A note on the Bellman function of the dyadic maximal operator

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Abstract

For each \( p > 1 \) we find an expression, for the main Bellman function of three variables associated to the dyadic maximal operator on \( \mathbb{R}^n \), that appears in [4], by using an alternative approach which was introduced in [5]. Actually we do that in the more general setting of tree-like maximal operators.

1 Introduction

The dyadic maximal operator is a useful tool in analysis and is defined by

\[
M_d \phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| \, du : \ x \in Q, \ Q \subseteq \mathbb{R}^n \text{ a dyadic cube} \right\},
\]

(1.1)

for every \( \phi \in L^1_{\text{loc}}(\mathbb{R}^n) \), where the dyadic cubes are those formed by the grids \( 2^{-N} \mathbb{Z}^n \) for \( N = 0, 1, 2, \ldots \). As is well known it satisfies the following weak-type (1,1) inequality

\[
|\{ x \in \mathbb{R}^n : M_d \phi(x) > \lambda \}| \leq \frac{1}{\lambda} \int_{\{ M_d \phi > \lambda \}} |\phi(u)| \, du,
\]

(1.2)

for every \( \phi \in L^1(\mathbb{R}^n) \) and every \( \lambda > 0 \), from which it is easy to get the following \( L^p \)-inequality:

\[
\|M_d \phi\|_p \leq \frac{p}{p-1} \|\phi\|_p,
\]

(1.3)

for every \( p > 1 \) and every \( \phi \in L^p(\mathbb{R}^n) \), which is known as Doob’s inequality.

It easy to see that the weak type inequality (1.2) is best possible. It has also been proved that (1.3) is best possible (see [1], [2] for the general martingales and [6] for dyadic ones).

For the study of the dyadic maximal operator it is more convenient to work with functions supported on the unit cube \([0,1]^n\), and for this reason we replace \( M_d \) by:

\[
M'_d \phi(x) = \sup \left\{ \frac{1}{|Q|} \int_Q |\phi(u)| \, du : \ x \in Q, \ Q \subseteq [0,1]^n \text{ a dyadic cube} \right\}.
\]

(1.4)
Actually we will work in a much more general setting of a non-atomic probability space \((X, \mu)\) equipped with a tree structure \(T\), which plays the same role as the dyadic cubes on \([0, 1]^n\) (for the precise definition, see the next section). Then we define the maximal operator corresponding to \(T\), by

\[
M_T \phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi| \, d\mu : x \in I \in T \right\},
\]

(1.5)

for every \(\phi \in L^1(X, \mu)\). Then (1.2) and (1.3) remain true and sharp even in this setting.

An approach for studying such maximal operators is the introduction of the so-called Bellman functions (see [4]) related to them. The main Bellman function of \(M_T\), of two variables is given by

\[
B_T^p(f, F) = \sup \left\{ \int_X (M_T \phi)^p \, d\mu : \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^p \, d\mu = F \right\},
\]

(1.6)

where \(f, F\) are variables satisfying \(0 < f^p \leq F\).

The evaluation of (1.6) has been given in [4], where an effective linearization technique was introduced for the operator \(M_T\), which allowed the author to precise compute it, and as it can be seen there is a difficult task. In fact it is proved that (1.6) is given by

\[
B_T^p(f, F) = F \omega_p \left( \frac{f^p}{F} \right),
\]

where \(\omega_p : [0, 1] \to \left[ 1, \frac{p}{p-1} \right]\), is the inverse function of \(H_p\), defined by \(H_p(z) = -(p-1)z^p + pz^{p-1}\).

In fact in [4] more general Bellman functions have been computed. The first is given by:

\[
B_T^p(f, F, L) = \sup \left\{ \int_X \left( \max(M_T \phi, L) \right)^p \, d\mu : \phi \geq 0, \int_X \phi \, d\mu = f, \int_X \phi^p \, d\mu = F \right\},
\]

(1.7)

where \(L \geq f\) and \(0 < f^p \leq F\).

The evaluation of (1.7), as is given in [4], uses the result of (1.6) and several complicated calculus arguments. At last the following function has also been evaluated in [4]. This is

\[
B_T^p(f, F, k) = \sup \left\{ \int_K (M_T \phi)^p \, d\mu : \phi \geq 0, \int_X \phi \, d\mu, \int_X \phi^p \, d\mu = F, \quad K : \mu - \text{measurable subset of } X \text{ with } \mu(K) = k \right\},
\]

(1.8)

for every \((f, F, k)\) that satisfy \(0 < f^p \leq F\) and \(0 < k \leq 1\).

