Optimal Degrees of Freedom Region for the Asymmetric MIMO Y Channel

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Abstract—This letter studies the optimal degrees of freedom (DoF) region for the asymmetric MIMO Y channel, where each user \(i (i = 1, 2, 3)\), equipped with \(M_i\) antennas, can exchange independent messages with the other two users via the help of a common relay, equipped with \(N\) antennas. To deal with the asymmetric data exchange pattern within each user pair, we introduce two data exchange patterns: pairwise data exchange (PDE) and cyclic data exchange (CDE). Our proposed achievable scheme is to align each PDE pattern with weight \(d\) into a \(d\)-dimensional subspace at the relay so as to form \(d\) network-coded symbols, and to align each CDE pattern with weight \(d\) into a \(2d\)-dimensional subspace so as to form \(2d\) network-coded symbols. With PDE and CDE patterns, we obtain a complete characterization of the optimal DoF region of the asymmetric MIMO Y channel.

I. INTRODUCTION

Degrees of freedom (DoF) characterizes the approximate capacity of a network. It specifies how the transmission rate scales as the power goes to infinity. The DoF analysis of the MIMO Y channel has attracted a lot of attention in the literature [1]–[13]. The main findings are summarized in TABLE II. Here, \(N\) denotes the number of antennas at the relay node, \(M_i\) denotes the number of antennas at each user \(i\) for the asymmetric antenna setting, and \(M\) denotes the number of antennas at each user for the symmetric antenna setting. From TABLE II the existing analysis of the optimal DoF region for MIMO Y channel under symmetric/asymmetric setting is not complete.

In this work, we present a complete characterization of the DoF region for the asymmetric three-user MIMO Y channel with antenna configuration \((M_1, M_2, M_3, N)\). We first obtain the DoF-region converse by using the cut-set theorem and the genie-message approach [2]. To show the DoF-region achievability, we introduce two data exchange patterns: pairwise data exchange (PDE) and cyclic data exchange (CDE). Our proposed achievable scheme is to align each PDE pattern with weight \(d\) into a \(d\)-dimensional subspace at the relay so as to form \(d\) network-coded symbols, and to align each CDE pattern with weight \(d\) into a \(2d\)-dimensional subspace so as to form \(2d\) network-coded symbols.

In [11], the DoF region of the MIMO Y channel with asymmetric antenna setting is studied by using signal alignment and detour scheme. But their scheme does not utilize the relay space efficiently. Thus, the achievable DoF region result in [11] is neither optimal nor complete. However, in this work, by introducing the CDE pattern, we can align more data streams at the relay and thus achieve a larger DoF region than the one in [11]. In [13], the DoF region of the \(K\)-user MIMO Y channel with symmetric antenna setting is studied by using channel diagonalization and cyclic communication techniques. Their transmission scheme is only applicable when the antenna configuration satisfies \(\frac{N}{M} \in (0, 1)\cup [K, +\infty)\). Compared with the scheme in [13], our transmission scheme is applicable to arbitrary antenna configurations.

Notations: \((\cdot)^T\) and \((\cdot)^H\) denote the transpose and the Hermitian transpose, respectively. \(\text{rank}(X)\) stands for the rank of \(X\). \(\text{span}(X)\) and \(\text{null}(X)\) stand for the column space and the null space of the matrix \(X\), respectively.

II. CHANNEL MODEL

Consider an asymmetric MIMO Y channel consisting of three users and one relay. Each user \(i\) is equipped with \(M_i\) antennas, for \(i = 1, 2, 3\), and the relay with \(N\) antennas. Each user intends to send one independent message to each of the other two users via the relay, and there is no direct link between any two users. Denote by \(H_{i,r}(t) \in \mathbb{C}^{N \times M_i}\) the channel matrix from user \(i\) to the relay for the channel use \(t\), and by \(H_{r,i}(t) \in \mathbb{C}^{M_i \times N}\) the channel matrix from the relay to user \(i\). It is assumed that the entries of the channel matrices are drawn independently from a continuous distribution, which guarantees that the channel matrices have full rank with probability one. Perfect channel knowledge is assumed at each node, and all the nodes in the network are assumed to be full duplex.

