SYMMETRY OF EIGENVALUES OF OPERATORS ASSOCIATED WITH REPRESENTATIONS OF COMPACT QUANTUM GROUPS

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Abstract. We ask the question whether for a given unitary representation \( U \) of a compact quantum group \( G \) the associated operator \( \rho_U \in \text{Mor}(U, U^{ccc}) \) has spectrum invariant under inversion – in this case we say that \( \rho_U \) has symmetric eigenvalues. This does not have to be the case. However, we give affirmative answer whenever a certain condition on the growth of dimensions of irreducible subrepresentations of tensor powers of \( U \) is imposed. This condition is satisfied whenever \( \hat{G} \) is of subexponential growth.

1. Introduction

Let \( G \) be a compact quantum group. We say that \( G \) is of Kac type if for every finite dimensional unitary representation \( U \) the contragradient representation \( U^c \) is unitary ([1] Definition 1.3.8]). There are various equivalent formulations of this property related to the Haar integral on \( G \), Haar integrals on \( \hat{G} \) (see [5]) or the antipode of \( G \) – see e.g. [3, 4, 9]. It is known that not all compact quantum groups are of Kac type – counterexamples are given e.g. by \( SU_q(2) \) groups when \( q \not\in \{-1, 1\} \) ([8]) or (some of) the free unitary and orthogonal groups ([7]). Even if \( U^c \) is not unitary, it is still a representation and we can form a second contragradient \( U^{ccc} \). It turns out that \( U^{ccc} \), as in the classical case, is equivalent to \( U \). This equivalence is given by a positive operator \( \rho_U \in \text{Mor}(U, U^{ccc}) \) which is characterized uniquely by the property \( \text{Tr}(T \rho_U) = \text{Tr}(T \rho_U^{-1}) \) \((T \in \text{End}(U)) \) ([4, Section 3, 4]). In this paper we consider a problem when the spectrum of \( \rho_U \) coincides with the spectrum of \( \rho_U^{-1} \) (counting with multiplicities) – this will be stated more precisely in Section 3. There are examples of representations
which does not have this property, however, we present some sufficient conditions for this to hold. Proposition 3.1 states that this symmetry condition holds when $U$ and its conjugate representation are equivalent, Theorem 3.4 relates this property to the rate of growth of dimensions of irreducible subrepresentations of tensor powers of $U$. Moreover, Corollary 3.6 states that $\rho_U$ has this property whenever $\hat{G}$ is of subexponential growth. Proof of Theorem 3.4 uses functions $d_t (t \in \mathbb{R})$ which are generalizations of classical and quantum dimension of a unitary representation and will be introduced in the next section.

We refer to [4, Chapter 1] for necessary definitions and basic theory of compact quantum groups. Furthermore, most of our notational conventions agree with those of [4].

2. Notation

Throughout the paper $G$ will stand for a fixed compact quantum group. We will denote by $\text{uRep}(G)$ the class of finite dimensional, unitary representations of $G$ and by $\text{Irr}(G)$ the set of equivalence classes of irreducible unitary representations. For each class $\alpha \in \text{Irr}(G)$ we choose a representative $U^\alpha \in \text{uRep}(G)$. We will write $\alpha$ instead of $U^\alpha$ in objects which depends only on equivalence class – e.g. $\dim(\alpha) = \dim(U^\alpha)$.

Let $U, V \in \text{uRep}(G)$ be two representations. Their tensor product will be denoted by $U \otimes V$. By [4, Proposition 1.4.4.], there exists a unique positive element of $\text{Mor}(U, U^c)$, $\rho_U$ such that

$$\text{Tr}(\cdot \rho_U) = \text{Tr}(\cdot \rho_U^{-1}) \quad \text{on } \text{End}(U) = \text{Mor}(U, U). \quad (2.1)$$

