ON TORIC VARIETIES AND ALGEBRAIC SEMIGROUPS

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Abstract. The main result of this paper is that every (separated) toric variety which has a semigroup structure compatible with multiplication on the underlying torus is necessarily affine. In the course of proving this statement, we also give a proof of the fact that every separated toric variety may be constructed from a certain fan in a Euclidean space. To our best knowledge, this proof differs essentially from the ones which can be found in the literature.

1. Introduction

In this paper we discuss two questions arising in the theory of toric varieties and their relation to the theory of algebraic semigroups. A toric variety is a normal algebraic variety which contains a torus as an open dense subset, such that the natural action of the torus on itself (by left multiplication) extends to an action of the torus on the variety. This definition is not very enlightening, so one needs a different description of a toric variety. It turns out that the geometry of toric varieties is closely related to such simple objects as polyhedral cones in Euclidean spaces. More precisely, given a fan in a Euclidean space, which is a collection of "strongly convex rational polyhedral cones" satisfying a certain condition, there exists a natural way of associating a toric variety to this fan (for details, see section 2.3). In principle, all questions concerning the resulting toric variety may be reformulated in terms of simple geometric statements about the fan we have started with. This makes the study of toric varieties much easier than the study of arbitrary algebraic varieties. However, to be able to use this theory, we must solve the reverse problem: does there always exist a fan such that the associated toric variety is isomorphic to the given one?

In this paper, we will see that the answer is affirmative in the most important case when the toric variety is separated\(^1\). To justify the use of this assumption, we will show that a toric variety associated to a fan is necessarily separated, and we will give an example of a toric variety which is not separated (and hence cannot be constructed from a fan). Of course, this result is far from being original. For example, the book \([2]\) gives a procedure\(^2\) of recovering a fan from a toric variety. However, Fulton assumes that the given toric variety can be constructed from a fan, and then shows how the fan can be reconstructed. In general, it seems that in most of the literature on toric varieties, the authors try to avoid giving all the details of this reconstruction. In the second part of this paper, we will give a complete proof of the fact that a separated toric variety can always be constructed from a fan. For

\(^1\)The author is thankful to Professor S. Shatz for pointing out that there exist nonseparated toric varieties.

\(^2\)but without checking the details
the only nontrivial result that we will state without a proof, the reader is referred to the paper [4].

It appears that our method differs essentially from the ones that may be found in the literature. There are many fewer technical details to check, and our argument makes it clear where the normality and separatedness assumptions are used.

We also address a question which connects the theories of toric varieties and algebraic semigroups. It is known that every affine toric variety has a natural structure of an algebraic semigroup such that the multiplication extends the multiplication on the torus.\(^3\) A natural question arises: is it true that every (separated) toric variety which is an algebraic semigroup whose multiplication is compatible with that on the torus is automatically affine?

The first part of the paper is devoted to proving that the answer is affirmative in the case when the "toric semigroup" can be constructed from a fan. Combining this result with the second part, we see that every separated toric semigroup\(^4\) is affine. Thus, the study of separated toric semigroups reduces to the study of affine toric semigroups, or, in the usual terminology, toric monoids. For a complete description of toric monoids, the reader is referred to the paper [1].

The exposition is kept as self-contained as possible. All the relevant definitions from the theories of toric varieties and algebraic semigroups are given. We use several basic results from the general theory of algebraic varieties, all of which may be found in [3]. We also use a few results on cones in Euclidean spaces. For the proofs the reader is referred to the book [2].

2. Basic Definitions

2.1. Algebraic semigroups and toric varieties. In this paper, \(k\) denotes an arbitrary algebraically closed field. We use the term algebraic variety in the sense of [3]. In particular, we do not assume that varieties are irreducible or separated. If \(X\) is a variety over \(k\), \(k[X]\) denotes its coordinate ring.

Definition 2.1. An algebraic semigroup is an algebraic variety \(S\) together with a morphism of algebraic varieties

\[ \mu : S \times S \rightarrow S, \]

denoted multiplicatively:

\[ (a, b) \mapsto ab, \]

such that the usual associativity condition holds: for all \(a, b, c \in S\),

\[ (ab)c = a(bc). \]

A zero of an algebraic semigroup \(S\) is such an element \(0 \in S\) (if it exists) that for all \(s \in S\), \(0 \cdot s = s \cdot 0 = 0\). If \(S\) has a unit, \(G(S)\) will denote the subgroup consisting of all invertible elements of \(S\).

Definition 2.2. A toric variety is a pair \((X, T)\) where \(X\) is a normal algebraic variety and \(T \subset X\) is an open dense subset isomorphic to an (algebraic) torus, such that the multiplication \(T \times T \rightarrow T\) extends to an action \(T \times X \rightarrow X\) of \(T\) on \(X\).

\(^3\)We show this in Lemma 3.8.

