The Projection Games Conjecture and the Hardness of Approximation of SSAT and related problems

Priyanka Mukhopadhyay *
Institute for Quantum Computing, University of Waterloo †

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Abstract

The Super-SAT or SSAT problem was introduced by Dinur, Kindler, Raz and Safra [DKRS03, Din02] to prove the NP-hardness of approximation of two popular lattice problems - Shortest Vector Problem (SVP) and Closest Vector Problem (CVP). They conjectured that SSAT is NP-hard to approximate to within factor $n^c$ for some constant $c > 0$, where $n$ is the size of the SSAT instance. In this paper we prove this conjecture assuming the Projection Games Conjecture (PGC), given by Moshkovitz [Mos12]. This implies hardness of approximation of SVP and CVP within polynomial factors, assuming the Projection Games Conjecture.

We also reduce SSAT to the Nearest Codeword Problem (NCP) and Learning Halfspace Problem (LHP), as considered by Arora, Babai, Stern and Sweedyk [ABSS97]. This proves that both these problems are NP-hard to approximate within factor $N^{c'/\log \log n}$ for some constant $c' > 0$ where $N$ is the size of the instances of the respective problems. Assuming the Projection Games Conjecture these problems are proved to be NP-hard to approximate within polynomial factors.

* mukhopadhyay.priyanka@gmail.com
† Much of this work was done while the author was in Centre for Quantum Technologies, National University of Singapore.
1 Introduction

1.1 SSAT and lattice problems

The Super-SAT or SSAT problem was introduced by Dinur et al. \cite{DKRS03} to prove the hardness of approximation of two popular lattice problems - the Shortest Vector Problem (SVP) and the Closest Vector Problem (CVP). An \(n\)-dimensional lattice \(L\) is the set of integral linear combination of \(n\) linearly independent vectors, called the basis of the lattice. The goal of SVP is to find the shortest non-zero lattice vector. Given a target vector \(\vec{t}\), CVP aims at finding the closest lattice vector to it.

Algorithms for these lattice problems are well-studied and have applications in factoring polynomials over rationals \cite{LLL82}, integer programming \cite{LJ83, EHNT11}, cryptanalysis \cite{NS01}, checking the solvability by radicals \cite{LM83}, solving low-density subset-sum problems \cite{CJLR92}, cryptography \cite{Ajt96, Gen09, Reg09, BLP13, BV14, DLL17}.

The SSAT problem is the gap version of SAT\([F]\), which is defined as follows: An instance of SAT\([F]\) consists of a set of constraints or Boolean functions, called tests. The variables in each test take values from a finite set \(F\) and each test has a set of satisfying assignments for its variables. The goal is to attach one assignment to each test such that consistency is maintained, i.e. each variable gets the same value in all the tests in which it appears.

In SSAT, we attach integer weights to each assignment and call it a super-assignment. It is consistent if for each variable the sum of weights on each value is the same in all the tests with this variable. Values which get non-zero net weight are said to be simultaneously assigned to the variable. If each variable gets at least one value assigned we call it a non-trivial super-assignment. If for at least one test there exists at least one non-zero weighted assignment then we call it not-all-zero super-assignment. This also gives rise to the notion of norm of a super-assignment and accordingly two variants of SSAT has been defined - the one in \cite{DKRS03} for \(\ell_1\) norm and another by Dinur in \cite{Din02} for \(\ell_\infty\) norm. An instance is accepted if each variable gets a single value everywhere giving a consistent super-assignment of norm 1. The rejection criteria is slightly different in the two variants. Roughly, an instance is rejected if every consistent super-assignment satisfying some conditions, has norm greater than \(g\). If after minimizing the norm of such a consistent super-assignment we get a value between 1 and \(g\), then the instance maybe accepted or rejected. A more detailed explanation of these concepts have been given in Section 2.2. The following hardness results have been proved in \cite{DKRS03, Din02}. Suppose \(n\) is the size of the SSAT instance, which is the encoding size of the number of variables, tests and satisfying assignments and this is polynomial in the number of variables (Section 2.2).

\textbf{Theorem 1.1 (SSAT Theorem \cite{DKRS03}).} There is some constant \(c > 0\), such that SSAT is NP-hard for \(g = n^{c/\log \log n}\).

\textbf{Theorem 1.2 (SSAT}_\infty \text{ Theorem \cite{Din02}).} SSAT\(_\infty\) is NP-hard for \(g = n^{c/\log \log n}\) for some \(c > 0\).

An approximation factor preserving reduction from SSAT to CVP\(_p\) (where distance is measured
in $\ell_p$ norm), for $1 \leq p < \infty$ was given in [DKRS03] and a similar reduction from $SSAT_\infty$ to $CVP_\infty$ and $SVP_\infty$ (where distance or length are measured in $\ell_\infty$ norm) was given in [Din02]. Thus the authors conjectured that the $SSAT$ problems are hard within a polynomial factor, which would imply NP-hardness of the above mentioned lattice problems within polynomial approximation factor.

**Conjecture 1.1.** $SSAT$ is NP-hard for $g = n^c$ for some constant $c > 0$.

**Conjecture 1.2.** $SSAT_\infty$ is NP-hard for $g = n^c$ for some constant $c > 0$.

### 1.2 Label Cover (LC) and Projection Games Conjecture (PGC)

An instance of a Label Cover (LC) problem (also referred to as Projection Games) consists of (i) a bipartite graph $G = (A, B, E)$; (ii) finite alphabets $\Sigma_A, \Sigma_B$ from which each vertex of $A$ and $B$ (respectively) are assigned a label; (iii) constraints or projections $\pi_e : \Sigma_A \rightarrow \Sigma_B$ for each $e \in E$. Given a labeling or assignment to the vertices, $\varphi_A : A \rightarrow \Sigma_A$ and $\varphi_B : B \rightarrow \Sigma_B$, we say an edge $e = (a, b)$ is satisfied if the corresponding projection constraint holds, i.e. $\pi_e(\varphi_A(a)) = \varphi_B(b)$. In the optimization version of this problem the task is to find a labeling that maximizes the number of satisfied edges. The decision version of this problem is of interest to us and by Label Cover (LC) we denote this problem of distinguishing between the (YES) case that all edges are satisfied and the (NO) case when at most $s$ (soundness error) fraction of the edges are satisfied. Note there are other variants of this problem (e.g. [ABSS97]) but in this paper we work with this one.

A PCP Theorem gives the hardness of LC as follows [AS98, ALM+98, Raz98]:

> Given an input of size $N$ for LC with alphabet size $k$, it is NP-hard to distinguish between the case where all edges can be satisfied and the case where at most $s$ fraction of the edges can be satisfied.

(s and $k$ maybe functions of $N$.) Equivalently it can be stated that there is a reduction from (exact) SAT to LC. Raz and Moshkovitz [MR08] proved the following result

**Theorem 1.3 (MR08).** There exists $c > 0$ such that for every $s \geq 1/N^c$, SAT on input of size $n$ can be reduced to LC of size $N = n^{1+o(1)}\text{poly}(1/s)$. The LC is over an alphabet of size exponential in $1/s$ and has soundness error $s$. The reduction can be computed in linear time in the size and the alphabet size of the LC instance. The LC is on a bi-regular graph whose degrees are poly$(1/s)$.

PCPs which achieve an LC instance of size $N = n^{1+o(1)}\text{poly}(1/s)$ are called almost-linear size PCP because of the exponent of $n$. The soundness error $s$ is at least $1/N$. Assuming $P \neq NP$ it can be shown that the alphabet size is at least $1/s$. Certain PCP constructions manage to have an alphabet size of poly$(1/s)$ at the cost of some other parameters [Raz98]. Thus Moshkovitz [Mos12] conjectured that a similar alphabet size maybe achieved in Theorem 1.3.

**Conjecture 1.3 (Projection Games Conjecture (PGC) [Mos12]).** There exists $c > 0$ such that for every $s \geq 1/N^c$, SAT on input of size $n$ can be efficiently reduced to LC of size $N = n^{1+o(1)}\text{poly}(1/s)$ over an alphabet of size poly$(1/s)$ and has soundness error $s$. 

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1.3 Related work

The Label Cover (LC) problem was introduced by Arora et al. [ABSS97] but with a slightly different formulation than what has been stated in this paper. Roughly, in their variant there is a “cost” attached to the labeling of each vertex. The approximation factor is given by the ratio of this cost between the NO and YES case. The authors proved this variant of LC is NP-hard up to the approximation factor of $2^{\log^{0.5-\varepsilon} n}$ where $\varepsilon > 0$ is some constant and $n$ is the size of LC instance, under the assumption that $\text{NP} \not\subseteq \text{DTIME}(n^{\text{poly}(\log n)})$. They gave an approximation factor preserving reduction from LC to a number of other problems like CVP, $\text{SVP}_\infty$, Nearest Codeword Problem (NCP), Min-Unsatisfy problem and Learning Halfspace Problem (LHP). They also proved that the above problems were NP-hard for constant approximation factors by a reduction from Set Cover.

In 2012, Moshkovitz [Mos12] reduced LC (the variant stated in Section 1.2) to Set Cover and proved that the latter is NP-hard to approximate within $(1 - \alpha) \ln n$ ($n$ being the instance size) for arbitrarily small $\alpha > 0$. She applied the Projection Games Conjecture to the reduction in [ABSS97] and concluded polynomial approximation factors are hard for CVP. Here the conjecture has also been used to study the behavior of CSPs around their approximability threshold.

The first NP hardness result for CVP in all $\ell_p$ norms and SVP in the $\ell_\infty$ norm was given by van Emde Boas [vEB81]. SVP was proven to be NP-hard to approximate within a constant factor in [Ajt98, CN98, Mic01]. Khot [Kho05] and later Haviv and Regev [HR12] improved the approximation factor to $2^{\log^{1-\varepsilon} n}$ under the assumption that $\text{NP} \not\subseteq \text{RTIME}(n^{\text{poly}(\log n)})$. In [BGSD17] it has been shown that for almost all $p \geq 1$, CVP in the $\ell_p$ norm cannot be solved in $2^{n(1-\varepsilon)}$ time under the Strong Exponential Time Hypothesis [IP99]. A similar hardness result has also been obtained for SVP [ASD18].

