Issues on the transport of one dimensional quantum systems

X. Zotos*

Department of Physics, University of Crete
and Foundation for Research and Technology-Hellas,
P. O. Box 2208, 71003 Heraklion, Crete, Greece

(Received September 27, 2018)

In this paper, we will critically discuss recent theoretical and experimental developments on the transport of one dimensional quantum systems. In particular, we will focus on open issues and controversial results related to the finite temperature dynamics of integrable quantum models. These singular systems are commonly used in the description of quasi-one dimensional materials. They have recently attracted a lot of interest in connection to the interpretation of experiments, specially after the discovery of unusual thermal conductivity in magnetic compounds.

KEYWORDS: quantum transport, integrability, quasi-one dimensional materials

1. An old story

In 1834 J. Scott-Russel observed\(^1\) a “singular and beautiful phenomenon” in the form of “a large solitary elevation...which continued his course along the channel apparently without change of form or diminution of speed...”. This phenomenon recently resurfaced in a quantum version remaining as exciting as well, as we will discuss extensively in this paper, singular.

Its mathematical description was formulated some years later by Korteweg and de Vries\(^2\) who proposed a one dimensional classical nonlinear wave equation to describe it. This equation (so called KdV) has nonlinear solutions in the form of localized, nondispersing waves. But it was only in the 1960’s, with the development of large scale computer simulations\(^3\) on dynamical systems, that it was realized that not only a single solitary wave solution was stable but that a set of “solitary waves” entering into collision they emerge after the collision with unchanged velocities and shapes and only suffering a phase shift.

Soon after this discovery, an impressive mathematical theory was proposed - the Inverse Scattering Method (ISM) - that solves the problem of time evolution of an initial configuration in an classical nonlinear but integrable system by a series of linear operations. In other words, this procedure is the analogue of the Fourier Transform for linear systems.\(^4\) Within this theory it was also discovered that integrable systems are characterized by a macroscopic number of conservation laws that we believe are the key to their singular transport properties.

At this point we should mention a distinction between systems with, (i) “topological” solitons (the stability of which is guaranteed by topological constraints) that in general do not exhibit particular transport properties and, (ii) “mathematical” solitons, the stability of which is guaranteed by the very specific structure of the Hamiltonian describing them.\(^5\) A possible source of confusion are systems as systems the sine-Gordon theory (classical and quantum) that are integrable but they also possess topological excitations.

The paradigm of classical integrable systems finds nowadays applications in all branches of science and engineering. We will just mention the example of nonlinear optics where the coding and transmission of information is by optical solitons in fibers described by the nonlinear Schrödinger equation,\(^6\) a classical integrable model. A key issue for applications, and as we will discuss below also for quantum ones, is the robustness of dissipationless transport to perturbations that destroy the integrability of the underlying system.

2. The quantum world

In parallel to these developments, the first quantum integrable system was discovered by H. Bethe in 1931\(^7\) who proposed an exact form for the eigenfunctions and eigenvalues for the prototype spin-1/2 one dimensional isotropic Heisenberg model. Over the years, a particular technique was developed, the Bethe ansatz (BA) method, that allows to extract physical information from the rather complex eigenfunctions for a large class of quantum integrable models; for instance the Heisenberg, Hubbard, supersymmetric t-J model,\(^8, 9\) to cite a few examples of interest in the field of Condensed Matter Physics. At the moment we can say that the thermodynamic properties of these models can analytically be studied and compared to experiment but that the finite temperature dynamic correlations, most relevant e.g. to transport experiments, are not easy to handle.

In a key development in 1979, by L. Fadeev and collaborators, it was realized that the Bethe ansatz method is nothing else but a quantum version of the ISM and thus there is a close analogy between classical and quantum integrable systems.\(^8\) In particular, they have developed a similar method, the Quantum Inverse Scattering Method (QISM). Within this formalism, it can be shown that the quantum systems are also characterized by a macroscopic number of conservation laws, that can be systematically constructed.

A fundamental issue emerging from this analogy, and being a driving force in this field, is the extent to which the physics and applications of classical integrable sys-

\(^{*}\)e-mail: zotos@physics.uoc.gr
tems can be carried over to the the quantum ones, for instance in the context of unconventional transport properties of (quasi-) one dimensional materials described by quantum integrable Hamiltonians.

