Weakly Clean Ideal

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Abstract: Motivated by the concept of clean ideals, we introduce the notion of weakly clean ideals. We define an ideal \( I \) of a ring \( R \) to be weakly clean ideal if for any \( x \in I \), \( x = u + e \) or \( x = u - e \), where \( u \) is a unit in \( R \) and \( e \) is an idempotent in \( R \). We discuss various properties of weakly clean ideals.

Key words: Clean ideals, weakly clean ideals, uniquely clean ideal, weakly uniquely clean ideal.

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1 INTRODUCTION

Here rings \( R \) are associative with unity unless otherwise indicated. The Jacobson radical, set of units, set of idempotents and centre of a ring \( R \) are denoted by \( J(R) \), \( U(R) \), \( Idem(R) \) and \( C(R) \) respectively. Nicholson\(^2\) called an element \( x \) of a ring \( R \), a clean element, if \( x = e + u \) for some \( e \in Idem(R) \), \( u \in U(R) \) and called the ring \( R \) as clean ring if all its elements are clean. Weakening the condition of clean element, M.S. Ahn and D.D. Anderson\(^1\) defined an element \( x \) as weakly clean if \( x \) can be expressed as \( x = u + e \) or \( x = u - e \), where \( u \in U(R) \), \( e \in Idem(R) \). H. Chen and M. Chen\(^2\), introduced the concept of clean ideals as follows: an ideal \( I \) of a ring \( R \) is called clean ideal if for any \( x \in I \), \( x = u + e \), for some \( u \in U(R) \) and \( e \in Idem(R) \). Motivated by these ideas we define an ideal \( I \) of a ring \( R \) as weakly clean ideal if for any \( x \in I \), \( x = u + e \) or \( x = u - e \), where \( u \in U(R) \) and \( e \in Idem(R) \). Also an ideal \( I \) of a ring \( R \) is called uniquely weakly clean ideal if for each \( a \in I \), there exists unique idempotent \( e \) in \( R \) such that \( a - e \in U(R) \) or \( a + e \in U(R) \). We discuss some interesting properties of weakly clean ideals.

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2 Weakly clean ideals

Definition 2.1. An ideal \( I \) of a ring \( R \) is called weakly clean ideal in case every element in \( I \) is a sum or difference of a unit and an idempotent of \( R \).

Clearly every ideal of a weakly clean ring is weakly clean ideal. But there exists non weakly clean rings which contains some weakly clean ideals. Let \( R_1 \) be weakly clean ring and \( R_2 \) be non weakly clean ring. Then \( R = R_1 \oplus R_2 \) is not a weakly clean ring. But clearly \( I = R_1 \oplus 0 \) is weakly clean ideal of \( R \).

Lemma 2.2. If every proper ideal of a ring \( R \) is clean(weakly clean) ideal then the ring \( R \) is also clean(weakly clean) ring.

Proof. Clearly every unit of a ring is clean. Let \( x \in R \setminus U(R) \) then the ideal \( <x> \) is proper ideal of \( R \), so \( x \) is clean in \( R \).

Corollary 2.3. \( R \) is clean(weakly clean) if and only if every proper ideal of \( R \) is clean(weakly clean).

The following is an example of weakly clean ideal which is not an clean ideal.

Example 2.4. For the ring \( R = \mathbb{Z}(3) \cap \mathbb{Z}(5) \), the ideal \( <2^{11}> \) generated by \( 2^{11} \) is weakly clean ideal but not a clean ideal of \( R \).

Following H. Chen and M. Chen\[2\], we define weakly exchange ideal as follows:

Definition 2.5. An ideal \( I \) of a ring \( R \) is called a weakly exchange ideal provided that for any \( x \in I \), there exists an idempotent \( e \in I \) such that \( e - x \in R(x - x^2) \) or \( e + x \in R(x + x^2) \).

Lemma 2.6. Every weakly clean ideal of a ring is a weakly exchange ideal.

Proof. Let \( I \) be a weakly clean ideal of \( R \) and \( x \in I \). Then \( x = u + e \) or \( x = u - e \), where \( u \in U(R) \) and \( e \in Idem(R) \). If \( x = u + e \) then by Lemma 1.2 \[2\], \( x \) satisfies the exchange property. If \( x = u - e \) then consider \( f = u^{-1}(1-e)u \) so that \( f^2 = f \). Now \( u(x + f) = x^2 + x \), so \( x + f \in R(x^2 + x) \).

