ON GREEN FUNCTIONS OF SECOND-ORDER ELLIPTIC OPERATORS ON RIEMANNIAN MANIFOLDS: THE CRITICAL CASE

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ABSTRACT. Let $P$ be a second-order, linear, elliptic operator with real coefficients which is defined on a noncompact and connected Riemannian manifold $M$. It is well known that the equation $Pu = 0$ in $M$ admits a positive supersolution which is not a solution if and only if $P$ admits a unique positive minimal Green function on $M$, and in this case, $P$ is said to be subcritical in $M$. If $P$ does not admit a positive Green function but admits a global positive solution, then such a solution is called a ground state of $P$ in $M$, and $P$ is said to be critical in $M$.

We prove for a critical operator $P$ in $M$, the existence of a Green function which is dominated above by the ground state of $P$ away from the singularity. Moreover, in a certain class of Green functions, such a Green function is unique, up to an addition of a product of the ground states of $P$ and $P^*$. This result extends and sharpens the celebrated result of P. Li and L.-F. Tam concerning the existence of a symmetric Green function for the Laplace-Beltrami operator on a smooth and complete Riemannian manifold $M$.

2000 Mathematics Subject Classification. Primary 35J08; Secondary 31C35, 35A08, 35B09, 58G03.

Keywords. Fundamental solution, Green function, critical operator, positive solutions.

1. Introduction

Let $M$ be a noncompact and connected manifold of dimension $N \geq 2$ and of class $C^2$. We assume that $\nu$ is a positive measure on $M$, satisfying $d\nu = f \, \text{vol}$, where $f$ is a strictly positive function and vol is the volume form of $M$. On $M$ we consider a second-order elliptic operator $P$ with real coefficients which (in any coordinate system $(U; x_1, \ldots, x_N)$) is of the divergence form

$$Pu := -\text{div} \left( (A(x)\nabla u + u b(x)) \right) + b(x) \cdot \nabla u + c(x)u. \quad (1.1)$$

Here, the minus divergence is the formal adjoint of the gradient with respect to the measure $\nu$. We assume that for every $x \in \Omega$ the matrix $A(x) := [a^{ij}(x)]$ is symmetric and that the real quadratic form

$$\xi \cdot A(x) \xi := \sum_{i,j=1}^{N} \xi_i a^{ij}(x) \xi_j \quad \xi \in \mathbb{R}^N$$

(1.2)
is positive definite. Moreover, throughout the paper it is assumed that $P$ is locally uniformly elliptic, and that locally, the coefficients of $P$ are sufficiently regular in $M$ such that standard elliptic (local) regularity results hold true. Our results hold for example when $A$ and $f$ are locally Hölder continuous, $b, \tilde{b} \in L^p_{\text{loc}}(M; \mathbb{R}^N, dx)$, and $c \in L^{p/2}_{\text{loc}}(M; \mathbb{R}, dx)$ for some $p > N$. Here $dx$ is the Lebesgue measure on $\mathbb{R}^N$ (note that locally $dv$ is equivalent to $dx$). In fact, we need to assume further local regularity on the coefficients that guarantee the existence of the limit

$$\lim_{x \to x_0} \frac{u(x)}{v(x)},$$

where $u$ and $v$ are positive solutions of the equation $Pu = 0$ in a punctured neighborhood of any $x_0 \in M$, and the limit might be $\infty$ (for sufficient conditions that guarantee it, see for example [8] and references therein).

The formal adjoint $P^*$ of the operator $P$ is defined on its natural space $L^2(M, dv)$. When $P$ is in divergence form (1.1) and $b = \tilde{b}$, the operator

$$Pu = -\text{div} \left[(Au + ub)\right] + b \cdot \nabla u + cu,$$

is symmetric in the space $L^2(M, dv)$. Throughout the paper, we call this setting the symmetric case.

By a solution $v$ of the equation $Pu = 0$ in a domain $\Omega \subset M$, we mean $v \in W^{1,2}_{\text{loc}}(\Omega)$ that satisfies the equation $Pu = 0$ in $\Omega$ in the weak sense. Subsolutions and supersolutions are defined similarly. We denote the cone of all positive solutions of the equation $Pu = 0$ in $\Omega$ by $C_P(\Omega)$. We say that $P$ is nonnegative in $\Omega$ (and denote it by $P \geq 0$ in $\Omega$) if $C_P(\Omega) \neq \emptyset$. We recall that in the symmetric case, by the Allegretto-Piepenbrink theorem, $P \geq 0$ in $\Omega$ if and only if the associated quadratic form is nonnegative on $C^\infty_0(\Omega)$ (see for example [9]).

**Throughout the paper we always assume that $P \geq 0$ in $M$.**

**Definition 1.1.** A function $G^M_P : M \times M \to [-\infty, \infty]$ is said to be a Green function (fundamental solution) of the operator $P$ in $M$ if for any $x, y \in M$

$$P(x, \partial_x)G^M_P(x, y) = \delta_y(x) \quad \text{and} \quad P^*(y, \partial_y)G^M_P(x, y) = \delta_x(y) \quad \text{in} \; M,$$

and

$$G^M_P(x, y) = G^M_P(y, x) \quad \forall x, y \in M,$$

where $\delta_z$ denotes the Dirac distribution at $z \in M$.

A positive Green function $G^M_P(x, y)$ is said to be a positive minimal Green function of $P$ in $M$ if any other positive Green function $\hat{G}^M_P(x, y)$ of $P$ in $M$ satisfies $0 < G^M_P(x, y) \leq \hat{G}^M_P(x, y)$ in $M \times M$.

The aim of the present article is to study the existence and the uniqueness of a certain type of Green functions for a general (not necessarily symmetric) nonnegative elliptic operator of the form (1.1) in $M$. We recall that the existence of a fundamental solution for differential operators with constant
coefficients has been proved by B. Malgrange and L. Ehrenpreis in [1, 5] and for elliptic operators with analytic coefficients by F. John [3] using the unique continuation property.

We are motivated by the celebrated paper of Peter Li and Luen-Fai Tam [4] who constructed a symmetric Green function for the Laplace-Beltrami operator $P := -\Delta$ on a complete, noncompact, smooth Riemannian manifold $M$.

In order to explain the Li–Tam result, we recall the standard construction of the positive minimal Green function for a nonnegative operator $P$ in $M$.

Let $\{M_j\}_{j=1}^{\infty}$ be a (compact) exhaustion of $M$, i.e. a sequence of smooth, relatively compact domains in $M$ such that $M_1 \neq \emptyset$, $M_j \subset M_{j+1}$ and $\bigcup_{j=1}^{\infty} M_j = M$. For every $j \geq 1$, let $G_P^{M_j}(x, y)$ be the Dirichlet Green function of $P$ in $M_j$. By the generalized maximum principle, $\{G_P^{M_j}(x, y)\}_{j=1}^{\infty}$ is an increasing sequence of positive functions which either converges in $M \times M$ to $G_P^{M}(x, y)$, the positive minimal Green function of the operator $P$ in $M$ (this is the subcritical case, see Definition 2.2), or

$$\lim_{j \to \infty} G_P^{M_j}(x, y) = \infty. \quad (1.5)$$

If (1.5) holds, then we say that $P$ is critical in $M$ (for more details see Section 2).

