ON THE DIFFERENTIABILITY OF HAIRS FOR ZORICH MAPS

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Abstract. Devaney and Krych showed that for the exponential family \( \lambda e^z \), where \( 0 < \lambda < 1/e \), the Julia set consists of uncountably many pairwise disjoint simple curves tending to \( \infty \). Viana proved that these curves are smooth. In this article we consider a quasiregular counterpart of the exponential map, the so-called Zorich maps, and generalize Viana’s result to these maps.

1. Introduction and main result

For an entire function \( f \) the Julia set \( J(f) \) of \( f \) is the set of all points in \( \mathbb{C} \) where the iterates \( f^k \) of \( f \) do not form a normal family in the sense of Montel. Given an attracting fixed point \( \xi \) of \( f \) we denote by \( A(\xi) := \{ z : \lim_{k \to \infty} f^k(z) = \xi \} \) the basin of attraction of \( \xi \). From the theory of complex dynamics it is well-known that \( J(f) = \partial A(\xi) \), see [Mi06, Corollary 4.12]. For further information on complex dynamics we refer to [Bea91, Ber93, Mi06, St93].

Devaney and Krych [DK84] showed that for \( f(z) = \lambda e^z \), where \( 0 < \lambda < 1/e \), there exists an attracting fixed point \( \xi \in \mathbb{R} \) such that \( J(f) = \mathbb{C} \setminus A(\xi) \) and gave a detailed description of the structure of \( J(f) \). We say that a subset \( H \) of \( \mathbb{C} \) (or \( \mathbb{R}^d \)) is a hair, if there exists a homeomorphism \( \gamma : [0, \infty) \to H \) such that \( \lim_{t \to \infty} \gamma(t) = \infty \). We call \( \gamma(0) \) the endpoint of the hair.

We only state the part of the result due to Devaney and Krych which is relevant for us.

**Theorem A.** For \( 0 < \lambda < 1/e \) the set \( J(\lambda e^z) \) is an uncountable union of pairwise disjoint hairs.

For a set \( X \) in \( \mathbb{C} \) (or in \( \mathbb{R}^d \)) we denote by \( \dim X \) the Hausdorff dimension of \( X \). The following result is due to McMullen [McM87].

**Theorem B.** Let \( \lambda \in \mathbb{C} \setminus \{0\} \). Then \( \dim J(\lambda e^z) = 2 \).

McMullen’s result implies that in the situation of Theorem A the union of the hairs has Hausdorff dimension 2. The following result of Karpińska [Ka99] is also known as Karpińska’s paradox, see e.g. [SZ03].

**Theorem C.** Let \( 0 < \lambda < 1/e \) and let \( \mathcal{C} \) be the set of endpoints of the hairs that form \( J(\lambda e^z) \). Then \( \dim \mathcal{C} = 2 \) and \( \dim( J(\lambda e^z) \setminus \mathcal{C} ) = 1 \).
The existence of hairs is not restricted to the situation considered by Devaney and Krych. Hairs appear for \( \lambda e^z \) for all \( \lambda \in \mathbb{C} \setminus \{0\} \) (see [DGH86, SZ03]) and also for more general classes of functions (see [Ba07, DT86, RRRS11]).

For further information on dynamics of exponential functions we refer to papers by Rempe [Re06] and Schleicher [Sch03].

Viana [Vi88] investigated the differentiability of hairs for exponential maps.

**Theorem D.** For all \( \lambda \in \mathbb{C} \setminus \{0\} \) the hairs of \( \lambda e^z \) are \( C^\infty \)-smooth.

In this paper we consider a higher-dimensional analog of exponential maps, the so-called Zorich maps. These maps are quasiregular maps, which can be considered as a higher-dimensional analog of holomorphic maps. Since we will not use any results about quasiregular maps, we do not give the definition here, but refer to Rickman’s monograph [Ri93]. We note, however, that the quasiregularity is an underlying idea in many of the arguments.

Every Zorich map depends on a bi-Lipschitz map which maps a square to some body, for example to a hemisphere or to the faces of a tetrahedron (see [IM01, page 121]). Recently, Nicks and Sixsmith [NS17] considered a bi-Lipschitz map which maps a square to the faces of a square based pyramid to obtain a quasiregular map \( f: \mathbb{R}^3 \to \mathbb{R}^3 \) of transcendental type, which has a periodic domain where all iterates of \( f \) tend locally uniformly to \( \infty \). In this paper we restrict ourself to the ‘standard’ case, i.e. our bi-Lipschitz maps always map a square to the upper or lower hemisphere.

For the definition of these maps we follow [IM01, page 119]. Subsequently we summarize some results of Bergweiler [Ber10], but we replace the dimension 3 by any dimension \( d \geq 3 \).

We define for \( d \in \mathbb{N} \) with \( d \geq 3 \) the hypercube
\[
Q := \{ x \in \mathbb{R}^{d-1} : \| x \|_\infty \leq 1 \} = [-1, 1]^{d-1},
\]
the upper hemisphere
\[
\mathbb{S}_+ := \{ x \in \mathbb{R}^d : \| x \|_2 = 1 \text{ and } x_d \geq 0 \}
\]
and for \( c \in \mathbb{R} \) the half-space
\[
\mathbb{H}_{\geq c} := \{ x \in \mathbb{R}^d : x_d \geq c \}.
\]
The half-spaces \( \mathbb{H}_{>c}, \mathbb{H}_{<c} \) and \( \mathbb{H}_{\leq c} \) are defined analogously. For a bi-Lipschitz map \( h: Q \to \mathbb{S}_+ \) we define
\[
F: Q \times \mathbb{R} \to \mathbb{H}_{\geq 0}, \quad F(x) = e^{x_d} h(x_1, \ldots, x_{d-1}).
\]
By reflection we get a function \( F: \mathbb{R}^d \to \mathbb{R}^d \) which we call a Zorich map.

If \( DF(x_1, \ldots, x_{d-1}, 0) \) exists, we have
\[
DF(x_1, \ldots, x_{d-1}, x_d) = e^{x_d} DF(x_1, \ldots, x_{d-1}, 0)
\]
and thus there exist \( \alpha, m, M \in \mathbb{R} \), \( \alpha \in (0, 1), m < M \), \( M \geq 1 \) such that
\[
\| DF(x) \| \leq \alpha \quad \text{a.e. for } x_d \leq m
\]
and

\[(1.3) \quad l(DF(x)) \geq \frac{1}{\alpha} \quad \text{a.e. for } x_d \geq M,\]

where $DF(x)$ denotes the derivative,

\[\|DF(x)\| = \sup_{\|h\|=1} \|DF(x)h\|_2\]

the operator norm of $DF(x)$ and

\[l(DF(x)) = \inf_{\|h\|=1} \|DF(x)h\|_2.\]

Consider now for $a \in \mathbb{R}$ such that

\[(1.4) \quad a \geq e^M - m\]

the map

\[f_a : \mathbb{R}^d \to \mathbb{R}^d, \quad f_a(x) = F(x) - (0, \ldots, 0, a).\]

In our context we call $f_a$ a Zorich map, too. The following theorem is due to Bergweiler [Ber10].

