ABSTRACT
We discuss the non-linear evolution of the angular momentum $L$ acquired by protostructures, like protogalaxies and protoclusters, due to tidal interactions with the surrounding matter inhomogeneities. The primordial density distribution is assumed to be Gaussian and the non-linear dynamics of the collisionless mass fluid is followed using Lagrangian perturbation theory. For a Cold Dark Matter spectrum, the inclusion of the leading-order Lagrangian correction terms results in a value of the rms ensemble average $\langle L^2 \rangle^{1/2}$ which is only a factor of 1.3 higher than the corresponding linear estimate, irrespective of the scale. Consequently, the predictions of linear theory are rather accurate in quantifying the evolution of the angular momentum of protostructures before collapse sets in. In the Einstein-de Sitter universe, the initial torque is a good estimate for the tidal torque over the whole period during which the object is spun up.

Key words: galaxies: formation – large–scale structure of the Universe

1 INTRODUCTION
The problem of the acquisition of angular momentum by protostructures in the universe is of considerable interest in theories of galaxy and cluster formation. A widely accepted view is that the present luminous structures acquired their spin via gravitational tidal interactions with the surrounding matter inhomogeneities (Hoyle 1949; Peebles 1969; Doroshkevich 1970; White 1984; Barnes & Ebstathiou 1987; Hoffman 1986, 1988; Heavens & Peacock 1988; Ryden 1988; Quinn & Binney 1992; Eisenstein & Loeb 1995; Catelan & Theuns 1996). So far, however, the theoretical analysis of the growth of the tidal galaxy angular momentum has been essentially limited to the linear regime, during which the galaxy spin grows proportionally to the cosmic time $t$ (Doroshkevich 1970; White 1984).

In this paper we examine analytically, for the first time, the question of how the galaxy tidal angular momentum evolves during the mildly non-linear regime. Previous attempts in this direction may be found in Peebles (1969) and White (1984). However, in contrast to their approach, we employ actual perturbative solutions of the dynamical equations that describe the motion of the fluid. More in detail, we apply Lagrangian perturbation theory.

The Lagrangian approach shows to be ideal in treating the evolution of the galaxy spin, because it is powerful in describing the non-linear growth of the mass-density fluctuations on one hand (Zel'dovich 1970a, b; Buchert 1992; Bouchet et al. 1992; Catelan 1995); and on the other hand the usual difficulty of inverting the mapping from Lagrangian coordinates $q$ to Eulerian coordinates $x$ is completely by-passed. This is because the angular momentum $L$ is invariant with respect to the Eulerian or Lagrangian description.

The layout of this paper is as follows: in the next section we first briefly review the basics of the Lagrangian theory and the perturbative solutions of the Lagrangian fluid equations. Next, we compute within this framework the perturbative corrections to the linear tidal angular momentum $L^{(1)}$ acquired by a protoobject. The resulting expressions are then simplified by calculating their averages over the ensemble of realisations of the linear Gaussian gravitational potential $\psi^{(1)}$ for objects with given inertia tensor. We then compare the non-linear spin corrections with the results of the linear analysis recently performed by Catelan & Theuns (1996). In the main text we restrict ourselves mainly to the case of a flat universe, leaving the more involved treatment of closed and open universes to Appendices.
2 NON-LINEAR EVOLUTION OF THE TIDAL ANGULAR MOMENTUM

We consider a Newtonian pressureless and irrotational self–gravitating fluid embedded in an expanding universe with arbitrary density parameter $\Omega$ but zero cosmological constant. Such a fluid is assumed to mimic the behaviour of matter on scales smaller than the horizon. Furthermore, we assume that luminous objects like galaxies and clusters of galaxies started to grow due to gravitational instability around primordial positive density fluctuations $\delta$ in this fluid.

We indicate by $x$ comoving Eulerian coordinates, from which physical distances may be obtained according to the law $r = a(t)x$, where $a(t)$ is the expansion scale factor and $t$ the standard cosmic time. We will use the temporal coordinate $\tau$ of which the differential is defined by

$$\,d\tau \equiv a^{-2} \,dt ,$$

instead of $t$, since this allows a considerable simplification of the formalism when dealing with the gravitational interactions in a generic non-flat Friedmann universe (Shandarin 1980). The peculiar velocity and the peculiar acceleration simplify when using $\tau$ instead of $t$ to:

$$\frac{dx}{d\tau} \equiv \dot{x} \equiv a(\tau) \,u(x, \tau) , \quad (2)$$

$$\frac{d^2x}{d\tau^2} \equiv \ddot{x} \equiv g(x, \tau) . \quad (3)$$

The dimensionless time $\tau$ is negative, as discussed in Shandarin (1980), and the initial cosmological singularity at $t = 0$ corresponds to $\tau = -\infty$. In the open models, the infinity of the cosmic time, $t = +\infty$, corresponds to $\tau = -1$; in the Einstein-de Sitter universe, $t = +\infty$ corresponds to $\tau = 0$; and in the closed models the contraction phase starts at $\tau = 0$. In terms of the density parameter $\Omega$, one has

$$\tau = -\sqrt{-k} \,(1 - \Omega)^{-1/2} ,$$

where $k$ is the curvature constant ($k = -1$ for open universes and $k = 1$ for closed universes). The case $\Omega = 1 \,(k = 0)$ is a singular point for the transformation (4) and in this case we take $\tau \equiv -(3t)^{-1/3}$, which corresponds to using $a(t) \equiv (3t)^{2/3}$ or $t_0 = t/a^{3/2} \equiv 1/3$, which defines the unit of time. The scale factor $a(\tau)$ may then be written for all Friedmann models as $a(\tau) = (\tau^2 + k)^{-1}$.

The linear evolution of angular momentum of protoobjects is most easily analysed using the Zel’dovich (1970a, b) formulation (see White 1984; Catelan & Theuns 1996) and the mildly non-linear spin growth is most easily analysed using the Lagrangian perturbation theory. We recall that the Zel’dovich approximation coincides with the linear Lagrangian description.

We will essentially adopt the formulation of the Lagrangian gravitational theory for a collisionless Newtonian fluid presented in Catelan (1995; see also references therein) but note that in the present paper, the variable $\tau$ has opposite sign and the growth factor of the linear density perturbation is normalised differently. An alternative formulation of the Lagrangian theory may be found in Buchert (1992).

2.1 Basic tools: Lagrangian theory

In Lagrangian formulation, the departure of the mass elements from the initial position $q$ is described in terms of the displacement vector $S$,

$$x(q, \tau) \equiv q + S(q, \tau) . \quad (5)$$

The trajectory $x(q, \tau)$ of the fluid element originally at $q$ satisfies the Lagrangian ‘irrotationality’ condition and the Poisson equation given by (Catelan 1995)

$$\epsilon_{\alpha\beta\gamma} x^C_{\beta\alpha} \ddot{x}_{\gamma\sigma} = 0 , \quad (6)$$

$$x^C_{\alpha\beta} \ddot{x}_{\gamma\alpha} = a(\tau)[J - 1] , \quad (7)$$

respectively, where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric Levi-Civita tensor of rank three, $\epsilon_{123} \equiv 1$, and summation over repeated Greek indices (where $\alpha = 1, 2, 3$) is understood. In these equations, $a(\tau) \equiv 6a(\tau)$ and $J \equiv 1/(1 + \delta)$ is the determinant of the Jacobian of the mapping $x \rightarrow q$. The determinant $J$ is non zero until the first occurrence of shell-crossing (see, e.g., Shandarin & Zel’dovich 1989). Furthermore, $x_{\alpha\beta} \equiv \partial x_{\alpha}/\partial q_{\beta}$ and $x^C_{\alpha\beta}$ denotes the cofactor of $x_{\alpha\beta}$: we recall that the latter is a quadratic function of $x_{\alpha\beta}$ and consequently, the master equations (5) and (6) are cubic in $x_{\alpha\beta}$. In general, $x_{\alpha\beta}$ is not a symmetric tensor: $x_{\alpha\beta} = x_{\beta\alpha}$ if, and only if, the Lagrangian motion is longitudinal in which case $x$ can be obtained from the gradient of a potential.

The irrotationality condition and the Lagrangian Poisson equation can be written in terms of the displacement field $S$ as:

$$\epsilon_{\alpha\beta\gamma} \left[(1 + \nabla \cdot S) \delta_{\beta\sigma} - S_{\beta\sigma} + S^C_{\beta\gamma} \right] \ddot{S}_{\gamma\sigma} = 0 , \quad (8)$$

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\[
[(1 + \nabla \cdot \mathbf{S}) \delta_{ab} - S_{a\beta} + S^C_{\alpha\beta}] \mathbf{S}_{\beta\alpha} = \alpha(\tau)[J(\mathbf{q}, \tau) - 1] ,
\]
where \( \nabla \equiv \nabla_\mathbf{q} \) and the symbol \( \delta_{ab} \) indicates the Kronecker tensor. These equations are the closed set of general dynamical equations for the displacement vector \( \mathbf{S} \) describing the motion of a collisionless fluid in the Lagrangian \( \{\mathbf{q}\} \)-space, embedded in an arbitrary Friedmann universe and subject to the Newtonian gravitational interaction of the mass fluctuations \( J^{-1} - 1 \). We briefly summarise their perturbative solutions in the next subsection.

### 2.2 Lagrangian perturbative solutions

The exact Lagrangian equations (8) and (9) are non-linear and non-local in the displacement \( \mathbf{S} \) (see the discussion in Kofman & Pogosyan 1995) and it is undoubtedly very difficult to solve them rigorously. A possible alternative is to seek for approximate solutions by expanding the trajectory \( \mathbf{S} \) in a perturbative series, the leading term being the linear displacement which corresponds to the Zel’dovich approximation: \( \mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \ldots \), where \( \mathbf{S}_n = O(S_1^n) \) is the \( n \)-th order approximation. Note that a perturbation series of this form needs to include at least the third-order term \( \mathbf{S}_3 \) to capture the essential physics contained in the cubic equations (8) and (9).

For the sake of simplicity we will limit ourselves to the Einstein-de Sitter universe in the main text. In this universe, \( a = \tau^{-2} \). The reader interested in more general Friedmann models is addressed to Appendix A. We neglect decreasing modes.

