A new Fibonacci identity and its associated summation identities

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Abstract
We derive a new Fibonacci identity. This single identity subsumes important known identities such as those of Catalan, Ruggles, Halton and others, as well as standard general identities found in the books by Vajda, Koshy and others. We also derive several binomial and ordinary summation identities arising from this identity; in particular we obtain a generalization of Halton’s general Fibonacci identity.

1 Introduction
As usual, the Fibonacci numbers, $F_n$, and the Lucas numbers, $L_n$, $n \in \mathbb{Z}$, are defined by:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geq 2), \quad F_{-n} = (-1)^{n-1}F_n$$

and

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad (n \geq 2), \quad L_{-n} = (-1)^nL_n.$$ 

Both $(F_n)_{n \in \mathbb{Z}}$ and $(L_n)_{n \in \mathbb{Z}}$ are examples of a Fibonacci-like sequence. We define a Fibonacci-like sequence, $(G_n)_{n \in \mathbb{Z}}$, as one having the same recurrence relation as the Fibonacci sequence, but with arbitrary initial terms. Thus, given arbitrary integers $G_0$ and $G_1$, not both zero, we define

$$G_n = G_{n-1} + G_{n-2} \quad (n \geq 2);$$

and also extend the definition to negative subscripts by writing the recurrence relation as

$$G_{-n} = G_{-n+2} - G_{-n+1}.$$ 

In section 2, it will be shown that

$$G_{-n} = (-1)^n(L_nG_0 - G_n).$$

In this paper, we will derive the following identity involving Fibonacci numbers and Fibonacci-like numbers:

$$F_{a-b}G_{n+m} = F_{m-b}G_{n+a} + (-1)^{a+b+1}F_{m-a}G_{n+b},$$

valid for all integers $a$, $b$, $n$ and $m$. 
Various summation identities emanating from this identity will be derived. In particular, we will derive (section 3, identity (3.9)) the following generalization of Halton’s identity (see Halton [3, identity (23)]):

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{a-b}^j F_{m+a}^{k-j} G_{n+(a-b)k+(m+b)j} = (-1)^{(a+b)k} F_{m+b}^{k} G_{n}.$$  

2 An identity involving Fibonacci and Fibonacci-like numbers

Theorem 1. The following identity holds for arbitrary integers $a$, $b$, $m$ and $n$:

$$F_{a-b} G_{n+m} = F_{m-b} G_{n+a} + (-1)^{a+b+1} F_{m-a} G_{n+b}.$$  

Proof. Since both sequences $(F_n)$ and $(G_n)$ have the same recurrence relation, we choose a basis set in $(F_n)$ and express the numbers from $(G_n)$ in this basis. We write

$$G_{n+m} = \lambda_1 F_{m-b} + \lambda_2 F_{m-a}, \quad (2.1)$$

where $a$, $b$, $n$ and $m$ are arbitrary integers and the coefficients $\lambda_1$ and $\lambda_2$ are to be determined. Setting $m = a$ and $m = b$, in turn, gives

$$G_{n+a} = \lambda_1 F_{a-b}, \quad G_{n+b} = \lambda_2 F_{b-a}. \quad (2.2)$$

Multiplying through identity $(2.1)$ by $F_{a-b} F_{b-a}$ gives

$$F_{a-b} F_{b-a} G_{n+m} = \lambda_1 F_{a-b} F_{b-a} F_{m-b} + \lambda_2 F_{b-a} F_{a-b} F_{m-a}. \quad (2.3)$$

Thus, we find

$$F_{a-b} F_{b-a} G_{n+m} = F_{b-a} F_{m-b} G_{n+a} + F_{a-b} F_{m-a} G_{n+b}$$

$$= F_{b-a} F_{m-b} G_{n+a} + (-1)^{a+b+1} F_{a-b} F_{m-a} G_{n+b};$$

so that the identity of the theorem is satisfied identically if $a = b$ and numerically if $a \neq b$.

Since the left hand side of the identity of Theorem 1 does not change under the interchange of $m$ and $n$ and the interchange of $a$ and $-b$ and $b$ and $-a$, we also have the following identities:

$$F_{a-b} G_{n+m} = F_{m+a} G_{n-b} + (-1)^{a+b+1} F_{m+b} G_{n-a}, \quad (2.4)$$

$$F_{a-b} G_{n+m} = F_{n-b} G_{m+a} + (-1)^{a+b+1} F_{n-a} G_{m+b} \quad (2.5)$$

and

$$F_{a-b} G_{n+m} = F_{n+a} G_{m-b} + (-1)^{a+b+1} F_{n+b} G_{m-a}. \quad (2.6)$$

If we set $a = 0 = m$, $b = -n$ in identity (2.5) and use the fact that $F_{2n} = F_n L_n$, we have

$$G_{-n} = (-1)^n (L_n G_0 - G_n),$$

providing a direct access to negative-index Fibonacci-like numbers.
The presumably new identity in Theorem 1 includes, as particular cases, most known three-term recurrence relations involving Fibonacci numbers, Lucas numbers and the generalized Fibonacci numbers. We will give a couple of examples to illustrate this.

