STABILITY OF STOCHASTIC SEMIGROUPS AND APPLICATIONS TO STEIN’S NEURONAL MODEL

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Abstract. A new theorem on asymptotic stability of stochastic semigroups is given. This theorem is applied to a stochastic semigroup corresponding to Stein’s neuronal model. Asymptotic properties of models with and without the refractory period are compared.

1. Introduction. Stochastic semigroups are positive $C_0$-semigroups on the space $L^1(X, \Sigma, m)$, which preserve the set of densities. They describe the evolution of densities of the distribution of Markov processes [1, 4, 8, 9, 14, 15]. One of the main problems in the theory of stochastic semigroups is their asymptotic stability [8], i.e., when there exists an invariant density $f^*$ such that $P(t)f \to f^*$ as $t \to \infty$ for every density $f$. If a physical or biological process is described by an asymptotically stable stochastic semigroup, then we can suppose that its distribution is approximately an invariant density, what can help to estimate its parameters.

Most of stochastic semigroups which appear in applications (e.g. related to diffusion or piecewise deterministic processes [5]) are partially integral, i.e., they have a nontrivial kernel part. If a partially integral semigroup has a unique invariant density supported on the whole space, then it is asymptotically stable (see [10]). However, it is not easy to prove the existence of invariant density, and we often need criteria for asymptotic stability without this assumption. In this paper we provide a result in this spirit, which might be useful in applications. This result is based on a decomposition theorem of a stochastic semigroup into asymptotically stable and sweeping components [11].

An application of this theorem to Stein’s model [16] is given in Section 3. This model describes neuronal membrane potential and spike generation. There are several papers devoted to study properties of this model and its generalizations, see [3, 7, 13, 17, 18, 19] and the references therein. Our aim is to describe and study the evolution of probability density for membrane potential using the semigroup approach. To our knowledge, this approach to Stein’s model have not received much

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attention (see [2, Chap. 30],[4, Chap. 6.4],[15]), but it could be useful in a precise formulation and study the model. In this section we shortly recall Stein’s model and we present it as a piecewise deterministic Markov process. Then we introduce a stochastic semigroup related to this model. Finally, we prove asymptotic stability of this semigroup and compare asymptotic properties of models with and without the refractory period.

2. Asymptotic properties of stochastic semigroups. Let a triple \((X, \Sigma, m)\) be a \(\sigma\)-finite measure space. Denote by \(D\) the subset of the space \(L^1 = L^1(X, \Sigma, m)\) which contains all densities

\[
D = \{ f \in L^1 : f \geq 0, \|f\| = 1 \}.
\]

A linear operator \(P: L^1 \rightarrow L^1\) is called stochastic if \(Pf \in D\) for \(f \in D\). A family \(\{P(t)\}_{t \geq 0}\) of linear operators on \(L^1\) is called a stochastic semigroup if it is a strongly continuous semigroup and all operators \(P(t)\) are stochastic.

We assume additionally that \(X\) is a separable metric space and \(\Sigma = \mathcal{B}(X)\) is the \(\sigma\)-algebra of Borel subsets of \(X\). We will consider a stochastic semigroup \(\{P(t)\}_{t \geq 0}\) such that for each \(t \geq 0\) we have

\[
P(t)f(x) \geq \int_X q(t,x,y)f(y)\,m(dy) \quad \text{for } f \in D,
\]

where \(q(t,\cdot,\cdot): X \times X \rightarrow [0,\infty)\) is a measurable function and the following condition holds:

(K) for every \(y_0 \in X\) there exist an \(\varepsilon > 0\), a \(t > 0\), and a measurable function \(\eta \geq 0\) such that \(\int \eta(x)\,m(dx) > 0\) and

\[
q(t,x,y) \geq \eta(x)1_{B(y_0,\varepsilon)}(y) \quad \text{for } x \in X,
\]

where \(B(y_0,\varepsilon) = \{ y \in X : \rho(y,y_0) < \varepsilon \}\).

Condition (K) is satisfied if, for example, for every point \(y \in X\) there exist a \(t > 0\) and an \(x \in X\) such that the kernel \(q(t,\cdot,\cdot)\) is continuous in a neighbourhood of \((x,y)\) and \(q(t,x,y) > 0\).

In [11] the following theorem is proved.

