Dual Reweighted $\ell_p$-Norm Minimization for Salt-and-pepper Noise Removal

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Abstract

The robust principle analysis (RPCA), which aims to estimate underlying low rank and sparse structures from the degraded observation data, has a wide range of applications in computer vision. It is usually replaced by the component analysis model (PCP) in order to pursue the convex property, leading to the undesirable overshrink problem. In this paper, we propose a dual reweighted $\ell_p$-norm (DWLP) model with a more reasonable weighting rule and weaker powers, which greatly generalizes previous works and provides a better approximation to the rank minimization problem for original matrix as well as the $\ell_0$-norm minimization problem for sparse noise. Moreover, an iterative reweighted algorithm is introduced to solve the proposed DWLP model by optimizing elements and weights alternatively. We then apply the DWLP model to remove salt-and-pepper noise by exploiting the image non-local self-similarity. Extensive experiments demonstrate that the proposed method outperforms other state-of-the-art methods in terms of both qualitative and quantitative evaluation. More precisely, our DWLP achieves about 6.814dB, 4.80dB, 3.142dB, 1.20dB and 0.1dB improvements over the current WSNM-RPCA in average under salt-and-pepper noise densities 10% to 50% with an interval 10% respectively.

Keywords: Image denoising, non-local self-similarity, robust principle analysis (RPCA), low rank, sparsity.
1. Introduction

Although, digital cameras and sensors have developed significantly in recent years, image data collected in practice is still liable to suffer from various annoying noise contamination due to high temperature, poor illumination, high ISO setting, or high capture rate, etc [1]. The presence of image noise often results in severe image degradation and make the post-processing more challenging. The purpose of image denoising is to remove the noise and recover the clean image from the degraded observation.

The salt-and-pepper noise as one of the most ordinary impulse noise, appears as black dots in the bright part like ‘pepper’ and white dots in the dark part like ‘salt’ in an image, also known as shot noise or spike noise, that usually introduced by dead pixels in sensor cell malfunction, faulty memory locations in hardware, or digital converter errors, etc [2, 3]. For an observation image whose pixels valued in the range of minimum \(a\) to maximum \(b\), its degradation model which corrupted by \(P/2\) ‘pepper’ pixels and \(P/2\) ‘salt’ pixels selected randomly can be formulated as,

\[
y_i = \begin{cases} 
  a, & \text{with probability } P/2 \\
  b, & \text{with probability } P/2 \\
  x_i, & \text{with probability } 1 - P 
\end{cases}
\]

where \(y_i\) and \(x_i\) is the \(i\)th pixel of the noisy image and the clean image, respectively.

We can observe that only part of pixels are corrupted by salt-and-pepper noise, while the rest remain as they are. Then, how to recover the noised pixels without affecting the clean pixels should be one of the most challenging problems. The non-linear filtering technique is widely used in salt-and-pepper noise removal due to their good performances and low computational complexity. The median filter [1] as the well-known non-linear noise removal and smoothing operator, has a poor performance when the density of the salt-and-pepper noise is higher than 20% [2]. For this reason, the weighted median filter (WMF)
and the center weighted median filter (CWMF) [5, 6] weight part of pixels inside the filtering window. However, these filters process every pixel without checking whether it is an impulse noise or not, leading to the image distortion and the loss of details and edges. The adaptive median filter (AMF) [7, 8] can handle impulse noise with higher probability by increasing the filtering window size during filter operation depending on whether the output at every pixel is an impulse noise or not.

Further, some improved versions are developed by detecting the corrupted pixels and then performing image restoration with appropriate filtering approaches. The decision-based algorithm (DBA) [9] removes and replaces the selected noisy pixel with the median from its corresponding neighborhood. It is an effective way to remove salt-and-pepper noise with high density, but usually at the expense of losing details in fine texture and causing blurring in output images. To better preserve image edges, the implementation of decision-based algorithm (IDBA) and the probabilistic decision-based filter (PDBF) are proposed to further improve the performance of impulse noise detection [10, 11]. However, those filter-based methods are still necessary to make further improvements for the image detail preservation, and improve the robustness in image restoration especially when there are some other types of noise contained.

Owing to the increasing popularity of image non-local self-similarity (NSS) in recent years, many important models and methods have emerged to further improve the performance of image denoising problems, such as BM3D [12, 13] and the state-of-the-art methods such as LSSC [14] and NCSR [15], etc. The NSS prior is based on the observation that for any reference patch from one natural image, there are many other patches similar to it across the image, which called ‘patch redundancy’. Even though one cannot visually perceive any obvious repetitive structure, such patch repetitions occur abundantly in natural images, this is due to the fact that very small patches often contain only an edge, a corner, etc [16]. The main idea of these NSS-based methods is to utilize the patch redundancy as a prior in the restoration process to reconstruct image self-similar structures.
Intuitively, for a clean natural image, by vectoring its non-local similar patches into a matrix form, the matrix should be approximate to be low rank and lie in a low-dimensional subspace [17, 18], since all the matched patches represent the same scene in each column vector. However, as for the observation image corrupted by sparse noise, its non-local self-similarity matrix may lose the low rank property, because the sparse noise destroys the similarity between column vectors. Based on this, the recovery of its low rank structures from the noisy observation data can be formulated as the robust principal component analysis (RPCA) problem in [17, 19, 20], which aims to estimate a low rank image matrix and a sparse noise matrix from one corrupted NSS matrix. The RPCA method is successfully applied in the signal reconstruction, matrix completion, image restoration and many other fields [21, 22, 23]. There is an remarkable performance obtained by the patch-based robust video restoration method [24], through grouping similar patches in a spatial-temporal domain and then formulating the restoration problem as a joint sparse and low rank matrix approximation.

The original RPCA model is a NP-hard problem and usually solved by its convex approximations. Many efficient solution algorithms are proposed for RPCA such as singular value thresholding (SVT) [25], accelerated proximal gradient (APG) algorithm [26], fast alternating linearization method, exact and inexact augmented Lagrangian multiplier (ALM) method etc [27, 28, 29, 30, 31]. Although these convex optimization algorithms have been used for a wide variety of tasks across different fields, they are still fail to recover the heavily polluted data matrix well, since the recovery capability is strictly bounded by the rank inherent in data matrix and/or the corrupted error density [20].

In [32], Candés et al., propose an iterative reweighted minimization algorithm to find the local minimum of the non-convex $\ell_0$-norm minimization penalty function that more closely resembles a democratic penalization, showing significant performance improvements on boosting the recoverable sparsity thresholds for certain types of signals. The weighted nuclear norm minimization proposed in [33] assigns different weights in a non-descending order, according to the dif-
ferent importance of rank components. Peng et al., propose the dual weighted model \cite{34} as a feasible extension way to improve the performance of RPCA, which integrates the weighted low rank minimization problem and the weighted sparse minimization problem into an iterative method. Moreover, the rigorous theoretical analysis for these weighted methods is provided, based on a high-dimensional geometrical analysis (Grassmann angle analysis) of the nullspace characterization, which demonstrates the iterative reweighted $\ell_1$-norm minimization can indeed deliver recoverable sparsity thresholds that allows to recover the low rank structures from the observations with more serious noise interference. Some differences and similarities between the $\ell_1$-norm minimization and the weighted $\ell_1$-norm minimization are analyzed in \cite{35}. Recently, the Schatten $\ell_p$-norm attracts the attention of many researchers because of its weakly restricted isometry properties \cite{32,33,36,37}. In \cite{36} and \cite{37}, the weighted Schatten $\ell_p$-norm minimization (WSNM) and its RPCA-based model is proposed to provide a good performance in video background subtraction and hyperspectral image restoration problems.

The contributions of this paper are summarized as follows: (i) we generalize the reweighted $\ell_p$-norm with the power $p$ valued in the range of $[0,1]$, yielding a better approximation of the original sparsity recovery problem; (ii) we propose an unified RPCA-based framework, called the dual reweighted $\ell_p$-norm (DWLP) model, which greatly integrates the weighted methods and the $\ell_p$-norm schemes to enhance the sparsity and the low rank simultaneously for matrix recovery; and (iii) the iterative reweighted algorithm is introduced to obtain the optimum solution of the proposed DWLP model, which provides a more accurate estimation of the joined weights. In this paper, we apply the DWLP method to remove the salt-and-pepper noise and provide extensive experimental results to demonstrate its state-of-the-art performance in view of both the quantitative evaluation and the subjective visual quality.

The reminder of this paper is organized as follows: Section 2 reviews the related works on low rank and sparse matrix recovery problems and the evolutions of the RPCA model. Section 3 presents the dual reweighted $\ell_p$-norm
(DWLP) minimization model and its effective iterative solution. Section 4 describes the application of the proposed DWLP model to salt-and-pepper noise removal based on the non-local self-similarity property in nature images, and validates its improvements of DWLP over various variational RPCA-based models. Section 5 illustrates experimental results and discusses the influence of key parameters on results. Finally, concluding remarks are given in Section 6.

2. Related Work

2.1. $\ell_0$-Norm Sparsity Recovery

Sparsity as one of the most common prior knowledge widely exists in observations, which defined as many or most elements in a matrix being zero. For an observation data $Y \in \mathbb{R}^{m \times n}$, the sparse recovery problem is to find a sparse matrix $X$, in which the sparse matrix $X$ is close to $Y$ under the $F$-norm fidelity term and can be enforced by the $\ell_0$-norm regularization term that is defined as follows,

$$\hat{X} = \arg \min_X \|X - Y\|_F^2 + \beta \|X\|_0$$  \hspace{1cm} (2)

where $\beta > 0$ is the trade-off parameter to balance the data fidelity induced by $\|.\|_F^2$ and the sparse regularization term by $\|.\|_0$.

