads to the greatest decrease in log-loss. This approach is computationally costlier, but as we only have two possible actions the increase in computational cost is constant and potentially affordable. However, when processing large amounts of data even a constant increase in computational cost can matter. Another flaw is that this approach is less amenable to extensions; in the next section we will discuss a bandit-based approach that can scale to an arbitrary number of other actions besides the two discussed here, or that can easily be adjusted to deal with non-stationary data, adversarial environments or incorporate contextual information.

Most importantly, however, the policy is memoryless and greedy. It does not encourage exploration and only exploits the action that reduced the log-loss the most on the last minibatch. Considering that unlike updating the ensemble parameters (which are geared towards reducing the classification error, leaving the final estimate poorly calibrated), the update of the calibrator parameters is explicitly performed with the objective of minimizing the log-loss, that action is almost guaranteed to be ‘calibrate’. Indeed, in our experiments with this technique, we saw that it reduced to always choosing to update the parameters of the calibrator. This caused the final probability estimates generated by this policy to be far worse than those produced by the other policies discussed here. We will therefore exclude it from further consideration.
5. Bandit Algorithms for Calibrated Online Boosting

Our results showed that employing a naive calibration policy is preferable to not calibrating the probability estimates at all. It is clear however, that the success of such a policy will depend on many factors: (1) the characteristics of the dataset, (2) the evaluation measure used (log-likelihood or Brier score for probability estimation, empirical risk for cost-sensitive classification, etc.), (3) the hyperparameters of the ensemble (weak learner, number of weak learners, scoring function), (4) the hyperparameters of the calibrator (choice of calibration function, optimization algorithm to train it). The interplay of these will determine the best value of $N_c$ for a fixed policy of the type discussed in the previous section.

Ideally, we would like to automate the process of learning a good policy of alternating between the two actions. This is already an issue in batch learning, where determining the correct fraction of the training data that will be used for calibration is not obvious and will depend on all the factors mentioned in the previous paragraph. But in an online setting, it becomes even more important. The value of $N_c$ cannot be determined by cross-validation as predictions need to be made on the fly. Moreover, the optimal value of $N_c$ might change during execution due to possible non-stationarity of the data (be it stochastic or adversarial).

To solve this problem we employed the different bandit optimization algorithms described in Section 2.3. The general methodology is described in Algorithm 3. Each of the two actions (train, calibrate) is associated with a reward distribution. After each action is taken, the parameters of its corresponding reward distribution are updated accordingly.

\begin{algorithm}
\caption{Deciding when to calibrate under a given BanditPolicy}
\For{each round $n$ do:}
\State Receive unlabelled examples of $\text{Minibatch}_n$
\State Make predictions on examples of $\text{Minibatch}_n$
\State Receive labels of examples of $\text{Minibatch}_n$
\State Evaluate performance on $\text{Minibatch}_n$
\If{$n < 2$}:
\State Update ensemble parameters on examples of $\text{Minibatch}_n$ (Sec. 2.1)
\Else:
\State Use reward $X_n$ to update the parameters of BanditPolicy
\State Use BanditPolicy to decide action $a_n$ (Sec. 2.3)
\If{$a_n == \text{‘TRAIN’}$}:
\State Update ensemble parameters on examples of $\text{Minibatch}_n$ (Sec. 2.1)
\Else:
\State Update calibrator parameters on examples of $\text{Minibatch}_n$ (Sec. 4.1)
\State Compute reward $X_n$ of performing action $a_n$ (Sec. 5)
\end{algorithm}
The reward for each action is defined as the resulting relative decrease in log-loss after the action is taken:

$$X_n(a_n) = -\frac{\mathcal{L}_n - \mathcal{L}_{n-1}}{\mathcal{L}_{n-1}} = \frac{\mathcal{L}_{n-1} - \mathcal{L}_n}{\mathcal{L}_{n-1}} = 1 - \frac{\mathcal{L}_n}{\mathcal{L}_{n-1}} \in \mathbb{R}, \quad (10)$$

where $\mathcal{L}_{n-1}$ is the log-loss of round $n - 1$, i.e. before performing action $a_n$ and $\mathcal{L}_n$ is the log-loss of round $n$, i.e. after performing action $a_n$.

For the cases of UCB1, UCB1-Improved and Gaussian Thompson Sampling, we also implemented versions employing discounted rewards to deal with the potentially non-stationary nature of online learning. More specifically, on each update, the cumulative rewards are multiplied by a discounting factor $\gamma < 1$, i.e.

$$R_n = \gamma R_{n-1} + r_n.$$ 

The result is that the influence of past rewards decays geometrically.

We should note here that the non-stationarity can be due to the distribution of the data changing, but also due to the actions performed, which might lead to the reward distributions of the two actions changing. For example, after many rounds of only performing one action (e.g. ‘training’), we would intuitively expect that the reward distributions of the two actions have changed considerably, the rewards for training becoming smaller and smaller and the actual reward of calibrating having increased considerably since last sampled. Especially the model we have for an action not taken for many rounds (‘calibrate’, in this example) is expected to be poor. Discounting can protect us to some extent from such behaviours.

6. Empirical Evaluation

6.1. Experimental Setup

In our experiments we compared uncalibrated OnlineBoost to its naively calibrated version –i.e. a fixed policy of calibrating every $N_c$ rounds– with $N_c \in \{2, 4, 6, 8, 10, 12, 14\}$, as well as to calibration under UCB1, UCB1-Improved and Gaussian Thompson Sampling policies and their discounted counterparts.

The uncalibrated probability estimates were of the form of Eq. (4). In the calibrated variants, logistic calibration was applied, by minimizing the loss of Eq. (3), with incremental BFGS steps\(^9\).

We experimented with different choices of weak learners, both lossless (Gaussian Naive Bayes) and lossy (logistic regression, linear SVM, perceptron –all trained with stochastic gradient descent\(^10\)). We also examined the effect of different ensemble sizes ($T \in \{10, 25, 50\}$) and explored different degrees of regularization on the weak learner ($\ell_1$-regularized logistic regression with a regularization parameter $\lambda \in \{10^{-1}, 10^{-2}, 10^{-3}, 0\}$). Unless otherwise specified the default parameters of scikit-learn\(^11\) were used.

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\(^9\)Although BFGS is not typically a very popular choice for online learning, as the calibration step here always consists of updating the two parameters of a sigmoid, the computational and memory cost of BFGS is constant and low.

\(^10\)In the case of Gaussian Naive Bayes we used the resampling version of OnlineBoost given in Algorithm 1 –i.e. the original from [18]. For all other learners, we used the faster reweighting version described in Footnote 3 as they supported it.

\(^11\)http://scikit-learn.org/stable/
The hyperparameters of the bandit algorithms were fixed, as the purpose of using these algorithms is to circumvent hyperparameter tuning. The Gaussian prior for Thompson Sampling was set to $\mathcal{N}(0,1)$. Discounted reward versions used a discount factor $\gamma = 0.95$.

The experiments were carried out on 10 real-world datasets, the characteristics of which are given in Appendix B. The examples in kruskp, landsat, splice, waveform, spambase, mushroom, musk2 are considered i.i.d., so they are used for simulating situations where online learning is employed to generate good predictions faster than a batch learning algorithm. For these, the minibatch size was set to $b = 50$. The datasets weather, electricity and forest were considered non-stationary. As these three datasets are also considerably larger than the other 7, the minibatch size was set to $b = 100$ for faster processing.

6.2. Experimental Results

We present the negative log-likelihood across the entire dataset as an overall measure of performance of each variant. We present the best and worst result attained on average by fixed policies (in the sense of final log-loss attained) and specify in each case the corresponding $N_e$ that produced it. Only some characteristic results are presented here. The remaining ones are given in Appendix C.

We also provide some characteristic learning curves on the negative log-likelihood (average negative log-likelihood across all past predictions versus number of minibatches seen). This allows us to observe how fast each algorithm can generate good probability estimates. In each case, we report average values and 95% confidence intervals across 10 runs$^{12}$.

6.2.1. Experiments on stationary datasets.

We shall first present the results under various choices of weak learners on stationary datasets. The results for varying ensemble size and degree of regularization can be found in Appendix C, as they are qualitatively similar. Tables 1–4 show the log-loss across the entire dataset. Figures 6.2.1 & 6.2.1 show the evolution of average log-loss during training for some selected combinations of dataset, base learner and policy.

