Inducibility in the hypercube

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Abstract
Let $Q_d$ be the hypercube of dimension $d$ and let $H$ and $K$ be subsets of the vertex set $V(Q_d)$, called configurations in $Q_d$. We say that $K$ is an exact copy of $H$ if there is an automorphism of $Q_d$ which sends $H$ onto $K$. Let $n \geq d$ be an integer, let $H$ be a configuration in $Q_d$ and let $S$ be a configuration in $Q_n$. We let $\lambda(H,d,n)$ be the maximum, over all configurations $S$ in $Q_n$, of the fraction of sub-$d$-cubes $R$ of $Q_n$ in which $S \cap R$ is an exact copy of $H$, and we define the $d$-cube density $\lambda(H,d)$ of $H$ to be the limit as $n$ goes to infinity of $\lambda(H,d,n)$. We determine $\lambda(H,d)$ for several configurations in $Q_3$ and $Q_4$ as well as for an infinite family of configurations. There are strong connections with the inducibility of graphs.

Keywords
density, extremal, graph, hypercube, inducibility

1 | INTRODUCTION

The $n$-cube, which we denote by $Q_n$, is the graph whose vertex set $V_n = V(Q_n)$ is the set of all binary $n$-tuples, with two vertices adjacent if and only if they differ in precisely one coordinate (so Hamming distance 1). Let $[n] = \{1, 2, \ldots, n\}$. We sometimes denote a vertex $(x_1, x_2, \ldots, x_n)$ of $Q_n$ by the subset $S$ of $[n]$ such that $i \in S$ if and only if $x_i = 1$. So if $n = 4$, then $\emptyset$ denotes (0000), and $\{1, 3\}$ or 13, denotes (1010) and $\{\{1\}, \{1, 3\}\}$ (or $\{1, 13\}$) denotes $\{(1000), (1010)\}$. The weight of a vertex is the number of 1s. For each positive integer $d$ less than or equal to $n$, $Q_n$ has $\binom{n}{d}2^{n-d}$ subgraphs which are isomorphic to $Q_d$ ($d$ coordinates can vary, while $n - d$ coordinates are fixed).
We sometimes refer to each of the \(d\) coordinates that can vary as a flip bit. A single flip bit determines an edge while a set of \(d\) flip bits, with the other \(n - d\) coordinates fixed, determines a particular sub-\(d\)-cube.

Let \(H\) and \(K\) be subsets of \(V_d\) (we call \(H\) and \(K\) configurations in \(Q_d\)). We say \(K\) is an exact copy of \(H\) if there is an automorphism of \(Q_d\) which sends \(H\) onto \(K\). For example, \(\emptyset, 12\) is an exact copy of \(\{2, 123\}\) in \(Q_3\), but \(\{2, 13\}\) is not (the vertices are distance 3 apart). So if \(K\) is an exact copy of \(H\) then they induce isomorphic subgraphs of \(Q_d\), but the converse may not hold.

Let \(d\) and \(n\) be positive integers with \(d \leq n\), let \(H\) be a configuration in \(Q_d\) and let \(S\) be a configuration in \(Q_n\). We let \(G(H, d, n, S)\) denote the number of sub-\(d\)-cubes \(R\) of \(Q_n\) in which \(S \cap R\) is an exact copy of \(H\) and define \(g(H, d, n, S)\) by

\[
g(H, d, n, s) = \frac{G(H, d, n, S)}{\binom{n}{d}2^{n-d}}.
\]

So \(g(H, d, n, S)\) is the fraction of all sub-\(d\)-cubes whose intersection with \(S\) is an exact copy of \(H\). We define \(\lambda(H, d, n)\) to be the maximum of \(g(H, d, n, S)\) over all configurations \(S\) in \(Q_n\).

Note that if \(n > d\), then \(\lambda(H, d, n)\) is the average of \(2n\) densities \(g(H, d, n-1, S_j)\), each of them the fraction of sub-\(d\)-cubes \(R\) in a sub-(\(n - 1\))-cube of \(Q_n\) in which \(R \cap S_j\) is an exact copy of \(H\), where \(S_j\) is the intersection of a maximizing configuration in \(Q_n\) with one of the \(2n\) sub-(\(n - 1\))-cubes. Hence \(\lambda(H, d, n)\) is the average of \(2n\) densities, each of them less than or equal to \(\lambda(H, d, n - 1)\), which means \(\lambda(H, d, n)\) is a nonincreasing function of \(n\), so we can define the \(d\)-cube density \(\lambda(H, d)\) of \(H\) by

\[
\lambda(H, d) = \lim_{n \to \infty} \lambda(H, d, n).
\]

We say a sub-\(d\)-cube is “good” if its intersection with \(S\) is an exact copy of \(H\). So \(\lambda(H, d)\) is the limit as \(n\) goes to infinity of the maximum fraction, over all \(S \subseteq V_n\), of “good” sub-\(d\)-cubes.

In [14] we initiated the investigation of \(d\)-cube density. There have been many papers on Turán and Ramsey-type problems in the hypercube. There has been extensive research on the minimum fraction of edges in \(Q_n\) one can take with no cycle of various lengths \([8, 10, 13, 23]\) and a few papers on vertex Turán problems in \(Q_n\) \([19–21]\). There has also been extensive work on which monochromatic cycles must appear in any edge coloring of a large hypercube with a fixed number of colors \([3, 4, 8, 9]\) and a few results on which vertex structures must appear \([16]\). In [2, 15], results were obtained on polychromatic colorings of \(Q_n\). These are edge colorings with \(p\) colors such that every sub-\(d\)-cube contains every color.

We wanted to investigate a different extremal problem in the hypercube: the maximum density of a certain kind of substructure. Instead of using graph isomorphism to determine if two substructures are the same, it seemed to capture the essence of the \(n\)-cube better if the small structures were “rigid” within a sub-\(d\)-cube, and that is what motivated our definitions of exact copy of a configuration and \(d\)-cube density. While \(d\)-cube density is not the same thing as graph inducibility, there are similarities between the two notions and, in fact, we use several results on inducibility (see Section 4) in our proofs.

In [14] we used a kind of blow-up to show that \(\lambda(H, d) \geq \frac{d!}{d^d}\) for every configuration \(H\) in \(Q_d\). We defined a perfect \(2d\)-cycle \(C_{2d}\) in \(Q_d\) to be a cycle with \(d\) pairs of vertices each Hamming distance \(d\) apart. We showed that \(\lambda(C_{2d}, 4) = \frac{4^d}{d!}\), achieving the smallest possible value for any configuration in \(Q_d\). We also showed \(\lambda(P_4, 3) = \frac{3}{8}\), where \(P_4\) is the induced path in \(Q_3\) with four vertices.
Finding $d$-cube density seems to be very difficult even for most small configurations. In this paper we find the $d$-cube density for three configurations in $Q_3$, two configurations in $Q_4$, and for an infinite family of configurations, $d − 1$ of them in $Q_d$ for each $d ≥ 2$. We find a construction to produce a lower bound and then find a matching upper bound by using known results on the inducibility of small graphs to show the local density cannot be larger. For all configurations in $Q_2$ and $Q_3$ for which we could not determine the $d$-cube density, we give a construction to produce a lower bound and give an upper bound computed by Rahil Baber [5] using flag algebras.

In Section 2 we define local $d$-cube density, the notion we use to find the upper bounds. In Section 3 we consider the possible configurations in $Q_2$. In Section 4 we summarize the results on the inducibility of graphs which will need. In Section 5 we discuss layered configurations in $Q_d$, those that are defined in terms of the weights of the $d$-vectors. In Section 6 we find $d$-cube density for a nontrivial infinite family of configurations. In Section 7 we consider $d$-cube density for configurations in $Q_3$, and in Section 8 we determine the $d$-cube density for two configurations in $Q_4$. In Section 9 we discuss some open problems and state some conjectures.

## 2 Local $d$-Cube Density

Let $H$ be a configuration in $Q_d$ and $S$ be a configuration in $Q_n$. For each vertex $v ∈ S$, we let $G_v(\text{in})(H, d, n, S)$ be the number of sub-$d$-cubes $R$ of $Q_n$ containing $v$ in which $S ∩ R$ is an exact copy of $H$ and $g_v(\text{in})(H, d, n, S) = \frac{G_v(\text{in})(H, d, n, S)}{\binom{n}{d}}$. So $g_v(\text{in})(H, d, n, S)$ is the fraction of sub-$d$-cubes containing $v$ which have an exact copy of $H$. We define $\lambda_{\text{local}}(H, d, n)$ to be the maximum of $g_v(\text{in})(H, d, n, S)$ over all $v$ and $S$, where $v ∈ S ⊆ V_n$. As with $\lambda(H, d, n)$, a simple averaging argument shows that $\lambda_{\text{local}}(H, d, n)$ is a nonincreasing function of $n$, so we define $\lambda_{\text{local}}(H, d)$ by

$$\lambda_{\text{local}}(H, d) = \lim_{n → ∞} \lambda_{\text{local}}(H, d, n)$$

For each $v ∉ S$, we perform a similar procedure to define $G_v(\text{out})(H, d, n, S)$, $g_v(\text{out})(H, d, n, S)$, and $\lambda_{\text{local}}(H, d)$. So $\lambda_{\text{local}}(H, d)$ and $\lambda_{\text{local}}(H, d)$ are the maximum local densities of sub-$d$-cubes with an exact copy of $H$ among all sub-$d$-cubes containing $v$ in $S$ and out of $S$, respectively. Finally, we define $\lambda_{\text{local}}(H, d)$ to be $\max\{\lambda_{\text{local}}(H, d), \lambda_{\text{local}}(H, d)\}$. Since the global density cannot be more than the maximum local density, we must have $\lambda(H, d) ≤ \lambda_{\text{local}}(H, d)$.

