THE Riemann Hypothesis Via The Mellin Transform, Power Series And The Reflection Relations

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Abstract. A proof of the Riemann hypothesis is proposed by relying on the properties of the Mellin transform. The function \( G_\eta(t) \) is defined on the set \( \mathbb{R}_+ \) of the non-negative real numbers, in term of a special power series, in such a way that the Mellin transform \( \hat{G}_\eta(s) \) of the function \( G_\eta(t) \) does not vanish in the fundamental strip \( 0 < \Re s < 1/2 \). In this strip every zero of the Riemann zeta function \( \zeta(1-s) \) is a zero of the function \( \hat{G}_\eta(s) \). Consequently, it is proved that no zero of the Riemann zeta function \( \zeta(s) \) exists in the strip \( 1/2 < \Re s < 1 \). The reflection relations, which hold around the line \( \Re s = 1/2 \) for \( s \neq 0, 1 \), prove that no zero of the Riemann zeta function \( \zeta(s) \) exists in the strip \( 0 < \Re s < 1/2 \). In conclusion, it is proved that no zero of the Riemann zeta function \( \zeta(s) \) exists in the strip \( 0 < \Re s < 1 \) for \( \Re s \neq 1/2 \).

1. Introduction

The Riemann zeta function (RZF) was introduced as a product of prime numbers by Leonhard Euler in his book which is entitled "'Introductio in analysin infinitarum'" and published in the year 1748. Bernhard Riemann has shown that the distribution of prime numbers is related to the zeros of this function in his article entitled "'On the number of primes less than a given magnitude'" and published in the year 1859 [1]. The RZF \( \zeta(s) \) is defined as a function of the complex variable \( s \) by the following Dirichlet series which converges for \( \Re s > 1 \) [1] [2] [3].

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.
\] (1.1)

For \( \Re s \leq 1 \), the RZF is defined from Eq. (1.1) by analytic continuation. This function is meromorphic and exhibits one singularity, in \( s = 1 \), with residue 1. The RZF vanishes for negative even integer values of the argument, \( \zeta(-2n) = 0 \) for every \( n \in \mathbb{N}_0 \). These zeros are referred to as trivial. The Riemann hypothesis (RH) states that the real part of every nontrivial zero of the RZF is equal to the value 1/2. The literature on the RZF and on the RH is enormously vast. The most various approaches have been proposed to prove the RH and continue to appear in literature, nowadays. A review of these studies is beyond the purposes of the present manuscript.

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For the sake of clarity and shortness, we report below uniquely the relations which concern the RZF and are adopted in the present manuscript to prove the RH. Beside Eq. (1.1), further representation of the RZF is given by the following series,

$$\zeta(s) = (1 - 2^{-s})^{-1} \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}, \quad (1.2)$$

which holds for $\Re s > 0$. The RZF exhibits reflection relations around the line $\Re s = 1/2$, for $s \neq 0, 1$. These relations are described by the following functional equations,

$$\zeta(1-s) = 2 (2\pi)^{-s} \cos (\pi s / 2) \Gamma(s) \zeta(s), \quad (1.3)$$

$$\zeta(s) = 2 (2\pi)^{s-1} \sin (\pi s / 2) \Gamma(1-s) \zeta(1-s), \quad (1.4)$$

which hold for $s \neq 0, 1$, where $\Gamma(s)$ is the Gamma function.

The RZF is obtained from Eqs. (1.1) and (1.2) as the result of various integral transforms and, particularly, via the Mellin transform (MT) such as follows,

$$(\hat{f}u) = \int_{0}^{\infty} t^{s-1} f(t) \, dt, \quad (1.5)$$

for every value of the complex variable $s$ such that the involved integral exists. The MT is properly defined in the linear space of the functions $f(t)$ such that $t^{\alpha-1} f(t)$ is summable on $\mathbb{R}^+$, $t^{\alpha-1} f(t) \in L^1(\mathbb{R}^+)$. Let the function $f(t)$ be locally summable, $f(t) \in L^1_{\text{loc}}(\mathbb{R}^+)$, and bounded as follows, $f(t) = O(t^{-\alpha})$ for every $t \in (0, 1]$ and $f(t) = O(t^{-\beta})$ for every $t \in (1, \infty]$. Under these conditions the integral which appears in Eq. (1.5) converges absolutely and uniformly in the complex variable $s$ in the strip $\alpha + \varepsilon \leq \Re s \leq \beta - \varepsilon$, for every $\varepsilon \in (0, (\beta - \alpha) / 2)$. The MT $\hat{f}(s)$ of the function $f(t)$ is analytic in this strip.

