Probabilistic characterisation of Besov-Lipschitz spaces on metric measure spaces

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Abstract

We give a probabilistic characterisation of the Besov-Lipschitz spaces $\text{Lip}(\alpha, p, q)(X)$ on domains which support a Markovian kernel with appropriate exponential bounds. This extends former results of [11, 14, 15, 7] which were valid for $\alpha = \frac{d_w}{2}, p = 2, q = \infty$, where $d_w$ is the walk dimension of the space $X$.

1 Introduction

There are several definitions of Besov-Lipschitz spaces on measure spaces. In this paper we will investigate the spaces $\text{Lip}(\alpha, p, q)(X)$, as defined in Jonsson [11]. Jonsson’s paper dealt with the Sierpiński gasket embedded in $\mathbb{R}^d$ only, but did not really use the embedding itself, and so this particular definition can be extended to general metric measure spaces (see e.g. [6], [13]). The most convenient to analyse are those spaces on which there exists a complete, symmetric Markovian kernel with appropriate exponential bounds. There are several results concerning such spaces, see e.g. [7], [15], [16], [10].

The existence of a Markovian kernel on $X$ of this type is equivalent to the existence of a fractional diffusion on $X$ (see [1] for the definition). Its generator, often called ‘the Laplacian’ on a general metric space, serves as a substitute for the bona fide differentiation operator, even though the differential itself is not-so-convenient to define in this generality. This is one of the reasons why the existence of such a kernel allows to prove certain properties of underlying spaces. In particular, in a series of papers([11, 14, 15, 7]) it has been proven that the spaces $\text{Lip}(\frac{d_w}{2}, 2, \infty)(X)$ (where $d_w$ is the walk dimension of $X$) are domains of the Dirichlet form associated with this particular diffusion – and

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so it can be described using the kernel \( p(\cdot, \cdot, \cdot) \). Also, the spaces \( \text{Lip}(\alpha, 2, 2) \) allow for a probabilistic characterisation (see [18]). In present work we extend these results and provide a characterisation of the spaces \( \text{Lip}(\alpha, p, q)(X) \) in terms of the Markovian kernel whose existence we are assuming, for \( \alpha > 0, p, q \geq 1 \) (nox excluding \( q = \infty \)). Our proof is entirely elementary and uses only a variant of discrete Hardy inequality (proven below).

Besov spaces, on the very same class of metric measure spaces, were also introduced by Hu and Zähle – in a different way – in their paper [10]. The way they are defined owes to the classical characterisation of Besov spaces from [5], [17]. Those spaces will be denoted by \( B^{p,q}_{\beta}(X) \); We will see that our characterisation of \( \text{Lip}(\alpha, p, q)(X) \) is consistent with \( B^{p,q}_{\beta}(X) \) for some range of parameters (see Section 4.2).

In the case of simple fractals (and the Sierpiński gasket in particular), yet another definition of Besov-Lipschitz spaces was given by Strichartz in [19]. This definition uses a discrete approximation of the space \( X \). The Strichartz spaces we think of are the spaces \( (\Lambda^{p,q}_{\alpha})^{1}(X) \) (in [19], one can find other spaces as well, corresponding to large values of \( \alpha \)). Strichartz spaces do not allow the smoothness parameter \( \alpha \) to be too small; the definition uses a discrete approximation of simple fractals, and so it is mandatory that the functions concerned be continuous. This is not necessarily true for small values of \( \alpha \). It is known (see [3]) that the Strichartz spaces and the Jonsson spaces agree for certain range of parameters. Therefore our characterisation remains valid for Strichartz spaces as well (see Section 4.1).

### 2 Preliminaries

**Convention.** In the sequel, \( c \) will denote a generic constant whose value is irrelevant and can change from line to line. The important constants will be denoted by upper case letters or by letters with subscripts: \( c_1, 2, \ldots \). When we write \( A \asymp B \), then we mean that for some \( c > 0, c^{-1}A \leq B \leq cA \).

Suppose \((X, \rho)\) is a locally compact metric space and that \( \mu \) is a Borel measure on \( X \) which is Ahlfors \( d \)-regular, i.e. such that

\[
\forall x \in X \forall 0 < r < \text{diam} X \quad C_1 r^d \leq \mu(B(x, r)) \leq C_2 r^d \tag{2.1}
\]

where \( d > 0 \) and \( C_1, C_2 \) are positive constants.

Following Jonsson, we define the Besov-Lipschitz spaces as follows.

**Definition 2.1** Suppose \( \alpha > 0, p, q \in [1, \infty) \). Then \( \text{Lip}(\alpha, p, q)(X) \) is, by definition, the collection of those \( f \in L^p(X, \mu) \) for which \( \| (a_m(f)) \|_q < +\infty \), where

\[
a_m(f) = 2^{m\alpha} \left( 2^{md} \int \int \chi_{\rho(x,y) \leq 2^{-m}} |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} \tag{2.2}
\]
When \( p = \infty \) or \( q = \infty \) then the usual modifications are needed.

The expression
\[
\|f\|_{\alpha, p, q} = \|(a_m(f))\|_q
\]
is a seminorm, which can be turned into a norm by adding \( \|f\|_{L^p} \). The space \( \text{Lip}(\alpha, p, q)(X) \), equipped with the norm
\[
\|f\|_{\text{Lip}(\alpha, p, q)} = \|f\|_{L^p} + \|f\|_{\alpha, p, q}
\]
is a Banach space.