For any real number $t \in \mathbb{R}$ we define

$$d_t(U) = \text{Tr}(\rho_U^{-t}) \in ]0, +\infty[ \quad (2.2)$$

(we know that $d_t(U)$ is a positive number because operator $\rho_U$ is positive and invertible – its spectrum lies in $]0, +\infty[$). Note that since operators $\rho_U$ satisfy ([4, Proposition 1.4.7., Theorem 1.4.9.])

$$\rho_{U \oplus V} = \rho_U \oplus \rho_V, \quad \rho_{U \otimes V} = \rho_U \otimes \rho_V, \quad \rho_{U^c} = (\rho_U^{-1})^\top,$$

we have

$$d_t(U \oplus V) = d_t(U) + d_t(V), \quad d_t(U \otimes V) = d_t(U)d_t(V), \quad d_t(U) = d_{-t}(U),$$

$$d_t(U^c) = d_{-t}(U).$$
where $\overline{U}$ is the conjugate of $U$ ([4, Definition 1.4.5.]). Moreover, $d_t(U)$ depends only on the equivalence class of $U$. It follows directly from the definition that $d_0(U)$ equals the dimension of $U$ and $d_1(U) = d_{-1}(U)$ is the quantum dimension of $U$ ([4, Definition 1.4.1.]). In general, behaviour of the function $t \mapsto d_t(U)$ allows us to infer information about the structure of eigenvalues of $\rho_U$.

Let us also define $\overrightarrow{\rho}_U$ to be the list of eigenvalues of $\rho_U$ (counting with multiplicities) in descending order and $\overleftarrow{\rho}_U$ to be the list of eigenvalues of $\rho_U^{-1}$ (counting with multiplicities) in the descending order. We will treat $\overrightarrow{\rho}_U$ and $\overleftarrow{\rho}_U$ as elements of $\mathbb{R}^{\dim U}$. Observe that for $t \geq 1$ we have

$$d_t(U) = (\|\overrightarrow{\rho}_U\|_t)^t,$$

where $\| \cdot \|_t$ is the $\ell_t$–norm on $\mathbb{R}^{\dim U}$.

3. Symmetry of eigenvalues

Let $U \in \text{uRep}(G)$. By (2.1) we have

$$\|\overrightarrow{\rho}_U\|_1 = \text{Tr}(\rho_U) = \text{Tr}(\rho_U^{-1}) = \|\overleftarrow{\rho}_U\|_1. \quad (3.1)$$

In this section we would like to address the question, when the following stronger property holds

$$\overrightarrow{\rho}_U = \overleftarrow{\rho}_U. \quad (3.2)$$

If (3.2) is true, we say that $\rho_U$ has symmetric eigenvalues. If $\dim U \in \{1, 2\}$ this is of course true, however if dimension of $U$ is larger, this need not be the case. Indeed, there exists compact quantum groups with fundamental representations which do not satisfy (3.2). For example, the construction of free quantum unitary groups begins with the choice of an invertible scalar matrix $F$ which satisfies $\text{Tr}(F^*F) = \text{Tr}((F^*F)^{-1})$. Then the operator $\rho_U$ for the fundamental representation is $(F^*F)^\top$ ([4, Example 1.4.2.]), hence one can easily choose matrix $F$ so that (3.2) fails (e.g. take $F = \text{diag}(y, x, x) \in \text{GL}_3(\mathbb{C})$, where $y > 1$ and $x > 0$ is the positive solution of the equation $\frac{2}{x^2} + \frac{1}{y^2} = 2x^2 + y^2$).

Nevertheless there are cases where one can prove (3.2). We start with a simple one.

**Proposition 3.1.** Let $U \in \text{uRep}(G)$ be a unitary representation of $G$. If the representations $U$ and $\overline{U}$ are equivalent then (3.2) holds.
Proof. This is an immediate consequence of [4, Proposition 1.4.7.] which says that $\rho_U^\top$ is the transpose of $\rho_U^{-1}$. □

**Corollary 3.2.** Equation (3.2) holds for every finite dimensional unitary representation $U$ of the free orthogonal quantum group $A_o(F)$. In particular, it holds for every $U \in u\text{Rep}(SU_q(2))$.

