COMPLETE LOGARITHMIC SOBOLEV INEQUALITY VIA RICCI CURVATURE BOUNDED BELOW II

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Abstract. Using a non-negative curvature condition, we prove the complete version of modified log-Sobolev inequalities for central Markov semigroups on various compact quantum groups, including group von Neumann algebras, free orthogonal group and quantum automorphism groups. We also prove that the “geometric Ricci curvature lower bound” introduced by Junge-Li-LaRacuente is stable under tensor products and amalgamated free products. As an application, we obtain the geometric Ricci curvature lower bound and complete modified logarithmic Sobolev inequality for word-length semigroups on free group factors and amalgamated free product algebras.

1. Introduction

For Riemannian manifolds, the Ricci curvature being bounded from below has many important applications in geometry and analysis. In recent years, progresses have been made to introduce a suitable notion of Ricci curvature lower bound for noncommutative spaces. Following the idea of $\Gamma$-calculus by Bakry-Emery, Junge and Zeng in [34] studied the noncommutative curvature dimension condition, called $\Gamma_2$-condition. They proved that the $\Gamma_2$-condition, similar to the classical cases, implies $L_p$-Poincaré type inequalities, as well as its consequences, including deviation inequalities and transport inequalities. On the other hand, using ideas from optimal transport theory, Carlen and Maas in [12] introduce a notion of Ricci curvature lower bound for quantum Markov semigroups. Their idea goes back to the famous works by Lott-Villani [37] and Sturm [42], which introduced a notion of Ricci curvature lower bound for metric measure spaces via certain convexity of entropy functionals. This curvature condition, also called entropy Ricci curvature lower bound (in short, ERic), recently has attracted a lot of attention. It is proved in [13] and [19] that for quantum Markov semigroups, ERic condition implies a modified log-Sobolev inequality, Talagrand’s transport cost inequality and also an $L_2$-Poincaré inequality. All these give a unified picture of functional and geometric inequalities in both classical and noncommutative settings.

A common point in Junge-Zeng [34] and Carlen-Maas [12] is to replace “geometry” by dynamics, described by a Markov semigroup. This idea were used earlier by Erbar and Maas in [24] to introduce the ERic condition for Markov semigroup on finite probability
spaces. In the noncommutative setting, quantum Markov semigroups are generalizations of classical Markov semigroups, where the underlying probability space is replaced by matrix algebras or operators algebras. Quantum Markov semigroups have been widely used in operator algebras for the study of approximation properties, structure theory and noncommutative analysis (see e.g. [14, 29]). They often serve as a replacement of classical tools that are not available in the quantum setting. From this point of view, introducing curvature conditions for quantum Markov semigroups is very relevant.

The $\Gamma_2$ condition and the ERic condition turns out to be closely related. Indeed, both of them can be viewed as gradient estimates on certain weighted $L^2$-spaces. For the $\Gamma_2$ condition, the weight is given by a double operator integral of arithmetic means, while the ERic condition corresponds to logarithmic mean. For the heat semigroup on a Riemannian manifold, both conditions are equivalent to the lower bound of the Ricci curvature tensor. For quantum Markov semigroups, they can differ due to noncommutativity issues. More recently, motivated by the Bochner formula, Li-Junge-LaRacuente introduce a notion of “geometric Ricci curvature lower bound” (in short, GRic). This GRic is a strong curvature condition that implies both $\Gamma_2$ and ERic, hence also the modified log-Sobolev inequality and its consequences.

In this paper, which is the second in a series of two papers (see [9]), we continue our study of the Ricci curvature condition and its connection to the complete version of modified log-Sobolev inequality (in short, CLSI). We focus on various concrete examples in operator algebras and prove the following results:

i) Central Markov semigroups on compact quantum groups always have $\text{GRic} \geq 0$.
Based on this, we show that under certain growth condition for the length functions, Fourier multiplier semigroups on group von Neumann algebras have positive CLSI constant. We also prove that the heat semigroups on free orthogonal group and quantum isomorphism groups (tracial case) has CLSI.

ii) The GRic condition is stable under tensor product and free product.

iii) The word-length semigroup on $q$-Gaussian and free group factors satisfy sharp $\text{GRic} \geq 1$ and 1-CLSI.

iv) The generalized depolarizing semigroup has $\text{GRic} \geq 1/2$.

v) Curvature lower bounds and positive CLSI constants for some natural semigroups on quantum tori.

There are two ingredients in our proof. The first one is the interwining relation from [12] which we use to prove our curvature condition. The second tool is the main result of our first paper [9], which enables us to obtain complete log-Sobolev inequality from non-negative curvature condition. This is essential for the examples in i) because the curvature lower bound is not strictly positive and the Bakry-Emery type theorem does not apply.
The paper is organized as follows. Section 2 reviews the definitions and previous results that will be used in the rest of the paper. In Section 3, we discuss central Markov semigroups on compact quantum groups, including group von Neumann algebras, free orthogonal group and quantum automorphisms groups. We prove in Section 4 that GRic is stable under tensor product and amalgamated free product. Section 5 is devoted to optimal GRic constant for word length semigroup on q-Gaussian and free group factors. Section 6 revisit generalized depolarizing semigroup and some semigroups on quantum tori.

Note added. While this manuscript was being prepared, the authors learned that several of the examples studied in this work were independently considered in the recent work of Wirth and Zhang [48] in the context of a complete version of the gradient estimate (corresponding to ERic). They obtained the complete gradient estimate results parallel to our study of GRic in this paper.

Acknowledgements. Michael Brannan was partially supported by NSF Grants DMS-2000331 and DMS-1700267. Marius Junge was partially supported by NSF grants DMS-1839177 and DMS-1800872. Parts of this work were completed at the 2019 Great Plains Operator Theory Symposium at Texas A&M University and the QLA Meets QIT 2019 conference at Purdue University. The authors thank the organizers of these conferences for the stimulating work environment. The authors also thank the organizers of the 48th Canadian Operator Symposium, where a preliminary version of these results were presented.

2. Preliminary

2.1. Quantum Markov semigroups. Throughout the paper, $\mathcal{M}$ will always be a finite von Neumann algebra equipped with a normal faithful tracial state $\tau$. For $0 < p < \infty$, the $L_p$-space $L_p(\mathcal{M})$ is defined as the completion of $\mathcal{M}$ with respect to the norm $\|a\|_p = \tau(|a|^p)^{1/p}$. We identify $L_\infty(\mathcal{M}) := \mathcal{M}$ and $L_1(\mathcal{M}) \cong \mathcal{M}_*$.

A quantum Markov semigroup is a family of linear maps $(T_t)_{t \geq 0}: \mathcal{M} \to \mathcal{M}$ satisfying the following properties

i) $T_t$ is a normal unital completely positive map for all $t \geq 0$.

ii) $T_t \circ T_s = T_{s+t}$ for any $t, s \geq 0$ and $T_0 = \text{id}$.

iii) for each $x \in \mathcal{M}$, $t \mapsto T_t(x)$ is continuous in ultra-weak topology.

We say a quantum Markov semigroups $(T_t)$ is symmetric if for any $t$, 

$$\tau(x^* T_t(y)) = \tau(T_t(x)^* y), \ x, y \in \mathcal{M}.$$
For symmetric \((T_t)\), the fixed-point subspace \(\mathcal{N} = \{x \in \mathcal{M} \mid T_t(x) = x, \forall t \geq 0\}\) forms a subalgebra and each \(T_t\) is an \(\mathcal{N}\)-bimodule map,

\[
T_t(axb) = aT_t(x)b, \quad \forall \ a, b \in \mathcal{N}, \ x \in \mathcal{M}
\]

In particular, we have

\[
T_t \circ E = E \circ T_t = E.
\]

where \(E : \mathcal{M} \to \mathcal{N}\) is the trace preserving conditional expectation onto \(\mathcal{N}\) given by

\[
\tau(xy) = \tau(xE(y)), \forall \ x \in \mathcal{N}, \ y \in \mathcal{M}.
\]

We say \((T_t)\) is ergodic if \(\mathcal{N} = \mathbb{C}1\) is trivial. This means the semigroup admits an unique invariant state as the identity element 1. Throughout the paper, we will focus on symmetric quantum Markov semigroups that are not necessarily ergodic. We refer to [20] for more information of symmetric quantum Markov semigroups.

Denote the generator of the semigroup as

\[
Ax = \lim_{t \to 0} \frac{x - T_t(x)}{t}, \quad T_t = e^{-At},
\]

For symmetric semigroups, \(T_t = T_t^\dagger\) are unital completely positive and trace preserving , and the generator \(A\) is a self-adjoint and positive operator on \(L_2(\mathcal{M})\). A symmetric quantum Markov semigroup is determined by its Dirichlet form

\[
\mathcal{E} : L_2(\mathcal{M}) \to [0, \infty], \quad \mathcal{E}(x, x) = \tau(x^*Ax).
\]

We write \(\text{dom}(A)\) for the domain of \(A\) and \(\text{dom}(A^{1/2})\) for the domain of \(\mathcal{E}\). The Dirichlet subalgebra \(\mathcal{A}_\mathcal{E} := \text{dom}(A^{1/2}) \cap \mathcal{M}\) is a dense \(*\)-subalgebra of \(\mathcal{M}\) and a core of \(A^{1/2}\) [20]. \(\mathcal{A}_\mathcal{E}\) is a core for \(\mathcal{E}\) (or \(A^{1/2}\)), i.e. closed under the graph norm \(\|x\|_\mathcal{E} = \|x\|_2 + \|A^{1/2}x\|_2\). In particular, we have \(A(\mathcal{N}) = 0\) and \(\mathcal{N} \subset \mathcal{A}_\mathcal{E}\).

\[\text{2.2. Modified logarithmic Sobolev inequalities.}\]

Let \((\mathcal{M}, \tau)\) be a finite von Neumann algebra. We say \(\rho \in L_1(\mathcal{M})\) is a density operator (or simply density) if \(\rho \geq 0\) and \(\tau(\rho) = 1\). Using the identification \(\mathcal{M}_* \cong L_1(\mathcal{M})\) via duality

\[
a \in L_1(\mathcal{M}) \leftrightarrow \phi_a \in \mathcal{M}_*, \quad \phi_a(x) = \tau(ax),
\]

density operators corresponds to the normal states of \(\mathcal{M}\). Throughout the paper, states always mean normal states and are identified with their density operators. We write \(S(\mathcal{M}) = \{\rho \in L_1(\mathcal{M}) \mid \rho \geq 0, \tau(\rho) = 1\}\) as the state space of \(\mathcal{M}\) and for a subalgebra \(\mathcal{A} \subset \mathcal{M}\), we write \(S(\mathcal{A}) := S(\mathcal{M}) \cap \mathcal{A}\) as the states with bounded density operators in \(\mathcal{A}\).

Recall that for two invertible densities \(\rho\) and \(\sigma\), the relative entropy is given by

\[
D(\rho||\sigma) = \tau(\rho \log \rho - \rho \log \sigma),
\]
and for general states, \( D(\rho||\sigma) := \lim_{\epsilon \to 0} D(\rho + \epsilon \eta || \sigma + \epsilon \eta) \). Let \( \mathcal{N} \subset \mathcal{M} \) be a von Neumann subalgebra and \( E : \mathcal{M} \to \mathcal{N} \) be the trace preserving conditional expectation on to \( \mathcal{N} \). For a state \( \rho \), the relative entropy with respect to \( \mathcal{N} \) is defined as follows

\[
D(\rho||\mathcal{N}) := \inf_{\sigma \in S(\mathcal{N})} D(\rho||\sigma) = D(\rho||E(\rho)),
\]

where the infimum is always attained by \( E(\rho) \). In the case \( \mathcal{N} = \mathbb{C}1 \), we write \( H(\rho) := D(\rho||1) \) as the entropy of \( \rho \). (Note that here \( H \) differs with the usual von Neumann entropy in information theory due to the normalization of the trace).

Let \( T_t = e^{-At} : \mathcal{M} \to \mathcal{M} \) be a quantum Markov semigroup with generator \( A \). The Fisher information of a state \( \rho \in S(\mathcal{A}_\xi) \) is given by

\[
I(\rho) = \tau(A \rho \log \rho) = \mathcal{E}(\rho, \log \rho).
\]

**Definition 2.1.** i) We say \( T_t \) satisfies \( \lambda \)-modified logarithmic Sobolev inequality (\( \lambda \)-MLSI) for \( \lambda > 0 \) if for any \( \rho \in S(\mathcal{A}_\xi) \),

\[
2\lambda D(\rho||\mathcal{N}) \leq I(\rho).
\]

ii) We say \( T_t \) satisfies \( \lambda \)-complete logarithmic Sobolev inequality (\( \lambda \)-CLSI) for \( \lambda > 0 \) if for all finite von Neumann algebra \( \mathcal{R} \), \( \text{id}_{\mathcal{R}} \otimes T_t \) satisfies \( \lambda \)-MLSI.

Let us recall that for ergodic \( T_t \), \( T_t \) satisfies \( \lambda \)-logarithmic Sobolev inequality (\( \lambda \)-LSI) for \( \lambda > 0 \) if for any \( \rho \in S(\mathcal{A}_\xi) \),

\[
\lambda H(\rho) \leq 2\mathcal{E}(\rho^{1/2}, \rho^{1/2}).
\]

It was proved in [40] that \( \lambda \)-LSI is equivalent to the hypercontractivity that \( \| T_t : L_2(\mathcal{M}) \to L_p(\mathcal{M}) \| \leq 1 \) if \( p \leq 1 + e^{2\lambda t} \), and for symmetric semigroup \( \lambda \)-LSI implies \( \lambda \)-MLSI by \( L_p - \)regularity.

### 2.3. Curvature conditions.

We now review different curvature conditions for quantum Markov semigroups. Recall that the gradient form (or carré du champ) of the generator \( A \) is given by

\[
\Gamma(x, y) := \frac{1}{2} \left( (Ax^*)y + x^*Ay - A(x^*y) \right), \forall x, y \in \text{dom}(A)
\]

and it be extended to \( x, y \in \text{dom}(A^{1/2}) \). We recall the following concept of derivation triple from [36, 9]. Recall that \( \mathcal{A}_\xi = \mathcal{M} \cap \text{dom}(A^{1/2}) \) is the Dirichlet algebra.

**Definition 2.2.** Let \( T_t : \mathcal{M} \to \mathcal{M} \) be a symmetric quantum Markov semigroup. A derivation triple of \( (\mathcal{A}, \mathcal{M}, \delta) \) of \( T_t \) consists of

i) a weak\(^*\)-dense subalgebra \( \mathcal{A} \subset \mathcal{M} \) such that \( T_t(\mathcal{A}) \subset \mathcal{A} \) and \( \mathcal{A} \subset \mathcal{A}_\xi \).

ii) a finite von Neumann algebra \( \mathcal{M} \) such that \( \mathcal{M} \subset \mathcal{M} \) with induced trace.
iii) a symmetric derivation \( \delta : \mathcal{A} \to L_2(\hat{M}) \), meaning that \( \delta(x^*) = \delta(x)^* \) and \( \delta \) satisfies the Leibniz rule:

\[
\delta(xy) = \delta(x)y + x\delta(y), \quad \forall x, y \in \mathcal{A}.
\]

Moreover, \( \delta \) and \( \Gamma \) are related through

\[
E_M(\delta(x)^*\delta(y)) = \Gamma(x, y), \quad x, y \in \mathcal{A}
\]

where \( E_M : \hat{M} \to M \) is the conditional expectation.

We say the derivation \( \delta \) has mean zero property if \( E_M(\delta(x)) = 0 \) for all \( x \in \mathcal{A} \).

A consequence of i) and iii) is that \( \mathcal{A} = \delta^* \delta \) for the closure \( \delta \) on \( \text{dom}(A^{1/2}) \). It was proved in the unpublished preprint [33] that \( T_t \) admits a derivation triple with \( \mathcal{A} = A_{E} \) if and only if \( T_t \) satisfies \( \Gamma \)-regularity that for all \( x \in \text{dom}(A^{1/2}) \), \( \Gamma(x, x) \in L_1(M) \).

Nevertheless, it is sufficient and often more convenient to work with subalgebra \( \mathcal{A} \subset \mathcal{A}_E \) with stronger regularity.

Derivation triple is the key concept ingredient in the following definition of geometric curvature lower bound. For an subalgebra \( \mathcal{A} \subset M \), we denote \( \mathcal{A}_0 = \bigcup_{t>0} T_t(\mathcal{A}) \). \( \mathcal{A}_0 \subset \text{dom}(A) \) and \( A(\mathcal{A}_0) \subset \text{dom}(A^{1/2}) \) because \( AT_t \) and \( A^{3/2}T_t \) are bounded operator on \( L_2(M) \).

We also write \( \Omega_\delta = A\delta(\mathcal{A}) \subset L_2(\hat{M}) \) as the \( \mathcal{A} \)-bimodule generated by the range of \( \delta \).

**Definition 2.3.** We say \((\mathcal{A}, \hat{M}, \delta)\) satisfies a geometric Ricci curvature lower bound \( \lambda \) for \( \lambda \in \mathbb{R} \) (in short \( GRic \geq \lambda \)) if there exists a symmetric quantum Markov semigroup \( \hat{T}_t = e^{-\hat{A}t} : \hat{M} \to \hat{M} \) with generator \( \hat{A} \) such that

i) \( \hat{T}_{t|M} = T_t \) for any \( t \geq 0 \).

ii) \( \delta(\mathcal{A}_0) \subset \text{dom}(\hat{A}) \) and there exists a \( \mathcal{A} \)-bimodule operator \( \text{Ric} : \Omega_\delta \to L_2(\hat{M}) \) such that for \( x \in \mathcal{A}_0 \),

\[
\text{Ric}(\delta(x)) = \delta A(x) - \hat{A}\delta(x).
\]

iii) for any \( y \in \Omega_\delta \),

\[
\langle y, \text{Ric}(y) \rangle \geq \lambda \langle y, y \rangle.
\]

where \( \langle \cdot, \cdot \rangle \) is the trace inner product of \((\hat{M}, \tau)\).

The motivation of above definition is of course the Bochner-Weitzenböck-Lichnerowicz formula (c.f. pp374 [43])

\[
\Delta(\nabla f) - \nabla(\Delta f) + \text{Ric}((\nabla f) = 0,
\]

where \( \Delta = \nabla^* \nabla \) is the Laplace-Beltrami operator on a Riemannian manifold and \( \nabla \) is the gradient operator. A special case that repeatedly occurs in our examples is \( \text{Ric} = \lambda \text{id} \), which is characterized by the following intertwining relation.
Proposition 2.4 (Theorem 3.25 of [9]). Let $T_t : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and let $(\mathcal{A}, \mathcal{M}, \delta)$ be a derivation triple of $T_t$. Suppose that there exists a symmetric quantum Markov semigroup $\hat{T}_t : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ such that for some $\lambda \in \mathbb{R}$ and any $t \geq 0$,

$$\hat{T}_t|_M = T_t, \quad \delta \circ T_t = e^{-\lambda t} \hat{T}_t \circ \delta.$$  

Then $T_t$ satisfies $\text{GRic} \geq \lambda$ with $\text{Ric} = \lambda \text{id}$ as a multiple of the identity operator.

We call the equation (5) as $\lambda$-GRic to specify the relation $\text{Ric} = \lambda \text{id}$.

Another curvature condition motivated from optimal transport is the entropy Ricci curvature lower bound introduced in [12] (see also [19, 13]). Such entropy Ricci curvature lower bound is defined via the $\lambda$-geodesic convexity of entropy $H$ with respect to Wasserstein distance. Here we recall a related condition of gradient estimate. Let $T_t : \mathcal{M} \to \mathcal{M}$ be a symmetric quantum Markov semigroup and $(\mathcal{A}, \hat{\mathcal{M}}, \delta)$ be a derivation triple for $T_t$. For simplicity of notation, we write $\tau$ for the trace on both $\mathcal{M}$ and $\hat{\mathcal{M}}$. For a state $\rho \in S(\mathcal{M})$, we define the weighted $L_2$-(semi)norm on $\hat{\mathcal{M}}$ as

$$\|x\|_{\rho} : = \left( \int_0^1 \tau(x^s \rho^{1-s} x \rho^s) ds \right)^{1/2}.$$  

We denote $L_2(\hat{\mathcal{M}}, \rho)$ as the completion of $\hat{\mathcal{M}}$ under this norm. We recall the following definitions from [47, 48].

Definition 2.5. We say $T_t$ satisfies $\lambda$-gradient estimate ($\lambda$-GE) if for any $\rho \in S(\mathcal{M})$ and $x \in \mathcal{A}_E$ with $E(x) = 0$,

$$\|\delta(T_t(x))\|_{\rho} \leq e^{-\lambda t} \|\delta(x)\|_{T_t(\rho)}, \quad \forall t \geq 0.$$  

We say $T_t$ satisfies $\lambda$-complete gradient estimate ($\lambda$-CGE) if for any finite von Neumann algebra $\mathcal{R}$, $\text{id}_\mathcal{R} \otimes T_t$ satisfies $\lambda$-GE.