\(^4\)For a formal definition, see 3.1.
Remark 2.3. In this paper we are interested in algebraic semigroups containing a torus as an open dense subset, such that the multiplication on the semigroup is compatible with that on the torus. Let $S \supset T$ be such a semigroup. Since the condition $s_1 s_2 = s_2 s_1$ is closed and is satisfied for all $s_1, s_2 \in T$, it must be satisfied for all $s_1, s_2 \in S$. Similarly, if $1$ denotes the unit element of $T$, then the condition $1 \cdot s = s \cdot 1 = s$, satisfied for all $s \in T$, must therefore be satisfied for all $s \in S$. Thus, $S$ is an irreducible commutative algebraic semigroup with a unit. Consequently, all elements of $T$ must be invertible. Now assume that $S$ is affine. We claim that, in fact, $G(S) = T$. To see this, note that for an affine semigroup $S$, there exists a closed embedding $S \hookrightarrow M_n(k)$ which is a homomorphism of semigroups. Moreover, if $S$ has a unit, we may assume that it maps to the identity matrix under this embedding (for details, see [5]). Hence $G(S)$ maps into $GL(n, k)$. In particular, $G(S)$ is itself an algebraic group (the map $g \mapsto g^{-1}$ is a morphism of algebraic varieties $G \rightarrow G$), and $G(S)$ is open in $S$. In our situation, this means that $G(S)$ contains $T$ as an open dense subgroup. Thus, $T = G(S)$ (see [6]).

2.2. Fans in euclidean spaces. Let $N$ be a lattice of finite rank $n$, i.e. $N \cong \mathbb{Z}^n$. We denote by $N_\mathbb{R}$ the Euclidean space $N \otimes \mathbb{Z} \mathbb{R}$. Let $M = \text{Hom}(N, \mathbb{Z})$ be the dual lattice and $M_\mathbb{R} = M \otimes \mathbb{Z} \mathbb{R}$ the corresponding dual space. Let $\sigma \subset N_\mathbb{R}$ be a cone (a subset closed under multiplication by nonnegative real numbers). We say that $\sigma$ is strongly convex if it contains no real line, polyhedral if it is an intersection of finitely many closed half-spaces, and rational if there exist finitely many elements $v_1, \ldots, v_k \in N$ such that

$$\sigma = \left\{ \sum_{i=1}^{k} \lambda_i v_i \mid \lambda_i \geq 0 \right\}.$$

We will denote by $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$ and $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \rightarrow \mathbb{R}$ the canonical pairings. Given a cone $\sigma \subset N_\mathbb{R}$, we may form the dual cone $\sigma^* := \{ u \in M_\mathbb{R} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}$. If $u \in M_\mathbb{R}$, we write $u^\perp = \{ v \in N_\mathbb{R} \mid \langle u, v \rangle = 0 \}$. Then a face of $\sigma$ is a subset which may be written as $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^*$.

Definition 2.4. A fan in $N_\mathbb{R}$ is a nonempty finite collection $\Delta$ of strongly convex rational polyhedral cones in $N_\mathbb{R}$ satisfying the following two conditions:

1. If $\sigma \in \Delta$ and $\tau$ is a face of $\sigma$, then $\tau \in \Delta$.
2. If $\sigma, \tau \in \Delta$, then $\sigma \cap \tau$ is a face of both $\sigma$ and $\tau$.

If there exists a cone $\sigma \in \Delta$ such that $\Delta$ consists of $\sigma$ together with all of its faces, we say that the fan $\Delta$ is generated by the cone $\sigma$.

2.3. Fans and toric varieties. Now we will show how to associate, to each fan $\Delta$ in $N_\mathbb{R}$, a toric variety. First, for each $\sigma \in \Delta$, we define an affine variety $U_\sigma$. Let $\sigma^*$ be the dual cone and let $S_\sigma$ be the subsemigroup of $M$ defined by $S_\sigma = \sigma^* \cap M$. We may form the semigroup ring $k[S_\sigma]^\mathbb{G}$.

Clearly, it is a finitely generated $k$-algebra without zero divisors, so we set $U_\sigma = \text{Spec}(k[S_\sigma])$ (technically, we should write $m$-Spec here, since we consider only the closed points of the corresponding scheme). Thus $U_\sigma$ is an irreducible algebraic variety over $k$.

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5 As usual, $M_n(k)$ denotes the set of $n \times n$ matrices with entries in $k$, and we make $M_n(k)$ into a semigroup under multiplication.

6 It is the commutative $k$-algebra generated by the “monomials” $\{ x^v \mid v \in S_0 \}$ subject to the relation $x^v \cdot x^w = x^{v+w}$. In the following discussion, we identify elements of $S_\sigma$ with the corresponding monomials in $k[S_\sigma]$. 

Next, we patch the affines $U_\sigma$ together to obtain a toric variety. For all $\sigma, \tau \in \Delta$, the inclusion maps $\sigma \cap \tau \hookrightarrow \sigma$ and $\sigma \cap \tau \hookrightarrow \tau$ correspond to maps $\sigma^* \hookrightarrow (\sigma \cap \tau)^*$, $\tau^* \hookrightarrow (\sigma \cap \tau)^*$, and hence to homomorphisms of semigroups $S_\sigma \hookrightarrow S_{\sigma \cap \tau}, S_\tau \hookrightarrow S_{\sigma \cap \tau}$. These induce homomorphisms of $k$-algebras $k[S_\sigma] \hookrightarrow k[S_{\sigma \cap \tau}], k[S_\tau] \hookrightarrow k[S_{\sigma \cap \tau}]$, and, finally, dominant morphisms of algebraic varieties $U_{\sigma \cap \tau} \to U_\sigma$ and $U_{\sigma \cap \tau} \to U_\tau$.