**Proof.** Assume that $F \in \text{GL}_n(\mathbb{C})$. Since every finite dimensional unitary representation is unitarily equivalent to a direct sum of irreducible ones, it is enough to assume that $U$ is irreducible. Irreducible representations of $A_o(F)$ are labeled (up to equivalence) by natural numbers: for every $r \in \mathbb{N}$ there is an irreducible unitary representation $V^{(r)} \in u\text{Rep}(A_o(F))$ of dimension $z_r = \frac{x^{r+1} - y^{r+1}}{x - y}$, where $x$ and $y$ are solutions of the equation $x^2 - nx + 1 = 0$. Moreover, every irreducible representation of $A_o(F)$ is equivalent to $V^{(r)}$ for some $r \in \mathbb{N}$ ([4, Remark 6.4.11., Corollary 6.4.12.]) since $z_r \neq z_{r'}$ when $r \neq r'$, it follows that for each $r \in \mathbb{N}$ the representation $\overline{V^{(r)}}$ is equivalent to $V^{(r)}$. In particular, $\overline{U}$ is equivalent to $U$. Second claim follows, because $SU_q(2) = A_o(F)$ for $F = \begin{bmatrix} 0 & 1 \\ -q^{-1} & 0 \end{bmatrix}$ ([6, Proposition 6.4.8.]). □

In order to state our main theorem we have to introduce a function $\mathbb{N} \ni n \mapsto P_U(n) \in \mathbb{R}$ which tells us how the maximal dimension of irreducible subrepresentations of $n$-th tensor power of $U$ grows as we increase $n$.

**Definition 3.3.** For $U \in u\text{Rep}(G)$ and $n \in \mathbb{N}$ put

$$P_U(n) = \max\{\dim \alpha_1, \ldots, \dim \alpha_N\},$$

where $\alpha_1, \ldots, \alpha_N \in \text{Irr}(G)$ are such that $U^{\otimes n}$ is equivalent to the direct sum $\bigoplus_{k=1}^N U^{\alpha_k}$.

The next theorem says that for any $U \in u\text{Rep}(G)$ eigenvalues of $\rho_U$ are symmetric (i.e. (3.2) holds) if $\mathbb{N} \ni n \mapsto P_U(n) \in \mathbb{R}$ grows subexponentially.

**Theorem 3.4.** Let $U \in u\text{Rep}(G)$ and assume that

$$\forall c > 1 \quad \lim_{n \to \infty} \frac{P_U(n)}{c^n} = 0. \quad (3.3)$$

Then eigenvalues of $\rho_U$ are symmetric, that is $\overline{\rho_U} = \overline{\rho_U}$.
Proof of Theorem 3.4.

Let \( n \in \mathbb{N} \) and \( k \) such that \( n \) for every \( t > 0 \) (we are not assuming the \( U \) lemma:

In the proof we will use standard inequalities for \( \ell_p \)-norms on \( \mathbb{R}^n \):

\[
\|x\|_{p'} \leq \|x\|_p \leq \|x\|_1^{\frac{p}{p'}}, \quad x \in \mathbb{R}^n, \ 1 \leq p \leq p' < +\infty
\]

which are a consequence of Hölder inequality, and the following elementary lemma:

**Lemma 3.5.** Let \( a_1 \geq a_2 \geq \cdots \geq a_n > 0 \) and \( b_1 \geq b_2 \geq \cdots \geq b_m > 0 \) be such that

\[
\sum_{i=1}^n a_i^t = \sum_{j=1}^m b_j^t
\]

for every \( t > 1 \). Then \( n = m \) and \( a_i = b_i \) for \( i \in \{1, \ldots, n\} \).

