On a Stroboscopic Approach to Quantum Tomography of Qudits Governed by Gaussian Semigroups

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Abstract
In this paper, we discuss the minimal number $\eta$ of observables $Q_1, \ldots, Q_\eta$, where expectation values at some time instants $t_1, \ldots, t_r$ determine the trajectory of a $d$-level quantum system (“qudit”) governed by the Gaussian semigroup

$$\Phi(t)\rho = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} ds e^{-s^2/(2t)} e^{-iHs} \rho e^{iHs}.$$ 

We assume that the macroscopic information about the system in question is given by the mean values $E_i(Q_i) = \text{Tr}(Q_i \rho(t_j))$ of $n$ selfadjoint operators $Q_1, \ldots, Q_n$ at some time instants $t_1 < t_2 < \ldots < t_r$, where $n < d^2 - 1$ and $r \leq \deg \mu(\lambda, L)$. Here $\mu(\lambda, L)$ stands for the minimal polynomial of the generator

$$\mathbb{L}\rho = -\frac{1}{2} [H, [H, \rho]]$$

of the Gaussian flow $\Phi(t)$.

1 Introduction

The ability to create, manipulate and characterize quantum states is becoming increasingly important in physical research with implications for other areas of science, such as: quantum information theory, quantum communication and computing. According to one of the basic assumptions of quantum mechanics, the achievable information about the state of a physical system is encoded in the density matrix $\rho$, which allows one to evaluate all possible expectation values of observables through the formula

$$\langle Q \rangle = \text{Tr}(\rho Q),$$

(11)
where $Q$ is a self-adjoint operator representing a particular physical quantity.

Thus, in order to have full information about a given quantum system we need to know its density matrix $\rho$. In particular, a very useful tool in this regard is quantum state tomography (QST) which provides means for the complete reconstruction of the density matrix for a qudit (or a set of qubits). The general procedure relies on the ability to reproduce a large number of identical states and to perform a series of measurements on complementary aspects of the state within an ensemble.

Suppose that we can prepare a quantum system repeatedly in the same state and make a series of experiments such that we can measure the expectation values

$$E_t(Q) = \text{Tr}(Q\rho(t))$$

of some observables $Q_1, \ldots, Q_n$ at different time moments $t_1 < t_2 < \ldots < t_r$. The fundamental question arises:

Can we find the average value of any desired operator from the set of measured outcomes of a given set $Q_1, \ldots, Q_n$

$$\begin{bmatrix} E_{t_1}(Q_1) & \cdots & E_{t_r}(Q_1) \\ \vdots & \cdots & \vdots \\ E_{t_1}(Q_n) & \cdots & E_{t_r}(Q_n) \end{bmatrix},$$

where $0 \leq t_1 < \ldots < t_r \leq T$, for an interval $[0, T]$?

Among the existing tomographic techniques for quantum systems, the so-called homodyne tomography has received much attention in the literature [1, 2, 3]. In the phase-space formulation of quantum mechanics there is a one-to-one relation between a quantum state and the so-called Wigner function. Its marginals are accessible experimentally, and an inverse (Radon) transformation allows one to reconstruct the phase-space distribution associated with the unknown quantum state.

The question of how to reconstruct states of $d$-level systems (qudits) is also natural. In this case various methods have been proposed to determine $\rho$ [4]. If the problem under consideration is static, then the state of a $d$-level open quantum system (a qudit) can be uniquely determined only if $n = d^2 - 1$ expectation values of linearly independent observables are at our disposal. However, if we assume that we know the dynamics of our system, i.e. we know the generator of the time evolution, then we can use the stroboscopic approach based on [13]. In general, the term “tomography” will be used collectively to denote any kind of state-reconstruction method.

With reference to the terminology used in the system theory, we assume the following definition:

**Definition 1** A $d$-level open quantum system $S$ is said to be $(Q_1, \ldots, Q_n)$-reconstructible on an interval $[0, T]$, if there exists at least one set of time instants $0 \leq t_1 < \ldots < t_r \leq T$ such that the state trajectory can be uniquely reconstructed.
determined by the correspondence

\[ [0, T] \ni t_j \mapsto \mathcal{E}_{t_j} = \text{Tr} (\rho(t_j) Q_i) \]  

for \( i = 1, \ldots, n, \) \( j = 1, \ldots, r. \)

The above definition is equivalent to the following one

**Definition 2** A \( d \)-level open quantum system \( S \) is said to be \((Q_1, \ldots, Q_n)\)-reconstructible on an interval \([0, T]\), if for every two trajectories with distinct initial states there exists at least one \( \hat{t} \in [0, T] \) and at least one operator \( Q_k \in \{Q_1, \ldots, Q_n\} \) such that

\[
\text{Tr} (Q_k \rho_1 (\hat{t})) \neq \text{Tr} (Q_k \rho_2 (\hat{t})).
\]  

**Remark 1** In the above definitions we assume that the time evolution of the system is given in terms of a completely positive semigroup of operators. Arguments in favour of completely positive semigroups as the foundation of non-Hamiltonian dynamics as well as the discussion of properties of such semigroups can be found in papers of Kraus [7], Lindblad [8], and Gorini *et al.* [9].

