A higher dimensional potential model approach in the study of the Isgur–Wise function for heavy–light mesons

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Abstract
The Nambu–Goto action of bosonic string predicts the quark–antiquark potential to be

\[ V(r) = \gamma r + \sigma r + \mu_0. \]

The coefficient \( \gamma = -\pi(d-2)/24 \) is the universal Lüscher coefficient of the Lüscher term \( \gamma r \), which depends upon the space–time dimension ‘d’. We take the linear term in the potential as the parent and the Lüscher term as perturbation for the generation of wave function for mesons in d space–time dimension. The wave function comes out in terms of Airy’s infinite polynomial series. With this wave function in higher dimension, we then study the Isgur–Wise function for heavy–light mesons and its derivatives.

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1. Introduction

It is well known that in the non-perturbative regime of quantum chromodynamics (QCD), i.e. in the low-energy regime, phenomenological models are very useful for the study of hadrons [1]. For a long time, potential models for mesons, with one heavy and one light quark, have been under theoretical investigation for the study of properties of hadrons, such as their mass, form factors, decay widths, etc.

It is worth mentioning here that, to get the mesonic wave function in such a potential model approach, the choice of potential is of utmost importance. There are several potentials modelling quark–antiquark bound states, such as the Martin potential [2], Cornell potential [3], Richardson potential [4] and Logarithmic potential [5]. Out of these, the Cornell potential, with \( \text{linear plus Coulombic} \) form, has been very popular and useful for such phenomenological study in QCD. The wave function for heavy–light mesons has been previously calculated with such a potential by applying the quantum mechanical perturbation technique [6]. This has been deduced both with a Coulombic term in the potential as parent [7–9] and also with a linear confinement term as parent [10, 11]. In the present work, we report the wave function for mesons developed within some higher-dimensional string inspired potential model approach.

In the classical string model for hadrons [12–14], proposed by Yoichiro Nambu and Tetsuo Goto, the two-quark potential can be expressed as [15] \( V(r) = \frac{\gamma}{r} + \sigma r + \mu_0 \). Here, the coefficient \( \gamma = -\frac{\pi(d-2)}{24} \) is the universal Lüscher coefficient of the Lüscher term \( \frac{\gamma}{r} \) [16], which depends upon the space–time dimension \( d \). \( \sigma \) is the string tension which characterizes the strength of the confining force between static charges. Its value is 0.178 GeV2 [17, 18]. \( \mu_0 \) is a regularization dependent constant. This Nambu–Goto potential in higher dimension is analogous to the linear plus Coulombic type Cornell potential in \((3 + 1)\)-dimensional QCD.

Previously we have developed a similar formalism considering the Lüscher term as parent in the perturbation technique [53]. However, the origin of the string potential model supports the idea of the linear confinement term as the main contributing part in the inter-quark potential and Lüscher’s term arises in this potential as a first order correction to the linear confinement term \( \sigma r \) [13]. In our present approach of developing the wave function for mesons using the perturbation technique, we consider the leading linear term as parent and Lüscher term as perturbation, and use the \( D \)-dimensional Schrödinger equation for solving the...
wave function of mesons. There exist several methods for a solution of higher dimensional Schrödinger equation for different potentials within the literature [19–29]. This we take as a generalization of our previous works in three-dimensional QCD [10, 11] to a higher dimension.

However, due to the lack of an exact solution for the $D$-dimensional Schrödinger equation with a linear plus Coulombic type potential, we opt to employ the perturbation method. With a linear term in the potential as parent, the wave function comes out in terms of Airy’s function [30], which is an infinite polynomial series. Previously, some approximated wave functions for mesons were deduced within the QCD framework, with a linear term in the Cornell potential as parent and a Coulombic term as perturbation, considering Airy’s polynomial function up to $O(r^3)$ [10], which was recently further improved upon by the authors considering the complete Airy series [52].

In this paper, we have reported the wave function for mesons in a higher space–time dimension considering the complete Airy’s infinite series, following Dalgarano’s method of perturbation [6, 31, 32]. Based on this wave function, we have then studied the Isgur–Wise function (IWF) for heavy–light mesons [33, 34] and its derivatives in a higher space–time dimension. The IWF is a universal function representing all the form factors of heavy–light mesons in the infinite mass limit in semileptonic transitions [35] of these mesons. It is a fundamental quantity in QCD, which can be determined non-perturbatively.

In such a study with a wave function containing Airy’s infinite series, the infinite upper limit of integration in the calculations of normalization constant and derivatives of the IWF gives rise to divergence. In our earlier works [11] we have successfully introduced some reasonable cut-off to the upper infinite limit of integration in the study of the IWF and its derivatives. Here we fix the cut-off to the infinite upper limit by applying the convergence condition of the wave function in perturbation theory. We also study the dimensional dependence of these parameters and compare our present results with our earlier ones, and also with the corresponding theoretical and experimental expectations for $D = 3$ [36–43, 53].

With this introduction, detailed formalism is reported in section 2, the calculations and results are in section 3. Final conclusion and remarks are reported in section 4.

