A Magnetite–Water-Based Nanofluid Three-Dimensional Thin Film Flow on an Inclined Rotating Surface with Non-Linear Thermal Radiations and Couple Stress Effects

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Abstract: This study is related to the heat energy transfer during 3D nanofluid (water-based) motion over a rotating surface by incorporating the combined impacts of thermal radiations and couple stress. The flow is modeled by a set of non-linear coupled PDEs, which is converted to a set of coupled non-linear ODEs by using suitable similarity transformations. The transformed equations are solved with the built-in NDSolve command. The effects of relevant interesting parameters on the nanofluid velocity components and temperature distribution are explained through various graphs. It is found that the velocity component $f(\eta)$ is increased with higher values of $\gamma$ and $A_0$ while it drops with an increasing rotation parameter and nanoparticle volume fraction. The fluid temperature increases with higher $\alpha_{nf}$, $Rd$, $\epsilon_2$, $A_1$ and drops with increasing $Pr$, $\epsilon_1$ and couple stress parameter ($A_0$). The Nusselt number remains constant at a fixed $Pr$ and $Rd$, whereas it increases with increasing $Pr$ and is reduced with rising $Rd$. A comparison between the achieved results is carried out with the analytical results through different tables. An excellent agreement is observed between these results.

Keywords: thermal radiations; magnetic field; Carreau fluid; stretching/shrinking surface; Hall effect; non-linear radiations; HAM

1. Introduction

Nanofluid is a mixture obtained by mixing nanoparticles with ordinary heat energy transfer fluids such as oil, glycol, water, ethylene glycol, etc. Nanoparticles can be prepared on a small scale (laboratories) as well as on a large scale (industries). The usual size of nanoparticles lies in the range 1–100 nm [1]. Nanoparticles can be made from metals such as Al, Cu, Au, and Ag, metal oxides such as Fe3O4, CuO, TiO2, and Al2O3, nitrides such as SiN and AlN, and carbides such as SiC, etc. These nanometer-sized particles can be added in a minute amount to augment the heat energy transfer rate due to their tremendous thermal conductivities [2,3]. Due to nanometer-sized geometry, the nanoparticles can easily mix with the base fluid [4]. A literature study suggests that there are two
main techniques to model nanofluids. In the first type, the nanoparticles are distributed uniformly throughout the host fluid. The nanofluids’ thermophysical characteristics lead to the boundary-layer equations, which can be used to study the nanoparticles’ effects. On the other hand, the models created to consider the nanoparticles’ interaction with the base fluid are known as two-phase models, and are known as the second case of the nanofluid model [5]. Tewari and Das [6] studied the single-phase model, while Buongiorno [7] has investigated the two-phase nanoparticle model. A study was also performed by Vajravelu and Mukhopadhyay [8] on a single-phase model.

The use of nanoparticles varies from a small scale in laboratories to a very large scale in industries. Due to high thermal conductivities, nanofluids are used for cooling purposes in transformers, cooling chambers, and in nuclear reactors. For medical purposes, they are used to design different surgery equipment to kill tumor cells. In electronic machines, the heat generated during their operation can also be reduced by using different nanofluids. Tewari and Das’s [6] model has been further investigated by many researchers to elaborate on the thermal features of different nanofluids [9–12].

The investigation of heat energy transfer during non-Newtonian fluid flow on stretched surfaces has recently gained significant attention [13]. Due to different kinds of flow in real situations, a single constitutive relation which relates the shear stress and shear rate is insufficient to investigate the non-Newtonian fluid properties [14]. There are some viscoelastic fluids which show polar effects, called couple stress fluids, which simplify the traditional theories for investigation [15]. For this kind of fluid, the constitutive relations associate the skew-symmetric portion of the stress tensor with the angular velocity, and the couple stress with the gradient in angular velocity [16]. Eringen [17] called the polar fluids micropolar fluids. The theory of dipolar fluids was developed by Bleustein and Green [18]. If the polar fluid is in such a state that the cosserat triad is rigidly attached to the medium, then it is known as a couple stress fluid. The polar and dipolar fluids are considered in the couple stress theory which Stokes created [19]. The Navier–Stokes equations cannot describe such fluids due to their stress tensor non-symmetric nature. Examples include lubricants that contain polymer preservative to some extent, synthetic fluids, blood, and electro-rheological fluids [20].