**Theorem E.** Let $f_a$ be as above with $a$ as in (1.4). Then there exists a unique fixed point $\xi = (\xi_1, \ldots, \xi_d)$ satisfying $\xi_d \leq m$, the set

\[J := \{x \in \mathbb{R}^d : f^{k}_{a}(x) \not\to \xi\}\]

consists of uncountably many pairwise disjoint hairs, and the set $C$ of endpoints of these hairs has Hausdorff dimension $d$, while $J \setminus C$ has Hausdorff dimension 1.

Thus Theorem E corresponds to Theorem A, B and C. For a set $\Omega \subset \mathbb{R}^{d-1}$ we denote by $\text{int}(\Omega)$ the interior of $\Omega$. The main result of this article is the following.

**Theorem 1.1.** Let $f_a$ be as above with $a$ as in (1.4) and assume that $h|_{\text{int}(Q)}$ is $C^1$ and $Dh$ is Hölder continuous. Then the hairs of $f_a$ are $C^1$-smooth.

This theorem corresponds to Theorem D concerning the differentiability of hairs. As in Viana’s result, we do not obtain differentiability of the hairs in the endpoints since they can spiral around a point. Rempe [Re03, 3.4.2 Theorem, page 31] gave a condition under which we obtain smoothness up to the endpoints for exponential maps. The proof of Theorem 1.1 will show that the conclusion of the theorem still holds if the assumptions on $h$ and $Dh$ are satisfied on a suitable subset of $\text{int}(Q)$ (see Section 4).

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2. Preliminaries

In this section we are recalling mainly results from [Ber10], formulating them for functions in \( \mathbb{R}^d \) with \( d \geq 3 \) instead of \( \mathbb{R}^3 \), however. For simplicity we write \( f = (f_1, \ldots, f_d) \) instead of \( f_a \). For \( r \in \mathbb{Z}^{d-1} \) we denote by

\[
P(r) := \{ x \in \mathbb{R}^{d-1} : \forall j \in \{1, \ldots, d-1\} : |x_j - 2r_j| < 1 \}
\]

the shifted and open square \( Q \) with centre \( 2r \). We put

\[
S := \left\{ r \in \mathbb{Z}^{d-1} : \sum_{j=1}^{d-1} r_j \in 2\mathbb{Z} \right\}.
\]

Then

\[
F \text{ maps } P(r) \times \mathbb{R} \text{ bijectively onto } \begin{cases} \mathbb{H}_{>0}, & \text{if } r \in S \\ \mathbb{H}_{<0}, & \text{if } r \notin S \end{cases}
\]

and thus

\[
(2.1) \quad f \text{ maps } P(r) \times \mathbb{R} \text{ bijectively onto } \begin{cases} \mathbb{H}_{>-a}, & \text{if } r \in S, \\ \mathbb{H}_{<-a}, & \text{if } r \notin S. \end{cases}
\]

Definition 2.1 (Tract). For \( r \in S \) we call the set

\[
T(r) := P(r) \times (M, \infty)
\]

the tract above \( P(r) \).

Now we want to understand the behaviour of our function \( f \) and collect some important facts about it. Since \( f(P(r) \times \mathbb{R}) = \mathbb{H}_{>-a} \) for \( r \in S \) and

\[
(2.2) \quad f_d(x_1, \ldots, x_d) = e^{x_d} h_d(x_1, \ldots, x_{d-1}) - a \leq e^{x_d} - e^{M} + m \leq m < M
\]

for \( x_d \leq M \) and hence \( f(P(r) \times (-\infty, M]) \subset \mathbb{H}_{<M} \), we have \( f(T(r)) \supset \mathbb{H}_{\geq M} \). Thus there exists a branch \( \Lambda^r : \mathbb{H}_{\geq M} \to T(r) \) of the inverse function of \( f \). Using the notation \( \Lambda := \Lambda^{(0, \ldots, 0)} \), we have

\[
(2.3) \quad \Lambda^{(r_1, \ldots, r_{d-1})}(x) = \Lambda(x) + (2r_1, \ldots, 2r_{d-1}, 0)
\]

for all \( x \in \mathbb{H}_{\geq M} \) and \( r \in S \).

In our proofs we will often use the derivative of \( \Lambda^r \). Together with (2.3) we obtain

\[
D\Lambda^r(x) = D\Lambda(x)
\]

for all \( x \in \mathbb{H}_{\geq M} \) for which the derivative exists. Because \( DF = Df \), we deduce from (1.2) that

\[
(2.4) \quad \|D\Lambda(x)\| \leq \alpha \quad \text{a.e.}
\]

for \( x \in \mathbb{H}_{\geq M} \). This implies for \( x, y \in \mathbb{H}_{\geq M} \) that

\[
(2.5) \quad \|\Lambda(x) - \Lambda(y)\|_2 \leq \|x - y\|_2 \text{ ess sup}_{z \in [x, y]} \|D\Lambda(z)\| \leq \alpha \|x - y\|_2.
\]
Since (2.2) and thus \( f(H \leq M) \subset H \leq m \), we obtain using (1.3)
\[
\|f(x) - f(y)\|_2 \leq \|x - y\|_2 \text{ ess sup}_{z \in [x,y]} \|Df(z)\| \leq \alpha \|x - y\|_2.
\]

So Banach’s fixed point theorem gives us the existence of a unique fixed point \( \xi \in H \leq m \) such that
\[
\lim_{n \to \infty} f^n(x) = \xi \quad \text{for all } x \in H \leq m.
\]
Together with (2.1) this yields
\[
J \subset \bigcup_{r \in S} T(r).
\]

In the following we collect some estimates for \( D\Lambda \). The equation (1.1) implies that there exist \( c_1, c_2 > 0 \) such that
\[
c_1 e^{xd} \leq l(Df(x)) \leq \|Df(x)\| \leq c_2 e^{xd} \quad \text{a.e.}
\]

Thus there are constants \( c_3 > 0 \) and \( c_4 \geq 1 \) such that for \( x \in H \geq M \)
\[
l(D\Lambda(x)) \geq \frac{c_3}{\|x\|_2} \quad \text{a.e.}
\]
\[
\|D\Lambda(x)\| \leq \frac{c_4}{\|x\|_2} \quad \text{a.e.} \quad (2.6)
\]

Moreover, there exist \( c_5 > 0 \) and \( c_6 \geq 1 \) such that
\[
\frac{c_5}{\|x\|_2^d} \leq J\Lambda(x) \leq \frac{c_6}{\|x\|_2^d} \quad \text{a.e.,}
\]

where \( J\Lambda(x) \) denotes the Jacobian determinant.

