Statistics of Quasiparticles in Fractional Quantum Hall States

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Abstract

We have considered here the statistics parameter for quasiparticles in FQH states from an analysis of these states in the framework of chiral anomaly and Berry phase. It is shown that we have a generalized relation such that the statistical phase of a quasiparticle for FQH states at $\nu = p/q$ with $p$ ($q$) even or odd, is given by $e^{i\pi \theta}$ with $\theta = p/q$ and the charge is $-e^{p/q}$. The statistics parameter for a quasihole is identical with that of a quasiparticle and the charge is of opposite sign.

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I. INTRODUCTION

In a seminal paper [1] Arovas, Schrieffer and Wilczek derived the statistics of the quasiholes at $\nu = 1/m$, $m$ odd in a Berry phase calculation. They examined the Berry phase corresponding to one quasihole encircling the origin and interpreted this as the Aharonov-Bohm phase. The charge was found to be $e/m$. Considering a pair of quasiholes encircling one another they found the statistics parameter to have the value $1/m$. Khongsberg and Myrheim [3] have argued that for FQH states the quasiparticles(qp) at $\nu = 1/m$, $m$ odd, do not possess well defined statistics. Their numerical studies show that at $\nu = 1/3$ the charge and statistics parameter for quasiholes(qh) are $e/3$ and $1/3$ respectively. However, for the quasiparticles though a slow convergence towards the expected value of the charge $-e/3$ is observed with a finite size correction for $N$ electrons, the statistics parameter has no well defined value even for 200 electrons but might possibly converge to $1/3$.

Johnson and Carrient [4] examined the exclusion statistics parameter considering state counting based on numerical simulations for interacting electrons on a sphere. It is found that the one dimensional exclusion statistics parameter for quasiholes is $1/m$ while for quasielectrons the value is $2 - 1/m$. The exclusion statistics parameter is the same parameter as is obtained from Berry phase analysis with an opposite sign for the quasielectrons since their charge is negative. This implies that the exclusion statistics parameter is $1/m$ for quasiholes and $(2 - 1/m)$ for quasielectrons. Though the values $1/m$ and $2 - 1/m$ represent the same particle statistics, their states are different, $1/m$ state singular and $2 + 1/m$ state nonsingular.

In the composite fermion (CF) scenario Khongsberg and Leinaas [5] calculated the statistics of the unprojected CF quasiparticle at $\nu = 1/m$, the wave function of which is different from Laughlin function, and observed a definite value though the sign is inconsistent with general considerations. Jeon, Graham and Jain [6] have confirmed that the statistics is robust to projection into the lowest Landau level and argued that the sign enigma has its origin in very small perturbations in the trajectory due to the insertion of an additional CF quasiparticle. These authors have numerically studied the statistics of the composite fermion quasiparticles at $\nu = 1/3$ and $\nu = 2/5$ by evaluating the Berry phase for a closed loop encircling another CF quasiparticle and the statistics parameters were found to be $-2/3$ and $-2/5$ respectively. The negative sign has been attributed to the fact that there is an inter-CF interaction which is weak and often attractive and involves a significant overlap of CF quasiparticles causing small perturbation in the trajectory.

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Here we shall address the problem of statistics in a heuristic manner on the basis of analysis of FQH states in the framework of chiral anomaly and Berry phase. From a calculation of the Berry phase of a quasiparticle (quasihole) encircling another quasiparticle (quasihole) we shall show that the charge and statistics parameter for quasiparticles at \( \nu = \frac{n}{2m+1} \) with \( m, n \) positive integers are given by \( -\nu e \) and \( \frac{n}{2m+1} \) respectively. For quasiholes the statistics parameter is identical with that of the quasiparticle and the charge has the opposite sign.

In sec. 2 we shall recapitulate certain features of the FQH states analysed from the Berry phase approach. In sec. 3 we shall calculate the statistics parameter for quasiparticles (quasiholes) for FQH states at different filling factors.

