A CHARACTERIZATION OF ALWAYS SOLVABLE TREES IN LIGHTS OUT GAME USING THE ACTIVATION NUMBERS OF VERTICES

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ABSTRACT. Lights out is a game that can be played on any simple graph \( G \). A configuration assigns one of the two states on or off to each vertex. For a given configuration, the aim of the game is to turn all vertices off by applying a push pattern on vertices, where each push switches the state of the vertex and its neighbors. If every configuration of vertices is solvable, then we say that the graph is always solvable. We introduce a concept which we call the activation numbers of vertices and we prove several characterization results of graphs by using this concept. We show that for every always solvable graph there exists a chain of always solvable subgraphs where each subgraph differs from the preceding one by a vertex. We also characterize always solvable trees by showing that all always solvable trees can be constructed from always solvable subtrees by some special types of connections. We call the dimension of the space of null-patterns, which leave configurations unchanged, the nullity of the graph \( G \). We show that the nullity of a tree can be characterized by the cardinality of its minimal partition into always solvable subtrees.

1. INTRODUCTION

Lights out game on an undirected graph \( G(V, E) \) is played as follows. Each vertex \( v \) has a state which is either on or off. When a vertex is pushed, the vertex itself and all of its neighbors switch state. This push is called the activation of vertex \( v \). For any given initial on/off configuration of the vertices, the aim of the game is to turn all the vertices off by a sequence of activations. It is easy to observe that the order of the activations has no importance nor the act of activating a vertex more than once. Hence, the sequence of activations can be identified by the set \( P \) of activated vertices, which we call an activation pattern (or simply a pattern). Similarly, each initial configuration can be identified by a set \( C \) such that \( v \in C \) if the state of \( v \) is on in the configuration. If there exists an activation pattern \( P \) which turns all lights off for a given initial configuration \( C \), then the configuration \( C \) is called solvable and \( P \) is called a solving pattern for \( C \) (we may also say \( P \) solves \( C \)). If every configuration is solvable, then the graph \( G \)

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is called *always solvable* (it is also called *all parity realizable, always winnable, universally solvable* by other authors [2], [12], [11]).

Let the order of $V$ be $n$ and let $\{v_1, ..., v_n\}$ be an enumeration of $V$. Then any subset $S \subseteq V$ can be represented by its characteristic column vector $s = (s_1, ..., s_n)^t$ where $s(v_i) := s_i = 1$ if $v_i \in S$, and $s_i = 0$ otherwise. Hence, patterns or configurations can be identified by characteristic vectors of sets as well.

$N[v] = \{u \in V \mid u = v$ or $u$ is adjacent to $v\}$ is called the closed neighborhood set of vertex $v$, and $n \times n$ matrix $N := N(G)$ whose $i$th column is the characteristic vector of $N[v_i]$ is called the *closed adjacency matrix* of $G$. We denote Kernel, column space, and row space of $N$ by $\text{Ker}(N)$, $\text{Col}(N)$, and $\text{Row}(N)$, respectively. Let $\nu(G) := \dim(\text{Ker}(N(G)))$ and $r(G) := \dim(\text{Col}(N(G)))$. We call $\nu(G)$ nullity of $G$ (Amin et al. [3] call it parity dimension of $G$). By rank nullity theorem, we have $\nu(G) + r(G) = n$. It was first observed by Sutner [13],[14] that an activation pattern $p$ is a solving pattern for an initial configuration $c$ iff

\begin{equation}
Np = c
\end{equation}

over the field $\mathbb{Z}_2$. This simple linear algebraic formulation of the game leads to several realizations [13],[14],[3]:

(R 1) A configuration $c$ is solvable iff $c \in \text{Col}(N)$. Hence, graph $G$ is always solvable iff $r(G) = n$, which is equivalent to say that $\nu(G) = 0$.

(R 2) Number of solving patterns of a given configuration is $2^\nu(G)$. Indeed, if $p$ solves $c$, then $p + \ell$ solves $c$ as well for every $\ell \in \text{Ker}(N)$. Members of $\text{Ker}(N)$ are called *null patterns* since they have no effect on a given configuration.

(R 3) Since $N$ is a symmetric matrix, we have $\text{Col}(N) = \text{Row}(N)$. Moreover, $\text{Row}(N) = \text{Kern}(N)^\perp$ where $S^\perp$ denotes the orthogonal complement of a set $S$ with respect to the dot product $x \cdot y := x^t y$. Hence a configuration is solvable iff it is orthogonal to every null pattern.

For a given graph, finding the pattern $p$ which solves all lights on configuration $c = 1$ is called *all-ones problem* and the configuration $1$ is called *all-ones configuration*. The reason why it deserves a special name is the fact that all-ones problem is solvable for all graphs [13] (see also [6], [8], [10]). In other words, for every graph $G$ there exists $p$ such that $N(G)p = 1$.