2. Formalism

2.1. Potential model

As expressed in equation (1), the quark–antiquark potential for bosonic strings, as enunciated by Nambu and Goto, has the standard form

$$V(r) = -{\gamma \over r} + \sigma r + \mu_0$$

with $\gamma$ as the Lüscher term given by

$$\gamma = -{\pi (d-2) \over 24}.$$  (2)

For the sake of simplification in formalism and comparison with the Cornell potential [3], we take the Nambu–Goto potential of equation (1) as

$$V(r) = -{\gamma \over r} + \sigma r + \mu_0$$

now with

$$\gamma = {\pi (d-2) \over 24}.$$  (3)

Here, $d$ is the space–time dimension with $d = D + 1$, $D$ is the spatial dimension.

We take $\sigma r$ as parent and $-{\gamma \over r}$ as perturbation, with the $\mu_0 = 0$. Our unperturbed Hamiltonian is [19]

$$H_0 = -{\hbar^2 \over 2\mu} {\partial^2 \over \partial r^2} + \sigma r.$$  (4)

The Schrödinger equation in $D$-dimension is [23]

$$- {\hbar^2 \over 2\mu} \nabla^2 + V_0(r) \Psi(r, \Omega_D) = E \Psi(r, \Omega_D)$$  (5)

with [55, 56]

$$- {\hbar^2 \over 2\mu} \nabla^2 + V_0(r) \Psi(r, \Omega_D) = E \Psi(r, \Omega_D)$$

and

$$\nabla^2 = {1 \over r^{D-1}} {d \over dr} \left[ r^{D-1} {d \over dr} \right] = {d^2 \over dr^2} + {D - 1 \over r} {d \over dr}$$

$$\nabla^2 = {1 \over r^{D-1}} {d \over dr} \left[ r^{D-1} {d \over dr} \right] = {d^2 \over dr^2} + {D - 1 \over r} {d \over dr}$$

with [55, 56]

$$\nabla^2 = {1 \over r^{D-1}} {d \over dr} \left[ r^{D-1} {d \over dr} \right] = {d^2 \over dr^2} + {D - 1 \over r} {d \over dr}$$

and

$$\Psi(r, \Omega_D) = R(r) Y(\Omega_D).$$  (7)

Here, $\Lambda^{\pm}_{\lambda}(\Omega_D)$ is a generalization of the centrifugal barrier [44] in $D$-dimension.

The eigenvalues of $\Lambda^{\pm}_{\lambda}(\Omega_D)$ are given by

$$\Lambda^{\pm}_{D}(\Omega_D) Y(\Omega_D) = l(l + D - 2) Y(\Omega_D).$$  (8)

Here $Y(\Omega_D)$ and $R(r)$ are the spherical harmonics and spherical coordinates; $l$ is the angular momentum quantum number and $E$ is the energy eigenvalue.

This gives equation (5) in terms of the radial part as

$$\left[ {d^2 \over dr^2} + {D - 1 \over r} {d \over dr} - l(l + D - 2) + {2\mu \over \hbar^2} (E - V_0) \right] R(r) = 0.$$  (9)

Taking $\hbar = 1$ and for the $l = 0$ state, we have

$$\left[ {d^2 \over dr^2} + {D - 1 \over r} {d \over dr} + 2\mu (E - V_0) \right] R(r) = 0,$$  (10)

$$R''(r) + {D - 1 \over r} R'(r) + 2\mu (E - \sigma r) R(r) = 0.\quad (11)$$

We take

$$R(r) = \Lambda^{\pm}_{\lambda}(\Omega_D) U(r).$$  (12)

Then our equation (11) transforms into

$$U''(r) - {D - 1(D - 3) \over 4r^2} U(r) + 2\mu (E - \sigma r) U(r) = 0\quad (13)$$

or

$$U''(r) - {\Lambda(\Lambda + 1) \over r^2} U(r) + 2\mu (E - \sigma r) U(r) = 0$$

with

$$\Lambda = {D - 3 \over 2}.$$  (14)
2.1.1. Wave function with only linear term in potential.

We take

\[ \varrho(r) = \varrho_1 r - \varrho_0. \tag{15} \]

So, equation (14) reduces to

\[
\left[ \frac{d^2}{dr^2} - \frac{\Lambda(\Lambda + 1)}{(\varrho + \varrho_0)^2} + \frac{2\mu E}{\varrho_1^2} - \frac{2\mu \varrho \varrho_0}{\varrho_1^2} - \frac{2\mu \sigma \varrho_0}{\varrho_1^2} \right] U(\varrho) = 0. \tag{16}
\]

Case-I. For the case \( \varrho \to \infty \), the \( \frac{1}{(\varrho + \varrho_0)^2} \) term vanishes, the above equation (16) becomes

\[
\left[ \frac{d^2}{dr^2} + \frac{2\mu E}{\varrho_1^2} - \frac{2\mu \varrho \varrho_0}{\varrho_1^2} - \frac{2\mu \sigma \varrho_0}{\varrho_1^2} \right] U(\varrho) = 0. \tag{17}
\]