Setting $a = 0$ and $b = m - n$ in the identity of Theorem 1 gives

$$F_{n-m}G_{n+m} = F_nG_n + (-1)^{n+m+1}F_mG_m,$$

(2.7)

which is a generalization of Catalan’s identity:

$$F_{n-m}F_{n+m} = F_n^2 + (-1)^{n+m+1}F_m^2.$$

(2.8)

Using $m = 0$ and $a = c + b$ in the identity and re-arranging the terms, we find

$$F_{c+b}G_{n+b} - F_bG_{n+b+c} = (-1)^{b+1}F_cG_n,$$

(2.9)

which is a generalization of Vajda [6, Formulas (19a) and (19b)].

Setting $a = 0$ and $b = -m$ in the identity of Theorem 1 gives

$$G_{n+m} + (-1)^mG_{n-m} = L_mG_n,$$

(2.10)

which is Vajda [6, Formula 10a].

Setting $b = 0$, $a = k$ and $m = 2k$ in the identity of Theorem 1 gives

$$F_{n+2k} = L_kF_{n+k} + (-1)^{n+k}F_kF_n,$$

(2.11)

which is Ruggles’ identity [2, 5].

Setting $b = -a$ in the identity of Theorem 1 gives

$$F_{2a}G_{n+a} = F_{m+a}G_{n+a} - F_{m-a}G_{n-a},$$

(2.12)

with the special case

$$G_{n+m} = F_{m+1}G_{n+1} - F_{m-1}G_{n-1},$$

(2.13)

which is a generalization of the following identity (Halton [3, Identity (63)], Koshy [4, Identity (44), page 89]):

$$F_{n+m} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1}.$$

(2.14)

Setting $b = 2k$, $a = 1$ and $b = 2k$, $a = 0$, in turn, in the identity of Theorem 1 produces

$$F_{2k-1}G_{n+m} = F_{m-2k}G_{n+1} + F_{m-1}G_{n+2k}$$

(2.15)

and

$$F_{2k}G_{n+m} = F_mG_{n+2k} - F_{m-2k}G_n.$$  

(2.16)

Identity (2.16) is a generalization of the following well-known addition formula (Vajda [6, Formula (8)]):

$$G_{n+m} = F_{m-1}G_n + F_mG_{n+1}.$$  

(2.17)

Setting $a = n$ and $b = -m$ in the identity of Theorem 1 produces

$$F_{2m}G_{2n} = F_{n+m}G_{n+m} - F_{n-m}G_{n-m}.$$  

(2.18)
3 Summation identities involving Fibonacci and Fibonacci-like numbers

3.1 Binomial summation identities

Lemma 1 ([1, Lemma 3]). Let \((X_n)\) be any arbitrary sequence. Let \(X_n, n \in \mathbb{Z}\), satisfy a three-term recurrence relation \(hX_n = f_1X_{n-\alpha} + f_2X_{n-\beta}\), where \(h, f_1\) and \(f_2\) are non-vanishing complex functions, not dependent on \(n\), and \(\alpha\) and \(\beta\) are integers. Then,

\[
\sum_{j=0}^{k} \binom{k}{j} f_2^{k-j} f_1^j X_{n-\beta k + (\beta-\alpha) j} = h^k X_n,
\]

(3.1)

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} f_2^{k-j} h^j X_{n+(\alpha-\beta)k + \beta j} = (-1)^k f_1^k X_n
\]

(3.2)

and

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} f_1^{k-j} h^j X_{n+(\beta-\alpha)k + \alpha j} = (-1)^k f_2^k X_n,
\]

(3.3)

for \(k\) a non-negative integer.

Theorem 2. The following identities hold for positive integer \(k\) and arbitrary integers \(a, b, n, m\):

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m-b}^j F_{m-a}^{k-j} G_{n-(m-b)k + (a-b)j} = F_{a-b}^k G_n,
\]

(3.4)

\[
\sum_{j=0}^{k} (-1)^{(a+b)} \binom{k}{j} F_{a-b}^j F_{m-a}^{k-j} G_{n-(a-b)k + (m-a)j} = (-1)^{(a+b)} k F_{m-b}^k G_n,
\]

(3.5)

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{a-b}^j F_{m-b}^{k-j} G_{n+(a-b)k + (m-a)j} = (-1)^{(a+b)} k F_{m-a}^k G_n,
\]

(3.6)

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m-a}^j F_{m-b}^{k-j} G_{n-(m-a)k + (a-b)j} = F_{a-b}^k G_n,
\]