The connection between the nullities of graphs related to each other by some type of graph operation such as edge/vertex join or removal was first considered by Amin et al. [4]. The same subject was investigated by Giffen et al. [12], Edwards et al. [9] and Ballard et al. [5] for a generalized version of the game where (1) is considered over $\mathbb{Z}_k$ for some integer $k \geq 2$. In [4] and [5] the difference $\nu(G - u) - \nu(G)$, where $u$ is a vertex of graph $G$, plays an important role in the analysis. We
adapt the terminology of \[5\] and call this difference null difference of vertex \(u\) and denote it by \(\text{nd}(u)\). It turns out that \(\text{nd}(u)\) can be either \(-1, 0,\) or \(1\); and as a result, vertices can be categorized by their null difference number into three different classes \([4],[5]\).

We show that these classes can be realized from a different point of view in the case of classical lights out game where \(k=2\). Indeed, there is another way of categorizing vertices by checking in how many solving patterns of all-ones configuration a vertex is activated. This categorization again leads to three different classes. We prove that the first set of classes determined by the null differences of vertices and the second one determined by the activation of the vertices under solving patterns of all-ones configuration, coincide with each other.

Seeing this set of classes from the second point of view has important advantages. First, it makes it easier to see the existence of a vertex from a specific class in a graph or in a subgraph. Using this observation we prove our first main result Theorem 2.9 whose corollary Corollary 2.10 states the existence of a chain of always solvable subgraphs of an always solvable graph differing from one another by a vertex.

Second, this point of view allows us to track how the class of some vertices changes under some special graph operations (see Theorem 3.3), and enables us to make two observations about trees. Our first observation is that the nullity of a tree can be characterized by the cardinality of its minimal partition into always solvable subtrees as we showed in Theorem 4.4. Our second observation is related to the characterization of always solvable trees. In Theorem 4.7 we showed that an always solvable tree can be characterized as some special join tree of its always solvable subtrees. This gives an alternative to the previous two characterizations of always solvable trees given in \([2],[4]\).

### 2. Activation numbers

**Definition 2.1.** We define the inverse configuration \(\overline{c}\) of a configuration \(c\) as \(\overline{c} = c + 1\). For a given vertex \(u\), we denote the configuration, where only the state of \(u\) is on by \(c_u\) i.e.; \(c_u(v) = 1\) iff \(v = u\).

Let \(u\) and \(w\) be two vertices of a graph \(G\) which are adjacent to each other. Let us denote the graph obtained by deleting the edge \(e = (u, w)\) between \(u\) and \(w\) by \(G - e\). Let \(V(G) = \{v_1,...,v_n\}\). Then the difference between the neighborhood matrices \(N(G - e)\) and \(N(G)\) is the symmetric matrix \(J\) where \(J_{ik} = J_{ki} = 1\) if \(v_i = u, v_k = w,\) and \(0\) otherwise. Let \(p\) be a solving pattern for a configuration \(c\) on \(G\). Then the same pattern applied to the graph \(G - e\) solves the configuration

\[
\overline{c} = N(G - e)p = N(G)p + Jp = c + p(w)c_u + p(u)c_w.
\]

In other words, for a given pattern \(p\), deletion of an edge \(e = (u, w)\) may only change the states of vertices \(u\) and \(w\) in the corresponding
solution configuration, and the state of \( u \) \((w)\) changes if and only if \( w \) \((u)\) is activated in \( p \).

**Lemma 2.2.** Let \( u \) be a vertex such that the configuration \( \overline{c_u} \) is solvable. Then \( u \) is not activated in any solving pattern \( p \) for \( \overline{c_u} \). In other words \( p(u) = 0 \) for all \( p \) satisfying \( Np = \overline{c_u} \).

**Proof.** We prove the lemma by applying induction on the size of the graph. Note that the lemma holds true for any graph without any edges. Assume that it holds true for any graph with size \( n \). Let \( G \) be a graph with size \( n + 1 \). Assume for a contradiction that there exists a pattern \( p \) such that \( p(u) = 1 \) and \( N(G)p = \overline{c_u} \) for some vertex \( u \in V(G) \). Since \( \overline{c_u}(u) = 0 \) and \( p(u) = 1 \), there must exist odd number of activated vertices adjacent to \( u \). Let \( w \) be one of those adjacent vertices to \( u \) such that \( p(w) = 1 \). Let \( e = (u, w) \). Then by (2) the pattern \( p \) applied to the graph \( G - e \) solves the configuration \( \overline{c_u} + c_u + c_w = 1 + c_w = \overline{c_w} \). Since \( G - e \) has \( n \) edges and \( p(w) = 1 \), this contradicts with the induction hypothesis. \( \square \)

**Definition 2.3.** We call a vertex \( v \) half-activated if \( \ell(v) = 1 \) for some null-pattern \( \ell \), and fixed otherwise.

By [3], Lemma 1] we know that for any vertex \( v \in V(G) \), either \( \ell(v) = 0 \) for all null-patterns \( \ell \) or \( \ell(v) = 1 \) for exactly half of the null-patterns which correspond to \( 2^{\nu(G)} - 1 \) patterns. Together with (R 2), this gives us the following lemma.

