Solvable subgroups of $Out(F_n)$ are virtually abelian

Mladen Bestvina, Mark Feighn, and Michael Handel*

June 1997

1 Introduction

Let $F_n$ denote a free group of rank $n$. The group $Out(F_n)$ contains mapping class groups of compact surfaces and maps onto $GL(n, \mathbb{Z})$. It is perhaps not surprising that $Out(F_n)$ behaves at times like a mapping class group and at times like a linear group. J. Birman, A. Lubotzky, and J. McCarthy [BLMS3] showed that solvable subgroups of mapping class groups are finitely generated and virtually abelian. Of course, $GL(3, \mathbb{Z})$ contains the Heisenberg group which is solvable but not virtually abelian. In this paper, we show that, with respect to the nature of solvable subgroups, $Out(F_n)$ behaves more like mapping class groups.

Theorem 1.1. Every solvable subgroup of $Out(F_n)$ has a subgroup of index at most $3^{5n^2}$ that is finitely generated and free abelian.

The rank of an abelian subgroup of $Out(F_n)$ is bounded by $vcd(Out(F_n)) = 2n - 3$ for $n > 1$ [CV86].

Since $Aut(F_n)$ embeds in $Out(F_{n+1})$, solvable subgroups of $Aut(F_n)$ are also virtually abelian. Theorem [1] complements [BFH96b] where we show that $Out(F_n)$ satisfies the Tits Alternative, i.e. that subgroups of $Out(F_n)$ are either virtually solvable or contain a free group of rank 2.

H. Bass and A. Lubotzky [BL94] have shown that solvable subgroups of $Out(F_n)$ are virtually polycyclic. In particular, they are finitely generated. We include an independent proof of this fact for completeness and because the ingredients of our proof are needed for the proof of Theorem [1].

The starting point for this paper is a short exact sequence from [BFH96a] and [BFH96b]. Begin with a solvable subgroup $\mathcal{H}$ of $Out(F_n)$. After passing to a finite index subgroup we may assume that $\mathcal{H}$ acts trivially on $H_1(F_n; \mathbb{Z}/3\mathbb{Z})$. By Theorem 8.1 of [BFH96a] and Proposition 3.5 of [BFH96b] there is an exact sequence

$$1 \to \mathcal{H}_0 \to \mathcal{H} \to \mathbb{Z}^b \to 1$$

*All three authors gratefully acknowledge the support of the National Science Foundation.
where \( \mathcal{H}_0 \) is UPG (defined below).

There are two parts to the proof of Theorem 1.1. First we show that \( \mathcal{H}_0 \) is abelian by constructing an embedding \( \Phi : \mathcal{H}_0 \to \mathbb{Z}^r \). Then we show that \( \Phi \) extends to a homomorphism \( \Phi' : \mathcal{H} \to \mathbb{Z}^r \). The direct sum of \( \Omega \) and \( \Phi' \) is an embedding of \( \mathcal{H} \) into \( \mathbb{Z}^{b+r} \) showing that \( \mathcal{H} \) is finitely generated and free abelian.

Our approach is motivated by the special case that \( \mathcal{H} \) is realized as a subgroup of the mapping class group of a compact surface \( S \). The surface \( S \) decomposes into a union of annuli \( A_1, \ldots, A_r \) and subsurfaces \( S_i \) of negative Euler characteristic; virtually every \( \eta \in \mathcal{H}_0 \) is represented by a homeomorphism \( f : S \to S \) that restricts to a Dehn twist on each \( A_j \) and that preserves each \( S_i \). If \( \eta \in \mathcal{H}_0 \) then each \( f|S_i \) is the identity. The homomorphisms \( \Phi \) and \( \Phi' \) are defined by taking their \( j \)th coordinates to be the number of twists that occurs across \( A_j \). For a further discussion of the geometric case, see Example 3.2.

We study an outer automorphism \( \eta \) through its lifts \( \hat{\eta} : C_\infty \to C_\infty \) to the Cantor set at infinity \( C_\infty \) or equivalently (see Subsection 2.1) through the automorphisms \( \hat{\Phi} : F_n \to F_n \) that represent \( \eta \). The \( C_\infty \) point of view simplifies certain proofs because it allows us to consider fixed ‘directions’ in \( F_n \) that are not periodic and therefore do not come from fixed elements of \( F_n \).

In the course of proving Theorem 1.1 we prove the following result which is of independent interest. We state it here in terms of automorphisms although we prove it in terms of \( C_\infty \).

**Proposition 1.2.** Every abelian subgroup \( \mathcal{H} \subset \text{Out}(F_n) \) has a virtual lift \( \hat{\mathcal{H}} \subset \text{Aut}(F_n) \). If \( \gamma \) is a non-trivial primitive element of \( F_n \) that is fixed, up to conjugacy, by each element of \( \mathcal{H} \) then \( \hat{\mathcal{H}} \) can be chosen so that each element of \( \hat{\mathcal{H}} \) fixes \( \gamma \).

The paper is organized as follows. In section 2 we establish notation and record known results for future reference. In sections 3 and 4 we prove that \( \mathcal{H}_0 \) is finitely generated and free abelian and that the above exact sequence is virtually central. In section 5 we prove Proposition 1.2 and in section 6 we prove Theorem 1.1.

### 2 Notation and Preliminaries

#### 2.1 Lifts to \( C_\infty \)

We assume that \( F_n \) is identified with \( \pi_1(R_n, \ast) \) where \( R_n \) is the rose with \( n \) petals and with vertex \( \ast \). Let \( \widehat{R}_n \) be the universal cover of \( R_n \) and let \( \hat{\ast} \) be a preferred lift of \( \ast \). The space of ends of \( \widehat{R}_n \) is a Cantor set that we denote \( C_\infty \).

A marked graph is a graph \( G \) with a preferred vertex \( v \) along with a homotopy equivalence \( \tau : (R_n, \ast) \to (G, v) \) that identifies \( \pi_1(G, v) \) with \( \pi_1(R_n, \ast) \) and so also with \( F_n \).
Denote the space of ends of $\Gamma$ by $E(\Gamma)$. The marking homotopy equivalence lifts to an equivariant map $\tilde{\tau} : (\tilde{\mathcal{R}}_n, \hat{\ast}) \to (\Gamma, \hat{v})$ that induces a homeomorphism from $C_\infty$ to $E(\Gamma)$ (See for example section 3.2 of [BFH96a]). We use this homeomorphism to identify $E(\Gamma)$ with $C_\infty$ and so for the rest of this paper refer to the space of ends of $\Gamma$ as $C_\infty$.

An outer automorphism $\eta$ of $F_n$ can be represented, in many ways, by a homotopy equivalence $f : G \to G$ of a marked graph. More precisely, $f : G \to G$ can be chosen so that when $\pi_1(G, v)$ is identified with $F_n$, the outer automorphism of $\pi_1(G, v)$ determined by $f$ agrees with $\eta$.

A preferred vertex $\hat{v}$ in the universal cover $\Gamma$ of $G$ provides an identification of the group $T$ of covering translations of $\Gamma$ with $\pi_1(G, v)$ and so with $F_n$. The action of $T$ on $\Gamma$ extends to an action (of $T$ group $Fix$) on $E(\Gamma)$ by homeomorphisms. If $T$ is non-trivial, then the endpoints of its axis are the only fixed points for the action of $T$ on $C_\infty$.

Suppose that $f : G \to G$ represents $\eta$ and that $\hat{f} : \Gamma \to \Gamma$ is a lift of $f$. For each $T \in T$ there exists a unique $T' \in T$ such that $\hat{f}T = T'\hat{f}$. This defines an automorphism $T \mapsto T'$ of $T$ and so an automorphism $\Phi : F_n \to F_n$. It is easy to check that the outer automorphism class of $\Phi$ is $\eta$ and that as $\hat{f}$ varies over all lifts of $f$, $\Phi$ varies over all automorphisms representing $\eta$. The subgroup $T(\hat{f}) \subset T$ of covering translations that commute with $\hat{f}$ corresponds to the fixed subgroup $Fix(\Phi) \subset F_n$. Since $Fix(\Phi)$ is quasiconvex and finitely generated, the closure in $C_\infty$ of the endpoints of axes of elements of $T(\hat{f})$ is identified with the space of ends of $Fix(\Phi)$ [Coo87].

Each $\hat{f} : \Gamma \to \Gamma$ extends to a homeomorphism $\hat{f} : C_\infty \to C_\infty$. Denote the group of $F_n$-equivariant homeomorphisms of $C_\infty$ by $EH(C_\infty)$. The composite $\Phi \mapsto \hat{f} \mapsto \hat{f}$ defines an injective homomorphism from $Aut(F_n)$ to $EH(C_\infty)$ that is independent of the choice of homotopy equivalence $f : G \to G$ representing $\eta$. (See for example section 3.2 of [BFH96a]). We will sometimes write $\hat{\eta}$ instead of $\hat{f}$ where $\eta \in Out(F_n)$ is the outer automorphism determined by $f : G \to G$.

If $\hat{L} \subset \Gamma$ is a line with endpoints $P$ and $Q$, we denote the line with endpoints $\hat{f}(P)$ and $\hat{f}(Q)$ by $\hat{f}_\#(\hat{L})$. If $\hat{L}$ is the axis of $T \in T$ then $\hat{f}_\#(\hat{L})$ is the axis of the covering translation $T' \in T$ satisfying $\hat{f}T = T'\hat{f}$.

We conclude this subsection by recording some facts for future reference. If $\hat{\mathcal{H}} \subset Aut(F_n)$ is a lift of $\mathcal{H} \subset Out(F_n)$ then we denote the corresponding lift to $EH(C_\infty)$ by $\hat{\mathcal{H}}$. For any subgroup $\mathbb{F} \subset F_n$, the closure in $C_\infty$ of the endpoints of the axes of elements in $\mathbb{F}$ is denoted by $C(\mathbb{F})$ and is naturally identified with the space of ends of $\mathbb{F}$. Although the UPG property referred to in the following lemma is not defined until the next section, it is convenient to place this result here. This part of the lemma is quoted from [BFH96b] so there is no danger of circular reasoning.

**Lemma 2.1.** Suppose that $\mathcal{H}$ is a subgroup of $Out(F_n)$ and that $\mathbb{F}$ is an $\mathcal{H}$-invariant (up to conjugacy) subgroup of $F_n$ that is its own normalizer. Then

1. There is a well-defined restriction $\mathcal{H}|\mathbb{F} \subset Out(\mathbb{F})$.
2. If $\mathcal{H}$ is UPG, then $\mathcal{H}|\mathbb{F}$ is UPG.
3. If $F$ has rank at least two, then any lift $\hat{H}|F \subset \text{Aut}(F)$ of $H|F$ extends uniquely to a lift $\hat{H} \subset \text{Aut}(F_n)$.

