Certifying Differential Equation Solutions from Computer Algebra Systems in Isabelle/HOL

Thomas Hickman, Christian Pardillo Laursen, and Simon Foster
University of York

Abstract. The Isabelle/HOL proof assistant has a powerful library for continuous analysis, which provides the foundation for verification of hybrid systems. However, Isabelle lacks automated proof support for continuous artifacts, which means that verification is often manual. In contrast, Computer Algebra Systems (CAS), such as Mathematica and SageMath, contain a wealth of efficient algorithms for matrices, differential equations, and other related artifacts. Nevertheless, these algorithms are not verified, and thus their outputs cannot, of themselves, be trusted for use in a safety critical system. In this paper we integrate two CAS systems into Isabelle, with the aim of certifying symbolic solutions to ordinary differential equations. This supports a verification technique that is both automated and trustworthy.

1 Introduction

Verification of Cyber-Physical and Autonomous Systems requires that we can verify both discrete control, and continuous evolution, as envisaged by the hybrid systems domain [1]. Whilst powerful bespoke verification tools exist, such as the KeYmaera X [2] proof assistant, software engineering requires a general framework, which can support a variety of notations and paradigms [3]. Isabelle/HOL [4] is such a framework. Its combination of an extensible frontend for syntax processing, and a plug-in oriented backend, based in ML, which supports a wealth of heterogeneous semantic models and proof tools, supports a flexible platform for software development, verification, and assurance [5,6,7].

Verification of hybrid systems in Isabelle is supported by several detailed libraries of Analysis, including Multivariate Analysis [8], Affine Arithmetic [9], and HOL-ODE [10], which supports reasoning for Systems of Ordinary Differential Equations (SODEs). These libraries essentially build all of calculus from the ground up, and thus provide the highest level of rigour. However, Isabelle currently does not offer many automated proof facilities for hybrid systems. KeYmaera X [2], in contrast, is highly automated and thus very usable, even by non-experts. This is partly due to the inclusion of efficient algorithms for differential equation solving and quantifier elimination, which are both vital techniques. Several of these techniques are supported by integration with Computer Algebra Systems (CAS), which support these and several other algorithms, in particular the Wolfram Engine, which is the foundation for Mathematica.
Nevertheless, whilst CAS systems are efficient, they do not achieve the same level of rigour as Isabelle, and thus the results cannot be used without care in a high assurance development. In Isabelle, all results are certified using a small kernel against the core axioms of the object logic, following the LCF architecture. The correctness of external tools does not need to be demonstrated, but only that particular results can be certified. This approach has been highly successful, and in particular has allowed the creation of the famous sledgehammer tool [11,12], which soundly integrates external automated theorem proving systems.

In this paper, we apply this approach to integration of CAS systems into Isabelle/HOL\(^1\). We focus on generation and certification of solutions to SODEs, though our approach is more generally applicable. We integrate two CAS systems: the Wolfram Engine and SageMath, the latter of which is open source. We show how SODEs and their solutions can be described and certified in Isabelle. We then show how we have integrated the two CAS systems, using their APIs, and several new high level Isabelle commands. We evaluate our approach using a large test set of SODEs, including a large fragment of the KeYmaera X example library. Our approach is largely successful, but we highlight some future work for improving the certification proof process in Isabelle.

The structure of our paper is as follows. In §2, we highlight related work and necessary context. In §3, we describe our tactic for certification of SODEs. In §4 and §5, we present our integrations with SageMath and Wolfram, respectively. In §6, we evaluate our approach using our test set, and in §7 we conclude.

## 2 Background

The dominant approach for CPS verification is differential dynamic logic (d\(\mathcal{L}\)), a proof calculus for reasoning about hybrid programs [13]. Hybrid programs allow modelling of hybrid systems by providing operators for discrete transitions, such as assignment and nondeterministic composition, together with modelling dynamics via continuous evolution of a SODE.

The most advanced tool for deductive verification of hybrid systems is KeYmaera X [2], a theorem prover for d\(\mathcal{L}\). Its capabilities have been shown in numerous case studies, such as in [14] for verifying various classes of robotic collision-avoidance algorithms, and in [15] for proving that the ACAS X aircraft collision avoidance system provides safe guidance under a set of assumptions. KeYmaera X uses the Wolfram Engine for SODE solving and quantifier elimination.

KeYmaera X is, however, restricted to reasoning about d\(\mathcal{L}\) hybrid programs, and cannot be applied directly to other notations. In particular, we cannot show that a controller specification is refined by a given implementation in a language like C [16], although tools such as VeriPhy [17] and ModelPlex [18] somewhat bridge this gap. It also cannot handle transcendental functions, such as sin and log, which are often used by control engineers.

\(^1\) The code supporting our approach can be found in the following GitHub repository: https://github.com/ThomasHickman/Isabelle-CAS-Integration
dL has also been implemented \cite{19,20,21} in the Isabelle proof assistant \cite{4}, as both a deep \cite{19} and shallow embedding \cite{20,21}. Verification in Isabelle brings the advantage of generality, whereby the hybrid systems proof could be used to show correctness of an implementation \cite{16}, or used in a larger proof about a complex system. It also allows integration with several notations in a single development, which is the goal of our target verification framework, Isabelle/UTP \cite{22}.

