Some Families of Directed Strongly Regular Graphs Obtained from Certain Finite Incidence Structures

Oktay Olmez · Sung Y. Song

Received: 24 February 2011 / Revised: 20 December 2012 / Published online: 18 September 2013 © Springer Japan 2013

Abstract We present many new directed strongly regular graphs by direct construction. We construct these graphs on the collections of antiflags of certain finite incidence structures. In this way, we confirm the feasibility of infinitely many parameter sets that was previously undetermined. We describe some examples of graphs together with their isomorphism classes to demonstrate the fact that our construction method is capable of producing many graphs with same parameters.

Keywords Tactical configuration · Doubly regular tournament · Association scheme

Mathematics Subject Classification (2000) 05E30

1 Introduction and Preliminaries

The concept of directed strongly regular graphs was introduced in 1988 by Duval [3] as a generalization of the concept of strongly regular graphs and doubly regular tournaments.1 The concept of strongly regular graphs was introduced by R. C. Bose in the early 1960s, although similar concept had been known earlier under the notion of association schemes. The interest in strongly regular graphs has been stimulated by the development of the theory of finite permutation groups and the classification of finite

1 A tournament is a loopless directed graph whose adjacency matrix $A$ satisfies $A + A^T + I = J$. A tournament is said to be doubly regular if $A$ satisfies $A^2 = \lambda A + \mu (J - I - A)$ for some positive integers $\lambda$ and $\mu$. 

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simple groups. It is well known that strongly regular graphs arise from many algebraic and geometric objects including finite fields, finite geometries, combinatorial designs and algebraic codes. The sources for directed strongly regular graphs (with $0 < t < k$) are also rich and diverse as reported by many researchers in [1–13]. The result of our work is to demonstrate this claim by showing another source through several explicit constructions.

A strongly regular graph with parameters $(v, k, \lambda, \mu)$ is defined as an undirected $k$-regular graph $G$ with $v$ vertices satisfying the properties that the number of common neighbors of vertices $x$ and $y$ is $\lambda$ if $x$ and $y$ are adjacent, and $\mu$ if $x$ and $y$ are non-adjacent distinct vertices. In terms of the adjacency matrix $A$ of a graph $G$, identity matrix $I$ and all-ones matrix $J$, the graph $G$ is a strongly regular graph with parameters $(v, k, \lambda, \mu)$ if and only if (i) $JA = AJ = kJ$ and (ii) $A^2 = kI + \lambda A + \mu (J - I - A)$.

A loopless directed graph $D$ with $v$ vertices is called directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ if and only if $D$ satisfies the following conditions:

(i) Every vertex has in-degree and out-degree $k$.
(ii) Every vertex $x$ has $t$ out-neighbors, all of which are also in-neighbors of $x$.
(iii) The number of directed paths of length two from a vertex $x$ to another vertex $y$ is $\lambda$ if there is an (directed) edge from $x$ to $y$, and is $\mu$ if there is no edge from $x$ to $y$.

In terms of adjacency matrix $A = A(D)$, $D$ is a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ if and only if (i) $JA = AJ = kJ$ and (ii) $A^2 = tI + \lambda A + \mu (J - I - A)$. A strongly regular graph and a doubly regular tournament may be viewed as a directed strongly regular graph with $t = k$ and $t = 0$, respectively. In what follows, a directed strongly regular graph with parameters $(v, k, t, \lambda, \mu)$ will be denoted by $\text{DSRG}-(v, k, t, \lambda, \mu)$. We will only consider $\text{DSRG}-(v, k, t, \lambda, \mu)$ with $0 < t < k$ throughout the paper.

In this paper, we prove existence (by explicit construction) of directed strongly regular graphs for families of parameter sets

1. $(v, k, t, \lambda, \mu) = (r(1 + ab)^2b, r(1 + ab)ab, ra^2b + a, ra^2b + a - ab - 1, ra^2b + a)$ for any positive integers $r, a$ and $b$ such that $r \geq 2$;  
2. $(v, k, t, \lambda, \mu) = ((1 + \frac{ls}{d})s, ls, ld, ld - d, ld)$ and  
3. $(v, k, t, \lambda, \mu) = ((1 + \frac{ls}{d})s, ls + s - 1, ld + s - 1, ld + s - 2, (l + 1)d)$ for any positive integers $d, l$ and $s$ such that $d|ls$ and $1 \leq l < \frac{ls}{d}$.

In this way, we confirm the existence of many infinite families of directed strongly regular graphs whose existence was previously undetermined. Examples of small orders include directed strongly regular graphs for parameter set $(v, k, t, \lambda, \mu)$ given by:

$(45, 30, 22, 19, 22), (54, 36, 26, 23, 26), (72, 48, 34, 31, 34), (75, 60, 52, 47, 52), (81, 54, 38, 35, 38), (90, 60, 11, 8, 11), (90, 60, 44, 38, 44), (99, 66, 46, 43, 46), (100, 40, 18, 13, 18), (108, 36, 13, 10, 13), (108, 72, 50, 47, 50), (108, 72, 52, 46, 52),$  

and $(108, 90, 80, 74, 80)$ among the feasible parameter sets for $v \leq 110$ listed in [7] by Hobart and Brouwer.
In our construction, each directed strongly regular graph is defined on a collection of antiflags of a tactical configuration. By definition, a tactical configuration with parameters \( (v, b, k, r) \) is a triple \( T = (P, B, I) \) where \( P \) is a \( v \)-element set, \( B \) is a collection of \( k \)-element subsets of \( P \) (called ‘blocks’) with \( |B| = b \), and \( I = \{(p, B) \in P \times B : p \in B\} \) such that each element of \( P \) (called a ‘point’) belongs to exactly \( r \) blocks. For the notational simplicity, we will denote the tactical configuration by \( T = (P, B) \) as incidence relation \( I \) is the natural incidence relation between the points and blocks. When the point set and the block set are clear from the context, we also denote a tactical configuration by \( T - (v, b, k, r) \).

The organization of the paper is as follows. Two major construction methods will be introduced in Sect. 2 and in Sect. 4. Discussions on two special cases for the first construction are discussed in Sect. 3. A variation of the second construction is discussed in Sect. 5. We then describe thirteen nonisomorphic graphs with parameters \( (10, 4, 2, 1, 2) \) that can be obtained from our method. The graph automorphism group of one of them acts transitively on its vertex set; and thus, we obtain a Schurian association scheme of class five which is a fission scheme of Johnson scheme \( J(5, 2) \). Finally, in Sect. 6, we make a few remarks and revisit the list of directed strongly regular graphs of small orders found in [7] and [9].

2 Construction of DSRG-\((r(1 + ab)^2b, r(1 + ab)ab, ra^2b + a, ra^2b + a - ab - 1, ra^2b + a)\)

In this section we describe the first construction method of the paper. We begin by describing the tactical configuration which will be used in this construction.

Let \( n, r \) and \( q \) be positive integers such that \( n = rq \) and \( q - 1 = ab \) for some positive integers \( a \) and \( b \) greater than 1.\(^2\) Let \( P = \{1, 2, \ldots, n\} \) be an \( n \)-element set with \( n = rq \). Let \( \{G_1, G_2, \ldots, G_r\} \) be a partition of \( P \) into \( r \) subsets (called ‘groups’) of size \( q \). For each \( j = 1, 2, \ldots, r \), let \( (G_j, \mathcal{P}_j) \) be a tactical configuration with parameters \( (v, b, k, r) = (q, q, a, a) \). We note that such a configuration always exists for given \( q \) and \( a \). For example, given a \( q \)-element set \( G = \{1, 2, \ldots, q\} \), we take \( \mathcal{P} = \{P_1, P_2, \ldots, P_q\} \) where \( P_i = \{i, i + 1, i + 2, \ldots, i + a - 1\} \) with addition modulo \( q \) to have the tactical configuration for our use. Let the blocks in \( \mathcal{P}_j \) be labeled by \( P_{j1}, P_{j2}, \ldots, P_{jq} \). It is clear that \( P_{ig} \cap P_{jh} = \emptyset \) if \( i \neq j \) since groups are disjoint.