2.1.1. $\ell_1$-Norm Minimization

Although the $\ell_0$-norm minimization model obtains an accurate description for sparse structure recovery, it is a highly non-convex problem and requires an intractable combinatorial search for solution [34]. Therefore, the $\ell_1$-norm as one of the tightest tractable relaxation of the $\ell_0$-norm, is adopted to formulate the $\ell_1$-norm minimization problem [35] as,

$$\hat{X} = \arg \min_X \|X - Y\|_F^2 + \beta \|X\|_1$$  \hspace{1cm} (3)

where $\beta > 0$ is the trade-off parameter. It is widely used to recover sparse structures and appeared in many sparse prior-based optimization problems as a sub-problem (e.g., robust principle component analysis (RPCA) [17, 19], sparse coding [38]). In general, the optimum solution of Eq. (3) can be obtained through
the soft-thresholding operation, i.e., \( \hat{x}_i = S_\beta(y_i, \beta) = \max(|y_i| - \beta, 0) \text{sgn}(y_i) \) where \( \hat{x}_i \) represents the recovered sparsest result of the \( i \)th sample \( y_i \).

2.1.2. Reweighted \( \ell_1 \)-Norm Minimization

The \( \ell_1 \)-norm minimization is not able to provide a perfect solution, for the convex relaxation problem is not equivalent to the original \( \ell_0 \)-norm problem [39]. In fact their solutions are supposed to be equal with high probability under some choice of the trade-off parameter \( \beta \), instead of exactly the same [21]. The original \( \ell_0 \)-norm minimization provides a more democratic penalization depending on the magnitude of elements while the \( \ell_1 \)-norm minimization shrinks each element with the same \( \beta \) as indicated in the soft-thresholding operation, leading to an inaccurate estimation of the location of non-zero elements in the recovered sparse matrix and thus failing to reconstruct sparse matrices exactly [34].

To address this imbalance, a more effective regularization norm is proposed to further improve the performance of the \( \ell_1 \)-norm minimization, called the weighted \( \ell_1 \)-norm [32, 34, 40], defined as \( \|W \odot X\|_1 = \sum_{i=1}^{mn} w_i |x_i| \), where \( \odot \) is the Hadamard product, and \( w_i \) denotes the weights assigned inversely proportional to the \( i \)th element \( x_i \), i.e., \( w_i = \frac{1}{|x_i| + \epsilon} \), and \( \epsilon \) is a very small constant to avoid the denominator being zero. This regularization term supplies a more reasonable weighting rule to the optimization problem, which enables different entries to make different contributions for the regularization term by discouraging the small but nonzero entries by large weights and encouraging large entries by small weights. Here, the weighted \( \ell_1 \)-norm minimization problem can be defined as follows,

\[
\hat{X} = \arg \min_X \|X - Y\|_F^2 + \beta \|W \odot X\|_1
\]

(4)

where \( \beta > 0 \) is the trade-off parameter.

The weighted \( \ell_1 \)-norm minimization has made an impressively quantitative improvement for recovering sparse signals [32], since the weights greatly generalize the typical \( \ell_1 \)-norm regularization term and represent a more democratic case similar to the original \( \ell_0 \)-norm. Moreover, there is an iterative reweighted algorithm [34, 41] which allows to further improve the accuracy of sparse signal
recovery, and the 're' means to estimate the elements and the weights alternatively.

2.2. Low Rank Recovery

For a low rank matrix \( X \in \mathbb{R}^{m \times n} \), its rank (defined as the number of non-zero singular values) is much less than the number of rows or columns, i.e., \( \text{rank}(X) \ll \min(m, n) \), which means the vector of the singular values is sparse. Suppose that \( Y \in \mathbb{R}^{m \times n} \) is the observation data, and \( X \) denotes its underlying low rank matrix. The matrix rank minimization problem aims to recover the underlying low rank structure from \( Y \) denoted as the matrix \( \hat{X} \), which is formulated as follows,

\[
\hat{X} = \arg \min_X \| X - Y \|_F^2 + \alpha \text{rank}(X)
\]

where \( \alpha > 0 \) is the trade-off parameter between the low rank regularization term and the fidelity term.

The matrix rank minimization problem designed to capture the intrinsic rank of the observation data, can be exactly formulated as the rank function, which is a highly non-convex and non-linear problem without efficient solution.

2.2.1. Nuclear Norm Minimization

Since the rank minimization problem cannot be solved directly, the rank function is usually replaced by its tightest relaxation—the nuclear norm, also called the trace norm or the Schatten \( \ell_1 \)-norm \([35, 38, 42]\) and the nuclear norm minimization (NNM) problem \([17, 18]\) can be formulated as,

\[
\hat{X} = \arg \min_X \| X - Y \|_F^2 + \alpha \| X \|_*
\]

where \( \alpha > 0 \) is the trade-off parameter, the nuclear norm regularization term of matrix \( X \) is defined as the sum of its singular values, i.e., \( \| X \|_* = \sum_{i=1}^{r} \sigma_i(X) \) and \( \sigma_i(X) \) is the \( i \)th singular value of \( X \), obtained by \( X = U \Sigma V^T \), \( \Sigma = \text{diag}(\sigma_1(X), \sigma_2(X), \ldots, \sigma_r(X)) \), \( r = \min(m, n) \). Its optimum solution can be effectively solved by the singular value thresholding (SVT) operation \([25]\) as
\( \hat{X} = U S_α(\Sigma, \alpha) V^T \), in which \( \Sigma \) denotes the singular value diagonal matrix of \( Y = U \Sigma V^T \), and \( S_α(\Sigma, \alpha) \) represents the soft-thresholding function of \( \Sigma \) with parameter \( \alpha \), defined as \( S_α(\Sigma, \alpha)_{ii} = \max(\Sigma_{ii} - \alpha, 0) \) for each diagonal element \( \Sigma_{ii} \).

### 2.2.2. Weighted Nuclear Norm Minimization

Unlike the original non-convex matrix rank minimization problem with a democratic penalization, the NNM suffers from the overshrink problem since it shrinks different components equally with the same value of \( \alpha \) in \( S_α(\Sigma, \alpha) \). However, this greatly restricts its capability and flexibility in practice since the singular values have clear physical meanings and should be treated differently.

To overcome the shortcoming of the NNM and better approximate the rank function, the truncated nuclear norm regularization term (TNNR) \([43]\) and the partial sum minimization (PSM) \([44]\) have been proposed to protect major components by keeping \( k \) largest singular values as they are and minimizing the remaining small singular values merely. However, it is hard to estimate the intrinsic rank determined by the matrix content, and there is only a binary decision to determine whether to regularize a specific singular value or not, let alone to treat each component differently. Thus, both TNNR and PSM are not flexible enough to incorporate the prior knowledge of different singular values. To solve this problem, a more reasonable variant of the NNM problems presented by Gu et al assigns different weights to different singular values and guarantees a more accurate recovery, called the weighted nuclear norm minimization (WNNM) \([33]\), which is defined as,

\[
\hat{X} = \arg \min_X \|X - Y\|_F^2 + \theta \|X\|_{w,*} \tag{7}
\]

where \( \theta > 0 \) is the trade-off parameter, \( \|X\|_{w,*} = \sum_{i=1}^{r} w_i \sigma_i(X) \) in which \( w_i = \frac{1}{\sigma_i(X)} + \epsilon \) denotes the weight of the \( i \)th singular value \( \sigma_i(X) \). In this way, the larger singular values could be penalized less than the smaller ones since they are generally associated with major projection orientations, and shrunk less to better preserve major data components.
Based on the WNNM, the weighted Schatten $\ell_p$-norm (WSNM) \cite{36} has been proposed to further improve the performance through the Schatten $\ell_p$-norm ($0 < p \leq 1$) in the low rank matrix recovery problem. For the observation matrix $Y$, the WSNM problem can be formulated as,

$$\hat{X} = \arg \min_X \| X - Y \|_F^2 + \theta \| X \|_{w,S}^p$$  \hspace{1cm} (8)

where $\theta > 0$ denotes the trade-off parameter, and the regularization term is defined as $\| X \|_{w,S}^p = \sum_{i=1}^r w_i \sigma_i(X)^p$ with power $0 < p \leq 1$ and weight $w_i = \frac{1}{\sigma_i(X)^p + \epsilon}$. Note that the weights are guaranteed to be in a non-descending order, since the singular values $\sigma_i(X)$ are always sorted in a non-ascending order \cite{33,36}.

The weighted Schatten $\ell_p$-norm provides a more feasible scheme to simulate the rank function through shrinking each singular value depending on its magnitude and combing with a weak restriction supported by setting the power $0 < p \leq 1$, that greatly improves the flexibility in many applications, e.g., image restoration \cite{45,46}. When the weights satisfy a non-descending order, the optimum solution of WSNM can be achieved by transforming Eq. (8) into a series of independent non-convex $\ell_p$-norm subproblems, which can be directly solved by the iterative singular value GST algorithm (refer to \cite{40} for more details).

