Weighted Colimits of 2-Representations and Star Algebras

Mateusz Stroiński

Received: 22 July 2022 / Accepted: 26 June 2023
© The Author(s) 2023

Abstract
We apply the theory of weighted bicategorical colimits to study the problem of existence and computation of such colimits of birepresentations of finitary bicategories. The main application of our results is the complete classification of simple transitive birepresentations of a bicategory studied previously by Zimmermann. The classification confirms a conjecture he has made.

Keywords 2-representations · Bicategorical colimits · Finitary 2-categories

1 Introduction
Systematic study of finitary 2-representations of finitary 2-categories was initiated by the series of papers [23–28], after a number of successful instances and applications of categorification, and, in particular, categorical actions, in various areas of mathematics. These include the advances in knot theory following the introduction of Khovanov homology in [21], the proof of Broué’s abelian defect group conjecture for symmetric groups in [4], the categorification of quantum groups developed in [18–20] as well as [7, 34, 36], and many more.

One of the main purposes of 2-representation theory is to gain a better abstract understanding of categorical actions of 2-categories (and, by extension, also of the 2-categories themselves), such as those in the above listed applications. Finitary 2-categories can be viewed as a 2-categorical counterpart to the classical notion of a finite-dimensional algebra, and from this point of view, 2-representation theory is analogous to classical representation theory of such algebras, which aims at a better understanding of the linear actions of the algebra.

One of the first problems to consider in the classical setting is the classification of simple modules. A 2-representation theoretic analogue of simple modules, known as simple transitive 2-representations, was introduced in [27], together with an associated weak Jordan–Hölder
theory. Classification of such 2-representations has since become one of the central problems of the theory, with many complete classification results obtained, such as those in [27] and [31]. See also [22] for a slightly outdated overview.

A common feature of most of these results is the quasi-fiatic structure of the 2-category considered. In the theory of tensor categories, this corresponds to rigidity of the tensor category. In the former case, one requires all 1-morphisms to have left and right adjoints; in the latter, one requires all objects to have left and right duals. Continuing the analogy with finite-dimensional algebras, a fiat 2-category can be viewed as analogous to a finite-dimensional algebra with an involution.

The most notable exception is the main result of [32], which classifies simple transitive 2-representations of a large family of finitary 2-categories which need not be fiat. This is done by embedding the studied 2-category in a 2-category with additional adjunctions and lifting 2-representations to this bigger 2-category.

The present paper classifies the simple transitive 2-representations of a non-fiat 2-category $\mathcal{B}^\text{str}_n$ which does not have the crucial cell symmetry of the non-fiat 2-categories considered by [32]. The classification for $\mathcal{B}^\text{str}_n$ confirms [38, Conjecture 1], establishing a bijection between equivalence classes of simple transitive 2-representations and set partitions of \( \{1, \ldots, n\} \). In contrast to [32], neither the construction nor the classification employs an auxiliary fiat 2-category. Instead, we use weighted bicategorical colimits of prior known, not necessarily simple transitive, 2-representations of $\mathcal{B}^\text{str}_n$, to construct new simple transitive 2-representations, and use the universal properties for the classification. The weighted colimits used can be thought of as a 2-representation theoretic categorification of quotient modules in classical representation theory. The prerequisite facts and a description of our application of such colimits is given in Sect. 3. To our best knowledge, this approach has not been considered previously in the study of 2-representations.

It was observed in [38] that for any simple transitive 2-representation $\mathbf{M}$ of $\mathcal{B}^\text{str}_n$, there is a distinguished 2-transformation $\Sigma : \mathbf{C} \to \mathbf{M}$, i.e. a functor intertwining the $\mathcal{B}^\text{str}_n$-actions, from the cell 2-representation $\mathbf{C}$. Further, [38] shows that $\Sigma$ sends indecomposable objects to indecomposable objects, possibly identifying certain isomorphism classes of such objects. The kind of potential identification observed there cannot be captured using the more familiar categorical constructions, such as orbit categories and skew group categories. Our construction uses the analogy with finite-dimensional algebras: if we instead considered a finite-dimensional algebra $A$ and an $A$-module $C$, given elements $x, y \in C$ we could universally construct a morphism $\varphi$ of modules from $C$, satisfying $\varphi(x) = \varphi(y)$, as the projection $C \to C/\langle x - y \rangle$, which is the coequalizer of the diagram $A \xrightarrow{1 \mapsto x} C \xrightarrow{1 \mapsto y} C$. It is this latter realization we mimic in the bicategorical setting, replacing the regular module by the (representable) principal 2-representation $\mathbf{P}$ and, using Yoneda lemma, viewing indecomposable objects $X, Y$ of $\mathbf{C}$ as parallel 2-transformations from $\mathbf{P}$ to $\mathbf{C}$. We then study the bicategorical colimit which universally renders the 2-transformations isomorphic (rather than equal).

This stands in stark contrast with the techniques employed in the theory of fiat 2-categories, where one heavily relies on 2-representations obtained from structures internal to the 2-category whose 2-representations are studied, such as (co)algebra 1-morphisms and internal Hom studied in [29], whereas our approach uses the structure of the 2-category of 2-representations. In fact, we need to embed the latter 2-category into the 2-category of all $\mathbb{k}$-linear 2-functors to the 2-category $\mathbf{Cat}_\mathbb{k}$ of $\mathbb{k}$-linear categories, which can be viewed as analogous to the embedding of the category $A$-mod of finitely generated modules over a $\mathbb{k}$-algebra $A$, to the category $A$-Mod of all its modules.
This new approach, which in a sense categorifies quotient modules, can prove useful in similar problems concerning construction and classification of categorical actions. Indeed, a different classification problem for 2-representations of a non-finitary (although in a sense locally finitary) 2-category has been solved in [15] using the above approach, confirming [14, Conjecture 2] and generalizing it to the setting of [13]. In fact, the 2-representations constructed in [15] and in this document are the first non-trivial non-cell 2-representations constructed in the theory of simple transitive 2-representations of non-fiat 2-categories.

We now briefly explain the definition of the 2-category $B_{str}^n$. First, consider the double quiver $R_n$ on the star graph $G_n$ on $n + 1$ vertices, as depicted below for $n = 5$:

$$
G_5 = \begin{array}{c}
2 \\
3 \\
0 \\
1 \\
4 \\
5
\end{array}
\quad \text{and} \quad
R_5 = \begin{array}{c}
2 \\
3 \\
0 \\
1 \\
4 \\
5
\end{array}.
$$

The zigzag algebra $A_n$ on $G_n$ is a quotient of the path algebra of $R_n$ by the ideal generated by paths $i \to j \to k$ with $i \neq k$, together with elements of the form $(i \to j \to i) - (i \to j' \to i)$, for $j \neq j'$. In particular, $A_n$ is weakly symmetric, hence also self-injective. Under the complete set of pairwise orthogonal, primitive idempotents $\{e_0, e_1, \ldots, e_n\}$, induced by the above labelling of the quiver, we may consider the set $L_0 = \{A_n\} \cup \{A_n e_i \otimes_k e_0 A_n \mid i = 0, 1, \ldots, n\}$ of $A_n$-$A_n$-bimodules. The additive, $k$-linear category $B_n := \text{add } L_0$ is a monoidal subcategory of $(A_n\text{-mod- } A_n, \otimes A_n)$. Observe that $B_n$ is not symmetric or braided. We denote by $B_n$ the delooping bicategory of $B_n$. The 2-category $B_{str}^n$ is a strictification of $B_n$. More precisely, it is obtained by delooping the strict monoidal category $B_{str}^n$ of right exact endofunctors of $A_n\text{-mod}$ isomorphic to those given by the objects of $B_n$.

We follow [30] in relaxing the 2-categorical setup to the bicategorical setup, which allows us to consider simple transitive birepresentations of $B_n$ rather than simple transitive 2-representations of $B_{str}^n$. As observed in [30], the resulting two classification problems are equivalent. The bicategorical setup allows for greater flexibility in the computation of colimits.

We can now state our main result precisely:

**Conjecture** ([38, Conjecture 1], Theorem 5.10) *Equivalence classes of simple transitive 2-representations of $B_{str}^n$ are in bijection with set partitions of $\{1, \ldots, n\}$.*

The paper is organized as follows. Section 2 contains the necessary preliminaries for the techniques of 2-representation theory we will employ, as well as a complete account of the notation we will use. In Sect. 3 we give an elementary account of weighted colimits, bicategorical cocompleteness of bicategories of $k$-linear pseudofunctors and preservation of weighted colimits by additive and Karoubi envelopes. Section 4 defines $B_n$, summarizes and extends the results of [38], giving necessary properties of simple transitive birepresentations of $B_n$ without proving their existence. Section 5 constructs the simple transitive birepresentations and uses the results of Sect. 4 to obtain the classification.
2 Preliminaries

Throughout the text we always require the structure 2-morphisms of bicategorical structures to be invertible. The resulting setting of bicategories, pseudofunctors, strong transformations and modifications is what we will call the bicategorical setting, and we will give our results in this setup. In particular, we will study birepresentations of finitary bicategories, following [30].

Most of our results also hold in what we call the 2-categorical setting, where we require the structure 2-morphisms to be the identities, thus working with 2-categories, 2-functors, 2-transformations and modifications. We will comment on possible differences between the bicategorical and the 2-categorical results, when suitable.

2.1 Notation

Our notational conventions largely follow those of [30], with one difference and a few additions.

- Bicategories are denoted by $\mathcal{B}, \mathcal{C}$, and the like. Pseudofunctors are denoted by $\mathcal{M}, \mathcal{N}$, and the like. Strong transformations are denoted by capital Greek letters, e.g. $\Sigma, \Theta$. Modifications are denoted by $m, n$ and the like.
- Structure 2-morphisms of bicategorical structures are denoted by lower case fraktur. In particular, associators are denoted by $\alpha, \beta$, left unitors are denoted by $l$, and right unitors are denoted by $r$. These 2-morphisms are denoted by $\alpha, \nu_l, \nu_r$ in [30].
- Categories are denoted by $\mathcal{C} \Rightarrow \mathcal{D}$, and the like; objects in a category $\mathcal{C}$ are denoted by capital letters, such as $X \in \text{Ob}\mathcal{C}$. Morphisms are denoted by lower case letters, such as $f \in \mathcal{C}(X, Y)$.
- Given bicategories $\mathcal{B}, \mathcal{C}$, we denote the bicategory of pseudofunctors from $\mathcal{B}$ to $\mathcal{C}$ by $[\mathcal{B}, \mathcal{C}]$.
- Objects in a bicategory are denoted by $i, j$, and the like. 1-morphisms in a bicategory are denoted by $F, G$, and the like. 2-morphisms are denoted by lower case Greek letters, e.g. $\alpha, \beta$. The identity 1-morphisms of objects will be denoted by $\mathbb{1}_i$ and the like, and the identity 2-morphisms of 1-morphisms will be denoted by $\text{id}_F$, and the like.
- We write $GF = G \circ h F$ for composition of 1-morphisms, $\beta \circ_h \alpha$ for horizontal composition of 2-morphisms, $\beta \circ_v \alpha$ for vertical composition of 2-morphisms. The whiskering of a 1-morphism $G$ with a 2-morphism $\alpha$, given by $\text{id}_G \circ_h \alpha$, will be denoted by $G \bullet \alpha$. Similarly for $\alpha \bullet G$.
- To emphasize the strictness of 2-categorical structures, we use the superscript $(-)^\text{str}$. For instance, if $\mathcal{C}, \mathcal{D}$ are 2-categories, we denote the 2-category of 2-functors from $\mathcal{C}$ to $\mathcal{D}$ by $[\mathcal{C}, \mathcal{D}]^\text{str}$, and, given a bicategory $\mathcal{C}$ and a 2-category $\mathcal{D}$ biequivalent to $\mathcal{C}$, we may denote $\mathcal{D}$ by $\mathcal{C}^\text{str}$.

Similarly to, [30], we only indicate the 1-morphisms indexing the structure 2-morphisms of a bicategorical structure, while omitting the indexing objects from the notation, thus writing $a_{H, G, F}$ rather than $a_{i, j, k, l}^{H, G, F}$ for the associator

$$(HG)F \sim H(GF)$$

of the composition $i \to F \to j \to G \to k \to H \to 1$.

Note that our notational conventions for objects, 1-morphisms and 2-morphisms of a bicategory do not apply to the pseudofunctor bicategory $[\mathcal{C}, \mathcal{D}]$, where we prioritize our
conventions for pseudofunctors, strong transformations and modifications. Similarly, our conventions do not apply to functor categories, or to the 2-category $\text{Cat}$, where we prioritize our separate conventions for categories and functors.

2.2 Finitary Bicategories and Their Birepresentations

Let $k$ be an algebraically closed field of characteristic zero. We remark that, for the statements that do not involve finitary categories, this assumption can be relaxed to the assumption that $k$ is a commutative ring.

Let $\text{Cat}_k$ denote the 2-category of small $k$-linear categories, $k$-linear functors and natural transformations. Under tensor product of $k$-linear categories, $\text{Cat}_k$ becomes a symmetric monoidal 2-category. For the general definition of a monoidal 2-category, we refer to [10, Definition 2.6].

We say that a bicategory $\mathcal{B}$ is $k$-linear if it is enriched in $\text{Cat}_k$. Similarly for $k$-linear pseudofunctors, strong transformations and modifications. For the general definition and extensive treatment of bicategories enriched in a monoidal bicategory, we refer to [11]. In the case of $\text{Cat}_k$, it follows that a $k$-linear bicategory is a bicategory $\mathcal{B}$ such that for any $i, j \in \text{Ob } \mathcal{B}$, the category $\mathcal{B}(i, j)$ is $k$-linear and horizontal composition $\circ_h$ is $k$-bilinear.

$\text{Cat}_k$ itself is a $k$-linear 2-category: for any $k$-linear categories $C, D$, the category $\text{Cat}_k(C, D)$ is $k$-linear under pointwise formation of $k$-linear combinations of natural transformations of $k$-linear functors.

Given $k$-linear bicategories $\mathcal{B}, \mathcal{C}$, a $k$-linear pseudofunctor $M : \mathcal{B} \to \mathcal{C}$ is a pseudofunctor of underlying ordinary bicategories, such that, for any objects $i, j$ of $\mathcal{B}$, the local functor $M_{i,j}$ is $k$-linear. A $k$-linear strong transformation of $k$-linear pseudofunctors is a strong transformation of underlying pseudofunctors, with no additional requirements. Similarly, $k$-linear modifications are just modifications of said strong transformations. We will omit specifying $k$-linearity of bicategorical structures whenever it is a vacuous condition.

Whenever speaking of an ambient bicategory $\mathcal{B}$, 2-category $\mathcal{C}$ or category $\mathcal{C}$, we implicitly assume it to be essentially small. Our main aim is to prove the existence of certain $k$-linear pseudofunctors from a fixed, essentially small, bicategory $\mathcal{B}$ to $\text{Cat}_k$. It will become clear that this result is invariant under $k$-linear biequivalence, so we may first construct such pseudofunctors from a biequivalent, small $\mathcal{B}'$ and then pass under biequivalence. Thus, for our purposes, we may further assume that said essentially small structures are in fact small.

Composition of two $k$-linear pseudofunctors is again a $k$-linear pseudofunctor, and so the collection of such pseudofunctors between the $k$-linear bicategories $\mathcal{B}$ and $\mathcal{C}$, together with strong transformations and modifications, forms a bicategory which we denote by $[\mathcal{B}, \mathcal{C}]_k$. In accordance with our notational conventions, if $\mathcal{B}, \mathcal{C}$ are $k$-linear 2-categories, we denote the corresponding 2-category of 2-functors, 2-transformations and modifications by $[\mathcal{B}, \mathcal{C}]_k^{str}$.

We say that a pair

$$(F, G), \quad F : \mathcal{C} \to \mathcal{D}, \quad G : \mathcal{D} \to \mathcal{C}$$

of pseudofunctors is bicategorically adjoint if there are equivalences of categories

$\mathcal{D}(Fc, d) \simeq \mathcal{C}(c, Gd), \quad \square \; \text{Springer}$$
strongly natural in \( c, d \). If \( C, D \) are 2-categories, \( F, G \) are 2-functors, and there are isomorphisms of categories

\[
D(FC, GD) \cong C(c, GD).
\]

2-natural in \( c, d \), we say that \((F, G)\) is 2-categorically adjoint. In particular, we avoid the common terminology calling bicategorical adjunctions biadjunctions, since 1-categorical ambidextrous adjunctions, abundant in 2-representation theory, are often referred to as biadjoint pairs. For an extensive account of bicategorical adjunctions, see [9].