Let us fix \( x, y \in H \geq M \). Then we can connect \( x \) and \( y \) by a path \( \gamma \) in
\[
H \geq M \cap \left\{ z \in \mathbb{R}^d : \|z\|_2 \geq \min\{|\|x\|_2, |\|y\|_2|\} \right\}
\]
with \( \text{length}(\gamma) \leq \pi \|x - y\|_2 \). Together with (2.6) this yields
\[
\|\Lambda(x) - \Lambda(y)\|_2 \leq \pi \|x - y\|_2 \text{ ess sup}_{z \in \gamma} \|D\Lambda(z)\| \leq c_4 \pi \frac{\|x - y\|_2}{\min\{|\|x\|_2, |\|y\|_2|\}}. \quad (2.7)
\]

Since \( h \) is bijective, there exists a unique point \( (v_1, \ldots, v_{d-1}) \in Q \) which is mapped under \( h \) to the north pole of the sphere, i.e.
\[
h(v_1, \ldots, v_{d-1}) = (0, \ldots, 0, 1). \quad (2.8)
\]
This implies \( F(v_1, \ldots, v_{d-1}, x_d) = (0, \ldots, 0, e^{xd}) \) and for \( r \in S \) and \( t \geq M \) this yields
\[
\Lambda^r(0, \ldots, 0, t) = (v_1 + 2r_1, \ldots, v_{d-1} + 2r_{d-1}, \log(t + a)).
\]
Moreover, the equation
\[
\|F(x)\|_2 = e^{xd}
\]
yields
\[
\|f(\Lambda(x)) + (0, \ldots, 0, a)\|_2 = e^{\Lambda_d(x)}
\]
and thus
\[
\Lambda_d(x) = \log(\|x + (0, \ldots, 0, a)\|_2). \quad (2.9)
\]
To discuss the existence of hairs, it is useful, following Schleicher and Zimmer [SZ03], to define a reference function

$$E: [0, \infty) \to [0, \infty), \ E(t) = e^t - 1.$$ 

This function has the following properties:

1. We have $E(0) = 0$ and $\lim_{k \to \infty} E^k(t) = \infty$ for $t > 0$.
2. For $b > 1$ we have
   \begin{equation}
   E^k(t) < \log (E^{k+1}(t) + b) \leq E^k(t) + \log(b).
   \end{equation}
3. If $0 < t' < t'' < \infty$, then
   \begin{equation}
   \lim_{k \to \infty} (E^k(t'') - E^k(t')) = \infty \quad \text{and} \quad \lim_{k \to \infty} \frac{E^k(t'')}{E^k(t')} = \infty.
   \end{equation}

We need an analogous definition as in the case of exponential maps:

**Definition 2.2** (External address/Symbolic space). For each $x \in J$ we call the sequence

$$s(x) := s_0 s_1 s_2 \cdots = (s_k)_{k \geq 0} \in S^\infty$$

such that $f^k(x) \in T(s_k)$ for all $k \geq 0$ the external address of $x$. Moreover, we call $\Sigma := S^\infty$ the symbolic space.

**Definition 2.3** (Admissibility/exponentially boundedness). We say that $s \in \Sigma$ is admissible (or exponentially bounded), if there exists a $t > 0$ such that

$$\limsup_{k \to \infty} \frac{\|s_k\|}{E^k(t)} < \infty.$$ 

Moreover, we denote by $\Sigma' \subset \Sigma$ the set of all admissible points.

With these definitions we obtain the following lemmas, see [Ber10, Propositions 3.1 and 3.2].

**Lemma 2.4.** Let $x \in J$. Then $s(x)$ is admissible.

**Lemma 2.5.** Let $s \in \Sigma'$. Then $\{x \in J : s(x) = s\}$ is a hair.

From Lemma 2.4 and Lemma 2.5 it follows that $J$ is the union of hairs as stated in Theorem E.

Fixing $s \in \Sigma'$, we denote

$$t_s := \inf \left\{ t > 0 : \limsup_{k \to \infty} \frac{\|s_k\|}{E^k(t)} < \infty \right\}.$$ 

Choosing $t_k \in [0, \infty)$ such that $2\|s_k\|_2 = E^k(t_k)$ and putting $\tau_k := \sup_{j \geq k} t_j$, we obtain

\begin{equation}
\limsup_{k \to \infty} t_k = \lim_{k \to \infty} \tau_k.
\end{equation}

Using the abbreviation

$$L_k := \Lambda^{s_k} = \Lambda^{(s_{k,1}, \ldots, s_{k,d-1})}$$

we define for $k \in \mathbb{N}_0$

\[(2.13) \quad g_k: [0, \infty) \to \mathbb{H}_{\geq M}, \quad g_k(t) = (L_0 \circ L_1 \circ \cdots \circ L_k)(0, \ldots, 0, E^{k+1}(t) + M)\]

The two main lemmas in the proof of Lemma 2.5 are the following [Ber10, Lemmas 3.1 and 3.2].

**Lemma 2.6.** The sequence $(g_k)_{k \geq 0}$ converges locally uniformly on $(t_\ast, \infty)$.

**Lemma 2.7.** The sequence $(g_k)_{k \geq 0}$ has a subsequence which converges uniformly on $[t_\ast, \infty)$ and thus $g$ extends to a continuous map $g: [t_\ast, \infty) \to \mathbb{H}_{\geq M}$.

3. Proof of Theorem 1.1

To have a chance for a $C^1$ condition for our hairs, we need enough regularity of the bi-Lipschitz mapping $h$. In this section we want to specify this condition and want to give precise $C^1$ estimates for the hairs.

Therefore we will introduce some new notations. For $n, m \in \mathbb{N}$ with $m < n$ and functions $f_1, \ldots, f_n: \mathbb{R}^d \to \mathbb{R}^d$ we denote

\[\bigcirc_{j=m}^n f_j := f_m \circ \cdots \circ f_n,\]

where we use the convention

\[\bigcirc_{j=m+1}^m f_j := id.\]

Then $g_k$ defined by (2.13) takes the form

\[g_k(t) = \left(\bigcirc_{j=0}^k L_j\right)(0, \ldots, 0, E^{k+1}(t) + M)\]

for all $k \in \mathbb{N}_0$ and $t \in [0, \infty)$. If $h$ is locally $C^1$, the derivative $g_k'$ then reads as

\[
g_k'(t) = \frac{d}{dt} \left((L_0 \circ \cdots \circ L_k)(0, \ldots, 0, E^{k+1}(t) + M)\right)
= \prod_{l=1}^{k+1} \left(D\Lambda \left(\left(\bigcirc_{j=l}^k L_j\right)(0, \ldots, 0, E^{k+1}(t) + M)\right)\right) \\
\cdot (0, \ldots, 0, (E^{k+1})'(t))^T\]

for $k \in \mathbb{N}$.

To prove Theorem 1.1, it is enough to show the following result.

**Theorem 3.1.** Let $f$ be as before and assume that $h|_{\text{int}(Q)}$ is $C^1$ and $Dh$ is H"older continuous. Then for all $z \in \Sigma'$ the sequence $(g_k)_{k \geq 0}$ consists of $C^1$-curves which converge (in $C^1$-sense) locally uniformly on $(t_\ast, \infty)$.

