A DEGENERATION OF MODULI OF HITCHIN PAIRS

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To C.S. Seshadri with respect and admiration

1. Introduction

The geometry of Hitchin pairs or Higgs bundles has been extensively studied for over twenty five years. The problem of constructing a natural theory of degenerations of the moduli space of Hitchin pairs on smooth curves is therefore of some significance. The purpose of this paper is to develop such a theory.

Let $R$ be a discrete valuation ring with quotient field $K$ and residue field an algebraically closed field $k$. Let $S = \text{Spec } R$, and $\text{Spec } K$ the

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References
generic point and let $s$ be the closed point of $S$. Let $X \to S$ be a proper, flat family with generic fibre $X_K$ a smooth projective curve of genus $g \geq 2$ and with closed fibre $X_s$ a stable singular curve $C$ with a single node $p \in C$. Let $(n, d)$ be a pair of integers such that $\gcd(n, d) = 1$.

Let $\mathcal{M}(n, d)_K^H$ be the moduli space of stable Hitchin pairs of rank $n$ and degree $d$ on the generic fibre $X_K$ of $X/S$. In this paper, we construct and study a degeneration of the moduli space $\mathcal{M}(n, d)_K^H$ of rank $n$ and degree $d$ with analytic normal crossing singularities. Central to this theory is the geometry of the Hitchin fibre which reveals a somewhat new aspect of the theory of compactifications of Picard varieties of curves, which at the same time yields a degeneration of the classical Hitchin picture. In contrast to the usual theory of Picard compactifications, the ones which arise here have analytic normal crossing singularities; recall that when the number of nodes of the curve is strictly bigger than 1, the singularities of the compactified Picard variety is a product of normal crossing singularities and therefore not a normal crossing singularities (cf. [5, Page 595], [22, Page 262, I]). In this process a very natural toric picture shows up, which in a certain sense underlies the so-called abelianization philosophy (Theorem 1.3).

Let $C$ be a projective curve of genus $g \geq 2$ over $k$ and let $\mathcal{L}$ be a line bundle on $C$. A Hitchin pair $(E, \theta)$, comprises of a torsion-free $\mathcal{O}_C$-module $E$ together with a $\mathcal{O}_C$-morphism $\theta : E \to E \otimes \mathcal{L}$ called the Higgs structure. The case when $C$ is smooth and when $\mathcal{L}$ is the dualizing sheaf $\omega_C$ was studied by Nigel Hitchin in the classic paper [10]. Hitchin gave an analytic construction of the moduli space and showed the properness of the Hitchin map; he also observed that the notion of spectral curves comes up naturally in the theory of Hitchin pairs and gives an abelianization of the non-abelian moduli space of stable vector bundles of rank 2 and odd degree. This theory on smooth projective curves has been generalized on numerous fronts, the most significant one being the work of Simpson ([24], [25] and [23]) where it is carried through for higher dimensional smooth projective varieties. The paper by Nitsure [16] was the first one to give a purely algebraic construction of the moduli space and he considered the more general situation of taking an arbitrary line bundle for the Higgs structure instead of the dualizing sheaf.

A natural approach to construct a degeneration would be to consider torsion-free Hitchin pairs on the family $X/S$ which is what is done when there is no Higgs structure involved. Quite surprisingly, unlike the usual case (where there is no Higgs structure), this method does
not quite give a degeneration with the essential properties that one desires, namely flatness and even more importantly, an analogue of the Hitchin map (which is only a rational map here). We therefore develop a theory analogous to that of the Gieseker construction of Hilbert stable bundles on a degenerating family (cf. [7]) which yields a flat degeneration (Theorem 1.1) and at the same time resolves the rational Hitchin map defined on the torsion-free moduli (Theorem 1.2).

Recall that Gieseker’s construction (when rank is 2) was based on the concept of $m$-Hilbert stability, which is a GIT stability condition for points in a certain Hilbert scheme of embeddings of curves in a Grassmannian. This is not quite amenable when one goes to ranks bigger than 2 and more so in the setting of Hitchin pairs. Nonetheless, an approach along the lines of [15] and [20] does work. The process becomes somewhat intricate and the analogy needs to be delicately carried hand in hand with the moduli of relative torsion-free Hitchin pairs on $X/S$.

The main sources for our tools are the papers by Simpson ([24], [25]), and Nagaraj-Seshadri [15] (see also Schmitt [20]).

Let $X \to S$ be a proper and flat fibered surface over $S = \text{Spec}(R)$, where $R$ is a local ring of a smooth curve over $k$ with generic fibre a smooth projective curve of genus $g$ and closed fibre a singular curve $C$ with a single node $p \in C$; assume that $X$ is regular over $k$. Let $\mathcal{L}$ be a relative line bundle on $X$ and we assume that $\text{deg}(\mathcal{L}|_C) > \text{deg}(\omega_C)$, where $\omega_C$ is the dualizing sheaf on $C$. The assumptions on $\mathcal{L}$ are essential only for the flatness of the degeneration. For much of the existence results, we do not need any ampleness assumptions on $\mathcal{L}$. Our principal results are the following:

**Theorem 1.1.**

1. There is a quasi-projective $S$-scheme $\mathcal{G}^H_s(n,d)$ of Gieseker-Hitchin pairs which is flat over $S$ and regular over $k$, with the closed fibre a divisor with (analytic) normal crossing singularities.
2. The generic fibre is isomorphic to the classical Hitchin space $\mathcal{M}^H_k(n,d)$.

**Theorem 1.2.**

1. We have a Hitchin map $\mathbf{g}_s : \mathcal{G}^H_s(n,d) \to A_s$ to an affine space over $S$ which is proper.
To a general section $\xi : S \to \mathcal{A}_s$ we can associate a spectral fibered surface $Y_\xi$ over $S$ with smooth projective generic fibre $Y_{\xi,K}$ and whose closed fibre $Y_{\xi,s}$ is an irreducible vine curve with $n$-nodes.

Let $P_{s,Y_\xi}$ denote the compactified relative Picard $S$-scheme of the spectral fibered surface $Y_\xi$ over $S$ (as constructed by Caporaso [5]). Then we have a proper birational morphism:

$$\nu_* : \mathfrak{g}^{-1}_s(\xi) \to P_{s,Y_\xi}$$

which is an isomorphism over the generic fibre and this map coincides with the classical Hitchin isomorphism of the Hitchin fibre with the Jacobian of $Y_{\xi,K}$.

The $S$-scheme $\mathfrak{g}^{-1}_s(\xi)$ gives a compactification of the Picard variety, whose fibre over $s$ is a divisor with analytic normal crossing singularities.

Theorem 1.3. The morphism $\nu_*$ is an isomorphism on the over subscheme of locally free sheaves of rank 1 and for each $j$, over the stratum $P_{s,Y_{\xi,s}}(j)$ (see (8.0.17)) the fibres are canonical toric subvarieties of the wonderful compactification $\overline{PGL(j)}$ obtained from the closures of the maximal tori of $PGL(j)$ (for details see Theorem 8.17). These are toric varieties associated to the Weyl chamber of $PGL(j)$ ([19]).

The theory generalizes without serious difficulty to reducible curves, which is the content of the last section. In principle it should generalize to any stable curve but the details need to be worked out.

The layout of the paper is as follows: in Section 2 we quickly rework Simpson’s theory for the family $X/S$. In Section 3 we introduce the Gieseker-Hitchin functor; in Section 5 and 6, the coarse moduli space $\mathcal{G}(n,d)_S$ of stable Gieseker-Hitchin pairs is constructed and we define the Gieseker-Hitchin map and prove its properness. Section 7 and 8 are devoted to the study of the geometry of the general fibre of the Gieseker-Hitchin map for a singular curve $C$ and we conclude with the main theorem of the paper. The final section shows how the results can be generalized for a reducible curve with a single node.

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2. Hitchin pairs on nodal curves

We will assume for the most part of this paper that the curve $C$ is an irreducible nodal curve with one node over an algebraically closed field $k$.

2.0.1. Hitchin pairs on $C$. Let $E$ be a coherent $\mathcal{O}_C$–module. Recall that $E$ has depth 1 (at each $x \in C$) if and only if $E$ is of pure dimension 1, i.e for all nonzero $\mathcal{O}_C$–submodules $\mathcal{F} \subset E$, $\dim(Supp(\mathcal{F})) = 1$. Recall that if $C$ is a singular curve with nodal singularities, a torsion-free $\mathcal{O}_C$-module is the same thing as a coherent $\mathcal{O}_C$-module which is of depth 1. Let $\mathcal{L}$ be an invertible sheaf over $C$.

**Definition 2.1.** A Hitchin pair $(E,\theta)$, comprises of a torsion-free $\mathcal{O}_C$-module $E$ together with a $\mathcal{O}_C$-morphism $\theta : E \to E \otimes \mathcal{L}$. The map $\theta$ is called a Higgs structure on $E$.

Let $\mu(E) := \frac{\chi(E)}{rk(E)}$, if $rk(E) \neq 0$. As $\dim(C) = 1$ for us, we see immediately that

$$p(E,m) = m \cdot deg(\mathcal{O}_C(1)) + \mu(E). \quad (2.0.1)$$

where $p(E,m) := \chi(E \otimes \mathcal{O}_C(m))$.

**Definition 2.2.** A Hitchin pair $(E,\phi)$, with $E$ of constant rank, is called $\mu$-(semi)stable if $\mu(E_1) \leq \mu(E)$ (resp. $\leq$), $\forall$ proper subsheaves $E_1 \subset E$ such that $\phi(E_1) \subset E_1 \otimes \mathcal{L}$.

2.0.2. The moduli spaces. We will in fact work with a family of smooth curves degenerating to an irreducible nodal curve $C$ with one node. More precisely, let $S = \text{Spec}(R)$, with $R$ a discrete valuation ring (which is the local ring of a smooth curve over $k$) with quotient field $\bar{K}$ and residue field $k$. We will denote by $s \in S$ the closed point and $\zeta \in S$ the generic point. Let $f : X \to S$ be an $S$-scheme of relative dimension 1, such that the generic fibre $X_\zeta$ is a smooth projective curve of genus $g = p_a(C) \geq 2$ and the closed fibre $X_s = C$. We will also assume that $X$ is regular as a scheme over $k$. Fix $\mathcal{L}$ an arbitrary invertible sheaf on $X$. We will make no ampleness assumptions on $\mathcal{L}$ till later, when we need it.

The aim of this section is to quickly summarize the construction of the moduli space $\mathcal{M}_S^H(n,d)$ of stable Hitchin pairs on $X$ of rank $n$ and degree $d$, as a quasi-projective scheme over $S$. This is done following
Simpson [24] and [25]; in fact, we need both the constructions in [25], the one in terms of \( \Lambda \)-modules as well as the one with pure sheaves.

2.0.3. The moduli space of Hitchin pairs. Let \( f : X \to S \), be as above. Let \( \Lambda = \text{Sym}(\mathcal{L}^*) \) as a sheaf of \( \mathcal{O}_X \)-algebras. A Hitchin pair on \( X \) over \( S \) is a coherent \( \mathcal{O}_X \)-module \( E \) together with an \( \mathcal{O}_X \)-morphism \( \theta : E \to E \otimes \mathcal{L} \). Giving \( \theta \) is equivalent to giving \( \theta : \mathcal{L}^* \to \mathcal{E}nd(E) \) (cf. [25, page 15]). That is, \( \theta \) gives \( E \) a structure of a sheaf of modules over \( \text{Spec}(\Lambda) \) which is the total space of \( \mathcal{L} \) (see [24, Lemma 2.13]). By a \( \Lambda \)-module, we will always mean a coherent \( \mathcal{O}_X \)-module with a structure as above.

Conversely (loc cit), giving a \( \mathcal{O}_X \)-coherent \( \Lambda \)-module is equivalent to giving a coherent \( \mathcal{O}_X \)-module \( E \) together with an \( \mathcal{O}_X \)-module map \( \theta : E \to E \otimes \mathcal{L} \).

Let \( \mathcal{O}_X (1) \) be the relative ample line bundle on \( X \) over \( S \), and for each point \( t \in S \), let \( X_t = X \times_S k(t) \) denote the fibre and \( \Lambda_t \) be the restriction of \( \Lambda \) to \( X_t \). A \( \Lambda \)-module \( E \) is \( p \)-semistable (resp. \( p \)-stable) if \( E \) is flat over \( S \) and if the restrictions \( E_t \) of \( E \) to \( X_t \) are \( p \)-semistable (resp. \( p \)-stable) of pure dimension 1. Recall [24] that \( E_t \) is a \( p \)-(semi)stable \( \Lambda_t \)-module if it is of pure dimension 1 (equivalently, torsion-free), and if for any \( \Lambda_t \)-submodule \( E_1 \subset E \), with \( 0 < \text{rk}(E_1) < \text{rk}(E) \), there exists an \( N \) such that

\[
\frac{p(E_1, m)}{\text{rk}(E_1)} \leq \frac{p(E, m)}{\text{rk}(E)}
\]  

(2.0.2)

(resp. \( \leq \)), for \( m \geq N \).

Remark 2.3. It follows from (2.0.1) that for a Hitchin pair on a curve \( C \), the notion of \( p \)-(semi)stability is the same as that of \( \mu \)-(semi)stability.

On the singular fibre \( X_s = C \), the notion of \( p \)-(semi)stability coincides with the notion of \( \mu \)-(semi)stability with respect to the polarization given by the ample line bundle \( \mathcal{O}_X (1)|_{X_s} \). The notion of pure dimension 1 is precisely the torsion-freeness of the sheaves.

Let \( \mathcal{M}^H_S(n, d) \) be the functor which associates for every \( S \)-scheme \( T \), the set \( \mathcal{M}^H_S(n, d)(T) \) of the equivalence classes of families of \( p \)-semistable Hitchin pairs \((E, \theta)\) on \( X_T := X \times_S T \) with Hilbert polynomial \( P \) given by \( n \) and \( d \), where \((E_T, \theta_T) \sim (E'_T, \theta'_T)\) if there exists a line bundle \( L_T \) on \( T \) such that \( E_T \simeq E'_T \otimes p^*_T(L_T) \) which sends \( \theta_T \) to \( \theta'_T \otimes \text{id} \).
By [24, Theorem 4.7], the set $\mathcal{M}_S^H(n,d)(T)$ can be viewed as the set of isomorphism classes of $p$-(semi)stable $\Lambda_T$-modules on $X_T$ over $T$. Thus, the functor $\mathcal{M}_S^H(n,d)$ has a coarse moduli scheme which is realized as a GIT quotient and we denote by $\mathcal{M}_S^H(n,d)$; by [24], this is quasi-projective over $S$ whose points correspond to $S$-equivalence classes (in the sense of Seshadri) of $p$-semistable torsion-free Hitchin pairs of rank $n$ and degree $d$ on the fibres $X_t$. Furthermore, there is an open subset of isomorphism classes of $p$-stable torsion-free Hitchin pairs and one has a separated open subfunctor $\mathcal{M}_S^{H,s}(n,d)$. In our situation, where we have assumed $\gcd(n,d) = 1$, one has $\mathcal{M}_S^H(n,d) = \mathcal{M}_S^{H,s}(n,d)$.