II. FRACTIONAL QUANTUM HALL STATES: A BERRY PHASE APPROACH

In some earlier papers \([7, 8]\) we have analyzed the sequence of quantum Hall states from the viewpoint of chiral anomaly and Berry phase. To this end, we have taken quantum Hall states on the two dimensional surface of a 3D sphere with a magnetic monopole of strength \( \mu \) at the centre. In this spherical geometry, we can analyze quantum Hall states in terms of spinor wave functions and take advantage of the analysis in terms of chiral anomaly which is associated with the Berry phase. In this geometry the angular momentum relation is given by

\[
J = r \times p - \mu \hat{r}, \quad \mu = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots
\]  

(1)

From the description of spherical harmonics \( Y_{\ell}^{m,\mu} \) with \( \ell = 1/2 \), \( |m| = |\mu| = 1/2 \), we can construct a two-component spinor \( \theta = \begin{pmatrix} u \\ v \end{pmatrix} \) where

\[
u = 1, \quad v = Y_{1/2}^{-1/2,1/2} = \cos \frac{\theta}{2} \exp[-i(\phi + \chi)/2]
\]

(2)

Here \( \mu \) corresponds to the eigenvalue of the operator \( i\frac{\partial}{\partial \theta} \).

The \( N \)-particle wave function for the quantum Hall fluid state at \( \nu = \frac{1}{m} \) can be written as

\[
\psi^{(m)}_N = \prod_{i<j} (u_i v_j - u_j v_i)^m
\]

(3)

\( m \) being an odd integer. Here \( u_i(v_j) \) corresponds to the \( i \)-th (\( j \)-th) position of the particle in the system.

It is noted that \( \psi^{(m)}_N \) is totally antisymmetric for odd \( m \) and symmetric for even \( m \). We can identify \( m \) as \( m = J_i + J_j \) for the \( N \)-particle system where \( J_i \) is the angular momentum of the \( i \)-th particle. It is evident from eqn. (1) that with \( r \times p = 0 \) and \( \mu = \frac{1}{2} \) we have \( m = 1 \) which corresponds to the complete filling of the lowest Landau level. From the Dirac quantization condition \( e\mu = \frac{1}{2} \), we note that this state corresponds to \( e = 1 \) describing the IQH state with \( \nu = 1 \).

The next higher angular momentum state can be achieved either by taking \( r \times p = 1 \) and \( \mu = \frac{1}{2} \) (which implies the higher Landau level) or by taking \( r \times p = 0 \) and \( |\mu_{eff}| = \frac{3}{2} \) implying the ground state for the Landau level. However, with \( |\mu_{eff}| = \frac{3}{2} \) we find the filling fraction \( \nu = \frac{1}{3} \) which follows from the condition \( e\mu = \frac{1}{2} \) for \( \mu = \frac{1}{2} \). Generalizing this, we can have \( \nu = \frac{1}{m} \) with \( |\mu_{eff}| = \frac{m}{2} \). In this way we can explain all the FQH states with \( \nu = \frac{1}{2m+1} \) with \( m \) an integer. It is noted that according to this formalism for a quantum Hall particle the charge is taken to be given by \( -\nu e \) when \( \nu \) is the filling factor.

As \( \mu \) here corresponds to the monopole strength, we can relate this with the Berry phase. Indeed \( \mu = \frac{1}{2} \) corresponds to one flux quantum and the flux through the sphere when there is a monopole of strength \( \mu \) at the centre is \( 2\mu \). The Berry phase of a fermion of charge \( q \) is given by \( e^{\phi_B} \) with \( \phi_B = 2\pi qN \) where \( N \) is the number of flux quanta enclosed by the loop traversed by the particle.

