THE KOHN-HÖRMANDER-MORREY FORMULA TWISTED BY A PSEUDODIFFERENTIAL OPERATOR

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Abstract. We establish a general, weighted Kohn-Hörmander-Morrey formula twisted by a pseudodifferential operator. As an application, we exhibit a new class of domains for which the \( \bar{\partial} \)-Neumann problem is locally hypoelliptic.

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References

1. Introduction

The Kohn-Hörmander-Morrey formula, together with the density of \( C^\infty \)- into \( L^2 \)-forms in the graph norm yields the closure of the range of \( \bar{\partial} \) and \( \Box \) \( [13], [6] \). By adding a weight, it also gives the global boundary regularity of the (non-canonical) solution of \( \bar{\partial} \). When it comes with a “gain” it gives in fact the regularity of the canonical solution of the \( \bar{\partial} \)-Neumann problem \( [8] \). For a gain consisting of the multiplication by a big constant, as in the compactness estimates, the regularity is global. For a gain such a subelliptic \( [13] \) or superlogarithmic \( [12] \) the regularity is stronger, that is, local: \( u \) is regular precisely in the portion of \( b\Omega \) where \( \Box u \) is. When these latter gains do not occur, but the points of failure are confined to a real curve transversal to the CR directions, local regularity still holds \( [11] \) and \( [1] \). This is an exquisitely geometric conclusion. In \( [1] \) it is shown that good estimates in full are not needed and what really counts is that for a system of cut-off \( \{ \eta \} \), the gradient \( \partial \eta \) and the Levi form \( \partial \bar{\partial} \eta \) are good multipliers in the sense of Kohn \( [9] \). If these are subelliptic multipliers, then \( \Box \) is hypoelliptic. The proof consists in modifying the Kohn-Hörmander-Morrey formula by a weight \( \phi = t|z|^2 - \log \eta^2 \); the exploitation of \( t|z|^2 \) is usual in controlling the commutators \( [\bar{\partial}, \Lambda^s] \) and \( [\bar{\partial}^*, \Lambda^s] \), but the one of \( - \log \eta^2 \) is new and is designed to avoid the commutators \( [\bar{\partial}, \eta] \) and \( [\bar{\partial}^*, \eta] \). Inserting the cut-off in the weight is not the only way to proceed and this could appear as well as a twisting term like
in the formula of [14]; what is crucial is not to apply the basic estimate to $\eta u$, for a testing form $u$, but the formula twisted by $\eta$ to the plain $u$. If the multipliers are weaker than subelliptic one needs a stronger modification of the Kohn-Hörmander-Morrey formula in which not only the cut-off but also a general pseudodifferential operator appears as already commutated with $\bar{\partial}$ and $\bar{\partial}^*$. The motivation of the present paper is to present, for the $\bar{\partial}$-Neumann problem, a general Kohn-Hörmander-Morrey formula with weight in which not only the cut-off but also a general pseudodifferential operator appears as a twisting term. This has already been done in [2] for the tangential system. The present paper serves therefore as the $\bar{\partial}$-Neumann version of the tangential twisted estimate established in [2]. It proves useful for the local regularity of $\Box$ when the multipliers are not subelliptic but only superlogarithmic (in the slightly stronger sense of (3.5) below). This requires to twist the formula not only by $\eta$ but also by $R^s$ where $R^s$ is the modification of the standard elliptic operator of order $s$ introduced by Kohn in [12]. As an application we get a new class of domains of infinite type for which $\Box$ is locally hypoelliptic.

2. The twisted basic estimate for the $\bar{\partial}$-Neumann problem

Let $\Omega$ be a domain of $\mathbb{C}^n$ with $C^\infty$-boundary $b\Omega$, $z_o$ a point of $b\Omega$, $U$ an open neighborhood of $z_o$ and $\Psi$ a tangential pseudodifferential operator with real symbol $S(\Psi)$. This is defined by introducing a local straightening $b\Omega \simeq \mathbb{R}^{2n-1} \times \{0\}$ and $\Omega \simeq \mathbb{R}^{2n-1} \times \mathbb{R}_r$ for a defining function $r < 0$ of $\Omega$ with $\partial r \neq 0$, taking coordinates $(x, r)$ or $(y, r)$ in $U$, dual coordinates $\xi$ of $x$ and setting

$$
\Psi(u) = \int e^{ix\xi}S(\Psi)(x, \xi) \left( \int e^{-iy\xi}u(y)dy \right) d\xi.
$$