2.0.4. Spectral construction. Let $Z = \mathbb{P}(L^* \oplus \mathcal{O}_X)$, be the projective completion of the total space of $L$ as a scheme over $S$; let $D_\infty = Z - L$ denote the divisor at $\infty$ and $\pi : Z \to X$ the projection which extends the map $\pi : L \to X$.

**Lemma 2.4.** (cf. [25, Lemma 6.8]) There is a functorial correspondence between the category of Hitchin pairs $(E_S, \theta_S)$ on $X$ and the category of coherent $\mathcal{O}_Z$-modules $\mathcal{E}$ such that $\text{Supp}(\mathcal{E}) \cap D_\infty = \emptyset$. The sheaf $E_S$ is flat over $S$ if and only if $\mathcal{E}$ is flat over $S$. Further,

1. $E_S$ is torsion-free if and only if $\mathcal{E}_S$ is pure of dimension 1.
2. $(E_S, \theta_S)$ is $\mu$-semistable (resp. $\mu$-stable) $\iff \mathcal{E}$ on $Z$ is $p$-semistable (resp. $p$-stable) in the sense of Gieseker-Maruyama-Simpson.

**Proof.** As we have seen earlier, since the map $\pi : L \to X$ is affine, giving a coherent $\mathcal{E}$ on the total space $\text{Spec}(\text{Sym}(L^*))$ of $L$, is equivalent to giving a coherent $\mathcal{O}_X$-module $\overline{E}_S = \pi_*(\mathcal{E})$ together with an action of the $\mathcal{O}_X$-Algebra $\Lambda$. Observe that $\Lambda = \pi_*(\mathcal{O}_L)$.

By what we have observed earlier, this is equivalent to giving a Hitchin pair $(E_S, \theta_S)$ on $X$. To $E$ one associates a coherent $\mathcal{E}$ on $L$ by letting (cf. [8, page 362]):

$$\mathcal{E} = \pi^{-1}(E_S) \otimes_{\pi^{-1}(\Lambda)} \mathcal{O}_L$$

The support of $\mathcal{E}$ is proper over $X$ and this condition is equivalent to saying that $\mathcal{E}$ is coherent on $L$ and the closure of the support of $\mathcal{E}$ on $Z$ does not meet $D$. If $E_S$ is torsion-free, then $\mathcal{E}$ is pure of dimension 1 and conversely.

By the equivalence of categories it follows that the subobjects also correspond to each other naturally and the equivalence of the semistable objects follows as in [25].
Choose $k$ such that $\mathcal{O}_Z(1) = \pi^*(\mathcal{O}_X(k)) \otimes \mathcal{O}_Z(D_\infty)$ is ample on $Z$; therefore $\mathcal{O}_Z(1)|_\mathcal{L} = \pi^*(\mathcal{O}_X(k))$. This way, for any coherent $\mathcal{E}$ on $Z$ whose support does not meet the divisor $D_\infty$, the Hilbert polynomials of the $\mathcal{O}_Z$-module $\mathcal{E}$ and that of $\pi_* (\mathcal{E})$ differ by a scaling factor, i.e $p(\mathcal{E}, m) = p(\pi_* (\mathcal{E}), km)$; hence, the notions of semistability remain intact.

Fix a polynomial $p$ as above of degree 1 (the relative dimension in our case) and let $p_k(m) = p(km)$. Then by [24, Theorem 1.19], we have a coarse moduli scheme $M(\mathcal{O}_Z, p_k)$ of $p$–semistable $\Lambda$–modules on the projective $\mathbb{S}$–scheme $Z$ with respect to the ample line bundle $\mathcal{O}_Z(1)$ and Hilbert polynomial $p_k$. Simpson shows that $M(\mathcal{O}_Z, p_k)$ is a projective $\mathbb{S}$–scheme. Furthermore, we have an open inclusion $\mathcal{M}^H(n,d) \subset M(\mathcal{O}_Z, p_k)$ since $\mathcal{M}^H(n,d)$ parametrizes $p$–semistable $\Lambda$–modules whose support avoids the divisor $D_\infty$.

3. The Gieseker-Hitchin functor

From now on $gcd(n, d) = 1$ and unless otherwise mentioned, from now on $C$ will be an irreducible nodal curve with a single node $p \in C$.

Let $\tilde{C}$ be its normalization and let $\nu: \tilde{C} \to C$ be the normalization map and let $\nu^{-1}(p) = \{p_1, p_2\}$.

**Definition 3.1.** A scheme $R^{(m)}$ is called a chain of projective lines if $R^{(m)} = \bigcup_{i=1}^m R_i$, with $R_i \simeq \mathbb{P}^1$, and if $i \neq j$,

$$R_i \cap R_j = \begin{cases} \text{singleton} & \text{if } |i - j| = 1 \\ \emptyset & \text{otherwise} \end{cases} \quad (3.0.1)$$

**Definition 3.2.** Let $E$ be a vector bundle of rank $n$ on a chain $R^{(m)}$. Let $E|_{R_i} = \bigoplus_{j=1}^n \mathcal{O}(a_{ij})$. Say that $E$ is standard if $0 \leq a_{ij} \leq 1, \forall i, j$. Say that $E$ is strictly standard if moreover, for every $i$ there is an index $j$ such that $a_{ij} = 1$.

**Definition 3.3.** Let $C^{(m)}$ denote the semi-stable curve which is semistably equivalent to $C$, i.e $\tilde{C}$ is a component of $C^{(m)}$ and if $\nu: C^{(m)} \to C$ is the canonical morphism, the fibre $\nu^{-1}(p)$ is a chain $R^{(m)}$ of projective lines of length $m$ cutting $\tilde{C}$ in $p_1$ and $p_2$ (Figure 7).

Let $p: X \to S$ be as before a family of smooth curves degenerating to the singular curve $C$. 
Definition 3.4. (cf. [12, Definition 3.8]) For every $S$-scheme $T$, a modification is a diagram:

$$
\begin{array}{ccc}
X^{\text{mod}}_T & \xrightarrow{\nu} & X_T \\
\downarrow p_T & & \downarrow p \\
T & & 
\end{array}
$$

(1) $p_T : X^{\text{mod}}_T \to T$ is flat,
(2) the $T$-morphism $\nu$ is finitely presented which is an isomorphism when $(X_T)_t$ is smooth,
(3) over each closed point $t \in T$ over $s \in S$, we have $(X^{\text{mod}}_T)_t = C^{(m)}$ for some $m$ and $\nu$ restricts to the morphism which contracts the $\mathbb{P}^1$’s on $C^{(m)}$.

Remark 3.5. We will reserve the notation $\nu$ for the modification morphism $\nu : X^{\text{mod}}_T \to X_T$ for any $S$-scheme $T$ and will not carry the subscript $T$ to avoid cumbersome notation.

Definition 3.6. A vector bundle $V$ on $C^{(m)}$ of rank $n$ is called a Gieseker vector bundle,

(1) If $m = 0$, i.e $C^{(0)} = C$ it is a vector bundle, else
(2) if, $m \geq 1$, $V|_{R^{(m)}}$ is strictly standard and the direct image $\nu_*(V)$ is a torsion-free on $\mathcal{O}_C$-module.

A Gieseker vector bundle on a modification $X^{\text{mod}}_T$ is a vector bundle such that its restriction to each $C^{(m)}$ in it is a Gieseker vector bundle.

Let $\mathcal{L}_{\text{mod}}$ be the line bundle on $X^{\text{mod}}_T$ defined by $\mathcal{L}_{\text{mod}} := \nu^*(\mathcal{L})$. In particular, $\mathcal{L}_{\text{mod}}|_{R^{(m)}} = \mathcal{O}_{R^{(m)}}$, on the chain $R^{(m)}$ in $C^{(m)}$. 

Figure 1. Semistable curve $C^{(2)}$ and $C^{(1)}$ over $C$
Remark 3.7. If $E$ is a vector bundle on $C^{(m)}$, then one has a local Mayer-Vietoris type computation to yield

$$H^i(E|_U) = H^i(E|_{V'}) \oplus H^i(E|_{R^{(m)}}), \forall i \geq 1$$

(3.0.3)

where $V$ is an affine neighbourhood of the node $p \in C$ and $U = \nu^{-1}(V)$ and $V' = U \cap \tilde{C}$ (cf. [15, page 170]).

**Definition 3.8.** A Gieseker-Hitchin pair on $X_T^{(\text{mod})}$ is a locally free Hitchin pair $(V_T, \phi_T)$, with an element $\phi_T \in H^0(X_T, (p_T)_! (\mathcal{L}_\text{mod} \otimes \mathcal{E}\text{nd}(V_T)))$, i.e a $\mathcal{O}_{X_T}$-morphism $\phi_T : V_T \to V_T \otimes \mathcal{L}_\text{mod}$ satisfying the following:

(1) $V_T$ is a Gieseker vector bundle on $X_T^{(\text{mod})}$ (Definition 3.6).
(2) For each closed point $t \in T$ over $s \in S$, the direct image $\nu_*(V_t, \phi_t)$ is a torsion-free Hitchin pair on $X_t = C$.

A Gieseker-Hitchin pair $(V_T, \phi_T)$ is called stable if the direct image $(\nu)_*(V_T, \phi_T)$ is a family of stable Hitchin pairs on $X_T$ over $T$.

**Definition 3.9.** Two families $(V_T, \phi_T)$ and $(V_T', \phi_T')$ parametrized by $T$ are called equivalent if there exists a $X_T$-automorphism $\sigma$, i.e:

$$X_T^{(\text{mod})} \xrightarrow{\sigma} X_T^{(\text{mod})} \xrightarrow{\nu} X_T \xrightarrow{\nu} X_T^{(\text{mod})}$$

(3.0.4)

and a line bundle $\mathcal{D}_T$ on the parameter space $T$ such that

$$\sigma^*((V_T, \phi_T) \otimes \mathcal{D}_T) \simeq (V_T', \phi_T').$$

(3.0.5)

Equivalently, for each closed point $t \in T$ over $s \in S$, if there exists an automorphism $g$ of $C^{(m)}$ which is identity on the normalization $\tilde{C}$, $g^*(V_t, \phi_t) \simeq (V'_t, \phi'_t)$.

Remark 3.10. This notion of equivalence is the natural generalization of the notion defined in [15, Definition 4, page 177].

**Definition 3.11.** The Gieseker-Hitchin functor $\mathcal{G}_S^H(n, d)(T)$ is defined as follows: for every $S$-scheme $T$,

$$\mathcal{G}_S^H(n, d)(T) := [X_T^{(\text{mod})}, (V_T, \phi_T)]$$

(3.0.6)

which are equivalence classes such that $(V_T, \phi_T)$ is a stable Gieseker-Hitchin pair on $X_T^{(\text{mod})}$ and $\nu_*(V_T, \phi_T) \in \mathcal{G}_S^H(n, d)(T)$. 

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4. Some auxiliary results

Let $T = \text{Spec } B$ be a $S$-scheme with $B$ a discrete valuation ring and let $L$ be the function field of $T$ which is assumed to be a finite extension of $K$. Assume that the closed point of $T$ maps to $s \in S$. Let $X_T = X \times_S T$ and let $p \in X_T$ be the node; let $U$ be a formal neighbourhood of $p$ in $X_T$. We recall ([15] Page 191) that $U$ is normal with an isolated singularity at $p$ of type $A$. By the generality of $A$-type singularities, one can realize $U$ as a cyclic quotient of the affine plane and we can write $U = \text{Spec } C$ where $C$ is:

$$C = \frac{k[[X_1, X_2, X_3]]}{(X_1X_2 - X_3^r)}$$

(4.0.1)

i.e. $U$ is a fibered surface over $\text{Spec } k[[X_3]]$.

Let $E_T$ be a family of torsion-free sheaves on $X_T$ which is locally free on $X_L$. Let us denote the restriction of $E_T$ to $U$ by $F$. Then we have the following general lemma from [15] which gives the complete description of such $F$’s. Recall that one can check that in this situation $F$ is not merely torsion-free but it is also reflexive.

**Lemma 4.1.** Let $F$ be a reflexive $C$-module which is free over the generic fibre. The $F$ is isomorphic as a $C$-module to a direct sum of ideal sheaves of the following kind:

$$F \cong \mathcal{O}_U^t \oplus \bigoplus_{i=1}^t (X_1, X_3^{\alpha_i})^{\oplus r_i}.$$  

(4.0.2)

where $(X_1, X_3^{\alpha_i})$ are ideals generated by two elements.

With this lemma in place we have the following key proposition.

**Proposition 4.2.** Let $E_T$ be a reflexive sheaf on $X_T$. Then there exists a modifications $\nu : X_T^{(\text{mod})} \to X_T$, such that $V_T = \nu^*(E_T)/\text{tors}$ obtained by going modulo torsion is locally free and is a Gieseker vector bundle; moreover, $\nu_*(V_T) \cong E_T$.

**Proof.** The bundle $E_T$ is locally free outside the single singularity $p \in X_T$, therefore we reduce to the case where we take a local formal neighbourhood $U$ of $p$ and we need to show that if $F$ is reflexive sheaf on $U$, then there exists a modifications $\nu : U^{(\text{mod})} \to U$, such that $V = \nu^*(F)/\text{tors}$ is locally free and is a local Gieseker vector bundle and in fact, $\nu_*(V) \cong F$. 

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Without loss of generality, we assume in the local description Lemma 4.1 that \( j = 0 \) and for simplicity of exposition we assume that the multiplicities \( r_i = 1, \forall i \) and

\[
F \simeq (X_1, X_3^a) \oplus (X_1, X_3^b), \ a < b \leq r
\]  

(4.0.3)

This case is sufficient to reflect the complexity of the general problem.