We refer to the following type of construction as a partition-modular construction. These are constructions generated by choosing a partition of $[n] = A_1 ∪ A_2 ∪ ⋯ ∪ A_l$ and taking as $S$ the set of vertices such that their binary $n$-tuples satisfy a chosen set of congruences for the weight of the coordinates within the parts. Sometimes we denote this as a list of $i$-tuples along with a list of their moduli for convenience. For example, the partition $A_1 ∪ A_2$ taking $01$ mod $(2, 2)$ would indicate the partition $[n] = A_1 ∪ A_2$ with $S$ being all vertices with weight $0$ mod $2$ in $A_1$ and weight $1$ mod $2$ in $A_2$. The fractional sizes of the $A_i$ which maximize the number of $Q_d$s having the configuration may also be indicated.

Note that the sets in the partition may be of any sizes, however, when $i = 1$ we call such a configuration layered since it is equivalent to choosing all vertices of particular weights modulo $a$ (i.e. entire “levels” of $Q_n$).
3 | CONFIGURATIONS IN $Q_2$

It is obvious that $\lambda(H, d) = \lambda(\overline{H}, d)$, where $\overline{H}$ is the complement of the configuration $H$ in $Q_d$, so we may restrict our consideration to only one configuration in each of the complementary pairs.

Sketches of all of the configurations in $Q_2$, subject to the above restriction, are given in Figure 1. In the figure, red vertices are in the configuration and open blue are not. The edges have been added for emphasis, but configurations are sets of vertices.

3.1 | Lower bounds by construction

Clearly $\lambda(Z_1, 2) = 1$, since we would simply take $S = \emptyset$. To show $\lambda(Z_4) = 1$, we take $S$ to be the layered configuration 0 mod 2 in $Q_n$ (all vertices with even weight).

We can find a lower bound for $Z_2$ by considering the layered configuration in $Q_n$ given by 0 mod 3. This gives $\frac{2}{3} \leq \lambda(Z_2, 2)$ since any sub-2-cube with the smallest weight of a vertex congruent to 0 or 1 mod 3 has an exact copy of $Z_2$. For an upper bound we have Baber’s flag algebra computation $0.68571$.

A construction for $Z_3$ is given by considering $[n] = A \cup B$ and then taking $S$ to be the set of all vertices given by binary $n$-tuples with weight 0 mod 2 in $A$. This gives a “good” $Q_2$ for each $Q_2$ with one flip bit in each of $A$ and $B$. If we take $|A| = \left\lfloor \frac{n}{2} \right\rfloor$ and $|B| = \left\lceil \frac{n}{2} \right\rceil$, this results in $2^{n-2}\left\lfloor \frac{n}{2} \right\rfloor \approx 2^{n-2}\frac{n^2}{4}$ many “good” $Q_2$s, which shows $\frac{1}{2} \leq \lambda(Z_3, 2)$.

Table 1 summarizes the best results obtained in $Q_2$.

3.2 | Upper bounds

To show that $\lambda(Z_3, 2) = \frac{1}{2}$, we use an argument that will be applied, in a slightly more general form, to an infinite family of configurations in Section 6.

**Theorem 3.1.** $\lambda(Z_3, 2) = \frac{1}{2}$.

**Proof.** We showed $\lambda(Z_3, 2) \geq \frac{1}{2}$ in Section 3.1. Let $v \in S \subseteq V_n$ be a vertex in $S$ and let $S_v = S \cap N(v)$, where $N(v)$ is the neighborhood of $v$ in $Q_n$. A $Q_2$ containing $v$ can be “good” only if one of the two vertices adjacent to $v$ is in $S_v$ and one is not. Hence

![Figure 1](image-url)
TABLE 1 Summary of the best results for configurations in $Q_2$.

| Configuration | Construction | Lower bound | Upper bound |
|---------------|--------------|-------------|-------------|
| $Z_1$         | $\emptyset$ | 1           | 1           |
| $Z_2$         | Layered: 0 mod 3 | $2/3$       | 0.685714286 |
| $Z_3$         | $A \cup B$ taking 0 mod 2 in $A$ | $1/2$       | $1/2$ (Theorem 3.1) |
| $Z_4$         | Layered: 0 mod 2 | 1           | 1           |

$$g_{v\text{-(in)}}(Z_3, 2, n, S) \leq \frac{|S_1| (n - |S_2|)}{\binom{n}{2}} \leq \frac{\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor}{\binom{n}{2}}$$

and

$$\lambda_{\text{local(in)}}(Z_3, 2) \leq \lim_{n \to \infty} \frac{\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor}{\binom{n}{2}} = \frac{1}{2}.$$ 

Since $Z_3$ is self-complementary in $Q_2$, $\frac{1}{2} \leq \lambda(Z_3, 2) \leq \lambda_{\text{local(out)}}(Z_3, 2) = \lambda_{\text{local(in)}}(Z_3, 2) = \lambda_{\text{local}}(Z_3, 2) \leq \frac{1}{2}$. □

4 | INDUCIBILITY

There are strong connections between $d$-cube density and inducibility of a graph. Given graphs $G$ and $H$, with $|V(G)| = n$ and $|V(H)| = k$, the density of $H$ in $G$, denoted $d_H(G)$, is defined by

$$d_H(G) = \frac{\# \text{induced copies of } H \text{ in } G}{\binom{n}{k}}.$$ 

Pippenger and Golumbic [22] defined the inducibility $i(H)$ of $H$ by

$$i(H) = \lim_{n \to \infty} \max_{|V(G)| = n} d_H(G).$$

Clearly $i(H) = i(\overline{H})$, where $\overline{H}$ is the complement of $H$. We summarize a few inducibility results, some of which we will use to prove upper bounds for $d$-cube density.

1. $i(K_{1,2}) = \frac{3}{4}$. The optimizing graph $G$ is $K_{\frac{n}{2}, \frac{n}{2}}$. That it cannot be larger than $\frac{3}{4}$ follows immediately from a theorem of Goodman [17] that says that in any 2-coloring of the edges of $K_n$, at least $\frac{1}{4}$ (asymptotically) of the $K_3$s are monochromatic.

2. $i(K_{2,2}) = \frac{3}{8}$. In [6], Bollobás et al. showed that the graph on $n$ vertices which has the most induced copies of $K_{r,r}$, for any $r \geq 2$, is $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor}$. 

TABLE 1

| Configuration | Construction | Lower bound | Upper bound |
|---------------|--------------|-------------|-------------|
| $Z_1$         | $\emptyset$ | 1           | 1           |
| $Z_2$         | Layered: 0 mod 3 | $2/3$       | 0.685714286 |
| $Z_3$         | $A \cup B$ taking 0 mod 2 in $A$ | $1/2$       | $1/2$ (Theorem 3.1) |
| $Z_4$         | Layered: 0 mod 2 | 1           | 1           |
(3) Since $2K_2$ is the complement of $K_{2,2}$, $i(2K_2) = \frac{3}{8}$. In [14], Goldwasser and Hansen showed that the bipartite graph on $n$ vertices with the most induced copies of $2K_2$ is two disjoint copies of $K_{\frac{n}{4}, \frac{n}{4}}$ (with the obvious modification if $n$ is not divisible by 4). This graph has $\frac{3}{32}$ of its subgraphs induced by four vertices isomorphic to $2K_2$. They needed this result to show that $\lambda(C_8, 4) = \frac{3}{32}$, where $C_8$ is the “perfect” 8-cycle.

(4) $i(K_{1,3}) = \frac{1}{2}$. In [7], Brown and Sidorenko showed that the graph on $n$ vertices which has the most induced copies of $K_{r,s}$, for any $r,s$ (except $r = s = 1$), is complete bipartite. The optimizing graph for $K_{1,3}$ is asymptotically, but not exactly, equibipartite; the sizes of the parts are roughly $\pm \frac{n}{2} \pm \sqrt{3n}/2$.

(5) In [18], Hirst used flag algebras to show that $i(K_{1,1,2}) = \frac{72}{125} = 0.576$ and $i(K_{PAW}) = \frac{3}{8}$, where $K_{1,1,2}$ is $K_4$ minus an edge and $K_{PAW}$ is $K_3$ plus a pendant edge, leaving the path $P_4$ as the only graph on four vertices whose exact inducibility has yet to be determined.

(6) In [11], Even-Zohar and Linial improve earlier best bounds [12, 24] for $i(P_4)$ and the inducibility of some graphs on five vertices.

5 | LAYERED CONFIGURATIONS

Recall that we say a configuration $H$ in $Q_d$ is layered if it is an exact copy of a configuration $K$ in $Q_d$ such that $v \in K$ if and only if $wt(v) \in Y_H$ for some subset $Y_H$ of $[0, d]$. For example, $H = \{1001, 1110, 0010, 0100, 0111\}$ is layered because there is an automorphism of $Q_4$ (interchange 0 and 1 in the second and third coordinates) which maps $H$ onto $K = \{1111, 1000, 0100, 0010, 0001\}$ and $K = \{v \in Q_4 : wt(v) = 1 \text{ or } 4\}$. We call $K$ a canonical layered configuration.