Let the general functions $f(t)$ and $g(t)$ be defined on $\mathbb{R}^+$, and let the function $h(t)$ be defined by the following integral form,

$$h(t) = \int_{0}^{\infty} f(tu) g(u) \, du. \quad (1.5)$$

The MT $\hat{h}(s)$ of the function $h(t)$ is given by the expression below,

$$\hat{h}(s) = \hat{f}(s) \hat{g}(1-s). \quad (1.6)$$

The present proof of the RH is based on the MT of the functions $\varphi_{\lambda}(t)$ and $\psi_{\mu}(t)$ which are defined in terms of exponential and circular-like function, respectively, as below,

$$\varphi_{\lambda}(t) \equiv \left(1 + e^{\lambda t}\right)^{-1}, \quad (1.7)$$

$$\psi_{\mu}(t) \equiv \cos \left(\mu t^{1/2}\right), \quad (1.8)$$

for every $t \in \mathbb{R}^+$ and for every $\lambda, \mu > 0$. The MT $\hat{\varphi}_{\lambda}(s)$ of the function $\varphi_{\lambda}(t)$ involves the RZF,

$$\hat{\varphi}_{\lambda}(s) = \lambda^{-s} \Gamma(s) \left(1 - 2^{1-s}\right) \zeta(s), \quad (1.9)$$
for \( \text{Re } s > 0 \). The MT \( \hat{\psi}_\mu(s) \) of the function \( \psi_\mu(t) \) is given by the following expression \([7, 8, 9, 10, 13]\),
\[
\hat{\psi}_\mu(s) = 2\mu^{-2s} \Gamma(2s) \cos(\pi s),
\]  
(1.10)
for \( 0 < \text{Re } s < 1/2 \). Below, it is shown how, and in which strip of the complex plane, the nonvanishing behavior of the MT \( \hat{\psi}_\mu(s) \) is related to the nonvanishing behavior of the RZF \( \zeta(s) \) via the MT \( \hat{\varphi}_\lambda(s) \).

2. THE POWER SERIES \( \mathfrak{G}_\eta(t) \) AND THE FUNCTION \( \mathfrak{G}_\eta(t) \)

Consider the following power series,
\[
\mathfrak{G}_\eta(t) \equiv \sum_{n=0}^{\infty} f_n t^n, \tag{2.1}
\]
where \( f_0 = 1 \), and
\[
f_n = \frac{(-1)^n \eta^n \ln 2}{n!(2n)! (1 - 2^{-n}) \zeta(n+1)}, \tag{2.2}
\]
for every \( n \in \mathbb{N}_0 \). The power series \( \mathfrak{G}_\eta(t) \) converges absolutely, and, consequently, converges, for every \( t \in \mathbb{R}_+ \). This series converges uniformly in the real variable \( t \) in every bounded and closed interval of \( \mathbb{R}_+ \), and is continuous for every \( t \geq 0 \) and differentiable for every \( t > 0 \). These properties hold for every \( \eta > 0 \).

Proof. The ratio \( |f_{n+1}/f_n| \) is given by the expressions below,
\[
\left| \frac{f_{n+1}}{f_n} \right| = \frac{\eta (1 - 2^{-n}) \zeta(n+1)}{2 (n+1)^2 (2n+1) (1 - 2^{-n-1}) \zeta(n+2)}, \tag{2.3}
\]
for every \( n \in \mathbb{N}_0 \), and \( f_1/f_0 = -6\pi^{-2}\eta \ln 2 \). As the RZF tends asymptotically to unity along the positive real axis \([1, 2, 3]\),
\[
\lim_{t \to +\infty} \zeta(t) = 1,
\]
the following limits holds,
\[
\lim_{n \to +\infty} (t |f_{n+1}/f_n|) = 0, \tag{2.4}
\]
for every \( t \in \mathbb{R}_+ \) and for every \( \eta > 0 \). Thus, according to the D’Alembert’s test \([14, 15]\), the power series \( \mathfrak{G}_\eta(t) \) converges absolutely, and, consequently, converges, for every \( t \in \mathbb{R}_+ \) and for every \( \eta > 0 \). The radius of convergence of the power series \( \mathfrak{G}_\eta(t) \) is infinite for every \( \eta > 0 \). The uniform convergence, the continuity and the differentiability in the real variable \( t \) of the series \( \mathfrak{G}_\eta(t) \) in every bounded and closed interval of \( \mathbb{R}_+ \) is a general property of the power series which holds inside the radius of convergence \([14, 15]\). \( \square \)

