A Fixed-Parameter Linear-Time Algorithm to Compute Principal Typings of Planar Flow Networks

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Abstract

We present an alternative and simpler method for computing principal typings of flow networks. When limited to planar flow networks, the method can be made to run in fixed-parameter linear-time – where the parameter not to be exceeded is what is called the edge-outerplanarity of the networks’ underlying graphs.
1 Introduction

Network typings are algebraic or arithmetic formulations of interface conditions that network components must satisfy in order to interconnect with each other safely and correctly. A particular use of network typings is to quantify desirable properties related to resource management (e.g., percentage ranges of channel utilization, mean delays between routers, etc., as well as flow conservation and capacity constraints along channels), and to enforce them as invariant properties across network interfaces. For a given network component \( N \), a principal typing for \( N \) is the most general – or also the most precise – in the sense that it subsumes all other sound typings of \( N \). More on this use of network typings is in several reports [3, 6, 7, and the references therein]. Computing efficiently principal typings of networks is an underlying concern in all these studies; new ways of computing them more efficiently, under various conditions, continue to be investigated.

In this report, we consider one version of network typings, here simplified to account for only one quantity (viz., flow) and under only one restriction (viz., flow must remain within pre-determined upper bounds along all channels). A formal definition of network typings that fits this simplified version is in Section 2. Our method for computing such network typings efficiently (and more simply) is based on what is called graph reassembling. When the underlying graph \( G \) of a network \( N \) is planar, our method runs in fixed-parameter linear time, where the parameter to be bounded is called the edge-outerplanarity of \( G \). We next explain these two notions: graph reassembling and edge-outerplanarity.

One way of understanding the reassembling of a simple undirected graph \( G \) is this: It is the process of cutting every edge of \( G \) in two halves, and then splicing the two halves of every edge, one by one in some order, in order to recover the original \( G \). We thus start from one-vertex components, with one component for each vertex \( v \) and each with \( \deg(v) \) dangling half edges, and then gradually reassemble larger and larger components of the original \( G \) until \( G \) is fully reassembled. One optimization associated with graph reassembling is to keep the number of dangling half edges of each reassembled component as small as possible. Graph reassembling and associated optimization problems are examined in earlier reports on network analysis [7, 13, 9, 11]. A formal definition of graph reassembling – different from, but equivalent to, the preceding informal definition – is in Section 4.

As for the notion of edge-outerplanarity of planar graphs, it is distinct but closely related to the usual notion of outerplanarity, and was introduced in earlier studies for other purposes (e.g., disjoint paths in sparse graphs, as in [2]). As with outerplanarity, for a fixed edge-outerplanarity \( k \), the number \( n \) of vertices in a planar graph can be arbitrarily large. Our main result can be re-phrased thus: Our main result can be re-phrased thus: For the class \( \cal C_k \) of planar flow networks whose edge-outerplanarity is bounded by a fixed \( k \geq 1 \), there is an algorithm which, given an arbitrary \( N \in \cal C_k \), computes a principal typing for \( N \) in time \( O(n) \) where \( n = |N| \).

2 Preliminary Notions

We review several standard notions, add new notions specially adapted to our needs in this paper, and fix our notational conventions.

Flow Networks:

A flow network is a pair of the form \( N = (G, c) \) where \( G \) is a directed graph without self-loops and without multi-edges (in the same direction), and \( c : E(G) \to \mathbb{R}_+ \) is a function that assigns an upper-bound capacity

\[ \deg(v) \] is the degree of vertex \( v \), i.e., the number of edges incident to \( v \), both entering and exiting \( v \).

However, \( G \) may contain two-edge cycles, i.e., two edges \( e_1 \) and \( e_2 \) such that \( \text{head}(e_1) = \text{tail}(e_2) \) and \( \text{tail}(e_1) = \text{head}(e_2) \).
to every edge $e$. We write $V(G)$ and $E(G)$ for the set of vertices and the set of edges of $G$, respectively.

For reasons that become clear later, we do not identify subsets of $V(G)$ as ‘sources’ and ‘sinks’ of $N$, following usual conventions. Instead, we allow some members of $E(G)$ to be ‘dangling’ edges. An edge $e \in E(G)$ is dangling if it is incident to only one vertex $v \in V(G)$, for which there are two cases, where we write ‘$\perp$’ to mean ‘undefined’:

- $\text{head}(e) = v$ and $\text{tail}(e) = \perp$, in which case $e$ is an input edge, or
- $\text{tail}(e) = v$ and $\text{head}(e) = \perp$, in which case $e$ is an output edge.

$E_{in}(G)$ denotes the set of input edges and $E_{out}(G)$ the set of output edges. An edge $e \in E(G)$ is not dangling if it is incident to two distinct vertices $v, w \in V(G)$ with $v = \text{tail}(e)$ and $w = \text{head}(e)$. The set of edges that are not dangling is denoted $E_{\#}(G)$. The three sets $\{E_{in}(G), E_{out}(G), E_{\#}(G)\}$ form a 3-part partition of $E(G)$, i.e., they are pairwise disjoint and:

$$E(G) = E_{in}(G) \cup E_{out}(G) \cup E_{\#}(G).$$

We write $E_{io}(G)$ for the union $E_{in}(G) \cup E_{out}(G)$.