Now, an alternative evaluation of (1.7) has been given in [5], where the authors avoided all the calculus arguments that appear in [4]. In this paper we
find an expression for (1.8), by using techniques that appear in [5], and by using new inequalities for the operator $M_T$. This expression already appears in [4], and helps us, as can be seen there, in the determination of the quantity (1.8). Our intension in this paper is to reach to this expression by an alternative way, that is by using the notion of the nonincreasing rearrangement of a measurable function defined on a probability nonatomic space. More precisely, we will prove the following

**Theorem A.** The following is true

$$B^*_T(f, F, k) = \sup \left\{ \left( F - \frac{(f - B)^p}{(1 - k)^p - 1} \right) \omega_p \left( \frac{B^p}{(1 - k)^p - 1} \right)^p : \right\}$$

for all $B \in [0, f]$ such that $h_k(B) \leq F$, where $h_k(B)$ is defined by, (1.9)

$$h_k(B) = \frac{(f - B)^p}{(1 - k)^p - 1} + \frac{B^p}{k^p - 1}.$$  (1.10)

## 2 Preliminaries

Let $(X, \mu)$ be a non-atomic probability space (i.e. $\mu(X) = 1$). We give the following.

**Definition 2.1.** A set $T$ of measurable subsets of $X$ will be called a tree if the following conditions are satisfied:

(i) $X \in T$ and for every $I \in T$ we have $\mu(I) > 0$.

(ii) For every $I \in T$, there corresponds a finite or countable subset $C(I) \subseteq T$ containing at least two elements such that

(a) the elements of $C(I)$ are disjoint subsets of $I$

(b) $I = \bigcup C(I)$

(iii) $T = \bigcup_{m \geq 0} T_{(m)}$, where $T_{(0)} = \{X\}$ and $T_{(m+1)} = \bigcup_{I \in T_{(m)}} C(I)$.

(iv) We have $\lim_{m \to \infty} \left[ \sup_{I \in T_{(m)}} \mu(I) \right] = 0$.

Then we define the dyadic maximal operator corresponding to $T$ by (1.5).

We now give the following which appears in [4]

**Lemma 2.1.** Let $p > 1$ be fixed. Then the function $\omega_p : [0, 1) \to \left[ \frac{p}{p - 1}, 1 \right]$ is strictly decreasing, and if we define $U$ on $(0, 1]$ by $U(x) = \frac{\omega_p(x)^p}{x}$, we have that $U$ is also strictly decreasing.
Lemma 2.2. For every $I \in \mathcal{T}$ and every $\alpha$ such that $0 < \alpha < 1$ there exists a subfamily $\mathcal{F}(I) \subseteq \mathcal{T}$ consisting of disjoint subsets of $I$ such that

$$\mu \left( \bigcup_{J \in \mathcal{F}(I)} J \right) = \sum_{J \in \mathcal{F}(I)} \mu(J) = (1 - \alpha) \mu(I).$$

We also present the following well known which can be found in [3].

Definition 2.2. Let $\phi : (X, \mu) \to \mathbb{R}^+$. Then $\phi^* : (0, 1] \to \mathbb{R}^+$ is defined as the unique non-increasing, left continuous and equimeasurable to $\phi$ function on $(0, 1]$.

There are several formulas that express $\phi^*$, by using $\phi$. Some of them, which can be found on [3], are as follows:

$$\phi^*(t) = \inf \{ y > 0 : \mu\{ x \in X : \phi(x) > y \} < t \}.$$

for every $t \in (0, 1]$. An equivalent formulation of the non-increasing rearrangement can be given as follows

$$\phi^*(t) = \sup_{\mu(\cdot) \geq t} \inf_{x \in X} |\phi(x)|,$$

for any $t \in (0, 1]$.

