Ahlfors Regular Conformal Dimension of Metrics on Infinite Graphs and Spectral Dimension of the Associated Random Walks

Kôhei Sasaya*

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Abstract

Quasisymmetry is a well-studied property of homeomorphisms between metric spaces, and Ahlfors regular conformal dimension is a quasisymmetric invariant. In the present paper, we consider the Ahlfors regular conformal dimension of metrics on infinite graphs, and show that this notion coincides with the critical exponent of $p$-energies. Moreover, we give a relation between the Ahlfors regular conformal dimension and the spectral dimension of a graph.

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*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan. JSPS Research Fellow (DC1). E-mail: ksasaya@kurims.kyoto-u.ac.jp
1 Introduction

Quasisymmetry is a well-studied property of homeomorphisms between metric spaces, and roughly speaking, means that the homeomorphism in question preserves ratios of distances. The Ahlfors regular conformal dimension is a quasisymmetric invariant of metric spaces, which gives a measure of the simplest (in a certain sense) quasisymmetrically equivalent space. The purpose of this paper is to study the Ahlfors regular conformal dimension of discrete unbounded metric spaces, and show relations between the Ahlfors regular conformal dimensions and spectral dimensions of such spaces.

Quasisymmetry was introduced by Tukia and Väisälä in [18] to generalize the notion of quasiconformal mappings on the complex plane. In [18], quasisymmetry was given as a property of a homeomorphism between two metric spaces. A specialization of this was given by Kigami [9], for the comparison of metrics on the same underlying space. This is the definition we will use.

Definition 1.1 (Quasisymmetry, Kigami’s). Let \( X \) be a set and \( d, \rho \) be metrics on \( X \), and let \( \theta : [0, \infty) \to [0, \infty) \) be a homeomorphism. Then we say \( d \) is \( \theta \)-quasisymmetric to \( \rho \) if for any \( x, y, z \in X \) with \( x \neq z \),

\[
\frac{\rho(x, y)}{\rho(x, z)} \leq \theta \left( \frac{d(x, y)}{d(x, z)} \right).
\]

Moreover, if \( d \) is \( \theta \)-quasisymmetric to \( \rho \) for some \( \theta \), then we say that \( d \) is quasisymmetric to \( \rho \) and write \( d \sim_{QS} \rho \).

For example, \( d \sim d^\alpha \) for any \( \alpha \in (0, 1) \) and any metric space \((X, d)\).

This notion was also called “quasisymmetrically related by the identity map” in [7], and “quasisymmetrically equivalent” in [4]. Note that if there exists a quasisymmetric map \( f : (X, d) \to (Y, \rho) \), then \( d \) is quasisymmetric to the pullback metric \( \rho^* \) in the sense of Kigami’s definition and we can identify \((Y, \rho)\) with \((X, \rho^*)\).

Quasisymmetry has been studied in various fields. For example, a quasi-isometric map (the definition is in [13], Definition 3.2.11, for example) between Gromov hyperbolic spaces induces a quasisymmetric map (see [13], Theorem 3.2.13 and Section 3.6), or [15]). There is also much research about quasisymmetry and Gromov hyperbolic spaces (see [13] for example). Quasisymmetry is a weaker notion of bi-Lipschitz equivalence, which has been studied extensively for decades, see [7] or [17] for example. From the viewpoint of global analysis, it is notable that quasisymmetric modification preserves the volume doubling property, which plays an important role in heat kernel estimates. This idea is used in [9], [10], and there is a recent application to circle packing graphs in [14].

Ahlfors regular conformal dimension is a relatively new quasisymmetric invariant. It was introduced by Bourdon and Pajot [3] (see also Bonk and Kleiner [2]), and is defined as follows:

Definition 1.2 (Ahlfors regularity). Let \((X, d)\) be a metric space, \( \mu \) be a Borel measure on \((X, d)\) and \( \alpha > 0 \). We say \( \mu \) is \( \alpha \)-Ahlfors regular with respect to
(X, d) if there exists C > 0 such that
\[ C^{-1}r^\alpha \leq \mu(B_d(x, r)) \leq Cr^\alpha \]
for any x \in X and r_x \leq r \leq \text{diam}(X, d)
where r_x = r_{x,d} = \inf_{y \in X \setminus \{x\}} d(x, y), and B_d(x, r) is the open ball \{ y \in X \mid d(x, y) < r \}. The space (X, d) is called \( \alpha \)-Ahlfors regular if there exists a Borel measure \( \mu \) such that \( \mu \) is \( \alpha \)-Ahlfors regular with respect to (X, d).

**Definition 1.3** (Ahlfors regular conformal dimension). Let (X, d) be a metric space. The Ahlfors regular conformal dimension (or ARC dimension in short) of (X, d) is defined by
\[
\dim_{\text{AR}}(X, d) = \inf \{ \alpha \mid \text{there exists a metric } \rho \text{ on } X \text{ such that } \rho \text{ is } \alpha \text{-Ahlfors regular and } d \sim_{\text{QS}} \rho \},
\]
where \( \inf \emptyset = \infty \).

The ARC dimension is related to the conformal dimension, another well-known quasisymmetric invariant, as introduced by Pansu [16] in 1989.

In this paper we will extend the notion of ARC dimension to discrete metric spaces. Note that the ARC dimension has mainly been studied on bounded metric spaces without isolated points, in which case \( r_x = 0 \) and \( \text{diam}(X, d) < \infty \).

ARC dimension is related to the well-known Cannon’s conjecture, which claims that for any hyperbolic group G whose boundary is homeomorphic to 2-dimensional sphere, there exists a discrete, cocompact and isometric action of G on the hyperbolic space \( \mathbb{H}^3 \). Bonk and Kleiner [2] proved Cannon’s conjecture is equivalent to the following: If G is a hyperbolic group whose boundary is homeomorphic to 2-dimensional sphere, then there exists a metric that attains the value of the ARC dimension of the boundary.

It is not easy to calculate the ARC dimension in general. Motivated by [4, 5], Kigami [11] gave a method to calculate ARC dimension as a critical exponent of a \( p \)-energy, which is defined by successive division of the original metric space. Furthermore, [11] gives inequalities between ARC dimension and \( p \)-spectral dimensions.

In this paper, we extend the results of [11] to infinite graphs and give a relation between spectral dimensions and ARC dimensions. Our main results need a lot of notations, so we postpone detailed definitions to Sections 2 and 4, and explain the main results through examples.

In our study, it will be useful to consider partitions of graphs that arise as edges are successively unified. One of the simplest cases is the unification of vertices of \( \mathbb{Z}_+ = \{ n \in \mathbb{Z} \mid n \geq 0 \} \). For \( a \in \mathbb{N} \) and \( n \in \mathbb{Z}_+ \), we identify \( 2^n \) edges \( \{(2^n(a-1), 2^n(a-1)+1), (2^n(a-1)+1, 2^n(a-1)+2), \ldots, (2^na-1, 2^na)\} =: K_{(n,a)} \) and consider unified graphs \( \{G_n, E_n\}_{n \geq 0} \) where \( G_n = \{(n,a) \mid a \in \mathbb{N} \} \) and \( E_n \) is the set of links between \( (n,a) \) and \( (n,a+1) \). Let \( (n,a) \sim (m,b) \) if \( n - m = 1 \) and \( K_{(n,a)} \supseteq K_{(m,b)} \) or \( m - n = 1 \) and \( K_{(n,a)} \subseteq K_{(m,b)} \). Consider \( T := \bigcup_{n \geq 0} (n,a) \) as a tree by \( \sim \), then we obtain a correspondence between \( \{G_n, E_n\}_{n \geq 0} \) and \( T \) (see Figure 1.1). We call such a correspondence between
unified graphs and a tree, a partition (see Definition 4.4 and note that we construct \( K \) by unification of edges but we treat \( K \) as a subsets of vertices because of technical reasons).

In this paper, we characterize the ARC dimension with a partition. For a given partition, we can define an upper \( p \)-energy \( \mathcal{E}_p \) of the partition as a certain limit of \( p \)-energies on unification graphs, see Definition 2.11, which is based on definitions of [11]. The \( p \)-energy enjoys a phase transition when \( p \) varies, that is, there exists a \( p_0 > 0 \) such that \( \mathcal{E}_p > 0 \) if \( p < p_0 \) and \( \mathcal{E}_p = 0 \) if \( p > p_0 \). We can also define a lower \( p \)-energy \( \mathcal{E}_p \), as well.

Our main result of this paper is the following.

**Theorem 1.4** (Theorem 4.14 (1)). Let \((G, E)\) be a graph and \( d \) be a metric on \( G \). Under some conditions about \( d \) and for partitions within a certain class,

\[
dim_{AR}(G, d) = \inf \{ p \mid \mathcal{E}_p = 0 \} = \inf \{ p \mid \mathcal{E}_p = 0 \}
\]

For detailed conditions, see Theorem 4.14. Let us give an interesting example for ARC dimension for an unbounded metric space.

**Example 1.5.** Let \( f(n) : \mathbb{Z}_+ \to \mathbb{Z}_+ \) be such that \( f(n) \leq n \) for any \( n \). For \( n \geq 0 \), divide \([2^n, 2^{n+1}] \times [0, 2^n]\) into \( 2^{f(n)} \times 2^{f(n)} \) blocks and call them \( G_n \), and consider \( G = \bigcup_{n \geq 0} G_n \cup \{(0, 0)\} \) as a subgraph of \( \mathbb{Z}^2 \) (see Figure 1.2 and Example 5.1 for precise definition).

Using Theorem 1.4, we obtain the following:

**Proposition 1.6** (Proposition 5.2).

1. If \( \limsup_{n \to \infty} f(n) = \infty \), then \( \dim_{AR}(G, d) = 2 \).
2. If \( \limsup_{n \to \infty} f(n) < \infty \), then \( \dim_{AR}(G, d) = 1 \).

It is remarkable that only \( \limsup_{n \to \infty} f(n) = \infty \) implies \( \dim_{AR}(G, d) = 2 \), although the size of boxes \( 2^{n-f(n)} \) may diverge.
We also compare the Ahlfors conformal dimension with the spectral dimension. For $p > 0$, we can define the upper and lower $p$-spectral dimension, $d^S_p$ and $d^S_p$ of a partition (see Definition 2.14). We can further obtain the following.

**Theorem 1.7** (Theorem 4.14(2) and (3)). Let $(G, E)$ be a graph and $d$ is a metric on $G$. Under the same conditions as in Theorem 1.4, if $\text{dim}_{AR}(G, d) < p$, then

$$\text{dim}_{AR}(G, d) \leq d^S_p \leq d^S_p < p.$$  

If $\text{dim}_{AR}(G, d) \geq p$, then

$$\text{dim}_{AR}(G, d) \geq d^S_p \geq d^S_p \geq p.$$  

When $p = 2$, the $p$-spectral dimension coincides in many example with the notion of the spectral dimension of random walks, where the latter is defined as follows:

$$d^S(G) = 2 \limsup_{n \to \infty} \frac{\log p_{2n}(x, x)}{\log n}, \quad d^S(G) = 2 \liminf_{n \to \infty} \frac{\log p_{2n}(x, x)}{\log n},$$

where $p_n(x, y)$ is the transition density of the associated random walk. Hence, where this occurs, the latter theorem will also relate the ARC dimension and the spectral dimension of random walks. See Theorem 4.27 and Corollary 4.30 for a sufficient condition that the 2-spectral dimension and the spectral dimension of random walks coincide. However, we prove that they can also be different - see Example 5.3 for an example where this is the case.
The outline of this paper is as follows. In Section 2, we introduce basic notation, and give the framework and results of [11] for perfect compact spaces, on which the main result of this paper is based. In Section 3, we extend the results of [11] to perfect \( \sigma \)-compact spaces. Section 4 is the main part of this paper, devoted to proving our main results. In Section 5, we give examples that illustrate properties of ARC dimension of graphs. In Section 6, we give a proof of a result of a heat kernel estimate that is used in Section 4.

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2 Notation and Kigami’s Results for Compact Metric Spaces

As the preparation of this paper, we introduce notation used in this paper, and we also introduce results of [11] on which the results of this paper are based.

2.1 Basic Notation

We use the following notation in this paper.

- Let \( A \) be a set and \( F \) be a map on \( A \) to itself. Then \( F^n \) denotes \( F \circ \cdots \circ F \) for any \( n > 0 \) and \( \text{id}_A \) for \( n = 0 \). Moreover, \( A^n \) denotes the product \( A \times \cdots \times A \).
- Let \( \{ A_\lambda \}_{\lambda \in \Lambda} \) be a family of sets, then \( \bigcup_{\lambda \in \Lambda} A_\lambda \) denotes \( \bigcup A_\lambda \) with \( A_\lambda \cup A_\tau = \emptyset \) for any \( \lambda, \tau \in \Lambda \) with \( \lambda \neq \tau \).
- Let \( f \) and \( g \) be functions with variables \( x_1, \ldots, x_n \). We use “\( f \asymp g \) for any \( (x_1, \ldots, x_n) \in A^n \)” if there exists \( C > 0 \) such that
  \[
  C^{-1}f(x_1, \ldots, x_n) \leq g(x_1, \ldots, x_n) \leq Cf(x_1, \ldots, x_n)
  \]
  for any \( (x_1, \ldots, x_n) \in A \).
- Let \( (X, \mathcal{O}) \) be a topological space and \( A \subseteq X \), then \( A^\circ \) and \( A^c \) denote the interior and complement, respectively.
• Let \((X, d)\) be a metric space and \(\mu\) is a (Borel) measure on \((X, d)\), then we write
\[
B_d(x, r) = \{y | d(x, y) < r\}
\]
and
\[
V_{d, \mu}(x, r) = \mu(B_d(x, r)).
\]
Moreover, let \(\rho\) be a distance on \(X\), then we write
\[
d_{\rho}(x, r) = \sup_{y \in B_{\rho}(x, r)} d(x, y).
\]
We also use the notation \(\text{diam}(X, d) = \sup\{d(x, y) | x, y \in X\}\). If no confusion may occur, we omit \(d, \rho\) or \(\mu\) in these notations.

• We also use the notations \([n, m] = \{k \in \mathbb{Z} | n \leq k \leq m\}\) for \(n, m \in \mathbb{Z}\), \(a \vee b = \max\{a, b\}\) and \(a \wedge b = \min\{a, b\}\).

The following definitions are basic notions of graph, but we write that for confirmation of our notations.

**Definition 2.1** (graph, tree). Let \(T\) be a (at most) countable set and let \(A \subseteq T \times T\) such that

- for any \(w \in T\), \((w, w) \notin A\).
- \((w, v) \in A\) if \((v, w) \in A\).

We call \((T, A)\) a simple graph. We write \(w \sim v\) if \((w, v) \in A\).

1. \((T, A)\) is called locally finite if \(#(\{y | y \sim x\}) < \infty\) for any \(x \in T\). \((T, A)\) is called bounded degree if \(\sup_{x \in T} #(\{y | y \sim x\}) < \infty\).

2. Let \(n \geq 0\). We call \((w_0, w_1, ..., w_n) \in T^n\) a \(n\)-path (between \(w_0\) and \(w_n\)) if \(w_i \sim w_{i-1}\) for any \(i \in [1, n]_{\mathbb{Z}}\). Especially, we call \((w_0, w_1, ..., w_n)\) a \(n\)-simple path (between \(w_0\) and \(w_n\)) if it is a \(n\)-path and \(w_i \neq w_j\) whenever \(i \neq j\). \((w_0, w_1, ..., w_n)\) is called a path if it is a \(n\)-path for some \(n \geq 0\), and called a simple path if it is a \(n\)-simple path for some \(n \geq 0\).

3. We call \((T, A)\) connected if there exists a path between \(w\) and \(v\) for any \(w, v \in T\). Moreover, we call \((T, A)\) a tree if there exists an unique simple path between \(w\) and \(v\) for any \(w, v \in T\).

4. Let \((T, A)\) be a simple graph. We define \(l_A\) by
\[
l_A(w, v) = \min\{n | \text{there exists an } n\text{-path between } w \text{ and } v\}.
\]
If \((T, A)\) is connected, then \(l_A\) is called the graph metric of \((T, A)\).

In this paper, we will consider only simple graphs.

**Definition 2.2** (rooted tree). Let \((T, A)\) be a tree and \(\phi \in T\). We call the triple \((T, A, \phi)\) a rooted tree.
(1) Define $|w| = L_A(\phi, w)$ and $(T)_n = \{w \mid |w| = n\}$ for any $n \geq 0$, and define

$$
\pi : T \times T \text{ by } \pi(w) = \pi_{T,A,\phi}(w) = \begin{cases} w_{n-1} & \text{if } w \neq \phi \text{ and } (\phi = w_0, \ldots, w_{n-1}, w_n = w) \text{ is} \\
\phi & \text{if } w = \phi.
\end{cases}
$$

$S$ denote the inverse of $\pi$ (excluding $\phi$), namely

$$
S(A) = \{w \in T \setminus \{\phi\} \mid \pi(w) \in A\}
$$

for any $A \subseteq X$, and we write $S(w)$ instead of $S(\{w\})$. Moreover, we define subtree $T_w = \{v \in T \mid \pi^n(v) = w \text{ for some } n \geq 0\}$.

(2) Define geodesics of $T$ (from $\phi$) by

$$
\Sigma = \{\omega = (\omega_n)_{n \geq 0} \mid \omega_n \in (T)_n, \pi(\omega_i+1) = \omega_i \text{ for all } i \geq 0\}
$$

and geodesics passing through $w$ by $\Sigma_w = \{\omega \in \Sigma | \omega_{|w|} = w\}$ for any $w \in T$.

Remark. In [11], Kigami used the terminology "tree with a reference point" instead of "rooted tree". However, we want to use the term "a reference point" to distinguish trees, which have no root but have a reference point of height, introduced in Section 3.

In the following, whenever $d \sim_{QS} \rho$ for metrics $d, \rho$ on a space, $\theta$ denotes a homeomorphism such that $d$ is $\theta$-quasisymmetric to $\rho$, if no confusion may occur.

We will use the same notations to a vertex and its equivalent class.

2.2 Kigami’s Results for Compact Metric Spaces

Throughout this section, $T = (T, A, \phi)$ is a locally finite rooted tree.

Definition 2.3 (partition). Let $(X, O)$ be a compact metrizable space having no isolated points, and let $\mathcal{C}(X, O)$ be the collection of nonempty compact subsets of $(X, O)$ without single points. A map $K : T \to \mathcal{C}(X, O)$, where we write $K_w$ instead of $K(w)$ for ease of notation, is called a partition of $(X, O)$ parametrized by $T$ if it satisfies the following conditions.

(P1) $K_{\phi} = X$ and for any $w \in T$,

$$
\bigcup_{v \in S(w)} K_v = K_w.
$$

(P2) For any $\omega \in \Sigma$, $\cap_{m \geq 0} K_{\omega_m}$ is a single point.
(1) Let $K$ be a partition of $X$. We define $O_w$ by

$$O_w = K_w \setminus \left( \bigcup_{v: v \in (T)_w \setminus \{w\}} K_v \right).$$

$K$ is called minimal if $O_w \neq \emptyset$ for any $w \in T$.

(2) For $m \geq 0$, we define $E^n_m \subseteq (T)_m \times (T)_m$ by

$$J^n_m = \{ (w, v) \mid w, v \in (T)_m, \ w \neq v \ \text{and} \ K_w \cap K_v \neq \emptyset \}$$

and $\Gamma_n(w) = \{ v \in (T)_w \mid l^n_{J_m}(w, v) \leq n \}$ for any $w \in T$.

We simply write $X$ for $(X, O)$ in notations if no confusion may occur.

**Definition 2.4 (weight function).** A function $g : T \rightarrow (0, 1]$ is called a weight function if it satisfies the following conditions.

(G1) $g(\phi) = 1$.

(G2) For any $w \in T$, $g(\pi(w)) \geq g(w)$.

(G3) For any $\omega \in \Sigma$, $\lim_{m \rightarrow \infty} g(\omega_m) = 0$.

(1) For $s > 0$, we define the scale $\Lambda^g_s$ associated $g$ by

$$\Lambda^g_s = \begin{cases} 
\{ w \in T | g(w) \leq s < g(\pi(w)) \} & \text{if } 0 < s < 1 \\
\{ \phi \} & \text{otherwise}
\end{cases}$$

and define $E^g_s \subseteq \Lambda^g_s \times \Lambda^g_s$ by

$$E^g_s = \{ (w, v) \mid w, v \in \Lambda^g_s, \ w \neq v \ \text{and} \ K_w \cap K_v \neq \emptyset \}.$$ 

(2) For $x \in X$, $s > 0$, $M \geq 0$ and $w \in \Lambda^g_s$, we define

$$\Lambda^g_{s,M}(w) = \{ v \in \Lambda^g_s \mid \lambda^g_s(w, v) \leq M \}, \ \Lambda^g_{s,M}(x) = \bigcup_{w \in \Lambda^g_s \text{ and } x \in K_w} \Lambda^g_{s,M}(w)$$

and

$$U^g_{M}(x, s) = \bigcup_{w \in \Lambda^g_{s,M}(x)} K_w.$$ 

**Definition 2.5.** Let $(X, O)$ be a compact metrizable space having no isolated point, and $K$ be a partition of $X$. Define

$$\mathcal{D}(X, O) = \{ d \mid d \text{ is a metric on } X \text{ inducing the topology } O \text{ and } \ \text{diam}(X, d) = 1 \}.$$ 

For $d \in \mathcal{D}(X, O)$, define $g_d : T \rightarrow (0, 1]$ by $g_d(w) = \text{diam}(K_w, d)$ for any $w \in T$. 

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**Proposition 2.6** ([11], Proposition 5.5(1)). Let \((X, \mathcal{O})\) be a compact metrizable space having no isolated point and \(K\) be a partition of \(X\). For any \(d \in \mathcal{D}(X, \mathcal{O})\), \(g_d\) is a weight function.

**Proof.**

(G1) By \((P1)\), \(g_d(\phi) = \text{diam}(X, d) = 1\).

(G2) By \((P1)\), \(K_{\pi(w)} \supseteq K_w\) and hence

\[
g_d(\pi(w)) = \text{diam}(K_{\pi(w)}, d) \geq \text{diam}(K_w, d) = g_d(w).
\]

(G3) Since \(\{g_d(\omega_n)\}_{n \geq 0}\) is decreasing, it must converge if \(n \to \infty\). Assume \(\lim_{n \to \infty} g_d(\omega_n) = c > 0\) for some \(\omega \in \Sigma\). Then there exist \(\{x_m\}_{m \geq 0}\) and \(\{y_m\}_{m \geq 0}\) such that \(x_m, y_m \in K_{\omega_m}\) and \(d(x_m, y_m) \geq c\) because \((G2)\) holds. We can take convergent subsequences \(\{x_{m_k}\}_{k \geq 0}\), \(\{y_{m_k}\}_{k \geq 0}\) because \(X\) is compact. Since \(\{K_{\omega_m}\}_{m \geq 0}\) is a decreasing sequence of compact sets and \(\{x_{m_k}\}_{k \geq n}, \{y_{m_k}\}_{k \geq n} \subseteq K_{\omega_m}\) for any \(n\), it follows that \(x, y \in \bigcap_{m \geq 0} K_{\omega_m}\) where \(x = \lim_{k \to \infty} x_{m_k}, y = \lim_{k \to \infty} y_{m_k}\). Since \((P2)\) holds, \(x = y\) and hence

\[
0 < c \leq \lim_{k \to \infty} d(x_{m_k}, y_{m_k}) = d(x, y) = 0.
\]

This is a contradiction.

**Example 2.7** (Sierpiński carpet). Let \(\{p_i\}_{i=1}^8 \subseteq \mathbb{C}\) such that

\[
\begin{align*}
p_1 &= 0, & p_2 &= \frac{1}{2}, & p_3 &= 1, & p_4 &= 1 + \frac{1}{2}i, \\
p_5 &= 1 + i, & p_6 &= \frac{1}{2} + i, & p_7 &= i, & p_8 &= \frac{1}{2}i
\end{align*}
\]

and let \(F_i = \frac{1}{3}(z - p_i) + p_i\) for any \(i \in [1, 8]_{\mathbb{Z}}\). It is well-known that there exists the unique compact set \(X\) such that \(\bigcup_{i=1}^8 F_i(X) = X\), called Sierpiński carpet. Let \(T = \bigcup_{n \geq 0} ([1, 8]_{\mathbb{Z}})^n\) where \(([1, 8]_{\mathbb{Z}})^0 = \{\phi\}\) and define \(\pi : T \setminus \{\phi\} \to T\) by

\[
\pi(w) = \begin{cases} 
(w_1, w_2, \ldots, w_{n-1}) & \text{if } w = (w_1, w_2, \ldots, w_{n-1}, w_n) \in \bigcup_{n \geq 2} ([1, 8]_{\mathbb{Z}})^n \\
\phi & \text{if } w \in [1, 8]_{\mathbb{Z}}^0
\end{cases}
\]

and \(A = \{(w, v) \mid w = \pi(v) \text{ or } v = \pi(w)\}\). Then \(T = (T, A, \phi)\) is a rooted tree. Moreover, for \(w \in T\) define \(F_w : \mathbb{C} \to \mathbb{C}\) by

\[
F_w = \begin{cases} 
F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_n} & \text{if } w = (w_1, w_2, \ldots, w_n) \in \bigcup_{n \geq 1} [1, 8]_{\mathbb{Z}}^n \\
\text{id}_{\mathbb{C}} & \text{if } w = \phi
\end{cases}
\]

and define \(K : T \to \mathcal{C}(X, \mathcal{O})\) by \(K_w = F_w(X)\). Then \(K\) is a partition of \(X\) parametrized by \(T\). We also let \(d(z, w) = \frac{\sqrt{2}}{2} |z - w|\), then \(d \in \mathcal{D}(X, \mathcal{O})\) and by Proposition 2.6, \(g_d\) is a weight function.
We denote $g_d$ by $d$ if no confusion may occur. For example, we will use notations $U^d_M(x, r)$ and $d(w)$ instead of $U^d_M(x, r)$ and $g_d(w)$, respectively. For the purpose of stating the main result of [11], we introduce some properties of weight functions and metrics.