Following our previous work [11], we take \( \varrho_1 = (2\mu \sigma)^{1/3} \) and \( \varrho_0 = \left( \frac{2\varrho}{\sigma^2} \right)^{1/3} E \), so that our construction of \( \varrho(r) \) becomes

\[ \varrho(r) = \varrho_1 r - \varrho_0 = (2\mu \sigma)^{1/3} r - \left( \frac{2\mu}{\sigma^2} \right)^{1/3} E. \tag{18} \]

Then equation (17) transforms into

\[ \frac{d^2U(\varrho)}{d\varrho^2} - \varrho U(\varrho) = 0. \tag{19} \]

Here, we mention that, \( \varrho_0 \) are the zeros of Airy’s function such that \( Ai[\varrho_0] = 0 \). The solution of equation (19) comes out in terms of Airy’s function \( Ai[\varrho] \), as [45]

\[ U(\varrho) \sim Ai[\varrho] = N Ai\left(2\mu \sigma \right)^{1/3} \left(2\mu^{1/3} \right)^{1/3} E. \tag{20} \]

This is the asymptotic solution of equation (17).

Case-II. If we now take \( \varrho \to 0 \), then the centrifugal term of \( \frac{1}{(\varrho + \varrho_0)^2} \) will be dominant so that equation (16) can be reduced as

\[ U''(r) - \frac{\Lambda(\Lambda + 1)}{(\varrho + \varrho_0)^2} U(\varrho) = 0. \tag{21} \]

The non-singular solution of this equation is

\[ U(\varrho) \sim (\varrho + \varrho_0)^{\sqrt{\Lambda(\Lambda + 1)}}. \tag{22} \]

We construct the approximate analytic solution of equation (16) as the product of two solutions of extreme cases [46] as

\[ U(\varrho) \sim (\varrho + \varrho_0)^{m} Ai[\varrho]. \tag{23} \]

Thus, the unperturbed ground state meson wave function with linear term in potential has the form

\[ \Psi^{0}(r, D) = N_r \left( \frac{\varrho_1}{\varrho_0} \right)^{m} Ai[\varrho_1 r - \varrho_0]. \tag{24} \]

Here, \( N_r \) is the normalization constant for the unperturbed wave function.

2.1.2. Wave function with the Lüscher term as perturbation.

Given this unperturbed wave function in equation (24), we now proceed to construct the perturbed (and hence total) wave function for mesons taking the Lüscher term in potential as perturbation.

Our \( D \)-dimensional Schrödinger equation is

\[ \left[ -\frac{1}{2\mu} \left( \frac{d^2}{dr^2} + \frac{D - 1}{r} \frac{d}{dr} \right) + (\sigma r - E) \right] \Psi'(r) = \left( \frac{W}{r} + W' \right) \Psi(\varrho, D). \tag{25} \]

Here \( W' \) is the first order perturbed energy eigenvalue and in \( D \) spatial dimension [54] it has the expression

\[ W' = - \int_{0}^{\infty} DC_{D} r^{D - 1} |\Psi(r)|^2 \frac{d\varrho}{r}, \tag{26} \]

where \( C_D = \frac{\pi^{D/2}}{\Gamma \left( \frac{D}{2} + 1 \right)} \).

In terms of the radial wave function \( R_1(r, D) \) equation (25) can now be expressed as

\[ \left[ -\frac{1}{2\mu} \left( \frac{d^2}{dr^2} + \frac{D - 1}{r} \frac{d}{dr} \right) + (\sigma r - E) \right] R_1(r, D) = \left( \frac{W}{r} + W' \right) \left( \frac{\varrho_1}{\varrho_0} \right)^{m} Ai[\varrho_1 r - \varrho_0]. \tag{27} \]

We assume

\[ R_1(r, D) = r^{\frac{m}{2}} F(r, D) \left( \frac{\varrho_1}{\varrho_0} \right)^m Ai[\varrho_1 r - \varrho_0]. \tag{28} \]

Employing Dalgaro’s method of perturbation, the unperturbed wave function comes out as (appendix A)

\[ \Psi'(r, D) = N_1 r^{\frac{m}{2}} F(r, D) \left( \frac{\varrho_1}{\varrho_0} \right)^m \left[A_1(r, D) + A_2(r, D) r^2 + A_3(r, D) r^3 + \cdots \right] Ai[\varrho_1 r - \varrho_0]. \tag{29} \]

\( A_1(r), A_2(r), A_3(r) \), etc are in the explicit form as given by equations (A.24)–(A.28) of appendix A. The total wave function comes out to be

\[ \Psi^{10}(r, D) = \Psi^{0}(r, D) + \Psi'(r, D) = N_1 r^{\frac{m}{2}} \left[1 + A_1(r, D) r + A_2(r, D) r^2 + A_3(r, D) r^3 + \cdots \right] \times \left( \frac{\varrho_1}{\varrho_0} \right)^m Ai[\varrho_1 r - \varrho_0]. \tag{30} \]

Here, \( N_1 \) is the normalization constant for the total wave function.