It turns out that these morphisms are open embeddings, so we may glue the affines $U_\sigma, \sigma \in \Delta$, along their open subvarieties $U_{\sigma \cap \tau}$, to obtain an irreducible variety $X$. Note that $\{0\}$ is a face of every strongly rational cone, so by condition 1 of definition 2.4, $\{0\} \in \Delta$. Moreover, for each $\sigma \in \Delta$, the inclusion $\{0\} \hookrightarrow \sigma$ induces an open embedding $U_{\{0\}} \hookrightarrow U_\sigma$.

But $S_{\{0\}} = M$, hence we have natural isomorphisms

$$k[S_{\{0\}}] \cong k[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]$$

and

$$U_{\{0\}} \cong T = (k^*)^n.$$ 

Thus all the affines $U_\sigma$ contain the torus $T$ as an open dense subset, and hence so does $X$. It remains to define the action of $T$ on $X$. We first construct this action on each of the $U_\sigma$’s. To show that the multiplication morphism $T \times T \to T$ extends to a morphism $T \times U_\sigma \to U_\sigma$, it suffices to prove the corresponding statement on the level of coordinate rings, namely, that we have the commutative diagram

$$
k[T] \quad \xrightarrow{\cup} \quad k[T] \otimes_k k[T] \\
k[U_\sigma] \quad \xrightarrow{\cup} \quad k[T] \otimes_k k[U_\sigma]
$$

Note that the lattice $M$ is naturally isomorphic to the lattice of characters of $T$. Hence $k[U_\sigma] = k[S_\sigma]$ has a basis consisting of characters of $T$. But if $\chi$ is a character of $T$, then the homomorphism $k[T] \to k[T] \otimes_k k[T]$ maps $\chi$ to $\chi \otimes \chi$. This argument shows that, in fact, we have a diagram

$$
k[T] \quad \xrightarrow{\cup} \quad k[T] \otimes_k k[T] \\
k[U_\sigma] \quad \xrightarrow{\cup} \quad k[U_\sigma] \otimes_k k[U_\sigma]
$$

Since the homomorphism in the top row of this diagram is coassociative, so is the homomorphism in the bottom row. Thus, $U_\sigma$ has a natural semigroup structure which is compatible with the multiplication on the torus.

It is easy to check that the actions of the torus on the different $U_\sigma$’s agree, and hence give an action of $T$ on $X$. This completes the construction of the toric variety $X$. For more details, see [2].

**Definition 2.5.** Given a finite dimensional lattice $N$ and a fan $\Delta$ in $N_\mathbb{R} = N \otimes_{\mathbb{Z}} \mathbb{R}$, the toric variety $X$ constructed above will be called the toric variety associated with the fan $\Delta$, written $X = X(\Delta)$. 

Observe that if the fan $\Delta$ is generated by a single cone $\sigma \in \Delta$, then the associated toric variety is affine. Later we will prove the converse of this statement (Corollary 3.9).

3. Main Results

In this section, we state and prove our main results and their corollaries.

3.1. Toric semigroups. First, we define the main object of our study.

Definition 3.1. A toric semigroup is a toric variety $X \supset T$ which has a structure of an algebraic semigroup compatible with the multiplication on the torus.

Remark 3.2. In modern literature (cf. [1]), affine toric semigroups are usually called toric monoids.

We intend to prove

**Theorem 3.1.** Let $N$ be a finite dimensional lattice, $\Delta$ a fan in $N_\mathbb{R}$ and $X = X(\Delta)$ the associated toric variety. If $X$ has a structure of a toric semigroup, then the fan $\Delta$ is generated by a single cone $\sigma \in \Delta$. In particular, $X$ is affine.

We will use several lemmas. We call a cone $\sigma \subset N_\mathbb{R}$ nondegenerate if its interior is nonempty, or, equivalently, if $N_\mathbb{R}$, as a real vector space, is spanned by $\sigma$. In the previous section, we have shown that for all $\sigma \in \Delta$, the corresponding affine variety $U_\sigma$ has a natural structure of a toric monoid. We have

**Lemma 3.2.** The toric monoid $U_\sigma$ has a zero if and only if the cone $\sigma$ is nondegenerate.

*Proof.* See Appendix, section 4.1.

To use this result, we would like to be able to relate the semigroup structure on $X$ with those on the open affines $U_\sigma$. For this, we apply

**Lemma 3.3.** The multiplication morphism $X \times X \to X$ maps $U_\sigma \times U_\sigma$ into $U_\sigma$, and its restriction to a morphism

$$U_\sigma \times U_\sigma \to U_\sigma$$

coincides with the natural monoid structure on $U_\sigma$. More generally, let $X \supset T$ be a separated toric variety and let $Y \subset X$ be a $T$-invariant subvariety containing $T$ (so $Y$ is naturally a toric variety). If both $X$ and $Y$ have structures of toric semigroups, the two structures agree on $Y$.

*Proof.* See Appendix, section 4.2.

