Anisotropic Moser-Trudinger inequality involving $L^n$ norm in the entire space $\mathbb{R}^n$

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Abstract

Let $F : \mathbb{R}^n \to [0, +\infty)$ be a convex function of class $C^2(\mathbb{R}^n \setminus \{0\})$ which is even and positively homogeneous of degree 1, and its polar $F^0$ represents a Finsler metric on $\mathbb{R}^n$. The anisotropic Sobolev norm in $W^{1,n}(\mathbb{R}^n)$ is defined by

$$||u||_{F} = \left( \int_{\mathbb{R}^n} F^n(\nabla u) + |u|^n \right)^{\frac{1}{n}}.$$ 

In this paper, the following sharp anisotropic Moser-Trudinger inequality involving $L^n$ norm

$$\sup_{u \in W^{1,n}(\mathbb{R}^n), ||u||_F \leq 1} \int_{\mathbb{R}^n} \Phi \left( \lambda_n |u|^n + (1 + \alpha ||u||_n)^{\frac{1}{n-1}} \right) dx < +\infty$$

in the entire space $\mathbb{R}^n$ for any $0 \leq \alpha < 1$ is established, where $\Phi (t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$, $\lambda_n = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}}$ and $\kappa_n$ is the volume of the unit Wulff ball in $\mathbb{R}^n$. It is also shown that the above supremum is infinity for all $\alpha \geq 1$. Moreover, we prove the supremum is attained, namely, there exists a maximizer for the above supremum when $\alpha > 0$ is sufficiently small. The proof of main results in this paper is based on the method of blow-up analysis.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. It is well known that $W^{1,n}_0(\Omega)$ is embedded into $L^p(\Omega)$ for any $p > 1$. Namely, using the Dirichlet norm $||u||_{W^{1,n}_0(\Omega)} = (\int_{\Omega} |\nabla u|^n dx)^{\frac{1}{n}}$ on
$W^{1,n}_0(\Omega)$, we have
\[
\sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} |u|^p \, dx < +\infty.
\]

But $W^{1,n}_0(\Omega)$ is not embedded into $L^\infty(\Omega)$. Hence, many mathematical researchers would like to look for a function $g(s) : \mathbb{R} \rightarrow \mathbb{R}^+$ with maximal growth such that
\[
\sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} g(u) \, dx < +\infty.
\]

The Moser-Trudinger inequality states that the maximal growth function is of exponential type, which was shown by Pohozhaev [31], Trudinger [40] and Moser [28]. This inequality says that
\[
(1.1) \quad \sup_{u \in W^{1,n}_0(\Omega), \|\nabla u\|_{L^n(\Omega)} \leq 1} \int_{\Omega} e^{\alpha |u|^n} \, dx < +\infty
\]
for any $\alpha \leq n \omega^{-1}_{n-1} := \alpha_n$, where $\omega_{n-1}$ is the surface area of the unit ball in $\mathbb{R}^n$. Also the inequality is optimal, that is, for any $\alpha > \alpha_n$ there exists a sequence of $\{u_\epsilon\}$ in $W^{1,n}_0(\Omega)$ with $\|\nabla u_\epsilon\|_{L^n(\Omega)} \leq 1$ such that
\[
\int_{\Omega} e^{\alpha |u_\epsilon|^n} \, dx \to +\infty \quad \text{as} \quad \epsilon \to 0.
\]

Whether extremal functions exist or not is another interesting question about Moser-Trudinger inequality. Carleson and Chang [9] firstly proved that the supremum is attained when $\Omega$ is a unit ball in $\mathbb{R}^n$. Then Struwe [35] got the existence of extremals for $\Omega$ close to a ball. Struwe’s technique was then used and extended by Flucher [15] to $\Omega$ which is the more general bounded smooth domain in $\mathbb{R}^2$. Later, Lin [21] generalized the existence result to a bounded smooth domain in $\mathbb{R}^n$.

Numerous generalizations, extensions and applications of the Moser-Trudinger inequality have been obtained due to important applications in partial differential equations and geometric analysis (see [2]-[4], [10]-[12], [14]-[27], [29], [30], [45]-[47] and references therein). We recall in particular the famous concentration-compactness result obtained by Lions [22], which says that if $\{u_k\}$ is a sequence of functions in $W^{1,n}_0(\Omega)$ with $\|\nabla u_k\|_{L^n(\Omega)} = 1$ such that $u_k \rightharpoonup u$ weakly in $W^{1,n}(\Omega)$, then for any $0 < p < (1 - \|\nabla u\|_{L^n(\Omega)}^{n/(n-1)})^{-1/(n-1)}$, it follows
\[
\sup_{k \to \infty} \int_{\Omega} e^{\alpha_n p |u_k|^n} \, dx < +\infty.
\]

Based on the result of Lions and the blowing up analysis method, Adimurthi and Druet [4] obtained an improved Moser-Trudinger type inequality in $\mathbb{R}^2$ on bounded domains $\Omega$, which
Anisotropic Moser-Trudinger inequality in $\mathbb{R}^n$

can be described as follows
\[
\sup_{\|\nabla u\|_2 \leq 1, u \in W^{1,2}(\Omega)} \int_{\mathbb{R}^2} e^{4\pi|u|^2(1+\alpha\|u\|_2^2)} \, dx < +\infty \text{ if and only if } \alpha < \inf_{u \in W^{1,2}_0(\Omega), u \neq 0} \frac{\|\nabla u\|_2^2}{\|u\|_2^2}.
\]

Later, this result was extended to high dimension and $L^p$ norm in two dimension or high dimension in Yang [43], Lu-Yang [24, 25] and Zhu [52].

Related Moser-Trudinger inequalities for unbounded domains have been first considered by Cao [8] in dimension two and for any dimension by do Ò [12] and Adachi-Tanaka [1]. In [32], Ruf showed that in the case of dimension two, one obtains that
\[
(1.2) \sup_{f_{\mathbb{R}^2}([\|u\|^2 + |\nabla u|^2]dx \leq 1, u \in W^{1,2}(\mathbb{R}^2)} \int_{\mathbb{R}^2} \phi (\alpha |u|^2) \, dx < +\infty \text{ if and only if } \alpha \leq 4\pi,
\]
where $\phi (t) = e^t - 1$. Li and Ruf [20] extended Ruf’s result to arbitrary dimension. Later, Souza and do Ò [34] obtained an Adimurthi-Druet type result in $\mathbb{R}^2$ for some weighted Sobolev space. Recently, Lu and Zhu [26] proved a sharp Moser-Trudinger inequality involving $L^n$ norm in $\mathbb{R}^n$.

The one interesting extension of (1.1) is to establish anisotropic Moser-Trudinger inequality which involves $n$-anisotropic Laplacian (or $n$-Finsler Laplacian) $Q_n$ as follows:
\[
Q_n u := \sum_{i=1}^n \frac{\partial}{\partial x_i} (F^{n-1}(\nabla u) F^n_{\xi_i}(\nabla u)).
\]

Here the function $F(x)$ is positive, convex and homogeneous of degree 1, and its polar $F^0(x)$ represents a Finsler metric on $\mathbb{R}^n$. The properties of the operator $Q_n$ was researched by Gong and the author of this paper in [44].

In 2012, Wang and Xia [42] proved the following anisotropic Moser-Trudinger inequality
\[
(1.3) \int_{\Omega} e^{\lambda |u|^{\frac{n}{n-1}}} \, dx \leq C(n)|\Omega|
\]
for all $u \in W^{1,n}_0(\Omega)$ and $\int_{\Omega} F^n(\nabla u) \, dx \leq 1$. Here $\lambda \leq \lambda_n := n^{\frac{n}{n-1}} \kappa_n^{\frac{n-1}{n}}$, where $\kappa_n$ is the volume of the unit Wulff ball in $\mathbb{R}^n$, i.e. $\kappa_n = |\{ x \in \mathbb{R}^n : F^0(x) \leq 1\}|$. $\lambda_n$ is optimal in the sense that if $\lambda > \lambda_n$ one can find a sequence $\{u_k\}$ such that $\int_{\Omega} e^{\lambda |u_k|^{\frac{n}{n-1}}} \, dx$ diverges. Later, Zhou and Zhou [48, 50] have shown that the supremum is attained when $\Omega$ is bounded domain in $\mathbb{R}^n$. Recently, Zhou [48] obtained the anisotropic Moser-Trudinger inequality involving $L^n$ norm in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ and Liu [23] extended the corresponding result to $L^p$ norm. On the unbounded domain in $\mathbb{R}^n$, Zhou and Zhou [51] established the anisotropic Moser-Trudinger inequality.
In this paper, we will research the anisotropic Moser-Trudinger type inequality involving $L^n$ norm and its extremal functions in the entire space $\mathbb{R}^n$. The isotropic Dirichlet norm $\|u\|_{W^{1,n}_0(\Omega)} = (\int_\Omega |\nabla u|^n dx)^{\frac{1}{n}}$ will be replaced by the anisotropic Dirichlet norm $\left(\int_\Omega F^n(\nabla u) dx\right)^{\frac{1}{n}}$ on $W^{1,n}_0(\Omega)$. Also, the isotropic Sobolev norm will be replaced by the anisotropic Sobolev norm $|| u ||_{F} = \left(\int_{\mathbb{R}^n} F^n(\nabla u) + |u|^n \right)^{\frac{1}{n}}$.

Now we stated the main results in this paper as follows.

**Theorem 1.1.** For any $0 \leq \alpha < 1$, we have

\[
\sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_F \leq 1} \int_{\mathbb{R}^n} \Phi \left( \lambda_n |u|^{\frac{n}{n-1}} (1 + \alpha \|u\|_n)^{\frac{1}{n-1}} \right) dx < +\infty,
\]

where $\Phi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$. Moreover, for any $\alpha \geq 1$, the supremum is infinite.

Set

\[ S = \sup_{u \in W^{1,n}(\mathbb{R}^n), \|u\|_F \leq 1} \int_{\mathbb{R}^n} \Phi \left( \lambda_n |u|^{\frac{n}{n-1}} (1 + \alpha \|u\|_n)^{\frac{1}{n-1}} \right) dx. \]

**Theorem 1.2.** There exists $u_\alpha \in W^{1,n}(\mathbb{R}^n)$ with $\|u_\alpha\|_F = 1$ such that

\[ S = \int_{\mathbb{R}^n} \Phi \left( \lambda_n |u_\alpha|^{\frac{n}{n-1}} (1 + \alpha \|u_\alpha\|_n)^{\frac{1}{n-1}} \right) dx \]

for sufficiently small $\alpha$.

This paper is organized as follows. In Section 2 we recall some notations and preliminaries which will be used later. Section 3 is devoted to proving the existence of radially symmetric maximizing sequence for the critical functional. In Section 4 we give the proof of Theorem 1.1. We prove the sharpness of the inequality in Theorem 1.1 i.e. the second part of Theorem 1.1 by constructing a appropriate test function sequence in Subsection 4.1. In Subsection 4.2, we prove the first part of Theorem 1.1 by considering the two cases. In Subsection 4.2.1, we prove the first part of Theorem 1.1 in the case of $\sup k c_k < +\infty$. The proof in the case of $\sup k c_k = +\infty$ is arranged in Subsection 4.2.2, we apply the blowing up analysis to analyze the asymptotic behavior of the maximizing sequence near and far away from the origin, and give the proof of the first part of Theorem 1.1 in this case. In Section 5, we also prove Theorem 1.2 by considering the two cases, which are $\sup k c_k < +\infty$ and $\sup k c_k = +\infty$. In Subsection 5.1, based on the concentration-compactness lemma, we give the proof of Theorem 1.2 in the case of $\sup k c_k < +\infty$. In Subsection 5.2, we prove the result...
in case of $\sup_k c_k = +\infty$ by contradiction. For this, we first establish the upper bound for critical functional when $\sup_k c_k = +\infty$, and then construct an explicit test function, which provides a lower bound for the supremum of our Moser-Trudinger inequality. Because this lower bound equals to the upper bound, one can obtain the contradiction and prove Theorem 1.2 in this case.

Throughout this paper, the letter $C$ denotes a constant independent of the main functions which may be different from line to line.

### 2 Notations and preliminaries

In this section, let us recall some important concepts and preliminaries which will be use later in this paper.

Throughout this paper, let $F : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative convex function of class $C^2(\mathbb{R}^n \setminus \{0\})$ which is even and positively homogenous of degree 1, i.e.

$$F(t\xi) = |t|F(\xi) \quad \text{for any} \quad t \in \mathbb{R}, \quad \xi \in \mathbb{R}^n.$$  

A typical example is $F(\xi) = (\sum_i |\xi_i|^q)^{\frac{1}{q}}$ for $q \in [1, \infty)$. We further assume that

$$F(\xi) > 0 \quad \text{for any} \quad \xi \neq 0.$$  

With the help of homogeneity of $F$, there exist two constants $0 < a \leq b < \infty$ such that

$$a|\xi| \leq F(\xi) \leq b|\xi|.$$  

Usually, we shall assume that the $\text{Hess}(F^2)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$. Then by Gong and the author of this paper in [44], $\text{Hess}(F^n)$ is also positive definite in $\mathbb{R}^n \setminus \{0\}$. Considering the minimization problem

$$\min_{u \in W^{1,n}_0(\Omega)} \int_{\Omega} F^n(\nabla u) dx,$$

its Euler-Lagrange equation contains an operator of the form

$$Q_n u := \sum_{i=1}^n \frac{\partial}{\partial x_i} (F^{n-1}(\nabla u) F_{\xi_i}(\nabla u)),$$

which is called as n-anisotropic Laplacian or n-Finsler Laplacian.