2.2. Isgur–Wise function and its derivatives

For mesons containing one heavy quark (c,b,t), the mass of heavy quark is much greater than the QCD scale parameter \( \Lambda_{QCD} \); the four-velocity of heavy quark is almost the same as the four velocity of meson and in the meson rest frame, the heavy quark appears as a static colour source [47]. This brings spin flavour symmetry and reduction in the number of form factors [48]. All six form factors in semi-leptonic decay are now expressible in terms of a single universal function depending only on the velocity transfer. This is known as the IWF. It measures the overlap of the wave function of light degrees of freedom in the initial and final mesons. This IWF \( \xi(v, v') \) is a function of four velocities \( v \) and \( v' \) of the heavy quark.
particle before and after decay. As $E_{\text{reci}} = \mu(v, v') - 1$ is the recoil energy of the final state meson $P'$ (of mass $\mu$) in the rest frame of initial meson $P$, the point $y = v, v' = 1$ is referred to as the zero recoil limit. The IWF is thus normalized at the zero recoil point, which is a consequence of current conservation [57]. If $y = v, v'$ (known as Lorentz boost), then, for zero recoil ($y = 1$), $\xi(y) = 1$. Knowledge of the IWF and its derivatives is essential in all calculations that lead to the estimation of branching ratios of semi-leptonic decays and of the elements of the Cabibbo–Kobayashi–Maskawa matrix (CKM).

The modelling of the IWF is guided by the condition of its extrapolation to zero recoil. Actually there is no unique formulation of the IWF and different ansatz produce different expressions of the IWF [49]. These expressions differ at large values of $y$, but agree within the range $y \approx 1$. Generally, the available experimental data fit to the linear form of the derivative term. But, as $\rho$ increases, higher order terms make a difference, as far as the shape of the IWF is concerned. For larger values of slope parameter ($\rho > 1$), it is important to include higher terms, in the expansion of the IWF about $y = 1$, in the analysis of experimental data. Also, retaining only the first term in the expansion leads to an underestimation of the curvature parameters. From the expression of total wave function in equation (37), we get

$$\xi(y) = 1 - \rho^2(y - 1) + C(y - 1)^2 + \cdots. \quad (31)$$

Here $\rho^2$ is the slope (charge radii) and $C$ is the curvature (convexity parameter) of the IWF, which are measured at zero recoil point as

$$\rho^2 = -\left.\frac{\delta \xi(y)}{\delta y}\right|_{y=1}, \quad C = \left.\frac{1}{2} \frac{\delta^2 \xi(y)}{\delta y^2}\right|_{y=1} \quad \cdot \quad (32)$$

The calculation of $\rho^2$ and $C$ provides a measure of the validity of the heavy quark effective theory in the infinite quark mass limit. There have been several attempts to calculate $\rho^2$ and $C$ from theory and models [36–43]. The corresponding results are shown in table 4. The curvature parameter $C$ gives the measure and direction of convexity of the shape of the IWF. A negative curvature would result in a concave IWF in the plot. However, in general, the IWF is expected to have a positive curvature for all $y > 1$. A quantitative reason in calculating the curvature along with the slope parameter is that beyond the first derivative, higher derivatives play a non-negligible role at $y > 1$, as discussed earlier. Also, as data are now coming out to be more and more precise, it is not only of an academic interest to analyse higher derivatives of the IWF.

In the non-relativistic quark model, the IWF describes the overlap between wave functions of light degrees of freedom. The calculation of this IWF is non-perturbative in principle [50] and this function depends upon the meson wave function and some kinematic factor as given below, in $D$ spatial dimension:

$$\xi(y) = \int_0^\infty DC_Dr^{D-1}|\Psi(r)|^2 \cos(pr) \, dr. \quad (33)$$

Here, $p^2$ is the square of virtual momentum transfer. Since the hadronic matrix element describing the weak decay process is invariant under complex conjugation along with an interchange of $v$ and $v'$, we get the squared invariant velocity transfer as $(v - v')^2 = 2(y - 1) [57]$. From this we get $p^2 = 2\mu^2(y - 1)$. From current conservation, it is now obvious that the IWF is normalized at $p^2 = 0$.

As $\cos(pr) = 1 - \frac{p^2r^2}{2} + \frac{p^4r^4}{4} + \cdots$, considering $\cos(pr)$ up to $D(r^2)$ we get

$$\xi(y) = DC_D\int_0^\infty r^{D-1}|\Psi(r)|^2 \, dr - \left[DC_D\mu^2 \int_0^\infty r^{D+1}|\Psi(r)|^2 \, dr\right]$$

$$\times \left[(y - 1) + \frac{1}{6} DC_D\mu^4 \int_0^\infty r^{D+3}|\Psi(r)|^2 \, dr\right](y - 1)^2. \quad (34)$$

Comparing (31) and (34) gives us

$$\rho^2 = DC_D\mu^2 \int_0^\infty r^{D+1}|\Psi(r)|^2 \, dr, \quad (35)$$

$$C = \frac{1}{6} DC_D\mu^4 \int_0^\infty r^{D+3}|\Psi(r)|^2 \, dr, \quad (36)$$

$$DC_D\int_0^\infty r^{D-1}|\Psi(r)|^2 \, dr = 1. \quad (37)$$

Equation (37) gives the normalization constants $N$ and $N_1$ for $\Psi^0(r, D)$ and $\Psi^{\text{total}}(r, D)$.

