ADM mass of the quantum-corrected
Schwarzchild black hole

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ABSTRACT

We study the hamiltonian and constraints of spherically symmetric dilaton gravity model. We find the ADM mass of the solution representing the Schwarzschild black hole in thermal equilibrium with the Hawking radiation.
1 Introduction

Quantum theory of gravity is expected to provide solutions for many problems of classical general relativity, such as the problem of singularities, or the problem of interpretation of thermodynamic quantities like temperature and entropy of the black hole. As the quantization of gravity is still missing, we do not have the full quantitative description of these phenomena. Yet, some insights come from studies of quantization of matter in the curved space and semiclassical treatments. One of the celebrated achievements on this line was the discovery of the Hawking radiation \( T = \frac{1}{4\pi a} \), and many papers followed it in an attempt to describe the backreaction of the radiation to the black hole geometry. Also, the Bekenstein-Hawking formula for entropy \( S = \frac{\pi a^2}{G} \) was a subject of many discussions.

The expressions for the temperature of the Schwarzschild black hole \( T = \frac{1}{4\pi a} \), its entropy \( S = \frac{\pi a^2}{G} \) and energy \( E = \frac{a}{2G} \) are all zero-th order or classical expressions and can be derived in various ways. (\( a \) is the radius of the horizon of Schwarzschild black hole, \( a = 2MG \).) The problem which is still open is to find the first quantum corrections to the above-mentioned quantities, as well as the correction of the metric. This problem was treated in the literature in various ways. One approach, done in extensively the eighties \(^3\), was to find the expectation value of the energy-momentum tensor (EMT) of the matter field from the symmetry arguments (trace anomaly), and to solve the coupled Einstein equations for the metric

\[
R_{\mu\nu} - g_{\mu\nu} R = <T_{\mu\nu}> .
\]

\( (1) \)

The other possibility is to integrate the matter fields in the path integral and determine the effective action \( S_{eff} \) to the one-loop order. This, unfortunately, has not been done in four dimensions by now. But in two dimensions (2D) this programme has been fulfilled for many 2D models. In the last years there are various attempts to find a 2D model which would successfully describe the properties of 4D spherically symmetric solutions and the quantum corrections of the Schwarzschild solution were examined \(^4\), \(^5\), \(^6\). Among others, much is expected from the dilaton spherically symmetric gravity model (SSG). In this model, the quantum correction is given by the effective action which is obtained in \(^7\), \(^8\) by evaluation of 2D path integral to the first order in \( \hbar \). The 2D classical action is obtained from the 4D Einstein-Hilbert action.
which interacts minimally with the scalar field by the spherically symmetric reduction. The one-loop correction terms are nonlocal, but can be written in the local form after introduction of the additional fields $\psi$ and $\chi$ [6]. The fields $\psi$ and $\chi$ are not auxiliary in the usual sense of the Hamiltonian analysis because they are dynamical, i.e. their equations of motion are of the second order. This is the remnance of the ”quantum origin” of these fields, i.e. of the fact that they describe the behaviour of the quantized radiated matter. In this picture, fixing of the integration constants in the zero’th order solutions for $\psi$ and $\chi$ corresponds to the choice of the quantum state of matter, and it can be done in such a way that the given solution describes thermalized Hawking radiation (Hartle-Hawking vacuum).

In our previous paper [6], we obtained the first quantum correction of the geometry of the Schwarzschild solution, its temperature and entropy. We also obtained the value of energy, assuming that it is defined by the thermodynamic relation $dE = TdS$. On the other hand, there are known methods for defining energy of the gravitational field which has a time-like Killing vector and specified asymptotic behaviour, e.g. the Arnowitt-Deser-Misner method [9]. As the effective action which we have used in [6] proved to be relatively simple in its local form and the quantum corrected solution is asymptotically flat, it is natural to try calculate the ADM mass of the mentioned solution and compare it to the thermodynamical result. The problem of finding energy and other conserved quantities in general relativity is known and well studied [10], also in the context of various 2D theories [11, 13]. It was applied in the case of nonlocal potential of the Polyakov-Liouville type in the paper of Blagojević et al [14], and we find their analysis very instructive for our problem, too.

