Bottlenecks, burstiness, and fat tails regulate mixing times of non-Poissonian random walks

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Abstract: A broad range of complex systems can conveniently be modeled as networks, with nodes representing their elements, and links their interactions. Over the years, researchers have looked for non-trivial patterns in complex networks, and studied how these patterns affect spreading. This focus on structure has shown, for instance, that the degree distribution and community structure significantly affect the dynamics of random walks, epidemics, and synchronization. These results implicitly rely on the assumption that diffusion takes place on static structures, i.e. the time scale of network evolution is slower than that of the process on it. However, there is growing evidence that the opposite actually takes place in several systems, i.e. the structure of the network changes faster than the dynamics on the network. Network evolution generally exhibits complex temporal patterns at the node and the link level, which put an additional constraint on dynamics. For instance, it has been shown that, with the same underlying topology, the burstiness of network activity tends to speed up the dynamics at early stages, and to slow it down in the asymptotic regime.

From a theoretical point of view, research has mainly focused on the effect of burstiness on diffusion, or that of the correlations between consecutive activations of edges. However, most theoretical results exclusively focus on temporality, e.g. by considering trivial network structures such as trees. This limitation fails at capturing the interplay between topology and time, as both factors may compensate each other or simply add up into a single direction. The main purpose of this Letter is to address this limitation and develop a theoretical framework combining time and topology. As a prototype for more general dynamical processes, we consider a random walker on a network with arbitrary waiting time distribution, i.e. distribution of times between the arrival and departure of the walker on the node. Our main result is an analytical expression for the mixing time, i.e. the inverse of the asymptotic convergence rate to the stationary distribution, and its validation on empirical networks.

In the case of standard random walks, in discrete or continuous time, the mixing time is determined by the spectral gap of the transition matrix, itself strongly related to the tightest bottleneck of the network (as quantified by the Cheeger inequality). In other words, the mixing/diffusion is as fast as allowed by the tightest bottleneck in the graph, and the probability of presence of the walker at time converges to the stationary distribution as . As we will show in this Letter, when the timings at which random walkers jump are non-Poissonian, the diffusion happens to be not only steered by the topology but also by the coefficient of burstiness, related to second moment of the waiting time distribution, and to the asymptotic decrease of the tail of the same distribution as . The slowest of these three factors, i.e. topology, burstiness, and tail, determines the time required for the process to reach stationary.

A random walk process on a network is mathematically governed by the so-called master equation describing the time evolution of the probability of finding a walker on each node. For a random walker starting on node at time zero, let be the probability that the first jump is to , and let be the probability to be at node at time . The scalars and can be arranged into the matrices and . In the simplest case when the jumps occur at discrete times , the master equation is written:
and its solution is given by \( p(t) = P^t \). In the case of continuous times, assuming that the random process is Poisson, and that the random walker is susceptible to jump in any time interval \([t, t + \Delta t]\) with probability \( \tau^{-1} \Delta t + o(\Delta t) \), the master equation becomes differential

\[
\dot{p}(t) = -p(t)\tau^{-1}(I - P),
\]

where \( I \) is the identity matrix, and where the solution is given by \( p(t) = \exp(-\tau^{-1}(I - P)) \). \( I - P \) is the normalised Laplacian of the underlying, possibly weighted and directed, network. Among its spectral properties, the spectral gap, or ‘algebraic connectivity’, plays a fundamental role. It is defined as the lowest positive eigenvalue and \( I - P \). This quantity is fully determined by the topology of the network and, in the particular case of undirected non-bipartite networks, it is associated to the tightest bottleneck separating two clusters in the network [15].

The case where the jump probability depends on the waiting time spent on a node after arriving on it is a generalization of the two previous cases. Let us place time zero at the instant where the walker has arrived a node \( \tau \), and \( P_{ij}(t) \) be the probability rate that the first jump at a positive time occurs at time \( t \) to node \( j \). Thus \( \int_0^\infty P_{ij}(t)dt = P_{ij} \). The walker is at node \( j \) at time \( t \), either because it has not jumped between 0 and \( t \), or because it has first jumped to some node \( k \) at some time \( t' < t \), and managed to reach \( j \) during the remaining time \( t - t' \) after this first jump. This recursive decomposition allows to compute \( p_{ij}(t) \) as the sum of \( \sum_k \int_0^\infty P_{ik}(r)dr \) and \( \sum_k \int_0^\infty P_{ki}(r)p_{kj}(t) \) [12]. Due to the convolution in the latter term, the master equation is best expressed in the Laplace domain

\[
p(s) = \frac{1}{s}(I - D(s)) + P(s)p(s),
\]

where we denote the Laplace transforms in boldface. The matrix \( D(t) \) is the diagonal matrix with \( D_{ii}(t) = \sum_j P_{ij}(t) \). This equation is akin to Montroll-Weiss equation in statistical physics and to the Bellman-Harris equation in epidemic processes, and has the solution

\[
p(s) = \frac{1}{s}(I - P(s))^{-1}(I - D(s)).
\]

