REALIZATIONS OF SIMPLE AFFINE VERTEX ALGEBRAS AND THEIR MODULES: THE CASES $sl(2)$ AND $osp(1, 2)$.

DRAŽEN ADAMOVIĆ

Abstract. We study the embeddings of the simple admissible affine vertex algebras $V_k(sl(2))$ and $V_k(osp(1, 2))$, $k \in \mathbb{Z}_{>0}$, into the tensor product of rational Virasoro and $N = 1$ Neveu-Schwarz vertex algebra with lattice vertex algebras. By using these realizations we construct a family of weight, logarithmic and Whittaker $sl(2)$ and $osp(1, 2)$–modules. As an application, we construct all irreducible degenerate Whittaker modules for $V_k(sl(2))$.

1. Introduction

Let $V^k(g)$ denotes the universal affine vertex algebra of level $k$ associated to a simple finite-dimensional Lie super algebra $g$. Let $J^k(g)$ be the maximal ideal in $V^k(g)$ and $V_k(g) = V^k(g)/J^k(g)$ its simple quotient. The representation theory of $V_k(g)$ depends on the structure of the maximal ideal $J^k(g)$. One sees that a $V^k(g)$–module $M$ is a module for the simple vertex algebra $V_k(g)$ if and only if $J^k(g).M = 0$. Such approach can be applied for a construction and classification of modules in the category $O$ and in the category of weight modules (cf. [10], [21], [22], [26], [46], [48]). But it seems that for a construction of logarithmic, indecomposable and Whittaker modules one needs different methods.

In this paper we explore the possibility that a simple affine vertex algebra can be realized as a vertex subalgebra of the tensor product:

$$V_k(g) \subset W(g) \otimes \Pi_{\tilde{g}}(0)$$

where $W(g)$ is a $W$-algebra associated to $g$ and $\Pi_{\tilde{g}}(0)$ is a lattice type vertex algebra. This can be treated as an inverse of the quantum Hamiltonian reduction (cf. [51]).

In this moment we can not prove that such inclusion exists in general, but we present a proof of (1) in the cases $g = sl(2)$ and $g = osp(1, 2)$. Let us describe our results in more details. Let $V^{Vir}(d_{p,p'}, 0)$ and $V^{ns}(c_{p,q}, 0)$ denote the universal Virasoro and $N=1$ Neveu-Schwarz vertex algebras with central charges: $d_{p,p'} = 1 - \frac{6(p-p')^2}{pp'}$ and $c_{p,q} = 3/2 - \frac{3(p-q)^2}{pq}$. Their simple quotients are denoted by $L^{Vir}(d_{p,p'}, 0)$ and $L^{ns}(c_{p,q}, 0)$. Let $\Pi(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c]$ and $\Pi^{1/2}(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c]$.
Let $F$ be the fermionic vertex algebra of central charge $c = \frac{1}{2}$ associated to a neutral fermion field.

We prove:

**Theorem 1.1.** There are non-trivial homomorphisms of simple vertex algebras at non-critical levels:

1. $\Phi_1 : V_k(sl(2)) \to L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)$ where $k + 2 = \frac{p}{p'}$ such that $p, p' \geq 2$, $(p, p') = 1$,

2. $\Phi_2 : V_k(osp(1, 2)) \to L^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)$, where $k + \frac{3}{2} = \frac{p}{q}$, such that $p, q \in \mathbb{Z}$, $p, q \geq 2$, $(\frac{p-2}{q}, q) = 1$.

Let us discuss some application of previous theorem in the case $V_k(sl(2))$:

- We show in Section 5 that all relaxed highest weight modules for the admissible vertex algebra $V_k(sl(2))$ have the form

  $$L^{Vir}(d_{p,p'}, h) \otimes \Pi_{-1}(\lambda)$$

  where $L^{Vir}(d_{p,p'}, h)$ is an irreducible $L^{Vir}(d_{p,p'}, 0)$–module and $\Pi_{-1}(\lambda)$ is a weight $\Pi(0)$–module. These modules were first detected in [10] by using the theory of Zhu’s algebras. We also show that the character of $L^{Vir}(d_{p,p'}, h) \otimes \Pi_{-1}(\lambda)$ coincides with the Creutzig-Ridout character formula presented in [26]. We should also say that a similar realization of irreducible relaxed highest weight modules were presented in [5, Section 9] in the case of critical level for $A_1^{(1)}$ and in [6, Corollary 7] in the case of affine Lie algebra $A_2^{(1)}$ at level $k = -3/2$.

- We prove in Section 6 that a family of degenerate Whittaker modules for $V_k(sl(2))$ have the form

  $$L^{Vir}(d_{p,p'}, h) \otimes \Pi_{\lambda}$$

  where $\Pi_{\lambda}$ is a Whittaker $\Pi(0)$–modules. This result is the final step in the classification and realization of Whittaker $A_1^{(1)}$–modules (all other Whittaker $A_1^{(1)}$–modules were realized in [9]). But our present result implies that affine admissible vertex algebra $V_k(sl(2))$ admits a family of Whittaker modules. On can expect a similar result in general.

- We present a vertex-algebraic construction of logarithmic modules by using the methods from [13] and the expressions for screening operators from [32, Section 5]. We prove that the admissible vertex algebra $V_k(sl(2))$, for arbitrary admissible $k \notin \mathbb{Z}_{\geq 0}$, admits logarithmic modules. The logarithmic modules were previously constructed only for levels $k = -1/2$ and $k = -4/3$ (cf. [42], [45], [13], [35]).

- We present in Section 8 a realization of the simple affine vertex algebra $W_k'(spo(2, 3), f_0)$ with central charge $c = -3/2$. It is realized on the tensor product of the simple supertriplet vertex algebra $SW(1)$ (introduced by the author and A. Milas in [11]) and a rank
one lattice vertex algebra. As a consequence we give a direct proof that the parafermion vertex algebra \( K(sl(2), -\frac{3}{2}) \) is a \( \mathbb{Z}_2 \)-orbifold of a super-singlet vertex algebra, also introduced in [11]. We should mention that a different approach based on the extension theory was recently presented in [23].

Some applications in the case \( V_k(osp(1, 2)) \) will be presented in our forthcoming paper [8]. Let us note here that we have the following realization at the critical level. We introduce a vertex algebra \( V_{ns} \) which is freely generated by \( G^{\text{crit}} \) and \( T \), such that \( T \) is central and the following \( \lambda \)-bracket relation holds:

\[
[G^{\text{crit}}_\lambda G^{\text{crit}}] = 2T - \lambda^2.
\]

We prove:

**Theorem 1.2.** Let \( k = -3/2 \). There is non-trivial homomorphism of vertex algebras:

\[
\Phi : V^k(osp(1, 2)) \to V_{ns}^{\text{crit}} \otimes F \otimes \Pi^{1/2}(0),
\]

such that \( T \) is a central element of \( V^k(osp(1, 2)) \).

In our forthcoming papers we plan to investigate a higher rank generalizations of the result discussed above.

**Acknowledgment.** We would like to thank T. Creutzig, A. Milas, G. Radobolja and D. Ridout on valuable discussions. The author is partially supported by the Croatian Science Foundation under the project 2634 and by the QuantiXLie Centre of Excellence, a project cofinanced by the Croatian Government and European Union through the European Regional Development Fund - the Competitiveness and Cohesion Operational Programme (KK.01.1.1.01).

2. **Wakimoto modules for \( \widehat{sl(2)} \)**

In this section we first recall the construction of the Wakimoto modules for \( \widehat{sl(2)} \) (cf. [28], [53]). Then by using the embedding of the Weyl vertex algebra into a lattice vertex algebra (also called FMS bosonization) we show that the universal affine vertex algebra \( V^k(sl(2)) \) can be embedded into the tensor product of a Virasoro vertex algebra with a vertex algebra \( \Pi(0) \) of a lattice type. This result is stated in Proposition 2.1 which is a vertex-algebraic interpretation of the result of A. M. Semikhatov from [50].

2.1. **Weyl vertex algebra** \( W \). Recall that the Weyl algebra \( Weyl \) is an associative algebra with generators \( a(n), a^*(n) \ (n \in \mathbb{Z}) \) and relations

\[
[a(n), a^*(m)] = \delta_{n+m,0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0 \quad (n, m \in \mathbb{Z}).
\]

Let \( W \) denotes the simple Weyl--module generated by the cyclic vector \( 1 \) such that

\[
a(n)1 = a^*(n+1)1 = 0 \quad (n \geq 0).
\]
As a vector space \( W \cong \mathbb{C}[a(-n), a^*(-m) \mid n > 0, m \geq 0] \). There is a unique vertex algebra \((W, Y, 1)\) where the vertex operator map is \( Y : W \to \text{End}(W)[[z, z^{-1}]] \) such that

\[
Y(a(-1)1, z) = a(z), \quad Y(a^*(0)1, z) = a^*(z),
\]

\[
a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n)z^{-n}.
\]

2.2. The Heisenberg vertex algebra \( M_\delta (\kappa, 0) \). Let \( \mathfrak{h} = \mathbb{C}\delta \) be the 1–dimensional commutative Lie algebra with a symmetric bilinear form defined by \((\delta, \delta) = 1\), and \( \hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c \) be its affinization. Set \( \delta(n) = \delta \otimes t^n \). Let \( M_\delta (\kappa, 0) \) denotes the simple \( \hat{\mathfrak{h}} \)–module of level \( \kappa \neq 0 \) generated by the vector \( 1 \) such that \( \delta(n)1 = 0 \) \( \forall n \geq 0 \). As a vector space \( M_\delta (\kappa, 0) = \mathbb{C}[\delta(n) \mid n \leq -1] \).

There is a unique vertex algebra \((M_\delta (\kappa, 0), Y, 1)\) generated by the Heisenberg field \( Y(\delta(-1)1, z) = \delta(z) = \sum_{n \in \mathbb{Z}} \delta(n)z^{-n-1} \) such that

\[
[\delta(n), \delta(m)] = \kappa n \delta_{n+m, 0} \quad (n, m \in \mathbb{Z}).
\]

Vector \( \omega = (\frac{1}{2\kappa} \delta(-1)^2 + a\delta(-2))1 \) is a conformal vector of central charge \( 1 - \frac{12\kappa^2}{\pi^2} \).

For \( r \in \mathbb{C} \), let \( M_\delta (\kappa, r) \) denotes the irreducible \( M_\delta (\kappa, 0) \)–module generated by the highest weight vector \( v_r \) such that

\[
\delta(n)v_r = r \delta_{n,0}v_r \quad (n \geq 0).
\]

We can consider lattice \( D_r = \mathbb{Z}(\frac{\delta}{\kappa}) \) and the generalized lattice vertex algebra \( V_{D_r} := M_\delta (\kappa, 0) \otimes \mathbb{C}[D_r] \) (cf. [27]). We have:

\[
M_\delta (\kappa, r) = M_\delta (\kappa, 0).e^{\frac{r\delta}{\kappa}}.
\]

Then the restriction of the vertex operator \( Y(e^{\frac{r\delta}{\kappa}}, z) \) on \( M_\delta (\kappa, 0) \) can be considered as a map \( M_\delta (\kappa, 0) \to M_\delta (\kappa, r)[[z, z^{-1}]] \).

2.3. The Wakimoto module \( W_{k, \mu} \). Assume that \( k \neq -2 \) and \( \mu \in \mathbb{C} \). Let

\[
W_{k, \mu} = W \otimes M_\delta (2(k + 2), \mu).
\]

Then \( W_{k, 0} \) has the structure of a vertex algebra and \( W_{k, \mu} \) is a \( W_{k, 0} \)–module.

Let \( V^k(sl(2)) \) be the universal vertex algebra of level \( k \) associated to the affine Lie algebra \( \widehat{sl(2)} \). There is a injective homomorphism of vertex algebras \( \Phi : V^k(sl(2)) \to W_{k, 0} \) generated by

\[
e(z) = a(z);
\]

\[
h(z) = -2 : a^*(z)a(z) : +\delta(z);
\]

\[
f(z) = - : a^*(z)^2a(z) : +kd_2a^*(z) + a^*(z)\delta(z).
\]

The screening operator is \( Q = \text{Res}_z : a(z)Y(e^{-\frac{1}{k+2}\delta}, z) := (a(-1)e^{-\frac{1}{k+2}\delta})_0 \) (cf. [28]).
2.4. Bosonization. Let $H$ be the lattice

$$H = \mathbb{Z}\alpha + \mathbb{Z}\beta, \quad \langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \quad \langle \alpha, \beta \rangle = 0,$$

and $V_H = M_{\alpha, \beta}(1) \otimes \mathbb{C}[L]$ the associated lattice vertex algebra, where $M_{\alpha, \beta}(1)$ denotes the Heisenberg vertex algebra generated by $\alpha$ and $\beta$.

The Weyl vertex algebra $W$ can be realized as a subalgebra of $V_H$ generated by

$$a = e^{\alpha+\beta}, \quad a^* = -\alpha(-1)e^{-\alpha-\beta}.$$

This gives a realization of the universal affine vertex algebra $V_k(sl(2))$ as a subalgebra of $V_H \otimes M_{\delta}(2(k+2), 0)$ generated by

$$e = e^{\alpha+\beta}, \quad h = -2\beta(-1) + \delta(-1), \quad f = [(k+1)(\alpha(-1)^2 - \alpha(-2)) + (k+2)\alpha(-1)\beta(-1) - \alpha(-1)\delta(-1)] e^{-\alpha-\beta}.$$

Screening operators are (cf. [28, Section 7]):

$$Q = \text{Res}_Z(Y(e^{\alpha+\beta-\frac{1}{k+2}\delta}, z)), \quad \tilde{Q} = \text{Res}_Z(Y(e^{-(k+2)(\alpha+\beta)+\delta}, z)).$$

2.5. Embedding of $V^k(sl(2))$ into vertex algebra $V^{Vir}(d_k, 0) \otimes \Pi(0)$. We shall first define new generators of the Heisenberg vertex algebra $M_{\alpha, \beta}(1) \otimes M_{\delta}(2(k+2))$. Let

$$\gamma = \alpha + \beta - \frac{1}{k+2}\delta, \quad \mu = -\beta + \frac{1}{2}\delta, \quad \nu = -\frac{k}{2}\alpha - \frac{k+2}{2}\beta + \frac{1}{2}\delta.$$

Then

$$\langle \gamma, \gamma \rangle = \frac{2}{k+2}, \quad \langle \mu, \mu \rangle = -\langle \nu, \nu \rangle = \frac{k}{2},$$

and all other products are zero. For our calculation, it is useful to notice that

$$\alpha = \nu + \frac{k+2}{2}\gamma, \quad \beta = -\frac{k+2}{2}\gamma + \frac{2}{k}\mu - \frac{k+2}{k}\nu, \quad \delta = -(k+2)\gamma + \frac{2(k+2)}{k}\mu - \frac{2(k+2)}{k}\nu.$$

Let $M(1) := M_{\mu, \nu}(1)$ be the Heisenberg vertex algebra generated by $\mu$ and $\nu$. Consider the rank one lattice $\mathbb{Z}c \subset M(1)$ where $c = \frac{2}{k}(\mu - \nu)$. Then

$$\Pi(0) := M(1) \otimes \mathbb{C}[\mathbb{Z}c]$$

has the structure of a vertex algebra. Some properties of $\Pi(0)$ will be discussed in Section 3.
Let $M_{\gamma}(\frac{2}{k+2})$ be the Heisenberg vertex algebra generated by $\gamma$. We obtain the following expression for the generators of $V^k(sl(2))$:

(7) $e = e_{\frac{2}{k+2}}^{2}(\mu-\nu)$,
(8) $h = 2\mu(-1)$
(9) $f = \left[\frac{1}{4}(k+2)^2\gamma(-1)^2 - \frac{1}{2}(k+1)(k+2)\gamma(-2) - (k+1)\nu(-2)\right] e^{-\frac{2}{k+2}(\mu-\nu)}$.