In the rest of this section, $(X, \mathcal{O})$ is a compact metrizable space and $d \in \mathcal{D}(X, \mathcal{O})$ (in other words, $(X,d)$ is a compact metric space with $\text{diam}(X,d) = 1$, and $\mathcal{O}$ is the induced topology). Moreover, $K$ is a partition of $X$.

**Definition 2.8.** Let $g$ be a weight function.

- $g$ is called uniformly finite if
  \[
  \sup \{ \#(\Lambda^g_{s,1}(w)) \mid s > 0, w \in \Lambda^g_s \} < \infty. \tag{2.1}
  \]

- $g$ is called thick (with respect to $K$) if there exists $\alpha > 0$ such that for any $w \in T$, $U^g_M(x, \alpha g(\pi(w))) \subseteq K_w$ for some $x \in K_w$.

$d$ is called uniformly finite and thick if $g_d$ is uniformly finite and thick, respectively.

**Definition 2.9.** Let $M \geq 1$. $d$ is called $M$-adapted if there exists $\alpha_1, \alpha_2 > 0$ such that
\[
B_d(x, \alpha_1 r) \subseteq U^d_M(x, r) \subseteq B_d(x, \alpha_2 r)
\]
for any $x \in X$ and $r \leq 1$. $d$ is called adapted if $d$ is $M$-adapted for some $M$.

**Example 2.10.** Let $X$ is the Sierpiński carpet and define $T, K, F$ and $d$ as in Example 2.7. We can see that $d(w) = \text{diam}(F_w(X), d) = 3^{-|w|}$, and hence $\Lambda^d_s = (T)^m_s$ for any $s \in (0,1)$ and $m \geq 1$ such that $3^{-m} \leq s < 3^{-(m-1)}$.

- $d(\pi(w)) = 3d(w)$ for any $w \in T \setminus \{\phi\}$.
- If $K_w \cap K_v \neq \emptyset$ and $|w| = |v|$, then
\[
K_v \in \{K_w + a3^{-|w|} + b3^{-|w|}i \mid (a,b) \in \mathbb{R} \setminus \{(0,0)\}\} \tag{2.2}
\]
where $K_w + z = \{x+z \mid x \in K_w\}$. Therefore $\Lambda^d_{s,1}(w) \leq 8$ for any $s \in (0,1)$ and $w \in \Lambda^d_s$, and hence $d$ is uniformly finite.

- Let $s \in (0,1)$ and $w \in \Lambda^d_s$. We also let $x_w = F_w(\frac{2}{3} + \frac{2}{3}i)$. Then by (2.2) and $\Lambda^d_{3-2s} = (T)^{|w|+2}$, $\Lambda^d_{4-2s,1}(x_w) \subseteq B_d(x_w, \sqrt{2}3^{-|w|+1}) \subseteq K_w$. Therefore $d$ is thick.

- By (2.2) and definition of $K$,
\[
B_d(x, \frac{\sqrt{2}}{6} s) \subseteq B_d(x, \frac{\sqrt{2}}{2} 3^{-m}) \subseteq U^d_M(x, s) \subseteq B_d(x, 3 \cdot 3^{-m}) \subseteq B_d(x, 3s)
\]
for any $x \in X, s \in (0,1)$ and $m \geq 1$ such that $3^{-m} \leq s < 3^{-(m-1)}$. Therefore $d$ is $(1)$-adapted.
Definition 2.11. Define

\[ J_{N,m}^h = \{(w,v) \mid w,v \in (T)_m \text{ such that } 0 < l_{P_m}(w,v) \leq N\} \]

and

\[
\mathcal{E}_{p,k,w}(N_1,N_2,N) = \inf\left\{ \frac{1}{2} \sum_{(x,y) \in \mathcal{P}_w} |f(x) - f(y)|^p \mid f : (T)_{|w|+k} \to \mathbb{R} \text{ such that } f|S^k(\Gamma_{N_1}(w)) \equiv 1, f|S^k(\Gamma_{N_2}(w)) \equiv 0 \right\}
\]

for any \( N \geq 1 \) and \( N_2 \geq N_1 \geq 0 \) (remark that \( J_{1,m}^h = J_{m}^h \)). We also define

\[
\mathcal{E}_{p,k}(N_1,N_2,N) = \sup_{w \in T} \mathcal{E}_{p,k,w}(N_1,N_2,N),
\]

\[
\overline{T}_p(N_1,N_2,N) = \limsup_{k \to \infty} \mathcal{E}_{p,k}(N_1,N_2,N),
\]

\[
\underline{T}_p(N_1,N_2,N) = \liminf_{k \to \infty} \mathcal{E}_{p,k}(N_1,N_2,N),
\]

\[
\mathcal{I}_p(N_1,N_2,N) = \inf\left\{ p \mid \mathcal{E}_p(N_1,N_2,N) = 0 \right\},
\]

\[
\mathcal{I}_p(N_1,N_2,N) = \inf\left\{ p \mid \mathcal{E}_p(N_1,N_2,N) = 0 \right\}.
\]

For ease of notations, we define the conditions of a partition, which repeatedly appears in this paper, basic framework same as [11], that is,

Definition 2.12. Let \((X,d)\) be a metric space and assume \(K\) is minimal. We say \(d\) satisfies basic framework (with respect to \(K\)) if the following conditions hold.

- \(\sup_{w \in T} \#(S(w)) < \infty\).
- \(d\) is uniformly finite, thick, \(M_*\)-adapted for some \(M_* \geq 1\).
- There exists \(r \in (0,1)\) such that \(d \asymp r^{|w|}\) for any \(w \in T\).

Theorem 2.13 ([11], Theorem 19.4 and 19.9.). If \(K\) is minimal and \(d\) satisfies basic framework, then

\[
\underline{T}_p(N_1,N_2,N) = \overline{T}_p(N_1,N_2,N) = \dim_{AR}(X,d)
\]

for any \(N,N_1,N_2\) with \(N_2 \geq N_1 + M_*\).

As a corollary of Theorem 2.13, we get the result for comparison between \(\dim_{AR}\) and "\(p\)-spectral" dimension, defined for any \(p > 0\), as follows.

Definition 2.14 (\(p\)-spectral dimension). Define

\[
\overline{N}_p = \limsup_{k \to \infty} \sup_{w \in T} \#(S^k(w))^{1/k}, \quad \overline{R}_p(N_1,N_2,N) = \limsup_{k \to \infty} \mathcal{E}_{p,k}(N_1,N_2,N)^{1/k},
\]

\[
\underline{R}_p(N_1,N_2,N) = \liminf_{k \to \infty} \mathcal{E}_{p,k}(N_1,N_2,N)^{1/k}.\]
and upper $p$-spectral dimension $d^S_p(N_1, N_2, N)$ and lower $p$-spectral dimension $d^L_p(N_1, N_2, N)$ by

$$d^S_p(N_1, N_2, N) = \frac{p \log N_*}{\log N_* - \log R_p(N_1, N_2, N)},$$

$$d^L_p(N_1, N_2, N) = \frac{p \log N_*}{\log N_* - \log R_p(N_1, N_2, N)}.$$

**Corollary 2.15** ([11] Theorem 20.8). If $K$ is minimal and $d$ satisfies basic framework, then

1. if $R_p(N_1, N_2, N) < 1$, then
   $$\dim_{AR}(X, d) \leq d^S_p(N_1, N_2, N) \leq d^L_p(N_1, N_2, N) < p$$

2. If $R_p(N_1, N_2, N) \geq 1$, then
   $$\dim_{AR}(X, d) \geq d^S_p(N_1, N_2, N) \geq d^L_p(N_1, N_2, N) \geq p$$

**Remark.** Originally in [11], these theorem are also proved for some broader framework, called proper system of horizontal networks.

## 3 Extension to Cases of $\sigma$-compact Metric Spaces

To use the former result for infinite graphs, we first extend the theory to $\sigma$-compact spaces. To do that, we introduce the notion of a bi-infinite tree.

**Definition 3.1.** Let $T$ be a countable set and $\pi_T : T \to T$ be a map which satisfies the following.

(T1) For any $w, v \in T$, there exists $n, m \geq 0$, $\pi^n(w) = \pi^m(v)$.

(T2) For any $n \geq 1$, and $w \in T$, $\pi^n(w) \neq w$.

Then we define $A_\pi = \{(w, v) \mid \pi(w) = v \text{ or } \pi(v) = w\}$ and consider the simple graph $(T, A_\pi)$. We call $(T, \pi)$ a bi-infinite tree.

Same as a rooted tree, we denote the inverse of $\pi$ by $S$ and write $S(w)$ instead of $S(\{w\})$. Moreover, we define subtree $T_w = \{v \in T \mid \pi^n(v) = w \text{ for some } n \geq 0\}$.

**Lemma 3.2.** Let $(T, \pi)$ be a bi-infinite tree and fix $w, v \in T$. Then $n - m$ is constant if $\pi^n(w) = \pi^m(v)$.

**Proof.** Assume $\pi^n(w) = \pi^{m_1}(v)$ for some $i = 1, 2$. Without loss of generality, we may assume $n_1 \leq n_2$, then

$$\pi^{m_2}(v) = \pi^{n_2}(w) = \pi^{n_2-n_1} \circ \pi^{n_1}(w) = \pi^{n_2-n_1} \circ \pi^{m_1}(v)$$

and so by (T2), we get $m_2 = (n_2 - n_1) + m_1$. It means $m_2 - n_2 = m_1 - n_1$. \qed
Definition 3.3. Let \((T, \pi)\) be a bi-infinite tree.

(1) Let \(\phi \in T\). We call the triple \((T, \pi, \phi)\) a bi-infinite tree with a reference point \(\phi\), and for any \(w \in T\) we define the height of vertices by \([w] = [w]_\phi = n - m\) such that \(\pi^n(w) = \pi^m(\phi)\). We also define \((T)_n\) by \((T)_n = \{w \in T \mid [w] = n\}\) for any \(n \in \mathbb{Z}\).

(2) Define \((\text{descending})\) geodesics of \(T\) by \(\Sigma^* = \{\omega = (\omega_n)_{n \in \mathbb{Z}} \mid \omega_n \in (T)_n, \pi(\omega_{n+1}) = \omega_n \text{ for all } n \in \mathbb{Z}\}\) and geodesics passing through \(w\) by \(\Sigma^*_w = \{\omega \in \Sigma^* \mid \omega[w] = w\}\) for any \(w \in T\).

Remark. (T1) and Lemma \(3.2\) ensure that \([w]\) is well-defined. Moreover, for fixed \(w, v \in T\), \(|[w]_\phi - [v]_\phi|\) is constant for every \(\phi \in T\) by Lemma 2.2 (that is, the difference of the height of a bi-infinite tree is determined only by \(\pi\) and does not depend on its reference point).

As the name shows, a bi-infinite tree is a tree.

Proposition 3.4. Let \((T, \pi)\) be a bi-infinite tree, then \((T, \mathcal{A}_\pi)\) is a tree.

Proof. For all \(w, v \in T\) there exists a path between them by (T1), and also a simple path exists.

Next we prove the uniqueness of a simple path. Fix any reference point \(\phi \in T\) and think \((T, \mathcal{A}, \phi)\). By definition, if \(w = \pi(v)\) then \([w] = [v] - 1\). Let \((w_0, w_1, ..., w_n)\) be a simple path. If \([w_i] > [w_{i+1}]\), then \(\pi(w_i) = w_{i+1}\) and so \(w_{i-1} \neq \pi(w_i)\), it means \([w_{i-1}] > [w_i]\). In the same way, we can see \([w_{i+1}] > [w_i]\) if \([w_i] > [w_{i-1}]\).

Now we let \((w, \pi(w), ..., \pi^n(w) = \pi^{m_1}(v), ..., \pi(v), v)\) and \((w, \pi(w), ..., \pi^{m_2}(w) = \pi^{m_2}(v), ..., \pi(v), v)\) be simple paths. If \(n_1 < n_2\) then \(m_1 < m_2\) by Lemma \(3.2\), so the latter simple path take \(\pi^{n_1}(w) = \pi^{m_1}(v)\) two times, which is contradiction. The case \(n_1 > n_2\) is the same. If \(n_1 = n_2\) then \(m_1 = m_2\) by Lemma \(3.2\), so the paths are equal. Therefore \((T, \mathcal{A})\) is a tree. \(\Box\)

The root of a bi-infinite tree does not exist, but \(\pi^\infty(w)\) is thought to be a virtual root.

Now we extend notions of partitions and weight functions to \(\sigma\)-compact spaces. In the rest of this section, let \(T = (T, \pi, \phi)\) be a locally finite bi-infinite tree with a reference point. Note that \((T, \pi)\) is locally finite if and only if \(\#(S(w)) < \infty\) for any \(w \in T\).

Definition 3.5 (partition). Let \((X, \mathcal{O})\) be a \(\sigma\) compact metrizable space having no isolated points, and let \(\mathcal{C}(X, \mathcal{O})\) be the collection of nonempty compact subsets of \((X, \mathcal{O})\) without single points. A map \(K : T \rightarrow \mathcal{C}(X, \mathcal{O})\), where we denote \(K(w)\) by \(k_w\) for ease of notation, is called a partition of \((X, \mathcal{O})\) parametrized by \(T\) if it satisfies the following conditions.

(P1) For any \(w \in T\),
\[
\bigcup_{v \in S(w)} K_v = K_w.
\]
For any $\omega \in \Sigma^*$, $\cap_{m \geq 0} K_{\omega_m}$ is a single point.

(P3) $\cup_{w \in (T)_0} K(w) = X$.

We say a partition $K$ is locally finite if it satisfies the following:

For any $w \in (T)_0$, there exists an open set $U_w$ which satisfies $K_w \subset U_w$ and $\# \{ v \in (T)_0 | K_v \cap U_w \neq \emptyset \} < \infty$.

We define $O_w, J^h_M, \Gamma_n(w)$ and minimality in the same way as compact cases, and similarly use $X$ instead of $(X, \mathcal{O})$.

Remark. (P3) is the counterparts of $K_{\phi} = X$ in Definition 2.3. The locally finiteness of a partition is used for reducing local properties of partitions of $\sigma$-compact spaces to compact cases.

**Definition 3.6** (weight function). A function $g : T \to (0, \infty)$ is called a weight function if it satisfies the following conditions.

(G1) $\lim_{n \to \infty} g(\pi^n(\phi)) = \infty$.

(G2) For any $w \in T$, $g(\pi(w)) \geq g(w)$.

(G3) For any $\omega \in \Sigma^*$, $\lim_{m \to \infty} g(\omega_m) = 0$.

For $s > 0$, we define the scale $\Lambda^g_s$ associated with $g$ by

$$\Lambda^g_s = \{ w \in T | g(w) \leq s < g(\pi(w)) \}$$

and define $E^g_s, \Lambda^g_{s,M}(w), \Lambda^g_{s,M}(x), U^g_M(x,s)$ in the same way as Definition 2.4.

**Definition 3.7.** Let $(X, \mathcal{O})$ be a $\sigma$-compact metrizable space having no isolated points and $K$ be a partition of $X$.

(1) Define

$$\mathcal{D}_\infty(X, \mathcal{O}) = \{ d \mid d \text{ is a metric on } X \text{ inducing the topology } \mathcal{O} \text{ and } \text{diam}(X, d) = \infty \}.$$  

For $d \in \mathcal{D}_\infty(X, \mathcal{O})$, define $g_d : T \to (0, \infty)$ by $g_d(w) = \text{diam}(X, d)$ for any $w \in T$.

(2) Define

$$\mathcal{M}_\infty(X, \mathcal{O}) = \{ \mu \mid \mu \text{ is a Radon measure on } (X, \mathcal{O}) \text{ such that } \mu(X) = \infty, \mu\{x\} = 0 \text{ for any } x \in X \text{ and } \mu(K_w) > 0 \text{ for any } w \in T \}.$$  

For $\mu \in \mathcal{M}_\infty(X, \mathcal{O})$, define $g_\mu : T \to (0, \infty)$ by $g_\mu(w) = \mu(K_w)$ for any $w \in T$. 

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Proposition 3.8. Let \((X, \mathcal{O})\) be a \(\sigma\)-compact metrizable space having no isolated point and \(K\) be a partition of \(X\).

(1) For any \(d \in \mathcal{D}_\infty(X, \mathcal{O})\), \(g_d\) is a weight function.

(2) For any \(\mu \in \mathcal{M}_\infty(X, \mathcal{O})\), \(g_\mu\) is a weight function.

Proof. (1) Fix \(x \in K_\phi\) and any \(r > 0\). Since \(\text{diam}(X, d) = \infty\), there exists \(y \in X\) such that \(d(x, y) > r\), and that there exists \(v \in (T)_0\) such that \(y \in K_v\) because of (P3). Since (T2) holds, there exists \(n_0 \geq 0\) such that \(\pi_{n_0}(\phi) = \pi_{n_0}(v)\), and then \(g_d(\pi_n(\phi)) \geq g_d(\pi_{n_0}(\phi)) \geq r\) for any \(n \geq n_0\), so (G1) holds. (G2) and (G3) can be proved in the same way as in the proof of Proposition 2.6.

(2) Since \(\mu\) is locally finite, so (2) immediately follows from general property of a measure and (P2).

We denote \(g_d\) by \(d\) if no confusion may occur. In the rest of this section, \((X, \mathcal{O})\) is a \(\sigma\)-compact metrizable space having no isolated points and \(d \in \mathcal{D}_\infty(X, \mathcal{O})\). Moreover, \(K\) is a partition of \(X\).

We introduce properties of \(d\) and a weight function \(g\) in the same way as Definition 2.8, 2.9 and 2.12, and also introduce variables in the same way as Definition 2.11 and 2.14. With using bi-infinite tree and these properties, we can extend the theory of [11] to \(\sigma\)-compact spaces. In particular, we get the following result.

Theorem 3.9. Assume \(K\) is locally finite and minimal. If \(d\) satisfies basic framework, then for any \(N, N_1, N_2\) with \(N_2 \geq N_1 + M_\mu\),

(1) \[L_\epsilon(N_1, N_2, N) = T_\epsilon(N_1, N_2, N) = \dim_{AR}(X, d).\]

(2) If \(R_p(N_1, N_2, N) < 1\), then
\[\dim_{AR}(X, d) \leq d_p^\delta(N_1, N_2, N) \leq d_p^\delta(N_1, N_2, N) < p.\]

(3) If \(R_p(N_1, N_2, N) \geq 1\), then
\[\dim_{AR}(X, d) \geq d_p^\delta(N_1, N_2, N) \geq d_p^\delta(N_1, N_2, N) \geq p.\]

Proof. Most of the theory in [11] does not use the property \(T = T_\phi\) or compactness of \(X\). Therefore if \(K\) is locally finite, then with using \(\Sigma^*, \mathcal{D}_\infty(X, \mathcal{O}), \mathcal{M}_\infty(X, \mathcal{O}), \omega_m\) and \((0, \infty)\) instead of \(\Sigma, \mathcal{D}(X, \mathcal{O}), \mathcal{M}_P(X, \mathcal{O}), [\omega]_m\) and \((0, 1]\) respectively, we can prove \(\sigma\)-compact version of all but following statements in [11] line by line in the same way:

- Lemma 4.2, 18.3,
- Proposition 3.8, 4.9(7.2), 5.2, 5.5, 5.7, 5.9,
- Theorem 4.7, 7.9, 7.12, 12.9, 15.2, 18.1,
Corollary 7.13.

We need some modification of statements or proofs for \(\sigma\)-compact version of these statements. Note that for proving our Theorem 3.9 (included in \(\sigma\)-compact versions of [11] Theorem 19.9 and Theorem 20.8), we do not need counterparts of [11] Theorem 4.2, Theorem 12.9 and statements in [11] Section 7, so we prove counterparts of the others. Moreover, we have proved the counterpart of [11] Proposition 5.5 as Proposition 3.8. Recall that \(\mathcal{T} = (T, \pi, \phi)\) is a bi-infinite tree, \((X, \mathcal{O})\) is a \(\sigma\)-compact space having no isolated points, \(K\) is a partition of \(X\) parametrized by \(T\) and \(d \in D_\infty(X, \mathcal{O})\).

Proposition 3.10 (the counterpart of [11] Proposition 3.8). Define \(\rho_*\) on \(\Sigma^* \times \Sigma^*\) by

\[
\rho_*(\omega, \tau) = \begin{cases} 
2^{- \max \{m \in \mathbb{Z} | \omega_m = \tau_m\}} & \text{if } \omega \neq \tau, \\
0 & \text{if } \omega = \tau.
\end{cases}
\]

Then \((\Sigma^*, \rho_*)\) is a totally disconnected \(\sigma\)-compact metric space. Moreover, if \(#(S(w)) \geq 2\) for any \(w \in T\), then \((\Sigma^*, \rho_*)\) is perfect.

The proof of this statement is standard.

Lemma 3.11. The following conditions are equal:

1. A partition \(K\) on \(X\) is locally finite.
2. For any \(w \in T\), there exists an open set \(U_w\) such that \(K_w \subset U_w\) and \(#\{v \in (T)_w | K_v \cap U_w \neq \emptyset\} < \infty\).
3. For any \(w \in T\), there exists an open set \(U_w\) such that \(K_w \subset U_w\) and \(#\{v \in (T)_0 | K_v \cap U_w \neq \emptyset\} < \infty\).

Since \(T\) is locally finite, this lemma is easily proven in (1) \(\Rightarrow\) (3) \(\Rightarrow\) (2) \(\Rightarrow\) (1).

Lemma 3.12 (the counterpart of [11] Lemma 4.2). Assume \(K\) is locally finite, then

1. For any \(w \in T\), \(O_w\) is an open set and \(O_v \subseteq O_w\) for any \(v \in S(w)\).
2. \(O_w \cap O_v = \emptyset\) if \(w, v \in T\) and \(\Sigma_* \cap \Sigma_* = \emptyset\).
3. If \(\Sigma_* \cap \Sigma_* = \emptyset\), then \(K_w \cap K_v = B_w \cap B_v\) where \(B_w = K_w \setminus O_w\).

Proof. (1) Since \(K\) is locally finite,

\[
O_w = K_w \setminus \left( \cup_{v \in (T)_w} K_v \cap U_w \neq \emptyset K_v \right) = U_w \setminus \left( \cup_{v \in (T)_w} K_v \cap U_w \neq \emptyset K_v \right)
\]

and \(O_w\) is open. The rest of the statement follows from (P2).

2. If \([v] \leq [w]\), \(\Sigma_* \cap \Sigma_* = \emptyset\) implies \(\pi[v] - [w](v) \neq w\) and by (1), \(O_w \cap O_v \subseteq O_w \cap O_{\pi[v] - [w](v)} = \emptyset\). The case \([v] \leq [w]\) is the same.

3. This immediately follows from (2).

\(\square\)
Lemma 3.13. Let $w \in T$. Then the triple $T_w = (T_w, A|_{T_w \times T_w}, w)$ is a rooted tree. Moreover, for any $v \in T_w \setminus \{w\}$, $\pi_T(v) = \pi_{T_v}(v).

Proof. It is trivial to prove $(T_w, A|_{T_w \times T_w}, w)$ to be a tree. Let $v \in T_w \setminus \{w\}$ then there is some $n \geq 1$ and $(v, \pi_T(v), \ldots, \pi_T^{(n)}(v) = w)$, which is the geodesic between $w$ and $v$. Therefore $\pi_T(v) = \pi_{T_v}(v)$.

Proposition 3.14 (the counterpart of [11 Proposition 5.2]). Suppose that $g : T \to (0, \infty)$ satisfies (G1) and (G2). Then $g \in G(T)$ if and only if $g$ satisfies the following condition:

(G3)' For any $w \in T$, $\lim_{n \to \infty} \sup_{v \in T_w \cap (T)|w|\in n}^{} g(v) = 0$.

Proof. (G3)' immediately shows (G3). On the other hand, if $g$ is a weight function on $T$, then $g'$ defined by $g'(v) = g(v)/g(w)$ is a weight function on $T_w = (T_w, A|_{T_w \times T_w}, w)$. Therefore [11 Proposition 5.2] shows (G3)'.

Remark. In [11], the definition of weight function on a rooted tree is given by the counterpart of (G3)' instead of (G3).

