Exploring the dynamics of three species delayed food
chain model with harvesting

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Abstract. This paper deals with the dynamics of a food chain model consisting of three
species (prey, middle-level predator and top-level predator). It is assumed that the top predator
grows by sexual reproduction and prey is harvested via nonlinear harvesting strategy. Further
delay is imposed to utilize the impact that prey takes some time to convert the food into its
growth. The essential conditions for local asymptotic stability (LAS) and Hopf bifurcation of
proposed model is derived. The derived results are examined through numerical examples.

Keywords: Food-chain model; Stability analysis; Hopf bifurcation; Prey harvesting.

1. Introduction
The relationship between prey and predator is quite common in nature and thus the field of
mathematical ecology has been receiving special attention among researchers in the past few
decades, see [1,2]. The interference among predator is common when predators compete for food
and which is lacking in most commonly used Holling-II functional response. To cope this issue,
Beddington-DeAngelis functional response is similar to Holling II but it has an extra term in the
denominator representing the mutual interference among predators, which was introduced by
Beddington and DeAngelis in [3,4]. Meantime, the existence of time delay in biological systems
is unavoidable due to maturation time, gestation time, handling and digesting time, etc. The
occurrence of delays in dynamical systems may enrich the complexity of its dynamics. Thus to
account for this in the predator-prey model, a variety of novel results has been given in [5,6].
On the other hand, owing to the view point of economic profit, harvesting of species commonly
exists in fishery and forestry, etc. Enthused by this fact, it is significant to analyze the harvesting
of species in prey-predator models and some novel results have been found in [7–9]. Among
different harvesting strategies discussed in literature, nonlinear harvesting is more realistic from
the economical as well as biological point of view rather than the others [9].

The interaction between more than two species (more than one prey and/or predators) is
modeled by food chain models. There are various types of food chain models consisting of three
species found in [10–14]. One type is as follows: the third species (top predator) feeds upon
the second one (intermediate predator), while the second species feeds on the first one (prey).
To study this model, Upadhyay et al. [10] discuss on the food chain model with consuming
rates for intermediate predator through Holling II type and top predator through Beddington-
DeAngelis functional response, where they derived the conditions for local and global stability.
By introducing feedback control mechanism, the problem of controlling chaos in food chain model
was attempted in [11]. The stability and Hopf bifurcation analysis for three species model with
gestation and Hutchinson delays have been studied in [12]. The problem of qualitative analysis
of three species model with Holling IV functional response and constant effort harvesting has
been investigated in [13]. Authors in [14] have considered the Holling-Tanner type of food
chain model with delays and they showed that the model undergoes periodic, multi-periodic
and chaotic motions via delay parameter. Main contribution of this paper is as follows: (1)
the nonlinear harvesting strategy associated with the impact of prey harvesting is taken into
account of this work. (2) The delay is incorporated to utilize the issue that prey takes some
time to convert the food into its growth. (3) Furthermore, to understand the more complex
qualitative behaviors of food chain model, more works are needed and this facts has motivated
our present study.

2. Mathematical Model
Let \( N(t), P(t) \) and \( Q(t) \) are the prey, middle-level predator and top-level predator densities,
respectively. We assume the following:

- The species \( N \) grows logistically with per capita growth rate \( R \) to its carrying capacity \( K \).
- The prey species are continuously harvested through nonlinear harvesting strategy with
catchability coefficient \( F \) and external applied effort \( E \).
- The delay \( \tau \) is taken into account to utilize the fact that the prey takes some
time lag to convert the food into its growth.
- The species \( P \) consumes the prey as \( N \) only at the bottom level according to Holling type II.
- The species \( Q \) grows quadratically because of sexual reproduction and it consumes the prey
as species \( P \) at the second level through Beddington-DeAngelis functional.

Then, the system to be discussed is:

\[
\begin{align*}
\frac{dN(t)}{dT} &= RN(t) \left( 1 - \frac{N(t-\tau)}{K} \right) - \frac{CN(t)P(t)}{A + N(t)} - \frac{FEN}{m_1E + m_2N}, \\
\frac{dP(t)}{dT} &= \frac{C_1N(t)P(t)}{A_1 + N(t)} - D_1P(t) - \frac{C_2P(t)Q(t)}{A_2 + P(t) + B_1Q(t)}, \\
\frac{dQ(t)}{dT} &= D_2Q(t)^2 - \frac{C_3Q(t)^2}{A_3 + P(t)},
\end{align*}
\]

