DIFFERENTIAL CALCULUS OVER DOUBLE LIE ALGEBROIDS

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Abstract. M. Van den Bergh [18] defined the notion of a double Lie algebroid and showed that a double quasi-Poisson algebra gives rise to a double Lie algebroid. We give new examples of double Lie algebroids and develop a differential calculus in that context. We recover the non commutative Karoubi–de Rham complex [9], [7] and the double Poisson–Lichnerowicz cohomology [15] as particular cases of our construction.

1. INTRODUCTION

Let $k$ be a field of characteristic 0. The pair $(L, \omega)$ is a Lie–Rinehart algebra [20] over a commutative $k$-algebra $A$ if $L$ is endowed with a $k$-Lie algebra structure and an $A$-module structure where the two structures are linked by a compatibility relation involving the anchor $\omega$. There is a one-to-one correspondence [10] between Lie–Rinehart algebra structures on $L$ and Gerstenhaber algebra structures on $L_{*}$ (where $L_{*}$ is the exterior algebra of $L$). If $A$ is smooth and $L$ is a finitely generated projective $A$-module, then $L$ is the $A$-module of global sections of a Lie algebroid. Lie–Rinehart algebras generalize at the same time $A$-algebras (case where $\omega = 0$) and the Lie algebra of derivations over $A$ (then $\omega = id$). A Poisson smooth algebra gives rise to a Lie–Rinehart algebra structure (on $L = \Omega^{1}$) and this is also true for quasi-Poisson smooth $G$-algebras [1], [7]. If $X$ is a Poisson manifold, then $L = \Omega_{X}^{1} \oplus \Gamma(X \times g)$. Lie algebroids have been extensively studied and used, in particular in Poisson geometry. A Lie–Rinehart algebra $L$ defines a differential $d_{L}$ on the graded algebra $L_{*}A$. In the case of an $A$-Lie algebra, we recover the Cartan–Eilenberg differential. In the case where $L = \text{Der}(A)$ (with $A$ smooth), we recover the de Rham differential. In the case where $L = \Omega^{1}(X)$ ($X$ being a Poisson manifold), we recover the Lichnerowicz–Poisson differential. More generally a differential calculus has been developed for Lie algebroids [8].

In this article, we are interested in the case where $A$ is not necessarily commutative. As in [17] and [18], we use a non commutative version of differential geometry based on an idea of Kontsevitch: For a property of a non commutative $k$-algebra to have a geometric meaning, it should induce (through a trace map) a standard geometric property on all representation spaces $\text{Rep}(A, N) = \text{Hom}(A, \text{Mat}_{N}(k))$, for all integer $N$. The coordinate ring of $\text{Rep}(A, N)$ is

$$\mathcal{O}_{\text{Rep}(A, N)} = \frac{k[a_{i,j} | a \in A, (i, j) \in [1, N]^2]}{< (ab)_{i,j} = a_{i,k}b_{k,j}, a, b \in A>},$$

If the $k$-algebra $A$ is smooth, then the commutative $k$-algebra $\mathcal{O}_{\text{Rep}(A, N)}$ is smooth for all $N$ [17], [18]. If $A = k < x_{1}, \ldots, x_{m} >$ is a free associative $k$-algebra generated by the variables $x_{1}, \ldots, x_{m}$, then

$$\text{Rep}_{N}(A) = \text{Mat}_{N}(k) \oplus \text{Mat}_{N}(k) \oplus \cdots \oplus \text{Mat}_{N}(k) \simeq k^{mN^{2}}$$

where, if $\alpha \in [1, m]$, $M_{\alpha}$ is the $N \times N$ matrix $(x^{j}_{i,\alpha})_{i,j}$ and $\text{Tr}(x_{\alpha_{1}} \cdots x_{\alpha_{r}}) = \text{Tr}(M_{\alpha_{1}}M_{\alpha_{2}} \cdots M_{\alpha_{r}})$.

The non commutative geometry notions defined according to the principle of Kontsevitch are often called by a name with the prefix "double". Thus derivations $\text{Der}(A)$ are replaced by double derivations $\text{Der}(A)$ (that are derivations from $A$ to $A \otimes A$ considered with its exterior $A$-bimodule structure), Poisson algebras are replaced by double Poisson algebras, etc ... The Karoubi–de Rham complex was defined in the non commutative setting [9], the contraction and the Lie derivative by a double vector field were defined in [2]. Double Poisson cohomology was defined [21], [15] and computed for some examples of double Poisson brackets associated with quivers.

In [18], double Lie algebroids or double Lie–Rinehart algebras were introduced and it was shown that a double quasi-Poisson algebra gives rise to a double Lie algebroid. A double Lie algebroid $L$ being an $A$-bimodule, one can construct the tensor algebra of $L$, $T_{A}(L)$. In [18], the double Lie algebroid structure has been defined by the Gerstenhaber double structure it induces on $T_{A}(L)$. 

1
We give a direct definition of a double Lie algebroid and show that there is a correspondence between double Lie algebroid structures on an $A$-bimodule and double Gerstenhaber algebra structures on $T_A(L)$. We give new examples of double Lie algebroids. Studying the case where $A = k$, we see that any associative $k$-algebra has a natural double Lie algebroid structure. Then, we develop a differential calculus for double Lie algebroids: definition of a differential $d_A$, for which we give an explicit formula, Lie derivative, contraction, etc... In the case where $L = \text{Der}(A)$ and $A$ is smooth, we recover the Karoubi–de Rham complex (17, 7) but some of our formulas are new even in that case. In the case where $A$ is a smooth double Poisson algebra and $L = \Omega_A^2$, the differential $d_A$ coincides with that of double Poisson cohomology (21, 15). In the case where $A = k$, $d_A$ is the differential computing cyclic cohomology. Thus we recover the classical picture. The theory of double Lie algebroids encompasses several theories.

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Convention :

We will use the same notation as [17].

If $(V_i)_{i=1,...,n}$ are $k$ vector spaces and $s \in S_n$, then for $a = a_1 \otimes \cdots \otimes a_n \in V_1 \otimes \cdots \otimes V_n$

$$\tau_s(a) = a_{s^{-1}(1)} \otimes \cdots \otimes a_{s^{-1}(n)}.$$  

If $(V_i)_{i=1,...,n}$ are $k$ graded vector spaces and $s \in S_n$, then for $a = a_1 \otimes \cdots \otimes a_n \in V_1 \otimes \cdots \otimes V_n$

$$\sigma_s(a) = (-1)^{t(a)} a_{s^{-1}(1)} \otimes \cdots \otimes a_{s^{-1}(n)}.$$

where $t = \sum_{i<j,s^{-1}(i)>s^{-1}(j)} |a_{s^{-1}(i)}||a_{s^{-1}(j)}|$. 

$\tau_{(12)}$ (respectively $\sigma_{(12)}$) will be also denoted $(x \otimes y)\circ = y \otimes x$ (respectively $(x \otimes y)\circ = (-1)^{[x,y]} y \otimes x$.)

Let $B$ be a fixed $k$-algebra that will be, most of the time, semisimple commutative of the form $B = k e_1 \oplus \cdots \oplus k e_n$ with $e_i^2 = e_i$. A $B$-algebra is a $k$-algebra $A$ equipped with a morphism of $k$-algebras $B \to A$. The notion of $B$-algebra allows to define relative versions.

2. Definitions and generalities

Most of the definition and results of this section come from [17].

Definition 2.1. An $n$-bracket is a linear map

$$\{\ldots,\ldots\} : A^\otimes_n \to A^\otimes_n$$

which is a derivation $A \to A^\otimes_n$ in its last argument for the outer bimodule structure on $A^\otimes_n$ i.e.

$$\{\{a_1, a_2, \ldots, a_{n-1}, a_n a'_n\} = a_n \{\{a_1, a_2, \ldots, a_{n-1}, a'_n\} + \{\{a_1, a_2, \ldots, a_{n-1}, a_n\}\} a'_n$$

and which is cyclically antisymmetric in the sense that

$$\tau_{(1\ldots n)} \circ \{\ldots,\ldots\} \circ \tau_{(1\ldots n)} = (-1)^{n+1}\{\ldots,\ldots\}.$$

If $A$ is a $B$-algebra, then an $n$-bracket is $B$-linear if it vanishes when its last argument is in the image of $B$.

As in [17], we set

$$\{a, b\}_L = \{a, b_1\} \otimes b_2 \otimes \cdots \otimes b_n$$

Associated to a double bracket $\{\ldots,\ldots\}$, we define a tri-ary operation $\{\ldots,\ldots\}$ as follows:

$$\{a, b, c\} = \{a, \{b, c\}\}_L + \tau_{(123)} \{b, \{c, a\}\}_L + \tau_{(132)} \{c, \{a, b\}\}_L$$

Proposition 2.2. ([17]) $\{\ldots,\ldots\}$ is a 3-bracket.

Definition 2.3. ([17]). Let $A$ be a $k$-algebra. A double bracket $\{\ldots,\ldots\}$ on $A$ is a double Poisson bracket if $\{\ldots,\ldots\} = 0$. An algebra with a double Poisson bracket is a double Poisson algebra.
Example 2.4. ([17], [16]) One may characterize the double Poisson brackets on $k[t]$. For $\lambda, \mu, \nu \in k$, 
$$\{(t, t)\} = \lambda(t \otimes 1 - 1 \otimes t) + \mu(t^2 \otimes 1 - 1 \otimes t^2) + \nu(t^2 \otimes t - t \otimes t^2)$$
defines a double Poisson structure if and only if $\lambda \nu - \mu^2 = 0$ and any double Poisson structure on $k[t]$ is of this form.

We will see many other examples of double Poisson algebras further.

The following proposition was proved in [17]:

Proposition 2.5. Assume that $(A, \{\cdot, \cdot\})$ is a double Poisson algebra. For any elements $a$ and $b$ of $A$, we set $\{a, b\} = \{(a, b)\}'$. Then the following holds:

1) $\{\cdot, \cdot\}$ is a derivation in its second argument and vanishes on commutators in its first argument.
2) $\{\cdot, \cdot\}$ is anti-symmetric modulo commutators.
3) $\{\cdot, \cdot\}$ makes $A$ into a left Loday algebra, i.e. $\{\cdot, \cdot\}$ satisfies the following version of Jacobi identity
$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + \{b, \{a, c\}\}$$
4) $\{\cdot, \cdot\}$ makes $A/[A, A]$ into a Lie algebra.

Definition 2.6. ([11], [15]) A Poisson structure on $A$ is a Lie bracket $\{-, -\}$ on $A$ such that for each $a \in A$, the map $[a, -] : \frac{A}{[A, A]} \to \frac{A}{[A, A]}$ is induced by a derivation on $A$.

Remarks 2.7. (i) In the case where $A$ is commutative, we recover the usual Poisson bracket.

(ii) It was shown in [17] (lemme 2.6.2) that a double Poisson bracket on $A$ induces a Poisson structure on $A$.

(iii) A Poisson structure on $A$ is also called a $H_0$-Poisson structure as it is a structure on $\frac{A}{[A, A]} = HC_0(A)$ which is the $0$th cyclic group of $A$. In [1], derived Poisson structure were defined on higher cyclic cohomology group.

