Number-conserving cellular automaton rules

Nino Boccara† and Henryk Fukš‡
† Department of Physics, University of Illinois, Chicago, USA
boccara@uic.edu
and
DRECAM/SPEC, CE Saclay, 91191 Gif-sur-Yvette Cedex, France
‡ The Fields Institute for Research in Mathematical Sciences,
Toronto ON M5T 2W1, Canada
hfuk@fields.utoronto.ca and
Department of Mathematics and Statistics, University of Guelph,
Guelph, ON N1G 2W1, Canada

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Abstract. A necessary and sufficient condition for a one-dimensional \( q \)-state \( n \)-input cellular automaton rule to be number-conserving is established. Two different forms of simpler and more visual representations of these rules are given, and their flow diagrams are determined. Various examples are presented and applications to car traffic are indicated. Two nontrivial three-state three-input self-conjugate rules have been found. They can be used to model the dynamics of random walkers.

1. Introduction

A one-dimensional cellular automaton (CA) is a discrete dynamical system, which may be defined as follows. Let \( s : \mathbb{Z} \times \mathbb{N} \rightarrow \mathcal{Q} \) be a function that satisfies the equation

\[
s(i, t + 1) = f(s(i - r_\ell, t), s(i - r_\ell + 1, t), \ldots, s(i + r_r, t)),
\]

(1)

for all \( i \in \mathbb{Z} \), and all \( t \in \mathbb{N} \). \( \mathbb{Z} \) is the set of all integers, \( \mathbb{N} \) the set of nonnegative integers, and \( \mathcal{Q} \) a finite set of states, usually equal to \( \{0, 1, 2, \ldots, q - 1\} \). \( s(i, t) \) represents the state of site \( i \) at time \( t \), and the mapping \( f : \mathcal{Q}^{r_\ell + r_r + 1} \rightarrow \mathcal{Q} \) is
the CA evolution rule. The positive integers \( r_\ell \) and \( r_r \) are, respectively, the left and right radii of the rule. In what follows, \( f \) will be referred to as an \( n \)-input rule, where \( n \) is the number \( r_\ell + r_r + 1 \) of arguments of \( f \). Following Wolfram [1], to each rule \( f \) we assign a rule number \( N(f) \) such that

\[
N(f) = \sum_{(x_1, x_2, \ldots, x_n) \in \mathbb{Q}^n} f(x_1, x_2, \ldots, x_n)q^{n-1}x_1 + q^{n-2}x_2 + \cdots + q^0x_n.
\]

Cellular automata (CAs) have been widely used to model complex systems in which the local character of the evolution rule plays an essential role [2, 3, 4, 5].

Recently, following Nagel and Schreckenberg [6], many authors proposed various CA models of highway traffic flow. In the simplest models of this type, \( \mathbb{Z}_L \) represents a one-lane circular highway, and \( \mathbb{Q} \) is equal to \( \{0, 1\} \). According to the value of \( s(i, t) \), it is said that, at time \( t \), site \( i \) is either empty, if \( s(i, t) = 0 \), or occupied by a car, if \( s(i, t) = 1 \). In order to complete the description of the model, an evolution rule has to be defined. For the sake of simplicity, we might assume that, at each time step, all cars move to the right neighboring site if, and only if, this site is empty. It is not difficult to verify that this rule coincides with Rule 184 defined by

\[
\begin{align*}
    f_{184}(0, 0, 0) &= 0, \quad f_{184}(0, 0, 1) = 0, \quad f_{184}(0, 1, 0) = 0, \quad f_{184}(0, 1, 1) = 1, \\
    f_{184}(1, 0, 0) &= 1, \quad f_{184}(1, 0, 1) = 1, \quad f_{184}(1, 1, 0) = 0, \quad f_{184}(1, 1, 1) = 1.
\end{align*}
\]