The message transmitted from user \(i\) to user \(j\) is denoted by \(W_{i,j}\). Each \(W_{i,j}\) is encoded using a codebook with size \(2^{nR_{i,j}}\), where \(n\) is the codeword length and \(R_{i,j}\) is the information rate of \(W_{i,j}\). Let \(\mathcal{W}_i\) be the set of messages transmitted from user \(i\), i.e., \(\mathcal{W}_i = \{W_{i,j} \mid \forall j \neq i\}\). Also, let \(\mathcal{V}_i\) be the set of messages desired by user \(i\), i.e., \(\mathcal{V}_i = \{W_{j,i} \mid \forall j \neq i\}\).

In the MAC phase, all the users transmit their signals to the relay. The received signal, denoted by \(y_r(t) \in \mathbb{C}^{N \times 1}\), at the relay is given by

\[
y_r(t) = \sum_{i=1}^{3} H_{i,r}(t)x_i(t) + n_r(t),
\]

where \(x_i(t) \in \mathbb{C}^{M_i \times 1}\) denotes transmitted signal from user \(i\) and \(n_r(t) \in \mathbb{C}^{N \times 1}\) denotes the additive white Gaussian noise (AWGN) vector for the channel use \(t\) with each element being independent and having zero mean and unit variance.
Upon receiving $y_r(t)$ in (1), the relay processes these messages to obtain a mixed signal $x_r(t) \in \mathbb{C}^{N \times 1}$, and broadcasts to all the users. The received signal at user $i$ is given by

$$y_i(t) = H_{r,i}(t)x_r(t) + n_i(t),$$  \hspace{1cm} (2)

where $n_i(t) \in \mathbb{C}^{M_i \times 1}$ denotes the AWGN vector for the channel use $t$ with each element being independent and having zero mean and unit variance.

Each user decodes its desired messages based on the received signals and its own transmitted messages. Let $R_{i,j}(P)$ be the achievable information rate of the message $W_{i,j}$ under the power constraint $P$. We say that a rate tuple $\{R_{i,j}(P) \mid \forall i, \forall j \neq i\}$ is achievable if

$$\lim_{n \to \infty} \Pr\left(\hat{W}_{i,j} \neq W_{i,j}\right) = 0, \quad \forall i, \forall j \neq i,$$  \hspace{1cm} (3)

where $\hat{W}_{i,j}$ is the estimate of $W_{i,j}$ at user $j$ based on the received signals and the self messages.

Then the DoF of each message is defined as

$$d_{i,j} \triangleq \lim_{P \to \infty} \frac{R_{i,j}(P)}{\log P}.$$ \hspace{1cm} (4)

The DoF tuple of the channel is given by

$$\mathbf{d} \triangleq (d_{1,2}, d_{1,3}, d_{2,1}, d_{2,2}, d_{3,1}, d_{3,2}).$$ \hspace{1cm} (5)

The sum DoF of the channel is given by

$$d_{\Sigma} = d_{1,2} + d_{1,3} + d_{2,1} + d_{2,2} + d_{3,1} + d_{3,2}.$$ \hspace{1cm} (6)

The DoF region is defined in (7) on the top of the next page, where $C(P)$ is the capacity region of the asymmetric three-user MIMO Y channel, i.e., the set of all achievable rate tuples $\{R_{i,j}(P) \mid \forall i, \forall j \neq i\}$.

III. MAIN RESULT

The main result of this letter is presented in the theorem below.

Theorem 1: For the asymmetric three-user MIMO Y channel with antenna configuration $(M_1, M_2, M_3, N)$, the optimal DoF region, denoted by $D^*$, can be expressed as

$$D^* = \left\{(d_{1,2}, d_{1,3}, d_{2,1}, d_{2,2}, d_{3,1}, d_{3,2}) \in \mathbb{R}_+^6 : \begin{array}{l}
d_{1,2} + d_{1,3} \leq M_1 \\
d_{2,1} + d_{2,2} \leq M_2 \\
d_{3,1} + d_{3,2} \leq M_3 \\
d_{1,2} + d_{1,3} + d_{2,1} \leq N \\
d_{1,2} + d_{1,3} + d_{2,2} + d_{3,1} \leq N \\
d_{1,2} + d_{1,3} + d_{2,2} + d_{3,1} + d_{3,2} \leq N \end{array} \right\}.$$ \hspace{1cm} (8a)

The DoF-region converse can be proved by using the cut-set theorem and the genie-message approach [2]. The achievability proof is presented in the next section.