4. If every element of a lift $\hat{H}|F \subset \text{EH}(C(F))$ of $H|F$ fixes at least three points then $\hat{H}|F$ extends uniquely to a lift $\hat{H} \subset \text{EH}(C_\infty)$.

**Proof of Lemma 2.1** Since $F$ is its own normalizer, two automorphisms of $F_n$ that preserve $F$ are conjugate by an element of $F_n$ if and only if they are conjugate by an element of $F$. Part (1) follows immediately. Part (2) is Lemma 4.13 of [BFH96b]. If $F$ has rank at least two, then there are no non-trivial inner automorphisms that pointwise fix $F$. It follows that the restriction homomorphism from the subgroup of $\text{Aut}(F_n)$ representing $\eta \in \text{Out}(F_n)$ to the subgroup of $\text{Aut}(F)$ representing $\eta|F$ is injective. Part (3) follows easily. For part (4), note that a non-trivial covering translation does not fix more than two points in $C_\infty$ and so two lifts $\hat{\eta}_1$ and $\hat{\eta}_2$ of $\eta$ that agree on three points must be equal. \hfill \Box

### 2.2 Relative train track maps

We study an element $\eta \in \text{Out}(F_n)$ through its lifts $\hat{\eta} : C_\infty \to C_\infty$. We analyze the $\hat{\eta}$’s by representing $\eta$ as a homotopy equivalence $f : G \to G$ of a marked graph with particularly nice properties and studying the corresponding $\hat{f}$’s. In this subsection we recall some properties of $f : G \to G$.

A **filtered graph** is a marked graph along with a filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_K = G$ where each $G_i$ is obtained from $G_{i-1}$ by adding a single edge $E_i$. We reserve the words path and loop for immersions of the interval and the circle respectively. If $\rho$ is a map of the interval or the circle into $G$ or $\Gamma$, then $[\rho]$ is the unique path or loop that is homotopic to $\rho$ rel endpoints if any. We say that a homotopy equivalence $f : G \to G$ respects the filtration if each $f(E_i) = E_i u_{i,f}$ for some loop $u_{i,f} \subset G_{i-1}$. Let $\text{FHE}(G, \mathcal{V})$ be the group (Lemma 6.2 of [BFH96b]) of homotopy classes, relative to vertices, of filtration respecting homotopy equivalences of $G$. There is a natural map from $\text{FHE}(G, \mathcal{V})$ to $\text{Out}(F_n)$. We say that $\eta \in \text{Out}(F_n)$ is $\text{UPG}$ (for unipotent with polynomial growth) if it is in the image of $\text{FHE}(G, \mathcal{V})$ for some $G$. We say that a subgroup of $\text{Out}(F_n)$ is $\text{UPG}$ if each of its elements is. The main theorem of [BFH96b] states that every $\text{UPG}$ subgroup $\mathcal{H}_0$ lifts to a subgroup $\mathcal{K}$ of $\text{FHE}(G)$ for some $G$. We say that $\mathcal{K}$ is a **Kolchin representative** of $\mathcal{H}_0$.

In general we will use $\mathcal{K}$ to denote a subgroup of $\text{FHE}(G)$.

If $\tilde{L} \subset \Gamma$ is a line and $l$ is the highest parameter value for which $\tilde{L}$ crosses a lift of $E_i$, then define the **highest edge splitting** of $\tilde{L} = \ldots \tilde{\sigma}_{-1} \cdot \tilde{\sigma}_0 \cdot \tilde{\sigma}_1 \ldots$ by subdividing at the initial vertex of each lift of $E_i$ that is crossed, in either direction, by $\tilde{L}$. We refer to the vertices that determine this decomposition as the **splitting vertices** of the highest edge splitting. If $\hat{f} : \Gamma \to \Gamma$ is a lift of $f \in \text{FHE}(G, \mathcal{V})$ and if $\tilde{L}$ is $\hat{f}_{\#}$-invariant, then Lemma 5.2 of [BFH96b] implies that $[\hat{f}(\tilde{\sigma}_j)] = \tilde{\sigma}_{j+r}$ for some $r$ and all $j$. Roughly speaking, $\hat{f}$ acts on $\tilde{L}$ by translating the highest edge splitting by $r$ units.
We will need the following fixed point results.

**Lemma 2.2.** • For any filtered graph $G$ and distinct $P_1, P_2, P_3 \in C_\infty$, there is a line $\tilde{L}$ in $\Gamma$ connecting $P_i$ to $P_j$ for some $1 \leq i < j \leq 3$ with the following property: If $f \in FHE(G, \mathcal{V})$ and $P_1, P_2, P_3 \in \text{Fix}(f)$ then $\hat{f}$ fixes each highest edge splitting vertex in $\tilde{L}$.

• If $f \in FHE(G, \mathcal{V})$ and $\tilde{f} : \Gamma \to \Gamma$ is fixed point free, then $\tilde{f}$ fixes exactly two points.

**Proof of Lemma 2.2** For the first item, let $\tilde{L}_{i,j} \subset \Gamma$ be the line connecting $P_i$ to $P_j$ and let $l_{i,j}$ be the highest parameter value for which $\tilde{L}_{i,j}$ crosses a lift of $E_{i,j}$. Assuming without loss that $l_{1,2} \geq l_{1,3}, l_{2,3}$, let $\tilde{L} = \tilde{L}_{1,2}$ and let $\tilde{v}$ be any highest edge splitting vertex of $\tilde{L}$. Suppose that $f \in FHE(G, \mathcal{V})$ and that $\hat{f}$ fixes each $P_i$. If $\tilde{f}$ does not fix $\tilde{v}$ then $\tilde{f}$ translates the highest edge splitting vertices of $\tilde{L}$ away from one endpoint of $\tilde{L}$, say $P_1$, and toward the other, $P_2$. It follows that the highest edge splitting of $\tilde{L}$ is bi-infinite and hence that $l_1 = l_2 = l_3$. This implies that $\tilde{f}$ translates the highest edge splitting vertices of $\tilde{L}_{2,3}$ away from $P_3$ and toward $P_2$. But now $\tilde{f}$ translates the highest edge splitting vertices of $\tilde{L}_{1,3}$ away from $P_1$ and away from $P_3$. This contradiction verifies the first item.

We assume now that $f \in FHE(G, \mathcal{V})$ and that $\hat{f}$ is fixed point free. By the first item, it suffices to show that $\hat{f}$ fixes at least two points. Implicit in Proposition 6.21 of [BFH96a] is the existence of a half-infinite ray $\tilde{R}_+ \subset \Gamma$ with highest edge splitting $\tilde{R}_+ = \tilde{\sigma}_0 \cdot \tilde{\sigma}_1 \ldots$ such that $[\tilde{f}(\tilde{\sigma}_j)] = \tilde{\sigma}_{j+r}$ for some $r > 0$ and all $j \geq 0$. Let $\tilde{v}_j$ be the initial vertex of $\tilde{\sigma}_j$. Since $\tilde{f}$ restricts to a bijection of vertices, there are unique vertices $\tilde{v}_j$ so that $\tilde{f}(\tilde{v}_j) = \tilde{v}_{j+r}$ for all $j \in \mathbb{Z}$. Let $\tilde{\sigma}_j$ be the path connecting $\tilde{v}_j$ to $\tilde{v}_{j+1}$ for $j < 0$ and note that $[\tilde{f}(\tilde{\sigma}_j)] = \tilde{\sigma}_{j+r}$ for all $j$. If $l$ is the highest parameter value for which $\tilde{R}_+$ crosses a lift of $E_l$, then each $\tilde{\sigma}_j$ contains exactly one lift of $E_l$ and these lifts are distinct for distinct $j$. (The $j < 0$ case follows from the $j \geq 0$ case which holds by construction.) It follows that $\ldots \tilde{\sigma}_1 \cdot \tilde{\sigma}_0 \cdot \tilde{\sigma}_1 \ldots$ is an embedded line whose endpoints are both fixed by $f$.

**Lemma 2.3.** Up to conjugation by covering translations, each $\eta \in \text{Out}(F_n)$ has only finitely many lifts $\tilde{\eta} : C_\infty \to C_\infty$ whose fixed point set contains at least three points.

**Proof of Lemma 2.3** Represent $\eta$ by a homotopy equivalence $f : G \to G$ of some marked graph and let $\tilde{f} : \Gamma \to \Gamma$ be a lift whose extension $\hat{f}$ fixes at least three points. Given $P_1, P_2, P_3 \in \text{Fix}(\tilde{f})$, let $\tilde{v} \in \Gamma$ be the unique vertex contained in each of the lifts connecting two of the $P_i$’s. The bounded cancellation lemma [Coo87] implies that there is a bound, depending on $f$ but not on the choice of the lift $\tilde{f}$ or the choice of points $P_i$, to the length of the path $\tilde{\sigma}$ connecting $\tilde{v}$ to $\tilde{f}(\tilde{v})$. In particular, the projected image $\sigma$ takes on only finitely many values as we vary the lift $\tilde{f}$ and the choice of points $P_i$. 


Suppose that \( \tilde{f}_i, i = 1, 2, \) are lifts of \( f \) and that there exist lifts \( \tilde{v}_i \) of a vertex \( v \) and \( \tilde{\sigma}_i \) of a path \( \sigma \) such that \( \tilde{\sigma}_i \) is the path connecting \( \tilde{v}_i \) to \( \tilde{f}_i(\tilde{v}_i) \). The covering translation \( T : \Gamma \to \Gamma \) that carries \( \tilde{v}_1 \) to \( \tilde{v}_2 \) satisfies \( \tilde{f}_2T = T\tilde{f}_1 \) and so conjugates \( \hat{f}_1 \) to \( \hat{f}_2 \).

\[ \square \]

### 3 Property A

In this section we prove that a finitely generated, solvable \( UPG \) subgroup \( \mathcal{H}_0 \) embeds in some \( \mathbb{Z}^r \) and hence that every solvable \( UPG \) subgroup is free abelian. Some of the arguments proceed by induction on the skeleta \( G_i \) of a filtered graph \( G \). For this reason, we do not assume in this section that \( K \) is a Kolchin representative of some \( \mathcal{H}_0 \) but only that \( K \) is a finitely generated solvable subgroup of \( \text{FHE}(G, V) \) with the following feature: if \( E_i \) is not a loop then some \( u_{i,f} \) is non-trivial. In particular, we do not assume that \( K \) injects into \( \text{Out}(\pi_1(G)) \) and we allow \( G \) to have valence one vertices. Note that if some \( u_{i,f} \) is non-trivial, then the terminal vertex of \( E_i \) is contained in a loop in \( G_{i-1} \) and so must have valence at least two in \( G_{i-1} \). Thus if \( E_i \) is the first edge to contain a vertex \( v \), then either \( E_i \) is a loop or \( E_i \) has \( v \) as initial vertex. In either case, \( v \) is the initial vertex of \( E_i \).

