QED in the Presence of Arbitrary Kramers–Kronig Dielectric Media

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The phenomenological Maxwell field is quantized for arbitrarily space- and frequency-dependent complex permittivity. The formalism takes account of the Kramers–Kronig relation and the dissipation-fluctuation theorem and yields the fundamental equal-time commutation relations of QED. Applications to the quantum-state transformation at absorbing and amplifying four-port devices and to the spontaneous decay of an excited atom in the presence of absorbing dielectric bodies are discussed.

I. INTRODUCTION

Quantization of the electromagnetic field in dispersive and absorbing dielectrics requires a concept which is consistent with both the principle of causality and the dissipation–fluctuation theorem and which necessarily yields the fundamental equal-time commutation relations of QED. In order to achieve this goal, several approaches are possible. The microscopic approach starts from the exact Hamiltonian of the coupled radiation–matter system and integrates out, in some approximation, the matter degrees of freedom to obtain an effective theory for the electromagnetic field. Since the procedure can hardly be performed for arbitrary media, simplified model systems are considered. A typical example is the use of harmonic-oscillator models for the matter polarization and the reservoir variables together with the assumption of bilinear couplings [1]. In the macroscopic approach, the phenomenological Maxwell theory, in which the effect of the medium is described in terms of constitutive equations, is quantized. Since this concept does not use any microscopic description of the medium, it has the benefit of being universally valid, at least as long as the medium can be regarded as a continuum.

Here we study the problem of quantization of the phenomenological Maxwell theory for nonmagnetic but otherwise arbitrary linear media at rest, starting from the classical Green function integral representation of the electromagnetic field. The method was first established for one-dimensional systems [2] and simple three-dimensional systems [3] and later generalized to arbitrary inhomogeneous dielectrics described in terms of a spatially varying permittivity which is a complex function of frequency [4].

In Sec. I we briefly review the quantization scheme and give an extension to anisotropic dielectrics (including amplifying media), which complete the class of nonmagnetic (local) media. In Sec. II we apply the method to the problem of quantum-state transformation at absorbing and amplifying four-port devices, and in Sec. III we give an application to the problem of spontaneous decay of an excited atom in the presence of absorbing bodies.
II. QUANTIZATION SCHEME

Let us first consider the electromagnetic field in isotropic dielectrics without external sources. The (operator-valued) phenomenological Maxwell equations in the temporal Fourier space read

\[ \nabla \cdot \mathbf{B}(\mathbf{r}, \omega) = 0, \quad \nabla \times \mathbf{E}(\mathbf{r}, \omega) = i\omega \mathbf{B}(\mathbf{r}, \omega), \]

\[ \nabla \cdot [\varepsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega)] = \mathbf{\hat{p}}(\mathbf{r}, \omega), \quad \nabla \times \mathbf{B}(\mathbf{r}, \omega) = -i(\omega/c^2)\varepsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega) + \mu_0 \mathbf{\hat{j}}(\mathbf{r}, \omega). \]

From the principle of causality it follows that the complex-valued permittivity \( \varepsilon(\mathbf{r}, \omega) = \varepsilon_R(\mathbf{r}, \omega) + i\varepsilon_i(\mathbf{r}, \omega) \) satisfies the Kramers–Kronig relations. Hence, it is a holomorphic function in the upper complex frequency plane without poles and zeros and approaches unity in the high-frequency limit. Consistency with the dissipation–fluctuation theorem requires the introduction of an operator noise charge density \( \mathbf{\hat{p}}(\mathbf{r}, \omega) \) and an operator noise current density \( \mathbf{\hat{j}}(\mathbf{r}, \omega) \) satisfying the equation of continuity. Quantization is performed by introducing bosonic vector fields \( \mathbf{f}(\mathbf{r}, \omega) \),

\[ \mathbf{\hat{j}}(\mathbf{r}, \omega) = \omega \sqrt{\hbar c \varepsilon_0 / \pi} \mathbf{\hat{E}}(\mathbf{r}, \omega), \]

which play the role of the fundamental variables of the theory. All relevant operators of the system such as the electric and magnetic fields and the matter polarization can be constructed in terms of them. For example, the operator of the electric field is given by the integral representation