The plan of the paper is the following: the second section contains the definition of the model, the analysis of the hamiltonian and constraints. The boundary term and the energy are found in the third section. The comparison of the results which are with the thermodynamic ones is given in the concluding, fourth, section of the paper.

### 2 Hamiltonian and constraints

As it is well known, the energy-momentum tensor of the gravitational field is not uniquely defined in general relativity. Also, the value of energy can
be obtained only for some classes of metrics. If the considered configuration of gravitational field has a time-like Killing vector, the corresponding conserved quantity can be identified with the energy of the system if the space is asymptotically Minkowskian. Also some other classes of metrics allow the identification of physical time and definition of energy (e.g. asymptotically flat, or asymptotically de Sitter spaces etc.) We will use the Arnowitt-Deser-Misner method in the hamiltonian formulation \[10, 11, 12\]. In order to obtain the ADM mass we need to find the hamiltonian and constraints of our system, and then, analyzing the variations of hamiltonian, find the correct boundary term. Let us first define the action and the lagrangian we are dealing with.

We start with Einstein-Hilbert gravity coupled minimally to \(N\) scalar fields \(f_i\) in four dimensions. This system is described by the action

\[
\Gamma_0 = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R^{(4)} + \frac{1}{8\pi} \sum_i \int d^4x \sqrt{-g} (\nabla f_i)^2 ,
\]

After the spherically symmetric reduction of all fields, we get 2D action

\[
\Gamma_0 = -\frac{1}{4G} \int d^2x \sqrt{-g} \left[ e^{-2\Phi} \left( R + 2(\nabla \Phi)^2 + 2e^{2\Phi} \right) - 2Ge^{-2\Phi} \sum_i (\nabla f_i)^2 \right] .
\]

Here, \(g_{\mu\nu}\) is 2D metric, \(\Phi\) is dilaton field, and \(f_i\) are the matter fields. In order to include the quantum effects to the first order in \(\hbar\), we add to this action the one-loop quantum correction which was found in \[7, 8\]. The effective action which we get is

\[
\Gamma = -\frac{1}{4G} \int d^2x \sqrt{-g} \left( e^{-2\Phi}(R + 2(\nabla \Phi)^2 + 2e^{2\Phi}) - 2Ge^{-2\Phi} \sum_i (\nabla f_i)^2 \right) + \frac{N}{8\pi} \int d^2x \sqrt{-g} \left( \frac{1}{12} R^{\Box} R + R\Phi - R \frac{1}{\Box} (\nabla \Phi)^2 \right) .
\]

We will calculate all quantities to the first order in \(\kappa\), as the effective action is also given to this order.

Since the matter fields enter the action only quadratically and we are analyzing the correction to the vacuum solution \(f_i = 0\), we can introduce \(f_i = 0\) directly into the action, prior to finding the equations of motion. It is convenient to rewrite the effective action in the local form, using the
auxilliary fields $\psi$ and $\chi$ and introducing the field $r = e^{-\Phi}$ instead of $\Phi$. We get, then

$$
\Gamma = -\frac{1}{4} \int d^2x \sqrt{-g} \left[ r^2 R + 2(\nabla r)^2 + 2 - \kappa [2R(\psi - 6\chi) + (\nabla \psi)^2 - 12(\nabla \psi)(\nabla \chi)] - 12\psi \frac{(\nabla r)^2}{r^2} - 12R \log r \right].
$$

(5)

Here we introduced the constant $\kappa = \frac{N_h}{24\pi}$. The auxilliary fields $\psi$ and $\chi$ satisfy the equations of motion:

$$
\Box \psi = R \tag{6}
$$

$$
\Box \chi = \frac{(\nabla r)^2}{r^2} \tag{7}
$$

The equations of motion for the other fields are given in [6].

The classical part ($\kappa = 0$) of the action (5) has the Schwarzschild black hole as a vacuum solution. It reads:

$$
f_i = 0 \ , \ r = x^1 ,
$$

$$
g_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & \frac{1}{f} \end{pmatrix} ; \ f = 1 - \frac{a}{r} .
$$

$a$ is the radius of the horizon of black hole, $a = 2MG$. The dilaton field $r$ has the role of radius. In the following, we will denote $x^0 = t$.

The quantum correction of this solution is given by the formula (34-37) and describes the black hole in equilibrium with its Hawking radiation. It was found in [6], and will be discussed in details in the next section in relation to the boundary conditions. We will now pass on finding the hamiltonian corresponding to the action (5).

