ON THE TRIPLE CORRELATIONS OF FRACTIONAL PARTS OF $n^2\alpha$

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Abstract. For fixed $\alpha \in [0,1]$, consider the set $S_{\alpha,N}$ of dilated squares $\alpha, 4\alpha, 9\alpha, \ldots, N^2\alpha$ modulo 1. Rudnick and Sarnak conjectured that for Lebesgue almost all such $\alpha$ the gap-distribution of $S_{\alpha,N}$ is consistent with the Poisson model (in the limit as $N$ tends to infinity). In this paper we prove a new estimate for the triple correlations associated to this problem, establishing an asymptotic expression for the third moment of the number of elements of $S_{\alpha,N}$ in a random interval of length $L/N$, provided that $L > N^{1/4+\epsilon}$. The threshold of $1/4$ is substantially smaller than the threshold of $1/2$ (which is the threshold that would be given by a naïve discrepancy estimate).

Unlike the theory of pair correlations, rather little is known about triple correlations of the dilations $(\alpha a_n \mod 1)_{n=1}^{\infty}$ for a non-lacunary sequence $(a_n)_{n=1}^{\infty}$ of increasing integers. This is partially due to the fact that second moment of the triple correlation function is difficult to control, and thus standard techniques involving variance bounds are not applicable. We circumvent this impasse by using an argument inspired by works of Rudnick–Sarnak–Zaharescu and Heath-Brown, which connects the triple correlation function to some modular counting problems.

In an appendix we comment on the relationship between discrepancy and correlation functions, answering a question of Steinerberger.

1. Introduction

Let $(a_n)_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. This paper is concerned with the distribution of $(\alpha a_n \mod 1)_{n=1}^{\infty}$ in short intervals, for a generic dilate $\alpha \in [0,1]$, with particular focus on the case when $a_n = n^2$.

We begin with the familiar notion of the discrepancy $D_N$, defined to be

$$D_N = D_N((\alpha a_n \mod 1)_{n=1}^{\infty}) = \sup_{0 < a < b < 1} \left| \frac{\{n \leq N : \alpha a_n \mod 1 \in (a, b)\}}{N} - (b - a) \right|. \quad (1.1)$$

It is an old result of Weyl, contained in his 1916 paper [27], that $D_N((\alpha n^2 \mod 1)_{n=1}^{\infty}) = o_\alpha(1)$ as $N \to \infty$ for any irrational $\alpha$. The sequence $(\alpha n^2 \mod 1)_{n=1}^{\infty}$ is then said to be equidistributed modulo 1. One way of viewing Weyl’s result is as demonstrating a pseudorandomness property for the sequence $(\alpha n^2 \mod 1)_{n=1}^{\infty}$. Indeed, in the random model in which $(\alpha n^2 \mod 1)_{n=1}^{N}$ is replaced by $N$ independent random variables $X_1, \ldots, X_N$ which are uniformly distributed on $[0,1)$ one has

$$\limsup_{N \to \infty} \frac{\sqrt{2N} D_N((X_n)_{n=1}^{\infty})}{\sqrt{\log \log N}} = 1$$

almost surely. In particular $\mathbb{E} D_N((X_n)_{n=1}^{\infty}) = o(1)$ as $N \to \infty$.

One might wonder, considering the strong quantitative decay enjoyed in the random model, whether Weyl’s result admits such a quantitative refinement. Unfortunately, as is well known, if $\alpha$ is well-approximated by rational numbers with small denominators then the

\[\frac{\sqrt{2N} D_N((X_n)_{n=1}^{\infty})}{\sqrt{\log \log N}} = 1\]
discrepancy $D_N((\alpha n^2 \mod 1)_{n=1}^\infty)$ can tend to zero extremely slowly. However, for a generic $\alpha$ the situation is much improved, and in fact for any strictly increasing sequence of positive integers $(a_n)_{n=1}^\infty$ one has the classical result of Erdős-Koksma [8], which implies that

$$D_N((\alpha a_n \mod 1)_{n=1}^\infty) = O_e(N^{-\frac{1}{2} + \varepsilon})$$

for Lebesgue almost all $\alpha \in [0, 1]$. This trivially implies that, for almost all $\alpha \in [0, 1]$, if $I \subset \mathbb{R}/\mathbb{Z}$ is a fixed interval of length $|I| > N^{-\frac{1}{2} + \varepsilon}$ then

$$|\{n \leq N : \alpha n^2 \mod 1 \in I\}| = (1 + o_e(1))N|I|.
\tag{1.2}$$

So, at least for a generic $\alpha$, pseudorandomness is enjoyed down to the scale $N^{-\frac{1}{2} + \varepsilon}$. A result of Aistleitner and Larcher [1, Cor. 1] implies that the exponent $1/2$ is optimal for the metric discrepancy problem: for generic $\alpha$ and for each $\varepsilon > 0$ there are infinitely many $N$ such that $D_N((\alpha a_n \mod 1)_{n=1}^\infty) > N^{-1/2-\varepsilon}$, provided $a_n = P(n)$ for some polynomial $P$, of degree at least two, with integer coefficients.

By allowing the interval $I$ to vary, one can develop a related notion of pseudorandomness at scales that are smaller than $N^{-1/2}$, one which concerns the ‘clustering’ of the points $\alpha n^2 \mod 1$. To introduce this notion, which will be the main focus of the paper, we let $Y$ be a random variable that is uniformly distributed on $[0, 1)$, and for a natural number $N$ and a parameter $L$ in the range $0 < L \leq N$ we let $W_{\alpha,L,N}$ be the random variable

$$W_{\alpha,L,N} := |\{n \leq N : \alpha n^2 \in [Y, Y + L/N] \mod 1\}|.
\tag{1.3}$$

It is easy to see that $\mathbb{E}W_{\alpha,L,N} = L$. But how should one expect $W_{\alpha,L,N}$ to be distributed in the limit $N \to \infty$ (for a generic dilate $\alpha$)? Consider the same random model as before, in which $(\alpha n^2 \mod 1)_{n=1}^\infty$ is replaced by $N$ independent random variables $X_1, \ldots, X_N$ which are uniformly distributed on $[0, 1)$. Then, if $L$ is constant as $N \to \infty$, letting

$$Z_{L,N} := |\{n \leq N : X_n \in [Y, Y + L/N] \mod 1\}|
\tag{1.4}$$

one may calculate that

$$Z_{L,N} \xrightarrow{\text{dist}} \text{Po}(L)$$

as $N \to \infty$, where $\text{Po}(L)$ is a Poisson-distributed random variable with parameter $L$.

Having described this random model, we can now state the following remarkable conjecture:

**Conjecture 1.1** (Rudnick-Sarnak\cite{21}). For almost all $\alpha \in [0, 1]$, for all fixed $L > 0$,

$$W_{\alpha,L,N} \xrightarrow{\text{dist}} \text{Po}(L)$$

as $N \to \infty$.

If true, this conjecture would represent a strong local notion of pseudorandomness for the sequence $(\alpha n^2 \mod 1)_{n=1}^\infty$, at least for a generic $\alpha$. In fact, a further conjecture [22, p. 38] posits more information about the full-measure set of suitable dilates $\alpha$. To state this conjecture, we recall that $\alpha$ is of type $\omega$ if there are only finitely many pairs $(a, q)$ with $|\alpha - a/q| < q^{-\omega}$.

**Conjecture 1.2** (Rudnick-Sarnak–Zaharescu). If $\alpha$ is of type $2 + \varepsilon$ for all $\varepsilon > 0$, and the convergents $a/q$ to $\alpha$ satisfy

$$\lim_{q \to \infty} \log \tilde{q}/\log q = 1,$$

where $\tilde{q}$ is the square-free part of $q$, then for all fixed $L > 0$, as $N \to \infty$, one has

$$W_{\alpha,L,N} \xrightarrow{\text{dist}} \text{Po}(L).$$

\cite{22}These authors refer to the ‘distribution of the spacings between the elements’, rather than mentioning the random variables $W_{\alpha,L,N}$ directly, but these are equivalent notions. For more on this other perspective, see the introduction to [22].
Remark: We remind the reader that, by Dirichlet’s approximation theorem, the type of a real number is never less than 2. Further, by Khintchine’s theorem, a generic number is of type $2 + \varepsilon$ for all $\varepsilon > 0$. One can also readily find explicit examples, like $\alpha = \sqrt{2}$, by using continued fractions.

Conjectures 1.1 and 1.2 appear to lie very deep. One hypothetical approach for showing the desired convergence in distribution would be to use the method of moments. More precisely, if one could show, for a generic $\alpha$, for a random variable $X_L \sim \text{Po}(L)$, and for all $k \in \mathbb{N}$, that $\mathbb{E}W_{\alpha,L,N}^k \to \mathbb{E}X_L^k$ as $N \to \infty$ then Conjecture 1.1 would follow. Rudnick–Zaharescu [23] used this approach to show that if $(a_n)_{n=1}^{\infty}$ is a lacunary sequence of natural numbers then for almost all $\alpha$ the sequence $(\alpha a_n \mod 1)_{n=1}^{\infty}$ has spacing statistics that agree with the Poisson model. For the squares, the first non-trivial case is $k = 2$, and this was settled by Rudnick–Sarnak some 22 years ago.

**Theorem 1.3** (Rudnick–Sarnak [21]). For almost all $\alpha \in [0, 1]$, for all fixed $L > 0$,

$$\mathbb{E}W_{\alpha,L,N}^2 = L + L^2 + o_{\alpha,L}(1)$$

as $N \to \infty$.

This gives an estimate on the so-called *number variance* $\text{Var}(W_{\alpha,L,N})$, namely

$$\text{Var}(W_{\alpha,L,N}) = L + o_{\alpha,L}(1)$$

as $N \to \infty$.

Very little is known regarding the higher moments, although certain results can be extracted from the literature. For larger $L$, the issue is settled by the aforementioned discrepancy bounds. Indeed, expression (1.2) implies that for almost all $\alpha \in [0, 1]$, for all integers $k \geq 1$, for all $N \in \mathbb{N}$ and for all $L \in \mathbb{R}$ in the range $N^{1/2+\varepsilon} < L \leq N$,

$$\mathbb{E}W_{\alpha,L,N}^k = L^k (1 + o_{\alpha,\varepsilon,k}(1))$$

(1.5)

as $N \to \infty$ (where the error term is independent of the choice of parameters $L$). We will give the simple proof of (1.5) in Appendix C alongside other consequences of the discrepancy bounds. In passing, we will answer a question of Steinerberger from [25].

One might wonder whether the methods that Rudnick–Sarnak introduced to tackle the second moment in Theorem 1.3 could be applied to higher moments. Unfortunately, in the case when $L$ is constant, Rudnick–Sarnak already noted in [21, Section 4] that their method faces a major obstruction when applied to higher moments. We will describe this obstruction in Appendix C where (for the third moment) we observe that the obstruction persists for $L$ as large as $N^{1/3}$.

We now present the main result of this paper.

**Theorem 1.4** (Main Theorem). Let $\varepsilon > 0$. Then, for almost all $\alpha \in [0, 1]$, for all $N \in \mathbb{N}$ and for all $L \in \mathbb{R}$ in the range $N^{1/4+\varepsilon} < L \leq N$ we have

$$\mathbb{E}W_{\alpha,L,N}^3 = L^3 (1 + o_{\alpha,\varepsilon}(1))$$

as $N \to \infty$, where the $o_{\alpha,\varepsilon}(1)$ term is independent of the choice of the parameters $L$.

Note in particular that $1/4 < 1/3$, so we successfully give an asymptotic expression for the third moment in part of the range in which the Rudnick–Sarnak obstruction holds (see Appendix C).

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We should remark that Rudnick–Sarnak phrased their result in terms of the pair correlation function, rather than explicitly mentioning $W_{\alpha,L,N}$, but the results are equivalent.
To prove Theorem 1.3, we will first perform a standard reduction to a statement concerning correlation functions. These functions are closely related to the moments $EW_{\alpha,L,N}^k$, but they can be more convenient to analyse.

**Definition 1.5 (Correlation functions).** Let $\alpha \in [0,1]$, let $k \geq 2$ be a natural number, and let $g : \mathbb{R}^{k-1} \to [0,1]$ be a compactly supported function. Then the $k$th correlation function $R_k(\alpha, L, N, g)$ is defined to be

$$
\frac{1}{N} \sum_{1 \leq x_1, \ldots, x_k \leq N \text{ distinct}} g\left(\frac{N}{L}(\alpha(x_1^2 - x_2^2))\right)_{\text{sgn}}, \frac{N}{L}\left(\alpha(x_2^2 - x_3^2)\right)_{\text{sgn}}, \ldots, \frac{N}{L}\left(\alpha(x_{k-1}^2 - x_k^2)\right)_{\text{sgn}},
$$

where

$$\{\cdot\}_{\text{sgn}} : \mathbb{R} \to (-1/2, 1/2)
$$

denotes the signed distance to the nearest integer, and $N$ and $L$ are real parameters.

In practice, one only needs to understand the correlation functions when the test function $g$ is sufficiently nice, e.g. smooth or the indicator function of a box.

When proving Theorem 1.3, Rudnick–Sarnak manipulated the pair correlation function $R_2(\alpha, L, N, g)$. We manipulate the triple correlation function $R_3(\alpha, L, N, g)$, proving the following result, which, for readers more familiar with correlation functions than with the moments of $W_{\alpha,L,N}$, might seem to be more natural.

**Theorem 1.6 (Triple correlations).** Let $\varepsilon > 0$. Then, for almost all $\alpha \in [0,1]$, for all compactly supported continuous functions $g : \mathbb{R}^2 \to [0,1]$, for all $N \in \mathbb{N}$ and for all $L \in \mathbb{R}$ in the range $N^{1/4+\varepsilon} < L < N^{1-\varepsilon}$ we have

$$R_3(\alpha, L, N, g) = (1 + o_{\alpha,\varepsilon,g}(1))L^2\left(\int g(w) \, dw\right)
$$

as $N \to \infty$, where the $o_{\alpha,\varepsilon,g}(1)$ term is independent of the choice of the parameters $L$.

Before continuing to survey other relevant papers, we should stop to explain why the spacing statistics of the sequence $\alpha n^2 \mod 1$ are of a particular interest (aside from as a part of the larger endeavour of finding pseudorandomness in arithmetic sequences). This is due to a connection between number theory and theoretical physics known as *arithmetic quantum chaos*. In brief, the spacing statistics between elements of the sequence $\alpha n^2 \mod 1$ correspond to the spacing statistics between the eigenvalues of a certain quantum system. (This system is a two-dimensional boxed oscillator, with a harmonic potential in one direction and hard walls in the other, as described in the introduction to [21].) A famous and far-reaching observation of Berry–Tabor [4] then suggests that such spacing statistics in the semi-classical limit (i.e. the distribution of $W_{\alpha,L,N}$ for constant $L$ as $N \to \infty$) should be determined by the dynamics of the corresponding classical system. Chaotic classical dynamics should correspond to spacing statistics in asymptotic agreement with the Poisson model.