This traffic rule does not allow cars to enter or exit the highway. Therefore, during the evolution, the number of cars should remain constant, that is, starting from any initial cyclic configuration of length \( L \), and for all \( t \in \mathbb{N} \), Rule 184 should satisfy the condition

\[
    f_{184}(s(1, t), s(2, t), s(3, t)) + f_{184}(s(2, t), s(3, t), s(4, t)) + \cdots + f_{184}(s(L, t), s(1, t), s(2, t)) = s(1, t) + s(2, t) + \cdots + s(L, t). \tag{2}
\]

Since Rule 184 may also be written

\[
    f_{184}(x_1, x_2, x_3) = x_2 + \min\{x_1, 1 - x_2\} - \min\{x_2, 1 - x_3\}, \tag{3}
\]

condition (3) is clearly verified. If we had assumed that cars, instead of moving to the right, had to move to the left, we would have found that the evolution rule would have been Rule 226. The relations

\[
\begin{align*}
    f_{226}(x_1, x_2, x_3) &= f_{184}(x_3, x_2, x_1) \\
    f_{226}(x_1, x_2, x_3) &= 1 - f_{184}(1 - x_1, 1 - x_2, 1 - x_3),
\end{align*}
\]

show that these two rules are not fundamentally distinct.

Rules 184 and 226 are the simplest nontrivial examples of number-conserving CAs. In a previous paper [2], in order to find all the deterministic car traffic rules, we determined, up to \( n = 5 \), all the 2-state number-conserving CA rules. This was done using a different technique than the one presented in this paper.
For \( n > 3 \), not all these rules did correspond to realistic car traffic rules since they allowed vehicles to move in both directions. We found then that it was more appropriate to interpret all these 2-state number-conserving CA rules as evolution operators of one-dimensional systems of distinguishable particles.

The purpose of this paper is to derive a necessary and sufficient condition for a one-dimensional \( q \)-state \( n \)-input CA rule to be number-conserving. We will then give examples of such rules and indicate some of their applications. Our result is an illustration of a general theorem on additive conserved quantities established by Hattori and Takesue [8].

2. Number-conserving rules

Definition 2.1. A one-dimensional \( q \)-state \( n \)-input CA rule \( f \) is number-conserving if, for all cyclic configurations of length \( L \geq n \), it satisfies

\[
f(x_1, x_2, \ldots, x_{n-1}, x_n) + f(x_2, x_3, \ldots, x_n, x_{n+1}) + \cdots + f(x_L, x_1 \ldots, x_{n-2}, x_{n-1}) = x_1 + x_2 + \cdots + x_L.
\] (4)

Lemma 2.1. If \( f \) is a number-conserving rule, then

\[
f(0, 0, \ldots, 0) = 0.
\] (6)

Write Condition (4) for a cyclic configuration of length \( L \geq n \) whose all elements are equal to zero. \( \square \)

To prove that Condition (4) is necessary, consider a cyclic configuration of length \( L \geq 2n - 1 \) which is the concatenation of a sequence \( (x_1, x_2, \ldots, x_n) \) and a sequence of \( L - n \) zeros, and express that the \( n \)-input rule \( f \) is number-conserving. We obtain

\[
f(0, 0, \ldots, 0, x_1) + f(0, 0, \ldots, 0, x_1, x_2) + \cdots + f(x_1, x_2, \ldots, x_n) + f(x_2, x_3, \ldots, x_n, 0) + \cdots + f(x_n, 0, \ldots, 0) = x_1 + x_2 + \cdots + x_n.
\] (7)
where all the terms of the form \( f(0, 0, \ldots, 0) \), which are equal to zero according to (3), have not been written. Replacing \( x_1 \) by 0 in (3) gives
\[
\begin{align*}
&f(0, 0, \ldots, 0, x_2) + \cdots + f(0, x_2, \ldots, x_n) \\
&+ f(x_2, x_3, \ldots, x_n, 0) + \cdots + f(x_n, 0, \ldots, 0) \\
&= x_2 + \cdots + x_n.
\end{align*}
\]
(8)
Subtracting (8) from (7) yields (5).

