ON THE LOGARITHMIC DERIVATIVE OF ZETA FUNCTIONS FOR COMPACT EVEN-DIMENSIONAL LOCALLY SYMMETRIC SPACES

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Abstract. We derive approximate formulas for the logarithmic derivative of the Selberg and Ruelle zeta functions over compact, even-dimensional, locally symmetric spaces of rank one. The obtained formulas are given in terms of the zeta-singularities.

1. Introduction

Let \( Y = \Gamma \backslash G / K = \Gamma \backslash X \) be a compact, \( n \)-dimensional (\( n \) even), locally symmetric Riemannian manifold with negative sectional curvature, where \( G \) is a connected semisimple Lie group of real rank one, \( K \) is a maximal compact subgroup of \( G \) and \( \Gamma \) is a discrete co-compact torsion free subgroup of \( G \). The covering manifold \( X \) is known to be a real, a complex or a quaternionic hyperbolic space or the hyperbolic Cayley plane, i.e. \( X \) is one of the following spaces:

\[
H^{k}, H^{m}, H^{m}, H^{2}.
\]

Here, \( n = k, 2m, 4m, 16 \), respectively.

We require \( G \) to be linear in order to have complexification available.

U. Bunke and M. Olbrich [4] derived the properties of the zeta functions of Selberg and Ruelle canonically associated with the geodesic flow of \( Y \).

In many applications it is often useful to have some approximate representation of the logarithmic derivative of an appropriate zeta function. Following traditional approach [5], (see also, [7]), we obtain such representations for the zeta functions described in [4].

2. Preliminaries

The notation that will be applied in the sequel follows [4] (see also [2], [3]).

Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the Cartan decomposition of the Lie algebra \( \mathfrak{g} \) of \( G \), \( \mathfrak{a} \) a maximal abelian subspace of \( \mathfrak{p} \) and \( M \) the centralizer of \( \mathfrak{a} \) in \( K \) with Lie
algebra $\mathfrak{m}$. We normalize the Ad $(G)$--invariant inner product $(.)$ on $\mathfrak{g}$ to restrict to the metric on $\mathfrak{p}$. Let $SX = G/M$ be the unit sphere bundle of $X$. Hence $SY = \Gamma\backslash G/M$.

Let $\Phi (\mathfrak{g}, \mathfrak{a})$ be the root system and $W = W (\mathfrak{g}, \mathfrak{a}) \cong \mathbb{Z}_2$ its Weyl group. Fix a system of positive roots $\Phi^+ (\mathfrak{g}, \mathfrak{a}) \subset \Phi (\mathfrak{g}, \mathfrak{a})$. Let $\Phi (g, a)$ be the root system and $W (\mathfrak{g}, \mathfrak{a}) \cong \mathbb{Z}_2$ its Weyl group.

Fixed a system of positive roots $\Phi^+ (\mathfrak{g}, \mathfrak{a}) \subset \Phi (\mathfrak{g}, \mathfrak{a})$. Let $n = \sum_{\alpha \in \Phi^+ (\mathfrak{g}, \mathfrak{a})} n_\alpha$ be the sum of the root spaces corresponding to elements of $\Phi^+ (\mathfrak{g}, \mathfrak{a})$. The decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus n$ corresponds to the Iwasawa decomposition of the group $G = KAN$. Define $\rho \in \mathfrak{a}^*_C$ by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+ (\mathfrak{g}, \mathfrak{a})} \dim (n_\alpha) \alpha.$$  

We normalize the metric on $Y$ to be of sectional curvature $-1$ if $X = H \mathbb{R}^k$. In all other cases, the normalized metric on $Y$ is such that the sectional curvature varies between $-1$ and $-4$. Hence, $\rho = \frac{1}{2} (k - 1)$, $m$, $2m + 1$, 11 if $n = k$, $2m$, $4m$, 16, respectively.

The positive Weyl chamber $a^+$ is the half line in $a$ on which the positive roots take positive values. Let $A^+ = \exp (a^+) \subset A$.

The symmetric space $X$ has a compact dual space $X_d = G_d/K$, where $G_d$ is the analytic subgroup of $GL (n, \mathbb{C})$ corresponding to $\mathfrak{g}_d = \mathfrak{k} \oplus \mathfrak{p}_d$, $\mathfrak{p}_d = i \mathfrak{p}$. We normalize the metric on $X_d$ in such a way that the multiplication by $i$ induces an isometry between $\mathfrak{p}$ and $\mathfrak{p}_d$.