If $Y_H$ is the set of weights for a canonical layered configuration $H$ in $Q_d$, and if $j + 2 \in Y_H$ if and only if $j \in Y_H$ for all $j \in [0, d - 2]$, then $H$ is either all vertices, none of the vertices, all even weight vertices, or all odd weight vertices in $Q_d$. We call these four types of configurations trivial. Each trivial configuration has $d$-cube density equal to 1 (using the obvious layered configuration $S$ in $Q_n$). The configurations $W_8$, $W_9$, $W_7$, $W_8$, $W_{12}$, and $W_{14}$ (and their complements) are layered configurations in $Q_3$, with $W_1$ and $W_{14}$ trivial (see Figure 2 and Table 2).

One can get a good lower-bound construction for any layered configuration in $Q_d$ by using an appropriate layered configuration $S$ in $Q_n$. For example, if we represent the configuration $W_8$ by $H = \{110, 101, 011\}$, we define $S$ by $S = \{v \in V_n : wt(v) \equiv 2 \mod 3\}$. Any sub-3-cube of $Q_n$ whose smallest weight vertex has weight congruent to 0 or 1 mod 3 is “good,” showing that $\lambda(W_8, 3) \geq \frac{2}{3}$. Baber’s flag algebra upper bound is 0.66666666673 so undoubtedly $\lambda(W_8, 3) = \frac{2}{3}$, but we have not proved it.

**Theorem 5.1.** If $H$ is a configuration in $Q_d$, then $\lambda_{local}(H, d) = 1$ if and only if $H$ is layered.

To prove Theorem 5.1 we will need the following lemma which can be proved using a “tower” of applications of Ramsey’s theorem.

**Lemma 5.2** (Goldwasser and Talbot [16]). For all positive integers $k$ and $d$, there exists a positive integer $N = N(k, d)$ such that for all $n \geq N$, if $c$ is any vertex coloring of $Q_n$ with $k$
colors, for each \( v \in V_n \), there exists a sub-\( d \)-cube \( D \) containing \( v \) such that the coloring induced by \( c \) on \( D \) is layered with “bottom” vertex \( v \) (so \( v \) plays the role of \( \emptyset \) in a canonical layered coloring).

**Proof of Theorem 5.1.** Suppose \( H \) is layered with weight set \( Y_H \subseteq [0, d] \). For each \( n \geq d \), if \( S \) is any layered configuration in \( Q_n \) with weight set \( Y_S \) such that \( Y_H = Y_S \cap [0, d] \), then every sub-\( d \)-cube containing \( \emptyset \) has an exact copy of \( H \), so \( \lambda_{\text{local}}(H, d) = 1 \).
Conversely, suppose \( \lambda_{\text{local}}(H, d) = 1 \). We can view a configuration \( S \) in \( Q_n \) as a 2-coloring of \( V_n \)—red if in \( S \), blue if not in \( S \). Hence, by Lemma 5.2, there exists an integer \( N \) such that for all \( n \geq N \) and all \( v \in V_n \), if \( S \) is any configuration in \( Q_n \) then there exists a sub-\( d \)-cube \( D \) containing \( v \) such that \( D \cap S \) is a layered configuration in \( D \). Since \( \lambda_{\text{local}}(H, d, n) \) is a nonincreasing function of \( n \), and since every vertex \( v \) is contained in a sub-\( d \)-cube which has an exact copy of a layered configuration, if \( H \) is not layered then \( \lambda_{\text{local}}(H, d) < 1 \).

Since \( \lambda_{\text{local}}(H, d) = 1 \) for every layered configuration \( H \) in \( Q_d \), our usual procedure of using \( \lambda_{\text{local}}(H, d) \) to get an upper bound for \( d \)-cube density cannot work and that is why we have not been able to obtain upper bounds by hand for any nontrivial layered configuration in \( Q_d \) for any \( d \). As with \( W_8 \), Baber’s flag algebra upper bounds for \( \lambda(W_7, 3) \) and \( \lambda(W_{12}, 3) \) are within 10⁻⁹ of our lower bounds \( (\frac{1}{3} \leq \lambda(W_7, 3) \text{ and } \frac{1}{2} \leq \lambda(W_{12}, 3)) \). The flag algebra upper bound for \( \lambda(W_5, 3) \) (one vertex in \( Q_3 \)) is 0.610043, not so close to our lower bound of \( \frac{1}{2} \).

**Conjecture 5.3.** \( \lambda(W_7, 3) = \frac{1}{3}, \lambda(W_5, 3) = \frac{2}{3}, \lambda(W_{12}, 3) = \frac{1}{2} \).

### 6 | AN INFINITE FAMILY

Theorem 3.1 can be generalized to apply to an infinite family of configurations containing \( Z_3, W_6, \) and \( W_{13} \). Let \( d \) and \( i \) be positive integers with \( 1 \leq i < d \). We define the configuration \( H(d, i) \) in \( Q_d \) by

\[
H(d, i) = \left\{ (v_1, v_2, ..., v_d) \in V_d \mid \sum_{j=1}^{i} v_j \text{ is even} \right\}.
\]

**Theorem 6.1.** \( \lambda(H(d, i), d) = \binom{d}{i} \frac{i^{d-i} (d-i)^i}{d^d} \).

**Proof:** Each vertex in \( H(d, i) \) has precisely \( d - i \) neighbors in \( H(d, i) \) (change any one of the last \( d - i \) coordinates). Since \( H(d, i) \) is self-complementary in \( Q_d \left( \sum_{j=1}^{i} v_j \text{ is odd} \right) \), \( \lambda_{\text{local}}(H(d, i), d) = \lambda_{\text{local(out)}}(H(d, i), d) = \lambda_{\text{local(in)}}(H(d, i), d) \).

If \( n \geq d \) and \( v \in S \subseteq V_n \) and \( R \) is a sub-\( d \)-cube of \( Q_n \) containing \( v \), then \( R \) can be good only if precisely \( d - i \) neighbors of \( v \) in \( R \) are in \( S \). If \( x \) is the fraction of neighbors of \( v \) in \( Q_n \) which are in \( S \), then the fraction of sub-\( d \)-cubes of \( Q_n \) containing \( v \) which have precisely \( d - i \) neighbors of \( v \) in \( S \) is asymptotically equal to \( f(x) = \binom{d}{i} x^{d-i} (1-x)^i \). By simple calculus, \( f(x) \) is maximized on \([0, 1]\) when \( x = \frac{d-i}{d} \), so \( \lambda_{\text{local(in)}}(H(d, i)) \leq \binom{d}{i} \frac{(d-i)^{d-i} i^i}{d^d} \).

To show this upper bound is a lower bound as well, let \( S = \{(x_1, x_2, ..., x_n) : \sum_{j=1}^{m} v_j \text{ is even, where } m = \left\lfloor \frac{in}{d} \right\rfloor \} \). Then any sub-\( d \)-cube of \( Q_n \) with precisely \( i \) flip bits in \([1, m]\) is good, and this is a fraction.
\[
\binom{d}{i} \frac{m!(n-m)^{d-i}}{n^d} = \binom{d}{i} \left( \frac{\left\lfloor \frac{m}{d} \right\rfloor}{n} \right)^i \left( \frac{\left\lfloor \frac{d-i}{d} \right\rfloor}{n} \right)^{d-i}
\]

of all sub-\(d\)-cubes, and the limit as \(n\) goes to infinity is \(\binom{d}{i} \frac{\epsilon^i}{d^d} \). \(\square\)

Note that the configuration \(W_6\) in \(Q_3\) is \(H(3, 1)\), \(W_{13}\) is \(H(3, 2)\), and \(Z_3\) in \(Q_2\) is \(H(2, 1)\). Further note that
\[
\lambda = \lambda(H) \leq \frac{\epsilon^i}{d^d}.
\]
In particular, when \(i = 1\), \(\lambda(H, 1, d) = \frac{1}{e}\). \((H(1, d)\) is a copy of \(Q_{d-1}\) in \(Q_d\).\)

7 CONFIGURATIONS IN \(Q_3\)

The configurations in \(Q_3\) are illustrated in Figure 2 and listed in Table 2. Recalling that
\[
\lambda = \lambda(\Omega), \text{ where } \Omega \text{ is the complement of } H \text{ in } Q_d,
\]
the table lists just one configuration in each complementary pair.

Six of these configurations are layered: the two trivial ones \((W_1\text{ and }W_{14})\) and four others
\((W_3, W_7, W_8, \text{ and } W_{12})\). Letting \(S\) be the layered configuration \(0 \mod 3\) in \(Q_n\) shows that
\[
\lambda(S, 3) \geq \frac{1}{3} \text{ and } \lambda(W_6, 3) \geq \frac{2}{3}.
\]
The layered configuration \(0 \text{ or } 1 \mod 4\) shows that
\[
\lambda(W_{12}, 3) \geq \frac{1}{2}.
\]
Baber’s flag algebra upper bound is within \(10^{-9}\) of the lower bound for each of these. For \(W_3\)—one vertex in \(Q_3\)—the layered configuration \(0 \mod 4\) in \(Q_n\) shows that
\[
\lambda(W_3, 3) \geq \frac{1}{2}, \text{ while the flag algebra upper bound is } 0.610043.
\]
The configurations \(W_6\) and \(W_{13}\) are \(H(3, 1)\) and \(H(3, 2)\), respectively, in the infinite family of
Section 6, so \(\lambda(W_6, 3) = \lambda(W_{13}, 3) = \frac{4}{9}\).