Following the observation [16, 2.29], given a \( k \)-linear category \( A \), the pair

\[
(- \otimes_k A, \text{Cat}_k(A, -))
\]

of 2-endofunctors of \( \text{Cat}_k \) is 2-categorically adjoint, and hence the symmetric monoidal 2-category \( \text{Cat}_k \) is closed. As remarked in [11, Section 5], given \( k \)-linear bicategories \( B, C, D \), we may form the \( k \)-linear bicategory \( B \otimes_k C \), given by products on the level of objects and 1-morphisms and by tensor product over \( k \) on the level of 2-morphisms and structure 2-morphisms, which yields the canonical \( k \)-linear bicequivalences

\[
[B, [C, D]_k]_k \simeq [B \otimes_k C, D]_k \simeq [C \otimes_k B, D]_k \simeq [C, [B, D]_k]_k.
\]

We may also form the \( k \)-linear bicategory \( B^{op} \), by reversing the direction of 1-morphisms in \( B \).

We say that a \( k \)-linear category \( C \) is finitary if it is equivalent to the category \( A \)-proj of finite dimensional projective modules over a finite dimensional associative \( k \)-algebra \( A \).

**Definition 2.1** A finitary bicategory is a bicategory \( B \) which is \( k \)-linear and such that the category \( B(i, j) \) is finitary, for all \( i, j \in \text{Ob } B \). For the remainder of this section, let \( B \) be a finitary bicategory.

Let \( \mathcal{A}^f_k \) denote the 2-category of finitary categories, \( k \)-linear functors and natural transformations. A finitary birepresentation of \( B \) is a \( k \)-linear pseudofunctor from \( B \) to \( \mathcal{A}^f_k \). We denote the bicategory of finitary birepresentations of \( B \) by \( \mathcal{B} \)-amod.

Given a finitary category \( A \), the split Grothendieck group \([A]_B \) of \( A \) is a free abelian group of finite rank. Let \( F : A \to B \) be a \( k \)-linear functor between finitary categories. Under a choice of bases for \([A]_B \) and \([B]_B \), the induced group homomorphism \([F]_B \) is represented by a matrix. Given a finitary birepresentation \( M \) of \( B \), together with choices of bases for \([M(i)]_B \) as \( i \in \text{Ob } B \), we obtain a matrix \([MF]_B \) for every 1-morphism \( F \) of \( B \)—the so-called action matrix of \( F \) under \( M \).

A \( k \)-linear, abelian category \( C \) is said to be finite if it is equivalent to the category \( A \)-mod of finite dimensional modules over a finite dimensional associative \( k \)-algebra \( A \). We let \( \mathcal{R}_k \) denote the 2-category of \( k \)-linear finite abelian categories, right exact functors and natural transformations. An abelian birepresentation of \( B \) is a \( k \)-linear pseudofunctor from \( B \) to \( \mathcal{R}_k \). Given a finitary birepresentation \( M \) of \( B \), its projective abelianization \( \overline{M} \) is an abelian birepresentation such that \( M \) can be recovered from \( \overline{M} \) by restricting to certain subcategories equivalent to \( \overline{M}(i) \)-proj, for \( i \in \text{Ob } B \). We refer the reader to [24] for details, and [29] for an improved construction. We will only use abelianization once in this document, and in that case the simpler construction of [24] can be used.

We say that a birepresentation of \( B \) is transitive if, for any \( i, j \in \text{Ob } B \) and any \( X \in \text{Ob } \mathcal{M}(i) \), \( Y \in \text{Ob } \mathcal{M}(j) \), there is a 1-morphism \( F \in B(i, j) \) such that \( Y \) is isomorphic to a direct summand of \( \mathcal{M}(F)(X) \).
A $\mathcal{B}$-stable ideal $I$ of a finitary birepresentation $M$ of $\mathcal{B}$ is a tuple $(I(i))_{i \in \text{Ob} \mathcal{B}}$ of ideals of $M(i)$ such that, for any $X \xrightarrow{f} Y \in I(i)$ and any $F \in \text{Ob} \mathcal{B}(i, j)$, we have $MF(f) \in I(j)$. We say that $I$ is proper if there is $i$ such that $\{0\} \subsetneq I(i) \subsetneq M(i)$. For any $\alpha \in \mathcal{B}(i, j)(F, F')$, we may define the evaluation of $\alpha$ at $f$ as

$$(M\alpha)_Y \circ MF(f) = MF'(f) \circ (M\alpha)_X \in I(j).$$

**Lemma 2.2** Let $M^\ominus \to N$ be a strong transformation of finitary birepresentations of a finitary bicategory $\mathcal{B}$. Let $I$ be a $\mathcal{B}$-stable ideal of $N$. Given $\hat{i} \in \text{Ob} \mathcal{B}$ and $X, Y \in M(\hat{i})$, let

$$(\Theta^{-1}I)(\hat{i})(X, Y) := \left\{ X \xrightarrow{f} Y \mid \Theta_2(\alpha) \in I \right\}.$$ 

Then the assignment $\Theta^{-1}I = (\Theta^{-1}I(\hat{i}))_{\hat{i} \in \text{Ob} \mathcal{B}}$ gives a $\mathcal{B}$-stable ideal of $M$.

**Proof** For every $\hat{i}$, the component $\Theta_2$ is a $k$-linear functor, and thus $$(\Theta^{-1}I)(\hat{i})$$ is an ideal of $M(\hat{i})$. To see that $\Theta^{-1}I$ is $\mathcal{B}$-stable, let $F \in \mathcal{B}(\hat{i}, j)$, and let $X \xrightarrow{f} Y$ be a morphism of $(\Theta^{-1}I)(\hat{i})$. We then have the commutative square

$$
\begin{array}{ccc}
NF\Theta_1(X) & \xrightarrow{NF\Theta_1(f)} & NF\Theta_1(X) \\
\downarrow{(\Theta_2)^X} & & \downarrow{(\Theta_2)^Y} \\
\Theta_2 MF(X) & \xrightarrow{\Theta_2 MF(f)} & \Theta_2 MF(Y)
\end{array}
$$

where the upper horizontal arrow lies in $I(\hat{j})$ by assumption, and hence the lower horizontal arrow also lies in $I(\hat{j})$. This proves that $MF(f) \in (\Theta^{-1}I)(\hat{j})$. \qed

**Definition 2.3** A finitary birepresentation $M$ of $\mathcal{B}$ is simple transitive if it has no proper $\mathcal{B}$-stable ideals. In particular, a simple transitive birepresentation is transitive.

### 2.3 Cells

The left preorder $\leq_L$ on the set of isomorphism classes of indecomposable 1-morphisms of $\mathcal{B}$ is defined by writing $F \leq_L G$ if there is a 1-morphism $H$ such that $G$ is a direct summand of $HF$. We denote the resulting equivalence relation by $\sim_L$, and refer to the equivalence classes as left cells. Similarly one defines the right and two-sided preorders $\leq_R, \leq_J$, together with the right and two-sided equivalence relations and right and two-sided cells.

Let $L$ be a left cell of $\mathcal{B}$. There is then a unique object $\hat{i}$ of $\mathcal{B}$ which is the domain of all 1-morphisms in $L$. To $L$ we associate a simple transitive subquotient $C_L$ of the principal 2-representation $P_L := \mathcal{B}(\hat{i}, -)$; see [27, Subsection 3.3] for details.

By [3, Lemma 1], given a transitive birepresentation $M$ of $\mathcal{B}$, the set of $J$-cells of $\mathcal{B}$ not annihilating $M$ admits a $J$-greatest element, which we call the apex of $M$. The apex is an invariant of transitive birepresentations.

### 2.4 Biideals

A two-sided biideal $I$ in a $k$-linear bicategory $C$ consists of a collection of ideals $(I_{j, k})_{j, k \in \text{Ob} \mathcal{C}}$ of $C(j, k)$, such that, for any 1-morphisms $F$ of $C(i, j)$ and $G$ of $C(k, 1)$,
and any 2-morphism $\gamma \in \mathcal{I}_{j,k}$, we have

$$G \cdot \gamma \in \mathcal{I}_{j,1}, \quad \gamma \cdot F \in \mathcal{I}_{1,k}.$$ 

As a consequence, given any 2-morphisms $\alpha$ of $\mathcal{C}(i, j)$ and $\beta$ of $\mathcal{C}(k, l)$, we have

$$\beta \circ \alpha \gamma \in \mathcal{I}_{j,1}, \quad \alpha \circ \beta \gamma \in \mathcal{I}_{1,k}$$

due to

$$\beta \circ \alpha \gamma = (\beta \circ \alpha \text{id}_{\text{cod} \gamma}) \circ (\text{id}_{\text{dom}} \beta \circ \alpha \gamma)$$

and similarly for $\alpha$. Given a biideal $\mathcal{I}$, the assignment $(\mathcal{I}^2)_{k,l} := (\mathcal{I}_{k,l})^2$ defines a biideal: given $\beta, \alpha \in \mathcal{I}_{1,j}$ such that $\beta \circ_v \alpha$ is defined, and $\gamma$ in $\mathcal{C}(j, k)$, we have

$$\gamma \circ \alpha \beta \circ_v \gamma = (\gamma \circ \alpha \beta) \circ_v (\gamma \circ \alpha \beta).$$

But $\gamma \circ \alpha \beta \gamma \circ \alpha \beta \alpha \in \mathcal{I}_{1,k}$, and so $\gamma \circ \alpha \beta \gamma \circ \alpha \beta \alpha \in \mathcal{I}^2_{1,k}$ by definition. In particular,

$$\mathcal{I}^2_{1,k} := \{ \alpha \in \mathcal{I}_{1,k} \mid \alpha = \gamma \circ \alpha \beta, \text{ for } j \in \text{Ob } \mathcal{C}, \beta \in \mathcal{I}_{1,j}, \gamma \in \mathcal{I}_{j,k} \} \subseteq \mathcal{I}^2_{1,k} \quad (1)$$

Inductively we may define $\mathcal{I}^m$, for any $m \in \mathbb{Z}_{\geq 0}$. We say that $\mathcal{I}$ is nilpotent if there is $m$ such that $\mathcal{I}^m_{1,1} = 0$, for all $1, j \in \mathcal{C}$.

Assume now that $\mathcal{C}$ is finitary, let $\mathcal{I}$ be a two-sided biideal of $\mathcal{C}$, let $\mathbf{M}$ be a finitary birepresentation of $\mathcal{C}$ and let $\mathbf{f}_{H,G} : \mathbf{MHMG} \xrightarrow{\sim} \mathbf{M}(HG)$ be the structure 2-morphisms of $\mathbf{M}$. The ideal $\text{ev}_{\mathbf{M}}(\mathcal{I})$ of $\mathbf{M}$ is defined by letting $\text{ev}_{\mathbf{M}}(\mathcal{I})(\hat{i})$ consist of $k$-linear combinations of morphisms of the form

$$W \xrightarrow{f} \mathbf{MF}(X) \xrightarrow{\mathbf{MF}(g)} \mathbf{MF}(Y)$$

$$\downarrow^{(\mathbf{M}s\mathbf{\alpha})_X} \quad \downarrow^{(\mathbf{M}s\mathbf{\alpha})_Y}$$

$$\mathbf{MG}(X) \xrightarrow{\mathbf{MG}(g)} \mathbf{MG}(Y) \xrightarrow{h} Z$$

where $f, h$ are arbitrary morphisms of $\mathbf{M}(\hat{i})$, whereas the 1-morphisms $F,G$, the 2-morphism $\alpha$ and the morphism $g$ are only required to be such that all the compositions and evaluations in the diagram are well-defined. It is clear that this collection is stable under vertical composition, since $f, h$ are arbitrary. To see that it is also closed under horizontal composition, let $H$ be such that the compositions $HG, HF$ are defined.

Applying $\mathbf{MH}$ on the defining diagram above, using the associators, we obtain:

$$\mathbf{MH}(W) \xrightarrow{\mathbf{MH}(f)} \mathbf{MHMF}(X) \xrightarrow{\mathbf{MHMF}(g)} \mathbf{MHMF}(Y)$$

$$\downarrow^{(\mathbf{MH}s\mathbf{\alpha})_X} \quad \downarrow^{(\mathbf{MH}s\mathbf{\alpha})_Y}$$

$$\mathbf{MHMG}(X) \xrightarrow{\mathbf{MHMG}(g)} \mathbf{MHMG}(Y) \xrightarrow{\mathbf{MH}(h)} \mathbf{MH}(Z)$$

$$\downarrow^{(\mathbf{M}s\mathbf{\alpha})_X} \quad \downarrow^{(\mathbf{M}s\mathbf{\alpha})_Y} \quad \downarrow^{(\mathbf{M}s\mathbf{\alpha})_Y}$$

$$\mathbf{M}(HG)(X) \xrightarrow{\mathbf{M}(HG)(g)} \mathbf{M}(HG)(Y)$$

The cube in the diagram commutes: its top and bottom faces commute by the naturality of $\mathbf{f}_{H,F}$, left and right by the naturality of $\mathbf{f}_{H,-}$, front by the naturality of $\mathbf{M}(H \circ \alpha)$, and back by
the naturality of $MH\bullet M\alpha$. The red part of the diagram indicates that the resulting morphism still belongs to $ev_M(I)$. Further, it shows that, if we modify the definition of $ev_M(I)$, so that instead of letting $\alpha$ be arbitrary, we require $\alpha = \gamma \circ h \beta$ as in (1), we again obtain an ideal of $M$, which we denote by $ev_M(I^{h,2})$, in view of the notation in (1). Inductively, we may define further ideals $ev_M(I^{h,k})$, for $k \in \mathbb{Z}_{>0}$. The above diagram also proves that $ev_M(I^{h,2})$ is the ideal generated by morphisms of the form $MH(f)$, for $f \in ev_M(I)$. If we express this by the suggestive notation

$$ev_M(I^{h,2}) = \mathcal{C} ev_M(I),$$

then, setting $I^{h,1} := I$, inductively we also have

$$ev_M(I^{h,k+1}) = \mathcal{C} ev_M(I^{h,k}), \text{ for } k \geq 1. \tag{2}$$

**Lemma 2.4** If $M$ is a simple transitive birepresentation of $\mathcal{C}$ and $I$ is a nilpotent ideal of $\mathcal{C}$, then $ev_M(I) = 0$. In other words, $I$ annihilates $M$.

**Proof** Since $M$ is simple transitive, we have $ev_M(I) = 0$ or $ev_M(I) = M$. Using (2) and the notation used in the paragraphs preceding it, the latter case would imply

$$ev_M(I^{h,2}) = \mathcal{C} ev_M(I) = \mathcal{C} M = M$$

and, inductively

$$ev_M(I^{h,k}) = M, \text{ for } k \geq 1.$$  

Let $m$ be such that $I^m = 0$. Similarly to (1), $I^m = 0$ implies $I^{h,m} = 0$. This yields

$$ev_M(I^{h,m}) = 0,$$

which contradicts the earlier conclusion, thus showing that $ev_M(I) = 0$. \hfill \Box

**Lemma 2.5** Let $I$ be a biideal of $\mathcal{C}$ not containing any identity 2-morphisms. Then $I$ is nilpotent.

**Proof** Since $\mathcal{C}(i, j)$ is finitary and $I_{i,j}$ contains no identity 2-morphisms, we have

$$I_{i,j} \subseteq \text{Rad } \mathcal{C}(i, j), \text{ for all } i, j.$$

Given $i, j \in \text{Ob } \mathcal{C}$, let $n_{i,j}$ be the nilpotency degree of $\text{Rad } \mathcal{C}(i, j)$. Since $\text{Ob } \mathcal{C}$ is finite, we may let $n := \max \{ n_{i,j} \mid i, j \in \text{Ob } \mathcal{C} \}$. Clearly then $I_{i,j}^n = 0$, for all $i, j \in \text{Ob } \mathcal{C}$, and hence $I^n = 0$. \hfill \Box

### 3 Weighted Colimits of 2-Representations

#### 3.1 Weighted Colimits of $k$-linear Pseudofunctors

We first recall the notion of a bicategorical weighted colimit. The simplified terminology we use here replaces the more precise terminology of pseudo-, bi- and lax colimits, which we will not need since we only use elementary bicategorical notions. Our choice of terminology here is the same as that in [1], and it is consistent with the notational conventions we made before.
Definition 3.1 Let $\mathcal{B}$ be a bicategory. Let $\mathcal{J}$ be a small bicategory and let $\mathcal{W}$ be a pseudofunctor from $\mathcal{J}^{\text{op}}$ to $\text{Cat}$. Given a pseudofunctor $F$ from $\mathcal{J}$ to $\mathcal{B}$, a weighted bicategorical colimit $\mathcal{W} \star F$ is an object of $\mathcal{B}$ together with a representation of the pseudofunctor

$$[\mathcal{J}^{\text{op}}, \text{Cat}] (\mathcal{W}, \mathcal{B}(F, i))$$

in $i$. In other words, there are equivalences of categories

$$\mathcal{B} (\mathcal{W} \star F, i) \simeq [\mathcal{J}^{\text{op}}, \text{Cat}] (\mathcal{W}, \mathcal{B}(F, i))$$

strongly natural in $i$. If $\mathcal{W} \star F$ exists, it is unique up to a compatible internal equivalence in $\mathcal{B}$. If $\mathcal{W} \star F$ exists for all choices of $\mathcal{J}$, $\mathcal{W}$, $F$, we say that $\mathcal{B}$ is bicategorically cocomplete.

