Phase transitions in $D$-dimensional Gauss-Bonnet-Born-Infeld $AdS$ black holes

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Abstract

In this paper, we have investigated the phase transition in black holes when Gauss-Bonnet corrections to the spacetime curvature and Born-Infeld extension in stress-energy tensor of electromagnetic field are considered in negative cosmological constant background. We found that the black hole undergoes second order phase transition as the specific heat capacity at constant potential shows discontinuities. Further, the Ehrenfest scheme and the Ruppeiner state space geometry analysis are carried out to check the validity of the second order phase transition. All analysis is done in general $D$ spacetime dimensions with $D > 4$, and specific computations have been carried out in $D = 5, 6, 7$ dimensions.

I. INTRODUCTION

Singularities are inevitable in Einstein’s gravity and the object called black hole has properties which are not fully explained by classical gravity. Hawking [1] and Bekenstein [2] considered these objects as thermodynamic systems and associated with it thermodynamic properties like entropy and temperature. The study of black hole thermodynamics has received a new impetus recently because it is believed that it may shed light on quantum gravity.

In seeking modifications to Einstein’s gravity in bottom-up approach, the very first idea was to see higher order effects of curvature in the action. This modification was brought in [3] where general spacetime lagrangian density has higher powers of curvature. Later it was generalized to adding some function of $R$ to the action and it is termed as $f(R)$ gravity [4]. These modifications are incorporated in such a way that the solutions resulting from these are consistent and are free from any ghost terms, hence arbitrary terms in action are not allowed. Gauss-Bonnet correction is one of those having first three terms of the Lovelock gravity and possess a consistent solution. It is believed that this simple modification may lead to interesting insights into the true nature of gravity.

Non-linear electrodynamics was first developed by Max Born and Leopold Infeld [5] in order to resolve the self-energy problem of a point charge. However, all such non-linear theories got soon replaced by the profound quantum description of Maxwell’s theory in the form of quantum electrodynamics. Later, Born-Infeld theory reappeared as a low energy limit of string theory in [6]–[7]. Since then an extensive study of black holes with these higher derivatives in gauge field has been done [8]–[10]. So it is worth studying the thermodynamics of black holes in higher curvature gravity theories with non-linear electrodynamics.

In this paper, we wish to study the black hole solutions obtained in Gauss-Bonnet gravity incorporating Born-Infeld theory in the action with negative cosmological constant. All analysis is done in $D$ spacetime dimensions. Thermodynamic quantities are calculated and singularities in heat capacity are registered for the fixed Gauss-Bonnet parameter, Born-Infeld parameter and charge. These divergences are shown in graphs when the heat capacity is plotted with the horizon radius for $D = 5, 6, 7$. This is a clear indication of the second order phase transition of the black hole system under consideration with specific values of the parameters. Ehrenfest scheme and the Ruppeiner state space geometry analysis [11]–[12] are trusted techniques in standard thermodynamics and are employed to study black hole thermodynamics as well [13]–[15]. We wish to employ these for our system and study the order of the phase transitions in the black holes. A parameter called Prigogine-Defay ratio [16] which determines the deviations from the second order nature of phase transition is also calculated using Ehrenfest equations. All results for Einstein-Born-Infeld black holes in [13] are recovered when the Gauss-Bonnet parameter is put to zero. Further, motivations to study these black hole phase transitions in AdS space comes from AdS/CFT correspondence [17]–[19]. Confinement problem, superconductivity and other strongly correlated systems are in the domain of this duality and has led to deep insights to these phenomena. Along the way, we shall also talk about the possibility of treating the cosmological constant,
Born-Infeld parameter and Gauss-Bonnet parameter as thermodynamical variables and derive a Smarr relation using scaling argument and first law of thermodynamics in $D$ spacetime dimensions.

The structure of this paper is as follows. In section II we shall be calculating the thermodynamic properties of the black holes considered in this paper and also calculate their heat capacity. We plot the heat capacity with the horizon radius to study its nature. Section III is devoted to carrying out Ehrenfest scheme analysis at the points of phase transitions in order to understand the order of phase transition. Section IV contains the Ruppeiner state space geometry analysis of the singularities. We conclude by summarizing our results in section V.