Let \( I := (X_1, X_3^a) \) and \( J := (X_1, X_3^b) \). Let \( f : U(I) = Bl_I(U) \to U \), the blow-up of \( U \) with centre the ideal \( I \). Its description is well-known but we need it fully. Because \( U \) is integral, the scheme \( U(I) \) is integral and is given as a closed subscheme of \( U \times \mathbb{P}^1 \). Note that \( U(I) \) can be realized over \( \text{Spec} \, \mathbb{C} \) as \( \text{Proj}(Gr_I(C)) \), where \( Gr_I(C) = \bigoplus_{n \geq 0} I^n \) (cf. [6, page 4.5]).

The blow-up scheme \( U(I) \) is therefore covered by two patches, \( U(I) = U_1 \cup U_2 \), where \( U_i = \text{Spec} \, A_{(t_i)}, i = 1, 2 \). A simple computation shows that:

\[
A_{(t_1)} = \frac{k[[X_1, X_2, X_3]][Y_2]}{(X_1 Y_2 - X_3^a Y_2 - X_2)}
\]  

(4.0.4)

where \( Y_2 = \frac{t_2}{t_1} \) and

\[
A_{(t_2)} = \frac{k[[X_1, X_2, X_3]][Y_1]}{(X_1 X_3^a Y_1 - Y_1 X_2)}
\]  

(4.0.5)

where \( Y_1 = \frac{t_1}{t_2} \). Observe that \( f^*(I)/\text{tors} \) is the image of \( f^*(I) \) in \( \mathcal{O}_{U(I)} \); more generally, for any ideal \( H \) in \( \mathcal{O}_U \) we will have the notation:

\[
f^#(H) := f^*(H)\mathcal{O}_{U(I)}.
\]  

(4.0.6)

It is well-known that \( f^#(I) \) is the relative ample bundle \( \mathcal{O}_{f}(1) \); explicitly this is given by \( X_1 \), and \( X_3^a \) on \( U_1 \) and \( U_2 \) respectively. We note that the generator \( X_3^a \) in the blow-up is considered an element of homogeneous degree 1.

Observe that since the open subsets \( U_i \) give the trivializing cover and the two branches \( E_i \subset U_i, i = 1, 2 \), hence

\[
f^#(I)|_{E_i} = \mathcal{O}_{E_i}.
\]  

(4.0.7)

We now examine the inverse image \( f^#(J) \) in \( U(I) \). It is easily seen that:

\[
J' = (X_1)
\]  

(4.0.8)
on $U_1$ and ,

$$J'' = (X_3^a)(Y_1, X_3^{b-a})$$

(4.0.9)
on $U_2$. Thus $J$ is principal on $U_1$ while its behaviour on the second
patch $U_2$ is similar to the initial situation, where for $C$ we have $C_{t_2}$
and the ideal $I$ gets replaced by the ideal $J''$ and we iterate. Let

$$U(I, J) := Bl_{f^#(J)}(U(I))$$

(4.0.10)
and $g : U(I, J) \to U(I)$ the blow-up morphism. By the local nature of
blow-ups, we can realize $U(I, J)$ by gluing the blow-ups of $J'$ and $J''$
on $U_1$ and $U_2$ respectively.

Now since $J'$ is locally principal on $U_1$, blowing it up gives us $U_1$
again. Hence we can glue $Bl_{J'}(U_2)$ with $U_1$ to obtain $U(I, J) =
Bl_J(U(I))$ and the blow-up morphism $g : U(I, J) \to U(I)$ is obtained
by gluing the blow-up morphism $g_{J'} : Bl_{J'}(U_2) \to U_2$ and the identity
map on $U_1$.

We now look closer at the picture which emerges in $U(I, J)$. Observe
that $U(I, J) = U^{(2)}$ and $\nu = g \circ f : U^{(2)} \to U$.

The special fibre of $U(I, J)$ over $k[[X_3]]$ is the same as $U$ except that
the singular point $p \in U$ is replaced by a scheme $R^{(2)} = R_1 \cup R_2$ which
is a union of two $\mathbb{P}^1$’s. We make the following observations which are
easily checked (see Figure 2):

**Figure 2. The blow-up picture**
(1) The inverse image \( g^*(J) \) on \( Bl_{j,n}(U_2) \) is now the relative ample \( \mathcal{O}_{j,n}(1) \).
(2) Hence the restriction \( g^*(J)|_{R_2} = \mathcal{O}_{R_2}(1) \).
(3) The restriction \( g^*(J)|_{R_1} = \mathcal{O}_{R_1} \).
(4) \( g^*(f^*(I))|_{R_2} = \mathcal{O}_{R_2} \).
(5) \( g^*(f^*(I))|_{R_1} = \mathcal{O}_{R_1}(1) \).
(6) Hence, \( (g^* \circ f^*)(I \oplus J)|_{R_2} = \mathcal{O}_{R_2} \oplus \mathcal{O}_{R_2}(1) \) and \( (g^* \circ f^*)(I \oplus J)|_{R_1} = \mathcal{O}_{R_1}(1) \oplus \mathcal{O}_{R_1} \).
(7) Let \( V = (g^* \circ f^*)(I \oplus J) \). In other words, \( V = \nu^*(I \oplus J) / tors \) which is locally free on \( U^{(2)} \) and on the closed fibre \( R^{(2)} \) over \( k[[X_i]] \), it is strictly standard.

We need to check that the direct image \( \nu_*(V) = \nu_*(\nu^*(I \oplus J) / tors) \) is firstly a flat family of torsion-free sheaves on \( U \) over \( k[[X_i]] \) and is in fact isomorphic to \( F \) we started with.

Clearly, it is enough to check this on the closed fibre of \( U(I, J) \) over \( k[[X_i]] \). We do this recursively in the manner in which we built up the blow-up by reversing the steps. By the criterion of [15, Lemma 2], we need to check that if \( s \) is a section of \( V|_{R^{(2)}} \) which vanishes at \( p_i \) and \( p_2 \), then it vanishes on the whole of the tree \( R^{(2)} \).

As \( s \) is a section of \( (g^* \circ f^*)(I \oplus J) \), it can be written as \((s_i \oplus s_j)\), such that \( s_i(p_i) = 0 = s_j(p_j), i = 1, 2 \). Now since \((g^* \circ f^*)(I) = g^*(f^*(I))\) and \( f^*(I) \) is locally free, by the projection formula, \( s_i \) gives a section of \( f^*(I) \) on \( R_i \) in \( U(I) \) and \( s_i(p_i) = 0 \) in \( R_i \).

As \( s_i(p_2) = 0 \) in \( R^{(2)} \), and \( g^*(f^*(I))|_{R_2} \) is \( \mathcal{O}_{R_2} \), it vanishes everywhere on \( R_2 \). In particular, \( s_i(q) = 0 \), where \( q = R_1 \cap R_2 \). This implies that when pushed down to \( U(I) \), since \( q \) maps to \( p_2 \in R_1 \), the section \( s_i(p_2) = 0 = s_i(p_i) \), i.e \( s_i = 0 \) since \( R_1 \simeq \mathbb{P}^3 \).

On the other hand, \( s_j(p_i) = 0 \) implies \( s_j|_{R_1} = 0 \), since \( g^*(J)|_{R_1} = \mathcal{O}_{R_1} \); in particular, \( s_j(q) = 0 \). On \( Bl_{j,n}(U_2) \) this is like the earlier picture, and hence \( s_j(q) = 0 = s_j(p_2) \), implying \( s_j = 0 \). This proves that \( \nu_*(\nu^*(I \oplus J) / tors) \) is firstly a flat family of torsion-free sheaves on \( U \).

Now we return to the global setting on \( X_T \). What we have is a Gieseker vector bundle \( V_T \) such that \( \nu_*(V_T) \) is a family of torsion-free sheaves on \( X_T \) which coincides with \( E_T \) on \( X_T - p \). Moreover, by taking
double duals it follows easily that $\nu_*(V_T)$ is reflexive (cf. [15 page 191]). That it is isomorphic to $E_T$ now follows as in loc.cit. □

Remark 4.3. The existence of a Gieseker vector bundle on a modification whose direct image is the torsion-free sheaf $F$ is proven in [15] and the result of Proposition 4.2 can be deduced with a little effort from [15]. However, the explicit form in which we prove this here is absolutely essential for the paper especially for the Hitchin pair situation. In fact, as we discovered later (see [13]), if $O$ is a 2-dimensional analytic local rational singularity, $\nu: \tilde{O} \to O$ the minimal resolution of singularities, and if $M$ is a reflexive $O$-module, then $\tilde{M} := \nu^*(M)/\text{tors}$ is locally free and $\nu_* (\tilde{M}) = M$. This holds in positive characteristics as well and this makes our entire set of results characteristic free.

4.0.5. A key properness result. We make an ad hoc definition for the purposes of this paper.

Definition 4.4. Let $F, G : \{S\text{-schemes}\} \to \{\text{Sets}\}$ be two functors with $S = \text{Spec } R$ as before. Suppose that $f : F \to G$ is a $S$-morphism of functors (more precisely, a natural transformation). We say, $f$ is horizontally proper if the following valuative property holds: let $A$ be a discrete valuation ring with function field $L$ such that $L$ is a finite extension of $K$ and $\text{Spec } A \to \text{Spec } R$ is surjective. Then for every map $\alpha \in F(L)$, if the composite $f(\alpha) \in G(L)$ extends to an element $G(A)$, then $\alpha$ also extends to an element in $F(A)$.

This definition becomes significant along with the following observation.

Lemma 4.5. Let $f : F \to G$ be a quasi-projective $S$-morphism of schemes of finite type such that $f_\zeta : F_\zeta \to G_\zeta$ over the generic point is proper. Suppose further that the structure morphisms $F, G \to S$ are surjective and that $f$ is horizontally proper. Then $f$ is proper.

Proof. The proof is essentially there in [15 Page 188]. Now since $f$ is quasi-projective, there is a projective morphism $\bar{f} : Z \to G$ and a diagram:

$$
\begin{array}{c}
F \\
\downarrow \nearrow \quad \downarrow \nearrow \\
Z \\
\downarrow f \\
G \\
\end{array}
$$

(4.0.11)

As $f_\zeta : F_\zeta \to G_\zeta$ is assumed to be proper, we may assume that $Z$ is the closure of $F_\zeta$. To show that $Z = F$, take a point in $z \in Z_\zeta$ over the
closed point \( s \in S \). Choose a smooth curve \( T \) connecting \( z \) to a point in \( F_{\zeta} \), which therefore surjects onto \( S \). Thus, we have an open subset \( \text{Spec}(L) = U \subset T \), such that \( \text{Im}(U) \subset F \) and since \( T \) maps to \( Z \), it maps by composition to \( G \). Now use the horizontal properness of \( f \) to conclude that \( \text{Im}(T) \subset F \) and hence \( z \in F \).

**Theorem 4.6.** Let \( \nu : \mathcal{G}_s^H(n,d) \rightarrow \mathcal{M}_s^H(n,d) \) be the morphism of functors induced by \( \nu_* \). Then the morphism \( \nu \) is horizontally proper.

**Proof.** In the notation of the proof of Lemma 4.5, suppose that we have a point \( g' \in \mathcal{G}_s^H(n,d)(U) \) such that \( \nu(g') \) extends to a point \( \mathcal{M}_s^H(n,d)(T) \). Then we need to show that \( g' \) is the image of a point \( g \in \mathcal{G}_s^H(n,d)(T) \).

Therefore, we may assume that the morphism \( g' \) gives a family \((V_U, \phi_U)\) of stable Hitchin pairs on the family of smooth curves \( X_U \). We need to show that there is a surface \( X_{\text{mod}}^T \) such that \( \lim_{u \to \tau} (V_u, \phi_u) = (V_{\tau}, \theta_{\tau}) \) exists as a stable Gieseker-Hitchin pair on \( C^{(l)} \) for some \( l \).

As \( \nu \) induces the identity map on \( U \), we may view \((V_U, \theta_U)\) as a family \((E_U, \theta_U)\) on \( X_U \). By assumption, this extends to a torsion-free Hitchin pair \( \lim_{u \to \tau} (E_u, \theta_u) = (E_{\tau}, \theta_{\tau}) \) on the singular curve \( C \). Let us denote this family by \((E_{\tau}, \theta_{\tau})\), a point of \( \mathcal{M}_s^H(n,d)(T) \).

By Proposition 4.2, the family \( E_{\tau} \) lifts to a family \( V_{\tau} \) on a suitable modification \( X_{\text{mod}}^T \) such that \( V_{\tau} \) is a Gieseker vector bundle and also a point of \( \mathcal{G}_s^H(n,d)(T) \). The choice of the number \( l \) which gives the length of the chain of \( \mathbb{P}^1 \)'s is dictated by the local type of the torsion-free sheaf \( E_{\tau} \).

In fact, by Proposition 4.2 we have an isomorphism

\[
\left( \frac{\nu^*(E_{\tau})}{\text{tors}} \right) \simeq V_{\tau}
\]  

(4.0.12)

Pulling back the Higgs structure \( \theta_{\tau} : E_{\tau} \rightarrow E_{\tau} \otimes \mathcal{L} \) by \( \nu \), we get a Higgs structure \( \nu^*(\theta_{\tau}) : \nu^*(E_{\tau}) \rightarrow \nu^*(E_{\tau}) \otimes \nu^*(\mathcal{L}) = \nu^*(E_{\tau}) \otimes \mathcal{L}_t \).

This gives a Higgs structure on \( \left( \frac{\nu^*(E_{\tau})}{\text{tors}} \right) \) and hence a morphism \( \phi_{\tau} : V_{\tau} \rightarrow V_{\tau} \otimes \mathcal{L}_t \) of \( \mathcal{O}_{X_{\text{mod}}^T} \)-modules.

To see that \( \nu_* (\phi_{\tau}) : E_{\tau} \rightarrow E_{\tau} \otimes \nu_* (\mathcal{L}_t) = E_{\tau} \otimes \mathcal{L} \), gives the original Hitchin pair \((E_{\tau}, \theta_{\tau})\), observe that on \( X_U \), they give the same point of \( \mathcal{M}_s^H(n,d)(U) \). In fact, they coincide on the whole of \( X_T - p \) where \( E_{\tau} \).
is a vector bundle. As $E_T$ is torsion-free, this implies immediately that they coincide on the whole of $X_T$.

