Quantum Groups, $q$-Oscillators and Covariant Algebras

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– Dedicated to the memory of M.C. Polivanov –

ABSTRACT

The physical interpretation of the main notions of the quantum group theory (co-product, representations and corepresentations, action and coaction) is discussed using the simplest examples of $q$-deformed objects (quantum group $F_q(\text{GL}(2))$, quantum algebra $sl_q(2)$, $q$-oscillator and $F_q$-covariant algebra.) Appropriate reductions of the covariant algebra of second rank $q$-tensors give rise to the algebras of the $q$-oscillator and the $q$-sphere. A special covariant algebra related to the reflection equation corresponds to the braid group in a space with nontrivial topology.

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1. Introduction

The formalism of the quantum inverse scattering method (the R-matrix approach), which was used in [1] in order to formulate the theory of quantum Lie groups and Lie algebras, contributed much to the growth of attention towards these new mathematical objects and to the use of the latter by theoretical physicists.

Convincing examples of the description of the symmetry properties of physical models by means of quantum groups and algebras have been constructed (see, e.g. [2-6]), both in the standard way, when Hamiltonian of the model H commutes with the quantum algebra generators $X_i : [H, X_i] = 0$, and in the more complicated situation of conformal field theory. Quantum groups and algebras are also used as a basis for the new approaches to the possible structure of space-time on small distances (see [7,8]), which gives us an opportunity to define naturally quantum homogeneous spaces.

However, despite intensive development of the mathematical theory of quantum groups (cf.,[9]), the physical interpretation of many statements in this field deserves, from our point of view, more attention. Naturally, it is much easier to keep the discussion to simple examples. For this purpose we will limit ourselves to the well-known quantum group $F_q(GL(2))$, the quantum algebra $sl_q(2)$, the $q$-deformed oscillator and (a new example, [10]) the simplest quadratic algebra $K$, which is defined by the reflection equation and is covariant with respect to the quantum group $F_q(GL(2))$. The notions discussed, include coproduct, representations and corepresentations, action and coaction.

Being the associative algebra, the quantum group $F_q(GL(2))$ is defined by four generators $a, b, c, d$, which satisfy the relations

\begin{align*}
ab &= qba, 
ac &= qca, 
[a, d] &= \omega bc , 
b d &= qdb, 
bd &= qdc, 
[b, c] &= 0 ,
\end{align*}

where $q$ is the complex deformation parameter and $\omega = q - q^{-1}$. The relations (1) define multiplication in algebra $F_q$ and can be written in a compact matrix form [1] by means of the $2 \times 2$ matrix $T$

\begin{equation}
T = \begin{pmatrix} a & b \\
c & d \end{pmatrix}
\end{equation}

and a $4 \times 4$ matrix $R$ with diagonal $(q, 1, 1, q)$ and the only under-diagonal element not
equal to zero $R_{21,12} = \omega$,

$$RT_1 T_2 = T_2 T_1 R$$  \hspace{1cm} (3)$$

where $T_1 = T \otimes I$ and $T_2 = I \otimes T$. Rows and columns of the $R$-matrix in $C^2 \otimes C^2$ are numerated by a pair of indexes, for example 11, 12, 21, 22. As a mathematical object, $F_q$ is a Hopf algebra. This means the existence of three more operations along with multiplication $\mu$: a coproduct $\Delta : F_q \rightarrow F_q \otimes F_q$, an antipode $s : F_q \rightarrow F_q$ and a counit $\epsilon : F_q \rightarrow \mathbb{C}$. According to the condition that $\Delta$ and $\epsilon$ are homomorphisms and $s$ is an antihomorphism ($s(ab) = s(b)s(a)$), these three maps are defined on generators and extended to the whole algebra $F_q$. It is convenient to write them in matrix form [1]

$$\Delta(T) = T(\otimes)T, \quad \Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}$$  \hspace{1cm} (4)$$

$$s(T)T = Ts(T) = I, \quad \epsilon(T) = I,$$

where, as above, $I$ is the unit $2 \times 2$ matrix. The operations $\mu, \Delta, \epsilon, s$ are connected by a set of axioms [1,11,15] the validity of which is almost obvious in the $R$-matrix approach.

The quantum algebra $sl_q(2)$ is defined by three generators $J, X_+, X_-$, which satisfy commutation relations

$$[J, X_\pm] = \pm X_\pm,$$  \hspace{1cm} (5)$$

$$[X_+, X_-] = (q^{2J} - q^{-2J})/(q - q^{-1}) = [2J]_q,$$

where the notation $[x]_q = (q^x - q^{-x})/\omega$ has been introduced. The three additional maps, which define the structure of Hopf algebra (quasitriangular one [11]) are fixed by the relations

$$\Delta(J) = J \otimes 1 + 1 \otimes J, \quad \Delta(X_\pm) = X_\pm \otimes q^{-J} + q^J \otimes X_\pm,$$  \hspace{1cm} (6)$$

$$s(J) = -J, \quad s(X_\pm) = -q^{\mp 1} X_\pm,$$

$$\epsilon(J) = \epsilon(X_\pm) = 0, \quad \epsilon(1) = 1.$$

Relations (5),(6) can also be written in the $R$-matrix form using upper + and low − triangular matrices $L^\pm[1,12]$.