II. THERMODYNAMICS OF GAUSS-BONNET-BORN-INFELD BLACK HOLES

In this section, we introduce the black hole spacetime we are interested in and study its thermodynamic properties. We consider the Gauss-Bonnet black hole spacetime with Born-Infeld electromagnetic charge in AdS background. The action of Gauss-Bonnet gravity with Born-Infeld electromagnetic field reads

$$I = \frac{1}{16\pi} \int d^D x \sqrt{-g} [R - 2\Lambda + \alpha L_{GB} + L(F)]$$  \hspace{1cm} (1)

where $G = c = 1$, the Gauss-Bonnet lagrangian density $L_{GB} = R^2 - 4R_{\gamma\delta}R^{\gamma\delta} + R_{\gamma\delta\lambda\sigma}R^{\gamma\delta\lambda\sigma}$, the Born-Infeld term $L(F) = 4b^2 \left(1 - \frac{1 + \frac{F_{\mu\nu}F_{\mu\nu}}{2b^2}}{2}\right)$, $\alpha$ is the Gauss-Bonnet parameter, $b$ is the Born-Infeld parameter, $\Lambda = -\frac{(D - 1)(D - 2)}{2l^2}$ with $l$ being AdS radius and $D$ is the spacetime dimensions greater than 4.

The solution following from the above theory reads

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega_{D-2}^2$$ \hspace{1cm} (2)

where the metric coefficient $f(r)$ is given by [20]

$$f(r) = 1 + \frac{r^2}{2\alpha'} \left(1 - \sqrt{g(r)}\right)$$ \hspace{1cm} (3)

with $g(r)$ given by

$$g(r) = 1 - \frac{4\alpha'}{r^2} + \frac{4\alpha'm}{r^{D-1}} - \frac{16\alpha' b^2}{(D - 1)(D - 2)} \left(1 - \sqrt{1 + \frac{(D - 2)(D - 3)q^2}{2b^2 r^{2D-4}}}\right)$$

$$-\frac{8(D - 2)\alpha' q^2}{(D - 1)r^{2D-2}} F_1 \left[ \frac{D - 3}{2D - 4}, \frac{1}{2}, \frac{3D - 7}{2D - 4}, \frac{(D - 2)(D - 3)q^2}{2b^2 r^{2D-4}} \right].$$ \hspace{1cm} (4)

Note that the actual black hole parameters such as the charge $Q$, mass $M$ and Gauss-Bonnet coefficient $\alpha$ are connected to $q$, $m$ and $\alpha'$ as

$$M = \frac{(D - 2)\omega m}{16\pi}, \quad \alpha = \frac{\alpha'}{(D - 3)(D - 4)}, \quad \omega = \frac{D - 1}{2\pi}$$

$$Q = \sqrt{2(D - 2)(D - 3)} \frac{\omega q}{8\pi}, \quad \omega = \frac{2\pi}{\Gamma \left(\frac{D - 1}{2} \right)}. \hspace{1cm} (5)$$

The black hole mass can be written in terms of the horizon radius ($r_+$) from the condition $f(r_+) = 0$ and reads

$$M = \frac{(D - 2)\omega \alpha'}{16\pi} r_+^{D-5} + \frac{(D - 2)\omega}{16\pi} r_+^{D-3} + \frac{(D - 2)\omega}{16\pi b^2} r_+^{D-1} + \frac{b^2 \omega}{4\pi(D - 1)} r_+^{D-1} \left(1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}}}\right)$$

$$+ \frac{4(D - 2)\pi Q^2}{(D - 1)(D - 3)\omega r_+^{D-3}} F_1 \left[ \frac{D - 3}{2D - 4}, \frac{1}{2}, \frac{3D - 7}{2D - 4}, \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}} \right].$$ \hspace{1cm} (6)
The Hawking temperature of the black hole spacetime can be calculated using eq.(3) as

\[
T = \frac{1}{4\pi} \left( \frac{df(r)}{dr} \right)_{r^+} = \frac{1}{4\pi r^+ (r^+_2 + 2\alpha')} \left[ \frac{(D-1)r^+_4}{l^2} + (D-3)r^+_2 + (D-5)\alpha' + \frac{4b^2r^+_4}{D-2} \left( 1 - \sqrt{1 + \frac{16\pi^2Q^2}{b^2\omega^2r^2_+}} \right) \right].
\] (7)