Let $i^*: R (K) \to R (M)$ be the restriction map induced by the embedding $i: M \hookrightarrow K$, where $R (K)$ and $R (M)$ are the representation rings over $\mathbb{Z}$ of $K$ and $M$, respectively.

Since $n$ is even, every $\sigma \in \hat{M}$ is invariant under the action of the Weyl group $W$ (see, [4, p. 27]). Let $\sigma \in \hat{M}$. We choose $\gamma \in R (K)$ such that $i^*(\gamma) = \sigma$ and represent it by $\Sigma a_i \gamma_i$, $a_i \in \mathbb{Z}, \gamma_i \in \hat{K}$. Set

$$V^\pm_\gamma = \sum_{\text{sign} (a_i) = \pm 1} \sum_{m=1}^{a_i} V_{\gamma_i},$$

where $V_{\gamma_i}$ is the representation space of $\gamma_i$. Define $V (\gamma) = G \times_K V^\pm_\gamma$ and $V_d (\gamma) = G_d \times_K V^\pm_\gamma$. To $\gamma$ we associate $\mathbb{Z}_2$--graded homogeneous vector bundles $V (\gamma) = V (\gamma)^+ \oplus V (\gamma)^-$ and $V_d (\gamma) = V_d (\gamma)^+ \oplus V_d (\gamma)^-$ on $X$ and $X_d$, respectively. Let

$$V_{Y, X} (\gamma) = \Gamma \backslash (V_X \otimes V (\gamma))$$
be a $\mathbb{Z}_2$-graded vector bundle on $Y$, where $(\chi, V_\chi)$ is a finite-dimensional unitary representation of $\Gamma$.

Reasoning as in [4, beginning of Subsection 1.1.2], we choose a Cartan subalgebra $t$ of $m$ and a system of positive roots $\Phi^+(m_C, t)$. Then, $\rho_m \in i t^*$, where

$$\rho_m = \frac{1}{2} \sum_{\alpha \in \Phi^+(m_C, t)} \alpha.$$

Let $\mu_\sigma \in i t^*$ be the highest weight of $\sigma$. Set

$$c(\sigma) = |\rho|^2 + |\rho_m|^2 - |\mu_\sigma + \rho_m|^2,$$

where the norms are induced by the complex bilinear extension to $g_C$ of the inner product $(.,.)$. Finally, we introduce the operators (see, [4, p. 28])

$$A_d(\gamma, \sigma)^2 = \Omega + c(\sigma) : C^\infty (X_d, V_d(\gamma)) \to C^\infty (X_d, V_d(\gamma)),$$
$$A_{Y,\chi}(\gamma, \sigma)^2 = -\Omega - c(\sigma) : C^\infty (Y, V_{Y,\chi}(\gamma)) \to C^\infty (Y, V_{Y,\chi}(\gamma)),$$

$\Omega$ being the Casimir element of the complex universal enveloping algebra $U(g)$ of $g$.

Let $m_\chi(s, \gamma, \sigma) = \text{Tr} E_{A_{Y,\chi}(\gamma, \sigma)}(\{s\})$, $m_d(s, \gamma, \sigma) = \text{Tr} E_{A_d(\gamma, \sigma)}(\{s\})$, where $E_A(\cdot)$ denotes the family of spectral projections of a normal operator $A$.

Now, we choose a maximal abelian subalgebra $t$ of $m$. Then, $h = t_C \oplus a_C$ is a Cartan subalgebra of $g_C$. Let $\Phi^+(g_C, h)$ be a positive root system having the property that, for $\alpha \in \Phi(g_C, h)$, $\alpha_{\parallel} \in \Phi^+(g, a)$ implies $\alpha \in \Phi^+(g_C, h)$.