The group of lifts \( \tilde{f} : \Gamma \to \Gamma \) [respectively \( \hat{f} : C_\infty \to C_\infty \)] of elements of \( K \) has a natural projection to \( K \). By an action of \( K \) on \( \Gamma \) [respectively \( C_\infty \)] by lifts we mean a section of this projection. In other words, an action by lifts is an assignment \( f \mapsto \tilde{f} \) [respectively \( f \mapsto \hat{f} \)] that respects composition. Every action \( s \) of \( K \) on \( \Gamma \) by lifts determines an action \( s \) of \( K \) on \( C_\infty \) by lifts and vice-versa.

For each edge \( E_i \subset G \) choose, once and for all, a lift \( \tilde{E}_i^* \subset \Gamma \). Define \( s_i(f) : \Gamma \to \Gamma \) to be the unique lift of \( f \in K \) that fixes the initial endpoint of \( \tilde{E}_i^* \) and note that \( s_i \) is an action of \( K \) on \( \Gamma \) by lifts. Denote the terminal endpoints of \( E_i \) and \( \tilde{E}_i^* \) by \( v_i \) and \( \tilde{v}_i \) respectively.

**Definition 3.1.** If \( E_i \) is not a component of \( G_i \), denote the component of \( G_{i-1} \) that contains \( v_i \) by \( B_i \) and the (necessarily \( s_i(f) \)-invariant) copy of the universal cover of \( B_i \) that contains \( \tilde{v}_i \) by \( \Gamma_{i-1} \).

We say that \( K \) (and the choice of the \( \tilde{E}_i^* \)'s) satisfies Property A (for abelian) if:

- For each \( i \), either \( E_i \) is a component of \( G_i \) (in which case \( E_i \) is a loop and \( u_{i,f} \) is trivial for all \( f \in K \)) or \( \tilde{v}_i \) is a highest edge splitting vertex in a line \( \tilde{L}_i \subset \Gamma_{i-1} \) that is \( s_i(f)_{\#} \)-invariant for all \( f \in K \).
- If \( \tilde{L}_i \) and \( \tilde{L}_j \) have the same projection in \( G \), then \( \tilde{L}_i = \tilde{L}_j \) and \( \tilde{v}_i = \tilde{v}_j \).
If $\tilde{L}_i$ projects to an indivisible loop $\alpha_i$ and if some $u_i.f$ is non-trivial, then we say (abusing notation slightly) that $\alpha_i$ is an essential axis and that $E_i$ is an essential edge. The set of essential axes is denoted $A(\mathcal{K})$ and the set of essential edges is denoted $E(\mathcal{K})$. If the essential axis $\alpha$ is associated to $m_\alpha$ essential edges, then we say that $m_\alpha$ is the multiplicity of $\alpha$. In the analogy with the mapping class group of a compact surface, $A(\mathcal{K})$ is the set of reducing curves and $\mathcal{K}$ can be chosen so that each $m_\alpha = 1$.

For each essential axis $\alpha$, let $\tilde{v}_\alpha$ be the preferred vertex of the preferred lift $\tilde{\alpha}$ ($= \tilde{v}_i \in \tilde{L}_i$ for any $\tilde{L}_i$ that projects to $\alpha$), let $T_\alpha$ be an indivisible covering translation whose axis is $\tilde{\alpha}$ and let $s_\alpha(f)$ be the lift of $f \in \mathcal{K}$ that fixes $\tilde{v}_\alpha$. Note that $s_\alpha$ defines an action of $\mathcal{K}$ on $\Gamma$ by lifts. Note also that if we think of $\alpha$ as a closed path with both endpoints at the projected image $v$ of $\tilde{v}$, then $[f(\alpha)] = \alpha$. It follows that $s_\alpha(f)(\tilde{\alpha}) = \tilde{\alpha}$ and hence that $s_\alpha(f)$ commutes with $T_\alpha$.

**Example 3.2.** We set notation for this geometric example as shown below: $M$ is the orientable genus two surface with one boundary component; $A \subset M$ is an embedded non-peripheral annulus with boundary components $X$ and $E_1$; $S = M \setminus A$; $G \subset M$ is the embedded graph (spine of $S$) shown below with vertices $v_1$ and $v_2$ and edges $E_1, \ldots, E_5$; $\mathcal{H}_0 \cong \mathbb{Z}$ is the subgroup of the mapping class group of $M$ generated by the Dehn twist across the annulus $A$; and $\mathcal{K} \cong \mathbb{Z} \subset FHE(G, \mathcal{V})$ is generated by $E_i \mapsto E_i$ for $i \neq 2$ and $E_2 \mapsto E_2 E_1$.

The universal cover of $M$ is identified with a convex subset of the hyperbolic plane $H$ in the Poincaré disk model. Choose a lift $\tilde{L}$ of $E_1$ and let $\tilde{A}$ be the lift of $A$ that contains $\tilde{L}$. On the left [respectively right] of $\tilde{A}$ is a copy $\tilde{S}_1$ [respectively $\tilde{S}_2$] of the universal cover of $S$. Given $D \in \mathcal{H}_0$, let $\tilde{D}_i : \tilde{M} \to \tilde{M}$ be the lift that is the identity on $\tilde{S}_i$. Then $\tilde{D}_i$ fixes the endpoints of $\tilde{S}_i$ and $\tilde{D}_1$ and $\tilde{D}_2$ are the only lifts of $D$ that fix the endpoints of $\tilde{L}$ and at least one other point. Note also that if $D$ is a Dehn twist of order $k$ around $A$ then $\tilde{D}_1 = T^k \tilde{D}_2$ where $T$ is the indivisible covering translation.
that preserves \( L \).

On the graph level, there is a single essential edge \( E_2 \) with essential axis \( \alpha = E_1 \). The lifts \( \tilde{D}_1 \) and \( \tilde{D}_2 \) correspond to \( s_\alpha(f) \) and \( s_2(f) \) respectively.

The free group \( \pi_1(S) \) is generated by \( E_2, E_1, \bar{E}_2, E_3, \bar{E}_4, \) and \( E_5 \). The group \( \mathcal{H}_0 \) can be extended to a non-UPG abelian group \( \mathcal{H} \) by adding a generator that restricts to a pseudo-Anosov homeomorphism on \( S \). This can be represented by a relative train map \( f : G \to G \) that is the identity on \( E_1 \) and \( E_2 \) and that has a single exponentially growing stratum with edges \( E_3, E_4 \) and \( E_5 \).

The following lemma justifies our notation.

**Lemma 3.3.** If \( \mathcal{K} \) satisfies property A, then there is an injective homomorphism \( \Phi_\mathcal{K} : \mathcal{K} \to \mathbb{Z}^r \) where \( r \) is the cardinality of \( E(\mathcal{K}) \). In particular, \( \mathcal{K} \) is free abelian.

**Proof of Lemma 3.3** If \( E_i \) is an essential edge with essential axis \( \alpha \) then \( s_i(f) \) and \( s_\alpha(f) \) are lifts of \( f \) that commute with \( T_\alpha \) and so differ by an iterate of \( T_\alpha \). Define a homomorphism \( \phi_\mathcal{K}^i : \mathcal{K} \to \mathbb{Z} \) by \( s_i(f) = T_\alpha^{\phi_\mathcal{K}^i} s_\alpha(f) \) and define \( \Phi_\mathcal{K} \) to be the product of the \( \phi_\mathcal{K}^i \)'s. Then \( f \) is in the kernel of \( \Phi_\mathcal{K} \) if and only if \( f(E_i) = E_i \) for each essential edge \( E_i \) if and only if \( u_i, f \) is trivial for each essential edge \( E_i \).

If \( f \neq \text{identity} \), then there is a smallest parameter value \( i > 1 \) for which \( u_i(f) \) is non-trivial. Since \( f|G_{i-1} \) is the identity, \( s_i(f) \) restricts to a non-trivial covering translation of \( \Gamma_{i-1} \). The line \( \tilde{L}_i \subset \Gamma_{i-1} \) is \( s_i(f) \)\( \neq \)-invariant so must be the axis of that covering translation. Thus \( E_i \in E(\mathcal{K}) \) and \( f \) is not in the kernel of \( \Phi_\mathcal{K} \). This proves that \( \Phi_\mathcal{K} \) is injective. \( \square \)

The following lemma produces a pair of fixed points in \( C_\infty \) or equivalently a fixed line in \( \Gamma \).

**Lemma 3.4.** Suppose that \( \hat{\psi}_1, \ldots, \hat{\psi}_m : C_\infty \to C_\infty \) are lifts of elements of a finitely generated UPG subgroup. If the \( \hat{\psi}_j \)'s commute, then \( \cap_{j=1}^m \text{Fix}(\hat{\psi}_j) \) contains at least two points.
Proof of Lemma 3.4 Choose a Kolchin representative $\mathcal{K}$ of the UPG subgroup. There are elements $f_j \in \mathcal{K}$ and commuting lifts $\tilde{f}_j : \Gamma \to \Gamma$ so that $\tilde{f}_j = \tilde{\psi}_j$ for $j = 1, \ldots, m$.

If $\tilde{f}_1$ is fixed point free then (Lemma 2.2) $\text{Fix}(\tilde{f}_1)$ is a pair of points $\{P, Q\}$. Since $\tilde{f}_j$ commutes with $\tilde{f}_1$, $\tilde{f}_j$ setwise preserves $\{P, Q\}$ and we need only show that $\tilde{f}_j$ does not reverse the orientation on the line $\tilde{L}$ connecting $P$ and $Q$. Let $\tilde{L} = \ldots \tilde{\sigma}_{-1} \cdot \tilde{\sigma}_0 \cdot \tilde{\sigma}_1 \ldots$ be highest edge splitting of $\tilde{L}$. If $\tilde{f}_j$ reverses the orientation on $\tilde{L}$, then for some $j$, $[\tilde{f}(\tilde{\sigma}_j)]$ equals $\tilde{\sigma}_j$ with its orientation reversed. The projected image $\sigma_j$ determines a conjugacy class in $F_n$ that is periodic but not fixed under the action of the outer automorphism determined by $f$. This contradicts Proposition 4.5 of [BFH96b].