Let $D$ be a graded algebra. There are two commuting $D^e$-module structures on $D \otimes D$ : For any homogeneous elements $\alpha, \beta, x, y$ in $D$,
$$\alpha(x \otimes y)\beta = \alpha x \otimes y\beta$$
$$\alpha * (x \otimes y) * \beta = (-1)^{|\alpha||\beta|+|\alpha||x|+|y||\beta|}x\beta \otimes \alpha y$$

Definition 2.8. ([13]) Let $d \in \mathbb{Z}$ and let $D$ be a graded algebra. $D$ is called a double Gerstenhaber algebra of degree $d$ if it is equipped with a graded bilinear map
$$\{\cdot, \cdot\} : D \otimes D \to D \otimes D$$
of degree $d$ such that the following identities hold:

1) $\{(\alpha, \gamma)\} = (-1)^{|\alpha||\gamma|}\gamma\{\alpha, \gamma\}$
1') $\{(\beta, \gamma, \alpha)\} = (-1)^{|\alpha||\beta|+|\beta||\gamma|}\gamma\{\alpha, \gamma\} + \{\alpha, \beta\} \cdot \gamma$
2) $\{\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+|\beta||\gamma|}\gamma\{\alpha, \gamma\} + \{\alpha, \beta\} \cdot \gamma$
2') $\{\{\alpha, \beta\} = (-1)^{|\alpha||\beta|+|\beta||\gamma|}\gamma\{\alpha, \gamma\} + \{\alpha, \beta\} \cdot \gamma$
3) $\{\{\alpha, \beta\} \} \{\{\alpha, \beta\} \} \{\{\alpha, \beta\} \} \{\{\alpha, \beta\} \} \{\{\alpha, \beta\} \}$

Remarks 2.9. 1) The definition of double Gerstenhaber algebra is given in [17]. It is extended to the case of double Gerstenhaber algebra of degree $d \in \mathbb{Z}$ in [13]. The case $d = -1$ corresponds to double Gerstenhaber algebras (see [17]) and the case $d = 0$ corresponds to double Poisson algebras ([17]).

2) Assertions 1) and 1') are equivalent if assertion 2) is satisfied ([13]).

If $D$ is a double Gerstenhaber algebra, we define the associated bracket $\{-, -\} : D \otimes D \to D$ by:
$$\forall (\alpha, \beta) \in D^2, \quad \{\alpha, \beta\} = \{(\alpha, \beta)\}'\{(\alpha, \beta)\}''.$$
The following proposition was partly stated in [17]:
Proposition 2.10. Let $D$ be a Gerstenhaber algebra of degree $d$ and let $\alpha, \beta, \gamma$ three homogeneous elements in $D$.

1) $\{\alpha \otimes (\beta - (-1)^{|\alpha||\beta|}\beta \otimes \alpha, \gamma\} = 0$.
2) $\{\alpha, \beta\} = (-1)^{|\alpha||\beta|}(\beta \otimes \alpha) \in [D, D]$
3) $\{\alpha, \{\beta, \gamma\}\} - \{\{\alpha, \beta\}, \gamma\} + (-1)^{|\alpha||\beta|+d}(\beta, \{\alpha, \gamma\}\}) = 0$

where $\{\alpha,\gamma\}$ acts on tensors by

$$\{\alpha, \{\beta, \gamma\}\} = \{\alpha, \{\beta, \gamma\}\} + \{\{\alpha, \beta\}, \gamma\} - \{\alpha, \{\beta, \gamma\}\}$$

4) $\frac{D}{[D, D]}[d]$ is a graded Lie algebra.

Remark 2.11. Proposition 2.10 is proved in the ungraded case in [17]. In this case, $d = 0$ and $D$ is a double Poisson algebra. Our proof is similar to that of [17] so that we only sketch the main lines of it.

Proof. 1) and 2) are straightforward computations if one uses the relation

$$\{\alpha, \beta\} = (-1)^{|\alpha||\beta|}((\beta \otimes \alpha) + ([\alpha, \beta])_T)$$

4) is a consequence of the previous statements. Let us now prove 3).

We will make use of the following lemma whose proof is left to the reader:

Lemma 2.12. Set $\{(\alpha, \{\gamma, \beta\})\} = (-1)^{|\alpha||\beta|}(\gamma, \{\beta, \alpha\}) \otimes (\alpha, \{\gamma, \beta\})$. The following equality holds

$$\{(\alpha, \{\gamma, \beta\})\} = (-1)^{|\gamma||\beta|+d}(\gamma, \{\beta, \alpha\}) \otimes (\alpha, \{\gamma, \beta\})$$

Let us now compute the three terms of the equality 3). Assertion 3) will follow from these computations.

$$\{(\alpha, \{\beta, \gamma\})\} = \{\alpha, \{\beta, \gamma\}\} + (-1)^{|\alpha||\beta|+d}((\beta, \gamma) \otimes (\alpha, \{\beta, \gamma\}))$$

$$= (m \otimes 1)\{(\alpha, \{\beta, \gamma\})\} + (1 \otimes m)\{(\alpha, \{\beta, \gamma\})\}$$

$$= (m \otimes 1)\{(\alpha, \{\beta, \gamma\})\} + (1 \otimes m)\{(\alpha, \{\beta, \gamma\})\}$$

$$\{(\alpha, \{\beta, \gamma\})\} = (\sigma_{(12)}(\gamma, \{\alpha, \beta\}) - (m \otimes 1)\{(\alpha, \{\beta, \gamma\})\})$$

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3. Double derivations

In this section, we recall results of [17].

Let $B$ be a $k$-algebra. A $B$-algebra is a pair $(A, \eta)$ where $\eta : B \to A$ is an algebra morphism.

Denote by $m : A \otimes_B A \to A$ multiplication on $A$. One sets $\Omega_B^1(A) := \text{Ker}(m)$. It is naturally endowed with an $A^e$-module structure. If $a \in A$, then $da = a \otimes 1 - 1 \otimes a$ belongs to $\Omega_B^1(A)$. If $A$ is finitely generated as a $B$-algebra, then $\Omega_B^1A$ is finitely generated as a left $A^e$-module.

Definition 3.1. A $B$-algebra $A$ is called smooth over $B$ if it is finitely generated as an $B$-algebra and $\Omega_B^1A$ is projective as a left $A^e$-module.

Example 3.2. Let $Q = (Q, I)$ be a finite quiver with vertex set $I = \{1, \ldots, n\}$ and edge set $Q$. Denote by $e_i$ the idempotent associated to the vertex $i$ and we put $B = \oplus_i k e_i$. The path algebra $A = k[Q]$ is smooth over $B$ ([17]).
Let $\text{Der}_B(A, A \otimes A)$ be the $B$-derivations from $A$ to $A \otimes A$ where we put the outer bimodule structure on $A \otimes A$. Such a derivation is called a double derivation. Inner bimodule structure on $A^e$ allows to endow $\text{Der}_B(A, A \otimes A)$ with an $A^e$-module structure:

$$\alpha \cdot D \cdot \beta(a) = D(a)\beta \otimes \alpha D(a)''$$

**Notation:** The $A^e$-module $\text{Der}_B(A, A \otimes A)$ will be denoted $\text{Der}_B(A)$.

**Example 3.3.** We assume that we are in the situation where $B = ke_1 \oplus \cdots \oplus ke_n$ with $e_i^2 = e_i$.

Let $E_i : A \rightarrow A \otimes A$ defined by $E_i(a) = ae_i \otimes e_i - e_i \otimes e_i a$ and $E = \sum_{i=1}^n E_i \in \text{Der}_B(A)$. 

**Proposition 3.4.** (17) Let $\delta, \Delta$ in $\text{Der}_B(A)$, then 

$$\{\{\delta, \Delta\}\} = (\delta \otimes 1)\Delta - (1 \otimes \Delta)\delta$$

$$\{\{\delta, \Delta\}\} := (1 \otimes \delta)\Delta - (\Delta \otimes 1)\delta = -\{\{\Delta, \delta\}\}$$

are derivation from $A$ to $A^{\otimes 3}$ where the $A^e$-bimodule structure on $A^{\otimes 3}$ is the outer structure.

We define 

$$\{\{\delta, \Delta\}\}_l = \tau_{(23)} \circ \{\{\delta, \Delta\}\}_r \in \text{Der}(A) \otimes A$$

$$\{\{\delta, \Delta\}\}_r = \tau_{(12)} \circ \{\{\delta, \Delta\}\}_r \in A \otimes \text{Der}(A)$$

We write 

$$\{\{\delta, \Delta\}\}_l = \{\{\delta, \Delta\}\} \otimes \{\{\delta, \Delta\}\}''$$

$$\{\{\delta, \Delta\}\}_r = \{\{\delta, \Delta\}\} \otimes \{\{\delta, \Delta\}\}''$$

with $\{\{\delta, \Delta\}\}_l, \{\{\delta, \Delta\}\}_r \in A$ and $\{\{\delta, \Delta\}\}_l, \{\{\delta, \Delta\}\}_r \in \text{Der}(A)$.

**Theorem 3.5.** (14) For $a, b \in A$ and $\delta, \Delta \in \text{Der}_B(A)$, the following equations 

$$\{a, b\} = 0$$

$$\{\delta, a\} = \delta(a) \in A \otimes A$$

$$\{\delta, \Delta\} = \{\{\delta, \Delta\}\}_l + \{\{\delta, \Delta\}\}_r$$

define a unique structure of double Gerstenhaber algebra on $T_A \text{Der}_B(A)$.

**Notation:** From now on, $T_A \text{Der}_B(A)$ will be denoted $D_B(A)$.

**Proposition 3.6.** (17) Assume that $A$ is a finitely generated $k$-algebra. The linear map 

$$\mu : (A^e)_n \rightarrow \{B \text{-linear } n \text{-brackets on } A\}$$

$$Q = \delta_1 \cdots \delta_n \rightarrow \{\{\ldots, \ldots, -\}\}_Q = \{\{-\ldots, -\}\} = \sum_{i=0}^{n-1} (-1)^{(n-1)\tau_{(1\ldots n)}} \circ \{\{-\ldots, -\}\}_Q \circ \tau_{(1\ldots n)}^{-1}$$

where 

$$\{a_1, \ldots, a_n\}_Q = \delta_n(a_1)'' \delta_1(a_2)'' \cdots \delta_{n-1}(a_{n-1})'$$

This maps factors through $\frac{D_B A}{[D_B A, D_B A]}$. The map $\mu$ is an isomorphism if $A$ is $B$-smooth.

In [17], the following expression is proved for $\{\{-\ldots, -\}\}_Q$.

**Proposition 3.7.** For $Q \in (A^e)_n$, the following identity holds:

$$\{a_1, \ldots, a_n\}_Q = (-1)^{\frac{n(n-1)}{2}} \{a_1, \ldots, a_n, \{Q, a_n\}\}_L \cdots \} \{a_1, \ldots, a_{n-2}\} \{Q, a_{n-1}\}$$

The isomorphism $\mu$ allows to characterize double Poisson algebras on a smooth algebra.

**Proposition 3.8.** Let $A$ be a smooth $B$-algebra. Double $B$-linear Poisson structures on $A$ are in bijection with the $P \in T^2 \text{Der}_B(A)$ such that $\{P, P\} = 0$ modulo $[D_B A, D_B A]$.

**Remark 3.9.** In the case of a double bivector field, the formula of proposition 3.8 is the double version of the formula $\{f, g\} = -[f, [P, g]]$ ([11]) for the Poisson bracket.