3. Framework

Most of the studies on transport properties of one dimensional quantum many body systems over the last few years were within the linear response theory (Kubo formalism\textsuperscript{10}). One fundamental quantity that attracted particular attention is the Drude weight $D$, defined as the weight of the zero-frequency contribution to the real part of the conductivity,

$$
\sigma'(\omega) = 2\pi D\delta(\omega) + \sigma_{\text{reg}}(\omega > 0)
$$

or equivalently as the pre-factor of the reactive response $\sigma''(\omega) = D/\omega|_{\omega \to 0}$ at low frequencies. As it follows from this definition, a finite $D$ implies a ballistic response. Note also for further use, that the d.c. conductivity is given by the $\omega \to 0$ limit of the regular part, $\sigma_{\text{dc}} = \sigma_{\text{reg}}(\omega \to 0)$.

The Drude weight was first proposed by W. Kohn in 1964 as a criterion of metallic or insulating behavior in the context of the Mott-Hubbard transition.\textsuperscript{11} In particular, it was related to the response of the ground state energy, of a system with periodic boundary conditions forming a ring, to a fictitious flux $\phi$ piercing the ring.

Recently, the Kohn expression for the Drude weight was generalized at finite temperature\textsuperscript{12} $T$. It is currently extensively studied in order to characterize the conduction of a system as ideal (ballistic - dissipationless) or diffusive. In an alternative, perhaps more physical, interpretation it can be shown that it is essentially proportional to the long time asymptotic value of the (e.g. charge, spin or energy) current-current correlations.\textsuperscript{13} The behavior of the Drude weight, although a seemingly simple quantity, still defies reliable characterization in several integrable models that have been studied so far. This is because it is not a thermodynamic quantity and thus difficult to analyze within known analytical methods.

Similarly, a thermal Drude weight $D_{\text{th}}$ can be defined that characterizes the thermal conductivity $\kappa(\omega)$ and it is proportional to the long time asymptotic of the energy current-energy current dynamic correlations.

The finite frequency behavior of the conductivity is still more involved and most known results are based on low energy effective theories of the Hamiltonians of interest or numerical simulations; for instance, variants of Exact Diagonalization (ED) via the Lanczos procedure (for a recent technique see the Microcanonical Lanczos Method\textsuperscript{14} and references therein) and the Quantum Monte Carlo method.\textsuperscript{15,16}

The numerical results are however limited to rather small size lattices considering the exponentially fast increasing number of states in quantum many body systems. This implies, on the one hand, limitations on the information that can be extracted on the long time, equivalently low frequency - $\omega$, behavior of the conductivities. On the other hand, it also implies limitations on the smallest wave-vector - $q$ correlations, of the order of $1/L$, that can be evaluated. Thus it is difficult to explore the $q, \omega \to 0$ region, relevant to the hydrodynamic regime. Notice for instance, that any finite size system shows a nonzero Drude weight (as there is always a part of coherent transport in such a system), that however scales to zero in the thermodynamic limit if the system shows normal transport. From this observation we can also conclude that in numerical simulations it is more advantageous and reliable to study the high temperature limit where we expect a relevant “mean free path” to be shorter, hopefully less, than the size of the lattice.

We should also mention a recent, promising development, where by the Density Matrix Re-normalization Group method\textsuperscript{17} (that has been proven so successful in the study of ground state properties) the time evolution of an initial state can evaluated on fairly large size lattices. Further development of this method might give access at the finite temperature, out-of equilibrium dynamics of one dimensional quantum lattice systems.

So we conclude that it remains crucial, in this field of strongly correlated systems, the development of reliable numerical simulation techniques for the study of finite temperature dynamic correlations. As they are related to transport are the most interesting for making contact with experiment.

4. A conjecture

A few years ago,\textsuperscript{12,18,19} it was conjectured that integrable quantum many body systems show fundamentally different finite temperature transport properties than the - generic - nonintegrable ones. According to the common scenario, illustrated in Fig. 1, a normal metallic system, with no disorder, at zero temperature, shows a finite Drude weight $D$ and thus a diverging d.c. conductivity. As the temperature rises, due to Umklapp scattering, the zero-frequency Drude $\delta$-function broadens to a peak of width of order $1/\tau$ where $\tau$ is a characteristic scattering time. Thus a finite d.c. conductivity $\sigma_{\text{dc}}$ is obtained that decreases with increasing temperature and decreasing $\tau$.

