EXISTENCE OF KIRILLOV-RESHETIKHIN CRYSTALS
OF TYPE $G_2^{(1)}$ AND $D_4^{(3)}$

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Abstract. In this paper we prove that every Kirillov-Reshetikhin module of type $G_2^{(1)}$ and $D_4^{(3)}$ has a crystal pseudobase (crystal base modulo signs), by applying the criterion for the existence of a crystal pseudobase due to Kang et al.

1. Introduction

Let $g$ be an affine Kac-Moody Lie algebra, and $U'_q(g)$ the quantum affine algebra without the degree operator. Among simple finite-dimensional $U'_q(g)$-modules there is a distinguished family called Kirillov-Reshetikhin (KR for short) modules, which are parametrized by two integers $r, \ell$ and denoted by $W_{r, \ell}$. Here $r$ corresponds to a node of the Dynkin diagram of $g$ except the prescribed 0, and $\ell$ is a positive integer. KR modules are known to have nice properties, such as $T(Q,Y)$-systems, fermionic character formulas, and so on (see [HKO+99, HKO+02, Nak03, Her06, DFK08, Her10] and references therein).

As another nice property, it was conjectured in [HKO+99, HKO+02] that a KR module has a crystal base in the sense of Kashiwara [Kas91]. This conjecture has been established by Okado and Schilling [OS08] when $g$ is of nonexceptional type, namely, when the subalgebra $g_0$, whose Dynkin diagram is obtained from that of $g$ by removing 0, is of classical type. More precisely, they have proved that every KR module of nonexceptional type has a crystal pseudobase. Here $B$ is said to be a crystal pseudobase if $B/\{\pm 1\}$ is a crystal base (for the precise definition, see Subsection 2.2 of the present paper). On the other hand, when $g$ is of exceptional type the existence of a crystal pseudobase has not been proved in most cases: it is known by general theory [KKM+92, Kas02] that $W_{r,\ell}$ has a crystal (pseudo)base if one of the following conditions holds; $W_{r,\ell}$ is simple as a $U_q(g_0)$-module, $r$ is a special node called the adjoint node, or $\ell = 1$. As far as the author knows, in exceptional types the existence of a crystal pseudobase has been proved for these $W_{r,\ell}$ only. A main difference between nonexceptional and exceptional types is that every KR module is multiplicity-free as a $U_q(g_0)$-module in the former types, but this is not true in the latter.

In this paper, we prove that every KR module of type $G_2^{(1)}$ and $D_4^{(3)}$ has a crystal pseudobase. In both cases $g_0$ is of type $G_2$, and this gives a first proof of the existence of a crystal pseudobase of $W_{r,\ell}$ for general $r,\ell$ in an exceptional type. Since one node of $g_0$ is adjoint in both cases, the KR modules associated with this node are already known to have crystal pseudobases. Hence we focus on the ones associated with the other node, and prove the following.
Theorem. Assume that \( g \) is of type \( G_2^{(1)} \) or \( D_4^{(3)} \), whose Dynkin diagram is labeled as in (3.1). Then for any positive integer \( \ell \), the KR module \( W^{2,\ell} \) has a crystal pseudobase.

Following [OS08], we prove this theorem by applying the criterion for the existence of a crystal pseudobase introduced in [KKM+92]. A KR module has a prepolarization (bilinear form having some properties) coming from the fusion construction, and the criterion reduces the existence of a crystal pseudobase to a statement concerning the values of the prepolarization of certain vectors (see Subsection 3.2). However it seems hard to calculate the explicit values, and therefore we use the following method. We consider \( W^{2,\ell} \) as a submodule of a twist of \( W^{2,\ell-1} \otimes W^{2,1} \), which enables us to express the values of the prepolarization on \( W^{2,\ell} \) using that on \( W^{2,\ell-1} \) and \( W^{2,1} \). Using this expression, we show the statement by the induction on \( \ell \).

The paper is organized as follows. In Subsections 2.1–2.3 we recall the basic notions. In Subsection 2.4 we recall the criterion for the existence of a crystal pseudobase, and in Subsection 2.5 we review the fusion construction of a KR module. We state the main theorem in Subsection 3.1, and then reduce it to a statement concerning the values of the prepolarization of certain vectors in Subsection 3.2. After collecting equalities used in the proof (Subsection 3.3), we show the statement for \( G_2^{(1)} \) and \( D_4^{(3)} \) separately in Subsections 3.4 and 3.5.

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2. Preliminaries

2.1. Quantum affine algebra. Let \( g \) be an affine Kac-Moody Lie algebra with index set \( I = \{0, 1, \ldots, n\} \) and Cartan matrix \( C = (c_{ij})_{0 \leq i, j \leq n} \). Here the indices are ordered as in [Kac90, Section 4.8]. For simplicity, we assume that \( g \) is not of type \( A_{2n}^{(1)} \) (later we further assume that \( g \) is of type \( G_2^{(1)} \) or \( D_4^{(3)} \)). Denote by \( \alpha_i \) and \( h_i \) \((i \in I)\) the simple roots and simple coroots respectively. Let \( \Lambda_i \) \((i \in I)\) be the fundamental weights, \( \delta \) the generator of null roots, and \( P = \bigoplus_i \mathbb{Z} \Lambda_i \oplus \mathbb{Z} \delta \) the weight lattice. Let \( W \) be the Weyl group, and \((\ , \ )\) a nondegenerate \( W \)-invariant bilinear form on \( P \) satisfying

\[
\min\{\langle \alpha_i, \alpha_i \rangle \mid i \in I\} = 2.
\]

Set

\[
\varpi_i = \Lambda_i - \langle K, \Lambda_i \rangle \Lambda_0 \quad \text{for } i \in I,
\]

where \( K \) denotes the canonical central element of \( g \). Set

\[
P_{\text{cl}} = P/\mathbb{Z} \delta,
\]

and let cl: \( P \to P_{\text{cl}} \) be the canonical projection. By abuse of notation, we also write \( \alpha_i, \Lambda_i, \varpi_i \) for cl(\( \alpha_i \)), cl(\( \Lambda_i \)), cl(\( \varpi_i \)).

Let \( q \) be an indeterminate, and set

\[
[q]^m = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [n]_q! = [n]_q[n - 1]_q \cdots [1]_q, \quad \begin{bmatrix} m \\ n \end{bmatrix}_q = [m]_q[m - 1]_q \cdots [m - n + 1]_q \frac{[n]_q!}{[n]_q!}
\]

for $m \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$. Set $q_i = q^{(\alpha_i,\alpha_i)/2}$ for $i \in I$. The quantum affine algebra (without the degree operator) $U'_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra generated by $e_i, f_i$ ($i \in I$), $q^h$ ($h \in P^*_\mathfrak{cl} = \text{Hom}(P_{\mathfrak{cl}}, \mathbb{Z})$), with the following relations;

$$
q^0 = 1, \quad q^{h_1} q^{h_2} = q^{h_1 + h_2} \quad \text{for } h_1, h_2 \in P^*_\mathfrak{cl},
$$

$$q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i \quad \text{for } h \in P^*_\mathfrak{cl}, i \in I,
$$

$$e_i f_j - f_j e_i = \delta_{ij} t_i - t_i^{-1} \quad \text{for } i, j \in I \text{ where } t_i = q^{(\alpha_i,\alpha_i)h_i/2},$$

and the Serre relations

$$\sum_{k=0}^{1-c_{ij}} (-1)^k e_i^{(k)} e_j^{(1-c_{ij}-k)} = 0, \quad \sum_{k=0}^{1-c_{ij}} (-1)^k f_i^{(k)} f_j^{(1-c_{ij}-k)} = 0 \quad \text{for } i, j \in I \text{ (} i \neq j \text{).}
$$

Here we set $e_i^{(k)} = e_i^k / [k]_q!, f_i^{(k)} = f_i^k / [k]_q!$.

Denote by $\Delta$ the coproduct of $U'_q(\mathfrak{g})$ defined by

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i \quad (2.1)$$

for $h \in P^*_\mathfrak{cl}, i \in I$.

For a $U'_q(\mathfrak{g})$-module $M$ and $\lambda \in P^*_\mathfrak{cl}$, we write

$$M_\lambda = \{ v \in M \mid q^h v = q^{(h,\lambda)} v \quad \text{for } h \in P^*_\mathfrak{cl} \}.$$

If $v \in M_\lambda$ with $v \neq 0$, we write $\text{wt}(v) = \lambda$.

