General solution for Hamiltonians with extended cubic and quartic potentials

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Abstract

We integrate with hyperelliptic functions a two-particle Hamiltonian with quartic potential and additional linear and nonpolynomial terms in the Liouville integrable cases $1 : 6 : 1$ and $1 : 6 : 8$.

1 Introduction

The generalized Hénon-Heiles Hamiltonian

$$H = \frac{1}{2}(P_X^2 + P_Y^2 + c_1 X^2 + c_2 Y^2) + aXY^2 - \frac{b}{3}X^3 + \mu Y^{-2}, \quad \mu \text{ arbitrary} \quad (1)$$

is known to be Liouville integrable \cite{1, 2, 3} for three sets of values of $(b/a, c_1, c_2)$ and to be related \cite{4} to the stationary reduction of the following fifth order soliton equations: KdV$_5$, Sawada-Kotera (SK) and Kaup-Kupershmidt (KK).

The canonical transformation \cite{5} between the SK and KK cases for $\mu \neq 0$ allowed us \cite{6} to define the separating variables of the Hamilton-Jacobi equations and to derive the general solution of the equations of motion with hyperelliptic functions.

The two-particle Hamiltonian with quartic potential

$$H = \frac{1}{2}(P_X^2 + P_Y^2) - \frac{1}{2}(aX^2 + bY^2) + CX^4 + BX^2Y^2 + AY^4, \quad (2)$$

is known to be Liouville integrable in four cases \cite{7, 8, 9}:

1. $A : B : C = 1 : 2 : 1, a, b \text{ arbitrary},$
2. $A : B : C = 1 : 6 : 1, a = b,$
3. $A : B : C = 1 : 6 : 8, a = 4b,$
4. \( A : B : C = 1 : 12 : 16, a = 4b \).

The extension of (2) to include some linear and non-polynomial terms which preserve the Liouville integrability [10, 9] and confirm their connection with some integrable soliton equations has been considered by various authors [11, 12]. The case 1 with extra inverse square terms is equivalent to the travelling wave reduction of the Manakov system [13] of two coupled NLS equations. It corresponds to the particular case for two particles of the extended Garnier system [14] which has been integrated by Wojciekowski [15]. The case 4 can be associated [5] with a coupled KdV system possessing a fifth order Lax pair [16].

The authors of [17, 5] have showed how cases 2 and 3 with some extra linear and inverse square terms are equivalent to the stationary reduction \( \xi = x - ct \) of the Hirota-Satsuma system [18]:

\[
\begin{align*}
    r_t &= \frac{1}{2}r_{xxx} + 3rr_x - 6ss_x, \\
    s_t &= -s_{xxx} - 3rs_x
\end{align*}
\]  

(3)

and an other coupled KdV system

\[
\begin{align*}
    f_t &= -\frac{1}{2}(2f_{xxx} + 3ff_{xx} + 3f_x^2 - 3f^2f_x + 6fg_x + 6gf_x), \\
    g_t &= \frac{1}{4}(2g_{xxx} + 12gg_x + 6fg_{xx} + 12gf_{xx} + 18f_xg_x - 6f^2g_x \\
    &\quad + 3f_{xxx} + 3ff_{xx} + 18f_xf_{xx} - 6f^2f_x - 6f^2_x),
\end{align*}
\]

(4)

In the present paper, using the canonical transformation [5] between these two extended cases, we show how the method we followed for the SK and KK cases of (1) to define the separating variables of the Hamilton-Jacobi equation can be applied here. Then the equations of motion associated with cases 2 and 3 of (2) with extra linear and inverse square terms are also integrated with hyperelliptic functions.

In section 2 we consider the SK and KK integrable cases of the Hénon-Heiles Hamiltonian [10] for \( \mu \neq 0 \) to explain the method we applied in [6] for integrating their equations of motion. In section 3 we consider the Hamiltonian (2) with extended quartic potential in case 2 and 3 and recall the canonical transformation [5], which relates the case 1 : 6 : 1 of (2) (with extra terms of the form \( \alpha X^{-2} + \beta Y^{-2} \)) and the case 1 : 6 : 8 (with extra terms like \( \gamma X + \delta Y^{-2} \)).

To the best of our knowledge, the results of sections 4 and 5 are new. In section 4 we explicitly integrate the equations of motion of the case 1 : 6 : 8 for \( \delta \neq 0, \gamma = 0 \). In section 5 we integrate the 1 : 6 : 1 case with \( \alpha = \beta \neq 0 \) and we use the canonical transformation to transport those solutions to the 1 : 6 : 8 case for \( \delta = 0, \gamma \neq 0 \).