Unfortunately if $\alpha$ is rational (or is very well approximated by rationals with square denominators), it is easy to prove that high moments of $W_{\alpha,L,N}$ do not agree with the Poisson model (see [22, Theorem 2])! However, excluding such zero-measure counter-examples\footnote{Zaharescu [28] showed that in a precise sense that, at least amongst all very well approximable $\alpha$, these were the only counter-examples.}, one might still hope for a metric result.

There are precious few examples of fixed sequences for which information is known about the spacing statistics, e.g. Elkies and McMullen [7], who used dynamical methods to establish the gap distribution of $(\sqrt{n} \mod 1)_{n=1}^{\infty}$ (which is not from a Poisson model), and results in [16] and [9] on the second moment of the gap distribution between values of quadratic forms. However, most of the results in the literature are metric in at least one parameter (e.g. [21], [23], [24], [2]), and such results still have substantial content.
To delve further into the relationship to theoretical physics and to other spacing statistics would be to digress too far from our main theme; we direct the interested reader to the articles of Marklof [15] and Rudnick [18] for more on these issues.

Returning to the study of the moments of \( W_{\alpha,L,N} \), and the discussion of relevant work, we continue with the paper [22]. Here Rudnick–Sarnak–Zaharescu develop tools to relate the diophantine approximation properties of \( \alpha \) to the size of the moments \( \mathbb{E}W_{\alpha,L,N}^k \). The main result of that paper can be phrased as follows:

**Theorem 1.7** (Rudnick-Sarnak-Zaharescu). Let \( \alpha \in [0,1] \), and suppose that there are infinitely many rationals \( b_j/q_j \), with \( q_j \) prime, satisfying

\[
|\alpha - \frac{b_j}{q_j}| < \frac{1}{q_j^3} \tag{1.6}
\]

Then there is a subsequence \( N_j \to \infty \), with \( \log N_j/\log q_j \to 1 \) for which, for all \( L > 0 \),

\[
W_{\alpha,L,N_j} \overset{\text{dist.}}{\to} \text{Po}(L)
\]

as \( j \to \infty \).

The authors of [22] sacrificed the genericness of \( \alpha \) (working instead with those \( \alpha \) which are unusually well-approximable) in favour of control over the moments \( \mathbb{E}W_{\alpha,L,N_j}^k \) for constant \( L \) and for all \( k \). One may switch objectives in their analysis, sacrificing the range of \( L \) in order to work with almost all \( \alpha \). Applying this switch in the context of the third moment calculation, their method shows the following result:

**Theorem 1.8** (Method of R–S–Z). Let \( \varepsilon > 0 \). Then, for almost all \( \alpha \in [0,1] \), for all \( N \in \mathbb{N} \) and for all \( L \in \mathbb{R} \) in the range \( N^{3/5+\varepsilon} < L \leq N \) we have

\[
\mathbb{E}W_{\alpha,L,N}^3 = L^3(1 + o_{\alpha,\varepsilon}(1))
\]

as \( N \to \infty \), where the \( o_{\alpha,\varepsilon}(1) \) term is independent of the choice of parameters \( L \). Moreover, one can give an explicit description of a suitable full-measure set of suitable \( \alpha \) (in terms of properties of rational approximations to \( \alpha \)).

Although the range of \( L \) in Theorem 1.7 is much smaller than the range in our result, the method of R–S–Z yields a more explicit description of the full-measure set of suitable \( \alpha \). We will indicate how to extract Theorem 1.8 from [22] in Section 3 below.

Our approach to proving Theorem 1.8 is inspired by this work of R–S–Z, but also by the work of Heath-Brown in [11], who introduced a related technique for studying the pair correlation function \( R_2(\alpha, L, N, g) \). The full description of the method will come in Section 3, but we sketch the idea here, so as to explain in a rough way how we extract an improvement over [22]. After having replaced \( \alpha \) by a rational approximation \( a/q \), one transforms the triple correlation function into an expression that counts the number of solutions to certain polynomial equations modulo \( q \). The equations which occur are of the form

\[
\{x, y, z \leq N : x^2 - y^2 \equiv c_1 \pmod{q}, y^2 - z^2 \equiv c_2 \pmod{q}\}, \tag{1.7}
\]

for certain ranges of \( N \) and \( q \) and for certain sets of coefficients \( c_1 \) and \( c_2 \). In [22] the number of solutions was estimated by using the ‘completion of sums’ technique to remove the \( N \) cut-off, followed by an implementation of the Hasse-Weil bound. In our work we manage to take advantage of the extra averaging over \( c_1 \) and \( c_2 \) which is present in the problem, together with some intricate (though elementary) exponential sum arguments, which ends up leading to a stronger bound for certain ranges of \( N \) and \( q \). Heath-Brown did something similar for the pair correlation function [11], but the analysis of the relevant exponential sums for the triple correlations is substantially more delicate.
Another relevant work is the paper of Kurlberg and Rudnick \[20\], in which those authors established that the spacing of quadratic residues mod \(q\) as the number of prime factors of \(q\) grows is in agreement with the Poisson model. Lemma 3.3 below could be viewed as a special case of the arguments of that paper. However, as will become evident, the work here necessarily concerns a rather sparse subset of the set of quadratic residues mod \(q\), as \(N \approx q^\theta\) with \(\theta < 1\), and so the work of \[20\] is not directly applicable. Pair correlations of rational functions mod \(q\) were also studied by Boca and Zaharescu \[5\], but again, we will not be able to use that paper directly.

The structure of the paper is as follows. In Section 2 we will give the standard argument (passing from moments to correlation functions) which reduces Theorem 1.4 to Theorem 1.6. The proof of Theorem 1.6 is then given in Section 3, in which it is resolved subject to three auxiliary results (one concerning diophantine approximation, the other two concerning the number of solutions to certain diophantine equations similar to (1.7)). The final three sections of the paper settle these results – one per section – thus concluding the main proof.

The appendices contain some arguments that are minor modifications of the literature (but which are nonetheless pertinent to the main paper). These are, respectively, a version of Theorem 1.3 in which \(L\) grows with \(N\); the proof of the asymptotic (1.5); and the discussion of the obstruction to the study of triple correlations that was identified by Rudnick-Sarnak.

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Notation: Most of our notation is standard, but perhaps we should highlight a few conventions. For a natural number \(q\) we let \(e_q(x)\) be a shorthand for \(e^{2\pi i x/q}\), and given a parameter \(M \geq 1\) we let \([M]\) denote the set \(\{m \in \mathbb{N} : 1 \leq m \leq M\}\). In particular \(1_{[M]}\) denotes the indicator function of all the natural numbers at most \(M\). If a range of summation is given as \(\sum_{m \leq M}\) then it is assumed that \(m\) is a natural number and that \(m \geq 1\). Finally, for \(x \in \mathbb{R}\), we will use \(\|x\|\) to denote the distance from \(x\) to the nearest integer, and \(\{x\}_{\text{sign}}\) to denote the signed distance to the nearest integer from \(x\).

2. Reduction to Correlation Functions

In this short section we will reduce Theorem 1.4 to Theorem 1.6. Firstly, since estimate (1.5) holds for large \(L\) we may assume without loss of generality that \(L < N^{1-\varepsilon}\). Then we use linearity of expectation to deduce that

\[
\mathbb{E}W_{\alpha,L,N}^3 = \sum_{x,y,z \leq N} \mathbb{P}(\alpha x^2, \alpha y^2, \alpha z^2 \in [Y,Y + L/N] \mod 1) \\
= \sum_{x \leq N} \mathbb{P}(\alpha x^2 \in [Y,Y + L/N] \mod 1) + 3 \sum_{x,y \leq N, x \neq y} \mathbb{P}(\alpha x^2, \alpha y^2 \in [Y,Y + L/N] \mod 1) \\
+ \sum_{x,y,z \leq N, \text{distinct}} \mathbb{P}(\alpha x^2, \alpha y^2, \alpha z^2 \in [Y,Y + L/N] \mod 1),
\]

(2.1)

where \(Y\) is a random variable that is uniformly distributed modulo 1. The first of the three terms in (2.1) is equal to \(L\), so may be absorbed into the error term of Theorem 1.4. The
Let Theorem 2.1. below by step functions, to prove Theorem 1.6 it will be enough to prove the following result:

Therefore expression (2.2) is equal to 3

\[ L \]

the following weaker result:

\[ = 3L \left( \frac{1}{N} \sum_{x,y \in N} \max_{x \not= y} \left( 0, 1 - \frac{N}{L} \| \alpha(x^2 - y^2) \| \right) \right) \]

\[ = 3LR_2(\alpha, L, N, f), \]  \hspace{1cm} (2.2)

where \( f : \mathbb{R} \to [0, 1] \) is the function \( f(x) = \max(0, 1 - |x|) \) and \( R_2(\alpha, L, N, f) \) is the pair correlation function as defined in Definition 1.5. In Appendix A we will show, by a trivial adaptation of the known techniques, that for almost all \( \alpha \in [0, 1] \) one has

\[ R_2(\alpha, L, N, f) = (1 + o_{\alpha, f}(1))L \left( \int f(x) \, dx \right) = (1 + o_{\alpha, f}(1))L. \]  \hspace{1cm} (2.3)

Therefore expression (2.2) is equal to \( 3L^2 + o_{\alpha, f}(L^2) \), which may also be absorbed into the error term of Theorem 1.4.

What remains is the third term of (2.1). This is equal to

\[ \frac{L}{N} \sum_{x,y,z \in N, \text{ distinct}} \max(0, 1 - \frac{N}{L} \| \alpha(x^2 - y^2) \|, \| \alpha(y^2 - z^2) \|, \| \alpha(z^2 - x^2) \|). \]  \hspace{1cm} (2.4)

Since \((x^2 - y^2) + (y^2 - z^2) = x^2 - z^2\) we see that (2.4) is equal to an expression of the form \( R_3(\alpha, L, N, g) \) for some continuous compactly supported function \( g : \mathbb{R}^2 \to [0, 1] \). Indeed,

\[ g(w_1, w_2) = \begin{cases} 
\max(0, 1 - w_1 - w_2) & w_1, w_2 \geq 0 \\
\max(0, 1 - \max(w_1, -w_2)) & w_1 \geq 0, w_2 \leq 0 \\
\max(0, 1 - \max(-w_1, w_2)) & w_1 \leq 0, w_2 \geq 0 \\
\max(0, 1 + w_1 + w_2) & w_1, w_2 \leq 0.
\end{cases} \]

An elementary calculation then demonstrates that

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(w_1, w_2) \, dw_1 \, dw_2 = 1. \]

Therefore, by Theorem 1.6 if \( L > N^{4+\varepsilon} \) then for almost all \( \alpha \in [0, 1] \) expression (2.4) is equal to \( L^3(1 + o_{\alpha, \varepsilon}(1)) \) as \( N \to \infty \). So Theorem 1.4 is proved. \( \square \)

We make the usual remark that, by approximating the continuous function \( g \) above and below by step functions, to prove Theorem 1.6 it will be enough to prove the following result:

**Theorem 2.1.** Let \( \varepsilon > 0 \). Then for almost all \( \alpha \in [0, 1] \), for all \( s_1, t_1, s_2, t_2 \in \mathbb{R} \) for which \( s_1 < t_1 \) and \( s_2 < t_2 \), and for all \( L \) in the range \( N^{1/4+\varepsilon} < L < N^{1-\varepsilon} \), we have

\[ R_3(\alpha, L, N, g_{s,t}) = (1 + o_{\alpha, \varepsilon, s,t}(1))L^2(t_1 - s_1)(t_2 - s_2) \]

as \( N \to \infty \), where \( s = (s_1, s_2) \), \( t = (t_1, t_2) \), and \( g_{s,t} \) is the indicator function of the box \([s_1, t_1] \times [s_2, t_2]\).

It will turn out to be crucial in our subsequent methods that the ratio \( \log L / \log N \) does not vary too wildly. To finish this section, we will show how to deduce Theorem 2.1 from the following weaker result:
Theorem 2.2. Let $\beta \in (0, 3/4)$ and let $\eta > 0$. Then, if $\eta \max(\beta^{-1}, (3/4 - \beta)^{-1})$ is small enough, the following holds: for almost all $\alpha \in [0, 1]$, for all $s_1, t_1, s_2, t_2 \in \mathbb{R}$ for which

$$s_1 < t_1 \text{ and } s_2 < t_2, \text{ for all } N \in \mathbb{N} \text{ and for all } L \in \mathbb{R} \text{ in the range } N^{1-\beta-\eta} < L < N^{1-\beta+\eta},$$

we have

$$R_3(\alpha, L, N, g_{s,t}) = (1 + o_{\alpha,\beta,\eta,s,t}(1))L^2(t_1 - s_1)(t_2 - s_2)$$

as $N \to \infty$, where $s = (s_1, s_2)$, $t = (t_1, t_2)$, and $g_{s,t}$ is the indicator function of the box $[s_1, s_1] \times [s_2, t_2]$. The error term is independent of the exact choice of the parameters $L$.

Deduction of Theorem 2.1 from Theorem 2.2. Let $\varepsilon > 0$ and choose $\eta > 0$ such that $\eta \varepsilon^{-1}$ is suitably small. Let $\{\beta_1, \ldots, \beta_R\}$ be a maximal $\eta$-separated subset of $[\varepsilon, 3/4 - \varepsilon]$. Then for each $\beta_i$ we get a full measure set $\Omega_i \subset [0, 1]$ such that, if $\alpha \in \Omega_i$, the conclusion of Theorem 2.2 holds with $\beta = \beta_i$ and with $\eta$ as chosen. We claim that $\Omega = \cap_{i \in R} \Omega_i$ is a suitable full measure set of values of $\alpha$ in Theorem 2.1.

Indeed, let $\alpha \in \Omega$ and let $s_1, t_1, s_2, t_2 \in \mathbb{R}$ with $s_1 < t_1$ and $s_2 < t_2$. From Theorem 2.2 we know that for all $i \leq R$, for any $\delta > 0$, for all $N \geq N_0(\alpha, \beta, \eta, \delta, s, t)$, and for any $L$ in the range $N^{1-\beta-\eta} < L < N^{1-\beta+\eta}$, we have

$$|R_3(\alpha, L, N, g_{s,t}) - L^2(t_1 - s_1)(t_2 - s_2)| < \delta L^2. \quad (2.5)$$

Now, given any $N$ and any $L$ in the range $N^{1/4+\varepsilon} < L < N^{1-\varepsilon}$, there exists an $i$ such that $N^{1-\beta-\eta} < L < N^{1-\beta+\eta}$. Therefore, if $N \geq \max_{i \in R} N_0(\alpha, \beta_i, \eta, \delta, s, t)$, the inequality (2.5) holds. Since $\delta$ is arbitrary, the conclusion of Theorem 2.1 holds.