(3.7)

\[
\sum_{j=0}^{k} (-1)^{(a+b)} \binom{k}{j} F_{a-b}^j F_{m-b}^{k-j} G_{n-(a-b)k + (m-b)j} = (-1)^{(a+b)} k F_{m-a}^k G_n
\]

(3.8)

and

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{a-b}^j F_{m-a}^{k-j} G_{n+(a-b)k + (m-b)j} = (-1)^{(a+b)} k F_{m-b}^k G_n.
\]

(3.9)

Proof. To derive identities (3.4) – (3.6), write the identity of Theorem 1 as

\[
F_{a-b} G_n = F_{m-b} G_{n-(m-a)} + (-1)^{a+b+1} F_{m-a} G_{n-(m-b)};
\]

identify \(h = F_{a-b}\), \(f_1 = F_{m-b}\), \(f_2 = (-1)^{a+b+1} F_{m-a}\), \(X_n = G_n\), \(\alpha = m-a\), \(\beta = m-b\) and use these in Lemma 1. Identities (3.7) – (3.9) are obtained from identities (3.4) – (3.6) by interchanging \(a\) and \(-b\) and \(b\) and \(-a\). □
Particular cases of identities (3.4) – (3.9) are the pure Fibonacci binomial summation identities
\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m-b}^j F_{m-a}^{k-j} F_{n-(m-b)k+(a-b)j} = F_{a-b}^k F_n,
\] (3.10)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} F_{a-b}^j F_{m-b}^{k-j} F_{n-(a-b)k+(m-b)j} = (-1)^{(a+b)k} F_{m-b}^k F_n,
\] (3.11)

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{a-b}^j F_{m-b}^{k-j} F_{n-(a-b)k+(m-a)j} = (-1)^{(a+b)k} F_{m-a}^k F_n,
\] (3.12)

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m+a}^j F_{m+b}^{k-j} F_{n-(m+a)k+(a-b)j} = F_{a-b}^k F_n,
\] (3.13)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} F_{a-b}^j F_{m+b}^{k-j} F_{n-(a-b)k+(m+a)j} = (-1)^{(a+b)k} F_{m+a}^k F_n
\] (3.14)

and
\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{a-b}^j F_{m+b}^{k-j} F_{n-(a-b)k+(m+b)j} = (-1)^{(a+b)k} F_{m+b}^k F_n
\] (3.15)

and the corresponding identities involving both Fibonacci and Lucas numbers:
\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m-b}^j L_{m-a}^{k-j} L_{n-(m-b)k+(a-b)j} = F_{a-b}^k L_n,
\] (3.16)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} F_{a-b}^j L_{m-a}^{k-j} L_{n-(a-b)k+(m-b)j} = (-1)^{(a+b)k} F_{m-b}^k L_n,
\] (3.17)

\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{a-b}^j L_{m-b}^{k-j} L_{n-(a-b)k+(m-a)j} = (-1)^{(a+b)k} F_{m-a}^k L_n,
\] (3.18)

\[
\sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} \binom{k}{j} F_{m+a}^j L_{m+b}^{k-j} L_{n-(m+a)k+(a-b)j} = F_{a-b}^k L_n,
\] (3.19)

\[
\sum_{j=0}^{k} (-1)^{(a+b)j} \binom{k}{j} F_{a-b}^j L_{m+b}^{k-j} L_{n-(a-b)k+(m+a)j} = (-1)^{(a+b)k} F_{m+a}^k L_n
\] (3.20)

and
\[
\sum_{j=0}^{k} (-1)^j \binom{k}{j} F_{a-b}^j L_{m+b}^{k-j} L_{n-(a-b)k+(m+b)j} = (-1)^{(a+b)k} F_{m+b}^k L_n
\] (3.21)

We remark that Halton’s identity [3, Identity 23], from which he derived a very large number of identities of different kinds, involving the Fibonacci numbers, is a particular case of identity (3.15), being an evaluation at \( b = 0 \).
3.2 Non-binomial summation identities

Lemma 2 ([1, Lemma 1]). Let \((X_n)\) and \((Y_n)\) be any two sequences such that \(X_n\) and \(Y_n\), \(n \in \mathbb{Z}\), are connected by a three-term recurrence relation \(hX_n = f_1X_{n-\alpha} + f_2Y_{n-\beta}\), where \(h\), \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(n\), and \(\alpha\) and \(\beta\) are integers. Then, the following identity holds for integer \(k\):

\[
\sum_{j=0}^{k} f_2^{k-j} h^j X_{n-ka-\beta+\alpha j} = h^{k+1} X_n - f_1^{k+1} X_{n-(k+1)\alpha}.
\]

Theorem 3. The following identities hold for \(a, b, m, n\) and \(k\) arbitrary integers:

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} G^{k-j}_{m+b} G_{m+a}^{-1} G_{n-(a-b)k+m+b+(a-b)j} = (-1)^{(a+b)k} G_{n}^{k+1} G_{m+a}^{-1} + (-1)^{a+b+1} F_{n-(a-b)(k+1)} G_{m+b}^{k+1},
\]

and

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} G^{k-j}_{m-a} G_{m-b}^{-1} G_{n-(a-b)k+m-a+(a-b)j} = (-1)^{(a+b)k} G_{n}^{k+1} G_{m-b}^{-1} + (-1)^{a+b+1} F_{n-(a-b)(k+1)} G_{m-a}^{k+1}.
\]

Proof. To prove identity (3.22), write identity (2.4) as

\[
G_{m+a} F_n = (-1)^{a+b} G_{m+b} F_{n-(a-b)} + F_{a-b} G_{n+m+b}.
\]

identify \(h = G_{m+a}\), \(f_1 = (-1)^{a+b} G_{m+b}\), \(f_2 = F_{a-b}\), \(X_n = F_n\), \(Y_n = G_{n+m+b}\), \(\alpha = a - b\) and \(\beta = 0\) and use these in Lemma 2. Identity (3.23) is obtained from identity (3.22) through the transformation \(a \to -b\), \(b \to -a\).

Lemma 3 ([1, Lemma 2]). Let \((X_n)\) be any arbitrary sequence, where \(X_n\), \(n \in \mathbb{Z}\), satisfies a three-term recurrence relation \(hX_n = f_1X_{n-\alpha} + f_2X_{n-\beta}\), where \(h\), \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(r\), and \(\alpha\) and \(\beta\) are integers. Then, the following identities hold for integer \(k\):

\[
\sum_{j=0}^{k} f_2^{k-j} h^j X_{n-ka-\beta+\alpha j} = h^{k+1} X_n - f_1^{k+1} X_{n-(k+1)\alpha},
\]

\[
\sum_{j=0}^{k} f_1^{k-j} h^j X_{n-ka-\beta+\alpha j} = h^{k+1} X_n - f_2^{k+1} X_{n-(k+1)\beta},
\]

and

\[
\sum_{j=0}^{k} (-1)^j f_2^{k-j} f_1^j X_{n-(\beta-\alpha)k+\alpha+(\beta-\alpha)j} = (-1)^k f_1^{k+1} X_n + f_2^{k+1} X_{n-(\beta-\alpha)(k+1)}.
\]
**Theorem 4.** The following identities hold for arbitrary integers \( a, b, n, m \) and \( k \):

\[
(-1)^{a+b+1} F_{m-a} \sum_{j=0}^{k} F_{m-b}^{j} F_{a-b}^{j} G_{n-(m-a)(k-(m-b)+(m-a))j} = F_{a-b}^{k+1} G_n - F_{m-b}^{k+1} G_{n-(m-a)(k+1)} ,
\]

(3.28)

\[
F_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m-a}^{j} F_{a-b}^{j} G_{n-(m-b)k-(m-a)+(m-b)j} = F_{a-b}^{k+1} G_n - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} G_{n-(m-b)(k+1)} ,
\]

(3.29)

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m-a}^{j} F_{a-b}^{j} G_{n-(a-b)k+(m-a)+(a-b)j}
\]

\[
= (-1)^{(a+b)k} F_{m-a}^{k+1} G_n + (-1)^{(a+b+1)} F_{m-a}^{k+1} G_{n-(a-b)(k+1)} ,
\]

(3.30)

\[
(-1)^{a+b+1} F_{m+b} \sum_{j=0}^{k} F_{m+a}^{j} F_{a-b}^{j} G_{n-(m+b)k-(m+a)+(m+b)j} = F_{a-b}^{k+1} G_n - F_{m+a}^{k+1} G_{n-(m+b)(k+1)} ,
\]

(3.31)

\[
F_{m+a} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m+b}^{j} F_{a-b}^{j} G_{n-(m+a)k-(m+b)+(m+a)j} = F_{a-b}^{k+1} G_n - (-1)^{(a+b+1)(k+1)} F_{m+b}^{k+1} G_{n-(a-b)(k+1)} ,
\]

(3.32)

and

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m+b}^{j} F_{m+a}^{j} G_{n-(a-b)k+(m+b)+(a-b)j}
\]

\[
= (-1)^{(a+b)k} F_{m+a}^{k+1} G_n + (-1)^{(a+b+1)} F_{m+b}^{k+1} G_{n-(a-b)(k+1)} ,
\]

(3.33)

**Proof.** In Lemma 3, with \( X_n = G_n \), use the \( h, f_1, f_2, \alpha \) and \( \beta \) obtained in the proof of Theorem 2.