Let

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+(g_C, h)} \alpha.$$

We set $\rho_m = \delta - \rho$. Define the root vector $H_\alpha \in a$ for $\alpha \in \Phi^+(g, a)$ by

$$\lambda(H_\alpha) = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle},$$

where $\lambda \in a^*$. For $\alpha \in \Phi^+(g, a)$, we define $\varepsilon_\alpha(\sigma) \in \{0, \frac{1}{2}\}$ by

$$e^{2\pi i \varepsilon_\alpha(\sigma)} = \sigma(e^{2\pi i H_\alpha}) \in \{\pm 1\}.$$
\[ \epsilon_\sigma \equiv \frac{|\rho|}{T} + \epsilon_\alpha (\sigma) \mod \mathbb{Z}. \]

We define the lattice \( L(\sigma) \subset \mathbb{R} \cong a^* \) by \( L(\sigma) = T(\epsilon_\sigma + \mathbb{Z}) \). Finally, for \( \lambda \in a_2^* \cong \mathbb{C} \) we set

\[ P_\sigma (\lambda) = \prod_{\beta \in \Phi^+ (g, h)} \frac{(\lambda + \mu_\sigma + \rho_{m, \beta})}{(\delta, \beta)}. \]

Since \( n \) is even, there exists a \( \sigma \)-admissible \( \gamma \in R(K) \) for every \( \sigma \in \hat{M} \) (see, [4, p. 49, Lemma 1.18]). Here, \( \gamma \in R(K) \) is called \( \sigma \)-admissible if \( i^\ast (\gamma) = \sigma \) and \( m_d (s, \gamma, \sigma) = P_\sigma (s) \) for all \( 0 \leq s \in L(\sigma) \).

3. Zeta functions and the geodesic flow

Since \( \Gamma \subset G \) is co-compact and torsion free, there are only two types of conjugacy classes - the class of the identity \( 1 \in \Gamma \) and classes of hyperbolic elements.

Let \( g \in G \) be hyperbolic. Then there is an Iwasawa decomposition \( G = NAK \) such that \( g = am \in A^+ M \). Following [4, p. 59], we define

\[ l(g) = |\log (a)|. \]

Let \( \Gamma_h \), resp. \( P\Gamma_h \) denote the set of the \( \Gamma \)-conjugacy classes of hyperbolic resp. primitive hyperbolic elements in \( \Gamma \).

Let \( \varphi \) be the geodesic flow on \( SY \) determined by the metric of \( Y \). In the representation \( SY = \Gamma \backslash G/M \), \( \varphi \) is given by

\[ \varphi : \mathbb{R} \times SY \ni (t, \Gamma g M) \to \Gamma g \exp (-tH) M \in SY, \]

where \( H \) is the unit vector in \( a^+ \). If \( V_\chi (\sigma) = \Gamma \backslash (G \times_M V_\sigma \otimes V_\chi) \) is the vector bundle corresponding to finite-dimensional unitary representations \( (\sigma, V_\sigma) \) of \( M \) and \( (\chi, V_\chi) \) of \( \Gamma \), then we define a lift \( \varphi_{\chi, \sigma} \) of \( \varphi \) to \( V_\chi (\sigma) \) by

(see, [4] p. 95)

\[ \varphi_{\chi, \sigma} : \mathbb{R} \times V_\chi (\sigma) \ni (t, [g, v \otimes w]) \to [g \exp (-tH), v \otimes w] \in V_\chi (\sigma). \]

For \( \text{Re} (s) > 2\rho \), the Ruelle zeta function for the flow \( \varphi_{\chi, \sigma} \) is defined by the infinite product

\[ Z_{R, \chi} (s, \sigma) = \prod_{\gamma_0 \in P\Gamma_h} \det \left( 1 - (\sigma (m) \otimes \chi (\gamma_0)) e^{-sl(\gamma_0)} \right)^{(-1)^{n-1}}. \]
The Selberg zeta function for the flow $\varphi_{\chi, \sigma}$ is given by

$$Z_{S, \chi}(s, \sigma) = \prod_{\gamma_0 \in \Gamma_0} \prod_{k=0}^{+\infty} \det \left( 1 - \left( \sigma (m) \otimes \chi (\gamma_0) \otimes S^k (\text{Ad} (ma)) \right) e^{-(s+\rho)\ell(\gamma_0)} \right),$$

for $\text{Re} (s) > \rho$, where $S^k$ denotes the $k$-th symmetric power of an endomorphism, $n = \theta n$ is the sum of negative root spaces of $a$ as usual, and $\theta$ is the Cartan involution of $g$.