**Corollary 3.4.** Under the assumptions of Theorem 3.1, the fan $\Delta$ contains at most one nondegenerate cone.

*Proof.* Suppose, to obtain a contradiction, that there are two distinct nondegenerate cones $\sigma, \tau \in \Delta$. By Lemma 3.3, the semigroup structure on $X$ induces monoid structures on $U_\sigma$ and $U_\tau$, and by Lemma 3.2, both have zeroes, say $0_\sigma \in U_\sigma$ and $0_\tau \in U_\tau$. The relation $x \cdot 0_\sigma = 0_\sigma \cdot x = 0_\sigma$, which holds for all $x \in T$ (because $T \subset U_\sigma$), must therefore hold for all $x \in X$. Thus, $0_\sigma$ is a zero of $X$. Similarly, $0_\tau$ is also a zero of $X$. But a semigroup cannot have two distinct zeroes, so $0_\sigma = 0_\tau$. Consequently,

$$0_\sigma \in U_\sigma \cap U_\tau = U_{\sigma \cap \tau},$$
and in particular $U_{\sigma \cap \tau}$ has a zero. But $\sigma, \tau$ are distinct cones, whence $\sigma \cap \tau$ is a proper face of $\sigma$, and is therefore degenerate. This contradicts Lemma 3.2.

This result already excludes many possibilities for the fan $\Delta$. It can be used indirectly to exclude even more possibilities. For example, suppose that all the cones in $\Delta$ are degenerate, and in fact are contained in a certain subspace $V \subset N_{\mathbb{R}}$. Then, clearly, $\Delta$ can be considered as a fan in $V$. Hence, if $\Delta$ contains two cones which are nondegenerate as cones in $V$, we may again apply Corollary 3.4.

Still, observe that there do exist fans $\Delta$ which are not generated by one cone, and to which our previous argument does not apply. For example, think of a collection of degenerate cones in $N_{\mathbb{R}}$ whose union spans $N_{\mathbb{R}}$. To exclude such possibilities, we apply

Lemma 3.5. Under the assumptions of Theorem 3.1, the union of all cones in $\Delta$, viewed as a subset on $N_{\mathbb{R}}$, is closed under addition.

Proof. See Appendix, section 4.3.

Now we may finish the proof of Theorem 3.1. First, note that we may assume that the union of all the cones in $\Delta$ spans the whole space $N_{\mathbb{R}}$. Indeed, let

$$V = \text{span}_{\mathbb{R}} (\bigcup_{\sigma \in \Delta} \sigma).$$

Since the cones in $\Delta$ are rational, $V$ has a basis consisting of elements on $N$. Thus, if we set $N_{\Delta} = N \cap V$, then $V = N_{\Delta} \otimes_{\mathbb{Z}} \mathbb{R}$, and we may consider $\Delta$ as a fan in $V$. Since $V$ is a vector space, $N_{\Delta}$ is a saturated subsemigroup of $N$, i.e. if $v \in N$, $m \in \mathbb{Z}$ and $m \cdot v \in N_{\Delta}$, then $v \in N_{\Delta}$. Therefore we may choose a sublattice $N'_{\Delta} \subset N_{\Delta}$ such that $N = N_{\Delta} \oplus N'_{\Delta}$. Corresponding to this decomposition, we have $T = T_{\Delta} \times T'_{\Delta}$ where $T_{\Delta}$ and $T'_{\Delta}$ are the tori associated to the lattices $N_{\Delta}$ and $N'_{\Delta}$, respectively. Hence, $X = Y \times T_{\Delta}$ where $Y$ is the toric variety associated to the fan $\Delta$ viewed as a fan in $V$. Since $T_{\Delta} \times \{1\}$ is dense in $Y \times \{1\}$, the multiplication morphism $X \times X \to X$ restricts to a morphism

$$(Y \times \{1\}) \times (Y \times \{1\}) \to (Y \times \{1\}),$$

thus making $Y$ into a toric semigroup. But now the fan $\Delta$ spans the corresponding space $V$, by construction.

Now write $\sigma$ for the union of all cones in $\Delta$. Then $\sigma$ is a cone, and it is convex because it is closed under addition (Lemma 3.5). Since $\sigma$ generates $N_{\mathbb{R}}$ as a real vector space, $\sigma$ must have nonempty interior. This already implies that $\Delta$ contains at least one nondegenerate cone, $\tau$, since otherwise $\sigma$ is a finite union of nowhere dense sets. In fact, since $\Delta$ has at most one nondegenerate cone (Corollary 3.4), we must have $\tau = \sigma$, otherwise $\sigma \setminus \tau$ has nonempty interior (because $\tau$ is closed). But now for each $\tau' \in \Delta$, we have $\tau' \subset \sigma = \tau$ by construction, whence, by condition 2 of Definition 2.2.1, $\tau'$ is a face of $\tau$. Consequently, $\Delta$ is generated by the cone $\tau$.

Q.E.D.

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7We will work out the details of this argument after Lemma 3.5.
3.2. **Construction of fans from toric varieties.** Here we consider the other problem mentioned in the introduction. Given a toric variety $X \supset T$, does there exist a fan $\Delta$ such that $X \cong X(\Delta)$? First let us show that it is necessary to assume that $X$ is separated:

**Proposition 3.6.** If $N$ is a finite dimensional lattice and $\Delta$ is a fan in $N_{R}$, then the associated toric variety $X(\Delta)$ is separated.

*Proof.* See Appendix, section 4.4.

On the other hand, there do exist nonseparated toric varieties. The easiest example to construct is the standard example of a nonseparated variety: the affine line with a doubled origin (see [3], the example after Lemma 3.3.4). We consider two copies of the affine line, $\mathbb{A}_{1}^{1}$ and $\mathbb{A}_{1}^{2}$, and glue them along their open subsets $\mathbb{A}_{1}^{1} \setminus \{0\}$ and $\mathbb{A}_{2}^{1} \setminus \{0\}$, with the identification morphism

$$\mathbb{A}_{1}^{1} \setminus \{0\} \to \mathbb{A}_{2}^{1} \setminus \{0\}$$

given by the identity map. The resulting nonseparated variety $X$ contains the one dimensional torus $T = \mathbb{A}_{1}^{1} \setminus \{0\}$, which acts by multiplication on both copies of the affine line, and hence on $X$.