Suppose next that $\text{Fix}(\tilde{f}_1) \neq \emptyset$ and that the group $\mathcal{T}(\tilde{f}_1)$ of covering translations that commute with $\tilde{f}_1$ is trivial. The fixed point set of $f_j$, and hence of $\tilde{f}_j$, is a union of vertices and edges. Since $\mathcal{T}(\tilde{f}_1)$ is trivial, each vertex $v \in G$ has at most one lift $\tilde{v} \in \text{Fix}(\tilde{f}_1)$. Since $\tilde{f}_j$ commutes with $\tilde{f}_1$, it preserves $\text{Fix}(\tilde{f}_1)$. We conclude that each vertex in $\text{Fix}(\tilde{f}_1)$ is fixed by each $\tilde{f}_j$. (Recall that $f_j$ fixes each vertex in $G$.)

The edges with initial vertex in $\text{Fix}(\tilde{f}_1)$ project to distinct edges in $G$. Let $\tilde{E}_k$ be the unique such edge with minimal $k$, let $B'$ be the component of $G_{k-1}$ that contains the terminal endpoint of $E_k$ and let $\Gamma'_{k-1} \subset \Gamma$ be the copy of the universal cover of $B'$ that contains the terminal endpoint of $\tilde{E}_k$. Since each $f_j$ maps an initial segment of $E_k$ to an initial segment of $E_k$, $\Gamma'_{k-1}$ is $\tilde{f}_j$-invariant for each $j$. By our choice of $k$, the restriction $\tilde{f}_1|\Gamma'_{k-1}$ is fixed point free. We can now repeat the argument of the first case on the restriction of the $\tilde{f}_j$’s to $\Gamma'_{k-1}$.

Finally, suppose that $\mathcal{T}(\tilde{f}_1)$ is non-trivial. Identify $\mathcal{T}(\tilde{f}_1)$ with a subgroup $\mathbb{F}$ of $F_n$. The space of ends of $\mathbb{F}$ is the closure $C(\mathbb{F}) \subset \text{Fix}(\tilde{f}_1) \subset C_\infty$ of the endpoints of axes for elements of $\mathcal{T}(\tilde{f}_1)$. Since $\tilde{f}_j$ commutes with $\tilde{f}_1$, the automorphism of $\mathcal{T}$ determined by $\tilde{f}_j$ preserves $\mathcal{T}(\tilde{f}_1)$ and $\tilde{f}_j = \tilde{\psi}_j$ preserves $C(\mathbb{F})$. By Lemma 2.1, the $\tilde{\psi}_j|\mathbb{F}$’s are contained in a UPG subgroup of $\text{Out}(\mathbb{F})$.

We argue by induction on $m$, the $m = 1$ case following from the fact that $C(\mathbb{F})$ contains at least two points. Suppose that $m > 1$. By the inductive hypothesis, there exist $P, Q \in C(\mathbb{F})$ that are fixed by the $m - 1$ maps $\tilde{f}_2|C(\mathbb{F}), \ldots, \tilde{f}_m|C(\mathbb{F})$. Since $\tilde{f}_1|C(\mathbb{F})$ is the identity, $P$ and $Q$ are fixed by each $\tilde{f}_j$. \qed
Corollary 3.5. Every finitely generated solvable UPG subgroup \( \mathcal{H}_0 \) has a Kolchin representative that satisfies property A; in particular, \( \mathcal{H}_0 \) is free abelian.

Proof of Corollary 3.5 Choose a Kolchin representative \( \mathcal{K} \subseteq FHE(G,V) \) for \( \mathcal{H}_0 \). We work our way up the strata \( E_i \), modifying \( \mathcal{K} \) so that it satisfies property A. Denote the restriction of \( \mathcal{K} \) to \( G_i \) by \( \mathcal{K}_i \). Since \( G_1 \) is a single edge, \( \mathcal{K}_1 \) has property A; we may assume by induction that \( \mathcal{K}_{i-1} \) satisfies property A and is therefore abelian.

If \( E_i \) is a component of \( G_i \) then \( \mathcal{K}_i \) satisfies property A. We may therefore assume that \( \Gamma_{i-1} \) and the action \( s_i \) of \( \mathcal{K} \) on \( C_\infty \) are defined. The ends of \( \Gamma_{i-1} \) define a subset \( C^*_\infty \subset C_\infty \).

Let \( R: \mathcal{K}_i \rightarrow \mathcal{K}_{i-1} \) be the restriction homomorphism. If \( R \) is an isomorphism, then \( \mathcal{K}_i \) is abelian. Choose generators \( g_1, \ldots, g_m \) for \( \mathcal{K}_i \) representing \( \psi_1, \ldots, \psi_m \in \mathcal{H}_0 \), and let \( \hat{s}_j = \hat{s}_i(g_j) | C^*_\infty \). Note that the \( \hat{s}_j \)'s are lifts of elements of the UPG subgroup \( \mathcal{H}_0|\pi_1(G_{i-1}) \) and that, since \( s_i \) is an action and the \( g_j \)'s commute, the \( \hat{s}_j \)'s commute. Lemma 3.4 therefore produces \( P_i, Q_i \in C^*_\infty \) that are fixed by each \( \hat{s}_i(g_j) \) and hence by \( \hat{s}_i(f) \) for each \( f \in \mathcal{K} \). The line \( L_i \) connecting \( P_i \) to \( Q_i \) is \( s_i(f) \# \)-invariant for each \( f \in \mathcal{K} \).

If \( \tilde{L}_i \) has the same image in \( G \) as \( \tilde{L}_j \) for some \( j < i \), then after replacing \( E^*_i \) by a translate if necessary, we may assume that \( \tilde{L}_i = \tilde{L}_j \). Choose a highest edge splitting vertex \( \tilde{w}_i \) of \( \tilde{L}_i \) and apply the sliding operation of \([3FH96]\) simultaneously to each \( f \in \mathcal{K} \) reattaching \( E^*_i \) so that its terminal vertex is \( \tilde{w}_i \). This does not change \( s_i(f)|\Gamma_{i-1} \) and respects the group structure of \( \mathcal{K} \). A new Kolchin representative (still called \( \mathcal{K} \)) is produced that agrees with the old one on \( G_{i-1} \) and has the additional feature that \( \tilde{w}_i \) is the terminal vertex of \( E^*_i \); in particular, the first condition of property A is satisfied. Since \( \tilde{w}_i \) can be any highest edge splitting vertex, we may assume without loss that the second condition of property A is also satisfied.

Suppose now that the kernel \( K \) of \( R \) is non-trivial. Each \( f^* \in K \) satisfies \( E_j \mapsto E_j \) for \( 1 \leq j \leq i-1 \) and \( E_i \mapsto E_i u_{i,f^*} \). Thus the restriction of \( s_i(f^*) \) to \( \Gamma_{i-1} \) is the covering translation determined by the lift of \( u_{i,f^*} \) beginning at \( \tilde{w}_i \). The assignment \( f^* \mapsto u_{i,f^*} \) defines a ‘suffix’ homomorphism from \( K \) into the free group \( \pi_1(G, v_i) \). Since \( \mathcal{K} \), and hence \( K \), is solvable, the image of this homomorphism is isomorphic to \( \mathbb{Z} \). Thus the non-trivial \( s_i(f^*)|\Gamma_{i-1} \)'s have a common axis \( \tilde{L}_i \). Choose \( f^* \) so that \( s_i(f^*)|\Gamma_{i-1} \) is non-trivial. For each \( f \in \mathcal{K} \), \( s_i(f f^* f^{-1})|\Gamma_{i-1} \) is a non-trivial covering translation with axis \( s_i(f) \#(\tilde{L}_i) \). Since \( K \) is normal in \( \mathcal{K} \), \( s_i(f) \#(\tilde{L}_i) = \tilde{L}_i \). We have verified that \( \tilde{L}_i \) is \( s_i(f) \# \)-invariant for each \( f \in \mathcal{K} \). The proof now concludes as in the previous case.

4 Abelian subgroups are finitely generated

In this section we prove that an abelian UPG subgroup \( \mathcal{H}_0 \) is finitely generated. Section 3 and the exact sequence \( 1 \rightarrow \mathcal{H}_0 \rightarrow \mathcal{H} \xrightarrow{\Omega} \mathbb{Z}^b \rightarrow 1 \) then implies that every solvable subgroup of \( \text{Out}(F_n) \) is finitely generated. This fact was originally proved by
H. Bass and A. Lubotzky \cite{BL94}. We also show that the above sequence is a virtually central extension.

If $\mathcal{K}$ is a Kolchin representative of $\mathcal{H}_0$ that satisfies property A, $f \in \mathcal{K}$ and $\Phi_{\mathcal{K}} : \mathcal{K} \rightarrow \mathbb{Z}^r$ is the embedding of Lemma 3.3, then the coordinates of $\Phi_{\mathcal{K}}(f)$ are defined in terms of the $s_i(f)$’s and the $s_\alpha(f)$’s. Our goal is to recognize these lifts of $f$ by their induced actions on $C_\infty$, thus removing their dependence on the choice of $\mathcal{K}$.

We begin placing an additional restriction (the second item below) on our Kolchin graphs and justifying our assumption from section 3 (the first item below).

**Lemma 4.1.** For every finitely generated UPG subgroup $\mathcal{H}_0$ there is a Kolchin representative $\mathcal{K}$ with the following properties.

- If $E_i$ is not a loop then $u_{i,f}$ is non-trivial for some $f \in \mathcal{K}$.
- Every vertex of $G$ is the initial vertex of at least two edges.

**Proof of Lemma 4.1** Start with any Kolchin representative $\mathcal{K}$. After restricting to a subgraph if necessary, we may assume that $G$ has no valence one vertices. We will show that if either of the two properties fail, then we can replace $G$ by a graph with fewer edges. This process terminates after finitely many steps to produce the desired Kolchin representative.

If the first property fails, then $E_i$ is pointwise fixed for each $f \in \mathcal{K}$ and we may collapse it to a point.

Suppose then that the first property holds, that $v$ is a vertex and that $E_i$ is the first edge that contains $v$. If $E_i$ is not a loop, then $u_{i,f}$ is a non-trivial loop in $G_{i-1}$ containing the terminal endpoint of $E_i$ for some $f \in \mathcal{K}$. Since loops are assumed to be immersed, they cannot pass through valence one vertices and $v$ must be the initial vertex of $E_i$. For the same reason, $v$ must also be the initial vertex of the second edge that is attached to it.