However, there are barriers for showing stronger inapproximability results. For example, a factor $n$ NP-hardness result would imply $\text{NP} = \text{coNP}$ [Has88, LLS90, Ban93], a $\sqrt{n/O(\log n)}$ factor NP-hardness result would imply $\text{coNP} \subseteq \text{AM}$ [GG00], a factor $\sqrt{n}$ NP-hardness for SVP would imply $\text{NP} = \text{coNP}$ [AR05], and thus the polynomial hierarchy collapses in all these cases.

1.4 Our results and techniques

Hardness of SSAT and SSAT$_\infty$

In this paper we prove Conjecture 1.1 and 1.2 assuming the Projection Games Conjecture (PGC). Specifically we prove the following:

**Theorem 1.4.** Assuming the Projection Games Conjecture SSAT is NP-hard for $g = n^c$ for some constant $c > 0$.

**Theorem 1.5.** Assuming the Projection Games Conjecture SSAT$_\infty$ is NP-hard for $g = n^c$ for some constant $c > 0$. 

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We give a reduction to both SSAT and SSAT$_\infty$ from a variant of Label Cover, introduced by Moshkovitz in [Mos12], let us call it the list-Label Cover, which we explain very briefly here. Here we can assign each $A$-vertex a list of labels. Two vertices $a_1, a_2 \in A$ agree on a label $y \in \Sigma_B$ for a vertex $b \in B$ if there exists at least one label in their respective lists such that these map to $y$ under the respective edge constraint functions. That is, there exists $x_1 \in \varphi_A(a_1)$ and $x_2 \in \varphi_A(a_2)$ such that $\pi_{e_1}(x_1) = \pi_{e_2}(x_2) = y$, where $e_1 = (a_1, b), e_2 = (a_2, b) \in E$. For this variant of LC a notion of list agreement soundness has been defined which gives the fraction of $B$-vertices on which the $A$-vertices totally disagree (i.e. no two $A$-vertices agree on any one label). A more detailed explanation of these concepts have been given in Section 2.1.

In our construction, each $A$-vertex corresponds to a variable in the SSAT instance and each $B$-vertex is a test. By a result of Moshkovitz [Mos12], we can bound the $B$-degree (degree of each $B$-vertex) by some large enough constant prime power. For simplicity and without much loss of generality, we assume that $\Sigma_A$ is in bijective correspondence to some finite field and the variables take values from $\Sigma_A$. For each test $\psi_b$ (corresponding to some $b \in B$) consider a $y \in \Sigma_B$ such that it has at least one pre-image in each of $b$’s neighbor in $A$, consider the following tuples:

$$R_y(\psi_b) = \{(x_1, \ldots, x_{D_B}) : x_j \in \pi_e^{-1}(y) \text{ where } e = (a_j, b) \text{ and } a_j \text{ is the } j^{th} \text{ neighbor of } b\}$$

Then the total set of satisfying assignments for $\psi_b$ is: $R(\psi_b) = \bigcup_{y \in \Sigma_B} R_y(\psi_b)$ and cardinality of this set is bounded by some polynomial in the number of variables.

Next, we argue that a YES case of LC maps to a YES case of SSAT. For soundness proof we need to show that a NO instance of LC maps to a NO case of SSAT. Instead, we give a contrapositive argument.

In the $\ell_1$ norm we show that if there exists a consistent non-trivial super-assignment of norm less than $g$ then there exists a labeling of the vertices such that a certain fraction of the $B$-vertices do not have total disagreement.

We show a similar result in the $\ell_\infty$ norm if there exists a consistent non-trivial (which is obviously not-all-zero) super-assignment. If a consistent super-assignment is not-all-zero but not non-trivial, i.e. some tests have all assignments with zero weight, then we show such a labeling exists if either (i) For most variables we can find a set of values of cardinality at most $g$ such that for at least a certain fraction of tests there exists at least one assignment where the respective variable has a value from this set, OR
(ii) $gD_A \approx \frac{1}{\sqrt{s}}$, where $D_A$ is the degree of the $A$-vertices and $s$ is the soundness error we are targeting.

Note if the norm of a not-all-zero consistent super-assignment is somewhere between 1 and $g$ then any answer is acceptable. By this contrapositive argument we show that in some cases we might get a YES reply from the oracle. So this is sufficient to prove the soundness in the $\ell_\infty$ norm.

**Hardness of lattice problems and some related problems**

Dinur et.al. [DKRS03, Din02] gave approximation factor preserving reduction from SSAT and SSAT$_\infty$ to some lattice problems like CVP, CVP$_\infty$, SVP$_\infty$ and a related problem - Short Integer
Solution (SIS). As a corollary of Theorem 1.4 and 1.5 we prove these problems are NP-hard to approximate within polynomial factors, provided Projection Games Conjecture holds (Corollary 4.1, 4.2, 4.3). These problems have been defined explicitly in Section 2.4.

**Hardness of Learning Halfspace Problem (LHP)**

We briefly describe the Learning Halfspace Problem (LHP) in the presence of malicious errors, as defined in [ABSS97]. It has been defined more precisely in Section 2.4. Roughly, the input to this problem is a set of points which are labeled by “+” or “-”, according to the side of a hyperplane they lie in a finite-dimensional space. Since this problem arises in the context of training a perceptron (a learning model) [MP17], so we say that the input is given to a “learner”, which has to output a hypothesis (i.e. a hyperplane) that correctly classifies as many points as possible. The error of an algorithm is the number of misclassifications by its hypothesis, and the noise of the sample (set of points) is the minimum error achievable by any algorithm. The failure ratio of an algorithm is the ratio of its error to noise.

We take the formulation of LHP given in [ABSS97] and give a reduction from SSAT, thus proving that approximating the minimum failure ratio of LHP is NP-hard up to a factor of $n^{c/\log \log n}$ (Theorem 4.2). Assuming PGC this factor can be improved to $n^{c}$ for some constant $c > 0$ (Corollary 4.5).

**Hardness of Nearest Codeword Problem (NCP)**

The input to the Nearest Codeword Problem (NCP) is the generator matrix of an error-correcting code of length $n$ over a $q$-ary alphabet. Given a target vector the goal is to find a codeword that is nearest to the target vector in Hamming distance. More detail description has been given in Section 2.4. Note that this problem is not exactly equivalent to CVP.

The fact that this problem maybe related to SSAT was hinted in [DMS03]. However, no proof was given there. We give a reduction from SSAT to NCP which proves that it is NP-hard to approximate the latter within a factor $n^{c/\log \log n}$ (Theorem 4.1), which can be improved to $n^{c}$ ($c > 0$ is a constant) assuming the Projection Games Conjecture (Corollary 4.4).

**1.5 Future directions**

One obvious direction would be to prove the hardness of SSAT and SSAT$_\infty$ (Conjecture 1.1 and 1.2) without assuming the Projection Games Conjecture. Alternately, one may prove the PGC (Conjecture 1.3). Either of these will imply improved NP-hard approximation factors for the problems considered in this paper, without any other assumptions.

To the best of our knowledge, apart from this work, the SSAT problem has only been studied to prove the hardness of approximation of lattice problems. Relating this problem to other problems
might give interesting hardness results as well as algorithms for them. It might even throw some light on the complexity of SSAT itself.

1.6 Overview of this paper

We give all necessary preliminary definitions and notations in Section 2. The reduction from LC to SSAT appears in Section 3, while the reduction from SSAT to other computational problems are in Section 4.

2 Preliminaries

Notations We write ln for natural logarithm and log for logarithm to the base 2. \( \mathbb{R}, \mathbb{Q}, \mathbb{Z} \) denotes the set of real numbers, rational numbers and integers respectively. \( \mathbb{F}_p \) denotes a field of order \( p \). We denote variables by bold letters. We denote arrays by letters (lower case letters for 1-dimensional arrays or vectors) with overhead arrow, e.g. \( \vec{v}^n \) and \( \vec{M}^{\ell \times m \times n} \) (or \( \vec{M}^{\ell \times m \times n} \)). We may drop the dimension in the superscript (or subscript) whenever it is clear from the context. The \( i^{th} \) co-ordinate of \( \vec{v} \) is denoted by \( v_i \) or \( (\vec{v})_i \). The \( (i,j,k)^{th} \) entry of \( \vec{M} \) is denoted by \( M_{ijk} \) or \( \vec{M}[i,j,k] \). Sometimes we represent a matrix (2-dimensional array) as a vector of column (vectors) (e.g. \( \vec{M}^m \times n = [\vec{m}_1^m \vec{m}_2^m \ldots \vec{m}_n^m] \) where each \( \vec{m}_i^m \) is an \( m \)-length vector).

2.1 Label Cover

Definition 2.1 (Label Cover (LC)). An instance of LC consists of (i) a bipartite graph \( G = (A, B, E) \); (ii) finite alphabets \( \Sigma_A, \Sigma_B \), such that each vertex in \( A \) and \( B \) is assigned a label from \( \Sigma_A \) and \( \Sigma_B \) respectively; (iii) a set \( \Pi \) of constraints consisting of projections \( \pi_e : \Sigma_A \to \Sigma_B \), \( \forall e \in E \). Given a labeling to the vertices \( \varphi_A : A \to \Sigma_A \) and \( \varphi_B : B \to \Sigma_B \), an edge \( e = (a, b) \) is satisfied if \( \pi_e(\varphi_A(a)) = \varphi_B(b) \). (With a slight abuse of notation, we sometimes drop the labelings and simply write \( \pi_e(a) = b \).)

We work with the promise problem where in a YES instance there exists a labeling that satisfies all edges and in a NO instance, for all possible labeling to the vertices at most \( s \) fraction of the edges can be satisfied. Such an instance is said to have soundness error \( s \). The goal is to distinguish between these two cases. The size of the label cover is \( N = |A| + |B| + |E| \) and the size of the alphabet is \( \max\{|\Sigma_A|, |\Sigma_B|\} \).

Feige [Fei98] defined a variant of LC (using the structure obtained from parallel repetition) where the soundness is determined by the fraction of \( B \)-vertices that have at least two neighbors from \( A \) that agree on a label for them. To be more precise, we define the following terms.