The function \( \mathfrak{G}_\eta(t) \) is defined on \( \mathbb{R}_+ \), in terms of the power series \( \mathfrak{G}_\eta(t) \) and of the function \( \varphi_\lambda(t) \), by the following integral form,
\[
\mathfrak{G}_\eta(t) \equiv \int_{0}^{\infty} \mathfrak{G}_\eta(tu) \varphi_\lambda_0(u) \, du, \tag{2.5}
\]
for every $\eta > 0$. The particular value $\ln 2$ of the parameter $\lambda$ is chosen, $\lambda_0 = \ln 2$. The function $G_\eta(t)$ results to be circular-like,

$$G_\eta(t) = \cos \left( \kappa t^{1/2} \right),$$  \hspace{1cm} (2.6)

for every $t \in \mathbb{R}^+$ and for every $\eta > 0$, where $\kappa = (\eta/\lambda_0)^{1/2}$.

**Proof.** The integrand function which appears in Eq. (2.5) is given by the expressions below,

$$f_\eta (tu) \varphi_{\lambda_0}(u) = \sum_{n=0}^{\infty} f_n (tu)^n \varphi_{\lambda_0}(u) = \sum_{n=0}^{\infty} f_n (tu)^n \left( 1 + e^{\lambda_0 u} \right)^{-1},$$  \hspace{1cm} (2.7)

for every $t,u \in \mathbb{R}^+$. The following inequality,

$$|f_n| (tu)^n \left( 1 + e^{\lambda_0 u} \right)^{-1} \leq 2^{-1} |f_n| (tu)^n,$$

holds for every $t,u \in \mathbb{R}^+$ and for every $\eta > 0$. According to Eqs. (2.1)-(2.4), the power series $\sum_{n=0}^{\infty} |f_n| (tu)^n$ converges uniformly in the real variable $u$ in every bounded and closed interval of $\mathbb{R}^+$, for every $t \in \mathbb{R}^+$. Consequently, the series $\sum_{n=0}^{\infty} f_n (tu)^n \varphi_{\lambda_0}(u)$ converges absolutely and uniformly in the real variable $u$ in every bounded and closed interval of $\mathbb{R}^+$, for every $t \in \mathbb{R}^+$.

The following term-by-term integrated series is evaluated via forms (2.2) and (1.9),

$$\sum_{n=0}^{\infty} f_n t^n \int_0^\infty u^n \varphi_{\lambda_0}(u) du = \sum_{n=0}^{\infty} f_n t^n \varphi_{\lambda_0}(n+1)$$  

$$= \sum_{n=0}^{\infty} \frac{(-\kappa^2 t)^n}{(2n)!} = \cos \left( \kappa t^{1/2} \right),$$  \hspace{1cm} (2.9)

These relations hold for every $t \in \mathbb{R}^+$, and for every $\eta > 0$, or, equivalently, for every $\kappa > 0$. Equalities (2.9) prove that the term-by-term integrated series under study is finite for every $t \in \mathbb{R}^+$ and for every $\eta > 0$, as $\cos (\kappa t^{1/2}) \in [-1,1]$ for every $t \in \mathbb{R}^+$ and $\kappa > 0$. Notice that the following particular values hold, $\hat{\varphi}_\lambda(1) = \lambda^{-1} \ln 2$ and $\hat{\varphi}_{\lambda_0}(1) = 1$.