Clearly, $(V_T, \phi_T)$ extends $(V_U, \phi_U)$ on $X_U^{(mod)}$ and furthermore, the identification $\nu_*(\phi_T) = \theta_T$ implies that $\phi_T : V_T \to V_T \otimes (L_L)_T$ give the desired limiting stable Gieseker-Hitchin pair on $C^{(t)}$ proving the theorem. □

5. Coarse moduli for Gieseker-Hitchin pairs

5.0.6. Some deformation theory. Fix a positive integer $m$ and let $C^{(m)}$ be a Gieseker curve. As before let $\nu : C^{(m)} \to C$ be the canonical morphism and let $L_m = \nu^*(\mathcal{L})$ be the line bundle on $C^{(m)}$ for defining the Higgs structure. Let $(V, \phi)$ be a Gieseker-Hitchin pair on $C^{(m)}$. Our primary interest in the subsection is some remarks on the infinitesimal deformations of the pair $(V, \phi)$.

We follow the papers [16, page 296] and [4].

Definition 5.1. The pair $(V, \phi)$ defines a complex

$$\mathcal{E} := \mathcal{E}^0 \xrightarrow{e(\phi)} \mathcal{E}^1 \to 0$$

(5.0.1)

where $\mathcal{E}^0 = \mathcal{E}nd(V)$ and $\mathcal{E}^1 = \mathcal{E}nd(V) \otimes L_m$ and the map $e(\phi)$ send a local section of $s$ of $\mathcal{E}nd(V)$ to $e(\phi)(s) = \phi \circ s - (Id_{\mathcal{E}nd(V)} \otimes s) \circ \phi$.

Remark 5.2. We have a short exact sequence:

$$0 \to \mathcal{E}nd(V) \otimes L_m[1] \to \mathcal{E} \to \mathcal{E}nd(V) \to 0$$

(5.0.2)

which gives the cohomology long exact sequence:

$$0 \to H^0(\mathcal{E}) \to H^0(\mathcal{E}nd(V)) \to H^0(\mathcal{E}nd(V) \otimes L_m) \to \mathbb{H}^1(\mathcal{E})$$

$$\to H^1(\mathcal{E}nd(V)) \to H^1(\mathcal{E}nd(V) \otimes L_m) \to \mathbb{H}^2(\mathcal{E}) \to 0$$

(5.0.3)

where the $\mathbb{H}^i(\mathcal{E})$ are the hypercohomologies of the complex and the map $H^0(\mathcal{E}nd(V)) \to H^0(\mathcal{E}nd(V) \otimes L_m)$ is the map $e(\phi)$.

Proposition 5.3. Let $\mathcal{L}$ be a line bundle on the curve $C$ such that $\deg(\mathcal{L}) > \deg(\omega_C)$. Then for any stable Gieseker-Hitchin pair $(V, \phi)$ on $C^{(m)}$, with $\phi \in H^0(\mathcal{E}nd(V) \otimes L_m)$, the hypercohomology group $\mathbb{H}^2(\mathcal{E}) = 0$.

Proof. Let $\omega_m := \omega_{C^{(m)}}$ denote the dualizing sheaf on $C^{(m)}$. We recall that for a Gieseker curve $C^{(m)}$, one has the property

$$\omega_m = \nu^*(\omega_C)$$

(5.0.4)
By the long exact sequence (5.0.3), it suffices to show that the map
\[ H^1(\mathcal{E}nd(V)) \to H^1(\mathcal{E}nd(V) \otimes \mathcal{L}_m) \]  
(5.0.5)
is surjective. Observe that by Serre duality this map is dual to the map:
\[ H^0(\mathcal{E}nd(V) \otimes \omega_m \otimes \mathcal{L}_m^{-1}) \to H^0(\mathcal{E}nd(V) \otimes \omega_m) \]  
(5.0.6)
and hence we need to show the injectivity of this canonical map which is given as follows:
\[ s \mapsto e(\phi)(s). \]  
(5.0.7)
Let \( s \in H^0(\mathcal{E}nd(V) \otimes \omega_m \otimes \mathcal{L}_m^{-1}) \) i.e \( s : V \to V \otimes \omega_m \otimes \mathcal{L}_m^{-1} \), and we suppose that \( e(\phi)(s) = 0 \).

By the projection formula, if \((\mathcal{E}, \theta) := (\nu_*(V), \nu_*(\phi))\), we have:
\[ \nu_*(s) : \mathcal{E} \to \mathcal{E} \otimes \omega_C \otimes \mathcal{L}^{-1} \]  
(5.0.8)
and it is checked easily that the condition \( e(\phi)(s) = 0 \) translates to \( e(\theta)(\nu_*(s)) = 0 \). By tensoring with \( \omega_C^{-1} \otimes \mathcal{L} \), the morphism \( \nu_*(s) \) gives rise to
\[ \psi : \mathcal{E} \otimes \omega_C^{-1} \otimes \mathcal{L} \to \mathcal{E} \]  
(5.0.9)
and since \( e(\theta)(\nu_*(s)) = 0 \), it follows that \( \text{Im}(\psi) \) is a Higgs subsheaf of the stable Hitchin pair \((\mathcal{E}, \theta)\) (since \((V, \phi)\) is assumed stable). Further, \((\mathcal{E} \otimes \omega_C^{-1} \otimes \mathcal{L}, \theta \otimes \text{Id})\) is also a stable Hitchin pair. Comparing degrees and observing that by assumption \( \text{deg}(\omega_C^{-1} \otimes \mathcal{L}) > 0 \), it follows that \( \mu(\text{Im}(\psi)) < \mu(\mathcal{E}) \) contradicting the stability of \((\mathcal{E}, \theta)\). Thus, \( \text{Im}(\psi) = 0 \) and hence \( \psi = 0 \). In other words, we conclude that \( \nu_*(s) = 0 \).

By [15] Remark 4(i), page 176], the direct image \( \nu_*(V) = \mathcal{E} \) completely determines the restriction \( V|_C \) to the normalization \( \tilde{C} \) of \( C \). Again by [15] Remark 4(iii)], since \( \nu_*(s) = 0 \), it follows that the restriction \( s|_C : V|_C \to V|_C \otimes (\omega_m \otimes \mathcal{L}_m^{-1})|_C \) is the zero map. This implies that
\[ s(p_1) = s(p_2) = 0. \]  
(5.0.10)
Viewing \( s \) as a section of \( \mathcal{E}nd(V) \otimes \omega_m \otimes \mathcal{L}_m^{-1} \) on \( C^{(m)} \), and noting that \( \omega_m \otimes \mathcal{L}_m^{-1}|_{R^{(m)}} = \mathcal{O}_{R^{(m)}} \), by restriction we get a section \( s \) of \( \mathcal{E}nd(V)|_{R^{(m)}} \) such that \( s(p_1) = s(p_2) = 0 \).

If \( s \) is a section of \( \mathcal{E}nd(V)|_{R^{(m)}} \) with \( s(p_1) = s(p_2) = 0 \) then by Lemma 5.5 below, it follows that \( s \) vanishes.
Now as $s|_C = 0$, we conclude that the section $s$ of $\mathcal{E}nd(V) \otimes \omega_m \otimes \mathcal{L}_m^{-1}$ obtained by gluing $s$ on $R^{(m)}$ with $s|_C$ is zero. This shows that the map (5.0.7) is injective proving the proposition. □

**Lemma 5.4.** Let $V$ be a Gieseker bundle of rank $n$ (Definition 3.6) on $C^{(m)}$. Let $R^{(m)} = \bigcup_{j=1}^m R_j$. Then

$$V|_{R^{(m)}} = \bigoplus_{i=1}^n \mathcal{L}_i$$

such that if $\mathcal{L}_{ij} = \mathcal{L}_i|_{R_j}$, then $\deg(\mathcal{L}_{ij}) \geq 0$ and $\sum_{j=1}^m \deg(\mathcal{L}_{ij}) \leq 1$.

**Proof.** The first statement (5.0.11) follows for Gieseker vector bundles from [15, Lemma 1]. More generally, by [26, Theorem 2.2], (5.0.11) is true for any vector bundle on a tree. The second statement follows immediately from [15, Proposition 5]. □

**Lemma 5.5.** Let $V$ be a Gieseker bundle of rank $n$ on $C^{(m)}$. Then

$$\mathcal{E}nd(V) = \bigoplus_{i,k=1}^n \mathcal{H}om(\mathcal{L}_i, \mathcal{L}_k)$$

(5.0.12)

such that $\mathcal{H}om(\mathcal{L}_i, \mathcal{L}_k)|_{R_j}$ is isomorphic to $\mathcal{O}(1)$ for at most one index $j$ and to $\mathcal{O}(-1)$ for at most one index $j$.

**Proof.** This follows immediately from Lemma 5.4. □

**Remark 5.6.** We observe that if $\mathcal{L} = \omega_C$, then $\dim \mathbb{H}^2(\mathcal{E}) = 1$.

5.0.7. **The total family construction.** We recall from [15, Page 179] the notion of Gieseker functor with respect to a choice of a bounded family of torsion-free sheaves on $X$. In [15] there is a slight mixing of terminologies which should become clear in our discussion. We are eventually interested in the Gieseker-Hitchin pairs therefore we begin by considering torsion-free Hitchin pairs.

We now recall from Simpson [24, Theorem 3.8], the construction of a parametrizing scheme $R^\Lambda_S$ for stable $\Lambda$-modules with fixed Hilbert polynomial $P$. There is a choice of a positive integer $N$, such that the functor which associates to each $S$-scheme $T$, the set of isomorphism classes of pairs $(\mathcal{E}, \alpha)$, with $\mathcal{E}$ a coherent $\Lambda$-module with Hilbert polynomial $P$ on $X_t$ and for each $t \in T$:

$$\alpha_t : k^{P(N)} \cong H^0(X_t, \mathcal{E}(N)_t)$$

(5.0.13)

is representable by a quasi-projective scheme $R^\Lambda_S$ over $S$. By the identification of torsion-free Hitchin pairs with $\Lambda$-modules, the $S$-scheme $R^\Lambda_S$ parametrizes torsion-free Hitchin pairs of rank $n$ and degree $d =$
Also we have an open subset of stable points, $R^\Lambda_s \subset R^\Lambda_s$ and by (cf. [24, Theorem 4.7]),

\[ M^H_s(n,d) \simeq R^\Lambda_s / \text{PGL}(\ell) \]  
(5.0.14)

with $\ell = P(N) = \text{dim}(k^{P(N)})$ and $d = \ell + n(g - 1)$; and since we have assumed that $gcd(n,d) = 1$, $R^\Lambda_s$ is a principal $\text{PGL}(\ell)$-bundle over $M^H_s(n,d)$.

We fix this total family $R^\Lambda_s$. Inside this quasi-projective variety we have a closed subscheme of torsion-free Hitchin pairs $(E,0)$ i.e. with the “0” Higgs structure. Let us denote this subset by $R^s_s \subset R^\Lambda_s$. Again we have an open subset $R^s_s \subset R^\Lambda_s$ of stable torsion-free Hitchin pairs $(E,0)$ and this is also invariant under the action of $\text{PGL}(\ell)$. Furthermore, the quotient $R^s_s / \text{PGL}(\ell) \simeq \mathcal{M}^H_s(n,d)$, is the moduli space of stable torsion-free sheaves on $X/S$ with rank $n$ and degree $d$ (without Higgs structure).

5.0.8. The relative functor. We now recall from [15, Definition 7] the definition of the Gieseker functor relative to $R^\Lambda_s$ (in [15], this functor is ambiguosly called Gieseker functor!).

**Definition 5.7.** Let $\mathcal{G}_{R^\Lambda_s} : (S\text{-schemes}) \to (\text{Sets})$ be the functor defined by:

\[ \mathcal{G}_{R^\Lambda_s}(T) = \{ \text{closed subschemes } \Delta_T \subset X \times_s T \times \text{Grass}(\ell,n) \} \]  
(5.0.15)

such that:

1. The projection $\Delta_T \to T \times \text{Grass}(\ell,n)$ is a closed immersion.
2. The projection $\Delta_T \to T$ is a flat family of curves $\Delta_t$, $t \in T$, such that $\Delta_T$ is a fibered scheme over $T$ of the form $X_T^{(\text{mod})}$.
3. The projection $\Delta_T \to X_T$ is the modification map $X_T^{(\text{mod})} \xrightarrow{\nu} X_T$. Furthermore, if $V_T$ is the pull-back and restrictio to $\Delta_T$ of the tautological quotient bundle of rank $n$ on $\text{Grass}(\ell,n)$, then $V_T$ is a Gieseker vector bundle on $\Delta_T = X_T^{(\text{mod})}$ of rank $n$ and degree $d = \ell + n(g - 1)$.

**Remark 5.8.** We see that the direct image $\nu_*(V_T) = E_T$ is a point of $R^\Lambda_s(T)$.

It is shown in [7] and [15, Proposition 8] that the functor $\mathcal{G}_{R^\Lambda_s}$ is represented by a $\text{PGL}(\ell)$-invariant open subscheme $\mathcal{Y}$ of the $S$-scheme $\text{Hilb}_{\mathcal{Y}}^\Lambda_s(X \times_s \text{Grass}(\ell,n))$. Let $\Delta_{\mathcal{Y}} \subset X \times_s \mathcal{Y} \times_s \text{Grass}(\ell,n)$ be the...
universal object defining the functor $\mathcal{G}_{RS}$. Then we have a canonical projection

$$\vartheta: \mathcal{Y} \to R_s$$

which for each $S$-scheme $T$, sends a point $(\Delta_T, V_T) = (X_T^{(\text{mod})}, V_T)$ to the direct image $\nu_*(V_T) = E_T$, with $E_T \in R_s(T)$.

The imbedding $\Delta_{\mathcal{Y}} \subset X \times_S \mathcal{Y} \times_S Grass(\ell,n)$ gives the natural projections of schemes over $S$:

$$\begin{tikzcd}
\Delta_{\mathcal{Y}} \arrow[rr, q] \arrow[dr, p] & & \mathcal{Y} \arrow[dl, r] \\
& S &
\end{tikzcd}$$

Let $f: \Delta_{\mathcal{Y}} \to X$ be the projection. Let $\mathcal{L}_\Delta = f^*(\mathcal{L})$ and $\mathcal{Y}$ be the universal vector bundle on $\Delta_{\mathcal{Y}}$ (obtained by pulling back the tautological quotient bundle from $Grass(\ell,n)$ and restricted to $\Delta_{\mathcal{Y}}$). Let $\mathcal{E} := \mathcal{E}_{\text{nd}}(\mathcal{Y}) \otimes \mathcal{L}_\Delta$.

