Caffarelli–Kohn–Nirenberg inequalities for curl-free vector fields and second order derivatives

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Abstract
The present work has as a first goal to extend the previous results in Cazacu et al. (J Funct Anal 283(10):109659, 2022) to weighted uncertainty principles with nontrivial radially symmetric weights applied to curl-free vector fields. Part of these new inequalities generalize the family of Caffarelli-Kohn-Nirenberg (CKN) inequalities studied by Catrina and Costa in Catrina and Costa (J Differ Equ 246(1):164–182, 2009) from scalar fields to curl-free vector fields. We will apply a new representation of curl-free vector fields developed by Hamamoto in Hamamoto and Takahashi (J Funct Anal 280(1):108790, 2021). The newly obtained results are also sharp and minimizers are completely described. Secondly, we prove new sharp second order interpolation functional inequalities for scalar fields with radial weights generalizing the previous results in Cazacu et al. (J Funct Anal 283(10):109659, 2022). We apply new factorization methods being inspired by our recent work (Cazacu et al. in J Differ Equ 302, 533–549, 2021). The main novelty in this case is that we are able to find a new independent family of minimizers based on the solutions of Kummer’s differential equations. We point out that the two types of weighted inequalities under consideration (first order inequalities for curl-free vector fields vs. second order inequalities for scalar fields) represent independent families of inequalities unless the weights are trivial.

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1 Introduction

The aim of this paper is twofold:

(a) We study the sharp constants in the weighted inequalities of Caffarelli–Kohn–Nirenberg (CKN) type having the form

\[ \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} \frac{|\nabla U|^2}{|x|^{2b}} \, dx \geq C_1(N, a, b) \left( \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{a+b+1}} \, dx \right)^2, \tag{1.1} \]

for any curl-free vector field (defined below) \( U \in \left( C_0^\infty(\mathbb{R}^N \setminus \{0\}) \right)^N \) and \( C_1(N, a, b) \) is the sharp constant in (1.1); the parameter \((a, b)\) may be any point in \( \mathbb{R}^2; N \geq 2 \) and \( C_0^\infty(\mathbb{R}^N \setminus \{0\}) \) denotes the space of smooth functions compactly supported in \( \mathbb{R}^N \setminus \{0\} \).

(b) We analyze the sharp second order CKN inequalities for scalar fields of the form

\[ \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} \frac{|x|^{2a+2}}{|x|} \cdot \frac{|\nabla u|^2}{|x|} \, dx \geq C_2(N, a) \left( \int_{\mathbb{R}^N} |
abla u|^2 \, dx \right)^2, \tag{1.2} \]

for any \( u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}); a \in \mathbb{R} \) is given and \( C_2(N, a) \) denotes the best constant in (1.2).

1.1 State of the art

The CKN inequalities were first introduced in 1984 by Caffarelli, Kohn and Nirenberg in the pioneering work [9] to generalize many well-known and important inequalities in analysis such as Gagliardo–Nirenberg inequalities, Hardy–Sobolev inequalities, Nash’s inequalities, Sobolev inequalities, etc. Since then, due to their important roles and applications in many areas of pure and applied mathematics, especially in analysis and PDE, the CKN type inequalities have been studied extensively in the literature by many authors; e.g., see [6, 7, 10, 12, 16, 20–23, 25, 26, 35] and the references therein. Afterwards, many researchers have become interested in studying finer properties such as sharp constants for CKN inequalities and their extremizers. An important subfamily consists of the extensively studied sharp \( L^2 \)-CKN inequalities given by

\[ \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} \, dx \geq C_2(N, a, b) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)^2, u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}). \tag{1.3} \]

Notice that inequality (1.3) represents the scalar version of the family (1.1). The best constant \( C_2(N, a, b) > 0 \) is known and the minimizers are fully described. We recall that these aspects of (1.3) were first studied by Costa in [18] for a particular range of parameters by using the expanding-the-square method, and then by Catrina and Costa in [11] for the full range of parameters using spherical harmonics decomposition and the Kelvin transform. (An argument without needing spherical harmonics is found in [25].) The obtained results depend on some
parameter regions defined as in

\[
\begin{align*}
A_1 &:= \{ (a, b) \mid b + 1 - a > 0, \ b \leq (N-2)/2 \} \\
A_2 &:= \{ (a, b) \mid b + 1 - a < 0, \ b \geq (N-2)/2 \} \\
A &:= A_1 \cup A_2 \\
B_1 &:= \{ (a, b) \mid b + 1 - a < 0, \ b \leq (N-2)/2 \} \\
B_2 &:= \{ (a, b) \mid b + 1 - a > 0, \ b \geq (N-2)/2 \} \\
B &:= B_1 \cup B_2
\end{align*}
\] (1.4)

More precisely, it was showed in [11] that in the region \(A\), the best constant is

\[
C(N, a, b) = \frac{N - (a + b + 1)}{2}
\]

and it is achieved by the nontrivial functions \(u(x) = \gamma \exp\left(\frac{t|x|^{b+1-a}}{b + 1 - a}\right)\), with \(t < 0\) in \(A_1\) and \(t > 0\) in \(A_2\), and \(\gamma\) a nonzero constant. In the region \(B\), the best constant is

\[
C(N, a, b) = \frac{|N - (3b - a + 3)|}{2}
\]

and it is achieved by the functions \(u(x) = \gamma |x|^{2(b+1)-N} \exp\left(\frac{t|x|^{b+1-a}}{b + 1 - a}\right)\), with \(t > 0\) in \(B_1\) and \(t < 0\) in \(B_2\). In addition, the only values of the parameters where the best constant is not achieved are those on the line \(a = b + 1\) where the best constant is \(C(N, b + 1, b) = \frac{|N - 2(b + 1)|}{2}\). In this latter case the CKN inequality degenerates into a weighted Hardy-Leray type inequality.

Very recently, the authors of the present paper provided in [15] a very simple and direct proof (bypassing spherical harmonics and the Kelvin transform) of the refined CKN inequality

\[
\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} \frac{|x \cdot \nabla u|^2}{|x|^{2b+2}} \, dx \geq \tilde{C}^2(N, a, b) \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} \, dx \right)^2, \quad u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}),
\] (1.5)

where \(\tilde{C}^2(N, a, b)\) denotes the sharp constant in (1.5). It was shown in [15] that \(\tilde{C}^2(N, a, b) = C^2(N, a, b)\) for the whole range of parameters \((a, b) \in \mathbb{R}^2\). However, inequality (1.5) requires only the radial derivative \(\partial_r := \frac{1}{|x|} \cdot \nabla\) on the left hand side instead of the full gradient. Of course, this makes (1.5) finer than (1.3) since \(|\partial_r u| \leq |\nabla u|\). We also characterized in [15] all the optimizers. The interesting fact which occurs is that the minimizers of (1.5) are not necessarily radially symmetric (as happens for (1.3)) and they differ from the minimizers of (1.3) by a multiplicative function depending only on the spherical component. It is important to emphasize that the obtained minimizers do not belong to the space \(C_0^\infty(\mathbb{R}^N \setminus \{0\})\) but to the functional spaces defined as the closure of \(C_0^\infty(\mathbb{R}^N \setminus \{0\})\) in the corresponding energy norm. This was an aspect which is not considered in [11] but we explained it in detail in [15].

Going back to the subclass (1.3) of CKN inequalities, we notice that it recovers the mathematical formulation for three famous inequalities of quantum mechanics, namely the Heisenberg Uncertainty Principle (HUP), the Hydrogen Uncertainty Principle (HyUP) and the Hardy Inequality (HI). The HUP is obtained when \(a = -1\) and \(b = 0\): for any \(N \geq 1\), there holds

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} |x|^2 |u|^2 \, dx \geq \frac{N^2}{4} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^2, \quad \forall u \in C_0^\infty(\mathbb{R}^N).
\] (1.6)

It can also be extended to functions \(u\) in the Schwartz space \(S(\mathbb{R}^N)\) or in appropriate Sobolev spaces. It can also be verified that the constant \(\frac{N^2}{4}\) in the (1.6) is sharp and is attainable in \(S(\mathbb{R}^N)\) by minimizers of the form \(u(x) = \gamma e^{-\beta|x|^2}, \ \gamma \in \mathbb{R}, \ \beta > 0\) (see, e.g. [27]).
\(a = b = 0\) we recover the HyUP: for any \(N \geq 2\), there holds
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \int_{\mathbb{R}^N} |u|^2 \, dx \geq \frac{(N - 1)^2}{4} \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x|} \, dx \right)^2, \quad u \in C_0^\infty (\mathbb{R}^N). \tag{1.7}
\]

The constant \(\frac{(N - 1)^2}{4}\) in (1.7) is also optimal and is attained by minimizers of the form
\(u(x) = \gamma e^{-\beta|x|}, \gamma \in \mathbb{R}, \beta > 0\) (see, e.g. [28]). Notice that in this case these minimizers are not in \(S(\mathbb{R}^N)\) but are in the Sobolev space \(W^{1,2}(\mathbb{R}^N)\).

In the case \(a = 1\) and \(b = 0\) we have a degenerate case of (1.3) which emerges as the famous HI: for any \(N \geq 3\), there holds
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx, \quad \forall u \in C_0^\infty (\mathbb{R}^N). \tag{1.8}
\]

The constant in (1.8) is sharp but not achieved. There has been considerable interest in the HI, its extension and their applications over the last few decades. One of the very first application of the HI (1.8) appeared for \(N = 3\) in Leray’s paper [36] when he studied the Navier–Stokes equations. Since the HI is not a priority of this paper we avoid to cite further references for (1.8). Inequalities (1.6)–(1.8) are independent but all of them can be deduced by applying the divergence theorem and Holder inequality. On the other hand, it is easy to check that the HI (1.8) implies (1.6) with a worse constant, i.e. \(\frac{(N - 2)^2}{4}\) instead of \(\frac{N^2}{4}\).

