\(\mu\tau\) symmetry, tribimaximal mixing and four zero neutrino Yukawa textures

Biswajit Adhikary\(^a,\,^{b,*}\), Ambar Ghosal\(^a,\uparrow\) and Proibir Roy\(^a,\downarrow\)

a) Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata 700064, India
b) Department of Physics, Gurudas College, Narkeldanga, Kolkata-700054, India

Abstract

Within the type-I seesaw framework with three heavy right chiral neutrinos and in the basis where the latter and the charged leptons are mass diagonal, a near \(\mu\tau\) symmetry in the neutrino sector is strongly suggested by the neutrino oscillation data. There is further evidence for a close to the tribimaximal mixing pattern which subsumes \(\mu\tau\) symmetry. On the other hand, the assumption of a (maximally allowed) four zero texture in the Yukawa coupling matrix \(Y_\nu\) in the same basis leads to a highly constrained and predictive theoretical scheme. We show that the requirement of an exact \(\mu\tau\) symmetry, coupled with observational constraints, reduces the seventy two allowed textures in such a \(Y_\nu\) to only four corresponding to just two different forms of the light neutrino mass matrix \(m_\nu\). The effect of each of these on measurable quantities can be described, apart from an overall factor of the neutrino mass scale, in terms of two real parameters and a phase angle all of which are within very constrained ranges. The additional input of a tribimaximal mixing reduces these three parameters to only one with a very nearly fixed value. Implications for both flavored and unflavored leptogenesis as well as radiative lepton flavor violating decays are discussed. We also investigate the stability of these conclusions under small deviations due to renormalization group running from a high scale where the four zero texture as well as \(\mu\tau\) symmetry or the tribimaximal mixing pattern are imposed.

PACS number(s): 14.60.Pq, 11.30.Hv, 98.80.Cq

1 Introduction

A lot is now known [1] about the masses and mixing angles of the three light neutrinos, based on the solid foundation of accumulated experimental evidence, while the remaining gaps are expected...
to be filled in the foreseeable future. Thus the task of pinning down the form of their Yukawa
coupling matrix $Y_\nu$ in flavor space, assuming the existence of three heavy right chiral neutrinos,
is very much at hand. The general structure of $Y_\nu$ is, however, intractable at the moment. One
needs concrete theoretical ideas to simplify it and then test such simplified forms by comparing
with extant data. Our present work is in such a spirit.

We try in this paper to bring together three theoretical ideas: (1) allowed four zero neutrino
Yukawa textures [2]-[3], (2) $\mu\tau$ symmetry [4]-[30] and (3) a tribimaximal mixing pattern$^1$ [31]-[34],
which actually subsumes the results of (2). Within the type-I seesaw framework [37]-[40] and in
the weak basis where the charged leptons $l_\alpha$ ($\alpha = 1,2,3$) and the heavy right chiral neutrinos $N_i$
($i=1,2,3$) have real and diagonal respective masses $m_\alpha$ and $M_i$, we explore the mutual compatibility
between (1) and (2) and further between (1) and (3). A drastic reduction of the allowed textures
and parameters under (1) ensues.

Let us start with (1). Assuming the absence$^2$ of any strictly massless neutrino as well as that of any
unnatural cancellation, the utilization of the observed lack of complete decoupling of any neutrino
flavor from the two others led to the demonstration [2] that four is the maximum number of zeroes
allowed in $Y_\nu$. All allowed four zero textures, seventy two configurations in total, were completely
classified in [2] into two categories: (A) fifty four textures with two (element by element) orthogonal
rows $i$ and $j$ say; (B) eighteen textures with nonorthogonal rows and one row having two zeroes
with the other two rows ($k$ and $l$, say) having one zero each. Let us write the complex symmetric
light neutrino Majorana mass matrix in our basis as

$$m_\nu = -Y_\nu \text{ diag.}(M_1^{-1}, M_2^{-1}, M_3^{-1}) \ Y_\nu^T v^2,$$  \hspace{1cm} (1.1)

$v$ being the relevant Higgs VEV. Now, for all textures of category (A), one has the condition [2]

$$(m_\nu)_{ij} = 0 : \text{category (A)},$$  \hspace{1cm} (1.2)

while, for those of category (B), the condition is [2]

$$\det \text{ cofactor}[(m_\nu)_{kl}] = 0 : \text{category (B)}.$$  \hspace{1cm} (1.3)

One very important and interesting feature of all these allowed four zero textures is that they
enable [2] the complete reconstruction of the neutrino Dirac mass matrix $m_D = vY_\nu$ in terms
of the physical masses of the light neutrinos as well as $M_{1,2,3}$ and the elements of the unitary
PMNS mixing matrix including the Majorana phase matrix factor. This means [2] that the high
scale CP violation required for leptogenesis gets specified exclusively [43]-[48] in terms of the CP

$^1$Such a pattern could be due to a flavor symmetry in the Lagrangian such as $A_4$ [35], $S_3$ [36].

$^2$Allowing one massless neutrino, five zeroes are allowed in $Y_\nu$ [41]-[42].
violation pertaining to laboratory energy neutrinos. Another striking feature of these textures is the
following. Conditions (1.2) and (1.3) on the corresponding neutrino mass matrix \( m_\nu \) are invariant
[3] under renormalization group running at the one loop level, though texture zeroes in general are
not. Thus if these conditions are the consequences of some symmetry operative at a high scale, they
would be approximately valid even at laboratory energies where neutrino oscillation experiments
are performed.

We next come to (2), i.e, \( \mu \tau \) symmetry [4]-[30]. For the purpose of implementing it, we find it
convenient to choose the following representation of the PMNS mixing matrix

\[
U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & -s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & -s_{23}c_{13} \\
-s_{12}s_{23} + c_{12}c_{23}s_{13}e^{i\delta} & c_{12}s_{23} + s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},
\]

(1.4)

with \( c_{ij} = \cos \theta_{ij}, \ s_{ij} = \sin \theta_{ij}, \ \delta \) being the Dirac phase and \( \alpha_M, \ \beta_M \) being the Majorana phases.

It is important to note that, with three real neutrino mass eigenvalues \( m_1, m_2, m_3 \), one has

\[
m_\nu = U \text{ diag.}(m_1, m_2, m_3) U^T.
\]

(1.5)

We can now define \( \mu \tau \) symmetry to be the invariance of all couplings and masses in the pure
neutrino part\(^3\) of the Lagrangian under the interchange of the flavor indices 2 and 3. As a result,

\[
(Y_\nu)_{12} = (Y_\nu)_{13},
\]

(1.6)

\[
(Y_\nu)_{21} = (Y_\nu)_{31},
\]

(1.7)

\[
(Y_\nu)_{23} = (Y_\nu)_{32},
\]

(1.8)

\[
(Y_\nu)_{22} = (Y_\nu)_{33}
\]

(1.9)

and

\[
M_2 = M_3.
\]

(1.10)

Using eq.(1.1), one then obtains

\[
(m_\nu)_{12} = (m_\nu)_{13},
\]

(1.11)

\[
(m_\nu)_{22} = (m_\nu)_{33}.
\]

(1.12)

We shall take eqs.(1.11, 1.12) as the statement of a custodial \( \mu \tau \) symmetry of the light neutrino
mass matrix \( m_\nu \). An automatic consequence of these two equations is the fixing of the two mixing
angles involving the third flavor at \( \theta_{23} = \pi/4, \ \theta_{13} = 0 \). Discarding unnatural cancellations, sixty
eight of the seventy two allowed four zero textures in \( Y_\nu \) are found to be incompatible with eqs.(1.11,
1.12) plus observational constraints. In particular, fifty two textures of category (A) and sixteen

\(^3\)This symmetry is, of course, badly broken in the charged lepton sector.
textures of (B) category are excluded. The two surviving textures of category A both lead to the
same light neutrino mass matrix with \((m_\nu)_{23} = 0\). On the other hand, each of the two surviving
category (B) textures turns out to have two zeroes in the first row and one each in the other rows
and they also lead to the same light neutrino mass matrix. For each surviving texture, \(m_\nu\) can be
described, apart from an overall neutrino mass scale, by two real parameters and one phase angle,
though their definitions are different for category A and category B. We call them \(k_1\), \(k_2\) and \(\alpha\) for
the former and \(l_1\), \(l_2\) and \(\beta\) for the latter. Their allowed ranges are found to be severely constrained
by the neutrino oscillation data.