The associative algebra $A(q)$ of the deformed oscillator can be defined even more easily: it has, like $sl_q(2)$, three generators $A, A^+, N$ which satisfy

$$[N, A] = -A, \quad [N, A^+] = A^+, \quad [A, A^+] = q^{-2N},$$  \hspace{1cm} (7)$$

but the corresponding Hopf algebra structure is not known.
Finally the associative algebra $K$, connected with the reflection equation [10], is defined by four generators $\alpha, \beta, \gamma, \delta$, which satisfy quadratic relations $(\omega = q - q^{-1})$

\[
[\alpha, \beta] = \omega \alpha \gamma, \quad \alpha \gamma = q^2 \gamma \alpha, \quad [\alpha, \delta] = \omega (q \beta + \gamma) \gamma,
\]

\[
[\beta, \gamma] = 0, \quad [\beta \delta] = \omega \gamma \delta, \quad \gamma \delta = q^2 \delta \gamma.
\]

(8)

The Hopf algebra structure for this is not known either. The $R$-matrix form of (8) uses a $2 \times 2$ matrix $K$ of generators $\alpha, \beta, \gamma, \delta$ and the $R$-matrix of the quantum group $F_q(GL(2))$ (3):

\[
RK_1 R^{t_1} K_2 = K_2 R^{t_1} K_1 R
\]

(9)

where $R^t$ is a matrix transposed in the first space of $C^2 \otimes C^2$. Thus the diagonal in $R^{t_1}$ is the same as in the R, and the only non zero non diagonal element is situated in the right upper corner of the matrix.

If we decide to use the above mentioned algebras as algebras of observables, it is necessary to introduce real structure (*- operation ) into them. For example from $sl_q(2, C)$ one can get $su_q(2), su_q(1, 1), sl_q(2, R)$ with suitable restrictions on the deformation parameter ($q \in R$ and $|q| = 1$ representively). However we will not discuss the real forms in this paper.

We should point out that all the above mentioned associative algebras $F_q(GL(2)), sl_q(2), \mathcal{A}(q)$ and $K$ have central elements

\[
det_q T = ad - qbc; \quad c_2 = X_- X_+ + [J_q][J + 1]_q; \quad c_2(q) = A^+ A - [N; q^{-2}]; \quad z_1 = \beta - q \gamma, \quad z_2 = \alpha \delta - q^2 \beta \gamma.
\]

(10)

where the notation $[n; q] = (q^n - 1)/(q - 1)$ is also used.

The main aim of the paper is to point out close connections between these algebras, some of which are well known, for example the duality of $sl_q(2)$ and $F_q(SL(2))$ [1], and to suggest the physical interpretation of these connections.
2. Connections between quantum algebras

The deformed oscillator algebra $\mathcal{A}(q)$ can be obtained from $sl_q(2)$ by means of the contraction [13,14]:

\[
A = \lim_{f \to 0} f \omega^{1/2} X_+, \quad A^+ = \lim_{f \to 0} f \omega^{1/2} X_-
\]

\[
q^{-N} = \lim_{f \to 0} f q^J,
\]

and for the Casimir operator (10) one gets $\lim f^2 \omega_{c2} = c_2(q) + q^2/(q^2 - 1)$. In particular relations (6), which define the Hopf algebra structure of $sl_q(2)$, do not survive in this limit. However, it is possible to obtain, a finite expression in the right hand side of (6) for coproduct $\Delta$,

\[
\psi(N) = N \otimes I - I \otimes J, \quad \\
\psi(A) = A \otimes q^{-J} + \omega^{1/2} q^{-N} \otimes X_+, \\
\psi(A^+) = A^+ \otimes q^{-J} + \omega^{1/2} q^{-N} \otimes X_-, 
\]

and to interpret it as a map $\psi$ from algebra $\mathcal{A}(q)$ into the tensor product $\mathcal{A}(q) \otimes sl_q(2)$. The latter preserves the commutation relations (7) of algebra $\mathcal{A}(q)$, for example,

\[
[\psi(A), \psi(A^+)] = q^{-2\psi(N)}. 
\]

This map, which satisfies the properties of consistency with the coproduct: $(\psi \otimes id) \circ \psi = (id \otimes \Delta) \circ \psi$ and counit $(id \otimes \epsilon) \circ \psi = id$ of the quantum algebra $sl_q(2)$, is called the coaction of $sl_q(2)$ on $\mathcal{A}(q)$, and the algebra $\mathcal{A}(q)$ is called the $sl_q(2)$-comodule algebra [15]. While the coproduct makes it possible to define the action of the Hopf algebra in the tensor product of its representations, the coaction, in this case $\psi$, defines the representation of the algebra $\mathcal{A}(q)$ in the tensor product of the representation of $\mathcal{A}(q)$ with a representation of $sl_q(2)$. Thus, with the coproduct for $sl_q(2)$ interpreted as an addition of $q$-spins, the coaction $\psi$ can be interpreted as a kind of addition of $q$-oscillator with $q$-spin, resulting in the reproducing of the $q$-oscillator algebra.