$^{12}$Datasets that are i.i.d. are shuffled on each run, thus changing the order in which examples are presented to the learner. On non-stationary datasets, we respect the order in which the examples arrive to preserve their non-stationary nature.
Table 1: Log-loss across the entire dataset. Lowest average value shown in bold. Results for Naive Bayes, $T = 10$ on stationary datasets.

| Dataset      | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | UCB1 | GTS | Disc. UCB1 | Disc. UCB1 Improved | Disc. GTS |
|--------------|--------------|------------|-------------|---------------|------|-----|------------|---------------------|----------|
| landsat      | 0.138        | 0.081      | 0.131       | 0.072         | **0.071** | 0.074 | 0.129      | 0.123               | 0.073    |
|              | ±            | ±          | ±           | ±              | ±    | ±   | ±          | ±                   | ±        |
|              | 0.002        | 0.002      | 0.004       | 0.001         | **0.001** | 0.002 | 0.010      | 0.011               | 0.002    |
| splice       | 0.444        | 0.280      | 0.410       | 0.234         | **0.229** | 0.282 | 0.353      | 0.376               | 0.232    |
|              | ±            | ±          | ±           | ±              | ±    | ±   | ±          | ±                   | ±        |
|              | 0.034        | 0.014      | 0.014       | 0.007         | **0.009** | 0.022 | 0.024      | 0.037               | 0.008    |
| musk2        | 0.723        | 0.343      | 0.427       | **0.320**     | 0.332 | 0.397 | 0.711      | 0.441               | 0.345    |
|              | ±            | ±          | ±           | ±              | ±    | ±   | ±          | ±                   | ±        |
|              | 0.027        | 0.008      | 0.013       | **0.006**     | 0.003 | 0.008 | 0.025      | 0.049               | 0.008    |
| kruskp       | 1.252        | 0.488      | 0.720       | 0.474         | 0.472 | 0.486 | 0.627      | 0.767               | **0.468** |
|              | ±            | ±          | ±           | ±              | ±    | ±   | ±          | ±                   | ±        |
|              | 0.154        | 0.014      | 0.075       | 0.019         | 0.012 | 0.016 | 0.114      | 0.114               | **0.020** |
| waveform     | 0.824        | 0.361      | 0.466       | 0.335         | **0.334** | 0.342 | 0.458      | 0.492               | 0.344    |
|              | ±            | ±          | ±           | ±              | ±    | ±   | ±          | ±                   | ±        |
|              | 0.012        | 0.005      | 0.009       | 0.002         | **0.004** | 0.007 | 0.015      | 0.024               | 0.006    |
| spambase     | 2.37         | 0.532      | 0.735       | 0.493         | 0.483 | **0.481** | 0.536      | 0.540               | 0.489    |
|              | ±            | ±          | ±           | ±              | ±    | ±   | ±          | ±                   | ±        |
|              | 0.026        | 0.006      | 0.017       | 0.006         | **0.008** | 0.028 | 0.037      | 0.037               | 0.005    |
| mushroom     | 0.770        | 0.358      | 0.503       | 0.375         | 0.335 | **0.318** | 0.570      | 0.570               | 0.327    |
|              | ±            | ±          | ±           | ±              | ±    | ±   | ±          | ±                   | ±        |
|              | 0.057        | 0.011      | 0.017       | 0.018         | 0.020 | **0.006** | 0.036      | 0.072               | 0.010    |
Table 2: Log-loss across the entire dataset. Lowest average value shown in bold. Results for logistic regression, $T = 10$ on stationary datasets.

| Dataset | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | GTS | Disc. UCB1 | Disc. UCB1 Improved | Disc. GTS |
|---------|-------------|------------|-------------|---------------|-----|------------|---------------------|-----------|
| landsat | 0.330       | **0.106**  | 0.158       | 0.107         | 0.108| 0.108      | 0.173               | 0.203     | 0.127 |
|         | ±           | ±          | ±           | ±             | ±   | ±          | ±                   | ±         |
|         | 0.011       | **0.002**  | 0.004       | 0.003         | 0.004| 0.004      | 0.019               | 0.018     | 0.013 |
| splice  | 1.779       | 0.626      | 0.895       | **0.506**     | 0.554| 0.636      | 0.711               | 1.155     | 0.584 |
|         | ±           | ±          | ±           | ±             | ±   | ±          | ±                   | ±         |
|         | 0.059       | 0.023      | 0.034       | **0.029**     | 0.020| 0.068      | 0.117               | 0.169     | 0.034 |
| musk2   | 1.276       | 0.331      | 0.398       | 0.321         | 0.330| **0.318**  | 0.536               | 0.514     | 0.322 |
|         | ±           | ±          | ±           | ±             | ±   | ±          | ±                   | ±         |
|         | 0.015       | 0.003      | 0.008       | 0.004         | 0.004| **0.003**  | 0.049               | 0.038     | 0.006 |
| kruskp  | 1.151       | 0.633      | 0.881       | **0.519**     | 0.528| 0.645      | 0.825               | 0.733     | 0.555 |
|         | ±           | ±          | ±           | ±             | ±   | ±          | ±                   | ±         |
|         | 0.073       | 0.041      | 0.042       | **0.037**     | 0.017| 0.048      | 0.074               | 0.041     | 0.033 |
| waveform| 1.700       | 0.444      | 0.596       | **0.427**     | 0.429| 0.428      | 0.503               | 0.519     | 0.464 |
|         | ±           | ±          | ±           | ±             | ±   | ±          | ±                   | ±         |
|         | 0.037       | 0.004      | 0.019       | **0.003**     | 0.007| 0.006      | 0.026               | 0.030     | 0.016 |
| spambase| 1.182       | 0.409      | 0.572       | 0.390         | 0.395| 0.407      | 0.540               | 0.477     | **0.389** |
|         | ±           | ±          | ±           | ±             | ±   | ±          | ±                   | ±         |
|         | 0.018       | 0.006      | 0.010       | 0.005         | 0.006| 0.007      | 0.029               | 0.024     | **0.006** |
| mushroom| 1.001       | 0.343      | 0.454       | 0.322         | 0.315| **0.312**  | 0.531               | 0.592     | 0.330 |
|         | ±           | ±          | ±           | ±             | ±   | ±          | ±                   | ±         |
|         | 0.023       | 0.007      | 0.017       | 0.007         | 0.006| **0.007**  | 0.026               | 0.026     | 0.006 |
Table 3: Log-loss across the entire dataset. Lowest average value shown in bold. Results for linear SVM, $T = 10$ on stationary datasets.

| Dataset            | Uncalibrated | Best       | Worst       | UCB1 Improved | Disc. UCB1 Improved | Disc. UCB1 Improved | Disc. UCB1 Improved | Disc. GTS |
|--------------------|--------------|------------|-------------|---------------|---------------------|---------------------|---------------------|-----------|
| landsat            | 0.245        | 0.130      | 0.184       | 0.118         | 0.121               | 0.127               | 0.250               | 0.243     | 0.111     |
| splice             | ±            | ±          | ±           | ±             | ±                   | ±                   | ±                   | ±         | ±         |
|                    | 0.019        | 0.009      | 0.010       | 0.008         | 0.008               | 0.010               | 0.025               | 0.034     | 0.004     |
| musk2              | 1.391        | 0.663      | 0.922       | 0.622         | 0.586               | 0.609               | 0.816               | 0.666     | 0.640     |
|                    | ±            | ±          | ±           | ±             | ±                   | ±                   | ±                   | ±         | ±         |
|                    | 0.064        | 0.018      | 0.040       | 0.031         | 0.022               | 0.020               | 0.105               | 0.088     | 0.038     |
| kruskp             | 0.953        | 0.385      | 0.432       | 0.400         | 0.390               | 0.406               | 0.639               | 0.622     | 0.386     |
|                    | ±            | ±          | ±           | ±             | ±                   | ±                   | ±                   | ±         | ±         |
|                    | 0.020        | 0.008      | 0.007       | 0.006         | 0.007               | 0.008               | 0.061               | 0.066     | 0.005     |
| waveform           | 1.081        | 0.671      | 0.897       | 0.608         | 0.648               | 0.677               | 0.706               | 0.811     | 0.667     |
|                    | ±            | ±          | ±           | ±             | ±                   | ±                   | ±                   | ±         | ±         |
|                    | 0.050        | 0.022      | 0.027       | 0.021         | 0.027               | 0.029               | 0.047               | 0.041     | 0.030     |
| spambase           | 1.156        | 0.487      | 0.615       | 0.453         | 0.451               | 0.451               | 0.611               | 0.603     | 0.464     |
|                    | ±            | ±          | ±           | ±             | ±                   | ±                   | ±                   | ±         | ±         |
|                    | 0.045        | 0.013      | 0.015       | 0.005         | 0.004               | 0.005               | 0.021               | 0.076     | 0.005     |
| mushroom           | 0.893        | 0.440      | 0.551       | 0.411         | 0.410               | 0.418               | 0.560               | 0.612     | 0.414     |
|                    | ±            | ±          | ±           | ±             | ±                   | ±                   | ±                   | ±         | ±         |
|                    | 0.019        | 0.008      | 0.010       | 0.005         | 0.007               | 0.009               | 0.028               | 0.053     | 0.0037    |
|                    | 0.673        | 0.374      | 0.489       | 0.359         | 0.363               | 0.352               | 0.506               | 0.573     | 0.357     |
|                    | ±            | ±          | ±           | ±             | ±                   | ±                   | ±                   | ±         | ±         |
|                    | 0.015        | 0.006      | 0.015       | 0.006         | 0.006               | 0.005               | 0.036               | 0.039     | 0.007     |
Table 4: Log-loss across the entire dataset. Lowest average value shown in bold. Results for perceptron, $T = 10$ on stationary datasets.