According to the dominated convergence theorem of Lebesgue [14, 15, 16], the term-by-term integration in the real variable $u$ of the functional series $\sum_{n=0}^{\infty} f_n (tu)^n \varphi_{\lambda_0}(u)$ can be performed over an infinite range. Therefore, Eq. (2.6) is proved by Eq. (2.9) and by the following equalities,

$$\int_0^\infty \tilde{f}_n (tu) \varphi_{\lambda_0}(u) du = \int_0^\infty \sum_{n=0}^{\infty} f_n (tu)^n \varphi_{\lambda_0}(u) du$$

$$= \sum_{n=0}^{\infty} f_n t^n \int_0^\infty u^n \varphi_{\lambda_0}(u) du = \cos \left( \kappa t^{1/2} \right),$$  \hspace{1cm} (2.10)

which hold for every $t \in \mathbb{R}^+$ and for every $\eta > 0$. □
3. A PROOF OF THE RIEMANN HYPOTHESIS

The MT $\hat{G}_\eta(s)$ of the function $G_\eta(t)$ is given by the expression below,

$$\hat{G}_\eta(s) = 2\kappa^{-2s} \Gamma(2s) \cos(\pi s), \quad (3.1)$$

for $0 < \text{Re} s < 1/2$. No zero of the function $\hat{G}_\eta(s)$ exists in the fundamental strip $0 < \text{Re} s < 1/2$.

$$\hat{G}_\eta(s) \neq 0, \quad \forall \; s \in \mathbb{C} \; | \; \text{Re} s \in (0, 1/2). \quad (3.2)$$

**Proof.** The MT $\hat{G}_\eta(s)$ is given by Eq. (3.1) and is obtained from Eqs. (2.6) and (1.10) for $\mu = \kappa$. The power $\kappa^{-2s}$ and the Gamma function $\Gamma(2s)$ do not vanish [3]. The cosine function $\cos(\pi s)$ vanishes for $s = s_n$, where $s_n = n + 1/2$ for every integer value of the variable $n$. These zeros do not belong to the fundamental strip, $\text{Re} s_n = s_n \notin (0, 1/2)$, for every $n \in \mathbb{Z}$. Consequently, the function $\hat{G}_\eta(s)$ does not vanish in the fundamental strip $0 < \text{Re} s < 1/2$. □

Every zero of the RZF $\zeta(1 - s)$ is a zero of the function $\hat{G}_\eta(s)$ in the strip $0 < \text{Re} s < 1/2$,

$$\zeta(1 - s) = 0 \Rightarrow \hat{G}_\eta(s) = 0 \quad \forall \; s \in \mathbb{C} \; | \; \text{Re} s \in (0, 1/2). \quad (3.3)$$

**Proof.** The MT $\hat{G}_\eta(s)$ is given by Eq. (3.1) for $0 < \text{Re} s < 1/2$. Consequently, the following integral,

$$\int_{0}^{\infty} t^{s-1} \left( \int_{0}^{\infty} \hat{F}_\eta(tu) \varphi_{\lambda_0}(u) \, du \right) \, dt, \quad (3.4)$$

exists and converges absolutely, and uniformly in the complex variable $s$, in the fundamental strip, $\varepsilon \leq \text{Re} s \leq 1/2 - \varepsilon$ for every $\varepsilon \in (0, 1/4)$. The integrand function $t^{s-1} \hat{F}_\eta(tu) \varphi_{\lambda_0}(u)$ of the two real variables $t$ and $u$ is locally integrable in $\mathbb{R}^2_+$,

$$t^{s-1} \hat{F}_\eta(tu) \varphi_{\lambda_0}(u) \in L^1_{\text{loc}}(\mathbb{R}^2_+),$$

for $0 < \text{Re} s < 1/2$. This property allows to change in Eq. (3.4) the order of the double integration [17, 15],

$$\hat{G}_\eta(s) = \int_{0}^{\infty} t^{s-1} \left( \int_{0}^{\infty} \hat{F}_\eta(tu) \varphi_{\lambda_0}(u) \, du \right) \, dt$$

$$= \int_{0}^{\infty} \left( \int_{0}^{\infty} t^{s-1} \hat{F}_\eta(tu) \varphi_{\lambda_0}(u) \, dt \right) \, du$$