Theorem 2.7. Let \( R \) be a ring and \( I \) an ideal in which every idempotent is central. Then the following are equivalent:

(i) \( I \) is weakly clean ideal.

(ii) \( I \) is weakly exchange ideal.
Proof. (1) ⇒ (2) is clear by Lemma 2.6.
(2) ⇒ (1) Given any \( x \in I \), we have an idempotent \( e \in Rx \) such that \( 1 - e \in R(1-x) \) or \( 1 - e \in R(1+x) \). If \( 1 - e \in R(1-x) \) then by Theorem 1.3 \([2]\), \( x \) is clean.

\( \Rightarrow \) Suppose, \( 1 - e \in R(1+x) \) then \( e = ax \) and \( 1 - e = b(1+x) \), for some \( a, b \in R \). Assume that \( ea = a \) and \( (1 - e)b = b \) so that \( axa = ea = a \) and \( b(1 + x)b = b \). Here \( ax, xa, b(1 + x), (1 + x)b \) all are central idempotents and \( ax \in (ax)(ax) = (ax)(xa) = x(ax)a = xa \), similarly \( (1 + x)b = b(1 + x) \). Now \( (a + b)(x + (1 - e)) = ax + bx + a(1 - e) + b(1 - e) = 1 \) so \( x + (1 - e) \) is a unit. Hence \( x \) is a weakly clean element.

Corollary 2.8. Every weakly exchange ideal of a ring without nonzero nilpotent elements is a weakly clean ideal.

Lemma 2.9. Let \( R \) be a commutative ring and let \( n \geq 1 \). If \( A \in M_n(R) \) and \( x \in R \), then \( \det (xE_{ij} + A) = xA_{ij} + \det(A) \).

Proof. See Lemma 7 \([3]\). \( \square \)

T. Koşan, S. Sahinkaya and Y. Zhou \([3]\), proved that for a commutative ring \( R \) and \( n \geq 2 \), \( M_n(R) \) is weakly clean if and only if \( R \) is clean. Motivated by this result we generalise the similar result for weakly clean ideals of \( M_n(R) \) as follows:

Theorem 2.10. Let \( I \) be an ideal of a commutative ring \( R \) and let \( n \geq 2 \). Then \( M_n(I) \) is weakly clean ideal of \( M_n(R) \) if and only if \( I \) is a clean ideal of \( R \).

Proof. Let \( I \) be a clean ideal of \( R \) then by Theorem 1.9 \([2]\), \( M_n(I) \) is clean ideal of \( M_n(R) \).

Conversely, Let \( M_n(I) \) is weakly clean ideal of \( M_n(R) \). If possible, assume that \( I \) is not clean ideal of \( R \). Then there exists \( x \in I \) such that \( x \neq u + e \), for any \( e \in Idem(R) \) and \( u \in U(R) \). Consider \( \mathcal{U} = \{ J \triangleleft R : \mathfrak{F} \in R/J \text{ is not clean} \} \). Notice that \( \mathcal{U} \) is non empty and \( \mathcal{U} \) is inductive set, so by Zorn’s Lemma, \( \mathcal{U} \) contains a maximal member, say \( I_1 \). The maximality of \( I_1 \) implies that \( R/I_1 \) is an indecomposable ring. So \( R/I_1 \) is an indecomposable ring and \( \mathfrak{F} \in R/I_1 \) is not clean.

For contradicting the assumption we show that \( A = xE_{11} - xE_{22} \) is not weakly clean in \( M_n(R) \). By Theorem 8 \([3]\), it is clear that \( A \in M_n(R) \) is not weakly clean in \( M_n(R) \). Hence \( I \) is clean ideal of \( R \). \( \square \)