Li and Tam [4] modified the above construction for the special case of a critical Laplace-Beltrami operator $P := -\Delta$ on a complete smooth manifold, by subtracting from the above sequence an appropriate sequence $\{a_j\}_{j=1}^{\infty}$ of positive numbers. It turns out that the sequence $\{G_P^{M_j}(x, y) - a_j\}_{j=1}^{\infty}$ admits a subsequence that converges to a symmetric Green function, which we call a Li–Tam Green function. Moreover, Li and Tam proved that such a Green function satisfies certain boundedness properties.

Using a modification of the Li–Tam’s construction, we obtain for a general critical operator $P$ of the form (1.1), the existence of a Green function that satisfies certain boundedness properties. Moreover, the obtained Green function is unique (up to an addition of a product of the unique ground states of $P$ and $P^\ast$) in a certain class of Green functions which we call the Li–Tam class of Green functions (See Definition 2.6). We note that the proof of Li and Tam has to be modified significantly, since in the general case, the constant function is not a solution, the weak maximum principle and the unique continuation property do not hold, and $P$ is not necessarily symmetric.

The outline of our paper is as follows. In Section 2 we recall some definitions and basic known results, and state the main results of the paper (Theorem 2.5 and Theorem 2.8). Section 3 and Section 4 are devoted to the proof of Theorem 2.5 and Theorem 2.8 respectively. Finally, the paper ends
in Section 5, where we present some explicit examples of Green functions, discuss some applications, and pose some questions related to our study.

We conclude this section with some notation. Throughout the paper, we write \( \Omega_1 \Subset \Omega_2 \) if \( \Omega_2 \) is open, \( \overline{\Omega_1} \) is compact and \( \Omega_1 \subset \Omega_2 \). Let \( f, g \in C(\Omega) \) be positive functions, we denote \( f \asymp g \) if there exists a positive constant \( C \) such that

\[
C^{-1}g(x) \leq f(x) \leq Cg(x) \quad \text{for all } x \in \Omega,
\]

and \( f \sim g \) as \( x \to p \) if \( \lim_{x \to p} \frac{g(x)}{f(x)} = 1 \). The oscillation of a function \( f \) in a set \( K \) is denoted by \( \text{Osc}(f) \) in \( K \).

Finally, the ideal point in the one-point compactification of \( M \) is denoted by \( \overline{\infty} \).

2. Statement of the main results

In the present section we state the main results of the paper. To this end, we first recall the definitions of critical and subcritical operators and of a ground state (for more details on criticality theory, see [6, 7, 9] and references therein).

**Definition 2.1.** Let \( K \Subset M \). We say that \( u \in \mathcal{C}_P(M \setminus K) \) is a positive solution of the operator \( P \) of minimal growth in a neighborhood of infinity in \( M \), if for any \( K \Subset K_1 \Subset M \) with a smooth boundary and any positive supersolution \( v \) of \( Pw = 0 \) in \( M \setminus K_1 \) satisfying \( v \in C((M \setminus K_1) \cup \partial K_1) \), the inequality \( u \leq v \) on \( \partial K_1 \) implies that \( u \leq v \) in \( M \setminus K_1 \).

A positive solution \( u \in \mathcal{C}_P(M) \) which has minimal growth in a neighborhood of infinity in \( M \) is called the (Agmon) ground state of \( P \) in \( M \).

**Definition 2.2.** The operator \( P \) is said to be critical in \( M \) if \( P \) admits a ground state in \( M \). The operator \( P \) is called subcritical in \( M \) if \( P \geq 0 \) in \( M \) but \( P \) is not critical in \( M \).

**Remark 2.3.** Let \( P \geq 0 \) in \( M \). It is well known that the operator \( P \) is critical in \( M \) if and only if the equation \( Pu = 0 \) in \( M \) has a unique (up to a multiplicative constant) positive supersolution (see [6, 7]). In particular, \( P \) is critical in \( M \) if and only if \( P \) does not admit a positive Green function in \( M \). Further, in the critical case \( \dim \mathcal{C}_P(M) = 1 \), and the unique positive solution (up to a multiplicative positive constant) is a ground state of \( P \) in \( M \). Moreover, \( P \) is critical in \( M \) if and only if \( P^* \) is critical in \( M \).

On the other hand, \( P \) is subcritical in \( M \) if and only if \( P \) admits a unique positive minimal Green function \( G^M_P(x, y) \) in \( M \). Moreover, for any fixed \( y \in M \), the function \( G^M_P(\cdot, y) \) is a positive solution of minimal growth in a neighborhood of infinity in \( M \). We recall that \( G^M_P(x, y) = G^M_P(y, x) \).

**Remark 2.4.** If \( P \geq 0 \) in \( M \), then \( P \) restricted to any subdomain \( \Omega \subset M \) is subcritical, and hence, \( P \) admits a unique positive minimal Green function \( G^\Omega_P \) of \( P \) in \( \Omega \).
Since in the subcritical case there exists a unique positive minimal Green function, our goal is in fact to establish the existence of a Green function in the critical case. Our main result reads as follows.

**Theorem 2.5.** Let $M$ be a $C^2$-smooth, noncompact, second countable, and connected Riemannian manifold of dimension $N \geq 2$, and let $P$ be a second-order elliptic operator of the form (1.1) which is critical in $M$. Denote by $\Phi$ and $\Phi^*$ the ground states of $P$ and $P^*$, respectively. Then

1. The operator $P$ admits a Green function $G^M_P(x, y)$ in $M$ obtained by (a modification of) the Li–Tam construction. In particular, in the symmetric case, $G^M_P$ is symmetric.
2. Any Green function $G^M_P(x, y)$ obtained by the Li–Tam construction satisfies the following boundedness property: For any $y \in M$ and any neighborhood $U_y$ of $y$ there exists $C > 0$ depending on $U_y$ such that
   \[ G^M_P(x, y) \leq C\Phi(x) \quad \text{and} \quad G^M_P(x, y) \leq C\Phi^*(x) \quad \forall x \in M \setminus U_y. \]  
   (2.1)
3. For any Green function $\hat{G}^M_P$ we have
   \[ \liminf_{x \to \infty} \frac{\hat{G}^M_P(x, y)}{\Phi(x)} = -\infty \quad \text{for each} \quad y \in M. \]  
   (2.2)
4. For any $z \in M$ there exists a Green function $\hat{G}^M_P(x, y)$ obtained by the Li–Tam construction such that in some neighborhood $U_z$ of $z$ we have
   \[ \hat{G}^M_P(x, z) < 0 \quad \forall x \in M \setminus U_z. \]  
   (2.3)

The proof of the above theorem is given in the next section.

**Definition 2.6.** For a critical operator $P$ in $M$, we define the Li–Tam class to be the set of all Green functions which can be obtained by the Li–Tam construction. We denote the Li–Tam class by $\mathcal{G}_{LT}$.

The set of all Green functions $\hat{G}^M_P(x, y)$ of the form

\[ \hat{G}^M_P(x, y) = G^M_P(x, y) + (\chi(x)\Phi^*(y) + \Phi(x)\chi^*(y)), \]  
   (2.4)
where $G^M_P(x, y) \in \mathcal{G}_{LT}$, the functions $\Phi$, and $\Phi^*$ are the ground states of $P$ and $P^*$, respectively, and $\chi(x)$ and $\chi^*(x)$ are solutions of the equation $Pu = 0$ and $P^*v = 0$ in $M$, respectively, is called the extended Li–Tam class and is denoted by $\mathcal{G}_{ELT}$.

