THREE NON-EQUIVALENT REALIZATIONS OF THE
ASSOCIAHEDRON

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ABSTRACT. We review three realizations of the associahedron that arise as secondary
polytopes, from cluster algebras, and as Minkowski sums of simplices, and show that
under any choice of parameters, the resulting associahedra are affinely non-equivalent.
Note: the results of this preprint have been included in a more comprehensive paper,
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1. INTRODUCTION

The associahedron, also known as the Stasheff polytope, is a “mythical” simple polytope
that was first described as a combinatorial object by Stasheff in 1963, and was used to
explore associativity of $H$-spaces. Three “conceptual” constructions of the associahedron
as a polytope, among numerous others, are: the associahedron as a secondary polytope due
to Gelfand, Zelevinsky and Kapranov [11] [12] (see also [10, Chap. 7]), the associahedron
associated to the cluster complex of type $A_n$ due to Chapoton, Fomin and Zelevinsky [4],
and the associahedron as a Minkowski sum of simplices introduced by Postnikov in [16].
Each one of these realizations depends on a large number of parameters that, a priori,
might be chosen appropriately so that the three constructions produce equivalent objects.
The main result of this paper is to show that regardless of how the parameters are chosen,
the three realizations are affinely non-equivalent.

The associahedron as a secondary polytope

The associahedron from the cluster complex of type $A_n$.

The associahedron as a Minkowski sum of simplices.

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The paper is organized as follows. Section 2 starts with a quick review of associahedra. In Section 2.1 we review the construction of the associahedron as a secondary polytope of a convex polygon and prove in Theorem 2.6 that it has no pairs of parallel facets. Section 2.2 is a review of the construction of associahedra using the cluster complex of type $A_n$, which yields an $n$-dimensional associahedron with $n$ pairs of parallel facets. Theorem 2.12 describes these pairs in terms of pairs of diagonals of an ($n + 3$)-gon. In Section 2.3 we review the construction of the $n$-dimensional associahedron as a Minkowski sum of simplices, and provide in Theorem 2.15 a precise description of a correspondence between faces of this polytope and subdivisions of an ($n + 3$)-gon. We prove in Theorem 2.16 that this associahedron has $n$ pairs of parallel facets, and we identify the corresponding pairs of diagonals of an ($n + 3$)-gon. Finally, in Section 3 we show that all three types of realizations are affinely non-equivalent for any choice of parameters.

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2. Three realizations of the associahedron

We start by recalling the definition of an $n$-dimensional associahedron in terms of polyhedral subdivisions of an ($n + 3$)-gon.

Definition 2.1. The associahedron $Ass_n$ is an $n$-dimensional simple polytope whose face lattice is isomorphic to the lattice of polyhedral subdivisions (without new vertices) of a convex ($n + 3$)-gon ordered by refinement.

Figure 1 illustrates the polytope $Ass_n$ for $n = 0, 1$ and 2. Such a polytope exists for every $n \geq 0$. We will present three constructions for it, which will be shown to be essentially different in the last section of this paper.

2.1. Construction I: The associahedron as a secondary polytope. The secondary polytope is an ingenious construction motivated by the theory of hypergeometric functions as developed by I.M. Gelfand, M. Kapranov and A. Zelevinsky [10]. In this section we recall the basic definitions and main results related to this topic, which yield in particular that the secondary polytope of any convex ($n + 3$)-gon is an $n$-dimensional associahedron. For more detailed presentations we refer to [18, Lect 9] and [5, §5].
2.1.1. The secondary polytope construction.

**Definition 2.2** (GKZ vector/secondary polytope). Let $Q$ be a $d$-dimensional convex polytope with $n$ vertices. The *GKZ vector* $v(t) \in \mathbb{R}^n$ of a triangulation $t$ of $Q$ is

$$v(t) := \sum_{i=1}^{n} \text{vol}(\text{star}_t(i))e_i = \sum_{i=1}^{n} \sum_{\sigma \in t : i \in \sigma} \text{vol}(\sigma)e_i$$

The *secondary polytope* of $Q$ is defined as

$$\Sigma(Q) := \text{conv}\{v(t) : t \text{ is a triangulation of } Q\}.$$  

**Proposition 2.3** (Gelfand–Kapranov–Zelevinsky [11]). Let $Q$ be a $d$-dimensional convex polytope with $n$ vertices. Then the secondary polytope $\Sigma(Q)$ has the following properties:

1. $\Sigma(Q)$ is an $(n - d - 1)$-dimensional polytope.
2. The vertices of $\Sigma(Q)$ are in bijective correspondence with the regular triangulations of $Q$ without new vertices.
3. The faces of $\Sigma(Q)$ are in bijective correspondence with the regular subdivisions of $Q$.
4. The face lattice of $\Sigma(Q)$ is isomorphic to the lattice of regular subdivisions of $Q$, ordered by refinement.

2.1.2. The associahedron as the secondary polytope of a convex $(n+3)$-gon.

**Definition 2.4.** $\text{Ass}^I_n(Q) \subset \mathbb{R}^{n+3}$ is defined as the $(n$-dimensional) secondary polytope of a convex $(n+3)$-gon $Q \subset \mathbb{R}^2$:

$$\text{Ass}^I_n(Q) := \Sigma(Q).$$

**Proposition 2.5** (GKZ). $\text{Ass}^I_n(Q)$ is an $n$-dimensional associahedron.

**Theorem 2.6.** $\text{Ass}^I_n(Q)$ has no parallel facets for $n \geq 2$.

*Proof.* The polytope $\text{Ass}^I_n(Q)$ satisfies the following correspondence:

- vertices $\leftrightarrow$ triangulations of $Q$
- facets $\leftrightarrow$ diagonals of $Q$

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**Figure 1.** The associahedron $\text{Ass}_n$ for $n = 1, 2$ and $3$. 

![Associahedron diagram](image-url)
For a given diagonal $\delta$ of $Q$, we denote by $F_\delta$ the facet of $\text{Ass}_n^1(Q)$ corresponding to $\delta$, and we define the $(n-1)$-dimensional subspace $V_\delta \subset \mathbb{R}^{n+3}$ by
\[
V_\delta = \text{span}\{u - v : u, v \text{ are vertices of } F_\delta\}.
\]
Thus for diagonals $\delta$ and $\delta'$ of $Q$ we have
\[
F_\delta \text{ is parallel to } F_{\delta'} \text{ if and only if } V_\delta = V_{\delta'}.
\]
Given two different diagonals $\delta$ and $\delta'$ of $Q$ there are two cases:

Case 1. $\delta$ and $\delta'$ do not cross, i.e. they do not intersect in an interior point of $Q$. In this case, $F_\delta$ and $F_{\delta'}$ intersect on the face of $\text{Ass}_n^1(Q)$ corresponding to the subdivision generated by the two non-crossing diagonals $\{\delta, \delta'\}$, and so $F_\delta$ and $F_{\delta'}$ are not parallel.

Case 2. $\delta$ and $\delta'$ cross in an interior point of $Q$. In this case, using $n+3 \geq 5$, we can relabel the vertices of $Q$ in either clockwise or counterclockwise direction so that $1 \in \delta$, $2 \notin \delta'$, so $\delta = \{1, j\}$ and $\delta' = \{i', j'\}$ with $3 \leq i' < j < j' \leq n + 3$. Thus there is a triangulation $t' = (\delta_1', \delta_2', \ldots, \delta_n')$ of $Q$ with $\delta_1' = \{2, n + 3\}$, $\delta_2' = \{3, n + 3\}$ and $\delta' \in \{\delta_2', \delta_3', \ldots, \delta_n'\}$ (see left hand side of Figure 2).

Let $t'_1$ be the triangulation obtained by flipping the diagonal $\delta'_1$ of $t'$ (to $\{1,3\}$), and let $w = v(t'_1) - v(t')$. Since the triangulations $t'_1$ and $t'$ contain the diagonal $\delta'$, the corresponding vertices $v(t'_1)$ and $v(t')$ are both vertices of $F_{\delta'}$, which implies that
\[
w \in V_{\delta'}.
\]
On the other hand, we prove that $w \notin V_\delta$. For this, let $t$ be the triangulation of $Q$ determined by the set of non-crossing diagonals of the form $\{k, j\}$ with $k \in S$, where $S = \{k \in [n + 3] : k \neq j - 1, j, j + 1\}$ (as shown in the right hand side of Figure 2), and let

\[\text{Figure 2.} \text{ Diagonals mentioned in Case 2.}\]
be the triangulation obtained by flipping the diagonal \( \{k, j\} \) of \( t \). We denote by \( v(t) \) and \( v(t_{k,j}) \) the vertices of \( \Ass^n \) corresponding to \( t \) and \( t_{k,j} \), respectively. Then we have

\[
V_\delta = \text{span}\{u_k = v(t_{k,j}) - v(t) : k \in S \text{ and } k \neq 1\},
\]