**Lemma 2.4.** Let \( c \) be a solvable configuration on a graph \( G \). If \( v \) is half-activated, then \( p(v) = 1 \) for exactly half of the solving patterns \( p \) for \( c \). If \( v \) is fixed, either

- Case i) \( p(v) = 1 \) for all solving patterns \( p \) for \( c \), or
- Case ii) \( p(v) = 0 \) for all solving patterns \( p \) for \( c \).

Above lemma motivates us to make the following definition.

**Definition 2.5.** For a given solvable configuration \( c \), we call a fixed vertex \( v \), \( c \)-always-activated or \( c \)-never-activated if \( v \) satisfies Case (i) or Case (ii), respectively. In the case of \( c = 1 \), instead of calling a vertex 1-always-activated (1-never-activated), we simply call it always-activated (never-activated). We also assign a number \( A(v) = A_G(v) \) to every vertex \( v \in V(G) \), which we call activation number of \( v \) as follows. We say \( A(v) = 1 \) if \( v \) is always-activated, \( A(v) = 0 \) if \( v \) is never-activated and \( A(v) = -1 \) if \( v \) is half-activated.

As a consequence of (R 3), a vertex \( v \) is fixed if and only if \( c_v \) is solvable; see [5], Proposition 2.4. Moreover, we have the following proposition.

**Proposition 2.6.** A vertex \( v \) is always-activated if and only if \( v \) is \( c_v \)-always-activated.
Proof. Let \( s \) and \( p \) be arbitrary solving patterns for \( 1 \) and \( c_v \), respectively. Then \( s + p \) solves \( \overline{c_v} \). By Lemma 2.8, \((s + p)(v) = 0\). Thus, \( s(v) = p(v) \). □

On the other hand, by [3, Proposition 2.8, Proposition 2.10] we have \( nd(v) = 0 \) (\( nd(v) = 1 \)) if and only if \( v \) is \( c_v \)-always-activated (\( c_v \)-never-activated). Thus, we reach the following identification which connects the null difference number of a vertex to its activation number in the case of classical lights out game.

**Proposition 2.7.** A vertex \( v \) is always-activated if and only if \( nd(v) = 0 \) and never-activated if and only if \( nd(v) = 1 \).

In terms of activation numbers, the above proposition reads \( A(v) = 1 - nd(v) \) if \( v \) is fixed and \( A(v) = nd(v) = -1 \) if \( v \) is half-activated.

**Lemma 2.8.** Every always solvable graph has an always-activated vertex.

Proof. Let \( G \) be an always solvable graph. Since the all-ones configuration is solvable in any graph and \( \nu(G) = 0 \), there is a unique solution \( p \) for configuration \( 1 \) on \( G \). Because \( p \neq 0 \) there must be a vertex \( w \in V(G) \) with \( p(w) \neq 0 \), i.e.; \( w \) is activated. Since \( p \) is the only solution, \( w \) is always-activated. □

The connection between the activation number and the null difference of a vertex allows us to state the following theorem. For convenience, we define \( \nu(K_0) = 0 \).

**Theorem 2.9.** Let \( G \) be a graph with order \( n \). Then there exits a sequence of vertices \( \{v_k\}_{k=1}^{k=n} \) which satisfies the following. Define \( G_0 := G \), and \( G_k := G_{k-1} - v_k \). Then \( \nu(G_k) = \nu(G) - k \) for \( k < \nu(G) \) and \( \nu(G_k) = 0 \) for \( k \geq \nu(G) \).

Proof. We construct the sequence \( \{v_k\}_{k=1}^{k=n} \) in two steps. In the first part of the construction we note that for any graph \( H \), \( \nu(H) \neq 0 \) if and only if there exists a vertex \( v \in V(H) \) such that \( v \) is half-activated. So if \( \nu(G) \neq 0 \), choose a half-activated vertex \( v \) of \( G \) and let \( v_1 = v \). Since \( nd(v_1) = -1 \), this implies \( \nu(G_1) = \nu(G) - 1 \). If \( \nu(G_1) \neq 0 \), choose a half-activated vertex of \( G_1 \) and call it \( v_2 \). As we continue this process, we obtain a sequence of graphs \( G_0, ..., G_m \), where \( m = \nu(G) \) and \( \nu(G_k) = \nu(G) - k \) for all \( k \in \{1, ..., m \} \). Note that \( \nu(G) \) can be at most \( n - 1 \) and this happens only if \( G = K_n \) [3, Proposition 1]. Hence, the sequence \( G_0, ..., G_m \) is well defined, i.e.; \( m \leq n - 1 \), and equality holds only if \( G_m = K_1 \).