As usual, a flow in the network $N$ is a function $f : E(G) \to \mathbb{R}_+$. If $X \subseteq E(G)$, we write $f(X)$ for the summation $\sum\{ f(e) \mid e \in X \}$. The flow $f$ is feasible if it satisfies the two standard conditions:

- flow conservation at every vertex $v \in V(G)$, i.e., if $X$ and $Y$ are all the edges entering $v$ and exiting $v$, respectively, then $f(X) = f(Y)$,
- capacity constraint at every edge $e \in E(G)$, i.e., $f(e) \leq c(e)$.

An input-output assignment (or an IO assignment) for the network $N = (G, c)$ is a function $g : E_{io}(G) \to \mathbb{R}_+$. The restriction of a flow $f : E(G) \to \mathbb{R}_+$ to the subset $E_{io}(G) \subseteq E(G)$, denoted $[f | E_{io}(G)]$, is an IO assignment. The value of the flow $f$, denoted $|f|$, is $f(E_{io}(G))$ or, equivalently, $f(E_{out}(G))$.

If $f, f' : E(G) \to \mathbb{R}_+$ are two flows in $N$, then $f + f'$ denotes their sum: $(f + f')(e) \triangleq f(e) + f'(e)$ for every edge $e \in E(G)$.

**Network Typings:**

Let $\mathcal{P}(E_{io}(G))$ be the powerset of $E_{io}(G)$ and $\mathcal{I}(\mathbb{R})$ the set of closed real intervals:

$$\mathcal{P}(E_{io}(G)) \triangleq \{ X \mid X \subseteq E_{io}(G) \} \quad \text{and} \quad \mathcal{I}(\mathbb{R}) \triangleq \{ [r_1, r_2] \mid r_1, r_2 \in \mathbb{R} \text{ and } r_1 \leq r_2 \}.$$

A typing for the network $N = (G, c)$ is a map $\tau$ of the form:

$$\tau : \mathcal{P}(E_{io}(G)) \to \mathcal{I}(\mathbb{R}).$$

If $X \subseteq E_{io}(G)$ with $A = X \cap E_{in}(G)$ and $B = X \cap E_{out}(G)$, we may write $\tau(A, B)$ instead of $\tau(X)$.

An IO assignment $g : E_{io}(G) \to \mathbb{R}_+$ satisfies the typing $\tau : \mathcal{P}(E_{io}(G)) \to \mathcal{I}(\mathbb{R})$ iff for every $A \subseteq E_{in}(G)$ and every $B \subseteq E_{out}(G)$ it holds that:

$$g(A) - g(B) \in \tau(A, B).$$

We can view the difference $g(A) - g(B)$ as expressing the excess flow that enters at $A$ but does not exit from $B$, which may be positive or negative. Only when $A = E_{in}(G)$ and $B = E_{out}(G)$ do we have $g(A) - g(B) = 0$.

A flow $f : E(G) \to \mathbb{R}_+$ satisfies the typing $\tau$ if its restriction $[f | E_{io}(G)]$ satisfies $\tau$. 
**Definition 1 (Principal Typings).** A typing \( \tau : \mathcal{P}(E_{\text{io}}(G)) \rightarrow \mathcal{I}(\mathbb{R}) \) for the network \( \mathcal{N} = (G, c) \) is principal iff two conditions are satisfied:

- If a flow \( f : E(G) \rightarrow \mathbb{R}_+ \) is feasible, then \( f \) satisfies \( \tau \).
- If an IO assignment \( g : E_{\text{io}}(G) \rightarrow \mathbb{R}_+ \) satisfies \( \tau \), then \( g \) can be extended to a feasible flow, i.e., there is feasible flow \( f \) such that \( g = [f \mid E_{\text{io}}(G)] \).

The first condition is the completeness of \( \tau \), the second condition is the soundness of \( \tau \). A minimum requirement on any typing \( \tau \) for \( \mathcal{N} \) is that it be sound; if \( \tau \) is also complete for \( \mathcal{N} \), and therefore principal for \( \mathcal{N} \), then \( \tau \) is the ‘most precise’ formulation of the condition for connecting \( \mathcal{N} \) with other networks.