In [3] one can see the following

Theorem 2.1. Let $g : (0, 1] \to \mathbb{R}^+$ be non-increasing and $G_1, G_2$ be non-decreasing non-negative functions defined on $[0, +\infty)$. Then the following is true:

$$\sup \left\{ \int_X G_1(M_T \phi) G_2(\phi) \, d\mu : \phi^* = g \right\} = \int_0^1 G_1 \left( \frac{1}{t} \int_0^t g \right) G_2(g(t)) \, dt.$$

An immediate consequence of Theorem 2.1 is the following

Corollary 2.1. For any $\phi : (X, \mu) \to \mathbb{R}^+$ integrable, and every $G : [0, +\infty) \to [0, +\infty)$ non-decreasing the following inequality is true:

$$\int_X G(M_T \phi) \, d\mu \leq \int_0^1 G \left( \frac{1}{t} \int_0^t \phi^*(u) \, du \right) \, dt.$$

We will also need the following, which can be seen in [3].

Lemma 2.3. Let $g_1, g_2 : [0, 1] \to \mathbb{R}^+$ be non-increasing functions, such that

$$\int_0^1 G(g_1(t)) \, dt \leq \int_0^1 G(g_2(t)) \, dt,$$

for every $G : [0, +\infty) \to [0, +\infty)$ non-decreasing. Then the inequality

$$g_1(t) \leq g_2(t),$$

holds almost everywhere on $(0, 1]$. 

4
3 Proof of Theorem A

**Lemma 3.1.** For any $\phi : (X, \mu) \to \mathbb{R}^+$ integrable, the following inequality is true

$$(\mathcal{M}_T \phi)^*(t) \leq \frac{1}{t} \int_0^t \phi^*(u) \, du, \text{ for every } t \in (0, 1].$$

**Proof.** By Corollary 2.1 we have that

$$\int_0^1 G(\mathcal{M}_T \phi) \, d\mu \leq \int_0^1 G\left(\frac{1}{t} \int_0^t \phi^*(u) \, du\right) \, dt.$$ (3.1)

But since $G$ is non-decreasing we have that

$$\left[G(\mathcal{M}_T \phi)^*(t)\right] = G\left(\left[\mathcal{M}_T \phi\right]^*(t)\right), \text{ for almost every } t \in (0, 1].$$

Thus

$$\int_0^1 G\left(\mathcal{M}_T \phi\right) \, d\mu \leq \int_0^1 G\left(\frac{1}{t} \int_0^t \phi^*(u) \, du\right) \, dt,$$ (3.2)

almost everywhere on $(0, 1]$. Since now $\phi^*$ and $(\mathcal{M}_T \phi)^*$ are left continuous, we conclude that (3.2) should hold everywhere on $(0, 1]$, and in this way we derive the proof our Corollary.

There is also a second, simpler proof of Lemma 3.1 which we present right below

2nd proof of Lemma 3.1

Suppose that we are given $\phi : (X, \mu) \to \mathbb{R}^+$ integrable and $t \in (0, 1]$ fixed. We set $A = \frac{1}{t} \int_0^t \phi^*(u) \, du$. Then obviously $A \geq \int_0^t \phi^*(u) \, du = f$, by the fact that $\phi^*$ is non-increasing on $(0, 1]$. We consider the set $E = \{\mathcal{M}_T \phi > A\} \subseteq X$.

Then by the weak type inequality (1.2) for $\mathcal{M}_T$, we have that

$$\mu(E) < \frac{1}{A} \int_E |\phi| \, d\mu \implies$$

$$A = \frac{1}{t} \int_0^t \phi^*(u) \, du < \frac{1}{\mu(E)} \int_E \phi \, d\mu \leq \frac{1}{\mu(E)} \int_0^{\mu(E)} \phi^*(u) \, du,$$ (3.3)

where the last inequality in (3.3) is due to the definition of $\phi^*$. Since $\phi^*$ is non-increasing we must have from (3.3), that $\mu(E) < t$. But $\mu(E) = |\{\mathcal{M}_T \phi > A\}|$ since $\mathcal{M}_T \phi$ and $(\mathcal{M}_T \phi)^*$ are equimeasurable. But since $(\mathcal{M}_T \phi)^*$ is non-increasing and because of the fact that $\mu(E) < t$ we conclude that $\{\mathcal{M}_T \phi > A\} = (0, \gamma)$ for some $\gamma < t$. Thus $t \notin \{\mathcal{M}_T \phi > A\} \implies (\mathcal{M}_T \phi)^*(t) \leq A = \frac{1}{t} \int_0^t \phi^*(u) \, du$, which is the desired result. □

