Chaotic iterations and topological chaos

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Abstract

Chaotic iterations have been introduced on the one hand by Chazan, Miranker [5] and Miellou [9] in a numerical analysis context, and on the other hand by Robert [11] and Pellegrin [10] in the discrete dynamical systems framework. In both cases, the objective was to derive conditions of convergence of such iterations to a fixed state. In this paper, a new point of view is presented, the goal here is to derive conditions under which chaotic iterations admit a chaotic behaviour in a rigorous mathematical sense. Contrary to what has been studied in the literature, convergence is not desired.

More precisely, we establish in this paper a link between the concept of chaotic iterations on a finite set and the notion of topological chaos [8], [6], [7]. We are motivated by concrete applications of our approach, such as the use of chaotic boolean iterations in the computer security field. Indeed, the concept of chaos is used in many areas of data security without real rigorous theoretical foundations, and without using the fundamental properties that allow chaos. The wish of this paper is to bring a bit more mathematical rigour in this field.

1 Introduction

Let us consider the system $B^2 = \{0; 1\}^2$, in which each of the two cells $c_i$ is characterized by a boolean state $e_i$. An evolution rule is, for example,

$$f : B^2 \rightarrow B^2 \quad (e_1, e_2) \rightarrow (e_1 + e_2, e_1)$$

These cells can be updated in a serial mode (the elements are iterated in a sequential mode, at each time only one element is iterated), in a parallel mode (at each time, all the elements are iterated), or by following a sequence $(S_n)_{n \in \mathbb{N}}$: the $n^{th}$ term $S_n$ is constituted by the block components to be updated at the $n^{th}$ iteration. This is the chaotic iterations, and $S$ is called the strategy. Let us notice that serial and parallel modes are particular cases of chaotic iterations. Until now, only the conditions of convergence have been studied.

A priori, the chaotic adjective means “in a disorder way”, and has nothing to do with the mathematical theory of chaos, studied by Li-Yorke [8], Devaney [6], Knudsen [7], etc. We asked ourselves what it really was.

In this paper we study the topological evolution of a system during chaotic iterations. To do so, chaotic iterations have been written in the field of discrete dynamical system:

$$\begin{cases} x^0 \in X \\ x^{n+1} = f(x^n) \end{cases}$$

where $(X, d)$ is a metric space (for a distance to be defined), and $f$ is continuous.

Thus, it becomes possible to study the topology of chaotic iterations. More exactly, the question: “Are the chaotic iterations a topological chaos ?” has been raised.

This study is the first of a series we intend to carry out. We think that the mathematical framework in which we are placed offers interesting new tools allowing the conception, the comparison and the evaluation of new algorithms where disorder, hazard or unpredictability are to be considered.

The rest of the paper is organised as follows.

The first next section is devoted to some recalls on the domain of topological chaos and the domain of discrete chaotic iterations. Third and fourth sections constitute the theoretical study of the present paper. In section 6, the computer and so the finite set of machine numbers is considered. The paper ends with some discussions and future work.
2 Basic recalls

This section is devoted to basic notations and terminologies in the fields of topological chaos and chaotic iterations.

2.1 Chaotic iterations

In the sequel $S^n$ denotes the $n^{th}$ term of a sequence $S$, $V_i$ denotes the $i^{th}$ component of a vector $V$, and $f^k = f \circ \ldots \circ f$ denotes the $k^{th}$ composition of a function $f$. Finally, the following notation is used: $[1; N] = \{1, 2, \ldots, N\}$.

Let us consider a system of a finite number $N$ of cells, so that each cell has a boolean state. Then a sequence of length $N$ of boolean states of the cells corresponds to a particular state of the system.

A strategy corresponds to a sequence $S$ of $[1; N]$. The set of all strategies is denoted by $S$.

Definition 1 Let $S \in S$. The shift function is defined by $\sigma : (S^n)_{n \in N} \in S \rightarrow (S^{n+1})_{n \in N} \in S$, and the initial function $i$ is the map which associates to a sequence, its first term: $i : (S^n)_{n \in N} \in S \rightarrow S^0 \in [1; N]$.