We showed \(\lambda(W_{11}, 3) = \frac{3}{8}\) in [14] \((W_{11}\) is the induced path with four vertices). The proof of
Theorem 7.2 shows that \(\lambda(W_2, 3) = \frac{3}{4}\).

For \(W_4\), with \([n] = A \cup B\), where \(|A| = \left\lfloor \frac{2n}{3} \right\rfloor\text{ and } |B| = \left\lfloor \frac{n}{3} \right\rfloor\) we take \(S\) to be the set of all vertices
\(v = (x_1, x_2, ..., x_n)\), where \(\sum_{i \in A} x_i\) is divisible by 3. Suppose a \(Q_3\) has precisely two flip bits whose
coordinate numbers are in \(A\). If the sum of all the other coordinates in \(A\) is \(m\), then the \(Q_3\) will
have configuration \(W_4\) if and only if \(m\) is congruent to 0 or 1 \(\mod 3\). For example, if \(m \equiv 0 \mod 3\) then a vertex in the \(Q_3\) will be in \(S\) if and only if the two coordinates in \(A\) are equal to 0, while
if \(m \equiv 1 \mod 3\) then a vertex in the \(Q_3\) will be in \(S\) if and only if the two coordinates in \(A\) are
equal to 1 (if \(m \equiv 2 \mod 3\) then the configuration will be \(W_{13}\)). In the limit as \(n\) goes to infinity, \(\frac{4}{9}\)
of the sub-3-cubes will have precisely two flip bits in \(A\), so \(\lambda(W_4, 3) \geq \frac{4}{9} \cdot \frac{2}{3} = \frac{8}{27} \approx 0.2963\).
Baber’s flag algebra upper bound is 0.304762.

For \(W_9\), we partition \([n]\) into sets \(A\) and \(B\) with \(|A| = \left\lfloor \frac{2n}{3} \right\rfloor\text{ and } |B| = \left\lfloor \frac{n}{3} \right\rfloor\) and let \(S\) be the set of all vertices
\(v = (x_1, x_2, ..., x_n)\) such that both \(\sum_{i \in A} x_i \equiv 0 \text{ or } 1 \mod 4\) and \(\sum_{j \in B} x_j \equiv 0 \mod 2\) or both \(\sum_{i \in A} x_i \equiv 2 \text{ or } 3 \mod 4\) and \(\sum_{j \in B} x_j \equiv 1 \mod 2\). Every sub-3-cube \(R\) which has two flip bits
with coordinate numbers in \(A\) and one in \(B\) will have an exact copy of \(W_9\). For example, if \(R\) is such that the sum of the nonflip bit coordinates in \(A\) is \(1 \mod 4\) and the sum of the nonflip bit
coordinates in \(B\) is \(1 \mod 2\) then the coordinates in the flip bits for \(R\) of the vertices in \(S\) with
the coordinate in \(B\) listed third are 001, 100, 010, 110, so \(R\) has an exact copy of \(W_9\). In the limit
as \( n \) goes to infinity, \( \frac{4}{9} \) of the sub-3-cubes will have precisely two flip bits with coordinate numbers in \( A \), so \( \lambda (W_9, 3) \geq \frac{4}{9} \).

For \( W_5 \), we partition \([n]\) into a set \( A \) of size \( \left\lfloor \frac{n}{2} \right\rfloor \) and \( B \) of size \( \left\lceil \frac{n}{2} \right\rceil \) and let \( S \) be all vectors \( v = (x_1, x_2, ..., x_n) \) such that \( \sum_{i \in A} x_i \) and \( \sum_{j \in B} x_j \) are both congruent to either 0 or 1 mod 3. If a \( Q_3 \) has two flip bits with coordinate numbers in \( A \) and one with coordinate number in \( B \), we let \( y \) be the sum of the nonflip bit coordinates in \( A \) and \( z \) be the sum of the nonflip bit coordinates in \( B \). This \( Q_3 \) will be “good” if \((y, z)\) is congruent to \((0, 1), (0, 2), (2, 1), \) or \((2, 2)\) mod 3. For example, if \((y, z)\) is congruent to \((0, 2)\), then the coordinate in the flip bits for the \( Q_3 \) of the vertices in \( S \) with the coordinate in \( B \) listed third are 001, 011, and 101 which is an exact copy of 000. If \( A \) goes to infinity, \( \frac{4}{9} \) of the \( Q_3 \)’s which have two flip bits in \( A \) and one in \( B \) will have exact copies of \( W_5 \). By symmetry, \( \frac{4}{9} \) of the \( Q_3 \)’s which have one flip bit in \( A \) and two in \( B \) will have exact copies of \( W_5 \), so \( \lambda (W_5, 3) \geq \frac{3}{8} \cdot \frac{4}{9} + \frac{3}{8} \cdot \frac{4}{9} = \frac{1}{3} \). Baber’s flag algebra upper bound is 0.333398.

In the remainder of this section, we will consider the two remaining configurations in \( Q_3 \): \( W_2 \) and \( W_{10} \).

The following lemma is used in the proof that \( \lambda (W_2, 3) = \frac{3}{4} \).

**Lemma 7.1.** Let \( G \) be a graph with \( n \) vertices where \( n \) is even. If \(|E(G)| = e\), then \( G \) has at most \( \min \left\{ n \binom{\frac{e}{2}}{2}, \frac{e}{2} (n - 2) \right\} \) induced copies of \( K_{1,2} \).

**Proof.** That it has at most \( n \binom{\frac{e}{2}}{2} \) was proved in [22]. The optimizing graph is \( K_{2,2} \).

Each edge of \( G \) can be in at most \( n - 2 \) induced \( K_{1,2} \)’s and summing over all edges \( uv \) counts each \( K_{1,2} \) twice.

Without the restriction on the parity of \( n \), the bound in Lemma 7.1 is \( \min \left\{ \frac{n^2}{4}, \frac{n-2}{2}, \frac{e}{2} (n - 2) \right\} \).

**Theorem 7.2.** \( \lambda (W_2, 3) = \frac{3}{4} \).

**Proof.** We partition \([n]\) into sets \( A \) and \( B \) of sizes \( \left\lfloor \frac{n}{2} \right\rfloor \) and \( \left\lceil \frac{n}{2} \right\rceil \), respectively. We define a configuration \( S \) in \( Q_n \) by \( v = (x_1, x_2, ..., x_n) \in S \) if and only if \( \sum_{i \in A} x_i \) and \( \sum_{j \in B} x_j \) are both even. If \( D \) is any sub-3-cube of \( Q_n \) with one or two flip bits in coordinates with numbers in \( A \) (and two or one in \( B \) then \( D \) has an exact copy of \( W_2 \). For example, if two flip bits have coordinate numbers in \( A \) and one in \( B \), and if the sum of the nonflip bit coordinates in \( A \) is even and of the nonflip bit coordinates in \( B \) is odd, then the vertices of \( D \) whose flip bit coordinates (with the flip bit in \( B \) listed third) are 001 and 111 are in \( S \), so \( D \) has an exact copy of \( W_2 \).

In the limit as \( n \) goes to infinity, \( \frac{1}{4} \) of the sub-3-cubes have one or two flip bits with coordinate numbers in \( A \), so \( \lambda (W_2, 3) \geq \frac{3}{4} \).
For the upper bound, suppose $\emptyset \notin S$, and let $\mathcal{M}$ be the set of good $Q_3$s containing $\emptyset$. Construct a graph $G_\delta$ with $V(G_\delta) = [n]$ and $E(G_\delta) = \{uv : \emptyset, uv\}$ are the vertices in $S$ for some $M \in \mathcal{M}$ (recall that $\emptyset$ is the all 0’s vector, $u$ and $v$ are elements of $[n]$ and also represent $n$-vectors of all 0’s except for a 1 in position $i$). If $u, v, x$ are flip bits for some $M \in \mathcal{M}$, and if $uv$ is in $M$, then neither $ux$ nor $vx$ can be in $E(G_\delta)$, so $|\mathcal{M}|$ is less than or equal to the number of induced copies of the graph with three vertices and a single edge in a graph with $n$ vertices. Equivalently, this is less than or equal to the number of induced copies of $K_{1,2}$ in a graph on $n$ vertices. By Remark 1 in Section 4, this means $\lambda_{\text{local(in)}}(W_2) \leq i(K_{1,2}) = \frac{3}{4}$.

Now suppose $\emptyset \notin S$. Let $A = \{i \in [n] : i \in S\}$ (so $i \in [n]$ and also represents the $n$-vector of all 0’s except for a 1 in position $i$), $B = [n] \setminus A$, $|A| = a$, and $|B| = b$. Let $\mathcal{M}$ be the set of all good $Q_3$s containing $\emptyset$. If $M \in \mathcal{M}$, then the two vertices of $M$ in $S$ have the structure of Type I, II, or III as shown in Figure 3, where $i$ and $j$ are vertices adjacent to $\emptyset$ which are in $S$, while $u, v, x, y$ are vertices adjacent to $\emptyset$ which are not in $S$. For example, in the Type II configuration the vertices $i$ and $i\bar{x}$ are in $S$ while $\emptyset, x, y, i\bar{x}, iy, xy$ are not in $S$.