Suppose that \( w \in V_\delta \). Then \( w \) can be written as a linear combination

\[
w = \sum_{k \in S \setminus \{1\}} c_k u_k
\]

Definition \( \ref{def:triangulation} \) yields that each of the vectors \( w \) and \( u_k \) has exactly 4 non-zero coordinates. More precisely, if \( w_\ell \) is the \( \ell \)-coordinate of \( w \) and \( u_\ell \) is the \( \ell \)-coordinate of \( u_k \), then

- \( w_\ell \neq 0 \) for \( \ell \in \{1, 2, 3, n + 3\} \) and \( w_\ell = 0 \) otherwise.
- \( u^n_{n+3} \neq 0 \) for \( \ell \in \{n + 2, n + 3, 1, j\} \) and \( u^n_{n+3} = 0 \) otherwise.
- for \( k \in S \setminus \{1, n + 3\} \), \( u_k \neq 0 \) for \( \ell \in \{k - 1, k, k + 1, j\} \) and \( u_k = 0 \) otherwise.

We will represent the system of linear equations (1) in matrix form. The columns of the matrix are the vectors \( \{u_k\}_{k \in S \setminus \{1\}} \); \( c = (c_k)_{k \in S \setminus \{1\}} \) is the vector of coefficients. The symbol \( * \) represents non-zero entries.

\[
\begin{pmatrix}
  u_2 & u_3 \\
  u_{j-2} & u_{j+3} & u_{n+2} & u_{n+3} \\
  0 & * & 0 & 0 \\
  0 & * & 0 & 0 \\
  0 & * & 0 & 0 \\
  0 & * & 0 & 0 \\
  0 & * & 0 & 0 \\
  0 & * & 0 & 0 \\
  0 & * & 0 & 0 \\
  0 & * & 0 & 0 \\
\end{pmatrix}
\]

One can easily see that the coefficients \( c_k \) for \( j + 2 \leq k \leq n + 3 \) must be equal to 0, and similarly for \( j - 2 \geq k \geq 3 \). Thus \( w = c_2 u_2 \), which yields a contradiction.

Thus we know that \( w \in V_\delta \) but \( w \notin V_{\delta'} \), so \( V_\delta \neq V_{\delta'} \), and hence \( F_\delta \) and \( F_{\delta'} \) are not parallel. \( \square \)
2.2. Construction II: The associahedron associated to a cluster complex. Cluster complexes are combinatorial objects that arose in the theory of cluster algebras [7] [8]. They correspond to the normal fans polytopes known as generalized associahedra. In this section we introduce the particular case of type $A_n$, which is related to the classical associahedron. Most of this theory was introduced by S. Fomin and A. Zelevinsky and can be found in [9], [6] and [4].

2.2.1. The cluster complex of type $A_n$. The root system of type $A_n$ is the set $\Phi := \Phi(A_n) = \{e_i - e_j, 1 \leq i \neq j \leq n + 1\}$. The simple roots of type $A_n$ are the elements of the set $\Pi = \{\alpha_i = e_i - e_{i+1}, i \in [n]\}$, the set of positive roots is $\Phi_{>0} = \{e_i - e_j : i < j\}$, and the set of almost positive roots is $\Phi_{\geq-1} := \Phi_{>0} \cup -\Pi$.

There is a natural correspondence between the set $\Phi_{\geq-1}$ and diagonals of the $(n+3)$-gon $P_{n+3}$: We identify the negative simple roots $-\alpha_i$ with the diagonals on the snake of $P_{n+3}$ as illustrated in Figure 3. Each positive root is a consecutive combination

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j, \quad 1 \leq i \leq j \leq n,$$

and thus is identified with the unique diagonal of $P_{n+3}$ crossing the (consecutive) diagonals that correspond to $-\alpha_i, -\alpha_{i+1}, \cdots, -\alpha_j$.