Definition 2.2 (Total disagreement). Let \( (G = (A, B, E), \Sigma_A, \Sigma_B, \Pi) \) be an LC instance. \( \varphi_A : A \to \Sigma_A \) is a labeling to the \( A \)-vertices. We say that the \( A \)-vertices totally disagree on a vertex
Definition 2.3 (Agreement soundness). Let $G = (G = (A, B, E), \Sigma_A, \Sigma_B, \Pi)$ be an LC instance for deciding whether a Boolean formula $\phi$ is satisfiable. We say that $G$ has agreement soundness error $s_{agr}$, if the following holds: If $\phi$ is satisfiable then there is a labeling $\varphi_A : A \to \Sigma_A$, $\varphi_B : B \to \Sigma_B$ that satisfies all edges, and if $\phi$ is unsatisfiable then for any labeling $\varphi_A : A \to \Sigma_A$, the $A$-vertices are in total disagreement on at least $1 - s_{agr}$ fraction of the $b \in B$.

Moshkovitz [Mos12] considered a variant of LC where each vertex can be assigned a list of $\ell$ labels and an agreement is interpreted as agreement on one of the labels in the list. We define the following terms related to this variant, which we call the list-Label Cover (list-LC).

Definition 2.4 (List total disagreement). Let $G = (G = (A, B, E), \Sigma_A, \Sigma_B, \Pi)$ be an LC instance. Let $\ell \geq 1$ and $\tilde{\varphi}_A : A \to (\frac{\Sigma_A}{\ell})$ is a labeling that assigns each $A$-vertex $\ell$ alphabet symbols. We say that the $A$-vertices totally disagree on a vertex $b \in B$ if there are no two neighbors $a_1, a_2 \in A$ of $b$, for which there exists $\sigma_1 \in \tilde{\varphi}_A(a_1)$, $\sigma_2 \in \tilde{\varphi}_A(a_2)$, such that $\pi_{e_1}(\sigma_1) = \pi_{e_2}(\sigma_2)$, where $e_1 = (a_1, b)$, $e_2 = (a_2, b) \in E$.

Definition 2.5 (List agreement soundness). Let $G = (G = (A, B, E), \Sigma_A, \Sigma_B, \Pi)$ be an LC instance for deciding whether a Boolean formula $\phi$ is satisfiable. We say that $G$ has list agreement soundness error $(\ell, s_{list})$, if the following holds: If $\phi$ is satisfiable, then there is a labeling $\varphi_A : A \to \Sigma_A$, $\varphi_B : B \to \Sigma_B$ that satisfies all edges, and if $\phi$ is unsatisfiable, then for any labeling $\tilde{\varphi}_A : A \to (\frac{\Sigma_A}{\ell})$, the $A$-vertices are in total disagreement on at least $1 - s_{list}$ fraction of the $b \in B$.

The following result relates agreement soundness and list agreement soundness.

Lemma 2.1 ([Mos12]). Let $\ell \geq 1$ and $0 < s_{agr} < 1$. An LC with agreement soundness error $s_{agr}$ has list agreement soundness error $(\ell, s_{agr}\ell^2)$.

2.2 SuperSAT (SSAT)

An SSAT instance has a set $\Psi = \{\psi_1, \ldots, \psi_m\}$ of tests over variables $V = \{v_1, \ldots, v_m\}$ which take values from a field $F$, called range of the variables. An assignment maps each variable to a value in $F$. For convenience, we can think of an assignment as a tuple of field values (for its variables) and it is satisfying if these values evaluate to some required value for the test. Each test $\psi$ has a list $R_\psi$ of satisfying assignments for its variables and we attach some “weight” to each such assignment.

Definition 2.6 (Super-assignment to tests). A super-assignment is a function $S$ mapping each $\psi \in \Psi$ to a value from $\mathbb{Z}^{R_\psi}$. Thus $\overline{S(\psi)}$ is a vector of integer coefficients, one for each $r \in R_\psi$.

Denote $\overline{S(\psi)}[r]$ as the $r^{th}$ co-ordinate of $\overline{S(\psi)}$. A natural super-assignment assigns each $\psi \in \Psi$ a unit vector $\overrightarrow{e}_r \in \mathbb{Z}^{R_\psi}$ with a 1 in the $r^{th}$ co-ordinate (i.e. $\overline{S(\psi)}[r] = 1$ and $\overline{S(\psi)}[r'] = 0$ for all $r' \neq r$). A super-assignment is not-all-zero if there is at least one test $\psi \in \Psi$ for which $\overline{S(\psi)} \neq \overrightarrow{0}$.
Definition 2.7 (Projection). Given a super-assignment \( S : \Psi \rightarrow \bigcup_{\psi} \mathbb{Z}^{\mathbb{R}^t} \), the projection of \( \overrightarrow{S(\psi)} \) on a variable \( x \) of \( \psi \), \( \pi_x(\overrightarrow{S(\psi)}) \in \mathbb{Z}^F \), is defined as follows:
\[
\forall a \in F : \pi_x(\overrightarrow{S(\psi)})[a] = \sum_{r \in \mathbb{R}^t, r|_x = a} \overrightarrow{S(\psi)}[r]
\]

The notion of projection facilitates the definition of consistency between tests.

Definition 2.8 (Consistency). A super-assignment \( S \) to the tests in \( \Psi \) is consistent if the projections of two tests on each mutual variable are equal, i.e. for every pair of tests \( \psi_i \) and \( \psi_j \) with a common variable \( x \),
\[
\pi_x(\overrightarrow{S(\psi_i)}) = \pi_x(\overrightarrow{S(\psi_j)})
\]

\( S \) is non-trivial if for every variable \( x \in V \) there is at least one test \( \psi \in \Psi \) that is not cancelled on \( x \), i.e. \( \pi_x(\overrightarrow{S(\psi)}) \neq \overrightarrow{0} \). For a variable \( x \) we think of all the values \( a \in F \) receiving non-zero coefficients in \( \pi_x(\overrightarrow{S(\psi)}) \) (i.e. values for which \( \pi_x(\overrightarrow{S(\psi)})[a] \neq 0 \)) as being “simultaneously” assigned to \( x \) by \( \psi \). The non-triviality requirement implies each variable must be assigned at least one value.

To formally define the SSAT problem we have to introduce the notion of norm of a super-assignment. Here we note that in [DKRS03] the problem was defined for \( \ell_1 \) norm (though the derived results work for all \( \ell_p \) norms where \( 1 \leq p < \infty \)) while Dinur [Din02] introduced a related problem called the SSAT\(_\infty \), where some definitions like norm of a super-assignment were modified.

Definition 2.9 (Norm of a Super-assignment). For the problem SSAT the norm of a super-assignment \( S \) is the average norm of its individual assignments:
\[
\|S\| = \frac{1}{|\Psi|} \sum_{\psi \in \Psi} \|\overrightarrow{S(\psi)}\|_1.
\]
We call \( \|\overrightarrow{S(\psi)}\|_1 = \sum_{r \in \mathbb{R}^t} |\overrightarrow{S(\psi)}[r]| \) as the norm of a test \( \psi \).

For the problem SSAT\(_\infty \) the norm of a super-assignment \( S \) is defined as:
\[
\|S\|_\infty = \max_{\psi \in \Psi} \|\overrightarrow{S(\psi)}\|_1.
\]

We now formally define both the \( g - \text{SSAT} \) and \( g - \text{SSAT}_\infty \) problem. The parameter \( g \) is an approximation factor for the norm of a super-assignment.

Definition 2.10 (\( g - \text{SSAT} \) and \( g - \text{SSAT}_\infty \)). The input instance
\[
\mathcal{I} = (\Psi = \{\psi_1, \ldots, \psi_n\}, V = \{v_1, \ldots, v_m\}, \{\mathcal{R}_{\psi_1}, \ldots, \mathcal{R}_{\psi_n}\})
\]
consists of (i) a set \( \Psi = \{\psi_1, \ldots, \psi_n\} \) of tests over a common set \( V = \{v_1, \ldots, v_m\} \) of variables that take values in a field \( F \); (ii) for each test \( \psi \in \Psi \) a list \( \mathcal{R}_{\psi} \) of satisfying assignments to its variables, called the range of the test \( \psi \). The size of an instance is the encoding size of the number of variables, tests and satisfying assignments. The parameters \( m, |F| \) and \( |\mathcal{R}_{\psi}| \) are always bounded by some polynomial in \( n \) and hence the size of an instance is also bounded by some polynomial in \( n \).

This is a promise problem where in the YES instance there is a consistent natural super-assignment for \( \Psi \). In the NO instance of SSAT, for every non-trivial consistent super-assignment
S for \( \Psi \), \( \|S\| \geq g \). While in the NO instance of SSAT\(_{\infty}\), for every not-all-zero consistent super-assignemnt \( S \) for \( \Psi \), \( \|S\|_{\infty} \geq g \). The goal is to distinguish between the YES and NO instance of the respective problems.

### 2.3 Lattice problems

**Definition 2.11.** A lattice \( \mathcal{L} \) is a discrete additive subgroup of \( \mathbb{R}^d \). Each lattice has a basis \( \mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n] \) where \( \mathbf{b}_i \in \mathbb{R}^d \) and \( \mathcal{L} = \mathcal{L}(\mathcal{B}) = \left\{ \sum_{i=1}^{n} x_i \mathbf{b}_i : x_i \in \mathbb{Z} \quad \text{for} \quad 1 \leq i \leq n \right\} \). We call \( n \) the rank of \( \mathcal{L} \) and \( d \) as the dimension.

We consider the following lattice problems. From here on, in all the problems \( c \geq 1 \) is some arbitrary approximation factor (usually specified as subscript), which can be a constant or a function of any parameter of the lattice (usually rank). For exact versions of the problems (i.e. \( c = 1 \)) we drop the subscript. Typically, we define length in terms of the \( \ell_p \) norm for some \( 1 \leq p \leq \infty \). Thus \( \|\mathbf{x}\|_p := (|x_1|^p + |x_2|^p + \cdots + |x_d|^p)^{1/p} \) for finite \( p \) and \( \|\mathbf{x}\|_{\infty} := \max |x_i| \).

**Definition 2.12 (Shortest Vector Problem (SVP\(_c(p)\)).** Given a lattice \( \mathcal{L} \) with rank \( n \) the goal is to find the shortest non-zero vector in the lattice.

In the promise version, usually denoted as GapSVP\(_c(p)\), the goal is to distinguish between the YES instance when \( \exists \mathbf{v} \in \mathcal{L} \setminus \{ \mathbf{0} \} \) such that \( \|\mathbf{v}\|_p \leq r \) (for some positive real \( r \) given as input) and the NO instance when \( \forall \mathbf{v} \in \mathcal{L} \), \( \|\mathbf{v}\|_p > c \cdot r \).