In finite dimensional cases, $\lambda$-GE is shown to be equivalent to the $\lambda$-geodesic convexity of $H$, i.e., the entropy Ricci curvature lower bound [13, 19]. On finite von Neumann algebras, $\lambda$-GE is a sufficient condition for $\lambda$-geodesic convexity of $H$ [17, Theorem 7.12]. It was proved in [36] that $\lambda$-GRic is stronger than $\lambda$-CGE.

Proposition 2.6 (Proposition 3.6 of [36]). For $\lambda \in \mathbb{R}$, $\lambda$-GRic implies $\lambda$-CGE

Up to this writing, it is not clear whether $\lambda$-GRic and $\lambda$-CGE are equivalent. One observation suggesting the negation is that CGE is independent of specific choice derivation triple and only determined by the semigroup $T_t$ [9, Proposition 3.14], while such independence is not clear for GRic.
The complete gradient estimates are introduced in \cite{[48]} and several examples in this paper are independently studied there. Here we emphasis the difference between our work and \cite{[48]}. In our discussion we always need a derivation triple into a larger von Neumann algebra $\mathcal{M}$, whereas \cite{[48]} uses derivation $\delta : \text{dom}(A^{1/2}) \to \mathcal{H}$ into a $\mathcal{M}$-bimodule $\mathcal{H}$, which is based on the representation theorem \cite{[17]} Theorem 8.2 & 8.3] by Cipriani and Sauvageot. The more special subalgebra structure enables us to prove geometric curvature lower bound GRic that is stronger than CGE. Most of examples in our discussion will be given with concrete construction of derivation triple.

On the other hand, $\lambda$-GRic implies the $\Gamma_2$-condition studied in \cite{[34]}. Assume that $A$ is a $w^*$-dense subalgebra invariant under the generator $A$, i.e., $A \subset \text{dom}(A)$ and $A(A) \subset A$. Recall that the $\Gamma_2$ operator of $T_t = e^{-At}$ is given by

$$
\Gamma_2(x, y) = \frac{1}{2} \left( \Gamma(Ax, y) + \Gamma(x, Ay) - A\Gamma(x, y) \right), \quad x, y \in A.
$$

We say $T_t$ satisfies $\Gamma_2 \geq \lambda \Gamma$ for $\lambda \in \mathbb{R}$ if for any finite sequence $(x_j)_{k=1}^n \subset A_x$

$$
[\Gamma_2(x_j, x_k)]_{j,k=1}^n \geq \lambda \Gamma(x_j, x_k)_{j,k=1}^n
$$

as elements in $M_n(\mathcal{M})$.

**Proposition 2.7.** Let $(\mathcal{A}, \mathcal{N}, \delta)$ be a derivation triple of $T_t = e^{-At}$. Assume that $A$ is a $w^*$-dense subalgebra invariant under the generator $A$. Then $\lambda$-GRic implies $\Gamma_2 \geq \lambda \Gamma$.

**Proof.** For $x \in A$ and $z \in \mathcal{M}$ positive,

$$
\tau(z\Gamma(x, x)) = \tau(z\delta(x)\delta(x))
$$

$$
2\tau(z\Gamma_2(x, x)) = \tau(z\delta(Ax)\delta(x)) + \tau(z\delta(x)\delta(Ax)) - \tau((Ax)\delta(x)\delta(x))
$$

$$
= \tau \left( z(\hat{A}\delta(x) + \text{Ric}(\delta(x)))\delta(x) \right) + \tau \left( z\delta(x)\delta(\hat{A}\delta(x) + \text{Ric}(\delta(x))) \right)
$$

$$
- \tau ( (Az)\delta(x)\delta(x) )
$$

$$
= \tau \left( z(\hat{A}\delta(x))\delta(x) \right) + \tau \left( z\delta(x)(\hat{A}\delta(x)) \right) - \tau ((Az)\delta(x)\delta(x))
$$

$$
+ \tau ( z(\text{Ric}(\delta(x))\delta(x) \right) + \tau ( z\delta(x)(\text{Ric}(\delta(x)))
$$

Note that

$$
\tau \left( z(\hat{A}\delta(x))\delta(x) \right) + \tau \left( z\delta(x)(\hat{A}\delta(x)) \right) - \tau ((Az)\delta(x)\delta(x))
$$

$$
= \lim_{t \to 0} \frac{1}{t} \tau \left( z(\hat{T}_t(\delta(x)\delta(x)) - \hat{T}_t(\delta(x))\hat{T}_t(\delta(x))) \right) \geq 0
$$

and

$$
\tau (z(\text{Ric}(\delta(x))\delta(x)) = \langle \text{Ric}(\delta(x)z^{1/2}), \delta(x)z^{1/2} \rangle
$$

$$
\geq \lambda \langle \delta(x)z^{1/2}, \delta(x)z^{1/2} \rangle
$$
\[ = \lambda \tau (z \delta(x)^* \delta(x)) = \lambda \tau (z \Gamma(x,x)) \]

and similarly for \( \tau (z \delta(x)^* (\text{Ric} \delta(x))) \). The same argument applies to \( \text{id} \otimes T_t \), which completes the proof. \[ \blacksquare \]

As observed in [48], \( \Gamma_2 \geq \lambda \Gamma \) corresponds to the (complete) gradient estimate similar to Definition 2.5 for the weighted norm \( |||x|||^\rho = \tau (x^*x\rho)^{1/2} \).

In the noncommutative case, this is unlikely to equivalent to GE.

2.4. CB-return time. We review the main theorem from [9], which the key ingredient that enables us to obtain CLSI from non-positive curvature lower bound is the CB-return time. Let \( M \) be a finite von Neumann algebra and \( N \subset M \) be a subalgebra. The conditional \( L_\infty \) space \( L^1_\infty(N \subset M) \) is defined as the completion of \( M \) with respect to the norm \[ \|x\|_{L^1_\infty(N \subset M)} = \sup_{a,b \in L_2(N), \|a\|_2=\|b\|_2=1} \|axb\|_1 , \]

where the supremum takes over all \( a, b \in L_2(N) \) with \( \|a\|_2 = \|b\|_2 = 1 \). It is clear that for \( N = \mathbb{C}1 \), \( L^1_\infty(N \subset M) \) is simply \( L_1(M) \). The operator space structure of \( L^1_\infty(N \subset M) \) is given by

\[ M_n(L^1_\infty(N \subset M)) = L^1_\infty(M_n(N) \subset M_n(M)) . \]

(see [32] and [28, Appendix]).

**Definition 2.8.** Let \( T_t : M \to M \) be a symmetric quantum Markov semigroup and \( N \) be its fixed point subalgebra with conditional expectation \( E : M \to N \). The complete bounded (CB) return time of semigroup \( T_t \) is defined as

\[ t_{cb} := \inf \{ t \geq 0 \mid \|T_t - E : L^1_\infty(N \subset M) \to L_\infty(M)\|_{cb} \leq 1/2 \} \]

If such \( t \) does not exist, we write \( t_{cb} = +\infty \).

Define the function

\[ \kappa(\lambda, t) = \begin{cases} \frac{1}{4t}, & \text{if } \lambda = 0 \\ \frac{\lambda}{2(1-e^{-2\lambda t})}, & \text{if } \lambda \neq 0. \end{cases} \]

The following is Corollary 3.28 from [9].

**Theorem 2.9.** Let \( T_t : M \to M \) be a symmetric quantum Markov semigroup. Suppose

i) \( T_t \) satisfies \( \text{GRic} \geq \lambda \) for some \( \lambda \in \mathbb{R} \);

ii) \( T_t \) has finite CB-return time \( t_{cb} < \infty \).

Then \( T_t \)-satisfies \( \kappa(\lambda, t_{cb}) \)-CLSI.
The condition i) can be weaken to $\lambda$-CGE as argued in \cite{GK99}, Corollary 3.20. In this paper, we will mostly argue through the stronger condition “G\text{Ric}”.

2.5. Compact quantum groups. We refer to \cite{Kad98, Wor87} as standard references for the basic facts on compact quantum groups. We write $\otimes_{\text{min}}$ for the $C^*$-minimal tensor product and $\widehat{\otimes}$ for the von Neumann algebra tensor product. A $C^*$-algebraic compact quantum group $(C\text{QG}) \mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital $C^*$-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\text{min}} C(\mathbb{G})$ is a unital $*$-homomorphism (called the comultiplication) which satisfies

i) co-associativity: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$

ii) cancellation property: $\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))$ and $\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$ are total in $C(\mathbb{G}) \otimes_{\text{min}} C(\mathbb{G})$.

There exists a unique Haar state $h : C(\mathbb{G}) \to \mathbb{C}$ such that

$$(id \otimes h)\Delta(a) = h(a)1 = (h \otimes id)\Delta(a), \forall a \in C(\mathbb{G}).$$

Let $\lambda : C(\mathbb{G}) \to B(L_2(\mathbb{G}))$ be left regular representation on the GNS Hilbert space $L_2(\mathbb{G}) = L_2(C(\mathbb{G}), h)$. We denote $\lambda(C(\mathbb{G}))$ by $C_r(\mathbb{G})$ and called it the reduced $C^*$-algebra of continuous functions on $\mathbb{G}$. We also denote by $L_\infty(\mathbb{G})$ the von Neumann algebra generated by $C_r(\mathbb{G})$ in $B(L_2(\mathbb{G}))$. Then $\Delta$ extends normally to $L_\infty(\mathbb{G})$ and $(L_\infty(\mathbb{G}), \Delta, h)$ is a von Neumann algebraic compact quantum group.

A (finite-dimensional) representation of a CQG $\mathbb{G}$ is an invertible element $u \in B(H) \otimes C(\mathbb{G})$ (where $H$ is a finite-dimensional Hilbert space) which satisfies

$$(id \otimes \Delta)u = u_{12}u_{13} \in B(H) \otimes C(\mathbb{G}) \otimes C(\mathbb{G}).$$

Here we use the standard leg numbering notation. Note that if we choose an a basis $(e_i)_{1 \leq i \leq d}$ for $H$ and write $u = [u_{ij}] \in M_d(C(\mathbb{G}))$ relative to this basis, then the above formula simply says that

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \ (1 \leq i, j \leq d).$$

We say that $u$ is unitary if $uu^* = u^*u = 1$. Given two representations of $\mathbb{G}$, say $u \in B(H_u) \otimes C(\mathbb{G})$ and $v \in B(H_v) \otimes C(\mathbb{G})$, we can define their direct sum $u \oplus v \in B(H_u \oplus H_v) \otimes C(\mathbb{G})$ in the obvious way and their tensor product as $u \otimes v = u_{13}v_{23} \in B(H_u \otimes H_v) \otimes C(\mathbb{G})$. We denote by $\text{Mor}(u, v) = \{ T \in B(H_u, H_v) : (T \otimes id)u = v(T \otimes id) \}$. Such a $T$ is said to intertwine $u$ and $v$. We say $u$ and $v$ are equivalent if $\text{Mor}(u, v)$ contains an invertible element. We say that $u$ is irreducible if $\text{Mor}(u, u) = \mathbb{C} \text{id}$. We also note that every representation $u$ is equivalent to a unitary representation, and every unitary representation is a direct sum of irreducible representations.

We denote by $\mathcal{O}(\mathbb{G}) \subset C(\mathbb{G})$ the collection of all matrix elements of finite dimensional representations of $\mathbb{G}$. I.e., $x \in \mathcal{O}(\mathbb{G})$ is and only if $x = (\omega_{\xi, \eta} \otimes id)u$ for some representation
\( \epsilon \otimes \text{id} \) \( u = \text{id}, \ (S \otimes \text{id}) u = u^{-1} \) for any representation \( u \in B(H_u) \otimes C(G) \).

Denote by \( \text{Irr}(G) \) the set of irreducible unitary representations of \( G \) up to unitary equivalence. Choosing representatives \( (u_\pi)_{\pi \in \text{Irr}(G)} \) and orthonormal bases \( (e^{\pi}_{ij})_{1 \leq i \leq \dim(\pi) \subset H_\pi} \), it follows that

\[
\left\{ u^{\pi}_{ij} : 1 \leq i, j \leq \dim(\pi) \right\}_{\pi \in \text{Irr}(G)}
\]

is a linear basis for \( \mathcal{O}(G) \).

In this paper we will focus on compact quantum group of Kac-type, that is, CQGs \( G \) for which the Haar state \( h \) is a trace. In this case we typically write \( h = \tau \). In this special situation, the above basis for \( \mathcal{O}(G) \) is an orthogonal basis for \( L^2(G) \). More precisely we have

\[
\tau((u^{\pi}_{ij})^* u^{\sigma}_{kl}) = \frac{\delta_{\pi,\sigma} \delta_{ik} \delta_{jl}}{\dim \pi}.
\]

Moreover, when \( G \) is of Kac type, the antipode \( S \) extends to a normal \( \ast \)-isomorphism \( S : L_\infty(G) \rightarrow L_\infty(G)^{\text{op}} \).

### 3. Central semigroups on compact quantum groups

Let \( G \) be a compact quantum group of Kac type. Whenever we speak of quantum Markov semigroups on \( L_\infty(G) \), we mean Markovian with respect to the canonical Haar trace \( \tau \) on \( L_\infty(G) \).

**Definition 3.1.** We say a quantum Markov semigroup \( T_t : L_\infty(G) \rightarrow L_\infty(G) \) is called central if for all \( t \geq 0 \), \( T_t \) is both left and right invariant, i.e.

\[
\Delta \circ T_t = (T_t \otimes \text{id}) \circ \Delta = (\text{id} \otimes T_t) \circ \Delta.
\]

Following the group case in [9], we show that central semigroups satisfy \( \text{GRic} \geq 0 \).

**Theorem 3.2.** Let \( G \) be a compact quantum group and let \( T_t = e^{-At} : L_\infty(G) \rightarrow L_\infty(G) \) be a symmetric quantum Markov semigroup. If \( T_t \) is central, then \( T_t \) satisfies \( \text{GRic} \geq 0 \). The CGE \( \geq 0 \) for central semigroups is independently obtained in [18, Example 3.12].

**Proof.** The central property \( \Delta \circ T_t = (\text{id}_G \otimes T_t) \circ \Delta = (T_t \otimes \text{id}_G) \circ \Delta \) translates to the following commutative diagram.

\[
\begin{array}{ccc}
L_\infty(G) \otimes L_\infty(G) & \overset{\text{id}_G \otimes T_t \text{ or } T_t \otimes \text{id}_G}{\longrightarrow} & L_\infty(G) \otimes L_\infty(G) \\
\uparrow \Delta & & \uparrow \Delta \\
L_\infty(G) & \overset{T_t}{\longrightarrow} & L_\infty(G)
\end{array}
\]
Let \((\mathcal{A}, \mathcal{M}, \delta)\) be a derivation triple for \(T_t\) such that
\[
E_G(\delta(x)^*\delta(y)) = \Gamma(x,y)
\]
where \(E_G : \mathcal{M} \to L_\infty(G)\) is the conditional expectation from \(\mathcal{M}\) to \(L_\infty(G)\). We show that \(\nabla = (\delta \otimes \text{id}) \circ \Delta : L_\infty(G) \to \mathcal{M} \otimes L_\infty(G)\) is also a derivation for \(T_t = e^{-At}\). First, for the generator \(A\) we have \(E_\Delta(A \otimes \text{id}_G)\Delta = A\) where \(E_\Delta : L_\infty(G) \otimes L_\infty(G) \to L_\infty(G)\) is the conditional expectation obtained as the adjoint of \(\Delta\). Then, for \(x, y \in \mathcal{A}\)
\[
\Gamma_A(x,y) = x^*Ay + (Ax)^*y - A(x^*y)
\]
\[
= x^*E_\Delta(A \otimes \text{id})\Delta(y) + (E_\Delta(A \otimes \text{id})\Delta(x))^*y - E_\Delta(A \otimes \text{id})\Delta(x^*y)
\]
\[
= E_\Delta(\Delta(x)^*A \otimes \text{id})\Delta(y) + (A \otimes \text{id})\Delta(x)^*\Delta(y) - (A \otimes \text{id})\Delta(x^*y))
\]
\[
= E_\Delta(\Gamma_{A \otimes \text{id}}(\Delta(x), \Delta(y))) = E_\Delta \circ (E_G \otimes \text{id})((\delta \otimes \text{id})\Delta(x), (\delta \otimes \text{id})\Delta(y))
\]
where we have used the fact \((\mathcal{A} \otimes L_\infty(G), \mathcal{M} \otimes L_\infty(G), \delta \otimes \text{id})\) is a derivation triple for \(T_t \otimes \text{id}_G\). Now for the new derivation \(\nabla = (\delta \otimes \text{id}_G) \circ \Delta\), we have 0-GRic relation
\[
\nabla \circ T_t = (\delta \otimes \text{id}_G) \circ \Delta \circ T_t = (\delta \otimes \text{id}_G)(\text{id}_G \otimes T_t) \circ \Delta = (\text{id}_\mathcal{M} \otimes T_t)(\delta \otimes \text{id}_G) \circ \Delta = (\text{id}_\mathcal{M} \otimes T_t)\nabla
\]
where \(\text{id}_\mathcal{M} \otimes T_t\) is the extension semigroup of \(T_t\) on \(\mathcal{M} \otimes L_\infty(G)\).

**3.1. Fourier multipliers on group von Neumann algebras.** In this subsection, we consider group von Neumann algebras as particular examples of co-commutative compact quantum groups. Let \(G\) be a discrete group. The left regular representation of \(G\) is given by
\[
\lambda : G \to B(l_2(G)), \lambda(g)|h\rangle = |gh\rangle
\]
where \(\{|h\rangle| h \in G\}\) is the standard orthonormal basis of \(l_2(G)\). The group von Neumann algebra
\[
\mathcal{L}(G) = \text{span}\{\lambda(g) | g \in G\}^{\text{w*}} \subset B(l_2(G))
\]
is a finite von Neumann algebra equipped with the canonical trace \(\tau(\sum_g a_g\lambda(g)) = a_e\)
where \(e\) is the identity element of \(G\). \(\mathcal{L}(G)\) has the structure of a von Neumann-algebraic compact quantum group (of Kac type) when equipped with the comultiplication map given by
\[
\Delta : \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G), \Delta(\lambda(g)) = \lambda(g) \otimes \lambda(g), \forall g \in G.
\]
Here the underlying C*-algebraic CQG is \(G = \hat{G} = (C^*_r(G), \Delta)\) with Haar trace \(h = \tau\). Then \(C_r(\hat{G}) = C^*_r(G), L_\infty(\hat{G}) = \mathcal{L}(G)\), and \(\mathcal{O}(\hat{G}) \cong \mathbb{C}G\), the group algebra of \(G\). We say that \(\hat{G}\) is the compact quantum group dual to \(G\). This generalizes the compact-discrete Pontryagin duality for abelian groups.

For a function \(\phi : G \to \mathbb{C}\), we associate the Fourier multiplier map
\[
T_\phi : \mathcal{L}(G) \to \mathcal{L}(G), T_\phi(\lambda(g)) = \phi(g)\lambda(g).
\]
In general, $T_\phi$ is of course only defined on the $\sigma$-weakly dense $*$-subalgebra $\lambda(\mathbb{C}G)$. Note that all Fourier multipliers are central in the sense of the previous section:

$$(T_\phi \otimes \text{id}) \circ \Delta(\lambda(g)) = (\text{id} \otimes T_\phi) \circ \Delta(\lambda(g)) = \phi(g) \lambda(g) \otimes \lambda(g).$$

Conversely, it is also clear that all central map $T : \lambda(\mathbb{C}G) \subseteq \mathcal{L}(G) \rightarrow \mathcal{L}(G)$ have to be Fourier multipliers.

Now let $T_t : \mathcal{L}(G) \rightarrow \mathcal{L}(G)$ be a semigroup of (bounded) Fourier multipliers. Then one can write

$$T_t(\lambda(g)) = e^{-t\psi(g)}\lambda(g), \ g \in G,$$

where the generator $A$ of the semigroup is given by $A(\lambda(g)) = \psi(g)\lambda(g)$, i.e., a (generally unbounded) multiplier associated to some function $\psi : G \rightarrow \mathbb{C}$. Recall that $\psi$ is called \textit{conditionally negative definite} if for any finite sequence $\sum_{i=1}^nc_i = 0, c_i \in \mathbb{C}$ and $g_1, \cdots, g_n \in G$,

$$\sum_{i,j=1}^nc_i\bar{c}_j\psi(g_i^{-1}g_j) \leq 0$$

It is known that $T_t$ is a symmetric quantum Markov semigroup if and only if $\psi$ is a real-valued conditionally negative definite function with $\psi(e) = 0$ and $\psi(g) = \psi(g^{-1})$.