In contrast, as it is illustrated in Fig. 2, in an integrable system this broadening with temperature does not occur. The Drude weight remains finite at all temperatures, but with a nontrivial temperature dependence. Thus the system remains an ideal conductor, exhibiting ballistic transport at all $T$. Note that we are concerned with the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Illustration of typical behavior of the conductivity in a nonintegrable - generic - system showing normal transport.}
\end{figure}
conductivity of bulk systems. Finite systems with boundaries would show a vanishing $D$ and a d.c. conductivity proportional to the scattering by the boundaries.

Two main issues, arising in relation to this conjecture, are debated at the moment.

First, it was proposed that even nonintegrable systems might also show ballistic transport at finite $T$; evidence so far indicates that it is probably not the case. To appreciate the difficulty of this question, note that despite extensive studies over the years in classical nonlinear one dimensional systems, it is still debated whether the thermal conductivity might be diverging even in nonintegrable ones. In particular, in several nonintegrable models, it was found that although the Drude weight (long time asymptotic of current correlations) vanishes, the integral over time of the dynamic correlation (that gives the d.c. conductivity) diverges with system size. Furthermore, the simulations were carried over on systems with thousands of lattice sites, that indicates the subtlety of the scaling procedure.

Second, it is unclear whether an integrable model in the gapped phase (insulating) at zero temperature (e.g. half-filled Hubbard, easy-axis $S = 1/2$ Heisenberg, nonlinear $\sigma$ model) turns to an ideal conductor or a normal metal at finite temperatures.\textsuperscript{15, 24–30}

Finally, we want to remark that although most of the studies focused on the behavior of the Drude weight, it still possible that $D$ might vanish but the low frequency behavior of the conductivity will turn out to be unconventional, e.g. power law, implying a diverging d.c. conductivity or of a non-diffusive form.

5. Three Heisenberg models

Spin models are prototype quantum many body systems that offer the possibility to test the above conjecture. Their study is also relevant in the interpretation of experiments in quasi-one dimensional magnetic compounds.

In this section we will discuss the transport behavior of three one dimensional spin models, (i) one integrable, the $S = 1/2$ Heisenberg chain and (ii) two nonintegrable ones, the $S = 1$ and $S = 1/2$ 2-leg ladder systems. We will focus on open issues and discuss contradictions in the results obtained by different approaches.

5.1 $S=1/2$

The Hamiltonian is given by the generic form,

$$H = J \sum_{i=1}^{L} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z),$$

where $S_i^a$ are components of $S = 1/2$ operators at site $i$ and we consider the, $J > 0$, anti-ferromagnetic regime. For $|\Delta| < 1$, the “easy-plane” case, the spectrum is gapless, while for $\Delta > 1$, the “easy-axis” case, it is gapped. By a Jordan-Wigner transformation, this spin model is equivalent to a spinless fermion system with hopping $-t$ and nearest-neighbor interaction $V$; the correspondence of parameters is $\Delta/J = V/2t$. The model is integrable and its Bethe ansatz algebraic structure\textsuperscript{3} and thermodynamic properties have been extensively studied.\textsuperscript{9} Regarding the finite temperature dynamics, the picture on the spin conductivity is still debated while the thermal one is rather clear.

First, it is easy to show that indeed this integrable system can exhibit ideal (ballistic) transport at all temperatures by using the fact that it is characterized by nontrivial conservation laws. In particular, it was shown,\textsuperscript{13} using an inequality proposed by Mazur and Suzuki,\textsuperscript{31} that $D$ at high temperatures, $\beta = 1/k_B T \to 0$, is bounded by the “overlap” of the spin current operator $j^z$ with at least the first nontrivial conservation law $Q_3$ as ($m = <S_i^z>$),

$$D(T) \geq \frac{\beta}{2L} < j^z Q_3 >^2 \geq \frac{1}{2k_B T} \frac{8\Delta^2 m^2 (1/4 - m^2)}{1 + 8\Delta^2 (1/4 + m^2)}.$$ 

This idea, although very powerful in proving that such a state - ideal conductor at all temperatures - exists, it provides at the moment a rather incomplete picture of the behavior of $D$. Indeed, we see from Eq. (2) that the r.h.s. vanishes at $\Delta = 0$, the point that corresponds to the XY model, where we know that the spin current commutes with $H$ and thus $D$ is finite. Also, at magnetization $m = 0$, the r.h.s. also vanishes according to Eq. (2) while independent BA\textsuperscript{32, 33} and numerical evaluations\textsuperscript{16, 21, 24, 34} show that it is finite in the $\Delta < 1$ regime.