**Definition 2.7** (Cluster complex of type $A_n$). Two roots $\alpha$ and $\beta$ in $\Phi_{\geq-1}$ are compatible if their corresponding diagonals do not cross. The cluster complex $\Delta(\Phi)$ of type $A_n$ is the clique complex of the compatibility relation on $\Phi_{\geq-1}$, i.e., the complex whose simplices correspond to the sets of almost positive roots that are pairwise compatible. Maximal simplices of $\Delta(\Phi)$ are called clusters.
In this case, the cluster complex satisfies the following correspondence, which is dual to the complex of the classical associahedron $\text{Ass}_n$:

- vertices $\leftrightarrow$ diagonals of a convex $(n+3)$-gon
- simplices $\leftrightarrow$ polyhedral subdivisions of the $(n+3)$-gon (viewed as collections of non-crossing diagonals)
- maximal simplices $\leftrightarrow$ triangulations of the $(n+3)$-gon (viewed as collections of $n$ non-crossing diagonals)

The following proposition is the particular case of type $A_n$ of [9, Thm. 1.10]. It allows us to think of the cluster complex as the complex of a complete simplicial fan.

**Proposition 2.8.** The simplicial cones $R_{\geq 0}C$ generated by all clusters $C$ of type $A_n$ form a complete simplicial fan in the ambient space $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1 + \cdots + x_{n+1} = 0\}$.

**Proposition 2.9.** The simplicial fan in Proposition 2.8 is the normal fan of a simple $n$-dimensional polytope $P$.

Proposition 2.9 was first conjectured by Fomin and Zelevinsky [9, Conj. 1.12] and proved by Chapoton, Fomin, and Zelevinsky [4]. For an explicit description of such a polytope by inequalities see [4, Cor. 1.9].

2.2.2. *The associahedron $\text{Ass}^\Pi_n(A_n)$.*

**Definition 2.10.** $\text{Ass}^\Pi_n(A_n)$ is any polytope whose normal fan is the fan with maximal cones $R_{\geq 0}C$ generated by all clusters $C$ of type $A_n$.

**Proposition 2.11.** $\text{Ass}^\Pi_n(A_n)$ is an $n$-dimensional associahedron.

A polytopal realization of the associahedron $\text{Ass}^\Pi_n(A_n)$ is illustrated in Figure 4; note how the facet normals correspond to the almost positive roots of $A_2$.

![Figure 4](image-url)

**Figure 4.** The simplicial fan of the cluster complex of type $A_2$ and the associahedron $\text{Ass}(A_2)$

**Theorem 2.12.** $\text{Ass}^\Pi_n(A_n)$ has exactly $n$ pairs of parallel facets. These correspond to the pairs of roots $\{\alpha_i, -\alpha_i\}$, for $i = 1, \cdots, n$, or equivalently, to the pairs of diagonals $\{\alpha_i, -\alpha_i\}$ as indicated in Figure 5.
2.3. Construction III: The associahedron as a Minkowski sum of simplices. The realization of the associahedron as a Minkowski sum of simplices generalizes Loday’s realization introduced in [15], and is a special case of generalized permutahedra, an important family of polytopes studied by Postnikov [16]. Loday’s realization is also equivalent to the associahedra described recently by Buchstaber [3].

Definition 2.13. For any vector \( a = \{a_{ij} > 0 : 1 \leq i \leq j \leq n+1 \} \) of positive parameters let

\[
\text{Ass}_{n}^{\text{III}}(a) := \sum_{1 \leq i \leq j \leq n+1} a_{ij} \Delta_{[i,...,j]},
\]

where \( \Delta_{[i,...,j]} \) denotes the simplex \( \text{conv}\{e_i, e_{i+1}, \ldots, e_j\} \) in \( \mathbb{R}^{n+1} \).

Proposition 2.14 (Postnikov [16, §8.2]). \( \text{Ass}_{n}^{\text{III}}(a) \) is an \( n \)-dimensional associahedron. In particular, for \( a_{ij} = 1 \) this yields the realization of Loday [15].