**Theorem 2.1.** Let \(\{P(t)\}_{t \geq 0}\) be a stochastic semigroup which satisfies (K). Then there exist an at most countable set \(J\), a family of invariant densities \(\{f_j^*\}_{j \in J}\) with disjoint supports \(\{A_j\}_{j \in J}\), and a family \(\{\alpha_j\}_{j \in J}\) of positive linear functionals defined on \(L^1\) such that

(i) for every \(j \in J\) and for every \(f \in L^1\) we have

\[
\lim_{t \rightarrow \infty} \|1_{A_j}P(t)f - \alpha_j(f)f_j^*\| = 0,
\]

(ii) if \(Y = X \setminus \bigcup_{j \in J} A_j\), then for every \(f \in L^1\) and for every compact set \(F\) we have

\[
\lim_{t \rightarrow \infty} \int_{F \cap Y} P(t)f(x)\,m(dx) = 0.
\]

**Remark 1.** The support of a measurable function \(f\) is defined up to a set of measure zero by the formula: \(\text{supp } f = \{ x \in X : f(x) \neq 0 \}\).

**Remark 2.** It is not difficult to check (see [12]) that the sets \(A_j, j \in J\), which occur in the formulation of Theorem 2.1, are not only disjoint but also their closures are disjoint.
Remark 3. Theorem 2.1 still holds if we assume that \( \{P(t)\}_{t \geq 0} \) is a substochastic semigroup [12], i.e., \( \|P(t)f\| \leq \|f\| \) and \( P(t)f \geq 0 \) for \( f \geq 0, t \geq 0 \).

Now, we formulate a theorem concerning asymptotic stability, which is a consequence of Theorem 2.1. We recall that a stochastic semigroup \( \{P(t)\}_{t \geq 0} \) is asymptotically stable if there exists a density \( f^* \) such that

\[
\lim_{t \to \infty} \|P(t)f - f^*\| = 0 \quad \text{for } f \in D. \tag{5}
\]

From (5) it follows immediately that \( f^* \) is invariant with respect to \( \{P(t)\}_{t \geq 0} \), i.e. \( P(t)f^* = f^* \) for each \( t \geq 0 \). We need two auxiliary conditions. Both of them are weak versions of well known notions: irreducibility and tightness.

(I) There exists a point \( x_0 \in X \) such that for each \( \varepsilon > 0 \) and for each density \( f \) we have

\[
\int_{B(x_0, \varepsilon)} P(t)f(x) \, m(dx) > 0 \quad \text{for some } t = t(\varepsilon, f) > 0. \tag{6}
\]

(T) There exists \( \kappa > 0 \) such that

\[
\sup_{F \in \mathcal{F}} \limsup_{t \to \infty} \int_F P(t)f(x) \, m(dx) \geq \kappa \tag{7}
\]

for \( f \in D_0 \), where \( D_0 \) is a dense subset of \( D \) and \( \mathcal{F} \) is the family of all compact subsets of \( X \).

Theorem 2.2. Let \( \{P(t)\}_{t \geq 0} \) be a stochastic semigroup. Assume that \( \{P(t)\}_{t \geq 0} \) satisfies conditions (K), (I), and (T). Then the semigroup \( \{P(t)\}_{t \geq 0} \) is asymptotically stable.

Proof. First, we observe that if a substochastic semigroup \( \{P(t)\}_{t \geq 0} \) satisfies conditions (K) and (I), then \( J \) is an empty set or a singleton. Indeed, if \( f \) is an invariant density and \( A = \text{supp} f \), then from condition (I) it follows that \( x_0 \in \text{cl} A \), where \( \text{cl} A \) is the closure of \( A \). But, as we have mentioned in Remark 2, the sets \( A_i, i \in J \), have disjoint closures, and consequently \( J \) contains at most one element.