We now use the relation

\[
dM = TdS + \Phi dQ
\] (8)

which is the first law of black hole thermodynamics. This form is analogous to the first law of thermodynamics

\[
dU = TdS - PdV
\] (9)

with the identification of the pressure \(P\) to the negative of the electrostatic potential \(\Phi\), the volume to the charge \(Q\) and the internal energy \(U\) to the mass of the black hole \(M\). From this relation, the black hole entropy \(S\) can be obtained as

\[
S = \int_0^{r^+} \frac{1}{T} \left( \frac{\partial M}{\partial r} \right)_Q dr_r = \frac{\omega}{4} r^{D-2}_+ \left[ 1 + \frac{D-2}{D-4} \frac{2\alpha'}{r^2_+} \right].
\] (10)

In principle, eq.(10) allows the Hawking temperature \(T\) of the black hole to be written in terms of the entropy \(S\). In Fig[1] we plot the Hawking temperature of the black hole with the horizon radius. The nature of the graphs for different spacetime dimensions are as follows.

From the above plots, we observe that there is no discontinuity in these graphs hence there is no first order phase transition. Further, we analyse the system and calculate the heat capacity at constant potential. This we do to find whether there is a chance of higher order phase transitions in this black hole spacetime.
The potential in \( D \) dimensions for the black hole spacetime can be calculated from the first law and eq. (10) to be

\[
\Phi = \left( \frac{\partial M}{\partial Q} \right) = \frac{4\pi Q}{\omega(D-3)r_+^{D-2}} F_1 \left[ \frac{D - 3}{2D - 4}, \frac{1}{2}, \frac{3D - 7}{2D - 4}, -\frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}} \right].
\]  

(11)

Considering the temperature of the black hole to be a function of entropy and charge \((T \equiv T(S, Q))\), we have

\[
\left( \frac{\partial T}{\partial S} \right)_\Phi = \left( \frac{\partial T}{\partial Q} \right)_S - \left( \frac{\partial T}{\partial Q} \right)_S \left( \frac{\partial S}{\partial Q} \right)_Q \left( \frac{\partial Q}{\partial \Phi} \right)_S.
\]  

(12)

Now using the thermodynamic identity

\[
\left( \frac{\partial Q}{\partial S} \right)_\Phi \left( \frac{\partial S}{\partial \Phi} \right)_Q \left( \frac{\partial S}{\partial Q} \right)_S = -1
\]

(13)

along with eq. (12), we obtain the heat capacity at constant potential to be

\[
C_\Phi = T \left( \frac{\partial S}{\partial T} \right)_\Phi = \frac{T \left( \frac{\partial \Phi}{\partial Q} \right)}{r_+} \left( \frac{\partial S}{\partial \Phi} \right)_Q \left( \frac{\partial Q}{\partial r_+} \right)_Q.
\]  

(14)

The partial derivatives involved in the above equation when calculated read

\[
\left( \frac{\partial \Phi}{\partial r_+} \right)_Q = \frac{4\pi Q}{\omega r_+^{D-2}} \left( 1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}} \right)^{-1/2}
\]  

(15)

\[
\left( \frac{\partial T}{\partial Q} \right)_r = -\frac{1}{r_+^2 + 2\alpha'} \frac{16\pi Q}{\omega^2(D-2)r_+^{2D-7}} \left( 1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}} \right)^{-1/2}
\]  

(16)

\[
\left( \frac{\partial S}{\partial r_+} \right)_Q = \frac{dS}{dr_+} = \frac{\omega}{4} (D-2) r_+^{D-3} \left( 1 + \frac{2\alpha'}{r_+} \right)
\]  

(17)

\[
\left( \frac{\partial \Phi}{\partial Q} \right)_r = \frac{4\pi}{(D-2)\omega r_+^{D-3}} \left[ \left( 1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}} \right)^{-1/2} + \frac{1}{D-3} F_1 \left[ \frac{D - 3}{2D - 4}, \frac{1}{2}, \frac{3D - 7}{2D - 4}, -\frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}} \right] \right]
\]  

(18)

\[
\left( \frac{\partial T}{\partial r_+} \right)_Q = \frac{1}{4\pi r_+ (r_+^2 + 2\alpha')} \left[ 4(D-1)r_+^3 + 2(D-3)r_+ + \frac{16b_+^2 r_+^3}{(D-2)} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}}} \right) + \frac{64\pi^2 Q^2}{\omega^2 r_+^{2D-7}} \left( 1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}} \right)^{-1/2} \right]
\]

\[-\frac{3r_+^2 + 2\alpha'}{4\pi r_+^2 (r_+^2 + 2\alpha')^2} \left( (D-1)r_+^4 + 2(D-3)r_+^2 + (D-5)\alpha' + \frac{4b_+^2 r_+^4}{D-2} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r_+^{2D-4}}} \right) \right). \]