**Remark 2.7.** By definition $\mathcal{G}_{LT} \subset \mathcal{G}_{ELT}$.

In the following theorem we establish a necessary and sufficient condition for a Green function in $\mathcal{G}_{ELT}$ to be in $\mathcal{G}_{LT}$.

**Theorem 2.8.** Let $P$ and $M$ satisfy the conditions of Theorem 2.5. Suppose that $G^M_P(x, y) \in \mathcal{G}_{LT}$, and $\hat{G}^M_P \in \mathcal{G}_{ELT}$. Then the following are equivalent:
(1) There exist \(x_0, y_0 \in M\) and a constant \(C\) such that
\[
\hat{G}_P^M(x, y_0) \leq G_P^M(x, y_0) + C\Phi(x) \quad \forall x \in M,
\]
and
\[
\hat{G}_P^M(x_0, y) \leq G_P^M(x_0, y) + C\Phi^*(y) \quad \forall y \in M.
\]

(2) There exists a constant \(C\) such that
\[
\hat{G}_P^M(x, y) = G_P^M(x, y) + C\Phi(x)\Phi^*(y) \quad \forall x, y \in M.
\]

(3) There exist \(x_0, y_0 \in M\) and a constant \(C\) such that
\[
G_P^M(x, y_0) \leq \hat{G}_P^M(x, y_0) + C\Phi(x) \quad \forall x \in M,
\]
and
\[
G_P^M(x_0, y) \leq \hat{G}_P^M(x_0, y) + C\Phi^*(y) \quad \forall y \in M.
\]

(4) \(\hat{G}_P^M \in \mathcal{G}_{LT}\).

Remark 2.9. Theorem 2.8 implies that \(\hat{G}_P^M \in \mathcal{G}_{ELT}\) is in \(\mathcal{G}_{LT}\) if and only if \(\hat{G}_P^M\) satisfies the “minimality” conditions (2.5) and (2.6).

As a direct application of Theorem 2.8, we have a sharp uniqueness result (cf. [4, Corollary 1]).

Corollary 2.10. Let \(P\) be a critical operator in \(M\), and let \(\hat{G}_P^M\) and \(G_P^M\) be two Green functions in \(\mathcal{G}_{LT}\). If \(\hat{G}_P^M(x_0, y_0) = G_P^M(x_0, y_0)\) for some \(x_0, y_0 \in M\), then \(\hat{G}_P^M = G_P^M\).

3. Proof of Theorem 2.5

The present section is devoted to the proof of Theorem 2.5. The proof hinges on Lemma 3.1 below.

We fix \(p \in M\) and an exhaustion \(\{M_j\}_{j=1}^{\infty}\) such that \(B(p, 1) \subset M_1\). By Remark 2.2, for any \(j \geq 1\), the operator \(P\) admits a unique minimal positive Green function \(G_P^{M_j}\) in \(M_j\).

Lemma 3.1. Let \(P\) be an elliptic operator of the form (1.1) satisfying \(P(1) = 0\) in \(M\), and let \(K \subset M \setminus \{p\}\) be a compact set such that \(K \subset M_{j_0}\) for some \(j_0\). Then the sequence of Green functions \(\{G_P^M(\cdot, p)\}_{j=j_0}^{\infty}\) has uniformly bounded oscillation in \(K\).

Remark 3.2. The proof of Lemma 3.1 is along the lines of the proof of [4, Lemma 1], but in contrast to [4, Lemma 1], our proof does not rely on the unique continuation property, and therefore, the proof applies to operators of the form (1.1) defined on \(M\).
Proof of Lemma 3.1. For $k \geq 1$, consider ‘annuli’ of the form $A_p(k) := M_k \setminus B(p, \frac{1}{k})$. For a fixed $k$ and $j > k$, denote by $\omega_j(k)$ the oscillation of $G_{P}^{M_j}(x, p)$ on $A_p(k)$ defined by

$$\omega_j(k) := \omega_j = \sup_{x \in A_p(k)} \{G_{P}^{M_j}(x, p)\} - \inf_{x \in A_p(k)} \{G_{P}^{M_j}(x, p)\}.$$  

(3.1)

Clearly $\omega_j \geq 0$. Since for any compact $K \subset M \setminus \{p\}$ there exists an annulus $A_p(k)$ such that $K \subset A_p(k)$, it suffices to prove that for any fixed $k \geq 1$, the sequence $\{\omega_j(k)\}_{j>k}$ is bounded.

We argue by contradiction. Suppose that there exists a subsequence of $\omega_j$ (that we do not rename), such that $\omega_j \to \infty$. This assumption clearly implies that $\lim_{j \to \infty} \omega_j(m) = \infty$ for any $m \geq k$.

Following [4, Lemma 1], we define for $j > k$ functions $h_j$ by

$$h_j(x) := \omega_j^{-1}G_{P}^{M_j}(x, p) - \omega_j^{-1}\inf_{z \in M_k} \{G_{P}^{M_j}(z, p)\}.$$  

(3.2)

Clearly, $Ph_j = 0$ in $M_j \setminus \{p\}$, and $\text{Osc}(h_j) = 1$ in $A_p(k)$. On the other hand, by our assumptions on the coefficients of $P$, and in light of [10, Théorème 9.6] we have $G_{P}^{M_j}(\cdot, p) \asymp G_{P}^{M_1}(\cdot, p)$ in a punctured neighborhood of $p$, and since by our assumption (3.3) holds true, we have

$$G_{P}^{M_j}(x, p) \sim_p G_{P}^{M_1}(x, p).$$  

(3.3)

Therefore,

$$h_j(x) \sim_p \omega_j^{-1}G_{P}^{M_1}(x, p).$$

Recall that by our assumption, $P(1) = 0$. Therefore, (3.3) and the maximum principle on the domain $M_k$ immediately imply

$$\inf\{h_j(x) : x \in M_k\} = \inf\{h_j : x \in \partial M_k\} = 0.$$  

Further, we claim that $h_j$ satisfies the following estimate

$$\omega_j^{-1}G_{P}^{M_k}(x, p) \leq h_j(x) \leq \omega_j^{-1}G_{P}^{M_k}(x, p) + 1 \quad \forall x \in M_k,$$  

(3.4)

where $G_{P}^{M_k}(x, p)$ is the positive minimal Green function of $P$ on $M_k$. Indeed, the right hand side of (3.4) can be easily verified by considering the following function

$$h_j^\alpha(x) := (1 - \alpha)h_j(x) - \omega_j^{-1}G_{P}^{M_k}(x, p) - 1,$$

where $0 < \alpha < 1$. Since $\alpha > 0$, it follows that $h_j^\alpha$ goes to $-\infty$ as $x \to p$. On the other hand, since the oscillation of $h_j$ on $\partial M_k$ is less or equal to $1$, it follows that $h_j^\alpha \leq 0$ on $\partial M_k$. Hence, by the maximum principle, for any $0 < \alpha < 1$, $h_j^\alpha \leq 0$ in $M_k \setminus \{p\}$. Consequently, by letting $\alpha \to 0$, we obtain $h_j(x) \leq \omega_j^{-1}G_{P}^{M_k}(x, p) + 1$ in $M_k$.