$$= \int_{0}^{\infty} \left( \int_{0}^{\infty} u^{-s} v^{s-1} \hat{F}_\eta(v) \varphi_{\lambda_0}(u) \, dv \right) \, du, \quad (3.5)$$

for $0 < \text{Re} s < 1/2$ and for every $\eta > 0$. The latter equality is obtained via the change of variable $v = ut$ [6, 7, 8, 9, 11, 12]. The integrand function $u^{-s} v^{s-1} \hat{F}_\eta(v) \varphi_{\lambda_0}(u)$ of the two real variables $u$ and $v$ is locally integrable in $\mathbb{R}^2_+$,

$$u^{-s} v^{s-1} \hat{F}_\eta(v) \varphi_{\lambda_0}(u) \in L^1_{\text{loc}}(\mathbb{R}^2_+),$$
for $0 < \text{Re } s < 1/2$ and for every $\eta > 0$. Consequently, the following relations hold \[17\],

$$
\hat{G}_\eta(s) = \lim_{U \to +\infty} \left\{ \lim_{V \to +\infty} \left[ \left( \int_0^U u^{-s} \varphi_{\lambda_0}(u) du \right) \left( \int_0^V v^{s-1} \hat{G}_\eta(v) dv \right) \right] \right\}, \tag{3.6}
$$

$$
\hat{F}_\eta(s) = \lim_{V \to +\infty} \left\{ \lim_{U \to +\infty} \left[ \left( \int_0^V v^{s-1} \hat{F}_\eta(v) dv \right) \left( \int_0^U u^{-s} \varphi_{\lambda_0}(u) du \right) \right] \right\}, \tag{3.7}
$$

for $0 < \text{Re } s < 1/2$ and for every $\eta > 0$, where $0 < U, V < +\infty$. Notice that the real variables $U$ and $V$ tend to infinity independently of one another.

Relations (3.6) and (3.7) and Eq. (3.1) prove that the following limit,

$$
\lim_{V \to +\infty} \int_0^V v^{s-1} \hat{F}_\eta(v) dv, \tag{3.8}
$$

exists and is finite for $0 < \text{Re } s < 1/2$ and for every $\eta > 0$, as the following integral function of the real variable $U$,

$$
\int_0^U u^{-s} \varphi_{\lambda_0}(u) du,
$$

does not vanish identically and does not diverge in domain $\mathbb{R}_+$ for every value of the complex variable $s$ in the strip $0 < \text{Re } s < 1/2$. Thus, the limit (3.8) provides the MT $\hat{F}_\eta(s)$ of the function $\hat{F}_\eta(t)$ for $0 < \text{Re } s < 1/2$.

According to Eq. (3.5), the following relation holds among the MTs $\hat{G}_\eta(s)$, $\hat{F}_\eta(s)$ and $\hat{\varphi}_{\lambda_0}(s)$ in the fundamental strip,

$$
\hat{G}_\eta(s) = \hat{F}_\eta(s) \hat{\varphi}_{\lambda_0}(1-s), \tag{3.9}
$$

for $0 < \text{Re } s < 1/2$ and for every $\eta > 0$. In this way, the RZF is connected to the MT $\hat{G}_\eta(s)$,

$$
\hat{G}_\eta(s) = \mathcal{K}_\eta(s) \zeta(1-s), \tag{3.10}
$$

for $0 < \text{Re } s < 1/2$. According to Eqs. (3.1) and (1.9), the function $\mathcal{K}_\eta(s)$ is defined in the fundamental strip as below,

$$
\mathcal{K}_\eta(s) \equiv \lambda_0^{s-1} (1 - 2^s) \Gamma(1-s) \hat{G}_\eta(s), \tag{3.11}
$$

for $0 < \text{Re } s < 1/2$ and for every $\eta > 0$. The Gamma function $\Gamma(1-s)$ is meromorphic and exhibits simple poles $s'_n = 1 + n$ which do not belong to the fundamental strip, $\text{Re } s'_n = s'_n \notin (0, 1/2)$ for every $n \in \mathbb{N} \[3\]$. Thus, the function $\mathcal{K}_\eta(s)$ is analytic and, therefore, exhibits no singularity in the strip $0 < \text{Re } s < 1/2$. Consequently, according to Eq. (3.10), every zero of the RZF $\zeta(1-s)$ which belongs to the strip $0 < \text{Re } s < 1/2$ is a zero of the function $\hat{G}_\eta(s)$. This property proves the implication (3.3). \[\square\]
No zero of the RZF \( \zeta(s) \) exists in the strip \( 1/2 < \text{Re} \, s < 1 \),
\[
\zeta(s) \neq 0 \quad \forall \ s \in \mathbb{C} \mid \text{Re} \, s \in (1/2, 1).
\] (3.12)