If we instead have a 2-category $\mathcal{C}$, as a small 2-category $\mathcal{J}$ and 2-functors $\mathcal{W}$, $F$, the weighted 2-categorical colimit $\mathcal{W} \star F$ is an object of $\mathcal{C}$ representing the 2-functor

$$[\mathcal{J}^{\text{op}}, \text{Cat}^{\text{str}}] (\mathcal{W}, \mathcal{C}(F, i)),$$

giving rise to isomorphisms of categories

$$\mathcal{C} (\mathcal{W} \star F, i) \simeq [\mathcal{J}^{\text{op}}, \text{Cat}^{\text{str}}] (\mathcal{W}, \mathcal{C}(F, i))$$

2-natural in $i$. It is then unique up to isomorphism in $\mathcal{C}$, and if it always exists, we say that $\mathcal{C}$ is 2-categorically cocomplete.

Bicategorical and 2-categorical limits are obtained as bicategorical and 2-categorical colimits in $\mathcal{B}^{\text{op}}$ and $\mathcal{C}^{\text{op}}$.

Proposition 3.2 The 2-category $\text{Cat}_k$ is complete and cocomplete, both bicategorically and 2-categorically.

Proof Cocompleteness can be shown by explicitly constructing certain colimits which can be used to obtain all colimits. A proof given by construction of coproducts, coinserters and coequifiers can be found in [1, Proposition 2.6]. For completeness, one easily verifies that the explicit constructions of products, cotensors, 2-equalizers and pseudoequalizers in $\text{Cat}$ given in [2] also apply in the case of $\text{Cat}_k$—this can be viewed as a consequence of preservation of limits by the forgetful 2-functor $\text{Cat}_k \to \text{Cat}$, which is 2-categorically right adjoint to the free $k$-linear category 2-functor. Completeness follows as a consequence of [37, 1.24]. $\Box$

Definition 3.3 Let $\mathcal{B}$ be a $k$-linear bicategory. Let $\mathcal{J}$ be a small $k$-linear bicategory and let $\mathcal{W}$ be a $k$-linear pseudofunctor from $\mathcal{J}^{\text{op}}$ to $\text{Cat}_k$. Given a $k$-linear pseudofunctor $F : \mathcal{J} \to \mathcal{B}$, a weighted $k$-linear bicategorical colimit $\mathcal{W} \star F$ is an object of $\mathcal{B}$ together with a representation of the $k$-linear pseudofunctor

$$[\mathcal{J}^{\text{op}}, \text{Cat}_k]_k (\mathcal{W}, \mathcal{B}(F, i))$$

in $i$. In other words, there are $k$-linear equivalences of categories

$$\mathcal{B} (\mathcal{W} \star F, i) \simeq [\mathcal{J}^{\text{op}}, \text{Cat}_k]_k (\mathcal{W}, \mathcal{B}(F, i))$$

Similarly one obtains the notion of a 2-categorical $k$-linear colimit, of $k$-linear bicategorical and 2-categorical limits and the resulting notions of $k$-linear bicategorical and 2-categorical cocompleteness and completeness.

The next statement can be viewed as a direct corollary of [11, Proposition 11.2], however, since that paper is written in a much more general setting, we give an explanation of how one views our particular case as an instance of the setting of [11], and how one uses the results therein to obtain the statement.
Proposition 3.4  Let $\mathcal{B}$ be a small $k$-linear bicategory. The 2-category $[\mathcal{B}, \text{Cat}_k]$ of $k$-linear pseudofunctors from $\mathcal{B}$ to $\text{Cat}_k$ is $k$-linear cocomplete, and the colimits are computed pointwise. In other words, given a diagram $F : \mathcal{J} \to [\mathcal{B}, \text{Cat}_k]_k$, a weight $W$ in $[\mathcal{J}, \text{Cat}_k]_k$, and an object $b \in \mathcal{B}$, the $k$-linear category $(W \ast F)(b)$ is the colimit $W \ast F_b$, where $F_b$ is obtained by evaluating at $b$ the $k$-linear pseudofunctor $F^\sim$ obtained by

$$J, [\mathcal{B}, \text{Cat}_k t_k)]_k \sim [\mathcal{B}, [\mathcal{J}, \text{Cat}_k]_k]_k$$

$F \mapsto F^\sim$

Proof  As we have observed before, we consider the case of enriching monoidal bicategory $\mathcal{V} = \text{Cat}_k$. As remarked in [11, Section 5], it is easy to verify that, given $k$-linear bicategories $\mathcal{A}$, $\mathcal{C}$, left $\mathcal{A}$-modules in the sense of [11] are $k$-linear pseudofunctors from $\mathcal{A}$ to $\text{Cat}_k$, and similarly, $\mathcal{A}$-$\mathcal{C}$-bimodules are $k$-linear pseudofunctors from $\mathcal{A} \otimes_k \mathcal{C}^{\text{op}}$ to $\text{Cat}_k$. Given $M, N \in [\mathcal{B}, \text{Cat}_k]_k$, their internal hom, as studied in the general case in [11, Section 7], is given by the $k$-linear category $[\mathcal{B}, \text{Cat}_k]_k(M, N)$, under the clear choice of the evaluation morphism $\xi$ of [11, Section 7.3]. Combining this with the fact that $\mathcal{B}$ is assumed to be small, we conclude that the $k$-linear bicategory of moderate right $\mathcal{B}$-modules studied in [11, Section 11] coincides with $[\mathcal{B}^{\text{op}}, \text{Cat}_k]_k$. Its $k$-linear cocompleteness follows from [11, Proposition 11.2]. The colimit $W \ast F$ is given by $W \otimes_\mathcal{J} \tilde{F}$, for $\tilde{F}$ and $\otimes_\mathcal{J}$ as defined in [11]. Similarly, the colimit $W \ast F_b$ is given by $W \otimes_\mathcal{J} \tilde{F}_b$. Yoneda Lemma implies that $\tilde{F}(b) \simeq \tilde{F}_b$. Taking all this into account, one sees that the result follows from [11, Corollary 6.10].

Using the categorification of [16, 3.8] in [11, 13.14], we conclude that ordinary bicategorical colimits in an $k$-linear bicategory $\mathcal{B}$ can be viewed as special cases of $k$-linear bicategorical colimits. We remark that in our case, the monoidal bicategorical adjunction obtained in [11, 13.14] is that given by the free $k$-linear bicategory $\mathcal{B}$ and the forgetful 2-functor. As a consequence of the last two observations, together with Proposition 3.2, we find the following:

Proposition 3.5 Let $\mathcal{B}$ be a $k$-linear bicategory. The $k$-linear pseudofunctor bicategory $[\mathcal{B}, \text{Cat}_k]_k$ is bicategorically complete and cocomplete, and the bicategorical limits and colimits are constructed pointwise from those in $\text{Cat}_k$.

For much more general results regarding weighted bicategorical limits, in a much more general setting of bicategories enriched in monoidal bicategories which are not necessarily symmetric or closed, see [11, Section 10, Section 11].

3.2 Additive and Karoubi Envelopes

Recall that a preadditive category is a category enriched in the category $\text{Ab}$ of abelian groups. Let $\text{Cat}_Z$ denote the 2-category of preadditive categories.

The additive envelope is the universal solution to the problem of adding direct sums to a preadditive category. Similarly, the Karoubi envelope universally makes a category idempotent split. These constructions are used, for instance, to define the category $\text{Rep}(S_t)$—see [6]. A more detailed account is given in [5], and a very detailed account, which will be our main reference, is given in [35].

Let $\text{Cat}_Z^\oplus, \text{Cat}_Z^K$ denote the 1, 2-full 2-subcategories of $\text{Cat}_Z$, with the respective object sets being that of categories with finite direct sums and that of idempotent split categories,
respectively. Let $\text{Cat}_k^\oplus, \text{Cat}_k^K$ be the $k$-linear variants thereof. Let $\text{Cat}_k^D$ denote the 1, 2-full 2-subcategory of $\text{Cat}_k$ given by the categories which are both additive and idempotent split. We have the following results:

- [35, 2.1.6, 2.2.3]: There are 2-functors $(-)^\oplus : \text{Cat}_Z \to \text{Cat}_Z^\oplus$ and $(-)^K : \text{Cat}_Z \to \text{Cat}_Z^K$.
- [35, 2.1.9, 2.2.6]: The above 2-functors are compatible with $k$-linear structures: restriction gives $k$-linear 2-functors $(-)^\oplus : \text{Cat}_k \to \text{Cat}_k^\oplus$ and $(-)^K : \text{Cat}_k \to \text{Cat}_k^K$.
- [35, Proposition 41]: The Karoubi envelope of an additive category is additive. We thus obtain the 2-functor $(-)^D := (-)^K \circ (-)^\oplus \in [\text{Cat}_k, \text{Cat}_k^D]_{\text{str}}$.
- [35, 2.1.3, 2.2.2]: Given $C \in \text{Ob Cat}_k$, there are canonical fully faithful functors $\Phi_1^C : C \to C^\oplus$ and $\Psi_1^C : C \to C^K$. Using the definitions of $\Psi_1^C, \Phi_1^C$, it is easy to verify that the following equations hold:

$$\Phi_1^D \circ F = F^\oplus \circ \Phi_1^C \quad \text{and} \quad \Psi_1^D \circ F = F^K \circ \Psi_1^C,$$

for any $k$-linear functor $F : C \to D$. Denote the respective inclusion 2-functors by $J^\oplus \in [\text{Cat}_k, \text{Cat}_k^\oplus]$ and $J^K \in [\text{Cat}_k, \text{Cat}_k^K]$. Equation (3) implies that we have 2-transformations

$$\Phi : J^\oplus \circ (-)^\oplus \quad \text{and} \quad \Psi : J^K \circ (-)^K.$$

- [35, Proposition 75, Theorem 113]: Given $C \in \text{Ob Cat}_k$ and $D \in \text{Ob Cat}_k^\oplus$, the functor

$$\text{Cat}_k(C^\oplus, D) \xrightarrow{\circ \Phi_1^C} \text{Cat}_k(C, D)$$

is an equivalence. Similarly, given $D \in \text{Cat}_k^K$, the functor

$$\text{Cat}_k(C^K, D) \xrightarrow{\circ \Psi_1^C} \text{Cat}_k(C, D)$$

is an equivalence.

Let $J^D : \text{Cat}_k \to \text{Cat}_k^D$ denote the indicated inclusion 2-functor. Using the above listed facts about $(-)^\oplus, (-)^K$, we make the following conclusion:

**Proposition 3.6** The pairs $((-)^\oplus, J^\oplus)$ and $((-)^K, J^K)$ are bicategorically adjoint. Composing the adjunctions, we further conclude that the 2-functors $((-)^D, J^D)$ are bicategorically adjoint.

**Proof** Given $C \in \text{Ob Cat}_k$ and $D \in \text{Ob Cat}_k^\oplus$, the 2-naturality of the equivalence

$$\text{Cat}_k(C^\oplus, D) \xrightarrow{\circ \Phi_1^C} \text{Cat}_k(C, D)$$

follows from Eq. (3). The 2-naturality in $D$ follows from the associativity of composition of functors, since the equivalences are given by precomposition. The proof for Karoubi envelope is completely analogous.

**Remark 3.7** Observe that the bicategorical adjunction above is almost 2-categorical, the only condition missing being that it is given by equivalences rather than isomorphisms of Hom-categories.
Combining Proposition 3.5 with Proposition 3.6, we obtain the following:

**Corollary 3.8** The 2-categories $\text{Cat}_k^\oplus$, $\text{Cat}_k^K$, $\text{Cat}_k^D$ are bicategorically cocomplete. Given a $k$-linear bicategory $B$, the bicategories

\[ [B, \text{Cat}_k^\oplus], [B, \text{Cat}_k^K], [B, \text{Cat}_k^D] \]

are bicategorically cocomplete.

**Proof** The first part of the statement is an immediate consequence of the preceding results. The second part follows since the bicategorical adjunctions

\[((-)^\oplus, J^\oplus), ((-)^K, J^K), ((-)^D, J^D)\]

give bicategorical adjunctions

\[ ([B, (-)^\oplus]_k, [B, J^\oplus]_k), ([B, (-)^K]_k, [B, J^K]_k), ([B, (-)^D]_k, [B, J^D]_k) \].

Corollary 3.8 gives two equivalent ways to compute colimits in $[B, \text{Cat}_k^D]$ (and the other two pseudofunctor bicategories considered): we may first compute the same colimit in $\text{Cat}_k$, apply the envelope $(-)^D$ and lift it pointwise to a colimit in $[B, \text{Cat}_k^D]_k$, or we may compute the colimit in $\text{Cat}_k$, lift it pointwise to a colimit in $[B, \text{Cat}_k^D]_k$ and then apply the envelope $(-)^D$.

Colimits in ordinary category theory give universal properties expressed in terms of unique factorization of morphisms. For bicategories, the universal properties are expressed in terms of equivalences of categories of 1-morphisms. Following [17], we call the resulting factorization of 1-morphisms up to invertible 2-morphisms the one-dimensional aspect of the universal property, and the resulting functorial bijections of 2-morphisms the two-dimensional aspect of the universal property. The next result further facilitates the computation of colimits. It can be shown by slightly modifying the proof of the analogous observation made in [16, Section 3], for non-enriched bicategories.

**Proposition 3.9** Let $B$ be bicategorically cocomplete. The one-dimensional aspect of the universal property of a weighted colimit in $B$ implies its two-dimensional aspect.

More precisely, to obtain the $k$-linear statement, one may argue directly on the level of cocones: given a colimiting cocone $W \star F$ and a cocone $H$ satisfying the one-dimensional universal property, one obtains 1-morphisms $H \to W \star F$ and $W \star F \to H$. The respective one-dimensional universal properties then suffice to conclude that these 1-morphisms are mutually quasi-inverse equivalences.

Ideally, we would like to further restrict our treatment of colimits to the 2-subcategory $\mathcal{A}_k^f$ of $\text{Cat}_k^D$, consisting of finitary categories. However, the condition of finite dimensional hom-spaces between objects is not preserved under taking weighted colimits—in fact, it fails already for conical colimits.

This is very different from the setting of classical representation theory: the category $\text{vec}_k$ of finite dimensional vector spaces over $k$ is abelian, and hence, in particular, cocomplete. As a consequence, if $C$ is a $k$-linear category, then the category $C\text{-mod}$, defined as $\text{Cat}_k(C, \text{vec}_k)$, also is $k$-linear and abelian. Our claim means that the categorification of this statement to finitary bicategories is false.
We now give an example of this phenomenon. Let $\text{Cat}_k^{f.d.}$ denote the $1, 2$-full $2$-subcategory of $\text{Cat}_k$ whose objects are $k$-linear categories with finite dimensional hom-spaces.

**Example 3.10** Let $k$ be a field. Consider the free $k$-linear categories $A_2, A_3$ on the quivers

$$A_2 = 1 \xrightarrow{a} 2, \quad A_3 = 1 \xrightarrow{b} 2 \xrightarrow{c} 3$$

and the $k$-linear functors $F_b, F_c : A_2 \to A_3$ determined by

$$F_b(a) = b, \quad F_c(a) = c.$$ 

The coequalizer $\text{coeq}(F_b, F_c)$ in $\text{Cat}_k$ is given by the free $k$-linear category on the quiver $X \xleftarrow{g} x$, together with the $k$-linear functor $A_3 \xrightarrow{C_{b=3}} \text{coeq}(F_b, F_c)$ determined by

$$C_{b=3}(b) = C_{b=3}(c) = x.$$

Assume that $\text{Cat}_k^{f.d.}$ admits a coequalizer

$$A_3 \xrightarrow{C_{b=3}^{f.d.}} \text{coeq}^{f.d.}(F_b, F_c)$$

of the above functors. The universal property of $\text{coeq}(F_b, F_c)$ gives a $k$-linear functor $K : \text{coeq}(F_b, F_c) \to \text{coeq}^{f.d.}(F_b, F_c)$ such that $C_{b=3}^{f.d.} = K \circ C_{b=3}$.