We then turn to the tribimaximal mixing (TBM) pattern \([31]-[34]\) which implies \(\theta_{13} = 0\), \(\theta_{23} = \pi/4\)
and \(\theta_{12} = \sin^{-1}/\sqrt{3}\). The effect of \(\mu\tau\) symmetry is thus subsumed here, but there is an additional
constraint on \(\theta_{12}\). Hence all configurations of \(m_\nu\) leading to TBM have not only to obey eqs.(1.11-
1.12) but also the extra requirement

\[
(m_\nu)_{11} + (m_\nu)_{13} = (m_\nu)_{22} + (m_\nu)_{23}.
\]

(1.13)

The four textures of \(Y_\nu\), allowed by \(\mu\tau\) symmetry, survive the imposition of eq.(1.13), but two
relations between \(k_1\), \(k_2\) and \(\alpha\) for category A and two between \(l_1\), \(l_2\) and \(\beta\) for category B emerge.
Consequently, one independent real parameter \(k_2\) for the former and one \(l_1\) for the latter suffice to
describe those textures after factoring out the overall mass scale. The allowed domains of \(k_2\) and
\(l_1\) are again found to be highly restricted.

A general nondiagonal Majorana mass matrix \(m_\nu\) in flavor space implies lepton flavor violation
as well as the nonconservation of lepton number. It is therefore interesting and important to
discuss the implications of the above forms of \(m_\nu\) for\(^4\) radiative lepton flavor violating decays
\((\tau \rightarrow \mu\gamma\), \(\tau \rightarrow e\gamma\), \(\mu \rightarrow e\gamma\)) and for realistic leptogenesis of both flavor independent and flavor
dependent varieties. The former are yet-to-be-observed processes \([49]\) for which the experimental
sensitivity is rapidly approaching theoretical expectations; the latter is a desirable theoretical goal
\([50]\) of any (high scale) seesaw-based model of light neutrino masses and mixing angles. In the
mSUGRA version \([51]\) of a supersymmetric scenario, the branching ratios for the three radiative
lepton flavor violating decays in question have rather simple flavor structures that are bilinear in
\(Y_\nu\) or \(m_D\). We are thus able to make some specific predictions for our allowed textures, namely,
the vanishing of \(\text{BR}(\tau \rightarrow \mu\gamma)\) for category A and the value of the ratio \(\text{BR}(\tau \rightarrow e\gamma)/\text{BR}(\mu \rightarrow e\gamma)\)
being \(\simeq 0.178\) for both categories. Concerning leptogenesis, the term contributing only to flavor
dependent lepton asymmetries vanishes for all flavor combinations in both categories. Regarding
the term, which contributes to the flavor summed lepton asymmetry, only the electron asymmetry

\(^4\)Nonradiative lepton flavor violating processes, such as \(\mu e\) conversion in nuclei and triple charged leptonic decays
of the \(\tau\) and the \(\mu\), are not considered here since current experimental limits on those yield considerably weaker
constraints than radiative lepton flavor violating decays.
gets generated in category A whereas the same always vanishes in category B. One can also make more definitive statements on specific flavor combinations of the latter term as well as on the effective mass for the washout of a particular flavor asymmetry.

One issue with $\mu\tau$ symmetry and TBM is that the former fixes $\theta_{13}$ and $\theta_{23}$ at 0 and $\pi/4$ respectively, while the latter further fixes $\theta_{12}$ at $\sin^{-1}\frac{1}{\sqrt{3}} \simeq 35.26^\circ$. Though these numbers lie within presently allowed $3\sigma$ ranges of those mixing angles, the true values of the latter may eventually turn out to be different. There are, in fact, hints already that such may be the case. Current best fit $1\sigma$ ranges for those angles, derived from global analyses of all neutrino oscillation data, are $[52]$

$$
\theta_{12} = 34.5^\circ \pm 1.4^\circ, \ \theta_{23} = 43.1^\circ \pm 4.4^\circ \pm 3.5^\circ \quad \text{and} \quad \theta_{13} = 8^\circ \pm 2^\circ.
$$

While it is premature to take these ranges too seriously, it is nonetheless interesting to consider deviations within a definitive theoretical framework by taking them to originate dynamically from radiative effects. We impose $\mu\tau$ symmetry or TBM on elements of the light neutrino mass matrix $m_\nu$ at a high scale of the order of the lowest heavy right chiral neutrino mass, i.e. at $\Lambda \sim \min(M_1, M_2, M_3) \sim 10^{12}$ GeV. We further assume the validity of the Minimal Supersymmetric Standard Model (MSSM) $[51]$ between this scale and the laboratory energy scale $\lambda \sim 10^4$ GeV. The elements of $m_\nu$ are then evolved from $\Lambda$ to $\lambda$ by one loop renormalization group running. Small deviations from the consequences of $\mu\tau$ symmetry or TBM, proportional to the square of the heaviest charged lepton mass divided by the Higgs VEV squared, are found to be generated. These lead to small but distinct extensions of the allowed values of $k_{1,2}$ in category A and $l_{1,2}$ in category B. Constrained deviations in the mixing angles also emerge.

The rest of the paper is organized as follows. Section 2 contains a discussion of the allowed four zero textures and their parameterization as a consequence of $\mu\tau$ symmetry and TBM. Radiative lepton flavor violating decays and leptogenesis are taken up for those textures in Section 3. In Section 4, radiatively induced small deviations in $m_\nu$ and their effects are discussed. The final Section 5 contains a summary of our results and the conclusions derived therefrom. The Appendix contains analytical expressions for the experimentally measured quantities utilized by us both without and with one loop RG evolution.

2 Allowed four zero textures

Category A

It is straightforward to see that only two of the fifty two four zero textures of category (A) are consistent with $\mu\tau$ symmetry, as implemented through eqs. (1.11, 1.2). The rest develop additional zeroes which are incompatible with known observational constraints and the assumption of no massless neutrino. The two allowed textures for the Dirac mass matrix $m_D = Y_\nu v$ can be given in
terms of three complex parameters \( a_1, a_2, b_1 \) as

\[
m^{(1)}_D = \begin{pmatrix}
a_1 & a_2 & a_2 \\
0 & 0 & b_1 \\
0 & b_1 & 0
\end{pmatrix},
\]

\[
m^{(2)}_D = \begin{pmatrix}
a_1 & a_2 & a_2 \\
0 & b_1 & 0 \\
0 & 0 & b_1
\end{pmatrix}.
\]

The corresponding light neutrino mass matrices are identical and can be written as

\[
m^{(A)}_\nu = -\begin{pmatrix}
a_2^2/M_1 + 2a_2^2/M_2 & a_2b_1/M_2 & a_2b_1/M_2 \\
a_2b_1/M_2 & b_1^2/M_2 & 0 \\
a_2b_1/M_2 & 0 & b_1^2/M_2
\end{pmatrix}.
\]

Let us now define \( m = -b_2^2/M_2 \), \( k_1 e^{(\alpha + \alpha')} = \frac{a_1}{b_1} \sqrt{M_2} \), \( k_2 e^{i\alpha'} = \frac{a_2}{b_1} \) and further absorb the phase \( \alpha' \) in the first family neutrino field \( \nu_e \). The latter is equivalent to rotating the mass matrix of eq.(2.3) by the phase matrix \( \text{diag.}(e^{-i\alpha'} , 1 , 1) \). This operation changes eq.(2.3) to

\[
m^{(A)}_\nu = m \begin{pmatrix}
k_1^2 e^{2i\alpha} + 2k_2^2 & k_2 & k_2 \\
k_2 & 1 & 0 \\
k_2 & 0 & 1
\end{pmatrix}.
\]

Apart from the overall mass scale factor \( m \), the light neutrino mass matrix now has two real parameters \( k_1, k_2 \) and the phase angle \( \alpha \).