Remark 1 (Comparison to the work in [13]): In the case with $M_1 = M_2 = M_3 = M$ and $N \leq M$, our result reduces to Theorem 1 in [14] with $K = 3$.

Remark 2 (Comparison to the work in [17]): In the case with $(M_1, M_2, M_3, N) = (3, 2, 2, 4)$, the DoF tuple $(2, 0, 0, 2, 2, 0)$ is achievable with cyclic signal alignment approach. Interestingly, this DoF tuple cannot be achieved using the scheme proposed in [11].

IV. PROOF OF DOF-REGION ACHIEVABILITY

In this section, we provide the achievability proof of Theorem 1. Due to user symmetry, we divide the DoF tuples in $D^*$ into two cases: (I) $d_{1,2} \geq d_{2,1}, d_{1,3} \geq d_{3,1}$ and $d_{2,2} \geq d_{3,2}$, (II) $d_{1,2} \geq d_{2,1}, d_{1,3} > d_{3,1}$ and $d_{2,2} \geq d_{3,2}$, and present the achievable scheme for each case. It is worth mentioning that each DoF tuple can be converted into these two cases by user index-reordering. Before discussing the achievability scheme, it is crucial to determine the transmission framework used in the communication to achieve this optimal DoF region. In what follows, we first define the message flow graph. Then we introduce two data exchange patterns: pairwise data exchange (PDE) and cyclic data exchange (CDE). Note that CDE has been previously studied in [12] for MIMO switching and also in [13] for the symmetric MIMO Y channel with $N \leq M$.

Definition 1: Every DoF tuple $\mathbf{d}$ defines a message flow graph as illustrated in Fig. [1] where the weight of each edge
\[ \mathcal{D} = \left\{ (d_{1,2}, d_{1,3}, d_{2,1}, d_{2,3}, d_{3,1}, d_{3,2}) \in \mathbb{R}_+^6 : \forall (\omega_{1,2}, \omega_{1,3}, \omega_{2,1}, \omega_{2,3}, \omega_{3,1}, \omega_{3,2}) \in \mathbb{R}_+^6, \right. \\
\sum_{i=1}^3 \sum_{j \neq i} \omega_{i,j} d_{i,j} \leq \lim \sup_{P \to \infty} \left[ \sup_{R(P) \in \mathcal{C}(P)} \left\{ \sum_{i=1}^3 \sum_{j \neq i} \omega_{i,j} R_{i,j}(P) \right\} \right] \left( P \right) \right\} 
\] (7)

Fig. 1. Message flow graph for the three-user asymmetric MIMO Y channel.

\[ i \rightarrow j \] represents the DoF of the message \( W_{i,j} \). A message flow \( i \rightarrow d \rightarrow j \) is called a PDE pattern \( i \rightarrow j \) with weight \( d \). A message flow \( i \rightarrow j \rightarrow k \rightarrow d \rightarrow i \) is called a CDE pattern \( i \rightarrow j \) with weight \( d \).

Remark 3: A directed edge with weight \( d_1 + d_2 \) can be split into two directed edges with weight \( d_1 \) and \( d_2 \).

A. Case I

In this subsection, we explain how to achieve the DoF tuple \( d \in D^* \) satisfying \( d_{1,2} \geq d_{2,1}, d_{3,1} \geq d_{1,3} \) and \( d_{2,3} \geq d_{3,2} \). Given the DoF tuple, we form the PDE pattern \( 1 \rightarrow 2 \rightarrow 1 \) with weight \( d_{2,1} \), \( 1 \rightarrow 3 \rightarrow 1 \) with weight \( d_{1,3} \), and \( 2 \rightarrow 3 \rightarrow 2 \) with weight \( d_{3,2} \). From the remaining data streams, we form the CDE pattern \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \) with weight \( \gamma \), where \( \gamma = \min\{d_{2,1} - d_{2,2}, d_{3,1} - d_{3,2}, d_{3,2} - d_{3,1}\} \). Without loss of generality, we assume \( \gamma = d_{2,2} \). Then, user 2 has \( d_{2,3} - d_{2,2} - \gamma \) additional data streams to transmit to user 3, and user 3 has \( d_{3,1} - d_{3,2} - \gamma \) additional data streams to transmit to user 1.

