QUINTIC SPLINE SOLUTIONS OF FOURTH ORDER
BOUNDARY-VALUE PROBLEMS

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Dedicated to the memory of Dr. M. Rafique

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Abstract
In this paper Quintic Spline is defined for the numerical solutions of the fourth order linear special case Boundary Value Problems. End conditions are also derived to complete the definition of spline. The algorithm developed approximates the solutions, and their higher order derivatives of differential equations. Numerical illustrations are tabulated to demonstrate the practical usefulness of method.

1 Introduction

Spline functions are used in many areas such as interpolation, data fitting, numerical solution of ordinary and partial differential equations. Spline functions are also used in curve and surface designing.

Riaz A. Usmani [1], considered the fourth order boundary value problem to be the problem of bending a rectangular clamped beam of length $l$ resting on an elastic foundation. The vertical deflection $w$ of the beam satisfies the system

$$L + \left( \frac{K}{D} \right) w = D^{-1} q(x) \quad \text{where} \quad L \equiv \frac{d^4}{dx^4}$$

$$w(0) = w(L) = w'(0) = w'(L) = 0$$

(1.1)

(1.2)

where $D$ is the flexural rigidity of the beam, and $k$ is the spring constant of the elastic foundation and the load $q(x)$ acts vertically downwards per unit length of the beam. The detail of the mechanical interpretation of (1.1) belongs to a general class of boundary value problems of the form
\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \frac{d^4}{dx^4} + f(x) \right)y(x) = g(x), \ x \in [a, b] \\
y(a) = \alpha_0, \ y(b) = \alpha_1 \\
y'(a) = \beta_0, \ y'(b) = \beta_1
\end{array} \right. 
\end{align*}
\] (1.3)

where \( \alpha_i, \beta_i ; i=0,1 \) are finite real constants and the functions \( f(x) \) and \( g(x) \) are continuous on \([a,b]\). The analytic solution of (1.3) for special choices of \( f(x) \) and \( g(x) \) are easily obtained, but for arbitrary choices, the analytic solution cannot be determined.

Numerical methods for obtaining an approximation to \( y(x) \) are introduced. Usmani [1] derived numerical techniques of order 2, 4 and 6 for solution of a fourth order linear boundary value problem. Usmani [2] derived cubic, quartic, quintic and sextic spline solution of nonlinear boundary value problems. Usmani and Manabu Sakai [3] developed a quartic spline for the approximation of the solution of certain two point boundary value problems involving third order linear differential equation.

N.Papamichael and A.J. Worsey [4] derived end conditions for cubic spline interpolation at equally spaced knots. N.Papamichael and A.J. Worsey [5] have developed a cubic spline method, similar to that proposed by Daniel and Swartz [6] for second order problems. In this paper a quintic spline method is described for the solution of (1.3). The end conditions for quintic spline interpolation, at equally spaced knots are derived which uniformly converge on \([a,b]\) to \( O(h^6) \), which is discussed in the next section.

\section{Quintic Spline}

Let \( Q \) be a quintic spline defined on \([a,b]\) with equally spaced knots

\[ x_i = a + ih; \ i = 0, 1, 2, ..., k \] (2.1)

where

\[ h = \frac{b - a}{k} \] (2.2)

Moreover for \( i=0,1,2,\ldots, k \), taking

\[ Q(x_i) = y_i ; \ Q^{(1)}(x_i) = m_i, \] (2.3)

\[ Q^{(2)}(x_i) = M_i ; \ Q^{(3)}(x_i) = n_i, \] (2.4)

and

\[ Q^{(4)}(x_i) = N_i \] (2.5)