where \( C, C_1, C_2, C_3 \) are the positive constants; \( A, A_2 \) and \( A_3 \) are the measure of environment
protection to the prey, intermediate predator and top predator \( Q \) respectively; \( B_1 \) represents the
interference among top predator. By using the transformations \( N \to Kn, P \to \frac{KP}{C}p, Q \to \frac{KQ}{CC_3}q \)
\( T \to \frac{1}{R}t \) and let \( \alpha = \frac{A}{K}, \alpha_1 = \frac{A_1}{K}, \alpha_2 = \frac{A_2C}{KR}, \alpha_3 = \frac{A_3C}{KK}, \beta_1 = \frac{BB_1}{C_2}, \gamma_1 = \frac{C_1}{N}, \gamma_3 = \frac{C_3}{C_2}, \delta_1 = \frac{B_1}{R}, \delta_2 = \frac{BB_2}{CC_3}, \mu = \frac{FE}{R}, h = \frac{m_1E}{m_2R} \) then the system (1) reduced to

\[
\begin{align*}
\frac{dn}{dt} &= n \left( 1 - n(t-\tau) - \frac{p}{\alpha + n} - \frac{\mu}{h + n} \right), \\
\frac{dp}{dt} &= \frac{\gamma_1 np}{\alpha_1 + n} - \frac{\delta_1 p}{\alpha_2 + p + \beta_1 q}, \\
\frac{dq}{dt} &= \delta_2 q^2 \left( 1 - \frac{\gamma_3}{\alpha_3 + p} \right),
\end{align*}
\]

subject to the initial conditions \( n_0 = \phi_1(\theta) > 0, p_0 = \phi_2(\theta) > 0, q_0 = \phi_3(\theta) > 0, \forall \theta \in [-\tau, 0] \).
2.1. Existence of Equilibria

Model (2) possesses four non-negative equilibrium points which are given by

\[ 1 - n - \frac{p}{\alpha + n} - \frac{\mu}{q + n} = 0, \]
\[ \frac{\gamma_1 n}{\alpha_1 + n} - \delta_1 - \frac{\alpha_2 + p + \beta_1 q}{\alpha_3 + p} = 0, \]
\[ 1 - \frac{\gamma_2 n}{\alpha_1 + n} = 0. \]

Solving the above equations, we get the following equilibria:

(i) The origin \( S_0(0, 0, 0) \), which is always exists.

(ii) The predators (intermediate and top) free equilibrium point \( S_1^T(n^T, 0, 0) \), where \( n^T = \frac{1}{2} [1 - h \pm \sqrt{(1 - h)^2 - 4(\mu - h)}] \). If \( \mu > h \), both equilibriums \( S_1^- \) and \( S_1^+ \) exist when \( 1 > h \) and \( (1 - h)^2 > 4(\mu - h) \) while if \( \mu < h \) then \( S_1^+ \) only exists.

(iii) The top-predator free equilibrium point \( S_2(\bar{n}, \bar{p}, 0) \), where \( \bar{p} = \frac{\alpha_1 + n}{n + h}((1 - \bar{n})(h + n) - \mu) \) and \( \bar{n} = \frac{\delta_1 \alpha_1}{\gamma_1 - \delta_1} \). If \( \gamma_1 > \delta_1 \) and \( h + \bar{n} > \frac{\mu}{1 - \bar{n}} \), then \( S_2 \) exists.

(iv) The coexistence equilibrium point \( S_+ (n^*, p^*, q^*) \), where \( p^* = \gamma_3 - \alpha_3 \) and \( n^* \) is a positive root of following equation

\[ -n^3 + (1 - \alpha - h)n^2 + (\alpha - \alpha h - \mu + h - p^*)n + \alpha h - \alpha \mu - h p^* = 0, \]

and \( q^* = \frac{\delta_1 (\alpha_1 + n^*)(\alpha_2 + p^*) - \gamma_1 (\alpha_2 + p^*)}{\gamma_1 \beta_1 - \beta_1 (\alpha_1 + n^*) - (\alpha_2 + p^*)} \). Hence the equilibrium point \( S_+ \) exists if: \( \alpha h > \alpha \mu - h p^* \), \( \gamma_3 > \alpha_3 \) and \( \frac{\gamma_1 n^*}{\alpha_3 + p} - \delta_1 < \frac{1}{\gamma_1} \).