## 2 Integration of the cubic Hénon-Heiles Hamiltonian

Let us consider the SK and KK Hénon-Heiles Hamiltonians with their second integral, denoted as \( K_2^2 \) and \( k_2^2 \), and their equations of motion [10, 9]:

\[
\text{SK: } \frac{b}{a} = -1, \quad c_1 = c_2, \quad a = \frac{1}{2}, \quad U = X + c_2, \quad V = Y, \quad c = c_1c_2
\]  

(5)
\[ H \equiv K_1 = \frac{1}{2}(P_U^2 + P_V^2) + \frac{1}{2}UV^2 + \frac{1}{6}U^3 - \frac{c}{2}U + \frac{\mu}{8V^2}, \quad (6) \]

\[ K_2^2 = K_{2,0}^2 + \frac{2}{3}\mu U + \mu \frac{P_U^2}{V^2}, \quad (7) \]

\[ K_{2,0} = -2PU PV - U^2 V - \frac{V^3}{3} + cV, \quad (8) \]

\[ U'' = -\frac{1}{2}(V^2 + U^2) - \frac{c}{2}, \quad V'' = -UV + \frac{\mu}{4V^3}, \quad (9) \]

\[ \frac{b}{a} = -16, \quad c_1 = 16c_2, \quad a = \frac{1}{4}, \quad c = c_1c_2, \quad u = X + 2c_2, \quad V = Y. \quad (10) \]

\[ H \equiv k_1 = \frac{1}{2}(P_a^2 + P_v^2) + \frac{1}{4}uv^2 + \frac{4}{3}u^3 - cu + \frac{1}{2}\frac{\mu}{v^2}, \quad (11) \]

\[ k_2^2 = k_{2,0}^2 + \frac{\mu}{3}u + 2\mu \frac{P_a^2}{v^2} + \frac{\mu^2}{v^4}, \quad (12) \]

\[ k_{2,0}^2 = P_v^4 - \frac{1}{72}v^6 - \frac{1}{12}u^2v^4 + up_a^2v^2 - \frac{1}{3}P_uP_vv^3 + \frac{c}{12}v^4, \quad (13) \]

\[ u'' = -\frac{1}{4}v^2 - 4u^2 + c, \quad v'' = -\frac{1}{2}uv + \frac{\mu}{v^3}. \quad (14) \]

which are equivalent to the stationary reduction \( \xi = x - ct \) of the two integrable PDE's:

\[ \text{SK:} \quad U_t + (Uxxxx + 5UU_{xx} + \frac{5}{3}U^3)_x = 0, \quad (15) \]

\[ \text{KK:} \quad u_t + (u_{xxxx} + 10uu_{xx} + \frac{20}{3}u^3 + 30u_x^2)_x = 0. \quad (16) \]

Both equations are connected to another fifth order integrable PDE \cite{19}:

\[ w_t + (w_{4x} - 5w_xw_{xx} - 5w_x^2w_{xx} - 5ww_x^2 + w^5)_x = 0. \quad (17) \]

by the Miura transformation

\[ U = w_x - w^2, \quad u = -w_x - \frac{1}{2}w^2. \quad (18) \]

Solving the stationary reduction of \cite{17} for \( w \) in function of \( V, V' \) and \( u, v' \) and defining the following expressions:

\[ \lambda^2 = -\mu, \quad \Gamma = 6(VK_{2,0} + \lambda P_U), \]

\[ \Omega = 48(3v^4k_{2,0}^2 + 6\lambda w^5p_v + 12\lambda p_v^3v^3 - \lambda v^6p_u \]
\[ + 3\lambda^2uv^4 + 18\lambda^2v^2p_v^2 + 12\lambda^3v^3p_v + 3\lambda^4), \quad (19) \]

one obtains the following canonical transformation \cite{5, 20}:

\[ u = -\frac{3}{2} \left( -\frac{P_v}{V} + \frac{\lambda}{2V^2} \right)^2 - U, \quad v = \sqrt{\frac{\Gamma}{V}}. \quad (20) \]
\[ p_u = \frac{1}{v^3} (3P_v^3 + 3UV^2P - P_U V^3) \]
\[ - \frac{3\lambda}{2v^6} \left( UV^4 + 3V^2P_U^2 - \frac{3}{2} \lambda VP + \frac{\lambda^2}{4} \right), \]  
\[ p_v = \frac{1}{4v^2} \left( -2P_U + \frac{\lambda}{V} \right) \sqrt{\Gamma} - \lambda \frac{1}{\sqrt{\Gamma}}, \]  
\[ U = -6 \left( \frac{p_v}{v} + \frac{\lambda}{v^2} \right)^2 - u, \quad V = \frac{\sqrt{\Omega}}{2v^4}, \]  
\[ P_U = \frac{1}{v^3} (12p_v^3 + 6uv^2p_v - v^3p_u) \]
\[ + \frac{3\lambda}{v^5} (2uv^4 + 12v^2p_v^2 + 12\lambda vp_v + 4\lambda^2), \]
\[ P_V = -\frac{1}{v^4} \left( p_v + \frac{\lambda}{v} \right) \sqrt{\Omega} + \lambda \frac{1}{\sqrt{\Omega}}. \]  

For \( \mu = 0 \), we introduce the expressions \([22]-[23]\) for \( U, V, P_U, P_V \) in the variables which separate the SK Hamiltonian
\[ Q_1 = U + V, \quad P_1 = \frac{1}{2}(P_U + P_V), \]
\[ Q_2 = U - V, \quad P_2 = \frac{1}{2}(P_U - P_V). \]  

This defines the change of variables \([24]\)
\[ q_1 = -6\frac{p_v^2 - k_{2,0}}{v^2} - u, \quad p_1 = \frac{1}{2v^3} (12p_v^3 + 6uv^2p_v - v^3p_u - 12p_vk_{2,0}), \]
\[ q_2 = -6\frac{p_v^2 + k_{2,0}}{v^2} - u, \quad p_2 = \frac{1}{2v^3} (12p_v^3 + 6uv^2p_v - v^3p_u + 12p_vk_{2,0}), \]  

that we apply on the KK Hamiltonian \([11]\), taking account that for \( \mu \neq 0 \), \( k_{2,0} \) is no more a constant of motion. Therefore, one has
\[ H \equiv k_1 = p_1^2 + p_2^2 + \frac{1}{12} (q_1^3 + q_2^3) - \frac{c}{4} (q_1 + q_2) + \frac{\mu}{24} \frac{q_1 - q_2}{k_{2,0}}, \]
\[ k_{2,0} = 2(p_2^2 - p_1^2) + \frac{1}{6} (q_2^3 - q_1^3) - \frac{c}{2} (q_2 - q_1), \]
\[ q_1' = 2p_1 + \frac{\mu}{6} \frac{q_1 - q_2}{k_{2,0}} \]
\[ q_2' = 2p_2 - \frac{\mu}{6} \frac{q_1 - q_2}{k_{2,0}}. \]

In this new setting of coordinates, it is important to note that, introducing the expression
\[ f(q_i, p_i) = 2p_i^2 + \frac{1}{6} q_i^3 - \frac{c}{2} q_i \]  
for \( i = 1, 2 \),

\[ \text{(31)} \]
in (27) and (28), the Hamilton-Jacobi equation is separated for \( \mu \) arbitrary. In the variables (26), the second invariant \( k_2^2 \) can be written in two equivalent ways:

\[
k_2^2 = -\frac{\mu}{3} q_1 + \left( k_{2,0} + \frac{\mu}{12} \frac{q_1 - q_2}{k_{2,0}} \right)^2, \tag{32}
\]

or

\[
k_2^2 = -\frac{\mu}{3} q_2 + \left( k_{2,0} - \frac{\mu}{12} \frac{q_1 - q_2}{k_{2,0}} \right)^2, \tag{33}
\]

which, after the elimination of \( \mu (q_1 - q_2) / k_{2,0} \) between (27)–(32) and (27)–(33), become

\[
k_2^2 = -\frac{\mu}{3} q_1 + \left( -4p_1^2 - \frac{q_1^3}{3} + cq_1 + 2k_1 \right)^2, \tag{34}
\]

or

\[
k_2^2 = -\frac{\mu}{3} q_2 + \left( 4p_2^2 + \frac{q_2^3}{3} - cq_2 - 2k_1 \right)^2. \tag{35}
\]

Then we can eliminate \( p_1 \) between (34) and (29), and \( p_2 \) between (35) and (30) to obtain

\[
q_1' = \sqrt{2k_1 - \frac{q_1^3}{3} + cq_1 - \sqrt{k_2^2 + \frac{\mu}{3} q_2}} \left( 1 + \frac{\mu}{3} \frac{q_1 - q_2}{(\sqrt{k_2^2 + \frac{\mu}{3} q_2} + \sqrt{k_2^2 + \frac{\mu}{3} q_1})^2} \right), \tag{36}
\]

\[
q_2' = \sqrt{2k_1 - \frac{q_2^3}{3} + cq_2 + \sqrt{k_2^2 + \frac{\mu}{3} q_1}} \left( 1 - \frac{\mu}{3} \frac{q_1 - q_2}{(\sqrt{k_2^2 + \frac{\mu}{3} q_2} + \sqrt{k_2^2 + \frac{\mu}{3} q_1})^2} \right). \tag{37}
\]