A present disadvantage of Isabelle is the lack of automated proof, in comparison to KeYmaera X. Consequently, our goal is to improve automation by safe integration of a CAS. Mathematica has previously been integrated into Isabelle for quantifier elimination problems over univariate polynomials \cite{23}. Here, we integrate two CAS systems for the purpose of certifying SODE solutions.

Plugins in Isabelle, like our CAS integration, are written using the ML language, and manipulate terms of the logic. Terms are used to encode a typed \(\lambda\)-calculus, and are encoded using the following ML data type.

\[\text{datatype term} = \text{Const of string} \times \text{typ} \mid \text{Free of string} \times \text{typ}\mid \var{Var} \text{ of indexname} \times \text{typ} \mid \text{Bound of int}\mid \text{Abs of string} \times \text{typ} \times \text{term} \mid \text{$ of term} \times \text{term}\]

The type \text{typ} describes Isabelle types. The basic constructors include constants (\text{Const}), free variables (\text{Var}), schematic variables (\text{Var}), and bound variables (\text{Bound}) with de Bruijn indices. With the exception of \text{Bound}, these all consist of a name and a type. The remaining two constructors represent \(\lambda\)-abstractions and applications of one term to another. For example, the term \(\lambda x \ y : \mathbb{R} . x + y\), a function that adds together two real numbers, is represented as follows:

\[\text{Abs ("x", "real", Abs ("y", "real", Const ("Groups.plus_class.plus", "real => real => real")$ Bound 1$ Bound 0))}\]

As usual, \(\lambda x \ y . e\) is syntactic sugar for \(\lambda x.\lambda y . e\). Predefined functions, such as +, are represented by constant terms are are fully qualified.

Our CAS plugin takes as input a SODE encoded as a term, which it turns into input for a CAS. The CAS returns a solution in its own internal representation, if one exists, and this is turned into another Isabelle term, and certification of the solution is attempted. Our approach builds on both on Immler’s library for representing SODEs and their solutions \cite{10,9} (HOL-ODE).

In the next section we describe the approach for SODE certification.

3 Certifying SODE Solutions

In this section we describe how SODE solutions can be certified using our \texttt{ode_cert} proof tactic. We assume a SODE of the form \(\dot{x}(t) = f(t(x(t))\), described by a function \(f : \mathbb{R} \to \mathbb{R}^n \to \mathbb{R}^n\), which gives a vector of derivatives for each continuous variable at time \(t\) and current state \(x(t)\). A candidate solution to this SODE is a function \(x : \mathbb{R} \to \mathbb{R}^n\), which can potentially have a restricted domain \(T \subseteq \mathbb{R}\) and range \(D \subseteq \mathbb{R}^n\). For example, consider the following SODE:
Example 1. 

\( \dot{x}(t), \dot{y}(t), \dot{z}(t) = (t, x(t), 1) \)

It has three continuous variables, and can be represented with the function \( g \triangleq (\lambda t (x, y, z). (t, x, 1)) \) whose type is \( \mathbb{R} \to \mathbb{R}^3 \to \mathbb{R}^3 \). Its representation in Isabelle is shown in Figure 1, where a name is introduced for it by an abbreviation.

The goal of the **ode_cert** tactic, then, is to prove conjectures of the form

\[ x \text{ solves-ode } f \ T \ D \]

which specifies that \( x \) is indeed a solution to \( f \), and is defined within the HOL-ODE package [10]. It requires that we solve the following two predicates:

\[ (\forall t \in T. (x \text{ has-vector-derivative } (f t (x t)) \text{ (at } t \text{ within } T))) \text{ and } x \in T \to D \]

We need to show that at every \( t \) in the domain, the derivative of \( x \) matches the one predicted by \( f \), and that \( x \) has the correct domain and range. The predicate **has-vector-derivative** is defined within the HOL-Analysis package [8], which also provides a large library of differentiation theorems. For brevity, we use the syntax \( f \triangleright f'[t \in T] \) to mean \( (f \text{ has-vector-derivative } f') \text{(at } t \text{ within } T) \).

**Theorem 1 (Derivative Introduction Theorems).**

\[
\begin{align*}
\frac{(\lambda x. c) \triangleright 0 [t \in T]}{(\lambda x. c) \triangleright 0 [t \in T]} & \quad (a) \\
\frac{(\lambda x. x) \triangleright 1 [t \in T]}{(\lambda x. x) \triangleright 1 [t \in T]} & \quad (b) \\
\frac{f \triangleright f'[t \in T] \quad g \triangleright g'[t \in T]}{(\lambda x. (f x, g x)) \triangleright (f', g') [t \in T]} & \quad (c) \\
\frac{f \triangleright f'[t \in T] \quad g \triangleright g'[t \in T]}{(\lambda x. f x + g x) \triangleright f' + g' [t \in T]} & \quad (d) \\
\frac{f \triangleright f'[t \in T] \quad f' \triangleright 0}{(\lambda x. \sin(f x)) \triangleright (f' \cdot \cos(f t)) [t \in T]} & \quad (e) \\
\frac{f \triangleright f'[t \in T] \quad f \triangleright f' [t \in T]}{(\lambda x. \sqrt{f x}) \triangleright (f' \cdot 1/(2 \cdot \sqrt{f t})) [t \in T]} & \quad (f) \\
\frac{f \triangleright f'[t \in T] \quad g \triangleright g'[t \in T] \quad g t \neq 0}{(\lambda x. f x/g x) \triangleright (-f \triangleright (1/(g t) \cdot g' \cdot 1/(g t)) + f'/g t) [t \in T]} & \quad (g)
\end{align*}
\]