The set \( J \) cannot be empty, because in this case \( \lim_{t \to \infty} \int_F P(t)f(x) \, m(dx) = 0 \) for each density \( f \) and each compact set \( F \), which contradicts (T). Thus, we have exactly one invariant density \( f^* \), supported on some set \( A \). Let \( Y = X \setminus A \) and \( \alpha(t, f) = \int_A P(t)f(x) \, m(dx) \) for \( f \in L^1 \). Then the function \( t \mapsto \alpha(t, f) \) is non-decreasing if \( f \geq 0 \), \( \alpha(f) = \lim_{t \to \infty} \alpha(t, f) \), and \( \lim_{t \to \infty} 1_A P(t)f = \alpha(f)f^* \) for \( f \in L^1 \). We claim that \( \alpha(f) = 1 \) for all \( f \in D \). In order to prove this, we define \( \beta = \inf \{\alpha(f) : f \in D\} \). We check that \( \beta = 1 \). If \( \beta \in (0,1) \), then for each \( \varepsilon > 0 \) we find \( f \in D \) and \( t_0 > 0 \), such that \( \alpha(t, f) \in (\beta - \varepsilon, \beta + \varepsilon) \) for \( t \geq t_0 \). Let \( f_1 = 1_A P(t_0)f \) and \( f_2 = 1_Y P(t_0)f \). Since \( \text{supp} f_1 \subseteq A \) we have \( \alpha(f_1) = \alpha(t_0, f) > \beta - \varepsilon \). The function \( g = (1 - \alpha(t_0, f))^{-1} f_2 \) is a density, and therefore

\[
\alpha(f_2) = (1 - \alpha(t_0, f)) \alpha(g) \geq (1 - \alpha(t_0, f)) \beta \geq (1 - \beta - \varepsilon) \beta.
\]

Since \( \alpha(f) = \alpha(f_1) + \alpha(f_2) \) we have \( \alpha(f) \geq (2 - \beta) \beta - \varepsilon (1 + \beta) \) but \( \alpha(f) \leq \beta + \varepsilon \), which is impossible for a sufficiently small \( \varepsilon \). If \( \beta = 0 \), then we have \( \int_A P(t)f(x) \, m(dx) \leq \kappa/2 \) for some \( f \in D_0 \), because the set \( D_0 \) is dense in \( D \). From this and from (4) it follows that \( \limsup_{t \to \infty} \int_F P(t)f(x) \, m(dx) \leq \kappa/2 \) for every compact set \( F \), which contradicts (T). Thus \( \beta = 1 \), and therefore \( \alpha(f) = 1 \) for \( f \in D \), which completes the proof of asymptotic stability. \( \square \)
3. Stein’s model of neural activity. Electrical activity of a neuron is described by its depolarization $x(t)$. Nerve cells may be excited or inhibited through neuron’s synapses — junctions between nerve cells (or between muscle and nerve cell) such that electrical activity in one cell may influence the electrical potential in the other. Synapses may be excitatory or inhibitory. We assume that there are two nonnegative constants $x_E$ and $x_I$ such that if at time $t$ an excitation occurs then $x(t^+) = x(t^-) + x_E$ and if an inhibition occurs then $x(t^+) = x(t^-) - x_I$. The jumps (excitations and inhibitions) may occur at random times according to two independent Poisson processes $N^E(t)$, $N^I(t)$, $t \geq 0$, with positive intensities $\lambda_E$ and $\lambda_I$, respectively. Between jumps depolarization $x(t)$ decays according to the equation $x'(t) = -\alpha x(t)$, $\alpha > 0$. When a sufficient (threshold) level $\theta > 0$ of excitation is reached, the neuron emits an action potential (fire). This will be followed by an absolute refractory period of duration $t_R$, during which $x \equiv 0$ and then the process starts again.

The neural activity can be described as a piecewise deterministic Markov process in the following way. We introduce an extra 0-phase, which begins at the end of the refractory period and finishes when depolarization jumps from 0 for the first time. After a first jump we have the subthreshold phase denoted by 1, and by 2 we denote the refractory phase of duration $t_R$. Therefore, the state space is $X = \{(0, 0)\} \cup (-\infty, \theta] \times \{1\} \cup [0, t_R] \times \{2\}$, and in the 0-phase we have only one state $(0, 0)$; if the neuron is in the 1-phase then it has a state $(x, 1)$, where $x$ is its depolarization; and in the 2-phase it has a state $(x, 2)$, where $x$ is the time since the moment of firing. We consider two types of jumps: when the neuron is excited or inhibited and at the end of the refractory period. Thus, we can have one or more jumps inside phase 1. Fig. 1 presents a schematic diagram of the model.