**Remark 3.3.** In the following discussion, we will implicitly assume the following criterion of separability:

an algebraic variety $X$ is separated if and only if a morphism $U \to X$ from an open dense subset $U$ of a smooth curve $C$ extends in at most one way to a morphism $C \to X$ (see [3], Proposition 7.2.2),

more precisely, the implication ($\Rightarrow$) for the special case $C = \mathbb{A}^{1}$, $U = \mathbb{A}^{1} \setminus \{0\}$.

According to the discussion above, the best statement we can try to prove is

**Theorem 3.7.** Let $X \supset T$ be a separated toric variety. Let $N = \text{Hom}(k^{*}, T)$ be the lattice of one-parameter subgroups of $T$, and $M = N^{*} = \text{Hom}(T, k^{*})$ the dual lattice of characters of $T$. Then there exists a fan $\Delta$ in $N_{R} = N \otimes_{Z} R$ such that the natural isomorphism

$$\text{Spec}(k[M]) \to T$$

extends to an isomorphism

$$X(\Delta) \cong X.$$ 

First, let’s prove the theorem in the particular case when $X$ is affine. The key idea is to observe that $X$ has a unique structure of a toric monoid (NB: earlier we proved this statement only assuming that $X = U_{\sigma}$ for some strongly convex rational polyhedral cone $\sigma$) and look at the "one-parameter subsemigroups" of $X$. We have

**Lemma 3.8.** Let $X \supset T$ be an affine toric variety. Then there exists a structure of a toric monoid on $X$.

*Proof.* See Appendix, section 4.5.
As a by-product of our discussion, we obtain a statement, which is not at all obvious from the definitions.

**Corollary 3.9.** Let $N$ be a finite dimensional lattice and $\Delta$ a fan in $N_{\mathbb{R}}$. The associated toric variety $X = X(\Delta)$ is affine if and only if $\Delta$ is generated by a single cone.

**Proof.** The direction ($\Leftarrow$) is clear. For ($\Rightarrow$), combine Lemma 3.8 and Theorem 3.1. 

Now we introduce an important

**Definition 3.4.** If $X$ is an algebraic semigroup, a one-parameter subsemigroup of $X$ is a homomorphism of algebraic semigroups $k \to X$, where $k$ is viewed as an algebraic semigroup under multiplication. The set of all such is denoted by $\text{Hom}(k, X)$. In case $X$ is commutative, $\text{Hom}(k, X)$ has a natural structure of an abelian semigroup under pointwise multiplication.

**Remark 3.5.** The notion of a one-parameter subsemigroup corresponds to the well-known notion of a one-parameter subgroup of an algebraic group (see, for example, [6]). In fact, if $S$ is an irreducible affine algebraic semigroup, so that the group of invertible elements of $S$, $G(S)$, is itself an algebraic group, then the restriction of every one-parameter subsemigroup $k \to S$ to $k^*$ is a one-parameter subgroup of $G(S)$. Moreover, a subsemigroup $k \to S$ is determined by its restriction to $k^*$ (since affine varieties are separated), and hence if $S$ is abelian, it is possible to view $\text{Hom}(k, S)$ as a subsemigroup of $\text{Hom}(k^*, G(S))$. With these tools, we may complete the proof of Theorem 3.7 in the affine case. In fact, we prove a slightly more precise statement.

**Proposition 3.10.** Under the conditions of Theorem 3.7, assume that $X$ is affine, and view $X$ as a toric monoid (cf. Lemma 3.8). Let $\sigma$ be the cone generated\(^8\) by the subsemigroup $\text{Hom}(k, X)$ of the lattice $N = \text{Hom}(k^*, T)$ (cf. Remark 3.2.1) and let $\Delta$ be the fan in $N_{\mathbb{R}}$ generated by the cone $\sigma$ (see Definition 2.2.1). Then the conclusion of Theorem 3.7 is valid for the fan $\Delta$.

**Proof.** See Appendix, section 4.6.

Now we attack the general case. The following result is very useful for us

**Proposition 3.11.** Let $T$ be a torus and let $X$ be a normal variety on which $T$ acts. Then, for any point $x \in X$, there is a $T$-invariant affine open neighborhood of $x$.

**Proof.** See [4], Corollary 2 of Lemma 8. 

**Corollary 3.12.** Under the assumptions of Theorem 3.7, there exists a finite open cover of $X$ by affine toric varieties $V_i \supset T$.

**Proof.** By Proposition 3.11, there exists a finite cover of $X$ by invariant open affine subsets. But every open subset of $X$ intersects $T$, since $T$ is dense in $X$, and an invariant subset of $X$ intersecting $T$ must contain $T$, because $T$ itself is an orbit under the action of $T$. 

\(^8\)That is, $\sigma$ is the convex hull of $\text{Hom}(k, X)$. 