### 1.2 HUP, HyUP and HI for vector fields

Motivated by questions and applications in hydrodynamics (see, e.g. [2, 37, 42]) and harmonic analysis (cf. [3, 40]) it is important and interesting to compute the best constants of functional inequalities such as HI, HUP, CKN-type inequalities, etc., for vector fields. Special classes of vector fields which appear frequently in applications are divergence-free vector fields (a restriction which enhances for instance the Stokes/Navier–Stokes equations for incompressible fluids, cf. [17, 39]) or curl-free vector fields (see e.g. Maxwell equations of electromagnetism [1, 5]). Here a divergence-free vector field \(U = (U_1, \ldots, U_N) \in (C_0^\infty (\mathbb{R}^N))^N\) satisfies \(\text{div} U = 0\). By a curl-free vector field we understand a vector field of the form \(U = \nabla u\), where \(u : \mathbb{R}^N \mapsto \mathbb{C}\) is a scalar potential field. As a consequence of Cauchy-Schwarz inequality, HUP, HyUP, HI, \(L^2\)-CKN inequalities for scalar fields transfer easily with the same best constant to non-restricted vector fields. A key question which arises is whether the new sharp constant remains the same as in the scalar case or improves in the case of vector fields with restrictions. Let us next recall the main results which have been done so far in this direction. If there is no restriction on the vector field, then we get from (1.8) that
\[
\int_{\mathbb{R}^N} |\nabla U|^2 \, dx \geq \frac{(N - 2)^2}{4} \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^2} \, dx \tag{1.9}
\]
and the constant \(\frac{(N - 2)^2}{4}\) is optimal. In [19], Costin and Maz’ya showed that for divergence-free vector fields with some additional restrictions (axisymmetric assumption) we have
\[
\int_{\mathbb{R}^N} |\nabla U|^2 \, dx \geq \frac{(N - 2)^2}{4} \left( 1 + \frac{8}{N^2 + 4N - 4} \right) \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^2} \, dx
\]
and \(\frac{(N-2)^2}{4} \left(1 + \frac{8}{N^2 + 4N - 4}\right)\) is sharp. Obviously, this new sharp constant improves the original optimal constant \(\frac{(N-2)^2}{4}\) of the Hardy inequality (1.9) without restrictions. Recently, Hamamoto proved in [30] that the additional assumption in [19] that \(U\) is axisymmetric can be removed and we still achieve the same best constant \(\frac{(N-2)^2}{4} \left(1 + \frac{8}{N^2 + 4N - 4}\right)\). The HI for curl-free vector fields is equivalent to the so-called Hardy-Rellich inequality

\[
\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \geq \lambda^2(N) \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^2} \, dx,
\]

where \(u\) is a potential scalar fields for \(U\), i.e. \(U = \nabla u\). The sharp constant \(\lambda^2(N)\) was determined progressively depending on the dimension \(N\). First it was shown in [41] that \(\lambda^2(N) = \frac{N^2}{N-1}\) for any \(N \geq 5\) by using spherical harmonics decomposition. Later, for \(N \in \{3, 4\}\) it was shown that \(\lambda^2(3) = \frac{25}{38}\) and \(\lambda^2(4) = 3\) by means of the Fourier transform (in [4]), Bessel pairs (in [29]) or an alternative proof using spherical harmonics decomposition in [13]. Hamamoto and Takahashi also investigated and determined in [32] the sharp constants of weighted Hardy type inequalities for curl-free vector fields.

In contrast with HI, to our knowledge HUP, HyUP or CKN inequalities for divergence-free and curl-free vector fields with best constants have been less investigated so far.

The problem of finding the best constant of HUP when one replaces \(u\) in (1.6) by a divergence-free vector field \(U\) was posed by Maz’ya in [38, Section 3.9]. Indeed, Maz’ya raised the following question in [38, Section 3.9]: determine the best constant \(\mu^*(N)\) in the following inequality

\[
\int_{\mathbb{R}^N} |\nabla U|^2 \, dx \int_{\mathbb{R}^N} |x|^2 |U|^2 \, dx \geq \mu^*(N) \left( \int_{\mathbb{R}^N} |U|^2 \, dx \right)^2, \quad \forall U \in \left(C_0^\infty(\mathbb{R}^N)\right)^N, \quad \text{div} U = 0.
\]  

(1.10)

In the recent paper [14], among other obtained results we answered in particular to the Maz’ya question for \(N = 2\) and we showed that \(\mu^*(2) = 4\). Motivated by the fact that in \(\mathbb{R}^2\), the divergence-free vector fields are isometrically isomorphic to the curl-free vector fields in the two-dimensional case, a divergence-free vector can be written in the form \(U = (-u_{x_2}, u_{x_1})\) where \(u\) is a scalar field. If \(U \in \left(C_0^\infty(\mathbb{R}^2)\right)^2\) then also \(u \in C_0^\infty(\mathbb{R}^2)\). Then, after integration by parts we get

\[
\int_{\mathbb{R}^2} |\nabla U|^2 \, dx = \int_{\mathbb{R}^2} |\Delta u|^2 \, dx.
\]

Therefore (1.10) is equivalent to the following second order CKN-type inequality in \(\mathbb{R}^2\):

\[
\int_{\mathbb{R}^2} |\Delta u|^2 \, dx \int_{\mathbb{R}^2} |x|^2 |\nabla u|^2 \, dx \geq \mu^*(2) \left( \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \right)^2.
\]

(1.11)

Then, by using spherical harmonic decomposition, we showed in [14] that \(\mu^*(2) = 4\) is sharp in (1.11) and is achieved by the Gaussian profiles of the form \(u(x) = \alpha e^{-\beta |x|^2}, \beta > 0\). Therefore \(\mu^*(2) = 4\) is sharp in (1.10) and is attained by the vector fields of the form \(U(x) = (-\alpha e^{-\beta |x|^2} x_2, \alpha e^{-\beta |x|^2} x_1), \beta > 0, \alpha \in \mathbb{R}\).
Very recently, Hamamoto answered Maz’ya’s open question in the remaining case \( N \geq 3 \) in [31]. More precisely, Hamamoto applied the poloidal-toroidal decomposition to prove that

\[
\mu^*(N) = \frac{1}{4} \left( \sqrt{N^2 - 4 (N - 3)} + 2 \right)^2 \quad \text{when } N \geq 3.
\]

It is worthy to mention that we also proved in [14] the following sharp HUP for curl-free vector fields when \( N \geq 1 \): for \( U \in \left( C^\infty_0(\mathbb{R}^N) \right)^N \), \( \text{curl} U = 0 \), there holds

\[
\int_{\mathbb{R}^N} |\nabla U|^2 \, dx \int_{\mathbb{R}^N} |x|^2 |U|^2 \, dx \geq \left( \frac{N + 2}{2} \right)^2 \left( \int_{\mathbb{R}^N} |U|^2 \, dx \right)^2,
\]

by improving the best constant \( \frac{N^2}{4} \) which corresponds to scalar fields in (1.6). Indeed, since in this case we can write \( U = \nabla u \) for some scalar potential \( u : \mathbb{R}^N \rightarrow \mathbb{C} \), (1.12) is equivalent to the following second order CKN inequality:

\[
\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \int_{\mathbb{R}^N} |x|^2 |\nabla u|^2 \, dx \geq \left( \frac{N + 2}{4} \right)^2 \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2.
\]

Then, by using spherical harmonics decomposition, we proved in [14] that the constant \( \frac{(N + 2)^2}{4} \) is optimal in (1.13) and is attained for Gaussian profiles of the form \( u(x) = ae^{-\beta |x|^2}, \beta > 0 \). Therefore, the constant \( \left( \frac{N + 2}{2} \right)^2 \) is sharp in (1.12) and is achieved by the vector fields \( U(x) = ae^{-\beta |x|^2} x, \beta > 0 \).

Along the same line of thought, we proved in addition in [14] that for \( N \geq 5 \), then for \( U \in \left( C^\infty_0(\mathbb{R}^N) \right)^N \), \( \text{curl} U = 0 \):

\[
\int_{\mathbb{R}^N} |\nabla U|^2 \, dx \int_{\mathbb{R}^N} |U|^2 \, dx \geq \frac{(N + 1)^2}{4} \left( \int_{\mathbb{R}^N} |U|^2 |x| \, dx \right)^2.
\]

Equivalently, that is

\[
\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N + 1)^2}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 |x| \, dx \right)^2.
\]

for a scalar potential \( u \) corresponding to \( U \). The constant \( \frac{(N + 1)^2}{4} \) is optimal and it is attained for functions of the form \( u(x) = \alpha (1 + \beta |x|) e^{-\beta |x|}, \beta > 0 \). Hence the constant \( \frac{(N + 1)^2}{4} \) is also optimal in (1.14) and is attained by \( U(x) = \alpha e^{-\beta |x|^2} x, \beta > 0 \). Thus, this new constant is larger than the constant \( \frac{(N-1)^2}{4} \) in (1.7).

### 2 Main results

The first principal goal of this article is to study the weighted versions of the HUP and HyUP for curl-free vector fields which are described in (1.1). The second goal is to analyze the weighted second order inequalities in (1.2). We note that, since we are dealing with weights, it is not true in general that for curl-free vector fields \( U \) with a scalar potential \( u \) (i.e. \( U = \nabla u \)) there holds

\[
\int_{\mathbb{R}^N} \frac{|\nabla U|^2}{|x|^{2\alpha}} \, dx = \int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2\alpha}} \, dx,
\]

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unless \( a = 0 \). Therefore, the family of inequalities in (1.1)–(1.2) are independent in general when \( a \neq 0 \). That is, the spherical harmonics decomposition used in [14] is not applicable in this situation for the family (1.1) and becomes more difficult to be applied for the family (1.2).

Therefore, a new approach needs to be used to establish the weighted cases in (1.1). In this paper, we will apply a new representation for curl-free vector fields developed in [33] to establish a very general \( L^2 \)-CKN inequality for curl-free vector fields that contains the HUP and HyUP for curl-free vector fields as specific cases. To prove the second order inequalities in (1.2) we apply the expanding square method in a new fashion way being inspired by our previous work [15] where we did it for first order inequalities.