Also, let \( y(x) \) be the exact solution of the system (1.3) and \( y_i \) be an approximation to \( y(x_i) \), obtained by the quintic spline \( Q(x_i) \). It may be noted that the \( Q_i(x) \), \( i = 1, 2, 3, \ldots, k \) can be defined on the interval \([x_{i-1}, x_i]\), integrating

\[ Q^{(4)}_i(x) = \frac{1}{h} [N_{i-1}(x_i - x) + N_i(x - x_{i-1})] \] (2.6)

four times w.r.t. \( x \), which gives

\[ Q_i(x) = \frac{1}{120h} [N_{i-1}(x_i - x)^5 + N_i(x - x_{i-1})^5] + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \] (2.7)
To calculate the constants of integrations, the following conditions are used.

\[ Q_i(x_i) = y_i ; \quad Q^2_i(x_i) = M_i \]
\[ Q_i(x_{i-1}) = y_{i-1} ; \quad Q^2_i(x_{i-1}) = M_{i-1} \]  \hspace{1cm} (2. 8)

The identities of quintic splines for the solution of (1.3) can be written as

\[ m_{i-2} + 26m_{i-1} + 66m_i + 26m_{i+1} + m_{i+2} = \frac{5}{h^2} \left[ -y_{i-2} - 10y_{i-1} + 10y_{i+1} + y_{i+2} \right] , \]
\[ i = 2, 3, \ldots, k - 2 \]  \hspace{1cm} (2. 9)

\[ M_{i-2} + 26M_{i-1} + 66M_i + 26M_{i+1} + M_{i+2} = \frac{20}{h^2} \left[ y_{i-2} - 2y_{i-1} - 6y_i + 2y_{i+1} + y_{i+2} \right] , \]
\[ i = 2, 3, \ldots, k - 2 \]  \hspace{1cm} (2. 10)

\[ n_{i-2} + 26n_{i-1} + 66n_i + 26n_{i+1} + n_{i+2} = \frac{60}{h^3} \left[ -y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2} \right] , \]
\[ i = 2, 3, \ldots, k - 2 \]  \hspace{1cm} (2. 11)

\[ N_{i-2} + 26N_{i-1} + 66N_i + 26N_{i+1} + N_{i+2} = \frac{120}{h^4} \left[ y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2} \right] , \]
\[ i = 2, 3, \ldots, k - 2 \]  \hspace{1cm} (2. 12)

\[ N_{i-1} + 4N_i + N_{i+1} - \frac{6}{h^2} \left[ M_{i-1} - 2M_i + M_{i+1} \right] = 0 , \]
\[ i = 1, 2, 3, \ldots, k - 1 \]  \hspace{1cm} (2. 13)

\[ 60h \{ m_{i-1} + 2m_i + m_{i+1} \} - h^3 \{ 3n_{i-1} + 14n_i + 3n_{i+1} \} = 120 \{ -y_{i-1} + y_{i+1} \} , \]
\[ i = 1, 2, 3, \ldots, k - 1 \]  \hspace{1cm} (2. 14)

\[ 8h \{ m_{i+1} - m_{i-1} \} - h^2 \{ M_{i-1} - 6M_i + M_{i+1} \} = 20 \{ y_{i-1} - 2y_i + y_{i+1} \} , \]
\[ i = 1, 2, 3, \ldots, k - 1 \]  \hspace{1cm} (2. 15)

\[ m_i = \frac{h}{6} \{ 2M_i + M_{i-1} \} - \frac{h^3}{360} \{ 8N_i + 7N_{i-1} \} + \frac{1}{h} \{ y_i - y_{i-1} \} , \]
\[ i = 1, 2, 3, \ldots, k \]  \hspace{1cm} (2. 16)

\[ m_i = \frac{h}{6} \{ 2M_i + M_{i+1} \} + \frac{h^3}{360} \{ 8N_i + 7N_{i+1} \} + \frac{1}{h} \{ y_{i+1} - y_i \} , \]
\[ i = 0, 1, \ldots, k - 1 \]  \hspace{1cm} (2. 17)