**Two Special Functions:**

Relative to a flow network \( \mathcal{N} = (G, c) \), we define two functions written as:

\[
\text{maxFromTo}_\mathcal{N}(A_1, B_1) \quad \text{and} \quad \text{maxFromToAft}_\mathcal{N}(A_1, B_1; A_2, B_2),
\]

where \( A_1, A_2 \subseteq E_{\text{in}}(G) \) and \( B_1, B_2 \subseteq E_{\text{out}}(G) \). Whenever ‘\( \mathcal{N} \)’ is understood from the context, we omit the subscript ‘\( \mathcal{N} \)’ and write instead:

\[
\text{maxFromTo}(A_1, B_1) \quad \text{and} \quad \text{maxFromToAft}(A_1, B_1; A_2, B_2).
\]

The meaning of the first function is given by (1) or (2):

1. \[
\text{maxFromTo}(A, B) \triangleq \max \left\{ f(A) \mid f : E(G) \rightarrow \mathbb{R}_+ \text{ is feasible and } f(\overline{A}) = f(\overline{B}) = 0 \right\},
\]
2. \[
\text{maxFromTo}(A, B) \triangleq \max \left\{ f(B) \mid f : E(G) \rightarrow \mathbb{R}_+ \text{ is feasible and } f(\overline{A}) = f(\overline{B}) = 0 \right\}.
\]

(1) and (2) are identical except for the highlighted parts. It is an easy exercise (omitted) to show (1) and (2) are equivalent definitions. Informally, \( \text{maxFromTo}(A, B) \) is the value of a maximum flow \( f : E(G) \rightarrow \mathbb{R}_+ \) from \( A \subseteq E_{\text{in}}(G) \) to \( B \subseteq E_{\text{out}}(G) \) when flow is blocked from entering \( \overline{A} \) and from exiting \( \overline{B} \).

In the case of the second function \( \text{maxFromToAft}(A_1, B_1; A_2, B_2) \), it will always be the case that:

- either \( A_1 \cap A_2 = \emptyset \) and \( B_1 = B_2 \),
- or \( A_1 = A_2 \) and \( B_1 \cap B_2 = \emptyset \).

For the first of these two cases, the meaning of \( \text{maxFromToAft}(A_1, B_1; A_2, B_2) \) is given by (3) or (4), where \( A_1, A_2 \subseteq E_{\text{in}}(G) \) and \( A_1 \cap A_2 = \emptyset \):

3. \[
\text{maxFromToAft}(A_1, B_1; A_2, B_2) \triangleq \max \left\{ f(A_1) \mid f + f' : E(G) \rightarrow \mathbb{R}_+ \text{ is feasible for some flow } f' \text{ such that } f'(A_2) = f'(B) = \text{maxFromTo}(A_2, B) \text{ and } (f + f')'(\overline{A_1 \cup A_2}) = (f + f')'(\overline{B}) = 0 \right\},
\]
4. \[
\text{maxFromToAft}(A_1, B_1; A_2, B_2) \triangleq \max \left\{ f(B) \mid f + f' : E(G) \rightarrow \mathbb{R}_+ \text{ is feasible for some flow } f' \text{ such that } f'(A_2) = f'(B) = \text{maxFromTo}(A_2, B) \text{ and } (f + f')'(\overline{A_1 \cup A_2}) = (f + f')'(\overline{B}) = 0 \right\}.
\]

There are different ways of proving the equivalence of (1) and (2). One particular simple way is by induction on the number \( m \) of edges for a fixed number \( n \) of vertices. Another simple way is to remove all input edges in \( \overline{A} \) and all output edges in \( \overline{B} \), then join all input edges in \( A \) to a fresh input edge \( e_{\text{in}} \) and all output edges in \( B \) to a fresh output edge \( e_{\text{out}} \), and then consider maximum flows from \( e_{\text{in}} \) to \( e_{\text{out}} \) in the thus-modified network.
(3) and (4) are identical except for the highlighted parts. For the second case of the function \( \text{maxFromToAft} \), the meaning of \( \text{maxFromToAft} (A, B_1; A, B_2) \) is given by (5) or (6), where \( B_1, B_2 \subseteq \mathbf{E}_{out}(G) \) and \( B_1 \cap B_2 = \emptyset \):

\[
\text{(5)} \quad \text{maxFromToAft} (A, B_1; A, B_2) \triangleq \max \left\{ f(B_1) \mid f + f' : \mathbf{E}(G) \to \mathbb{R}_+ \text{ is feasible for some flow } f' \text{ such that } f'(A) = f'(B_2) = \text{maxFromTo} (A, B_2) \text{ and } (f + f')(\overline{A}) = (f + f')(B_1 \cup B_2) = 0 \right\},
\]

\[
\text{(6)} \quad \text{maxFromToAft} (A, B_1; A, B_2) \triangleq \max \left\{ f(A) \mid f + f' : \mathbf{E}(G) \to \mathbb{R}_+ \text{ is feasible for some flow } f' \text{ such that } f'(A) = f'(B_2) = \text{maxFromTo} (A, B_2) \text{ and } (f + f')(\overline{A}) = (f + f')(B_1 \cup B_2) = 0 \right\},
\]

(5) and (6) are identical except for the highlighted parts. Just as (1) and (2) are equivalent, so too (3) and (4) are equivalent, and (5) and (6) are equivalent, and by the same reasoning.

Informally, the meaning of \( \text{maxFromToAft} (A_1, B; A_2, B) \) and \( \text{maxFromToAft} (A, B_1; A, B_2) \) is as follows:

- \( \text{maxFromToAft} (A_1, B; A_2, B) \) returns the value of a maximum flow from \( A_1 \) to \( B \), after a maximum flow has been already directed from \( A_2 \) to \( B \),
- \( \text{maxFromToAft} (A, B_1; A, B_2) \) returns the value of a maximum flow from \( A \) to \( B_1 \), after a maximum flow has been already directed from \( A \) to \( B_2 \).