For the proof of the theorem we will compare $g_k'$ and $g_{k-1}'$ in a suitable way. Therefore we define as a preparation for all $k \in \mathbb{N}_0$ the auxilary function

\[(3.2) \quad \phi_k: [0, \infty) \to \mathbb{H}_{\geq M}, \quad \phi_k(t) = L_k(0, \ldots, 0, E^{k+1}(t) + M).\]

Then we obtain for all $t \geq 0$

\[\phi_k(t) = (v_1 + 2s_{k,1}, \ldots, v_{d-1} + 2s_{k,d-1}, \log(E^{k+1}(t) + M + a)).\]
Thus
\[ \phi'_k(t) = \left(0, \ldots, 0, \frac{1}{E^{k+1}(t) + M + a} E'(E^k(t)) \cdot (E^k)'(t) \right) \]

and
\[ \phi'_k(t) = DA \left(0, \ldots, 0, E^{k+1}(t) + M \right) \cdot \left(0, \ldots, 0, (E^{k+1})'(t) \right)^T. \]

Noticing that
\[ a \geq e^M - m \geq 1 + M - m > 1, \]
we obtain
\[ \|\phi'_k(t)\|_2 = \frac{E^{k+1}(t) + 1}{E^{k+1}(t) + M + a} \cdot (E^k)'(t) \leq (E^k)'(t). \]

**Lemma 3.1.** For all \( k \in \mathbb{N}_{\geq 2} \) and \( l \in \{1, \ldots, k-1\} \) and for all \( t \geq 0 \) we have
\[ \left\| \left( \bigcap_{j=l}^{k-1} L_j \right) \phi_k(t) \right\|_2 \geq \left( \bigcap_{j=l}^{k-1} L_j \right) d \phi_k(t) \geq E^l(t). \]

**Proof.** From (2.9), (2.10) and (3.5) we deduce that
\[ \|L_{k-1}(\phi_k(t))\|_2 \geq |L_{k-1,d}(\phi_k(t))| \]
\[ = |\Lambda_d(\phi_k(t))| \]
\[ = \log(\|\phi_k(t) + (0, \ldots, 0, a)\|_2) \]
\[ \geq \log(\phi_k,d(t) + a) \]
\[ = \log(\log(E^{k+1}(t) + M + a) + a) \]
\[ \geq \log(E^k(t) + a) \]
\[ \geq E^{k-1}(t). \]

Take now \( l < k-1 \) such that (3.7) is true with \( l \) replaced by \( l+1 \). Then we obtain
\[ \left\| \left( \bigcap_{j=l}^{k-1} L_j \right) \phi_k(t) \right\|_2 = \left\| L_l \circ \left( \bigcap_{j=l+1}^{k-1} L_j \right) \phi_k(t) \right\|_2 \]
\[ \geq \left\| L_{l,d} \left( \bigcap_{j=l+1}^{k-1} L_j \right) \phi_k(t) \right\|_2 \]
\[ = \log \left( \left\| \left( \bigcap_{j=l+1}^{k-1} L_j \right) \phi_k(t) \right\|_2 \right) \]
\[ \geq \log \left( \left( \bigcap_{j=l+1}^{k-1} L_j \right) d \phi_k(t) + a \right) \]
\[ \geq \log(E^{l+1}(t) + a) \]
\[ \geq E^l(t) \]
which proves the result. \( \square \)
Lemma 3.2. If \( \phi_k(t) \) is replaced by \((0, \ldots, 0, E^k(t) + M)\).

Since the operator norm of \(DF(x)\) is comparable to the maximum of all entries of this matrix, there exists a constant \(C > 0\) such that

\[
\|DF(x)\| \leq C \max_{1 \leq j \leq d} |DF_{jk}(x)|.
\]

In the following let \(\beta \in (0, 1]\) and let \(Dh\) be \(\beta\)-Hölder continuous, i.e. there is a constant \(H_\beta > 0\) such that

\[
\|Dh(x) - Dh(y)\| \leq H_\beta \|x - y\|_2^\beta
\]

for all \(x, y \in \mathbb{H}_{\geq M}\). Moreover, we denote by \(L_h\) the Lipschitz constant of \(h\).

We want to use this to prove the following Lemma:

**Lemma 3.2.** If \(h\) is as in Theorem 3.1, then there is a constant \(c_7 > 0\) such that for all \(x, y \in \mathbb{H}_{\geq M}\)

\[
\|DF(\Lambda(x)) - DF(\Lambda(y))\| \leq c_7 \min \left\{ \|x\|_2^{1 - \beta}, \|y\|_2^{1 - \beta} \right\} \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}.
\]

**Proof.** We have for \(x = (\tilde{x}, x_d), y = (\tilde{y}, y_d) \in \mathbb{H}_{\geq M}\) using (3.8)

\[
\|DF(x) - DF(y)\| \leq C \max_{1 \leq j \leq d} |DF_{jk}(x) - DF_{jk}(y)|
\]

\[
\leq C \max_{1 \leq j \leq d} \left| e^{x_d} DF_{jk}(\tilde{x}, 0) - e^{y_d} DF_{jk}(\tilde{y}, 0) \right|.
\]

Because \(|DF_{jk}(x) - DF_{jk}(y)|\) is symmetric in \(x\) and \(y\), we assume without loss of generality that \(y_d \leq x_d\). Then we obtain

\[
|e^{x_d} DF_{jk}(\tilde{x}, 0) - e^{y_d} DF_{jk}(\tilde{y}, 0)|
\]

\[
\leq e^{y_d} |DF_{jk}(\tilde{x}, 0) - DF_{jk}(\tilde{y}, 0)| + (e^{x_d} - e^{y_d}) \cdot \max \{|DF_{jk}(\tilde{x}, 0)|, |DF_{jk}(\tilde{y}, 0)|\}.
\]

Since \(h\) is as in Theorem 3.1 and for all \(z \in \mathbb{R}^d\) and \(j, k \in \{1, \ldots, d\}\)

\[
DF_{jk}(\tilde{z}, 0) = \begin{cases} 
\frac{\partial}{\partial x_k} h_j(\tilde{z}), & \text{if } k \leq d - 1, \\
h_j(\tilde{z}), & \text{if } k = d,
\end{cases}
\]

we have with \(\tilde{C} = \max\{L_h, H_\beta\} \geq 1\), noting that \(\left| \frac{\partial}{\partial x_k} h_j(\tilde{z}) \right| \leq L_h\),

\[
\|DF(x) - DF(y)\| \leq \tilde{C} \left( e^{y_d} \cdot \max \left\{ \|\tilde{x} - \tilde{y}\|_2, \|\tilde{x} - \tilde{y}\|_2^\beta \right\} + L_h \cdot (e^{x_d} - e^{y_d}) \right)
\]

\[
\leq C \tilde{C} \left( e^{y_d} \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\} + (e^{x_d} - e^{y_d}) \right).
\]
Using the fact that \(\min\{\|x\|_2, \|y\|_2\} \geq M \geq 1\) and (2.7) we obtain

\[
\tilde{M} := \max\left\{\|\Lambda(x) - \Lambda(y)\|_2, \|\Lambda(x) - \Lambda(y)\|_2^{\beta}\right\} \\
\leq \max\left\{c_4^{\pi} \frac{\|x - y\|_2}{\min\{\|x\|_2, \|y\|_2\}}, c_4^\beta \frac{\|x - y\|_2^{\beta}}{\min\{\|x\|_2, \|y\|_2^{\beta}\}} \right\} \\
\leq c_4^{\pi} \frac{\|x - y\|_2}{\min\{\|x\|_2, \|y\|_2\}}.
\]