One possibility to analyze 2D gravity lagrangians is to fix the gauge partially and use the lapse and shift functions as variables [14]. We will proceed along the lines of [14] in order to keep trace of all symmetries. This means that for variables we take all components of the metric tensor $g_{00}$, $g_{01}$, $g_{11}$, and, along with them, $r$, $\psi$ and $\chi$. The conjugated momenta are denoted $\pi^{00}$, $\pi^{01}$, $\pi^{11}$, $\pi_r$, $\pi_\psi$ and $\pi_\chi$. In order to have only the derivatives of the first order in the lagrangian, we perform a suitable partial integration. Up
to surface terms (which are at this stage of the procedure not important and will be fixed at the end), the lagrangian density corresponding to the action (3) is:

\[ 4G\sqrt{-g} \mathcal{L} = 2g + \frac{g_{01}}{g_{11}}(\dot{Q}g'_{11} - Q'\dot{g}_{11}) + g_{11}\dot{Q} + g'_{00}Q' - 2g'_{01}\dot{Q} \]

\[ + 2(1 + 6\psi\frac{\kappa}{r^2})(g_{11}\dot{r}^2 + g_{00}r'^2 - 2g_{01}\dot{r}r') \]

\[ - \kappa(g_{11}\dot{\psi}^2 + g_{00}\psi'^2 - 2g_{01}\dot{\psi}\psi') \]

\[ + 12\kappa(g_{11}\dot{\psi}\dot{\chi} + g_{00}\psi'\chi' - g_{01}(\dot{\psi}\chi' + \dot{\chi}\psi')) \] . (8)

Dot and prime denote temporal and spatial derivatives and, to simplify the expression (8), the function \( Q = r^2 + 12\kappa \log r - 2\kappa(\psi - 6\chi) \) is introduced.

The lagrangian density (8) does not contain the velocities \( \dot{g}_{00} \) and \( \dot{g}_{01} \) and therefore the system is constrained. Using the definition

\[ \pi_\Phi = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} = \frac{\delta \mathcal{L}}{\delta \dot{\Phi}} , \] (9)

where

\[ L = \int dx^3 \mathcal{L} , \] (10)

we obtain the generalized momenta:

\[ \pi_{00} = 0 \] (11)

\[ \pi_{01} = 0 \] (12)

\[ \pi_{11} = \frac{1}{4G\sqrt{-g}}[2rA(\dot{r} - \frac{g_{01}}{g_{11}}r') - 2\kappa(\dot{\psi} - \frac{g_{01}}{g_{11}}\psi') + 12\kappa(\dot{\chi} - \frac{g_{01}}{g_{11}}\chi')] \] (13)

\[ \pi_r = \frac{2}{4G\sqrt{-g}}[rA(\frac{g_{01}}{g_{11}}g'_{11} + \dot{g}_{11} - 2g'_{01}) + 2Bg_{11}(\dot{r} - \frac{g_{01}}{g_{11}}r')] \] (14)

\[ \pi_\chi = \frac{12\kappa}{4G\sqrt{-g}}[(\frac{g_{01}}{g_{11}}g'_{11} + \dot{g}_{11} - 2g'_{01}) + g_{11}(\dot{\psi} - \frac{g_{01}}{g_{11}}\psi')] \] (15)

\[ \pi_\psi = \frac{-2\kappa}{4G\sqrt{-g}}[(\frac{g_{01}}{g_{11}}g'_{11} + \dot{g}_{11} - 2g'_{01}) + g_{11}(\dot{\psi} - \frac{g_{01}}{g_{11}}\psi') - 6g_{11}(\dot{\chi} - \frac{g_{01}}{g_{11}}\chi')] \] (16)
A and B are defined as $A = 1 + 6\kappa r^2$, $B = 1 + 6\psi \frac{r}{r}$.

