Field theory for zero sound and ion acoustic wave in astrophysical matter
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We set up a field theory model to describe the longitudinal low energy modes in high density matter present in white dwarf stars. At the relevant scales, ions – the nuclei of oxygen, carbon and helium – are treated as heavy point-like spin-0 charged particles in an effective field theory approach, while the electron dynamics is described by the Dirac Lagrangian at the one-loop level. We show that there always exists a longitudinal gapless mode in the system irrespective whether the ions are in a plasma, crystal, or quantum liquid state. For certain values of the parameters, the gapless mode can be interpreted as a zero sound mode and, for other values, as an ion acoustic wave; we show that the zero sound and ion acoustic wave are complementary to each other. We discuss possible physical consequences of these modes for properties of white dwarfs.

INTRODUCTION AND SUMMARY

Consider a neutral system of particles made of quantum degenerate electrons, and either Oxygen (Z = 8), or Carbon (Z = 6), or Helium (Z = 2) nuclei, at mass densities $\rho \sim (10^6 - 10^7) \text{ g/cm}^3$, and temperatures $T < (a \text{ few}) \times 10^8 \text{ K}$. For such high densities, average inter-particle separations in the system are much smaller than the atomic scale – hence no atoms would form, even if the system were cold. On the other hand, the separations are much larger than the nuclear scale, hence, one can regard the O, C, and H nuclei as point-like, positively charged, spin-0 particles; we’ll refer them as ions in the present work, to emphasize that one doesn’t need to care about their detailed nuclear structure.

For the O, C, and H ions the baryon number, A, equals twice their charge Z, and therefore the ion mass densities, $\rho \sim (10^6 - 10^7) \text{ g/cm}^3$, translate into the electron number density that can be estimated as $n_e \approx (\rho/2m_p) \sim (0.1 - 0.3 \text{ MeV})^3$, where $m_p$ is the protons mass. The corresponding Fermi momenta are $p_F \sim (0.3 - 0.9)\text{MeV}$, and hence the electrons are (nearly) relativistic. Their Fermi energy significantly exceeds their thermal energy, as well as their two-body Coulomb interaction energies; therefore, the electron system can be regarded as a quantum degenerate gas.

As for the ions, they’re much heavier than the electrons, and hence their thermal de Broglie wavelengths much shorter, so that at $T \sim (a \text{ few})\times10^8 \text{ K}$ they generically form a classical gas (see [1], and references therein). However, at lower temperatures two effects need to be taken into account: (a) Their two body Coulomb interaction energies start to dominate over their thermal energy; (b) Their thermal de Broglie wavelengths become comparable to their average separation, and they enter a quantum regime. As a result, below $T \sim (a \text{ few})\times10^6 \text{ K}$, the ions may form either a classical or quantum crystal [2], or a quantum liquid [3], depending on concrete values of $\rho$ and $Z$.

The above described matter is believed to exist in the interiors of white dwarf stars (WD’s). These are stars that finished their thermonuclear burning process, and are sitting in the sky to radiate away the heat stored in them. They can be regarded as retired stars, slowly evolving from being part of luminous matter to become baryonic dark matter. There are abundant numbers of such WD’s observed in our galaxy alone, and the interiors of the majority of them consist of Carbon, or Oxygen, or a mixture of the two, with mass density $\rho \sim (10^6 - 10^7) \text{ g/cm}^3$, although higher densities are also present in many of them (see, e.g., [1] and references therein). A typical WD starts off at temperature $T \sim (a \text{ few})\times10^8 \text{ K}$, and takes from a few to 10 Gyrs to cool down to about $10^5 \text{ K}$ or below, becoming then directly unobservable.

The WD cooling rate is strongly influenced by thermodynamic properties of the state of matter in the bulk of these stars, and in particular by its specific heat. The latter can be calculated if one knows dispersion relations of low energy excitations of the substance in WD’s [1]. As mentioned above, depending on concrete values of the temperature $T$, density $\rho$, and ion charge $Z$, the ions could end up being in a classical gas state, or may create a classical or quantum crystal, or may condense into a quantum liquid state. These different substances will have different low energy excitations, and hence, different specific heats and cooling rates. Knowing accurate values of these rates is important, e.g., for precise determination of the age of the Universe.

Irrespective of the microscopic structure of the resulting state of the ions in WD’s, the ion system may be treated as a uniform substance described by an appropriate equation of state, at length scales much greater than the average inter-ion separation. We’d like to study long-wavelength collective fluctuations in this neutral system.
Furthermore, we’ll be solely interested in the longitudinal low energy excitations for reasons that will become clear below. In particular, we’d like to understand the interplay between the zero sound mode, that typically exists in degenerate fermionic systems, and the ion acoustic wave that is usually present in a neutral plasma. What we’ll show is that these two modes are complementary to each other: when one of these modes is present the other one is absent. However, one of them is always present. We will also show that the cores of O and C WD’s with \( \rho \sim (10^8-10^9) \text{g/cm}^3 \) support only the ion acoustic wave; the zero sound mode could be present in those WD’s in a relatively narrow spherical shell, away from the cores if density in those domains is \( \rho < 10^5 \text{g/cm}^3 \).

In Section 1 we will set up a prototype effective field theory model to calculate dispersion relations for the low energy excitations. Our calculations will focus on temperatures below \((a \text{ few}) \times 10^7 \text{K}\). The two-body Coulomb energy of the ions is of the order of, \((10^4-10^6) \text{eV} = (10^8-10^9) \text{K}\). While the Coulomb interactions are screened by the electrons, the screening length is greater than the average inter-ion separation, thus leading only to a small reduction, at the level of 10% or so, of the unscreened two-body Coulomb interaction energy [4]. Therefore, for \( T < (a \text{ few}) \times 10^7 \text{K} \), the screened Coulomb energy dominates over the ion thermal energies, by at least an order of magnitude. Hence, we neglect the thermal effects. Likewise, finite temperature effects are negligible for the calculation of dispersion relations in the fermionic part of the system, since \( T/J_0^{1/3} \ll 1 \). Thus, in Section 1, we formulate an effective field theory at zero temperature, to calculate the dispersion relations for low energy modes. One can then use the standard formalism of finite temperature statistical mechanics to evaluate the effects of the dispersion relations on the values of the specific heat. This is a self-consistent procedure, as long as \( T/J_0^{1/3} \ll 1 \).

In Section 2 we recover all the results of Section 1 in a Coulomb gauge. The advantage of the latter is that the properties of the longitudinal collective modes are captured by the phase of a scalar field describing the system of spin-0 charged ions. It is in this section that we show that the O and C WD’s will dominantly support the ion acoustic wave in their cores, even before the ions turn into a crystal state.