The ratio \( R = \Delta m^2_{21}/\Delta m^2_{32} \) and the solar/reactor mixing angle \( \theta_{12} \) are now given by

\[
R = 2(X_1^2 + X_2^2)^{1/2} [X_3 - (X_1^2 + X_2^2)^{1/2}]^{-1},
\]

\[
\tan 2\theta_{12} = \frac{X_1}{X_2}
\]

with

\[
X_1 = 2\sqrt{2}k_2[(1 + 2k_2^2)^2 + k_1^4 + 2k_1^2(1 + 2k_2^2) \cos 2\alpha]^{1/2},
\]

\[
X_2 = 1 - k_1^4 - 4k_2^4 - 4k_1^2k_2^2 \cos 2\alpha,
\]

\[
X_3 = 1 - 4k_1^2 - k_1^4 - 4k_1^2k_2^2 \cos 2\alpha - 4k_2^4.
\]

The observables of eqs.(2.5) and (2.6) can be compared with the available data. We see right away that the expression for \( R \) is incompatible with a normal mass ordering (\( \Delta m^2_{32} > 0 \)) and can only accommodate an inverted one (\( \Delta m^2_{32} < 0 \)). This is consistent with the conclusion of Merle and Rodejohann [53] who had shown that the condition \((m_\nu)_{23} = (m_\nu)_{32} = 0\) is compatible only with an inverted mass ordering. The allowed ranges are given respectively by \( R = -3.476 \times 10^{-2} \text{eV}^2 \).
Figure 1: Variation of $k_1$ and $k_2$ in category A with $\mu\tau$ symmetry over the $3\sigma$ allowed ranges of $R$ and $\theta_{12}$. 

To $-2.972 \times 10^{-2}$ eV$^2$ at the $1\sigma$ level and $-4.129 \times 10^{-2}$ eV$^2$ to $-2.534 \times 10^{-2}$ eV$^2$ at the $3\sigma$ level and by $\tan 2\theta_{12} = 3.045$ - 2.278 at $1\sigma$ and 4.899 - 1.828 at $3\sigma$. The angle $\alpha$ is immediately found to be correspondingly restricted to be between $89^\circ$ and $90^\circ$. We find that there is no acceptable solution for the $1\sigma$-allowed range of $R$. For the $3\sigma$-allowed range of the latter, a very narrow strip is allowed in the $k_1$-$k_2$ plane for the allowed domain of $\alpha$, as shown in Fig.1 with $2 < k_1 < 5.3$ and $1.2 < k_2 < 3.7$. Thus $m_\nu^{(A)}$ may quite possibly be excluded by further improvements of error in the data on $R$ and $\tan 2\theta_{12}$.

On further assuming tribimaximal neutrino mixing, i.e, eq.(1.13), one obtains the relation

$$k_1^2 e^{2i\alpha} + 2k_2 + k_2 = 1.$$  \tag{2.10}

Given eq.(2.10), $\alpha$ is now fixed$^6$ to be $\pi/2$ and the two real parameters $k_{1,2}$ are therefore reduced to one, which we take to be $k_2$ fixing $k_1$ at

$$k_1 = (2k_2^2 + k_2 - 1)^{1/2}. \tag{2.11}$$

Now that $\tan 2\theta_{12}$ is fixed at $2\sqrt{2}$, the ratio $R$ is given by

$$R = \frac{3(k_2 - 2)}{k_2 + 2}. \tag{2.12}$$

The range of $k_2$ restricted by the $3\sigma$ allowed domain of $R$ is now $1.95 \leq k_2 \leq 1.97$ so that its value is fixed to the first decimal place.

---

$^5$We are using the range of $R$ extracted $^{[54]}$ by assuming an inverted mass-ordering.

$^6$The solution $\alpha = 0$ is incompatible with the allowed range of $R$ and the reality of $k_{1,2}$.
Again, in this case, only two of the original eighteen textures are allowed by \( \mu \tau \) symmetry. These may be written in terms of three complex parameters \( a_1, b_1, b_2 \) as

\[
m_D^{(3)} = \begin{pmatrix}
a_1 & 0 & 0 \\
b_1 & b_2 & 0 \\
b_1 & 0 & b_2
\end{pmatrix},
\]

\[
m_D^{(4)} = \begin{pmatrix}
a_1 & 0 & 0 \\
b_1 & 0 & b_2 \\
b_1 & b_2 & 0
\end{pmatrix},
\]

with the corresponding light neutrino mass matrices both being

\[
m_\nu^{(B)} = -\begin{pmatrix}
a_1^2/M_1 & a_1b_1/M_1 & a_1b_1/M_1 \\
a_1b_1/M_1 & b_1^2/M_1 + b_2^2/M_2 & b_1^2/M_1 \\
a_1b_1/M_1 & b_1^2/M_1 & b_1^2/M_1 + b_2^2/M_2
\end{pmatrix}.
\]

Now, we choose to define \( m = -\frac{b_2^2}{M_2}, l_1e^{i\beta'} = \frac{a_1\sqrt{M_1}}{b_2\sqrt{M_1}}, l_2e^{i\beta} = \frac{b_1\sqrt{M_1}}{b_2\sqrt{M_1}} \) and absorb the phase \( \beta' \) in \( \nu_e \). We are then left with

\[
m_\nu^{(B)} = m \begin{pmatrix}
l_1^2 & l_1l_2e^{i\beta} & l_1l_2e^{i\beta} \\
l_1l_2e^{i\beta} & l_2^2e^{2i\beta} + 1 & l_2^2e^{2i\beta} \\
l_1l_2e^{i\beta} & l_2^2e^{2i\beta} & l_2^2e^{2i\beta} + 1
\end{pmatrix}.
\]

The measurable quantities \( R \) and \( \tan 2\theta_{12} \) are still given by eqs.(2.5) and (2.6), but now the functions \( X_{1,2,3} \) are given in terms of the parameters \( (l_1, l_2, \beta) \) as

\[
X_1 = 2\sqrt{2}l_1l_2[(l_1^2 + 2l_2^2)^2 + 1 + 2(l_1^2 + 2l_2^2)\cos 2\beta]^{1/2},
\]

\[
X_2 = 1 + 4l_2^2\cos 2\beta + 4l_2^4 - l_1^4,
\]

\[
X_3 = 1 - (l_1^2 + 2l_2^2)^2 - 4l_2^2\cos 2\beta.
\]

In this case we see that the expression for \( R \) admits only a normal mass ordering and disallows an inverted one. A comparison with data fixes \( \beta \) in the ranges 89° to 90° and 87° to 90° respectively for the values of \( R = 3.329 \times 10^{-2} \text{ eV}^2 \) to \( 2.858 \times 10^{-2} \text{ eV}^2 \) at the 1\( \sigma \) level and \( 3.915 \times 10^{-2} \text{ eV}^2 \) to \( 2.455 \times 10^{-2} \text{ eV}^2 \) at the 3\( \sigma \) level with the allowed values of \( \tan 2\theta_{12} \) as previously mentioned.

\footnote{We are using the range of R extracted \cite{54} by assuming a normal mass-ordering.}
Figure 2: Variation of $l_1$ and $l_2$ in category B with $\mu \tau$ symmetry over the $3\sigma$ allowed ranges of $R$ and $\theta_{12}$.

The corresponding allowed values of $l_{1,2}$ are shown in Fig.2 for the $3\sigma$-allowed range. Unlike category A, a substantial region of the parameter space, consisting of two branches, is allowed here.

The imposition of the tribimaximal mixing condition of eq.(1.13) now leads to

$$l_1^2 + l_1 l_2 e^{i\beta} = 2 l_2^2 e^{2i\beta} + 1$$

(2.20)

which fixes $\beta$ by

$$\cos \beta = \frac{l_1}{4 l_2}.$$  

(2.21)

Moreover, $l_{1,2}$ can now be reduced to a single real parameter $l_1$ with $l_2$ given by

$$l_2 = \frac{1}{2} \left( 1 - l_1^2 \right)^{1/2}.$$  

(2.22)

Again, tan $2\theta_{12}$ being $2\sqrt{2}$, $R$ is given by

$$R = \frac{3 l_1^2}{2 - 4 l_1^2}.$$  

(2.23)

In consequence, the allowed $1\sigma$ and $3\sigma$ ranges of $l_1$ get restricted to $0.12 \leq l_1 \leq 0.13$ and $0.11 \leq l_1 \leq 0.15$ respectively. Once again, the value of this surviving one parameter is fixed to the first decimal place.