It then follows from Theorem 3.2 that any symmetric Markov semigroup of Fourier multipliers satisfies $\text{GRic} \geq 0$. In the following, we give a explicit construction of a derivation triple for $T_t$. The idea is inspired from the Markov dilation of Fourier multiplier semigroups from [1]. Recall that by Schoenberg Theorem [41], there exist a real Hilbert space $\mathcal{H}$ and an affine isometric action $\beta : G \rightarrow \text{Isom}(\mathcal{H}), g \mapsto \beta_g$ such that

$$\psi(g) = \|\beta_g(0)\|_{\mathcal{H}}^2, \ g \in G.$$ 

Here $0$ is the zero vector. For any $v \in \mathcal{H}$, one can write

$$\beta_g(v) = \pi_g(v) + b(g)$$

where $\pi : G \rightarrow O(\mathcal{H}), g \mapsto \pi_g$ is an orthogonal representation of $G$ and $b : G \rightarrow \mathcal{H}$ is a 1-cocycle with respect to $\pi$,

$$b(gh) = b(g) + \pi_g(b(h)) g, h \in G.$$ 

Then $\psi(g) = \|b(g)\|_{\mathcal{H}}^2$. Then the gradient form can be expressed as

$$\Gamma(\lambda(g), \lambda(h)) = \frac{1}{2} \left( (A\lambda(g))^*\lambda(h) + \lambda(g)^*(A\lambda(h)) - A(\lambda(g^{-1}h)) \right)$$

$$\begin{align*}
&= \frac{1}{2} \left( \|b(g)\|^2 + \|b(h)\|^2 - \|b(g^{-1}h)\|^2 \right)\lambda(g^{-1}h) \\
&= \frac{1}{2} \left( \|b(g^{-1})\|^2 + \|\pi_{g^{-1}}(b(h))\|^2 - \|b(g^{-1}) + \pi_{g^{-1}}(b(h))\|^2 \right)\lambda(g^{-1}h)
\end{align*}$$
Here \( L_0(\Omega) \) is the space of measurable functions on \( \Omega \). From the properties of the Gaussian distribution,
\[
E_{\Omega}(e^{-itW(v)}) = e^{-\frac{1}{2}|t|^2 v_2}, \quad t \in \mathbb{R}, \, v \in \mathcal{H}.
\]

Given an orthogonal transformation \( T : \mathcal{H} \to \mathcal{H} \), the quantization map
\[
\Gamma(T) : L_\infty(\Omega) \to L_\infty(\Omega), \quad \Gamma(T)(e^{iW(v)}) = e^{iW(Tv)},
\]
\[
\Gamma(T)(W(v_1) \cdots W(v_n)) = W(Tv_1) \cdots W(Tv_n).
\]

is a normal *-automorphism. Then the orthogonal representation \( \pi : G \to O(\mathcal{H}) \) induces an action \( \alpha \) of \( G \) on \( L_\infty(\Omega) \) as follows
\[
\alpha_s(e^{iW(v)}) = \Gamma(\pi_s)(e^{iW(v)}) = e^{iW(\pi_s v)}, \quad s \in G.
\]

Let \( \mathcal{M} = L_\infty(\Omega) \rtimes_\alpha G \) be the crossed product given by the action \( \alpha \). \( \mathcal{M} \) is again a finite von Neumann algebra equipped with the extension trace
\[
\tau_{\mathcal{M}}(a \rtimes \lambda(g)) = \begin{cases} E_{\Omega}(a), & \text{if } g = e \\ 0, & \text{otherwise.} \end{cases}
\]

Denote \( \mathbb{C}G = \text{span}\{\lambda(g)\} \subset \mathcal{L}(G) \) as the group algebra. We define the following derivation \( \delta : \mathbb{C}G \to L_2(\mathcal{M}) \cong (L_2(\Omega) \otimes L_2(\mathcal{L}(G))) \) by
\[
\delta(\lambda(g)) = iW(b(g)) \rtimes \lambda(g).
\]

This is a derivation because
\[
\delta(\lambda(g))\lambda(h) + \lambda(g)\delta(\lambda(h)) = (W(b(g)) \rtimes \lambda(g))\lambda(h) + \lambda(g)(W(b(h)) \rtimes \lambda(h))
\]
\[
= W(b(g)) \rtimes \lambda(gh) + W(\pi_g b(h)) \rtimes \lambda(gh)
\]
\[
= (W(b(g)) + W(\pi_g(b(h)))) \rtimes \lambda(gh) = W(gh) \rtimes \lambda(gh).
\]

Moreover,
\[
E_G(\delta(\lambda(g))^*\delta(\lambda(h))) = E(\lambda(g)^*W(b(g))W(b(h))\lambda(h))
\]
\[
= \lambda(g)^*E(W(b(g))W(b(h)))\lambda(h)
\]
The fixed-point subalgebra is $\mathcal{N}$.

**Proof.** It suffices to prove the following estimate that for $t > 0$:

$$T_t : \mathcal{L}(G) \to \mathcal{L}(G) \text{ admits an extension } \hat{T}_t : L_\infty(\Omega) \rtimes_G \to L_\infty(\Omega) \rtimes_G,$$

Then $T_t$ is complete positive because $\hat{T}_t$ is unital and $\|\hat{T}_t\|_{cb} = \|T_t\|_{cb} = 1$. It is clear that $\hat{T}_t$ forms a symmetric Markov semigroup satisfying the algebraic relation

$$\hat{T}_t \circ \delta = \delta \circ T_t.$$

This verifies that $T_t$ has GRic $\geq 0$ (actually 0-GRic). To ensure the CB-return time is finite, we need some growth condition in $\psi$.

**Theorem 3.3.** Let $G$ be a discrete group and $T_t : \mathcal{L}(G) \to \mathcal{L}(G)$ be a symmetric quantum Markov semigroup of Fourier multipliers

$$T_t : \mathcal{L}(G) \to \mathcal{L}(G), \quad T_t(\lambda(g)) = e^{-t\psi(g)}\lambda(g)$$

given by a conditionally negative definite function $\psi : G \to \mathbb{R}$. Then $T_t$ satisfies GRic $\geq 0$. The fixed-point subalgebra is $\mathcal{N} = \lambda(H)' \cong \mathcal{L}(H)$ where $H$ is the subgroup $\{g \in G \mid \psi(g) = 0\}$. If in additional, $\psi$ satisfies

i) the growth condition: for some $r > 0$, $C_r = \sum_{g \notin H} r^{\psi(g)} < +\infty$,

ii) the spectral gap condition: $\sigma = \inf_{g \notin H} \psi(g) > 0$.

Then $T_t$ satisfies $\lambda$-CLSI for

$$\lambda = \left(4\tau_{-1}(2C_r) - 4\log r \right)^{-1}$$

**Proof.** It suffices to prove the following estimate that for $t > -\log r$,

$$\|T_t - E_{\mathcal{N}} : L_1(\mathcal{L}(G)) \to L_\infty(\mathcal{L}(G))\|_{cb} \leq e^{-\sigma(t + \log r)}C_r$$

(7)

where $E_{\mathcal{N}}$ is the conditional expectation onto $\mathcal{N}$. Note that $\{\lambda(g) \mid g \in G\}$ is an ONB of $L_2(\mathcal{L}(G))$. Then the Choi operator of $T_t$ and $E_{\mathcal{N}}$ is

$$C(T_t) = \sum_{g \in G} e^{-\psi(g)t} \lambda(g)^{op} \otimes \lambda(g) \in \mathcal{L}(\mathcal{G})^{op \otimes \mathcal{L}(G)}$$

$$C(E_{\mathcal{N}}) = \sum_{g \in H} \lambda(g)^{op} \otimes \lambda(g) \in \mathcal{L}(\mathcal{G})^{op \otimes \mathcal{L}(G)}$$

Then by Effros-Ruan Theorem [23],

$$\|T_t - E_r : L_1(\mathcal{L}(G)) \to L_\infty(\mathcal{L}(G))\|_{cb} = \|C(T_t) - C(E_r)\|_{\mathcal{L}(\mathcal{G})^{op \otimes \mathcal{L}(G)}}$$

$$= \|\sum_{g \notin H} e^{-\psi(g)t}(g^{-1})^{op} \otimes \lambda(g)\|_{\mathcal{L}(\mathcal{G})^{op \otimes \mathcal{L}(G)}} \leq \sum_{g \notin H} e^{-\psi(g)t}$$
provided the sum is finite. By the growth condition and spectral gap condition, we have
\[
\| T_t - E_N : L_1(\mathcal{L}(G)) \to \mathcal{L}(G) \|_{cb} \leq \sum_{g \notin H} e^{-\psi(g)t} = \sum_{g \notin H} e^{\psi(g) \log r} e^{-\psi(g)(t + \log r)} = e^{-\sigma(t + \log r) C_r}
\]
is finite for \( t > -\log(r) \). Therefore we have the cb-return estimate
\[
t_{cb} \leq \sigma^{-1} \log(2C_r) - \log r
\]
The CLSI constant follows from Theorem 2.9.

**Remark 3.4.** For ergodic semigroups, this above theorem is comparable to [30, Theorem B]. On one hand, we know that hypercontractivity, or equivalently LSI, implies MLSI. On the other hand, our growth condition here is weaker to [30, Theorem B], and moreover, the above Theorem 3.3 implies MLSI for \( T_t \otimes \text{id}_\mathcal{R} \) for any finite von Neumann algebra \( \mathcal{R} \).

Following [30], we have the following variant of the assumptions on the growth of \( \psi \).

**Corollary 3.5.** Let \( G \) be a countable discrete group and let \( \psi : G \to \mathbb{R} \) be a real conditionally negative definite function. Suppose \( \psi \) satisfies the following two conditions.

i) Exponential growth condition: there exists a \( R > 0 \) such that for any \( s > 0 \)
\[
|\{g \notin H | \psi(g) \leq s + 1\}| \leq CR^s
\]

ii) Spectral gap condition:
\[
\sigma = \inf_{g \notin H} \psi(g) > 0
\]
Then \( T_t \) satisfies \( \lambda \)-CLSI for \( \lambda = \sigma \left( 4 \log \left( 2C + 2C \left( \frac{2}{R} \right)^\sigma \right) \right)^{-1} \).

**Proof.** It sufficient to note that for \( 0 < r < R^{-1} < 1 \),
\[
\sum_{g \notin H} r^{\psi(g)} \leq \sum_{\sigma \leq \psi(g) \leq 1} r^\sigma + \sum_{m=1}^\infty \sum_{g \notin H, m \leq \psi(g) \leq (m+1)} r^m \leq Cr^\sigma + \sum_{m=1}^\infty CR^m r^m \leq Cr^\sigma + C \frac{Rr}{1 - Rr}.
\]
Then by Theorem 3.3 \( T_t \) satisfies \( \lambda \)-CLSI with
\[
\lambda = \sigma \left( 4 \log \left( 2C + 2C \left( \frac{2}{R} \sigma \right) \right) - \log r \right)^{-1} = \sigma \left( 4 \log \left( 2C + 2C \left( \frac{Rr}{1 - Rr} \right) \right) \right)^{-1}
\]
Choose \( r = R/2 \), we obtain \( \lambda = \sigma \left( 4 \log \left( 2C + 2C \left( \frac{2}{R} \sigma \right) \right) \right)^{-1} \).

**Example 3.6.** Let \( G \) be a finitely generated group and \( | \cdot | : G \to \mathbb{Z}_+ \) be the word length function to relative to some fixed finite symmetric generating set \( S \). Suppose \( | \cdot | \) is conditionally negative definite. We have

i) the fixed point subgroup \( H = \{ e \} \) is trivial.
ii) \( G \) is at most exponential growth with respect to \(|\cdot|\): \( \{ g | g| \leq n+1 \} \leq |S|(|S|-1)^n \).

iii) the spectral gap \( \sigma = 1 \).

Then the word length semigroup

\[ P_t : \mathcal{L}(G) \to \mathcal{L}(G), \lambda(g) \mapsto e^{-|g|t} \lambda(g) \]

satisfies \( \lambda \)-CLSI with \( \lambda = \left( 4 \log \left( 2 |S| \left( \frac{|S|+1}{|S|-1} \right) \right) \right)^{-1} \).

**Remark 3.7.** The above estimates can be potentially improved if \( G \) has the property of rapid decay (RD) with respect to \(|\cdot|\). That is, there exists a polynomial \( P \in \mathbb{R}_+ [x] \) such that for any \( d \geq 0 \),

\[ \| \lambda(f) \|_{\mathcal{L}(G)} \leq P(k) \| f \|_2 \quad \forall f \in \mathbb{C}G, \text{ supp}\, f \subseteq W_d, \]

where \( W_d = \{ g \in G : |g| = d \} \) are the words of length \( d \). Then instead of using triangle inequality for all \( g \in G \), one can have

\[ \| P_t - E_\tau : L_1(\mathcal{L}(G)) \to \mathcal{L}(G) \|_{cb} \leq \| \sum_{g \neq e} e^{-|g|t} \lambda(g)^{op} \otimes \lambda(g) \| \]

\[ \leq \sum_{d \geq 1} e^{-dt} \| \sum_{|g|=d} \lambda(g) \| \]

\[ = \sum_{d \geq 1} e^{-dt} P(d) |W_d|^{1/2}. \]

Here in the second inequality we have used that the map \( \lambda(g) \mapsto \lambda(g^{-1})^{op} \) is a \( * \)-isomorphism and the comultiplication \( \Delta(\lambda(g)) = \lambda(g) \otimes \lambda(g) \) is an injective \( * \)-homomorphism. Then \( T_t = e^{-|\cdot|t} \) has finite CB-return time whenever \( |W_d| \) is at most exponential growth.

### 3.2. Free orthogonal quantum groups

In this section, we consider the free orthogonal quantum groups \( O_N^+ \). The free orthogonal quantum groups were introduced by Wang and Van Daele \cite{15, 18}, and their representation theory was later studied in detail by Banica \cite{2, 3}. Fix \( N \geq 2 \). Let \( O_N^+ = (C(O_N^+), \Delta) \) be the compact quantum group where \( C(O_N^+) \) be the universal \( C^* \)-algebra generated by \( \{ u_{ij} \} \) such that \( u_{ij} = u_{ij}^* \) and \( u = \sum_{ij} e_{ij} \otimes u_{ij} \) is unitary. The comultiplication \( \Delta : C(O_N^+) \to C(O_N^+) \otimes_{\min} C(O_N^+) \) is given by

\[ \Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}. \]

We note that \( u \) as defined above is a unitary representation of \( O_N^+ \), called the fundamental representation. The quantum groups \( O_N^+ \) should be thought of as free analogues of the classical real orthogonal groups \( O_N \). Indeed, for each \( N \) we have a surjective \( * \)-homomorphism \( C(O_N^+) \to C(O_N) \) given by \( u_{ij} \mapsto v_{ij} \), where \( v_{ij} \in C(O_N) \) is the standard
coordinate function on $O_N$. This morphism respects the (quantum) group structures and naturally realizes $O_N$ as a quantum subgroup of $O_N^+$. We now briefly summarize the representation theory of $O_N^+$, ($N \geq 2$) [2]: It is known that $\text{Irr}(O_N^+)$ can be indexed by non-negative integers $k \in \mathbb{N}_0$ in such a way that and the fusion rules are given by

$$k \otimes m \cong |k - m| \oplus (|k - m| + 2) \oplus \cdots \oplus (k + m), k, m \in \mathbb{N}_0.$$ 

The labeling $k \mapsto u^k$ can be chosen so that $u^0 = 1$ (the trivial representation) and $u^1 = u \in B(\mathbb{C}^N) \otimes C(O_N^+)$ is the fundamental representation. Denote by $n_k = \dim k = \dim u^k$ the dimension of $k$-th irreducible representation, we then have from the fusion rules recursive relation

$$n_k N = n_{k+1} + n_{k-1}.$$ 

Recall that the (dilated) Chebyshev polynomials $(U_k)_{k \in \mathbb{N}_0} \subset \mathbb{R}[x]$ of the second kind are defined by the recursion

$$U_0(x) = 1, U_1(x) = x, U_{1+1}U_k = U_{k-1} + U_{k+1}.$$ 

Comparing the above two recursion relations, one obtains

$$n_k = \dim(k) = U_k(N), (k \in \mathbb{N}_0)$$

Let $H_k$ be the Hilbert space associated to $u^k$. We fix an orthonormal basis $(e_{ij}^k)^{1 \leq i \leq \dim k}$ for $H_k$ and hence get a corresponding matrix representation $u^k = [u_{ij}^k] \in M_{\dim k}(C(O_N^+))$ for each $k$. Then we have that the Hopf-algebra $O(O_N^+)$ is spanned by the basis $\{u_{ij}^k\}$, and this basis is orthogonal with respect to Haar trace $\tau$ (which is faithful on $O(O_N^+)$):

$$\tau((u_{ij}^{k'})^*u_{ij}^k) = \frac{1}{\dim(k)}\delta_{ii'}\delta_{jj'}\delta_{kk'}.$$ 

In what follows we will slightly abuse notation and identify $O(O_N^+) \subseteq L_\infty(O_N^+)$ and $O(O_N^+) \subseteq L_2(O_N^+)$ via the usual GNS maps.

3.2.1. The heat semigroup on $O_N^+$. In [16], the symmetric central quantum Markov semigroups on $T_t = e^{At} : L_\infty(O_N^+) \rightarrow L_\infty(O_N^+)$ were characterized in terms of their generators

$$A : O(O_N^+) \rightarrow O(O_N^+); Au_{ij}^k = \lambda_j(A)u_{ij}^k, \lambda_k(A) \in \mathbb{C}.$$ 

We recall the following theorem from [16] on central semigroups.

**Theorem 3.8.** [16] Corollary 10.3] Let $(\lambda_k)_{k \in \mathbb{N}_0} \subset \mathbb{C}$ and define a central semigroup on $O(O_N^+)$ via the formula

$$T_t : O(O_N^+) \rightarrow O(O_N^+); T_t(u_{ij}^k) = e^{-\lambda_k t}u_{ij}^k.$$
Then $T_t$ extends uniquely to a symmetric central quantum Markov semigroup $T_t : L_\infty(O_N^+) \to L_\infty(O_N^+)$ if and only if there is a constant $b \geq 0$ and a finite positive Borel measure $\nu$ supported on $[-N,N]$ satisfying $\nu\{N\} = 0$, so that

$$\lambda_k = \frac{1}{U_k(N)} \left( bU_k'(N) + \int_{-N}^{N} \frac{U_k(x) - U_k(N)}{x - N} d\nu(x) \right).$$ (9)

As explained in [16], the above formula can be regarded as a quantum analogue of Hunt’s formula for generating functionals of central (=conjugation invariant) Lévy processes on compact Lie groups. The measure $\nu$ in the above theorem plays the role of the Lévy measure in such processes and accounts for the drift term in the Lévy process. As in the classical case of compact (connected) Lie groups, we obtain the Laplace-Beltrami operator (or Casimir operator) by setting $\nu = 0$ and choosing a normalization $b = 1$. This led the authors in [16, 25] to define the heat semigroup on $O_N^+$ to be central quantum Markov semigroup $T_t = e^{-At} : L_\infty(O_N^+) \to L_\infty(O_N^+)$ defined by

$$T_t(u_{ij}^k) = e^{-\lambda_k t u_{ij}^k}, \quad A(u_{ij}^k) = \lambda_k u_{ij}^k, \quad \lambda_k = \frac{U_k'(N)}{U_k(N)}.$$

It is well-known that [25] $T_t$ is ergodic. The heat semigroup on $O_N^+$ has been the subject of intensive study in recent years [16, 25, 26, 14, 10], where hypercontractivity properties and connections to deformation/rigidity for von Neumann algebras were considered.

We note that it is an immediate consequence of Theorem 3.2 that the heat semigroup on $O_N^+$ satisfies $\text{G Ric} \geq 0$. In the following subsection, we construct a concrete derivation of $T_t$ by uncovering an unexpected connection to the connection between the heat semigroup on $O_N^+$ and the Laplace-Beltrami operator on the classical orthogonal group $O_N$.

### 3.2.2. An explicit derivation triple for the heat semigroup on $O_N^+$: Connections to the Laplace operator on $O_N$.

Let $O_N$ be the $N \times N$ real orthogonal group in $M_N(\mathbb{R})$. Its Lie algebra $\mathfrak{so}_N$ consists of skew-symmetric matrices and the Lie bracket is given by the commutator $[A,B] = AB - BA$. Denote by $L = \sum_j X_j^2$ as the Casimir operator on $O_N$, where $X_j$ is an orthonormal basis for the negative Killing form $-K(a,b) = (N-2)\text{Tr}(a^tb)$. For the ease of notation, we write $\mathcal{M} = L_\infty(O_N^+)$ in the following. We denote by $E_\Delta : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M}$ the conditional expectation given by the adjoint of the comultiplication $\Delta$. Define a right action $\alpha : O_N \to \text{Aut}(\mathcal{M})$ by the formula

$$\alpha_g(u_{ij}) = \sum_{1 \leq l \leq N} u_{il}g_{jl}.$$

A priori, $\alpha$ is well-defined as a right action $O_N \to \text{Aut}(C(O_N^+))$, but one readily sees that this action is $\tau$-preserving, and hence extends to a right action $\alpha : O_N \to \text{Aut}(\mathcal{M})$. We
note that for each \( k \in \mathbb{N}_0 \), we have
\[
\alpha_g(u^k_{ij}) = \sum_{l=1}^{n_k} u^k_{il}(g^{-1})^k_{lj} = \sum_{l=1}^{n_k} u^k_{il}g^k_{jl},
\]
where \( g \mapsto [g^k_{ij}] \in U(H_k) \) is the corresponding representation of \( O_N \) on \( H_k \) obtained by restricting the representation \( u^k \) of \( O_N^+ \) to the subgroup \( O_N \). In other words, if \( \rho : O(O_N^+) \to O(O_N) \) is the canonical quotient map, then \( g^k_{ij} := \rho(u^k_{ij})(g) \).