In order to state the main results we need to introduce some notations and definitions regarding the functional framework. Thus, let \( X_{a,b}(\mathbb{R}^N) \) be the set of vector fields \( U \in C^\infty(\mathbb{R}^N \setminus \{0\})^N \) such that

\[
\int_{\mathbb{R}^N} \frac{|\nabla U|^2}{|x|^{2a}} \, dx < \infty, \quad \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{2b}} \, dx < \infty, \quad \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{a+b+1}} \, dx < \infty,
\]

and

\[
\lim_{|x| \to 0, \infty} |x|^{-b+\frac{N}{2}} U(x) = 0.
\]

The first main result of this paper is the following \( L^2 \)-CKN inequality for curl-free vector fields:

**Theorem 2.1** Let \( N \geq 1 \) and \( a, b \) be real numbers such that \( \left( \frac{N}{2} - a \right)^2 \geq N + 1 \). We have for \( U \in X_{a,b}(\mathbb{R}^N) \) with \( \text{curl } U = 0 \) that

1. if \( a - b + 1 > 0 \), then

\[
\int_{\mathbb{R}^N} \frac{|\nabla U|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{2b}} \, dx \geq C^2(N, a, b) \left( \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{a+b+1}} \, dx \right)^2,
\]

where

\[
C(N, a, b) = \sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1 + \frac{a - b + 1}{2}}
\]

is sharp and is attained by the curl-free vector fields

\[
U(x) = \gamma |x|^{-\frac{N}{2} + a + \sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1}} e^{-\frac{a}{a-b+1} |x|^{a-b+1}} x, \quad \gamma \in \mathbb{R}, \quad \beta > 0;
\]

2. if \( a - b + 1 < 0 \), then

\[
\int_{\mathbb{R}^N} \frac{|\nabla U|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{2b}} \, dx \geq C^2(N, a, b) \left( \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{a+b+1}} \, dx \right)^2,
\]

where

\[
C(N, a, b) = \sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1 - \frac{a - b + 1}{2}}
\]
is sharp and is attained by the curl-free vector fields

\[ U(x) = \gamma |x|^{-\frac{N}{2} + a + \sqrt{(1 - \frac{N}{2} + a)^2 + N - 1}} e^{-\frac{b}{a + 1 - 1}|x|^{a+b+1}} x, \quad \gamma \in \mathbb{R}, \quad \beta < 0. \]

Here are some direct consequences of our Theorem 2.1. Let \( b = -a - 1 \), we get the following weighted HUP for curl-free vector fields:

**Corollary 2.1** Let \( N \geq 1 \) and \( a \) be such that \( a > -1 \) and \( \left( \frac{N}{2} - a \right)^2 \geq N + 1 \). Then we have for \( U \in X_{a,-a-1} (\mathbb{R}^N) \) with \( \text{curl } U = 0 \) that

\[
\int_{\mathbb{R}^N} \frac{|
abla U|^2}{|x|^{2a}} dx \int_{\mathbb{R}^N} |x|^{2a+2} |U|^2 dx \geq C^2 (N, a, -a - 1) \left( \int_{\mathbb{R}^N} |U|^2 dx \right)^2,
\]

where

\[
C(N, a, -a - 1) = \sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1 + 1 + a}
\]

is sharp and is attained by curl-free vector fields

\[ U(x) = \gamma |x|^{-\frac{N}{2} + a + \sqrt{(1 - \frac{N}{2} + a)^2 + N - 1}} e^{-\frac{b}{a + 1 - 1}|x|^{a+b+1}} x, \quad \gamma \in \mathbb{R}, \quad \beta > 0. \]

When \( b = -a \), we obtain the weighted HyUP for curl-free vector fields:

**Corollary 2.2** Let \( N \geq 1 \) and \( a \) be such that \( a > -\frac{1}{2} \) and \( \left( \frac{N}{2} - a \right)^2 \geq N + 1 \). Then we have for \( U \in X_{a,-a} (\mathbb{R}^N) \) with \( \text{curl } U = 0 \) that

\[
\int_{\mathbb{R}^N} \frac{|
abla U|^2}{|x|^{2a}} dx \int_{\mathbb{R}^N} |x|^{2a} |U|^2 dx \geq C^2 (N, a, -a) \left( \int_{\mathbb{R}^N} \frac{|U|^2}{|x|} dx \right)^2,
\]

where

\[
C(N, a, -a) = \sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1 + \frac{1}{2} + a}
\]

is sharp and is attained by curl-free vector fields

\[ U(x) = \gamma |x|^{-\frac{N}{2} + a + \sqrt{(1 - \frac{N}{2} + a)^2 + N - 1}} e^{-\frac{b}{a + 1 - 1}|x|^{2a+1}} x, \quad \gamma \in \mathbb{R}, \quad \beta > 0. \]

In particular, when \( a = 0 \) and \( N \geq 5 \), in view of Corollaries 2.1–2.2 we recover the HUP (1.12) and HyUP (1.14) for curl-free vector fields proven in [14].

Obviously, using \( U = \nabla u \), Theorem 2.1 yields

**Corollary 2.3** Let \( N \geq 1 \) and \( a, b \) be real numbers such that \( \left( \frac{N}{2} - a \right)^2 \geq N + 1 \). We have for \( u \) such that \( \nabla u \in X_{a,b} (\mathbb{R}^N) \) that

(1) if \( a - b + 1 > 0 \), then

\[
\int_{\mathbb{R}^N} \frac{|D^2 u|^2}{|x|^{2a}} dx \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{2b}} dx \geq C^2 (N, a, b) \left( \int_{\mathbb{R}^N} \frac{|
abla u|^2}{|x|^{a+b+1}} dx \right)^2,
\]
where

\[ C(N, a, b) = \sqrt{ \left(1 - \frac{N}{2} + a\right)^2 + N - 1 + \frac{a-b+1}{2}} \]

is sharp and is attained by the functions \( u \) such that

\[ \nabla u = \gamma \left| x \right|^{-\frac{N}{2} + a + \sqrt{\left(1 - \frac{N}{2} + a\right)^2 + N - 1}} e^{-\frac{\beta}{a-b+1} \left| x \right|^{(a-b+1)}} x, \quad \gamma \in \mathbb{R}, \quad \beta > 0; \]

\( \text{(2) if } a-b+1 < 0, \text{ then} \)

\[ \int_{\mathbb{R}^N} \frac{|D^2 u|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq C^2(N, a, b) \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2, \]

where

\[ C(N, a, b) = \sqrt{ \left(1 - \frac{N}{2} + a\right)^2 + N - 1 + \frac{a-b+1}{2}} \]

is sharp and is attained by the functions \( u \) such that

\[ \nabla u = \gamma \left| x \right|^{-\frac{N}{2} + a - \sqrt{\left(1 - \frac{N}{2} + a\right)^2 + N - 1}} e^{-\frac{\beta}{a-b+1} \left| x \right|^{(a-b+1)}} x, \quad \gamma \in \mathbb{R}, \quad \beta < 0. \]

In particular, when \( a = 0 \), we have that

**Corollary 2.4** Let \( N \geq 5 \). We have for \( u \) such that \( \nabla u \in X_{0,b} \left( \mathbb{R}^N \right) \) that

(1) if \( b < 1 \), then

\[ \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \left( \frac{N-b+1}{2} \right)^2 \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2, \]

where the constant \( \left( \frac{N-b+1}{2} \right)^2 \) is sharp and is attained by the functions \( u \) such that

\[ \nabla u = \gamma e^{-\frac{\beta}{N-b} \left| x \right|^{1-b}} x, \quad \gamma \in \mathbb{R}, \quad \beta > 0; \]

(2) if \( b > 1 \), then

\[ \int_{\mathbb{R}^N} |\Delta u|^2 \, dx \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \left( \frac{N+b-1}{2} \right)^2 \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2, \]

where the constant \( \left( \frac{N+b-1}{2} \right)^2 \) is sharp and is attained by the functions \( u \) such that

\[ \nabla u = \gamma \left| x \right|^{-N} e^{-\frac{\beta}{N-b} \left| x \right|^{1-b}} x, \quad \gamma \in \mathbb{R}, \quad \beta < 0. \]

As mentioned earlier, it is not true in general that for \( U = \nabla u \)

\[ \int_{\mathbb{R}^N} |\nabla U|^2 \, dx = \int_{\mathbb{R}^N} |\Delta u|^2 \, dx. \]

Therefore, Theorem 2.1 does not imply weighted versions of the second order HUP (1.13) or HyUP (1.15). That is, Theorem 2.1 does not imply that

\[ \int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} |x|^{2a+2} |\nabla u|^2 \, dx \geq C(N, a) \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2. \]
Motivated by this observation, the second goal of our paper is to study the weighted second order inequalities in (1.2). First, let \( Y_a (\mathbb{R}^N) \) be the set of scalar-valued functions \( u \in C^\infty (\mathbb{R}^N \setminus \{0\}) \) such that

\[
\int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx < \infty, \quad \int_{\mathbb{R}^N} \frac{1}{|x|^{2a+2}} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx < \infty, \quad \int_{\mathbb{R}^N} |\nabla u|^2 \, dx < \infty.
\] (2.1)

\[
\lim_{|x| \to 0, \infty} |x|^{N-1} |u(x)|^2 = 0,
\]

\[
\lim_{|x| \to 0, \infty} |x|^N \left| \frac{x}{|x|} \cdot \nabla u (x) \right|^2 = 0.
\]

and

\[
\lim_{|x| \to 0, \infty} |x|^{2a+N} |u(x)|^2 = 0.
\]

Then we will prove that

**Theorem 2.2** Let \( N \geq 1 \) and \( a \in \mathbb{R} \). For all \( u \in Y_a (\mathbb{R}^N) \), there holds

\[
\int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} \frac{1}{|x|^{2a+2}} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx \geq \frac{(N + 2 + 4a)^2}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2.
\] (2.2)

If either \( \min \{a + 1, N + 2 + 4a\} > 0 \) or \( \max \{a + 1, N + 2 + 4a\} < 0 \), the constant \( \frac{(N + 2 + 4a)^2}{4} \) is optimal in (2.2) and is attained by functions of the form

\[
u (x) = \gamma e^{-\beta |x|^{2(1+a)}} , \quad \gamma \in \mathbb{R} , \quad \beta > 0.
\] (2.3)