If \( \mu \) is an integer, we can have a relation of the form

\[
J = r \times p - \mu \hat{r} = r' \times p'
\]

(4)
which indicates that the Berry phase associated with $\mu$ may be unitarily removed to the dynamical phase. Evidently, the average magnetic field may be considered to be vanishing in these states. The attachment of $2m$ vortices ($m$ an integer) to an electron effectively leads to the removal of Berry phase to the dynamical phase. So, FQH states with $2\mu_{eff} = 2m + 1$ can be viewed as if one vortex is attached to the electron. Now we note that for a higher Landau level we can consider the Dirac quantization condition $e\mu_{eff} = \frac{1}{2}n$, with $n$ being a vortex of strength $2\ell + 1$. This can generate FQH states having the filling factor of the form $\frac{n}{2\mu_{eff}}$ where both $n$ and $2\mu_{eff}$ are odd integers. Indeed, we can write the filling factor as $\frac{\nu}{2}$.

$$\nu = \frac{n}{2\mu_{eff}} = \frac{n}{(2\mu_{eff} \mp 1) \pm 1} = \frac{n}{2m' \pm 1} = \frac{n}{2mn \pm 1}$$

(5)

where $2\mu_{eff} \mp 1$ is an even integer given by $2m' = 2mn$. The particle-hole conjugate state can be generated with the filling factor given by

$$\nu = 1 - \frac{n}{2mn \pm 1} = \frac{n(2m - 1) \pm 1}{2mn \pm 1} = \frac{n'}{2mn' \pm 1}$$

(6)

where $n(n')$ is an odd(even) integer.

Recently, some novel generation of filling factors for FQH fluid have been observed which do not satisfy the primary Jain sequence. An analysis of these states from a Berry phase approach suggests that the observed filling factors like $\nu = 4/11, 5/13, 6/17, 4/13$ and $5/17$ corresponds to the lowest Landau level and are given by

$$\nu = \frac{n}{2\mu_{eff}} = \frac{n}{2m' \pm 1}$$

where $m'$ is such that it cannot be split into the form $mn$. The FQH state $\nu = 7/11$ is found to be particle-hole conjugate state.

Besides, there are some FQH states with even denominator filling factors which also needs explanation. Indeed in some earlier papers, it has been pointed out that FQH states at even denominator filling factor appear as pairs and correspond to non-Abelian Berry phase. From our previous discussions we note that the Dirac quantization condition $e\mu = 1/2$ suggests that $\nu = 1/2$ FQH state corresponds to the factor $\mu = 1$. However, from the angular momentum relation (4) we note that in case $\mu$ is an integer the Berry phase may be unitarily removed to the dynamical phase. This implies that the average magnetic field may be taken to be vanishing in these states. However, the effect of the Berry phase may be observed when we split the state into a pair with each component having the constraint of representing the state with $\mu = 1/2$. This corresponds to non-Abelian Berry phase for these states.

### III. STATISTICS OF QUASIPARTICLES IN FQH STATES

In an earlier paper, it has been pointed out that the quantization of a Fermi field is achieved when we introduce an anisotropy in the internal space through the introduction of a direction vector attached to the space-time point. Indeed, to have quantization in Minkowski space we have to take into account a complex manifold when the coordinate is given by $x^\mu = x^\mu + i\xi^\mu$ where $\xi^\mu$ is the direction vector attached to the space-time point $x^\nu$. The two opposite orientations of the direction vector correspond to particle and anti-particle states. This direction vector corresponds to the internal helicity when $\xi^\mu$ is written in terms of the spinorial variable $\xi^\mu = \lambda^\mu, \alpha \theta^\alpha (\alpha = 1, 2)$ where $\theta$ is a two component spinor. The corresponding metric $g_{\mu\nu}(x, \theta, \bar{\theta})$ eventually gives rise to a gauge theoretic extension when the position and momentum variables can be written as

$$Q_\mu = -i(\frac{\partial}{\partial p_\mu} + B_\mu) \quad P_\mu = i(\frac{\partial}{\partial q_\mu} + C_\mu)$$

(7)

where $B_\mu(C_\mu)$ corresponds to a $SL(2, C)$ gauge field. Here $q_\mu(p_\mu)$ denotes the mean position (momentum) in the external observable space. In fact, the direction vector effectively represents a vortex line which is topologically equivalent to a magnetic flux quantum and the gauge field theoretical extension
corresponds to the magnetic field associated with the internal extension of the particle \[12\]. This leads to the description of a massive fermion as a Skyrmion.

