Thermodynamic Properties of a Quantum Group Boson Gas

$GL_{p,q}(2)$

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Abstract

An approach is proposed enabling to effectively describe the behaviour of a bosonic system. The approach uses the quantum group $GL_{p,q}(2)$ formalism. In effect, considering a bosonic Hamiltonian in terms of the $GL_{p,q}(2)$ generators, it is shown that its thermodynamic properties are connected to deformation parameters $p$ and $q$. For instance, the average number of particles and the pressure have been computed. If $p$ is fixed to be the same value for $q$, our approach coincides perfectly with some results developed recently in this subject. The ordinary results, of the present system, can be found when we take the limit $p = q = 1$.

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1 Introduction

In the past few years, the application of the quantum groups \([1, 2]\) in the study of the physical models has attracted several researchers. Indeed, many surprising results have appeared, which involve this mathematical background. For example, in quantum Hall effect \([3]\), a fascinating relation between deformation parameter of \(U_q(sl(2))\) and the so-called filling factor is established \([4 – 10]\). Furthermore, it is shown that the deformation gives some information about the statistics of anyon systems \([11]\) and can solve some problems like Azbel-Hofstadter one \([12 – 14]\).

Another kind of application has appeared recently. In effect, the thermodynamic properties of the fermions and bosons are investigated in the framework of the quantum groups. For this matter, we note the works developed in \([15 – 17]\), where the author has obtained some more important results, like the generalization of the ordinary physical quantities of the fermions and bosons. Exactly, he has proved the role that quantum group \(SU_q(2)\) plays in the thermodynamic system at high temperature \((T)\), corresponding to the case \(z = e^{\beta \mu} \ll 1\), where \(\beta = \frac{1}{k_B T}, k_B\) Boltzman’s constant, and \(\mu\) the chemical potential.

This paper is based on the comprehensive work of Ubriaco \([17]\), which we refer to for a list of references concerning the history of this subject. Thus, we wish to extend the Ubriaco’s theory, developed in \([17]\) concerning the bosonic gas, to cover some other interesting results. For this, we introduce a mathematical background. The latter includes that of quantum group \(GL_{p,q}(2)\). Otherwise, we are looking for the role that group, instead of \(SU_q(2)\), plays in the thermodynamic system at high temperature. Effectively, we show that it is possible to obtain the thermodynamic properties in terms of two deformation parameters \(p\) and \(q\), when we work with the same case of temperature like in \([17]\). We mean at high temperature condition, namely \(z = e^{\beta \mu} \ll 1\). Furthermore, we find the results derived in \([17]\) just by limiting to the case \(p = q\).
We shall present, in the next section, the quantum group $SU_q(N)$-bosons background and in particular case where $N = 2$. By this we mean a set of definitions and notations concerning the generators and $R$-matrix of $SU_q(N)$. We shall especially insist on the terminology in use in our generalization. Section 3 is devoted to the description of the quantum group $GL_{p_{ij},q_{ij}}(N)$-bosons and especially the $GL_{p,q}(2)$ one. In section 4, we introduce an explicit physical model. In effect, we consider a Hamiltonian, describing two bosonic types, in terms of the $GL_{p,q}(2)$ generators. In that case, we present a new representation of the generators from which a new Hamiltonian is obtained. In order to control to what extent or new Hamiltonian, we compute in section 5 the thermodynamic properties like the average number of particles and the pressure. We conclude and give a brief comment on the extension of our methods to Bose-Einstein condensation (BEC) in the final section.

2 Quantum group $SU_q(N)$-bosons

In this section, we give a short review concerning the quantum group $SU_q(N)$-bosons, including the generators, $R$-matrix and some elementary properties. We begin by establishing the commutation relations of $SU_q(N)$ as follows [17]

$$\Phi_j \Phi_i = \delta_{ij} + q R_{kijl} \Phi_l \Phi_k,$$

$$\Phi_i \Phi_k = q^{-1} R_{jikl} \Phi_j \Phi_i, \quad i, j = 1, 2, \ldots, N,$$

which generalize the ordinary ones

$$\phi_i \phi_j^+ - \phi_j^+ \phi_i = \delta_{ij}. \quad \text{(2)}$$

$\delta_{ij} = 1$ for $i = j$ and $0$ for $i \neq j$. The $R$-matrix is [18]

$$R_{jikl} = \delta_{jk} \delta_{il} (1 + (q - 1) \delta_{ij}) + (q - q^{-1}) \delta_{ik} \delta_{jl} \theta(j - i), \quad \text{(3)}$$