Now we fix a finite open cover \( \{ V_i \} \) of \( X \) by affine toric varieties as in Corollary 3.12. By Proposition 3.10, for each of the \( V_i \) we have the corresponding strongly convex rational polyhedral cone \( \sigma_i \subset N_\mathbb{R} \). Let \( \Delta \) be the finite set of cones consisting of the cones \( \sigma_i \) together with all their faces. We claim that \( \Delta \) is a fan and that the conclusion of Theorem 3.7 is valid for \( \Delta \).

**Remark 3.6.** One needs to be a little careful here. In this construction, we implicitly use the assumption that \( X \) is separated. The reason is that it makes sense to consider the cones \( \sigma_i \) simultaneously only if we know that every one-parameter subgroup of \( T \) extends in *at most one way* to a one-parameter subsemigroup of some \( V_i \). To illustrate this phenomenon, consider again the affine line with a doubled origin (see the example after Proposition 3.6). It has an open cover by affine toric varieties, namely, the two copies of the affine line that were used for gluing. But the cones corresponding to these two affine varieties coincide! So in the nonseparated case, intersection of cones does not necessarily correspond to intersection of open affine varieties. In a moment we’ll see that this phenomenon never occurs in the separated case.

To show that \( \Delta \) is a fan, it will suffice to see that for all \( i, j \), \( \sigma_i \cap \sigma_j \) is a face of both \( \sigma_i \) and \( \sigma_j \). Consider \( V_{ij} = V_i \cap V_j \). It is certainly an open subset of \( X \) containing \( T \). But in fact, since \( X \) is separated, \( V_{ij} \) is affine, and the natural \( k \)-algebra homomorphism \( k[V_i] \otimes_k k[V_j] \to k[V_{ij}] \) is surjective (see [3], Proposition 3.3.5). We know that \( k[V_i] = k[\sigma_i] \) where we use the same notation as in section 2.3. We also have the cone \( \sigma_{ij} \) corresponding to the affine toric variety \( V_{ij} \) (Proposition 3.10). By separability, a one-parameter subgroup of \( T \) extends to a one-parameter subsemigroup of \( V_{ij} \) if and only if it extends to one-parameter subsemigroups both of \( V_i \) and of \( V_j \). That is,

\[
\text{Hom}(k, V_{ij}) = \text{Hom}(k, V_i) \cap \text{Hom}(k, V_j),
\]

as subsemigroups of \( \text{Hom}(k^*, T) \). Consequently, \( \sigma_{ij} = \sigma_i \cap \sigma_j \). Say we already know that \( \sigma_{ij} \) is a face of both \( \sigma_i \) and \( \sigma_j \). Then \( \Delta \) is a fan, and we are done, because the toric variety \( X(\Delta) \) associated to \( \Delta \) is obtained by gluing the affine varieties corresponding to the cones \( \sigma_i \) along their open subvarieties corresponding to the cones \( \sigma_{ij} \), and the induced morphism \( X(\Delta) \to X \) (which exists by the universal property of gluing) is clearly an isomorphism.

To see that \( \sigma_{ij} \) must be a face of \( \sigma_i \) (and hence also \( \sigma_{ij} \), by symmetry), we make the following observation. In any case, the inclusion \( \sigma_{ij} \subset \sigma_i \) corresponds to the inclusion \( \sigma_{ij}^* \subset \sigma_i^* \), hence to the inclusion \( S_{\sigma_{ij}} \subset S_{\sigma_i} \), which induces a dominant morphism \( U_{\sigma_{ij}} \to U_{\sigma_i} \). We know that if \( \sigma_{ij} \) is a face of \( \sigma_i \), then this morphism is an open embedding. We conclude the proof with

**Lemma 3.13.** If \( \sigma_{ij} \) is not a face of \( \sigma_i \), then the morphism \( U_{\sigma_{ij}} \to U_{\sigma_i} \) is not an open embedding.

Note that this observation automatically implies that \( \sigma_{ij} \) is a face of \( \sigma_i \), because in our case the morphism \( U_{\sigma_{ij}} \to U_{\sigma_i} \) is just the inclusion \( V_{ij} \to V_i \). For the proof of the lemma, see Appendix, section 4.7.

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9Here we implicitly use the fact that the structures of toric semigroups on \( V_i \) and \( V_j \) agree on \( V_{ij} \). In fact, this follows from the second statement of Lemma 3.3.
4. Appendix

4.1. Proof of Lemma 3.2. The key idea here is to write down the definition of a zero of an affine semigroup in terms of homomorphisms of coordinate rings. Since $U_\sigma$ is commutative, an element of $U_\sigma$ is a zero if an only if it is a left zero. Note that an element of $U_\sigma$ can be naturally viewed as a morphism $\{\text{pt}\} \to U_\sigma$ where $\{\text{pt}\} = \text{Spec} k$ is the one-point variety. Therefore we may restate our definition of zero as follows: it is a morphism $0 : \{\text{pt}\} \to U_\sigma$ such that the diagram

\[
\begin{array}{ccc}
\{\text{pt}\} \times U_\sigma & \longrightarrow & U_\sigma \\
\| & & \| \\
\{\text{pt}\} \times U_\sigma & \longrightarrow & \{\text{pt}\} \\
\end{array}
\]

commutes, where the arrows are (from the left to the right) in the top row: $0 \times \text{id}_{U_\sigma}$, multiplication; in the bottom row: projection onto the factor $\{\text{pt}\}$, 0. In terms of coordinate rings, a zero of $U_\sigma$ is such a homomorphism of $k$-algebras $f : k[U_\sigma] \to k$ that the following diagram commutes:

\[
\begin{array}{ccc}
k \otimes_k k[U_\sigma] & \longleftarrow & k[U_\sigma] \\
\| & & \| \\
k \otimes_k k[U_\sigma] & \longleftarrow & k \\
\end{array}
\]

and now the arrows are (from the left to the right) in the top row: $f \otimes \text{id}_{k[U_\sigma]}$, comultiplication; in the bottom row: natural inclusion of $k$ into a $k$-algebra, $f$.