Given $m \in \mathbb{Z}_{>0}$, let $T_m$ be the $k$-linear category with a unique object $Y$ and a morphism $y \in \text{End}_{T_m}(Y)$ such that $\text{End}_{T_m}(Y) = k[y]/(y^m)$. Let $T_m : A_3 \to T_m$ be the $k$-linear functor determined by $T_m(b) = T_m(c) = y$. In particular, $\tilde{T}_m$ coequalizes $F_b, F_c$. Applying the universal properties of the respective coequalizers, we obtain the following commutative diagram of $k$-linear functors:

$$\begin{array}{ccc}
A_3 & \xrightarrow{C_{b=3}^{f.d.}} & \text{coeq}^{f.d.}(F_b, F_c) \\
\downarrow{T_m} & & \downarrow{T_m} \\
\tilde{T}_m & \xrightarrow{\tilde{T}_m^{f.d.}} & \tilde{T}_m
\end{array}$$

where $\tilde{T}_m, \tilde{T}_m^{f.d.}$ are obtained using the respective universal properties of their domains and hence are the unique $k$-linear functors making the left inner triangle and the outer triangle commute, respectively. The functor $\tilde{T}_m$ is determined by $\tilde{T}_m(x) = y$. The right triangle part of the above diagram gives the following diagram of associative $k$-algebras:

$$\begin{array}{ccc}
\mathbb{K}[x] & \xrightarrow{K_{X,X}} & \text{End}_{\text{coeq}^{f.d.}(F_b, F_c)}(K(X)) \\
\downarrow{1 \mapsto y} & & \downarrow{(\tilde{T}_m^{f.d.})_{K(X), K(X)}} \\
\mathbb{K}[y]/(y^m) & \xrightarrow{z} & (\tilde{T}_m^{f.d.})_{K(X), K(X)}
\end{array}$$

Using image factorization, we replace $\text{End}_{\text{coeq}^{f.d.}(F_b, F_c)}(K(X))$ with its subalgebra generated by $K_{X,X}(x) = z$. Since $\text{End}_{\text{coeq}^{f.d.}(F_b, F_c)}(K(X))$ is finite dimensional by assumption, the aforementioned subalgebra is a finite dimensional quotient of $k[x]$, hence isomorphic to $k[w]/(w^k)$, for some $k$. Since $z$ is mapped to $y$ under $(\tilde{T}_m^{f.d.})_{K(X), K(X)}$, we see that $z$ must be a radical element with nilpotency degree greater than or equal to $m$. This implies $k \geq m$. But, while we may vary $m$, the coequalizer, and thus also the integer $k$, remain unchanged, so the existence of a coequalizer in $\text{Cat}_k^{f.d.}$ would imply the existence of $k$ such that $k \geq m$ for all $m > 0$, which is a contradiction.
In view of the Krull-Schmidt theorem for finitary categories, we see from Corollary 3.8 that colimits of finitary birepresentations, although themselves not necessarily finitary, may be computed starting from the indecomposable objects of the underlying categories. Formally, we have the following:

**Proposition 3.11** Let $C$ be a finitary category. Let $C\text{-indec}$ be the full subcategory of indecomposable objects of $C$. Then $C \cong (C\text{-indec})^\oplus$.

Let $X$ be a set of objects of $C$, and let $\mathcal{X}$ be the full subcategory of $C$ satisfying $\text{Ob } \mathcal{X} = X$. Then $\text{add } X \cong \mathcal{X}^D$.

**Proof** Since $C$ is additive, by [35, Proposition 99], the functor $\Phi_C : C \to C^\oplus$ is an equivalence. Let $I$ be the inclusion functor $C\text{-indec} \hookrightarrow C$. Since $I$ is full and faithful, by [35, Lemma 101], so is $I^\oplus : (C\text{-indec})^\oplus \to C^\oplus$. The Krull-Schmidt theorem implies that all the isomorphism classes of objects of $C$ are given by those of finite direct sums of objects in $C\text{-indec}$. Thus, $I^\oplus$ is essentially surjective, and hence an equivalence.

Similarly, since $\text{add } X$ is additive and idempotent split, we have $\text{add } X \cong X^D$.

For a more extensive study of zigzag algebras, see for example [8].

**4 Projective Bimodules for Star Algebras**

**4.1 The Bicategories $\tilde{B}$ and $B$**

Given $a, b \in \mathbb{Z}$ with $a \leq b$, we denote by $[a, b]$ the set $\{a, a + 1, \ldots, b\}$.

Let $S_n$ denote the star graph on $n + 1$ vertices. Label the unique internal node of $S_n$ by 0, and the leaves by 1, 2, ..., $n$. Let $\Lambda_n$ denote the zigzag algebra on $S_n$, i.e. the quotient of the path algebra of

```
    a_1 \quad 0 \quad a_n

  1 \quad b_1 \quad \ldots \quad b_n \quad n
```

by the ideal given by the sum of the third power of the arrow ideal and the ideal given by relations $a_i b_j = 0$, for $i \neq j$, and $b_1 a_1 = \cdots = b_n a_n = 0$. Let $e_0, \ldots, e_n$ denote the complete set of pairwise orthogonal, primitive idempotents induced by the labelling on $S_n$.

For $k \in [1, n]$, denote by $e_k$ the element given by the 2-cycle $a_k b_k$.

For a more extensive study of zigzag algebras, see for example [8].

Consider the $k$-linear monoidal subcategory $\tilde{B}_n$ of $(\Lambda_n\text{-mod} \otimes \Lambda_n, \otimes_{\Lambda_n})$ given by the additive closure $\text{add } \{\Lambda_n, \Lambda_n \otimes_{\Lambda_n} \Lambda_n, \Lambda_n e_k \otimes_k e_0 \Lambda_n \mid k \in [0, n]\}$. Viewed as a bicategory with a unique object, $\tilde{B}_n$ is a finitary bicategory. We denote its unique object by $i$. In particular, $\tilde{B}_n$ is biequivalent to its essential image in $\text{Cat}_k(\Lambda_n\text{-mod}, \Lambda_n\text{-mod})$, under the pseudofunctor sending a bimodule $M$ to the functor $M \otimes_{\Lambda_n} -$. Simple transitive 2-representations of the 2-category $\tilde{B}_n^{\text{str}}$ given by this essential image were studied in [38]. In this section, we use the partial results of [38] to give a description of the underlying categories and action
matrices for simple transitive birepresentations of $\tilde{\mathcal{B}}_n$. As observed in [30, 2.3], studying the simple transitive birepresentations of $\tilde{\mathcal{B}}_n$ is equivalent to studying the simple transitive 2-representations of $\tilde{\mathcal{B}}_n^{\text{str}}$.

To simplify the notation, we now fix $n \in \mathbb{Z}_{>0}$ and denote $\Lambda_n$ by $\Lambda$, denote $\tilde{\mathcal{B}}_n^{\text{str}}$ by $\tilde{\mathcal{B}}^{\text{str}}$, and finally write $\tilde{\mathcal{B}}$ for $\tilde{\mathcal{B}}_n$. We will introduce the subscripts again whenever there is a risk of ambiguity, in particular when different values of $n$ need to be considered simultaneously.

Using the canonical isomorphism $\text{End}_{\Lambda} (\Lambda) \simeq \Lambda^{\text{op}}$, we identify morphisms of indecomposable projective modules with elements of $\Lambda$. For example, given $i \in [1, n]$, this yields $\text{Hom}_{\Lambda} (\Lambda e_i, \Lambda e_0) = \mathbb{k} \{ a_i \}$. This applies also to the indecomposable bimodules $\Lambda e_k \otimes_k e_0 \Lambda$, where we identify morphisms with the images of the generators $e_k \otimes e_0$. We obtain

$$\text{Hom}_{\Lambda} (\Lambda e_j \otimes_k e_0 \Lambda, \Lambda e_k \otimes_k e_0 \Lambda) = \Lambda e_j \Lambda e_k \otimes_k e_0 \Lambda.$$  \hspace{1cm} (4)

Similarly, we identify morphisms from $\Lambda$ with the images of the generator 1. An easy but tedious calculation yields

$$\text{End}_{\Lambda} (\Lambda) = \mathbb{k} \{ 1, c, c_k \ | \ k \in [1, n] \} ;$$
$$\text{Hom}_{\Lambda} (\Lambda e_j \otimes_k e_0 \Lambda, \Lambda) = e_j \Lambda e_0 ;$$
$$\text{Hom}_{\Lambda} (\Lambda, \Lambda e_j \otimes_k e_0 \Lambda) = \mathbb{k} \{ b_j \otimes c + c_j \otimes b_j, c_i \otimes c \} , j \neq 0 ;$$
$$\text{Hom}_{\Lambda} (\Lambda, \Lambda e_0 \otimes_k e_0 \Lambda) = \mathbb{k} \left\{ e_0 \otimes c + c \otimes e_0 + \sum_{j=1}^{n} a_j \otimes b_j, c \otimes c \right\} .$$ \hspace{1cm} (5)

Let $\mathcal{I}$ denote the ideal of add $\{ \Lambda \Lambda, \Lambda e_k \otimes_k e_0 \Lambda \ | \ k \in [0, n] \}$ determined by

$$\mathcal{I} (\Lambda, \Lambda) = \mathbb{k} \{ c_k \ | \ k \in [1, n] \} , \mathcal{I} (\Lambda e_j \otimes_k e_0 \Lambda, \Lambda) = 0 ,$$
$$\mathcal{I} (\Lambda, \Lambda e_0 \otimes_k e_0 \Lambda) = 0 , \mathcal{I} (\Lambda, \Lambda e_j \otimes_k e_0 \Lambda) = \mathbb{k} \{ c_i \otimes c \} , j \neq 0$$
$$\mathcal{I} (\Lambda e_j \otimes_k e_0 \Lambda, \Lambda e_k \otimes_k e_0 \Lambda) = \mathbb{k} \{ c_j \otimes e_0, c_j \otimes c \} \text{ if } j = k \neq 0 , \text{ otherwise } 0 .$$ \hspace{1cm} (6)

**Lemma 4.1** The ideal $\mathcal{I}$ gives a biideal of $\tilde{\mathcal{B}}$.

**Proof** It suffices to show that indecomposable non-identity 1-morphisms send the 2-morphisms given by $\mathcal{I}$ back to $\mathcal{I}$ under horizontal composition. Since, for any $v \in e_0 \Lambda$ and any $j \in [1, n]$, we have $vc_j = 0$, it follows that

$$\text{id}_{\Lambda e_j \otimes_k e_0 \Lambda} \otimes \alpha = 0 , \text{ for } \alpha \in \mathcal{I} .$$

To see that $\mathcal{I}$ is also a right biideal, consider the case of the morphism

$$c_i \otimes e_0 \in \mathcal{I} (\Lambda e_0 \otimes_k e_0 \Lambda, \Lambda e_0 \otimes_k e_0 \Lambda) .$$

Identifying isomorphic bimodules and identifying morphisms between decomposable bimodules with matrices of those given in (5), we may write

$$(c_i \otimes e_0) \otimes \text{id}_{\Lambda e_j \otimes_k e_0 \Lambda} = \begin{cases} (c_i \otimes e_0) & \text{if } n = 0 ; \\ c_i \otimes e_0 & \text{otherwise} \end{cases} ,$$

from which it is clear that the resulting morphism remains in $\mathcal{I}$. The remaining cases are similar. $\square$
Since $I$ consists of radical morphisms, Lemma 2.5 implies that it is nilpotent. Consider the unique cell birepresentation $C$ of $\tilde{B}$, and the unique cell 2-representation $C^{\str}$ of $\tilde{B}^{\str}$. From the definition, we know that $C(\tilde{i})$ is the quotient of the category $\add \{ \Lambda e_k \otimes_k e_0 \Lambda \mid k \in [1, n] \}$ by the unique maximal nilpotent ideal stable under the left $\tilde{B}$-action given by tensor products over $\Lambda$. Under the identification in (4), it corresponds to the maximal ideal $I$ of $\Lambda \otimes_k e_0 \Lambda e_0$ such that $e_0 \Lambda \otimes \Lambda I$ belongs to $\Lambda \otimes_k \Rad(e_0 \Lambda e_0)$. A simple calculation yields

$$I = \kappa \{ c_k \mid k \in [1, n] \} \otimes_k e_0 \Lambda e_0 + \Lambda \otimes_k \Rad(e_0 \Lambda e_0),$$

which implies that for the ideal $J = \kappa \{ c_i \mid i \in [1, n] \}$, we have

$$C(\tilde{i}) \simeq \Lambda/J\text{-proj}, \text{ under } \Lambda e_k \otimes_k e_0 \Lambda \mapsto (\Lambda/J)e_k. \quad (7)$$

Let $A := \Lambda/J$. Remembering the index $n$, we obtain an algebra $A_n$, for every $n \in \mathbb{Z}_{>0}$. It follows that

$$C(\Lambda e_k \otimes_k e_0 \Lambda) \simeq Ae_k \otimes_k e_0 A \otimes_A -.$$

Since $C$ is equivalent to $C^{\str}$ as a birepresentation of $\tilde{B}$, we obtain

$$C^{\str}(\Lambda e_k \otimes_k e_0 \Lambda \otimes_A -) \simeq Ae_k \otimes_k e_0 A \otimes_A -. \quad (8)$$

Let $B$ denote the bicategory given by

$$(\add \{ A A_A, \ Ae_k \otimes_k e_0 A \mid k \in [0, n] \}, - \otimes_A -).$$

Consider its strictification $B^{\str}$, defined similarly to $\tilde{B}^{\str}$. Remembering the index, we obtain the bicategories $B_n$ and the 2-categories $B_n^{\str}$. From (8), we see that the assignments

$$\tilde{i} \mapsto \tilde{i}, \ F \mapsto C^{\str}F, \ \alpha \mapsto C^{\str}\alpha$$

give a 2-functor

$$Q^{\str} : \tilde{B}^{\str} \to B^{\str}.$$ 

Passing under biequivalences, this gives a pseudofunctor $Q : \tilde{B} \to B$. As described earlier, $Q$ maps indecomposable 1-morphisms to the corresponding indecomposable 1-morphisms, and hence is essentially surjective.

Abusing notation, we identify elements of $\Lambda$ and their images of the projection from $\Lambda$ onto $A$, whenever such images are non-zero. A simple but tedious calculation yields

$$\begin{align*}
\End_{A\text{-mod-}A}(A) & = \kappa \{ 1, c \}; \\
\Hom_{A\text{-mod-}A}(Ae_j \otimes_k e_0 A, A) & = e_j Ae_0; \\
\Hom_{A\text{-mod-}A}(A, Ae_j \otimes_k e_0 A) & = \kappa \{ b_j \otimes c \}, \ j \neq 0; \\
\Hom_{A\text{-mod-}A}(A, Ae_0 \otimes_k e_0 A) & = \kappa \left\{ e_0 \otimes c + c \otimes e_0 + \sum_{j=1}^{n} a_j \otimes b_j, \ c \otimes c \right\}. \quad (9)
\end{align*}$$

**Proposition 4.2** The pseudofunctor $Q$ induces a biequivalence $\tilde{B}/I \to B$.

**Proof** $I$ annihilates $C^{\str}$ by Lemma 2.4, so, by construction, $Q$ sends $I$ to zero. There is thus an induced pseudofunctor $\tilde{Q} : \tilde{B}/I \to B$. It is essentially surjective, since $Q$ is such. Further, as a consequence of the calculations following Lemma 4.1, the cell birepresentation
C is equivalent to the pullback birepresentation \( Q_A\text{-proj} \), where \( A\text{-proj} \) has the structure of the defining birepresentation of \( \mathcal{B} \). The latter is a faithful birepresentation (it is given by a locally faithful pseudofunctor) and so \( \text{Ker} \, Q = I \). Indeed, if the kernel properly contained \( I \), its image under \( Q \) would give a non-zero biideal of \( \mathcal{B} \) annihilating \( A\text{-proj} \). Thus, \( \mathcal{Q} \) is locally faithful.

To see that it is also locally full, observe that, for bimodules \( M, N \in \mathcal{B}(\hat{i}, \hat{j}) \), using (5),(6), and (9), we obtain

\[
\dim \text{Hom}_{A\text{-mod-}\Lambda}(M, N) = \dim \text{Hom}_{A\text{-mod-}\Lambda}(QM, QN) + \dim I(M, N).
\]

The statement follows from the fact that an injective map of equidimensional finite dimensional vector spaces is an isomorphism. \( \square \)

4.2 Prior Results

Since \( I \) is nilpotent, Lemma 2.4 shows that simple transitive birepresentations of \( \mathcal{B} \) are the same as the simple transitive birepresentations of \( \mathcal{B} \). Hence, all the analysis given in [38] applies also if we make this replacement. On the level of the underlying \( k \)-algebras, we replace \( \Lambda \) by \( A \). That the results of [38] apply is even clearer if one observes the following:

- For \( j \in [0, n] \), we have \( e_0 A e_j \cong e_0 A e_j \). Since \( A \) is a quotient of \( \Lambda \) by a nilpotent ideal, this implies that the multisemigroup given by composition of indecomposable 1-morphisms is the same for \( \mathcal{B} \) and \( \mathcal{B} \).
- The module \( A e_0 \) is projective-injective, and so the bimodule \( A e_0 \otimes_k e_0 A \) gives a self-adjoint endofunctor of \( A\text{-proj} \). Following [38, Remark 3.2] one may apply [33, Lemma 8] to show that, for a simple transitive birepresentation \( \mathcal{M} \) of \( \mathcal{B} \), the functor \( MF \) is a projective functor for any 1-morphism \( F \).

Using these observations, one may prove [38, Theorem 4.1, Theorem 5.1] for \( \mathcal{B} \) by verbatim repeating the proofs for \( \mathcal{B} \) given therein.

From now on, the apex of any transitive birepresentation of \( \mathcal{B} \) we consider is implicitly assumed to be the \( J \)-cell given by \( \{ A e_j \otimes_k e_0 A \mid j \in [0, n] \} \). Up to equivalence, there is a unique simple transitive birepresentation with apex given by \( \{ A e_j \} \). This is a consequence of [27, Theorem 18].