One of the most common choice of $\Psi$ is the elliptic standard operator of degree $s$ with symbol

$$
S(\Lambda^s) = \left( 1 + |\xi|^2 \right)^{\frac{s}{2}}.
$$

It is also of great interest its local modification by means of a cut-off $\sigma \in C_c^\infty(U)$, which is 1 in a neighborhood of $z_o$,

$$
S(R^s) = \left( 1 + |\xi|^2 \right)^{s\sigma(x)}.
$$

The operators we have in mind are $\Psi = \eta \Lambda^s \eta_o$ for a pair of cut-off $\eta_o \prec \eta$ in $C_c^\infty(U)$ or $\Psi = \eta R^s \eta$ for $\eta_o \prec \sigma \prec \eta$. However, our formula applies to a general symbol and what we ask is that this is already subject to the multiplication by a cut-off, that is, $\Psi = \eta \Psi$ for some $\eta \in C_c^\infty(U)$.

To give our formula a bigger flexibility, we also consider spaces $L^2_\phi$ weighted by a weight $e^{-\phi}$ with norm defined by $||u||_\phi^2 = \int e^{-\phi}|u|^2dV$. In particular, the action of $\Psi^2 = \eta^2$ can
be achieved by means of the weight $\phi = -\log r^2$ and it is under this appearance that the basic formula was introduced in \cite{H}. We use the notation

$$Q_\phi^\circ(u, u) = ||\Psi \partial u||_\phi^2 + ||\Psi \partial^* u||_\phi^2, \quad u \in D_{\partial} \cap D_{\partial^*}.$$  

We choose a smooth orthonormal basis of $(1, 0)$ forms $\omega_1, \ldots, \omega_{n-1}, \omega_n = \partial r$ and the dual basis of vector fields $\partial_{\omega_1}, \ldots, \partial_{\omega_n}$; to simplify notation, we also write $\partial_j$ instead of $\partial_{\omega_j}$. We use the notation $\phi_i$ for $\partial_i \phi$, $\phi_i$ for $\bar{\partial}_i \phi$ and $(\phi_{ij})$ for the matrix of the Levi form $\partial \partial \phi$. Note that $\phi_{ij}$ differs from $\bar{\partial}_j (\phi)$ because of the presence of the derivatives of the coefficients of the forms $\partial_j$. We define various constants $c_{ij} = c_{ij}^n$ and $c_{ij}^0$, $i, j = 1, \ldots, n - 1$ by means of the identity

$$[\partial_i, \bar{\partial}_j] = c_{ij}(\partial_n - \bar{\partial}_n) + \sum_{h=1}^{n-1} c_{ij}^h \partial_h - \sum_{h=1}^{n-1} e_{ij}^h \bar{\partial}_h. \quad (2.2)$$

Thus $(c_{ij})$ is the matrix of the Levi form $\partial \partial r|_{T^c b\Omega}$ in the basis $\{\omega_j\}$. (Here we have used the notation $T^c b\Omega = T b\Omega \cap i T b\Omega$.) We denote by $\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}_{\partial}$, resp. $\text{Op}^0$, an operator of order $\text{ord}(\Psi) - \frac{1}{2}$, resp. 0. We assume that $\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}_{\partial}$ has symbol with support in a neighborhood of that of $\Psi$ and that $\text{Op}^0$ only depends on the $C^2$-norm of $b\Omega$ and not on $\phi$ nor $\Psi$. Here is the substance of the paper.

**Theorem 2.1.** For any $u \in D^k_{\partial} \cap D^k_{\partial^*} \cap C^\infty_c(U \cap \Omega)$, $k \in [1, n - 1]$, we have

$$\int_{\Omega} e^{-\phi} (c_{ij})(\Psi u, \overline{\Psi u}) dV + \int_{\Omega} e^{-\phi} \partial \bar{\partial} \phi (\Psi u, \overline{\Psi u}) dV + \sum_{j=1}^{n-1} ||\tilde{L}_j u||^2_\phi \leq Q_\phi^\circ(u, \bar{u})$$
$$+ \int_{\Omega} e^{-\phi} [\partial, [\bar{\partial}, \Psi^2]](u, \bar{u}) dV + ||[\partial, \phi] \nabla \Psi u||^2_\phi + ||[\partial, \Psi] \nabla u||^2_\phi$$
$$+ \sum_{h} \int_{\Omega} (c_{ij}) (\partial h, \Psi)(u, \bar{u}) dV + Q_\phi^\circ_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}}(u, \bar{u}) + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} u||^2_\phi + ||\Psi u||^2_\phi. \quad (2.3)$$