We assume that the relay only utilizes \( J = d_{2,1} + d_{2,3} + d_{3,1} \) antennas for this case by the relay antenna deactivation. Note that \( J \leq N \) from (5). In the \( J \)-dimensional subspace of the relay, the first \( d_{2,1} \) dimensions are used for the PDE pattern \( 1 \rightarrow 2 \rightarrow 1 \) with weight \( d_{2,1} \) to form network-coded symbols. Similarly, the second \( d_{1,3} \) dimensions are used for the PDE pattern \( 1 \rightarrow 3 \rightarrow 1 \) with weight \( d_{1,3} \), and the third \( d_{3,2} \) dimensions are used for the CDE pattern \( 2 \rightarrow 3 \rightarrow 2 \) with weight \( d_{3,2} \). The fourth \( 2 \gamma \) dimensions are used for the CDE pattern \( 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \) with weight \( \gamma \). The remaining \( (d_{3,2} - d_{2,3} - \gamma) \) and \( (d_{3,1} - d_{1,3} - \gamma) \) dimensions are used to decode the additional \( (d_{2,3} - d_{3,2} - \gamma) \) data streams sent from user 2 to user 3 and the additional \( (d_{3,1} - d_{1,3} - \gamma) \) data streams sent from user 3 to user 1, respectively.

Remark 4: By comparing with complete decoding of all the data streams at the relay, each PDE pattern with weight \( d \) reduces \( d \) dimension required at the relay, and each CDE pattern with weight \( d \) also reduces \( d \) dimension required at the relay.

We present the signal alignment design to realize the above patterns. We first consider the MAC phase. During the MAC phase, the signal received at the relay can be rewritten as

\[ y_r = \sum_{i=1}^3 \sum_{j \neq i} H_{i,r} V^p_{i,j} s^p_{i,j} + \sum_{(i,j) \in S} H_{i,r} V^c_{i,j} s^c_{i,j} + H_{2,r} V^c_{2,3} s^c_{2,3} + H_{3,r} V^c_{3,1} s^c_{3,1} + n_r, \] (9)

where \( S = \{(1,2), (2,3), (3,1)\} \). Here, \( s^p_{2,3}, s^p_{3,3}, \) and \( s^p_{3,3} \) are the precoding matrices for \( s^p, s^c, \) and \( s^c \), respectively. Similar definitions apply to the other \( (i,j) \).

From the aforementioned relay space division scheme, we aim to design all the precoding matrices jointly such that

\[ H_{1,r} V^c_{1,2} = H_{2,r} V^c_{2,1} \triangleq B_1, \] (10a)
\[ H_{1,r} V^p_{1,3} = H_{3,r} V^p_{3,1} \triangleq B_2, \] (10b)
\[ H_{2,r} V^c_{2,3} = H_{3,r} V^c_{3,2} \triangleq B_3, \] (10c)
\[ \text{rank}(V^p_{2,3} V^p_{2,3}) = d_{2,3}, \] (10d)
\[ \text{rank}(V^p_{3,1} V^p_{3,1}) = d_{3,1}. \] (10f)

Here, condition (10a) requires that the relay aligns the signal pair \( (s^c_{1,2}, s^p_{2,1}) \) in a subspace to form network-coded symbols; condition (10b) and (10c) require to align the signal pair \( (s^p_{1,3}, s^c_{3,1}) \) and \( (s^p_{2,3}, s^p_{3,2}) \), respectively. The condition (10d) requires that the relay aligns the signal \( s^p_{2,3} \) to the subspace spanned by \( (s^p_{1,2}, s^c_{2,3}) \) to form network-coded symbols. Condition (10e) is to ensure the separability of \( s^p_{2}, s^c_{2}, \) and \( s^p_{2} \) and at user 2, and likewise condition (10f) is to ensure the separability of \( s^p_{3,1}, s^c_{3,1}, \) and \( s^p_{3,1} \) at user 3.

We rewrite (10a) and (10f) as

\[ \begin{bmatrix} V^p_{i,j} \quad V^p_{j,i} \end{bmatrix}^T \subseteq \null \left( [H_{i,r} - H_{j,r}] \right). \] (11)

We have the following lemma.

Lemma 1 (Sufficient condition for (10a)-(10f)): There exist \( V^p_{i,j} \) and \( V^c_{i,j} \) satisfying (10a)-(10f) with probability one if

\[ M_r + M_j - J \geq \min\{d_{i,j}, d_{i,j}\} \forall i, \forall j \neq i. \] (12)

Proof: The proof follows directly from the rank-nullity theorem and the channel randomness. We refer interested readers to [1] for details.