| Dataset  | Uncalibrated | Best   | Worst  | UCB1   | Improved | GTS   | Disc. UCB1   | Disc. UCB1 Improved | Disc. GTS |
|----------|--------------|--------|--------|--------|----------|-------|--------------|---------------------|-----------|
| landsat  | 0.258 ± 0.012 | 0.120  | 0.190  | 0.125  | 0.103    | 0.115 | 0.164 ± 0.014 | 0.198 ± 0.028       | 0.120     |
| splice   | 1.453 ± 0.037 | 0.683  | 0.879  | 0.566  | 0.624    | 0.642 | 0.759 ± 0.005 | 0.651 ± 0.025       | 0.607     |
| musk2    | 0.945 ± 0.030 | 0.392  | 0.443  | 0.390  | 0.385    | 0.419 | 0.591 ± 0.014 | 0.648 ± 0.067       | 0.408     |
| krvskp   | 1.021 ± 0.042 | 0.745  | 0.909  | 0.632  | 0.609    | 0.777 | 0.812 ± 0.014 | 0.811 ± 0.067       | 0.611     |
| waveform | 1.106 ± 0.027 | 0.498  | 0.589  | 0.468  | 0.492    | 0.486 | 0.573 ± 0.026 | 0.686 ± 0.066       | 0.456     |
| spambase | 0.885 ± 0.012 | 0.430  | 0.575  | 0.400  | 0.415    | 0.415 | 0.521 ± 0.027 | 0.620 ± 0.043       | 0.418     |
| mushroom | 0.670 ± 0.012 | 0.383  | 0.489  | 0.359  | 0.350    | 0.367 | 0.502 ± 0.032 | 0.599 ± 0.033       | 0.355     |

In Tables 1-4 we show on bold the best policy on average for each dataset. Note that in many situations the confidence intervals overlap, in which situation no clear winning policy can be determined.

We see that regardless of the weak learner used, it is almost always the case that applying some calibration (even under the worst fixed policy) produces significantly better probability estimates than applying no calibration. This is in vein with results batch boosting [11, 14] and agrees with both our theoretical intuitions and the empirical analysis of Section 3.

Moreover, it is almost always the case that certain bandit policies –more specifically the non-discounted
versions of UCB1 and UCB1-Improved closely followed by Gaussian Thompson Sampling, especially in its discounted version—significantly outperform the best fixed calibration policy. Even when they don’t, the best fixed calibration policy does not significantly outperform them. We can conclude that these policies are producing at least as good probability estimates as the best fixed policy in each case.

To get a clearer picture of this, and see how fast the log-loss reduces under each policy, in Figures 6.2.1 & 6.2.2 we provide some characteristic learning curves. As the bandit policies produce comparably good results, to prevent cluttering, we only include one bandit policy per figure (UCB1-Improved or discounted Gaussian Thompson Sampling), compared against the best and worst fixed policy and the uncalibrated online boosting ensemble. We specifically chose to visualize the results on the datasets for which the best fixed policy is competitive—for some choice of weak learner—with bandit policies when its predictions are evaluated across the entire dataset. In other words, we only provide learning curves for the datasets in which the confidence intervals of the best fixed policy overlap with those of the winning policy in at least one of the Tables 1–4.

6.2.2. Experiments on non-stationary datasets.

Next, we present experiments on the non-stationary datasets for Naive Bayes with $T = 25$ in Table 5 and Figure 6.2.2. The general pattern we observed in the previous set of experiments also appears here. Inspecting the largest dataset used in our study, forest, we can see that the different calibration policies produce similar results with one another. This appears to be because ample datapoints are available and the feature space is relatively small (the dataset consists of 581,012 datapoints and only 54 features) to allow learning both good ensemble parameters and calibrator parameters regardless of the relative amount of data used for each of these learning tasks.
Figure 3: Log-loss versus number of minibatches seen. Lower values correspond to better probability estimation. The best and worst fixed calibration policies are compared against uncalibrated OnlineBoost and to calibration under the non-discounted UCB1-Improved policy. UCB1, discounted Gaussian Thompson Sampling performed similarly. Results for Naive Bayes, $T = 10$ on stationary datasets. Only datasets for which the best fixed policy is competitive with bandit policies are shown. Note how fast the log-loss reduces under the bandit policies.
Figure 4: Log-loss versus number of minibatches seen. Lower values correspond to better probability estimation. The best and worst fixed calibration policies are compared against uncalibrated OnlineBoost and calibration under the non-discounted UCB1-Improved policy. UCB1, discounted Gaussian Thompson Sampling performed similarly. Results for logistic regression, $T = 10$ on stationary datasets. Only datasets for which the best fixed policy is competitive with bandit policies are shown. Note how fast the log-loss reduces under the bandit policies.
Table 5: Log-loss across the entire dataset. Lowest average value shown in bold. Results for Naive Bayes, $T = 25$ on nonstationary datasets. The low variance is due to (i) not shuffling the datapoints on each run, respecting the order in which they appear & (ii) the larger ratio of datapoints to features compared to the stationary datasets used.

| Dataset | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | Disc. UCB1 Improved | Disc. GTS |
|---------|--------------|------------|-------------|---------------|--------------------|----------|
| weather | 1.599 ± 0.062 | 0.583 ± 0.002 | 0.683 ± 0.008 | 0.579 ± 0.001 | 0.579 ± 0.007 | 0.001 ± 0.083 | 0.739 ± 0.036 | **0.572** |
| electricity | 4.662 ± 0.126 | 0.604 ± 0.001 | 0.631 ± 0.004 | 0.599 ± 0.001 | 0.594 ± 0.002 | 0.608 ± 0.002 | 0.612 ± 0.002 | 0.603 ± 0.009 | **0.609** |
| forest | 5.618 ± 0.010 | 0.645 ± 0.001 | 0.660 ± 0.001 | 0.645 ± 0.002 | 0.643 ± 0.001 | 0.647 ± 0.001 | 0.677 ± 0.015 | 0.648 ± 0.006 | **0.657** |

6.3. General observations

Overall, calibration under any policy almost always improves the probability estimation w.r.t. uncalibrated OnlineBoost. As far as fixed policies are concerned, the policy with $N_c = 2$ and that with $N_c = 4$ dominated the rest in our experiments, while $N_c = 12$ and $N_c = 14$ were the values that led to the poorest probability estimates. This was to be expected, as it suggests that the more frequently we calibrate, the better the resulting probability estimates will be (averaged across all rounds).

Moreover, we saw that certain bandit policies (UCB1 policies without reward discounting and discounted-reward Gaussian Thompson Sampling) consistently exceed or at least match the probability estimation performance of the best fixed calibration policy. These results are robust to the choice of weak learner and the presence or absence of non-stationarity in the dataset but also to the ensemble size and degree of weak learner regularization (see Appendix C).

It should also be noted that the computational cost of the bandit policies is the same as that of the fixed ones (assuming a given value of $N_c$ for the latter) and that finding the best fixed policy requires a search over the possible values of $N_c$, something impossible in an online setting without parallelization or increasing the computational cost. Finally, bandit policies allow the ratio of training over calibrating steps to be adaptive, unlike, fixed policies. All these reasons make bandit policies superior to naive calibration.

Overall, discounting rewards—at least as applied to our experiments—considerably improves Thompson Sampling, but greatly deteriorates UCB-based policies. In Thompson Sampling, discounting increases the
Figure 5: Log-loss versus number of minibatches seen. Lower values correspond to better probability estimation. The best and worst fixed calibration policies are compared against calibration under the non-discounted UCB1-Improved policy [LEFT] or the discounted Gaussian Thompson Sampling policy [RIGHT]. Results for Naive Bayes, $T = 25$ on nonstationary datasets. Uncalibrated OnlineBoost curves were omitted due to their large log-loss (see Table 5).
variance of the reward posteriors. This increases the probability of the currently non-optimal arm being pulled. On the other hand, we observed that discounted UCB policies tended to get ‘trapped’ to situations where only one action (either ‘train’ or ‘calibrate’) was taken. In discounted UCB policies, the padding function’s value for an action shrinks with the number of times it has been performed. It appears that in the situations examined, the padding function’s value for the action performed shrinks more and more slowly and never gets to the point where it eventually allows the upper confidence bound of the other action to overtake its own.

7. Conclusion and Future Work

We examined probability estimation in online boosting ensembles and found that the scores they generate are distorted in a systematic fashion. We saw that –as in the case of batch boosting– calibration can greatly improve the probability estimates. We resolved the problem of deciding when to train the ensemble and when to calibrate with the use of bandit optimization. More specifically, UCB1 policies\cite{22, 23, 24} without reward discounting and Thompson Sampling\cite{25, 26, 27}, especially with reward discounting were found to perform at least as well as the best naive calibration policy in terms of probability estimation in every experiment.