#### 2.2.1 First-order approximation: Zel’dovich approximation

The first-order solution to equations (8) and (9) is separable in space and time and corresponds to the Zel’dovich approximation (Zel’dovich 1970a, b):

\[
\mathbf{S}_1(\mathbf{q}, \tau) = D(\tau) \mathbf{S}^{(1)}(\mathbf{q}) \equiv D(\tau) \nabla \psi^{(1)}(\mathbf{q}) ;
\]

the function \( D(\tau) \) is the growth factor of linear density perturbations and is given in the Einstein-de Sitter universe by:

\[
D(\tau) = \tau^{-2} = a(\tau) .
\]

The expression of \( D(\tau) \) valid in a generic Friedmann model is reported in Appendix A. The function \( \psi^{(1)}(\mathbf{q}) \) is the (initial) gravitational potential. For later use, we define its Fourier transform, \( \tilde{\psi}^{(1)}(\mathbf{p}) = \int d\mathbf{q} \psi^{(1)}(\mathbf{q}) e^{-i\mathbf{p} \cdot \mathbf{q}} \), where \( \mathbf{p} \) is the comoving Lagrangian wave vector. The Fourier transform of the linear density field, \( \delta^{(1)}(\mathbf{p}, \tau) = D(\tau)\delta_1(\mathbf{p}) \), is related to \( \tilde{\psi}^{(1)}(\mathbf{p}) \) via the Poisson equation, \( \tilde{\psi}^{(1)}(\mathbf{p}) = p^{-2}\delta_1(\mathbf{p}) \).

#### 2.2.2 Second-order approximation

The second-order solution is again separable with respect to the spatial and temporal variables and describes a longitudinal motion in Lagrangian space:

\[
\mathbf{S}_2(\mathbf{q}, \tau) = E(\tau) \mathbf{S}^{(2)}(\mathbf{q}) \equiv E(\tau) \nabla \psi^{(2)}(\mathbf{q}) .
\]

The growing mode \( E(\tau) \) is for the Einstein-de Sitter model given by:

\[
E(\tau) = -\frac{3}{7} \tau^{-4} .
\]

The analytic expression for \( E(\tau) \) in a non-flat universe and the expression for \( \tilde{\psi}^{(2)}(\mathbf{p}) \) are reported in Appendix A.

#### 2.2.3 Third-order approximation

The third-order solution \( \mathbf{S}_3 \) corresponds to three separable modes, two longitudinal and one transverse, denoted by subscripts \( a, b \) and \( c \), respectively:

\[
\mathbf{S}_3(\mathbf{q}, \tau) = F_a(\tau) \mathbf{S}^{(3)}_a(\mathbf{q}) + F_b(\tau) \mathbf{S}^{(3)}_b(\mathbf{q}) + F_c(\tau) \mathbf{T}^{(3)}(\mathbf{q}) = F_a(\tau) \nabla \psi^{(3)}_a(\mathbf{q}) + F_b(\tau) \nabla \psi^{(3)}_b(\mathbf{q}) + F_c(\tau) \nabla \times \mathbf{A}^{(3)}(\mathbf{q}) .
\]

The growing modes \( F_a, F_b \) and \( F_c \) for a flat universe are, respectively,

\[
F_a(\tau) = -\frac{1}{3} \tau^{-6} , \quad F_b(\tau) = +\frac{10}{21} \tau^{-6} , \quad F_c(\tau) = -\frac{1}{7} \tau^{-6} .
\]
2.3 Non-linear spin dynamics

We stress that the transverse mode $T(3)$ does not describe any physical vorticity in the fluid since the latter is assumed to be irrotational. Rather, the occurrence of a transverse component is due to the fact that a Lagrangian frame of reference is not inertial. Consequently, this ‘fictitious’ term is required to obtain a correct physical description of the motion. Unfortunately, this transverse mode is forgotten in some of the relevant literature on the subject (e.g. Bouchet et al. 1995).

The general expressions for $F_a, F_b$ and $F_c$ for a non-flat universe as well as the potentials $\psi_a^{(3)}(p)$ and $\psi_b^{(3)}(p)$ and the transverse components $T_a$ are given in Appendix A.

Comparing equations (11), (13) and (15-17), it is clear that the perturbative expansion for $S$ is in fact a Taylor series in the variable $D(\tau) = \tau^{-2}$. As the general expressions reported in the Appendix A testify, this is no longer true in a non-flat universe. However, it is shown there that the higher-order growth factors can be approximated exceedingly well by powers of $D: E \propto D^2$ and $F \propto D^3$, so the expansion is still ‘close’ to a Taylor expansion. We will consider next how the perturbative series in $S$ translates into a perturbative series for the angular momentum $L$.

2.3 Non-linear spin dynamics

Let us consider some volume $V$ of the Eulerian $x$-space. The Cartesian coordinate system is assumed to be centred at the centre of mass of $V$. Since we are interested in the intrinsic angular momentum of the mass contained in $V$, we disregard the centre of mass motion.

The angular momentum $L$ of the matter contained at the time $t$ in the volume $V$ is

$$L(t) = \rho_0(t) a(t)^4 \int_{V(t)} dx \{ [1 + \delta(x, t)] \times u(x, t) = \eta_0 \int_{\eta} dQ \left[ q + S(q, \tau) \right] \times dS(q, \tau) \},$$

(18)

where we have substituted the time variable $\tau$ in favour of the standard cosmic time $t$ in the second integral. Here, the matter density field is $\rho = \rho_0(1 + \delta)$, where $\rho_0(\tau)$ is the background mean density and $\delta$ is the density fluctuation field, which is assumed to be initially Gaussian distributed and $\eta_0 \equiv a^3 \rho_0$; the peculiar velocity field is denoted by $u$ (see, e.g., Peebles 1980).

The second equality in equation (18) stresses the important fact that the integral over the Eulerian volume $V$ may be written equally well as an integral over the corresponding (initial) Lagrangian volume $\Gamma$. This enables us to apply the Lagrangian description of the Newtonian gravity previously reviewed. The linear regime (Zel’dovich approximation) has been fully analysed in this way by Doroshkevich (1970), White (1984) and Catelan & Theuns (1996), whereas its Eulerian counterpart was studied by Heavens and Peacock (1988). We can extend the Lagrangian analysis of the evolution of the angular momentum $L(\tau)$ to the non-linear regime by applying perturbation theory to equation (18). Perturbative corrections to $S(q, \tau)$ (Bouchet et al. 1992; Buchert 1994; Catelan 1995 and references therein) then give perturbative corrections to $L(\tau)$:

$$L(\tau) = \eta_0 \int_{\eta} dQ \left[ \sum_{l=0}^{\infty} S_l(q, \tau) \right] \times \frac{d}{d\tau} \left[ \sum_{l=0}^{\infty} S_l(q, \tau) \right] = \sum_{h=0}^{\infty} L^{(h)}(\tau),$$

(19)

where we have defined

$$L^{(h)}(\tau) = \sum_{j=0}^{h} \eta_0 \int_{\eta} dQ S_j(q, \tau) \times \frac{dS_{h-j}(q, \tau)}{d\tau},$$

(20)

with $S_0 \equiv q$, hence $L^{(0)} = 0$. Since we will be interested in calculating second-order corrections to the ensemble average $\langle L^2 \rangle$, we need to compute corrections to $L$ up to third-order. After briefly reviewing the results of the linear theory, we summarise the final expressions of the corrections $L^{(2)}$ and $L^{(3)}$. The reader interested in the details of the calculations is addressed to the Appendix B.

2.3.1 Linear approximation

The linear Lagrangian theory corresponds to the Zel’dovich approximation and the first-order term in equation (19) is given by:

$$L^{(1)}(\tau) = \eta_0 \hat{D}(\tau) \int_{\eta} dQ q \times \nabla \psi^{(1)}(q).$$

(21)

If $\psi^{(1)}(q)$ is adequately represented in the volume $\Gamma$ by the first three terms of the Taylor series about the origin $q = 0$, then each component $L^{(1)}(t)$ may be written in a compact form as (White 1984):

$$L^{(1)}_{\alpha}(\tau) = \hat{D}(\tau) \epsilon_{\alpha\beta\gamma} T^{(1)}_{\beta\gamma} J_{\gamma},$$

(22)

where we introduced the deformation tensor at the origin:
\[ D_{\beta\sigma}^{(1)} = D_{\beta\sigma}^{(1)}(0) = \partial_\beta \partial_\sigma \psi^{(1)}(0) , \]  
and the inertia tensor of the mass contained in the volume \( \Gamma \)
\[ J_{\sigma\gamma} = \eta_0 \int_\Gamma dq \, q_\sigma \, q_\gamma . \]  

Equation (22) shows that the linear angular momentum \( L^{(1)} \) is in general non-zero because the principal axes of the inertia tensor \( J_{\alpha\beta} \), which depends only on the (irregular) shape of the volume \( \Gamma \), are not aligned with the principal axes of the deformation tensor \( D_{\alpha\beta\gamma}^{(1)} \), which depends on the location of neighbour matter fluctuations. The temporal growth of tidal angular momentum is completely contained in the function \( \dot{D}(\tau) \), which behaves as \( \dot{D}(\tau) = -2\tau^{-3} \sim t \) in the Einstein-de Sitter universe, as first noted by Doroshkevich (1970). Finally, if \( \Gamma \) is a spherical Lagrangian volume, then \( L_{\alpha}^{(1)} \sim \epsilon_{\alpha\beta\gamma} \, D_{\beta\gamma}^{(1)} = 0 \). Consequently, the matter contained initially in a spherical volume does not gain any tidal spin during the linear regime (see also the discussion in White 1984).

