Singularity of the Dual Curve of a Certain Plane Curve in Positive Characteristic

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Abstract. It is well known that the Gauss map for a complex plane curve is birational, whereas the Gauss map in positive characteristic is not always birational. Let $q$ be a power of a prime integer. We study a certain plane curve $C$ of degree $q^2 + q + 1$ for which the Gauss map is inseparable with inseparable degree $q$. As a special case, we show a relation between the dual curve of the Fermat curve of degree $q^2 + q + 1$ and the Ballico-Hefez curve.

1. Introduction

Let $p$ be a prime integer, and $q$ a power of $p$. We work over an algebraically closed field $k$ of characteristic $p$. We consider a plane curve $C$ of degree $q^2 + q + 1$ defined by a homogeneous polynomial of the form

$$F = \sum_{i,j,k} a_{ijk} x_i x_j^q x_k^{q^2},$$

where $a_{ijk}$ are coefficients in $k$, and $[x_0 : x_1 : x_2]$ is a homogeneous coordinate system in $\mathbb{P}^2$. If $a_{ijk}$ are general, then the plane curve $C$ is smooth. The condition that the defining polynomial of $C$ is of the form (1) is independent of the choice of homogeneous coordinates of $\mathbb{P}^2$ (see Proposition 2.1).

Let $C^\vee$ be the dual curve of the plane curve $C$. The Gauss map

$$\Gamma: C \to C^\vee; [x_0 : x_1 : x_2] \mapsto \left[ \frac{\partial F}{\partial x_0} : \frac{\partial F}{\partial x_1} : \frac{\partial F}{\partial x_2} \right]$$

is an inseparable morphism. For every $i$, the partial derivative of $F$ with respect to $x_i$ is

$$\frac{\partial F}{\partial x_i} = \sum_{j,k} a_{ijk} x_j^q x_k^{q^2} = \left( \sum_{j,k} \alpha_{ijk} x_j^q x_k^{q^2} \right)^q,$$

where $\alpha_{ijk} = a_{ijk}^{1/q}$. Thus, if $a_{ijk}$ are general, then the inseparable degree of the Gauss map is $q$. The purpose of this paper is to study singularities of the dual curve $C^\vee$ of a plane curve $C$ defined by a polynomial of the form (1).

We define $\mathcal{C}$ to be a set of all the projective plane curves defined by homogeneous polynomials of the form (1). Note that $\mathcal{C}$ is identified with $\mathbb{P}^{26}$.

Note that all tangent lines of the curve $C \in \mathcal{C}$ intersect $C$ with multiplicity at least $q$ at the tangent points. In our case, a double tangent and a flex are defined as following:

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Definition 1.1. Let $m$ be an integer at least 2. We define an $m$-ple tangent to be a tangent line of $C$ which has distinct $m$ tangent points with multiplicity $q$, and a flex to be a point at which the tangent line intersects $C$ with multiplicity $q + 1$. A 2-ple tangent is called a double tangent.

Theorem 1. Suppose that $C$ is a general member of $\mathcal{C}$. Then

(i) the degree of the dual curve $C^\vee$ is $(q^2 + q + 1)(q + 1)$,

(ii) the dual curve $C^\vee$ has only ordinary nodes as its singularities,

(iii) the number of ordinary nodes of $C^\vee$ i.e. double tangent lines of $C$, is

$$\frac{q(2q^2 + q + 1)(q^3 + 3q^2 + 3q - 1)}{2},$$

and

(iv) the number of flexes of $C$ is

$$q^5 + 2q^4 + q^3 + 2q^2 + 2q + 1.$$

We compare our theorem with the classical situation. Let $\tilde{C}$ be a general complex plane curve of degree $d$. Then the degree of $\tilde{C}^\vee$ is $d(d - 1)$. Moreover, each flex of $\tilde{C}$ corresponds to a cusp of $\tilde{C}^\vee$, whereas each flex of $C \in \mathcal{C}$ corresponds to a smooth point of $C^\vee$. The singularities of $\tilde{C}^\vee$ consist of $3d(d - 2)$ ordinary nodes and $\frac{1}{2}d(d - 2)(d - 3)(d + 3)$ cusps.

As a special case, we consider the singularities of the dual curve of the Fermat curve $C_0 \in \mathcal{C}$ of degree $q^2 + q + 1$. We will show that the dual curve $C_0^\vee$ is related to the Ballico-Hefez curve.

Let $\gamma_d : \mathbb{P}^2 \to \mathbb{P}^2$ be a morphism defined by $\gamma_d([x_0 : x_1 : x_2]) = [x_0^d : x_1^d : x_2^d]$, and $l_0$ be a line $x_0 + x_1 + x_2 = 0$ in $\mathbb{P}^2$.