These are standard laws, but in a deductive rather than equational form. A constant function \( \lambda x. c \) has derivative 0 (a), and the identity function \( \lambda x. x \) has derivative 1 (b). If a derivative is a composed of a pair \( (f', g') \) then it can be decomposed into two derivative proofs (c). This law is particularly useful for decomposing a SODE into its component ODEs. A function composed of two summed components can similarly be composed (d). The derivative of \( \sin \) is \( \cos \) (e). The remaining two rules are for square root (f) and division (g). They both have additional provisos to avoid undefinedness. Square root \( \sqrt{x} \)
can be differentiated only when $x > 0$. Similarly, a division requires that the denominator is non-zero, hence the extra proviso.

The strategy employed by ode_cert is as follows:

Algorithm 1 (SODE Certification Method)

1. Decompose a SODE in $n$ variables to $n$ subgoals of the form $f_i \triangleright f'_i [t \in T]$ for $1 \leq i \leq n$;
2. Replace every such goal with two goals: $f_i \triangleright X_i [t \in T]$ and $X_i = f'_i$ using a fresh meta-variable $X_i$. The latter goal is used to prove equivalence between the expected and actual derivative in $f'$;
3. For each remaining derivative goal, recursively apply the derivative introduction laws (Theorem 1). If any derivative goals remain, the method fails;
4. The remaining subgoals are equalities and inequalities in the real variables. Attempt to discharge them all using real arithmetic and field laws using the simplifier tactic for recursive equational rewriting.
5. If no goals remain, the ODE is certified.

We exemplify this method with Example 1, using $f \triangleq (\lambda t \ (x,y,z). (t,x,1))$. A proposed solution is $x \triangleq (\lambda t \ (t^2/2 + x_0, t^3/3 + x_0 \cdot t + y_0, z_0 + t))$, where $x_0, y_0, z_0$ are integration constants, or initial values for variables. We form the goal $x \text{ solves-ode } f \ R \ R^3$ and execute ode_cert. The domain and range constraints are trivial in this case. Following step (1), we obtain 3 subgoals:

1. $(\lambda t \ t^2/2 + x_0) \triangleright t [t \in T]$;
2. $(\lambda t \ t^3/3 + x_0 \cdot t + y_0) \triangleright x [t \in T]$;
3. $(\lambda t \ z_0 + t) \triangleright 1 [t \in T]$

We focus on the second subgoal. Having applied the derivative introduction laws, we receive two proof obligations. The first is $6 \neq 0$, which is required since 6 is the denominator in a division, and is trivial. The second is the following equality:

$$-(t^3) \cdot (1/6 \cdot 0 \cdot 1/6) + 3 \cdot 1 \cdot t^3^{-1}/6 + (x_0 \cdot 1 + 0 \cdot t) + 0 = t^2/2 + x_0$$

Though seemingly complex, it simplifies to give the desired result, since all but one of the summands reduce to 0. This, and more complex goals, can be solved using the built-in simplification sets algebra_simps and field_simps. The other two derivative subgoals similarly reduce, and so the solution is certified.

The interface for the CAS tools is through two Isabelle commands:

```
ode_solve <SODE>
ode_solve_thm (<NAME>:)? <SODE>? <DOM>? <CODOM>? <ASSM>?
```

The node_sole command takes a SODE in the form used in Example 1, and sends this to the CAS for processing. If a solution is found, the tool suggests a lemma that can be inserted of the form $x \text{ solves-ode } f \ T \ D$, with a concrete solution $x$, in the style of the sledgehammer tool [11,12]. The given lemma is proved using ode_cert. node_solve_thm produces a lemma directly, with the given
name. It also optionally allows specification of an explicit domain, codomain, and assumption. The assumption is necessary if the SODE contains constants that are locally constrained. Different CAS systems can be selected using the Isabelle variable SODE_solver, which can take the value fricas, maxima, sympy, or wolfram. An example showing our tool can be seen in Figure 4.

In the next two sections we describe our integrations of Isabelle with SageMath and the Wolfram Engine.

4 SageMath Integration

SageMath [24] is an open source competitor to the Wolfram Engine. Its functionality is accessed via calls to a Python API. It integrates several open source CAS systems in order to provide its functionality, in each case choosing the best implementation for a particular symbolic computation. This makes SageMath an ideal target for integration with Isabelle. In the latest version of SageMath (version 9.1), Maxima [25] is the default CAS for solving SODEs. FriCAS [26] is also an option, though this is not bundled with SageMath by default. Our plugin also supports the CAS SymPy [27], which is implemented using the SymPy to SageMath translation functions.