In particular, we have the pure Fibonacci summation identities

\[
(-1)^{a+b+1} F_{m-a} \sum_{j=0}^{k} F_{m-b}^{j} F_{a-b}^{j} G_{n-(m-a)k-(m-b)+(m-a)j} = F_{a-b}^{k+1} F_n - F_{m-b}^{k+1} F_{n-(m-a)(k+1)} ,
\]

(3.34)

\[
F_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m-a}^{j} F_{a-b}^{j} G_{n-(m-b)k-(m-a)+(m-b)j} = F_{a-b}^{k+1} F_n - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} F_{n-(m-b)(k+1)} ,
\]

(3.35)

\[
F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m-a}^{j} F_{m-b}^{j} G_{n-(a-b)k+(m-a)+(a-b)j}
\]

\[
= (-1)^{(a+b)k} F_{m-b}^{k+1} F_n + (-1)^{(a+b+1)} F_{m-a}^{k+1} F_{n-(a-b)(k+1)} ,
\]

(3.36)
\[ (-1)^{a+b+1} F_{m+b} \sum_{j=0}^{k} F_{m+a}^{k-j} F_{a-b}^{j} F_{-(m+b)k-(m+a)+(m+b)j} \]
\[ = F_{a-b}^{k+1} F_{n} - F_{m+a}^{k+1} F_{-(m+b)(k+1)} \; , \]  
(3.37)

\[ F_{m+a} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m+b}^{k-j} F_{a-b}^{j} F_{-(m+a)k-(m+b)+(m+a)j} \]
\[ = F_{a-b}^{k+1} F_{n} - (-1)^{(a+b+1)(k+1)} F_{m+b}^{k+1} F_{-(m+a)(k+1)} \; , \]  
(3.38)

and

\[ F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m+b}^{k-j} F_{m+a}^{j} F_{-(a-b)k+(m+b)+(a-b)} \]
\[ = (-1)^{(a+b)k} F_{m+a}^{k+1} F_{n} + (-1)^{a+b+1} F_{m+b}^{k+1} F_{-(a-b)(k+1)} \; , \]  
(3.39)

and the corresponding results involving Fibonacci and Lucas numbers:

\[ (-1)^{a+b+1} F_{m-a} \sum_{j=0}^{k} F_{m-b}^{k-j} F_{a-b}^{j} L_{-(m-a)k-(m-b)+(m-a)j} \]
\[ = F_{a-b}^{k+1} L_{n} - F_{m-b}^{k+1} L_{n-(m-a)(k+1)} \; , \]  
(3.40)

\[ F_{m-b} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m-a}^{k-j} F_{a-b}^{j} L_{-(m-b)k-(m-a)+(m-b)j} \]
\[ = F_{a-b}^{k+1} L_{n} - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} L_{n-(m-b)(k+1)} \; , \]  
(3.41)

\[ F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m-a}^{k-j} F_{m-b}^{j} L_{-(a-b)k+(m-a)+(a-b)} \]
\[ = (-1)^{(a+b)k} F_{m-b}^{k+1} L_{n} + (-1)^{a+b+1} F_{m-a}^{k+1} L_{n-(a-b)(k+1)} \; , \]  
(3.42)

\[ (-1)^{a+b+1} F_{m+b} \sum_{j=0}^{k} F_{m+a}^{k-j} F_{a-b}^{j} L_{-(m+b)k-(m+a)+(m+b)j} \]
\[ = F_{a-b}^{k+1} L_{n} - F_{m+a}^{k+1} L_{n-(m+b)(k+1)} \; , \]  
(3.43)

\[ F_{m+a} \sum_{j=0}^{k} (-1)^{(a+b+1)(k-j)} F_{m+b}^{k-j} F_{a-b}^{j} L_{-(m+a)k-(m+b)+(m+a)j} \]
\[ = F_{a-b}^{k+1} L_{n} - (-1)^{(a+b+1)(k+1)} F_{m+b}^{k+1} L_{n-(m+a)(k+1)} \; , \]  
(3.44)

and

\[ F_{a-b} \sum_{j=0}^{k} (-1)^{(a+b)j} F_{m+b}^{k-j} F_{m+a}^{j} L_{-(a-b)k+(m+b)+(a-b)} \]
\[ = (-1)^{(a+b)k} F_{m+a}^{k+1} L_{n} + (-1)^{a+b+1} F_{m+b}^{k+1} L_{n-(a-b)(k+1)} \; . \]  
(3.45)
3.3 Sums involving products of Fibonacci or Fibonacci-like numbers in the denominator of the summand

Lemma 4. Let \((X_n)\) and \((Y_n)\) be any two sequences such that \(X_n\) and \(Y_n\), \(n \in \mathbb{Z}\), are connected by a three-term recurrence relation \(hX_n = f_1X_{n-\alpha} + f_2Y_{n-\beta}\), where \(f_1\) and \(f_2\) are arbitrary non-vanishing complex functions, not dependent on \(n\), and \(\alpha\), \(\beta\) and \(k\) are integers. Then,

\[
X_nX_{n-\alpha(k+1)}f_2\sum_{j=0}^{k} h^{k-j} f_1^j \frac{Y_{n-\beta-ak+\alpha j}}{X_{n-\alpha(k+1)}X_{n-\alpha-ak+\alpha j}} = h^{k+1}X_n - f_1^{k+1}X_{n-\alpha(k+1)}.
\]