Let \( \{B_1, B_2, \ldots, B_q\} \) be a family of \( ra \)-element subsets of \( P \) satisfying that

(i) every \( B_i \) contains exactly one block from every \( \mathcal{P}_j \), and  
(ii) each block in each \( \mathcal{P}_j \) is contained in \( B_i \) for exactly one \( i \).

For each \( g \in G_h \), let \( \{X_{g1}, X_{g2}, \ldots, X_{gb}\} \) be a partition of \( G_h \backslash \{g\} \) with \( |X_{gl}| = a \) for all \( l \in \{1, 2, \ldots, b\} \). That is, \( G_j \backslash \{g\} = X_{g1} \cup X_{g2} \cup \cdots \cup X_{gb} \) (disjoint union) and \( |X_{gl}| = a \) for every \( l \). Then we have the following tactical configuration.

\(^2\) We assume that all these integers are greater than 1 in this section, and will consider the case of \( a = 1 \) or \( b = 1 \) separately in the subsequent section.
Lemma 2.1 For each point \( g \in G_h \), and each \( l \in \{1, 2, \ldots, b\} \), if we define

\[ B_{g,l,j} = X_{gl} \cup (B_j \setminus P_{hj}) \]

where \( B_j = P_{1j} \cup P_{2j} \cup \cdots \cup P_{rj} \), for \( j = 1, 2, \ldots, q \),

\[ B_g = \{ B_{g,l,j} : 1 \leq l \leq b, \ 1 \leq j \leq q \}, \]

and

\[ B = \bigcup_{g=1}^{rq} B_g = \{ B_{g,l,j} : 1 \leq g \leq rq, \ 1 \leq l \leq b, \ 1 \leq j \leq q \}, \]

then the pair \( (P, B) \) forms a tactical configuration with parameters

\[ (v, b, k, r) = (rq, rq^2b, ra, rq(q - 1)). \]

Proof From the definition, \( v, b \) and \( k \) are clear. For \( r \), given a point \( g \in G_h \), we claim that the size of the set \( \{ B' \in B : g \in B' \} \) is the sum of \( q(q - 1) \) and \( (r - 1)qab \). The first summand \( q(q - 1) \) comes from the fact that \( g \) is a member of an \( X_{il} \) for each \( i \in G_h \setminus \{ g \} \), and each \( X_{il} \) is contained in \( q \) blocks in \( B_i \). The second summand \( (r - 1)q \cdot ab \) is the number of blocks \( B' \) such that \( g \in B' \in B \setminus \left( \bigcup_{i \in G_h} B_i \right) \) since \( g \) belongs to \( B_j \) for a different \( j \)'s (because \( g \) belongs to \( a \) blocks of \( G_h \), \( P_h \)), and each \( B_j \) is contained in \( b \) blocks of \( B_i \) for each of \( (r - 1)q \) points \( i \in P \setminus G_h \). This completes the proof. \( \square \)

We now use this tactical configuration to construct a directed strongly regular graph as follows.

Theorem 2.2 Let \( T \) be the above tactical configuration \( (P, B) \). Let \( D = D(T) \) be the directed graph defined on the vertex set

\[ V(D) = \{ (g, B) : B \in B_g, \ g \in P \} \]

with adjacency between vertices \((g, B)\) and \((g', B')\) defined by \((g, B) \rightarrow (g', B')\) if and only if \( g \in B' \). Then \( D \) is a directed strongly regular graph with parameters

\[ (v, k, t, \lambda, \mu) = (rq^2(q - 1)/a, \ rq(q - 1), \ r(q - 1)a + a, \]

\[ q(a - 1) + (r - 1)(q - 1)a, \ r(q - 1)a + a). \]

Proof It is clear that \( v = |B| = rq^2b = rq^2(q - 1)/a \). The parameter \( k \) is the size of the set \( \{ (g', B') \in V(D) : g \in B' \} \) for a given vertex \((g, B) \in V(D)\), and it equals to \( r = rq(q - 1) \). To compute \( t \), let \((g, B) \in V(D)\) with \( g \in G_h \) and let \( B = B_{g,l,j} = X_{gl} \cup (B_j \setminus P_{hj}) \) for some \( l \) and \( j \). Then \( t = |\{(g', B') \in V(D) : g' \in B, \ g \in B'\}| \).

We see that for each \( g' \in X_{gl} \subseteq B \), there are \( q \) blocks in \( B_{g'} \) all of which contain \( g \). On the other hand, for each of \( g' \in B_j \setminus P_{hj} \subseteq B \), there are \( ab = q - 1 \) blocks in \( B_{g'} \) containing \( g \). Together, we have \( t = qa + (q - 1)(r - 1)a \) as desired since \( |X_{gl}| = a \) and \(|B_j \setminus P_{hj}| = (r - 1)a\).
Let \((g, B)\) and \((g', B')\) be two adjacent vertices with \(g \in B'\). Suppose \(g' \in G_f\) and \(B' = B_{g', l, j} = X_{g' l} \cup (B_j \setminus P_{fj})\). In order to show that \(\lambda = \left| \{(g^*, B^*) \in V(D) : g^* \in B', B^* \ni g\} \right|\) is constant, we consider two cases:

Case 1. Suppose \(g \in G_f\), that is, \(g \in X_{g' l}\). Then (i) for each element, say \(g^*\), of \(X_{g' l} \setminus \{g\}\) there are \(ab\) blocks of \(B_{g^*}\) containing \(g\); while (ii) for each element \(g^* \in B_j \setminus P_{fj}\), there are \(ab\) blocks of \(B_{g^*}\) containing \(g\). Therefore, \(\lambda = (a - 1)q + (r - 1)a(q - 1)\) in this case.

Case 2. If \(g \notin G_f\), then \(g\) must be an element of \(B_j \setminus P_{fj}\). Suppose \(g \in P_{hj} \subset G_h\). Then (i) for each choice of \(g^* \in X_{g' l}\) there are \(ab = q - 1\) blocks possessing \(g\) (so available for \(B^*\)) in \(B_{g^*}\); (ii) for each choice of \(g^* \in P_{hj} \setminus \{g\}\), there are \(q\) blocks possessing \(g\) in \(B_{g^*}\); and (iii) for each element \(g^*\) of the remaining \((r - 2)a\) elements in \(B'\), there are \((q - 1)\) blocks available for \(B^*\) in \(B_{g^*}\). Hence together we have \(\lambda = a(q - 1) + (a - 1)q + (r - 2)a(q - 1)\) as well.

Hence \(\lambda\) has constant value \((a - 1)q + (r - 1)a(q - 1)\).