Section 3 is dedicated to the He WD’s. There are WD’s that have a helium core due to removal of matter and energy from them by their binary companions (see discussions and references in [5]). Among these, furthermore, there is a very small subclass of the dwarf stars for which the temperature \( T_c \), at which the de Broglie wavelengths of the nuclei begin to overlap, is higher than the would-be crystallization temperature. Then, right below \( T_c \), the quantum-mechanical uncertainty in the position of the charged nuclei is greater than the average inter-nuclear separation. This is exactly opposite to the crystallized state where the nuclei would have well-localized positions with slight quantum-mechanical fuzziness due to their zero-point oscillations.

It was argued in Refs. [3, 6] that such a system, instead of forming a crystalline lattice, would condense owing to the quantum-mechanical probabilistic attraction of Bose particles to occupy one and the same zero-momentum state, and leading to a quantum liquid in which the charged spin-0 nuclei would form a macroscopic quantum state with a large occupation number the charged condensate.

The dispersion relations for the quasi-modes of the charged condensate were derived in [3, 6]. These results were obtained by using the unitary gauge, and the Thomas-Fermi (TF) approximation for the electrons. Using these results the cooling of the He core WD stars was studied in [5, 7], with the conclusion that they’d cool faster due to condensation, and that this prediction could be tested if a large enough sample of He core WD’s existed.

However, it was subsequently shown in [8], the TF approximation combined with the unitary gauge, misses one gapless mode, which could affect the cooling calculation. In particular, instead of using the TF approximation, Ref. [8], took into account the one-loop fermion effects, and unveiled the gapless mode. The same results were later confirmed in a gauge independent way in [9]. Furthermore, it was also shown in [8], that the gapless mode makes a negligible contribution to the specific heat at relevant temperatures and densities for He WD’s, and therefore the predictions for the fast cooling rates obtained in [5, 7] remain unchanged. The contribution of the gapless mode to cooling via neutrinos was discussed in [10], however this is also not a significant effect at the relevant temperatures.

In the present work we’ll confirm the results of Ref. [8] on the existence of the gapless mode in charged condensate, by doing calculations in both unitary and Coulomb gauges. The latter calculation turns out to be simpler, and does not require going beyond the TF approximation to establish the existence of the gapless mode (the one-loop result is needed though in the Coulomb gauge to calculate the imaginary part of this mode). Furthermore, the Coulomb gauge calculations lead us to the arguments that the gapless mode found in [8] is an ion acoustic wave, that is present in charged condensate even when the finite temperature effects are ignored.

A brief outlook is presented in the last section of the paper.

1. A PROTOTYPE MODEL

Consider a high density medium composed of nuclei and degenerate electrons, as discussed in the previous section. Depending on the temperature and/or composition, the nuclei could be in a state of plasma, crystal, or charged condensate. In all the three cases Lorentz invariance is broken by the medium; for a crystal the rotation and translation symmetries are also broken to their discrete subgroups. We consider this medium at length
scales larger than the average inter-particle separations, and would like to study long wavelength longitudinal collective modes. We use an effective field theory technique to write down the Lagrangian that respects symmetries of the system at hand. In this section we discuss a prototype Lagrangian.

The ions are much heavier than the electrons, and at low momenta the effects they produce are presumed to be modeled by introducing in the Maxwell Lagrangian the “electric” and “magnetic” masses, denoted by \( m_0 \) and \( m_\gamma \), respectively (electrons and their effects will be accounted for a bit later)

\[
\mathcal{L}_A = \frac{1}{2} \left( E_j^2 - H_j^2 + m_0^2 A_0^2 - m_\gamma^2 A_j^2 \right),
\]

where \( E_j \) and \( H_j, j = 1, 2, 3 \), are the components of the electric and magnetic fields, and unless stated otherwise we use the units \( c = \hbar = 1 \), for which the vacuum values of the dielectric constant and magnetic permeability are set to unity.

The presence of the “electric” and “magnetic” mass terms in (1) captures the phase of the spin-0 ion matter, as will be more clear below. This phase is what’s capturing the relevant low energy physics of longitudinal modes. In reality, the parameters \( m_0 \) and \( m_\gamma \) would be scale dependent quantities, and should account for renormalization of the vacuum values of the dielectric constant and magnetic permeability, due to the quantum effects of the ions. However, we are interested in the low momentum and frequency limit, where these parameters can be approximated by constant \( m_0 \) and \( m_\gamma \). The scale dependent renormalization of these quantities due to the electrons will be explicitly included below. We’ll show that the outlined approximations give a reasonable description for longitudinal modes in the plasma and crystal (Sections 1 and 2), however, they break down for the charged condensate (Section 3), where the explicit scale dependence of the “electric mass” cannot be ignored.

In particular, the prototype model allows for a simple and clear description of the conditions for the existence of a zero sound and ion acoustic wave in the degenerate plasma and crystal; with a certain modification, derived in Section 3, this model can also describe the collective fluctuations of the charged condensate.

The Lagrangian (1) is not gauge invariant. The only physical meaning of the latter statement is that (1) describes more degrees of freedom than the Maxwell theory. Local gauge invariance can always be restored at the expense of introducing new fields, and therefore, it is a redundancy of the description (although a convenient one). In particular, (1) can always be regarded as a gauge-fixed version of a gauge invariant Lagrangian obtained from (1) by the substitution, \( A_\mu \rightarrow B_\mu = A_\mu - \partial_\mu \alpha \), with the invariance transformation, \( \delta A_\mu = \partial_\mu \gamma, \delta \alpha = \gamma \), where \( \gamma \) is a gauge transformation parameter, and \( \mu = 0, 1, 2, 3 \).

The phase field \( \alpha \) makes explicit the presence of the degree(s) of freedom beyond the two transverse states that can be attributed to the gauge field \( A_\mu \), when a nonzero \( \alpha \) is retained.

Conversely, (1) can be regarded as the Lagrangian in the so-called unitary gauge, \( \alpha = 0 \). We use this gauge in the present Section, while in Section 2 we restore back a nonzero \( \alpha \), and use instead the Coulomb gauge for the gauge field, \( \partial^j A_j = 0 \). The results in the two gauges will naturally be the same, but the two derivations are different, each having its own advantages for understanding of the final results.

Irrespective of the gauge choice, the system described by (1) contains an extra longitudinal degree of freedom, in addition to the two transverse modes of a photon. The longitudinal mode can be thought of as a collective low energy excitation of the charged ion background. Since we have not introduced yet the neutralizing electrons in (1), the spectrum of excitations is gapped by the parameter \( m_\gamma \). As long as the latter is smaller than the inverse inter-particle distance, all three gapped modes can still be meaningfully described by the low energy Lagrangian.

