Investigation of the properties of reactions-diffusion model solutions with double nonlinearity

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Abstract. Analyses show that the central place in the mathematical description of the spatial-temporal dynamics of one or several interacting populations take the task of research of interaction of the migration population processes with demographic. Population models that take into account only the demographic processes are well known and fairly well-developed. This assumption is true if modelled area is quite small compared with the mean free path of individuals, or, equivalently, with a radius of individual activity. If this provision violated, in population models must account migration.

1. Introduction

In problems of mathematical optimization, the preference between possible solutions of nonlinear differential equations is always accompanied by considerable difficulties, because it is possible to solve them in analytical form only in some cases. Establishing new properties of solutions requires painstaking research. Therefore, in order to analyze the properties of solutions of nonlinear differential equations, various exact and approximate methods are used. In [1] and others, it was shown that self-similar solutions corresponding to certain values of parameters play a very important role. Therefore, great importance is attached to the study of nonlinear differential equations of parabolic type, describing various processes, on the basis of the self-similar and approximate-self-similar approach.

The simplest and most widely used in the present situation is the hypothesis of randomness “walk” of individuals in space. This assumption allows substantiating of using as a tool of modeling equations of reaction-diffusion type, where as a reaction part was used the right part of the point models, and diffusion coefficients (mobility of individuals) are assumed to be constant.

In the framework of these models it is possible to explain such effects as wave propagation of population at settling the area and existence of complex spatio-temporal dynamics of population.

Consider in $Q_T = [0, T] \times \mathbb{R}^N$ generalized reaction–diffusion task of Kolmogorov–Fisher type in the following form

$$\frac{\partial u}{\partial t} = \nabla \left( D |\nabla u|^p \nabla u \right) + k(t)u(1-u^\beta),$$

$$u|_{t=0} = u_0(x), x \in \mathbb{R}^N, \sup_x u_0(x) < +\infty,$$

which describes the process of biological populations with double nonlinearity, which diffusion coefficient is equal to $D |\nabla u|^p$. Here $D, m, p, \beta \geq 1$ - are given constants,
Equation (1) is a generalization of the simplest diffusion model for the logistic model of population growth [1], [2]. It can be regarded as the equation of nonlinear filtration, thermal conductivity in the presence of simultaneous influence of a source and acquisitions.

This equation in the case of $m=0$, $k(t)=k >0$ constant suggested (as well as the cubic equation instead quadratic nonlinearity in the right side) Fisher [2] as a stochastic model of the deterministic version of the gene in a favorable distribution of diploid populations. He examined in detail the equation and obtained some useful results. Heuristic and based on genetics conclusion of the equation have also led by A. N. Kolmogorov, I. G. Petrovskii and N.S.Piskunov, classical work [4] which served as the basis for a more rigorous analytical approach to the Fisher equation.

Properties of the solution of the Cauchy problem and the solution of equation (1) in the case of a homogeneous environment, when (1) $m=0$ and $k(t)=k$ - constant studied in detail by many other authors (see for example [2], where you can find links to other work). Suppose that in (1) $m=0$, $k(t)=k$ -constant. In this case, for speed of propagation of the wave with wave solutions of (1)

$$Df'' - cf' + f(1-f^p) = 0,$$

$$u(t,x) = f(\xi) \cdot \xi = ct + x,$$

authors of [3] were rated an estimate $c \geq 2\sqrt{kD}$.

With regard to the properties of solutions of the initial task for the equation (1) is unknown and so it is interesting to trace the evolution of the process of reaction diffusion in heterogeneous environment and explore the impact of heterogeneity and case of dependences of the reaction coefficient from the time, i.e. $k= k(t)$.

This paper is devoted to the study of properties of solutions of the task (1), (2). Clarified conditions on parameter $m$ and diffusion coefficient $k(t)$, where take the place finite speed of movement of a wave. Proved also, bilateral estimation of a solution of the task (1).

We construct self-similar equations for (1)-(2)-more simple for research equations. We construct self-similar equation by the method of nonlinear splitting [1].

Below we propose an algorithm based on the splitting of the original equation of parabolic type with which assesses estimations of lower and upper solutions of the task (1)-(2).