Lemma 3.15. Let $g$ be a weight function on $T$. Then for any $s \in (0, \infty), w \in \Lambda^g_{s,M}$, there exists $v \in T$ such that $\Lambda^g_{s,M}(w) \subset T_v$.

Proof. Let $\hat{\Lambda}^g_{s,n}(w) := \{\pi^{[0,\infty]}_T(v)|v \in \Lambda^g_{s,n}(w)\}$, then $\#\hat{\Lambda}^g_{s,n}(w) \leq \infty$ because $K$ is locally finite. And inductively we define

\[
\hat{\Lambda}^g_{s,n+1}(w) := \bigcup\{\hat{\Lambda}^g_{s,n}(w')|v \in \hat{\Lambda}^g_{s,n}(w), v' \in \Lambda^g_{g(v)}(w)\} \text{ and } v \in T_{v'},
\]

then $\#\hat{\Lambda}^g_{s,M}(w) \leq \infty$ for all $M$.

Moreover, for any $v \in \Lambda^g_{s,n}(w)$ there exists $v' \in \hat{\Lambda}^g_{s,n}(w)$ such that $v \in T_{v'}$. Therefore $\Lambda^g_{s,M}(w) \cup \bigcup_{v \in \hat{\Lambda}^g_{s,M}(w)} T_v \subset T_u$ for some $u$ because of (T1).

Corollary 3.16. Let $g$ be a weight function on $T$, then for any $x \in X$ and $M \geq 0$, $\#(\Lambda^g_{s,M}(x)) \leq \infty$ for any $s > 0$ and $\min_{w \in \Lambda^g_{s,M}(x)} |w| \to \infty$ as $s \to 0$.

Proof. $\Lambda^g_{s,M}(x) \subset \Lambda^g_{s,M+1}(w) \subset \Lambda^g_{s,M} \cap T_v$ for some $w, v \in T$. Since $\Sigma^g_u$ is compact, $\Sigma^g_u$ is open for any $u \in T$ and $\Sigma^g_u = \bigcup_{u \in \Lambda^g_{s,M} \cap T_v} \Sigma^g_u$, it follows that $\#(\Lambda^g_{s,M}(x)) \leq \#(\Lambda^g_{s,M} \cap T_v) \leq \#(\Lambda^g_{s,M} \cap T_v) < \infty$. The rest of this corollary follows from $\Lambda^g_{s,M}(x) \subset \Lambda^g_{s,M} \cap T_v$ and (G3)'.

By Lemma 3.15 and Corollary 3.16, with using $\Lambda^g_{s,M}(x)$ instead of $\Lambda^g_{s,M}$, we can prove following two propositions in the same way as [11].

Proposition 3.17 (the counterpart of [11 Propostion 5.7]). Let $g : T \to (0, \infty)$ be a weight function. Then for any $s \in (0, \infty), x \in X, U^g_M(x, s)$ is a neighborhood of $x$. Furthermore, $\{U^g_M(x, s)\}_{s \in (0, \infty)}$ is a fundamental system of neighborhood of $x$ for any $x \in X$. 

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Proposition 3.18 (the counterpart of [11 Proposition 5.9]). For any $M \geq 0$ and $x, y \in X$, define $\delta^g_M(x, y)$ by
\[
\delta^g_M(x, y) = \inf\{s | y \in U^g_M(x, s)\},
\]
then the infimum attains for any $M \geq 0$ and $x, y \in X$. In particular, for any $M \geq 0$ and $s \in (0, \infty)$,
\[
U^g_M(x, s) = \{y | \delta^g_M(x, y) \leq s\}.
\]

Definition 3.19 (exponential). Let $g$ be a weight function. $g$ is called exponential if there exist $c \in (0, 1)$ and $m_0 \geq 0$ such that
\[
\begin{align*}
cg(\pi(w)) &\leq g(w) \quad \text{for any } w \in T \text{ and } \quad (3.1) \\
cg(w) &\geq g(v) \quad \text{for any } m \geq m_0, w \in T \text{ and } v \in S^m(w). \quad (3.2)
\end{align*}
\]

$d \in D_\infty(X, O)$ is called exponential if $g_d$ is exponential.

Definition 3.20 (gentle). Let $g$ be a weight function on $T$. Function $g : T \to (0, \infty)$ is called gentle with respect to $g$ if there exists $C > 0$ such that
\[
f(v) \leq Cf(w) \quad \text{whenever } K_v \cap K_w \neq \emptyset \text{ and } K_v, K_w \in \Lambda_g^s \text{ for some } s > 0.
\]
We write $f \sim_{GE} g$ if $f$ is gentle with respect to $g$.

Theorem 3.21 (the counterpart of [11 Theorem 15.2.]). Assume $\sup(\#S(w)) < \infty$ and $d$ is exponential, thick, uniformly finite and $d \sim_{GE} 2^{-[w]}$. Let $\alpha > 0$. Then there exist a metric $\rho \in D_\infty(X, O)$ and a measure $\mu \in \mathcal{P}_\infty(X, O)$ such that $\rho \sim QS d$ and $\mu$ is $\alpha$-Ahlfors regular with respect to $\rho$ if and only if there exists an exponential weight function on $T$ such that
\[
\begin{itemize}
\item $g \sim_{GE} d$,
\item there exists $c > 0$ and $M \geq 1$ such that
\[
\begin{align*}
&cD^g_M(x, y) \leq D^g(x, y) \\
\text{for any } x, y \in X \text{ where}
\end{align*}
\]
\[
D^g_M(x, y) = \inf\left\{\sum_{i=0}^n g(w(i)) \mid n \leq m, x \in K(w(0)), y \in K(w(n)) \right. \\
\left. \quad \text{and } K_{w(i-1)} \cap K_{w(i)} \neq \emptyset \text{ for any } i \in [1, n]_Z\right\},
\]
\item there exists $c > 0$ such that for any $w \in T$ and $n \geq 0$,
\[
c^{-1}g(w)^\alpha \leq \sum_{v \in S^n(w)} g(v)^\alpha \leq cg(w)^\alpha.
\end{itemize}
\]
\]
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Proof. The “only if” part follows from in the same way of [11, Theorem 15.2] by replacing \( \rho(x,y) \) by \( D^\gamma(x,y) \). On the other hand, applying [11, Theorem 15.2] to \( T_w \) and \( g(\cdot)/g(w) \) for \( w \in (T)_{0} \), we obtain a probability measure \( \mu_w \) on \( K_w \) which is \( \alpha \)-Ahlfors regular with respect to \((K_w, D^\gamma)\). Define \( \mu = \sum_{w \in (T)_{0}} g(w)\mu_w \), then \( \mu \) satisfies \( \mu(w) \asymp g(w)^\alpha \) for any \( w \in T \), and in the same way of the original proof, we can see that \( \mu \) is \( \alpha \)-Ahlfors regular with respect to \( D^\gamma \).

**Theorem 3.22** (the counterpart of [11, Theorem 18.1]). Assume \( d \) satisfies basic framework. Let \( M_1 \in \mathbb{N} \) and \( k \) be a sufficiently large number. If there exists \( \varphi : T^{(k)} = \cup_{n \in \mathbb{Z}} (T)_{nk} \to (0,1] \) such that

\[
\sum_{i=1}^{m} \varphi(w(i)) \geq 1 \quad \text{and} \quad \sum_{v \in S^k(w)} \varphi(v)^p < \frac{1}{2} (\sup_{w \in \mathcal{T}} \#(\Gamma_1(w)))^{-2(M_1 + M_\ast)}
\]

for any \( w \in T^{(k)} \) and any path \( (w(1), \ldots, w(m)) \) in \( J_{[w]+k}^h \) such that

- \( w(i) \in S^k(\Gamma_{M_1}(w)) \) for any \( i \in [1, m]_\mathbb{Z} \),
- \( \Gamma_{M_1}(w(1)) \cap S^k(w) \neq \emptyset \) and \( \Gamma_{M_1}(w(m)) \setminus S^k(\Gamma_{M_1}(w)) \neq \emptyset \),

then there exists an exponential metric \( \rho \in \mathcal{D}_\infty(X, \mathcal{O}) \) such that \( \rho \sim d \) and \( \rho \) is \( \alpha \)-Ahlfors regular.

This theorem needs the following lemma.

**Lemma 3.23** (the counterpart of [11, Lemma 18.3]). Let \( k \) be sufficiently large, \( \kappa_0 \in (0,1) \) and let \( f : T^{(k)} \to [\kappa_0, 1) \). Then there exists \( g : T^{(k)} \to (0, \infty) \) such that

\[
g(u) \geq \kappa_0 g(v) \quad \text{for any} \quad (u,v) \in \bigcup_{m \in \mathbb{Z}} J_{mk}^h,
\]

\[
f(u) \leq \frac{g(u)}{g(\pi^k(u))} \leq \max_{v \in \Gamma_{M_\ast}(u)} f(v) \quad \text{for any} \quad u \in T^{(k)} \text{ and}
\]

\[
\sum_{v \in S^k(w)} \left( \frac{g(v)}{g(\pi^k(v))} \right)^p \leq \left( \sup_{w \in \mathcal{T}} \#(\Gamma_1(w)) \right)^{2M_\ast} \sup_{w' \in \Gamma_{M_\ast}(w)} \left( \sum_{u \in S^k(w')} f(u)^p \right)
\]

for any \( p > 0 \) and \( w \in T^{(k)} \).

Proof. Apply [11, Lemma 18.3] to \( T_{\pi nk}^{(k)}(\phi) \) and get \( g_n : T_{\pi nk}^{(k)}(\phi) \to (0,1] \). Let \( \tilde{g}_n := \frac{g_n}{g_n(\phi)} \) then \( \tilde{g}_n(\phi) = 1 \) for all \( n \). Let \( w \in T \), then we take \( l, m \) such that \( \pi_{mk}(\phi) = \pi_{mk}(w) \) and then for any \( n \geq l \),

\[
\left( \prod_{i=0}^{l-1} \max_{v \in \Gamma_{M_\ast}(\pi_{ik}(\phi))} f(v) \right)^{-1} \leq g_n(\pi_{lk}(\phi)) \leq \left( \prod_{i=0}^{l-1} f(\pi_{ik}(\phi)) \right)^{-1}
\]

\[
\left( \prod_{j=0}^{m-1} \max_{v \in \Gamma_{M_\ast}(\pi_{jk}(w))} f(v) \right)^{-1} \leq g_n(\pi_{mk}(w)) \leq \left( \prod_{j=0}^{m-1} f(\pi_{jk}(w)) \right)^{-1}
\]

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therefore there exist \( C_w, \overline{C}_w \) with \( 0 < C_w < \overline{C}_w < \infty \) such that for any \( n \geq l, g_n(w) \in [C_w, \overline{C}_w] \). Using diagonal sequence argument, we get a subsequence of \((g_n)\) and its limit \( g \), which also satisfies all required conditions. 

**proof of Theorem 3.22.** In the same way as proof of [11, Theorem 18.1], we obtain \( \sigma : T^{(k)} \to [n_0, 1) \). We inductively define \( \tilde{g}(\phi) = 1 \) and

\[
\tilde{g}(w) = \prod_{i=1}^{m(w)} \sigma(\pi^{(i-1)k}(w)) \left( \prod_{j=1}^{l(w)} \sigma(\pi^{(j-1)k}(\phi)) \right)^{-1}
\]

where \( l(w) = \min\{l \mid \pi^{lk}(\phi) = \pi^{mk}(w) \text{ for some } k \} \) and \( m(w) = \min\{m \mid \pi^{lk}(\phi) = \pi^{mk}(w) \text{ for some } l \} \). Note that then \( \pi^{l(w)k}(\phi) = \pi^{m(w)k}(w) \) by Lemma 3.2. Using this \( \tilde{g} \), we can prove this Theorem in the same way as [11, Theorem 18.1].

We have modify all statements which is necessary for proving Theorem 3.9. Therefore it follows.

### 4 Ahlfors Regular Conformal Dimension of Infinite Graphs

In this section, we give results about the ARC dimensions and the spectral dimensions of metrics on infinite graphs, which are the main results of this paper. To get these results, the cable systems of graphs play an important role. Technically, the main contribution of this paper is to show ARC dimension and a partition of a graph coincides with those of its cable system. Cable systems do NOT appear in statements of main results, but we use them and adapt the results of former sections and lead results for graphs. Throughout this section, \( G \) is a countable (infinite) set, \((G, E)\) is a connected, bounded degree graph and \( T = (T, \pi, \phi) \) is a bi-infinite tree with a reference point.

#### 4.1 Ahlfors Regular Conformal Dimension of Metrics on Infinite Graphs

We first denote a class of metrics on \((G, E)\), which we consider in this paper.

**Definition 4.1 (fitting metric).** We say a metric \( d \) on \( G \) fits to \((G, E)\) if it satisfies the following conditions.

(F1) There exists \( C > 0 \) such that \( d(x, y) \leq Cd(x, z) \) for any \( x, y, z \in G \) with \( x \sim y \) and \( x \neq z \).

(F2) For any \( \epsilon > 0 \), there exists \( r > 0 \), \( n \geq 1 \) and \( \{x_i\}_{i=0}^n \subseteq G \) such that

- \( x_i \in B_d(x_0, r) \) for any \( i \in [0, n-1] \) and \( x_n \notin B_d(x_0, r) \).
- \( d(x_i, x_{i-1}) \leq \epsilon r \) and \( x_i \sim x_{i-1} \) for any \( i \in [1, n] \).

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If the graph \((G, E)\) is fixed or clear, we simply say \(d\) is fitting when \(d\) fits to \((G, E)\).

The condition (F1) is natural condition for metrics on \(G\). For example, the graph distance \(l_E\) and “gently weighted” graph distances satisfy (F1). Moreover, the effective resistance of a weighted graph with controlled weight, which we will introduce later, also satisfies (F1). The condition (F2) is a little technical, which is needed to evaluate \(\text{dim}_{\text{AR}}(G, d)\).

Example 4.2. Let \(G = \mathbb{Z}\) and \(E = \{(n, m) \mid |n - m| = 1\}\). For any \(k \geq 1\), let \(x_i = i\) for \(i \in [0, k]_{\mathbb{Z}}\) then \(l_E(i + 1, i) \leq \frac{k}{2}\) and \(x_k \notin B_{l_E}(x_0, k)\), so \(l_E\) satisfies (F2). On the other hand, let \(d(n, m) := 2^n - 2^m\) then for any simple path \((n, n + 1, \ldots, n + k)\), \(d(n + k - 1, n + k) = 2^{n+k-1} \geq \frac{d(n, n + k)}{2}\). This shows \(d\) does not satisfy (F2) (for \(\epsilon \geq 1/2\)).

Remark that for any \((G, E)\), \(l_E\) satisfies (F2).

Lemma 4.3. Let \(d, \rho\) are metrics on \(G\) and \(d \sim_{\text{QS}} \rho\). If \(d\) fits to \((G, E)\), then \(\rho\) fits to \((G, E)\).

For this lemma and later statements, now we recall basic properties of quasisymmetry. Let \((X, d)\) and \((X, \rho)\) be metric spaces.

(1) Let \(\theta : [0, \infty) \to [0, \infty)\) be a homeomorphism, then the following conditions are equal:

i) \(d\) is \(\theta\)-quasisymmetric to \(\rho\).

ii) \(\rho(x, z) \leq \theta(t)\rho(x, z)\) whenever \(d(x, y) \leq td(x, z)\).

iii) \(\rho(x, z) < \theta(t)\rho(x, z)\) whenever \(d(x, y) < td(x, z)\).

(2) If \(d \sim_{\text{QS}} \rho\) and \(\text{diam}(X, d) = \infty\), then \(\text{diam}(X, \rho) = \infty\).

(3) \(\sim_{\text{QS}}\) is an equivalence relation between metrics on \(X\).

(4) If \(d \sim_{\text{QS}} \rho\), then both \((X, d)\) and \((X, \rho)\) induce the same topology (in other words, \(\text{id}_X\) is a homeomorphism between \((X, d)\) and \((X, \rho)\)).

(1) follows from monotonicity of \(\theta\). For (2)~(4), see [7, Section 10] for example.

proof of Lemma 4.3 Since \(d\) satisfies (F1) and \(d \sim_{\text{QS}} \rho\), \(\rho(x, y) \leq \theta(C)\rho(x, z)\) for any \(x, y, z \in G\) with \(x \sim y\) and \(x \neq z\), so \(\rho\) satisfies (F1). Next we show \(\rho\) satisfies (F2). Fix any \(\epsilon > 0\). Let \(\delta < 1/2\) such that \(\theta(2\delta)\theta(3) < \epsilon\). Since \(d\) satisfies (F2), there exists \(\{x_i\}_{i=0}^n \subseteq G\) such that

- \(x_i \in B_d(x_0, r)\) for any \(i \in [0, n - 1]_{\mathbb{Z}}\) and \(x_n \notin B_d(x_0, r)\).
- \(d(x_i, x_{i-1}) \leq \delta r\) and \(x_i \sim x_{i-1}\) for any \(i \in [1, n]_{\mathbb{Z}}\).
Let \( i \in [0, n - 1] \). Since \( d(x_0, x_n) \geq r \) and \( x_i, x_n \in B_d(x_0, (1 + \delta)r) \),
\[
\frac{r}{2} \leq d(x_0, x_i) \lor d(x_i, x_n) < 3r
\]
and hence
\[
\rho(x_i, x_{i+1}) \leq \theta(2\delta)(\rho(x_0, x_i) \lor \rho(x_i, x_n)) \leq \theta(2\delta)\theta(3)\rho(x_0, x_n).
\]
Let \( m = \min\{ i \mid x_i \notin B_\rho(x_0, \rho(x_0, x_n)) \} \), then \( r = \rho(x_0, x_n) \) and \( \{x_i\}_{i=0}^m \) satisfies the conditions of (F2) for \( \epsilon \).

Next we introduce partitions of infinite graphs.

**Definition 4.4** (partition). A map \( K : T \to \{ A \subseteq G \mid \#(A) < \infty \} \) is called a partition of \((G, E)\) parametrized by \( T \) if it satisfies following conditions.

(PG1) \( \bigcup_{v \in \mathcal{S}(w)} K_v = K_w \) for any \( w \in T \).

(PG2) For any \( \omega \in \Sigma_e \), there exist \( n_0(\omega) \in \mathbb{Z} \) and \( x, y \in G \) such that \( x \sim y \) and \( K_{\omega_n} = \{x, y\} \) for any \( n \geq n_0(\omega) \).

(PG3) For any \( (x, y) \in E \), there exists \( w \in T \) such that \( K_w = \{x, y\} \).

In the rest of this section, \( K \) is a partition of \((G, E)\) parametrized by \( T \).

**Lemma 4.5.** Let \( \Lambda_e = \{ w \in T \mid \#(K_w) = 2 \text{ and } \#(K_{\pi(w)}) > 2 \} \), then
\( \sqcup_{w \in \Lambda_e} \Sigma^*_w = \Sigma^* \).

**Proof.** \( \sqcup_{w \in \Lambda_e} \Sigma^*_w = \Sigma^* \) directly follows from (PG2). By (PG1), \( \#(K_{\omega_n}) \) is non-increasing for any \( \omega \in \Sigma^* \), so there exists a unique \( n \in \mathbb{Z} \) such that \( \omega_n \in \Lambda_e \). This shows \( \Sigma^*_w \cap \Sigma^*_v = \emptyset \) for any \( w, v \in \Lambda_e \) with \( w \neq v \).

**Definition 4.6.** (1) We denote \( \omega_{n_0(\omega)} \) by \( \omega_e \), where \( n_0(\omega) \) is in the condition (PG2). We also define \( T_e \) by
\[
T_e = \{ w \in T \mid T_w \cap \Lambda_e \neq \emptyset \} = \{ w \in T \mid \#(K_{\pi(w)}) > 2 \}
\]
and for \( w \in (T \setminus T_e) \cup \Lambda_e \), we define \( w_e \in \Lambda_e \) such that \( w \in T_{w_e} \).

(2) \( K \) is called minimal if \( K_w \neq K_v \) for any \( w, v \in \Lambda_e \) with \( w \neq v \).

**Definition 4.7** (discrete weight function). Recall that \( K \) is a partition of \((G, E)\). A function \( g : T_e \to (0, \infty) \) is called a discrete weight function (with respect to \( K \)) if it satisfies following conditions.

(GG1) For some \( w \in T_e \), \( \lim_{n \to \infty} g(\pi^n(w)) = \infty \).

(GG2) For any \( w \in T_e, g(\pi(w)) \geq g(w) \).

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For $s > 0$, we define the scale $\Lambda^g_s$ associated by

$$
\Lambda^g_s = \{ w \in T_e \mid g(w) \leq s < g(\pi(w)) \}
$$

and define $E^g_s, \Lambda^g_{s,M}(w), \Lambda^g_{s,M}(x)$ in the same way as compact cases. We also define $U^g_M(x, s)$ for $M \geq 0, x \in G$ and $s > 0$ by

$$
U^g_M(x, s) = \begin{cases}
\{ x \} & \text{if } \Lambda^g_{s,M}(x) = \emptyset \\
\bigcup_{w \in \Lambda^g_{s,M}(x)} K_w & \text{otherwise.}
\end{cases}
$$

Remark. Different from compact and $\sigma$-compact cases, $\Sigma^*$ is not necessarily equal to $\bigsqcup_{w \in \Lambda^g_s} \Sigma^*_w$ since they are restricted to $T_e$. The difference also appears in the definition of $U^g_M(x, s)$.

Lemma 4.8. Define

$$
\mathcal{D}_\infty(G) = \{ d \mid d \text{ is a metric on } G \text{ such that } \text{diam}(G, d) = \infty \}
$$

and let $d \in \mathcal{D}_\infty(G)$. We also define $g_d : T_e \to (0, \infty)$ by $g_d(w) = \max_{x, w \in K_w} d(x, y)$, then $g_d$ is a discrete weight function.

These lemmas are proved in the same way as Proposition 3.8. We denote $g_d$ by $d$ if no confusion may occur.

Definition 4.9. Let $g$ be a discrete weight function.

- $g$ is called uniformly finite if (2.1) holds.
- $g$ is called thick (with respect to $K$) if there exists $\alpha > 0$ such that for any $w \in T_e, \Lambda^g_{\alpha g(\pi(w)),1}(x) \subset T_w$ for some $x \in K_w$.

$d \in \mathcal{D}_\infty(G)$ is called uniformly finite and thick if $g_d$ is uniformly finite and thick, respectively.

We define $(M_*)$-adapted in the same way as compact cases, and

Definition 4.10. Let $(G, d)$ be a metric space and assume $K$ is minimal. We say $(G, d)$ satisfies basic framework (with respect to $K$) if the following conditions hold.

- $\sup_{w \in T_e \setminus \Lambda^g_s} \#(S(w)) < \infty$.
- $d$ is uniformly finite, thick, $M_*$-adapted for some $M_* \geq 1$.
- There exists $r \in (0, 1)$ such that $d \asymp r^{[w]}$ for any $w \in T_e$.

The difference between these definition and those of compact cases are $T_e$s in the notations.
Definition 4.11. Let $r \in (0, 1)$. For $w \in \Lambda_e$ and $n \geq 0$, let
\[
\mathfrak{S}_{w,m} = \{ ((x,k),(y,2^n(1-m) - 1 - k))_{w,m} \mid k \in [0,2^n - 1] \}
\]
where $x, y \in G$, $n(m) \in \mathbb{N}$ such that $K_w = \{x,y\}$ and $2^{-n(m)} \leq r^m < 2^{1-n(m)}$.
Then we define $T_r = T_e \cup (\bigcup_{w \in \Lambda_e} \bigcup_{m \geq 1} \mathfrak{S}_{w,m})$ and $\pi' : T_r \to T_r$ by
\[
\pi'(w) = \begin{cases} 
\pi(w) & \text{if } w \in T_e \\
\{(x, l), (y, 2^n(1-m) - 1 - l)\}_{v,n-1} & \text{if } w = \{(x,k),(y,2^n(m) - 1 - k)\}_{v,m} \\
& \text{and } [\frac{k}{2^n(m-1)}, \frac{k+1}{2^n(m-1)}] \subseteq [\frac{l-1}{2^n(m-1)}, \frac{l}{2^n(m-1)}]
\end{cases}
\]
Moreover, we define $K'$ by $K'_w = K_w$ (if $w \in T_e$), $K_v$ (if $w \in \bigcup_{m \geq 1} \mathfrak{S}_{w,m}$).

Lemma 4.12. $(T_r, \pi')$ is a by-infinite tree and $K'$ is a partition of $(G,E)$. Moreover, if we write $[w]'$ for the height of $(T_r, \pi', \phi')$ for $\phi' \in T_r$ and $\Lambda'_e$ for $K'$ version of $\Lambda_e$, then $\Lambda_e = \Lambda'_e$ and we can take $\phi' \in T_r$ such that $[w] = [w]'$.

Proof. For any $w \in T_r \setminus T_e$, by definition of $\pi'$, we have
- $\pi^n(w) \in \Lambda_e$ for some $n > 0$.
- For any $n > 0$, $\pi^n(w) \neq w$.