**Definition 2.13 (Closest Vector Problem (CVP\(_c(p)\)).** Given a lattice \( \mathcal{L} \) with rank \( n \) and a target vector \( \mathbf{t} \in \mathbb{R}^n \) the goal is to find the closest lattice vector to \( \mathbf{t} \).

In the promise version, usually denoted as GapCVP\(_c(p)\), the goal is to distinguish between the YES instance when \( \exists \mathbf{v} \in \mathcal{L} \) such that \( \|\mathbf{v} - \mathbf{t}\|_p \leq r \) (for some positive real \( r \) given as input) and the NO instance when \( \forall \mathbf{v} \in \mathcal{L} \), \( \|\mathbf{v} - \mathbf{t}\|_p > c \cdot r \).

In this paper, with a slight abuse of notation we denote both the optimization and the promise versions of the above problems by the same notation, i.e. SVP and CVP respectively.

### 2.4 Other computational problems

**Shortest Integer Solution**

**Definition 2.14 (Shortest Integer Solution (SIS\(_c\) [DKRS03]).** Given (i) a matrix \( \mathbf{B} \in \mathbb{Z}^{m \times n} \); (ii) target vector \( \mathbf{t} \in \mathbb{Z}^m \) such that \( \mathbf{t} \in \{ \mathbf{B} \mathbf{x} : \mathbf{x} \in \mathbb{Z}^n \} \) and (iii) \( d \in \mathbb{Z} \), the goal is to distinguish between the YES instance when \( \exists \mathbf{z} \in \mathbb{Z}^n \) such that \( \mathbf{B} \mathbf{z} = \mathbf{t} \) and \( \|\mathbf{z}\|_p \leq d \) and the NO instance when \( \forall \mathbf{z} \in \mathbb{Z}^n \) where \( \mathbf{B} \mathbf{z} = \mathbf{t} \) we have \( \|\mathbf{z}\|_p > c \cdot d \).
Nearest Codeword Problem

An error-correcting code \( A \) of block length \( n \) over a \( q \)-ary alphabet \( \Sigma \) (\( = \mathbb{F}_q \)) is a collection of strings (vectors) from \( \Sigma^n \), called codewords. A linear code \( A \) is a linear subspace of \( \mathbb{F}_q^n \) over base field \( \mathbb{F}_q \) and it can be compactly represented by a generator matrix \( \overrightarrow{A} \in \mathbb{F}_q^{m \times n} \) such that \( A = \{ \overrightarrow{A} \overrightarrow{x} : \overrightarrow{x} \in \mathbb{F}_q^n \} \).

For any \( \overrightarrow{v} \in \Sigma^m \), the Hamming weight of \( \overrightarrow{v} \) is denoted by \( \text{wt}(\overrightarrow{v}) = |\{ i : v_i \neq 0 \}| \). The Hamming distance between two vectors \( \overrightarrow{u}, \overrightarrow{v} \in \Sigma^m \) is \( \| \overrightarrow{u} - \overrightarrow{v} \|_H = \text{wt}(\overrightarrow{u} - \overrightarrow{v}) \).

Definition 2.15 (Nearest Codeword Problem (NCP) [ABSS97]). Given (i) a matrix \( \overrightarrow{A} \) over \( \mathbb{F}_q^{m \times n} \); (ii) a target vector \( \overrightarrow{t} \in \mathbb{F}_q^m \) and (iii) an integer \( d \), the goal is to distinguish between the YES instance when \( \exists \overrightarrow{z} \in \mathbb{F}_q^n \) such that \( \| \overrightarrow{A} \overrightarrow{z} - \overrightarrow{t} \|_H \leq d \), and the NO instance when \( \forall \overrightarrow{z} \in \mathbb{F}_q^n \), \( \| \overrightarrow{A} \overrightarrow{z} - \overrightarrow{t} \|_H > c \cdot d \).

Learning Halfspace Problem

We consider a popular problem in learning theory: learning a halfspace in the presence of malicious errors, as described in [ABSS97].

The input to the learner consists of a set of \( k \) points in \( \mathbb{R}^m \), each labeled with a + (positive examples of a concept) or − (negative examples of a concept). The learner’s output is a hyperplane, \( \langle \overrightarrow{a}, \overrightarrow{x} \rangle = b \), where \( \overrightarrow{a} \in \mathbb{R}^m \) and \( b \in \mathbb{R} \). The hypothesis correctly classifies a point \( \overrightarrow{y} \) marked + (or −) if it satisfies \( \langle \overrightarrow{a}, \overrightarrow{y} \rangle > b \) (or \( \langle \overrightarrow{a}, \overrightarrow{y} \rangle < b \) respectively). Else, it misclassifies the point.

Finding a hypothesis that minimizes the number of misclassifications is the open hemispheres problem, which is NP-hard. The error of the algorithm is the number of misclassifications by its hypothesis, and the noise of the sample is the error of the best possible algorithm. The failure ratio of the algorithm is the ratio of the error to noise.

The LHP can be formulated in the following way.

Definition 2.16 (Learning Halfspace Problem (LHP) [ABSS97]). Given a set of linear inequalities in \( n \) variables over \( \mathbb{F}_q \) and an integer \( d \), distinguish between the YES instance when there exists an assignment to the \( n \) variables which does not satisfy at most \( d \) inequalities, and the NO instance when every assignment to the \( n \) variables does not satisfy at least \( c \cdot d \) inequalities.

3 Hardness result for SSAT

We now prove the hardness of SSAT (Theorem 1.4) and SSAT\(_\infty\) (Theorem 1.5). In Section 3.1 we reduce Label Cover (LC) to SSAT and then appropriately adopt this reduction for SSAT\(_\infty\) in Section 3.2.

Let \( G' = (G' = (A', B', E'), \Sigma_A, \Sigma_B, \Pi') \) be an LC instance obtained after applying the Projection...
Games Conjecture (Conjecture 1.3). Let it has size $N'$, soundness $s \geq 1/N'^\beta$ (for some $\beta > 0$) and alphabet size $\text{poly}(1/s)$. We assume without loss of generality that the LC instance is bi-regular [MR08, Mos12]. That is, every $A$ vertex has the same degree $D_L$ (which we call the left degree or $A$-degree) and every $B$ vertex has the same degree $D_R$ (which we call the right degree or $B$-degree).

We use the following lemma from [Mos12] which relates the soundness error to agreement soundness error.

**Proposition 3.1** ([Mos12] (re-phrased)). Let $D_B \geq 2$ be a prime power and let $D_R$ be a power of $D_B$. Let $\epsilon > 0$. From an LC instance with soundness error $\epsilon^2 D_B^2$ and $B$-degree $D_R$, we can construct an LC instance with agreement soundness error $2\epsilon D_B^2$ and $B$-degree $D_B$. The transformation preserves the alphabets. The size is raised to a constant power.

Thus after applying this lemma, we can assume we have an LC instance $G = (G = (A, B, E), \Sigma_A, \Sigma_B, \Pi)$ with size $N = N'^\gamma$ (for some constant $\gamma > 0$), right degree $D_B$ (constant prime power), left degree $D_A$ and agreement soundness error $s_{agr} = \frac{2\epsilon}{D_B}$, where $\epsilon = \frac{1}{D_B}$. Expressing in terms of $N$, we can write $s \geq 1/N^c$ for some $c > 0$. Let for each edge $e \in E$, $\pi_e$ is a $p \rightarrow 1$ projection where $p \leq |\Sigma_A|$.

Here we note that our proof works even without the bi-regularity condition, but what is crucial is the fact that degree of each $B$-vertex is bounded by some large enough constant. However, for convenience, we assume we have a bi-regular graph.

### 3.1 Reduction from LC to SSAT

We reduce the above LC instance $G$ to a SSAT instance $I = (V, \Psi, \mathcal{R}_\Psi)$ as follows. To each $A$-vertex $a$ we associate a variable $a$, i.e. $|V| = |A|$. To each $B$-vertex $b$ we associate a test $\psi_b$, i.e. $|\Psi| = |B|$. The variables in a test $\psi_b$ are the neighbors of $b$ in $A$. Thus each test has $D_B$ variables and each variable appears in $D_A$ tests.

**Values of variables**: Without much loss of generality we assume that $\Sigma_A$ is in bijective correspondence with some field $\mathcal{F}$, which is the range of the variables in $V$. We use the letters $x$ and $y$ (with subscript and superscript as required) for the elements of $\Sigma_A$ (or $\mathcal{F}$) and $\Sigma_B$ respectively.

**Satisfying assignments for tests**: Consider a $\psi_b \in \Psi$. For each label $y \in \Sigma_B$ such that it has at least one pre-image in each of $b$’s neighbors in $A$, consider the following tuples:

$\mathcal{R}_y(\psi_b) = \{(x_1, \ldots, x_{D_B}) : x_j \in \pi_e^{-1}(y) \text{ where } e = (a_j, b) \text{ and } a_j \text{ is the } j^{th} \text{ neighbor of } b\}$

Thus the total set of satisfying assignments for $\psi_b$ is : $\mathcal{R}(\psi_b) = \bigcup_{y \in \Sigma_B} \mathcal{R}_y(\psi_b)$. And cardinality of this set is at most $|\Sigma_B|p^{D_B}$, which is polynomially bounded by $|V|$.
Completeness

Lemma 3.1. If there exists a labeling that satisfies all the edges in $G$ then there exists a consistent natural super-assignment $A : V \rightarrow F$ satisfying all the tests in $\Psi$.

Proof. Suppose there is a labeling $\phi_A : A \rightarrow \Sigma_A$, and $\phi_B : B \rightarrow \Sigma_B$ that satisfies all edges. Then for any $b \in B$, let $r_b := (\phi_A(a_1), \ldots, \phi_A(a_{D_B}))$, where $a_1, \ldots, a_{D_B}$ are neighbors of $b$. Note that since the labeling $(\phi_A, \phi_B)$ satisfies all edges, we have that $r_b \in R_{\phi_B(b)}(\psi_b)$, and hence $r_b \in R(\psi_b)$.