The merits of using bandit policies over naive calibration are manifold. Not only is the overall probability estimation performance superior, but it also converges much faster than the latter (in terms of minibatches seen). Moreover, to find the best naive calibration policy (i.e. fixed policy of calibrating on every $N_c$ rounds), we need to either determine the value of $N_c$ in advance, which is not possible in an online –possibly non-stationary– setting. Furthermore, a fixed policy would be unable to adapt to non-stationarity: the optimal ratio of train and calibration actions might need to change during the course of training. The memory and computational complexity of the Bandit policies we examine here is constant w.r.t. the number of examples seen (as is required for online learning) and is the same as that of the fixed policies. We found that the superiority of these policies is robust to the choice of weak learner, ensemble size, degree of regularization and across datasets –both stationary and non-stationary. Finally, it would be straightforward to apply the same techniques to other types of learning tasks e.g. cost-sensitive or imbalanced class learning (situations in which good probability estimates are necessary for making decisions), simply by changing the reward function.

The purpose of this work was not to determine the best calibration method for an online boosting ensemble, but rather to identify fast, flexible and successful policies for balancing between training the ensemble and the calibrator (regardless of their specifics) in a hyperparameter-free fashion. In future work, to improve probability estimation, we can use the online version of isotonic regression \cite{50}, or produce online adaptations of spline calibration \cite{51} or beta calibration \cite{52} to calibrate the scores of the ensemble. Isotonic
regression is a non-parametric method that can capture score distortions of any non-decreasing form. It is prone to overfitting in the presence of small samples, but is expected to outperform logistic calibration as the number of available data grows. Spline calibration is a smoothed version of the –piecewise-linear– former. Beta calibration is an improvement over logistic calibration, especially when score distributions are heavily skewed or when the scores happen to already be calibrated.

An alternative direction for future work could be to explore different bandit policies based on more relaxed learning assumptions. These could include the use of contextual bandits [53] that take into account the feature vector $x$ of new instances when selecting the next action. Another family of bandit policies worth exploring is that of adversarial bandits [35], which can handle environments that adapt to our actions.

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Appendix A: Proof sketch of Theorem 1

Theorem. OnlineBoost greedily minimizes the exponential loss of the margin $L(y, F(x)) = e^{-yF(x)}$ via stochastic gradient descent steps in the space of functions $F(x)$.

Proof Sketch. In Algorithm 4 we repeat the OnlineBoost algorithm for convenience.

Algorithm 4 OnlineBoost (Original version)

| Input: Number of weak learners $T$, training examples $\{(x_i, y_i) | i = 1, \ldots, N\}$ presented one at a time |
| For each $i$ do: |
| Set example weight $\lambda = 1$ |
| For each $t \in 1, 2, \ldots, T$ do: |
| Set $k$ according to $\text{Poisson}(\lambda)$ |
| Do $k$ times: |
| $h_t \leftarrow \text{OnlineLearnAlg}(h_t, (x_i, y_i))$ |
| If $h_t(x_i) = y_i$: |
| $\lambda_t^{ec} \leftarrow \lambda_t^{ec} + \lambda$ |
| $\epsilon_t = \frac{\lambda_t^{sw}}{\lambda_t^{sw} + \lambda_t^{ec}}$ |
| $\lambda \leftarrow \lambda \times \frac{1}{2(1 - \epsilon_t)}$ |
| Else: |
| $\lambda_t^{sw} \leftarrow \lambda_t^{sw} + \lambda$ |
| $\epsilon_t = \frac{\lambda_t^{sw}}{\lambda_t^{sw} + \lambda_t^{ec}}$ |
| $\lambda \leftarrow \lambda \times \frac{1}{2\epsilon_t}$ |

Prediction: On example $(x, y)$, predict $H(x) = \text{sign}\left[\sum_{t=1}^{T} h_t(x) \log \frac{1-\epsilon_t}{\epsilon_t}\right]$}

In Algorithm 5 we rewrite OnlineBoost in such a way that the weight assigned to each example $i \in \{1, \ldots, N\}$ for the purposes of updating the parameters of each base learner $t \in \{1, \ldots, T\}$ is stored in a separate variable $\lambda_t(i)$, rather than overwritten on each update on the variable $\lambda$. Storing so many variables is –of course– prohibitive in the online setting, but here we do this merely for illustrative purposes.

The equivalence of Algorithm 5 to Algorithm 4 is trivial, thus we will not give an explicit proof. It is based on the fact that $\lambda$ in Algorithm 4 will equal to $\lambda_{t-1}(i)$ of Algorithm 5 before updating the weights of the $t$-th weak learner according to the $i$-th datapoint and to $\lambda_t(i)$ after. Once all $T$ weak learners are trained on the $i$-th datapoint, $\lambda$ will become equal to $\lambda_t(i)$ and so on.

Using the notation introduced in Algorithm 5 we can denote the sums of weights corresponding to
Algorithm 5 OnlineBoost (storing all weights)

**Input:** Number of weak learners \( T \), training examples \( \{(x_i, y_i) | i = 1, \ldots, N\} \) available one at a time

For each \( i \) do:
- Set example weight \( \lambda_0(i) = 1 \)

For each \( t \in \{1, 2, \ldots, T\} \) do:
  - Set \( k \) according to \( \text{Poisson}(\lambda_{t-1}(i)) \)
  - Do \( k \) times:
    - If \( h_t(x_i) = y_i \):
      - \( \lambda^c_t \leftarrow \lambda^c_{t-1}(i) + \lambda_{t-1}(i) \)
      - \( \epsilon_t = \frac{\lambda^w_t}{\lambda^w_t + \lambda^c_t} \)
      - \( \lambda_t(i) \leftarrow \lambda_{t-1}(i) \times \frac{1}{\lambda^c_t} \)
    - Else:
      - \( \lambda^w_t \leftarrow \lambda^w_{t-1}(i) + \lambda_{t-1}(i) \)
      - \( \epsilon_t = \frac{\lambda^w_t}{\lambda^w_t + \lambda^c_t} \)
      - \( \lambda_t(i) \leftarrow \lambda_{t-1}(i) \times \frac{1}{2\epsilon_t} \)

**Prediction:** On example \((x, y)\), predict \( H(x) = \text{sign} \left[ \sum_{t=1}^{T} h_t(x) \log \frac{1 - \epsilon_t}{\epsilon_t} \right] \)

We now claim that Algorithm 6 is another reformulation of OnlineBoost, thus equivalent to Algorithm 5. This is less straightforward to see, so we shall prove that the steps of the two algorithms are equivalent. Notice that the changes w.r.t. Algorithm 5 occur in the weight update rule, the calculation of the confidence coefficients, and the final decision rule. Let us inspect these to verify that the two algorithms are equivalent.

**Confidence coefficients:** Algorithm 5 does not explicitly store the confidence coefficients (i.e. the voting weights \( \beta_t \)) of the weak learners. It does not need to, as \( \beta_t \) is calculated based on \( \epsilon_t \), which in turn only requires \( \lambda^c_t \) and \( \lambda^w_t \). However, as we will see, in Algorithm 6 we can define the quantity,

\[
\beta_t = \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t},
\]

where—as we will see—\( \epsilon_t \) is the same in both algorithms, and use it to produce equivalent weight update and prediction rules.
Algorithm 6 OnlineBoost (Reformulated –resampling version)

**Input:** Number of weak learners \( T \), training examples \( \{(x_i, y_i) | i = 1, \ldots, N\} \) available one at a time

**For each** \( i \): Set example weight \( \lambda_0(i) = 1 \)

**For each** \( t \in 1, 2, \ldots, T \): Do \( k \) times:

- \( h_t \leftarrow \text{OnlineLearnAlg}(h_{t-1}(x_i)) \)
- \( \epsilon_t = \sum_{j: h_t(x_j) \neq y_j} \lambda_{t-1}(j) / \sum_{j: h_t(x_j) = y_j} \lambda_{t-1}(j) \)
- \( \beta_t = \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t} \)
- \( \lambda_t(i) = \frac{1}{Z_t} \lambda_{t-1}(i) e^{-y_i \beta_t h_t(x_i)} \), where \( Z_t = \frac{2(1 - \epsilon_t)}{e^{\beta_t}} \) (constant w.r.t. \( i \))

**Prediction:** On example \( (x, y) \), predict \( H(x) = \text{sign} \left[ \sum_{t=1}^{T} \beta_t h_t(x) \right] \)

**Weight updates:** In Algorithm 6, the weight of the \( i \)-th example for the purposes of updating the parameters of the next weak learner is given by the equation

\[
\lambda_t(i) = \frac{1}{Z_t} \lambda_{t-1}(i) e^{-y_i \beta_t h_t(x_i)}, \quad \text{where} \quad Z_t = \frac{2(1 - \epsilon_t)}{e^{\beta_t}}
\]

is used for normalization. This weight update rule can be rewritten as

\[
\lambda_t(i) = \frac{e^{\beta_t}}{2(1 - \epsilon_t)} \lambda_{t-1}(i) e^{-y_i \beta_t h_t(x_i)} = \frac{e^{\beta_t} \log \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)^{\frac{1}{2}}}{2(1 - \epsilon_t)} \lambda_{t-1}(i) e^{-y_i \beta_t h_t(x_i)} = \frac{\lambda_{t-1}(i) \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)^{\frac{1}{2}} \left( \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)^{\frac{1}{2}} \right)^{-y_i \beta_t h_t(x_i)}}{2(1 - \epsilon_t)} = \begin{cases} 
\lambda_{t-1}(i) \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)^{\frac{1}{2}} \left( \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)^{\frac{1}{2}} \right)^{-y_i \beta_t h_t(x_i)} & , h_t(x_i) = y_i \\
\lambda_{t-1}(i) \left( \frac{1 - \epsilon_t}{\epsilon_t} \right)^{\frac{1}{2}} & , h_t(x_i) \neq y_i.
\end{cases}
\]