The inequality $\omega_j^{-1}G_{P}^{M_k}(x, p) \leq h_j(x)$ follows by a similar argument using

$$\tilde{h}_j^\alpha(x) := (1 + \alpha)h_j(x) - \omega_j^{-1}G_{P}^{M_k}(x, p)$$

with $\alpha > 0$. 

Note that (3.4) holds for all \( m \geq k \) with \( \omega_j(m) \) and \( M_m \) replacing \( \omega_j(k) \) and \( M_k \), respectively. Therefore, our assumption that \( \omega_j \to \infty \), estimate (3.4), and a standard diagonalization argument imply that (up to a subsequence) \( h_j \) converges locally uniformly in \( M \setminus \{ p \} \) to a function \( h \) which satisfies \( Ph = 0 \) in \( M \setminus \{ p \} \) and \( 0 \leq h \leq 1 \). By a removable singularity theorem (see for example [2]), it follows that \( h \) can be extended to a nonnegative function \( \tilde{h} \) which satisfies \( P \tilde{h} = 0 \) in \( M \). By the criticality of \( P \) in \( M \), it follows that \( \tilde{h} = \text{const} \). But this contradicts the fact that \( \text{Osc}(\tilde{h}) = \text{Osc}(h_j) = 1 \) in \( A_p(k) \). □

We turn now to the proof of our main theorem.

**Proof of Theorem 2.5.** First, we make a simple reduction step. Recall that \( P \) is critical in \( M \) if and only if \( P^* \) is critical in \( M \). Using a modified ground state transform, we transform the operator \( P \) into an elliptic operator \( L \) of the form (1.1) defined by

\[
L(u) := \Phi^* P(\Phi u),
\]

where \( \Phi \) and \( \Phi^* \) denote the ground state of the operator \( P \) and \( P^* \), respectively.

Clearly, \( L(1) = L^*(1) = 0 \), and hence, \( L \) and \( L^* \) satisfy the weak and the strong maximum principle. Moreover, since \( P \) is critical in \( M \) it follows that \( L \) and \( L^* \) are critical in \( M \), and 1 is the unique (up to a multiplicative constant) ground state of \( L \) and of \( L^* \) in \( M \). Furthermore, \( G_M^P(x, y) \) is a Green function of \( P \) in \( M \) if and only if \( G_M^L(x, y) := \frac{G_M^P(x, y)}{\Phi^*(y)\Phi(x)} \) is a Green function of \( L \) in \( M \).

Therefore, it is enough to prove the theorem for a critical elliptic operator \( L \) of the form (1.1) that satisfies

\[
L(1) = L^*(1) = 0 \quad \text{with ground states } \Phi = \Phi^* = 1.
\]

1. The proof of the existence of a Green function for the operator \( L \) is divided into two steps.

**Step 1:** In this step we claim that for \( p \in M \) fixed, there exist a subsequence of \( \{G_L^{M_j}(x, p)\} \) (that we do not rename) and a sequence of real number \( \alpha_j^{(p)} \) which depends on \( p \) such that (up to a subsequence), the sequence of functions defined by

\[
J_L^{M_j}(x, p) := G_L^{M_j}(x, p) - \alpha_j^{(p)}
\]

converges locally uniformly in \( M \setminus \{ p \} \).

Let \( k > 1 \), and denote for \( j > k \)

\[
I_j(k) := \inf_{x \in \partial M_k} G_L^{M_j}(x, p), \quad S_j(k) := \sup_{x \in \partial M_k} G_L^{M_j}(x, p),
\]

and let \( \alpha_j^{(p)} := I_j(1) \).
By Lemma 3.1 in $A_p(k)$, there exists a constant $C = C(k) > 0$ such that for large $j$ there holds
\[
\sup_{x \in \partial B(p, \frac{1}{j})} G^{M_j}_L(x, p) \leq C + \alpha_j^{(p)} \quad \text{and} \quad \alpha_j^{(p)} \leq C + I_j(k).
\] (3.6)

We claim that (3.6) implies that
\[
-C \leq J^{M_j}_L(x, p) \leq C \quad \text{in} \quad A_p(k).
\] (3.7)

Indeed, by the maximum principle in $A_p(k)$, we obtain in light of (3.6) that
\[
G^{M_j}_L(x, p) - I_j(1) + C \geq I_j(k) - I_j(1) + C \geq 0 \quad \forall x \in A_p(k),
\]
and hence, $J^{M_j}_L(x, p) \geq -C$ in $A_p(k)$.

On the other hand, since $S_j(k)$ is a decreasing function of $k$, the maximum principle and (3.6) imply
\[
G^{M_j}_L(x, p) - I_j(1) - C \leq \sup_{x \in \partial B(p, \frac{1}{j})} G^{M_j}_L(x, p) - I_j(1) - C \leq 0,
\]
and hence, $J^{M_j}_L(x, p) \leq C$ in $A_p(k)$. So, (3.7) is proved.

Hence, the sequence $\{J^{M_j}_L(\cdot, p)\}$ is locally uniformly bounded, and by standard elliptic regularity it is also locally equicontinuous in $M \setminus \{p\}$. Arzelà–Ascoli theorem and again elliptic regularity imply that there exists a subsequence of $\{J^{M_j}_L(\cdot, p)\}$ which converges locally uniformly in $M \setminus \{p\}$ to a solution $J^M_L(x, p)$ of the equation $Lu = 0$ in $M \setminus \{p\}$.

**Claim:** $J^M_L(x, p)$ has an irremovable singularity at $p$, and
\[
J^M_L(x, p) \xrightarrow{\sim} G^{M_1}_L(x, p).
\] (3.8)

Moreover, $LJ^M_L(x, p) = \delta_p(x)$.

Indeed, fix $0 < \alpha < 1$, and $k > 0$. Let $G^{M_{2k}}_L$ be the Dirichlet Green function in $M_{2k}$, and let $\bar{\omega}$ be the upper bound for the oscillation of $G^{M_j}_L$ on $M_{2k} \setminus M_1$, where $j > 2k$. For such $j$ consider the function
\[
f_{\alpha, j}(x) := G^{M_j}_L(x, p) - \alpha_j^{(p)} + \bar{\omega} - (1 - \alpha) G^{M_{2k}}_L(x, p).
\]

Clearly, $\lim_{x \to p} f_{\alpha, j}(x) = \infty$. On the other hand, $f_{\alpha, j}(x) \geq 0$ on $\partial M_{2k}$. The maximum principle yields that $f_{\alpha, j} > 0$ in $M_{2k}$. Passing to the limit, first with $\alpha \to 0$ and then with $j \to \infty$, we obtain that
\[
G^{M_{2k}}_L(x, p) \leq J^M_L(x, p) + \bar{\omega} \quad \text{in} \quad M_{2k}.
\] (3.9)

Similarly, we obtain
\[
J^M_L(x, p) \leq G^{M_{2k}}_L(x, p) + C \quad \text{in} \quad M_{2k},
\] (3.10)
where $C$ is some positive constant. Hence, $J^M_L(\cdot, p)$ is a positive solution in $M_1 \setminus \{p\}$ which has a nonremovable singularity near $p$, and satisfies (3.8). Therefore, by the Riesz representation theorem, we have $LJ^M_L(x, p) = \delta_p(x)$, and the claim is proved.
Step 2: In this step we establish the existence of a Green function $J_L^M(x,y)$ of $L$ in $M \times M$.