We now introduce the electrons. Instead of writing an effective Lagrangian for them, we use the fundamental description in terms of the Dirac theory. This is justified: the ions are much heavier than the electrons, and at energy scales below the ion mass their collective longitudinal mode is captured into an effective Lagrangian (1), while the electrons can be kept fundamental as long as they are weakly interacting, and as long as their loop effects will in the end be taken into account (see below). Hence, the total low-energy prototype Lagrangian for the electron-ion system reads as follows:

\[
\mathcal{L} = \mathcal{L}_A + \mathcal{L}_F, \quad \mathcal{L}_F = \bar{\psi}(i\gamma^\mu D_\mu - m_e)\psi.
\]

Here we have introduced the chemical potential for the electrons \( \mu_e \) via the usual prescription on the covariant derivative \( D_0 = D_0 - i\mu_e \). We will also package the electric and magnetic fields in a Lorentz-invariant Maxwell form, with the Lorentz-violating effects of the nuclei summarized by \( m_0 \) and \( m_\gamma \); this is pending additional Lorentz-violating effects to arise due to the electrons. To account for the latter, we look at the contribution of the electrons to the Lagrangian at the one-loop level, namely via the photon self-energy \( \Pi^{\mu\nu} \). Then, the Lagrangian density for the fluctuations in the quadratic approximation is given by

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m_0^2 A_0^2 - \frac{1}{2} m_\gamma^2 A_j^2 + \frac{1}{2} A_\mu \Pi^{\mu\nu} A_\nu.
\]

Note that the inclusion of a one-loop expression for \( \Pi^{\mu\nu} \) in the effective Lagrangian (3) to determine the dispersion relations, yields the same results for the dispersion relations, as if they were deduced from the poles of Green’s functions in which the one-loop bubble diagrams have been resummed (such a resummation is often referred as the random phase approximation).

It will be convenient to work in the momentum space, and use the Fourier transform of \( \Pi^{\mu\nu} \) that we denote by \( \Pi^{\mu\nu} \). Due to the conservation of the fermion number, and rotation symmetry, \( \Pi^{\mu\nu} \) can be expressed in terms of two functions, \( \Pi(\omega, k) \) and \( \Pi^-(\omega, k) \), where \( k = |\vec{k}| \),
and takes the form

\[ \Pi^{\mu\nu} = \left( \begin{array}{cc} \frac{\omega_k}{k^2} & \frac{\omega_k}{k}\Pi \\ \omega_k & -\frac{\omega_k}{k} - \frac{k}{m_e} \end{array} \right)\Pi \right) . \quad (4) \]

If we decompose the photon into transverse, longitudinal, and time-like components, \( A_0, A^x, A_\omega \), the Lagrangian for the transverse modes decouples from that of the longitudinal and time-like components. In momentum space, we have

\[ L^\perp = \frac{1}{2} A^\perp (\omega^2 - k^2 - \Pi^\perp - m_e^2) A^\perp , \]

\[ L^\perp_2 = \frac{1}{2} (A_0 A^L) \cdot \mathcal{M} \left( \begin{array}{c} \omega_k \\ \omega \end{array} \right) , \quad (6) \]

where

\[ \mathcal{M} = \left( \begin{array}{cc} k^2 (1 + \frac{\Pi}{k^2}) + m_e^2 & \omega k \left( 1 + \frac{\Pi}{k^2} \right) \\ \omega k (1 + \frac{\Pi}{k^2}) & \omega^2 \left( 1 + \frac{\Pi}{k^2} \right) - m_e^2 \end{array} \right) \right) . \quad (7) \]

The dispersion relations of the two transverse modes are clearly given by

\[ \omega^2 - k^2 - \Pi^\perp (\omega, k) - m_e^2 = 0 . \]

The dispersion relations for the remaining modes are given by the zeros of the determinant of the matrix \( \mathcal{M} \),

\[ \det \mathcal{M} = (m_e^2 \omega^2 - m_e^2 k^2) \left( 1 + \frac{\Pi(\omega, k)}{k^2} \right) - m_e^2 m_e^2 = 0 . \]

If we were to neglect the contribution of the electrons, i.e., to set \( \Pi (\omega, k) = 0 \), we would have found from (9) one massive longitudinal mode

\[ \omega^2 (k \to 0) = m_e^2 , \]

(10)

alongside with two massive transverse modes described by (8). Let us see now how the effects of the electrons, encoded in \( \Pi (\omega, k) \) and \( \Pi^\perp (\omega, k) \), modify this spectrum.

The expression for the non-vacuum contribution to the one-loop self energy of the photon due to electrons at finite chemical potential is standard and can be found, for example, in [11, 12]. In the approximation of small \( \omega \) and \( k \) as compared to the Fermi energy and momenta, and for \( \omega \not= v_F k \), this expression is given by

\[ \Pi (\omega, k) \approx \frac{\mu_e^2}{\pi^2} \left( 1 - \frac{1}{2} k \omega \right) \ln \left( \frac{\omega + k \omega}{\omega + k v_F} \right) , \]

(11)

where the Fermi velocity \( v_F \) is defined as \( v_F = k_F / m_e \). The usual Debye screening mass is denoted by \( m_s \)

\[ m_s^2 \equiv \Pi (\omega = 0, k \to 0) = \frac{\mu_e^2}{\pi^2} . \]

(12)

The above expression is for relativistic electrons, while in the non-relativistic case one has: \( m_s^2 = c^2 m_e k_F / \pi^2 \).

Using equation (11), we can now check for both massive and massless longitudinal modes. For massive modes we take \( k \to 0 \) while keeping \( \omega \) finite. We find,

\[ \omega^2 (k \to 0) = m_s^2 + \frac{1}{4} m_s^2 v_F^2 . \]

(13)

Thus the Debye screening mass contributes to the longitudinal mode of the massive photon.

To check for a massless pole, we set \( \omega = x v_F k \), and also introduce the following notations:

\[ a^2 = \frac{m_s^2}{v_F^2 m_0^2} = \frac{v_F^2}{V_F^2} , \quad b^2 = \frac{m_s^2}{v_F^2 m_s^2} = \frac{v_F^2}{V_F^2} . \]

(14)

We then take the \( k \to 0 \) limit while keeping \( x \) fixed. Then \( \det \mathcal{M} = 0 \) corresponds to,

\[ \left( x^2 - a^2 \right) \left( 1 - \frac{x}{2} \ln \left[ \frac{x + 1}{x - 1} \right] \right) = b^2 . \]

(15)

The right hand side of the above expression is clearly both real and positive. Thus, in order for a solution to this expression to exist, there must be some value of \( x \), either real or complex, for which the left hand side is both real and positive. We’ll investigate these solutions below.

1.1 Zero Sound

Consider real \( x \) that is also large, \( x \gg 1 \). Then equation (15) takes the form

\[ \left( x^2 - a^2 \right) \left( 1 - \frac{1}{3x^2} - \frac{1}{5x^4} + ... \right) = b^2 , \]

and the solution is

\[ x^2 \approx \frac{5a^2 - 3}{15b^2 + 5} . \]

(16)

Our assumption of \( x \gg 1 \) is valid as long as \( a \gg \sqrt{1 + 3b^2} \). In that case, \( x^2 \approx a^2 / (1 + 3b^2) \), and the dispersion relation for the massless mode is

\[ \omega \approx \frac{v_0 k}{\sqrt{1 + 3m_e^2 / m_0^2}} . \]

(17)

Thus, the velocity of the massless mode (both phase and group) is given by \( v_0 / \sqrt{1 + 3b^2} \). The mode exists when \( v_0 > \sqrt{v_F^2 + 3m_s^2 / m_0^2} \), because only in that case \( x \gg 1 \). Furthermore, in our system of electrons and ions, \( v_F \gg v_s \), and (17) can be approximated by the dispersion relation \( \omega \approx v_0 k \).