**Remark 2.2.** In our application in Section 3 below, $[\partial, [\bar{\partial}, \Psi^2]]$ and $[\partial, \Psi] \nabla$ have good estimates. Also, $\phi$ has “selfbounded gradient”, that is

$$[\partial, \phi] \nabla < \partial \bar{\partial} \phi,$$

where inequality is meant in the operator sense. In particular, the term in the right of (2.3) which involves $[\partial, \phi] \nabla$ is absorbed in the left.

**Remark 2.3.** Formula (2.3) is also true for complex $\Psi$. In this case, one replaces $[\partial, [\bar{\partial}, \Psi^2]]$ by $[\partial, [\bar{\partial}, ||\Psi||^2]]$ and add an additional error term $[\partial, \Psi] \nabla$.

**Proof.** We start from

$$e^{\phi} \Psi^{-2} [\partial_i, e^{-\phi} \Psi^2] = -\phi_i + 2\frac{[\bar{\partial}_i, \Psi]}{\Psi} + \frac{\text{Op}^{\text{ord}(\Psi)-1}_2}{\Psi^2}, \quad (2.4)$$
whose sense is fully clear when both sides are multiplied by $\Psi^2$. In other terms, we have

$$\bar{\partial}_{e^{-\phi} \Psi^2} = \bar{\partial}^* + \partial \phi \perp - 2 \frac{[\partial_i, \Psi]}{\Psi} \perp + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2}. \quad (2.5)$$

We are thus lead to define the transposed operator to $\bar{\partial}$ by

$$\delta_i := \partial_i - \phi_i + 2 \frac{[\partial_i, \Psi]}{\Psi} \perp + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0. \quad (2.6)$$

Using the trivial identity $\partial \bar{\partial} = -\bar{\partial} \partial$, we have

$$[\delta_i, \bar{\partial}_j] = [\partial_i, \bar{\partial}_j] + \phi_{ij} - \sum_{h=1}^n c_{ij}^h \phi_h - 2 \frac{[\partial_i, [\bar{\partial}_j, \Psi]]}{\Psi} + 2 \frac{[\partial_i, \Psi] \otimes [\bar{\partial}_j, \Psi]}{\Psi^2} + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0$$

$$= c_{ij} (\delta_n - \bar{\partial}_n) - \sum_{h=1}^{n-1} c_{ij}^h \delta_h + \sum_{h=1}^{n-1} c_{ij}^h \phi_h + \phi_{ij} - 2 \frac{c_{ij}^h [\partial_h, \Psi]}{\Psi} \perp$$

$$- 2 \frac{[\partial_i, [\bar{\partial}_j, \Psi]]}{\Psi} + 2 \frac{[\partial_i, \Psi] \otimes [\bar{\partial}_j, \Psi]}{\Psi^2} + \frac{\text{Op}^{2\text{ord}(\Psi)-1}}{\Psi^2} + \text{Op}^0. \quad (2.7)$$

We also have to observe that (cf. [2])

$$||\Psi \bar{\partial}^{(\star)} u||^2_{\phi} = \int_{\Omega} e^{-\phi} \Psi^2 |\bar{\partial}^{(\star)} u|^2 dV + Q_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}} (u, \bar{u}) + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} u||^2_{\phi} + ||\text{Op}^0 \Psi u||^2_{\phi}. \quad (2.8)$$

This yields the “basic estimate with weight $e^{-\phi} \Psi^2$”

$$\int_{\Omega} e^{-\phi} (c_{ij})(\Psi u, \overline{\Psi u}) dV + \int_{\Omega} [\partial, [\bar{\partial}, e^{-\phi} \Psi^2]](u, \bar{u}) dV$$

$$- ||[\partial, \phi] \perp \Psi u||^2_{\phi} - ||[\partial, \Psi] \perp u||^2_{\phi} + \sum_j ||\Psi \bar{\partial}_j u||^2_{\phi}$$