### 2.2. Crystal bases and pseudobases

We briefly recall the definition of crystal bases (see [HK02] for more details), and crystal pseudobases. Let $A$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions without poles at $q = 0$. Let $M$ be an integrable $U'_q(\mathfrak{g})$-module, and $\tilde{e}_i, \tilde{f}_i$ ($i \in I$) the Kashiwara operators. A free $A$-submodule $L$ of $M$ is called a crystal lattice of $M$ if

(a) $M \cong \mathbb{Q}(q) \otimes A L$,  \quad (b) $L = \bigoplus_{\lambda \in P^*_\mathfrak{cl}} L_\lambda$ where $L_\lambda = L \cap M_\lambda$,

(c) $\tilde{e}_i L \subseteq L$, $\tilde{f}_i L \subseteq L$ ($i \in I$).

A pair $(L, B)$ is called a crystal base of $M$ if

(i) $L$ is a crystal lattice of $M$,  \quad (ii) $B$ is a $Q$-basis of $L/qL$,

(iii) $B = \bigsqcup A_{\lambda \in P^*_\mathfrak{cl}} B_\lambda$ where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,  \quad (iv) $\tilde{e}_i B \subseteq B \cup \{0\}$, \quad $\tilde{f}_i B \subseteq B \cup \{0\}$,

(v) for $b, b' \in B$ and $i \in I$, $\tilde{f}_i b = b'$ if and only if $\tilde{e}_i b' = b$.

We call $(L, B)$ a crystal pseudobase of $M$ if they satisfy the conditions (i), (iii)–(v), and

(iii) $B = B' \cup (-B')$ with $B'$ a $Q$-basis of $L/qL$.

### 2.3. Polarization

Let $M$ and $N$ be $U'_q(\mathfrak{g})$-modules. A bilinear pairing $(\ ,\ ) : M \otimes_{\mathbb{Q}(q)} N \to \mathbb{Q}(q)$ is said to be admissible if it satisfies

$$(q^h u, v) = (u, q^h v), \quad (e_i u, v) = (u, q^{-1}_i t_i^{-1} f_i v), \quad (f_i u, v) = (u, q^{-1}_i t_i e_i v) \quad (2.2)$$

for $h \in P^*_\mathfrak{cl}, i \in I, u \in M, v \in N$. A bilinear form $(\ ,\ )$ on $M$ is called a prepolarization if it is symmetric and satisfies (2.2) for $u, v \in M$. A prepolarization is called a polarization.
Proposition 2.1. Let $M$ be a finite-dimensional integrable $U_q'(g)$-module, and assume that $M$ has a prepolarization $(\ ,\ )$ and a $U_q'(g)_{K_Z}$-submodule $M_{K_Z}$ such that $(M_{K_Z}, M_{K_Z}) \subseteq K_Z$. Let $\lambda_1, \ldots, \lambda_m \in P_0^+$, and assume further that the following conditions hold:

(i) $M \cong \bigoplus_{\ell=1}^m V_0(\lambda_{\ell})$ as $U_q(g_0)$-modules,

(ii) there exist $u_k \in (M_{K_Z})_{\lambda_k}$ ($1 \leq k \leq m$) such that $(u_k, u_l) \in \delta_{kl} + qA$, and $||e_i u_k||^2 \in q^{-2(h_i, \lambda_k) - 2} qA$ for all $i \in I_0$.

Then $(\ ,\ )$ is a polarization. Moreover, if we set

$L = \{ u \in M \mid ||u||^2 \in A \}$ and $B = \{ b \in (M_{K_Z} \cap L)/(M_{K_Z} \cap qL) \mid (b, b)_0 = 1 \}$

where $(\ ,\ )_0$ is the $Q$-valued bilinear form on $L/qL$ induced by $(\ ,\ )$, then $(L, B)$ is a crystal pseudobase of $M$.

2.4. Criterion for the existence of a crystal pseudobase. In this subsection, we recall a criterion for the existence of a crystal pseudobase given by [KKM+92].

Let $g_0$ denote the simple Lie algebra whose Dynkin diagram is obtained from that of $g$ by removing the node 0. Let $I_0 = I \setminus \{0\}$, $P_0$ be the weight lattice of $g_0$, and $P_0^+ \subseteq P_0$ the subset of dominant integral weights. We identify $P_0$ with a subgroup of $P_{cl}$ as follows:

$$P_0 = \sum_{i \in I_0} \mathbb{Z}c_i \subseteq P_{cl}.$$ 

Let $U_q(g_0)$ be the $\mathbb{Q}(q)$-subalgebra of $U_q'(g)$ generated by $e_i, f_i, q^{h_i}$ ($i \in I_0$), which is the quantized enveloping algebra of $g_0$. For $\lambda \in P_0^+$, let $V_0(\lambda)$ denote the simple integrable $U_q(g_0)$-module with highest weight $\lambda$.

Let $A_Z$ and $K_Z$ be the subalgebras of $\mathbb{Q}(q)$ defined respectively by

$$A_Z = \{ f(q)/g(q) \mid f(q), g(q) \in \mathbb{Z}[q], g(0) = 1 \}, \quad K_Z = A_Z[q^{-1}].$$

Let $U_q'(g)_{K_Z}$ denote the $K_Z$-subalgebra of $U_q'(g)$ generated by $e_i, f_i, q^h$ ($i \in I, h \in P_{cl}$).

Now the following proposition, which immediately follows from [KKM+92] Propositions 2.6.1 and 2.6.2], is a key tool to show our main theorem.

Proposition 2.1. Let $M$ be a finite-dimensional integrable $U_q'(g)$-module, and assume that $M$ has a prepolarization $(\ ,\ )$ and a $U_q'(g)_{K_Z}$-submodule $M_{K_Z}$ such that $(M_{K_Z}, M_{K_Z}) \subseteq K_Z$. Let $\lambda_1, \ldots, \lambda_m \in P_0^+$, and assume further that the following conditions hold:

(i) $M \cong \bigoplus_{\ell=1}^m V_0(\lambda_{\ell})$ as $U_q(g_0)$-modules,

(ii) there exist $u_k \in (M_{K_Z})_{\lambda_k}$ ($1 \leq k \leq m$) such that $(u_k, u_l) \in \delta_{kl} + qA$, and $||e_i u_k||^2 \in q^{-2(h_i, \lambda_k) - 2} qA$ for all $i \in I_0$.

Then $(\ ,\ )$ is a polarization. Moreover, if we set

$L = \{ u \in M \mid ||u||^2 \in A \}$ and $B = \{ b \in (M_{K_Z} \cap L)/(M_{K_Z} \cap qL) \mid (b, b)_0 = 1 \}$

where $(\ ,\ )_0$ is the $Q$-valued bilinear form on $L/qL$ induced by $(\ ,\ )$, then $(L, B)$ is a crystal pseudobase of $M$.

2.5. Fundamental representations and fusion construction of KR modules. For $r \in I_0$, denote by $W(\varpi_r)$ the fundamental representation introduced by [Kas02]. $W(\varpi_r)$ is known to have the following properties.

Proposition 2.2 ([OS08] Propositions 2.4 and 2.6]).

(i) $W(\varpi_r)$ is a finite-dimensional simple integrable $U_q'(g)$-module.

(ii) $\dim W(\varpi_r)_{\varpi_r} = 1$. 

if it is positive definite with respect to the following total order on $\mathbb{Q}(q)$:

$$f > g \text{ if and only if } f - g \in \bigcup_{n \in \mathbb{Z}} \{ q^n(c + qA) \mid c \in \mathbb{Q}_{>0} \},$$

and $f \geq g$ if $f = g$ or $f > g$. Throughout the paper, we use the notation $||u||^2 = (u, u)$ for $u \in M$. 

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(iii) The weight set of $W(\varpi_r)$, which is a subset of $P_{cl}$, is contained in the intersection of $\varpi_r + \sum_{i \in I} \mathbb{Z}e_i$ and the convex hull of $W(\varpi_r)$.

(iv) $W(\varpi_r)$ has a crystal base.

(v) $W(\varpi_r)$ has a polarization $(\ , \ )$.

(vi) There exists a $U_q'(\mathfrak{g})_{K_Z}$-submodule $W(\varpi_r)_{K_Z}$ of $W(\varpi_r)$ such that

$$\{W(\varpi_r)_{K_Z}, W(\varpi_r)_{K_Z}\} \subseteq K_Z.$$  

For a $U_q'(\mathfrak{g})$-module $M$, denote by $M_{aff}$ the $U_q'(\mathfrak{g})$-module $\mathbb{Q}(q)[z, z^{-1}] \otimes M$ on which $e_i$ and $f_i$ act by $z^{\delta_{0i}} \otimes e_i$ and $z^{-\delta_{0i}} \otimes f_i$, respectively. For $a \in \mathbb{Q}(q)$, set $M_a = M_{aff}/(z-a)M_{aff}$. For $v \in M$, denote by $v_a \in M_a$ the image of $1 \otimes v$ under the projection $M_{aff} \rightarrow M_a$. We simply write $v$ for $v_a$ when no confusion is possible. The following lemma is a consequence of [Kas02, Proposition 9.3] (note that it is also proved in the paper that $W(\varpi_r)$ is a “good” module).