The set $B$ denotes $\{0, 1\}$, let $f : B^N \rightarrow B^N$ be a function, and $S \in S$ be a strategy. Then, the so called chaotic iterations are defined by

$$x^0 \in B^N,$$

$$\forall n \in N^+, \forall i \in [1; N], x^n_i = \begin{cases} x^{n-1}_i & \text{if } S^n \neq i \\ (f(x^n))_{S^n} & \text{if } S^n = i. \end{cases}$$

In other words, at the $n^{th}$ iteration, only the $S^n-$th cell is “iterated”. Note that in a more general formulation, $S^n$ can be a subset of components, and $f(x^n)_{S^n}$ can be replaced by $f(x^n)_{S_{k,n}}$, where $k \leq n$, modeling, for example, delay transmission (see e.g. [2]). For the general definition of such chaotic iterations, see, e.g. [3].

2.2 Devaney’s chaotic dynamical systems

Consider a metric space $(\mathcal{X}, d)$, and a continuous function $f : \mathcal{X} \rightarrow \mathcal{X}$.

Definition 2 $f$ is said to be topologically transitive if, for any pair of open sets $U, V \subset \mathcal{X}$, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

Definition 3 $(\mathcal{X}, f)$ is said to be regular if the set of periodic points is dense in $\mathcal{X}$.

Definition 4 $f$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in \mathcal{X}$ and any neighbourhood $V$ of $x$, there exists $y \in V$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

$\delta$ is called the constant of sensitivity of $f$.

Let us now recall the definition of a chaotic topological system, in the sense of Devaney [6]:

Definition 5 $f : \mathcal{X} \rightarrow \mathcal{X}$ is said to be chaotic on $\mathcal{X}$ if $(\mathcal{X}, f)$ is regular, topologically transitive, and has sensitive dependence on initial conditions.

3 A topological approach for chaotic iterations

In this section we will put our study in a topological context by defining a suitable metric set.

3.1 The iteration function and the phase space

Denote by $\delta$ the discrete boolean metric, $\delta(x, y) = 0 \iff x = y$. Given a function $f$, define the function

$$F_f : [1; N] \times B^N \rightarrow B^N$$

$$(k, E) \mapsto \left( E_j, \delta(k, j) + f(E)_k \delta(k, j) \right)_{j \in [1; N]},$$

where $+$ and $.$ are boolean operations.

Consider the phase space: $\mathcal{X} = [1; N]^N \times B^N$, which has the cardinality of the continuum, and the map

$$G_f (S, E) = (\sigma(S), F_f (i(S), E)),$$

then the chaotic iterations defined in [1] can be described by the following iterations

$$\begin{cases} X^0 \in \mathcal{X} \\ X^{k+1} = G_f (X^k). \end{cases}$$
3.2 A new distance

We define a new distance between two points \((S, E), (\tilde{S}, \tilde{E}) \in \mathcal{X}\) by 
\[
d(S, E; (\tilde{S}, \tilde{E})) = d(N, E, \tilde{E}) + d_s(S, \tilde{S}),
\]
where
\[
d_s(S, \tilde{S}) = \sum_{k=1}^{N} \delta(E_k, \tilde{E}_k)
\]
and
\[
d(N, E, \tilde{E}) = \frac{9}{N} \sum_{k=1}^{\infty} |S^k - \tilde{S}^k| / 10^k.
\]

3.3 The topological framework

It can be easily proved that,

**Theorem 1** \(G_f\) is continuous on \((\mathcal{X}, d)\).

Then chaotic iterations can be seen as a dynamical system in a topological space. In the next section, we will show that chaotic iterations are a case of topological chaos, in the sense of Devaney [6].

4 Discrete chaotic iterations and topological chaos

To prove that we are in the framework of Devany’s topological chaos, we have to check the regularity and transitivity conditions.

4.1 Regularity

**Theorem 2** Periodic points of \(G_f\) are dense in \(\mathcal{X}\).

**Proof** Let \((S, E) \in \mathcal{X}\), and \(\varepsilon > 0\). We are looking for a periodic point \((S', E')\) satisfying \(d((S, E); (S', E')) < \varepsilon\).