Second part of the construction is as follows. Since \( \nu(G_m) = 0 \) there is an always-activated vertex \( w \) of \( G_m \) by Lemma 2.8. Thus, \( nd(w) = 0 \) by Proposition 2.7. So let \( v_{m+1} = w \). Then \( \nu(G_{m+1}) = 0 \) and if \( G_{m+1} \) is not the empty graph, this time we can choose an always-activated vertex of \( G_{m+1} \) to determine the next vertex of the sequence.
and continue the process. The process is terminated when we reach the empty graph.

Corollary 2.10. Let \( G \) be an always solvable graph with order \( n \). Then there exists a chain of always solvable subgraphs \( G = G_0 \supset G_1 \supset \ldots \supset G_{n-1} = K_1 \) where each subgraph differs from the preceding one by only a vertex.

3. Join of Graphs

Let \( G_1 \) and \( G_2 \) be two nonempty disjoint graphs and \( H := G_1uwG_2 \) be the join graph constructed by joining the vertices \( u \) of \( G_1 \) and \( w \) of \( G_2 \) by an edge. Let \( |G_i| = n_i \) where \( i \in \{1, 2\} \). Then we can enumerate the vertices of \( H \) as \( \{v_1, \ldots, v_{n_1+n_2}\} \) such that \( V(G_1) = \{v_1, \ldots, v_{n_1}\} \) with \( v_{n_1} = u \) and \( V(G_2) = \{v_{n_1+1}, \ldots, v_{n_1+n_2}\} \) with \( v_{n_1+1} = w \). This way, we can represent every pattern \( p \) of \( H \) as \( p^t = (p_1^t, p_2^t) \) where \( p_i \) is the restriction of \( p \) on \( V(G_i) \).

Let \( s \) be a solving pattern for all-ones configuration on \( H \), i.e.; \( N(H)s = 1 \). Note that by (2) we have

\[
N(G_1)s_1 = \begin{cases} 
1 & \text{if } s(w) = 0 \\
\overline{c_u} & \text{if } s(w) = 1
\end{cases},
\]

and similarly

\[
N(G_2)s_2 = \begin{cases} 
1 & \text{if } s(u) = 0 \\
\overline{c_w} & \text{if } s(u) = 1
\end{cases},
\]

where we use the same notation \( 1 \) to denote the all-ones configurations of different graphs.

We already know that in any solving pattern for the all-ones configuration on a tree, adjacent vertices cannot be both activated [7]. Moreover, we have the following lemma.

Lemma 3.1. Let \( H \) be a graph with a cut edge \( e = (u, w) \). Then in any solving pattern \( s \) for all-ones configuration on \( H \), vertices \( u \) and \( w \) cannot be both activated i.e.; either \( s(u) = 0 \) or \( s(w) = 0 \).

Proof. Without loss of generality we can assume \( H \) is connected. Further, we can see \( H \) as \( H = G_1uwG_2 \) where \( G_1 \) (\( G_2 \)) is the connected component containing \( u \) (\( w \)) in \( G - e \). Assume for a contradiction that there exists a solving pattern \( s \) for the all-ones configuration on \( H \) such that \( s(u) = s(w) = 1 \). Then by (3) \( N(G_1)s_1 = \overline{c_u} \) with \( s_1(u) = s(u) = 1 \), which contradicts with Lemma 2.2.

Definition 3.2. We define a Type-\((a, b)\) connection of disjoint graphs \( G_1 \) and \( G_2 \), respectively as the join graph \( H = G_1uwG_2 \), where \( u \in V(G_1) \), \( w \in V(G_2) \) with \( A_{G_1}(u) = a, A_{G_2}(w) = b \).
**Theorem 3.3.** Let $G_1$ and $G_2$ be disjoint graphs with $u \in V(G_1)$, $w \in V(G_2)$, and $H = G_1uwG_2$. Let $s$ be a solving pattern for the all-ones configuration on $H$, and $\Delta \nu := \nu(H) - \nu(G_1) - \nu(G_2)$. Then we have the following table:

**Table 1. Join Graph $H = G_1uwG_2$**

| $A_{G_1}(u)$ | $A_{G_2}(w)$ | $A_H(u)$ | $A_H(w)$ | $\Delta \nu$ | $N(G_1)s_1$ | $N(G_2)s_2$ | when  |
|--------------|--------------|----------|----------|--------------|--------------|--------------|-------|
| 0            | 0            | 0        | 0        | 0            | 1            | 1            |       |
| 0            | -1           | 0        | -1       | 0            | $\frac{1}{c_u}$ | 1            | $s(w) = 0$ |
| 1            | 1            | -1       | -1       | 1            | $\frac{1}{c_v}$ | 1            | $s(u) = 0, s(w) = 1$ |
| 1            | -1           | 0        | 1        | -1           | $\frac{1}{c_u}$ | 1            | $s(u) = 1, s(w) = 0$ |
| -1           | -1           | 0        | 0        | -2           | 1            | 1            |       |

where the not written cases can be obtained by symmetry. Some of the above information can be expressed more compactly as

$$
\Delta \nu = \begin{cases} 
-2 & \text{if } A_{G_1}(u)A_{G_2}(w) = -1 \\
A_{G_1}(u)A_{G_2}(w) & \text{otherwise}
\end{cases},
$$