The double Poisson - Lichnerowicz cohomology was defined in [21], [15]:
Definition 3.10. Let $A$ be a $k$-double Poisson algebra with Poisson double bracket defined by the Poisson bivector $P \in T^2 \text{Der}_k(A)$ such that $\{ P, P \} = 0 \text{ mod } [D_k A, D_k A]$. The double Poisson-Lichnerowicz cohomology of $A$ is the cohomology of the complex $\left( \frac{D_k A}{[D_k A, D_k A]}, \{ P, - \} \right)$.

The concept of quasi-Poisson G-manifolds was introduced in [1] and Massuyeau and Turaev ([13]) gave an algebraic formulation of it. Double quasi-Poisson algebras were defined in [17]. They also give rise to $H_0$-Poisson structures.

Definition 3.11. We assume that we are in the situation where $B = ke_1 \oplus \cdots \oplus ke_n$ with $e_i^2 = e_i$.

Let $E_i : A \rightarrow A \otimes A$ defined by $E_i(a) = ae_i \otimes e_i - e_i \otimes e_i a$ and $E = \sum_{i=1}^{\infty} E_i$.

A double quasi-Poisson bracket on $A$ is a $B$-linear bracket $\{ \cdot, \cdot \}$ such that

$$\{ [-,-,-] \} = \{ [-,-,-] \}_{E^3}.$$

Proposition 3.12. ([17]) Let $(A, \{ [-,-] \})$ be a quasi-Poisson algebra. Then

1) $(A, \{ [-,-] \})$ is a left Loday algebra.

2) $(A, \{ [-,-] \})$ induces a $H_0$-Poisson structure on $A$.

Definition 3.13. $P \in (D_B A)_2$ is a differential double quasi Poisson bracket if

$$\{ P, P \} = \frac{1}{6} E^3 \text{ mod } [D_B A, D_B A]$$

Remark 3.14. If $A$ is smooth, then quasi Poisson bracket and differential quasi Poisson bracket are equivalent notions.

Examples 3.15. 1) Examples of Poisson double brackets and of quasi-Poisson double brackets over the path algebra of a double of a quiver are given in [17].

2) Let $A$ be the free associative algebra on $n$ variables, $A = k < x_1, \ldots, x_n >$. Linear double Poisson structures are studied in [15]. Define $\{ \{ x_i, x_j \} \} = b_{ij} x_i \otimes 1 - b_{ji} 1 \otimes x_j$ and extend this skew symmetric bracket to a biderivation. This linear skew symmetric bracket is a double Poisson bracket if and only if $x_i x_j = b_{ij} x_i \otimes x_j$ is an associative multiplication.

3) Quadratic double Poisson brackets on $C < X_1, X_2, \ldots, X_n >$ have been studied in [19]. Define $\{ \{ x_i, x_j \} \} = k_{ij} x_i \otimes x_j - k_{ji} x_j \otimes x_i + 1 - b_{ij} 1 \otimes x_j$ and extend this bracket to a bi-derivation. For this quadratic double bracket to be Poisson, $r$ has to satisfy the associative Yang-Baxter equation.

4) Given an oriented surface $\Sigma$ with boundary $\partial \Sigma$ and base point $* \in \partial \Sigma$, a quasi-Poisson double algebra structure is constructed on the group algebra $A = \mathbb{K}[\pi]$ of $\pi_1(\Sigma, *)$ in [13]. The Lie bracket on $\frac{A}{[A, A]}$ in this case is twice the Goldman Lie bracket.

5) Double Poisson structures on a semi-simple algebra (over an algebraically closed field) are described in [21].

4. Double Lie Algebroids

V. Roubtsov drew my attention to the fact that the definition of a double Lie algebroid is given in [18]. But it is defined by its characterization in terms of double Gerstenhaber algebras as explained in proposition 4.1 below. The definition we give is more explicit.

Definition 4.1. A double Lie algebroid is a quadruple $(L, A, \omega, \{ [-,-] \})$ where

- $L$ is an $A^e$-module.
- $\{ [-,-] \} : L \otimes L \rightarrow L \otimes A \oplus A \otimes L$
- $(D, \Delta) \mapsto [(D, \Delta)]^0 \otimes [(D, \Delta)]^0 = [(D, \Delta)]^0 \otimes [(D, \Delta)]^0 + [(D, \Delta)]^0 \otimes [(D, \Delta)]^0$ (with $[(D, \Delta)]^0, [(D, \Delta)]^0 \in L$ and $[(D, \Delta)]^0, [(D, \Delta)]^0 \in A$) is a map satisfying $[(D, \Delta)]^0 = -[(\Delta, D)]^0$.
- $\omega : L \rightarrow \text{Der}_B(A)$ is a morphism of $A^e$-modules.
- $\{ \omega(D), \omega(D) \} = \omega(\{ D, D \})$ where we also denote by $\omega$ the extension of $\omega$ to a map from $A \otimes L \oplus L \otimes A$ to $A \otimes \text{Der}_B(A) \oplus \text{Der}_B(A) \otimes A$. 

• Jacobi identity: If \( D_1, D_2, D_3 \) are elements of \( \mathbb{L} \), one has:

\[
\left\{ \left[ \left[ D_1, D_2, D_3 \right] \right] + \tau_{(123)} \right\} + \tau_{(123)} = 0
\]

where

\[
\left\{ \left[ \left[ D_1, \left[ D_2, D_3 \right] \right] \right] \right\} L := \left\{ \left[ \left[ D_1, \left[ D_2, D_3 \right] \right] \right] \right\} \otimes \left\{ \left[ D_2, D_3 \right] \right\}^
u
\]

\( \forall (D, a) \in \mathbb{L} \times A \rightarrow \left\{ D, a \right\} = \omega(D)(a) \)

• If \( (D, \Delta) \in \mathbb{L}^2 \), one has

\[
\left\{ \left[ \left[ D, a\Delta \right] \right] = D(a)\Delta + a\left\{ \left[ D, \Delta \right] \right\} \right\} \)

\[
\left\{ \left[ \left[ D, \Delta \right] \right] \right\} + \left\{ \left[ D, a \right] \right\} \Delta + a\left\{ \left[ D, \Delta \right] \right\} = 0
\]

Remarks 4.2.

(i) When there will be ambiguity, the bracket \( \left\{ \left[ \left[ D, \Delta \right] \right] \right\} \) will be denoted \( \left\{ \left[ \left[ D, \Delta \right] \right] \right\}_L \).

(ii) The Jacobi identity is equivalent to the following: Let \( D_1, D_2, D_3 \) be elements of \( \mathbb{L} \):

\[
\left\{ \left[ \left[ D_1, \left[ D_2, D_3 \right] \right] \right] + \tau_{(123)} \right\} = 0
\]

where

\[
\left\{ \left[ \left[ D_1, \left[ D_2, D_3 \right] \right] \right] \right\} L := \left\{ \left[ \left[ D_1, \left[ D_2, D_3 \right] \right] \right] \right\} \otimes \left\{ \left[ D_2, D_3 \right] \right\} \]

(iii) For differential calculus, we will assume that \( A \) is a smooth \( B \)-algebra and that \( \mathbb{L} \) is a finitely generated projective \( A^* \)-module.

Notation: We set \( \left\{ \left[ D, \Delta \right] \right\} \) for the component of \( \left\{ \left[ D, \Delta \right] \right\} \) that is \( \mathbb{L} \otimes A \) and \( \left\{ \left[ D, \Delta \right] \right\} \) for its component in \( \mathbb{L} \otimes \mathbb{L} \). Adopting a Sweedler’s type notation, we set

\[
\left\{ \left[ D, \Delta \right] \right\}_L = \left\{ \left[ D, \Delta \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \]

with \( \left\{ \left[ D, \Delta \right] \right\}_L, \left\{ \left[ D, \Delta \right] \right\}_L \in \mathbb{L} \) and \( \left\{ \left[ D, \Delta \right] \right\}_L, \left\{ \left[ D, \Delta \right] \right\}_L \in A \). One has

\[
\left\{ \left[ D, \Delta \right] \right\}_L = -\left\{ \left[ D, \Delta \right] \right\}_L \]

Lemma 4.3. Let \( D_1, D_2, D_3 \) be three elements of \( \mathbb{L} \). One has:

\[
\left\{ \left[ \left[ D_1, \left[ D_2, D_3 \right] \right] \right] \right\} \otimes \left\{ \left[ D_1, \left[ D_2, D_3 \right] \right] \right\} \otimes \left\{ \left[ D_2, D_3 \right] \right\} \]

\( \forall (D, a) \in \mathbb{L} \times A \rightarrow \left\{ D, a \right\} = \omega(D)(a) \)

Proof. The lemma follows from the Jacobi identity, taking the component on \( \mathbb{L} \otimes A \).

Proposition 4.4. Let \( (D, \Delta) \in \mathbb{L}^2 \) and \( (\alpha, \beta) \in A^2 \), one has:

\[
\left\{ \left[ \left[ D, \Delta \right] \right] \right\}_L = \left\{ \left[ \left[ D, \Delta \right] \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \]

\[
\left\{ \left[ D, \Delta \right] \right\}_L = \left\{ \left[ D, \Delta \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \]

\[
\left\{ \left[ D, \Delta \right] \right\}_L = \left\{ \left[ D, \Delta \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \]

\[
\left\{ \left[ D, \Delta \right] \right\}_L = \left\{ \left[ D, \Delta \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \]

\[
\left\{ \left[ D, \Delta \right] \right\}_L = \left\{ \left[ D, \Delta \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \]

\[
\left\{ \left[ D, \Delta \right] \right\}_L = \left\{ \left[ D, \Delta \right] \right\}_L \otimes \left\{ \left[ D, \Delta \right] \right\}_L \]

Proof. The proof follows from the properties of the double Lie algebroid bracket and easy computations.

Proposition 4.5. Let \( (\mathbb{L}, \Delta, \omega, \{., .\}_L) \) be a quadruple such that \( \{., .\}_L \) is a map from \( \mathbb{L} \otimes \mathbb{L} \) to \( A \otimes \mathbb{L} \) and \( \omega \) a map from \( \mathbb{L} \) to \( \text{Der}(A) \). There is a unique graded bilinear map \( \{., .\}_L : \text{T}_A(\mathbb{L}) \otimes \text{T}_A(\mathbb{L}) \rightarrow \text{T}_A(\mathbb{L}) \) of degree \(-1\) satisfying the following conditions:

• For all \( (\alpha, \beta, D, \Delta) \in A^2 \otimes L^2 \),

\[
\{\alpha, \beta\} = 0
\]

\[
\{D, \alpha\} = \omega(D)(\alpha) \otimes \omega(D)(\alpha)
\]

\[
\{D, \Delta\} = \{D, \Delta\}_L
\]
• for all \((a, b, c) \in T_A(\mathbb{L})\)

\[
\{\{a, be\}\} = (-1)^{(|a|-1)|b|} b\{\{a, c\}\} + \{\{a, b\}\} c
\]

\[
\{\{a, b\}\} = -(-1)^{(|a|-1)(|b|-1)} \sigma_{(12)} \{\{b, a\}\}
\]

\((\mathbb{L}, A, \omega)\) is a double Lie–Rinehart algebra over \(A\) if and only if \(T_A(\mathbb{L})\) is a double Gerstenhaber algebra.