In this Letter, we focus on the homogeneous case when \( p_{ij}(t) \) has the same expression \( \rho(t) \) for each available link, in which case \( P(t) = \rho(t)P \). Let us note that the Laplace transform \( \rho(s) \) is known as the moment-generating function of the waiting time distribution, because its Taylor series, if defined, writes \( 1 - ts + (\tau^2 + \sigma^2)s^2/2! - \ldots \), where \( \tau^2 + \sigma^2 \) is the second moment (i.e. expected square waiting time) and \( \sigma^2 \) is the variance. In this case of homogeneous waiting time distributions, the solution of the master equation simplifies to

\[
p(s) = \frac{1 - \rho(s)}{s}(I - \rho(s)P)^{-1}.
\]

We observe that for \( \rho(t) = \delta(\tau) \) (discrete-time) and \( \rho(t) = e^{-t/\tau} \) (memoryless continuous time, for a Poissonian walker), we recover Eqs. 1 and 2 in the Laplace domain.

The matrix \( \rho(t) \) converges towards \( \rho(\infty) \), every row of which is the same unique stationary distribution, provided that the random walk is mixing, and which happens to coincide with the asymptotic solution \( P(\infty) \) independently of the exact form of \( \rho(t) \). The mixing time \( \tau_{\text{mix}} \) can be characterized in the Laplace domain, since \( p(t) - p(\infty) \sim \exp(-t/\tau_{\text{mix}}) \) if and only if the Laplace transform \( p(s) - p(\infty)/s \) is analytic in the half plane \( \Re(s) > -1/\tau_{\text{mix}} \) but has a point of non-analyticity (i.e. a point where no analytic continuation exists) on the line \( \Re(s) = -1/\tau_{\text{mix}} \). The right-hand side of Eq. 5 shows that non-analyticity occurs either if \( \rho(s) \) is non-analytic, or at a singularity of \((I - \rho(s)P)^{-1} \), and only in those two circumstances. The latter case involves finding the right-most solution of \( \det(I - \rho(s)P) = 0 \), which reduces to

\[
\rho(s) = 1/\lambda
\]

where \( \lambda \) is any eigenvalue of \( P \). Call \( -1/\tau_0 \) the real part of the right-most solution \( s \) of Eq. 6 (set to \( -\infty \) in case no such \( s \) exists), and call \( -1/\tau_{\text{tail}} \) the real part of the right-most point of non-analyticity of \( \rho(s) \). The tail characteristic time \( \tau_{\text{tail}} \) is such that \( \rho(t) \) decreases asymptotically as \( \exp(-t/\tau_{\text{tail}}) \). The random walker’s mixing time is therefore

\[
\tau_{\text{mix}} = \max(\tau_0, \tau_{\text{tail}}).
\]

In the case of a reversible mixing random walk, i.e. when \( P \) describes a simple random walk on an undirected connected graph [15], as we suppose from now on, the optimal \( \lambda \) in Eq. 6 is the second eigenvalue \( 1 - \epsilon \) of \( P \), where \( \epsilon \) is the spectral gap of the network. Therefore we find \( \tau_0 \) as the solution of

\[
\rho(-1/\tau_0) = \frac{1}{1 - \epsilon}
\]

An approximate solution of this equation is obtained from a second-order Padé (i.e. rational) approximation

\[
\rho(s) \approx \frac{2\tau + (\sigma^2 - \tau^2)s}{2\tau + (\sigma^2 + \tau^2)s}
\]
from which one finds
\[ \tau_0 \approx \frac{\tau}{\epsilon} (1 + \beta \epsilon) = \tau (\epsilon^{-1} + \beta). \] (10)

The coefficient \( \beta = (\sigma^2 - \tau^2)/2\tau^2 \) is a measure of the burstiness of the process, as it is zero for Poisson waiting times, positive for a larger-than-Poisson variance, and negative for smaller-than-Poisson variance, taking minimal value \(-1/2\) for discrete time, i.e. deterministic waiting times. The approximation in Eq. (9) is exact for Poisson random walkers, and otherwise acceptable for small \( s \), i.e. large \( \tau_0 \). Outside the range of validity of the approximation, higher moments have to be involved to get a more accurate estimate.