This yields together with (2.7), (2.9), \(\|x + (0, \ldots, 0, a)\|_2 \leq \|x\|_2 + a\) and replacing \(x\) and \(y\) by \(\Lambda(x)\) and \(\Lambda(y)\)

\[
\|DF(\Lambda(x)) - DF(\Lambda(y))\| \\
\leq C\tilde{C} \left(\min\{\|x\|_2, \|y\|_2\} + a\right) \cdot \tilde{M} + \|x\|_2 - \|y\|_2 \\
\leq C\tilde{C} \left(c_4^{\pi} \frac{\min\{\|x\|_2, \|y\|_2\} + a}{\min\{\|x\|_2^{\beta}, \|y\|_2^{\beta}\}} + 1\right) \cdot \max\{\|x - y\|_2, \|x - y\|_2^{\beta}\} \\
\leq C\tilde{C}c_4^{\pi} \left(\min\{\|x\|_2, \|y\|_2\} + a\right) \cdot \max\{\|x - y\|_2, \|x - y\|_2^{\beta}\}.
\]

Since for \(x, y \in \mathbb{H}_{\geq M}\) we have \(\min\{\|x\|_2, \|y\|_2\} \geq \min\{\|x\|_2^{\beta}, \|y\|_2^{\beta}\} \geq 1\) and

\[
\frac{\min\{\|x\|_2, \|y\|_2\} + a}{\min\{\|x\|_2^{\beta}, \|y\|_2^{\beta}\}} + 1 = \frac{\min\{\|x\|_2, \|y\|_2\} + a + \min\{\|x\|_2^{\beta}, \|y\|_2^{\beta}\}}{\min\{\|x\|_2^{\beta}, \|y\|_2^{\beta}\}} \\
\leq (2 + a) \cdot \min\{\|x\|_2^{1-\beta}, \|y\|_2^{1-\beta}\},
\]

there exists \(c_7 > 0\) such that we have for all \(x, y \in \mathbb{H}_{\geq M}\)

\[
\|DF(\Lambda(x)) - DF(\Lambda(y))\| \\
\leq c_7 \min\{\|x\|_2^{1-\beta}, \|y\|_2^{1-\beta}\} \cdot \max\{\|x - y\|_2, \|x - y\|_2^{\beta}\},
\]

which proves the Lemma. \(\square\)

**Lemma 3.3.** For all \(k \in \mathbb{N}\) and \(t \in (0, \infty)\) we have

\[
\left\|\frac{d}{dt} (L_{k-1} \circ \phi_k)(t) - \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M))\right\|_2 \leq (I_{1,k}(t) + I_{2,k}(t)) \cdot (E^k)'(t)
\]

with

\[
I_{1,k}(t) := \frac{ac_4}{E^{k+1}(t) \cdot E^k(t)}
\]
and
\[ I_{2,k}(t) := c_8 \cdot \frac{2\|s_k\|_2 + 1}{E^k(t)^{1+\beta}}, \]

where \( c_8 := c_2^2 c_7 (d + \log(M + a) + M). \)

**Proof.** By the triangle inequality we have
\[
\left\| \frac{d}{dt} (L_{k-1} \circ \phi_k)(t) - \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M)) \right\|_2 \\
= \left\| DA(\phi_k(t)) \cdot \phi_k'(t) - DA(0, \ldots, 0, E^k(t) + M) \cdot (0, \ldots, 0, (E^k)'(t))^T \right\|_2 \\
\leq \left\| DA(\phi_k(t)) \right\| \cdot \left\| \phi_k'(t) - (0, \ldots, 0, (E^k)'(t))^T \right\|_2 \\
+ \left\| DA(\phi_k(t)) \cdot (0, \ldots, 0, (E^k)'(t))^T \right\|_2.
\]

We estimate the first part on the right hand side. Since, by (3.3)
\[
\left\| \phi_k'(t) - (0, \ldots, 0, (E^k)'(t))^T \right\|_2 = \left\| \left( 0, \ldots, 0, \left( \frac{E'(E^k(t))}{E^{k+1}(t) + M + a} - 1 \right) \cdot (E^k)'(t) \right) \right\|_2 \\
= \left\| \left( 1 - \frac{E^{k+1}(t) + 1}{E^{k+1}(t) + M + a} \right) \cdot (E^k)'(t) \right\|_2 \\
= \frac{M + a - 1}{E^{k+1}(t) + M + a} \cdot (E^k)'(t) \\
\leq \frac{a}{E^{k+1}(t)} \cdot (E^k)'(t),
\]

and, by (2.6) and (2.10)
\[
\left\| DA(\phi_k(t)) \right\| \leq \frac{c_4}{\|\phi_k(t)\|_2} \\
\leq \frac{c_4}{\phi_{k,d}(t)} \\
= \frac{c_4}{\log (E^{k+1}(t) + M + a)} \\
\leq \frac{c_4}{E^k(t)}.
\]

Now we estimate the second part on the right hand side. Since the estimates in (3.9) are symmetric in \( x \) and \( y \) we may assume without loss of generality that \( \|x\|_2 \leq \|y\|_2. \)

Using the fact that for all \( A, B \in GL(d, \mathbb{R}) \)
\[
A^{-1} - B^{-1} = B^{-1} \cdot (B - A) \cdot A^{-1}
\]
and for all \( x \in \mathbb{H}_{\geq M} \)
\[
DA(x) \cdot DF(\Lambda(x)) = I,
\]
we obtain from Lemma 3.2 for all \( x, y \in \mathbb{H}_{\geq M} \)
\[
\|D\Lambda(x) - D\Lambda(y)\|
= \|DF(\Lambda(x))^{-1} - DF(\Lambda(y))^{-1}\|
= \|DF(\Lambda(y))^{-1} \cdot (DF(\Lambda(y)) - DF(\Lambda(x))) \cdot DF(\Lambda(x))^{-1}\|
\leq \|DF(\Lambda(y))^{-1}\| \cdot \|DF(\Lambda(y)) - DF(\Lambda(x))\| \cdot \|DF(\Lambda(x))^{-1}\|
\leq c_7 \min \left\{ \|x\|_2^{1-\beta}, \|y\|_2^{1-\beta} \right\} \cdot \|D\Lambda(x)\| \cdot \|D\Lambda(y)\| \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}
= c_7 \|x\|_2^{1-\beta} \cdot \|D\Lambda(x)\| \cdot \|D\Lambda(y)\| \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\}
\]