**Proof.** Adopting a Sweedler type notation for \(\{\{,\}\}\), one has for any \(a_1, \ldots, a_m, b_1, \ldots, b_n \in T_A(\mathbb{L})\):

\[
\{a_1 \cdots a_m, b_1 \cdots b_n\} = \sum_{p,q} b_1 \cdots b_{q-1} \{a_p, b_q\} a_{p+1} \cdots a_m \otimes a_1 \cdots a_{p-1} \{a_p, b_q\}'' b_{q+1} \cdots b_n
\]

One sets

\[
\{\{a, b, c\}\} = \{\{a, \{b, c\}\}\}_L + (-1)^{|a|-1(|b|+|c|)} \sigma_{(123)} \{\{b, \{c, a\}\}\}_L + (-1)^{|c|-1(|a|+|b|)} \sigma_{(123)}^2 \{\{c, \{a, b\}\}\}_L.
\]

\(\{[-, -, -]\}\) is a triple bracket. We want to show that it is zero. We need to check it on generators.

**First case:** If two of the \(a, b, c\) are in \(A\) then all the terms are zero.

**Second case:** If all the \(a, b, c\) are in \(\mathbb{L}\), it is 0 by hypothesis.

**Third case:** We check Jacobi identity in the case of a triple \((\alpha, D, \Delta)\) in \(A \times \mathbb{L} \times \mathbb{L}\). We need to show the equality:

\[
\{\{\alpha, \{D, \Delta\}\}\}_L + \sigma_{(123)} \{\{D, \{\Delta, \alpha\}\}\}_L + \sigma_{(123)}^2 \{\{\Delta, \{\alpha, D\}\}\}_L = 0
\]

Before computing each of these terms, let us remark that

\[
\omega(\{\{D, \Delta\}\}_L) = \{\{\omega(D), \omega(\Delta)\}\}_L.
\]

This equality can be written more simply as follows:

\[
\omega(\{\{D, \Delta\\}'_L \otimes \{\{D, \Delta\\}\}'_L) = \{\{\omega(D), \omega(\Delta)\\}'_L \otimes \{\{\omega(D), \omega(\Delta)\\}'_L
\]

one has

\[
\{\{\alpha, \{D, \Delta\}\}\}_L = \{\{\alpha, \{\{D, \Delta\\}'_L \otimes \{\{D, \Delta\\}'_L\}_L = -\tau_{(12)} \omega(\{\{D, \Delta\\}'_L \otimes \{\{D, \Delta\\}'_L\}_L
\]

\[
= -\tau_{(12)} \omega(\{\{D, \Delta\\}'_L \otimes \{\{D, \Delta\\}'_L\}_L
\]

\[
= -\tau_{(12)} \omega(\{\{D, \Delta\\}'_L \otimes \{\{D, \Delta\\}'_L\}_L
\]

\[
\sigma_{(123)} \{\{D, \{\Delta, \alpha\}\}\}_L = \sigma_{(123)} \{\{\omega(D), \omega(\Delta)\}'_L \otimes \{\{\omega(D), \omega(\Delta)\}'_L\}_L = \tau_{(123)} (\{\omega(D) \otimes 1\} \omega(\Delta) ) (\alpha)
\]

\[
\sigma_{(123)} \{\{\Delta, \{\alpha, D\}\}\}_L = -\tau_{(123)} \{\Delta, \{\omega(\Delta), \omega(D)\}'_L \otimes \{\{\omega(\Delta), \omega(D)\}'_L\}_L
\]

In the next proposition, we study the case where the anchor \(\omega\) is zero.

**Proposition 4.6.** Let \((\mathbb{L}, [-, -, -], \omega)\) be a double Lie–Rinehart algebra over \(A\) such that \(\omega = 0\).

Set \(X \bullet Y = -\{X, Y\}_r = -\{\{X, Y\}'_L, \{X, Y\}'_L\}_r\) and \(X \bullet Y = \{X, Y\}_l = \{\{X, Y\}'_L, \{X, Y\}'_L\}_r\).

The laws \(\bullet_r\) and \(\bullet_l\) are associative so that \([- , -] : \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}\) is the difference of two associative products. The laws induces by \(\bullet_r\) and \(\bullet_l\) in \(\mathbb{L} / [A, \mathbb{L}]\) are opposite from each other. The bracket \([-, -]\) induces a Lie bracket on \(\mathbb{L} / [A, \mathbb{L}]\) that comes from an associative product.

**Proof.** The associativity of \(\bullet_l\) follows from lemma \([33]\) and proposition \([34]\). The proof of the associativity of \(\bullet_r\) is similar. Let \(X\) and \(Y\) in \(\mathbb{L}\).

\[
X \bullet Y = -\{\{X, Y\}'_L, \{X, Y\}'_L\}_r = \{\{X, Y\}'_L, \{X, Y\}'_L\}_r = \{\{X, Y\}'_L, \{X, Y\}'_L\}_r \text{ mod } [A, \mathbb{L}]
\]

\(Y \bullet_l X \text{ mod } [A, \mathbb{L}]\).
Let us now give several examples of double Lie algebroids.

**Example 1:**
\( (\mathcal{D}er(A), A, id) \) is a double Lie algebroid.

**Example 2:** If \( A = k \), double Lie algebroid structures on \( \mathbb{L} \) over \( k \) are in bijection with associative \( k \)-algebras structures over \( \mathbb{L} \). Indeed, If \( \mathbb{L} \) is a double Lie algebroid over \( k \), then \( \{\{\cdot, \cdot\}\}_t \) is an associative product. Conversely, an associative product over \( k \) gives rise to a double Lie–Rinehart algebra over \( k \) as follows: \( \{\{X, Y\}\}_t = XY \otimes 1, \{\{X, Y\}\}_r = -1 \otimes XY \) and \( \{\{X, Y\}\} = XY \otimes 1 - 1 \otimes YX \). The bracket \( \{\{\cdot, \cdot\}\} \) identifies to the Lie bracket coming from the associative structure.

**Exemple 3:**
Let us recall the definition of a double Lie algebra:

**Definition 4.7.** Let \( \mathfrak{g} \) be a vector space of finite dimension. A double Lie bracket over \( \mathfrak{g} \) is a map \( \{\{\cdot, \cdot\}\} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \) satisfying the Jacobi identity.

**Remark 4.8.** Let \( V \) be a \( m \) dimensional \( \mathbb{C} \)-vector space. Double Lie algebra structures on \( V \) are in bijection (\([2]\)) with operators \( r \in \text{End}(V \otimes V) \) such that \( r(v \otimes u) = -r(u \otimes v) \) (skew-symmetry) and solution of the classical Yang Baxter equation
\[
r^{23}_{12} + r^{31}_{23} + r^{12}_{31} = 0
\]
where \( r^{ij} \) acts on \( V^\otimes 3 \) non trivially on \( (i,j) \) spaces and as identity elsewhere.

Let \( \mathfrak{g} \) be a double Lie algebra with double bracket \( \{\{\cdot, \cdot\}\}_g \). This latter induces a Poisson bracket on \( T(\mathfrak{g}) \). There is a unique double Lie algebroid structure on \( L = T(\mathfrak{g}) \otimes \mathfrak{g} \otimes T(\mathfrak{g}) \) such that
\[
\forall (X,Y) \in \mathfrak{g}, \quad \forall a \in T(\mathfrak{g}),
\{\{X,Y\}\}_L = \{\{X,Y\}\}_g
\omega(X)(a) = \{\{X,a\}\}_T(\mathfrak{g})
\]
Then \( T(\mathfrak{g}) \otimes \mathfrak{g} \otimes T(\mathfrak{g}) \) is a Lie double algebroid over \( T(\mathfrak{g}) \).

**Example 4:** Let \( V \) be an \( A^c \)-module of finite type. For \( \lambda, \mu \) in \( \text{Hom}_{A^c}(V, V \otimes V) \) (the exterior \( A^c \)-module structure on \( V \otimes V \) is used for the \( \text{Hom}_{A^c} \)), one sets:
\[
\{\{\lambda, \mu\}\}^\sim = (\lambda \otimes 1)\mu - (1 \otimes \mu)\lambda
\{\{\lambda, \mu\}\}_t = (1 \otimes \lambda)\mu - (\mu \otimes 1)\lambda
\]
One sets
\[
\{\{\lambda, \mu\}\}_l = \tau_{23}\{\{\lambda, \mu\}\}^\sim
\{\{\lambda, \mu\}\}_r = \tau_{12}\{\{\lambda, \mu\}\}^\sim
\{\{\lambda, \mu\}\}_t = \{\{\lambda, \mu\}\}_l + \{\{\lambda, \mu\}\}_r
\]
Then
\[
\{\{\lambda, \mu\}\} \in \text{Hom}_{A^c}(V, V \otimes V) \otimes_k V + V \otimes_k \text{Hom}_{A^c}(V, V \otimes V)
\]
Then
\[
\{\{\lambda, \mu\}\} \in \text{Hom}_{A^c}(V, V \otimes V) \otimes_k T_A(V) \oplus T_A(V) \otimes_k \text{Hom}_{A^c}(V, V \otimes V)
\]
There is a unique double Lie algebroid structure on \( T_A(V) \otimes_k \text{Hom}(V, V \otimes V) \otimes_k T_A(V) \) with anchor map
\[
\omega : T_A(V) \otimes_k \text{Hom}(V, V \otimes V) \otimes_k T_A(V) \to \mathcal{D}er(T(V))
\alpha \odot \lambda \odot \beta \implies \alpha \{\{\lambda, \cdot\}\}_t \beta
\]
where \( \{\{\lambda, \cdot\}\} \) is the unique double derivation of \( T_A(V) \) such that for all \( v \in V \), \( \{\{\lambda, v\}\} = \lambda(v) \) and, for all \( a \in A \), \( \{\{\lambda, a\}\} = 0 \).

**Example 5:**
Let \( A \) be a \( B \)-algebra endowed with a \( B \)-linear double Poisson bracket. The quadruple \( (\Omega^1_B A, A, \omega, \{\cdot, \cdot\}) \) is a double Lie–Rinehart algebra with \( \omega \) and \( \{\cdot, \cdot\} \) defined as follows:
\[ \Omega_1 \circ A \rightarrow \text{Der}_B(A) \]
\[ da \mapsto \{(a, -)\} \]
\[ \{(da, db)\} = d\{(a, b)\}' \otimes \{(a, b)\}'' + \{(a, b)\}' \otimes d\{(a, b)\}'' \]

**Example 6:** (15) Let \((A, P)\) a double quasi-Poisson algebra and let \(E\) be the double derivation of \(A\) defined by \(E(a) = a \otimes 1 - 1 \otimes a\). Then \(\tilde{\Omega}_A = \Omega_A \otimes AEA\) has the structure of a double Lie algebroid where the double bracket is defined as follows
\[ \{(da, db)\} = d\{(a, b)\}' \otimes \{(a, b)\}'' \]
\[ \{(da, db)\}_{\tilde{\Omega}_A} = \{(a, b)\} + \frac{1}{4} [b, [a, E \otimes 1 - 1 \otimes E]] \]
\[ \{(E, X)\}_{\tilde{\Omega}_A} = X \otimes 1 - 1 \otimes X \]
for \(a, b \in A, X \in \mathcal{T}_A,\tilde{\Omega}_A\) where \([-,-]\) denotes the commutator for the inner \(A\)-bimodule structure on \(AEA \otimes AEA\). Futhermore the anchor is the \(A^r\)-bimodule morphism defined by:
\[ \Omega_A \otimes AEA \rightarrow \text{Der}(A) \]
\[ (du, \delta) \rightarrow \{(u, -)\} + \delta \]

is surjective.

