Patterns of primes

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1 Introduction

A few years ago Green and Tao [GT] proved their striking result about patterns in primes.

**Theorem** (Green–Tao). The primes contain arbitrarily long arithmetic progressions.

The method of proof immediately gave that the same result is true for any subset $P'$ of the primes $P = \{p_n\}_{n=1}^{\infty}$ with positive relative upper density, that is with

$$\limsup_{N \to \infty} \frac{|P' \cap [1, N]|}{\pi(N)} > 0$$

(1.1)

(where $\pi(N)$ denotes the number of primes less or equal to $N$, $|A|$ the number of elements of a set $A$, and the fact that the number of $m$-term arithmetic progressions obtained below $N$ is $\gg N^2 (\log N)^{-m}$.

Another, albeit conditional result of Goldston, Yıldırım and the author [GPY2] yielded the existence of other patterns.

**Theorem** ([GPY2]). If the primes have a distribution level $\vartheta > 1/2$, that is, if for any positive $\varepsilon$ and $A$ we have

$$\sum_{q \leq N^{\vartheta - \varepsilon}} \max_{(a,q)=1} \left| \sum_{p \leq x, \quad p \equiv a \pmod{q}} \log p - \frac{N}{\varphi(q)} \right| \lesssim_{\varepsilon, A} \frac{N}{\log A} N,$$

(1.2)

then there exists a positive even $d \leq C_1(\vartheta)$ and infinitely many pairs of primes

$$p, p + d \in P.$$  

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The author showed recently that a combination of the two above results is possible, showing thereby new patterns of primes.

**Theorem** [Pin]. *If the primes have a distribution level \( \vartheta > 1/2 \) then there exists a positive even \( d \leq C_1(\vartheta) \) such that the set \( P(d) \) of primes \( p \) satisfying (1.3) contains arbitrarily long arithmetic progressions.*

**Remark.** The above (conditionally existing) patterns form two-dimensional arithmetic progressions with one difference being bounded.

**Remark.** In the above two theorems we have \( 0 < d \leq 16 \) if \( \vartheta > 0.971 \), in particular, if the Elliott–Halberstam conjecture [EH] \( \vartheta = 1 \) is true. On the other hand, the best unconditional result \( \vartheta = 1/2 \), the celebrated Bombieri–Vinogradov theorem, unfortunately does not imply the existence of infinitely many bounded gaps between consecutive primes.

However, beside Selberg’s sieve, the Bombieri–Vinogradov theorem played a crucial role in the proof [GPY2] of

\[
\Delta_1 = 0, \quad \text{where} \quad \Delta_\nu = \liminf_{n \to \infty} (p_{n+\nu} - p_n) / \log p_n,
\]

thereby improving the best known bound

\[
\Delta_1 < 0.2486
\]

of Helmut Maier [Mai].

One important question which remained open after the work [GPY1] was whether the small gaps of size \( < \eta \log p \) appear with a positive density for any \( \eta > 0 \). Since the existence of some patterns in a subset of primes can be deduced from information about the relative density of the subset, this gives an extra interest to problems asking whether some “events” as short gaps between consecutive primes or short blocks of gaps between consecutive primes appear in a positive proportion of all cases or not. This motivates the definition of the quantities

\[
\Delta^*_\nu = \inf \left\{ c_\nu; \liminf_{x \to \infty} \left\{ \frac{\{p_n \leq x; p_{n+\nu} - p_n \leq c_\nu \log p_n\}}{\pi(x)} > 0 \right\} \right\}.
\]

The methods of Hardy–Littlewood, [HL, Ran] Erdős [Erd], Bombieri–Davenport [BD] and its refinements by Huxley [Hux1, Hux2, Hux3] yielded always a positive proportion of small gaps; however, the ingenious improvement (1.5) by H. Maier [Mai] just showed the existence of a rare set of small
gaps or blocks of gaps. Thus, our knowledge in the time of Maier’s work was as follows:

\[
\begin{align*}
\Delta_1^* &\leq 0.4425 \ldots \text{[Hux2]},
\Delta_1 \leq e^{-\gamma} \cdot 0.4425 \ldots \approx 0.2486 \ldots \text{[Mai]}, \\
\Delta_\nu^* &\leq \nu - \frac{5}{8} + o(1) \text{ [Hux2]},
\Delta_\nu \leq e^{-\gamma} \left( \nu - \frac{5}{8} + o(1) \right) \text{ [Mai]}. 
\end{align*}
\]

**Remark.** A slight improvement over (1.7)–(1.8) is contained in [Hux3]. However, Maier refined the version (1.7)–(1.8) of [Hux2].