Let the reference point $p \in M$, and the converging sequence $J_L^{M_j}(x,p)$, be as in Step 1. For a fixed $y \neq p$, consider, as in Step 1, a new sequence

$$J_L^{M_j}(x,y) := G_L^{M_j}(x,y) - \alpha_j^{(y)},$$

(3.11)

with a sequence $\{\alpha_j^{(y)}\}$ of appropriate real numbers such that a subsequence of $J_L^{M_j}(x,p)$ and of $J_L^{M_j}(x,y)$ (which we do not rename) converge to a solution in $M \setminus \{p\}$ and $M \setminus \{y\}$, respectively.

Recall that $G_L^{M_j}(x,y) = G_L^{M_j}(y,x)$, where $L^*$ denotes the formal adjoint of $L$. Using the fact that $L^*(1) = 0$ and Lemma 3.1 we deduce as above that for a fixed $x \in M$ there exists a sequence of real numbers $\bar{\alpha}_j^{(x)}$ such that

$$G_L^{M_j}(y,x) - \bar{\alpha}_j^{(x)}$$

converges (up to subsequence) as a function of $y$ to a solution to the equation $L^*u = 0$ in $M \setminus \{x\}$. Therefore,

$$J_L^M(x,p) = \lim_{j \to \infty} J_L^{M_j}(x,p)$$

$$= \lim_{j \to \infty} \{G_L^{M_j}(x,p) - \alpha_j^{(p)}\}$$

$$= \lim_{j \to \infty} \{G_L^{M_j}\}(p,x) - \alpha_j^{(p)}\}$$

$$= \lim_{j \to \infty} \{G_L^{M_j}(p,x) - \tilde{\alpha}_j^{(x)}\} + \lim_{j \to \infty} \{\bar{\alpha}_j^{(x)} - \alpha_j^{(p)}\}. $$

Hence, the sequence $\{\bar{\alpha}_j^{(x)} - \alpha_j^{(p)}\}$ converges (up to a subsequence) to a constant $C$. Also

$$G_L^{M_j}(x,y) - \alpha_j^{(p)} = G_L^{M_j}(y,x) - \alpha_j^{(p)} = \{G_L^{M_j}(y,x) - \bar{\alpha}_j^{(x)}\} + \{\bar{\alpha}_j^{(x)} - \alpha_j^{(p)}\},$$

converges in $M \setminus \{x\}$ (up to a subsequence), and again as above $G_L^{M_j}(x,y) - \alpha_j^{(p)}$ converges as a function of $x$ (up to a subsequence) for all $x \neq y$ to a function $J_L^M(x,y)$.

The proof will be completed if we can show that if there is another subsequence $\alpha_j^{(p)}$ of $\alpha_j^{(p)}$ such that

$$G_L^{M_j}(x,y) - \alpha_j^{(p)}$$

converges in $M \setminus \{y\}$, then it must converge to $J_L^M(x,y)$. Let us assume that

$$\lim_{l \to \infty} \{G_L^{M_{jl}}(x,y) - \alpha_j^{(p)}\} = K_L^M(x,y).$$

(3.12)

Our aim is to prove that $J_L^M(x,y) = K_L^M(x,y)$. To this end, let us first assume that $J_L^M(\cdot, y) - K_L^M(\cdot, y)$ is a bounded function on $M \setminus \{y\}$, and
hence a removable singularity theorem and the criticality of the operator $L$ (with 1 as the unique ground state) readily imply that
\[ J_M^L(x, y) - K_M^L(x, y) = \text{constant} \quad \text{in } M. \]
Furthermore, we have
\[ K_M^L(y, p) + \text{constant} = J_M^L(y, p) = \lim_{j \to \infty} \{ G_L^{M_j}(y, p) - \alpha_j^{(p)} \} \]
\[ = \lim_{l \to \infty} \{ G_L^{M_{ji}}(y, p) - \alpha_j^{(p)} \} = \lim_{l \to \infty} \{ G_L^{M_{ji}}(y, p) - \alpha_j^{(p)} \} = K_M^L(y, p). \]
Hence, $J_M^L(x, y) = K_M^L(x, y)$ for all $x \in M$.

Next we show that $J_M^L(\cdot, y) - K_M^L(\cdot, y)$ is indeed a bounded function on $M$. In fact, the proof of this statement follows as in [3]. For the sake of completeness, we provide the proof.

Consider the difference between the two functions $J_M^L(x, y) - K_M^L(x, y)$ as a function of $x \in M \setminus M_k$ for some fixed $k$ with $1 \leq 2k < j_l$ and for a fixed $y \in M_k$.

\[
J_M^L(x, y) - K_M^L(x, y) = \lim_{l \to \infty} \{ G_L^{M_{ji}}(x, p) - \alpha_j^{(p)} \} - \lim_{l \to \infty} \{ G_L^{M_{ji}}(x, y) - \alpha_j^{(p)} \} \\
= \lim_{l \to \infty} \{ G_L^{M_{ji}}(x, p) - G_L^{M_{ji}}(x, y) \}. \tag{3.13}
\]

Applying the maximum principle on $M_j \setminus M_{2k}$ for large $j_l > 2k$, we see
\[
\sup_{x \in M_j \setminus M_{2k}} \{ |G_L^{M_{ji}}(x, p) - G_L^{M_{ji}}(x, y)| \} \leq \sup_{x \in \partial M_{2k}} \{ |G_L^{M_{ji}}(x, p) - G_L^{M_{ji}}(x, y)| \}. \tag{3.14}
\]

Therefore, (3.13) and (3.14), Lemma 3.1 (for $L^*$ and then for $L$) imply for $y \in M_k$,
\[
\sup_{x \in M \setminus M_{2k}} \{ |J_M^L(x, p) - K_M^L(x, y)| \} \leq C, \tag{3.15}
\]
where $C$ is a constant depending on $k$ and $p$. A similar argument shows that $J_M^L(x, y) - J_M^L(x, p)$ is bounded on $M \setminus M_{2k}$ if $y \in M_k$. Consequently, for all $x \in M \setminus M_{2k}$ and each fixed $y \in M_k$, we have
\[
|J_M^L(x, y) - K_M^L(x, y)| \leq |J_M^L(x, y) - J_M^L(x, p)| + |J_M^L(x, p) - K_M^L(x, y)| \leq C_1,
\]
where $C_1$ is a constant depends on $k$ and $p$.

On the other hand, it follows from the proof of the Claim that $J_M^L(\cdot, y) - K_M^L(\cdot, y)$ is a bounded function on $M_{2k}$. This together with the above imply that $J_M^L(\cdot, y) - K_M^L(\cdot, y)$ is bounded in $M$.

Since $y \in M$ is an arbitrary point, this completes the proof that $J_M^L(x, y)$ is a well defined Green function of $L$ satisfying
\[
LJ_M^L(x, y) = \delta_y(x) \quad \text{in } M.
\]
Moreover, from the construction it follows that $J_M^L(x, y) = J_M^L(y, x)$.
By the maximum principle, the sequence \( \{G^M_L(x, y) - S_j(1)\} \) is bounded above away from the pole \( y \) (that is, \( J^M_L \) satisfies (2.1)).