Consider the super-assignment for the resulting SSAT instance that sets for all $b \in B$, $S(\psi_b)[r_b] = 1$ and $S(\psi_b)[r'] = 0$ for all $r' \neq r_b$. This assignment is natural by definition and it is consistent since for each $a \in V$ and for each neighbor $b$ of $a$, we have that $\pi_a(S(\psi_b))[\phi_A(a)] = 1$ and $\pi_a(S(\psi_b))[\alpha] = 1$ for any $\alpha \neq \phi_A(a)$. Thus we get a consistent natural super-assignment.

Soundness

We can make some observations about the structure of the SSAT instance we constructed. These are not essential for our soundness proof, so we have moved these to Appendix A and only state the following result.

Lemma 3.2 (Corollary A.1 in Appendix A). For each test with non-zero norm, in the set of non-zero weighted assignments either there exists at least one assignment such that it has at least two variables with assigned values or all its assignments have exactly one variable with assigned value.

Note that the non-triviality condition does not guarantee the existence of at least one variable with assigned value in “each” assignment.

Lemma 3.3. Let $D_B$ be a constant prime power such that $N$ is a power of $D_B$ and $0 < c < 1$. Let $s \geq 1/N^c$ and $\ell' < \frac{1}{\sqrt{2D_B s^2}}$, where $d < 1/4$. Assume $s_{\text{list}} = \sqrt{s} D_B \ell'^2$.

If $G$ has list agreement soundness error $(\ell, s_{\text{list}})$ then every non-trivial consistent super-assignment for $I$ has norm at least $g = N^{c'}(1 - 2s_{\text{list}})$, where $c' \leq d$ and the expected value of $\ell$ is at most $2N^{c'}(1 - s_{\text{list}})$. Thus with high probability $\ell$ remains bounded by $\ell'$.

Proof. We prove the contrapositive. Let $I$ has a non-trivial consistent super-assignment $A$ of norm at most $g$. Then we prove that there exists a labeling such that for at least $s_{\text{list}}$ fraction of $B$-vertices, the $A$-vertices do not totally disagree.

If the average norm is at most $g$ then by Markov’ inequality there exists at least $s_{\text{list}}$ fraction of tests for which the norm, and hence the number of non-zero weight satisfying assignments, is at most $g_1 = N^{c'}(1 - s_{\text{list}})$. Let us denote this set of tests by $\Psi'(\subseteq \Psi)$. For each assignment
(which is a tuple of values), if a value is assigned to the corresponding variable then we call the corresponding co-ordinate “good”. Since $A$ is non-trivial each test must have at least one non-zero weight assignment with at least one good co-ordinate.

We now define an assignment $\hat{\phi}_A : A \rightarrow (\Sigma_\ell^1)$ to the $A$-vertices. For convenience, let us call this the procedure of List-Construction.

1. For each variable we include all its assigned values in its list.

2. For any test $\psi \in \Psi'$ if there are no assignment with at least two good co-ordinates, then we consider any one assignment with one good co-ordinate (existence of such an assignment is guaranteed by the non-triviality condition) and include each of the non-assigned values with probability $p = \min\{1, \frac{g_1}{D_A}\}$ in the lists of the corresponding variables.

(Lemma 3.2 exerts that in such a case each assignment will have exactly one good co-ordinate.)

**Expected list size** Consider a variable $a$ (or corresponding vertex) and let $N_a(\leq g_1)$ is the number of values assigned to it. Now $a$ can appear in $D_A$ tests, in each it can have at most $g_1 - N_a$ non-assigned values. By step (2) of List-Construction, from each test one such non-assigned value can be in the list with probability $p$. If $L_a$ is the variable for the size of the list of labels for $a$, then

$$\mathbb{E}[L_a] \leq N_a + D_A \cdot \frac{g_1}{D_A} \leq 2g_1$$

This is true for every variable. Thus expected list size ($\ell$) is at most $2g_1$.

Now consider any test $\psi \in \Psi'$ with $R_{\neq 0}(\psi) = \{r_1, r_2, \ldots, r_{g'}\}$ as the set of non-zero weighted assignments, where $g' \leq g_1$. Let $P_0(\psi)$ is the probability that for each $r \in R_{\neq 0}(\psi)$, not a single value appears in the list of its corresponding variable. By Lemma 3.2 we can say $P_0(\psi) = 0$ by Step 1 of List-Construction. (We can come to similar conclusion simply by noting the non-triviality condition, without using Lemma 3.2.) Let $P_{\geq 2}(\psi)$ is the probability of the event that for any $r \in R_{\neq 0}(\psi)$ there exists at least two values that appear in the list of their variables. Now there can be two cases.

In Case (i) there exists at least one $r \in R_{\neq 0}(\psi)$ such that it has at least two good co-ordinates. Then $P_{\geq 2}(\psi) = 1$, by Step 1 of List-Construction.

In Case (ii) for each $r \in R_{\neq 0}(\psi)$ there is at most one good co-ordinate. Then by Step 2 of List-Construction we know there is an $r \in R_{\neq 0}(\psi)$ from which we took some non-assigned values. Probability that only one value (assigned value) of $r$ is in the list of its corresponding variable is $P_1(r) = \prod_a (1 - p) \leq (1 - \frac{g_1}{D_A})^{D_B - 1} \approx (1 - \frac{g_1}{D_A})^{D_B}$, where the product is over all the variables with non-assigned values in $r$. Thus $P_{\geq 2}(\psi) \geq 1 - (1 - \frac{g_1}{D_A})^{D_B} > \frac{g_1}{D_A}$.

Let $P_{\geq 2}(\Psi')$ is the probability that for each test $\psi \in \Psi'$ there exists at least one assignment such that at least two of its values appear in the list of their corresponding variable. Let $P_{\geq 2}(\psi_1|\psi_1, \ldots, \psi_{i-1})$ is the probability that such an event happens for $\psi_i \in \Psi'$ after conditioning
on the fact that this happened for the previous tests considered in $\Psi'$, i.e. $\psi_1, \ldots, \psi_{i-1}$. Since the above two cases remain true for each event, even after conditioning, so

$$P_{\geq 2}(\Psi') = \prod_{\psi_i \in \Psi'} P_{\geq 2}(\psi_i | \psi_1, \ldots, \psi_{i-1}) \geq \left( \frac{g_1}{D_A} \right)^{s_{\text{list}} | B |} > 0.$$  

Thus there exists a labeling of the $A$-vertices (variables) such that for every $b \in \Psi'$ (tests), we have at least two vertices $a_1, a_2$ in $A$ such that $\sigma_1 \in \widehat{\phi}_A(a_1)$, $\sigma_2 \in \widehat{\phi}_A(a_2)$ and $\pi_{e_1}(\sigma_1) = \pi_{e_2}(\sigma_2)$, where $e_1 = (a_1, b), e_2 = (a_2, b) \in E$, i.e. both the vertices agree (values appear in the same assignment). Hence we do not have an agreement soundness of $(\ell, s_{\text{list}})$.

Thus we prove Theorem 1.4: Assuming the Projection Games Conjecture SSAT is NP-hard for $g = n^c$ for some constant $c > 0$.

### 3.2 Reduction from LC to SSAT$_\infty$

Given an LC instance $\mathcal{G}$ we construct an SSAT$_\infty$ instance $\mathcal{I}$, as done in Section 3.1. The completeness proof is also similar to Lemma 3.3, so we do not repeat it again here.

#### Soundness

We need to show that a “NO” instance of LC, i.e. when it has a list-agreement soundness of $(\ell, s_{\text{list}})$ (corresponding to unsatisfiable SAT), maps to a “NO” instance of SSAT$_\infty$, i.e. when every consistent not-all-zero super-assignment has norm at least $g$. Note that when the norm is between 1 and $g$ then any outcome (YES/NO) is fine.

If we assume that the constructed SSAT$_\infty$ instance has a consistent non-trivial solution of norm at most $g$, then similar to Lemma 3.3 we can prove that there exists a labeling such that the given LC instance does not have a list agreement soundness error $(\ell, s_{\text{list}})$. The only difference is that since here $\ell_\infty$ norm is at most $g$, so norm of every test remains bounded by $g$. Note that a non-trivial solution is also not-all-zero.

**Lemma 3.4.** Let $D_B$ is a constant prime power such that $N$ is a power of $D_B$ and $0 < c < 1$. Let $s \geq 1/N^c$ and $\ell' < \frac{1}{\sqrt{2D_B}s^2}$, where $d < 1/4$. Assume $s_{\text{list}} = \sqrt{s}D_B\ell'^2$

If $\mathcal{I}$ has a consistent non-trivial super-assignment of $\ell_\infty$ norm at most $g = N^{c'}$ then $\mathcal{G}$ has list agreement soundness error $(\ell, s_{\text{list}})$. Here $c' \leq d$ and the expected value of $\ell$ is at most $2N^{c'}$. Thus with high probability $\ell$ remains bounded by $\ell'$.

Now suppose the constructed SSAT$_\infty$ instance of norm at most $g$ does not have a non-trivial solution, i.e. every variable does not have at least one value assigned, but the solution is consistent
and not-all-zero, i.e. some variables do not have any assigned value. Let \( \Psi' \subseteq \Psi \) is the set of tests with non-zero norm and \( V' \subset V \) is the set of variables with no assigned values. Note when a value has projection weight 0, it can be due to the weights of the super-assignments (and how they cancelled) or because the value never appeared in any super-assignment (e.g. if the projection functions are partial functions). Thus in the \( \ell_\infty \) case even if consistency is maintained, for variables with no assigned values, we cannot say that one particular value appears in all its tests. Then we can change the List-Construction procedure of Lemma 3.3 as follows. Here it might be convenient to note Lemma 3.2 which remains true even in the \( \ell_\infty \) norm.

1. For each variable we include all its assigned values in its list.

2. For any test in \( \psi \in \Psi' \) if each of its non-zero assignment has exactly one good co-ordinate, then we consider any one such assignment and include each of the non-assigned values with probability \( p = \min \{ 1, \frac{g}{D_A} \} \) in the list of the corresponding variables. Once values have been included we consider the tests as “marked”.

3. If \( \Psi \setminus \Psi' \neq \emptyset \) then we do the following sequentially. Consider any variable in \( V' \) (say \( a_1 \)) and let \( \Psi'_{a_1} \subseteq \Psi' \) are the tests in which it appear. We fix any one assignment in one of them and include its value in the list. We include the values of the other variables with probability \( p \). We consider this test “marked” and go to the next one in \( \Psi'_{a_1} \). If this same value appears here we fix any assignment with that value or we fix some other assignment and repeat the same process. This continues till we have taken at most \( g \) values of \( a_1 \) (we call these “marked values”). Let \( \Psi'' \subset \Psi \setminus \Psi' \) is the set of tests marked in this way. We repeat this procedure again with another variable in \( V' \) and the remaining “unmarked” tests. This continues till we exhaust all variables in \( V' \).