3. The model without delay

Now we focuses on deriving the sufficient conditions for LAS and Hopf bifurcation of (2) without time-delay. Then the model (2) becomes

\[
\begin{align*}
\frac{dx}{dt} &= n \left( 1 - n - \frac{p}{\alpha + n} - \frac{\mu}{q + n} \right), \\
\frac{dy}{dt} &= \gamma_1 np - \delta_1 p - \frac{\beta_1 q}{\alpha_3 + p}, \\
\frac{dz}{dt} &= \frac{\gamma_2 q}{\alpha_1 + n} \left( 1 - \frac{\gamma_3}{\alpha_3 + p} \right). 
\end{align*}
\]

To study the LAS of model (5) about equilibrium points, the Jacobian matrix of (5) at any point \( S(n, p, q) \) is given by

\[
J = \begin{pmatrix}
1 - 2n - \frac{ap}{(\alpha + n)^2} - \frac{\mu h}{(h + n)^2} & -\frac{\gamma_1 n}{\alpha_1 + n} - \delta_1 - \frac{q(\alpha_2 + \beta_1 q)}{(\alpha_2 + p + \beta_1 q)^2} & 0 \\
\gamma_1 n & 0 & \frac{\gamma_2 q^2}{(\alpha_3 + p)^2} \\
0 & 2\delta_1 & 0
\end{pmatrix}
\]

3.1. Local Stability

The local dynamics of \( S_0, S_1 \) and \( S_2 \) are given as follows:

i. The variational matrix at \( S_0(0, 0, 0) \) is given by \( J_{S_0} = \begin{pmatrix} 1 - \frac{\mu}{h} & 0 & 0 \\
0 & -\delta_1 & 0 \\
0 & 0 & 0 \end{pmatrix} \). The corresponding eigenvalues are \( \frac{h - \mu}{h}, -\delta_1, 0 \). The non-hyperbolic equilibrium \( E_0 \) is having unstable manifold along \( p \)-direction if \( h > \mu \).
ii. Evaluation of $J$ at $S_1^\pm$ is given by $J_{S_1^\pm} = \begin{pmatrix} n^+ \left(1 - \frac{\mu}{(h+n^+)^2}\right) & -\frac{1}{\alpha+n^+} & 0 \\ 0 & -\frac{1}{\alpha+n^+} - \delta_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The eigenvalues of $J_{S_1^-}$ are $n^- \sqrt{(1-h)^2 - 4(\mu-h)}$, $\frac{\gamma_1}{\alpha_1+n^-} - \delta_1 < 0$. The non-hyperbolic equilibria $S_1^-$ are having stable manifold along $p$-direction if $\frac{\gamma_1}{\alpha_1+n^-} < \delta_1$. Similarly the eigenvalues of $J_{S_1^+}$ are $-n^+ \sqrt{(1-h)^2 - 4(\mu-h)}$, $\frac{\gamma_1}{\alpha_1+n^+} - \delta_1 < 0$. Hence non-hyperbolic equilibrium $S_1^+$ is having stable manifold along both $n$ and $p$-directions if $\frac{\gamma_1}{\alpha_1+n^+} < \delta_1$, while $S_1^+$ is having stable manifold on $n$-direction only.

iii. Calculating $J$ at $S_2$ is given by $V_{E_2} = \begin{pmatrix} n\left(-1 + \frac{\bar{p}}{(a+n)^2} + \frac{\mu}{(h+n)^2}\right) & -\frac{n}{\alpha+n} & 0 \\ 0 & 0 & -\frac{\bar{p}}{\alpha_2+p} \\ 0 & 0 & 0 \end{pmatrix}$ and its characteristic polynomial is $\omega^3 - n \left(-1 + \frac{\bar{p}}{(a+n)^2} + \frac{\mu}{(h+n)^2}\right) \omega^2 + \frac{n}{\alpha+n} \frac{\alpha_1\bar{p}}{(a_1+n)^2} \omega = 0$. Note that the above equation must have one zero eigenvalue, say $\omega_3 = 0$ and the remaining two eigenvalues will be negative if $\frac{\bar{p}}{(a+n)^2} + \frac{\mu}{(h+n)^2} < 1$. Thus non-hyperbolic equilibrium $S_2$ is having stable manifold along both $n$ and $p$-directions if $\frac{\bar{p}}{(a+n)^2} + \frac{\mu}{(h+n)^2} < 1$.