Suppose then that $E_i$ is a loop and that $E_{j_0}, \ldots, E_{j_m}$ are the other edges that contain $v$. If the second property fails then $E_{j_0}, \ldots, E_{j_m}$ are non-loops with $v$ as terminal endpoint and $u_{j_j}(f) = E^{k_j}_{i_j}(f)$ for $j = j_0, \ldots, j_m$. Redefine $\mathcal{K}$ by replacing each $k_j(f)$ with $k_j(f) - k_{j_0}(f)$. This can be achieved on the graph level by sliding $v$ around $E_{j_0}(f)$ times. This has no effect on the outer automorphism determined by $f$ and respects the group structure on $\mathcal{K}$. The edge $E_{j_0}$ is now fixed by each $f \in \mathcal{K}$ and so can be collapsed to point.

**Definition 4.2.** We say that a Kolchin representative $\mathcal{K}$ for $\mathcal{H}_0$ is an abelian Kolchin representative if it satisfies property A and the conclusions of Lemma 4.1.

We now turn to the task of determining, from the action of $\mathcal{K}$ on $C_\infty$ by lifts, if the axis of a given covering translation projects to an element of $A(\mathcal{K})$.

Recall that in the analogy with the mapping class group of a compact surface $M$, $A(\mathcal{K})$ corresponds to the set of reducing curves in the minimal reduction. Such
reducing curves are completely characterized as follows (See Example 3.2, the proof of Proposition 1.2 or [HT85]): The free homotopy class determined by a closed curve \( \alpha \subset M \) is an element of the minimal reducing set for the mapping class represented by a homeomorphism \( h : M \to M \) if and only if for some (and hence each) covering translation \( T : \tilde{M} \to \tilde{M} \) corresponding to \( \alpha \), there are two lifts \( \tilde{h}_1, \tilde{h}_2 : \tilde{M} \to \tilde{M} \) that commute with \( T \) and whose extensions \( \hat{h}_i \) over the ‘circle at infinity’ fix at least three points. (If the free homotopy class of \( \alpha \) is fixed by \( h \) but \( \alpha \) is not one of the reducing curves then there is one such lift \( \tilde{h}_i \).) The analogous result in the non-geometric case is given in Corollary 4.6.

**Definition 4.3.** For any \( \psi \in H_0 \) and covering translation \( T \), define \( IL(\psi, T) \) (for Interesting Lifts) to be the set of lifts \( \hat{\psi} : C_\infty \to C_\infty \) that commute with \( T \) and fix at least three points.

Recall that \( \alpha \in A(K) \) has a preferred lift \( \tilde{\alpha} \) and an indivisible covering translation \( T_\alpha \) with axis equal to \( \tilde{\alpha} \). The next lemma states that the actions \( s_\alpha \) and \( s_i \) produce interesting lifts.

**Lemma 4.4.** Suppose that \( K \) is an abelian Kolchin representative and that \( \alpha \) is the essential axis for \( E_i \in E(K) \). Then \( \bigcap_{f \in K} \text{Fix}(\hat{s}_\alpha(f)) \) and \( \bigcap_{f \in K} \text{Fix}(\hat{s}_i(f)) \) each contain at least three points. In particular, if \( f \) represents \( \psi \in H_0 \), then \( \hat{s}_\alpha(f), \hat{s}_i(f) \in IL(\psi, T_\alpha) \).

**Proof of Lemma 4.4** There is a preferred topmost splitting vertex \( \tilde{v}_\alpha \in \tilde{\alpha} \). Choose an edge \( \tilde{E}_j \) with initial vertex \( \tilde{v}_\alpha \); by Lemma 4.1, we may assume that \( E_j \neq \alpha \). If \( u_{j,f} \) is trivial for all \( f \in K \), then \( E_j \) is a loop that is fixed by each \( f \in K \). In this case, let \( R_\alpha \) be an endpoint of the axis for \( E_j \) that contains \( \tilde{E}_j \). If some \( u_{j,f} \) is non-trivial, then \( \tilde{L}_j \) is defined and we choose \( R_\alpha \) to be an endpoint of the translate \( \tilde{L}_j' \) of \( \tilde{L}_j \) associated to \( \tilde{E}_j \). In either case, each \( \hat{s}_\alpha(f) \) fixes \( R_\alpha \).

The proof for \( s_i(f) \) is similar. By Lemma 4.1 there exists an edge \( \tilde{E}_i \neq \tilde{E}_i^* \), with
the same initial endpoint as $\tilde{E}_i^*$. Define $R_i \in \text{Fix}(\hat{s}_i(f))$ as in the previous case using $E_i$ in place of $E_j$.

If $f \in K$ represents $\psi \in H_0$, then we use $\hat{s}(f)$ and $\hat{s}(\psi)$ interchangably. We refer to the $\hat{s}_i(\psi)$’s and the $\hat{s}_\alpha(\psi)$’s as the canonical lifts of $\psi$. The next lemma and corollary show that $A(K)$ depends only on $H_0$ and not on the choice of $K$ and that one can decide if $\psi$ is canonical from its action on $C_\infty$.

**Lemma 4.5.** Suppose that $K$ is an abelian Kolchin representative and that $\tilde{L}$ is a line with endpoints $P, Q \in C_\infty$. Suppose further that:

- $\tilde{f} : \Gamma \to \Gamma$ is a lift of some $f \in K$
- $\text{Fix}(\hat{f})$ contains $P, Q$ and at least one other point.
- $\tilde{f}$ does not fix the highest edge splitting vertices of $\tilde{L}$.

Then there exists $i$ and a covering translation $T$ such that $T(\tilde{L}) = \tilde{L}_i$ and $\tilde{f} = T^{-1}s_i(f)T$.

**Proof of Lemma 4.5** By Lemma 2.2, $\text{Fix}(\tilde{f}) \neq \emptyset$. Choose an arc $\tilde{\sigma}$ that intersects $\text{Fix}(\tilde{f})$ only in its initial vertex, say $\tilde{p}_1$, and that intersects the splitting vertices of $\tilde{L}$ only in its terminal vertex, say $\tilde{v}$. Let $\tilde{E}_i'$ be the first edge of $\tilde{\sigma}$ and let $\tilde{p}_2$ be its terminal endpoint. Since $\tilde{f}$ fixes $\tilde{p}_1$ but not $\tilde{p}_2$, $u_{i,f}$ is non-trivial and $\tilde{L}_i$ is defined. Let $T$ be the covering translation that carries $\tilde{E}_i'$ to $\tilde{E}_i^*$ and let $\tilde{L}_i' = T^{-1}(\tilde{L}_i)$ be the translate of $\tilde{L}_i$ associated to $\tilde{E}_i'$. By construction, the ends of $\tilde{L}_i'$ are $\tilde{f}$-invariant.

If $\tilde{L}_i' \neq \tilde{L}$ then there is an endpoint, say $S$, of $\tilde{L}_i'$ that is neither $P$ nor $Q$. Lemma 2.2 implies that the highest edge splitting vertices of either $\tilde{L}_{PS}$ (= the line connecting $P$ to $S$) or $\tilde{L}_{SQ}$ are fixed. But $\tilde{L}_{PS}$ consists of a segment of $\tilde{L}$, a segment of $\tilde{L}_i'$ and perhaps a segment of $(\tilde{\sigma} \setminus \tilde{E}_i)$. The last segment contains no fixed vertices. By construction, the highest edge splitting vertices of the other segments are not fixed. Thus the highest edge splitting vertices of $\tilde{L}_{PS}$ are not fixed. The symmetric argument applies to $\tilde{L}_{SQ}$ and yields the desired contradiction. We conclude that $\tilde{L}_i' = \tilde{L}$ and that $T(\tilde{v}_i) = \tilde{p}_2 = \tilde{v}$. Since $T^{-1}s_i(f)T$ and $\tilde{f}$ both fix $\tilde{p}_1$, they must be equal.

13
Corollary 4.6. If $\mathcal{K}$ is an abelian Kolchin representative of $\mathcal{H}_0$, then:

- A covering translation $T$ corresponds to an essential axis of $\mathcal{K}$ if and only if $IL(\psi, T)$ contains at least two elements for some $\psi \in \mathcal{H}_0$.
- For each $\alpha \in A(\mathcal{K})$ and each $\psi \in \mathcal{H}_0$, $IL(\psi, T_\alpha) = \{s_i(\psi) : E_i$ is an essential edge with axis $\alpha\} \cup \{s_\alpha(\psi)\}$
- $A(\mathcal{K}) = A(\mathcal{H}_0)$ depends only on $\mathcal{H}_0$ and not on $\mathcal{K}$.

Proof of Corollary 4.6 The first and second items are a direct consequence of Lemma 4.3 and Lemma 4.5. The third item follows from the first. \hfill $\square$

Lemma 4.7. Every abelian UPG subgroup $\mathcal{H}_0$ is finitely generated.

Proof of Lemma 4.7 Choose an increasing sequence $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots$ of finitely generated subgroups whose union is $\mathcal{H}_0$. The cardinality of $A(\mathcal{H}_j)$ and the multiplicities of the elements of $A(\mathcal{H}_j)$ are uniformly bounded (by the maximum number of edges in a marked graph with the property that the terminal vertex of each edge has valence at least three.) Since $A(\mathcal{H}_j) \subset A(\mathcal{H}_{j+1})$, we may assume after passing to a subsequence, that $A(\mathcal{H}_j)$ and the multiplicities are independent of $j$. In particular, there is a fixed $r$ so that $\Phi_j : \mathcal{H}_j \to \mathbb{Z}^r$ where $\Phi_j$ is the embedding of Lemma 3.3 and where $\mathcal{H}_j$ has been identified with a Kolchin representative $\mathcal{K}_j$.

Fix $\psi \in \mathcal{H}_1$, $\alpha \in A(\mathcal{H}_1)$ and $j \geq 1$; Let $IL(\psi, T_\alpha) = \{b_1, b_2, \ldots\}$. For each $b_k$ and $b_l$, there is an integer $p(k, l)$ such that $b_kb_l^{-1} = T_\alpha^{p(k, l)}$. For any fixed $f \in \mathcal{K}$, Corollary 4.4 implies that each coordinate $\phi^j_i(f)$ of $\Phi_j(f)$ is one of the $p(k, l)$’s. In particular, $\Phi_j(f)$ takes on only finitely many values as $j$ varies.