\[ \mathbf{\hat{E}}_k(\mathbf{r}) = i\mu_0 \sqrt{\hbar c \varepsilon_0 / \pi} \int_0^\infty d\omega \int d^3 \mathbf{r}' \omega^2 \sqrt{\varepsilon_f(\mathbf{r}', \omega)} G_{kk'}(\mathbf{r}, \mathbf{r}', \omega) \mathbf{\hat{f}}_{k'}(\mathbf{r}', \omega) + \text{H.c.}, \]

with \( G_{kk'}(\mathbf{r}, \mathbf{r}', \omega) \) being the classical dyadic Green function. This representation together with the fundamental relation

\[ \int d^3 \mathbf{s} (\omega/c)^2 \varepsilon_f(\mathbf{s}, \omega) G_{ik}(\mathbf{r}, \mathbf{s}, \omega) G^{*}_{jk}(\mathbf{r}', \mathbf{s}, \omega) = \text{Im} G_{ij}(\mathbf{r}, \mathbf{r}', \omega), \]

which follows directly from the partial differential equation for the dyadic Green function, leads to the equal-time commutation relation \[ [\varepsilon_0 \mathbf{\hat{E}}_k(\mathbf{r}), \mathbf{\hat{B}}_l(\mathbf{r}')] = (\hbar / \pi) \varepsilon_{lmkl} \partial_{m}' \delta(\mathbf{r} - \mathbf{r}'), \]

Using general properties of the Green function, it can be shown that Eq. (10) reduces, for arbitrary \( \varepsilon(\mathbf{r}, \omega) \), to the well-known QED commutation relation

\[ [\varepsilon_0 \mathbf{\hat{E}}_k(\mathbf{r}), \mathbf{\hat{B}}_l(\mathbf{r}')] = i\hbar \varepsilon_{klm} \partial_{m}' \delta(\mathbf{r} - \mathbf{r}'). \]

The extension to anisotropic and amplifying media is straightforward, since we may assume the medium to be reciprocal, so that the permittivity tensor \( \varepsilon_{ij}(\mathbf{r}, \omega) \) is necessarily symmetric. In particular, \( \varepsilon_{ij}(\mathbf{r}, \omega) \) can be diagonalized by an orthogonal matrix \( O_{kl}(\mathbf{r}, \omega) \). With regard to amplifying media, we note that amplification requires the role of the noise creation and annihilation operators to be exchanged. The calculation then shows that the fundamental relation (3) can be generalized to

\[ \mathbf{\hat{j}}(\mathbf{r}, \omega) = \omega \sqrt{\hbar c \varepsilon_0 / \pi} \left[ \gamma_{ij}^- (\mathbf{r}, \omega) \mathbf{\hat{f}}_j(\mathbf{r}, \omega) + \gamma_{ij}^+ (\mathbf{r}, \omega) \mathbf{\hat{f}}_j^+(\mathbf{r}, \omega) \right], \]

with

\[ \gamma_{ij}^\pm (\mathbf{r}, \omega) = O_{ik}(\mathbf{r}, \omega) \sqrt{\varepsilon_{kl} I(\mathbf{r}, \omega)} \left[ O_{lj}^{-1}(\mathbf{r}, \omega) \Theta [\pm \varepsilon_{kl} I(\mathbf{r}, \omega)] \right], \]

\[ \varepsilon_{ij} I(\mathbf{r}, \omega) = \delta_{ij} \varepsilon_i^+(\mathbf{r}, \omega) = O_{ik}^{-1}(\mathbf{r}, \omega) \varepsilon_{kl} I(\mathbf{r}, \omega) O_{lj}(\mathbf{r}, \omega). \]

Equation (8) completes the quantization scheme for the electromagnetic field in arbitrary linear, nonmagnetic (local) media.
III. QUANTUM-STATE TRANSFORMATIONS BY ABSORBING AND AMPLIFYING FOUR-PORT DEVICES

Let us first apply the theory to the problem of quantum-state transformation at absorbing and amplifying four-port devices such as beam-splitter-like devices. Specifying the formulas to the one-dimensional case for simplicity and rewriting the integral representation (7) in terms of amplitude operators \( \hat{a}_j(\omega) \) and \( \hat{b}_j(\omega) \) for the incoming and outgoing waves \((j = 1, 2)\), the action of an absorbing device can be given by the (vector) operator transformation

\[
\hat{b}(\omega) = \mathbf{T}(\omega)\hat{a}(\omega) + \mathbf{A}(\omega)\hat{g}(\omega),
\]