Let $\xi(t) = (\xi_1(t), \xi_2(t)), t \geq 0$, be a stochastic process with values in $X$ describing the state of the neuron. Between jumps the process $\xi(t)$ satisfies the following system of equations

$$\xi'_1(t) = \begin{cases} 
0, & \text{if } \xi_2(t) = 0, \\
-\alpha \xi_1(t), & \text{if } \xi_2(t) = 1, \\
1, & \text{if } \xi_2(t) = 2.
\end{cases}$$
If \( \xi(t) = (0,0) \) then the process \( \xi \) jumps with intensity \( \lambda_E \) to \((x_E,1)\), and with intensity \( \lambda_I \) to \((-x_I,1)\), i.e. the probability of jumping to \((x_E,1)\) or to \((-x_I,1)\) in the time interval \([t,t+\Delta t]\) is, respectively, \( \lambda_E \Delta t + o(\Delta t) \) and \( \lambda_I \Delta t + o(\Delta t) \). Consider the 1-phase. Then the process \( \xi(t) = (\xi_1(t),1) \) moves according to the equation

\[
\frac{d}{dt} \xi_1(t) = -\alpha \xi_1(t)
\]

and has a jump at some time \( s \in [t,t+\Delta t] \) to \((\xi_1(s) - x_I,1)\) with probability \( \lambda_I \Delta t + o(\Delta t) \) and with probability \( \lambda_E \Delta t + o(\Delta t) \) to \((\xi_1(s) + x_E,1)\) if \( \xi_1(s) \leq \theta - x_E \) and to \((0,2)\) if \( \xi_1(s) > \theta - x_E \). If \( \xi(t) = (0,2) \) then \( \xi(t+s) = (s,2) \) for \( s \leq t_R \) and at the point \((t_R,2)\) the process \( \xi \) jumps to \((0,0)\) with probability one.

Define a measure \( m \) on the \( \sigma \)-algebra \( \mathcal{B}(X) \) of the Borel subsets of \( X \) by \( m = \delta_{(0,0)} + m_1 + m_2 \), where \( \delta_{(0,0)} \) is the Dirac measure at \((0,0)\), \( m_1 \) is the Lebesgue measure on the segment \((-\infty,\theta] \times \{1\}\), and \( m_2 \) is the Lebesgue measure on the segment \([0,t_R] \times \{2\}\). The family of PDMPs described above induces a stochastic semigroup \( \{P(t)\}_{t \geq 0} \) on the space \( L^1(X,\mathcal{B}(X),m) \) in the following way. If the distribution of \( \xi(0) \) has density \( f \) then \( P(t)f \) is the density of \( \xi(t) \).

One can ask about the infinitesimal generator \( A \) of the semigroup \( \{P(t)\}_{t \geq 0} \). In order to find it, we first determine the infinitesimal generator \( L \) of the process \( \xi(t) \), and since the operator \( A \) is formally adjoint to \( L \), we will be able to set down \( A \). We additionally assume that \( \theta > x_E > 0 \), and \( x_I > 0 \) (for other cases see Remark 4).

Let us recall that

\[
L f(x,i) = \lim_{t \downarrow 0} \frac{1}{t} (E f(\xi(t)) - f(x,i)),
\]

with \( \xi(0) = (x,i) \). It is easy to check that if \( f \) is a \( C^1 \) function then

\[
L f(0,0) = \lambda_E f(x_E,1) + \lambda_I f(-x_I,1) - (\lambda_E + \lambda_I) f(0,0),
\]

\[
L f(x,1) = \begin{cases} 
\lambda_E f(x + x_E,1) + Bf(x,1), & \text{if } x \leq \theta - x_E, \\
\lambda_E f(0,2) + Bf(x,1), & \text{if } x > \theta - x_E,
\end{cases}
\]

\[
L f(x,2) = \frac{\partial f(x,2)}{\partial x},
\]

where \( Bf(x,1) = -\alpha x \frac{\partial f(x,1)}{\partial x} + \lambda_I f(x - x_I,1) - (\lambda_E + \lambda_I) f(x,1) \), and \( f \) satisfies the boundary condition \( f(0,0) = f(t_R,2) \). Since \( A \) is adjoint to \( L \) we find that

\[
A f(0,0) = f(t_R,2) - (\lambda_E + \lambda_I) f(0,0), \tag{8}
\]

\[
A f(x,1) = \frac{\partial (\alpha x f(x,1))}{\partial x} - (\lambda_E + \lambda_I) f(x,1) + \lambda_E f(x - x_E,1) + \lambda_I f(x + x_I,1) 1_{(-\infty,\theta-x_I]}(x), \tag{9}
\]

\[
A f(x,2) = -\frac{\partial f(x,2)}{\partial x}. \tag{10}
\]