(5)

$$
A_H(u) = A_{G_1}(u)(1 + A_{G_2}(w)) \mod 3,
$$

(6)

$$
A_H(w) = A_{G_2}(w)(1 + A_{G_1}(u)) \mod 3.
$$

(7)

**Proof.** Let $s$ be a solving pattern for the all-ones configuration on $H$. Note that by Lemma 3.1 there are three cases we need to consider:

- **Case a:** $s(u) = 0$, $s(w) = 0$
- **Case b:** $s(u) = 0$, $s(w) = 1$
- **Case c:** $s(u) = 1$, $s(w) = 0$.

We will investigate each case under a specific Type-$(a, b)$ connection of $G_1$ and $G_2$. By symmetry we need to consider total of six types of connection:

Type-$(0, 0)$ where $A_{G_1}(u) = 0, A_{G_2}(w) = 0$: Case $b$ and Case $c$ are not possible. Indeed, for example, if $s(u) = 1, s(w) = 0$ then $N(G_1)s_1 = 1$ with $s_1(u) = 1$, which contradicts with the fact that $u$ is a never-activated vertex of $G_1$. Thus, $s(u) = 0, s(w) = 0$ and $N(G_1)s_1 = 1, N(G_2)s_2 = 1$. So the number of solutions is $2^{\nu(H)} = 2^{\nu(G_1)2^{\nu(G_2)}}$ and $A_H(u) = 0, A_H(w) = 0$.

Type-$(0, 1)$ where $A_{G_1}(u) = 0, A_{G_2}(w) = 1$: Case $a$ is not possible. Otherwise, $N(G_2)s_2 = 1$ with $s_2(w) = 0$, which contradicts with always-activatedness of $w$. Case $c$ is not possible because of the
same reason in Type-\((0,0)\) connection. Thus, \(s(u) = 0, s(w) = 1\) and \(N(G_1)s_1 = \overline{c_u}, \, N(G_2)s_2 = 1\). So the number of solutions is \(2^{\nu(H)} = 2^{\nu(G_1)}2^{\nu(G_2)}\) and \(A_H(u) = 0, \, A_H(w) = 1\).

Type-\((0,-1)\) where \(A_{G_1}(u) = 0, A_{G_2}(w) = -1\): Case \(c\) is not possible. Otherwise, \(N(G_2)s_2 = \overline{c_w}\), which implies \(c_w = \overline{c_w} + 1\) is solvable, which contradicts with the half-activatedness of \(w\) \([5\text{, Proposition 2.4}]\). In Case \(a\), \(N(G_1)s_1 = 1, \, N(G_2)s_2 = 1\) with \(s_2(w) = 0\). Since \(w\) is not activated in exactly half of the solutions for any configuration on \(G_2\), there are \(2^{\nu(G_1)}2^{\nu(G_2)-1}\) solutions of the all-ones configuration on \(H\) in Case \(a\). And in Case \(b\), \(N(G_1)s_1 = \overline{c_u}, \, N(G_2)s_2 = 1\) with \(s_2(w) = 1\) which satisfies \(2^{\nu(G_1)}2^{\nu(G_2)-1}\) solutions of the all-ones configuration on \(H\). In total, we have \(2^{\nu(H)} = 2^{\nu(G_1)}2^{\nu(G_2)}\) solutions in which \(u\) is never-activated and \(w\) is half-activated. Hence \(A_H(u) = 0, \, A_H(w) = -1\).

Type-\((1,1)\) where \(A_{G_1}(u) = 1, A_{G_2}(w) = 1\): Case \(a\) is not possible. Otherwise, \(N(G_1)s_1 = 1\) with \(s_1(u) = 0\), which contradicts with always-activatedness of \(u\). In Case \(b\), \(N(G_1)s_1 = \overline{c_u}\) and \(N(G_2)s_2 = 1\). Hence, there are \(2^{\nu(G_1)}2^{\nu(G_2)}\) solutions \(s\) on \(H\), which satisfies Case \(b\). In Case \(c\), \(N(G_1)s_1 = 1, \, N(G_2)s_2 = \overline{c_w}\), which again corresponds \(2^{\nu(G_1)}2^{\nu(G_2)}\) solutions \(s\) on \(H\). In total, there are \(2^{\nu(H)} = 2^{\nu(G_1)}2^{\nu(G_2)}\) solutions. Moreover, each vertex is activated only in one of the cases. Hence, \(A_H(u) = -1, \, A_H(w) = -1\).