Let \( \mathcal{P} \) be a set partition of a set \( X \). We will associate a function to \( \mathcal{P} \), which, abusing notation, we denote by \( \mathcal{P} : X \rightarrow 2^X \). This function sends an element \( x \in X \) to the subset \( \mathcal{P}(x) \) of \( X \) it belongs to in the partition. [38, Theorem 5.1] implies that, for every simple transitive birepresentation \( \mathcal{M} \) of \( \mathcal{B} \), there is a set partition \( \mathcal{P}_M \) of \( [0, n] \) such that \( [\mathcal{M}(A e_k \otimes_k e_0 A)] \mathcal{P}_M = [\mathcal{M}(A e_k' \otimes_k e_0 A)] \mathcal{P}_M \) if and only if \( \mathcal{P}(k) = \mathcal{P}(k') \), and such that \( \mathcal{P}_M(0) = \{0\} \).

Completely in analogy to the proof of [38, Theorem 4.1.], there is a strong transformation \( \Sigma \) from the cell birepresentation \( \mathcal{C} \) to \( \mathcal{M} \), induced from the action of \( \mathcal{B} \) on the simple top \( L_0 \) of the unique up to isomorphism object \( Q_0 \) of the essential image of \( \mathcal{M}(A e_0 \otimes_k e_0 A) \). Recall that \( \mathcal{C}(\hat{i}) \cong A\text{-proj} \) and, under that equivalence, we have \( \mathcal{C}(A e_k \otimes_k e_0 A) \cong A e_k \otimes_k e_0 A \otimes A \).

For the remainder of this section, let \( \mathcal{M} \) be a simple transitive birepresentation of \( \mathcal{B} \), let \( \Sigma \) be the above described strong transformation, and let \( \mathcal{P}_M \) partition \( [0, n] \) into \( r + 1 \) subsets; recall that \( \mathcal{P}(0) = \{0\} \). We now give a very short summary of the results of [38] we will use.

**Proposition 4.3** Given \( k, k' \in [0, n] \), let \( X, Y \) be indecomposable objects of \( \mathcal{C}(\hat{i}) \) belonging to the isomorphism classes of \( A_n e_k \), \( A_n e_{k'} \) in \( A_n\text{-proj} \) under the identification \( \mathcal{C}(\hat{i}) \cong A_n\text{-proj} \) given in (7).
• The objects $\Sigma(X)$, $\Sigma(Y)$ are indecomposable, with $\Sigma(X) \simeq \Sigma(Y)$ if and only if $\Psi_M(k) = \Psi_M(k')$.
• If $k = k'$ or $\Psi(k) \neq \Psi(k')$, then $\Sigma_{X,Y}$ gives a bijection of $\text{Hom}$-spaces.
• As a consequence, if $U = \{u_1, \ldots, u_r\} \subseteq [0, n]$ is a transversal of $\Psi_M$, then the restriction of $\Sigma$ to add $\{A_n e_k \mid k \in U\}$ is an equivalence of categories. In particular, $M(\hat{i}) \simeq A_r$-proj.

We obtain an induced complete set of representatives of isomorphism classes of indecomposable objects of $M(\hat{i})$, which we denote by $\{A_r^M e_{\Psi(u_1)}, \ldots, A_r^M e_{\Psi(u_r)}\}$. Using $M(\hat{i}) \simeq A_r$-proj and identifying bimodules and endofunctors, we have

$$M(A_n e_k \otimes_k e_0 A_n) \simeq A_r^M e_{\Psi(u_k)} \otimes_k e_0 A_r^M.$$  

Proof For $n = 1$, all the claims follow from the proof of [38, Theorem 4.1]. The exact same proof works for $n > 1$ after a modification of indices, since all the claims follow from the self-adjointness of $A_n e_0 \otimes_k e_0 A_n$, which is independent of $n$, and the action matrices, which are completely determined by $\Psi_M$. □

Since action matrices are an invariant of finitary birepresentations, the following statement can be concluded already from their characterization in [38, Theorem 5.1], but it is even clearer in view of Proposition 4.3:

Corollary 4.4 Let $M$, $M'$ be simple transitive birepresentations of $\mathcal{B}$. If $\Psi_M \neq \Psi_{M'}$, then $M \nRightarrow M'$.

In particular, when classifying simple transitive 2-representations of $\mathcal{B}$, we may do so for each set partition of $[0, n]$ separately.

4.3 Equifying Modifications

Since $\mathcal{B}$ admits a unique object $\hat{i}$, we denote the principal birepresentation $P_{\hat{i}} = \mathcal{B}(\hat{i}, -)$ simply by $P$. It admits a transitive subbirepresentation $N$, given by the subcategory $\{A_n e_k \otimes_k e_0 A_n \mid k \in [0, n]\}$ of $\mathcal{B}(\hat{i}, \hat{i})$.

For any birepresentation $M$ of $\mathcal{B}$, Yoneda lemma gives

$$\mathcal{B}\text{-afmod}(P, M) \simeq M(\hat{i}).$$

Given an object $X \in M(\hat{i})$, we denote by $\Theta_X : P \rightarrow M$ the strong transformation that sends a 1-morphism $F$ to the object $MF(X)$. Denote the strong transformation $N \rightarrow P$, given by inclusion, by $\Gamma$. Let $\Theta_j := \Theta_{A_n e_j \otimes_k e_0 A_n}$ be the indicated strong transformation from $P$ to $C$. The identification $C(\hat{i}) \simeq A$-proj of (7) sends the object $A e_j \otimes_k e_0 A$ of the quotient category to $A e_j$. We may thus identify $\Theta_j$ with the functor $- \otimes_{A_{\hat{i}}} A_{\hat{i}} e_j$.

Lemma 4.5 Given $j, k \in [1, n]$, there is an invertible modification

$$s_{j,k} : \Theta_j \circ \Gamma \sim \Theta_k \circ \Gamma.$$

Proof Under the identifications above, a modification $m$ from $\Theta_j \circ \Gamma$ to $\Theta_k \circ \Gamma$ is given by a natural transformation $m : - \otimes_A A e_j \rightarrow - \otimes_A A e_k$ of functors from $N(\hat{i}) =$
add \([Ae_k \otimes_k e_0 A \mid k \in \{0, n\}]\) to \(A\)-proj, such that, for any \(M, N \in \mathcal{N}(\mathfrak{i})\), s

\[
\begin{array}{c}
(M \otimes_A N) \otimes_A Ae_j \xrightarrow{m_{M\otimes_A N}} (M \otimes_A N) \otimes_A Ae_k \\
\downarrow^{a_{M,N,Ae_j}} \quad \downarrow^{a_{M,N,Ae_k}} \\
M \otimes_A (N \otimes_A Ae_j) \xrightarrow{M \otimes_A m_N} M \otimes_A (N \otimes_A Ae_k)
\end{array}
\]  

(10)

commutes, where \(a\) denotes the associator. Let \(c^j_M : M \otimes_A Ae_j \xrightarrow{\sim} Me_j\) denote the canonical isomorphism given by \(m \otimes a \mapsto ma\). Since \((M \otimes_A N)e_j = M \otimes_A Ne_j\), we have \(c^j_M \otimes N = (M \otimes_A c^j_N) \circ a_{M,N,Ae_j}\).

For \(N \in \mathcal{N}(\mathfrak{i})\), the right \(A\)-module \(N\) lies in add \([e_0 A]\), and thus, for \(j \in [1, n]\), the map \(Ne_j \xrightarrow{(a_j)_N} Nc\) is an isomorphism of left \(A\)-modules, since it is such an isomorphism for \(N = e_0 A\). Let \(\varphi^j_N\) denote the isomorphism \((a_k)_N^{-1} \circ (a_j)_N\). Since \(\varphi^j_N\) is defined in terms of right action, we have \(\varphi^{j,k}_{M \otimes_A N} = M \otimes_A \varphi^j_N\). We define \((s)_{j,k}\) as \((\varphi^j_N)^{-1} \circ \varphi^j_N\). From the earlier statements, it follows that the diagram

\[
\begin{array}{c}
(M \otimes N) \otimes_A Ae_j \xrightarrow{c^j_{M\otimes N}} M \otimes Ne_j \xrightarrow{\varphi^j_{M\otimes N}} M \otimes Ne_k \xrightarrow{(c^j_{M\otimes N})^{-1}} (M \otimes N) \otimes_A Ae_k \\
\downarrow^{\beta_{M,N,Ae_j}} \quad \downarrow^{\varphi^j_{M\otimes N}} \quad \downarrow^{(c^j_{M\otimes N})^{-1}} \quad \downarrow^{\alpha_{M,N,Ae_k}} \\
M \otimes (N \otimes_A Ae_j) \xrightarrow{M \otimes c^j_N} M \otimes Ne_j \xrightarrow{M \otimes \varphi^j_N} M \otimes Ne_k \xrightarrow{M \otimes (c^j_N)^{-1}} M \otimes (N \otimes_A Ae_k)
\end{array}
\]

commutes, which proves that \(s\) satisfies the axiom (10). Further, \(s_N\) is natural in \(N\), since it is defined in terms of the right \(A\)-action on \(N\), which commutes with left \(A\)-module morphisms.

\(\square\)

**Lemma 4.6** Let \(i, i' \in [1, n]\) be such that \(\mathfrak{P}_M(i) = \mathfrak{P}_M(i')\). The strong transformations \(\Sigma \circ \Theta_i\) and \(\Sigma \circ \Theta_{i'}\) are isomorphic.

**Proof** From the description of \(\Sigma\) in Proposition 4.3, we have

\[\Sigma \circ \Theta_i(A) \simeq \Sigma(Ae_i \otimes e_0 A) \simeq A^M e_{\mathfrak{P}_M(i)} \simeq A^M e_{\mathfrak{P}_M(i')} \simeq \Sigma \circ \Theta_{i'}(A)\]

The result follows by Yoneda lemma applied on \(\mathfrak{P}\). \(\square\)

Similarly to the argument preceding (7), under the identification (4), the unique maximal \(\mathcal{B}\)-stable ideal of \(\mathfrak{N}\), corresponds to \(A \otimes_k \text{Rad}(e_0 Ae_0)\). Using the identification of \(\Theta_i\) with \(- \otimes_A Ae_i\), we see that \(\Theta_i\) sends this ideal to zero, which implies that \(\Theta_i \circ \Gamma\) factors canonically through the projection \(\mathfrak{N} \rightarrow \mathfrak{C}\). Let \(\Theta_i \circ \Gamma : \mathfrak{C} \rightarrow \mathfrak{C}\) denote the resulting transformation. Since \(\mathfrak{C}\) is simple transitive, the strong transformation \(\Sigma \circ \Theta_i \circ \Gamma\) is faithful. Since \(\Theta_i\) sends the isomorphism class represented by \(Ae_j \otimes_k e_0 A\) to that represented by \(Ae_j\), we see that \(\Sigma \circ \Theta_i \circ \Gamma\) sends indecomposables to indecomposables, as prescribed by \(\mathfrak{P}_M\). Thus, the underlying functor of \(\Sigma \circ \Theta_i \circ \Gamma\) is determined by a faithful, \(k\)-linear functor from \(A_n\)-proj to \(A\)-proj, which maps indecomposable objects to indecomposable objects, as prescribed by \(\mathfrak{P}_M\). Let \(T\) denote the set of isomorphism classes of such functors. Recall that \(\text{Rad End}_{A_n\text{-mod}}(A_ne_0) = \mathbb{k}\{c\}\), independently of \(n\) (abusing notation by identifying \(c \in A_n\) for varying \(n\)). From the above description it follows that the restriction of a functor \(F\) in \(T\) to that subspace corresponds to an endomorphism of \(\mathbb{k}\{c\}\), hence a scalar, which we denote by \(\chi_F\).
Lemma 4.7 The map $T \to \mathbb{k}\setminus\{0\}$, sending $F$ to $\chi_F$, is a bijection. An automorphism $\tau$ of $F$ in $T$ is determined uniquely by $\tau_{A_n e_0}$.

Proof As a consequence of Proposition 3.11, an additive functor of finitary categories is determined, up to natural isomorphism, by its restriction to the full subcategory of indecomposable objects. Further, a natural transformation between such functors is uniquely determined by its components indexed by indecomposable objects. Since we do not consider or assume any monoidal or strict monoidal structure on our categories and functors, we may simplify further by replacing the domain and codomain categories by equivalent, skeletal categories $\mathcal{N}, \mathcal{R}$. We write $\text{Ob}\mathcal{N} = \{A_n e_0, \ldots, A_n e_n\}$ and $\text{Ob}\mathcal{R} = \{A_r e_0, \ldots, A_r e_r\}$. Choose $F, F' \in T$. Identifying $c \in A_n$ with $c \in A_r$, the map $F_{A_n e_0, A_n e_0}$ corresponds to an algebra endomorphism of $\mathbb{k}[c]/(c^2)$. Since $F$ is faithful, it sends $c$ to $\chi_F c$ with $\chi_F \neq 0$. Assume $\chi_F \neq \chi_{F'}$, and let $\sigma : F \to F'$ be a natural transformation. We have $F(A_n e_0) = F'(A_n e_0) = A_r e_0$.

We write $\sigma_0(e_0) = \sigma_0 e_0 + \sigma_c c$. Naturality implies the commutativity of

$$
\begin{array}{ccc}
A_r e_0 & \xrightarrow{c \mapsto \chi'_c} & A_r e_0 \\
\downarrow & & \downarrow \\
A_r e_0 & \xrightarrow{c \mapsto \chi_c} & A_r e_0 \\
\end{array}
$$

and $\sigma_0 \neq 0$ implies $\chi = \chi'$. But $\sigma_0 = 0$ implies $\sigma_{A_n e_0} \in \text{Rad End } A_r e_0$, so $\sigma$ is not an isomorphism.

For the remaining values of $i$, we have $F(A_n e_i) = A_r e_0 = F'(A_n e_i)$, and we may write $F(a_i) = \lambda_i a_{i}(i)$ and $F(b_i) = \mu_i b_{i}(i)$, for $\mathbb{k}$-scalars $\mu_i, \lambda_i$. Functoriality gives $\lambda_i \mu_i = \chi_F$. Similarly for $F'$ and $\mu_i', \lambda_i'$. Further, a natural isomorphism $\sigma : F \to F'$ would yield a diagram

$$
\begin{array}{ccc}
A_r e_0 & \xrightarrow{e_{p(i)} \mapsto \lambda_i a_{p(i)}} & A_r e_0 \\
\downarrow & & \downarrow \\
A_r e_0 & \xrightarrow{e_{p(i)} \mapsto \sigma i e_{p(i)}} & A_r e_0 \\
\end{array}
$$

and so, for a fixed $\sigma_0$, we may set $\sigma_i = \frac{\sigma_0 \lambda_i}{\lambda_i'}$. Given $i, i' \in [1, n]$ such that $i \neq i'$, the Hom-space $\text{Hom}_{A_n\text{-proj}}(A_n e_i, A_n e_{i'})$ is zero, so the commutativity of the above diagram, together with the commutativity of

$$
\begin{array}{ccc}
A_r e_0 & \xrightarrow{e_{p(i)} \mapsto \lambda_i b_{p(i)}} & A_r e_0 \\
\downarrow & & \downarrow \\
A_r e_0 & \xrightarrow{e_{p(i)} \mapsto \sigma_i e_{p(i)}} & A_r e_0 \\
\end{array}
$$

suffice to conclude that $\sigma_i = \frac{\sigma_0 \lambda_i}{\lambda_i'}$ does define a natural isomorphism. In particular, we see that $\sigma$ is completely determined by its component indexed by $A_n e_0$. This concludes the proof. \qed

Lemma 4.8 Let $B$ be a finite-dimensional $\mathbb{k}$-algebra and let $F$ be an indecomposable projective endofunctor of $B$-mod. Let $M \in B$-mod, $f, f' \in \text{End}_B(M)$ and let $X$ be an indecomposable summand of $FM$ such that the restrictions of $Ff, Ff'$ to $X$ are automorphisms. There is $\lambda \in \mathbb{k}\setminus\{0\}$ such that $F f'_|X = \lambda F f|_X$. 

\[ Springer \]
Proof Clearly, if the statement holds for a functor $F$, it also holds for any functor $F'$ isomorphic to $F$. We may thus set $F = B e_i \otimes_k e_j B \otimes_B -$ for some primitive idempotents $e_i, e_j \in B$. Hence $F M = B e_i \otimes_k e_j M \cong B e_i^{\oplus \dim e_j M}$, and $X \cong B e_i$. For any $f \in \text{End}_B(M)$, the morphism $F f$ maps $k \{ e_i \otimes v \mid v \in e_j M \}$ to itself, and so $F f$ cannot be a radical morphism. If $F f|_X$, $F f'|_X$ are linearly independent automorphisms of $X$, then, since the top of $X$ is simple, there is a linear combination

$$\mu F f|_X + \mu' F f'|_X = F(\mu f + \mu' f')|_X \in \text{Rad End}_B(X),$$

which is a contradiction. \qed

**Lemma 4.9** Let $m, m' : \Sigma \circ \Theta_i \circ \Gamma \rightarrow \Sigma \circ \Theta_i \circ \Gamma$ be a pair of invertible modifications. There is $\lambda \in \mathbb{k}_i(0)$ such that $m' = \lambda m$.

