ON THE RATIONAL HOMOTOPICAL NILPOTENCY INDEX OF PRINCIPAL BUNDLES

YANLONG HAO AND XIUGUI LIU

Abstract. Let Aut(p) denote the space of all self-fibre homotopy equivalences of a principal G-bundle p : E → X of simply connected CW complexes with E finite. When G is a compact connected topological group, we show that there exists an inequality

$$n - N(p) \leq \text{Hnil}_Q(\text{Aut}(p)_0) \leq n$$

for any space X, where n is the number of non-trivial rational homotopy groups of G and N(p) is defined in Section 2. In particular, Hnil_Q(\text{Aut}(p)_0) = n if p is a fibre-homotopy trivial bundle and X is finite.

Key words: Fibre-homotopy equivalence, Samelson-Lie algebra, Sullivan minimal model, Derivation

1. Introduction

Let G be a compact connected topological group, and p : E → X be a principal G-bundle of connected CW complexes with E finite. The identity component Aut(p)_0 of the monoid of self-fibre homotopy equivalences of E is known to be a grouplike space of CW type [5, Prop. 2.2].

We first recall the general definition of homotopy nilpotency index for grouplike spaces introduced by Berstein and Ganea [1].

Suppose (G, µ) is a grouplike space with homotopy inverse ν : G → G. Let

$$\varphi_2 : G \times G \rightarrow G$$

denote the commutator map

$$\varphi_2(g_1, g_2) = \mu(\mu(g_1, g_2), \mu(\nu(g_1), \nu(g_2))).$$

Extend this definition to \(\varphi_n : G^n \rightarrow G\) by the rule

$$\varphi_n = \varphi_2(\varphi_{n-1} \times id_G).$$

for \(n > 2\).

The homotopical nilpotency index Hnil(G) is the least integer (or infinite) such that \(\varphi_{n+1}\) is nullhomotopic. The rational homotopical nilpotency index Hnil_Q(G) is set to be Hnil(G_Q) where G_Q denotes classical rationalization.

In [5] Thm. 5.2, it is proven that Hnil_Q(\text{Aut}(p)_0) \leq n, where n is the number of non-trivial rational homotopy groups of G. Here we find a lower bound which in particular generalizes the main result of [6].

2010 Mathematics Subject Classification. 55P62, 55P10.
The second author was supported in part by the National Natural Science Foundation of China (No. 11171161), Program for New Century Excellent Talents in University (No. NCET-08-0288), and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.
Theorem 1.1. Let $p : E \to X$ be a principal $G$-bundle of simply connected CW complexes with classifying map $f : X \to BG$, where $E$ is finite. Then we have the following inequality:
\[ n - N(p) \leq \text{Hnil}_Q(\text{Aut}(p)_0) \leq n, \]
where $n$ is the number of non-trivial rational homotopy groups of $G$ and the quantity $N(p)$ is defined in Section 2.

In particular, we have

Corollary 1.2. When the fibre bundle is trivial, $\text{Hnil}_Q(\text{Aut}(p)_0) = n$.

2. Background, tools and proofs

For the sake of completeness, we first review some background and notations. We assume that the reader is familiar with the basics of rational homotopy theory. The book [4] is a standard and excellent reference.

In [7], Sullivan defined a functor $A_{PL}(-)$ from topological spaces to commutative differential graded algebras over $\mathbb{Q}$ (CDGA for short) that is connected to the cochain algebra functor $C^*(-; \mathbb{Q})$ by a sequence of natural quasi-isomorphisms. Let $\wedge V$ denote the free commutative graded algebra on the graded rational vector space $V$. A CDGA $(A, d)$ is a Sullivan algebra if $A \cong \wedge V$ and $V$ admits a basis $(v_i)$ indexed by a well ordered set such that $d(v_i) \in \wedge (v_j, j < i)$. If the differential $d$ is a decomposable differential, that is, $d(V) \subseteq \wedge^{\geq 2} V$, we say $(A, d)$ is minimal. A CDGA $(A, d)$ is a Sullivan model for $X$ if $(A, d)$ is a Sullivan algebra and there is a quasi-isomorphism $(A, d) \to A_{PL}(X)$. If $(A, d)$ is minimal then it is the minimal Sullivan model of $X$.