Type-\((-1,1)\) where \(A_{G_1}(u) = 1, A_{G_2}(w) = -1\): Case \(c\) is not possible because of the same reason in Type-\((1,1)\) and Type-\((0,-1)\) connections, respectively. In Case \(b\), \(N(G_1)s_1 = \overline{c_u}\) and \(N(G_2)s_2 = 1\) with \(s_2(w) = 1\). Since \(w\) is not activated in exactly half of the solutions for any configuration on \(G_2\), there are \(2^{\nu(H)} = 2^{\nu(G_1)}2^{\nu(G_2)-1}\) solutions of the all-ones configuration on \(H\) with \(A_H(u) = 0, \, A_H(w) = 1\).

Type-\((-1,-1)\) where \(A_{G_1}(u) = -1, A_{G_2}(w) = -1\): Case \(c\) is not possible because of the same reason in Type-\((0,-1)\) connection. Case \(b\) corresponds to the symmetric situation of Case \(c\), where \(u\) and \(w\) are switched. So it is not possible either. In Case \(a\), \(N(G_1)s_1 = 1\) with \(s_1(u) = 0\). Since \(u\) is half-activated there are \(2^{\nu(G_1)}-1\) such solutions. Similarly, \(N(G_2)s_2 = 1\) with \(s_2(w) = 0\) with \(2^{\nu(G_2)}\) such solutions. In total, there are \(2^{\nu(H)} = 2^{\nu(G_1)}2^{\nu(G_2)}\) solutions of the all-ones configuration on \(H\) with \(A_H(u) = 0, \, A_H(w) = 0\).

We want to note that the fifth column of Table 1 is also deducible from \([5\text{, Theorem 2.18}]\) using the relation between activation numbers and null differences. We have the following corollary.

**Corollary 3.4.** Let \(F, G_1, ..., G_n\) be always solvable graphs. Let \(u \in V(F)\), and \(v_i \in V(G_i)\) for \(1 \leq i \leq n\). Consider the join graph \(H\) obtained by joining \(v_i\) to \(u\) for \(1 \leq i \leq n\) by an edge. If \(A_F(u) = 0\), then \(H\) is always solvable. If \(A_F(u) = 1\), then \(H\) is always solvable if and only if \(A_{G_i}(v_i) = 1\) for even number of vertices.
Proof. Note that $H$ can be obtained by applying Type-$(a, b)$ connections successively to the graphs $F, G_1, \ldots, G_n$, where at each connection one of the vertices is always taken as $u$. The first part of the corollary follows from the fact that Type-$(0, b)$ connections never change the activation numbers nor $\Delta \nu$. On the other hand, single Type-$(1, 1)$ connection increases $\Delta \nu$ by 1. However, joining even number of $v_i$ to $u$ corresponds a series of Type-$(1, 1)$ connection followed by a Type-$(−1, 1)$ connection, which remains $\Delta \nu$ unchanged. \hfill \Box

4. Always solvable trees

In this section, we use the following observation: Since every edge $e$ of a tree $T$ is a cut edge, $T$ can always be seen as a join graph of the connected components of $T − e$. As a result, every adjacent pair of vertices $u$ and $w$ can only have the pair of activation numbers determined by the third and fourth column of Table 1. And the possible values of their activation numbers with respect to the connected components of $T − (u, w)$ can be one of those corresponding pairs in the first and second column of Table 1.

From [1], Lemma 2, we know that any graph can only have even number of half-activated vertices. Moreover, proof of [1], Lemma 2 also implies the following.

**Lemma 4.1.** Let $u$ be a half-activated vertex of a graph. Then there exists a half-activated vertex adjacent to $u$.

**Proof.** By definition, $u$ is a half-activated vertex of a graph $G$ if $\ell(u) = 1$ for some null-pattern $\ell$. Let us denote the characteristic vector of the closed neighborhood set of $u$ by $N[u]$. Since $\ell$ is a null-pattern $N[u] \cdot \ell = (N(G)\ell)(u) = 0(u) = 0$. Since $\ell(u) = 1$ this implies $\ell$ activates odd number of vertices adjacent to $u$, i.e.; there exists at least one vertex $w$ adjacent $u$ such that $\ell(w) = 1$. Hence, $w$ is half activated. \hfill \Box

**Definition 4.2.** Let $G$ be a nonempty graph and $P = \{G_1, \ldots, G_k\}$ be a set of subgraphs of $G$. We say $G$ is partitioned into $P$ (or $P$ partitions $G$) if $V(G_i)$'s are pairwise disjoint and $V(G) = \bigcup_{i=1}^{k} V(G_i)$. If all $G_i$'s are always solvable, we say $P$ is a partition into always solvable subgraphs (PASS) of $G$. We say a PASS $M$ is minimal if for all PASS’s $P$ of $G$, $|M| \leq |P|$. We define the number $\pi(G)$ as the cardinality of a minimal PASS of $G$.

**Remark 4.3.** Note that since $K_1$ is always solvable, every graph $G$ has an at least one PASS which consists of single vertex subgraphs of $G$. Consequently, every graph has a minimal PASS and each minimal PASS has the same cardinality. Hence, $\pi(G)$ is well defined for all graphs $G$. 
Theorem 4.4. For a nonempty tree \( T \), \( \pi(T) = \nu(T) + 1 \). In other words, \( T \) has nullity \( n \) if and only if there exists a minimal PASS of \( T \) with cardinality \( n + 1 \).