*Proof.* The implication (3.3) is logically equivalent to the corresponding counternominal implication,
\[
\hat{G}_\eta(s) \neq 0 \Rightarrow \zeta(1 - s) \neq 0 \quad \forall \ s \in \mathbb{C} \mid \text{Re} \, s \in (0, 1/2).
\] (3.13)

As the function \( \hat{G}_\eta(s) \) does not vanish in the strip \( 0 < \text{Re} \, s < 1/2 \), the property (3.2) proves the property (3.12) via the implication (3.13). \( \square \)

No zero of the RZF \( \zeta(s) \) exists in the strip \( 0 < \text{Re} \, s < 1/2 \),
\[
\zeta(s) \neq 0 \quad \forall \ s \in \mathbb{C} \mid \text{Re} \, s \in (0, 1/2).
\] (3.14)

*Proof.* Consider the reflection relation (1.4). The power \((2\pi)^{-2s}\) and the Gamma function \(\Gamma(1 - s)\) do not vanish \(\Box\). The sine function \(\sin(\pi s/2)\) vanishes for \(s = s'_n\), where \(s'_n = 2n\) for every integer value of the variable \(n\). These zeros do not belong to the fundamental strip, \(\text{Re} \, s'_n \notin (0, 1/2)\) for every \(n \in \mathbb{Z}\). The RZF \(\zeta(1 - s)\) does not vanish in the strip \(0 < \text{Re} \, s < 1/2\). Consequently, the reflection relation (1.4) and the property (3.12) prove the property (3.14). Similarly, the same proof is provided by the reflection relation (1.3). \( \square \)

In conclusion, no zero of the RZF \( \zeta(s) \) exists in the strip \( 0 < \text{Re} \, s < 1 \) for \( \text{Re} \, s \neq 1/2 \),
\[
\zeta(s) \neq 0 \quad \forall \ s \in \mathbb{C} \mid \text{Re} \, s \in (0, 1/2) \cup (1/2, 1).
\] (3.15)

*Proof.* The properties (3.12) and (3.14) prove the property (3.15). \( \square \)

### 4. Summary and Conclusions

A proof of the RH is proposed by relying on the properties of the MT and by introducing a special power series \( \hat{F}_\eta(t) \) which converges in \( \mathbb{R}_+ \). The function \( \hat{G}_\eta(t) \) is defined via an integral form which involves the power series \( \hat{F}_\eta(t) \) and the function \( \varphi_{\lambda_0}(t) \). The RZF is related to the MT \( \hat{\varphi}_{\lambda_0}(s) \) of the function \( \varphi_{\lambda_0}(t) \).

The present proof is based on the existence of the MT \( \hat{G}_\eta(s) \) of the function \( \hat{F}_\eta(t) \) in the strip \( 0 < \text{Re} \, s < 1/2 \) and, thus, on the absence of singularities of the function \( \hat{G}_\eta(s) \) in this strip. First, it is proved that the MT \( \hat{G}_\eta(s) \) of the function \( \hat{G}_\eta(t) \) does not vanish in the strip \( 0 < \text{Re} \, s < 1/2 \). Then, it is proved that, in this strip, every zero of the RZF \( \zeta(1 - s) \) is a zero of the function \( \hat{G}_\eta(s) \). Consequently, no zero of the RZF \( \zeta(s) \) is proved to exist in the strip \( 1/2 < \text{Re} \, s < 1 \). Finally, the reflection relations prove that no zero of the RZF \( \zeta(s) \) exists in the strip \( 0 < \text{Re} \, s < 1/2 \). In conclusion, no zero of the RZF \( \zeta(s) \) is proved to exist in the strip \( 0 < \text{Re} \, s < 1 \) for \( \text{Re} \, s \neq 1/2 \).
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