3. Main proof

Our task is now to prove Theorem 2.2. Let us fix $\beta$ and $\eta$, which is assumed to be sufficiently small, and for the time being let us also fix some $\alpha \in [0, 1]$ and some $s_1, t_1, s_2, t_2 \in \mathbb{R}$ with $s_1 < t_1$ and $s_2 < t_2$. We may also assume, without loss of generality, that $N$ is sufficiently large in terms of $\alpha, \beta, \eta, s$ and $t$.

We begin by replacing $\alpha$ with a suitably good rational approximation $a/q$. Assume that there exists some rational $a/q$, with $q$ prime, for which

$$|\alpha - \frac{a}{q}| \leq \frac{1}{q^{2-\eta}} \quad (3.1)$$

and

$$N^{2+\beta+10\eta} \leq q \leq 2N^{2+\beta+10\eta}. \quad (3.2)$$

We will show in Section 4 that almost all $\alpha$ admit such an approximation. Then, for such a pair $(N, q)$, we define

$$A(N, q, c_1, c_2) := |\{x, y, z \leq N : x^2 - y^2 \equiv c_1 \text{ (mod } q), y^2 - z^2 \equiv c_2 \text{ (mod } q)\}|. \quad (3.3)$$

We then claim that

$$\frac{1}{N} \sum_{(r_1, r_2) \in S^{-}} A(N, q, \overline{r_1}, \overline{r_2}) \leq R_3(\alpha, L, N, g_{s,t}) \leq \frac{1}{N} \sum_{(r_1, r_2) \in S^{+}} A(N, q, \overline{r_1}, \overline{r_2}), \quad (3.4)$$

where $\overline{r}$ denotes the inverse of $r$ modulo $q$, and

$$S^{-} = \{(r_1, r_2) \in \mathbb{Z}^2 : \frac{s_1 q L}{N} + \frac{N^2}{q^{1-\eta}} \leq r_i \leq \frac{t_1 q L}{N} - \frac{N^2}{q^{1-\eta}}, i = 1, 2\}, \quad (3.5)$$

and

$$S^{+} = \{(r_1, r_2) \in \mathbb{Z}^2 : \frac{s_1 q L}{N} - \frac{N^2}{q^{1-\eta}} \leq r_i \leq \frac{t_1 q L}{N} + \frac{N^2}{q^{1-\eta}}, i = 1, 2\}. \quad (3.6)$$
Indeed, given $x, y$ in the range $1 \leq x, y \leq N$ let us consider $r_1 \in \mathbb{Z}$ to be defined by the relation

$$a(x^2 - y^2) \equiv r_1 \pmod{q}$$

and $-q/2 < r_1 < q/2$. Suppose that

$$\frac{s_1 qL}{N} + \frac{N^2}{q^{1-\eta}} \leq r_1 \leq \frac{t_1 qL}{N} - \frac{N^2}{q^{1-\eta}}.$$ 

Then $|r_1| \leq \varepsilon q$, since $N$ is large enough, and so we have the inequalities

$$\{\alpha(x^2 - y^2)\}_{\text{sgn}} \leq \left\{ \frac{a}{q}(x^2 - y^2) \right\}_{\text{sgn}} + \left| \left( \alpha - \frac{a}{q} \right) (x^2 - y^2) \right| \leq \frac{r_1}{q} + \frac{N^2}{q^{2-\eta}} \leq t_1 \frac{L}{N},$$

and

$$\{\alpha(x^2 - y^2)\}_{\text{sgn}} \geq \left\{ \frac{a}{q}(x^2 - y^2) \right\}_{\text{sgn}} - \left| \left( \alpha - \frac{a}{q} \right) (x^2 - y^2) \right| \geq \frac{r_1}{q} - \frac{N^2}{q^{2-\eta}} \geq s_1 \frac{L}{N}.$$

Suppose instead that

$$\frac{s_1 L}{N} \leq \{\alpha(x^2 - y^2)\}_{\text{sgn}} \leq t_1 \frac{L}{N}.$$ 

Then, similarly to the above, we have

$$r_1 = q\left\{ \frac{a}{q}(x^2 - y^2) \right\}_{\text{sgn}} \leq q \left( \{\alpha(x^2 - y^2)\}_{\text{sgn}} + \left| \left( \alpha - \frac{a}{q} \right) (x^2 - y^2) \right| \right) \leq t_1 \frac{Lq}{N} + \frac{N^2}{q^{1-\eta}}$$

and

$$r_1 = q\left\{ \frac{a}{q}(x^2 - y^2) \right\}_{\text{sgn}} \geq q \left( \{\alpha(x^2 - y^2)\}_{\text{sgn}} - \left| \left( \alpha - \frac{a}{q} \right) (x^2 - y^2) \right| \right) \geq s_1 \frac{Lq}{N} - \frac{N^2}{q^{1-\eta}}.$$

Finally, take $1 \leq z \leq N$ and define $r_2$ by the relation

$$a(y^2 - z^2) \equiv r_2 \pmod{q}$$

with $-q/2 < r_2 < q/2$. Then, since $N < q/2$, we have that $x, y, z$ are distinct if and only if $r_1 r_2 \neq 0$ and $r_1 + r_2 \neq 0$.

From all these observations taken together, claim (3.4) is settled.

**Remark:** The reader might find it helpful to note, at this early stage, that the relative sizes of $N$ and $q$ were chosen so that the two terms $qL/N$ and $N^2/q^{1-\eta}$ which appear in (3.5) and (3.6) are of approximately the same magnitude, namely $q^{(2-\beta)/(2+\beta)}$.

A substantial portion of this paper will involve estimating the quantity $A(N, q, c_1, c_2)$, on average over $c_1$ and $c_2$. To this end, we let

$$A_0(q, c_1, c_2) = |\{x, y, z \leq q : x^2 - y^2 \equiv c_1 \pmod{q}, y^2 - z^2 \equiv c_2 \pmod{q}\}|$$

be the number of solutions to the key congruences, in which the variables $x, y, z$ may range over the entire field $\mathbb{Z}/q\mathbb{Z}$. One might reasonably expect that

$$A(M, q, c_1, c_2) \approx (M/q)^3 A_0(q, c_1, c_2)$$

as long as $M$ is large enough, and so we introduce the difference

$$\Delta(M, q, c_1, c_2) := \left| A(M, q, c_1, c_2) - \left( \frac{M}{q} \right)^3 A_0(q, c_1, c_2) \right|. \quad (3.7)$$

The key technical lemma of our paper is the following:
Lemma 3.1. If $q$ is an odd prime and $M < q$, then
\[
\sum_{c_1,c_2 \leq q} \Delta(M, q, c_1, c_2)^2 \ll (\log q)^3 M^3 + (\log q)^6 q^2
\]  
(3.8)
as $q \to \infty$.

Lemma 3.1 can be used in turn to prove the following lemma on the average size of the error terms $\Delta(N, q, \overline{ar}_1, \overline{ar}_2)$, which is useful for analysing the relation (3.4).

Lemma 3.2. Let $\beta \in (0, 3/4)$ and let $\eta > 0$. Then, if $\eta \max(\beta^{-1}, (3/4 - \beta)^{-1})$ is small enough, the following holds: for almost all $x \in [0, 1]$, for all $s_1, t_1, s_2, t_2 \in \mathbb{R}$ for which $s_1 < t_1$ and $s_2 < t_2$, for all $N \in \mathbb{N}$ and for all $L \in \mathbb{R}$ in the range $N^{1-\beta-\eta} < L < N^{1-\beta+\eta}$, and for all prime $q$ and $a/q$ satisfying (3.3) and (3.2), we have
\[
\frac{1}{N} \sum_{(r_1, r_2) \in S^+} \Delta(N, q, \overline{ar}_1, \overline{ar}_2) \ll_{\alpha, \beta, \eta, s, t} L^2 q^{-\eta},
\]  
(3.9)
where $S^+$ is as defined in (3.6).

At this point it is worth us taking a small diversion from the main proof to discuss the numerology in Lemma 3.1 and why the bound is close to best-possible. We begin with an easy lemma concerning $A_0(q, c_1, c_2)$ itself. A more sophisticated version of this lemma was worked out by Kurlberg and Rudnick in [20, Prop. 4], but to make our exposition self-contained we decided to include a direct proof for the triple correlation case.

Lemma 3.3. Let $q$ be an odd prime. If $c_1, c_2$ are not both divisible by $q$, then
\[
A_0(q, c_1, c_2) = \begin{cases} 
q + O(\sqrt{q}) & \text{if } c_1 + c_2 \not\equiv 0 \pmod{q}; \\
2q - 1 - \left(\frac{c_2}{q}\right) & \text{if } c_1 \equiv 0 \pmod{q} \\
2q - 1 - \left(\frac{c_1}{q}\right) & \text{if } c_2 \equiv 0 \pmod{q} \\
2q - 1 - \left(\frac{c_1}{q}\right) & \text{if } c_1 + c_2 \equiv 0 \pmod{q}
\end{cases}
\]
where $\left(\frac{\cdot}{q}\right)$ denotes the Legendre symbol modulo $q$. Further, $A_0(q, 0, 0) = 4q - 3$.

Remark: For this paper, it would have been enough to consider only the non-degenerate case of Lemma 3.3. However, the arguments of Section 5 are cleaner if we allow ourselves to include the degenerate cases, which is the reason why we do so.

Proof. In the trivial case $c_1 \equiv c_2 \equiv 0 \pmod{q}$, we note that $y = \pm x \pmod{q}$ and $z = \pm y \pmod{q}$. So, except for when $x = q$, there are 4 choices for $y, z$ given a fixed $x$, thus showing that $A_0(0, 0) = 4(q - 1) + 1 = 4q - 3$ as claimed.

Thus we assume in the following that at least one of $c_1, c_2$ is not divisible by $q$. We have that
\[
A_0(q, c_1, c_2) = \frac{1}{q^2} \sum_{0 \leq j, k \leq q-1} \sum_{0 \leq x, y, z \leq q-1} e_q(j(x^2 - y^2 - c_1) + k(y^2 - z^2 - c_2))
\]
The terms when $j = 0$, $k = 0$, or $j = k$ contribute
\[
\frac{1}{q^2} \left( \sum_{0 \leq x, y \leq q-1} \sum_{y^2 \equiv x^2 - c_1 \pmod{q}} 1 + q^2 \sum_{0 \leq y, z \leq q-1} \sum_{y^2 \equiv x^2 + c_1 \pmod{q}} 1 + q^2 \sum_{0 \leq x, z \leq q-1} \sum_{x^2 \equiv y^2 + c_2 \pmod{q}} 1 - 2q^3 \right),
\]
where the final term is used to correct for the overcounting of $(j, k) = (0, 0)$. By factorising the ranges of summation using the difference of two squares, this expression is equal to
\[
q + (q - 1)(1_{q|c_1} + 1_{q|c_2} + 1_{q|(c_1+c_2)}).
\]
Recall the Gauss sum evaluation

\[
\sum_{0 \leq x \leq q-1} e_q(jx^2) = \left(\frac{j}{q}\right) \varepsilon_q \sqrt{q},
\]  

(3.10)

when \( q \nmid j \) and where \( \varepsilon_q = 1 \) if \( q \equiv 1 \) (mod 4) and \( \varepsilon_q = i \) if \( q \equiv 3 \) (mod 4), see [12, Thm. 3.3]. Therefore, the contribution from the remaining frequencies \( j, k \) is exactly

\[
\frac{\varepsilon_q^3}{q^{1/2}} \sum_{\substack{1 \leq j, k \leq q-1 \atop j \neq k}} \left(\frac{j}{q}\right) \left(\frac{k-j}{q}\right) \left(\frac{-k}{q}\right) e_q(-jc_1)e_q(-kc_2).
\]

By introducing a change of variables \( k = lj \), this expression is equal to

\[
\frac{\varepsilon_q^3}{q^{1/2}} \left(\frac{-1}{q}\right) \sum_{2 \leq l \leq q-1} \left(\frac{l-1}{q}\right) \sum_{1 \leq j \leq q-1} \left(\frac{j}{q}\right) e_q(j(-c_1 - lc_2)).
\]  

(3.11)

One can evaluate the inner sum of (3.11) using (3.10). Indeed, for an arbitrary integer \( m \) and an arbitrary quadratic non-residue \( h \) we have

\[
\sum_{1 \leq j \leq q-1} \left(\frac{j}{q}\right) e_q(jm) = \frac{1}{2} \left( \sum_{1 \leq n \leq q-1} e_q(mn^2) - \sum_{1 \leq n \leq q-1} e_q(mn^2h) \right)
\]

\[
= \frac{1}{2} \left( \sum_{0 \leq n \leq q-1} e_q(mn^2) - \sum_{0 \leq n \leq q-1} e_q(mn^2h) \right)
\]

\[
= \frac{1}{2} \left( \left(\frac{m}{q}\right) - \left(\frac{hm}{q}\right) \right) \varepsilon_q \sqrt{q}
\]

\[
= \left(\frac{m}{q}\right) \varepsilon_q \sqrt{q}.
\]

Plugging this expression into (3.11) we establish that (3.11) is equal to

\[
\sum_{2 \leq l \leq q-1} \frac{l(l-1)(lc_1 + c_2)}{q}.
\]

This is \( O(\sqrt{q}) \) by Hasse (see [12, (14.32)]), provided that neither \( q|c_1, q|c_2 \), nor \( q|(c_1 + c_2) \).

In the singular cases, if \( q \) divides \( c_1 \) we end up with

\[
\left(\frac{c_2}{q}\right) \sum_{2 \leq l \leq q-1} \left(\frac{l-1}{q}\right),
\]

which is equal to \(-\left(\frac{c_2}{q}\right)\). If \( q|(c_1 + c_2) \), we end up with

\[
\left(\frac{c_2}{q}\right) \sum_{2 \leq l \leq q-1} \left(\frac{l}{q}\right),
\]

which is equal to \(-\left(\frac{c_2}{q}\right)\). The final case, when \( q|c_2 \), follows easily from the case \( q|c_1 \) after permuting the variables \( x, y, z \) in the original expression for \( A_0(q, c_1, c_2) \). \( \square \)

Therefore \( (M/q)^3 A_0(q, c_1, c_2) \propto M^3 q^{-2} \), and this is the expected size of \( A(M, q, c_1, c_2) \). We note, then, that the \( M^3 \) term in (3.8) represents 'square-root cancellation on average' for the size of \( \Delta(M, q, c_1, c_2) \).

Moreover, Lemma 3.1 is close to best possible, at least for certain ranges of \( M \). We would like to emphasise that here and throughout, \( M \) denotes a positive integer. Indeed, if \( M < \lambda q^{2/3} \) and \( \lambda \) is a suitably small constant, then we have the matching lower bound

\[
\sum_{c_1, c_2 \leq q} \Delta(M, q, c_1, c_2)^2 \gg M^3.
\]  

(3.12)
Proof of (3.12). Certainly
\[ \sum_{c_1,c_2 \leq q} A(M, q, c_1, c_2) = M^3. \]
Furthermore,
\[ \sum_{c_1,c_2 \leq q} A(M, q, c_1, c_2)^2 = \sum_{x,y,z \leq M} 1. \]
By using the divisor bound to control the terms arising when \( x^2 - (x')^2 \neq 0 \), this sum is at most \( O(q^{o(1)}M^2q^{-1} + M^3) \), which is certainly at most \( O(M^3) \).