(19)

The form of heat capacity is a large expression to write, hence we study its behaviour through plots. We plot \( C_\Phi \) with horizon radius \( r_+ \) for fixed values of the Gauss-Bonnet parameter \( \alpha' \), Born-Infeld parameter \( b \) and charge \( Q \) for \( D = 5, 6, 7 \).

The plots of heat capacity have multiple points of discontinuities and hence corresponds to second order phase transition. We calculate the exact points of discontinuities of heat capacity. We also calculate the horizon radius of extremal black hole \( r_+^{(e)} \) for the same fixed parameters for each dimension in order to check the physical validity of the singularities in heat capacity. The following Table shows extremal black hole radii \( r_+^{(e)} \) and singularities in heat capacity at radii \( r_+^{(e)} \).
Table II shows that the black hole in $D=5$ spacetime dimensions has one unphysical singularity ($r_{+0}$) in the heat capacity as the point is below the extremal radius whereas the other two at points $r_{+1}$ and $r_{+2}$ are actual phase transition points. For $D=6,7$ there are two singularities each which are actual phase transition points as both the points lie above extremal radius values for respective dimensions. These plots also reveal that for a particular spacetime dimension there are three phases of the black hole, namely phase I ($r_{+}^{(e)} < r_+ < r_{+1}$), phase II ($r_{+1} < r_+ < r_{+2}$), and phase III ($r_+ > r_{+2}$).

In the following section, we shall show that the system follows Ehrenfest second order phase transition equations. We shall also carry out the Ruppeiner’s state space geometry analysis to confirm the second order phase transition exhibited by this black hole spacetime.

Before ending this section, we would like to discuss about the possibility of interpreting the cosmological constant $\Lambda$ as a thermodynamic pressure $p$ and also treating the Born-Infeld parameter $b$ and the Gauss-Bonnet parameter $\alpha$ as new thermodynamic variables, as it has been proposed recently in [21]-[22]. The argument put forward in these studies was that since $\Lambda$, $b$ and $\alpha$ are dimensionful quantities, the corresponding terms would definitely appear in the Smarr formula. To write down the Smarr formula, we first note that the first law of black hole thermodynamics takes the form

$$dM = TdS + pdV + \Phi dQ + Bdb + \Omega d\alpha$$

where $B$ and $\Omega$ are the conjugate variables to $b$ and $\alpha$ respectively defined by

$$B = \frac{\partial M}{\partial b} \quad \text{and} \quad \Omega = \frac{\partial M}{\partial \alpha}$$
Considering the black hole mass \( M = M(S, Q, p, b, \alpha) \) and performing dimensional analysis, we find that \( [M] = L^{D-3}, \quad [S] = L^{D-2}, \quad [p] = L^{-2}, \quad [Q] = L^{D-3}, \quad [b] = L^{-1} \) and \( [\alpha] = L^2 \). Using these along with Euler’s theorem\(^1\), we obtain

\[
(D - 3)M = (D - 2)S \left( \frac{\partial M}{\partial S} \right) - 2p \left( \frac{\partial M}{\partial p} \right) + (D - 3)Q \left( \frac{\partial M}{\partial Q} \right) - b \left( \frac{\partial M}{\partial b} \right) + 2\alpha \left( \frac{\partial M}{\partial \alpha} \right). \tag{22}
\]

Now using eq.(20), we get

\[
\left( \frac{\partial M}{\partial S} \right) = T, \quad \left( \frac{\partial M}{\partial p} \right) = v, \quad \left( \frac{\partial M}{\partial Q} \right) = \Phi, \quad \left( \frac{\partial M}{\partial b} \right) = B, \quad \left( \frac{\partial M}{\partial \alpha} \right) = \Omega. \tag{23}
\]

Substituting this in eq.(22) yields the Smarr formula

\[
M = \frac{D - 2}{D - 3}TS - \frac{2}{D - 3}pv + \Phi Q - \frac{1}{D - 3}bB + \frac{2}{D - 3}\alpha\Omega. \tag{24}
\]

It would be interesting to work with the first law of black hole thermodynamics given in eq.(20) and study phase transitions. However, we shall not carry out this investigation in this work and we intend to do it in future.