\[ m_i = -\frac{h^2}{120} \{ n_{i-1} + 18n_i + n_{i+1} \} + \frac{1}{2h} \{ y_{i+1} - y_i \} , \]
\[ i = 1, 2, 3, \ldots, k - 1 \]  \hspace{1cm} (2. 18)
\begin{align}
M_i &= -\frac{h^2}{120} \{ N_{i-1} + 8 N_i + N_{i+1} \} + \frac{1}{h^2} \{ y_{i-1} - 2 y_i + y_{i+1} \}, \\
i &= 1, 2, 3, \ldots, k - 1 
\tag{2.19}
\end{align}

\begin{align}
M_i &= \frac{1}{32h} \{ m_{i-2} + 32 m_{i-1} - 32 m_{i+1} - m_{i+2} \} \\
&\quad + \frac{5}{32h^2} \{ y_{i-2} + 16 y_{i-1} - 34 y_i + 16 y_{i+1} + y_{i+2} \}, \\
i &= 2, 3, \ldots, k - 2 
\tag{2.20}
\end{align}

\begin{align}
N_i &= -\frac{3}{2h^2} \{ M_{i-1} + 18 M_i + M_{i+1} \} + \frac{30}{h^2} \{ y_{i-1} - 2 y_i + y_{i+1} \}, \\
i &= 1, 2, 3, \ldots, k - 1 
\tag{2.21}
\end{align}

The relations (2.9)–(2.21) can be derived from the results of Albasiny and Hoskins (1971), Fyfe (1971), Ahlberg, Nilson and Walsh (1966) and Sakai (1970) discussed by N. Papamichael [7]. The uniqueness of Q can be established showing that any of the four \((k+1) \times (k+1)\) linear systems, obtained, using one of the relations (2.9), (2.10), (2.11) and (2.12) together with the four end conditions. In this paper the linear system corresponding to (2.12) has been chosen and has a unique solution for \(N_i\), \(i = 0, 1, \ldots, k\). Equation (2.13) and (2.19), or (2.19) and (2.21) give the parameters \(M_i\), \(i = 0, 1, \ldots, k\). Consider the system (2.12)

\begin{align}
N_{i-2} + 26 N_{i-1} + 66 N_i + 26 N_{i+1} + N_{i+2} &= \frac{120}{h^4} \{ y_{i-2} - 4 y_{i-1} + 6 y_i - 4 y_{i+1} + y_{i+2} \}, \\
i &= 2, 3, \ldots, k - 2 
\tag{2.22}
\end{align}

where

\begin{equation}
N_i = -f_i y_i + g_i 
\tag{2.23}
\end{equation}

The above system gives \((k - 3)\) linear algebraic equations in the \((k - 1)\) unknowns \((y_i, i = 1, 2, \ldots, k - 1)\). Two more equations are needed to have complete solution of \(y_i\)s which are derived in section 3.

It may be noted that Papamichael [5] needed two consistency systems for the solution of boundary value problem (1.3) but the method developed in this paper, needs only one such system.

Taking forward difference operator \(E = e^{hD}\), equation (2.12) can be rewritten as

\begin{align}
(E^{-2} + 26 E^{-1} + 66 I + 26 E^1 + E^2) N_i &= \frac{120}{h^4} \{ E^{-2} - 4 E^{-1} + 6 I - 4 E^1 + E^2 \} y_i, \\
i &= 2, 3, \ldots, k - 2 
\tag{2.24}
\end{align}

\begin{align}
N_i &= \frac{120}{h^4} \left( e^{-2hD} + 26 e^{-hD} + 66 I + 26 e^{hD} + e^{2hD} \right)^{-1} \\
&\quad \left( e^{-2hD} - 4 e^{-hD} + 6 I - 4 e^{hD} + e^{2hD} \right) y_i, \\
i &= 2, 3, \ldots, k - 2 
\tag{2.25}
\end{align}
Expanding the R.H.S. of above, in terms of power series and dividing give,

\[ N_i = y_i^{(4)} - \frac{h^2}{12} y_i^{(6)} + \frac{h^4}{240} y_i^{(8)} + O(h^6) \]  \hspace{1cm} (2.26)

Papamichael [7] proved the following lemma to determine the end conditions in terms of first derivative of quintic spline

**Lemma 1**

Let \( \lambda_i = m_i - y_i^{(1)} \). If \( y \in C^7[a, b] \) then

\[ \lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = \beta_i \ , \ i = 2, 3, ..., k - 2 \]  \hspace{1cm} (2.27)

where

\[ |\beta_i| \leq \frac{11}{21} h^6 \| y^{(7)} \| , \ i = 2, 3, ..., k - 2 \]

Using the above lemma along with equation (2.12) and with Taylor series expansion about the point \( x_i \), the following lemma can easily be proved, for the case discussed in this paper.