3. Calculations and results

With the wave function constructed, we now proceed to study the IWF and its derivatives, like slope and curvature parameters. From the expression of total wave function in equation (30), we find that it contains two infinite series—one of power series in $r$ with coefficients $A_1(r, D), A_2(r, D), A_3(r, D), \ldots$ and another of infinite Airy series $A_3[0]$. As the infinite limit of integration in the calculations of the IWF and its derivatives, involving Airy’s infinite series, gives rise to divergence, we opt for some reasonable cut-off to this infinite upper limit of integration. In principle, this cut-off $r_0$ should be greater than the size of hadrons, i.e., $r_0$ should be greater than $\frac{1}{m^2} m_0$, being the mass of hadron. The convergence condition of the total wave function obtained through the perturbation technique implies that $\Psi'(r) < \Psi(r)$. From this we obtain

$$\left|A_1(r, D) + A_2(r, D)r^2 + A_3(r, D)r^3 + \cdots\right| < 1. \quad (38)$$

This condition gives us the limiting values of cut-off $r_0$ for different $D$ values. Considering two terms in equation (38), we obtain the equation relating cut-off $r_0$ and dimension $D$ as

$$A_1(r_0, D)r_0 + A_2(r_0, D)r_0^2 = 1. \quad (39)$$
Using expressions for $A_1(r, D)$, $A_2(r, D)$ and on simplification, equation (39) transforms into the polynomial series in $r_0$:

$$K_{11}(D) r_0^3 + K_{12}(D) r_0^2 + K_{13}(D) r_0 + K_{14}(D) = 0. \quad (40)$$

Here, coefficients $K_{11}(D), K_{12}(D), \text{etc}$ are independent of $r_0$, having the following explicit $D$-dependence in terms of the function $f(D) = \frac{1 - D e 3}{2}$:

$$K_{11}(D) = 2 \mu W'[f(D)(f(D) - 1) - (D - 1) f(D)] + 2 f(D) + (D - 1) q_1. \quad (41)$$

$$K_{12}(D) = -4 \mu^2 [2 f(D) + (D - 1) + 2 q_1] + 2 \mu W'[2 q_1 k] - 2 \mu q_1 [f(D)(f(D) - 1) - (D - 1) f(D)] + 4 \mu^2 [f(D) + (D - 1) q_1] + 2 \mu W'[f(D) - 1 q_1 k], \quad (42)$$

$$K_{13}(D) = 2 \mu W'[f(D) + (D - 1) + 2 q_1] - 4 \mu q_1 [2 q_1 k] - 2 \mu q_1 [f(D) - 1 q_1 k] + 2 \mu W'[q_1 k^2] + 1, \quad (43)$$

$$K_{14}(D) = -2 \mu q_1 [q_1 k^2]. \quad (44)$$

Equation (40) is a cubic equation having only one real root which is given by

$$r_0 = \frac{-K_{12}}{3 K_{11}} - 2^{1/3} \left[ -K_{12}^2 + 3 K_{12} K_{13} \right]^{1/3} + \frac{1}{3} \left[ 2^{1/3} K_{11} \right]^{2/3} \quad \text{with}$$

$$g = 9 K_{11} K_{12} K_{13} - 2 K_{12}^2 - 27 K_{11}^2 K_{13}^2$$

$$+ \sqrt{4(3 K_{11} K_{13} - K_{12}^2)^3 + (9 K_{11} K_{12} K_{13} - 2 K_{12}^2 - 27 K_{11}^2 K_{13}^2)^2}. \quad (45)$$

Following equations (45) and (41)–(44), the numerical values of cut-off $r_0$ for D and B mesons for different dimensions are shown in table 1.

Consideration of such cut-off to the upper limit of integrations will not sacrifice the nature and value of the IWF and its derivatives because Airy’s function falls very sharply (almost exponentially) and almost dies out with increasing $r$-value [45, 51]. In fact, the Airy’s function value becomes negligibly small for $r > 4$ (AiryAi [4] = 0.000952). Further, we restrict our calculation up to Airy order $r^{10}$ and term $A_5(r, D)$ in the second infinite series. The values of unperturbed energy $E$ and of perturbed energy $W'$ for different $D$ values are shown in table 2. The table reflects that the perturbed energy $W'$ is less than the unperturbed energy $E$, as it should be.

It is to be mentioned here that, $A_1(r, D)$, $A_2(r, D)$, $A_3(r, D)$, etc appearing in the second infinite series of the wave function are functions of $C_1(r)$ and $C_2(r)$. Now, for the simplification of calculation, we consider the lowest Airy order in computing $C_1(r)$ and $C_2(r)$ (appendix B).

