On generalized stochastic fractional integrals and related inequalities

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Received: 29 May 2018, Revised: 13 September 2018, Accepted: 13 September 2018, Published online: 24 September 2018

Abstract The generalized mean-square fractional integrals $J_{\sigma, \rho, \lambda, u, \omega}^+$ and $J_{\sigma, \rho, \lambda, v, \omega}^-$ of the stochastic process $X$ are introduced. Then, for Jensen-convex and strongly convex stochastic processes, the generalized fractional Hermite–Hadamard inequality is established via generalized stochastic fractional integrals.

Keywords Hermite–Hadamard inequality, stochastic fractional integrals, convex stochastic process

2010 MSC 26D15, 26A51, 60G99

1 Introduction

In 1980, Nikodem [11] introduced convex stochastic processes and investigated their regularity properties. In 1992, Skwronski [17] obtained some further results on convex stochastic processes.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \rightarrow \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \rightarrow \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t, \cdot)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \rightarrow \mathbb{R}$ is called
(i) continuous in probability in interval $I$, if for all $t_0 \in I$ we have

$$P\lim_{t \to t_0} X(t, .) = X(t_0, .),$$

where $P\lim$ denotes the limit in probability.

(ii) mean-square continuous in the interval $I$, if for all $t_0 \in I$

$$\lim_{t \to t_0} E[(X(t) - X(t_0))^2] = 0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t, .)$.

Obviously, mean-square continuity implies continuity in probability, but the converse implication is not true.

**Definition 1.** Suppose we are given a sequence $\{\Delta^m\}$ of partitions, $\Delta^m = \{a_{m,0}, \ldots, a_{m,n_m}\}$. We say that the sequence $\{\Delta^m\}$ is a normal sequence of partitions if the length of the greatest interval in the $n$-th partition tends to zero, i.e.,

$$\lim_{m \to \infty} \sup_{1 \leq i \leq n_m} |a_{m,i} - a_{m,i-1}| = 0.$$

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties see [18].

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E[X(t)^2] < \infty$ for all $t \in I$. Let $[a, b] \subset I$, $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a, b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, \ldots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process $X$ on $[a, b]$, if we have

$$\lim_{n \to \infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0$$

for all normal sequences of partitions of the interval $[a, b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, n$. Then, we write

$$Y(\cdot) = \int_{a}^{b} X(s, \cdot) \, ds \text{ (a.e.).}$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process $X$.

Throughout the paper we will frequently use the monotonicity of the mean-square integral. If $X(t, \cdot) \leq Y(t, \cdot)$ (a.e.) in some interval $[a, b]$, then

$$\int_{a}^{b} X(t, \cdot) \, dt \leq \int_{a}^{b} Y(t, \cdot) \, dt \text{ (a.e.).}$$

Of course, this inequality is an immediate consequence of the definition of the mean-square integral.
Definition 2. We say that a stochastic processes $X : I \times \Omega \to \mathbb{R}$ is convex, if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality

$$X(\lambda u + (1 - \lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) \quad (\text{a.e.})$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process $X$ is Jensen-convex or $\frac{1}{2}$-convex. A stochastic process $X$ is concave if $-(X)$ is convex. Some interesting properties of convex and Jensen-convex processes are presented in [11, 18].

Now, we present some results proved by Kotrys [6] about Hermite–Hadamard inequality for convex stochastic processes.

Lemma 1. If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t, \cdot) = A(\cdot)t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E[A^2] < \infty$, $E[B^2] < \infty$ and $[a, b] \subset I$, then

$$\int_a^b X(t, \cdot)dt = A(\cdot)\frac{b^2 - a^2}{2} + B(\cdot)(b - a) \quad (\text{a.e.}).$$

Proposition 1. Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in \text{int} I$. Then there exists a random variable $A : \Omega \to \mathbb{R}$ such that $X$ is supported at $t_0$ by the process $A(\cdot)(t - t_0) + X(t_0, \cdot)$. That is

$$X(t, \cdot) \geq A(\cdot)(t - t_0) + X(t_0, \cdot) \quad (\text{a.e.})$$

for all $t \in I$.

Theorem 1. Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex, mean-square continuous in the interval $I$ stochastic process. Then for any $u, v \in I$ we have

$$X\left(\frac{u + v}{2}, \cdot\right) \leq \frac{1}{v - u} \int_u^v X(t, \cdot)dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad (\text{a.e.})$$

In [7], Kotrys introduced the concept of strongly convex stochastic processes and investigated their properties.