(3.13)

With inequality (2.6) this yields for all \( x, y \in \mathbb{H}_{\geq M} \)
\[
\|D\Lambda(x) - D\Lambda(y)\| \leq c_4^2 c_7 \|x\|_2^{1-\beta} \cdot \max \left\{ \|x - y\|_2, \|x - y\|_2^\beta \right\} \frac{\|x\|_2 \cdot \|y\|_2}{\|x\|_2^\beta \cdot \|y\|_2}.
\]

Notice that we obtain from (3.7) for all \( k \in \mathbb{N} \)
\[
\frac{1}{\|\phi_k(t)\|_2^\beta \cdot (E^k(t) + M)} \leq \frac{1}{E^k(t)^{1+\beta}}.
\]

Recall that by (2.10) we have \( \log(E^{k+1}(t) + M + a) \leq E^k(t) + \log(M + a) \) for all \( k \in \mathbb{N} \). With
\[
c_8 := c_4^2 c_7 (d + \log(M + a) + M)
\]

and putting \( x := \phi_k(t) = (v_1 + 2s_{k,1}, \ldots, v_{d-1} + 2s_{k,d-1}, \log(E^{k+1}(t) + M + a)) \) and \( y := (0, \ldots, 0, E^k(t) + M) \) we obtain
\[
\|D\Lambda(\phi_k(t)) - D\Lambda(0, \ldots, 0, E^k(t) + M)\|
\leq c_4^2 c_7 \cdot \max \left\{ \|(v + 2s_k, \log(M + a) + M)\|_2, \|(v + 2s_k, \log(M + a) + M)\|_2^\beta \right\} \frac{E^k(t)^{1+\beta}}{E^k(t)^{1+\beta}}
\leq c_4^2 c_7 \cdot \max \left\{ d + 2 \|s_k\|_2 + \log(M + a) + M, (d + 2 \|s_k\|_2 + \log(M + a) + M)^\beta \right\} \frac{E^k(t)^{1+\beta}}{E^k(t)^{1+\beta}}
\leq c_4^2 c_7 \cdot \frac{d + 2 \|s_k\|_2 + \log(M + a) + M}{E^k(t)^{1+\beta}}
\leq c_8 \cdot \frac{2 \|s_k\|_2 + 1}{E^k(t)^{1+\beta}}
for all \( k \in \mathbb{N} \). Together this yields

\[
\| D\Lambda(\phi_k(t)) - D\Lambda(0, \ldots, 0, E^k(t) + M) \| \cdot \| (0, \ldots, 0, (E^k)')(t) \|_2 \\
\leq c_8 \cdot \frac{2}{E^k(t)^{1+\beta}} \cdot (E^k)'(t)
\]

(3.14)

\[
= I_{2,k}(t) \cdot (E^k)'(t).
\]

Finally we obtain the conclusion from (3.11), (3.12) and (3.14).

From (3.1) we know that

\[
\left\| g_{k-1} - g_k \right\| = \frac{1}{t} \int_0^t \left\| g_{k-1}'(s) - g_k'(s) \right\| ds
\]

for all \( k \in \mathbb{N} \). Moreover we obtain with (3.2) and (3.4)

\[
g_k(t) = \prod_{l=1}^{k+1} D\Lambda \left( \left. \frac{k}{j=1} L_j \right| (0, \ldots, 0, E^{k+1}(t) + M) \right) \cdot \left( 0, \ldots, 0, (E^{k+1})'(t) \right)^T
\]

\[
= \prod_{l=1}^{k-1} D\Lambda \left( \left. \frac{k}{j=1} L_j \right| \phi_k(t) \right) \cdot D\Lambda \left( L_k(0, \ldots, 0, E^{k+1}(t) + M) \right) \\
\cdot \left( 0, \ldots, 0, (E^{k+1})'(t) \right)^T
\]

\[
= \prod_{l=1}^{k-1} D\Lambda \left( \left. \frac{k}{j=1} L_j \right| \phi_k(t) \right) \cdot D\Lambda(\phi_k(t)) \cdot \phi_k'(t)
\]

\[
= \prod_{l=1}^{k-1} D\Lambda \left( \left. \frac{k}{j=1} L_j \right| \phi_k(t) \right) \cdot \frac{d}{dt}(L_{k-1} \circ \phi_k(t)).
\]

Putting

\[
A_k(t) := \prod_{l=1}^{k-1} D\Lambda \left( \left. \frac{k}{j=1} L_j \right| \phi_k(t) \right)
\]

\[
B_k(t) := \prod_{l=1}^{k-1} D\Lambda \left( \left. \frac{k}{j=1} L_j \right| (0, \ldots, 0, E^k(t) + M) \right)
\]

for all \( k \in \mathbb{N} \) and \( t \in [0, \infty) \), we obtain

\[
g_k'(t) - g_{k-1}'(t) = A_k(t) \cdot \frac{d}{dt}(L_{k-1} \circ \phi_k)(t) - B_k(t) \cdot \frac{d}{dt}(L_{k-1}(0, \ldots, 0, E^k(t) + M)).
\]

(3.15)

At this point we need suitable estimates for \( A_k(t) \) and \( B_k(t) \).
Lemma 3.4. For all $k \in \mathbb{N}_{\geq 2}$ and $t \in [0, \infty)$ we have

$$\|A_k(t) - B_k(t)\| \leq \sum_{r=1}^{k-1} \left( \prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \|X_{r,k-1}(t) - Y_{r,k-1}(t)\| \cdot \prod_{s=r+1}^{k-1} \|Y_{s,k-1}(t)\| \right),$$

where for $l \in \{0, \ldots, k-1\}$

$$X_{l,k-1}(t) := D\Lambda \left( \left( \bigwedge_{j=l}^{k-1} L_j \right) \phi_k(t) \right),$$

$$Y_{l,k-1}(t) := D\Lambda \left( \left( \bigwedge_{j=l}^{k-1} L_j \right) (0, \ldots, 0, E_k(t) + M) \right).$$

Proof. By the triangle inequality and the submultiplicity of the operator norm we obtain

$$\|A_k(t) - B_k(t)\| = \left\| \prod_{l=1}^{k-1} X_{l,k-1}(t) - \prod_{l=1}^{k-1} Y_{l,k-1}(t) \right\|$$

$$= \left\| \prod_{l=1}^{k-1} X_{l,k-1}(t) - \left( \prod_{l=1}^{k-2} X_{l,k-1}(t) \right) \cdot Y_{k-1,k-1}(t) + \left( \prod_{l=1}^{k-2} X_{l,k-1}(t) \right) \cdot Y_{k-1,k-1}(t) - \prod_{l=1}^{k-1} Y_{l,k-1}(t) \right\|$$

$$\leq \prod_{l=1}^{k-2} \|X_{l,k-1}(t)\| \cdot \|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\|$$

$$+ \left\| \prod_{l=1}^{k-2} X_{l,k-1}(t) - \prod_{l=1}^{k-2} Y_{l,k-1}(t) \right\| \cdot \|Y_{k-1,k-1}(t)\|.$$  