$$\leq ||\Psi \bar{\partial} u||^2_{\phi} + ||\Psi \bar{\partial}_{e^{-\phi} \Psi^2} u||^2_{\phi} + sc||\Psi \nabla u||^2_{\phi} + \sum_h \int_{\Omega} \left( c_{ij}^h \right) \left( [\partial_h, \Psi](u, \bar{u}) \right) dV$$

$$+ Q^{\phi}_{\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}}} (u, \bar{u}) + ||\text{Op}^{\text{ord}(\Psi)-\frac{1}{2}} u||^2_{\phi} + ||\Psi u||^2_{\phi}. \quad (2.8)$$
In (2.8) we absorb the term which comes with sc and rewrite \([\partial, [\bar{\partial}, e^{-\phi}\Psi^2]]\) by the aid of \((2.4)\); what we get is

\[
\int_{b\Omega} e^{-\phi}\Psi^2 c_{ij}(u, \bar{u})dV + \int_{\Omega} e^{-\phi}\Psi^2 \phi_{ij}(u, \bar{u})dV - \|[[\partial, \phi] \Psi u]\|_\phi^2 \\
+ \int_{\Omega} e^{-\phi}\Psi^2 [\partial, [\bar{\partial}, \Psi]](u, \bar{u})dV - \|[[\partial, \Psi] \Psi u]\|_\phi^2 + \|\Psi \nabla u\|_\phi^2 \\
\lesssim \|\Psi \partial u\|_\phi^2 + \|\Psi \bar{\partial} e^{-\phi}\Psi^2 u\|_\phi^2 + Q^\phi_{\text{Op}_{\text{ord}(\Psi)} - \frac{1}{2}}(u, \bar{u}) + \left| \sum_h \int_{\Omega} (c_{ij}^h)([\partial_h, \Psi](u, \bar{u}))dV \right| \\
+ \|\text{Op}_{\text{ord}(\Psi)} - \frac{1}{2} u\|_\phi^2 + \|\Psi u\|_\phi^2.
\]  

(2.9)

To carry out our proof we need to replace \(\bar{\partial} e^{-\phi}\Psi^2\) by \(\bar{\partial}\). We have from (2.5)

\[
\|\Psi \bar{\partial} e^{-\phi}\Psi^2 u\|_\phi^2 \lesssim \|\Psi \bar{\partial} u\|_\phi^2 + \|\Psi \partial \phi u\|_\phi^2 + \|[\bar{\partial}, \Psi] \Psi u\|_\phi^2 \\
+ \left| \sum_h \text{Re}(\Psi \bar{\partial} u, \bar{\partial} \phi \Psi u) \right| + \left| \text{Re}(\Psi \bar{\partial} u, [\partial, \Psi] \Psi u) \right| + \left| \text{Re}(\Psi \partial \phi u, [\bar{\partial}, \Psi] \Psi u) \right|.
\]

(2.10)

We next estimate by Cauchy-Shwarz inequality

\[
\# \lesssim \|\Psi \bar{\partial} u\|_\phi^2 + \|\Psi \partial \phi u\|_\phi^2 + \|[\bar{\partial}, \Psi] \Psi u\|_\phi^2.
\]

(2.11)

We move the third, forth and fifth terms from the left to the right of (2.9), use (2.10) and (2.11) and end up with (2.3).

\[
\square
\]

3. \textit{F TYPE, TWISTED f ESTIMATE AND HYPOELLIPTICITY OF } \Box.

We start by recalling a result by [7]. In our presentation it contains a specification of the estimate by the Levi form which is important for further application. We consider a smoothly bounded pseudoconvex domain \(\Omega \subset \mathbb{C}^n\). For a form \(u \in D_{\bar{\partial}}\), let \(u = \sum_k \Gamma_k u\) be the decomposition into wavelets (cf. [12]), and \(Q(u, \bar{u}) = \|\partial u\|^2 + \|\bar{\partial} u\|^2\) the energy. We use the notation \((c_{ij})\) and \((\phi_{ij})\) for the Levi form of the boundary \(b\Omega\) and of a function \(\phi\) respectively. We introduce a real function \(F\) such that \(\frac{F(d)}{d} \searrow 0\) as \(d \searrow 0\) and set \(f(t) := (F^*(t^{-\frac{1}{2}}))^{-1}\) where \(F^*\) denotes the inverse.