It is well known that in the isotropic case, i.e. $F(\xi) = |\xi|$, when $n = 2$, $Q_n$ is the ordinary Laplacian operator; when $n > 2$, $Q_n$ is the $n$-Laplacian operator. In the anisotropic case, when $n = 2$, $Q_n$ is anisotropic Laplacian operator. The operator $Q_n$ was studied by many researchers, see [13, 12, 5, 7, 44] and their references therein.

Consider the map

$$\phi : S^{n-1} \to \mathbb{R}^n, \quad \phi(\xi) = F_{\xi}(\xi).$$
Its image $\phi(S^{n-1})$ is a smooth, convex hypersurface in $\mathbb{R}^n$, which is called Wulff shape of $F$. Let $F^o$ be the support function of $K := \{ x \in \mathbb{R}^n : F(x) \leq 1 \}$, which is defined by

$$F^o(x) := \sup_{\xi \in \mathbb{R}^n} \langle x, \xi \rangle.$$  

It is easy to prove that $F^o : \mathbb{R}^n \mapsto [0, +\infty)$ is also a convex, homogeneous function of class $C^2(\mathbb{R}^n \setminus \{0\})$. Actually, $F, F^0$ are polar to each other in the sense that

$$F^o(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)}, \quad F(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F^o(\xi)}.$$  

One can see easily that $\phi(S^{n-1}) = \{ x \in \mathbb{R}^n : F^o(x) = 1 \}$. Let $W_F := \{ x \in \mathbb{R}^n : F^0(x) \leq 1 \}$ and $\kappa_n = |W_F|$, which is the Lebesgue measure of $W_F$. Also, denote $W_r(x_0)$ by the Wulff ball of center at $x_0$ with radius $r$, i.e. $W_r(x_0) = \{ x \in \mathbb{R}^n : F^0(x - x_0) \leq r \}$.  

Next, we summarize the properties on $F$ and $F^0$, which can be proved easily by the assumption on $F$, also see [11, 13, 6].

**Lemma 2.1.** We have

(i) $|F(x) - F(y)| \leq F(x + y) \leq F(x) + F(y)$;

(ii) $\frac{1}{C} \leq |\nabla F(x)| \leq C$, and $\frac{1}{C} \leq |\nabla F^o(x)| \leq C$ for some $C > 0$ and any $x \neq 0$;

(iii) $\langle \xi, \nabla F(\xi) \rangle = F(\xi), \langle x, \nabla F^o(x) \rangle = F^o(x)$ for any $x \neq 0$, $\xi \neq 0$;

(iv) $F(\nabla F^o(x)) = 1$, $F^o(\nabla F(\xi)) = 1$ for any $x \neq 0$, $\xi \neq 0$;

(v) $F^o(x)F(\nabla F^o(x)) = x$ for any $x \neq 0$;

(vi) $F_\xi(t\xi) = \text{sgn}(t)F_\xi(\xi)$ for any $\xi \neq 0$ and $t \neq 0$.

Next we give the co-area formula and isoperimetric inequality in the anisotropic situation. For a bounded domain $\Omega \subset \mathbb{R}^n$ and a function of bounded variation $u \in BV(\Omega)$, denote the anisotropic bounded variation of $u$ with respect to $F$ by

$$\int_\Omega |\nabla u|_F = \sup \left\{ \int_\Omega u \text{div} \sigma dx : \sigma \in C_0^1(\Omega), F^0(\sigma) \leq 1 \right\},$$

and anisotropic perimeter of $E$ with respect to $F$ by

$$P_F(E) := \int_\Omega |\nabla \chi_E|_F,$$

where $E$ is a subset of $\Omega$ and $\chi_E$ is the characteristic function of $E$. The co-area formula and isoperimetric inequality can be expressed by

$$\int_\Omega |\nabla u|_F = \int_0^\infty P_F(|u| > t) dt, \tag{2.1}$$
and

\[(2.2) \quad P_F(E) \geq N k^{\frac{1}{n}} |E|^{1 - \frac{1}{n}} \]

respectively. Moreover, the equality in (2.2) holds if and only if $E$ is a Wulff ball.

In the sequel, we will use the convex symmetrization with respect to $F$. The convex symmetrization generalizes the Schwarz symmetrization (see [37]). It was defined in [5] and will be an essential tool for establishing the Lions type concentration-compactness theorem under the anisotropic Dirichlet norm. Let us consider a measurable function $u$ on $\Omega \subset \mathbb{R}^n$. The one dimensional decreasing rearrangement of $u$ is defined as

$$u^*(s) = \sup \{ t \geq 0 : |\{ x \in \Omega : |u(x)| > s \}| > t \}, \quad \text{for } t \in \mathbb{R}.$$  

The convex symmetrization of $u$ with respect to $F$ is

$$u^*(x) = u^*(\kappa_n F^\alpha(x)^n), \quad \text{for } x \in \Omega^*.$$  

Here $\kappa_n F^\alpha(x)^n$ is just Lebesgue measure of a homothetic Wulff ball with radius $F^\alpha(x)$ and $\Omega^*$ is the homothetic Wulff ball centered at the origin having the same measure as $\Omega$. Throughout this paper, we assume that $\Omega$ is bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$.

Now let us recall some important results which can be found in [50, 51]. Lemma 2.3 is also called the concentration-compactness lemma.

**Lemma 2.2.** Assume that $u \in W^{1,n}_0(\Omega)$ is a solution of the equation

\[(2.3) \quad -Q_n(u) = f.\]

If $f \in L^q(\Omega)$ for some $q > 1$, then $\|u\|_{L^\infty(\Omega)} \leq C \|f\|_{L^q(\Omega)}^{\frac{1}{n}}$.

**Lemma 2.3.** Let $\{u_k\}$ be a sequence in $W^{1,n}(\mathbb{R}^n)$ such that $\|u_k\|_F = 1$ and $u_k \rightharpoonup u \neq 0$, weakly in $W^{1,n}(\mathbb{R}^n)$. If

$$0 < p < p_n(u) := \frac{1}{(1 - \|u\|_F^n)^{1/(n-1)}},$$

then

$$\sup_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \lambda_n p \left| u_k \right|^\frac{n}{n-1} \right) dx < +\infty.$$ 

3 The maximizing sequence

Let $\{\beta_k\}$ an increasing sequence which converges to $\lambda_n$ and $\{R_k\}$ be an increasing sequence which diverges to infinity as $k \to \infty$. Denote

$$I^\alpha_{\beta_k} (u) = \int_{W_{R_k}} \Phi \left( \beta_k \left| u \right|^\frac{n}{n-1} \left( 1 + \alpha \left| u \right|_F^n \right)^\frac{1}{n-1} \right) dx$$

and

$$H = \{ u \in W^{1,n}_0(W_{R_k}) \mid \|u\|_F = 1 \}.$$
Lemma 3.1. For any $0 \leq \alpha < 1$, there exists an extremal function $u_k \in H$ such that

$$I_{\beta_k}^\alpha (u_k) = \sup_{u \in H} I_{\beta_k}^\alpha (u).$$

Proof. There exists a sequence $\{v_i\} \in H$ such that

$$\lim_{i \to \infty} I_{\beta_k}^\alpha (v_i) = \sup_{u \in H} I_{\beta_k}^\alpha (u).$$

Since $v_i$ is bounded in $W^{1,n} (\mathbb{R}^n)$, there exists a subsequence which will still be denoted by $v_i$, such that

$$v_i \rightharpoonup u_k \text{ weakly in } W^{1,n} (\mathbb{R}^n),$$

$$v_i \to u_k \text{ strongly in } L^s (\mathcal{W}_{R_k}),$$

$$v_i \to u_k \text{ a.e. in } \mathbb{R}^n$$

for any $1 < s < \infty$ as $i \to \infty$. Therefore

$$g_i = \Phi \left\{ \beta_k |v_i| \frac{n}{n-1} (1 + \alpha \|v_i\|_n^{n})^{\frac{1}{n-1}} \right\}$$

$$\to g_k = \Phi \left\{ \beta_k |u_k| \frac{n}{n-1} (1 + \alpha \|u_k\|_n^{n})^{\frac{1}{n-1}} \right\}$$

a.e. in $\mathbb{R}^n$. Next we claim that $u_k \neq 0$. If not, we have $1 + \alpha \|v_i\|_n^{n} \to 1$. Thus $g_i$ is bounded in $L^r (\mathcal{W}_{R_k})$ for some $r > 1$, then $g_i \to 0$. Hence $\sup_{u \in H} I_{\beta_k}^\alpha (u) = 0$, which is impossible. For any $p < p_n (u_k) := \frac{1}{(1 - \|u_k\|_F^{n})^{1/(n-1)}}$, it follows from Lemma 2.3 that

$$\limsup_{i \to \infty} \int_{\mathbb{R}^n} \Phi \left( \lambda_n p |v_i|^{\frac{n}{n-1}} \right) dx < +\infty.$$

Since $0 \leq \alpha < 1$, it is easy to see that

$$(1 + \alpha \|u_k\|_n^{n})^{\frac{1}{n-1}} < (1 + \|u_k\|_F^{n})^{\frac{1}{n-1}} < \frac{1}{(1 - \|u_k\|_F^{n})^{1/(n-1)}} = p_n (u_k),$$

then $g_i$ is bounded in $L^s$ for some $s > 1$ and $g_i \to g_k$ strongly in $L^1 (\mathcal{W}_{R_k})$ as $i \to \infty$. Thus the extremal function is attained for the case $\beta_k < \lambda_n$ and $\|u_k\|_F = 1$.

Similar as in [20, 26, 51], we have the following results.

Lemma 3.2. Let $u_k$ be as above, then

(i) $u_k$ is a maximizing sequence for $S$;

(ii) $u_k$ may be chosen to be radially symmetric and decreasing with respect to $F^0(x).$
Proof. (i) Let $\eta$ be a cut-off function which is 1 on $\mathcal{W}_1$ and 0 on $\mathbb{R}^n \setminus \mathcal{W}_2$. Then for any given $\varphi \in W^{1,n}(\mathbb{R}^n)$ with $\|\varphi\|_F = 1$, it follows that

$$\tau^n(L) := \int_{\mathbb{R}^n} (F^n(\nabla(\eta(\frac{x}{L})\varphi)) + \eta \left(\frac{x}{L}\right) \varphi \right)^n dx \to 1, \quad \text{as } L \to +\infty.$$  

Thus for a fixed $L$ and $R_k > 2L$, we have

$$\int_{W_L} \Phi \left( \beta_k \left| \frac{\varphi}{\tau(L)} \right|^{\frac{n}{n-1}} \left( 1 + \alpha \left\| \frac{\eta(\frac{x}{L}) \varphi}{\tau(L)} \right\|_n^{\frac{1}{n-1}} \right) \right) dx \leq \int_{W_{2L}} \Phi \left( \beta_k \left| \frac{\eta(\frac{x}{L}) \varphi}{\tau(L)} \right|^{\frac{n}{n-1}} \left( 1 + \alpha \left\| \frac{\eta(\frac{x}{L}) \varphi}{\tau(L)} \right\|_n^{\frac{1}{n-1}} \right) \right) dx \leq \int_{W_{R_k}} \Phi \left( \beta_k |u_k|^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_n^{\frac{1}{n-1}}) \right) dx.$$

Then it follows from Levi Lemma that

$$\int_{W_L} \Phi \left( \lambda_n \left| \frac{\varphi}{\tau(L)} \right|^{\frac{n}{n-1}} \left( 1 + \alpha \left\| \frac{\eta(\frac{x}{L}) \varphi}{\tau(L)} \right\|_n^{\frac{1}{n-1}} \right) \right) dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \beta_k |u_k|^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_n^{\frac{1}{n-1}}) \right) dx.$$

Taking limits $L \to +\infty$,

$$\int_{\mathbb{R}^n} \Phi \left( \lambda_n |\varphi|^{\frac{n}{n-1}} (1 + \alpha \|\varphi\|_n^{\frac{1}{n-1}}) \right) dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \beta_k |u_k|^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_n^{\frac{1}{n-1}}) \right) dx.$$

Thus

$$\lim_{k \to \infty} \int_{W_{R_k}} \Phi \left( \beta_k |u_k|^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_n^{\frac{1}{n-1}}) \right) dx = \sup_{\|u\|_F = 1} \int_{\mathbb{R}^n} \Phi \left( \lambda_n |u|^{\frac{n}{n-1}} (1 + \alpha \|u\|_n^{\frac{1}{n-1}}) \right) dx.$$

Let $u^*_k$ be convex symmetric rearrangement of $u_k$ with respect to $F^0(x)$, then

$$\tau^n_k := \int_{\mathbb{R}^n} (F^n(\nabla u^*_k) + |u^*_k|^n) dx \leq \int_{\mathbb{R}^n} (F^n(\nabla u_k) + |u_k|^n) dx = 1.$$

Therefore
\[
\int_{W_{R_k}} \Phi \left( \beta_k \left| \frac{u^*_k}{\tau_k} \right|^{\frac{n-1}{n}} \left( 1 + \alpha \left| \frac{u^*_k}{\tau_k} \right|^{n} \right)^{\frac{1}{n-1}} \right) \, dx \geq \int_{W_{R_k}} \Phi \left( \beta_k \left| u^*_k \right|^{\frac{n}{n-1}} \left( 1 + \alpha \| u^*_k \|^n \right)^{\frac{1}{n-1}} \right) \, dx.
\]

It is easy to see that
\[
\int_{W_{R_k}} \Phi \left( \beta_k \left| u^*_k \right|^{\frac{n}{n-1}} \left( 1 + \alpha \| u^*_k \|^n \right)^{\frac{1}{n-1}} \right) \, dx = \int_{W_{R_k}} \Phi \left( \beta_k \left| u_k \right|^{\frac{n}{n-1}} \left( 1 + \alpha \| u_k \|^n \right)^{\frac{1}{n-1}} \right) \, dx.
\]

Then one can obtain \( \tau_k = 1 \). Also we know the fact that \( \tau_k = 1 \) if and only if \( u_k \) is radial. Thus
\[
\int_{W_{R_k}} \Phi \left( \beta_k \left| u^*_k \right|^{\frac{n}{n-1}} \left( 1 + \alpha \| u^*_k \|^n \right)^{\frac{1}{n-1}} \right) \, dx = \sup_{\| u \|_{F} = 1} \int_{W_{R_k}} \exp \left\{ \beta_k \left| u \right|^{\frac{n}{n-1}} \left( 1 + \alpha \| u \|^n \right)^{\frac{1}{n-1}} \right\} \, dx.
\]

Therefore one can assume \( u_k = u_k(r) \) and \( u_k(r) \) is decreasing with respect to \( r = F^0(x) \).