**Definition 5.9.** Let $T$ be a $\mathcal{Y}$-scheme and $f: T \to \mathcal{Y}$. Let $q_T: \Delta_T \to T$ be the induced projection and $\mathcal{L}_T$ the corresponding line bundle. Let $\mathcal{E}_T = \mathcal{E}_{\text{nd}}(\mathcal{Y}) \otimes \mathcal{L}_T$. Consider the functor:

$$\mathcal{G}^H_{RS}: \mathcal{Y} - \text{schemes} \to \text{Groups}$$

which associates to each point $f \in \mathcal{Y}(T)$, the group $H^a(T, (q_T)_*(\mathcal{E}))$.

As $\mathcal{Y}$ is a reduced scheme ([15]), by [16] Lemma 3.5, the functor $\mathcal{G}^H_{RS}$ is representable; moreover, there exists a linear $\mathcal{Y}$-scheme $\mathcal{Y}^H$ which represents it.

**Remark 5.10.** By Definition [5.9] for an $S$-scheme $T$, a point in $\mathcal{G}^H_{RS}(T)$ is given by $(\Delta_T, V_T, \phi_T)$, where

1. $(\Delta_T, V_T) = (X_T^{(\text{mod})}, V_T) \in \mathcal{G}_{RS}(T)$, and
2. $(V_T, \phi_T)$ is a Gieseker-Hitchin pair on $X_T^{(m)}$.

Thus by taking direct images and by the universal property of $R^\Lambda_s$, we see that the direct image torsion-free Hitchin pair $(\nu_*(V_T, \phi_T)) = (E_T, \theta_T)$ gives a point in $R^\Lambda_s(T)$ and we get a canonical morphism (see [5.0.16])

$$\vartheta: \mathcal{Y}^H \to R^\Lambda_s.$$
Proposition 5.11. The morphism \( \vartheta : \mathcal{Y}^H \to R_s^A \) obtained in (5.0.19) is proper.

Proof. The schemes involved are quasi-projective schemes over \( S \) and hence the morphism \( \vartheta \) is a quasi-projective morphism. Also the morphism \( \vartheta : \mathcal{Y}^H \to R_s^A \) is an isomorphism over the generic point of \( S \). Hence by Lemma [4.5] we need to check only horizontal properness of \( \vartheta \). This follows now from Theorem [4.6]. \( \square \)

Let \( \mathcal{Y}^H_{st} := \vartheta^{-1}(R_s^{A,s}) \). (5.0.20)

and let \( \mathcal{G}_{R_s}^{H, st} \) be the corresponding subfunctor of \( \mathcal{G}_{R_s}^H \).

Let \( \mathcal{G}'_{R_s} \) be the functor obtained from \( \mathcal{G}_{R_s}^H \), by forgetting the imbeddings into the Grassmannians (see [15, Appendix, page 197]). Then we have a canonical morphism

\[
\mathcal{G}_{R_s}^{H, st} \to \mathcal{G}'_{R_s} \quad (5.0.21)
\]

Proposition 5.12. Let \( L \) be a line bundle on \( X \) with the assumptions of Proposition 5.3. Then the morphism (5.0.21) is formally smooth. In particular, the \( S \)-scheme \( \mathcal{Y}^H_{st} \) is regular over \( k \) with a divisor \( (\mathcal{Y}^H_{st})_s \) with normal crossing singularities.

Proof. Let \( T \) be the spectrum of an Artin local ring, and \( T_o \subset T \) the subscheme defined by an ideal of dimension 1. Let \( \theta \in \mathcal{G}'_{R_s}(T) \) be such that the restriction \( \theta_o \in \mathcal{G}'_{R_s}(T_o) \) can be lifted to an element of \( \mathcal{G}_{R_s}^{H, st}(T_o) \), then we need to show that \( \theta \) itself can be lifted to an element of \( \mathcal{G}_{R_s}^{H, st}(T) \). If \( \theta \) is defined by \( \Delta_T \to T \), with \( \Delta_T \subset X \times_s T \times Grass(\ell, n) \), then the lift of \( \theta_o \) defines a stable Gieseker-Hitchin pair \( (V_{T_o}, \phi_{T_o}) \) on \( \Delta_{T_o} \) of \( T_o \) to \( T_o \).

The problem is to extend the pair \((V_{T_o}, \phi_{T_o})\) to a stable Gieseker-Hitchin pair \((V_T, \phi_T)\) on \( \Delta_T \) as well as the sections of \( V_{T_o} \) to those of \( V_T \). The second issue is taken care of as in [15]. The key issue for us is the first one. Let \((V, \phi)\) be the restriction of \((V_{T_o}, \phi_{T_o})\) to the closed fibre of \( \Delta_{T_o} \to T_o \). Then by [11, Theorem 3.1], the obstruction to extending the pair \((V_{T_o}, \phi_{T_o})\) lies in the second hypercohomology \( \mathbb{H}^2(\mathcal{C}) \) of the complex \( \mathcal{C} \) defined in (5.0.1). By Proposition 5.3 together with the assumptions on the line bundle \( \mathcal{L} \), it follows that \( \mathbb{H}^2(\mathcal{C}) = 0 \). This
implies the formal smoothness of the morphism \(5.0.21\). Now following the arguments in [15, Appendix] it follows in much the same manner that the \(S\)-scheme \(Y^H_{st}\) which represents the functor \(G^H_{st}\) is regular over \(k\) and the closed fibre \((Y^H_{st})_s\) has normal crossing singularities.

\[\square\]

**Proposition 5.13.** The action of \(PGL(\ell)\) on \(R^\Lambda_s\) lifts to \(Y^H_{st}\) and the geometric quotient \(Y^H_{st} \rightarrow Y^H_{st}/PGL(\ell)\) exists. The quotient

\[G^H_S(n,d) := Y^H_{st}/PGL(\ell); \quad (5.0.22)\]

gives the moduli scheme for the Gieseker-Hitchin functor \(G^H_S(n,d)\).

**Proof.** It is seen easily that \(\vartheta : Y^H \rightarrow R^\Lambda_s\) is generically an isomorphism and on the closed fibre it is birational over the open subset of locally free Hitchin pairs on \(C\). Under these conditions, we have a natural algorithm formulated in [15, page 179-180] to show the existence of a quotient which is quasi-projective over \(S\).

Now since \(R^\Lambda_s \rightarrow R^\Lambda_s/PGL(\ell)\) is a principal \(PGL(\ell)\)-bundle, it follows that the GIT quotient \(Y^H_{st} \rightarrow Y^H_{st}/PGL(\ell)\) is also a principal bundle and gives a natural quasi-projective \(S\)-scheme structure on \(Y^H_{st}/PGL(\ell)\).

\[\square\]

**Corollary 5.14.** The map \(\vartheta\) descends to give a proper and birational morphism

\[\nu_* : G^H_S(n,d) \rightarrow \mathcal{M}^H_S(n,d). \quad (5.0.23)\]

**Proof.** Properness follows immediately from Proposition 5.11. Birationality follows from the isomorphism over smooth curves.

\[\square\]

**Corollary 5.15.** The Gieseker-Hitchin moduli space \(G^H_S(n,d)\) of stable Gieseker-Hitchin pairs is flat over \(S\). Furthermore, as a scheme over \(k\) it is regular and the closed fibre \(G^H_S(n,d)_s\) is a divisor with analytic normal crossing singularities.

**Proof.** As \(Y^H_{st} \rightarrow Y^H_{st}/PGL(\ell)\) is a principal bundle, by Proposition 5.12 it follows that \(G^H_S(n,d)\) has all the properties stated in the corollary.

\[\square\]

**Remark 5.16.** It is however not clear whether the moduli space of torsion-free Hitchin pairs \(\mathcal{M}^H_S(n,d)\) is even flat over \(S\). Of course, one does not expect good singularities on it.
6. The Gieseker-Hitchin map over a base

Let $\mathcal{A}_S \to S$ be the affine $S$–scheme representing the functor $T \mapsto \bigoplus_{i=1}^n H^0(X_T, \mathcal{L}_i)$, where $X_T = X \times_S T$. Recall that we have chosen $\mathcal{L}$ to be sufficiently very ample on $X/S$ so that the higher direct images $R^1 f_*(\mathcal{L}^i)$ are zero. Then, it is not hard to see by the projection formula this is an affine $S$-scheme representing the functor $T \mapsto \bigoplus_{i=1}^n H^0(S, f_*(\mathcal{L}_i^i))$. This is the relative version of the space of characteristic polynomials. The points of this space $\mathcal{A}_S$ are polynomials $t^n + \sum_{i=1}^n q_i t^{n-i}$, with $q_i \in H^0(S, f_*(\mathcal{L}^i))$.

Equivalently, a point of $\mathcal{A}_S$ can be viewed as a polynomial written $t^n + \sum_{i=1}^n q_i t^{n-i}$, with $q_i \in H^0(X, \mathcal{L}^i)$.

We observe that for each modification $X_T^{(\text{mod})}$ we have a canonical identification:

$$\mathcal{A}_S(T) = \bigoplus_{i=1}^n H^0(X_T^{(\text{mod})}, \mathcal{L}_i^{\text{mod}})$$

(6.0.1)

Remark 6.1. Obstructions to defining the Hitchin map.

- Let $(E, \theta)$ be a torsion-free Hitchin pair on the surface $X$ and let $\mathcal{E}$ be the pure sheaf on $Z$ which corresponds to $(E, \theta)$. Because the generic fibre is smooth and projective curve one can define the Hitchin map $h$ as is shown in [25].
- However, in general the Hitchin map $h$ does not extend in a well-defined manner to the moduli functor of torsion-free Hitchin pairs on $X$; for instance, if $T$ is an arbitrary $S$-scheme, then it is not clear why the values of the characteristic polynomial coincide when we approach the node on the closed fibre through distinct curves on $T$.
- Secondly, for an arbitrary $S$-scheme $T$, the singularities of $X_T$ are no longer of the $A$-type.
- One of the important points of this paper is that the Hitchin map is well-defined on the Gieseker-Hitchin functor (Definition 6.2); indeed, the Gieseker-Hitchin space resolves the rational morphism $h$.

Definition 6.2. Define the morphism:

$$g_s : \mathcal{G}_s^H(n, d) \to \mathcal{A}_S$$

(6.0.2)

for each $S$-scheme $T$, $g_T[X_T^{(\text{mod})}, (V_T, \phi_T)] = (q_1(\phi_T), \ldots, q_{n-1}(\phi_T))$. 

24
Remark 6.3. The morphism \( g_s \) respects the equivalence (Definition 3.9) of families. The action of the automorphism \( g \) (which leaves the end points \( p_1 \) and \( p_2 \) fixed) does not affect the gluing, while on the bundle \( V_1 \mid R \) restricted to the tree \( R \), \( g \) acts as an automorphism and the Higgs structure \( \phi \) is simply an endomorphism (\( \mathcal{L}_{\text{mod}} \) is trivial on \( R \)). Thus, the Higgs structure is simply conjugated by the automorphism \( g \) and hence the characteristic polynomial is well-defined.

6.0.9. The Spectral variety.

Proposition 6.4. Let \( X \to S \) be as before with a singular fibre \( C \). Let \( T \to S \) be a smooth curve over \( S \) equipped with a marked point \( \tau \in T \) over the closed point \( s \in S \). Let \( U = T - \{ \tau \} \) and suppose that \((E_u, \theta_u)\) is a family of stable Hitchin pairs on the family of smooth curves \( X_u \) parametrized by \( U \). Assume that the characteristic polynomial \( h(\theta_u) \) has a limit \( h_o \) in \( \mathbb{A}^T \). Then, \( \lim_{u \to \tau} (E_u, \theta_u) \) exists as a stable torsion-free Hitchin pair with \( h(\theta_\tau) = h_o \).

Proof. The proof follows [25] closely, but we need to keep in mind two points; firstly, the closed fibre is singular and secondly the Hitchin map is not defined on the moduli space of torsion-free Hitchin pairs.

Let \( \eta_i(u) = q_i(h(\theta_u)) \), the coefficients of the characteristic polynomial. By assumption, these functions extends to the whole of \( T \). Now consider the function \( c : T \times_s \mathcal{L} \to \mathcal{L}^n \) defined by \( c(z, t) = t^n + \sum_{i=1}^n \eta_i(z) t^{n-i} \). By Cayley-Hamilton theorem, one knows that \( c(u, \theta_u) = 0 \) for all \( u \in U \).

Observe that the family \((E_u, \theta_u)\) gives a morphism \( g' : U \to \mathcal{M}_S^H \). Using the compactification \( \mathcal{M}_S^H \subset M(O_Z, P_k) \) (see Lemma 2.4), we extend the map \( g' \) to a map \( g : T \to M(O_Z, P_k) \). By possibly going to a finite covering of \( U \) and using the GIT construction of \( M(O_Z, P_k) \), via \( g \), we get a family \( \mathcal{E} \) on \( Z \times_S T \), such for each \( u \in U \), this family \( \mathcal{E}_u \) gives a point of \( \mathcal{M}_S^H(T) \).

By Lemma 2.4, we have \( \text{Supp}(\mathcal{E}_u) \cap D = \emptyset \) for each \( u \in U \), i.e \( \text{Supp}(\mathcal{E}_u) \subset U \times_s \mathcal{L} \). Since \( c(u, \theta_u) = 0 \) for all \( u \in U \) the support is contained in the zero scheme \( Z(c) \subset T \times_s \mathcal{L} \).