An overview of the SageMath pipeline is shown in Figure 2. The distinct steps that the SageMath integration uses are Steps 2 to 6.

**Step 2 and 6: Conversion.** In Step 2, the SageMath integration code receives a term for the input SODE. This is traversed, converted to a string containing Python code, passed to a Python integration script over the command line, and evaluated using Python’s eval function. In Step 6 the opposite happens: the Python integration script traverses the expression, converts it to a string containing Isabelle code, and returns it on standard output. This string is then evaluated using the Isabelle function Syntax.read_prop : context -> string -> term, which parses and type checks a proposition term in a given proof context.

Converting between the two representations is mostly a task of mapping between function names. However, there are several exceptions to this rule:

1. Numbers in Isabelle are in a decomposed binary format. The Isabelle function HOLogic.dest_number is used to convert these to integer values.
2. There are different operators in Isabelle for integer, rational and real powers, whereas SageMath uses one operator. When converting from SageMath to Isabelle, the plugin chooses the simplest type of power function.
Initialise `simple-equations` to an empty array;
repeat
  foreach equation in the `sode` do
    if equation is of the form \( \dot{y} = x \) for any variable \( x \) or \( y \) then
      replace any occurrence of \( x \) in `sode` with \( \dot{y} \);
      append equation to `simple-equations`;
      remove equation from `sode`
    end
    if equation is of the form \( \dot{y} = f(t, x) \) (the equation is solely a function of its independent and dependent variables) then
      `solved-equation` ← `SolveODE(equation)`;
      replace any occurrence of \( y \) in `sode` with `solved-equation`;
      remove equation from `sode`;
      output `solved-equation` as a solution to `equation`
    end
  end
until `sode` is unchanged;
solve `sode` and output the solutions;
foreach equation as \( \dot{y} = x \) in `simple-equations` do
  find the solution to \( \dot{x} \) from the existing outputs, differentiate it and output this as a solution to \( \dot{y} \)
end

Algorithm 2: Preprocessing Step Algorithm

3. Isabelle does not contain a representation of the mathematical constant \( e \), but rather the exponential function \( e^n \). When \( e \) is used on its own in SageMath, this is converted into \( \exp(1) \).

**Step 3: Preprocessing.** In many CAS systems, the SODE solving functionality is less powerful than the single equation ODE solving functionality. This often means that SODEs can be solved by the CAS system only when rewritten as ODEs. The preprocessing step, described in Algorithm 2, takes advantage of this by rewriting two different types of SODEs:

1. SODEs formed from a higher order ODE, where a variable is introduced to represent a higher derivative, so that the SODE can conform to the format specified in `ode_cert` (for example \((\dot{x}, \dot{y}) = (2x + y, x))\). This can be preprocessed back into the higher order ODE which was originally intended.
2. SODEs formed from two distinct system. For example, the SODE of a particle acting under gravity with a constant horizontal velocity - \((\dot{v}_x, \dot{v}_y) = (2, -g))\). This can be preprocessed into multiple independent ODEs, and solved using the CAS’s ODE solving functionality.

As an example of Algorithm 2, consider again Example 1 \((\dot{x}, \dot{y}, \dot{z}) = (t, x, 1))\). We can apply the first rewriting rule, as the equation \( \dot{y} = x \) exists in this SODE. This means we can transform the equation \( \dot{x} = t \) into \( \dot{y} = t \), yielding a new SODE of the form \((\dot{y}, \dot{z}) = (t, 1))\). The two equations \((\dot{y} = t \text{ and } \dot{z} = 1))\) in this SODE are
expressed solely in terms of their independent and dependent variables, so they can be solved without considering the other equations. This yields the solution:

$$(y, z) = \left( \frac{t^3}{6} + c_0 t + c_1, t + c_2 \right)$$

We can now find $x$ by calculating $\dot{y}$. The final solution is:

$$(x, y, z) = \left( \frac{t^2}{3} + c_0, \frac{t^3}{6} + c_0 t + c_1, t + c_2 \right)$$

**Step 4: Solving.** In Step 4, the input SODE is fed into one of three SODE solvers: SymPy, Maxima and FriCAS. These three solvers were all considered as potential CAS systems to use, in the order of: SymPy then Maxima then FriCAS. FriCAS performs best on the test set, but the option to use the other CAS systems is preserved.

**Step 5: Domain finding.** When verifying the solution of a SODE, ode_cert requires a domain for which the solution is valid. When SageMath returns a solution, it does not return this information, therefore this domain needs to be generated from the solution. The strategy we have taken is to assume the domain for which the solution is valid to be the maximal domain of the solution.

SageMath does not have any maximal domain finding functionality, so we have used SymPy for this part of the pipeline. The function that finds maximal domains in SymPy was also patched to ensure that greater maximal domains can be calculated.\(^2\)

We evaluate our SageMath integration in §6, but first, in the next section, we describe our Wolfram integration.