For \(\mu\), let \((g, B) \mapsto (g', B')\), (so \(g \notin B'\)). Let \(g\) belong to \(G_h\) for some \(h\). Then by the similar counting argument, we can verify that the number of vertices (\(g^*, B^*\)) such that \((g, B) \to (g^*, B^*) \to (g', B')\) (or equivalently the number of choices for \(g^*\) and \(B^*\) such that \(g^* \in B'\) and \(g \in B^*\)) is \(aq + (r - 1)a(q - 1)\) whether \(g\) and \(g'\) belong to the same group \(G_h\) for some \(h\) or not as \(a\) vertices in \(B'\) can be paired with \(q\) blocks while the rest can be paired with \(ab = (q - 1)\) blocks. This completes the proof. \(\Box\)

**Example 2.1** To illustrate the above construction, we consider the case when \(r = 2, q = 5,\) and \(a = b = 2\). This will provide us a DSRG-(100, 40, 18, 13, 18), which confirms the feasibility of the parameter set (cf. [7]).

Let \(P = \{0, 1, \ldots, 9\}, \ G_1 = \{1, 2, 3, 4, 5\}, \ G_2 = P \setminus G_1, \ P_1 = \{12, 23, 34, 45, 15\}\) and \(P_2 = \{67, 78, 89, 90, 60\}\). Then one example of tactical configuration that produces a DSRG-(100, 40, 18, 13, 18) may be described as in the following Table 1. In this table entries 23, 45 and 2367 represent the sets \{2, 3\}, \{4, 5\} and \{2, 3, 6, 7\} respectively. \(^{3}\)

As a consequence of Theorem 2.2 and the result of Duval [3, Theorem 7.1] on directed strongly regular graphs with \(t = \mu\), we have the following corollary.

**Corollary 2.3** Let \(r, q\) and \(a\) be positive integers such that \(a\) divides \((q - 1)\) as before. Then, for all positive integer \(m\), there exist directed strongly regular graphs with parameters \((v, k, t, \lambda, \mu) = (mrq^2(q - 1)/a, mrq(q - 1), m[r(q - 1)a + a], m[q(a - 1) + (r - 1)(q - 1)a], m[r(q - 1)a + a])\).  

\(^{3}\) For the notational simplicity, we will remove the brackets and commas between the elements when we list sets in a table throughout the paper.
Table 1  The blocks $B_i$ for each point $i$

| $i$ | $X_{i1}, X_{i2}$ | $B_{i,1,1}$ | $B_{i,1,2}$ | $B_{i,1,3}$ | $B_{i,1,4}$ | $B_{i,1,5}$ | $B_{i,2,1}$ | $B_{i,2,2}$ | $B_{i,2,3}$ | $B_{i,2,4}$ | $B_{i,2,5}$ |
|-----|------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1   | 23, 45           | 2367        | 2378        | 2389        | 2390        | 2360        | 4567        | 4578        | 4589        | 4590        | 4560        |
| 2   | 13, 45           | 1367        | 1378        | 1389        | 1390        | 1360        | 4567        | 4578        | 4589        | 4590        | 4560        |
| 3   | 12, 45           | 1267        | 1278        | 1289        | 1290        | 1260        | 3567        | 3578        | 3589        | 3590        | 3560        |
| 4   | 12, 35           | 1267        | 1278        | 1289        | 1290        | 1260        | 3467        | 3478        | 3489        | 3490        | 3460        |
| 5   | 12, 34           | 1267        | 1278        | 1289        | 1290        | 1260        | 4767        | 4789        | 4790        | 4760        | 4789        |
| 6   | 78, 90           | 7812        | 7823        | 7834        | 7845        | 7815        | 9012        | 9023        | 9034        | 9045        | 9015        |
| 7   | 89, 60           | 8912        | 8923        | 8934        | 8945        | 8915        | 6012        | 6023        | 6034        | 6045        | 6015        |
| 8   | 79, 60           | 7912        | 7923        | 7934        | 7945        | 7915        | 6012        | 6023        | 6034        | 6045        | 6015        |
| 9   | 67, 80           | 6712        | 6723        | 6734        | 6745        | 6715        | 8012        | 8023        | 8034        | 8045        | 8015        |
| 0   | 67, 89           | 6712        | 6723        | 6734        | 6745        | 6715        | 8912        | 8923        | 8934        | 8945        | 8915        |

3 Construction of Two Particular Classes of DSRGs

In this section we consider two families of directed strongly regular graphs obtained by the above construction for the particular cases with (i) $a = q - 1$ and $b = 1$ and (ii) $a = 1$ and $b = q - 1$.

3.1 DSRG-$(r(1 + a)^2, r(1 + a)a, ra^2 + a, ra^2 - 1, ra^2 + a)$

Let $r$ and $q$ be positive integers greater than 1, and $P = \{1, 2, \ldots, rq\}$ a set of $rq$ elements. Let $\{G_1, G_2, \ldots, G_r\}$ be a partition of $P$ into $r$ groups of size $q$. For each $j = 1, 2, \ldots, r$, let $\mathcal{P}_j$ be the family of all $(q - 1)$-element subsets of $G_j$. Let $B_1, B_2, \ldots, B_q$ be $(r(q - 1))$-element subsets of $P$ defined as follows:

1. Select one set from each family to have $B_1 = \bigcup_{j=1}^{r} P_{j1}$ where $P_{j1} \in \mathcal{P}_j$ for $j = 1, 2, \ldots, r$.
2. For $B_2$, select one set from each $\mathcal{P}_j \setminus \{P_{j1}\}$, for $j = 1, 2, \ldots, r$, so that $B_2 = \bigcup_{j=1}^{r} P_{j2}$.
3. Continue this process to have $B_i = P_{i1} \cup P_{i2} \cup \cdots \cup P_{ri}$ where $P_{ji} \in \mathcal{P}_j \setminus \{P_{j1}, P_{j2}, \ldots, P_{j(i-1)}\}$ for $i = 3, 4, \ldots, q$.

Then for each point $g \in G_h$, define $B_{g,j} = (G_h \setminus \{g\}) \cup (B_j \setminus P_{hj})$ for $j = 1, 2, \ldots, q$ and have $B_g = \{B_{g,1}, B_{g,2}, \ldots, B_{g,q}\}$. Then with $B_g = \bigcup_{g \in P} B_g = \{B_{g,j} : 1 \leq g \leq rq, 1 \leq j \leq q\}$.
the pair \((P, \mathcal{B})\) becomes a tactical configuration with parameters

\[(v, b, k, r) = (rq, rq^2, r(q - 1), rq(q - 1)).\]

**Corollary 3.1** Let \(T\) be the above tactical configuration \((P, \mathcal{B})\). Let \(D = D(T)\) be the directed graph defined on the vertex set

\[V(D) = \{(g, B) : B \in \mathcal{B}_g, g \in P\}\]

with adjacency between vertices \((g, B)\) and \((g', B')\) defined by \((g, B) \rightarrow (g', B')\) if and only if \(g \in B'\). Then \(D\) is a directed strongly regular graph with parameters

\[(v, k, t, \lambda, \mu) = (rq^2, rq(q - 1), (q - 1)(rq - r + 1),
\quad r(q - 1)^2 - 1, (q - 1)(rq - r + 1)).\]

**Proof** Omitted. \(\square\)

**Example 3.1** Let \(r = 2, q = 3, P = \{1, 2, 3, 4, 5, 6\}, G_1 = \{1, 2, 3\}\) and \(G_2 = \{4, 5, 6\}, G_2 = \{4, 5, 6\}\). With the tactical configuration described in Table 2 below, we have a DSRG-(18, 12, 10, 7, 10). This graph is shown to be nonisomorphic to its orientation-reversing conjugate. Here the orientation-reversing conjugate of a graph \(D\), we mean the graph whose adjacency matrix is the transpose of the adjacency matrix of \(D\). By Jørgensen [10] we know that these are the two nonisomorphic graphs with the parameters (18, 12, 10, 7, 10) and there is no more.