### 2.3.2 Second-order approximation

The second-order term in equation (19) involves the second-order displacement \( S^{(2)} \):
\[ L^{(2)}_{\alpha}(\tau) = \eta_0 \int_\Gamma dq \, \mathbf{q} \times \frac{dS^{(2)}}{d\tau} = \eta_0 \dot{E}(\tau) \int_\Gamma dq \, \mathbf{q} \times \nabla \psi^{(2)}(\mathbf{q}) . \]  

Note that the second-order term which follows from the product \( S_{1} \times dS_{1}/d\tau \) is identically zero since is involves the product \( \nabla \psi^{(1)} \times \nabla \psi^{(1)} \). The potential \( \psi^{(2)} \) is determined by the potential \( \psi^{(1)} \) through the equation (64) in Appendix A. Note that, since \( \dot{E} \propto \tau^{-4} \), one has \( \dot{E} \propto \tau^{-5} \) hence the second-order terms grows \( \propto t^{7/3} \) in the Einstein-de Sitter universe. This growth rate was first derived by Peebles (1969). If we represent \( \psi^{(2)}(\mathbf{q}) \) in \( \Gamma \) by the first three terms of a Taylor series around \( \mathbf{q} = 0 \), as we did before for \( \psi^{(1)} \), we obtain for the \( \alpha \)-component:
\[ L^{(2)}_{\alpha}(\tau) = \dot{E}(\tau) \, \epsilon_{\alpha\beta\gamma} \, D_{\beta\gamma}^{(2)} \, J_{\sigma\gamma} , \]  
where
\[ D_{\beta\sigma}^{(2)} = D_{\beta\sigma}^{(2)}(0) = \partial_\beta \partial_\sigma \psi^{(2)}(0) . \]  

We can call \( D_{\alpha\beta}^{(2)} \) the second-order deformation tensor. The component \( L^{(2)}_{\alpha}(\tau) \) may be written in terms of the second-order shear tensor \( \epsilon^{(2)}_{\alpha\beta\gamma} \) as
\[ L^{(2)}_{\alpha}(\tau) = \dot{E}(\tau) \, \epsilon_{\alpha\beta\gamma} \, \epsilon^{(2)}_{\beta\gamma} \, J_{\sigma\gamma} , \]  
where
\[ \epsilon^{(2)}_{\beta\sigma} = D_{\beta\sigma}^{(2)} - \frac{1}{3} (\nabla \cdot S^{(2)}) \delta_{\beta\sigma} = (\partial_\beta \partial_\sigma - \frac{1}{3} \delta_{\beta\sigma} \nabla^2) \psi^{(2)} . \]  

An analogous relation is valid for the linear term (see Catelan & Theuns 1996). Note that the non-linear dynamical evolution modifies only the deformation tensor and not the Lagrangian inertia tensor. Furthermore, if \( \Gamma \) is a sphere, then again \( L^{(2)} = 0 \). This is in contrast to Eulerian perturbation theory, where the angular momentum of an Eulerian sphere does grow in second-order perturbation theory (Peebles 1969; White 1984).

### 2.3.3 Third-order approximation

The two longitudinal modes \( S^{(3)}_a, S^{(3)}_b \) and the transverse one \( T^{(3)} \), and the coupling between first and second order displacements originate the following third-order spin corrections:
\[ L^{(3)}_{\alpha}(\tau) = \eta_0 \int_\Gamma dq \, \mathbf{q} \times \frac{dS^{(1)}}{d\tau} + \eta_0 \int_\Gamma dq \left( S_1 \times \frac{dS_{2}}{d\tau} + S_2 \times \frac{dS_{1}}{d\tau} \right) = L^{(3)}_{\alpha}(\tau) + L^{(3)}_{b}(\tau) + L^{(3)}_{c}(\tau) + L^{(12)}(\tau) , \]  
where
\[ L^{(3)}_{b}(\tau) = \eta_0 \dot{F}_b(\tau) \int_\Gamma dq \, \mathbf{q} \times \nabla \psi^{(3)}_b(\mathbf{q}) , \]  
\[ L^{(3)}_{c}(\tau) = \eta_0 \dot{F}_c(\tau) \int_\Gamma dq \, \mathbf{q} \times \mathbf{T}^{(3)}(\mathbf{q}) , \]  
and \( h = a, b \). Here, the temporal functions are such that \( \dot{F}_b \propto \dot{F}_b \propto \dot{F}_c \propto \tau^{-7} \propto t^{7/3} \) in the Einstein-de Sitter universe. Proceeding as in the previous cases by expanding the fields \( \psi^{(3)}_a = \psi^{(3)}_b = T^{(3)}_a \) in Taylor series around \( \mathbf{q} = 0 \), one gets the expressions,
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\[ L_{h\alpha}^{(3)}(\tau) = \hat{F}_h(\tau) \epsilon_{\alpha\beta\gamma} D_{h,\beta\gamma}^{(3)} J_{\sigma\gamma} , \]  
(33)

\[ L_{c\alpha}^{(3)}(\tau) = \hat{F}_c(\tau) \epsilon_{\alpha\beta\gamma} T_{\gamma\delta}^{(3)} J_{\beta\delta} . \]  
(34)

In these last equations, we defined the third-order deformation tensors

\[ D_{h,\beta\gamma}^{(3)} = \mathcal{D}_{h,\beta\gamma}^{(3)}(0) = \partial_\beta \partial_\gamma \psi_h^{(3)}(0) , \]  
(35)

and again \( h = a, b \) and \( T_{\alpha\beta}^{(3)} = \partial T_{\alpha\beta}^{(3)}/\partial q_\beta \). In addition, one has:

\[ \mathbf{L}^{(12)}(\tau) = \eta_0 [\eta(0) - E(\tau) - \mathcal{E}(\tau) T(\tau)] \int_\Gamma dq \nabla \psi^{(1)} \times \nabla \psi^{(2)} , \]  
(36)

where \( (\mathcal{D}E - E\mathcal{D}) \propto \tau^{-7} \propto \tau^{7/3} \) in the Einstein-de Sitter universe. Taylor expanding as before results in

\[ L^{(12)}_{\alpha} = (\mathcal{D}E - E\mathcal{D}) \epsilon_{\alpha\beta\gamma} \left[ \eta_0 \partial_\beta \psi^{(1)}(0) \partial_\gamma \psi^{(2)}(0) + T_{\beta\gamma}^{(3)} \mathcal{J}_{\gamma\delta} \right] . \]  
(37)

A more explicit expression for the contribution \( L^{(12)}_{\alpha} \) in terms of integrals in Fourier space may be found in the Appendix B. Note that if one considers the volume \( \Gamma \) to be centered on a peak of the underlying density distribution, as is presumably appropriate when studying the formation of collapsing objects, the first term is zero since \( \psi^{(1)} \) is an extremum at the origin in that case.

For a single collapsing region enclosed in a volume \( \Gamma \) it is enough to evaluate equation (19) at the time of maximum expansion \( \tau_{\gamma} \) to compute its angular momentum. After \( \tau_{\gamma} \) the angular momentum essentially stops growing since the collapsed object is less sensitive to tidal couplings (Peebles 1969). However it is more useful to compute the mean angular momentum of the object averaged over an ensemble of realisations of the gravitational potential random field \( \psi^{(1)} \); this is particularly important in order to compare the theory against statistical results obtained from \( N \)-body simulations or from observations. This programme is carried out in the next section.

3 ENSEMBLE AVERAGES

We simplify the previous results by considering the expectation value over the ensemble of realisations of \( \psi^{(1)} \) of the square of \( \mathbf{L} \), \( \langle \mathbf{L}^2 \rangle \equiv \langle \mathbf{L}^2 \rangle \), for objects with preselected inertia tensor. The underlying motivation for this is that it gives the appropriate analytical estimate to compare against numerical simulations that make use of the Hoffman-Ribak algorithm to set up a constrained density field that contains an object with given inertia tensor (Hoffman & Ribak 1991; van de Weygaert & Bertschinger 1996). Moreover, the resulting expectation value is still a good estimate for the angular momentum of Gaussian peaks in case where the correlation between gravitational potential field and inertia tensor – neglected here – can be properly taken into account, at least in the linear regime (see the discussion in Catelan & Theuns 1996): the exact calculation in the non-linear regime appears intractable analytically. In addition, the procedure could give some insight into perturbative spin corrections in the case of more generic primordial non-Gaussian statistics.

Taking into account the mildly non-linear corrections, one has as leading terms:

\[ \langle \mathbf{L}^2 \rangle = \langle \mathbf{L}^{(12)} \rangle + \langle \mathbf{L}^{(22)} \rangle + 2 \langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle + o(\tau^{-11}) . \]  
(38)

Note that the term \( \langle \mathbf{L}^{(1)} \cdot \mathbf{L}^{(2)} \rangle \) is zero if the linear potential is assumed to be Gaussian distributed, as it involves an odd number of random fields. This term would represent the lowest-order perturbative correction to the linear term for the more general case of non-Gaussian statistics (work in progress).

3.1 Perturbative corrections

3.1.1 Linear approximation

The linear ensemble average \( \langle \mathbf{L}^{(12)} \rangle \) is computed and discussed extensively in Catelan & Theuns (1996):

\[ \langle \mathbf{L}^{(12)} \rangle = \frac{2}{15} \mathcal{D}(\tau)^2 (\mu^2_1 - 3\mu_2) \sigma(R)^2 , \]  
(39)

where the quantity \( \sigma(R)^2 \) is the mass variance on the scale \( R \), explicitly given by \( \sigma(R)^2 \equiv (2\pi)^{-7} \int_0^\infty dp p^6 P_\sigma(p) \tilde{W}(pR)^2 \). In this expression, \( \tilde{W} \) denotes a filter applied to the input power spectrum to remove any ultra-violet divergence. We will use the Gaussian smoothing function \( \tilde{W}(pR) = \exp(-p^2 R^2/2) \). Equation (39) holds for any power spectrum \( \langle \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2) \rangle_{\phi} \equiv (2\pi)^3 \delta_\phi(p_1 + p_2) P_\phi(p) \) and the value of \( \langle \mathbf{L}^{(12)} \rangle \) depends on the normalisation of the spectrum; the symbol \( \delta_\phi \) indicates the Dirac function. The general expression [39] is independent of the details of the shape of the boundary surface of the volume.
assuming a power-law spectrum. The calculation of the term (equation 39).

For a spherical volume, \( \mu_1 = \mu_2 = \mu_3 \), hence \( \mu_1^2 - 3\mu_2 = 0 \), as we stressed before. For any volume \( \Gamma \) one has \( \mu_1^2 - 3\mu_2 \geq 0 \).