2.3. Robust Principle Component Analysis

The RPCA \cite{17,19,20} model is proposed based on the classical principal component analysis (PCA). The classical PCA model remains optimal in the low rank matrix recovery problem when the noise is subject to Gaussian distribution, but non-Gaussian noise, even a single gross error, could deviate the estimation far from its ground truth. Thus, the RPCA is modeled as a more robust optimization problem combined with the sparse regularization term, and transforms the recovery problem into a joint low rank and sparse matrix approximation.

Suppose that a corrupted observation data $D \in \mathbb{R}^{m \times n}$ is composed of a low rank matrix $A \in \mathbb{R}^{m \times n}$ and an error matrix $E \in \mathbb{R}^{m \times n}$ (assumed to be sparse
with arbitrarily large magnitude), i.e., \( D = A + E \). Here, the robust principle component analysis model is defined as:

\[
\min_{A, E} \text{rank}(A) + \lambda \|E\|_0 \quad \text{s.t. } A + E = D
\]

where parameter \( \lambda > 0 \) is the trade-off parameter between enforcing the low rank property and separating the sparse error appropriately. The low rank regularization term is induced by the rank function, i.e., the number of non-zero singular values in \( A \), and the sparsity penalty of matrix \( E \) by \( \ell_0 \)-norm, i.e., the number of non-zero entries in \( E \).

Unfortunately, both \( \text{rank}(A) \) and \( \|E\|_0 \) are highly non-convex and non-linear problems that cannot be solved directly. Thus, under rather weak assumptions, the RPCA problem can usually be approximated by the principal component pursuit (PCP) problem \cite{17,19,42}, which defined as follows:

\[
\min_{A, E} \|A\|_* + \lambda \|E\|_1 \quad \text{s.t. } A + E = D
\]

where \( \lambda > 0 \) is the trade-off parameter, \( \|A\|_* \) is the nuclear norm of matrix \( A \), the tightest convex relaxation of \( \text{rank}(A) \), defined as the sum of its singular values, i.e., \( \|A\|_* = \sum_{i=1}^{r} \sigma_i(A) \), where \( \sigma_i(A) \) is the \( i \)-th singular value of \( A \); and \( \|E\|_1 \) is the \( \ell_1 \)-norm of matrix \( E \) that replaces \( \|E\|_0 \) as its tightest convex relaxation, \( \|E\|_1 = \sum_{i=1}^{mn} |e_i| \), where \( e_i \) means the \( i \)-th element in \( E \).

It has been proved that if the PCP satisfies the following conditions: (i) \( \lambda = O \left( \frac{1}{\sqrt{t_{(1)}}} \right) \), \( t_{(1)} = \min(m,n) \); (ii) \( \text{rank}(A) \leq O \left( \frac{t_{(2)}}{\log(t_{(1)})} \right) \), \( t_{(2)} = \max(m,n) \); and (iii) \( \|E\|_0 \leq O(mn) \), it would be able to estimate the unique \( A \) and \( E \) exactly from the observation data \( D \) with a high probability more than \( 1 - O \left( t_{(1)}^{-10} \right) \) \cite{17}.

Note that, either the \( \ell_1 \)-norm minimization problem (convex approximation of the \( \ell_0 \)-norm minimization problem) to recover the sparse matrix, or the nuclear norm minimization problem (convex approximation of the rank function minimization problem) to recover the low rank matrix, can be regarded as the subproblem included in the process of solving the PCP model and involved to better approximate the RPCA problem. Specifically, taking the WSNM model
mentioned in Section 2.2.2 as the low rank regularization term, the WSNM-RPCA method could give a better approximation to the original low rank assumption through the weighted nuclear norm and the weaker power $0 < p \leq 1$, which is given as follows,

$$\min_{A,E} \|A\|_{w,S}^p + \lambda \|E\|_1 \quad \text{s.t.} \quad A + E = D$$

(11)

where $\lambda > 0$ is the trade-off parameter, and weight $w_i$ is defined as $w_i = \frac{1}{\sigma_i(A)^p + \epsilon}$ in which is the $i$th singular value of $A$.

In addition to the low rank regularization term, another important factor that affects performance of the RPCA should be the sparse regularization term. Peng et al., propose the weighted low rank matrix recovery scheme to enhance the low rank constraint penalty and the sparse constraint penalty simultaneously and greatly improve the performance of the low rank matrix recovery problem [34]. Here, the model is defined as,

$$\min_{A,E} \|A\|_{\Omega^*,\ast} + \lambda \|W \odot E\|_1 \quad \text{s.t.} \quad D = A + E$$

(12)

where $\lambda > 0$ is the trade-off parameter, $\|A\|_{\Omega^*,\ast} = \sum_{i=1}^{r} \omega_i \sigma_i(A)$ denotes the low rank penalty with diagonal weight matrix $\Omega = \text{diag}(\omega_1, \omega_2, \cdots, \omega_r)$ and the weight assigned to the $i$th singular value being $\omega_i = \frac{1}{\sigma_i(A)^p + \epsilon}$, and $\|W \odot E\|_1 = \sum_{i=1}^{mn} w_i |e_i|$ is the sparse constraint with the weight of the $i$th element being $w_i = \frac{1}{|e_i| + \epsilon}$.

3. Dual Reweighted $\ell_p$-Norm Model

Note that, the matrix sparsity defined as only a few non-zero elements contained, can be used to reconstruct a sparse matrix, as well as a low rank matrix by regularizing the vector of the singular values to be sparse [47]. The matrix rank function can be represented as the $\ell_0$-norm of its singular values, i.e., $\text{rank}(X) = \|\sigma(X)\|_0$. Therefore, the overshrink problem of both the $\ell_1$-norm minimization and the nuclear norm minimization can be attributed to the difference between the $\ell_1$-norm and the $\ell_0$-norm, and filling the gap between them would help us to better solve the overshrink problem in the approximation.
3.1. Reweighted $\ell_p$-Norm Minimization

The $\ell_p$-norm is perceived as the general case of $\ell_1$-norm with the power $p$ valued in the range of $(0,1]$, provides a better performance owing to its weaker restricted isometry property for approximating the $\ell_0$-norm \cite{46, 36, 37}. Therefore, we generalize the weighted $\ell_1$-norm minimization described in Section 2.1.2 as the following weighted $\ell_p$-norm minimization formulated in Eq. (13), which provides a feasible solution to fill the gap between the $\ell_0$-norm and the $\ell_1$-norm,

$$\hat{X} = \arg \min_X \|X - Y\|_F^2 + \varphi \|W \odot X\|_q^q$$  \hspace{1cm} (13)

where $\varphi > 0$ is the trade-off parameter. The sparse regularization term is defined as $\|W \odot X\|_q^q = \sum_{i=1}^{mn} w_i |x_i|^q$, in which the weight for the $i$th element is $w_i = \frac{1}{|x_i|^{q+\epsilon}}$, with power $0 < q \leq 1$ and small constant $\epsilon$ to avoid a zero denominator.

The widely used $\ell_1$-norm can be represented as a special case of the weighted $\ell_p$-norm by setting both the weights $w_i$ and the power $q$ to 1, i.e., $\|W \odot X\|_q^q = \sum_{i=1}^{mn} w_i |x_i|^q = \sum_{i=1}^{mn} |x_i| = \|X\|_1$, while $w_i = 1$ and $q = 1$.

3.2. Our DWLP Model

Here, we incorporate the weighted Schatten $\ell_p$-norm of the low rank matrix and the weighted $\ell_p$-norm model of the sparse matrix into the RPCA problem to form the DWLP optimization, which is given as follows,

$$\min_{A,E} \|A\|_{\Omega,S_p}^p + \lambda \|W \odot E\|_q^q \quad \text{s.t.} \quad D = A + E$$ \hspace{1cm} (14)

where $\lambda > 0$ is the trade-off parameter, $\|A\|_{\Omega,S_p}^p = \sum_{i=1}^{r} \omega_i \sigma_i(A)^p$ denotes the low rank penalty with diagonal weight matrix $\Omega = \text{diag} \left( \omega_1, \omega_2, \ldots, \omega_r \right)$, and the weight assigned to the $i$th singular value being $\omega_i = \frac{1}{\sigma_i(A)^{p+\epsilon}}$, and $\|W \odot E\|_q^q = \sum_{i=1}^{mn} w_i |e_i|^q$ is the sparse constraint, in which the weight of the $i$th element is $w_i = \frac{1}{|e_i|^{q+\epsilon}}$ with power $0 < q \leq 1$.

Our DWLP model reduces to the original RPCA model when the weights $\omega_i, w_i$ are equal to 1, and the powers $p, q$ are equal to 0 where $0^0=0$. Moreover,
the WSNM-RPCA model in [36, 37] can be considered as a special case of the proposed DWLP model that only enhances the low rank property, while the low rank matrix recovery model in [34] is the special case of the proposed DWLP with powers $p=q=1$ (denoted as DWLP($p=q=1$) in the following). For the reasonable weighting rule and weak power contained, the DWLP method could provide a more accurate estimation of low rank and sparse structures, which obtains a significant improvement over the widely used PCP to solve the low rank and sparse matrix recovery problem.