**Lemma 2**

Let

\[ \lambda_i = y_i^{(4)} - \frac{h^2}{12} y_i^{(6)} + \frac{h^4}{240} y_i^{(8)} - N_i \]  \hspace{1cm} (2.28)

If \( y \in C^{10}[a, b] \) then

\[ \lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = \beta_i \ , \ i = 2, 3, ..., k - 2 \]  \hspace{1cm} (2.29)

where

\[ |\beta_i| \leq \frac{2665920}{10!} h^6 \| y^{(10)} \| , \ i = 2, 3, ..., k - 2 \]

Finally the required end conditions are derived in the following section.

### 3 End Conditions

Consider end conditions of the form

\[ N_0 + \alpha N_1 + \beta N_2 + \gamma N_3 + N_4 = \frac{1}{h^4} \left[ \sum_{i=0}^{3} a_i y_i + bh y_0^{(1)} + h^4 \sum_{i=0}^{4} c_i y_i^{(4)} \right] \]  \hspace{1cm} (3.1)

and

\[ N_k + \alpha N_{k-1} + \beta N_{k-2} + \gamma N_{k-3} + N_{k-4} = \frac{1}{h^4} \left[ \sum_{i=0}^{3} a_i y_{k-i} + bh y_k^{(1)} + h^4 \sum_{i=0}^{4} c_i y_{k-i}^{(4)} \right] \]  \hspace{1cm} (3.2)
where the scalars \( \alpha, \beta, \gamma, a_i, i=0,1,2,3 \) and \( c_i, i=0,1,...,4 \) can be determined with the assumption that \( Q \) exists uniquely and

\[
\| Q^r - y^r \| = O(h^{6-r}), \quad r = 0, 1, ..., 5
\]

For this let

\[
\lambda_i = y_i^{(4)} - \frac{h^2}{12} y_i^{(6)} + \frac{h^4}{240} y_i^{(8)} - N_i, \quad i = 0, 1, ..., k.
\]

Since it is supposed that \( y \in C^{10}[a,b] \), therefore the equations (2.12), (3.1) and (3.2) give,

\[
\lambda_0 + \alpha \lambda_1 + \beta \lambda_2 + \gamma \lambda_3 + \lambda_4 = \beta_1
\]

\[
\lambda_{i-2} + 26 \lambda_{i-1} + 66 \lambda_i + 26 \lambda_{i+1} + \lambda_{i+2} = \beta_i, \quad i = 2,3,...,k-2
\]

\[
\lambda_k + \alpha \lambda_{k-1} + \beta \lambda_{k-2} + \gamma \lambda_{k-3} + \lambda_{k-4} = \beta_{k-1}
\]

where

\[
\beta_1 = \frac{1}{h^4} \left[ -\sum_{i=0}^{3} a_i y_i - bh y_0^{(1)} - \sum_{i=0}^{4} c_i y_i^{(4)} + h^4 \left( y_0^{(4)} + \alpha y_1^{(4)} + \beta y_2^{(4)} + \gamma y_3^{(4)} + y_4^{(4)} \right) \right.
\]

\[
- \frac{h^6}{12} \left( y_0^{(6)} + \alpha y_1^{(6)} + \beta y_2^{(6)} + \gamma y_3^{(6)} + y_4^{(6)} \right)
\]