### 5. Differential calculus

\(A \otimes A\) is endowed with a \(A^r \otimes A^r\)-module structure. Let \(M\) be an \(A^r\)-module. We choose to set
\[ M^* = \{ \lambda : M \rightarrow A \otimes A \mid \lambda(\alpha \cdot D \cdot \beta) = \lambda(D)' \otimes \alpha \lambda(D)'' \} \]
\(M^*\) is itself an \(A^r\)-module as follows:
\[ \forall \lambda \in M^*, \quad \forall (a, b) \in A^2, \quad \forall D \in M, \quad (a \cdot \lambda \cdot b)(D) = a \lambda(D)' \otimes \lambda(D)'' b. \]

**Remark 5.1.** We can exchange the role of the two \(A^r\)-module structures on \(A \otimes A\) and define
\[ M_\ast = \{ \lambda : M \rightarrow A \otimes A \mid \lambda(\alpha \cdot D \cdot \beta) = \alpha \lambda(D)' \otimes \lambda(D)'' \beta \}. \]

Composition by \(\tau_{(12)}\) is an isomorphism of \(A^r\)-modules from \(M^*\) to \(M_\ast\).

Let \(M\) be an \(A^r\)-module. Let us endow \(A^{\otimes n+1}\) with the \((A^r)^{\otimes n}\)-module structure where the ith copy of \(A^r\) acts as follows:
\[ a' \otimes a'' \cdot (a_1 \otimes \cdots \otimes a_{n+1}) = a_1 \otimes \cdots \otimes a_n a'' \otimes a_{n+1} \otimes \cdots \otimes a_{n+1}, \]
then the map (17)
\[ \Psi : M^* \otimes A^n \rightarrow \text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n+1}) \]
\[ \Psi(\lambda_1 \otimes \lambda_2 \otimes \cdots \lambda_n)(m_1 \otimes \cdots \otimes m_n) = \lambda_1(m_1) \otimes \lambda_1(m_1)'' \lambda_2(m_2) \otimes \cdots \otimes \lambda_n(m_n)'' \]
is well defined.

If \(M\) is a finitely generated \(A^r\)-projective module, then \(\Psi\) is an isomorphism of \(A^r\)-modules.

The cyclic group \(C_n\) acts on \(\text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n})\) as follows:
\[ \forall \omega \in \text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n}), \quad \tau_{(1 \ldots n)} \cdot \omega = \tau_{(1 \ldots n)} \circ \omega \circ \tau_{(1 \ldots n)}^{-1} \]
The set of signed invariants of \(\text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n})\) under the action of \(C_n\) is
\[ s - \text{inv} \text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n}) := \{ \omega \in \text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n}) \mid \tau_{(1 \ldots n)} \cdot \omega = (\omega) \} \]
If \(M\) is a finitely generated projective \(A^r\)-module, M. van den Bergh showed (17) that \(T^*(M^*)\) is isomorphic to \(s - \text{inv} \text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n})\). He constructed the following isomorphism \(\mu\) between the two spaces:

\[ \mu : T^*(M^*) \rightarrow s - \text{inv} \text{Hom}_{(A^r)^{\otimes n}}(M^{\otimes n}, A^{\otimes n}) \]

\[ \text{our map is slightly different from that of M. Van den Bergh due to different conventions for the dual of L} \]

makes use of \(M_\ast\), we make use of \(M^*\).
\[ \mu : \frac{T^\ast(M^\ast)}{[T^\ast(M^\ast), T^\ast(M^\ast)']} \rightarrow s - \text{inv} \operatorname{Hom}_{A^\ast} (M^\otimes n, A^\otimes n) \]

\[ \mu(\lambda_1 \otimes \cdots \otimes \lambda_n) = \{[-, \ldots, -]\} \lambda_1 \ldots \lambda_n = \sum_i (-1)^{(n-1)\delta_{1(i-1)} n} \circ \Phi \circ \tau_{(1) n}^i \]

with
\[ \Phi(\lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n)(m_1 \otimes \cdots \otimes m_n) = \lambda_n(m_n)'' \lambda_1(m_1)' \otimes \lambda_1(m_1)' \lambda_2(m_2)' \otimes \cdots \lambda_{n-1}(m_{n-1})'' \lambda_n(m_n)' . \]

**Notation**: In the computation, \( \Phi(\lambda_1 \otimes \cdots \otimes \lambda_n) \) will be denoted \( \lambda_1 \otimes \cdots \otimes \lambda_n \).

Let \( C \) be a graded algebra and let
\[ \Theta : C \rightarrow C \otimes C \]
\[ c \mapsto \Theta(c)' \otimes \Theta(c)'' \]
be a double derivation. One sets
\[ \circ \Theta : C \rightarrow C \]
\[ c \mapsto \Theta(c)'' \Theta(c)' . \]

\( \circ \Theta \) is an endomorphism of \( C \) and induces an endomorphism of \( \frac{C}{[C; C]} \). We will be mostly interested in the case where \( C = T_A(\mathbb{L}^\ast) \) or \( C = T_A(\mathbb{L}) \) where \( \mathbb{L} \) is a double Lie–Rinehart algebra.

**Proposition 5.2.** 1) Let \( d : C \rightarrow C \) be a derivation of degree \( |d| \). It induces a derivation still denoted \( d : C \otimes C \rightarrow C \otimes C \) :
\[ \forall c_1 \otimes c_2 \in C \otimes C , \quad d(c_1 \otimes c_2) = d(c_1) \otimes c_2 + (-1)^{|c_1| c_1} \circ d(c_2) \]
Let \( \tilde{d} : C \rightarrow C \otimes C \) be a double derivation. If \( c \in C \), we will write \( \tilde{d}(c) = \tilde{d}(c)' \otimes \tilde{d}(c)'' \) and we set \( \iota = \circ \tilde{d} \). Then \( \circ(d \circ \iota) = d \circ \iota \) and \( \circ(\iota \circ d) = \iota \circ d \).

2) (17) Let \( \delta : C \rightarrow C \otimes C \) and \( \Delta : C \rightarrow C \otimes C \) be two double derivations.
\[ \delta \circ \delta = \tau_{12} \Delta \circ \delta = \circ \iota \{\delta, \Delta\} + \circ \tau \{\delta, \Delta\} \]
where \( \circ \iota (\epsilon' \otimes \epsilon'') = \circ \iota \epsilon' \otimes \epsilon'' \) and \( \circ \tau (\epsilon' \otimes \epsilon'') = \epsilon' \otimes \circ \iota \epsilon'' \)

**Proof.** 1) \( (d \circ \iota)(c) = d((\tilde{d}(c)')' \otimes \tilde{d}(c)''') + (-1)^{|\tilde{d}(c)|} |\tilde{d}(c)'| \tilde{d}(c)'' \circ d(\tilde{d}(c)''') . \)
Hence
\[ \circ(d \circ \iota)(c) = \tilde{d}(c)'' d(\tilde{d}(c)')(1 + |\tilde{d}(c)'| + |\tilde{d}(c)''|) + (d(\tilde{d}(c)''')) \tilde{d}(c)'(1 + |\tilde{d}(c)'| + |\tilde{d}(c)'''|) \]
\[ = d \left[ \tilde{d}(c)'' d(\tilde{d}(c)')'(1 + |\tilde{d}(c)'| + |\tilde{d}(c)'''|) + (d(\tilde{d}(c)''')) \tilde{d}(c)'(1 + |\tilde{d}(c)'| + |\tilde{d}(c)'''|) \right] \]
\[ = (d \circ \iota)(c) \]
The equality \( \circ(d \circ \iota) = \iota \circ d \) is obvious.
2) is stated in [17].

Let us now see examples of this situation.

**The contraction** (similar to [17])
Set \( \mathbb{L}^\ast = \operatorname{Hom}_{A^\ast} (L, A \otimes A) \) and let \( D \in \mathbb{L} \). The element \( D \) defines a degree -1 double derivation \( \tilde{d}_D : T_A(\mathbb{L}^\ast) \rightarrow T_A(\mathbb{L}^\ast) \otimes T_A(\mathbb{L}^\ast) \)
\[ \forall c \in \mathbb{L}^\ast , \quad \tilde{d}_D(c) = \alpha(D)' \otimes \alpha(D)''' . \]
More explicitly, the map \( \tilde{d}_D : T^\ast(\mathbb{L}^\ast) \rightarrow T^\ast(\mathbb{L}^\ast) \otimes T^\ast(\mathbb{L}^\ast) \) is given by
\[ \tilde{d}_D(a_1 \ldots a_n) = \sum_{i=1}^n (-1)^{i-k} a_1 \ldots a_k(D)' \otimes \alpha_k(D)'' \alpha_{k+1} \ldots a_n \]

**Lemma 5.3.** (17) For all \( \Phi, \Theta \) in \( \mathbb{L} \), one has :
\[ \iota_{\Phi} \circ \iota_{\Theta} = \iota_{\Phi \circ \Theta} = \iota_{\Phi} \circ \iota_{\Theta} = 0 \]
From now on, we will assume that $L$ is a finitely generated and projective $A^r$-module.

The Karoubi-de Rham complex was defined in [9]. It corresponds to the case where $L = \mathbb{D}er(A)$. In this case, differential calculus was treated in [7] (differential, Lie derivative etc...) but the formulas don’t adjust to any double Lie algebroid. Even if $L \cong \mathbb{D}er(A)$, some of our formulas are different but, most of the time, we make use of the hypothesis $A$ smooth.

The differential
For a general double Lie algebroid, the definition of the differential is more complicated than in the case where $L = \mathbb{D}er(A)$.

**Theorem 5.4.** One defines $d_{\triangle}: T_A(L^*) \to T_A(L^*)$ as the degree one derivation determined by:

- For $D \in L$, $d_{\triangle}(D) = D(a)\otimes D(a)''$
- For $\lambda \in L^*$, $d_{\triangle}(\lambda) \in \mathbb{L}^* \otimes_A L^* \cong \text{Hom}_A(L \otimes \Lambda, A^\otimes 3)$ is given by for all $D, \Delta \in L$:

$\quad \quad d_{\triangle}(\lambda)(D, \Delta) = D(\lambda(D)) \otimes \lambda(\Delta)'' - \lambda(\Delta) \otimes D(\lambda(D)'') - \tau_{23}([\lambda([D, \Delta])]_1) \otimes \{D, \Delta]\}_3''$

One can easily compute the general formula for $d_{\triangle}$: $\forall \psi \in \text{Hom}_A(\mathbb{L}^{\otimes n}, A^\otimes n+1)$

\[
d_{\triangle}(\psi)(D_1, \ldots, D_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i-1} (id^{n-i} \circ d_1 \circ id^{n-i}) \psi(D_1, \ldots, \widehat{D_i}, \ldots, D_{n+1})
\]

where if $\psi = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_n \in \mathbb{L}^{\otimes n \otimes A}$ with $\psi_i : L \to A^\otimes 2$, then $\psi_i([D_1, D_i+1]) = \tau_{23} (\psi_i([D_1, D_i+1]) \otimes \{D_1, D_i+1\})_3''$

But $d_{\triangle}$ is a derivation whose square is zero (that is to say a differential).

**Proof:** In the proof, we will make use of the following notation: If $\lambda \in L^*$, then $d_{\triangle}(\lambda(1)) \otimes \cdots \otimes d_{\triangle}(\lambda(n+1))$ where $d_{\triangle}(\lambda(1))$ takes values in $A$.