We also define the \(*\)-monomorphism
\[
\pi : \mathcal{M} \otimes \mathcal{M} \to L_\infty(O_N, \mathcal{M} \otimes \mathcal{M}), \pi(x)(g) = (\alpha_g^{-1} \otimes \text{id})(x).
\]
and its adjoint conditional expectation
\[
E_\pi : L_\infty(O_N, \mathcal{M} \otimes \mathcal{M}) \to \mathcal{M} \otimes \mathcal{M}, E_\pi(f) = \int_{O_N} (\alpha_g \otimes \text{id})(f(g)) dg.
\]

**Proposition 3.9.** Let \( L = \sum_j X_j^2 \) be the Casimir operator on \( O_N \) and \( \nabla \) be the gradient of \( L \) given by
\[
\nabla : C^\infty(O_N) \to \bigoplus_{j=1}^{N(N-1)/2} C^\infty(O_N), \nabla(f) = \bigoplus_{j=1}^{N(N-1)/2} X_j f.
\]
Then
(i) On \( O(O_N^+) \), we have the identity
\[
E_\Delta \circ E_\pi \circ (L \otimes \text{id}_\mathcal{M} \otimes \mathcal{M}) \circ \pi \circ \Delta = \frac{N(N-1)}{2(N-2)} A.
\]
(ii) Denote by \( \hat{\mathcal{M}} := \bigoplus_{j=1}^{N(N-1)/2} L_\infty(O_N, \mathcal{M} \otimes \mathcal{M}) \) and
\[
\delta := \left( \frac{N(N-1)}{2(N-2)} \right)^{-1/2} \left( \nabla \otimes \text{id}_\mathcal{M} \otimes \mathcal{M} \right) \circ \pi \circ \Delta : O(O_N^+) \to \hat{\mathcal{M}}.
\]
Then \( (O(O_N^+), \hat{\mathcal{M}}, \delta) \) is a derivation triple of \( T_l \).
(iii) \( \nabla \) satisfies the G\text{Ric} \( \geq 0 \) relation: On \( O(O_N^+) \),
\[
\delta \circ T_l = (\text{id} \otimes L_\infty(O_N) \otimes \text{id}_\mathcal{M} \otimes T_l) \delta
\]

**Proof.** In the following, we will abuse notation slightly and write \( g^k_{ij} \) to mean both the scalar coefficient of the associated \( O_N \)-representation and also the coefficient function \( O_N \ni g \mapsto g^k_{ij} \). Let \( S_h \) be the left shift operator on \( C(O_N) \) by \( h \in O_N \), so that
\[
S_h g^k_{ij} = (hg)^k_{ij} = \sum_{1 \leq l \leq n_k} h^k_{il}g^k_{lj}.
\]
Because $L$ is left invariant, 
\[
L(g^k_{ij})|_{g=\mathbf{h}} = S_h(L|_e(g^k_{ij})) = L(S_h(g^k_{ij}))|_e = \sum_l h^k_{il} L(g^k_{ij})|_e,
\]
where $e$ is the identity element in $O_N$. Therefore, 
\[
\left( (L \otimes \text{id}) \circ \pi \circ \Delta(u^k_{ij}) \right)|_h = \left( \sum_l (L \otimes \text{id}) \circ \pi \left( u^k_{il} \otimes u^k_{lj} \right) \right)|_h
\]
\[
= \left( \sum_l (L \otimes \text{id})(u^k_{im} g^k_{mi} \otimes u^k_{lj}) \right)|_h
\]
\[
= \sum_{l,m} (u^k_{im} L(g^k_{ml})|_h \otimes u^k_{lj})
\]
\[
= \sum_{l,m,n} (u^k_{im} h^k_{ln} L(g^k_{nl})|_e \otimes u^k_{lj})
\]
\[
= \sum_{l,n} (\alpha_{h^{-1}}(u^k_{in} L(g^k_{nl})|_e) \otimes u^k_{lj})
\]
\[
= \alpha_{h^{-1}} \otimes \text{id} \left( (L \otimes \text{id}) \circ \pi \circ \Delta(u^k_{ij}) \right)|_e
\]
Note that the range of $\pi$ is $\text{ran}(\pi) = \{ f \in L_\infty(O_N, M^k \otimes M) \mid f(g) = \alpha_{g^{-1}} \otimes \text{id}(f(e)) \}$ and 
\[
E_\pi(f) = \int_{O_N} \alpha_g \otimes \text{id}(f) dg.
\]
Then $(L \otimes \text{id}) \circ \pi \circ \Delta(u^k_{ij}) \in \text{ran}(\pi)$, and 
\[
E_\Delta \circ E_\pi \circ (L \otimes \text{id}) \circ \pi \circ \Delta(u^k_{ij}) = E_\Delta \left( (L \otimes \text{id}) \circ \pi \circ \Delta(u^k_{ij}) \right)|_e
\]
\[
= E_\Delta \left( \sum_{l,n} u^k_{in} L(g^k_{nl})|_e \otimes u^k_{lj} \right)
\]
\[
= \sum_{l,n} u^k_{il} L(g^k_{il})|_e \otimes u^k_{lj}
\]
\[
= \frac{L(\chi_k)|_e}{U_k(N)} u^k_{ij}
\]
where $g \mapsto \chi_k(g) = \sum_l g^k_l$ is the character function of the representation of $O_N$ on $H^k$.

Now it is sufficient to verify that there is a constant $C(N) \neq 0$ (depending only on $N$ and not $k$) so that 
\[
L(\chi_k)|_e = C(N) U_k'(N)
\]
By the recursive relation $xU_k = U_{k-1} + U_{k+1}$ we have at $x = N$, 
\[
U_k(N) + N U_k'(N) = U_{k-1}'(N) + U_{k+1}'(N)
\]
On the other hand, by the same fusion rule 
\[
\chi_1(g) \chi_k(g) = \chi_{k-1}(g) + \chi_{k-1}(g)
\]
Applying the Casimir operator and evaluating at $g = e$, we thus obtain
\[ \left. L(\chi_1\chi_k) \right|_e = \left. L(\chi_{k+1}) \right|_e + \left. L(\chi_{k-1}) \right|_e \] (10)

By the Leibniz rule, we have
\[ \left. L(\chi_1\chi_k) \right|_e = \sum_j X_j^2 (\chi_1\chi_k) \]
\[ = \sum_j X_j \left( (X_j\chi_1)\chi_k + \chi_1 (X_j\chi_k) \right) \]
\[ = \sum_j (X_j^2\chi_1)\chi_k + (X_j\chi_1)(X_j\chi_k) + \chi_1 (X_j^2\chi_k) \]
\[ = \left. L(\chi_1)\chi_k \right|_e + \sum_j (X_j\chi_1)(X_j\chi_k) + \chi_1 \left. L(\chi_k) \right|_e \]

Note that for $k = 1$, $g \mapsto \chi_1(g) = \text{Tr}(g)$ is just the character of the fundamental representation of $O_N$, so at $g = e$,
\[ X_j\chi_1|_e = \frac{d}{dt} \text{Tr}(\exp(tX_j))|_{t=0} = \text{Tr}(X_j) = 0, \]
where the last equality follows because $X_j \in \mathfrak{so}_N$ is skew-symmetric. Also,
\[ \left. L(\chi_1) \right|_e = \sum_j X_j X_j (\chi_1) = \sum_j \frac{\partial^2}{\partial s \partial t} \text{Tr}(\exp(sX_j) \exp(tX_j))|_{s,t=0} \]
\[ = \sum_j \text{Tr}(X_j X_j) = \frac{N(N-1)}{2(N-2)}. \]

Here the constant $\frac{N(N-1)}{2(N-2)}$ comes from the fact that $\dim(\mathfrak{so}_N) = N(N-1)/2$ and $\{X_j\}$ is an orthonormal basis for negative Killing form of $\mathfrak{so}_N$,
\[ B(X,Y) = (N-2)\text{Tr}(XY). \]

Note that $\chi_1(e) = N, \chi_k(e) = \dim(H_k) = U_k(N)$. Then the (10) becomes
\[ \frac{N(N-1)}{2(N-2)} U_k(N) + N L(\chi_k)|_e = \left. L(\chi_{k+1}) \right|_e + \left. L(\chi_{k-1}) \right|_e \]

This shows that the sequences \{\frac{N(N-1)}{2(N-2)} (L\chi_k)|_e\} and \{U'_k(N)\} coincide because they satisfy the same recursive relation. This verifies i).

For ii), we denote by $J = \pi \circ \Delta : \mathcal{M} \to L^\infty(O_N, \mathcal{M} \otimes \mathcal{M})$ the $*$-monomorphism and $E = E_\Delta \circ E_\pi$ the adjoint of $J$. Let $\nabla$ be the gradient of $L$,
\[ \nabla : C^\infty(O_N) \to \bigoplus_{j=1}^{N(N-1)/2} C^\infty(O_N), \quad \nabla(f) = \bigoplus_{j=1}^{N(N-1)/2} X_j f. \]
Finally, for iii) we verify that \( \delta \circ T_t = \left( \frac{N(N-1)}{2(N-2)} \right)^{-1/2} \left( \nabla \otimes \operatorname{id} \right) \circ \pi \circ \Delta \circ T_t \)

\[ = \left( \frac{N(N-1)}{2(N-2)} \right)^{-1/2} \left( \nabla \otimes \operatorname{id} \right) \circ \pi \circ (\operatorname{id}_\mathcal{M} \otimes T_t) \circ \Delta \]

\[ = \left( \frac{N(N-1)}{2(N-2)} \right)^{-1/2} (\operatorname{id} \oplus \mathcal{L}_\infty(O_N) \otimes \operatorname{id}_\mathcal{M} \otimes T_t) \circ (\nabla \otimes \operatorname{id}) \circ \pi \circ \Delta \]

\[ = (\operatorname{id} \oplus \mathcal{L}_\infty(O_N) \otimes \operatorname{id}_\mathcal{M} \otimes T_t) \circ \delta. \]

This completes the proof. \( \square \)

We now turn to estimating the CB-return time. We first recall some estimates of the growth of the dimensions \( n_k = \dim k = U_k(N) \) and the eigenvalues \( \lambda_k \) for the heat semigroup.

**Lemma 3.10** (Lemma 1.7 of \([25]\)). Denote by \( n_k = U_k(N) \) and \( \lambda_k = \frac{\bar{U}_k(N)}{U_k(N)} \). Then for \( k \geq 0 \),

\[ n_k \leq N^k, \quad \frac{k}{N-2} \geq \lambda_k \geq \frac{k}{N}. \]

We are now ready to prove CLSI for \( T_t \). We write \( L_1(O_N^+) := L_1(L_\infty(O_N^+), \tau) \).

**Theorem 3.11.** Let \( T_t : L_\infty(O_N^+) \to L_\infty(O_N^+) \) be the heat semigroup defined as above. Let \( E_\tau(a) = \tau(a)1 \) be the conditional expectation onto the scalars. Then for \( t > N \log N \),

\[ \| T_t - E_\tau : L_1(O_N^+) \to L_\infty(O_N^+) \|_{cb} \leq \frac{2e^{(tN^{-1}-\log N)} - e^{2(tN^{-1}-\log N)}}{(1 - e^{(tN^{-1}-\log N)})^2} \]
As a consequence, $T_t$ satisfies $\lambda$-CLSI for $\lambda = \left(4N \log \left(\frac{N}{1-\frac{1}{\sqrt{3}}}\right)\right)^{-1}$.

**Proof.** Denote $C(T_t) \in L_\infty(O^+_N)^{\text{op} \otimes L_\infty(O^+_N)}$ (resp. $C(E_\tau)$) as the Choi operator of $T_t$ (resp. $E_\tau$). Note that $\{u_{ij}^k\}$ is an orthogonal set with $\tau((u_{ij}^k)^* u_{ij}^k) = \frac{1}{n_k}$. We have

$$C(T_t) = \sum_{k \geq 0} e^{-\lambda_k t} n_k \sum_{1 \leq i,j \leq n_k} (u_{ij}^k)^{\text{op} \otimes u_{ij}^k}$$

Then

$$\| T_t - E_\tau : L_1 \rightarrow L_\infty \|_{cb} = \| C(T_t) - C(E_\tau) \|_{L_\infty(O^+_N)^{\text{op} \otimes L_\infty(O^+_N)}}$$

$$= \| \sum_{k \geq 1} e^{-\lambda_k t} n_k \sum_{1 \leq i,j \leq n_k} (u_{ij}^k)^{\text{op} \otimes u_{ij}^k} \|_{L_\infty(O^+_N)^{\text{op} \otimes L_\infty(O^+_N)}}$$

$$\leq \sum_{k \geq 1} e^{-\lambda_k t} n_k \| \sum_{1 \leq i,j \leq n_k} (u_{ij}^k)^{\text{op} \otimes u_{ij}^k} \|_{L_\infty(O^+_N)^{\text{op} \otimes L_\infty(O^+_N)}}$$

$$= \sum_{k \geq 1} e^{-\lambda_k t} n_k \| \sum_{1 \leq i,j \leq n_k} (S u_{ij}^k)^{\text{op} \otimes u_{ij}^k} \|_{L_\infty(O^+_N)^{\text{op} \otimes L_\infty(O^+_N)}}$$

$$= \sum_{k \geq 1} e^{-\lambda_k t} n_k \| \sum_{1 \leq i,j \leq n_k} (S \otimes \text{id}) \Delta(u_{jj}^k) \|_{L_\infty(O^+_N)^{\text{op} \otimes L_\infty(O^+_N)}}$$

But since $S : L_\infty(O^+_N) \rightarrow L_\infty(O^+_N)^{\text{op}}$ is a $\ast$-isomorphism and hence a complete isometry, we have

$$\| \sum_{1 \leq j \leq n_k} (S \otimes \text{id}) \Delta(u_{jj}^k) \|_{L_\infty(O^+_N)^{\text{op} \otimes L_\infty(O^+_N)}}$$

$$= \| \sum_{j} u_{jj}^k \|_{L_\infty(O^+_N)}$$

$$= \| \chi_k \|_{L_\infty(O^+_N)}$$

$$= k + 1.$$  

Note that in last equality, we have used the fact that $\chi_k = U_k(\chi_1)$ and the spectrum of $\chi_1$ relative to the Haar state is $[-2, 2]$. Combining the above and Lemma 3.10, we have

$$\| T_t - E_\tau : L_1 \rightarrow L_\infty \|_{cb} \leq \sum_{k \geq 1} e^{-\lambda_k t} (k + 1) n_k$$

$$\leq \sum_{k \geq 1} (k + 1) e^{-\frac{1}{N} N^k}$$

$$= \sum_{k \geq 1} (k + 1) e^{k\left(-\frac{1}{N}\right) + \log N}$$
Now, provided $t > N \log N$, we can let $r = e^{-\frac{t}{N} + \log N} < 1$ and get
\[
\| T_t - E_{\tau} : L_1 \to L_\infty \|_{cb} \leq \sum_{k \geq 1} (k + 1) r^k
\]
\[
= \frac{d}{dr} \sum_{k \geq 1} r^{k+1}
\]
\[
= \frac{d}{dr} \left( \frac{r^2}{1 - r} \right)
\]
\[
= \frac{2r - r^2}{(1 - r)^2}
\]
\[
= \frac{1}{2} \quad \text{(provided } r = 1 - \sqrt{\frac{2}{3}})\].

This shows that $t_{cb}$ is given by
\[
e^{-\frac{t_{cb}}{N} + \log N} = r = 1 - \sqrt{\frac{2}{3}} \iff t_{cb} = N \log \left( \frac{N}{1 - \sqrt{\frac{2}{3}}} \right).
\]

From this, we see that $T_t$ has $\lambda$-CLSI for $\lambda = \frac{1}{4r_{cb}}$, as claimed. □

**Remark 3.12.** It was proved in [10] that $T_t$ satisfies $\lambda$-LSI for $\lambda = 2/N$. This is asymptotically different with our CLSI-constant.

### 3.2.3. Transference semigroup of small dimensional irreducible representation.

In this part we investigate the transference semigroup of small dimensional irreducible representation. Let $U \in B(H) \otimes C(O_N^+)$ be a unitary representation on a finite dimensional Hilbert space $H$. Consider the (left) coaction induced by $U$,
\[
\pi_U : B(H) \to B(H) \otimes C(O_N^+) \, , \, \pi(x) = U^* (x \otimes 1) U
\]

The transference semigroup $S_t : B(H) \to B(H)$ is defined by the following commuting diagram,
\[
\begin{array}{ccc}
B(H) \otimes L_\infty(O_N^+) & \xrightarrow{T_t \otimes \text{id}} & B(H) \otimes L_\infty(O_N^+) \\
\uparrow \pi_U & & \uparrow \pi_U \\
B(H) & \xrightarrow{S_t} & B(H)
\end{array}
\] (11)

Here, the induced map $S_t$ is well-defined because $\text{ran}(T_t \otimes \text{id} \circ \pi_U) \subset \text{ran}(\pi_U)$. Indeed, let $E_U$ be the conditional expectation as adjoint of $\pi_U$,
\[
\text{id} \otimes T_t (\pi_U(x)) = \text{id} \otimes T_t \otimes \tau (\pi_U(x) \otimes 1)
\]
\[
= \text{id} \otimes T_t \otimes \tau (U_{12}^* (x \otimes 1 \otimes 1) U_{12})
\]
\[
= \text{id} \otimes T_t \otimes \tau (U_{13} U_{13}^* U_{12}^* (x \otimes 1 \otimes 1) U_{12} U_{13} U_{13}^*)
\]
Here we use the fact $\tau = 0$ for any $j, l$ if $s \neq r$. For the matrix unit $e_{rr},$ 

$$= T_t \left( \sum_{ij} (u^1_{ij} \otimes e_{ji}) (1 \otimes e_{rr}) \left( \sum_{kl} u^1_{kl} e_{kl} \right) \right)$$

$$= T_t \left( \sum_{j,l} (u^1_{s j} \otimes u^1_{r l}) \otimes e_{ji} \right)$$

$$= e^{-\lambda t} \sum_{j,l} ((u^1_{s j})^* u^1_{r l}) \otimes e_{ji}$$

$$= \pi_1(e_{ss}^*) \sum_{j,l} ((u^1_{s j})^* u^1_{r l}) \otimes e_{ji}$$

Thus $S_t$ is a quantum Markov semigroup on $B(H)$. By the interwine relation $\pi_U \circ S_t = (\text{id}_{B(H)} \otimes T_t) \circ \pi_U$, $S_t$ can be viewed as a sub-system of $\text{id}_{B(H)} \otimes T_t$. In particular, if $T_t$ has $\text{GRic} \geq \lambda$ or $\lambda$-CLSI, so does $S_t$.

Now we consider the transference semigroup induced by the 1-st irreducible representation given by $U = \sum_{i,j} u^1_{ij} \otimes e_{ij} \in C(O^+_N) \otimes M_N$. To calculate $S_t$ for $U^1$, we recall the fusion rule $1 \otimes 1 = 2 \otimes 0$. That is, for each level-1 coefficients $a^1, b^1$, 

$$a^1 b^1 = (a^1 b^1) \oplus (a^1 b^1) 1$$

where $(a^1 b^1) \in \text{span}\{u^2_{ij} | 1 \leq i, j \leq n_2\}$ is a level-2 coefficient. Let 

$$\pi_1 : M_N \rightarrow L_{\infty}(O^+_N) \otimes M_N; \pi_1(x) = (U^1)^*(x \otimes 1) U^1$$

be the coaction of $U^1$. Then for the matrix unit $e_{sr}$ with $s \neq r,$ 

$$T_t(\pi_1(e_{sr})) = T_t \left( \sum_{ij} (u^1_{ij})^* \otimes e_{ji} \right) \left( \sum_{kl} u^1_{kl} e_{kl} \right)$$

$$= T_t \left( \sum_{j,l} (u^1_{s j})^* u^1_{r l} \otimes e_{ji} \right)$$

$$= T_t \left( \sum_{j,l} ((u^1_{s j})^* u^1_{r l}) \otimes e_{ji} \right)$$

$$= e^{-\lambda t} \sum_{j,l} ((u^1_{s j})^* u^1_{r l}) \otimes e_{ji}$$

$$= \pi_1(e^{-\lambda t} e_{sr})$$

Here we use the fact $\tau((u^1_{s j})^* u^1_{r l}) = 0$ for any $j, l$ if $s \neq r$. For the matrix unit $e_{rr},$ 

$$T_t(\alpha(e_{rr}))$$

$$= T_t \left( \sum_{ij} (u^1_{ij})^* \otimes e_{ji} \right) \left( 1 \otimes e_{rr} \right) \left( \sum_{kl} u^1_{kl} e_{kl} \right)$$

$$= T_t \left( \sum_{j,l} (u^1_{r j})^* u^1_{r l} \otimes e_{ji} \right)$$
$T_t \left( \sum_{j,l} (u^1_{rj} \ast u^1_{rl}) \otimes e_{jl} + \sum_l \tau((u^1_{rl})^* u^1_{rl}) e_{ll} \right)$

$= e^{-\lambda_2 t} \sum_{j,l} ((u^1_{rj})^* u^1_{rl}) \otimes e_{jl} + \sum_l \tau((u^1_{rl})^* u^1_{rl}) e_{ll}$

$= e^{-\lambda_2 t} \left( \sum_j ((u^1_{rj})^* u^1_{rl}) \otimes e_{jl} + \sum_l \tau((u^1_{rl})^* u^1_{rl}) e_{ll} \right) + \left( 1 - e^{-\lambda_2 t} \right) \left( \sum_l \tau((u^1_{rl})^* u^1_{rl}) e_{ll} \right)$

$= e^{-\lambda_2 t} \pi_1(e_{rr}) + \left( 1 - e^{-\lambda_2 t} \right) \frac{1}{N} 1$

Thus $S_t : M_N \to M_N$ is exactly the depolarizing semigroup

$S_t(\rho) = e^{-\lambda_2 t} (\rho - tr(\rho) \frac{1}{N}) + tr(\rho) \frac{1}{N}$.

which has $\lambda_2/2$-CGE by [9, Section 3.3]. We will revisit the depolarizing semigroups in Section 6.1.