Definition 3. Let $C : \Omega \to \mathbb{R}$ denote a positive random variable. The stochastic process $X : I \times \Omega \to \mathbb{R}$ is called strongly convex with modulus $C(\cdot) > 0$, if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality

$$X(\lambda u + (1 - \lambda)v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda)X(v, \cdot) - C(\cdot)\lambda(1 - \lambda)(u - v)^2 \quad \text{a.e.}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process $X$ is strongly Jensen-convex with modulus $C(\cdot)$.

In [5], Hafiz gave the following definition of stochastic mean-square fractional integrals.
Definition 4. For the stochastic process $X : I \times \Omega \to \mathbb{R}$, the concept of stochastic mean-square fractional integrals $I_{u+}^{\alpha}$ and $I_{v-}^{\alpha}$ of $X$ of order $\alpha > 0$ is defined by

\[
I_{u+}^{\alpha}[X](t) = \frac{1}{\Gamma(\alpha)} \int_{u}^{t} (t-s)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t > u,
\]

and

\[
I_{v-}^{\alpha}[X](t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{v} (s-t)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t < v.
\]

Using this concept of stochastic mean-square fractional integrals $I_{u+}^{\alpha}$ and $I_{v-}^{\alpha}$, Agahi and Babakhani proved the following Hermite–Hadamard type inequality for convex stochastic processes:

Theorem 2. Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process that is mean-square continuous in the interval $I$. Then for any $u, v \in I$, the following Hermite–Hadamard inequality

\[
X \left( \frac{u + v}{2}, \cdot \right) \leq \frac{\Gamma(\alpha + 1)}{2(v-u)^{\alpha}} \left[ I_{u+}^{\alpha}[X](v) + I_{v-}^{\alpha}[X](u) \right] \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \quad (a.e.)
\]

holds, where $\alpha > 0$.

For more information and recent developments on Hermite–Hadamard type inequalities for stochastic processes, please refer to [2–4, 9–11, 14, 16, 15, 20, 19].

2 Main results

In this section, we introduce the concept of the generalized mean-square fractional integrals $J_{\rho,\lambda,u+;\omega}^{\sigma}$ and $J_{\rho,\lambda,v-;\omega}^{\sigma}$ of the stochastic process $X$.

In [13], Raina studied a class of functions defined formally by

\[
F_{\rho,\lambda}^{\sigma}(x) = F_{\rho,\lambda}^{\sigma(0),\sigma(1),\ldots}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda > 0; |x| < R),
\]

where the coefficients $\sigma(k)$ ($k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) make a bounded sequence of positive real numbers and $R$ is the set of real numbers. For more information on the function (4), please refer to [8, 12]. With the help of (4), we give the following definition.

Definition 5. Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process. The generalized mean-square fractional integrals $J_{\rho,\lambda,u+;\omega}^{\sigma}$ and $J_{\rho,\lambda,v-;\omega}^{\sigma}$ of $X$ are defined by

\[
J_{\rho,\lambda,u+;\omega}^{\sigma}[X](x) = \int_{u}^{x} (x-t)^{\lambda-1} F_{\rho,\lambda}^{\sigma}[\omega(x-t)^{\rho}] X(t, \cdot)dt, \quad (a.e.) \ x > u,
\]

and

\[
J_{\rho,\lambda,v-;\omega}^{\sigma}[X](x) = \int_{x}^{v} (t-x)^{\lambda-1} F_{\rho,\lambda}^{\sigma}[\omega(t-x)^{\rho}] X(s, \cdot)dt, \quad (a.e.) \ x < v,
\]

where $\lambda, \rho > 0, \omega \in \mathbb{R}$. 

Many useful generalized mean-square fractional integrals can be obtained by specializing the coefficients \( \sigma(k) \). Here, we just point out that the stochastic mean-square fractional integrals \( I_{a+}^\alpha \) and \( I_{b+}^\alpha \) can be established by coosing \( \lambda = \alpha, \sigma(0) = 1 \) and \( w = 0 \).

Now we present Hermite–Hadamard inequality for generalized mean-square fractional integrals \( \mathcal{I}_{\rho,\lambda}^{\sigma,a+\omega} \) and \( \mathcal{I}_{\rho,\lambda}^{\sigma,a-\omega} \) of \( X \).