We rewrite (10f) as

\[ \begin{bmatrix} V^c_{1,2} \quad V^c_{2,3} \quad V^c_{3,1} \end{bmatrix}^T \subseteq \null \left( [PH_{1,r} PH_{2,r} - PH_{3,r}] \right). \] (13)
Then we have the following lemma.

**Lemma 2 (Sufficient condition for (10d))**: There exist $V_{i,2}^c$, $V_{2,3}$ and $V_{3,1}$ satisfying (10d) with probability one if

$$M_1 + M_2 + M_3 - J \geq d_{1,2} - d_{2,1}. \quad (14)$$

**Proof**: It is worth mentioning that the column rank of $[H_{i,1}r H_{2,r} - H_{3,r}]$ is less than or equal to $J$. This implies that the dimension of the null space of $[H_{i,1}r H_{2,r} - H_{3,r}]$ is greater than or equal to $M_1 + M_2 + M_3 - J$ with probability one. Hence, we can find at least $d_{1,2} - d_{2,1}$ linear independent vectors in the null space of $[H_{i,1}r H_{2,r} - H_{3,r}]$ if (14) holds.

From (8a)-(8c), we obtain (i) $(M_1 + M_2) - J \geq d_{2,1}$; (ii) $(M_1 + M_3) - J \geq d_{1,3}$; (iii) $(M_2 + M_3) - J \geq d_{3,2}$; (iv) $(M_1 + M_2 + M_3) - J \geq d_{1,2} - d_{2,1}$. Thus, from Lemma 1 and Lemma 2, we design the precoding matrices $V_{i,j}^p$ and $V_{i,j}^C$ such that (10a)-(10b) hold. The remaining two precoding matrices $V_{2,3}$ and $V_{3,1}$ can be designed randomly provided that (10c) and (10d) hold.

The signal received at the relay can be expressed as

$$y_r = B_1 (s_{1,2}^p + s_{1,1}^p) + B_2 (s_{2,3}^p + s_{2,2}^p) + B_3 (s_{3,1}^p + s_{3,2}^p) + B_4 (s_{1,2}^c + s_{1,1}^c) + B_5 (s_{2,3}^c + s_{2,2}^c) + B_6 (s_{3,1}^c + s_{3,2}^c) + n_r, \quad (15)$$

where $B_1 \triangleq H_{i,1}r^T V_{1,2}$, $B_2 \triangleq H_{2,r}^TV_{2,3}$, $B_3 \triangleq H_{3,r}^TV_{3,1}$. Thus far, the relay is able to decode the network-coded symbols, $\{s_{1,2}^p, s_{1,1}^p, s_{2,3}^p, s_{2,2}^p, s_{3,1}^p, s_{3,2}^p, s_{1,2}^c, s_{1,1}^c, s_{2,3}^c, s_{2,2}^c, s_{3,1}^c, s_{3,2}^c\}$, together with the remaining symbols, $\{s_{2,3}, s_{3,1}\}$, by using a $J \times J$ zero-forcing matrix

$$W = ([B_1 B_2 B_3 B_4 B_5 B_6 B_7])^{-1}. \quad (16)$$

We next introduce the transmission scheme for the BC phase. The signal received at user $i$ with receiving matrix $U_1 \in \mathbb{C}^{d_{i,1} \times M_i}$ can be expressed as

$$\hat{s}_i = U_1 y_i + U_1 n_i = U_1 H_{r,i}^T T s_r + U_1 H_{r,i}^T T W n_r + U_1 n_i, \quad (17)$$

where $T \in \mathbb{C}^{N \times J}$ denotes a zero-forcing matrix in the BC phase.