In the sequel, we assume that there exists a symmetric Markovian kernel \( \{p(t, x, y)\}_{t>0} \) on \( X \), i.e. a family of measurable functions \( p(t, \cdot, \cdot) : X \times X \to \mathbb{R}_+ \), which satisfies:

(A1) \( \forall t>0 \forall x,y \in X \quad p(t, t, y) = p(t, y, x) \) (symmetry),

(A2) \( \forall t>0 \forall x \in X \quad \int_X p(t, x, y) d\mu(y) = 1 \) (normalisation or stochastic completeness),

(A3) \( \forall s,t>0 \forall x,y \in X \quad p(s+t, x, y) = \int_X p(s, x, z)p(t, z, y) d\mu(z) \) (the Chapman-Kolmogorov identity, or the Markov property),

(A4) \( \forall t>0 \forall x,y \in X \quad p(t, x, y) > 0 \) (irreducibility),

(A5) \( \forall f \in L^2(X), \quad P_tf \to f \) when \( t \to 0 \), strongly in \( L^2(X) \), where \( P_tf(x) = \int_X f(y)p(t, x, y) d\mu(y) \) (strong continuity).

These conditions allow us to freely use the Dirichlet form theory for Markov processes.

And, finally, our main assumption:

(A6)
\[
\frac{c_1}{d/dw} e^{-c_2 \left( \frac{\rho(x,y)}{\left( t^{\frac{1}{d+1}} \right)^{d/dw}} \right)^{\frac{d/dw}{d/dw}}} \leq p(t, x, y) \leq \frac{c_3}{d/dw} e^{-c_4 \left( \frac{\rho(x,y)}{\left( t^{\frac{1}{d+1}} \right)^{d/dw}} \right)^{\frac{d/dw}{d/dw}}}. \tag{2.4}
\]

The parameter \( d/w \) is usually called the walk dimension of \( X \), as it controls the weak time/space scaling of the Markov process with transition density \( p(\cdot, \cdot, \cdot) \). Such a process will be denoted by \( (B_t, \mathbb{P}_x)_{t \geq 0, x \in X} \). We know that the parameter \( d/w \) is the same for all possible Markov processes sharing the estimate (2.4). When the space \( (X, \rho) \) satisfies the chain condition:

(CC) there exists a constant \( C > 0 \) such that for any \( x, y \in X \) and any positive integer \( n \), there exists a chain \( x = x_0, x_1, \ldots, x_n = y \) of points from \( X \) s.t. \( \rho(x_i, x_{i+1}) \leq \frac{C}{n} \rho(x, y) \),

then
\[
d \leq d/w \leq d + 1.
\]
The estimate (2.4) ensures that this process is a diffusion. In fact it is known (see [8]) that when the process is a diffusion, then the only possible function $\Phi$ in the estimate of the form $\frac{c}{t^{d/2w}} \Phi \left( \frac{t^{d/2w}}{1^{d/2w}} \right)$ is the exponential function as in (2.4).

Among the examples, we can list

- the Euclidean space $\mathbb{R}^d$ with the Gaussian kernel, $g(t, x, y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x-y|^2}{2t}\right)$, certain manifolds with nonnegative Ricci curvature,

- simple fractals, where $p(t, x, y)$ is the transition density of the Brownian motion, and can be bounded from both above and below by $\frac{c}{t^{d/2w}} \exp(\frac{c}{t^{d/2w}} d_w/(d_w-1))$, see [12],

- the Brownian motion on p.c.f. self similar sets and on the Sierpiński carpets, where we have an estimate analogous to that on simple nested fractals (see [9] and [2]).

3 The main theorem

In a series of papers ([11, 14, 15, 7]) it has been proven that the domain of the Dirichlet form associated with the Markovian kernel satisfying (1)-(6) is equal to the space $\text{Lip} \left( \frac{d_w}{2}, 2, \infty \right) (X)$, and that the norms: of the Dirichlet space $D(E)$ and of the Besov-Lipschitz space $\text{Lip} \left( \frac{d_w}{2}, 2, \infty \right) (X)$ are equivalent.

The domain of the Dirichlet form, $D(E)$, consists of those functions $f \in L^2(X, \mu)$, for which $s(f) < \infty$, where

$$s(f) = \sup_{t>0} \frac{1}{2t} \int_X \int_X (f(x) - f(y))^2 p(t, x, y) d\mu(x) d\mu(y). \quad (3.1)$$

We will give a similar characterisation of the spaces $\text{Lip}(\alpha, p, q)(X)$, for general parameters $\alpha > 0$, $p, q \geq 1$. Namely, we show:

**Theorem 3.1** Suppose $\alpha > 0$ and $p, q \in [1, \infty)$. Then $f$ belongs to the Lipschitz-Besov space $\text{Lip}(\alpha, p, q)(X)$ if and only if:

1. $f \in L^p(X, \mu)$,

2. $I^{(\alpha)}(f) := \int_0^1 \frac{1}{t^{d/2w}} \left( \int_X \int_X |f(x) - f(y)|^p p(t, x, y) d\mu(x) d\mu(y) \right)^{\frac{1}{p}} \frac{dt}{t} < \infty$.

Moreover, we have

$$\|f\|_{\text{Lip}(\alpha, p, q)} \approx \|f\|_{L^p} + \left( I^{(\alpha)}(f) \right)^{\frac{1}{q}}. \quad (3.2)$$
Proof. Part 1. Suppose that $f \in L^p(X, \mu)$ and that $I^{(\alpha)}(f) < \infty$.