For \( \mu \neq 0 \), setting:

\[
s_1 = \sqrt{\frac{3k_2^2}{\mu} + q_1}, \quad s_2 = -\sqrt{\frac{3k_2^2}{\mu} + q_2}, \tag{38}
\]

and defining

\[
P(s) = 2k_1 - \frac{1}{3} \left( s^2 - 3 \frac{k_2^2}{\mu} \right)^3 + c \left( s^2 - 3 \frac{k_2^2}{\mu} \right) - \sqrt{\frac{\mu}{3}} s, \tag{39}
\]

the equations (36)–(37) become:

\[
s_1' = \frac{P(s_1)}{s_1 - s_2}, \quad s_2' = -\frac{P(s_2)}{s_1 - s_2}. \tag{40}
\]

They can be solved by inverting the hyperelliptic integrals

\[
\int_{\infty}^{s_1} \frac{ds}{\sqrt{P(s)}} + \int_{\infty}^{s_2} \frac{ds}{\sqrt{P(s)}} = k_3, \tag{41}
\]

\[
\int_{\infty}^{s_1} \frac{ds}{\sqrt{P(s)}} + \int_{\infty}^{s_2} \frac{ds}{\sqrt{P(s)}} = \xi + k_4. \tag{42}
\]
and the general solution for the KK system is
\[ u = -\frac{1}{2}(s_1^2 + s_2^2) + \frac{3}{\mu} k_2^2 - \frac{3}{2} \left( \frac{s_1' + s_2'}{s_1 + s_2} \right)^2, \quad v^2 = \frac{2\sqrt{3\mu}}{s_1 + s_2}. \] (43)

For \( \mu = 0 \) from (36)–(37) and (26), one easily recovers the known solution [21] expressed with Weierstrass elliptic functions:
\[ u = 6(\wp_1 + \wp_2) - \frac{3}{2} \left( \frac{\wp_1' - \wp_2'}{\wp_1 - \wp_2} \right)^2, \quad v^2 = \frac{k_{2,0}}{\wp_2 - \wp_1}, \] (44)

where \( \wp_i \) satisfies the equation
\[ \wp_i'^2 = 4\wp_i^3 - g_2 \wp_i - g_3^{(i)}, \quad i = 1, 2 \] (45)
\[ g_2 = \frac{c}{12}, \quad g_3^{(1)} = -\frac{1}{144}(2k_{1,0} - k_{2,0}), \quad g_3^{(2)} = -\frac{1}{144}(2k_{1,0} + k_{2,0}). \]

In the SK case, from (43) and the canonical transformation (22), the general solution writes
\[ U = \sqrt{-3(s_1' + s_2') + s_1^2 + s_1 s_2 + s_2^2} - \frac{3}{2} \alpha_{K_2}^2, \]
\[ V^2 = -2\sqrt{-3(s_1 + s_2)(s_1 s_1' + s_2 s_2')} + 2(s_1 + s_2)^2 \left( s_1^2 + s_2^2 - \frac{9\alpha_{K_2}^2}{2\mu} \right), \] (46)

which, in the particular case \( \mu = 0 \), is merely
\[ U = -6(\wp_1 + \wp_2), \quad V = -6(\wp_1 - \wp_2). \] (47)

3 Canonical transformation between extended 1 : 6 : 1 and 1 : 6 : 8 Hamiltonians

Let us consider the quartic Hamiltonian [2] with extra linear and inverse quadratic terms in the two following cases [3]:

1 : 6 : 1 \( H = K_1 = \frac{1}{2}(P_U^2 + P_V^2) - \frac{1}{32}(U^4 + 6U^2V^2 + V^4) \)
\[ -\frac{c}{2}(U^2 + V^2) + \frac{1}{2}(\frac{\alpha}{U^2} + \frac{\beta}{V^2}), \] (48)
\[ K_2^2 = K_{2,0}^2 - \beta U^2 - \alpha V^2 + 4(\alpha \frac{P_U^2}{U^2} + \beta \frac{P_V^2}{V^2}) + \frac{4\alpha\beta}{U^2V^2} - 4c(\alpha + \beta). \] (49)
\[ K_{2,0} = 2P_UP_V - \frac{1}{4}UV(U^2 + V^2 + 8c), \] (50)
\[ U' = P_U, \quad V' = P_V, \] (51)
\[ U'' = \frac{1}{8}U^3 + \frac{3}{8}U^2V + cU + \frac{\alpha}{U^3}, \quad V'' = \frac{1}{8}V^3 + \frac{3}{8}U^2V + cV + \frac{\beta}{V^3}. \] (52)
$H \equiv k_1 = \frac{1}{2}(p_u^2 + p_v^2) - \frac{1}{16}(8u^4 + 6u^2v^2 + v^4)$
\[\frac{c}{2}(4u^2 + v^2) - \gamma u + \frac{\delta}{2v^2}, \quad (53)\]

$k_2^2 = k_{2,0}^2 + \frac{\delta^2}{v^2} - \gamma v^2u^3 - \frac{\gamma}{2}uv^4 - \frac{\gamma}{2}u^2 - 2\gamma^2v^2 + 2\delta p_v^2 - \frac{\gamma}{4}v^2$
\[+ 4\gamma(vpu_v - up_v) - 4\gamma\delta \frac{v}{v^2} - 2\delta c - 4c\gamma v^2 u, \quad (54)\]

$k_{2,0}^2 = (-p_v^2 + \frac{1}{8} u^4 + \frac{1}{4} u^2v^2 - uvp_v + \frac{1}{2} p_u v^2 + cv^2)$
\[\times (-p_v^2 + \frac{1}{8} u^4 + \frac{1}{4} u^2v^2 + uvp_v - \frac{1}{2} p_u v^2 + cv^2), \quad (55)\]

$u' = p_u, \quad v' = p_v, \quad (56)$

$u'' = 2u^3 + \frac{3}{4} uv^2 + 4cu + \gamma, \quad v'' = \frac{1}{4} v^3 + \frac{3}{4} u^2v + cv + \frac{\delta}{v^3}, \quad (57)$

where $K_1, K_2$ and $k_1, k_2$ are the constants of motion of both systems.

The equations of motion are respectively equivalent to the stationary reduction
\[\xi = x - ct\] of the PDE’s (3) and (4) with the correspondence:
\[r = -\frac{1}{4}(U^2 + V^2 + 4c), \quad s = \frac{1}{8}(U^2 - V^2), \quad (58)\]
\[f = u, \quad g = -\frac{1}{4}(v^2 + 2u^2 + 2p_u + 4c). \quad (59)\]

Both PDE’s (3) and (4) possess a fourth order Lax pair with scattering operators $L$ and $\tilde{L}$ which factorize as follows (5) (7),

$1 : 6 : 1 \quad L = (\partial_x^2 + r + s)(\partial_x^2 + r - s)$
\[= (\partial_x - v_1)(\partial_x + v_1)(\partial_x + v_2)(\partial_x - v_2), \quad (60)\]

$1 : 6 : 8 \quad \tilde{L} = (\partial_x^2 + f \partial_x + f_x + g)(\partial_x - f \partial_x + g)$
\[= (\partial_x + v_1)(\partial_x + v_2)(\partial_x - v_2)(\partial_x - v_1), \quad (61)\]

and yield the Miura maps:

$1 : 6 : 1 \quad r = \frac{1}{2}(v_{1,x} - v_{2,x} - v_1^2 - v_2^2), \quad s = \frac{1}{2}(v_{1,x} + v_{2,x} - v_1^2 + v_2^2); \quad (62)$

$1 : 6 : 8 \quad f = v_1 + v_2, \quad g = v_1v_2 - v_{1,x}; \quad (63)$

while $v_1$ and $v_2$ are solutions of the system of PDE’s

$v_{1,t} = \frac{1}{8}(-2v_{1,xx} - 6v_{2,xx} - 12v_{1}v_{2,xx} - 12v_{1}v_{2,x} - 12v_{1}v_2^2 + 4v_1^2), \quad (64)$

$v_{2,t} = \frac{1}{8}(-2v_{2,xx} - 6v_{1,xx} + 12v_{1}v_{1,x} + 12v_{2}v_{1,xx} - 12v_{2}v_1^2 + 4v_2^2).$

From the relations (62)–(63), the transformations (58)–(59) and the equations of motion (51)–(52), (56)–(57), the stationary reduction of (64) can be solved for $v_1$ and $v_2$ in function of the canonical variables of the two extended systems:

$1 : 6 : 1 \quad v_1 = \frac{P_U}{U} + \frac{\sqrt{-\alpha}}{U^2}, \quad v_2 = \frac{P_V}{V} + \frac{\sqrt{-\beta}}{V^2}, \quad (65)$