Define a graph $G_\delta$ by $V(G_\delta) = B$ and $E(G_\delta) = \{uv : uv\}$ are the vertices in $S$ of some Type III $M \in \mathcal{M}$ for some $x$ with flip bits $u, v, x$. For such an $M$, $ux$ cannot be in $S$ so the number of Type III $Q_3$s in $\mathcal{M}$ is at most the number of induced copies of $K_{1,2}$ in $G_\delta$.

If $L$ is a Type I $Q_3$ in $\mathcal{M}$ with flip bits $i, j, x$ and with $i, j$ the vertices of $L$ in $S$, then $i, j \in A$ and $x \in B$. So the number of Type Is in $\mathcal{M}$ is at most $b\left(\frac{a}{2}\right)$.

If $L$ is a Type II $Q_3$ in $\mathcal{M}$ where $i, i\bar{x}$ are the vertices of $L$ in $S$, then $i \in A$ and $x, y \in B$, but $xy \notin E(G_\delta)$. So, if $e = |E(G_\delta)|$, then the number of Type II $Q_3$s in $\mathcal{M}$ is at most $a\left(b\left(\frac{b}{2}\right) - e\right)$. By Lemma 7.1, we have that the number of Type III $Q_3$s in $\mathcal{M}$ is at most $\min\left\{b\left(\frac{b}{2}\right), \frac{e}{2}(b - 2)\right\}$.

If $|E| = e$, we then have

$$|\mathcal{M}| \leq b\left(\frac{a}{2}\right) + a\left(b\left(\frac{b}{2}\right) - e\right) + \min\left\{b\left(\frac{b}{2}\right), \frac{e}{2}(b - 2)\right\} \tag{\star}$$

where the three summands on the right-hand side are the maximum number of “good” $Q_3$s of Types I, II, and III, and we have used Lemma 7.1. If $e \geq \frac{b^2}{4}$ then $\min\left\{b\left(\frac{b}{2}\right), \frac{e}{2}(b - 2)\right\} = b\left(\frac{b}{2}\right)$, so the right-hand side of inequality (\star) is a

$$i\bar{x}y$$

$$\begin{array}{ccc}
  j & i & uv \quad vx \\
  Type I & Type II & Type III
\end{array}$$

**Figure 3** Three possible structures of vertices in $S$ for $M \in \mathcal{M}$ where $\emptyset \notin S$. 
decreasing function of $e$. Hence to maximize $|\mathcal{M}|$ we can assume $e \leq \frac{b^2}{4}$, so that

$$\min \left\{ b \left( \frac{b}{2} \right), \frac{e}{2} (b - 2) \right\} = e(b - 2)/2.$$

**Case 1:** If $\frac{b - 2}{2} \leq a$, then

$$|\mathcal{M}| \leq b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} - e \right) + \frac{e}{2} (b - 2)$$

$$\leq b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} - e \right) + ea$$

$$= b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} \right).$$

We remark that equality holds if $e = 0$ and $S = \{i : i \in A\} \cup \{i y x : i \in A \text{ and } x, y \in B\}$ because then any $Q_3$ containing $\emptyset \in S$ which has 1 or 2 flipbits in $A$ (and 2 or 1 in $B$) is good.

**Case 2:** If $\frac{b - 2}{2} > a$, then

$$|\mathcal{M}| \leq b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} - e \right) + \frac{e}{2} (b - 2)$$

$$= b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} \right) + e \left( \frac{b - 2}{2} - a \right)$$

$$\leq b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} \right) + \frac{b^2}{4} \left( \frac{b - 2}{2} - a \right)$$

$$= b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} \right) + b\left( \frac{b}{2}\right) - \frac{b^2}{4} a$$

$$= b \left( \frac{a}{2} \right) + a \left( \frac{b}{2} - \frac{b^2}{4} \right) + b\left( \frac{b}{2}\right)$$

$$= b \left( \frac{a}{2} \right) + 2a\left( \frac{b}{2}\right) + b\left( \frac{b}{2}\right)$$

$$= b \left( \frac{a}{2} \right) + (2a + b)\left( \frac{b}{2}\right).$$

We remark that equality holds if $G_S = K_{\frac{b}{2}, \frac{b}{2}}$ and $S = \{i j x : i, j \in A \text{ and } x \in B\} \cup \{i y x : i \in A \text{ and } x, y \in B \text{ but } xy \notin E(G_S)\} \cup \{x y z : \{x, y, z\} \text{ induces } K_{1,2} \text{ in } G_S\}$. This expression can be rewritten as
\[
\frac{b}{2} \binom{a}{2} + \frac{b}{2} \binom{b}{2} + \frac{b}{2} \binom{b}{2} + \frac{a}{2} \binom{b}{2} + \frac{a}{2} \binom{b}{2} + \frac{b}{2} \binom{a}{2}
\]

which is equal to

\[
x \binom{y}{2} + x \binom{z}{2} + y \binom{x}{2} + y \binom{z}{2} + z \binom{x}{2} + z \binom{y}{2} \quad (**)
\]

when \( x = z = \frac{b}{2} \) and \( y = a \). The expression in (**) is the number of induced \( K_{1,2} \)'s in a complete tripartite graph with part sizes \( x, y, \) and \( z \). We know that \( K_{\frac{n}{2}, \frac{n}{2}} \) is the graph with \( n \) vertices which has the maximum number of induced \( K_{1,2} \)'s (Remark 1. in Section 4), so (**) attains its maximum value when \( x = z = \frac{n}{2} \) and \( y = 0 \), so \( b = n \) and \( a = 0 \). The upper bound in Case 1, \( a \binom{b}{2} + b \binom{a}{2} \), is the value of (**) when \( x = a, y = b, \) and \( z = 0 \), so it attains its maximum value when \( a = b = \frac{n}{2} \) and both upper bounds equal

\[
2 \cdot \frac{n}{2} \binom{n}{2} = \frac{n^2(n - 2)}{8} = \frac{3}{4} \binom{n}{3} \frac{n}{n - 1}
\]

and \(|\mathcal{M}|\) cannot be bigger. Hence

\[
\frac{3}{4} \leq \lambda(W_2, 3) \leq \lambda_{\text{local}}(W_2, 3) \leq \frac{3}{4}.
\]

We remark that in the construction we have with density \( \frac{3}{4} \), of the vertices not in \( S \), \( \frac{2}{3} \) of them are in good \( Q_3 \)'s only of the type where equality holds in Case 1 (those vertices which have an odd sum in precisely one of \( A \) or \( B \)) and \( \frac{1}{3} \) are in good \( Q_3 \)'s only of the type where equality holds in Case 2 (those vertices with an odd sum in both \( A \) and \( B \)). The local density at all vertices is \( \frac{3}{4} \).

We suspect that no nontrivial \( d \)-cube density can be bigger than \( \frac{3}{4} \) and that this is the only one bigger than \( \frac{2}{3} \).

**Conjecture 7.3.** If \( H \) is a configuration in \( Q_d \) such that \( \frac{2}{3} < \lambda(H, d) < 1 \) then \( d = 3 \) and \( H \) is an exact copy of \( W_2 \).

We will use a blow-up (introduced in [14]) of the miraculous configuration \( T \) in \( Q_6 \) shown in Figure 4 to get a lower bound for \( \lambda(W_{6n}, 3) \). Let \( H \) be a configuration in \( Q_d \). We say the configuration \( S \) in \( Q_n \) is a blow-up of \( H \) if it is obtained as follows. We partition \([n]\) into parts \( A_1, A_2, ..., A_d \). For each \( v = (x_1, x_2, ..., x_n) \) in \( V_n \) we define a vector \( u = (y_1, y_2, ..., y_d) \) in \( V_d \) by \( y_j \equiv \sum_{i \in A_j} x_i \mod 2 \), and put \( v \) in \( S \) if and only if \( u \in H \). Any sub-\( d \)-cube of \( Q_n \) which has one flip...
bit in each \( A_j \) will have an exact copy of \( H \) (and perhaps others will as well). Using an equipartition of \([n]\) into \( d \) parts shows that \( \lambda(H, d) \geq \frac{d!}{d^d} \) for each \( H \) in \( Q_d \) (Proposition 6 of [14]).

**Proposition 7.4.** \( \lambda(W_{10}, 3) \geq \frac{5}{12} \)

**Proof.** Let \( S \) be a blow-up in \( Q_n \) of the configuration \( T \) in \( Q_6 \) shown in Figure 4. If \( n \) is large and \([n]\) is equipartitioned into six parts, then the probability that a randomly chosen sub-3-cube has its 3 flip bits in different parts is \( \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} = \frac{5}{9} \). To complete the proof we need to show that \( \frac{3}{4} \) of the \( \binom{6}{3} \cdot 2^3 = 160 \) sub-3-cubes of \( Q_6 \) have an intersection with \( T \) which is an exact copy of \( W_{10} \), since \( \frac{5}{9} \cdot \frac{3}{4} = \frac{5}{12} \).