For later use, introduce the notation

$$I^{(\alpha)}_m(f) = \int \int_{|x-y| \leq 2^{-m}} |f(x) - f(y)|^p \, d\mu(x) d\mu(y).$$

(3.3)

Clearly, we have

$$I^{(\alpha)}(f) \geq \int_0^1 \frac{1}{t^{\alpha w}} \left( \int \int_{\rho(x,y) \leq t^{\alpha w}} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \right)^{\frac{q}{p}} \frac{dt}{t}.$$ 

When $\rho(x,y) \leq t^{\alpha w}$, then $p(t,x,y)$ is nearly constant, and so (2.4) gives

$$I^{(\alpha)}(f) \geq c \sum_{m=0}^\infty \int_{2^{-(m+1)d_w}}^{2^{-md_w}} \frac{1}{t^{\alpha w}} \left( \int \int_{\rho(x,y) \leq t^{\alpha w}} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \right)^{\frac{q}{p}} \frac{dt}{t}.$$ 

$$\geq c \sum_{m=0}^\infty \sum_{m=0}^{2^{m}q} (2^{md_w} I^{(\alpha)}_m(f))^{\frac{q}{p}} = c \|(a^{(\alpha)}_m(f))\|_{L^q} = c \|f\|_{L^{p,q}}.$$ 

This proves the inequality

$$\|f\|_{L^p} + (I^{(\alpha)}(f))^{1/q} \geq c \|f\|_{L^{p,q}}.$$ 

Part 2. Now we prove the opposite inequality. Suppose that $f \in Lip(\alpha,p,q)(X)$. Similarly as before, write

$$I^{(\alpha)}(f) = \int \int_{Z_m} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) + \int \int_{\bar{Z}_m} |f(x) - f(y)|^p \, d\mu(x) d\mu(y).$$

For given $m$, we split the inner integral into two parts: over the set $Z_m = \{\rho(x,y) \leq 2^{-m/2}\}$, and over its complement $\bar{Z}_m = \{\rho(x,y) > 2^{-m/2}\}$, i.e.

$$I^{(\alpha)}(f) \leq c \sum_{m=0}^\infty \int_{2^{-(m+1)d_w}}^{2^{-md_w}} \frac{1}{t^{\alpha w}} \left( \int \int_{Z_m} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \right)^{\frac{q}{p}} \frac{dt}{t}$$

$$+ c \sum_{m=0}^\infty \int_{2^{-(m+1)d_w}}^{2^{-md_w}} \frac{1}{t^{\alpha w}} \left( \int \int_{\bar{Z}_m} |f(x) - f(y)|^p \, d\mu(x) d\mu(y) \right)^{\frac{q}{p}} \frac{dt}{t},$$

$$=: I^{(\alpha)}_1 + I^{(\alpha)}_2.$$
The integral over $Z_m^c$ is not bigger than (use symmetry)

\[2^p \int_{Z_m^c} |f(x)|^p p(t, x, y) \, d\mu(x) \, d\mu(y) = 2^p \int_X |f(x)|^p \left( \int_{\{y \in \mathbb{R}^d : |y - \bar{y}| > 2^{-m/2}\}} p(t, x, y) \, d\mu(y) \right) \, d\mu(x)\]

\[= 2^p \|f\|_{L^p}^p \sup_{x \in X} \mathbb{P}_x[\rho(x, B_t) > 2^{-m/2}] \leq 2^p \|f\|_{L^p}^p \exp \left( -c 2^{-m/2} \right) \frac{d\mu}{\bar{\mu} - 1}\]

(we have used the estimate $\mathbb{P}_x[\rho(x, B_t) \geq \delta] \leq \exp(-c(\delta t^{-1/d_w} d_w - 1))$, valid under our assumption (A6), see [1]).

Observe that while integrating over $Z_m^c$, the values of $t$ are confined to $[2^{-(m+1)d_w}, 2^{-md_w}]$, and so the integral we are estimating does not exceed

\[c\|f\|_{L^p}^p \exp(-c(2^{m/2} d_w - 1)),\]

thus

\[I_2^{(\alpha)}(f) \leq c \sum_{m=0}^{\infty} \int_{2^{-(m+1)d_w}}^{2^{-md_w}} \frac{1}{t^{\bar{\mu} - 1}} \left( \|f\|_{L^p}^p \exp(-c(2^{m/2} d_w - 1)) \right)^p \frac{dt}{t} \leq c\|f\|_{L^p}^q \sum_{m=0}^{\infty} 2^{m\alpha \rho} \exp(-c(2^{m/2} d_w - 1)) \leq c\|f\|_{L^p}^q.\]

We are left with estimating $I_1^{(\alpha)}(f)$, which requires subtler tools.

The double integral over the set $Z_m$ can be written as

\[\sum_{k=m/2}^{\infty} \int \int_{2^{-(k+1)} \leq \rho(x, y) \leq 2^k} |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) \, d\mu(y), \quad (3.4)\]

and again, in this integral we have $t \in [2^{-(m+1)d_w}, 2^{-md_w}]$, so from the basic estimate (2.4) for the transition density, we get that (3.4) is not bigger than

\[c \sum_{k=m/2}^{\infty} 2^{md} \left( \int \int_{2^{-(k+1)} \leq \rho(x, y) \leq 2^k} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y) \right) \exp(-c_4(2^{m-2k} d_w - 1)) \]

\[\leq c \sum_{k=m/2}^{\infty} 2^{md} \exp(-c_4(2^{m-k} d_w - 1)) i_k^{(\alpha)}(f)\]

($i_k^{(\alpha)}(f)$ was defined by (3.3)). Consequently,

\[I_1^{(\alpha)}(f) \leq c \sum_{m=0}^{\infty} 2^{m\rho q} \left( \sum_{k=m/2}^{\infty} 2^{md} \exp(-c_4(2^{(m-k) d_w - 1})) i_k^{(\alpha)}(f) \right)^{\frac{2}{p}}\]
To estimate these double sums we will use the discrete Hardy inequalities: the classical Hardy inequality (3.6) for the second sum, and the modified Hardy inequality (3.7) for the first one.