3.3. Solution

Now, let us focus on Eq. (14) and invoke the augmented Lagrange multiplier (ALM) algorithm [48] to approximate its optimum solution alternatively. We first translate Eq. (14) into its Lagrange form defined as follows,

$$L(A, E, Z, \mu) = \lambda_a \|A\|_{\Omega, S_p}^p + \lambda_e \|W \odot E\|_q^q + \frac{\mu}{2} \|D - A - E\|_F^2$$

(15)

where $\lambda_a$, $\lambda_e$ denote the non-negative trade-off parameters, $\langle \cdot , \cdot \rangle$ is the inner product, and $\mu$ and $Z$ represent the Lagrange multiplier and the augmented Lagrange multiplier respectively. Further, although the parameter $\lambda_a$ can be absorbed into $\lambda_e$, the low rank regularization term and the sparsity regularization term are solved separately during the procedure of the iterative algorithm [37]. Thus, we prefer to maintain these two parameters for more flexibility. Then, we alternatively update the low rank matrix $A$, the sparse matrix $E$, the augmented Lagrange multiplier $Z$ and the Lagrange multiplier $\mu$. The process of solving the Lagrange function of DWLP is summarized in Algorithm 1 where $k$ counts the iteration.
Algorithm 1 ALM for DWLP-RPCA

Input: Observed data $D$, low-rank power $p$, sparse power $q$, and trade-off parameters $\lambda_a, \lambda_e$;

1: Initialize: $\mu_0 > 0$, $\rho > 0$, $k=0$, $A=D$, $Y=0$;
2: while not converged do
3: \[ E_{k+1} = \arg \min_E \lambda_e \| W \odot E \|_q + \frac{\mu_k}{2} \| D + \mu_k^{-1}Z_k - A_k - E \|_F^2; \]
4: \[ A_{k+1} = \arg \min_A \lambda_a \| A \|_{\Omega_S^p}^p + \frac{\mu_k}{2} \| D + \mu_k^{-1}Z_k - E_{k+1} - A \|_F^2; \]
5: \[ Z_{k+1} = Z_k + \mu_k (D - A_{k+1} - E_{k+1}); \]
6: \[ \mu_{k+1} = \rho \times \mu_k; \]
7: \[ k = k + 1; \]
8: end while

Output: $\hat{A} = A_{k+1}$ and $\hat{E} = E_{k+1}$

The minimization of Eq. (15) involves two minimization subproblems i.e., $E$ and $A$ subproblem. Next, we will present their solutions to the subproblems of $E$ and $A$ as follows,

1) $E$ subproblem: Given the weight matrix $W$, the power $q$ and the trade-off parameter $\lambda_e$, the $E$ subproblem for each observation data $D$ is,

\[ E_{k+1} = \arg \min_E \lambda_e \| W \odot E \|_q + \frac{\mu_k}{2} \| D + \mu_k^{-1}Z_k - A_k - E \|_F^2; \]  

(16)

this is the Lagrange function of the weighted $\ell_p$-norm minimization stated in Section [3.1] which as the $E$ subproblem aims to estimate the sparsest solution of DWLP that can be simplified as the following equation,

\[ E_{k+1} = \arg \min_E \frac{1}{2} \| E - F \|_F^2 + \| \Phi \odot E \|_q^2 \]  

(17)

where $F = D + \mu_k^{-1}Z_k - A_k$ and $\Phi = \lambda_e \mu_k^{-1}W$.

Unfortunately, the Eq. (17) cannot be solved by the traditional soft-thresholding operation, since the added weights destroy the convexity, and the included $\ell_p$-norm makes the non-convex relaxation problem much more difficult to be optimized. Thus, we decompose the problem into $m \times n$ independent $\ell_p$-norm subproblems to reduce the challenge and obtain the global optimum, and give.
the $i$th subproblem as follows,

$$
\hat{e}_i = \arg \min_{e_i} \frac{1}{2} (e_i - f_i)^2 + \phi_i |e_i|^q \tag{18}
$$

where $e_i$ denotes the $i$th element in $E_{k+1}$, $f_i$ is the $i$th elements in $F$, i.e., $f_i = d_i + \mu^{-1} z_i - a_i$ and $\phi_i$ is the $i$th elements in $\Phi$, i.e., $\phi_i = \lambda \mu^{-1} w_i$. To avoid confusion, the subscript $k$ of $\mu_k$ is omitted for conciseness.

According to [41], Eq. (18) can be directly solved by the generalized soft-thresholding algorithm (GST) summarized in Algorithm 2.

**Algorithm 2 Generalized Soft-Thresholding (GST)**

**Input:** $f_i$, parament $\phi_i$, power $q$;

1: Initialize: $J = 2$ or 3;
2: $\tau^\text{GST}_{q}(\phi_i) = (2\phi_i(1 - q))^{\frac{1}{2-q}} + \phi_i q(2\phi_i(1 - q))^{\frac{q-1}{2-q}}$;
3: if $|f_i| \leq \tau^\text{GST}_{q}(\phi_i)$ then
4: $S^\text{GST}_{q}(f_i; \phi_i) = 0$;
5: else
6: $j = 0, e^{(j)} = |f_i|$;
7: Iterate on $j = 0, 1, \cdots, J$
8: $e^{(j+1)} = |f_i| - \phi_i q(e^{(j)})^{q-1}$;
9: $j \leftarrow j + 1$;
10: $S^\text{GST}_{q}(f_i; \phi_i) = \text{sgn}(f_i)e^{(j)}$;
end if

**Output:** $\hat{e}_i = S^\text{GST}_{q}(f_i; \phi_i)$

However, there is still a remaining problem, that is, the weights $w_i$ are not available without knowing $e_i$ at first. To solve this problem, an iterative reweighted GST algorithm (IRGA) has been proposed [40] to optimize each element $e_i$ in $E_{k+1}$ by Eq. (19) and update its weight $w_i$ by Eq. (20) in turn.

$$
e^{(l)}_i = \text{GST}(f_i, \lambda_e w^{(l)}_i \mu^{-1}, q) \tag{19}
$$

$$
w^{(l+1)}_i = \frac{1}{|e^{(l)}_i|^q + \epsilon} \tag{20}
$$
where $l$ counts the reweighting iteration for the sparse minimization.

We summarize the complete IRGA in Algorithm 3 where in each inner iteration, $e_i$ is calculated with the current weight $w_i$, and then update $w_i$ according to the current $e_i$. Even though the early iterations may find inaccurate estimates, the largest values are most likely to be identified as non-zeros to better estimate the non-zero elements successively. As the number of iterations increases, the effect of weights will become more sensitive during the iteration, so as to have a better prediction for the rest small but non-zero elements [32].

**Algorithm 3 Iterative Reweighted GST Algorithm (IRGA)**

**Input:** $f_i$, parameter $\lambda \mu^{-1}$, power $q$;

1. Initialize: $w_i^{(0)} = 1$, $L_s > 0$;

2. for $l = 0 : L_s$ do

   3. $e_i^{(l)} = \text{GST}(f_i, \lambda \mu^{-1} w_i^{(l)}, q)$;

   4. $w_i^{(l+1)} = \frac{1}{|e_i^{(l)}| + \epsilon}$;

3. end for

**Output:** $\hat{e}_i = e_i^{(L_s)}$

2) A subproblem: Given the weight matrix $\Omega$, the power $p$ and the parameter $\lambda$, the A subproblem for each observation data $D$ can be written as,

$$A_{k+1} = \arg \min_A \lambda ||A||_p^p \Omega, S_p + \frac{\mu}{2} ||D + \mu^{-1} Z_k - E_{k+1} - A||_F^2. \tag{21}$$

This is the Lagrange function of the weighted Schatten $\ell_p$-norm minimization stated in Section 2.2.2 before. In the non-descending weight order, the WSNM can be effectively solved by the GST algorithm as described in the following theorem.

**Theorem 1 [36]:** Let $D + \mu^{-1} Z_k - E_{k+1} = U \Sigma V^T$, $\Sigma = \text{diag} \left( \sigma_1, \sigma_2, \ldots, \sigma_r \right)$. Suppose that all singular values are in non-ascending order and all the weights satisfy $0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_r$, the optimum solution Eq. (21) will be $A = U \Delta V^T$, with $\Delta = \text{diag} \left( \delta_1, \delta_2, \ldots, \delta_r \right)$, where the $r$th independent sub-
problem of \( \delta_i \) is given in the following,

\[
\hat{\delta}_i = \arg\min_{\delta_i} (\delta_i - \sigma_i)^2 + \lambda_a \mu^{-1} \omega_i \delta_i^p
\]  

(22)

In order to further improve the accuracy of the low rank matrix recovery model, we adopt the more efficient iterative reweighted GST algorithm (IRGA) as \( \delta_i = \text{IRGA}(\sigma_i, \lambda_a \mu^{-1}, p) \) to estimate each singular value in \( \Delta \) by Eq. (23) and its weight \( \omega_i \) by Eq. (24) alternatively.

\[
\delta_i^{(l)} = \text{GST}(\sigma_i, \lambda_a \mu^{-1} \omega_i^{(l)}, p)
\]

(23)

\[
\omega_i^{(l+1)} = \frac{1}{|\delta_i^{(l)}|_p + \epsilon}
\]

(24)

where \( l \) counts the reweighting iteration for the low rank minimization.

We have shown the process of solving the weighted Schatten \( \ell_p \)-norm minimization by IRGA in Algorithm 4. Note that the optimum solution of each subproblem would be in a non-ascending order so as to preserve the major components, since the smaller weights could guarantee that components with larger singular values shrunk less than smaller ones [36, 37].