4 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. Firstly, we prove the second part of Theorem 1.1 by the test functions argument. Then we prove the first part of Theorem 1.1 by considering two cases. Let \( c_k = \max u_k(x) = u_k(0) \). When \( \sup_k c_k < +\infty \), it can be proved by using the concentration-compactness lemma. When \( \sup_k c_k = +\infty \), we perform a blow-up procedure and prove the corresponding results.

4.1 Proof of the second part of Theorem 1.1

In this subsection, we will show that the supremum in Theorem 1.1 is infinity if \( \alpha \geq 1 \). Namely, we prove the sharpness of the inequality in Theorem 1.1. The proof of the second part of Theorem 1.1 is based on a test function argument. Unlike in the case for bounded domains, we cannot construct the test function by the eigenfunction of the first eigenvalue problem:
\[
\inf_{u \in W^{1,n}(\Omega), u \neq 0} \frac{\| F(\nabla u) \|^n}{\| u \|^n},
\]

since the above infimum is actually not attained when \( \Omega = \mathbb{R}^n \). To overcome this difficulty, we will construct a new test function sequence.
Proof of the Second Part of Theorem 1.1. Let

\[ u_k = \begin{cases} 
\frac{1}{(n\kappa)^n} (\log k)^{\frac{n-1}{n}}, & \text{if } 0 < F^0(x) \leq \frac{R_k}{k}, \\
\frac{1}{(n\kappa)^n} (\log k)^{-\frac{1}{n}} \log \frac{R_k}{F^0(x)}, & \text{if } \frac{R_k}{k} < F^0(x) \leq R_k, \\
0, & \text{if } F^0(x) > R_k,
\end{cases} \]

where \( R_k := \left(\frac{\log k}{\log \log k}\right)^{1/2n} \to +\infty \) as \( k \to \infty \). It is easy to verify that

\[ \int_{\mathbb{R}^n} F^n(\nabla u_k) dx = 1. \]

Also we have

\[ \|u_k\|_n^n = \int_{W_{R_k}/k} |u_k|^n dx + \int_{W_{R_k}/k} |u_k|^n dx \]
\[ = \frac{R_k^n}{\log k} \int_{\frac{1}{k}}^1 (\log r)^n r^{n-1} dr + \frac{(\log k)^{n-1}}{n} \left(\frac{R_k}{k}\right)^n \]
\[ = C_n \frac{R_k^n}{\log k} (1 + o(1)) \to 0 \text{ as } k \to \infty, \]

where \( C_n = \int_{\frac{1}{k}}^1 (\log r)^n r^{n-1} dr \). Thus

\[ \|u_k\|_F^n = 1 + \frac{C_n R_k^n}{\log k} (1 + o(1)). \]

Using the following fact

\[ 1 + \frac{\|u_k\|_n^n}{\|u_k\|_F^n} = \frac{1 + 2 \|u_k\|_n^n}{1 + \|u_k\|_n^n}, \]

then on the Wulff ball \( W_{R_k/k} \), it follows

\[ \lambda_n \frac{|u_k|^{\frac{n}{n-1}}}{\|u_k\|_F^{\frac{1}{n-1}}} \left(1 + \frac{\|u_k\|_n^n}{\|u_k\|_F^n}\right) \]
\[ = n^{\frac{n}{n-1}} \kappa_n^{\frac{1}{n-1}} |u_k|^{\frac{n}{n-1}} \frac{1 + 2 \|u_k\|_n^n}{\left(1 + \|u_k\|_n^n\right)^{\frac{1}{n-1}}} \]
\[ = n \log k \left(1 - \frac{1}{n-1} \|u_k\|_n^{2n} + \frac{2}{n-1} \|u_k\|_n^{3n} (1 + o(1))\right). \]

Therefore

\[ \sup_{\|u\|_F = 1} \int_{\mathbb{R}^n} \Phi \left(\lambda_n |u|^{\frac{n}{n-1}} (1 + \|u\|_n^n)\right) dx \]
\[ \geq C \int_{W_{R_k/k}} \exp \left( \lambda \frac{\| u_k \|^2_{H^1}}{\| u_k \|^2_F} \left( 1 + \frac{\| u_k \|^2_F}{\| u_k \|^2_F} \right) \right) dx \]
\[ \geq C \exp \left( n \log k \left( - \frac{1}{n-1} \| u_k \|^2_n + \frac{2}{n-1} \| u_k \|^3_n (1 + o(1)) \right) + n \log R_k \right) \]
\[ = C \exp \left( n \log k \left( - \frac{1}{n-1} \| u_k \|^2_n + \frac{2}{n-1} \| u_k \|^3_n (1 + o(1)) \right) + n \log R_k \right), \]

here we have used the following result

\[ |W_{R_k/k}| = \kappa_n \left( \frac{R_k}{k} \right)^n = \kappa_n \exp(n \log R_k - n \log k). \]

Since

\[ n \log R_k = n \log \left( \frac{(\log k)^{1/2n}}{\log \log k} \right) = \frac{1}{2} \log \log k - n \log \log k \]

and

\[ n \log k \left( - \frac{1}{n-1} \| u_k \|^2_n + \frac{2}{n-1} \| u_k \|^3_n (1 + o(1)) \right) \]
\[ = -n \frac{C^2_n R_k^{2n}}{\log k} (1 + o(1)) \]
\[ = -n \frac{C^2_n}{\log k} \left( \frac{1}{(\log \log k)^{2n}} (1 + o(1)) \right), \]

then

\[ \int_{\mathbb{R}^n} \Phi \left( \lambda \frac{|u_k|^2}{n-1} (1 + \| u_k \|^2_n) \right) dx \]
\[ \geq C \exp \left( n \log k ( - \| u_k \|^2_n (1 + o(1)) ) + n \log R_k \right) \]
\[ = C \exp \left( \frac{1}{2} \log \log k - n \log \log \log k - \frac{nC^2_n}{n-1} \frac{1}{(\log \log k)^{2n}} (1 + o(1)) \right) \]
\[ \to + \infty \quad \text{as } k \to \infty. \]

The proof of the second part of Theorem 1.1 has been completed.

4.2 Proof of the first part of Theorem 1.1

Now we consider two cases for the proof of the first part of Theorem 1.1. Denote \( c_k = \max u_k(x) = u_k(0) \). In the case of \( \sup_k c_k < +\infty \), the proof of Theorem 1.1 is indirect and easy. In the case of \( \sup_k c_k = +\infty \), we will use the blowing up analysis method, which is based on a blowing up analysis of sequences of solutions to \( n \)-anisotropic Laplacian in \( \mathbb{R}^n \) with exponential growth. The method has been successfully applied in the proof of the Moser-Trudinger inequalities and related extremal functions existence results in bounded domains (see [4, 52, 24, 25]) and in the unbounded domains (see [32, 26, 51]).
4.2.1 Proof in the case of \( \sup k \epsilon_k < +\infty \)

By the variational calculation, the Euler-Lagrange equation for the extremal function \( u_k \in W_0^{1,n} (\mathcal{W}_{R_k}) \) of \( I_{\beta_k}^\alpha (u) \) can be written as

\[
- Q_n(u_k) = \mu_k \lambda_k^{-1} u_k^{n-1} - (\gamma_k - 1) u_k^{n-1},
\]

where

\[
\begin{align*}
\mu_k &= (1 + \alpha \|u_k\|_n^n)/(1 + 2\alpha \|u_k\|_n^n), \\
\gamma_k &= \alpha/(1 + 2\alpha \|u_k\|_n^n), \\
\lambda_k &= \int_{\mathcal{W}_{R_k}} u_k^{n-1} \Phi' \left( \alpha_k u_k^{n-1} \right) dx.
\end{align*}
\]

Let us give the following important results firstly.

**Lemma 4.1.** \( \inf_k \lambda_k > 0 \).

**Proof.** We prove this result by contradiction. Assume \( \lambda_k \to 0 \) as \( k \to \infty \), then

\[
\lambda_k = \int_{\mathbb{R}^n} u_k^{n-1} \Phi' \left( \alpha_k u_k^{n-1} \right) dx = \int_{\mathbb{R}^n} u_k^{n-1} \sum_{j=n-2}^{\infty} \frac{(\alpha_k u_k^{n-1})^j}{j!} dx.
\]

(4.2)

Since \( u_k(r) \) is decreasing, we have \( u_k^n (L) \|\mathcal{W}_L\| \leq \int_{\mathcal{W}_L} u_k^n dx \leq 1 \), thus

\[
u_k^n (L) \leq \frac{1}{\kappa_n L^n}.
\]

Set \( \varepsilon^n = \frac{1}{\kappa_n L^n} \), then we get \( u_k \leq \varepsilon \) for any \( x \not\in \mathcal{W}_L \). Thus

\[
\int_{\mathbb{R}^n \setminus \mathcal{W}_L} \Phi \left( \alpha_k u_k^{n-1} \right) dx \leq C \int_{\mathbb{R}^n \setminus \mathcal{W}_L} u_k^n dx \leq C\lambda_k \to 0.
\]

It is easy to see that

\[
\Phi \left( \alpha_k u_k^{n-1} \right) = \sum_{j=n-1}^{\infty} \frac{(\alpha_k u_k^{n-1})^j}{j!} \leq \sum_{j=n-2}^{\infty} \frac{\alpha_k u_k^{n-1} (\alpha_k u_k^{n-1})^j}{(j+1) j!} \leq \alpha_k u_k^{n-1} \Phi' \left( \alpha_k u_k^{n-1} \right),
\]

then we obtain

\[
\lim_{k \to \infty} \int_{\mathcal{W}_L} \Phi \left( \alpha_k u_k^{n-1} \right) dx
\]
\[
= \lim_{k \to \infty} \left( \int_{W_L \cap \{u_k \geq 1\}} + \int_{W_L \cap \{u_k < 1\}} \right) \Phi \left( \alpha_k u_k^n \right) dx
\]
\[
\leq \lim_{k \to \infty} \left( C \int_{W_L} u_k^n dx + \int_{W_L \cap \{u_k < 1\}} \Phi \left( \alpha_k u_k^n \right) dx \right)
\]
\[
\leq C \lim_{k \to \infty} \left( \lambda_k + \int_{W_L} u_k^n dx \right).
\]

By \((4.2)\), it follows that \(\int_{W_L} u_k^n dx \to 0\). Thus for any \(q > 1\), we obtain
\[
\lim_{k \to \infty} \int_{W_L} \Phi \left( \alpha_k u_k^n \right) dx = 0.
\]

It is impossible and the proof of Lemma 4.1 is finished. \(\square\)

Now we recall the concept of Sobolev-normalized concentrating sequence and concentration-compactness principle as in [32].

**Definition 4.1.** A sequence \(\{u_k\} \in W^{1,n}(\mathbb{R}^n)\) is a Sobolev-normalized concentrating sequence, if

i) \(\|u_k\|_F = 1\);

ii) \(u_k \rightharpoonup 0\) weakly in \(W^{1,n}(\mathbb{R}^n)\);

iii) there exists a point \(x_0\) such that for any \(\delta > 0\),

\[
\int_{\mathbb{R}^n \setminus W_L}(F^n(\nabla u_k) + |u_k|^n) dx \to 0.
\]

From Lemma 2.3 in this paper, we have the following result.

**Lemma 4.2.** Let \(\{u_k\}\) be a sequence satisfying \(\|u_k\|_F = 1\), and \(u_k \rightharpoonup u\) weakly in \(W^{1,n}(\mathbb{R}^n)\). Then either \(\{u_k\}\) is a Sobolev-normalized concentrating sequence, or there exists \(\gamma > 0\) such that \(\Phi \left( (\lambda_n + \gamma) |u_k|^\frac{n}{n-1} \right)\) is bounded in \(L^1(\mathbb{R}^n)\).