\[
+ \frac{h^8}{240} \left( y_0^{(8)} + \alpha y_1^{(8)} + \beta y_2^{(8)} + \gamma y_3^{(8)} + y_4^{(8)} \right)
\]

\[
(3.6)
\]

and

\[
\beta_{k-1} = \frac{1}{h^4} \left[ -\sum_{i=0}^{3} a_i y_{k-i} - bh y_0^{(1)} - \sum_{i=0}^{4} c_i y_{k-i}^{(4)} + h^4 \left( y_{k}^{(4)} + \alpha y_{k-1}^{(4)} + \beta y_{k-2}^{(4)} \right.ight.
\]

\[
+ \gamma y_{k-3}^{(4)} + y_{k-4}^{(4)} \left. \right) - \frac{h^6}{12} \left( y_{k}^{(6)} + \alpha y_{k-1}^{(6)} + \beta y_{k-2}^{(6)} + \gamma y_{k-3}^{(6)} + y_{k-4}^{(6)} \right)
\]

\[
+ \frac{h^8}{240} \left( y_{k}^{(8)} + \alpha y_{k-1}^{(8)} + \beta y_{k-2}^{(8)} + \gamma y_{k-3}^{(8)} + y_{k-4}^{(8)} \right)
\]

\[
(3.7)
\]

and, from lemma 2

\[
|\beta_i| \leq \frac{2665920}{10!} h^6 \| y^{(10)} \|, \quad i = 2,3,...,k-2
\]

\[
(3.8)
\]

Following Papamichael \[7\], the required end conditions may be written as

\[
N_0 + N_1 = \frac{1}{h^4} \left[ \frac{220}{3} y_0 - 120 y_1 + 60 y_2 - \frac{40}{3} y_3 + 40 h y_0^{(1)} \right]
\]

\[
+ h^4 \left( \frac{2519}{2520} y_0^{(4)} + \frac{11822}{1260} y_1^{(4)} + \frac{223}{210} y_2^{(4)} + \frac{176}{180} y_3^{(4)} \right.
\]

\[
+ \frac{1769}{2520} y_4^{(4)} \right]
\]

\[
(3.9)
\]
and

\[ N_k + N_{k-1} = \frac{1}{h^4} \left[ \frac{220}{3} y_k - 120y_{k-1} + 60y_{k-2} - \frac{40}{3} y_{k-3} + 40h y_k^{(1)} \
+ h^4 \left( \frac{2519}{2520} y_k^{(4)} + \frac{11822}{1260} y_{k-1}^{(4)} + \frac{223}{210} y_{k-2}^{(4)} + \frac{176}{180} y_{k-3}^{(4)} \
+ \frac{1769}{2520} y_{k-4}^{(4)} \right) \right] \quad (3.10) \]

The quintic spline solution of the system (1.3) is defined in the next section.

4 Quintic Spline Solution

The quintic spline solution of (1.3) is based on the linear equations (2.12), (3.9) and (3.10). Let \( Y = y_i, \ C = c_i, \ e = e_i. \) Then the parameters \( y_i \) of \( Q \) satisfy the following matrix equation

\[(A + h^4 BF)Y = C + e\]

where \( Y, C, e \) are \((k-1)\) dimensional column vectors and \( A, B, F \) are \((k-1) \times (k-1)\) matrices, where

\[
A = \begin{bmatrix}
9 & \frac{-9}{2} & 1 \\
-4 & 6 & -4 & 1 \\
1 & -4 & 6 & -4 & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & -4 & 6 & -4 & 1 \\
1 & -4 & 6 & -4 \\
1 & \frac{-9}{2} & 9 
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
\frac{31686}{5040} & \frac{669}{840} & \frac{528}{7200} & \frac{5307}{100800} \\
\frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} \\
\frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} \\
& & & \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} \\
\frac{5307}{100800} & \frac{528}{7200} & \frac{669}{8400} & \frac{31686}{5040} \\
\end{bmatrix}
\]

\[F = \text{diag}(f_i)\]