Let $n_C$ be the complexification of $n$. For $\lambda \in \mathbb{C} \cong a_\lambda^\ast$, let $\mathbb{C}_\lambda$ denote the one-dimensional representation of $\mathfrak{a}$ given by $\mathfrak{a} \ni a \rightarrow a \lambda$. Let $p \geq 0$. There exist sets

$$I_p = \left\{ (\tau, \lambda) \mid \tau \in \hat{M}, \lambda \in \mathbb{R} \right\}$$

such that $A^p n_C$ as a representation of $MA$ decomposes with respect to $MA$ as

$$A^p n_C = \sum_{(\tau, \lambda) \in I_p} V_\tau \otimes \mathbb{C}_\lambda,$$

where $V_\tau$ is the space of the representation $\tau$. Bunke and Olbrich proved that the Ruelle zeta function $Z_{R, \chi}(s, \sigma)$ has the following representation (see, [4, p. 99, Prop. 3.4])

$$(3.1) \quad Z_{R, \chi}(s, \sigma) = \prod_{p=0}^{n-1} \prod_{(\tau, \lambda) \in I_p} Z_{S, \chi}(s + \rho - \lambda, \tau \otimes \sigma)^{(-1)^p}.$$ 

Let $d_Y = -(-1)^{\frac{d}{2}}$. The following theorem holds true (see, [4, p. 113, Th. 3.15]).

**Theorem A.** The Selberg zeta function $Z_{S, \chi}(s, \sigma)$ has a meromorphic continuation to all of $\mathbb{C}$. If $\gamma$ is $\sigma-$admissible, then the singularities (zeros and poles) of $Z_{S, \chi}(s, \sigma)$ are the following ones:

1. at $\pm s$ of order $m_\chi(s, \gamma, \sigma)$ if $s \neq 0$ is an eigenvalue of $A_{Y, \chi}(\gamma, \sigma)$,
2. at $s = 0$ of order $2m_\chi(0, \gamma, \sigma)$ if $0$ is an eigenvalue of $A_{Y, \chi}(\gamma, \sigma)$,
3. at $-s$, $s \in T(N - \epsilon)$ of order $2d_Y \frac{\dim(\chi) \vol(Y)}{\vol(X_d)} m_d(s, \gamma, \sigma)$. Then $s > 0$ is an eigenvalue of $A_d(\gamma, \sigma)$.

If two such points coincide, then the orders add up.
Note that the shifts $\rho - \lambda$ that appear in (3.1) are always contained in the interval $[-\rho, \rho]$, (see, [3]).

In [2], we proved that there exist entire functions $Z_{\chi,1}^{\alpha}(s)$, $Z_{\chi,2}^{\alpha}(s)$ of order at most $n$ such that

$$Z_{\chi,1}(s, \sigma) = \frac{Z_{\chi,1}^{1}(s)}{Z_{\chi,2}^{1}(s)}.$$  

Here, $\gamma$ is $\sigma$-admissible, the zeros of $Z_{\chi,1}^{1}(s)$ correspond to the zeros of $Z_{\chi,1}(s, \sigma)$ and the zeros of $Z_{\chi,2}^{1}(s)$ correspond to the poles of $Z_{\chi,1}(s, \sigma)$. The orders of the zeros of $Z_{\chi,1}^{1}(s)$ resp. $Z_{\chi,2}^{1}(s)$ equal the orders of the corresponding zeros resp. poles of $Z_{\chi,1}(s, \sigma)$. Furthermore, (see, [3]),

$$|Z_{\chi}^{1}(\sigma_1 + it)| = e^{O(t^{n-1})}$$

uniformly in any bounded strip $b_1 \leq \sigma_1 \leq b_2$ for $i = 1, 2$.

Denote by $N(t)$ the number of singularities of $Z_{\chi,1}(s, \sigma)$ on the interval $ix$, $0 < x < t$. In [3], we proved that

$$N(t) = \frac{\dim(\chi) \text{ vol}(Y)}{nT \text{ vol}(X_d)} t^n + O(t^{n-1}).$$

Moreover, we proved that there exists a constant $C$ such that

$$N_R(t) = Ct^n + O(t^{n-1}),$$

where $N_R(t)$ denotes the number of singularities of $Z_{R,\chi}(s, \sigma)$ in the rectangle $a \leq \text{Re}(s) \leq b$, $0 < \text{Im}(s) < t$. Here, $-\rho \leq a \leq b \leq \rho$.

Recall that $\gamma$ is assumed to be $\sigma$-admissible in (3.4) and (3.5).