**Theorem 3.** Let \( X : I \times \Omega \rightarrow \mathbb{R} \) be a Jensen-convex stochastic process that is mean-square continuous in the interval \( I \). For every \( u, v \in I \), \( u < v \), we have the following Hermite–Hadamard inequality:

\[
X \left( \frac{u + v}{2} \right) \leq \frac{1}{2(v - u)\mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v - t)^\rho]} \left[ \mathcal{I}_{\rho,\lambda}^{\sigma,a+\omega} [X](t) + \mathcal{I}_{\rho,\lambda}^{\sigma,a-\omega} [X](t) \right] \leq \frac{X(u, \cdot) + X(v, \cdot)}{2}, \text{ a.e. (7)}
\]

**Proof.** Since the process \( X \) is mean-square continuous, it is continuous in probability. Nikodem [11] proved that every Jensen-convex and continuous in probability stochastic process is convex. Since \( X \) is convex, then by Proposition 1, it has a supporting process at any point \( t_0 \in int I \). Let us take a support at \( t_0 = \frac{u + v}{2} \), then we have

\[
X(t, \cdot) \geq A(\cdot) \left( t - \frac{u + v}{2} \right) + X \left( \frac{u + v}{2}, \cdot \right), \text{ a.e. (8)}
\]

Multiplying both sides of (8) by \([ (v - t)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v - t)^\rho] + (t - u)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(t - u)^\rho]) \], then integrating the resulting inequality with respect to \( t \) over \([u, v]\), we obtain

\[
\int_u^v [(v - t)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v - t)^\rho] + (t - u)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(t - u)^\rho] ] X(t, \cdot) dt \geq A(\cdot) \int_u^v [(v - t)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v - t)^\rho] \\
+ (t - u)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(t - u)^\rho] ] \left( t - \frac{u + v}{2} \right) dt \\
+ X \left( \frac{u + v}{2}, \cdot \right) \int_u^v [(v - t)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v - t)^\rho] \\
+ (t - u)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(t - u)^\rho] ] dt \\
= A(\cdot) \int_u^v [t(v - t)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v - t)^\rho] + t(t - u)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(t - u)^\rho] ] dt \\
- A(\cdot) \frac{u + v}{2} \int_u^v [(v - t)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(v - t)^\rho] + (t - u)\lambda^{-1} \mathcal{F}_{\rho,\lambda}^{\sigma} [\omega(t - u)^\rho] ] dt
\]
\[ + X \left( \frac{u + v}{2}, \cdot \right) \int_u^v [(v - t)^{\lambda - 1} F^{\sigma}_{\rho, \lambda} [\omega(v - t)^\rho] + (t - u)^{\lambda - 1} F^{\sigma}_{\rho, \lambda} [\omega(t - u)^\rho]] dt. \]

Calculating the integrals, we have

\[ \int_u^v t(v - t)^{\lambda - 1} F^{\sigma}_{\rho, \lambda} [\omega(v - t)^\rho] dt \]

\[ = \int_u^v (v - t)^\lambda F^{\sigma}_{\rho, \lambda} [\omega(v - t)^\rho] dt + v \int_u^v (v - t)^{\lambda - 1} F^{\sigma}_{\rho, \lambda} [\omega(v - t)^\rho] dt \]

\[ = -(v - u)^{\lambda + 1} F^{\sigma_1}_{\rho, \lambda} [\omega(v - u)^\rho] + v(v - u)^\lambda F^{\sigma}_{\rho, \lambda + 1} [\omega(v - u)^\rho] \]

and similarly,

\[ \int_u^v t(t - u)^{\lambda - 1} F^{\sigma}_{\rho, \lambda} [\omega(t - u)^\rho] dt \]

\[ = \int_u^v (t - u)^\lambda F^{\sigma}_{\rho, \lambda} [\omega(t - u)^\rho] dt + u \int_u^v (t - u)^{\lambda - 1} F^{\sigma}_{\rho, \lambda} [\omega(t - u)^\rho] dt \]

\[ = (v - u)^{\lambda + 1} F^{\sigma_1}_{\rho, \lambda} [\omega(v - u)^\rho] + u(v - u)^\lambda F^{\sigma}_{\rho, \lambda + 1} [\omega(v - u)^\rho] \]

where \( \sigma_1(k) = \frac{\sigma(k)}{\rho k + \lambda + 1}, k = 0, 1, 2, \ldots \) Using the identities (10) and (11) in (9), we obtain

\[ J^{\sigma}_{\rho, \lambda, u+\omega}[X](t) + J^{\sigma}_{\rho, \lambda, v-\omega}[X](t) \]

\[ \geq A(\cdot)(u + v)(v - u)^\lambda F^{\sigma}_{\rho, \lambda + 1} [\omega(v - u)^\rho] \]

\[ - A(\cdot) \frac{u + v}{2} 2(v - u)^\lambda F^{\sigma}_{\rho, \lambda + 1} [\omega(v - u)^\rho] \]

\[ + X \left( \frac{u + v}{2}, \cdot \right) 2(v - u)^\lambda F^{\sigma}_{\rho, \lambda + 1} [\omega(v - u)^\rho] \]

\[ = X \left( \frac{u + v}{2}, \cdot \right) 2(v - u)^\lambda F^{\sigma}_{\rho, \lambda + 1} [\omega(v - u)^\rho]. \]