Let $g_1, \ldots, g_q$ be generators of $\mathcal{H}_1$. After passing to a subsequence we may assume that $\Phi_j(g_i)$ is independent of $j$ for $i = 1 \ldots q$. The lattice $\Phi_j(\mathcal{H}_1)$ is therefore independent of $j$. It is contained with finite index in a maximal lattice $L$ of rank $q$. The lattice $\Phi_j(\mathcal{H}_j)$ has rank $q$ and contains $\Phi_j(\mathcal{H}_1)$ so is contained in $L$. In particular, the index of $\mathcal{H}_1$ in $\mathcal{H}_j$ is uniformly bounded. It follows that $\mathcal{H}_j = \mathcal{H}_{j+1}$ for all sufficiently large $j$. \hfill $\square$

We close this section by showing that $1 \to \mathcal{H}_0 \to \mathcal{H} \to \mathcal{H}_0 \to 1$ is a virtually central extension; the proof is a variation on that of Lemma 1.7.

Lemma 4.8. $A(\mathcal{H}_0)$ is $\mathcal{H}$ invariant (up to conjugacy). There is a finite index subgroup $\mathcal{H}' \subset \mathcal{H}$ whose actions on $\mathcal{H}_0$ and on $A(\mathcal{H}_0)$ are trivial.

Proof of Lemma 4.8 For each $\psi \in \mathcal{H}_0$, each $\alpha \in A(\mathcal{H}_0)$ and each lift $\tilde{\eta}$ of each $\eta \in \mathcal{H}$, conjugation by $\tilde{\eta}$ sends $IL(\psi, T_\alpha)$ to $IL(\psi', T')$ where $\psi' = \eta\psi\eta^{-1}$ and $T' = \tilde{\eta}T_\alpha\tilde{\eta}^{-1}$. In particular, $IL(\psi, T_\alpha)$ and $IL(\psi', T')$ have same cardinality. Corollary 4.5 therefore implies that the axis of $T'$ projects to an element of $A(\mathcal{H}_0)$ and so $A(\mathcal{H}_0)$ is invariant.
under the action of \( \eta \). Since \( A(\mathcal{H}_0) \) is finite, there is a finite index subgroup for which this action is trivial.

We assume now that the action on \( A(\mathcal{H}_0) \) is trivial. Choose \( \hat{\eta} \) so that \( T' = T_\alpha \) or equivalently so that \( \hat{\eta} \) commutes with \( T_\alpha \). Choose an abelian Kolchin representative \( K \) for \( \mathcal{H}_0 \) and let \( \Phi : \mathcal{H}_0 \to \mathbb{Z}^r \) be the embedding of Lemma 3.3. For any pair of elements \( b_k, b_l \in IL(\psi, T_\alpha) \), there is an integer \( p_{k,l} \) such that \( b_l b_k^{-1} = T_\alpha^{p_{k,l}} \). Let \( P(\psi) \) be the finite collection of integers that occur as \( p_{k,l} \)'s. Since conjugation by \( \hat{\eta} \) carries \( IL(\psi, T_\alpha) \) to \( IL(\psi', T_\alpha) \) and since \( \hat{\eta} \) commutes with \( T_\alpha \), \( P(\psi) = P(\psi') \). By construction, \( \Phi(\psi') \) therefore takes on only finitely many values as \( \eta \) varies over \( \mathcal{H} \) and \( \psi \in \mathcal{H}_0 \) is fixed. Since \( \Phi \) is an embedding, \( \psi' \) takes on only finitely many values. After passing to a finite index subgroup we may assume that the action of \( \mathcal{H} \) by conjugation on \( \psi \) is trivial. After applying this argument to a finite generating set for \( \mathcal{H}_0 \), we see that the action of \( \mathcal{H} \) on \( \mathcal{H}_0 \) is virtually trivial. \( \square \)

## 5 Proof of Proposition 1.2

The following lemma produces interesting lifts for (iterates of) individual elements of \( Out(F_n) \).

**Lemma 5.1.** Suppose that \( n \geq 2 \) and that \( \eta \in Out(F_n) \). After replacing \( \eta \) by an iterate if necessary, there is a lift \( \hat{\eta} \in EH(C_\infty) \) that fixes at least three points. If \( \gamma \) is a non-trivial primitive element of \( F_n \) that is fixed (up to conjugacy) by \( \eta \) and if \( T \) is a covering translation corresponding to \( \gamma \), then we may choose \( \hat{\eta} \) to commute with \( T \).

G. Levitt and M. Lustig inform us that they are developing techniques to understand the dynamics of automorphisms of hyperbolic groups on the boundary of the group, and that they are able to provide an alternate proof of this lemma.

**Proof of Lemma 5.1** We assume at first that \( \gamma \) and \( T \) are given.

The case that \( \eta \) is realized as an isotopy class of a surface homeomorphism \( h : S \to S \) is well known (see for example Lemma 3.1 of [HT85]): The Thurston classification theorem implies, after replacing \( h \) by an iterate if necessary, that \( S \) divides along annuli into subsurfaces with negative Euler characteristic on which \( h \) is either the identity or is pseudo-Anosov. Assuming that the reduction is done along the minimal number of annuli, we may choose the curve representing \( \gamma \) to lie in one of the subsurfaces \( S_i \). If \( h|S_i \) is the identity, then it has a lift that fixes the ends determined by \( S_i \) (See Example 52). If \( h|S_i \) is pseudo-Anosov then \( \gamma \) determines a boundary component of \( S_i \) and there is a lift fixing \( \tilde{\gamma} \) and the endpoints of the singular leaves of the pseudo-Anosov foliations associated to that boundary component.

We now turn to the general case and argue by induction. Since every outer automorphism of \( F_2 \) is realized as a surface isotopy class, the preceding argument handles the \( n = 2 \) case. We may therefore assume that the lemma holds for free groups of rank less than \( n \).
The smallest free factor $\mathcal{F}(\gamma)$ that contains $\gamma$ \cite{BFH96a} is $\eta$-invariant (up to conjugacy). If $1 < \text{rank}(\mathcal{F}(\gamma)) < n$, then the inductive hypothesis provides a lift of $\eta|\mathcal{F}(\gamma)$ with the desired properties. Extending this lift (Lemma 2.1) to all of $F_n$ completes the proof. We may therefore assume that $\text{rank}(\mathcal{F}(\gamma))$ is either 1 or $n$.

Choose an improved relative train track map $f : G \rightarrow G$ representing an iterate of $\eta$. Theorem 6.4 of \cite{BFH96a} contains a list of all the properties of $f : G \rightarrow G$ that are used in this proof. The conjugacy class of $\gamma$ determines a loop in $G$ that we also call $\gamma$.

If $\mathcal{F}(\gamma)$ has rank $n$, then $\gamma$ must cross an edge in the highest stratum of $G$. If this stratum is exponentially growing, then Theorem 6.4 of \cite{BFH96a} implies that there is an $\eta$-invariant (up to conjugacy) subgroup $\mathbb{F}$ that is its own normalizer and with the following additional property: There is a conjugacy between the the outer automorphism $\eta|\mathbb{F}$ and a pseudo-Anosov mapping class $h : S \rightarrow S$. Moreover, the conjugacy carries $\gamma$ to a boundary component of $S$. Since $\mathbb{F}$ has rank at least two, lifts of $\eta|\mathbb{F}$ extend uniquely to lifts of $\eta$. We may therefore assume that $\mathbb{F} = F_n$ and hence that $\eta$ is represented by $h$. We are now reduced to a previous case.

If the top stratum $G_m$ is not exponentially growing, then $G_m$ is a single edge $E_m$. The loop $\gamma$ splits (Lemma 5.2 of \cite{BFH96a}) at the initial vertex $v$ of $E_m$ each time that it crosses $E_m$ in either direction. We may therefore think of $\gamma = \gamma_1 \cdots \gamma_r$ as a concatenation of Nielsen paths based at $v$ (i.e. each $[f(\gamma_i)] = \gamma_i$). We claim that there are two distinct $\gamma_i$'s. If not, then, since $\gamma$ is indivisible, $\gamma = \gamma_1$ is of the form $E_m\delta, \delta E_m$ or $E_m\delta E_m$ for some path $\delta \subset G_{m-1}$, where $E_m$ is $E_v$ with its orientation reversed. In the first two cases $\gamma$ is a free factor and in the last case $\gamma$ is freely homotopic to $\delta$. Each of these contradicts our assumption that $\mathcal{F}(\gamma)$ has rank $n$ and so verify our claim.

The axis $Ax(T) = \ldots \tilde{\gamma}_1 \tilde{\gamma}_2 \ldots$ of $T$ decomposes as a concatenation of lifts of the $\gamma_i$'s. Let $\tilde{v}$ be the initial vertex of $\tilde{\gamma}_1$ and let $\tilde{f}$ be the lift of $f$ that fixes $\tilde{v}$. Since the $\gamma_i$'s are Nielsen paths, $\tilde{f}$ fixes each concatenation point in the decomposition $Ax(T) = \ldots \tilde{\gamma}_1 \tilde{\gamma}_2 \ldots$. In particular, $\tilde{\eta} = \tilde{f}$ fixes the endpoints of $Ax(T)$ and so commutes with $T$. Now think of $\gamma_1$ as a loop and extend $\tilde{\gamma}_1$ to the axis $Ax(T_1)$ of a covering translation by concatenating translates of $\tilde{\gamma}_1$. Then $\tilde{\eta} = \tilde{f}$ also fixes the endpoints of $Ax(T_1)$.

It remains to consider the case that $\mathcal{F}(\gamma)$ has rank one. We may assume that the first stratum $G_1 = \gamma$ is a single edge; let $v$ be the vertex of $G_1$.

Suppose that an edge of an exponentially growing stratum $H_r$ is attached to $v$. After replacing $f$ by an iterate if necessary there is an edge $E$ of $H_r$ with initial vertex $v$ such that $f(E) = E \cdot \beta$ splits into the concatenation of $E$ with some non-trivial path $\beta$. Let $\tilde{v}$ be a lift of $v$ in the axis of $T$, let $\tilde{f} : \Gamma \rightarrow \Gamma$ be the lift of $f$ that fixes $\tilde{v}$ and let $\tilde{E}$ be the lift of $E$ with initial vertex $\tilde{v}$. Then $\tilde{f}(\tilde{E}) = \tilde{E} \cdot \tilde{\beta}$ splits into the concatenation of $\tilde{E}$ and some other non-trivial path and so each $[\tilde{f}^k(\tilde{E})]$ is a proper initial segment of $[\tilde{f}^{k+1}(\tilde{E})]$. It follows that the $[\tilde{f}^k(\tilde{E})]$'s converge to an invariant ray whose endpoint $R$ is fixed by $\tilde{f}$. Let $\tilde{\eta} = \tilde{f}$. \vspace{1cm}
If there are no exponentially growing strata attached to \( v \), then there are no zero strata attached to \( v \) and each edge \( E_i \) that is attached to \( v \) is its own stratum \( H_i \) and satisfies \( f(E_i) = E_i u_i \) for some path \( u_i \subset G_{i-1} \). If \( v \) is the initial vertex of \( E_i \) and either \( E_i \) is a loop or \( u_i \) is non-trivial, then we define \( \tilde{\eta} \) as in the proof of Lemma 1.4.