**Theorem 4.1.** If \(\sup_{k} c_k < +\infty\), then the first part of Theorem 1.1 holds.

**Proof.** For any \(\varepsilon > 0\), by using \((4.3)\), there exist some \(L\) such that \(u_k(x) \leq \varepsilon\) when \(x \notin W_L\). It is easy to see that
\[
\int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^n \right) - \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \right) dx = \left( \int_{W_L} + \int_{\mathbb{R}^n \setminus W_L} \right) \Phi \left( \alpha_k u_k^n \right) - \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \right) dx.
\]

Also
\[
\int_{\mathbb{R}^n \setminus W_L} \Phi \left( \alpha_k u_k^{n-1} \right) - \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \right) dx = C \int_{\mathbb{R}^n \setminus W_L} u_k^{n-1} dx \leq C \varepsilon^{\frac{n^2}{n-1}} \int_{\mathbb{R}^n} u_k^n dx = C \varepsilon^{\frac{n^2}{n-1}} - n.
Then
\[ \int_{\mathbb{R}^n} \left( \Phi (\alpha_k u_k^{n-1}) - \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \right) \, dx = \int_{W_L} \left( \Phi (\alpha_k u_k^{n-1}) - \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \right) \, dx + O \left( \varepsilon^{\frac{2}{n-1}} n \right). \]

By sup\limits_k c_k < +\infty, we have
\[ \int_{\mathbb{R}^n} \Phi (\alpha_k u_k^{n-1}) \, dx = \int_{W_L} \left( \Phi (\alpha_k u_k^{n-1}) - \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \right) \, dx + \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \, dx + O \left( \varepsilon^{\frac{2}{n-1}} n \right). \]

Thus the proof of Theorem 1.1 in the case of sup\limits_k c_k < +\infty is finished.

4.2.2 Blow-up analysis and proof in the case of sup\limits_k c_k = +\infty

In the following, we assume sup\limits_k c_k = +\infty and perform a blow-up procedure. The method of blow-up analysis will be used to analyze the asymptotic behavior of the maximizing sequence \{u_k\}, and the first part of Theorem 1.1 in the case of sup\limits_k c_k = +\infty will be proved.

First, we denote
\[ r_k^n = \frac{\lambda_k}{\mu_k c_k^{\frac{n}{1 + \beta}}}. \]

By (4.3), one can find a sufficiently large \( L \) such that \( u_k \leq 1 \) on \( \mathbb{R}^n \setminus W_L \). Then \((u_k - u_k(L))^+ \in W^{1,n}_0(W_L)\) and
\[ \int_{W_L} F^n (\nabla (u_k - u_k(L))^+) \, dx \leq 1. \]

By Theorem 1.1 in [48], we know that if
\[ \beta < \inf_{u \in W^{1,n}_0(W_L)} \frac{\|F(\nabla u)\|_{n}^{n}}{\|u\|_{n}^{n}}, \]

then
\[ \int_{W_L} \exp \left\{ \lambda_n (u_k - u_k(L))^{\frac{n}{n-1}} (1 + \beta \|u_k - u_k(L)\|_{n}^{\frac{1}{n-1}}) \right\} \, dx \leq C(L). \]

For any \( q < \lambda_n (1 + \beta \|u_k - u_k(L)\|_{n}^{\frac{1}{n-1}}) \), there exists a constant \( C(q) \) such that
\[ qu_k^{\frac{n}{n-1}} \leq \lambda_n ((u_k - u_k(L))^+)^{\frac{n}{n-1}} (1 + \beta \|u_k - u_k(L)\|_{n}^{\frac{1}{n-1}}) + C(q). \]

Then
\[
\int_{\mathcal{W}_L} \exp \left\{ q \frac{n}{k} \right\} dx \leq C (L, q).
\]

Taking some \(0 < A < 1\) such that

\[(1 - A) \beta_k (1 + \alpha \|u_k\|^n_n)^{\frac{1}{n-1}} < \lambda_k (1 + \beta \|u_k - (L)\|^n_n)^{\frac{1}{n-1}},\]

then

\[
\lambda_k e^{-A\beta_k (1+\alpha\|u_k\|^n_n)^{\frac{1}{n-1}}} c_k^{\frac{n}{n-1}} = e^{-A\beta_k (1+\alpha\|u_k\|^n_n)^{\frac{1}{n-1}}} c_k^{\frac{n}{n-1}} \left[ \left( \int_{\mathbb{R}^n \setminus \mathcal{W}_L} + \int_{\mathcal{W}_L} \right) u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx \right]
\]

\[
\leq C e^{-A\beta_k (1+\alpha\|u_k\|^n_n)^{\frac{1}{n-1}}} c_k^{\frac{n}{n-1}} \left( \int_{\mathbb{R}^n \setminus \mathcal{W}_L} u_k^n dx + \int_{\mathcal{W}_L} u_k^{\frac{n}{n-1}} \beta_k (1+\alpha\|u_k\|^n_n)^{\frac{1}{n-1}} u_k^{\frac{n}{n-1}} dx \right)
\]

\[
\leq C \int_{\mathcal{W}_L} u_k^{\frac{n}{n-1}} e^{(1-A)\beta_k (1+\alpha\|u_k\|^n_n)^{\frac{1}{n-1}}} c_k^{\frac{n}{n-1}} dx + o(1).
\]

Since \(u_k\) converges strongly in \(L^s (\mathcal{W}_L)\) for any \(s > 1\), by (4.5), it follows that

\[
\lambda_k \leq C e^{A\alpha_k c_k^{\frac{n}{n-1}}}.
\]

Then for any \(q > 0\), we have

\[
(4.6) \quad r_k^n \leq C e^{(A-1)\alpha_k c_k^{\frac{n}{n-1}}} = o \left( c_k^{-q} \right).
\]

Set

\[
\begin{align*}
m_k (x) &= u_k (r_k x), \\
\phi_k (x) &= \frac{m_k (x)}{r_k}, \\
\psi_k (x) &= \frac{n-1}{n-1} \alpha_k c_k^{\frac{n}{n-1}} (m_k - c_k),
\end{align*}
\]

where \(m_k, \phi_k\) and \(\psi_k\) are defined on \(\Omega_k := \{ x \in \mathbb{R}^n : r_k x \in \mathcal{W}_1 \}\). From (4.1) and (4.6), it is easy to see that \(\phi_k (x)\) and \(\psi_k (x)\) respectively satisfy

\[
(4.7) \quad -Q_n \phi_k (x) = \frac{r_k^n}{c_k^{n-1}} \left( \mu_k \lambda_k^{-1} m_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k m_k^{\frac{n}{n-1}} \right) + (\gamma_k - 1) m_k^{n-1} \right)
\]

\[
= \left( \frac{1}{c_k} \phi_k^{\frac{n}{n-1}} (x) \right) \Phi' \left( \alpha_k \left( m_k^{\frac{n}{n-1}} - c_k^{\frac{n}{n-1}} \right) \right) + o(1),
\]
\[-Q_n \psi_k(x) = \left( \frac{n\alpha_k}{n-1} \right)^{n-1} c_k \rho_k^n \left( \frac{\mu_k \lambda_k^{1/n} m_k^{n-1} \Phi'}{\alpha_k m_k^{n-1}} + (\gamma_k - 1) m_k^{n-1} \right) \]

\[(4.8) = \left( \frac{n\alpha_k}{n-1} \right)^{n-1} \left( \frac{m_k}{c_k} \right)^{n-1} e^{\frac{n}{n-1} (c_k m_k^{n-1} - c_k^{n-1})} + o(1). \]

Now let us analyze the limit function of \( \phi_k(x) \) and \( \psi_k(x) \). Because \( u_k \) is bounded in \( W^{1,n}(\mathbb{R}^n) \), there exists a subsequence such that \( u_k \rightharpoonup u \) weakly in \( W^{1,n}(\mathbb{R}^n) \). Since the right side of \( (4.7) \) vanishes as \( k \to \infty \), then \( \phi_k \to \phi \) in \( C^1_{loc}(\mathbb{R}^n) \) as \( k \to \infty \), by applying the classical estimates \[39\], we have

\[-Q_n \phi(x) = 0 \text{ in } \mathbb{R}^n.\]

Since \( \phi(0) = 1 \), Liouville type theorem (see \[16\]) asserts that \( \phi \equiv 1 \) in \( \mathbb{R}^n \).

Now we analyze the asymptotic behavior of \( \psi_k \). By \( (4.6) \) and \( \phi_k(x) \leq 1 \), we can rewrite \( (4.8) \) as

\[-Q_n \psi_k(x) = O(1).\]

By Theorem 7 in \[33\], we have \( \text{osc}_{W^L} \psi_k \leq C(L) \) for any \( L > 0 \). Then from the result of \[39\], one can get \( \| \psi_k \|_{C^{1,\delta}(W_L)} \leq C(L) \) for some \( \delta > 0 \). Thus \( \psi_k \) converges in \( C^1_{loc}(W_L) \) and \( m_k - c_k \to 0 \) in \( C^1_{loc}(W_L) \).

It is easy to see that

\[ m_k^{n-1} = c_k^{n-1} \left( 1 + \frac{m_k - c_k}{c_k} \right)^{n-1} = c_k^{n-1} \left( 1 + \frac{n}{n-1} \frac{m_k - c_k}{c_k} + O \left( \frac{1}{c_k^2} \right) \right), \]

then

\[ (4.9) \alpha_k \left( m_k^{n-1} - c_k^{n-1} \right) = \alpha_k c_k^{n-1} \left( \frac{n}{n-1} \frac{m_k - c_k}{c_k} + O \left( \frac{1}{c_k^2} \right) \right) \]

\[ = \psi_k(x) + o(1) \to \psi(x) \text{ in } C^0_{loc}(\mathbb{R}^n). \]

Thus

\[ (4.10) - Q_n \psi(x) = \left( \frac{nc_n}{n-1} \right)^{n-1} e^{\psi(x)}, \]

where \( c_n = \lim_{k \to \infty} \alpha_k = \lambda_n (1 + \alpha \lim_{k \to \infty} \| u_k \|_n^{\frac{1}{n-1}}). \)

Since \( \psi \) is radially symmetric and decreasing, we know that \( (4.10) \) has only one solution. Thus we have

\[ \psi(x) = -n \log \left( 1 + \frac{c_n}{n^{n-1}} F^0(x)^{\frac{n}{n-1}} \right). \]
Therefore

\[
\int_{\mathbb{R}^n} e^\psi(x) \, dx = (n - 1) \kappa_n \left( \frac{n^{n-1}}{c_n} \right)^{n-1} \int_0^\infty (1 + t)^{-n} t^{n-2} dt
\]

(4.11)

\[
= (n - 1) \kappa_n \left( \frac{n^{n-1}}{c_n} \right)^{n-1} \cdot \frac{1}{n - 1} = \frac{1}{1 + \alpha \lim_{k \to \infty} \|u_k\|_n^n}.
\]

For any \( A > 1 \), denote \( u^A_k = \min \{ u_k, \frac{c_k}{A} \} \).

**Lemma 4.3.** For any \( A > 1 \), we have

\[
\limsup_{k \to \infty} \int_{\mathbb{R}^n} \left( F^n \left( \nabla u^A_k \right) + |u^A_k|^n \right) dx \leq 1 - \frac{A - 1}{A} \frac{1}{1 + \alpha \lim_{k \to \infty} \|u_k\|_n^n}.
\]

**Proof.** Since \( \left\{ x : u_k \geq \frac{c_k}{A} \right\} \left| \frac{c_k}{A} \right|^n \leq \int_{\mathbb{R}^n} |u_k|^n dx \leq 1 \), there exists a sequence \( \rho_k \to 0 \) such that

\( \left\{ x : u_k \geq \frac{c_k}{A} \right\} \subset W_{\rho_k} \).