These and (T1),(T2) conditions of $\pi$ show (T1),(T2) conditions of $\pi'$, so $(T_r, \pi')$ is a bi-infinite tree. Fix $w \in T_e$ and let $\phi' \in S^{[w]}$ then $[w] = [w]'$ and by Lemma 3.1, $[v] = [v]'$ for any $v \in T_e$ because $\pi = \pi'$ on $T_e$. The rest of this lemma is clear by definition.

Remark that the definition of discrete weight function, and its properties are defined only on $T_e$, so they do not change if we replace $(T, \pi, \phi)$ by $(T_r, \pi', \phi')$. In the rest of this section, assume $(T, \pi, \phi) = (T_r, \pi', \phi')$.

Definition 4.13. Now we formally define $K$ on $T_r$ by
\[
K_w = \begin{cases} 
K_w & \text{if } w \in T_e \\
\{x\} & \text{if } w = \{(x,0),(y,2^n(m) - 1)\}_{m,v} \text{ for some } m, v \\
\emptyset & \text{otherwise}
\end{cases}
\]
and define $J^h_m \subset (T)_m \times (T)_m$ by
\[
J^h_m(K) = \{(w, u) \mid w, u \in (T)_m, K_w \cap K_v \neq \emptyset \text{ or there exists } v \in T_e, i \geq 0 \text{ such that } w = \{(x,i),(y,2^n(m-[v]) - 1 - i)\}_{w,m-[v]} \\
u = \{(x,i-1),(y,2^n(m-[v]) - i)\}_{w,m-[v]} \}
\]
\( J^h_m \) is thought to be constructed as follows: define \( J^h_m = \{(w,v) \mid w,v \in ((T)_m \cap T_e) \cup \Lambda_{e,m}, K_w \cap K_v \neq \emptyset \} \) where \( \Lambda_{e,m} = \{w_e \mid w \in (T)_m \setminus T_e\} \), and replace each \( w \in \Lambda_{e,m} \) by a \( 2^{m-|v|} \)-path. We will justify this idea later in the cable system. We define variables in the same way as Definition 2.11 and 2.14.

The following is the one of two main theorems of this paper.

**Theorem 4.14.** Let \( K \) be a minimal partition of \((G,E)\). If \( d \in D_\infty(G) \) satisfies basic framework and fits to \((G,E)\), then for any \( N, N_1, N_2 \) with \( N_2 \geq N_1 + M_* + 1 \),

\[
(1) \quad L^c_e(N_1, N_2, N) = T^c_e(N_1, N_2, N) = \dim_{AR}(G,d).
\]

\[
(2) \text{If } R_p(N_1, N_2, N) < 1, \text{ then } \dim_{AR}(G,d) \leq d_p^S(N_1, N_2, N) \leq d_p^S(N_1, N_2, N) < p.
\]

\[
(3) \text{If } R_p(N_1, N_2, N) \geq 1, \text{ then } \dim_{AR}(G,d) \geq d_p^S(N_1, N_2, N) \geq d_p^S(N_1, N_2, N) \geq p.
\]

To show that, we introduce the notion of cable system

**Definition 4.15** (cable system). Let \( \simeq \) be the minimal equivalence relation on \( G \times [0,1] \) which satisfies

- \( ((x,y),0) \simeq ((x,z),0) \) for any \( x,y,z \in G \),
- \( ((x,y),t) \simeq ((y,x),1-t) \) for any \( (x,y) \) and \( t \in [0,1] \).

Then we define the cable system \( \mathcal{C}_G \) of \((G,E)\) by \( \mathcal{C}_G := (E \times [0,1]) / \simeq \).
For \((x,y) \in E\), we also define \( \iota(x,y) = (x,y) \times [0,1] / \simeq \). Moreover, for any \( x \in G \), \( \tau(x) = ((x,y),0) / \simeq \), where \((x,y) \in E\), is well-defined because of the definition of \( \simeq \). We equate \( \tau(x) \) with \( x \) and regard \( G \) as a subset of \( \mathcal{C}_G \).

**Definition 4.16** (induced cable metric). Let \( \alpha \in (0,1) \) and \( d \in D_\infty(G) \). We define an induced cable metric \( d_{\mathcal{C},\alpha} : \mathcal{C}_G \times \mathcal{C}_G \to [0,\infty) \) by

\[
d_{\mathcal{C},\alpha}(x,y) = \begin{cases} |t-s|^\alpha d(x_0,x_1) & \text{if } (x \simeq ((x_0,x_1),t) \text{ and } y \simeq ((x_0,x_1),s) \text{ for some } (x_0,x_1) \in E) \\ \min\{t^\alpha d(x_1,x_0) + d(x_0,y_0) + s^\alpha d(y_0,y_1) & \text{if } x \simeq ((x_0,x_1),t) \text{ and } y \simeq ((y_0,y_1),s) \} \end{cases}
\]

We write \( d_\mathcal{C} \) instead of \( d_{\mathcal{C},1} \). Note that

**Lemma 4.17.** \((\mathcal{C}_G,d_{\mathcal{C},\alpha})\) is a metric space.
Proof. If $d_{E,\alpha}(x,y) = 0$, then $x \simeq ((x_0,x_1), t)$ for some $x_0, x_1, t$. By the definition of $d_E(y, x)$ is obvious. Moreover, since $t^\alpha + s^\alpha \geq (t+s)^\alpha$ and the triangle inequality for $d$ holds, we can write

$$d_{E,\alpha}(x,y) = \min\left\{\sum_{i=1}^n |s_i-t_i|^\alpha d(x_i, y_i) \mid n \geq 1, x_i, y_i \in G\right\}$$

such that $x \simeq ((x_1,y_1), s_1)$, $y \simeq ((x_n,y_n), t_n)$ and $(x_i, y_i), t_i \simeq ((x_{i+1}, y_{i+1}), s_{i+1})$ for any $i \in [1, n-1]$. (remark that $(x_i, y_i)$ is not necessarily in $E$), and hence the triangle inequality for $d_{E,\alpha}$ also holds.

Note that $d(x,y) = d_{E,\alpha}(x,y)$ for any $x, y \in G$ since $d$ satisfies the triangle inequality.

For the notation of a partition, we write $K_w = \{ w^+, w^- \}$ for each $w \in \Lambda_e$.

**Definition 4.18.** Define $K : T = T_e \to \mathcal{C}_G$ by

$$K_w = \begin{cases} \bigcup_{\omega \in \Sigma_w^+} (\omega_e^+, \omega_e^-) & \text{if } w \in T_e \\ (x, y) \times \left[\frac{k}{2m(w)}, \frac{k+1}{2m(w)}\right] & \text{if } w = \{ (x, k), (y, 2^{m(w)} - 1 - k) \}_{w,m}. \end{cases}$$

**Lemma 4.19.** (1) $K$ is a partition of $(\mathcal{C}_G, d_{E,\alpha})$.

(2) For any $w, v \in T_e$, $K_w \cap K_v \neq \emptyset$ if and only if $K_v \cap K_w \neq \emptyset$.

Proof. (1) By the definition of $d_{E,\alpha}$, $\mathcal{C}_G$ has no isolated point, and $((x,y) \times [0,1], d_{E,\alpha})$ is isomorphic to $([0,1], \| \cdot \|_x)$ and compact. Since $\{ \omega_e^+, \omega_e^- \} \subseteq K_w$ for any $\omega \in \Sigma_w^+$ and $K_w$ is a finite set, $K_w$ is a compact set for any $w$.

(P1) By the definition of $K$, (P1) follows from (PG1).

(P2) For any $w \in \Sigma^+, \cap_{n \geq 0} K_{\omega_n^w} = (\omega_e^+, \omega_e^-) \times \cap \left[\frac{k_n}{2m(w)}, \frac{k_n+1}{2m(w)}\right]$ for some $\{k_n\}_{n \geq 0}$ and is a single point.

(P3) By definition of $\Sigma^+$, $\Sigma^+ = \sqcup_{w \in (T_0)_{\Sigma_w^+}}$. Let $w \notin T_e$, then for any $w \in \Sigma_w^+$, $\omega_e = w_e$ and

$$\begin{align*} 
\bigcup_{v \in (T_0)_{\Sigma_w^+}} K_v & = \bigcup_{k \in [0, 2m(w)-|w_e| - 1]} (w_e^+, w^-_e) \times \left[\frac{k}{2m(w)-|w_e|}, \frac{k+1}{2m(w)-|w_e|}\right] \\
\ & = (w_e^+, w^-_e) \times [0, 1].
\end{align*}$$

Therefore

$$\bigcup_{w \in (T_0)} K_w = \bigcup_{w \in (T_0)} \bigcup_{\omega \in \Sigma_w^+} (\omega_e^+, \omega_e^-) \times [0,1] = \bigcup_{\omega \in \Sigma^+} (\omega_e^+, \omega_e^-) \times [0,1] = \mathcal{C}_G.$$

(The last equation follows from (PG2) and (PG3)).
(2) Let \( w \in T_e \), then (PG1) ensures \( K_w \supseteq \bigcup_{\omega \in \Sigma_w} K_\omega \). On the other hand, since (PG1) holds, for any \( v \in T \) and \( x \in K_v \) we can take \( v' \in S(v) \) such that \( x \in K_{v'} \). Inductively use this fact, we can get \( \omega \in \Sigma_w \) such that \( x \in \omega \) for any \( n \), therefore \( K_w = \bigcup_{\omega \in \Sigma_w} K_\omega \). Hence by definition of \( K \), we get the desired result.

\[ \square \]

The following proposition plays the key role in the proof of the main theorem.

**Proposition 4.20.** Let \( d \in D_\infty(G) \) and fitting to \((G, E)\). Then

\[ 1 \leq \dim_{\text{AR}}(G, d) = \dim_{\text{AR}}(\mathcal{C}_G, d). \]

**Proof.** We first show \( \dim_{\text{AR}}(G, d) \geq 1 \). Let \( \rho \) be a metric on \((G, E)\) and \( m \) be a measure on \( G \) such that \( d \sim QS^\alpha \) and \( \rho \) is \( \alpha \)-Ahlfors regular with respect to \( m \) for some \( \alpha > 0 \). Then by Lemma 4.3, \( \rho \) fits to \((G, E)\) and so for any \( n \), there exist \( m \geq 1 \) and \( \{x_i\}_{i=0}^m \subseteq G \) such that

- \( x_i \in B_d(x_0, r) \) for any \( i \in [0, m - 1] \) and \( x_m \notin B_d(x_0, r) \),
- \( d(x_i, x_{i-1}) \leq r/2n \) and \( x_i \sim x_{i-1} \) for any \( i \in [1, m] \).

Let \( y_0 = x_0 \) and inductively choose \( y_k \) by \( y_k = x_{i_k} \) where

\[ i_k = \min \{ i \mid x_i \notin \bigcup_{j=0}^{k-1} B_\rho(y_j, r/2n) \}. \]

Note that

\[ \min_{0 \leq j \leq k-1} \rho(y_j, y_k) < \frac{2r}{2n} \] because \( \rho(y_k, x_{i_k-1}) < \frac{r}{2n} \)

and so

\[ \rho(y_0, y_k) < \frac{k}{n} r \] and \( \bigcup_{j=0}^{k} B_\rho(y_j, r/2n) \subseteq B_\rho(y_0, r) \) for any \( k < n \).

Therefore we can take \( \{y_k\}_{k=0}^n \). By definition, \( \bigcup_{k=0}^{n-1} B_\rho(y_k, r/4n) \subseteq B_\rho(y_0, r) \), so

\[ Cr^\alpha \geq m(B_\rho(x, r)) \geq \sum_{k=0}^{n-1} m(B_\rho(y_k, r/4n)) \]

\[ \geq \sum_{k=0}^{n-1} C^{-1}(r/4n)^\alpha \]

\[ \geq 4^{-\alpha} C^{-1} n^{1-\alpha} r^\alpha \]

for some \( C > 0 \). Taking sufficiently large \( n \), this inequality contradicts if \( \alpha < 1 \).

Therefore \( \dim_{\text{AR}}(G, d) \geq 1 \).

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Next we show \( \dim_{\mathcal{AR}}(G, E) \leq \dim_{\mathcal{AR}}(\mathcal{C}_G, d_{\mathcal{E}}) \). Let \( \rho_X \) be a metric on \( \mathcal{C}_G \) and \( m_{\mathcal{E}} \) be a Borel measure on \( (\mathcal{C}_G, d_{\mathcal{E}}) \) such that \( d_{\mathcal{E}} \sim \rho_X \) and \( \rho_X \) is \( \alpha \)-Ahlfors regular with respect to \( m_{\mathcal{E}} \).

Define \( \rho \) on \( G \) by \( \rho(x, y) = \rho_X(x, y) \) for any \( x, y \in G \) and it is clear that \( d \sim \rho \) because \( d(x, y) = d_{\mathcal{E}}(x, y) \).

Define \( m \) by
\[
m(\{x\}) = \sum_{y \sim x} m_{\mathcal{E}}(\iota(x, y)).
\]

Let \( r > 0, x, y, z \in G \) such that \( y \in B_\rho(x, r) \setminus \{x\} \) and \( y \sim z \). Then by (F1), there exists \( C > 1 \), \( C d_{\mathcal{E}}(x, y) \geq d_{\mathcal{E}}(y, z) \geq d_{\mathcal{E}}(y, z') \) and hence \( \theta(C)\rho_X(x, y) \geq \rho_X(y, z') \) for any \( z' \in \iota(y, z) \). Moreover, if \( r > r_{x, \rho} \) then there exists \( o_0 \in B_\rho(x, r) \) and \( d(x, o') \leq d(x, o) \leq Cd(x, o_0) \) for any \( o' \) such that \( o' \in \iota(x, o) \) for some \( o \in G \) with \( o \sim x \). Hence \( o' \in B_{\rho_X}(x, \theta(C)r) \) and so for some \( C' > 0 \),
\[
m(B_\rho(x, r)) = \sum_{y \in B_\rho(x, r)} \sum_{z \sim y} m_{\mathcal{E}}(\iota(y, z)) \leq 2m_{\mathcal{E}}(B_{\rho_X}(x, (1 + \theta(C))r)) \leq 2C'(1 + \theta(\theta(C)))^\alpha r^\alpha.
\]

(Remark that \( m_{\mathcal{E}}\{x\} = 0 \) for any \( x \in G \) because of Ahlfors regularity). On the other hand, for any \( y' \in B_{\rho_X}(x, r) \) there exists \( y, z \in G \) such that \( y' \in \iota(y, z) \) and \( d_{\mathcal{E}}(x, y) \leq d_{\mathcal{E}}(x, y') \), and hence \( \rho_X(x, y) \leq \theta(1)\rho_X(x, y') \). Therefore
\[
m(B_\rho(x, r)) = \sum_{y \in B_\rho(x, r)} \sum_{z \sim y} m_{\mathcal{E}}(\iota(y, z)) \geq B_{\rho_X}(x, \theta(1)r) \geq C'^{-1}\theta(1)^{\alpha}r^\alpha.
\]

We have shown \( \rho \) is \( \alpha \)-Ahlfors regular with respect to \( m \), so \( \dim_{\mathcal{AR}}(G, E) \leq \dim_{\mathcal{AR}}(\mathcal{C}_G, d_{\mathcal{E}}) \).

Next we show \( \dim_{\mathcal{AR}}(G, E) \geq \dim_{\mathcal{AR}}(\mathcal{C}_G, d_{\mathcal{E}}) \). Let \( \rho \) be a metric on \( G \) and \( m \) be a measure on \( G \) such that \( d \sim \rho \) and \( \rho \) is \( \alpha \)-Ahlfors regular with respect to \( m \). Note that \( \alpha \geq 1 \) because \( \dim_{\mathcal{AR}}(G, d) \geq 1 \).

**Claim.** \( d_{\mathcal{E}} \sim \rho_{\mathcal{E}, 1/\alpha} \)

**proof of claim.** Let \( t > 0 \) and \( x, y, z \in \mathcal{C}_G \) with \( d_{\mathcal{E}}(x, y) \leq td_{\mathcal{E}}(x, z) \). We first consider the conditions

There exist \( e \in E \) and \( s, t \in (0, 1) \) such that \( x = (e, s) \) and \( y = (e, t) \). \( (y*) \)
There exist \( e \in E \) and \( s, t \in (0, 1) \) such that \( x = (e, s) \) and \( z = (e, t) \). \( (z*) \)

Let (Ch) be the condition such that neither \( (y*) \) nor \( (z*) \) hold. Under the condition (Ch), we consider the following condition \( (d_1) \):

There exist \( x_0, x_1, y_0 \in G \) such that \( x_0 \sim x_1, x_1 \sim y_0, \)
\[ x \in \iota(x_0, x_1), y \in \iota(y_0, x_1) \text{ and } d_{\mathcal{E}}(x, y) = d_{\mathcal{E}}(x, x_1) + d_{\mathcal{E}}(x_1, y). \]
If \( (d_*) \) does not hold, then

there exist \( x_0, x_1, y_0, y_1 \in G \) such that \( x_1 \neq y_1, x_0 \sim x_1, y_1 \sim y_0, \)
x \( \in \iota(x_0, x_1) \), \( y \in \iota(y_0, x_1) \) and \( d_\varepsilon(x, y) = d_\varepsilon(x, x_1) + d_\varepsilon(x_1, y_1) + d_\varepsilon(y_1, y). \)

We also consider the following similar condition \( (\rho_*) \):

There exist \( x_2, x_3, z_0 \in G \) such that \( x_2 \sim x_3, x_3 \sim z_0, \)
x \( \in \iota(x_2, x_3) \), \( z \in \iota(x_3, z_0) \) and \( \rho_{\varepsilon,1/\alpha}(x, z) = \rho_{\varepsilon,1/\alpha}(x, x_3) + \rho_{\varepsilon,1/\alpha}(x_3, z). \)

Otherwise,

there exist \( x_2, x_3, z_0, z_1 \in G \) such that \( x_3 \neq z_1, x_2 \sim x_3, z_1 \sim z_0, x \in \iota(x_2, x_3), \)
z \( \in \iota(x_3, z_0) \) and \( \rho_{\varepsilon,1/\alpha}(x, z) = \rho_{\varepsilon,1/\alpha}(x, x_3) + \rho_{\varepsilon,1/\alpha}(x_3, z_1) + \rho_{\varepsilon,1/\alpha}(z_1, z). \)

We first prove the claim with \( (\text{Ch}) \) in four cases with these conditions. Recall that \( d \) fits to \( (G, E) \), so there exists \( C > 0 \) such that for any \( o, p, q \in G \) with \( o \sim p \sim q, d(o, p) \leq C d(p, q) \) and hence \( \rho(o, p) \leq \theta(C) \rho(p, q). \)

**Case 1. Both \( (d_*) \) and \( (\rho_*) \) hold.** Let

\[
 u = \left( \frac{d_\varepsilon(x, x_3)}{d(x_2, x_3)} + \frac{d_\varepsilon(x_3, z)}{d(x_3, z_0)} \right),
\]

then

\[
 (d_\varepsilon(x, x_1) \lor d_\varepsilon(x_1, y_1)) \leq d_\varepsilon(x, y) \leq td_\varepsilon(x, z)
\]

\[
 \leq 2t(d_\varepsilon(x, x_3) \lor d_\varepsilon(x_3, z)) \leq 2tu(d(x_2, x_3) \lor d(x_3, z_0)).
\]

Since \( x \in \iota(x_0, x_1) \cup \iota(x_2, x_3), l_E(x_1, x_3) \leq 2 \) and so

\[
 (d(x_2, x_3) \lor d(x_3, z_0)) \leq C^2 (d(x_0, x_1) \lor d(x_1, y_0)).
\]

Hence

\[
 \left( \frac{d_\varepsilon(x, x_1)}{d(x_0, x_1)} \lor \frac{d_\varepsilon(x_1, y_0)}{d(x_1, y_0)} \right) \leq C^2 tu,
\]

therefore

\[
 \rho_{\varepsilon,1/\alpha}(x, y) \leq \rho_{\varepsilon,1/\alpha}(x, x_1) + \rho_{\varepsilon,1/\alpha}(x_1, y)
\]

\[
 \leq 2(2C^2tu)^{1/\alpha}(\theta(C))^2(\rho(x_2, x_3) \lor \rho(x_3, z_0))
\]

\[
 \leq 2(2C^2tu)^{1/\alpha}(\theta(C))^2 u^{-1/\alpha}(\rho_{\varepsilon,1/\alpha}(x, x_3) \lor \rho_{\varepsilon,1/\alpha}(x_3, z))
\]

\[
 \leq \eta_1(t) \rho_{\varepsilon,1/\alpha}(x, z)
\]

where \( \eta_1(t) = 2^{(\alpha+1)/\alpha} C^{2/\alpha}(\theta(C))^{2t^{1/\alpha}} \).

**Case 2. Neither \( (d_*) \) nor \( (\rho_*) \) hold.** Then,

\[
 d(x_1, y_1) \leq d_\varepsilon(x, y) \leq td_\varepsilon(x, z)
\]

\[
 \leq (1 + 2C)td(x_3, z_1) \leq f_1(C)td(x_1, z_1)
\]
where $f_1(C) = (1 + C + C^2)(1 + 2C)$. Therefore
\[
\rho_{\varepsilon,1/\alpha}(x,y) \leq (1 + 2\theta(C))\rho(x_1, y_1) \\
\leq (1 + 2\theta(C))\theta(f_1(C)t)\rho(x_1, z_1) \leq \eta_2(t)\rho_{\varepsilon,1/\alpha}(x,z)
\]
where $\eta_2(t) = f_1(\theta(C))\theta(f_1(C)t)$.

**Case 3.** Only $(d_u)$ holds. Then
\[
(d\varepsilon(x,x_1) \vee d\varepsilon(x_1, y_0)) \leq d\varepsilon(x,y) \leq t d\varepsilon(x,z) \leq (1 + 2C)td(x_3,z_1).
\]

Let $u = \left( \frac{d\varepsilon(x_1, y_0)}{d(x_0, x_1)} \vee \frac{d\varepsilon(x, y)}{d(x, x_1)} \right)$, then
\[
\frac{1}{C}(d(x_0, x_1) \vee d(x_1, y_0)) \leq (d(x_0, x_1) \wedge d(x_1, y_0)) \leq \frac{t}{u}d\varepsilon(x,z) \\
\leq (1 + 2C)\frac{t}{u}d(x_3, z_0) \leq f_1(C)\frac{t}{u}d(x_1, z_0)
\]
and hence
\[
\rho_{\varepsilon,1/\alpha}(x,y) \leq 2u^{1/\alpha}(\rho(x_0, x_1) \vee \rho(x_1, y_0)) \\
\leq 2u^{1/\alpha}(\rho(\theta(C) \wedge \theta(Cf_1(C)\frac{t}{u})))\rho(x_1, z_0) \\
\leq 2u^{1/\alpha}f_1(\theta(C))(\theta(C) \wedge \theta(Cf_1(C)\frac{t}{u})))\rho_{\varepsilon,1/\alpha}(x,z).
\]

Since $u \wedge \frac{t}{u} \leq \sqrt{t}$ and $u \leq 1$, $\rho_{\varepsilon,1/\alpha}(x,y) \leq \eta_3(t)\rho_{\varepsilon,1/\alpha}(x,z)$ where $\eta_3(t) = 2f_1(\theta(C))(t^{1/2\alpha}\theta(C) \wedge \theta(Cf_1(C)\sqrt{t})).$

**Case 4.** Only $(\rho_u)$ holds. Then,
\[
d\varepsilon(x_1, y_1) \leq d\varepsilon(x,y) \leq t d\varepsilon(x,z) \leq 2t(d\varepsilon(x,x_3) \vee d\varepsilon(x_3, z)).
\]

Let $u = \left( \frac{d\varepsilon(x,x_3)}{d(x_2,x_3)} \vee \frac{d\varepsilon(x_3,z_0)}{d(x_3,z_0)} \right)$, then
\[
d(x_3, y_1) \leq (1 + C + C^2)d(x_1, y_1) \leq (1 + C + C^2)2tu(d(x_2, x_3) \vee d(x_3, z_0) (4.1)
\]
and hence
\[
\rho_{\varepsilon,1/\alpha}(x,y) \\
\leq f_1(\theta(C))\rho(x_3, y_1) \\
\leq f_1(\theta(C))\theta(C)((1 + C + C^2)2tu)(\rho(x_2, x_3) \wedge \rho(x_3, z_0)) \\
\leq f_1(\theta(C))\theta(C)u^{-1/\alpha}((1 + C + C^2)2tu)(\rho_{\varepsilon,1/\alpha}(x,x_3) \wedge \rho_{\varepsilon,1/\alpha}(x_3, z))
\]
Since $Cd(x_3, y_1) \geq d(x_2, x_3) \vee d(x_3, z_0)$ and $(4.1)$ hold, $1 \leq u^{-1} \leq C(1 + C + C^2)2t$. Therefore $\rho_{\varepsilon,1/\alpha}(x,y) \leq \eta_4(t)\rho_{\varepsilon,1/\alpha}(x,z)$ where $\eta_4(t) = f_1(\theta(C))\theta(C)(C + C^2 + C^3)^{1/\alpha}((1 + C + C^2)2t)^{1/\alpha}$. 31
Finally take \( \eta = \eta_1 \lor \eta_2 \lor \eta_3 \lor \eta_4 \), we have shown \( d_\varepsilon \gtrsim \rho_{1,\alpha} \) under (Ch). Next we prove general case with using the result for (Ch).