Set

$$\omega^{(k)} = \left(\frac{k+2}{4}\gamma(-1)^2 - \frac{k+1}{2}\gamma(-2)\right) 1.$$ 

Then

$$f = \left[(k+2)\omega^{(k)} - \nu(-1)^2 - (k+1)\nu(-2)\right] e^{-\frac{2}{k+2}(\mu-\nu)}.$$ 

Note that $\omega^{(k)}$ generates the universal Virasoro vertex algebra $V^{Vir}(d_k,0)$ where $d_k = 1 - 6\frac{(k+1)^2}{(k+2)}$, which is realized as a subalgebra of the Heisenberg vertex algebra $M_{\gamma}(\frac{2}{k+2},0)$.

As usual we set $L(n) = \omega_{n+1}$ and denote the Virasoro field by $L(z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$.

We get the following result:

**Proposition 2.1.** [50] Let $\omega$ be the conformal vector in $V^{Vir}(d_k,0)$. There is an injective homomorphism of vertex algebras

$$\Phi : V^k(sl(2)) \rightarrow V^{Vir}(d_k,0) \otimes \Pi(0) \subset M_{\gamma}(\frac{2}{k+2},0) \otimes \Pi(0)$$

such that

(10) $e \mapsto e_{\frac{2}{k+2}}^{2}(\mu-\nu)$,
(11) $h \mapsto 2\mu(-1)$,
(12) $f \mapsto \left[(k+2)\omega - \nu(-1)^2 - (k+1)\nu(-2)\right] e^{-\frac{2}{k+2}(\mu-\nu)}$.

**Remark 1.** The realization in Proposition 2.1 had first obtained by A. M. Semikhatov in [50] using slightly different notations.

A critical level version of this proposition was obtained in [27]. Let $M_T(0) = \mathbb{C}[T(-n), n \geq 2]$ be the commutative vertex algebra generated by the commutative field

$$T(z) = \sum_{n\leq-2} T(n)z^{-n-2}.$$ 

**Proposition 2.2.** Let $k = -2$. There is an injective homomorphism of vertex algebras

$$\Phi : V^k(sl(2)) \rightarrow M_T(0) \otimes \Pi(0)$$
such that

\begin{align*}
(13) & \quad e \mapsto e^{2k(\mu-\nu)}, \\
(14) & \quad h \mapsto 2\mu(-1), \\
(15) & \quad f \mapsto [T(-2) - \nu(-1)^2 - (k + 1)\nu(-2)] e^{-\frac{2}{k}(\mu-\nu)} \tag{13}
\end{align*}

3. SOME $\Pi(0)$–MODULES.

We consider the integral lattice $L = \mathbb{Z}c + \mathbb{Z}d \subset M(1)$, where $c = \alpha + \beta = \frac{2}{k}(\mu - \nu)$, and $d = \mu + \nu$. Then

$$\langle c, c \rangle = (d, d) = 0, \quad \langle c, d \rangle = 2.$$ 

The vertex algebra $\Pi(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c]$ is generated by $c(-1), d(-1), u = e^c, u^{-1} = e^{-c}$. Its representation theory was studied in [24]. Let us recall some steps in the construction of $\Pi(0)$–modules. Let $A$ be the associative algebra generated by $d, e^{nc}$, where $n \in \mathbb{Z}$ and relations

$$[d, e^{nc}] = 2ne^{nc}, \quad e^{nc}e^{mc} = e^{(n+m)c}, \quad (n, m \in \mathbb{Z}).$$

(We use the convention $e^{0} = 1$). By using results from [24, Section 4] we see that for any $A$–module $U$ and any $r \in \mathbb{Z}$, there exists a unique $\Pi(0)$–module structure on the vector space

$$\mathcal{L}_{r}(U) = U \otimes M(1)$$

such that $c(0) \equiv r \text{Id}$ on $\mathcal{L}_{r}(U)$. Moreover $\mathcal{L}_{r}(U)$ is irreducible $\Pi(0)$–module if and only if $U$ is irreducible $A$–module. By using this method one can construct the weight $\Pi(0)$–modules from [24] a Whittaker $\Pi(0)$–modules from [9] (see Proposition 3.1 below).

In the present paper we shall need the following simple current extension of $\Pi(0)$:

$$\Pi^{1/2}(0) = M(1) \otimes \mathbb{C}[\mathbb{Z}c^{\frac{1}{2}}] = \Pi(0) \oplus \Pi(0)e^{\frac{\pi i}{2}}.$$ 

$\Pi^{1/2}(0)$ is again the vertex algebra of the same type and it is generated by $c(-1), d(-1), u^{1/2} = e^{c/2}, u^{-1/2} = e^{-c/2}$. Note that $g = \exp[\pi i \mu(0)]$ is an automorphism of order two of the vertex algebra $\Pi^{1/2}(0)$ and that $g = \text{Id}$ on $\Pi(0)$.

In order to construct $\Pi^{1/2}(0)$ we need to consider slightly larger associative algebra. Let $A^{1/2}$ be the associative algebra generated by $d, e^{nc}$, $n \in \frac{1}{2}\mathbb{Z}$, and relations

$$[d, e^{nc}] = 2ne^{nc}, \quad e^{nc}e^{mc} = e^{(n+m)c}, \quad (n, m \in \frac{1}{2}\mathbb{Z}).$$

For any $A^{1/2}$–module $U'$ and any $r \in \mathbb{Z}$, there exists a unique $(g$–twisted) $\Pi^{1/2}(0)$–module structure on the vector space

$$\mathcal{L}_{r}(U') = U' \otimes M(1)$$

such that $c(0) \equiv r \text{Id}$ on $\mathcal{L}_{r}(U')$. Module $\mathcal{L}_{r}(U')$ is untwisted if $r$ is even and $g$–twisted if $r$ is odd. We omit details, since arguments are completely analogous to those of [24]. In this way we get a realization of a family of irreducible modules for the vertex algebras $\Pi(0)$ and $\Pi^{1/2}(0)$. 
Proposition 3.1. 

1. [24] For every \( r \in \mathbb{Z} \) and \( \lambda \in \mathbb{C} \), \( \Pi_r(\lambda) := \Pi(0)e^{r\mu + \lambda c} \) is an irreducible \( \Pi(0) \)-module on which \( c(0) \) acts as \( r \text{Id} \).

2. Assume that \( r \in \mathbb{Z} \) is even (resp. odd) and \( \lambda \in \mathbb{C} \). Then \( \Pi_{1/2}(\lambda) := \Pi^{1/2}(0)e^{r\mu + \lambda c} \) is an irreducible untwisted (resp. g-twisted) \( \Pi^{1/2}(0) \)-module on which \( c(0) \) acts as \( r \text{Id} \).

3. [9] For every \( \lambda \in \mathbb{C} \setminus \{0\} \) there is an irreducible \( \Pi(0) \)-module \( \Pi_\lambda \) so that \( c(0) \) acts on \( \Pi_\lambda \) as \(-\text{Id}\), and that \( \Pi_\lambda \) is generated by cyclic vector \( w_\lambda \) satisfying

\[
e_0^c w_\lambda = \lambda w_\lambda, \quad e_0^{-c} w_\lambda = \frac{1}{\lambda} w_\lambda.
\]

As a vector space, \( \Pi_\lambda = M(1) \otimes \mathbb{C}[d(0)] \).

4. For every \( \lambda \in \mathbb{C} \setminus \{0\} \) \( \Pi_\lambda \) has the structure of an irreducible \( g \)-twisted \( \Pi^{1/2}(0) \)-module generated by cyclic vector \( w_\lambda \) such that

\[
e_0^{c/2} w_\lambda = \sqrt{\lambda} w_\lambda, \quad e_0^{-c/2} w_\lambda = \frac{1}{\sqrt{\lambda}} w_\lambda.
\]

As usual for a vector \( V \) is a vertex algebra \( V \) we define

\[
\Delta(v, z) = z^w \exp \left( \sum_{n=1}^{\infty} \frac{v_n}{-n} (-z)^{-n} \right).
\]

The following lemma follows from [40, Proposition 3.4].

Lemma 3.2. For \( \ell, r \in \mathbb{Z} \) we have

\[
(\Pi_{\ell+r}(\lambda), Y_{\Pi_{\ell+r}(\lambda)}(\Delta(\ell \mu, z), z)) = (\Pi_{\ell}(\lambda), Y_{\Pi_{\ell}(\lambda)}(\Delta(\ell \mu, z), z))
\]

We also have the following important observation which essentially follows from the analysis of \( \Pi(-1) \) as a module for the Heisenberg-Virasoro vertex algebra at level zero [18].

Lemma 3.3. The operator \( e_0^c \) acts injectively on \( \Pi(-1) \).

4. Realization of the Admissible Affine Vertex Algebra \( V_k(sl(2)) \)

In this section we use the realization from Proposition [24] and get a realization of the admissible affine vertex algebra \( V_k(sl(2)) \) and its ordinary modules.

Assume now that \( k \) is admissible and \( k \notin \mathbb{Z} \). Then

\[
k + 2 = \frac{p'}{p}, \quad d_k = 1 - 6 \frac{(p - p')^2}{pp'} = d_{p,p'}.
\]

\[1\] It was proved in [18] that \( \Pi(-1) \) is a direct sum of irreducible modules for the Heisenberg-Virasoro vertex algebra at level zero, and \( e_0^c \) is a homomorphism which acts non-trivially on each irreducible component.
The Virasoro vertex algebra $L^{\text{Vir}}(d_{p,p'},0)$ is rational and its irreducible modules are $\{L^{\text{Vir}}(d_{p,p'}, h) \mid h \in S_{p,p'}\}$ where

$$S_{p,p'} = \{ h^{r,s}_{p,p'} = \frac{(sp - rp')^2 - (p - p')^2}{4pp'} \mid 1 \leq r \leq p - 1, 1 \leq s \leq p'-1 \}.$$ 

Let now $\varphi = p' \gamma$. Since $\langle \varphi, \varphi \rangle = 2pp'$, we set $M_{\varphi}(2pp',0) = M_{\gamma}(\frac{2}{k+2},0)$ and

$$\omega(k) = \left( \frac{1}{4pp'} \varphi(-1)^2 + \frac{p-p'}{2pp'} \varphi(-2) \right) 1.$$ 

The universal vertex algebra $V^{\text{Vir}}(d_{p,p'},0)$ is not simple and it contains a non-trivial ideal generated by singular vector $\Omega^{\text{Vir}}_{p,p'}$ of conformal weight $(p-1)(p'-1)$. Moreover,

$$L^{\text{Vir}}(d_{p,p'},0) = \frac{V^{\text{Vir}}(d_{p,p'},0)}{U(V^{\text{Vir}})\Omega^{\text{Vir}}_{p,p'}}$$

is a simple vertex algebra (minimal model). The singular vector $\Omega^{\text{Vir}}_{p,p'}$ can be constructed in the free–field realization using screening operators.

**Proposition 4.1.** [52, 47] There exist a unique, up to a scalar factor, Vir–homomorphism

$$\Phi^{\text{Vir}}_{p,p'} : M_{\varphi}(2pp',0)e^{-pp'-1} \varphi \rightarrow M_{\varphi}(2pp',0)$$

$$e^{-pp'-1} \varphi \rightarrow \Omega^{\text{Vir}}_{p,p'}.$$ 

There is a cycle $\Delta_{p'-1}$ and a non-trivial scalar $c_{p-1}$ such that $\Phi^{\text{Vir}}_{p,p'}$ can be represented as

$$\frac{1}{c_{p-1}} \int_{\Delta_{p'-1}} Y(e^{pp'}, z_1) \cdots Y(e^{pp'}, z_{p'-1}) dz_1 \cdots dz_{p'-1}.$$ 

Then $\omega_{p,p'} = \omega^{(k)} + U(V^{\text{Vir}})\Omega^{\text{Vir}}_{p,p'}$ is the conformal vector in $L^{\text{Vir}}(d_{p,p'},0)$.

The universal affine vertex algebra $V^k(sl(2))$ also contains a non-trivial maximal ideal generated by $sl(2)$–singular vector $\Omega^k_{sl(2)}$ of conformal weight $p(p'-1)$. Moreover,

$$V_k(sl(2)) = \frac{V^k(sl_2)}{U(sl(2)) \Omega^k_{sl(2)}}$$

is a simple, admissible vertex algebra. Let $\omega_{\text{sug}}$ denotes the Sugawara Virasoro vector in $V_k(sl(2))$ od central charge $\frac{3k}{k+2}$. The singular vector $\Omega^k_{sl(2)}$ can be also constructed using screening operators. The proof was presented in [48 Theorem 3.1] for $sl(2)$ and in [20 Proposition 6.14] in a more general setting (see also [21] for some applications).
Proposition 4.2. [48, 20] There exist a unique, up to a scalar factor, $\widehat{\text{sl}(2)}$–homomorphism

$$
\Phi^{\text{sl}(2)}_k : W_{k,2(p'-1)} \rightarrow W_{k,0}
$$

$$
e^{-\frac{p'-1}{p'-1}}\varphi+(p'-1)(\alpha+\beta) \mapsto \Omega^{\text{sl}2}_k.
$$

By [48, Theorem 3.1] $\Phi^{\text{sl}(2)}_k$ can be represent as

$$
\frac{1}{e^{p'-1}}\int_{\Delta_{p'-1}} U(z_1) \cdots U(z_{p'-1}) dz_1 \cdots dz_{p'-1},
$$

where $U(z) = Y(a(-1)e^{-\frac{a}{k+2}}, z)$ and the cycle $\Delta_{p'-1}$ is as in Proposition 4.1. But since $U(z) = Y(e^{\frac{\phi}{c}}, z)$ we get the following consequence:

Corollary 4.3. $\Phi^{\text{sl}(2)}_k$ can be extended to a $\widehat{\text{sl}(2)}$–homomorphism

$$
M^{V\text{ir}}_{\varphi,p,p'} e^{-\frac{p'-1}{p'}} \otimes \Pi(0) \rightarrow M^{V\text{ir}}_{\varphi,p,p'} \otimes e^{(p'-1)c},
$$

such that $\Phi^{\text{sl}(2)}_k = \Phi^{V\text{ir}}_{p,p'} \otimes \text{Id}$ and $\Omega^{\text{sl}2}_k = \Omega^{V\text{ir}}_{p,p'} \otimes e^{c}$.