Definition 1.2. The Ballico-Hefez curve is the image of the line $l_0$ of the morphism $\gamma_{q+1}$.

In [5], Hoang and Shimada define the Ballico-Hefez curve to be the image of the morphism $\mathbb{P}^1 \to \mathbb{P}^2$ defined by

$$[s : t] \mapsto [s^{q+1} : t^{q+1} : st^q + st^q].$$

Note, however, that the image of this morphism is projectively isomorphic to the image of the line $l_0$ of the morphism $\gamma_{q+1}$.

Theorem 2. Let $B$ be the Ballico-Hefez curve. Let $\gamma_{q^2+q+1} : \mathbb{P}^2 \to \mathbb{P}^2$ be a morphism defined by the above. If $C_0 \in \mathcal{C}$ is the Fermat curve of the degree $q^2 + q + 1$, then

(i) the dual curve $C_0^\vee$ is $\gamma_{q^2+q+1}^{-1}(B)$, and

(ii) the singularities of $C_0^\vee$ consist of $(q^2 + q + 1)^2(q^3 - 1)/2$ ordinary nodes, and $3(q^2 + q + 1)$ singular points with the Milnor number $q^2(q + 1)$.

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2. Preliminaries

From now, let $k$ be an algebraically closed field of characteristic $p > 0$.

**Proposition 2.1.** Let $C$ be a plane curve. The defining polynomial of $C$ being of the form (1) is a property independent of the choice of homogeneous coordinates.

*Proof.* Under the coordinates change $x_i = \sum t_i y_i$ $(t_i \in k)$, a homogeneous polynomial $F$ of the form (1) is transformed into

$$F = \sum_{i,j,k} a_{ijk} \left( \sum_{l=0}^{2} t_i y_i \right) \left( \sum_{m=0}^{2} t_{jm} y_m \right)^q \left( \sum_{n=0}^{2} t_{kn} y_n \right)^{q^2}$$

where $b_{lmn} = \sum_{i,j,k} a_{ijk} t_i t_{jm} t_{kn} y_i y_m y_n^2$.

**Claim 0.** The reduced Gauss map $\Gamma_{\text{red}}$ of $C$ is a morphism of separable degree 1.

*Proof.* See the section 5. □

We put the degree of a curve $C \in \mathcal{C}$ into $d = q^2 + q + 1$. If $C \in \mathcal{C}$ is general, then the Gauss map $\Gamma$ is an inseparable morphism of inseparable degree $q$ by \[\Box\]

Thus the degree of $C^\vee$ is

$$\frac{d(d-1)}{q} = \frac{(q^2 + q + 1)(q^2 + q)}{q} = (q^2 + q + 1)(q + 1).$$

3. Proof of the first half of Theorem 1

We define the **reduced Gauss map** $\Gamma_{\text{red}} : C \to (\mathbb{P}^2)^\vee$ of $C \in \mathcal{C}$ by

$$\Gamma_{\text{red}}([x_0 : x_1 : x_2]) = \left[ \left( \frac{\partial F}{\partial x_0}(x_0, x_1, x_2) \right)^{1/q} : \left( \frac{\partial F}{\partial x_1}(x_0, x_1, x_2) \right)^{1/q} : \left( \frac{\partial F}{\partial x_2}(x_0, x_1, x_2) \right)^{1/q} \right].$$

**Claim 0.** The reduced Gauss map $\Gamma_{\text{red}}$ is the morphism of separable degree 1.

*Proof.* See the section 5. □

We put the degree of a curve $C \in \mathcal{C}$ into $d = q^2 + q + 1$. If $C \in \mathcal{C}$ is general, then the Gauss map $\Gamma$ is an inseparable morphism of inseparable degree $q$ by \[\Box\]

Thus the degree of $C^\vee$ is

$$\frac{d(d-1)}{q} = \frac{(q^2 + q + 1)(q^2 + q)}{q} = (q^2 + q + 1)(q + 1).$$

In order to prove (ii) of Theorem 1, first we prove the following:

**Claim 1.** If $C \in \mathcal{C}$ is general, then the curve $C$ has no $m$-ple tangent line for $m \geq 3$. 

Proof. We define a variety $\mathcal{X}_1$ by
\[
\mathcal{X}_1 = \left\{ (Q_0, Q_1, Q_2, l) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \mid Q_0 \in l, Q_1 \in l, Q_2 \in l \text{ and } Q_i \neq Q_j \text{ for } i \neq j \right\}.
\]
Then the action of $\text{PGL}_3(k)$ on $\mathcal{X}_1$ is transitive. Let $(P_0, P_1, P_2, l_0)$ be a point of $\mathcal{X}_1$ and let $[x_0 : x_1 : x_2]$ be a homogeneous coordinate system such that $P_0 = [0 : 0 : 1]$, $P_1 = [0 : 1 : 0]$, $P_2 = [0 : 1 : 1]$ and $l_0 = \{x_0 = 0\}$. Let $C$ be a plane curve in $\mathcal{C}$. We define a subspace $\mathcal{D}_1$ of $\mathcal{C}$ by
\[
\mathcal{D}_1 = \left\{ Y \in \mathcal{C} \mid P_0, P_1, P_2 \text{ are smooth points of } Y, \text{ and } T_{P_0}Y = T_{P_1}Y = T_{P_2}Y = l_0 \right\}.
\]
Then a curve $C \in \mathcal{C}$ is in $\mathcal{D}_1$ if and only if
\[
a_{222} = 0, \ a_{122} = 0, \ a_{111} = 0, \ a_{221} = 0, \ a_{212} + a_{221} = 0, \ a_{112} + a_{121} = 0, \ a_{022} \neq 0, \ a_{011} \neq 0 \text{ and } a_{012} + a_{021} + a_{022} \neq 0.
\]
Therefore $\mathcal{D}_1$ is of codimension 6 in $\mathcal{C}$. Since $\dim \mathcal{X}_1 = 5$, we have $\dim \mathcal{X}_1 + \dim \mathcal{D}_1 < \dim \mathcal{C}$.

Thus if the curve $C$ is general in $\mathcal{C}$, then $C$ does not have any $m$-ple tangent line for $m \geq 3$.

Second we prove the following:

Claim 2. If $C \in \mathcal{C}$ is general, then $\Gamma_{\text{red}}$ is an immersion at every point of $C$.

Proof. Let $P_0$ be the point $[0 : 0 : 1]$, and let $l_0$ be the line $\{x_0 = 0\}$. By linear change of coordinates, we can assume that $P_0 \in C$ and $T_{P_0}C = l_0$. Let $(x, y)$ be affine coordinates such that $P_0 = (x, y)$ and $l_0 = \{x = 0\}$. Then up to multiple constant, the polynomial $F$ can be written as
\[
F(x, y, 1) = f(x, y) = x + a_{202}x^2 + a_{212}y^2 + a_{002}x^q + a_{102}x^qy + a_{012}xy^q + a_{112}y^q + (\text{terms of degree } \geq q^2).
\]
Then we have a local parametrization $x = \phi(t), \ y = t$ of $C$ at $P_0$ such that the power series $\phi(t)$ is written as
\[
\phi(t) = -a_{212}t^q - a_{112}t^{q+1} + a_{012}a_{212}t^{2q} + \cdots.
\]
We consider the Gauss map given by (2). Let $(\eta, \zeta)$ be the affine coordinates of $(\mathbb{P}^2)^\vee$ with the origin $l_0$ in $(\mathbb{P}^2)^\vee$ such that the point $(\eta, \zeta)$ corresponds to the line $x + \eta y + \zeta = 0$. Then the tangent line of $C$ at $P_t = (\phi(t), t)$ is
\[
\frac{\partial f}{\partial x}(P_t)x + \frac{\partial f}{\partial y}(P_t)y - \frac{\partial f}{\partial x}(P_t)\phi(t) - \frac{\partial f}{\partial y}(P_t)t = 0
\]
Therefore the Gauss map locally around $P_0$ is written as
\[
\Gamma((\phi(t), t)) = \left( \frac{f_y(P_t)}{f_x(P_t)}, \frac{-f_y(P_t)}{f_x(P_t)}t - \phi(t) \right) \quad = \left( \frac{d\phi}{dt}(t), -\frac{d\phi}{dt}(t) - \phi(t) \right).
\]
Since
\[
-\frac{d\phi}{dt}(t) = a_{112}t^q + (\text{terms of degree } > q)
\]
and
\[ t \frac{d \phi}{dt}(t) - \phi(t) = a_{212}t^q + \text{(terms of degree } q), \]
the reduced Gauss map \( \Gamma_{\text{red}} \) locally around \( P_0 \) is
\[ t \mapsto (\alpha_{112}t + \text{(terms of degree } > 1), a_{212}t + \text{(terms of degree } > 1)), \]
where \( \alpha_{ijk} = a_{ijk}^{1/q} \). The reduced Gauss map \( \Gamma_{\text{red}} \) is not smooth at the point \( P_0 \) if and only if \( \alpha_{112} = \alpha_{212} = 0 \). Since the codimension of the space
\[ \{ C \in \mathcal{C} | \alpha_{112} = \alpha_{212} = 0 \} \]
is 2 in \( \mathcal{C} \), the reduced Gauss map \( \Gamma_{\text{red}} \) is locally immersion at every point of a general member \( C \) of \( \mathcal{C} \).