A function \( f \in L^1(X) \) belongs to the domain of the operator \( A \) if it is absolutely continuous on the segments \([0,t_R] \times \{2\}\), \((-\infty,-x_I) \times \{1\}\), \((-x_I,x_E) \times \{1\}\), \((x_E,\theta] \times \{1\}\), \( A f \in L^1(X) \), and \( f \) satisfies the following conditions

\[
f(x_E,1) - f(x_E^+,1) = \frac{\lambda_E}{\alpha x_E} f(0,0), \tag{11}
\]

\[
f((-x_I)^+,1) - f((-x_I)^-,1) = \frac{\lambda_I}{\alpha x_I} f(0,0). \tag{12}
\]
\[ f(\theta, 1) = 0, \quad (13) \]
\[ f(0, 2) = \lambda E \int_{x_E}^{\theta} f(x, 1) \, dx. \quad (14) \]

Conditions (11)–(14) also follow from the fact that \( A \) is adjoint to \( L \). We omit here the formal proof that \( A \) is indeed the generator of the stochastic semigroup related to Stein’s model.

**Theorem 3.1.** The stochastic semigroup introduced by Stein’s model is asymptotically stable.

**Proof.** First we need to find \( q \) such that conditions (1) and (K) hold. We set \( q(t, x, y) = 0 \) for \( x = (0, 0) \), \( x = (x, 2) \), and all \( y \in X \). Therefore it remains to define \( q \) for \( x = (x, 1) \) and \( y \in X \). Consider the process \( \xi(t) \) starting from a point \( y = (y_1, 1) \), where \( y \in (-\infty, \theta] \). If in the time interval \([0, t]\) the process \( \xi \) has exactly one jump and this jump is to the left and it occurs at time \( s \) then

\[ \xi_1(t) = (ye^{-\alpha s} - x_I)e^{-\alpha(t-s)} = ye^{-\alpha t} - x_Ie^{-\alpha(t-s)}. \]

Since \( e^{-(\lambda_I + \lambda_E)t}\lambda_I \Delta s + o(\Delta s) \) is the probability that the jump is in the time interval \([s, s + \Delta s]\), we have

\[ E f(\xi_1(t)) \geq \int_0^t f \left( ye^{-\alpha t} - x_Ie^{-\alpha(t-s)} \right) e^{-(\lambda_I + \lambda_E)t} \lambda_I \, ds \]
\[ = \lambda_I e^{-(\lambda_I + \lambda_E)t} \int_{ye^{-\alpha t} - x_I}^{(y-x_I)e^{-\alpha t}} \frac{f(x) \, dx}{\alpha(ye^{-\alpha t} - x)} \]

for any measurable function \( f: (-\infty, \theta] \to [0, \infty) \). From (15) it follows that we can define \( q \) for \( y = (y_1, 1) \) by

\[ q(t, (x, 1), (y, 1)) = \frac{\lambda_I e^{-(\lambda_I + \lambda_E)t}}{\alpha(ye^{-\alpha t} - x)} \mathbf{1}_{[ye^{-\alpha t} - x_I, (y-x_I)e^{-\alpha t}]}(x). \quad (16) \]

We fix a point \( y_0 = (y_0, 1) \in X \) and take \( t_0 = \frac{1}{2}\ln 2 \) and \( x_0 = (x_0, 1) \), where \( x_0 = \frac{y_0}{T} - \frac{2}{3}x_I \). Then the function \( q(t, \cdot, \cdot) \) is continuous in some neighborhood of \((x_0, y_0)\) and \( q(t, x_0, y_0) > 0 \), and consequently (2) holds. If \( y_0 = (0, 0) \), then \( y_0 \) is an isolated point in \( X \) and condition (2) holds with

\[ q(t, (x, 1), (0, 0)) = \lambda_I e^{-(\lambda_I + \lambda_E)t}(-\alpha t)^{-1} \mathbf{1}_{[-x_I, -x_Ie^{-\alpha t}]}(x) \]

and \( \eta(x) = \lambda_I e^{-(\lambda_I + \lambda_E)t}(\alpha x_I)^{-1} \mathbf{1}_{[-x_I, -x_Ie^{-\alpha t}]}(x) \). If \( y = (y_2, 2) \) then we define \( q \) by

\[ q(t, (x, 1), (y, 2)) = q(t - (t_R - y), (x, 1), (0, 0)) \quad (18) \]

for \( y \in [0, t_R] \) and \( t > t_R \). From (17) and (18) it follows that condition (2) also holds for \( y_0 = (y_0, 2) \).