Example 4.4. Let us illustrate the above analysis in the simplest case $p' = 2$. Then we have that $k+2 = \frac{2}{p}$ where $p$ is odd natural number, $p \geq 3$. Moreover, we have $\langle \varphi, \varphi \rangle = 4p$. The construction of the Virasoro singular vectors was obtained in [14] by using lattice vertex algebras.

The singular vector in $V^k(\text{sl}(2))$ is given by

$$
Q e^{\frac{\delta}{k+2}} = (a(-1)e^{-\frac{\delta}{k+2}}) e^{\frac{\delta}{k+2}}
$$

$$
= S_{p-1}(\alpha + \beta - \frac{\delta}{k+2}) a(-1) 1
$$

$$
= S_{p-1}(\frac{\varphi}{2}) e^{\alpha + \beta}
$$

$$
= Q e^{-\frac{\varphi}{2}} \otimes e^{\alpha + \beta} = \Omega^{V\text{ir}}_{p,2} \otimes e^c.
$$

Here $S_n(\gamma)$ denotes the $n$-th Schur polynomial in $(\gamma(-1), \gamma(-2), \ldots)$. In particular, $Q e^{-\frac{\varphi}{2}} = S_{p-1}(\frac{\varphi}{2})$ is a singular vector in $V^{V\text{ir}}(d_{p,2},0) \subset M_{\varphi}(4p)$ (cf. [14]).

Finally we get the realization of $V_k(\text{sl}(2))$:

Theorem 4.5. There exist a non-trivial $\widehat{\text{sl}(2)}$–homomorphism

$$
\Phi : V_k(\text{sl}(2)) \rightarrow L^{V\text{ir}}(d_{p,p'},0) \otimes \Pi(0)
$$

defined by the relations (10)-(12). Moreover,

$$
\Phi(\omega_{\text{sug}}) = \omega_{p,p'} + \frac{1}{k} \mu(-1)^2 - \frac{1}{k} \nu(-1)^2 - \nu(-2)
$$

$$
= \omega_{p,p'} + \frac{1}{2} c(-1)d(-1) - \frac{1}{2} d(-2) + \frac{k}{4} c(-2).
$$
Proof. We have constructed homomorphism $\Phi : V^k(sl(2)) \to V^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)$ and showed in Corollary 4.3 that $\Phi(\Omega_k^{sl(2)}) = \Omega_k^{Vir} \otimes e^{(p'-1)c}$. The claim follows.  

In what follows, we identify $\omega_{sug}$ with $\Phi(\omega_{sug})$ and denote the Sugawara Virasoro field by

$$L_{sug}(z) = \sum_{n \in \mathbb{Z}} L_{sug}(n) z^{-n-2}, \quad L_{sug}(n) = (\omega_{sug})_{n+1}.$$

Remark 2. Note that $\Phi(\omega_{sug}) = \omega_{p,p'} + \omega_{\Pi(0)}$ where

$$\omega_{\Pi(0)} = \frac{1}{2} c((1)d(-1) - \frac{1}{2} d(-2) + \frac{k}{4} c(-2)$$

is a Virasoro vector in the vertex algebra $\Pi(0)$ of central charge $6k + 2$. In particular, we have

$$c_{sug} = \frac{3k}{k+2} = d_{p,p'} + 6k + 2.$$

The next result gives a realization of all ordinary, irreducible $V_k(sl(2))$–modules, i.e., the $V_k(g)$–modules having finite-dimensional $L_{sug}(0)$–eigenspaces. Recall [10] that the set

$$\{ \mathcal{L}_s := L_{A_1}((k + 1 - s)A_0 + (s - 1)A_1) | s = 1, \ldots, p'-1 \}$$

provides all irreducible, ordinary $V_k(g)$–modules.

Proposition 4.6. Let $s \in \mathbb{Z}, 1 \leq s \leq p' - 1$. We have

$$\mathcal{L}_s \cong V_k(sl(2)).(v_{1,s} \otimes e^{\frac{s+1}{2}c}) \subset L^{Vir}(d_{p,p'}, h_{p,p'}^{1,s}) \otimes \Pi^{1/2}(0).$$

Proof. Since $L^{Vir}(d_{p,p'}, h_{p,p'}^{1,s}) \otimes \Pi^{1/2}(0)$ is a $V_k(sl(2))$–module, it suffices to show that $w_s = v_{1,s} \otimes e^{\frac{s+1}{2}c}$ is a singular vector for $\Pi(2)$. For $n \geq 0$ we have:

$$e(n)w_s = f(n+2)w_s = 0, \quad h(n)w_s = (s-1)\delta_{n,0}w_s.$$

It remains to prove that $f(1)w_s = 0$. Since

$$(w_s)_1f = (k+2)(v_{1,s})_1\omega_{p,p'} \otimes e^{\frac{s+1}{2}c} - v_{1,s} \otimes e^{\frac{s+1}{2}c}(\nu(-1)^2 + (k+1)\nu(-2))e^{-c}$$

(18)

$$= \left((k+2)h_{p,p'}^{1,s} - \frac{(s-1)^2}{4} + \frac{(k+1)(s-1)}{2}\right) v_{1,s} \otimes e^{\frac{s+1}{2}c} = 0,$$

we get $f(1)w_s = -[(w_s)_1, f(1)]1 = (w_s)_1f = 0$. The proof follows.

5. Explicit realization of relaxed highest weight $V_k(sl(2))$–modules

We say that a $\mathbb{Z}_{\geq 0}$–graded $V^k(sl(2))$–module $M = \bigoplus_{m=0}^\infty M(m)$ is a relaxed highest weight module if the following conditions hold:

- Each graded component $M(m)$ is an eigenspace for $L_{sug}(0)$;
- $M = V^k(sl(2)).M(0)$;
Creutzig-Ridout character formulas \cite{26}. Itly construct these modules and see from the realization that their characters are given by the modules also appeared in \cite{22}, \cite{29}, \cite{33}, \cite{26}, \cite{48}, \cite{46}, \cite{49}. In this section we shall explic-

Using Lemma 3.2 we get

$$L^{\mu, \nu} = (\ell^{\mu}) \otimes \Pi(0)$$

The subspace \(M(0)\) is usually called the top component of \(M(0)\) (although it has lowest conformal weight with respect to \(L_{sug}(0)\)).

By using the classification of irreducible \(V_k(sl(2))\)–modules from \cite{10} (see also \cite{48}), we conclude that any irreducible \(\mathbb{Z}_{\geq 0}\)-graded \(V_k(sl(2))\)–module belongs to one of the following series:

1. The ordinary modules \(L_{sug}\) (cf. Proposition \ref{prop:ordinary_modules}) with lowest conformal weight \(h_{p,p'}^{1,0} + \frac{s-1}{2}\).
2. The \(\mathbb{Z}_{\geq 0}\) graded \(V_k(sl(2))\)–modules \(D_{r,s}^{\pm}\), \(1 \leq r \leq p - 1, 1 \leq s \leq p' - 1\), where
   - \(D_{r,s}^+\) is an irreducible \(\mathbb{Z}_{\geq 0}\)-graded \(V_k(sl(2))\)–module such that \(D_{r,s}(0)\) is an irreducible highest weight \(sl(2)\)–module with highest weight \(\mu_{r,s} = (s - 1 - (k + 2)r)\omega_1\), where \(\omega_1\) is the fundamental weight for \(sl(2)\).
   - \(D_{r,s}^-\) is an irreducible \(\mathbb{Z}_{\geq 0}\)-graded \(V_k(sl(2))\)–module such that \(D_{r,s}(0)\) is an irreducible lowest weight \(sl(2)\)–module with highest weight \(-\mu_{r,s}\).
3. Relaxed highest weight modules \(M = \bigoplus_{m=0}^{\infty} M(m)\), such that the top component \(M(0)\) has conformal weight \(h_{p,p'}^{0,0} + k/4\).

In this section we construct a family of relaxed highest weight modules for \(V_k(sl(2))\). These modules also appeared in \cite{22}, \cite{29}, \cite{33}, \cite{26}, \cite{48}, \cite{46}, \cite{49}. In this section we shall explicitly construct these modules and see from the realization that their characters are given by the Creutzig-Ridout character formulas \cite{26}.

For every \(\lambda \in \mathbb{C}\) and \(r, s \in \mathbb{Z}\), \(0 < r < p, 0 < s < p'\) we define the \(L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0)\)–module

\[ E^\lambda_{r,s} = L^{Vir}(d_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi_{-1}(\lambda) = L^{Vir}(d_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi(0), e^{-\mu + \lambda + \frac{2}{n}(\nu - \mu)}. \]

Let \(\ell \in \mathbb{Z}\) and \(\pi_\ell\) be the (spectral flow) automorphism of \(V_k(sl(2))\) defined by

\[ \pi_\ell(c(n)) = c(n + \ell), \quad \pi_\ell(f(n)) = f(n - \ell), \quad \pi_\ell(h(n)) = h(n) + \ell k_0. \]

By using the realization of \(V_k(sl(2))\) one can see that the spectral-flow automorphism \(\pi_\ell\) can be realized as the lattice element \(e^{\ell\mu}\) acting on \(\Pi(0)\)–modules:

**Proposition 5.1.** We have: \(\pi_\ell(E^\lambda_{r,s}) = L^{Vir}(d_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi_{-1}(\lambda)\).

**Proof.** Using \cite[Proposition 2.1]{5} we get that if \((M, Y_M(\cdot, z))\) is a \(V_k(sl(2))\)–module then

\[ (\pi_\ell(M), Y_{\pi_\ell(M)}(\cdot, z)) := (M, Y_M(\Delta(\frac{\ell h}{2}, z), z)) \]

Using Lemma \ref{lem:automorphism} we get

\[ (\Pi_{-1}(\lambda), Y_{\Pi_{-1}(\cdot, z)}) = (\Pi_{-1}(\lambda), Y_{\Pi_{-1}(\lambda)}(\Delta(\ell \mu, \cdot, z), z)) \]

which implies \(\pi_\ell(E^\lambda_{r,s}) = L^{Vir}(d_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi_{-1}(\lambda)\). The proof follows.

Let

\[ E^\lambda_{r,s} = v_{r,s} \otimes e^{-\mu + \lambda + \frac{2}{n}(\nu - \mu)} = v_{r,s} \otimes e^{\beta - \frac{\delta}{2} + \lambda (\alpha + \beta)}. \]
Then \( E_{r,s}^\lambda \) is a primary vector with conformal weight \( k/4 + h_{p,p'}^{r,s} \), i.e.

\[
L_{\text{sug}}(n) E_{r,s}^\lambda = (k/4 + h_{p,p'}^{r,s}) \delta_{n,0} E_{r,s}^\lambda, \quad (n \geq 0).
\]

The \( sl(2) \) action on these vectors is as follows:

\[
e(0) E_{r,s}^\lambda = E_{r,s}^{\lambda+1},
\]

\[
h(0) E_{r,s}^\lambda = (-k + 2\lambda) E_{r,s}^\lambda,
\]

\[
f(0) E_{r,s}^\lambda = \left( (k + 2) h_{p,p'}^{r,s} - \lambda^2 + \lambda(k + 1) \right) E_{r,s}^{\lambda-1}
\]

\[
= \left( \frac{(sp - tp')^2}{4p^2} - (\lambda - \frac{p'}{2p})^2 \right) E_{r,s}^{\lambda-1}.
\]

**Remark 3.** Note that \( f(0) E_{r,s}^\lambda = 0 \) iff \( \lambda = \lambda_{r,s}^\pm \) where \( \lambda_{r,s}^\pm = \frac{s-1}{2} - \frac{r-1}{2}(k + 2), \lambda_{r,s} = \lambda_{p-r,p'-s}^+ \).

If \( \lambda = \lambda_{r,s}^+ \), then \( E_{r,s}^\lambda \) is an indecomposable \( \mathbb{Z}_{\geq 0} \)-graded \( V_k(sl(2)) \)-module which appears in the non-split extension

\[
0 \to D_{p-r,p'-s}^- \to E_{r,s}^\lambda \to D_{r,s}^+ \to 0.
\]

where \( D_{r,s}^\pm \) are irreducible \( V_k(sl(2)) \)-modules described above. This extension was also constructed in [26] (see [26, Section 4] and their formula (4.3)). In Section 7, we shall see that indecomposable modules \( E_{r,s}^\lambda \) appear in the construction of logarithmic modules.

Assume that \( \lambda \notin \lambda_{r,s}^\pm + \mathbb{Z} \).

**Theorem 5.2.** We have:

1. \( E_{r,s}^\lambda \) is \( \mathbb{Z}_{\geq 0} \)-graded \( V_k(sl(2)) \)-module.
2. The top component is \( E_{r,s}^\lambda(0) = \text{span}_\mathbb{C} \{ E_{r,s}^{\lambda+j} \mid j \in \mathbb{Z} \} \) and it has conformal weight \( k/4 + h_{p,p'}^{r,s} \). If \( \lambda \notin (\lambda_{r,s}^\pm + \mathbb{Z}) \), then \( E_{r,s}^\lambda(0) \) is an irreducible \( sl(2) \)-module.
3. The character of \( E_{r,s}^\lambda \) is given by

\[
\text{ch}[E_{r,s}^\lambda](q, z) = z^{-k+2\lambda} \chi_{r,s}(q) \frac{\delta(z^2)}{\eta(\tau)^2},
\]

where \( \chi_{r,s}(q) = \text{ch}[L^{Vir}(d_{p,p'}, h_{p,p'}^{r,s})](q), \delta(z^2) = \sum_{\ell \in \mathbb{Z}} z^{2\ell}, \eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n). \)

**Proof.** By using (19) we see that \( E_{r,s}^\lambda \) is \( \mathbb{Z}_{\geq 0} \)-graded, and by (20)-(22) we get that the top component \( E_{r,s}^\lambda(0) \) is an irreducible \( sl(2) \)-module. This proves assertions (1) and (2). Recall that
Proof. If quotient of $E_M$ that $V$ sation of irreducible
Assume thatLemma 5.3.
\[\square\]
The proof follows.