Theorem 2.11. Let \( \{ R_\alpha \} \) be a family of rings and \( I_\alpha \)'s are ideals of \( R_\alpha \), then the ideal \( I = \prod I_\alpha \) of \( R = \prod R_\alpha \) is weakly clean ideal if and only if each \( I_\alpha \) is weakly clean ideal of \( \{ R_\alpha \} \) and at most one \( I_\alpha \) is not clean ideal.
Proof. Let \( I \) be weakly clean ideal of \( R \). Then being homomorphic image of \( I \) each \( I_\alpha \) is weakly clean ideal of \( R_\alpha \). Suppose \( I_{\alpha_1} \) and \( I_{\alpha_2} \) are not clean ideal, where \( \alpha_1 \neq \alpha_2 \). Since \( I_{\alpha_1} \) is not clean ideal, so not all elements \( x \in I_{\alpha_1} \) is of the form \( x = u - e \), where \( u \in U(R_{\alpha_1}) \) and \( e \in \text{Idem}(R_{\alpha_1}) \). As \( I_{\alpha_1} \) is weakly clean ideal of \( R_{\alpha_1} \), so there exists \( x_{\alpha_1} \in I_{\alpha_1} \) with \( x_{\alpha_1} = u_{\alpha_1} + e_{\alpha_1} \), where \( u_{\alpha_1} \in U(R_{\alpha_1}) \) and \( e_{\alpha_1} \in \text{Idem}(R_{\alpha_1}) \), but \( x_{\alpha_1} \neq u - e \), for any \( u \in U(R_{\alpha_1}) \) and \( e \in \text{Idem}(R_{\alpha_1}) \). Similarly there exists \( x_{\alpha_2} \in I_{\alpha_2} \) with \( x_{\alpha_2} = u_{\alpha_2} - e_{\alpha_2} \), where \( u_{\alpha_2} \in U(R_{\alpha_2}) \) and \( e_{\alpha_2} \in \text{Idem}(R_{\alpha_2}) \), but \( x_{\alpha_2} \neq u + e \), for any \( u \in U(R_{\alpha_2}) \) and \( e \in \text{Idem}(R_{\alpha_2}) \). Define \( x = (x_\alpha) \in I \) by

\[
x_\alpha = x_\alpha \quad \text{if} \quad \alpha \in \{\alpha_1, \alpha_2\}
\]

\[
= 0 \quad \text{if} \quad \alpha \notin \{\alpha_1, \alpha_2\}
\]

Then clearly \( x \neq u \pm e \), for any \( u \in U(R) \) and \( e \in \text{Idem}(R) \). Hence at most one \( I_\alpha \) is not clean ideal.

\((\Leftarrow)\) If each \( I_\alpha \) is clean ideal of \( R_\alpha \) then \( I = \prod I_\alpha \) is clean ideal of \( R \) and hence weakly clean ideal of \( R \). Assume \( I_{\alpha_0} \) is weakly clean ideal but not clean ideal of \( I_{\alpha_0} \) and that all other \( I_\alpha \)'s are clean ideals of \( R_\alpha \). If \( x = (x_\alpha) \in I \) then in \( I_{\alpha_0} \), we can write \( x_{\alpha_0} = u_{\alpha_0} + e_{\alpha_0} \) or \( x_{\alpha_0} = u_{\alpha_0} - e_{\alpha_0} \), where \( u_{\alpha_0} \in U(R_{\alpha_0}) \) and \( e_{\alpha_0} \in \text{Idem}(R_{\alpha_0}) \). If \( x_{\alpha_0} = u_{\alpha_0} + e_{\alpha_0} \), then for \( \alpha \neq \alpha_0 \) let, \( x_\alpha = u_\alpha + e_\alpha \) and if \( x_{\alpha_0} = u_{\alpha_0} - e_{\alpha_0} \), then for \( \alpha \neq \alpha_0 \) let, \( x_\alpha = u_\alpha - e_\alpha \) then \( u = (u_\alpha) \in U(R) \) and \( e = (e_\alpha) \in \text{Idem}(R) \), such that \( x = u + e \) or \( x = u - e \) and consequently \( I \) is weakly clean ideal of \( R \).

Next we define the concept of uniquely weakly clean ideal of a ring.

Definition 2.12. An ideal \( I \) of a ring \( R \) is called uniquely weakly clean ideal if for each \( a \in I \), there exists a unique idempotent \( e \) in \( R \) such that \( a - e \in U(R) \) or \( a + e \in U(R) \).

Lemma 2.13. Every idempotent in a uniquely weakly clean ideal is a central idempotent.