**Theorem 3.1.** Assume that \(b\Omega\) has type \(F^2\) along a submanifold \(S \subset b\Omega\) of CR dimension 0 in the sense that \(\langle c_{ij} \rangle \sim \left( \frac{F(d_\Sigma)}{d_S} \right)^2 \text{Id}\) where \(d_S\) is the Euclidean distance to \(S\) and \(\text{Id}\) the identity of \(T^\Sigma b\Omega\). Then there is a uniformly bounded family of weights \(\{\phi^k\}\) which yield
The “f estimate”

\[
||f(\Lambda)u||^2_0 \lesssim \int_{\partial \Omega} (c_{ij})(u, \bar{u}) \, dV + \sum_k \int_{\Omega} (\phi^k_{ij})(\Gamma_k u, \bar{\Gamma_k u}) + \sum_{j=1}^n ||\bar{L}_j u||^2_0 + ||u||^2_0
\]

(3.1)

\[\lesssim Q^\Omega(u, \bar{u}) \text{ for any } u \in D^k_\partial \cap D^k_{\bar{\partial}}, \cap C^\infty_c(U \cap \bar{\Omega}), k \in [1, n-1].\]

The estimate \[||f(\Lambda)u||^2_0 \lesssim Q^\Omega(u, \bar{u})\] is stated in [7] as a combination of Theorems 1.4 and 2.1 therein. For the purpose of the present paper (Theorem 3.3 below), the two separate estimates which occur in (3.1), with the intermediate term which carries the Levi forms, are essential.

**Proof.** We give two parallel proofs inspired to [7], resp. [2], which use the families of weights

\[\psi^k := -\log\left(\frac{-r}{2-k} + 1\right) + \chi\left(\frac{d_s}{a_k}\right) \log\left(\frac{d_s^2}{a_k^2} + 1\right)\]

and

\[\phi^k := \chi\left(\frac{d_s}{a_k}\right) \log\left(\frac{d_s^2}{a_k^2} + 1\right)\]

Here \(r = 0\) is an equation for \(b\Omega\) with \(r < 0\) on \(\Omega\), \(a_k := F^s(2^{-k})\) and \(\chi\) is a cut-off such that \(\chi \equiv 1\) in \([0, 1]\) and \(\chi \equiv 0\) for \(s \geq 2\). We also use the notation \(S_{a_k}\) for the strip \(S_{a_k} := \{z \in \Omega : d_{b\Omega}(z) < a_k\}\). Following word by word the proof of [7], resp. [2], we conclude

\[||f(\Lambda)u||^2_0 \lesssim \sum_k \int_{\partial \Omega \setminus S_{a_k}} (c_{ij})(d_{\partial\Omega}^{-\frac{1}{2}} u, d_{\partial\Omega}^{-\frac{1}{2}} \bar{u}) \, dV + \sum_k \int_{\Omega} \phi^k_{ij}(\Gamma_k u, \bar{\Gamma_k u}) + ||u||^2_0\]

resp.

\[||f(\Lambda)u||^2_0 \lesssim \int_{\Omega} (c_{ij})\left((\Lambda^\frac{1}{2} u, \Lambda^\frac{1}{2} \bar{u}) \right) \, dV + \sum_k \int_{\Omega} \phi^k_{ij}(\Gamma_k u, \bar{\Gamma_k u}) + ||u||^2_0.\]

Finally, from

\[||d_{\partial\Omega}^{-\frac{1}{2}} u||_0 \sim ||u||^b_0 + \sum_{j=1}^n ||\bar{L}_j u||_0, \quad \text{resp. } ||\Lambda^\frac{1}{2} u||_0 \sim ||u||^b_0 + \sum_{j=1}^n ||\bar{L}_j u||_0,\]

(cf. [12] Section 8) we get the first estimate in (3.1). The second is a basic estimate weighted by the \(\phi^k\)'s in which the weights has been removed from the norms on account of their uniform boundedness.

\[\square\]

We modify the weights \(\phi^k\) to \(\phi^k + t|z|^2\) so that their Levi form releases an additional \(tId\) for \(t\) big. They are absolutely uniformly bounded with respect to \(k\) and to \(t\) provided that we correspondingly shrink the neighborhood \(U = U_t\). Possibly by raising to exponential, boundedness implies “selfboundedness of the gradient” when the weight is plurisubharmonic. In our case, in which to be positive is not \((\phi^k_{ij})\) itself but \(2^k(c_{ij}) + (\phi^k_{ij})\), we have,
for $|z|$ small

$$|\partial_b \phi|^2 = |\partial_b (\phi^k + t |z|^2)|^2$$

$$\lesssim |\partial_b \phi|^2 + t^2 |z|^2$$

(3.2)