14 DRAŽEN ADAMOVIĆ

\[c_{sug} = \frac{2k}{k+2} = d_{p,p'} + 2 + 6k\] (see Remark[2]). From the realization we get
\[\text{ch}[E_{r,s}^\lambda](q, z) = \text{Tr}_{E_{r,s}^\lambda}q^{L_{sug}(0)-c_{sug}/24}z^{h(0)}\]
\[= \text{Tr}_{Vir}(d_{p,p',h_{p,p'}})q^{L(0)}d_{p,p'}/24q^{-\frac{1}{4}} - \frac{1}{\eta(q)} \prod_{n=1}^{\infty}(1 - q^n)^2 z^{-k+2\lambda}\delta(z^2)/\eta(\tau)^2.\]
The proof follows. \[\square\]

Lemma 5.3. Assume that $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n)$ is an irreducible $\mathbb{Z}_{\geq 0}$-graded $V_k(sl(2))$-module such that $M(0)$ is an irreducible, infinite-dimensional weight $sl(2)$-module. Then $M$ is isomorphic to a sub-quotient of $E_{r,s}^\lambda$ for certain $1 \leq r \leq p' - 1, 1 \leq s \leq p - 1, \lambda \in \mathbb{C}$ and
\[L_{sug}(0) \equiv \langle h_{p,p'} + k/4\rangle \text{Id} \text{ on } M(0).

Proof. If $M(0)$ is an irreducible highest (resp. lowest) weight module for $sl(2)$, then the classification of irreducible $V_k(sl(2))$-modules gives that $M \cong D_{r,s}^\pm$ (resp. $M \cong D_{p-r,p'-s}^\pm$). By Remark[3] $M$ can be realized as a submodule or a quotient of the indecomposable module $E_{r,s}^\lambda$. If $M$ is an irreducible relaxed $V_k(sl(2))$-module, then $M(0) \cong E_{r,s}^\lambda(0)$ for certain $r, s, \lambda$, and therefore $M$ is isomorphic to a (quotient) of $E_{r,s}^\lambda$. The proof follows. \[\square\]

Remark 4. Modules $E_{r,s}^\lambda$ are irreducible for $\lambda \notin \lambda_{r,s}^\pm + \mathbb{Z}$. This follows from the fact that they have the same characters as irreducible quotients of relaxed Verma modules presented by T. Creutzig and D. Ridout in[26]. We should mention that a new proof of irreducibility of a large family of relaxed highest weight modules will be presented in[37] using Mathieu’s coherent families[3]. K. Sato in[49] presented a proof of irreducibility of certain typical modules for the $N = 2$ superconformal algebra which are related to the relaxed $sl(2)$-modules via the anti Kazama–Suzuki mapping[2].[33].

In the present paper we will not use the general irreducibility result. Instead of that, we will see how the irreducibility can be proved using Lemma 5.3 in some special cases.

Proposition 5.4. Let $r_0, s_0$ such that $1 \leq r_0 \leq p' - 1, 1 \leq s_0 \leq p - 1$ and $\lambda \notin (\lambda_{r_0,s_0}^\pm + \mathbb{Z})$.

Assume that
\[h - h_{p,p'}^{r_0,s_0} \notin \mathbb{Z}_{>0} \quad \forall h \in S_{p,p'}.
\]
Then $E_{r_0,s_0}^\lambda$ is an irreducible $V_k(sl(2))$-module. In particular, $E_{r_0,s_0}^\lambda$ is irreducible if $h_{p,p'}^{r_0,s_0}$ is maximal in the set $S_{p,p'}$.

Proof. Assume that $E_{r_0,s_0}^\lambda$ is reducible. By Lemma 5.3 the operator $e(0) = e^c_0$ acts injectively on the module $E_{r_0,s_0}^\lambda$ and therefore there are no submodules of $E_{r_0,s_0}^\lambda$ with finite-dimensional

---

2Talk presented by K. Kawasetsu at the conference Affine, vertex and W-algebras, Rome, December 11-15, 2017
This contradicts the choice of \( E \).

Let \( L \) be arbitrary admissible level. As a consequence, we will see that admissible affine vertex algebra submodule \( M \) is generated by the Whittaker vector \( w_{\lambda,\mu,k} \) satisfying

\[
\begin{align*}
  e(n)w_{\lambda,\mu,k} &= \delta_{n,0}\lambda w_{\lambda,\mu,k} \quad (n \in \mathbb{Z}_{\geq 0}), \\
  f(m)w_{\lambda,\mu,k} &= \delta_{m,1}\mu w_{\lambda,\mu,k} \quad (m \in \mathbb{Z}_{\geq 1}).
\end{align*}
\]

If \( \mu \cdot \lambda \neq 0 \), then the Whittaker module is called non-degenerate. It was proved in [9] that at the non-critical level the universal non-degenerate Whittaker module is automatically irreducible.

In this section we extend result from [9] and construct all degenerate Whittaker modules at an arbitrary admissible level. As a consequence, we will see that admissible affine vertex algebra \( V_k(sl(2)) \) contains \( \mathbb{Z}_{\geq 0} \)-graded modules of the Whittaker type.

Let us first recall some notation from [9].

For a \( (\lambda, \mu) \in \mathbb{C}^2 \), let \( \tilde{W} h_{sl(2)}(\lambda, \mu, k) \) denotes the universal Whittaker module at level \( k \) which is generated by the Whittaker vector \( w_{\lambda,\mu,k} \) satisfying

\[
\begin{align*}
  e(n)w_{\lambda,\mu,k} &= \delta_{n,0}\lambda w_{\lambda,\mu,k} \quad (n \in \mathbb{Z}_{\geq 0}), \\
  f(m)w_{\lambda,\mu,k} &= \delta_{m,1}\mu w_{\lambda,\mu,k} \quad (m \in \mathbb{Z}_{\geq 1}).
\end{align*}
\]

6. Whittaker Modules for \( V_k(sl(2)) \)

Let \( \mathbb{Z}_{\geq 0} \)-eigenspaces. Since the top component \( \mathcal{E}^\lambda_{r_0,s_0}(0) \) is an irreducible \( sl(2) \)-module, we conclude that \( \mathcal{E}^\lambda_{r_0,s_0} \) has a non-trivial \( \mathbb{Z}_{\geq 0} \)-graded irreducible submodule \( M = \bigoplus_{m=0}^{\infty} M(m) \), such that \( \dim M(0) = \infty \). By Lemma 5.3, the conformal weight of \( M(0) \) is \( h + k/4 \), such that

\[
  h \in \mathcal{S}_{p,p'}, \ h > h_{p,p'}^{r_0,s_0}, \ h - h_{p,p'}^{r_0,s_0} \in \mathbb{Z}_{>0}.
\]

This contradicts the choice of \((r_0, s_0)\). The claim holds. \( \square \)

For all \( k, \ h, \lambda, \mu \in \mathbb{C} \), \( \lambda \neq 0 \) we have:

\[
Wh_{sl(2)}(\lambda, 0, k) = L^{Vir}(d_k, h) \otimes \Pi_{\lambda}.
\]

Proof. The proof will use [9, Lemma 10.2] which says that \( \Pi_{\lambda} \) is an irreducible \( \hat{\mathfrak{b}}_1 \)-module, where \( \hat{\mathfrak{b}}_1 \) is a Lie subalgebra of \( sl(2) \) generated by \( e(n), h(n), n \in \mathbb{Z} \).

On \( L^{Vir}(d_k, h) \) we have the weight decomposition:

\[
L^{Vir}(d_k, h) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} L^{Vir}(d_k, h)_{h+m}, \quad L^{Vir}(d_k, h)_{h+m} = \{v \in L^{Vir}(d_k, h) | L(0)v = (h + m)v\}.
\]

Let \( v_h \) be the highest weight vector in \( L^{Vir}(d_k, h) \), and define \( \tilde{w}_{\lambda,0,k} = v_h \otimes w_{\lambda,0,k} \).

\[
\begin{align*}
  e(n)\tilde{w}_{\lambda,0,k} &= \delta_{n,0}\lambda \tilde{w}_{\lambda,0,k} \quad (n \in \mathbb{Z}_{\geq 0}), \\
  f(m)\tilde{w}_{\lambda,0,k} &= 0 \quad (m \in \mathbb{Z}_{\geq 1}), \\
  L^{Vir}(n)\tilde{w}_{\lambda,0,k} &= \delta_{n,0}(h + k/4)w_{\lambda,0,k} \quad (n \in \mathbb{Z}_{\geq 0}).
\end{align*}
\]
we conclude that $\widehat{W} = U(sl(2)) \cdot \bar{w}_\lambda, 0, k$ is a certain quotient of the Whittaker module $M_{\bar{sl}(2)}(\lambda, 0, k, h + k/4)$.

Let us first prove that $\widehat{W} = L^{Vir}(d_k, h) \otimes \Pi_\lambda$. It suffices to prove that for every $m \in \mathbb{Z}_{\geq 0}$ we have that

$$v \otimes w \in \widehat{W} \quad \forall v \in L^{Vir}(d_k, h)_{h+m}, \quad \forall w \in \Pi_\lambda.$$  

For $m = 0$, the claim follows by using the irreducibility of $\Pi_\lambda$ as a $\hat{b}_1$–module. Assume now that $v' \otimes w \in \widehat{W}$ for all $v' \in L^{Vir}(d_k, h)_{h+m'}$ such that $m' < m$ and all $w \in \Pi_\lambda$. Let $v \in L^{Vir}(d_k, h)_{h+m}$. It suffices to consider homogeneous vectors

$$v = L(-n_0)L(-n_1) \cdots L(-n_s)v_h, \quad n_0 \geq \cdots \geq n_s \geq 1, \quad n_0 + \cdots + n_s = m.$$  

Then by inductive assumption we have that $L(-n_1) \cdots L(-n_s)v_h \otimes w \in \widehat{W}$ for all $w \in \Pi_\lambda$. By using the formulæ for the action of $f(m)$, $m \in \mathbb{Z}$, we get

$$f(-n_0)(L(-n_1) \cdots L(-n_s)v_h \otimes w_\lambda) = AL(-n_0)L(-n_1) \cdots L(-n_s)v_h \otimes w_\lambda + z$$

where $A \neq 0$ and

$$z = \sum_i v_i \otimes w_i, \quad v_i \in L^{Vir}(d_k, h)_{h+m'_i}, \quad m'_i < m, \quad w_i \in \Pi_\lambda.$$  

By using inductive assumption we get that $z \in \widehat{W}$, and therefore $v \otimes w_\lambda \in \widehat{W}$. Using the fact that $\Pi_\lambda$ is an irreducible $\widehat{b}_1$–module, we get that $v \otimes w \in \widehat{W}$ or every $w \in \Pi_\lambda$. The claim (30) now follows by induction.

Now the irreducibility result will be a consequence of the following claim:

$$v \otimes w \text{ is cyclic vector in } L^{Vir}(d_k, h) \otimes \Pi_\lambda \quad \forall v \in L^{Vir}(d_k, h)_{h+m}, \quad m \in \mathbb{Z}_{\geq 0}, \quad \forall w \in \Pi_\lambda.$$  

For $m = 0$, the claim (31) follows by using irreducibility of $\Pi_\lambda$ as a $\hat{b}_1$–module and (30). Assume now that $v \in L^{Vir}(d_k, h)_{h+m}$ for $m > 0$. Then there is $m_0, 0 < m_0 \leq m$ so that $L(m_0)v \neq 0$ and $L(m_0)v \in L^{Vir}(c_k, h)_{h+m-m_0}$. Since

$$f(m_0)(v \otimes w_\lambda) = (k + 2)\lambda L(m_0)v \otimes w_\lambda,$$

by induction we have that $L(m_0)v \otimes w_\lambda$ is a cyclic vector. So $v \otimes w_\lambda$ is also cyclic. By using again the irreducibility of $\Pi_\lambda$ as $\hat{b}_1$–module, we see that $v \otimes w$ is cyclic for all $w \in \Pi_\lambda$. The proof follows.

As a consequence, we shall describe the structure of simple Whittaker module $W h_{\bar{sl}_2}(\lambda, 0, k, a)$ at admissible levels, and show that these modules are $V_k(sl_2)$–modules.

**Theorem 6.2.** Assume that $k$ is admissible, non-integral, and $\lambda \neq 0$. Then we have:

1. $W h_{\bar{sl}_2}(\lambda, 0, k, a) \cong L^{Vir}(d_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi_\lambda$, where $a = h_{p,p'}^{r,s} + k/4$.  

(2) The set
\[ \{ Wh_{sl(2)}(\lambda, 0, k, h + k/4) \mid h \in \mathcal{S}_{p,p'} \} \]
provides all irreducible \( \mathbb{Z}_{\geq 0} \)-graded \( V_k(sl(2)) \)-modules which are Whittaker \( sl(2) \)-modules.

Proof. Since \( L^{Vir}(d_{p,p'}, h) \) for \( h \in \mathcal{S}_{p,p'} \) is a \( L^{Vir}(d_{p,p'}, 0) \)-module, we conclude that \( L^{Vir}(d_{p,p'}, h) \otimes \Pi(0) \) is a \( L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0) \)-module and therefore a \( V_k(sl(2)) \)-module.

Assume that \( Wh_{sl(2)}(\lambda, 0, k, h + k/4) \) is a \( V_k(sl(2)) \)-module. We proved in Theorem 6.1 that \( Wh_{sl(2)}(\lambda, 0, k, h + k/4) \cong L^{Vir}(d_{p,p'}, h) \otimes \Pi(0) \) for certain \( h \in \mathbb{C} \) and that \( L_{sug}(0) \) acts on lowest weight component as \( h + \frac{k}{2} \). By using description of Zhu’s algebra for \( V_k(sl(2)) \) (cf. [10], [48]) we see that \( L(0) \) must act on lowest component as \( h \cdot \text{Id} \) for \( h \in \mathcal{S}_{p,p'} \). Therefore, \( Wh_{sl(2)}(\lambda, 0, k, h + k/4) \cong L^{Vir}(d_{p,p'}, h) \otimes \Pi(0) \) for \( h \in \mathcal{S}_{p,p'} \). The proof follows.

\[ \Box \]

7. Screening Operators and Logarithmic Modules for \( V_k(sl(2)) \)

This section gives a vertex-algebraic interpretation of the construction of logarithmic modules from [32, Section 5]. By using the embedding of \( V_k(sl(2)) \) in the vertex algebra \( L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0) \subset L^{Vir}(d_{p,p'}, 0) \otimes V_L \), we are able to use methods [13] to construct logarithmic modules for admissible affine vertex algebra \( V_k(sl(2)) \). Formula for the screening operator \( S \) appeared in [32]. In the case \( k = -\frac{4}{3} \), the construction of logarithmic modules reconstructs modules from [13, Section 8] and [35].