We include them as lemmas. Lemma 3.1 is classical so we omit its proof.

**Lemma 3.1 (classical discrete Hardy inequality)** Suppose $r > 0$, $t > 1$, $x_m > 0$, $m = 1, 2, \ldots$. Then

$$
\sum_{m=0}^{\infty} t^m \left( \sum_{k=m}^{\infty} x_k \right)^r \leq K \sum_{m=0}^{\infty} t^m x_m^r,
$$

where $K$ is a constant depending on $r, t$ only.

**Lemma 3.2 (modified discrete Hardy inequality)** Suppose $r > 0$, $t > 1$, $x_m > 0$, $\kappa > 1$, $\lambda > 0$. Then

$$
\sum_{m=0}^{\infty} t^m \left( \sum_{k=m/2}^{m} x_k e^{-\lambda \kappa^{m-k}} \right)^r \leq K \sum_{m=0}^{\infty} t^m x_m^r,
$$

where the constant $K$ depends on $r, t, \kappa, \lambda$ only.

**Proof of Lemma 3.2.**

**Case 1.** $r \leq 1$. Starting with the elementary inequality

$$(y_1 + \ldots + y_n)^r \leq y_1^r + \ldots + y_n^r,$$

which is valid for all $n = 1, 2, \ldots$, $y_1, \ldots, y_n \geq 0$ and $r \in (0, 1]$, we have

$$
\sum_{m=0}^{\infty} t^m \left( \sum_{k=m/2}^{m} x_k \exp(-\lambda \kappa^{m-k}) \right)^r \leq \sum_{m=0}^{\infty} t^m \left( \sum_{k=m/2}^{m} x_k^r \exp(-\lambda r \kappa^{m-k}) \right) = \sum_{k=0}^{\infty} x_k^r \left( \sum_{m=k}^{2k} t^m \exp(-\lambda r \kappa^{m-k}) \right) = \sum_{k=0}^{\infty} x_k^r k^{2k} \left( \sum_{m=k}^{m-k} t^m \exp(-\lambda r \kappa^{m-k}) \right).
$$
\[
\leq \sum_{k=0}^{\infty} x_k^r t^k \left( \sum_{m=0}^{\infty} t^m \exp(-\lambda r \kappa^m) \right) = K(t, r, \kappa) \sum_{k=0}^{\infty} x_k^r t^k,
\]

because the series \( \sum_{m=0}^{\infty} t^m \exp(-\lambda r \kappa^m) \) is convergent.

**Case 2.** \( r > 1 \).
First, we extend the inner sum to \( k \) from 0 to \( m \). Also, to avoid problems with summability, we replace the infinite series \( \sum_{m=0}^{\infty} t^m \exp(-\lambda r \kappa^m) \) with a finite one \( \sum_{m=0}^{M} t^m \exp(-\lambda r \kappa^m) \), prove the appropriate inequality with a constant \( K \) not depending on \( M \) and pass to the limit \( M \to \infty \) afterwards.

Let \( a = 2 \ln \frac{t}{\ln \kappa} \) (so that \( \kappa^a r = t^2 \)). There exists a constant \( C > 0 \) such that \( e^{-\lambda x} \leq C x^{-a} \), for \( x > 0 \). It follows:

\[
\sum_{m=0}^{M} t^m \left( \sum_{k=0}^{m} x_k^r \kappa^{m-k} \right)^r \leq C^r \sum_{m=0}^{M} t^m \left( \sum_{k=0}^{m} x_k^r \frac{1}{\kappa^{(m-k)a}} \right)^r = C^r \sum_{m=0}^{M} \frac{t^m}{\kappa^{amr}} \left( \sum_{k=0}^{m} x_k^r \kappa^{ka} \right)^r = C^r \sum_{m=0}^{M} \frac{1}{t^m} \left( \sum_{k=0}^{m} x_k^r \kappa^{ka} \right)^r. \tag{3.8}
\]

Denote \( S_m = \sum_{k=0}^{m} x_k^r \kappa^{ka} \) (for completeness, set \( S_{-1} = 0 \)) and let \( \tau = \frac{1}{r} (< 1) \). We have:

\[
\sum_{m=0}^{M} \tau^m (S_m^r - S_{m-1}^r) = \sum_{m=1}^{M-1} \tau^m S_m^r (1 - \tau) + \tau^M S_M^r \geq (1 - \tau) \sum_{m=0}^{M} \tau^m S_m^r,
\]

and so

\[
(3.8) \quad = \quad C^r \sum_{m=0}^{M} \tau^m S_m^r 
\leq \quad \frac{C^r}{1 - \tau} \sum_{m=0}^{M} \tau^m (S_m^r - S_{m-1}^r)
\leq \quad \frac{C^r}{1 - \tau} \sum_{m=0}^{M} \tau^m x_m \kappa^{ma} S_{m-1}^r
\]

(this is so because for \( x, y > 0 \) we have \((x + y)^r - x^r \leq r y (x + y)^{r-1}\)).