Note the expected list size for variables with assigned values remain at most \( 2g \), similar to Lemma 3.3. For each variable \( a_1 \in V' \) there can be at most \( g \) “marked” values. In each of the \( D_A \) tests that \( a_1 \) appears one value (apart from the marked one) can be in the list with probability \( p \). So expected list size is at most \( g + D_A \cdot \frac{g}{D_A} = 2g \).

In Step (3) of List-Construction we see that each variable \( a_1 \in V' \) can mark at least \( g \) tests in \( \Psi' \). Each variable with assigned values can mark \( D_A \) tests each. By the condition of not-all-zero there is at least one variable with assigned values. Thus at least \( g|A| \) tests can be marked. So \( s_{\text{list}} = \frac{g|A|}{|B|} = \frac{gD_B}{D_A} \). Next, we can argue in the same way as in Lemma 3.3 that for at least \( s_{\text{list}} \) fraction of the \( B \)-vertices there will not be total disagreement.

Now, one crucial thing here is, for the parameters to make sense, \( gD_A \approx \frac{1}{\sqrt{s}} \). Or, the projection functions are such that more tests can be marked in Step (3). For example if they are total functions. Or they are partial functions but for most variables we can find a set of values of cardinality at most \( g \) such that for at least \( s_{\text{list}} \) fraction of the tests (with that variable) there exists at least one pre-image in that set.

Thus we can state the following lemma.

**Lemma 3.5.** Let \( D_B \) is a constant prime power such that \( N \) is a power of \( D_B \) and \( 0 < c < 1 \). Let \( s \geq 1/N^c \) and \( \ell' < \frac{1}{\sqrt{2D_B s}} \), where \( d < 1/4 \). Assume \( s_{\text{list}} = \sqrt{sD_B\ell'^2} \).
If $I$ has a consistent not-all-zero (but not non-trivial) super-assignment of $\ell_\infty$ norm at most $g = N^{c'}$ then $G$ has list agreement soundness error $(\ell, s_{\text{list}})$ if either

(i) $gDA \approx \frac{1}{\sqrt{s}}$; OR  
(ii) For most variables we can find a set of values of cardinality at most $g$ such that for at least $s_{\text{list}}$ fraction of tests (with that variable) there exists at least one pre-image in that set.

Here $c' \leq d$ and the expected value of $\ell$ is at most $2N^{c'}$. Thus with high probability $\ell$ remains bounded by $\ell'$.

This proves Theorem 1.5: Assuming the Projection Games Conjecture SSAT is NP-hard for $g = n^c$ for some constant $c > 0$.

4 Applications : Reduction from SSAT to other problems

4.1 Complexity of lattice problems

Dinur et.al. [DKRS03] reduced $g - \text{SSAT}$ to $g - \text{SIS}$ and $g - \text{CVP}$. In a separate paper Dinur [Din02] reduced $g - \text{SSAT}_\infty$ to $g - \text{SVP}_\infty$. Thus using Theorem 1.4 and 1.5 we have the following corollaries.

**Corollary 4.1.** Assuming the Projection Games Conjecture SIS is NP-hard to approximate within a factor $g = N^c$, for some constant $c > 0$.

**Corollary 4.2.** Assuming the Projection Games Conjecture CVP on $N$-dimensional lattice is NP-hard to approximate within a factor $g = N^c$, for some constant $c > 0$.

**Corollary 4.3.** Assuming the Projection Games Conjecture SVP\(_\infty\) on $N$-dimensional lattice is NP-hard to approximate within a factor $g = N^c$, for some constant $c > 0$.

4.2 Complexity of Nearest Codeword Problem (NCP)

Here we give an approximation factor preserving reduction from SSAT to NCP and prove the following.

**Theorem 4.1.** Approximating NCP within factor $n^{c/\log\log n}$ is NP-hard.

**Proof.** Let $I = \langle \Psi = \{\psi_1, \ldots, \psi_n\}, V = \{v_1, \ldots, v_m\}, \{R_{\psi_1}, \ldots, R_{\psi_n}\} \rangle$ be an SSAT instance. From this we efficiently construct a matrix $\vec{B}'$ (SIS matrix) and a target vector $\vec{t}'$, as given in [DKRS03]. Here we give a brief description of this matrix and the relevant facts that we require. For completeness we give the reduction from $g - \text{SSAT}$ to $g - \text{SIS}$ and prove these facts in Appendix B.

$\vec{B}'$ is a $0-1$ matrix and its dimension is $n' \times m'$ where $n', m' \in \text{poly}(n)$. $\vec{t}'$ is an all-1 column vector of length $n'$. If the SSAT instance has a natural consistent super-assignment (YES) then
there exists a coefficient vector $\overrightarrow{z}$ such that $B'\overrightarrow{z} = \overrightarrow{t}$ and $\|\overrightarrow{z}\|_1 = n$. If every non-trivial consistent super-assignment of the SSAT instance has norm greater than $g$ (NO) then for all coefficient vectors $\overrightarrow{z}$ with $B'\overrightarrow{z} = \overrightarrow{t}$ we have $\|\overrightarrow{z}\|_1 > gn$.

Let $D > gn$ is some integer. We construct the matrix $\overrightarrow{A} \in \{0, 1\}^{n'D + m' \times m'}$ for NCP as follows: We can think of $\overrightarrow{A}$ as consisting of two parts - the first $n'D$ rows in the upper part ($\overrightarrow{A}'$, say) and the rest $m'$ rows in the lower part. In the upper part $\overrightarrow{A}[ik, j] = B'[i, j]$ for $1 \leq i \leq n', 1 \leq k \leq D, 1 \leq j \leq m'$. The lower part is the identity matrix $\overrightarrow{I}_m' \times m'$.

The target vector $\overrightarrow{t}$ is as follows : $t_{ik} = 1$ for $1 \leq i \leq n', 1 \leq k \leq D$. Let us call these rows as $\overrightarrow{t}'$. The rest of $\overrightarrow{t}$ is 0.

Let $q$ be a prime number greater than $g$ times $\max\{n', m'\}$. The NCP instance obtained by reduction is $(\overrightarrow{A}, \overrightarrow{t}', n)$ over $\mathbb{F}_q$. For any coefficient vector $\overrightarrow{z}$ we note that $\|A\overrightarrow{z} - \overrightarrow{t}\| = \|A'\overrightarrow{z} - \overrightarrow{t}'\| + \|\overrightarrow{z}\|$.

**Completeness**: There exists $\overrightarrow{z}$ such that $B'\overrightarrow{z} = \overrightarrow{t}$ and $\|\overrightarrow{z}\|_1 = n$. This implies $\|A\overrightarrow{z} - \overrightarrow{t}\| = \|\overrightarrow{z}\|_1 = n$ and NCP oracle outputs YES.

**Soundness**: For all coefficient vectors $\overrightarrow{z}$ where $B'\overrightarrow{z} = \overrightarrow{t}$ we have $\|\overrightarrow{z}\|_1 > gn$. In the NCP instance if $A'\overrightarrow{z} - \overrightarrow{t}' \neq \overrightarrow{0}$ then it implies that at least $D$ co-ordinates are non-zero, else it implies $B'\overrightarrow{z} = \overrightarrow{t}'$. In both the cases $\|A\overrightarrow{z} - \overrightarrow{t}\| > gn$ and the NCP oracle outputs NO.

We can get an improved hardness of approximation factor using Theorem 4.4

**Corollary 4.4.** Assuming the Projection Games Conjecture, approximation of NCP upto factors $n^c$ is NP-hard, for some constant $c > 0$.

### 4.3 Complexity of Learning Halfspaces Problem (LHP)

**Theorem 4.2.** Approximating the minimum failure ratio of Learning Halfspace Problem (LHP) within factor $n^{c/\log \log n}$ is NP-hard.

**Proof.** From an SSAT instance $\mathcal{I} = (\mathcal{P} = \{\psi_1, \ldots, \psi_n\}, V = \{v_1, \ldots, v_m\}, \{\mathcal{R}_{\psi_1}, \ldots, \mathcal{R}_{\psi_n}\})$, we construct an SIS instance $(B', \overrightarrow{t}, D)$ as in [DKRS03] and derive inequalities, somewhat similar to [ABSS97]. A brief description of the SIS-matrix $B'$ and the relevant facts for this proof have been given in Theorem 4.1. More details can be found in Appendix B. Observe that an SIS instance $(B', \overrightarrow{t}, d)$ can be viewed as a set of linear equations $B'\overrightarrow{z} - \overrightarrow{t} = \overrightarrow{0}$ for some variable vector $\overrightarrow{z}$. Note that while learning a halfplane $(\overrightarrow{a}, \overrightarrow{z}) = b$, the unknown variables are $\overrightarrow{a}$ and $b$. So it is a homogeneous system and the coefficient of one of the variables (here $b$) is always $\pm 1$. We use a standard technique for homogenization of the SIS equations- use a new variable $y$ and replace every constant $c$ by $cy$.

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Let $U = n^{c/\log\log n}|\Psi|$ and $d = |\Psi|$. Following are the linear inequalitites for LHP instance.

1. We make $U$ copies of the linear inequality: $\frac{-y}{U} < \delta < \frac{y}{U}$.

2. For each equation from SIS of the form $\sum_{i=1}^{n} a_i x_i = c$, we make $U$ copies of each of the following two inequalities:
   \[ \sum_{i=1}^{n} a_i x_i - cy + \delta > 0, \quad \sum_{i=1}^{n} a_i x_i - cy - \delta < 0 \]

3. For every variable make $U$ copies of each of the following:
   \[ x_i - 2y < 0, \quad x_i + 2y > 0 \]

4. For every variable make a copy of the following:
   \[ x_i + \delta > 0, \quad x_i - \delta < 0 \]

5. Make $U$ copies of the inequality: $y > 0$

**Completeness**: There exists a \{0,1\} coefficient vector (variables $\vec{x}$) with norm $d$ that satisfies the linear equations. In the LHP instance assign the same values to the $x_i$ variables and put $y = 1$ and $\delta$ a very small number tending to 0. This will satisfy all the inequalities of type 1, 2, and 3. Among inequalities of type 4 only when $x_i = 1$ the second inequality will not be satisfied. Thus the number of unsatisfied linear inequalities are $d$.