In particular, in Lindblad’s paper [8] the general form of the generator of an arbitrary completely positive semigroup was derived. A linear operator \( \mathbb{L} \) on a set of linear operators \( B(\mathcal{H}) := M(\mathbb{C}^d) \), where \( \mathcal{H} \simeq \mathbb{C}^d \) is a \( d \)-dimensional Hilbert space and \( M \) denotes the set of matrices with complex entries, proves to be the generator of a completely positive semigroup if and only if it can be represented in the form

\[
\mathbb{L} \rho = -i[H, \rho] + \frac{1}{2} \sum_{j=1}^\kappa \left( [V_j \rho, V_j^*] + [V_j, \rho V_j^*] \right),
\]  

where \( V_j \in B(\mathcal{H}) \) for \( j = 1, \ldots, \kappa \), and \( H \) is a self-adjoint operator also belonging to \( B(\mathcal{H}) \) (cf. also [10]).

**Remark 2** It is important that for the number \( r \) of time instants \( t_1, \ldots, t_r \) we do not formulate any restriction (except that it is finite).

**Remark 3** The question of obvious physical interest is to find the minimal number of observables \( Q_1, \ldots, Q_\eta \) for which the \( d \)-level quantum system \( S \) with the generator \( \mathbb{L} \) can be \((Q_1, \ldots, Q_\eta)\)-reconstructible. It can be shown that if the time evolution of the system \( S \) is described by the master equation

\[
\frac{d}{dt} \rho(t) = \mathbb{L} \rho(t),
\]  

then there exists \([4, 5]\) a set of observables \( Q_1, \ldots, Q_\eta \), where

\[
\eta = \max_{\lambda \in \sigma(\mathbb{L})} \left\{ \dim \text{Ker} (\lambda \mathbb{I} - \mathbb{L}) \right\}
\]  

such that the system \( S \) is \((Q_1, \ldots, Q_\eta)\)-reconstructible. Moreover, if we have another set of observables \( \tilde{Q}_1, \ldots, \tilde{Q}_{\tilde{\eta}} \) with this property, then \( \tilde{\eta} \geq \eta \). The number \( \eta \) given by \([13, 5]\) we will call the index of cyclicity of the system \( S \).
2 Polynomial Representation of Flows

The main idea of the stroboscopic approach to quantum tomography is based on the polynomial representation of the flow defined by the general master equation. Namely, we have

\[ \Phi(t) = \exp(\mathbb{L}t) = \sum_{k=0}^{m-1} \alpha_k(t)\mathbb{L}^k, \quad (21) \]

where by Cauchy’s theorem

\[ \alpha_k(t) := \frac{1}{2\pi i} \oint_{\partial D} \mu_k(z) \exp(tz) \frac{dz}{\mu(z,\mathbb{L})}. \quad (22) \]

In the above expression \( \partial D \) is any simple closed contour enclosing the spectrum of the operator \( \mathbb{L} \) in the complex plane and

\[ \mu(z,\mathbb{L}) = \sum_{k=0}^{m-1} d_k z^k \]

denotes the minimal polynomial of the generator \( \mathbb{L} \). It is interesting that there are ways to compute the functions \( \alpha_k(t) \) in (21) without summing the exponential series or without knowing the Jordan canonical form of \( \mathbb{L} \). Namely, differentiating (21) with respect to \( t \) and using the minimal polynomial of \( \mathbb{L} \) one finds that the functions \( \alpha_k(t) \) for \( k = 0, \ldots, m-1 \) satisfy the system of equations

\[
\begin{align*}
\frac{d\alpha_0(t)}{dt} &= d_0\alpha_{m-1}(t), \\
\frac{d\alpha_1(t)}{dt} &= \alpha_0(t) + d_1\alpha_{m-1}(t), \\
&\quad \ldots \\
\frac{d\alpha_{m-1}(t)}{dt} &= \alpha_{m-2}(t) + d_{m-1}\alpha_{m-1}(t),
\end{align*}
\]

(24)

with initial conditions \( \alpha_k(0) = \delta_{ik} \). It can be shown that functions \( \alpha_k(t) \) are mutually linearly independent, therefore for a given \( T \) there exists at least one set of time instants \( t_1, \ldots, t_m \) (\( m = \deg \mu(\lambda,\mathbb{L}^*) \)) such that \( 0 \leq t_1 < \ldots < t_m \leq T \) and \( \det [\alpha_k(t_j)] \neq 0 \).