The same conclusion holds even including all conservation laws derived from the algebraic structure of this integrable model\textsuperscript{3} as it can be seen by invoking their “electron-hole” symmetry.\textsuperscript{13}

Regarding the BA calculation\textsuperscript{32} at $m = 0$, it is based on the assumption of a “rigid string” picture for the excitations proposed by Fujimoto and Kawakami\textsuperscript{25} in the context of the Hubbard model; notice that it can be done at the special values of $\Delta = \cos(\pi/\nu)$, $\nu = \text{integer}$. It predicted a nonzero $D$ for $|\Delta| < 1$ with a power law behavior at low temperatures as it can be seen in Fig. 3. In particular:

- at zero magnetization, in the easy plane antiferromagnetic regime ($0 < \Delta < 1$), the Drude weight decreases at low temperatures as $D(T) = D_0 - \text{const.} T^\alpha$, $\alpha = 2/(\nu - 1)$, $\Delta = \cos(\pi/\nu)$;
- in the ferromagnetic regime, $-1 < \Delta < 0$, $D(T)$
decreases quadratically with temperature (as in a noninteracting, XY-system);

- the same low temperature quadratic behavior is true at any finite magnetization;

- for $\beta \to 0$, $D(T) = \beta C_{jj}$ and it can be shown that $D(-\Delta) = D(\Delta)$ by applying a unitary transformation;

- a closed expression for $C_{jj}$ has be obtained analytically,$^{33}$ $C_{jj} = (\pi/\nu - 0.5 \sin(2\pi/\nu))/(16\pi/\nu)$ for $|\Delta| < 1$, while $C_{jj} = 0$ for $\Delta > 1$;

- at the isotropic anti-ferromagnetic point ($\Delta = 1$), $D(T)$ seems to vanish, implying non ballistic transport at all finite temperatures.

The above low temperature behavior for $|\Delta| < 1$ is challenged by an alternative BA “spinon” approach,$^{33}$ QMC simulations,$^{16}$ and an effective field theory approach,$^{22}$ although all agree on a finite $D$ in this regime. On this disagreement, we can comment that, in the QMC simulations the temperature is very low (so as to obtain reliable real time data from imaginary time ones) of the order of the level spacing and thus not necessarily in the bulk regime - level spacing much less than $T$. Second, in effective field theories the dispersion relation is linearized and it is not clear whether curvature effects might affect the temperature behavior of $D$; note that, for free fermions on a lattice, $D$ decreases as $T^2$ at low $T$ while with a linearized dispersion is constant.

Regarding low energy effective theories, the canonical approach to one dimensional quantum systems, the Luttinger liquid theory, was applied to transport.$^{35}$ In the context of this approach, it was recently stressed the importance of, (i) conservation laws$^{36}$ in altering the behavior of transport - from diffusive to ballistic - and, (ii) correctly accounting perturbations irrelevant to thermodynamics but crucial for transport.$^{37}$ We should emphasize that it remains an outstanding problem, first, to match the descriptions in terms of the original (non)integrable Hamiltonian and its effective field theory, second, to account for curvature effects and, third, to trace the fate of conservation laws - present at all energies in BA systems - to low energies.

For $\Delta > 1$ the Heisenberg model is gapped and thus $D = 0$ at zero temperature. Numerical evidence in the $\beta \to 0$ limit$^{27}$ (shown in Fig. 4) and Bethe ansatz analytical results$^{32,33}$ suggest that $D$ remains zero at all temperatures. The data follow from Exact Diagonalization calculations for the smallest systems and the Microcanonical Lanczos method (MCLM) for the largest ones, while FM denotes a frequency moment analysis. Note the crucial role of the MCLM technique in extracting sufficient information from the conductivity spectra. Very large finite size effects are observed in the low frequency behavior of the conductivity that scale as $1/L$ and the significance of which is not clear at the moment. It is interesting to examine whether they signify an unconventional low frequency-small wave-vector behavior that is not of the standard diffusive form. They also render particularly difficult the analysis of the bulk system conductivity and the extraction, if there is one, of a value for the d.c. conductivity. Also note that, one cannot exclude the convergence of the finite frequency peak to a Drude $\delta$-function in the thermodynamic limit, although such a scenario is rather unlikely. We should remind that a semi-classical theory for gapped systems by Sachdev and collaborators predicts a normal diffusive behavior.$^{28}$

Finally, we will conclude with some open questions related to the issue of conservation laws and transport. From the Mazur inequality one could argue that the presence of only one conserved quantity is enough to guarantee a finite $D$ and that integrability is not a necessary condition. However, there are arguments that, at least within the BA systems, the presence of one conservation law guarantees the existence of a macroscopic number.
Notice that for the $S = 1/2$ Heisenberg\(^{38}\) and the Hubbard model\(^{39}\) the conservation laws can be constructed recursively using a “boost” operator. Conversely, from the case $m = 0$, it is not clear that the conservation laws, in the context of the Mazur-Suzuki (in)equality, exhausts the Drude weight or even that the presence of at least one is necessary for a finite $D$. Finally, on this problem, we can also argue that if the BA analysis\(^{32}\) turns out to be valid and $D$ shows a non-analytic behavior at $\Delta = 1$, then one, or a finite number of conservation laws within the Mazur inequality cannot reproduce this behavior.