### 3.1.2 Higher-order approximation: \( \langle \mathbf{L}^{(2)} \rangle \)

The calculation of the term \( \langle \mathbf{L}^{(2)} \rangle \) takes advantage of the results of the second-order approximation. The final expression, assuming a power-law spectrum \( P_b(p) = A p^{n} = P_v p^{n} \), may be written as:

\[
\langle \mathbf{L}^{(2)} \rangle = \frac{2}{15} \hat{E}(\tau)^2 (\mu_1^2 - 3\mu_2) \Sigma^{(2)}(R; n),
\]

where the function \( \Sigma^{(2)} \) depends on the smoothing scale \( R \) and the normalisation of the spectrum \( A \), as

\[
\Sigma^{(2)}(R; n) \equiv \frac{A^2}{(2\pi)^7} \int_0^\infty dp \, p^{n+2} |\hat{W}(pR)|^2 \Delta^{(2)}(p; n),
\]

\[
\Delta^{(2)}(p; n) \equiv \frac{1}{2^{n+3}} \int_0^\infty dp_2 \, p_2^{n+2} \int_{-1}^{-1} d\theta \, (1 - \theta^2)^2 \left[ (p_1 p_2^{-1} + p_2 p_1^{-1})^2 - 4\theta^2 \right]^{(n-4)/2}.
\]

Typically, these integrals have to be evaluated numerically. The results for more physical power spectrum, like the Cold Dark Matter (CDM) spectrum, are discussed in Appendix B, where the details of the derivation of the equation (42) are given as well. Surprisingly, the average \( \langle \mathbf{L}^{(2)} \rangle \) factorises the same invariant \( \mu_1^2 - 3\mu_2 \) of the inertia tensor \( J \) as appeared in the linear term (equation 39).

### 3.1.3 Higher-order approximation: \( \langle \mathbf{L}^{(1)} \rangle, \langle \mathbf{L}^{(3)} \rangle \)

The calculation of the correction \( \langle \mathbf{L}^{(1)} \rangle, \langle \mathbf{L}^{(3)} \rangle \) takes advantage of the results of the third-order Lagrangian theory. The displacement \( \mathbf{S}_b^{(3)} \) does not induce on average any higher-order correction to the angular momentum, since it corresponds to radial motions of the fluid patches (see Appendix B for an explicit derivation).

\[
\langle \mathbf{L}^{(1)} \rangle, \langle \mathbf{L}^{(3)} \rangle = 0.
\]

Assuming again a scale-free power spectrum \( P_b(p) = A p^{n-4} \), the corrections due to the third-order displacements \( \mathbf{S}_b^{(3)} \) and \( \mathbf{T}^{(3)} \) are (see Appendix B for details):

\[
\langle \mathbf{L}^{(1)} \rangle = \frac{2}{15} \hat{E}(\tau)^2 \hat{F}(\tau)(\mu_1^2 - 3\mu_2) \Sigma_b^{(3)}(R; n),
\]

\[
\langle \mathbf{L}^{(3)} \rangle = -\frac{2}{15} \hat{E}(\tau)^2 \hat{F}(\tau)(\mu_1^2 - 3\mu_2) \Sigma_c^{(3)}(R; n),
\]

where the functions \( \Sigma_b^{(3)} \) and \( \Sigma_c^{(3)} \) are respectively

\[
\Sigma_b^{(3)}(R; n) \equiv \frac{A^2}{(2\pi)^7} \int_0^\infty dp \, p^{n+4} |\hat{W}(pR)|^2 \Delta_b^{(3)}(p; n),
\]

\[
\Sigma_c^{(3)}(R; n) \equiv \frac{A^2}{(2\pi)^7} \int_0^\infty dp \, p^{n+4} |\hat{W}(pR)|^2 \Delta_c^{(3)}(p; n),
\]

and the integrands \( \Delta_b^{(3)} \) and \( \Delta_c^{(3)} \)

\[
\Delta_b^{(3)}(p; n) \equiv \int_0^\infty dp_2 \, p_2^{n+1} \int_{-1}^{-1} d\theta \, (1 - \theta^2)^2 \left[ (p_1 p_2^{-1} + p_2 p_1^{-1})^2 - 2\theta \right]^{-1},
\]

\[
\Delta_c^{(3)}(p; n) \equiv \int_0^\infty dp_2 \, p_2^{n} \int_{-1}^{-1} d\theta \, (1 - \theta^2)^2 \left( \theta - p_2 p_1^{-1} \right) \left[ (p_1 p_2^{-1} + p_2 p_1^{-1}) - 2\theta \right]^{-1}.
\]

Again, these integrals have to be calculated numerically. The results for a more physical power spectrum are discussed in Appendix B. Note that once more these averages factorise out the invariant \( \mu_1^2 - 3\mu_2 \) of the inertia tensor.
3.1.4 Higher-order approximation: \( \langle L^{(1)}, L^{(12)} \rangle \)

The perturbative correction \( \langle L^{(1)}, L^{(12)} \rangle \) originates from the coupling between first- and second-order displacements. The calculation of this term is cumbersome and is detailed in Appendix B: we restrict ourselves to giving the final result:

\[
\langle L^{(1)}, L^{(12)} \rangle = \frac{2}{15} \dot{D}(\tau) \left[ D(\tau)E(\tau) - \dot{D}(\tau)E(\tau) \right] (\mu^2_1 - 3\mu_2) \Sigma^{(12)}(R; n),
\]

where

\[
\Sigma^{(12)}(R; n) = \frac{15}{2} \int \frac{dp_1 dp_2}{(2\pi)^6} \tilde{W}(p_1 R) \tilde{W}(p_2 R) \tilde{W}(p_1 - p_2 | R) P_\sigma(p_1) P_\sigma(p_2) \kappa^{(2)}(p_1, p_2) \frac{M}{(p_1 - p_2)^2} p_{14}(p_1 - p_2)_z [p_{14} p_{24} - p_{15} p_{24}],
\]

where \( \kappa^{(2)} \) is the second-order kernel defined in Appendix A. This integral needs to be computed numerically both for scale-free and CDM spectra. Note that as previously this perturbative correction term to the spin is proportional to \( \mu^2_1 - 3\mu_2 \). This remarkable property enables us to calculate the relative contribution of linear and higher-order spin terms.

The time dependent growth factors of the various terms in equation (38) are illustrated in Fig. 1 and the results of numerically integrating the momentum contributions \( \Sigma \) for both power law and the CDM spectrum are shown in Fig. 2 as a function of the spectral index \( n \). These wavevector integrations generally diverge at small wavelengths for scale-free spectra; to obtain finite expressions we have filtered out this ultra-violet divergence by smoothing the integrals over \( p_2 \) (in equations 44, 50 and 51) artificially with a Gaussian filter of width \( 0.5 \times h^{-1} \) Mpc. The CDM integrals, in contrast, are finite. The numerical values of these momentum integrals will be used in the next section to estimate the relative contributions of the higher-order terms to the linear estimate of \( \langle L^2 \rangle \). The dependence of these relative momentum contributions on the smoothing radius \( R \) for a CDM spectrum is shown in Fig. 3, from which it is clear that, although the various \( \Sigma \)'s depend strongly on \( R \), the normalised contributions \( \Sigma / \sigma^4 \) are more weakly dependent on the smoothing scale.

4 ANGULAR MOMENTUM AT THE MAXIMUM EXPANSION TIME

In this section we quantify the relative perturbative corrections to the linear angular momentum by computing:

\[
\Upsilon \equiv \frac{\langle L^2 \rangle}{\langle L^{(2)} \rangle^2} = 1 - \frac{\langle L^{(2)} \rangle}{\langle L^{(1)} \rangle^2} + 2 \frac{\langle L^{(1)}, L^{(3)} \rangle}{\langle L^{(1)} \rangle^2} = \Upsilon^{(22)} + \Upsilon^{(13)} + \Upsilon^{(13)} + \Upsilon^{(112)},
\]

where we recall that \( L^{(3)} = L^{(3)} + L^{(3)} + L^{(3)} + L^{(12)} \) hence \( L^{(3)} \) gives rise to four correction terms. From equation (45) we find immediately that \( \Upsilon^{(22)} = 0 \). We compute the other correction terms at the time defined by \( D(\tau)M = 1 \) on the mass scale \( M \), which is close to the maximum expansion time found from extrapolating the spherical model (e.g., Peebles 1980). After \( \tau_M \), the protoobject starts collapsing and tidal torques are much less efficient in spinning up its matter content (Peebles 1969; Barnes & Efstathiou 1987). Therefore, we assume that the growth of the spin effectively ceases after maximum expansion of the object and identify the angular momentum at that time with the “final” angular momentum. We understand that this is a partial description of the real world (see the discussion in Catelan & Theuns 1996).

Collecting the expressions for the various corrections, we find:

\[
\Upsilon^{(22)} = \left( \frac{\dot{E}}{D^2} \right)^2 \frac{\Sigma^{(3)}(M)}{\sigma(M)^4} [D(\tau_M)M] = 0.22
\]

\[
\Upsilon^{(13)} = 2 \left( \frac{\dot{F}_0}{D^2} \right) \frac{\Sigma^{(3)}(M)}{\sigma(M)^4} [D(\tau_M)M] = 0.44
\]

\[
\Upsilon^{(13)} = -2 \left( \frac{\dot{F}_0}{D^2} \right) \frac{\Sigma^{(3)}(M)}{\sigma(M)^4} [D(\tau_M)M] = -0.12
\]

\[
\Upsilon^{(112)} = 2 \left( \frac{\dot{E} - \dot{D}E}{D^2} \right) \frac{\Sigma^{(2)}(M)}{\sigma(M)^4} [D(\tau_M)M] = 0.04
\]

where the numerical values are calculated for the (flat) standard CDM model when filtered on the scale of \( R = 0.5 h^{-1} \) Mpc and at the maximum expansion time, i.e., when \( D = 1 \). The factors \( \dot{E} / D^2 \dot{D}, \dot{F}_0 / D^2 \dot{D}, \dot{F}_o / D^2 \dot{D} \) and \( (\dot{E} - \dot{D}E) / D^2 \dot{D} \) do not depend on \( D \) for a flat universe; for a non-flat universe, their \( \tau \) dependence is extremely weak, in view of the excellent approximations \( E \propto D^2, \dot{F}_0 \propto D^3 \) and \( D \dot{E} - \dot{D}E \propto D^3 \), as illustrated in Fig. 5. Note that, with these approximations, the various \( \Upsilon \)'s do not depend on the normalisation of the power spectrum.