Now, we check condition (1). We take \( x_0 = (-x_I, 1) \). Then from (16), (17), and (18) it follows that for \( \varepsilon > 0 \) and for each point \( y \in X \) there exists \( t > 0 \) such that

\[ \int_{B(x_0, \varepsilon)} q(t, x, y) \, m(dx) > 0, \quad \text{which implies (6).} \]

In order to prove condition (T) we fix a density \( f \) such that

\[ \int_{-\infty}^{0} |x| f(x, 1) \, dx < \infty. \quad (19) \]

Since the set of all densities which satisfy (19) is dense in \( D \) it is sufficient to check (7) for densities from this set. Consider a process \( \xi(t) \) with initial value \( \xi(0) \)
with density \( f \). We define a new process \( \zeta(t) \) with values in the interval \((-\infty, 0]\) with jumps \( \zeta(t^+) = \zeta(t^-) - x_I \) which occur at random times according to Poisson processes \( N(t), t \geq 0 \), with intensity \( \lambda_I \), \( \zeta(t) \) satisfies between jumps the equation \( \zeta'(t) = -\alpha \zeta(t) \), and assume that

\[
\zeta(0)(\omega) = \begin{cases} 
\xi_1(0)(\omega), & \text{if } \xi_1(0)(\omega) \leq 0 \text{ and } \xi_2(0)(\omega) = 1, \\
0, & \text{in other cases.}
\end{cases}
\]

The process \( \zeta(t) \) is isomorphic to the shot noise process [6] and from its definition it follows that \( \zeta(t) \leq \xi_1(t) \) for all \( t \geq 0 \) and \( \mathbb{E}\zeta(0) = \int_{-\infty}^{0} xf(x, 1) \, dx > -\infty \). Let the measure \( \mu_I \) be the distribution of \( \zeta(t) \). Since the process \( \zeta(t), t \geq 0 \), has the infinitesimal generator

\[
L_\zeta f(x) = -\alpha x f'(x) + \lambda_I f(x - x_I) - \lambda_I f(x),
\]

we have

\[
\mathbb{E}\zeta(t) = \int_{-\infty}^{0} x \mu_I(dx) = \int_{-\infty}^{0} u(t, x) \mu_0(dx),
\]

where \( u(t, x) \) is the solution of the equation

\[
\frac{\partial u(t, x)}{\partial t} = -\alpha x \frac{\partial u(t, x)}{\partial x} + \lambda_I u(t, x - x_I) - \lambda_I u(t, x)
\]

with initial condition \( u(0, x) = x \). One can find that

\[
u(t, x) = e^{-\alpha t} \left( x + \frac{\lambda_I x_I}{\alpha} \right) - \frac{\lambda_I x_I}{\alpha},
\]

and consequently

\[
\mathbb{E}\zeta(t) = e^{-\alpha t} \left( \mathbb{E}\zeta(0) + \frac{\lambda_I x_I}{\alpha} \right) - \frac{\lambda_I x_I}{\alpha} \geq \min \left\{ \mathbb{E}\zeta(0), -\frac{\lambda_I x_I}{\alpha} \right\} := -c > -\infty.
\]

From Markov’s inequality we obtain \( \text{Prob}(\zeta(t) \leq -n) \leq \mathbb{E}|\zeta(t)|/n \leq c/n \), and since \( \zeta(t) \leq \xi_1(t) \), we have \( \int_{-\infty}^{-n} P(t)(x, 1) \, dx \leq c/n \). Let \( F_n = X \setminus (-\infty, -n) \times \{1\} \) for \( n = 1, 2, \ldots \). Then \( F_n \) is a sequence of compact subsets of \( X \),

\[
\int_{F_n} P(t)(x) \, dx \geq 1 - c/n,
\]

and therefore condition (T) holds.