3.3. Quantum Automorphism Groups. We briefly discuss here another class of examples of quantum groups given by the quantum automorphism groups of finite-dimensional C*-algebras. Let $B$ be a finite-dimensional C*-algebra, and let $\psi : B \to \mathbb{C}$ be the canonical trace-state on $B$. Namely, $\psi$ is the restriction of the unique normalized trace on the endomorphism algebra $\text{End}(B)$, where $B \hookrightarrow \text{End}(B)$ via the left-regular representation of $B$. Given any pair $(B, \psi)$, one can define the quantum automorphism group of $(B, \psi)$, which we denote by $G^+(B, \psi)$. The construction goes as follows. Put $H = L^2(B, \psi)$ and fix any orthonormal basis $(e_i)_i \subset H$ and let $e_{ij} \in B(H)$ be the corresponding matrix units. Then $C^u(G^+(B, \psi))$ is the universal C*-algebra with generators $u_{ij}$ subject to the following relations:

1. The matrix $u = \sum_{i,j} u_{ij} \otimes e_{ij} \in C^u(G^+(B, \psi)) \otimes B(H)$ is unitary.
2. $(1 \otimes m) u_{12} u_{13} = u(1 \otimes m)$, where $m : H \otimes H \to H$ is the multiplication map on $B \cong H$.
3. $u(1 \otimes \eta) = 1 \otimes \eta$, where $\eta : \mathbb{C} \to B \cong H; \alpha \mapsto \alpha 1_B$ is the unit map.

We equip $C^u(G^+(B, \psi))$ with the coproduct given by the formula

$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$,

and then the pair $G^+(B, \psi) = (C^u(G^+(B, \psi)), \Delta)$ is a compact quantum group.

Example 3.13 (Quantum Permutation Groups). When $B = C(X)$, where $X$ is a finite set with $N$ elements, we have that $\psi$ corresponds to the uniform probability on $X$ and
$G^+(B, \psi)$ is nothing other than the \textit{quantum permutation group} [16], which is commonly denoted by $S_N^+$. In this case the generators $u_{ij} \in C^u(S_N^+)$ satisfy the relations $u_{ij} = u_{ij}^* = u_{ij}^2$ and $\sum_k u_{ik} = 1 = \sum_k u_{ki}$ for all $1 \leq i \leq N$. When $N \leq 3$, one can show that $C^u(S_N^+) \cong C(S_N)$, but when $N \geq 4$, $C^u(S_N^+)$ is infinite dimensional and non-commutative.

When $B$ is a finite-dimensional C$^*$-algebra with $\dim B \geq 4$, it turns out that the representation theory of $G^+(B, \psi)$ is very similar to that of $O_N^+$. The irreducible representations $\text{Irr}(G^+(B, \psi))$ are indexed by the non-negative integers $\mathbb{N}_0$ and the fusion rules are given by

$$k \otimes m = |k - m| \oplus (|k - m| + 1) \oplus \cdots \oplus (k + m - 1) \oplus (k + m), k, m \in \mathbb{N}.$$  

Here, the label 0 corresponds to the trivial representation and 1 \oplus 0 corresponds to the fundamental representation $u = [u_{ij}] \in C^u(G^+(B, \psi)) \otimes B(H)$. Recall that the Chebyshev polynomials of second kind are denoted by $(U_k)_{k \in \mathbb{N}_0}$ (cf. (28)). We then have the following analogue of Theorem 3.8 characterizing the generators of the central Markov semigroups on $L_\infty(G^+(B, \psi))$. The following result (to the best of our knowledge) does not appear explicitly in the literature. The corresponding version for $S_N^+$ was proved in [26, Theorem 10.10], and the general statement can be proved using methods from [14], the classification of central states on $SU_q(2)$ and $SO_q(3)$ from [21], and the fact that every $G^+(B, \psi)$ with $\dim B \geq 4$ is monoidally equivalent to $SO_q(3)$ with $q + q^{-1} = \sqrt{\dim B}$ [22].

In the following, we let $v_{ij}^k$ denote a generic coefficient of the $k$th irreducible representation $v^k$ of $G^+(B, \psi)$.

**Theorem 3.14.** [16 Corollary 10.3] Let $d = \dim B \geq 4$. Let $(\xi_k)_{k \in \mathbb{N}_0} \subset \mathbb{C}$ and define a central semigroup on $\mathcal{O}(G^+(B, \psi))$ via the formula

$$T_t : \mathcal{O}(G^+(B, \psi)) \to \mathcal{O}(G^+(B, \psi)); T_t(v_{ij}^k) = e^{-\xi_k t} v_{ij}^k.$$  

Then $T_t$ extends uniquely to a symmetric central quantum Markov semigroup on $L_\infty(G^+(B, \psi))$ if and only if there is a constant $b \geq 0$ and a finite positive Borel measure $\nu$ supported on $[0, d]$ satisfying $\nu\{d\} = 0$, so that

$$\xi_k = \frac{1}{U_{2k}(\sqrt{d})} \left( bU'_{2k}(\sqrt{d}) \frac{2\sqrt{d}}{2\sqrt{d}} + \int_0^d \frac{U_{2k}(\sqrt{d}) - U_{2k}(\sqrt{x})}{x - d} d\nu(x) \right). \quad (12)$$  

By analogy with the case of $O_N^+$, we define the heat semigroup on $G^+(B, \psi)$ (with $d = \dim B \geq 4$) as $T_t = e^{-At} : L_\infty(G^+(B, \psi)) \to L_\infty(G^+(B, \psi))$ given by

$$T_t(v_{ij}^k) = e^{-\xi_k t} v_{ij}^k, A(v_{ij}^k) = \xi_k v_{ij}^k, \xi_k = \frac{U'_{2k}(\sqrt{d})}{2\sqrt{d}U_{2k}(\sqrt{d})};$$  

See [16 Section 1.4]. It follows from Theorem 3.2 that $T_t$ satisfies $\text{GRic} \geq 0$. We also have the following estimates.
Lemma 3.15. [16] Lemma 1.8] Let \( m_k = \dim(v^k) = U_{2k}(\sqrt{d}) \) and \( \xi_k = \frac{U_{1k}(\sqrt{d})}{2\sqrt{d}U_{2k}(\sqrt{d})} \). Then for \( k \geq 0 \)

\[
m_k \leq (d - 1)^k, \quad \frac{k}{d} \leq \xi_k \leq \frac{k}{\sqrt{d}(\sqrt{d} - 2)}.
\]

Theorem 3.16. Let \( T_t : L_\infty(G^+(B, \psi)) \to L_\infty(G^+(B, \psi)) \), \( T_t(v_{ij}^k) = e^{-\xi_{ij}t}v_{ij}^k \) be the heat semigroup defined as above. Let \( E_\tau(a) = \tau(a)1 \) be the conditional expectation onto the scalars. Then for \( t \geq d \log(d - 1) \),

\[
\| T_t - E_\tau : L_1(G^+(B, \psi)) \to L_\infty(G^+(B, \psi)) \|_{cb} \leq \frac{4r - 2r^2}{(1 - r)^2} \frac{r}{1 - r},
\]

where \( r = e^{\frac{d}{4} + \log(d - 1)} \). As a consequence \( T_t \) satisfies \( \lambda \text{-CLSI} \) for \( \lambda = \left( 4d \log \left( \frac{3(d - 1)}{4 - \sqrt{d}3} \right) \right)^{-1} \).

Proof. The argument is analogous to the proof of Theorem 3.11 so will sketch the main arguments. Denote by \( C(T_t) \in L_\infty(G^+(B, \psi))^{op} \otimes L_\infty(G^+(B, \psi)) \) (resp. \( C(E_\tau) \)) the Choi operator of \( T_t \) (resp. \( E_\tau \)). Note that \( \{v_{ij}^k\} \) is an orthogonal basis with \( \tau((v_{ij}^k)^*v_{ij}^k) = \frac{1}{m_k} \).

We have

\[
C(T_t) = \sum_{k \geq 0} e^{-\xi_{ij}t}m_k \sum_{1 \leq i,j \leq m_k} (v_{ij}^k)^{op} \otimes v_{ij}^k
\]

\[
C(E_\tau) = 1^{op} \otimes 1
\]

Then

\[
\| T_t - E_\tau : L_1 \to L_\infty \|_{cb} = \| C(T_t) - C(E_\tau) \|_{L_\infty(G^+(B, \psi))^{op} \otimes L_\infty(G^+(B, \psi))} \\
= \| \sum_{k \geq 1} e^{-\xi_{ij}t}m_k \sum_{1 \leq i,j \leq m_k} (v_{ij}^k)^{op} \otimes v_{ij}^k \|_{L_\infty(G^+(B, \psi))^{op} \otimes L_\infty(G^+(B, \psi))} \\
\leq \sum_{k \geq 1} e^{-\xi_{ij}t}m_k \| \sum_{1 \leq i,j \leq m_k} (v_{ij}^k)^{op} \otimes v_{ij}^k \|_{L_\infty(G^+(B, \psi))^{op} \otimes L_\infty(G^+(B, \psi))} \\
= \sum_{k \geq 1} e^{-\xi_{ij}t}m_k \| \chi_k \|_{L_\infty(G^+(B, \psi))} \\
= \sum_{k \geq 1} e^{-\xi_{ij}t}m_k (2k + 1) \\
\leq \sum_{k \geq 1} (2k + 1)e^{-\frac{k}{d} + k \log(d - 1)}.
\]

Here, we used Lemma 3.15 and the equality \( \| \chi_k \|_{L_\infty(G^+(B, \psi))} = \sup_{t \in [0,4]} |U_{2k}(\sqrt{t})| \). We then get, for \( t > d \log(d - 1) \) and \( r = e^{\frac{d}{4} + \log(d - 1)} \),
\[ \| T_t - E_\tau : L_1 \to L_\infty \|_{cb} \leq \sum_{k \geq 1} (2k + 1) r^k \]

\[ = \frac{4r - 2r^2}{(1-r)^2} - \frac{r}{1-r} \]

\[ = \frac{1}{2} \left( \text{provided } r = \frac{1}{3} \left( 4 - \sqrt{13} \right) \right). \]

This shows that \( t_{cb} \) is given by

\[ e^{-\frac{t_{cb} + \log(d-1)}{d}} = r = \frac{1}{3} \left( 4 - \sqrt{13} \right) \iff t_{cb} = d \log \left( \frac{d - 1}{\frac{1}{3} \left( 4 - \sqrt{13} \right)} \right). \]

By Theorem 2.9 we see that \( T_t \) has \( \lambda \)-CLSI for \( \lambda = \frac{4}{t_{cb}} \), as claimed.

**Remark 3.17.** It is desired to have a concrete derivation triple for the heat semigroup on \( L_\infty(G^+(B, \psi)) \). Nevertheless, for a special case, one can show that the heat semigroup on quantum permutation group \( L_\infty(S_N^+) \) does not admit factorization through any classical Markov semigroup on \( l_\infty(S_N) \) as in Proposition 3.9.

4. **Tensorization and Free Product**

In this section, we discuss tensorization and free product of CLSI and geometric Ricci curvature bound. The similar discussion for CGE is in [18].

4.1. **Commuting Semigroup.** Let \( T_t, S_t : \mathcal{M} \to \mathcal{M} \) be two symmetric quantum Markov semigroups and \( A \) (resp. \( B \)) be the generator of \( T_t \) (resp. \( S_t \)). We say \( T_t \) and \( S_t \) are commuting if \( T_t \circ S_s = S_s \circ T_t \) for any \( s, t \geq 0 \). For commuting \( T_t \) and \( S_t \), \( S_s \circ T_t \) is again an symmetric quantum Markov semigroup because

\[ (S_s \circ T_s) \circ (S_t \circ T_t) = (S_s \circ S_t) \circ (T_s \circ T_t) = S_{s+t} \circ T_{s+t}. \]

Let \( \mathcal{N}_T \) (resp. \( \mathcal{N}_T \)) be the fixed point algebra of \( S_t \) (resp. \( S_t \)). Then \( \mathcal{N} = \mathcal{N}_S \cap \mathcal{N}_T \) is the fixed point subalgebra of \( S_t \circ T_t \). We write \( E_T, E_S \) and \( E \) as the condition expectation respectively onto \( \mathcal{N}_T, \mathcal{N}_S \) and \( \mathcal{N} \). The following lemma is inspired from [35, Corollary 4.2] by LaRacuente.

**Proposition 4.1.** Let \( T_t, S_t : \mathcal{M} \to \mathcal{M} \) be two symmetric quantum Markov semigroups. Suppose \( T_t \) and \( S_t \) are commuting. Then \( S_t \circ T_t \) is a symmetric quantum Markov semigroup. If in additional, both \( T_t \) and \( S_t \) satisfies \( \lambda \)-MLSI (resp. \( \lambda \)-CLSI).

i) \( E_S \circ E_T = E_T \circ E_S = E \) forms a commuting square.

ii) \( T_t \circ S_t \) satisfies \( \lambda \)-MLSI (resp. \( \lambda \)-CLSI).
Proof. Because $T_t$ and $S_t$ satisfies $\lambda$-MLSI, we have the mixing time estimate that for any density operator $\rho$ with finite entropy $H(\rho) < \infty$ (see [4]),

$$\lim_{t \to \infty} \|T_t(\rho) - E_T(\rho)\|_1 \leq \lim_{t \to \infty} \sqrt{2D(T_t(\rho)||E_T(\rho))} \leq \lim_{t \to \infty} \sqrt{2e^{-2Mt}D(\rho||E(\rho))} = 0 .$$

and similar $\lim_{t \to \infty} \|S_t(\rho) - E_S(\rho)\|_1 = 0$ This implies for any $x \in L_1 \cap L_\infty(\mathcal{M})$

$$T_t \circ E_S(x) = \lim_{s \to \infty} T_t \circ S_s(x) = \lim_{s \to \infty} S_s \circ T_t(x) = E_S(x) \circ T_t ,$$

By continuity the same equality extends to $\mathcal{M}$. Thus we have $T_t \circ E_S = E_S \circ T_t$ and by the same argument for $T_t \to E_T$, we have $E = E_T \circ E_S = E_S \circ E_T$ forms a commuting square. It follows that

$$D(\rho||E(\rho)) = H(\rho) - H(E(\rho)) = H(\rho) - H(E_S(\rho)) + H(E_S(\rho)) - H(E(\rho))$$

$$= D(\rho||E_S(\rho)) + D(E_S(\rho)||E(\rho)) .$$

By data processing inequality,

$$D(T_t \circ S_t(\rho)||E(\rho)) = D(S_t(T_t\rho)||E_S(T_t\rho)) + D(T_t(E_S\rho)||E_T \circ E_S(\rho))$$

$$\leq e^{-2Mt}D(T_t\rho||T_tE_S\rho) + e^{-2Mt}D(E_S(\rho)||E(\rho))$$

$$\leq e^{-2Mt}D(\rho||E_S(\rho)) + e^{-2Mt}D(E_S(\rho)||E(\rho))$$

$$= e^{-2Mt}D(\rho||E(\rho)) .$$

which implies that $T_t \circ S_t$ has $\lambda$-MLSI. The same argument for $T_t \otimes \text{id}_\mathcal{R}, S_t \otimes \text{id}_\mathcal{R} : \mathcal{M} \otimes \mathcal{R} \to \mathcal{M} \otimes \mathcal{R}$ yield the assertion for CLSI.

4.2. Tensor product semigroup. Let $T_t : \mathcal{M}_1 \to \mathcal{M}_1$ and $S_t : \mathcal{M}_2 \to \mathcal{M}_2$ be two symmetric quantum Markov semigroups. The tensor product $T_t \otimes S_t : \mathcal{M}_1 \otimes \mathcal{M}_2 \to \mathcal{M}_1 \otimes \mathcal{M}_2$ is again a symmetric Markov semigroup.

Corollary 4.2. Let $T_t : \mathcal{M}_1 \to \mathcal{M}_1$ and $S_t : \mathcal{M}_2 \to \mathcal{M}_2$ be two quantum Markov semigroups. If $T_t$ and $S_t$ satisfy $\lambda$-CLSI, $T_t \otimes S_t$ satisfy $\lambda$-CLSI.

Proof. By definition of CLSI, both $T_t \otimes \text{id}_{\mathcal{M}_2}$ and $\text{id}_{\mathcal{M}_1} \otimes S_t$ satisfy $\lambda$-CLSI. The assertion follows from Proposition 4.1. ■

We shall now discuss the tensorization of GRic. Let $A$ (resp. $B$) be the generator of $T_t$ (resp. $S_t$). Let $L$ be the generator of $T_t \otimes S_t$. Then $T$ is an closed extension of $L = A \otimes \text{id} + \text{id} \otimes B$ and dom($A$) $\otimes$ dom($B$) $\subset$ dom($L$). Namely, for $x \in \text{dom}(A), y \in \text{dom}(B)$, $L(x \otimes y) = Ax \otimes y + x \otimes By$. The gradient form of $T_t \otimes S_t$ is

$$\Gamma(x \otimes y, x \otimes y) = \Gamma_A(x, x) \otimes y^*y + x^*x \otimes \Gamma_B(y, y) .$$

where $\Gamma_A$ (resp. $\Gamma_B$) is the gradient form of $A$ (resp. $B$).
Lemma 4.3. Let $T_t : \mathcal{M}_1 \to \mathcal{M}_1$ and $S_t : \mathcal{M}_2 \to \mathcal{M}_2$ be two symmetric quantum Markov semigroups. Let $(\mathcal{A}, \mathcal{M}_1, \delta_1)$ (resp. $(\mathcal{B}, \mathcal{M}_2, \delta_2)$) be a derivation triple of $T_t$ (resp. $S_t$) with mean zero property. Define the derivation
\[
\delta : \mathcal{A} \otimes \mathcal{B} \to L_2(\hat{\mathcal{M}}_1 \otimes \hat{\mathcal{M}}_2), \delta(x \otimes y) = \delta_1(x) \otimes y + x \otimes \delta_2(y).
\]
Then $(\mathcal{A} \otimes \mathcal{B}, \hat{\mathcal{M}}_1 \otimes \hat{\mathcal{M}}_2, \delta)$ is derivation triple of $T_t \otimes S_t$.