The following well known lemma will be used in the sequel (see [8, p. 56])

**Lemma B.** If $f(s)$ is regular, and

$$\left| \frac{f(s)}{f(s_0)} \right| < e^M \quad (M > 1)$$

in the circle $|s - s_0| \leq r$, then

$$\left| \frac{f'(s)}{f(s)} - \sum \frac{1}{s - \rho} \right| < \frac{AM}{r} \quad \left( |s - s_0| \leq \frac{1}{4} r \right),$$

where $\rho$ runs through the zeros of $f(s)$ such that $|\rho - s_0| \leq \frac{1}{2} r$. 

4. Main result

The main result of the paper is the following theorem.

**Theorem 4.1.** Let $\gamma$ be $\sigma$ - admissible. Suppose $t \gg 0$ is selected so that $\rho + it$ is not a singularity of $Z_{R,\chi}(s, \sigma)$. Then,

(a) \[
\frac{Z_{R,\chi}'(s, \sigma)}{Z_{R,\chi}(s, \sigma)} = O(t^{n-1}) + \sum_{|t-t_R|\leq 1} \frac{1}{s-s_R},
\]

where $s = \sigma_1 + it$, $\rho \leq \sigma_1 \leq \frac{1}{2}t + \rho$ and $s_R = \rho + i t_R$ is a singularity of $Z_{R,\chi}(s, \sigma)$ along the line $\Re(s) = \rho$.

(b) \[
\frac{Z_{R,\chi}'(s, \sigma)}{Z_{R,\chi}(s, \sigma)} = O(t^{n-1}),
\]

where $s = \sigma_1 + it$, $\rho + u \leq \sigma_1 \leq \frac{1}{2}t + \rho$ and $u > 0$.

**Proof.** (a) Let $(\tau, 2\rho) \in I_p$ for some $p \in \{0, 1, ..., n-1\}$. Then,

$$Z_{S,\chi}(s-\rho, \tau \otimes \sigma)^{(-1)^p}$$

is the corresponding factor in the representation (3.1). By (3.2) and (3.3),

(4.1) \[
|Z_{S,\chi}(s-\rho, \tau \otimes \sigma)| = e^{O(t_1^{n-1})}
\]

uniformly in any bounded half-strip $b_1 \leq \Re(s) \leq b_2$, $s = \sigma_1 + it_1$, $t_1 > 0$.

Let $8\rho \leq r < t$. We choose $c$, $2\rho < c < \frac{1}{4}r+\rho$ and put $s_0 = c+i t$. It follows immediately that the circles $|s-s_0| \leq r$, $|s-s_0| \leq \frac{1}{2}r$ and $|s-s_0| \leq \frac{1}{4}r$ cross the line $\Re(s) = \rho$.

Denote the set of poles of $Z_{S,\chi}(s-\rho, \tau \otimes \sigma)$ lying in the circle $|s-s_0| \leq r$ by $P$. Then, the function

$$\mathcal{H}(s) = Z_{S,\chi}(s-\rho, \tau \otimes \sigma) \cdot \prod_{\rho_1 \in P} (s-\rho_1)$$

is regular in $|s-s_0| \leq r$. By (4.1),

$$|Z_{S,\chi}(s-\rho, \tau \otimes \sigma)| = e^{O(t_1^{n-1})}$$

uniformly in the half-strip $c-r \leq \Re(s) \leq c+r$, $s = \sigma_1 + it_1$, $t_1 > 0$. Hence,
\[
|Z_{S,\chi} (s - \rho, \tau \otimes \sigma)| = e^{O(t_{n-1})}
\]

for \( s = \sigma_1 + it_1, |s - s_0| \leq r \). Specially,

\[
|Z_{S,\chi} (s_0 - \rho, \tau \otimes \sigma)| = e^{O(t_{n-1})}.
\]

Having in mind that \( t_1 \leq t + r < 2t \) for \( s = \sigma_1 + it_1, |s - s_0| \leq r \), the relations (4.2) and (4.3) imply

\[
\left| \frac{Z_{S,\chi} (s - \rho, \tau \otimes \sigma)}{Z_{S,\chi} (s_0 - \rho, \tau \otimes \sigma)} \right| = e^{O(t_{n-1})}.
\]

for \( s = \sigma_1 + it_1, |s - s_0| \leq r \).