If \( N \geq 2 \) and either \( a + 1 > 0 \) or \( N + 2a < 0 \), the equality also happens in (2.2) for infinitely many nonradial functions of the form

\[
u (x) = |x|^\alpha {}_1 F_1 \left( \frac{a + N + 2a + N}{2a + 2}, \frac{2a + 2 + N}{2a + 2}; \frac{t}{2a + 2} |x|^{2a+2} \right) g \left( \frac{x}{|x|} \right),
\]

where \( \alpha = 2 - N + \text{sgn}(a + 1)\sqrt{(N - 2)^2 - 4\lambda} \),

\[
\text{with } t \in \mathbb{R} \text{ such that } \frac{t}{2a+2} > 0, \Delta_\sigma g = \lambda g \text{ for some } \lambda = -k (N + k - 2), k = 1, 2, ..., \text{ and } {}_1 F_1 (A; B; z) \text{ are the Kummer's confluent hypergeometric functions, that are the solutions to Kummer's equation}
\]

\[
 \frac{d^2 w}{dz^2} + (B - z) \frac{dw}{dz} - Aw = 0.
\] (2.4)

In the particular case \( a = 0 \), we get

\[
\int_{\mathbb{R}^N} |\Delta u|^2 \, dx \int_{\mathbb{R}^N} \frac{1}{|x|^2} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx \geq \frac{(N + 2)^2}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2, \quad \forall u \in C^\infty_0 (\mathbb{R}^N),
\] (2.5)

which implies the second order HUP (1.11) since \( \left| \frac{x}{|x|} \cdot \nabla u \right| \leq |\nabla u| \).
Remark 2.1 In this paper we describe explicitly one family of radially symmetric minimizers for the sharp constant in Theorem 2.1 and two independent (radially symmetric and non-radially symmetric, respectively) families of minimizers for the best constant in Theorem 2.2. To describe the entire family of minimizers for a sharp inequality with weights is a very difficult issue in general. However, due to our method used in the proofs based on expanding squares, we can say that any possible minimizer must satisfy a second order partial differential equation pointwise in $\mathbb{R}^N \setminus \{0\}$ and certain integrability conditions (see, for example (5.3)). In general such an equation is very difficult to integrate unless we are seeking for radially symmetric solutions.

We note that in the process of preparing our manuscript, we have noticed that (2.2) has been investigated in [24]. Nevertheless, our approach in this paper is different and much simpler than the one in [24]. Our method also provides nonradial optimizers of (2.2).

As a consequence of Theorem 2.2, we obtain

**Corollary 2.5** Let $N \geq 2$ and $a \in \mathbb{R}$. We have for $u \in Y_a(\mathbb{R}^N)$ that

$$
\int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx \int_{\mathbb{R}^N} |x|^{2a+2} |\nabla u|^2 \, dx \geq \frac{(N + 2 + 4a)^2}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2.
$$

(2.6)

If either $a + 1 > 0$ or $N + 2 + 4a < 0$, then the constant $\frac{(N + 2 + 4a)^2}{4}$ is optimal and is attained by optimizers of the form

$$
u(x) = \gamma e^{-\beta |x|^{2(1+a)}}, \quad \gamma \in \mathbb{R}, \quad \beta > 0.
$$

Remark 2.2 Following the proofs it is easy to notice that the CKN-type inequalities in Theorems 2.1–2.2 can be directly proven for functions belonging to $C_0^\infty(\mathbb{R}^N \setminus \{0\})$. However, the optimizers that we obtain for the sharp constants are not in $C_0^\infty(\mathbb{R}^N \setminus \{0\})$. For that reason we choose to prove Theorems 2.1–2.2 for functions in $X_{a,b}(\mathbb{R}^N)$ and $Y_a(\mathbb{R}^N)$, respectively, which are larger spaces than $C_0^\infty(\mathbb{R}^N \setminus \{0\})$. The advantage is that these new spaces contain our minimizers and the sharpness of the inequalities under consideration makes sense.

It turns out that the limiting conditions at zero and infinity in the definitions of $X_{a,b}(\mathbb{R}^N)$ and $Y_a(\mathbb{R}^N)$ are necessary in the proofs when integrating by parts. In principle, these limits can be eliminated by showing that our inequalities hold for functions lying in the functional spaces obtained as the completion of $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ in the norms

$$
\|U\|_{a,b} := \left( \int_{\mathbb{R}^N} \frac{|
abla U|^2}{|x|^{2a}} \, dx + \int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{2b}} \, dx \right)^{1/2},
$$

and our obtained minimizers belong to these spaces.

Such a rigorous approximation procedure using regularization through convolution was performed in our previous paper [15].

3 Preliminary

For the sake of clarity let us recall few important aspects about the curl operator. While the divergence operator is clearly defined in any dimension $N \geq 2$ through the formula

$$
\text{div} U := \sum_{j=1}^N \frac{\partial U_j}{\partial x_j},
$$

the curl operator can be directly defined only in dimensions 2 and 3; for
higher dimensions it is well-known that it is understood via differential forms. The curl of a vector field $U = (U_1, ..., U_N) \in \left( C^\infty (\mathbb{R}^N) \right)^N$ is defined as the differential 2-form
\[
\text{curl } U = d (U \cdot dx) = d \left( \sum_{j=1}^{N} U_j dx_j \right),
\]
where $d$ denotes the exterior differential. In the standard Euclidean coordinates, we can write
\[
d (U \cdot dx) = \sum_{j=1}^{N} dU_j \wedge dx_j = \sum_{j<k} \left( \frac{\partial U_k}{\partial x_j} - \frac{\partial U_j}{\partial x_k} \right) dx_j \wedge dx_k.
\]
Therefore, we have that $\text{curl } U = 0$ if and only if for all $1 \leq j, k \leq N$:
\[
\frac{\partial U_k}{\partial x_j} = \frac{\partial U_j}{\partial x_k}.
\]
This also implies that any curl-free vector field $U$ has a scalar potential $u \in C^\infty (\mathbb{R}^N)$ satisfying $U = \nabla u$. Indeed, we can always choose $u(x) = \int_{0}^{\frac{|x|}{|x|}} U \left( \rho \frac{x}{|x|} \right) d\rho$.

Let $U = (U_1, ..., U_N)$ be a vector field. Then we can write
\[
U = \sigma U_R + U_S
\]
for all $x = r \sigma$ where $r = |x|$, $\sigma = \frac{x}{|x|}$, $U_R = U_R(x)$ is the radial scalar component and $U_S = U_S(x)$ is the spherical vector part. In particular, $\sigma \cdot U_S = 0$.

We also denote $\partial_r \varphi = \sigma \cdot \nabla \varphi$ and $\nabla_\sigma \varphi = r (\nabla \varphi)_S$ and so
\[
\nabla = \sigma \partial_r + \frac{1}{r} \nabla_\sigma.
\]
Then it is known that
\[
\Delta = \partial_{rr} + \frac{N-1}{r} \partial_r + \frac{1}{r^2} \Delta_\sigma,
\]
where $\Delta_\sigma$ is the Laplace-Beltrami operator on the sphere.

In [33], the following representation of curl-free fields has been established. Let $\lambda \in \mathbb{R}$ and let $V = r^{1-\lambda} U$. Then $U \in C^\infty_0 (\mathbb{R}^N)^N$ is curl-free if and only if there exist two scalar fields $f, \varphi \in C^\infty (\mathbb{R}^N \setminus \{0\})$ satisfying $f$ is radially symmetric,
\[
\int_{S^{N-1}} \varphi (r \sigma) \, d\sigma = 0 \text{ for all } r > 0,
\]
and
\[
V = \sigma (f + (\lambda + \partial_t) \varphi) + \nabla_\sigma \varphi \text{ on } \mathbb{R}^N \setminus \{0\}.
\]
Moreover, if $U$ has a compact support on $\mathbb{R}^N \setminus \{0\}$, then so do $f$ and $\varphi$.

Now, we proceed as in [33] and let $V = r^{1-\lambda} U$ with $\lambda = 2 - \frac{N}{2} + a$. Then we have with $t = \ln r$ that
\[
|x|^\gamma \, dx = r^{\gamma + N - 1} dr d\sigma = r^{\gamma + N} \frac{dr}{r} d\sigma = r^{\gamma + N} dt d\sigma.
\]
Therefore
\[
\int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{2b}} \, dx = \int_{\mathbb{R}^N} |x|^{-2b} |x|^{2\lambda-2} |V|^2 \, dx
\]
\[
= \int_{\mathbb{R}^N \times S^{N-1}} r^{2\lambda-2-2b+N-1} |V|^2 \, dr \, d\sigma
\]
\[
= \int_{\mathbb{R}^N \times S^{N-1}} e^{(2a-2b+2)r} |V|^2 \, dr \, d\sigma
\]
and
\[
\int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} |x|^{-a-b-1} |x|^{2\lambda-2} |V|^2 \, dx
\]
\[
= \int_{\mathbb{R}^N \times S^{N-1}} r^{2\lambda-2-a-b-1+N-1} |V|^2 \, dr \, d\sigma
\]
\[
= \int_{\mathbb{R}^N \times S^{N-1}} e^{(a-b+1)r} |V|^2 \, dr \, d\sigma.
\]
Also,
\[
\int_{\mathbb{R}^N} \frac{|
abla U|^2}{|x|^{2a}} \, dx = \int_{\mathbb{R}^N \times S^{N-1}} \left( \left| \partial_r U \right|^2 + \frac{1}{r^2} |\nabla_{\sigma} U|^2 \right) r^{N-1-2a} \, dr \, d\sigma
\]
\[
= \int_{\mathbb{R}^N \times S^{N-1}} \left( \left| \partial_r \left( r^{\lambda-1} V \right) \right|^2 + \frac{1}{r^2} \left| \nabla_{\sigma} \left( r^{\lambda-1} V \right) \right|^2 \right) r^{3-2\lambda} \, dr \, d\sigma
\]
\[
= \int_{\mathbb{R}^N \times S^{N-1}} \left( (\lambda - 1)^2 |V|^2 + |\partial_r V|^2 + |\nabla_{\sigma} V|^2 \right) \, dtd\sigma.
\]
Now, using \( V = \sigma (f + (\lambda + \partial_t) \phi) + \nabla_{\sigma} \phi \), we have
\[
\Delta_{\sigma} V = \Delta_{\sigma} \left( \sigma (f + (\lambda + \partial_t) \phi) + \nabla_{\sigma} \phi \right)
\]
\[
= (\sigma \Delta_{\sigma} + 2 \nabla_{\sigma} - (N - 1) \sigma) (f + (\lambda + \partial_t) \phi)
\]
\[
+ (\nabla_{\sigma} \Delta_{\sigma} + (N - 3) \nabla_{\sigma} - 2 \sigma \Delta_{\sigma}) \phi
\]
\[
= \sigma ((\partial_t + \lambda + 2) \Delta_{\sigma} \phi) - (N - 1) \sigma (f + (\lambda + \partial_t) \phi)
\]
\[
+ \nabla_{\sigma} (2 \partial_t + \Delta_{\sigma} + 2 \lambda + N - 3) \phi
\]
\[
= \sigma ((\partial_t + \lambda + 2) \Delta_{\sigma} \phi) + \nabla_{\sigma} (2 \partial_t + \Delta_{\sigma} + 2 \lambda + 2N - 4) \phi
\]
\[
- (N - 1) V.
\]
Therefore we get by integration by parts that
\[
\int_{\mathbb{R} \times S^{N-1}} |\nabla_{\sigma} V|^2 \, dtd\sigma
\]
\[
= - \int_{\mathbb{R} \times S^{N-1}} V \cdot (\Delta_{\sigma} V) \, dtd\sigma
\]
\[
= - \int_{\mathbb{R} \times S^{N-1}} (f + (\lambda + \partial_t) \phi) (\partial_t + \lambda - 2) \Delta_{\sigma} \phi \, dtd\sigma
\]
\[
+ \int_{\mathbb{R} \times S^{N-1}} (- \nabla_{\sigma} \phi \cdot \nabla_{\sigma} (2 \partial_t + \Delta_{\sigma} + 2 \lambda + 2N - 4) \phi + (N - 1) |V|^2) \, dtd\sigma
\]
\[= \int_{\mathbb{R} \times S^{N-1}} ((\Delta_\sigma \varphi)^2 + (\lambda^2 - 4\lambda - 2N + 4) |\nabla_\sigma \varphi|^2) \, dt \, d\sigma + \int_{\mathbb{R} \times S^{N-1}} (|\partial_t \nabla_\sigma \varphi|^2 + (N - 1) |V|^2) \, dt \, d\sigma.\]