**Case 5. Both \((y^*)\) and \((z^*)\) hold.** Then \( \rho_{\varepsilon,1/\alpha}(x,y) \leq \eta_5(t)\rho_{\varepsilon,1/\alpha}(x,z) \) whenever \( d_\varepsilon(x,y) \leq t d_\varepsilon(x,z) \) where \( \eta_5(t) = t^{\alpha} \).

**Case 6. Only \((y^*)\) holds.** We can take \( w \in G \) such that \( \rho_{\varepsilon,1/\alpha}(x,w) + \rho_{\varepsilon,1/\alpha}(w,z) = \rho_{\varepsilon,1/\alpha}(x,z) \). If \( \sqrt{t}d_\varepsilon(w,z) \geq d_\varepsilon(x,w) \) then

\[
d_\varepsilon(x,w) \lor d_\varepsilon(y,w) \leq d_\varepsilon(x,w) + d_\varepsilon(x,y) \leq (\sqrt{t} + t(1 + \sqrt{t} )d_\varepsilon(w,z)
\]

and hence

\[
\rho_{\varepsilon,1/\alpha}(x,y) \leq \rho_{\varepsilon,1/\alpha}(x,w) \lor \rho_{\varepsilon,1/\alpha}(w,y) \leq \eta(\sqrt{t} + t(1 + \sqrt{t})\rho_{\varepsilon,1/\alpha}(w,z).
\]

Otherwise \( d_\varepsilon(x,y) \leq (\frac{1+\sqrt{t}}{\sqrt{t}})d_\varepsilon(x,w) \). Therefore

\[
\rho_{\varepsilon,1/\alpha}(x,y) \leq \eta_6(t)\rho_{\varepsilon,1/\alpha}(w,z) \lor \rho_{\varepsilon,1/\alpha}(x,w) \leq \eta_6(t)\rho_{\varepsilon,1/\alpha}(x,z)
\]

where \( \eta_6(t) = \eta(\sqrt{t} + t(1 + \sqrt{t}) \lor (1 + \sqrt{t})\sqrt{t})^{1/\alpha} \).

**Case 7. Only \((z^*)\) holds.** We can take \( w \in G \) such that \( d_\varepsilon(x,w) + d_\varepsilon(w,y) = d_\varepsilon(x,y) \). Since

\[
d_\varepsilon(w,y) \leq d_\varepsilon(x,w) \leq t d_\varepsilon(x,z) \leq t(d_\varepsilon(x,w) \lor d_\varepsilon(z,w)),
\]

we get \( \rho_{\varepsilon,1/\alpha}(w,y) \leq \eta(t)\rho_{\varepsilon,1/\alpha}(x,w) \lor \rho_{\varepsilon,1/\alpha}(x,z)) \). Moreover,

\[
d_\varepsilon(x,w) \lor d_\varepsilon(z,w) \leq d_\varepsilon(x,z) + d_\varepsilon(y,z) \leq (1 + t)d_\varepsilon(x,z),
\]

hence \( \rho_{\varepsilon,1/\alpha}(x,w) \lor \rho_{\varepsilon,1/\alpha}(x,w) \leq (1 + t)^{1/\alpha}\rho_{\varepsilon,1/\alpha}(x,z) \). Therefore

\[
\rho_{\varepsilon,1/\alpha}(x,y) \leq \rho_{\varepsilon,1/\alpha}(x,w) + \rho_{\varepsilon,1/\alpha}(w,y) \leq \eta_7(t)\rho_{\varepsilon,1/\alpha}(x,z)
\]

where \( \eta_7(t) = t^{1/\alpha} + (1 + t)^{1/\alpha}\eta(t) \). Let \( \eta' = \eta \lor \eta_5 \lor \eta_6 \lor \eta_7 \), then we have shown \( d_\varepsilon \) is \( \eta' \)-quasisymmetric to \( \rho_{\varepsilon,1/\alpha} \).

Finally, we prove \( \rho_{\varepsilon,1/\alpha} \) is \( \alpha \)-Ahlfors regular. Define \( m_\varepsilon \) by \( m_\varepsilon \{ (x,y,t) \in \mathcal{C}_0 \mid a \leq t \leq b \} = (b - a)(m(\{x,y\})) \) for any \( (x,y) \in E \) and \( 0 \leq a < b \leq 1 \). First we consider the case \( x \in G \).

- Let \( x \in G \). Since \( r_x = r_{x,\rho} \lesssim r_y \sim \rho(x,y) \) for any \( (x,y) \in E \) and \( \rho \) is \( \alpha \)-Ahlfors regular with respect to \( m \),

\[
V_{\rho_{\varepsilon,1/\alpha},m_\varepsilon}(x,r) = \sum_{y:y \sim x} \left( \frac{r}{\rho_{\varepsilon,1/\alpha}(x,y)} \right)^\alpha m(\{x,y\}) \sim \sum_{y:y \sim x} \left( \frac{r}{r_x} \right)^\alpha \lesssim r_x \sim r^\alpha
\]

for any \( x \in G \) and \( r \leq r_x \) (the last inequality is because \( G \) is bounded degree). Next we consider global cases.
\( \circ \) Let \((y, z) \in E\). If \(y \in B_\rho(x, r)\) for some \(r > 0\), then for any \(y' \in \iota(y, z)\),
\[
\rho_{\mathcal{G}, 1/\alpha}(x, y') \leq \rho(x, y) + \rho(y, z) < (1 + \theta(C))r.
\]

\( \circ \) Let \(y' \in B_{\rho_{\mathcal{G}, 1/\alpha}}(x, r)\), then there exists \((y, z) \in E\) such that \(y' \in \iota(y, z)\) and \(y \in B_\rho(x, r)\).

Therefore there exists \(C_1, C_2 > 0\) such that
\[
V_{\rho_{\mathcal{G}}, 1/\alpha, m_\mathcal{E}}(x, r) \geq \frac{1}{2} \sum_{y \in B_\rho(x, r/(1 + \theta(C)))} \sum_{z \sim y} m_\mathcal{E}(\iota(y, z))
\]
\[
= \frac{1}{2} \sum_{y \in B_\rho(x, r/(1 + \theta(C)))} \sum_{z \sim y} m(\{y, z\}) \geq V_{\rho, m}(x, r/(1 + \theta(C))) \geq C_1 r^\alpha
\]
for any \(x \in G\) and \(r > (1 + \theta(C))r_x\), and
\[
V_{\rho_{\mathcal{G}}, 1/\alpha, m_\mathcal{E}}(x, r) \leq \sum_{y \in B_\rho(x, r)} \sum_{z \sim y} m(\{y, z\})
\]
\[
\leq (\sup_{y \in G} \#\{z \mid z \sim y\}) V_{\rho, m}(x, (1 + \theta(C))r) \leq C_2 r^\alpha
\]
for any \(x \in G\) and \(r > r_x\). Adjusting constants, we get \(V_{\rho_{\mathcal{G}, 1/\alpha, m_\mathcal{E}}}(x, r) \asymp r^\alpha\) for any \(r \in G\) and \(r > 0\).

\( \bullet \) Let \(x \in \mathcal{G}\) and \(y, z \in G\) such that \(x \in \iota(y, z)\). We also let \(r_0 = \rho_{\mathcal{G}, 1/\alpha}(x, y)\). If \(r \leq r_0/(1 + \theta(C))\), then \(B_{\rho_{\mathcal{G}, 1/\alpha}}(x, r) \subseteq \cup_{w \sim y} \iota(w, y) \cup \cup_{w \sim z} \iota(w, z)\), so there exist \(C', C_3\) such that
\[
V_{\rho_{\mathcal{G}, 1/\alpha, m_\mathcal{E}}}(x, r)
\]
\[
\leq \sum_{w \sim y} \left( \frac{r}{\rho_{\mathcal{G}, 1/\alpha}(y, w)} \right)^\alpha m(\{w, y\}) + \sum_{w \sim z} \left( \frac{r}{\rho_{\mathcal{G}, 1/\alpha}(w, z)} \right)^\alpha m(\{w, z\})
\]
\[
\leq \sum_{w \sim y} \left( \frac{\theta(C)r_x}{r_y} \right)^\alpha C'(r_y^\alpha + (\theta(C)r_y)^\alpha) + \sum_{w \sim z} \left( \frac{\theta(C)^2 r_x}{r_y} \right)^\alpha C'(2(\theta(C)^2 r_y)^\alpha)
\]
\[
\leq C_3 \left( \frac{r}{r_y} \right)^\alpha r_y^\alpha = C_3 r^\alpha
\]
because \((G, E)\) is bounded degree. Otherwise, since
\[
B_{\rho_{\mathcal{G}, 1/\alpha}}(x, r) \subseteq B_{\rho_{\mathcal{G}, 1/\alpha}}(y, r) \cup B_{\rho_{\mathcal{G}, 1/\alpha}}(x, r)\text{ if }r > r_0,
\]
there exists \(C_4 > 0\) such that \(V_{\rho_{\mathcal{G}, 1/\alpha, m_\mathcal{E}}}(x, r) \leq C_4 r^\alpha\) for any \(r > r_0\) and \(x \in \mathcal{G}\). On the other hand, there exist \(C_5, C_6\) such that
\[
V_{\rho_{\mathcal{G}, 1/\alpha, m_\mathcal{E}}}(x, r) \geq \left( \frac{r}{\rho_{\mathcal{G}, 1/\alpha}(y, z)} \right)^\alpha m(\{y, z\}) \geq C_5 \left( \frac{r}{r_y} \right)^\alpha r_y^\alpha = C_5 r^\alpha
\]
for any $x \in C_G$ and $r < r_0(x)$, and
\[ V_{r_{ɛ,1/α},m_ɛ}(x,r) \geq V_{r_{ɛ,1/α},m_ɛ}(y,r/2) \geq C_6 \left( \frac{r}{2} \right)^α \]
for any $x \in C_G$ and $r \geq 2r_0(x)$. Therefore
\[ \left( \frac{C_5 \vee C_6}{2α} \right)^r \leq V_{r_{ɛ,1/α},m_ɛ}(x,r) \leq (C_3 \vee (1 + θ(C))^αC_4)^rα \]
for any $r > 0$ and $x \in C_G$.

\[ \square \]

**Lemma 4.21.** Under the same assumption of Theorem 4.14, $d_ɛ$ satisfies the assumptions of Theorem 3.9 with respect to the partition $K_0$ hold by replacing $M_σ$ with $M_σ + 1$.

**Proof.**

- (locally finite) Since $(G, E)$ is locally finite and $d$ fits to $(G, E)$, for any $x \in G$, there exists $r > 0$ such that $B_ɛ(x, r) \subseteq \sum_{y, y\sim x} εt(x, y)$ and then $\# \{w \in \{T\}_{0} \mid B_ɛ(x, r) \cap K_ω \neq ∅\} < \infty$. Let $U_w := K_ω \cup \{x \in K_ω B_ɛ(x, r)\}$, we get $\# \{v \in \{T\}_{0} \mid U_w \cap K_ω \neq ∅\} < \infty$.

- (minimal) Since $K$ is minimal, it directly follows from Lemma 4.19 (2) and the definition of $K$.

- ($r^{[w]} \asymp d_ɛ$) Let $w \in T_ɛ$. For any $x, y \in K_ω$, there exists $x_0, y_0, x_1, y_1 \in K_ω$ such that $x \in t(x_0, x_1), y \in t(y_0, y_1)$ and then $d_ɛ(x, y) \leq d_ɛ(x_0, x_1) + d_ɛ(x_1, y_1) + d_ɛ(y_1, y_0)$, so $d_ɛ(w) \leq d_ɛ(w) \leq 3d_ɛ(w)$ and so $d_ɛ(w) \asymp r^{[w]}$ for any $w \in T_ε$. Moreover, by the definition of $K_ω$ and $d_ɛ$,
\[ 2^{-1}r^{[w]-[w]}d_ɛ(w) \leq d_ɛ(w) = \text{diam}(K_ω, d_ɛ) \leq r^{[w]-[w]}d_ɛ(w) \]
for any $w \not\in T_ε$. Combining them, we get $r^{[w]} \asymp d_ɛ(w)$ for any $w \in T_ε$.

- (uniformly finite) Let $w \in T_ε \cup \Lambda_ɛ^{d_ɛ}$ for some $s > 0$. Since $d_ɛ \asymp d \asymp r^{[w]}$ and $d_ɛ(w) \leq d_ɛ(w)$ for $w \in T_ε$, there exist $c < 1$ and $m > 0$ such that if $v \in \Lambda_ɛ^{d_ɛ}(w) \cup T_ε$ then $cs \leq d(v) \leq s$ and hence there exists $v' \in \Lambda_ɛ^{d}$ such that $v \in T_ε$ and $|v| - |v'| < m$. On the other hand, let $v \in \Lambda_ɛ$ such that $K_ε \cap K_ω \neq ∅$ and $d_ε(v) = d(v) > s$. Then $s < d(v) \leq C_δ(w) \leq Cs$ by (F1), and similarly, there exists $m_1 > 0$, which is independent of $v$, and $v' \in \Lambda_ɛ^{d_ε}$ such that $v \in T_ε'$ and $|v| - |v'| < m_1$. Therefore
\[ \#(\Lambda_ɛ^{d_ε}(w)) \]
\[ = \#(\Lambda_ɛ^{d_ɛ}(w) \cup T_ε) + \#(\Lambda_ɛ^{d_ε}(w) \setminus T_ε) \]
\[ \leq \#(\Lambda_ɛ^{d_ɛ}(w) \cup T_ε) + 2\#(v \in \Lambda_ɛ \mid d(v) > s \text{ and } K_ε \cap K_ω \neq ∅) \]
\[ \leq (\sup_{v \in T_ε} #(S(v)))^m \#(\Lambda_ɛ^{d_ε}(w')) + 2(\sup_{v \in T_ε} #(S(v)))^m \#(\Lambda_ɛ^{d_ɛ}(w'')) \]
where \( w' \in \Lambda^d_s \) and \( w'' \in \Lambda^d_C \) such that \( w \in T_{w'} \subseteq T_{w''}. \) Since \( d \) is uniformly finite, \( \#(\Lambda^d_{s+1}(w)) \) is bounded. Moreover, since \( \#(\Lambda^d_{s+1}(w)) \leq \sup_{x \in G} \#(\{y \mid y \sim x\}) + 1 \) for \( s > 0 \) and \( w \in \Lambda^d_s \setminus T_v, \) \( d \) is uniformly finite.

- (thick) Let \( w \in \Lambda^d_s \setminus T_v, \) then \( K_w = \{(x,y,t) \mid a \leq t \leq b\} \) for some \( (x,y) \in E, 0 \leq a < b \leq 1, \) and then \( U^d_{1}(((x,y),(t+s)/2),(r/8)d_{e}(\pi(w))) \) \( \subseteq K_w. \) Next we let \( w \in T_v. \) Since \( d \) is thick and \( d \approx d_{e}, \) there exists \( \alpha, \) independent of \( w, \) and \( x \in \mathcal{K}_w \) such that \( U^d_{1}(x,\alpha d_{e}(\pi(w))) \subseteq \mathcal{K}_w. \)

  - If \( \Lambda^d_{\alpha d_{e}(\pi(w)),1}(x) \setminus T_v \neq \emptyset, \) then there exists \( v \in \Lambda_v \) such that \( K_v \cap K_w \neq \emptyset \) and \( d(v) > \alpha d_{e}(\pi(w)), \) and hence by (F1), there also exists \( w' \in \Lambda_v \) such that \( w' \in T_w \) and \( d(w') > (\alpha/C)d_{e}(\pi(w)). \) Then similar to the former case, \( U^d_{1}(x',(\alpha/C)d_{e}(\pi(w))) \subseteq \mathcal{K}_{w'} \subseteq \mathcal{K}_w \) for some \( x' \in \mathcal{K}_w. \)

  - If \( \Lambda^d_{\alpha d_{e}(\pi(w)),1}(x) \subseteq T_v, \) then for any \( v \in \Lambda^d_{\alpha d_{e}(\pi(w)),1}(x) \), there exists \( v' \in \Lambda^d_{\alpha d_{e}(\pi(w)),1}(x) \) such that \( v \in T_{v'} \) because \( d(v) \leq d_{e}(v) \) and \( v \in T_v, \) so \( U^d_{1}(x,\alpha \pi(d_{e}(w))) \subseteq \mathcal{K}_w. \)

Therefore \( d_{e} \) is thick.

- ((\( M_s + 1 \))-adapted) Let \( x \in \mathcal{C}_G. \) If \( y \in U^d_{M_s+1}(x,r), \) then there exist \( w_0, w_1, ..., w_{M_s+1} \in \Lambda^d_v \) such that \( x \in K_{w_0}, \) \( y \in K_{w_1}, \) and \( K_{w_1} \cup K_{w_1} \neq \emptyset \) for any \( i \in [0, M_s], \) so \( d_{e}(x,y) \leq \sum_{i=0}^{M_s+1} d_{e}(w_i) \leq (M_s + 2)r \) and hence \( U^d_{M_s}(x,r) \subseteq B_d(x, (M_s + 3)r). \) To show inverse direction, we take \( y, z \in G \) such that \( x \in \mathcal{L}(y,z). \) If \( r < d(y,z)/C, \) then

\[
B_{d_{e}}(x,r) \subseteq \left( \bigcup_{w : w\sim z} \iota(w,z) \right) \cup \left( \bigcup_{w : w\sim y} \iota(w,y) \right),
\]

so \( B_{d_{e}}(x,r) \subseteq U^d_{1}(x,2r) \subseteq U^d_{d_{e}}(x,2r) \) by the definition of \( K \) on \( T \setminus T_v. \) If \( r > d(y,z), \) then \( B_{d_{e}}(x,r) \subseteq B_{d_{e}}(y,r) \cup B_{d_{e}}(z,r). \) Recall that if \( p \in B_{d_{e}}(x,r) \) and \( p \in \iota(p_1, p_2) \) for \( p \in \mathcal{C}_G \) and \( p_1, p_2 \in G, \) then \( \{p_1, p_2\} \subseteq B_{d_{e}}(x, (1 + C)r), \)

\[
B_{d_{e}}(y,r) \subseteq \{\iota(p_1, p_2) \mid p_1, p_2 \in B_{d_{e}}(y, (1 + C)r)\}
\]

because \( d \) is \( (\alpha + 1)-\text{adapted}, \) (note that the last inclusion follows from \( d_{e}(w) \geq d(w), \) similar to the proof of thick). Since \( r > d_{e}(y,z), \) \( y, z \in U^d_{M_s}(x,r) \) and hence \( B_{d_{e}}(x,r) \subseteq U^d_{M_s}(y, (\alpha + C)r). \) Adjusting constants, we get \( d_{e} \) is \( (M_s + 1)-\text{adapted}. \)
proof of Theorem 4.14 Let \((w, v) \in T_m\). then \(K_w \cap K_v \neq \emptyset\) if and only if \(K_w \cap K_v \cap G \neq \emptyset\) or

\[
\begin{align*}
  w &= \{(x, i), (y, 2^{n(m-[w_\infty]) - 1 - i})\}_{w,m-[w_\infty]} \quad \text{and} \\
  v &= \{(x, j), (y, 2^{n(m-[v_\infty]) - 1 - j})\}_{w,m-[v_\infty]}
\end{align*}
\]

with \(w_\infty = v_\infty\) and \(|i - j| = 1\). Therefore

\[
J^h_M = J^h_M(K) = J^h_M(K).
\]

and so the other variants defined only by \(T_r\) and \(J^h_M\) also coincide. Especially, \(J^h_M(N_1, N_2, N) = O_K(N_1, N_2, N)\) of \(K\) and \(K\) coincide respectively. Therefore by Lemma 4.21 and Theorem 3.9, we get

\[
J^h_M(N_1, N_2, N) = dim_{AR}(C_G, d_G)
\]

for \(N_2 \geq N_1 + M^* + 1\). Combining it with Proposition 4.20, we get

\[
J^h_M(N_1, N_2, N) = dim_{AR}(G, d_G)
\]

Since \(d^S_p(N_1, N_2, N) = \overline{d}^S_p(N_1, N_2, N)\) also coincide respectively, we also obtain

\[
dim_{AR}(G, d) \leq d^S_p(N_1, N_2, N) \leq \overline{d}^S_p(N_1, N_2, N) < p
\]

if \(R_p(N_1, N_2, N) < 1\), and

\[
dim_{AR}(G, d) \geq d^S_p(N_1, N_2, N) \geq \underline{d}^S_p(N_1, N_2, N) \geq p
\]

if \(\overline{R}_p(N_1, N_2, N) \geq 1\).

4.2 Spectral Dimension and Ahlfors Regular Conformal Dimension of Weighted Graphs

In Theorem 4.14 we saw a relation between the ARC dimension and the p-spectral dimension of associated metrics on graphs. On the other hand, the spectral dimension of the associated random walks on graphs can be determined. In this subsection, we see the relation between these dimensions. Recall that \((G, E)\) is a connected, bounded degree simple graph and \(T = (T, \pi, \phi) = (T_r, \pi', \phi')\) is a bi-infinite tree with a reference point. Throughout this section, let \(K\) be a partition of \((G, E)\) parametrized by \(T\).

Definition 4.22 (Weighted graph). Let \(\mu\) be a positive symmetric function on \(E\), then we call \((G, \mu)\) a weighted graph and \(\mu\) a conductance (or weights) on \((G, E)\). Moreover, we treat \(\mu\) as a measure on \(G\) defined by

\[
\mu_x := \sum_{y:y \sim x} \mu_{xy} \quad \text{and} \quad \mu(A) := \sum_{x \in A} \mu_x
\]

for any \(x \in G\) and \(A \subset G\).
• (controlled weight) We say \((G, \mu)\) has controlled weight, or satisfies condition \((p_0)\) if there exists \(p_0 > 0\) such that

\[
p(x, y) := \frac{\mu_{xy}}{\mu_x} \geq p_0 \text{ for any } x, y \in G \text{ with } x \sim y.
\]

\((p_0)\)

Note that if \((G, \mu)\) has controlled weight, then \(#\{y \mid y \sim x\} \leq \lfloor p_0^{-1} \rfloor\) for any \(x \in T\). (It shows that \((G, E)\) must be a bounded degree graph).

• (heat kernel) We inductively define

\[
p_0(x, y) = \delta_{x,y}, \quad p_n(x, y) = \sum_{z \in G} p_{n-1}(x, z)p(z, y).
\]

\(p_n(x, y)\) is also thought as transition function of associated random walk; that is,

\[
P^x(X_n = y) = p_n(x, y).
\]

Additionally, we define the heat kernel of this random walk (with respect to \(\mu\)) by \(h_n(x, y) = p_n(x, y)/\mu_y\). It is easy that \(h_n(x, y) = h_n(y, x)\).

• (effective resistance) For \(f \in \mathbb{R}^G\), we define

\[
\mathcal{E}_\mu(f) = \mathcal{E}(f) := \sum_{x, y \in G} (f(x) - f(y))^2 \mu_{xy}
\]

and define the effective resistance of \((G, \mu)\) by

\[
R(A, B) = (\inf\{\mathcal{E}(f) \mid f|A = 1, f|B = 0\})^{-1}
\]

for any \(A, B \subseteq G\), where \(\inf \emptyset = \infty\). We write \(R(x, A)\) and \(R(x, y)\) instead of \(R(\{x\}, A)\) and \(R(\{x\}, \{y\})\), respectively.

It is known that the infimum of \(R(A, B)^{-1}\) is attained and that \(R(x, y)\) is a distance on \(G\) (for example, see [8]).

In the rest of this paper, let \((G, \mu)\) be a weighted graph and \(R\) be the associated effective resistance.

**Definition 4.23** (Spectral dimension). Fix \(x \in G\) and define

\[
\dS(G, \mu) = 2 \limsup_{n \to \infty} \frac{\log p_{2n}(x, x)}{\log n}
\]

and

\[
\dlS(G, \mu) = 2 \liminf_{n \to \infty} \frac{\log p_{2n}(x, x)}{\log n}.
\]

We can see that \(\dS(G, \mu)\) and \(\dlS(G, \mu)\) are independent of \(x\). We call \(\dS(G, \mu)\) the upper spectral dimension of \((G, \mu)\), and \(\dlS(G, \mu)\) the lower spectral dimension of \((G, \mu)\). If \(\dS(G, \mu) = \dlS(G, \mu)\), then we call \(\dS(G, \mu) = \dS(G, \mu)\) the spectral dimension of \((G, \mu)\).

We introduce other notions of a partition.
Definition 4.24. We say \( K \) is connected if for any \( w \in T_e \) and \( x, y \in K \), there exists a path between \( x \) and \( y \) in \( K_w \), in other words, \( (K_w, E|_{K_w \times K_w}) \) is connected for any \( w \in T_e \).