*The solution $\beta = 0$ is not compatible with real $l_{1,2}$ and the allowed range of $R$. 

9
3 Radiative lepton flavor violation and leptogenesis

Radiative lepton flavor violating decays \( l_\alpha \to l_\beta \gamma \) (flavor indices \( \alpha, \beta \) spanning \( 1 = e, 2 = \mu, 3 = \tau \) with the constraint \( \alpha > \beta \)) together with the required generation of a lepton asymmetry at a high scale, provide powerful tools to check and test \[55\]-\[73\] any proposed seesaw-based scheme of neutrino mixing and masses. There already exist lower bounds on the partial lifetimes of the former processes; moreover, forthcoming experiments with higher sensitivity will hope to observe some of the decay channels. Coming to leptogenesis as a route to baryogenesis, a fair amount of theoretical understanding exists for high scale leptogenesis - both of the flavored and unflavored varieties. In this section, we explore the implications of the allowed four zero texture configurations, with tribimaximal mixing or at least \( \mu \tau \) symmetry, for these two types of phenomena.

We note first the one-loop expression \[74\] for \( \text{BR}(l_\alpha \to l_\beta \gamma) \) which is valid in mSUGRA scenarios with universal boundary conditions on the masses of scalar particles at a high scale \( M_X \):

\[
\text{BR}(l_\alpha \to l_\beta \gamma) = \text{const.} \text{BR}(l_\alpha \to l_\beta \nu \bar{\nu}) |(m_D L m_D^\dagger)_{\alpha \beta}| \tag{3.1}
\]

with

\[
L_{kl} = \ln \frac{M_X}{M_k} \delta_{kl}, \tag{3.2}
\]

\( M_k \) being the mass of the \( k \)th heavy right chiral neutrino. The matrix \( L \) takes care of the RG running from \( M_X \) to \( M_k \). We can now discuss what happens with our four allowed configurations for \( m_D \).

**Category A**

For both the allowed textures \( m_D^{(1)} \) and \( m_D^{(2)} \), we have

\[
(m_\nu)_{23} = -(m_D M_R^{-1} m_D^T)_{23} = 0 \tag{3.3}
\]

in a basis in which \( M_R \) is diagonal. Since \( L \) is a diagonal matrix, it follows that

\[
(m_D L m_D^\dagger)_{23} = 0. \tag{3.4}
\]

Consequently,

\[
\text{BR}(\tau \to \mu \gamma) = 0. \tag{3.5}
\]

and any observation of the \( \tau \to \mu \gamma \) process will rule out these configurations. It has moreover been shown \[53\] from the twin requirements of two nonzero neutrino masses and mixing angles that in such a case \( (m_\nu)_{12} \neq 0 \neq (m_\nu)_{13} \). As a result, \( (m_D L m_D^\dagger)_{12} \) and \( (m_D L m_D^\dagger)_{13} \) are also both nonzero, leading to nonvanishing rates for the decays \( \mu \to e \gamma \) and \( \tau \to e \gamma \) respectively. There is moreover a
relation between them. On account of \( \mu \tau \) symmetry, \( M_2 = M_3 \) and \( (m_D L m_D^\dagger)_{12} = (m_D L m_D^\dagger)_{13} \), so that we have

\[
\frac{\text{BR}(\tau \rightarrow e\gamma)}{\text{BR}(\mu \rightarrow e\gamma)} \approx \frac{\text{BR}(\tau \rightarrow \nu e\bar{\nu})}{\text{BR}(\mu \rightarrow \nu e\bar{\nu})} \approx 0.178. \tag{3.6}
\]

**Category B**

For both the allowed textures \( m_D^{(3)} \) and \( m_D^{(4)} \), the matrix \( m_D L m_D^\dagger \) is identical with all elements nonvanishing. Thus, all the three radiative modes \( \mu \rightarrow e\gamma, \tau \rightarrow \mu\gamma, \tau \rightarrow e\gamma \) are possible. However, \( \mu \tau \) symmetry has the same consequence as in category A, i.e eqn.(3.6) holds here too.

We next turn to leptogenesis at the scale \( \sim \min(M_1, M_2, M_3) \) which for simplicity we take to be \( M_1 \). Most pertinent for this are the lepton asymmetries generated by the decay of a heavy right chiral neutrino \( N_i \) into a lepton of flavor \( \alpha (= e, \mu, \tau) \) and a Higgs \( \phi \)

\[
\epsilon_i^\alpha = \frac{\Gamma(N_i \rightarrow \phi l_\alpha) - \Gamma(N_i \rightarrow \phi \bar{l}_\alpha)}{\sum_\beta [\Gamma(N_i \rightarrow \phi l_\beta) + \Gamma(N_i \rightarrow \phi \bar{l}_\beta)]} \approx \frac{g^2}{16\pi M_W^2} \frac{1}{(m_D^\dagger m_D)_{ii}} \sum_{j \neq i} \left[ I_{ij}^\alpha f\left(\frac{M_j^2}{M_i^2}\right) + J_{ij}^\alpha \left(1 - \frac{M_j^2}{M_i^2}\right)^{-1}\right] \tag{3.7}
\]

where we have neglected \( O(M_W^2/M_i^2) \) terms. Here

\[
I_{ij}^\alpha = \text{Im}(m_D^\dagger_{i\alpha}(m_D)_{\alpha j}(m_D^\dagger m_D)_{ij}) = -I_{ji}^\alpha,
J_{ij}^\alpha = \text{Im}(m_D^\dagger_{i\alpha}(m_D)_{\alpha j}(m_D^\dagger m_D)_{ji}) = -J_{ji}^\alpha. \tag{3.8}
\]

The function \( f(x) \) has the form

\[
f(x) = \sqrt{x} \left[ \frac{2}{1-x} - \ln \frac{1+x}{x} \right] \tag{3.9}
\]

in the MSSM. For \( M_1 << M_{2,3} \), \( f(M_{2,3}^2/M_i^2) \approx -3M_1/M_{2,3} \) in which case the \( J_{ij}^\alpha \) term in \( \epsilon_i^\alpha \) gets suppressed by \( M_1/M_{2,3} \). Another interesting quantity is the effective mass for the washout of a flavor asymmetry. This is given by [75]-[77]

\[
\tilde{m}_1^\alpha = |(m_D)_{\alpha 1}|^2/M_1 \tag{3.10}
\]

and controls the magnitude of the final baryon asymmetry \( Y_B \) in the way shown in Ref. [75] - [77]. Summing over all lepton flavors \( \alpha \), the \( J_{ij}^\alpha \) term drops out since \( \sum_\alpha J_{ij}^\alpha = 0 \). Utilizing the result that \( I_{ij} = \sum_\alpha I_{ij}^\alpha = \text{Im}[(m_D^\dagger m_D)_{ij}]^2 \), we have

\[
\epsilon_i = \sum_\alpha \epsilon_i^\alpha = \frac{g^2}{16\pi M_W^2} \frac{1}{(m_D^\dagger m_D)_{ii}} \sum_{j \neq i} \left[ (m_D^\dagger m_D)_{ij} \right]^2 f\left(\frac{M_j^2}{M_i^2}\right). \tag{3.11}
\]
Though the above expressions are valid in the MSSM, their flavor structure is just that of the Standard Model.