Proof. Let \( I \) be a uniquely weakly clean ideal of a ring \( R \) and \( e \) be any idempotent in \( I \). For any \( x \in R \), since \( -e = -(e + ex(1 - e)) + ex(1 - e) = (1 - (e + ex(1 - e))) - (1 - ex(1 - e)) = (1 - e) - 1 \), so \( 1 - (e + ex(1 - e)) = 1 - e \Rightarrow ex = exe \). Similarly we can show that \( xe = exe \). Hence \( xe = ex \).
$m, m' \in M$ and $n, n' \in N$. These conditions ensure that the set of matrices 
$\begin{pmatrix} r & n \\ m & s \end{pmatrix}$, where $r \in R$, $s \in S$, $m \in M$ and $n \in N$ forms a ring denoted by $T$, called the ring of the context. H. Chen and M. Chen[2], showed that for rings $R$ and $S$, if $T$ be the ring of Morita context $\begin{pmatrix} R & M \\ N & S \end{pmatrix}$ with zero pairing and $I$ and $J$ are clean ideals of rings $R$ and $S$ respectively, then $\begin{pmatrix} I & M \\ N & J \end{pmatrix}$ is a clean ideal of $T$. Here we prove the similar result for weakly clean ideal.

**Theorem 2.14.** Let $T = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ be a Morita context. If $I$ and $J$ be weakly clean ideals of $R$ and $S$ respectively and either $I$ or $J$ is clean ideal, then the ideal $\begin{pmatrix} I & M \\ N & J \end{pmatrix}$ is weakly clean ideal of $T$.

**Proof.** Without loss of generality, we can assume that $J$ is clean ideal of $S$. To show $\begin{pmatrix} I & M \\ N & J \end{pmatrix}$ is weakly clean ideal of $T$. Let $A = \begin{pmatrix} a & m \\ n & b \end{pmatrix} \in \begin{pmatrix} I & M \\ N & J \end{pmatrix}$, where $a \in I$, $b \in J$, $m \in M$ and $n \in N$. As $I$ is weakly clean ideal of $R$, so $a = e + u$ or $a = -e + u$, where $e \in \text{Idem}(R)$ and $u \in U(R)$.

**Case I:** If $a = e + u$, then set $b = f + v$, where $f \in \text{Idem}(S)$ and $v \in U(S)$. Let, $E = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ and $U = \begin{pmatrix} u & m \\ n & v \end{pmatrix}$. It is easy to verify that $E = E^2 \in T$ and

$U \begin{pmatrix} u^{-1} & -u^{-1}mv^{-1} \\ -v^{-1}nu^{-1} & v^{-1} \end{pmatrix} = \begin{pmatrix} u^{-1} & -u^{-1}mv^{-1} \\ -v^{-1}nu^{-1} & v^{-1} \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

So $U \in U(T)$.

**Case II:** If $a = -e + u$, then we set $b = -f + v$, where $f \in \text{Idem}(S)$ and $v \in U(S)$. Let, $E = - \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$ and $U = \begin{pmatrix} u & m \\ n & v \end{pmatrix}$. Similar as above $E^2 = E \in \text{Idem}(T)$ and $U \in U(T)$. \qed

Let $A_1$, $A_2$ and $A_3$ be associative rings with identities and $A_{21}$, $A_{31}$ and $A_{32}$ be $(A_2, A_1)$-, $(A_3, A_1)$- and $(A_3, A_2)$-bimodules respectively. Let $\phi : A_{32} \odot A_2 \rightarrow A_{31}$ be an $(A_3, A_1)$-homomorphism then $T = \begin{pmatrix} A_1 & 0 & 0 \\ A_{21} & A_2 & 0 \\ A_{31} & A_{32} & A_3 \end{pmatrix}$ is a lower triangular matrix ring with usual matrix operations.
Theorem 2.15. If \( I, J \) and \( K \) are weakly clean ideals of rings \( A_1, A_2 \) and \( A_3 \) respectively, where at least two of them are clean ideals then the formal triangular matrix ideal \( \begin{pmatrix} I & 0 & 0 \\ A_{21} & J & 0 \\ A_{31} & A_{32} & K \end{pmatrix} \) is a weakly clean ideal of \( \begin{pmatrix} A_1 & 0 & 0 \\ A_{21} & A_2 & 0 \\ A_{31} & A_{32} & A_3 \end{pmatrix} \).