We are now in position to state and prove
Lemma 3.2. Let \( \phi : (X, \mu) \to \mathbb{R}^+ \) such that \( \int_X \phi \, d\mu = f \) and \( \int_X \phi^p \, d\mu = F \) where \( 0 < F^p \leq F \). Suppose also that we are given a measurable subset \( K \) of \( X \) such that \( \mu(K) = k \), where \( k \) is fixed such that \( k \in (0, 1] \). Then the following inequality is true:

\[
\int_K (\mathcal{M}_T \phi)^p \, d\mu \leq \int_0^k [\phi^*(u)]^p \, du \cdot \omega_p \left( \frac{\left( \int_0^k \phi^*(u) \, du \right)^p}{k^{p-1} \int_0^k [\phi^*(u)]^p \, du} \right)^p.
\]

Proof. We obviously have that

\[
\int_K (\mathcal{M}_T \phi)^p \, d\mu \leq \int_0^k [(\mathcal{M}_T \phi)^*]^p(t) \, dt.
\]

We evaluate the right-hand side of (3.4). We have:

\[
\int_0^k [(\mathcal{M}_T \phi)^*]^p(t) \, dt \leq \int_0^k \left( \frac{1}{t} \int_0^t \phi^*(u) \, du \right)^p \, dt,
\]

by using Corollary 3.2. On the other hand we have by using Fubini’s theorem that

\[
\int_0^k \left( \frac{1}{t} \int_0^t \phi^*(u) \, du \right)^p \, dt = \int_{\lambda=0}^{+\infty} p \lambda^{p-1} \left\{ \left\{ \lambda \in (0, k] : \frac{1}{t} \int_0^t \phi^* \, du \geq \lambda \right\} \right\} \, d\lambda = \int_{\lambda=0}^{f_k} + \int_{\lambda=f_k}^{+\infty} p \lambda^{p-1} \left\{ \left\{ \lambda \in (0, k] : \frac{1}{t} \int_0^t \phi^* \, du \geq \lambda \right\} \right\} \, d\lambda,
\]

where \( f_k \) is defined by \( f_k = \frac{1}{k} \int_0^k \phi^*(u) \, du > f = \int_0^1 \phi^*(u) \, du \).

The first integral in (3.6) is obviously equal to \( k(f_k)^p = \frac{1}{k^{p-1}} (\int_0^k \phi^*)^p \). We suppose now that \( \lambda > f_k \) is fixed. Then there exists \( \alpha(\lambda) \in (0, k] \) such that \( \frac{1}{\alpha(\lambda)} \int_0^{\alpha(\lambda)} \phi^*(u) \, du = \lambda \) and as a consequence \( \left\{ \lambda \in (0, k] : \frac{1}{t} \int_0^t \phi^*(u) \, du \geq \lambda \right\} = (0, \alpha(\lambda)] \), thus \( \int \left\{ \lambda \in (0, k] : \frac{1}{t} \int_0^t \phi^*(u) \, du \geq \lambda \right\} \, d\lambda = \alpha(\lambda) \). So the second integral in (3.6) equals

\[
\int_{\lambda=f_k}^{+\infty} p \lambda^{p-1} \alpha(\lambda) \, d\lambda = \int_{\lambda=f_k}^{+\infty} p \lambda^{p-1} \left( \int_0^{\alpha(\lambda)} \phi^*(u) \, du \right) \, d\lambda,
\]

by the definition of \( \alpha(\lambda) \). The last new integral, equals

\[
\int_{\lambda=f_k}^{+\infty} p \lambda^{p-2} \left( \left\{ t \in (0, k] : \int_0^t \phi^*(u) \, du \geq \lambda \right\} \phi^*(u) \, du \right) \, d\lambda = \int_{t=0}^k \frac{p}{p-1} \phi^*(t) \left( \int_0^k \phi^* \, du \right)^p \, dt,
\]

by a use of Fubini’s theorem. As a consequence (3.6) gives

\[
\int_0^k \left( \frac{1}{t} \int_0^t \phi^*(u) \, du \right)^p \, dt = -\frac{1}{p-1} k^{p-1} \left( \int_0^k \phi^* \right)^p + \frac{p}{p-1} \int_0^k \phi^*(t) \left( \frac{1}{t} \int_0^t \phi^* \right)^{p-1} \, dt.
\]
Then by Hölder’s inequality applied in the second integral we have that:

\[
\int_0^k \left( \frac{1}{t} \int_0^t \phi^* \right)^p \, dt \leq -\frac{1}{p-1} \frac{1}{k^{p-1}} \left( \int_0^k \phi^* \right)^p + \frac{p}{p-1} \left( \int_0^k \left( \frac{1}{t} \int_0^t \phi^* \right)^p \, dt \right)^{\frac{p-1}{p}}. \tag{3.9}
\]