Selecting the $\mu\tau$ symmetric four zero texture configurations of $m_D$, we find that $\mathcal{J}^\alpha_{ij}$ vanishes in every case for all $\alpha, i, j$. Thus we need not consider the second term in eqn.(3.7) at all. Regarding $\mathcal{T}^\alpha_{ij}$, both allowed textures in category A yield the same result, namely, $\mathcal{T}^\alpha_{12} = \mathcal{T}^\alpha_{13} \neq 0$ while the other combinations vanish. Therefore, only the electron asymmetry gets generated in this case. Turning to the effective washout mass, only the electron one, namely $\tilde{m}_1^e$, is nonvanishing for both textures of category A. For those of category B, all the washout masses $\tilde{m}_1^e, \tilde{m}_1^\mu, \tilde{m}_1^\tau$ are nonzero with $\tilde{m}_1^\mu = \tilde{m}_1^\tau$. We provide a table containing the relevant information on leptogenesis parameters for each of our allowed four texture zero configurations.

| configuration | $\mathcal{T}^\alpha_{ij}$ | $\mathcal{J}^\alpha_{ij}$ | $\tilde{m}_1^e$ | $\tilde{m}_1^\mu$ | $\tilde{m}_1^\tau$ |
|--------------|-----------------|-----------------|--------------|--------------|--------------|
| $m_D^{(1)}$  | $\mathcal{T}^e_{12} = \mathcal{T}^e_{13} \neq 0$, rest zero | 0 | nonzero | 0 | 0 |
| $m_D^{(2)}$  | do | 0 | nonzero | 0 | 0 |
| $m_D^{(3)}$  | $\mathcal{T}^\mu_{12} = \mathcal{T}^\mu_{13} \neq 0$, rest zero | 0 | nonzero | nonzero | equals $\tilde{m}_1^\mu$ |
| $m_D^{(4)}$  | $\mathcal{T}^\tau_{12} = \mathcal{T}^\tau_{13} \neq 0$, rest zero | 0 | nonzero | nonzero | equals $\tilde{m}_1^\mu$ |

Table 1: Leptogenesis Table

4 Radiatively induced deviations

We mentioned in the previous section that the results $\theta_{13} = 0$ and $\theta_{23} = \pi/4$ follow from a custodial $\mu\tau$ symmetry in $m_\nu$. A breaking of this symmetry would in general result in a nonzero value of $\theta_{13}$ as well as a departure of $\theta_{23}$ from $\pi/4$. The goals of many ongoing and planned experiments are to measure their actual values [78]. Another interesting consequence of a nonzero $\theta_{13}$ would be the presence of a CKM-type of CP violation in the lepton sector. Our previous expressions for $R$ and $\tan 2\theta_{12}$ will be modified if $\mu\tau$ symmetry is indeed broken.

In this section we invoke the dynamical origin of such a symmetry breaking due to the Renormalization Group (RG) evolution of the elements of the neutrino mass matrix. Our basic idea is to posit that $\mu\tau$ symmetry (or more restrictively whichever symmetry, say $A_4$ or $S_3$ is responsible for TBM) is valid at a high energy scale $\Lambda \sim 10^{12}$ GeV which characterizes the heavy right chiral neutrinos $N_i$. We then consider the radiative breaking of such a symmetry through charged lepton mass terms, induced at the one loop level, as one evolves by RG running to the lower energy scale.
The proportionality involves a scale factor which is not relevant to our present analysis. The factor $\Delta_\tau$ is due to one loop RG evolution and we can neglect $m^2_e$ and $m^2_\mu$ terms as compared to $m^2_\tau$. $\Delta_\tau$ is given approximately by

$$\Delta_\tau \simeq \frac{m^2_\tau}{8\pi^2 v^2} (\tan^2 \beta + 1) \ln \left( \frac{\Lambda}{\lambda} \right),$$

(4.2)

where $\tan \beta$ is the ratio of the VEVs of the up-type and down-type neutral Higgs fields in the MSSM and $v^2$ is twice the sum of their squares. Suppose the $\mu\tau$ symmetric form of $m_\nu$ is written as

$$m_\nu = m \left( \begin{array}{ccc} P & Q & Q \\ Q & R & S \\ Q & S & R \end{array} \right),$$

(4.3)

where the complex quantities $P, Q, R, S$ are to be identified from the neutrino mass matrices given in eqn.(2.4) or (2.16). Then the corresponding neutrino mass matrix at the low energy scale $\lambda$ comes out as

$$m_\nu^\lambda = m \left( \begin{array}{ccc} P & Q & (1 - \Delta_\tau) \\ Q & R & S(1 - \Delta_\tau) \\ Q(1 - \Delta_\tau) & Q(1 - \Delta_\tau) & R(1 - 2\Delta_\tau) \end{array} \right).$$

(4.4)

From eqn.(4.4) we can calculate $R^\lambda$ as well as sin $\theta_{12}^\lambda$, sin $\theta_{23}^\lambda$ and sin $\theta_{13}^\lambda$ for the allowed textures of category A and category B. The corresponding analytic expressions are given in the Appendix. There is now a slight extension of the allowed regions in the $k_1$-$k_2$ plane for category A and in the $l_1$-$l_2$ plane for category B are shown in Figs.3 and 4 respectively. For the allowed category A textures, we find that any value of $\theta_{23}^\lambda$ greater than $45^\circ$ is disallowed. Then the experimentally allowed $3\sigma$ ranges $30.7^\circ \leq \theta_{12}^\lambda \leq 39.2^\circ$, $36^\circ \leq \theta_{23}^\lambda \leq 45^\circ$ and the maximum allowed value $\simeq 60$ of $\tan \beta$ [51] restrict $\theta_{13}^\lambda$ to $0^\circ \leq \theta_{13}^\lambda \leq 2.7^\circ$. Similarly, for the allowed category B textures, we find that any value of $\theta_{23}^\lambda$ less than $45^\circ$ is excluded. For the $3\sigma$ allowed ranges $45^\circ \leq \theta_{23}^\lambda \leq 54^\circ$ and $30.7^\circ \leq \theta_{12}^\lambda \leq 39.2^\circ$, $\theta_{13}^\lambda$ is found to be in the interval $0^\circ \leq \theta_{13}^\lambda \leq 0.85^\circ$.

\footnote{In terms of $Y_\nu$ with which we started, $Y_\nu^\lambda \simeq \text{diag}(1, 1, 1 - \Delta_\tau)Y_\nu$.}
Figure 3: The allowed variation of $k_1$ vs $k_2$ including radiative deviation within $3\sigma$ allowed ranges of $R^\lambda$ and $\theta_{12}^\lambda$. The phase angle $\alpha$ does not change significantly to $O(\Delta_{\tau})$.

Figure 4: The allowed variation of $l_1$ vs $l_2$ including radiative deviation within $3\sigma$ allowed ranges of $R^\lambda$ and $\theta_{12}^\lambda$. The phase angle $\beta$ does not change significantly to $O(\Delta_{\tau})$. 
5 Concluding summary

This paper has investigated the effect of $\mu\tau$ symmetry and (more restrictively) TBM on the maximally allowed four zero neutrino Yukawa textures within the type I seesaw in the weak basis where charged leptons and the three heavy right chiral neutrino are mass diagonal. Only two textures (leading to the same from of $m_\nu$) out of fifty four in category A and two textures (again leading to an identical $m_\nu$ form) out of eighteen in category B survive the imposition of $\mu\tau$ symmetry. Each $m_\nu$ can be characterized by two real parameters and one phase: chosen to be $k_1$, $k_2$, $\alpha$ for category A and $l_1$, $l_2$, $\beta$ for category B. All are severely constrained by extant neutrino oscillation data. In each category, the additional requirement of TBM reduces the three parameters to a single real constant with a nearly fixed value.

We have further looked at radiative lepton flavor violating decays $l_\alpha \to l_\beta \gamma$ (with $\alpha > \beta=1,2,3$) in the mSUGRA version of the MSSM. Our conclusion is that $\text{BR}(\tau \to \mu \gamma) = 0$ for category A and $\text{BR}(\tau \to e \gamma)/\text{BR}(\mu \to e \gamma) \simeq 0.178$ for both categories. Leptogenesis has also been considered at the energy scale $\min(M_1, M_2, M_3)$ with the following result. The term $J_{ij}$, which does not contribute to the flavor-summed lepton asymmetry, vanishes in either category. The term $I_{ij}$, which can cause such an asymmetry, is constrained. In particular, (1) $I_{12}^e$ and $I_{13}^e$ are nonzero while the other contributions vanish in category A; (2) either $I_{12}^\mu$, $I_{13}^\tau$, $I_{12}^\tau$ or $I_{13}^\mu$, $I_{12}^\mu$ are nonzero with the rest vanishing in category B. Regarding effective washout masses, only $\tilde{m}_1^e$ is nonvanishing in category A, while all of $\tilde{m}_1^e$, $\tilde{m}_1^\mu$, $\tilde{m}_1^\tau$ are nonzero in category B with $\tilde{m}_1^\mu = \tilde{m}_1^\tau$.