We choose \(E' = E\), and we reproduce enough entries from \(S\) to \(S'\) so that the distance between \((S', E)\) and \((S, E)\) is strictly less than \(\varepsilon\): a number \(k = \lfloor \log_{10}(\varepsilon) \rfloor + 1\) of terms is sufficient.

After \(k\) iterations, the new common state is \(E\), and strategy \(S'\) is shifted of \(k\) positions: \(\sigma^k(S')\). Then we have to complete strategy \(S'\) in order to make \((E', S')\) periodic (at least for sufficiently large indices). To do so, we put an infinite number of 1 to the strategy \(S'\).

Then, either the first state is conserved after one iteration, so \(E\) is unchanged and we obtain a fixed point. Or the first state is not conserved, then: if the first state is not conserved after a second iteration, then we will be again in the first case above (due to the fact that a state is a boolean). Otherwise the first state is conserved, and we have indeed a fixed (periodic) point.

Thus, there exists a periodic point into every neighbourhood of any point, so \((\mathcal{X}, G_f)\) is regular, for any map \(f\).

4.2 Transitivity

Contrary to the regularity, the topological transitivity condition is not automatically satisfied by any function \((f = \text{Identity} \text{ is not topologically transitive})\). Let us denote by \(T\) the set of maps \(f\) such that \((\mathcal{X}, G_f)\) is topologically transitive.

**Theorem 3** \(T\) is a nonempty set.

**Proof** We will prove that the vectorial logical negation function \(f_0\)
\[
f_0 : \mathcal{B}^N \quad \rightarrow \quad \mathcal{B}^N
\]
\[
(x_1, \ldots, x_N) \quad \mapsto \quad (\overline{x_1}, \ldots, \overline{x_N})
\]
is topologically transitive.

Let \(B_A = B(X_A, r_A)\) and \(B_B = B(X_B, r_B)\) be two open balls of \(\mathcal{X}\), where \(X_A = (S_A, E_A)\), and \(X_B = (S_B, E_B)\). Our goal is to start from a point of \(B_A\) and to arrive, after some iterations of \(G_{f_0}\), in \(B_B\).

We have to be close to \(X_A\), then the starting state \(E\) must be \(E_A\): it remains to construct the strategy \(S\). Let \(S^n = S^n_A, \forall n \leq n_0\), where \(n_0\) is chosen in such a way that \((S, E_A) \in B_A\), and \(E'\) be the state of \(G_{f_0}^n(S, E_A)\).

\(E'\) differs from \(E_B\) by a finite number of cells \(c_1, \ldots, c_n\). Let \(S^{n_0+n} = c_n, \forall n \leq n_1\). Then the state of \(G_{f_0}^n(S, E)\) is \(E_B\).

Last, let \(S^{n_0+n_1+n} = S^n_B, \forall n \leq n_2\), where \(n_2\) is chosen in such a way that \(G_{f_0}^n(S, E)\) is at a distance less than \(r_B\) from \((S_B, E_B)\). Then, starting from a point \((S, E)\) close to \(X_A\), we are close to \(X_B\) after \(n_0 + n_1 + n_2\) iterations: \((\mathcal{X}, G_{f_0})\) is transitive.
4.3 Sensitive dependence on initial conditions

Theorem 4 \((\mathcal{X}, G_{f_0})\) has sensitive dependence on initial conditions, and its constant of sensitiveness is equal to \(N\).

Proof Let \((S, E) \in \mathcal{X}\), and \(\delta > 0\). A new point \((S', E')\) is defined by: \(E' = E\), \(S'^m = S^n, \forall n \leq n_0\), where \(n_0\) is chosen in such a way that \(d((S, E); (S', E')) < \delta\), and \(S'^{n_0+k} = k, \forall k \in [1;N]\).