Concerning energy transport the situation is far more clear and simple. It was early on realized\(^{40}, 41\) that the energy current in the $S = 1/2$ Heisenberg model commutes with the Hamiltonian, at it happens to coincide with the so-called $Q_3$ conservation law in the BA analysis. Thus it follows that the energy current correlations do not decay in time and thus the thermal conductivity diverges. The “thermal Drude weight” has been exactly evaluated using Bethe ansatz techniques\(^{42}\) for all values of the anisotropy $\Delta$.

Based on the same idea, of a commuting energy current with the Hamiltonian, we finally mention a magneto-thermal effect that has recently been proposed.\(^{43}\) In this phenomenon, a spin current would flow in the presence of a temperature gradient and a magnetic field, a phenomenon analogous to the thermoelectric effect for electronic systems. The study, using ED and QMC methods, found a diverging transport coefficient resulting from the conservation of the energy current.

5.2 $S=1$

A natural question coming up is, what about the transport behavior of the same Heisenberg model Eq. (1) but for higher spin, e.g. $S = 1$, that is not an integrable model. The $S = 1$ case is known to be a gapped system\(^{44}\) and thus an insulator at $T = 0$ with $D = 0$.

Its low energy physics is described by the quantum nonlinear $\sigma$-model ($q\text{NL}\sigma\text{M}$). A semiclassical treatment of this effective theory by Sachdev and collaborators,\(^{45}\) that essentially maps this system to a classical gas of different species impenetrable particles, concluded to diffusive dynamics. Note, that the classical model is also integrable and can be analytically solved. On the other hand, as the nonlinear $\sigma$ model is also an integrable field theory, the Bethe ansatz method has been applied\(^{29, 30}\) but there it was concluded that the Drude weight is finite and thus the transport ballistic. This discrepancy raises the question whether, within the $q\text{NL}\sigma\text{M}$ model description, subtle quantum effects due the finite tunneling of the quasi-particles alter the transport behavior from diffusive to ballistic.

Motivated by this problem and the theoretical and experimental interest for the dynamics of $S = 1$ compounds, a numerical simulation in the high temperature limit\(^{46}\) was performed. It indicates that both the spin and energy conductivities have finite d.c. values. The frequency dependence of the data however do not seem to fit a standard Lorenzian/diffusive form. The results for $\kappa(\omega)$ and $\sigma(\omega)$ are shown in Fig. 6 and 7 respectively. It became possible to obtain them for the largest lattices
finite size effects are limited to exponentially small, with the Hamiltonian, the conclusions are drawn from numerical studies on that the integrability-transport conjecture seems to hold. However, we should keep in mind though it seems that the integrability-transport conjecture will shrink to zero in the thermodynamic limit. Also note that in lattices with an odd number of sites space is signaled by an enhanced conductivity and this presence of an integrable point in interaction parameter space is an open issue at the moment.

In this context, frustrated spin chain and ladder models are components of $S = 1/2$ operators at site $\ell$ and we consider the, $J > 0$, anti-ferromagnetic regime. The interchain coupling $J_\perp$ controls the perturbation breaking the integrability of the separate Heisenberg chains. Indeed, for $J_\perp \to 0$ the two chains decouple and the thermal conductivity of the ladder is the sum of the conductivities of each chain, that converge as the energy current of the Heisenberg chain model commutes with the Hamiltonian.

This model is also of experimental interest as the finite temperature transport of several compounds described by this Hamiltonian are recently investigated. The low energy physics of this gapped system is also described by the nonlinear $\sigma$ model and extensive studies based on this approach have been performed. In particular, the analysis by Sachdev and collaborators concluded to a finite d.c. conductivity, the significance of this difference in scaling is an open issue at the moment.

