1 Introduction

This article deals with an attempt of proof of the Riemann Hypothesis with some elementary geometric, algorithmic aspects and a use of complex analysis limited to Taylor series of some analytic functions.

1.1 The aim of the article.

Let $\zeta(z)$ be the Riemann Zeta function. Let $T > 10^{10}$ arbitrarily large. Let $\Omega_T$ be the region

$$\Omega_T = \{ z = x + iy \mid \frac{1}{2} < x < 1, \ 0 < y < T \}. \quad (1)$$

where $i$ is for the imaginary unit ($i^2 + 1 = 0$). There is a finite number of roots $N_T = \mathcal{O}(T \log(T))$ of roots of the equation $\zeta(z) = 0$ in $\Omega_T$. The aim of this paper is to prove that $N_T = 0$. We suppose that $N_T > 0$ and we will search for a contradiction.

1.2 Some definitions

1. Let $\rho_j = \frac{1}{2} + u_j + i\gamma_j \in \Omega_T$, $u_j > 0$, $j = 1, \ldots, N_T$ be the roots of $\zeta(z) \in \Omega_T$.
2. Let $u = \max\{u_j \mid j = 1, \ldots, N_T\}$. Let us denote a corresponding root:

$$\rho = \frac{1}{2} + u + i\gamma,$$  \hfill (2)

so with no other root of $\Omega_T$ satisfying

$$\rho_j = \frac{1}{2} + u_j + i\gamma_j = 0 \ \text{with} \ \frac{1}{2} > u_j > u. \quad (3)$$

1 A preliminary remark: I am fully aware of the history of RH and of all the failed attempts of proof given by a lot of serious (or less serious) mathematicians (see for instance \cite{2} p. 69), therefore it is possible that this paper contain an irrecoverable error. In that case, it would be important for me to know from the reader where is the error in the paper and its nature.

2 2010 AMS Subject classification: 11M06; 11M26
1.3 A sketch of the proof

Let $v \geq \frac{3}{2}$. Let $\varepsilon > 0$ arbitrarily small. We prove that $f(z) = \frac{\zeta'(z)}{\zeta(z)}$ is analytic in the open disk $\Omega_\varepsilon = \left\{ \left| z - \left( \rho + \frac{\varepsilon}{2} + v \right) \right| < v \right\}$. Let $s = \rho + \varepsilon$. We prove, from the Taylor series of $\zeta(s)$, that $f(s) \sim \frac{1}{\varepsilon} \to \infty$ when $\varepsilon \to 0$, and that, through the representation of $f(s)$ as a Taylor series,

$$f(s) = f(c_0) - (v - \varepsilon)^2f'(c_0) + \frac{(v - \varepsilon)^3}{2!}f''(c_0) - \frac{(v - \varepsilon)^3}{3!}f^{(3)}(c_0) + \ldots$$

for $c_0 = \rho + \frac{\varepsilon}{2} + v$, in $\Omega_\varepsilon$, that $f(s) \not\to \infty$ when $\varepsilon \to 0$, a contradiction which allows us to prove RH.

2 On Riemann Hypothesis (RH)

We study separately the two cases $\rho$ is a simple root and $\rho$ is a multiple root of $\zeta(z)$ in the critical strip.

Theorem 2.1. An attempt

If $\rho$ is a simple root of the Zeta function in the critical strip, then $\rho$ is on the critical line.

Proof.

1. Assume that $N_T > 0$ and that there are some roots $\rho \in \Omega_T$. We will search for a contradiction. Then there exists at least one root $\rho$

$$0 < u < \frac{1}{2} \text{ and } \rho = \frac{1}{2} + u + i\gamma$$

satisfying (2) and (3). Let $\varepsilon > 0$ arbitrarily small and

$$s = \rho + \varepsilon = \frac{1}{2} + u + i\gamma + \varepsilon. \quad (4)$$

2. Let $v$ be a constant satisfying

$$v \in \mathbb{R}, \ v \geq \frac{3}{2}. \quad (5)$$

Let us define the region open disk of radius $v$ and of center $\rho + \frac{\varepsilon}{2} + v$.

$$\Omega_\varepsilon = \left\{ \left| z - \left( \rho + \frac{\varepsilon}{2} + v \right) \right| < v \right\}.$$
4. Let us introduce the function
\[ f(z) = \frac{\zeta'(z)}{\zeta(z)}. \]

5. At first:
\[ \zeta(z) \] is analytic in the open upper space \( \Im(z) > 0 \) and we have successively the Taylor developments, observing that \( \zeta'(\rho) \neq 0 \) because \( \rho \) is a simple root:

for \( s = \rho + \varepsilon \) and \( \zeta'(\rho) = 0 \):

\[ \zeta(s) = \varepsilon \zeta'(\rho) + \frac{\varepsilon^2}{2} \zeta''(\rho) + O(\varepsilon^3) = \varepsilon \zeta'(\rho) \left[ \left( 1 + \frac{\varepsilon}{2} \right) \zeta'(\rho) + O(\varepsilon^2) \right], \]

\[ \zeta'(s) = \zeta'(\rho) + \varepsilon \zeta''(\rho) + O(\varepsilon^2) = \zeta'(\rho) \left[ \left( 1 + \frac{\varepsilon \zeta''(\rho)}{\zeta'(\rho)} \right) + O(\varepsilon^2) \right], \]

\[ \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{\varepsilon} \left\{ 1 + \frac{\varepsilon \zeta''(\rho)}{\zeta'(\rho)} + O(\varepsilon^2) \right\}; \]

\[ \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{\varepsilon} \left\{ 1 + \frac{\varepsilon \zeta''(\rho)}{2 \zeta'(\rho)} + O(\varepsilon^2) \right\}; \]

\[ \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{\varepsilon} + \frac{1}{2} \zeta''(\rho) + O(\varepsilon); \]

\[ \zeta'(s) = \zeta'(\rho) + \varepsilon \zeta''(\rho) + O(\varepsilon^2) = \zeta'(\rho) \left[ \left( 1 + \frac{\varepsilon \zeta''(\rho)}{\zeta'(\rho)} \right) + O(\varepsilon^2) \right], \]