Lemma 2.3. Assume that $a, b \in \mathbb{Q}(q)^*$ satisfy $a/b \in A$. Then for any $r \in I_0$, there exists a unique nonzero $U_q'(\mathfrak{g})$-module homomorphism (the normalized $R$-matrix)

$$R(a, b) : W(\varpi_r)_a \otimes W(\varpi_r)_b \rightarrow W(\varpi_r)_b \otimes W(\varpi_r)_a$$

satisfying $R(a, b)((v_1)_a \otimes (v_1)_b) = (v_1)_b \otimes (v_1)_a$, where $v_1$ is a nonzero vector of $W(\varpi_r)$ with weight $\varpi_r$.

Now let us recall the fusion construction of Kirillov-Reshetikhin modules. Fix $r \in I_0$ and $\ell \in \mathbb{Z}_{\geq 0}$, and let $v_1 \in W(\varpi_r)_{K_Z}$ be a vector with weight $\varpi_r$ and satisfying $||v_1||^2 = 1$. Let $\mathfrak{S}_\ell$ be the $\ell$-th symmetric group, and $s_i \in \mathfrak{S}_\ell$ the adjacent transposition interchanging $i$ and $i + 1$. For $w \in \mathfrak{S}_\ell$, denote by $\ell(w) \in \mathbb{Z}_{\geq 0}$ the length of $w$. Let $x_1, \ldots, x_\ell \in \mathbb{Q}(q)^*$ be such that $x_i/x_j \in A$ for all $i < j$. Then for any $w \in \mathfrak{S}_\ell$, we can construct a well-defined $U_q'(\mathfrak{g})$-module homomorphism

$$R_w(x_1, \ldots, x_\ell) : W(\varpi_r)_{x_1} \otimes \cdots \otimes W(\varpi_r)_{x_\ell} \rightarrow W(\varpi_r)_{x_w(1)} \otimes \cdots \otimes W(\varpi_r)_{x_w(\ell)}$$

by

$$R_1(x_1, \ldots, x_\ell) = \text{id}$$

$$R_{s_i}(x_1, \ldots, x_\ell) = \left( \bigotimes_{j < i} \text{id}_{W(\varpi_r)_{x_j}} \right) \otimes R(x_i, x_{i+1}) \otimes \left( \bigotimes_{j > i+1} \text{id}_{W(\varpi_r)_{x_j}} \right)$$

$$R_w'(x_1, \ldots, x_\ell) = R_w'(x_{w(1)}, \ldots, x_{w(\ell)}) \circ R_w(x_1, \ldots, x_\ell)$$

for $w, w' \in \mathfrak{S}_\ell$ such that $\ell(w'w) = \ell(w') + \ell(w)$.

Set

$$k = \begin{cases} 
(\alpha_r, \alpha_r)/2 & \text{if } \mathfrak{g} \text{ is of nontwisted affine type,} \\
1 & \text{if } \mathfrak{g} \text{ is of twisted affine type.}
\end{cases}$$

Let $w_0 \in \mathfrak{S}_\ell$ denote the longest element, and define a map $R_\ell$ by

$$R_\ell = R_{w_0} q^{k(\ell-1)} \cdot q^{k(\ell-3)} \cdot \cdots \cdot q^{k(1-\ell)} ;$$

$$W(\varpi_r)_{q^k(\ell-1)} \otimes \cdots \otimes W(\varpi_r)_{q^k(1-\ell)} \rightarrow W(\varpi_r)_{q^k(1-\ell)} \otimes \cdots \otimes W(\varpi_r)_{q^k(\ell-1)}.$$
We denote the image of $R_\ell$ by $W^{r,\ell}$. Then $W^{r,\ell}$ is a simple $U'_q(G)$-module by [Kas02, Theorem 9.2], and it is easily seen that the Drinfeld polynomials (see [CP95] for the nontwisted case and [CP98] for the twisted case) of $W^{r,\ell}$ are

$$P_i(u) = \begin{cases} (1 - a_i^\dagger q_i^{-1-\ell}u)(1 - a_i^\dagger q_i^{-3-\ell}u) \cdots (1 - a_i^\dagger q_i^{-1-\ell}u) & \text{if } i = r, \\ 1 & \text{otherwise,} \end{cases}$$

where $a_i^\dagger \in \mathbb{Q}(q)^*$ is the element such that the Drinfeld polynomials of $W(\varpi_r)$ are

$$P_r(u) = 1 - a_i^\dagger u, \quad P_j(u) = 1 \ (j \neq r).$$

The module $W^{r,\ell}$ is called the Kirillov-Reshetikhin module (KR module for short) associated with $r, \ell$.

Let us recall how to define a prepolarization on $W^{r,\ell}$. We begin with recalling the following lemma.

**Lemma 2.4** ([KKM+92, Lemma 3.4.1]). Let $M_j$ and $N_j$ ($j = 1, 2$) be $U'_q(G)$-modules, and assume for $j = 1, 2$ that there exists an admissible pairing $(\ , \ )_j$ between $M_j$ and $N_j$. Then the pairing $(\ , \ )$ between $M_1 \otimes M_2$ and $N_1 \otimes N_2$ defined by

$$(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2) \quad \text{for } u_j \in M_j, \ v_j \in N_j$$

is admissible.

By the lemma and Proposition 22 (v), there exists an admissible pairing $(\ , \ )_0$ between $W(\varpi_r)_{q^{k(\ell-1)}} \otimes \cdots \otimes W(\varpi_r)_{q^{k(1-\ell)}}$ and $W(\varpi_r)_{q^{k(1-\ell)}} \otimes \cdots \otimes W(\varpi_r)_{q^{k(1-\ell)}}$. Now a bilinear form $(\ , \ )$ on $W^{r,\ell}$ is defined by

$$(R_\ell u, R_\ell v) = (u, R_\ell v)_0$$

for $u, v \in W(\varpi_r)_{q^{k(\ell-1)}} \otimes \cdots \otimes W(\varpi_r)_{q^{k(1-\ell)}}$. We set

$$(W^{r,\ell})_{KZ} = R_\ell(W(\varpi_r)_{KZ}^\otimes_{\ell}) \cap W(\varpi_r)_{KZ}^\otimes_{\ell}.$$

**Proposition 2.5** ([KKM+92, Proposition 3.4.3]).

(i) $(\ , \ )$ is a nondegenerate prepolarization on $W^{r,\ell}$.

(ii) $||R_\ell(v^\otimes_{\ell})||^2 = 1$.

(iii) $((W^{r,\ell})_{KZ}, (W^{r,\ell})_{KZ}) \subseteq KZ$.

### 3. Main theorem

#### 3.1. Statement of the main theorem.

In the rest of this paper, we assume that $\mathfrak{g}$ is either of type $G_2^{(1)}$ or $D_4^{(3)}$, whose Dynkin diagram is as follows;

$$G_2^{(1)}: \quad o \quad o \quad o \quad o \quad D_4^{(3)}: \quad o \quad o \quad o \quad 0 \quad 1 \quad 2 \quad 2$$

Note that in both types $\mathfrak{g}_0$ is of type $G_2$, but the labeling of the simple roots are inverted.

Our main theorem is the following.

**Theorem 3.1.** Assume that $\mathfrak{g}$ is either of type $G_2^{(1)}$ or $D_4^{(3)}$. For any $\ell \in \mathbb{Z}_{>0}$, the KR module $W^{2,\ell}$ has a crystal pseudobase.
Since the existence of a crystal pseudobase of $W^{1,\ell}$ for any $\ell \in \mathbb{Z}_{>0}$ has already been proved by [Yam98] in type $G_2^{(1)}$ and by [KMOY07] in type $D_4^{(3)}$, this theorem implies that every KR module of these types has a crystal pseudobase.

3.2. Reduction to calculations on the prepolarization. Hereafter, we write $W^\ell$ for $W^{2,\ell}$. In order to apply Proposition 2.4, let us recall the branching rule of $W^\ell$. There is an explicit formula for the decomposition of general KR modules as $U_q(\mathfrak{g}_0)$-modules, called ($q = 1$) fermionic formula, see [HKO+93, HKO+02, Nak03, Her06, DFK08, Her10]. For $W^\ell$ in type $G_2^{(1)}$ and $D_4^{(3)}$, the formulas are rewritten as follows.

**Proposition 3.2.** For a positive integer $\ell$, define a set $S_\ell \subseteq \mathbb{Z}_{\geq 0}^4$ by

$$S_\ell = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4 \mid 3b \leq c \leq b + d, \ a \leq b, \ -c + 3d \leq \ell\} \text{ if } \mathfrak{g} \text{ is of type } G_2^{(1)},$$

and

$$S_\ell = \{(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4 \mid 3c \leq b + d, \ a \leq b \leq c, \ -c + d \leq \ell\} \text{ if } \mathfrak{g} \text{ is of type } D_4^{(3)}.$$ Then we have

$$W^\ell \cong \bigoplus_{(a, b, c, d) \in S_\ell} V_0(\ell \varpi_2 + (a + d)\alpha_0 + (b + d)\alpha_1 + c\alpha_2)$$
as $U_q(\mathfrak{g}_0)$-modules.