Suppose that \( C \in \mathcal{C} \) is general. We prove that the singular points of the dual curve \( C^\vee \) are only ordinally nodes. Let \( P_0 \) and \( P_1 \) be the points in the proof of claim 1, and let \( l_0 \) be the line \( \{ x_0 = 0 \} \). Suppose that \( P_0 \) and \( P_1 \) are smooth points of \( C \) and \( T_{P_0}C = T_{P_1}C = l_0 \). Let \( (x', y') \) be affine coordinates such that \( P_1 = (0, 0) \) and \( l_0 = \{ x' = 0 \} \). Similar to the proof of the claim 2, up to multiple constant, the polynomial \( F \) can be written as
\[ F(x', 1, y') = g(x', y') = x' + a_{101}x'^q + a_{121}y'^q + a_{001}x'^{q+1} + a_{021}x'y'^q + a_{201}x'y'^q + a_{211}y'^{q+1} + \text{(terms of degree } \geq q^2) \]
Then we have a local parametrization \( x' = \psi(t), y' = t, \) of \( C \) at \( P_0 \) such that the power series \( \psi(t) \) is written as
\[ \psi(t) = -a_{121}t^q - a_{221}t^{q+1} + a_{201}a_{121}t^{2q} + \cdots. \]
Let \( (\eta', \zeta') \) be the affine coordinates of \( (\mathbb{P}^2)^\vee \) with the origin \( l_0 \in (\mathbb{P}^2)^\vee \) such that the point \( (\eta', \zeta') \) corresponds to the line \( x' + \eta'y' + \zeta' = 0 \). The tangent line of \( C \) at \( P'_1 = (\psi(t), t) \) is
\[ \frac{\partial g}{\partial x'}(P'_1)x' + \frac{\partial g}{\partial y'}(P'_1)y' - \frac{\partial g}{\partial x'}(P'_1)\psi(t) - \frac{\partial g}{\partial y'}(P'_1)t = 0. \]
Therefore the Gauss map \( \Gamma \) locally around \( P_1 \) is written as
\[ \Gamma((\psi(t), t)) = \left( \frac{g_{y'}(P'_1)}{g_{x'}(P'_1)}t - \psi(t), -\frac{g_{y'}(P'_1)}{g_{x'}(P'_1)}t - \psi(t) \right) = \left( -\frac{d\psi}{dt}(t), t\frac{d\psi}{dt}(t) - \psi(t) \right). \]
Since
\[ -\frac{d\psi}{dt}(t) = a_{221}t^q + \text{(terms of degree } q) \]
and
\[ t\frac{d\psi}{dt}(t) - \psi(t) = a_{121}t^q + \text{(terms of degree } q), \]
we describe the reduced Gauss map
\[ t \mapsto (a_{221}t + \text{(terms of degree } > 1), a_{121}t + \text{(terms of degree } > 1)) \]
locally around \( P_1 \). We define a variety \( \mathcal{X}_2 \) by
\[ \mathcal{X}_2 = \{ (Q_0, Q_1, l) | Q_0 \in l, Q_1 \in l \text{ and } Q_0 \neq Q_1 \}. \]
Then the action of $\text{PGL}_3(k)$ on $\mathcal{X}$ is transitive and $\dim \mathcal{X} = 4$. Let $(P_0, P_1, l_0)$ be the point of $\mathcal{X}$ such that $P_0 = [0 : 0 : 1]$, $P_1 = [0 : 1 : 0]$ and $l_0 = \{x_0 = 0\}$. We define a subspace $\mathcal{D}_2$ of $\mathcal{C}$ by

$$\mathcal{D}_2 = \{Y \in \mathcal{C} \mid P_0 \text{ and } P_1 \text{ are smooth points of } Y, \text{ and } T_{P_0}Y = T_{P_1}Y = l_0\}$$

Then $C \in \mathcal{D}_2$ if and only if

$$a_{222} = 0, \ a_{122} = 0, \ a_{111} = 0, \ a_{211} = 0, \ a_{022} \neq 0, \ a_{011} \neq 0.$$

Thus the codimension of $\mathcal{D}_2$ is 4. For $C \in \mathcal{D}_2$, by (4) and (5), the singularities of $C^\vee$ at the point $l_0$ is not an ordinary node if and only if

$$\begin{vmatrix}
\alpha_{112} & \alpha_{212} \\
\alpha_{211} & \alpha_{121}
\end{vmatrix} = 0.$$

We define a subspace $\mathcal{D}_2'$ of $\mathcal{C}$ by

$$\mathcal{D}_2' = \left\{ Y \in \mathcal{C} \mid P_0 \text{ and } P_1 \text{ are smooth points of } Y, \ T_{P_0}Y = T_{P_1}Y = l_0, \text{ and } Y^\vee \text{ does not have ordinary node at } l_0 \right\}.$$

Since the codimension of $\mathcal{D}_2'$ is 5,

$$\dim \mathcal{D}_2' + \dim \mathcal{X} < \dim \mathcal{C}.$$ 

Therefore, since $a_{ijk}$ are general, the dual curve $C^\vee$ has only ordinary nodes as its singularities.