First we notice that \( L^{Vir}(d_{p,p'}, h^{2,1}_{p,p'}) \) is an irreducible \( L^{Vir}(d_{p,p'}, 0) \)-module generated by lowest weight vector \( v_{2,1} \) of conformal weight
\[ h^{2,1} := h^{2,1}_{p,p'} = \frac{3p' - 2p}{4p} = \frac{3}{4} k + 1. \]

Let us now consider \( L^{Vir}(c_{p,p'}, 0) \otimes \Pi(0) \)-module
\[ \mathcal{M}_{2,1} = L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0). (v_{2,1} \otimes c') = L^{Vir}(d_{p,p'}, h^{2,1}_{p,p'}) \otimes \Pi(1)(-\frac{k}{2}). \]

As in [13] we have extended vertex algebra
\[ \mathcal{V} = L^{Vir}(d_{p,p'}, 0) \otimes \Pi(0) \bigoplus \mathcal{M}_{2,1}. \]

Note also
\[
\begin{align*}
L(-2)v_{2,1} & = \frac{1}{k+2}L(-1)^2v_{2,1}.
[ L(n), (v_{2,1})_m ] & = ((h^{2,1}) - 1)(n + 1) - m)(v_{2,1})_{m+n} \quad (m, n \in \mathbb{Z}).
[ L(-2), (v_{2,1})_{-1} ] & = (2 - h^{2,1})(v_{2,1})_{-3}
[ L(-2), (v_{2,1})_0 ] & = (1 - h^{2,1})(v_{2,1})_{-2}
[ L(-2), (v_{2,1})_1 ] & = -h^{2,1}(v_{2,1})_{-1}
\end{align*}
\]
18 DRAŽEN ADAMOVIĆ

Let $s = v_{2,1} \otimes e^\nu$. By using formula (16) we get

$$L_{\text{sug}}(n)s = \delta_{n,0}s \quad (n \geq 0).$$

Therefore

$$S = s_0 = \text{Res}_z Y(s, z)$$

commute with the action of the Virasoro algebra $L_{\text{sug}}(n), n \in \mathbb{Z}$.

We want to see that $S$ commutes with $\hat{\text{sl}}(2)$–action. The arguments for claim were essentially presented in [32]. The following lemma can be proved by direct calculation in lattice vertex algebras.

**Lemma 7.1.** [32] We have

$$s_2 f = 2(k + 1)v_{2,1} \otimes e^{\nu - \frac{2}{k}(\mu - \nu)}$$

$$s_1 f = kL(-1)v_{2,1} \otimes e^{\nu - \frac{2}{k}(\mu - \nu)} + (k + 2)v_{2,1} \otimes e^{\nu}$$

$$s_0 f = Sf = 0.$$

**Proposition 7.2.** [32] We have:

$$[S, \hat{\text{sl}}(2)] = 0,$$

i.e., $S$ is a screening operator.

**Proof.** Since

$$s_n e = s_n h = 0 \quad (n \geq 0),$$

we get $[S, e(n)] = [S, h(n)] = 0$.

By using Lemma 7.1 we get $[S, f(n)] = (Sf)(n) = 0$. The claim follows. □

**Lemma 7.3.** Assume that $\ell \in \mathbb{Z}, 1 \leq s \leq p - 1, 1 \leq r \leq p - 2$ and

$$\lambda \equiv \lambda^+_{r,s} = \frac{1}{2}(s - 1 - (k + 2)(r - 1)) \mod(\mathbb{Z}).$$

Then we have that

$$\mathcal{M}^\ell_{r,s}(\lambda) = L^{\text{Vir}}(c_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi(\ell)(\lambda) \bigoplus L^{\text{vir}}(c_{p,p'}, h_{p,p'}^{r+1,s}) \otimes \Pi(\ell + 1)(-\frac{k}{2} + \lambda)$$

is a $\mathcal{V}$–module.

**Proof.** In general, $L^{\text{Vir}}(c_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi(\ell)(\lambda)$ is a $\mathbb{Z}$–graded module whose conformal weights are congruent mod$(\mathbb{Z})$ to $h_{p,p'}^{r,s} + \frac{1}{4}(k\ell^2 + 4(\ell + 1)\lambda)$. By direct calculation we see that

$$h_{p,p'}^{r+1,s} + \frac{1}{4}(\ell + 1)^2k + 4(\ell + 2)(\lambda - \frac{k}{2}) \equiv h_{p,p'}^{r,s} + \frac{1}{4}(\ell^2k + 4(\ell + 1)\lambda) \mod(\mathbb{Z})$$

if and only if (32) holds. Therefore we conclude $L^{\text{Vir}}(c_{p,p'}, h_{p,p'}^{r,s}) \otimes \Pi(\ell)(\lambda)$ and $L^{\text{Vir}}(c_{p,p'}, h_{p,p'}^{r,s+1}) \otimes \Pi(\ell + 1)(-\frac{k}{2} + \lambda)$ have conformal weights congruent mod$(\mathbb{Z})$ if and only if (32) holds. By using the
fusion rules, we conclude that in this case there is a non-trivial intertwining operator $\mathcal{Y}(\cdot, z)$ of type
\[
L^{ir}(c_{p,p'}, h^{r+1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2} + \lambda)
\]
\[
L^{ir}(c_{p,p'}, h^1_{p,p'}) \otimes \Pi_{(1)}(-\frac{k}{2}) \quad L^{ir}(c_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda)
\]
with integral powers of $z$. Now, the assertion follows by applying [15, Lemma 3.2].

For $v \in V$ we define
\[
\Delta(v, z) = z^{v_0} \exp \left( \sum_{n=1}^{\infty} \frac{v_n}{n} (-z)^{-n} \right)
\]

Next result shows that at a non-integral admissible levels, logarithmic modules always exist. By applying methods from [13] and taking $v = s$ we get:

**Proposition 7.4.** Assume that $(\mathcal{M}, \mathcal{Y}_\mathcal{M})$ is any $\mathcal{V}$-module. Then
\[
(\widetilde{\mathcal{M}}, \widetilde{\mathcal{Y}}_{\mathcal{M}}(\cdot, z)) := \mathcal{M}, \mathcal{Y}_{\mathcal{M}}(\Delta(s, z) \cdot, z)
\]
is a $V_k(sl(2))$-module such that
\[
\widetilde{L}_{sug}(0) = L_{sug}(0) + S.
\]
In particular $\widetilde{\mathcal{V}}$ is a logarithmic $V_k(sl(2))$-module of nilpotent rank two.

Using Lemma 7.3 we obtain:

**Corollary 7.5.** Assume that $\ell \in \mathbb{Z}$, $1 \leq s \leq p' - 1$, $1 \leq r \leq p - 2$ and $\lambda = \lambda^{+}_{r,s}$. Then the logarithmic module $M^\ell_{r,s}(\lambda)$ appears in the following extensions of weight $V_k(sl(2))$-modules:
\[
0 \rightarrow L^{ir}(c_{p,p'}, h^{r+1,s}_{p,p'}) \otimes \Pi_{(\ell+1)}(-\frac{k}{2} + \lambda) \rightarrow M^\ell_{r,s}(\lambda) \rightarrow L^{ir}(c_{p,p'}, h^{r,s}_{p,p'}) \otimes \Pi_{(\ell)}(\lambda) \rightarrow 0.
\]

8. A realization of the $N = 3$ superconformal vertex algebra $W_{k'}(spo(2, 3), f_\theta)$ for $k' = -1/3$

The cases $k \in \{-1/2, -4/3\}$ were already studied in the literature. In these cases the quantum Hamiltonian reduction maps $V_k(sl(2))$ to the trivial vertex algebra, and therefore the affine vertex algebra $V_k(sl(2))$ is realized as a vertex subalgebra of $\Pi(0)$. In the case $k = -1/2$, $V_k(sl(2))$ admits a realization as a subalgebra of the Weyl vertex algebra and it is also related with the triplet vertex algebra $W(2)$ with central charge $c = -2$ (cf. [31], [44]). In [4], the author related $V_{-4/3}(sl(2))$ with the triplet vertex algebra $W(3)$ at central charge $c = -7$.

Let $k = -2/3$. Then $V_k(sl(2))$ is realized as a subalgebra of $L^{ir}(d_{3,4}, 0) \otimes \Pi(0)$. But $L^{ir}(d_{3,4}, 0)$ is exactly the even subalgebra of the fermionic vertex superalgebra $F = L^{ir}(d_{3,4}, 0) \oplus L^{ir}(d_{3,4}, \frac{1}{2})$...
generated by the odd field \( \Psi(z) = \sum_{m \in \mathbb{Z}} \Psi(m + \frac{1}{2})z^{-m-1} \) (see Section 9). The Virasoro vector is \( \omega_F = \frac{1}{2} \Psi(-\frac{3}{2}) \Psi(-\frac{1}{2}) \mathbf{1} \). Let \( \gamma = \frac{2}{\tau} \nu, \varphi = \frac{2}{\tau} \mu. \) Then
\[
(\gamma, \gamma) = -(\varphi, \varphi) = 3.
\]

Let \( D = \mathbb{Z}\gamma. \) Then \( V_D = M_{\gamma}(1) \otimes \mathbb{C}[D] \) is the lattice vertex superalgebra, where \( M_{\gamma}(1) \) is the Heisenberg vertex algebra generated by \( \gamma \) and \( \mathbb{C}[D] \) the group algebra of the lattice \( D. \) The screening operator \( S \) is then expressed as
\[
S = \text{Res}_z : \Psi(z)e^{\nu}(z) := \text{Res}_z : \Psi(z)e^{-\frac{1}{3}\gamma}(z) :.
\]
Define also
\[
Q = \text{Res}_z : \Psi(z)e^{\gamma}(z) :.
\]

We have:

**Proposition 8.1.** \(^{[11]}\)

(1) \( SW(1) \cong \text{Ker}_{F \otimes V_{D}} S \) is isomorphic to the \( N = 1 \) super-triplet vertex algebra at central charge \( c = -5/2 \) strongly generated by
\[
X = e^{-\gamma}, \ H = QX, \ Y = Q^2X, \ \tilde{X} = \Psi(-1/2)X, \ \tilde{H} = Q\tilde{X}, \ \tilde{Y} = Q^2\tilde{X}
\]
and superconformal vector \( \tau = \frac{1}{\sqrt{2}} (\Psi(-\frac{1}{2}) \gamma(-1) + 2\Psi(-\frac{3}{2})) \mathbf{1} \) and corresponding conformal vector
\[
\omega_{N=1} = \frac{1}{2} \tau_0 \tau = \frac{1}{6} \left( \gamma(-1)^2 + 2\gamma(-2) \right) \mathbf{1} + \frac{1}{2} \Psi(-\frac{3}{2}) \Psi(-\frac{1}{2}) \mathbf{1}.
\]

(2) \( SM(1) \cong \text{Ker}_{F \otimes M_{\gamma}(1)} S \) is isomorphic to the \( N = 1 \) super singlet vertex algebra at central charge \( c = -5/2 \) strongly generated by \( \tau, \omega_{N=1}, H, \tilde{H}. \)

Consider the lattice vertex algebra \( F_{-3} = V_{2\gamma}. \) We shall now see that the admissible affine vertex algebra \( V_{-2/3}(sl(2)) \) is a vertex subalgebra of \( SW(1) \otimes F_{-3}. \) Note that \( \gamma(0) - \varphi(0) \) acts semisimply on \( SW(1) \otimes F_{-3} \) and we have the following vertex algebra
\[
\mathcal{U} = \{ v \in SW(1) \otimes F_{-3} | (\gamma(0) - \varphi(0))v = 0 \}.
\]

Following \(^{[36]}\), we identify the \( N = 3 \) superconformal vertex algebra with affine \( W \)–algebra \( W_{k'}(spo(2, 3), f_\theta). \) By applying results on conformal embeddings from \(^{[17]} \) Theorem 6.8 (12)), we see that the vertex algebra \( W_{k'}(spo(2, 3), f_\theta) \) for \( k' = -1/3 \) is isomorphic to the simple current extension of \( V_{-2/3}(sl(2)) \):
\[
W_{k'}(spo(2, 3), f_\theta) = L_{A_1}(-\frac{2}{3} \Lambda_0) \bigoplus L_{A_1}(-\frac{8}{3} \Lambda_0 + 2 \Lambda_1).
\]

**Theorem 8.2.** We have:

(1) \( \mathcal{U} \cong W_{k'}(spo(2, 3), f_\theta) \) for \( k' = -1/3. \)

(2) \( \text{Com}(M_{\bar{h}}(1), W_{k'}(spo(2, 3), f_\theta)) \cong SM(1). \)
(3) $\text{Ker}(sl(2), -\frac{3}{2}) \cong \frac{SM(1)}{V}$, where $\frac{SM(1)}{V}$ is even subalgebra of the supersinglet vertex algebra $SM(1)$.

**Proof.** Since

$$Y = Q^2 X = \left(-6\Omega(-\frac{3}{2})\Omega(-\frac{1}{2}) + \gamma(-1)^2 - \gamma(-2)\right) e^\varphi$$

$$= -9 ((k + 2)\omega - \nu(-1)^2 - (k + 1)\nu(-2)) e^\varphi \quad (k = -2/3)$$

we have that

$$e = X \otimes e^\varphi = e^{\varphi - \gamma},$$

$$h = -\frac{2}{3} \varphi,$$

$$f = -\frac{1}{9} Y \otimes e^{-\varphi} = -\frac{1}{9} Q^2 e^{-\varphi - \gamma},$$

$$\omega_{\text{sug}} = \omega_{N = 1} - \frac{1}{6} \varphi(-1)^2 1.$$  

This implies that $V_{-2/3}(sl(2))$ is a vertex subalgebra of $U$. Therefore $U$ is a $V_{-2/3}(sl(2))$–module which is $\frac{1}{2}\mathbb{Z}_{\geq 0}$–graded $U = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}_{\geq 0}} U_m$ with respect to $L_{\text{sug}}(0)$. One directly sees that $U_{1/2} = \{0\}$ and that $U_{3/2} = \text{span}_C \{X \otimes e^{-\varphi}, \tau, Y \otimes e^{-\varphi}\}$. Then $U_{3/2}$ generates a $V_{-2/3}(sl(2))$–module isomorphic to $L_{A_1}(-\frac{8}{3} \Lambda_0 + 2\Lambda_1)$. Since $U$ is completely reducible as $V_{-2/3}(sl(2))$–module we easily conclude that

$$U \cong L_{A_1}(-\frac{2}{3} \Lambda_0) \bigoplus L_{A_1}(-\frac{8}{3} \Lambda_0 + 2\Lambda_1).$$

Since $U$ is simple and the extension of $V_{-2/3}(sl(2))$ by its simple current module $L_{A_1}(-\frac{8}{3} \Lambda_0 + 2\Lambda_1)$ is unique, we get the assertion (1). Assertion (2) follows from

$$\text{Com}(M_{k}(1), W_{k'}(\text{spo}(2, 3), f_{0})) = \{v \in SW(1) \otimes F_{-3} | \varphi(n)v = (\gamma(0) - \varphi(0))v = 0, n \geq 0\} \cong \text{Ker}_{SW(1)} \gamma(0) = \frac{SM(1)}{V}.$$

(3) easily follows from (2). \hfill \Box

9. **Realization of $V_k(osp(1, 2))$**

A free field realization of $\widehat{osp(1, 2)}$ of the Wakimoto type was presented in [30]. In this section we study an explicit realization of affine vertex algebras associated to $\widehat{osp(1, 2)}$ which generalize realizations for $\widehat{sl(2)}$ from previous sections. Since the quantum Hamiltonian reduction of $V_k(osp(1, 2))$ is the $N = 1$ Neveu-Schwarz vertex algebra $V_{ns}(c_k, 0)$ where $c_k = \frac{3}{2} - 12\frac{(k+1)^2}{2k+3}$ (cf. [36 Section 8.2]), one can expect that inverse of the quantum Hamiltonian reduction (assuming that it should exist) gives a realization of the form $V_{ns}(c_k, 0) \otimes F$, where $F$ is a certain vertex algebra of free-fields. In this section we show that for $F$ one can take the tensor product
of the fermionic vertex algebra $F$ at central charge $1/2$ and the lattice type vertex algebra $\Pi^{1/2}(0)$ introduced in Section 3.