**Proof.** First assume that $\mathfrak{g}$ is of type $G_2^{(1)}$. In [CM07, Subsection 2.4], the fermionic formula of $W^\ell$ is rewritten as follows:

$$W^\ell \cong \bigoplus_{(r_1, r_2, r_3, r_4) \in T_\ell} V_0((r_2 + r_3 - r_4)\varpi_1 + (\ell - r_1 - 3r_2 - 3r_3)\varpi_2),$$

where $T_\ell \subseteq \mathbb{Z}_{\geq 0}^4$ is defined by

$$T_\ell = \{(r_1, r_2, r_3, r_4) \in \mathbb{Z}_{\geq 0}^4 \mid r_4 \leq r_2, \ 2r_1 + 3r_2 + 3r_3 \leq \ell\}.$$ Then the assertion follows by the following substitution;

$$r_1 = -3b + c, \ r_2 = a + b - c + d, \ r_3 = -a + b, \ r_4 = a,$$

since $\alpha_0 = -\varpi_1$, $\alpha_1 = 2\varpi_1 - 3\varpi_2$, $\alpha_2 = -\varpi_1 + 2\varpi_2$.

Next assume that $\mathfrak{g}$ is of type $D_4^{(3)}$. In this case the branching rule given in *loc. cit.* is as follows:

$$W^\ell \cong \bigoplus_{(r_1, r_2, r_3, r_4) \in T_\ell} V_0((r_1 + r_2 - r_3)\varpi_1 + (\ell - r_1 - r_2 - r_4)\varpi_2)$$

with $T_\ell = \{(r_1, r_2, r_3, r_4) \in \mathbb{Z}_{\geq 0}^4 \mid r_3 \leq r_1, \ r_1 + r_2 + r_3 + r_4 \leq \ell\}$. Then the assertion is proved via

$$r_1 = -2c + d, \ r_2 = -a + b, \ r_3 = -b + c, \ r_4 = a,$$

since $\alpha_0 = -\varpi_1$, $\alpha_1 = 2\varpi_1 - \varpi_2$, $\alpha_2 = -3\varpi_1 + 2\varpi_2$. □
For a quadruple \((a, b, c, d) \in \mathbb{Z}^4\) of integers, set
\[
e^{(a, b, c, d)} = e_0^{(a)} e_1^{(b)} e_2^{(c)} e_3^{(d)} \in U'_q(\mathfrak{g}),
\]
where we use the convention \(e_i^{(k)} = 0\) for \(k < 0\). For each \(\ell \in \mathbb{Z}_{>0}\), let \(v_\ell \in W^\ell\) denote the image of \(v_1^{\otimes \ell}\) under \(R_\ell\). From Propositions 3.1, 3.2 and 3.3, we easily see that the following proposition implies Theorem 3.1.

**Proposition 3.3.** Let \(\ell\) be a positive integer, and \(S_\ell \subseteq \mathbb{Z}^4_{\geq 0}\) the set defined in Proposition 3.2. Then the vectors \(\{e^{(a, b, c, d)} v_\ell \mid (a, b, c, d) \in S_\ell\}\) in \(W^\ell\) satisfy
\[
(e^{(a, b, c, d)} v_\ell, e^{(a', b', c', d')} v_\ell) \in \delta_{(a, b, c, d), (a', b', c', d')} + qA,
\]
and
\[
||e_i^{(a, b, c, d)} v_\ell||^2 \leq q_i^{-2(h_i, \text{wt}(e^{(a, b, c, d)} v_\ell))} qA \quad \text{for} \quad i = 1, 2.
\]

3.3. *Collection of equalities.* Here we collect equalities which are repeatedly used in the proof of Proposition 3.3.

The following equality is easily proved from (2.1): for \(i \in I\) and \(k \in \mathbb{Z}_{\geq 0}\),
\[
\Delta(e_i^{(k)}) = \sum_{j=0}^{k} j^{(k-j)} e_i^{(k-j)} \otimes e_i^{(j)}.
\]

It is easily proved from the defining relations of \(U'_q(\mathfrak{g})\) that, for \(i \in I\), \(r, s \in \mathbb{Z}_{\geq 0}\),
\[
f_i^{(r)} e_i^{(s)} = \sum_{k=0}^{\min\{r, s\}} e_i^{(s-k)} f_i^{(r-k)} \prod_{j=1}^{k} q_i^{r-s-j+1} t_i^j - q_i^{-(r-s-j+1)} t_i^j q_i^2 - q_i^{-j}.
\]

In particular, if \(M\) is a \(U'_q(\mathfrak{g})\)-module we have
\[
f_i^{(r)} e_i^{(s)} v = \sum_{k=0}^{\min\{r, s\}} \left[ r - s - \langle h_i, \lambda \rangle \right]_{q_i} e_i^{(s-k)} f_i^{(r-k)} v \quad \text{for} \quad v \in M_\lambda.
\]

Assume that a given \(U'_q(\mathfrak{g})\)-module \(M\) has a prepolarization \((\ ,\ )\). Then for \(i \in I\), \(u, v \in M\) and \(k \in \mathbb{Z}_{\geq 0}\), we have from (2.2) that
\[
(e_i^{(k)} u, v) = (u, q_i^{-k^2} t_i^k f_i^{(k)} v), \quad (f_i^{(k)} u, v) = (u, q_i^{-k^2} t_i^k e_i^{(k)} v).
\]

In particular, it holds that
\[
(e_i^{(k)} u, v) = q_i^{k(k-\langle h_i, \lambda \rangle)} (u, f_i^{(k)} v), \quad (f_i^{(k)} u, v) = q_i^{k(k+\langle h_i, \lambda \rangle)} (u, e_i^{(k)} v) \quad \text{for} \quad v \in M_\lambda.
\]

The following lemma is immediate from (3.3) and (3.6).

**Lemma 3.4.** Let \(M\) be a finite-dimensional \(U'_q(\mathfrak{g})\)-module with a prepolarization \((\ ,\ )\), \(u \in M_\lambda\), \(i \in I\) and assume that \(f_iu = 0\). Then
\[
||e_i^{(k)} u||^2 = q_i^{-k(k+\langle h_i, \lambda \rangle)} \left( -\langle h_i, \lambda \rangle \right)_{q_i} ||u||^2.
\]

In particular, \(||e_i^{(k)} u||^2 \in (1 + qA)||u||^2\) holds for \(0 < k \leq -\langle h_i, \lambda \rangle\).
3.4. **Proof in type \( G_2^{(1)} \) case.** Throughout this subsection we assume that \( \mathfrak{g} \) is of type \( G_2^{(1)} \), and we will show Proposition 3.3 in this type (the type \( D_4^{(3)} \) will be treated in the next subsection). Note that

\[
q_0 = q_1 = q^3, \quad q_2 = q.
\]

We write \( e^{(b,c,d)} \) for \( e^{(0,b,c,d)} = e_1^{(b)} e_2^{(c)} e_0^{(d)} \). For each \( \ell \in \mathbb{Z}_{>0} \), let us consider the following collection of statements (G1)–(G6) concerning the vector \( v_\ell \in W_\ell \):

(G1) \( \ell e^{(b,c,d)} v_\ell \neq 0 \) if and only if \( 2b \leq c \leq 2d \leq 2\ell \).

(G2) \( \| e^{(b,c,d)} v_\ell \|^2 \in 1 + qA \) if \( 2b \leq c \leq 2d \leq 2\ell \).

(G3) For \((b, c, d), (b', c', d') \in \mathbb{Z}_{\geq 0}^3\), \((e^{(b,c,d)} v_\ell, e^{(b',c',d')} v_\ell) = 0\) if \((b, c, d) \neq (b', c', d')\).

(G4) For \((b, c, d) \in \mathbb{Z}_{\geq 0}^3\), \(\|e_2 e^{(b,c,d)} v_\ell\|^2 \in q^{\min\{0, 6b-2c, -2c+6d-2\ell\}} A\).

(G5) For \((b, c, d) \in \mathbb{Z}_{\geq 0}^3\), \((e^{(b-1,c-2,d-1)} v_\ell, e_2 e^{(b,c,d)} v_\ell) \in q^{-c+3d-\ell} A\).

(G6) For \((b, c, d) \in \mathbb{Z}_{\geq 0}^3\), \((e^{(b,c,d)} v_\ell, e_2 e^{(b,c-1,d)} v_\ell) \in q^{3b-c+1} A\).