We conclude that, since at maximum expansion

\[
\langle L^2 \rangle = (1 + \Upsilon) \langle L^{(12)} \rangle \approx 1.6 \langle L^{(12)} \rangle
\]
Non-linear evolution of the tidal angular momentum

Figure 1. Time dependencies, for open, flat and closed universes, (top, middle and bottom curves respectively) $\langle L^{(1)} \rangle^{1/2} \propto \dot{D}$, $\langle L^{(2)} \rangle^{1/2} \propto |\dot{E}|$, $\langle L^{(1)} \cdot L^{(3)} \rangle^{1/2} \propto (-\dot{D}\dot{F})^{1/2}$ and $\langle L^{(1)} \cdot L^{(12)} \rangle^{1/2} \propto \dot{D}(\dot{D}E - \dot{D}\dot{E})^{1/2}$. The latter four have been scaled by the indicated numerical factors to give the asymptotic behaviour $(-\tau)^{-5}$ for $\tau \to -\infty$ and are practically indistinguishable on the plot. The vertical line at $\tau = -1$ denotes the infinity of physical time $t$ for open universes. The term corresponding to $\dot{D}\dot{F}$ is not plotted since $\langle L^{(1)} \cdot L^{(3)} \rangle = 0$.

the linear estimate of $\langle L^2 \rangle$ is roughly a factor 1.6 times smaller than the value obtained when taking into account the lowest-order non-linear corrections, for a standard CDM spectrum. Hence, $\sqrt{\langle L^2 \rangle} \approx 1.3 \sqrt{\langle L^{(1)} \rangle^{2}}$. From this we conclude that the predictions of linear theory are surprisingly accurate and that the dynamical perturbative corrections appear converged.

5 SUMMARY AND CONCLUSIONS

In this paper we analysed the growth of the tidal angular momentum $L$ acquired by a protoobject (protogalaxy or protocluster) during the mildly non-linear evolution of the matter density perturbations, assuming the latter to be Gaussian distributed. The dynamics of the collisionless matter fluid is described using the Lagrangian approach in the formulation given by Catelan (1995). This formulation is very suitable to study the problem at hand, because the Lagrangian expressions are considerably simpler than their Eulerian counterparts, yet the protogalaxy’s tidal spin is a vector invariant under the change of Eulerian to Lagrangian spatial coordinates, $x$ and $q$ respectively. Specifically, the difficult problem of inverting the mapping $x = q + S$, where $S$ is the displacement vector, in order to recover the Eulerian quantities from the Lagrangian ones, is completely avoided.

The strategy we follow is straightforward. The non-linear spin corrections $L^{(n)}$, where $L^{(1)}$ is the linear angular momentum, are calculated approximating the fluid elements’ trajectories $S$ by the perturbative solutions $S_n$ of the Lagrangian fluid equations (8) and (9). This leads to the expression (20). Since we are interested in computing the lowest-order perturbative corrections to the ensemble average $\langle L^{(1)} \rangle$ for objects with given inertia tensor, we need to calculate corrections to $L$ up to third-order. This has the added advantage that we take account of the full physical content of equations (8) and (9), since the latter are cubic in the displacement. The calculation is summarised as follows: from the knowledge of $S = S_1 + S_2 + S_3$ (where $S_1$ corresponds to the displacement in Zel’dovich approximation), we deduce the corresponding corrections $L = L^{(1)} + L^{(2)} + L^{(3)}$ and finally get the perturbative expansion $\langle L^2 \rangle = \langle L^{(1)} \rangle^2 + \langle L^{(2)} \rangle + 2 \langle L^{(1)} \cdot L^{(3)} \rangle$. The term $\langle L^{(1)} \cdot L^{(2)} \rangle$ is zero for an underlying...
Figure 2. Wave vector parts of higher-order corrections to $\langle L^2 \rangle$ computed for power law spectra $P_\psi(p) \propto p^n$, filtered with a Gaussian smoothing function at $R = 0.5 \, h^{-1} \, \text{Mpc}$, as a function of spectral index $n$, in units of the square of the mass variance $\sigma^2$ in order to eliminate the dependence on the normalisation of the spectrum. The points corresponding to the CDM spectrum are indicated by stars and positioned arbitrarily at $n = -1.5$ which resembles the slope of CDM power spectrum at galactic scales.

Gaussian matter distribution, but it should be taken into account in the framework of more general non-Gaussian statistics (work in progress). Assuming Gaussian statistics here, we disregard it. In sections 2.1 and 2.2 (for the Einstein-de Sitter universe; in Appendix A for a more general Friedmann universe) we reviewed the Lagrangian theory and the perturbative solutions $S_1, S_2$ and $S_3$ of the Lagrangian fluid equations. Using these results we calculate the corrections $\langle L^{(2)} \rangle$ and $\langle L^{(1)} \cdot L^{(3)} \rangle$ in section 3, after summarising the results of linear theory (i.e., the term $\langle L^{(1)} \rangle$). The final expressions are rather cumbersome (the details of the calculations have been deferred to Appendix B), but we can summarise the main features of our results as follows: for an Einstein-de Sitter universe,

- $\langle L^{(1)} \rangle \propto \langle L^{(2)} \rangle \propto \langle L^{(1)} \cdot L^{(3)} \rangle \propto \mu_2^2 - 3 \mu_2$, where $D$ is the growth factor of the density perturbations, $E$ and $F_h$ ($h = a, b, c$) are the growing modes of the second- and third-order Lagrangian displacements, respectively. We see that the perturbative corrections to $\langle L^{(1)} \rangle^{1/2}$ grow proportionally to $t^{5/3}$ in the Einstein-de Sitter universe, in agreement with Peebles (1969). The expressions between square brackets give the generalisations of the results for a generic Friedmann universe. Interestingly, all the corrections we have analysed are proportional to the same invariant of the inertia tensor $J$ of the matter contained in the homogeneous Lagrangian volume $\Gamma$, a result we can express as

\begin{align*}
\bullet & \quad \langle L^{(1)} \rangle^{1/2} \propto \langle L^{(2)} \rangle^{1/2} \propto \langle L^{(1)} \cdot L^{(3)} \rangle^{1/2} \propto \mu_2^2 - 3 \mu_2, \\
\bullet & \quad \langle L^{(2)} \rangle^{1/2} \propto \langle L^{(3)} \rangle^{1/2} \propto \langle L^{(1)} \cdot L^{(3)} \rangle^{1/2} \propto \mu_2^2 - 3 \mu_2, \\
\bullet & \quad \langle L^{(1)} \cdot L^{(3)} \rangle^{1/2} \propto \langle L^{(2)} \rangle^{1/2} \propto \langle L^{(3)} \rangle^{1/2} \propto \mu_2^2 - 3 \mu_2,
\end{align*}
where $\mu_1$ and $\mu_2$ are the first and the second invariant of the inertia tensor (see equations (40) and (41)). This invariant $\mu_1^2 - 3\mu_2$ has been thoroughly investigated in Catelan & Theuns (1996). As a consequence of this factorisation we have been able to express the order of magnitude of the non-linear corrections to $\langle L^2 \rangle$ in terms of the linear contribution, $\langle L^2 \rangle = (1 + \Upsilon) \langle L^{(1)} \rangle^2$ (equation (54)), where $\Upsilon \approx 0.6$ for the standard CDM spectrum at galactic scales. Taking into account that the non-linear correction is small, we conclude that linear theory gives a good description of the angular momentum up to maximum expansion. Since in addition linear theory predicts, in the Einstein-de Sitter model, a growth rate $L \propto t$, it follows that the initial torque is a good estimate for the tidal torque over the whole period during which the object is spun up: $dL(t)/dt \approx dL(0)/dt$.

Finally, as is the case with almost any analytic calculation, comparison of these results against observations is hampered by the fact that the very final stages of galaxy formation are likely to be highly non-linear and in addition dissipative processes may play an important role as well. Analytic investigations are not able to take such highly complex phenomena into account.

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APPENDIX A

In this first appendix we report the expressions of the Lagrangian perturbative solutions $S_n$ explicitly up to third-order and valid for a generic non-flat universe. Specifically, we give the expressions for the growing modes $D, E, F_a, F_b$ and $F_c$ for linear, second- and third-order terms and the expressions for their wave vector dependence, i.e. the Fourier transforms of the longitudinal potentials $\psi^{(2)}, \psi^{(3)}_a, \psi^{(3)}_b$ and of the transverse components $T_\alpha$. For brevity we use the symbol $\Theta(\tau) \equiv \ln(\sqrt{(\tau - 1)/(\tau + 1)}$ for the open universe case ($k = -1$) and $\Lambda(\tau) \equiv \arctan(1/\tau) = -i\Theta(i\tau)$ for the closed universe case ($k = +1$).

A.1 Zel’dovich approximation

The growing mode of the density fluctuations $D(\tau)$ is given by (Shandarin 1980):

$$D(\tau) = \frac{5}{2} \left[ 1 + 3 (\tau^2 - 1) [1 + \tau \Theta(\tau)] \right],$$

for the open universe, and

$$D(\tau) = \frac{5}{2} \left[ -1 + 3 (\tau^2 + 1) [1 - \tau \Lambda(\tau)] \right],$$

for the closed universe. The solution (61) can be obtained from (60) by substituting in the latter $\tau$ by $i\tau$ and reversing the sign to make the growing mode positive. Note that, in contrast to Bouchet et al. (1992) and Catelan (1995), we normalised $D(\tau)$ according to the suggestion of Shandarin (1980): the coefficient $5/2$ is such that $D(\tau) \rightarrow \tau^{-2}$ in the limit $\tau \rightarrow -\infty$, which coincides with the Einstein-de Sitter case. $D(\tau)$ for the different universes is plotted in Fig. 4.

A.2 Second-order approximation

The time dependence of second-order growing mode $E(\tau)$ corresponding to the normalisation chosen for $D$ is:

$$E(\tau) = -\frac{25}{8} - \frac{225}{8} (\tau^2 - 1) \left[ 1 + \tau \Theta(\tau) + \frac{1}{2} \left( \tau + (\tau^2 - 1) \Theta(\tau) \right)^2 \right],$$

for the open universe and

$$E(\tau) = -\frac{25}{8} + \frac{225}{8} (\tau^2 + 1) \left[ 1 - \tau \Lambda(\tau) - \frac{1}{2} \left( \tau - (\tau^2 + 1) \Lambda(\tau) \right)^2 \right],$$

for the closed universe. These solutions have been first derived by Bouchet et al. (1992). The extra factor $25/4$ of the present version is due to the different normalisation of the first-order solution $D$. An excellent approximation of the second-order growing mode is $E = -\frac{5}{2} D^2$ (see Fig. 4 for a plot of $E(\tau)$ and Fig. 5 for a plot of the approximation). In the limit $\tau \rightarrow -\infty$ one has $E = -\frac{5}{2} \tau^{-4}$ which corresponds to the flat case.
The Fourier transform of the second-order potential $\psi^{(2)}$ is (Catelan 1995):

$$\tilde{\psi}^{(2)}(p) = -\frac{1}{p^2} \int \frac{dp_1 dp_2}{(2\pi)^6} \left[ (2\pi)^3 \delta_D(p_1 + p_2 - p) \right] \kappa^{(2)}(p_1, p_2) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2),$$

(64)

where we have defined the kernel

$$\kappa^{(2)}(p_1, p_2) = \frac{1}{2} \left[ p_1^2 p_2^2 - (p_1 \cdot p_2)^2 \right].$$

(65)

This kernel describes the effects of non-linearity on the second-order Lagrangian motion of the mass fluid elements.