Proof. Assume that \( I \) and \( K \) are clean ideals \( A_1 \) and \( A_3 \) and \( J \) is weakly clean ideal of \( A_2 \). Let, \( B = \begin{pmatrix} A_2 & 0 \\ A_{32} & A_3 \end{pmatrix} \) and \( M = \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} \). As \( J \) is weakly clean ideal of \( A_2 \) and \( K \) is clean ideal of \( A_3 \), so by Theorem 2.14, we see that \( P = \begin{pmatrix} J & 0 \\ A_{32} \end{pmatrix} \) is a weakly clean ideal of \( B \). Again by Theorem 2.14, \( \begin{pmatrix} I & 0 \\ M & P \end{pmatrix} \) is a weakly clean ideal of \( \begin{pmatrix} A_1 & 0 & 0 \\ A_{21} & J & 0 \\ A_{31} & A_{32} & K \end{pmatrix} \).

Theorem 2.16. Let \( A_1, A_2 \) and \( A_3 \) are rings. If the formal triangular matrix ideal \( \begin{pmatrix} I & 0 & 0 \\ A_{21} & J & 0 \\ A_{31} & A_{32} & K \end{pmatrix} \) is a weakly clean ideal of \( T = \begin{pmatrix} A_1 & 0 & 0 \\ A_{21} & A_2 & 0 \\ A_{31} & A_{32} & A_3 \end{pmatrix} \) then \( I, J \) and \( K \) are weakly clean ideals of \( A_1, A_2 \) and \( A_3 \) respectively.

Proof. For \( x \in I \), we have \( \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \begin{pmatrix} I & 0 & 0 \\ A_{21} & J & 0 \\ A_{31} & A_{32} & K \end{pmatrix} \). Thus,

\[
\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e_1 & 0 & 0 \\ * & e_2 & 0 \\ * & * & e_3 \end{pmatrix} + \begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix}
\]

or

\[
\begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} e_1 & 0 & 0 \\ * & e_2 & 0 \\ * & * & e_3 \end{pmatrix} + \begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix}
\]

where \( \begin{pmatrix} e_1 & 0 & 0 \\ * & e_2 & 0 \\ * & * & e_3 \end{pmatrix} \in \text{Idem}(T) \) and \( \begin{pmatrix} u_1 & 0 & 0 \\ * & u_2 & 0 \\ * & * & u_3 \end{pmatrix} \in \text{U}(T) \). It is clear that \( e_1^2 = e_1 \in \text{Idem}(A_1) \) and \( u_1 \in U(A_1) \). Also \( x = e_1 + u_1 \) or \( x = -e_1 + u_1 \), so \( I \) is
weakly clean ideal of \( A_1 \). Similarly we can show that \( J \) and \( K \) are weakly clean ideals of \( A_2 \) and \( A_3 \) respectively.

A finite orthogonal set of idempotents \( e_1, \ldots, e_n \) in a ring \( R \) is said to be complete set if \( e_1 + \cdots + e_n = 1 \).

**Proposition 2.17.** Let \( R \) be a ring and \( I \) an ideal of \( R \). Then the following are equivalent:

(i) \( I \) is a weakly clean ideal of \( R \).

(ii) There exists a complete set \( \{ e_1, e_2, \ldots, e_n \} \) of idempotents such that \( e_i I e_i \) is a weakly clean ideal of \( e_i R e_i \), for all \( i \) and at most one \( e_i I e_i \) is not clean ideal of \( e_i R e_i \).

**Proof.** (1) \( \Rightarrow \) (2) is trivial by taking \( n = 1 \) and \( e_1 = 1 \).

(2) \( \Rightarrow \) (1) It is enough to show the result for \( n = 2 \). Without loss of generality assume that \( e_1 I e_1 \) is weakly clean ideal of \( e_1 R e_1 \) and \( e_2 I e_2 \) is clean ideal of \( e_2 R e_2 \).