**Proof** From Lemma 4.7 and the discussion preceding it, we conclude that it suffices to show that $m'_{A e_0 \otimes_k e_0 A_{n}} = \lambda m_{A e_0 \otimes_k e_0 A_{n}}$.

Since $m$ is a modification, for every 1-morphism $F \in \mathcal{B}(\mathfrak{i}, \mathfrak{i})$ and any object $X \in \mathcal{N}(\mathfrak{i})$, we have

$$M(F) \circ (\Sigma \circ \Theta_i \circ \Gamma)(X) \rightarrow (\Sigma \circ \Theta_i \circ \Gamma)(X)$$

and similarly for $m'$. Let $F = A e_0 \otimes_k e_0 A = X$. We have

$$(A e_0 \otimes_k e_0 A)^{\otimes 2} := (A e_0 \otimes_k e_0 A) \otimes_A (A e_0 \otimes_k e_0 A) \cong (A e_0 \otimes_k e_0 A)^{\otimes 2}.$$ We fix a split monomorphism $\iota : A e_0 \otimes_k e_0 A \rightarrow (A e_0 \otimes_k e_0 A) \otimes_A (A e_0 \otimes_k e_0 A)$ and a split epimorphism $\pi : (A e_0 \otimes_k e_0 A) \otimes_A (A e_0 \otimes_k e_0 A) \rightarrow A e_0 \otimes_k e_0 A$ such that $\pi \circ \iota = \text{id}_{A e_0 \otimes_k e_0 A}$. Due to the naturality in $X$, we obtain the commutative diagram

Thus the restriction of $M(A e_0 \otimes_k e_0 A) \left((m)_{A e_0 \otimes_k e_0 A}\right)$ to the indecomposable projective summand $\text{Im} \left((\Sigma \circ \Theta_i \circ \Gamma)^{-1}_{A e_0 \otimes_k e_0 A} \circ (\Sigma \circ \Theta_i \circ \Gamma)(\iota)\right)$ gives the automorphism $m_{A e_0 \otimes_k e_0 A}$. The same holds for $m'$. The fact that $m'_{A e_0 \otimes_k e_0 A} = \lambda m_{A e_0 \otimes_k e_0 A}$, for some non-zero $\lambda$, follows from Lemma 4.8. \qed

**Corollary 4.10** Given $j, k \in [1, n]$ such that $\mathcal{P}_M(j) = \mathcal{P}_M(k)$, there is an invertible modification $t_{j, k} : \Sigma \circ \Theta_j \rightarrow \Sigma \circ \Theta_k$ such that $t_{j, k} \circ \Gamma = \Sigma \circ s_{j, k}$, for $s_{j, k}$ defined in Lemma 4.5.

**Proof** Using Lemma 4.6, choose $t_{j, k}' : \Sigma \circ \Theta_j \rightarrow \Sigma \circ \Theta_k$. From Lemma 4.9 it follows that there is a non-zero scalar $\lambda$ such that $t_{j, k} \circ \Gamma = \lambda \Sigma \circ s_{j, k}$. The result follows by letting $t_{j, k} := \frac{1}{\lambda} t_{j, k}'$. \qed
5 Simple Transitive Birepresentations of $B_n$

We will use the theory of bicategorical weighted colimits to construct and classify simple transitive birepresentations of $B$. The bicategories in which we will consider colimits are $\text{Cat}_k$ and the pseudofunctor bicategory $[B, \text{Cat}_k]_k$. Observe that both these bicategories are, in fact, 2-categories. This simplifies the notions below.

Let $C$ be a 2-category. Let $J_1$ be the 2-category given by $\bullet \xrightarrow{p_2} \bullet$. In particular, $J_1$ has no non-identity 2-morphisms. Let $F$ be the 2-functor $J_1 \to C$ given by $i \overset{G}{\rightarrow} j$, for objects $i, j$ and 1-morphisms $F,G$ of $C$.

Let $W : J_1^{\text{op}} \to \text{Cat}$ be the 2-functor given by the diagram $\text{Iso} \xrightarrow{1} P_1$, where 1 denotes the terminal category with a unique object and only its identity morphism, and $\text{Iso}$ is the walking isomorphism category, with two objects $P_1, P_2$ and, as its only non-identity morphisms, two mutually inverse morphisms between the two objects.

The bicategorical coisoinserter of $F, G$ above is the bicategorical colimit $W \star F$. One may verify that it is given by an object $w$ together with a 1-morphism $W : j \to w$ and an invertible 2-morphism $\zeta : W \circ F \overset{\sim}{\rightarrow} W \circ G$, such that, given a 1-morphism $H : j \to k$ and an invertible 2-morphism $\gamma : HF \overset{\sim}{\rightarrow} HG$, there is a 1-morphism $\hat{H} : w \to k$ and an invertible 2-morphism $\hat{\gamma} : H \overset{\sim}{\rightarrow} \hat{H} W$, such that the diagram

$$
\begin{array}{ccc}
HF & \xleftarrow{\gamma} & \hat{H}W F \\
\downarrow & & \downarrow \hat{H} \zeta \\
HG & \xleftarrow{\hat{\gamma}} & \hat{H} W G
\end{array}
$$

commutes. Further, the pair $(\hat{H}, \hat{\gamma})$ is unique up to an invertible 2-morphism compatible with $\hat{\gamma}$. This is the one-dimensional aspect of the universal property of $(w, W, \zeta)$. In our applications we will not need the two-dimensional aspect, and hence we omit describing it.

Let $J_2$ be the 2-category given by the diagram $\bullet \xrightarrow{\alpha, \beta} \bullet$. Let $F$ be the 2-functor from $J_2$ to $C$ given by the diagram $i \overset{F}{\rightarrow} j$.

In $C$. Let $W : J_2^{\text{op}} \to \text{Cat}$ be the 2-functor given by the diagram $\text{Arr} \xrightarrow{Q_1} \text{Arr} Q_2 \xrightarrow{1} 1$, where $\text{Arr}$ is the walking arrow category, with two objects $Q_1, Q_2$ and a unique morphism $Q_1 \to Q_2$. In particular, the two 2-cells in the last diagram coincide.

The bicategorical coequifier of $\alpha$ and $\beta$ above is the colimit $W * F$. One may verify that it is given by an object $\gamma$ together with a 1-morphism $R : j \to \gamma$ such that there are equivalences

$C(j, \gamma) \overset{\alpha R}{\rightarrow} C(j, \gamma)_{\text{eqf}}$,

strongly natural in $\gamma$. Here, $C(j, \gamma)_{\text{eqf}}$ denotes the full subcategory of $C(j, \gamma)$ given by 1-morphisms $H$ such that $H\alpha = H\beta$.

Let $\mathcal{P}$ be a set partition of $[0, n]$ satisfying $\mathcal{P}(0) = \{0\}$, given by

$[0, n] = \{0\} \cup \{i_1^1, i_2^1, \ldots, i_{k_1}^1\} \cup \cdots \cup \{i_1^r, i_2^r, \ldots, i_{k_r}^r\}$. 
Similarly to the notation $[1, k] = \{1, \ldots, k\}$, we use the interval notation for subsets given by $\mathfrak{P}$, writing for example $[i_1^2, i_3^2] := \{i_1^2, i_2^2, i_3^2\}$.

We associate particular kinds of weighted colimits to $\mathfrak{P}$. Let $\mathcal{J}_n$ be the 2-category with $n$ parallel 1-morphisms and no non-identity 2-morphisms. Let $\mathbf{W}(\mathfrak{P})$ be the 2-functor from $\mathcal{J}_n$ to $\mathbf{Cat}$ given by $\text{Iso}(\mathfrak{P}) \to \mathbf{1}$ where $\text{Iso}(\mathfrak{P})$ is the category presented by

$$\begin{array}{cccccc}
i_1^1 & \xleftarrow{\psi_1^1} & i_2^1 & \xleftarrow{\psi_2^1} & \cdots & \xleftarrow{\psi_{k_1}^1} & i_{k_1}^1 \\
i_2^2 & \xleftarrow{\psi_1^2} & i_2^2 & \xleftarrow{\psi_2^2} & \cdots & \xleftarrow{\psi_{k_2}^2} & i_{k_2}^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
i_r^r & \xleftarrow{\psi_1^r} & i_2^r & \xleftarrow{\psi_2^r} & \cdots & \xleftarrow{\psi_{k_r}^r} & i_{k_r}^r
\end{array}$$

We now return to the setting of the bicategory $\mathcal{B}_n$ studied in the previous sections. $\mathbf{F}^{\mathfrak{P}}(\mathfrak{P})$ be the 2-functor from $\mathcal{J}_n$ to $[\mathcal{B}_n, \mathbf{Cat}_{\mathfrak{P}}]$ given by the diagram

$$\begin{array}{ccc}
P & \xleftarrow{\Theta_1} & C \\
\Theta_1 \downarrow & & \downarrow \Theta_k \\
\Theta_k \downarrow & & \downarrow \Theta_{k_r} \\
\Theta_{k_r} & \downarrow & \downarrow \Theta_{k_r}
\end{array}$$

The colimit $\mathbf{W}(\mathfrak{P}) \ast \mathbf{F}^{\mathfrak{P}}(\mathfrak{P})$ can be obtained by iterating coisoinserters, and is given by a $\mathfrak{P}$-linear pseudofunctor $\mathbf{C}^{\mathfrak{P}}(\mathfrak{P})$, a strong transformation $\Omega^{\mathfrak{P}}(\mathfrak{P})$ and a set of invertible modifications $\{\widehat{x}_l^m : \Omega^{\mathfrak{P}}(\mathfrak{P}) \circ \Theta_{i_l}^m \sim \Omega^{\mathfrak{P}}(\mathfrak{P}) \circ \Theta_{i_{l+1}}^m, | l \in [i_1^m, i_{k_m-1}], m \in [1, r]\}$.

The one-dimensional property is analogous to that of coisoinserters given above: given a $\mathfrak{P}$-linear pseudofunctor $\mathbf{M}$, a strong transformation $\Upsilon : \mathbf{C} \to \mathbf{M}$ and a family of modifications $\eta_l^m$ indexed as above, there is a 1-morphism $\Upsilon : \mathbf{C}^{\mathfrak{P}}(\mathfrak{P})$ such that diagrams analogous to (11) commute for all choices of indices. In particular we have $\Upsilon \circ \Omega^{\mathfrak{P}}(\mathfrak{P}) \simeq \Upsilon$.

Recall the birepresentation $\mathbf{N}$ of $\mathcal{B}_n$, the strong transformation $\Gamma : \mathbf{N} \to \mathbf{P}$ and the invertible modifications $s_{j,k} : \Theta_j \circ \Gamma \sim \Theta_k \circ \Gamma$, for $j, k \in [1, n]$. For any $m \in [1, r]$ and any $l \in [i_1^m, i_{k_m-1}]$ we now have the parallel invertible modifications $\chi_l^m \ast \Gamma$ and $\Omega^{\mathfrak{P}}(\mathfrak{P}) \ast s_{i_l^m,i_{l+1}^m}$, from $\Omega^{\mathfrak{P}}(\mathfrak{P}) \circ \Theta_{i_l^m} \circ \Gamma$ to $\Omega^{\mathfrak{P}}(\mathfrak{P}) \circ \Theta_{i_{l+1}^m} \circ \Gamma$.

We consider the multiple coequifier $\mathbf{C} \xrightarrow{\Omega^{\mathfrak{P}}(\mathfrak{P})} \mathbf{C}^{\mathfrak{P}}(\mathfrak{P})$ coequifying all such pairs. In other words, we obtain equivalences

$$[\mathcal{B}_n, \mathbf{Cat}_{\mathfrak{P}}](\mathbf{C}^{\mathfrak{P}}(\mathfrak{P}), \mathbf{M}) \xrightarrow{\Omega^{\mathfrak{P}}(\mathfrak{P})} [\mathcal{B}_n, \mathbf{Cat}_{\mathfrak{P}}]^{\mathfrak{P}}(\mathfrak{P}), \mathbf{M}),$$
such that

\[ \tilde{\Psi} \]

We write \( \chi_i^m = \Omega^{\forall}(\Psi) \circ \tilde{x}_i^m \) and \( \Omega^{\forall^R}(\Psi) := \Omega^R \circ \tilde{\Psi} \).

**Lemma 5.1** Let \( M \in [\mathcal{B}, \text{Cat}_k]_k \). Given a strong transformation \( C \xrightarrow{\gamma} M \) together with invertible modifications \( m_i^m : \gamma \circ \Theta_{i_1}^m \rightarrow \gamma \circ \Theta_{i_2}^m \), for all \( m \in [1, r] \) and for all \( l \in [i_1, i_{km-1}], \) such that \( m_i^m \bullet \gamma = \gamma \circ s_i^m \), there is a strong transformation \( \tilde{\gamma} : C^{\forall^R} \rightarrow M \) such that \( \tilde{\gamma} \circ \Omega^{\forall^R} \simeq \gamma \). Further, \( \tilde{\gamma} \) is unique up to an invertible modification.

**Proof** The statement follows from the one-dimensional aspect of the universal property of \( (C^{\forall^R}(\Psi), \Omega^{\forall^R}(\Psi), x_i^m) \) obtained by combining the universal properties of the iterated coisoinserter \( C^{\forall}(\Psi) \) and the coequifier \( C^R(\Psi) \). Note that the statement of the lemma does not exhaust the one-dimensional universal property.

We now describe the underlying category for \( C^{\forall}(\Psi) \). Since it is constructed pointwise, it suffices to find the coisoinserter of the underlying diagram in \( \text{Cat}_k \). We will only determine the underlying category up to equivalence, and thus, using Proposition 3.6, Corollary 3.8, and Proposition 3.11 we may restrict the domain category \( \text{P}(\tilde{\cdot}) \) to its full subcategory \( \text{P}(\tilde{\cdot}) \)-indec of indecomposable objects. We may also restrict the codomain to the common essential image of the underlying functors of \( \Theta_i \), for \( i \in [1, n] \), which is exactly \( C(\tilde{\cdot}) \)-indec. This is clear under the identification of \( C(\tilde{\cdot}) \) with \( A_n \)-proj and of \( \Theta_i \) with \( - \otimes_{A_n} A_n e_i \). We have thus reduced the problem of finding the underlying coisoinserter to that of finding \( W(\Psi) \bullet D \), where \( D \) is the diagram

\[
\begin{array}{ccc}
\text{P}(\tilde{\cdot}) \text{-indec} & : & A_n \text{-indec. proj.} \\
\downarrow & & \downarrow \\
- \otimes_{A_n} A_n e_i & \rightarrow & - \otimes_{A_n} A_n e_{ikr}
\end{array}
\]

**Remark 5.2** We recommend the reader to first consider the case \( n = 2 \) and \( \Psi = [0, 2] \) given by \([0, 2] = \{0\} \cup \{1, 2\} \). In fact, the only aspect of the remaining arguments that does not carry over verbatim from that case to the general case is taken care of by the explicit description of the weight \( W(\Psi) \) above. We will see that, in the case of \( \Psi = [0, 2] \), one only needs to, in a sense, adjoin a single isomorphism to the category. In the general case, the form of the weight \( W(\Psi) \) tells us to add as few isomorphisms as possible to still obtain the sought identifications of isomorphism classes. This considerably facilitates the next step (using coequifiers), which is to ensure that the result is finitary.