(11)

where \( \hat{g}_j(\omega) \) are the operators of device excitations and \( \mathbf{T}(\omega) \) and \( \mathbf{A}(\omega) \) are the characteristic transformation and absorption matrices of the device given in terms of its complex refractive-index profile \( \omega \). Note that \( \hat{a}_j(\omega) \) and \( \hat{g}_j(\omega) \) are independent bosonic operators. Further, it can be shown that the relation \( \mathbf{T}(\omega)\mathbf{T}^+(\omega) + \mathbf{A}(\omega)\mathbf{A}^+(\omega) = \mathbf{I} \) is satisfied, which ensures bosonic commutation relations for \( \hat{b}_j(\omega) \). In order to construct the unitary transformation, we introduce some auxiliary (bosonic) device variables \( \hat{h}_j(\omega) \), combine the two-vectors \( \hat{a}(\omega) \) and \( \hat{g}(\omega) \) to the four-vector \( \hat{\alpha}(\omega) \), and accordingly \( \hat{b}(\omega) \) and \( \hat{h}(\omega) \) to \( \hat{\beta}(\omega) \). The four-vectors \( \hat{\alpha}(\omega) \) and \( \hat{\beta}(\omega) \) are related to each other as

\[
\hat{\beta}(\omega) = \Lambda(\omega)\hat{\alpha}(\omega), \quad \Lambda(\omega) \in \text{SU}(4).
\]

(12)

Introducing the positive Hermitian matrices \( \mathbf{C}(\omega) = \sqrt{\mathbf{T}(\omega)\mathbf{T}^+(\omega)} \) and \( \mathbf{S}(\omega) = \sqrt{\mathbf{A}(\omega)\mathbf{A}^+(\omega)} \), the four-matrix \( \Lambda(\omega) \) can be written in the form \( \mathbf{1} \)

\[
\Lambda(\omega) = \begin{pmatrix}
\mathbf{T}(\omega) & \mathbf{A}(\omega) \\
-\lambda\mathbf{S}(\omega)^{-1}\mathbf{C}^{-1}(\omega)\mathbf{T}(\omega) & \mathbf{C}(\omega)\mathbf{S}^{-1}(\omega)\mathbf{A}(\omega)
\end{pmatrix}
\]

(13)

(\( \lambda = 1 \)). The input–output relation \( \mathbf{1} \) can then be expressed in terms of a unitary operator transformation \( \hat{\beta}(\omega) = \hat{U}\hat{\alpha}(\omega)\hat{U} \). Equivalently, \( \hat{U} \) can be applied to the density operator of the input quantum state \( \hat{\rho}_{\text{in}}[\hat{\alpha}(\omega), \hat{\alpha}^\dagger(\omega)] \), and tracing over the device variables yields

\[
\hat{\rho}_{\text{out}}^\text{(Field)} = \mathbf{T}\mathbf{r}(\text{Device}) \left \{ \hat{\rho}_{\text{in}}[\mathbf{A}^+\hat{\alpha}(\omega), \mathbf{A}^T\hat{\alpha}^\dagger(\omega)] \right \}.
\]

(14)

To give an example, let us consider the case when one input channel is prepared in an \( n\)-photon Fock state and the device and the second input channel are left in vacuum, i.e., \( \hat{\rho}_{\text{in}} = |n, 0, 0, 0\rangle\langle n, 0, 0, 0| \). Applying Eq. \((\mathbf{14})\), after some algebra we derive for the density operator of the \( i\)-th output channel

\[
\hat{\rho}_{\text{out},i}^\text{(Field)} = \sum_{k=0}^{n} \binom{n}{k} |T_{i1}|^{2k} \left(1 - |T_{i1}|^2\right)^{n-k} |k\rangle\langle k|.
\]

(15)