We set now

\[
J(k) = \int_0^k \left( \frac{1}{t} \int_0^t \phi^* \right)^p \, dt, \quad A(k) = \int_0^k [\phi^*]^p \quad \text{and} \quad B(k) = \int_0^k \phi^*.
\]

Then we conclude by (3.9) that

\[
J(k) \leq -\frac{1}{p-1} \frac{1}{k^{p-1}} |B(k)|^p + \frac{p}{p-1} [A(k)]^{\frac{p}{p-1}} [J(k)]^{\frac{p-1}{p}} \iff \frac{J(k)}{A(k)} \leq -\frac{1}{p-1} \left( \frac{|B(k)|^p}{k^{p-1} [A(k)]^{\frac{p}{p-1}}} \right) + \frac{p}{p-1} \left( \frac{J(k)}{A(k)} \right)^{\frac{p-1}{p}}. \tag{3.10}
\]

We set now in (3.10) \( \Lambda(k) = \left[ \frac{J(k)}{A(k)} \right]^{\frac{p}{p-1}} \), thus we get

\[
\Lambda(k)^p \leq -\frac{1}{p-1} \left( \frac{|B(k)|^p}{k^{p-1} [A(k)]^{\frac{p}{p-1}}} \right) + \frac{p}{p-1} \Lambda(k)^{p-1} \iff p\Lambda(k)^{p-1} - (p-1)|\Lambda(k)|^p \geq \frac{\left( \int_0^k \phi^* \right)^p}{k^{p-1} \int_0^k [\phi^*]^p} \iff H_p(\Lambda(k)) \geq \left( \int_0^k \phi^* \right)^p \quad \text{and} \quad \Lambda(k) \leq \omega_p \left( \frac{\left( \int_0^k \phi^* \right)^p}{k^{p-1} \int_0^k [\phi^*]^p} \right) \Rightarrow J(k) \leq \int_0^k [\phi^*]^p \omega_p \left( \frac{\left( \int_0^k \phi^* \right)^p}{k^{p-1} \int_0^k [\phi^*]^p} \right)^{\frac{1}{p}}. \tag{3.11}
\]

At last by (3.4), (3.5) and (3.11) we derive the proof of our Lemma.

We fix now \( k \in (0, 1] \), and \( K \subseteq X \) measurable such that \( \mu(K) = k \). Then if \( A = A(k), \ B = B(k) \) as in the proof of the previous Lemma we conclude that

\[
\int_K (\mathcal{M}_T \phi)^p \, d\mu \leq A \omega_p \left( \frac{B^p}{k^{p-1} A} \right). \tag{3.12}
\]

Note now that the \( A, B \) must satisfy the following conditions

i) \( B^p \leq k^{p-1} A \), because of Hölder’s inequality for \( \phi^* \) on the interval \( (0, k] \)

ii) \( A \leq F, \ B \leq f \)
iii) \((f - B)^p \leq (1 - k)^{p-1}(F - A)\), because of Hölder’s inequality for \(\phi^*\) on the
interval \([k, 1]\).

From all the above we conclude the following

**Corollary 3.1.**

\[ B^{T}_p(f, F, k) \leq \sup \left\{ A \omega_p \left( \frac{B^p}{k^{p-1}A} \right)^p : A, B \text{ satisfy } i), ii) \text{ and } iii) \text{ above} \right\}. \]

*Proof.* Immediate.

For the next Lemma we fix \(0 < k < 1\) and we consider the function \(h_k(B)\),
defined as in the statement of Theorem A, for \(0 \leq B \leq f\). Now by Lemma 2.1
and the condition iii) for \(A, B\) we immediately conclude the following:

**Corollary 3.2.**

\[ B^{T}_p(f, F, k) \leq \sup \left\{ \left( \frac{F - (f - B)^p}{(1 - k)^{p-1}} \right) \omega_p \left( \frac{B^p}{k^{p-1}(F - (f - B)^p)} \right)^p : \right\}, \]

for all \(B \in [0, f]\) such that \(h_k(B) \leq F\), where \(h_k(B)\) is defined above.