$1 : 6 : 8 \quad v_1 = \frac{u}{2} + \frac{p_v}{v} + \frac{\sqrt{-\delta}}{v^2}, \quad v_2 = \frac{u}{2} - \frac{p_v}{v} - \frac{\sqrt{-\delta}}{v^2}, \quad (66)$
such that, defining

\[ \alpha = -\kappa_1^2, \quad \beta = -\kappa_2^2, \quad \delta = -(\kappa_2 - \kappa_1)^2, \quad \gamma = \frac{1}{2}(\kappa_1 + \kappa_2), \]  

\( \Omega = -\frac{1}{4}(U^2 + V^2) + \frac{2}{UV}(P_U P_V + \kappa_2 P_U V - \kappa_1 P_V - \kappa_1 \kappa_2 UV) - 2c, \)  

\[ \Gamma_\mp = -4\left( \mp 2p_u + \frac{1}{2}v^2 + u^2 - \frac{p_v^2}{v^2} \pm \frac{4up_v}{v} + 8(\kappa_2 - \kappa_1)P_v \right. \]

\[ \left. \pm 4(\kappa_2 - \kappa_1)\frac{u}{v^2} - 4\left(\frac{2\kappa_2 - \kappa_1}{v^4}\right) + 4c \right), \]

the canonical transformation between the 1 : 6 : 1 and 1 : 6 : 8 cases becomes

\[ u = -\frac{P_U}{U} + \frac{P_V}{V} + \frac{\kappa_1}{U^2} + \frac{\kappa_2}{V^2}, \]

\[ p_u = -\frac{1}{4}(V^2 - U^2) + \frac{P_U^2}{U^2} - \frac{P_V^2}{V^2} - 2\kappa_1 \frac{P_U}{U^3} - 2\kappa_2 \frac{P_V}{V^3} + \frac{\kappa_1^2}{U^4} - \frac{\kappa_2^2}{V^4}, \]

\[ v = 2\sqrt{\Omega}, \quad p_v = \sqrt{\Omega}\left( -\frac{P_U}{U} - \frac{P_V}{V} + \frac{\kappa_1}{U^2} - \frac{\kappa_2}{V^2} \right) - \frac{\kappa_2 - \kappa_1}{2\sqrt{\Omega}}, \]

\[ U = \frac{1}{2}\sqrt{\Gamma_-}, \quad P_U = -\frac{1}{2}\sqrt{\Gamma_-}\left( \frac{u}{2} + \frac{p_v}{v} - \frac{\kappa_2 - \kappa_1}{v^2} \right) + 2\kappa_1\sqrt{\Gamma_-}, \]

\[ V = \frac{1}{2}\sqrt{\Gamma_+}, \quad P_V = \frac{1}{2}\sqrt{\Gamma_+}\left( \frac{u}{2} - \frac{p_v}{v} + \frac{\kappa_2 - \kappa_1}{v^2} \right) - 2\kappa_2\sqrt{\Gamma_+}. \]

### 4 General solution of the 1 : 6 : 8 Hamiltonian for \( \gamma = 0 \)

For \( \alpha = \beta = 0 \), one introduces the transformation [72]-[73] in the variables

\[ Q_1 = \frac{1}{2}(U + V)^2, \quad P_1 = \frac{1}{2}\frac{P_U + P_V}{U + V}, \]

\[ Q_2 = \frac{1}{2}(U - V)^2, \quad P_2 = \frac{1}{2}\frac{P_U - P_V}{U - V}, \]

which separated the 1 : 6 : 1 Hamiltonian [18] in the polynomial case. This defines the following change of variables:

\[ q_1 = 4\frac{k_{2,0} + p_v^2}{v^2} - \frac{v^2}{2} - u^2 - 4c, \]

\[ p_1 = \frac{-8vp_v p_u + 8up_v^2 + 2v^2 u^3 + u(v^4 + 8k_{2,0}) + 8c u v^2}{16v(vp_u - 2up_v)}, \]

\[ q_2 = 4\frac{k_{2,0} + p_v^2}{v^2} - \frac{v^2}{2} - u^2 - 4c, \]

\[ p_2 = \frac{-8vp_v p_u + 8up_v^2 + 2v^2 u^3 + u(v^4 - 8k_{2,0}) + 8c u v^2}{16v(vp_u - 2up_v)}, \]