Now use the following discrete Hölder inequality:
for $M = 1, 2, \ldots, \tau > 0$, $A_m, B_m \geq 0$ for $m = 0, 1, 2, \ldots, M$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$
\sum_{m=0}^{M} A_m B_m \tau_m \leq \left( \sum_{m=0}^{M} A_m^p \tau_m \right)^{\frac{1}{p}} \left( \sum_{m=0}^{M} B_m^q \tau_m \right)^{\frac{1}{q}}.
$$

(3.9)

Applying (3.9) with $p = r, q = \frac{r}{r-1}, A_m = x_m \kappa^m a, B_m = S_m^{r-1}$ we get:

$$
(3.8) \leq \frac{rC_r}{1-\tau} \left( \sum_{m=0}^{M} \tau_m x_m \kappa^{mar} \right)^{\frac{1}{r}} \left( \sum_{m=0}^{M} \tau_m S_m^r \right)^{\frac{1}{r-1}},
$$

which results in

$$
\sum_{m=0}^{M} \tau_m S_m^r \leq \frac{rC_r}{1-\tau} \left( \sum_{m=0}^{M} \tau_m x_m \kappa^{mar} \right)^{\frac{1}{r}} \left( \sum_{m=0}^{M} \tau_m S_m^r \right)^{\frac{1}{r-1}},
$$

and further in

$$
\left( \sum_{m=0}^{M} \tau_m S_m^r \right)^{\frac{1}{r}} \leq \frac{rC_r}{1-\tau} \left( \sum_{m=0}^{M} \tau_m x_m \kappa^{mar} \right)^{\frac{1}{r}}.
$$

Since $S_m = \sum_{k=0}^{m} x_k \kappa^{ka}, \tau = \frac{1}{r}, \kappa^{ar} = t^r$, from (3.8) it follows that we are done. \(\square\)

Conclusion of the proof of Theorem 3.1.} The first sum in (3.5) is estimated by (3.7), with $t = 2^{q(\alpha + \frac{d}{p})}, x_k = i_k^{(\alpha)}(f), \kappa = 2^{\frac{d}{p} - 1}$. For the second sum in (3.5), first forget about the exponential factor (which is smaller than 1 anyway), and then use (3.6) with $t$ and $x_k$ as in the first sum. What we get is:

$$
I^{(\alpha)}_1(f) \leq c \sum_{m=1}^{\infty} 2^{mq(\alpha + \frac{d}{p})} \left( i_m^{(\alpha)}(f) \right)^\frac{q}{p} = c \sum_{m=1}^{\infty} 2^{mq} \left( 2^{md} i_m^{(\alpha)}(f) \right)^\frac{q}{p} = \| (a_m^{(\alpha)}(f)) \|_q^q.
$$

Collecting all the estimates obtained, we get that

$$
I^{(\alpha)}(f) \leq c \| f \|_{L^p}^q + c \| (a_m(f)) \|_q^q,
$$

and further

$$
(I^{(\alpha)}(f))^{\frac{1}{q}} \leq c \| f \|_{L^p} + c \| (a_m(f)) \|_q.
$$

This concludes the proof. \(\square\)

This theorem has a natural extension to the case $q = \infty$. In For $\alpha = \frac{dp}{2}, p = 2$ and $q = \infty$ it has been proven in [11, 15, 7]. To obtain the desired result, we basically follow the lines of [15] and [7].
Theorem 3.2 [extension to the case \(q = \infty\)] Let \(\alpha > 0, p \geq 1, f \in L^p(X, \mu)\) and let \(a^{(\alpha)}_m(f)\) be defined as before. Then \(\sup_{m > 0} a^{(\alpha)}_m(f)\) is finite if and only if

\[
\sup_{t \in (0, 1)} \frac{1}{t^{\alpha \beta}} \int_X \int_X |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) \, d\mu(y) < \infty.
\]

Moreover,

\[
\|(a^{(\alpha)}_m(f))\|_\infty \asymp \|f\|_p + \sup_{t > 0} \frac{1}{t^{\alpha \beta}} \int_X \int_X |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) \, d\mu(y),
\]

Before we start the proof of Theorem 3.2, let us state and prove the following simple lemma.

Lemma 3.3 Suppose \(C, \alpha, \beta, \gamma > 0\). Then there exist constants \(K_1, K_2 = K_{1,2}(C, \alpha, \beta, \gamma)\) such that

\[
K_1 t^{\alpha \beta} \leq \sum_{m=0}^{\infty} 2^{-m\alpha} \exp\left(- \left(\frac{C}{2m^\beta}\right)^\gamma\right) \leq K_2 t^{\alpha \beta}.
\]  

(3.10)

Proof of the lemma. Consider

\[
I_C := \int_0^1 \exp(-\left(\frac{C x^{\frac{1}{\beta}}}{t^\beta}\right)^\gamma) \, dx = \sum_{m=0}^{\infty} \int_{2^{-(m+1)\alpha}}^{2^{-m\alpha}} \exp(-\left(\frac{C x^{\frac{1}{\beta}}}{t^\beta}\right)^\gamma) \, dx =: \sum_{m=0}^{\infty} I_m.
\]