Equations (13-16) can be solved in velocities $g_{11}'$, $\dot{r}$, $\dot{\psi}$ and $\dot{\chi}$, while the equations (11-12) are primary constraints. The canonical Hamiltonian density $\mathcal{H} = \sum \pi_\Phi \dot{\Phi} - L$, obtained from (8) is

$$4G\mathcal{H} = 2\sqrt{-g} \left( 2Br^2 - \kappa \psi'^2 + 12\kappa \psi' \chi' \right)$$

$$+ \frac{2}{\sqrt{-g}} \left( -\frac{g_1}{g_{11}} \right)' (Arr' - \kappa \psi' + 6\kappa \chi')$$

$$+ 4Gg_{11}' \left( 2g_{11}' - g_{11} g_{11}' \right) + 4Gg_{11}'g_{11} (\pi r' + \pi \psi' + \pi \chi')$$

$$+ (4G)^2 \sqrt{-g} \left( \frac{1}{144\kappa} \pi^2 \pi_r^2 - \frac{\kappa}{4r^2F^2} \pi_r^2 - \frac{B}{2r^2F} (g_{11} \pi_{11} - \pi_\psi)^2 \right)$$

$$+ \frac{1}{12\kappa} \pi_\psi \pi_\chi + \frac{rA}{2r^2F} \pi_r (g_{11} \pi_{11} - \pi_\psi) \right),$$

where $F = A^2 - \frac{2}{r} B$.

In order to find the secondary constraints, we calculate the Poisson brackets of the primary constraints with the Hamiltonian, $H = \int dx^1 \mathcal{H}$. The Poisson brackets give:

$$\{\pi^{00}, H\} = -\frac{1}{2\sqrt{-g}} \mathcal{H}_0$$

$$\{\pi^{01}, H\} = \frac{1}{g_{11}} \left( \frac{g_{01}}{\sqrt{-g}} \mathcal{H}_0 - \mathcal{H}_1 \right),$$

where the secondary constraints $\mathcal{H}_0$ and $\mathcal{H}_1$ are given by

$$4G\mathcal{H}_0 = -\frac{1}{4G} \left( 2g_{11}' + (2Br^2 + \kappa \psi'^2 + 12\kappa \psi' \chi') \right)$$

$$+ 4(Arr' - \kappa \psi' + 6\kappa \chi')' - 2\frac{g_1}{g_{11}} (Arr' - \kappa \psi' + 6\kappa \chi')$$

$$- 4G \left( \frac{1}{144\kappa} \pi^2 \pi_r^2 - \frac{\kappa}{4r^2F^2} \pi_r^2 - \frac{B}{2r^2F} (g_{11} \pi_{11} - \pi_\psi)^2 \right)$$

$$+ \frac{1}{12\kappa} \pi_\psi \pi_\chi + \frac{rA}{2r^2F} \pi_r (g_{11} \pi_{11} - \pi_\psi) \right),$$

$$4G\mathcal{H}_1 = -\pi_{11}' g_{11}' - 2g_{11} \pi_{11}' + \pi_r r' + \pi \psi' + \pi \chi' \right).$$
Note that the canonical Hamiltonian can be written as the sum of the constraints

\[ H = -\frac{\sqrt{-g}}{g_{11}} H_0 + \frac{g_{01}}{g_{11}} H_1, \]  

(22)

which tells us that the only nonvanishing contribution to the energy comes from the surface terms which we are to determine. Also note that the structure of constraints and Hamiltonian is completely analogous to the one obtained from the Liouville model and PGT \([13, 14]\). This reflects the fact that all the considered models have the same symmetries, namely 2D diffeomorphisms. We will not analyse the symmetry aspects further (algebra of constraints, generators of symmetry), but concentrate on on the boundary terms.

Let us review the main idea shortly. The Hamilton’s equations of motion are obtained from the variational principle

\[ \delta L = \delta \int dx^1 \left( \sum \pi_{\Phi} \dot{\Phi} - H \right) = 0, \]  

(23)

when the variations are well defined, i.e. when they are of the form

\[ \delta H = \sum \left( \frac{\delta H}{\delta \pi_{\Phi}} \delta \pi_{\Phi} + \frac{\delta H}{\delta \dot{\Phi}} \delta \Phi \right). \]  

(24)

In the case of the Hamiltonian densities of the type \([17]\) which contain the spatial derivatives of fields and momenta, the terms of the type \(\delta \Phi'\) might occur in the variation \(\delta H\), and this produces terms \(\delta \Phi'_{\text{bound}}\) in \(\delta L\). In order to make the variational procedure consistent, one adds boundary term to the Hamiltonian to cancel the unwanted variations in \(\delta L\) and get the Hamilton’s equations of motion. The boundary term may not always be defined and its existence depends on the asymptotic behaviour of the class of the fields in which we are performing the variations. This is the point where the asymptotic behaviour of the fields enters the definition of the conserved quantities. In the cases where only the matter fields are varied the asymptotic conditions are such that the fields and their derivatives vanish in the asymptotic region and therefore the boundary term is unimportant. But gravity is not such a case.