Therefore, by Cauchy-Schwarz
\[ \sum_{c_1,c_2 \leq q} 1_{A(M,q,c_1,c_2) \geq 1} \geq \left( \sum_{c_1,c_2 \leq q} A(M, q, c_1, c_2) \right)^2 \left( \sum_{c_1,c_2 \leq q} A(M, q, c_1, c_2)^2 \right)^{-1} \gg M^3. \]
If \( Mq^{-2/3} \) if sufficiently small then \((M/q)^3A_0(q, c_1, c_2) < 1/2 \) for all \( c_1, c_2 \). Hence
\[ \sum_{c_1,c_2 \leq q} \Delta(M, q, c_1, c_2)^2 \gg \sum_{c_1,c_2 \leq q} 1_{A(M,q,c_1,c_2) \geq 1} \gg M^3 \]
as claimed. \( \square \)

Rudnick–Sarnak–Zaharescu [22] also considered \( \Delta(M, q, c_1, c_2) \). By Fourier-expanding the cut-off \( x, y, z \leq M \), they derived
\[ \Delta(M, q, c_1, c_2) \leq \sum_{0 \leq b_1,b_2,b_3 \leq q-1} \prod_{i=1}^{3} \left| \hat{1}_{\left| \mathcal{M} \right|}(b_i) \right| \sum_{x,y,z \leq q-1} \sum_{x^2 - y^2 \equiv c_1 \pmod{q}} \sum_{y^2 - z^2 \equiv c_2 \pmod{q}} e_q(b_1x + b_2y + b_3z), \]
where
\[ \hat{1}_{\left| \mathcal{M} \right|}(b) = \frac{1}{q} \sum_{x \leq M} e_q(-bx). \] (3.13)
In expression (9.15) of [22] they used the Weil bound\(^5\) ending up with a bound of
\[ \Delta(M, q, c_1, c_2) \ll q^{1/2}(\log q)^3, \] (3.14)
in the non-degenerate cases. Comparing this result to Lemma 3.1 estimate (3.14) implies
\[ \sum_{c_1,c_2 \leq q} \Delta(M, q, c_1, c_2)^2 \ll q^{3+o(1)}, \] (3.15)
which is weaker than Lemma 3.1. We note, therefore, that the proof of Lemma 3.1 must utilise the extra averaging in \( c_1 \) and \( c_2 \) in a critical way.

In the introduction we promised to explain how the threshold \( N^{3/5} \) in Theorem 1.8 arises from the arguments of [22], and now seems to be an appropriate moment. Indeed, in order to analyse (3.4), it would be enough to show that
\[ \Delta(N, q, \overline{a}r_1, \overline{a}r_2) = o((N/q)^3A_0(q, \overline{a}r_1, \overline{a}r_2)), \]
since then one could replace \( A(N, q, \overline{a}r_1, \overline{a}r_2) \) with \((N/q)^3A_0(q, \overline{a}r_1, \overline{a}r_2) \) (and then evaluate these latter terms explicitly using Lemma 3.3). Using the bound (3.14), this is only possible when \( N^3q^{-2} > q^{1/2}(\log q)^3 \), i.e. provided that \( N \geq q^{5/6+o(1)} \). This, one notes, is the same as the threshold from Theorem 4 of [22] taken with \( m = 3 \) (for triple correlations). Noting that \( N \approx q^{2/5} \), this approach succeeds provided that \( 2/(2 + \beta) > 5/6 \), i.e. provided that \( \beta < 2/5 \). From the definition of \( \beta \), this implies that \( L \) must be at least \( N^{3/5} \).

\(^5\)For triple correlations the relevant curve has genus 1, so this is in fact the same Hasse bound as we used in Lemma 3.3.
Having finished our diversion on the subject of Lemma 3.1, let us return to the main argument. From now on, we assume \( \alpha \) satisfies Lemma 3.2. Putting this information into expression (3.4), we derive

\[
N^2q^{-3} \sum_{(r_1,r_2) \in S^\pm, r_1+r_2 \neq 0} A_0(q, \overline{a}r_1, \overline{a}r_2) - O_{\alpha, \eta, s, \kappa}(L^2q^{-\eta}) \leq R'_3(\alpha, L, N, g_{s, \kappa})
\]

\[
\leq N^2q^{-3} \sum_{(r_1,r_2) \in S^+, r_1+r_2 \neq 0} A_0(q, \overline{a}r_1, \overline{a}r_2) + O_{\alpha, \eta, s, \kappa}(L^2q^{-\eta}).
\]

(3.16)

To estimate the terms in the expression (3.16), we use Lemma 3.3. Indeed

\[
N^2q^{-3} \sum_{(r_1,r_2) \in S^\pm, r_1+r_2 \neq 0} A_0(q, \overline{a}r_1, \overline{a}r_2) = N^2q^{-3} \sum_{(r_1,r_2) \in S^\pm, r_1+r_2 \neq 0} (q + O(q^{1/2})).
\]

(3.17)

The size of \( S^\pm \) is given by

\[
(t_1 - s_1)qL N \pm 2N^2/q^{1-\eta} + O(1) \left( t_2 - s_2 \right) qL N \pm 2N^2/q^{1-\eta} + O(1).
\]

From relation (3.2), one observes that

\[
N^2q^{1-\eta} = o_{\beta, \eta}(qL N)
\]

as \( N \to \infty \), and therefore the size of \( S^\pm \) is seen to be

\[
|S^\pm| = (1 + o_{\beta, \eta, s, \kappa}(1))(t_1 - s_1)(t_2 - s_2)q^2L^2 N^{-2}
\]

(3.18)

as \( N \to \infty \). The estimate (3.18) remains after we remove those pairs \((r_1, r_2) \in S^\pm\) with \( r_1r_2 = 0 \) or \( r_1 + r_2 = 0 \).

Thus, returning to (3.17), we conclude that

\[
N^2q^{-3} \sum_{(r_1,r_2) \in S^\pm, r_1+r_2 \neq 0} A_0(q, \overline{a}r_1, \overline{a}r_2) = (1 + o_{\beta, \eta, s, \kappa}(1))L^2 (t_1 - s_1)(t_2 - s_2).
\]

Substituting this estimate into (3.16), we derive Theorem 2.2 as required.

What remains is to verify that almost all \( \alpha \) admit an approximation \( a/q \) of the form required in (3.1) and (3.2), and to prove Lemma 3.1 and Lemma 3.2.

4. Approximation with Prime Denominator

To derive a suitable approximation \( a/q \) with prime denominator, we use the following quantitative version of (a generalized) Khintchine’s theorem, due to Harman:

**Theorem 4.1.** [10 Thm. 4.2] Let \( \psi : \mathbb{N} \to (0, 1) \) be a non-increasing function such that

\[
\Psi(N) = \sum_{n \leq N} \psi(n)
\]

is unbounded. For \( \mathcal{B} \) an infinite set of integers, let \( S(\mathcal{B}, \alpha, N) \) denote the number of \( n \leq N \), with \( n \in \mathcal{B} \), such that \( \|n\alpha\| < \psi(n) \). Then, for almost all \( \alpha \), we have

\[
S(\mathcal{B}, \alpha, N) = 2\Psi(N, \mathcal{B}) + O_{\varepsilon}((\Psi(N))^2 \log \Psi(N))^{2+\varepsilon}
\]

(4.1)
for each $\varepsilon > 0$ where

$$
\Psi(N, B) = \sum_{n \in B \cap [N]} \psi(n).
$$

Moreover, the implied constant in (4.1) is uniform in $\alpha$.

From this we deduce the following:

**Lemma 4.2.** Let $\eta \in (0, 1)$. Then, for almost all $\alpha$, for all $N \geq N_0(\eta)$ there is a prime $q$ satisfying

$$
\|q\alpha\| < \frac{1}{N^{1-\eta}}, \quad \text{and} \quad N \leq q \leq 2N.
$$

(*Proof.* We let $B$ denote the set of primes, and $\psi(n) = n^{1+\eta}$. Then $\Psi(N) \sim \eta^{-1}N^\eta$ and (by the prime number theorem) $\Psi(N, B) \sim \eta^{-1}N^\eta(\log N)^{-1}$. So, the asymptotic formula (4.1) shows that for almost all $\alpha$ one has that

$$
S(B, \alpha, N) = 2N^\eta \left(1 + o(1)\right) \eta \log N + O(\eta^{-1}N^\eta(\log N)^3),
$$

with the implied constant uniform in $\alpha$. From this it immediately follows that if $N$ is sufficiently large in terms of $\eta$ then

$$
S(B, \alpha, 2N) - S(B, \alpha, N) > 0,
$$

as required. \hspace{1cm} \Box

Therefore an approximation $a/q$ may be found, with $q$ prime, that satisfies (3.1) and (3.2).

5. **Proof of Lemma 3.1**

Expanding the square we have

$$
\sum_{c_1, c_2 \leq q} \Delta(N, q, c_1, c_2)^2 = S_1 - 2S_2 + S_3,
$$

where

$$
S_1 = \sum_{c_1, c_2 \leq q} A(M, q, c_1, c_2)^2,
$$

$$
S_2 = \left(\frac{M}{q}\right)^3 \sum_{c_1, c_2 \leq q} A(M, q, c_1, c_2)A_0(q, c_1, c_2),
$$

$$
S_3 = \left(\frac{M}{q}\right)^6 \sum_{c_1, c_2 \leq q} A_0(q, c_1, c_2)^2.
$$

We can write each $S_i$ as the number of solutions to certain equations, namely...
where terms when $j$ and $A$ we did when estimating $S$ the same expansion on with the bound but since $q$ is prime and $b$ we will be able to improve on this bound substantially. This is

$$S_1 = \sum_{1 \leq x, y, z \leq M \atop 1 \leq x', y', z' \leq M} 1, \quad x^2 - y^2 \equiv (x')^2 - (y')^2 \pmod{q} \quad y^2 - z^2 \equiv (y')^2 - (z')^2 \pmod{q}$$

$$S_2 = \left(\frac{M}{q}\right)^3 \sum_{1 \leq x, y, z \leq M \atop 1 \leq x', y', z' \leq q} 1, \quad x^2 - y^2 \equiv (x')^2 - (y')^2 \pmod{q} \quad y^2 - z^2 \equiv (y')^2 - (z')^2 \pmod{q}$$

$$S_3 = \left(\frac{M}{q}\right)^6 \sum_{1 \leq x, y, z \leq q \atop 1 \leq x', y', z' \leq q} 1, \quad x^2 - y^2 \equiv (x')^2 - (y')^2 \pmod{q} \quad y^2 - z^2 \equiv (y')^2 - (z')^2 \pmod{q}$$

Expanding the cut-offs $1 \leq x, y, z, x', y', z' \leq M$ in terms of additive characters we have

$$S_1 = \sum_{b=(b_1, b_2, b_3, b_4, b_5, b_6), \atop 0 \leq b_1, b_2, b_3, b_4, b_5, b_6 \leq q-1} S(b, q) \prod_{i=1}^{6} \widehat{1_{[M]}}(b_i),$$

where

$$S(b, q) := \sum_{x, y, z \leq q \atop x', y', z' \leq q} e_q(b \cdot (x, y, z, x', y', z')) \quad x^2 - y^2 \equiv (x')^2 - (y')^2 \pmod{q} \quad y^2 - z^2 \equiv (y')^2 - (z')^2 \pmod{q}$$

and $\widehat{1_{[M]}}$ is as in (3.13). The contribution from the term with $b = 0$ is equal to $S_3$. Performing the same expansion on $S_2$, we see that the terms arising from $b = 0$ cancel, and we are left with the bound

$$S_1 - 2S_2 + S_3 \leq \sum_{0 \leq b_1, b_2, b_3, b_4, b_5, b_6 \leq q-1} \left( \prod_{i=1}^{6} |\widehat{1_{[N]}}(b_i)| \right) |S(b, q)|. \quad (5.1)$$

Our task moves to bounding $|S(b, q)|$. We have the trivial bound

$$|S(b, q)| \leq q^6,$$

but since $q$ is prime and $b \neq 0$ we will be able to improve on this bound substantially.

Our approach will be elementary. We begin with writing

$$S(b, q) = \frac{1}{q^2} \sum_{j, k \leq q \atop x, y, z, x', y', z' \leq q} e_q(b \cdot (x, y, z, x', y', z')) \quad j(x^2 - y^2 - (x')^2 + (y')^2) + k(y^2 - z^2 - (y')^2 + (z')^2). \quad (5.3)$$

As we did when estimating $A_0(q, c_1, c_2)$, let us first consider the contribution from those terms when $j = 0$, $k = 0$, or $j = k$. This is
\[
\frac{1}{q^2} \sum_{k \leq q} \sum_{x,y,z \leq q \atop x',y',z' \leq q} e_q(b \cdot (x, y, z, x', y', z') + k(y^2 - z^2 - (y')^2 + (z')^2)) + 1 \sum_{j \leq q} \sum_{x,y,z \leq q \atop x',y',z' \leq q} e_q(b \cdot (x, y, z, x', y', z') + j(x^2 - y^2 - (x')^2 + (y')^2)) + \frac{1}{q^2} \sum_{j \leq q} \sum_{x,y,z \leq q \atop x',y',z' \leq q} e_q(b \cdot (x, y, z, x', y', z') + j(x^2 - z^2 - (x')^2 + (z')^2)) \]

\[
- \frac{2}{q^2} \sum_{x,y,z \leq q \atop x',y',z' \leq q} e_q(b \cdot (x, y, z, x', y', z')).
\]

The first three of these terms devolve into the exponential sums involved in the pair correlations of the fractional parts of \(\alpha n^2\) considered by Heath-Brown in [11]. Indeed, note that by completing the square in the variables \(y, y', z, z'\) we have

\[
\frac{1}{q^2} \sum_{k \leq q} \sum_{x,y,z \leq q \atop x',y',z' \leq q} e_q(b \cdot (x, y, z, x', y', z') + k(y^2 - z^2 - (y')^2 + (z')^2))
\]

is equal to

\[
1_{q|b_1} 1_{q|b_4} \sum_{k \leq q-1} |G(k)|^4 e_q(\frac{-b_2^2 + b_3^2 + b_5^2 - b_6^2}{4k}),
\]

where

\[
G(k) = \sum_{x \leq q} e_q(kx^2)
\]

is the Gauss sum, as before. Here we use \(\frac{1}{q}\) to refer to the multiplicative inverse of \(k\) modulo \(q\). Since \(|G(k)| = q^{1/2}\) by the standard evaluation (3.10), the term (5.5) is equal to

\[
\begin{cases} 
q^3 - q^2 & \text{if } q|b_1, q|b_4, q|(-b_2^2 + b_3^2 + b_5^2 - b_6^2) \\
-q^2 & \text{if } q|b_1, q|b_4, q \nmid (-b_2^2 + b_3^2 + b_5^2 - b_6^2) \\
0 & \text{if } q \nmid b_1 \text{ or } q \nmid b_4.
\end{cases}
\]

We compute that the overall size of (5.4) is

\[
\begin{cases} 
O(q^3) & \text{if } q|b_1, q|b_4, q|(-b_2^2 + b_3^2 + b_5^2 - b_6^2) \\
O(q^3) & \text{if } q|b_2, q|b_5, q|(-b_2^2 + b_3^2 + b_5^2 - b_6^2) \\
O(q^3) & \text{if } q|b_3, q|b_6, q|(-b_2^2 + b_3^2 + b_5^2 - b_6^2) \\
O(q^2) & \text{otherwise.}
\end{cases}
\]

Now consider the contribution to (5.2) when \(j \neq 0, k \neq 0, j \neq k\). By completing the square again, this contribution is equal to

\[
\frac{1}{q^2} \sum_{j,k \leq q-1 \atop j \neq k} \left|G(j)\right|^2 \left|G(k-j)\right|^2 \left|G(-k)\right|^2 e_q\left(\frac{-b_2^2}{4j} - \frac{b_3^2}{4(k-j)} - \frac{b_5^2}{4k} + \frac{b_4^2}{4j} + \frac{b_6^2}{4(k-j)} + \frac{b_7^2}{4k}\right).
\]

\[
(5.8)
\]
This is equal to

\[ q \sum_{\substack{j, k \leq q - 1 \atop j \neq k}} e_q \left( \frac{f(j, k)}{g(j, k)} \right) \]

where

\[ f(j, k) = j^2(b_2^2 - b_6^2) + jk(b_1^2 - b_2^2 - b_3^2 - b_4^2 + b_5^2) + k^2(-b_1^2 + b_2^2) \]

and

\[ g(j, k) = 4jk(-j + k). \]

By letting \( l = j/k \) (mod \( q \)), we can reparametrize this exponential sum as

\[ q \sum_{k \leq q - 1 \atop 2 \leq l \leq q - 1} e_q \left( -\frac{f(l, 1)}{kg(l, 1)} \right). \quad (5.9) \]

Estimating (5.9) splits into several cases.