In what follows, we will first show \((G1)_{\ell-}(G6)_{\ell}\) by the induction on \( \ell \), and then prove Proposition 3.3 using them.

The fundamental module \( W^1 = W(\varpi_2) \) is isomorphic to \( V_0(\varpi_2) \) as a \( U_q'(\mathfrak{g}_0) \)-module and all the weight spaces are 1-dimensional. Then its \( U_q'(\mathfrak{g}) \)-module structure is automatically determined from the fact that \( W(\varpi_2) \) has a global basis \([\text{Kas02}]\) and we describe the module structure explicitly here. Let \( W \) be the 7-dimensional \( \mathbb{Q}(q) \)-vector space with a basis \( \{1, \cdots, 7\} \) \( j = 1, 2, 3 \} \cup \{0\} \). Define a \( U_q'(\mathfrak{g}) \)-module structure on \( W \) by

\[
\begin{align*}
e_0 \quad &1 = \overline{7}, \quad e_0 \quad &2 = \overline{1}, \quad \text{and } e_0 \quad &* = 0 \text{ otherwise,} \\
e_1 \quad &3 = \overline{2}, \quad e_1 \quad &2 = \overline{3}, \quad \text{and } e_1 \quad &* = 0 \text{ otherwise,} \\
e_2 \quad &1 = \overline{3}, \quad e_2 \quad &0 = \overline{2}, \quad e_2 \quad &\overline{1} = \overline{5}, \quad e_2 \quad &\overline{3} = \overline{0}, \quad \text{and } e_2 \quad &* = 0 \text{ otherwise,} \\
f_0 \quad &\overline{3} = \overline{3}, \quad f_0 \quad &\overline{2} = \overline{1}, \quad \text{and } f_0 \quad &* = 0 \text{ otherwise,} \\
f_1 \quad &\overline{1} = \overline{3}, \quad f_1 \quad &\overline{2} = \overline{7}, \quad \text{and } f_1 \quad &* = 0 \text{ otherwise,} \\
f_2 \quad &\overline{0} = \overline{2}, \quad f_2 \quad &\overline{3} = \overline{1}, \quad f_2 \quad &\overline{0} = \overline{4}, \quad f_2 \quad &\overline{1} = \overline{7}, \quad \text{and } f_2 \quad &* = 0 \text{ otherwise,} \\
\text{wt}(j) = \delta_{j1} \varpi_2 + \delta_{j2} (\varpi_1 - \varpi_2) + \delta_{j3} (-\varpi_1 + 2\varpi_2), \quad \text{wt}(\overline{j}) = -\text{wt}(j).
\end{align*}
\]

Then we have \( W^1 \cong W \) and identify \( W^1 \) with \( W \) hereafter. To illustrate the actions, we give the crystal graph of \( W^1 \) below:

![Crystal Graph](attachment:image.png)
Using Lemma 3.4, we easily check that
\[ \| \mathbf{I} \| = 1 \quad \text{for } j \in \{1, 2, 3, 4, 5, 6\}, \quad \| \mathbf{0} \| = 1 + q^2. \] (3.8)

Now (G1) \( \rightarrow \) (G6) \( \rightarrow \) are easily checked, and the induction begins.

Let \( \ell > 1 \), and assume that (G1) \( \ell-1 \rightarrow \) (G6) \( \ell-1 \) hold. Since \((W^{\ell-1})_{q-1}\) is a submodule of
\[ \left((W^1)_{q^2-\ell} \otimes \cdots \otimes (W^1)_{q^\ell-2}\right)_{q-1} \cong (W^1)_{q^{\ell-\ell}} \otimes \cdots \otimes (W^1)_{q^{\ell-3}}, \]
we have an injection
\[ (W^{\ell-1})_{q-1} \otimes (W^1)_{q^{\ell-1}} \hookrightarrow (W^1)_{q^{\ell-\ell}} \otimes \cdots \otimes (W^1)_{q^{\ell-3}} \otimes (W^1)_{q^{\ell-1}}. \]

By the simplicity, we see that \((W^\ell)_{q-1} \otimes (W^1)_{q^{\ell-1}}\) is generated by \((v_{\ell-1})_{q-1} \otimes (v_1)_{q^{\ell-1}}\). Hence for \((b, c, d) \in \mathbb{Z}_{\geq 0}^3\),
\[ e^{(b, c, d)} v_\ell \neq 0 \quad \text{if and only if} \quad e^{(b, c, d)} ((v_{\ell-1})_{q-1} \otimes (v_1)_{q^{\ell-1}}) \neq 0 \quad \text{in} \quad (W^{\ell-1})_{q-1} \otimes (W^1)_{q^{\ell-1}} \]
if and only if \( e^{(b, c, d)} v_{\ell-1} \otimes (v_1)_q \neq 0 \) in \((W^\ell)_{q^{\ell-1}} \otimes (W^1)_{q^{\ell-1}}\). (3.9)

The following lemma is proved by straightforward calculations using (3.4) (note that we have \( v_1^{(k)} e_0^{(m)} v_{\ell-1} = 0 \) for \( k > m \) by the Serre relations).

**Lemma 3.5.** For any \( m \in \mathbb{Z} \), we have
\[ e^{(b, c, d)} (v_{\ell-1} \otimes (v_1)_q^m) = q^{-c+3d} e^{(b, c, d)} v_{\ell-1} \otimes 1 + q^m e^{(b-1, c-2, d-1)} v_{\ell-1} \otimes 2 \]
\[ + q^{3b+m} e^{(b, c-2, d-1)} v_{\ell-1} \otimes 3 + q^{c+m-1} e^{(b-1, c-1, d-1)} v_{\ell-1} \otimes 0 \]
\[ + q^{-3b+2c+m} e^{(b, c, d-1)} v_{\ell-1} \otimes 3. \]

Now (G1) \( \ell \) follows from this lemma, (3.3) and (G1) \( \ell-1 \). Next we will show (G2) \( \ell \). By the definition, the composition of the maps
\[ (W^1)_{q^{\ell-1}} \otimes \cdots \otimes (W^1)_{q^{\ell-\ell}} \xrightarrow{R'_{\ell-1} \otimes \operatorname{id}_{(W^1)_{q^{\ell-\ell}}}} (W^1)_{q^{\ell-\ell}} \otimes \cdots \otimes (W^1)_{q^{\ell-1}} \otimes (W^1)_{q^{\ell-1}} \]
\[ \xrightarrow{R_{q^2-q^{\ell-1}}} (W^1)_{q^{\ell-\ell}} \otimes \cdots \otimes (W^1)_{q^{\ell-1}} \]
coincides with \( R_\ell \), where
\[ R'_{\ell-1} : (W^1)_{q^{\ell-1}} \otimes \cdots \otimes (W^1)_{q^{\ell-\ell}} \rightarrow (W^1)_{q^{\ell-\ell}} \otimes \cdots \otimes (W^1)_{q^{\ell-1}} \]
is defined similarly to \( R_{\ell-1} \). Since the image of the first map is \((W^{\ell-1})_q \otimes (W^1)_{q^{\ell-\ell}}\) and that of \( R_\ell \) is \( W^\ell \), the second map induces a \( U'_q(\mathfrak{g}) \)-module homomorphism
\[ R' : (W^{\ell-1})_q \otimes (W^1)_{q^{\ell-\ell}} \rightarrow W^\ell. \]

Let \( \langle , \rangle \) be the admissible pairing between \((W^{\ell-1})_q \otimes (W^1)_{q^{\ell-\ell}}\) and \((W^{\ell-1})_{q^{\ell-1}} \otimes (W^1)_{q^{\ell-1}}\) which is obtained by Lemma 2.4. Now the following lemma is obvious from the construction of the prepolarization \( \langle , \rangle \) on \( W^\ell \).

**Lemma 3.6.** For \( u, v \in (W^{\ell-1})_q \otimes (W^1)_{q^{\ell-\ell}} \), we have
\[ \langle R'(u), R'(v) \rangle = \langle u, R'(v) \rangle \]
where $R'(v)$ in the right-hand side is regarded as a vector of $(W^{\ell-1})_{q^{-1}} \otimes (W^1)_{q^{-1}}.$ via the embedding $W^\ell \hookrightarrow (W^{\ell-1})_{q^{-1}} \otimes (W^1)_{q^{-1}}$ mapping $v_\ell$ to $(v_{\ell-1})_{q^{-1}} \otimes (v_1)_{q^{-1}}$.