Goldston and Yıldırım [GY] worked out a method which yielded

\[
\Delta_1^* \leq 1/4
\]

and it remained unclear whether the method of [GPY2] proving \( \Delta_1 = 0 \) is able to yield \( \Delta_1^* = 0 \) too. Very recently, this question was answered positively.

**Theorem** [GPY3]. *Unconditionally we have \( \Delta_1^* = 0 \); further the Elliott–Halberstam conjecture [EH] implies \( \Delta_2^* = 0 \).*

Taking into account the stronger form of the Green–Tao Theorem (cf. (1.1)) the above theorem implies

**Corollary 1.** Let \( \eta > 0 \) be arbitrary, \( p' \) be the prime following \( p \). Then the set

\[
P'(\eta) = \{ p \in \mathcal{P}; \ p' - p \leq \eta \log p \}
\]

contains arbitrarily long arithmetic progressions. The same is true under EH for

\[
P''(\eta) = \{ p \in \mathcal{P}; \ p_{n+2} - p_n \leq \eta \log p_n \}.
\]

The method of proof of [GPY3] can also yield that the best unconditional bound of [GPY2],

\[
\Delta_\nu \leq \left( \sqrt{\nu} - \sqrt{2\theta} \right)^2,
\]

can be refined to

\[
\Delta_\nu^* \leq \left( \sqrt{\nu} - \sqrt{2\theta} \right)^2.
\]
Remark. The unconditional result

\[ \Delta_\nu \leq e^{-\gamma} (\sqrt{\nu} - 1)^2 \]

of the work [GYP3] cannot be modified to yield the same estimate to \( \Delta^*_\nu \) as well, since it uses Maier’s matrix method too (as can be guessed from the factor \( e^{-\gamma} \)), which in general yields just a negligible portion of primes with a given property.

The aim of this note is to show that the method of the mentioned work [GYP3] can be modified to yield for any fixed \( \eta > 0 \) for \( N \to \infty \) many \( \nu + 1 \)-dimensional patterns of type \( (d_1, \ldots, d_\nu) \) with \( 0 < d_1 < \cdots < d_\nu \) such that

\[ |\mathcal{P}(d_1, \ldots, d_\nu)| = \left| \{ p \in \mathcal{P}, p \in [N, 2N), p + d_i \in \mathcal{P} (i=1,\ldots,\nu) \} \right| \geq \frac{c_1(\nu, \eta)N}{(\log N)^{\nu+1}}, \]

\[ d_\nu \leq (\Delta^*_\nu + \eta) \log N, \]

where we choose \( c_1(\nu, \eta) \) sufficiently small, depending on \( \eta \) and \( \nu \).

The exact formulation of our result to be proved is as follows.

**Theorem.** Let \( \eta > 0 \) be any positive constant, \( \nu \) and \( m \) natural numbers. Then we have a positive constant \( c(\eta, \nu) \) depending on \( \eta \) and \( \nu \) such that for any \( N > N_0(\eta, \nu, m) \) we have a set \( \mathcal{D}_N^\nu \) of \( \nu \)-tuples \( (d_1, \ldots, d_\nu) \) with \( 0 < d_1 < \cdots < d_\nu \) such that

\[ |\mathcal{D}_N^\nu| \geq c(\eta, \nu) \log^\nu N \]

and every element of \( \mathcal{D}_N^\nu \) satisfies \((1.15)\) and \((1.16)\).

**Corollary** Under the above conditions, if \( (d_i)_{i=1}^\nu \in \mathcal{D}_N^\nu \) then the set \( \mathcal{P}(d_1, \ldots, d_\nu) \) of primes contains at least \( c'(\eta, \nu, m) \frac{N^2}{\log N} \) arithmetic progressions of length \( m \).

Remark. In such a way we actually obtain a large number of \( \nu + 1 \)-dimensional arithmetic progressions, more exactly a positive proportion of all \( \nu \)-tuples \( (d_1, \ldots, d_\nu) \) with \( 0 < d_1 < \cdots < d_\nu \leq (\Delta^*_\nu + \eta) \log N \) will appear as a configuration of primes \( p(j) + d_i \in \mathcal{P}, p(j) \in \mathcal{P} \) where \( \{p(j)\}_{j=1}^m \) forms an \( m \)-term arithmetic progressions (and consequently so do the primes \( p(j) + d_i \) for all \( i \in [1, \nu] \)).
2 Proof of the Theorem