**FIG. 1.** (Color online) a) Impact of the burstiness and the spectral gap in the slowdown factor \( 1 + \beta \epsilon \) of Eq. (10) b) Estimated burstiness and spectral gap of empirical networks. (■) Original network; (○) original topology, randomized time-stamps; (●) fixed time-stamps, randomized topology conserving degree; (△) randomized time-stamps and topology. Randomization conserves the number of links. Poissonian walkers sit on the level curve 1, while the level curve 2, in the absence of tail effect, separates the bottleneck and burstiness driven phases of Eq. (11).

Eq. (10) identifies the factor \( 1 + \beta \epsilon \) by which burstiness affects the standard Poisson mixing time \( \tau/\epsilon \). The fact that this factor combines temporal and structural information has several consequences. In particular it shows that burstiness slows down the walker more efficiently on random networks that have no bottlenecks, such as the Erdős-Rényi and configuration models, or small diameter graphs with no communities. For instance, a \( \sigma^2/\tau^2 \) ratio of 10 has virtually no effect on the slowdown factor for \( \epsilon \approx 0.1 \), yet has a significant impact for larger values of \( \epsilon \) (Fig. 1a). These competing mechanisms lead to contrasting effects at different network scales. In a network with community structure (e.g. high modularity), and thus with tight bottlenecks between communities, random walkers quickly reach a quasi-stationary distribution within the community, and at larger time scales only the global stationary distribution \( \tau_{\text{mix}} \). In other words, the temporal activity regulates the slowdown inside the community while globally the network topology determines the convergence to the stationary state.

The link to empirical temporal networks is done by performing a random walk on the time-stamped empirical data. We start a large number of walkers on each node at \( t = 0 \). As time goes by, a walker remains in the node until one of its links becomes active and then jumps to one of the neighbors. The distribution of inter-jump or waiting times provides us with empirical measures of \( \tau \), \( \sigma^2 \) and \( \tau_{\text{tail}} \), assumed to be homogeneous across the network. The process is applied to 5 different categories of empirical networks: face-to-face interactions between visitors in a museum (SPM) and between conference attendees (SPC) \[18\]; email communication within a university (EMA) \[19\]; sexual contacts between sex-workers and -buyers (SEX) \[20\]; and communication between members of a dating site (POK) \[21\] (see Table I).

Figure 1b shows the estimated parameters for different networks. In the original networks, the slowdown factor is smaller for EMA and roughly the same for SPC and SPM. The impact of both topology and temporal randomization is stronger in SPC and SPM. In particular, in those cases, the drop on \( \beta \) created by a time stamps randomization is partly compensated by the rise of \( \epsilon \) induced by a link randomization. These values are highly dependent on the randomization protocol and on which network characteristics are conserved.

| \( N \) | \( L \) | \( \epsilon \) | \( \tau \) | \( \sigma^2 \) | \( \tau_{\text{tail}} \) | \( \delta t \) | \( D \) |
|---|---|---|---|---|---|---|---|
| SPC 113 | 20,818 | 0.073 | 16.2 | 10,238 | 182.5 | min | Bu |
| SPM 72 | 6,980 | 0.142 | 3.3 | 247 | 48.8 | min | Ta |
| EMA 3,186 | 309,120 | 0.074 | 14.5 | 3,528 | 177.4 | hour | To |
| SEX 11,416 | 33,645 | 0.012 | 18 | 1,037 | 20.2 | week | To |
| POK 28,295 | 529,890 | 0.005 | 214 | 1,189 | 1,021,675 | hour | To |

**TABLE I.** Summary statistics of the empirical networks: number of vertices (\( N \)); number of links (\( L \)); spectral gap (\( \epsilon \)) for the aggregated weighted network; the mean (\( \tau \)) and variance (\( \sigma^2 \)) of the distribution of waiting times; \( \tau_{\text{tail}} \) is the least-square estimate of the tail characteristic time of the cumulative distribution of waiting times; \( \delta t \) is the temporal resolution; and \( D \) refers to the slowdown driving mechanism according to Eq. (11) (Bu) burstiness, (Ta) tail, or (To) topology. We use the largest connected component.