Theorem 5. The following identities hold for values of \(a, b, m, n, k\) for which the summand is non-singular in the summation interval:

\[
F_nF_{n-(a-b)(k+1)}F_{a-b}\sum_{j=0}^{k} (-1)^{(a+b)j} \frac{G_{m+a}^{k-j}G_{m+b}^{j}G_{n+m+b-(a-b)k+(a-b)j}}{F_{n-(a-b)k+(a-b)j}F_{n-a+b-(a-b)k+(a-b)j}} = F_nG_{m+a}^{k+1} - (-1)^{(a+b)(k+1)}F_{n-(a-b)(k+1)}G_{m+b}^{k+1},
\]

(3.46)

\[
F_nF_{n-(a-b)(k+1)}F_{a-b}\sum_{j=0}^{k} (-1)^{(a+b)j} \frac{G_{m-a}^{k-j}G_{m-b}^{j}G_{n+m-a-(a-b)k+(a-b)j}}{F_{n-(a-b)k+(a-b)j}F_{n+b-(a-b)k+(a-b)j}} = F_nG_{m-b}^{k+1} - (-1)^{(a+b)(k+1)}F_{n-(a-b)(k+1)}G_{m-a}^{k+1},
\]

(3.47)

Proof. In Lemma 4, make the identification \(X_n = F_n\) and \(Y_n = G_{n+m+b}\) and use the \(f_1, f_2, h, \alpha\) and \(\beta\) obtained in the proof of Theorem 3. 

Particular cases of identities (3.46) and (3.47) are the following:

\[
F_nF_{n-(a-b)(k+1)}F_{a-b}\sum_{j=0}^{k} (-1)^{(a+b)j} \frac{F_{m+a}^{k-j}F_{m+b}^{j}F_{n+m+b-(a-b)k+(a-b)j}}{F_{n-(a-b)k+(a-b)j}F_{n-a+b-(a-b)k+(a-b)j}} = F_nF_{m+a}^{k+1} - (-1)^{(a+b)(k+1)}F_{n-(a-b)(k+1)}F_{m+b}^{k+1},
\]

(3.48)

\[
F_nF_{n-(a-b)(k+1)}F_{a-b}\sum_{j=0}^{k} (-1)^{(a+b)j} \frac{F_{m-a}^{k-j}F_{m-b}^{j}F_{n+m-a-(a-b)k+(a-b)j}}{F_{n-(a-b)k+(a-b)j}F_{n+b-a-(a-b)k+(a-b)j}} = F_nF_{m-b}^{k+1} - (-1)^{(a+b)(k+1)}F_{n-(a-b)(k+1)}F_{m-a}^{k+1},
\]

(3.49)

and

\[
F_nF_{n-(a-b)(k+1)}F_{a-b}\sum_{j=0}^{k} (-1)^{(a+b)j} \frac{L_{m+a}^{k-j}L_{m+b}^{j}L_{n+m+b-(a-b)k+(a-b)j}}{F_{n-(a-b)k+(a-b)j}F_{n-a+b-(a-b)k+(a-b)j}} = F_nL_{m+a}^{k+1} - (-1)^{(a+b)(k+1)}F_{n-(a-b)(k+1)}L_{m+b}^{k+1},
\]

(3.50)

\[
F_nF_{n-(a-b)(k+1)}F_{a-b}\sum_{j=0}^{k} (-1)^{(a+b)j} \frac{L_{m-a}^{k-j}L_{m-b}^{j}L_{n+m-a-(a-b)k+(a-b)j}}{F_{n-(a-b)k+(a-b)j}F_{n+b-a-(a-b)k+(a-b)j}} = F_nL_{m-b}^{k+1} - (-1)^{(a+b)(k+1)}F_{n-(a-b)(k+1)}L_{m-a}^{k+1},
\]

(3.51)
Lemma 5. Let \((X_n)\) be any arbitrary sequence. Let \(X_n, n \in \mathbb{Z}\), satisfy a three-term recurrence relation \(hX_n = f_1X_{n-\alpha} + f_2X_{n-\beta}\), where \(f_1\) and \(f_2\) are non-vanishing complex functions, not dependent on \(n\), and \(\alpha, \beta\) and \(k\) are integers. Then, the following identities hold for arbitrary integers \(n\), \(\alpha\), \(\beta\) and \(k\) for which the summand is not singular in the summation interval:

\[
X_nX_{n-\alpha(k+1)}f_2 \sum_{j=0}^{k} h^{k-j} f_1^j \frac{X_{n-\beta+\alpha+j}}{X_{n-\alpha+\alpha+j}X_{n-\alpha+\alpha+j}} = h^{k+1}X_n - f_1^{k+1}X_{n-\alpha(k+1)} ,
\]