Due to the relevance of couple stress fluids in the chemical trade and various machineries, a good number of researchers have given attention to the study of their flow and associated properties. Hayat et al. [21] worked out the melting process in investigating the heat energy transfer for this kind of fluid at the stagnation point. Ramzan et al. [22] examined the accumulative effects of heating and magnetic field on the couple stress fluid three-dimensional flow. Srinivasacharya and Kaladhar [23] have studied both Duofour and Soret effects on the convective flow in a porous medium. Turkylmazoglu [24] analytically investigated two-dimensional flow on an extended surface. Furthermore, Hayat et al. [25] employed the two-phase nanofluid model to investigate the combined impacts of convective boundary conditions and magnetic field on the couple stress fluid three-dimensional flow on a non-linear enlarged surface. Pordanjani et al. [26] examined the impact of radiation and magnetic field on entropy generation and convective thermal energy nanofluid flow rate through a rectangular container and investigated the influence of pertinent parameters on the flow features. Additionally, Ramzan [27] has investigated the effects of Joule heating, viscous dissipation, and applied magnetic field on the 3D couple stress nanofluid flow. The heat transfer augmentation characteristics of alumina water nanofluid in the presence of a homogeneous magnetic field flowing through a rectangular channel was investigated by Bagherzadeh et al. [28].

There are different mechanisms by which energy can transfer from one place to another. In different manufacturing techniques, the heat transfer through radiation is more efficient and practical than convective heat transfer when there is a large temperature difference between the ambient fluid and the surface [29]. To simplify the situation, the majority of researchers have applied the Rosseland approximation to study the effects of linear radiation [30]. Jamshed et al. [31] studied the impacts of solar radiation transport
and slip condition on the convective thermal features of an unsteady flow of Casson nanofluid. Sheikholeslami [32] investigated the effects of linear thermal radiations on nanofluid flow through an elliptic cylinder. Sajid et al. [33] examined the effects of multiple convective surface boundary conditions, non-linear thermal radiations, exponential heat source, and viscous dissipation on incompressible micropolar fluid motion through a porous stretching surface. Dogonchi et al. [34] used a porous channel to study the impacts of thermal radiations on the heat transfer during nanofluid migration. In the majority of these investigations, the researchers applied linear thermal radiation terms to make the situation simple. To obtain the complete knowledge of the heat transfer during nanofluid flow, it is very important to take into account the impacts produced due to the inclusion of non-linear thermal radiations [35–39].

The wide range of applications of nanofluids depend upon the geometry of the problem under consideration. Heat transfer analysis of the motion of different nanofluids has been performed by many researchers using different geometries. Devakar et al. [40] studied non-Newtonian fluid motion through a square duct past a permeable medium. Srinivasacharya and Shafeeurrahman [41] investigated the nanofluid flow between concentric cylinders. Rashidi et al. [42] investigated the fluid flow on a rotatory sheet, and derived a semi-analytic solution. Khan and Pop [43] studied the nanofluid flow over a stretching sheet. The flow over a stretching sheet has attracted significant attention in the last few years due to its unique nature and industrial applications. Different studies on stretching sheets can be found in detail in the papers [44–49]. Gosh et al. [50] used the Tiwari and Das model to investigate the couple stress magnetite–water-based nanofluid motion near a bi-directional stretchable surface incorporating the effects due to non-linear thermal radiations. Lund et al. [51] discussed the dual and symmetrical solution of hybrid nanofluid flow past a stretching surface. They analyzed the rotating frame’s impact on the hybrid nanofluid flow. A more recent survey of the hybrid nanofluid flow is discussed by Ali et al. [52,53].

The present article deals with the three-dimensional nanofluid (water-based) flow of the boundary layer over a rotating surface considering the impacts of both thermal radiations and couple stress. The study of three-dimensional problem gives more insight into the actual scenario as compared to two-dimensional investigations. The Tewari and Das [6] model is used. The modeled equations are converted to a simple set of non-linear ODEs through suitable similarity transformations. From the obtained solution, we have found that the numerical results are in agreement with the analytical results obtained through HAMS. The results of the study are explained with the help of different graphs as well as tables.

2. Mathematical Model of the Problem

The impacts of thermal radiations on the three-dimensional and steady boundary-layer laminar flow of couple stress water-based nanofluid (incompressible) over a rotating surface are considered in the current study. The geometry of the problem is sketched in Figure 1. The nanoparticles are supposed to be in equilibrium with the base fluid. Furthermore, no slip at the boundary occurs.
The model equations for the nanofluid flow are [54]:

\[ \nabla \cdot \mathbf{V} = 0, \]  

\[ 0 = \rho_{nf} \mathbf{F} + \frac{1}{2} \nabla \times (\rho_{nf} \mathbf{C}) - \mu_{nf} \nabla \times \nabla \times \mathbf{V} - \nabla p + (\lambda_1 + 2\mu_{nf}) \nabla (\nabla \cdot \mathbf{V}) - n \nabla \times \nabla \times \nabla \times \mathbf{V}. \]  