**Algorithm 4 Weighted Schatten \( \ell_p \)-norm Minimization via IRGA**

**Input:** \( \Psi = \mathbf{D} + \mu^{-1} \mathbf{Z}_k - \mathbf{E}_{k+1} \), power \( p \), parameters \( \lambda_a \) and \( \mu^{-1} \);

1: \( \Psi = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^\top \), \( \mathbf{\Sigma} = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r) \);

2: \( \Delta = \text{IRGA}(\mathbf{\Sigma}, \lambda_a \mu^{-1}, p) \);

3: \( \mathbf{A} = \mathbf{U} \Delta \mathbf{V}^\top \), \( \Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_r) \);

**Output:** Matrix \( \mathbf{A} \)

Finally, terminate on convergence or when IRGA attains a specified maximum number of iterations.

4. Non-local-based DWLP for salt-and-pepper noise Removal

Many non-local-based low rank models have received much attention in the field of image restoration [3, 16, 24] in recent years. In this part, we apply
the DWLP to remove salt-and-pepper noise by employing image non-local self-similarity.

4.1. General Process

In general, the low rank and sparse matrix recovery problems combined with the non-local method for image restoration include three main steps as follows: (i) Match and group patches to obtain the NSS matrix, (ii) Decompose the NNS matrix into one low rank matrix and one sparse matrix, and (iii) Aggregate these patches into the reconstructed image.

Following [24], we search for similar patches in the prefiltered images obtained by smoothing the noisy image with a median filter, so as to produce a much more accurate patch matching result than directly in corrupted images. This is because the performance of patch matching will seriously degrade in the presence of severe salt-and-pepper noise. It is noted that the prefiltered data is only used for patch matching, while the observation data is used as the input of the denoising process.

Next, for each patch \( d_i \in \mathbb{R}^c \) vectorized in lexicographic ordering, we extract its corresponding prefiltered patch according to its index location and search for \( K - 1 \) similar patches of this prefiltered patch across the prefiltered image based on Euclidean distance. Letting \( \mathcal{S}_i \) denotes the collection of these \( K \) prefiltered patches from the noisy image at index location \( j \in \mathcal{S}_i \) are grouped into a matrix \( \mathbf{D}_i = \{d_{i,1}, d_{i,2}, \ldots, d_{i,K}\} \in \mathbb{R}^{c \times K} \), and the matrix consisting of all the patches with similar structures is called a NNS matrix, where \( d_{i,K} \) represents the \( K-1 \)th similar patch of the \( i \)th NNS matrix. In our case, \( \mathbf{D}_i \) can be considered as a combination of the original image NSS matrix \( \mathbf{A}_i \) and the noise matrix \( \mathbf{E}_i \), respectively, i.e., \( \mathbf{D}_i = \mathbf{A}_i + \mathbf{E}_i \), which is usually simplified as \( \mathbf{D} = \mathbf{A} + \mathbf{E} \). Intuitively, \( \mathbf{A} \) is a low rank matrix and \( \mathbf{E} \) is a sparse matrix, and thus the low rank and sparse matrix approximation would be suitable for the problem that estimates \( \mathbf{A} \) and \( \mathbf{E} \) under the observation data \( \mathbf{D} \) unknown.

In this way, the salt-and-pepper noise removal problem can be done for each NSS matrix sequentially. In this non-local-based method, image patches
are always overlapped and most of the pixels are covered by several recovered patches. Thus, each estimated pixel of the final reconstructed image would be generated by taking the average of the recovered patches at each location. Such a kind of operation is referred to as the patch aggregation, which is used to suppress possible artifacts caused by block discontinuities in the neighborhood of the boundaries of patches [37].

4.2. Analysis

We randomly select two NSS matrices and compute their singular values from Fig. 1(a) shows a scene of streets, houses, and terraces in the distance, and Fig. 2(a) shows a portrait of a woman. One can observe from Figs. 1(b) and 2(b) that the singular values decay very fast, according the low rank structure of nature images. The NSS matrix can usually be projected into a lower dimensional space owing to the similarity between its column vectors. As shown in Figs. 1(c) and 2(c), we add 10% and 30% salt-and-pepper noise to these two images respectively, where the reference patch is marked by the red box and its similar patches are marked by green boxes. It is obvious to see from Figs. 1(d) and 2(d) that the NSS matrices extracted from the corrupted images are usually full rank. This is due to the fact that the added noise disturbs the similarity between the column vectors in the NNS matrix, and destroys its original low rank property.

The low rank property widely exists in natural images and can help us to recover the potential low-dimensional subspace of the NSS matrix. In addition, combining with the sparsity constraint of the salt-and-pepper noise, the salt-and-pepper noise removal problem can be solved as the joint low rank and sparse matrix approximation as a kind of prior knowledge. In order to demonstrate the superiority of the proposed DWLP model to the other non-local-based RPCA methods, we conduct a matrix rank singular value decomposition (SVD) experiment, which helps to observe low rank structures of NSS matrices and illustrate the overshrink problem visually.

In this experiment, we select 10 random NSS matrices composed of 64 sim-
Figure 1: Low rank analysis of a natural image. (a) The test image (Goldhill); (b) The singular value distribution of clean NSS matrix; (c) Noisy image with 10% salt-and-pepper noise; (d) The singular value distribution of noisy NSS matrix.
Figure 2: Low rank analysis of a natural image. (a) The test image (*Barbara*); (b) The singular value distribution of clean NSS matrix; (c) Noisy image with 30% salt-and-pepper noise; (d) The singular value distribution of noisy NSS matrix.
ilar patches of size $8 \times 8$ from noisy images shown in Figs. 1(c) and 2(c), and decompose each NSS matrix into a low rank matrix and a noise matrix through PCP [17], WNNM-RPCA [33, 37], WSNM-RPCA [37], DWLP ($p=q=1$) [34], and DWLP respectively. It can be observed from their singular value distributions plotted in Fig. 3 that the classical PCP (denoted by green line) deviates far from the ground truth (denoted by red line), meaning that the overshrinkage is serious. While the large singular values of the WNNM-RPCA method (denoted by blue line) are shrunk less to preserve the major data components with the shrinkage being proportional to the non-descending weights [33, 36, 37]. When the power $p$ less than 1, the non-zero singular values estimated by the WSNM-RPCA method are getting closer to those of the ground truth matrix (marked by cyan line), indicating the effectiveness of the weak power in recovery methods. However, the overshrink problem not only occurs in the low rank reconstruction but also in the sparse noise estimation, and thus the DWLP($p=q=1$) achieves a better assumption of the original low rank structure thanks to its dual weighted scheme (see the yellow line), while the proposed DWLP (denoted by magenta line) further improves the performance by integrating the weights and the $\ell_p$-norm into both the low rank and sparse regularization term.

![Figure 3: Singular value decomposition (SVD) for NSS matrices. (a) SVD results of NSS matrices sampled from Fig. 1(c) (b) SVD results of NSS matrices sampled from Fig. 2(c).](image-url)
We come to the conclusion through this experiment that our DWLP could better deal with the overshrink problem, and provide a more satisfactory solution of the low rank and sparse matrix approximation problem.

5. Experimental Results and Discussions

In this section, we validate the performance of the proposed DWLP method for image denoising under different salt-and-pepper noise levels and present both qualitative and quantitative comparisons with other state-of-the-art methods. To this end, we analyse the influence of several important parameters on the proposed DWLP method. All experiments were run in Matlab R2016a on a personal computer with an Intel Xeon E3-1245 v6 CPU with 3.7GHz and 16GB RAM.

5.1. Experiments on 10 Test Images

We evaluate the competing methods on 10 widely used test images of size 256 × 256 shown in Fig. 4. To quantitatively evaluate the performance of the proposed method, the salt-and-pepper noise with various probability densities is added to those test images to the noisy observations. Typically, the results are shown on three noise levels, ranging from low noise level $P=10\%$, to medium noise level $P=30\%$ and to high noise level $P=50\%$. Tables 1 and 2 show these peak signal to noise ratio (PSNR) and structural similarity (SSIM) results of our denoising method compared with the competing denoising methods.