Finally, deviations from $\mu\tau$ symmetry, that are radiative in origin, have been considered. First, this symmetry has been imposed on $m_\nu$ at $\Lambda \sim 10^{12}$ GeV which typifies an energy scale that is characteristic of the heavy right chiral neutrino masses. Then the deviations in the elements of $m_\nu$, caused by one-loop RG running from $\Lambda$ to the laboratory scale $\lambda \sim 10^3$ GeV, have been computed in the MSSM with the largest allowed value of $\tan \beta$. Using the experimental $3\sigma$ ranges of $R$ and $\theta_{12}$, we have found the following results: (1) category A allows only an inverted neutrino mass ordering ($\Delta m_{32}^2 < 0$) with $\theta_{23} \leq 45^\circ$ and $0^\circ \leq \theta_{13} \leq 2.7^\circ$; (2) only a normal mass ordering ($\Delta m_{32}^2 > 0$) with $\theta_{23} \geq 45^\circ$ and $0^\circ \leq \theta_{13} \leq 0.85^\circ$ are allowed in category B. These predictions will face crucial future tests of the allowed four zero neutrino Yukawa textures in our scenario. Our bottom line is that $m_\nu^{(A)}$ is on the verge of exclusion, while $m_\nu^{(B)}$ is a good candidate for the true $m_\nu$ occurring in nature. A measured value of $\theta_{13}$ will provide a crucial test of the latter’s viability.
A Expressions for measurable quantities

\( \mu \tau \) symmetric case

Eqn. (4.3) leads to

\[
\begin{pmatrix}
|P|^2 + 2|Q|^2 & PQ^* + Q(R^* + S^*) & PQ^* + Q(R^* + S^*) \\
P^*Q + Q^*(R + S) & |Q|^2 + |R|^2 + |S|^2 & |Q|^2 + RS^* + R^*S \\
P^*Q + Q^*(R + S) & |Q|^2 + R^*S + RS^* & |Q|^2 + |R|^2 + |S|^2 \\
\end{pmatrix}
\]

(A.1)

The diagonalization of \( h \) yields \( \text{diag.}(m_1^2, m_2^2, m_3^2) \) and also expressions for five relevant measurable quantities. The latter are: (1) \( \Delta m_{21}^2 = m_2^2 - m_1^2 \), i.e the light neutrino mass squared difference relevant to solar/reactor experiments, (2) the corresponding mixing angle \( \theta_{12} \), (3) \( \Delta m_{32}^2 = m_3^2 - m_2^2 \), i.e the neutrino mass squared difference pertaining to atmospheric/long-baseline studies, (4) the corresponding mixing angle \( \theta_{23} \) and (5) the remaining mixing angle \( \theta_{13} \).

The last five quantities can all be expressed in terms of three real functions \( X_1, X_2, X_3 \) of the complex quantities \( P, Q, R, S \) appearing in \( m_\nu \). These are defined as

\[
\begin{align*}
X_1 &= 2\sqrt{2}|PQ^* + Q(R^* + S^*)|, \\
X_2 &= |R + S|^2 - |P|^2, \\
X_3 &= |R + S|^2 - |P|^2 - 4(|Q|^2 + RS^* + R^*S).
\end{align*}
\]

(A.2)

We then have

\[
\begin{align*}
\Delta m_{21}^2 &= m_2^2(X_1^2 + X_2^2)^{1/2}, \\
\theta_{12} &= \frac{1}{2}\tan^{-1}\frac{X_1}{X_2}, \\
\Delta m_{32}^2 &= \frac{m_2^2}{2}[X_3 - (X_1^2 + X_2^2)^{1/2}], \\
\theta_{23} &= \frac{\pi}{4}, \\
\theta_{13} &= 0.
\end{align*}
\]

(A.3) - (A.7)

Case with RG- broken \( \mu \tau \) symmetry

We work to one loop and ignore \( O(\Delta_\tau^2) \) terms. Now from eqn.(4.4) one derives that

\[
\begin{pmatrix}
2|Q|^2 & 2QS^* & PQ^* + QS^* + 3QR^* \\
2Q^*S & 2|S|^2 & |Q|^2 + RS^* + 3R^*S \\
P^*Q + Q^*S + 3Q^*R & |Q|^2 + R^*S + 3RS^* & 2(|Q|^2 + |S|^2) + 4|R|^2 \\
\end{pmatrix}
\]

(A.8)
The cumbersome diagonalization of $m_\nu^\dagger m_\nu$ is avoidable since the algebra simplifies in the specific cases of category A and category B. Let us reintroduce $c_{12} = \cos \theta_{12}$, $s_{12} = \sin \theta_{12}$ where $\theta_{12}$ is given in eqn.(A.4). We now define five functions $F_1,...,5$ in terms of $c_{12}$, $s_{12}$ and elements of the $m_\nu$ matrix $P$, $Q$, $R$ and $S$. The five functions $F$ are

\[
F_1 = \frac{-\sqrt{2} c_{12} \{P^* Q + 3 Q^* (R + S)\} \{P Q^* + Q (R^* + S^*)\}}{|P Q^* + Q (R^* + S^*)|} + 4 c_{12} s_{12} |R + S|^2
\]

\[
+ \sqrt{2} s_{12} \{P Q^* + 3 Q (R^* + S^*)\} \{P^* Q + Q^* (R + S)\},
\]

\[
F_2 = \frac{-\sqrt{2} c_{12} \{P^* Q + Q^* (3 R - S)\} \{P Q^* + Q (R^* + S^*)\}}{|P Q^* + Q (R^* + S^*)|} + 2 s_{12} \left(|Q|^2 + 2 |R|^2 + |S|^2 - S R^*\right),
\]

\[
F_3 = \frac{-4 \sqrt{2} c_{12} s_{12} \left(|P Q^* + Q (R^* + S^*)|^2 + 2 |Q|^2 |R + S|^2 + Q^2 P^* (R^* + S^*) + Q^* P (R + S)\right)}{|P Q^* + Q (R^* + S^*)|}
\]

\[
- 4 (c_{12}^2 - s_{12}^2) |R + S|^2
\]

\[
F_4 = \frac{-\sqrt{2} s_{12} \{P^* Q + Q^* (3 R - S)\} \{P Q^* + Q (R^* + S^*)\}}{|P Q^* + Q (R^* + S^*)|} - 2 c_{12} \left(|Q|^2 + 2 |R|^2 + |S|^2 - S R^*\right),
\]

\[
F_5 = \frac{2 \sqrt{2} c_{12} s_{12} \left(|P Q^* + Q (R^* + S^*)|^2 + 2 |Q|^2 |R + S|^2 + Q^2 P^* (R^* + S^*) + Q^* P (R + S)\right)}{|P Q^* + Q (R^* + S^*)|}
\]

\[
- 4 |R - S|^2 + 4 s_{12}^2 |Q|^2 + 4 c_{12}^2 \left(|Q|^2 + |R + S|^2\right).
\]

Thus $F_3$ and $F_5$ are real, while $F_1$, $F_2$ and $F_4$ are in general complex. We now list the changed values of the earlier mentioned five measurable quantities.

\[
(\Delta m^2_{21})^\lambda = \Delta m^2_{21} + \frac{1}{2} m^2 F_3 \Delta \tau,
\]

\[
\theta^\lambda_{12} = \sin^{-1} \left| s_{12} + \frac{m^2 c_{12}}{2 \Delta m^2_{21}} F_1^* \Delta \tau\right|,
\]

\[
(\Delta m^2_{32})^\lambda = \Delta m^2_{32} + \frac{1}{2} m^2 F_5 \Delta \tau,
\]

\[
\theta^\lambda_{23} = \sin^{-1} \left| \frac{1}{2} \sqrt{\frac{\Delta \tau}{2 \sqrt{2} m^2}} \left( \frac{\Delta m^2_{21} + \Delta m^2_{31}}{\Delta m^2_{21} + \Delta m^2_{32}} \right) - \frac{c_{12} F_1^*}{\Delta m^2_{32}} \right|,
\]

\[
\theta^\lambda_{13} = \frac{\Delta \tau}{2 \sqrt{2} m^2} \left| \frac{c_{12} F_2^*}{\Delta m^2_{32}} - \frac{s_{12} F_4^*}{\Delta m^2_{32}} \right|
\]