It is not hard to check that for each \( u, v \) in \( T \) (possibly \( u = v \)), there are 10 automorphisms of \( Q_6 \) which maps \( u \) to \( v \) and \( T \) to itself. For example, if \( \psi \) is such an automorphism such that \( \psi(\emptyset) = \emptyset \), then \( \psi(1) = 1, \psi(2) = j \), where \( j \in \{2, 3, 4, 5, 6\} \), \( \psi(3) = j + 1 \) (\( \psi(3) = 2 \) if \( j = 6 \) or \( \psi(3) = j - 1 \) (\( \psi(3) = 6 \) if \( j = 2 \)), and that completely determines \( \psi \). That means the group of automorphisms of \( Q_6 \) which fix the set \( T \) is vertex transitive within \( T \) and outside of \( T \). Hence it suffices to show that 15 of the 20 sub-3-cubes containing a vertex in \( T \) (say \( \emptyset \)) have exact copies of \( W_{10} \) and that 15 of the 20 sub-3-cubes containing a vertex not in \( T \) (say 2) have exact copies of \( W_{10} \).

Remarkably, every sub-3-cube contains precisely three vertices in \( T \). Of the 20 sub-3-cubes containing \( \emptyset \), only the ones containing 234, 345, 456, 562, and 623 do not have exact copies of \( W_{10} \). For example, the one with flip bits 2, 3, and 5 contains the vertices \( \emptyset, 23, \) and 235 of \( T \), the one with flip bits 1, 2, and 3 contains the vertices \( \emptyset, 1, \) and 23 of \( T \), and the one with flip bits 1, 2, and 4 contains the vertices \( \emptyset, 1, \) and 124 of \( T \).

Of the 20 sub-3-cubes containing 2, the five which do not have exact copies of \( W_{10} \) are the ones with flip bits 1, 3, 6; 1, 4, 5; 2, 3, 4; 2, 3, 6; 2, 5, 6. For example, the one with flip bits 1, 3, 5 contains the vertices 23, 125, and 235 of \( T \) and the one with flip bits 2, 3, 5 contains the vertices \( \emptyset, 23, \) and 235 of \( T \). □
We remark that the generic lower bound $\lambda(H, d) \geq \frac{d^4}{d^d}$ is attained by $Z_3$ in $Q_2$ and by the perfect 8-cycle $C_4$ in $Q_4$, but not by any configuration in $Q_3$, since $\lambda(H, 3) \geq \frac{8}{27}$ for all configurations $H$ in $Q_3$.

8  |  CONFIGURATIONS IN $Q_4$

In [14] we determined the $d$-cube density of the perfect 8-cycle $C_8$ in $Q_4$. In this section, we will determine the $d$-cube density for two other configurations in $Q_4$.

**Theorem 8.1.** If $Y$ is the configuration $\{0000, 1100, 0011, 1111\}$ in $Q_4$ (see Figure 5), then $\lambda(Y, 4) = \frac{3}{8}$.

**Proof:** First we give a construction to show $\lambda(Y, 4) \geq \frac{3}{8}$. Partition $[n]$ into sets $A$ and $B$ and let $S$ be the set of all vertices in $Q_n$ given by binary $n$-tuples with even weight in both $A$ and $B$. This gives a “good” $Q_4$ for each $Q_4$ with two flip bits in each of $A$ and $B$. If it is an equipartition and $n$ is large then $\frac{3}{8}$ of the $Q_4$s are “good.”

Suppose $\emptyset \in S$ and let $\mathcal{M}$ be the set of all good $Q_4$s containing $\emptyset$. We construct a graph $G_s$ with $V(G_s) = [n]$ and $E(G_s) = \{uv : \emptyset, uv, xy, uvxy\}$ are the vertices in $S$ of some $M \in \mathcal{M}$. If $uv$ and $xy$ are in $M \in \mathcal{M}$, then neither $ux$, $uy$, $vx$, nor $vy$ can be in $E(G_s)$, so $|\mathcal{M}|$ is less than or equal to the number of induced copies of $2K_2$ in $G_s$. That means $\lambda_{local}(Y) \leq i(2K_2) = \frac{3}{8}$.

Now suppose $\emptyset \not\in S$. Let $A = \{i \in [n] : i \in S\}$, $B = [n]\setminus A$, $|A| = a$, and $|B| = b$. Let $\mathcal{M}$ be the set of all good $Q_4$s containing $\emptyset$. If $M \in \mathcal{M}$, then the four vertices of $M$ in $S$ have the structure of Type I or Type II in Figure 6, where $i, j, u, v, x, y \in [n]$ with $i$ and $j$ in $S$, but $u, v, x, y$ not in $S$.

**FIGURE 5** Configuration $Y$.

**FIGURE 6** Two possible structures of vertices in $S$ for $M \in \mathcal{M}$ where $\emptyset \not\in S$. 

\[ iuvjux \quad uvvxxyyyu \]

\[ i \quad j \]

Type I  Type II

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Define a graph $G$ by $V(G) = B$ and $E(G) = \{uv : uv, vx, xy, yu\}$ are the vertices in $S$ of some Type II $M \in \mathcal{M}$ for some $x, y \in [n]$. For such an $M$, neither $ux$, nor $vy$ can be in $S$, so the number of Type II $Q_4$s in $\mathcal{M}$ is at most the number of induced copies of $K_{2,2}$ in $G$. □

**Lemma 8.2.** Let $G$ be a graph with $n$ vertices where $n$ is even. If $|E(G)| = e$, then $G$ has at most

\[
\min \left\{ \left( \frac{n}{2} \right)^2, \frac{e(n-2)^2}{4} \right\}
\]

induced copies of $K_{2,2}$.

**Proof.** That it has at most $\left( \frac{n}{2} \right)^2$ copies of $K_{2,2}$ is proved in [7, 6] (the optimizing graph is $K_{\frac{n}{2}, \frac{n}{2}}$). If $uv \in E(G)$, define an auxiliary graph $F$ with $V(F) = V(G) \setminus \{u, v\}$ and $E(F) = \{xy : \{u, v, x, y\} \text{ induces } K_{2,2}\}$. The graph $F$ is triangle-free since if $\{u, v, x, y\}$ and $\{u, v, x, z\}$ both induce $K_{2,2}$, then either $\{uv, uz\} \subseteq E(G)$ or $\{vy, vz\} \subseteq E(G)$. In either case, $\{u, v, y, z\}$ induces $K_{1,3}$. Since $F$ is triangle-free, by Turán’s theorem, $uv$ is in at most $\frac{(n-2)^2}{4}$ induced $K_{2,2}$s. Finally, summing over all edges $uv$ counts each $K_{2,2}$ four times. □

If $M$ is a good Type I in $\mathcal{M}$ where $i, j, iux$, and $iju$ are the vertices of $M$ in $S$, then $i, j \in A, u, x \in B$, but $ux \notin E(G_i)$. If $|E(G_i)| = e$, then the number of Type I $Q_4$s in $\mathcal{M}$ is at most

\[
\left( \frac{b}{2} \right)^2 - e \left( \frac{a}{2} \right)^2 + \min \left\{ \left( \frac{b}{2} \right)^2, \frac{e(b-2)^2}{4} \right\}
\]

and of Type II, by Lemma 8.2, is at most

\[
\min \left\{ \left( \frac{b}{2} \right)^2, \frac{e(b-2)^2}{4} \right\}
\]

(with slight modification if $b$ is odd).

Hence, we have $|\mathcal{M}| \leq \left( \frac{b}{2} \right)^2 - e \left( \frac{a}{2} \right)^2 + \min \left\{ \left( \frac{b}{2} \right)^2, \frac{e(b-2)^2}{4} \right\}$.

**Case 1:** If $e \geq \frac{b^2}{4}$, then

\[
|\mathcal{M}| \leq \left( \frac{b}{2} \right)^2 - \left( \frac{b^2}{4} \right) \left( \frac{a}{2} \right)^2 + \left( \frac{b}{2} \right)^2
\]

\[
= \frac{b(b-2)}{2} \left( \frac{a}{2} \right) + \left( \frac{b}{2} \right)^2.
\]

**Case 2:** If $e < \frac{b^2}{4}$, then

\[
|\mathcal{M}| \leq \left( \frac{b}{2} \right)^2 - e \left( \frac{a}{2} \right)^2 + \frac{e}{16} (b - 2)^2
\]

\[
= \left( \frac{a}{2} \right) \left( \frac{b}{2} \right) + e \left[ \frac{1}{16} (b - 2)^2 - \left( \frac{a}{2} \right) \right].
\]
If \( \frac{1}{16}(b-2)^2 \leq \binom{a}{2} \), then \( \mathcal{M} \leq \binom{a}{2}\binom{b}{2} \).

If \( \frac{1}{16}(b-2)^2 > \binom{a}{2} \), then

\[
\mathcal{M} \leq \binom{a}{2}\binom{b}{2} + \frac{b^2}{4}\left[\frac{1}{16}(b-2)^2 - \binom{a}{2}\right]
\]

\[
= \binom{a}{2}\binom{b}{2} - \frac{b^2}{4} + \frac{b(b-2)}{8}^2
\]

\[
= \frac{b(b-2)}{4}\binom{a}{2} + \left(\frac{b}{2}\right)^2
\]

the same upper bound as in Case 1.

Clearly the maximum value of \( \binom{a}{2}\binom{b}{2} \) is \( \binom{n}{2}^2 \) (with a slight modification if \( n \) is odd).