Here, we have the following observations. First, the original PCP method has a poor performance when the image is heavily corrupted by salt-and-pepper noise, since the dense noise destroys sparse and low rank priors we illustrated in Section 2.3. Second, the proposed DWLP method achieves the highest PSNR and SSIM consistently on all the five noise levels. It achieves 7.03dB-9.26dB, 5.10dB-8.74dB, 4.75dB-9.40dB and 0.81dB-1.60dB improvements over PCP, WNNM-RPCA, WSNM-RPCA and DWLP $(p=q=1)$ under 10% salt-and-pepper noise level, and 1.57dB-4.45dB, 1.75dB-4.92dB, 1.63dB-4.72dB, and 0.1dB-1.34dB under 30% salt-and-pepper noise level, respectively. Third, as the noise
Table 1: Quantitative comparison of denoised results with different methods in terms of PSNR.

| Method     | PCP | WNNM-RPCA | WSNM-RPCA | DWLP | DWLP | PCP | WNNM-RPCA | WSNM-RPCA | DWLP | DWLP |
|------------|-----|-----------|-----------|------|------|-----|-----------|-----------|------|------|
|            |     |           |           |      |      |     |           |           |      |      |
| couple     | 27.046 | 27.291 | 28.084 | 33.195 | 34.083 | 26.342 | 26.031 | 26.294 | 30.227 | 31.150 |
| elaine     | 31.441 | 32.165 | 32.462 | 40.021 | 40.905 | 30.348 | 30.234 | 30.480 | 36.816 | 36.641 |
| flower     | 28.659 | 28.882 | 29.223 | 34.882 | 35.520 | 27.938 | 27.461 | 27.673 | 31.440 | 31.930 |
| goldhill   | 29.419 | 29.294 | 29.575 | 36.699 | 37.445 | 28.637 | 28.002 | 28.158 | 33.066 | 33.027 |
| house      | 32.004 | 33.223 | 33.415 | 38.609 | 39.427 | 30.292 | 31.366 | 31.432 | 35.882 | 36.298 |
| lin        | 29.190 | 30.761 | 31.004 | 35.408 | 36.220 | 28.605 | 29.512 | 29.538 | 32.940 | 33.120 |
| monarach   | 25.304 | 28.272 | 28.616 | 32.181 | 32.181 | 24.514 | 26.377 | 26.549 | 29.255 | 29.194 |
| pentagon   | 26.784 | 27.130 | 27.408 | 32.905 | 33.875 | 25.839 | 25.510 | 25.661 | 30.636 | 30.656 |
| plants     | 31.829 | 31.809 | 32.058 | 37.798 | 39.271 | 31.085 | 30.664 | 30.883 | 35.834 | 36.688 |
| tank       | 32.215 | 30.749 | 30.914 | 38.715 | 40.369 | 31.674 | 30.122 | 30.231 | 35.371 | 36.131 |
|            |     |           |           |      |      |     |           |           |      |      |
|            |     |           |           |      |      |     |           |           |      |      |
| couple     | 25.319 | 24.408 | 24.641 | 26.851 | 28.145 | 23.431 | 21.527 | 23.808 | 23.846 | 24.870 |
| elaine     | 28.923 | 28.443 | 28.652 | 32.538 | 33.372 | 25.538 | 22.739 | 27.030 | 28.039 | 29.669 |
| flower     | 27.260 | 26.284 | 26.433 | 28.241 | 28.834 | 25.782 | 24.913 | 25.600 | 25.555 | 26.140 |
| goldhill   | 25.570 | 26.536 | 26.687 | 29.475 | 29.677 | 24.787 | 22.840 | 25.921 | 25.807 | 26.999 |
| house      | 28.876 | 29.509 | 29.760 | 32.181 | 33.239 | 24.827 | 22.185 | 27.917 | 28.618 | 29.415 |
| lin        | 27.544 | 27.803 | 27.926 | 29.462 | 29.560 | 25.425 | 23.975 | 26.152 | 25.996 | 26.364 |
| monarach   | 22.929 | 24.269 | 24.468 | 26.215 | 26.357 | 21.339 | 21.572 | 23.218 | 23.364 | 23.629 |
| pentagon   | 24.499 | 23.965 | 24.175 | 26.851 | 28.190 | 22.344 | 21.126 | 22.957 | 22.914 | 24.054 |
| plants     | 29.841 | 28.894 | 29.113 | 31.767 | 32.476 | 28.185 | 27.883 | 28.023 | 28.666 | 30.259 |
| tank       | 30.817 | 29.566 | 29.640 | 31.787 | 33.074 | 27.574 | 28.682 | 29.188 | 29.085 | 30.429 |
|            |     |           |           |      |      |     |           |           |      |      |
|            |     |           |           |      |      |     |           |           |      |      |
| couple     | 19.599 | 23.284 | 23.531 | 23.020 | 23.268 | 18.892 | 26.167 | 26.618 | 26.608 |
| elaine     | 19.652 | 24.912 | 26.167 | 26.618 | 26.608 | 22.185 | 25.316 | 24.876 | 24.680 |
| flower     | 20.598 | 24.623 | 25.433 | 25.273 | 25.495 | 18.712 | 24.305 | 26.367 | 26.840 |
| goldhill   | 18.945 | 24.684 | 25.111 | 25.191 | 25.180 | 18.791 | 22.351 | 22.540 | 22.128 | 22.065 |
| house      | 18.583 | 22.344 | 22.566 | 22.489 | 23.118 | 24.635 | 27.834 | 27.780 | 27.551 | 27.598 |
| lin        | 20.841 | 28.337 | 28.838 | 28.681 | 28.850 | 18.583 | 22.344 | 22.566 | 22.489 | 23.118 |
| monarach   | 18.791 | 22.351 | 22.540 | 22.128 | 22.065 | 18.583 | 22.344 | 22.566 | 22.489 | 23.118 |
| pentagon   | 18.583 | 22.344 | 22.566 | 22.489 | 23.118 | 24.635 | 27.834 | 27.780 | 27.551 | 27.598 |
| plants     | 20.841 | 28.337 | 28.838 | 28.681 | 28.850 | 20.841 | 28.337 | 28.838 | 28.681 | 28.850 |
| tank       | 20.841 | 28.337 | 28.838 | 28.681 | 28.850 | 20.841 | 28.337 | 28.838 | 28.681 | 28.850 |
Table 2: Quantitative comparison of denoised results with different methods in terms of SSIM.

|         | PCP | WNNM-PCA | WNNM-PCA | DWLP | DWLP | PCP | WNNM-PCA | WNNM-PCA | DWLP | DWLP |
|---------|-----|----------|----------|------|------|-----|----------|----------|------|------|
|         |     | (p=q=1)  | (p=q=1)  |      |      |     | (p=q=1)  | (p=q=1)  |      |      |
| couple  | 0.838 | 0.801 | 0.812 | 0.966 | 0.976 | 0.840 | 0.736 | 0.748 | 0.927 | 0.950 |
| elaine  | 0.928 | 0.890 | 0.896 | 0.887 | 0.899 | 0.909 | 0.884 | 0.864 | 0.972 | 0.972 |
| flower  | 0.856 | 0.821 | 0.831 | 0.967 | 0.977 | 0.845 | 0.775 | 0.782 | 0.928 | 0.946 |
| goldhill| 0.857 | 0.799 | 0.809 | 0.974 | 0.978 | 0.836 | 0.753 | 0.759 | 0.943 | 0.945 |
| house   | 0.926 | 0.891 | 0.893 | 0.983 | 0.988 | 0.908 | 0.875 | 0.875 | 0.967 | 0.975 |
| lin     | 0.908 | 0.888 | 0.893 | 0.977 | 0.979 | 0.897 | 0.866 | 0.868 | 0.957 | 0.951 |
| monarch | 0.889 | 0.897 | 0.903 | 0.974 | 0.980 | 0.869 | 0.860 | 0.864 | 0.952 | 0.952 |
| pentagon| 0.805 | 0.767 | 0.783 | 0.953 | 0.967 | 0.773 | 0.691 | 0.703 | 0.914 | 0.937 |
| plants  | 0.912 | 0.866 | 0.872 | 0.981 | 0.988 | 0.897 | 0.840 | 0.844 | 0.965 | 0.978 |
| tank    | 0.859 | 0.765 | 0.770 | 0.969 | 0.980 | 0.841 | 0.748 | 0.751 | 0.934 | 0.951 |

|         | PCP | WNNM-PCA | WNNM-PCA | DWLP | DWLP | PCP | WNNM-PCA | WNNM-PCA | DWLP | DWLP |
|---------|-----|----------|----------|------|------|-----|----------|----------|------|------|
|         |     | (p=q=1)  | (p=q=1)  |      |      |     | (p=q=1)  | (p=q=1)  |      |      |
| couple  | 0.769 | 0.669 | 0.662 | 0.830 | 0.899 | 0.650 | 0.534 | 0.614 | 0.647 | 0.756 |
| elaine  | 0.880 | 0.822 | 0.825 | 0.935 | 0.946 | 0.749 | 0.602 | 0.793 | 0.831 | 0.901 |
| flower  | 0.811 | 0.730 | 0.737 | 0.842 | 0.888 | 0.739 | 0.673 | 0.701 | 0.721 | 0.780 |
| goldhill| 0.793 | 0.693 | 0.696 | 0.855 | 0.884 | 0.687 | 0.594 | 0.673 | 0.687 | 0.780 |
| house   | 0.881 | 0.851 | 0.855 | 0.920 | 0.949 | 0.692 | 0.566 | 0.820 | 0.846 | 0.892 |
| lin     | 0.875 | 0.833 | 0.835 | 0.907 | 0.901 | 0.780 | 0.708 | 0.802 | 0.814 | 0.859 |
| monarch | 0.829 | 0.806 | 0.811 | 0.897 | 0.913 | 0.750 | 0.678 | 0.770 | 0.800 | 0.844 |
| pentagon| 0.725 | 0.603 | 0.616 | 0.804 | 0.888 | 0.621 | 0.523 | 0.556 | 0.552 | 0.815 |
| plants  | 0.871 | 0.795 | 0.801 | 0.914 | 0.936 | 0.820 | 0.774 | 0.775 | 0.809 | 0.886 |
| tank    | 0.813 | 0.731 | 0.733 | 0.840 | 0.909 | 0.673 | 0.683 | 0.720 | 0.713 | 0.803 |