\[c_1 = \frac{11}{2} y_0 + 3hy_0^{(1)} + \frac{h^4}{280} \left( -\frac{3}{360} g_0 + \frac{3}{360} f_0 y_0 + \frac{31686}{180} g_1 \right) + \frac{669}{30} g_2 + \frac{3696}{180} g_3 + \frac{5307}{360} g_4 \] (4.1)

\[c_2 = \frac{h^4}{120} \left( g_0 - f_0 y_0 + 26 g_1 + 66 g_2 + 26 g_3 + g_4 \right) - y_0 \] (4.2)

\[c_i = \frac{h^4}{120} \left( g_{i-2} + 26 g_{i-1} + 66 g_i + 26 g_{i+1} + g_{i+2} \right) \quad i = 3, 4, \ldots, k - 3 \] (4.3)

\[c_{k-2} = \frac{h^4}{120} \left( g_k - f_k y_k + 26 g_{k-1} + 66 g_{k-2} + 26 g_{k-3} + g_{k-4} \right) - y_k \] (4.4)

\[c_{k-1} = \frac{11}{2} y_k - 3hy_k^{(1)} + \frac{h^4}{280} \left( -\frac{3}{360} g_k + \frac{3}{360} f_k y_k + \frac{31686}{180} g_{k-1} \right) + \frac{669}{30} g_{k-2} + \frac{3696}{180} g_{k-3} + \frac{5307}{360} g_{k-4} \] (4.5)

and \[e_i = O(h^6) \quad i = 1, \ldots, k - 1\]

**Extending the method for the solution of sixth, eighth and higher order boundary value problems, is in process.**

To implement the method for the quintic spline solution of the boundary value problem (1.3), three examples are discussed in the following section.
5 Numerical Examples

In this section numerical technique discussed in section 4 is illustrated, by the following three boundary value problems of the type (1.3).

Example 1

Consider the following boundary value problem

\[
\begin{align*}
&y^{iv} + 4y = 1, \quad y(-1) = y(1) = 0, \\
&y'(-1) = -y'(1) = \frac{\sinh(2) - \sin(2)}{4\cosh(2) + \cos(2)}
\end{align*}
\]

The analytic solution of the above problem is

\[
y(x) = 0.25[1 - 2\sin(1) \sinh(1) \sin(x) \sinh(x) + \cos(1) \cosh(1) \cos(x) \cosh(x)] / (\cos(2) + \cosh(2))
\]

The results are summarized in Table 1.

| h   | $y_i$ | $(y_i)^{(1)}$ | $(y_i)^{(2)}$ | $(y_i)^{(3)}$ | $(y_i)^{(4)}$ |
|-----|-------|---------------|---------------|---------------|---------------|
| $\frac{1}{8}$ | $2.1 \times 10^{-3}$ | $5.5 \times 10^{-3}$ | $9.7 \times 10^{-3}$ | $1.06 \times 10^{-2}$ | $8.5 \times 10^{-3}$ |
| $\frac{1}{16}$ | $9.10 \times 10^{-5}$ | $3.34 \times 10^{-4}$ | $8.86 \times 10^{-4}$ | $1.4 \times 10^{-3}$ | $3.64 \times 10^{-4}$ |
| $\frac{1}{32}$ | $5.12 \times 10^{-6}$ | $1.77 \times 10^{-5}$ | $1.20 \times 10^{-4}$ | $2.60 \times 10^{-4}$ | $2.05 \times 10^{-5}$ |
| $\frac{1}{64}$ | $2.85 \times 10^{-6}$ | $4.50 \times 10^{-6}$ | $2.44 \times 10^{-5}$ | $6.03 \times 10^{-5}$ | $1.14 \times 10^{-5}$ |
| $\frac{1}{128}$ | $7.87 \times 10^{-7}$ | $1.18 \times 10^{-6}$ | $5.83 \times 10^{-6}$ | $1.47 \times 10^{-5}$ | $3.23 \times 10^{-6}$ |
| $\frac{1}{256}$ | $1.98 \times 10^{-7}$ | $2.79 \times 10^{-7}$ | $1.44 \times 10^{-6}$ | $3.59 \times 10^{-6}$ | $8.73 \times 10^{-7}$ |
| $\frac{1}{512}$ | $4.15 \times 10^{-8}$ | $4.04 \times 10^{-8}$ | $3.05 \times 10^{-7}$ | $6.81 \times 10^{-7}$ | $2.46 \times 10^{-7}$ |
| $\frac{1}{1024}$ | $1.07 \times 10^{-7}$ | $1.91 \times 10^{-7}$ | $8.01 \times 10^{-7}$ | $2.12 \times 10^{-6}$ | $3.49 \times 10^{-7}$ |
Example 2