9.1. **Affine vertex algebra** $V_k(osp(1, 2))$. Recall that $\mathfrak{g} = osp(1, 2)$ is the simple complex Lie superalgebra with basis $\{e, f, h, x, y\}$ such that the even part $\mathfrak{g}^0 = \text{span}_\mathbb{C}\{e, f, h\}$ and the odd part $\mathfrak{g}^1 = \text{span}_\mathbb{C}\{x, y\}$. The anti-commutation relations are given by

\[
[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f
\]

\[
[h, x] = x, \quad [e, x] = 0, \quad [f, x] = -y
\]

\[
[h, y] = -y, \quad [e, y] = -x, \quad [f, y] = 0
\]

\[
\{x, x\} = 2e, \quad \{x, y\} = h, \quad \{y, y\} = -2f.
\]

Choose the non-degenerate super-symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ such that non-trivial products are given by

\[
(e, f) = (f, e) = 1, \quad (h, h) = 2, \quad (x, y) = -(y, x) = 2.
\]

Let $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}K$ be the associated affine Lie superalgebra, and $V_k(\mathfrak{g})$ (resp. $V_k(\mathfrak{g})$) the associated universal (resp. simple) affine vertex algebra. As usually, we identify $x \in \mathfrak{g}$ with $x(-1)$.

9.2. **Clifford vertex algebras.** Consider the Clifford algebra $Cl$ with generators $\Psi_i(r), r \in \frac{1}{2} + \mathbb{Z}$, $i = 1, \cdots, n$ and relations

\[
\{\Psi_i(r), \Psi_j(s)\} = \delta_{r+s,0}\delta_{i,j}, \quad (r, s \in \frac{1}{2} + \mathbb{Z}, \ 1 \leq i, j \leq n).
\]

Then the fields

\[
\Psi_i(z) = \sum_{m \in \mathbb{Z}} \Psi_i(m + \frac{1}{2})z^{-m-1} \quad (i = 1, \cdots, n)
\]

generate on $F_n = \bigwedge(\Psi_i(-n - 1/2) \mid n \in \mathbb{Z}_{\geq 0})$ a unique structure of the vertex superalgebra with conformal vector

\[
\omega_{F_n} = \sum_{i=1}^{n} \frac{1}{2} \Psi_i(-\frac{3}{2})\Psi_i(-\frac{1}{2})1
\]

of central charge $n/2$. Let $F = F_1$. Then $F$ is a vertex operator superalgebra of central charge $c = d_{3,4} = 1/2$. Moreover $F = F_{\text{even}} \oplus F_{\text{odd}}$ and

\[
F_{\text{even}} = LVir(d_{3,4}, 0), \quad F_{\text{even}} = LVir(d_{3,4}, 1/2).
\]

Let $\sigma$ be the canonical automorphism of $F$ of order two. The vertex algebra $F$ has precisely two irreducible $\sigma$–twisted modules $M^\pm$. Twisted modules can be also constructed explicitly as an exterior algebra

\[
M^\pm = \bigwedge(\Psi(-n) \mid n \in \mathbb{Z}_{>0})
\]
which is an irreducible module for twisted Clifford algebra $CL^{tw}$ with generators $\Psi(r), r \in \mathbb{Z}$, and relations

$$\{\Psi(r), \Psi_j(s)\} = \delta_{r+s,0}, \quad (r, s \in \mathbb{Z}).$$

$\Psi(0)$ acts on top component of $M^\pm$ as $\pm \frac{1}{\sqrt{2}} \text{Id}.$

As a $L^{Vir}(d_{3,4}, 0)$–module, we have that $M^\pm \cong L^{Vir}(d_{3,4}, \frac{1}{16}).$

9.3. **The general case.** Let $V^{ns}(c_{p,q}, 0)$ be the universal $N = 1$ Neveu-Schwarz vertex superalgebra with central charge $c_{p,q} = \frac{3}{2}(1 - \frac{2(p-q)^2}{pq})$. Let $L^{ns}(c_{p,q}, 0)$ be its simple quotient. If

$$p, q \in \mathbb{Z}, \quad p, q \geq 2, \quad (\frac{p-q}{2}, q) = 1,$$

then $L^{ns}(c_{p,q}, 0)$ is called a minimal $N = 1$ Neveu-Schwarz vertex operator superalgebra. It is a rational vertex operator superalgebra $[1].$

**Proposition 9.1.** Let $p, q \in \mathbb{Z}, p, q \neq 0$. We have:

1. Assume that $p, q, p + q \neq 0$. The Virasoro vertex operator algebra $V^{Vir}(d_{\frac{p+q}{2}}, 0) \otimes V^{Vir}(d_{\frac{p+q}{2}}, 0)$ is conformally embedded in $V^{ns}(c_{p,q}, 0) \otimes F$ and $\omega_{p,q} + \omega_F = \omega_{\frac{p+q}{2}} + \omega_{\frac{p+q}{2}}$ where

$$\omega_{\frac{p+q}{2}} = \frac{p}{p+q} \omega_{p,q} + i \frac{\sqrt{pq}}{p+q} G(-\frac{3}{2}) \Psi(-\frac{1}{2}) \mathbf{1} + \frac{2q-p}{p+q} \omega_F$$

$$\omega_{\frac{p+q}{2}} = \frac{q}{p+q} \omega_{p,q} - i \frac{\sqrt{pq}}{p+q} G(-\frac{3}{2}) \Psi(-\frac{1}{2}) \mathbf{1} + \frac{2p-q}{p+q} \omega_F$$

2. Assume that $p + q = 0$. Then

$$t_{p,q} := \frac{1}{2} \left( -\omega_{p,q} - G(-\frac{3}{2}) \Psi(-\frac{1}{2}) \mathbf{1} + 3\omega_F \right) \in L^{ns}(c_{p,q}, 0) \otimes F$$

is a commutative vector in the vertex algebra $L^{ns}(\frac{27}{2}, 0) \otimes F.$ The vertex subalgebra generated by $t_{p,q}$ is isomorphic to the commutative vertex algebra $M_T(0).$

3. Assume that $p, q$ satisfy condition (33). Then the rational Virasoro vertex operator algebra $L^{Vir}(d_{\frac{p}{2}}, 0) \otimes L^{Vir}(d_{\frac{q}{2}}, 0)$ is conformally embedded in $L^{ns}(c_{p,q}, 0) \otimes F.$

**Proof.** Assertions (1) follows by direct calculation. One can also directly show that $T = t_{p,q}$ is a commutative vector in $L^{ns}(\frac{27}{2}, 0) \otimes F$. Let $\langle T \rangle$ be the vertex subalgebra generated by $T$. By using fact that vectors

$$L(-n_1) \cdots L(-n_r) \mathbf{1} \quad (n_1 \geq \cdots \geq n_r \geq 2)$$

are linearly independent in $L^{ns}(\frac{27}{2}, 0)$ and expression for $T$ one can easily show that the vectors

$$T(-n_1) \cdots T(-n_r) \mathbf{1} \quad (n_1 \geq \cdots \geq n_r \geq 2)$$

provide a basis of $\langle T \rangle$. So $\langle T \rangle \cong M_T(0).$

Assertion (3) was proved in [3] (see also [39], [41]).
Theorem 9.2. Assume that \( k + \frac{3}{2} = \frac{p}{2q} \neq 0 \). There exists a non-trivial vertex superalgebra homomorphism

\[
\Phi : V^k(osp(1, 2)) \to V^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)
\]

such that

\[
\begin{align*}
e & \mapsto e^{2\frac{(\mu-\nu)}{k}}, \\
h & \mapsto 2\mu(-1), \\
f & \mapsto \left[ \Omega_{p,q} - \nu(-1)^2 - (k+1)\nu(-2) \right] e^{-2\frac{(\mu-\nu)}{k}}, \\
x & \mapsto \sqrt{2}\Psi\left(-\frac{1}{2}\right)e^{\frac{1}{k}(\mu-\nu)}, \\
y & \mapsto \sqrt{2}\left[ -\frac{\sqrt{-2k-3}}{2}G(-3/2) + \Psi(-1)\nu(-1) + \frac{2k+1}{2}\Psi\left(-\frac{3}{2}\right) \right] e^{-\frac{1}{k}(\mu-\nu)},
\end{align*}
\]

where \( \Omega_{p,q} = (k+2)\omega_{\frac{p+q}{2}, q} \) if \( k \neq -2 \) and \( \Omega_{p,q} = t_{p,q} \) if \( k = -2 \).

The proof of Theorem 9.2 will be presented in Section 10.

Theorem 9.3. Assume that \( k + \frac{3}{2} = \frac{p}{2q} \neq 0 \) and that \( p, q \) satisfy condition (33).

(1) There exists a non-trivial vertex superalgebra homomorphism

\[
\Phi : V_k(osp(1, 2)) \to L^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0)
\]

such that

\[
\begin{align*}
e & \mapsto e^{2\frac{(\mu-\nu)}{k}}, \\
h & \mapsto 2\mu(-1), \\
f & \mapsto \left[ (k+2)\omega_{\frac{p+q}{2}, q} - \nu(-1)^2 - (k+1)\nu(-2) \right] e^{-2\frac{(\mu-\nu)}{k}}, \\
x & \mapsto \sqrt{2}\Psi\left(-\frac{1}{2}\right)e^{\frac{1}{k}(\mu-\nu)}, \\
y & \mapsto \sqrt{2}\left[ -\frac{\sqrt{-2k-3}}{2}G(-3/2) + \Psi(-1)\nu(-1) + \frac{2k+1}{2}\Psi\left(-\frac{3}{2}\right) \right] e^{-\frac{1}{k}(\mu-\nu)},
\end{align*}
\]

(2) \( \omega_{\frac{p+q}{2}, q} \in \text{Com}(V_k(sl(2)), V_k(osp(1, 2))) \).

Proof. (1) Using Theorem 9.2, we get a homomorphism \( \tilde{\Phi} : V^k(osp(1, 2)) \to L^{ns}(c_{p,q}, 0) \otimes F \otimes \Pi^{1/2}(0) \). Then Proposition 9.1 implies that \( \omega_{\frac{p+q}{2}, q} \) generates a subalgebra of \( L^{ns}(c_{p,q}, 0) \otimes F \) isomorphic to the minimal Virasoro vertex algebra \( L_{\text{Vir}}(c_{\frac{p+q}{2}, q}, 0) \). Therefore Theorem 4.5 gives that \( e, f, h \) generate the simple admissible affine vertex algebra \( V_k(sl(2)) \).

At admissible level \( k \), the vertex algebra \( V^k(osp(1, 2)) \) contains a unique singular vector, i.e., the maximal ideal of \( V^k(osp(1, 2)) \) is simple. So we have two possibilities:

\[
\text{Im}(\tilde{\Phi}) = V^k(osp(1, 2)) \quad \text{or} \quad \text{Im}(\tilde{\Phi}) = V_k(osp(1, 2)).
\]
But, if $\text{Im}(\tilde{\Phi}) = V^k(osp(1,2))$, then the subalgebra generated by the embedding $sl(2)$ into $osp(1,2)$ must be universal affine vertex algebra $V^k(sl(2))$. A contradiction. So $\text{Im}(\tilde{\Phi}) = V^k(osp(1,2))$, and first assertion holds.

(2) By using relation
\[ x(-1)y - \omega^{}_{s\omega} - \frac{1}{2} h(-2) = -\frac{p}{q} \omega^{\frac{p+q}{2}} \]
we see that $\omega^{\frac{p+q}{2}} \in V^k(osp(1,2))$. Since $V^k(sl(2)) \subset L^{vir}(c_{\omega}^{\frac{p+q}{2}},0) \otimes \Pi(0)$ we get that $\omega^{\frac{p+q}{2}}$ commutes with the action of $V^k(sl(2))$. The claim (2) follows.

\[ \square \]

Remark 5. T. Creutzig and A. Linshaw studied the coset \(\text{Com}(V^k(sl(2)), V^k(osp(1,2)))\), and proved in [25, Theorem 8.2] that if $k$ is admissible, then the coset is isomorphic to a minimal Virasoro vertex algebra. This can be also directly proved from Theorem 9.3.

9.4. Realization of $V^k(osp(1,2))$ at the critical level. Let $M(0) = C[b(-n) \mid n \geq 1]$ be the commutative vertex algebra generated by the field $b(z) = \sum_{n<1} b(n) z^{-n-1}$.

Let $\text{NS}_{cri}$ the infinite-dimensional Lie superalgebra with generators
\[ C, T(n), G^{cri}(n + \frac{1}{2}) \ (n \in \mathbb{Z}) \]
such that $T(n), C$ are in the center and
\[ \{G^{cri}(r), G^{cri}(s)\} = 2T(r + s) + \frac{p^2 - 1}{3} \delta_{r+s,0} C \ (r, s \in \frac{1}{2} + \mathbb{Z}). \]

Let $V^{ns}_{cri}$ be the universal vertex superalgebra associated to $\text{NS}_{cri}$ such that $C$ acts as scalar $C = -3$. $V^{ns}_{cri}$ is freely generated by odd field $G^{cri}(z) = \sum_{n \in \mathbb{Z}} G^{cri}(n + \frac{3}{2}) z^{-n-2}$ and even vector $T(z) = \sum_{n \in \mathbb{Z}} T(n) z^{-n-2}$ such that $T$ is in central and that the following $\lambda$–bracket relation holds:
\[ [G^{cri}_\lambda G^{cri}] = 2T - \lambda^2. \]

$V^{ns}_{cri}$ can by realized as the vertex subalgebra of $F_2 \otimes M(0)$ generated by
\[ G^{cri} = b(-1) \Psi_2(-\frac{1}{2}) + \Psi_2(-\frac{3}{2}), \ T = \frac{1}{2} (b(-1)^2 + b(-2)). \]

By direct calculation we get that
\[ \omega_{1,2} = T(-2) + G^{cri}(-\frac{3}{2}) \Psi(-\frac{1}{2}) + 2\omega_F \]
is a Virasoro vector of central charge $c_{1,2} = -2$.