### A.3 Third-order approximation

The third-order solution $S_3$ actually corresponds to three separable modes, two longitudinal and one transverse, as was discussed previously (section 2.2). The expressions of the growing modes $F_a$, $F_b$ and $F_c$ for a non-flat universe are known up to quadratures in terms of the lower-order solutions $D$ and $E$. Explicitly (Catelan 1995):

$$F_a(\tau) = -2 D(\tau) \int_{-\infty}^{\tau} d\tau_1 D(\tau_1)^{-2} \int_{-\infty}^{\tau_1} d\tau_2 \alpha(\tau_2) D(\tau_2)^{4},$$

(66)

$$F_b(\tau) = -2 D(\tau) \int_{-\infty}^{\tau} d\tau_1 D(\tau_1)^{-2} \int_{-\infty}^{\tau_1} d\tau_2 \alpha(\tau_2) D(\tau_2)^{2} \left[ E(\tau_2) - D(\tau_2)^2 \right],$$

(67)

$$F_c(\tau) = -\int_{-\infty}^{\tau} d\tau_1 \int_{-\infty}^{\tau_1} d\tau_2 \alpha(\tau_2) D(\tau_2)^{3}. $$

(68)

Excellent fits of these functions are $F_a(\tau) = -\frac{1}{4} D^3$, $F_b(\tau) = \frac{3}{2} D^3$ and $F_c(\tau) = -\frac{1}{4} D^3$ (see again Fig. 4 for a plot of the functions and Fig. 5 for a plot of the approximations; these functions were not shown in Catelan 1995). In the limit $\tau \to -\infty$ one recovers the flat solutions as required: $F_a(\tau) = -\frac{1}{4} \tau^{-6}$, $F_b(\tau) = \frac{3}{2} \tau^{-6}$ and $F_c(\tau) = -\frac{1}{4} \tau^{-6}$.

The Fourier transforms of the third-order potentials $\tilde{\psi}^{(3)}_a$ and $\tilde{\psi}^{(3)}_b$ and of the transverse components $T_\alpha$ are respectively (Catelan 1995)

$$\tilde{\psi}^{(3)}_a(p) = -\frac{1}{p^2} \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \left[ (2\pi)^3 \delta_D(p_1 + p_2 + p_3 - p) \right] \kappa^{(3)}_a(p_1, p_2, p_3) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2) \tilde{\psi}^{(1)}(p_3),$$

(69)

$$\tilde{\psi}^{(3)}_b(p) = \frac{1}{p^2} \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \left[ (2\pi)^3 \delta_D(p_1 + p_2 + p_3 - p) \right] \kappa^{(3)}_b(p_1, p_2, p_3) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2) \tilde{\psi}^{(1)}(p_3),$$

(70)

$$\tilde{T}_\alpha(p) = i \int \frac{dp_1 dp_2 dp_3}{(2\pi)^3} \left[ (2\pi)^3 \delta_D(p_1 + p_2 + p_3 - p) \right] \iota^{(3)}_\alpha(p_1, p_2, p_3) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2) \tilde{\psi}^{(1)}(p_3),$$

(71)

where we have introduced the following kernels:

$$\kappa^{(3)}_a(p_1, p_2, p_3) \equiv \frac{1}{6} \varepsilon_{\alpha\gamma\delta} \varepsilon_{\beta\sigma\tau} p_\alpha p_1^\beta p_2^\gamma p_3^\tau,$$

(72)

$$\kappa^{(3)}_b(p_1, p_2, p_3) \equiv \frac{1}{6} \left[ p \cdot p_1 - \frac{p_1 \cdot (p_2 + p_3) p \cdot (p_2 + p_3)}{p_2 + p_3} \right] \kappa^{(2)}(p_2, p_3),$$

(73)

$$\iota^{(3)}_\alpha(p_1, p_2, p_3) \equiv \frac{1}{6} \left[ p_1 \cdot (p_2 + p_3) \right] \kappa^{(2)}(p_2, p_3) (p_2 + p_3 - p_1)_\alpha.$$

(74)

These expressions are more suitable to compute correction terms to $L$ than the original equations as derived in Catelan (1995).

### APPENDIX B

In this appendix we give explicit derivations for the ensemble averages $\langle L^{(2)}_a \rangle$, $\langle L^{(1)}_b \rangle$ and $\langle L^{(1)}_c \rangle$. We will use the perturbative corrections to the Lagrangian displacement as reviewed previously.

- Starting with the former one, we use the expression (26) for $L^{(2)}_a$ to find:

$$\langle L^{(2)}_a \rangle = \dot{E}(\tau)^2 \varepsilon_{\alpha\beta\gamma} \varepsilon_{\alpha\beta\gamma'} J_{\sigma\gamma} J_{\sigma'\gamma'} \langle D^{(2)}_{\beta\sigma} D^{(2)}_{\beta'\sigma'} \rangle.$$

(75)

The second-order deformation tensor $D^{(2)}_{\alpha\beta}$ may be written in terms of the second-order potential $\psi^{(2)}$ as
D_{\alpha \beta} \equiv \partial_\alpha \partial_\beta \psi^{(2)}(0) = - \int \frac{dp}{(2\pi)^6} p_\alpha p_\beta \tilde{\psi}^{(2)}(p) \tilde{W}(pR),

\text{(76)}

where the field \( \psi^{(2)} \) is now assumed to be filtered on scales \( R \) using the smoothing function \( W_R \). Inserting the expression (64) for the Fourier transform \( \tilde{\psi}^{(2)}(p) \) of \( \psi^{(2)} \) we find

\[
D^{(2)}_{\alpha \beta} = \int \frac{dp_1 dp_2}{(2\pi)^6} \frac{(p_1 + p_2)_{\alpha} (p_1 + p_2)_{\beta}}{|p_1 + p_2|^2} \tilde{W}(|p_1 + p_2|R) \kappa^{(2)}(p_1, p_2) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2).
\]

\text{(77)}

From this last expression we obtain:

\[
\langle D^{(2)}_{\alpha \beta} D^{(2)}_{\beta' \sigma'} \rangle = \int \frac{dp_1 dp_2}{(2\pi)^6} \frac{dp'_1 dp'_2}{(2\pi)^6} \tilde{W}(|p_1 + p_2|R) \tilde{W}(|p'_1 + p'_2|R) \kappa^{(2)}(p_1, p_2) \kappa^{(2)}(p'_1, p'_2)
\times \frac{(p_1 + p_2)_{\beta} (p_1 + p_2)_{\beta'}}{|p_1 + p_2|^2} \frac{(p'_1 + p'_2)_{\sigma} (p'_1 + p'_2)_{\sigma'}}{|p'_1 + p'_2|^2} \langle \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p'_1) \tilde{\psi}^{(1)}(p'_2) \rangle.
\]

\text{(78)}

Since the primordial gravitational potential is assumed to be Gaussian distributed, one has (see, e.g., Peebles 1980)

\[
\langle \tilde{\psi}^{(1)}(p_1) \tilde{\psi}^{(1)}(p_2) \tilde{\psi}^{(1)}(p_1') \tilde{\psi}^{(1)}(p_2') \rangle = (2\pi)^3 \delta_D(p_1 + p_2) P_\psi(p_1) (2\pi)^3 \delta_D(p'_1 + p'_2) P_\psi(p'_1)
+ (2\pi)^3 \delta_D(p_1 + p'_1) P_\psi(p_1) (2\pi)^3 \delta_D(p_2 + p'_2) P_\psi(p_2)
+ (2\pi)^3 \delta_D(p_1 + p'_2) P_\psi(p_1) (2\pi)^3 \delta_D(p_2 + p'_1) P_\psi(p_2).
\]

\text{(79)}

The first term does not contribute to the integral since \( \kappa^{(2)}(p, -p) = 0 \). The remaining two terms give identical contributions, hence:

\[
\langle D^{(2)}_{\alpha \beta} D^{(2)}_{\beta' \sigma'} \rangle = 2 \int \frac{dp_1 dp_2}{(2\pi)^6} \frac{(p_1 + p_2)_{\alpha} (p_1 + p_2)_{\sigma} (p_1 + p_2)_{\alpha'} (p_1 + p_2)_{\sigma'}}{|p_1 + p_2|^4} \kappa^{(2)}(p_1, p_2)^2 \tilde{W}(|p_1 + p_2|R)^2 P_\psi(p_1) P_\psi(p_2).
\]

\text{(80)}

The trick now is to reduce this integral in such a way that we can apply the rule,
Figure 5. Ratios between the higher-order growth factors $E$, $F_\alpha$, $F_\beta$ and $F_c$ and their approximations in terms of powers of $D$, for open (upper curves), flat (middle curves) and closed (lower curves) universes. The third-order ratios have been offset artificially by 0.1, 0.2 and 0.3 for clarity. The second-order ratio $E/(-3D^2/T)$ is shown here for completeness; it has been first shown as a function of $\Omega$ in Bouchet et al. (1992).