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\[ \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{\varepsilon} \left\{ 1 + \frac{\varepsilon \zeta''(\rho)}{\zeta'(\rho)} + O(\varepsilon^2) \right\}. \]

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\[ \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{\varepsilon} \left\{ 1 + \frac{\varepsilon \zeta''(\rho)}{\zeta'(\rho)} + O(\varepsilon^2) \right\}. \]
where $c$ is a positive constant depending only on $\zeta'(\rho)$ and $\zeta''(\rho)$. Thus we have, for $\varepsilon$ sufficiently small, the sufficiently good approximation for the sequel

$$f(s) = \frac{\zeta'(s)}{\zeta(s)} = \frac{1}{\varepsilon} + \frac{1}{2} \frac{\zeta''(\rho)}{\zeta'(\rho)} + O(\varepsilon) \sim \frac{1}{\varepsilon} \text{ for } \varepsilon \to 0,$$

so $f(s) \to \infty$ when $\varepsilon \to 0$.

6. In the other hand:

The center of the open disk $\Omega_\varepsilon$ is $c_0 = \rho + \frac{\varepsilon}{2} + v$. We have

$$s = \rho + \varepsilon = \rho + \frac{\varepsilon}{2} + v - (v - \frac{\varepsilon}{2}) = c_0 - (v - \frac{\varepsilon}{2}).$$

- From the relation (6) the function $\zeta(z)$ has no root in $\Omega_\varepsilon$, therefore $f(z)$ is analytic in $\Omega_\varepsilon$, so we have for $s \in \Omega_\varepsilon$ the representation of $f(s)$ by the Taylor series

$$f(s) = f(c_0) - (v - \frac{\varepsilon}{2}) f'(c_0) + \frac{(v - \frac{\varepsilon}{2})^2}{2!} f''(c_0) - \frac{(v - \frac{\varepsilon}{2})^3}{3!} f^{(3)}(c_0) + \ldots, \quad (9)$$

where $f(c_0) = \frac{\zeta'(c_0)}{\zeta(c_0)}$, because $s \in \Omega_\varepsilon$ by construction and that $f(z)$ is analytic in $\Omega_\varepsilon$ (see for instance Ahlfors [1] thm 3. p. 179).

When $\varepsilon \to 0$,

$$f^{(k)}(c_0) \to f^{(k)}(\rho + v),$$

so when $\varepsilon \to 0$

$$f(s) \to \sum_{k=0}^{\infty} (-1)^k \frac{v^k}{k!} f^{(k)}(\rho + v),$$

which is finite and contradicts relation (8). Therefore we have eliminated the root with the greatest real part $\rho = \frac{1}{2} + u + i \gamma$ and reiterating, we can eliminate successively all the possible roots one by one.

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4 The verbatim of the theorem 3 p.179 of Ahlfors is: Theorem 3.: If $f(z)$ is analytic in the region $\Omega$, containing $z_0$, then the representation

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!} (z - z_0) + \ldots + \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n + \ldots$$

is valid in the largest open disk of center $z_0$ contained in $\Omega$. 

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Therefore there is no root
\[ \rho = \frac{1}{2} + u + i\gamma \text{ for any } u > 0, \]
and the only possible roots are of form \[ \rho_0 = \frac{1}{2} + i\gamma_0 \text{ and } 1 - \rho_0 = \frac{1}{2} - i\gamma_0, \]
therefore all the roots of \( \zeta(z) \), with \( 0 < \Im(\rho) < T \) for any \( T \) arbitrarily large, are on the critical line. Therefore all the roots are with \( 0 < \Im(\rho) \) are on the critical line: if there was a root \( \rho_1 \) not on the critical line, it should contradicts that all the roots \( \rho \) with \( 0 < \Im(\rho) < T_1 \) are on the critical line for any \( T_1 \) with \( 0 < \Im(\rho_1) < T_1 \). By conjugation, all the roots are on the critical line.

\[ \square \]

**Theorem 2.2.** An attempt
RH is true.

**Proof.** If \( \zeta'(\rho) = 0 \) (\( \rho \) non simple root) there exists an integer \( K > 1 \) such that
\[ \zeta^{(k)}(\rho) \text{ for } k=0, \ldots, K-1 \text{ and } \zeta^{(K)}(\rho) \neq 0. \]
We can always apply the previous method of proof considering now the function \( f(s) = \frac{\zeta^{(K)}(s)}{\zeta(s)} \) instead of \( \frac{\zeta'(s)}{\zeta(s)} \), which implies that RH is true.

\[ \square \]

**Remark 1.** If there was no irrecoverable error in this paper the same method of proof should be applicable to the Dirichlet \( L \)-functions analytic continuation of the associated Dirichlet \( L \)-series \( \sum_{n=1}^{\infty} \chi_k(n)n^{-s} \) for a proof of the Generalised Riemann Hypothesis GRH.

**References**

[1] L.V. Ahlfors, *Complex Analysis*, Mac Graw Hill (1979).

[2] P. Borwein, S. Choi, B. Rooney, A. Weirathmueller, *The Riemann Hypothesis, A Ressource for the Afficionado and Virtuoso Alike*, Canadian Mathematical Society (2008).

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