It is clear that \( I \cong \left( \begin{array}{cc} e_1 I e_1 & e_1 I e_2 \\ e_2 I e_1 & e_2 I e_2 \end{array} \right) \) and \( R \cong \left( \begin{array}{cc} e_1 R e_1 & e_1 R e_2 \\ e_2 R e_1 & e_2 R e_2 \end{array} \right) \) as \( \{ e_1, e_2 \} \) be a complete set. Let \( A = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \in I \). As \( e_1 I e_1 \) is weakly clean ideal, so \( a_{11} = u + e \) or \( a_{11} = u - e \), where \( e \in Idem(e_1 R e_1) \) and \( u \in U(e_1 R e_1) \). Also \( a_{22} - a_{21} u^{-1} a_{12} \in e_2 I e_2 \).

Case I: If \( a_{11} = e + u \), then we can set \( a_{22} - a_{21} u^{-1} a_{12} = f + v \), where \( f \in Idem(e_2 R e_2) \) and \( v \in U(e_2 R e_2) \) then by Proposition 1.15 [2], \( A \) is a clean element of \( \left( \begin{array}{cc} e_1 R e_1 & e_1 R e_2 \\ e_2 R e_1 & e_2 R e_2 \end{array} \right) \).

Case II: If \( a_{11} = -e + u \) then we can set \( a_{22} - a_{21} u^{-1} a_{12} = -f + v \), where \( f \in Idem(e_2 R e_2) \) and \( v \in U(e_2 R e_2) \). Set \( E = \left( \begin{array}{cc} e & 0 \\ 0 & f \end{array} \right) \) and \( U = \left( \begin{array}{cc} u & a_{12} \\ a_{21} & v + a_{21} u^{-1} a_{12} \end{array} \right) \).

By Proposition 1.15 [2], \( E^2 = E \) and \( U \) is a unit in \( \left( \begin{array}{cc} e_1 R e_1 & e_1 R e_2 \\ e_2 R e_1 & e_2 R e_2 \end{array} \right) \). Also \( A = -E + U \), as required.

**\( \Box \)**

**Proposition 2.18.** Let \( I \) be an ideal of a commutative ring \( R \). Then \( I \) is weakly clean ideal of \( R \) if and only if the ideal \( I[[x]] \) is weakly clean ideal of \( R[[x]] \).

**Proof.** Let \( I \) be a weakly clean ideal of \( R \). Let \( f(x) = \sum a_i x^i \in I[[x]] \), clearly \( a_0 \in I \), so \( a_0 = u_0 + e_0 \) or \( a = u_0 - e_0 \), where \( e_0 \in Idem(R) \) and \( u_0 \in U(R) \).

If \( a_0 = u_0 + e_0 \), then \( f(x) = \sum a_i x^i = e_0 + u_0 + a_1 x + a_2 x^2 + \cdots \), where \( u_0 + a_1 x + a_2 x^2 + \cdots \in U(R[[x]]) \) and \( e_0 \in Idem(R) \subseteq Idem(R[[x]]) \). Similarly for
Let $R$ be a commutative ring and $M$ be a $R$-module. Then the idealization of $R$ and $M$ is the ring $R(M)$ with underlying set $R \times M$ under coordinatewise addition and multiplication given by $(r, m)(r', m') = (rr', rm' + r'm)$, for all $r, r' \in R$ and $m, m' \in M$. It is obvious that if $I$ is an ideal of $R$ then for any submodule $N$ of $M$, $I(N) = \{(r, n) : r \in I$ and $n \in N\}$ is an ideal of $R(M)$. We mention basic existing result about idempotent and unit element in $R(M)$ and study the weakly clean ideals of the idealization $R(M)$ of $R$ and $R$-module $M$.

**Lemma 2.19.** Let $R$ be a commutative ring and $R(M)$ be the idealization of $R$ and $R$-module $M$. Then the following hold:

(i) $(r, m) \in \text{Idem}(R(M))$ if and only if $r \in \text{Idem}(R)$ and $m = 0$.

(ii) $(r, m) \in U(R(M))$ if and only if $r \in U(R)$.

**Proposition 2.20.** Let $R$ be a commutative ring and $R(M)$ is a idealization of $R$ and $R$-module $M$. Then an ideal $I$ of $R$ is weakly clean ideal(clean ideal) of $R$ if and only if $I(N)$ is weakly clean ideal(clean ideal) of $R(M)$, for any submodule $N$ of $M$.