Note that the slope of the linear dispersion relation of the mode (17) is larger than \( v_F \) since \( x \gg 1 \), and thus the \( \omega (k) \) line never intersects a region in which the near-the-fermi-surface modes live (see Fig. 1). Hence, dumping of the mode into electron-hole pairs is negligible.

This excitation corresponds to the zero sound mode that exists in interacting degenerate Fermi systems, e.g., in repulsively interacting Fermi liquids and gases alike [13]. More will be said on this in the next section.
For both real $x$ and purely imaginary $x$, we see that $W(x)$ is real. However, only when $a > 1$ can $W(x)$ be positive and thus solve (15).

Finally, we check for complex $x$. Setting $x = \sigma + i\beta$, with $\sigma, \beta \in \mathbb{R}$ and $\sigma, \beta \neq 0$, and using the representation (21) we find through some basic algebra that either $\text{Im} W(x) \neq 0$ or $\text{Re} W(x) < 0$. Thus only in the case that $a > 1$ is $W(x)$ both real and positive. Otherwise, we cannot solve (15) and no massless pole exists on the first Riemann sheet. This completes the proof.

Thus we look for solutions on the second Riemann sheet, and rewrite the equation (15) in the following form

$$\left(x^2 - a^2\right) \left(1 - \frac{x}{2} \left(\ln \left[\frac{1+x}{1-x}\right] - i\pi\right)\right) = b^2. \quad (22)$$

In the approximation of small $x$, this has a solution:

$$x \simeq \sqrt{a^2 + b^2} - i\frac{\pi b^2}{4}. \quad (23)$$

Thus the physical mode at hand has the following dispersion relation

$$\omega \simeq \left(\sqrt{v_s^2 + v_s^2} - i\frac{\pi v_s^2}{4v_F}\right) k. \quad (24)$$

The sign of the imaginary parts above is such that the mode is on the second Riemann sheet of the complex energy plane. Thus, it's a resonance, with a momentum dependent width. In what follows we will argue that this is nothing but the ionic acoustic wave.

Consider the $m_0 \to \infty$ limit; the above mode does exist in this limit and its dispersion relation reads

$$\omega = (v_s - i\frac{\pi v_s^2}{4v_F}) k. \quad (25)$$

It is instructive to go back to the Lagrangian and calculate the exact $k$ dependence of the mode, not only its low $k$ limit. The $m_0 \to \infty$ limit settles us in the regime $A_0 = 0$. The Lagrangian then takes the form:

$$\mathcal{L} = \frac{1}{2} (\partial_i A_j)^2 - \frac{1}{4} F_{ij}^2 - \frac{1}{2} m^2 A_j^2 + \frac{1}{2} A_i \Pi^{ij} A_j. \quad (26)$$

This Lagrangian describes four degrees of freedom: two massive helicity-1, one massive helicity-0, and one massless helicity-0 states; the latter is the mode with the dispersion relation (25). To see all this we proceed by splitting the transverse and longitudinal parts of $A_j$ as follows:

$$A_j = A^+_j + \frac{\partial_j \pi}{m_\gamma}. \quad (27)$$

Upon this substitution the Lagrangian splits into two separate parts, one for $A^+_j$,

$$\frac{1}{2} (\partial_i A^+_j)^2 + \frac{1}{4} (A^+_j)^2 - \frac{1}{2} m^2 (A^+_j)^2 + \frac{1}{2} A^+_i \Pi^{ij} A^+_j,$$
grangian (28) are determined by the equation
\[ \omega_\perp^2 = k^2 + m_\gamma^2 + \Pi_\perp(\omega_\perp, k), \]  
while the dispersion relations that follow from the Lagrangian (28) are determined by the equation
\[ \omega^2 = \frac{m_\gamma^2 k^2}{k^2 + \Pi(k, \omega)}. \]

In the absence of \( \Pi \) (i.e., if the electrons are assumed to be frozen) we get from (30)
\[ \omega^2 = \frac{2Ze^2 J_0}{m_H}. \]

The latter is a plasmon dispersion of a charged ion gas. In this limit of “frozen” electrons (\( \Pi \to 0 \)) there is nothing to compensate for a displaced change of ions upon their perturbation, and therefore the mode is gapped. Note that in the \( \Pi = \Pi^\perp = 0 \) limit the longitudinal and transverse modes have the same gap, \( \omega(k = 0) = \omega_\perp(k = 0) = m_\gamma \), consistent with the rotation symmetry.

Once the dynamical electrons are returned back, then the Lagrangian (28) and the dispersion relation (30) describe two modes, one massive and one massless. Let us start with the massive mode, taking \( k \to 0 \) in (30) with fixed and nonzero \( \omega \), we find \( \Pi \simeq -m_\gamma^2 v_F k^2/3\omega^2 \), and get a solution \( \omega^2 = m_\gamma^2 + m_s^2 v_F^2/3 \), which agrees with (13). The latter sets the gap for the longitudinal photon.

For small \( \omega \) and \( k \), on the other hand, we get a massless mode as follows: in this regime the real part of \( \Pi \), \( Re \Pi \simeq m_s^2 \), and therefore, the real part of the corresponding dispersion relation reads
\[ \omega = \sqrt{m_\gamma^2 k^2/k^2 + m_s^2} \simeq v_s k. \]

This dispersion relation coincides with the \( m_0 \to \infty \) limit of the one in (24), up to the imaginary part that can also be straightforwardly recovered by keeping it in \( \Pi \). Therefore, this mode is an ion acoustic wave; sometimes it’s also referred as an ionic sound [14]. It corresponds to a collective excitation of the ions and neutralizing fast electrons, producing a gapless longitudinal mode.

A few comments before we turn to the next section. We reiterate that the propagator in the considered regime has no low energy poles on the first Riemann sheet, neither real nor complex ones. The obtained complex pole describing the ionic sound is on the second sheet. The function that the propagator is proportional to
\[ \frac{1}{(x^2 - a^2)\Pi - m_\gamma^2}. \]

has a branch cut and exhibits a “bump-like” behavior for real values of energy and momentum. Why does this mode have a resonance-like nature? The reason is simple: the phase velocity of the mode equals to \( xv_F \), that is less than \( v_F \). The range of allowed energies of the electron-hole pairs near the Fermi surface (depicted in Fig. 1) have the upper boundary with the slope near the origin that equals to \( v_F \). Thus, a wave with the dispersion \( \omega \approx xv_F k \) would be damped due to near-the-Fermi-surface excitations, as long as \( x < 1 \).