**Lemma 8.3.** If \( n \) is even and \( x, y, \) and \( z \) are nonnegative integers such that \( x + y + z = n \), then the maximum value of

\[
\left(\binom{x}{2}\binom{y}{2}\right) + \left(\binom{x}{2}\binom{z}{2}\right) + \left(\binom{y}{2}\binom{z}{2}\right)
\]

is \( \binom{n}{2}^2 \).

**Proof.** This function counts the number of induced copies of \( K_{2,2} \) in a complete tripartite graph with parts \( X, Y, \) and \( Z \) with part sizes \( x, y, \) and \( z, \) respectively, subject to the constraint \( x + y + z = n \). In [7], it was shown that for all \( n \geq 4 \) the maximum number of induced copies of \( K_{2,2} \) in any graph is \( \binom{n}{2}^2 \). \( \Box \)

If \( x = a \) and \( y = z = \frac{b}{2} \), then (**) reduces to (*), so the maximum of (*) occurs when \( a = 0 \) and \( b = n \) and is equal to \( \frac{3}{8}\left(\binom{n}{4}\frac{n(n-2)}{(n-1)(n-3)} \right) \). Hence, \( \frac{3}{8} \) is an upper bound for \( \lambda_{\text{local(out)}}(Y) \) and \( \lambda_{\text{local(in)}}(Y) \), so \( \frac{3}{8} \leq \lambda(Y, 4) \leq \lambda_{\text{local}}(Y, 4) \leq \frac{3}{8} \).

**Theorem 8.4.** If \( Z \) is the configuration \{0000, 1100, 1010, 0110\} in \( Q_4 \) (see Figure 7), then \( \lambda(Z, 4) = \frac{1}{2} \).

**Proof.** The construction to show \( \lambda(Z, 4) \geq \frac{1}{2} \) is similar to the one for \( Y \) in Theorem 8.1. Partition \( [n] \) into sets \( A \) and \( B \) of size \( \left\lfloor \frac{n}{2} \right\rfloor \) and \( \left\lceil \frac{n}{2} \right\rceil \) and let \( S \) be the set of all vertices in \( Q_n \) given by binary \( n \)-tuples with even weight in both \( A \) and \( B \). This gives a good \( Q_4 \) for each
Q₄ with three flip bits in A and one in B or one flip bit in A and three in B. In the limit as n goes to infinity, \( \frac{4}{16} + \frac{4}{16} = \frac{1}{2} \) of the Q₄'s will be “good,” so \( \lambda(H, 4) \geq \frac{1}{2} \).

For the upper bound, suppose \( \emptyset \in S \) and let \( \mathcal{M} \) be the set of good Q₄s containing \( \emptyset \). We define a graph \( G_S \) with \( V(G_S) = [n] \) and \( E(G_S) = \{ xy : \emptyset, xy, yz, xz \} \) are the vertices in \( S \) of some \( M \in \mathcal{M} \) for some \( z \in [n] \). If \( x, y, z, w \) are the coordinates of a good Q₄ where \( \emptyset, xy, yz, xz \) are the vertices in \( S \), then \( wx, wy, wz \) are not in \( E(G_S) \), so \( \{w, x, y, z\} \) induces \( K_3 \) plus an isolated vertex in \( G_S \). Since this is the complement of \( K_3, \lambda_{\text{local(out)}}(Z, 4) \leq i(K_{1,3}) = \frac{1}{2} \).

Now suppose \( \emptyset \not\in S \). Let \( A = \{i \in [n] : i \in S\}, B = [n] \setminus A, |A| = a, |B| = b \). Let \( \mathcal{M} \) be the set of all good Q₄s containing \( \emptyset \). If \( M \in \mathcal{M} \) then the four vertices of \( M \) in \( S \) have the structure of Type I, Type II, or Type III in Figure 8 (where \( i, j, k \in A \) and \( w, x, y, z \in B \)).

Define a graph \( G \) by \( V(G) = A \cup B \) and \( E(G) = \{ ix : i \in A \text{ and } x \in B \} \cup \{ wx, wy, wz, wxyz \} \) are the vertices in \( S \) of a Type III \( M \in \mathcal{M} \) for some \( y, z \in B \). If \( M \) is a Type I Q₄ with coordinates \( i, j, k, x \), then \( \{i, j, k, x\} \) induces \( K_{1,3} \) in \( G \). If \( M \) is a Type III Q₄ with vertices \( wx, wy, wz, wxyz \) in \( S \), then \( \{w, x, y, z\} \) induces \( K_{1,3} \) in \( G \) (since \( xy, yz, xz \) are not edges in \( G \)) and \( \{i, x, y, z\} \) also induces \( K_{1,3} \) since \( ix, iy, iz \) are all edges. That means that the number of Type II Q₄s in \( \mathcal{M} \) is at most the number of \( K_{1,3} \)s in \( G \) with one vertex in \( A \) and three vertices in \( B \), since if \( i, x, y, z \) are the coordinates of a Type II \( M \), then \( xy, yz, xz \) are all noneges. Thus \( |\mathcal{M}| \) is at most the number of \( K_{1,3} \)s in \( G \) which have precisely 3, 1, or 0 vertices in \( A \), so is certainly at most the maximum number of \( K_{1,3} \)s in a graph with \( n \) vertices. Hence \( \lambda_{\text{local(out)}}(Z, 4) \leq \lambda_{\text{local(out)}}(K_{1,3}) = \frac{1}{2} \), and \( \frac{1}{2} \leq \lambda(Z, 4) \leq \lambda_{\text{local(out)}}(Z, 4) \leq \frac{1}{2} \).

We remark that since the only optimizing host graph which maximizes the number of induced \( K_{1,3} \) subgraphs is complete bipartite, the graph \( G \) defined above can only be optimal if either there are no Type III \( M \in \mathcal{M} \) (so both \( A \) and \( B \) are independent sets), or \( A = \emptyset \), each \( M \in \mathcal{M} \) is Type III, and \( B \) induces a complete bipartite graph with parts asymptotically, but not exactly equal in size (Remark 4 in Section 4).
In this section, we will restate our conjectures stated earlier in the paper, state some other ones, and discuss some other problems for possible further research.

9.1 | Conjectures on \(d\)-cube density

Baber’s flag algebra upper bounds are within \(10^{-9}\) of our lower bounds for five of the configurations in \(Q_3\), so we conjecture equality holds for all of them.

**Conjecture 9.1.** The lower bounds for \(W_7, W_5, W_9, W_{10},\) and \(W_{12}\) given in Table 2 are, in fact, the exact 3-cube densities for these configurations.

It follows from Theorem 5.1 that if a configuration \(H\) in \(Q_d\) is not layered, then \(\lambda(H, d) < 1\). It is easy to show that the only layered configurations with \(d\)-cube density equal to 1 are the trivial ones and that for each \(d \geq 3\), there is a layered configuration in \(Q_d\) with \(d\)-cube density equal to \(\frac{2}{3}\). We have an example of a configuration with 5-cube density equal to at least the inducibility of \(K_{2,3}\), which is \(\frac{5}{8}\) (configuration \(E(5, 2)\) in Section 9.2).

**Conjecture 9.2.** If \(H\) is a configuration in \(Q_d\) such that \(\frac{5}{8} < \lambda(H, d) < 1\) then either

1. \(H\) is layered and \(\lambda(H, d) = \frac{2}{3}\) or
2. \(d = 3, H = W_2,\) and \(\lambda(H, d) = \frac{3}{4}\).

9.2 | Another infinite family

Let \(d\) and \(i\) be positive integers with \(1 \leq i < d\). We define the configuration \(E(d, i)\) in \(Q_d\) by

\[
E(d, i) = \left\{ (x_1, x_2, \ldots, x_d) \in V_d : \sum_{j=1}^{i} x_j \text{ and } \sum_{j=i+1}^{d} x_j \text{ are both even} \right\}.
\]

So two weights must be even for each vertex in \(E(d, i)\), whereas only one must be even for each vertex in the configuration \(H(d, i)\) of Section 6. Note that \(E(d, d - i)\) is an exact copy of \(E(d, i)\) for all \(i\) and \(d\). We remark that \(E(3, 1)\) is the configuration \(W_2\) in \(Q_3\) and \(E(4, 1)\) and \(E(4, 2)\) are the configurations \(Z\) and \(Y\) in \(Q_4\) of Section 8.

To get a lower bound for \(\lambda(E(d, i), d)\) we let \(x\) be a real number in \((0, \frac{1}{2}]\), \(m = [xn] \in \left[1, \frac{n}{2}\right]\), and we define a configuration \(S_x\) in \(Q_n\) by \(S_x = \{(x_1, x_2, \ldots, x_n) \in V_n : \sum_{j=1}^{m} x_j \text{ and } \sum_{j=m+1}^{n} x_j \text{ are both even}\}\). Then any sub-\(d\)-cube of \(Q_n\) with precisely \(i\) or \(d - i\) flip bits in \([1, m]\) is “good,” and in the limit as \(n\) goes to infinity this is a fraction \(f_{d,i}(x) = \frac{d}{i} \left[ x^i (1 - x)^{d-i} + x^{d-i} (1 - x)^i \right] \) of all sub-\(d\)-cubes. The function \(f_{d,i}(x)\) is also the fraction of subsets of \(d\) vertices of \(K_{m,n-m}\) which induce \(K_{i,d-i}\). As shown in [7], \(i(K_{i,d-i})\) is the maximum value of \(f_{d,i}(x)\), where \(x \in \left(0, \frac{1}{2}\right]\), so we have shown the following.
**Proposition 9.3.** For all integers $i$ and $d$ with $1 \leq i < d$, $\lambda(E(d, i), d) \geq i (K_{i,d-i})$.