Since for any \( s > 1 \), \( u_k \) converges in \( L^s (W_1) \), then we obtain

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \left| u^A_k \right|^s dx \leq \lim_{k \to \infty} \int_{\mathbb{R}^n} |u_k|^s dx = 0.
\]

Thus for any \( s > 0 \), it follows

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^s dx = 0.
\]

Testing (4.1) with \( \left( u_k - \frac{c_k}{A} \right)^+ \), we obtain

\[
\int_{\mathbb{R}^n} \left( F^n \left( \nabla u_k - \frac{c_k}{A} \right) \right)^+ dx + \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^n dx
\]

\[
= \int_{\mathbb{R}^n} \left( u_k - \frac{c_k}{A} \right)^+ \mu_k \lambda_k^{-1} u_k^{n-1} \Phi \left\{ \alpha_k u_k^{n-1} \right\} dx + o(1)
\]

\[
\geq \int_{W_{R \rho_k}} \left( u_k - \frac{c_k}{A} \right)^+ \mu_k \lambda_k^{-1} u_k^{n-1} \exp \left\{ \alpha_k u_k^{n-1} \right\} dx + o(1)
\]

\[
= \int_{W_R} \left( \frac{m_k - c_k}{c_k} \right)^+ \left( \frac{m_k - c_k}{c_k} + 1 \right)^{n-1} \exp \left\{ \psi_k (x) + o(1) \right\} dx + o(1)
\]

\[
\geq \frac{A - 1}{A} \int_{W_R} e^\psi(x) dx.
\]
Taking limits \( R \to \infty \) and \( k \to \infty \), then it follows from (4.11) that
\[
\liminf_{k \to \infty} \int_{\mathbb{R}^n} \left( F^n \left( \nabla \left( u_k - \frac{c_k}{A} \right)^+ \right) + \left( u_k - \frac{c_k}{A} \right)^+ |u_k|^{n-1} \right) dx \geq \frac{A-1}{A} \frac{1}{1 + \alpha \lim_{k \to \infty} \|u_k\|^n_n}.
\]

Hence
\[
\int_{\mathbb{R}^n} \left( F^n(\nabla u_k^A) + |u_k^A|^n \right) dx \leq 1 - \frac{1}{A} \frac{1}{1 + \alpha \lim_{k \to \infty} \|u_k\|^n_n} + o(1).
\]

Then the proof of Lemma 4.3 is completed. \( \square \)

**Lemma 4.4.** \( \lim_{k \to \infty} \|u_k\|^n_n = 0. \)

**Proof.** If \( \{u_k\} \) is a Sobolev-normalized concentrating sequence, then \( \lim_{k \to \infty} \|u_k\|^n_n = 0. \) If \( \{u_k\} \) is not a Sobolev-normalized concentrating sequence, and \( \lim_{k \to \infty} \|u_k\|^n_n \neq 0. \) For \( A \) large enough, there exist some constant \( \varepsilon_0 > 0 \) such that
\[
\int_{\mathbb{R}^n} \left( F^n(\nabla u_k^A) + |u_k^A|^n \right) dx = 1 - \frac{1}{1 + (\alpha + \varepsilon_0) \lim_{k \to \infty} \|u_k\|^n_n} < 1.
\]

By Theorem 1.2 in [51], we have \( \int_{\mathbb{R}^n} \Phi(q\lambda_n |u_k^A|^\frac{n}{n-1}) dx < +\infty, \) provided
\[
q < \left( \frac{1 + (\alpha + \varepsilon_0) \lim_{k \to \infty} \|u_k\|^n_n}{(\alpha + \varepsilon_0) \lim_{k \to \infty} \|u_k\|^n_n} \right)^{\frac{1}{n-1}}.
\]

Since \( \alpha < 1, \|u_k\|_F^n = 1 \) and \( \lim_{k \to \infty} \|u_k\|^n_n \neq 0, \) one can take some \( \varepsilon_0 \) such that \( (\alpha + \varepsilon_0) \lim_{k \to \infty} \|u_k\|^n_n < 1, \) thus
\[
\left( 1 + \alpha \lim_{k \to \infty} \|u_k\|^n_n \right)^{\frac{1}{n-1}} < \left( \frac{1 + (\alpha + \varepsilon_0) \lim_{k \to \infty} \|u_k\|^n_n}{(\alpha + \varepsilon_0) \lim_{k \to \infty} \|u_k\|^n_n} \right)^{\frac{1}{n-1}}.
\]

Thus for some \( t > 1, \) we have
\[
\int_{\mathbb{R}^n} \Phi \left( t\alpha_k \left| u_k^A \right|^\frac{n}{n-1} \right) dx < +\infty.
\]
Next we claim that $Q_n u_k \in L^r$ for some $r > 1$. When $\int_{\{u_k > \frac{c_k}{n}\}} F^n(\nabla u_k) dx \to 0$ as $k \to \infty$, it is easy to prove the above claim by (4.12) and the classical anisotropic Moser-Trudinger inequalities on bounded domains. When $\int_{\{u_k > \frac{c_k}{n}\}} F^n(\nabla u_k) dx \geq C$ for some $C > 0$, we split $u_k$ as $u_k^1 + u_k^2$ with $u_k^1 \to C\delta_0$ and $\int_{\{u_k > \frac{c_k}{n}\}} F^n(\nabla u_k^2) dx \to 0$. Then it follows from $\alpha < 1$ that

$$1 + \alpha \|u_k\|^n_n = 1 + \alpha \|u_k^2\|^n_n + o_k(1) < \frac{1}{1 - \|u_k^2\|^n_F} + o_k(1) \leq \frac{1}{\|F(\nabla u_k)\|^n_n} + o_k(1).$$

Thus there exist some constant $s > 1$ such that $(1 + \alpha \|u_k\|^n_n)^s \leq \frac{1}{\|F(\nabla u_k)\|^n_n(\{u_k > \frac{c_k}{n}\})}$. Therefore by (4.12) and the classical anisotropic Moser-Trudinger inequality on the bounded domain, the claim is proved.

Based on the claim above and Lemma 2.2, we obtain that $u_k$ is bounded near 0, which contradicts the assumption that $\sup_k c_k = +\infty$. Thus $\lim_{k \to \infty} \|u_k\|^n_n = 0$ and the proof of Lemma 4.4 has been finished.

**Remark 4.1.** From Lemma 4.4, one can obtain the following results.

$$\lim_{k} \alpha_k = \lambda_n, \quad \lim_{k} \mu_k = 1,$$

$$\limsup_{k \to \infty} \int_{\mathbb{R}^n} (F^n(\nabla u_k^1) + |u_k|^n) dx = \frac{1}{A},$$

$$\psi(x) = -n \log \left(1 + \frac{1}{n} F^0(x)^{\frac{n}{n-1}}\right),$$

and

$$\lim_{R \to \infty} \lim_{k \to \infty} \frac{1}{\lambda_k} \int_{\mathcal{W}_{R\mu_k}} u_k^{\frac{n}{n-1}} \exp \left(\alpha_k u_k^{\frac{n}{n-1}}\right) dx = \lim_{R \to \infty} \lim_{k \to \infty} \frac{1}{\mu_k} \int_{\mathcal{W}_R} e^{\psi(x)} dx$$

$$= \lim_{k \to \infty} \frac{1}{\mu_k} \left(1 + \alpha \lim_{k} \|u_k\|^n_n\right) = 1. \tag{4.13}$$

**Corollary 4.1.** We have $\lim_{k \to \infty} \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} (F^n(\nabla u_k) + |u_k|^n) dx = 0$ for any $\delta > 0$, and then $\lim_{k \to \infty} u_k \equiv 0$.

**Lemma 4.5.** There holds

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left(\alpha_k u_k^{\frac{n}{n-1}}\right) dx \leq \lim_{R \to \infty} \lim_{k \to \infty} \int_{\mathcal{W}_{R\mu_k}} \left(\exp \left(\alpha_k u_k^{\frac{n}{n-1}}\right) - 1\right) dx = \limsup_{k \to \infty} \frac{\lambda_k}{\epsilon_c^{\frac{n}{n-1}}}.$$
Moreover,

\[(4.15) \quad \frac{\lambda_k}{c_k} \to \infty \text{ and } \sup_{k \to \infty} \frac{c_k}{\lambda_k} \leq \infty.\]

**Proof.** For any \(A > 1\), it follows from the expression of \(\lambda_k\) that

\[
\int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx \leq \int_{u_k < \frac{A}{c_k}} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx + \int_{u_k \geq \frac{A}{c_k}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \Phi \left( \alpha_k \left( u_k^A \right)^{\frac{n}{n-1}} \right) \, dx + \int_{u_k \geq \frac{A}{c_k}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \Phi \left( \alpha_k \left( u_k^A \right)^{\frac{n}{n-1}} \right) \, dx + \left( \frac{A}{c_k} \right)^{\frac{n}{n-1}} \lambda_k \int_{u_k \geq \frac{A}{c_k}} u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx.
\]

By Remark 4.1 and Theorem 1.1 in [51], we obtain \(\Phi \left( \alpha_k \left( u_k^A \right)^{\frac{n}{n-1}} \right)\) is bounded in \(L^r\) for some \(r > 1\). Since \(u_k^A \to 0\) a.e. in \(\mathbb{R}^n\) as \(k \to \infty\), it follows

\[
\int_{\mathbb{R}^n} \Phi \left( \alpha_k \left( u_k^A \right)^{\frac{n}{n-1}} \right) \, dx \to \int_{\mathbb{R}^n} \Phi \left( 0 \right) \, dx = 0, \text{ as } k \to \infty.
\]

By (4.13), then we have

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx \leq \left( \frac{A}{c_k} \right)^{\frac{n}{n-1}} \lambda_k \int_{u_k \geq \frac{c_k}{A}} u_k^{\frac{n}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx + o \left( 1 \right)
\]

\[
= \lim_{k \to \infty} A^{\frac{n}{n-1}} \frac{\lambda_k}{c_k^{\frac{n}{n-1}}} + o \left( 1 \right).
\]

Letting \(A \to 1\) and \(k \to \infty\), we get (4.14).

If \(\frac{\lambda_k}{c_k}\) is bounded or \(\sup_{k \to \infty} \frac{c_k}{\lambda_k} = \infty\), it follows from (4.14) that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx = 0,
\]

which is impossible and the proof of Lemma 4.5 is completed. \(\square\)

**Lemma 4.6.** For any \(\varphi \in C_0^\infty (\mathbb{R}^n)\), there holds

\[
\int_{\mathbb{R}^n} \varphi \mu_k \lambda_k^{-1} c_k u_k^{\frac{1}{n-1}} \Phi' \left( \alpha_k u_k^{\frac{n}{n-1}} \right) \, dx = \varphi \left( 0 \right).
\]
Proof. We adopt the method for the proof of Lemma 3.6 in [20]. Split the integral as follows

\[
\int_{\mathbb{R}^n} \varphi \mu_k \lambda_k^{-1} c_k u_k^\frac{1}{n-1} \Phi' \left( \alpha_k \left( u_k \right)^\frac{n}{n-1} \right) \, dx
\]

Then

\[
I_1 \leq A \| \varphi \|_{L^\infty} \int_{\{ u_k \geq \frac{c_k}{n} \}} \mu_k \lambda_k^{-1} c_k u_k^\frac{1}{n-1} \Phi' \left( \alpha_k \left( u_k \right)^\frac{n}{n-1} \right) \, dx
\]

\[
\leq A \| \varphi \|_{L^\infty} \left( \int_{\mathbb{R}^n} - \int_{W_{rk}} \right) \mu_k \lambda_k^{-1} u_k^\frac{1}{n-1} \Phi' \left( \alpha_k \left( u_k \right)^\frac{n}{n-1} \right) \, dx
\]

\[
\leq A \| \varphi \|_{L^\infty} \left( 1 - \int_{W_k} \exp \left( \alpha_k m_k^\frac{n}{n-1} - \alpha_k c_k^\frac{n}{n-1} \right) \right)
\]

\[
= A \| \varphi \|_{L^\infty} \left( 1 - \int_{W_k} \exp \left( \psi_k (x) + o (1) \right) \right).
\]

For \( I_2 \), we have

\[
I_2 = \int_{W_{rk}} \varphi \mu_k \lambda_k^{-1} c_k u_k^\frac{1}{n-1} \Phi' \left( \alpha_k \left( u_k \right)^\frac{n}{n-1} \right) \, dx
\]

\[
= \int_{W_k} \varphi \left( r_k x \right) \left( m_k \right)^\frac{1}{n-1} \exp \left( \alpha_k m_k^\frac{n}{n-1} - \alpha_k c_k^\frac{n}{n-1} \right) \, dx + o (1)
\]

\[
= \varphi \left( 0 \right) \int_{W_k} \exp \left( \psi_k (x) + o (1) \right) \, dx + o (1) = \varphi \left( 0 \right) + o (1) \to \varphi \left( 0 \right), \text{ as } k \to \infty.
\]

By Lemma 4.5 and Hölder’s inequality, it follows

\[
I_3 = \int_{\{ u_k < \frac{c_k}{n} \}} \varphi \mu_k \lambda_k^{-1} c_k u_k^\frac{1}{n-1} \Phi' \left( \alpha_k \left( u_k \right)^\frac{n}{n-1} \right) \, dx
\]

\[
= \int_{\mathbb{R}^n} \varphi \mu_k \lambda_k^{-1} c_k u_k^\frac{1}{n-1} \Phi' \left( \alpha_k \left( u_k \right)^\frac{n}{n-1} \right) \, dx
\]

\[
\leq c_k \| \varphi \|_{L^\infty} \lambda_k^{-1} \left( \int_{\mathbb{R}^n} u_k^\frac{q-1}{n-1} \, dx \right) \left( \int_{\mathbb{R}^n} \Phi' \left( q' \alpha_k \left( u_k^\frac{n}{n-1} \right) \right) \, dx \right) \frac{1}{q'} \to 0, \text{ as } k \to \infty,
\]

for any \( q' < A \frac{1}{n-1} \) such that \( q = \frac{q'}{q'-1} \) large enough. Letting \( R \to +\infty \), by Remark 4.1 then Lemma 4.6 is proved. \( \square \)
Lemma 4.7. On any \( \Omega \subset \subset \mathbb{R}^n \setminus \{0\} \), we have \( c_k^{-1} u_k \to G_\alpha \) in \( C^1(\Omega) \), where \( G_\alpha \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) is a Green function satisfying the following equation

\[
-Q_n(G_\alpha) = \delta_0 + (\alpha - 1) G_\alpha^{n-1}.
\]

Proof. The idea of the proof is from Struwe [36] (also see [20]). Denote \( U_k = c_k^{-1} u_k \), by (4.1), \( U_k \) satisfy the following equation

\[
-Q_n U_k = \mu_k c_k \lambda_k^{-1} u_k^{\frac{n}{n-1}} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} + (\gamma_k - 1) U_k^{n-1}.
\]