2. Algorithm of splitting of the parabolic type equation

We solve the equation

$$\bar{u}_t = k\bar{u}(1-\bar{u}^\beta) , \quad \bar{u}(t) = \left[ \frac{\beta e^{\beta t}}{(1+e^{\beta t})^\frac{1}{\beta}} \right]$$

and then search solutions of the equation (1) in the form

$$u(t,x) = \bar{u}(t)w(\tau,x).$$

Then supplying (4) into (1) we have again equation of the form (1)

$$\frac{\partial \bar{w}}{\partial \tau} = \nabla \left( D|\nabla \bar{w}|^{p-2} \nabla \bar{w} \right) + k_1(t)w(1-w^p),$$

where $k_1(t) = k(t)\bar{u}^{m(p-1)}$

only difference is that instead of the $t$ stands $\tau(t)$,and instead of $k(t) - k_1(t)$.
Hence, if \( m(p-1)=0 \) equation (5) has exactly the form (1), so again, we get an equation of the form (1)
\[
\frac{\partial w}{\partial t} = \nabla \left( D \left| \nabla w^m \right|^{p-2} \nabla w^m \right) + k(t) w(1-w^\beta),
\]
however, instead of a variable by the time \( t \) stand the function \( \tau(t) = \int \left( \nabla (t)^{(m-1)}/w \right) dt \).

Now consider equation
\[
\frac{\partial \bar{w}}{\partial \tau} = \nabla \left( | \nabla \bar{w}^m |^{p-2} \nabla \bar{w}^m \right).
\]

Equation (7), which is called the etalon for (1), has six types of self-similar solutions, one of which has the form:
\[
\bar{w}(\tau, x) = \bar{f}(\xi), \quad \xi = x \sqrt{1/\tau^{1/p}},
\]
where \( \bar{f}(\xi) \) satisfies the equation
\[
\xi^{1-N} \frac{d}{d\xi} \left( \xi^{N-1} \frac{d}{d\xi} \left| \bar{f} \right|^{p-2} \frac{d}{d\xi} \bar{f} \right) + \frac{\xi}{p} \frac{df}{d\xi} + k(t)(f-f^\beta) = 0.
\]
at \( \xi < (a)^{1/(p-1)/p} \). From (6) it is clear that if selected as the comparable function \( z(t, x) = \bar{u}(t) \bar{w}(\tau, x) \) and \( \bar{w} \) is a solution of (6), \( \bar{u}(t) \) given by (3), it will be a lower solution of the original task, if \( u_0(x) \geq z(0, x) \). This result can be improved by applying again algorithm of nonlinear splitting applied above.

Now after putting \( w(\tau, x) = f(\xi), \quad \xi = x \sqrt{1/\tau^{1/p}} \), equation (6) will turn into a self-similar equation
\[
\xi^{1-N} \frac{d}{d\xi} \left( \xi^{N-1} \frac{d}{d\xi} \left| f \right|^{p-2} \frac{d}{d\xi} f \right) + \frac{\xi}{p} \frac{df}{d\xi} + k(t)(f-f^\beta) = 0.
\]

Now if we construct a function \( z(t, x) = \bar{u}(t) \bar{w}(\tau, x) \), where \( \bar{w} \) is a solution of the etalon equation (7), for which runs conditions \( AZ \leq 0 \) in \( Q_T \) and \( u_0(x) \leq z(0, x) \), then function \( z(t, x) \) will be an upper solutions of the task (1), (2). Here
\[
Au = \frac{\partial u}{\partial t} - \nabla \left( D \left| \nabla u^m \right|^{p-2} \nabla u^m \right) - k(t) u(1-u^\beta).
\]

On these considerations are based algorithm for the construction of upper and lower solutions of the Cauchy task (1), (2), the global solvability of the initial task and definition of the critical value of the parameter \( \beta \), where solution of the task becomes unlimited, and changes view of the asymptotic representation.

Let’s introduce notations \( Q_e = Q \), \( D=\{(t,x): t>0, |x|<l(t)\} \), where \( l(t) > 0 \) at \( t \geq 0 \) - continuous function.