**Case 1:** \( q \nmid (b_1^2 - b_2^2)(b_2^2 - b_6^2)(b_3^2 - b_5^2), \) \( q \nmid \text{disc}(f(l, 1)) \). In this case the equation \( f(l, 1) = 0 \) (mod \( q \)) has exactly two solutions in the range \( 2 \leq l \leq q - 1 \), and hence (5.9) is equal to exactly \( q(q + 2) \).

**Case 2:** \( q \nmid (b_1^2 - b_2^2)(b_2^2 - b_6^2)(b_3^2 - b_5^2), \) \( q| \text{disc}(f(l, 1)) \). In this case the equation \( f(l, 1) = 0 \) (mod \( q \)) has only one solution in the range \( 2 \leq l \leq q - 1 \), and hence (5.9) is equal to exactly \( 2q \).

**Case 3:** \( q \) divides exactly one of the differences \( b_1^2 - b_2^2, b_2^2 - b_3^2, \) and \( b_3^2 - b_5^2 \). If we have \( q| (b_3^2 - b_5^2) \) then \( f(l, 1) \) is linear, and since \( q \nmid (b_2^2 - b_6^2)(b_3^2 - b_5^2) \) we must have \( f(1, 1)f(0, 1) \neq 0 \) (mod \( q \)). In the other cases \( f(l, 1) \) is quadratic with one root \( \alpha \in \{0, 1\} \) and another root \( \beta \in \{2, \ldots, q-1\} \). In both options we have exactly one solution to \( f(l, 1)/g(l, 1) \equiv 0 \) (mod \( q \)) with \( 2 \leq l \leq q - 1 \), and hence (5.9) is equal to exactly \( 2q \).

**Case 4:** \( q \) divides exactly two of difference \( b_1^2 - b_2^2, b_2^2 - b_3^2, \) and \( b_3^2 - b_5^2 \). In this case there are no solutions to \( f(l, 1)/g(l, 1) \equiv 0 \) (mod \( q \)) with \( 2 \leq l \leq q - 1 \). Hence (5.9) is equal to exactly \( -q(q-2) \).

**Case 5:** \( q \mid (b_2^2 - b_4^2), q \mid (b_2^2 - b_5^2), \) and \( q \mid (b_3^2 - b_6^2) \). In this case (5.9) is exactly \( q(q-1)(q-2) \).

If we combine our knowledge of the sum (5.9) with our knowledge of the terms with \( j = 0, k = 0, j = k \) from (5.7), this yields:

\[ |S(b, q)| = \begin{cases} O(q^2) & \text{if } q|b_1, q|b_4, q((-b_2^2 + b_3^2 + b_5^2 - b_6^2) \\
O(q^3) & \text{if } q|b_2, q|b_5, q((-b_1^2 + b_2^2 + b_4^2 - b_7^2) \\
O(q^3) & \text{if } q|b_3, q|b_6, q((-b_1^2 + b_2^2 + b_4^2 - b_7^2) \\
O(q^2) & \text{if } q \text{ divides all of } (b_1^2 - b_2^2), (b_2^2 - b_3^2), \text{ and } (b_3^2 - b_5^2) \\
O(q^2) & \text{otherwise.} \end{cases} \quad (5.10) \]

Plugging this bound into (5.11), we infer that

\[ S_1 - 2S_2 + S_3 \ll q^3(T_1 + T_2) + q^2T_3 \quad (5.11) \]
where (after a straightforward relabelling of the variables)

\[ T_1 = \sum_{-q/2 < b_1, b_2, b_3, b_4, b_5, b_6 < q/2 \atop b \neq 0} \left( \prod_{i=1}^{6} |\widehat{1}_{[M]}(b_i)| \right), \]

\[ T_2 = \sum_{-b/2 < b_1, b_2, b_3, b_4 < q/2 \atop b \neq 0} \frac{M^2}{q^2} \left( \prod_{i=1}^{4} |\widehat{1}_{[M]}(b_i)| \right), \]

\[ T_3 = \sum_{-q/2 < b_1, b_2, b_3, b_4, b_5, b_6 \leq q/2 \atop b \neq 0} \left( \prod_{i=1}^{6} |\widehat{1}_{[M]}(b_i)| \right). \]

Indeed, since \( q \) is prime, if \( q | b_1 - b_2 \), then \( q | (b_1 + b_4) \) or \( q | (b_1 - b_4) \). It is at this point when the assumption that \( q \) is prime becomes particularly convenient.

For \(-q/2 < b < q/2\), recall the standard bound

\[ |\widehat{1}_{[M]}(b)| \ll \min \left( \frac{M}{q}, \frac{1}{|b|} \right) \]

which produces the estimate

\[ \sum_{-q/2 < b < q/2} |\widehat{1}_{[M]}(b)| \ll \sum_{|b| \leq q/M} \frac{M}{q} + \sum_{q/M < |b| < q/2} \frac{1}{|b|} \ll \log q. \]

In the sum defining \( T_1 \), if \((b_1, b_2, b_3)\) are fixed then there are \( O(1) \) possibilities for \((b_4, b_5, b_6)\). Similarly, in \( T_2 \), once \((b_1, b_2, b_3)\) are fixed then there are at most 2 possibilities for \( b_4 \). Hence

\[ T_1, T_2 \ll \frac{M^3}{q^3} (\log q)^3 \quad \text{and} \quad T_3 \ll (\log q)^6, \]

since \( M < q \). Substituting these bounds into \((6.1)\), the proof of Lemma 3.1 is complete. \( \square \)

6. Proof of Lemma 3.2

Let \( \beta \) and \( \eta \) be as in the statement of Lemma 3.2. We begin the proof with another auxiliary lemma, which concerns the quantity

\[ \Delta^*_{\beta}(q, c_1, c_2) = \max_{M \leq q^{2/3+\varepsilon}} |\Delta(M, q, c_1, c_2)|. \]

**Lemma 6.1.** Let \( q \) be an odd prime. Then we have

\[ \sum_{c_1, c_2 \leq q} \Delta^*_{\beta}(q, c_1, c_2)^2 \ll q^{o(1)} q^{7-\beta/2}. \]

**Proof.** We will deduce this result from Lemma 3.1. Taking \( 1 \leq K \leq M \leq q \), we begin by seeking a bound on

\[ \sum_{c_1, c_2 \leq q} (A(M + K, q, c_1, c_2) - A(M, q, c_1, c_2))^2. \quad (6.1) \]
Firstly, \(A(M + K, q, c_1, c_2) - A(M, q, c_1, c_2)\) is at most

\[
|\{ x \in (M, M + K), y, z \leq M + K : x^2 - y^2 \equiv c_1 \pmod{q}, y^2 - z^2 \equiv c_2 \pmod{q}\}|
\]

\[
+ |\{ y \in (M, M + K), x, z \leq M + K : y^2 - z^2 \equiv c_1 \pmod{q}, y^2 - x^2 \equiv c_2 \pmod{q}\}|
\]

\[
+ |\{ z \in (M, M + K), x, y \leq M + K : x^2 - y^2 \equiv c_1 \pmod{q}, y^2 - z^2 \equiv c_2 \pmod{q}\}|
\].

Let \(E_1(q, c_1, c_2)\) refer to the first quantity, \(E_2(q, c_1, c_2)\) refer to the second quantity, and \(E_3(q, c_1, c_2)\) refer to the third quantity. We have that expression (6.1) is at most a constant times

\[
\sum_{c_1, c_2 \leq q} (E_1(q, c_1, c_2)^2 + E_2(q, c_1, c_2)^2 + E_3(q, c_1, c_2)^2).
\]

By a change of variables, we may reduce consideration just to \(E_2\). Indeed, if \(x^2 - y^2 \equiv c_1 \pmod{q}\) and \(y^2 - z^2 \equiv c_2 \pmod{q}\) then \(z^2 - x^2 \equiv -c_1 - c_2 \pmod{q}\). Hence \(E_1(q, c_1, c_2) = E_2(q, -c_1 - c_2, c_1)\), and so

\[
\sum_{c_1, c_2 \leq q} E_1(q, c_1, c_2)^2 = \sum_{c_1, c_2 \leq q} E_2(q, c_1, c_2)^2.
\]

A similar argument works for \(E_3(q, c_1, c_2)\).

Now, \(\sum_{c_1, c_2 \leq q} E_2(q, c_1, c_2)^2\) is equal to

\[
\left| \left\{ y_1, y_2 \in (M, M + K), x_1, x_2, z_1, z_2 \leq M + K : x_1^2 - y_1^2 \equiv x_2^2 - y_2^2 \pmod{q}, y_1^2 - z_1^2 \equiv y_2^2 - z_2^2 \pmod{q} \right\} \right|. \tag{6.2}
\]

Fixing integers \(k_1, k_2\) and integers \(y_1, y_2 \in (M, M + K)\), we will now bound the number of solutions \((x_1, x_2, z_1, z_2)\) to the system of equations

\[
x_1^2 - x_2^2 = y_1^2 - y_2^2 + k_1 q
\]

\[
z_1^2 - z_2^2 = y_1^2 - y_2^2 + k_2 q. \tag{6.3}
\]

If both \(y_1^2 - y_2^2 + k_1 q \neq 0\) and \(y_1^2 - y_2^2 + k_2 q \neq 0\) then, using the divisor bound, we get \(q^{o(1)}\) solutions. If \(y_1^2 - y_2^2 + k_1 q = 0\) and \(y_1^2 - y_2^2 + k_2 q \neq 0\) we get \(O(q^{o(1)} M)\) solutions. If both \(y_1^2 - y_2^2 + k_1 q = 0\) and \(y_1^2 - y_2^2 + k_2 q = 0\) we get \(O(M^2)\) solutions. Note that in such a case we must have \(k_1 = -k_2\).

Regarding the other conditions on \((y_1, y_2, k_1, k_2)\), we note that the variables \(k_1, k_2\) are both restricted to intervals of length \(O(M^2/q + 1)\). Further, if \(k_1 \neq 0\) then the total number of solutions of \(y_1, y_2, k_1\) to \(y_1^2 - y_2^2 = k_1 q\), with \(y_1, y_2 \in (M, M + K)\), is \(O(q^{o(1)}(KM/q + 1))\), since \(|y_1^2 - y_2^2| = O(KM)\). If, however, \(k_1 = 0\) then of course \(y_1 = y_2\) so there are \(O(K)\) solutions here. Thus, summing over \((y_1, y_2, k_1, k_2)\), we bound (6.2) above by

\[
q^{o(1)} \left( K^2 \left( \frac{M^2}{q} + 1 \right)^2 + M \left( \frac{M^2}{q} + 1 \right) \left( \frac{KM}{q} + 1 \right) + KM \left( \frac{M^2}{q} + 1 \right) + M^2 \left( \frac{KM}{q} + 1 \right) + KM^2 \right). \tag{6.4}
\]

Since \(K \leq M \leq q\) we may simplify the above and conclude that

\[
\sum_{c_1, c_2 \leq q} (A(M + K, q, c_1, c_2) - A(M, q, c_1, c_2))^2 \ll q^{o(1)}(K^2 M^4 q^{-2} + KM^2). \tag{6.4}
\]

One may think of this upper bound as being given by the sum of an average contribution and a diagonal contribution.

From (6.4), together with previous lemmas, it turns out that one can control

\[
\sum_{1 \leq c_1, c_2 \leq q} \max_{0 \leq H \leq K} \Delta(M + H, q, c_1, c_2)^2.
\]

Indeed, since

\[|(a_1 - a_2)^2 - (b_1 - b_2)^2| \ll (a_1 - b_1)^2 + (a_2 - b_2)^2\]
for all reals $a_1, b_1, a_2, b_2$, we have that
$$
\sum_{c_1, c_2 \leq q} \max_{0 \leq H \leq K} \Delta(M + H, q, c_1, c_2)^2 - \sum_{c_1, c_2 \leq q} \Delta(M, q, c_1, c_2)^2
$$
\hspace{1cm} (6.5)

is as most a constant times
$$
\sum_{c_1, c_2 \leq q} (A(M + K, q, c_1, c_2) - A(M, q, c_1, c_2))^2
$$
$$
+ \max_{0 \leq H \leq K} \left( \frac{(M + H)^3 - M^3}{q^3} \right)^2 \sum_{c_1, c_2 \leq q} A_0(q, c_1, c_2)^2.
$$
\hspace{1cm} (6.6)

From expression (6.4) and Lemma 3.3, we conclude that (6.6) is at most
$$
q^{o(1)}(K^2 M^4 q^{-2} + K M^2)
$$
again. Finally, using Lemma 3.1 combined with (6.5), we end up with the bound
$$
\sum_{c_1, c_2 \leq q} \max_{0 \leq H \leq K} \Delta(M + H, q, c_1, c_2)^2 \ll q^{o(1)}(M^3 + q^2) + q^{o(1)} K^2 M^4 q^{-2}.
$$
\hspace{1cm} (6.7)

(The $K M^2$ term has been absorbed into the $M^3$ term.)