And from Eq. (11), we have that

\[
\epsilon_t = \frac{\sum_{j: h_t(x_j) \neq y_j} \lambda_{t-1}(j)}{\sum_{j: h_t(x_j) \neq y_j} \lambda_{t-1}(j) + \sum_{j: h_t(x_j) = y_j} \lambda_{t-1}(j)} = \frac{\lambda_t^{sw}}{\lambda_t^{sw} + \lambda_t^{sc}}.
\]

Therefore, the weight update rules of the two algorithms are identical.
**Prediction Rule.** In the new formulation of Algorithm 6, the predictions are given by

\[
    H(x) = \text{sign} \left[ \sum_{t=1}^{T} \beta_t h_t(x) \right] = \text{sign} \left[ \sum_{t=1}^{T} \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t} \right] = \text{sign} \left[ \frac{1}{2} \sum_{t=1}^{T} h_t(x) \log \frac{1 - \epsilon_t}{\epsilon_t} \right] = \text{sign} \left[ \sum_{t=1}^{T} h_t(x) \log \frac{1 - \epsilon_t}{\epsilon_t} \right].
\]

So, from Eq. (14), we have that Algorithm 5 and Algorithm 6 use the same prediction rule. Therefore all the steps of Algorithm 5 and Algorithm 6 are equivalent, and since Algorithm 5 is a reformulation of OnlineBoost, so is Algorithm 6.

Algorithm 7 is a final reformulation of OnlineBoost. WeightedOnlineLearnAlg\(h_t, (x_i, y_i), \lambda_{t-1}(i)\) is a weighted online learning algorithm that updates the parameters of the weak learner \(h_t\) on example \((x_i, y_i)\) using a weight of \(\lambda_{t-1}(i)\). The rationale is that instead of training the new classifier \(k\) times on the \(i\)-th example, we can train it once with a weight \(k\) (i.e. taking the reweighting rather than the resampling approach to boosting). Moreover, as each time \(k \sim \text{Poisson}(\lambda_{t-1}(i))\), for the purposes of updating the parameters of the \(t\)-th weak learner on the \(i\)-th example, we have that \(\lambda_{t-1}(i) = E[k]\). So by training the \(t\)-th weak learner on the \(i\)-th example with a weight of \(\lambda_{t-1}(i)\) we get—in expectation—the same updates. This is the only change w.r.t. Algorithm 6.

**Algorithm 7** OnlineBoost (Reformulated—rewriting version using expected weights \(\lambda_{t-1}(i)\))

**Input:** Number of weak learners \(T\), training examples \(\{(x_i, y_i)\mid i = 1, \ldots, N\}\) available one at a time

**For each** \(i\) **do:**

- Set example weight \(\lambda_0(i) = 1\)

**For each** \(t \in 1, 2, \ldots, T\) **do:**

\[
    h_t \leftarrow \text{WeightedOnlineLearnAlg}(h_t, (x_i, y_i), \lambda_{t-1}(i))
\]

\[
    \epsilon_t = \frac{\sum_{j \in \{1, \ldots, k\} \mid j \neq y_i} \lambda_{t-1}^{(j)}}{\sum_{j \in \{1, \ldots, k\} \mid j \neq y_i} \lambda_{t-1}^{(j)} + \sum_{j \in \{1, \ldots, k\} \mid j = y_i} \lambda_{t-1}^{(j)}}
\]

\[
    \beta_t = \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t}
\]

\[
    \lambda_t(i) = \frac{1}{Z_t} \lambda_{t-1}(i) e^{-y_i \beta_t h_t(x_i)}, \text{ where } Z_t = \frac{2(1 - \epsilon_t)}{e^{\epsilon_t}} \text{ (constant w.r.t. } i)\]

**Prediction:** On example \((x, y)\), predict \(H(x) = \text{sign} \left[ \sum_{t=1}^{T} \beta_t h_t(x) \right] \)

Finally, in Algorithm 8 we present (the reweighting version of) batch AdaBoost with a fixed ensemble size \(T\). It uses a batch learning algorithm to train the weak learner \(h_t\) on the full dataset using weights \(\lambda_{t-1}(i), \forall i\), denoted as WeightedBatchLearnAlg\(\{(x_i, y_i), \lambda_{t-1}(i)\)}, \forall i\)).

---

13 This step of the proof is somewhat redundant as simply establishing the equivalence of OnlineBoost to AdaBoost-by-reweighting would have been sufficient for the purposes of this paper.
Algorithm 8 AdaBoost (Fixed ensemble size $T$)

**Input:** Number of weak learners $T$, training examples $\{(x_i, y_i) | i = 1, \ldots, N\}$

Set example weight $\lambda_0(i) = 1$, $\forall i$

**For each** $t \in \{1, 2, \ldots, T\}$ **do:**

- $h_t \leftarrow \text{WeightedBatchLearnAlg}(\{(x_i, y_i) | \forall i\}, \{\lambda_{t-1}(i) | \forall i\})$
- $\epsilon_t = \frac{\sum_{i : h_t(x_i) \neq y_i} \lambda_{t-1}(i)}{\sum_{i : h_t(x_i) = y_i} \lambda_{t-1}(i) + \sum_{i : h_t(x_i) \neq y_i} \lambda_{t-1}(i)}$
- $\beta_t = \frac{1}{2} \log \frac{1 - \epsilon_t}{\epsilon_t}$
- $\lambda_t(i) = \frac{1}{Z_t} \lambda_{t-1}(i) e^{-y_i \beta_t h_t(x_i)}$, $\forall i$, where $Z_t$ is constant w.r.t. $i$

**Prediction:** On example $(x, y)$, predict $H(x) = \text{sign} \left[ \sum_{t=1}^{T} \beta_t h_t(x) \right]$

Comparing Algorithm 8 to Algorithm 7 (equivalent to OnlineBoost), we see that the only difference is that all parameters estimated in the latter are updated only on the examples seen so far, rather than on the entire dataset (something natural for the online setting).

Since the steps of AdaBoost can be derived as (batch) gradient descent on an exponential loss of the margin $L(y, F(x)) = e^{-y F(x)}$ in the space of functions $F(x)$ \[54 \text{[4]}\], and since OnlineBoost uses the same steps, but updates all parameters based on one example at a time, we can conclude that OnlineBoost minimizes the same loss function as AdaBoost (i.e. the exponential loss), but taking stochastic gradient descent steps in the space of functions $F(x)$.
Appendix B: Datasets used

Table 6: Characteristics of the datasets used in our experiments; number of instances used, number of features, number of classes and presence or not of dataset shift. All stationary datasets can be found in the UCI repository. The non-stationary datasets were taken from [55].

| Dataset   | # Instances | # Features | # Classes | Considered Stationary |
|-----------|-------------|------------|-----------|-----------------------|
| landsat   | 1,252       | 36         | 6         | Yes                   |
| splice    | 1,524       | 60         | 3         | Yes                   |
| musk2     | 2,034       | 166        | 2         | Yes                   |
| kruskp    | 3,054       | 36         | 2         | Yes                   |
| waveform  | 3,306       | 40         | 3         | Yes                   |
| spambase  | 3,626       | 57         | 2         | Yes                   |
| mushroom  | 7,832       | 21         | 2         | Yes                   |
| weather   | 18,159      | 8          | 2         | No                    |
| electricity| 45,312      | 8          | 2         | No                    |
| forest    | 581,012     | 54         | 7         | No                    |
Appendix C: Additional results

Here we provide tables of log-loss across all examples, for each method, for all the experiments mentioned in the paper, along with learning curves of log-loss versus number of minibatches seen.

**Effect of weak learner choice**

We provide results on the stationary datasets for ensembles of size $T = 10$ for four different types of weak learners: Gaussian Naive Bayes, logistic regression, linear SVM and perceptron.