Theorem 9.4.

(1) Assume that $k = -3/2$. There exists a non-trivial homomorphism
\[ \Phi : V^k(osp(1,2)) \to V^{ns}_{cri} \otimes F \otimes \Pi^{1/2}(0) \]
such that

\[ e \mapsto e^{\frac{2}{k}(\mu - \nu)}, \]
\[ h \mapsto 2\mu(-1), \]
\[ f \mapsto \left[(k + 2)\omega_{1,2} - \nu(-1)^2 - (k + 1)\nu(-2)\right]e^{-\frac{2}{k}(\mu - \nu)}, \]
\[ x \mapsto \sqrt{2}\Psi(-\frac{1}{2})e^{\frac{1}{k}(\mu - \nu)}, \]
\[ y \mapsto \sqrt{2}\left[-\frac{i}{2}G^\text{cri}(\frac{3}{2}) + \Psi(-\frac{1}{2})\nu(-1) + \frac{2k + 1}{2}\Psi(-\frac{3}{2})\right]e^{-\frac{1}{k}(\mu - \nu)}. \]

(2) \( T = \frac{1}{2}G^\text{cri}(\frac{1}{2})G^\text{cri}(\frac{-3}{2})1 \) is a central element of \( V^k(osp(1, 2)) \).

**Remark 6.** In the same way as in [36 Section 8.2] one can show that for \( k = -\frac{3}{2} \):

\[ \mathcal{W}^k(osp(1, 2), f_\theta) \cong V^\text{ns}_{\text{cri}}. \]

T. Arakawa proved in [19] that when \( g \) is a Lie algebra and \( f \) nilpotent element, then

\[ \mathfrak{z}(\mathcal{W}^{-h^\vee}(g, f)) = \mathfrak{z}(V^{-h^\vee}(g)). \]

We believe that the results from [19] hold for \( g = osp(1, 2) \), which would prove the following:

- \( \mathfrak{z}(V^k(osp(1, 2))) \cong \mathfrak{z}(V^\text{ns}_{\text{cri}}) \cong M_T(0); \)
- the homomorphism \( \Phi \) from Theorem 9.2 is injective.

**10. Proof of Theorem 9.2**

We shall first prove an important technical lemma.

**Lemma 10.1.** Let \( \tilde{y} = \left[\Psi(-\frac{1}{2})\nu(-1) + \frac{2k + 1}{2}\Psi(-\frac{3}{2})\right]e^{-\frac{1}{k}(\mu - \nu)} \). We have:

1. \( \tilde{y}(2)\tilde{y} = -\frac{1}{4}(2k + 1)(4k + 5)e^{-\frac{2}{k}(\mu - \nu)}, \)
2. \( \tilde{y}(1)\tilde{y} = \frac{2k + 1}{4k}(\mu(-1) - \nu(-1))e^{-\frac{2}{k}(\mu - \nu)} = -\frac{(2k + 1)(4k + 5)}{8}De^{-\frac{2}{k}(\mu - \nu)}, \)
3. \( \tilde{y}(0)\tilde{y} = \frac{2k + 1}{4}\Psi(-\frac{3}{2})\Psi(-\frac{1}{2}) + \frac{2k + 1}{2}\nu(-1)^2 + (k + 1)\nu(-2) - \frac{(2k + 1)(4k + 5)}{4}S_2(\frac{\nu - \mu}{k})e^{-\frac{2}{k}(\mu - \nu)}, \)

where \( S_2(\gamma) = \frac{1}{2}(\gamma(-1)^2 + \gamma(-2)). \)

**Proof.** Let \( \overline{\tau} = \sqrt{-\frac{2}{k}}\left(\Psi(-\frac{1}{2})\nu(-1) + (k + 1)\Psi(-\frac{3}{2})\right)1, b^\tau = e^{\frac{r}{k}(\mu - \nu)}. \) Then \( \overline{\tau} \) generates a \( N = 1 \) superconformal vertex algebra of central charge \( \overline{\tau} = \frac{3}{2k}(4(k + 1)^2 + k). \) We have

\[ \overline{\omega} = \frac{1}{2}\overline{\tau}_0\overline{\tau} = -\frac{1}{k}(\nu(-1)^2 + (k + 1)\nu(-2)) + \omega_{\text{fer}}. \]

\[ \tilde{y}(p) = \sqrt{-\frac{k}{2}}(\tau_1b^-)_p = \sqrt{-\frac{k}{2}}\sum_{j=0}^{\infty} \tau_{-1-j}b_{-p+j}^{-1} + b_{-1-j+p}^{-1}\tau_j. \]
By applying formulas

\[
\begin{align*}
\tau_n b^{-1} &= -\frac{1}{2} \sqrt{\frac{-2}{k}} \delta_n \Psi \left( -\frac{1}{2} \right) b^{-1} \quad (n \geq 0), \\
\tau_j \tau_{-1} b^{-1} &= 0 \quad (j \geq 3), \\
\tau_2 \tau_{-1} b^{-1} &= \left( \frac{4(k+1)^2 + k}{k} - \frac{2k + 3}{2k} \right) b^{-1} = \frac{8(k+1)^2 - 3}{2k} b^{-1} \\
\tau_1 \tau_{-1} b^{-1} &= 2 \nu(-1) b^{-1} \\
\tau_0 \tau_{-1} b^{-1} &= -\frac{2}{k} \left( \nu(-1)^2 + k \nu(-2) \right) b^{-1} + \Psi \left( -\frac{3}{2} \right) \Psi \left( -\frac{1}{2} \right) b^{-1} \\
&\quad - \frac{1}{k} \nu(-2) b^{-1} - \frac{2k + 1}{2k} \Psi \left( -\frac{3}{2} \right) \Psi \left( -\frac{1}{2} \right) b^{-1} \\
&= -\frac{2}{k} \left( \nu(-1)^2 + \frac{2k + 1}{2} \nu(-2) \right) b^{-1} - \frac{1}{2k} \Psi \left( -\frac{3}{2} \right) \Psi \left( -\frac{1}{2} \right) b^{-1}
\end{align*}
\]

we get

\[
\begin{align*}
\bar{y}(2) \bar{y} &= -\frac{k}{2} \left( b_{-1}^{-1} \tau_2 \tau_{-1} b^{-1} + b_0^{-1} \tau_1 \tau_{-1} b^{-1} + b_1^{-1} \tau_0 \tau_{-1} b^{-1} \right) \\
&= -k \left( \frac{4(k+1)^2 + k}{k} - \frac{2k + 3}{2k} + \frac{1}{k} - \frac{k + 1}{k} \right) b^{-2} \\
&= -\frac{1}{4} (2k + 1)(4k + 5)b^{-2}.
\end{align*}
\]

\[
\begin{align*}
\bar{y}(1) \bar{y} &= -\frac{k}{2} \left( b_{-3}^{-1} \tau_2 \tau_{-1} b^{-1} + b_{-1}^{-1} \tau_1 \tau_{-1} b^{-1} + b_0^{-1} \tau_0 \tau_{-1} b^{-1} \right) \\
&= -\frac{k}{2} \left( -\frac{8(k+1)^2 - 3}{2k^2} (\mu(-1) - \nu(-1)) b^{-2} \\
&\quad + \frac{2}{k} \nu(-1) b^{-2} - \frac{1}{k^2} (\mu(-1) - \nu(-1)) b^{-2} \\
&\quad + \frac{k + 1}{k^2} (\mu(-1) - \nu(-1)) b^{-2} - \frac{2}{k} \nu(-1) b^{-2} \right) \\
&= \frac{(4k + 5)(2k + 1)}{4k} (\mu(-1) - \nu(-1)) b^{-2} = -\frac{(2k + 1)(4k + 5)}{8} De^{-\frac{2}{k}(\mu - \nu)}.
\end{align*}
\]
\[ y(0) \bar{y} = \frac{-k}{2} \left( b_{-3}\tau_1 b_{-1} + b_{-2}\tau_1 b_{-1} + b_{-1}\tau b_{-1} + \tau b_0 \tau_{-1} b_{-1} \right) \]
\[ = \frac{-k}{2} \left( \frac{8(k+1)^2 - 3}{2k} S_2 (\nu - \mu) b^{-2} - \frac{2}{k^2} \nu(-1)(\mu(-1) - \nu(-1)) b^{-2} \right) \]
\[ + \frac{1}{k} S_2 (\nu - \mu) b^{-2} - \frac{2}{k} \left( \nu(-1)^2 + \frac{2k+1}{2} \nu(-2) \right) b^{-2} - \frac{1}{2k} \Psi(\frac{3}{2}) \Psi(\frac{1}{2}) b^{-2} \]
\[ - \frac{k+1}{k} S_2 (\nu - \mu) b^{-2} + \frac{2}{k^2} \nu(-1)(\mu(-1) - \nu(-1)) b^{-2} \]
\[ - \frac{1}{k} \left( \nu(-2) + k \Psi(\frac{3}{2}) \Psi(\frac{1}{2}) b^{-2} \right) \]
\[ = \left( \frac{(2k+1)(4k+5)}{4} S_2 (\nu - \mu) b^{-2} + \frac{2k+1}{4} \Psi(\frac{3}{2}) \Psi(\frac{1}{2}) b^{-2} \right) \]
\[ + (\nu(-1)^2 + (k+1) \nu(-2)) b^{-2} \]

The proof follows. \[ \square \]

10.1. **Proof of Theorem 9.2:** First we notice that \( c_{p,q} = \frac{3(4k+5)(2k+1)}{2k+3} \).

Assume that \( k \neq -2 \). Since

\[ e, f, h \in V^{vir}(c_{p,q},0) \otimes \Pi(0) \subset V^{ns}(c_{p,q},0) \otimes F \otimes \Pi^{1/2}(0), \]

then Proposition 2.1 implies that vectors \( e, f, h \) generate a vertex subalgebra of \( V^{ns}(c_{p,q},0) \otimes F \otimes \Pi^{1/2}(0) \) isomorphic to \( V^k(sl(2)) \). In the case \( k = -2 \), vector \( \Omega_{p,q} \) generates a commutative vertex algebra isomorphic to \( M_T(0) \), and therefore Proposition 2.2 implies that \( e, f, h \) generate a quotient of \( V^k(sl(2)) \).

Next we need to prove that for \( n \geq 0 \) the following relations hold:

\[ h(n)x = \delta_{n,0} x, \ e(n)x = 0, \ x(n)f = \delta_{n,0} y \]
\[ h(n)y = -\delta_{n,0} y, \ e(n)y = -\delta_{n,0} x, \ f(n)y = 0 \]
\[ x(n)x = 2\delta_{n,0} e, \ y(n)y = -2\delta_{n,0} y, \ x(0)y = h, \ x(1)y = 2k1. \]
Let us first prove that \( x(n)f = \delta_{n,0}y \) for \( n \geq 0 \). Clearly \( x(n)f = 0 \) for \( n \geq 2 \). We have:

\[
x(1)f = \sqrt{2} \left( \Psi \left( \frac{3}{2} \right) \Omega_{p,q} + \frac{2k + 1}{4} \Psi \left( \frac{1}{2} \right) \right) e^{-\frac{1}{k} \left( \mu - \nu \right)}
\]

\[
= \sqrt{2} \left( -\frac{2k + 1}{4} \Psi \left( \frac{3}{2} \right) \Psi \left( \frac{3}{2} \right) \Psi \left( \frac{1}{2} \right) + \frac{2k + 1}{4} \Psi \left( \frac{1}{2} \right) \right) e^{-\frac{1}{k} \left( \mu - \nu \right)} = 0,
\]

\[
x(0)f = \sqrt{2} \left( \Psi \left( \frac{1}{2} \right) \Omega_{p,q} + \frac{2k + 1}{4} \Psi \left( \frac{3}{2} \right) \right) e^{-\frac{1}{k} \left( \mu - \nu \right)}
\]

\[
+ \sqrt{2} \Psi \left( \frac{3}{2} \right) \Omega_{p,q} \frac{1}{k} (\mu - \nu)(-1) e^{-\frac{1}{k} \left( \mu - \nu \right)}
\]

\[
+ \sqrt{2} \frac{2k + 1}{4} \Psi \left( \frac{3}{2} \right) \frac{1}{k} (\mu - \nu)(-1) e^{-\frac{1}{k} \left( \mu - \nu \right)} + \sqrt{2} \Psi \left( \frac{1}{2} \right) \nu(-1) e^{-\frac{1}{k} \left( \mu - \nu \right)}
\]

\[
= \sqrt{2} \left[ \frac{-\sqrt{2k} - \frac{3}{2} G(-3/2) + \Psi \left( \frac{1}{2} \right) \nu(-1) + \frac{2k + 1}{2} \Psi \left( \frac{3}{2} \right) } \right] e^{-\frac{1}{k} \left( \mu - \nu \right)} = y.
\]

By using an easy calculation we get:

\[
x(1)y = 2 \left( \frac{2k + 1}{2} - \frac{1}{2} \right) 1 = 2k1,
\]

\[
x(0)y = 2k \frac{1}{k} (\mu - \nu)(-1) + 2 \nu(-1) = 2 \mu(-1) = h,
\]

\[
e(0)y = \delta_{n,0} x,
\]

\[
x(n)x = 2 \delta_{n,0} e.
\]

Finally, we will check relation \( y(n)y = -2 \delta_{n,0} \). Clearly, \( y(n)y = 0 \) for \( n \geq 3 \). For the cases \( n = 0,1,2 \) we need to use Lemma 10. We have:

\[
y(2)y = -\frac{1}{2} (2k + 1)(4k + 5) - \frac{2k + 3}{3} c_{p,q} e^{-\frac{2}{k} \left( \mu - \nu \right)} = 0,
\]

\[
y(1)y = \left( \frac{(2k + 1)(4k + 5)}{2k} + \frac{2k + 3}{3} c_{p,q} \right) (\mu(-1) - \nu(-1)) e^{-\frac{2}{k} \left( \mu - \nu \right)} = 0,
\]

\[
y(0)y = \left( \frac{(2k + 1)(4k + 5)}{4} - \frac{2k + 3}{3} c_{p,q} \right) S_{2} \left( \frac{\nu - \mu}{k} \right) e^{-\frac{2}{k} \left( \mu - \nu \right)}
\]

\[
+ \left( (2k + 3) \omega_{p,q} + (2k + 1) \omega_{F} - i \sqrt{2k + 3} G \left( \frac{3}{2} \right) \Psi \left( \frac{1}{2} \right) \right) e^{-\frac{2}{k} \left( \mu - \nu \right)}
\]

\[
+ \left( 2 (\nu(-1)^2 + (k + 1) \nu(-2)) \right) e^{-\frac{2}{k} \left( \mu - \nu \right)}
\]

\[
= -2 \left( (k + 2) \omega_{p,q} + (k + 1) \omega_{p,q} - (\nu(-1)^2 + (k + 1) \nu(-2)) \right) e^{-\frac{2}{k} \left( \mu - \nu \right)} = -2 f.
\]

In this way we have checked relations (34)-(36). This finishes the proof of Theorem. □
11. Example: Weight and Whitaker Modules for $k = -5/4$

As we have seen in previous sections (see also [9, 15, 7]) for the analysis of weight, Whitaker and logarithmic modules, the explicit free-field realization is very useful.