**Proof.** ($\Rightarrow$) Consider $(x, n) \in I(N)$. For $x \in I$, $x = u + e$ or $x = u - e$, where $u \in U(R)$ and $e \in \text{Idem}(R)$, so $(x, n) = (e, 0) + (u, n)$ or $(x, n) = -(e, 0) + (u, n)$, where $(e, 0) \in \text{Idem}(R(M))$ and $(u, n) \in U(R(M))$, by Lemma 2.20.

($\Leftarrow$) Let $r \in I$, for $(r, n) \in I(N)$, $(r, n) = (e, 0) + (u, n')$ or $(r, n) = -(e, 0) + (u, n')$, where $(e, 0) \in \text{Idem}(R(M))$, $(u, n') \in U(R(M))$ and $n, n' \in M$. Hence $r = e + u$ or $r = -e + u$, where $e \in \text{Idem}(R)$ and $u \in U(R)$ by Lemma 2.20, as required.

**Theorem 2.21.** Let $I$ be an ideal of a ring $R$ containing $J(R)$ and idempotents can be lifted modulo $J(R)$. Then $I$ is weakly clean ideal of $R$ if and only if $I/J(R)$ is weakly clean ideal of $R/J(R)$.

**Proof.** ($\Leftarrow$) Let, $x \in I$, so $\overline{x} = \overline{e} + \overline{u}$ or $\overline{x} = -\overline{e} + \overline{u}$, where $\overline{e} \in \text{Idem}(R/J(R))$ and $\overline{u} \in U(R/J(R))$. Hence, $x - e - u \in J(R)$ or $x + e - u \in J(R)$, so $x = e + u + r$ or $x = -e + u + r$, where $r \in J(R)$. Since idempotents can be lifted modulo $J(R)$, we may assume that $e$ is an idempotent of $R$. So $I$ is weakly clean ideal of $R$.

Converse is clear because if $u \in U(R)$ then $u + J(R) \in U(R/J(R))$ and $e + J(R) \in \text{Idem}(R/J(R))$, for $e \in \text{Idem}(R)$. $\square$
If $I + J$, sum of two ideals $I$ and $J$, is weakly clean ideal of $R$ then $I$ and $J$ are also weakly clean ideal of $R$, as $I, J \subseteq I + J$. The converse is not true as shown by the example given below.

Example 2.22. For $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$, the ring $R \times R$ is not weakly clean ring by Theorem 1.7 [1]. Clearly the ideals $< \frac{2}{11} >$ and $< \frac{4}{7} >$ generated by $\frac{2}{11}$ and $\frac{4}{7}$ respectively are weakly clean ideals but not clean ideals of $R$. Let $I_1 = < \frac{2}{11} > \times \{0\}$ and $I_2 = \{0\} \times < \frac{4}{7} >$, then $I_1$ and $I_2$ are weakly clean ideals of $R \times R$ but not clean ideals of $R \times R$. Hence $I_1 + I_2 = < \frac{2}{11} > \times < \frac{4}{7} >$ is not weakly clean ideal of $R \times R$ by Theorem 2.11.

However we have a partial converse as follows.

Proposition 2.23. If $I$ and $J$ are two weakly clean ideals of a ring $R$ and any one of $I$ and $J$ is contained in $J(R)$ then $I + J$ is also weakly clean ideal of $R$.

Proof. Without loss of generality assume $J \subseteq J(R)$ and $x \in I + J$. Then $x = a + b$, where $a \in I$ and $b \in J \subseteq J(R)$. So, there exist $e \in Idem(R)$ and $u \in U(R)$ such that $a = u + e$ or $a = u - e$. Hence $x = e + u + b$ or $x = -e + u + b$, which gives $x$ is a weakly clean element of $R$. \qed

References

[1] Ahn, Myung-Sook and Anderson, DD, Weakly clean rings and almost clean rings. *Rocky Mountain J. Math.*, 6(3):783 – 798, 2006.

[2] Chen, Huanyin and Chen, Miaosen, On clean ideals, *International Journal of Mathematics and Mathematical Sciences* 2003(62):3949 – 3956, 2003.

[3] Nicholson, W Keith, Lifting idempotents and exchange rings, *Transactions of the American Mathematical Society*, 229:269 – 278, 1977.

[4] Tamer Koan and Serap Sahinkaya and Yiqiang Zhou, On weakly clean rings. *Communications in Algebra*, 45(8):3494 – 3502, 2017.