Varying the Hamiltonian \([17]\), we get

\[ 4G \delta H = \text{Reg} + \Delta' = \text{Reg} + \]
\[ + \left[ \frac{\sqrt{-g}}{g_{11}} (4Br'\delta r - 2\kappa \psi'\delta \psi + 12\kappa \psi'\delta \chi + 12\kappa \chi'\delta \psi) \right. \]
\[ + \frac{2}{\sqrt{-g}} \delta (\frac{-g}{g_{11}}) (Arr' - \kappa \psi' + 6\kappa \chi') + \frac{2}{\sqrt{-g}} \left( \frac{-g}{g_{11}} \right)' (Ar \delta r - \kappa \delta \psi + 6\kappa \delta \chi) \]
\[ + 4G \pi^{11} (2\delta g_{01} - \frac{g_{01}}{g_{11}} \delta g_{11}) + 4G \frac{g_{01}}{g_{11}} (\pi_r \delta r + \pi_\psi \delta \psi + \pi_\chi \delta \chi) \right] ', \quad (25) \]

where Reg denotes the regular terms of the type \((...)\delta \pi_\Phi + (...)\delta \Phi\). We have written explicitly only the terms that give contribution on the boundary. Now we have to examine that contribution for the fields which asymptotically behave as the Schwarzschild black hole in the Hartle-Hawking vacuum.

3 Boundary term end energy

In order to specify the class of functions \(i\) which we are varying, let us write the exact solution of the SSG model. The static solution is given by:

\[
\begin{align*}
  r &= x^1 \\
  g_{00} &= -fe^{2\Phi} \\
  g_{01} &= 0 \\
  g_{11} &= \frac{1}{f} \\
  \psi &= -2\Phi + C \int \frac{dr}{fe^\Phi} \\
  \chi &= \frac{D}{fe^\Phi} + \frac{1}{fe^\Phi} \int \frac{fe^\Phi}{r^2} dr .
\end{align*}
\]

The dilaton field \(r\) plays the same role of radial coordinate as before. The functions \(f\) and \(\Phi\) are given by

\[
\begin{align*}
  f(r) &= 1 - \frac{a}{r} + \kappa \frac{m(r)}{r} , \\
  \Phi(r) &= \kappa (F(r) - F(L)) ,
\end{align*}
\]

where

\[
m(r) = \frac{11a}{4r^2} + \frac{1}{2r} + \frac{5r}{2a^2} + \log \frac{r}{t} \left( \frac{5}{2a} - \frac{6}{r} + \frac{3a}{r^2} \right) ,
\]
\[ F(r) = \frac{3}{4r^2} + \frac{5}{ar} + \log \frac{r}{l} \left( -\frac{5}{2a^2} + \frac{3}{r^2} \right). \] (35)

As the functions \( \psi \) and \( \chi \) enter the hamiltonian always multiplied by \( \kappa \), it suffices to take the zero'th order solution in their asymptotic behaviour. It reads:

\[ \psi = \frac{r}{a} + \log \frac{r}{l} \] (36)
\[ \chi = \frac{r}{2a} - \frac{1}{2} \log \frac{r}{l}. \] (37)

(34-37) are written in the form obtained after the fixing of the integration constants \( C \) and \( D \) of (30-31). The question of integration constants \( C \) and \( D \) was analyzed in details in [6]. The choice \( C = \frac{1}{a} \) and \( D = \frac{1}{2a} \) which was taken in (34-37), ensures that all functions \( \psi, \chi \) and \( g_{\mu\nu} \) and therefore also the corrections of the curvature, energy-momentum tensor, temperature, etc. are regular on the horizon \( r = a \), which is precisely the definition of the Hartle-Hawking vacuum. The constants \( l \) and \( L \) have the dimension of length; \( L \) defines the boundary of the space and is taken to be large, \( a \ll L \).