The realization of $V_k(osp(1,2))$ is simpler in the cases when $L^{ns}(c_{p,q},0)$ is a 1-dimensional vertex algebra, and therefore $V_k(osp(1,2))$ can be realized on the vertex algebra $F \otimes \Pi^{1/2}(0)$. This happens only in the cases $k = -1/2$ and $k = -1/4$. In the case $k = -1/2$, $V_k(osp(1,2))$ can be realized on the tensor product of the Weyl vertex algebra $W$ with the fermionic vertex algebra $F$ of central charge $c = 1/2$. But this is essentially known in the literature, as a special case of the realization of $V_{-1/2}(osp(1,2n))$ (cf. [31]).

In this section we specialize our realization to the case $k = -5/4$. We get a realization of the vertex algebra $V_k(osp(1,2))$, which was investigated by D. Ridout, J. Snadden and S. Wood [46] by using different methods. It is also important to notice that the vertex algebra $V_k(osp(1,2))$ is a simple current extension of $V_k(sl(2))$:

$$V_{-5/4}(osp(1,2)) = L_{A_1}(-\frac{5}{4}\Lambda_0) + L_{A_2}(-\frac{9}{4}\Lambda_0 + \Lambda_1),$$

which can be also proved from our realization. Then $k + 2 = \frac{p}{q}$ for $p = 2, q = 4$. Since $c_{p,q} = 0$, we have that $L^{ns}(c_{p,q},0)$ is a 1-dimensional vertex algebra.

**Remark 7.** Note that $k = -h^\vee/6 - 1$ is a collapsing level for $g = osp(1,2)$ [16].

We have the following realization of $V_k(osp(2,1))$.

**Corollary 11.1.** Assume that $k = -\frac{5}{4}$.

1. There exists a non-trivial vertex superalgebra homomorphism
   $$\Phi : V_k(osp(1,2)) \to F \otimes \Pi^{1/2}(0)$$
   such that
   
   $$e \leftrightarrow e^\frac{2}{k}(\mu-\nu),$$
   $$h \leftrightarrow 2\mu(-1),$$
   $$f \leftrightarrow [(k+2)\omega_{3,4} - \nu(-1)^2 - (k+1)\nu(-2)] e^{-\frac{2}{k}(\mu-\nu)}$$
   $$x \leftrightarrow \sqrt{2}\Psi(-\frac{1}{2}) e^\frac{1}{k}(\mu-\nu)$$
   $$y \leftrightarrow \sqrt{2} \left[ \Psi(-\frac{1}{2}) \nu(-1) + \frac{2k+1}{2} \Psi(-\frac{3}{2}) \right] e^{-\frac{1}{k}(\mu-\nu)},$$

   where $\omega_{3,4} = \frac{1}{2} \Psi(-\frac{3}{2}) \Psi(-\frac{1}{2}) 1$.

2. Assume that $U$ (resp. $U^{tw}$) is any untwisted (resp. g-twisted) $\Pi^{1/2}(0)$–module. Then
   - $F \otimes U$ and $M^\pm \otimes U^{tw}$ are untwisted $V_k(osp(1,2))$–modules.
   - $F \otimes U^{tw}$ and $M^\pm \otimes U$ are Ramond twisted $V_k(osp(1,2))$–modules.
A classification of irreducible untwisted and twisted $V_k(\mathfrak{osp}(2, 1))$–modules were obtained [46, Theorem 9] by using Zhu’s algebra approach. All representations can be constructed using our free-field realization. Maybe most interesting examples are relaxed highest weight modules. We shall consider here only Neveu-Schwarz sector, i.e, non-twisted $V_k(\mathfrak{osp}(2, 1))$–modules.

Consider the $\sigma \otimes g$–twisted module $F \otimes \Pi^{1/2}(0)$–module $\mathcal{F}^\lambda := M^\pm \otimes \Pi^{-1/2}(\lambda)$ for $\lambda \in \mathbb{C}$. Then $\mathcal{F}^\lambda$ is an untwisted $V_k(\mathfrak{osp}(1, 2))$–module. As in Section 1, we define $E_{1,2}^\lambda = 1 \otimes e^{-\mu + \frac{1}{2} \frac{\mathbf{h}}{8} (\mu - \nu)}$. Then the action of $\mathfrak{osp}(1, 2)$ is given by
\[
e(0)E_{1,2}^\lambda = E_{1,2}^{\lambda+1},
\]
\[
h(0)E_{1,2}^\lambda = (-k + 2\lambda)E_{1,2}^\lambda,
\]
\[
f(0)E_{1,2}^\lambda = \left(\frac{1}{16} - (\lambda + \frac{1}{8})^2\right)E_{1,2}^{\lambda-1} = \left(\frac{3}{8} + \lambda\right)\left(1 - \lambda\right)E_{1,2}^{\lambda-1}
\]
\[
x(0)E_{1,2}^\lambda = \pm E_{1,2}^{\lambda+\frac{1}{2}}
\]
\[
y(0)E_{1,2}^\lambda = \mp \left(\frac{3}{8} + \lambda\right)E_{1,2}^{-\lambda-\frac{1}{2}}.
\]
Moreover, we have
\[
L_{sug}(n)E_{1,2}^\lambda = -\frac{1}{4} \delta_{n,0}E_{1,2}^\lambda \quad (n \geq 0).
\]

**Theorem 11.2.** Assume that $\lambda \notin \frac{1}{8} + \frac{1}{2}\mathbb{Z}$. Then $\mathcal{F}^\lambda$ is an irreducible $\mathbb{Z}_{\geq 0}$–graded $V_k(\mathfrak{osp}(1, 2))$–module whose character is
\[
\chi[\mathcal{F}^\lambda](q, z) = \text{Tr}_{\mathcal{F}^\lambda} q^{L_{sug}(0) - c/24} z^h(0) = q^{\frac{1}{8} \sum_{n=1}^{\infty} (1 + q^n)} \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{1 - q^{2n}} z^{2\lambda} 2^k \delta(z).
\]
(In the terminology of [15], $\mathcal{F}^\lambda$ corresponds to $\mathcal{C}_{\lambda, 0}$ where $\Lambda = 2\lambda + \frac{5}{4}$).

**Proof.** Note first that $\mathcal{F}^\lambda$ is $\mathbb{Z}_{\geq 0}$–graded and that its lowest component is $\mathcal{F}^\lambda(0) = \text{span}_\mathbb{C}\{E_{1,2}^{\lambda+i}, i \in \frac{1}{2}\mathbb{Z}\}$. The $\mathfrak{osp}(1, 2)$–action obtained above implies that $\mathcal{F}^\lambda(0)$ is irreducible for $\lambda \notin \frac{1}{8} + \frac{1}{2}\mathbb{Z}$. By using realization, we see that as $V_k(\mathfrak{sl}(2))$–module we have $\mathcal{F}^\lambda = \mathcal{E}_{1,2}^\lambda \otimes \mathcal{E}_{1,2}^{\lambda+1/2}$, where $\mathcal{E}_{1,2}^\lambda = L^{vir}(d_{3,4}, \frac{1}{16}) \otimes \Pi_{-1}(r)$. By Proposition 5.4, $\mathcal{E}_{1,2}^\lambda$ is irreducible for $r \notin \frac{1}{8} + \frac{1}{2}\mathbb{Z}$. Therefore, $\mathcal{F}^\lambda$ is a direct sum of two irreducible $V_k(\mathfrak{sl}(2))$–modules, which easily gives irreducibility result since $V_k(\mathfrak{osp}(1, 2))$ is a simple current extension of $V_k(\mathfrak{sl}(2))$. The character formula follows directly from the realization. $\square$

We also have the following result on the irreducibility of some Whittaker modules.

**Corollary 11.3.** We have: $M^\pm \otimes \Pi_\lambda$ is irreducible $V_k(\mathfrak{osp}(1, 2))$–module.

**Proof.** $M^\pm \otimes \Pi_\lambda$ is a $V_k(\mathfrak{osp}(1, 2))$–module by Corollary 11.1. The irreducibility follows from the fact that $M^\pm \otimes \Pi_\lambda$ is, as a $V_k(\mathfrak{sl}(2))$–module, isomorphic to the Whittaker module $L^{vir}(d_{3,4}, \frac{1}{16}) \otimes \Pi_\lambda$, which is irreducible. $\square$
Remark 8. A generalization of modules constructed above is as follows. Let $L^R(c, h)^\pm$ be the irreducible Ramond twisted modules for the simple $\mathcal{N} = 1$ Neveu-Schwarz vertex algebra $L^{ns}(c, 0)$ (cf. [12], [43], [34]).

For an arbitrary admissible level $k$, we have the following family of $\mathbb{Z}_{\geq 0}$–graded relaxed and Whittaker $V_k(osp(1, 2))$–modules:

$$L^R(c_{p,q}, h)^\pm \otimes M^\pm \otimes \Pi_{-1}^{1/2}(\lambda), \quad L^R(c_{p,q}, h)^\pm \otimes M^\pm \otimes \Pi_{\lambda}.$$
[21] T. Arakawa, Rationality of admissible affine vertex algebras in the category $O$, Duke Math. J., Volume 165, Number 1 (2016), 67–93. arXiv:1207.4857.

[22] T. Arakawa, V. Futorny, L. E. Ramirez, Weight representations of admissible affine vertex algebras, Comm. Math. Phys. 353 (2017), no.3, 1151–1178.

[23] J. Auger, T. Creutzig and D. Ridout, Modularity of logarithmic parafermion vertex algebras, arXiv:1704.05168v1.

[24] S. Berman, C. Dong and S. Tan, Representations of a class of lattice type vertex algebras, J. Pure Appl. Algebra 176 (2002) 27–47.

[25] T. Creutzig, A. Linshaw, Cosets of affine vertex algebras inside larger structures, arXiv:1407.8516v4 (2017).

[26] T. Creutzig, D. Ridout, Modular Data and Verlinde Formulae for Fractional Level WZW Models II, Nuclear Physics, B875:423-458, 2013, arXiv:1306.4388 [hep-th].

[27] C. Dong, J. Lepowsky, Generalized vertex algebras and relative vertex operators, Birkhäuser, Boston, 1993.

[28] E. Frenkel, Lectures on Wakimoto modules, opers and the center at the critical level, Adv. Math 195 (2005) 297-404.

[29] C. Eicher, Relaxed Highest Weight Modules from $D$-Modules on the Kashiwara Flag Scheme, arXiv:1607.0834.

[30] I.P. Ennes, A.V. Ramallo, J. M. Sanchez de Santos, On the free field realization of the osp(1, 2) current algebra, Phys.Lett. B389 (1996) 485–493, arXiv:hep-th/9606180.

[31] A.J. Feingold, I.B. Frenkel, Classical affine algebras, Adv. in Math., 56 (1985), pp. 117–172.

[32] J. Fjelstad, J. Fuchs, S. Hwang, A.M. Semikhatov and I. Yu. Tipunin, Logarithmic conformal field theories via logarithmic deformations, Nuclear Phys. B 633 (2002).

[33] B.L. Feigin, A.M. Semikhatov and I. Yu. Tipunin, Equivalence between chain categories of representations of affine $sl(2)$ and $N = 2$ superconformal algebras, J. Math. Phys. 39 (1998), 3865–3905.

[34] O. Blondeau-Fournier, F. Mathieu, D. Ridout and S. Wood, Superconformal minimal models and admissible Jack polynomials, Advances in Mathematics 314, pp. 71–123.

[35] M. Gaberdiel, Fusion rules and logarithmic representations of a WZW model at fractional level, Nuclear Phys. B 618 (2001) 407–436.

[36] V. G. Kac, M. Wakimoto, Quantum reduction and representation theory of superconformal algebras, Adv. Math., 185 (2004), pp. 400–458.

[37] K. Kawasetsu, D. Ridout, to appear.

[38] K Iohara and Y Koga, Representation theory of the Virasoro algebra. Springer Monographs in Mathematics, Springer-Verlag, London, 2011.

[39] M. Yu. Lashkevich, Superconformal 2D minimal models and an unusual coset constructions, Modern Physics Letter A (1993) 851–860, hep-th/9301093.

[40] H. Li, The physical superselection principle in vertex operator algebra theory, J. Algebra 196 (1997) 436–457.

[41] C. Lam, H. Yamauchi, 3-dimensional Griess algebras and Miyamoto involutions, arXiv:1604.04470.

[42] F. Lesage, P. Mathieu, J. Rasmussen, H. Saleur, Logarithmic lift of the $su(2)_{-1/2}$ model, Nucl. Phys. B 686:313-346, 2004, arXiv:hep-th/0311039.

[43] A. Milas, Characters, supercharacters and Weber modular functions, Journal fur die reine und angewandte Mathematik (Crelle’s Journal), 608 (2007), 35–64.

[44] D. Ridout, $sl(2)_{-1/2}$ and the Triplet Model, Nucl. Phys. B 835: 314-342, 2010.

[45] D. Ridout, Fusion in Fractional Level $sl(2)$-Theories with $k = -1/2$, Nucl. Phys. B 848:216–250, 2011.

[46] D. Ridout, J. Snadden and S. Wood, An admissible level $osp(1, 2)$–model: modular transformations and the Verlinde formula, arXiv:1706.04882.

[47] D. Ridout and S. Wood, From Jack Polynomials to Minimal Model Spectra, Journal of Physics, A48:045201, 2015.

[48] D Ridout and S Wood, Relaxed Singular Vectors, Jack Symmetric Functions and Fractional Level $sl(2)$ Models Nuclear Physics, B894:621-664, 2015.

[49] R. Sato, Modular invariant representations of the $N = 2$ superconformal algebra, to appear in Internat. Math. Res. Notices, arXiv:1706.04882.

[50] A. Semikhatov, The MFF Singular Vectors in Topological Conformal Theories, Modern Physics Letters A, June 1994, Vol. 09, No. 20 : pp. 1867-1896.

[51] A. Semikhatov, Inverting the Hamiltonian Reduction in String Theory, arXiv:hep-th/9410109.
[52] A. Tsuchiya and Y. Kanie, *Fock space representations of the Virasoro algebra – Intertwining operators* Publ. Res. Inst. Math. Sci, 22:259–327, 1986.

[53] M. Wakimoto, *Fock representations of affine Lie algebra \( A_1^{(1)} \),* Comm. Math. Phys. 104 (1986) 605–609.

Current address: Faculty of Science, Department of Mathematics, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia

E-mail address: adamovic@math.hr