Table 7: Log-loss across the entire dataset. Lowest average value shown in bold. Results for Naive Bayes, $T = 10$ on stationary datasets.

| Dataset | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | GTS | Disc. UCB1 Improved | Disc. GTS |
|---------|--------------|------------|-------------|---------------|-----|---------------------|-----------|
| landsat | 0.138        | 0.081      | 0.131       | 0.072         | 0.071 | 0.074               | 0.129     | 0.123               | 0.073     |
|         | ± ±          | ± ±        | ± ±         | ± ±           | ± ±   | ± ±                 | ± ±       | ± ±                 | ± ±       |
|         | 0.002        | 0.002      | 0.004       | 0.001         | 0.001 | 0.002               | 0.010     | 0.011               | 0.002     |
| splice  | 0.444        | 0.280      | 0.410       | 0.234         | 0.229 | 0.282               | 0.353     | 0.376               | 0.232     |
|         | ± ±          | ± ±        | ± ±         | ± ±           | ± ±   | ± ±                 | ± ±       | ± ±                 | ± ±       |
|         | 0.034        | 0.014      | 0.014       | 0.007         | 0.009 | 0.022               | 0.024     | 0.037               | 0.008     |
| musk2   | 0.723        | 0.343      | 0.427       | 0.320         | 0.332 | 0.397               | 0.711     | 0.441               | 0.345     |
|         | ± ±          | ± ±        | ± ±         | ± ±           | ± ±   | ± ±                 | ± ±       | ± ±                 | ± ±       |
|         | 0.027        | 0.008      | 0.013       | 0.006         | 0.003 | 0.008               | 0.025     | 0.049               | 0.008     |
| kruskp  | 1.252        | 0.488      | 0.720       | 0.474         | 0.472 | 0.486               | 0.627     | 0.767               | 0.468     |
|         | ± ±          | ± ±        | ± ±         | ± ±           | ± ±   | ± ±                 | ± ±       | ± ±                 | ± ±       |
|         | 0.154        | 0.014      | 0.075       | 0.019         | 0.012 | 0.016               | 0.114     | 0.114               | 0.020     |
| waveform| 0.824        | 0.361      | 0.466       | 0.335         | 0.334 | 0.342               | 0.458     | 0.492               | 0.344     |
|         | ± ±          | ± ±        | ± ±         | ± ±           | ± ±   | ± ±                 | ± ±       | ± ±                 | ± ±       |
|         | 0.012        | 0.005      | 0.009       | 0.002         | 0.004 | 0.007               | 0.015     | 0.024               | 0.006     |
| spambase| 2.37         | 0.532      | 0.735       | 0.493         | 0.483 | 0.481               | 0.536     | 0.540               | 0.489     |
|         | ± ±          | ± ±        | ± ±         | ± ±           | ± ±   | ± ±                 | ± ±       | ± ±                 | ± ±       |
|         | 0.026        | 0.006      | 0.017       | 0.006         | 0.005 | 0.008               | 0.028     | 0.037               | 0.005     |
| mushroom| 0.770        | 0.358      | 0.503       | 0.375         | 0.335 | 0.318               | 0.570     | 0.570               | 0.327     |
|         | ± ±          | ± ±        | ± ±         | ± ±           | ± ±   | ± ±                 | ± ±       | ± ±                 | ± ±       |
|         | 0.057        | 0.011      | 0.017       | 0.018         | 0.020 | 0.006               | 0.036     | 0.072               | 0.010     |
Table 8: Log-loss across the entire dataset. Lowest average value shown in bold. Results for logistic regression, $T = 10$ on stationary datasets.

| Dataset | Uncalibrated | Best  | Worst | UCB1 | UCB1 Improved | GTS | Disc. UCB1 | Disc. UCB1 Improved | Disc. GTS |
|---------|--------------|-------|-------|------|---------------|-----|------------|---------------------|----------|
| landsat | 0.330        | 0.106 | 0.158 | 0.107| 0.108         | 0.108| 0.173      | 0.203               | 0.127    |
|         | ±            | ±     | ±     | ±    | ±             | ±   | ±          | ±                   | ±        |
|         | 0.011        | 0.002 | 0.004 | 0.003| 0.004         | 0.004| 0.019      | 0.018               | 0.013    |
| splice  | 1.779        | 0.626 | 0.895 | 0.506| 0.554         | 0.636| 0.711      | 1.155               | 0.584    |
|         | ±            | ±     | ±     | ±    | ±             | ±   | ±          | ±                   | ±        |
|         | 0.059        | 0.023 | 0.034 | 0.029| 0.020         | 0.068| 0.117      | 0.169               | 0.034    |
| musk2   | 1.276        | 0.331 | 0.398 | 0.321| 0.330         | 0.318| 0.536      | 0.514               | 0.322    |
|         | ±            | ±     | ±     | ±    | ±             | ±   | ±          | ±                   | ±        |
|         | 0.015        | 0.003 | 0.008 | 0.004| 0.004         | 0.003| 0.049      | 0.038               | 0.006    |
| kruskp  | 1.151        | 0.633 | 0.881 | 0.519| 0.528         | 0.645| 0.825      | 0.733               | 0.555    |
|         | ±            | ±     | ±     | ±    | ±             | ±   | ±          | ±                   | ±        |
|         | 0.073        | 0.041 | 0.042 | 0.037| 0.017         | 0.048| 0.074      | 0.041               | 0.033    |
| waveform| 1.700        | 0.444 | 0.596 | 0.427| 0.429         | 0.428| 0.503      | 0.519               | 0.464    |
|         | ±            | ±     | ±     | ±    | ±             | ±   | ±          | ±                   | ±        |
|         | 0.037        | 0.004 | 0.019 | 0.003| 0.007         | 0.006| 0.026      | 0.030               | 0.016    |
| spambase| 1.182        | 0.409 | 0.572 | 0.390| 0.395         | 0.407| 0.540      | 0.477               | 0.389    |
|         | ±            | ±     | ±     | ±    | ±             | ±   | ±          | ±                   | ±        |
|         | 0.018        | 0.006 | 0.010 | 0.005| 0.006         | 0.007| 0.029      | 0.024               | 0.006    |
| mushroom| 1.001        | 0.343 | 0.454 | 0.322| 0.315         | 0.312| 0.531      | 0.592               | 0.330    |
|         | ±            | ±     | ±     | ±    | ±             | ±   | ±          | ±                   | ±        |
|         | 0.023        | 0.007 | 0.017 | 0.007| 0.006         | 0.007| 0.026      | 0.026               | 0.006    |
Table 9: Log-loss across the entire dataset. Lowest average value shown in bold. Results for linear SVM, $T = 10$ on stationary datasets.

| Dataset  | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | Disc. UCB1 | Disc. UCB1 Improved | Disc. GTS |
|----------|--------------|------------|-------------|---------------|-------------|---------------------|-----------|
| landsat  | 0.245        | 0.130      | 0.184       | 0.118         | 0.127       | 0.250               | 0.243     | **0.111**           |
|          | ±            | ±          | ±           | ±             | ±           | ±                   | ±         |
|          | 0.019        | 0.009      | 0.010       | 0.008         | 0.008       | 0.010               | 0.025     | 0.034 **0.004**     |
| splice   | 1.391        | 0.663      | 0.922       | 0.622         | **0.586**   | 0.609               | 0.816     | 0.666 0.640         |
|          | ±            | ±          | ±           | ±             | ±           | ±                   | ±         |
|          | 0.064        | 0.018      | 0.040       | 0.031         | **0.022**   | 0.020               | 0.105     | 0.088 0.038         |
| musk2    | 0.953        | **0.385**  | 0.432       | 0.400         | 0.390       | 0.406               | 0.639     | 0.622 0.386         |
|          | ±            | ±          | ±           | ±             | ±           | ±                   | ±         |
|          | 0.050        | 0.010      | 0.027       | **0.021**     | 0.027       | 0.029               | 0.047     | 0.041 0.030         |
| kruskp   | 1.081        | 0.671      | 0.897       | **0.608**     | 0.648       | 0.677               | 0.706     | 0.811 0.667         |
|          | ±            | ±          | ±           | ±             | ±           | ±                   | ±         |
|          | 0.050        | 0.010      | 0.027       | **0.021**     | 0.027       | 0.029               | 0.047     | 0.041 0.030         |
| waveform | 1.156        | 0.487      | 0.615       | 0.453         | **0.451**   | **0.451**           | 0.611     | 0.603 0.464         |
|          | ±            | ±          | ±           | ±             | ±           | ±                   | ±         |
|          | 0.045        | 0.013      | 0.015       | **0.004**     | **0.005**   | 0.021               | 0.076     | 0.005               |
| spambase | 0.893        | 0.440      | 0.551       | 0.411         | **0.410**   | 0.418               | 0.560     | 0.612 0.414         |
|          | ±            | ±          | ±           | ±             | ±           | ±                   | ±         |
|          | 0.019        | 0.008      | 0.010       | **0.007**     | **0.005**   | 0.009               | 0.028     | 0.053 0.0037        |
| mushroom | 0.673        | 0.374      | 0.489       | 0.359         | 0.363       | **0.352**           | 0.506     | 0.573 0.357         |
|          | ±            | ±          | ±           | ±             | ±           | ±                   | ±         |
|          | 0.015        | 0.006      | 0.015       | 0.006         | **0.005**   | 0.036               | 0.039     | 0.007               |
Table 10: Log-loss across the entire dataset. Lowest average value shown in bold. Results for perceptron, \( T = 10 \) on stationary datasets.