With these more explicit forms of functions $A_i(r, D)$ in the meson wave function as in equation (39), we now explore the IWF and its derivatives for Airy’s polynomial order $r^{10}$, considering specific cut-off values for specific dimension value $D$. Results for B and D mesons are shown in table 3.

The variation of the IWF with Lorentz boost $y$ for different $D$ values are shown in figure 1. The figure is self-explanatory.

| $D$ | $E$ | $W'$ |
|-----|-----|------|
| 3   | 0.345938 | 0.137536 |
| 4   | 0.637435 | 0.141397 |
| 5   | 0.832819 | 0.148987 |
| 6   | 1.04202 | 0.16323 |
| 7   | 1.2732 | 0.18567 |
| 8   | 1.5177 | 0.20905 |
| 9   | 1.77193 | 0.233483 |

| $D$ | $E$ | $W'$ |
|-----|-----|------|
| 3   | 0.352764 | 0.139463 |
| 4   | 0.708288 | 0.143216 |
| 5   | 1.0102 | 0.149692 |
| 6   | 1.66759 | 0.16872 |
| 7   | 1.74195 | 0.20805 |
| 8   | 1.74838 | 0.24205 |
| 9   | 1.76108 | 0.306265 |

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Figure 1. Variation of $\xi(y)$ with $y$ for different $D$. (a) $\xi(y)$ versus $y$ for D meson and (b) $\xi(y)$ versus $y$ for B meson.
In this work, we have developed a wave function for mesons in $d = D+1$ space–time dimension considering the linear confinement term of the Nambu–Goto potential as parent and the Lüscher term as perturbation. We have then further extended our analysis in finding the derivatives of the IWF, which are here obviously dimension dependent.

In calculating the slope and curvature of the IWF, we have introduced a cut-off value to the infinite upper limit of integrations to overcome divergences; this cut-off is found to be dimension dependent. We put forward our observation that our choice of specific cut-off value in these calculations does not compromise the accuracy of our results. This will be more established, if we consider the asymptotic form of Airy’s function (at $D = 3$) [51]

$$ A_i[\varrho]_\text{asympt} \sim \frac{\exp\left(-\frac{1}{2}\varrho^{3/2}\right)}{2^{1/4}\varrho^{1/4}}. \quad (46) $$

With this asymptotic form (for $D = 3$) we have also calculated the derivatives of $\xi(\varrho)$ considering the limit of integration from $r_0$ to $\infty$. The results for different cut-off values are shown in table 5. The values of $\rho^2$ and $C$ are exceptionally low here as compared to our calculated values thus justifying our consideration of restricting the upper limit of integration to some reasonable finite value.

While studying the compatibility of our results for $D = 3$ with the standard results of different models in three dimensional QCD (table 4), we find that our results are comparatively lower than the expectations. However, while comparing our results with that of our previous work [53], we put forward our comment that in [53] the results are exceptionally higher than the expectations and in our present approach we have overcome this limitation. Although our results are comparatively lower than the expectations, we find that $\rho^2$ and $C$ values gradually go on increasing with the increase of dimension $D$, as is evident from table 3, which is further confirmed from figure 2.

Lastly, we make the following comments on our present formalism and calculations.

1. In this work, we have developed a meson wave function with $n$ higher dimensional outlook and made studies of the IWF and its derivatives within such approach.
We opt for the quantum mechanical perturbation technique in deducing the meson wave function, due to the constraint in getting an exact analytic solution of the Schrödinger equation involving linear plus Coulombic potential.

As far as numerical accuracy is concerned, the present perturbation approach appears short of the numerical solution of the Schrödinger equation or the calculations of Lattice QCD. Still, we believe that the potential model approach gives sufficiently reasonable physical insight into the problem.

Lastly, terms $A_1(r, D)$, $A_2(r, D)$, $A_3(r, D)$, etc of the second infinite series in the meson wave function are truncated up to $A_5(r, D)$ in our calculation. Improvement in the numerical analysis can be carried out by considering higher terms in both infinite polynomial series involved in the wave function.

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Appendix A. Determination of equation (29) using Dalgarno’s method of perturbation

From equation (28), we have

$$R_1(r) = \phi_1^m r^{1-\beta-\gamma} F(r) A_i[\varrho].$$

(A.1)

We obtain

$$R'_1(r) = \phi_1^m r^{1-\beta-\gamma} \left[ \frac{1-D+2m}{2} \frac{1}{r} F(r) A_i[\varrho] \right. \left. + F'(r) A_i[\varrho] + \varrho_1 F(r) A''_i[\varrho] \right].$$