The following lemma is used in the induction in Section 3.

**Lemma 2.** The functions \( \text{maxFromTo} \) and \( \text{maxFromToAft} \) are related by the following equalities:

\[
\text{(‡) For all } A_1, A_2 \subseteq \mathbf{E}_{in}(G) \text{ and } B \subseteq \mathbf{E}_{out}(G) \text{ such that } A_1 \cap A_2 = \emptyset:\n\]

\[
\text{maxFromToAft} (A_1, B; A_2, B) = \text{maxFromTo} (A_1 \cup A_2, B) - \text{maxFromTo} (A_2, B)
\]

\[
\text{(†) For all } A \subseteq \mathbf{E}_{in}(G) \text{ and } B_1, B_2 \subseteq \mathbf{E}_{out}(G) \text{ such that } B_1 \cap B_2 = \emptyset:\n\]

\[
\text{maxFromToAft} (A, B_1; A, B_2) = \text{maxFromTo} (A, B_1 \cup B_2) - \text{maxFromTo} (A, B_2).
\]

**Proof Sketch.** The proof of (‡) and (†) are essentially the same, and it suffices to focus on (‡). Hence, from (3) we need to show that (‡) is true; in fact, what is more, (3) and (‡) imply each other. This is easy to see by conservation of flow through the network. A more formal proof is to prove the equivalence of (3) and (‡) for every component of \( \mathcal{N} \) as it is reassembled inductively in Section 3 – said differently still, given the definition in (3), the equality (‡) is an invariant of the induction – starting with the one-vertex components and finishing with the full network \( \mathcal{N} \).

The next lemma is used in the proof of our main result, Theorem 9.

**Lemma 3.** Let \( \tau : \mathcal{P} (\mathbf{E}_{in}(G)) \to \mathcal{T} (\mathbb{R}) \) be the principal typing of the flow network \( \mathcal{N} = (G, c) \). For all \( A \subseteq \mathbf{E}_{in}(G) \) and \( B \subseteq \mathbf{E}_{out}(G) \), it holds that:

\[
\tau(A, B) = [r_1, r_2] \iff r_1 = -\text{maxFromTo} (\overline{A}, B) \text{ and } r_2 = \text{maxFromTo} (A, \overline{B}).
\]

**Proof Sketch.** Somewhat informally, using flow conservation through the network, this is a straightforward consequence of the definitions of 'network typings' and the function \( \text{maxFromTo}_N \). More formal, but less transparent, is a proof by induction, as \( \mathcal{N} \) is reassembled inductively from the one-vertex components to the full network \( \mathcal{N} \), as in Section 3. All formal details omitted. 

\[\square\]
3 Reassembling the Network

Given a flow network \( N = (G, c) \), let \( m = |E\#(G)| \) and \( n = |E(G)| \). Note that \( m \) does not include a count of the edges in \( E_{\text{in}}(G) \cup E_{\text{out}}(G) \). Starting from \( n \) one-vertex components, which we denote:

\[
N_1 = (G_1, c), \quad N_2 = (G_2, c), \ldots, \quad N_n = (G_n, c),
\]

with one for each of the \( n \) vertices, we splice the two halves of each of the \( m \) edges in \( E\#(G) \), one by one in some order, until the full network \( N = (G, c) \) is reassembled:

\[
N_{n+1} = (G_{n+1}, c), \quad N_{n+2} = (G_{n+2}, c), \ldots, \quad N_{n+m} = (G_{n+m}, c),
\]

where \( N_{n+m} = N \). For every \( i \geq n + 1 \), the graph \( G_i \) is directed and connected, though not necessarily strongly connected, and has at least two vertices.

For every \( k = n + 1, \ldots, n + m \), the new network component \( N_k \) is the result of splicing the two dangling halves of some non-dangling edge \( e \) in the initial \( G \). If the two halves of \( e \) are \( e_1 \) and \( e_2 \), then the new \( N_k \) is related to the preceding network components \( \{N_1, \ldots, N_{k-1}\} \) in one of two ways:

**Case 1:** There are two distinct network components \( N_i \) and \( N_j \) such that \( i < j < k \), with \( e_1 \) an input (or output) edge in \( N_i \) and \( e_2 \) an output (or, resp., input) edge in \( N_j \).

**Case 2:** There is one network component \( N_i \) such that \( i < k \), with both \( e_1 \) an input (or output) edge and \( e_2 \) an output (or, resp., input) edge in \( N_i \).

For every \( i \in \{1, \ldots, n + m\} \), define the quantities:

\[
p_i \triangleq |E_{\text{in}}(G_i)| \quad \text{and} \quad q_i \triangleq |E_{\text{out}}(G_i)|.
\]

Thus, \( p_i + q_i \) is the total number of dangling edges (input edges and output edges) in \( G_i \), what is also called the edge-boundary degree of \( G_i \) (the number of edges that connect vertices inside \( G_i \) with vertices outside \( G_i \)).