**Example 3.2** Let \(r = 3, q = 2, P = \{1, 2, 3, 4, 5, 6\}, G_1 = \{1, 2\} G_2 = \{3, 4\}\) and \(G_3 = \{5, 6\}, G_3 = \{5, 6\}\). With the tactical configuration described in Table 3 below, we have a

| \(i\) | \(B_{i,j}, j = 1, 2, 3\) |
|---|---|
| 1 | 2356, 2346, 2345 |
| 2 | 1356, 1346, 1345 |
| 3 | 1256, 1246, 1245 |
| 4 | 2356, 1356, 1256 |
| 5 | 2346, 1346, 1246 |
| 6 | 2345, 1345, 1245 |

**Table 3** \(T = (6, 12, 3, 6)\)

| \(i\) | \(B_{i,j}, j = 1, 2\) |
|---|---|
| 1 | 235, 246 |
| 2 | 135, 146 |
| 3 | 415, 426 |
| 4 | 315, 326 |
| 5 | 613, 624 |
| 6 | 513, 524 |
It is easy to see that there are $2^6 = 64$ different tactical configurations available for the given combinations of $r = 3$ and $q = 2$. These 64 tactical configurations yield seven nonisomorphic graphs. (Their adjacency matrices are given below.) It is easy to verify that the orientation-reversing conjugates of the seven graphs are all nonisomorphic. Therefore, our construction provides us 14 distinct graphs with parameters $(12, 6, 4, 2, 4)$. The table showing the description of the automorphism groups of these graphs and the size of the isomorphism classes are followed by the adjacency matrices (Table 4).

**Remark 3.3** In Example 3.2, we produced fourteen directed strongly regular graphs with parameters $(12, 6, 4, 2, 4)$. However, Jørgensen has shown that there exist exactly twenty nonisomorphic graphs with parameters $(12, 5, 3, 2, 2)$, which are the complementary graphs of directed strongly regular graphs with parameters $(12, 6, 4, 2, 4)$. Therefore, there are six graphs that are not obtained from the above construction.

**Remark 3.4** Due to the above construction, the existence of the following feasible parameter sets listed on the table in [7] with $v \leq 110$, is now realized (Table 5).

### 3.2 DSRG-$(r(1 + b)^2b, r(1 + b)b, rb + 1, rb - b, rb + 1)$

The graphs constructed in this subsection may be obtained from Sect. 2 with $a = 1$. However, we construct them in a different way to demonstrate their connection to the graphs constructed in [13]. This construction produces many more graphs than the method reported in [13] including an infinite family of new graphs which were previously unknown. For example, DSRG-$(90, 30, 11, 8, 11)$ (for $r = 5$ and $q = 3$) and DSRG-$(108, 36, 13, 10, 13)$ (for $r = 6$ and $q = 3$) are the new graphs among the unknown graphs listed in [7].

Let $r$ and $q$ be positive integers greater than 1 such that $r \leq qr - 3$, and let $P$ be a set of $rq$ elements. Let $\mathcal{P} = \{G_1, G_2, \ldots, G_r\}$ be a partition of $P$ into $r$ groups of size $q$. Let

$$B = \{B \subset P : |B \cap G_i| = 1 \quad \text{for all} \quad i = 1, 2, \ldots, r\}.$$ 

Then $B$ consists of $q^r$ subsets (which will be called ‘blocks’) of $P$ of size $r$. For each $i \in P$, let

$$B_i = \{B \in B : i \in B\}.$$ 

Then $|B_i| = q^{r-1}$. Let $B_i$ be partitioned into $q^{r-2}$ parts each of which consists of $q$ blocks such that no two blocks in the same part share any other common point besides $i$. To be precise, let $B_{i,1}, B_{i,2}, \ldots, B_{i,w}$, where $w = q^{r-2}$, denote the parts of the partition of $B_i$, so that

$$B_i = \bigcup_{j=1}^{w} B_{i,j}.$$
Table 4 The adjacency matrices of the graphs with parameters (12, 6, 4, 2, 4) constructed in Corollary 3.1

\[
N_1 = 
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
N_2 = 
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
N_3 = 
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
N_4 = 
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
N_5 = 
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
N_6 = 
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Table 4 continued

\[
N_7 = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
\end{pmatrix}
\]

Table 5 The new DSRGs (with \( v \leq 110 \)) constructed by Corollary 3.1

| Graph | Automorphism Group | Size of Isomorphism Class |
|-------|-------------------|--------------------------|
| \( N_1 \) | \( D_12 \) | 4 |
| \( N_2 \) | \( D_6 \) | 6 |
| \( N_3 \) | \( C_2 \times C_2 \) | 12 |
| \( N_4 \) | \( C_2 \times C_2 \) | 12 |
| \( N_5 \) | \( D_{12} \) | 4 |
| \( N_6 \) | \( S_4 \) | 2 |
| \( N_7 \) | \( C_2 \) | 24 |

\[
q \quad r \quad v \quad k \quad t \quad \lambda \quad \mu \quad m
\]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 3 | 5 | 45 | 30 | 22 | 19 | 22 | 1 |
| 3 | 6 | 54 | 36 | 26 | 23 | 26 | 1 |
| 3 | 8 | 72 | 48 | 34 | 31 | 34 | 1 |
| 3 | 9 | 81 | 54 | 38 | 38 | 38 | 1 |
| 3 | 11 | 99 | 66 | 46 | 43 | 46 | 1 |
| 3 | 12 | 108 | 72 | 50 | 47 | 50 | 1 |
| 5 | 3 | 75 | 60 | 52 | 47 | 52 | 1 |
| 6 | 3 | 108 | 90 | 80 | 74 | 80 | 1 |
| 3 | 5 | 90 | 60 | 44 | 38 | 44 | 2 |
| 3 | 6 | 108 | 72 | 52 | 46 | 52 | 2 |

where (i) \( B_{i,j} \cap B_{i,h} = \emptyset \), for any distinct \( j, h \in \{1, 2, \ldots, w\} \); (ii) \( |B_{i,j}| = q \), for every \( j \in \{1, 2, \ldots, w\} \); and (iii) \( B \cap C = \{i\} \) for any distinct \( B, C \in B_{i,j} \) for each \( j \in \{1, 2, \ldots, w\} \).

Given any injective map \( \pi : \{1, 2, \ldots, rq\} \to \{1, 2, \ldots, w\} \), if \( g \in G_h \), let \( C^\pi_g \) denote the collection of all blocks in the \( B_{i,\pi(i)} \) for all \( i \in G_h \setminus \{g\} \), and let \( B^\pi \) be the union of \( C^\pi_g \) over all points in \( P \). That is, for each given injection \( \pi \), define

\[
B^\pi = \bigcup_{g \in P} C^\pi_g \quad \text{where} \quad C^\pi_g = \bigcup_{i \in G_h \setminus \{g\}} B_{i,\pi(i)} \quad \text{for} \quad g \in G_h.
\]
Then $T^\pi = (P, B^\pi)$ becomes a tactical configuration with parameters\(^4\)

$$(v, b, k, r) = (rq, rq^2(q - 1), r, rq(q - 1)).$$

We obtain a directed strongly regular graph from this tactical configuration as follows.