Let \( \mathcal{E}_\tau = \lim_{u \to \tau} \mathcal{E}_u \). By flatness, the support of \( \mathcal{E}_\tau \) is contained in the closure of \( \text{Supp}(\mathcal{E}_u) \) in \( T \times_s Z \). But this is contained in the closed subscheme \( Z(c) \) and hence contained in \( T \times_s \mathcal{L} \). In other words, the limiting sheaf \( \mathcal{E}_\tau \) also has the property that \( \text{Supp}(\mathcal{E}_\tau) \cap D = \emptyset \). Thus, by Lemma 2.4 we get a limiting Hitchin pair \((E_\tau, \theta_\tau)\). \( \square \)
Remark 6.5. From the proof of Proposition 6.4 it follows that, if for some \( T \), and a family \((E_T, \theta_T)\) of torsion-free Hitchin pairs, if the characteristic polynomial of \( \theta_T \) is defined as a \( T \)-valued point of \( \mathcal{A}_S \), then, one can consider the notion of a spectral scheme associated to a family of Hitchin pairs \((E_T, \theta_T)\). Once the characteristic polynomial, say \( u \), is fixed in \( \mathcal{A}_S(T) \), it defines a closed subscheme \( Y_u \subset \mathcal{L} \) and the family \( \mathcal{E}_T \) of pure sheaves on \( Z_T \) which corresponds to \((E_T, \theta_T)\) is set-theoretically supported on \( Y_u \). If moreover, \( Y_u \) is reduced and irreducible, it is precisely the scheme theoretic support of the pure sheaf \( \mathcal{E}_T \) on \( Z_T \).

**Theorem 6.6.** The Hitchin map (6.2), \( g_S : \mathcal{G}_S^H(n,d) \rightarrow \mathcal{A}_S \) is proper over \( S \).

*Proof.* Observe that over the generic point \( \zeta \in S \), the conditions of Lemma 4.5 hold good by the classical properness of the Hitchin map ([25, Theorem 6.11]). Thus, we need to check only the horizontal properness of the Hitchin map.

Let \( T \rightarrow S \) be a smooth curve over \( S \) equipped with a marked point \( \tau \in T \) over \( s \in S \). Let \( U = T - \{\tau\} \) which maps to the generic point of \( S \).

Suppose that we are given a morphism \( \alpha : U \rightarrow \mathcal{G}_S^H(n,d) \) such that \( g_S \circ \alpha \) extends to the whole of \( T \). We need to check that \( \alpha \) itself extends to \( T \). Now since over the subset \( U \) the surface is isomorphic to \( X_U \), the map \( \alpha \) gives a family of stable Hitchin pairs \((E_u, \phi_u)\) on \( X_U \). Observe that since the line bundle \( \mathcal{L}_{mod} \) is a pull-back of \( \mathcal{L} \) from \( X \), the characteristic polynomial \( \lim_{u \rightarrow \tau} g_S(E_u, \phi_u) \) exists in \( \mathcal{A}_S(T) \) (since \( g_S \circ \alpha \) extends to the whole of \( T \)).

By Proposition 6.4, the family extends to a torsion-free stable Hitchin pair \((E_\tau, \phi_\tau)\). Now applying Corollary 5.14, we see that \( \alpha \) extends to \( T \) completing the proof. \( \square \)

In summary we have the following theorem:

**Theorem 6.7.** Let \( X/S \) be as before.

1. We have a quasi-projective \( S \)-scheme \( \mathcal{G}_S^H(n,d) \) which is flat over \( S \) which is regular as a scheme over \( k \).
2. The generic fibre \((\mathcal{G}_S^H(n,d))_\zeta \) is the classical Hitchin moduli space of stable Hitchin pairs on \( X_\zeta \) of rank \( n \) and degree \( d \), and the closed fibre \((\mathcal{G}_S^H(n,d))_s \) is a divisor with analytic normal crossing singularities.
(3) We have a natural Hitchin morphism $\varphi_S : \mathcal{G}_S^H(n,d) \to \mathcal{A}_S$ which is proper over $S$ and extends the classical Hitchin morphism on $(\mathcal{G}_S^H(n,d))_\zeta$.

Recall the $S$-morphism $\nu_* : \mathcal{G}_S^H(n,d) \to \mathcal{M}_S^H(n,d)$, which is an isomorphism over the generic point $\zeta \in S$.

**Proposition 6.8.** The Hitchin map $\varphi_S : \mathcal{G}_S^H(n,d) \to \mathcal{A}_S$ descends to a well-defined set-theoretic map on the image $\text{Im}(\nu_*)$ and we have a diagram:

$$
\begin{array}{ccc}
\mathcal{G}_S^H(n,d) & \xrightarrow{\nu_*} & \text{Im}(\nu_*) \\
\varphi_S \downarrow & & \downarrow h_S \\
\mathcal{A}_S & & \\
\end{array}
$$

**Proof.** As $\nu_*$ is an isomorphism outside the closed point on $S$, it is enough to check the statement on the closed fibre $X_s = C$. Let $(E, \theta)$ is a stable Hitchin pair on $C$ so that $\theta : E \to E \otimes L$. Let the local type of $E$ at the maximal ideal be $O_a \oplus m_b$; choose any $(V, \phi)$, on the semistable curve $C^{(b)}$ such that $\nu_*(V, \phi) = (E, \theta)$ (by assumption $(E, \theta) \in \text{Im}(\nu_*)$). Note that $\phi : V \to V \otimes \nu^*(L)$.

As $\nu^*(L)|_R = O_R$, it follows that the coefficients of the characteristic polynomial $q_i(\phi)$ give sections of $\mathcal{L}_C^i$ with an identification at the points $p_1, p_2$. This gives sections of $\mathcal{L}^i$ on $C$. Clearly these are independent of the $(V, \phi)$ chosen on a semistable curve above $C$ since the characteristic polynomial is defined in terms of the restrictions of $V$ to $C$, and by [H], Remark 4(i), page 176], the direct image $\nu_*(V) = E$ determines $V|_C$. Hence they define the characteristic polynomial of $(E, \theta)$. \qed

7. The Hitchin fibre

Let $X \to S$ be as before a fibered surface with a singular fibre $C$ which is irreducible with a single node. Also, $X$ is regular over $k$. By Theorem 6.6, the Hitchin morphism is well-defined and proper on the Gieseker-Hitchin scheme $\mathcal{G}_S^H(n,d)$ and it is not well-defined on the space of Hitchin pairs $\mathcal{M}_\zeta^H(n,d)$. We recall briefly well-known facts from Hitchin’s work and highlight facts which are specially relevant for our discussion. We make the following general observations first.

**Lemma 7.1.** Let $p : X \to S$ be as before. Let $\mathcal{L}$ be a line bundle such that $\mathcal{L}$ is relatively very ample. For a general section $\xi : S \to \bigoplus_{i=1}^np_*(\mathcal{L}^i)$,
we get a spectral surface $\psi_\xi : Y_\xi \to X$, which is a ramified $n$-sheeted cover over $X$ such that it is unramified over the nodes of the special fibre $X_s$.

**Proof.** The proof is essentially from [3, page 172]. We quickly recall that the construction of the spectral surface $Y_\xi$ with the associated properties. Let $W = \text{Spec}(\text{Sym}(\mathcal{L}^*))$ be the total space of the line bundle $\mathcal{L}$ and $\varrho : W \to X$ the projection. The pull-back $\varrho^*(\mathcal{L})$ then gets a tautological section $x$. Now take the sections of $\varrho^*(\mathcal{L}^n)$ on $W$ of the form

$$\xi = x^n + \sum_{i=1}^{n} \varrho^*(\xi_i).x^{n-i} \quad (7.0.1)$$

for $\xi_i : S \to p_*(\mathcal{L}^i)$. The zero scheme $Y_\xi$ of the sections $\xi$ of $\varrho^*(\mathcal{L}^n)$ will be the spectral surface we desire.

We need to show the existence of one such which satisfies the properties of the Proposition. Consider the embedding $X \hookrightarrow \mathbb{P}^N \times S$ given by the line bundle $\mathcal{L}$. Restrict this to the closed fibre $X_s \subset \mathbb{P}^N \times \{s\}$ and choose a section $h \in H^0(X_s, \mathcal{L}^n)$ such that the hyperplane in $\mathbb{P}(H^0(X_s, \mathcal{L}^n))$ defined by $h$ has the following properties:

1. $h$ does not meet the nodes on $X_s$. Because $\mathcal{L}_s$ is very ample, this is always possible. In other words,
   $$ (h)_0 \cap \{\text{nodes of } X_s\} = \emptyset \quad (7.0.2) $$

2. $h$ does not have multiple zeros
3. $Y_h$ is irreducible.

Again, we can choose a section $\xi_n : S \to p_*(\mathcal{L}^n)$ such that $\xi_n(s) = h$ and by choice, we see that:

$$ (\xi_n(s))_0 \cap \{\text{nodes of } X_s\} = \emptyset \quad (7.0.3) $$

We now consider the special section $\xi = x^n + \varrho^*(\xi_n)$. By the properties of the section $h$, it follows that the spectral curve $Y_h$ defined by $x^n + \varrho^*(h)$ is smooth except for nodal singularities, with exactly $n$-nodes over each node of $X_s$. The smoothness is the consequence of the fact that $h$ has no multiple zeroes and the discriminant of this polynomial is (upto a sign) simply $n^nx^n\varrho^*(h)^{n-1} \in H^0(X_s, \mathcal{L}^{n(n-1)})$.

By the choice of $\xi_n$ as a generalization of $h$ and by the openness of smoothness, the generic spectral curve $Y_{\xi_n(\zeta)} \to X_\zeta$ defined by the section $\xi_n(\zeta)$ is smooth. Thus the spectral surface $\psi_\xi : Y_\xi \to X$ has all
the properties that was claimed. This shows that the set of spectral surfaces with these properties is non-empty. It is clearly open and the result follows. □

We now define the genericity condition to analyze the general Hitchin fibre, which by Lemma 7.1 is non-empty.

**Definition 7.2.** We define the subspace:

\[
\Gamma(A^w_s) := \{ \text{sections } \xi : S \to A_s \mid \xi \text{ is as in Lemma 7.1} \} \tag{7.0.4}
\]

...to be the set of general sections of \(A_s \to S\).

**Remark 7.3.** Let \(\xi : S \to A_s\) be a general section. Then there is a canonically defined spectral fibered surface \(Y_\xi\) over \(S\) together with a covering \(S\)-morphism \(\psi_\xi : Y_\xi \to X\); over the generic point \(\zeta \in S\), \((\psi_\xi)_{K} : (Y_\xi)_K \to X_K\) it is the classical spectral curve which is smooth and irreducible and over the closed fibre \(s \in S\) when for instance the fibre \(X_s = C\) is a nodal curve with a single node the spectral cover \(\psi_u : Y_u \to C\), for \(u = \xi(s)\) is as in Figure 3 where the fibre \(Y_u\) is a vine curve.

**Remark 7.4.** Let \(\xi\) be as above and \(u = \xi(s)\). Let \(\psi_u : Y_u \to C\) be the spectral curve over the closed point \(s \in S\). The direct image \((\psi_u)_*(\mathcal{O}_{Y_u}) = \mathcal{O}_Y \oplus \mathcal{L}^* \ldots \oplus (\mathcal{L}^*)^n\) and the genus \(g_{Y_u}\) of \(Y_u\) is equal to \(n(g - 1) + 1 + deg(\mathcal{L}) \cdot \frac{n(n-1)}{2}\), where \(g\) is the genus of \(C\).

By [10] (or [3, Proposition 3.6]), we have a bijective correspondence between torsion-free sheaves \(\eta\) on \(Y_\xi\) of rank 1, relative degree \(\delta\) and families of stable torsion-free Hitchin pairs \((E, \theta)\) on \(X\), where \(E\) is torsion-free of rank \(n\) and degree \(d = \delta - deg(\mathcal{L}) \cdot \frac{n(n-1)}{2}\) and \(\theta : E \to E \otimes \mathcal{L}\) is a homomorphism with characteristic coefficients \(\xi_i : S \to p_*(\mathcal{L}^i)\). The correspondence is given as follows. Let \(W = \text{Spec}(\text{Sym}(\mathcal{L}^*))\) be the total space of the line bundle \(\mathcal{L}\). Recall the diagram:

\[
\begin{align*}
Y_\xi & \xrightarrow{\subset} W \\
\psi_\xi & \downarrow \pi \\
X & \to
\end{align*}
\tag{7.0.5}
\]
The line bundle \( \pi^*(L) \) has a tautological section \( t \) which induces the canonical map
\[
\eta \xrightarrow{1 \otimes t} \eta \otimes \psi^*(\xi) (7.0.6)
\]
Pushing this down gives the map \( \theta : E \to E \otimes \mathcal{L} \), where \( E := \pi_*(\eta) \). The correspondence sends \( \eta \mapsto (E, \theta) \). The stability of the Hitchin pair \((E, \theta)\) is easily checked.

If \( \xi : S \to A_S \) is any section then we get a spectral surface \( Y_\xi \subset W = \text{Spec}(\text{Sym}(\mathcal{L}^*)) \subset Z \), where \( Z = \mathbb{P}(\mathcal{L}^* \oplus \mathcal{O}_X) \), is the projective completion of the total space \( W \) of \( \mathcal{L} \) as a scheme over \( S \) (see Lemma [2.4]).

**Proposition 7.5.** Let \( \xi : S \to A_S \) be a general section. Then the compactified Picard variety \( P_{s,Y_\xi} \) of spectral fibered surface \( Y_\xi \subset Z \) can be canonically identified with the subscheme of the moduli space \( M(\mathcal{O}_Z, P_k) \) of pure sheaves \( \mathcal{E} \) on \( Z \) such that the scheme theoretic support \( \text{Supp}(\mathcal{E}) = Y_\xi \).

**Proof.** Recall (Lemma [2.4]) that a family \((E, \theta)\) of stable torsion-free Hitchin pairs on \( X \), canonically defines a family of stable pure sheaves \( \mathcal{E} \) on the scheme \( Z \) over \( S \). As \( \xi \) is generic by Remark [7.4], rank 1 torsion-free sheaves of relative degree \( \delta \) on \( Y_\xi \) give points of \( \mathcal{M}^H_S(n, d) \) or equivalently pure sheaves on \( Z \). Again by the genericity of \( \xi \), the scheme \( Y_\xi \) is reduced and irreducible and hence by Remark [6.5] these pure sheaves \( \mathcal{E} \) have scheme theoretic support \( \text{Supp}(\mathcal{E}) = Y_\xi \), i.e the compactified Picard variety \( P_{s,Y_\xi} \) (cf. Caporaso [5]) gets realized as a subscheme of \( M(\mathcal{O}_Z, P_k) \) which parametrizes \( p \)-semistable pure sheaves on \( Z \) with fixed Hilbert polynomial. \( \square \)

Recall that the scheme structure on \( \mathcal{M}^H_S(n, d) \) was realized as an open subscheme of \( M(\mathcal{O}_Z, P_k) \) which parametrizes \( p \)-semistable pure sheaves on \( Z \) with fixed Hilbert polynomial.