Taking into account the above conditions one finds that the state \( \rho(0) \) can be determined uniquely (and the trajectory \( \Phi(t)\rho(0) \) can be reconstructed) if and only if operators of the form \( (\mathbb{L}^*)^k Q_i \) for \( i = 1, \ldots, n \) and \( k = 0, 1, \ldots, m-1 \) span the space \( B(\mathcal{H}) \).

If the dynamical semigroup is completely positive, then the general form of the generator \( \mathbb{L} \) is given by (10). In this case the criterion for reconstructibility of a \( d \)-level quantum system can be formulated using the operators \( H \) and \( V_j \). In particular, if we consider an isolated quantum system characterized by a
Hamiltonian $H_0$ and $V_j = 0$ for $j = 1, \ldots, \kappa$, then the minimal number of observables $Q_1, \ldots, Q_\eta$ for which the system is $(Q_1, \ldots, Q_\eta)$-reconstructible is given by
\[ \eta = n_1^2 + n_2^2 + \cdots + n_r^2, \]
where $n_i = \dim \ker (\lambda_i I - H_0)$ for all $\lambda_i \in \sigma(H_0)$, $i = 1, \ldots, r$ (for details cf. [5]).

3 The Minimal Number of Observables for Qu-dits Governed by Gaussian Semigroups

Let us assume that the time evolution of a $d$-level quantum system $S$ is described by the generator $L$ given by
\[
L \rho = \frac{1}{2} \{ [H, \rho H] + [H, \rho H] \} = -\frac{1}{2} [H, [H, \rho]]
\]
that is, the semigroup $\Phi(t)$ has the form (cf. e.g. [11])
\[
\Phi(t) \rho = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} ds e^{-s^2/(2t)} e^{-iHs} \rho e^{iHs}.
\]

The symbol $H$ in (31) and (32) denotes a self-adjoint operator with the spectrum
\[ \sigma(H) = \{\lambda_1, \ldots, \lambda_m\}. \] (33)

In the sequel $n_i$ stands for the multiplicity of the eigenvalue $\lambda_i$ for $i = 1, \ldots, m$. One can assume that the elements of the spectrum of $H$ are numbered in such a way that the inequalities $\lambda_1 < \lambda_2 < \ldots < \lambda_m$ are fulfilled. The following theorem holds:

**Theorem 1** The index of cyclicity of the Gaussian semigroup with a generator $L$ given by (31) is expressed by the formula
\[ \eta = \max\{\kappa, \gamma_1, \ldots, \gamma_r\}, \] (34)
where $r = (m - 1)/2$ if $m$ is odd or $r = (m - 2)/2$ if $m$ is even, and
\[
\kappa := n_1^2 + n_2^2 + \cdots + n_m^2, \quad \gamma_k := 2 \sum_{i=1}^{m-k} n_i n_{i+k}.
\] (35) (36)

**Proof.** In order to determine the value of $\eta$ for the generator $L$ defined by (31) we must find the number of nontrivial invariant factors of the operator $L$. Let
us observe that if \( \sigma(H) = \{\lambda_1, \ldots, \lambda_m\} \) then the spectrum of the operator \( L \) is given by

\[
\sigma(L) = \left\{ \nu_{ij} \in \mathbb{R} ; \nu_{ij} = (\lambda_i - \lambda_j)^2, \ i, j = 1, \ldots, m \right\}.
\]

(37)

The above statement follows from the fact that the operator \( L \) can also be represented as

\[
L = H^2 \otimes 1_l + 1_l \otimes H^2 - 2H \otimes H,
\]

(38)

where \( 1_l \) denotes the identity in the space \( \mathcal{B}(H) \). Since \( H \) is self-adjoint therefore the algebraic multiplicity of \( \lambda_i \), i.e. the multiplicity of \( \lambda_i \) as the root of the characteristic polynomial of \( H \), is equal to the geometric multiplicity of \( \lambda_i \), i.e. \( n_i = \dim \ker (\lambda_i I - H) \). Of course, we have \( n_1 + \ldots + n_m = \dim H \).

From (38) we can see that the multiplicities of the eigenvalues of the operator \( L \) are not determined uniquely by the multiplicities of \( \lambda_i \in \sigma(H) \). But if we assume that \( \lambda_1 < \ldots < \lambda_m \) and \( \lambda_k = (k-1)c + \lambda_1 \), where \( k = 1, \ldots, m \), and \( c = \text{const} > 0 \), then the multiplicities of all eigenvalues of \( L \) are given by

\[
\gamma_{|i-j|} = \dim \ker [(\lambda_i - \lambda_j)^2 1_l - L]
\]

(39)

for \( i \neq j \) and

\[
\dim \ker (L) = n_1^2 + \ldots + n_m^2 = \kappa
\]