Definition 4.25. We introduce notions \( \overline{N}, \overline{N}, \overline{R}_p \) by

\[
\overline{N} = \sup \limsup_{w \in T} \limsup_{k \to \infty} \#(\{S^k(\pi^k(w))\})^{1/k}, \quad \overline{N} = \sup \liminf_{w \in T} \liminf_{k \to \infty} \#(\{S^k(\pi^k(w))\})^{1/k},
\]

\[
\bar{R}_p(N_1, N_2, N) = \sup \liminf_{w \in T} E_{p,k,\pi^k(w)}(N_1, N_2, N)^{1/k}.
\]

Remark that the difference between \( \overline{N}_* \), \( R_* \) and \( \overline{N}, \overline{R}_p \), respectively, is the order of the supremum over \( w \in T \) and the limit as \( k \), the index of scales, approaches to infinity. By definition, \( \overline{N}_* \geq \overline{N} \geq \overline{N} \) and \( R_* \geq \bar{R}_p \).

Lemma 4.26. Assume \( \sup_{w \in T_\Lambda_ \setminus \Lambda_*} \#(S(w)) < \infty \), then

(1) \( \overline{N} = \limsup_{k \to \infty} \#(\{S^k(\pi^k(w))\})^{1/k} \) and \( \overline{N} = \liminf_{k \to \infty} \#(\{S^k(\pi^k(w))\})^{1/k} \) for any \( w \in T \).

(2) \( \bar{R}_p(N_1, N_2, N) = \sup \liminf_{l \geq 0} \liminf_{k \to \infty} E_{p,k,\pi^{k+l}(w)}(N_1, N_2, N)^{1/k} \) for any \( w \in T \).

Proof. Let \( N_* = \sup_{w \in T_\Lambda_ \setminus \Lambda_*} \#(S(w)) \)

(1) Let \( w \in T \) and \( l \geq 0 \). Then for any \( k \geq l \), \( \#(S^{k-l}(\pi^k(w))) \leq \#(S^k(\pi^k(w))) \leq N_*^{1/k} \) and so

\[
N_*^{-1/(k-l)} \#(S^{k-l}(\pi^{k-l}(\pi^l(w))))^{1/(k-l)} \leq N_*^{-1/(k-l)} \#(S^k(\pi^k(w)))^{1/(k-l)} \\
\leq \#(S^k(\pi^k(w)))^{1/k} \\
\leq N_*^{1/k} \#(S^{k-l}(\pi^{k+l}(w)))^{1/(k-l)}
\]

because \( \#(S^k(\pi^k(w)))^{1/(k-l)-1/k} \leq (N_*^{1/k})^{1/k} = N_*^{1/(k-l)} \). Therefore \( \limsup_{k \to \infty} \#(S^k(\pi^k(w)))^{1/k} = \limsup_{k \to \infty} (\#(S^{k-l}(\pi^{k+l}(w))))^{1/k} \). By (P1), for any \( w,v \in T \), there exists \( n,m \geq 0 \) such that \( \pi^n(w) = \pi^m(v) \) and hence

\[
\overline{N} = \sup \limsup_{v \in T} \limsup_{k \to \infty} \#(\{S^k(\pi^k(v))\})^{1/k} \\
= \sup \limsup_{l \geq 0} \limsup_{k \to \infty} \#(\{S^k(\pi^{k+l}(w))\})^{1/k} \\
= \limsup_{k \to \infty} \#(\{S^k(\pi^k(v))\})^{1/k}.
\]

The case of \( \overline{N} \) is the same.
(2) Let $w \in T$ and $k \geq l \geq 0$. We also let $f$ be a function on $(T)_{[w] - l}$ such that $f \equiv 1$ on $S^{k-l}(\Gamma_{N_1}(\pi^{k}(w)))$ and $f \equiv 0$ on $S^{k-l}(\Gamma_{N_2}(\pi^{k}(w)))$. Then for $\overline{f}(w) = f(\pi^i(w))$, $\overline{f} \equiv 1$ on $S^k(\Gamma_{N_1}(\pi^k(w)))$, $\overline{f} \equiv 0$ on $S^k(\Gamma_{N_2}(\pi^k(w)))$ and

$$
\sum_{(u,v) \in J_{N,[w]}^{k}} |\overline{f}(u) - \overline{f}(v)|^p \leq \sum_{(u,v) \in J_{N,[w]}^{k}} \sum_{w' \in S(u)} \sum_{w'' \in S(v)} |\overline{f}(u') - \overline{f}(v')|^p
$$

because $l_{p,[w]}(u,v) \leq N$ implies $l_{p,[w]-l}(\pi(u), \pi(v)) \leq N$. Therefore

$$(N_s)^{-2l} \mathcal{E}_{p,k,\pi^k(w)}(N_1, N_2, N) \leq \mathcal{E}_{p,k-l,\pi^{k-l}(\pi^i(w))}(N_1, N_2, N)$$

and same as the former case, we get

$$\liminf_{k \to \infty} (\mathcal{E}_{p,k,\pi^k(w)}(N_1, N_2, N))^{1/k} \leq \liminf_{k \to \infty} (\mathcal{E}_{p,k,\pi^{k-l}(\pi^i(w))}(N_1, N_2, N))^{1/k}$$

and

$$\mathcal{R}_p(N_1, N_2, N) = \sup_{v \in T} \liminf_{k \to \infty} (\mathcal{E}_{p,k,\pi^k(v)}(N_1, N_2, N))^{1/k} \leq \liminf_{k \to \infty} (\mathcal{E}_{p,k,\pi^k(v)}(N_1, N_2, N))^{1/k} \leq \mathcal{R}_p(N_1, N_2, N).$$

We will consider the case that weight is uniformly bounded. In the following theorem, we evaluate $\mathcal{F}_S$ and $d_S$ by a partition.

**Theorem 4.27.** Assume $\mu_{xy} > 1$ for any $(x, y) \in E$ and $K$ is minimal and connected. Let $d \in D_\infty(G)$, fitting to $(G, E)$ and satisfying basic framework. If $d$ satisfies

- $d(x, y) \asymp 1$ for any $(x, y) \in E$,
- $h_{2n}(x, x) \asymp \frac{e}{V_d(x, x^{n/\beta})}$ for any $n$, (DHK(\beta))
- There exists $\lambda, C > 0$ such that $R(B_d(x, \lambda r), B_d(x, r^\beta)) V(x, r) \geq C r^\beta$ for any $r > r_x$, (ARL(\beta))
- There exists $C' > 0$ such that $R(x, B_d(x, r)^\beta)V(x, r) \leq C' r^\beta$ for any $r > r_x$, (BRU(\beta))
- There exists $C'' > 0$ such that $V_d(x, r) \leq C'' r^{\alpha} / s^\alpha$ for any $x \in G$ and $r > s > 0$ (VG(\alpha))
for some \( \beta > \alpha \geq 1 \), then for any \( N, N_1 \geq 0 \) and sufficiently large \( N_2 = N_2(N_1) \),

\[
d_S(G, \mu) = 2 \frac{\log N}{\log N - \log R_2(N_1, N_2, N)}
\]

and \( d_S(G, \mu) = 2 \frac{\log N}{\log N - \log R_2(N_1, N_2, N)} \)

The assumption of \( d \) seems to be too strong, but we can justify the above assumption in the following way.

**Definition 4.28** (Volume doubling condition). Let \((X, d)\) be a metric space and \( \mu \) be a measure on \( X \). We say \( \mu \) satisfies volume doubling condition with respect to \( d \), we will write \( \mu \) satisfies \((VD)_d\) in short, if there exists \( C > 0 \) such that

\[
V_d,\mu(x, 2r) \leq CV_d,\mu(x, r) \quad \text{for any} \quad x \in X \quad \text{and} \quad r > 0.
\]

If \( \mu \) is fixed or obvious, we also say \((VD)_d\) holds if \( \mu \) satisfies \((VD)_d\).

**Theorem 4.29.** Assume \((G, \mu)\) satisfies condition \((p_0)\) and \((VD)_R\) holds. If \( R \in D_\infty(G) \) and \( V_R(X, r) < \infty \) for any \( r > 0 \) and \( x \in G \), then there exists a fitting metric \( d \) such that \( d \sim R, d(x, y) \approx 1 \) for any \( (x, y) \in E \) and satisfy \((DHK(\beta)),(ARL(\beta)),(BRU(\beta))\) and \((VG(\alpha))\) for some \( 1 \leq \alpha < \beta \).

This theorem is a discrete version of results in [9], and also based on [1]. For the completeness, we give the proof of this theorem in section 6. Combining these theorem, we get the following corollary.

**Corollary 4.30.** Assume \( \mu \) satisfies \((VD)_R\), \((G, \mu)\) satisfies \( \mu_{xy} \approx 1 \) for any \( (x, y) \in E \), \( V_R(X, r) < \infty \) for any \( r > 0 \), \( x \in G \) and \( \text{diam}(X, R) = \infty \). If the metric \( d \), taken in Theorem 4.29 satisfies basic framework (with respect to some minimal connected partition \( K \)) and

\[
\frac{\log R_2(N_1, N_2, N)}{\log N} \leq \frac{\log R_2(N_1, N_2, N)}{\log N} \quad (4.2)
\]

for some \( N, N_1 \geq 0 \) and sufficiently large \( N_2 > N_1 \). Then

\[
\dim_{AR}(G, R) \leq d_S(G, \mu) \leq d_S(G, \mu) < 2
\]

The condition \((4.2)\) holds in natural settings, including Sierpiński carpets or \( n \)-gaskets. We give an interesting example in Example 5.3 such that assumptions of Corollary 4.30 but \((4.2)\) hold. It helps to understand the difference between \( R \) and \( \bar{R} \).

**proof of Corollary 4.30.** Since \( d \) satisfies \((VG(\alpha))\) and \((DHK(\beta))\),

\[
d_S(G, \mu) \leq 2 \limsup_{n \to \infty} \frac{\log V_d(x, n^{1/\beta})}{\log n}
\]

\[
\leq 2 \limsup_{n \to \infty} \frac{\log V_d(x, 1) + \log n^{n/\beta}}{\log n} = 2 \frac{\alpha}{\beta} < 2.
\]
On the other hand, since by definition and Theorem 4.27,
\[ d_S^2(N_1, N_2, N) = \left(1 - \frac{\log R_2(N_1, N_2, N)}{\log N_*}\right)^{-1}, \]
\[ d_S(G, \mu) = \left(1 - \frac{\log R_2(N_1, N_2, n)}{\log N}\right)^{-1}, \]
(note that \( \text{diam}(G, d) = \infty \) because \( R \sim QSd \)) and hence
\[ d_S^2(N_1, N_2, N) \leq d_S(G, \mu) \leq d_S(G, \mu) < 2. \]

Since \( d_S^2(N_1, N_2, N) < 2 \) and Theorem 4.14, \( \dim_{\text{AR}}(G, d) \leq d_S^2(N_1, N_2, N) \).
Moreover, \( \dim_{\text{AR}}(G, R) = \dim_{\text{AR}}(G, d) \) because \( R \sim QSd \), so this shows
\[ \dim_{\text{AR}}(G, R) \leq d_S(G, \mu) \leq d_S(G, \mu) < 2. \]

In the rest of this section, we prove Theorem 4.27. First we give a general lemmas and definition.

**Lemma 4.31.** Let \( g \) be a thick discrete weight function, then for any \( M \geq 1 \), there exists \( \eta = \eta_M > 0 \) such that for any \( w \in T_e \), there exists \( x \in K_w \) such that
\[ U_d^d(x, \eta g(\pi(w))) \subseteq K_w. \]

**Proof.** If \( M = 1 \), the statement follows from the definition of \( U_1^d(x, s) \). We prove the rest using induction.

Assume the statement for \( M \geq 1 \) holds. Let \( w \in T_e \) and \( x \in K_w \) such that \( U_d^d(x, \eta g(\pi(w))) \subseteq K_w \).

- If \( \Lambda^g_{\eta g(\pi(w)), 0}(x) = \emptyset \), then \( U_d^d(x, \eta g(\pi(w))) = \{x\} \subseteq K_w \).
- Otherwise, we can take \( v \in \Lambda^g_{\eta g(\pi(w)), 0}(x) \) and \( y \in K_v \) such that
  \[ U_d^d_M(y, \eta^2 g(\pi(w))) \subseteq U_d^d_M(y, \eta g(\pi(v))) \subseteq K_v \]
  and hence
  \[ U_d^d_M+1(y, \eta^2 g(\pi(w))) \subseteq \{K_u \mid u \in \Lambda^g_{\eta g(\pi(w)), 1}(v)\} \subseteq K_w. \]

**Lemma 4.32.** Let \( d \in D_\infty(G) \). If \( d \) is \( M_0 \)-adapted, then \( M \)-adapted for any \( M \geq M_0 \).
Definition 4.33. For \( \alpha_1 \)
\[
B_d(x, \alpha_1) \subseteq U^{d}_{\alpha_0}(x, r) \subseteq U^d_M(x, r).
\]

On the other hand, if \( y \in U^d_M(x, r) \setminus \{x\} \), then there exists \( \{w_i\}_{i=0}^n \subseteq \Lambda^d_r \) with \( n \leq M \) such that \( x \in K^d_{w_0}, y \in K^d_{w_n} \) and \( K^d_{w_{i-1}} \cap K^d_{w_i} \neq \emptyset \) for any \( i \in [1, n] \). Hence \( d(x, y) \leq \sum_{i=0}^n d(w_i) \leq (M + 1)r \), therefore \( U^d_M(x, r) \subseteq B_d(x, (M + 2)r) \).

**Proof.** Since \( d \) is \( M_0 \) adapted, there exists \( \alpha_1 \),
\[
B_d(x, \alpha_1) \subseteq U^{d}_{\alpha_0}(x, r) \subseteq U^d_M(x, r).
\]

In the rest of this section, we assume \( d \) satisfies basic framework, \( K \) is connected, \( \mu_{xy} \neq 1 \) and \( d(x, y) \neq 1 \) for any \( (x, y) \in E \). We write \( \eta_0 > 0 \) such that \( \eta_0^{-1} \mu[w] \leq d(w) \leq \eta_0 \mu(w) \) and \( N_* = \sup_{w \in T} \#(S(w)) \) (remark that \( N_* \leq \sup_{w \in T \setminus \Lambda_r} \#(S(w)) \vee 2r^{-1} < \infty \)). Moreover, since \( d(x, y) \geq 1 \) for any \((x, y) \in E \) and \( d(w) \approx r^{|w|} \) for \( w \in \Lambda_e \), there exist \( m_0, m_1 \in \mathbb{Z} \) such that \( m_0 \leq |w| \leq m_1 \) for any \( w \in \Lambda_e \).

**Lemma 4.34.** \( \sup_{w \in T} \#(\Gamma_1(w)) < \infty \).

**Proof.** If \( w \in T_e \), then for \( k > 2 \log \eta_0 / \log 2 \),
\[
(\Gamma_1(w)) \leq \#(\{v \in (T)[w] \mid \text{there exists } v', w' \in \Lambda^d_r[w] \text{ such that } v \in T_{v'}, w \in T_{w'} \text{ and } K_{v'} \cap K_{w'} \neq \emptyset\})
\]
\[
\leq N_*^k \sup_{w'} \#(\Lambda^d_{\eta_0 r'[w]}(w')).
\]

Otherwise, \( \#(\Gamma_1(w)) \leq \sup_{x \in G} \#(\{y \mid y \sim x\}) + 1 \). Since \( d \) is uniformly finite and \((G,E)\) is bounded degree, these values are bounded. \( \square \)

We write \( L_* = \sup_{w \in T} \#(\Gamma_1(w)) \).

**Lemma 4.35.** Let \( N_1 \geq 0 \) and \( \lambda \in (0, 1) \), then there exists \( N_2 \) and \( \xi > 0 \) such that for any \( x \in G \) and \( w \in T_e \) such that \( x \in K_w \),
\[
U_{N_1}(w) \subseteq B_d(x, \lambda \xi r[w]), \quad B_d(x, \xi r[w]) \subseteq U_{N_2}(w)
\]  \hspace{1cm} (4.3)

**Proof.** Since \( d \) is \( M_* \) adapted, there exists \( \eta_1 \) such that
\[
U_{N_1}(w) \subseteq U^d_{N_1 \vee M_*}(x, \eta_0 r[w]) \subseteq B_d(x, \eta_1 r[w])
\]
and
\[
B_d(x, \lambda^{-1} \eta_1 r) \subseteq U^d_{N_1 \vee M_*}(x, \lambda^{-1} \eta^2_1 r)
\]
for any \( x \in G \) and \( w \in T \) such that \( x \in K_w \). Let \( m = \lceil -\log(\lambda^{-1} \eta_0 \eta_1^2) / \log r \rceil \), then
\[
U^d_{N_1 \vee M_*}(x, \lambda^{-1} \eta^2_1 r[w]) \subseteq U_{(N_1 \vee M_*)+1}(\pi^m(w)) \subseteq U_{N_2}(\pi_{(N_1 \vee M_*)+1}(w))
\]
because \( K \) is connected. Adjusting constants, we get desired result. \( \square \)
Lemma 4.36. Assume \((\text{VG}(\alpha))\) holds. Then \(\#(S^{m_1-w}(w)) \approx V_d(x, r[w])\) for any \(w \in T_e\) and \(x \in K_w\).

Proof. Since \(\mu \asymp 1\), \(K\) is minimal and \((G, E)\) has bounded degree,

\[
\mu(K_w) \approx \sum_{v \in T_v \cap \Lambda} \mu(K_v) \asymp \#(\{v \mid v \in T_v \cap \Lambda\}).
\]

Moreover, since \(m_0 \leq |v| \leq m_1\) for any \(v \in \lambda_e\),

\[
\#(S^{m_1-w}(w)) \geq \#(\{v \mid v \in T_v \cap \Lambda\})
\]

\[
\geq \#(S^{(m_0-w)\cap 0}) \geq N^2 \#(S^{m_1-w}(w)).
\]

On the other hand, let \(w \in T_e\) and \(x \in K_w\). Then there exist \(\eta_1 > 0\) such that

\[
K_w \subseteq U^d_{M_*}(x, \eta r[w]) \subseteq B_d(x, \eta_1 r[w])
\]

because \(d\) is \(M_*\)-adapted. This and \((\text{VG}(\alpha))\) shows there exists \(C_1 > 0\) such that \(\mu(K_w) \leq C_1 V_d(x, r[w])\).

Moreover, since \(d\) is thick, there exists \(\eta_2, \eta_3 > 0\) and \(x' \in K_w\) such that

\[
K_w \supseteq U^d_{M_*}(x', \eta_2 r^{-1}[w]-1) \supseteq B_d(x', \eta_3 r[w]).
\]

Note that \(d(x, x') \leq d(w) \leq \eta r[w]\), this and \((\text{VG}(\alpha))\) shows there exists \(C_2 > 0\) such that

\[
V_d(x, r[w]) \leq V_d(x', (1 + \eta)r[w]) \leq C_2 V_d(x', \eta_3 r[w]) \leq C_2 \mu(K_w).
\]

Therefore

\[
\#(S^{m_1-w}(w)) \asymp \mu(K_w) \asymp V_d(x, r[w])
\]

for any \(w \in T_e\) and \(x \in K_w\). \(\square\)

Lemma 4.37. Fix any \(w \in T\) such that \([w] \leq m_0\). Then

\[
R(U_{N_1}(\pi^k(w)), U_{N_2}(\pi^k(w))c)^{-1} \asymp \mathcal{E}(N_1, N_2, N)
\]

for any \(k \geq 0\).

Proof. Let \(u, v \in (T)_w\). If \(x \in K_u, y \in K_v\) and \(x \sim y\), then there exists \(\omega \in \Sigma^*\) such that \(K_\omega = \{x, y\}\). Since \([w] \leq m_0\), \(\omega[w] \in T_e\). Then \(K_\omega \cap K_v \neq \emptyset\) and \(K_\omega \cap K_v \neq \emptyset\), so \(L_{(w)}(u, v) \leq 1\). Now, let \(f\) be a function on \((T)_w\) such that \(f \equiv 1\) on \(S^k(\Gamma_{N_1}(\pi^k(w)))\) and \(f \equiv 0\) on \(S^k(\Gamma_{N_2}(\pi^k(w)))^c\). Define \(\mathcal{F}\) on \(G\) by \(\mathcal{F}(x) = \max_{w \in K_w} f(w)\), then \(\mathcal{F} \equiv 1\) on \(U_{N_1}(\pi^k(w))\) and \(\mathcal{F} \equiv 0\) on \(U_{N_2}(\pi^k(w))^c\) because

- if \(x \in U_{N_1}(\pi^k(w))\), then there exists \(v \in S^k(\Gamma_{N_1}(\pi^k(w)))\) such that \(x \in K_v\) by (PG1).
- If \(x \not\in U_{N_2}(\pi^k(w))\), then for any \(v \in S^k(\Gamma_{N_2}(\pi^k(w)))\), \(x \not\in K_v\) by (PG1).
Hence, (since $\mu_{xy} \cong 1$) there exists $C > 0$,
\[
\frac{1}{2} \sum_{x \sim y} \left| \mathcal{J}(x) - \mathcal{J}(y) \right|^2 \mu_{xy} \leq \frac{1}{2} C \sum_{x \sim y} \sum_{u \in K_u} \sum_{v \in K_v} |f(u) - f(v)|^2 \\
\leq \frac{1}{2} C \sum_{(u,v) \in J^h_{[w]}} \sum_{x \in K_u} \sum_{y \in K_v} |f(u) - f(v)|^2 \\
\leq \frac{1}{2} C N_*^2(m_1 - |w|) \sum_{(u,v) \in J^h_{[w]}} |f(u) - f(v)|^2,
\]
so $R(U_{N_1}(\pi^k(w)), U_{N_2}(\pi^k(w)))^{-1} \leq C N_*^2(m_1 - |w|) \mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N)$.

On the other hand, let $h$ be a function on $G$ such that $h \equiv 1$ on $U_{N_1}(\pi^k(w))$ and $h \equiv 0$ on $U_{N_2}(\pi^k(w))^c$. Define $h(w) = \min_{x \in K_w} h(w)$, then similarly we get $h \equiv 1$ on $S^k(\Gamma_{N_1}(w))$ and $h \equiv 0$ on $S^k(\Gamma_{N_2}(w))^c$.