**Corollary 3.2** Let $D = D(T^\pi)$ be the directed graph defined on the vertex set

$$V(D) = \{(g, B) : B \in C^\pi_g, \; g \in P\}$$

with adjacency defined by $(g, B) \to (g', B')$ if and only if $g \in B'$. Then $D$ is a directed strongly regular graph with parameters

$$(v, k, t, \lambda, \mu) = (rq^2(q - 1), \; rq(q - 1), \; rq - r + 1, \; rq - r - q + 1, \; rq - r + 1).$$

**Proof** Omitted. \(\square\)

**4 Construction of DSRG-$\langle ns, ls + s - 1, ld + s - 1, ld + s - 2, ld + d \rangle$ and DSRG-$\langle ns, ls, ld, ld - d, ld \rangle$ with $d(n - 1) = ls$**

In this section we present our second construction method which is slightly different from any of previous constructions. This construction does not produce previously unknown graphs, but it can produce graphs coming from many different methods.\(^5\)

Let $n, d, l$ and $s$ be positive integers such that $d(n - 1) = ls$, or equivalently, $n = 1 + \frac{ls}{d}$. Let $P = \{1, 2, \ldots, n\}$. For each $i \in P$, suppose $\mathcal{P}_i = (P \setminus \{i\}, B_i)$ is a tactical configuration with parameters $(v, b, k, r) = (n - 1, s, l, d)$. Clearly, there exists such a tactical configuration subject to the conditions: $d \mid ls, \; 1 \leq d < s, \; 1 \leq l \leq n - 2$.

We define the tactical configuration $T = (P, B)$ with $B = \bigcup_{i=1}^n B_i$ by collecting the blocks of all configurations $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$. Then $T = (P, B)$ has parameters $(v, b, k, r) = (n, ns, l, ls)$. Using this configuration, we now construct two directed strongly regular graphs on the set

$$V = \{(g, B) : B \in B_g, \; g \in P\}.$$

**Theorem 4.1** Let $T = (P, B)$ be the above tactical configuration $T - (n, ns, l, ls)$ where $n = 1 + \frac{ls}{d}$. Let $D_1 = D_1(T)$ be the directed graph with its vertex set

$$V = \{(g, B) : B \in B_g, \; g \in P\}$$

and adjacency defined by

$$(g, B) \to (g', B') \text{ if and only if } g \in B'.$$

Then $D_1$ is a directed strongly regular graph with the parameters

---

\(^4\) Note that $T^\pi$ may have repeated blocks unless the disjointness of the union defining $B^\pi$ and $C^\pi_g$ are guaranteed.

\(^5\) See the examples in the Table 11 in Sect. 6.
\((v, k, t, \lambda, \mu) = (ns, ls, ld, (l - 1)d, ld)\).

**Proof** It is clear that \(v = \sum_{g \in P} |B_g| = ns\). A vertex \((g', B')\) is to be an out-neighbor of \((g, B), g'\) can be any point different from \(g\), and \(B'\) can be any member of \(B_g\) containing \(g\). Since there are \(d\) blocks in \(B_g\) containing \(g\), \(k = (n - 1)d\).

A vertex \((g', B')\) is to be a (in and out)-neighbor of \((g, B), g'\) should be one of \(l = |B|\) points while \(B'\) must be any one of \(d\) blocks containing \(g\) and belonging to \(B_g\). Hence \(t = ld\).

Given \((g, B) \rightarrow (g', B')\), (and so \(g \in B'\)), the number of vertices \((g^*, B^*) \in V(D)\) such that \(g^* \in B', B^* \in B_g^*\) and \(B^* \ni g, (l - 1)d\) since there are \(l - 1\) choices for \(g^*\) in \(B' \setminus \{g\}\) and for any \(g^*\), there are \(d\) blocks in \(B_g^*\) that contain \(g\). Thus, \(\lambda = (l - 1)d\).

For \(\mu\), let \((g, B) \rightarrow (g', B')\), (and so \(g \notin B'\)). For any point \(g^*\) in \(B'\), there are \(d\) blocks in \(B_g^*\) that contain \(g\). Hence we have \(\mu = ld\). This completes the proof. \(\square\)

**Theorem 4.2** Let \(T = (P, B)\) be the same tactical configuration as in the above theorem. Let \(D_2 = D_2(T)\) be the directed graph with its vertex set

\[V = \{(g, B) : B \in B_g, g \in P\}\]

and adjacency defined by

\[(g, B) \rightarrow (g', B') \quad \text{if and only if either} \quad g \in B' \quad \text{or} \quad g = g' \quad \text{and} \quad B \neq B'.\]

Then \(D_2\) is a directed strongly regular graph with the parameters

\[(v, k, t, \lambda, \mu) = (ns, ls + s - 1, ld + s - 1, ld + s - 2, (l + 1)d).\]

**Proof** It can be proved by the routine counting argument.

**Corollary 4.3** Let \(T = (P, B)\) be the tactical configuration, and let \(D_1 = D_1(T)\) and \(D_2 = D_2(T)\) as in the above theorems. In the constructions for \(D_1\) and \(D_2\), if we take the multi-set consisting of \(m\) copies of the vertex set \(V\) as its vertex set, we can obtain the directed strongly regular graphs with parameters

\[(v, k, t, \lambda, \mu) = (mns, mls, mld, m(l - 1)d, mld)\]

and

\[(v, k, t, \lambda, \mu) = (mns, m(ls + s) - 1, m(ld + s) - 1, m(ld + s) - 2, m(l + 1)d),\]

respectively.

**Proof** Omitted. (cf. [3, 7.1]) \(\square\)
In this section, as a concrete realization of the construction method discussed in the previous section, we consider the particular case with \( d = 1 \). Let \( l \) and \( s \) be positive integers. Consider the \((ls + 1)\)-element set \( P = \{1, 2, \ldots, ls + 1\} \). For each \( i \in P \), let \( B_i = \{B_{i1}, B_{i2}, \ldots, B_{is}\} \) be a partition of \( P \setminus \{i\} \) into \( s \) parts (blocks) of equal size \( l \). Let

\[
B = \bigcup_{i=1}^{ls+1} B_i = \{B_{ig} : 1 \leq g \leq s, 1 \leq i \leq ls + 1\}.
\]

Then the pair \((P, B)\) forms a tactical configuration \( T - (ls + 1, s(ls + 1), l, ls) \). We construct directed strongly regular graphs on the set

\[
V = \{(i, B) : B \in B_i, i \in P\}
\]

in two ways.

**Theorem 5.1** Let \((P, B)\) be \( T - (ls + 1, s(ls + 1), l, ls) \). Let \( D_1 = D_1(T) \) be the directed graph with its vertex set

\[
V = \{(i, B_{ig}) \in P \times B : 1 \leq i \leq ls + 1, 1 \leq g \leq s\}
\]

and adjacency defined by

\[
(i, B_{ig}) \to (j, B_{jh}) \quad \text{if and only if} \quad i \in B_{jh}.
\]

Then \( D_1 \) is a directed strongly regular graph with the parameters

\[
(v, k, t, \lambda, \mu) = (ls^2 + s, ls, l, l - 1, l).
\]

**Proof** Straightforward.

**Theorem 5.2** Let \((P, B)\) be \( T - (ls + 1, ls^2 + s, l, ls) \). Let \( D_2 = D_2(T) \) be the directed graph with its vertex set

\[
V = \{(i, B_{ig}) \in P \times B : 1 \leq i \leq ls + 1, 1 \leq g \leq s\}
\]

and adjacency defined by

\[
(i, B_{ig}) \to (j, B_{jh}) \quad \text{if and only if} \quad \text{either} \quad i \in B_{jh} \quad \text{or} \quad i = j \quad \text{and} \quad B_{ig} \neq B_{jh}.
\]