and the function $\theta(j - i)$ is given by

$$\theta(j - i) = \begin{cases} 
1 & \text{if } j > i, \\
0 & \text{otherwise.}
\end{cases} \quad \text{(4)}$$
With the help of the following linear transformation
\[
\Phi'_i = \sum_{j=1}^{N} T_{ij} \Phi_i,
\]  
(5)

it is shown that eqs.(1) are covariant under the SU\(_q(N)\) transformation. The matrix \(T_{ij}\) is the \(N\)-dimensional representation of the present group, which verifies the algebraic relations [19]
\[
RT_1 T_2 = T_2 T_1 R,
\]
\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12},
\]
(6)

where \(T_1 = T \otimes 1\) and \(T_2 = 1 \otimes T \in V \otimes V\) and \((R_{23})_{ijk,j'k'} = \delta_{ii'} R_{jk,j'k'} \in V \otimes V \otimes V\).

For a given unitary quantum group matrix \(T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) and from RTT equation it is found [20]
\[
ab = q^{-1} ba, \quad ac = q^{-1} ca,
\]
\[
bc = cb, \quad dc = qcd,
\]
\[
db = qbd, \quad da - ad = (q - q^{-1})bc,
\]
\[
det_q T = ad - q^{-1} bc = 1,
\]
(7)

where \(q \in \mathbb{R}\) and the adjoint matrix of \(T\)
\[
\bar{T} = \begin{pmatrix} d & -qb \\ -q^{-1} c & a \end{pmatrix},
\]
(8)

are taken in to consideration.

The commutation relations generating the quantum group SU\(_q(2)\)-bosons are given by
\[
\Phi_2 \bar{\Phi}_2 - q^2 \bar{\Phi}_2 \Phi_2 = 1,
\]
\[
\Phi_1 \bar{\Phi}_1 - q^2 \bar{\Phi}_1 \Phi_1 = 1 + (q^2 - 1) \bar{\Phi}_2 \Phi_2,
\]
\[
\Phi_2 \Phi_1 = q \Phi_1 \Phi_2,
\]
\[
\Phi_2 \bar{\Phi}_1 = q \bar{\Phi}_1 \Phi_2,
\]
(9)

which can be obtained by replacing \(N = 2\) in eqs.(1).

Having given some basic notions related to the quantum group SU\(_q(N)\)-bosons. Now we turn to generalize the above definitions and properties. For this matter, we introduce the multiparametric quantum group \(GL_{p,q}(N)\) tools in the next section.
3 Quantum group $GL_{p_{ij},q_{ij}}(N)$-bosons

Let us reomtne to another quantum group. In effect, we would like to establish a set of the background useful in the next, which leads to obtain a quantum group bosons. For this matter, we start by recalling $GL_{p_{ij},q_{ij}}(N)$, $i, j = 1, ..., N$. The latter is defined by the commutation relations [21]

$$ R(T \otimes 1)(1 \otimes T) = (1 \otimes T)(T \otimes 1)R, $$

which can be written otherwise

$$ R^{ik}_{rs}T^{r}_{v}T^{s}_{w} = T^{f}_{v}R^{ik}_{ef}T^{f}_{w}, $$

where $k, r, s, v, w, e, f = 1, ..., N$ and the $R$-matrix is [21]

$$ (R^{ij}_{p_{ij},q_{ij}})_{kl} = \delta^i_k \delta^j_l \delta^{ij} + \frac{\theta(j - i)}{p_{ij}} + \theta(i - j) \frac{1}{q_{ij}} + (1 - p_{ij}^{-1}q_{ij}^{-1})\delta^i_k \delta^j_l \theta(i - j). $$

The latter satisfies the Yang-Baxter equation

$$ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, $$

where the $\theta$-function is given by Eq.(4). The inverse of $R^{ij}_{p_{ij},q_{ij}}$ is

$$ (R^{ij}_{p_{ij},q_{ij}})^{-1} = R^{1}_{p_{ij}^{-1}q_{ij}}. $$

In this step, we would like to introduce a differential calculus for the reason which will appear in the next. To do this, we would like to remember a work of Schirrmacher done in [21]. In this reference, the author has constructed a differential calculus on the $N$-dimensional vector space associated to the multiparametric deformation of $GL(N)$, which generalized the one-parameter differential calculus on the Manin plane [22]. Indeed, the coordinates $x_i$ and the derivatives $\partial_i$ verify the following relations

$$ x_i x_j = q_{ij} x_j x_i, $$
$$ \partial_i \partial_j = p_{ij}^{-1} \partial_j \partial_i, $$
$$ \partial_i x_i = 1 + p_{ij} q_{ij} x_i \partial_i + (p_{ij} q_{ij} - 1) \sum_{j=i+1}^{N} x_j \partial_j, $$
$$ \partial_i x_j = p_{ij} x_j \partial_i, $$
$$ \partial_j x_i = q_{ij} x_i \partial_j, $$

(15)
with \( i, j = 1, ..., N \) and \( i < j \). We note two remarks. The first one, the above differential calculus is covariant under the \( GL_{p_{ij},q_{ij}}(N) \) transformation. The second when \( N = 2 \), equations (15) coincide perfectly with two-parameter generalization of the Wess and Zumino differential calculus on the Manin plane \[23\].