Here, \( \rho_{nf} \), \( \mathbf{V} \), and \( p \) are, respectively, the density, velocity, and pressure of the assumed fluid. The symbols \( \mathbf{F} \) and \( \mathbf{C} \) represent the body force and body couple per unit mass, respectively. The symbols \( \lambda_1 \) and \( \mu_{nf} \) are the coefficients of viscosity. The stress tensor \( t_{ij} \) and the deformation rate tensor \( d_{ij} \) [54] are related by the following constitutive relation:

\[ t_{ij} = (-p + \lambda_1 \nabla \cdot \mathbf{V}) \delta_{ij} + 2\mu d_{ij} - \frac{1}{2} \epsilon_{ijk} \left( m_{jk} + 4n \omega_{kpr} + \rho C_k \right). \]  

The constitutive relation for the couple stress tensor is

\[ m_{ij} = \frac{1}{3} m \delta_{ij} + 4n' \omega_{ij} + 4n \omega_{ij}. \]  

These model relations for the incompressible fluid flow, in the absence of body force and couple stress effects [54] reduce to the following relations:

\[ \nabla \cdot \mathbf{V} = 0, \]  

\[ 0 = -\nabla p - n \nabla \times \nabla \times \nabla \times \mathbf{V} - \mu_{nf} \nabla \times \nabla \times \mathbf{V}. \]  

By using \( \mathbf{V} = (u, v, w) \) and then employing the Tiwari and Das model [6], the model equations in the final form are [54]:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \]  

\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - v_{nf}^4 \frac{\partial^4 u}{\partial z^4} - 2\omega w = v_{nf}^4 \frac{\partial^2 u}{\partial z^2} + g \sin \beta, \]  

\[ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = v_{nf}^4 \frac{\partial^4 v}{\partial z^4} - v_{nf}^4 \frac{\partial^2 v}{\partial z^2}, \]  

\[ u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} + 2\omega u = v_{nf}^4 \frac{\partial^2 w}{\partial z^2} - v_{nf}^4 \frac{\partial^4 w}{\partial z^4} + g \cos \beta - \frac{p}{\rho_f}, \]  

\[ u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = \frac{k_{nf}}{(\rho c_p)_{nf}} \frac{\partial^2 T}{\partial z^2} - \frac{1}{(\rho c_p)_{nf}} \frac{\partial g_r}{\partial z}. \]
Here, \( \nu_{nf} = \frac{\mu_{nf}}{\rho_{nf}}, \nu'_{nf} = \frac{\mu_{nf}}{\rho_{nf}} \) are the nanofluid kinematic and couple stress viscosities, \( \mu_{nf} (\rho_{nf}) \) is the nanofluid viscosity (density), \( \kappa_{nf} (\rho C_p)_{nf} \) is the nanofluid thermal conductivity (specific heat capacity).

The boundary conditions for Equations (8)–(12) are

\[
\begin{align*}
&u_z = 0, \quad v_z = 0, \quad w = -W, \quad T = T_0, \quad P = P_0 \quad \text{at} \quad z = h, \\
&u = -\Omega y, \quad v = \Omega x, \quad w = 0, \quad T = T_w \quad \text{at} \quad z = 0.
\end{align*}
\]

\[ (13) \]

3. Similarity Transformations

The similarity transformations which are used to transform Equations (8)–(12) are given here [55]. The use of similarity transformation relations converts the coupled system of developed PDEs to a simplified system of coupled ODEs. The system of ODEs can then be easily solved by using the NDSolve command built into Mathematica. The similarity transformations are defined as the following:

\[
\begin{align*}
w &= -2\sqrt{\Omega \nu_{nf}} f(\eta), \quad v = \Omega xf'(\eta) + gS(\eta) \sin\left(\frac{\beta}{\Omega}\right), \\
u &= \Omega yg(\eta) + \Omega xf'(\eta) + gK(\eta) \sin\left(\frac{\beta}{\Omega}\right), \quad T = (T_0 - T_w)\theta(\eta) + T_w,
\end{align*}
\]

where, \( \eta = z\sqrt{\frac{\Omega}{\nu_{nf}}} \).