Then \( D_2 \) is a directed strongly regular graph with the parameters

\[
(v, k, t, \lambda, \mu) = (ls^2 + s, ls + s - 1, l + s - 1, l + s - 2, l + 1).
\]
Proof Straightforward. □

Corollary 5.3 Let \((P, B)\) be the tactical configuration \(T - (ls + 1, ls^2 + s, l, ls)\) as in the above. Let \(D_1 = D_1(T)\) and \(D_2 = D_2(T)\). In the constructions for \(D_1\) and \(D_2\), if we take the multi-set consisting of \(m\) copies of the vertex set \(V\) as its vertex set, we can obtain the directed strongly regular graph with parameters

\[(v, k, t, \lambda, \mu) = (m(ls^2 + s), mls, ml, m(l - 1), ml)\]

and

\[(v, k, t, \lambda, \mu) = (m(ls^2 + s), m(ls + s) - 1, m(l + s) - 1, m(l + s) - 2, m(l + 1)),\]

respectively.

Proof Omitted. □

In the above constructions in Theorems 5.1 and 5.2, different tactical configurations coming from different partitions of \(P\) may produce nonisomorphic graphs with the same parameters as before. For example, for \(l = s = 2\), we obtain 13 different directed strongly regular graphs with the same parameter set \((v, k, t, \lambda, \mu) = (10, 4, 2, 1, 2)\).

To illustrate the above claim and to show the connections to other combinatorial structures, we will describe them in detail in the remainder of the current section.

5.1 Isomorphism Classes of DSRG-\((10, 4, 2, 1, 2)\)

When \(l = s = 2\), the number of ways to form tactical configurations with parameters \((v, b, k, r) = (5, 10, 2, 4)\) is 243. Let \(F\) be the set of these tactical configurations.

Each tactical configuration \(T = (P, B) \in F\) gives rise to a directed strongly regular graph \(D(T)\) with its vertex set \(V(T) = \{(i, Bij) : i \in P, \ B_{ij} \in B\}\) by Theorem 5.1. Consider the action of \(S_5\) on \(F\) under the rule that \(T_1 = T_2\) if and only if \(V(T_1) = V(T_2)\) with natural action on \(B_{ij}\); i.e., \((B_{ij})^\sigma = \{x^\sigma, y^\sigma\}\) if \(B_{ij} = \{x, y\}\). Under this action \(F\) is partitioned into seven orbits. The tactical configurations belong to the same orbit produce isomorphic directed strongly regular graphs. Let \(T_1, T_2, \ldots, T_7\) denote the representatives of the orbits. The block sets of these representatives are given in Table 6.

The Table 7 shows the group structure of each stabilizer of \(T_i, i = 1, 2, \ldots, 7\) and its generators. The last row of the table indicates the size of the orbit represented by the corresponding tactical configuration.

Let \(D(T_i), i = 1, 2, \ldots, 7\) be the directed strongly regular graphs with parameters \((10, 4, 2, 1, 2)\) obtained from the seven orbit representatives given in Table 6 by Theorem 5.1. Then it is shown that the orientation-reversing conjugates of \(D(T_i)\) for
Table 6 The block sets of the representatives of seven orbits

| i     | $B(T_1)$ | $B(T_2)$ | $B(T_3)$ | $B(T_4)$ | $B(T_5)$ | $B(T_6)$ | $B(T_7)$ |
|-------|----------|----------|----------|----------|----------|----------|----------|
| 1     | 23, 45   | 23, 45   | 23, 45   | 23, 45   | 23, 45   | 23, 45   | 23, 45   |
| 2     | 13, 45   | 13, 45   | 13, 45   | 14, 35   | 13, 45   | 13, 45   | 13, 45   |
| 3     | 12, 45   | 14, 25   | 12, 45   | 15, 24   | 14, 25   | 14, 25   | 14, 25   |
| 4     | 12, 35   | 12, 35   | 12, 35   | 13, 25   | 12, 35   | 13, 25   | 13, 25   |
| 5     | 12, 34   | 12, 34   | 13, 24   | 12, 34   | 14, 23   | 13, 24   | 14, 23   |

Table 7 Stabilizers and the size of orbits for the action of $S_5$ on $F$

| $T_1$   | $T_2$   | $T_3$   | $T_4$   | $T_5$   | $T_6$   | $T_7$   |
|---------|---------|---------|---------|---------|---------|---------|
| $D_8$   | $C_2 \times C_2$ | $C_2$   | $C_5 \rtimes C_4$ | $C_2$   | $C_2$   | $D_{10}$   |
| (1524), (15)(24) | (12)(45), (15)(24) | (23)(45) | (15234), (1345) | (15)(23) | (15)(34) | (12435), (12)(45) |
| 15  | 30   | 60   | 6   | 60  | 60  | 12   |

$i = 1, 2, \ldots, 6$ are nonisomorphic to any of the seven. The graph $D(T_7)$ is isomorphic to its orientation-reversing conjugate. Therefore, together with their conjugates, our construction produces thirteen directed strongly regular graphs for the given parameter set. However, by Jørgensen [10] it is known that there are sixteen graphs for the given parameter set (Table 8).

We now describe the three graphs that are not produced by our construction. Their adjacency matrices are given by $J_8$ and $J_9$ below, and the transpose of $J_8$ gives for the third. The graph of $J_9$ is self-transpose and has the trivial automorphism group. The automorphism groups for the graphs of $J_8$ and its transpose are isomorphic to $C_2$.

\[
J_8 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

\[
J_9 = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}
\]

6 This graph was constructed in [11, Ex. 4.2].

7 Jørgensen kindly provided us the adjacency matrices of all sixteen graphs.
Table 8  The adjacency matrices for seven graphs, $D(T_1)$, $D(T_2)$, \ldots, $D(T_7)$

|   | $D(T_1)$          |   | $D(T_2)$          |
|---|------------------|---|------------------|
| 1 | 0 1 1 0 0 0 1 0 1 | 1 | 0 1 0 1 0 1 0 0 1 |
| 2 | 1 0 1 0 0 0 1 0 1 | 2 | 1 0 1 0 0 0 1 0 1 |
| 3 | 1 1 0 0 1 1 0 0 0 | 3 | 0 1 0 1 1 0 1 0 0 |
| 4 | 1 1 0 0 1 1 0 0 0 | 4 | 1 0 1 0 0 1 0 1 0 |
| 5 | 0 0 0 1 1 0 1 0 1 | 5 | 0 1 0 1 1 0 1 0 0 |
| 6 | 1 0 1 0 0 0 1 0 1 | 6 | 0 0 0 1 1 0 1 1 0 |
| 7 | 0 0 0 1 1 0 1 0 1 | 7 | 0 0 0 1 1 0 1 1 0 |

|   | $D(T_3)$          |   | $D(T_4)$          |
|---|------------------|---|------------------|
| 1 | 0 1 1 0 1 0 0 0 1 | 1 | 0 1 0 1 0 1 0 0 1 |
| 2 | 1 0 1 0 1 0 1 0 0 | 2 | 1 0 1 0 0 0 1 0 1 |
| 3 | 1 1 0 0 0 0 0 1 0 1 | 3 | 0 1 0 1 0 1 0 1 0 |
| 4 | 1 1 0 0 0 0 0 1 0 1 | 4 | 0 1 0 1 0 1 0 1 0 |
| 5 | 0 1 1 0 1 0 0 0 1 | 5 | 0 0 1 0 1 1 0 0 1 |
| 6 | 0 0 0 1 0 1 1 0 1 | 6 | 0 1 0 1 0 0 1 0 1 |
| 7 | 1 0 1 0 0 0 0 1 0 1 | 7 | 0 0 0 1 1 0 0 1 0 |
| 8 | 0 0 0 1 0 1 1 0 1 | 8 | 0 1 1 0 1 0 0 1 0 |