We recall that $k[U_\sigma] = k[S_\sigma]$. I claim that the diagram above exists if and only if there exists a $k$-algebra homomorphism $f : k[S_\sigma] \to k$ which kills all nonzero elements of $S_\sigma$. Indeed, let $v \in S_\sigma$, then in the top row it maps first to $v \otimes v$, and then to $f(v) \otimes v$. In the bottom row, it maps first to $f(v)$, and then to $f(v) \otimes 1$. Clearly, $f(v) \otimes v = f(v) \otimes 1$ in $k \otimes_k k[S_\sigma]$ if and only if either $v = 0$, $f(v) = 1$, or $f(v) = 0$. So if the diagram above commutes, then $f$ has the required property. Conversely, if $f$ has the required property, then the diagram commutes, because $S_\sigma$ is a basis of $k[S_\sigma]$.

On the other hand, there exists a homomorphism $f$ with the required property if and only if there exists a map $g : S_\sigma \to k$ such that $g(0) = 1$, $g(v) = 0$ for all $v \neq 0$, and $g(v + w) = g(v)g(w)$ for all $v, w \in S_\sigma$. If $\sigma$ is degenerate, then there are nonzero elements $v, w \in S_\sigma$ such that $v + w = 0$, and then such a map $g$ cannot exist (applying $g$ to the equality $v + w = 0$ gives $0 \cdot 0 = 1$, absurd). On the other hand, if $\sigma$ is nondegenerate, then such elements $v, w$ do not exist, so if we define $g$ by $g(0) = 1$, $g(v) = 0$ for $v \neq 0$, then $g$ satisfies $g(v + w) = g(v)g(w)$ for all $v, w$. This completes the proof.

4.2. Proof of Lemma 3.3. By Proposition 3.6 (whose proof does not depend on this lemma), $X$ is separated, so the first statement is a special case of the second. We have two morphisms

\[
\phi_1, \phi_2 : Y \times Y \to X,
\]

denote, $\phi_1$ is the composition

\[
Y \times Y \hookrightarrow X \times X \to X,
\]

the second morphisms being the multiplication on $X$, and $\phi_2$ is the composition

\[
Y \times Y \to Y \hookrightarrow X,
\]

This completes the proof.
the first morphism being the multiplication on $Y$. Since the multiplications on both $X$ and $Y$ agree with the multiplication on the torus, $\phi_1$ and $\phi_2$ agree on $T \times T$. But $X$ is separated and $T \times T$ is dense in $Y \times Y$, which implies that $\phi_1 = \phi_2$. Thus, the diagram

$$
\begin{array}{ccc}
Y \times Y & \longrightarrow & Y \\
\cap & \cap & \\
X \times X & \longrightarrow & X
\end{array}
$$

commutes, which is exactly the statement of Lemma 3.3.

4.3. Proof of Lemma 3.5. Choose $\sigma \in \Delta$ and a point $v \in N$. Note that since we constructed the torus $T$ as $\text{Spec} k[M]$, the lattice $M$ is naturally identified with the character lattice of $T$, and then $N$ is identified with the lattice of one-parameter subgroups of $T$. Thus, $v$ naturally corresponds to a one-parameter subgroup, $s_v$, of $T$. Let us first check that $v \in \sigma$ if and only if $s_v$ extends to a one-parameter subsemigroup of $U_\sigma$ (see Definition 3.2.1). On the level of coordinate rings, $s_v$ extends to a one-parameter subsemigroup of $U_\sigma$ if and only if under the homomorphism $s_v^*: k[T] \rightarrow k[k^*] = k[x, x^{-1}]$, the subring $k[U_\sigma]$ of $k[T]$ is mapped into $k[x]$. We also note that on elements of $M$ (naturally viewed as a subset of $k[T] = k[M]$), the homomorphism $s_v^*$ is given by $\chi \mapsto x^{(x, v)}$. Thus $s_v$ extends to a one-parameter subsemigroup of $U_\sigma$ if and only if $v$ has nonnegative pairing with all elements of $S_\sigma$. By definition, this happens if and only if $v \in (\sigma^*)^*$. But $\sigma = (\sigma^*)^*$ (see [2]).

Now it’s not hard to complete the proof. We know that if elements $v, w \in N$ correspond to one-parameter subgroups $s_v, s_w$ of $T$, then $v + w$ corresponds to $s_v \cdot s_w$ (pointwise product of the subgroups). Now if $v \in \sigma, w \in \tau$ for some cones $\sigma, \tau \in \Delta$, we know that both $s_v$ and $s_w$ extend to one-parameter subsemigroups of $U_\sigma$ and $U_\tau$, respectively, and hence to subsemigroups of $X$, because the multiplication on $X$ is compatible with those on $U_\sigma$ and $U_\tau$ (see Lemma 3.3). We denote the extensions again by $s_v$ and $s_w$. Then, clearly, $s_{v+w}$ also extends to a subsemigroup of $X$, namely, $s_v \cdot s_w$. There is a cone $\psi \in \Delta$ such that $(s_v \cdot s_w)(0) \in U_\psi$, and since $s_{v+w}(k^*) \subseteq T$, the image of the extension, $s_v \cdot s_w$, of $s_{v+w}$ is contained in $U_\psi$. Thus, $v + w \in \psi$, completing the proof.