In this section we present two main results. Theorem 2.15 explains a correspondence between faces of \( \text{Ass}_{n}^{\text{III}}(a) \) and subdivisions of a convex \( (n+3) \)-gon, and Theorem 2.16 is a combinatorial description of the pairs of parallel facets of \( \text{Ass}_{n}^{\text{III}}(a) \).

2.3.1. Correspondence between faces of the polytope and subdivisions of an \( (n+3) \)-gon. Each face of \( \text{Ass}_{n}^{\text{III}}(a) \) can be represented as the set of points in \( \text{Ass}_{n}^{\text{III}}(a) \) maximizing a linear function. Figure 6 illustrates an example of a natural correspondence between faces of \( \text{Ass}_{n}^{\text{III}}(a) \) maximizing a linear function, and subdivisions of an \( (n+3) \)-gon. This correspondence works as follows. Let \( w = (w_1, \ldots, w_{n+1}) \in (\mathbb{R}^{n+1})^* \) be a linear function and \( F_w \) be the face of \( \text{Ass}_{n}^{\text{III}}(a) \) that maximizes it. The subdivision \( S_w \) that correspond to \( F_w \) is a subdivision of an \( (n+3) \)-gon with vertices labeled in counterclockwise direction from 0 to \( n+2 \); it is obtained as the common refinement of the subdivisions that are induced by the polygons \( H_k, k = 0, \ldots, n+2, \) where \( H_k = \text{conv}\{i \in \{0, \ldots, n+2\} : w_i \geq w_k\} \) with \( w_0 = w_{n+2} = \infty \) (see Figure 6).
The face $F_w$ of $\text{Ass}^{III}_7(\mathbf{a})$ that maximizes the linear function $w = (w_1, \ldots, w_8)$ with $w_1 = w_4 = w_7 > w_2 = w_6 > w_3 = w_5 = w_8$.

Figure 6. Example of the correspondence between faces of $\text{Ass}^{III}_7(\mathbf{a})$ maximizing a linear function and subdivisions of a convex (7 + 3)-gon.

**Theorem 2.15.** Let $L\text{Ass}^{III}_n(\mathbf{a})$ be the face lattice of $\text{Ass}^{III}_n(\mathbf{a})$ and $L\text{Ass}_n$ be the lattice of subdivisions of a convex $(n + 3)$-gon ordered by refinement. Then, the correspondence

$$\varphi : L\text{Ass}^{III}_n(\mathbf{a}) \rightarrow L\text{Ass}_n$$

$$F_w \rightarrow S_w$$

defines an order-preserving bijection. In particular, $\text{Ass}^{III}_n(\mathbf{a})$ is an associahedron.

2.3.2. Combinatorial description of the pairs of parallel facets.

**Theorem 2.16.** $\text{Ass}^{III}_n(\mathbf{a})$ has $n$ pairs of parallel facets. They correspond to the pairs of diagonals ($\{n + 2, i\}, \{0, i + 1\}$) for $1 \leq i \leq n$, as illustrated in Figure 7.

![Diagram](image)

Figure 7. Diagonals of the $(n+3)$-gon corresponding to the pairs of parallel facets of $\text{Ass}^{III}_n(\mathbf{a})$.

**Proof.** A convex $(n + 3)$-gon has three types of diagonals:

1. The diagonals of the form $\{n + 2, i\}$, $1 \leq i \leq n$. They correspond to facets of $\text{Ass}^{III}_n(\mathbf{a})$ maximizing the linear functions $w = (w_1, \ldots, w_{n+1})$ with $w_1 = \cdots = w_i > w_{i+1} = \cdots = w_{n+1}$. These facets are parallel to the affine spaces generated by $\Delta_{[1, \ldots, i]} + \Delta_{[i+1, \ldots, n+1]}$. 

2. [Further details...]

3. [Further details...]

4. [Further details...]

5. [Further details...]


Theorem 3.1. For polytopes, for all choices of parameters.