Let $p : E \to X$ be a principal $G$-bundle of simply connected CW complexes, with $G$ a compact connected topological group. Let $(A, d)$ be a CDGA model of $X$ and recall that the minimal model of $G$ is necessarily of the form $(\Lambda V, d)$ with $V$ finite dimensional and concentrated in odd degrees. Then, see [4, §15.f], the bundle $p$ is modeled by an inclusion $(A, d) \hookrightarrow (A \otimes \Lambda V, D)$ for which $DV \subset A$. In other words, $(A \otimes \Lambda V, D)$ is a relative Sullivan algebra with base algebra $(A, d)$.

We say that a set $\{v_i\}_{i \in I}$ generates the relative Sullivan algebra $(A \otimes \Lambda V, D)$, if $v_i \notin A$, $Dv_i \in A$ for each $i$, $A$ and the linear space $W$ spanned by $\{v_i\}_{i \in I}$ generate multiplicatively $(A \otimes \Lambda V, D)$. For each generator of $(A \otimes \Lambda V, D)$, say $v \in W$, the cohomology class $\alpha_v \in H^*(A, d) = H^*(X, \mathbb{Q})$ represented by $Dv$ is called a characteristic class of the bundle $p$. These classes characterize the rational homotopy type of the bundle as they encode the rational homotopy type of the classifying map $X \to BG$.

Remark 2.1. Note that the definition of the characteristic class $\alpha_v$ depends on the choice of generators of $(A \otimes \Lambda V, D)$. For example, let $p$ be an $S^1 \times S^3$-fibration with minimal Sullivan model
\[ \varphi : (\Lambda e, 0) \to (\Lambda (e, x, y), D), \]
where $\deg(e) = 2$, $\deg(x) = 1$, $\deg(y) = 3$, $Dx = e$ and $Dy = e^2$. On the one hand, $\{x, y\}$ is a set of generators of $(A \otimes \Lambda V, D)$ with $\alpha_x \neq 0$ and $\alpha_y \neq 0$, on the other hand, $\{x' := x, y' := y - ex\}$ is also a set of generators of $(A \otimes \Lambda V, D)$ with $\alpha_{x'} \neq 0$ and $\alpha_{y'} = 0$. 
Let \( W \) always be the linear space spanned by a set of generators of \((A \otimes A V, D)\). Then we can define a linear map of degree 1,
\[
f : W \to H^*(A, d), \quad f(v) = [Dv].
\]
Let \( k \) be the number of dimensions \( m \) such that \( f^m : W^m \to H^{m+1}(A, d) \) is injective.

**Definition 2.2.** \( N(p) \) is the minimum of the different values \( k \) resulting from different choices of generators of \((A \otimes A V, D)\).

Now we recall two fundamental facts. On the one hand, based on [2 Thm. 10], it is proven in [3 Thm. 1] that a Lie model of \( \text{BAut}(p) \) is given by the differential graded Lie algebra (DGL henceforth) formed by the positive degree \( A \)-derivations of \((A \otimes A V, D)\). Recall that an \( A \)-derivation \( \theta \in \text{Der}_A^p(A \otimes A V) \) of degree \( p \) is a self linear map of degree \( -p \) of \( A \otimes A V \) which vanishes on \( A \) and satisfies \( \theta(xy) = \theta(x)y + (-1)^{|\theta||x|}x\theta(y) \) for \( x, y \in A \otimes A V \). The graded vector space \( \text{Der}_A^p(A \otimes A V) \) is a DGL with the commutator bracket \([\theta_1, \theta_2] = \theta_1\theta_2 - (-1)^{|\theta_1||\theta_2|}\theta_2\theta_1\) and differential \( D(\theta) = [D, \theta] \). In particular,
\[
\pi_*(\text{Aut}(p)_0) \otimes \mathbb{Q} \cong H_*(\text{Der}_A(A \otimes A V))
\]
as Lie algebras, where we are considering the Samelson bracket in \( \pi_*(\text{Aut}(p)_0) \otimes \mathbb{Q} \).

On the other hand, in [5 Prop. 2.3] it is proven that
\[
\text{Hnil}_Q(\text{Aut}(p)_0) = \text{nil}(\pi_*(\text{Aut}(p)_0))
\]
the latter being the usual nilpotency index of a Lie algebra. In conclusion,
\[
\text{Hnil}_Q(\text{Aut}(p)_0) = \text{nil}(H_*(\text{Der}_A(A \otimes A V))).
\]

Now we are in a position to prove Theorems 1.1.