After applying the similarity transformations (14), we obtain:

\[
\begin{align*}
A_0f^{(iv)} - A_1f''' + (f')^2 - g^2 - 2ff'' - 2R_0f &= 0, \\
A_0K^{(iv)} - A_1K'' + f'K - gS - 2fK' - 1 &= 0, \\
A_0g^{(iv)} - A_1g'' - 2R_0(f'g - f'g) &= 0, \\
A_0S^{(iv)} - A_1S'' - 2fS' - f'S + gK - 2R_0(f' + g) &= 0, \\
a + \frac{4}{3 A_{nf}} + 2Pr(\frac{\epsilon_2\epsilon_3}{\epsilon_1\epsilon_4}) \theta'f &= 0.
\end{align*}
\]

The boundary conditions are:

\[
\begin{align*}
f(0) &= f'''(0) = 0, f'(0) = 1, f''(\gamma) = 0 = f'(\gamma), g(0) = 0, g'(0) = 1, g'''(0) = 0, \\
g''(\gamma) &= 0, g'(\gamma) = 0, K(0) = 0, K'(0) = 1, K''(0) = 0, K''(\gamma) = 0, K'(\gamma) = 0, \\
S(0) &= 0, S'(0) = 0, S'''(0) = 1, S''(\gamma) = 0 = S'(\gamma), \theta(0) = 1, \theta(\gamma) = 0.
\end{align*}
\]

where,

\[
\begin{align*}
Rd &= \frac{4T_0^3}{Kk}, \epsilon_1 = \frac{\rho_{nf}}{\rho_f}, \epsilon_2 = \frac{\mu_{nf}}{\mu_f}, \epsilon_3 = \frac{(\rho C_p)_{nf}}{\rho C_p f}, \epsilon_4 = \frac{k_{nf}}{k_f}, \\
Pr &= \frac{\nu_f}{\kappa_{nf}} \gamma = h\sqrt{\frac{\Omega}{\nu_{nf}}} f(\gamma) = \frac{W}{2\sqrt{\Omega \nu_{nf}}} = \frac{\nu_f}{(T_0 - T_w)}, \\
N_u &= \frac{\nu_f}{k_f (T_0 - T_w)} = \frac{\nu_f}{\kappa_{nf}}, \\
N_{\Omega} &= \frac{\nu_f}{\kappa_{nf}}, N_{\nu_f} = \frac{\nu_f}{(T_0 - T_w)} = \frac{\nu_f}{(T_0 - T_w)}.
\end{align*}
\]

\[ (21) \]

4. Numerical Solution

The solution of the model in Equations (15)–(19) with the boundary conditions (20) is obtained by using the built-in NDSolve command in Wolfram Mathematica. NDsolve is a widely used mathematical solver for the solution of ordinary differential equations and some partial differential equations. This technique can also handle some algebraic
differential equations, which are a mix of algebraic and differential equations. NDSolve can solve both initial and boundary value problems iteratively. The schematic diagram for the solution procedure of the implemented technique is displayed in Figure 2.

![Solution methodology schematic diagram.](image)

Figure 2. Solution methodology schematic diagram.

5. Results and Discussion

This section is devoted to discussing the achieved results. The results obtained from the solution of the system of coupled ODEs are explained by displaying different graphs. In these plots, the variations in the velocity components, fluid temperature, and Nusselt number with respect to increasing values of the pertinent parameters are displayed.

The effect of $\gamma$ (thickness parameter) on the velocity profile is displayed in Figure 3. The increasing values of the thickness parameter augment the fluid velocity. These variations are more effective beyond $\eta = 0.1$, and are maximum at $\eta = 0.75$. Physically, when $\gamma$ increases, the kinematic viscosity decreases and the rotation rate increases, which increases the flow along the x-direction, and as a result the velocity profile increases.

![Plot showing $f(\eta)$ dependence on $\gamma$.](image)

Figure 3. Plot showing $f(\eta)$ dependence on $\gamma$. 
The impact of $\gamma$ on the draining velocity in the x-direction is displayed in Figure 4. The rising values of the thickness parameter decrease the velocity profile due to the higher values of the rotation rate. The higher values of $\Omega$ intensify the flow in the x-direction, but the draining flow reverses, and as a result the velocity profile declines. These variations are much more effective beyond $\eta = 0.2$, and change with higher values of $\eta$.

![Figure 4](image)

**Figure 4.** Plot displaying $K(\eta)$ dependence on $\gamma$.

The influence of $\gamma$ on the induced flow in the y-direction is displayed in Figure 5. The figure shows that the larger values of the thickness parameter increase the velocity profile, and these variations are more effective beyond $\eta = 0.2$. The spacing between different curves increases with changing $\eta$. Physically, the increasing values of the thickness parameter increase due to larger values of $\Omega$, which further increases the rotation in the flow, and results an increase in the induced flow in the y-direction.

![Figure 5](image)

**Figure 5.** Graph depicting $S(\eta)$ dependence on $\gamma$.

Figure 6 plots the fluid temperature $\theta(\eta)$ for varying $\alpha_{nf}$. At fixed $\alpha_{nf}$, there is approximately a direct relation between $\theta(\eta)$ and $\eta$ up to about $\eta = 0.7$. The figure also shows that there is an increasing trend in $\theta(\eta)$ with the higher values of $\alpha_{nf}$. This increasing trend is more prominent for the highest value of $\alpha_{nf}$ in the range from $\eta = 0.3$ up to $\eta = 0.8$.