(310)

when \( i = j \). Now, as we know, the minimal number of observables \( Q_1, \ldots, Q_\eta \) for which the qudit \( S \) can be \( (Q_1, \ldots, Q_\eta) \)-reconstructible is given by \( \kappa \), so in our case

\[
\eta = \max_{i,j=1,\ldots,m} \left\{ \dim \ker [(\lambda_i - \lambda_j)^2 1_l - L] \right\},
\]

(311)

where \( \lambda_i \in \sigma(H) \). Using the above formulae and the inequality \( \gamma_k < \kappa \) for \( k > r \), where \( r \) is given by \((m-1)/2\) if \( m \) is odd and \((m-2)/2\) if \( m \) is even, we can observe that also without the assumption \( \lambda_k = (k-1)c + \lambda_1 \) one obtains

\[
\eta = \max\{\kappa, \gamma_1, \ldots, \gamma_r\}.
\]

(312)

This completes the proof.

According to Theorem 1 if the quantum system governed by a Gaussian semigroup is \( (Q_1, \ldots, Q_n) \)-reconstructible then the number \( n \) of observables must satisfy the inequality \( n \geq \eta \). In this case there exists a set of time instants \( t_1 < t_2 < \ldots < t_m \) (\( m = \deg \mu(\lambda, L) \)) such that the knowledge of the expectation values \( \mathcal{E}_i(Q_i) = \text{Tr} (\rho(t_j)Q_i) \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) uniquely determines the trajectory of the system.

The natural question arises: what are the criteria governing the choice of time instants \( t_1, \ldots, t_m \)? The following theorem holds:

**Theorem 2** Let us assume that \( 0 \leq t_1 < t_2 < \ldots < t_m \leq T \). Suppose that the mutual distribution of time instants \( t_1, \ldots, t_m \) is fixed, i.e. a set of nonnegative
numbers \( c_1 < \ldots < c_m \) is given and \( t_j := c_j t \) for \( j = 1, \ldots, m \), and \( t \in \mathbb{R}_+ \). Then for \( T > 0 \) the set

\[
\tau(T) := \{(t_1, \ldots, t_m) : \ t_j = c_j t, \ 0 \leq t \leq \frac{T}{c_m}\}
\]

contains almost all sequences of time instants \( t_1, \ldots, t_m \), i.e. all of them except a finite number.

Proof. As one can check, the expectation values \( \mathcal{E}_{t_j}(Q_i) \) and the operators \((\mathbb{L}^*)^k Q_1 i\) are related by the equality

\[
\mathcal{E}_{t_j}(Q_i) = \sum_{k=0}^{m-1} \alpha_k(c_j t) \left( \langle \mathbb{L}^* \rangle^k Q_1, \rho_0 \right),
\]

where we assume that \( t_j = c_j t \) and the bracket \( \langle \cdot, \cdot \rangle \) denotes the Hilbert-Schmidt product in \( \mathcal{B}(\mathcal{H}) \). One can determine \( \rho_0 \) from (313) for all those values \( t \in \mathbb{R}_+ \) for which the determinant \( \alpha(t) \) is different from zero, i.e.

\[
\alpha(t) := \det [\alpha_k(c_j t)] \neq 0.
\]

One can prove that the range of the parameter \( t \in \mathbb{R}_+ \) for which \( \alpha(t) = 0 \) consists only of isolated points on the semiaxis \( \mathbb{R}_+ \), i.e. does not possess any accumulation points on \( \mathbb{R}_+ \). To this end let us note that since the functions \( t \to \alpha_k(t) \) for \( k = 0, 1, \ldots, m-1 \), are analytic on \( \mathbb{R} \), the determinant \( \alpha(t) \) defined by (314) is also an analytic function of \( t \in \mathbb{R} \). If \( \alpha(t) \) can be proved to be nonvanishing identically on \( \mathbb{R} \), then, making use of its analyticity, we shall be in position to conclude that the values of \( t \), for which \( \alpha(t) = 0 \), are isolated points on the axis \( \mathbb{R} \).

It is easy to check that for \( k = m(m-1)/2 \)

\[
\left. \frac{d^k \alpha(t)}{dt^k} \right|_{t=0} = \prod_{1 \leq j < i \leq m} (c_i - c_j).
\]

According to the assumption \( c_1 < c_2 < \ldots < c_m \), we have \( \alpha^{(k)}(0) \neq 0 \) if \( k = m(m-1)/2 \). This means that the analytic function \( t \to \alpha(t) \) does not vanish identically on \( \mathbb{R} \) and the set of values of \( t \) for which \( \alpha(t) = 0 \) cannot contain accumulation points. In other words, if we limit ourselves to an arbitrary finite interval \([0, T]\), then \( \alpha(t) \) can vanish only on a finite number of points belonging to \([0, T]\). This completes the proof.

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