**Proof of Theorem 1.1.** In [5 Theorem 5.2], it is proven that \( \text{Hnil}_Q(\text{Aut}(p)_0) \leq n \).

Write \( l = n - N(p) \). By definition, there exists a set of generators of \( A \otimes A V \) such that there are \( l \) different dimensions, say \( m_1 < \cdots < m_l \), for which \( f^{m_i} \) is not injective. In other words, for each \( i = 1, 2, \ldots, l \) there is a non zero element \( v_i \in W^{m_i} \) such that \( Dv_i = d\Phi_i \) with \( \Phi_i \in A \). Then, replace \( v_i \) by \( v_i - \Phi_i \) so that \( Dv_i = 0 \) and we will assume that henceforth. Enlarge \( v_1, \ldots, v_l \) to a set of generators of \( A \otimes A V \), denote \( v_0 = 1 \) and proceed as follows: For each \( 0 \leq i < j \leq l \), define \( \theta_{j,i} \in \text{Der}_A(A \otimes A V) \) as the \( A \)-derivation which sends \( v_j \) to \( v_i \) and vanishes in any other element of the fixed basis of \( V \).

It is immediate to see that \( \theta_{j,i} \) is a cycle and not a boundary (as \( v_j \) will never be a boundary). Moreover, a straightforward computation shows that if \( i < j < h \),
\[
[\theta_{h,j}, \theta_{j,i}] = \theta_{h,i}.
\]
In particular,
\[
[[\ldots[\theta_{l,t-1}, \theta_{l-1,t-2}], \theta_{l-2,t-3}], \ldots], \theta_{1,0}] = \theta_{1,0}.
\]
The desired result follows.

**Proof of Corollary 1.2.** When the fibre bundle is trivial, \( N(p) = 0 \). Hence we have
\[
\text{Hnil}_Q(\text{Aut}(p)_0) = n.
\]
Example 2.3. Let \( p : E \to \mathbb{C}P^m \) be a principal \( G \)-bundle with \( G \) a compact connected topological group. We show that \( N(p) \leq 1 \), which means
\[
n - 1 \leq \text{Hnil}_Q(\text{Aut}(p)_0) \leq n.
\]
If \( N(p) \neq 0 \), let \( k \) be the smallest integer such that \( f : W^k \to H^{k+1}(\mathbb{C}P^m, \mathbb{Q}) \) is injective. So \( W^k \) is generated by one element \( v \) and \( f(v) = a^i \) where \( a \) is a generator of \( H^2(\mathbb{C}P^m, \mathbb{Q}) \). For each generator of \( W^{>k} \), say \( w \), we have \( f(w) = \lambda a^j \). Replace \( w \) by \( w - \lambda a^j v \) so that \( f(w) = 0 \), so \( N(p) \leq 1 \), which means \( N(p) = 1 \).

In fact, it is easy to show that in this case \( \text{Hnil}_Q(\text{Aut}(p)_0) = n - 1 \) if and only if \( N(p) = 1 \) and the dimension \( k \) used above is not the top dimension of the rational homotopy groups of \( G \).

Example 2.4. When \( G = S^1 \), the model of a \( G \)-fibration is just \((A \otimes \Lambda v, D)\). Independently of the value of \( N(p) \), we have
\[
\text{Hnil}_Q(\text{Aut}(p)_0) = 1,
\]
since there is a non-trivial derivation that sends the generator \( v \) to 1.

References

[1] I. Berstein, T. Ganea, Homotopical nilpotency, Illinois J. Math. 5 (1961), 99–130.
[2] U. Buijs, A. Murillo, Lie models for the components of the space of sections of a nilpotent fibration, Trans. Amer. Math. Soc. 361 (10) (2009), 5601-5614.
[3] U. Buijs, S. Smith, Rational homotopy type of the classifying space for fibre-wise self-equivalences, Proc. Amer. Math. Soc. 141 (6) (2013), 2153-2167.
[4] Y. Félix, S. Halperin, J.-C. Thomas, Rational homotopy theory, Graduate Texts in Mathematics, Vol. 205, Springer-Verlag, New York, 2001.
[5] Y. Félix, G. Lupton, S. B. Smith, The rational homotopy type of the space of self-equivalences of a fibration, Homology, Homotopy Appl. 12 (2) (2010), 371–400.
[6] X. Liu, R. Huang, The rational homotopical nilpotency of principal \( U_n(\mathbb{C}) \)-bundles, Topology Appl. 160 (2013), 1466-1475.
[7] D. Sullivan, Infinitesimal computations in topology, Publ. I.H.E.S. 47 (1977), 269–331.