Thus by Proposition [7.5] we have an inclusion:
\[
P_{s,Y_\xi} \subset \mathcal{M}^H_S(n, d) (7.0.7)
\]

**Theorem 7.6.** Let \( \xi \) be a general section as above and let the fibre of \( g_s \) over \( \xi \) be denoted by \( g_s^{-1}(\xi) \). Then we have a proper birational morphism of \( S \)-schemes:
\[
g_s^{-1}(\xi) \to P_{s,Y_\xi} (7.0.8)
\]
which is an isomorphism over Spec $K$; more precisely, it coincides over Spec $K$ with the classical Hitchin isomorphism of the Hitchin fibre with the Jacobian of a smooth spectral curve $(Y_\xi)_K$.

Proof. By Proposition 6.8 we see that for every $S$-scheme $T$,

$$\nu_*(g_s^{-1}(\xi))(T) = \{(E, \theta) \in \mathcal{M}_S^\mu(n, d)(T) \mid \text{Supp}(\mathcal{E}) = Y_\xi \times_S T\}$$  \hspace{1cm} (7.0.9)

and by the observation (7.0.7), we get a proper birational surjective morphism:

$$\nu_* : g_s^{-1}(\xi) \to \mathcal{P}_{s,Y_\xi}$$ \hspace{1cm} (7.0.10)\qed

Remark 7.7. By Zariski’s Main theorem, since $\mathcal{P}_{s,Y_\xi}$ is normal ([5]), $\nu_*$ has connected fibres. Since the morphism $\nu_*$ is an isomorphism over the generic fibre, we need to look closely on the phenomenon over the closed fibre i.e the nodal curve $C$.

8. Geometry of the degenerate Hitchin fibre

The aim of this section is to give a description of the geometry of the Hitchin fibre and prove a statement which can be described as a quasi-abelianization of the moduli space of Hitchin pairs.

8.0.10. A review of the compactified Picard variety. We begin by a variation in the description of the compactification of the Picard variety of a stable curve. For the sake of simplicity we work with an irreducible vine curve $Y$ with $n$-nodes (which occurs as our spectral curve) and take a re-look at the compactification of the Picard variety of $Y$.

Recall that since $Y$ is irreducible, there is a natural choice of the compactification. In [1], we find a comparison of various approaches to the compactification, beginning with the one by Oda-Seshadri [17], Caporaso [5] and Simpson [24]. In fact, all three approaches give the same object. Recall that in [17], the compactification is described as a moduli of torsion-free sheaves on the curve $Y$ with fixed slope while in [5], following Gieseker, the description is in terms of embeddings of semistable curves stably equivalent to the curve $Y$.

The description we wish to give here is closer in spirit to the one in [5] and comes from the paper of Nagaraj-Seshadri [15]. Following the approach in [15] (see also [22] page 15) we realize the compactification of $Pic Y$, as Gieseker line bundles on a ladder curve semistably equivalent to $Y$ (see Proposition 8.0). This approach is essential in our
description of the geometry of the Gieseker-Hitchin fibre which reveals more interesting phenomena and a new compactification of the Picard variety (see Theorem 8.17 and Remark 8.18 below for details).

Let \( Y^{(\ell)} \) be the semistable curve obtained from \( Y \) by attaching trees of length \( \ell \) to the normalization of \( Y \) at the points \( a_i, b_i, i = 1, \ldots, n \) over the \( n \)-nodes on \( Y \). Let \( p : Y^{(\ell)} \to Y \) be the morphism contracting the trees \( \mathbb{P}^1(a_i, b_i) \) joining the pairs of points \( a_i, b_i, i = 1, 2, \ldots, n \) (see Figure 4).

**Figure 4. The contraction of \( \mathbb{P}^1 \)'s**

**Definition 8.1.** A quasi-Gieseker line bundle \( \mathcal{N} \) on \( Y^{(\ell)} \) is a line bundle such that on each tree \( \{ \mathbb{P}^1(a_i, b_i) \}_{i=1}^n \), we have the line bundle \( \mathcal{N}|_{\mathbb{P}^1(a_i, b_i)} \) is standard and such that the direct image \( p_* (\mathcal{N}) \) is torsion free on \( Y \).

**Remark 8.2.** The key difference between this definition of a quasi-Gieseker line bundle and a Gieseker vector bundle earlier in Definition 3.6 is that in Definition 3.6 we impose the strict standardness. In the quasi-Gieseker line bundle we allow the line bundle to be trivial on some \( \mathbb{P}^1 \)'s in some trees while in the Gieseker vector bundle definition this is not allowed. Also, the considerations in Definition 3.6 were for a nodal curve with a single node.

**Definition 8.3.** A vertical automorphism \( \sigma : Y^{(\ell)} \to Y^{(\ell)} \) which leaves the end-points \( a_i \) and \( b_i \) fixed is an automorphism:

\[
\begin{array}{c}
Y^{(\ell)} \xrightarrow{\sigma} Y^{(\ell)} \\
\downarrow p \quad \downarrow p \\
Y \quad Y
\end{array}
\]  

(8.0.1)
which acts on the trees $R^{(\ell)}(a_i, b_i)$ leaving the end-points $a_i$ and $b_i$ fixed. The group of vertical automorphisms leaving the end-points of the trees fixed will be denoted by $\text{Aut}_{epf}(Y^{(\ell)})$.

**Remark 8.4.** Take the case when $\ell = 1$. As an automorphism $\lambda$ of a tree $R^{(1)}$ which fixes two points is the multiplicative group $\mathbb{G}_m$, we see that $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{G}_m \times \ldots \times \mathbb{G}_m$, i.e

$$\text{Aut}_{epf}(Y^{(1)}) \simeq \mathbb{G}_m \times \ldots \times \mathbb{G}_m.$$  \hspace{1cm} (8.0.2)

Each component $\lambda_i$ act trivially on $\mathcal{O}_{R^{(1)}}$, while on $\mathcal{O}_{R^{(1)}}(1)$ it acts as multiplication by $\lambda_i$ on the fibre $\mathcal{O}_{R^{(1)}}(1)_{a_i}$ and by $\lambda_i^{-1}$ on $\mathcal{O}_{R^{(1)}}(1)_{b_i}$ for each $i$. In other words, we have a canonical action of such automorphisms on the set of quasi-Gieseker line bundles on the curve $Y^{(1)}$.

Similarly we see that:

$$\text{Aut}_{epf}(Y^{(\ell)}) \simeq \mathbb{G}_m^\ell \times \ldots \times \mathbb{G}_m^\ell.$$  \hspace{1cm} (8.0.3)

In the earlier setting of Definition 3.9, when we work with the curve $C^{(\ell)}$, with a single tree $R^{(\ell)}(p_1, p_2)$ joining $p_1$ and $p_2$, the group of vertical automorphisms is simply $\text{Aut}_{epf}(C^{(\ell)}) \simeq \mathbb{G}_m^\ell$. The equivalence of Gieseker-Hitchin pairs is defined via the orbits of this group (see Figure 5).

![Figure 5. The covering $Y^{(3)} \to C^{(3)}$](image)

**Definition 8.5.** Let

$$\mathcal{G}_Y(1, \delta) := \frac{\text{quasi-Gieseker line bundles on } Y^{(1)} \text{ of deg } \delta}{\text{Aut}_{epf}(Y^{(1)})}.$$  \hspace{1cm} (8.0.4)

the isomorphism classes of quasi-Gieseker line bundles modulo the action of $\text{Aut}_{epf}(Y^{(1)})$ as in Definition 8.3.
Following the strategy of [15] or Section 6 of this paper, one can give a natural scheme structure to the set $G_Y(1, \delta)$ which we will call the moduli space of quasi-Gieseker line bundles on $Y^{(1)}$.

**Proposition 8.6.** The compactified Picard variety $P_{\delta,Y}$ of $Y$ is isomorphic to the moduli space $G_Y(1, \delta)$ of quasi-Gieseker line bundles on $Y^{(1)}$. The isomorphism is induced by the direct image morphism $p_*$.

**Proof.** This result is proven by Pandharipande [18]. In the context of the present paper, the proof can be given as in [15, Theorem 2, page 196] (see also Section 6 above), where this isomorphism is shown more generally for the case when the rank and degree are coprime, except that in [15], the case is when the curve $Y$ has a single node. The generalization to all stable curves has been carried out in [20]. □

**Remark 8.7.** When the number of nodes is strictly bigger than 1, the singularities of the compactified Picard variety is a product of normal crossing singularities and therefore not a normal crossing singularities (cf. [5, Page 595], [22, Page 262, I]). This is erroneously written in [20, Theorem 3.3.1].

**8.0.11. Towards the geometry of the Gieseker-Hitchin fibre.**

**Lemma 8.8.** Let $\xi \in \Gamma(A^w_S)$ as in Definition 7.2. Then, for each $\ell$, the section $\xi$ defines a (spectral) covering surface $\psi^{(\ell)}_\xi : Y^{(\ell)}_\xi \to X^{(\ell)}$, where $Y^{(\ell)}_\xi$ is a (semistable) modification of $Y_\xi$. Over the closed point $s \in S$ the fibre $Y^{(\ell)}_{\xi,s}$ is a curve with $n$ pairs of marked points each pair being joined by a tree $R^{(\ell)}$ and the covering morphism $Y^{(\ell)}_{\xi,s} \to C^{(\ell)}$ as in Figure 5.

**Proof.** By the choice of $\xi$, we have a covering morphism $\psi_\xi : Y_\xi \to X$ which has the good properties given by Lemma 7.1. Now by [9, Corollary 7.15, Chapter II], we have a natural morphism $\psi^{(\ell)}_\xi : Y^{(\ell)}_\xi \to X^{(\ell)}$ and a diagram:

$$
\begin{array}{ccc}
X^{(\ell)} & \xrightarrow{\psi^{(\ell)}_\xi} & Y^{(\ell)}_\xi \\
\downarrow{\nu} & & \downarrow{p} \\
X & \rightarrow & Y_\xi \\
\end{array}
$$

where $p$ and $\nu$ contracts the $R^{(\ell)}$s to the respective stable curves (see Figure 5).
Consider the morphism $\nu : X^{(\ell)} \to X$ and \( L^\ell \) the pull-back $\nu^*(\mathcal{L})$. The generic section $\xi$ as in Lemma 7.1 pulls back to give a section $\xi : S \to \mathbb{G}_{\mathbb{A}}^n \mathcal{P}_{\mathbb{A}}(\mathcal{L}^\ell)$. Let $W_\ell^\xi = \text{Spec}(\text{Sym}(\mathcal{L}^\ell))$ be the total space of the line bundle $\mathcal{L}^\ell$. As in Lemma 7.1 we can take the spectral surface $Y_{\xi^\ell}$ defined by the section $\xi^\ell$ as a subscheme of $W_\ell^\xi$. We have a canonical diagram:

\[
\begin{array}{ccc}
X^{(\ell)} & \xrightarrow{\psi_{\xi^\ell}} & Y_{\xi^\ell} \\
\downarrow \nu & & \downarrow p \\
X & \xleftarrow{\psi_\xi} & Y_\xi
\end{array}
\]

(8.0.6)

By the universal property of blow-ups, it is easy to see that $Y_{\xi^\ell} \cong Y^{(\ell)}_{\xi^\ell}$. The remaining claims in the Lemma are easily established. \qed

Remark 8.9. Let $T$ be a $S$-scheme and let $\sigma : Y^{(\ell)}_{\xi,T} \to Y^{(\ell)}_{\xi,T}$ be an automorphism as follows:

\[
\begin{array}{ccc}
Y^{(\ell)}_{\xi,T} & \xrightarrow{\sigma} & Y^{(\ell)}_{\xi,T} \\
p & & p \\
Y_{\xi,T} & & Y_{\xi,T}
\end{array}
\]

(8.0.7)

In other words, $\sigma$ is essentially given by an automorphism over a point $t \in T$ above the closed point $s \in S$. i.e., the entire information is given by the following diagram:

\[
\begin{array}{ccc}
Y^{(\ell)}_{\xi,t} & \xrightarrow{\sigma_t} & Y^{(\ell)}_{\xi,t} \\
p_t & & p_t \\
Y_{\xi,t} & & Y_{\xi,t}
\end{array}
\]

(8.0.8)

and by Figure 6 and Definition 8.3 above, giving $\sigma_t \in Aut_{epf}(Y^{(\ell)}_{\xi,t})$ is giving a tuple of $n$-automorphisms of the trees $R^{(\ell)}(a_i, b_i)$’s joining the points $a_i$ and $b_i$ leaving the points $a_i$ and $b_i$ fixed. By Remark 8.4 we get a point of $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{G}_m^\ell \times \cdots \times \mathbb{G}_m^\ell$.

Remark 8.10. The morphism $Y^{(\ell)}_{\xi,t} \to X^{(\ell)}_t$ is a covering morphism which is unramified over the $R^{(\ell)}(p_1, p_2)$ (see Figure 6). Hence the automorphism $\lambda$ of the $R^{(\ell)}(p_1, p_2)$ lifts to an automorphism of $Y^{(\ell)}_{\xi,t}$ as a diagonal element $\lambda := (\lambda, \lambda, \ldots, \lambda)$ in $\mathbb{G}_m^\ell \times \cdots \times \mathbb{G}_m^\ell$. 

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8.0.12. The stratification of the Gieseker-Hitchin spaces. We work over the nodal curve $C$. Recall that the Gieseker-Hitchin space $G^H_C(n,d)$ was constituted of Gieseker-Hitchin pairs $(V,\phi)$ on various curve $C^{(\ell)}$ which were modifications of the nodal curve $C$. This can be expressed as giving the following stratification of the moduli spaces:

$$G^H_C(n,d) = \bigsqcup G^H_C(n,d)_{(\ell)}$$

(8.0.9)

where

$$G^H_C(n,d)_{(\ell)} := \{(V,\phi) \mid V \text{ a Gieseker bundle on } C^{(\ell)} \} / \text{Aut}_{\text{epf}}(C^{(\ell)})$$

(8.0.10)

(see Remark 8.4).