Condition (5) is obviously sufficient since, when summed on a cyclic configuration, all the left-hand side terms except the first cancel. □

**Remark 2.1.** The above proof shows that if we can verify that a CA rule \( f \) is number-conserving for all cyclic configurations of length \( 2n - 1 \), then it is number-conserving for all cyclic configurations of length \( L > 2n - 1 \).

The following corollaries are simple necessary conditions for a CA rule to be number-conserving.

**Corollary 2.1.** If \( f \) is a one-dimensional \( q \)-state \( n \)-input number-conserving CA rule, then, for all \( x \in \mathbb{Q} \),
\[
f(x, x, \ldots, x) = x.
\]
(9)
To prove (9), which is a generalization of (6), write Condition (5) for \( x_1 = x_2 = \cdots = x_n = x \). □

**Corollary 2.2.** If \( f \) is a one-dimensional \( q \)-state \( n \)-input number-conserving CA rule, then,
\[
\sum_{(x_1, x_2, \ldots, x_n) \in \mathbb{Q}^n} f(x_1, x_2, \ldots, x_n) = \frac{1}{2} (q - 1) q^n.
\]
(10)
When we the sum (9) over \((x_1, x_2, \ldots, x_n) \in \mathbb{Q}^n\), all the left-hand side terms except the first cancel, and the sum over the remaining term is equal to \((0 + 1 + 2 + \cdots + (q - 1))q^{n-1} = \frac{1}{2} (q - 1) q^n\). □

It is possible to give an interesting alternative proof of Relation (10). Consider a De Bruijn cycle of length \( q^n \). Such a cycle contains all the different \( n \)-tuples \((x_1, x_2, \ldots, x_n) \in \mathbb{Q}^n\). The existence of De Bruijn cycles is related to the existence of Eulerian circuits on a De Bruijn graph \( G_{n-1} \). Such a graph has \( q^{n-1} \) vertices and \( q^n \) arcs. The number of arcs leaving a vertex is the same as the number which arrive. Each vertex is labeled by an \((n-1)\)-tuple over \( \mathbb{Q} \), and the arc joining the vertex \((x_1, x_2, \ldots, x_{n-1}) \) to the vertex \((x_2, \ldots, x_{n-1}, x_n) \) is labeled \((x_1, x_2, \ldots, x_{n-1}, x_n) \). For example,
\[
(0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1)
\]
is a De Bruijn cycle for \( q = 2 \) and \( n = 4 \). It can be shown that there exist \( q^{-n}(q!q^{n-1}) \) distinct De Bruijn cycles. Since all the elements of \( Q \) appear an equal number of times in a De Bruijn cycle, Condition (1) implies

\[
\sum_{(x_1,x_2,\ldots,x_n) \in Q^n} f(x_1,x_2,\ldots,x_n) = (0 + 1 + 2 + \cdots + (q-1)) q^{n-1},
\]

and (10) follows. \( \square \)

### 3. Examples of number-conserving rules

Using Condition (3) we can determine all the number-conserving CA rules for fixed values of \( q \) and \( n \). These rules will be considered as operators governing the discrete dynamics of systems of particles whose total number is conserved. The particles occupy the cells of a one-dimensional periodic lattice subject to the condition that, at a given time, no more than \( q-1 \) particles may occupy the same cell. Among these rules some have similar properties, which can be exhibited using the operators of reflection and conjugation. These two operators denoted, respectively by \( R \) and \( C \), are defined on the set of all one-dimensional \( q \)-state \( n \)-input cellular automaton rules by

\[
R f(x_1,x_2,\ldots,x_n) = f(x_n,x_{n-1},\ldots,x_1)
\]

\[
C f(x_1,x_2,\ldots,x_n) = q - 1 - f(q-1-x_1,q-1-x_2,\ldots,q-1-x_n).
\]

It is clear that, if \( f \) is number-conserving, then \( R f, C f, \) and \( RC f = CR f \) have the same property. Rules \( f \) and \( R f \) govern identical dynamics, the only difference is that, if for one rule particles flow to the right, for the other rule, they will flow in the opposite direction. To understand the difference between rules \( f \) and \( C f \), let us assume that a cell, which contains \( k \) particles \( (1 \leq k \leq q-1) \) contains \( q-1-k \) holes. Then, conjugation may be viewed as exchanging the roles of particles and holes. That is, if \( f \) describes a specific motion of particles then \( C f \) describes the same rule, but for the motion of holes.