If $D_1, D_2, D_3$ are three elements of $L$, one has:

\[
(d_{\triangle} \circ d_{\triangle})(\lambda)(D_1, D_2, D_3) = D_1 [D_2 \lambda(D_3)']' \otimes D_2 (\lambda(D_3)')'' \otimes \lambda(D_3)' - D_1 (\lambda(D_2)')' \otimes D_3 (\lambda(D_2)')''
\]

But

\[
-\frac{1}{12} \sum_{i=1}^{n+1} (-1)^{i-1} (id^{n-i} \circ d_1 \circ id^{n-i}) \psi(D_1, \ldots, \widehat{D_i}, \ldots, D_{n+1})
\]

A lot of terms cancel, and we are left with

\[
\frac{1}{12} \psi(D_1, D_2) \otimes \psi(D_3) = \lambda([D_1, D_3])' \otimes D_2 (\lambda(D_3)')'' \otimes \lambda([D_1, D_3])_3''
\]

\[
+ \lambda([D_1, D_2])' \otimes \psi(D_3) = \lambda([D_1, D_3])'' \otimes \psi(D_3)
\]

\[
- \lambda([D_1, D_3])'' \otimes \psi(D_3)
\]

\[
- \psi(D_1, D_2) \otimes \psi(D_3) = \frac{1}{12} \psi(D_1, D_3) = \lambda([D_1, D_3])'' \otimes \psi(D_3)
\]

\[
+ \lambda([D_1, D_2])' \otimes \psi(D_3) = \lambda([D_1, D_3])'' \otimes \psi(D_3)
\]

\[
- \lambda([D_1, D_3])'' \otimes \psi(D_3)
\]

A lot of terms cancel, and we are left with

\[
\frac{1}{12} \psi(D_1, D_2) \otimes \psi(D_3) = \lambda([D_1, D_3])'' \otimes D_2 (\lambda(D_3)')'' \otimes \lambda([D_1, D_3])_3''
\]

\[
+ \lambda([D_1, D_2])' \otimes D_3 (\lambda(D_3)')'' \otimes \lambda([D_1, D_3])_3''
\]

\[
- \lambda([D_1, D_3])'' \otimes \psi(D_3)
\]

A lot of terms cancel, and we are left with

\[
\frac{1}{12} \psi(D_1, D_2) \otimes \psi(D_3) = \lambda([D_1, D_3])'' \otimes D_2 (\lambda(D_3)')'' \otimes \lambda([D_1, D_3])_3''
\]

\[
+ \lambda([D_1, D_2])' \otimes D_3 (\lambda(D_3)')'' \otimes \lambda([D_1, D_3])_3''
\]

\[
- \lambda([D_1, D_3])'' \otimes \psi(D_3)
\]
The equality \(d_L \circ d_L(\lambda)(D_1, D_2, D_3) = 0\) follows now from the lemma [3].

**Remark 5.5.** In [7], it is shown that for \(L = \mathcal{D}er(A)\) and \(A\) smooth, the complex \((T^*_A(L), d_L)\) is acyclic in strictly positive degree.

**Definition 5.6.** The differential \(d_L\) induces a differential

\[
d_L : \frac{T_A(L^*)}{[T_A(L^*), T_A(L^*)]} \to \frac{T_A(L^*)}{[T_A(L^*), T_A(L^*)]}
\]

We set \(DR(L)^* := \frac{T_A(L^*)}{[T_A(L^*), T_A(L^*)]}\) and \(d_L\) will be called "the differential of the double Lie algebroid" \(L\).

**Remarks 5.7.** 1) If \(L = \mathcal{D}er_k(A)\) and \(A\) smooth, the complex \((DR^*(L), d_L)\) is the Karoubi–de Rham complex ([2], [7]).

2) If \(A\) is a smooth double Poisson algebra and \(L = \Omega^1(A)\), we will see that the complex \((DR^*(L), d_L)\) is the complex computing the non commutative Poisson cohomology ([15]).

In low degree, the expression of \(d_L\) is the following: For all \(a \in A\), \(\phi \in L\) and \(\tilde{\phi} \in s - \text{inv} \text{Hom}(A, L, A)\)

\[
d_L(a)(D) = D(a)^\nu D(a)^\nu D(a)^\nu \Delta - \tilde{\phi} \Delta (\tilde{\phi}(D)) - \tilde{\phi}[\{\{D, \Delta\}^\nu, \{D, \Delta\}^\nu\}] - \{\{D, \Delta\}^\nu, \tilde{\phi}(\{\{D, \Delta\}^\nu\})
\]

We now give an expression for \(d_L\) in any degree.

**Theorem 5.8.** Let \(\mu_1, \ldots, \mu_n \in \mathbb{R}^*\). We have the following formula where indices should be understood modulo \(n + 1\):

\[
d_L (\{\{-, \ldots, -\}\}_{\mu_1, \ldots, \mu_n})(D_1 \otimes \cdots \otimes D_{n+1}) = \{L\} + \{LL\}
\]

where \(\{L\} = \mu_n(D_{n+1})^{\nu} \mu_1(D_1)^{\nu} \mu_2(D_2)^{\nu} \cdots \otimes \mu_{n-1}(D_{n-1})^{\nu} \mu_n(D_{n+1})^{\nu} + \cdots + \sum_{i=1}^{n}(-1)^{(n-1)(i-1)}\mu_i(D_i)^{\nu} \cdots \otimes \mu_{i-1}(D_{i-1})^{\nu} \mu_1(D_1)^{\nu} \cdots \otimes \mu_n(D_{n+1})^{\nu}
\]

\[
\{LL\} = \sum_{i=1}^{n}(-1)^i \mu_n(D_{n+1}) \nu \mu_1(D_1)^{\nu} \cdots \mu_2(D_2)^{\nu} \cdots \otimes \mu_{i-1}(D_{i-1}) \nu \mu_1(D_1)^{\nu} \cdots \mu_{n-1}(D_{n-1}) \nu \mu_n(D_{n+1})^{\nu}
\]

and \(\{\{\cdot, \ldots, \cdot\}\}_{\mu_1, \ldots, \mu_n} = \sum_{i=1}^{n}(-1)^{(n-1)(i-1)} \mu_n(\{\{D_1, D_i\}^\nu, \{D_i, D_{i+1}\}^\nu\}) \nu \mu_1(\{\{D_1, D_i\}^\nu, \{D_i, D_{i+1}\}^\nu\}) \nu \mu_2(\{\{D_1, D_i\}^\nu, \{D_i, D_{i+1}\}^\nu\}) \nu \cdots \otimes \mu_{n-1}(D_{n-1}) \nu \mu_n(D_{n+1})^{\nu} + \cdots + \sum_{i=2}^{n}(-1)^{(n-1)(i-1)} \mu_n(D_{n+1}) \nu \mu_1(D_1)^{\nu} \cdots \otimes \mu_{i-1}(D_{i-1}) \nu \mu_1(D_1)^{\nu} \cdots \otimes \mu_n(D_{n+1})^{\nu}
\]

For simplicity, we write \(\sigma := \tau_{(1 \ldots n)}\) and \(\tau := \tau_{(1 \ldots n+1)}\). Then

\[
\{\{-, \ldots, -\}\}_{\mu_1, \ldots, \mu_n} = \sum_{i=1}^{n}(-1)^{(n-1)(i-1)} \mu_n(\{\{D_1, D_i\}^\nu, \cdots \otimes \{D_1, D_{i+1}\}^\nu\}) \nu \mu_1(\{\{D_1, D_i\}^\nu, \cdots \otimes \{D_1, D_{i+1}\}^\nu\}) \nu \cdots \otimes \mu_{n-1}(D_{n-1}) \nu \mu_n(D_{n+1})^{\nu}.
\]

Using the formulas of \(L\) and \(LL\) one writes \(d_L((\{-, \ldots, -\}_{\mu_1, \ldots, \mu_n})(D_1 \otimes \cdots \otimes D_{n+1}) = (I) + (II)\).

In the computation of \((I)\), the terms of the form \(D_1(\cdot) \otimes \ldots \otimes \cdot\) give \(\{\{D_1, \{D_2, \ldots, D_{n+1}\}\}_{\mu_1, \ldots, \mu_n}\}L\).
More precisely:

\[
(I) \quad D_1 \sum_{n=1}^{\infty} \mu_n (D_n, D_{n+1})^\prime \mu_2 (D_2) \mu_2 (D_2) \cdots \mu_{n-1} (D_{n-1}) \mu_n (D_n, D_{n+1}) + \ldots \\
+ \sum_{i=1}^{n} \sum_{j=1}^{i-1} (-1)^{(i-1)(j-1)} (i-1)^{\mu_{i-1}(m_{i-1}(D_i L), D_l \in D_{r-1}(1))}
\]

= \left\{ D_1, \{D_2, \ldots, D_n\} \right\} L + \ldots \\
+ \sum_{n=1}^{k} (-1)^{n-1} \tau_1 \left\{ D_1, \left\{ D_{n+1}, \ldots, D_{n+1} \right\} \right\} L + \ldots \\
= \sum_{n=1}^{k} (-1)^{n-1} \tau_1 \left\{ D_1, \left\{ D_{n+1}, \ldots, D_{n+1} \right\} \right\} L + \ldots
\]

where the notation \( \mu_k \leftarrow \mu_{i-1}(m_{i-1}(D_i L), D_l \in D_{r-1}(1)) \) means that we reproduce the previous term, replacing \( \mu_k \) by \( \mu_{i-1}(m_{i-1}(D_i L), D_l \in D_{r-1}(1)) \).

When computing \((II)\), the terms finishing by \( \{D_n, D_{n+1}\} \) give

\[
(-1)^n \left\{ D_1, \ldots, D_{n-1}, \{D_n, D_{n+1}\} \right\} \mu_1 \cdots \mu_n \times \{D_n, D_{n+1}\}
\]

More precisely:

\[
(II) = \sum_{n=1}^{k} (-1)^{n} \mu_n \left\{ \{D_1, \ldots, D_{n+1}\} \right\} \mu_1 (D_1) \mu_2 (D_2) \cdots \mu_{n-1} (D_{n-1}) \mu_n (\{D_n, D_{n+1}\}) \times \{D_n, D_{n+1}\}
\]

\[
+ \sum_{i=1}^{k} \sum_{j=1}^{i-1} (-1)^{(i-1)(j-1)} (i-1)^{\mu_{i-1}(m_{i-1}(D_i L), D_l \in D_{r-1}(1))}
\]

\[
= (-1)^n \left\{ D_1, \ldots, D_{n-1}, \{D_n, D_{n+1}\} \right\} \mu_1 \cdots \mu_n + \ldots + (-1)^{2n-1} (D_k \leftarrow D_{r+1}(D_k))
\]

\[
= \sum_{n=1}^{k} (-1)^{n-1} \tau_1 \left\{ D_1, \ldots, D_{n+1} \right\} \left\{ D_{n+1}, D_{n+2}, \ldots, \{D_{n+1}, D_{n+1}\} \right\} \mu_1 \cdots \mu_n L
\]

\[\square\]

**Remark 5.9.** In the case where \( A = k, L = B \) is a finite dimensional algebra and \( \{a, b\} = ab - ba \), it is easy to see (using theorem 5.8) that \( T^* (L^*) \) is isomorphic to the cochains of the cyclic cohomology

\[ C^n \bigl( \{ f : B^\otimes n \to k, f \circ \tau^{-1}_{(1, n)} = (-1)^{n-1} f \} \) and \( d_L \) is the differential of the Hochschild complex. Thus the complex \( \left( T^* (L^*), \frac{\text{Der}_k (A)}{(1, k)}, d_L \right) \) computes the cyclic cohomology of \( B \).