Denote by $(\ , \ )_2$ the admissible pairing between $W^{\ell-1} \otimes (W^1)_{q^2}$ and $W^{\ell-1} \otimes (W^1)_{q^{-\ell}}$. For $(b, c, d) \in \mathbb{Z}^3 \geq 0$, we have from Lemma 3.6 that

$$
||e^{(b,c,d)}v_{\ell-1}||^2 = \left(e^{(b,c,d)}((v_{\ell-1})_q \otimes (v_1)_{q^{-1}}), e^{(b,c,d)}((v_{\ell-1})_q \otimes (v_1)_{q^{-1}})\right)_1
$$

and the right-hand side is equal to

$$
q^{-2c+6d}||e^{(b,c,d)}v_{\ell-1}||^2 + ||e^{(b-1,c-2,d-1)}v_{\ell-1}||^2 + q^{6b}||e^{(b,c-2,d-1)}v_{\ell-1}||^2
$$

and hence we have from Lemma 3.5 that

$$
||e^{(b,c,d)}v_\ell||^2 = q^{-c+3d-1}||e^{(b,c,d)}v_\ell||^2 + q^m||e^{(b-1,c-2,d-1)}v_\ell||^2
$$

Next we will show (G4) $\ell$. By (G1) we may assume $2b \leq c \leq 2d \leq 2\ell$. When $b = 0$, (G2) implies $||e^{(b,c,d)}v_{\ell-1}||^2 \in q^{-2c}A$, and (G4) $\ell$ holds. Hence we may further assume that $b > 0$. Writing $v = v_{\ell-1}$ for short, we have from Lemma 3.5 that

$$
\begin{aligned}
e & e^{(b,c,d)}v(\otimes (v_1)_{q^m}) \\
= & (e^{(b,c,d)}v(\otimes (v_1)_{q^{-1}}))_2 + q^{m+1}e^{(b-1,c-2,d-1)}v(\otimes 1) + q^{m}e^{(b,c-2,d-1)}v(\otimes 2) \\
& + q^{m}(q^{3b-2}e^{(b,c-2,d-1)}v + q^{c-1}e^{(b,c-2,d-1)}v(\otimes 1)) + q^{m}(q^{c-1}e^{(b,c-2,d-1)}v + q^{-3b+2c}e^{(b,c-2,d-1)}v(\otimes 0) + q^{-3b+2c+m+2}e^{(b,c,d-1)}v(\otimes 3)
\end{aligned}
$$

and hence we have from Lemma 3.4 that

$$
||e^{(b,c,d)}v_{\ell}||^2 = q^{-c+3d-1}(q^{\ell} + q^{-c+3d-1})(e^{(b-1,c-2,d-1)}v, e^{(b,c,d)}v)
$$

We need to show that all (3.11)-(3.14) belong to $q^{\min(0,6b-2c,-2c+6d-2\ell)}A$. (G5) $\ell$ implies (3.11) $\in q^{-2c+6d-2\ell}A$, and (G2) $\ell$ implies (3.14) $\in A$ since we are assuming $2 \leq 2b \leq c$. It is easily proved from (G6) $\ell$ that (3.13) $\in A$. Finally let us show that (3.14) $\in q^{\min(0,6b-2c,-2c+6d-2\ell)}A$. Since $2 \leq c \leq 2d$, it follows from (G4) $\ell$ that

$$
q^{-2c+6d-2}(e^{(b,c,d)}v_{\ell})^2 \in q^{-2c}q^{\min(0,6b-2c,-2c+6d-2\ell+2)}A \subseteq q^{\min(0,6b-2c,-2c+6d-2\ell)}A.
$$

It is proved by the same argument that all the other terms of (3.14) also belong to $q^{\min(0,6b-2c,-2c+6d-2\ell)}A$, and hence (G4) $\ell$ holds.
The proof of (G5)$_{\ell}$ is similar. Writing $v = v_{\ell-1}$, we have from Lemmas 3.3, 3.3, and 8.10 that

$$(e^{b-1,c-2,d-1})v_{\ell+1}e_2e^{b,c,d}v_{\ell+1}$$

$$= \left( e^{b-1,c-2,d-1}(a_{q^a} \otimes (v_{1})_{q_{a-1}}), e_2e^{b,c,d}((v)_{q-1} \otimes (v_{1})_{q^{a-1}}) \right)_1$$

$$= q^{-1}\left( e^{b-1,c-2,d-1}(v \otimes (v_{1})_{q-1}), e_2e^{b,c,d}(v \otimes (v_{1})_{q}) \right)_2$$

$$= q^{-2c+6d-3}(e^{b-1,c-2,d-1})v_{\ell+1}e_2e^{b,c,d}(v)_{q-1} + q^{-c+3d+\ell-2}(e^{b-1,c-2,d-1})v_{\ell+1}e_2e^{b,c,d}(v)_{q-1}$$

$$+ (e^{b-2,c-4,d-2})v_{\ell+1}e_2e^{b,c,d}(v)_{q-1} + q^{6b-6}(e^{b-1,c-4,d-2})v_{\ell+1}e_2e^{b,c,d}(v)_{q-1}$$

$$+ q^{3+3c-3}(2)_{q}v^{2}(e^{b-1,c-4,d-2})v_{\ell+1}e_2e^{b,c,d}(v)_{q-1} + q^{3c-3}(2)_{q}v^{2}(e^{b-1,c-4,d-2})v_{\ell+1}e_2e^{b,c,d}(v)_{q-1}.$$ 

All the terms are easily seen to belong to $q^{c+3d-\ell}A$ by applying (G1)$_{\ell-1}$ (G3)$_{\ell-1}$ and (G5)$_{\ell-1}$, and hence (G5)$_{\ell}$ holds. The proof of (G6)$_{\ell}$ is similar, and we omit the details.

Now let us prove Proposition 3.3 by using (G1)$_{\ell}$ (G4)$_{\ell}$. Let $(a, b, c, d), (a', b', c', d') \in S_{\ell}$, and assume first that $a \neq a'$. We may assume $a > a'$, and then by (3.6) we have

$$(e^{a,b,c,d})v_{\ell}, e^{a',b',c',d'}(v) = q^{3a(a-2a+b'-d'+\ell)}(e^{a,b,c,d}v_{\ell}, f_0(a)(e^{a',b',c',d'}v_{\ell}) = 0,$$

since $f_0v_{\ell} = 0$ and $e_1^{(d')}e_0^{(d'')}v_{\ell} = 0$ if $d'' < d'$. Therefore (3.2) holds in this case. On the other hand, if $a = a'$, it follows from (3.3) and (3.6) that

$$(e^{a,b,c,d})v_{\ell}, e^{a',b',c',d'}(v) = q^{3a(a-2a+b'-d'+\ell)}(e^{a,b,c,d}v_{\ell}, f_0(a)(e^{a',b',c',d'})v_{\ell})$$

$$= q^{3a(a-2a+b'-d'+\ell)}b' - d' + \ell a q^1(e^{a,b,c,d}v_{\ell}, e^{b',c',d'}(v),$$

and hence (3.2) holds by (G2)$_{\ell}$ and (G3)$_{\ell}$.

Assume that $(a, b, c, d) \in S_{\ell}$. By Lemma 3.3 and (G4)$_{\ell}$, we have

$$||e^{2}e^{a,b,c,d}(v)_{\ell}||^2 = ||e^{a}(e^{a,b,c,d}(v)_{\ell})||^2 \leq (1 + qA)||e^{a,b,c,d}(v)_{\ell}||^2 \leq q^{\min\{6b-2c-2c+6d-2\ell\}}A,$$

which implies (3.3) with $i = 2$. Finally it remains to show (3.3) with $i = 1$, namely,

$$||e^{1}(e^{a,b,c,d}(v)_{\ell}||^2 \leq q^{6(a-2b+c-d-1)}A,$$

which needs a little bit more calculations. By applying (3.6) and (3.5), we have

$$||e^{1}(e^{a,b,c,d}(v)_{\ell}||^2 = q^{3(a-2b+c-d-1)}(e^{a,b,c,d}v_{\ell}, f_1e^{1}(e^{a,b,c,d}(v)_{\ell})$$

$$= q^{3(a-2b+c-d-1)}(a - 2b + c - d)q^2||e^{a,b,c,d}(v)_{\ell}||^2 + (e^{a,b,c,d}v_{\ell}, e_1f_1e^{1}(e^{a,b,c,d}(v)_{\ell})$$

$$\equiv q^{6(a-2b+c-d-1)}||f_1e^{a,b,c,d}(v)_{\ell}||^2 (mod q^{6(a-2b+c-d-1)}A).$$

Hence it suffices to show that $||f_1e^{a,b,c,d}(v)_{\ell}||^2 \leq A$. Set $p = b - d + \ell$. Note that $(a, b, c, d) \in S_{\ell}$ implies $p > 0$. Since $f_0(k)f_1e^{1}(e^{a,b,c,d}(v)_{\ell} = 0 (k > 1)$ by the Serre relations,
we have
\[
\|f_1e^{(a,b,c,d)}v_\ell\|^2 = q^{3a(p-a-1)}(f_1e^{(b,c,d)}v_\ell, f_0e^{(a)}f_1e^{(b,c,d)}v_\ell) \\
= q^{3a(p-a-1)}[|p-1|q^a_1||f_1e^{(b,c,d)}v_\ell||^2 + |p-1|q^a_1(f_1e^{(b,c,d)}v_\ell, e_0f_1e^{(b,c,d)}v_\ell)] \\
\leq ||f_1e^{(b,c,d)}v_\ell||^2 A + q^{6(p-a)}||f_0f_1e^{(b,c,d)}v_\ell||^2 A.
\]
(3.15)