When the number of states and number of inputs are not very small, the dynamics of the particles is not clearly exhibited by the rule table of a number-conserving CA rule. A simpler and more visual picture of the rule is given by its motion representation. Such a motion representation may be defined as follows. List all the neighborhoods of a given occupied site represented by its site value \( s \). Then, for each neighborhood, indicate the displacements of the \( s \) particles by a nondecreasing sequence of \( s \) integers \( (v_1, v_2, \ldots, v_s) \) representing the different velocities of the \( s \) particles. Velocities are positive if particles move to the right and negative if they move to the left. Alternatively, to the sequence \( (v_1, v_2, \ldots, v_s) \), we can substitute arrow(s) joining the site where are initially located the \( s \) particles to the final positions of the particle(s). A number above the arrow indicates how many particles are moving to this final position. To simplify a bit more these representations, we only list neighborhoods for which, at least one velocity \( v_j \) \( (j = 1, 2, \ldots, s) \) is different from zero. For example, these two forms of the motion representation of 2-state 3-input Rule 184 are

\[
10 \ (1) \quad \text{and} \quad 10.
\]
Since, for Rule 184, particles move only to the right, there is no need to indicate the state of the left neighboring site of the particle. Many other examples of motion representations for 2-state 4- and 5-input rules are given in [7].

**Remark 3.1.** In many applications, as one-way road car traffic, which prohibits passing, we have to assume that the particles are distinguishable. We have, therefore, to label them using an increasing sequence of integers, in order to be able to follow the motion of each individual particles. Instead of describing formally the labeling process, we will give a simple example. Suppose we have the configuration:

\[
\cdots 0 \ 1 \ 0 \ 2 \ 0 \ 2 \ 0 \ \cdots
\]

We first replace each particle in each occupied cell by the symbol •

\[
\cdots \bullet \bullet \bullet \bullet \ \cdots
\]

and then label consecutively all the • to obtain:

\[
\cdots 1 \ 2 \ 3 \ 4 \ 5 \ \cdots
\]

If now, we assume that the motion representation is

\[
0 \ 1 \ (-1) \ 0 \ 2 \ (-1,1) \ 0 \ 2 \ (-1,1) \ 1 \ 2 \ (0,1) \ 12 \ (0,1)
\]

\[
2 \ 2 \ (0,1) \ 2 \ 2 \ (1,0) \ 0 \ 2 \ (-1,0) \ 0 \ 2 \ (-1,0)
\]

then, applying this rule, we obtain the new labeled configuration:

\[
\cdots 1 \ 2 \ 3 \ 4 \ 5 \ \cdots
\]

which corresponds to the new configuration:

\[
\cdots 1 \ 0 \ 1 \ 0 \ 2 \ 0 \ 1 \ \cdots
\]

Note that the above labeling convention assumes that particles cannot jump above each other, and, therefore, the ordering of labels remains unchanged after each iteration. We use this convention to ensure uniqueness of the motion representation for number-conserving CA rules.

### 3.1. Three-state two-input number-conserving rules

There are only 4 three-state two-input number-conserving rules, which are listed in Table 1.

For radii pair \( (r_t, r_r) \), we have two possible choices, either \((0, 1)\) or \((1, 0)\). In each case, we have chosen the pair simplifying most the motion representation.
| rule number | base 3 representation | \((r_L, r_R)\) |
|-------------|------------------------|-------------|
| 19305       | 222111000              | (0,1)       |
| 15897       | 210210210              | (1,0)       |
| 18561       | 221110110              | (1,0)       |
| 16641       | 211211100              | (0,1)       |

Table 1. Three-state two-input minimal number-conserving rules.