Using the fact that the spectrum of \(-\Delta_\sigma\) is the set \(\{\kappa (N + \kappa - 2), \kappa = 0, 1, \ldots\}\), we obtain that for all \(\varphi\) such that \(\int_{S^{N-1}} \varphi \, d\sigma = 0\):

\[\int_{\mathbb{R} \times S^{N-1}} ((\Delta_\sigma \varphi)^2 + (\lambda^2 - 4\lambda - 2N + 4) |\nabla_\sigma \varphi|^2) \, dt \, d\sigma \geq \min_\kappa \{\kappa^2 (N + \kappa - 2)^2 + (\lambda^2 - 4\lambda - 2N + 4) \kappa (N + \kappa - 2)\} \int_{\mathbb{R} \times S^{N-1}} |\varphi|^2 \, dt \, d\sigma \]

\[= (N - 1) ((\lambda - 2)^2 - N - 1) \int_{\mathbb{R} \times S^{N-1}} |\varphi|^2 \, dt \, d\sigma.\]

Also, since \(\int_{S^{N-1}} \varphi \, d\sigma = 0\), we get \(\int_{S^{N-1}} \partial_t \varphi \, d\sigma = 0\) and therefore

\[\int_{S^{N-1}} |\nabla_\sigma (\partial_t \varphi)|^2 \, d\sigma \geq (N - 1) \int_{S^{N-1}} |\partial_t \varphi|^2 \, d\sigma.\]

Thus, we get

\[\int_{\mathbb{R} \times S^{N-1}} |\nabla_\sigma V|^2 \, dt \, d\sigma \geq (N - 1) \int_{\mathbb{R} \times S^{N-1}} (|V|^2 + (\lambda - 2)^2 - N - 1) |\varphi|^2 + |\partial_t \varphi|^2 \, dt \, d\sigma.\]

Hence

\[\int_{\mathbb{R}^N} \frac{|\nabla U|^2}{|x|^{2\alpha}} \, dx \geq (\lambda - 1)^2 + N - 1 \int_{\mathbb{R} \times S^{N-1}} |V|^2 \, dt \, d\sigma + \int_{\mathbb{R} \times S^{N-1}} |\partial_t V|^2 \, dt \, d\sigma + (N - 1) \int_{\mathbb{R} \times S^{N-1}} ((\lambda - 2)^2 - N - 1) |\varphi|^2 + |\partial_t \varphi|^2 \, dt \, d\sigma.\]

**4 Weighted \(L^2\)-Caffarelli–Kohn–Nirenberg inequalities for curl-free vector fields—Proof of Theorem 2.1**

**Proof of Theorem 2.1** Let \(U \in X_{a,b} (\mathbb{R}^N)\). Then we can find two scalar fields \(f, \varphi \in C^\infty (\mathbb{R}^N \setminus \{0\})\) satisfying \(f\) is radially symmetric,

\[\int_{S^{N-1}} \varphi (r \sigma) \, d\sigma = 0 \text{ for all } r > 0,\]

and

\[V = \sigma (f + (\lambda + \partial_t) \varphi) + \nabla_\sigma \varphi \text{ on } \mathbb{R}^N \setminus \{0\},\]

where \(V = r^{1-\lambda} U\) with \(\lambda = 2 - \frac{N}{2} + a\). Let \(t = \ln r\). We have

\[|x|^{\gamma} \, dx = r^{\gamma+N-1} \, d\sigma = r^{\gamma+N} \frac{dr}{r} \, d\sigma = r^{\gamma+N} \, dt \, d\sigma.\]
As in the previous section, we get
\[
\int_{\mathbb{R}^N} \frac{|VU|^2}{|x|^{2a}} \, dx \geq ((\lambda - 1)^2 + N - 1) \int_{\mathbb{R}^N} |V|^2 \, dt \sigma + \int_{\mathbb{R}^N} |\partial_t V|^2 \, dt \sigma \\
+ (N - 1) \int_{\mathbb{R}^N} ((\lambda - 2)^2 + N - 1) |\varphi|^2 + |\partial_t \varphi|^2 \, dt \sigma \\
\geq ((\lambda - 1)^2 + N - 1) \int_{\mathbb{R}^N} |V|^2 \, dt \sigma + \int_{\mathbb{R}^N} |\partial_t V|^2 \, dt \sigma
\]
if \((\frac{N}{2} - a)^2 \geq N + 1\).

Also
\[
\int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{2b}} \, dx = \int_{\mathbb{R}^N} e^{(2a - 2b + 2)t} |V|^2 \, dt \sigma
\]
and
\[
\int_{\mathbb{R}^N} \frac{|U|^2}{|x|^{a+b+1}} \, dx = \int_{\mathbb{R}^N} e^{(a-b+1)t} |V|^2 \, dt \sigma.
\]

**Case 1 \(a - b + 1 > 0\)**

We aim to show that
\[
\left[ \int_{\mathbb{R}^N} |\partial_t V|^2 \, dt \sigma + \left( \left( 1 - \frac{N}{2} + a \right) + N - 1 \right) \int_{\mathbb{R}^N} |V|^2 \, dt \sigma \right] \\
\times \left[ \int_{\mathbb{R}^N} e^{(2a - 2b + 2)t} |V|^2 \, dt \sigma \right] \\
\geq \left( \sqrt{\left( 1 - \frac{N}{2} + a \right) + N - 1 + \frac{a - b + 1}{2}} \right)^2 \left[ \int_{\mathbb{R}^N} e^{(a-b+1)t} |V|^2 \, dt \sigma \right]^2.
\]

Indeed, we have
\[
\int_{\mathbb{R}^N} |\partial_t V (t \sigma) + a V (t \sigma) + \beta e^{(a-b+1)t} V (t \sigma)|^2 \, dt \sigma \\
= \int_{\mathbb{R}^N} |\partial_t V (t \sigma)|^2 \, dt \sigma + a^2 \int_{\mathbb{R}^N} |V (t \sigma)|^2 \, dt \sigma + \beta^2 \int_{\mathbb{R}^N} e^{(2a - 2b + 2)t} |V (t \sigma)|^2 \, dt \sigma \\
+ a \int_{\mathbb{R}^N} \partial_t |V (t \sigma)|^2 \, dt \sigma + \beta \int_{\mathbb{R}^N} e^{(a-b+1)t} \partial_t |V (t \sigma)|^2 \, dt \sigma \\
+ 2a\beta \int_{\mathbb{R}^N} e^{(a-b+1)t} |V (t \sigma)|^2 \, dt \sigma.
\]

Note that since \(U \in X_{a,b} (\mathbb{R}^N)\), we have
\[
\lim_{r \to 0, \infty} r^{-b+\frac{N}{2}} U(r \sigma) = \lim_{r \to 0, \infty} r^{-1+\frac{N}{2}-a} U(r \sigma) = 0.
\]
That is,
\[
\lim_{r \to 0, \infty} r^{a-b+1} V (r \sigma) = \lim_{r \to 0, \infty} V (r \sigma) = 0.
\]
Equivalently
\[
\lim_{t \to -\infty, \infty} e^{(a-b+1)t} V (t \sigma) = \lim_{t \to -\infty, \infty} V (t \sigma) = 0.
\]
Therefore, by integrations by parts, we obtain
\[
\int_{\mathbb{R} \times S^{N-1}} \left| \partial_t V(t\sigma) + \alpha V(t\sigma) + \beta e^{(a-b+1)t} V(t\sigma) \right|^2 \, dt\,d\sigma
\]
\[
= \int_{\mathbb{R} \times S^{N-1}} \left| \partial_t V(t\sigma) \right|^2 \, dt\,d\sigma + \alpha^2 \int_{\mathbb{R} \times S^{N-1}} \left| V(t\sigma) \right|^2 \, dt\,d\sigma + \beta^2 \int_{\mathbb{R} \times S^{N-1}} e^{2(a-b+2)t} \left| V(t\sigma) \right|^2 \, dt\,d\sigma
\]
\[
+ \beta (2a - (a - b + 1)) \int_{\mathbb{R} \times S^{N-1}} e^{(a-b+1)t} \left| V(t\sigma) \right|^2 \, dt\,d\sigma.
\]