Let $x, y \in G$. If $x \in K_u$ and $y \in K_v$ for some $(u,v) \in J^h_{[w]}.N$, then $l_E(x, y) \leq (N + 1) \sup_{\nu \in \mathcal{F}_{[w]}} \#(K_\nu) \leq 2(N + 1)N_*^{m_1 - |w|}$ because $K$ is connected. Therefore
\[
\frac{1}{2} \sum_{(u,v) \in J^h_{[w]}.n} |h(u) - h(v)|^2 \leq \frac{1}{2} \sum_{(u,v) \in J^h_{[w]}.n} \sum_{x \in K_u} \sum_{y \in K_v} |h(x) - h(y)|^2 \\
\leq \frac{1}{2} \sum_{x : y \in E(x,y) \leq l_0} \sum_{x \in K_u} \sum_{y \in K_v} |h(x) - h(y)|^2 \\
\leq \frac{1}{2} N_*^{2(m_1 - |w|)} \sum_{x : y \in E(x,y) \leq l_0} |h(x) - h(y)|^2,
\]
where $l_0 = 2(N + 1)N_*^{m_1 - |w|}$. Remark that if $l_E(x, y) \leq l_0$, then $|h(x) - h(y)|^2 \leq l_0 \sum_{i=1}^n |h(x_{i-1}) - h(x_i)|^2$ for some $n$-path $\{x_i\}_{i=0}^n$ between $x$ and $y$ with $n \leq l_0$,

so for $E_{x,y,l_0} = \{ (p,q) \mid p \sim q, p \in B_{l_E}(x,l_0), \text{ and } q \in B_{l_E}(y,l_0) \}$, $E_{x,y,l_0} \subseteq E$,
\[
\sum_{x,y \in E(x,y) \leq l_0} |h(x) - h(y)|^2 \leq l_0 \sum_{x,y \in E(x,y) \leq l_0} |h(p) - h(q)|^2 \\
\leq l_0 \sum_{p \sim q} \sum_{x \in B_{l_E}(x,l_0)} |h(p) - h(q)|^2 \\
\leq l_0 (\sup_{x \in G} \#(\{ y \mid y \sim x \}))^2 l_0 \sum_{p \sim q} |h(p) - h(q)|^2.
\]
These inequalities with $\mu_{xy} \cong 1$ shows
\[
C'R(U_{N_1}(\pi^k(w)), U_{N_2}(\pi^k(w)))^{-1} \geq \mathcal{E}_{2,k,\pi^k(w)}(N_1, N_2, N) \text{ for some } C' > 0.
\]

\[\square\]

**proof of Theorem 4.27** Fix $N_1, N \geq 0$ and let $w \in T$ such that $|w| \leq m_0$. Then by Lemma 4.35 and $d$ is adapted, there exists $\xi$ and $\zeta$ such that for sufficiently
large $N_2$, for any $x \in K_w$,

$$x \in U_{N_1}(w) \subseteq B_d(x, \lambda \xi r^{[w]})$$

and

$$B_d(x, \xi r^{[w]}) \subseteq U_{N_2}(w) \subseteq U_d^{[w]}(\eta r^{[w]}) \subseteq B_d(x, \xi r^{[w]}).$$

Hence

$$R(x, B_d(x, \xi r^{[w]})^c) \geq R(U_{N_1}(w), U_{N_2}(w)^c) \geq R(B_d(x, \lambda \xi r^{[w]}), B_d(x, \xi r^{[w]})^c).$$

These with Lemmas 4.36 and 4.37 show there exist $C_1, C_2$ such that

$$C_1 R(x, B_d(x, \xi r^{[w]})^c)V_d(x, \xi r^{[w]}) \geq (E_{2,k,\pi^k(w)}(N_1, N_2, N))^{-1}#(S^{m_1-[w] + k}(\pi^k(w)))$$

$$\geq C_2 R(B_d(x, \lambda \xi r^{[w]}), B_d(x, \xi r^{[w]})^c)V_d(x, \xi r^{[w]})$$

for any $k \geq 0$. This and (ARL(β)),(BRU(β)) imply that there exist $\delta > 0$ such that

$$-k\beta \log r - \delta \leq \log #(S^{m_1-[w] + k}(\pi^k(w))) - \log E_{2,k,\pi^k(w)}(N_1, N_2, N)$$

$$\leq -k\beta \log r + \delta$$

and hence by Lemma 4.26

$$\log N' + \beta \log r = \liminf_{k \to \infty} \log E_{2,k,\pi^k(w)}(N_1, N_2, N)^{1/k}$$

because $N_1^{m_1-[w]}#(S^k(\pi^k(w))) \geq #((S^{m_1-[w]} + k(\pi^k(w))) \geq #(S^k(\pi^l(w))))$. This equation also holds for $\pi^l(w)$ with $l \geq 0$, so again using Lemma 4.26 we obtain

$$\log N' + \beta \log r = \sup_{l \geq 0} \liminf_{k \to \infty} \log E_{2,k,\pi^k(w)}(N_1, N_2, N)^{1/k} = \log R_2(N_1, N_2, N).$$

Now, by (DHK) and (VG)$_\alpha$,

$$\frac{d_S}{2} = \liminf_{n \to \infty} \log V_d(x, n^{1/\beta}) \log n = \liminf_{r \to \infty} \log V_d(x, r^{1/\beta}) \log r = \liminf_{k \to \infty} \log V_d(x, r^{-k}) \log r^{-\beta k}$$

and by Lemma 4.36

$$d_S = 2 \liminf_{k \to \infty} \frac{1}{\beta} \log V_d(x, r^{-k}) \frac{\log N'}{\log N - \log R_2(N_1, N_2, N)}.$$

In the same way, we also get $\overline{d}_S = 2 \frac{\log N'}{\log N - \log R_2(N_1, N_2, N)}$. □
5 Examples

We first see an example of calculation of ARC dimension with Theorem 4.14

**Example 5.1.** Let \( f(n) : \mathbb{Z}_+ \to \mathbb{Z}_+ \) such that \( f(n) \leq n \) for any \( n \). For \( n \geq 0 \), define \( B_n, L_n, X_n \in \mathbb{R}^2 \) by

\[
B_n = [2^n, 2^{n+1}] \times [0, 2^n],
\]

\[
L_n = \bigcup_{j \in \mathbb{Z}} \left( \{(x, y) \mid x = 2^n - f(n)j \} \cup \{(x, y) \mid y = 2^n - f(n)j \} \right),
\]

\[
X_n = B_n \cap L_n.
\]

We also define \( X, G, E \) by

\[
X = \{(t, 0) \mid 0 \leq t \leq 1\} \bigcup \left( \bigcup_{n \geq 0} X_n \right),
\]

\[
G = X \cap \mathbb{Z}^2,
\]

\[
E = \{(p, q) \in G \times G \mid d_2(p, q) = 1\},
\]

where \( d_2 \) is the Euclidean metric in \( \mathbb{R}^2 \). See examples for Figure 5.1 or Figure 5.2.

Next we introduce a partition of \((G, E)\). For \( m, a, b \in \mathbb{Z} \), define

\[
S_{m, a, b} = \{(x, y) \mid 2^m a \leq x + y \leq 2^m (a + 1), \ 2^m b \leq x - y \leq 2^m (b + 1)\}
\]

\[
T_m = \{S_{m, a, b} \mid (S_{m, a, b})^o \cap X \neq \emptyset\}
\]

\[
T = \bigcup_{m \in \mathbb{Z}} T_m
\]

and for \( w \in T_m \), define \( \pi(w) \) as the unique element in \( T_{m-1} \) such that \( w \subseteq \pi(w) \) as subsets of \( \mathbb{R}^2 \). Then \((T, \pi)\) is a bi-infinite tree, and let \( \phi = S_{0, 0, 0} \) then \( T_m = (T)_m \). We also define \( K : T \to \{ \text{finite subsets of } G \} \) by

\[
K_w = \begin{cases} 
 w \cap G \text{ (as subsets of } \mathbb{R}^2) & \text{if } [w] \leq 0 \\
 \pi[w](w) \cap G \text{ (as subsets of } \mathbb{R}^2) & \text{if } [w] > 0
\end{cases}
\]
then \( K_w \) is a partition of \((G, E)\). Moreover, \( \Lambda_c = (T)_0 \) and \( T = T_{1/2} \). Now we let \( d = l_E \), and calculate \( \dim_{AR}(G,d) \).

**Proposition 5.2.**

1. If \( \limsup_{n \to \infty} f(n) = \infty \), then \( \dim_{AR}(G,d) = 2 \).
2. If \( \limsup_{n \to \infty} f(n) < \infty \), then \( \dim_{AR}(G,d) = 1 \).

**Proof.** (1) First we check \( d, K \) satisfy assumptions of Theorem 4.14

- By definition, \( \#(S(w)) \leq 4 \) for any \( w \in T \) and \( K \) is minimal. It is easily seen that \( d \) fits to \((G, E)\).
- \( d(w) = 2^{-m} = (1/2)^m \) for any \( m \leq 0 \) and \( w \in (T)_m \).
- (uniformly finite) Similar to Example 2.7.

\[
\Lambda_d^m = \begin{cases} 
(T)_{-m} & \text{if } 2^m \leq s < 2^{m+1}, \\
\emptyset & \text{if } s < 1.
\end{cases}
\]

Hence \( \#(\Lambda_d^m(w)) \leq \#(\{(v \in (T)_{|w|} \mid v \cap w \neq \emptyset \text{ as subsets of } \mathbb{R}^2\}) \leq 9 \) for any \( s > 0 \) and \( w \in \Lambda_d^m \). This shows \( d \) is uniformly finite.

- (thick) Let \( w = S_{m,a,b} \in (T)_{-m} \) for some \( m \geq 0 \).
  - If \( m \geq 1 \), then \( \Lambda_{d(\pi(w))/8,1}^d(x_w) = \Lambda_{2^{m-2},1}^d(x_w) = S^2(w) \) for \( x_w = (2^{m-1}(a+b+1), 2^{m-1}(a-b)) \).
  - If \( m = 0 \), then either \( (\frac{a+b}{2}, \frac{a-b}{2}) \) or \( (\frac{a+b+1}{2}, \frac{a-b+1}{2}) \) belongs to \( K_w \).

Let \( x_w \) be such a point, then \( \Lambda_{d(\pi(w))/4,1}^d(x_w) = \emptyset \).

Hence \( d \) is thick.

- (1-adapted) Similar to Lemma 4.32. \( U_d^d(x, r) \subseteq B_d(x, 3r) \). On the other hand, if \( r \geq 1 \), then \( U_d^d(x, r) = U_1^1(x, 2^n) \supseteq B_d(x, 2^n) \supseteq B_d(x, r/2) \) for some \( n \), hence \( d \) is 1-adapted. (See Figure 5.3).

Therefore \( d, K \) satisfy the assumptions of Theorem 4.14. Now we adapt Theorem 4.14 and show \( \dim_{AR}(G,d) = 2 \).

The first step is to show \( \dim_{AR}(G,d) \geq 2 \). Since \( \sup_n f(n) = \infty \), for any \( k \geq 0 \), there exists \( m \in \mathbb{N} \) and \( w = S_{m,a,b} \in (T)_{-m} \) such that

\[
\{S_{m-k,i,j} \mid i \in [2^k(a-2) - 1, 2^k(a+2) + 1]_Z, \ j \in [2^k(b-2) - 1, 2^k(b+2) + 1]_Z \} \\
\subseteq (T)_{-(m-k)}.
\]

(5.1)
Let $g$ be a function on $(T)-(m-k)$ such that $g \equiv 1$ on $S^k(\Gamma_0(w))$ and $g \equiv 0$ on $S^k(\Gamma_2(w))^c$. We also let $\tilde{g} = (g \lor 0) \land 1$, then for any $p \geq 1$,

$$\sum_{(u,v) \in E^h_{(m-k)}} |g(u) - g(v)|^p \geq \sum_{(u,v) \in E^h_{(m-k)}} |\tilde{g}(u) - \tilde{g}(v)|^p$$

$$\geq \sum_{i \in [2^k a, 2^{k(a+1)}]} \sum_{j \in [2^k(b-2), 2^k b] \setminus 2^k \mathbb{Z}} |\tilde{g}(S_{(m-k),i,j}) - \tilde{g}(S_{(m-k),i,j-1})|^p$$

$$\geq \sum_{i} (2^k+1)^{-p} \geq C 2^{(2-p)k}$$

for some $C > 0$, because of the Jensen’s inequality, $\tilde{g}(S_{(m-k),i,2^k b}) = 1$ and $\tilde{g}(S_{(m-k),i,2^k(b-2)-1}) = 0$ for any $i \in [2^k a, 2^k(a+1)] \setminus \mathbb{Z}$. Moreover for $p < 1$,

$$\sum_{(u,v) \in E^h_{(m-k)}} |g(u) - g(v)|^p \geq \sum_{(u,v) \in E^h_{(m-k)}} |\tilde{g}(u) - \tilde{g}(v)|^p$$

$$\geq \sum_{j \in [2^k(b-2), 2^k b] \setminus 2^k \mathbb{Z}} |\tilde{g}(S_{(m-k),2^k a},j) - \tilde{g}(S_{(m-k),2^k a,j-1})| \geq 1.$$ 

Therefore $\lim_{q \to \infty} E_{p,k}(0,2,1) > 0$ for any $p \leq 2$, hence $\dim_{AR}(G,d) \geq 2$.

On the other hand, define $g = g_w$ on $E^h_{(m-k)}$ by

$$g(S_{m-k,i,j}) = \left( \frac{(2^k(a+2) - i) \land (i - 2^k(a-2)) \land (2^k(b+2) - j) \land (j - 2^k(b-2))}{2^k} \lor 0 \right) \land 1,$$

then $f \equiv 1$ on $S^k(\Gamma_0(w))$, $f \equiv 0$ on $S^k(\Gamma_2(w))^c$ and

$$\sum_{(u,v) \in E^h_{(m-k)}} |f(u) - f(v)|^p \leq \sum_{v \in S^k(\Gamma_2(w))} 8 \cdot 2^{-kp} \leq C' 2^{(2-p)k}$$

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for some $C' > 0$, hence $\mathcal{E}_{p,k,w}(0, 2, 1) \leq C''2^{(2-p)k}$. Moreover, for any $v \in T$, this upper bound holds because of the definitions of $T$ and $K$. Therefore $\lim_{k \to \infty} \mathcal{E}_{p,k,w}(0,1,2) = 0$ for any $p > 2$ and hence $\dim_{\text{AR}}(G, d) = 2$.

(2) We show $\dim_{\text{AR}}(G, d) = 1$. Let $m(A) = \#(A)$ for any $A \subset G$, then for any $n \geq 0$,

$$m(B_d(x, r) \cap G_n) \leq 2(2^f(n) + 1)(\text{diam}(B_d(x, r) \cap G_n, d) + 1)$$

where $G_n = X_n \cap G$, because $G_n$ consists of $2(2^f(n) + 1)$ segments whose length are $2^n$. Hence there exists $C$ such that for any $x \in G$ and $r \geq 1$,

$$r \leq V_d(x, r) \leq 1 + \sum_{n \geq 0} m(B_d(x, r) \cap G_n) \leq Cr,$$

because $\sum_{n \geq 0} \text{diam}(B_d(x, r) \cap G_n, d) \leq 2r$ and $\sup_n f(n) < \infty$. Therefore $d$ is 1-Ahlfors regular. On the other hand, $\dim_{\text{AR}}(G, d) \geq 1$ by Proposition 4.20 and hence $\dim_{\text{AR}}(G, d) = 1$. \hfill $\square$

Remark. If we use a partition parallel to axes, that is, a partition $K'$ defined by $S'_{m,a,b} = [2^m a, 2^m(a + 1)] \times [2^m b, 2^m(b + 1)]$ in similar way to $K$, then $K'$ is not minimal. For example, both $S'_{0,0,0}$ and $S'_{0,1,0}$ include a edge $((1, 0), (1, 1)) \in E$. So we need some modification to apply 4.14 to $d, K'$.

The next example is that $d_s(G, \mu) \neq d^{S}_2 = d^{S}_2$ although $d$ satisfies (DHK($\beta$)), (ARL($\beta$)) and (BRU($\beta$)).

Example 5.3. Let $f : \mathbb{N} \to \{0, 1\}$, $G_0 = \{0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i\} \in \mathbb{C}$ and $E_0 = \{(x, y) \in G_0 \times G_0 \mid x \neq y\}$. We inductively define $G_n, E_n$ and other notations by $|G_n| = \max_{z \in G_n} |z|$,

\begin{align*}
F_{n,1}(z) &= z, & F_{n,2}(z) &= z + |G_{n-1}|, \\
F_{n,3}(z) &= z + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) |G_{n-1}|, & F_{n,4}(z) &= z + 2 |G_{n-1}|, \\
F_{n,5}(z) &= z + \left(1 + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right) |G_{n-1}|, & F_{n,6}(z) &= z + 2 \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) |G_{n-1}|, \\
F_n(z) &= \begin{cases} 
\cup_{i=1}^{3} F_{n,i} & \text{if } f(n) = 0, \\
\cup_{i=1}^{6} F_{n,i} & \text{if } f(n) = 1,
\end{cases}
\end{align*}

$G_n = F_n(G_{n-1})$ and

$E_n = \{(x, y) \in G_n \times G_n \mid \text{there exist } x', y' \in G_{n-1} \text{ and } i \geq 0$

such that $(x', y') \in E_{n-1}$ and $x = F_{n,i}(x'), \ y = F_{n,i}(y')\}$.

Note that $|G_{n}| = 2^{n-m(n)} \cdot 3^{m(n)}$ where $m(n) = \#(\{k \mid k \leq n, f(k) = 1\})$. Let $G = \cup_{n \geq 0} G_n$ and $E = \cup_{n \geq 0} E_n$. We also let $\mu \equiv 1$ on $E$ and consider the effective resistance $R$ of $(G, \mu)$.
Figure 5.5: $G_n$ (if $f(n) = 1$).  Figure 5.6: $G_n$ (if $f(n) = 0$).

Note that
\[
R(x, y)^{-1} \geq 1 \quad \text{for any } (x, y) \in E,
\]
\[
R(x, y)^{-1} \leq \mathcal{E}(\chi_{\{x\}}) \leq 6 \quad \text{for any } x, y \in G \text{ with } x \neq y,
\]
so $R$ fits to $(G, E)$. We will check properties of $R$ in order to use Theorem 4.29.

For this purpose, we first introduce a partition. For $n \geq 0$ and $a, b \in \mathbb{Z}$, define
\[
\triangle_{0,0,0} = \left\{ s + \left( \frac{1}{2} + \frac{\sqrt{3}}{2} i \right) t \mid s \geq 0, t \geq 0, s + t \leq 1 \right\},
\]
\[
\triangle_{n,a,b} = \{ [G_n](\triangle_{0,0,0} + a + (\frac{1}{2} + \frac{\sqrt{3}}{2} i)b) \mid a, b \in \mathbb{Z} \},
\]
\[
T_{-n} = \{ \triangle_{n,a,b} \mid \triangle_{n,a,b} \subseteq \bigcup_{m \geq n} F_m \circ F_{m-1} \circ \cdots \circ F_n(\triangle_{0,0,0}) \},
\]
\[
K_{n,a,b} = \triangle_{n,a,b} \cap G \quad \text{(as subsets of } \mathbb{C}).
\]

For any $n \geq 1$, we let $T_n = \bigcup_{w \in T_0} \cup_{x, y \in K_w} \{ x, y \}$ and $K_w = w$ for any $w \in T_n$. Define $T = \bigcup_{n \in \mathbb{Z}} T_n$ and $\pi(w)$ for $w \in T_n$ as the unique elements in $T_{n-1}$ such that $K_w \subseteq K_{\pi(w)}$. Then $(T, \pi)$ is a bi-infinite tree, $(T)_n = T_n$ with $\phi = \triangle_{0,0,0}$. $K$ is minimal connected partition and $\Lambda_e = (T)_1$. If necessary, we replace $T, K$ by $T_r, K'$ for $r \in (0, 1)$ in the way of Definition 4.11.

**Lemma 5.4.** Let $\mathfrak{R}(n) = \left( \frac{5}{3} \right)^{n-m(n)} \left( \frac{13}{17} \right)^{m(n)}$ for any $n \geq 0$ and let
\[
\min \{ n \geq 0 \mid \text{there exist } w, v \in (T)_{-n} \text{ such that } x \in K_w, y \in K_v \text{ and } K_w \cap K_v \neq \emptyset \}
\]
for $(x, y) \in G$. Then $R(x, y) \approx \mathfrak{R}(n(x, y))$ for any $x, y \in G$ with $x \neq y$.

**Proof.** We first evaluate $R(w)$. By the method of Laplacian on finite set (see [8]), $R(0, [G_n]) = \frac{2}{3} \mathfrak{R}(n)$ and
\[
\min \left\{ \frac{1}{2} \sum_{(x,y) \in E_n} |f(x) - f(y)|^2 \mid f : G_n \to \mathbb{R}, f(0) = 1, \quad f([G_n]) = f\left( \left( \frac{1}{2} + \frac{\sqrt{3}}{2} i \right) [G_n] \right) = 0 \right\} = 2 \mathfrak{R}(n)^{-1}. \quad (5.2)
\]
Hence for any $\triangle_{n,a,b} \in T$,

$$\frac{2}{3} \mathfrak{R}(n) \geq R((a + \frac{1}{2} + \sqrt{3}/2)i)b|G_n|, (a + 1 + \frac{1}{2} + \sqrt{3}/2)i)b|G_n|) \geq \frac{1}{6} \mathfrak{R}(n),$$

and since $\mathfrak{R}(n - 1) \leq \frac{2}{3} \mathfrak{R}(n)$, we obtain $R(w) \asymp R(n)$ for any $n \geq -1$ and $w \in (T)_{-n}$ by using the sum of a geometric series. Fix any $x, y \in G$, and let $w, v \in (T)_{-n(x,y)}$ such that $x \in K_w, y \in K_v$ and $K_w \cap K_v \neq \emptyset$. Then by (5.2), there exists $C > 0$ such that

$$R(w) + R(v) \geq R(x, y) \geq \frac{\mathfrak{R}(n - 1)}{2|(G_{w'})|} \geq C\mathfrak{R}(n)$$

where $w' \in S(w)$ such that $x \in K_{w'}$. Therefore $R(x, y) \asymp \mathfrak{R}(n(x, y))$. 

This lemma also implies $R$ is 1-adapted and $V(x, R(x, y)) \asymp \mathfrak{V}(n(x, y))$ where $\mathfrak{V}(n) = 3^{n - \mu(n)} \cdot 6^{\mu(n)}$. This inequality also shows (VD)$_R$, and $(G, \mu)$ satisfies the conditions of Theorem 4.29.

Next we modify $T$ in order to satisfy $d(w) \asymp r^{[w]}$ for some $r \in (0, 1)$, where $d$ is the metric obtained by Theorem 4.29. For $j \geq 0$, let $n(j) \geq 0$ such that $\mathfrak{R}(n(j))\mathfrak{V}(n(j)) \leq \left(\frac{90}{7}\right)^j < \mathfrak{R}(n(j) + 1)\mathfrak{V}(n(j) + 1)$ and for $j < 0$, let $n(j) = j$.

We consider $\widetilde{T} = \bigcup_{j \in \mathbb{Z}} (T)_{-n(j)}$, and $\tilde{\pi}(w) = \pi^{n(j+1)-n(j)}$ for $w \in (T)_{-n(j)}$. Then $(\widetilde{T}, \tilde{\pi}, \triangle_{0,0,0})$ is a bi-infinite tree with a reference point, $(\widetilde{T})_j = (T)_{n(j)}$. 

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and $K|_{T}$ is minimal, connected partition, $A_e = (T)$. Moreover, for any $w \in \cup_{j \geq 0} (T)_{n(j)}$,

$$
\sup_{x, y \in K_w} R(x, y)V(x, R(x, y)) = \mathcal{R}(n(j)) \mathcal{R}(n(j)) \approx \left( \frac{90}{7} \right)^j
$$

and hence $d(w) \approx \left( \frac{90}{7} \right)^{\|w\|/\beta}$ for any $w \in T$, where $\beta$ is the constant in Theorem 4.29. Comparing with $R(x, y)V(x, R(x, y))$, we can also see that $d$ is uniformly finite, thick and 1-adapted because of Lemma 5.4 and (5.2).

Now we let $f(n) = \begin{cases} 1 & \text{if } l(l^2 - 1) < n \leq l^3 \text{ for some } l \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

Then,

**Proposition 5.5.** $d_2^S(G, \mu) = 2 \log 3 / \log 5$ and $\mathcal{R}_2^S(0, 2, 1) = \mathcal{R}_2^S(0, 2, 1) = 2 \log 6 / (\log 90 - \log 7)$.

**Proof.** Let $w = \Delta_{n(j), 0, 0}$ for some $j \geq 0$. With the $\Delta$-Y transform (see [8, Lemma 2.1.15]), we can see that $\mathcal{E}_{p, k, \pi^k(w)}(0, 2, 1) \approx \mathcal{R}(n(k + j))/\mathcal{R}(n(j))$ (see Figure 6), so

$$
\lim_{k \to \infty} \frac{1}{k} \log \mathcal{E}_{p, k, \pi^k(w)}(0, 2, 1) = \lim_{k \to \infty} \frac{1}{k} \log \frac{\mathcal{R}(n(k + j))}{\mathcal{R}(n(j))} = \lim_{k \to \infty} \frac{1}{k} \log \mathcal{R}(n(k)) = \lim_{k \to \infty} \frac{1}{k} \left( n(k) \log \frac{3}{5} + m(n(k)) \log \frac{7}{15} \right).
$$

Now we consider $\lim_{k \to \infty} n(k)/k$. By definition, we obtain

$$
k \geq n(k) \geq k \log 90 - \log 7 \log 5 - m(n(k)) - C
$$

for some $C > 0$. Note that $\lim_{k \to \infty} m(k)/k = \lim_{k \to \infty} k^{-1/3} = 0$ because $m(k^3) = \sum_{j=1}^k j = k(k - 1)/2$, hence $\lim_{k \to \infty} k/n(k) = \log 5 / (\log 90 - \log 7)$. Therefore by Lemma 4.26

$$
\mathcal{R}_2(0, 2, 1) = \sup_{j \geq 0} \lim_{k \to \infty} \frac{1}{k} \log \mathcal{E}_{p, k, \pi^k(\Delta_{n(j), 0, 0})}(0, 2, 1) = \frac{\log 90 - \log 7}{\log 5} \log \frac{3}{5}.
$$

Figure 5.8: $\mathcal{E}_{p, k, \pi^k}(0, 2, 1) \approx \mathcal{R}(n(k + j))/\mathcal{R}(n(j))$. 
Similarly we get

\[ \mathcal{N} = \mathcal{N} = \lim_{k \to \infty} \frac{1}{k} \log \#(S^k(\triangle_{0,0,0})) = \lim_{k \to \infty} \frac{1}{k} (n(k) \log 3 + m(n(k)) \log 6) = \frac{\log 90 - \log 7}{\log 5} \log 3. \]

Therefore by Theorem 4.27, \( d_S(G, \mu) = 2 \log 3 / (\log 3 - \log \frac{2}{3}) = 2 \log 3 / \log 5. \)

On the other hand, since

\[ \sup \{ k \mid \text{there exist } a \in \mathbb{N} \text{ such that } f(b) = 1 \text{ for any } b \in [a, a + k] \} = \infty, \]

it follows that

\[ \log N_* = \lim_{k \to \infty} \frac{1}{k} \left( \log 6^k \lor \log 3^{(\log 3 / \log 5)k} \right) = \log 6 \lor \frac{\log 90 - \log 7}{\log 5} \log 3 = \log 6 \]

because \( \log_{10} 6 > 0.77 > 0.76 > \frac{\log 90 - \log 7}{\log 5} \log_{10} 3. \) Similarly,

\[ \log R_2(0, 1, 2) = \log R_2(0, 1, 2) = \log 7 \lor \frac{\log 90 - \log 7}{\log 5} \log \frac{3}{5} = \log \frac{7}{15}. \]

Therefore

\[ \overline{d}_2^S(0, 2, 1) = \overline{d}_2^S(0, 2, 1) = 2 \cdot \frac{\log 6}{\log 6 - \log \frac{7}{15}} = 2 \cdot \frac{\log 6}{\log 90 - \log 7}. \]

\[ \square \]

Remark. In the same way, we can see that \( d_S(G, \mu) = d_2^S(0, 2, 1) = d_2^S(0, 2, 1) = 2 \log 3 / \log 5 \) if \( f \equiv 0 \) (Sierpiński gasket graph) and \( d_S(G, \mu) = d_2^S(0, 2, 1) = \overline{d}_2^S(0, 2, 1) = 2 \log 6 / (\log 90 - \log 7) \) if \( f \equiv 1. \) Clearly the assumptions of Corollary 4.30 holds in these cases.