The function \(x \mapsto \exp(-\left(\frac{C x^{\frac{1}{\beta}}}{t^\beta}\right)^\gamma)\) is monotone decreasing, and so

\[
\frac{1}{2m\alpha} \left(1 - \frac{1}{2^\alpha}\right) \exp(-\left(\frac{C}{2m^\beta}\right)^\gamma) \leq I_m \leq \frac{1}{2m\alpha} \left(1 - \frac{1}{2^\alpha}\right) \exp(-\left(\frac{1}{2^\alpha} \left(\frac{C}{2m^\beta}\right)^\gamma\right)).
\]

Summing up over \(m\), we get

\[
\left(1 - \frac{1}{2^\alpha}\right) \sum_{m=0}^{\infty} \frac{1}{2^{m\alpha}} \exp(-\left(\frac{C}{2m^\beta}\right)^\gamma) \leq I_C \leq \left(1 - \frac{1}{2^\alpha}\right) \sum_{m=0}^{\infty} \frac{1}{2^{m\alpha}} \exp(-\left(\frac{1}{2^\alpha} \left(\frac{C}{2m^\beta}\right)^\gamma\right))
\]

and it follows that

\[
\frac{2^\alpha}{2^\alpha - 1} I_C / 2^\alpha \leq \sum_{m=0}^{\infty} 2^{-m\alpha} \exp\left(- \left(\frac{C}{2m^\beta}\right)^\gamma\right) \leq \frac{2^\alpha}{2^\alpha - 1} I_C.
\]

The last thing we have to do is to single out the dependence of \(I_C\) on \(t\). Substitute \(z = x^{\gamma/\alpha} t^{-\beta \gamma}\) in the integral, so that

\[
I_C = t^{\alpha \beta} \int_0^{t^{-\beta \gamma}} \exp(-C^\gamma z) z^{\alpha/\gamma - 1} \, dz,
\]

10
which is integrable as long as \( \alpha / \gamma > 0 \). We are done. \( \square \)

**Proof of Theorem 3.2.**  
**Part 1. The lower bound.** Suppose that \( f \in L^p(X, \mu) \) is such that

\[
s(f) = \sup_{t \in (0, 1)} \frac{1}{t^{d/\omega}} \int_X \int_X |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) d\mu(y) < \infty.
\]

Similarly to what we have done before, restrict the area of integration to the set \( \{\rho(x, y) \leq t^{\gamma/\omega}\} \), and use the bound for the transition density, so that

\[
s(f) \geq c \sup_{t \in (0, 1)} \frac{1}{t^{d/\omega}} \int_X \int_\rho(x, y) \leq t^{\gamma/\omega} |f(x) - f(y)|^p \, d\mu(x) d\mu(y)
\]

\[
\geq c \sup_m 2^{mp\alpha} 2^{md} \int_X \int_\rho(x, y) \leq 2^{-m} |f(x) - f(y)|^p \, d\mu(x) d\mu(y),
\]

where the last inequality was obtained by using \( t_m = 2^{-md\omega} \).

**Part 2. The upper bound.** Now suppose that \( f \in L^p \) and that \( \|(a_m^{(\alpha)}(f))\|_\infty < \infty \).

Write the integral in the definition of \( s(f) \) as

\[
\sum_{m=0}^{\infty} \int_{X} \int_{2^{-(m+1)} < \rho(x, y) \leq 2^{-m}} |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) d\mu(y)
\]

\[
\leq c t^{-d/\omega} \sum_{m=0}^{\infty} \exp \left( -c_4 \left( \frac{1}{2m^{d/\omega}} \right)^{d/\omega} \right) \int_X \int_{\rho(x, y) \leq 2^{-m}} |f(x) - f(y)|^p \, d\mu(x) d\mu(y)
\]

\[
\leq c \| (a_m^{(\alpha)}(f)) \|_\infty t^{-d/\omega} \sum_{m=0}^{\infty} 2^{-m(d+p\alpha)} \exp \left( -c_4 \left( \frac{1}{2m^{d/\omega}} \right)^{d/\omega} \right).
\]

Lemma 3.3 allows us to estimate the last sum by \( c t^{-d/\omega} \), and so we get

\[
\frac{1}{t^{d/\omega}} \int_X \int_X |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) d\mu(y) \leq c \|(a_m^{(\alpha)}(f))\|_\infty.
\]

We are done. \( \square \)

**Remark 3.1** \((q < \infty)\) For a function \( f \in L^p(X, \mu) \) it is immediate to establish that

\[
\int_1^{\infty} \frac{1}{t^{d/\omega}} \left( \int_X \int_X |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) d\mu(y) \right)^{\frac{q}{p}} \, dt \leq c \|f\|_{L^p}^q.
\]

(3.11)

Indeed, from symmetry we have

\[
\int_X \int_X |f(x) - f(y)|^p \, d\mu(x) d\mu(y)
\]

\[
\leq 2^p \int_X \int_X |f(x)|^p p(t, x, y) \, d\mu(x) d\mu(y)
\]

\[
= 2^p \int_X |f(x)|^p \left( \int_X p(t, x, y) \, d\mu(y) \right) d\mu(x) = 2^p \|f\|_{L^p}^p.
\]
and (3.11) follows.