Now, since \( \int_{\mathbb{R} \times S^{N-1}} \left| \partial_t V(t\sigma) + \alpha V(t\sigma) + \beta e^{(a-b+1)t} V(t\sigma) \right|^2 \, dt\,d\sigma \geq 0 \) for all \( \beta \in \mathbb{R} \), we deduce that
\[
\left[ \int_{\mathbb{R} \times S^{N-1}} \left| \partial_t V(t\sigma) \right|^2 \, dt\,d\sigma + \alpha^2 \int_{\mathbb{R} \times S^{N-1}} \left| V(t\sigma) \right|^2 \, dt\,d\sigma \right] \int_{\mathbb{R} \times S^{N-1}} e^{(2a-2b+2)t} \left| V(t\sigma) \right|^2 \, dt\,d\sigma \]
\[
\geq \left( \alpha - \frac{a - b + 1}{2} \right)^2 \left( \int_{\mathbb{R} \times S^{N-1}} e^{(a-b+1)t} \left| V(t\sigma) \right|^2 \, dt\,d\sigma \right)^2.
\]

If we choose \( \alpha^2 = \left( 1 - \frac{N}{2} + a \right)^2 + N - 1 \), that is \( \alpha = -\sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1} \), then the best constant is
\[
\left( \alpha - \frac{a - b + 1}{2} \right)^2 = \left( \sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1 + \frac{a - b + 1}{2}} \right)^2.
\]

The equality happens when \( \partial_t V(t\sigma) + \alpha V(t\sigma) + \beta e^{(a-b+1)t} V(t\sigma) = 0 \) with
\[
\alpha = -\sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1}
\]
and
\[
\beta = \left( \sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1 + \frac{a - b + 1}{2}} \right) \frac{\int_{\mathbb{R} \times S^{N-1}} e^{(a-b+1)t} \left| V(t\sigma) \right|^2 \, dt\,d\sigma}{\int_{\mathbb{R} \times S^{N-1}} e^{(2a-2b+2)t} \left| V(t\sigma) \right|^2 \, dt\,d\sigma} > 0.
\]

For instance,
\[
V(t\sigma) = e^{-\alpha t - \frac{\beta}{(a-b+1)} e^{(a-b+1)t}} \sigma.
\]

Hence \( V(r\sigma) = r^{-\alpha} e^{-\frac{\beta}{(a-b+1)} r^{(a-b+1)}} \sigma \). That is \( U(r\sigma) = r^{1 - \frac{N}{2} + a - \alpha} e^{-\frac{\beta}{(a-b+1)} r^{(a-b+1)}} \sigma \) and \( U(x) = |x|^{\frac{N}{2} + a - \alpha} e^{-\frac{\beta}{(a-b+1)} |x|^{(a-b+1)}} x \). This is curl-free since \( \frac{\partial U_i}{\partial x_j} = \frac{\partial U_j}{\partial x_i} \). Note that \( U \in X_{a,b}(\mathbb{R}^N) \). Indeed, it is obvious that \( \lim_{r \to \infty} r^k U(r\sigma) = 0 \) for all \( k \) since \( a - b + 1 > 0 \) and \( \beta > 0 \). Now,
\[
\lim_{r \to 0} r^{1 + \frac{N}{2} - a} U(r\sigma) = \lim_{r \to 0} r^{1 + \frac{N}{2} - a} r^{1 - \frac{N}{2} + a - \alpha} e^{-\frac{\beta}{(a-b+1)} r^{(a-b+1)}} \sigma
\]
\[
= \lim_{r \to 0} r\sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1} = 0
\]
and
\[
\lim_{r \to 0} r^{-b + \frac{N}{2}} U(r\sigma) = \lim_{r \to 0} r^{-b + \frac{N}{2}} r^{-1 + \frac{N}{2} - a - \alpha} e^{-\frac{\beta}{(a-b+1)} r^{(a-b+1)}} \sigma
\]
The equality happens when 

\[ \alpha = \sqrt{\left(1 - \frac{N}{2} + a\right)^2 + N - 1} \]

Case 2 \( a - b + 1 < 0 \).

As in Case 1, we get

\[
\int_{\mathbb{R}^N} \left| \partial_x V(t \sigma) + \alpha V(t \sigma) + \beta e^{(a-b+1)t} V(t \sigma) \right|^2 \, dt \, d\sigma
\]

\[
= \int_{\mathbb{R}^N} \left| \partial_x V(t \sigma) \right|^2 \, dt \, d\sigma + \alpha^2 \int_{\mathbb{R}^N} \left| V(t \sigma) \right|^2 \, dt \, d\sigma + \beta^2 \int_{\mathbb{R}^N} e^{(a-b+2)t} \left| V(t \sigma) \right|^2 \, dt \, d\sigma
\]

\[
+ \alpha \int_{\mathbb{R}^N} \partial_x \left| V(t \sigma) \right|^2 \, dt \, d\sigma + \beta \int_{\mathbb{R}^N} e^{(a-b+1)t} \partial_x \left| V(t \sigma) \right|^2 \, dt \, d\sigma
\]

\[
+ 2 \alpha \int_{\mathbb{R}^N} e^{(a-b+1)t} \left| \partial_x V(t \sigma) \right|^2 \, dt \, d\sigma.
\]

Now, since \( U \in X_{a,b} (\mathbb{R}^N) \), we have

\[
\lim_{r \to 0, \infty} r^{-a} U(r \sigma) = \lim_{r \to 0, \infty} r^{-b+N} U(r \sigma) = 0.
\]

Therefore

\[
\lim_{t \to -\infty, \infty} e^{(a-b+1)t} V(t \sigma) = \lim_{t \to -\infty, \infty} V(t \sigma) = 0.
\]

Therefore, by integrations by parts, we get

\[
\int_{\mathbb{R}^N} \left| \partial_x V(t \sigma) + \alpha V(t \sigma) + \beta e^{(a-b+1)t} V(t \sigma) \right|^2 \, dt \, d\sigma
\]

\[
= \int_{\mathbb{R}^N} \left| \partial_x V(t \sigma) \right|^2 \, dt \, d\sigma + \alpha^2 \int_{\mathbb{R}^N} \left| V(t \sigma) \right|^2 \, dt \, d\sigma + \beta^2 \int_{\mathbb{R}^N} e^{(a-b+2)t} \left| V(t \sigma) \right|^2 \, dt \, d\sigma
\]

\[
+ \beta \left( 2a - (a-b+1) \right) \int_{\mathbb{R}^N} e^{(a-b+1)t} \left| V(t \sigma) \right|^2 \, dt \, d\sigma.
\]

and

\[
\left[ \int_{\mathbb{R}^N} \left| \partial_x V(t \sigma) \right|^2 \, dt \, d\sigma \right]^2 + \alpha^2 \left[ \int_{\mathbb{R}^N} \left| V(t \sigma) \right|^2 \, dt \, d\sigma \right]^2 \geq \left( \alpha - \frac{a-b+1}{2} \right)^2 \left( \int_{\mathbb{R}^N} e^{(a-b+1)t} \left| V(t \sigma) \right|^2 \, dt \, d\sigma \right)^2.
\]

Now, we choose \( \alpha = \sqrt{\left(1 - \frac{N}{2} + a\right)^2 + N - 1} \) and get

\[
\left[ \int_{\mathbb{R}^N} \left| \partial_x V(t \sigma) \right|^2 \, dt \, d\sigma \right]^2 \left[ \int_{\mathbb{R}^N} e^{(a-b+1)t} \left| V(t \sigma) \right|^2 \, dt \, d\sigma \right]^2 \]

\[
\times \left[ \int_{\mathbb{R}^N} e^{(a-b+1)t} \left| V(t \sigma) \right|^2 \, dt \, d\sigma \right]^2 \geq \left( \sqrt{\left(1 - \frac{N}{2} + a\right)^2 + N - 1 - \frac{a-b+1}{2}} \right)^2 \left[ \int_{\mathbb{R}^N} e^{(a-b+1)t} \left| V(t \sigma) \right|^2 \, dt \, d\sigma \right]^2.
\]

The equality happens when \( \partial_x V(t \sigma) + \alpha V(t \sigma) + \beta e^{(a-b+1)t} V(t \sigma) = 0 \) with

\[
\alpha = \sqrt{\left(1 - \frac{N}{2} + a\right)^2 + N - 1}
\]
and
\[
\beta = \left( -\sqrt{\left( 1 - \frac{N}{2} + a \right)^2 + N - 1 + \frac{a - b + 1}{2}} \right) \frac{\int_{\mathbb{R} \times S^{N-1}} e^{(a-b+1)t} |V(t\sigma)|^2 \, dt \, d\sigma}{\int_{\mathbb{R} \times S^{N-1}} e^{(2a-2b+2)t} |V(t\sigma)|^2 \, dt \, d\sigma} < 0.
\]
That is
\[
V(t\sigma) = e^{-\alpha t} - \frac{\beta}{(a+b+1)} e^{(a-b+1)t} \sigma
\]
and \(V(r\sigma) = r^{-\alpha} e^{-\frac{\beta}{(a+b+1)} r^{(a-b+1)} \sigma}. \) Therefore \( U(r\sigma) = r^{1-N/2 + a} \beta e^{-\frac{\beta}{(a+b+1)} r^{(a-b+1)} \sigma}. \) That is \( U(x) = |x|^{-N/2 + a} \beta e^{-\frac{\beta}{(a+b+1)} |x|^{(a-b+1)}} x. \) Note that \( U \) is curl-free since \( \frac{\partial U_j}{\partial x_k} = \frac{\partial U_k}{\partial x_j}. \) Also, it is easy to check that \( U \in X_{a,b} (\mathbb{R}^N). \) Indeed, since \( a - b + 1 < 0, \) we can see easily that \( \lim_{r \to 0} r^k U(r\sigma) = 0 \) for all \( k. \) Now,
\[
\lim_{r \to \infty} r^{-1+\frac{N}{2}-a} U(r\sigma) = \lim_{r \to \infty} r^{-1+\frac{N}{2}-a} r^{1-N/2 + a} e^{-\frac{\beta}{(a+b+1)} r^{(a-b+1)} \sigma}
\]
\[
= \lim_{r \to \infty} r \left[ \left( 1 - \frac{N}{2} + a \right)^2 + N - 1 \right] = 0
\]
and
\[
\lim_{r \to \infty} r^{-b+\frac{N}{2}} U(r\sigma) = \lim_{r \to \infty} r^{-b+\frac{N}{2}} r^{1-N/2 + a} e^{-\frac{\beta}{(a+b+1)} r^{(a-b+1)} \sigma}
\]
\[
= \lim_{r \to \infty} r^{a-b+1} r \left[ \left( 1 - \frac{N}{2} + a \right)^2 + N - 1 \right] = 0.
\]