(3.52)

\[
X_nX_{n-\beta(k+1)}f_1 \sum_{j=0}^{k} h^{k-j} f_2^j \frac{X_{n-\alpha-\beta+j}}{X_{n-\beta+\beta+j}X_{n-\beta+\beta+j}} = h^{k+1}X_n - f_2^{k+1}X_{n-\beta(k+1)} ,
\]

(3.53)

and

\[
X_nX_{n-(\alpha-\beta)(k+1)}h \sum_{j=0}^{k} (-1)^j f_1^{k-j} f_2^j \frac{X_{n+\alpha-(\beta-\alpha)+j}}{X_{n-(\beta-\alpha)+j}X_{n-(\beta-\alpha)+j}} = f_1^{k+1}X_n + (-1)^k f_2^{k+1}X_{n-(\beta-\alpha)(k+1)} .
\]

(3.54)

Theorem 6. The following identities hold for values of \(a, b, m, n, k\) for which the summand is non-singular in the summation interval:

\[
(-1)^{a+b+1} F_{m-a}G_nG_{n-(m-a)(k+1)} \sum_{j=0}^{k} \frac{F_{a-b}^j F_{m-b}^j G_{n-(m-b)+j}}{G_{n-(m-a)+j}G_{n-(m-a)-j}} = F_{a-b}^{k+1}G_n - F_{m-b}^{k+1}G_{n-(m-a)(k+1)} ,
\]

(3.55)

\[
F_{m-b}G_nG_{n-(m-b)(k+1)} \sum_{j=0}^{k} (-1)^{(a+1)+j} F_{a-b}^j F_{m-a}^j G_{n-(m-a)} = F_{a-b}^{k+1}G_n - (-1)^{a+b+1}(k+1) F_{m-a}^{k+1}G_{n-(m-a)(k+1)} ,
\]

(3.56)

\[
F_{a-b}G_nG_{n-(a-b)(k+1)} \sum_{j=0}^{k} (-1)^{(a+b)+j} F_{m-b}^j F_{m-a}^j G_{n-(a-b)+j} = F_{m-b}^{k+1}G_n - (-1)^{(a+b)(k+1)} F_{m-a}^{k+1}G_{n-(a-b)(k+1)} ,
\]

(3.57)

\[
(-1)^{a+b+1} F_{m+b}G_nG_{n-(m+b)(k+1)} \sum_{j=0}^{k} \frac{F_{a-b}^j F_{m+a}^j G_{n-(m+a)+j}}{G_{n-(m+b)+j}G_{n-(m+b)-j}} = F_{a-b}^{k+1}G_n - F_{m+a}^{k+1}G_{n-(m+b)(k+1)} ,
\]

(3.58)

\[
F_{m+a}G_nG_{n-(a+m)(k+1)} \sum_{j=0}^{k} (-1)^{(a+b+1)+j} F_{a-b}^j F_{m+b}^j G_{n-(a+m)+j} = F_{a-b}^{k+1}G_n - (-1)^{(a+b+1)(k+1)} F_{m+b}^{k+1}G_{n-(a+m)(k+1)} ,
\]

(3.59)

and

\[
F_{a-b}G_nG_{n-(a-b)(k+1)} \sum_{j=0}^{k} (-1)^{(a+b)+j} F_{m+a}^j F_{m+b}^j G_{n-(a-b)+j} = F_{m+a}^{k+1}G_n - (-1)^{(a+b)(k+1)} F_{m+b}^{k+1}G_{n-(a-b)(k+1)} .
\]

(3.60)
**Proof.** In Lemma 5, make the identification $X_n = G_n$ and use the $f_1$, $f_2$, $h$, $\alpha$ and $\beta$ obtained in the proof of Theorem 4.

In particular, we have the pure Fibonacci identities:

\[
(-1)^{a+b+1} F_{m-a} F_n F_{n-(m-a)(k+1)} \sum_{j=0}^{k} \frac{F_{a-b}^j F_{m-b}^j F_{-m+b-(m-a)k+(m-a)j}}{F_{n-(m-a)k+(m-a)j}} 
= F_{a-b}^{k+1} F_n - (-1)^{a+b+1}(k+1) F_{m-a}^{k+1} F_{n-(m-a)(k+1)},
\]

(3.61)

\[
F_{m-b} F_n F_{n-(m-b)(k+1)} \sum_{j=0}^{k} \frac{(-1)^{(a+b+1)} j F_{a-b}^j F_{m-a}^j F_{n-(m-a)-(m-b)j}}{F_{n-(m-b)k+(m-b)j}} 
= F_{a-b}^{k+1} F_n - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} F_{n-(m-b)(k+1)},
\]