Therefore the norm in $\text{Lip}(\alpha, p, q)(X)$ is also equivalent to

$$\|f\|_{L_p} + (\bar{I}(\alpha)(f))^{\frac{1}{q}},$$

where

$$\bar{I}(\alpha)(f) = \int_0^\infty \frac{1}{t^{\frac{\alpha}{p}}} \left( \int_X \int_X |f(x) - f(y)|^p p(t, x, y) \, d\mu(x) d\mu(y) \right)^{\frac{1}{q}} \, dt.$$

\textbf{Remark 3.2} ($q = \infty$) Similarly, in this case we can take the supremum over $t > 0$ instead of $t \in (0, 1)$ and still get an equivalent norm.

\textbf{3.1 Range of parameters allowed}

The definition of Besov-Lipschitz spaces, as well as our characterisation, work for arbitrary values of $\alpha > 0$, $p, q \geq 1$. However, for some triples of parameters the resulting spaces are trivial, and consist of constant functions only.

So far we have satisfactory results for $p = 2$ only. It is known that (see [11], Cor. 3 for the Sierpiński gasket, [15], Prop. 2 for the general case) the spaces $\text{Lip}(\alpha, 2, \infty)(X)$ are degenerate when $\alpha > \frac{d_w}{2}$. From here, it is immediate to see that $\text{Lip}(\alpha, 2, q)(X)$ are degenerate as well when $\alpha > \frac{d_w}{2}$. On the other hand, when $\alpha \leq \frac{d_w}{2}$, then the spaces $\text{Lip}(\alpha, 2, \infty)(X)$ are dense in $L^2(X)$.

Therefore $\alpha \leq \frac{d_w}{2}$ is the natural threshold for the spaces $\text{Lip}(\alpha, 2, \infty)$.

Consider now the spaces $\text{Lip}(\alpha, p, p)(X)$. It is clear that for $f \in L^p(X)$ we have

$$\|f\|_{\alpha, p, p} \propto \int_X \int_X \frac{|f(x) - f(y)|^p}{\rho(x, y)^{d+\alpha}} d\mu(x) d\mu(y) + \|f\|_{L_p}^p$$

(when diam $X < \infty$, then the term $\|f\|_{L_p}^p$ can be omitted).

In [16] it has been proven that the finiteness of the integral in (3.12), when $p = 2$ and $\alpha \geq \frac{d_w}{2}$ implies that $f = \text{const}$ (and for $\alpha < \frac{d_w}{2}$ we get dense subspaces of $L^2(X)$, which are domains of the stable processes on $X$, see [18]). For the Sierpiński gasket, this result (degeneracy) was earlier proved in [11]. When diam $X < \infty$, then by an application of Jensen’s inequality we get the same conclusion for $p \geq 2$ (i.e. $\text{Lip}(\alpha, p, p)$ degenerate when $\alpha > \frac{d_w}{2}$). We do not know whether the value $\alpha = \frac{d_w}{2}$ is critical when $p > 2$, and where should the threshold be placed when $1 \leq p < 2$.

For open subsets of the Euclidean space it is known that $\alpha = 1 = \frac{d_w(\mathbb{R}^d)}{2}$ works for all values of $p \geq 1$, see [4]. We do not expect this to hold in general.
4 Links with other definitions of Besov-Lipschitz spaces

4.1 Strichartz Besov spaces on simple fractals

Strichartz in [19] introduced the definition of three types of Besov spaces on the Sierpiński gasket first, and then on p.c.f. self-similar fractals.

For the Sierpiński gasket, the spaces \((\Lambda_{p,q}^{\alpha}(1,X)\) are defined for \(\frac{d}{p} < \alpha \leq \frac{d}{p} + \alpha_1\), where \(\alpha_1 = \frac{\log 2}{\log 5/3} = \frac{1}{d_w - \frac{1}{2}}\), and consist of those bounded continuous functions on the gasket (this is why there is a restriction to \(\frac{d}{p} < \alpha\)), for which

\[
\|(2^{d_w - 2}m^\alpha \delta_{m,p}(f))_m\|_q < \infty,
\]

where

\[
\delta_{m,p}(f) = \begin{cases} 
(2^{-md} \sum_{x \sim y} |f(x) - f(y)|^p)^{\frac{1}{p}} & \text{if } p < \infty, \\
\sup\{|f(x) - f(y)| : x \sim y\} & \text{if } p = \infty.
\end{cases}
\]

The relation \(x \sim y\) is the neighbouring relation on the \(m\)-th approximation of the gasket, \(V_m\). Namely, \(x \sim y\) if and only if \(x, y \in V_m\) and \(|x - y| = \frac{1}{2^m}\) (for a precise definition we refer to [19]).

Theorem 3.18 in [3] and the discussion thereafter assert that on simple fractals one has \((\Lambda_{p,q}^{\alpha}(1,X) = Lip(\alpha/(d_w - d), p, q)(X)\). Jonsson spaces do not carry the restriction \(\frac{d}{p} < \alpha\), but on the other hand we know that \(Lip(\alpha, p, p)(X)\) are empty when \(p \geq 2\) and \(\alpha \geq \frac{d_w}{2}\) (when \(p > 2\), then the proof requires that \(\text{diam } X < \infty\)). Therefore the admissible \(\alpha\)'s in the definition (4.1) should be restricted to \(\alpha \leq \frac{(d_w - d) d_w}{2} \approx 0.920042\), which is a smaller number than \(\frac{d}{p} + \frac{1}{d_w - d}\) (e.g. when \(p = 2\), then \(\frac{d}{2} + \frac{1}{d_w - d} \approx 2.14939665\)).

Since Theorems 3.1 and 3.2 give a characterisation of the spaces \(Lip(\alpha, p, q)(X)\), they simultaneously characterise the spaces \((\Lambda_{p,q}^{\alpha}(1,X)\).