Next, we deal the LAS of the positive equilibrium $S_*$. The Jacobian matrix of model (5) at $S_*(n^*, p^*, q^*)$ is given by

$$J_{S_*} = \begin{pmatrix} -n^* + \frac{n^*p^*}{(a+n^*)^2} & -\frac{n^*}{\alpha+n^*} & (\alpha_2+p^*)q^* \\ 0 & -\frac{p^*q^*}{(a_2+p^*)^2} & 0 \\ \frac{\mu n^*}{(a+n^*)^2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} & 0 \\ \theta_{21} & \theta_{22} & \theta_{23} \\ 0 & 0 & \theta_{32} \end{pmatrix}. $$

The characteristic equation of the above matrix is given by

$$\omega^3 + A\omega^2 + B\omega + C = 0,$$  

where $A = -(\theta_{11} + \theta_{22})$, $B = \theta_{11}\theta_{22} - \theta_{12}\theta_{23} - \theta_{21}\theta_{12}$, $C = \theta_{11}\theta_{23}\theta_{32}$. According to Routh-Hurwitz’s criteria $S_*(n^*, p^*, q^*)$ is LAS if

$$A > 0, C > 0 \text{ and } AB - C > 0.$$  

Straightforward calculation shows that $A > 0$, if $-(\theta_{11} + \theta_{22}) > 0$, that is,

$$\frac{p^*}{(a+n^*)^2} + \frac{\mu}{(h+n^*)^2} + \frac{p^*q^*}{n^*(a_2+p^* + \beta_1q^*)^2} < 1$$

and $C > 0$ automatically if condition (7) holds. In addition

$$AB - C = (\theta_{11} + \theta_{22})(-\theta_{11}\theta_{22} + \theta_{12}\theta_{23} + \theta_{21}\theta_{12}) - \theta_{11}\theta_{23}\theta_{32} = (\theta_{11} + \theta_{22})(\theta_{12}\theta_{21} - \theta_{11}\theta_{22}) + \theta_{23}\theta_{23}\theta_{32}.$$  

Thus the necessary condition for $AB - C > 0$ means $(\theta_{12}\theta_{21} - \theta_{11}\theta_{22}) < 0$. The above results summarized as follows:

**Theorem 1.** Suppose that $S_*$ exists in $\mathbb{R}^3_+$ and $\theta_{11} + \theta_{22} > 0$, $(\theta_{12}\theta_{21} - \theta_{11}\theta_{22}) < 0$ hold. Then the equilibrium $S_*$ is LAS.
3.2. Hopf bifurcation

Now we derive the conditions for existence of periodic solutions at $S_3$. For this purpose, let harvesting parameter $\mu$ as bifurcation parameter. The necessary conditions for existence of Hopf bifurcation at $\mu = \mu^*$ are:

i. $A(\mu^*) > 0$, $C(\mu^*) > 0$,

ii. $A(\mu^*)B(\mu^*) - C(\mu^*) = 0$,

iii. $\frac{d[R(\omega(\mu))]}{d\mu}|_{\mu=\mu^*} \neq 0$.

**Theorem 2.** Assume that $\mu = \mu^*$ and $\frac{d}{d\mu}[A(\mu)B(\mu)]|_{\mu=\mu^*} \neq \frac{d}{d\mu}[C(\mu)]|_{\mu=\mu^*}$. Then the system (5) undergoes Hopf bifurcation around $E_\ast$.

**Proof.** Suppose that $A(\mu)B(\mu) = C(\mu)$ which gives $\mu = \mu^*$. Then equation (6) becomes

$$(\omega^2 + B(\mu)(\omega + A(\mu)) = 0. \tag{8}$$

The above equation has three roots: $\omega_1(\mu) = i\sqrt{B(\mu)}$, $\omega_2(\mu) = -i\sqrt{B(\mu)}$ and $\omega_3(\mu) = -A(\mu)$.

The roots are in the following form

$$\omega_1(\mu) = a(\mu) + ib(\mu), \quad \omega_2(\mu) = a(\mu) - ib(\mu), \quad \omega_3(\mu) = -A(\mu).$$