Now we are looking for a realization of the quantum group bosons. To do this let us proceed as follows

\[
  x_i \rightarrow \bar{\Phi}_i, \quad \partial_i \rightarrow \Phi_i. \tag{16}
\]

These correspondences lead us to obtain

\[
\begin{align*}
  \bar{\Phi}_i \bar{\Phi}_j &= q_{ij} \bar{\Phi}_j \bar{\Phi}_i, \\
  \Phi_i \Phi_j &= \frac{1}{p_{ij}} \Phi_j \Phi_i, \\
  \Phi_i \bar{\Phi}_i &= 1 + p_{ij} q_{ij} \Phi_i \bar{\Phi}_i + (p_{ij} q_{ij} - 1) \sum_{j=i+1}^N \bar{\Phi}_j \Phi_j, \\
  \Phi_i \bar{\Phi}_j &= p_{ij} \bar{\Phi}_j \Phi_i, \\
  \Phi_j \bar{\Phi}_i &= q_{ij} \bar{\Phi}_i \Phi_j,
\end{align*}
\]

which generate the quantum group \( GL_{p_{ij},q_{ij}}(N) \)-bosons. The latter admits the following involution (Hermitian conjugation)

\[
\Phi_i \rightarrow \Phi_i, \quad q_{ij} \rightarrow p_{ij}. \tag{18}
\]

To reproduce eqs.(1), we can fix \( p_{ij} = q_{ij} = q \).

Now we focus ourselves on the interesting case \( N = 2 \), \( q_{ij} = q \) and \( p_{ij} \), which will be useful in the next of our study. Introducing these conditions in eqs.(17), we find the \( GL_{p,q}(2) \) commutation relations

\[
\begin{align*}
  \Phi_1 \Phi_2 &= p^{-1} \Phi_2 \Phi_1, \\
  \Phi_1 \Phi_1 &= 1 + pq \Phi_1 \Phi_1 + (pq - 1) \Phi_2 \Phi_2, \\
  \Phi_2 \Phi_2 &= 1 + pq \Phi_2 \Phi_2, \\
  \Phi_1 \Phi_2 &= p \Phi_2 \Phi_1, \\
  \Phi_2 \Phi_1 &= q \Phi_1 \Phi_2, \\
  \Phi_1 \Phi_2 &= q \Phi_2 \Phi_1. \tag{19}
\end{align*}
\]
When \( p = q \), these equations coincide perfectly with eqs.(9). The algebraic relations analogue to eqs.(7) can be obtained from eq.(11)

\[
\begin{align*}
ab & = pba, \quad ac = qca, \\
pbc & = qcb, \quad cd = pdc, \\
bd & = qdb, \quad ad - da = (p - q^{-1})bc,
\end{align*}
\]

(20)
corresponding to the quantum group \( GL_{p,q}(2) \), where \( 0 \leq q < \infty \) and \( 0 \leq p < \infty \).

This concludes this section. In the next section we would like to analyse a physical model in the framework of the above tools. For this, we consider a bosonic Hamiltonian involving the generators of the quantum group \( GL_{p,q}(2) \).

4 \( GL_{p,q}(2) \)-boson model

Before going on, we would like to make a remark. Indeed, our approach is different from that known in the history of this subject, for example [24 – 27]. So, our main goal is to generalize the work done recently in [17]. For this matter, let us consider a Hamiltonian describing two types of bosons with the same energy like

\[
H = \sum_k \epsilon_k (\hat{D}_{1,k} + \hat{D}_{2,k}),
\]

(21)
where the operators \( \hat{D}_{1,k} \) and \( \hat{D}_{2,k} \) are given by

\[
\hat{D}_{1,k} = \Phi_{1,k} \Phi_{1,k}, \quad \hat{D}_{2,k} = \Phi_{2,k} \Phi_{2,k},
\]