This cannot be achieved at all: The three constructions produce affinely non-equivalent results. The main results of this paper, given by Theorem 3.1 and Theorem 3.2, show that parameters in an appropriate way so that these constructions would produce the same descriptions of the pairs of parallel facets. Each of the three constructions depends on a large number of parameters:

- \( \text{Ass}_{n}^{I}(Q) \) depends on \( 2(n+3) \) parameters corresponding to the coordinates of the convex \( (n+3) \)-gon \( Q \subset \mathbb{R}^{2} \).
- \( \text{Ass}_{n}^{II}(A_{n}) \) depends on \( \binom{n+1}{2} + n = \frac{n(n+3)}{2} \) parameters corresponding to the almost positive roots of type \( A_{n} \), which determine the distances of the corresponding hyperplanes from the origin.
- \( \text{Ass}_{n}^{III}(a) \) depends on \( \binom{n+2}{2} \) parameters corresponding to the values of \( a_{ij} \), \( 1 \leq i \leq j \leq n+1 \).

One may wonder (and the second author has repeatedly asked) how can one choose the parameters in an appropriate way so that these constructions would produce the same result. The main results of this paper, given by Theorem 3.1 and Theorem 3.2 show that this cannot be achieved at all: The three constructions produce affinely non-equivalent polytopes, for all choices of parameters.

**Theorem 3.1.** For \( n \geq 2 \), \( \text{Ass}_{n}^{I}(Q) \) is affinely equivalent to neither \( \text{Ass}_{n}^{II}(A_{n}) \) nor \( \text{Ass}_{n}^{III}(a) \).

**Proof.** By Theorem 2.6, \( \text{Ass}_{n}^{I}(Q) \) has no pairs of parallel facets, while \( \text{Ass}_{n}^{II}(A_{n}) \) and \( \text{Ass}_{n}^{III}(a) \) do. Since affine maps preserve pairs of parallel facets, \( \text{Ass}_{n}^{I}(Q) \) is affinely equivalent to neither \( \text{Ass}_{n}^{II}(A_{n}) \) nor \( \text{Ass}_{n}^{III}(a) \).

**Theorem 3.2.** \( \text{Ass}_{n}^{II}(A_{n}) \) and \( \text{Ass}_{n}^{III}(a) \) are not affinely equivalent for \( n \geq 3 \).

**Proof.** We say that a facet \( F \) of a polytope \( P \) is special if there is an other facet \( F' \) of \( P \) which is parallel to \( F \). Theorem 3.2 follows from the following lemma.

**Lemma 3.3.**

1. \( \text{Ass}_{n}^{II}(A_{n}) \) and \( \text{Ass}_{n}^{III}(a) \) both have 2\( n \) special facets.
2. There are two special facets of \( \text{Ass}_{n}^{III}(a) \) which intersect exactly \( n-1 \) other special facets.
3. For \( n > 3 \), every special facet of \( \text{Ass}_{n}^{II}(A_{n}) \) intersects more than \( n-1 \) special facets.

For \( n = 3 \), there is only one facet which intersects exactly \( n-1 = 2 \) special facets.
Proof.

(1) This has already been proved in Theorem 2.12 and Theorem 2.16.

(2) Two special facets intersect if and only if the corresponding special diagonals do not cross. Figure 7 shows the special diagonals associated to the construction of Ass$_n^{III}$($\mathbf{3}a$).

Among these, \{n + 2, 1\} and \{0, n + 1\} have the property that they do not cross exactly n − 1 other special diagonals. Therefore, the two special facets of Ass$_n^{III}$($\mathbf{3}a$) corresponding to these two diagonals satisfy the desired condition.

(3) The special facets of Ass$_n^{II}$($\mathbf{3}A_n$) correspond to the diagonals of the form $\alpha_i$ or $-\alpha_i$, $i = 1, \ldots, n$, that are shown in Figure 5. A diagonal $-\alpha_i$ does not cross the 2(n − 1) diagonals of the form $\alpha_j$ or $-\alpha_j$ with $j \neq i$. On the other hand, when $n > 3$, a diagonal $\alpha_i$ does not cross the $n − 1$ diagonals $-\alpha_j$ with $j \neq i$ and neither one between $\alpha_1$ and $\alpha_n$. Therefore, when $n > 3$ any special facet of Ass$_n^{II}$($\mathbf{3}A_n$) intersects more than $n − 1$ other special facets. In the case $n = 3$, the facet corresponding to the diagonal $\alpha_2$ is the only special facet that intersects exactly $n − 1 = 2$ other special facets. □

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