Let \( \mathcal{P} \subset \text{Cat}_k \) be the skeletal subcategory of \( \text{P}(\tilde{\cdot}) \)-indec with

\[
\text{Ob} \mathcal{P} = \{A, \ 	ext{Ae}_k \otimes_{\text{Bk}} e_0 A \mid k \in [0, n]\}
\]

and let \( _{\text{P} \subset \text{Cat}_k} \) be its inclusion functor. Further, consider the full subcategories \( \mathcal{C}_A, \mathcal{C}_S \) of \( A \)-indec. proj given by

\[
\text{Ob} \mathcal{C}_A = \{ A \otimes A \text{Ae}_j, (\text{Ae}_k \otimes_{\text{Bk}} e_0 A) \otimes A \text{Ae}_j \mid k \in [0, n], j \in [1, n]\} \phantom{,}
\text{and}
\text{Ob} \mathcal{C}_S = \{ \text{Ae}_k \mid k \in [0, n]\}.
\]
Using $e_0 A \otimes_A A e_j \simeq e_0 A e_j = \mathbb{k} \{ b_j \}$ for $j \in \{1, \ldots, n\}$, let $F^{\mathcal{A}, \mathcal{S}} : \mathcal{C}^\mathcal{A} \to \mathcal{C}^\mathcal{S}$ be the $\mathbb{k}$-linear functor induced by $A$-module isomorphisms

$$ Ae_k \otimes_k e_0 A e_j \sim Ae_k \quad \text{and} \quad A \otimes_A A e_j \sim Ae_j. $$

In particular, $F^{\mathcal{A}, \mathcal{S}}((A e_k \otimes_k e_0 A) \otimes_A A e_j) = A e_k$, $F^{\mathcal{A}, \mathcal{S}}(A \otimes_A A e_j) = A e_j$, and $F^{\mathcal{A}, \mathcal{S}}$ is an equivalence. The full images of $(- \otimes_A A e_j) \circ I_{P^S}$, for $j \in \{1, \ldots, n\}$, are contained in $\mathcal{C}^\mathcal{A}$, and so we may consider the diagram

$$ \begin{array}{ccc}
F^{\mathcal{A}, \mathcal{S}}(\otimes_A e_0) & \sim & I_{P^S} \\
\circ & \sim & \circ \downarrow \sim \downarrow \\
\mathcal{P}^S & \longrightarrow & \mathcal{C}^\mathcal{S} \\
\circ \downarrow \sim \downarrow & & \circ \downarrow \sim \downarrow \\
F^{\mathcal{A}, \mathcal{S}}(\otimes_A e_0) & \sim & I_{P^S}
\end{array} \quad (13) $$

in $\mathbf{Cat}_k$. Since the inclusions of the subcategories $\mathcal{C}^\mathcal{S}$, $\mathcal{P}^S$ are both equivalences of categories, Diagram 12 and Diagram 13 are equivalent as 2-functors to $\mathbf{Cat}_k$. We may thus compute the colimit of this diagram in order to obtain an underlying category equivalent to the colimit of diagram 12, and, similarly, the universal cones will be compatible under such equivalence. This again reduces the problem of finding the colimit $\mathcal{C}^\mathcal{G}(\mathfrak{R})$. To simplify the notation, let $F_i := F^{\mathcal{A}, \mathcal{S}} \circ (- \otimes_A A_n e_i) \circ I_{P^S}$, for $i \in \{1, \ldots, n\}$. It is easy to describe $F_i$ explicitly. For example, we have

$$ \left( A e_1 \otimes e_0 A \xrightarrow{e_1 \otimes e_0 \mapsto a_1 \otimes 2 e_0} A e_0 \otimes_k e_0 A \right) \xrightarrow{F_i} \left( A e_1 \xrightarrow{e_1 \mapsto 2 a_1} A e_0 \right). $$

Observe that all objects of $\mathcal{P}^S$ and $\mathcal{C}^\mathcal{S}$ are cyclic modules. Similarly to Equations (5), we use the images of the cyclic generators $e_j \otimes e_0$ and 1 to denote the morphisms in these categories.

**Notation 5.3** Due to the abundance of subscripts and superscripts in our notation coming from the enumerations induced by $\mathfrak{R}$, in Proposition 5.4, Lemma 5.6 and their respective proofs we deviate from the standard notation, denoting the components of a natural transformation as if they were the argument of a function, rather than using a subscript. In other words, given functors $F : \mathcal{C} \to \mathcal{D}$, a natural transformation $\tau : F \to G$ and some $X \in \text{Ob} \mathcal{C}$, we denote the component of $\tau$ at $X$ by $\tau(X)$ rather than $\tau_X$.

**Proposition 5.4** The weighted colimit $\mathcal{C}^{\mathcal{G}}$ of the diagram (13) is obtained from $\mathcal{C}^\mathcal{S}$ by freely adjoining invertible morphisms

$$ \xi^m_l(j) : Ae_j \to Ae_j \quad \text{and} \quad \xi^m_l(A) : Ae_{i^m_l} \to Ae_{i^m_{l+1}}, $$

for every $j \in \{0, \ldots, n\}$, $m \in \{0, \ldots, r\}$ and $l \in \{i^m_l, \ldots, i^m_{l+1} - 1\}$, and then imposing the relations giving

$$ c \xi^m_l(0) = \xi^m_l(0)c; \quad b_{i^m_{l+1}} \xi^m_l(0) = \xi^m_l(A)b_{i^m_l}; \quad a_{i^m_{l+1}} \xi^m_l(A) = a_{i^m_l} \xi^m_l(0)a_{i^m_l}; \quad a_k \xi^m_l(0) = \xi^m_l(0)a_k; \quad b_k \xi^m_l(0) = \xi^m_l(k)b_k, \quad \text{for } k \in \{0, \ldots, n\} \text{ and } l, m \text{ as above.} \quad (14) $$

The universal cone $\mathcal{C}^\mathcal{S} \xrightarrow{W} \mathcal{C}^{\mathcal{G}}$ is given by inclusion.

**Proof** Recall that in order to satisfy the universal property, we need natural isomorphisms $\xi^m_l : W \circ F_{i^m_l} \sim W \circ F_{i^m_{l+1}}$ which are universal in the sense of (11).
Let $\mathcal{D}$ be a $k$-linear category and let $G : \mathcal{C}^S \to \mathcal{D}$ be a $k$-linear functor. Given $l, m$ as above, a natural isomorphism $\tau_l^m : G \circ F_l^m \sim \sim G \circ F_{l+1}^m$ consists of $n + 2$ isomorphisms, indexed by the objects of $\mathcal{P}^S$. We write $\tau_l^m(j)$ for $\tau_l^m(Ae_j \otimes_k e_0 A)$.

For $j \in [0, n]$, we have

$$\tau_l^m(j) : (G \circ F_l^m)(Ae_j \otimes_k e_0 A) = G(Ae_j) \sim G(Ae_j) = (G \circ F_{l+1}^m)(Ae_j \otimes_k e_0 A).$$

We denote the remaining component by $\tau_l^m(A)$ and have

$$\tau_l^m(A) : (G \circ F_l^m)(A) = G(Ae_l^m) \sim G(Ae_{l+1}^m) = (G \circ F_{l+1}^m)(A).$$

Eqs. 9 explicitly describe the morphisms in $\mathcal{P}^S$ and so the conditions for naturality of $\tau_l^m$ can be found using these. We claim that $\tau_l^m$ is natural if and only if it satisfies the equations

1. $G(a_j) \tau_l^m(j) = \tau_l^m(0)G(a_j)$, $G(b_j) \tau_l^m(j) = \tau_l^m(j)G(b_j)$ for $j \in [1, n]$;
2. $G(c) \tau_l^m(0) = \tau_l^m(0)G(c)$; $G(b_{l+1}^m) \tau_l^m(0) = \tau_l^m(0)G(b_{l+1}^m)$;
3. $G(a_{l+1}^m) \tau_l^m(A) = \tau_l^m(0)G(a_l^m)$.

We show that the last equation is necessary for the naturality of $\tau_l^m$—the remaining cases are very similar, and, in fact, easier.

Consider the morphism $z : A \rightarrow Ae_0 \otimes_k e_0 A$ given by

$$z(1) = e_0 \otimes c + c \otimes e_0 + \sum_{j=1}^n a_j \otimes b_j$$

Tensoring with $Ae_{l+1}^m$ from the right gives

$$A \otimes_{A_n} Ae_{l+1}^m \rightarrow (Ae_0 \otimes_k e_0 A) \otimes_{A_n} Ae_{l+1}^m$$

$$1 \otimes_{A_n} e_{l+1}^m \rightarrow (e_0 \otimes_k c + c \otimes_k e_0 + \sum_{j=1}^n a_j \otimes_k b_j) \otimes_{A_n} e_{l+1}^m = (a_{l+1}^m \otimes_k b_{l+1}^m) \otimes_{A_n} e_{l+1}^m$$

and, to find $F^A_{l+1}(z)$, we identify $(a_{l+1}^m \otimes_k b_{l+1}^m) \otimes_{A_n} e_{l+1}^m$ with $a_{l+1}^m \otimes_k b_{l+1}^m$. By definition, $F^A_{l+1}$ sends this morphism to the morphism $Ae_{l+1}^m \rightarrow Ae_0$. Thus, $F_l^m(z) = a_l^m$, and similarly, $F^m_{l+1}(z) = a_{l+1}^m$. Naturality of $\tau_l^m$ requires

$$G(a_{l+1}^m) \circ \tau_l^m(A) = (G \circ F_{l+1}^m)(z) \circ \tau_l^m(A) = \tau_l^m(0) \circ (G \circ F_l^m)(z) = \tau_l^m(0) \circ G(a_l^m)$$

which proves the necessity of the last equation in Eqs. 15. Comparing Eqs. 15 with Eqs. 14, we see that

$$\{ \xi_l^m(A) \} \cup \{ \xi_l^m(j) \mid j \in [0, n] \}$$

gives a natural isomorphism $\xi_l^m : W \circ F_l^m \rightarrow W \circ F_{l+1}^m$.

Let $G : \mathcal{C}^S \to \mathcal{D}$ be a $k$-linear functor and let $(\tau_l^m)$ be a collection of natural isomorphisms $\tau_l^m : G \circ F_l^m \sim \sim G \circ F_{l+1}^m$, indexed by $l, m$ as above.

We may extend $G$ to $\hat{G} : \mathcal{C}^\mathcal{Ob} \rightarrow \mathcal{D}$ by setting $\hat{G}(\xi_l^m(X)) := \tau_l^m(X)$, for $X \in \mathrm{Ob} \mathcal{P}^S$. From the definition of $(\mathcal{C}^\mathcal{Ob}, W)$, it follows that $\hat{G}$ is the unique functor which satisfies $\hat{G} \circ W = G$ and $\tau_l^m = \hat{G} \circ \xi_l^m$, for all $l, m$. 
This shows the one-dimensional aspect of the universal property of \((C^W, \mathcal{W}, (\xi^m_{\ell,j})_{\ell,m})\). From Proposition 3.9 it now follows that this triple indeed gives the required colimit. \(\square\)

**Remark 5.5** We have only proved the universal property with respect to functors and collections of natural isomorphisms giving 2-transformations from \(\mathcal{W}(\mathcal{P})\), rather than all strong transformations, as we generally should when considering a bicategorical weighted colimit. Further, since \(\tilde{G}\) is unique (rather than unique up to a compatible natural isomorphisms), we obtain isomorphisms rather than just equivalences. In other words, the above determines the weighted 2-categorical colimit. However, the bicategorical colimit we were looking for is a PIE colimit, and bicategorical PIE colimits are equivalent to 2-categorical PIE colimits, as observed for example in [12, Remark 1.10.29], so the 2-categorical colimit above is also the corresponding bicategorical colimit. This is not necessarily the case for weighted colimits which are not PIE colimits.

Recall the above auxiliary categories \(\mathcal{P}^S\), \(C^A\), \(C^S\) and the auxiliary functors \(F^{A,S}_{\mathcal{P}}, F_1, I_{\mathcal{P}^W}\). We further consider the additional auxiliary category \(\mathcal{N}^S\), which is the full subcategory of \(\mathcal{P}^S\) with \(\text{Ob } \mathcal{N}^S = \{ Ae_j \otimes_k e_0 A \mid j \in [0, n]\}\), and also its inclusion functor \(I_{\mathcal{N}^S}\).

We will calculate the multiple coequifier of all pairs
\[(W \circ F^{A,S}) \bullet s^m_{i_{l+1},i_l} \bullet I_{\mathcal{N}^S} \text{ and } \tilde{\xi}^m_l \bullet I_{\mathcal{N}^S}, \text{ for } l, m \text{ as above,}\]
of modifications from \(W \circ F^{A,S} \circ (- \otimes_A A_n e^m_{i_j}) \circ I_{\mathcal{N}^S}\) to \(W \circ F^{A,S} \circ (- \otimes_A A_n e^m_{i_{l+1}}) \circ I_{\mathcal{N}^S}\), which we denote by \(s^m_l\) and \(\tilde{\xi}^m_l\), respectively.

Similarly to the previous calculation, this coequifier \(C^W \overset{R}{\to} C^N_{\mathcal{P}R}\) in \(\mathcal{C}^A\) gives the underlying category and the cone functor of \(C^W \overset{R}{\to} C^N_{\mathcal{P}R}\), on the level of indecomposable objects.

**Lemma 5.6** The above described coequifier \(C^N_{\mathcal{P}R}\) is obtained from \(C^N\) by adjoining invertible morphisms \(\tilde{\xi}^m_l(A) : Ae_j \to Ae_{i_{l+1}}\), satisfying the equations
\[b^m_{i_{l+1}} = b^m_l(A)b^m_{i_j}; \quad a^m_{i_{l+1}} \tilde{\xi}^m_l(A) = a^m_{i_j} \text{ for } l \in [i_j, i_{k-1}] \text{ and } m \in [1, r]. \quad (16)\]

The universal cone \(C^N \overset{R}{\to} C^N_{\mathcal{P}R}\) is the functor given by

- sending \(C^S\), viewed as a subcategory of \(C^N\) under the embedding described in Proposition 5.4, to itself, viewed as a subcategory of \(C^N_{\mathcal{P}R}\), as described above;
- for all \(l, m\) as above and any \(j \in [0, n]\), sending \(\xi^m_j(j)\) to \(\text{id}_{Ae_j}\) and sending \(\tilde{\xi}^m_l(A)\) to itself.

In particular, \((C^N_{\mathcal{P}R})^D \simeq A_\tau\)-proj, where \((-)^D\) is the envelope considered in Sect. 3.

**Proof** Since \(\tau^m_l\) is just the restriction of \(\tilde{\xi}^m_l\) to the subset of objects of \(\mathcal{P}^S\) excluding the regular bimodule \(A\), we know its components. To find the components of \(s^m_{i_l} e^m_{i_j}\), let \(j, j' \in [1, n]\) and \(k \in [0, n]\), and recall the definition of \(s^m_{i_l} e^m_{i_{l+1}}\) given in Lemma 4.5 to find that, for any \(j \in [0, n]\) and any \(x \in Ae_j\), we have
\[
(Ae_j \otimes_k e_0 A) \otimes_A Ae_j \overset{s^m_{i_{l+1}} e^m_{i_j}}{\longrightarrow} (Ae_j \otimes_k e_0 A) \otimes_A Ae_j \overset{F^S}{\longrightarrow} Ae_j \overset{W}{\longrightarrow} Ae_j
\]
\[
(x \otimes b^m_{i_l}) \otimes e^m_{i_j} \overset{\xi^m_{i_{l+1}}}{\longrightarrow} (x \otimes b^m_{i_{l+1}}) \otimes e^m_{i_j} \overset{\tilde{\xi}^m_l(A)}{\longrightarrow} x
\]
and the image is exactly the same if we omit $s_{i_1}^m \cdots s_{i_{n-1}}^m$ from the diagram. We conclude that $s_{i_1}^m(Ae_j \otimes e_0 A) = \text{id}_{Ae_j}$. Observe that, after substituting $\xi_{i_1}^m(k) = \text{id}_{Ae_k}$ in 14, the equations still hold, so the functor described in the lemma is well-defined, and the assignment $\xi_{i_1}^m(Ae_k \otimes_k e_0 A) := \text{id}_{Ae_k}$, $(\xi_{i_1}^m)(A) = \xi_{i_1}^m(A)$ gives an isomorphism coinciding with $s_{i_1}^m$ on $Ae_j \otimes_k e_0 A$, for $j \in [0, n]$.

We conclude that a $k$-linear functor $G : C^{\Omega_1} \rightarrow D$ coequifies $s_{i_1}^m$ and $s_{i_1}^m$ if and only if it sends $\xi_{i_1}^m(k)$ to $\text{id}_{G(Ae_k)}$. It is now clear that there is a unique functor $G : C^{\Omega_1} \rightarrow D$ such that $G \circ R = G$. If we view $C^{\Omega_1}$ as a subcategory of $C^{\Omega_2}$, under the clear embedding, we may describe $G$ as the corresponding restriction of $G$. That $C^{\Omega_1}$ is the coequifier now follows from Proposition 3.9.

To see that $(C^{\Omega_1})^D \cong A_r$-proj, note that $C^{\Omega_1}$ is obtained from the category of indecomposable $A_n$-projectives by adjoining isomorphisms that do not increase the dimensions of the Hom-spaces, due to the relations given in the lemma and the fact that we only adjoin isomorphisms along the linear orders $[i_1, i_{n-1}]$, so no new automorphisms can be obtained by composing the adjoined isomorphisms.

**Proposition 5.7** The $k$-linear pseudofunctor $C^{\Omega_1}(\mathcal{P})$ is a finitary birepresentation of $\mathcal{B}$, with $C^{\Omega_1}(\mathcal{P})(1) \cong A_r$-proj. Further, it satisfies the following properties:

- The canonical strong transformation $C \xrightarrow{\Omega_1(\mathcal{P})} C^{\Omega_1}(\mathcal{P})$ is faithful.
- Given indecomposable objects $X, Y \in C(1)$, the map $\Omega_{X,Y}^{\Omega_1}(\mathcal{P})$ is a bijection unless $X \not\cong Y$ and $\Omega_{X,Y}^{\Omega_1}(\mathcal{P})(X) \cong \Omega_{X,Y}^{\Omega_1}(\mathcal{P})(Y)$.
- If $U$ is a transversal of $\mathcal{P}$, then the restriction of $\Omega_{X,Y}^{\Omega_1}(\mathcal{P})$ to add $\{Ae_k : k \in U\}$ is an equivalence of categories.