Substituting $\omega_j(\mu) = a(\mu) \pm ib(\mu)$ into (8) and evaluating the total derivative with respect to $\mu$, then by separating real and imaginary parts we obtain

$$\Delta_2(\mu)a'(\mu) - \Delta_1(\mu)b'(\mu) + \Delta_3(\mu) = 0,$$

$$\Delta_1(\mu)a'(\mu) + \Delta_3(\mu)b'(\mu) + \Delta_2(\mu) = 0,$$ \hspace{1cm} \tag{9}

where

$$\Delta_1(\mu) = 6a(\mu)b(\mu) + 2A(\mu)b(\mu),$$

$$\Delta_2(\mu) = 2a(\mu)b(\mu)A'(\mu) + B'(\mu)b(\mu),$$

$$\Delta_3(\mu) = 3a^2(\mu) + 2A(\mu)a(\mu) + B(\mu) - 3b^2(\mu),$$

$$\Delta_4(\mu) = a^2(\mu)A'(\mu) + B'(\mu)a(\mu) + C'(\mu) - A'(\mu)b^2(\mu).$$

From (9), we have

$$\left.\frac{d[R(\omega(\mu))]}{d\mu}\right|_{\mu=\mu^*} = -\frac{\Delta_1(\mu)\Delta_2(\mu) + \Delta_3(\mu)\Delta_4(\mu)}{\Delta_1^2(\mu) + \Delta_2^2(\mu)}. \tag{10}$$

In Hopf bifurcation threshold, we have $a(\mu) = 0$, $b(\mu) = \sqrt{B(\mu)}$, which implies $\Delta_1(\mu) = 2A(\mu)\sqrt{B(\mu)}$, $\Delta_2(\mu) = B'(\mu)\sqrt{B(\mu)}$, $\Delta_3(\mu) = -2B(\mu)$, $\Delta_4(\mu) = C'(\mu) - A'(\mu)B(\mu)$. Thus from (10), we get

$$\left.\frac{d[R(\omega(\mu))]}{d\mu}\right|_{\mu=\mu^*} = \frac{C'(\mu) - A(\mu)B'(\mu) - B(\mu)A'(\mu)}{2(A^2(\mu) + B(\mu))} \neq 0,$$ \hspace{1cm} \tag{11}

if $\frac{d}{d\mu}[A(\mu)B(\mu)]|_{\mu=\mu^*} \neq \frac{d}{d\mu}[C(\mu)]|_{\mu=\mu^*}$. According to Liu’s criterion, the system (5) undergoes Hopf bifurcation around $E_\ast$. Hence the proof.
4. The model with time delay

For simplicity, we let $u = n - n^*$, $v = p - p^*$ and $w = q - q^*$. By using Taylor series expansion of (2) at $(n^*, p^*, q^*)$, we obtain

$$
\begin{align*}
\dot{u}(t) &= \hat{\theta}_{11} u + \hat{\theta}_{12} v + \hat{\theta}_{13} u(t - \tau),
\dot{v}(t) &= \hat{\theta}_{21} u + \hat{\theta}_{22} v + \hat{\theta}_{23} w,
\dot{w}(t) &= \hat{\theta}_{32} v,
\end{align*}
$$

(12)

where $\hat{\theta}_{11} = \frac{n^*p^*}{(n^* + n^*)^2} + \frac{\mu n^*}{(n^* + n^*)^2}$, $\hat{\theta}_{12} = \frac{-n^*}{\alpha + n^*}$, $\hat{\theta}_{21} = \frac{\alpha^2 n^*}{(\alpha + n^*)^2}$, $\hat{\theta}_{22} = \frac{p^* q^*}{(\alpha_2 + p^* + \beta_1 q^*)^2}$, $\hat{\theta}_{23} = -\frac{\alpha_2 p^* q^*}{(\alpha_2 + p^* + \beta_1 q^*)^2}$, $\hat{\theta}_{31} = \frac{\beta_1 q^*}{(\alpha_3 + p^*)^2}$, $\hat{\theta}_{13} = -n^*$. The characteristic equation of (12) is

$$
\omega^3 + A\omega^2 + C\omega + E + e^{-\omega\tau}(B\omega^2 + D\omega + F) = 0,
$$

(13)

where $A = -(\hat{\theta}_{11} + \hat{\theta}_{22})$, $B = -\hat{\theta}_{13}$, $C = \hat{\theta}_{11}\hat{\theta}_{22} - \hat{\theta}_{32}\hat{\theta}_{23} - \hat{\theta}_{21}\hat{\theta}_{12}$. Substituting $\omega = i\eta$ ($\eta > 0$) into (13) and then separating the real and imaginary parts, we get

$$
\begin{align*}
A\eta^2 - E &= (F - B\eta^2) \cos \eta\tau + D\eta \sin \eta\tau, \\
\eta^3 - C\eta &= D\eta \cos \eta\tau - (F - B\eta^2) \sin \eta\tau.
\end{align*}
$$

From the above equations, we get

$$
\eta^6 + (A^2 - 2C - B^2)\eta^4 + (C^2 - 2AE + 2BF - D^2)\eta^2 + E^2 - F^2 = 0.
$$