4. Proof of (iii), (iv) of Theorem 1

4.1. **Number of the ordinary nodes of $C^\vee$.** Let $g$ and $g^\vee$ be the genera of a general curve $C \in \mathcal{C}$ and its dual curve $C^\vee$, respectively. Let $\delta$ be the number of the ordinary nodes of $C^\vee$. Then

$$g = \frac{(d-1)(d-2)}{2} = \frac{(q^2 + q + 1)(q^2 + q + 1)}{2}$$

and

$$g^\vee = \frac{(d^\vee - 1)(d^\vee - 2)}{2} - \delta \frac{(q^2 + q + 1)(q + 1)}{2} \frac{(q^2 + q + 1)(q + 1) - 2}{2}.$$ 

where $d$ and $d^\vee$ are the degree of $C$ and $C^\vee$, respectively, because, by the previous section, $C^\vee$ has only ordinary nodes. By claim 2 of section 3, the reduced Gauss map $\Gamma_{\text{red}}$ is birational onto its image. Thus $g = g^\vee$ and hence we have

$$\delta = \frac{(q^2 + q + 1)(q + 1) - 1}{2} \frac{(q^2 + q + 1)(q + 1) - 2}{2}$$

$$= \frac{g(q^2 + q + 1)(q^3 + 3q^2 + 3q - 1)}{2}$$

$$= q(q^2 + q + 1)(q^3 + 3q^2 + 3q - 1)$$

$$= \frac{g(q^2 + q + 1)(q^3 + 3q^2 + 3q - 1)}{2}.$$
4.2. Number of the flexes. We denote by mult_P(D_1, D_2) the intersection multiplicity of projective plane curves D_1 and D_2 at a point P ∈ D_1 ∩ D_2.

Lemma 4.1. We suppose that C ∈ ℂ is a general plane curve in ℂ. If the multiplicity mult_u(T_uC, C) is more than q at u ∈ C, then the multiplicity mult_u(T_uC, C) is q + 1 at u ∈ C and all other intersection points of T_uC and C are not tangent points.

Proof. We use the same notation as in Section 3. We define a variety ℳ_0 by

\[ ℳ_0 = \{ (Q, l) ∈ ℙ^2 × (ℙ^2)′ \mid Q ∈ l \}. \]

Then the action of PGL_3(k) on ℳ_0 is transitive and dim ℳ_0 = 3. We recall that \([x_0 : x_1 : x_2]\) are homogeneous coordinates, \(P_0 = [0 : 0 : 1], P_1 = [0 : 1 : 0]\) and \(l_0 = \{x_0 = 0\}\). We define two subspaces ℳ and \(\tilde{ℳ}_0\) of ℂ by

\[ ℳ = \left\{ Y ∈ ℂ \mid P_0 \text{ is the smooth point of } Y, T_{P_0}Y = l_0 \text{ and } \operatorname{mult}_{P_0}(T_{P_0}Y, Y) = q + 1 \right\}. \]

\[ \tilde{ℳ}_0 = \left\{ Y ∈ ℂ \mid P_0 \text{ is the smooth point of } Y, T_{P_0}Y = l_0 \text{ and } \operatorname{mult}_{P_0}(T_{P_0}Y, Y) > q + 1 \right\}. \]

Then the curve C ∈ ℳ if and only if

\[ a_{222} = 0, a_{122} = 0, a_{212} = 0, a_{110} ≠ 0 \text{ and } a_{022} ≠ 0, \]

and C ∈ \(\tilde{ℳ}_0\) if and only if

\[ a_{222} = 0, a_{122} = 0, a_{212} = 0, a_{110} = 0 \text{ and } a_{022} ≠ 0. \]

Therefore codimension of ℳ is 3 and \(\tilde{ℳ}_0\) is more than 3 in ℂ. Thus we have

\[ \dim ℳ_0 + \dim \tilde{ℳ}_0 < \dim ℂ. \]

We proved the first half of the lemma. We define a subspace \(\tilde{ℳ}_2\) of ℂ by

\[ \tilde{ℳ}_2 = \left\{ Y ∈ ℂ \mid P_0 \text{ and } P_1 \text{ are the smooth points of } Y, T_{P_0}Y = l_0, T_{P_1}Y = l_0 \text{ and } \operatorname{mult}_{P_0}(T_{P_0}Y, Y) = q + 1 \right\}. \]