8
that we apply on the Hamiltonian system \[63\] for \(\kappa_1\kappa_2 \neq 0\), i.e. when \(k_2,0\) is no
more a constant of motion. This transformation restricted to \(\kappa_2 = -\kappa_1\ (\delta = -4\kappa_1^2,\ 
\gamma = 0)\) defines the coordinates which separate \[114\] [12] the Hamilton-Jacobi equation.
Indeed, one has
\[
H \equiv k_1 = \frac{1}{16} \left(32q_1p_1^2 - q_1^2 + 32q_2p_2^2 - q_2^2 - 8c(q_1 + q_2)\right) - \kappa_1^2 \frac{(q_1 - q_2)}{4k_{2,0}}, 
\]
(79)
with
\[
k_{2,0} = \frac{1}{8} \left(32q_1p_1^2 - q_1^2 - 32q_2p_2^2 + q_2^2 + 8c(q_2 - q_1)\right),
\]
(80)
\[
q_1' = 4q_1p_1 + 2\kappa_2^2 \frac{(q_1 - q_2)q_1p_1}{k_{2,0}},
\]
(81)
\[
q_2' = 4q_2p_2 - 2\kappa_2^2 \frac{(q_1 - q_2)q_2p_2}{k_{2,0}},
\]
(82)
such that defining
\[
f(q_i, p_i) \equiv 32q_ip_i^2 - q_i^2 - 8cq_i, \quad i = 1,2
\]
we have the following separated Hamilton-Jacobi equation:
\[
f(q_1, p_1)^2 - f(q_2, p_2)^2 - 32\kappa_2^2(q_1 - q_2) = 16k_1 \left(f(q_1, p_1) - f(q_2, p_2)\right), \quad p_i = \frac{\partial S}{\partial q_i}. \quad (83)
\]
In analogy with the Kaup-Kupershmidt case, we can write the second invariant in two equivalent ways:
\[
k_2^2 = -2\kappa_1^2 q_1 + \left(k_{2,0} + \frac{\kappa_1^2 (q_1 - q_2)}{2k_{2,0}}\right)^2, \quad (84)
\]
or
\[
k_2^2 = -2\kappa_1^2 q_2 + \left(k_{2,0} - \frac{\kappa_1^2 (q_1 - q_2)}{2k_{2,0}}\right)^2, \quad (85)
\]
which allow us to eliminate \(\kappa_1^2 (q_1 - q_2)/k_{2,0}\) between \(69\)–\(81\) and \(69\)–\(85\). Next, we eliminate \(p_1\) and \(p_2\) between those two resulting expressions and the equations \(81\), \(82\) and obtain:
\[
q_1' = q_1 \sqrt{\frac{q_1}{2} + 4c + \frac{k_1}{q_1}} + \frac{2}{q_1} \sqrt{k_2^2 + 2\kappa_1^2 q_1} \left(1 + \frac{2\kappa_1^2 (q_1 - q_2)}{\sqrt{k_2^2 + 2\kappa_1^2 q_1} + \sqrt{k_2^2 + 2\kappa_1^2 q_2}}\right) \quad (86)
\]
\[
q_2' = q_2 \sqrt{\frac{q_2}{2} + 4c + \frac{k_1}{q_2}} - \frac{2}{q_2} \sqrt{k_2^2 + 2\kappa_1^2 q_2} \left(1 - \frac{2\kappa_1^2 (q_1 - q_2)}{\sqrt{k_2^2 + 2\kappa_1^2 q_1} + \sqrt{k_2^2 + 2\kappa_1^2 q_2}}\right) \quad (87)
\]
For \(\kappa_1 = 0\) the general solution of equations \(86\)–\(87\) can be expressed in terms of
Weierstrass elliptic functions:
\[
q_1 + \frac{8}{3} c = 8\wp \left(\xi - \xi_1, \frac{4}{3} c^2 - \frac{k_1}{2} - \frac{k_2}{4}, \frac{c}{12} (2k_1 + k_2 - \frac{32}{9} c^2)\right) \equiv 8\wp_1, \quad (88)
\]
\[
q_2 + \frac{8}{3} c = 8\wp \left(\xi - \xi_2, \frac{4}{3} c^2 - \frac{k_1}{2} + \frac{k_2}{4}, \frac{c}{12} (2k_1 - k_2 - \frac{32}{9} c^2)\right) \equiv 8\wp_2. \quad (89)
\]
For $\kappa_1 \neq 0$ setting in (86)–(87):

$$s_1 = \sqrt{\frac{1}{2\kappa_1^2} k_2^2 + q_1} \quad \text{and} \quad s_2 = -\sqrt{\frac{1}{2\kappa_1^2} k_2^2 + q_2},$$