### III. ANALYSIS OF PHASE TRANSITION USING EHRENFEST SCHEME

Ehrenfest’s approach to studying phase transitions is the standard technique in thermodynamics to determine the nature of phase transitions for various thermodynamical systems\[^2, 24\]. It simply says that the order of the phase transition corresponds to the discontinuity in the order of the derivative of Gibb’s potential. For the second order phase transition the second derivative encounters discontinuities, however first derivative and Gibb’s potential at those points are continuous. These conditions with Maxwell’s relations give two equations which have to be satisfied for the second order phase transition.

The first and second Ehrenfest equations in thermodynamics are given by

\[
\left( \frac{\partial P}{\partial T} \right)_S = \frac{1}{VT} C_{P_2} - C_{P_1} = \frac{\Delta C_P}{VT \Delta \beta}, \tag{25}
\]

\[
\left( \frac{\partial P}{\partial T} \right)_V = \frac{\beta_2 - \beta_1}{\kappa_2 - \kappa_1} = \frac{\Delta \beta}{\Delta \kappa}. \tag{26}
\]

where subscripts 1 and 2 denote two distinct phases of the system. Now we use the correspondence between the pressure \((P)\) to the negative of the electrostatic potential difference \((\Phi)\) and the volume \((V)\) to the charge \((Q)\) of the black hole. These identifications lead to the following equations

\[
-\left( \frac{\partial \Phi}{\partial T} \right)_S = \frac{1}{QT} C_{\Phi_2} - C_{\Phi_1} = \frac{\Delta C_{\Phi}}{QT \Delta \beta}, \tag{27}
\]

\[
-\left( \frac{\partial \Phi}{\partial T} \right)_Q = \frac{\beta_2 - \beta_1}{\kappa_2 - \kappa_1} = \frac{\Delta \beta}{\Delta \kappa}. \tag{28}
\]

Note that \( \beta \) is the volume expansion coefficient and \( \kappa \) is the isothermal compressibility of the system and are defined as

\[
\beta = \frac{1}{Q} \left( \frac{\partial Q}{\partial T} \right)_\Phi, \quad \kappa = \frac{1}{Q} \left( \frac{\partial Q}{\partial \Phi} \right)_T. \tag{29}
\]

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\(^1\) Given a function \( g(x, y) \) satisfying \( g(\alpha^m x, \alpha^n y) = \alpha^r g(x, y) \), we have \( \frac{\partial g}{\partial x} = mx \left( \frac{\partial g}{\partial x} \right)_x + ny \left( \frac{\partial g}{\partial y} \right)_y \).
Now we proceed to check whether the black hole phase transition satisfies the Ehrenfest equations. In other words, we investigate the validity of the Ehrenfest equations at the points of discontinuities $r_{+i}$, $i = 1, 2$. Here we denote the critical values of temperature by $T_{ci}$, charge by $Q_i$, and entropy by $S_i$. The left hand side of the first Ehrenfest equation (27) at the critical point can be written as

\[ - \left[ \left( \frac{\partial \Phi}{\partial T} \right)_{T=S_i} \right]_{r=r_{+i}} = - \left[ \left( \frac{\partial \Phi}{\partial Q} \right)_{Q=S_i} \right]_{r=r_{+i}} - \left[ \left( \frac{\partial Q}{\partial T} \right)_{T=S_i} \right]_{r=r_{+i}}.
\]

Using eq. (s) (13), the above equation can be written in the form

\[ \frac{\Delta C_{\Phi}}{T_i Q_i \Delta \beta} = - \left[ \left( \frac{\partial \Phi}{\partial Q} \right) \right]_{Q=S_i} \left( \frac{\partial Q}{\partial S} \right)_{S=S_i} \left( \frac{C_{\Phi}}{T_i} \right).
\]

which implies

\[ \frac{\Delta C_{\Phi}}{T_i Q_i \Delta \beta} = - \left[ \left( \frac{\partial \Phi}{\partial Q} \right) \right]_{Q=S_i} \left( \frac{\partial Q}{\partial S} \right)_{S=S_i} \left( \frac{C_{\Phi}}{T_i} \right).
\]