Lemma. Let \( u(t, x) \geq 0 \) - generalized solution of the task (1), (2) and functions \( u_e(t, x) \) such that
\[
u_e \geq 0 \text{ in } Q, \quad u_e = 0 \text{ in } QD, \quad u_e \in C^1_{t,e}(D) \cap C(\bar{D}), \quad \left| \nabla u^e \right|^{p-2} \nabla u^e \in C(Q) \quad \text{and in } D \text{ runs an inequality } Au_e \leq 0, \quad Au_e \geq 0, \quad u_0(x) \leq u_e(0,x), \quad u_0(x) \geq u_e(0,x) \in R^N.\]
Then \( u_0(x) \leq u_*(t,x) \), \( u_0(x) \geq u_*(t,x) \) in \( Q \).

Functions \( u_*, u_* \) are called, respectively, the upper and lower solution of (1), (2).

By direct calculation we can ensure that the function

\[
f_0(\xi) = \left[ a - \left| \xi^{p} \right| \right]^{\frac{(p-1)}{m(p-1)+1}},
\]

is generalized solution of the equation

\[
\xi^{p-N} \frac{d}{d\xi} \left( \xi^{N-1} \left| \xi^{p} \right| \frac{d^{m}}{d\xi^{m}} \right) + \frac{\xi}{p} \frac{df}{d\xi} + \frac{N}{p} f = 0,
\]

where constant \( b > 0 \).

3. Evaluation of the problem solution

Theorem 1. Let in (1)-(2) \( u_0(x) \leq u_* (0,x) \), \( x \in R^N, \xi = x / \tau^{\mu p} \) and \( \frac{\xi}{p} \frac{df}{d\xi} + \frac{N}{p} f \leq 0 \).

Then in \( Q \) there is a global solution of (1), (2), for which we have the estimate \( u(t,x) \leq u_*(t,x) \), where

\[
u_*(t,x) = u(t)f_0(\xi),
\]

here \( u(t), f_0(\xi) \) given by formulas (3) and (11).

Suggested that, \( \beta \geq 1 \), \( 0 \leq u_0(x) \leq 1 \), \( 0 < k(t) \in C(0,\infty) \).

Proof. The top solution to problem (1), (2) will be sought in the form \( u_*(t,x) = \overline{u}(t)\xi(\xi) \), where \( \tau(t) = \int [\overline{u}(t)]^{m(p-2)} dt \).

By direct verification, you can verify that the function

\[
\overline{f}(\xi) = \left[ a - \left| \xi^{p} \right| \right]^{\frac{(p-1)}{m(p-1)+1}},
\]

is a generalized solution to the equation

\[
\xi^{p-N} \frac{d}{d\xi} \left( \xi^{N-1} \left| \xi^{p} \right| \frac{d^{m}}{d\xi^{m}} \right) + \frac{\xi}{p} \frac{d\overline{f}}{d\xi} + \frac{N}{p} \overline{f} = 0,
\]

having the property of continuity \( \overline{f} \) and \( \left| \frac{d\overline{f}}{dx} \right| \frac{d^{m}}{dx^{m}} \).

In order to prove the theorem, we use the decision comparison theorem.

Show that

\[
Au_* \leq 0 \quad D = \left\{ (t,x) : t > 0, \left| x \right| \leq l(t) = a^{\frac{1}{p}} \left[ \tau(t) \right]^{\frac{1}{p}} \right\},
\]

where \( u_* \) the above defined function, which allows one to apply the decision comparison theorem [4].

To do this, we show that really \( Au_* \leq 0 \) in \( D \).
Putting \( w(\tau, x) = \tilde{f}(\xi) \), we have

\[
\xi^{1-N} \frac{d}{d\xi} \left( \xi^{N-1} \frac{d\tilde{f}}{d\xi} \right) + \xi \frac{d\tilde{f}}{p d\xi} = -\frac{N}{p} \tilde{f}_1.
\]

In order to be \( Au_+ \leq 0 \), enough condition \( \xi \frac{d\tilde{f}}{p d\xi} + \frac{N}{p} \leq 0 \).

By virtue of the condition of the theorem, it is satisfied.

It follows that \( u(t, x) \) is bounded for all \( t > 0 \), and thus the global solvability of problem (1), (2) is established. Theorem 1 is proved.