When the direction vector \( \xi^\mu \) is attached to the space-time point \( x^\mu \), the corresponding field function is given by \( \phi(x^\mu, \xi^\mu) \) which can be treated to describe a particle moving in an anisotropic space and the wave function takes into account the polar coordinate \( r, \theta, \phi \) for the space component of the vector \( x^\mu \) along with the angle \( \chi \) specifying the rotational orientation around the direction vector. The eigenvalue of the operator \( i\partial_\chi \) just corresponds to the internal helicity associated with the vortex line (magnetic flux quantum). Indeed, the angular momentum operator is now given by

\[
J = r \times p - \mu \mathbf{r} \quad \mu = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots
\]

which is similar to that of a charged particle moving in the field of a magnetic monopole of strength \( \mu \).

The spherical harmonics incorporating the term \( \mu \) is given by \[13\ 14\]

\[
Y_l^{\mu, \nu} = (1 + x)^{- (m - \mu)/2} (1 - x)^{-(m + \mu)/2} \frac{d^{l+m}}{d^{l-m} x} [(1 + x)^{l-\mu} (1 - x)^{l+\mu}] e^{i\mu \phi} e^{-i\mu \chi}
\]

where \( x = \cos \theta \). Here \( m \) and \( \mu \) represents the eigenvalues of the operators \( i\partial_\rho \) and \( i\partial_\omega \) respectively.

In this formalism, a fermion is represented by a boson attached with a magnetic flux quantum. The Berry phase acquired by the boson when traversing a loop enclosing the flux quantum is given by \( e^{i2\pi \mu} \) \[15\]. Indeed, the angular part associated with the angle \( \chi \) in the spherical harmonics \( Y_l^{m, \mu} \) is given by \( e^{-i\mu \chi} \) where we have

\[
i \frac{\partial}{\partial \chi} = \mu e^{-i\mu \chi}
\]

Thus when \( \chi \) is changed to \( \chi + \delta \chi \) we have

\[
i \frac{\partial}{\partial (\chi + \delta \chi)} e^{-i\mu \chi} = i \frac{\partial}{\partial (\chi + \delta \chi)} e^{-i\mu (\chi + \delta \chi)} e^{i\mu \delta \chi}
\]

which implies that the wave function will acquire an extra factor \( e^{i\mu \delta \chi} \) due to infinitesimal change of the angle \( \chi \). As the angle \( \chi \) is changed over the closed path \( 0 \leq \chi \leq 2\pi \), for one such complete rotation, the wave function will acquire, the required phase

\[
\exp \left\{ i\mu \int_0^{2\pi} \delta \chi \right\} = e^{i2\pi \mu}
\]

As \( \mu = 1/2 \) corresponds to one flux quantum, when a boson traverses a closed path enclosing one flux quantum we have the phase \( e^{i\pi} \) which suggests that the system represents a fermion implying that the wave function changes sign after \( 2\pi \) rotation. The effect of the flux tube is to induce an appropriate Aharonov-Bohm phase which simulates the statistical phase factor. Indeed, as a fermion is described by a scalar particle carrying one flux quantum, we note that when a fermion traverses a loop enclosing another fermion we can view it as if a boson traverses a loop enclosing two magnetic flux quanta so that the Berry phase is given by \( e^{i2\pi \mu} \) with \( \mu = 1 \) i.e. \( e^{i2\pi} \). This implies that when a fermion encloses \( N \) number of flux quanta the Berry phase is \( 2\pi N \) \[16\]. Now a process which exchanges two fermions can be viewed as if one of the fermions moving about the other in a half circle. Thus the statistical phase \( e^{i\pi \theta} \) is given by \( \theta = 1 \). This is the statistical phase factor when two fermions are adiabatically exchanged. This suggests that when a boson (scalar particle) is dressed with \( n \) vortices (magnetic flux quanta) the statistical phase is given by \( e^{i\pi \theta} \) with \( \theta = n \).