Using the identity, the above equation can be written in the form

\[ \frac{\Delta C_{\Phi}}{T_i Q_i \Delta \beta} = - \left[ \left( \frac{\partial \Phi}{\partial Q} \right) \right]_{Q=S_i} \left( \frac{\partial Q}{\partial S} \right)_{S=S_i} \left( \frac{C_{\Phi}}{T_i} \right).
\]

Now using eq. (s) (15), (17), (18), we calculate the right hand side of the above relation to be

\[ \frac{\Delta C_{\Phi}}{T_i Q_i \Delta \beta} = - \left[ \left( \frac{\partial \Phi}{\partial Q} \right) \right]_{Q=S_i} \left( \frac{\partial Q}{\partial S} \right)_{S=S_i} \left( \frac{C_{\Phi}}{T_i} \right).
\]

Eq. (31) and eq. (35) show the validity of the first Ehrenfest’s equation for the black hole spacetime under consideration. We now proceed to check the second Ehrenfest equation. To calculate the left hand side of the second Ehrenfest equation, we use the thermodynamic relation

\[ \left( \frac{\partial T}{\partial \Phi} \right)_{Q} = \left( \frac{\partial T}{\partial S} \right)_{\Phi} \left( \frac{\partial S}{\partial \Phi} \right)_{Q} + \left( \frac{\partial T}{\partial \Phi} \right)_{S}, \tag{36} \]

taking $T = T(S, \Phi)$. 
Since the heat capacity diverges at the critical points, hence \( \left( \frac{\partial T}{\partial S} \right)_\Phi S=S_i = 0 \) and \( \left( \frac{\partial S}{\partial \Phi} \right)_Q \) are finite at the critical point. Therefore, the above equation becomes
\[
\left[ \left( \frac{\partial T}{\partial \Phi} \right)_Q \right]_{S=S_i} = \left[ \left( \frac{\partial T}{\partial \Phi} \right) \right]_{S=S_i}
\] (37)
which implies
\[
\left[ \left( \frac{\partial T}{\partial \Phi} \right)_Q \right]_{r_+ = r_{+i}} = \left[ \left( \frac{\partial T}{\partial \Phi} \right) \right]_{r_+ = r_{+i}}.
\] (38)

Therefore, the left hand side of the second Ehrenfest equation is equal to left hand side of the first Ehrenfest equation. Hence, we have
\[
\kappa Q_i = \left[ \left( \frac{\partial Q}{\partial \Phi} \right) \right]_{T=S_i}.
\] (40)

Using the thermodynamic identity \( \left( \frac{\partial Q}{\partial \phi} \right)_T \left( \frac{\partial \Phi}{\partial T} \right)_Q \left( \frac{\partial T}{\partial Q} \right) \Phi = -1 \) and the definition of \( \beta \) in eq. (29), we find
\[
\kappa Q_i = \left[ \left( \frac{\partial T}{\partial \Phi} \right)_Q \right]_{S=S_i} Q_i \beta.
\] (41)

Therefore, the right hand side of the second Ehrenfest equation (28) reduces to
\[
\frac{\Delta \beta}{\Delta \kappa} = - \left[ \left( \frac{\partial \Phi}{\partial Q} \right)_r \right]_{S=S_i}.
\] (42)

This can further be written as
\[
- \left[ \left( \frac{\partial \Phi}{\partial T} \right)_Q \right]_{S=S_i} = - \left[ \left( \frac{\partial \Phi}{\partial S} \right)_Q \right]_{S=S_i} \left[ \left( \frac{\partial S}{\partial T} \right)_Q \right]_{S=S_i}
\] (43)
\[
- \left[ \left( \frac{\partial \Phi}{\partial T} \right)_Q \right]_{S=S_i} = - \left[ \left( \frac{\partial \Phi}{\partial S} \right)_Q \right]_{S=S_i} \left[ \left( \frac{\partial S}{\partial T} \right)_Q \right]_{S=S_i}
\]
which in turn implies
\[
- \left[ \left( \frac{\partial \Phi}{\partial T} \right)_Q \right]_{S=S_i} = - \left[ \left( \frac{\partial \Phi}{\partial T} \right)_Q \right]_{r_+ = r_{+i}} = - \left[ \left( \frac{\partial \Phi}{\partial r_+} \right)_Q \right]_{r_+ = r_{+i}}.
\] (44)