We consider a generalized problem of reaction with diffusion of the Kolmogorov-Petrovsky-Piskunov-Fisher type. In this case, the equation describing such a diffusion-reaction process can be written as

\[
Au \equiv -\rho(x) \frac{\partial u}{\partial t} + \nabla(D|x|^m \nabla u) + k \rho(x) u(1-u^\beta) = 0, \quad (14)
\]

\[
u/t_{10} = u_0(x) > 0, \ x \in R^N, \ N \geq 1, \quad (15)
\]

where \( D, \kappa > 0, \beta \geq 1 \) are respectively the diffusion and reaction coefficients, and \( \rho(x) = |x|^{2-n} \).

For the solution of problem (14), (15) fair

Theorem 2. Let \( 0 \leq u_0(x) \leq 1, \ x \in R \).

Then to solution of the problem (14), (15) in \( Q = \{ (t, x); t > 0, x \in R \} \) there is a two-way assessment

\[
\tilde{u}(t)(T + t)^{-s/2} e^{-\frac{(x)^{2-m}}{4D(T+t)^s}} \leq u(t, x) \leq e^{s} (T + t)^{-s/2} e^{-\frac{(x)^{2-m}}{4D(T+t)^s}}, \quad (16)
\]

where \( \tilde{u}(t) \) the function defined above, and the number \( s = \frac{2}{2-m} \).

Proof. To prove the theorem, we first obtain an estimate from above. To this end, in (14) will make the replacement

\[
u(t, x) = \tilde{f}\gamma(t, x).
\]

Then for \( \gamma(t, x) \) we have equation

\[
Aw_\gamma \equiv -\frac{\partial \gamma}{\partial t} + \frac{\partial}{\partial t} \left( D x^n \frac{\partial \gamma}{\partial x} \right) - k(t) \gamma^\beta = 0. \quad (17)
\]

It is easy to see that equation (17) by transformation \( u(t, x) = w(t, \varphi(x)) \), where \( \varphi(x) = 2x^{-m} \)

conducted to the "radially symmetric" form

\[
Aw_\gamma = -\frac{\partial \gamma}{\partial t} + \varphi^{1-m} \frac{\partial}{\partial \varphi} \left( D \varphi^{1-m} \frac{\partial \gamma}{\partial \varphi} \right) - k(t) \gamma^\beta = 0. \quad (18)
\]
Therefore, it is obvious that the function \( w_+ (t, x) = (T + t)^{-\alpha/2} e^{-\frac{x^2}{4D(T + t)}} \) is the upper solution of equation (17), since for \( w_+ (t, x) \) have

\[
A w_+ = -k(t)(T + t)^{-\alpha/2} e^{-\frac{x^2}{4D(T + t)}} \leq 0 \text{ in } Q \text{ for any constant } T > 0.
\]

Consequently, by the decision comparison theorem [4] we have the upper bound

\[
u(t, x) \leq e^{\int_{k(t)d(t)}^{} w_+ (t, x)}
\]

in \( Q \) if \( w_+ (0, x) \leq u_0 (x), x \in \mathbb{R} \)

In order to obtain a lower bound for the future lower solution, we apply the nonlinear splitting method [6]. According to this method, the lower solution is searched as

\[
u(t, x) = -\bar{u}(t)w_-(t, x),
\]

we \( \bar{u}(t) \) function defined by formula (20).

Then for \( Aw_- \) from (14) we have

\[
Aw_- = -\frac{\partial w_-}{\partial t} + \frac{\partial}{\partial x} \left( D x_m \frac{\partial w_-}{\partial x} \right) + \bar{u}(t)k(t)w_-(1 - w_-^\beta) = 0.
\]

For the function \( w_0 = (T + t)^{-\alpha/2} e^{-\frac{x^2}{4D(T + t)}} \)

\[
A w_0 = u(t)k(t)w_0(1 - w_0^\beta) \geq 0 \text{ in } Q, \text{ if constant } T \geq 1.
\]

Then applying the decision comparison theorem [3] by virtue of (20) we have

\[
u(t, x) \geq -\bar{u}(t)(T + t)^{-\alpha/2} e^{-\frac{x^2}{4D(T + t)}}
\]

which, in view of (18), proves the validity of theorem 2.

Theorem 2 is proved.