5 Weighted second order Heisenberg Uncertainty Principle—Proof of Theorem 2.2

\textbf{Proof of Theorem 2.2} We have for \( u \in Y_a(\mathbb{R}^N) \) and any \( s, t \in \mathbb{R} \) that
\[
\int_{\mathbb{R}^N} \left| \frac{\Delta u}{x^a} + t \frac{x}{|x|} \cdot \nabla u + s \frac{x}{|x|} u \right|^2 \, dx
\]
\[
= \int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx + 2t \int_{\mathbb{R}^N} \frac{|x|^{2a+2} \frac{x}{|x|} \cdot \nabla u}{|x|^{a+1}} \, dx + s^2 \int_{\mathbb{R}^N} \frac{|x|^{2a} |u|^2}{|x|^{a+1}} \, dx
\]
\[
+ 2s \int_{\mathbb{R}^N} \Delta u (x \cdot \nabla u) \, dx + 2s \int_{\mathbb{R}^N} u \Delta u \, dx + 2ts \int_{\mathbb{R}^N} |x|^{2a} u (x \cdot \nabla u) \, dx.
\]
Since \( u \in Y_a(\mathbb{R}^N), \) we have by integration by parts that
\[
\int_{\mathbb{R}^N} \Delta u (x \cdot \nabla u) \, dx = \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx,
\]
\[
\int_{\mathbb{R}^N} u \Delta u \, dx = - \int_{\mathbb{R}^N} |\nabla u|^2 \, dx
\]
and
\[
\int_{\mathbb{R}^N} |x|^{2a} u (x \cdot \nabla u) \, dx = \sum_{j=1}^N \int_{\mathbb{R}^N} |x|^{2a} u x_j \partial_j u \, dx
\]
\[\square\] Springer
By choosing $s = 2t\left(\frac{N}{2} + a\right)$, we get that for all $t \in \mathbb{R}$:

$$
\int_{\mathbb{R}^N} \left[ \frac{\Delta u}{|x|^a} + t \frac{x}{|x|^{a+1}} \cdot \nabla u + s \frac{x}{|x|^a} u \right]^2 \, dx
= \int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx + t^2 \int_{\mathbb{R}^N} |x|^{2a+2} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx + s^2 \int_{\mathbb{R}^N} |x|^{2a} |u|^2 \, dx
+ t(N - 2) \int_{\mathbb{R}^N} |
\nabla u|^2 \, dx - 2s \int_{\mathbb{R}^N} |
\nabla u|^2 \, dx - 2ts \left(\frac{N}{2} + a\right) \int_{\mathbb{R}^N} |x|^{2a} |u|^2 \, dx.
$$

By choosing $s = 2t\left(\frac{N}{2} + a\right)$, we get that for all $t \in \mathbb{R}$:

$$
\int_{\mathbb{R}^N} \left[ \frac{\Delta u}{|x|^a} + t \frac{x}{|x|^{a+1}} \cdot \nabla u + s \frac{x}{|x|^a} u \right]^2 \, dx
= \int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx + t^2 \int_{\mathbb{R}^N} |x|^{2a+2} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx - (N + 2 + 4a) t \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx.
$$

From here, we deduce that

$$
\left(\int_{\mathbb{R}^N} \frac{|\Delta u|^2}{|x|^{2a}} \, dx\right) \left(\int_{\mathbb{R}^N} |x|^{2a+2} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx\right) \geq \left(\frac{N + 2 + 4a}{2}\right)^2 \left(\int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^2.
$$

Moreover, if the quadratic (5.1) has a real root $t$ for a nontrivial $u$, then

$$
t = \frac{N + 2 + 4a}{2} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\int_{\mathbb{R}^N} |x|^{2a+2} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx},
$$

from which we have that $\text{sgn } t = \text{sgn}(N + 2 + 4a)$. In particular, such a $u$ must satisfy

$$
\frac{\Delta u}{|x|^a} + t \frac{x}{|x|^{a+1}} \frac{x}{|x|} \cdot \nabla u + s |x| = 0
$$

for this given $t$. We note that when $N \geq 2$, $a + 1 > 0$ implies $N + 2 + 4a > 0$ (and hence $N + 2 + 4a < 0$ implies $a + 1 < 0$).

We show that $u(x) = ae^{-\frac{\beta}{a+1} |x|^{2(a+1)}}$ is a radial extremizer. We begin by showing that $u$ belongs to $Y_a(\mathbb{R}^N)$. First note that, if $\frac{\beta}{a+1} < 0$, then $u$ behaves exponentially at $0$ (when $a + 1 < 0$) and at $\infty$ (when $a + 1 > 0$), and so it must hold that $\frac{\beta}{a+1} > 0$. In case $\frac{\beta}{a+1} > 0$, $u$ satisfies the following limits:

$$
\text{if } a + 1 > 0, \text{ then } u(x) \to \begin{cases} 0 & \text{as } |x| \to \infty, \\ \alpha & \text{as } |x| \to 0 \end{cases},
$$
Therefore, by (5.1), the quadratic equation as desired, thereby showing that \( u \) for some \( t \) has exactly one real root.

Similarly, if \( a + 1 < 0 \), then there needs to hold \( N + 2 + 4a < 0 \). (This can be conclude from the computations below.) All that is left to check is that equality in (2.2) holds for this particular \( u \).

To this end, we compute

\[
\nabla u = -\alpha \beta e^{-\frac{\beta}{\alpha+\gamma} |x|^{2(1+a)}} |x|^{1+2a} \frac{x}{|x|},
\]

\[
\frac{x}{|x|} \cdot \nabla u = -\alpha \beta e^{-\frac{\beta}{\alpha+\gamma} |x|^{2(1+a)}} |x|^{1+2a},
\]

and

\[
\Delta u = \alpha \beta |x|^{2a} e^{-\frac{\beta}{\alpha+\gamma} |x|^{2(1+a)}} \left( \beta |x|^{2(1+a)} - 2a - 1 \right) - \alpha \beta (N - 1) e^{-\frac{\beta}{\alpha+\gamma} |x|^{2(1+a)}} |x|^{2a}.
\]

Therefore for any \( t \in \mathbb{R} : \)

\[
\frac{\Delta u}{|x|^a} + \frac{t}{|x|^{a+1}} \frac{x}{|x|} \cdot \nabla u + 2t \left( \frac{N}{2} + a \right) |x|^a u
\]

\[
= \left[ \beta |x|^a \left( \beta |x|^{2(1+a)} - 2a - 1 \right) - \beta (N - 1) |x|^a - \beta t |x|^{2+3a} + t (N + 2a) |x|^a \right] u
\]

\[
= (\beta - t) \left[ \beta |x|^{2+3a} - (N + 2a) |x|^a \right] u.
\]

Therefore, by (5.1), the quadratic equation

\[
t^2 \int_{\mathbb{R}^N} |x|^{2a+2} |\nabla u|^2 \, dx - (N + 2 + 4a) t \int_{\mathbb{R}^N} \frac{x}{|x|} \cdot \nabla u \, dx + \int_{\mathbb{R}^N} \frac{\Delta u^2}{|x|^{2a}} \, dx = 0
\]

has exactly one real root \( t = \beta \). Note that, by (5.2), if \( a + 1 > 0 \) or if \( N + 2 + 4a < 0 \), then \( \frac{\beta}{\alpha+\gamma} > 0 \). We can now conclude that

\[
\left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \left( \int_{\mathbb{R}^N} |x|^{2a+2} \left| \frac{x}{|x|} \cdot \nabla u \right|^2 \, dx \right) = \left( \frac{N + 2 + 4a}{2} \right)^2 \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2,
\]

as desired, thereby showing that \( u(x) = \alpha e^{-\frac{\beta}{\alpha+\gamma} |x|^{2(a+1)}} \) is a radial extremizer.

For \( N \geq 2 \), we will show that there exist infinitely many nonradial optimizers of (2.2). Indeed, as observed above, equality happens in (2.2) if and only if

\[
\frac{\Delta u}{|x|^a} + \frac{t}{|x|^{a+1}} \frac{x}{|x|} \cdot \nabla u + s |x|^a u = 0
\]

(5.3) for some \( t \) and \( s = 2t \left( \frac{N}{2} + a \right) \) such that \( \text{sgn} \, t = \text{sgn}(N + 2 + 4a) \). We will show that (5.3) has nonradial solutions.
To begin, we recall Kummer’s confluent hypergeometric functions $1 F_1(A; B; z)$ which are solutions to Kummer’s equation

$$z \frac{d^2 w}{dz^2} + (B - z) \frac{dw}{dz} - Aw = 0. \quad (5.4)$$

We will show that solutions of (5.3) may be expressed in terms of Kelvin-like transforms of Kummer’s confluent hypergeometric functions, i.e., written in terms of the functions of the form $z^\alpha 1 F_1(A; B; cr^\beta)$ for appropriately chosen $A, B, \alpha, \beta, c \in \mathbb{R}$. To be precise, we show

$$u(x) = |x|^{\alpha} 1 F_1 \left( \begin{array}{c} \alpha + N + 2a; \frac{2\alpha + 2a + N}{2a + 2}; - \frac{t}{2a + 2} |x|^{2a+2} \end{array}; \right),$$

where $\lambda$ is an eigenvalue of the spherical Laplacian $\Delta_\sigma$. We will suppose solutions of (5.3) take the form $u(x) = f(r)g(\sigma)$ with $\Delta_\sigma g = \lambda g$ for some $\lambda \in \mathbb{R}$ (this is justified by writing $u$ in terms of spherical harmonics and therefore $\lambda = -c_k = -k (N + k - 2), k = 0, 1, \ldots$). Then, by using (3.1), (5.3) may be reformulated as

$$r^{-a} \left[ f''(r) g(\sigma) + (N - 1) r^{-1} f'(r) g(\sigma) + \lambda r^{-2} f(r) g(\sigma) \right] + t r^{a+1} f'(r) g(\sigma) + sr^a f(r) g(\sigma) = 0.$$ 