This stratification of the total space induces a stratification of the Gieseker-Hitchin fibre $g^{-1}_C(\xi_s)$ given as follows:

$$g^{-1}_C(\xi_s) = \bigsqcup g^{-1}_C(\xi_s)_{(\ell)}$$

(8.0.11)

where

$$g^{-1}_C(\xi_s)_{(\ell)} := g^{-1}_C(\xi_s) \cap G^H_C(n,d)_{(\ell)}$$

(8.0.12)

Let $\mathcal{N}$ be a a quasi-Gieseker line bundle on $Y^{(\ell)}_{\xi_s}$ in the sense of Definition 8.1. For simplicity of notation, let us denote the morphism $\psi^{(\ell)}_{\xi_s} : Y^{(\ell)}_{\xi_s} \to C^{(\ell)}$ for each $\ell$ simply by $\psi^{(\ell)}$. 

Figure 6. The spectral picture
Definition 8.11. Let \( G^{(\ell)}(1, \delta) \) denote the set of isomorphism classes of quasi-Gieseker line bundles with degree \( \delta \) on the curve \( Y^{(\ell)}_{\xi,s} \).

Remark 8.12. Note that we are not quotienting out by any equivalence of the automorphism action.

Proposition 8.13. If \( \mathcal{N} \in G^{(\ell)}(1, \delta) \) is a quasi-Gieseker line bundle on \( Y^{(\ell)}_{\xi,s} \), then, \( \psi^{(\ell)}_*(\mathcal{N}) \) is a stable Gieseker vector bundle on \( C^{(\ell)}_\xi \) equipped with a Higgs structure \( \phi \) making \((\psi^{(\ell)}_*(\mathcal{N}), \phi)\) a stable Gieseker-Hitchin pair. Moreover, this induces an identification:

\[
g_C^{-1}(\xi) \simeq G^{(\ell)}(1, \delta)/\Delta,
\]

where \( \Delta \) is the induced diagonal action of \( G^{(\ell)}_m \) (see Remark 8.10).

Proof. As in Remark 7.4, we see that in the spectral situation such as \( \psi^{(\ell)} : Y^{(\ell)}_{\xi,s} \to C^{(\ell)}_\xi \) we have a corresponding diagram:

\[
\begin{array}{ccc}
Y^{(\ell)}_{\xi,s} & \xrightarrow{\subset} & W^{(\ell)}_\xi \\
\psi^{(\ell)}_\xi & \downarrow & \downarrow f^{(\ell)}_\xi \\
C^{(\ell)}_\xi & \xrightarrow{} & \end{array}
\]

and it follows that if \( \mathcal{N} \) is any quasi-Gieseker line bundle on \( Y^{(\ell)}_{\xi,s} \), then, \( V = (\psi^{(\ell)}_\xi)_*(\mathcal{N}) \) is a vector bundle on \( C^{(\ell)}_\xi \) canonically equipped with a Higgs structure \( \psi : V \to V \otimes \mathcal{L}_\xi \) since it arises from a spectral construction.

The commutativity of the diagram (8.0.5), gives the identification

\[(\psi^{(\ell)}_\xi)_*(\mathcal{N}) = (\nu^{(\ell)}_\xi)_* p_*(\mathcal{N}) = (\nu^{(\ell)}_\xi)_* V \]

from which we conclude that \( \nu^{(\ell)}_\xi \) is the underlying torsion-free sheaf of a stable torsion-free Hitchin pair on \( C \). Hence by the definition of a stable Gieseker-Hitchin pair, it follows that \( (V, \psi) \) obtained above is a stable Gieseker-Hitchin pair.

The action of \( \text{Aut}_{epf}(C^{(\ell)}_\xi) = G^{(\ell)}_m \) to determine the open startum of the Gieseker-Hitchin fibre (see (8.0.10)) lifts to give the diagonal action on \( G^{(\ell)}_m(1, \delta) \), and we get the identification. \( \Box \)

The compactified Picard variety \( P_{\delta,Y^{(\ell)}_{\xi,s}} \) of \( Y^{(\ell)}_{\xi,s} \) (which is an irreducible vine curve has \( n \)-nodes) also has a stratification in terms of the complexity of the torsion-freeness of the sheaves. This can be given as
follows:

\[ P_{δ,Yξ,s} = \bigsqcup P_{δ,Yξ,s}(j) \]  

(8.0.16)

where

\[ P_{δ,Yξ,s}(j) := \{ η \mid η \text{ is non-free at exactly } j \text{ nodes} \} \]  

(8.0.17)

In this description \( P_{δ,Yξ,s}(0) \) corresponds to the open subset of line bundles on \( Y_{ξ,s} \) of degree \( δ \).

8.0.13. The big stratum. Recall that we have a proper birational morphism \( ν : g_-(ξ_s) → P_{δ,Yξ,s} \).

Proposition 8.14. Let \( η ∈ P_{δ,Yξ,s}(n) \), in the worst stratum, i.e \( η \) is given by the maximal ideal sheaf on each of the \( n \)-nodes. The part of the fibre \( ν_-(η) \) in the open stratum \( g_-(ξ_s) \) can be described as follows:

\[ g_-(ξ_s) \cap ν_-(η) ≃ n \bigg( \prod_{m} G_m \times \ldots × G_m \bigg) / \Delta(G_m) \]  

(8.0.18)

Proof. By Proposition 8.13 we have an identification

\[ g_-(ξ_s) ≃ G^{(1)}(1, δ) / \Delta(G_m) \]  

(8.0.19)

and by Proposition 8.6 we see that

\[ P_{δ,Yξ,s}(n) ≃ G^{(1)}(1, δ) / \bigg( \prod_{m} G_m \times \ldots × G_m \bigg) \]  

(8.0.20)

Thus we get the required identification (8.0.18) of the big stratum of the general Gieseker-Hitchin fibre. □

Remark 8.15. A similar description clearly holds for the other strata as well.

Remark 8.16. Proposition 8.14 should be viewed in the light of the following remarks. Let \( E \) be a torsion-free \( O_C \)-module such that the local structure at the node on \( C \) is of type \( m^r \). Then by [22, Remark 5.2], the fibre \( ν_-(E) \) can be identified with the so-called wonderful compactification of \( PGL(n) \).

Consider \( P_{δ,Yξ,s} \) the compactified Picard variety of the spectral curve \( Y_{ξ,s} \) over the closed fibre \( C \). We view \( P_{δ,Yξ,s} \) as a subscheme of \( M_C^H(1, δ) \) of torsion-free Hitchin pairs on \( C \). Under this identification, a point
\( \eta \in P_{\delta, \nu_{\xi,s}}(n) \) gives a torsion-free Hitchin pair \((E, \theta)\), such that the local structure at the node on \( C \) is of type \( m^n \).

Proposition 8.13 shows that the inclusion

\[
\mathfrak{g}_c^{-1}(\xi) \cap \nu_{-1}(\eta) \subset \nu_{-1}(E)
\]

is in fact the inclusion

\[
\begin{array}{c}
\mathfrak{g}_m \times \ldots \times \mathfrak{g}_m \\
\Delta(\mathfrak{g}_m)
\end{array} \subset PGL(n)
\]

i.e the standard inclusion of the maximal torus of \( PGL(n) \).

Let \( \tilde{g} = g(n-1) + 1 + \deg(\mathcal{L}) \cdot \frac{n(n-1)}{2} \) be the arithmetic genus of the spectral vine curve \( Y_{\xi,s} \) (see Remark 7.4).

By Corollary 5.15, the structure morphism Gieseker-Hitchin scheme \( \mathcal{G}_s^H(n, d) \rightarrow S \) is flat and the closed fibre \( \mathcal{G}_c^H(n, d) \) has analytic normal crossing singularities. Because we have shown the Gieseker-Hitchin morphism is proper and because we are in characteristic zero, the general fibre also has analytic normal crossing singularities.

In summary we have proven the following main theorem.

**Theorem 8.17.** (Quasi-abelianization) Consider the restriction of the proper birational morphism \( \nu_* : \mathfrak{g}_c^{-1}(\xi) \rightarrow P_{\delta, \nu_{\xi,s}} \).

1. Let \( \eta \in P_{\delta, \nu_{\xi,s}}(j) \). The fibre \( \nu_{-1}(\eta) \) can be identified with the projective toric variety \( \overline{T}_j \), which is the closure of the maximal torus \( T_j \subset PGL(j) \) in the wonderful compactification \( \overline{PGL(j)} \). This is in fact the toric variety associated to the Weyl chamber of \( PGL(j) \) \( [19] \).
2. The scheme \( \mathfrak{g}_s^{-1}(\xi) \) provides a relative compactification of the Jacobian of smooth (spectral) curves of genus \( \tilde{g} \) which has a divisor \( \mathfrak{g}_c^{-1}(\xi) \) with analytic normal crossing singularities.

**Remark 8.18.** The last statement poses the interesting general problem of giving a modular construction of a compactified Picard variety for stable curves which has analytic normal crossing singularities. For the case of a vine curve with \( n \)-nodes \( n \geq 2 \), one needs to consider quasi-Gieseker line bundles on curves \( \{Y^{(t)}\}_{t=1}^n \), unlike the Caporaso compactification which requires only quasi-Gieseker line bundles on the ladder curve alone.
9. The reducible curve case

The above theory goes through in the case when the closed fibre of \( X \to S \) is a reducible curve \( C = C_1 \cup C_2 \) with a single node at a point \( p \in C_1 \cap C_2 \), where \( C_1 \) and \( C_2 \) are two smooth curves over an algebraically closed field \( k \) of genus \( g_1 \) and \( g_2 \) respectively.

If \( C \) is irreducible, and \( \mathcal{L} \) be an invertible sheaf over \( C \) then \( \mathcal{L} \) is obtained by giving line bundles \( \mathcal{L}_i \) on \( C_i \) together with a gluing isomorphism \( \ell : \mathcal{L}_{1,p} \simeq \mathcal{L}_{2,p} \).

A polarization on the reducible nodal curve \( C \) can be thought of as giving a pair \( a = (a_1, a_2) \) with \( a_i > 0 \) positive rational numbers with \( a_1 + a_2 = 1 \). Let \( \mathcal{L} \) be an ample invertible sheaf on \( C \); this in turn gives a pair of ample invertible sheaves \( \mathcal{L}_i \) on \( C_i \). Equivalently, we say that \( \mathcal{L} \) gives a polarization on \( C \) if in terms of \( \mathcal{L}_i \) on \( C_i \), one has \( \frac{\deg(\mathcal{L}_1)}{\deg(\mathcal{L}_2)} = \frac{a_1}{a_2} \). To ensure that under the assumption \( \gcd(n, d) = 1 \) we have the condition semistable = stable for the Hitchin pairs, we need to impose a genericity condition, namely we assume that \( a_1 \chi \notin \mathbb{Z} \).

Under these hypotheses, we will be dealing only with stable objects in this paper. Note that if the curve \( C \) is irreducible, only the condition \( \gcd(n, d) = 1 \) would do since no polarization figures in the definition of stability. We will make these assumptions in this section.

The sheaf \( E \) is of rank \( (n_1, n_2) \), if \( \text{rank}(E_i) = n_i \), where \( E_i := E|_{C_i} \). Say \( E \) is of rank \( n \) if \( n = n_1 = n_2 \). Note that for a torsion-free \( \mathcal{O}_C \)-module, at least one of the \( n_i \neq 0 \).

For a torsion-free sheaf \( E \) on \( C \) and the polarization \( a \), define the \( a \)-rank and \( a \)-slope of \( E \) as follows:

\[
\text{rk}_a(E) := a_1 \cdot \text{rk}(E_1) + a_2 \cdot \text{rk}(E_2) \quad (9.0.1)
\]

\[
\mu_a(E) := \frac{\chi(E)}{\text{rk}_a(E)}, \text{if } \text{rk}_a(E) \neq 0 \quad (9.0.2)
\]

Since \( \dim(C) = 1 \) for us, we see immediately that

\[
\frac{p(E, m)}{\text{rk}_a(E)} = m \cdot \deg(\mathcal{L}) + \mu_a(E). \quad (9.0.3)
\]

where \( p(E, m) := \chi(E \otimes \mathcal{L}^m) \).

Almost all of the general theory developed above works without change for this case also and the proofs are really no different. The only new feature which emerges is that the choice of a polarization is needed to define the notion of stability of Hitchin pairs as we saw in
Section 2. This naturally leads to a suitable notion of stability of the pure sheaves on the \( Z = \mathbb{P}(L^* \oplus \mathcal{O}_C) \). The interesting new feature is that in the description of the Hitchin fibre the choice of polarization enters the definition of the compactified Picard variety when seen from the stand-point of Oda-Seshadri. The generic spectral curve is again a vine curve with \( n \)-nodes and two irreducible components.

As one knows, Caporaso’s construction of the compactification of the Picard variety does not need any polarization; this can be explained by showing that the Oda-Seshadri compactification for a generic polarization is isomorphic to Caporaso’s compactification (see [1]). We summarize the results in the reducible curve case in the following theorem whose proof we omit since it is similar to the proof of previous theorem:

**Theorem 9.1.** The Gieseker-Hitchin moduli space \( \mathcal{G}_S^H(n,d) \) is well-defined quasi-projective scheme, flat over \( S \) with generic fibre isomorphic to the classical Hitchin space. We have a Hitchin map \( g_s : \mathcal{G}_S^H(n,d) \to \mathcal{A}_S \) which is proper. There is a proper birational morphism from the Hitchin fibre over a general section \( \sigma : S \to \mathcal{A}_S \) to the compactified relative Picard \( S \)-scheme \( P_{s,Y_{\sigma}} \) of the spectral surface \( Y_{\sigma} \).

Remark 9.2. The spectral surface \( Y_{\sigma} \) is a fibered surface over \( S \) with smooth generic fibre and the closed fibre is a reducible vine curve with two components and \( n \)-nodes.

Remark 9.3. We remark that in the special case when \( n = 2 \) and \( d = 1 \) and \( C \) is reducible with two components, in contrast to the general phenomenon discussed in the paper, it so happens that there is an isomorphism

\[
\mathcal{G}_S^H(2,1) \simeq \mathcal{M}_S^H(2,1) \tag{9.0.4}
\]

in this special situation, which is somewhat misleading since in this case the Hitchin map is well-defined even on the moduli of torsion-free Hitchin pairs. The choice of a polarization allows the discarding of torsion-free sheaves which locally look like \( m \oplus m \) and this is the reason for the isomorphism (9.0.4) in the rank 2 case. Details of this will appear in the second author’s doctoral thesis ([2]).

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