(90)

and defining

$$\tilde{P}(s) \equiv \frac{1}{2} \left( s^2 - \frac{k_2^2}{2\kappa_1^2} \right)^3 + 4c \left( s^2 - \frac{k_2^2}{2\kappa_1^2} \right)^2 + (4k_1 + 2\sqrt{2}\kappa_1 s) \left( s^2 - \frac{k_2^2}{2\kappa_1^2} \right),$$

(91)

we obtain the equations,

$$s_1' = \sqrt{\tilde{P}(s_1)} \frac{s_1}{s_1 - s_2}, \quad s_2' = -\sqrt{\tilde{P}(s_2)} \frac{s_1}{s_1 - s_2},$$

(92)

which are solved with the inversion of the hyperelliptic integrals

$$\int_{s_1}^{s_1'} \frac{ds}{\sqrt{\tilde{P}(s)}} + \int_{s_2}^{s_2'} \frac{ds}{\sqrt{\tilde{P}(s)}} = k_3,$$

(93)

$$\int_{s_1}^{s_1'} \frac{ds}{s^2 \sqrt{\tilde{P}(s)}} + \int_{s_2}^{s_2'} \frac{ds}{s^2 \sqrt{\tilde{P}(s)}} = \xi + k_4.$$  

(94)

Therefore, for $\delta$ arbitrary, $\gamma = 0$, the general solution of the $1 : 6 : 8$ Hamiltonian is defined with symmetric combinations of $s_1, s_2$

$$u^2 = -\frac{1}{2} (s_1^2 + s_2^2) + \frac{k_2^2}{2\kappa_1^2} + \left( \frac{s_1' + s_2'}{s_1 + s_2} \right)^2 - \frac{2\sqrt{2}\kappa_1}{s_1 + s_2} - 4c, \quad v^2 = \frac{4\sqrt{2}\kappa_1}{s_1 + s_2},$$

(95)

and is a single-valued function of the complex variable $\xi$. For $\kappa_1 = 0$ it degenerates into

$$u^2 = -4(\varphi_1 + \varphi_2) + \left( \frac{\varphi_1' - \varphi_2'}{\varphi_1 - \varphi_2} \right)^2 + \frac{k_{2,0}}{2(\varphi_2 - \varphi_1)} - \frac{4}{3}c, \quad v^2 = \frac{k_{2,0}}{\varphi_1 - \varphi_2}.$$  

(96)

5 General solution of the extended $1 : 6 : 1$ Hamiltonian

Now, we again use the canonical transformation (72)–(73) to find the general solution of the $1 : 6 : 1$ Hamiltonian.

For $\kappa_1 = \kappa_2 = 0$, we introduce the expressions (96) in (72)–(73) and obtain the solution of the equations of motion

$$U^2 + V^2 = 8(\varphi_1 + \varphi_2 - \frac{2}{3}c), \quad UV = 4(\varphi_1 - \varphi_2).$$  

(97)
Next, for $\alpha \beta \neq 0$, with the restriction $\alpha = \beta$, we obtain the expressions

$$U^2 + V^2 = -2\sqrt{2}(s'_1 + s'_2) + 2(s^2_1 + s^2_2 + s_1s_2) - \frac{K_2^2}{\kappa_1},$$

(98)

$$U^2V^2 = -2\sqrt{2}(s_1 + s_2)(s_1s'_1 + s_2s'_2 - 2\kappa_1) + 2(s_1 + s_2)^2 \left( s^2_1 + s^2_2 - \frac{3K_2^2}{4\kappa_1} + 4c \right),$$

(99)

which depend on symmetric combinations of $s_1, s_2$ and therefore are single-valued functions of $\xi$.

Since the Hamiltonian (48) for $\alpha = \beta$ is an even function of $\kappa_1$, we apply on (98)–(99) the canonical transformation (70)–(71) for $\kappa_1 = \kappa_2$, and obtain the general solution of the 1 : 6 : 8 Hamiltonian in the extended case $\delta = 0$.

6 Conclusion

Romeiras [22] described a procedure which relates the extended cubic potential $\tilde{V}_3(u, v) = V_3(u, v) + \mu v^{-2}/2$ in the KK case with the extended quartic potential $\tilde{V}_4(u, v) = V_4(u, v) + \delta v^{-2}/2$ in the 1 : 6 : 8 case. This explains why the combination $u^2 + v^2/2 + 4c$ of the solutions (35) is identical to the solution (43). The question remains open to extrapolate the present results to the case $\alpha \neq \beta$, i.e. $\gamma \delta \neq 0$.

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