4.4. Proof of Proposition 3.6. Write $\text{diag}: X \hookrightarrow X \times X$ for the diagonal embedding. We must prove that $\text{diag}(X)$ is closed in $X \times X$. By construction, $X = X(\Delta)$ is covered by the affines $U_\sigma$. Hence, it is enough to show that for all $\sigma, \tau \in \Delta$, $\text{diag}(U_\sigma \cap U_\tau)$ is closed in $U_\sigma \times U_\tau$. We know that $U_\sigma \cap U_\tau = U_{\sigma \cap \tau}$ is affine, so it suffices to see that the induced $k$-algebra homomorphism

$$
\text{diag}^*: k[U_\sigma] \otimes_k k[U_\tau] \rightarrow k[U_{\sigma \cap \tau}]
$$

is surjective. But, in the notation of section 2.3, $k[U_\sigma] = k[S_\sigma]$ and $k[U_\tau] = k[S_\tau]$, and since addition in the lattice $M$ corresponds to multiplication in the algebras $k[S_\sigma], k[S_\tau]$, the image of $\text{diag}^*$ is $k[S_\sigma + S_\tau]$. Since $(\sigma \cap \tau)^* = \sigma^* + \tau^*$, the proof is complete.
4.5. **Proof of Lemma 3.8.** It suffices to prove that there exists a commutative diagram

\[
\begin{array}{c}
k[T] \rightarrow k[T] \otimes_k k[T] \\
\cup & \\
k[X] \rightarrow k[X] \otimes_k k[X]
\end{array}
\]

because this will give a morphism \( X \times X \rightarrow X \), compatible with the multiplication on the torus, and associativity will follow from the coassociativity of the bottom row of the diagram, which in turn is implied by the coassociativity of the top row.

We know, by definition, that there exists a commutative diagram of the required form if we replace \( k[X] \otimes_k k[X] \) by \( k[T] \otimes_k k[X] \). But the torus is commutative, so the left action of \( T \) on \( X \) induces a right action of \( T \) on \( X \), which is also compatible with the multiplication on the torus. Hence, in the diagram above, \( k[X] \otimes_k k[X] \) may also be replaced by \( k[X] \otimes_k k[T] \). But, by simple linear algebra,

\[
k[X] \otimes_k k[T] \cap k[T] \otimes_k k[X] = k[X] \otimes_k k[X],
\]

as subalgebras of \( k[T] \otimes_k k[T] \). This completes the proof.

4.6. **Proof of Proposition 3.10.** The left action of \( T \) on itself (by multiplication) naturally induces an action of \( T \) on the coordinate ring \( k[T] \). The latter decomposes into a direct sum of one-dimensional \( T \)-invariant subspaces, namely, those spanned by the characters of \( T \). Since the action of \( T \) on itself extends to an action of \( T \) on \( X \), the subalgebra \( k[X] \) of \( k[T] \) is \( T \)-invariant, and hence itself is a direct sum of one-dimensional subspaces spanned by characters of \( T \). Let \( S \) denote the set of all characters of \( T \) contained in \( k[X] \), so that \( k[X] = k[S] \). Since \( k[X] \) is closed under multiplication, \( S \) is closed under addition. Thus, if we let \( \sigma \) be the cone in \( M_k \) generated by \( S \) and \( \tau = \sigma^* \), then \( \sigma = \tau^* \) and therefore \( X = U_\tau \). We already know (see the proof of Lemma 3.5) that \( \sigma \) is generated by the set of one-parameter subsemigroups of \( U_\sigma \). This completes the proof.

4.7. **Proof of Lemma 3.13.** To simplify notation, write \( \sigma \) for \( \sigma_i \) and \( \tau \) for \( \sigma_j \).

We recall that, in general, given an affine variety \( V \), every open affine subset of \( V \) has the form \( \{x \in V \mid f(x) \neq 0\} \) for some \( f \in k[V] \). For us, this says that \( U_\tau \rightarrow U_\sigma \) is an open embedding if and only if the corresponding homomorphism of coordinate rings \( k[S_\sigma] \rightarrow k[S_\tau] \) may be realised as an embedding \( k[S_\sigma] \rightarrow k[S_\tau](f) \) for some \( f \in k[S_\sigma] \), that is (if we now view \( k[S_\sigma] \) as a subalgebra \( k[S_\tau] \)), if and only if

\[
k[S_\tau] = k[S_\sigma, f^{-1}]
\]

for some \( f \in k[S_\sigma] \). Now we assume that such \( f \) exists. Then, in particular, \( f \) is nonzero on \( T \), and hence, multiplying \( f \) by a nonzero constant, we may assume that \( f \) is a character of \( T \). Consequently, we have \( f \in S_\sigma \). Now \( S_\tau = S_\sigma + \mathbb{Z}f \).

Therefore,

\[
\tau = \{v \in \sigma \mid (-f, v) \geq 0\} = \{v \in \sigma \mid (f, v) = 0\}
\]

is a face of \( \sigma \), completing the proof.
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