**Soundness**: We give a contrapositive argument. Suppose there exists an assignment such that less than $U$ inequalities are unsatisfied. So the assignment must satisfy all inequalities of type 1, 2, and 3. Now in the SIS instance simply assign the variables the corresponding values divided by value of $y$. This implies the variables in SIS get values from \{-1,0,1\} and satisfies all the SIS equations. Hence we have a solution with norm less than $U$ for SIS. This would imply the existence of a non-trivial consistent super-assignment of norm at most $g$ for the SSAT instance.

\[ \square \]

Using Theorem 1.4 we get the following corollary.

**Corollary 4.5.** Assuming the Projection Games Conjecture, approximating the minimum failure ratio of LHP upto factors $n^{c}$ is NP-hard, for some constant $c > 0$.

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References

[ABSS97] Sanjeev Arora, László Babai, Jacques Stern, and Z Sweedyk. The hardness of approximate optima in lattices, codes, and systems of linear equations. *Journal of Computer and System Sciences*, 54(2):317–331, 1997.

[Ajt96] Miklós Ajtai. Generating hard instances of lattice problems. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 99–108. ACM, 1996.

[Ajt98] Miklós Ajtai. The shortest vector problem in $l_2$ is np-hard for randomized reductions. In *Proceedings of the thirtieth annual ACM symposium on Theory of computing*, pages 10–19. ACM, 1998.

[ALM+98] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. Proof verification and the hardness of approximation problems. *Journal of the ACM (JACM)*, 45(3):501–555, 1998.

[AR05] Dorit Aharonov and Oded Regev. Lattice problems in np ∩ comp. *Journal of the ACM (JACM)*, 52(5):749–765, 2005.

[AS98] Sanjeev Arora and Shmuel Safra. Probabilistic checking of proofs: A new characterization of np. *Journal of the ACM (JACM)*, 45(1):70–122, 1998.

[ASD18] Divesh Aggarwal and Noah Stephens-Davidowitz. (gap/s) eth hardness of svp. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 228–238. ACM, 2018.

[Ban93] Wojciech Banaszczyk. New bounds in some transference theorems in the geometry of numbers. *Mathematische Annalen*, 296(1):625–635, 1993.

[BGSD17] Huck Bennett, Alexander Golovnev, and Noah Stephens-Davidowitz. On the quantitative hardness of cvp. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 13–24. IEEE, 2017.

[BLP+13] Zvika Brakerski, Adeline Langlois, Chris Peikert, Oded Regev, and Damien Stehlé. Classical hardness of learning with errors. In *STOC*, pages 575–584, 2013.

[BV14] Zvika Brakerski and Vinod Vaikuntanathan. Lattice-based FHE as secure as PKE. In *ITCS*, pages 1–12, 2014.

[CJL+92] Matthijs J Coster, Antoine Joux, Brian A LaMacchia, Andrew M Odlyzko, Claus-Peter Schnorr, and Jacques Stern. Improved low-density subset sum algorithms. *computational complexity*, 2(2):111–128, 1992.

[CN98] Jin-Yi Cai and Ajay Nerurkar. Approximating the svp to within a factor $(1 + 1/dim^ε)$ is np-hard under randomized conditions. In *Proceedings. Thirteenth Annual IEEE Conference on Computational Complexity (Formerly: Structure in Complexity Theory Conference) (Cat. No. 98CB36247)*, pages 46–55. IEEE, 1998.
Irit Dinur. Approximating svp to within almost-almost-polynomial factors is np-hard. *Theoretical Computer Science*, 285(1):55–71, 2002.

Irit Dinur, Guy Kindler, Ran Raz, and Shmuel Safra. Approximating cvp to within almost-almost-polynomial factors is np-hard. *Combinatorica*, 23(2):205–243, 2003.

Léo Ducas, Tancrède Lepoint, Vadim Lyubashevsky, Peter Schwabe, Gregor Seiler, and Damien Stehlé. Crystals–dilithium: Digital signatures from module lattices. Technical report, IACR Cryptology ePrint Archive, 2017: 633, 2017.

Ilya Dumer, Daniele Micciancio, and Madhu Sudan. Hardness of approximating the minimum distance of a linear code. *IEEE Transactions on Information Theory*, 49(1):22–37, 2003.

Friedrich Eisenbrand, Nicolai Hähnle, and Martin Niemeier. Covering cubes and the closest vector problem. In *Proceedings of the twenty-seventh annual symposium on Computational geometry*, pages 417–423. ACM, 2011.

Uriel Feige. A threshold of ln n for approximating set cover. *Journal of the ACM (JACM)*, 45(4):634–652, 1998.

Craig Gentry. Fully homomorphic encryption using ideal lattices. In *STOC’09—Proceedings of the 2009 ACM International Symposium on Theory of Computing*, pages 169–178. ACM, New York, 2009.

Oded Goldreich and Shafi Goldwasser. On the limits of nonapproximability of lattice problems. *Journal of Computer and System Sciences*, 60(3):540–563, 2000.

Johan Hastad. Dual vectors and lower bounds for the nearest lattice point problem. *Combinatorica*, 8(1):75–81, 1988.

Ishay Haviv and Oded Regev. Tensor-based hardness of the shortest vector problem to within almost polynomial factors. *Theory of Computing*, 8(23):513–531, 2012. Preliminary version in STOC’07.

Russell Impagliazzo and Ramamohan Paturi. Complexity of k-sat. In *Computational Complexity, 1999. Proceedings. Fourteenth Annual IEEE Conference on*, pages 237–240. IEEE, 1999.

Subhash Khot. Hardness of approximating the shortest vector problem in lattices. *Journal of the ACM*, 52(5):789–808, September 2005. Preliminary version in FOCS’04.

Hendrik W Lenstra Jr. Integer programming with a fixed number of variables. *Mathematics of operations research*, 8(4):538–548, 1983.

A. K. Lenstra, H. W. Lenstra, Jr., and L. Lovász. Factoring polynomials with rational coefficients. *Math. Ann.*, 261(4):515–534, 1982.

Jeffrey C Lagarias, Hendrik W Lenstra, and Claus-Peter Schnorr. Korkin-zolotarev bases and successive minima of a lattice and its reciprocal lattice. *Combinatorica*, 10(4):333–348, 1990.
Here we prove some results about the SSAT instance obtained in Section 3.1 by a reduction from an LC instance. We give a quick recap of some relevant facts.

We are given an LC instance $\mathcal{G} = (G = (A,B,E), \Sigma_A, \Sigma_B, \Pi)$ with size $N$, right degree $D_B$ (constant prime power), left degree $D_A$. For each edge $e \in E$, $\pi_e$ is a $p-$to$-1$ projection where $p \leq |\Sigma_A|$. 

We reduce $\mathcal{G}$ to a SSAT instance $\mathcal{L} = (V, \Psi, \mathcal{R}_\Psi)$ as follows. To each $A$-vertex $a$, we associate a variable $a$, i.e. $|V| = |A|$. To each $B$-vertex $b$ we associate a test $\psi_b$, i.e. $|\Psi| = |B|$. The variables in a test $\psi_b$ are the neighbors of $b$ in $A$.

**Values of variables**: Without much loss of generality we assume that the variables take values from a field $\mathcal{F}$, which is in bijective correspondence to $\Sigma_A$. We use the letters $x$ and $y$ (with subscript and superscript as required) for the elements of $\Sigma_A$ (or $\mathcal{F}$) and $\Sigma_B$ respectively.
**Satisfying assignments for tests:** Consider a \( \psi_b \in \Psi \). For each label \( y \in \Sigma_B \) such that it has at least one pre-image in each of \( b \)'s neighbors in \( A \), consider the following tuples:

\[
R_y(\psi_b) = \left\{(x_1, \ldots, x_{DB}) : x_j \in \pi_e^{-1}(y) \text{ where } e = (a_j, b) \text{ and } a_j \text{ is the } j^{th} \text{ neighbor of } b\right\}
\]

Thus the total set of satisfying assignments for \( \psi_b \) is:

\[
\mathcal{R}(\psi_b) = \bigcup_{y \in \Sigma_B} R_y(\psi_b).
\]

And cardinality of this set is at most \(|\Sigma_B|p^{DB}\), which is polynomially bounded by \(|V|\).

Consider a test \( \psi \in \Psi \) consisting of the variables \( a_1, a_2, \ldots, a_{DB} \). We can partition its set of satisfying assignments \( \mathcal{R}(\psi) \) as follows:

For each \( y \in \Sigma_B \), \( R_y(\psi) \) can be viewed as a \( DB \)-dimensional array \( \overrightarrow{M}_y \) where \( i^{th} \) dimension or co-ordinate corresponds to variable \( a_i \). The number of values of \( a_i \) in its dimension is the number of pre-images of \( y \). Thus there can be at most \( p \) values in each dimension. With a slight abuse of notation, we denote array elements by \( M^y_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{DB}} \), where \( x_j \in \pi_e^{-1}(y) \), \( e = (a_j, b) \) and \( a_j \) is the \( j^{th} \) neighbor of \( b \). This is the weight of the corresponding assignment \((x_1, \ldots, x_{DB})\). Thus the projection of a super-assignment \( \overrightarrow{S}(\psi) \) on a variable \( a_j \) for any pre-image of \( y \) is:

\[
\forall x \in \pi_e^{-1}(y) : \quad \pi_{a_j}(\overrightarrow{S}(\psi))[x] = \sum_{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{DB}} M^y_{x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{DB}}
\]

where the summation is over all the pre-images of \( y \) for each variable \( a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{DB} \).

Note, since each edge constraint are functions, so the values for each variable that appear in one array cannot come in the other. In this case the arrays can be considered as “disjoint” or “non-interfering”. While taking projection on a variable for any value, it is sufficient to consider only the array in which this value appears, and not the other arrays.

We can build these arrays in polynomial time and for convenience we can think of giving these arrays as input to the SSAT oracle, so that it can fill the arrays with weights according to some well-defined criteria. A co-ordinate of an array is good (with respect to it) if at least one of the values is assigned to its corresponding variable, else it is bad.