## 5 Wolfram Engine Integration

Using the Wolfram Engine over SageMath comes with the main disadvantage of losing open-source status, but it also has a few advantages. In our implementation, it is notably faster at producing solutions, and the SODEs require no preprocessing before solving. Our Wolfram interface is written entirely in SML, which makes it easier for those familiar with Isabelle system programming to use and extend.

The implementation of the Wolfram interface is illustrated in Figure 3. First, the Isabelle term that represents the SODE is translated to an equivalent Wolfram expression. This is passed to the Wolfram Engine for solving. The Wolfram solution to the SODE is lexed and parsed, and stored as an ML datatype. The Wolfram interface is used to retrieve and parse the solution domain, and all parsed expressions are translated to Isabelle. Finally, the plugin combines the domain, the solution, and the original SODE to produce the solution theorem.

\(^2\) The pull request for this can be found at https://github.com/sympy/sympy/pull/19024. This is merged at https://github.com/sympy/sympy/pull/19947
Default Wolfram expressions are typeset and difficult to parse, so we instead retrieve solutions from the Wolfram Engine in “full form”\(^3\). This format presents the expression in a similar style to an algebraic datatype, with explicit constructors, and is implemented in our tool as the following ML datatype.

\[
\text{datatype} \ \text{expr} \ = \ \text{Int} \ \text{of} \ \text{int} \ | \ \text{Real} \ \text{of} \ \text{real} \ | \ \text{Id} \ \text{of} \ \text{string} \ | \\
\quad \text{Fun} \ \text{of} \ \text{string} * \ \text{expr} \ \text{list} \ | \ \text{CurryFun} \ \text{of} \ \text{string} * \ \text{expr} \ \text{list} \ \text{list}
\]

We distinguish between functions with only one set of arguments (Fun) and those with several (CurryFun), as the latter are uncommon and dealing with them clutters the code.

To illustrate the implementation stages, the internal representation at each stage is shown for Example 1. The approach for translation of the SODE to Wolfram is the following:

1. Generate an alphabetically ordered variable mapping, for each of the SODE variables, to avoid name clashes and ease solution reconstruction.
2. Translate the term to an equivalent Wolfram expression by traversing the expression tree.
3. Construct a DSolve call using the expression.

\(^3\) Please see [https://reference.wolfram.com/language/ref/FullForm.html](https://reference.wolfram.com/language/ref/FullForm.html).
DSolve is the general differential equation solver for the Wolfram Engine [28], which can solve a list of differential equations for given dependent and independent variables.

We exemplify the translation, again using Example 1. To represent this system, the following variable mapping is used:

\[ t \rightarrow a, x \rightarrow b, y \rightarrow c, z \rightarrow d \]

Using this mapping, the system is translated to the following DSolve call:

\[
\text{DSolve} \[
\begin{align*}
\{ & b'[a]==a, \\
& c'[a]==b[a], \\
& d'[a]==1, \\
& b[a], c[a], d[a]
\}\]
\]

The Wolfram engine is called using its command-line interface wolframscrip, which takes a function call as an argument and returns the result. Warnings are suppressed to facilitate parsing. The Wolfram engine represents solutions as a list of rules, which are simply functions on expressions. Many solutions may be returned, but we only use the first one, which is lexed and parsed. The maximal domain of this solution is retrieved in another call to the Wolfram Engine, similar to the SageMath integration.

The solution to the test ODE after lexing and parsing is the following:

\[
\text{Fun} \("\text{List}"\),
[\text{Fun} \("\text{List}"\),
[\text{Fun} \("\text{Rule}"\),
[\text{Fun} \("b", [\text{Id} \"a\"]\), \text{Fun} \("\text{Plus}", \ldots\)],
\text{Fun} \("\text{Rule}"\),
[\text{Fun} \("c", [\text{Id} \"a\"]\), \text{Fun} \("\text{Plus}", \ldots\)],
\text{Fun} \("\text{Rule}"\),
[\text{Fun} \("d", [\text{Id} \"a\"]\), \text{Fun} \("\text{Plus}", \ldots\)]]]]
\]

Here, the inner most list gives values for each of the continuous variables. The translation from such a Wolfram expression to an Isabelle term is done by reversing the variable mapping and then traversing the expression tree. There are special cases for the constant \(e\), which is translated to the exponent function, and negative powers, similar to the SageMath integration. Solutions may be provided by the Wolfram Engine which use functions not available in Isabelle, such as those containing integrals. These are reported as errors. Finally, the lemma is assembled by combining the domain, solution, and original SODE.

This completes our description of the two CAS integrations. In the next section we evaluate them both.