5.3 $S=1/2$ 2-leg ladder

The next question that must be addressed is the robustness of ideal transport of an integrable system to perturbations breaking integrability. Namely, whether the presence of an integrable point in interaction parameter space is signaled by an enhanced conductivity and this over a finite region and not an infinitesimal one, e.g. one that shrinks to zero as the system size increases to the bulk limit.

In this context, frustrated spin chain and ladder models have been studied by numerical diagonalization techniques with conclusions that are still debated, although it seems that the integrability-transport conjecture seems to hold. However, we should keep in mind that the conclusions are drawn from numerical studies on rather small size lattices and thus extreme care is needed in drawing definite conclusions for the bulk system; no exact results are known so far.

A system on which the question of robustness can be studied is the $S = 1/2$ 2-leg ladder model described by the Hamiltonian,

$$H = J \sum_{\ell = 1, L} (S_{1,\ell+1} \cdot S_{1,\ell} + S_{2,\ell+1} \cdot S_{2,\ell}) + \frac{J_\perp}{J} S_{1,\ell} \cdot S_{2,\ell}, \quad (3)$$

where $S_{i,\ell}$ are components of $S = 1/2$ operators at site $\ell$ and we consider the, $J > 0$, anti-ferromagnetic regime. The interchain coupling $J_\perp$ controls the perturbation breaking the integrability of the separate Heisenberg chains. Indeed, for $J_\perp \to 0$ the two chains decouple and the thermal conductivity of the ladder is the sum of the conductivities of each chain, that diverge as the energy current of the Heisenberg chain model commutes with the Hamiltonian.

This model is also of experimental interest as the finite temperature transport of several compounds described by this Hamiltonian are recently investigated. The low energy physics of this gapped system is also described by the nonlinear $\sigma$ model and extensive studies based on this approach have been performed. In particular, the analysis by Sachdev and collaborators concluded to a fairly complete picture of diffusive dynamics.

A recent numerical simulation study in the high temperature limit, again using the MCLM, also indicates a finite d.c. thermal conductivity as shown in Fig. 8. The frequency dependence, as in the $S = 1$ model, does not seem to conform to a diffusive form. Furthermore, as shown in the inset, the d.c. thermal conductivity scales as $(J/J_\perp)^2$, diverging as it should in the limit of decoupled chains. This result is interesting because, first, it indicates that the influence of an integrable point $(J_\perp = 0)$ extends to a finite, of order one, region in interaction parameter space and, second, the quadratic dependence on the interaction is similar to that expected from a perturbative result. Notice however that the perturbation now is around not an independent particle system, but around a fully interacting, albeit integrable, one.

Thus the conclusion of a finite d.c. conductivity at high temperatures does not contradict the semiclassical treatment of the qNLoM at low temperatures, but the same problematic as the one for the $S = 1$ model applies in this case.

Fig. 8. Thermal conductivity of the $S = 1/2$ 2-leg ladder model in the high temperature limit for $L = 16$. 

$$H = J \sum_{\ell = 1, L} (S_{1,\ell+1} \cdot S_{1,\ell} + S_{2,\ell+1} \cdot S_{2,\ell}) + \frac{J_\perp}{J} S_{1,\ell} \cdot S_{2,\ell}, \quad (3)$$

where $S_{i,\ell}$ are components of $S = 1/2$ operators at site $\ell$ and we consider the, $J > 0$, anti-ferromagnetic regime. The interchain coupling $J_\perp$ controls the perturbation breaking the integrability of the separate Heisenberg chains. Indeed, for $J_\perp \to 0$ the two chains decouple and the thermal conductivity of the ladder is the sum of the conductivities of each chain, that diverge as the energy current of the Heisenberg chain model commutes with the Hamiltonian.
6. A new mode of transport

Quasi-one dimensional materials have been synthesized and experimentally studied since the 60's. Recently there has been renewed interest in their transport properties as new materials where discovered - some in the class of cuprates - with very low disorder, very weak interchain coupling and extraordinary transport properties.

In relation to the above theoretical developments we should mention the NMR experiments by Takigawa and Thurber\(^{49,50}\) on $S = 1/2$ compounds (e.g. $Sr_2CuO_3$) that show diffusive spin dynamics but with a diffusion constant orders of magnitude larger than that expected from a classical moment analysis. Similarly, the spin dynamics of the $S = 1$ compound $AgVP_2S_6$ has been investigated, indicating diffusive behavior in this case,$^{51}$ at least at elevated temperatures.