Figure 7 displays the variation in $\theta(\eta)$ with changing Prandtl number ($Pr$) values. The figure shows a decreasing behavior with higher values of the Prandtl number. The decreasing trend is more significant for the highest value of $Pr$ only for the intermediate values of $\eta$. The higher values of the Prandtl number correspond to less thermal diffusivity, which results in a decline in the fluid temperature. The temperature function $\theta(\eta)$ shows the opposite behavior with respect to the variation in $Pr$ as compared to the variation with respect to $\alpha_{nf}$. 
Figure 6. Plot displaying $\theta(\eta)$ dependence on $\alpha_{nf}$.

Figure 7. Plot depicting $\theta(\eta)$ dependence on $Pr$.

Figure 8 displays the effect of the radiation parameter $Rd$ on the fluid temperature $\theta(\eta)$. The figure shows that $\theta(\eta)$ increases with the higher $Rd$ values. The increasing trend in the fluid temperature for the higher values of $Rd$ is evident for the values of $\eta$ from 0.2 to 0.9. Therefore, it is concluded that the presence of thermal radiation augments the fluid temperature.

Figure 8. Plot showing $\theta(\eta)$ dependence on $Rd$.

Figure 9 depicts $\theta(\eta)$ for varying scalar parameter $\epsilon_1$ values. It is evident that the temperature field $\theta(\eta)$ drops with the changing $\epsilon_1$ values. We see that at $\epsilon_1 = 0.2$, $\theta(\eta)$ increases with a uniform rate up to about $\eta = 0.5$ and then decreases beyond it. The spacing between different curves is reduced with increasing $\epsilon_1$. Furthermore, the $\theta(\eta)$ profiles overlap with one another at larger values of $\eta$. The increasing $\epsilon_1$ is associated with higher nanoparticle density, which decreases the temperature due to increasing thermal conduction.
Figure 9. Graph displaying $\theta(\eta)$ dependence on $\epsilon_1$.

The dependence of the fluid temperature $\theta(\eta)$ on $\eta$ for increasing values of $\epsilon_2$ is shown in Figure 10. An increasing trend in the temperature is observed at the intermediate values of $\eta$ for increasing $\epsilon_2$. Thus, the changing dynamic nanofluid viscosity, which is associated with increasing $\epsilon_2$, results in an increase in the temperature distribution of the nanofluid.

Figure 10. Plot depicting $\theta(\eta)$ dependence on $\epsilon_2$.

The dependence of $\theta(\eta)$ on $\eta$ for varying $\epsilon_3$ is displayed in Figure 11. We observed the same behavior as in the case of Figure 10. This increase in the temperature with increasing values of $\epsilon_3$ is due to increasing nanofluid-specific heat.

Figure 11. Figure showing $\theta(\eta)$ dependence on $\epsilon_3$. 
The dependence of \( f(\eta) \) on the rotation parameter \( (R_0) \) is displayed in Figure 12. The figure shows that \( f(\eta) \) increases with increasing \( \eta \) at fixed \( R_0 \). The velocity profile drops with increasing \( R_0 \). Thus, the increasing rotation rate increases the opposition to the flow in the axial direction, and hence decreases the velocity in that direction. The spacing between different curves is reduced with increasing \( R_0 \).

![Figure 12. Graph showing \( f(\eta) \) dependence on \( R_0 \).](image)

\[ R_0 = 1 \quad R_0 = 3 \quad R_0 = 5 \quad R_0 = 7 \]

Figure 13 depicts the variation in \( g(\eta) \) with changing \( R_0 \). It is observed that \( g(\eta) \) shows completely opposite behavior with increasing \( R_0 \), as shown by \( f(\eta) \) in Figure 12. This means that the changing rotation rate associated with the higher values of \( R_0 \) increases \( g(\eta) \).

![Figure 13. Graph displaying \( g(\eta) \) dependence on \( R_0 \).](image)

The effect of \( A_0 \) (couple stress parameter) on \( f(\eta) \) is plotted in Figure 14. It is observed that the velocity profile changes with higher \( A_0 \) values. Thus, the increasing couple stress parameter changes the nanofluid velocity in the axial direction. Near the surface, the viscosity effect is dominant, and in a particular region away from the surface, the higher values of \( A_0 \) decrease the viscous effects and, as a result, the velocity profile increases in the axial direction.