Rules 19305 and 15897 coincide with the identity. The motion representations of Rules 18561 and 16641, which are obtained from one another by reflection, are, respectively,

\[ 20 \ (0,1), \ 21 \ (0,1) \text{ and } 02 \ (-1,0), \ 12 \ (-1,0), \]

or

\[ \overset{\wedge}{20}, \ \overset{\wedge}{21}, \text{ and } \overset{\wedge}{02}, \ \overset{\wedge}{12}. \]

For both rules, if a site is occupied by 2 particles, one of them move to a neighboring site, except if this site is already occupied by 2 particles.

### 3.2. Three-state three-input number-conserving rules

There are 144 three-state three-input number-conserving rules. They can be divided into 48 equivalence classes under the dihedral group generated by the operators \( R \) and \( C \). Each class will be represented by the rule having the smallest rule number, and which will be called the minimal rule of the class. Tables 2 and 3 list rule numbers and corresponding base-3 representations of all 48 minimal rules.

Among these minimal rules, 18 eventually emulate the identity. Their reference numbers are: 27, 28, 29, 30, 31, 32, 33, 35, 37, 39, 40, 42, 43, 44, 45, 46, 47, 48. The motion representations of the remaining 30 rules are given in Table 4.

In order to have a global view of the properties of the various minimal rules, Figures 1 and 2 represent the flow diagrams of the 30 minimal rules whose motion representations are listed in Table 4. If \( \rho \) is the average particles density and \( v_{av} \) the average particles velocity, a flow diagram shows how the flow \( \rho v_{av} \) varies as a function of \( \rho \). They are common practice in car traffic theory.

Concerning these flow diagrams, one point has to be stressed. For \( q > 2 \), that is, if more than one particle can occupy a site, it is not always possible to choose the left and right radii of a rule so as to have \( v_{av} = 0 \) for \( \rho = 1 \). This is due to the fact that, choosing \((r_L, r_R) = (1,1)\), for some rules, \( \rho v_{av} = -0.5 \) when \( \rho = 1 \). Any other choice will either increase or decrease \( v_{av} \) by one unit.

The flow of the rules which emulate the identity, is equal to zero for all values of \( \rho \).

**Remark 3.2.** All the rules for which the flow is non-negative for all values of the average density, could be *a priori* considered as deterministic one-lane high-
| reference number | rule number     | base-3 representation             |
|------------------|----------------|-----------------------------------|
| 1                | 6159136430181  | 210210210210210210210210210210210 |
| 2                | 6159523870341  | 2102102112102102112102111021021100 |
| 3                | 6169984499133  | 210211211210211100210211100211100 |
| 4                | 6171146819301  | 21021121212102111102101001100000 |
| 5                | 6171534259461  | 21021122210211112112101100000000 |
| 6                | 6181994888253  | 21021222221021200021010100000000 |
| 7                | 6201360244413  | 21021111021110211102112112111000 |
| 8                | 6202522564581  | 21021111021111021100211211121100 |
| 9                | 6202910004741  | 21021111121111021111021121110000 |
| 10               | 6213370633533  | 21021111121111021111121112110000 |
| 11               | 6436931290101  | 21121010021121021121121010000000 |
| 12               | 6447391918893  | 2112111002111211102112111100110 |
| 13               | 6448554239061  | 2112111110211211211021111000110 |
| 14               | 6448941679221  | 21121111121121111121111211100000 |
| 15               | 6450878870973  | 21121200211211211210001100000000 |
| 16               | 6452428631301  | 21121121121112112110010010000000 |
| 17               | 6459402308013  | 21121111121121120002111010000000 |
| 18               | 6462889260093  | 21121121211212120010010100000000 |
| 19               | 6478767664173  | 211212100211111002121212121 |
| 20               | 6480317424501  | 211211112111101112111100000000 |
| 21               | 6482254616253  | 21121200211102001002212000000000 |
| 22               | 6483804376581  | 21121211211102111001010000000000 |
| 23               | 6490778053293  | 21122211211110002111110000000000 |
| 24               | 6491940373461  | 21122212121111010211000000100000 |