The condition that heat capacity in eq. (14) diverges at the critical point gives
\[
\left[ \left( \frac{\partial \Phi}{\partial Q} \right) \right]_{r_+ = r_{+i}} \left[ \left( \frac{\partial T}{\partial r_+} \right)_Q \right]_{r_+ = r_{+i}} = \left[ \left( \frac{\partial T}{\partial Q} \right) \right]_{r_+ = r_{+i}} \left[ \left( \frac{\partial \Phi}{\partial r_+} \right)_Q \right]_{r_+ = r_{+i}}.
\] (45)
Using the above condition and eq.(44) we obtain the right hand side of the second Ehrenfest equation to be
\[
\Delta \beta = \omega (r_{+}^2 + 2\alpha') r_{+}^{D-4} Q_{i}[^{16}\pi^2 Q_i^2]^2 F_1 \left[ \frac{D - 3}{2 D - 4}, \frac{1}{2}, \frac{1}{4}, \frac{3 D - 7}{2 D - 4}, \frac{-16 \pi^2 Q_i^2}{b^2 \omega^2 r_{+}^{2D-4}} \right].
\]  
(46)

Eq.(39) and eq.(46) shows the validity of the second Ehrenfest equation at critical points.

Finally, we calculate the Prigogine-Defay (PD) ratio [16] for the system. Using eq.(27) and eq.(37), we have
\[
\left[ \frac{\partial \Phi}{\partial T} \right]_{Q_i} S = \left[ \frac{\partial \Phi}{\partial T} \right]_{S} S_i = \Delta C_{\Phi} T_{i} Q_{i} \Delta \beta.
\]  
(47)

Now eq.(42) with the above equation gives
\[
\Pi = \frac{\Delta C_{\Phi} \Delta \kappa}{T_{i} Q_{i} (\Delta \beta)^2} = 1.
\]  
(48)

These results show the exact second order nature of phase transition in this black hole.

IV. RUPPEINER STATE SPACE GEOMETRY ANALYSIS

Ruppeiner’s state space geometry technique is another approach to study thermodynamic systems. The idea is to write the abstract manifold with the notion of distance from thermodynamic variables and study the curvature in order to study phase transition. This technique has been applied to black hole systems in [25]-[26]. Definitions of metric coefficients are quite well known [27].

The Ruppeiner metric coefficients for the manifold are given by
\[
g^R_{ij} = - \frac{\partial^2 S(x)}{\partial x^i \partial x^j},
\]  
(49)

where \(x^i = x^i(M,Q); i = 1, 2\) are extensive variables of the manifold. The calculation of the Weinhold metric coefficients is convenient for computational purpose. These are given by
\[
g^W_{ij} = \frac{\partial^2 M(x)}{\partial x^i \partial x^j},
\]  
(50)

where \(x^i = x^i(S,Q); i = 1, 2\). It is to be noted that Weinhold geometry is connected to the Ruppeiner geometry through the following relation
\[
dS^2_R = \frac{dS^2_W}{T}.
\]  
(51)

For the black hole spacetime with which we are working, the Ruppeiner metric coefficients can be calculated to be
\[
g_{SS} = \frac{1}{T} \left[ \frac{\partial^2 M}{\partial r^2} Q + \left( \frac{\partial M}{\partial r} \right)_{Q} \left( \frac{dS}{dr} \right) \right],
\]  
\[
g_{SQ} = \frac{1}{T} \left( \frac{\partial^2 M}{\partial r \partial Q} \right)_{S},
\]  
\[
g_{QQ} = \frac{1}{T} \left( \frac{\partial^2 M}{\partial Q^2} \right)_{r_+}.
\]  
(52)

From eq.(s)(6, 10), we calculate first and second order partial derivatives appearing in eq.(52) to be
\[
\frac{\partial^2 M}{\partial r_+ \partial Q} = - \frac{4 \pi Q}{\omega r_+^{D-2}} \left( 1 + \frac{16 \pi^2 Q_i^2}{b^2 \omega^2 r_{+}^{2D-4}} \right)^{-1/2}
\]  
(53)
\[
\frac{d^2 S}{dr^2} = \frac{\omega}{4} (D-2)r^{D-4} \left( (D-3) + \frac{2(D-5)\alpha'}{r^2} \right)
\]