Proof. Let $E_1 : \hat{\mathcal{M}}_1 \to \mathcal{M}_1$ and $E_2 : \hat{\mathcal{M}}_2 \to \mathcal{M}_2$ the conditional expectation. Then $E = E_1 \otimes E_2$ is the conditional expectation from $\hat{\mathcal{M}}_1 \otimes \hat{\mathcal{M}}_2$ to $\mathcal{M}_1 \otimes \mathcal{M}_2$. It is clear that $\delta$ is a $*$-preserving derivation and mean zero $E(\delta(x \otimes y)) = E_1(\delta_1(x)) \otimes y + x \otimes E_2(\delta_2(y)) = 0$. For $x \in \mathcal{A}, y \in \mathcal{B}$, we have
\[
E(\delta(x \otimes y)^* \delta(x \otimes y)) = E(\delta_1(x)^* \delta_1(x) \otimes y^*y + \delta_1(x)^*x \otimes y^*\delta_2(y) + x^*\delta_1(x) \otimes y^*\delta_2(y) + x^*x \otimes \delta_2(y)^*\delta_2(y))
\]
\[
= E(\delta_1(x)^* \delta_1(x) \otimes y^*y + E_1(\delta_1(x))^*x \otimes E_2(y^*\delta_2(y))
\]
\[
\quad + E_1(x^*\delta_1(x)) \otimes E_2(y^*\delta_2(y)) + x^*x \otimes E_2(\delta_2(y)^*\delta_2(y))
\]
\[
= \Gamma_1(x \otimes y^*y + E_1(\delta_1(x))^*x \otimes y^*E_2(\delta_2(y))
\]
\[
\quad + x^*E_1(\delta_1(x)) \otimes E_2(\delta_2(y)^*y + x^*x \otimes x^*x \otimes \Gamma_2(y, y)
\]
\[
= \Gamma(x \otimes y, x \otimes y).
\]
which coincides with the gradient form of $T_t \otimes S_t$. Here in the second last step we used the mean zero property $E_1(\delta_1(x)) = E_2(\delta_2(x)) = 0$. It follows that for any $\xi \in \mathcal{A} \otimes \mathcal{B}$,
\[
(\delta(\xi), \delta(\xi))_{L_2(\hat{\mathcal{M}}_1 \otimes \hat{\mathcal{M}}_2)} = \langle \xi, (A \otimes \text{id} + \text{id} \otimes B)\xi \rangle_{L_2(\hat{\mathcal{M}}_1 \otimes \hat{\mathcal{M}}_2)} = \langle \xi, L\xi \rangle_{L_2(\hat{\mathcal{M}}_1 \otimes \hat{\mathcal{M}}_2)}.
\]
Note that $\mathcal{A} \otimes \mathcal{B}$ is a subalgebra satisfying
\[
T_t \otimes S_t(\mathcal{A} \otimes \mathcal{B}) \subset T_t(\mathcal{A}) \otimes S_t(\mathcal{B}) \subset \mathcal{A} \otimes \mathcal{B},
\]
\[
\mathcal{A} \otimes \mathcal{B} \subset \text{dom}(A^{1/2}) \otimes \text{dom}(B^{1/2}) \subset \text{dom}(L^{1/2}).
\]
Moreover, $\mathcal{A} \otimes \mathcal{B}$ is dense in $\mathcal{A}_\xi \otimes \mathcal{B}_\xi$ with respect to the graph norm, which is a core for $\text{dom}(L^{1/2})$. Thus $\delta$ admits an closed extension $\bar{\delta}$ such that $\delta^*\bar{\delta} = L$. That completes proof.

Proposition 4.4. Let $T_t : \mathcal{M}_1 \to \mathcal{M}_1$ and $S_t : \mathcal{M}_2 \to \mathcal{M}_2$ be two symmetric quantum Markov semigroups.

i) If both $S_t, T_t$ has GRic $\geq \lambda$, $S_t \otimes T_t$ has GRic $\geq \lambda$;

ii) If both $S_t, T_t$ has $\lambda$-GRic, $S_t \otimes T_t$ has $\lambda$-GRic.
Proof. Let \((A, \hat{M}_1, \delta_1)\) (resp. \((B, \hat{M}_2, \delta_2)\)) be a derivation triple of \(T_t\) (resp. \(S_t\)) and \(\hat{T}_t = e^{-\hat{A}t} : \hat{M}_1 \to \hat{M}_1\) and \(\hat{S}_t = e^{-\hat{B}t} : \hat{M}_2 \to \hat{M}_2\) be the extension Markov semigroups giving \(G\text{Ric} \geq \lambda\) respectively. That is, \(\hat{T}_t|_{\hat{M}_1} = T_t\) and \(\hat{S}_t|_{\hat{M}_2} = S_t\)

\[
\hat{A}\delta_1(x) - \delta_1 A(x) = \text{Ric}_A(\delta_1(x)), \ x \in A_0 \\
\hat{B}\delta_2(y) - \delta_2 B(y) = \text{Ric}_B(\delta_2(y)), \ y \in B_0.
\]

with bimodule Ricci operator \(\text{Ric}_A \geq \lambda\) and \(\text{Ric}_B \geq \lambda\). Here \(A, B\) (resp. \(\hat{A}, \hat{B}\)) are the generator of \(T_t, S_t\) (resp. \(\hat{T}_t, \hat{S}_t\)). In particular \(\hat{A}|_{\text{dom}(A)} = A\) and \(\hat{B}|_{\text{dom}(B)} = B\).

We take the derivation triple \((A \otimes B, \hat{M}_1 \hat{\otimes} \hat{M}_2, \delta)\) for \(T_t \otimes S_t\) in Lemma 4.3. Consider the tensor product semigroup \(\hat{T}_t \otimes \hat{S}_t : \hat{M}_1 \hat{\otimes} \hat{M}_2 \to \hat{M}_1 \hat{\otimes} \hat{M}_2\). We have \(\hat{T}_t \otimes \hat{S}_t|_{\hat{M}_1 \hat{\otimes} \hat{M}_2} = T_t \otimes S_t\) and its generator is \(\hat{L} := \hat{A} \otimes \text{id} + \text{id} \otimes \hat{B}\). For any \(x \in A_0, y \in B_0\),

\[
\hat{L}\delta(x \otimes y) - \delta(L(x \otimes y)) = \left(\hat{A}\delta_1(x) \otimes y + \delta_1(x) \otimes By + Ax \otimes \delta_2(y) + x \otimes \hat{B}\delta_2(y)\right) \\
\quad - \left(\delta_1(Ax) \otimes y + \delta_1(x) \otimes By + Ax \otimes \delta_2(y) + x \otimes \delta_2(By)\right) \\
= \left(\hat{A}\delta_1(x) - \delta_1(Ax)\right) \otimes y + (\hat{B}\delta_2(y) - \delta_2(By)) \\
= \text{Ric}_A(\delta_1(x)) \otimes y + x \otimes \text{Ric}_B(\delta_2(y))
\]

Note that \(\Omega_\delta = \overline{(A \otimes B)\delta(A \otimes B)} \subset L_2(\hat{M}_1 \hat{\otimes} \hat{M}_2)\) and

\[
(A \otimes B)\delta(A \otimes B) = A \otimes B\delta_2(B) + A\delta_1(A) \otimes B.
\]

Moreover, by the mean zero property, \(E_1(A\delta_1(A)) = E_2(B\delta_2(B)) = 0\). Indeed, for any \(\xi_1 = x_1\delta_1(y_1) \in \Omega_{\delta_1}\),

\[
E_1(\xi_1) = E_1(x_1\delta_1(y_1)) = x_1 E_1(\delta_1(y_1)) = 0.
\]

Then \(A \otimes B\delta_2(B)\) and \(A\delta_1(A) \otimes B\) are mutually orthogonal in \(L_2(\hat{M}_1 \hat{\otimes} \hat{M}_2)\). We can define

\[
\text{Ric} : \Omega_\delta \to L_2(\hat{M}_1 \hat{\otimes} \hat{M}_2), \ \text{Ric} = (\text{Ric}_A \otimes \text{id}) + (\text{id} \otimes \text{Ric}_B).
\]

It is clear that \(\text{Ric}\) is a \(A \otimes B\)-bimodule operator. For any \(\xi \in \Omega_\delta\), we can write \(\xi = \xi_1 + \xi_2\) with \(\xi_1 \in \Omega_{\delta_1} \otimes B\) and \(\xi_2 \in A \otimes \Omega_{\delta_2}\). Then

\[
\langle \xi, \text{Ric}(\xi) \rangle = \langle \xi_1 + \xi_2, \text{Ric}_A \otimes \text{id}(\xi_1) + \text{id} \otimes \text{Ric}_B(\xi_2) \rangle = \\
= \langle \xi_1, \text{Ric}_A \otimes \text{id}(\xi_1) \rangle + \langle \xi_2, \text{id} \otimes \text{Ric}_B(\xi_2) \rangle \\
\geq \lambda \left\| \xi_1 \right\|_2 + \lambda \left\| \xi_2 \right\|_2 = \lambda \left\| \xi \right\|_2.
\]

That completes the proof. 

\[\square\]
4.3. **Free product semigroup.** Let $\mathcal{M}_1, \mathcal{M}_2$ be two finite von Neumann algebra and $\mathcal{N} \subset \mathcal{M}_1, \mathcal{N} \subset \mathcal{M}_2$ be a common subalgebra. We refer to [44] for the definition amalgamated free product $\mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2$. Denote $E_{\mathcal{N}} : \mathcal{M}_i \to \mathcal{N}, i = 1, 2$ as the conditional expectation onto $\mathcal{N}$ and write $\mathcal{M}_i = \{a \in \mathcal{M}_i | E_{\mathcal{N}}(a) = 0\}$ as the mean zero part. It was proved in [5, Theorem 3.1] that for two Markov semigroup with fixed-point algebra $\mathcal{N}$ was proved in [5, Theorem 3.1] that for two $\mathcal{N}$-bimodule UCP maps $T_1 : \mathcal{M}_1 \to \mathcal{M}_1$ and $T_2 : \mathcal{M}_2 \to \mathcal{M}_2$, the free product map $T_1 \ast T_2 : \mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2 \to \mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2$,

$$T_1 \ast T_2(a_1a_2 \cdots a_n) = T_{i_1}(a_1)T_{i_2}(a_2) \cdots T_{i_n}(a_n), a_k \in \mathcal{M}_{i_k}, i_1 \neq i_2 \neq \cdots \neq i_n.$$ is again UCP.

Let $T_{1,t} = e^{-At} : \mathcal{M}_1 \to \mathcal{M}_1$ and $T_{2,t} = e^{-Bt} : \mathcal{M}_2 \to \mathcal{M}_2$ be two symmetric Markov semigroup with fixed-point algebra $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. Suppose $\mathcal{N} \subset \mathcal{N}_1, \mathcal{N}_2$ as a common subalgebra. Then the free product map $T_t = T_{1,t} \ast T_{2,t} : \mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2 \to \mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2$ is a symmetric quantum Markov semigroup that for any $a_i \in \mathcal{M}_{i_n}$

$$T_{1,t} \ast T_{2,t}(a_1 \cdots a_n) = T_{i_1,t}(a_1)T_{i_2,t}(a_2) \cdots T_{i_n,t}(a_n), a_k \in \mathcal{M}_{i_k}, i_1 \neq i_2 \neq \cdots \neq i_n$$

This map is well-defined because $T_{i,t}(x) = x$ for $x \in \mathcal{N}, i = 1, 2$.

**Definition 4.5.** We say a Markov semigroup $T_t : \mathcal{M} \to \mathcal{M}$ satisfies $\lambda$-free logarithmic Sobolev inequality ($\lambda$-FLSI) for $\lambda \in \mathbb{R}$ if for any finite von Neumann algebra $\mathcal{R}$ with $\mathcal{N} \subset \mathcal{R}, T_t \ast \text{id}_\mathcal{R} : \mathcal{M} \ast_{\mathcal{N}} \mathcal{R} \to \mathcal{M} \ast_{\mathcal{N}} \mathcal{R}$ has $\lambda$-MLSI with respect to $\mathcal{R} \simeq \mathcal{N} \ast_{\mathcal{N}} \mathcal{R} \subset \mathcal{M} \ast_{\mathcal{N}} \mathcal{R}$.

**Proposition 4.6.** Let $T_t : \mathcal{M}_1 \to \mathcal{M}_1$ and $S_t : \mathcal{M}_2 \to \mathcal{M}_2$ be two symmetric Markov semigroup with same fixed-point subalgebra $\mathcal{N}$. If $T_t, S_t$ satisfies $\lambda$-FLSI, $T_t \ast S_t$ satisfies $\lambda$-FLSI.

**Proof.** Let $\mathcal{R}$ be an arbitrary finite von Neumann algebra. We have

$$T_t \ast S_t \ast \text{id}_\mathcal{R} : \mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2 \ast_{\mathcal{N}} \mathcal{R} \to \mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2 \ast_{\mathcal{N}} \mathcal{R}$$

is a symmetric Markov semigroup satisfying

$$T_t \ast S_t \ast \text{id} = (T_t \ast \text{id}_{\mathcal{M}_2} \ast \text{id}_\mathcal{R}) \circ (\text{id}_{\mathcal{M}_1} \ast S_t \ast \text{id}_\mathcal{R}) = (\text{id}_{\mathcal{M}_1} \ast S_t \ast \text{id}_\mathcal{R}) \circ (T_t \ast \text{id}_{\mathcal{M}_2} \ast \text{id}_\mathcal{R}).$$

Then the assertion follows from applying Proposition 4.1 for By assumption of FLSI,

$$T_t \ast \text{id}_{\mathcal{M}_2} \ast \text{id}_\mathcal{R} = T_t \ast \text{id}_{\mathcal{M}_2 \ast_{\mathcal{N}} \mathcal{R}}, (\text{id}_{\mathcal{M}_1} \ast S_t \ast \text{id}_\mathcal{R}) = S_t \ast \text{id}_{\mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{R}}$$

satisfy $\lambda$-MLSI. Then the assertion follows from applying Proposition 4.1 to the above to semigroup on $\mathcal{M}_1 \ast_{\mathcal{N}} \mathcal{M}_2 \ast_{\mathcal{N}} \mathcal{R}$.

Let $T_{1,t} = e^{-A_1t} : \mathcal{M}_1 \to \mathcal{M}_1$ and $T_{2,t} = e^{-A_2t} : \mathcal{M}_2 \to \mathcal{M}_2$ be two symmetric Markov semigroup with fixed-point algebra $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively. Let $\mathcal{N} \subset \mathcal{N}_1, \mathcal{N}_2$ be a
Proposition 4.7. Let \( T \) be a derivation triple of \( T_{1,t} \) (resp. \( T_{2,t} \)). Denote \( A_1 \ast_N A_2 \) as the algebraic free product. Define the closable derivation \( \delta : A_1 \ast_N A_2 \to L_2(\hat{M}_1 \ast_N \hat{M}_2) \)

\[
\delta(a) = \delta_1(a), \delta(b) = \delta_2(b), a \in A_1, b \in A_2 \\
\delta(a_1 \cdots a_n) = \sum_{k=1}^n a_1 \cdots \delta_{i_k}(a_k) \cdots a_n, a_k \in \hat{A}_{i_k}.
\]

Then \( (A_1 \ast_N A_2, \hat{M}_1 \ast_N \hat{M}_2, \delta) \) is a derivation triple for \( T_i = T_{1,t} \ast T_{2,t} \).

Proof. Let \( E_i : M_i \to N_i, i = 1, 2 \) be the conditional expectation to the fixed point subalgebra. Then \( T_{1,t} \ast T_{2,t} \) has fixed point subalgebra \( N_1 \ast_N N_2 \) and conditional expectation \( E = E_1 \ast E_2 \). It is clear from the definition that \( \delta \) is \( * \)-preserving and satisfy Leibniz rule.

For \( a_k \in \hat{A}_{i_k}, i_1 \neq i_2 \neq \cdots \neq i_n, \)

\[
E(\delta(a_1 \cdots a_n)) = \sum_{k=1}^n E_1 \ast E_2(a_1 \cdots \delta_{i_k}(a_k) \cdots a_n) = \sum_{k=1}^n E_{i_k}(a_1) \cdots E_{i_k}(\delta_{i_k}(a_k) \cdots E_{i_k}(a_n) = 0
\]

To verify the \( \Gamma_L(v, w) = E(\delta(v) \ast \delta(w)) \). It suffices to consider the free word \( v = b_1 \cdots b_m \) and \( w = a_1 \cdots a_n \), where \( a_k \in \hat{A}_{i_k} \) and \( b_l \in \hat{A}_{i_l} \). We argue it by induction. Denote \( \hat{a} = a - E_N(a) \) for the mean zero part. The initial step is \( v = b \) with \( b \in A_j \) and \( w = a_1 \cdots a_n \),

\[
2\Gamma_L(b, w) = b^*Lw + (Lb)^*w - L(b^*w) \\
= \sum_{k=1} b^*a \cdots La_k \cdots a_n + (Lb^*)a_1 \cdots a_n - L(b^*a_1 \cdots a_n)
\]

For the last term we have two cases: if \( j = i_1, \)

\[
L(b^*a_1 \cdots a_n) = L\left((b^*a_1)a_2 \cdots a_n + E_N(b^*a_1)a_2 \cdots a_n\right) \\
= L(b^*a_1)a_2 \cdots a_n + (b^*a_1) \sum_{k=2}^n a_2 \cdots La_k \cdots a_n + E_N(b^*a_1) \sum_{k=2}^n a_2 \cdots La_k \cdots a_n
\]

common subalgebra of \( N_1 \) and \( N_2 \). The generator of free product semigroup \( T_t = T_{1,t} \ast T_{2,t} \) on \( M_1 \ast_N M_2 \) is

\[
La = A_1a, Lb = A_2b, a \in \text{dom}(A_1), b \in \text{dom}(A_2) \\
L(a_1 \cdots a_n) = \sum_k a_1 \cdots La_k \cdots a_n
\]

where \( a_k \in \text{dom}(A_{i_k}) \cap \hat{M}_{i_k}, i_1 \neq i_2 \neq \cdots \neq i_n. \)
In particular, if $T$ (resp. $\lambda$)

**Proposition 4.8.** Let $L_n(a_1, a_2, \ldots, a_n) = L(b^*a_1) a_2 \cdots a_n + \sum_{k=2}^{n} a_2 \cdots La_k \cdots a_n$.

If $j \neq i_1$, 

$$L(b^*a_1 \cdots a_n) = (Lb^*)a_1 a_2 \cdots a_n + \sum_{k} a_1 \cdots La_k \cdots a_n$$

In total, $L(b^*a_1 \cdots a_n)$ equals

$$\begin{cases} 
L(b^*a_1) a_2 \cdots a_n + \sum_{k=2}^{n} a_2 \cdots La_k \cdots a_n, & \text{if } j = i_1 \\
(Lb^*) a_1 a_2 \cdots a_n + \sum_{k=2}^{n} a_2 \cdots La_k \cdots a_n, & \text{otherwise.}
\end{cases}$$

For both case, the last term cancels in $\Gamma_L(b, w)$. We have

$$\Gamma_L(b, a_1 a_2 \cdots a_n) = \begin{cases} 
\Gamma_j(b, a_1) a_2 \cdots a_n, & \text{if } j = i_1 \\
0 & \text{otherwise.}
\end{cases}$$

Let $E_{\mathcal{M}_i} : \mathcal{M}_i \to \mathcal{M}_i$, $i = 1, 2$ be the conditional expectation. Now we calculate that

$$E_{\mathcal{M}}(\delta(b)^*\delta(w)) = E_{\mathcal{M}}(\delta_j(b)^* \sum_{k=1}^{n} a_1 \cdots \delta(a_k) \cdots a_n)$$

where $E_{\mathcal{M}} = E_{\mathcal{M}_1} * E_{\mathcal{M}_2}$. Because $E_{\mathcal{M}_j}(\delta_j(a)) = 0$ if $a \in A_j$, we know the only nonzero case is $i_1 = j$ and

$$E_{\mathcal{M}}(\delta(b)^*\delta(w)) = E_{\mathcal{M}_1}(\delta_j(b)^* \delta_j(a_1) \cdots a_n)$$

$$= E_{\mathcal{M}_1}(\delta_j(b)^* \delta_j(a_1)) E_{\mathcal{M}_2}(a_2) \cdots E_{\mathcal{M}_{i_n}}(a_n)$$

$$= E_{\mathcal{M}_1}(\delta_j(b)^* \delta_j(a_1)) a_2 \cdots a_n$$

$$= \Gamma_j(b, a_1) a_2 \cdots a_n.$$

which coincides with $\Gamma_L(b, a_1 a_2 \cdots a_n)$. Then the induction step can be done using the product rule

$$\Gamma(xy, z) = y^*\Gamma(x, z) + \Gamma(y, x^*z) - \Gamma(y, x^*)z.$$ 

That completes the proof.

The above discussion naturally extends to free product of $n$ algebras.

**Proposition 4.8.** Let $T_1 : \mathcal{M}_1 \to \mathcal{M}_1$ and $S_1 : \mathcal{M}_2 \to \mathcal{M}_2$ be two symmetric Markov semigroup with same fixed-point subalgebra respectively $\mathcal{N}_1$ and $\mathcal{N}_2$. Then

i) if $T_{1,t}$ and $T_{2,t}$ satisfies $GRic \geq \lambda$, $T_{1,t} * T_{2,t}$ on $\mathcal{M}_1 *_{\mathcal{N}} \mathcal{M}_2$ satisfies $GRic \geq \lambda$.

ii) if $T_{1,t}$ and $T_{2,t}$ satisfies $\lambda$-$GRic$, $T_{1,t} * T_{2,t}$ satisfies $\lambda$-$GRic$.