Consider the sequence
\[
G^M_L(x, y) - S_j(1) = (G^M_L(x, y) - I_j(1)) - (S_j(1) - I_j(1)).
\]

Since the sequence \( \{S_j(1) - I_j(1)\} \) is bounded by Lemma 3.1, we deduce that \( \{G^M_L(x, y) - S_j(1)\} \) converges (up to a subsequence) uniformly on compact subsets of \( M \setminus \{y\} \) to a Green function, denoted by \( G^M_L \), that satisfies
\[
G^M_L(x, y) := J^M_L(x, y) + C.
\]

Define for \( k \geq 1 \),
\[
\tilde{S}_j(k) := \sup\{G^M_L(x, p) : x \in \partial M_k\} - S_j(1).
\]

By the maximum principle, the sequence \( \{\tilde{S}_j(k)\} \) is decreasing as a function of \( k \), hence, \( \tilde{S}_j(k) \leq \tilde{S}_j(1) = 0 \) for all \( k \geq 1 \). Therefore, the maximum principle implies that
\[
G^M_L(x, p) - S_j(1) \leq 0 \quad \forall x \in M_j \setminus M_1.
\]

By passing to the limit, we conclude that \( G^M_L(x, p) \leq 0 \) for all \( x \in M \setminus M_1 \). Fix \( y \in M_{k-1} \). In light of Lemma 3.1, we have for all \( x \in M_j \setminus M_k \), and \( j > k \),
\[
G^M_L(x, y) - S_j(1) = (G^M_L(x, y) - G^M_L(x, p)) + (G^M_L(x, p) - S_j(1))
\]
\[
\leq |G^M_L(x, y) - G^M_L(x, p)| \leq \sup_{x \in M_j \setminus M_k} |G^M_L(x, y) - G^M_L(x, p)|
\]
\[
\leq \sup_{x \in \partial M_k} |G^M_L(x, y) - G^M_L(x, p)| \leq C.
\]

Therefore, we have
\[
G^M_L(x, y) \leq C \quad \forall x \in M \setminus M_k, \forall y \in M_{k-1}.
\]

and this implies (2.1).

(3) Suppose to the contrary that there exists \( y \in M \) and \( C \in \mathbb{R} \) such that
\[
\hat{G}^M_L(x, y) > C \quad \forall x \in M \setminus U_y,
\]
where \( U_y \) is a bounded neighborhood of \( y \). Then, the maximum principle implies that \( \hat{G}^M_L(\cdot, y) - C > 0 \) in \( M \). So, \( \hat{G}^M_L(\cdot, y) - C > 0 \) is a positive supersolution of the equation \( Lu = 0 \) in \( M \) which is not a solution. But this contradicts the criticality of \( L \) in \( M \).

(4) Let \( \hat{G}^M_L(x, y) \) be a Green function that satisfies (2.1), and fix \( z \in M \). Let \( C_z \) a positive constant and \( U_z \) a neighborhood of \( z \) such that
\[
\hat{G}^M_L(x, z) < C_z \quad \forall x \in M \setminus U_z,
\]
and define $\hat{G}^M_L(x, y) := \tilde{G}^M_L(x, y) - C_z$. Then $\hat{G}^M_L(x, y)$ is a Green function that satisfies (2.1) and satisfies
\[ \hat{G}^M_L(x, z) < 0 \quad \forall x \in M \setminus U_z. \]
\[ \square \]

**Remark 3.3.** It follows from the proof of Theorem 2.5 that for any $p \in M$ we may construct a Green function $G^M_p$ such that $G^M_p(x, p) \leq 0$ for all $x \in M \setminus M_1$.

### 4. Proof of Theorem 2.8

This short section is devoted to the proof of Theorem 2.8. As in the proof of Theorem 2.5, it is enough to prove the theorem for the operator $L := \Phi^* P \Phi$.

We first prove a simple lemma.

**Lemma 4.1.** Let $\tilde{G}^M_L \in \mathcal{G}_{\xi LT}$ and $G^M_L \in \mathcal{G}_{\xi LT}$. Then there exist $\chi$, and $\chi^*$ such that $L \chi = 0$, and $L^* \chi^* = 0$ in $M$, and
\[ \tilde{G}^M_L(x, y) = G^M_L(x, y) + \chi(x) + \chi^*(y) \quad \text{in } M \times M. \]

**Proof.** By definition, $\tilde{G}^M_L \in \mathcal{G}_{\xi LT}$ means that there exist $\tilde{G}^M_L \in \mathcal{G}_{\xi LT}$, and $\chi_1$, $\chi_1^*$ such that
\[ \tilde{G}^M_L(x, y) = \tilde{G}^M_L(x, y) + \chi_1(x) + \chi_1^*(y), \]
where $\tilde{G}^M_L \in \mathcal{G}_{\xi LT}$, $L \tilde{\chi}_1 = 0$, and $L^* \tilde{\chi}_1^* = 0$ in $M$.

Therefore, in order to prove (1.1), it is enough to show that there exist $\check{\chi}$ and $\check{\chi}^*$ with $L \check{\chi} = 0$ and $L^* \check{\chi}^* = 0$ in $M$ such that
\[ \check{G}^M_L(x, y) - G^M_L(x, y) = \check{\chi}(x) + \check{\chi}^*(y) \quad \text{in } M \times M. \]

To see this, let us consider the difference $\hat{f}(x, y) := \check{G}^M_L(x, y) - \check{G}^M_L(x, y)$. Clearly, $\hat{f}(\cdot, y)$ and $\hat{f}(x, \cdot)$ are solutions to $L u = 0$ and $L^* v = 0$, respectively. Following Li-Tam [4, Theorem 2], we define a new Green function
\[ G^1_L(x, y) := \check{G}^M_L(x, y) + \hat{f}(x, p) + \hat{f}(p, y) - \hat{f}(p, p), \]
where $p \in M$ is fixed.

We assert that $G^1_L(x, y) = \check{G}^M_L(x, y)$, which clearly implies (1.1), and hence (1.1).

Indeed, let $\hat{f}(x, y) := \check{G}^M_L(x, y) - G^1_L(x, y)$. Then
\[ \hat{f}(x, y) = \hat{f}(x, y) - \hat{f}(p, y) - \hat{f}(p, y) + \hat{f}(p, p), \]
and $\hat{f}(x, p) = \hat{f}(p, y) = 0$ for any $x, y \in M$. Since $\check{G}^M_L, G^M_L \in \mathcal{G}_{\xi LT}$, it follows (as in the proof of Theorem 2.5) that for any $k \in \mathbb{N}$, $q \in M_k$, and $p \in M \setminus M_{2k}$
\[ |\hat{f}(p, y) - \hat{f}(q, y)| \leq |\check{G}^M_L(p, y) - \check{G}^M_L(q, y)| + |G^M_L(p, y) - G^1_M(p, q, y)| \]
\[ - \hat{f}(p, p) + \hat{f}(q, p) \leq C, \]
where \( C \) depends on \( k, p, q \). On the other hand, as a function \( y \), the difference 
\[ |\hat{f}(p, y) - \tilde{f}(q, y)| \]
is bounded in \( M_{2k} \). Therefore, the criticality of \( L^* \) and the arbitrariness of \( k \) implies that
\[ \hat{f}(p, y) - \tilde{f}(q, y) = C(p, q) \quad \forall y, q \in M, \]
where \( C(p, q) \) is a constant depending only on \( p, q \). By substituting \( y = p \) in the above equation, we see that \( C(p, q) = 0 \) and consequently, \( \hat{f}(q, y) = 0 \) for all \( y, q \in M \). This proves the assertion and therefore also (4.1).