Then the curve C ∈ \(\tilde{ℳ}_2\) if and only if

\[ a_{222} = 0, a_{122} = 0, a_{111} = 0, a_{211} = 0, a_{212} = 0, a_{110} ≠ 0, a_{022} ≠ 0, a_{011} ≠ 0. \]

Therefore codimension of \(\tilde{ℳ}_2\) is 5, and we recall dim \(\tilde{ℳ}_2\) = 4. Thus, since we have

\[ \dim \tilde{ℳ}_2 + \dim \tilde{ℳ}_0 < \dim ℂ, \]

the second half of the lemma is proved. \(\square\)

Let g be the genus of a general curve C ∈ ℂ. We use the notion and notation about the correspondence of a curve introduced in [2], Chap. 2, Section 5]. Let T : C → C be correspondence defined by T(u) = T_uC.C − qu, D ⊂ C × C its curve of correspondence, i.e. D = \{(u, v) \mid u ≠ v, v ∈ T_uC\}. Then the degree of T is

\[ \deg T = (q^2 + q + 1) - q = q^2 + 1. \]

In order to find the degree of T⁻¹, we have to calculate the number of tangent lines to C other than T_uC passing through a general point v ∈ C. We consider the projection \(π_v : C → ℙ^1\) from the center v ∈ C onto a line. Let \(Ω_{C/ℙ^1}\) be the sheaf of
the relative differential of $C$ over $\mathbb{P}^1$. By Hurwitz-formula \cite[Chap. IV, Corollary 2.4]{4},
\[
2g - 2 = -2(q^2 + q) + \deg R,
\]
where the divisor $R$ is the ramification divisor of $\pi_v$ i.e. $R = \sum_{u \in C} \text{length}(\Omega_{C/\mathbb{P}^1})_u u$.
Hence
\[
\deg R = q^4 + 2q^3 + 2q^2 + q - 2.
\]
If $\pi_v$ is ramified at $u$, then we have
\[
\text{length}(\Omega_{C/\mathbb{P}^1})_u = \begin{cases} q & (u \neq v), \\ q - 2 & (u = v). \end{cases}
\]
Hence, we have
\[
\deg T^{-1} = \frac{(q^4 + 2q^3 + 2q^2 + q - 2) - (q - 2)}{q} = q^3 + 2q^2 + 2q.
\]

**Lemma 4.2.** Let $\pi_1, \pi_2 : C \times C \to C$ be the projections on first and second factors, respectively. The divisor $D$ on $C \times C$ is algebraically equivalent to
\[
(q^3 + 2q^2 + 3q)E_u + (q^2 + q + 1)F_v - q\Delta,
\]
where $E_u = \pi_1^{-1}(u), F_v = \pi_2^{-1}(v)$ and $\Delta \subset C \times C$ is the diagonal.

**Proof.** For some $u_0, v_0 \in C$, we write
\[
T(u_0) + qu_0 = \sum b_i v_i
\]
and
\[
T^{-1}(v_0) + qv_0 = \sum a_i u_i.
\]
Let $L$ be the line bundle
\[
L = D - \sum a_i E_{u_i} - \sum b_i F_{v_i} + q\Delta.
\]
For any $x \in C$, the restriction of $L$ to $E_x$ is trivial because the divisor $T(x) + qx$ is linearly equivalent to $T(u_0) + qu_0$. The restriction of $L$ to $F_{v_0}$ is also trivial. Let $s$ be a global nonzero regular section of the restriction of $L$ to $F_{v_0}$. Then, for any $u \in C$, there is a unique global regular section $t_u$ of the restriction of $L$ to $E_u$ such that $t_u(u, v) = s(u, v_0)$. Set
\[
t(u, v) = t_u(u, v).
\]
Then $t$ is a global nonzero regular section of $L$. Thus $D$ is linearly equivalent to
\[
\sum a_i E_{u_i} + \sum b_i F_{v_i} - q\Delta.
\]
For any $u, v \in C$, the divisors $E_{u_i}$ (resp. $F_{v_i}$) are algebraically equivalent to $E_u$ (resp. $F_v$). Note that the degrees of $T(u_0) + qu_0$ and $T^{-1}(v_0) + qv_0$ are
\[
\deg(T(u_0) + qu_0) = q^2 + q + 1
\]
and
\[
\deg(T^{-1}(v_0) + qv_0) = q^3 + 2q^2 + 3q,
\]
and hence the result is proved. \qed
In order to find the number of the flexes, we should calculate the intersection number \((D \cdot \Delta)\). Since the self-intersection number of \(\Delta\) is \(2 - 2g\), the intersection number \((D \cdot \Delta)\) is
\[
(D \cdot \Delta) = \{(q^3 + 2q^2 + 3q)E_u + (q^2 + q + 1)F_v - q\Delta \} \cdot \Delta
= q^3 + 3q^2 + 4q + 1 - q(2 - 2g)
= q^6 + 2q^4 + q^3 + 2q^2 + 2q + 1.
\]

5. **Fermat curve**

Calculation method of the Milnor number for a formal power series in characteristic zero is well known. (For example, see [6].)