For \( t \geq 1 \), let \( U_k^t = \min \{ U_k, t \} \) and \( \Omega_k^t = \{ 0 \leq U_k \leq t \} \). Testing (4.17) with \( U_k^t \), it follows

\[
\int_{\mathbb{R}^n} -U_k^t Q_n(U_k) dx + (1 - \gamma_k) \int_{\mathbb{R}^n} U_k^t U_k^{n-1} dx \leq \int_{\mathbb{R}^n} U_k^t \mu_k c_k \lambda_k^{-1} u_k^{\frac{n}{n-1}} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} dx.
\]

By the fact that \( \gamma_k \to \alpha < 1 \) as \( k \to \infty \), we obtain

\[
\int_{\Omega_k^t} F^n(\nabla U_k^t) dx + \int_{\Omega_k^t} |U_k^t|^n dx \leq \int_{\mathbb{R}^n} \left( -U_k^t Q_n(U_k) dx + U_k^t U_k^{n-1} dx \right) \leq C \int_{\mathbb{R}^n} U_k^t \mu_k c_k \lambda_k^{-1} u_k^{\frac{n}{n-1}} \Phi' \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} dx \leq Ct.
\]

Let \( \eta \) be a radially symmetric cut-off function which is 1 on \( \mathcal{W}_{R/2} \) and 0 on \( \mathcal{W}_R^c \), and satisfy \( F(\nabla \eta) \leq \frac{C'}{R} \). Then when \( R \) large enough, we obtain

\[
\int_{\mathcal{W}_R} F^n(\nabla(\eta U_k^t)) dx \leq \int_{\mathcal{W}_R} F^n(\nabla \eta) \left| U_k^t \right|^n dx + \int_{\mathcal{W}_R} \eta^n F^n(\nabla U_k^t) dx \leq C_1(R) t + C_2(R).
\]

Taking \( t \) large enough such that \( C_1(R) t > C_2(R) \), then

\[
\int_{\mathcal{W}_R} F^n(\nabla(\eta U_k^t)) dx \leq 2C_1(R) t.
\]

Let \( |\mathcal{W}_\rho| = |\{ x \in \mathcal{W}_R : U_k > t \}| \), then

\[
\min_{\substack{\psi \in W_0^{1,n}(\mathcal{W}_R), \, \psi|_{\mathcal{W}_\rho} = t}} \int_{\mathcal{W}_R} F^n(\nabla \psi) dx \leq \int_{\mathcal{W}_R} F^n(\nabla(\eta U_k^t)) dx \leq 2C_1(R) t.
\]

The above infimum can be attained (see [50]) by

\[
\psi_1(x) = \begin{cases} 
t \log \frac{R}{R_0(x)}/ \log \frac{R}{\rho}, & \text{in } \mathcal{W}_R \setminus \mathcal{W}_\rho, \\
t, & \text{in } \mathcal{W}_\rho. 
\end{cases}
\]
By computing $\|F(\nabla \psi_1)\|_{L^n(W_R)}$, then it follows from (4.18) that $\rho \leq CR^{-C_3 t}$. Thus
\[ \{|x \in W_R : U_k \geq t\| = |\mathcal{W}_\rho| \leq \kappa n R^n e^{-nC_3 t}.\]

For any $0 < \delta < nc_3$, we have
\[
\int_{W_R} e^{\delta U_k} dx \leq e^\delta |W_R| + \sum_{m=1}^{\infty} e^{(m+1)\delta} |\{x \in W_R : m \leq U_k \leq m + 1\}|
\leq e^\delta |W_R| + \kappa n R^n e^{\delta} \sum_{m=1}^{\infty} e^{-(nc_3-\delta)m} \leq C.
\]

Testing (4.17) with $\log \frac{1+2U_k}{1+U_k}$, we get
\[
\int_{W_R} F^n(\nabla U_k) \frac{1}{(1+2U_k)(1+U_k)} dx \leq \log 2 \int_{W_R} \mu_k c_k \lambda_k^{-1} u_k^{-\alpha} \phi_{\beta_k u_k} \frac{n}{n-1} dx + \int_{W_R} (\gamma_k - 1) U_k^{n-1} \log \frac{1+2U_k}{1+U_k} dx \leq C.
\]

For any $1 < q < n$, it follows by the Young inequality that
\[
\int_{W_R} F^q(\nabla U_k) dx \leq \int_{W_R} F^n(\nabla U_k) \left(\frac{1}{(1+2U_k)(1+U_k)} \right)^{\frac{n}{n-q}} dx + \int_{W_R} \left\{\left(1+2U_k\right)(1+U_k)\right\}^{\frac{n}{n-q}} dx \leq C(1 + \int_{W_R} e^{\delta U_k} dx) \leq C.
\]

Then we can obtain that $\|F(\nabla U_k)\|_{L^p(W_R)} \leq C$ for any $1 < q < n$, thus $\|U_k\|_{L^p(W_R)} \leq C$ for any $0 < p < +\infty$. By Corollary 4.1 we know that $\exp \left\{\alpha_k u_k^{\frac{n}{n-1}}\right\}$ is bounded in $L^r(\Omega \setminus \{W_\delta\})$ for any $r > 0$ and $\delta > 0$. Applying Theorem 2 in [33] and Theorem 1 in [39], we have $\|U_k\|_{C^{1,\alpha}(W_R)} \leq C$, then $c_k^{\frac{n}{n-1}} u_k \to G_\alpha$ in $C^1(W_R)$. So we complete the proof of Lemma 4.7.

Similar as Lemma 3.8 in [20] or Lemma 4.9 in [26], we can obtain the following asymptotic representation of $G_\alpha$.

**Lemma 4.8.** $G_\alpha \in C^{1,\beta}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ for some $\beta > 0$, and near 0 we can write
\[
G_\alpha = -\frac{n}{\lambda_n} \log r + A + O\left(r^n \log^n r\right),
\]
where $A$ is a constant and $r = F^0(x)$. Moreover, for any $\delta > 0$, it holds
\[
\lim_{k \to \infty} \left(\int_{\mathbb{R}^n \setminus W_\delta} F^n(\nabla U_k) dx + (1-\alpha) \int_{\mathbb{R}^n \setminus W_\delta} U_k^n dx\right) = G_\alpha(\delta) \left(1 + (\alpha - 1) \int_{W_\delta} G_\alpha^{n-1} dx\right).
\]
Proof. The proof of (4.19) is similar as Lemma 4.7 in [51], here we omit the details. Now we give the proof of (4.20). By Corollary 4.1 we have

\[(4.21) \quad \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} u_k^n \Phi' \{ \alpha_k u_k^{n-1} \} \, dx \leq C \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} u_k^n \, dx \to 0.\]

Testing (4.17) with \( U_k \),

\[
\int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} F^n(\nabla U_k) \, dx + \int_{\partial \mathcal{W}_\delta} F^{n-1}(\nabla U_k) U_k \frac{\partial U_k}{\partial n} \, dx = \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} \mu_k c_k^{n-1} \lambda_k^{-1} u_k^{n-1} \Phi' \{ \alpha_k u_k^{n-1} \} \, dx + \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} (\gamma_k - 1) U_k^n \, dx.
\]

By (4.21), (4.15), it follows that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} F^n(\nabla U_k) \, dx = -\lim_{k \to \infty} \int_{\partial \mathcal{W}_\delta} F^{n-1}(\nabla U_k) U_k \frac{\partial U_k}{\partial n} \, dx + (\alpha - 1) \lim_{k \to \infty} \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} U_k^n \, dx = -G_\alpha(\delta) \int_{\partial \mathcal{W}_\delta} F^{n-1}(\nabla G_\alpha) \frac{\partial G_\alpha}{\partial n} \, dx + (\alpha - 1) \lim_{k \to \infty} \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} U_k^n \, dx = G_\alpha(\delta) \left( 1 + (\alpha - 1) \int_{\mathcal{W}_\delta} G_\alpha^{n-1} \, dx \right) + (\alpha - 1) \lim_{k \to \infty} \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} U_k^n \, dx.
\]

Thus

\[
\lim_{k \to \infty} \left( \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} F^n(\nabla U_k) \, dx + (1 - \alpha) \int_{\mathbb{R}^n \setminus \mathcal{W}_\delta} U_k^n \, dx \right) = G_\alpha(\delta) \left( 1 + (\alpha - 1) \int_{\mathcal{W}_\delta} G_\alpha^{n-1} \, dx \right).
\]

The proof of Lemma 4.8 is completed. \( \square \)

Proof of the first part of Theorem 1.1. By (4.3), there exist some \( L > 0 \) such that \( u_k(L) < 1 \), then

\[
\int_{\mathbb{R}^n \setminus \mathcal{W}_L} \exp \left\{ \beta_k |u_k|^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_{n}^2) \right\} \, dx \leq C \int_{\mathbb{R}^n \setminus \mathcal{W}_L} |u_k|^n \, dx \leq C.
\]

Since \( (u_k - u_k(L))^+ \in W^{1,n}_0 (B_L) \), then

\[
\frac{n}{n-1} u_k^{\frac{n}{n-1}} = \left( (u_k - u_k(L))^+ + u_k(L) \right)^{\frac{n}{n-1}} \leq \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} + C \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} u_k(L) + u_k(L)^{\frac{n}{n-1}}.
\]

By Lemma 4.7 we obtain \( c_k^{-\frac{1}{n-1}} u_k \to G_\alpha \), then \( u_k(L) = \frac{G_\alpha(L)}{c_k^{\frac{1}{n-1}}} \). Thus

\[
\frac{n}{n-1} u_k^{\frac{n}{n-1}} \leq \left( (u_k - u_k(L))^+ \right)^{\frac{n}{n-1}} + C \left( \frac{(u_k - u_k(L))^+}{c_k} \right)^{\frac{1}{n-1}} + u_k(L)^{\frac{n}{n-1}}.
\]
\[
\leq \left( (u_k - u_k (L))^+ \right)^{\frac{n}{n-1}} + C.
\]

Therefore
\[
\int_{W_L} \exp \left\{ \beta_k \frac{n}{n-1} |u_k|^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_n^\frac{n}{n-1}) \right\} \, dx
\leq C \int_{W_L} \exp \left\{ \beta_k \left( (u_k - u_k (L))^+ \right)^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_n^\frac{n}{n-1} - 1) \right\} \exp \left( \beta_k \left( (u_k - u_k (L))^+ \right)^{\frac{n}{n-1}} \right) \, dx
\leq C \exp \left\{ \beta_k c_k^{\frac{n}{n-1}} \left( (1 + \alpha \|u_k\|_n^\frac{n}{n-1} - 1) \right) \right\} \int_{W_L} \exp \left( \beta_k \left( (u_k - u_k (L))^+ \right)^{\frac{n}{n-1}} \right) \, dx.
\]

From Lemma 4.7 and Lemma 4.8, we obtain that \( \|c_k^{\frac{1}{n-1}} u_k\|_n \) is bounded. Applying the anisotropic Moser-Trudinger inequality (see [42]), by the fact that \( \|u_k\|_n \to 0 \), we have
\[
\int_{W_L} \exp \left\{ \beta_k |u_k|^{\frac{n}{n-1}} (1 + \alpha \|u_k\|_n^\frac{n}{n-1}) \right\} \, dx
\leq C \exp \left\{ \frac{\alpha \beta_k c_k^{\frac{n}{n-1}}}{n-1} \|u_k\|_n^n \right\} \int_{W_L} \exp \left( \beta_k \left( (u_k - u_k (L))^+ \right)^{\frac{n}{n-1}} \right) \, dx
= C \exp \left\{ \frac{\alpha \beta_k}{n-1} \|c_k^{\frac{1}{n-1}} u_k\|_n^n \right\} \int_{W_L} \exp \left( \beta_k \left( (u_k - u_k (L))^+ \right)^{\frac{n}{n-1}} \right) \, dx
\leq C.
\]

Thus we complete the proof of the first part of Theorem 1.1 in the case of \( \sup_k c_k = +\infty \). \( \square \)

## 5 Proof of Theorem 1.2

In this section, we prove the proof of Theorem 1.2 in this paper by considering the two cases. When \( \sup_k c_k < +\infty \), the proof is based on the concentration-compactness lemma. When \( \sup_k c_k = +\infty \), we prove the result by contradiction. We first establish the upper bound for critical functional when \( \sup_k c_k = +\infty \), and then construct an explicit test function, which provides a lower bound for the supremum of our Moser-Trudinger inequality. Because this lower bound equals to the upper bound, one can obtain the contradiction.

### 5.1 Proof in the case of \( \sup_k c_k < +\infty \)

**Theorem 5.1.** If \( \sup_k c_k < +\infty \), then Theorem 1.2 holds.
Anisotropic Moser-Trudinger inequality in $\mathbb{R}^n$

**Proof.** By Lemma 4.1 and applying the elliptic estimate in [39] to equation (4.1), we can obtain $u_k \to u$ in $C^1_{loc}(\mathbb{R}^n)$. Next we will prove $u \neq 0$. We prove this result by contradiction.