**Proof of Theorem 3.4.** Let \( n \in \mathbb{N} \) and \( \alpha_1, \ldots, \alpha_N \in \text{Irr}(G) \) be such that \( U^\otimes n \) and \( \bigoplus_{k=1}^N U^{\alpha_k} \) are unitarily equivalent (we are not assuming the \( U^{\alpha_k} \)'s are pairwise non-equivalent). For each \( k \in \{1, \ldots, N\} \) and \( t > 1 \) we have

\[
d_t(\alpha_k) = \left(\|\overline{\rho}_{\alpha_k}\|_t\right)^t \leq \left(\|\overline{\rho}_{\alpha_k}\|_1\right)^t = \left(\dim\overline{\rho}_{\alpha_k}\right)^t \leq P_U(n)^{t-1}d_{-t}(\alpha_k),
\]

where in the third step we used (3.1).

Similarly for any \( k \in \{1, \ldots, N\} \) and \( t > 1 \)

\[
d_{-t}(\alpha_k) = \left(\|\overline{\rho}_{\alpha_k}\|_t\right)^t \leq \left(\|\overline{\rho}_{\alpha_k}\|_1\right)^t = \left(\dim\overline{\rho}_{\alpha_k}\right)^t \leq P_U(n)^{t-1}d_t(\alpha_k).
\]

Summing these inequalities over \( i \) yields

\[
d_t(U^n) = d_t(U^\otimes n) = \sum_{k=1}^N d_t(\alpha_k) \leq \sum_{k=1}^N P_U(n)^{t-1}d_{-t}(\alpha_k) = P_U(n)^{t-1}d_{-t}(U^\otimes n) = P_U(n)^{t-1}d_{-t}(U)^n,
\]

\[
d_{-t}(U^n) = d_{-t}(U^\otimes n) = \sum_{k=1}^N d_{-t}(\alpha_k) \leq \sum_{k=1}^N P_U(n)^{t-1}d_t(\alpha_k) = P_U(n)^{t-1}d_t(U^\otimes n) = P_U(n)^{t-1}d_t(U)^n.
\]

Thus

\[
1 \leq \left(\frac{P_U(n)}{(d_t(U)/d_{-t}(U))^{1/t}}\right)^{t-1}
\]
and
\[ 1 \leq \left( \frac{P_U(n)}{(d_{t-i}(U)/d_t(U))^{\frac{1}{t+i}}} \right)^{t-1} \]
for all \( t > 1 \).

Now, if \( d_{t-i}(U) \neq d_t(U) \) for some \( t > 1 \), then either \( (d_{t-i}(U)/d_t(U))^{\frac{1}{t+i}} \) or \( (d_{t-i}(U)/d_t(U))^{\frac{1}{t+i}} \) is strictly greater then 1. Setting \( c \) to be this number we get
\[ 1 \leq \left( \frac{P_U(n)}{c^n} \right)^{t-1}. \]

Now taking limit \( n \to \infty \) gives a contradiction. It follows that we must have \( d_{t-i}(U) = d_t(U) \) for all \( t > 1 \) and consequently \( \bar{\rho}_U = \bar{\rho}_U \) by Lemma 3.3.

For any unitary representation \( U \in \text{uRep}(G) \) and \( n \in \mathbb{N} \) let us define
\[ b(U, n) = \sum (\dim \alpha)^2, \]
where \( \alpha \) ranges over all (nonequivalent) irreducible subrepresentations of \( \bigoplus_{k=0}^{n} U^{\otimes k} \). It is natural to say that the dual \( \hat{G} \) has subexponential growth if
\[ \lim_{n \to \infty} b(U, n)^{1/n} = 1 \quad (3.4) \]
holds for every \( U \in \text{uRep}(G) \) (cf. [2, Section 3] and [1]).

**Corollary 3.6.** Assume that \( \hat{G} \) has subexponential growth. Then for every \( U \in \text{uRep}(G) \) eigenvalues of \( \rho_U \) are symmetric i.e. \( \bar{\rho}_U = \bar{\rho}_U \).

**Proof.** We have \( P_U(n) \leq b(U, n) \) for every \( n \in \mathbb{N} \), hence the equality (3.4) implies that condition (3.3) from Theorem 3.4 holds. \( \square \)

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