(A.2)

$$R''_1(r) = \phi_1^m r^{1-\beta-\gamma} \left[ \frac{1-D+2m}{2} \left( \frac{1-D+2m}{2} \right) - 1 \right. \times \frac{1}{r^2} F(r) A_i[\varrho] + \frac{1-D+2m}{2} \frac{1}{r} F'(r) A_i[\varrho] \left. + 2 \frac{1-D+2m}{2} \frac{1}{r} \varrho_1 F(r) A''_i[\varrho] + F''(r) A_i[\varrho] \right. \left. + 2 \varrho_1 F'(r) A''_i[\varrho] + \varrho_1^2 F(r) A''_i[\varrho] \right].$$

(A.3)

We take

$$A_i'[\varrho] = Z(r) A_i[\varrho]$$

(A.4)

so that

$$A_i''[\varrho] = Z^2(r) A_i[\varrho] + Z'(r) A_i[\varrho].$$

(A.5)

We then have

$$R'_1(r) = \phi_1^m r^{1-\beta-\gamma} A_i[\varrho] \left[ \frac{1-D+2m}{2} \frac{1}{r} F(r) + F'(r) \right. \left. \right. \left. + \varrho_1 F(r) Z \right].$$

(A.6)

$$R''_1(r) = \phi_1^m r^{1-\beta-\gamma} A_i[\varrho] \left[ \frac{1-D+2m}{2} \left( \frac{1-D+2m}{2} \right) - 1 \right. \times \frac{1}{r^2} F(r) + \frac{1-D+2m}{2} \frac{1}{r} F'(r) \left. + 2 \frac{1-D+2m}{2} \frac{1}{r} \varrho_1 F(r) Z + F''(r) + 2 \varrho_1 F'(r) Z \right. \left. \right. \left. + \varrho_1^2 F(r) Z^2 + \varrho_1^2 F(r) Z' \right].$$

(A.7)

Putting equations (A.1), (A.6) and (A.7) into equation (27), we obtain

$$F''(r) + F'(r) K_1(r, D) + F(r) K_2(r, D) - 2 \mu (\sigma r - E) F(r) = -2 \mu \left( \frac{Z}{r} + W' \right)$$

(A.8)

with

$$K_1(r, D) = 2 \frac{1-D+2m}{2} \frac{1}{r} + 2 \varrho_1 Z + \frac{D-1}{r}.$$ 

(A.9)

We take

$$Z(r) = \frac{C_1(r)}{r},$$

(A.11)

and

$$Z^2(r) + Z'(r) = \frac{C_2(r)}{r^2}.$$ 

(A.12)

Our $K_1(r, D)$ and $K_2(r, D)$ transform to

$$K_1(r, D) = \frac{2}{r} \left( \frac{1-D+2m}{2} + (D-1) + 2 \varrho_1 C_1(r) \right) \frac{1}{r} = M_1(r, D) \frac{1}{r^2}. $$

(A.13)

$$K_2(r, D) = \frac{1}{r^2} \left( - \frac{(D-1)}{2} + \frac{1-D+2m}{2} \right) \left( 2 \varrho_1 C_1(r) + \varrho_1^2 C_2(r) \right) \frac{1}{r^2} = M_2(r, D) \frac{1}{r^2}. $$