Before we use (6.7) to prove the lemma, we need the bound
$$
\sum_{c_1, c_2 \leq q} \max_{0 \leq H \leq K} \Delta(H, q, c_1, c_2)^2 \ll q^{o(1)}(K^6 q^{-2} + K^3).
$$
\hspace{1cm} (6.8)

This may be proved by an identical analysis to the one above, proceeding with $M = 0$.

Now, returning to the original object of the lemma, we may cover
$$
\{ n \in \mathbb{N} : 1 \leq n \leq q^{2/3} \}
$$
by the interval $[1, K]$ together with at most $q^{2/3} K^{-1}$ other intervals each of the form $[M, M + K]$ with $K \leq M$. Thus,
$$
\sum_{c_1, c_2 \leq q} \Delta^\beta(q, c_1, c_2)^2 \leq q^{o(1)}(K^6 q^{-2} + K^3) + \sum_{l \leq q^{2/3}} \sum_{1 \leq l \leq K} \max_{X \in [lK, (l+1)K]} \Delta(X, q, c_1, c_2)^2
$$
$$
\leq q^{o(1)}(K^6 q^{-2} + K^3) + q^{o(1)} \sum_{l \leq q^{2/3}} (l^3 K^3 + q^2 + l^4 K^6 q^{-2})
$$
$$
\leq q^{o(1)}(K^6 q^{-2} + K^3) + q^{o(1)}(q^{8/3} K^{-1} + q^{2/3} K^{-1} + q^{10/3} K^{-2}).
$$

Since $\beta \leq 1$ always, we have $q^{2/3} K^{-2} \ll q^{8/3}$, which allows us to reduce matters to the bound
$$
q^{o(1)}(K^6 q^{-2} + K^3 + q^{2/3} K^{-1} + q^{10/3} K^{-2} K).
$$

Optimising in $K$, we find that the minimum value is achieved when $K \asymp q^{1+\beta}$, in which case the third and fourth terms have the same order of magnitude. We conclude that
$$
\sum_{c_1, c_2 \leq q} \Delta^\beta(q, c_1, c_2)^2 \ll q^{o(1)} \frac{q^{7-\beta}}{q^{2/3}}
$$
and the lemma is proved.
Proof of Lemma 3.2. Let $C$ be a suitably large absolute constant. For all odd prime $q$, and integers $a$ such that $(a, q) = 1$, define

$$D_{\beta,\eta}(a, q) := \sum_{\substack{r_1, r_2 \neq 0 \\ |r_1| \leq q^{2-\frac{4\alpha}{2+\beta}} + C\eta \\ |r_2| \leq q^{2-\frac{4\alpha}{2+\beta}} + C\eta}} \Delta^*_\beta(q, \overline{ar_1}, \overline{ar_2}). \quad (6.9)$$

Now, suppose $\alpha \in [0, 1]$ and fractions $a/q$ and $N$ satisfy (3.1) and (3.2). We claim that, if $C$ is large enough, and if $q$ is large enough in terms of $s$, $t$, and $\eta$, it follows that

$$\sum_{(r_1, r_2) \in S^+, \ r_1 \neq 0, \ r_1 + r_2 \neq 0} \Delta(N, q, \overline{ar_1}, \overline{ar_2}) \leq D_{\beta,\eta}(a, q). \quad (6.10)$$

Indeed, since $N \leq q^{2+\beta}$, we have

$$\Delta(N, q, \overline{ar_1}, \overline{ar_2}) \leq \Delta^*_\beta(q, \overline{ar_1}, \overline{ar_2}).$$

Also, we have both

$$\frac{qL}{N} \ll N^{\frac{2+\beta}{2} - \beta + 11\eta} \ll q^{(\frac{2+\beta}{2} + 11\eta) \frac{2}{2+\beta}} \ll q^{2-\frac{4\alpha}{2+\beta} + C\eta}$$

and

$$N^2 q^{\eta - 1} \ll N^{2(1-\eta)(\frac{2+\beta}{2} + 10\eta)} \ll q^{(\frac{2+\beta}{2} + C\eta) \frac{2}{2+\beta}} \ll q^{2-\frac{4\alpha}{2+\beta} + C\eta}.$$ 

Referring to the definition (3.6) of $S^+$, we have thus settled the inequality (6.10).

Therefore, to prove Lemma 3.2, it suffices to show that, for almost all $\alpha \in [0, 1]$, for all $(a, q, N)$ satisfying (3.1) and (3.2),

$$D_{\beta,\eta}(a, q)L^{-2}N^{-1} \ll_{\alpha,\beta,\eta} q^{-\eta}. \quad (6.11)$$

In fact, by (3.2), it suffices to show that, for almost all $\alpha \in [0, 1]$ and for all $(a, q)$ satisfying (3.1),

$$D_{\beta,\eta}(a, q)q^{\frac{6+4\beta}{2+\beta}} \ll_{\alpha,\beta,\eta} q^{-C\eta}. \quad (6.12)$$

This expression makes no mention of $N$ or $L$, which will be a technical necessity in the Borel–Cantelli argument to come.

To show (6.12), for each $q$ we define

$$\text{Bad}_{\beta,\eta}(q) = \{a \leq q : (a, q) = 1, D_{\beta,\eta}(a, q)q^{\frac{6+4\beta}{2+\beta}} \geq q^{-C\eta}\}.$$ 

It will be enough, then, to show that for almost all $\alpha \in [0, 1]$, only finitely many of the fractions $(a, q)$ satisfying (3.1) also satisfy $a \in \text{Bad}_{\beta,\eta}(q)$.

To bound the size of $\text{Bad}_{\beta,\eta}(q)$, we first note that

$$|\text{Bad}_{\beta,\eta}(q)| \leq q^{\frac{6+4\beta}{2+\beta} + C\eta} \sum_{1 \leq a \leq q \ (a, q) = 1} D_{\beta,\eta}(a, q).$$

Further, we define $f_{\beta,\eta}(q, c_1, c_2)$ to be the number of triples $(a, r_1, r_2)$ such that

$$\overline{ar_1} \equiv c_1 \pmod{q}$$

$$\overline{ar_2} \equiv c_2 \pmod{q},$$

for which $1 \leq a \leq q$, $(a, q) = 1$, $r_1r_2 \neq 0$, and $|r_1|, |r_2| \leq q^{2-\frac{4\alpha}{2+\beta} + C\eta}$. We then have

$$|\text{Bad}_{\beta,\eta}(q)| \leq q^{\frac{6+4\beta}{2+\beta} + C\eta} \sum_{0 \leq c_1, c_2 \leq q-1} f_{\beta,\eta}(q, c_1, c_2) \Delta^*_\beta(q, c_1, c_2). \quad (6.13)$$
To estimate (6.13), we need the following simple lemma about the size of \( f_{\beta, \eta}(q, c_1, c_2) \):

**Lemma 6.2.** We have

\[
\sum_{c_1, c_2 \leq q} f_{\beta, \eta}(q, c_1, c_2)^2 \ll q^{4 - 2\beta} + 2C\eta^2(q + q^{4 - 2\beta} + 2C\eta)q^{o(1)}.
\]

**Proof.** We have that \( \sum_{c_1, c_2 \leq q} f_{\beta, \eta}(q, c_1, c_2)^2 \) is equal to the number of pairs of triples \((a, r_1, r_2), (b, s_1, s_2)\) such that

\[
\begin{aligned}
\bar{a} r_1 &\equiv \bar{b} s_1 \pmod{q} \\
\bar{a} r_2 &\equiv \bar{b} s_2 \pmod{q},
\end{aligned}
\]

with \(1 \leq a, b \leq q\), \((a, q) = 1\), \((b, q) = 1\), \(r_1 r_2 s_1 s_2 \neq 0\) and \(|r_1|, |r_2|, |s_1|, |s_2| \leq q^{2 - \frac{2}{3\beta}} + C\eta\). By multiplying the first equation by \(br_2\) and the second equation by \(br_1\), one sees that for every such solution we must also have

\[
s_1 r_2 = s_2 r_1 \pmod{q}.
\]

The mapping between the solutions to (6.14) and (6.15) is at most \(q\)-to-1 because if the variables \(a, r_1, r_2, s_1, s_2\) are fixed then \(b\) is uniquely determined in (6.14).

Solutions to (6.15) are given by solutions \((r_1, r_2, s_1, s_2, k)\) to

\[
s_1 r_2 - s_2 r_1 = kq,
\]

with \(k\) in the range \(0 \leq |k| \ll q^{2 - \frac{2}{3\beta}} - 1 + 2C\eta\). By the divisor bound, after \(k, s_1, r_2\) are fixed there at most \(q^{o(1)}\) valid choices of \(s_2\) and \(r_1\), so the total number of solutions to (6.15) is at most

\[
q^{4 - 2\beta} + 2C\eta^2(1 + q^{4 - 2\beta} - 1 + 2C\eta).
\]

Multiplying by \(q\) to get the number of solutions to (6.14), we prove the lemma.

The reader may wish to note that if \(\beta > 2/3\) then the only valid value of \(k\) in the above is \(k = 0\), which simplifies the remainder of the analysis for these cases.

Returning to (6.13), we have

\[
|\text{Bad}_{\beta, \eta}(q)| \ll q^{\frac{6 + 6\beta}{2 + 3\beta} + C\eta} \cdot \left( \sum_{c_1, c_2 \leq q} f_{\eta, \beta}(q, c_1, c_2)^2 \right)^{1/2} \cdot \left( \sum_{c_1, c_2 \leq q} \Delta^*_\beta(q, c_1, c_2)^2 \right)^{1/2} \\
\ll q^{\frac{6 + 6\beta}{2 + 3\beta} \cdot \left( q^{4 + 2\beta - \frac{4 - 2\beta}{2 + 3\beta}} + q^{4 - 2\beta} \right) \cdot q^{\frac{7 - 2\beta}{2 + 3\beta}} \cdot q^{C\eta + o(1)}} \\
\ll (q^{\frac{1 + 6\beta}{2 + 3\beta}} + q^{\frac{3 + 3\beta}{2 + 3\beta}})q^{C\eta + o(1)}.
\]

Here we have used Lemma 6.1 and Lemma 6.2 to go from the first line to the second line.

Now, recall that \(\beta < 3/4\), which implies that

\[
\frac{1 + 6\beta}{2(2 + \beta)} < 1.
\]

The other term is less severe, and in fact \(\beta < 1\) implies

\[
\frac{3 + 3\beta}{2(2 + \beta)} < 1.
\]

Since \(\eta\) is small enough, and \(C\) is absolute, we conclude that

\[
|\text{Bad}(q, \beta, \eta)| \ll_{\beta, \eta} q^{1 - 2\eta}.
\]

(6.17)
Now we can finally complete the proof of Lemma 3.2 by using the first Borel–Cantelli lemma. Indeed, pick $\alpha$ uniformly at random in $[0, 1]$ and for each prime $q \geq 3$ let $E_{q, \beta, \eta}$ be the event that there exists an $a$ with $1 \leq a \leq q$, $(a, q) = 1,$ $|\alpha - \frac{a}{q}| < \frac{1}{q^2 - \eta},$ and $a \in \text{Bad}_{\beta, \eta}(q)$. Then

$$P(E_{q, \beta, \eta}) \leq \sum_{a \in \text{Bad}_{\beta, \eta}(q)} \mu\left(\left|\frac{a}{q} - \frac{1}{q^{2 - \eta}}\right|, \frac{a}{q} + \frac{1}{q^{2 - \eta}}\right) \ll_{\beta, \eta} q^{-1 - \eta}.$$

Then $\sum_{q \geq 3} P(E_{q, \beta, \eta}) < \infty$, and so with probability 1 only finitely many of the events $E_{q, \beta, \eta}$ occur. Thus, by our long chain of reductions, Lemma 3.2 follows. □

7. Concluding Remarks

The proof of all of our main theorems is now complete. However, before concluding the paper, it is certainly worth us discussing whether $\beta < 3/4$ represents a natural limit of our approach.

The chain of inequalities (6.10) is the critical moment of the entire proof, and this particular application Cauchy–Schwarz is the main source of our loss in the range of $\beta$. Suppose that instead we had used the bound

$$|\text{Bad}_{\beta, \eta}(q)| \leq q^{6 + 4 \beta} + Cq\left(\sum_{c_1, c_2 \leq q} f_{\beta, \eta}(q, c_1, c_2)\right)^{1/2}\left(\sum_{c_1, c_2 \leq q} f_{\beta, \eta}(q, c_1, c_2)\Delta_{\beta}^*(q, c_1, c_2)^2\right)^{1/2}.$$

(7.1)

For simplicity of exposition here, we will assume that $\beta > 2/3$, that $C = 0$, and we will ignore all $q^{o(1)}$ terms. It is then easy to see that

$$\sum_{c_1, c_2 \leq q} f_{\beta, \eta}(q, c_1, c_2) \approx q^{1 + \frac{2(2 - \beta)}{2 + \beta}}.$$

Combining this bound with Lemma 6.2 one may conclude that $f_{\beta, \eta}(q, c_1, c_2) \approx 1$ for $q^{1 + \frac{2(2 - \beta)}{2 + \beta}}$ pairs $(c_1, c_2)$, and is 0 otherwise. So, given what we know from Lemma 6.1 about the value of $\Delta_{\beta}^*(q, c_1, c_2)^2$ averaged over all pairs $c_1$ and $c_2$, it is not utterly unreasonable to hope that one could prove

$$\sum_{c_1, c_2 \leq q} f_{\beta, \eta}(q, c_1, c_2)\Delta_{\beta}^*(q, c_1, c_2)^2 \ll_{\beta, \eta} q^{7 - \beta} \cdot q^{1 + \frac{2(2 - \beta)}{2 + \beta}} \cdot q^{-2} = q^{\frac{9 - 4\beta}{2 + \beta}},$$

(7.2)

provided that the weight of $\Delta_{\beta}^*(q, c_1, c_2)^2$ does not concentrate on the support of $f$.

Putting this bound into (7.1) one would then get

$$|\text{Bad}_{\beta, \eta}(q)| \ll_{\beta, \eta} q^{\frac{3 + 3\beta}{2(2 + \beta)}},$$

i.e. only the second term from (6.10) would occur. As we have already remarked, we would then derive

$$|\text{Bad}_{\beta, \eta}(q)| \ll_{\beta, \eta} q^{1 - 2\eta},$$

provided $\beta < 1$ and $\eta$ is small enough. This estimate would expand the range of Theorem 1.4 all the way to $L > N^\varepsilon$. Unfortunately, we have not been able to prove a version of Lemma 3.1 which includes the weight $f_{\beta, \eta}(q, c_1, c_2)$ in the manner of expression (7.2).
One also recalls that in our application of Borel–Cantelli we did not need to bound \( |\text{Bad}_{\beta,\eta}(q)| \) uniformly for all \( q \). One would be satisfied with
\[
\sum_{q \text{ prime}} \frac{|\text{Bad}_{\beta,\eta}(q)|}{q^2} < \infty.
\]

Thoughts move towards expressing the relevant exponential sum \( s \) as an average of Kloosterman-type sums over the modulus \( q \), which might be another route for future research.