Assume $u = 0$, we claim that $\{u_k\}$ is not a Sobolev-normalized concentrating sequence. If not, i.e. $\{u_k\}$ is a Sobolev-normalized concentrating sequence, by iii) of Definition 4.1 and the fact that $|u_k|$ is bounded, we have for any $\delta > 0$,

$$
\int_{\mathbb{R}^n} u_k^n dx \leq \int_{W_k} u_k^n dx + \int_{\mathbb{R}^n \setminus W_k} u_k^n dx \leq C\delta^n + o_k(1).
$$

Letting $\delta \to 0$, it follows $\int_{\mathbb{R}^n} u_k^n dx \to 0$ as $k \to \infty$. When $L$ is large enough, for any $\epsilon > 0$, it follows by (4.4) that

$$
S + o_k(1) = \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k \frac{n}{n-1} \right) dx
$$

$$
= \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} dx + \int_{\mathbb{W}_L} \left( \Phi \left( \alpha_k \cdot \frac{n}{n-1} \right) - \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} \right) dx + O \left( \varepsilon \frac{n^2}{n-1} \right).
$$

Thus

$$
S \leq \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} u_k^n}{(n-1)!} dx \to 0,
$$

which is impossible. Therefore the claim is proved, i.e. when $u = 0$, we have $\{u_k\}$ is not a Sobolev-normalized concentrating sequence. By Lemma 4.2 it follows that $\int_{\mathbb{R}^n} \Phi \left( \alpha_k \frac{u_k^n}{n} \right) dx \to \int_{\mathbb{R}^n} \Phi \left( \alpha_n \frac{u^n}{n} \right) dx = 0$, which is still impossible. Thus $u \neq 0$.

Next we will prove that $\int_{\mathbb{R}^n} u_k^n \to \int_{\mathbb{R}^n} u^n$. By (4.4), we get

$$
S = \lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k \frac{u_k^n}{n} \right) dx
$$

$$
= \int_{\mathbb{R}^n} \left( \Phi \left( \lim_{k \to \infty} \alpha_k \frac{u_k^n}{n-1} \right) \right) dx + \lim_{k \to \infty} \int_{\mathbb{R}^n} \frac{\alpha_k^{n-1} (u_k^n - u^n)}{(n-1)!} dx.
$$

Denote

$$
\tau^n = \lim_{k \to +\infty} \frac{\int_{\mathbb{R}^n} u_k^n}{\int_{\mathbb{R}^n} u^n}.
$$

By the Levi Lemma, it is easy to see that $\tau \geq 1$. Set $\tilde{u} = u \left( \frac{\cdot}{\tau} \right)$, then it follows

$$
\int_{\mathbb{R}^n} F^n(\nabla \tilde{u}) dx = \int_{\mathbb{R}^n} F^n(\nabla \tilde{u}) dx \leq \int_{\mathbb{R}^n} F^n(\nabla u_k) dx
$$

and

$$
\int_{\mathbb{R}^n} |\tilde{u}|^n dx = \tau^n \int_{\mathbb{R}^n} |u|^n dx \leq \int_{\mathbb{R}^n} |u_k|^n dx.
$$
Thus
\[ \|\tilde{u}\|_F = \int_{\mathbb{R}^n} (F^n(\nabla \tilde{u}) + |\tilde{u}|^n) \, dx \leq 1. \]

By (5.1), it follows that
\[
S \geq \int_{\mathbb{R}^n} \Phi \left( \alpha_k \tilde{u}^{n-1} \left( 1 + \alpha \tilde{u}^{n-1} \right) \right) \, dx
\]
\[
= \tau^n \int_{\mathbb{R}^n} \Phi \left( \lambda_n \tilde{u}^{n-1} \left( 1 + \alpha \tilde{u}^{n-1} \right) \right) \, dx
\]
\[
\geq \tau^n \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k \tilde{u}^{n-1} \right) \, dx + o(1)
\]
\[
= \tau^n - 1 \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k \tilde{u}^{n-1} \right) \, dx + o(1)
\]
\[
\geq (\tau^n - 1) \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k \tilde{u}^{n-1} \right) \, dx + o(1)
\]
\[
\geq (\tau^n - 1) \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k \tilde{u}^{n-1} \right) - \frac{\lim_{k \to \infty} \alpha_k^{n-1} u^n}{(n-1)!} \, dx + o(1)
\]
\[
\geq \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k \tilde{u}^{n-1} \right) \, dx + o(1)
\]
\[
= S + (\tau^n - 1) \int_{\mathbb{R}^n} \Phi \left( \lim_{k \to \infty} \alpha_k \tilde{u}^{n-1} \right) - \frac{\lim_{k \to \infty} \alpha_k^{n-1} u^n}{(n-1)!} \, dx + o(1).
\]

Since \( \Phi \left( \lim_{k \to \infty} \alpha_k \tilde{u}^{n-1} \right) - \frac{\lim_{k \to \infty} \alpha_k^{n-1} u^n}{(n-1)!} > 0 \), we get \( \tau = 1 \), thus
\[
\lim_{k} \int_{\mathbb{R}^n} \Phi \left( \alpha_k \tilde{u}^{n-1} \right) \, dx = \int_{\mathbb{R}^n} \Phi \left( \lambda_n \left( 1 + \alpha \tilde{u}^{n-1} \right) \right) \, dx.
\]

So \( u \) is an extremal function and the proof of Theorem 5.1 is finished.

5.2 Proof in the case of \( \sup_k c_k = +\infty \)

In this subsection, we will show that the existence of the extremal functions of Moser-Trudinger inequality involving \( L^n \) norm in \( \mathbb{R}^n \) in the case of \( \sup_k c_k = +\infty \). In order to prove the existence of the extremal functions, we need the following result due to Zhou and Zhou [50], which often plays a key role in the proof of existence result. This method
has been widely used to prove the existence of the extremal functions of many kinds of Moser-Trudinger inequality (see [20, 46, 25, 52]).

**Lemma 5.1.** Assume that \( \{u_k\} \) is a normalized concentrating sequence in \( W^{1,n}_0(W_1) \) with a blow up point at the origin, i.e. \( \int_{W_1} F^n(\nabla u_k)dx = 1, \) \( u_k \rightharpoonup 0 \) weakly in \( W^{1,n}_0(W_1) \) and \( \lim_{k \to +\infty} \int_{W_1 \setminus W_r} F^n(\nabla u_k)dx = 0 \) for any \( 0 < r < 1, \) then

\[
\limsup_{k \to \infty} \int_{W_1} e^{\lambda_n|u_k|^\frac{n}{n-1}}dx \leq \kappa_n \left( 1 + \exp \left\{ 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\} \right).
\]

**Lemma 5.2.** If \( S \) can not be attained, then

\[
S \leq \kappa_n \exp \left\{ \lambda_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\}.
\]

where \( A \) is the constant in (4.14).

**Proof.** By Lemma 4.8, it follows that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^n \setminus W_\delta} \left( F^n(\nabla u_k) + |u_k|^n \right) dx
\]

\[
= c_k \left( \alpha \int_{\mathbb{R}^n \setminus W_\delta} U^n_k dx + G_\alpha(\delta) \left( 1 + (\alpha - 1) \int_{W_\delta} G^{n-1}_\alpha dx \right) \right)
\]

\[
= c_k \left( \alpha \lim_{k \to \infty} \|U_k\|^n_n + G_\alpha(\delta) \left( 1 + (\alpha - 1) \int_{W_\delta} G^{n-1}_\alpha dx \right) \right).
\]

Set \( \tilde{u}_k(x) = (u_k(x) - u_k(\delta))^+, \) then

\[
\int_{W_\delta} F^n(\nabla \tilde{u}_k)dx \leq \int_{W_\delta} F^n(\nabla u_k)dx = \tau_k := 1 - \int_{\mathbb{R}^n \setminus W_\delta} \left( F^n(\nabla u_k) + |u_k|^n \right) dx - \int_{W_\delta} |u_k|^n dx
\]

(5.2)

\[
= 1 - c_k \left( \alpha \lim_{k \to \infty} \|U_k\|^n_n - \frac{n}{\lambda_n} \log \delta + A + o_k(1) + O_\delta(1) \right).
\]

When \( x \in W_{Lr_k}, \) by Lemma 4.7 and (5.2), it follows that

\[
\alpha_k \tilde{u}_k \leq \lambda_n \left( 1 + \alpha \|u_k\|^n_n \right)^{\frac{1}{n-1}} (\tilde{u}_k + u_k(\delta))^{\frac{1}{n-1}}
\]

\[
\leq \lambda_n \|\tilde{u}_k\|^\frac{1}{n-1} + \frac{n \lambda_n}{n-1} |\tilde{u}_k|^{\frac{1}{n-1}} |u_k(\delta)| + \frac{\lambda_n \alpha}{n-1} \lim_{k \to \infty} \|U_k\|^n_n + o_k(1)
\]

\[
\leq \lambda_n \|\tilde{u}_k\|^\frac{1}{n-1} + \frac{n \lambda_n}{n-1} |c|^{\frac{1}{n-1}} |u_k(\delta)| + \frac{\lambda_n \alpha}{n-1} \lim_{k \to \infty} \|U_k\|^n_n + o_k(1)
\]

\[
\leq \lambda_n \|\tilde{u}_k\|^\frac{1}{n-1} + \frac{n \lambda_n}{n-1} |G_\alpha(\delta)| + \frac{\lambda_n \alpha}{n-1} \lim_{k \to \infty} \|U_k\|^n_n + o_k(1)
\]
\[
= \lambda_n |\tilde{u}_k|^{\frac{n}{n-1}} - \frac{n^2}{n-1} \log \delta + \frac{n \lambda_n}{n-1} A + \frac{\lambda_n \alpha}{n-1} \lim_{k \to \infty} \|U_k\|^n_n + o_k(1) + o_\delta(1)
\]
\[
\leq \frac{\lambda_n |\tilde{u}_k|^{\frac{n}{n-1}}}{\tau_k^{\frac{1}{n-1}}} + \lambda_n A - \log \delta^n + o_k(1) + o_\delta(1).
\]

Integrating the above estimates on \(W_{Lr_k}\), we obtain
\[
\int_{W_{Lr_k}} \left( \exp \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} - 1 \right) dx \leq \delta^{-n} \exp \left\{ \lambda_n A + o_k(1) \right\}
\]
\[
\cdot \int_{W_{Lr_k}} \left( \exp \left\{ \frac{\alpha_k u_k^{\frac{n}{n-1}}}{\tau_k^{\frac{1}{n-1}}} \right\} - 1 \right) dx + o_k(1).
\]

By Lemma 5.1, we get
\[
\int_{W_{Lr_k}} \left( \exp \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} - 1 \right) dx \leq \kappa_n \exp \left\{ \lambda_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\}.
\]

By Lemma 4.3, we obtain
\[
\lim_{k \to \infty} \int_{\mathbb{R}^n} \Phi \left( \alpha_k u_k^{\frac{n}{n-1}} \right) dx \leq \lim_{L \to \infty} \lim_{k \to \infty} \int_{W_{Lr_k}} \left( \exp \left\{ \alpha_k u_k^{\frac{n}{n-1}} \right\} - 1 \right) dx
\]
\[
\leq \kappa_n \exp \left\{ \lambda_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\}.
\]

Thus the conclusion of Lemma 5.2 holds.

In the following, we will construct a function sequence \(\{u_\varepsilon\} \subset W^{1,n}(\mathbb{R}^n)\) with \(\|u_\varepsilon\|_F = 1\) such that
\[
\int_{\mathbb{R}^n} \Phi \left( \lambda_n u_\varepsilon^{\frac{n}{n-1}} \right) dx > \kappa_n \exp \left\{ \lambda_n A + 1 + \frac{1}{2} + \ldots + \frac{1}{n-1} \right\}.
\]

**Proof of Theorem 1.2 in the case of \(\sup_k c_k = +\infty\).** Let
\[
\begin{align*}
\varepsilon = \begin{cases}
- \frac{C - C^{\frac{1}{n-1}} \left( \frac{n}{n-1} \log \left( 1 + c_n \left( \frac{F^0(x)}{\varepsilon} \right)^{\frac{n}{n-1}} \right) - B_\varepsilon \right)}{F^0(x) - B_\varepsilon} &, & F^0(x) \leq R_\varepsilon, \\
- \frac{\left( 1 + \frac{n}{n-1} \|G_\alpha\|_{F^0(x)} \right)^{\frac{1}{n-1}}}{\left( \right)} &, & F^0(x) > R_\varepsilon,
\end{cases}
\end{align*}
\]

where \(c_n = \frac{1}{\kappa_n^{\frac{1}{n-1}}}, B_\varepsilon, R\) and \(C\) depending on \(\varepsilon\) will also be determined later, such that

(i) \(R_\varepsilon \to 0, R \to \infty\) and \(C \to \infty\), as \(\varepsilon \to 0\),
Anisotropic Moser-Trudinger inequality in $\mathbb{R}^n$.