Going back to (2.3) under this choice of $\phi$, we have that $\|\partial_b \phi \Psi u\|^2$ can be removed from the right side. We combine Theorem 2.3 with Theorem 3.1 formula (3.1), observe again that the weights $\phi^k$ can be removed from the norms by uniform boundedness, and get the proof of the following

**Theorem 3.2.** Let $b\Omega$ have type $F^2$ along $S$ of CR dimension 0. Then we have the $f$ estimate

$$||f(\Lambda)\Psi u||_0^2 \lesssim \int_{b\Omega} (c_{ij})(\Psi u, \overline{\Psi u}) dV + \sum_k \int_{\Omega} (\phi^k_{ij})(\Gamma_k \Psi u, \overline{\Gamma_k \Psi u}) dV + \sum_j ||L_j \Psi u||_0^2 + t||\Psi u||_0^2$$

$$\lesssim Q\Psi(u, \bar{u}) + ||[\partial, \Psi] \Psi u||_0^2 + \left| \int_{\Omega} [\partial, [\bar{\partial}, \Psi] \Psi u, \bar{\Psi} u] dV \right|$$

$$+ \sum_h \int_{\Omega} (c^h_{ij}) ([\partial_h, \Psi] \Psi u, \bar{\Psi} u) dV + Q^\phi_{Op^{ord}(\Psi)-\frac{1}{2}}(u, \bar{u}) + ||Op^{ord}(\Psi)-\frac{1}{2} u||^2_0 + ||\Psi u||^2_0.$$  (3.3)

We have as application a criterion of regularity for the Neumann operator $N$ in a new class of domains. Let $b\Omega$ be a “block decomposed”, rigid, boundary, that is, defined by $x_n = \sum_{j=1}^m h^I_j (z_I)$ where $z = (z_I, ..., z_{I_m}, z_n)$ is a decomposition of coordinates.

**Theorem 3.3.** Assume that

\begin{align*}
(a) \ h^I_j \text{ has infraexponential type along a totally real } S^I_j \setminus \{0\} \\
\text{where } S^I_j \text{ is totally real in } C^I_j,
\end{align*}

(3.4)

(b) $h^I_j$ are superlogarithmic multipliers at $z_I = 0$.

Then, we have local hypoellipticity of $\Box$ at $z_o = 0$.

In the same class of domains, it is proved in [2] the hypoellipticity of the Kohn-Laplacian $\Box_b$.

**Proof.** The proof is the same as in [2] Theorem 1.11. The argument is that, for a system $\{\eta\}$ of cut-off, and with the decomposition $\partial \eta = (\partial_\tau \eta, \partial_\nu \eta)$, the vectors $\partial_\tau \eta$ are superlogarithmic multipliers in the sense that for any $\epsilon$, suitable $c_\epsilon$ and for a bounded family of weights $\phi^k$, we have

$$||\log(\Lambda)\partial_\tau \eta \Psi u|| \leq \epsilon \left( \int_{b\Omega} (c_{ij}(u_\tau, \overline{u_\tau})dV + \sum_k \int_{\Omega} (\phi^k_{ij})(\Gamma_k u_\tau, \overline{\Gamma_k u_\tau})dV \right) + c_\epsilon ||u_\tau||^2_0.$$  (3.5)
This is an immediate consequence of the hypotheses (a) and (b) of Theorem 3.3 combined with (3.3) for \( f = \varepsilon^{-1} \log \) (with an error \( c_\varepsilon \)) and \( \Psi = \text{id} \). (Note that this notion of superlogarithmic multiplier is a little more restrictive than in the literature where the term between brackets in the right of (3.5) is replaced by \( Q \).) \( \partial_\nu \eta \) is 1 but it hits \( u_\nu \) which is 0 at \( b\Omega \) and therefore enjoys elliptic estimates. In the same way, \([\partial_\Sigma, [\partial_\tau, \eta]]\) are superlogarithmic multipliers, whereas \([\partial_\nu, [\partial_\tau, \eta]]\) and \([\partial_\nu, [\partial_\nu, \eta]]\) give \( \frac{1}{2} \)-subelliptic and elliptic estimates respectively. We also take \( \eta_0 < \sigma < \eta \) and recall the operator \( R^s \) whose symbol has been defined by (2.1); note that \( \partial_\tau \sigma \) is a superlogarithmic multiplier. We also have to notice that, \( b\Omega \) being rigid, then \( (c_{ij}^b) \lesssim (c_{ij}) \) are \( \frac{1}{2} \)-subelliptic matrix multipliers (cf. (2.2) which defines these matrices). We also have to remark that

\[
[\partial, \eta R^s \eta] = \eta \partial \sigma \log(\Lambda) R^s \eta + \text{Op}^{-\infty}.
\]  