That is,

\[ X \left( \frac{u + v}{2}, \right) \]

\[ \leq \frac{1}{2(v - u)^\lambda F^{\sigma}_{\rho, \lambda + 1} [\omega(v - u)^\rho]} [J^{\sigma}_{\rho, \lambda, u+\omega}[X](t) + J^{\sigma}_{\rho, \lambda, v-\omega}[X](t)] \quad \text{a.e.}, \]

which completes the proof of the first inequality in (7).
By using the convexity of $X$, we get

$$X(t, \cdot) = X\left(\frac{v-t}{v-u}u + \frac{t-u}{v-u}v, \cdot\right) \leq \frac{v-t}{v-u}X(u, \cdot) + \frac{t-u}{v-u}X(v, \cdot)$$

$$= \frac{X(v, \cdot) - X(u, \cdot)}{v-u} + \frac{X(u, \cdot) + X(v, \cdot)u}{v-u}$$

for $t \in [u, v]$. Using the identities (10) and (11), it follows that

$$\int_u^v \left[ (v-t)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [\omega(v-t)^\rho] + (t-u)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [\omega(t-u)^\rho] \right] X(t, \cdot) dt$$

$$\leq \frac{X(v, \cdot) - X(u, \cdot)}{v-u} \left( u + v \right) \frac{(v-t)^{\lambda} \mathcal{F}_{\rho, \lambda}^\sigma [\omega(v-u)^\rho]}{v-u}$$

$$\times \int_u^v \left[ (v-t)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [\omega(v-t)^\rho] + (t-u)^{\lambda-1} \mathcal{F}_{\rho, \lambda}^\sigma [\omega(t-u)^\rho] \right] dt$$

$$\frac{X(u, \cdot) + X(v, \cdot)u}{v-u} \left( 2(v-u)^\lambda \mathcal{F}_{\rho, \lambda}^\sigma [\omega(v-u)^\rho] \right)$$

$$= \left[ X(u, \cdot) + X(v, \cdot) \right] \left( v - u \right)^\lambda \mathcal{F}_{\rho, \lambda}^\sigma [\omega(v-u)^\rho].$$

That is,

$$\frac{1}{2(v-u)^\lambda \mathcal{F}_{\rho, \lambda}^\sigma [\omega(v-u)^\rho]} \left[ J_{\rho, \lambda, \omega}^\sigma [X](t) + J_{\rho, \lambda, \omega}^\sigma [X](t) \right]$$

$$\leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \text{ a.e.},$$

which completes the proof. \hfill \Box

**Remark 1.** i) Choosing $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 3, the inequality (7) reduces to the inequality (3).

ii) Choosing $\lambda = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 3, the inequality (7) reduces to the inequality (2).

**Theorem 4.** Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process, which is strongly Jensen-convex with modulus $C(\cdot)$ and mean-square continuous in the interval $I$ so that $E[C^2] < \infty$. Then for any $u, v \in I$, we have

$$X\left( \frac{u + v}{2}, \cdot \right)$$
\[
-C(\cdot) \left\{ 2(v-u)^{\lambda+2} F_{\rho,\lambda}^{\sigma_2} [\omega(v-u)^{\rho}] - 2(v-u)^{\lambda} F_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}] \right\} \\
+ (u^2 + v^2)(v-u)^{\lambda} F_{\rho,\lambda+1}^{\sigma_1} [\omega(v-u)^{\rho}] - \left( \frac{u+v}{2} \right)^2 \\
\leq \frac{1}{2(v-u)^{\lambda} F_{\rho,\lambda+1}^{\sigma_1} [\omega(v-u)^{\rho}]} \left[ J_{\rho,\lambda,u+\omega}^{\sigma} [X](t) + J_{\rho,\lambda,v-\omega}^{\sigma} [X](t) \right] \\
\leq \frac{X(u,\cdot) + X(v,\cdot) - C(\cdot) \left( \frac{u^2 + v^2}{2} \right)(v-u)^{\lambda} F_{\rho,\lambda+1}^{\sigma_1} [\omega(v-u)^{\rho}] - 2(v-u)^{\lambda} F_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}] }{2} \quad \text{a.e.}
\]