**Proof of Theorem 2.8.** Obviously, (2) implies (1) and (3). Next we show that (1) \( \Rightarrow \) (2). Set
\[ f(x, y) := \hat{G}^M_L(x, y) - G^M_L(x, y). \]
By Lemma 4.1,
\[ f(x, y) = \chi(x) + \chi^*(y) \quad \text{in } M \times M, \]
where \( \chi \) and \( \chi^* \) satisfy \( L\chi = 0 \), and \( L^*\chi^* = 0 \) in \( M \), respectively. We need to prove that \( f = \text{constant} \).

Next we claim for any \( q \in M \) there exists \( k \in \mathbb{N} \) such that for \( y \in M_k \), the function
\[ \hat{G}^M_L(x, q) - G^M_L(x, y) \]
is a bounded function of \( x \) on \( M \setminus M_{2k} \). Indeed we write for a fixed \( p \in M \)
\[ |\hat{G}^M_L(x, q) - G^M_L(x, y)| \leq |\hat{G}^M_L(x, q) - G^M_L(x, p)| + |G^M_L(x, p) - G^M_L(x, y)| \]
\[ + |\chi^*(q) - \chi^*(y)|, \quad (4.3) \]
and the boundedness of (4.3) follows similarly to the proof of (3.15) in Step 2 of Theorem 2.5. Therefore, using the boundedness of (4.3) and (2.5) we have for a fixed \( y \in M_k \) and for all \( x \in M \setminus M_{2k} \)
\[ f(x, y) = \hat{G}^M_L(x, y) - G^M_L(x, y) \leq |\hat{G}^M_L(x, y) - \hat{G}^M_L(x, y_0)| \]
\[ + |G^M_L(x, y_0) - G^M_L(x, y)| + \hat{G}^M_L(x, y_0) - G^M_L(x, y_0) \leq C, \]
for some \( y_0 \in M \). This implies, for a fixed \( y \in M_k \), the function \( f(\cdot, y) \) is a bounded above solution of the equation \( Lu = 0 \) in \( M \). On the other hand, by the criticality of \( L \) in \( M \) it follows that any nonconstant solution \( v \) of the equation \( Lu = 0 \) in \( M \) satisfies
\[ \liminf_{x \to \infty} v(x) = -\infty, \quad \text{and} \quad \limsup_{x \to \infty} v(x) = \infty. \]

Therefore,
\[ \liminf_{x \to \infty} f(x, y) = -\infty, \quad \text{and} \quad \limsup_{x \to \infty} f(x, y) = \infty, \]
but this contradicts the fact that \( f(\cdot, y) \) is bounded above. Hence, \( f(x, y) := F(y) \). A similar consideration concerning \( f(x, y) \) as a function of \( y \) keeping \( x \) fixed implies \( f(x, y) := G(x) \). So, \( f(x, y) := G(x) = F(y) = \text{constant} \).
(3) \Rightarrow (2) follows immediately by considering 
\( \hat{f}(x, y) := G^M_L(x, y) - G^M_L(x, y) \) and using similar arguments as above.

(2) \Leftrightarrow (4) is obvious. This completes the proof of the theorem. \hfill \Box

5. Examples and Concluding Remarks

In this section we present several examples of Green functions which satisfies (2.1), and discuss some questions that arise in our study.

Definition 5.1. We call the set of Green functions that satisfy (2.1) the class of relatively bounded above Green functions and denote by \( \mathcal{G}_{BA} \).

Clearly, by Theorem 2.5 we have \( \mathcal{G}_{LT} \subset \mathcal{G}_{BA} \).

Example 5.2. Consider the critical operator \( P := -\Delta = -\frac{d^2}{dz^2} \) in \( M = \mathbb{R}^1 \).

A straightforward computation shows that a Green function \( G_{\mathbb{R}^1}^{-\Delta}(x, y) \in \mathcal{G}_{BA} \) is given by
\[
G_{\mathbb{R}^1}^{-\Delta}(x, y) = -\frac{1}{2}|x - y| + C,
\]
where \( C \in \mathbb{R} \) is a constant (but note that Theorem 2.5 is proved only for \( N \geq 2 \)).

Similarly, a Green function \( G_{\mathbb{R}^2}^{-\Delta}(x, y) \in \mathcal{G}_{LT} \) is given by
\[
G_{\mathbb{R}^2}^{-\Delta}(x, y) = -\frac{1}{2\pi}\log|x - y| + C.
\]

Next, we present a 1-dimensional example. Although Theorem 2.5 is proved only for \( N \geq 2 \), the 1-dimensional example below gives us the idea how to construct a nontrivial behavior of a Green function in higher dimension.

Example 5.3. Let \( M = (0, \infty) \) and consider the critical Hardy operator \( P := -\frac{d^2}{dz^2} - \frac{1}{4z^2} \). Note that \( \Phi(x) = x^{1/2} \) is the ground state of \( P \) in \( M \). We construct a Green function for \( P \) in \( M \).

Define an exhaustion \( \{M_j\}_{j=1}^{\infty} \) of \( M \) by \( M_j := (\frac{1}{j}, j) \). It can be shown easily that the Dirichlet Green function \( G^{M_j}(x, 1) \) of \( P \) in \( M_j \) is given by
\[
G^{M_j}_P(x, 1) = \frac{1}{2}(\log j - |\log x|)x^{1/2}. \tag{5.1}
\]
Clearly \( G^{M_j}_P(x, 1) \to \infty \) as \( j \to \infty \). Therefore, we need to subtract a sequence of the form \( \{a_j(x) = a_j x^{1/2}\} \) of constants times the ground state \( \Phi(x) = x^{1/2} \) such that
\[
\{G^{M_j}_P(x, 1) - a_j(x)\}_{j=1}^{\infty}
\]
converges to a Green function of \( P \) in \( M \). Note that \( Pa_j(x) = 0 \). Choose \( a_j(x) = \frac{1}{2}(\log j)x^{1/2} \), then the above condition is satisfied, and we obtain a Green function \( G^M_P \in \mathcal{G}_{LT} \) given by
\[
G^M_P(x, 1) = -\frac{1}{2}|\log x|x^{1/2}. \tag{5.2}
\]
Clearly, \( \lim_{x \to \infty} \frac{G^M_P(x,1)}{x^{1/2}} = -\infty \), and \( \lim_{x \to 0} \frac{G^M_P(x,1)}{x^{1/2}} = -\infty \).