(A.14)
where
\[
M_1(r, D) = 2\frac{1-D+2m}{2} + (D-1) + 2\varrho_1 C_1(r), \quad (A.15)
\]
\[
M_2(r, D) = \frac{1-D+2m}{2} \left( \frac{1-D+2m}{2} - 1 \right) - (D-1) \frac{1-D+2m}{2} + (\frac{1-D+2m}{2} + (D-1))\varrho_1 C_1(r) + \varrho_1^2 C_2(r).
\]
\[
(A.16)
\]
It is to be noted that at \( D = 3 \),
\[
K_1(r) = 2\varrho_1 C_1(r)\frac{1}{r}, \quad (A.17)
\]
\[
K_2(r) = \varrho_1^2 C_2(r)\frac{1}{r^2}.
\]
(A.18)
And equation (A.8) becomes
\[
F''(r) + 2\varrho_1 C_1(r)\frac{1}{r} F'(r) + \varrho_1^2 C_2(r)\frac{1}{r^2} F(r) - 2\mu\sigma r - EF(r)
\]
\[
= -2\mu \left( \frac{\nu}{r} + W \right), \quad (A.19)
\]
which is similar to the equation obtained with the three-dimensional QCD potential. We take
\[
F(r, D) = \sum l A_l(r, D) r^l
\]
(A.20)
so that
\[
F'(r, D) = l \sum A_l(r, D) r^{l-1}, \quad (A.21)
\]
\[
F''(r, D) = l(l-1) \sum A_l(r, D) r^{l-2}. \quad (A.22)
\]
Applying equations (A.20)–(A.22) into equation (A.8), we obtain
\[
\sum l A_l(r, D)[l(l-1) + M_1(r, D)l + M_2(r, D)] r^{l-2}
\]
\[
+ 2\mu E \sum A_l(r, D) r^{l-2} = -2\mu \gamma \frac{2\mu W}{r} - 2\mu W'.
\]
(A.23)
Equating power of \( r^{-2} \) on both sides of equation (A.23), we get \( A_0 M_2 = 0 \), which imply that \( A_0 = 0 \).
Further, equating powers of \( r^{-1}, r^0, r^2 \) and \( r^3 \) of equation (A.23), we obtain
\[
A_1(r, D) = -\frac{2\mu\gamma}{M_1 + M_2}, \quad (A.24)
\]
\[
A_2(r, D) = -\frac{2\mu W}{2 + 2M_1 + M_2}, \quad (A.25)
\]
\[
A_3(r, D) = \frac{4\mu^2 E\gamma}{6 + 3M_1 + M_2}(M_1 + M_2), \quad (A.26)
\]
\[
A_4(r, D) = \frac{4\mu^2 EW'(M_1 + M_2) - 4\mu^2 \sigma \gamma (2 + 2M_1 + M_2)}{12 + 4M_1 + M_2}(2 + 2M_1 + M_2)(M_1 + M_2),
\]
(A.27)
\[
A_5(r, D) = \frac{8\mu^3 E^2\gamma (2 + 2M_1 + M_2) + 4\mu^2 \gamma W'(6 + 3M_1 + M_2)(M_1 + M_2)}{(20 + 5M_1 + M_2)(6 + 3M_1 + M_2)(2 + 2M_1 + M_2)(M_1 + M_2)}.
\]
(A.28)
With these expressions of \( A_1(r, D), A_2(r, D), A_3(r, D), \) etc we can now construct \( F(r, D) \) as
\[
F(r, D) = A_1(r, D) r + A_2(r, D) r^2 + A_3(r, D) r^3 + A_4(r, D) r^4 + A_5(r, D) r^5 + \ldots.
\]
(A.29)
And our radial wave function \( R_1(r, D) \) comes out to be
\[
R_1(r, D) = r^{1/2} \varrho_1(r) r^m F(r, D) \text{Ai} [\varrho_1]. \quad (A.30)
\]
The perturbed wave function \( \Psi'(r) \) and total wave function \( \Psi^{tot}(r) \) are constructed as below:
\[
\Psi'(r, D) = N_1 r^{1/2} \varrho_1(r)^m [A_1(r, D) r + A_2(r, D) r^2 + A_3(r, D) r^3 + \ldots] \text{Ai} [\varrho_1 r - \varrho_0].
\]
(A.31)
\[
\Psi^{tot}(r, D) = \Psi^0 (r, D) + \Psi'(r, D)
\]
\[
= N_1 r^{1/2} \varrho_1(r)^m [1 + A_1(r, D) r + A_2(r, D) r^2 + A_3(r, D) r^3 + \ldots] \text{Ai} [\varrho_1 r - \varrho_0].
\]
(A.32)
\[
\textbf{Appendix B. Calculation of } C_1(r) \text{ and } C_2(r)
\]
Airy’s infinite series as a function of \( \varrho = \varrho_1 r - \varrho_0 \) can be expressed as [51]
\[
\text{Ai} [\varrho_1 r - \varrho_0] = a_1 \left[ 1 + \frac{(\varrho_1 r - \varrho_0)^3}{6} + \frac{(\varrho_1 r - \varrho_0)^6}{180} + \frac{(\varrho_1 r - \varrho_0)^9}{12960} + \ldots \right] - b_1 [\varrho_1 r - \varrho_0] + \frac{(\varrho_1 r - \varrho_0)^3}{12} + \frac{(\varrho_1 r - \varrho_0)^7}{504} + \frac{1}{45360} \ldots \quad (B.1)
\]
with \( a_1 = \frac{1}{3^{5/3}(2/3)} = 0.355 \, 028 \, 1 \) and \( b_1 = \frac{1}{3^{5/3}(1/3)} = 0.258 \, 819 \, 4 \).
To find $C_1(r)$ and $C_2(r)$, we take a truncated Airy series up to the lowest order, so that we have

$$Z(r) = \frac{C_1(r)}{r} = \frac{Ai'(q_1r - q_0)}{Ai(q_1r - q_0)} = \frac{-b_1q_1}{a_1 - b_1(q_1r - q_0)}$$

(B.2)

$$= \frac{b_1q_1}{1 - b_1(q_1r - q_0)}$$

(B.3)

$$= \frac{1}{r} \left[ 1 - k + \frac{1}{r} \right]^{-1}$$

(B.5)

Therefore,

$$C_1(r) = 1 + \frac{k}{r}$$

(B.7)

with

$$k = \frac{a_1 + b_1q_0}{b_1q_1}.$$  

(B.8)

Also,

$$Z'(r) = \frac{1}{r^2} \left( 1 + \frac{2k}{r^2} \right) = \frac{1}{r^2} + \frac{2k}{r^3} + \frac{k^2}{r^4}$$

(B.9)

and

$$Z'(r) = -\frac{1}{r^2} - \frac{2k}{r^3}$$

(B.10)

so that

$$\frac{C_2(r)}{r^2} = Z(r) + Z'(r) = \frac{k^2}{r^4}.$$  

(B.11)

We thus obtain

$$C_2(r) = \frac{k^2}{r^2}.$$  

(B.12)

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