We do not whether the other spaces defined in [19]: \((\Lambda_{p,q}^{\alpha}(2,X)\) and \((\Lambda_{p,q}^{\alpha}(3,X)\) (defined through higher order differences) can be characterised in a similar manner.

4.2 Hu-Zähle Besov spaces

In [10], the authors introduce the following Besov-Lipschitz spaces on \((X, \rho, \mu)\), under identical assumptions on \((X, \rho, \mu)\) as those in the present paper:

for \(\beta > 0, p, q \geq 1\),

\[
B_{p,q}^{\beta}(X) = \{f \in L^p(X, \mu) : \sqrt[p]{\int_0^\infty \left(\int_0^t (t^{k-\frac{\beta}{\gamma}} \|\partial^{k} \partial_t^k P_t f\|_p) \frac{dt}{t}\right)^{\frac{\gamma}{\beta}} < \infty}\},
\]

where \(k = \left[\frac{\beta}{\gamma}\right] + 1 \ ((P_t)_{t \geq 0} \text{ is the semigroup with kernel } p(\cdot, \cdot, \cdot))\). When \(k = 1\) (i.e. \(\beta < 2\)), they also establish that (see Theorem 5.2 of [10]) \(B_{2,2}^{\beta}(X) = H_{2}^{\beta}(X)\), with
equivalent norms, where

\[ H_2^\beta(X) = \{ f \in L^2(X, \mu) : \int_0^\infty (1 + \lambda)^{\beta/2} d\langle E_\lambda f, f \rangle < \infty \}. \]

\((E_\lambda)_{\lambda > 0}\) is the resolution of identity of the generator of the semigroup \((P_t)_{t > 0}\) of s.a. operators on \(L^2(X, \mu)\), (the actual definition was different, but this equivalent condition is also given in [10]).

We know that when \(\beta < 2\), then \(H_2^\beta(X) = Lip(\frac{\beta d_\mu}{4}, 2, 2)\), with equivalent norms. It follows from the earlier results of Stós [18], and also from the results in [10]. It can also be derived from our Theorem 3.1 together with Remark 3.1.

Indeed, the condition \(f \in Lip(\frac{\beta d_\mu}{4}, 2, 2)\) means that

\[
\infty > \int_0^\infty \frac{1}{t^{\beta/2}} \int_X \int_X |f(x) - f(y)|^2 p(t, x, y) \frac{dt}{t} = 2 \int_0^\infty \frac{1}{t^{\beta/2}} \frac{1}{t} (f - P_t f, f) dt
\]

\[
= \int_0^\infty \int_0^\infty \frac{1 - e^{-\lambda t}}{t} d\langle E_\lambda f, f \rangle dt
\]

\[
= \int_0^\infty \left( \int_0^\infty \frac{1 - e^{-\lambda t}}{t^{1+\beta/2}} dt \right) d\langle E_\lambda f, f \rangle.
\]

After the substitution \(s = \lambda t\), the inner integral becomes \(C_\beta \lambda^{\beta/2}\), with \(C_\beta = \int_0^\infty \frac{1 - e^{-s}}{s^{1+\beta/2}} ds\), which is finite as long as \(\beta < 2\). And since we are dealing with \(f \in L^2(X)\), the finiteness of \(\int_0^\infty \lambda^{\beta/2} d\langle E_\lambda f, f \rangle\) implies the finiteness of \(\int_0^\infty (1 + \lambda)^{\beta/2} d\langle E_\lambda f, f \rangle\), so that \(f \in H_2^\beta(X)\).

**Open question.** We do not know what is the relation of spaces \(Lip(\alpha, p, q)(X)\) to the spaces \(B^{\alpha, q}_p(X)\), in general – even when \(p = 2\). From our characterisation we see that \(f \in Lip(\alpha, 2, q)(X)\) if and only if \(f \in L^2(X)\) and

\[
\infty \geq \int_0^\infty \frac{1}{t^{\alpha/2}} \left( \int_X \int_X |f(x) - f(y)|^2 p(t, x, y) d\mu(x) d\mu(y) \right)^{q/2} \frac{dt}{t}
\]

\[
= \int_0^\infty \frac{1}{t^{\alpha/2}} ((f - P_t f, f))^2 \frac{dt}{t} < \infty
\]

\[
= \int_0^\infty \frac{1}{t^{\alpha/2}} \left( \frac{1}{t} \int_0^\infty (1 - e^{-t\lambda}) d\langle E_\lambda f, f \rangle \right)^{q/2} \frac{dt}{t}.
\]

On the other hand, \(f \in B^{\alpha, q}_p(X)\) if and only if

\[
\int_0^\infty \left( t^{k-2} \left( \int_0^\infty \lambda^{2k} e^{-2t\lambda} d\langle E_\lambda f, f \rangle \right)^{1/2} \right)^{q/2} \frac{dt}{t} < \infty
\]

(this is so because \(\|\frac{\partial}{\partial t} P_t f\|_2^2 = \int_0^\infty \lambda^{2k} e^{-2t\lambda} d\langle E_\lambda f, f \rangle\)).
Guided by the case $q = 2$ we expect to have some kind of relationship between the two conditions when $\beta < 2$ (i.e. when $k = 1$). It is unclear whether they yield the same sets of functions, and what should be the dependence between $\beta$ and $\alpha$ in general. When $p \neq 2$, we can no longer use the spectral representation, and so the situation is even more unaccountable.

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