This also explains why this mode does not exist for \( x \gg 1 \); if it existed it would have been faster than the near-the-Fermi-surface excitations, which then would not be able to catch up with the mode to give rise to a neutral acoustic wave; instead, in the \( x \gg 1 \) regime only the zero-sound mode exists. Thus, the zero sound and ion acoustic wave are complementary to each other.

2. THE PHASE FIELD AND LONGITUDINAL MODES

The derivations in the previous section were performed in the unitary gauge, i.e., the phase of the charged scalar field describing collective motion of the ions was gauge-fixed to zero. This was reflected in the gauge non-invariant terms in Lagrangian (1). The latter was regarded as a gauge-fixed version on a gauge invariant Lagrangian, obtained from (1) by a substitution, \( A_\mu \to B_\mu \equiv A_\mu - \partial_\mu \alpha \), with the unitary gauge corresponding to \( \alpha = 0 \). In this section we’d like to keep nonzero \( \alpha \), but we use instead the Coulomb gauge \( \partial^\mu A_\mu = 0 \). This will enable us to obtain and understand the results of the previous section in a more clear way.

2.1 A Warmup Example

We start with a field theory containing a complex scalar field, degenerate fermions, and an Abelian gauge field that couples to both the scalar and fermions in a conventional way. The charged scalar field is thought to model properties of charged nuclei, while fermions model the electrons, and the Abelian gauge boson models a photon. For illustrative purposes, we consider the case when the \( U(1) \) gauge symmetry is spontaneously broken, and the gauge field acquires the mass term as a result of this breaking. Furthermore, we consider the parameter space for which the scalar field radial mode is a heavy state that can be decoupled from the rest of the fields. In this approximation, and in the Coulomb gauge, the relevant part of the Lagrangian reads as follows:
\[ \frac{1}{2} m_s^2 B_\mu^2 + \frac{1}{2} A_\mu \left( -\Delta + \Pi \right) A^\mu. \]

The first term is the gauge boson mass term, while the second term contains a part of the Maxwell Lagrangian,
as well as the term generated due to one-loop renormalization of the polarization operator via the fermion-antifermion pair. These are all the terms in the Lagrangian that contain $A_0$ and $\alpha$.

Integrating out $A_0$ we get for $m^2 \neq 0$

$$A_0 = \frac{m^2}{(-\Delta + m^2 + \Pi)} \partial_0 \alpha.$$  \hfill (34)

(we neglect an irrelevant zero mode of the inverse of the operator multiplying $\partial_0 \alpha$ on the right hand side of (34)). Substituting (34) back into the Lagrangian (33), and calculating the dispersion relation for the remaining field $\alpha$, we obtain the following expression

$$\omega^2 \left( k^2 + \Pi(\omega, k) \right) = k^2.$$ \hfill (35)

Let us now study this relation in two different regimes, $x \gg 1$, and $x \ll 1$, where as before we introduce $\omega = xv_Fk$. For the $x \gg 1$ case, one gets $\Pi \approx -m_s^2/(3x^2)$. Hence, the solution to (35) reads:

$$\omega = \frac{v_F}{\sqrt{v_F^2 + 3m^2/m_s^2}} k.$$ \hfill (36)

When $m_s \gg m$ and $v_F \ll 1$, we get the condition, $x \gg 1$, required by the approximation made. As per the arguments of the previous section, this is the dispersion relation of the zero sound mode

$$\omega = \frac{v_F}{\sqrt{v_F^2 + 3m^2/m_s^2}} k.$$ \hfill (37)

We see that in the Coulomb gauge the zero sound mode is described by the phase $\alpha$. Since $x \gg 1$, the phase/group velocity of this mode is greater than $v_F$, hence, this mode experiences no damping in the approximation we use.

In the opposite regime, $x \ll 1$, the gapless mode does not exist, as shown by reducito ad absurdum: assuming that $x \ll 1$, one gets $\Pi \approx m_s^2 (1 + \mathcal{O}(x))$, and the solution $x = v_F^{-1} \sqrt{1 + m_s^2/m^2} > 1$, which contradicts the initial assumption that $x$ were small.

The lack of the ion sound wave in this model has a reason: we used the approximation when the Lorentz-invariant vacuum condensate of the charge scalar is the only source for the mass term (the first term in (33)); this implies that the number of dynamical scalars that can fluctuate is zero, in the approximation used (the Lorentz invariant Higgs vacuum has zero scalar number). Hence, one should not expect to have the ion acoustic wave, in the approximation when the number of ions that can fluctuate is ignored (the charge that neutralizes the electrons in this approximation is not dynamical).

### 2.2 A More Realistic Model

We now consider the case when the “electric” and “magnetic” masses of the photon are different, and the phase is explicitly kept, while the Coulomb gauge is assumed for $A_1$, as in the previous Subsection. The relevant part of the Lagrangian in this gauge then reads:

$$\frac{1}{2} \left( m_0^2 B_0^2 - m_s^2 B_s^2 \right) + \frac{1}{2} A_0 \left( -\Delta + \Pi \right) A_0.$$ \hfill (38)

As before, we integrate out $A_0$ to get

$$A_0 = \frac{m_0^2}{(-\Delta + m_0^2 + \Pi)} \partial_0 \alpha.$$ \hfill (39)

Substituting this back into (38), and deducing the dispersion relation for $\alpha$ we obtain

$$\omega^2 \left( k^2 + \Pi(\omega, k) \right) = v_0^2 k^2, \quad v_0^2 = \frac{m_s^2}{m_0^2}.$$ \hfill (40)

In a realistic system such as a plasma and solid, $m_s \gg m$, and the ion sound speed, $v_0 \ll 1$.

For $x \ll 1$, we know that $\Pi \approx -m_s^2/(3x^2)$, and in this approximation the solution to (40) reads

$$x^2 = \frac{1}{v_F^2/v_0^2 + 3m_0^2/m_s^2}. \quad (41)$$

Hence, when $m_s > m_0$ and $v_0 > v_F$, we get $x \gg 1$, and the zero sound dispersion relation

$$\omega \approx \frac{v_F}{\sqrt{v_F^2/v_0^2 + 3m_0^2/m_s^2}} k.$$

This coincides with the dispersion relations for the zero sound mode found in (17).

Unlike in the previous Subsection, however, a solution also exists in the opposite limit, $x \ll 1$: indeed, assuming that $x \ll 1$, one gets $\Pi \approx m_s^2 (1 + \mathcal{O}(x))$, and hence the solution $x = \frac{m_s}{v_0} \sqrt{1 + m_0^2/m_s^2}$, which if, $v_0 \sqrt{1 + m_0^2/m_s^2} \ll v_F$, gives rise to the following physical mode

$$\omega \approx v_0 \sqrt{1 + \frac{m_0^2}{m_s^2}} k.$$ \hfill (43)

This is nothing but the dispersion relation for the longitudinal ion sound wave discussed in Section 1.