6 Proof of Theorem 4.29

To show Theorem 4.29, we first prepare some condition.

Definition 6.1 (uniformly shrinking). Let \((X, d)\) be a metric space. We call \((X, d)\) is uniformly shrinking if there exists \(\alpha \in (0, 1)\) such that for any \(x \in G\) and \(r > 0\), \(B(x, r) \setminus B(x, \alpha r) \neq \emptyset\) whenever \(B(x, r) \neq X\) and \(B(x, r) \neq \{x\}\).

Uniformly shrinking condition is an extension of uniformly perfect condition to discrete metric spaces, but clearly it does not imply perfectness of a space.

Lemma 6.2. Let \((X, d), (X, \rho)\) be metric spaces such that \(d \sim QS \rho\). Moreover, assume \((X, d)\) is uniformly shrinking, then

(1) \((X, \rho)\) is uniformly shrinking.
 Lemma 6.3.  Let \((G, \mu)\) be a weighted graph and \(R\) be the associated effective resistance.

(1) \(R(x,y) \geq \max\{\mu_x^{-1}, \mu_y^{-1}\}\) for any \(x, y \in G\) such that \(x \neq y\).

(2) \(R(x,y) \leq \mu_y^{-1}\) for any \(x, y \in G\) such that \(x \sim y\).

(3) If \((G, \mu)\) satisfies the condition \((p_0)\), then \((X, R)\) is fitting and uniformly shrinking.

Proof. Since \((X, d)\) is uniformly shrinking, there exists \(\alpha \in (0, 1)\) such that \(B(x, \alpha d(x, y)) \neq \emptyset\) whenever \(B(x, r) \neq X\) and \(B(x, r) \neq \{x\}\).

(1) Fix \(x \in X\) and \(r > 0\) such that \(B_\rho(x, r) \neq X\) and \(B_\rho(x, r) \neq \{x\}\). Choose \(\delta \in (0, 1)\) such that \(\theta(\delta) < 1\). Let \(y_1 \in X \setminus B_\rho(x, r)\). If \(B_\rho(x, \alpha \delta d(x, y_1)) \neq \{x\}\), then there exists \(y_2\) such that \(\alpha \delta d(x, y_1) \leq d(x, y_2) < \delta d(x, y_1)\) and then \(\lambda_1 \rho(x, y_1) \leq \rho(x, y_2) < \lambda_2 \rho(x, y_1)\) where \(\lambda_1 = (1/\theta(\delta^{-1})\alpha^{-1}) > 0, \lambda_2 = \theta(\delta) < 1\).

We can inductively choose \(y_n\) such that \(\lambda_1 \rho(x, y_n) \leq \rho(x, y_{n+1}) < \lambda_2 \rho(x, y_n)\) whenever \(B_\rho(x, \delta d(x, y_n)) \neq \{x\}\).

- If there exists \(n\) such that \(\rho(x, y_{n+1}) < r \leq \rho(x, y_n)\), then \(y_{n+1} \in B_\rho(x, r) \setminus B(x, \lambda_1 r)\).

- Assume \(r \leq \rho(x, y_n)\) and \(B_\rho(x, \delta d(x, y_n)) = \{x\}\). Let \(y \in B_\rho(x, r) \setminus \{x\}\), then \(\rho(x, y) \geq (1/\theta(\delta^{-1}))\rho(x, y_n)\) because \(d(x, y) \geq \delta d(x, y_n)\). Therefore \(y \in B_\rho(x, r) \setminus B_\rho(x, r/\theta(\delta^{-1}))\).

Therefore \(\rho\) is uniformly shrinking.

(2) We first consider the case \(B_\rho(x, r) \neq \{x\}\). Then by (1), there exist \(\alpha' \in (0, 1)\) and \(z \in B_\rho(x, r)\) such that \(\rho(x, y) < (2/\alpha')\rho(x, z)\) for any \(y \in B_\rho(x, 2r)\) where \(\alpha'\) is independent of \(x\) and \(r\).

Therefore \(B_\rho(x, 2r) \subset B_d(x, 2\overline{\rho}_\rho(x, 2r)) \subset B_d(x, \lambda_3 \overline{\rho}_\rho(x, r))\)

for some \(\lambda_3\). Moreover, if \(d(x, y) < \theta^{-1}(1/\overline{\rho}_\rho(x, r))\), then \(\rho(x, y) < r\). Since \((V D)_d\) holds, there exists \(C > 0\) and

\[V_\rho(x, 2r) \leq V_d(x, \lambda_3 \overline{\rho}_\rho(x, r)) \leq CV_d(x, \theta^{-1}(1/\overline{\rho}_\rho(x, r)) \leq CV_\rho(x, r).\]

Next we assume \(B_\rho(x, r) = \{x\}\). Choose \(y \in B_\rho(x, 2r)\) such that \(d(x, y) \geq (1/2)\overline{\rho}_\rho(x, 2r)\). Then for any \(z \in X \setminus \{x\}, d(x, z) \geq \theta^{-1}(1)d(x, y)\) because \(B_\rho(x, r) = \{x\}\). Since \((V D)_d\) holds, there exists \(C > 0\) such that

\[V_\rho(x, 2r) \leq V_d(x, 2\overline{\rho}_\rho(x, 2r)) \leq CV_d(x, \theta^{-1}(1)d(x, y))\]

and

\[B_\rho(x, r) = \{x\} = B_d(x, \theta^{-1}(1)d(x, y)),\]

so we get desired condition. \(\Box\)
Proof. For any \(x, y \in G\), \(E(\chi(x), \chi(x)) = \mu(x)\) and so \(R(x, y) \geq \max\{\mu_x^{-1}, \mu_y^{-1}\}\). On the other hand, if \(x \sim y\), then \(E(f, f) \geq (1 - 0)^2 \mu_{xy}\) for any \(f\) such that \(f(x) = 1, f(y) = 0\), so \(R(x, y) \leq \mu_{xy}^{-1}\). (1) and (2) also show that \(R\) fits to \((G, E)\) under the condition \((p_0)\). We prove uniformly shrinking condition in two cases.

- Assume \(B(x, r) \neq X\) and \(r > 2 \max_{y:y-x} \mu_{xy}^{-1}\). Choose \(\{x_n\}_{n=1}^k\) such that \(x_1 = x, x_k \notin B(x, r)\) and \(x_n \sim x_{n+1}\) for any \(n\). Then for some \(n < k\), \(x_n \in B(x, r)\) and \(x_{n+1} \notin B(x, r)\). Note that \(x \neq x_n\) because \(x \neq x_{n+1}\).

Since \((G, \mu)\) satisfies the condition \((p_0)\), there exists \(C > 0\) such that \(R(x_n, x_{n+1}) \leq \mu_{x_n,x_{n+1}}^{-1} \leq C \mu_{x_n}^{-1} \leq CR(x, x_n)\), so \((1 + C)R(x, x_n) \geq r\) and \(B(x, r) \setminus B(x, (1 + C)^{-1}r) \neq \emptyset\).

- Assume \(B(x, r) \neq \{x\}\) and \(r \leq 2 \max_{y:y-x} \mu_{xy}^{-1}\). By (1), \(B(x, \mu_x^{-1}) = \{x\}\) so \(B(x, r) \setminus B(x, 2^{-1} \mu_x^{-1}(\max_{y:y-x} \mu_{xy})r) \neq \emptyset\). Since \((p_0)\) holds, \(B(x, r) \setminus B(x, C'r) \neq \emptyset\) for some \(C' \in (0, 1/2]\).

Take \((1 + C)^{-1} \land C'\), we get the uniformly shrinking condition. \(\square\)

We prove Theorem 4.29 in two steps. The first step is to show existence of suitable distance \(d\).

In the rest of this section, we assume that

- \((G, \mu)\) be a weighted graph.
- \(E(f)\) is the associated energy and \(R\) is the associated effective resistance to \(\mu\).

**Proposition 6.4.** Let \((G, \mu)\) be a weighted graph and assume \(R\) satisfies the condition of Theorem 4.29(1),

(1) then there exists a distance \(d\) on \(G\), which satisfies following conditions for some \(\beta > \alpha \geq 1\).

- \(\triangleright\) \(d \sim R\) (\(Q_{\delta}\))
- \(R(x, y)V_d(x, d(x, y)) \geq d^3(x, y)\) for any \(x, y \in G\). (\(R(\beta)\))
- \(V_d(x, r) \leq (Cr^\alpha/s^\alpha)V_d(x, s)\) for any \(x \in G\) and \(r > s > 0\). (\(VG(\alpha)\))

(2) Let \(d\) be a metric in (1). Then we can choose \(d\) also satisfies following conditions.

- \(\inf_{x,y \in G} d(x, y) = 0 > 0\). (dL)
- There exists \(C_+ > 0\) such that \(d(x, y) \leq C_+\) for any \(x, y \in G\) with \(x \sim y\). (NdU)
- For any \(x \in G\) and \(r > 0\), \(B_d(x, r)\) is a finite set. (BF)

To prove this proposition, we prepare a lemma.
Lemma 6.5. Assume \((X, R)\) is uniformly shrinking and \((VD)_R\) holds. Then there exists a homeomorphism \(\eta : [0, \infty) \to [0, \infty)\) such that for any \(t > 0\),

\[
V_R(r, R(x, y)) < \eta(t) V_R(x, R(x, z)) \quad \text{whenever } R(x, y) < tR(x, z)
\]

Proof. If \(t \geq 1\), then there exists \(C_1\) and \(\tau_1\) such that \(V_R(x, R(x, y)) < C_1 t^{\tau_1}\) whenever \(R(x, y) < tR(x, z)\) because of \((VD)_R\). Since \(R\) is uniformly shrinking, there exists \(c \in (0, 1)\) such that \(B_R(x, r/2) \setminus B_{R(x, cr/2)} \neq \emptyset\) whenever \(B(x, r/2) \neq \{x\}\). Choose \(\xi \in B_R(x, r/2) \setminus B_{R(x, cr/2)}\). Then by \((VD)_R\), there exists \(\gamma\), which is independent of \(x, \xi\) and \(r\) such that \(\gamma V_R(x, cr/4) \leq V_R(\xi, cr/4)\).

This implies

\[
(1 + \gamma) V_R(x, cr/4) \leq V_R(x, cr/4) + V_R(\xi, cr/4) \leq V_R(x, r).
\]

Therefore there exists \(C_2\) and \(\tau_2\) such that

\[
V_R(x, R(x, y)) < C_2 t^{\tau_2} V_R(x, R(x, z)) \quad \text{whenever } R(x, y) < tR(x, z)
\]

for \(t \leq 1\) and we obtain desired \(\eta(t) := (C_1 \lor C_2)(t^{\tau_1} \lor t^{\tau_2})\). \(\square\)

Proof of Proposition 6.4. (1) Let \(\varphi(x, y) = R(x, y)(V(x, R(x, y)) + V(y, R(x, y)))\) and \(\eta\) be a function given in Lemma 6.5. Note that for any \(x, y, z \in G\), \(R(x, z)/2 \leq \max\{R(x, y), R(x, z)\}\). Assume \(R(x, z)/2 \leq R(x, y)\), then

\[
R(x, z)V_R(x, R(x, z)) \leq 2\eta(2) R(x, y)V_R(x, R(x, y)) \leq 2\eta(2) (\varphi(x, y) + \varphi(x, z))
\]

and by \((VD)_R\), there exists \(C_1\), which is independent of \(x, y, z\), such that

\[
\varphi(x, z) \leq 2C_1 \eta(2)(\varphi(x, y) + \varphi(x, z)).
\]

So, by \([8\text{ Proposition 14.5}]\), there exist a metric \(d\) on \(G\) and \(\beta > 1\) such that \(\varphi \sim d^\beta\) for any \(x, y \in G\), and \((VD)_R\ shows \(d^\beta(x, y) \sim V_R(x, R(x, y))R(x, y)\). Moreover, this inequality and the above lemma shows that for some \(C_2 > 0\),

\[
d(x, y) \leq C_2 (R(x, y)V_R(x, y))^{1/\beta} \leq C_2 \eta^{1/\beta}(t)(R(x, z)V_R(x, z))^{1/\beta} \leq C_2 d(x, z)
\]

whenever \(R(x, y) < tR(x, z)\). This shows \(R \sim d\).

Next we show \(R(\beta)\). Because \(d \sim R\), there exists \(c > 0\) such that \(d(x, z) < cd(x, y)\) whenever \(R(x, z) < R(x, y)\), and then \(B_R(x, R(x, y)) \subset B_d(x, cd(x, y))\).

Using the same way, we get \(B_d(x, d(x, y)) \subset B_R(x, c'R(x, y))\) for some \(c'\). By Lemma 6.2 \(\mu\) satisfies both \((VD)_d\) and \((VD)_R\). By using them and the above conditions, we get \(V_R(x, R(x, y)) \sim V_d(x, d(x, y))\) for any \(x, y \in G\) and therefore \(R(x, y)V_d(x, d(x, y)) \sim d^\beta(x, y)\).

Finally, we show \((VG)\). Recall that by Lemma 6.2 there exists \(\gamma \in (0, 1)\) such that \(B_d(x, r) \setminus B_d(x, \gamma r) \neq \emptyset\) whenever \(B_d(x, r) \neq \{x\}\). Fix \(x \in G\) and assume \(r > s > 2r_{x,d}\), then there exist \(y \in B_d(x, s) \setminus B_d(x, \gamma s)\) and \(z \in B_d(x, r) \setminus B_d(x, \gamma r)\).

Therefore

\[
\frac{V_d(x, r)}{V_d(x, s)} \leq C_3 \frac{V_d(x, d(x, z))}{V_d(x, d(x, y))} \leq C_4 \left(\frac{r}{s}\right)^\beta \frac{R(x, y)}{R(x, z)}
\]
Lemma 6.2, (VD)

for some \(C_3, C_4 > 0\), because of \((VD)_d\) and \(R(x, y) V_d(x, d(x, y)) \propto d^\beta(x, y).\)

To evaluate \(R(x, z)\), we take \(\tau, \nu > 1\) such that \(\nu R(o, p) \leq R(o, q)\) whenever \(\tau d(o, p) \leq d(o, q)\). Now we let \(y = y_0\) and choose \(y_n \in B_d(x, \gamma^{-1} \tau d(x, y_{n-1})) \setminus B_d(x, \tau d(x, y_{n-1}))\), inductively. Then there exists \(C_5 > 0\) such that \(\nu^n R(x, y) \leq R(x, y_n) \leq C_5 R(x, z)\) for any \(n\) with \((\tau^n / \gamma^n) \leq (d(x, z) / d(x, y))\). Therefore there exists \(C_6, \iota > 0\) such that

\[
\frac{R(x, y)}{R(x, z)} \leq C_6 \left(\frac{s}{r}\right)\iota,
\]

and so

\[
\frac{V_R(x, r)}{V_R(x, s)} \leq C \left(\frac{r}{s}\right)^{(\beta-1)\iota 1}.
\]

If \(s < 2r_x, d\), then \(B_d(x, s) \supset \{x\} = B_d(x, \inf_{y \neq x} d(x, y))\) and \((VD)_d\) imply that this inequality holds by modifying \(C\). Because \(\iota, C\) are determined only by \((VD)_d\) and independent of \(x, r\) and \(s\), we get the desired inequality.

(2) Lemma 6.2, (VD)_d, and \((p_0)\) imply

\[
V_R(x, R(x, y)) R(x, y) \geq \frac{\mu_x}{\mu_x} = 1
\]

for any \(x, y \in G\) and there exists \(C_1, C_2 > 0\) such that

\[
V_R(x, R(x, y)) R(x, y) \leq C_1 V_R(x, \mu_x^{-1}) \mu_x^{-1} = C_1 \frac{\mu_x}{\mu_x} \leq C_2
\]

for any \(x, y \in G\) such that \(x \sim y\). Since \(V_R(x, R(x, y)) R(x, y) \propto d^\beta(x, y)\), we get desired inequalities for \(d(x, y)\). Since \((VD)_d\) holds, for any \(x \in G, c > 0\) and \(r > c, B_d(x, r) \subset \bigcup_{i=1}^n B_d(x, c)\) for some \(n = n(c)\) and \(\{x_i\}_{i=1}^n\) (see [7] for example). Let \(c = \inf_{x, y \in G} d(x, y)\) then this imply \(#\{B_d(x, r)\} < \infty.\)

The second step of proof of Theorem 4.29 is applying the method of [1], but we need some modifications because \(d\) is not the graph metric. For the sake of completeness, we do not omit the proofs unless no modification is needed.

Properties of \(d\) in Proposition 6.4 (2) and uniformly shrinking condition help our modification.

In the rest of this section, let \(d\) be a metric on \(G\) and define

\[
\tau(x, r) = \tau_d(x, r) = \min\{n | X_n \notin B_d(x, r)\}
\]

where \(X_n\) is associated random walk.

**Lemma 6.6.** Assume \(d\) is uniformly shrinking and satisfies \((dL)_d, (NdU), (BF), (R(\beta))\) and \((VG(\alpha))\) for some \(\beta > \alpha \geq 1\). Then there exists \(\lambda, C > 0\) such that

\[
R(B_d(x, \lambda r), B_d(x, r)^c) V(x, r) \geq C r^\beta \quad \text{(ARL(\beta))}
\]

for any \(x \in G\) and \(r > r_0.\)
Proof. Fix $x \in G$ and $r > r_0$. Let $c_*= (C_+ + r_0)/r_0, A = B_d(x, c_r) \setminus B_d(x, r)$. Since $z_1 \not= z_2$ for any $z_1 \in B_d(x, r)$ and $z_2 \in B_d(x, r + c_r)$, $R(B_d(x, \lambda r), A) = R(B_d(x, \lambda r), B_d(x, r))^c$ for any $\lambda < 1$, so we consider $R(B_d(x, \lambda r), A)$. Let $f_y$ be functions for $y \in A$ such that $f_y(x) = 1, f_y(y) = 0$ and $\mathcal{E}(f_y, f_y) = R(x, y)$. We first show there exists $\lambda$ such that $f_y(z) \geq 2/3$ for any $z \in B_d(x, \lambda r)$ and $y \in A$. Since $f_y$ is harmonic on (finite set) $B_d(x, \lambda r) \setminus \{x\}$,

$$(1 - f_y(z))^2 \leq (1 - f_y(z_0))^2 \leq C \frac{d(x, z_0)^\beta}{V_d(x, d(x, z_0))} \mathcal{E}(f_y, f_y)$$

for some $C, C' > 0, z_0 \not\in B_d(x, \lambda r)$ such that $z_0 \sim z'$ for some $z' \in B_d(x, \lambda r)$ and any $z \in B_d(x, \lambda r)$ by (R($\beta$)). Since $d(x, z_0) \geq (\lambda r - c_+) \lor r_0 \geq c_*^{-1} \lambda r$ and (VG($\alpha$)) holds, for sufficiently small $\lambda$, $f_y(z) \geq 2/3$, for any $z \in B_d(x, \lambda r)$ and $y \in A$. In the same way, we also get $f_y(z) \leq 1/3$ for any $z \in B_d(y, \lambda r)$. We assume $\lambda < 1/4$.

Since (VD)$_d$ holds, there exists $n = n_\lambda > 0$ such that for any $x \in G$ and $r > 0$, there exist $\{y_i\}_{i=1}^n \subset A$ such that $\sum_{i=1}^n B_d(y_i, \lambda r) \supset A$. Let $g = \min_{1 \leq i \leq n} f_i$ and $h = (1 \lor (3g - 1)) \land 0$. Then $h|_{B_d(x, \lambda r)} = 1, h|_A \equiv 0$ and so

$$R(B_d(x, \lambda r), A)^{-1} \leq \mathcal{E}(h) \leq \mathcal{E}(3g - 1) = 9\mathcal{E}(g).$$

Let $z_1, z_2 \in X$ such that $z_1 \sim z_2$ and $g(z_1) \geq g(z_2) = h_j(z_2)$ for some $j$, then

$$(g(z_1) - g(z_2))^2 \leq (g(z_1) - h_j(z_2))^2 \leq \sum_{i=1}^n (h_i(z_1) - h_i(z_2))^2$$

and it follows that

$$\mathcal{E}(g) \leq \sum_{i=1}^n \mathcal{E}(h_i).$$

Moreover, (VD)$_d$ implies

$$\mathcal{E}(h_i) = R(x, y_i)^{-1} \leq C'' \frac{V_d(x, d(x, y_i))}{d(x, y_i)^\beta}$$

for some $C''$. By these inequalities, we obtain (ARL($\beta$)).

In the following theorem and proposition, we can apply the original proof in [1].

**Theorem 6.7.** ([1] Theorem 3.1) Assume (VG($\alpha$)) and (R($\beta$)) hold for $\beta > \alpha \geq 1$. Then there exists $C > 0$ such that for any $x \in X$ and $n \in \mathbb{N},$

$$h_n(x, x) \leq \frac{C}{V_d(x, n^{1/\beta})},$$

(DUHK($\beta$))

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Proposition 6.8. \(\text{([1] Proposition 3.4)}\) Assume (BF), (VG\((\alpha))\) and \((R(\beta))\) hold for some \(\beta > \alpha \geq 1\). If there exists \(C, r_0 > 0\) such that
\[
R(x, B_d(x, r)) V(x, r) > Cr^\beta
\]
(BRL\((\beta))\) for any \(x \in G\) and \(r > r_0\). Then
\[
\mathbb{E}^x[\tau(x, r)] \asymp r^\beta
\]
(E\((\beta))\) for any \(x \in G\) and \(r > r_0\).

Now we give on-diagonal lower heat kernel estimate. The following type of result is well-known, but we prove it for completeness.

Proposition 6.9. Assume \((E(\beta))\) and \((VG(\alpha))\), then there exists \(C > 0\) such that for any \(x \in G\) and \(n \in \mathbb{Z}_+\),
\[
\sum_{y \notin B_d(x, Cn^{1/\beta})} h_n^2(x, y) \mu_y \geq \sum_{y \in B_d(x, Cn^{1/\beta})} h_n^2(x, y) \mu_y \geq \frac{1}{V_d(x, Cn^{1/\beta})} \left( \sum_{y \in B_d(x, Cn^{1/\beta})} h_n(x, y) \mu_y \right)^2 \geq \left( \frac{1 - p}{2} \right)^2 \frac{1}{V_d(x, Cn^{1/\beta})}
\]
and (VG\((\alpha))\) implies desired result.

Proof. In the same way as \([1\) Lemma 3.7], we can prove that there exist \(p \in (0, 1)\) and \(A > 0\) such that
\[
\mathbb{P}^x(\tau(x, r) \leq n) \leq p + An/r^\beta
\]
for any \(x \in G, r > r_0\), and \(n \in \mathbb{Z}_+\). Hence
\[
\sum_{y \notin B_d(x, Cn^{1/\beta})} h_n(x, y) \mu_y = \mathbb{P}^x(X_n \notin B_d(x, Cn^{1/\beta})) \leq \mathbb{P}^x(\tau(x, Cn^{1/\beta}) \leq n) \leq \frac{1 + p}{2}
\]
for some \(C > 0\). Therefore
\[
h_{2n}(x, x) = \sum_{y \in G} h_n(x, y) \mu_y \geq \sum_{y \in B_d(x, Cn^{1/\beta})} h_n^2(x, y) \mu_y \geq \frac{1}{V_d(x, Cn^{1/\beta})} \left( \sum_{y \in B_d(x, Cn^{1/\beta})} h_n(x, y) \mu_y \right)^2 \geq \left( \frac{1 - p}{2} \right)^2 \frac{1}{V_d(x, Cn^{1/\beta})}
\]
and (VG\((\alpha))\) implies desired result.

proof of Theorem 4.29 With using (VG\((\alpha))\), all but (BRU\((\beta))\) have shown in above statements (recall that \(d(x, y) \asymp 1\) follows from \(dL)\) and \(NdU)\), and note that (BRL\((\beta))\) for Proposition 6.8 immediately follows from (ARL\((\beta))\). Since \(d\) is uniformly shrinking, there exists \(\alpha \in (0, 1)\) such that for any \(r > r_x\), there exists \(y \in B_d(x, \alpha^{-1} r) \setminus B_d(x, r)\), so \(R(x, B_d(x, r)) \leq R(x, y)\) with \(r \leq d(x, y) < \alpha^{-1} r\). This with \((R(\beta))\) and (VG\((\alpha))\) implies (BRU\((\beta))\).

Remark. We have seen that if (BF), (VG\((\alpha))\), (R\((\beta))\) and (BRL\((\beta))\) hold, then (DUHK\((\beta))\) and (DLHK\((\beta))\) hold without \(dL)\) nor \(NdU)\). This was already used, for example, see \([12\).
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