The number of $\nu+1$-tuple of primes satisfying $p + d_i \in \mathcal{P}$ for a concrete $\mathcal{D} = (d_1, \ldots, d_\nu)$ can be estimated from above by Selberg’s sieve (cf. Theorem 5.1 of [HR] or Theorem 2 in § 2.2.2 of [Gre])

\begin{equation}
|\mathcal{P}(d_1, \ldots, d_\nu)| \ll_N \frac{N \mathcal{G}(\mathcal{D}^+)}{(\log N)^{\nu+1}}, \quad \mathcal{D}^+ = \mathcal{D} \cup \{0\}.
\end{equation}

This would be immediately sufficient to prove a positive proportion of the required prime $\nu+1$-tuples if $\mathcal{G}(\mathcal{H})$ were bounded for $k$-tuples $\mathcal{H}$ of a given size, which is not the case. However, using the definition (1.6) of $\Delta^*_\nu$, with the notation

\begin{equation}
H := \lfloor (\Delta^*_\nu + \eta) \log N \rfloor,
\end{equation}

we have (with $c_i$ depending always on $\eta$ and $\nu$) by the definition of $\Delta^*_\nu$ and (2.1)

\begin{equation}
\frac{c_1 N}{\log N} \leq \sum_{\mathcal{D} \subseteq [1,H]} \frac{c_2 N \mathcal{G}(\mathcal{D}^+)}{(\log N)^{\nu+1}} \leq c_3 \frac{N}{\log N} \cdot \frac{1}{H^\nu} \sum_{\mathcal{D} \subseteq [1,H]} \mathcal{G}(\mathcal{D}^+).
\end{equation}

Deleting from the summation those $\mathcal{D}$’s for which with a sufficiently small $c_4$ we have

\begin{equation}
|\mathcal{P}(d_1, \ldots, d_\nu)| \leq \frac{c_4 N \mathcal{G}(\mathcal{D}^+)}{(\log N)^{\nu+1}},
\end{equation}

we obtain for a subset $\mathcal{D}$ of all $\mathcal{D} \subseteq [1,H]$ (we denote summation over this subset by $\sum^*$)

\begin{equation}
\frac{c_1 N}{2 \log N} \leq c_3 \frac{N}{\log N} \cdot \frac{1}{H^\nu} \sum_{\mathcal{D} \subseteq \mathcal{D}^*} \mathcal{G}(\mathcal{D}^+).
\end{equation}

In order to prove our theorem it is clearly sufficient to show

\begin{equation}
\sum_{\mathcal{D} \subseteq \mathcal{D}^*} 1 \geq c_5 H^\nu.
\end{equation}

Now, using Cauchy’s inequality, (2.5) implies

\begin{equation}
H^\nu \leq c_6 \sum_{\mathcal{D} \subseteq \mathcal{D}^*} \mathcal{G}(\mathcal{D}^+) \leq c_6 \left( \sum_{\mathcal{D} \subseteq \mathcal{D}^*} 1 \sum_{\mathcal{D} \subseteq [1,H]} \mathcal{G}(\mathcal{D}^+) \right)^{1/2}.
\end{equation}
Hence, in order to show (2.6), thereby our Theorem, it is sufficient to show the following

**Lemma 1.** For fixed \( \nu \) and any \( H > H_0(\nu) \) we have

\[
(2.8) \quad \sum_{D \subset [1,H], |D|=\nu} \mathcal{S}^2(D^+) \leq c_7(\nu)H^\nu.
\]

**Remark.** The parameter \( H \) can be arbitrary here, not just that given in (2.2).

**Remark.** The above lemma is somewhat analogous to Gallagher’s theorem

\[
(2.9) \quad \sum_{D \subset [1,H], |D|=\nu} \mathcal{S}(D) \sim H^\nu,
\]

the difference being the non-essential appearance of \( D^+ = D \cup \{0\} \) in place of \( D \) and the more essential change in the exponent: two instead of one.

Since the singular series is interesting in itself and appears often in additive number theory, it might be interesting to prove with the same effort a more general form of it as

**Lemma 2.** For fixed \( \nu r \) and \( H > H_0(\nu, r) \) we have

\[
(2.10) \quad S(\nu, r) = \sum_{D \subset [1,H], |D|=\nu} \mathcal{S}^2(D^+) \leq c_8(\nu, r)H^\nu.
\]

**Remark.** The condition \( H > H_0(\nu, r) \) and \( H > H_0(\nu) \) is naturally not necessary if we do not care about the values of the constants \( c_7(\nu) \) and \( c_8(\nu, r) \).