We will perform the variation of the Hamiltonian in the class of the static configurations with fixed magnitude of space, \( L \) and variable mass, \( a \). More precisely, we are considering all configurations which asymptotically tend to the given solution i.e. differ from it for the terms that decrease like \( \frac{1}{r} \) or faster as \( r \to L \). This means that the asymptotic behavior of the fields we consider is

\[ r = L + O\left( \frac{1}{L} \right) \]
\[ \psi = \frac{L}{a} + \log \frac{L}{l} + O\left( \frac{1}{L} \right) \]
\[ \chi = \frac{L}{2a} - \frac{1}{2} \log \frac{L}{l} + O\left( \frac{1}{L} \right). \] (38)

The variations and derivatives behave as

\[ \delta r = 0, \quad r' = 1 \]
\[ \delta \psi = -\frac{L}{a^2} \delta a + O\left( \frac{1}{L} \right), \quad \psi' = \frac{1}{a} + \frac{1}{L} + O\left( \frac{1}{L^2} \right) \]
\[ \delta \chi = -\frac{L}{2a^2} \delta a + O\left( \frac{1}{L} \right), \quad \chi' = \frac{1}{2a} - \frac{1}{2L} + O\left( \frac{1}{L^2} \right). \] (39)
The behaviour of the components of the metric tensor is given by:

\[
\begin{align*}
g_{00} &= -f + O\left(\frac{1}{L}\right) \\
g_{01} &= O\left(\frac{1}{L}\right) \\
g_{11} &= \frac{1}{f} + O\left(\frac{1}{L}\right),
\end{align*}
\]

(40)

where we have taken \(e^{\Phi(L)} = 1\), as \(\Phi(r) = \kappa(F(r) - F(L))\). Also, \(\Psi' = O\left(\frac{1}{L}\right)\), etc. The functions \(f(r) = 1 - \frac{a}{r} + \kappa \frac{m(r)}{r}\) and \(m(r)\) for large \(r\) behave as

\[
\begin{align*}
m &= \frac{5}{2a^2}L + \frac{5}{2a} \log \frac{L}{l} + O\left(\frac{1}{L}\right) \\
\delta m &= \left(-\frac{5L}{a^3} - \frac{5}{2a^2} \log \frac{L}{l} - \frac{1}{2a^2 L}\right) \delta a + O\left(\frac{1}{L^2}\right) \\
f &= 1 + \kappa \frac{5}{2a^2} - \frac{a}{L} + \kappa \frac{5}{2aL} \log \frac{L}{l} + O\left(\frac{1}{L^2}\right). \quad (41)
\end{align*}
\]

As it can easily be checked, the given solution has the vanishing momenta, \(\pi_\Phi = 0\), and therefore we can take

\[
\pi_\Phi = O\left(\frac{1}{L}\right). 
\]

(42)

Entering the given behaviour of fields and momenta into the formula (25) for the boundary term \(\Delta\) we get

\[
\begin{align*}
4G\Delta &= 2f(-\kappa \psi' \delta \psi + 6\kappa \psi' \delta \chi + 6\kappa \chi' \delta \psi) \\
&+ 2\delta f(Ar - \kappa \psi' + 6\kappa \chi') + 2f'(-\kappa \delta \psi + 6\kappa \delta \chi))\bigg|_L.
\end{align*}
\]

(43)

Leaving only the terms of the highest order in \(L\), we obtain

\[
4G\Delta = -\delta a \left(2 + \kappa \left(\frac{20L}{a^3} - \frac{12}{a^2} + \frac{5}{a^2} \log \frac{L}{l}\right)\right).
\]

(44)

This, obviously, can be written as a variation of a function \(H_b\) defined on the boundary: \(\Delta = -\delta H_b\). This function is given by

\[
4GH_b = 2a + \kappa \left(-\frac{10L}{a^2} + \frac{12}{a} - \frac{5}{a} \log \frac{L}{l}\right).
\]

(45)
Note that the classical limit $\kappa = 0$ of (45) gives that $H_b$ is equal to the mass of the Schwarzschild black hole, $H_b = \frac{2a}{4G} = M$. Now we can get the complete hamiltonian $H_c$ adding the boundary term $H_b$ to the canonical hamiltonian $H$. It reads:

$$H_c = \int dx^1 (\sqrt{-g} H_0 + \frac{g_{01}}{g_{11}} H_1) + H_b . \quad (46)$$

$H_c$ gives the correct equations of motion because its variation is regular

$$\delta H_c = \text{Reg} + \int dx^1 \Delta' - \Delta = \text{Reg} .$$

As we discussed earlier, the fact that $H_0$ and $H_1$ are the constraints implies that the value of $H_c$ equals to the value of $H_b$.