We do not worry now about the order in which the reassembling is carried out in this section. Later we specify an order with which we obtain the result claimed in the report’s title. Define:

\[
\delta \triangleq \max \{ p_i + q_i \mid 1 \leq i \leq n + m \}.
\]

Thus, \( \delta \) is the least upper bound on the edge-boundary degrees of \( \{G_1, \ldots, G_{n+m}\} \). Our next task is to determine the function \( \text{maxFromTo}_{N_i} \) for every \( i = 1, \ldots, n + m \). We do this by induction on \( i \).

**Basis step.** Let \( i \in \{1, \ldots, n\} \). Each \( N_i = (G_i, c) \) is a one-vertex component. If \( V(G_i) = \{v\} \), then \( p_i + q_i = \deg(v) \). It is straightforward to compute \( \text{maxFromTo}_{N_i}(A, B) \) for every \( A \subseteq E_{\text{in}}(G_i) \) and every \( B \subseteq E_{\text{out}}(G_i) \). All details omitted.

**Induction hypothesis (IH).** Let \( k \in \{n+1, \ldots, n+m-1\} \). For every \( i \leq k-1 \), assume \( \text{maxFromTo}_{N_i}(A, B) \) has been already determined for every \( A \subseteq E_{\text{in}}(G_i) \) and every \( B \subseteq E_{\text{out}}(G_i) \).

**Induction step.** Let \( k \in \{n+1, \ldots, n+m-1\} \). We determine \( \text{maxFromTo}_{N_k} \) using IH. Let \( N_k \) be obtained from \( \{N_1, \ldots, N_{k-1}\} \) by splicing the two halves, \( e_1 \) and \( e_2 \), of some original edge \( e \in E\#(G) \). We consider the two cases identified earlier in this section separately.
Case 1: With no loss of generality, suppose \( e_1 \in E_{in}(G_i) \) and \( e_2 \in E_{out}(G_j) \). Hence:

\[
E_{in}(G_k) = (E_{in}(G_i) - \{e_1\}) \cup E_{in}(G_j),
\]
\[
E_{out}(G_k) = E_{out}(G_i) \cup (E_{out}(G_j) - \{e_2\}),
\]
\[
E_{\#}(G_k) = E_{\#}(G_i) \cup E_{\#}(G_j) \cup \{e\}.
\]

Consider arbitrary \( A \subseteq E_{in}(G_k) \) and \( B \subseteq E_{out}(G_k) \) and let:

\[
A = A_1 \cup A_2 \quad \text{and} \quad B = B_1 \cup B_2,
\]
\[
A_1 \subseteq E_{in}(G_i) - \{e_1\} \quad \text{and} \quad B_1 \subseteq E_{out}(G_i),
\]
\[
A_2 \subseteq E_{in}(G_j) \quad \text{and} \quad B_2 \subseteq E_{out}(G_j) - \{e_2\}.
\]

We then define:

\[
\max_{\text{FromTo}_{N_k}}(A, B) \triangleq \max_{\text{FromTo}_{N_i}}(A, B_1) + \max_{\text{FromTo}_{N_j}}(A_2, B_2) + \min \left\{ \max_{\text{FromToAft}_{N_i}}(e_1, B_1; A_1, B_1), \max_{\text{FromToAft}_{N_j}}(A_2, e_2; A_2, B_2) \right\}
\]

The first line after ‘\( \triangleq \)’ is the part of the maximum flow from \( A \) to \( B \) that does not use the edge \( e \); the second line after ‘\( \triangleq \)’ is the part of the maximum flow from \( A \) to \( B \) that does use the edge \( e \). We have thus defined \( \max_{\text{FromTo}_{N_k}} \) in terms of the already-defined, by IH, the functions \( \max_{\text{FromTo}_{N_i}} \) and \( \max_{\text{FromTo}_{N_j}} \), also invoking Lemma 2 which gives us \( \max_{\text{FromToAft}} \) in terms of \( \max_{\text{FromTo}} \). The value of \( \max_{\text{FromTo}_{N_k}}(A, B) \) is obtained by using twice ‘\( + \)’, once ‘\( \min \)’, and twice ‘\( - \)’ for the invocations of \( \max_{\text{FromToAft}_{N_i}}(e_1, B_1; A_1, B_1) \) and \( \max_{\text{FromToAft}_{N_j}}(A_2, e_2; A_2, B_2) \) (see Lemma 2).