Note that, up to order $\Delta \tau$, we can write the changed value of the ratio $R$ as

\[
R^\lambda = \frac{\Delta m^2_{21}}{\Delta m^2_{32}} + \frac{1}{2} m^2 \Delta \tau \left( F_3 \frac{\Delta m^2_{21}}{\Delta m^2_{32}} - F_5 \frac{\Delta m^2_{21}}{(\Delta m^2_{32})^2} \right).
\]
For convenience, we list the quantities $P$, $Q$, $R$ and $S$ in each category below:

**Category A** From elements of $m^{(A)}_\nu$ in eqn.(2.4)

\[ P = k_1^2 e^{2i\alpha} + 2k_2^2, \]
\[ Q = k_2, \]
\[ R = 1, \]
\[ S = 0. \]  
(A.16)

**Category B** From elements of $m^{(B)}_\nu$ in eqn.(2.16)

\[ P = l_1^2, \]
\[ Q = l_1 l_2 e^{i\beta}, \]
\[ R = l_2^2 e^{2i\beta} + 1, \]
\[ S = l_2^2 e^{2i\beta}. \]  
(A.17)

**References**

[1] L. Camilleri, E. Lisi, J. F. Wilkerson, Ann. Rev. Nucl. Part. Sci. 58, 343 (2008).

[2] G. C. Branco, D. Emmanuel-Costa, M. N. Rebelo and P. Roy, Phys. Rev. D 77, 053011 (2008).

[3] S. Choubey, W. Rodejohann and P. Roy, Nucl. Phys. B 808, 272 (2009).

[4] T. Fukuyama and H. Nishiura, arXiv:hep-ph/9702253.

[5] W. Rodejohann, Prepared for 12th International Workshop on Neutrinos Telescopes: Twenty Years after the Supernova 1987A Neutrino Bursts Discovery, Venice, Italy, 6-9 Mar 2007

[6] J. C. Gomez-Izquierdo and A. Perez-Lorenzana, Phys. Rev. D 77, 113015 (2008).

[7] T. Baba, Int. J. Mod. Phys. E 16, 1373 (2007).

[8] N. Nimai Singh, H. Zeen Devi and M. Patgiri, arXiv:0707.2713 [hep-ph].

[9] A. S. Joshipura and B. P. Kodrani, Phys. Lett. B 670, 369 (2009).

[10] B. Adhikary, Phys. Rev. D 74, 033002 (2006) [arXiv:hep-ph/0604009].

[11] T. Baba and M. Yasue, Phys. Rev. D 75, 055001 (2007).
[12] W. Grimus, arXiv:hep-ph/0610158.

[13] Z. z. Xing, H. Zhang and S. Zhou, Phys. Lett. B 641, 189 (2006).

[14] N. Haba and W. Rodejohann, Phys. Rev. D 74, 017701 (2006).

[15] R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 636, 114 (2006).

[16] Y. H. Ahn, S. K. Kang, C. S. Kim and J. Lee, Phys. Rev. D 73, 093005 (2006).

[17] K. Fuki and M. Yasue, Phys. Rev. D 73, 055014 (2006).

[18] S. Nasri, Int. J. Mod. Phys. A 20, 6258 (2005).

[19] I. Aizawa and M. Yasue, Phys. Rev. D 73, 015002 (2006).

[20] R. N. Mohapatra and W. Rodejohann, Phys. Rev. D 72, 053001 (2005).

[21] I. Aizawa, M. Ishiguro, M. Yasue and T. Kitabayashi, J. Korean Phys. Soc. 46, 597 (2005).

[22] T. Kitabayashi and M. Yasue, Phys. Lett. B 621, 133 (2005).

[23] R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 615, 231 (2005).

[24] R. N. Mohapatra and S. Nasri, Phys. Rev. D 71, 033001 (2005).

[25] R. N. Mohapatra, JHEP 0410, 027 (2004).

[26] I. Aizawa, M. Ishiguro, T. Kitabayashi and M. Yasue, Phys. Rev. D 70, 015011 (2004).

[27] I. de Medeiros Varzielas and G. G. Ross, Nucl. Phys. B 733, 31 (2006).

[28] P. F. Harrison and W. G. Scott, Phys. Lett. B 547, 219 (2002).

[29] T. Kitabayashi and M. Yasue, Phys. Rev. D 67, 015006 (2003).

[30] A. Ghosal, Mod. Phys. Lett. A 19, 2579 (2004).

[31] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002).

[32] P. F. Harrison and W. G. Scott, arXiv:hep-ph/0402006.

[33] Z. z. Xing, Phys. Lett. B 533, 85 (2002).

[34] P. F. Harrison and W. G. Scott, Phys. Lett. B 535, 163 (2002).
[35] G. Altarelli, arXiv:0905.3265 [hep-ph], arxiv:0905.2350 [hep-ph], E. Ma and G. Rajasekaran, Phys. Rev. D 64, 113012 (2001), K. S. Babu, E. Ma and J. W. F. Valle, Phys. Lett. B 552, 207 (2003), M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle and A. Villanova del Moral, arXiv:hep-ph/0312244, M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle and A. Villanova del Moral, Phys. Rev. D 69, 093006 (2004), E. Ma, Phys. Rev. D 70, 031901 (2004), New J. Phys. 6, 104 (2004), Mod. Phys. Lett. A 20, 2601 (2005), Phys. Rev. D 72, 037301 (2005), S. L. Chen, M. Frigerio and E. Ma, Nucl. Phys. B 724, 423 (2005), M. Hirsch, A. Villanova del Moral, J. W. F. Valle and E. Ma, Phys. Rev. D 72, 091301 (2005), [Erratum-ibid. D 72, 119904 (2005)], K. S. Babu and X. G. He, arXiv:hep-ph/0507217, A. Zee, Phys. Lett. B 630, 58 (2005), X. G. He, Y. Y. Keum and R. R. Volkas, JHEP 0604, 039 (2006), B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M. K. Parida, Phys. Lett. B 638, 345 (2006), E. Ma, Phys. Rev. D 73, 057304 (2006), Mod. Phys. Lett. A 21, 2931 (2006), Mod. Phys. Lett. A 22, 101 (2007), S. F. King and M. Malinsky, Phys. Lett. B 645, 351 (2007), S. Morisi, M. Picariello and E. Torrente-Lujan, Phys. Rev. D 75, 075015 (2007), F. Yin, Phys. Rev. D 75, 073010 (2007), F. Bazzocchi, S. Kaneko and S. Morisi, JHEP 0803, 063 (2008), F. Bazzocchi, S. Morisi and M. Picariello, Phys. Lett. B 659, 628 (2008), M. Honda and M. Tanimoto, Prog. Theor. Phys. 119, 583 (2008), B. Brahmachari, S. Choubey and M. Mitra, Phys. Rev. D 77, 073008 (2008), [Erratum-ibid. D 77, 119901 (2008)], B. Adhikary and A. Ghosal, Phys. Rev. D 78, 073007 (2008), A. Ghosal, arXiv:hep-ph/0612245, B. Adhikary and A. Ghosal, Phys. Rev. D 75, 073020 (2007), G. Altarelli, F. Feruglio and C. Hagedorn, JHEP 0803, 052 (2008), F. Bazzocchi, S. Morisi, M. Picariello and E. Torrente-Lujan, J. Phys. G 36, 015002 (2009), M. Hirsch, S. Morisi and J. W. F. Valle, Phys. Rev. D 78, 093007 (2008), P. H. Frampton and S. Matsuzaki, arXiv:0806.4592 [hep-ph], C. Csaki, C. Delaunay, C. Grojean and Y. Grossman, JHEP 0810, 055 (2008), F. Bazzocchi, M. Frigerio and S. Morisi, Phys. Rev. D 78, 116018 (2008), S. Morisi, arXiv:0901.1080 [hep-ph], P. Ciafaloni, M. Picariello, E. Torrente-Lujan and A. Urbano, Phys. Rev. D 79, 116010 (2009), M. C. Chen and S. F. King, JHEP 0906, 072 (2009), G. Altarelli and F. Feruglio, Nucl. Phys. B 720, 64 (2005), G. Altarelli and F. Feruglio, Nucl. Phys. B 741, 215 (2006), G. Altarelli, F. Feruglio and Y. Lin, Nucl. Phys. B 813, 91 (2009), Y. Lin, arXiv:0903.0831 [hep-ph], G. Altarelli and D. Meloni, J. Phys. G 36, 085005 (2009).