Consider the following boundary value problem

\[
\begin{align*}
y'' + xy &= -(8 + 7x + x^3)e^x, \\
y(0) &= y(1) = 0, \quad y'(0) = 1, \quad y'(1) = -e
\end{align*}
\]

The analytic solution of the above differential system is

\[y(x) = x(1 - x)e^x\]

The observed maximum errors (in absolute value) associated with \(y_i^{(\mu)}\), \(\mu = 0, 1, 2, 3, 4\), for the system 5.2 are briefly summarized in Table 2.

Table 2: Maximum absolute errors for Problem 5.2 in \(y_i^{(\mu)}\), \(\mu = 0, 1, 2, 3, 4\)

| h    | \(y_i\)          | \(y_i^{(1)}\)  | \(y_i^{(2)}\)  | \(y_i^{(3)}\)  | \(y_i^{(4)}\)  |
|------|------------------|----------------|----------------|----------------|----------------|----------------|
| \(\frac{1}{8}\) | 5.3828 \times 10^{-4} | 1.9 \times 10^{-3} | 6.0 \times 10^{-3} | 3.8 \times 10^{-2} | 3.2768 \times 10^{-4} |
| \(\frac{1}{16}\) | 8.8114 \times 10^{-5} | 2.9045 \times 10^{-4} | 1.4 \times 10^{-3} | 9.9 \times 10^{-3} | 5.3779 \times 10^{-5} |
| \(\frac{1}{32}\) | 1.636 \times 10^{-5} | 5.1554 \times 10^{-5} | 3.3586 \times 10^{-4} | 2.6 \times 10^{-3} | 9.824 \times 10^{-6} |
| \(\frac{1}{64}\) | 3.3894 \times 10^{-6} | 1.058 \times 10^{-5} | 8.2872 \times 10^{-5} | 6.6254 \times 10^{-4} | 1.991 \times 10^{-6} |
| \(\frac{1}{128}\) | 7.5919 \times 10^{-7} | 2.388 \times 10^{-6} | 2.269 \times 10^{-5} | 1.6824 \times 10^{-4} | 4.3958 \times 10^{-7} |
| \(\frac{1}{256}\) | 1.7884 \times 10^{-7} | 5.671 \times 10^{-7} | 5.9302 \times 10^{-6} | 4.2401 \times 10^{-5} | 1.0264 \times 10^{-7} |
| \(\frac{1}{512}\) | 3.7364 \times 10^{-8} | 1.1843 \times 10^{-7} | 1.3068 \times 10^{-6} | 9.4838 \times 10^{-6} | 2.1304 \times 10^{-8} |
| \(\frac{1}{1024}\) | 6.3664 \times 10^{-8} | 2.001 \times 10^{-7} | 1.9244 \times 10^{-6} | 9.969 \times 10^{-6} | 3.5439 \times 10^{-8} |
Example 3

Consider the differential system

\[
\begin{align*}
y'' - y &= -4(2x \cos(x) + 3 \sin(x)), \\
y(0) &= y(1) = 0, \quad y'(-1) = 2 \sin(1), \quad y'(1) = 2 \sin(1)
\end{align*}
\]

(5.3)

The analytic solution of the above system is

\[y(x) = (x^2 - 1) \sin(x)\]