|   | $D(T_5)$          |   | $D(T_6)$          |
|---|------------------|---|------------------|
| 1 | 0 1 0 0 0 1 0 1 0 1 | 1 | 0 1 0 1 0 1 0 0 1 |
| 2 | 1 0 1 1 0 1 0 0 0 0 | 2 | 1 0 1 0 0 1 0 1 0 |
| 3 | 1 1 0 1 0 0 0 0 1 0 | 3 | 1 1 0 1 0 0 0 1 0 |
| 4 | 0 0 1 0 1 0 1 0 1 0 | 4 | 0 1 0 1 0 1 0 1 0 |
| 5 | 0 0 0 1 0 1 1 0 1 0 | 5 | 0 0 1 0 1 0 1 1 0 |
| 6 | 1 0 1 0 1 0 0 0 0 0 | 6 | 0 1 0 1 0 1 0 0 1 |
| 7 | 0 0 0 1 0 1 1 0 1 | 7 | 0 0 0 1 1 0 1 1 0 |
| 8 | 1 1 0 1 0 0 0 0 1 0 | 8 | 1 1 0 1 0 0 0 1 0 |

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Table 8  continued

\[
D(T_i)
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

The rows of the matrices are indexed by the vertices of corresponding graphs

Table 9  The block sets of six tactical configurations that are isomorphic to \(T_4\)

|    | (23), (45) | (14), (35) | (15), (24) | (13), (25) | (12), (34) |
|----|-------------|-------------|-------------|-------------|-------------|
| 23 | 23, 45      | 25, 34      | 25, 34      | 24, 35      | 24, 35      |
| 14 | 15, 34      | 14, 35      | 13, 45      | 15, 34      | 13, 45      |
| 15 | 14, 25      | 12, 45      | 15, 24      | 12, 45      | 14, 25      |
| 13 | 12, 35      | 15, 23      | 12, 35      | 13, 25      | 15, 23      |
| 12 | 13, 24      | 13, 24      | 14, 23      | 14, 23      | 12, 34      |

Top row indicates the isomorphism \(\sigma \in S_5\) to the first tactical configuration

Remark 5.1  For each graph \(D(T_i), i = 1, 2, \ldots, 7\), we can see that the full automorphism group of \(D(T_i)\) is determined by the stabilizer of \(T_i\) under the action of the symmetric group \(S_5\) on \(F\). Hence from the knowledge of the orbits or the stabilizers of the permutation action of \(S_5\) on \(F\), we can obtain the number of distinct graphs produced by our construction. For instance, as we have seen it in Table 7 we have the following six different tactical configurations all of which produce graph \(D(T_4)\) (Table 9).

Therefore, the graph \(D(T_4)\) is isomorphic to the graphs obtained from the following vertex sets.

| \(V = V_1\) | \(V_2\) | \(V_3\) | \(V_4\) | \(V_5\) | \(V_6\) |
|-------------|---------|---------|---------|---------|---------|
| (1), (23), (14) | (1), (23), (14) | (1), (25), (13) | (1), (25), (13) | (1), (24), (13), (15) | (1), (24), (13), (15) |
| (1), (24), (23) | (1), (24), (23) | (1), (25), (14) | (1), (25), (14) | (1), (24), (15), (23) | (1), (24), (15), (23) |
| (1), (35), (34) | (1), (35), (34) | (1), (35), (15) | (1), (35), (15) | (1), (35), (14), (23) | (1), (35), (14), (23) |
| (1), (45), (43) | (1), (45), (43) | (1), (45), (1) | (1), (45), (1) | (1), (45), (13), (24) | (1), (45), (13), (24) |
| (1), (54), (53) | (1), (54), (53) | (1), (54), (13) | (1), (54), (13) | (1), (54), (14), (25) | (1), (54), (14), (25) |
5.2 Association Schemes and a SRG Arising from a DSRG-(10, 4, 2, 1, 2)

Let $A$ be the adjacency matrix of $D(T_4)$, and let $\tilde{A}$ be the matrix given by

$$\tilde{A}_{ij} = \begin{cases} 1 & \text{either } A_{ij} = 1 \text{ or } A_{ji} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Let $G$ be the graph whose adjacency matrix is $\tilde{A}$. Then $G$ is the strongly regular graph with parameters $(v, k, \lambda, \mu) = (10, 6, 3, 4)$, which is known as Johnson graph $J(5, 2)$. We note that $J(5, 2)$ is also obtained from the Jørgensen’s graph of $J_9$ by ‘symmetrizing’ the matrix $J_9$. In fact, this is the only strongly regular graph that can be obtained from any of the directed strongly regular graphs with parameters $(10, 4, 2, 1, 2)$ through the symmetrization process.

Among the directed strongly regular graphs with parameters $(10, 4, 2, 1, 2)$, $D(T_4)$ has the largest automorphism group. It is the only one that has vertex transitive automorphism group. The automorphism group $H = \text{Aut}(D(T_4))$ is isomorphic to the group $C_5 \rtimes C_4$ of order 20. From the transitive permutation group $H$ on the vertex set of $D(T_4)$, we obtain a 5-class association scheme. Let $\mathcal{X}(H, V(T_4))$ denote this association scheme. Then its association relation table is given by the matrix on the left below (Table 10).

It is observed that $\mathcal{X}(H, V(T_4))$ is isomorphic to the 5-class non-commutative association scheme labeled as $Y$ on [14, p.255]. This scheme has two fusion schemes of class 3; they are $K_2 \times K_5$, the direct product of two trivial schemes of order 2 and 5, and $C_5 \wr K_2$, the wreath product of the scheme coming from pentagon and the trivial scheme of order 2. The scheme $\mathcal{X}(H, V(T_4))$ also has three symmetric fusion schemes of class 2, $K_2 \wr K_5$, $K_5 \wr K_2$ and the Johnson scheme $J(5, 2)$. The Johnson scheme $J(5, 2)$ is obtained from $\mathcal{X}(H, V(T_4))$ by fusing the relations $R_1$, $R_3$, and $R_4$ together, and fusing $R_2$ and $R_5$ together as easily observed from the above relation tables.

The edge set of $D(T_4)$ coincides with $R_1 \cup R_3$. The orientation-reversing conjugate of $D(T_4)$ is the graph with edge set $R_1 \cup R_4$. The edge set of Johnson graph $J(5, 2)$ is

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\[8\] This graph was constructed in [3, Sec. 5] and [11].
Table 10  Relation matrices of $X(H, V(T_4))$ and its 2-class symmetric fusion scheme

$$
\begin{pmatrix}
0 & 3 & 2 & 2 & 3 \\
3 & 0 & 3 & 2 & 2 \\
2 & 3 & 0 & 3 & 2 \\
2 & 2 & 3 & 0 & 3 \\
3 & 2 & 2 & 3 & 0 \\
\end{pmatrix}
\begin{pmatrix}
5 & 1 & 4 & 4 & 4 \\
4 & 1 & 5 & 1 & 5 \\
1 & 4 & 4 & 1 & 5 \\
2 & 2 & 3 & 0 & 3 \\
4 & 1 & 5 & 1 & 5 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 2 & 1 \\
1 & 0 & 1 & 2 & 2 \\
2 & 1 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 1 \\
1 & 2 & 2 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 2 & 1 \\
1 & 2 & 1 & 0 & 1 \\
1 & 1 & 2 & 1 & 1 \\
1 & 2 & 2 & 1 & 0 \\
\end{pmatrix}
$$

$R_1 \cup R_3 \cup R_4$, while its complement, the Petersen graph has edge set $R_2 \cup R_5$. As we have mentioned earlier, although both graphs $D(T_4)$ and $J_9$ give rise to Johnson graph $J(5, 2)$ via the symmetrization process, $D(T_4)$ is the one which yields $X(H, V(T_4))$.