[36] C. H. Albright, arxiv:0905.0146 [hep-ph], E. Ma, Phys. Rev. D 44, 587 (1991), Y. Koide, Phys. Rev. D 60, 077301 (1999), M. Tanimoto, Phys. Lett. B 483, 417 (2000), J. Kubo, Phys. Lett. B 578, 156 (2004), [Erratum-ibid. B 619, 387 (2005)], F. Caravaglios and S. Morisi, arXiv:hep-ph/0503234, S. Morisi and M. Picariello, Int. J. Theor. Phys. 45, 1267 (2006), P. F. Harrison and W. G. Scott, Phys. Lett. B 557, 76 (2003), W. Grimus and L. Lavoura, JHEP 0508, 013 (2005), R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 639, 318 (2006), N. Haba
and K. Yoshioka, Nucl. Phys. B 739, 254 (2006), C. Y. Chen and L. Wolfenstein, Phys. Rev. D 77, 093009 (2008), S. Kaneko, H. Sawanaka, T. Shingai, M. Tanimoto and K. Yoshioka, arXiv:hep-ph/0703250, Y. Koide, Eur. Phys. J. C 50, 809 (2007), Phys. Rev. D 73, 057901 (2006), T. Teshima, Phys. Rev. D 73, 045019 (2006), L. Lavoura and E. Ma, Mod. Phys. Lett. A 20, 1217 (2005), T. Araki, J. Kubo and E. A. Paschos, Eur. Phys. J. C 45, 465 (2006), N. Haba, A. Watanabe and K. Yoshioka, Phys. Rev. Lett. 97, 041601 (2006), F. Feruglio and Y. Lin, Nucl. Phys. B 800, 77 (2008), A. S. Joshipura and S. D. Rindani, Eur. Phys. J. C 14, 85 (2000), R. N. Mohapatra, A. Perez-Lorenzana and C. A. de Sousa Pires, Phys. Lett. B 474, 355 (2000), Q. Shafi and Z. Tavartkiladze, Phys. Lett. B 482, 145 (2000), L. Lavoura, Phys. Rev. D 62, 093011 (2000), W. Grimus and L. Lavoura, Phys. Rev. D 62, 093012 (2000), T. Kitabayashi and M. Yasue, Phys. Rev. D 63, 095002 (2001), A. Aranda, C. D. Carone and P. Meade, Phys. Rev. D 65, 013011 (2002), K. S. Babu and R. N. Mohapatra, Phys. Lett. B 532, 77 (2002), Phys. Lett. B 536, 83 (2002), H. S. Goh, R. N. Mohapatra and S. P. Ng, Phys. Lett. B 542, 116 (2002), G. K. Leontaris, J. Rizos and A. Psallidas, Phys. Lett. B 597, 182 (2004).

[37] P. Minkowski, Phys. Lett. B 67, 421 (1977).

[38] M. Gell-Mann, P. Ramond and R. Slansky, in 'Supergravity', eds. D. Friedman and P. van Nieuwenhuizen (North-Holland, Amsterdam, 1979), p.315.

[39] T. Yanagida in Proc. Workshop 'Unified Theory and Baryon Number in the Universe', eds. O. Sawada and A. Sugamoto (KEK, Tsukuba, Japan, 1979), p.95.

[40] R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. 44, 912 (1980).

[41] S. Goswami, S. Khan and W. Rodejohann, arXiv:0905.2739 [hep-ph].

[42] A. Dighe and N. Sahu, arXiv:0812.0695 [hep-ph].

[43] P. H. Frampton, S. L. Glashow and D. Marfatia, Phys. Lett. B 536, 79 (2002).

[44] A. Ibarra and G. G. Ross, Phys. Lett. B 591, 285 (2004).

[45] S. Kaneko and M. Tanimoto, Phys. Lett. B 551, 127 (2003).

[46] G. C. Branco, R. Gonzalez Felipe, F. R. Joaquim, I. Masina, M. N. Rebelo and C. A. Savoy, Phys. Rev. D 67, 073025 (2003).

[47] E. Molinaro, S. T. Petcov, T. Shindou and Y. Takanishi, Nucl. Phys. B 797, 93 (2008).

[48] K. S. Babu, Y. Meng and Z. Tavartkiladze, arXiv:0812.4419 [hep-ph].

[49] M. Raidal et al., Eur. Phys. J. C 57, 13 (2008).
[50] G. C. Branco and M. N. Rebelo, Nucl. Phys. Proc. Suppl. 188, 325 (2009).

[51] 'Theory And Phenomenology Of Sparticles: An Account Of Four-dimensional N=1 Supersymmetry In High Energy Physics', by M. Drees, R. Godbole, P. Roy, World Scientific, Singapore, 2006.

[52] G. L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A. M. Rotunno, Phys. Rev. Lett. 101, 141801 (2008); arXiv:0905.3549 [hep-ph].

[53] A. Merle and W. Rodejohann, Phys. Rev. D 73, 073012 (2006).

[54] M. Maltoni and T. Schwetz, arXiv:0812.3161 [hep-ph].

[55] S. Davidson and A. Ibarra, JHEP 0109, 013 (2001).

[56] J. R. Ellis, J. Hisano, S. Lola and M. Raidal, Nucl. Phys. B 621, 208 (2002).

[57] F. Deppisch, H. Pas, A. Redelbach, R. Ruckl and Y. Shimizu, Eur. Phys. J. C 28, 365 (2003).

[58] J. R. Ellis and M. Raidal, Nucl. Phys. B 643, 229 (2002).

[59] S. Pascoli, S. T. Petcov and C. E. Yaguna, Phys. Lett. B 564, 241 (2003).

[60] S. Pascoli, S. T. Petcov and W. Rodejohann, Phys. Rev. D 68, 093007 (2003).

[61] W. Rodejohann, Eur. Phys. J. C 32, 235 (2004).

[62] S. T. Petcov, W. Rodejohann, T. Shindou and Y. Takanishi, Nucl. Phys. B 739, 208 (2006).

[63] F. Deppisch, H. Pas, A. Redelbach and R. Ruckl, Phys. Rev. D 73, 033004 (2006).

[64] S. T. Petcov and T. Shindou, Phys. Rev. D 74, 073006 (2006).

[65] S. Antusch, E. Arganda, M. J. Herrero and A. M. Teixeira, JHEP 0611, 090 (2006).

[66] G. C. Branco, A. J. Buras, S. Jager, S. Uhlig and A. Weiler, JHEP 0709, 004 (2007).

[67] S. Antusch and A. M. Teixeira, JCAP 0702, 024 (2007).

[68] E. J. Chun, J. L. Evans, D. E. Morrissey and J. D. Wells, Phys. Rev. D 79, 015003 (2009).

[69] M. Endo and T. Shindou, arXiv:0805.0996 [hep-ph].

[70] S. Davidson, J. Garayoa, F. Palorini and N. Rius, JHEP 0809, 053 (2008).

[71] M. Hirsch, S. Kaneko and W. Porod, Phys. Rev. D 78, 093004 (2008).

[72] K. Kojima and H. Sawanaka, Phys. Lett. B 678, 373 (2009).
[73] D. Ibanez, S. Morisi and J. W. F. Valle, arXiv:0907.3109 [hep-ph].

[74] F. Borzumati and A. Masiero, Phys. Rev. Lett. 57, 961 (1986).

[75] L. Covi, E. Roulet and F. Vissani, Phys. Lett. B 384, 169 (1996).

[76] A. Abada, S. Davidson, F. X. Josse-Micheaux, M. Losada and A. Riotto, JCAP 0604, 004 (2006).

[77] E. Nardi, Y. Nir, E. Roulet and J. Racker, JHEP 0601, 164 (2006).

[78] T. Kajita, Pramana J.Phys. 72, 109 (2009).

[79] A. Dighe, S. Goswami and P. Roy, Phys. Rev. D 76, 096005 (2007).

[80] A. Dighe, S. Goswami and P. Roy, Phys. Rev. D 73, 071301 (2006).