Let \( \tilde{v} \) be a lift of \( v \) in the axis of \( T \), let \( \tilde{f} : \Gamma \to \Gamma \) be the lift of \( f \) that fixes \( \tilde{v} \) and let \( \tilde{E} \) be the lift of \( E \) with initial vertex \( \tilde{v} \). If \( u_i \) is trivial then \( R \) is the endpoint of the axis for \( E_i \) that contains \( \tilde{E}_i \). If \( u_i \) is non-trivial then \( R \) is the endpoint of the invariant ray \( \tilde{u}_i \cdot [\tilde{f}(\tilde{u}_i)] \cdot [\tilde{f}^2(\tilde{u}_i)] \cdots \).

In the remaining case, every \( E_i \) attached to \( v \) is either a fixed non-loop or has \( v \) as its terminal endpoint. Proceeding as in the proof of Lemma 1.1 we can reduce the number of edges in \( G \) and arrive at one of our previous cases.

This completes the proof when \( \gamma \) and \( T \) are given. It remains to consider the case that \( \eta \) does not act periodically on any conjugacy class in \( F_n \). By induction on \( n \), we may assume that \( \eta \) does not act periodically on the conjugacy class of any proper free factor in \( F_n \). We may therefore assume our improved relative train track map \( f : G \to G \) has only one stratum and that this stratum is exponentially growing.

After passing to a further iterate if necessary, we may assume that there is a vertex \( v \) and two edges \( E_1 \) and \( E_2 \) initiating at \( v \) such that \( f(E_i) = E_i \cdot \beta_i \) splits into the concatenation of \( E_i \) with some non-trivial path \( \beta_i \). Choose a lift \( \tilde{v} \) of \( v \), let \( \tilde{f} \) be the lift of \( f \) that fixes \( \tilde{v} \) and let \( \tilde{E} \) be the lift of \( E_i \) initiating at \( \tilde{v} \). As above, \( [\tilde{f}^k(\tilde{E}_i)] \) converges to an invariant ray whose endpoint \( R_i \) is fixed by \( \tilde{f} \). Moreover, the bounded cancellation lemma \( [\text{Coo87}] \) and the fact that the lengths of \( [\tilde{f}^k(\tilde{E}_i)] \) grow exponentially in \( k \) imply that \( R_i \) is an attracting fixed point for the action of \( \tilde{f} \) on \( C_\infty \).

We have shown that some iterate of \( \eta \) has a lift with at least two attracting fixed points. Applying this to \( \eta^{-1} \), we conclude (suppressing the iterate for notational simplicity) that some \( \tilde{\eta} \) has at least two repelling fixed points. If \( \tilde{\eta} \) has a fixed point, then the preceding argument shows, after passing to an iterate if necessary, that there are also at least two attracting fixed points. Suppose then that \( \tilde{\eta} \) is fixed point free. During the proof of Proposition 6.21 of \( [\text{BFH96}] \) we show that there exists \( \tilde{x} \in \Gamma \) such that \( \tilde{x}, \tilde{\eta}(\tilde{x}), \tilde{\eta}^2(\tilde{x}) \ldots \) is an infinite sequence in an embedded ray \( \tilde{B} \); the endpoint of \( \tilde{B} \) is fixed by \( \tilde{\eta} \) and is not one of the repelling fixed points.

We prove Proposition 1.2 in the following equivalent form.

**Proposition 1.2** Every abelian subgroup \( \mathcal{H} \subset \text{Out}(F_n) \) has a virtual lift \( \hat{\mathcal{H}} \subset E\text{H}(F_n) \). If \( \gamma \) is a non-trivial primitive element of \( F_n \) that is fixed, up to conjugacy, by each element of \( \mathcal{H} \) and if \( T \) is a covering translation corresponding to \( \gamma \), then \( \hat{\mathcal{H}} \) can be chosen so that each element commutes with \( T \).

**Proof of Proposition 1.2** We argue by induction on the rank of the free abelian group \( \mathcal{H} \). If \( \mathcal{H} \) has rank one with generator \( \eta \), then \( \mathcal{H} \) is determined by choosing a lift \( \hat{\eta} \) that commutes with \( T \) if \( T \) is given. We may now assume that the rank of \( \mathcal{H} \) is at least two and that the lemma holds for all ranks less than that of \( \mathcal{H} \).

17
By Lemma 5.1, there is an element \( \eta \in \mathcal{H} \) and a lift \( \hat{\eta} \) that fixes at least three points and that commutes with \( T \) if \( T \) is given. Let \( C = \text{Fix}(\hat{\eta}) \) and let \( \mathcal{T}(C) \) be the group of covering translations that preserve \( C \). As described in subsection 2.1, \( \hat{\eta} \) determines an automorphism \( \Phi : F_n \to F_n \) whose fixed subgroup \( \mathcal{F} \) corresponds to \( \mathcal{T}(C) \) under the identification of \( F_n \) with \( \mathcal{T} \). (See the proof of Lemma 6.2 for further details.) If \( \gamma \) and \( T \) are given, then \( T \in \mathcal{T}(C) \) and \( \gamma \in \mathcal{F} \).

We first show, after passing to a subgroup of \( \mathcal{H} \) with finite index, that every \( \mu \in \mathcal{H} \) has a lift \( \hat{\mu} \) that commutes with \( \hat{\eta} \) and with \( T \) if \( T \) is given. In particular, \( C \) is \( \hat{\mu} \)-invariant, \( \mathcal{F} \) is \( \mathcal{H} \)-invariant (up to conjugacy) and, if \( \gamma \) is given then each element of \( \mathcal{H} \), \( \mathcal{F} \) fixes \( \gamma \) up to conjugacy.

Suppose at first that \( \gamma \) and \( T \) are not given. We say that two lifts \( \hat{\eta}_1 \) and \( \hat{\eta}_2 \) of \( \eta \) are equivalent if \( \hat{\eta}_1 = T_1 \hat{\eta}_2 T_1^{-1} \) for some covering translation \( T_1 \). Let \( IL(\eta) \) be the set of equivalence classes of lifts \( \hat{\eta} \) that fix at least three points. By Lemma 2.3, \( IL(\eta) \) is finite. Since \( \mathcal{H} \) is abelian, \( \mathcal{H} \) acts by conjugation on \( IL(\eta) \). After passing to a subgroup of finite index, we may assume that this action is trivial. Thus for any lift \( \hat{\mu}_1 \) of \( \mu \in \mathcal{H} \), there is a covering translation \( T_1 \) such that \( \hat{\mu}_1 \hat{\eta} \hat{\mu}_1^{-1} = T_1 \hat{\eta} T_1^{-1} \). Thus \( \hat{\mu} = T_1^{-1} \hat{\mu}_1 \) commutes with \( \hat{\eta} \).

Suppose now that \( \gamma \) and \( T \) are given. Each \( \mu \in \mathcal{H} \) has a lift \( \hat{\mu} \) that commutes with \( T \). Since \( \mathcal{H} \) is abelian, \( \hat{\mu} \hat{\eta} \hat{\mu}^{-1} \) is a lift of \( \eta \) that commute with \( T \) and so \( \hat{\mu} \hat{\eta} \hat{\mu}^{-1} = T^a \hat{\eta} \) for some \( a \). Let \( \hat{\eta}_k = \hat{\mu}^k \hat{\eta} \hat{\mu}^{-k} = \hat{\mu}^k \hat{\eta}_1 \hat{\mu}^{-k} = T^{ak} \hat{\eta} \). Then each \( \hat{\eta}_k \) is conjugate to \( \hat{\eta} \) and so fixes at least three points. On the other hand, there are only finitely many values of \( l \) for which \( \text{Fix}(T^l \hat{\eta}) \neq \text{Fix}(T) \). (This follows from: (i) \( \hat{\eta} \) fixes \( \text{Fix}(T) = \{ P, Q \} \) and so cannot move points very near \( P \) to points very near \( Q \); and (ii) \( T \) acts cocompactly on \( C_\infty \setminus \{ P, Q \} \).) We conclude that \( a = 0 \) and hence that \( \hat{\mu} \) commutes with \( \hat{\eta} \).

The proof now divides into cases, depending on the rank of \( \mathcal{T}(C) \). If \( \mathcal{T}(C) \) is the trivial group, then each \( \mu \) has a unique lift \( \hat{\mu} \) that preserves \( C \) and so the assignment \( \mu \mapsto \hat{\mu} \) defines \( \mathcal{H} \).

Suppose next that \( \mathcal{T}(C) \) has rank one. Since \( \mathcal{F} \) is \( \mathcal{H} \)-invariant (up to conjugacy), we may assume that \( T \) is given and generates \( \mathcal{T}(C) \).

We claim that \( C \) contains only finitely many \( T \)-orbits. This is a special case of the main theorem of [Coo87]; the argument in this case is short so we include it for completeness. Let \( P \) and \( Q \) be the endpoints of the axis of \( T \). Since \( T \) acts cocompactly on \( C_\infty \setminus \{ P, Q \} \), it suffices to show that \( P \) and \( Q \) are the only accumulation points of \( C \). Suppose to the contrary that \( S \) is an accumulation point other than \( P \) and \( Q \). Arguing as in the proof of Corollary 2.3, using triples of points in \( C \) limiting on \( S \), we find lifts \( \tilde{v}_1 \in \Gamma \) of a vertex \( v \in G \) so that the covering translation \( T_i \) that carries \( \tilde{v}_1 \) to \( \tilde{v}_i \) commutes with \( \hat{\eta} \). But then \( T_i \) is a multiple of \( T \) in contradiction to the assumption that \( T_i(\tilde{v}_1) \to S \). This verifies our claim.