Case 2: With \( e_1 \in E_{in}(G_i) \) and \( e_2 \in E_{out}(G_i) \), we have in this case:

\[
E_{in}(G_k) = E_{in}(G_i) - \{e_1\},
\]
\[
E_{out}(G_k) = E_{out}(G_i) - \{e_2\},
\]
\[
E_{\#}(G_k) = E_{\#}(G_i) \cup \{e\}.
\]

Consider arbitrary \( A \subseteq E_{in}(G_k) \) and \( B \subseteq E_{out}(G_k) \). Since \( E_{in}(G_k) \subseteq E_{in}(G_i) \) and \( E_{out}(G_k) \subseteq E_{out}(G_i) \), we also have \( A \subseteq E_{in}(G_i) \) and \( B \subseteq E_{out}(G_i) \). We then define:

\[
\max_{\text{FromTo}_{N_k}}(A, B) \triangleq \max_{\text{FromTo}_{N_i}}(A, B) + \min \left\{ \max_{\text{FromToAft}_{N_i}}(A \cup \{e_1\}, B; A, B), \max_{\text{FromToAft}_{N_j}}(A, B \cup \{e_2\}; A, B) \right\}
\]

The first line after ‘\( \triangleq \)’ is the part of the maximum flow from \( A \) to \( B \) that does not use the edge \( e \); the second line after ‘\( \triangleq \)’ is the part of the maximum flow from \( A \) to \( B \) that does use the edge \( e \). We have again defined \( \max_{\text{FromTo}_{N_k}} \) in terms of the already-defined, by IH, functions \( \max_{\text{FromTo}_{N_i}} \) and \( \max_{\text{FromTo}_{N_j}} \), and again invokingLemma 2 which gives us \( \max_{\text{FromToAft}} \) in terms of \( \max_{\text{FromTo}} \). The value of \( \max_{\text{FromTo}_{N_k}}(A, B) \) is obtained by using ‘\( + \)’ once, ‘\( \min \)’ once, and twice ‘\( - \)’ for the invocations of \( \max_{\text{FromToAft}_{N_i}}(A \cup \{e_1\}; A, B) \) and \( \max_{\text{FromToAft}_{N_j}}(A, B \cup \{e_2\}; A, B) \) (see Lemma 2).

This completes the induction step and the definition of the function \( \max_{\text{FromTo}_{N_i}} \) for every \( i = 1, \ldots, n+m \).

The next lemma is used in the proof of Theorem 9.
Lemma 4. Consider the reassembling of the flow network $N = (G, c)$ described in the opening paragraph of Section 3. Let $\delta$ be the least upper bound of the resulting edge boundary degrees $\{p_i + q_i\mid 1 \leq i \leq n + m\}$. Then the function $\max_{\text{FromTo}} = \max_{\text{FromTo}_{N_{n+m}}}$ is computed in time $O((m + n) \cdot 2^\delta)$ using only three arithmetic operations $\{\min, +, -\}$.

Proof. For each $i = 1, \ldots, n + m$, the number of arguments $(A, B) \in E_{\text{in}}(G_i) \times E_{\text{out}}(G_i)$ at which the function $\max_{\text{FromTo}}$ has to be determined is $2^{p_i} \cdot 2^{q_i} = 2^{p_i+q_i} \leq 2^\delta$. And each such determination is carried out using at most four times an operation in $\{+,-\}$ and at most twice an operation in $\{\max, \min\}$. A subtraction with ‘−’ is involved with each invocation of the function $\max_{\text{FromToAft}}$ (Lemma 2). Both ‘+’ and ‘min’ are involved in the determination of $\max_{\text{FromTo}_{N_{i-1}}}, \ldots, \max_{\text{FromTo}_{N_{hi}}}$, and both ‘−’ and ‘min’ are involved in the determination of $\max_{\text{FromTo}_{N_{n+1}}}, \ldots, \max_{\text{FromTo}_{N_{n+m}}}$. \hfill $\square$

4 The Main Result

In order to use the algorithm whose existence is asserted in Lemma 8 in our main result (Theorem 9), we need to transform the underlying graph $G$ of the network $N = (G, c)$ according to Lemma 7. Part 5 of the latter uses the notion of edge-outerplanarity, which we next define and compare to the standard notion of outerplanarity.

We make a distinction between planar graphs and plane graphs. $G$ is a plane graph if it is drawn on the plane without any edge crossings. $G$ is a planar graph if it is isomorphic to a plane graph; i.e., it is embeddable in the plane in such a way that its edges intersect only at their endpoints. To keep the distinction between the two notions, we define the outerplanarity index of a planar graph and the outerplanarity of a plane graph.

If $G$ is a plane graph, directed or undirected, then the outerplanarity of $G$ is the number $k$ of times that all the vertices on the outer face (together with all their incident edges) have to be removed in order to obtain the empty graph. In such a case, we say that the plane graph $G$ is $k$-outerplanar.

If $G$ is a planar graph, directed or undirected, then the outerplanarity index of $G$ is the minimum of the outerplanarities of all the plane embeddings $G'$ of $G$.

Deciding whether an arbitrary graph is planar can be carried out in linear time $O(n)$ and, if it is planar, a plane embedding of it can also be carried out in linear time [12]. Given a planar graph $G$, the outerplanarity index $k$ of $G$ and a $k$-outerplanar embedding of $G$ in the plane can be computed in time $O(n^2)$, and a 4-approximation of its outerplanarity index can be computed in linear time [5].