Proof. We first show \( \pi(T) \leq \nu(T) + 1 \), which is equivalent to prove that there exists a PASS of \( T \) with cardinality \( \nu(T) + 1 \). We prove the claim by applying induction on the nullity of the tree. Taking the partition as \( T \) itself, we see that the claim trivially holds true for \( \nu(T) = 0 \). Assume that it holds true for all trees \( S \) with \( \nu(S) < n \). If \( T \) is a tree with \( \nu(T) = n \), then it has a half-activated vertex. Moreover, by Lemma 4.3 there exists an adjacent pair of half-activated vertices \( x \) and \( y \) of \( T \). Let \( X \) and \( Y \) be the components of \( T - \{x, y\} \) containing \( x \), and \( y \), respectively. From Table 1 we see that \( T \) must be a Type-(1, 1) connection of \( X \) and \( Y \), and \( \nu(X) + \nu(Y) = \nu(T) - \Delta \nu = n - 1 \). So both \( \nu(X) \) and \( \nu(Y) \) are less than \( n \). By induction hypothesis there exist PASS’s \( \{X_1, \ldots, X_r\} \) and \( \{Y_1, \ldots, Y_s\} \) of \( X \) and \( Y \), respectively with \( r = \nu(X) + 1 \) and \( s = \nu(Y) + 1 \). Note that since \( X \) and \( Y \) are disjoint, \( X_i \) and \( Y_j \) are disjoint as well for all \( 1 \leq i < r \), \( 1 \leq j < s \). Moreover, \( V(T) = V(X) \cup V(Y) \). Thus \( \{X_1, \ldots, X_r, Y_1, \ldots, Y_s\} \) is a PASS of \( T \) with cardinality \( r + s = \nu(X) + 1 + \nu(Y) + 1 = n - 1 + 2 = n + 1 \). This proves \( \pi(T) \leq \nu(T) + 1 \).

To prove the converse inequality let \( M = \{T_1, \ldots, T_m\} \) be a minimal PASS of \( T \) with \( m = \pi(T) \). Since \( T \) is a tree, one can easily observe that there cannot be more than one edge between different subtrees \( T_i \) and \( T_j \) in \( T \). Moreover, each subtree \( T_i \) is connected by an edge to at least one other subtree \( T_j \); and every edge between subtrees is a cut edge. These observations allow us to realize that the number of edges between subtrees \( T_i \)’s equals to \( m - 1 \), and \( T \) can be seen as a join graph obtained by applying \( m - 1 \) successive Type-(a, b) connection on \( T_i \)’s. Since initially all subgraphs has nullity 0, and since the nullity of any join graph obtained by a single Type-(a, b) connection exceeds the sum of the nullities of joined components at most by 1, we conclude that \( \nu(T) \) can at most be \( m - 1 = \pi(T) - 1 \). Thus, \( \nu(T) + 1 \leq \pi(T) \). □

Remark 4.5. Note that the above identity cannot be generalized to arbitrary graphs. Indeed, if we consider the cycle \( C_6 \), we see that it has a minimal PASS which consists of two subgraphs isomorphic to \( P_3 \). Thus \( \pi(C_6) = 2 \). However, \( \nu(C_6) = 2 \) as well.

Definition 4.6. Let \( G_1, G_2, \) and \( G_3 \) be disjoint graphs with vertices \( x \in V(G_1) \), \( y \in V(G_2) \) and \( z \in V(G_3) \). Let \( K := G_1xyG_2yzG_3 \) be the join graph obtained by joining both \( x \) and \( z \) to \( y \) by edges. Then we call \( K \) a Type-(a, b, c) connection of graphs \( G_1, G_2, G_3 \), respectively if \( A_{G_1}(x) = a \), \( A_{G_2}(y) = b \), and \( A_{G_3}(z) = c \).

Theorem 4.7. Let \( T \) be a tree different than \( K_1 \). Then the followings are equivalent.
A CHARACTERIZATION OF ALWAYS SOLVABLE TREES

(1) $T$ is always solvable.

(2) $T$ is obtained from always solvable trees by a Type-(0, 1) connection or a Type-(1, 1, 1) connection.

(3) $T$ is obtained from always solvable trees by a Type-(0, 1) connection or a Type-(1, 1) connection followed by a Type-(1, −1) connection.

Proof. Proof of (3) $\Rightarrow$ (1): From Table 1, we see Type-(0, 1) connection of two always solvable trees is always solvable. Now take three always solvable trees. By Lemma 2.8, they all have always-activated vertices. Connect two of them by a Type-(1, 1) connection. This produces a tree $X$ with $\nu(X) = 1$. Hence, it has a half-activated vertex $x$. On the other hand, the remaining tree, call it $Y$, has an always-activated vertex $y$. So it is possible to join $Y$ and $X$ by a Type-(1, −1) connection, and resulting tree $T := YyxX$ has nullity $\nu(T) = \Delta \nu + \nu(Y) + \nu(X) = -1 + 0 + 1 = 0$. Hence, $T$ is always solvable.