We thus need to find solutions to the ODE

$$y'' + [(N - 1)r^{-1} + tr^{2a+1}] y' + [\lambda r^{-2} + sr^{2a}] y = 0. \quad (5.5)$$

Comparing (5.5) to (5.4), we are led to take $y = r^\alpha v(cr^\beta)$, where $v(z) = 1 F_1(A; B; z)$ for some to be determined $A$ and $B$. We compute

$$y = r^\alpha v(cr^\beta), \quad y' = \alpha r^{\alpha-1} v(cr^\beta) + c \beta r^{\alpha+\beta-1} v'(cr^\beta),$$

$$y'' = \alpha (\alpha - 1) r^{\alpha-2} v(cr^\beta) + (2\alpha + \beta - 1) \beta c r^{\alpha+\beta-2} v'(cr^\beta) + c^2 \beta^2 r^{\alpha+2\beta-2} v''(cr^\beta).$$

Therefore, (5.5) becomes

$$\alpha (\alpha - 1) r^{\alpha-2} v(cr^\beta) + (2\alpha + \beta - 1) \beta c r^{\alpha+\beta-2} v'(cr^\beta) + c^2 \beta^2 r^{\alpha+2\beta-2} v''(cr^\beta) + \left[ (N - 1)r^{-1} + tr^{2a+1} \right] \left( \alpha r^{\alpha-1} v(cr^\beta) + c \beta r^{\alpha+\beta-1} v'(cr^\beta) \right) + \left[ \lambda r^{-2} + sr^{2a} \right] r^\alpha v(cr^\beta) = 0.$$ 

Simplifying and collecting terms, we obtain

$$c^2 \beta^2 r^{2\beta-2} v''(cr^\beta) + \left[ 2\alpha + \beta + N - 2 + tr^{2a+2} \right] \beta c r^{\beta-2} v'(cr^\beta) + \left[ (\alpha(\alpha + N - 2) + \lambda) r^{-2} + (\alpha t + s) r^{2\beta} \right] v(cr^\beta) = 0.$$ 

Letting $\alpha$ solve $\alpha(\alpha + N - 2) + \lambda = 0$, i.e.,

$$\alpha = \frac{2 - N \pm \sqrt{(N - 2)^2 - 4\lambda}}{2},$$
and further simplifying (recall $s = 2t \left( \frac{N}{2} + a \right)$), we obtain

\[
c^2 \beta^2 r^{2\beta - 2a - 2} v''(cr^\beta)
+ \left[ 2\alpha + \beta + N - 2 + tr^{2a+2} \right] \beta cr^{\beta - 2a - 2} v'(cr^\beta)
+ t(\alpha + N + 2a)v(cr^\beta) = 0.
\]

By choosing

\[
\beta = 2a + 2
\]
\[
A = \frac{\alpha + N + 2a}{2a + 2}
\]
\[
B = \frac{2\alpha + 2a + N}{2a + 2}
\]
\[
z = -\frac{t}{2a + 2} r^{2a+2}
\]
\[
c = -\frac{t}{2a + 2},
\]

we conclude that (5.6) is equivalent to

\[
z v''(z) + (B - z)v'(z) - Av(z) = 0,
\]

and so

\[
v(r) = \, _1F_1(A; B; z) = \, _1F_1\left(\frac{\alpha + N + 2a}{2a + 2}; \frac{2\alpha + 2a + N}{2a + 2}; -\frac{t}{2a + 2} r^{2a+2}\right),
\]

with $\alpha$ as above. In conclusion, we obtain as solutions for (5.3) functions of the form

\[
u(x) = |x|^\alpha \, _1F_1\left(\frac{\alpha + N + 2a}{2a + 2}; \frac{2\alpha + 2a + N}{2a + 2}; -\frac{t}{2a + 2} |x|^{2a+2}\right) g\left(\frac{x}{|x|}\right)
\]

where $\Delta_\sigma g = \lambda g$ and

\[
\alpha = 2 - N \pm \sqrt{(N - 2)^2 - 4\lambda}.
\]

Using superposition, we may obtain more solutions. The radial solutions are obtained by taking $\lambda = 0$, which implies $\alpha = 0$ or $2 - N$. Indeed, choosing $\alpha = 0$, we conclude

\[
u(x) = \, _1F_1\left(\frac{N + 2a}{2a + 2}; \frac{N + 2a}{2a + 2}; -\frac{t}{2a + 2} |x|^{2a+2}\right) = e^{-\frac{t}{2a + 2} |x|^{2a+2}},
\]

which recovers the radial extremizers given above.

If $a + 1 > 0$, then $N + 2 + 4a > N + 2a > 0$. Therefore, $\frac{t}{2a + 2} > 0$. We will now show that $U \in Y_a(\mathbb{R}^N)$ where

\[
U(x) = |x|^\alpha \, _1F_1\left(\frac{\alpha + N + 2a}{2a + 2}; \frac{2\alpha + 2a + N}{2a + 2}; -\frac{t}{2a + 2} |x|^{2a+2}\right) g\left(\frac{x}{|x|}\right)
\]

and

\[
\alpha = 2 - N + \sqrt{(N - 2)^2 - 4\lambda}.
\]
Indeed, using the following asymptotic behavior of Kummer’s function of the first kind (see, for instance, [8]): for \( r \to -\infty \)

\[
1 F_1 (a, b, r) \sim \Gamma (b) \frac{(-r)^{-a}}{\Gamma (b - a)},
\]

we have that as \( |x| \to \infty \):

\[
1 F_1 \left( \frac{\alpha + N + 2a}{2a + 2} ; \frac{2a + 2 + N}{2a + 2} ; -\frac{t}{2a + 2} |x|^{2a+2} \right) = O \left( |x|^{-(\alpha + N + 2a)} \right).
\]

Therefore, as \( |x| \to \infty \):

\[
U(x) = O \left( |x|^{-N-2a} \right).
\]

Also, using the formula \( \frac{d}{dr} \left( 1 F_1 (a, b, r) \right) = \frac{a}{b} \left( 1 F_1 (a + 1, b + 1, r) \right) \) (see [8]), we get

\[
\left| \frac{x}{|x|} \cdot \nabla U (x) \right| = O \left( |x|^{\alpha-1-(\alpha+N+2a)} \right) + O \left( |x|^{\alpha+2a+1-(\alpha+N+2a)-(2a+2)} \right) = O \left( |x|^{-N-2a-1} \right).
\]

Hence

\[
\lim_{|x| \to \infty} |x|^{N-1} |U (x)|^2 = \lim_{|x| \to \infty} |x|^{-N-4a-1} = 0 \text{ since } N + 4a + 1 > 0,
\]

\[
\lim_{|x| \to \infty} |x|^N \left| \frac{x}{|x|} \cdot \nabla u (x) \right|^2 = \lim_{|x| \to \infty} |x|^{-N-4a-2} = 0 \text{ since } N + 4a + 2 > 0,
\]

and

\[
\lim_{|x| \to \infty} |x|^{2a+N} |U (x)|^2 = \lim_{|x| \to \infty} |x|^{-N-2a} = 0 \text{ since } N + 2a > 0.
\]

As \( r \to 0 \), then

\[
1 F_1 (a, b, r) = 1 + O (r).
\]

Therefore, it is easy to see that

\[
\lim_{|x| \to 0} |x|^{N-1} |U (x)|^2 = 0
\]

\[
\lim_{|x| \to 0, \infty} |x|^N \left| \frac{x}{|x|} \cdot \nabla U (x) \right|^2 = 0
\]

and

\[
\lim_{|x| \to 0} |x|^{2a+N} |U (x)|^2 = 0.
\]

If \( N + 2a < 0 \), then \( a + 1 < 0 \) and \( N + 4a + 2 < 0 \). Hence, \( \frac{t}{2a+2} > 0 \). In this case, we choose

\[
\alpha = \frac{2 - N - \sqrt{(N - 2)^2 - 4\lambda}}{2}.
\]

Now, when \( |x| \to \infty \), then \( z = -\frac{t}{2a+2} |x|^{2a+2} \to 0 \). Hence,

\[
1 F_1 \left( \frac{\alpha + N + 2a}{2a + 2} ; \frac{2a + 2 + N}{2a + 2} ; -\frac{t}{2a + 2} |x|^{2a+2} \right) = 1 + O \left( |x|^{2a+2} \right)
\]
and $U(x) = O (|x|^{-1})$. Therefore

$$|x|^{N-1} |U(x)|^2 = O \left( |x|^{N-1+2\alpha} \right) = o(1)$$

$$|x|^{2\alpha+N} |U(x)|^2 = O \left( |x|^{2\alpha+N+2\alpha} \right) = o(1)$$

$$|x|^{N} \left| \frac{x}{|x|} \cdot \nabla U(x) \right|^2 = O \left( |x|^{N+2\alpha-2} \right) = o(1).$$

When $|x| \to 0$, then $z = -\frac{t^2}{2\alpha+2} |x|^{2\alpha+2} \to -\infty$. In this case, as above, we have

$$\lim_{|x| \to 0} |x|^{N-1} |U(x)|^2 = \lim_{|x| \to 0} |x|^{-N-4\alpha-1} = 0 \text{ since } N + 4\alpha + 1 < 0,$$

$$\lim_{|x| \to 0} |x|^{N} \left| \frac{x}{|x|} \cdot \nabla u(x) \right|^2 = \lim_{|x| \to 0} |x|^{-N-4\alpha-2} = 0 \text{ since } N + 4\alpha + 2 < 0,$$

and

$$\lim_{|x| \to 0} |x|^{2\alpha+N} |U(x)|^2 = \lim_{|x| \to 0} |x|^{-N-2\alpha} = 0 \text{ since } N + 2\alpha < 0.$$

\[\square\]

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