Definition 5 (Edge-Outerplanarity). Let $G$ be a plane graph, directed or undirected. If $E(G) = \emptyset$ and $G$ is a graph of isolated vertices, the edge outerplanarity of $G$ is 0. If $E(G) \neq \emptyset$, we pose $G_0 := G$ and define $K_0$ as the set of edges lying on OuterFace($G_0$).

For every $i > 0$, we define $G_i$ as the plane graph obtained after deleting all the edges in $K_0 \cup \cdots \cup K_{i-1}$ from the initial $G$ and $K_i$ the set of edges lying on OuterFace($G_i$).

The edge outerplanarity of $G$, denoted $E$-outerplanarity($G$), is the least integer $k$ such that $G_k$ is a graph without edges, i.e., the edge outerplanarity of $G_k$ is 0. This process of peeling off the edges lying on the outer face $k$ times produces a $k$-block partition of $E(G)$, namely, $\{K_0, \ldots, K_{k-1}\}$.

To keep outerplanarity and edge outerplanarity clearly apart, we call the first vertex outerplanarity, or more simply $V$-outerplanarity, and the second edge outerplanarity, or more simply $E$-outerplanarity.

4There is an unessential difference between our definition here and the definition in [2]. In Section 2.2 of that reference, “a $k$-edge-outerplanar graph is a planar graph having an embedding with at most $k$ layers of edges.” In our presentation, we limit the definition to plane graphs and say “a $k$-edge-outerplanar plane graph has exactly $k$ layers of edges.” Our version simplifies a few things later.
There is a close relationship between $V$-outerplanarity and $E$-outerplanarity (Theorem 4 in Section 5.1 in [2]). In the case of three-regular plane graphs, the relationship is much easier to state. This is Proposition 6 next, not needed for our main result (Theorem 9) but included here for completeness.

**Proposition 6.** If $G$ is a 3-regular plane graph, directed or undirected, then:

$$V\text{-outerplanarity}(G) \leq E\text{-outerplanarity}(G) \leq 1 + V\text{-outerplanarity}(G).$$

Thus, for 3-regular plane graphs, V-outerplanarity and E-outerplanarity are “almost the same”.

**Proof Sketch.** For a 3-regular plane graph, the difference between $V$-outerplanarity($G$) and $E$-outerplanarity($G$) occurs in the last stage in the process of repeatedly removing (in the case of standard $V$-outerplanarity) all vertices on the outer face and all their incident edges. The corresponding last stage in the case of $E$-outerplanarity may or may not delete all edges; if it does not, then one extra stage is needed to delete all remaining edges. □

**Lemma 7.** There is an algorithm which, given an arbitrary flow network $\mathcal{N} = (G, c)$, returns a flow network $\mathcal{N}^* = (G^*, c^*)$ in time $O(n + m)$, where $n = |V(G)|$ and $m = |E_\#(G)|$, such that:

1. $E_{in}(G^*) = E_{in}(G)$ and $E_{out}(G^*) = E_{out}(G)$, so that also $E_{io}(G^*) = E_{io}(G)$.
2. $|V(G^*)| = O(n)$ and $|E_\#(G^*)| = O(m)$.
3. $\mathcal{N}$ and $\mathcal{N}^*$ are equivalent flow networks, in particular, $\tau : \mathcal{P}(E_{io}(G)) \rightarrow \mathcal{I}(\mathbb{R})$ is a principal typing for $\mathcal{N}$ iff it is a principal typing for $\mathcal{N}^*$.
4. $G^*$ is a 3-regular directed graph without two-edge cycles.\(^5\)

Moreover, if $G$ is a plane graph, then:

5. $G^*$ is a plane graph such that $E$-outerplanarity($G^*$) = $E$-outerplanarity($G$).

It is worth pointing out that the hidden constants in the big-O notations above are small integers, each a single-digit number.

**Proof.** This is shown in Section 3 of the earlier report [8]. The 5-part conclusion of the lemma here is divided into several lemmas in the earlier report. □

Let $G$ be a simple undirected graph. A **reassembling** of $G$ is a rooted binary tree $\mathcal{B}$ whose nodes are subsets of $V(G)$ and whose leaf nodes are singleton sets, with each of the latter containing a distinct vertex of $G$. The parent of two nodes in $\mathcal{B}$ is the union of the two children’s vertex sets. The root node of $\mathcal{B}$ is the full set $V(G)$. If $n = |V(G)|$, there are thus $n$ leaf nodes in $\mathcal{B}$ and a total of $(2n - 1)$ nodes in $\mathcal{B}$. We denote the reassembling of $G$ according to $\mathcal{B}$ by writing $(G, \mathcal{B})$.\(^6\)

The **edge-boundary degree** of a node in $\mathcal{B}$ is the number of edges that connect vertices in the node’s set to vertices not in the node’s set. Following a terminology used in earlier reports, the $\alpha$-measure of the reassembling $(G, \mathcal{B})$, denoted $\alpha(G, \mathcal{B})$, is the largest edge-boundary degree of any node in the tree $\mathcal{B}$. We say $\alpha(G, \mathcal{B})$ is **optimal** if it is minimum among all $\alpha$-measures of $G$’s reassemblings, in which case we also say $\mathcal{B}$ is $\alpha$-optimal.\(^7\)

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\(^5\)See footnote 2 on page 1.