Note that the space of the above described quantum corrected solution (34-37) is asymptotically flat but not asymptotically Minkowskian, due to the existence of the Hawking radiation. The values of the components of metric tensor at infinity are

$$g_{00} = -f = -(1 + \frac{5\kappa}{2a^2}) + O(\frac{1}{L}) \quad (47)$$

$$g_{11} = f^{-1} = 1 - \frac{5\kappa}{2a^2} + O(\frac{1}{L}) . \quad (48)$$

This means that the value of the energy is not simply equal to the boundary term $H_b$ which we obtained. In order to find the energy we have to define the coordinates which are asymptotically Minkowskian, and express $H_b$ in that coordinate system. The new coordinate system is defined by the conditions

$$\tilde{g}_{00} = -1 , \quad \tilde{g}_{01} = 0 , \quad \tilde{g}_{11} = 1 .$$

As $g_{00} = (\frac{\partial^2}{\partial x^0})^2 g_{00}$, we have $\frac{\partial^2}{\partial x^0} = \sqrt{-g_{00}}$ and similarly for the 11-component.

The desired transformation in the first order in $\kappa$ is:

$$\tilde{t} = (1 + \frac{5\kappa}{4a^2}) t , \quad \tilde{r} = (1 - \frac{5\kappa}{4a^2}) r . \quad (49)$$

Now we get the boundary term $\tilde{H}_b$ in the asymptotically flat coordinates

$$4G\tilde{H}_b = 2a + \kappa(-\frac{10L}{a^2} + \frac{19}{2a} - \frac{5}{a} \log \frac{L}{l}) . \quad (50)$$
which gives the value of energy. The term proportional to $L$ is the energy of the hot gas. Note, in the case of the null-dust model, where the one-loop correction is the Polyakov-Liouville term only, the boundary term $\Delta$ is given by

$$4G\Delta = \left. \left( -2\kappa\psi\delta\psi + 2\delta f(L - \kappa\psi') - 2\kappa f'\delta\psi \right) \right|_{L}$$

This expression is obtained from (43) by taking $A = B = 1, \chi = 0$. It easy to see that the energy is given by

$$4G\tilde{H}_b = 2a + \kappa \left( \frac{2L}{a^2} + \frac{1}{2a} + \frac{1}{a} \log \frac{L}{l} \right).$$

This result is in agreement with [4, 16].

4 Conclusions

The main conclusion of our calculation is that the value of energy obtained from the ADM analysis

$$\tilde{H}_b = M \left( 1 - \kappa \frac{5L}{a^2} - \frac{19}{4} + \frac{5}{2} \log \frac{L}{l} \right)$$

is not equal to the thermodynamic energy which we got in [4].

$$E = \int TdS = M \left( 1 - \frac{\kappa}{a^2} \left( \frac{35}{2} + 5 \log \frac{L}{l} \right) \right).$$

Note that the expression (52) contains a term proportional to $L$, while this term is absent from (53). It is not new that the various definitions of energy in general relativity do not coincide, even in the classical theory. In the recent paper [17] Fursaev show that the hamiltonian of the matter fields differs from their energy (defined as $\int T_{00}dV$) for the spaces which have bifurcate Killing horizons. The matters should be more complicate when the gravitational energy is also taken into account. The other reason for the discrepancy might be that the validity of the thermodynamical approach [18] might not be extrapolated beyond the zero’th order level naively. Really, it is easy to see that the expressions for energy, temperature and entropy obtained in [4] for null-dust model and here for SSG do not satisfy the thermodynamic relation $dE = TdS$. The temperature is obtained by the conical singularity.
method in both models. And finally, it is also possible that the hamiltonian analysis does not give the same result when it is applied to the local and the nonlocal forms of the same action. In any case, this question is interesting and deserves further attention and clarification.

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