(3.13)

We now prove that we have equality in Corollary 3.2. Fix \(k \in (0, 1]\) and \(B\)
satisfy the conditions stated in Corollary 3.2. We set \(A = F - \frac{(f - B)^p}{(1 - k)^{p-1}}\) and we
fix a \(\delta \in (0, 1)\).

We use now Lemma 2.2 to pick a family \(\{I_1, I_2, \ldots\}\) of pairwise disjoint elements
of \(T\) such that \(\sum_j \mu(I_j) = k\) and since \(\frac{B^p}{k^{p-1}} \leq A\), using the value of (1.6), for
each \(j\) we choose a non-negative \(\phi_j \in L^p \left( I_j, \frac{1}{\mu(I_j)}\mu \right)\) such that

\[ \int_{I_j} \phi^p d\mu = \frac{A}{k} \mu(I_j), \quad \int_{I_j} \phi d\mu = \frac{B}{k} \mu(I_j), \quad (3.14) \]

and

\[ \int_{I_j} (M_{T(I_j)}(\phi_j))^p d\mu \geq \delta \frac{A}{k} \omega_p \left( \frac{B^p}{k^{p-1}A} \right)^p \mu(I_j), \quad (3.15) \]

where \(T(I_j)\) is the subtree of \(T\), defined by

\[ T(I_j) = \{ I \in T : I \subseteq I_j \}. \]

Next we choose \(\psi \in L^p(X \setminus K, \mu)\) such that \(\int_{X \setminus K} \psi^p d\mu = F - A > 0\) and
\(\int_{X \setminus K} \psi d\mu = f - B > 0\), which in view of the value of \(A\) \((= F - \frac{(f - B)^p}{(1 - k)^{p-1}})\)
must be in fact constant and equal to \(\frac{F - B}{1 - k} = \left( \frac{(F - A)}{(1 - k)^{p-1}} \right)^{\frac{1}{p}}\). Here \(K\) stands for
\(K = \cup I_j \subseteq X\). Then we define \(\phi = \psi \chi_{X \setminus K} + \sum_j \phi_j \chi_{I_j}\), and we obviously have

\[ \int_X \phi^p d\mu = F \quad \text{and} \quad \int_X \phi d\mu = f. \quad (3.16) \]
Additionally we must have by (3.15) that

\[
\int_K (\mathcal{M}_T \phi)^p \, d\mu \geq \delta A \omega_p \left( \frac{B^p}{k^{p-1} A} \right)^p = \\
\delta \left( F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right) \omega_p \left( \frac{B^p}{k^{p-1} \left( F - \frac{(f - B)^p}{(1 - k)^{p-1}} \right)} \right)^p.
\]  (3.17)

Letting \( \delta \to 1^- \) we obtain the equality we need in Corollary 3.2 thus proving Theorem A.

References

[1] D. L. Burkholder, \textit{Martingales and Fourier Analysis in Banach spaces}, C.I.M.E. Lectures, Varenna, Como, Italy, 1985, Lecture Notes Math. 1206 (1986) 81–108.

[2] D. L. Burkholder, \textit{Explorations in martingale theory and its applications}, École d’Été de Probabilités de Saint-Flour XIX–1989, Lecture Notes Math. 1464 (1991) 1–66.

[3] G. H. Hardy, J. E. Littlewood and G. Polya, \textit{Inequalities}, Cambridge University Press, Cambridge (1934).

[4] A. D. Melas, \textit{The Bellman functions of dyadic-like maximal operators and related inequalities}, Adv. in Math. 192 (2005) 310–340.

[5] E. N. Nikolidakis, A. D. Melas, \textit{A sharp integral rearrangement inequality for the dyadic maximal operator and applications}, Appl. and Comput. Harm. Anal., Vol 38, Issue 2, (March 2015), 242–261.

[6] G. Wang, \textit{Sharp maximal inequalities for conditionally symmetric martingales and Brownian motion}, Proc. Amer. Math. Soc. 112 (1991) 579–586.