The action of \( \mathcal{H} \) on the finitely many \( T \)-orbits of \( C \) is well defined. After passing to a finite index subgroup, this action is trivial. Choose a point \( R \in C \setminus \{ P, Q \} \). Composing \( \hat{\mu} \) with an iterate of \( T \), we may assume that \( \hat{\mu} \) fixes \( R \). The assignment
Finally suppose that $T(C)$ has rank at least two. As noted above, if $\gamma$ is given then it is fixed, up to conjugacy, by each element of $H^* = H|F$. Since $F$ is the fixed subgroup of an automorphism $\Phi$ representing $\eta$, the image of $\eta$ in $H^*$ is trivial. Thus the rank of $H^*$ is less than that of $H$ and by induction, there is a lift $\tilde{H}^* \subset EH(C)$ to elements that commute with $T$ if $T$ is given. By Lemma 2.3, $\tilde{H}^*$ extends to the desired lift $\tilde{H}$.

\section{Proof of Theorem 1.1}

We may assume without loss that $H_0$ is non-trivial and, by Lemma 4.8, that $H$ acts trivially on $H_0$ and on $A(H_0)$. Let $K$ be an abelian Kolchin representative for $H_0$. Throughout this section $\hat{s} = \hat{s}_i$ or $\hat{s}_\alpha$ and $T_\alpha$ is its associated covering translation.

Before proving the next lemma we show that it implies Theorem 1.1.

**Lemma 6.1.** The lift $\hat{s}(H_0) \subset EH(C_\infty)$ virtually extends to a lift $\hat{S}(H) \subset EH(C_\infty)$ all of whose elements commute with $T_\alpha$.

**Proof of Theorem 1.1** We may assume that the extensions $\hat{S}_i$ and $\hat{S}_\alpha$ of $\hat{s}_i$ and $\hat{s}_\alpha$ produced by Lemma 5.1 are defined on all of $H$. Define a homomorphism $\phi'_i : H \to \mathbb{Z}$ by $S_i(\psi) = T_\alpha^{S_i(\psi)} S_\alpha(\psi)$ and note that the product $\Phi' : H \to \mathbb{Z}^r$ of the $\phi'_i$’s extends the embedding $\Phi : H_0 \to \mathbb{Z}^r$ of Lemma 3.3. Define $\Psi : H \to \mathbb{Z}^{b+r}$ to be the product of $\Omega$ and $\Phi'$. Since $H_0$ is the kernel of $\Omega$ and $\Phi'|H_0 = \Phi$ is an embedding, $\Psi$ is an embedding.

We now know that solvable subgroups $H$ of $Out(F_n)$ are finitely generated and virtually abelian. By passing to a subgroup of index at most $D(n)$ where $D(n) := |GL(n, \mathbb{Z}/3\mathbb{Z})| < 3^{n^2}$, we may assume that the image of $H$ in $GL(n, \mathbb{Z}/3\mathbb{Z})$ is trivial. Thus, $H$ has a subgroup of index at most this number that is a torsion free Bieberbach group of vcd at most $vcd(Out(F_n)) = 2n - 3$ (see [CV86]). Also, a Bieberbach group of vcd at most $n$ has a subgroup of index at most $D(n)$ that is free abelian (see, for example, [Cha86]). Thus, $H$ has a free abelian subgroup of index at most $D(n)D(2n - 3) < 3^{5n^2}$. This completes the proof of the Theorem 1.1.

We are now reduced to Lemma 6.1. The proof uses Proposition 1.2 and follows the general line of the proof of Proposition 1.2.

Let $C = \bigcap_{f \in K} Fix(\hat{s}(f))$ and let $T(C)$ be the group of covering translations that preserves $C$. Note that $T(C)$ contains $T_\alpha$ and so has rank at least one. Let $F$ be the subgroup of $F_n$ corresponding to $T(C)$ and let $\tilde{s}(H_0) \subset Aut(F_n)$ be the lift of $H_0$ corresponding to $\hat{s}(H_0) \subset EH(C_\infty)$.

**Lemma 6.2.** $F$ is the fixed subgroup $\{ \gamma \in F_n : \gamma(\gamma) = \gamma \text{ for each } \gamma \in \tilde{s}(H_0) \}$ of $\tilde{s}(H_0)$.
Proof of Lemma 6.2 If the endpoints of the axis of \( T \in \mathcal{T} \) are contained in \( C \), then they are fixed by each \( \hat{s}(f) \) and so \( T \) commutes with each \( s(f) : \Gamma \to \Gamma \). Thus \( T \) commutes with each \( \hat{s}(f) \) and each \( \text{Fix}(\hat{s}(f)) \) is \( T \)-invariant. It follows that \( T \in \mathcal{T}(C) \).

Conversely, if \( P \) and \( Q \) are, respectively, the backward and forward endpoints of the axis of \( T \) then \( \lim_{n \to \infty} T^n(R) = Q \) and \( \lim_{n \to \infty} T^{-n}(R) = P \) for all \( R \in C_\infty \setminus \{P, Q\} \). If \( \mathcal{T}(C) = C \) then \( C \) must contain \( P \) and \( Q \).

We have shown that \( T \in \mathcal{T}(C) \) if and only if the endpoints of the axis of \( T \) are contained in \( C \). By construction, the latter condition is equivalent to the endpoints of the axis of \( T \) being fixed by each \( \hat{s}(f) \). The lemma now follows from the definition of \( \hat{s}(H_0) \).

Proof of Lemma 6.1 Given \( \mu \in \mathcal{H} \), choose a lift \( \hat{\mu} \) that commutes with \( T_\alpha \). Suppose also that \( \hat{\eta} \in \hat{s}(\mathcal{H}_0) \) is given. Since \( \mathcal{H} \) acts trivially on \( \mathcal{H}_0 \), \( \hat{\mu} \hat{\eta} \hat{\mu}^{-1} \) is a lift of \( \eta \) that commutes with \( T_\alpha \) and so \( \hat{\mu} \hat{\eta} \hat{\mu}^{-1} = T_\alpha^a \hat{\eta} \) for some \( a \). Arguing exactly as in the proof of Proposition 1.2, we conclude that \( \hat{\mu} \) commutes with \( \hat{\eta} \). It follows that \( C \) is \( \hat{\mu} \)-invariant, that \( \mathbb{F} \) is \( \mathcal{H} \)-invariant up to conjugacy and that each element of \( \mathcal{H}[\mathbb{F}] \) fixes \( \alpha \) up to conjugacy.

The proof now divides into cases, depending on the rank of \( \mathcal{T}(C) \). Suppose that \( \mathcal{T}(C) \) has rank one. We claim that \( \mathcal{H}[\mathbb{F}] \) contains only finitely many \( T_\alpha \)-orbits. Let \( P \) and \( Q \) be the endpoints of the axis of \( T_\alpha \). Since \( T_\alpha \) acts co-compactly on \( C_\infty \setminus \{P, Q\} \), it suffices to show that \( P \) and \( Q \) are the only accumulation points of \( S \). Suppose to the contrary that \( S \) is an accumulation point other than \( P \) and \( Q \). Lemma 2.2 applied to triples of points in \( C \) limiting on \( S \), implies that there are vertices \( \hat{v}_i \in \Gamma \) that limit on \( S \) and that are fixed by \( \hat{s}(f) \) for each \( f \in \mathcal{K} \). There is no loss in assuming that the \( \hat{v}_i \)'s are all lifts of the same vertex in \( G \). The covering translation \( T_i \) that carries \( \hat{v}_i \) to \( \hat{v}_i \) commutes with each \( s(f) \) and so must be a multiple of \( T_\alpha \). But this contradicts the assumption that \( T_i(\hat{v}_i) \to S \). This verifies our claim.

There is a well defined action of \( \mathcal{H} \) on the finitely many \( T \)-orbits of \( C \). After passing to a finite index subgroup, this action is trivial. By Lemma 1.4 there exists \( R \in C_\infty \) that is not an endpoint of the axis of \( T_\alpha \). Composing \( \hat{\mu} \) with an iterate of \( T \), we may assume that \( \hat{\mu} \) fixes \( R \). The assignment \( \eta \mapsto \hat{\eta} \) defines \( \hat{S} \).

We may now assume that \( \mathcal{T}(C) \) has rank at least two. By Lemma 2.1, the restriction \( \mathcal{H}^* \) of \( \mathcal{H} \) to \( \text{Out}(\mathbb{F}) \) is well defined. By Lemma 6.2, the restriction \( \mathcal{H}^*_0 \) of \( \mathcal{H}_0 \) to \( \text{Out}(\mathbb{F}^*) \) is trivial. Restricting the exact sequence \( 1 \to \mathcal{H}_0 \to \mathcal{H} \xrightarrow{\Omega} \mathbb{Z}_b \to 1 \) to \( \mathbb{F}^* \) we see that \( \mathcal{H}^* \) is abelian. By Proposition 1.2, there is a virtual lift \( \hat{S}^* \subset EH(C) \) of \( \mathcal{H}^* \) such that each element of \( \hat{S}^* \) commutes with \( T_\alpha \). Let \( \hat{S} \) be the unique extension of \( \hat{S}^* \) to a virtual lift of \( \mathcal{H} \). Since \( \mathcal{H}^*_0 \) is trivial, \( \hat{S}(\psi) \) restricts to the identity on \( C \) for each \( \psi \in \mathcal{H}_0 \). Thus \( \hat{S}(\psi) \) and \( \hat{s}(\psi) \) agree on \( C \) and so must agree everywhere. \( \square \)
References

[BFH96a] M. Bestvina, M. Feighn, and M. Handel, *The Tits Alternative for Out(F_n) I: Dynamics of Exponentially Growing Automorphisms*, preprint, 1996.

[BFH96b] M. Bestvina, M. Feighn, and M. Handel, *The Tits Alternative for Out(F_n) II: a Kolchin Theorem*, preprint, 1996.

[BL94] H. Bass and A. Lubotzky, *Linear-central filtrations on groups*, The mathematical legacy of Wilhelm Magnus: groups, geometry and special functions (W. Abikoff, J. Birman, and K. Kuiken, eds.), Contemp. Math., vol. 169, American Mathematical Society, 1994, pp. 45–98.

[BLM83] J.S. Birman, A. Lubotzky, and J. McCarthy, *Abelian and solvable subgroups of the mapping class group*, Duke Math. J. 50 (1983), 1107–1120.

[Cha86] L. S. Charlap, *Bieberbach groups and flat manifolds*, Springer-Verlag, 1986.

[Coo87] D. Cooper, *Automorphisms of free groups have finitely generated fixed point sets*, J. Algebra 111 (1987), 453–456.

[CV86] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. 84 (1986), 91–119.

[HT85] M. Handel and W.P. Thurston, *New proofs of some results of Nielsen*, Adv. in Math. 56 (1985), 173–191.