Since \( P \) is a finite set and \(|s - \rho_1| \leq 2r, |s_0 - \rho_1| > \rho \) for all \( \rho_1 \in P \) and \( s = \sigma_1 + it_1, |s - s_0| \leq r \), it follows from (4.4) that

\[
\left| \frac{\mathcal{H}(s)}{\mathcal{H}(s_0)} \right| = \left| \frac{Z_{S,\chi} (s - \rho, \tau \otimes \sigma)}{Z_{S,\chi} (s_0 - \rho, \tau \otimes \sigma)} \right| \cdot \prod_{\rho_1 \in P} \frac{|s - \rho_1|}{|s_0 - \rho_1|} = e^{O(t_{n-1})} \cdot O(1) = e^{O(t_{n-1})}
\]

for \( s = \sigma_1 + it_1, |s - s_0| \leq r \). Hence, there exists a constant \( C \) such that

\[
\left| \frac{\mathcal{H}(s)}{\mathcal{H}(s_0)} \right| < e^{Ct_{n-1}}
\]

for \( s = \sigma_1 + it_1, |s - s_0| \leq r \). Putting \( M = Ct_{n-1} \) and applying Lemma B, we obtain

\[
\frac{\mathcal{H}'(s)}{\mathcal{H}(s)} = O(t_{n-1}) + \sum_{\rho_2 \in Q} \frac{1}{s - \rho_2}
\]

for \( s = \sigma_1 + it_1, |s - s_0| \leq \frac{1}{4}r \), where \( Q \) denotes the set of zeros of \( \mathcal{H}(s) \) lying in \(|s - s_0| \leq \frac{1}{2}r \). It follows from the definition of \( \mathcal{H}(s) \) that

\[
\frac{Z_{S,\chi} (s - \rho, \tau \otimes \sigma)}{Z_{S,\chi} (s - \rho, \tau \otimes \sigma)} = O(t_{n-1}) + \sum_{\rho_2 \in Q} \frac{1}{s - \rho_2} - \sum_{\rho_1 \in P} \frac{1}{s - \rho_1}
\]

for \( s = \sigma_1 + it_1, |s - s_0| \leq \frac{1}{4}r \). In particular, (4.5) remains valid for \( s = \sigma_1 + it, \rho \leq \sigma_1 < c + \frac{1}{4}r \). However, \( c < \frac{1}{4}r + \rho \). Hence, we get
\begin{align}
\frac{Z'_{S,\chi}(s-\rho, \tau \otimes \sigma)}{Z_{S,\chi}(s-\rho, \tau \otimes \sigma)} &= O \left( t^{n-1} \right) + \sum_{\rho_2 \in Q} \frac{1}{s-\rho_2} - \sum_{\rho_1 \in P} \frac{1}{s-\rho_1} \\
\text{for } s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2}r + \rho.
\end{align}

One can see from the definition of $H(s)$ that $Q$ is the set of zeros of $Z_{S,\chi}(s-\rho, \tau \otimes \sigma)$ lying in $|s-s_0| \leq \frac{1}{2}r$. Put $\rho_2 = \rho + i \gamma_1$. We have

$$|\rho_2 - s_0| \leq \frac{1}{2}r$$

if and only if

$$t - \sqrt{\frac{1}{4}r^2 - (c-\rho)^2} \leq \gamma_1 \leq t + \sqrt{\frac{1}{4}r^2 - (c-\rho)^2}.$$

Note that

$$2\rho < c < \frac{1}{4}r + \rho$$

if and only if

$$\frac{3\sqrt{3}}{4}r < \sqrt{\frac{1}{4}r^2 - (c-\rho)^2} < \sqrt{\frac{1}{4}r^2 - \rho^2}.$$

Taking into account our normalization of the metric on $Y$, we obtain

$$\sqrt{\frac{1}{4}r^2 - (c-\rho)^2} > \frac{3\sqrt{3}}{4}r \geq 2\sqrt{3}\rho > 1.$$

Hence, the first sum on the right hand side of (4.6) can be written as

\begin{align}
(4.7) \quad \sum_{\rho_2 \in Q} \frac{1}{s-\rho_2} &= \sum_{|s-\gamma_1| \leq 1} \frac{1}{s-\rho_2} + \\
&\quad \sum_{t+1 < \gamma_1 \leq t + \sqrt{\frac{1}{4}r^2 - (c-\rho)^2}} \frac{1}{s-\rho_2} + \sum_{t - \sqrt{\frac{1}{4}r^2 - (c-\rho)^2} \leq \gamma_1 < t-1} \frac{1}{s-\rho_2}
\end{align}

for $s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2}r + \rho$.