One can give heuristic arguments for the complementarity of the two modes that we’ve derived above. For this, we consider the regime in which $v_s$ can be neglected as compared with $v_0$ and $v_F$. This is a meaningful approximation for a plasma and crystal since $m_s$ is suppressed by the heavy ion mass scale, while $m_s$ is determined by the electron mass and chemical potential. Then, the phase velocity of both the zero sound mode and the ion acoustic wave is approximately $v_0$. When $v_0 \ll v_F$, we get $x \ll 1$ and the ionic sound is present; this makes sense – the electrons are faster than the wave and thus they can readily follow a perturbation of the ions to screen its charge and create a gapless neutral mode, the ionic sound. In the opposite limit, $v_0 \gg v_F$, the electrons are slower than the would be ionic sound...
wave, therefore they cannot be effectively screening ion perturbations, and it makes sense that the ion acoustic wave does not exist. Instead, the electron fluctuations themselves — which are now effectively screened by the ambient mobile ion charge distribution — form a collective mode, the zero sound.

Having the dispersions relations derived, let us apply them to the system of the Oxygen and Carbon ions at densities \( \rho \sim (10^5 - 10^7) \text{ g/cm}^3 \). As we’ve already discussed the corresponding Fermi momentum for the electrons is \( p_F \sim (0.3 - 0.9) \text{ MeV} \). Therefore, the electrons are (nearly) relativistic, with \( v_F \sim 1 \), to a good accuracy. This implies that the value of \( x \) given in (41) can never be greater than the unity, since \( v_0 < 1 \). Therefore, we conclude that the zero sound cannot be supported in the cores of the \( O \) and \( C \) WD’s. Instead, the dominant longitudinal mode in this case is the ion acoustic wave (43), with the sounds speed approximated by \( v_0 \ll 1 \). The exact value of the sound speed cannot be calculated in our effective Lagrangian approach, however, our finding confirms the suggestion made in [2] that a longitudinal wave, the one that’s similar to the longitudinal acoustic wave of a crystal, can be used to study the cooling of the \( O \) and \( C \) WD’s even when the interior is in a strongly interacting plasma state, and the crystal is not yet formed (i.e., from temperatures \( (a \ \text{few}) \times 10^7 \ K \), down to \( (a \ \text{few}) \times 10^6 \ K \), after which the crystal forms).

As to the zero sound mode, its existence requires lower densities. While WD’s have fairly uniform density profiles in their bulks (excluding their “atmospheres” that are dominantly made of \( H \) and \( He \), with some small fractions of “metals”), nevertheless the interiors are not exactly uniform. Typically one can have variation of density amounting to a factor of 5 in the ratio of the maximal density to average density. Therefore, it might not be foolish to think of relatively narrow spherical shells, away from the cores of low density WD’s, in which densities might be \( \rho \sim 10^3 \text{ g/cm}^3 \). In such shells the zero sound mode could be supported instead of the ion acoustic wave. It could also be interesting to look for the existence of the zero sound mode in the low mass brown dwarfs [15], where densities are \( \rho \sim 10^3 \text{ g/cm}^3 \).

A classical nonzero vacuum expectation value of a field \( \Phi \) can serve as an order parameter for the condensation of the helium-4 nuclei, describing a state with a large occupation number. Fluctuations of the order parameter describe the collective modes of the condensate. In the non-relativistic approximation, the effective Lagrangian for the nuclei \( \Phi \) and electrons \( \psi \), can be written as follows [6]:

\[
\mathcal{L} = \mathcal{P}(X) - \frac{1}{4} F_{\mu \nu}^2 + \psi (i \partial \Phi - m_\epsilon) \psi, \tag{44}
\]

where

\[
X = \frac{i}{2} (\Phi^* D_\mu \Phi - (D_\mu \Phi)^*) - \frac{m_\epsilon^2}{2 m_H^2}. \tag{45}
\]

The covariant derivative of the scalar field is given by \( D_\mu = \partial_\mu - e \epsilon A_\mu \). \( \mathcal{P}(X) \) stands for a general polynomial function of its argument. The coefficients of this polynomial are dimensionful numbers that are inversely proportional to powers of a short-distance cutoff of the effective field theory. We normalize the first coefficient to one so that

\[
\mathcal{P}(X) = X + c_2 X^2 + \ldots. \tag{46}
\]

The Lagrangian is invariant under global \( U_e(1) \) transformations, responsible for the conservation of the number of scalars. Another global \( U_e(1) \) guarantees the electron number conservation. Accordingly, we can introduce chemical potentials for both the scalars and electrons, \( \mu_s \) and \( \mu_e \) respectively, via the usual prescription on the covariant derivative \( D_\mu \rightarrow D_\mu - i \mu \).

We could also have included a quartic interaction for the scalars \( \lambda (\Phi^* \Phi)^2 \). However, as long as the quartic coupling is small, \( \lambda n \ll m_H^2 \), our results will not be affected by this term.

It is useful to represent the scalar as \( \Phi = \Sigma e^{i \tau} \), and to work first in the unitary gauge where the phase of the scalar is set to zero, \( \Gamma = 0 \). At a later stage, we’ll move instead to the Coulomb gauge where things will become easier.

When the scalar chemical potential is zero, \( \mu_s = 0 \), there is a solution to the equations of motion of (44) with a nonzero expectation value for the scalar field

\[
\langle \Sigma \rangle = \sqrt{\frac{J_0}{2}}, \tag{47}
\]

where \( J_0 \) is the background electron density. Let us consider the quadratic action around this background solution. We introduce perturbations of the scalar field as follows:

\[
\Sigma(x) = \sqrt{\frac{J_0}{2}} + \sqrt{m_H^2} \tau(x). \tag{48}
\]

The factor of \( \sqrt{m_H^2} \) has been introduced for convenience. Let us also integrate out the electrons as was done in [8]. Thus the Lagrangian density for the fluctuations in the quadratic approximation reads

\[
\mathcal{L}_2 = -\frac{1}{4} F_{\mu \nu}^2 + \frac{1}{2} m_0^2 A_0^2 - \frac{1}{2} m_0^2 A_0^2 + \frac{1}{2} A_\mu \Pi^{\mu \nu} A_\nu - \frac{1}{2} (\partial_0 \tau)^2 + 2 m_H m_\epsilon A_0 \tau. \tag{49}
\]
Here
\[ m_0^2 = \left(2e\right)^2 \frac{J_0}{2m_H}, \quad m_0^2 = c_2 J_0 m_H m_s^2, \] (50)
When the coefficient \(c_2\) is such that \(m_0 = m_\gamma\), the dispersion relations coincide with those of a relativistic theory, however, in general \(m_0 \neq m_\gamma\).