|         | PCP | WNNM-PCA | WNNM-PCA | DWLP | DWLP | PCP | WNNM-PCA | WNNM-PCA | DWLP | DWLP |
|---------|-----|----------|----------|------|------|-----|----------|----------|------|------|
|         |     | (p=q=1)  | (p=q=1)  |      |      |     | (p=q=1)  | (p=q=1)  |      |      |
| couple  | 0.341 | 0.708 | 0.707 | 0.701 | 0.711 | 0.630 | 0.768 | 0.769 | 0.775 | 0.621 |
| elaine  | 0.407 | 0.539 | 0.546 | 0.534 | 0.621 | 0.529 | 0.697 | 0.736 | 0.771 | 0.774 |
| flower  | 0.475 | 0.767 | 0.750 | 0.791 | 0.792 | 0.357 | 0.610 | 0.731 | 0.790 | 0.788 |
| goldhill| 0.501 | 0.745 | 0.745 | 0.767 | 0.774 | 0.501 | 0.745 | 0.745 | 0.777 | 0.776 |
| house   | 0.434 | 0.723 | 0.766 | 0.803 | 0.822 | 0.499 | 0.591 | 0.603 | 0.625 | 0.679 |
| lin     | 0.504 | 0.685 | 0.690 | 0.702 | 0.721 | 0.581 | 0.745 | 0.745 | 0.777 | 0.777 |
| monarch | 0.501 | 0.745 | 0.745 | 0.767 | 0.774 | 0.501 | 0.745 | 0.745 | 0.777 | 0.777 |
| pentagon| 0.434 | 0.723 | 0.766 | 0.803 | 0.822 | 0.499 | 0.591 | 0.603 | 0.625 | 0.679 |
| plants  | 0.434 | 0.723 | 0.766 | 0.803 | 0.822 | 0.499 | 0.591 | 0.603 | 0.625 | 0.679 |
| tank    | 0.504 | 0.685 | 0.690 | 0.702 | 0.721 | 0.581 | 0.745 | 0.745 | 0.777 | 0.777 |
level increases, some methods such as DWLP\( (p=q=1) \) may get a higher PSNR than DWLP, our DWLP method achieves a higher SSIM, indicating a more satisfactory visual performance.

Then we compare the visual quality of the denoised images by the com-
Figure 6: Denoised results on image house by different methods (noise level 30%). (a) Ground Truth; (b) Noisy image (11.715dB/0.075); (c) PCP [17] (28.876dB/0.881); (d) WNNM-RPCA [33, 37] (29.509dB/0.851); (e) WSNM-RPCA [37] (29.769dB/0.855); (f) DWLP (p=q=1) [34] (32.181dB/0.920); (g) DWLP (33.239dB/0.949).

peting methods under different noise levels. The ground-truth images of size 256×256 pixels are shown in Figs. 5(a), 6(a) and 7(a). The input images to be restored are corrupted by slight sparse noise on 10% pixels randomly as shown in Fig. 5(b) moderate sparse noise on 30% pixels in Fig. 6(b) and gross sparse noise on 50% pixels in Fig. 7(b). Moreover, the recovered images obtained by PCP, WNNM-RPCA, WSNM-RPCA, DWLP(p=q=1) and DWLP are shown in Figs. 5, 6 and 7(c)-(g), respectively, the regions of size 50×50 marked with red boxes are shown in zoom-in windows to have a close-up observation data.

According to the recovery experiments of the image plants with 10% salt-and-pepper noise in Fig. 5 and the image house with 30% salt-and-pepper noise in Fig. 6 we observe that much more appealing results are obtained by our DWLP. One can clearly discern the plant branches recovered by DWLP and DWLP(p=q=1) shown in the zoom-in windows of Figs. 5(f) and 5(g) which are hard to be distinguished in the other methods. Since the DWLP
Figure 7: Denoised results on image *monarch* by different methods (noise level 50%). (a) Ground Truth; (b) Noisy image (9.135dB/0.082); (c) PCP [17] (18.791dB/0.501); (d) WNNM-RPCA [33, 37] (22.351dB/0.745); (e) WSNM-RPCA [37] (22.340dB/0.745); (f) DWLP \((p=q=1)\) [34] (22.128dB/0.767); (g) DWLP (22.065dB/0.771).

and DWLP\((p=q=1)\) are able to enhance low rank and sparsity simultaneously for matrix recovery which helps to capture data structure. Compared with DWLP\((p=q=1)\), our method not only obtains a higher PSNR and SSIM, but also preserves more details, because of the weaker power and the balanced manner. When the noise level is increased to 30%, it can be seen in the zoom-in window of Fig. 6 that our method can better preserve the brick structure of the wall, while the other methods tend to over-smooth the textures or have some noise points remained. Although the PSNR of DWLP is lower than that of DWLP\((p=q=1)\), our DWLP could obtain a higher SSIM and better preserve wing veins of the butterfly, as shown in Figs. 7(f) and 7(g).

5.2. Discussions

In the proposed DWLP method some parameters can be readily fixed by experience as follows. For some parameters of size 256 × 256 grayscale images, the step size is 4 (denotes the neighborhood image patches are extracted in
every 4 pixels); the patch size $h$ is 8; and the number of iterations is set to 30, since the PSNR increment becomes quite small after 30 iterations.

Next, we discuss the setting of the other parameters ($K$, $\lambda_a$, $\lambda_e$ and powers $p$, $q$) through several experiments. Different from the empirical value $K = 9$ based on the observation that more than 80% of image patches recur 9 or more times in the original image scale [16], the number of similar patches $K$ is suggested to be $h^2$ in our method. The parameter experiments are based on 10 random test images and the average PSNR and SSIM results are listed in Table 3. Since it can be observed from Table 3 that increments of the PSNR and the SSIM become quite small when more than 64 similar patches are used, we set $K$ as 64 to avoid unnecessary computation. The setting $K = h^2 = 64$ helps enhance the low rank constraint as a square matrix. As for the norm powers $p$ and $q$, we determine their optimal values by investigating the influence of their difference value upon the quality of restored results. In this test, the powers $p$ and $q$ are uniformly sampled range from 0.1 to 1 with an interval 0.05, and Fig. 8 shows the PSNR of denoised results with different powers under both low and medium noise levels. Moreover, we find that DWLP is less sensitive to either $\lambda_a$ or $\lambda_e$ than their ratio $\lambda_a/\lambda_e$, for the optimal values of $\lambda_a/\lambda_e$ can guarantee to achieve a perfect balance between enforcing low-rank and separating sparse noise. Fixing other optimization parameters, we analyze the PSNR versus different values of the trade-off parameter ratio $\lambda_a/\lambda_e$ in Fig. 9 and note that the ratio is increasing with noise density. The DWLP achieves the highest PSNR at $\lambda_a/\lambda_e = 6.358$ under the low noise level, and $\lambda_a/\lambda_e = 10$ under the medium noise level, Table 4 summarizes all the suggested values of $p$, $q$ and $\lambda_a/\lambda_e$ under different noise level.

| $K$ | 8   | 22  | 36  | 50  | 64  | 78  | 92  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| PSNR| 6.372 | 11.524 | 21.993 | 28.722 | 29.372 | 29.605 | 29.490 |
| SSIM| 0.112 | 0.506 | 0.803 | 0.891 | 0.904 | 0.906 | 0.900 |
Table 4: The optimal parameter values of the DWLP on different noise levels

|       | 10%   | 20%   | 30%   | 40%   | 50%   |
|-------|-------|-------|-------|-------|-------|
| \(p\) | 0.651 | 0.765 | 0.800 | 0.905 | 0.916 |
| \(q\) | 0.340 | 0.393 | 0.419 | 0.570 | 0.595 |
| \(\lambda_a/\lambda_e\) | 6.358 | 7.738 | 10.003| 10.792| 13.866|

Figure 8: PSNR values with different \(p\) and \(q\) on both low and medium noise levels. (a) low noise level (10%); (b) medium noise level (30%).

Figure 9: PSNR values with different \(\lambda_a/\lambda_e\) on both low and medium noise levels. (a) low noise level (10%); (b) medium noise level (30%).
6. Conclusion

In this paper, we propose the DWLP model to improve the performance of the RPCA method, which integrates the weighting method and $\ell_p$-norm into the low rank regularization term and the sparsity regularization term simultaneously. The iterative reweighted algorithm is introduced to solve the proposed DWLP, which provides a better estimation of the joined weights through optimizing the elements and weights alternatively. The DWLP makes impressively quantitative improvements not only on the low rank approximation of NSS matrices to deal with the overshrink problem, but also on the image denoising experiments which preserves much better image structures and details with less visual artifacts on visual quality, outperforming the original PCP optimization, the WNNM-RPCA, the WSNM-RPCA and the DWLP($p=q=1$) greatly. In light of promise of DWLP, deeper investigations of our method remain: theoretical analysis of the developed method, such as global convergence, is considered as one of the future work.

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References

[1] C. Khare, K. K. Nagwanshi, Image restoration technique with non linear filter, Int. J. Adv. Sci. Technol 39 (2012) 67C74.

[2] M. S. A. Alias, N. Ibrahim, Z. M. Zin, Salt and pepper noise removal by using improved decision based algorithm, 2017 IEEE 15th Student Conference on Research and Development (SCOReD) (2017) 487–492.

[3] I. Djurović, Bm3d filter in salt-and-pepper noise removal, EURASIP Journal on Image and Video Processing 2016 (1) (2016) 13.
[4] H. C. Bandala Hernández, J. Rocha-Pérez, A. Díaz-sanchez, J. Lemus-López, H. Vázquez-Leal, A. Díaz-Armendariz, J. Ramírez-Angulo, Weighted median filters: An analog implementation, in: Integration, the VLSI Journal, Vol. 55, 2016, pp. 227–231.