6 Evaluation

In this section, we evaluate our approach to certifying SODEs. We consider two test sets, to which we apply both CAS integrations, and then evaluate the results.
The first test set is generated programmatically by searching the KeYmaera X example repository for any lines containing fragments of the form `<EQUATION>`, which describes a SODE in KeYmaera X. Any duplicate SODEs are then combined. In total, 148 SODEs were found, 79 were duplicate pairs, leaving 69 unique SODEs to add to the test set. To represent these equations, and from now on, $t$ is the independent variable, meaning $(\dot{x}, \dot{y}) = (dx/dt, dy/dt)$: The KeYmaera X tests contains many “simple” SODEs. For example, the basic gravity SODE:

$$ (\dot{h}, \dot{v}) = (v, -g) $$

was present in many variations. A few of the test cases also contains more complex SODEs, such as the following example taken from the test set:

$$ (q_x, q_y, f_x, f_y) = \left( f_x \frac{K q_x}{D}, f_y \frac{K q_y}{D}, f_{xp}, f_{yp} \right) $$

We class complex SODEs as those that contain at least one operator, excluding unary minus (for example $-2$). For example, Equation 4 contains four: two multiplication operators and two division operators. Under this classification, this test set contains 20 “complex” SODEs and 49 “simple” SODEs.

The KeYmaera X test set is restricted by the capabilities of KeYmaera X, which doesn’t support transcendental functions, and also its typical applications – cyber-physical systems. In order to test the full capabilities of our tool, we construct a second test set. For this, we try to cover a wide variety of cases, to highlight any areas where our tool might have problems. Table 1 contains this test set, together with the rationale behind why each expression was chosen. Figure 4 shows some of these test cases running in the tool.

Most of the KeYmaera test cases were successfully solved by both SageMath/FriCAS+ and Wolfram using `ode_cert`, apart from five test cases:

$$ \dot{x} = x^2 + x^4 $$

$$ (\dot{x}, \dot{y}) = (x - 3^4 + y^5, y^2) $$

$$ \dot{x} = x - 3^4 + a $$

$$ (\dot{x}, \dot{y}) = (v, g + dv^2) $$

$$ (\dot{x}, \dot{y}) = (y, -w^2 x - 2dw y) $$

Both SageMath/FriCAS+ and Wolfram could not solve the first three equations. In KeYmaera X [2] and also Isabelle/HOL [21], such SODEs can be verified using

4 These examples can be found here: [https://github.com/LS-Lab/KeYmaera-release/tree/master/examples/hybrid](https://github.com/LS-Lab/KeYmaera-release/tree/master/examples/hybrid)
Table 1. Test cases

| Number | ODE System: \( \dot{x}; (\dot{y}, \ldots) = \ldots \) | Rationale |
|--------|--------------------------------------------------|------------|
| 1      | \( x + t \)                                     | Inhomogeneous polynomial |
| 2      | \( \tan(t) \)                                   | Tangent function |
| 3      | \( x^2 \)                                       | Second order polynomial |
| 4      | \( -y, x \)                                     | Trigonometric solution |
| 5      | \( 1/t \)                                       | Domain issues at 0 |
| 6      | \( 1/(2x - 1) \)                                | Has two solutions |
| 7      | \( xy, 3 \)                                     | Contains a factor of \( xy \) |
| 8      | \( 2x + y, x \)                                 | Homogeneous 2\(^{nd}\) order SODE |
| 9      | \( 2x + y + t^2, x \)                           | Inhomogeneous 2\(^{nd}\) order SODE |
| 10     | \( \arcsin(t) \)                                | Inverse trigonometric function |
| 11     | \( \sqrt{t} \)                                  | Square root |
| 12     | \( \sqrt[3]{t} \)                               | Higher roots |
| 13     | \( i\sqrt{2} \)                                 | Non-rational powers |
| 14     | \( x + y, y + 2z, x^2 + 1 \)                    | Higher dimensional SODE |
| 15     | \( x^2 - t \)                                   | Bessel function |
| 16     | \( y, e^{t^2} \)                                | Imaginary error function |
| 17     | \( \sin(x)/\ln(x) \)                           | Impossible to solve |
| 18     | \( \ln(t), x \)                                | Logarithmic |

differential induction [13] rather than explicit solutions. Equation 8 was solved correctly only by Wolfram Engine, but \texttt{ode\_cert} was unable to prove this and Equation 9 was solved by both Wolfram and SageMath, but \texttt{ode\_cert} was again unable to certify it. In all the cases where \texttt{ode\_cert} was unable to prove the result, this was due to a failure to prove a large algebraic proposition containing more than 50 operator applications.

In addition, 7 of the test cases required an assumption to be specified in the \texttt{ode\_solve\_thm} statement. For example, the SODE

\[
(\dot{x}; \dot{t}) = (c + b(u - x), 1)
\]

does not require the assumption \( b > 0 \). This means that out of the 20 “complex” test cases, 15 could be automatically solved and verified and all of the 49 “simple” test cases could be automatically solved and verified.

The results from our additional SODE test cases are presented in Table 2. In these results, 10 of the 18 test cases could not be automatically proved by Isabelle. There are four distinct reasons behind these failures:

1. The tactic \texttt{ode\_cert} cannot automatically prove the stated theorem. In all of the test cases where this occurs, this is due to \texttt{ode\_cert} failing to prove an algebraic proposition. This occurs in test case 2 for SageMath’s result; and 8, 9, 10, 13 for both CAS systems. We have been able to prove test cases 8, 9 and 2 correct manually with the help of the \texttt{sledgehammer} tool\(^5\) [12]. The goals left to prove in cases 8, 9, 10 and 13 contained more than 50 operators.