(14)

Let $v = \eta^2$. Then (14) becomes

$$
v^3 + (A^2 - 2C - B^2)v^2 + (C^2 - 2AE + 2BF - D^2)v + E^2 - F^2 = 0.
$$

(15)

Assume that (15) has at least one real positive root. We always assume that (15) has three real positive roots, say $v_1, v_2$ and $v_3$. Then we have $\eta_l = \sqrt{v_l}$, $l = 1, 2, 3$. Let

$$
\tau_l^j = \frac{1}{\eta_l} \arccos \left\{ \frac{(\eta_l^3 - C\eta_l)D\eta_l + (A\eta_l^2 - E)(F - B\eta_l^2)}{(F - B\eta_l^2)^2 + (D\eta_l)^2} + 2j \pi \right\}, \quad l = 1, 2, 3, \quad j = 0, 1, \cdots 
$$

(16)

Define

$$
\tau_0 = \tau_l^0 = \min_{l=1,2,3} \tau_l^0 \quad \text{and} \quad \eta_0 = \eta_l.
$$

In addition, taking derivative of equation (13) with respect to $\tau$, we have

$$
\frac{d\omega}{d\tau} = \frac{\omega(B\omega^2 + D\omega + F)e^{-\omega\tau}}{3\omega^2 + 2A\omega + C + (2B\omega + D)e^{-\omega\tau} - \tau(B\omega^2 + D\omega + F)e^{-\omega\tau}}.
$$

Then the transversality condition

$$
\left[ \Re \left\{ \frac{d\omega}{d\tau} \right\} \right]^{-1} = \frac{2\eta_0^6 + (A^2 - 2C - B^2)\eta_0^4F^2 - E^2}{\eta_0^2(F - B\eta_0^2)^2 + w^2(D\eta_0)^2} \neq 0,
$$

is holds if $2\eta_0^6 + (A^2 - 2C - B^2)\eta_0^4F^2 - E^2 \neq 0$ holds. Now we have the following theorem:

**Theorem 3.** Assume that $\tau_0 \neq 0$ and $\eta_0$ is the positive root of (14). Then the equilibrium $S_*$ is LAS for $\tau < \tau_0$ and unstable for $\tau > \tau_0$. Furthermore the system (2) undergoes a Hopf bifurcation at $S_*$ when $\tau = \tau_0$. 

6
5. Numerical Example
In this subsection, to illustrate our proposed theoretical results we perform numerical simulations. First we consider the system (5) with the following fixed parameters $\alpha = 0.48$, $h = 0.4, \delta_1 = 1.15, \gamma_1 = 2.93, \alpha_1 = 0.54, \beta_1 = 0.21, \alpha_2 = 0.1, \gamma_3 = 0.57, \delta_2 = 0.8, \alpha_3 = 0.25$ and $\mu$ may vary. With the same parameters and $\mu = 0.08$, the stability conditions given in Theorem 1 are well satisfied, which is shown in figure 1. Let $\mu$ be a bifurcation parameter. When $\mu$ crosses its critical value $\mu^*$, system (5) loses its stability and Hopf bifurcation occurs around $S_*$, see figure 2. The system (5) undergoes multi-periodic attractors by varying $\mu$ which is shown in figure 3. When $\tau = 0.4$, by varying $\mu$, the system (2) exhibits multi-periodic and chaotic attractors, see figure 4.

![Figure 1.](image1)

**Figure 1.** a. The positive equilibrium $S_*$ is LAS when $\mu = 0.08$.

![Figure 2.](image2)

**Figure 2.** The system (5) exhibits Hopf bifurcation when $\mu = 0.0765$.

6. Conclusion
In this paper, we have considered and analyzed the dynamics of three species food chain model consisting of prey, intermediate predator and top predator. The effect of mutual
interference among generalist type top predator is considered in terms of Beddington-DeAngelis functional response which grows by sexual reproduction. We have discussed the local dynamics of considered system at equilibrium points. Also we showed that the model (5) experiences Hopf bifurcation around interior equilibrium point when the parameter crosses its critical value. Numerical examples are given to verify the effectiveness of developed theoretical results. Through varying harvesting parameter, we observed that the delayed system exhibits complex dynamical behaviors. In the near future, we consider the three species food chain model with diffusion and herd behavior.

Figure 3. The multi-periodic attractors of system (5) by varying $\mu$.

Figure 4. The multi-periodic and chaotic attractors of system (2) by varying $\mu$. 
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