We will now study more in detail the case where \( A \) is a smooth double Poisson algebra with double bracket defined by the double biderivation \( P = \delta \Delta \) and \( L = \Omega^1_A \). Note that \( \Omega^1_A = \text{Der}_k (A)^* \) and \( \text{Der}_k (A) = \Omega^1_A^* \) so that if \( D \) is in \( \text{Der}_k (A) \), then \( \sigma D \in \Omega^1_A^* \).

**Proposition 5.10.** 1) If \( a, f \in A \), then \( d_L (a)(df) = \{ f, [P, a] \} \).

2) If \( D_1, \ldots, D_n \) are in \( L \), then

\[ d_L (\sigma D_1, \ldots, \sigma D_n) (da_1 \otimes \cdots \otimes da_{n+1}) = \{ a_1, \{a_2, \ldots, \{a_{n+1}, \{P, D_1 \ldots D_n\}\} \} \}
\]

**Proof.** Let us prove that \( d_L (a)(df) = \{ f, \{\delta \Delta, a\} \} \). With our definition, we get

\[ d_L (a)(df) = \{ f, [\delta] \} \otimes \{ f, [\delta] \} \otimes \Delta (\delta(f)) \otimes \Delta (\delta(f)) \otimes \Delta (\delta(f)) - \delta(f) \Delta (\delta(f)) \otimes \Delta (\delta(f)) \]

On the other hand:

\[ \{ \delta \Delta, a \} = \{ a, \delta \Delta \} + \{ a, \delta \Delta \} + \{ a, \delta \} \Delta (\delta(a)) - \delta(a) \Delta (\delta(a)) \]

Hence \( \{ \delta \Delta, a \} (df) = \delta(f) \delta(a) \Delta (\delta(a)) + \delta(a) \Delta (\delta(a)) \Delta (\delta(a)) \neq 0 \).

The proof of 2) is a consequence of the following lemmas.
Lemma 5.11. \( d_L(\sigma D)(da \otimes db) = -\{\{a, \{b, \{P, D\}\}\}\}_L \)

Proof.
\[
d_L(\sigma D)(da \otimes db) = -D(a)^{\nu}(b, D(a^\nu)^{\nu})P + \{\{a, D(b)^{\nu}\}P \otimes D(b)^{\nu} - \tau(23)\} \{\{\sigma D, \{\{a, \{P, b\}\}\}\}\}_L
\]
\[
= -D(a)^{\nu}(b, D(a^\nu)^{\nu})P + \{\{a, D(b)^{\nu}\}P \otimes D(b)^{\nu} - D(\{\{a, b\}\}^\nu \otimes \{\{a, b\}\}^\nu) \otimes D(\{\{a, b\}\}^\nu) \}^\nu
\]
\[
= -D(a)^{\nu}(b, D(a^\nu)^{\nu})P + \{\{a, D(b)^{\nu}\}P \otimes D(b)^{\nu} - \tau(132)\} \{\{D, \{a, \{P, b\}\}\}\}_L
\]
\[
= -D(a)^{\nu}(b, D(a^\nu)^{\nu})P + \{\{a, D(b)^{\nu}\}P \otimes D(b)^{\nu} - \{\{a, \{\{P, b\}_L, \{D, a\}\}_L\}_{L + \tau(132)}\} \{\{b, \{\{D, a\}\}^\nu\}P \otimes \{\{D, a\}\}^\nu\}
\]
\[
= +\{\{a, D(b)^{\nu}\}P \otimes D(b)^{\nu} + \{\{a, \{\{P, b\}_L, \{D, a\}\}_L\}_{L + \tau(132)}\} \{\{b, \{\{D, a\}\}^\nu\}P \otimes \{\{D, a\}\}^\nu\}
\]
\[
= -\{\{\{a, b, \{P, D\}\}\}_L\}_{L + \tau(132)}\}
\]
where we used the formula \(\{\{a, b\}\} = \{\{a, \{P, b\}\}\}\).

Lemma 5.12. If \(\delta_1, \ldots, \delta_n\) are in \(\mathcal{D}(A)\) and \(a_1, \ldots, a_n \in A\), then
\[
\{\{\{a_1, \{\{a_2, \ldots, a_n\}\}_L\}_L \ldots \}_L = \delta_1(a_1)^{\nu} \otimes \delta_1(a_1)^{\nu} \otimes \delta_2(a_2)^{\nu} \otimes \delta_3(a_3)^{\nu} \otimes \cdots \delta_{n-1}(a_{n-1})^\nu \otimes \delta_n(a_n)^\nu \otimes \delta_n(a_n)^\nu
\]
Proof by induction on \(n\).

Proposition 5.13. Let \(A\) be a smooth double Poisson algebra. If \(L = \Omega^*_A\), the differential \(d_L\) coincides with the double Poisson cohomology defined by Pichereau and Van Weyer. ([13]).

This proposition follows from the previous theorem.

The Lie derivative:

The definition of the Lie derivative is more complicated for a general double Lie algebroid than in the case where \(L = \mathcal{D}(A)\) ([7]).

Let \(D\) be an element of \(L\). The Lie derivative along \(D\) is the map \(L_D : T_A(L^*) \to T_A(L^*) \otimes T_A(L^*)\) defined by the following conditions:

- \(L_D(a) = D(a)\)
- If \(\lambda \in L^*\),
\[
L_D(\lambda)(\Delta) = \lambda(\Delta)^{\nu} \otimes D(\lambda(\Delta)^{\nu}) \otimes \tau(23) \{\lambda(\{\{D, \lambda(\Delta)^{\nu}\} \otimes \{\{D, \Delta\}\}^\nu\}_{L + \tau(23)}\}_{L + \tau(23)}\}
\]
- \(L_D(\lambda) \in A \otimes L^* \otimes L^* \otimes A\) as the map
\[
\Delta \to \lambda(\Delta)^{\nu} \otimes D(\lambda(\Delta)^{\nu}) \otimes \tau(32) \{\lambda(\{\{D, \Delta\}\}^\nu) \otimes \{\{D, \Delta\}\}^\nu\}_{L + \tau(32)}\}
\]

belongs to \(L^* \otimes A\) and the map
\[
\Delta \to D(\lambda(\Delta)^{\nu}) \otimes \lambda(\Delta)^{\nu} \otimes \tau(32) \{\lambda(\{\{D, \Delta\}\}^\nu) \otimes \{\{D, \Delta\}\}^\nu\}_{L + \tau(32)}\}
\]
is in \(A \otimes L^*\).

- \(L_D\) is a degree preserving double derivation from \(T_A(L^*)\) to \(T_A(L^*) \otimes T_A(L^*)\).

Standard formulas involving the differential \(d_L\), contraction and Lie derivative hold in the case where \(L = \mathcal{D}(A)\) ([7, 17]). We prove now that they hold for any double Lie algebroids. In particular, one has the Cartan identity:

Proposition 5.14. The map \(d_L\) is a degree one derivation of \(T_A(L^*)\) and can be extended to a degree one derivation of \(T_A(L^*) \otimes T_A(L^*)\) as follows:
\[
\forall (\alpha, \beta) \in T_A(L^*_L), \quad d_L(\alpha \otimes \beta) = d_L(\alpha) \otimes \beta + (-1)^{\alpha} \alpha \otimes d_L(\beta).
\]

One has the following properties: For any \(D\) and \(\Delta\) in \(L\):
1) \(d_L(\alpha_L) \otimes \alpha_L + \alpha_L \otimes d_L(\alpha_L) = L_D\)
2) \(\{L_D, \alpha_L\}_L = \{L_D, \alpha_L\}_L = 0\).

Remark 5.15. Consequences of these formulas will be seen later.
Proof. 2) It is enough to prove the relation \( \partial_{b_{D}} \circ \tilde{b_{D}} + \tilde{b_{D}} \circ \partial_{b_{D}} = L_{D} \) on elements of \( A \) and \( L^{*} \).

On elements of \( A \), it is obvious. On elements \( \lambda \in L^{*} \), we give the main steps of the computation:

\[
(\tilde{b_{D}} \circ \partial_{b_{D}} + \partial_{b_{D}} \circ \tilde{b_{D}})(\lambda)(\Delta) = \frac{\partial_{b_{D}}(\lambda(D), \Delta)}{\tilde{b_{D}}(\lambda(D), \Delta) + \partial_{b_{D}}(\lambda(D)) (\Delta)} = \frac{\lambda(D)(\Delta)}{L_{D}(\lambda)(\Delta)}
\]

3) The relations \( \{[\tilde{b_{D}}, b_{D}]\}_{1} = 0 \) and \( \{[b_{D}, \tilde{b_{D}}]\}_{r} = 0 \) are easy to check on elements of \( A \) and \( \mathbb{L} \).

\[
\square
\]

6. FROM DOUBLE TO CLASSICAL

We now explain how to go from the double picture to the classical picture by the use of representation spaces \([\ref{17}]\). In the section, we assume by simplicity that \( B = k \).

Let \( A \) be a \( k \)-algebra an \( N \in \mathbb{N} \). Denote by \( \text{Rep}_{N}(A) \) the representation space \( \text{Hom}_{alg}(A, M_{N}(k)) \) The coordinate ring of \( \text{Rep}_{N}(A) \) is

\[
O_{N}(A) := k[\text{Rep}_{N}(A)] = \frac{k[a_{p,q}, (p, q) \in [1, N], a \in A]}{<a_{p,q}b_{q,r} - (ab)_{p,r}>}
\]

For any element \( x \in \text{Rep}_{N}(A) \) one has

\[
a_{p,q}(x) = x(a)_{p,q}
\]

Examples 6.1. 1) If \( A = \frac{k[t]}{[t]} \), then \( \text{Rep}_{N}(A) = \{Q \in M_{N}(k) \mid Q^{n} = (0)\} \).

2) If \( A = k < x_{1}, \ldots, x_{q} > \), then \( \text{Rep}_{N}(A) = M_{N}(k) \oplus \cdots \oplus M_{N}(k) \).

Theorem 6.2. \([\ref{17}]\) If \( (A, \{\{-, -\}\}) \) is a double Poisson algebra, then \( O_{N}(A) \), endowed with the bracket determined by

\[
\{a_{i,j}, b_{u,v}\} = \{(a, b)'_{u,j}, (a, b)'_{v,i}\}
\]

is a Poisson algebra.

If \( a \in A \), one introduces \( X(a) \) the \( M_{n}(k) \) valued function on \( \text{Rep}_{N}(A) \) defined by \( X(a)_{i,j} = a_{i,j} \).

One has the relation \( X(ab) = X(a)X(b) \) and one defines \( \text{Tr} : \frac{A}{[A, A]} \rightarrow O_{N}(A), \quad \varpi \rightarrow \text{Tr}(X(a)) ) = \sum_{i} a_{i,i} \)

\( GL_{N}(k) \) acts by conjugation on \( \text{Rep}_{N}(A) \). Using a result of Lebruyn and Procesi \([\ref{12}]\), the following theorem is shown in \([\ref{8}]\).

Theorem 6.3. If the \( k \)-algebra \( A \) is finitely generated, the map

\[
\text{Tr} : \frac{A}{[A, A]} \rightarrow O_{N}(A)^{GL_{N}(k)} \quad \varpi \rightarrow \text{Tr}(X(a)) = \sum_{i} a_{i,i}
\]

is an isomorphism of Lie algebras.