Our final remark is that if \( L \to \infty \) and \( N/L \to \infty \) as \( N \to \infty \) then, in the random model (1.4), the asymptotics are governed by the Central Limit Theorem. One can derive
\[
Z_{L,N} - \frac{L}{\sqrt{L}} \xrightarrow{\text{dist}} N(0,1)
\]
as \( N \to \infty \). Theorem 1.4 could then be considered as a first step towards showing that for almost all \( \alpha \) the skewness of \( W_{\alpha,L,N} \) satisfies \( \mathbb{E}((W_{\alpha,L,N} - L)/\sqrt{L})^3 \to 0 \) as \( N \to \infty \), with \( L \) in a certain range. However, to show this asymptotic one would need to be able to extract the lower degree main-term from \( \mathbb{E}W_{\alpha,L,N}^3 \) (which is \( 3L^2 \), as in Section 2) and then subsequently show that the error term in Theorem 2.1 is in fact \( o(L^{3/2}) \), rather than merely \( o(L^3) \).

**Appendix A. Pair Correlations of the dilated squares at scale \( N^{-\beta} \)**

In this section, we briefly indicate how one can deduce the following fact from the methods of the literature.

**Theorem A.1.** Let \( \varepsilon \in (0,1/4) \). Then for almost all \( \alpha \in [0,1] \), for all \( 1 \leq L \leq N^{1-\varepsilon} \) and for all \( (\log N)^{-1} \leq s \leq \log N \) we have
\[
R_2(\alpha, L, N, 1_{[-s,s]}) = 2L s (1 + O_{\varepsilon, \alpha}(N^{-\varepsilon/13})).
\]

Note that this result immediately implies the estimate (2.3), by approximating the function \( f \) in (2.3) with a suitable step function.

We have made no attempt to obtain the best possible error term in Theorem A.1, nor the largest admissible ranges for \( s \) and for \( L \). One will observe from the proof that rather better bounds would certainly follow if one assumed at the outset that \( L \) were a slowly varying function of \( N \).

A version of Theorem A.1 follows from arguments of Aistleitner, Larcher, and Lewko [2] as well as from arguments of Rudnick [19] and Rudnick–Sarnak [21], by changing the relevant parameters. If one wanted an explicit characterisation of the set of suitable \( \alpha \) in terms of properties of its rational approximations then one could also adapt the (much more involved) methods of Heath-Brown [11]. In particular, the material of the present section is in no way novel. However, we decided to add some explanations on how to deduce Theorem A.1 partly in order to make the exposition of our previous arguments complete and self-contained, and partly in order to describe explicitly a suitable ‘sandwiching argument’ for this result (expanding upon the description in [2]).

We begin with the following auxiliary lemma (where again no attempt was made to obtain the best possible error term):

**Lemma A.2.** For each \( m \in \mathbb{N} \), let \( N_m = m^4 \). Letting \( \varepsilon \in (0,1/4) \), for each \( i \) in the range \(-m^{\varepsilon/3}/10 \leq i \leq m^{\varepsilon/3} \) let \( \beta_{m,i} = i/m^{\varepsilon/3} \) and \( L_{m,i} = N_m^{\beta_{m,i}} \). Let \( s_m \) be a real quantity that satisfies \( m^{-\varepsilon/10} < s_m < m^{\varepsilon/10} \) for large enough \( m \). Then, for almost all \( \alpha \in [0,1] \), for all \( m \)
and for all \( i \) in the range \(-m^{\epsilon/3}/10 \leq i \leq m^{\epsilon/3}\) we have

\[
R_2(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}) = 2s_m L_{m,i} + O_{\alpha,e} \left( \frac{L_{m,i}}{m^{1/5 - \delta}} \right). \tag{A.1}
\]

**Proof of Lemma A.2** Let \( I = (\gamma, \delta) \) be an arc on the torus \( \mathbb{R}/\mathbb{Z} \) such that \( 0 < \gamma - \delta < 1 \). Let \( J \geq 1 \) be an integer. To proceed we introduce trigonometric polynomials \( S_j^\pm(x) \), of degree \( J \), which approximate the indicator function \( \chi_I \) from above and below. Selberg, and also Vaaler, constructed such polynomials, cf. Montgomery \[17, p. 5–6\]. Indeed, there exists

\[
S_j^+(x) = \sum_{|j| \leq J} s_j^+(j)e(jx)
\]

satisfying

\[
S_j^-(x) \leq \chi_I(x) \leq S_j^+(x) \quad (x \in \mathbb{R}/\mathbb{Z})
\]

such that

\[
s_j^+(0) = \delta - \gamma \pm \frac{1}{J + 1}, \quad |s_j^+(j)| \leq \frac{1}{J + 1} + \min \left( \delta - \gamma, \frac{1}{\pi |j|} \right) \quad (0 < |j| \leq J).
\]

For our purposes, given \( N_m \) and some growth function \( w(N_m) \geq 1 \), to be specified later, we specify \( J_{m,i} = \lfloor N_m w(N_m)/L_{m,i} \rfloor \), \( \gamma_{m,i} = -s_m L_{m,i}/N_m \), and \( \delta_{m,i} = s_m L_{m,i}/N_m \). Note that the definitions of \( s_m \) and \( L_{m,i} \) in the statement of the theorem imply that these choices yield a valid arc on the torus. Then, defining

\[
R_2^+(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}) := \frac{1}{N_m} \sum_{x \neq y \leq N_m} S_{m,i}^+(\alpha(x^2 - y^2)),
\]

we can control \( R_2 \) from above and below via

\[
R_2(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}) \leq R_2(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}) \leq R_2^+(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}). \tag{A.2}
\]

Let

\[
E^\pm(L_{m,i}, N_m, s_m) := \int_0^1 R_2^\pm(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}) \, d\alpha
\]

be the expected value of \( R_2^\pm(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}) \). It is easy to see that

\[
E^\pm(L_{m,i}, N_m, s_m) = 2s_m L_{m,i} + O(L_{m,i}/w(N_m)) + O(s_m L_{m,i}/N_m). \tag{A.3}
\]

Furthermore, by using orthogonality, we also see that the variance

\[
\int_0^1 (R_2^\pm(\alpha, L_{m,i}, N_m, 1_{[-s_m, s_m]}) - E^\pm(L_{m,i}, N_m, s_m))^2 \, d\alpha
\]

of \( R_2^\pm \) is at most

\[
\leq \frac{1}{N_m^2} \sum_{x_1 \neq y_1 \leq N_m} \sum_{x_2 \neq y_2 \leq N_m} |s_{m,i}^+(j_1)s_{m,i}^\pm(j_2)|.
\]

(Note that there are no contributions from terms in which \( j_1 = 0 \) and \( j_2 \neq 0 \), since the condition \( j_2(x_2^2 - y_2^2) = 0 \) cannot be satisfied.) Since \( |s_{m,i}^\pm(j)| \leq (2s_m + 1)L_{m,i}/N_m \) for all \( j \) in the range \( 0 < |j| \leq J_{m,i} \), we can bound the variance above by \( O((s_m + 1)^2) \) times

\[
\frac{L_{m,i}^2}{N_m^4} \left\{ (j_1, x_1, y_1, j_2, x_2, y_2) \in \mathbb{Z}^6 : j_1(x_1^2 - y_1^2) = j_2(x_2^2 - y_2^2), 0 < |j_k| \leq \frac{N_m w(N_m)}{L_{m,i}}, \right\}. \tag{A.4}
\]
Let us fix the first three variables above, that is \(j_1, x_1, y_1\). Then by writing \(x_2^2 - y_2^2 = d_1 d_2\), with \(d_1 = x_2 - y_2\) and \(d_2 = x_2 + y_2\), we deduce that \(d_k \mid j_1(x_k^2 - y_k^2)\) for \(k = 1, 2\). By the divisor bound, that there are at most \(N_m^{o(1)}\) many possibilities for \(d_1, d_2\) (provided that \(w(N_m) \leq N_m^{O(1)}\)). Moreover, any choice of \(d_1, d_2\) uniquely determines the variables \(x_1, y_1\) via \(x_2 = (d_1 + d_2)/2\) and \(y_2 = (d_2 - d_1)/2\). Further, we note that \(j_2\) is determined up to \(\ll N_m^{o(1)}\) many choices. The upshot is that given one of the \(O(N_m^3 w(N_m)/L_m)\) many admissible choices for \(j_1, x_1, y_1\), the second block of variables \(j_2, x_2, y_2\) is determined up to \(O(N_m^{o(1)})\) many possibilities. Therefore the variance of \(R_2^\pm\) is at most \(O((s_m + 1)^2 L_m, w(N_m) N_m^{-1+o(1)})\).

For the ease of exposition, we let \(\kappa_{m,i}^\pm = |R_2^\pm(\alpha, L_m, N_m, 1_{[-s_m, s_m]} - E^\pm(L_m, N_m, s_m))|\). We infer, by Chebychev’s inequality, that

\[
\mathbb{P}\left(\alpha \in [0, 1] : \kappa_{m,i}^\pm \geq \frac{L_{m,i}}{w(N_m)}\right) \leq \frac{(s_m + 1)^2 w(N_m)^3 N_m^{o(1)}}{N_m L_{m,i}} \leq \frac{(s_m + 1)^2 w(N_m)^3 N_m^{o(1)}}{N_m^{9/10}}.
\]

Choose the growth function \(w(N_m)\) to be

\[
w(N_m) = \left(\frac{N_m^{9/10}}{m^{1+\varepsilon}}\right)^{1/3}.
\]

Therefore, since \(s_m < m^{\varepsilon/10}\), we conclude that

\[
\mathbb{P}\left(\alpha \in [0, 1] : \kappa_{m,i}^\pm \geq \frac{L_{m,i}}{w(N_m)}\right) \ll \frac{1}{m^{1+\varepsilon}}.
\]

Summing over \(i\) in the range \(-m^{\varepsilon/3}/10 \leq i \leq m^{\varepsilon/3}\), with the union bound we get

\[
\mathbb{P}\left(\alpha \in [0, 1] : \exists i \in [-m^{\varepsilon/3}/10, m^{\varepsilon/3}] \text{ s.t. } \kappa_{m,i}^\pm \geq \frac{L_{m,i}}{w(N_m)}\right) \ll \frac{1}{m^{1+\varepsilon}}.
\]

Recalling our estimate [A.3], the first Borel–Cantelli lemma implies that, for almost every \(\alpha \in [0, 1]\), the following relation holds for all \(m \geq 1\) and all admissible \(i \in [-m^{\varepsilon/3}/10, m^{\varepsilon/3}]\):

\[
\frac{1}{N_m} \sum_{x \neq y \ll N_m} R_2^\pm(\alpha, L_m, N_m, 1_{[-s_m, s_m]}) = 2s_m L_m i + O_{\alpha, \varepsilon} \left(\frac{L_{m,i}}{w(N_m)}\right) + O_\alpha \left(\frac{s_m}{N_m}\right).
\]

From [A.2], and substituting in the explicit growth function \(w(N_m)\), the lemma follows. □

**Proof of Theorem [A.7].** We begin with the trivial observation that, by combining \(s\) and \(L\) into a single parameter, it is enough to show that for almost all \(\alpha \in [0, 1]\), for all \(N\) and for all \(L\) in the range \(N^{-1/11} \leq L \leq N^{1-\varepsilon/2}\),

\[
R_2(\alpha, L, N, 1_{[-1, 1]}) = 2L (1 + O_{\alpha, \varepsilon}(N^{-\varepsilon/2})). \tag{A.5}
\]

Now, for each \(N\), choose \(m\) such that \(N_m \leq N < N_{m+1}\), where \(N_m = m^4\) as in Lemma [A.2]. We put \(\theta_m = N_{m+1}/N_m\). Then, for any \(L\),

\[
R_2(\alpha, L, N, 1_{[-1, 1]}) \leq \frac{N_{m+1}}{N} R_2(\alpha, L, N_{m+1}, 1_{\frac{N_{m+1}}{N_{m+1}}[-1, 1]}) \leq \theta_m R_2(\alpha, L, N_{m+1}, 1_{\theta_m[-1, 1]}),
\]

and similarly

\[
R_2(\alpha, L, N, 1_{[-1, 1]}) \geq \frac{N_m}{N} R_2(\alpha, L, N_m, 1_{\theta_m[-1, 1]}) \geq \theta_m^{-1} R_2(\alpha, L, N_m, 1_{\theta_m[-1, 1]}).
\]

For each \(N^{-1/11} \leq L \leq N^{1-\varepsilon/2}\), there exists an \(i\) in the range \(-m^{\varepsilon/3}/10 \leq i \leq m^{\varepsilon/3}\) such that \(L_{m,i} \leq L \leq L_{m,i+1}\), where \(L_{m,i}\) is as in Lemma [A.2]. Now we record that the upper and lower bounds above satisfy

\[
R_2(\alpha, L, N_{m+1}, 1_{\theta_m[-1, 1]}) \geq R_2(\alpha, L_{m,i}, N_m, 1_{\theta_m[-1, 1]}),
\]
\[
R_2(\alpha, L, N_{m+1}, 1_{\theta_m[-1, 1]}) \leq R_2(\alpha, L_{m,i+1}, N_{m+1}, 1_{\theta_m[-1, 1]}).
\]
Moreover, Lemma A.2 implies that there is a set \( \Omega \subset [0, 1] \) with full measure, such that, if \( \alpha \in \Omega \) and \( \varepsilon \in (0, 1/4] \), then for all \( m \geq 1 \) and \( i \) in the range \( -m^{4/3}/10 \leq i \leq m^{4/3} \),

\[
R_2(\alpha, L_{m,i}, N_m, 1_{\theta_m[-1,1]}) = 2\theta_m L_{m,i}(1 + O_{\alpha,\varepsilon}(m^{4/13} + \frac{\varepsilon}{4})).
\]

By using that \( L_{m,i+1}/L_{m,i} = 1 + O_2(N^{-\varepsilon/13}) \) and also that \( \theta_m = 1 + O(m^{-1}) \), we infer that

\[
R_2(\alpha, L, N, 1_{[-1,1]}) \leq 2L(1 + O_{\alpha,\varepsilon}(N^{-1/4}))(1 + O_{\alpha,\varepsilon}(N^{-1/4}))(1 + O_{\alpha,\varepsilon}(m^{4/13} + \frac{\varepsilon}{4}))
\]

\[
\leq 2L(1 + O_{\alpha,\varepsilon}(N^{-\frac{1}{4}})).
\]

Similarly, we conclude that

\[
R_2(\alpha, L, N, 1_{[-1,1]}) \geq 2L(1 + O_{\alpha,\varepsilon}(N^{-\frac{1}{4}})).
\]

Combining these two estimates shows (A.5), thus completing the proof of Theorem A.1. \( \square \)

**Appendix B. Discrepancy and k-point correlation functions at scale \( N^{-\beta} \)**

The purpose of the present section is to record a few simple observations concerning the relationship between discrepancy and k-point correlation functions.