| Dataset    | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | Disc. UCB1 Improved | Disc. GTS | Disc. UCB1 Improved | Disc. GTS |
|------------|--------------|------------|-------------|---------------|---------------------|-----------|---------------------|-----------|
| landsat    | 0.258        | 0.120      | 0.190       | 0.125         | **0.103**           | 0.115     | 0.164               | 0.198     |
|            | ±            | ±          | ±           | ±             | ±                   | ±         | ±                   | ±         |
|            | 0.012        | 0.005      | 0.009       | 0.013         | **0.004**           | 0.005     | 0.014               | 0.028     |
| splice     | 1.453        | 0.683      | 0.879       | **0.566**     | 0.624               | 0.642     | 0.759               | 0.651     |
|            | ±            | ±          | ±           | ±             | ±                   | ±         | ±                   | ±         |
|            | 0.037        | 0.033      | 0.050       | **0.021**     | 0.028               | 0.025     | 0.105               | 0.022     |
| musk2      | 0.945        | 0.392      | 0.443       | 0.390         | **0.385**           | 0.419     | 0.591               | 0.648     |
|            | ±            | ±          | ±           | ±             | ±                   | ±         | ±                   | ±         |
|            | 0.030        | 0.006      | 0.012       | 0.006         | **0.006**           | 0.014     | 0.044               | 0.067     |
| krskp      | 1.021        | 0.745      | 0.909       | 0.632         | **0.609**           | 0.777     | 0.812               | 0.811     |
|            | ±            | ±          | ±           | ±             | ±                   | ±         | ±                   | ±         |
|            | 0.042        | 0.031      | 0.043       | 0.037         | **0.022**           | 0.079     | 0.069               | 0.062     |
| waveform   | 1.106        | 0.498      | 0.589       | 0.468         | 0.492               | 0.486     | 0.573               | 0.686     |
|            | ±            | ±          | ±           | ±             | ±                   | ±         | ±                   | ±         |
|            | 0.027        | 0.010      | 0.021       | 0.010         | **0.034**           | 0.026     | 0.029               | 0.066     |
|            |              |            |             |               |                     |           |                     |           |
| spambase   | 0.885        | 0.430      | 0.575       | **0.400**     | 0.415               | 0.415     | 0.521               | 0.620     |
|            | ±            | ±          | ±           | ±             | ±                   | ±         | ±                   | ±         |
|            | 0.012        | 0.006      | 0.010       | **0.004**     | 0.004               | 0.007     | 0.027               | 0.043     |
| mushroom   | 0.670        | 0.383      | 0.489       | 0.359         | **0.350**           | 0.367     | 0.502               | 0.599     |
|            | ±            | ±          | ±           | ±             | ±                   | ±         | ±                   | ±         |
|            | 0.012        | 0.006      | 0.011       | 0.003         | **0.006**           | 0.004     | 0.032               | 0.033     |

7.1. Effect of ensemble size

Next we examined the effect of different ensemble sizes, using \( T \in \{10, 25, 50\} \). For this experiment we picked logistic regression as the weak learner. We present results on the stationary datasets.
Table 11: Log-loss across the entire dataset. Lowest average value shown in bold. Results for logistic regression, $T = 25$ on stationary datasets.

| Dataset  | Uncalibrated | Best | Worst | UCB1   | UCB1   | Disc. | Disc. | Disc. |
|----------|--------------|------|-------|--------|--------|-------|-------|-------|
|          |              | Fixed| Fixed | Improved| GTS    | UCB1  | UCB1  | GTS   |
|          |              |      |       |        |        |       |       |       |
| landsat  | 0.323        | 0.120| 0.182 | 0.118  | **0.113**| 0.124 | 0.202 | 0.251 | 0.120 |
|          | ±            | ±    | ±     | ±      | ±      | ±     | ±     | ±     |
|          | 0.009        | 0.003| 0.001 | 0.004  | **0.004**| 0.004 | 0.030 | 0.029 | 0.004 |
| splice   | 1.553        | 0.684| 0.975 | 0.607  | 0.607  | 0.723 | 0.667 | 0.747 | **0.594**|
|          | ±            | ±    | ±     | ±      | ±      | ±     | ±     | ±     |
|          | 0.064        | 0.021| 0.042 | 0.016  | 0.018  | 0.073 | 0.058 | 0.121 | **0.021**|
| musk2    | 1.515        | 0.343| 0.401 | 0.343  | **0.342**| 0.344 | 0.565 | 0.461 | **0.342**|
|          | ±            | ±    | ±     | ±      | ±      | ±     | ±     | ±     |
|          | 0.020        | 0.003| 0.007 | 0.003  | **0.002**| 0.005 | 0.037 | 0.037 | **0.002**|
| kruskp   | 1.155        | 0.698| 0.895 | 0.605  | **0.583**| 0.667 | 0.774 | 0.765 | 0.603 |
|          | ±            | ±    | ±     | ±      | ±      | ±     | ±     | ±     |
|          | 0.060        | 0.020| 0.040 | 0.039  | **0.023**| 0.030 | 0.037 | 0.042 | 0.030 |
| waveform | 1.427        | 0.470| 0.572 | **0.441**| 0.457  | 0.453 | 0.517 | 0.551 | 0.453 |
|          | ±            | ±    | ±     | ±      | ±      | ±     | ±     | ±     |
|          | 0.047        | 0.009| 0.011 | **0.006**| 0.019  | 0.011 | 0.042 | 0.048 | 0.009 |
| spambase | 1.226        | 0.430| 0.590 | 0.410  | **0.407**| 0.412 | 0.519 | 0.524 | **0.407**|
|          | ±            | ±    | ±     | ±      | ±      | ±     | ±     | ±     |
|          | 0.019        | 0.007| 0.010 | **0.003**| 0.005  | 0.035 | 0.033 | 0.007 |
| mushroom | 1.043        | 0.362| 0.457 | **0.324**| 0.333  | 0.353 | 0.431 | 0.547 | 0.347 |
|          | ±            | ±    | ±     | ±      | ±      | ±     | ±     | ±     |
|          | 0.013        | 0.005| 0.012 | **0.004**| 0.004  | 0.010 | 0.031 | 0.043 | 0.007 |
Table 12: Log-loss across the entire dataset. Lowest average value shown in bold. Results for logistic regression, $T = 50$ on stationary datasets.

| Dataset | Uncalibrated Best | Fixed Worst | UCB1 Fixed | UCB1 Improved | GTS | Disc. UCB1 Improved | Disc. UCB1 Improved | GTS |
|---------|-------------------|------------|------------|---------------|-----|---------------------|---------------------|-----|
| landsat | 0.296             | 0.132      | 0.176      | 0.133         | 0.122 | 0.129               | 0.151               | 0.180 | 0.125 |
|         | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± |
|         | 0.012             | 0.005      | 0.006      | 0.005         | 0.005 | 0.007               | 0.017               | 0.014 | 0.004 |
| splice  | 1.637             | 0.647      | 0.907      | 0.548         | 0.577 | 0.607               | 0.698               | 0.652 | 0.587 |
|         | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± |
|         | 0.069             | 0.023      | 0.050      | 0.010         | 0.026 | 0.034               | 0.066               | 0.030 | 0.038 |
| musk2   | 1.520             | 0.350      | 0.403      | 0.350         | 0.346 | 0.353               | 0.452               | 0.550 | 0.348 |
|         | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± |
|         | 0.013             | 0.004      | 0.008      | 0.003         | 0.004 | 0.003               | 0.036               | 0.045 | 0.003 |
| kruskp  | 1.224             | 0.675      | 0.844      | 0.582         | 0.658 | 0.616               | 0.701               | 0.715 | 0.597 |
|         | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± |
|         | 0.045             | 0.024      | 0.032      | 0.010         | 0.033 | 0.029               | 0.024               | 0.020 | 0.009 |
| waveform| 1.439             | 0.470      | 0.591      | 0.482         | 0.444 | 0.482               | 0.556               | 0.563 | 0.441 |
|         | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± |
|         | 0.033             | 0.005      | 0.015      | 0.031         | 0.006 | 0.028               | 0.030               | 0.027 | 0.005 |
| spambase| 1.241             | 0.429      | 0.584      | 0.397         | 0.399 | 0.410               | 0.527               | 0.527 | 0.414 |
|         | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± |
|         | 0.018             | 0.005      | 0.011      | 0.006         | 0.006 | 0.007               | 0.028               | 0.032 | 0.005 |
| mushroom| 1.082             | 0.363      | 0.481      | 0.345         | 0.344 | 0.340               | 0.464               | 0.529 | 0.345 |
|         | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± | ± ± ± ± ± ± ± ± ± ± |
|         | 0.017             | 0.004      | 0.010      | 0.005         | 0.006 | 0.009               | 0.040               | 0.044 | 0.008 |

7.2. Effect of weak learner regularization

We now investigate different degrees of regularization on the weak learner. We used $\ell_1$-regularized logistic regression with a regularization parameter $\lambda \in \{10^{-1}, 10^{-2}, 10^{-3}, 0\}$, with $T = 10$. We present experiments on the stationary datasets.
Table 13: Log-loss across the entire dataset. Lowest average value shown in bold. Results for $\ell_1$-regularized logistic regression, $T = 10$ and $\lambda = 10^{-3}$ on stationary datasets.