If \( M \) is an \( A^{*} \)-module, one defines \([\ref{13}]\) the \( O_{N}(A) \)-module \( (M)_{N} \)

\[
(M)_{N} = \frac{k[m_{i,j}, \ m \in M]}{<a_{i,u}m_{u,j} - (am)_{i,j}, a_{u,j}m_{i,u} - (ma)_{i,j}>}
\]

If \( m \in M \), one introduces \( \text{Tr}(m) = \sum_{i=1}^{N} m_{i,i} \in (M)_{N} \).

We will be interested in the case where \( M = \mathbb{L} \) is a double Lie Rinehart algebra.

If \( \delta \in \text{Der}(A) \), one defines \([\ref{17}]\) the corresponding derivation on \( O_{N}(A) \) by

\[
\delta_{i,j} (a_{u,v}) = \delta(a)'_{u,j} \delta(a)'_{v,i}
\]

If \( \delta = \delta_{1} \ldots \delta_{n} \), one sets \( \delta_{i,j} = \delta_{1, a_{1}, i, j} \delta_{2, a_{1}, 2, j} \cdots \delta_{n, a_{n-1}, j} \in \wedge \text{Der}(O_{N}(A)) \). In other words, this can be rewritten by the relation \( X(\delta) = X(\delta_{1}) \ldots X(\delta_{n}) \).
Proposition 6.4. ([17]) If \( P, Q \in DA \), then the following relation holds:
\[
\{P_{ij}, Q_{uv}\} = \{[P, Q]\}_{ij}^{uv}
\]
where \( \{\cdot, \cdot\} \) denotes the Schouten bracket on \( DA \) and \( \{\cdot, \cdot\} \) the Schouten bracket between poly-vector fields on \( \text{Rep}_N(A) \).

Proposition 6.5. The Trace map \( \text{Tr} \) ([17])
\[
\text{Tr} : \frac{DA}{[DA, DA]} \rightarrow \wedge \text{Der} (\mathcal{O}_N(A))
\]
is a Lie algebra homomorphism if both side are equipped with the Schouten bracket.

Theorem 6.6. ([13])
There exists a unique Lie algebroid structure \( (\mathbb{L}_N, \mathcal{O}_N(A), [\cdot, \cdot], \omega) \) with bracket \([\cdot, \cdot]\) and anchor \( \omega \) determined below by the equalities below:
\[
\omega(X_{i,j})(a_{u,v}) = X(a)(a)_{i,j}^{uv},
\]
\[
[X_{i,j}, Y_{u,v}] = \{X, Y\}_{i,j}^{uv}
\]

Proof. The only thing that is not obvious is the Jacobi identity and the fact that \( \omega \) is a Lie algebra morphism. We need to prove the identities
\[
[X_{i,j}, Y_{u,v}, Z_{k,m}] + [Y_{u,v}, Z_{k,m}, X_{i,j}] + [Z_{k,m}, X_{i,j}, Y_{u,v}] = 0
\]
These two identities follows from the double Jacobi identities by a straightforward computation. \( \square \)

1. If \( D \in L \), we consider the matrix \( X(D) = (D_{i,j}) \) as being with values in \( \wedge \mathcal{O}_N(A)(\mathbb{L})_N \) and we set
\[
\forall D_1 \cdots \cdot D_n, \quad X(D_1 \cdots D_n) = X(D_1) \cdots X(D_n) \in \wedge \mathcal{O}_N(A)(\mathbb{L})_N.
\]
One defines the trace map (as in [17]) by \( \text{Tr} : T_A(L) \rightarrow \wedge \mathcal{O}(L)_N \) by \( \text{Tr}(D_1 \cdots \cdot D_n) = \text{Tr}X(D) \).

Proposition 6.7. If \( D, \Delta \) are in \( L \), one has the following equality \( [\text{Tr}(D), \text{Tr}(\Delta)] = \text{Tr}([D, \Delta]) \) where the left hand side involves the Schouten bracket on \( \wedge \mathcal{O}_N(A)(\mathbb{L})_N \) and the right hand side involves the double Schouten bracket. \( \square \)

Proof. It is a straightforward computation.

Let \( M \) be an \( A^* \)-module. To the \( A^* \)-module
\[
M^* = \{ \lambda : M \rightarrow A \otimes A : \lambda(amb) = \lambda(m)b \otimes a\lambda(m)^n \},
\]
on one associates
\[
(M^*)_N = \{ \lambda_{i,j}, \lambda \in M^* \}
\]
If \( \lambda \in M^* \) and \( m \in M, \lambda_{i,j} \) defines an element of \( \text{Hom}_{\mathcal{O}_N(A)}((M)_N, \mathcal{O}_N(A)) \) by
\[
\lambda_{i,j}(m_{u,v}) = \lambda(m)_{i,j}^{uv}
\]
For us, \( M \) will be a double Lie–Rinehart algebra \( L \) or its dual \( L^* \).

Remark 6.8. If \( M = \wedge \text{Der}(A) \), one recovers the equality \( (da)_{i,j} = da_{i,j} \) (17).

We go on mimicking Van den Berg’s construction:

If \( \Lambda = \lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n \in T^n(M^*) \), we define
\[
\Lambda_{i,j} = \lambda_{1,i_1} \lambda_{2,i_2} \cdots \lambda_{n,i_n-j} \in \wedge ((M)_N^*)
\]
and \( X(\Lambda) = (\Lambda_{i,j})_{i,j} \). The latter is a matrix with values in \( \wedge ((M)_N^*) \).

Lemma 6.9. Let us identify \( \Lambda = \lambda_1 \otimes \lambda_2 \otimes \cdots \otimes \lambda_n \) to \( \Phi \in \text{Hom}_{A^*}(\mathbb{L}_N^a, A^{\otimes n+1}) \). If we write
\[
\Phi(D_1 \otimes \cdots \otimes D_n) = \Phi^{(1)}(D_1) \otimes \cdots \otimes \Phi^{(n+1)}(D_n),
\]
then
\[
\Phi_{i,j}(D_{1,u_{i,v_1}} \otimes \cdots \otimes D_{u_{n,v_n}}) = \sum_{\sigma \in S_n} (-1)^n \Phi^{(1)}(D_{\sigma(1)} \otimes \cdots \otimes D_{\sigma(n)})_{i_1,v_1} \Phi^{(2)}(D_{\sigma(1)} \otimes \cdots \otimes D_{\sigma(n)})_{u_1,v_2} \cdots \Phi^{(n+1)}(D_{\sigma(1)} \otimes \cdots \otimes D_{\sigma(n)})_{u_{n-1},v_{n-1}} \Phi^{(n)}(D_{\sigma(1)} \otimes \cdots \otimes D_{\sigma(n)})_{u_{n},v_{n}}
\]
Theorem 6.12.

Proof.

\[ \Phi(1)^{(D)}(\sigma(1) \cdots \sigma(n))_{uv} \Phi(2)^{(D)}(\sigma(1) \cdots \sigma(n))_{uv} \Phi(n-1)^{(D)}(\sigma(1) \cdots \sigma(n))_{uv} \Phi(n)^{(D)}(\sigma(1) \cdots \sigma(n))_{uv} \]

Let us now prove (ii). In the following computation \((D \leftrightarrow \Delta, u \leftrightarrow k, v \leftrightarrow p)\) means the same expression as before exchanging \(D\) with \(\Delta\), \(u\) with \(k\), \(v\) with \(p\).

\[ d_{(\lambda),j}^i(D_{k,p}, \Delta_{u,v}) = \lambda(D) \Delta(\lambda(D))_{k,p} \Delta_{u,v} \]

Proposition 6.11. Let \(Tr\) be the Trace map : if \(\Phi \in T_A(\mathbb{L}^*)\), \(Tr(\Phi) = Tr(X(\Phi))\)

\( a)\) If \(\lambda \in \mathbb{L}^*\) and \(D \in \mathbb{L}\), then \(Tr(\lambda(D)) = Tr(\lambda)(Tr(D))\).

\( b)\) If \(\Phi \in T_A(\mathbb{L}^*)\), one has \(Tr_d(\Phi) = d_{(\lambda),j}((Tr\Phi))\)

Proof. a) is an easy computation :

\[ Tr(\lambda)(Tr(D)) = \sum_{i,j} \lambda_{i,j}(D_{i,j}) = \lambda(D)_{i,j} \lambda(D)_{j,i} = Tr(\lambda(D)) \]

Let us now prove b).

Let us first prove by induction on the degree of \(\Phi\) that \(d_{(\lambda)}(\Phi)_{i,j} = d_{(\lambda)}(\Phi_{i,j})\).

For \(deg(\Phi) = 1\), we have already proved it in a previous lemma.

Assume that it is proved for \(deg(\Phi) = n\) and let us use it for \(\Phi\lambda\) if \(\lambda \in \mathbb{L}^*\).

Then \(Tr[d_{(\lambda)}(\Phi)] = \sum_{i,j} d_{(\Phi)}(d_{(\lambda)}(\Phi))_{i,j} = d_{(\lambda)}(\Phi))\).
7. Reduced contraction and Lie derivative

In this section, we generalize results of [7].

Definition 7.1. Let $L$ be a double Lie algebroid. If $\Theta$ is in $L$, one defines the reduced Lie derivative and the reduced contraction by:

$$\iota_\Theta : T^2_A(L^*) \to T^1_A(L^*), \quad \alpha \mapsto \circ \iota_\Theta \alpha$$

$$L_\Theta : T^2_A(L^*) \to T^3_A(L^*), \quad \alpha \mapsto \circ L_\Theta \alpha$$

Explicitly, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ are in $L^*$, one has

$$\iota_\Theta (\alpha_1 \alpha_2 \ldots \alpha_n) = \sum_{k=1}^n (-1)^k (n-k+1) \alpha_k(\Theta)'' \cdot \alpha_{k+1} \ldots \alpha_n \alpha_1 \ldots \alpha_{k-1} \cdot \alpha_k(\Theta)'$$

$$L_\Theta (\alpha_1 \alpha_2 \ldots \alpha_n) = \sum_{k=1}^n (-1)^k (n-k+1) L_\Theta (\alpha_k)'' \cdot \alpha_{k+1} \ldots \alpha_n \alpha_1 \ldots \alpha_{k-1} \cdot L_\Theta (\alpha_k)'$$

Proposition 7.2. 1) For any $\Theta \in L$, the following equalities of endomorphisms of $T_A(L^*)$ hold:

$$d_L \circ \iota_\Theta + \iota_\Theta \circ d_L = L_\Theta,$$

$$d_L \circ L_\Theta = L_\Theta \circ d_L$$

2) The maps $d_L$, $L_\Theta$ and $\iota_\Theta$ descend to maps from $DR^*(L)$ to $DR^*(L)$ denoted respectively $d_L$, $L_\Theta$ and $\iota_\Theta$. One has

$$d_L \circ \iota_\Theta + \iota_\Theta \circ d_L = L_\Theta,$$

$$d_L \circ L_\Theta = L_\Theta \circ d_L$$

3) For any $\delta$ and $\Delta$ in $L$, one has $L_\Theta (\Delta + \sigma_{12} \iota_\Theta \delta) = 0$ (as maps from $T_A(L^*)$ to $T_A(L^*) \otimes T_A(L^*)$).

Remark 7.3. The previous proposition is proved for $L = \mathcal{D}er(A)$ in [7] but our proof is different.

Proof. 1) follows by applying $\circ()$ to the relation

$$L_\Theta = d_L \circ \iota_\Theta + \iota_\Theta \circ d_L$$

and using proposition 5.2 and from proposition 5.2.

2) followed from proposition 5.2 and from proposition 5.2.

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