6 Concluding Remarks

Remark 6.1  We note that the determination of the graph automorphisms is deduced to the investigation of the isomorphism classes of the underlying tactical configurations as different tactical configurations may produce isomorphic graphs. In the examples discussed in the last section, we have seen that the number of nonisomorphic graphs is determined by the orbit structure of the permutation group $S_{ls+1}$ on the set of all tactical configurations $T - (ls + 1, s(ls + 1), l, ls)$ for given $l$ and $s$. Although it is involved as the order of a graph gets large, it is routine to calculate the automorphism groups.

Remark 6.2  All our constructions are based on tactical configurations which arise in many structures. For example, interesting particular cases of the construction methods in Theorems 5.1 and 5.2 occur when we consider a $2 - (v, k, 1)$ design, especially, a $2 - (n^2 + n + 1, n + 1, 1)$ design, the symmetric design coming from a projective plane of order $n$. Let $P$ be the point set of this projective plane. For a point $p \in P$, let $L_{p0}, L_{p1}, \ldots, L_{pn}$ denote the $n + 1$ lines passing through $p$. Since $p$ is the unique common intersecting point for any two of these lines, if we set $B_{pi} = L_{pi} - \{p\}$ for $i = 0, 1, \ldots, n$, then with $B = \{B_{pi} : p \in P, i \in \{0, 1, \ldots, n\}\}$, the pair $(P, B)$ forms a tactical configuration with parameters $(v, b, k, r) = (n^2 + n + 1, (n + 1)(n^2 + n + 1), n, n(n + 1))$. For each prime power $n$, using this tactical configuration, we can obtain directed strongly regular graphs with parameters 

$$(v, k, t, \lambda, \mu) = ((n + 1)(n^2 + n + 1), n(n + 1), n, n - 1, n)$$
Table 11  The list of small directed strongly regular graphs ($v \leq 20$) revisited

| $v$ | $k$ | $t$ | $\lambda$ | $\mu$ | Known construction recorded in [7] | New construction with $d, m, s, r, q, l$ | Known # |
|-----|-----|-----|-----------|-------|-----------------------------------|-----------------------------------------------|---------|
| 6   | 2   | 1   | 0         | 1     | T1, T5, T8, T12                  | Theorem 5.1 $m = 1, s = 2, l = 1$              | 1       |
| 8   | 3   | 2   | 1         | 1     | T4, T6, T7                      | Theorem 5.1 $m = 1, s = 2, l = 1$              | 1       |
| 10  | 4   | 1   | 3         | 1     | T17                              | Theorem 3.2 $m = 1, q = 2, r = 2$              | 1       |
| 10  | 4   | 2   | 0         | 2     | T8, T10, T12                    | Theorem 5.1 $m = 2, s = 2, l = 1$              | 1       |
| 6   | 8   | 5   | 1         | 2     | T2, T4                           | Theorem 3.2 $m = 2, q = 2, r = 3$              | 1       |
| 14  | 5   | 1   | 2         | 1     | DNE by [11]                     |                                               |         |
| 14  | 6   | 3   | 3         | 2     | T5, T12, M6                     | Theorem 5.1 $m = 1, s = 2, l = 3$              | 16,495  |
| 15  | 4   | 2   | 1         | 1     | T2, T4                           | Theorem 5.1 $m = 1, s = 2, l = 3$              | 16,495  |
| 15  | 5   | 2   | 1         | 2     | M5                               | Theorem 3.2 $m = 1, q = 2, r = 3$              | 20      |
| 16  | 7   | 4   | 3         | 3     | T4, T6, T15                      | Theorem 3.1 $m = 2, q = 2, r = 2$              |         |
| 16  | 7   | 5   | 4         | 2     | T11                              | Theorem 3.2 $m = 1, q = 2, r = 4$              | 1       |
| 18  | 4   | 3   | 0         | 1     | M3                               | Theorem 5.1 $m = 1, s = 2, l = 3$              | 1       |
| 18  | 5   | 3   | 2         | 1     | T7                               | Theorem 5.1 $m = 3, s = 2, l = 1$              | 1       |
| 18  | 6   | 3   | 0         | 3     | T8, T10, T12, T17                | Theorem 5.1 $m = 3, s = 2, l = 1$              | 1       |
| 18  | 7   | 5   | 2         | 3     | T16, M4                          | Theorem 5.1 $m = 1, s = 2, l = 3$              | 1       |
| 18  | 8   | 4   | 3         | 4     | T3, T5, T12                     | Theorem 5.1 $m = 1, s = 2, l = 4$              | 1       |
| 18  | 8   | 5   | 4         | 3     | T11                              | Theorem 5.1 $m = 1, s = 2, l = 4$              | 1       |
| 20  | 4   | 1   | 0         | 1     | T1, T8, T12                      | Theorem 5.1 $m = 1, s = 4, l = 1$              | 1       |
| 20  | 15  | 12  | 11        | 12    | T8                               | Theorem 5.1 $m = 1, s = 4, l = 1$              | 1       |
| 20  | 7   | 4   | 3         | 2     | T8, T9                           | Theorem 5.1 $m = 1, s = 4, l = 1$              | 1       |
| 12  | 9   | 6   | 9         | 6     | T8, T12                          | Theorem 5.1 $m = 2, s = 2, l = 2$              | 1       |
| 20  | 8   | 4   | 2         | 4     | T10, T12                         | Theorem 5.1 $m = 2, s = 2, l = 2$              | 1       |
| 11  | 7   | 6   | 6         | 4     | T11                              | Theorem 5.1 $m = 2, s = 2, l = 2$              | 1       |
| 20  | 9   | 5   | 4         | 4     | T4, T6, T11                      | Theorem 5.1 $m = 2, s = 2, l = 2$              | 1       |

For some parameter sets, only one of a complementary pair of graphs is listed in this Table.
and

\[(v, k, t, \lambda, \mu) = ((n + 1)(n^2 + n + 1), n(n + 2), 2n, 2n - 1, n + 1).\]

For these directed strongly regular graphs, we have complete information on their automorphism groups from the knowledge of automorphism groups of projective planes.

**Remark 6.3** Finally, we close our paper by revisiting the table of small directed strongly regular graphs \((v \leq 20)\) found in [7] and [9] to recall the current status of their existence, enumeration results, known construction methods and our constructions. For undefined symbols in the table we refer the readers to [7] (Table 11). (See also the tables provided in [3] and [11] for other characteristics of some of these graphs.)

**Acknowledgments** We are grateful to Leif Jørgensen for various useful inputs relevant to our work. We thank to the anonymous referees for numerous helpful comments. We are also thankful to the Department of Mathematics at Iowa State University for providing Summer Research Assistantship and Faculty Research Support over two summers during which this research was carried on.

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