Similarly,

$$|\rho_1 - s_0| \leq r$$

if and only if

$$t - \sqrt{r^2 - (c-\rho)^2} \leq \gamma_2 \leq t + \sqrt{r^2 - (c-\rho)^2},$$

where $\rho_1 = \rho + i \gamma_2$. Therefore,
We have
\[\frac{Z_{S,\chi}^\prime (s - \rho, \sigma \otimes \sigma)}{Z_{S,\chi} (s - \rho, \tau \otimes \sigma)} = O (t^{n-1}) + \sum_{|t-\gamma_1| \leq 1} \frac{1}{s - \rho_2} - \sum_{|t-\gamma_2| \leq 1} \frac{1}{s - \rho_1} + \sum_{t+1 < \gamma_2 \leq t + \sqrt{t^2 - (c - \rho)}^2} \frac{1}{s - \rho_2} - \sum_{t+1 < \gamma_2 \leq t + \sqrt{r^2 - (c - \rho)}^2} \frac{1}{s - \rho_1} \]
for \(s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2} \tau + \rho\). Combining (4.6), (4.7) and (4.8), we obtain

\[\sum_{t+1 < \gamma_2 \leq t + \sqrt{t^2 - (c - \rho)}^2} \frac{1}{s - \rho_2} - \sum_{t+1 < \gamma_2 \leq t + \sqrt{r^2 - (c - \rho)}^2} \frac{1}{s - \rho_1} \]
for \(s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2} \tau + \rho\).

Corresponding to the pair \((\tau, 2\rho) \in I_p\), singularities of \(Z_{S,\chi} (s - \rho, \tau \otimes \sigma)\) along the line \(\text{Re}(s) = \rho\) resp. the number of singularities on the interval \(\rho + ix, 0 < x < t\) will be denoted by \(\rho_{S,p,\tau} = \rho + i\gamma_{S,p,\tau}\) resp. \(N_{S,p,\tau} (t)\).

We have
\[
\left| \sum_{t+1 < \gamma_1 \leq t + \sqrt{\frac{1}{t^2} - (c - \rho)}^2} \frac{1}{s - \rho_2} - \sum_{t+1 < \gamma_2 \leq t + \sqrt{r^2 - (c - \rho)}^2} \frac{1}{s - \rho_1} \right| \leq \sum_{t+1 < \gamma_1 \leq t + \sqrt{\frac{1}{t^2} - (c - \rho)}^2} \frac{1}{|s - \rho_2|} + \sum_{t+1 < \gamma_2 \leq t + \sqrt{r^2 - (c - \rho)}^2} \frac{1}{|s - \rho_1|} < 1 + \sum_{t+1 < \gamma_1 \leq t + \sqrt{\frac{1}{t^2} - (c - \rho)}^2} \frac{1}{t+1 < \gamma_2 \leq t + \sqrt{r^2 - (c - \rho)}^2} =
\]
\[N_{S,p,\tau} \left( t + \sqrt{r^2 - (c - \rho)}^2 \right) - N_{S,p,\tau} (t + 1) \]
for \(s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2} \tau + \rho\). Hence, it follows from (4.4) that
\[(4.10) \sum_{t+1<\gamma_1 \leq t+\sqrt{r^2-(c-\rho)^2}} \frac{1}{s-\rho_2} - \sum_{t+1<\gamma_2 \leq t+\sqrt{r^2-(c-\rho)^2}} \frac{1}{s-\rho_1} = O \left( t^{n-1} \right) \]

for \( s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2} t + \rho \). Similarly,

\[(4.11) \sum_{t-\sqrt{r^2-(c-\rho)^2} \leq \gamma_1 < t-1} \frac{1}{s-\rho_2} - \sum_{t-\sqrt{r^2-(c-\rho)^2} \leq \gamma_2 < t-1} \frac{1}{s-\rho_1} = O \left( t^{n-1} \right) \]

for \( s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2} t + \rho \). Combining (4.9), (4.10) and (4.11), we conclude

\[ Z'_S,\chi (s-\rho, \tau \otimes \sigma) Z_S,\chi (s-\rho, \tau \otimes \sigma) = O \left( t^{n-1} \right) + \sum_{|t-\gamma_{S,p,\tau}| \leq 1} \frac{1}{s-\rho_{S,p,\tau}} \]

for \( s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2} t + \rho \). However, \( r < t \). Hence,

\[(4.12) \frac{Z'_S,\chi (s-\rho, \tau \otimes \sigma)}{Z_S,\chi (s-\rho, \tau \otimes \sigma)} = O \left( t^{n-1} \right) + \sum_{|t-\gamma_{S,p,\tau}| \leq 1} \frac{1}{s-\rho_{S,p,\tau}} \]

for \( s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2} t + \rho \).