It follows that
\[
||f_1e^{(b,c,d)}v_\ell||^2 = q^{6b-3c+3d-3}\langle e^{(b,c,d)}v_\ell, e_1f_1e^{(b,c,d)}v_\ell \rangle \\
= q^{6b-3c+3d-3}(2b-c+d)q^a_1||e^{(b,c,d)}v_\ell||^2 + (e^{(b,c,d)}v_\ell, f_1e_1e^{(b,c,d)}v_\ell) \\
= q^{6b-3c+3d-3}2b-c+d)q^a_1||e^{(b,c,d)}v_\ell||^2 + q^{12b-6c+6d}||e^{(b+1,c,d)}v_\ell||^2 \in A.
\]

Moreover, since
\[
f_0f_1e^{(b,c,d)}v_\ell = f_0(e^{(b)}e^{(c)}e_1^{(d-1)}e_0^{(d)}v_\ell + \alpha e^{(b-1,c,d)}v_\ell) = [\ell - d + 1]q^a_1e^{(b,c,d-1)}v_\ell
\]
where \( \alpha \in \mathbb{C}(q) \) is a certain element, we have
\[
q^{6(p-a)}||f_0f_1e^{(b,c,d)}v_\ell||^2 = q^{6(p-a)}[\ell - d + 1]q^a_1||e^{(b,c,d-1)}v_\ell||^2 \in A.
\]

Now (3.15) implies \( ||f_1e^{(a,b,c,d)}v_\ell||^2 \in A \), as required. The proof of Proposition 3.3 is complete.

3.5. **Proof in type \( D_{4}^{(3)} \) case.** Throughout this subsection we assume that \( g \) is of type \( D_{4}^{(3)} \), and we will show Proposition 3.3 in this type. A large part of the proof is parallel to the case of \( G_{2}^{(1)} \). Note that
\[
g_0 = q_1 = q, \quad q_2 = q^3.
\]

Let us consider the following collection of statements:

(D1) \( \ell e^{(b,c,d)}v_\ell \neq 0 \) if and only if \( 2b \leq 3c \leq 2d \leq 6\ell \).

(D2) \( ||e^{(b,c,d)}v_\ell||^2 \in 1 + qA \) if \( 2b \leq 3c \leq 2d \leq 6\ell \).

(D3) For \( (b, c, d), (b', c', d') \in \mathbb{Z}_{\geq 0}^3 \), \( (e^{(b,c,d)}v_\ell, e^{(b',c',d')}v_\ell) = 0 \) if \( (b, c, d) \neq (b', c', d') \).

(D4) For \( (b, c, d) \in \mathbb{Z}_{\geq 0}^3 \), \( ||e_2e^{(b,c,d)}v_\ell||^2 \leq \begin{cases} q^{\min\{0,6b-6c\}}A & (b < 3), \\
q^{\min\{0,6b-6c-6d+6\ell\}}A & (b \geq 3). \end{cases} \)

(D5) For \( (b, c, d) \in \mathbb{Z}_{\geq 0}^3 \), \( (e^{(b-3,c-2,2d-3)}v_\ell, e_2e^{(b,c,d)}v_\ell) \in q^{-3c+3d-3\ell}A \).

(D6) For \( (b, c, d) \in \mathbb{Z}_{\geq 0}^3 \), \( (e^{(b,c,d)}v_\ell, e_2e^{(b-1,c,1,d)}v_\ell) \in q^{3b-3c+3}A \).

First let us show (D1)–(D6) (unlike \( G_{2}^{(1)} \) case, we do not give the explicit module structure here). Write \( w = v_1 \). (D2) follows from Lemma 4.4 except for the case
\((b, c, d) = (1, 1, 3)\), in which \((D2)_1\) is proved by the following calculation:

\[
||e^{(1,1,3)} w||^2 = q^{-1}(e^{(0,1,3)} w, f_1 e^{(1,1,3)} w) = q^{-1}(e^{(0,1,3)} w, e_1 e_2 e_1^{(2)} e_0^{(3)} w) = (f_1 e^{(0,1,3)} w, e_2 e_1^{(2)} e_0^{(3)} w) = ||e_2 e_1^{(2)} e_0^{(3)} w||^2 = 1 + qA.
\]

Then to show \((D1)_1\), it suffices to check that \(e^{(b,c,d)} w = 0\) unless \(2b \leq 3c \leq 2d \leq 6\). By the Serre relations we have

\[
e^{(2,1,2)} w = e_1^{(2)} e_2 e_1^{(2)} e_0^{(2)} w = \sum_{k=0,1,3,4} (-1)^{k+1} e_1^{(k)} e_2 e_1^{(4-k)} e_0^{(2)} w = e_1^{(3)} e_2 e_1 e_0^{(2)} w
\]

and in all the other cases \(e^{(b,c,d)} w = 0\) is proved from Proposition 2.2 (iii) and \(e_1^{(k)} e_0^{(m)} w = 0\) for \(k > m\). Hence \((D1)_1\) holds. In order to show \((D3)_1\), it is enough to check by the weight consideration that

\[
(e^{(0,0,1)} w, e^{(1,1,2)} w) = (e^{(0,0,1)} w, e^{(2,2,3)} w) = (e^{(1,1,2)} w, e^{(2,2,3)} w) = (e^{(0,0,2)} w, e^{(1,1,3)} w) = (e^{(0,0,2)} w, e^{(1,2,3)} w) = 0,
\]

and these are all proved by applying \(\mathbb{A}\). The statements \((D4)_1\)–\((D6)_1\) are all proved from the following claims:

(a) \(e_2 e^{(b,c,d)} w = 0\) unless \(b = 0\) or \((b, c, d) \in \{(1, 1, 3), (3, 2, 3)\}\),
(b) \(e_2 e^{(1,1,3)} w = e^{(1,2,3)} w\),
(c) \(e_2 e^{(3,2,3)} w = w\).

The claim (a) follows from Proposition 2.2 (iii), (b) from the Serre relations, and (c) from the construction of \(W^1 = W(\varpi_2)\) (see [Kas02, Section 5]). Now the proofs of \((D1)_1\)–\((D6)_1\) are complete.

Let \(\ell > 1\), and assume that \((D1)_{\ell-1}\)–\((D6)_{\ell-1}\) hold. The following lemma is proved by straightforward calculations.

**Lemma 3.7.** Let \(m \in \mathbb{Z}\), and write \(v = v_{\ell-1}\), \(w = v_1\). Then the following equalities hold, where we simply write \(e^{(b,c,d)} w\) for \((e^{(b,c,d)} w)^{q^m}\) in the right-hand sides.

\[
(i) \quad e^{(b,c,d)} (v \otimes (w)^{q^m}) = q^{-3c+3d} e^{(b,c,d)} v \otimes w + q^{-b+2d+m} e^{(b,c,d-1)} v \otimes e^{(0,0,1)} w
\]

\[
+ q^{-2b+3c+d} e^{(b,c,d-2)} v \otimes e^{(0,0,2)} w + q^{b+d+2m} e^{(b,c-1,d-2)} v \otimes e^{(0,1,2)} w
\]

\[
+ q^{d+2m} e^{(b-1,c,d-2)} v \otimes e^{(1,1,2)} w + q^{3b+6c+3m} e^{(b,c,d-3)} v \otimes e^{(0,0,3)} w
\]

\[
+ q^{3c+3m-3} e^{(b-1,c-1,d-3)} v \otimes e^{(0,1,3)} w + q^{-b+3c+3m-2} e^{(b-1,c-1,d-3)} v \otimes e^{(1,1,3)} w
\]

\[
+ q^{3b+3m} e^{(b-2,c,d-3)} v \otimes e^{(0,2,3)} w + q^{2b+3m} e^{(b-1,c-2,d-3)} v \otimes e^{(1,2,3)} w
\]

\[
+ q^{b+3m-2} e^{(b-2,c-2,d-3)} v \otimes e^{(2,2,3)} w + q^{3m} e^{(b-3,c-2,d-3)} v \otimes e^{(3,2,3)} w.
\]
(ii) \( e_2 e^{(b,c,d)} (v \otimes (w)_q^m) = (q^{-3c+3d-3} e_2 e^{(b,c,d)} v + q^3 m e^{(b-3,c-2,d-3)} v) \otimes w + q^{-b+2d+m-2} e_2 e^{(b,c,d-1)} v \otimes e^{(0,0,1)} w + q^{-2b+3c+d+2m+1} e_2 e^{(b,c,d-2)} v \otimes e^{(0,0,2)} w \)