In particular, if $T_t$ satisfies $GRic \geq \lambda$ (resp. $\lambda$-$GRic$), then $T_t * id$ satisfies $GRic \geq \lambda$ (resp. $\lambda$-$GRic$).
Proof. Let \((A_1, \hat{M}_1, \delta_1)\) (resp. \((A_2, \hat{M}_2, \delta_2)\)) be a derivation triple of \(T_{1,t}\) (resp. \(T_{2,t}\)) and \(\hat{T}_{1,t} : \hat{M}_1 \to \hat{M}_1\) (resp. \(\hat{T}_{2,t} : \hat{M}_2 \to \hat{M}_2\)) be the semigroup gives the GRic \(\geq \lambda\) relation of \(T_{1,t}\) (resp. \(T_{2,t}\)). Namely, \(\hat{T}_{1}|_{\hat{M}_1} = T_t, \hat{T}_{2}|_{\hat{M}_2} = S_t\) and

\[
\hat{A}\delta_1(x) - \delta_1A(x) = \text{Ric}_1(\delta_1(x)) , \hat{B}\delta_2(x) - \delta_2B(x) = \text{Ric}_2(\delta_2(x)) .
\]

with bimodule Ricci operator \(\text{Ric}_A \geq \lambda\) and \(\text{Ric}_B \geq \lambda\). Here \(A, B\) (resp. \(\hat{A}, \hat{B}\)) are the generator of \(T_t, S_t\) (resp. \(\hat{T}_t, \hat{S}_t\)). Consider the free product semigroup \(\hat{T}_t = \hat{T}_{1,t} \ast \hat{T}_{2,t} : \hat{M}_1 \ast \hat{M}_2 \to \hat{M}_1 \ast \hat{M}_2\). It follows that \(\hat{T}_{1,t} \ast \hat{T}_{2,t}|_{\hat{M}_1 \ast \hat{M}_2} = T_{1,t} \ast T_{2,t}\). The generator of \(\hat{T}_{1,t} \ast \hat{T}_{2,t}\) is

\[
\hat{L}a = \hat{A}a , \hat{L}b = \hat{B}b , a \in \hat{M}_1 , b \in \hat{M}_2
\]

where \(a_k \in \text{dom}(\hat{A}_{i_k}) \cap \hat{M}_{i_k}, i_1 \neq i_2 \neq \ldots \neq i_n\) and similarly for the generator \(L\) of \(T_{1,t} \ast T_{2,t}\). Since \(\hat{L}|_{\hat{M}_1 \ast \hat{M}_2} = L\), for \(a_k \in \hat{A}_{i_k}\),

\[
\begin{align*}
\hat{L}\delta(a_1 \ldots a_n) - \delta L(a_1 \ldots a_n) &= \sum_{1 \leq k \neq l \leq n} \left( a_1 \ldots \hat{A}_{i_k} a_k \delta_{i_k}(a_l) \cdots a_n - a_1 \ldots \hat{A}_{i_k} a_k \delta_{i_k}(a_l) \cdots a_n \right) \\
&+ \sum_{1 \leq k \leq n} \left( a_1 \ldots \hat{A}_{i_k} \delta_{i_k}(a_l) \cdots a_n - a_1 \ldots \delta_{i_k}(A_{i_k} a_k) \cdots a_n \right) \\
&= \sum_{1 \leq k \leq n} a_1 \ldots \text{Ric}_{i_k}(\delta_{i_k}(a_k)) \cdots a_n .
\end{align*}
\]

Note that \(\Omega_\delta = (A_1 \ast \hat{A}_2)\delta(A_1 \ast \hat{A}_2) \subset L_2(\mathcal{M}_1 \ast \hat{M}_2)\) and by Leibniz rule

\[
(A_1 \ast \hat{A}_2)\delta(A_1 \ast \hat{A}_2) = \bigoplus \bigoplus \hat{A}_{i_1} \hat{A}_{i_2} \cdots \hat{A}_{i_k} \delta(A_{i_k}) \cdots \hat{A}_{i_n} .
\]

Moreover, the above decomposition are mutually orthogonal. Now we define Ric : \(\Omega_\delta \to L_2(\hat{M}_1 \ast \hat{M}_2)\) as

\[
\text{Ric}(a_1 \ldots a_k \delta_{i_k}(b_k) \cdots a_n) = a_1 \ldots \text{Ric}_{i_k}(a_k \delta_{i_k}(a_k)) \cdots a_n .
\]

which is clearly a \(A_1 \ast \hat{A}_2\)-bimodule operator. Now let us focus on a vector \(h = \sum_{j=1}^m \eta_j \xi_j \in \hat{A}_{i_1} \hat{A}_{i_2} \cdots \hat{A}_{i_k} \delta(A_{i_k}) \cdots \hat{A}_{i_n}\) with

\[
\eta_k \in \hat{A}_{i_1} \hat{A}_{i_2} \cdots \hat{A}_{i_{k-1}} \hat{A}_{i_k} \delta(A_{i_k}) , \xi_k \in \hat{A}_{i_k} \delta(A_{i_k}) , \gamma_k \in \hat{A}_{i_{k+1}} \cdots \hat{A}_{i_n} .
\]
Then
\[ \| h \|_2^2 = \sum_{j,l=1}^m \tau \left( \gamma_j^* \xi_j^* \eta_l^* \eta_l \gamma_l^* \right) = \sum_{j,l} \tau \left( E_N(\gamma_l^* \gamma_j^*) \xi_j^* E_N(\eta_j^* \eta_l) \xi_l \right) \]

Denote
\[ X = \sum_{j,l=1}^m e_{j,l} \otimes E_N(\eta_l^* \eta_l) \in M_m(N), \quad Y = \sum_{j,l=1}^m e_{l,j} \otimes E_N(\gamma_j^* \gamma_j) \in M_m(N), \]
\[ Z = \sum_{j=1}^m e_{j,j} \otimes \xi_j \in M_n(\hat{M}_i). \]

We have
\[ \| h \|_2^2 = \text{tr} \otimes \tau(YZ^*XZ) = \| X^{1/2}ZY^{1/2} \|_2 \]

Since Ric is \( A_1 \ast_N A_2 \)-bimodule operator and \( N \subset A_1 \ast_N A_2 \), we have
\[ \langle h, \text{Ric}(h) \rangle = \sum_{j,l} \tau \left( E_N(\gamma_l^* \gamma_j^*) \xi_j^* E_N(\eta_j^* \eta_l) \text{Ric}(\xi_l) \right) \]
\[ = \text{tr} \otimes \tau(YZ^*X \text{id} \otimes \text{Ric}(Z)) \]
\[ = \langle X^{1/2}ZY^{1/2}, \text{id} \otimes \text{Ric}(X^{1/2}ZY^{1/2}) \rangle \geq \lambda \| X^{1/2}ZY^{1/2} \|_2 = \| h \|_2^2 \]

It then follows from orthogonality that Ric \( \geq \lambda \) as an operator on \( \Omega \subset L_2(\hat{M}_1 \ast_N \hat{M}_2) \).

That completes the proof. \( \blacksquare \)

5. Optimal curvature for word length semigroups

In this section, we discuss the optimal GRic conditions for several word length semigroups. This implies the corresponding CGE and CLSI results that are also independently obtained in \[48\].

5.1. The \( q \)-Gaussian algebras. The \( q \)-deformed Gaussian variable was introduced by Frisch and Bourret in \[27\]. We refer to \[7, 6\] for the more information about \( q \)-Gaussian operator algebra. Let \( H \) be a separable real Hilbert space and \( H_C \) be its complexification. Let \( F(H) = \Omega \oplus (\oplus_{n \geq 1} H_C^{\otimes n}) \) be the algebraic Fock space over \( H_C \) where \( \Omega \) is the distinguished unit vector for the vacuum state. Let \( -1 \leq q \leq 1 \). We equipped \( F(H) \) with \( q \)-deformed sesquilinear form,
\[ \langle h_1 \otimes \cdots \otimes h_n, k_1 \otimes \cdots \otimes k_m \rangle_q = \delta_{n,m} \sum_{\sigma \in S_n} q^{i(\sigma)} \Pi_{j=1}^n \langle h_{\sigma(j)}, k_{\sigma(j)} \rangle, \quad h_j, k_j \in H_C \quad (14) \]
Here $S_n$ denotes the permutation group on $n$ characters and $\iota(\sigma)$ denotes the inversion number of $\sigma \in S_n$. This form is nonnegative definite and strictly positive definite for $-1 < q < 1$. Denote $\mathcal{F}_q(H)$ be the Hilbert space completion of $\mathcal{F}(H)$ with respect to $\langle \cdot, \cdot \rangle_q$. Define the left creation operator that for $h \in H_C$,

$$l_q(h)h_1 \otimes \cdots \otimes h_n = h_0 \otimes h_1 \otimes \cdots \otimes h_n .$$  \hspace{1cm} (15)

Its adjoint is the left annihilation operator

$$l_q^*(h)h_1 \otimes \cdots \otimes h_n = \sum_{j=1}^{n} q^{j-1} \langle h, h_j \rangle h_1 \otimes \cdots \otimes \hat{h}_j \otimes \cdots h_n ,$$

where $\hat{h}_j$ means the $j$ component is missing. For $-1 \leq q < 1$, $l_q(h) \in B(\mathcal{F}_q(H))$ and satisfy the $q$-commutation relation.

$$l_q(h_1)l_q(h_2)^* - ql_q(h_1)^*l_q(h_2) = \langle h_1, h_2 \rangle \cdot 1 .$$

Let $s_q(h) = l_q(h) + l_q(h)^*$. The $q$-Gaussian von Neumann algebra is defined as

$$\Gamma_q(H) := \{ s_q(h) | h \in H \}'' \subset B(\mathcal{F}_q(H))$$

For ease of notation, we will suppress the “$q$” in the generator $s_q(h)$. The canonical trace is given by the vacuum state, $\tau(x) = \langle \Omega, x\Omega \rangle_q$ where $\Omega$ is the vacuum vector in $F(H)$. The distribution of $q$-Gaussian variables is given by the following formula [7] that for $h_1, \cdots, h_n$,

$$\tau(s(h_1) \cdots s(h_n)) = \begin{cases} 
\sum_{\sigma \in P_2(n)} q^{c(\sigma)} \prod_{\{i,j\} \in \sigma} \langle h_i, h_j \rangle, & \text{if } n \text{ even} \\
0, & \text{if } n \text{ odd}
\end{cases}$$

where $P_2(n)$ denotes the pair partition of the set $\{1, \cdots, n\}$ and $c(\sigma)$ is the crossing number of the partition $\sigma$. For each $\xi \in \mathcal{F}(H)$, the Wick word of $\xi$ is the unique element $W(\xi) \in \Gamma_q(H)$ such that $W(\xi)\Omega = \xi$. Given a contraction $T : H \rightarrow K$, it induces a quantization contraction

$$\mathcal{F}(T) : \mathcal{F}_q(H) \rightarrow \mathcal{F}_q(H) , \ F(T)(h_1 \otimes \cdots \otimes h_n) = Th_1 \otimes \cdots \otimes Th_n .$$

and a normal completely positive unital map

$$\Gamma(T) : \Gamma_q(T) \rightarrow \Gamma_q(T) , \ \Gamma(T)W(h_1 \otimes \cdots \otimes h_n) = W(Th_1 \otimes \cdots \otimes Th_n) .$$

Moreover,

$$\mathcal{F}(T^*) = \mathcal{F}(T)^* , \mathcal{F}(ST) = \mathcal{F}(S)\mathcal{F}(T) , \Gamma(S^*) = \Gamma(S)^\dagger \Gamma(ST) = \Gamma(S)\Gamma(T)$$

where $\Gamma(S)^\dagger$ is the adjoint map of $\Gamma(S)$ with respect to trace inner product. If $T$ is an isometry, $\Gamma(T)$ is an injective *-homomorphism. If $T$ is self-adjoint, $\Gamma(T)$ is symmetric.
Let $t \geq 0$ and $V_t : H \to H$ be the contraction

$$V_t(h) = e^{-t}h, \ h \in H.$$ 

This induces the word length semigroup $T_t : \Gamma_q(H) \to \Gamma_q(H)$ that

$$T_t(W(\xi)) = e^{-mt}W(\xi), \ \xi \in H^{\otimes m}.$$ 

$T_t$ is an ergodic symmetric quantum Markov semigroup. Denote $E_m$ as the space of Wick words of length $m$. Note $E_m$ are mutually orthogonal subspace of $L_2(\Gamma_q(H \otimes K))$. The generator of $T_t$ is the number operator

$$NW(\xi) = mW(\xi), \ \xi \in E_m.$$ 

The Dirichlet algebra is then $\mathcal{A}_\xi = \{W(\xi) | \| N^{1/2}W(\xi) \|_2 < \infty \}$. Let $\mathcal{A}_q(H) = \{W(\xi) \in \mathcal{F}(H)\}$ be the $*$-algebra of Wick words of finite length. $\mathcal{A}_q(H)$ is clearly a $w^*$-dense subalgebra of $\Gamma_q(H)$ and a norm-dense subalgebra of $\mathcal{A}_\xi$ with respect to the graph norm $\| x \|_\xi = \| x \|_2 + \| N^{1/2}x \|_2$. The gradient form of $T_t$ is that for $\xi \in E_m, \eta \in E_n$

$$2\Gamma(W(\xi), W(\eta)) = W(\xi)(NW(\eta)) + (NW(\xi))^*W(\eta) - N(W(\xi)^*W(\eta))$$

$$= \sum_{|n-m| \leq l \leq n+m} (n + m - l)P_l(W(\xi)^*W(\eta)).$$

where $P_l : L_2(\Gamma_q(H)) \to E_l$ is the projection onto Wick words of length $l$.

**Lemma 5.1.** Define the map

$$\delta : \mathcal{A}_q(H) \to \mathcal{A}_q(H \oplus H), \ \delta(W(h_1 \otimes \cdots \otimes h_m)) = \sum_{j=1}^m W(h_1 \otimes \cdots \hat{h}_j \otimes \cdots \otimes h_m).$$

where $h_j \in H$ and $\hat{h}_j \in 0 \oplus H$ is the vector corresponding to $h_j$. Then

i) $\delta$ is a $*$-preserving closable derivation such that $E(\delta(x)) = 0$ for any $x \in \mathcal{A}_q(H)$. Here $E : \Gamma(H \oplus H) \to \Gamma(H)$ is the conditional expectation induced by the projection $H \oplus H \to H \oplus 0$.

ii) $(\mathcal{A}_q(H), \Gamma_q(H \oplus H), \delta)$ is a derivation triple for word length semigroup $T_t$.

**Proof.** We will repeatedly use the relation

$$s(h_0)W(h_1 \otimes \cdots \otimes h_m) = W(h_0 \otimes h_1 \otimes \cdots \otimes h_m) + \sum_{j=1}^m (h_0, h_j)q^{j-1}W(h_1 \otimes \cdots \hat{h}_j \otimes \cdots \otimes h_m).$$

where $\hat{h}_j$ means the $j$-th component is missing. Since the Wick words are polynomials of the generators $s(h) = W(h)$, it suffices to verify the Leibniz rule that for $h_0, \cdots, h_m \in H$

$$\delta(W(h_0)W(h_1 \otimes \cdots \otimes h_m)) = \delta(W(h_0))W(h_1 \otimes \cdots \otimes h_m) + W(h_0)\delta(W(h_1 \otimes \cdots \otimes h_m))$$.
We prove this by induction. First, note that $W(h) = s(h)$ and
\[ W(h_0)W(h_1) = s(h_0)s(h_1) = W(h_0 \otimes h_1) - \langle h_0, h_1 \rangle, \]
\[ W(\hat{h}_0)W(h_1) = W(\hat{h}_0 \otimes h_1), \quad W(h_0)W(\hat{h}_1) = W(h_0 \otimes \hat{h}_1). \]
Then the Leibniz rule is satisfied for $m = 1$,
\[ \delta(W(h_0)W(h_1)) = \delta(W(h_0 \otimes h_1) + \langle h_0, h_1 \rangle) = W(\hat{h}_0 \otimes h_1) + W(h_0 \otimes \hat{h}_1) \]
\[ = W(\hat{h}_0)W(h_1) + W(h_0)W(\hat{h}_1) = \delta(W(h_0))W(h_1) + W(h_0)\delta(W(h_1)) \]
Assume that the Leibniz rule is satisfied for all $m \leq n$. By induction,
\[ \delta(s(h_0)W(h_1 \otimes \cdots \otimes h_n)) \]
\[ = \delta(W(h_0 \otimes h_1 \otimes \cdots \otimes h_n) + \sum_{1 \leq j \leq n} \langle h_0, h_j \rangle q^{-1}W(h_1 \otimes \cdots \hat{h}_j \otimes \cdots \otimes h_n)) \]
\[ = \sum_{0 \leq k \leq n} W(h_0 \otimes h_1 \otimes \cdots \hat{h}_k \otimes \cdots \otimes h_n) \]
\[ + \sum_{1 \leq j \leq n} \langle h_0, h_j \rangle q^{-1} \sum_{1 \leq k \leq n, k \neq j} W(h_1 \otimes \cdots \hat{h}_j \cdots \hat{h}_k \otimes \cdots \otimes h_n) \]
\[ = s(\hat{h}_0)W(h_1 \otimes \cdots \otimes h_n) + s(h_0) \sum_{1 \leq k \leq n} W(h_1 \otimes \cdots \hat{h}_k \otimes \cdots \otimes h_n) \]
\[ = \delta(s(h_0))W(h_1 \otimes \cdots \otimes h_n) + s(h_0)\delta(W(h_1 \otimes \cdots \otimes h_n)). \]
This verifies $\delta$ is a derivation. To verify the form $\Gamma(x, y) = E(\delta(x)^*\delta(y))$, we first consider $x = W(h_0)$ and $y = W(\xi) = W(h_1 \otimes \cdots \otimes h_m)$. We denote
\[ \hat{\xi}_j = h_1 \otimes \cdots \otimes \hat{h}_j \otimes \cdots \otimes h_m, \quad \hat{\xi}_j = h_1 \otimes \cdots \hat{h}_j \otimes \cdots \otimes h_m, \]
where $\hat{h}$ is the copy of $h$ in $0 \oplus H$. Thus,
\[ E(\delta(s(h_0))^*\delta(W(\xi))) = \sum_{j=1}^n E(s(\hat{h}_0)W(\hat{\xi}_j)) = \sum_{j=1}^n \langle h_0, h_j \rangle q^{-1}W(\hat{\xi}_j) \]
On the other hand,
\[ \Gamma(s(h_0), W(\xi)) = \frac{1}{2} \sum_{j=n-1, n+1} (n + 1 - j) P_j(s(h_0)W(\xi_j)) \]
\[ = \frac{1}{2} \sum_{j=n-1, n+1} (n + 1 - j) P_j(s(h_0)W(\xi_j)) \]
\[ = \sum_{j=1}^n \langle h_0, h_j \rangle q^{-1}W(\hat{\xi}_j) = E(\delta(s(h_0))^*\delta(W(\xi))). \]
which verifies the case $x = s(h_0), y = W(\xi)$. Note that both $\Gamma(x, y)$ and $E(\delta(x)^*\delta(y))$ satisfies product the rule

$$\Gamma(xy, z) = y^*\Gamma(x, z) + \Gamma(y, x^*z) - \Gamma(y, x^*)z.$$ 

Then by induction, the desired equality holds for products of $s(h_0)$ which spans all Wick words of finite length. Finally, the mean zero property follows from $E(\delta(W(\xi))) = E(W(P\xi)) = 0$. That completes the proof. ■

**Theorem 5.2.** Let $H$ be a real separable Hilbert space. Let $T_t$ be word length semigroup on $q$-Gaussian algebra $\Gamma_q(H)$. Then $T_t$ satisfies optimal 1-GRic. As a consequence, $T_t$ satisfies optimal 1-CGE and 1-CLSI.

**Proof.** Consider the Hilbert space contraction

$$\hat{O}_t : H \oplus H \to H \oplus H, \hat{O}_t(h_1 \oplus h_2) = e^{-t}h_1 \oplus h_2,$$

Let $\hat{T}_t : \Gamma_q(H \oplus H) \to \Gamma_q(H \oplus H)$ be the quantization $\hat{O}_t$ as a symmetric quantum Markov semigroup. For $h_1 \otimes \cdots \otimes h_m \in E_m$, we have

$$\delta \circ T_t(W(h_1 \otimes \cdots \otimes h_m)) = e^{-mt}\delta(W(h_1 \otimes \cdots \otimes h_m))$$

$$= e^{-mt} \sum_{k=1}^m W(h_1 \otimes \cdots \hat{h}_k \cdots \otimes h_m)$$

$$= e^{-t}\hat{T}_t(\sum_{k=1}^m W(h_1 \otimes \cdots \hat{h}_k \cdots \otimes h_m))$$

$$= e^{-t}\hat{T}_t(\delta(W(h_1 \otimes \cdots \otimes h_m))).$$

This verifies $T_t \circ \delta = e^{-t}\hat{T}_t \circ \delta$ on $A$. The assertion follows from Proposition 2.4. The constant is optimal because the spectral gap of the number operator $N$ is also 1. ■

5.2. **CAR algebras revisited.** The CAR algebras (also called Clifford algebra) can be viewed as special case of $q$-Gaussian algebras for $q = -1$. The word length semigroup are the special case of Orstein-Unlenbeck semigroup at temperature zero, which are studied in [13, 11] by Carlen and Maas for entropy Ricci curvature lower bound. The GRic result in this section is essentially due to discussion in [13, Section 8]. Because their result will be used in the next example, we briefly revisit their results in our setting.