\(^6\)To keep apart $\mathcal{B}$ and $G$, we reserve the words ‘node’ and ‘branch’ for the tree $\mathcal{B}$, and the words ‘vertex’ and ‘edge’ for the graph $G$.

\(^7\)The reassembling process described in the Introduction, Section 1, and again in the opening paragraph of Section 3, is a lazy version of the reassembling defined here. We can call the latter the eager version of reassembling. The difference is that, in the lazy version, only one edge’s two halves are spliced at any given time; in the eager version defined in this section, the two halves of all the edges between two disjoint components (i.e., two sibling nodes in the tree $\mathcal{B}$) are spliced simultaneously. Hence, if we carry out the reassembling $(G, \mathcal{B})$ lazily, then a least upper bound on the edge-boundary degrees of all the components is $2 \cdot \alpha(G, \mathcal{B}) - 1$. 

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8
The problem of constructing an \( \alpha \)-optimal reassembling \((G, B)\) of a simple undirected graph \(G\) in general was already shown NP-hard \([9, 11, \text{among others}]\). However, restricting attention to plane graphs, we have the following positive result.

**Lemma 8.** There is an algorithm which, given a plane 3-regular simple undirected graph \(G\), returns a reassembling \((G, B)\) in time \(O(n)\) such that \(\alpha(G, B) \leq 2k\), where \(k = E\text{-outerplanarity}(G)\) and \(n = |V(G)|\).

**Proof.** This is Theorem 9 and Corollary 20 in the report \([10]\).

**Theorem 9.** There is an algorithm which, given a flow network \(\mathcal{N} = (G, c)\) where \(G\) is planar, computes a principal typing for \(\mathcal{N}\) in time \(O(n \cdot 2^\delta)\), where \(n = |V(G)|\), \(\delta = \max\left\{ 2k, |E_{\text{in}}(G) \cup E_{\text{out}}(G)| \right\}\) and \(k = E\text{-outerplanarity}(H)\) where \(H\) is a plane embedding of \(G\).

**Proof.** We start by computing a plane embedding \(H\) of \(G\), which can be done in time \(O(n)\), as pointed out at the beginning of this section. After this embedding, we refer to the network \((H, c)\) by the same name ‘\(\mathcal{N}\)’. Next, we use Lemma 7 to transform the network \(\mathcal{N} = (H, c)\) into an equivalent network \(\mathcal{N}^* = (H^*, c^*)\) where \(H^*\) is a 3-regular plane graph such that \(k = E\text{-outerplanarity}(H) = E\text{-outerplanarity}(H^*)\). The transformation \(\mathcal{N} \mapsto \mathcal{N}^*\) is carried out in time \(O(n + m)\) and therefore in time \(O(n)\), because \(H\) is a plane graph.

Next, we compute a reassembling \((H^*, B)\) in time \(O(n)\), by invoking Lemma 8, with \(\alpha(H^*, B) \leq 2k\). We now use Lemma 4 to compute the function \(\max_{\mathcal{N}^*} \text{FromTo}_{\mathcal{N}^*} = \max_{\mathcal{N}^*} \text{FromTo}_{\mathcal{N}}\) in time \(O((m + n) \cdot 2^\delta)\) and therefore in time \(O(n \cdot 2^\delta)\) where \(\delta = \max\left\{ 2k, |E_{\text{in}}(H^*) \cup E_{\text{out}}(H^*)| \right\}\).

Finally, we use Lemma 3 to return a principal typing \(\tau\) for \(\mathcal{N}\), simultaneously with the computation of the function \(\max_{\mathcal{N}^*} \text{FromTo}_{\mathcal{N}^*}\).

It is worth pointing out that the computation of principal typings in Theorem 9 involves only three arithmetic operations \{\(\min, +, -\}\), according to Lemma 4.

## 5 Future Work

Flow networks in this report are the simplest possible and are of the form \(\mathcal{N} = (G, c)\), where the function \(c : E(G) \to \mathbb{R}_+\) assigns an upper-bound capacity to every edge. The method proposed in this report to compute principal typings for such networks, in fixed-parameter linear time, is a ‘template’ for further extensions to more general forms of flow networks.

The next extension of the method considers flow networks of the form \(\mathcal{N} = (G, \underline{c}, \overline{c})\), where the two functions \(\underline{c}, \overline{c} : E(G) \to \mathbb{R}_+\) assign a lower-bound capacity and an upper-bound capacity, respectively, to every edge. And there are still other extensions under consideration, including the following:

- **multicommodity flows** (formal definitions in \([1, \text{Chapt. 17]}\),
- **minimum-cost flows, minimum-cost max flows**, and variations (definitions in \([1, \text{Chapt. 9-11]}\),
- **flows with multiplicative gains and losses**, also called **generalized flows** (definitions in \([1, \text{Chapt. 15]}\),
- **flows with additive gains and losses** (definitions in \([4]\)).

This is on-going work requiring various refinements, not all the same for the different extensions.
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