The phenomenon that attracted most interest recently$^{52-57}$ is the strongly anisotropic and large in magnitude heat conductivity of quasi-one dimensional compounds attributed to magnetic excitations. These experiments where partly motivated by the prospect of ballistic conductivity, as predicted for the $S = 1/2$ Heisenberg model. Large thermal conductivities however were reported in ladder materials, e.g. $(La, Sr, Ca)_{14}Cu_{24}O_{41}$. Note that the $S = 1/2$ Heisenberg chain compounds offer the unique possibility to analyze directly the effect of spin-phonon scattering on thermal transport in a strongly interacting system as the spin-spin scattering is totally ineffective.

Thus, in connection to these experiments, we can conclude that:

- They established a new, very efficient, mode of energy transport through magnetic excitations, similar in magnitude to metals but in insulators; the reason is that the characteristic magnetic exchange couplings are in the $eV$ range.
- These novel materials have potential for technological applications.
- They promote the study of prototype integrable models as the description of many of these compounds is in terms of, for example, the $S = 1/2$ Heisenberg, Hubbard, sine-Gordon, nonlinear $\sigma$ model.

7. Perspectives

From the above discussion we can conclude with a series of open issues and probable developments:

- The transport properties of integrable models are bound to be exactly evaluated following the development of appropriate Bethe ansatz techniques. These models offer the unique possibility of an analytical solution and it is ironic that because of their solvability they also present unconventional transport properties.
- The transport of low energy effective theories, some of them integrable models, is to be settled.
- The connection of high energy integrable models to their low energy effective field theories is to be clarified.
- Of fundamental interest for the interpretation of experiments and applications is the role of perturbations in breaking the ideal transport. Apparently unavoidable perturbations are phonons, disorder, interchain coupling.
- New experimental systems for the study of one dimensional quantum transport will be developed. For the time being magnetic compounds are the most promising, the electronic ones do not seem so ideal for fundamental studies. But work is in progress in developing artificial one dimensional structures as superlattices or self-assembled 1D systems on surfaces that will offer far more controllable systems for the study of quantum transport.

Acknowledgements

On this occasion I would like to thank the organizers of the Conference *Statistical Physics of Quantum Systems - novel orders and dynamics* held in Sendai in July 2004 for giving me the opportunity to present these ideas. I would also like to thank my long standing collaborators in this field, P. Prelovšek, M. Long, J. Karadamoglou, H. Castella and F. Naef.