### 3.1 Spectrum in the unitary gauge

We now consider the spectrum of these fluctuations. We decompose the photon into transverse, longitudinal and time-like components. In addition, let us also integrate out the scalar mode \(\tau\) so that we can easily compare our results to those of the previous section. Again, the transverse modes of the photon decouple entirely. Their dispersion relations are given by
\[ \omega^2 - k^2 - \Pi^4(\omega, k) - m_s^2 = 0. \] (51)

For the remaining modes, we have
\[ \mathcal{L}^2 = \frac{1}{2} \left( A_0, A_L \right) \cdot \mathbb{M} \cdot \left( A_0, A_L \right), \] (52)
where now
\[ \mathbb{M} = \left( \begin{array}{cc} k^2 \left(1 + \frac{\Pi}{\omega} \right) + \frac{4M^4}{m^2} & \omega k \left(1 + \frac{\Pi}{\omega} \right) \\ \omega k \left(1 + \frac{\Pi}{\omega} \right) & \omega^2 \left(1 + \frac{\Pi}{\omega} \right) - m_s^2 \end{array} \right), \] (53)
with \(M \equiv \sqrt{m_\gamma m_\pi}\). The matrix \(\mathbb{M}\) differs from the analogous matrix in our prototype model only via the appearance of the \(4M^4/k^2\) term.

The dispersion relations of these modes are again given by the zeros of the determinant of the matrix \(\mathbb{M}\). To check for the presence of massive modes, we use expression (11) in the matrix (53) and take the \(k \to 0\) limit, keeping \(\omega\) finite. Setting \(\det \mathbb{M} = 0\), we find a massive mode
\[ \omega^2(k \to 0) = m_\gamma^2 + \frac{1}{4} m_s^2 v_F^2. \] (54)
This can be though as the mass of the longitudinal component of the photon, that now carries three degrees of freedom. The mass receives two contributions, one from the charged nuclei, \(m_\gamma^2\), and another one from charged electrons, \(m_s^2 v_F^2/3\). Hence, we find that the one-loop contribution shifts up the mass of the longitudinal mode. This mode was argued to be heavy to contribute to specific heat of the charged condensate at relevant temperatures, [6]; we see that the one loop contribution makes it even heavier, justifying further that it can be neglected.

To determine the possibility of a massless pole, we set \(\omega = x v_F k\) and then take the \(k \to 0\) limit. Then \(\det \mathbb{M} = 0\) corresponds to,

\[ x^2 \left(1 - \frac{x}{2} \ln \left[ \frac{x + 1}{x - 1} \right] \right) = \frac{m_\gamma^2}{m_s^2}. \] (55)
Thus we see that the charged condensate corresponds to our prototype model of Section 1 in the particular case with \(a = 0\). As argued above, there is a massless resonance solution, with the dispersion relation \(\omega \simeq (\nu_s - \frac{\pi^2}{M_\pi^3}) k\), in agreement with [8]. This is also in agreement with the dispersion relation obtained in Section 1, describing the ion acoustic wave in a plasma. In our view, this mode has the same physical origin and interpretation as the ion sound wave of ordinary degenerate plasma [14]. We further strengthen this latter point in Subsection 3.3 by unveiling the hydrodynamics origin of the ion acoustic wave in charged condensate.

### 3.2 Spectrum in the Coulomb gauge

Here, we derive the results of the previous subsection in the Coulomb gauge. This makes the connection to the calculations in the prototype model presented in Section 2 clearer, and helps to explain the origin of the ion sound wave in the charged condensate.

As before, we use \(B_\mu = A_\mu - \partial_\mu \alpha\). The relevant part of the Lagrangian in the Coulomb gauge reads as follows:

\[ \frac{1}{2} \left( m_0 B_0^2 - m_s^2 B_x^2 \right) + \frac{1}{2} A_0 \left( -\Delta + \Pi \right) A_0, \] (56)
where the key difference from the model of Sections 1 and 2 is that “electric” mass in the charged condensate has essential dependence on the momentum:

\[ m_0 = \left( m_0^2 + \frac{4M^4}{-\Delta} \right). \] (57)
This suggests that for small momenta, we always end up in the regime of large “electric” mass for the photon; as discussed in Section 1, this implies that the zero sound mode will be absent, but the ionic acoustic wave should be present, in agreement with the results of Subsection 3.1. Let us see this explicitly in the Coulomb gauge:

We first integrate out \(A_0\)

\[ A_0 = \frac{m_0}{\left(-\Delta + m_0^2 + \frac{4M^4}{\Delta} + \Pi\right)} \partial_\alpha \alpha. \] (58)
Substituting this back into the Lagrangian (56), and deducing the dispersion relation for \(\alpha\) we get

\[ \omega^2 \frac{k^2 + \Pi(\omega, k)}{k^2 + \Pi(\omega, k) + m_0^2 + \frac{4M^4}{k^2}} = v_F^2 \frac{m_0^2}{m_0^2 + \frac{4M^4}{k^2}} k^2. \] (59)
As before, we use the notation \(\omega = x v_F k\). For both \(k\) and \(\omega\) approaching zero, we only get one dispersion relation, and that is with, \(x \ll 1\). This dispersion relation reads:

\[ \omega \simeq \frac{m_\gamma}{m_s} k. \] (60)
This is the ionic acoustic wave, in agreement with previously obtained results.
A final comment before we move to the hydrodynamical considerations. In the approach adopted above the fermions were treated in a one loop approximation, while the scalars in terms of a low energy effective field theory. The quantum effects of the condensed scalars were captured by the order parameter Lagrangian. It is instructive to check that the same results are obtained if the scalars are also treated via the one-loop calculations, as was done in [16, 17]. We briefly use this method in the Coulomb gauge. Then, the poles of the full propagator are determined by
\[ k^2 + \Pi^B(\omega, k) + \Pi(\omega, k) = 0, \]  
where the polarization operator for the bosons, \( \Pi^B(\omega, k) \) can be calculated straightforwardly via the corresponding one loop diagrams [16, 17]. The part of the loops that is due to the existence of the condensate reads as follows:
\[ \Pi^B(\omega, k) = \frac{m_{\pi}^2}{2} \left[ \frac{(2m_H - \omega)^2}{(\omega - m_H)^2 - k^2 - m_H^2} \right. \]
\[ \left. + \frac{(2m_H + \omega)^2}{(\omega + m_H)^2 - k^2 - m_H^2} \right] - 2 \].  
Substituting this into (61) and taking the small momentum limit, we get the pole at
\[ \omega \simeq \frac{m_{\pi}^2}{m_s} k. \]  
The latter coincides with the result already obtained above (60).

### 3.3 Hydrodynamic considerations

The purpose of this subsection is to demonstrate that the ionic sound found in the prototype model, as well as its counterpart emerging in charged condensate, can be understood in terms of standard hydrodynamics, with the only difference that the charged condensate hydrodynamics equations need to retain the pressure gradient term even when finite temperature effects are ignored, as will be shown below.

We are looking at a degenerate plasma of electrons and positively charged nuclei (or ions); as already mentioned, depending on temperature \( T \), the ionic mass \( m_H \) and charge \( Z \), the system of nuclei could be in a classical gas state, or may create a Wigner crystal, or may be in a condensed quantum liquid state. In either case, at length scales much greater than the inter-ion separation, the ion system may be treated as a uniform substance described by an appropriate equation of state. We’d like to understand the spectrum of long wavelength longitudinal collective fluctuations in this system.