\(^5\)This work can be found here: [https://github.com/ThomasHickman/Isabelle-CAS-Integration/blob/master/manually\_solved\_cases.thy](https://github.com/ThomasHickman/Isabelle-CAS-Integration/blob/master/manually\_solved\_cases.thy)
Table 2. Results from the SODE tests.

| Number | Solved by the CAS | Correct domain found | Proved automatically in Isabelle | Solved by the CAS | Correct domain found | Proved automatically in Isabelle |
|--------|-------------------|----------------------|-----------------------------------|-------------------|----------------------|-----------------------------------|
| 1      | ✅                | ✅                   | ✅                                | ✅                | ✅                   | ✅                                |
| 2      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 3      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 4      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 5      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 6      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 7      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 8      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 9      | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 10     | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 11     | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 12     | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 13     | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |
| 14     | ✗                | N/A                 | N/A                              | ✗                | N/A                 | N/A                              |
| 15     | ✗                | N/A                 | N/A                              | ✗                | N/A                 | N/A                              |
| 16     | ✗                | N/A                 | N/A                              | ✗                | N/A                 | N/A                              |
| 17     | ✗                | N/A                 | N/A                              | ✗                | N/A                 | N/A                              |
| 18     | ✅                | ✗                   | ✗                                | ✗                | ✗                   | ✗                                |

1 This refers to using SageMath/FriCAS with the preprocessing step.

2. The CAS system cannot produce the correct answer, but if it did, Isabelle could not prove the answer, as appropriate derivative laws have not been implemented. This occurs in test cases 15 and 16.
3. The CAS system cannot produce the correct answer, and it is theoretically impossible for it to do so. This occurs in test case 17.
4. The CAS system cannot produce the correct answer, but Isabelle does contain the derivative laws to prove the answer, if one was produced. This occurs in test case 14.

Consequently, if we exclude the cases where the CAS system cannot provide a solution, and include those where a proof using sledgehammer was required, the rate of success is 11 out of 14, with 3 uncertifiable solutions.

7 Conclusions

In this paper, we described our work on integrating Isabelle with SODE solving in SageMath and Wolfram, to support verification of hybrid systems. In §3 we introduced the tactic ode_cert for the automatic certification of SODE solutions. In §4 and §5 we described our integration with SageMath and Wolfram, respectively. In §6 we evaluated our plugin using two different test sets: one generated from the KeYmaera X examples, and one with more complex examples.
We consider our approach to be successful. Our integration managed to solve and prove most of the test cases that we have devised. This means that projects using SODEs with a similar scope to those in the KeYmaera X examples should be able to use our plugin to generate certified solutions. This largely down to the impressive library of theorems developed in HOL-Analysis [8] and HOL-ODE [10,9], which allow certification to be substantially automated.

However, as we have noted five of our test cases (two from KeYmaera X, and three of our own) produced solutions that could not be certified. This could either be due to lack of proof rules for derivation and real arithmetic in Isabelle. Alternatively, it could be that the solutions returned by the CAS systems are in reality approximations, as indicated by their size compared to the actual SODE. We plan to investigate this further in the future. Either way, we note that Isabelle places a high bar on the artifacts that are accepted as mathematically sound, which gives confidence that they can be used in safety critical applications.

In future work, we plan to use our integration as part of a Isabelle-based hybrid systems verification tool, using our implementation of dL and related calculi [21,29]. We aim to apply to a number of example projects, such as the KeYmaera X examples. This may expose areas in the Isabelle/HOL hybrid systems infrastructure that require improvement. In addition, we will investigate the integration of other CAS features into Isabelle. One example of this is quantifier elimination, which would further improve automation.

Acknowledgements. This work is supported by the EPSRC-UKRI Fellowship project CyPhyAssure, grant reference EP/S001190/1.