**Remark.** In case of \( r = 1 \) we will additionally show, similarly to (2.9), \( S(\nu, r) \sim H^\nu \) as \( H \to \infty \). This slightly modified form implies easily the original Gallagher’s theorem too, by dividing all possible \( \nu + 1 \)-tuples according to the smallest element of it and using that \( \mathcal{S}(H) \) is invariant under translation.

**Proof of Lemma 2** We will prove in fact a little bit more. Namely, the fact that extending every concrete admissible \( D \cup \{0\} \) of size \( t + 1 \geq 1 \) with just one element running over \([1,H]\) the square of the singular series will be
larger at most by a factor depending on \( t \). In such a way, (2.10) follows by induction from

\[
(2.11) \quad S^*(t, r, D) := \sum_{1 \leq h \leq H, h \notin D} \left( \frac{S(D^+ \cup \{h\})}{S(D^+)} \right)^r \ll H
\]

where \( D^+ \) is any admissible set of size \( t + 1 \) and, as in the following, we will not mark the dependence of the constants implied by \( \ll \) or \( \ll 0 \) symbols on \( t \) and \( r \). We can start with \( D^+ = D \cup \{0\} = \{0\} \), that is, with the case \( t = 0 \).

In order to investigate (2.11), we study the ratio in (2.11) for any single \( h \) and denote

\[
(2.12) \quad \nu'_p = \nu_p(D^+ \cup \{h\}), \quad \nu_p = \nu_p(D^+), \quad y = \frac{\log H}{2}, \quad P = \prod_{p \leq y} p, \quad \Delta := \prod_{i=1}^{\nu} (h - d_i).
\]

With these notations we can write

\[
(2.13) \quad \frac{S(D^+ \cup \{h\})}{S(D^+)} = \prod_p \frac{1 - \nu'_p}{1 - \nu_p} = \prod_{p \leq y} p \cdot \prod_{p \mid \Delta \atop p > y} \prod_{p \mid \Delta \atop p > y} \prod_{p \mid \Delta}.
\]

For \( p \nmid \Delta \) we have \( \nu'_p = \nu_p + 1 \), otherwise \( \nu'_p = \nu_p \), hence

\[
(2.14) \quad \prod_{p > y} \left( 1 + O \left( \frac{1}{p^2} \right) \right) = 1 + O \left( \frac{1}{y} \right),
\]

\[
(2.15) \quad \log \prod_{p \mid \Delta \atop p \leq y} \ll \sum_{p \mid \Delta \atop p > y} \frac{1}{p} \ll \sum_{p \mid \Delta} \frac{\log p}{y \log y} \ll \frac{\log \Delta}{y \log y} \ll \frac{1}{\log y}.
\]

If \( H = RP + r, \ 0 \leq r < P \) then \( \prod_1(h) \) is periodic with period \( P \). For any \( p \leq y \) we have exactly \( \nu_p \) possibilities for \( h \) with \( \nu'_p = \nu_p \mod p \), and \( p - \nu_p \) possibilities with \( \nu'_p = \nu_p + 1 \). Consequently

\[
(2.16) \quad \frac{1}{P} \sum_{h=1}^{P} \prod_1(h) = \prod_{p \mid P} \frac{\left( \frac{\nu_p}{p} \left( 1 - \frac{\nu_p}{p} \right)^r + \left( 1 - \frac{\nu_p}{p} \right) \left( 1 - \frac{\nu_p + 1}{p} \right)^r \right)}{\left( 1 - \frac{\nu_p}{p} \right)^r \left( 1 - \frac{1}{p} \right)^r}
\]

\[
= \prod_{p \mid P} \frac{\nu_p + 1 - \frac{\nu_p}{p} - \frac{r(\nu_p + 1)}{p} + O \left( \frac{1}{p^2} \right)}{1 - \frac{r(\nu_p + 1)}{p} + O \left( \frac{1}{p^2} \right)}
\]
\[
\prod_{p | P} \left( 1 + O \left( \frac{1}{p^2} \right) \right) = O(1).
\]

(2.14)–(2.16) together prove the lemma, while for \( r = 1 \), in order to obtain \( \sim \) instead of \( \ll \), it is enough to observe that the numerator after the product sign equals exactly 1 for each prime \( p \), and the contribution of the incomplete period, the interval \([RP + 1, RP + r]\), is \( \leq P = 0(H) \) by the prime number theorem, since \( y = \log H/2 \).

Hence, as mentioned earlier, Lemma 2 implies the Theorem by (2.1)–(2.7).

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