Table 2. Three-state three-input minimal number-conserving rules, Part 1.
| reference number | rule number | base-3 representation |
|------------------|-------------|------------------------|
| 25               | 6494265005373 | 211222211211111100100111100 |
| 26               | 6495473235541 | 212222211211111110100011100 |
| 27               | 6726349098981 | 21221100021221111212100000 |
| 28               | 6729836051061 | 212211100212211110110111100 |
| 29               | 673689727773 | 212212000212212000212101000 |
| 30               | 6740296679853 | 212212100212212100101011100 |
| 31               | 6757724844261 | 212221000212111110011111100 |
| 32               | 6761211796341 | 212221000212111110011111100 |
| 33               | 6768185473053 | 212222000212111000212111100 |
| 34               | 6769347793221 | 212222010212111010212000100 |
| 35               | 6771672425133 | 2122221002121111100111111100 |
| 36               | 6881331565845 | 220100211220211100220211100 |
| 37               | 689341954965 | 220101222220212000220101000 |
| 38               | 6912707311125 | 2201102120110100220221100 |
| 39               | 6924717700245 | 2201122222011000220111000 |
| 40               | 6956093445525 | 2201212222001000220121000 |
| 41               | 7155789856805 | 2211101002121111002212111100 |
| 42               | 7170649374725 | 22111011121221200221110100 |
| 43               | 7174263626805 | 221110121221212010110111100 |
| 44               | 7202125120005 | 221111111221111000221111000 |
| 45               | 7205612072085 | 221111212221111101101101100 |
| 46               | 7235005865285 | 22112111221010002212121000 |
| 47               | 7448156794485 | 222101000222212000222101000 |
| 48               | 7479532539765 | 222111000222111000222111000 |

Table 3. Three-state three-input minimal number-conserving rules, Part 2.
| reference number | motion representation |
|------------------|-----------------------|
| 1                | 1 (-1) 2 (-1,-1)      |
| 2                | 10 (1) 20 (-1,1) 21 (-1,0) 22 (-1,0) |
| 3                | 20 (-1,1) 21 (-1,0) 22 (-1,0) |
| 4                | 01 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0) |
| 5                | 10 (1) 11 (1) 20 (1,1) 21 (0,1) |
| 6                | 20 (1,1) 11 (1) 21 (0,1) |
| 7                | 11 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0) |
| 8                | 01 (-1) 11 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0) |
| 9                | 10 (1) 20 (1,1) 21 (0,1) |
| 10               | 20 (1,1) 21 (0,1) |
| 11               | 10 (1) 2 (-1,0) |
| 12               | 2 (-1,0) |
| 13               | 01 (-1) 2 (-1,0) |
| 14               | 10 (1) 11 (1) 20 (0,1) 21 (0,1) |
| 15               | 02 (-1,-1) 12 (-1,0) 22 (-1,0) |
| 16               | 10 (1) 11 (1) 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0) |
| 17               | 11 (1) 20 (0,1) 21 (0,1) |
| 18               | 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 11 (1) 022 (-1,0) |
| 19               | 11 (-1) 2 (-1,0) |
| 20               | 10 (1) 20 (0,1) 21 (0,1) |
| 21               | 11 (-1) 02 (-1,-1) 12 (-1,0) 22 (-1,0) |
| 22               | 10 (1) 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0) |
| 23               | 20 (0,1) 21 (0,1) |
| 24               | 01 (-1) 20 (0,1) 21 (0,1) |
| 25               | 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0) |
| 26               | 01 (-1) 020 (-1,1) 021 (-1,1) 120 (0,1) 121 (0,1) 220 (0,1) 221 (0,1) 022 (-1,0) |
| 27               | 01 (1) 21 (0,1) |
| 28               | 21 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0) |
| 29               | 11 (-1) 21 (-1) 20 (-1,1) 21 (-1,0) 22 (-1,0) |
| 30               | 21 (-1) 2 (-1,0) |