1) J. Scott-Russel: Proc. Roy. Soc. Edinburgh (1844) 319.
2) D.J. Korteweg and G. deVries: Phil. Mag. 39 (1895) 422.
3) N.J. Zabusky and M.D. Kruskal: Phys. Rev. Lett. 15 (1965) 240.
4) C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura: Phys. Rev. Lett. 19 (1967) 1095.
5) M.J. Segur, H. Ablowitz, M.J. Ablowitz and H. Segur: *Solitons and Inverse Scattering Transform* (SIAM Studies in Applied Mathematics, No. 4, 2000).
6) Optical solitons: theoretical challenges and industrial perspectives editors: V.E. Zakharov and S. Wabnitz, *Les Houches Workshop* (Springer, 1998).
7) H. Bethe: Zeit. Phys. 71 (1931) 205.
8) V.E. Korepin, N.M. Bogoliubov and A.G. Izergin: *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge Univ. Press, 1993).
9) M. Takahashi: *Thermodynamics of One-Dimensional Solvable Models* (Cambridge: Cambridge University Press, 1999).
10) R. Kubo: J. Phys. Soc. Jpn. 12 (1957) 570.
11) W. Kohn: Phys. Rev. 133 (1964) A171.
12) H. Castella, X. Zotos, P. Prelovšek: Phys. Rev. Lett. 74 (1995) 972.
13) X. Zotos, F. Naef and P. Prelovšek: Phys. Rev. B55 (1997) 11029.
14) M. W. Long, P. Prelovšek, S. El Shawish, J. Karadamoglou and X. Zotos: Phys. Rev. B68 (2003) 235106.
15) S. Kirchner, H.G. Evertz, and W. Hanke: Phys. Rev. B59 (1999) 7382.
16) J. V. Alvarez and C. Gros: Phys. Rev. Lett. 88 (2002) 077203.
17) U. Schollwöck: Rev. Mod. Phys.: scheduled to appear January 2004.
18) K. Saito, S. Takesue and S. Miyashita: Phys. Rev. E54 (1996) 2404.
19) B.M. McCoy: *Statistical Mechanics and Field Theory*, ed. V.V. Bazhanov and C.J. Burden (World Scientific, 1995).
20) J.V. Alvarez and C. Gros: Phys. Rev. Lett. 89 (2002) 156603.
21) F. Heidrich-Meisner, A. Honecker, D. C. Cabra and W. Brenig: Phys. Rev. B66 (2002) 140406; F. Heidrich-Meisner, A. Honecker, D. C. Cabra, and W. Brenig, ibid. B68 (2003) 134436.
22) S. Fujimoto and N. Kawakami: Phys. Rev. Lett. 90 (2003) 197202.
23) S. Lepri, R. Livi and A. Politi: Phys. Rep. 377 (2003) 1.
24) X. Zotos and P. Prelovšek: Phys. Rev. B53, 983 (1996).
25) S. Fujimoto and N. Kawakami: J. Phys. A. 31, 465 (1998).
26) N.M.R. Peres, R.G. Dias, P.D. Sacramento, and J.M.P. Carmelo: Phys. Rev. B61 (2000) 5169.
27) P. Prelovsek, S. El Shawish, X. Zotos, and M. Long: to appear in Phys. Rev. B (2004).
28) C. Buragohain and S. Sachdev: Phys. Rev. B59 (1999) 9285.
29) S. Fujimoto, J. Phys. Soc. Jpn.: 68 (1999) 2810.
30) R. M. Konik: Phys. Rev. B68 (2003) 104435.
31) P. Mazur: Physica 43 (1969) 533; M. Suzuki: Physica 51 (1971) 277.
32) X. Zotos: Phys. Rev. Lett. 82 (1999) 1764.
33) A. Klümp: private communication.
34) B.N. Narozhny, A.J. Millis and N. Andrei: Phys. Rev. B58 (1998) 2921.
35) T. Giamarchi: Physica B230-232 (1997) 975.
36) A. Rosch and N. Andrei: Phys. Rev. Lett. 85 (2000) 1092.
37) E. Shimshoni, N. Andrei and A. Rosch: Phys. Rev. B68 (2003) 104401.
38) M. P. Grabowski and P. Mathieu: Ann. Phys. 243 (1996) 299.
39) J. Links, H. Zhou, R.H. McKenzie and M.D. Gould: Phys. Rev. Lett. 86 (2001) 5096.
40) D.L. Huber and J.S. Semura: Phys. Rev. B182 (1969) 602.
41) T. Niemeijer and H.A.W. van Vianen: Phys. Lett. 34A (1971) 401.
42) A. Klümp and K. Sakai: J. Phys. A35 (2002) 2173; ibid. A36 (2003) 11617.
43) K. Louis and C. Gros: Phys. Rev. B67 (2003) 224410.
44) F.D.M. Haldane: Phys. Lett. 81A (1981) 153.
45) S. Sachdev and K. Damle: Phys. Rev. Lett. 78 (1997) 943; K. Damle and S. Sachdev: Phys. Rev. B57 (1998) 8307.
46) J. Karadamoglou and X. Zotos: to appear in Phys. Rev. Lett. (2004).
47) E. Orignac, R. Chitra and R. Citro: Phys. Rev. B67 (2003) 134426.
48) X. Zotos: Phys. Rev. Lett. 92 (2004) 067202.
49) M. Takigawa et al.: Phys. Rev. Lett. 76 (1996) 4612.
50) K.R. Thurber et al.: Phys. Rev. Lett. 87 (2001) 247202.
51) M. Takigawa et al.: Phys. Rev. Lett. 76 (1996) 2173.
52) A. V. Sologubenko, K. Giannó, H. R. Ott, U. Ammerahl and A. Revcolevschi: Phys. Rev. Lett. 84 (2000) 2714.
53) A. V. Sologubenko, K. Giannó, H. R. Ott, A. Vietkine and A. Revcolevschi: Phys. Rev. B64 (2001) 054412.
54) C. Hess and C. Baumann and U. Ammerahl and B. Büchner and F. Heidrich-Meisner and W. Brenig and A. Revcolevschi: Phys. Rev. B64 (2001) 184305.
55) K. Kudo, S. Ishikawa, T. Noji, T. Adachi, Y. Koike, K. Maki, S. Tsuji and K. Kumagai: J. Phys. Soc. Jpn. 70 (2001) 437.
56) A. V. Sologubenko, H. R. Ott and A. Revcolevschi: Eur. Lett. 62 (2003) 540.
57) A. V. Sologubenko, S. M. Kazakov and H. H. Ott: Phys. Rev. B68 (2003) 094432.