Then the point \((S', E')\) is as close as we want than \((S, E)\), and systems of \(G_{f_0}^k(N, S, E)\) and \(G_{f_0}^{k+N}(S', E')\) have no cell presenting the same state: distance between this two points is greater or equal than \(N\).

Remark 1 This sensitive dependence could be stated as a consequence of regularity and transitivity (by using the theorem of Banks [3]). However, we have preferred proving this result independently of regularity, because the notion of regularity must be redefined in the context of the finite set of machine numbers (see section 5).

4.4 Chaos

In conclusion, if \(f \in \mathcal{T}\), then \((\mathcal{X}, G_f)\) is topologically transitive, regular and has sensitive dependence on initial conditions. Then we have the result.

Theorem 5 \(\forall f \in \mathcal{T} \neq \emptyset, G_f\) is a chaotic map on \((\mathcal{X}, d)\) in the sense of Devaney.

We have proven that under the transitivity condition of \(f\), chaotic iterations generated by \(f\) can be described by a chaotic map on a topological space in the sense of Devaney. We have considered a finite set of states \(B^N\) and a set \(S\) of strategies composed by an infinite number of infinite sequences. In the following section we will discuss the impact of these assumptions in the context of the finite set of machine numbers.

5 The case of finite strategies

In the computer science framework, we also have to deal with a finite set of states of the form \(B^N\) and the set \(S\) of sequences of \([1;N]\) is infinite (countable), so in practice the set \(\mathcal{X}\) is also infinite. The only difference with respect to the theoretical study comes from the fact that the sequences of \(S\) are of finite but not fixed length in the practice.

The proof of the continuity, the transitivity and the sensitivity conditions are independent of the finitude of the length of strategies (sequences of \(S\)), so even in the case of finite machine numbers, we have the two fundamental properties of chaos: sensitivity and transitivity, which respectively implies unpredictability and indecomposability (see [4], p.50). The regularity property has no meaning in the case of finite systems because of the notion of periodicity. We propose a new definition in order to bypass the notion of periodicity in practice.

Definition 6 A strategy \(S = (S^1, ..., S^L)\) is said cyclic if a subset of successive terms is repeated from a given rank, until the end of \(S\). A point of \(\mathcal{X}\) that admits a cyclic strategy is called a cyclic point.

For example, \((1, 3, 2, 4, 1, 2, 1, 2)\) and \((1, 3, 2, 4, 1, 2, 2, 2)\) are cyclic, but \((1, 3, 2, 4, 1, 2)\) and \((1, 3, 2, 1, 3)\) are not cyclic. This definition can be interpreted as the analog of periodicity on finite sets. Then, following the proof of regularity (section 4.1), it can be proved that the set of cyclic points is dense on \(\mathcal{X}\), hence obtaining a desired element of regularity in finite sets, as quoted by Devaney ([4], p.50): two points arbitrary close to each other could have different behaviours, the one could have a cyclic behaviour as long as the system iterates while the trajectory of the second could "visit" the whole phase space. It should be recalled that the regularity was introduced by Devaney in order to counteract the transitivity and to obtain such a property: two points close to each other can have fundamental different behaviours.

It is worthwhile to notice that even if the set of machine numbers is finite, we deal with strategies that have a finite but unbounded length. Indeed, it is not necessary to store all the terms of the strategy in the memory, only the \(n^{th}\) term (an integer less than or equal to \(N\)) of the strategy has to be stored at the \(n^{th}\) step, as it is illustrated in the following example. Let us suppose that a given text is input from the outside world in the computer character by character, and that the current term of the strategy is given by the ASCII code of the current stored character. Then, as the set of all possible texts of the outside world is infinite and the number of their characters is unbounded, we have to deal with an infinite set of finite but unbounded strategies. Of course, the preceding example is a simplistic illustrating example. A chaotic procedure should to be introduced to generate the terms of the strategy from the stream of characters.