Let us first ignore the temperature effects and consider the case when the ions are in the plasma or crystal state. All the hydrodynamic equations presented below for this case are well known, but are given to emphasize the difference of these states from the charged condensate, to which we’ll turn by the end of the section.

The continuity equations for the electron and ion number densities – denoted respectively by \( n_e \) and \( n_H \) – read as:
\[ \partial_t n_{e,H} + \partial_j (n_{e,H} v_j) = 0, \]  
while the momentum equation for the ions is
\[ \partial_t v_j + (v_k \partial_k) v_j = -\frac{Ze}{m_H} E_j. \]  
Consider small localized perturbations, small overdensities \( \delta n_{e,H} \) of the electrons and ions over their background values set by \( J_0 \) and \( J_0/Z \), respectively
\[ n_e = J_0 + \delta n_e, \quad n_H = \frac{1}{Z} J_0 + \delta n_H. \]  
For both the plasma and crystal state at long wavelength, the Poisson equation for the electrostatic potential created by these perturbations reads as follows
\[ \Delta A_0 = -Ze\delta n_H + e\delta n_e. \]  
For simplicity we consider relativistic electrons here. The electron over-density can be related to the local potential via the Thomas-Fermi approximation:
\[ E_F \equiv \mu = -eA_0 + p_F, \]  
and using that \( n_e = p_F^3/3\pi^2 \), we find
\[ \delta n_e \simeq 3\mu^2 eA_0. \]  
Substituting this expression into (67), one gets:
\[ (\Delta - m_s^2) A_0 = -Ze\delta n_H, \]  
where \( m_s^2 \equiv e^2\mu^2/\pi^2 \) is the Debye screening mass squared, due to the electrons. We now look at the system of three equations (64), (65) and (70), and consider their linearization above the background, \( (bg) \), with
\[ n_{eb} = J_0, \quad n_{Hb} = \frac{J_0}{Z}, \quad A_{0b} = 0, \quad v_{jb} = 0. \]  
In the linearized equations we transform to the Fourier modes for all perturbations, as for instance,
\[ \delta n_e(x, t) = \int d^3k d\omega \tilde{\delta n}_e(k, \omega) e^{i(\omega t - k_j x_j)}. \]  
As a result get the following dispersion relation from the linearized system of equations (64), (65) and (70):
\[ \omega^2 = \frac{m_{\pi}^2 k^2}{k^2 + m_s^2}, \quad \omega^2 = \frac{(Ze)^2 (J_0/Z)}{m_H}. \]  
This describes a gapless collective mode; at small momentum the dispersion relations reads
\[ \omega \simeq \frac{m_{\pi}^2 k}{m_s}, \]  
that is the dispersion relations of the ion acoustic wave (ionic sound) for plasma, while for a crystal it describes a longitudinal acoustic phonon.
Things are a bit different, however, for the case of charged condensate. While the continuity and Poisson equations remain unchanged, the momentum equation gets modified due to the pressure term on the right hand side; this can be shown from the Lagrangian formulation of the charged condensate given in Section 3 (the pressure term would certainly exist in ordinary plasma as a finite temperature effect, but in charge condensate it is nonzero even when the finite temperature effects are ignored.) The corresponding momentum equation reads:

$$\partial_t v_j + (v_k \partial_k)v_j = -\partial_j b_0 - \frac{Ze}{m_H} E_j. \quad (75)$$

The difference is due to the fact that a gradient of the pressure is not negligible for perturbations in the charged condensate; the respective term is kept as the first term on the right hand side in (75); it is determined by the gradient of the gauge invariant potential, $b_0 = (m^2_x - 4F^2/\Delta)\delta n_H$.

We now combine this new momentum equation (75), with the Poisson (70) and continuity (64) equations, and easily derive the modified dispersion relation for the longitudinal mode in charged condensate:

$$\omega^2 \sim \frac{k^4}{4m_H^2} + \frac{m^2_x k^2}{k^2 + m^2_x}. \quad (76)$$

The result coincides with the real part of the dispersion relation for the massless mode found in [8]. It also coincides, in the low momentum approximation, with the dispersion relations for the ion acoustic waves found in Sections 1, 2, and 3 of the present work.

An important approximation made above is the Thomas-Fermi method. This method will in general miss fermion dynamics near the Fermi surface. In particular, as we see, it does not capture the width of the ion acoustic wave due to its dumping by the fermionic quasiparticles. However, this effect was already taken into account by the one-loop consideration in [8], and above in the present work.

4. OUTLOOK

White dwarf stars constitute interesting physical objects, and are also important for inferring key astrophysical and cosmological data. The theory of cooling of $\mathcal{O}$ and $\mathcal{C}$ WD’s is well known [2], and agrees well with observations [18, 19]. While at temperatures above $\sim 10^7$ K the ions can be regarded as being in a classical gas state, and below $\sim 10^6$ being in a bcc crystal state, between these two temperatures – from $\sim 10^7$ K, down to $\sim 10^6$ K – one is dealing with a strongly interacting plasma of ions that is not easy to study using fundamental electromagnetic interactions. Instead, we used an effective Lagrangian approach that is in general well suited to study long-wavelength excitations, even for strongly interacting systems. Our finding of the absence of the zero sound mode, and the presence of the longitudinal ion acoustic wave in the interacting plasma regime, confirms a suggestion made in [2] that the longitudinal acoustic wave can be used to describe physics of WD’s in this interval of temperatures where neither gas nor crystal descriptions are valid.

As to the zero sound mode, its existence requires somewhat lower densities, $\rho < 10^5 \text{ g/cm}^3$. While WD’s have fairly uniform density profiles in their interiors, they’re not exactly uniform and are described by an adiabatic equation of state. Thus, the density can vary from the core to the outskirts of the bulk by a factor of 5 or more. Therefore, there might exist a subclass of low density WD’s, with relatively narrow spherical shells away from the cores, in which densities might be $\rho < 10^5 \text{ g/cm}^3$; so if so then the zero sound mode could be supported in those domains instead of the ion acoustic wave. It could also be interesting to see if the zero sound mode could be supported in the low mass brown dwarfs stars studied in [15], where densities are $\rho \sim 10^3 \text{ g/cm}^3$.

Finally, the interaction between the electrons due to the exchange of the zero sound mode (or the ion acoustic wave) would be strong if the momentum transfer in the two-by-two electron scattering amplitude is near the pole of the zero sound (or the ion acoustic wave), i.e., is at $\omega \simeq xv_F k$, with $x \gg 1$ (or with $x \ll 1$ for the ion acoustic wave). This interaction, if attractive for a certain domain of momenta, might lead to formation of bound states, or loosely bound states such as Cooper pairs. While naive kinematical arguments suggest that the momentum transfer is not close to the zero sound pole as long as $x \gg 1$ (or with $x \ll 1$ for the ion acoustic wave), the issue needs careful study, presumably via the Schwinger - Dyson equation, to see if such interactions can be used to trigger Cooper instability and produce a gap of a physically meaningful magnitude.

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