Next, let us assume that the two input channels are prepared in single-photon Fock states, i.e., \( \hat{\rho}_{\text{in}} = |1, 1, 0, 0\rangle\langle 1, 1, 0, 0| \). We derive for the density operator of the \( i\)-th output channel

\[
\hat{\rho}_{\text{out},i}^\text{(Field)} = \left[1 - |T_{i1}|^2(1 - |T_{i2}|^2) - |T_{i2}|^2(1 - |T_{i1}|^2)\right] |0\rangle\langle 0| + \left(|T_{i1}|^2 + |T_{i2}|^2 - 4|T_{i1}|^2|T_{i2}|^2\right) |1\rangle\langle 1| + 2|T_{i1}|^2|T_{i2}|^2 |2\rangle\langle 2|.
\]

(16)

The extension to amplifying devices is straightforward. One has to replace the annihilation operators \( \hat{g}_j(\omega) \) in Eq. \((\mathbf{11})\) by the corresponding creation operators \( \hat{g}_j^\dagger(\omega) \). This leads again to an input–output relation of the form \( \mathbf{12} \) but with \( \lambda = -1 \) in Eq. \((\mathbf{13})\), the matrix \( \Lambda(\omega) \) being now an element of the noncompact group \( \text{SU}(2,2) \).
IV. SPONTANEOUS DECAY NEAR DIELECTRIC BODIES

Spontaneous decay of an excited atom is a process that is directly related to the quantum vacuum noise, which in the presence of absorbing bodies is drastically changed and so is the rate of spontaneous decay, because of the additional noise introduced by absorption. To study a radiating (two-level) atom in the presence of dielectric media, we start from the following Hamiltonian in dipole and rotating wave approximations:

\[
\hat{H} = \int d^3r \int_0^\infty d\omega \ h \hat{f}^\dagger(\mathbf{r}, \omega) \cdot \hat{f}(\mathbf{r}, \omega) + \sum_{\alpha=1}^2 \hbar \omega_\alpha \hat{A}_{\alpha\alpha} - \left[ i \omega_{21} \hat{A}^{(+)}(\mathbf{r}_A) \cdot \mathbf{d}_{21} + \text{H.c.} \right]. \tag{17}
\]

Here, the atomic operators \(\hat{A}_{\alpha\alpha'} = |\alpha\rangle\langle\alpha'|\) are introduced, and \(\hat{A}^{(+)}(\mathbf{r}_A)\) is the (positive-frequency part of the) vector potential (in Weyl gauge) at the position of the atom. Note that the first term in Eq. (17) is the (diagonal) Hamiltonian of the system that consists of the electromagnetic field and the medium (including the dissipative system) and is expressed in terms of the fundamental variables \(\hat{f}(\mathbf{r}, \omega)\). Solving the resulting equations of motion in Markov approximation, the well-known Bloch equations for the atom are recognized, where the decay rate is given by [8]

\[
\Gamma = 2 \omega_2^2 \mu_k \mu_k' / (\hbar \varepsilon_0 c^2) \text{Im} G_{kk'}(\mathbf{r}_A, \mathbf{r}_A, \omega_A) \tag{18}
\]

\([\mu_k \equiv (d_{21})_k, \omega_A \equiv \omega_{21}]\). Note that from Eq. (4) together with Eq. (3) it follows that

\[
\text{Im} G_{kk'}(\mathbf{r}, \mathbf{r}', \omega) \delta(\omega - \omega') = \pi \varepsilon_0 c^2 / (\hbar \omega^2) \langle 0 | [\hat{E}_k(\mathbf{r}, \omega), \hat{E}^\dagger_{k'}(\mathbf{r}', \omega')] | 0 \rangle \tag{19}
\]

in full agreement with the dissipation-fluctuation theorem.

Equation (18) is valid for any absorbing dielectric body. For example, when the atom is sufficiently near to an absorbing planar interface, then purely nonradiative decay is observed, with [8]

\[
\Gamma = \Gamma_0 \left( 1 + \frac{\mu^2}{\mu'^2} \right) \frac{\epsilon_f(\omega_A)}{\epsilon(\omega_A) + 1} \frac{3c^3}{(2\omega_A z)^3}, \tag{20}
\]

where \(z\) is the distance between the atom and the interface, and \(\Gamma_0\) is the spontaneous emission rate in free space (for a guest atom embedded in an absorbing dielectric, see [7]).

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