In Eq. (10) the approximation \( \epsilon^{-1} + \beta \approx \max(\epsilon^{-1}, \beta) \) is valid, provided that the two terms are positive and dissimilar in order of magnitude. Under those conditions, the mixing time is given by

\[ \tau_{\text{mix}} \approx \max(\tau/\epsilon, \tau \beta, \tau_{\text{tail}}). \] (11)

Eq. (11) highlights three competing factors regulating the mixing time. While \( \tau/\epsilon \) is essentially a topological factor, capturing the effect of the structural bottleneck, the second term captures the burstiness-driven slowdown (as in \[3\]) and \( \tau_{\text{tail}} \) quantifies the ‘fatness’ of the tail of the distribution of waiting times \[7, 9\]. Burstiness in the waiting times should not be confounded with the burstiness in inter-contact times, in case of a random walker on a stochastic temporal network jumping to a
neighbor as soon as a contact is established. Application of the bus paradox in renewal theory shows the latter’s impact on the average waiting time \[ τ_{12} \leq \frac{1}{\rho} \] , while we focus on the slowdown due to the variance of the waiting time.

Burstiness and tail characteristic time capture two unrelated aspects of the waiting time distribution \[ τ_{tail} \]. The burstiness depends on the ‘bulk’ of the distribution and on the low order moments, however, the tail characteristic time depends on the asymptotically high moments. A large \( τ_{tail} \) translates into quick asymptotic growth of high moments. For instance a distribution \( ρ(t) \propto t^{-γ} \) for \( t \geq 1 \) and constant between 0 and 1 may exhibit a positive or a negative burstiness, yet has in all cases a fat tail with \( τ_{tail} = ∞ \). Conversely, \( ρ(t) \propto (t+1)^{-γ} \) (for \( 1 \leq t \leq τ^* \)), and \( ρ(t) \propto \exp(-t/τ_{tail}) \) for \( t > τ^* \), has an arbitrary tail characteristic time that has arbitrarily little influence on burstiness, which is mainly determined by the shape of \( ρ \) until \( τ^* \).

In the extreme case of waiting times following a pure power law, i.e. \( τ_{tail} = ∞ \), infinite mixing time occurs irrespective of the topology. After any time \( t \), there will be a probability \( 1 - \int_0^t ρ(r)dr \), polynomially decreasing, that the random walker has not moved at all from its initial node. Therefore no exponential convergence to stationarity is possible, and \( τ_{mix} = ∞ \). On the other hand, if an soft (exponential) cutoff is added so that \( ρ(t) \propto (t+1)^{-γ} \exp(-t/τ_{tail}) \), the slowdown is either driven by the tail or by the topology (Fig. 2(d)). Above a certain \( ϵ^* \), the tail completely determines the mixing time, which is not surprising considering that large \( ϵ \) corresponds to the absence of bottlenecks. Below this threshold, for instance at \( ϵ = 0.05 \) (Fig. 2(a)), there is a strong dependence on the topology for a range of exponents \( γ \) and \( τ_{tail} \). In this scenario, burstiness never dominates, and the power law exponent \( γ \) does not play a direct role other than influencing \( τ \). Nevertheless, if the distribution of waiting times is modelled by a power law with sharp cutoff, e.g. \( ρ(t) \propto (t+1)^{-γ} \exp(-t/τ_{sharp}) \), and \( ρ(t) = 0 \) for \( t > τ_{sharp} \), there is no tail, and only burstiness and topology compete to regulate the slowdown (Fig. 2(f)). We have compared our approximation to the exact solution of Eq. 8 and it generally works very well (error < 5%). In the case of the power law with sharp cutoff, the approximation is also valid for \( γ \leq 2 \).

If we plug in the estimated values of the empirical networks in Eq. 11 we identify that with the exception of the face-to-face networks SPM and SPC, in the other cases the slowdown is driven by the topology (Table 1). This result is consistent with Fig. 1(b), where the slowdown factor significantly drops for SPM and SPC when the time-stamps are randomized, suggesting a stronger dependence on time rather than topology.

In conclusion, we have presented a unified framework to quantify the importance of temporal patterns and network topology in the convergence of random walk processes toward stationarity. One key observation is that mixing time is more sensitive to temporal bursts in the absence of topological bottlenecks. In contrast, in the presence of bottlenecks, the walker dynamics becomes robust to variations in burstiness. These results are valid for any type of shape for the waiting time distribution. We have also highlighted two contrasting temporal aspects of the distribution of waiting times, their burstiness and their tail, and shown that they regulate the dynamics on networks in different ways. Finally, our measurements on empirical data suggest that topology remains a dominant mechanism for the dynamics of random walks on several real-life networks.

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**FIG. 2.** (Color online) The slowdown factor \( τ_{mix}/τ \) (1st column) and the mixing time (2nd column) for a given distribution of waiting times. Panels (a,b) correspond to fixed \( ϵ = 0.05 \) and (c,d) to fixed \( γ = 3 \) for soft cutoff; and (e,f) correspond to fixed \( γ = 2 \) for sharp cutoff. Black curves divide the phases where either topology, or burstiness, or tail dominates.
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