| Dataset | Uncalibrated | Best | Worst | Fixed | Fixed | UCB1 Improved | GTS | Disc. UCB1 Improved | Disc. | Disc. GTS |
|---------|--------------|------|-------|-------|-------|---------------|-----|--------------------|--------|----------|
| landsat |              | 0.335 | 0.105 | 0.153 | 0.101 | 0.103         | 0.105 | 0.150              | 0.172  | 0.118    |
|         | ±            | ±     | ±     | ±     | ±     | ±             | ±   | ±                  | ±      | ±        |
|         | 0.006        | 0.003 | 0.004 | 0.003 | 0.002 | 0.003         | 0.015 | 0.027              | 0.005  |          |
| splice  |              | 1.761 | 0.604 | 0.885 | 0.520 | 0.576         | 0.502 | 0.720              | 0.707  | 0.517    |
|         | ±            | ±     | ±     | ±     | ±     | ±             | ±   | ±                  | ±      | ±        |
|         | 0.093        | 0.024 | 0.040 | 0.022 | 0.070 | 0.013         | 0.135 | 0.076              | 0.019  |          |
| musk2   |              | 1.334 | 0.335 | 0.424 | 0.320 | 0.319         | 0.317 | 0.480              | 0.683  | 0.324    |
|         | ±            | ±     | ±     | ±     | ±     | ±             | ±   | ±                  | ±      | ±        |
|         | 0.022        | 0.006 | 0.008 | 0.004 | 0.004 | 0.004         | 0.004 | 0.048              | 0.122  | 0.003    |
| kruskp  |              | 1.126 | 0.592 | 0.805 | 0.479 | 0.605         | 0.546 | 0.781              | 0.823  | 0.581    |
|         | ±            | ±     | ±     | ±     | ±     | ±             | ±   | ±                  | ±      | ±        |
|         | 0.036        | 0.007 | 0.029 | 0.016 | 0.051 | 0.039         | 0.075 | 0.077              | 0.055  |          |
| waveform |              | 1.879 | 0.457 | 0.690 | 0.440 | 0.440         | 0.437 | 0.523              | 0.642  | 0.459    |
|         | ±            | ±     | ±     | ±     | ±     | ±             | ±   | ±                  | ±      | ±        |
|         | 0.044        | 0.007 | 0.012 | 0.007 | 0.006 | 0.007         | 0.032 | 0.135              | 0.011  |          |
| spambase|              | 0.994 | 0.406 | 0.567 | 0.373 | 0.375         | 0.383 | 0.506              | 0.556  | 0.386    |
|         | ±            | ±     | ±     | ±     | ±     | ±             | ±   | ±                  | ±      | ±        |
|         | 0.024        | 0.006 | 0.005 | 0.006 | 0.006 | 0.005         | 0.024 | 0.034              | 0.009  |          |
| mushroom|              | 0.891 | 0.317 | 0.461 | 0.305 | 0.312         | 0.314 | 0.462              | 0.590  | 0.319    |
|         | ±            | ±     | ±     | ±     | ±     | ±             | ±   | ±                  | ±      | ±        |
|         | 0.009        | 0.005 | 0.014 | 0.005 | 0.006 | 0.008         | 0.045 | 0.033              | 0.006  |          |
Table 14: Log-loss across the entire dataset. Lowest average value shown in bold. Results for $\ell_1$-regularized logistic regression, $T = 10$ and $\lambda = 10^{-2}$ on stationary datasets.

| Dataset | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | UCB1 GTS | Disc. UCB1 Improved | Disc. UCB1 GTS |
|---------|--------------|------------|------------|--------------|----------|---------------------|----------------|
| landsat | 0.345        | 0.110      | 0.157      | 0.103        | 0.106    | 0.107               | 0.144          |
|         | ±            | ±          | ±          | ±            | ±        | ±                   | ±              |
|         | 0.009        | 0.003      | 0.004      | 0.003        | 0.005    | 0.002               | 0.012          |
| splice  | 1.483        | 0.572      | 0.806      | 0.529        | 0.489    | 0.528               | 0.556          |
|         | ±            | ±          | ±          | ±            | ±        | ±                   | ±              |
|         | 0.054        | 0.025      | 0.033      | 0.015        | 0.011    | 0.041               | 0.022          |
| musk2   | 1.501        | 0.352      | 0.455      | 0.348        | 0.349    | 0.347               | 0.481          |
|         | ±            | ±          | ±          | ±            | ±        | ±                   | ±              |
|         | 0.016        | 0.004      | 0.003      | 0.004        | 0.004    | 0.002               | 0.038          |
| kruskm  | 1.357        | 0.756      | 1.007      | 0.658        | 0.628    | 0.654               | 0.782          |
|         | ±            | ±          | ±          | ±            | ±        | ±                   | ±              |
|         | 0.071        | 0.05       | 0.05       | 0.039        | 0.027    | 0.029               | 0.064          |
| waveform| 1.491        | 0.456      | 0.629      | 0.434        | 0.434    | 0.444               | 0.494          |
|         | ±            | ±          | ±          | ±            | ±        | ±                   | ±              |
|         | 0.031        | 0.006      | 0.020      | 0.006        | 0.006    | 0.006               | 0.024          |
| spambse | 0.887        | 0.405      | 0.553      | 0.383        | 0.380    | 0.400               | 0.475          |
|         | ±            | ±          | ±          | ±            | ±        | ±                   | ±              |
|         | 0.026        | 0.007      | 0.007      | 0.004        | 0.006    | 0.009               | 0.024          |
| mushroom| 0.743        | 0.330      | 0.465      | 0.310        | 0.310    | 0.318               | 0.790          |
|         | ±            | ±          | ±          | ±            | ±        | ±                   | ±              |
|         | 0.012        | 0.003      | 0.011      | 0.005        | 0.004    | 0.007               | 0.031          |
Table 15: Log-loss across the entire dataset. Lowest average value shown in bold. Results for $\ell_1$-regularized logistic regression, $T = 10$ and $\lambda = 10^{-1}$ on stationary datasets.

| Dataset | Uncalibrated | Best Fixed | Worst Fixed | UCB1 Improved | GTS | Disc. UCB1 Improved | Disc. UCB1 Improved |
|---------|--------------|------------|-------------|---------------|-----|--------------------|--------------------|
| landsat | 0.308        | 0.113      | 0.153       | 0.100         | 0.100 | 0.099              | 0.141              |
|         | ±            | ±          | ±           | ±             | ±    | ±                  | ±                  |
|         | 0.010        | 0.002      | 0.006       | 0.003         | 0.002 | 0.002              | 0.013              |
|         |              |            |             |               |       |                    | 0.017              |
|         |              |            |             |               |       |                    | 0.004              |
| splice  | 3.703        | 0.930      | 1.607       | 0.778         | 0.770 | 0.927              | 1.187              |
|         | ±            | ±          | ±           | ±             | ±    | ±                  | ±                  |
|         | 0.119        | 0.037      | 0.080       | 0.022         | 0.023 | 0.091              | 0.288              |
|         |              |            |             |               |       |                    | 0.267              |
|         |              |            |             |               |       |                    | 0.098              |
| musk2   | 2.244        | 0.510      | 0.794       | 0.462         | 0.466 | 0.503              | 0.664              |
|         | ±            | ±          | ±           | ±             | ±    | ±                  | ±                  |
|         | 0.038        | 0.006      | 0.012       | 0.008         | 0.008 | 0.007              | 0.035              |
|         |              |            |             |               |       |                    | 0.161              |
|         |              |            |             |               |       |                    | 0.009              |
|         |              |            |             |               |       |                    | 0.013              |
| kruskp  | 2.857        | 1.109      | 1.483       | 0.775         | 0.797 | 0.749              | 0.924              |
|         | ±            | ±          | ±           | ±             | ±    | ±                  | ±                  |
|         | 0.263        | 0.044      | 0.084       | 0.053         | 0.037 | 0.030              | 0.143              |
|         |              |            |             |               |       |                    | 0.149              |
|         |              |            |             |               |       |                    | 0.030              |
| waveform| 1.198        | 0.557      | 0.698       | 0.500         | 0.552 | 0.488              | 0.569              |
|         | ±            | ±          | ±           | ±             | ±    | ±                  | ±                  |
|         | 0.050        | 0.014      | 0.025       | 0.011         | 0.030 | 0.005              | 0.023              |
|         |              |            |             |               |       |                    | 0.030              |
|         |              |            |             |               |       |                    | 0.007              |
| spambase| 0.639        | 0.563      | 0.671       | 0.491         | 0.516 | 0.511              | 0.597              |
|         | ±            | ±          | ±           | ±             | ±    | ±                  | ±                  |
|         | 0.014        | 0.007      | 0.015       | 0.008         | 0.008 | 0.008              | 0.011              |
|         |              |            |             |               |       |                    | 0.020              |
|         |              |            |             |               |       |                    | 0.015              |
|         |              |            |             |               |       |                    | 0.008              |
| mushroom| 0.521        | 0.428      | 0.564       | 0.394         | 0.378 | 0.404              | 0.553              |
|         | ±            | ±          | ±           | ±             | ±    | ±                  | ±                  |
|         | 0.009        | 0.005      | 0.007       | 0.007         | 0.005 | 0.007              | 0.018              |
|         |              |            |             |               |       |                    | 0.021              |
|         |              |            |             |               |       |                    | 0.009              |