In [7], Brown and Sidorenko determined for which complete bipartite graphs $K_{d,3+t}$ the optimizing host complete bipartite graph on $n$ vertices, as $n$ goes to infinity, can be taken to be asymptotically equibipartite. It follows from their work that for $i \in \left[ 1, \frac{d}{2} \right]$, $f_{d,i}(x)$ is maximized when $x = \frac{1}{2}$ if and only if $i \geq \frac{d-\sqrt{d}}{2}$. So $\lambda(E(4,1), 4) \geq f_{4,1}(\frac{1}{2}) = i(K_{1,3}) = \frac{1}{2}$ and $\lambda(E(4,2), 4) \geq f_{4,2}(\frac{1}{2}) = i(K_{2,2}) = \frac{3}{8}$ (and Theorems 8.1 and 8.4 say that equality holds).

However, $f_{5,1}(x)$ is maximized when $x = \frac{3-\sqrt{3}}{6}$, so $\lambda(E(5,1), 5) \geq f_{5,1}(\frac{3-\sqrt{3}}{6}) = i(K_{1,4}) = \frac{5}{12}$. We believe equality holds.

**Conjecture 9.4.** Equality holds in Proposition 9.3.

### 9.3 One vertex in $Q_d$

Let $U_d$ be the configuration in $Q_d$ consisting of a single vertex. We have been unable to determine $\lambda(U_d, d)$ for any $d \geq 2$. Since $U_d$ is a layered configuration (with weight set $\{0\}$), it makes sense to use a layered configuration $S$ in $Q_n$ to get a construction for a lower bound. Letting $S$ be the layered configuration $0 \mod (d+1)$ in $Q_n$ shows that $\lambda(U_d, d) \geq \frac{2}{d+1}$. This is the best lower bound we have for $\lambda(U_2, 2)$ ($Z_2$ in Table 1) and $\lambda(U_3, 3)$ ($W_3$ in Table 2). The flag algebra upper bounds are somewhat larger: $\frac{2}{3} \leq \lambda(U_2, 2) \leq 0.685714286$ and $\frac{1}{2} \leq \lambda(U_3, 3) \leq 0.6100431$. We remark that 0.685714286 is about $3 \times 10^{-10}$ larger than 24/35, but we do not have a construction which gives a lower bound greater than 2/3.

A Hamming perfect code can be used to construct a nonlayered configuration $S$ in $Q_n$ which produces a lower bound for $\lambda(U_3, 3)$ which is almost as good. Let $H$ be the configuration in $Q_7$ consisting of the 16 vectors in the length 7 dimension 4 Hamming perfect code. (One realization of it is $\emptyset$ (the 0 vector), the seven incidence vectors of a Fano plane, say the seven cyclic permutations of 124, and the complements of these eight vectors.) The minimum distance in this code is 3, and each vector is Hamming distance 3 from seven other vectors. So the number of sub-3-cubes which contain two vertices of $H$ is $\frac{1}{2} \cdot 16 \cdot 7 = 56$. Since the average number of vertices of $H$ in a sub-3-cube is $\frac{24}{2} \cdot 8 = 1$, there must be 56 sub-3-cubes with no vertices of $H$. Hence there are $\left( \frac{7}{3} \right) \cdot 2^4 - 112 = 448$ sub-3-cubes which have an exact copy of $U_3$, a fraction $\frac{448}{560} = \frac{4}{5}$ of the sub-3-cubes of $Q_7$. If $n$ is large and $S$ is a blow-up of $H$ in $Q_n$ (equipartition of $[n]$ into seven parts) then a sub-3-cube of $Q_n$ with flip bits in different parts will have an exact copy of a sub-3-cube of the $Q_7$ (with configuration $H$), and $\frac{4}{5}$ of these will have an exact copy of $U_3$. Hence $\lambda(U_3, 3) \geq \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{5} = \frac{24}{49}$, not quite as good as the lower bound $\frac{1}{2}$ using the layered configuration $0 \mod 4$ in $Q_n$.

Another way to get a lower bound for $\lambda(U_d, d)$ is to let $R$ be the configuration in $Q_n$ obtained by the following random process: for each $v \in V_n$, put $v \in R$ with (uniform independent) probability $\frac{1}{2^d}$. The probability that a sub-$d$-cube has an exact copy of $U_d$ is $2^d \cdot \frac{1}{2^d} \cdot \left( \frac{2^d-1}{2^d} \right)^{2d-1}$
so $\lambda(U_d, d) \geq \left( \frac{2d^2 - 1}{2d} \right)^{2d-1}$. Of course this is larger than $\frac{1}{e}$ and, as noted in Section 6, is equal to $\frac{1}{e}$ in the limit as $d$ goes to infinity. This lower bound is larger than $\frac{2}{d+1}$ if $d > 4$.

**Conjecture 9.5.** If $d \geq 5$ then $\lambda(U_d, d) = \left( \frac{2d^2 - 1}{2d^2} \right)^{2d-1}$.

This has the same flavor as a special case of the edge-statistics conjecture of Alon et al. [1] which says (though formulated differently) that the limit as $k$ goes to infinity of the inducibility of a graph with $k$ vertices and one edge is $\frac{1}{e}$.

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**REFERENCES**

1. N. Alon, D. Hefetz, M. Krivelevich, and M. Tyomkyn, *Edge-statistics on large graphs*, Combin. Probab. Comput. 29 (2020), no. 2, 163–189.
2. N. Alon, A. Krech, and T. Szabó, *Turán’s theorem in the hypercube*, SIAM J. Discrete Math. 21 (2007), 66–72.
3. N. Alon, R. Radoičić, B. Sudakov, and J. Vondrák, *A Ramsey-type result for the hypercube*, J. Graph Theory. 23 (2006), 196–208.
4. M. Axenovich and R. Martin, *A note on short cycles in a hypercube*, Discrete Math. 306 (2006), 2212–2218.
5. R. Baber, Private communication, 2014.
6. B. Bollobás, C. Nara, and S. Tachibana, *The maximal number of induced complete bipartite graphs*, Discrete Math. 62 (1986), no. 3, 271–275.
7. J. I. Brown and A. Sidorenko, *The inducibility of complete bipartite graphs*, J. Graph Theory. 18 (1994), no. 6, 629–645.
8. F. R. K. Chung, *Subgraphs of the hypercube containing no small even cycles*, J. Graph Theory. 16 (1992), 273–286.
9. M. Conder, *Hexagon-free subgraphs of hypercubes*, J. Graph Theory. 17 (1993), 477–479.
10. D. Conlon, *An extremal theorem in the hypercube*, Electron. J. Combin. 17 (2010), R111.
11. C. Even-Zohar and N. Linial, *A note on the inducibility of 4-vertex graphs*, Graphs Combin. 31 (2015), 1367–1380.
12. G. Exoo, *Dense packings of induced subgraphs*, Ars Combin. 22 (1986), 5–10.
13. Z. Füredi and L. Ozkahya, *On 14-cycle-free subgraphs of the hypercube*, Combin. Probab. Comput. 18 (2009), 725–729.
14. J. Goldwasser and R. Hansen, *Maximum density of vertex-induced perfect cycles and paths in the hypercube*, Discrete Math. 344 (2021), no. 11, 112585. https://doi.org/10.1016/j.disc.2021.112585
15. J. Goldwasser, B. Lidicky, R. Martin, D. Offner, J. Talbot, and M. Young, *Polychromatic colorings on the hypercube*, J. Combin. 9 (2018), 631–657.
16. J. Goldwasser and J. Talbot, *Vertex Ramsey problems in the hypercube*, SIAM J. Discrete Math. 26 (2012), 838–853.
17. A. W. Goodman, *Triangles in a complete chromatic graph with three colors*, Discrete Math. 57 (1985), no. 3, 225–235.
18. J. Hirst, *The inducibility of graphs on four vertices*, J. Graph Theory. 75 (2014), no. 3, 231–243.
19. J. R. Johnson and J. Talbot, *Vertex Turan problems in the hypercube*, J. Combin. Theory Ser. A. 117 (2010), 454–465.
20. K. A. Johnson and R. Entringer, *Largest induced subgraphs of the n-cube that contain no 4-cycles*, J. Combin. Theory Ser. B. 46 (1989), no. 3, 346–355.

21. E. A. Kostochka, *Piercing the edges of the n-dimensional unit cube*, Diskret. Anal. Vyp. 28 (1976), no. 55–64, 223.

22. N. Pippenger and M. C. Golumbic, *The inducibility of graphs*, J. Combin. Theory Ser. B. 19 (1975), 189–203.

23. M. S. Rahman, M. Kaykobad, and Md. T. Kaykobad, *Bipartite graphs, Hamiltonicity and Z graphs*, Electron. Notes Discrete Math. 44 (2013), 307–312.

24. E. Vaughan, *Flagmatic: A tool for researchers in extremal graph theory* (Version 2.0), 2013, http://flagmatic.org/graph.html

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