We note that
\[
\lim_{x \to \infty} G^M_P(x,1) = -\infty, \quad \text{while} \quad \lim_{x \to 0} G^M_P(x,1) = 0.
\]

Recall that by Theorem 2.5, for any \( G^M_P(x,y) \in \mathcal{G}_{BA} \), we have
\[
\liminf_{x \to \bar{\infty}} \frac{G^M_P(x,y)}{\Phi(x)} = -\infty,
\]
where \( \Phi \) is the ground state of \( P \) in \( M \). On the other hand, in all the above examples, the Green functions are not only finally negative, but also
\[
\lim_{|x| \to \bar{\infty}} \frac{G^M_P(x,y)}{\Phi(x)} = -\infty.
\]

Let us consider one more example.

**Example 5.4.** Let \( M = \mathbb{R}^N \setminus \{0\} \), where \( N \geq 3 \), and consider the critical Hardy operator
\[
P := -\Delta - \frac{(N-2)^2}{4} \frac{1}{|x|^2}.
\]

The two linearly independent, positive, radial solutions of the equations \( Pu = 0 \) near 0 and near \( \infty \) are \( v_1(x) = |x|^{(2-N)/2} \), \( v_2(x) = \log |x| |x|^{(2-N)/2} \), and \( v_1 \) is the corresponding ground state. It follows from [8, Lemma 8.5] that any positive solution \( v \) of the equation \( Pu = 0 \) in a punctured neighborhood of 0 or \( \infty \) satisfies
\[
\lim_{x \to 0} v(x) = \infty, \quad \text{or} \quad \lim_{x \to \infty} v(x) = 0, \quad \text{respectively}. \quad (5.3)
\]

On the other hand, Theorem 2.5 implies that for any point \( x_0 \in M \), there exists a Green function \( G^M_P(x,x_0) \in \mathcal{G}_{BA} \) and a neighborhood \( \mathcal{U}_{x_0} \) of \( x_0 \) such that \( G^M_P(x,x_0) < 0 \) for all \( x \in M \setminus \mathcal{U}_{x_0} \). Therefore, \( -G^M_P(x,x_0) \) is a positive solution of \( Pu(x) = 0 \) near zero and near \( \infty \). Hence, \( -G^M_P(x,x_0) \) exists (in the generalized sense), where \( \zeta = 0 \) or \( \zeta = \infty \), and by Theorem 2.5, the limit is equal to \( -\infty \) at least at one of these points. But, we do not know whether the limit is equal to \( -\infty \) at both points.

On the other hand, \( -G^M_P(x,x_0) \) is a positive solution of \( Pu(x) = 0 \) near \( \infty \) and near 0. Hence, \( -G^M_P(x,x_0) \to 0 \) as \( x \to \infty \), while \( G^M_P(x,x_0) \to -\infty \) as \( x \to 0 \). This is in contrast with the behavior of Green function of \( -\Delta \) in \( \mathbb{R}^2 \).

This leads us to formulate the following two problems.

**Problem 5.5.** Let \( G^M_P \in \mathcal{G}_{BA} \) or \( G^M_P \in \mathcal{G}_{LT} \). Does the following assertion hold true?
\[
\lim_{x \to \infty} \frac{G^M_P(x,y)}{\Phi(x)} = -\infty.
\]
Problem 5.6. Let $G^M_P \in \mathcal{G}_{\mathcal{B}_A}$ or $G^M_P \in \mathcal{G}_{\mathcal{L}T}$, and let $\{M_j\}$ be a compact exhaustion of $M$. Does the following assertion hold true?

For any $j \geq 1$ there exists $k_j > j$ such that

$$G^M_P(x, y) < 0 \quad \forall x \in M_j \text{ and } \forall y \in M \setminus M_{k_j}. \quad (5.4)$$

Remark 5.7. It is well known that for a subcritical operator $P$ on a noncompact manifold $M$, the celebrated Martin compactification gives an integral representation for all $u \in \mathcal{C}_P(M)$. Such a compactification is not available for a critical operator since a critical operator does not admit a positive Green function.

Nevertheless, we may define a Martin kernel for a critical operator $P$ with respect to a Green function $G^M_P \in \mathcal{G}_{\mathcal{B}_A}$. Let $x_0 \in M_1$ be a fixed reference point. There exists a Green function $G^M_P \in \mathcal{G}_{\mathcal{B}_A}$ and a neighborhood $U_{x_0}$ of $x_0$ such that $G^M_P(x_0, y) < 0$ for all $y \in M \setminus U_{x_0}$. Therefore, the following Martin kernel

$$K^M_P(x, y) := \frac{G^M_P(x, y)}{G^M_P(x_0, y)} \quad \forall y \in M \setminus U_{x_0}, \ x \in M \setminus \{y\}.$$ is well defined.

If there exists a Green function $G^M_P \in \mathcal{G}_{\mathcal{B}_A}$ which in addition satisfies (5.4), then we have

Corollary 5.8. Let $M$ be a $C^2$-smooth noncompact Riemannian manifold of dimension $N$, and let $P$ be an operator of the form (1.1) which is critical in $M$. Suppose there exists a Green function $G^M_P \in \mathcal{G}_{\mathcal{B}_A}$ such that (5.4) holds true. Then the corresponding Martin kernel $K^M_P$ satisfies

$$\lim_{y \to \Phi(\bar{x})} K^M_P(x, y) = \Phi(x), \quad (5.5)$$

where $\Phi$ is the ground state of $P$ satisfying $\Phi(x_0) = 1$.

Proof. Let $G^M_P \in \mathcal{G}_{\mathcal{B}_A}$ such that (5.4) holds true. Let $x \in M_j$ for some $j \in \mathbb{N}$. By our assumption, there exists $k_j > j$ such that $G^M_P(x, y)$ are negative for all $x \in M_j$ and $y \in M \setminus M_{k_j}$. Consider the Martin kernel

$$K^M_L(x, y) = \frac{G^M_L(x, y)}{G^M_L(x_0, y)}.$$ Clearly, $K^M_L(x, y_0) = 1$. Moreover, for any fixed $j$ and $y \in M \setminus M_{k_j}$, the function $K^M_L(\cdot, y)$ is a positive solution of the equation $Pu = 0$ in $M_j$.

Since $P$ is critical, it follows from the Harnack principle and a standard diagonalization argument that for any sequence $y_n \to \infty$, there exists a subsequence $\{y_{n_\ell}\}$, such that the sequence

$$\{K^M_P(x, y_{n_\ell})\}_{\ell=1}^\infty$$
converges to a positive solution of the equation $Pu = 0$ in $M$, and (5.5) follows by the uniqueness of the ground state. \qed
We conclude the paper with the following problems.

**Problem 5.9.** For the critical case, find an (alternative) approach to obtain in a unique way a relatively bounded above Green function. In particular, we would like to obtain $\mathcal{G}^{*}_P \in \mathcal{G}_{RA}$ via a method that avoids the need of the extraction of a subsequence.

By Theorem [25] we have $\mathcal{G}_{ELT} \subset \mathcal{G}_{RA}$, and hence $\mathcal{G}_{ELT} \cap \mathcal{G}_{RA} \neq \emptyset$. It is natural to pose the following problem.

**Problem 5.10.** Characterize the class $\mathcal{G}_{RA}$.

**Acknowledgments**

D. G. was supported in part at the Technion by a fellowship of the Israel Council for Higher Education. The authors acknowledge the support of the Israel Science Foundation (grants No. 963/11 and 970/15) founded by the Israel Academy of Sciences and Humanities.

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