[5] T. Chen, H. R. Wu, Adaptive impulse detection using center-weighted median filters, IEEE Signal Processing Letters 8 (1) (2001) 1–3.

[6] S. Yazdi, F. Homayouni, Modified adaptive center weighted median filter for suppressing impulse noise in image, in: Int J Res Rev Appl Sci, Vol. 1, 2009, pp. 218–227.

[7] T. Loupas, W. N. McDicken, P. L. Allan, An adaptive weighted median filter for speckle suppression in medical ultrasonic images, IEEE Transactions on Circuits and Systems 36 (1) (1989) 129–135.

[8] S. Shrestha, Image denoising using new adaptive based median filter, in: Signal & Image Processing : An International Journal (SIPIJ), Vol. 5, 2014, pp. 1–13.

[9] K. S. Srinivasan, D. Ebenezer, A new fast and efficient decision-based algorithm for removal of high-density impulse noises, IEEE Signal Processing Letters 14 (3) (2007) 189–192.

[10] R. N. KULKARNI, P. C. Prof. BHASKAR, Implementation of decision based algorithm for median filter to extract impulse noise, International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering.

[11] C. A. V. S. Balasubramanian, G., Probabilistic decision based filter to remove impulse noise using patch else trimmed median, in: AEU - International Journal of Electronics and Communications, Vol. 70, 2016.

[12] I. Djurović, Bm3d filter in salt-and-pepper noise removal, EURASIP Journal on Image and Video Processing 2016 (1) (2016) 13.
[13] K. Dabov, A. Foi, V. Katkovnik, K. Egiazarian, Image denoising by sparse 3-d transform-domain collaborative filtering, IEEE Transactions on Image Processing 16 (8) (2007) 2080–2095.

[14] J. Mairal, F. Bach, J. Ponce, G. Sapiro, A. Zisserman, Non-local sparse models for image restoration, in: 2009 IEEE 12th International Conference on Computer Vision, 2009, pp. 2272–2279.

[15] W. Dong, L. Zhang, G. Shi, X. Li, Nonlocally centralized sparse representation for image restoration, IEEE Transactions on Image Processing 22 (4) (2013) 1620–1630.

[16] D. Glasner, S. Bagon, M. Irani, Super-resolution from a single image, in: 2009 IEEE 12th International Conference on Computer Vision, 2009, pp. 349–356.

[17] J. Wright, A. Ganesh, S. Rao, Y. Peng, Y. Ma, Robust principal component analysis: Exact recovery of corrupted low-rank matrices via convex optimization, in: Advances in Neural Information Processing Systems 22, 2009, pp. 2080–2088.

[18] P. L. Guangcan Liu, Recovery of coherent data via low-rank dictionary pursuit, in: Advances in Neural Information Processing Systems (NIPS), Vol. 20, 2014, pp. 2080–2088.

[19] E. J. Candés, X. Li, Y. Ma, J. Wright, Robust principal component analysis?, J. ACM 58 (3) (2011) 11:1–11:37.

[20] E. J. Candés, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Transactions on Information Theory 52 (2) (2006) 489–509.

[21] E. J. Candés, B. Recht, Exact matrix completion via convex optimization, Foundations of Computational Mathematics 9 (6) (2009) 717–772.
[22] K. Zhao, Z. Zhang, Successively alternate least square for low-rank matrix factorization with bounded missing data, Comput. Vis. Image Underst. 114 (10) (2010) 1084–1096.

[23] J.-F. Cai, E. J. Candès, Z. Shen, A singular value thresholding algorithm for matrix completion, SIAM J. on Optimization 20 (4) (2010) 1956–1982.

[24] H. Ji, S. Huang, Z. Shen, Y. Xu, Robust video restoration by joint sparse and low rank matrix approximation, SIAM Journal on Imaging Sciences 4 (4) (2011) 1122–1142.

[25] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm with application to wavelet-based image deblurring, in: 2009 IEEE International Conference on Acoustics, Speech and Signal Processing, 2009, pp. 693–696.

[26] K.-C. Toh, S. Yun, An accelerated proximal gradient algorithm for nuclear norm regularized least squares problems, in: Pacific Journal of Optimization, Vol. 6, 2010.

[27] Z. Lin, M. Chen, L. Wu, Y. Ma, The augmented lagrange multiplier method for exact recovery of corrupted low-rank matrices technical report, in: UIUC UILU-ENG-09-2215. ArXiv.

[28] X. Yuan, J. Yang, Sparse and low rank matrix decomposition via alternating direction method, in: Pacific Journal of Optimization, Vol. 9, 2013.

[29] N. Srebro, T. Jaakkola, Weighted low-rank approximations, in: Proceedings of the Twentieth International Conference on International Conference on Machine Learning, ICML’03, AAAI Press, 2003, pp. 720–727.

[30] F. Nie, H. Huang, C. Ding, Low-rank matrix recovery via efficient schatten p-norm minimization, in: Proceedings of the Twenty-Sixth AAAI Conference on Artificial Intelligence, AAAI’12, AAAI Press, 2012, pp. 655–661.
[31] K. Mohan, M. Fazel, Iterative reweighted algorithms for matrix rank minimization, J. Mach. Learn. Res. 13 (1) (2012) 3441–3473.

[32] M. B. W. E. J. Candes, S. P. Boyed, Enhancing sparsity by reweighted $\ell_1$ minimization, Journal of Fourier Analysis Applications 14 (5-6) (2008) 877–905.

[33] S. Gu, L. Zhang, W. Zuo, X. Feng, Weighted nuclear norm minimization with application to image denoising, in: 2014 IEEE Conference on Computer Vision and Pattern Recognition, 2014, pp. 2862–2869.

[34] Y. Peng, J. Suo, Q. Dai, W. Xu, Reweighted low-rank matrix recovery and its application in image restoration, IEEE Transactions on Cybernetics 44 (12) (2014) 2418–2430.

[35] W. Xu, M. A. Khajehnejad, A. S. Avestimehr, B. Hassibi, Breaking through the thresholds: an analysis for iterative reweighted $\ell_1$ minimization via the grassmann angle framework, in: 2010 IEEE International Conference on Acoustics, Speech and Signal Processing, 2010, pp. 5498–5501.

[36] Y. Xie, S. Gu, Y. Liu, W. Zuo, W. Zhang, L. Zhang, Weighted schatten $p$-norm minimization for image denoising and background subtraction, IEEE Transactions on Image Processing 25 (2016) 4842–4857.

[37] Y. Xie, Y. Qu, D. Tao, W. Wu, Q. Yuan, W. Zhang, Hyperspectral image restoration via iteratively regularized weighted schatten $p$-norm minimization, IEEE Transactions on Geoscience and Remote Sensing 54 (8) (2016) 4642–4659.

[38] Z. Gu, M. Shao, L. Li, Y. Fu, Discriminative metric: Schatten norm vs. vector norm, in: Proceedings of the 21st International Conference on Pattern Recognition (ICPR2012), 2012, pp. 1213–1216.

[39] Z. Zha, X. Liu, X. Huang, H. Shi, Y. Xu, Q. Wang, L. Tang, X. Zhang, Analyzing the group sparsity based on the rank minimization methods,
in: 2017 IEEE International Conference on Multimedia and Expo (ICME), 2017, pp. 883–888.

[40] C. Lu, C. Zhu, C. Xu, S. Yan, Z. Lin, Generalized singular value thresholding, in: Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, AAAI’15, AAAI Press, 2015, pp. 1805–1811.

[41] W. Zuo, D. Meng, L. Zhang, X. Feng, D. Zhang, A generalized iterated shrinkage algorithm for non-convex sparse coding, in: 2013 IEEE International Conference on Computer Vision, 2013, pp. 217–224.

[42] Z. Zhou, X. Li, J. Wright, E. Candes, L. Yu, Stable principal component pursuit, in: IEEE International Symposium on Information Theory - Proceedings.

[43] D. Zhang, Y. Hu, J. Ye, X. Li, X. He, Matrix completion by truncated nuclear norm regularization, in: 2012 IEEE Conference on Computer Vision and Pattern Recognition, 2012, pp. 2192–2199.

[44] T. Oh, H. Kim, Y. Tai, J. Bazin, I. S. Kweon, Partial sum minimization of singular values in rpca for low-level vision, in: 2013 IEEE International Conference on Computer Vision, 2013, pp. 145–152.

[45] Z. Zha, X. Yuan, B. Wen, J. Zhou, C. Zhu, Joint patch-group based sparse representation for image inpainting, in: J. Zhu, I. Takeuchi (Eds.), Proceedings of The 10th Asian Conference on Machine Learning, Vol. 95 of Proceedings of Machine Learning Research, PMLR, 2018, pp. 145–160.

[46] L. Liu, W. Huang, D.-R. Chen, Exact minimum rank approximation via schatten p-norm minimization 267 (2014) 218C227.

[47] Z. Zha, X. Zhang, Q. Wang, Y. Bai, L. Tang, Image denoising using group sparsity residual and external nonlocal self-similarity prior, in: 2017 IEEE International Conference on Image Processing (ICIP), 2017, pp. 2956–2960.
[48] Z. Lin, M. Chen, Y. Ma, The augmented lagrange multiplier method for exact recovery of corrupted low-rank matrices, Mathematics.