The variation in \( g(\eta) \) with changing values of \( (A_0) \) is displayed in Figure 15. A similar trend is also shown in Figure 14 for the axial velocity. The couple stress parameter is linked with the viscosity, so increasing the couple stress parameter reduces the viscosity effects and, as a result, the velocity profile increases. On the other hand, far away from the surface where the viscosity effects are negligible, the velocity profile declines with higher values of the couple stress parameter.
The impact of the nanofluid volume fraction parameter on the axial velocity $f(\eta)$ is displayed in Figure 16. It is clear from the figure that at intermediate values of the independent variable ($\eta = 0.5$), the velocity profile declines at a rapid rate as compared to the boundary values. Physically, when the nanoparticle volume fraction increases, the mass per unit volume also increases, and as a result, when the nanofluid volume fraction increases from 0.01 to 0.04, the axial velocity profile declines.
The impact of the nanoparticle volume fraction parameter $A_1$ on the draining velocity $g(\eta)$ in the x-direction is displayed in Figure 17. A similar trend is displayed in Figure 16 for the axial velocity. The larger values of the nanoparticle volume fraction reduce the velocity profile, and it is maximum at the intermediate values of $\eta$. On the other hand, the nanoparticle volume fraction decreases the boundary layer thickness as well. In conclusion, we can say that the velocity of the ordinary fluid (water in our case) is higher as compared to nanofluids. These results are also reported by Gireesha et al. [56].

![Figure 17](image1.png)

**Figure 17.** Plot depicting $\theta(\eta)$ dependence on $A_1$.

The variation in the thermal boundary layer for various values of the nanoparticle volume fraction is portrayed in Figure 18. As we have stated earlier, the larger values of the nanoparticle concentration decrease the boundary layer thickness. That is why the larger values of the nanoparticles increase the temperature profile. The variation in the thermal boundary layer is larger at $\eta = 0.56$, when the nanoparticle volume fraction values increase from 0.01 to 0.04.

![Figure 18](image2.png)

**Figure 18.** Graph depicting $\theta(\eta)$ dependence on $A_1$.

The dependence of the Nusselt number ($Nu$) on $\delta$ for changing values of $Pr$ is plotted in Figure 19. The figure shows that for a fixed value of $Pr$, $Nu$ does not change with the variation in the values of $\delta$. Furthermore, the plot also shows that by increasing $Pr$ to larger values, $Nu$ also increases in approximately equal proportions. Thus, the increasing $Pr$, which is due to the higher values of momentum diffusivity, augment the convective thermal energy flow of the nanofluid.

![Figure 19](image3.png)
Figure 19. Plot displaying the impact of $Pr$ on $Nu$.

Figure 20 displays the Nusselt number ($Nu$) against $\delta$ for varying radiation parameter ($Rd$) values. It is clear that at a fixed value of $Rd$, $Nu$ remains constant when changing the values of $\delta$. By increasing $Rd$ to larger values, $Nu$ drops to smaller values, maintaining the same constant trend with increasing $\delta$. This shows that the changing radiation parameter values decrease the convective thermal energy flow.

Figure 20. Graph displaying $Nu$ dependence on $Rd$.

6. Discussion of Tables

This section is devoted to the comparison of the present (numerical) and analytical (HAM calculation of the same study) results for the velocity components and temperature function. This comparison is described in Tables 1–5. We can see that the HAM and numerical results agree with one another with a good degree of accuracy. Table 1 shows the comparison for the velocity component $f(\eta)$. The value of $\eta$ changes from 0.0 to 1.0, with a step size equal to 0.1. From the table, it is clear that after a few iterations, the numerical results match with HAM results up to four decimal places, as is evident from the absolute error on the rightmost column. The relative difference between the two computations decreases to three decimal places beyond $\eta = 0.8$. Table 2 shows the comparison of the velocity component $K(\eta)$ for the two computations. The step size is chosen to be 0.1, whereas $\eta$ varies from 0.0 to 1.0. The HAM and the numerical calculations are in excellent agreement and the absolute error is very small in this case. The maximum relative difference for the two results decreases to six significant figures beyond $\eta = 0.7$. The comparison between analytical and numerical computations for the component $g(\eta)$ is given in Table 3. The independent variable $\eta$ ranges from 0.0 to 1.0 with a step size equal to 0.1. We see that the numerical and the analytical calculations are in agreement after a few iterations. The absolute error calculation in the right column demonstrates that both results are in agreement up to four significant figures. Similarly, from Table 4, we note that HAM and numerical calculations are in excellent agreement for $S(\eta)$ as well. We have compared the results of both procedures (HAM and numerical) for the evaluation of the temperature function ($\theta(\eta)$) in Table 5. It is evident that there is a very small absolute error between
numerical results and HAM calculations. The absolute error for the largest value of the independent variable, \( \eta = 1 \), is \( 2.220450 \times 10^{-16} \), which is a negligible number. Thus, we can see that the numerical calculations match completely with the analytical (HAM) results.