Let $H$ be a real separable Hilbert space and $\{e_j\}$ be a ONB of $H$. The Clifford algebra $cl(H)$ is the unital $*$-algebra generated by a family of generators $\{c_j \mid 1 \leq j \leq \dim H\}$ satisfying the anti-commutation (CAR) relation

$$c_ic_j + c_jc_i = 2\delta_{ij}1, c_i = c_i^*.$$
Denote $[H] = \{1, 2, \cdots, \dim(H)\}$ if $\dim(H) < \infty$ and $[H] = \mathbb{N}_+$ for $\dim(H) = \infty$. For any finite subset $A \subset [H]$, we define the monomial

$$c_A = c_{j_1}c_{j_2}\cdots c_{j_m}, A = \{j_1, \cdots, j_m\}$$

The canonical trace on $\text{cl}(H)$ is given by

$$\tau(c_A) = \begin{cases} 1, & \text{if } A = \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Define $Cl(H)$ as the von Neumann algebra generated by the GNS representation. It is well-known that $Cl(\mathbb{R}^d)$ is $2^n$-dimensional $C^*$-algebra and for $\dim(H) = \infty$, $Cl(H) = \mathcal{R}$ is the hyperfinite II$_1$ factor. In particular $\{c_A | A \subset [H]\}$ forms a orthonormal basis for $L_2(Cl(H), \tau)$. Define the number operator

$$Nc_A = |A|c_A.$$
5.3. Free groups. In this part we prove the optimal Ricci curvature and CLSI for word length semigroup on free group factor. Our idea is to use Theorem 4.8 that GRic ≥ λ condition is stable under free product. We denote \( \mathbb{Z}_2 \) as the order 2 group, \( \mathbb{Z} \) as the integer group, and \( \mathbb{F}_d \) as the free group of \( d \) generator. We write \( G_N = \mathbb{Z}_2 \ast \mathbb{Z}_2 \cdots \ast \mathbb{Z}_2 \) as the free product of \( N \)-copies of \( \mathbb{Z}_2 \). We start with a corollary of Example (5.4) and Theorem 4.8.

**Proposition 5.5.** Let
\[
P_{G_N, t} : \mathcal{L}(G_N) \rightarrow \mathcal{L}(G_N), P_{G_N, t}(\lambda(g)) = e^{-|g|t} \lambda(g)
\]
be the word length semigroup on \( \mathcal{L}(G_N) \), where \( g \) is a word of \( a_j = a_j^{-1}, 1 \leq j \leq d \) and \( |g| \) is the word length of \( g \). Then \( P_{G_N, t} \) satisfies optimal GRic ≥ 1 and 1-CLSI.

**Proof.** Note that the \( \mathcal{L}(G_d) = \mathcal{L}(\mathbb{Z}_2 \ast \mathbb{Z}_2 \cdots \ast \mathbb{Z}_2) = \mathcal{L}(\mathbb{Z}_2) \ast \cdots \ast \mathcal{L}(\mathbb{Z}_2) \) and the semigroup
\[
P_{G_N, t} = P_{Z_2, t} \ast P_{Z_2, t} \ast \cdots \ast P_{Z_2, t},
\]
where \( P_{Z_2, t} \) is the word length semigroup on the two-points space \( \mathcal{L}(\mathbb{Z}_2) \cong \mathbb{C} \oplus \mathbb{C} \) discussed in Example (5.4). The assertion follows from Theorem 4.8.

**Corollary 5.6.** Let
\[
P_{Z, t} : \mathcal{L}(\mathbb{Z}) \rightarrow \mathcal{L}(\mathbb{Z}), P_{Z, t}(\lambda(u)) = e^{-|m|t} u^m
\]
be the word length semigroup on \( \mathcal{L}(\mathbb{Z}) \), where \( u = \lambda(1) \) is the unitary generator of \( \mathcal{L}(\mathbb{Z}) \). Then \( P_{Z, t} \) satisfies optimal GRic ≥ 1 and 1-CLSI.

**Proof.** Recall the [31, Lemma 3.3] that
\[
\pi \circ P_{Z, t} = (id_{M_2} \otimes P_{Z_2 \ast Z_2, t}) \otimes \pi
\]
where \( \pi \) is the \(*\)-homomorphism
\[
\pi : L(\mathbb{Z}) \rightarrow M_2 \otimes L(\mathbb{Z}_2 \ast \mathbb{Z}_2), \pi(u) = \begin{bmatrix} 0 & \lambda(a_1) \\ \lambda(a_2) & 0 \end{bmatrix}
\]
Here \( u \) is the generator of \( \mathbb{Z} \) and \( a_1 \) (resp. \( a_2 \)) is the nontrivial element in the first (resp. second) copy of \( \mathbb{Z}_2 \) in \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \). This implies \( P_{Z, t} \) is a transference semigroup of \( P_{Z_2 \ast Z_2, t} \). Hence \( P_{Z, t} \) satisfies GRic ≥ 1 and 1-CLSI. This is optimal because the spectral gap of \( P_{Z, t} \) is also 1.

**Remark 5.7.** By \( \mathcal{L}(\mathbb{Z}) \cong L_\infty(\mathbb{T}) \), \( P_{Z, t} \) is the Possion semigroup on the torus \( \mathbb{T} \).

**Corollary 5.8.** Let
\[
P_{F_d, t} : \mathcal{L}(\mathbb{F}_d) \rightarrow \mathcal{L}(\mathbb{F}_d), P_{F_d, t}(\lambda(g)) = e^{-|g|t} \lambda(g)^m
\]
be the word length semigroup on \( \mathcal{L}(\mathbb{F}_d) \), where \( g \) is the free word of \( d \) generators and \( |g| \) is the word length of \( g \). Then \( P_{F_d, t} \) satisfies optimal GRic ≥ 1 and 1-CLSI.
Proof. Note that the $L(\mathbb{F}_d) = L(\mathbb{Z} \ast \mathbb{Z} \ast \cdots \ast \mathbb{Z}) = L(\mathbb{Z}) \ast \cdots \ast L(\mathbb{Z})$ and the semigroup

$$P_{\mathbb{F}_d,t} = P_{\mathbb{Z},t} \ast P_{\mathbb{Z},t} \ast \cdots \ast P_{\mathbb{Z},t}.$$ 

Then by Corollary 5.6 and Theorem 4.8, $P_{\mathbb{F}_d,t}$ satisfies GRic $\geq 1$ and 1-CLSI. The optimality follows from that the spectral gap is 1. □

Remark 5.9. The best known LSI constant proved in [31] is $(1 + \log 2)^{-1} < 1$, which implies $(1 + \log 2)^{-1} \text{-MLSI}$. Here we obtained the sharp 1-CLSI. The sharp 1-LSI remains open.

6. Other examples

6.1. Generalized depolarizing channel. Let $\mathcal{N} \subset \mathcal{M}$ be a subalgebra and $E : \mathcal{M} \to \mathcal{N}$ be the conditional expectation. Define the generalized depolarizing channel

$$T_t(\rho) = e^{-\lambda t} \rho + (1 - e^{-t})E(\rho)$$

It was proved in [9] and [48] that $T_t$ satisfies $1/2 \text{-CGE}$. Here we use free product to show that $T_t$ satisfies the stronger condition GRic $\geq 1/2$.

Proposition 6.1. The generalized depolarizing channel $T_t(\rho) = e^{-\lambda t} \rho + (1 - e^{-t})E(\rho)$ satisfies GRic $\geq 1/2$.

Proof. Let $\mathcal{M}_1 = \mathcal{M}$ and $\mathcal{M}_2 = L(\mathbb{Z}_2) \otimes \mathcal{N}$. Consider the semigroup $P_t = P_{\mathbb{Z}_2,t} \otimes \text{id}_\mathcal{N}$ on $L(\mathbb{Z}_2) \otimes \mathcal{N}$ where

$$P_{\mathbb{Z}_2,t}(a1 + b\epsilon) = a1 + e^{-t}b\epsilon$$

is the semigroup on $\mathbb{Z}_2$. Now consider the embedding

$$\pi : \mathcal{M} \to \mathcal{M}_1 \ast_\mathcal{N} \mathcal{M}_2 , \pi(x) = \epsilon x \epsilon .$$

$\pi$ is an injective trace-preserving *-homomorphism because $\epsilon$ is uniatry. Note that for $x = E(x) + \hat{\chi} \epsilon$, $\pi(x) = \epsilon x \epsilon = \epsilon E(x) \epsilon + \epsilon \hat{\chi} \epsilon = E(x)$ because $L(\mathbb{Z}_2)$ and $\mathcal{N}$ commute in $\mathcal{M}_1 \ast_\mathcal{N} \mathcal{M}_2$. Then we have the intertwining relation

$$P_t \ast \text{id}_\mathcal{M}(\pi(x)) = P_t \ast \text{id}_\mathcal{M}(E(x) + \epsilon \hat{\chi} \epsilon) = E(x) + e^{-2t} \epsilon \hat{\chi} \epsilon = \pi(T_{2t}(x)).$$

Note that $P_{\mathbb{Z}_2,t}$ and hence $P_t \ast \text{id}_\mathcal{M} = (P_{\mathbb{Z}_2,t} \otimes \text{id}_\mathcal{N}) \ast \text{id}_\mathcal{M}$ has GRic $\geq 1$. Then $T_{2t}$ as a subsystem of $P_t \ast \text{id}_\mathcal{M}$ has GRic $\geq 1$. Therefore $T_t$ has GRic $\geq 1/2$. □

Proposition 6.2. The above constant $1/2$ is sharp for general conditional expectation $E$.

Proof. Note that GRic $\geq \lambda$ implies $\Gamma_2 \geq \lambda \Gamma$. We show that $\Gamma_2 \geq \lambda \Gamma$ is optimal for a general conditional expectation. Let $G$ be a discrete group and $L(G)$ be its group von Neumann algebra. Consider the Fourier multiplier discussed in Section 3.1,

$$T_t(\lambda(g)) = e^{-\psi(g)t} \lambda(g) , A(\lambda(g)) = \psi(g)\lambda(g) .$$
The gradient form and $\Gamma_2$ operator are
\[
\Gamma(\lambda(g), \lambda(h)) = K(g, h)\lambda(g)\lambda(h)^*, \Gamma(\lambda(g), \lambda(h)) = K(g, h)^2\lambda(g)^*\lambda(h),
\]
where $2K(g, h) = \psi(g) + \psi(h) - \psi(g^{-1}h)$. Thus $\text{GRic} \geq \lambda$ implies $\Gamma_2 \geq \lambda\Gamma$, which is
\[
[K(g, h)^2]_{g, h \in G} \geq \lambda[K(g, h)]_{g, h \in G}
\]
as matrices on $l_2(G)$. Indeed, take $\mathcal{M} = \mathcal{L}(\mathbb{Z})$ and $\mathcal{N} = \mathbb{C}1$. The depolarizing semigroup has generator
\[
A(\lambda(g)) = 1_{g \neq 0}\lambda(g), g \in \mathbb{Z}.
\]
which corresponds to the Fourier multiplier of indicator function $\psi(g) = 1_{g \neq 0}$. Then
\[
K(g, h) = \frac{1}{2}1_{gh \neq 0} - \frac{1}{2}1_{g = h \neq 0} + \frac{1}{2}\int_{gh \neq 0}1_{gh \neq 0} + \frac{1}{2}\int_{g = h \neq 0}1\int_{g \neq 0}1\int_{h \neq 0}1.
\]
and
\[
K(g, h)^2 = \frac{1}{4}1_{gh \neq 0} + \frac{1}{4}\int_{g = h \neq 0}1\int_{g \neq 0}1\int_{h \neq 0}1
\]
Therefore $1/2K^2 \geq K$ is the best possible constant by checking vector $v = \sum_{j=1}^N |j\rangle \in l_2(\mathbb{Z})$ for arbitrary large $N$.

**Corollary 6.3.** Let $\mathcal{M}_1, \cdots, \mathcal{M}_n$ be $n$ finite von Neumann algebras and $\mathcal{N}$ is a common subalgebra of $\mathcal{M}_j, 1 \leq j \leq n$. Then the word length semigroup $P_t: \mathcal{M}_1 \ast \cdots \ast \mathcal{M}_n \to \mathcal{M}_1 \ast \mathcal{N} \ast \cdots \ast \mathcal{N} \ast \mathcal{M}_n$
\[
P_t(a_1 \cdots a_m) = e^{-mt}a_1 \cdots a_m, a_i \in \mathcal{M}_j, i_1 \neq i_2 \neq \cdots \neq i_m
\]
satisfies $\text{GRic} \geq 1/2$.

**Proof.** Use Proposition 6.1 and 4.8.

6.2. **Quantum Tori.** Quantum tori are prototype examples in noncommutative geometry. Let $d \geq 2$ and $(\theta_{jk})_{j, k=1}^d$ be a $d \times d$ skew-symmetric real matrix. The $n$-dimensional quantum torus $\mathcal{A}_\theta$ is the universal $C^*$-algebra generated by $d$-tuple of unitaries $(u_1, u_2, \cdots, u_d)$ satisfying the commutation relation
\[
u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j, \ j, k = 1, 2, \cdots, d,
\]
We denote $u = (u_1, u_2, \cdots, u_d), m = (m_1, m_2, \cdots, m_d)$ and use the standard notation of multiple Fourier series as follows,
\[
u^m = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}.
\]
The canonical tracial state is
\[
\tau(\sum_{m \in \mathbb{Z}^d} \alpha_m u^m) = \alpha_0.
\]
The monomials \( \{u^m | m \in \mathbb{Z}^d\} \) forms a ONB of \( L_2(\mathcal{A}_\theta, \tau) \). Denote \( \mathcal{R}_\theta \) be the von Neumann algebra as the \( w^* \)-closure of the GNS representation \( \mathcal{A}_\theta \subset B(L_2(\mathcal{A}_\theta, \tau)) \). Let \( \mathbb{T}^d = \{(z_1, z_2, \cdots, z_d) \in \mathbb{C}^d \mid |z_j| = 1, \forall j\} \) be the \( d \)-torus. When \( \theta = 0 \), \( \mathcal{A}_\theta \cong C(\mathbb{T}^d) \) and \( \mathcal{R}_\theta \cong L_\infty(\mathbb{T}^d, dm) \). The heat semigroup on \( \mathcal{R}_\theta \) is defined as
\[
T_t : \mathcal{R}_\theta \rightarrow \mathcal{R}_\theta , \quad T_t(u^m) = e^{-|m|^2t}u^m ,
\]
where the generator is the Laplacian
\[
\Delta(u^m) = |m|^2u^m , \quad |m| = \sqrt{m_1^2 + \cdots + m_n^2} .
\]
It is known that \( T_t \) is the transference of heat semigroup on \( \mathbb{T}^d \). This property has been used in [15] for harmonic analysis on \( \mathcal{R}_\theta \). For each \( z = (z_1, z_2, \cdots, z_d) \in \mathbb{T}^d \), the associated transference action is given by
\[
\alpha_z(u_m) = z^m u^m = z_{m_1}z_{m_2}\cdots z_{m_d}u_{m_1}u_{m_2}\cdots u_{m_d} .
\]
Define the trace preserving \(*\)-monomorphism
\[
\alpha : \mathcal{R}_\theta \rightarrow L_\infty(\mathbb{T}^d, \mathbb{R}_\theta) , \quad \alpha(x)(z) = \alpha_z(x) .
\]
Let \( S_t : L_\infty(\mathbb{T}^d) \rightarrow L_\infty(\mathbb{T}^d) \), \( S_t(z^m) = e^{-|m|^2t}z^m \) be the heat semigroup on \( \mathbb{T}^d \). The following diagram commutes
\[
\begin{array}{ccc}
L_\infty(\mathbb{T}^d, \mathbb{R}_\theta) & \xrightarrow{S_t \otimes \text{id}_{\mathcal{R}_\theta}} & L_\infty(\mathbb{T}^d, \mathbb{R}_\theta) \\
\uparrow \alpha & & \uparrow \alpha \\
\mathcal{R}_\theta & \xrightarrow{T_t} & \mathcal{R}_\theta
\end{array}
\]
Namely, \((S_t \otimes \text{id}) \circ \alpha = \alpha \circ T_t\). This means the semigroup \( T_t \) is the restriction of \( S_t \otimes \text{id}_{\mathcal{R}_\theta} \) on \( \alpha(\mathcal{R}_\theta) \). Denote \( \mathcal{P}_\theta = \text{span}\{u^m | m \in \mathbb{Z}^d\} \) as the algebra of polynomials. Then \((\mathcal{P}_\theta, L_\infty(\mathbb{T}^d, \mathbb{R}_\theta), \delta = (\nabla_{\mathbb{T}^d} \otimes \text{id}) \circ \alpha)\) gives a derivation triple of \( T_t \), where \( \nabla_{\mathbb{T}^d} : C^\infty(\mathbb{T}^d) \rightarrow T\mathbb{T}^d \cong \oplus_{j=1}^d C^\infty(\mathbb{T}^d) \) is the gradient operator on \( \mathbb{T}^d \). More explicitly,
\[
\delta(u^m)(z) = (m_1\alpha_z(u^m), m_2\alpha_z(u^m), \cdots, m_d\alpha_z(u^m)) .
\]
Conversely, consider the trace preserving \(*\)-homomorphism
\[
\alpha_\theta : L_\infty(\mathbb{T}^d) \rightarrow \mathcal{R}_\theta \overline{\otimes} \mathcal{R}_\theta^{\text{op}}, \quad \alpha_\theta(z^m) = u^m \otimes (u^m)^{\text{op}}
\]
where \( \mathcal{R}_\theta^{\text{op}} \) is the opposite algebra of \( \mathcal{R}_\theta \). \( \alpha_\theta \) is a \(*\)-homomorphism because \( \alpha_\theta(z_j) = u_j \otimes u_j^{\text{op}}, 1 \leq j \leq d \) are \( d \)-commuting unitaries. We have another commuting diagram
\[
\begin{array}{ccc}
\mathcal{R}_\theta \overline{\otimes} \mathcal{R}_\theta^{\text{op}} & \xrightarrow{T_t \otimes \text{id}_{\mathcal{R}_\theta^{\text{op}}}} & \mathcal{R}_\theta \overline{\otimes} \mathcal{R}_\theta^{\text{op}} \\
\uparrow \alpha_\theta & & \uparrow \alpha_\theta \\
L_\infty(\mathbb{T}^d) & \xrightarrow{S_t} & L_\infty(\mathbb{T}^d)
\end{array}
\]
Namely, $\alpha \circ S_t = (T_t \otimes \text{id}_{\mathcal{R}^{op}}) \circ \alpha$. This means $S_t$ and $T_t$ are the transference semigroup of each other. By the completeness of the definition of GRic, CGE and CLSI, we know that $T_t$ on $\mathcal{R}$ and $S_t$ on $\mathbb{T}_d$ have same GRic, CGE and CLSI constant. Similar equivalence holds for other corresponding semigroup on $\mathcal{R}$ and $\mathbb{T}_d$.

**Corollary 6.4.** Let $\mathcal{R}$ be the $d$-dimensional quantum tori. Consider the quantum Markov semigroups

$$
T_t : \mathcal{R} \to \mathcal{R}, \quad T_t(u^m) = e^{-|m|^2 t} u^m,
$$

$$
P_t : \mathcal{R} \to \mathcal{R}, \quad P_t(u^m) = e^{-|m| t} u^m,
$$

$$
Q_t : \mathcal{R} \to \mathcal{R}, \quad Q_t(u^m) = e^{-\|m\|_1 t} u^m,
$$

where $\|m\|_1 = \sum_{j=1}^{d}|m_j|$. Then

i) $T_t$ has GRic $\geq 0$ and $(4 \ln 3)^{-1}$-CLSI.

ii) $P_t$ has GRic $\geq 0$ and $(4 \pi)^{-1} \left( \frac{2(d-1)!}{\Gamma(d/2)} \right)^{-\frac{1}{2}}$-CLSI.

iii) $Q_t$ has optimal GRic $\geq 1$ and 1-CLSI.

**Proof.** By the transference trick above, it suffices to consider the corresponding semigroup on torus $\mathbb{T}_d$. i) follows from [9, Theorem 4.12]. iii) follows from Lemma 5.6 and $Q_t$ is a tensor product semigroup of Possion semigroup on $\mathbb{T}$. ii) corresponds to the Possion semigroup on $\mathbb{T}_d$, which is also central. Then $P_t$ has GRic $\geq 0$. The CB-return time estimate is

$$
\| P_t - E_t : L_1(\mathbb{T}^d) \to L_\infty(\mathbb{T}^d) \| = \| \sum_{m \in \mathbb{Z}^d, m \neq 0} e^{-|m|t} z^m w^{-m} \|_{L_\infty(\mathbb{T}^d \times \mathbb{T}^d)} \\
\leq \sum_{m \in \mathbb{Z}^d, m \neq 0} e^{-|m|t} \\
\leq \int_{\mathbb{R}^d} e^{-|x|t} dx = s_d \int_0^\infty e^{-rt} r^{d-1} dr = \frac{s_d (d-1)!}{t^d}
$$

where $s_d = \frac{2\pi^d}{\Gamma(d/2)}$ is the surface are of $(d-1)$-dimensional unit sphere. The CLSI constant follows from Theorem 2.9.

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