Proof of (1) $\Rightarrow$ (2): Let $T$ be an always solvable tree with order greater than 1. By Lemma 2.8, it has an always-activated vertex. Moreover, since adjacent vertices of trees cannot be activated at the same time [7], $T$ must have an adjacent pair $u, w$ such that $A_T(u) = 0$ and $A_T(w) = 1$. Note that $T = UuwS$ where $U$ and $S$ are the connected components of $T - (u, w)$ containing $u, w$, respectively. From Table 1, we see that $T$ must be either Type-(0, 1) or Type-(1, 1, 1) connection of $U$ and $S$, respectively. If the former holds $\nu(U) + \nu(S) = \nu(T) - \Delta \nu = 0 - 0 = 0$, which implies $U$ and $S$ are always solvable. So we are done. If the latter holds $\nu(T) + \nu(S) = \nu(H) - \Delta \nu = 0 - (-1) = 1$. Moreover, $A_U(u) = 1$ and $A_S(w) = -1$. Hence, $U$ is always solvable and $S$ has nullity 1. By the Lemma 4.1 there exists a half-activated vertex $z$ of $S$ which is adjacent to $w$. Let $S = WwzZ$ where $W$ and $Z$ are the connected components of $S - (w, z)$ containing $w, z$, respectively. From Table 1, we see $S$ is a Type-(1, 1) connection of $W$ and $Z$. Consequently, $A_W(w) = A_Z(z) = 1$ and $\nu(W) + \nu(Z) = \nu(S) - \Delta \nu = 1 - 1 = 0$, which implies both $W$ and $Z$ are always solvable. We see that $H = UuwWwzZ$ with $A_U(u) = A_W(w) = A_Z(z) = 1$. Thus, $T$ is a Type-(1, 1, 1) connection of always solvable trees $U, W,$ and $Z$, respectively.

Proof of (2) $\Rightarrow$ (3): Just observe that every Type-(1, 1, 1) connection is a Type-(1, 1) connection followed by a Type-(1, −1) connection.

□

References

[1] A. T. Amin and P. J. Slater. Neighborhood domination with parity restrictions in graphs. In Proceedings of the Twenty-third Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1992), volume 91, pages 19–30, 1992.

[2] A. T. Amin and P. J. Slater. All parity realizable trees. J. Combin. Math. Combin. Comput., 20:53–63, 1996.
[3] Ashok T. Amin, Lane H. Clark, and Peter J. Slater. Parity dimension for graphs. *Discrete Math.*, 187(1-3):1–17, 1998.

[4] Ashok T. Amin, Peter J. Slater, and Guo-Hui Zhang. Parity dimension for graphs—a linear algebraic approach. *Linear Multilinear Algebra*, 50(4):327–342, 2002.

[5] Laura E. Ballard, Erica L. Budge, and Darin R. Stephenson. Lights out for graphs related to one another by constructions. *Involve*, 12(2):181–201, 2019.

[6] Yair Caro. Simple proofs to three parity theorems. *Ars Combin.*, 42:175–180, 1996.

[7] Margaret M. Conlon, Maria Falidas, Mary Jane Forde, John W. Kennedy, Susan McIlwaine, and Joseph Stern. Inversion numbers of graphs. *Graph Theory Notes N. Y.*, 37:42–48, 1999.

[8] Robert Cowen, Stephen H. Hechler, John W. Kennedy, and Alex Ryba. Inversion and neighborhood inversion in graphs. *Graph Theory Notes N. Y.*, 37:37–41, 1999.

[9] Stephanie Edwards, Victoria Elandt, Nicholas James, Kathryn Johnson, Zachary Mitchell, and Darin Stephenson. Lights out on finite graphs. *Involve*, 3(1):17–32, 2010.

[10] Henrik Eriksson, Kimmo Eriksson, and Jonas Sjöstrand. Note on the lamp lighting problem. volume 27, pages 357–366. 2001. Special issue in honor of Dominique Foata’s 65th birthday (Philadelphia, PA, 2000).

[11] Rudolf Fleischer and Jiajin Yu. A survey of the game “Lights Out!”. In *Space-efficient data structures, streams, and algorithms*, volume 8066 of Lecture Notes in Comput. Sci., pages 176–198. Springer, Heidelberg, 2013.

[12] Alexander Giffen and Darren B. Parker. On generalizing the “lights out” game and a generalization of parity domination. *Ars Combin.*, 111:273–288, 2013.

[13] K. Sutner. Linear cellular automata and the Garden-of-Eden. *Math. Intelligencer*, 11(2):49–53, 1989.

[14] Klaus Sutner. The $\sigma$-game and cellular automata. *Amer. Math. Monthly*, 97(1):24–34, 1990.

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