References

1. Alur, R.: Formal verification of hybrid systems. In: Proc. 9th. ACM Intl. Conf. on Embedded Software (EMSOFT), New York, NY, USA, ACM (2011) 273–278
2. Fulton, N., Mitsch, S., Quesel, J.D., VölP, M., Platzer, A.: KeYmaera X: An axiomatic tactical theorem prover for hybrid systems. In Felty, A.P., Middeldorp, A., eds.: CADE. Volume 9195 of LNCS., Springer (2015) 527–538
3. Gleirscher, M., Foster, S., Woodcock, J.: New opportunities for integrated formal methods. ACM Comput. Surv. 52(6) (2019)
4. Nipkow, T., Wenzel, M., Paulson, L.C.: Isabelle/HOL: A Proof Assistant for Higher-Order Logic. Volume 2283 of LNCS. Springer (2002)
5. Wenzel, M., Wolff, B.: Building formal method tools in the Isabelle/Isar framework. In: TPHOLs. Volume 4732 of LNCS., Springer (2007)
6. Brucker, A., Wolff, B.: Using ontologies in formal developments targeting certification. In: iFM. Volume 11918 of LNCS., Springer (2019) 65–82
7. Foster, S., Nenouchi, Y., O’Halloran, C., Tudor, N., Stephenson, K.: Formal model-based assurance cases in Isabelle/SACM: An autonomous underwater vehicle case study. In: FormaliSE, ACM (2020)
8. Harrison, J.: A HOL theory of Euclidean space. In Hurd, J., Melham, T., eds.: Theorem Proving in Higher Order Logics, 18th International Conference, TPHOLOGs 2005. Volume 3603 of LNCS., Oxford, UK, Springer (August 2005)
9. Immler, F.: A verified ODE solver and the Lorenz attractor. J. Autom. Reasoning 61(1) 73–111
10. Fabian, I., Hötzl, J.: Numerical analysis of ordinary differential equations in Isabelle/HOL. In Beringer, L., Felty, A., eds.: ITP. Volume 7406 of LNCS., Springer (2012) 377–392
11. Blanchette, J.C., Bulwahn, L., Nipkow, T.: Automatic proof and disproof in Isabelle/HOL. In: FroCoS. Volume 6989 of LNCS., Springer (2011) 12–27
12. Blanchette, J.C., Kaliszyk, C., Paulson, L.C., Urban, J.: Hammering towards QED. Journal of Formalized Reasoning 9(1) (2016)
13. Platzer, A.: Differential dynamic logic for hybrid systems. J. Autom. Reas. 41(2) (2008) 143–189
14. Mitsch, S., Ghorbal, K., Vogelbacher, D., Platzer, A.: Formal verification of obstacle avoidance and navigation of ground robots. The International Journal of Robotics Research 36(12) (2017) 1312–1340
15. Jeannin, J.B., Ghorbal, K., Kouskoulas, Y., Schmidt, A., Gardner, R., Mitsch, S., Platzer, A.: A formally verified hybrid system for safe advisories in the next-generation airborne collision avoidance system. Software Tools for Technology Transfer 19(6) 717–741
16. Tuong, F., Wolff, B.: Deeply integrating C11 code support into Isabelle/PIDE. In: F-IDE. Volume 310 of EPTCS. (2019) 13–28
17. Bohrer, B., Tan, Y.K., Mitsch, S., Myreen, M.O., Platzer, A.: VeriPhy: Verified controller executables from verified cyber-physical system models. SIGPLAN Not. 53(4) (2018) 617–630
18. Mitsch, S., Platzer, A.: ModelPlex: Verified runtime validation of verified cyber-physical system models. Form. Methods Syst. Des. 49(1) (2016) 33–74 Special issue of selected papers from RV’14.
19. Bohrer, B., Rahli, V., Vukotic, I., Platzer, A.: Formally verified differential dynamic logic. In Bertot, Y., Vafeiadis, V., eds.: Proc 6th ACM SIGPLAN Conf. on Certified Programs and Proofs (CPP), ACM (2017) 208–221
20. Munive, J.H., Struth, G.: Verifying hybrid systems with modal Kleene algebra. In: RAMICS. Volume 11194 of LNCS., Springer (2018)
21. Munive, J.H., Struth, G., Foster, S.: Differential Hoare logics and refinement calculi for hybrid systems with Isabelle/HOL. In: RAMiCS. Volume 12062 of LNCS., Springer (April 2020)
22. Foster, S., Baxter, J., Cavalcanti, A., Woodcock, J., Zeyda, F.: Unifying semantic foundations for automated verification tools in Isabelle/UTP. Science of Computer Programming 197 (October 2020)
23. Li, W., Passmore, G., Paulson, L.: Deciding univariate polynomial problems using untrusted certificates in Isabelle/HOL. J. Autom. Reasoning 62 (2019) 29–91
24. The Sage Developers: SageMath, the Sage Mathematics Software System (Version 9.0). (2020)
25. Maxima: Maxima, a computer algebra system. version 5.34.1 (2014) Available at http://maxima.sourceforge.net/.
26. FriCAS team: FriCAS—an advanced computer algebra system (2019) Available at http://fricas.sf.net.
27. Meurer, A., Smith, C.P., Paprocki, M., Čertík, O., Kirpichev, S.B., Rocklin, M., Kumar, A., Ivanov, S., Moore, J.K., Singh, S., Rathnayake, T., Vig, S., Granger, B.E., Muller, R.P., Bonazzi, F., Gupta, H., Vats, S., Johansson, F., Pedregosa, F., Curry, M.J., Terrel, A.R., Roučka, v., Saboo, A., Fernando, I., Kulal, S., Cinmran, R., Scopatz, A.: SymPy: symbolic computing in Python. PeerJ Computer Science 3 (January 2017) e103
28. Wolfram Research, Inc.: Wolfram language documentation Available at https://reference.wolfram.com.
29. Foster, S., Gleirscher, M., Calinescu, R.: Towards deductive verification of control algorithms for autonomous marine vehicles. In: 25th Proc. Intl. Conf. on Engineering of Complex Computer Systems (ICECCS), IEEE (March 2021)