In conclusion, even in the computer science framework our previous theory applies.
6 Discussion and future work

We proved that discrete chaotic iterations are a particular case of Devaney’s topological chaos if the iteration function is topologically transitive, and that the set of topologically transitive functions is non void. This theory has a lot of applications, because of the high number of situations that can be described with the chaotic iterations: neural networks, cellular automata, multi-processor computing, and more generally any aggregation of cells (such as pictures, movies, sounds). If this system is requested to evolve in an apparently disorderly manner, e.g. for security reasons (encryption, watermarking, pseudo-random number generation, hash functions, etc.), our results could be useful. Moreover, the theory brings another way to compare two given algorithms concerned by disorder (evaluation of theirs constants of sensitivity, expansivity, etc.), which can be seen as a complement of existing statistical evaluations. In future work, other forms of chaos (such as Li-York chaos [8]) will be studied, other quantitative and qualitative tools such as expansivity or entropy (see e.g. [4] or [3]) will be explored, and the domain of applications of our theoretical concepts will be enlarged.

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Appendix: Continuity of $G_f$

Theorem 6 $G_f$ is a continuous function for $(X, d)$.

Proof We use the sequential continuity (we are in a metric space).

Let $(S^n, E^n)_{n \in \mathbb{N}}$ be a sequence of the phase space $X$, which converges to $(S, E)$. We will prove that $(G_f(S^n, E^n))_{n \in \mathbb{N}}$ converges to $G_f(S, E)$. Let us recall that for all $n$, $S^n$ is a strategy, thus, we consider a sequence of strategy (i.e. a sequence of sequences).

As $d((S^n, E^n); (S, E))$ converges to 0, each distance $d_e(E^n, E)$ and $d_s(S^n, S)$ converges to 0. But $d_e(E^n, E)$ is an integer, so $\exists n_0 \in \mathbb{N}, d_e(E^n, E) = 0$ for any $n \geq n_0$.

In other words, there exists threshold $n_0 \in \mathbb{N}$ after which no cell will change its state:

$$\exists n_0 \in \mathbb{N}, n \geq n_0 \implies E^n = E.$$

In addition, $d_s(S^n, S) \to 0$, so $\exists n_1 \in \mathbb{N}, d_s(S^n, S) < 10^{-1}$ for all indices greater than or equal to $n_1$. This means that for $n \geq n_1$, all the $S^n$ have the same first term, which is $S_0$:

$$\forall n \geq n_1, S^n_0 = S_0.$$

Thus, after the $\text{max}(n_0, n_1)$-th term, states of $E^n$ and $E$ are the same, and strategies $S^n$ and $S$ start with the same first term.

Consequently, states of $G_f(S^n, E^n)$ and $G_f(S, E)$ are equal, then distance $d$ between this two points is strictly less than 1 (after the rank $\text{max}(n_0, n_1)$).

We now prove that the distance between $(G_f(S^n, E^n))$ and $(G_f(S, E))$ is convergent to 0. Let $\varepsilon > 0$.

- If $\varepsilon \geq 1$, then we have seen that the distance between $(G_f(S^n, E^n))$ and $(G_f(S, E))$ is strictly less than 1 after the $\text{max}(n_0, n_1)$-th term (same state).

- If $\varepsilon < 1$, then $\exists k \in \mathbb{N}, 10^{-k} \geq \varepsilon \geq 10^{-(k+1)}$. But $d_s(S^n, S)$ converges to 0, so

$$\exists n_2 \in \mathbb{N}, \forall n \geq n_2, d_s(S^n, S) < 10^{-(k+2)},$$

after $n_2$, the $k + 2$ first terms of $S^n$ and $S$ are equal.

As a consequence, the $k + 1$ first entries of the strategies of $G_f(S^n, E^n)$ and $G_f(S, E)$ are the same (because $G_f$ is a shift of strategies), and due to the definition of $d_s$, the floating part of the distance between $(S^n, E^n)$ and $(S, E)$ is strictly less than $10^{-(k+1)} \leq \varepsilon$.

In conclusion, $G_f$ is continuous,

$$\forall \varepsilon > 0, \exists N_0 = \text{max}(n_0, n_1, n_2) \in \mathbb{N}, \forall n \geq N_0, d(G_f(S^n, E^n); G_f(S, E)) \leq \varepsilon.$$