We define the norm of each array \( \overrightarrow{M}_y \) as

\[
\|\overrightarrow{M}_y\| = \sum_{x_1, \ldots, x_{DB}} |M_{x_1, \ldots, x_{DB}}|
\]

The norm of a super-assignment \( \overrightarrow{S}(\psi) \) is:

\[
\|\overrightarrow{S}(\psi)\| = \sum_{y \in \Sigma_B} \|\overrightarrow{M}_y\|
\]

(If the reduction is to SSAT∞ then \( \|\overrightarrow{S}(\psi)\|_\infty = \max_{y \in \Sigma_B} \|\overrightarrow{M}_y\| \).)

**Claim A.1.** With a consistent super-assignment if a test has an array with non-zero norm but all bad co-ordinates, then we can always find a consistent super-assignment with a lesser norm.
Proof. This follows from the “disjoint”-ness of the arrays, as has been explained above. Since the weights cannot cancel from entries in other arrays, we can put all-zero weight in this array (with all bad co-ordinates). We get a super-assignment of smaller norm, but consistency is maintained. □

Claim A.2. In any array if any variable has all non-assigned values, i.e. any co-ordinate is bad, then the total sum of all the entries is zero.

Proof. Let in array $\tilde{M}^y$ the first variable $a_1$ has all non-assigned values. Thus for each value $x$ of $a_1$

$$\pi_{a_1}[x] = \sum_{x_2,\ldots,x_{DB}} M^y_{x_1,x_2,\ldots,x_{DB}} = 0.$$ 

The sum of all array entries is

$$\sum_{x_1,\ldots,x_{DB}} M^y_{x_1,\ldots,x_{DB}} = \sum_{x_1} \sum_{x_2,\ldots,x_{DB}} M^y_{x_1,x_2,\ldots,x_{DB}} = 0.$$ 

Claim A.3. In an array with not-all bad co-ordinates, there can be either at least two good co-ordinates or one good co-ordinate with at least two values as signed to the corresponding variable.

If there is only one good co-ordinate then a consistent super-assignment can be made with a constant (one should suffice) number of non-assigned values in the other co-ordinates.

Proof. If possible let there is only one good co-ordinate corresponding to variable $a_1$ (say) and it has only one assigned value, say $x$. Thus $\sum_{x_2,\ldots,x_{DB}} M_{x_1,x_2,\ldots,x_{DB}} = w \neq 0$ and $\sum_{x_2,\ldots,x_{DB}} M_{x',x_2,\ldots,x_{DB}} = 0$, for all $x' \neq x$ (which appear in this matrix).

Hence $\sum_{x_1,\ldots,x_{DB}} M_{x_1,\ldots,x_{DB}} = w \neq 0$, which cannot be true by Claim A.2

For the second part of the claim, w.l.o.g. let $a_1$ is the co-ordinate with assigned values $1, 2, \ldots, x$ and corresponding projection weights $w_1, w_2, \ldots w_x$. The following conditions must hold:

$$\sum_{x_2,\ldots,x_{DB}} M_{i,x_2,\ldots,x_{DB}} = w_i, \quad i = 1, \ldots, x$$

By Claim A.2 $\sum_i w_i = 0$.

Thus we can have a super-assignment as follows: Fix $a_2, \ldots, a_{DB}$ to some value $x_2, \ldots, x_{DB}$ respectively. Assign $M_{i,x_2,\ldots,x_{DB}} = w_i$, for all $i = 1, \ldots, x$. It is easy to see that consistency is maintained and this gives the minimum norm. □

Claim A.4. If an array has more than one good co-ordinates then either (i) there exists at least one assignment with at least two assigned values or (ii) each assignment has one assigned value and there are only a constant number of non-assigned values for each variable.

Proof. Easy to see from Claim A.3 □
As a corollary of the above claims we can conclude that

**Corollary A.1.** For each test with non-zero norm, in the set of non-zero weighted assignments either there exists at least one assignment such that it has at least two variables with assigned values or all its assignments have exactly one variable with assigned value.

### B. $g\text{-SSAT}$ to $g\text{-SIS}$ reduction

In this section we give a brief outline of the approximation factor preserving reduction from $SSAT$ to $SIS$ given by Dinur et al. [DKRS03]. Given a $g\text{-SSAT}$ instance $I = (\Psi = \{\psi_1, \ldots, \psi_n\}, V = \{v_1, \ldots, v_m\}, \{R_{\psi_1}, \ldots, R_{\psi_n}\})$ we construct a $g\text{-SIS}$ instance $S = (B, \vec{t}, d)$ as follows.

The target vector $\vec{t}$ is an all-1 vector and $d = |\Psi| = n$.

The $SIS$-matrix $B$ has a column for every pair $(\psi, r)$ where $\psi \in \Psi$ is a test and $r \in R_\psi$ is a satisfying assignment for it. Thus there are $\sum_{i=1}^{n} |R_{\psi_i}|$ columns. It can be divided into two parts: the upper part consists of consistency rows to take care of consistency and the lower part consists of non-triviality rows to take care of non-triviality.

**Non-triviality rows:** There is a row for each test. In the row for $\psi$ all the columns associated with $\psi$ have 1, and all the other columns have 0. Thus there are $|\Psi|$ non-triviality rows.

**Consistency rows:** There are $|F|$ rows for each pair of tests $\psi_i$ and $\psi_j$ and common variable $x$ shared by them. If $a_{ij}$ is the number of variables shared by $\psi_i$ and $\psi_j$ then the number of consistency rows is $\sum_{i,j} a_{ij} \cdot |F|$. These rows contain a consistency-ensuring gadget and only the columns for $\psi_i$ and $\psi_j$ will have non-zero values in these rows.

The consistency-ensuring gadget for pair of tests $\psi_i$ and $\psi_j$ with common variable $x$ ensures that the super-assignments to these tests are consistent on $x$. It consists of a pair of matrices $G_1^{[F] \times |R_{\psi_i}|}$ and $G_2^{[F] \times |R_{\psi_j}|}$. The $|F|$ rows of each matrix correspond to the possible assignments for the variable $x$. The $r^{th}$ column in $G_1$ is the characteristic function of $r|_x$, i.e. it has 1 on the value of $x$ in $r$, and 0 everywhere else. For $G_2$ the $r^{th}$ column is a “sort of” negation of the characteristic function of $r'|_x$, i.e. it has 0 on the value of $x$ in $r'$ and 1 everywhere else.

**Correctness** We show that the YES instance of $g\text{-SSAT}$ maps to the YES instance of $g\text{-SIS}$.

**Lemma B.1.** If there is a consistent natural super-assignment to the $g\text{-SSAT}$ instance $I$ then there exists a solution of $\ell_1$ norm $|\Psi|$ to the $g\text{-SIS}$ instance $S$.

**Proof.** Let $S$ is a consistent natural super-assignment. We will construct a solution $\vec{z}$ to the $g\text{-SIS}$ as follows: Note each $S(\psi_i)$ is a $|R_{\psi_i}|$-length vector. $\vec{z}$ is a $\sum_{i=1}^{n} |R_{\psi_i}|$-long vector consisting of the concatenation of the vectors $S(\psi_1), S(\psi_2), \ldots, S(\psi_n)$. 

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Since $S$ is natural, it assigns a $+1$ to exactly one assignment of every test. Thus the target vector is reached in the non-triviality rows.

To show that the target vector is reached in the consistency rows, consider the $|F|$ rows belonging to a pair of tests $\psi_i$ and $\psi_j$ with common variable $x$. Let $\overrightarrow{S(\psi_i)}[r_1]$ and $\overrightarrow{S(\psi_j)}[r_2]$ are the single 1’s in $\overrightarrow{S(\psi_i)}$ and $\overrightarrow{S(\psi_j)}$ respectively. Since $S$ is consistent so $r_1|x = r_2|x$. By the construction of the gadget matrices in $\overrightarrow{B}$ we see that the sum of these two columns gives a all-1 vector. So the target vector is reached in the consistency rows as well.

$$\|\overrightarrow{z}\|_1 = \sum_{i=1}^n \|\overrightarrow{S(\psi_i)}\|_1 = |\Psi|$$, since $\|S\| = 1$.

**Soundness** We need to show that a NO instance of $g$–SSAT maps to a NO instance of $g$–SIS. Instead we give a contrapositive argument and prove the following.

**Lemma B.2.** If there exists a solution $\overrightarrow{z}$ of the $g$–SIS instance $S$ such that $\|\overrightarrow{z}\|_1 \leq g|\Psi|$, then there exists a non-trivial consistent super-assignment $S$ of norm at most $g$ for the $g$–SSAT instance $I$.

**Proof.** Given $\overrightarrow{z}$ we construct a super-assignment $S$ as follows : Note $\overrightarrow{z}$ is of length $\sum_{i=1}^n |R_{\psi_i}|$. We break $\overrightarrow{z}$ into $|\Psi|$ pieces of length $|R_{\psi_1}|, \ldots, |R_{\psi_n}|$, one for each test $\psi \in \Psi$. We obtain a super-assignment of norm $\|S\| = \frac{1}{|\Psi|}\|\overrightarrow{z}\|_1 \leq g$.

Since for each $\psi \in \Psi$, the target vector is reached in the $\psi^{th}$ row of the non-triviality rows, so

$$\sum_{r \in R_\psi} \overrightarrow{S(\psi)}[r] = 1$$ (1)

and $S$ is non-trivial.

Let $\psi_i, \psi_j \in \Psi$ with common variable $x$. Consider the $|F|$ rows that correspond to $\psi_i, \psi_j, x$. In each of these rows the sum of the vectors is 1, i.e. for any $f \in F$,

$$\sum_{r: r|x = f} \overrightarrow{S(\psi_i)}[r] + \sum_{r: r|x \neq f} \overrightarrow{S(\psi_j)}[r] = 1$$ (2)

Subtracting Eq.(1) for $\psi_j$ from Eq.(2) gives,

$$\sum_{r: r|x = f} \overrightarrow{S(\psi_i)}[r] = \sum_{r: r|x = f} \overrightarrow{S(\psi_j)}[r]$$

which implies $\pi_x(\overrightarrow{S(\psi_i)}) = \pi_x(\overrightarrow{S(\psi_j)})$.

Thus we have a consistent non-trivial super-assignment of norm at most $g$. 

\[\square\]