The observed maximum errors (in absolute value) associated with \(y_i^{(\mu)}\), \(\mu = 0, 1, 2, 3, 4\), for the system 5.3 are briefly summarized in Table 3.

| h      | \(y_i\)    | \(y_i^{(1)}\) | \(y_i^{(2)}\) | \(y_i^{(3)}\) | \(y_i^{(4)}\) |
|--------|------------|---------------|---------------|---------------|---------------|
| \(\frac{1}{2}\) | \(1.4 \times 10^{-3}\) | \(9.9 \times 10^{-3}\) | \(4.3 \times 10^{-2}\) | \(1.0 \times 10^{-1}\) | \(1.4 \times 10^{-3}\) |
| \(\frac{1}{16}\) | \(1.42 \times 10^{-4}\) | \(1.2 \times 10^{-3}\) | \(7.0 \times 10^{-3}\) | \(2.26 \times 10^{-2}\) | \(1.42 \times 10^{-4}\) |
| \(\frac{1}{32}\) | \(8.18 \times 10^{-6}\) | \(1.34 \times 10^{-7}\) | \(1.2 \times 10^{-3}\) | \(5.1 \times 10^{-3}\) | \(8.81 \times 10^{-6}\) |
| \(\frac{1}{64}\) | \(5.05 \times 10^{-6}\) | \(1.95 \times 10^{-5}\) | \(2.51 \times 10^{-4}\) | \(1.2 \times 10^{-3}\) | \(5.05 \times 10^{-6}\) |
| \(\frac{1}{128}\) | \(1.6 \times 10^{-6}\) | \(5.88 \times 10^{-6}\) | \(5.68 \times 10^{-5}\) | \(3.01 \times 10^{-4}\) | \(1.61 \times 10^{-6}\) |
| \(\frac{1}{256}\) | \(4.42 \times 10^{-7}\) | \(1.58 \times 10^{-6}\) | \(1.35 \times 10^{-5}\) | \(7.48 \times 10^{-5}\) | \(4.42 \times 10^{-7}\) |
| \(\frac{1}{512}\) | \(1.15 \times 10^{-7}\) | \(4.09 \times 10^{-7}\) | \(3.31 \times 10^{-6}\) | \(1.86 \times 10^{-5}\) | \(1.15 \times 10^{-7}\) |
| \(\frac{1}{1024}\) | \(3.19 \times 10^{-8}\) | \(1.05 \times 10^{-7}\) | \(8.51 \times 10^{-7}\) | \(5.01 \times 10^{-6}\) | \(3.19 \times 10^{-8}\) |

References

[1] Riaz A. Usmani, *Discrete Variable Methods For A Boundary Value Problem With Engineering Applications*, Mathematics of Computation, Vol. 32, (1978) No. 144, pp. 1087-1096.

[2] Riaz A. Usmani, *Spline Solutions For Nonlinear Two Point Boundary Value Problems*, Internat. J. Math and Math. Sci. Vol. 3 No. 1 (1980) 151-167.
[3] Riaz A. Usmani and Manabu Sakai, *Quartic Spline Solutions For Two-Point Boundary Value Problems Involving Third Order Differential Equations*, Journal Mathematical Phy. Sci. Vol 18, No. 4, (1984), India.

[4] N. Papamichael and A. J. Worsey, *End Conditions For Improved Cubic Spline Derivative Approximations*, Technical Report TR/100, July(1980)

[5] N. Papamichael and A. J. Worsey, *A Cubic Spline Method For The Solution Of A Linear Fourth-Order Two-Point Boundary Value Problem*, Technical Report TR/101 (1980).

[6] J. W. Daniel and B. K. Swartz, *Extrapolated Collocation For Two-Point Boundary-Value Problems Using Cubic Splines*, J. Inst. Maths Applics., Vol 16, (1975) , pp 161-174.

[7] N. Papamichael and G. H. Behforooz, *End Conditions For Interpolatory Quintic Splines*, Technical Report TR/95 1980.