\[
\left( \frac{\partial M}{\partial r^+} \right)_Q = \frac{(D-2)(D-5)\alpha'}{16\pi} r^{D-6} + \frac{(D-2)(D-3)\omega}{16\pi} r^{D-4} + \frac{(D-1)(D-2)\omega}{16\pi} r^{D-2} + \frac{b^2 \omega}{4\pi} r^{D-2} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r^{2D-4}}} \right)
\]

\[
\left( \frac{\partial^2 M}{\partial r^2} \right)_Q = \frac{(D-2)(D-5)(D-6)\alpha'}{16\pi} r^{D-7} + \frac{(D-2)(D-3)(D-4)\omega}{16\pi} r^{D-5} + \frac{(D-1)(D-2)\omega}{16\pi} r^{D-3} + \frac{b^2(D-2)\omega}{4\pi} r^{D-3} \left( 1 - \sqrt{1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r^{2D-4}}} \right) \frac{4(D-2)\pi Q^2}{\omega r^{D-1}} \left( 1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r^{2D-4}} \right)^{-1/2}
\]

\[
\frac{\partial^2 M}{\partial Q^2} \bigg|_{r^+} = \frac{4\pi}{\omega(D-2)} r^{D-3} \left[ \left( 1 + \frac{16\pi^2 Q^2}{b^2 \omega^2 r^{2D-4}} \right)^{-1/2} + \frac{1}{D-3} \right] F_1 \left[ \frac{D-3}{2D-4}, \frac{1}{2} \frac{3D-7}{2D-4}, \frac{16\pi^2 Q^2}{b^2 \omega^2 r^{2D-4}} \right].
\]

**FIG. 3:** Ruppeiner curvature vs. horizon radius

\((Q=0.13, b=10, \alpha'=0.01)\)
With these metric coefficients, we can calculate the curvature of the two dimensional manifold, and singularities in the curvature indicate second order phase transition. The formula for the curvature reads

\[
R = -\frac{1}{\sqrt{|g|}} \left[ \frac{\partial}{\partial r_+} \left( g_{SQ} \frac{\partial g_{SS}}{\partial Q} - \frac{1}{\sqrt{|g|}} g_{QQ} \left( \frac{dr_+}{dS} \right) \right) \left( \frac{dr_+}{dS} \right) \right.
\]

\[
\left. + \frac{\partial}{\partial Q} \left( \frac{2}{\sqrt{|g|}} \frac{\partial g_{SQ}}{\partial Q} - \frac{1}{\sqrt{|g|}} \frac{\partial g_{SS}}{\partial Q} - g_{SQ} \frac{\partial g_{SS}}{\partial r_+} \left( \frac{dr_+}{dS} \right) \right) \right]
\]

(58)

where \( g \) is \( g_{SS} \times g_{QQ} - g_{SQ}^2 \). We have calculated and plotted \( R \) with the horizon radius for fixed values of other parameters in fig. 3. These show multiple points of discontinuities, however, all are not physical. It is interesting to note that the Ruppeiner curvature diverges at all the values where the heat capacity is singular. This indicates the second order nature of phase transition in this black hole.

V. CONCLUSION

In this work, we have analysed black holes in Gauss-Bonnet gravity with Born-Infeld electrodynamics in AdS background. We calculated the thermodynamic properties associated with the black hole spacetime and calculated the heat capacity. It has been obtained that there are infinite discontinuities in the heat capacity when plotted with the horizon radius with other parameters kept constant. These are the signs of second order phase transitions. Ehrenfest scheme is then employed to determine the authenticity of the second order phase transition. We find that the system under consideration followed both Ehrenfest equations. We calculated the Prigogine-Defay ratio using Ehrenfest equations which indicated that the divergences are exactly of second order. Ruppeiner’s state space geometry technique also helped further confirming the second order nature of these black hole phase transitions. Ruppeiner curvature is calculated and again plotted with the horizon radius for the same fixed parameters. Singularities are observed exactly at those points where the heat capacity diverged. Both these methods reconfirm the second order nature of phase transition in these black holes. Reassuringly, we recover the Einstein-Born-Infeld black holes when the Gauss-Bonnet parameter is put to zero. We have also derived a Smarr relation in \( D \)-spacetime dimensions using scaling arguments and first law of black hole thermodynamics in which the cosmological constant, Born-Infeld parameters and Gauss-Bonnet parameter are treated as thermodynamic variables.

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