**Lemma 5.1.** Let \(a\) and \(b\) be elements in \(k \setminus \{0\}\), and let \(f \in k[[x, y]]\) be a formal power series defined by
\[
f(x, y) = ax^\alpha + by^\beta + \sum_{\alpha \beta < \alpha + \beta r} c_{r,s} x^r y^s,
\]
where \(\alpha\) and \(\beta\) satisfy \(p \not| \alpha, p \not| \beta\) and relatively prime. Then the Milnor number \(\mu(f)\) of \(f\) is
\[
\mu(f) = (\alpha - 1)(\beta - 1).
\]

**Proof.** By Puiseux’s theorem, \(f(x, y) = 0\) is expressed by a parametrization
\[
(t^\beta, c_0t^\alpha + \text{(terms of degree > } \alpha)),
\]
where \(c_0\) is a \(\beta\)-th root of \(-\frac{b}{a}\). Here, the partial derivative of \(f\) by \(y\) is
\[
\frac{\partial f}{\partial y} = b\beta y^{\beta - 1} + \sum_{\alpha \beta < \alpha + \beta r} sc_{r,s} x^r y^{s - 1}
= b\beta (c_0 t^\alpha + \text{(terms of degree > } \alpha))^{\beta - 1}
+ \sum_{\alpha \beta < \alpha + \beta r} sc_{r,s} t^{\beta r} (c_0 t^\alpha + \text{(terms of degree > } \alpha))^{s - 1}.
\]
Since \(\alpha \beta < \alpha s + \beta r\) in the second summation, the order of \(\frac{\partial f}{\partial y}\) is \(\text{ord}_y \frac{\partial f}{\partial y} = \alpha \beta - \alpha \)
For two power series \(g(x, y)\) and \(h(x, y)\), we denote \(\dim k[[x, y]]/ < g, h > \) by \((g, h)_0\).
Then
\[
\left( f, \frac{\partial f}{\partial y} \right)_0 = \alpha \beta - \alpha,
\]
and
\[
(f, x)_0 = \beta \not\equiv 0 \pmod{p}.
\]
By Teissier’s lemma in a positive characteristic in [2],
\[
\mu(f) = \left( f, \frac{\partial f}{\partial y} \right)_0 - (f, x)_0 + 1
= \alpha \beta - \alpha - \beta + 1
= (\alpha - 1)(\beta - 1).
\]

□
Proof of Theorem 2. The morphisms $\gamma_{q^2+q+1}$ and $\gamma_{q+1}$ satisfy

$$\gamma_{q^2+q+1} \circ \gamma_{q+1} = \gamma_{q+1} \circ \gamma_{q^2+q+1} = \gamma((q^2+q+1)(q+1)).$$

By the definition of the Ballico-Hefez curve and the line $l = \gamma_{q^2+q+1}(C_0)$, we have

$$B = \gamma_{q+1}(l) = \gamma_{q+1}(\gamma_{q^2+q+1}(C_0)) = \gamma_{q^2+q+1}(\gamma_{q+1}(C_0)) = \gamma_{q^2+q+1}(C'_0),$$

and hence (i) is proved.

We define $X \subset \mathbb{P}^2$ by

$$X = \{x_0 = 0\} \cup \{x_1 = 0\} \cup \{x_2 = 0\}.$$

The Ballico-Hefez curve $B$ has $\frac{q^2+q+1}{2}$ ordinary nodes on $\mathbb{P}^2 \setminus X$ (see [1]), and no singular points on $X$. Let $G$ and $g$ be the defining polynomials of $C'_0$ and $B$, respectively. Using Proposition 1.6 of [5], if $p = 2$, then

$$g = x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_0^q x_2 + x_0^q x_2 + x_0 x_2^q + x_1 x_2^q$$

$$+ \sum_{i=0}^{\nu-1} x_0^i x_1^i (x_0 + x_1 + x_2)^{q+1-2^{i+1}},$$

whereas if $p$ is odd, then

$$g = x_0^{q+1} + x_1^{q+1} + x_2^{q+1} + x_0^q x_2 - x_0^q x_2 - x_0 x_2^q - x_0 x_2^q - x_1 x_2^q$$