(22)
and \( \epsilon_k \) is the spectrum of energy, \( k = 0, 1, 2... \). We show that the following relations are satisfied for a given \( k \)

\[
\begin{align*}
\hat{D}_2 \Phi_1 - \Phi_1 \hat{D}_2 & = 0, \\
q \hat{D}_1 \Phi_1 - p^{-1} \Phi_1 \hat{D}_1 & = 0.
\end{align*}
\]

(23)
The normalized states of the Hamiltonian eq.(21) can be built as follows

\[
|d_1, d_2> = \frac{\Phi_{2,d2} \Phi_{1,d1}}{\sqrt{|d_1\rangle_{p,q}! |d_2\rangle_{p,q}!}} |0, 0>,
\]

(24)
where $|0,0>$ is the ground state of $H$ and we propose that the symbol $\{x\}_{p,q}$ takes the form

$$\{x\}_{p,q} = \frac{1 - q^p p^q}{1 - qp},$$  \hspace{1cm} (25)$$

and $\{x\}_{(p,q)!}$ is

$$\{x\}_{p,q}! = \{x\}_{p,q} \{x - 1\}_{p,q}...1.$$  \hspace{1cm} (26)

Our goal in the next section is to compute the thermodynamic properties of the above Hamiltonian. To do this, let us introduce a new representation. Indeed, we would like to establish a relation between the new operators and the old ones, namely the ordinary boson operators. In order to reply to this request, we propose the following representation for a given $k$

$$\Phi_2 = (\phi_2^+)^{-1}\{D_2\}_{p,q},$$

$$\bar{\Phi}_2 = \phi_2^+,$$

$$\Phi_1 = (\phi_1^+)^{-1}\{D_1\}_{p,q} \rho^{D_2},$$

$$\bar{\Phi}_1 = \phi_1^+ \rho^{D_2}.$$  \hspace{1cm} (27)

where $D_{1,k} = \phi_{1,k}^+ \phi_{1,k}, D_{2,k} = \phi_{2,k}^+ \phi_{2,k}$ and $\phi_i, \phi_i^+$ are the ordinary boson operators. It easy to show that these equations satisfy the commutation relations given by eqs.(19). Furthermore, by using the last set of equations and eq.(23), we can rewrite the above Hamiltonian as follows

$$H = \sum_k \epsilon_k \{D_{1,k} + D_{2,k}\}_{p,q}.$$  \hspace{1cm} (28)

Note that this representation led us to obtain an interacting Hamiltonian instead of the original one eq.(21).

Having obtained the Hamiltonian in terms of the ordinary boson operators, let us now calculate explicitly its thermodynamic properties.

## 5 Thermodynamic properties

In this section, we are looking for the thermodynamic properties of the bosonic system described by the Hamiltonian eq.(28). To get these properties, we begin by evaluating the
corresponding grand partition function. The latter can be written as follows

$$Z = Tr \exp[-\beta\epsilon_k(\bar{\Phi}_{1,k}\Phi_{1,k} + \bar{\Phi}_{2,k}\Phi_{2,k})] \exp[\beta\mu(\bar{\phi}_{1,k}\phi_{1,k} + \bar{\phi}_{2,k}\phi_{2,k})].$$  \hspace{1cm} (29)$$

Using the above tools, we show that the last equation becomes

$$Z = \prod_k \sum_{d_{1,k}=0}^{\infty} \sum_{d_{2,k}=0}^{\infty} \exp[-\beta\epsilon_k\{d_{1,k} + d_{2,k}\}_{p,q}] \exp[\beta\mu(d_{1,k} + d_{2,k})],$$  \hspace{1cm} (30)

which can be written otherwise

$$Z = \prod_k \sum_{d_k=0}^{\infty} (d_k + 1) \exp[-\beta\epsilon_k\{d_k\}_{p,q}] z^{d_k},$$  \hspace{1cm} (31)

where \(z = e^{\beta\mu}\) is called the fugacity. It is clear that a direct computation of this equation is very hard. However, there is a limit case which leads to obtain the thermodynamic properties of eq.(28). For this, let us consider the limit \(z \ll 1\), which corresponds to the high temperature case. Then, taking into account this approximation, we can write eq.(31) as follows

$$\ln Z = \frac{4\pi V}{h^3} \int_0^\infty p^2 dp[2e^{-\beta\epsilon}z + (6e^{-\beta\epsilon(2)}_{p,q} - 4e^{-2\beta\epsilon})\frac{z^2}{2} +$$

\(24e^{-\beta\epsilon(3)}_{p,q} - 36e^{-\beta\epsilon(2)}_{p,q} + 8e^{-3\beta\epsilon})\frac{z^3}{3!}...].$$ \hspace{1cm} (32)