Table 1. Comparison between the analytical (HAM) and numerical results for \( f(\eta) \).  

| \( \eta \) | HAM Calculation | Numerical Calculation | Absolute Error |
|----------|----------------|----------------------|---------------|
| 0.0      | 0.000000       | 4.577760 \times 10^{-10} | 4.577760 \times 10^{-10} |
| 0.1      | 0.003486       | 0.003485              | 1.510100 \times 10^{-6} |
| 0.2      | 0.013168       | 0.013162              | 6.028170 \times 10^{-6} |
| 0.3      | 0.027989       | 0.027975              | 0.000014       |
| 0.4      | 0.047020       | 0.046996              | 0.000024       |
| 0.5      | 0.069445       | 0.069408              | 0.000037       |
| 0.6      | 0.094543       | 0.094490              | 0.000053       |
| 0.7      | 0.121673       | 0.121602              | 0.000071       |
| 0.8      | 0.150260       | 0.150170              | 0.000091       |
| 0.9      | 0.179784       | 0.179671              | 0.000112       |
| 1.0      | 0.209761       | 0.209626              | 0.000135       |

Table 2. Comparison between the analytical (HAM) and numerical results for \( K(\eta) \).  

| \( \eta \) | HAM Calculation | Numerical Calculation | Absolute Error |
|----------|----------------|----------------------|---------------|
| 0.0      | 0.000000       | 3.072460 \times 10^{-8} | 3.072460 \times 10^{-8} |
| 0.1      | 0.077339       | 0.077339              | 1.116850 \times 10^{-7} |
| 0.2      | 0.145234       | 0.145234              | 1.893820 \times 10^{-7} |
| 0.3      | 0.204221       | 0.204221              | 2.839270 \times 10^{-7} |
| 0.4      | 0.254771       | 0.254771              | 4.077460 \times 10^{-7} |
| 0.5      | 0.297259       | 0.297259              | 5.588600 \times 10^{-7} |
| 0.6      | 0.331947       | 0.331948              | 7.476420 \times 10^{-7} |
| 0.7      | 0.358986       | 0.358787              | 9.603290 \times 10^{-7} |
| 0.8      | 0.378418       | 0.378419              | 1.172660 \times 10^{-6} |
| 0.9      | 0.390187       | 0.390189              | 1.344810 \times 10^{-6} |
| 1.0      | 0.394153       | 0.394154              | 1.430440 \times 10^{-6} |

Table 3. Comparison between the analytical (HAM) and numerical results for \( g(\eta) \).  

| \( \eta \) | HAM Calculation | Numerical Calculation | Absolute Error |
|----------|----------------|----------------------|---------------|
| 0.0      | 1.000000       | 1.000000              | 1.961610 \times 10^{-8} |
| 0.1      | 0.939014       | 0.939000              | 0.000013      |
| 0.2      | 0.882296       | 0.882270              | 0.000026      |
| 0.3      | 0.830862       | 0.830824              | 0.000038      |
| 0.4      | 0.785467       | 0.785417              | 0.000050      |
| 0.5      | 0.746649       | 0.746589              | 0.000060      |
| 0.6      | 0.714763       | 0.714694              | 0.000069      |
| 0.7      | 0.690009       | 0.689932              | 0.000077      |
| 0.8      | 0.672450       | 0.672367              | 0.000083      |
| 0.9      | 0.662036       | 0.661949              | 0.000087      |
| 1.0      | 0.658614       | 0.658526              | 0.000088      |
Table 4. Comparison between the analytical (HAM) and numerical results $S(\eta)$.

| $\eta$ | HAM Calculation | Numerical Calculation | Absolute Error |
|--------|-----------------|----------------------|---------------|
| 0.0    | 0.000000        | $9.953480 \times 10^{-9}$ | $9.953480 \times 10^{-9}$ |
| 0.1    | $-0.022955$     | $-0.022957$          | $1.915760 \times 10^{-6}$ |
| 0.2    | $-0.045051$     | $-0.045054$          | $3.801180 \times 10^{-6}$ |
| 0.3    | $-0.065602$     | $-0.065608$          | $5.642030 \times 10^{-6}$ |
| 0.4    | $-0.084090$     | $-0.084098$          | $7.408600 \times 10^{-6}$ |
| 0.5    | $-0.100131$     | $-0.100140$          | $9.053530 \times 10^{-6}$ |
| 0.6    | $-0.113455$     | $-0.113466$          | $0.000011$     |
| 0.7    | $-0.123888$     | $-0.123900$          | $0.000012$     |
| 0.8    | $-0.131335$     | $-0.131348$          | $0.000013$     |
| 0.9    | $-0.135770$     | $-0.135784$          | $0.000013$     |
| 1.0    | $-0.137231$     | $-0.137245$          | $0.000014$     |