It would be interesting to find some properties of the invariant density \( f^* \) of the semigroup induced by Stein’s model. Since \( Af^* \equiv 0 \), from (8) – (10), and (14) we have

\[
\begin{align*}
f^*(x, 2) &= (\lambda_E + \lambda_I) f^*(0, 0), \\
f^*(0, 0) &= \frac{\lambda_E}{\lambda_E + \lambda_I} \int_{\theta - x_E}^{\theta} f^*(x, 1) \, dx,
\end{align*}
\]

\[
\frac{\partial(\alpha x f^*(x, 1))}{\partial x} = (\lambda_E + \lambda_I) f^*(x, 1) - \lambda_E f^*(x - x_E, 1)
\]

\[
- \lambda_I f^*(x + x_I, 1),
\]

where we formally assume that \( f^*(x, 1) = 0 \) for \( x \in [\theta, \theta + x_I] \). Moreover, \( f^* \) satisfies conditions (11) and (12).
If we consider Stein’s model without the refractory period (i.e. \( t_R = 0 \)), then the phase space is \( X = \{(0, 0)\} \cup (-\infty, \theta] \times \{1\} \) and the infinitesimal generator \( \tilde{A} \) of the stochastic semigroup \( \{\tilde{P}(t)\}_{t \geq 0} \) satisfies (9), but (8) should be replaced by

\[
\tilde{A} f(0, 0) = \lambda_E \int_{\theta - x_E}^\theta f(x, 1) \, dx - (\lambda_E + \lambda_I) f(0, 0).
\]  

(23)

Moreover, functions from the domain of the operator \( \tilde{A} \) satisfy conditions (11)–(13). If \( \tilde{f}^* \) is an invariant density of \( \{\tilde{P}(t)\}_{t \geq 0} \), then \( \tilde{A} \tilde{f}^* = 0 \), and consequently \( \tilde{f}^* \) satisfies conditions (21) and (22). It means that there exists \( c > 0 \) such that \( f^*(0, 0) = c \tilde{f}^*(0, 0) \), \( f^*(x, 1) = c \tilde{f}^*(x, 1) \) for \( x \in (-\infty, \theta] \), and \( f^*(x, 2) = c(\lambda_E + \lambda_I) \tilde{f}^*(0, 0) \) for \( x \in [0, t_R] \). Knowing that \( f^* \) and \( \tilde{f}^* \) are densities we can check that

\[
c = \left( 1 + t_R(\lambda_E + \lambda_I) \tilde{f}^*(0, 0) \right)^{-1}.
\]

Remark 4. In Stein’s model we have assumed that \( \theta > x_E \), to omit the case when we can have a jump from the state \( (0, 0) \) immediately to the the refractory period. If \( \theta \in (0, x_E] \), then the model also induces a stochastic semigroup and the same proof of asymptotic stability remains valid for this case, but some formulas for operators \( L \) and \( A \) should be changed. Namely, in the definition of \( L f(0, 0) \) we need to replace \( \lambda_E f(x_E, 1) \) with \( \lambda_E f(0, 2) \), we do not have condition (11), and instead of (14) we have

\[
f(0, 2) = \lambda_E f(0, 0) + \lambda_E \int_{\theta - x_E}^\theta f(x, 1) \, dx.
\]

We can also consider a simpler version of the model without jumps to the left. Then depolarization is a number from the interval \([0, \theta] \). We put \( \lambda_I = 0 \) and all formulas for \( A \) and \( L \) remain the same. We should also change the proof of Theorem 3.1. The sketch of the proof is following. We assume additionally that \( \theta > x_E \). In order to check (K) we start from a point \( y = (y, 1) \) and we find time \( \tau \) such that \( y_\tau = e^{-\alpha\tau} y < \theta - x_E \). We assume that we have no jump until \( \tau \). Now, treating \((y_\tau, 1)\) as a starting point of the process \( \xi \) we replace the first jump to the left with a jump to the right and we proceed analogously to the proof of Theorem 3.1. We check that condition (I) holds for \( x_0 = (x_E, 1) \). Condition (T) is automatically fulfilled because \( X \) is compact. The case \( \lambda_I = 0 \) and \( \theta \leq x_E \) needs extra analysis, but the semigroup is also stable in this case and the invariant density \( f^* \) is given by the formula \( f^*(0, 0) = \gamma, f^*(x, 1) = 0, \) and \( f^*(x, 2) = \lambda_E \gamma, \) where \( \gamma = (1 + \lambda_E t_R)^{-1} \). The model without jumps to the right does not make physical sense because then the neuron cannot emit an action potential.

Remark 5. As we have mentioned in Introduction, a partially integral stochastic semigroup is asymptotically stable if the semigroup has a unique invariant density supported on the whole space. One can try to prove Theorem 3.1 by using this criterion. In order to do it we need to find directly a density \( f^* \) such that \( Af^* = 0 \). This task seems to be difficult because \( f^* \) is a solution of a complex system of equations including a differential equation with both delayed and advanced arguments.

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