Table 4. Motion representations of the 30 minimal which do not emulate the identity.
way car traffic. Consider for instance Rule 6171534259461 (reference number 5), the second form of its motion representation is

\[
\begin{align*}
\hat{1}_{10} & \quad \hat{1}_{11} \\
\hat{2}_{20} & \quad \hat{1}_{21}.
\end{align*}
\]

That is, as many particles as possible move from one site to the right neighboring site. This rule, which is the simplest three-state rule generalizing Rule 184, may be written

\[
f(x_1, x_2, x_3) = x_2 + \min\{x_1, 2 - x_2\} - \min\{x_2, 2 - x_3\}.
\]

This expression suggests a further generalization. The \(q\)-state rule defined by, for all \((x_1, x_2, x_3) \in \{0, 1, \ldots, q - 1\}^3\),

\[
f(x_1, x_2, x_3) = x_2 + \min\{x_1, q - 1 - x_2\} - \min\{x_2, q - 1 - x_3\},
\]

can be viewed as the following car traffic rule: Each cell represents a section of a one-way road between two traffic lights. The number of states \(q\) measures the maximum capacity of that section. The above evolution rule describes the way cars move from one section to the next. As for elementary CA Rule 184,
the flow is maximal for $\rho = 0.5$. Of course, and this goes beyond our purpose here, we could, as in the Nagel–Schreckenberg model \cite{6}, introduce some noise and say that some cars, which could move to the next section, do not do so with a probability $p$. The existence of traffic lights could be used to monitor traffic in order to increase the flow, and, with this in mind, we could extend to $q$-state rules classes of models we studied in a recent paper \cite{10}.

**Remark 3.3.** As mentioned above, conjugation exchanges the roles of particles and holes. Therefore, if a rule $f$ is self-conjugate, we can always choose the rule radii $(r_+, r_-)$ such that the point $(0.5, 0)$ is a center of symmetry of its flow diagram. Rules whose reference numbers are 26 and 34 are self-conjugate and their flow diagrams have this property. In \cite{7} we studied a few 2-state self-conjugate number-conserving rules. Some of these rules, which allow motion in both directions, govern the dynamics of ensembles of one-dimensional pseudo-random walkers. Starting from a random initial configuration, whose density $\rho$ is exactly equal to 0.5, we followed, in the limit set, which is reached after a maximum of $L/2$ iterations, the position of a specific particle as a function of time on a ring of length $L = 5000$ for 5000 time steps. The walks are...
represented in Figures 3 and 4. Since $L$ is finite, the random walks are periodic in time. The first walk has a period equal to $L$, while the period of the second one is $2L$. Note that these walks are deterministic, their random character comes from the randomness of the initial configuration.

![Random walker evolving according to Rule 6495427325541 (no 26).](image1)

**Figure 3.** Random walker evolving according to Rule 6495427325541 (no 26).

![Random walker evolving according to Rule 6769347793221 (no 34).](image2)

**Figure 4.** Random walker evolving according to Rule 6769347793221 (no 34).

4. **Conclusion**

We have established a necessary and sufficient condition for a $q$-state $n$-input CA rule to be number-conserving. For given values of $q$ and $n$, this result allows to find all the rules possessing this property. As an example, we have determined all the three-state three-input number-conserving CA rules. We have listed their motion representation and studied their flow diagrams. All the rules for which the flow is non-negative for all values of the average density, can be considered as deterministic one-way road car traffic rules in which the
cells represent road sections whose car capacity is equal to $q - 1$. Among the 148 three-state three-input rules there exist two nontrivial self-conjugate rules which mimic the dynamics of an ensemble of random walkers.

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