Let \( u > 0 \). One has

\[ \left| \sum_{|t-\gamma_{S,p,\tau}| \leq 1} \frac{1}{s-\rho_{S,p,\tau}} \right| \leq \sum_{|t-\gamma_{S,p,\tau}| \leq 1} \frac{1}{|s-\rho_{S,p,\tau}|} < \frac{1}{u} \sum_{|t-\gamma_{S,p,\tau}| \leq 1} \frac{1}{|t-\gamma_{S,p,\tau}|} = \]

\[ \frac{1}{u} \left( N_{S,p,\tau} (t+1) - N_{S,p,\tau} (t-1) \right) \]

for \( s = \sigma_1 + it, \rho + u \leq \sigma_1 < \frac{1}{2} t + \rho \). Therefore, it follows from (3.4) and (4.12) that

\[(4.13) \frac{Z'_S,\chi (s-\rho, \tau \otimes \sigma)}{Z_S,\chi (s-\rho, \tau \otimes \sigma)} = O \left( t^{n-1} \right) \]

for \( s = \sigma_1 + it, \rho + u \leq \sigma_1 < \frac{1}{2} t + \rho \).

Finally, by (3.1), (4.12) and (4.13)
\( Z'_{R,\chi}(s, \sigma) \) and \( D\) ˇzena\'n Gu\'si\'c

\[
\frac{Z'_{R,\chi}(s, \sigma)}{Z_{R,\chi}(s, \sigma)} = \sum_{p=0}^{n-1} (-1)^p \sum_{(\tau, \lambda) \in I_p} \frac{Z'_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma)}{Z_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma)} = \\
\sum_{p=0}^{n-1} (-1)^p \sum_{(\tau, \lambda) \in I_p} \frac{Z'_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma)}{Z_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma)} + \\
\sum_{p=0}^{n-1} (-1)^p \sum_{(\tau, 2\rho) \in I_p} \frac{Z'_{S,\chi}(s - \rho, \tau \otimes \sigma)}{Z_{S,\chi}(s - \rho, \tau \otimes \sigma)} = \\
\sum_{p=0}^{n-1} (-1)^p \sum_{(\tau, \lambda) \in I_p} O\left(t^{n-1}\right) + \\
\sum_{p=0}^{n-1} (-1)^p \sum_{(\tau, 2\rho) \in I_p} \frac{1}{s - s_R} \\
\text{for } s = \sigma_1 + it, \rho \leq \sigma_1 < \frac{1}{2}t + \rho. \text{ This proves (a).} \\
(b) \text{ Let } u > 0. \text{ Obviously,} \\
\left| \sum_{|t-t_R| \leq 1} \frac{1}{s - s_R} \right| \leq \sum_{|t-t_R| \leq 1} \frac{1}{|s - s_R|} < \frac{1}{u} \sum_{|t-t_R| \leq 1} 1 \\
\text{for } s = \sigma_1 + it, \rho + u \leq \sigma_1 < \frac{1}{2}t + \rho. \text{ Hence, it follows from (3.5) and (4.14) that} \\
\frac{Z'_{R,\chi}(s, \sigma)}{Z_{R,\chi}(s, \sigma)} = O\left(t^{n-1}\right) \\
\text{for } s = \sigma_1 + it, \rho + u \leq \sigma_1 < \frac{1}{2}t + \rho. \text{ This completes the proof.} \]

Remark 4.2. Approximate formulas for the logarithmic derivative of the zeta functions were quite often exploited by many authors (see, e.g., [5]-[8]), not always for the same underlaying space. Usually, they were applied to obtain error terms in the prime number resp. prime geodesic theorem, where the search for the optimal error bound is widely open (see, e.g., [11],[6]).
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