Now to study the statistics parameter for FQH states, we note that according to the formalism as described in the previous section, the FQH states at \( \nu = \frac{1}{2m+1} \) corresponds to the fact that an electron is attached with \( (2m+1) \) vortices of which \( 2m \) vortices contribute to the dynamical phase. Now taking into account the duality principle \[17\] which interchanges the role of charges and vortices we may consider vortices as bosons which see each original particle as a flux tube carrying one Dirac flux quantum. That is, an electron attached with \( m \) vortices may be viewed as if \( m \) hard-core bosons share one vortex (magnetic flux quantum). So quasiparticles describing FQH states at \( \nu = \frac{1}{2m+1} \) may be viewed as such that \( (2m+1) \) hard-core boson are attached with one flux quantum implying that each hard-core boson shares the vortex...
density $\frac{1}{2n+1}$. So from our above analysis we note that the statistical phase will be given by $e^{i\pi \theta}$ with $\theta = \frac{1}{2n+1}$. So for the FQH state at $\nu = 1/3$ and $1/5$, we will have the statistics parameter $\theta = 1/3$ and $1/5$ respectively. As under the duality principle, the charge is now shared by the hard-core bosons the charge of the quasiparticle will be given by $-\nu e$, i.e. for $\nu = 1/3$ and $1/5$, the charge will be $-e/3$ and $-e/5$ respectively. This also follows from our analysis in sec. 2 where we have pointed out that from the Dirac quantization condition $\epsilon \mu = 1/2$, with $\mu = 3/2(5/2)$ we obtain $\nu = 1/3(1/5)$ implying that the charge is $-\nu e$.

To consider the statistics parameter for FQH states at $\nu = \frac{n}{2mn \pm 1}$ with $n > 1$ and odd, we have noted in the earlier section that this corresponds to the higher Landau level which follows from the Dirac quantization condition $\epsilon \mu = \frac{q}{\nu}$, where $n$ corresponds to the vortex of strength $2l + 1$. This implies that under the duality principle, $n$ units of flux density $\frac{1}{2mn \pm 1}$ are attached with each hard core boson. So from our above analysis, we find that the statistical phase $e^{i\pi \theta}$ will be given by $\theta = \frac{n}{2mn \pm 1}$. Also the charge will be given by $-\nu e$. This suggests that for the FQH states at $\nu = 3/5, 3/7, 5/9...$ we have $\theta = 3/5, 3/7, 5/9$ and so on.

For FQH states at $\nu = \frac{n}{2mn \pm 1}$ with $n$ an even integer, we discussed in sec. 2 that these correspond to particle-hole conjugate states. Indeed, these are given by the relation

$$\nu = 1 - \frac{n}{2mn \pm 1} = \frac{n'}{2mn' \pm 1}$$

where $n(n')$ is an odd (even) integer. So, under the duality principle, the effective flux density associated with each hard-core boson will be $1 - \frac{n}{2mn \pm 1}$, such that the statistical phase $e^{i\pi \theta}$ is given by $\theta = 1 - \frac{n}{2mn \pm 1} = \frac{n'}{2mn' \pm 1}$. Thus we find that the statistical phase $e^{i\pi \theta}$ for a quasiparticle in FQH states at filling factor $\nu = \frac{n}{2mn \pm 1}$, $n$ odd or even corresponds to $\theta = \nu = \frac{n}{2mn \pm 1}$ and charge is $-\nu e$. For the quasihole, the statistics parameter will be identical to that of a quasiparticle and the charge will be of opposite sign.

The above result for statistics parameter can be generalized to the newly discovered states also and we will have the statistical phase given by $e^{i\pi \theta}$ with $\theta = \nu$ and the charge for the quasiparticles (quasiholes) is $-\nu e$.