**Proof** As observed earlier, applying $(-)^D$ to the diagram

$$C^{\mathcal{G}} \xrightarrow{W} C^{\Omega_1} \xrightarrow{R} C^{\Omega_1}$$  \hspace{1cm} (17)

gives a diagram in $\text{Cat}_k^D$ equivalent to that underlying

$$C \xrightarrow{\Omega_1} C^{\Omega_1}(\mathcal{P}) \xrightarrow{\Omega_1} C^{\Omega_1}(\mathcal{P})$$.

Since $(-)^D$ sends faithful functors to faithful functors, and the remaining statements concern indecomposable objects and are preserved under equivalences of categories, the result follows from the explicit description of Diagram (17). \qed

**Proposition 5.8** The birepresentation $C^{\Omega_1}(\mathcal{P})$ is simple transitive.

**Proof** Let $I$ be a $\mathcal{B}$-stable ideal of $C^{\Omega_1}(\mathcal{P})$. Consider the $\mathcal{B}$-stable ideal $\Omega^{\Omega_1}(\mathcal{P})^{-1}I$ of $C$, as defined in Lemma 2.2. Since $C$ is simple transitive, this latter ideal is either zero, or coincides with all of $C$. In the former case, it is immediate that $I = 0$.

Since $I = C$. Since $\Omega^{\Omega_1}$ restricted to add $\{Ae_k : k \in U\}$, where $U$ is a transversal of $\mathcal{P}$, is an equivalence of categories, it follows that the functor $\Omega^{\Omega_1}$ is surjective on morphisms—its essential image is all of $C^{\Omega_1}(\mathcal{P})(1)$. It follows that $\Omega^{\Omega_1}(\mathcal{P})^{-1}I = C$ implies $I = C^{\Omega_1}(\mathcal{P})$. We conclude that $I = 0$ or $I = C^{\Omega_1}(\mathcal{P})$, which proves that $C^{\Omega_1}(\mathcal{P})$ is simple transitive. \qed

**Theorem 5.9** Let $M$ be a simple transitive birepresentation of $\mathcal{B}$. We then have $M \cong C^{\Omega_1}(\mathcal{P})_M$.\[Springer]
**Proof** Write $\mathcal{P}_M$ as

$$[0, n] = \{0\} \cup \{i_1^1, i_2^1, \ldots, i_k^1\} \cup \cdots \cup \{i_1^r, i_2^r, \ldots, i_k^r\}.$$  

Consider the strong transformation $\Sigma : C \to M$ described in Proposition 4.3. Using Corollary 4.10, we choose, for $m \in [1, r]$ and $l \in [i_{km-1}^m, i_{km}^m]$, an invertible modification $t^m_l$ from $\Sigma \circ \Theta^m_{i_{km}}$ to $\Sigma \circ \Theta^m_{i_{km+1}}$ such that $t^m_l \cdot \Gamma = \Sigma \cdot s^m_{i_{km}^m, i_{km+1}^m}$.

This choice provides a strong transformation $\tilde{\Sigma} : C^{\text{WR}}(\mathcal{P}) \to M$, using the universal property of $C^{\text{WR}}(\mathcal{P})$. In particular, $\tilde{\Sigma} \circ \Omega^{\text{WR}}(\mathcal{P}) \simeq \Sigma$. Let $U$ be a transversal for $\mathcal{P}$. Denote by $\Sigma|_U$ the restriction of $\Sigma$ to add $\{Ae_k \mid k \in U\}$. Using similar notation for $\tilde{\Sigma} \circ \Omega^{\text{WR}}(\mathcal{P})$, we obtain

$$\tilde{\Sigma} \circ \Omega^{\text{WR}}(\mathcal{P})|_U \simeq \Sigma|_U.$$  

But Propositions 4.3 and 5.7 prove that $\Sigma|_U$ and $\Omega^{\text{WR}}(\mathcal{P})|_U$ are equivalences of categories. It follows that $\tilde{\Sigma}$ is an equivalence of categories, and thus also an equivalence of birepresentations. \hfill \Box

**Theorem 5.10** The map

$$\left\{\text{Set partitions of } [1, n]\right\} \longrightarrow \left\{\text{Simple transitive birepresentations of } \mathcal{B}_n\right\}/\simeq$$

is a bijection.

**Proof** From Proposition 5.8 we have that, for any $\mathcal{P}$, the birepresentation $C^{\text{WR}}(\mathcal{P})$ is simple transitive. From Corollary 4.4 it follows that for different set partitions $\mathcal{P}, \mathcal{P}'$, we have $C^{\text{WR}}(\mathcal{P}) \not \simeq C^{\text{WR}}(\mathcal{P}')$. Finally, Theorem 5.9 shows that any simple transitive birepresentation $M$ is equivalent to $C^{\text{WR}}(\mathcal{P})$, for $\mathcal{P} = \mathcal{P}_M$. \hfill \Box

**Proposition 5.11** Let $\mathcal{P}$ be a set partition, let $\mathcal{P}'$ be a refinement of $\mathcal{P}$ and let $\mathcal{P}''$ be a refinement of $\mathcal{P}'$. There is a strong transformation $\widetilde{\Omega}(\mathcal{P}, \mathcal{P}') : C^{\text{WR}}(\mathcal{P}) \to C^{\text{WR}}(\mathcal{P}')$ such that

$$\Omega^{\text{WR}}(\mathcal{P}) \simeq \widetilde{\Omega}(\mathcal{P}, \mathcal{P}') \circ \Omega^{\text{WR}}(\mathcal{P}).$$  

Further, we have

$$\tilde{\Omega}(\mathcal{P}, \mathcal{P}'') \simeq \widetilde{\Omega}(\mathcal{P}, \mathcal{P}') \circ \tilde{\Omega}(\mathcal{P}, \mathcal{P}'').$$

**Proof** Recall the universal property of $C^{\text{WR}}(\mathcal{P})$ described in Lemma 5.1. Applying it on the strong transformation $\Omega^{\text{WR}}(\mathcal{P}')$, we obtain $\widetilde{\Omega}(\mathcal{P}, \mathcal{P}')$ satisfying

$$\Omega^{\text{WR}}(\mathcal{P}') \simeq \widetilde{\Omega}(\mathcal{P}, \mathcal{P}') \circ \Omega^{\text{WR}}(\mathcal{P}).$$  

Further, we conclude that a strong transformation satisfying the above property is unique up to invertible modification.

Similarly, we have a strong transformation $\tilde{\Omega}(\mathcal{P}, \mathcal{P}'')$, unique up to invertible modification, such that

$$\tilde{\Omega}(\mathcal{P}, \mathcal{P}'') \circ \Omega^{\text{WR}}(\mathcal{P}) \simeq \Omega^{\text{WR}}(\mathcal{P}'').$$

We have

$$\tilde{\Omega}(\mathcal{P}', \mathcal{P}'') \circ \tilde{\Omega}(\mathcal{P}, \mathcal{P}') \circ \Omega^{\text{WR}}(\mathcal{P}) \simeq \widetilde{\Omega}(\mathcal{P}', \mathcal{P}'') \circ \Omega^{\text{WR}}(\mathcal{P}') \simeq \Omega^{\text{WR}}(\mathcal{P}'').$$

\hfill \Box
which, by the above described uniqueness of $\tilde{\Omega}(\mathcal{P}, \mathcal{P}''')$, proves that

$$\tilde{\Omega}(\mathcal{P}, \mathcal{P''}) \simeq \tilde{\Omega}(\mathcal{P}, \mathcal{P}')(\tilde{\Omega}(\mathcal{P}, \mathcal{P}')).$$

\[\square\]

**Remark 5.12** The above construction of finitary birepresentations using coequifiers and coisoinserters essentially relies on the properties of the bijection

$$e_0 Ae_i \sim e_0 Ae_j, \quad b_i \mapsto b_j$$

which may be viewed as induced by the corresponding automorphism of $A$ sending $e_k$ to $e_k$, for $k \neq i, j, e_i$ to $e_j$ and $e_j$ to $e_i$. One is led to speculate that the construction can be mimicked for any basic finite dimensional algebra $B$ with a fixed complete set of pairwise orthogonal, primitive idempotents $\{f_1, \ldots, f_s\}$ and an automorphism $\psi$ permuting $\{f_1, \ldots, f_s\}$ with a fixed point. However, this is not the case. Consider the quotient $B$ of the path algebra of

\[
\begin{array}{cccc}
2 & \xleftarrow{a_2} & 0 & \xrightarrow{a_1} \\
\xrightarrow{b_2} & & \xrightarrow{b_1} & 1
\end{array}
\]

by $\mathcal{R}^4$, where $\mathcal{R}$ is the arrow ideal.

The quiver automorphism fixing 0 and swapping 1 and 2 gives an automorphism $\psi$ of $B$. Denote the induced complete set of pairwise orthogonal, primitive idempotents by $\{f_0, f_1, f_2\}$. For $\psi$ to give a natural isomorphism $- \otimes_B Bf_1 \sim - \otimes_B Bf_2$ on $\{f_0B\}$ extending $f_0Bf_1 \xrightarrow{\psi/f_0Bf_1} f_0Bf_2$, the diagram

$$f_0Bf_1 \xrightarrow{b_1a_1} f_0Bf_2$$

$$\downarrow{\psi} \quad \downarrow{\psi}$$

$$f_0Bf_2 \xrightarrow{b_1a_1} f_0Bf_2$$

would have to commute. However, chasing the element $b_1$, we obtain

$$b_1 \mapsto b_1a_1b_1$$

$$b_2 \mapsto b_1a_1b_2 \neq b_2a_2b_2$$

One can also construct examples where the automorphism of the algebra does indeed give a natural isomorphism as above, which is further monoidal—and thus gives a modification of strong transformations. One example is the quotient $B'$ of the path algebra of

\[
\begin{array}{cccc}
3 & \xleftarrow{b_3} & a_1 & \xrightarrow{1} \\
\xleftarrow{c_3} & & \xrightarrow{a_1} & c_1
\end{array}
\]

\[
\begin{array}{cccc}
4 & \xleftarrow{a_4} & b_2 & \xrightarrow{2} \\
\xleftarrow{c_4} & & \xrightarrow{a_3} & b_4
\end{array}
\]

by the ideal generated by paths

$$a_3b_1, a_4b_1, a_3b_2, a_4b_2, a_1b_3, a_1b_4, a_2b_3, a_2b_4, a_kb_k, b_kb_k, \text{ for } k \in [1, 4]$$

together with the relations making all the 3-cycles at a vertex coincide, and sending the fourth power of the arrow ideal to zero. Labelling the complete set of pairwise orthogonal, primitive
idempotents coming from the quiver by \{ f_0, \ldots, f_4 \}, we find that we have \( \text{End}_{B'}(B f_0) \simeq \mathbb{F}[x]/(x^2) \) and that \( \text{Rad} \text{End}_{B'}(B f_0) \otimes_{B'} B' f_i = 0 \), for \( i \neq 0 \). For the permutation \( \sigma = (12)(34) \), the elements \( x_i = a_i + c_{\sigma(i)} a_{\sigma(i)} \in f_i B' f_0 \) are such that \( f_0 B' f_1 \xrightarrow{x_i} f_0 B' f_0 \) is a \( k \)-linear isomorphism. The quiver automorphism swapping 1, 2 and fixing the remaining vertices gives an automorphism \( \psi_{(12)} \) of \( B' \). The resulting linear isomorphism \( f_0 B' f_1 \xrightarrow{\psi} f_0 B' f_2 \) coincides with the composition \( f_0 B' f_1 \xrightarrow{x_1} f_0 B' f_0 \xrightarrow{(x_2+1)} f_0 B' f_2 \). These properties are completely analogous to those of star algebras, and using them one may verify that there is a finitary birepresentation of \( (\text{add} \{ B' f_1, B' f_k \otimes_k f_0 B' \}, \otimes_{B'}) \) given by an analogous (multiple) coequifier of (multiple) coisoinserter.

However, for such examples, the analysis of the action matrices and Cartan matrices such as that for star algebras in [38] has not been conducted, and so we cannot easily generalize our classification of simple transitive birepresentations to these cases.

**Acknowledgements** This research is partially supported by Göran Gustafssons Stiftelse. The author would like to thank his advisor, Volodymyr Mazorchuk, for numerous helpful discussions, and also for providing and explaining the proof of Lemma 2.4. The author would also like to thank Richard Garner, and the anonymous referee, for their valuable remarks.

**Funding** Open access funding provided by Uppsala University.

**Declarations**

**Conflict of interest** The author has no conflict of interest.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**References**

1. Brandenburg, M.: Bicategorical colimits of tensor categories. Preprint, arXiv:2001.10123
2. Canevali, N.: 2-filtered bicolimits and finite weighted bilimits commute in \textbf{Cat}. Bachelor’s thesis, Universidad de Buenos Aires (2016)
3. Chan, A., Mazorchuk, V.: Diagrams and discrete extensions for finitary 2-representations. Math. Proc. Camb. Philos. Soc. 166(2), 325–352 (2019)
4. Chuang, J., Rouquier, R.: Derived equivalences for symmetric groups and \( \text{sl}_2 \)-categorification. Ann. Math. (2) 167(1), 245–298 (2008)
5. Comes, J., Wilson, B.: Deligne’s category \( \text{Rep}(GL_\delta) \) and representations of general linear supergroups. Represent. Theory 16, 568–609 (2012)
6. Deligne, P.: La catégorie des représentations du groupe symétrique \( S_t \), lorsque \( t \) n’est pas un entier naturel. Algebraic groups and homogeneous spaces, pp. 209–273, Tata Inst. Fund. Res. Stud. Math., 19, Tata Inst. Fund. Res., Mumbai (2007)
7. Etingof, P., Nikshych, D., Ostrik, V.: On fusion categories. Ann. Math. (2) 162(2), 581–642 (2005)
8. Ehrig, M., Tubbenhauer, D.: Algebraic properties of zigzag algebras. Commun. Algebra 48(1), 11–36 (2020)
9. Fiore, T.: Pseudo limits, biadjoints, and pseudo algebras: categorical foundations of conformal field theory. Mem. Amer. Math. Soc. 182(860), x+171 pp (2006)
10. Gordon, R., Power, A.J., Street, R.: Coherence for tricategories. Mem. Amer. Math. Soc. 117(558), vi+81 pp (1995)
11. Garner, R., Shulman, M.: Enriched categories as a free cocompletion. Adv. Math. 289, 1–94 (2016)
12. Hazratpour, S.: A logical study of some 2-categorical aspects of topos theory. Ph.D. thesis, University of Birmingham (2019)
13. Jonsson, H.: Cell structure of bimodules over radical square zero Nakayama algebras. Commun. Algebra 48(2), 573–588 (2020)
14. Jonsson, H.: On simple transitive 2-representations of bimodules over the dual numbers. Alg. Rep, Th (2022)
15. Jonsson, H., Stroński, M.: Simple transitive 2-representations of bimodules over radical square zero Nakayama algebras via localization. J. Alg. 612, 87–114 (2023)
16. Kelly, G.M.: Elementary Observations on 2-categorical limits. Bull. Austral. Math. Soc. 39(2), 301–317 (1989)
17. Kelly, G.M.: Basic Concepts of Enriched Category Theory. London Math. Soc. Lec. Note Ser., vol. 64. Cambridge University Press (1982)
18. Khovanov, M., Lauda, A.: A diagrammatic approach to categorification of quantum groups I. Represent. Theory 13, 309–347 (2009)
19. Khovanov, M., Lauda, A.: A categorification of quantum sl_3. Quantum Topol. 1(1), 1–92 (2010)
20. Mazorchuk, V., Miemietz, V.: Cell 2-representations of finitary 2-categories. Compositio Math. 147(5), 1519–1545 (2011)
21. Mazorchuk, V., Miemietz, V.: Additive versus abelian 2-representations of fiat 2-categories. Mosc. Math. J. 14(3), 595–615, 642 (2014)
22. Mazorchuk, V., Miemietz, V.: Morita theory for finitary 2-categories. Quantum Topol. 7(1), 1–28 (2016)
23. Mackaay, M., Mazorchuk, V., Miemietz, V., Tubbenhauer, D.: Simple transitive 2-representations via (co)algebra 1-morphisms. Indiana Univ. Math. J. 68(1), 1–33 (2019)
24. Mazorchuk, V., Miemietz, V., Tubbenhauer, D., Zhang, X.: Finitary birepresentations of finitary bicategories. Forum Math. 33, 1261–1320 (2021)
25. Ostrick, V.: Module categories, weak Hopf algebras and modular invariants. Transform. Groups 8(2), 177–206 (2003)
26. Ritter, M.: On universal properties of preadditive and additive categories. Bachelor’s thesis, University of Stuttgart (2016)
27. Rouquier, R.: 2-Kac-Moody algebras. Preprint, arXiv:0812.5023
28. Zimmermann, J.: Simple transitive 2-representations of left cell 2-subcategories of projective functors for star algebras. Commun. Algebra 47(3), 1222–1237 (2019)