$$+ (x_0^q + x_1^q + x_2^q - 2x_0 x_1 - 2x_1 x_2 - 2x_2 x_0)^{q+1}.$$

By (i), the polynomial $G$ satisfies $G(x_0, x_1, x_2) = g(x_0^q + q, x_1^q + q, x_2^q + q) + 1$, and two polynomials $G$ and $g$ are symmetric under the permutation of coordinates $x_0, x_1$ and $x_2$. First we consider the singularities of $C'_0$ on $\mathbb{P}^2 \setminus X$. The morphism $\gamma_{q^2+q+1} : \mathbb{P}^2 \setminus X \to \mathbb{P}^2 \setminus X$ is étale of degree $(q^2 + q + 1)^2$. Thus, the ordinary nodes of $C'_0$ on $\mathbb{P}^2 \setminus X$ are $(q^2 + q + 1)^2(q^2 - q)/2$.

Next, we consider the singularities of $C'_0$ on $X$. $g(0, x_1, x_2) = 0$ if and only if $x_1 = x_2$ by (6) and (7). Moreover, the polynomial $G$ and its partial derivatives $\partial G/\partial x_i = x_i^{q^2+q+1}(\partial g/\partial x_i)$ vanish at a point in $\{x_0 = 0\}$. Thus all the points on $C'_0 \setminus \{x_0 = 0\}$ are singular points of $C'_0$. The morphism $\gamma_{q^2+q+1}|_{\{x_0=0\}}$ restricted to $\{x_0 = 0\}$ is degree $q^2 + q + 1$. Thus the number of the singular points of $C'_0$ on $\{x_0 = 0\}$ are $q^2 + q + 1$. Therefore, by the polynomial $G$ is symmetric, the number of the singular points of $C'_0$ on $X$ are $3(q^2 + q + 1)$.

Finally, since all Milnor numbers at points in $\gamma_{q^2+q+1}(\{0 : 1 : 1\})$ are equal, we should calculate the Milnor number at the point $\{0 : 1 : 1\} \in C'_0$. If $p = 2$,

$$g(x_0^{q^2+q+1}, x_1 + 1, 1) = x_0^{q^2+q+1} + x_1^{q+1} + x_0^{q(q^2+q+1)} + x_0^{(q+1)(q^2+q+1)}$$

$$+ \sum_{i=0}^{\nu-1} (x_0^{q^2+q+1})^i (x_1 + 1)^{q^2+q+1} + x_1^{q+1-2^{i+1}},$$

whereas if $p$ is odd,

$$g(x_0^{q^2+q+1}, x_1 + 1, 1) = -2x_0^{q^2+q+1} + x_1^{q+1} + x_0^{q(q^2+q+1)}$$

$$- x_0^{q(q^2+q+1)} x_1 - 2x_0^{q(q^2+q+1)} - x_0^{q^2+q+1} q - 2x_0^{q^2+q+1} q x_1$$

$$+ (x_0^{2(q^2+q+1)} + x_1^{2} - 2x_0^{q^2+q+1} x_1 - 4x_0^{q^2+q+1})^{+1}.$$
By Lemma 3.1, the Milnor number of \( g(x_0^{q^2+q+1}, x_1 + 1, 1) \) is

\[
g(q^2 + q) = q^2(q + 1).
\]

We confirm that the genus of the Fermat curve agree with the genus of its dual curve. The genus \( g \) of the Fermat curve \( C_0 \) of the degree \( d = q^2 + q + 1 \) is

\[
g = \frac{(d-1)(d-2)}{2} = \frac{(q^2 + q)(q^2 + q - 1)}{2}.
\]

Let \( \mu_P \) be the Milnor number and \( r_P \) be the number of the branches at a singular point of the dual curve \( C_0^\vee \). If a point \( P \in C_0^\vee \) is ordinary node, then \( \mu_P = 1 \) and \( r_P = 2 \), whereas if a point \( P \) is in \( C_0^\vee \cap X \), then \( \mu_P = q^2(q + 1) \) and \( r_P = 1 \). Thus the degree \( d^\vee \) of \( C_0^\vee \) is \((q + 1)(q^2 + q + 1)\), and the genus \( g^\vee \) of \( C_0^\vee \) is

\[
g^\vee = \frac{(d^\vee - 1)(d^\vee - 2)}{2} - \frac{1}{2} \sum_{P \in \text{Sing}C_0^\vee} (\mu_P + r_P - 1)
\]

\[
= \frac{1}{2} \frac{((q^2 + q + 1)(q + 1) - 1)((q^2 + q + 1)(q + 1) - 2)}{2}
\]

\[
- \frac{1}{2} \frac{(q^2 + q + 1)^2(q^2 - q) + 3(q^2 + q + 1)q^2(q + 1)}{2}
\]

\[
= \frac{(q^2 + q)(q^2 + q - 1)}{2}.
\]

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