The manipulation of the integral in the 3-dimensional momentum space leads to

$$\ln Z = \frac{4\pi V}{h^3}[\frac{\sqrt{\pi}}{2} \left(\frac{2m}{\beta}\right)^\frac{3}{2} z + \sqrt{\pi} \left(\frac{2m}{\beta}\right)^\frac{3}{2} \delta(p,q) z^2 + ...],$$ \hspace{1cm} (33)

where the function \(\delta(p,q)\) is given by

$$\delta(p,q) = \frac{1}{4} \left[\frac{3}{(1 + pq)^\frac{3}{2}} - \frac{1}{\sqrt{2}}\right].$$ \hspace{1cm} (34)

We note that when we take the limit \(p = q\), we find for the grand function partition coincides with the one derived in ref.[17] concerning the boson gas.

Having obtained eq.(33), we return now to investigate the thermodynamic properties related to the bosonic system described by the above Hamiltonian. For these, we begin by the average number of particles. The latter is given by

$$\langle N \rangle = \frac{1}{\beta} \left(\frac{\partial \ln Z}{\partial \mu}\right)_{T,V}.$$ \hspace{1cm} (35)
Using eq.(34), we find
\[ \langle N \rangle = \frac{4\pi V}{h^3} \left[ \frac{\sqrt{\pi}}{2} \left( \frac{2m}{\beta} \right)^{\frac{3}{2}} z + 2 \sqrt{\pi} \left( \frac{2m}{\beta} \right)^{\frac{3}{2}} \delta(p, q) z^2 + ... \right]. \] (36)

By inverting this equation, we prove that the fugacity can be written as
\[ z \approx \frac{1}{2} \left( \frac{h^2}{2m\pi kT} \right)^{\frac{3}{2}} \frac{\langle N \rangle}{V} - \delta(p, q) \left( \frac{h^2}{2m\pi kT} \right)^{\frac{3}{2}} \frac{\langle N \rangle^2}{V}. \] (37)

In the same way, the pressure is related to the grand function partition by
\[ P = \frac{1}{\beta} \left( \frac{\partial \ln Z}{\partial V} \right)_{\mu, T}. \] (38)

All calculations done, we get
\[ P = \frac{4\pi V}{h^3} \left[ \frac{\sqrt{\pi}}{2} \left( \frac{2m}{\beta} \right)^{\frac{3}{2}} z + \sqrt{\pi} \left( \frac{2m}{\beta} \right)^{\frac{3}{2}} \delta(p, q) z^2 + ... \right]. \] (39)

The above relations led us to obtain an equation of state like
\[ PV = kT \langle N \rangle \left( 1 - \delta(p, q) \left( \frac{h^2}{2m\pi kT} \right)^{\frac{3}{2}} \frac{\langle N \rangle}{V} + ... \right). \] (40)

Note that this expression depends on deformation parameters \( p \) and \( q \).

For a system of particles living in two dimension space, the above equation of state can be written as follows
\[ PV = kT \langle N \rangle \left( 1 - \eta(p, q) \left( \frac{h^2}{2m\pi kT} \right)^{\frac{3}{2}} \frac{\langle N \rangle}{A} + ... \right), \] (41)

where \( A \) is the surface in which confined the bosonic system and the function \( \eta(p, q) \) is
\[ \eta(p, q) = \frac{2 - pq}{4(1 + pq)}. \] (42)

Note that the last two equations are obtained in the same way as above for a 3-dimension space.

This analysis generalizes the bosonic results derived in ref.[17]. The latter can be found in the limit where \( p = q \). Furthermore, for \( p = q = 1 \) we obtain the classical results, i.e. the thermodynamic properties of the ordinary bosons.
6 Conclusion

In this paper, we have investigated a bosonic system in the framework of the quantum group $GL_{p,q}(2)$. To do this, we have first introduced a differential calculus. The latter leads us to obtain the multiparametric quantum group $GL_{p_{ij},q_{ij}}(N)$ and then the $GL_{p,q}(2)$ one. Second, we have considered a bosonic Hamiltonian in terms of $GL_{p,q}(2)$ generators. Subsequently, we have calculated the thermodynamic properties starting from the corresponding grand partition function. Therefore, we have shown, for example, that the average number of particles and the pressure depending on deformation parameters $p$ and $q$. The equation of state is also obtained in this way. This task generalized the recent one [17] concerning the boson gas. The latter results are reproduced when we take the limit $p = q$ in our analysis. Also the ordinary results can be found for the case where $p = q = 1$.

The low temperature case will be considered in a subsequent work [28]. This will include a study of the Bose-Einstein condensation (BEC) in the framework of the quantum group $GL_{p,q}(2)$-bosons.

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