Repeating this procedure we obtain our result. \qed

Proof of Theorem 3.1. Let $\varepsilon > 0$. We will show that there is a constant $C_1 > 0$ such that

$$\|g'_k(t) - g'_{k-1}(t)\|_2 \leq C_1 \alpha^{k-1}$$

for $t \in [t_2 + \varepsilon, \infty)$ and large $k \in \mathbb{N}$, where $\alpha$ is the constant given in (1.2). This implies that $(g'_k)_k$ converges locally uniformly on $(t_2, \infty)$ and thus the hairs of $f$ are $C^1$-smooth which yields Theorem 1.1.
First of all we deduce from (3.15) that
\[
\begin{align*}
\|g'_k(t) - g'_{k-1}(t)\|_2 &= \left\| A_k(t) \cdot \frac{d}{dt} (L_{k-1} \circ \phi_k)(t) - B_k(t) \cdot \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M)) \right\|_2 \\
&= \left\| A_k(t) \cdot \frac{d}{dt} (L_{k-1} \circ \phi_k)(t) - A_k(t) \cdot \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M)) \right\|_2 \\
&\quad + (A_k(t) - B_k(t)) \cdot \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M)) \\
&\leq \|A_k(t)\| \cdot \left\| \frac{d}{dt} (L_{k-1} \circ \phi_k)(t) - \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M)) \right\|_2 \\
&\quad + \|A_k(t) - B_k(t)\| \cdot \left\| \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M)) \right\|_2.
\end{align*}
\]

With inequality (3.6) we obtain
\[
\left\| \frac{d}{dt} (L_{k-1}(0, \ldots, 0, E^k(t) + M)) \right\|_2 = \|\phi'_{k-1}(t)\|_2 \leq (E^{k-1})'(t).
\]

Thus using Lemma 3.3 we obtain
\[
\begin{align*}
\|g'_k(t) - g'_{k-1}(t)\|_2 &\leq \|A_k(t)\| \cdot (I_{1,k}(t) + I_{2,k}(t)) \cdot (E^k)'(t) + \|A_k(t) - B_k(t)\| \cdot (E^{k-1})'(t).
\end{align*}
\]

For simplicity we split the last step up into three parts which we will estimate separately. Therefore we define
\[
\begin{align*}
J_{1,k}(t) &:= \|A_k(t)\| \cdot I_{1,k}(t) \cdot (E^k)'(t) \\
J_{2,k}(t) &:= \|A_k(t)\| \cdot I_{2,k}(t) \cdot (E^k)'(t) \\
J_{3,k}(t) &:= \|A_k(t) - B_k(t)\| \cdot (E^{k-1})'(t).
\end{align*}
\]

Then the upper inequality reads as
\[
\|g'_k(t) - g'_{k-1}(t)\|_2 \leq J_{1,k}(t) + J_{2,k}(t) + J_{3,k}(t).
\]

1. **Estimate of** $J_{1,k}(t)$.

Using inequality (2.4) we obtain
\[
J_{1,k}(t) \leq \frac{ac \cdot k}{E^{k+1}(t) \cdot E^k(t)} \cdot (E^k)'(t).
\]

Since $t \in \left[ t_2 + \varepsilon, \infty \right) \subset (0, \infty)$ this implies
\[
\lim_{k \to \infty} \frac{(E^k)'(t)}{E^{k+1}(t)} = \lim_{k \to \infty} \frac{\prod_{j=1}^{k} (E^j(t) + 1)}{E^{k+1}(t)} = 0
\]
and hence
\[
\lim_{k \to \infty} \frac{(E^k)'(t)}{E^{k+1}(t) \cdot E^k(t)} = 0.
\]
So there exists $c_9 > 0$ such that

$$J_{1,k}(t) \leq c_9 \alpha^{k-1}.$$ 

2. **Estimate of $J_{2,k}(t)$**.

We obtain by the submultiplicivity of the operator norm, Lemma 3.1 and inequality (2.6)

$$\|A_k(t)\| \leq \prod_{l=1}^{k-1} \left\| D \Lambda \left( \left( \bigcup_{j=l}^{k-1} L_j \right) \phi_k(t) \right) \right\| \leq c_4^{k-1} \cdot \left( \prod_{l=1}^{k-1} E^l(t) \right)^{-1}.$$ 

With (3.10) we have

$$J_{2,k}(t) \leq c_4^{k-1} \cdot \frac{(E^k)'(t)}{\prod_{l=1}^{k-1} E^l(t)} \cdot I_{2,k}(t) \leq c_4^{k-1} c_8 \frac{2 \|s_k\|_2 + 1}{E^k(t)^{3/2}} \cdot \prod_{l=1}^{k} \left( 1 + \frac{1}{E^l(t)} \right) \leq c_8 \frac{c_4^{k-1}}{E^k(t)^{3/2}} \cdot \frac{E^k(t_k) + 1}{E^k(t)^{3/2}} \cdot \prod_{l=1}^{k} \left( 1 + \frac{1}{E^l(t)} \right).$$

At this point notice that it follows from (2.11) and (2.12), since $t > t_u$, that

$$\lim_{k \to \infty} \frac{E^k(t_k)}{E^k(t)^{3/2}} = 0$$

and

$$\prod_{l=1}^{\infty} \left( 1 + \frac{1}{E^l(t)} \right) < \infty$$

for $t > t_u$. Moreover

$$\frac{c_4^{k-1}}{E^k(t)^{3/2}} \leq \alpha^{k-1}$$

for large $k \in \mathbb{N}$. Since $t \in [t_u + \varepsilon, \infty)$ there is a constant $c_{10} > 0$ such that we have for large $k \in \mathbb{N}$

$$J_{2,k}(t) \leq c_{10} \alpha^{k-1}.$$ 

3. **Estimate of $J_{3,k}(t)$**.