(ii) \[ \frac{c_{n-1}}{\lambda_n} \log \left( 1 + c_n R \right) + \frac{1}{\lambda_n} R \beta \left( 1 + \alpha c_{n-1} \| \alpha \|^n \right) \left( \frac{\alpha}{\lambda_n} \right) \frac{\lambda_n}{\lambda_n} = \frac{G_\alpha (R \epsilon)}{\left( \frac{\alpha}{\lambda_n} \right) \frac{\lambda_n}{\lambda_n}}. \]

We can obtain the information of $B \epsilon$, $C$ and $R$ by normalizing $u \epsilon$. By Lemma 4.8, it can check that

\[ \int_{\mathbb{R}^n \setminus W_{R \epsilon}} \left( F^n (\nabla u \epsilon) + |u \epsilon|^n \right) dx \]

\[ = \frac{1}{C_{n-1} + \alpha \| G \alpha \|^n} \int_{\mathbb{R}^n \setminus W_{R \epsilon}} \left( F^n (\nabla G \alpha) + |G \alpha|^n \right) dx \]

\[ = \frac{1}{C_{n-1} + \alpha \| G \alpha \|^n} \left( -G \alpha (R \epsilon) \int_{\partial W_{R \epsilon}} \left( F^{n-2} (\nabla G \alpha) \frac{\partial G \alpha}{\partial n} \right) dx + \alpha \int_{\mathbb{R}^n \setminus W_{R \epsilon}} |G \alpha|^n dx \right) \]

\[ = \frac{n \kappa_a G \alpha (R \epsilon) |G' (R \epsilon)|^{n-1} (R \epsilon)^{n-1} + \alpha \int_{\mathbb{R}^n \setminus W_{R \epsilon}} |G \alpha|^n dx}{C_{n-1} + \alpha \| G \alpha \|^n}, \]

and

\[ \int_{W_{R \epsilon}} F^n (\nabla u \epsilon) dx = \frac{n-1}{\lambda_n \left( C_{n-1} + \alpha \| G \alpha \|^n \right)} \int_0^{c_n R_{n-1}} \frac{u^{n-1}}{(1 + u)^n} du \]

\[ = \frac{n-1}{\lambda_n \left( C_{n-1} + \alpha \| G \alpha \|^n \right)} \int_0^{c_n R_{n-1}} \frac{(1 + u - 1)^{n-1}}{(1 + u)^n} du \]

\[ = \frac{n-1}{\lambda_n \left( C_{n-1} + \alpha \| G \alpha \|^n \right)} \left( \sum_{k=0}^{n-2} C_{n-1} (-1)^{n-1-k} \frac{(-1)^{n-1-k}}{n-k-1} \right) + \log \left( 1 + c_n R_{n-1} \right) + O \left( R_{n-1} \right). \]

Using the fact that

\[ E := \sum_{k=0}^{n-2} C_{n-1} (-1)^{n-1-k} \frac{(-1)^{n-1-k}}{n-k-1} = - \left( 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \right), \]

we obtain

\[ \int_{W_{R \epsilon}} F^n (\nabla u \epsilon) dx = \frac{n-1}{\lambda_n \left( C_{n-1} + \alpha \| G \alpha \|^n \right)} \left( E + \log \left( 1 + c_n R_{n-1} \right) + O \left( R_{n-1} \right) \right). \]

It is easy to check that

\[ \int_{W_{R \epsilon}} |u \epsilon|^n dx = O (C_n (R \epsilon)^n), \]
then
\[
\int_{\mathbb{R}^n} (F^n(\nabla u_\varepsilon) + |u_\varepsilon|^n) \, dx = \frac{1}{\lambda_n \left( C_{n-1}^n + \alpha \|G_\alpha\|^n_n \right)} \left( (n-1) E + (n-1) \log \left( 1 + c_n R_n^{n-1} \right) \right) - \log (R\varepsilon)^n + \lambda_n A + \alpha \lambda_n \|G_\alpha\|^n_n + O(\phi),
\]
where
\[
\phi = C^n (R\varepsilon)^n + (R\varepsilon)^n \log^n (R\varepsilon) + R_n^{-n-1} + C_{n-1}^{n-2} + C_{n-1}^{n-2} (R\varepsilon)^n.
\]
Because \( \int_{\mathbb{R}^n} (F^n(\nabla u_\varepsilon) + |u_\varepsilon|^n) \, dx = 1 \), it follows that
\[
\lambda_n \left( C_{n-1}^n + \alpha \|G_\alpha\|^n_n \right) = (n-1) E + (n-1) \log \left( 1 + c_n R_n^{n-1} \right) - \log (R\varepsilon)^n + \lambda_n A + \alpha \lambda_n \|G_\alpha\|^n_n + O(\phi),
\]
namely,
\[
(5.4) \quad \lambda_n C_{n-1}^n = (n-1) E + \log \kappa_n - \log \varepsilon^n + \lambda_n A + O(\phi).
\]
By (ii) we obtain
\[
C - C_{n-1}^n \left( \frac{n-1}{\lambda_n} \log \left( 1 + c_n |R|^{n-1}_n \right) - B_\varepsilon \right) = -\frac{n}{\lambda_n} \log (R\varepsilon) + A + O(\phi).
\]
Thus
\[
(5.5) \quad C_{n-1}^n = -\frac{n}{\lambda_n} \log \varepsilon + \log \kappa_n - B_\varepsilon + A + O(\phi).
\]
Combining (5.4) and (5.5), it is easy to see that
\[
(5.6) \quad B_\varepsilon = -\frac{n-1}{\lambda_n} E + O(\phi).
\]
Letting \( R = -\log \varepsilon \), which satisfies \( R\varepsilon \to 0 \) as \( \varepsilon \to 0 \), then
\[
(5.7) \quad \|u_\varepsilon\|^n_n = \frac{\|G_\alpha\|^n_n + O \left( C_{n-1}^{n-2} R^n \varepsilon^n \right) + O \left( ((R\varepsilon)^n (- \log (R\varepsilon)) \right)}{C_{n-1}^{n-1} + \alpha \|G_\alpha\|^n_n}.
\]
It is easy to check that when \( |t| < 1 \),
\[
(1 - t)^{n-1} \geq 1 - \frac{n}{n-1} t, \quad (1 + t)^{-\frac{1}{n-1}} \geq 1 - \frac{t}{n-1}.
\]
By using the above inequalities and (5.7), we deduce that for any \( x \in W_{R\varepsilon} \),
Anisotropic Moser-Trudinger inequality in $\mathbb{R}^n$

By (5.4) and (5.6), we obtain

$$
\lambda_n |u_\varepsilon|^\frac{n}{n-1} (1 + \alpha \|u_\varepsilon\|_n^{\frac{n}{n-1}}) \\
\geq \lambda_n C_n \left( 1 - C_n \left( \frac{n-1}{\lambda_n} \log \left( 1 + c_n \left( \frac{F^0(x)}{\varepsilon} \right)^{\frac{n}{n-1}} \right) - B_\varepsilon \right) \right)^{\frac{n}{n-1}} (1 + \alpha \|u_\varepsilon\|_n^{\frac{1}{n-1}}) \\
\geq \lambda_n C_n \left( 1 - \frac{n}{n-1} C_n \left( \frac{n-1}{\lambda_n} \log \left( 1 + c_n \left( \frac{F^0(x)}{\varepsilon} \right)^{\frac{n}{n-1}} \right) - B_\varepsilon \right) \right)^{\frac{1}{n-1}} (1 + \alpha \|u_\varepsilon\|_n^{\frac{1}{n-1}}) \\
\cdot \left( 1 - \alpha C_n \|G_\alpha\|_n^{\frac{1}{n-1}} \right)^{\frac{1}{n-1}} (1 + \alpha \|u_\varepsilon\|_n^{\frac{1}{n-1}}) \\
\geq \lambda_n C_n \left( 1 - \frac{n}{n-1} C_n \left( \frac{n-1}{\lambda_n} \log \left( 1 + c_n \left( \frac{F^0(x)}{\varepsilon} \right)^{\frac{n}{n-1}} \right) - B_\varepsilon \right) \right)^{\frac{1}{n-1}} \\
\cdot \left( 1 - \frac{n}{n-1} C_n \left( \frac{n-1}{\lambda_n} \log \left( 1 + c_n \left( \frac{F^0(x)}{\varepsilon} \right)^{\frac{n}{n-1}} \right) - B_\varepsilon \right) \right)^{\frac{1}{n-1}} \\
\cdot \left( 1 - \alpha C_n \|G_\alpha\|_n^{\frac{1}{n-1}} - n \log \left( 1 + c_n \left( \frac{F^0(x)}{\varepsilon} \right)^{\frac{n}{n-1}} \right) + \frac{n \lambda_n}{n-1} B_\varepsilon - \frac{\lambda_n \alpha^2 \|G_\alpha\|_n^{2n}}{(n-1) C_n^{\frac{2n}{n-1}}} + O(\phi) \right).
$$

By (5.4) and (5.6), we obtain

$$
\lambda_n |u_\varepsilon|^\frac{n}{n-1} (1 + \alpha \|u_\varepsilon\|_n^{\frac{n}{n-1}}) \\
\geq - E + \log \kappa_n - \log \varepsilon^n - n \log \left( 1 + c_n \left( \frac{F^0(x)}{\varepsilon} \right)^{\frac{n}{n-1}} \right) \\
- \frac{\lambda_n \alpha^2 \|G_\alpha\|_n^{2n}}{(n-1) C_n^{\frac{2n}{n-1}}} + \lambda_n A + O(\phi).
$$

Therefore

$$
\int_{\mathcal{W}_\varepsilon} \Phi \left( \lambda_n |u_\varepsilon|^\frac{n}{n-1} (1 + \alpha \|u_\varepsilon\|_n^{\frac{1}{n-1}}) \right) dx
$$
≥ \exp \left\{-E + \lambda_n A + \log \kappa_n - \log \epsilon^n - \frac{\lambda_n \alpha^2 \|G_\alpha\|_{\kappa}^{2n}}{(n-1) C^{\frac{n-1}{n}}} + O(\phi)\right\} \\
\cdot \int_{\mathcal{W}_{R\epsilon}} \exp \left\{-n \log \left(1 + c_n \left(\frac{F^0(x)}{\epsilon} \frac{n}{n-1}\right)\right)\right\} \\
≥ c_n^{-1} \epsilon^{-n} \exp \left\{-E + \lambda_n A - \frac{\lambda_n \alpha^2 \|G_\alpha\|_{\kappa}^{2n}}{(n-1) C^{\frac{n-1}{n}}} + O(\phi)\right\} \int_{\mathcal{W}_{R\epsilon}} \left(1 + c_n \left(\frac{F^0(x)}{\epsilon} \frac{n}{n-1}\right)\right)^{-n} dx \\
≥ (n-1) \kappa_n \exp \left\{-E + \lambda_n A - \frac{\lambda_n \alpha^2 \|G_\alpha\|_{\kappa}^{2n}}{(n-1) C^{\frac{n-1}{n}}} + O(\phi)\right\} \int_0^{\kappa_n R^{-\frac{n-1}{n}}} u^{n-2} \frac{(1+u)^n du}{(1+u)^n} \\
≥ (n-1) \kappa_n \exp \left\{-E + \lambda_n A\right\} \left(1 - \frac{\lambda_n \alpha^2 \|G_\alpha\|_{\kappa}^{2n}}{(n-1) C^{\frac{n-1}{n}}} + O(\phi)\right) .

Also

\int_{\mathbb{R}^n \setminus \mathcal{W}_{R\epsilon}} \Phi \left(\lambda_n u_{\epsilon}^{\frac{n}{n-1}}\right) \frac{dx}{(n-1)! C^{\frac{n-1}{n}}} ≥ \frac{\lambda_n^{-1} \|G_\alpha\|^n}{(n-1)! C^{\frac{n-1}{n}}} \int_{\mathbb{R}^n \setminus \mathcal{W}_{R\epsilon}} |G_\alpha|^n dx \\
= \frac{\lambda_n^{-1} \|G_\alpha\|^n + O \left(R^n \epsilon^n \left(\log (R\epsilon)^n)\right)\right)}{(n-1)! C^{\frac{n-1}{n}}},

thus

\int_{\mathbb{R}^n} \Phi \left(\lambda_n \left|u_{\epsilon}\right|^{\frac{n}{n-1}} \left(1 + \alpha \left|u_{\epsilon}\right| \frac{n}{n-1}\right)\right) dx \\
≥ \kappa_n \exp \left\{-E + \lambda_n A\right\} \left(1 - \frac{\lambda_n \alpha^2 \|G_\alpha\|_{\kappa}^{2n}}{(n-1) C^{\frac{n-1}{n}}} + O(\phi)\right) + \frac{\lambda_n^{-1} \|G_\alpha\|^n}{(n-1)! C^{\frac{n-1}{n}}} .

Since \( R = \log \frac{1}{\epsilon} \), by (5.5) one can obtain \( R \sim C^{\frac{n}{n-1}} \), then it is easy to verify that \( \phi = o \left(C^{\frac{n}{n-1}}\right) \). Thus when \( \alpha \) small enough, we have

\int_{\mathbb{R}^n} \Phi \left(\lambda_n \left|u_{\epsilon}\right|^{\frac{n}{n-1}} \left(1 + \alpha \left|u_{\epsilon}\right| \frac{n}{n-1}\right)\right) dx > \kappa_n \exp \left\{-E + \lambda_n A\right\} .

Then the proof of Theorem 1.2 with \( \sup_k c_k = +\infty \) has been completed. \square

**Acknowledgement.** The author would like to thank the supervisor Professor Jiayu Li for his continuous guidance and encouragement. The research was partially supported by Natural Science Foundation of China (Nos.11526212, 11721101, 11971026), Natural Science Foundation of Anhui Province (No.1608085QA12), Natural Science Foundation of Education.
Committee of Anhui Province (Nos.KJ2016A506, KJ2017A454) and Excellent Young Talents Foundation of Anhui Province (No.GXYQ2017070).

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