(3.6)

After this preliminary, and under the choice \( \Psi = \eta R^s \eta \), we have readily the following chain of estimates

\[
t ||\Lambda^s \eta_\nu u||_0^2 \lesssim t ||\eta R^s \eta u||_0^2 + ||u||_0^2
\]

since \( (c_{ij}) \geq 0 \)

\[
\lesssim Q_{\Omega R^s \eta}(u, \bar{u}) + ||[\partial, \eta R^s \eta] \sigma \eta u||_0^2 + \left| \int_{\Omega} [\partial, [\partial, \eta R^s \eta]](u, \bar{u}) \, dV \right|
\]

by (3.6) and because \( (c_{ij}^b) \) are subelliptic multipliers

\[
\lesssim Q_{\Omega R^s \eta}(u, \bar{u}) + ||\partial(\sigma) \log(\Lambda) R^s \eta u||_0^2 + Q_{\Lambda^{s - \frac{1}{2} \eta'}}(u, \bar{u}) + ||\eta' u||_{s - \frac{1}{2}}
\]

by (3.6) and elliptic estimate for \( u^{s'} \)

\[
\lesssim Q_{\Omega R^s \eta}(u, \bar{u}) + \epsilon \left( \int_{\Omega} (c_{ij})(\eta R^s \eta u, \bar{R^s \eta u}) \, dV + \sum_k \int (\phi_{ij}^k)(\eta R^s \eta \Gamma_k u, \bar{R^s \eta \Gamma_k u}) \, dV \right) + c_\epsilon ||\eta R^s \eta u||_0^2 + Q_{\Lambda^{s - \frac{1}{2} \eta'}}(u, \bar{u}) + ||\eta' u||_{s - \frac{1}{2}}
\]

absorption in the second line for \( \epsilon \approx \frac{1}{2} \) and \( t \sim c_\varepsilon^{-1} \)

(3.7)
Now, the \( s - \frac{1}{2} \) norm is reduced to 0 by induction and the various \( Q^0_{\Lambda_s^*} \eta \) and \( Q^0_{\Lambda_s^{*\frac{1}{2}}} \eta \) are estimated by a common \( Q^0_{\Lambda_s^*} \eta \) for a new \( \eta \). Thus we end up with

\[
\| \eta_0 u \|_s \lesssim \| \eta \bar{\partial} u \|_s + \| \eta \bar{\partial}^* u \|_s + \| u \|_0. \tag{3.8}
\]

We observe that (3.8) is an a-priori estimate, that is, it only holds in principle for smooth forms. To overcome this restraint, we use the Kohn microlocal decomposition \( u = u^+ + u^- + u^0 \); since the \( \bar{\partial} \)-Neumann problem is elliptic for \( u^- \) and \( u^0 \), we only have to prove Theorem 3.3 for \( u^+ \). We then use an approximation of the identity in the \( y_n \)-variable by smooth functions \( \chi_\nu(y_n) \) and smoothen \( u^+ \) by \( u^+ = u^+ \chi_\nu \). Now, since \( b\Omega \) is rigid, then 

\[
\bar{\partial}(u^+ \chi_\nu) = (\bar{\partial}^*u)^+ \chi_\nu + \tilde{u}^0,
\]

where \( \tilde{u}^0 \) is an error term supported by the elliptic microlocal region. Then, from (3.8) we get the following. If \( \bar{\partial} u, \bar{\partial}^* u \) belong to \( H^s \) in a neighborhood of \( \text{supp} \eta \), we have that \( \eta_0 u \in H^s \) (and, moreover, \( \| \eta_0 u \|^2_2 \lesssim Q_{\Lambda^0}(u, u) + \| u \|^2_0 \)). In particular, \( u \in H^s \) in a neighborhood of \( \{ z : \eta_0(z) \equiv 1 \} \). This concludes the proof of the theorem.