**Proof.** It is known that if \( X \) is strongly convex process with the modulus \( C(\cdot) \), then the process \( Y(t,\cdot) = X(t,\cdot) - C(\cdot)t^2 \) is convex [7, Lemma 2]. Applying the inequality (7) for the process \( Y(t,\cdot) \), we have

\[
Y \left( \frac{u+v}{2}, \cdot \right) \leq \frac{1}{2(v-u)^{\lambda} F_{\rho,\lambda+1}^{\sigma_1} [\omega(v-u)^{\rho}]} \int_{u}^{v} \left[ (v-t)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(v-t)^{\rho}] + (t-u)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(t-u)^{\rho}] \right] Y(t,\cdot) dt \\
\leq \frac{X(u,\cdot) + X(v,\cdot) - C(\cdot) \left( \frac{u^2 + v^2}{2} \right)(v-u)^{\lambda} F_{\rho,\lambda+1}^{\sigma_1} [\omega(v-u)^{\rho}] - 2(v-u)^{\lambda} F_{\rho,\lambda}^{\sigma_1} [\omega(v-u)^{\rho}] }{2} \quad \text{a.e.}
\]

That is

\[
X \left( \frac{u+v}{2}, \cdot \right) - C(\cdot) \left( \frac{u+v}{2} \right)^2 \\
\leq \frac{1}{2(v-u)^{\lambda} F_{\rho,\lambda+1}^{\sigma_1} [\omega(v-u)^{\rho}]} \left\{ \int_{u}^{v} \left[ (v-t)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(v-t)^{\rho}] + (t-u)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(t-u)^{\rho}] \right] X(t,\cdot) dt \\
- C(\cdot) \int_{u}^{v} \left[ t^2 (v-t)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(v-t)^{\rho}] + t^2 (t-u)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(t-u)^{\rho}] \right] dt \right\} \\
\leq \frac{X(u,\cdot) - C(\cdot)u^2 + X(v,\cdot) - C(\cdot)v^2}{2} \quad \text{a.e.}
\]

Calculating the integrals, we obtain

\[
\int_{u}^{v} t^2 (v-t)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(v-t)^{\rho}] dt \\
= \int_{u}^{v} t^2 (v-t)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(v-t)^{\rho}] dt + \int_{u}^{v} t^2 (v-t)^{\lambda-1} F_{\rho,\lambda}^{\sigma_2} [\omega(v-t)^{\rho}] dt
\]
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Choosing Remark 2.

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Then

Theorem 7 in

This completes the proof.

and similarly,

where \( \sigma_2(k) = \frac{\sigma(k)}{\rho k + \lambda + 2} \), \( k = 0, 1, 2, \ldots \). Then it follows that

\[
X \left( \frac{u + v}{2}, \cdot \right) \leq \frac{1}{2(v - u)^\lambda} \mathcal{F}_{\rho, \lambda, 1}[\omega(v - u)^\rho] \{ J_{\rho, \lambda, a + \cdot; \omega}[X](t) + J_{\rho, \lambda, b - \cdot; \omega}[X](t) \\ - C(\cdot) \left[ 2(v - u)^\lambda \mathcal{F}_{\rho, \lambda, 1}[\omega(v - u)^\rho] \right] + 2(v - u)^\lambda \mathcal{F}_{\rho, \lambda, 1}[\omega(v - u)^\rho] \} \] a.e.

Then

\[
X \left( \frac{u + v}{2}, \cdot \right) - C(\cdot) \left\{ u^2 + v^2 \right\} \leq \frac{1}{2(v - u)^\lambda} \mathcal{F}_{\rho, \lambda, 1}[\omega(v - u)^\rho] \{ J_{\rho, \lambda, u + \cdot; \omega}[X](t) + J_{\rho, \lambda, v - \cdot; \omega}[X](t) \\ - C(\cdot) \left\{ u^2 + v^2 \right\} \right\} \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} - C(\cdot) \left\{ u^2 + v^2 \right\} a.e.

This completes the proof.

Remark 2. Choosing \( \lambda = \alpha \), \( \sigma(0) = 1 \) and \( w = 0 \) in Theorem 4, it reduces to Theorem 7 in [1].
Acknowledgments

Authors thank the reviewer for his/her thorough review and highly appreciate the comments and suggestions.

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