Consider a problem of Kolmogorov-Fisher type with a power nonlinearity.

\[
Au \equiv -\frac{\partial u}{\partial t} + D \Delta u + k(t)u(1 - u^\beta) \text{ in } t > 0, x \in \mathbb{R}^N,
\]

\[
u \mid_{t=0} = u_0 (x), \ x \in \mathbb{R}^N.
\]

It is suggested that \( \beta \geq 1, \ 0 \leq u_0 (x) \leq 1, \ 0 < k(t) \in C(0, \infty) \). To solve this problem is fair

Theorem 3. Let \( 0 \leq u_0 (x) \leq 1, \ x \in \mathbb{R}^N \). Then to solve the problem (21), (22) \( u(t, x) \) in \( Q \) we have an assessment

\[
-\bar{u}(t)(T_0 + t)^{-N/2} e^{-\frac{h^T}{4d(T + t)}} \leq u(t, x) \leq e^{\int_{k(t)d(t)}^{} w_+ (t, x)} (T_0 + t)^{-N/2} e^{-\frac{h^T}{4d(T + t)}}
\]

where \( \bar{u}(t) \) - the solution of equation (21) without the diffusion part, and the constant \( T_0 \geq 1 \).
Proof.
We note that the last estimate (23) of the solution to problem (21), (22) differs from the previous estimate in Theorem 2 in that instead of (24) in this estimate, the dimension is \( N \) and other \( \overline{u}(t) \), given above.

Theorem 3 is proved.

Study various properties of the solution of the Kolmogorov-Petrovsky-Piskunov-Fisher task with diffusion with reaction applied to the study of biological populations attracts the attention of many researchers, as it will be of interest from the point of view of mathematics as a nonlinear, which has an important application task. Therefore, it’s important generalisation this task on different cases. In particular, it has not been studied in case of a heterogeneous environment (diffusion coefficient is a function of the spatial variable), reaction coefficient depends on time or has a more complicated character. Consider the task of reaction diffusion of Kolmogorov – Fisher type

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \phi \left[ \left( \frac{\partial u}{\partial x} \right)^{p-2} \frac{\partial u}{\partial x} \right] \right) + k(t)u(1-u^p),
\]

(24)

\[
u |_{t=0} = u_0(x), x \in \mathbb{R}^N, \sup_{x} u_0(x) < +\infty,
\]

(25)

describing the process of diffusion-reaction in heterogeneous (anisotropic) environment with heterogeneity coefficient \( \chi \), where \( 0 \neq n \in \mathbb{R} \).

Propose an algorithm for constructing upper solutions of the Cauchy problem (16),(17) by the method of nonlinear splitting and etalon equations.

For the solutions of the Cauchy task (16),(17) fair following theorem:

Theorem 4. Suppose that in (24) performed following conditions

\[
u(0,x) \leq z(0,x), \text{ where } z(t,x) = \overline{u}(t) \cdot f(\xi),
\]

\( \overline{u}(t) \) is a solution of the equation \( \overline{u}_t = k(t)\overline{u}(1-\overline{u}^p) \),

\[
f(\xi) = \left( a - b \xi^p \right)^{p-1} \left[ \frac{\phi(\xi)}{\phi(x)} \right]^\frac{1}{p}, \text{ where } \phi(x) = 2^{2-m} x^\frac{2-m}{p}, a>0, b = \frac{a \tau^{\frac{1}{p-1}}}{\phi(x)^{\frac{1}{p}}}.
\]

Then the problem (24),(25) is globally solvable and for the solution in the field of \( Q \) take place following estimation \( u(t,x) \leq z(t,x) \).

Proof is similar to Theorem 1.

Presence of coefficient \( \ln \) in (24) makes difficult of study the task (24), (25) in terms of building self-similar or some partial solutions, as well as numerical modeling. Turning to the new variables, assuming \( u(t,x) = w(t,\phi(x)) \) and choosing \( \phi(x) \), reduce (16) to a form which in the case \( \geq \beta \) previously more detail studied by many authors [1], [5], [6].

4. Conclusion

Thus, the proposed nonlinear mathematical model of a biological population with a double nonlinearity correctly reflects the process under study. A self-similar equation is constructed in order to study an equation that is simpler for research. The self-similar equation is constructed by the nonlinear splitting method [7]. Two-sided estimates are obtained for the solution of equations with a double nonlinearity. An analysis of the results based on the resulting estimates of the solutions provides a comprehensive picture of the process.
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