We want to estimate the right hand side of (3.16) step by step. From (2.6) and (3.7) we obtain for all $r \in \{1, \ldots, k-1\}$

$$\prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \leq c_4^{r-1} \cdot \left( \prod_{l=1}^{r-1} \left\| \left( \bigcup_{j=l}^{k-1} L_j \right) \phi_k(t) \right\|_2 \right)^{-1} \leq c_4^{r-1} \cdot \left( \prod_{l=1}^{r-1} E^l(t) \right)^{-1}.$$ 

Similarly Remark 1 yields

$$\prod_{l=r+1}^{k-1} \|Y_{l,k-1}(t)\| \leq c_4^{k-r-1} \cdot \left( \prod_{l=r+1}^{k-1} E^l(t) \right)^{-1}. $$
Together this implies that
\[
\prod_{l=1}^{r-1} \|X_{l,k-1}(t)\| \cdot \prod_{l=r+1}^{k-1} \|Y_{l,k-1}(t)\| \leq c_4^{k-2} \cdot \left( \prod_{l=1}^{k-1} E^l(t) \right)^{-1}.
\]

Recall that for
\[
x = \left( \bigcup_{j=r}^{k-1} L_j \right) (\phi_k(t)) \quad \text{and} \quad y = \left( \bigcup_{j=r}^{k-1} L_j \right) (0, \ldots, 0, E^k(t) + M)
\]
the inequalities (3.7) and (3.13) ensure that
\[
\|D\Lambda(x) - D\Lambda(y)\| \leq c_7 c_{11} E_k(t)^{1+\beta}.
\]

Notice that there is a constant \(c_{11} \geq 1\) such that for large \(k \in \mathbb{N}\)
\[
\|L_{k-1}(\phi_k(t)) - L_{k-1}(0, \ldots, 0, E^k(t) + M)\|_2 \leq c_{11}.
\]

For \(r = k - 1\) we obtain with (3.7) and (3.13)
\[
\|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\| = \|D\Lambda(L_{k-1}(\phi_k(t))) - D\Lambda(L_{k-1}(0, \ldots, 0, E^k(t) + M))\| \\
\leq c_7 c_{11} E_k(t)^{1+\beta}.
\]

Thus
\[
\prod_{l=1}^{k-2} \|X_{l,k-1}(t)\| \cdot \|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\| \cdot \prod_{l=k}^{k-1} \|Y_{l,k-1}(t)\| \cdot (E^{k-1})'(t) \\
\leq c_7 c_{11} \frac{c_4^k}{E_k(t)^{1+\beta}} \cdot \left( \prod_{l=1}^{k-1} E^l(t) \right)^{-1} \cdot (E^{k-1})'(t) \\
= c_7 c_{11} \frac{c_4^k}{E_k(t)^{1+\beta}} \cdot \prod_{l=1}^{k-1} \left( 1 + \frac{1}{E^l(t)} \right).
\]

Since
\[
\frac{c_4^k}{E_k(t)^{1+\beta}} \leq \alpha^{k-1} \frac{1}{k}
\]
for large \(k \in \mathbb{N}\) and \(t > t_s\), we obtain for \(r = k - 1\) from (3.17)
\[
(3.18) \quad \prod_{l=1}^{k-2} \|X_{l,k-1}(t)\| \cdot \|X_{k-1,k-1}(t) - Y_{k-1,k-1}(t)\| \cdot (E^{k-1})'(t) \\
\leq c_{12} \alpha^{k-1}
\]
for a constant \(c_{12} > 0\) and large \(k \in \mathbb{N}\).
For $r \neq k - 1$ notice that with (2.5) and (2.7) we obtain
\[
\left\| \left( \bigcap_{j=r}^{k-1} L_j \right) \left( \phi_k(t) \right) - \left( \bigcap_{j=r}^{k-1} L_j \right) \left( 0, \ldots, 0, E^k(t) + M \right) \right\|_2 \\
\leq \alpha^{k-r-2} \left\| L_{r-2}(L_{r-1}(\phi_k(t))) - L_{r-2}(L_{r-1}(0, \ldots, 0, E^k(t) + M)) \right\|_2 \\
\leq \alpha^{k-r-2} c_{4\pi} \frac{1}{c_{4\pi}} \left\| L_{k-1}(\phi_k(t)) - L_{k-1}(0, \ldots, 0, E^k(t) + M) \right\|_2 \\
\leq \frac{c_{4\pi}^2}{c_{4\pi}^2} \frac{1}{E^r(t)^{1-\beta}} \cdot \frac{c_{4\pi}^2}{E^{k}(t)^{\beta}} \cdot \frac{c_{4\pi}^2}{E^{k}(t)^{\beta}} \cdot \frac{1}{E^{k}(t)^{\beta}} \cdot \left( \prod_{l=1}^{k-1} E^l(t) \right) - (E^{k-1})'(t) \\
\leq \frac{c_{12}}{k} \alpha^{k-1}.
\]

Therefore we get again the estimate from above for large $k \in \mathbb{N}$ which finally yields with (3.18) and (3.19)
\[
J_{3,k}(t) = \| A_k(t) - B_k(t) \| \cdot (E^{k-1})'(t) \\
\leq \sum_{r=1}^{k-1} \frac{c_{12}}{k} \alpha^{k-1} \\
\leq c_{12} \alpha^{k-1}
\]

for large $k \in \mathbb{N}$. Altogether we obtain with $c_{13} := c_9 + c_{10} + c_{12}$
\[
(3.20) \quad \left\| g_k'(t) - g_{k-1}'(t) \right\|_2 \leq J_{1,k}(t) + J_{2,k}(t) + J_{3,k}(t) \leq c_{13} \alpha^{k-1}.
\]

Since for all $k \in \mathbb{N}$
\[
g_k'(t) = g_0'(t) + \sum_{l=1}^{k} (g_l'(t) - g_{l-1}'(t))
\]

we obtain from (3.20) the uniform convergence of $(g_k')$ on $[t_2 + \varepsilon, \infty)$. This yields for $\varepsilon \to 0$ the locally uniform convergence of $(g_k')$ on $[t_2, \infty)$ which proves Theorem 3.1.

\[\square\]

Remark 2. It seems plausible that this result generalizes to the case of higher derivatives using similar methods, but in a more technical way.
4. Remarks

In Theorem 1.1 we assumed that the function $h$ is $C^1$ in the interior of $Q$ and $Dh$ is Hölder continuous there. Karpińska [Ka99] proved her result using the fact that points in $J \setminus C$ escape to $\infty$ in a comparatively small and parabolic-like domain. To put it in a precise form in our context, Bergweiler [Ber10] considered the function

$$\psi: [1, \infty) \to [1, \infty), \psi(x) = \exp \left( \sqrt{\log(x)} \right).$$

Then for all $\varepsilon > 0$ we obtain

$$\lim_{x \to \infty} \frac{\psi(x)}{x^\varepsilon} = \lim_{x \to \infty} \exp \left( -\varepsilon \log(x) + \sqrt{\log(x)} \right) = 0.$$

The parabolic-like domain from above than has the form

$$\Omega = \{ x \in \mathbb{R}^d : x_d > M \text{ and } \|\tilde{x}\|_2 < \psi(x_d)^2 \},$$

where $x = (\tilde{x}, x_d)$. In [Ber10] it was shown that if $x \in J \setminus C$, then $f^k(x) \in \Omega$ for large $k \in \mathbb{N}$. Using this information it is enough to require that $h$ satisfies the conditions of Theorem 1.1 on a neighborhood of the point $v$ from (2.8) to obtain differentiability of the ‘tails’ of the hairs [SZ03, §3]. Moreover it can be shown that if $h$ is $C^1$ on a suitable compact subset of int($Q$), we obtain the differentiability of the hairs except for the endpoints. This shows that one can relax the condition on $Dh$ in such a way that $Dh$ needs to be only locally Hölder continuous.

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