\[
\int_{\text{sphere}} dp_\alpha p_\beta p_\gamma \frac{1}{15} (\delta_{\alpha \beta} \delta_{\gamma \delta} + \delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma}) \int dp p^6 F(p),
\]

which holds for any function $F(p)$ which depends only on the modulus $p$ of $p$. Let us define the new variables $k_1 \equiv p_1 + p_2$ and $k_2 \equiv p_1 - p_2$. The determinant of the Jacobian of this transformation is 1/8. At this point, noting that

\[
P_1^{-4} P_2^{-4} |k^{(2)}(p_1, p_2)|^2 = 4 \left[ \frac{1 - \theta^2}{(k_1 k_2^{-1} + k_2 k_1^{-1})^2 - 4\theta^2} \right]^2,
\]

where $\theta \equiv k_1 \cdot k_2 / k_1 k_2$ and $k_h \equiv |k_h|$, the integral (80) may be written as

\[
\langle D_{\alpha \sigma}^{(2)} D_{\beta \sigma'}^{(2)} \rangle = \frac{\int dk_1 dk_2 k_{1\beta} k_{2\sigma} k_{1\sigma'} k_{2\beta'}}{k_1^4} \left[ \frac{1 - \theta^2}{(k_1 k_2^{-1} + k_2 k_1^{-1})^2 - 4\theta^2} \right]^2 [W(k_1 R)]^2 P_3(|k_1 + k_2|/2) P_3(|k_1 - k_2|/2).
\]

This expression holds for any power spectrum $P_3(p) = p^4 P_\nu(p)$, but let us assume for simplicity that the power spectrum is scale-free, $P_\nu(p) = A p^{n-4}$. The calculation can be continued as follows. Since one has:

\[
|k_1 + k_2|^n |k_1 - k_2|^n = (k_1 k_2)^n \left[ (k_1 k_2^{-1} + k_2 k_1^{-1})^2 - 4\theta^2 \right]^{n/2},
\]

the integral in (83) may be simplified to

\[
\langle D_{\alpha \sigma}^{(2)} D_{\beta \sigma'}^{(2)} \rangle = \frac{A^2}{4^n} \int \frac{dk_1}{(2\pi)^3} k_{1\beta} k_{2\sigma} k_{1\sigma'} k_{2\beta'} \frac{W(k_1 R)}{k_1^{n+4}} \int_0^\infty \frac{dk_2}{(2\pi)^2} k_2^{n+2} \int_{-1}^{+1} d\theta \left( 1 - \theta^2 \right)^2 \left[ (k_1 k_2^{-1} + k_2 k_1^{-1})^2 - 4\theta^2 \right]^{(n-4)/2}.
\]

We note at this point that the function

\[
2^{2n-4} \Delta^{(2)}(k_1; n) \equiv \int_0^\infty dk_2 k_2^{n+2} \int_{-1}^{+1} d\theta \left( 1 - \theta^2 \right)^2 \left[ (k_1 k_2^{-1} + k_2 k_1^{-1})^2 - 4\theta^2 \right]^{(n-4)/2},
\]
which typically has to be evaluated numerically, depends only on the modulus $|k_1|$ and on the spectral index $n$, as indicated. We can therefore apply the rule in (81) since the smoothing function depends only on the modulus of $k_1$ as well, to do the transformation:

$$
\langle D_{\beta\sigma}^{(2)} D_{\delta'\sigma'}^{(2)} \rangle = \frac{A^2}{2(2\pi)^2} \int \frac{dk}{2(2\pi)^3} k_3 k_a k_{3'} k_{a'} [\widetilde{W}(kR)]^2 k^{n-4} \Delta^{(2)}(p_1; n)
$$

$$
= \frac{1}{15} (\delta_{\beta\sigma} \delta_{\delta'\sigma'} + \delta_{\beta\delta'} \delta_{\sigma\sigma'} + \delta_{\beta\sigma} \delta_{\delta'\sigma'}) \Sigma^{(2)}(R; n),
$$

where $\Sigma^{(2)}$ is given in equation (43). Finally, following along the lines of the derivation of $\langle L^{(1)2} \rangle$ reported in Appendix A of Catelan and Theuns (1996), one ends up with the result (42) in the main text.

The last expressions are not valid for more physical, non-power law, spectra. However, the appropriate expressions can be found by following the same strategy. Let us consider for example the case of the CDM power spectrum (see, e.g., Efstathiou 1989), where the power spectrum is parametrised by:

$$
P_v(p) \equiv Ap^{-3}[T(p)]^2 = Ap^{-3} \left[ 1 + \left( \frac{a p + b p}{(a p^2 + (c p)^2)^{3/2}} \right) \right]^{2/n},
$$

where $a = 6.4 (\Omega h^2)^{-1} \text{ Mpc}$, $b = 3.0 (\Omega h^2)^{-1} \text{ Mpc}$, $c = 1.7 (\Omega h^2)^{-1} \text{ Mpc}$, and $\nu = 1.13$; $A$ is the normalisation of the spectrum, as before. In this case, one obtains instead of equation (85):

$$
\langle D_{\beta\sigma}^{(2)} D_{\delta'\sigma'}^{(2)} \rangle = \frac{A^2}{4} \left[ \int \frac{dk_1}{2(2\pi)^3} k_{1\beta} k_{1\sigma} k_{1\beta'} k_{1\sigma'} [\widetilde{W}(k_1 R)]^2 \right]
$$

$$
\times \int_0^\infty \frac{dk_2}{2(2\pi)^2} k_2^2 \int_{-1}^{+1} d\theta (1 - \theta^2)^2 \left[ (k_1 k_2^{-1} + k_2 k_1^{-1})^2 - 4\theta^2 \right]^{-3/2} \left[ T(k_1, k_2; +\theta) \right] \left[ T(k_1, k_2; -\theta) \right] ^2.
$$

where we have defined the transfer functions,

$$
T(k_1, k_2; \pm\theta)^2 = \left\{ 1 + \left( a G(\pm\theta) + \left( c G(\pm\theta) \right) \right)^{3/2} \left( c G(\pm\theta) \right)^{2} \right\}^{2/\nu},
$$

and $G(\pm\theta) = \frac{1}{\sqrt{k_2 k_3}} (k_1 k_3^{-1} + k_2 k_1^{-1})^{1/2}$. As a consequence of the change of variables, $(p_1, p_2) \rightarrow (k_1, k_2)$ we have, for example, $T(k_1, k_2; \theta) = T(p_1)$. Finally, proceeding as earlier we get:

$$
\Sigma^{(2)}_{CDM}(R) \equiv \frac{A^2}{(2\pi)^3} \int_0^\infty dp \ p^3 [\widetilde{W}(pR)]^2 \Delta^{(2)}_{CDM}(p),
$$

and

$$
\Delta^{(2)}_{CDM}(k_1) \equiv \frac{1}{2} \int_0^\infty dk_2 k_2^2 \int_{-1}^{+1} d\theta (1 - \theta^2)^2 \left[ (k_1 k_2^{-1} + k_2 k_1^{-1})^2 - 4\theta^2 \right]^{-3/2} \left[ T(k_1, k_2; +\theta) \right] \left[ T(k_1, k_2; -\theta) \right] ^2.
$$

• Let us now summarise how to deal with the simpler case of the averages $\langle L^{(1)}, L^{(3)} \rangle$. We first show that $\langle L^{(1)}, L^{(3)} \rangle = 0$. One has:

$$
\langle L^{(1)}, L^{(3)} \rangle = \sum_\alpha \langle L^{(1)}_\alpha, L^{(3)}_\alpha \rangle = \hat{D}(\tau) \tilde{F}(\tau) \epsilon_{\alpha\beta\gamma} \epsilon_{\alpha\beta'\gamma'} \mathcal{J}_{\sigma\sigma'} \langle D^{(1)}_{\beta\sigma} D^{(3)}_{\beta'\sigma'} \rangle.
$$

Writing $D^{(3)}_{\alpha\beta'\sigma'}$, as

$$
D^{(3)}_{\alpha\beta'\sigma'} \equiv \partial_{\beta'} \partial_{\sigma'} \psi^{(3)}(0) = - \int \frac{dp}{(2\pi)^3} P_{\beta' \sigma'} \tilde{\psi}^{(3)}(p) \tilde{W}(pR),
$$

and inserting the expression (64) for $\tilde{\psi}^{(3)}(p)$, one gets

$$
\langle D^{(2)}_{\beta\sigma} D^{(3)}_{\beta'\sigma'} \rangle = \frac{1}{6} \epsilon_{\alpha\beta'\gamma'} \epsilon_{\beta\delta'} \epsilon_{\delta'\alpha'\sigma} \int \frac{dp_1 dp_2}{(2\pi)^3} p^{2-2} [\tilde{W}(p_2 R)]^2 P_v(p_1) P_v(p_2) P_{\beta\sigma} P_{\beta'\sigma'} P_{2\alpha\sigma} P_{2\beta\sigma} P_{2\beta'\sigma'} P_{2\alpha\sigma'},
$$

which is zero for any $W$ and power spectrum $P_v$, since symmetric tensors saturate antisymmetric tensors. This completes the proof.

• Next, let us compute:

$$
\langle L^{(1)}, L^{(3)} \rangle = \sum_\alpha \langle L^{(1)}_\alpha, L^{(3)}_\alpha \rangle = \hat{D}(\tau) \tilde{F}(\tau) \epsilon_{\alpha\beta\gamma} \epsilon_{\alpha\beta'\gamma'} \mathcal{J}_{\sigma\sigma'} \langle D^{(1)}_{\beta\sigma} D^{(3)}_{\beta'\sigma'} \rangle.
$$

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The third-order deformation tensor $D_{b \alpha \beta}^{(3)}$ may be written as

$$D_{b \alpha \beta}^{(3)} = \partial_{\alpha} \partial_{\beta} \psi_b^{(3)} (0) = - \int \frac{dp}{(2\pi)^3} p_\alpha p_\beta \psi_b^{(3)} (p) \widetilde{W}(pR),$$

(97)

where the field $\psi_b^{(3)}$ is now filtered on scale $R$ as previously. Inserting the expression (65) for $\psi_b^{(3)} (p)$, we get:

$$D_{b \alpha \beta}^{(3)} = - \int \frac{dp_1 dp_2 dp_3}{(2\pi)^9} \frac{(p_1 + p_2 + p_3, \alpha)(p_1 + p_2 + p_3, \beta)}{|p_1 + p_2 + p_3|^2} \widetilde{W}(|p_1 + p_2 + p_3| R) \times \frac{1}{2} p_1^2 \left[ 1 - \left( \frac{p_1 \cdot (p_2 + p_3)}{|p_1| p_2 + p_3|} \right)^2 \right] \kappa^{(2)} (p_2, p_3) \psi^{(1)} (p_1) \psi^{(1)} (p_2) \psi^{(1)} (p_3).$$

(98)

The average can now be computed easily by following the same steps as in the derivation of $\langle L^{(2)} \rangle$. One ends up with the expression

$$\langle D_{b \alpha \beta}^{(3)} D_{b' \alpha' \beta'}^{(3)} \rangle = \int \frac{dp_2}{(2\pi)^3} \int p_{2\beta} p_{2\beta'} p_{2\sigma'} \widetilde{W}(p_2 R)^2 p_2 P_\psi (p_2) \int_0^\infty \frac{dp_1}{(2\pi)^2} \int p_1^3 P_\psi (p_1) \int d\theta \frac{1}{p_1^2} \left( 1 - \theta^2 \right)^2 \frac{1}{p_1^2 + p_2 p_1^2 - 2\theta}. \quad (99)$$

We stress that this result holds for any spectrum $P_\psi (p)$, including the CDM spectrum (88). Limiting ourselves to a scale-free power spectrum $P_\psi (p) = A p^n$, applying the rule (81), one recovers the equation (46) in the main text. The generalisation to a CDM power spectrum is straightforward, since there is no coupling between different wave vectors in the power spectrum.