Table 5. Comparison between the analytical (HAM) and numerical results for $\theta(\eta)$.

| $\eta$ | HAM Calculation | Numerical Calculation | Absolute Error |
|--------|-----------------|----------------------|---------------|
| 0.0    | 0.000000        | $-2.18660 \times 10^{-9}$ | $2.186660 \times 10^{-9}$ |
| 0.1    | 0.101538        | 0.101537             | $2.957580 \times 10^{-7}$ |
| 0.2    | 0.203047        | 0.203046             | $5.885800 \times 10^{-7}$ |
| 0.3    | 0.304459        | 0.304459             | $8.787530 \times 10^{-7}$ |
| 0.4    | 0.405671        | 0.405670             | $1.160980 \times 10^{-6}$ |
| 0.5    | 0.506549        | 0.506547             | $1.421310 \times 10^{-6}$ |
| 0.6    | 0.606936        | 0.606934             | $1.632020 \times 10^{-6}$ |
| 0.7    | 0.706660        | 0.706658             | $1.737560 \times 10^{-6}$ |
| 0.8    | 0.805537        | 0.805536             | $1.634890 \times 10^{-6}$ |
| 0.9    | 0.903380        | 0.903379             | $1.149780 \times 10^{-6}$ |
| 1.0    | 1.000000        | 1.000000             | $2.220450 \times 10^{-16}$ |

7. Conclusions

In this section, we conclude our work. We have studied the impacts produced by various physical parameters of interest ($\alpha_n f$, $Pr$, $Rd$, $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, $A_0$, $R_0$, $A_1$) on the temperature profile and the velocity components. The dependence of the Nusselt number ($Nu$) on the radiation parameter ($Rd$), and Prandtl number ($Pr$) are also investigated. The comparison between the numerical and analytical (HAM) results obtained for different velocity components and temperature functions is tabulated in different tables. It is observed that the velocity components $f(\eta)$ and $S(\eta)$ increase, while $K(\eta)$ falls with higher values of the thickness parameter. It has also been observed that the temperature function $\theta(\eta)$ increases with changing values of $\alpha_n f$, $Rd$ and $A_1$, whereas it shows a decreasing trend with increasing values of $Pr$. Furthermore, $\theta(\eta)$ decreases with rising values of $\epsilon_1$, and shows an increasing behavior with the changing $\epsilon_2$ and $\epsilon_3$. The velocity component $f(\eta)$ increases with higher $A_0$ values while it decreases with increasing $R_0$. Both the axial and the draining velocities decline with the increasing values of $A_1$. The Nusselt number remains invariant at a particular value of $Pr$, but increases with higher $Pr$ values. In addition, the Nusselt number shows an opposite behavior with higher values of $Rd$. We also observed from the size of the absolute error that the numerical results of different velocity components and temperature distribution are in complete agreement with the analytical results of the same study obtained through the standard procedure of HAM.
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Abbreviations
The following abbreviations are used this article:

- $\sigma$: Electrical conductivity
- $T_w$: Surface temperature
- $R_\infty$: Free stream temperature
- $A_1$: Couple stress parameter
- $\nu'_{nf}$: Couple stress viscosity of the nanofluid
- $Re_x$: Local Reynolds number
- $C_{fx}$: Local skin friction
- $\tau_{ij}$: Stress tensor
- $d_{ij}$: Deformation rate tensor
- $\infty$: Condition at infinity
- $x$, $y$, and $z$: Coordinates
- $0$: Reference condition
- $\eta$: Similarity variable
- $\eta'$: Couple stress viscosity coefficient
- $\gamma$: Thickness parameter
- $\lambda_1$: Viscosity coefficient
- $k_{nf}$: Thermal conductivity of the nanofluid
- $\rho_{nf}$: Nanofluid density
- $\mu_{nf}$: Nanofluid viscosity
- $k$: Thermal conductivity
- $\alpha$: Thermal diffusivity
- $Pr$: Prandtl number
- $T$: Fluid temperature
- $\rho$: Density
- $m$: Trace of the couple stress
- $\nu$: Kinematic viscosity
- $\mu$: Dynamic viscosity
- $t$: Time
- $c_p$: Specific heat
- $R_0$: Rotation Parameter
- $Rd$: Radiation Parameter
- $f$: Dimensionless axial velocity $x$-direction
- $k$: Dimensionless draining velocity $x$-direction
- $S$: Dimensionless induced velocity along $y$-direction
- $g$: Dimensionless induced velocity along $z$-direction
- $\theta$: Dimensionless temperature
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