\(+ (q^{-2b+3c+d+2m-2} e^{(b,c,d-2)} v + q^{b+d+2m-5} e_2 e^{(b-1,c-1,d-2)} v) \otimes e^{(0,1,2)} w + q^{d+2m-2} e_2 e^{(b-1,c-1,d-2)} v \otimes e^{(1,1,2)} w + q^{-3b+6c+3m+6} e_2 e^{(b,c,d-3)} v \otimes e^{(0,0,3)} w \)

\(+ (q^{-3b+6c+3m} e^{(b,c,d-3)} v + q^{3c+3m-3} e_2 e^{(b,c-1,d-3)} v) \otimes e^{(0,1,3)} w + q^{b+3c+3m+1} e_2 e^{(b-1,c-1,d-3)} v \otimes e^{(1,1,3)} w \)

\(+ (q^{3c+3m-3} e^{(b,c-1,d-3)} v + q^{3b+3m-6} e_2 e^{(b,c-2,d-3)} v) \otimes e^{(0,2,3)} w + q^{b+3c+3m-2} e_2 e^{(b-1,c-1,d-3)} v + q^{2b+3m-5} e_2 e^{(b-1,c-2,d-3)} v) \otimes e^{(1,2,3)} w \)

\(+ q^{b+3m-2} e_2 e^{(b-2,c-2,d-3)} v \otimes e^{(2,2,3)} w + q^{3m+3} e_2 e^{(b-3,c-2,d-3)} v \otimes e^{(3,2,3)} w. \)

By the same arguments, (3.3) and Lemma 3.6 are also proved in type \( D_4^{(3)} \). Then (D1) follows from Lemma 3.7 (i), (3.3), and (D1)\( \ell \rightarrow 1 \). (D2) and (D3)\( \ell \rightarrow 1 \) are proved by calculating the values of the prepolazation by using Lemmas 3.4 and 3.7 (i), and applying (D2)\( \ell \rightarrow 1 \) and (D3)\( \ell \rightarrow 1 \) respectively.

Let us prove (D4)\( \ell \). Let \((b,c,d) \in \mathbb{Z}^{3}_0 \). We may assume \(2 \leq 2b \leq 3c \leq 2d \leq 6\ell \). Set \( B = 1 + qA \). By Lemmas 3.4 and 3.7 (ii), we have

\[ || e_2 e^{(b,c,d)} v ||^2 \leq X + Y + 2Z + W, \]

where

\[ X = q^{-3c+3d-3} (q^{-3\ell} + q^{3\ell}) (e^{(b-3,c-2,d-3)} v, e_2 e^{(b,c,d)} v) B, \]

\[ Y = || e^{(b-3,c-2,d-3)} v ||^2 B + q^{-4b+6c+2d-4} || e^{(b,c-2,d)} v ||^2 B + q^{-6b+12c} || e^{(b,c,d-3)} v ||^2 B, \]

\[ Z = q^{-b+3c+2d-7} (e^{(b,c,d-2)} v, e_2 e^{(b,c-1,d-2)} v) B + q^{-3b+9c-3} (e^{(b,c,d-3)} v, e_2 e^{(b,c-1,d-3)} v) B + q^{3b+3c-12} (e^{(b,c-1,d-3)} v, e_2 e^{(b,c-2,d-3)} v) B + q^{b+3c-7} (e^{(b-1,c-1,d-3)} v, e_2 e^{(b-1,c-2,d-3)} v) B, \]

\[ W = q^{-6c+6d-6} || e_2 e^{(b,c,d)} v ||^2 B + q^{-2b+4d-4} || e^{(b,c,d-1)} v ||^2 B \]

\[ + q^{-4b+6c+2d+2} || e^{(b,c,d-2)} v ||^2 B + q^{2b+2d-10} || e_2 e^{(b,c-1,d-2)} v ||^2 B \]

\[ + q^{2d-4} || e_2 e^{(b-1,c-1,d-2)} v ||^2 B + q^{-6b+12c+12} || e_2 e^{(b,c,d-3)} v ||^2 B \]

\[ + q^{6c-6} || e_2 e^{(b,c-1,d-3)} v ||^2 B + q^{2b+6c+2} || e^{(b-1,c-1,d-3)} v ||^2 B \]

\[ + q^{6b-12} || e_2 e^{(b,c-2,d-3)} v ||^2 B + q^{4b-10} || e_2 e^{(b-1,c-2,d-3)} v ||^2 B \]

\[ + q^{2b-4} || e_2 e^{(b-2,c-2,d-3)} v ||^2 B + q^{6} || e_2 e^{(b-3,c-2,d-3)} v ||^2 B. \]

Note that \( X = 0 \) if \( b < 3 \), and \( X \subseteq q^{-6c+6d-6}\ell A \) if \( b \geq 3 \) by (D5)\( \ell \rightarrow 1 \). \( Y \subseteq A \) holds by (D1)\( \ell \rightarrow 1 \) and (D2)\( \ell \rightarrow 1 \), and \( Z \subseteq A \) holds by (D6)\( \ell \rightarrow 1 \). Finally \( W \subseteq q^{\min(0,6b-6c)} \) for \( b < 3 \) and \( W \subseteq q^{\min(0,6b-6c,6(d+6d-6))} \) for \( b \geq 3 \) are proved from (D1)\( \ell \rightarrow 1 \) and (D6)\( \ell \rightarrow 1 \). Hence (D4)\( \ell \) follows. The proofs of (D5)\( \ell \) and (D6)\( \ell \) are similar, and we omit the details.
Now Proposition 3.3 is deduced from (D1)_ℓ—(D4)_ℓ by exactly the same arguments as in the case of type $G_2^{(1)}$.

REFERENCES

[CM07] V. Chari and A. Moura. Kirillov-Reshetikhin modules associated to $G_2$. In *Lie algebras, vertex operator algebras and their applications*, volume 442 of *Contemp. Math.*, pages 41–59. Amer. Math. Soc., Providence, RI, 2007.

[CP95] V. Chari and A. Pressley. Quantum affine algebras and their representations. In *Representations of groups (Banff, AB, 1994)*, volume 16 of *CMS Conf. Proc.*, pages 59–78. Amer. Math. Soc., Providence, RI, 1995.

[CP98] V. Chari and A. Pressley. Twisted quantum affine algebras. *Commun. Math. Phys.*, 196(2):461–476, 1998.

[DFK08] P. Di Francesco and R. Kedem. Proof of the combinatorial Kirillov-Reshetikhin conjecture. *Int. Math. Res. Not. IMRN*, (7):Art. ID rnm006, 57, 2008.

[Her06] D. Hernandez. The Kirillov-Reshetikhin conjecture and solutions of $T$-systems. *J. Reine Angew. Math.*, 596:63–87, 2006.

[Her10] D. Hernandez. Kirillov-Reshetikhin conjecture: the general case. *Int. Math. Res. Not. IMRN*, (1):149–193, 2010.

[HK02] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[HKO+99] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Y. Yamada. Remarks on fermionic formula. In *Recent developments in quantum affine algebras and related topics (Raleigh, NC, 1998)*, volume 248 of *Contemp. Math.*, pages 243–291. Amer. Math. Soc., Providence, RI, 1999.

[HKO+02] G. Hatayama, A. Kuniba, M. Okado, T. Takagi, and Z. Tsuboi. Paths, crystals and fermionic formulæ. In *MathPhys odyssey, 2001*, volume 23 of *Prog. Math. Phys.*, pages 205–272. Birkhäuser Boston, Boston, MA, 2002.

[Kac90] V.G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[Kas02] M. Kashiwara. On crystal bases of the $Q$-analogue of universal enveloping algebras. *Duke Math. J.*, 112(1):117–175, 2002.

[KKM+92] S.-J. Kang, M. Kashiwara, K. C. Misra, T. Miwa, T. Nakashima, and A. Nakayashiki. Perfect crystals of quantum affine Lie algebras. *Duke Math. J.*, 68(3):499–607, 1992.

[KMOY07] M. Kashiwara, K.C. Misra, M. Okado, and D. Yamada. Perfect crystals for $U_q(D_4^{(3)})$. *J. Algebra*, 317(1):392–423, 2007.

[Nak03] H. Nakajima. $t$-analog of $q$-characters of Kirillov-Reshetikhin modules of quantum affine algebras. *Represent. Theory*, 7:259–274 (electronic), 2003.

[OS08] M. Okado and A. Schilling. Existence of Kirillov-Reshetikhin crystals for nonexceptional types. *Represent. Theory*, 12:186–207, 2008.

[Yam98] S. Yamane. Perfect crystals of $U_q(G_2^{(1)})$. *J. Algebra*, 210(2):440–486, 1998.

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