This analysis can be generalized to study the statistics parameter for the FQH states with even denominator filling factor. In this case, we observe that for FQH state at $\nu = 1/2$, as this corresponds to $\mu = 1$ the quasiparticle corresponding to the pair state effectively represent that this is dressed with two vortices. Indeed, the quasiparticle for FQH liquid at $\nu = 1/2$ corresponds to a singlet characterized by a flux $\phi_0 = \frac{\pi}{2}$. So under the duality principle, this implies that two hard-core bosons share one Dirac flux quantum. So from our above analysis we find that the statistical phase $e^{i\pi \theta}$ is given by $\theta = 1/2$. This also implies that the charge of the quasiparticle will be given by $-\frac{1}{2}e$. For the Haldane-Rezayi FQH state at $\nu = 5/2$, it has been observed that this corresponds to higher Landau level ($l = 2$). The Dirac quantization condition $\epsilon \mu = n/2$ with $n = 2l + 1$ suggests that for $l = 2$ we have $\nu = 5/2$ FQH state. Generalizing our above analysis we note that under duality principle this implies that 5 units of flux density $1/2$ are attached with each hard-core boson and as such the statistical phase $e^{i\pi \theta}$ will be here characterized by the parameter $\theta = 5/2$. Following our above argument we note that the charge here will also be given by $-\frac{5}{2}e$. The same argument suggests that the newly observed even denominator FQH state such as $\nu = 3/8$ and $\nu = 3/10$ will also have the statistical phase $e^{i\pi \theta}$ with $\theta = 3/8$ and $\theta = 3/10$ respectively.

IV. DISCUSSION

Our above analysis suggests that we have a generalized relation such that the statistical phase of a quasiparticle (quasihole) for FQH states at $\nu = p/q$ with $p(q)$ even or odd is given by $e^{i\pi \theta}$ with $\theta = p/q$ and the charge for a quasiparticle is $-\frac{e}{q}$, whereas for a quasihole the charge will be of opposite sign. This is consistent with that observed in numerical studies with the Laughlin’s wave function for the quasiholes at $\nu = 1/3$ though for quasiparticles the result is not straightforward for finite number of electrons and such that it may possibly converge to $1/3$. This discrepancy is due to the fact that for Laughlin wave functions the inverse of the operator used to create a quasielectron is not simply the conjugate of the inverse quasihole operator. In the composite fermion (CF) scenario a general analysis suggests that for
the FQH state at $\nu = \frac{n}{2m+1}$ the statistics parameter is $\theta = \frac{2n}{2m+1}$. However numerical studies have suggested that $\theta$ has negative sign. Jeon et al. have conjectured that the sign enigma is due to a small perturbation caused by the insertion of an additional CF quasiparticle. Evidently our above result for $\theta$ is at variance with the prediction of the CF formalism.

Based on the hierarchical model, Halperin has derived a recurrence relation for the statistical angle of the elementary charged excitations. This recurrence relation is found to be equivalent to the explicit analysis of Su based on adiabatic theorem. Though the results obtained in Ref.[23] are consistent with our results for values at $\nu = 1/m$, it is not so for other states. The discrepancy has its origin in the fact that unlike the hierarchical model, in our formalism quasiparticles are not considered as composites of elementary excitations.

In the present framework we note that quasiparticles (quasiholes) of FQH states are anyons when projected in 2D but these anyons are not ideal point particles. Indeed these are extended objects when a magnetic flux density is dressed with a hard core boson. It may be mentioned here that in some recent papers we have pointed out that the quasiparticles at $\nu = 1/m$ may be considered as underlying fields in noncommutative manifold $M_4 \times Z_N$ with $N > 2$ and odd. In fact it is pointed out that while the noncommutative manifold $M_4 \times Z_2$ has its underlying field as fermion, the manifold $M_4 \times Z_N$ with $N > 2$ and odd has its underlying fields with fractional statistics having the statistics parameter $\theta = 1/N$. The discrete space when considered as the internal space the system corresponds to an extended body dressed with magnetic flux density.

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