Due to the symmetry between the MAC and BC phases, we partition $U_1$ as

$$U_1 = \begin{bmatrix} U_{2,1}^p & U_{3,1}^p & U_{3,1}^c \end{bmatrix} \begin{bmatrix} T & T & T \end{bmatrix}^T, \quad (18)$$

where $U_{2,1}^p \in \mathbb{C}^{d_{2,1} \times M_1}$, $U_{3,1}^p \in \mathbb{C}^{d_{3,1} \times M_1}$, $U_{3,1}^c \in \mathbb{C}^{d_{3,1} \times M_1}$, and $U_{3,1}^r \in \mathbb{C}^{d_{3,1} \times d_{3,1}}$. $U_2$ and $U_3$ are partitioned similarly. Then we design $\{U_{i,j}\}_{i=1}^3$ and $Q$ such that

$$U_{2,1}^p H_{r,1} = U_{1,2}^p H_{r,2}, \quad (19a)$$
$$U_{3,1}^p H_{r,1} = U_{1,3}^p H_{r,3}, \quad (19b)$$
$$U_{3,1}^c H_{r,2} = U_{2,3}^c H_{r,3}, \quad (19c)$$
$$U_{3,1}^r H_{r,1} + U_{1,2}^r H_{r,2} = U_{2,3}^r H_{r,3}. \quad (19d)$$

Comparing (19a)-(19d) with (10a)-(10d), we see the symmetry between the design of $\{V_{i,j}^p, V_{i,j}^c, V_{i,j}^r\}$ and that of $\{U_{i,j}^p, U_{i,j}^c, U_{i,j}^r\}$. Then the zero-forcing matrix $T$ in the BC phase can be designed as

$$T = \begin{bmatrix} U_{2,1}^p H_{r,1} & U_{3,1}^p H_{r,1} & U_{3,1}^c H_{r,2} & U_{3,1}^r H_{r,1} \\ U_{2,1}^p H_{r,1} & U_{3,1}^c H_{r,2} & U_{3,1}^r H_{r,1} & U_{3,1}^r H_{r,1} \\ U_{2,1}^p H_{r,1} & U_{3,1}^p H_{r,1} & U_{3,1}^c H_{r,2} & U_{3,1}^r H_{r,1} \\ U_{2,1}^p H_{r,1} & U_{3,1}^p H_{r,1} & U_{3,1}^c H_{r,2} & U_{3,1}^r H_{r,1} \end{bmatrix}^{-1}. \quad (20)$$

The signal received at user $i$ in (17) can be rewritten as

$$\hat{s}_1 = \begin{bmatrix} s_{1,2}^p + s_{1,1}^p \\ s_{1,2}^p + s_{1,1}^p \\ s_{1,2}^c + s_{1,1}^c \end{bmatrix} + U_1 H_{r,1}^T T W n_r + U_1 n_i, \quad (21)$$
$$\hat{s}_2 = \begin{bmatrix} s_{2,3}^p + s_{2,2}^p \\ s_{2,3}^p + s_{2,2}^p \\ s_{2,3}^c + s_{2,2}^c \end{bmatrix} + U_2 H_{r,2}^T T W n_r + U_2 n_i, \quad (22)$$
$$\hat{s}_3 = \begin{bmatrix} s_{3,1}^p + s_{3,1}^p \\ s_{3,1}^p + s_{3,1}^p \\ s_{3,1}^c + s_{3,1}^c \end{bmatrix} + U_3 H_{r,3}^T T W n_r + U_3 n_i. \quad (23)$$

Finally, each user decodes its desired signal after self-interference cancellation. The DoF tuple $d \in \mathbb{D}^*$ satisfying $d_{1,2} \geq d_{2,1}$, $d_{1,3} \geq d_{2,3}$ and $d_{3,2} \geq d_{3,1}$ is thus achievable.

**Remark 5**: The design of the zero-forcing matrix in the MAC and BC phase is not symmetric.

**B. Case II**

For this case, we only consider the PDE pattern to achieve the DoF tuple $d \in \mathbb{D}^*$ satisfying $d_{1,2} \geq d_{2,1}$, $d_{1,3} \geq d_{2,3}$ and $d_{3,2} \geq d_{3,1}$ by using signal alignment and antenna deactivation scheme. Let the relay only utilize $J = d_{1,2} + d_{1,3} + d_{2,3}$ antennas. Then, the method is similar to Case I and thus omitted here.

**V. Conclusion**

In this letter, we have presented a complete characterization of the optimal DoF region of the asymmetric MIMO Y channel. The proposed transmission scheme takes into account both PDE and CDE patterns and designs the source precoding matrices for each pattern respectively. We have shown that there is symmetry between the source precoding matrices in the MAC phase and the source combining matrices in the BC phase. However, the design of the zero-forcing matrix in the MAC and BC phase is not symmetric. In the future work, it is interesting to extend these two patterns to the analysis of the asymmetric $K$-user MIMO Y channel.

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