- In a similar fashion, one computes the average

$$\langle L^{(1)} \rangle = \sum_\alpha \langle L^{(1)}_\alpha \rangle D_{\alpha \gamma}^{(3)} \kappa_{\alpha \beta}^{(3)} \mathcal{J}_{\gamma \gamma} \mathcal{J}_{\sigma \sigma} \end{array} \langle D^{(3)}_{b \alpha \beta} D^{(3)}_{b' \alpha' \beta'} \rangle.$$ (100)

From the expression (71) for $\widetilde{T}^{(3)}_\alpha (p)$ we have:

$$T^{(3)}_{\alpha \beta} = \int \frac{dp}{(2\pi)^3} p_\beta \widetilde{T}^{(3)}_\alpha (p) \widetilde{W}(pR).$$

$$= - \int \frac{dp_1 dp_2 dp_3}{(2\pi)^9} \left( p_1 + p_2 + p_3, \beta \right) \epsilon^{(3)}_\alpha (p_1, p_2, p_3) \widetilde{W}(|p_1 + p_2 + p_3| R) \psi^{(1)} (p_1) \psi^{(1)} (p_2) \psi^{(1)} (p_3). \quad (101)$$

One now proceeds as earlier. For any power spectrum $P_\psi (p)$, the result is:

$$\langle D^{(3)}_{b \alpha \beta} D^{(3)}_{b' \alpha' \beta'} \rangle = \int \frac{dp_2}{(2\pi)^3} \int p_{2\beta} p_{2\beta'} p_{2\sigma'} \widetilde{W}(p_2 R)^2 p_2 P_\psi (p_2) \int_0^\infty \frac{dp_1}{(2\pi)^2} \int p_1^3 P_\psi (p_1) \int d\theta \frac{1}{p_1^2} \left( 1 - \theta^2 \right)^2 \frac{1}{p_1^2 + p_2 p_1^2 - 2\theta}. \quad (102)$$

Assuming a scale-free power spectrum $P_\psi (p) = A p^n$ and applying again the rule (81), one recovers the equation (47) in the main text. Once again, the generalisation to a CDM power spectrum is straightforward.

- We conclude this appendix by discussing the third-order correction $L^{(12)}$:

$$L^{(12)} = \mathcal{J}_{\alpha \beta} \left[ \eta_0 (D E - E D) \int dq \nabla \psi^{(1)} \times \nabla \psi^{(2)} \right]. \quad (103)$$

Expanding the potentials in Taylor series around $q = 0$, one obtains for the $\alpha$-component

$$L^{(12)}_{\alpha \alpha} = \left[ (D E - E D) \right] \kappa^{(3)} \left[ \eta_0 \Gamma \partial_\beta \psi^{(1)} (0) \partial_\gamma \psi^{(2)} (0) + \kappa^{(1)} \phi^{(1)} \mathcal{J}_{\gamma \gamma} \right] \equiv \mathcal{L}_{A \alpha}^{(12)} + \mathcal{L}_{B \alpha}^{(12)}. \quad (104)$$

In general, the first of these two terms may be written as

$$L^{(12)}_{A \alpha} = \int \frac{dp_1 dp_2 dp_3}{(2\pi)^9} \left( p_2 + p_3, \gamma \right) \widetilde{W}(p_1 R) \widetilde{W}(p_2 + p_3 R) \times \kappa^{(2)}(p_2, p_3) \psi^{(1)}(p_1) \psi^{(1)}(p_2) \psi^{(1)}(p_3), \quad (105)$$

and the second term as

$$L^{(12)}_{B \alpha} = -\left( (D E - E D) \right) \kappa^{(3)} \left[ \eta_0 \Gamma \partial_\beta \psi^{(1)} (0) \partial_\gamma \psi^{(2)} (0) + \kappa^{(1)} \phi^{(1)} \mathcal{J}_{\gamma \gamma} \right] \equiv \mathcal{L}_{A \alpha}^{(12)} + \mathcal{L}_{B \alpha}^{(12)}. \quad (106)$$

With these informations, one can compute the ensemble average $\langle L^{(1)} L^{(12)} \rangle = \langle L^{(1)} L_{A \alpha}^{(12)} \rangle + \langle L^{(1)} L_{B \alpha}^{(12)} \rangle$. The resulting
expression are, respectively,

\[
\langle L^{(1)}, L_A^{(12)} \rangle = -\eta_0 \Gamma (D\dot{E} - ED) \int \frac{dp_1 dp_2}{(2\pi)^6} \tilde{W}(p_1 R) \tilde{W}(p_2 R) \tilde{W}((p_1 - p_2)|R) P_\nu(p_1) P_\nu(p_2) \\
\times |p_1 - p_2|^{-2} \kappa^{(2)}(p_1, p_2) p_{2\alpha}(p_{1\alpha} p_{2\beta} - p_{2\alpha} p_{1\beta}) p_{2\gamma} \mathcal{J}_{\beta\gamma}
\]

\[
\equiv -\eta_0 \mathcal{F} [p_{2\alpha}(p_{1\alpha} p_{2\beta} - p_{2\alpha} p_{1\beta}) p_{2\gamma} \mathcal{J}_{\beta\gamma}]
\]

(107)

\[
\langle L^{(1)}, L_B^{(12)} \rangle = \dot{D}(D\dot{E} - ED) \int \frac{dp_1 dp_2}{(2\pi)^6} \tilde{W}(p_1 R) \tilde{W}(p_2 R) \tilde{W}((p_1 - p_2)|R) P_\nu(p_1) P_\nu(p_2) \\
\times |p_1 - p_2|^{-2} \kappa^{(2)}(p_1, p_2) p_{2\alpha}(p_{1\alpha} p_{2\beta} - p_{2\alpha} p_{1\beta}) p_{2\gamma} \mathcal{J}_{\beta\gamma}(p_{1\alpha}(p_2 - p_1)_\alpha \mathcal{J}_{\beta\gamma})
\]

\[
\equiv \mathcal{F} [p_{2\alpha}(p_{1\alpha} p_{2\beta} - p_{2\alpha} p_{1\beta}) p_{2\gamma} \mathcal{J}_{\beta\gamma}(p_{1\alpha}(p_2 - p_1)_\alpha \mathcal{J}_{\beta\gamma})]
\]

(108)

where we have introduced the integral operator \( \mathcal{F} \) for conciseness. We proceed by showing that the first term \( \langle L^{(1)}, L_A^{(12)} \rangle \) is zero for any power spectrum. In the eigenframe of the inertia tensor and taking advantage of the fact that \( \mathcal{F} \) is a linear operator, we find:

\[
\langle L^{(1)}, L_A^{(12)} \rangle = -\eta_0 \Gamma t_\alpha \mathcal{F} [A_\alpha]
\]

(109)

where we defined \( A_\alpha \equiv (p_1 \cdot p_2)p_2^\alpha - p_2^2 p_{1\alpha} p_{2\alpha} \). Using the fact that, because of rotational symmetry, \( \mathcal{F}[A_\alpha] = \mathcal{F}[A_\beta] \) for any Cartesian axes \( \alpha \neq \beta \), the result simplifies to

\[
\langle L^{(1)}, L_A^{(12)} \rangle = -\eta_0 \Gamma t_\alpha \mathcal{F} [A_\alpha]
\]

(110)

Since \( \Sigma_\alpha A_\alpha = 0 \) we finally get \( \mathcal{F} [\Sigma_\alpha A_\alpha] = 0 = \frac{1}{2} \mathcal{F} [A_\alpha] \), which completes the proof.

We can use a similar trick to simplify the second term: defining \( B_\alpha \equiv p_{1\alpha}(p_2 - p_1)_\alpha \) (no sum over \( \alpha \)!) it is straightforward to show that in the eigenframe of \( \mathcal{F} \) one gets

\[
\langle L^{(1)}, L_B^{(12)} \rangle = \mathcal{F} [t_\alpha A_\alpha t_\beta B_\beta]
\]

\[
= \mathcal{F} [t_\alpha^2 A_\alpha B_\beta + t_\alpha^2 t_\beta B_\gamma + t_\alpha^2 t_\gamma A_\alpha B_\beta + t_\alpha t_\beta t_\gamma (A_\alpha B_\beta + A_\beta B_\alpha) + t_\alpha t_\beta t_\gamma (A_\gamma B_\alpha + A_\alpha B_\gamma)]
\]

\[
= (\mu^2 - 2\mu_2) \mathcal{F} [A_\alpha B_\beta] + 2\mu_2 \mathcal{F} [A_\alpha B_\beta]
\]

\[
= (\mu^2 - 3\mu_2) \mathcal{F} [A_\alpha B_\beta].
\]

(111)

The last equality is obtained as follows: we first use the same trick as before to prove that \( \mathcal{F} [\Sigma_\alpha B_\alpha A_\alpha] = \frac{1}{2} \mathcal{F} [\Sigma_\alpha B_\beta \Sigma_\beta A_\alpha] = 0 \). Next, since because of symmetry \( \mathcal{F} [A_\alpha B_\beta] = \mathcal{F} [A_\beta B_\alpha] \) we get \( \mathcal{F} [A_\alpha B_\beta] = -\frac{1}{2} \mathcal{F} [A_\alpha B_\beta] \). The integral \( \mathcal{F} [A_\alpha B_\beta] \) has been computed numerically. As in all previously discussed correction terms, \( \mu^2 - 3\mu_2 \) factors out. One recovers equation (53).