(3.62)

\[
F_{a-b} F_n F_{n-(a-b)(k+1)} \sum_{j=0}^{k} \frac{(-1)^{(a+b)} j F_{m-b}^j F_{m-a}^j F_{n-(a-b)+(a-b)j}}{F_{n-(a-b)k+(a-b)j}} 
= F_{a-b}^{k+1} F_n - (-1)^{(a+b)(k+1)} F_{m-a}^{k+1} F_{n-(a-b)(k+1)},
\]

(3.63)

\[
(-1)^{a+b+1} F_{m+b} F_n F_{n-(m+b)(k+1)} \sum_{j=0}^{k} \frac{F_{a-b}^j F_{m+a}^j F_{n-m-a-(m+b)j}}{F_{n-(m+b)k+(m+b)j}} 
= F_{a-b}^{k+1} F_n - (-1)^{(a+b+1)(k+1)} F_{m+b}^{k+1} F_{n-(m+b)(k+1)},
\]

(3.64)

\[
F_{m+a} F_n F_{n-(m+a)(k+1)} \sum_{j=0}^{k} \frac{(-1)^{(a+b+1)} j F_{a-b}^j F_{m+b}^j F_{n-(m+b)-(m+a)j}}{F_{n-(m+a)k+(m+a)j}} 
= F_{a-b}^{k+1} F_n - (-1)^{(a+b+1)(k+1)} F_{m+b}^{k+1} F_{n-(m+a)(k+1)},
\]

(3.65)

and

\[
F_{a-b} F_n F_{n-(a-b)(k+1)} \sum_{j=0}^{k} \frac{(-1)^{(a+b)} j F_{m+a}^j F_{m+b}^j F_{n-(a-b)+(a-b)j}}{F_{n-(a-b)k+(a-b)j}} 
= F_{a-b}^{k+1} F_n - (-1)^{(a+b)(k+1)} F_{m+b}^{k+1} F_{n-(a-b)(k+1)},
\]

(3.66)

and the corresponding identities involving Lucas and Fibonacci numbers:

\[
(-1)^{a+b+1} F_{m-a} L_n L_{n-(m-a)(k+1)} \sum_{j=0}^{k} \frac{F_{a-b}^j F_{m-b}^j L_{n-m+b-(m-a)k+(m-a)j}}{L_{n-(m-a)k+(m-a)j}} 
= F_{a-b}^{k+1} L_n - (-1)^{(a+b+1)(k+1)} F_{m-b}^{k+1} L_{n-(m-a)(k+1)},
\]

(3.67)

\[
F_{m-b} L_n L_{n-(m-b)(k+1)} \sum_{j=0}^{k} \frac{(-1)^{(a+b+1)} j F_{a-b}^j F_{m-a}^j L_{n-(m-a)-(m-b)j}}{L_{n-(m-b)k+(m-b)j}} 
= F_{a-b}^{k+1} L_n - (-1)^{(a+b+1)(k+1)} F_{m-a}^{k+1} L_{n-(m-b)(k+1)},
\]

(3.68)

\[
F_{a-b} L_n L_{n-(a-b)(k+1)} \sum_{j=0}^{k} \frac{(-1)^{(a+b)} j F_{m-b}^j F_{m-a}^j L_{n-(a-b)+(a-b)j}}{L_{n-(a-b)k+(a-b)j}} 
= F_{m-b}^{k+1} L_n - (-1)^{(a+b)(k+1)} F_{m-a}^{k+1} L_{n-(a-b)(k+1)},
\]

(3.69)
\begin{align*}
(-1)^{a+b+1}F_{m+b}L_nL_{n-(m+b)(k+1)} &= \sum_{j=0}^{k} \frac{F_a^{k-j}F_{m+a}^{j}L_{n-(m+b)k+(m+b)j}}{F_{m+a}^{k}L_{n-(m+b)(k+1)}}, \\
F_{m+a}L_nL_{n-(m+a)(k+1)} &= \sum_{j=0}^{k} \frac{(-1)^{(a+b+1)j}F_a^{k-j}F_{m+b}^{j}L_{n-(m+b)-(m+a)k+(m+a)j}}{L_{n-(m+a)k+(m+a)j}L_{n-(m+a)(k+1)}}, \\
&= F_{a-b}^{k+1}L_n - (-1)^{(a+b+1)(k+1)}F_{m+b}^{k+1}L_{n-(m+a)(k+1)}, \\
\text{and} \\
F_{a-b}L_nL_{n-(a-b)(k+1)} &= \sum_{j=0}^{k} \frac{(-1)^{(a+b)j}F_{m+a}^{k-j}F_{m+b}^{j}L_{n-(a-b)k+(a-b)j}}{L_{n-(a-b)k+(a-b)j}L_{n-(a-b)(k+1)}}, \\
&= F_{m+a}^{k+1}L_n - (-1)^{(a+b)(k+1)}F_{m+b}^{k+1}L_{n-(a-b)(k+1)}.
\end{align*}

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