**Definition B.1.** Let \( (x_n)_{n=1}^\infty \) be a sequence of points in \([0, 1)\). Let \( k \geq 2 \) be a natural number, and let \( g : \mathbb{R}^{k-1} \to [0, 1] \) be a compactly supported function. Then the \( k \)-th correlation function \( R_k((x_n)_{n=1}^\infty, L, N, g) \) is defined to be

\[
R_k((x_n)_{n=1}^\infty, L, N, g) := \frac{1}{N} \sum_{\substack{n_1, \ldots, n_k \leq N \\
\text{distinct}}} g\left(\frac{N}{L}\{x_1 - x_2\}_{\text{sgn}}, \frac{N}{L}\{x_2 - x_3\}_{\text{sgn}}, \ldots, \frac{N}{L}\{x_k-1 - x_k\}_{\text{sgn}}\right),
\]

where

\[
\{\cdot\}_{\text{sgn}} : \mathbb{R} \to (-1/2, 1/2)
\]

denote the signed distance to the nearest integer.

The main point we are conveying here is that, as expected, the correlations are controlled on the scales in which the discrepancy allows us to count points asymptotically.

**Lemma B.2.** (a) Let \( D_N \) denote the discrepancy of the sequence \( (x_n)_{n=1}^\infty \) in \([0, 1)\), and suppose \( \sup\{g > 0 : D_N \ll_N N^{-g} \text{ for all } N\} = \gamma > 0 \). Let \( k \geq 2 \) be a natural number, and let \( Y \) be a uniformly distributed random variable modulo 1. Let \( \varepsilon > 0 \) be suitably small in terms of \( \gamma \), and for all \( N \in \mathbb{N} \) and \( L \in \mathbb{R} \) satisfying \( L \leq N \) let

\[
W((x_n)_{n=1}^\infty, L, N) := \{n \leq N : x_n \in [Y, Y + L/N] \text{ mod } 1\}.
\]

Then, if \( L \) is in the range \( N^{1-\gamma+\varepsilon} < L \leq N \), we have

\[
\mathbb{E}W((x_n)_{n=1}^\infty, L, N) = L^k(1 + O(\varepsilon)N^{-\varepsilon/2})). \quad (B.1)
\]

Furthermore, for all continuous functions \( g : \mathbb{R}^{k-1} \to [0, 1] \) and for all \( L \) in the range \( N^{1-\gamma+\varepsilon} < L < N^{1-\varepsilon} \),

\[
R_k((x_n)_{n=1}^\infty, L, N, g) = (1 + o_{g,\varepsilon,k}(1))L^{k-1} \int g(w) \, dw \quad (B.2)
\]

as \( N \to \infty \), where the error term is independent of the choice of parameters \( L \).

(b) If \( (a_n)_{n=1}^\infty \) is a strictly increasing sequence of positive integers, for almost every \( \alpha \in [0, 1] \), for all \( N \in \mathbb{N} \) and for all \( L \in \mathbb{R} \) in the range \( N^{1/2+\varepsilon} < L \leq N \), the sequence

\[
(a_n \text{ mod } 1)_{n=1}^\infty
\]

satisfies estimates (B.1) and (B.2).

**Remark:** Part (b) of the lemma proves our earlier assertion (1.5).
Proof. Fix a small $\varepsilon > 0$ throughout this proof. For part (a), the proof of (B.1) is trivial. Indeed, by the discrepancy estimate we have

$$|\{n \leq N : x_n \in [Y, Y+L/N] \mod 1\}| = L + O(ND_N) = L + O_\varepsilon(N^{1-\gamma + \varepsilon/2}) = L(1 + O_\varepsilon(N^{-\varepsilon/2})).$$

Raising to the $k^{th}$ power and averaging over $Y$, we obtain (B.1).

To prove the correlation estimate (B.2), by approximating the function $g$ by step functions we see that it is enough to prove it in the case when $g$ is the indicator function of a box $[s_1, t_1] \times \cdots \times [s_{k-1}, t_{k-1}]$. We may assume without loss of generality that $N$ is large enough so that $(t_i - s_i)L/N \leq 1$ for all $i \leq k - 1$. (This is why it is important for the correlation estimate to preclude the case $L = N$.) Then, fixing $n_k$, we see that $R_k((x_n)_{n=1}^\infty, L, N, g)$ counts the number of $x_{nk-1} \neq x_{nk}$ such that $x_{nk-1} \in [s_{k-1}L/N + x_{nk}, t_{k-1}L/N + x_{nk}] \mod 1$, times the number of $x_{nk-2} \neq x_{nk-1}, x_{nk}$ such that $x_{nk-2} \in [s_{k-2}L/N + x_{nk-1}, t_{k-2}L/N + x_{nk-1}] \mod 1$, etc. By the discrepancy estimate, the total number of choices is

$$((t_{k-1} - s_{k-1})L + O_\varepsilon(LN^{-\varepsilon/2})) \times ((t_{k-2} - s_{k-2})L + O_\varepsilon(LN^{-\varepsilon/2})) \times \cdots \times ((t_1 - s_1)L + O_\varepsilon(LN^{-\varepsilon/2})).$$

Summing over all $n_k$ and then normalising by $1/N$, we have

$$R_k((x_n)_{n=1}^\infty, L, N, g) = \left(\prod_{i=1}^{k-1} (t_i - s_i)\right) L^{k-1}(1 + O_\varepsilon(kN^{-\varepsilon/2}))$$

as desired.

This proves part (a) of the assertion. The remaining part follows by recalling that a classical (and far more general) result of Erdős and Koksma [8, Thm. 2] furnishes an upper bound on the discrepancy of $(\alpha a_n \mod 1)_{n=1}^\infty$ of the quality $N^{-1/2 + \varepsilon}$, for each fixed $\varepsilon > 0$ and for almost every $\alpha \in [0, 1]$. \hfill \Box

Steinerberger [25] raised the question of whether ‘most sequences\footnote{The precise meaning of the word ‘most’ was left open for interpretation by Steinerberger, and was already put in quotation marks in the original paper.} have uniform pair correlations at some scale $0 < \beta < 1$. The above part (b) answers Steinerberger’s question in a strong sense. Further, it seems worthwhile to record the following consequence.

**Corollary B.3.** Let $\alpha_n$ be the $n^{th}$ partial quotient of $\alpha \in [0, 1]$. Given $N \geq 1$, let $i(N)$ be such that the convergent denominator $q_{i(N)}$ of $\alpha$ satisfies $q_{i(N)} \leq N < q_{i(N)+1}$. If for each $\varepsilon > 0$ we have

$$A_N(\alpha) = \sum_{j \leq i(N)} \alpha_j \ll N^\varepsilon,$$

then the Kronecker sequence $(\alpha n)_{n=1}^\infty$ satisfies (B.3), for any $k \geq 2$ and for any scale $L$ such that $N^k < L < N$ for any fixed $\delta \in (0, 1)$ (the $o(1)$ term in (B.2) then also depends on $\delta$).

**Proof.** As $D_N((\alpha n \mod 1)_{n=1}^\infty) \ll_\varepsilon A_N(\alpha)$, cf. [14, Eq. (3.18)], Lemma [B.2] completes the proof. \hfill \Box

It is well known (also with a higher-dimensional generalisations due to Beck [3]) that for almost every $\alpha \in [0, 1]$ the discrepancy of the Kronecker sequence is $\ll (\log N)^{1+\varepsilon}$, for each $\varepsilon > 0$. Thus the above corollary generalizes and sharpens a result of Skill and Weiß [26] stating that $(\phi n)_{n=1}^\infty$ has Poissonian pair correlations on each scale $\beta < 1$ where $\phi$ denotes the Golden ratio $\sqrt{5}+1/2$. Further, the condition on $A_N$ is known to be true for algebraic $\alpha$, due to Roth’s famous approximation theorem.

Finally we remark that, if one so wished, one could readily replace the $N^\varepsilon$-terms in Corollary B.3 by appropriate powers of logarithms.
In this final appendix, we detail an obstruction to studying the higher order correlation function $R_k^d(\alpha, L, N, g)$. This obstruction is a generalisation of a fundamental observation of Rudnick–Sarnak [21, Section 4], which those authors made in the context of constant $L$ and for the triple correlation function of $(\alpha n^d \mod 1)_{n=1}^\infty$. We address the more general situation of sequences of the shape $(\alpha n^d \mod 1)_{n=1}^\infty$, where $d \geq 2$ is a fixed integer. We are also interested in identifying the full range of $L$ in which the obstruction persists.

To this end, for a compactly supported smooth test function $g : \mathbb{R}^{k-1} \to \mathbb{R}$, we define the correlation function $R_k^d(\alpha, L, N, g)$ by

$$R_k^d(\alpha, L, N, g) := \frac{1}{N} \sum_{1 \leq x_1, \ldots, x_k \leq N \text{ distinct}} g\left(\frac{N}{L} (\alpha (x_1^d - x_2^d))_{\text{sgn}}, \ldots, N \frac{L}{N} (\alpha (x_{k-1}^d - x_k^d))_{\text{sgn}}\right),$$

where

$$\{ \cdot \}_{\text{sgn}} : \mathbb{R} \to (-1/2, 1/2]$$

denotes the signed distance to the nearest integer. Rudnick–Sarnak’s approach to pair correlations, like in Appendix A, involves showing that

$$\int_0^1 \left( R_k^d(\alpha, L, N, g) - L \frac{N - 1}{N} \hat{g}(0) \right)^2 d\alpha = o_d(L^2)$$

as $N \to \infty$. Therefore, in order to make a similar approach work for the higher $k$-point correlation functions, one would need

$$\int_0^1 \left( R_k^d(\alpha, L, N, g) - L^{k-1} \frac{(N)_k}{N^k} \hat{g}(0) \right)^2 d\alpha = o_{d,k}(L^{2(k-1)})$$

(C.1)

as $N \to \infty$, where $\mathbf{0}$ is the zero-vector in $\mathbb{R}^{k-1}$ and $(N)_k = N(N-1) \ldots (N-k+1)$ abbreviates the $k^{\text{th}}$ falling factorial. Our purpose here is to show that for certain ranges of $d$, $k$ and $L$, equation (C.1) cannot hold.

Indeed, by applying the Poisson summation formula one may expand $R_k^d(\alpha, L, N, g)$ into a Fourier series

$$R_k^d(\alpha, L, N, g) = \sum_{\ell \in \mathbb{Z}} c_{k,\ell}(L, N, g)e(\ell\alpha),$$

with certain Fourier coefficients $c_{k,\ell}(L, N, g)$. One may compute $c_{k,\ell}(L, N, g)$ explicitly. For a given vector $\mathbf{a} \in \mathbb{Z}^{k-1}$, let $S_{\mathbf{a},k,\ell}(N)$ denote the set of integer vectors $\mathbf{x} \in \mathbb{N}^k$, with distinct components $1 \leq x_i \leq N$, satisfying the Diophantine equation

$$a_1(x_1^d - x_2^d) + \ldots + a_{k-1}(x_{k-1}^d - x_k^d) = \ell.$$  

One readily verifies that $c_{k,\ell}(L, N, g)$ is of the special form

$$c_{k,\ell}(L, N, g) = \frac{L^{k-1}}{N^k} \sum_{\mathbf{a} \in \mathbb{Z}^{k-1}} |S_{\mathbf{a},k,\ell}(N)| \hat{g}\left(\frac{L}{N}\mathbf{a}\right).$$

(C.2)

From Parseval, we conclude that

$$\int_0^1 \left( R_k^d(\alpha, L, N, g) - L^{k-1} \frac{(N)_k}{N^k} \hat{g}(0) \right)^2 d\alpha = \sum_{\ell \neq 0} |c_{k,\ell}(L, N, g)|^2.$$
Now let $\rho = N^{d+1+\varepsilon}/L$. We observe that if $|\ell| \geq \rho$ and $S^d_{a,k}(N) \neq 0$ then $\|a\|_\infty \geq N^{1+\varepsilon}/L(k-1)$. Therefore, from the rapid decay of $\tilde{g}$ and the formula (C.2), we conclude that $|c_{k,\ell}(L, N, g)| = O_{\varepsilon, g, k, K}(N^{-K})$ for such $\ell$. Hence

$$\sum_{\ell \neq 0} |c_{k,\ell}(L, N, g)|^2 = \sum_{0 < |\ell| \leq \rho} |c_{k,\ell}(L, N, g)|^2 + O_{\varepsilon, g, k, K}(N^{-K}).$$

To estimate the right hand side, we note that by the Cauchy–Schwarz inequality

$$\rho \sum_{0 < |\ell| \leq \rho} |c_{k,\ell}(L, N, g)|^2 \geq \left( \sum_{0 < |\ell| \leq \rho} c_{k,\ell}(L, N, g) \right)^2.$$ 

The sum inside the absolute value on the right-hand side equals, up to a term of size $O_{\varepsilon, g, k, K}(N^{-K})$, the quantity

$$\sum_{\ell \leq 2} c_{k,\ell}(L, N, g)e(0) - L^{k-1}(N)\frac{k}{N^k}g(0),$$

which is

$$R^d_k(0, L, N, g) + O_g(L^{k-1}).$$

Assuming that $g(0) \neq 0$ and $L/N = o(1)$ as $N \to \infty$, this is equal to

$$N^{k-1}(1 + o(1))g(0)$$

as $N \to \infty$. Now, by combining these considerations, we conclude that

$$\int_0^1 \left( R^d_k(\alpha, L, N, g) - L^{k-1}(N)\frac{k}{N^k}g(0) \right)^2 d\alpha \gg_{\varepsilon, g, k} \frac{1}{\rho} N^{2(k-1)}|g(0)|^2 = LN^{2k-3-\varepsilon}|g(0)|^2.$$ 

If (C.1) is to hold for all functions $g$, for each $\varepsilon > 0$ we must have

$$L \gg_{g, \varepsilon, k} N^{\frac{2k-3-\varepsilon}{2k-3}} = N^{1-\frac{d+\varepsilon}{d-3}}.$$ 

In particular, for $k = 3$ and $d = 2$ the convergence (C.1) fails unless $L \gg N^{\frac{6+\varepsilon}{6-3}} = N^{1-\frac{\varepsilon}{3}}$. This justifies the statements we made in the introduction to the effect that the Rudnick–Sarnak obstruction for triple correlations extends throughout the range $L < N^{1/3}$.

We also note, however, that as soon as

$$d > 2k - 3$$

there is no such obstruction (for $L$ constant in terms of $N$).

The failure of (C.1) is an artefact of the integrand having a large spike when $\alpha \approx 0$ (and more generally the integrand has large spikes when $\alpha$ is very well-approximated by rationals with small denominators). One wonders whether $L^2$-convergence (C.1) can be recovered by restricting to the ‘minor arcs’, but this also appears to be a difficult problem.

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