In case of a single block \( x_n = h^j \) we regain [4] which transfers [11] from the tangential system to the \( \bar{\partial} \)-Neumann problem and also gives a more general statement. The proof is far more efficient because it uses the elementary decomposition \( \partial \eta = (\partial b \eta, \partial \nu \eta) \) instead of \( Q = Q^r \oplus \bar{L}_n \) (over tangential forms \( u^r \)) which requires the heavy technicalities of the harmonic extension.

Remark 3.4. the above proof shows a general criterion. If \( \partial_b \eta \) is a superlogarithmic multiplier for \( \Box_b \), then it is also for the \( \bar{\partial} \)-Neumann problem; thus we have hypoellipticity of \( \Box_b \) and \( \Box \) at the same title.

Example Let \( b\Omega \) be defined by

\[
x_n = \sum_{j=1}^{n-1} e^{-\frac{1}{|x_j|^a}} e^{-\frac{1}{|x_j|^b}} \quad \text{for any } a \geq 0 \text{ and for } b < 1.
\]

Then, (3.3) (a) is obtained starting from \( h_{z_j}^j e^{-\frac{1}{|x_j|^b}} \), that is, the condition of type

\[
F_j^2 := e^{-\frac{1}{|x_j|^a}} \quad \text{along } S_j = \mathbb{R} x_j \setminus \{0\}. \text{ Since } b < 1, \text{ this is infraexponential (and yields a superlogarithmic estimate for } f = \log^{rac{1}{b}}). \text{ (3.3) (b) follows from } |h_{z_j}^j|^2 \lesssim h_{z_j}^j, \text{ which says that the } h_{z_j}^j \text{'s are not only superlogarithmic, but indeed } \frac{1}{2} \text{ subelliptic, multipliers. Altogether we have that } \Box \text{ is hypoelliptic according to Theorem 3.3.}

References

[1] L. Baracco, T.V. Khanh and G. Zampieri—Hypoellipticity of the \( \bar{\partial} \)-Neumann problem at a point of infinite type, Asian J. Math. (2014) to appear.
[2] L. Baracco, T.V. Khanh, S. Pinton and G. Zampieri—Local regularity of the Green operator in a CR manifold of general “type”, (2014) Preprint

[3] H. P. Boas and E. J. Straube—Sobolev estimates for the \( \bar{\partial} \)-Neumann operator on domains in \( \mathbb{C}^n \) admitting a defining function that is plurisubharmonic on the boundary, Math. Z. 206 (1) (1991) 81–88

[4] L. Baracco, S. Pinton and G. Zampieri—Hypoellipticity of the \( \bar{\partial} \)-Neumann problem by means of subelliptic multipliers, Math. Annalen (2014) to appear

[5] D. Catlin—Subelliptic estimates for the \( \bar{\partial} \)-Neumann problem on pseudoconvex domains, Ann. of Math. 126 (1987), 131-191

[6] G.B. Folland and J.J. Kohn—The Neumann problem for the Cauchy-Riemann complex, Ann. Math. Studies, Princeton Univ. Press, Princeton N.J. 75 (1972)

[7] T.V. Khanh and G. Zampieri—Regularity of the \( \bar{\partial} \)-Neumann problem at a flat point, J. Funct. Anal. 259 no. 11 (2010), 2760-2775

[8] J. J. Kohn—Global regularity for \( \bar{\partial} \) on weakly pseudo-convex manifolds, Trans. of the A.M.S. 181 (1973), 273–292

[9] J.J. Kohn—Subellipticity of the \( \bar{\partial} \)-Neumann problem on pseudoconvex domains: sufficient conditions, Acta Math. 142 (1979), 79–122

[10] J. J. Kohn—The range of the tangential Cauchy-Riemann operator, Duke Math. J. 53 (1986), 525–545

[11] J.J. Kohn—Hypoellipticity at points of infinite type, Contemporary Math. 251 (2000), 393–398

[12] J.J. Kohn—Superlogarithmic estimates on pseudoconvex domains and CR manifolds, Annals of Math. 156 (2002), 213–248

[13] J.J. Kohn and L. Nirenberg—Non-coercive boundary value problems, Comm. Pure Appl. Math. 18 (1965), 443–492

[14] E. Straube—Lectures on the \( L^2 \)-Sobolev theory of the \( \bar{\partial} \)-Neumann problem, ESI Lect. in Math. and Physics (2010)

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