If we limit ourselves to the Fock representation of $\mathcal{A}(q)$ (the other representations are obtained in [13]), which is $\mathcal{H}_F$ with basic vectors : $|0\rangle$ (such that $A|0\rangle = 0$) and $|n\rangle \simeq$
\((A^+)^n|0\rangle\) and choose the Hamiltonian of the \(q\)-oscillator to be
\[
H = A^+ A = [N; q^{-2}] = (1 - q^{-2N})/(1 - q^{-2}),
\]
\[
\text{spec}H = [n; q^{-2}], \quad n = 0, 1, 2, \ldots
\]
then the Hamiltonian \(H_I\) of the interacting in a special way \(q\)-oscillator and \(q\)-spin will have the same structure of spectrum (with additional multiplicities) : \(\text{spec}H_I = \{[n; q^{-2}], \nu_n(j), n = -j, -j + 1, \ldots\}\), where \(j\) is the value of the \(q\)-spin and \(\nu_n(j)\) correspondant multiplicity \(\nu_n(j) = n + j + 1, n \in -j, -j + 1, \ldots, j; \:\nu_n(j) = 2j + 1, n > j.\)

The corresponding expression for the interacting Hamiltonian can be obtained by introducing the right hand side of (12) into (14)
\[
H_I = \psi(A^+)\psi(A)
\]
\[
H_I = A^+ A q^{-2J} + \omega q^{-2N} X_- X_+ + \omega^{1/2} q^{-J-N} (A^+ X_+ + A X_-)
\]
The Hamiltonian (15) acts in the \(\mathcal{H}_F \otimes V_j\) space, where \(V_j\) is an irreducible representation of \(sl_q(2)\) with \(\text{dim} V_j = 2j + 1\). This space is decomposed into a direct sum of \(2j + 1\) irreducible Fock space representations of the \(q\)-oscillator algebra \(\mathcal{A}(q)\). Due to the consistency condition of the coaction \(\psi\) with coproduct \(\Delta\) of \(sl_q(2)\), it is possible to consider the interaction of the \(q\)-oscillator with any number of \(q\)-spins whilst preserving the spectrum structure of the Hamiltonian (14). In frame of the \(q\)-Clebsch-Gordan-Wigner-Racah calculus it corresponds to the contraction procedure w.r.t. one of the \(q\)-spin variables.

The algebra \(\mathcal{K}\) (8) and the quantum group \(F_q(GL(2))\) are connected in the same way [10]. The map (coaction) \(\varphi : \mathcal{K} \to F_q \otimes \mathcal{K}\) is given on the generators \(\alpha, \beta, \gamma, \delta\) by matrix multiplication :
\[
\varphi : K \to K_T = TKT^t
\]
for example \(\varphi(\beta) = aac\alpha + b\alpha\gamma + ad\beta + bd\delta.\) As in the previous case (12) the coaction (16) is consistent with the comultiplication \(\Delta\) and counit \(\epsilon\) of \(F_q\). Like \(\psi\), it also preserves the commutation relations (8) for the transformed generators \(\varphi(\alpha), \varphi(\beta), \ldots\). This connection between the algebras \(\mathcal{K}\) and \(F_q\) enables us to characterize \(\mathcal{K}\) as \(F_q\)-comodule algebra [15]. Thus, if \(F_q\) and \(\mathcal{K}\) are considered as algebras of observables, the existence of the coaction \(\varphi\) makes it possible to reproduce the structure of \(\mathcal{K}\) in the combined system \(F_q \otimes \mathcal{K}\) and with fixed representations \(V\) and \(W\) for \(F_q\) and \(\mathcal{K}\) to look for the expansion of \(V \otimes W\) on irreducible representations of algebra \(\mathcal{K}\).
In a Hopf algebra, e.g. \( F_q(GL(2)) \), there are two main maps: multiplication \( \mu \) and comultiplication \( \Delta \). Hence one can consider representations of \( F_q \) as an associative algebra by appropriate linear operators in a linear space. Then \( \mu \) corresponds to multiplication of the operators and the existence of the comultiplication \( \Delta \) in \( F_q \) defines a representation of \( F_q \) in the tensor product of given representations \( V_1 \otimes V_2 \) (cf. [17]). One can consider also corepresentations of \( F_q \) when the defined map (comultiplication) \( \Delta \) is consistent with the corepresentation map \( \varphi \) (coaction). For example, the space \( V \) with basic elements \( x, y \) is a corepresentation for \( F_q(GL(2)) \) with coaction \( \varphi : V \rightarrow F_q \otimes V \) given by

\[
\varphi(x) = ax + by, \quad \varphi(y) = cx + dy
\]

for these maps \( \varphi, \Delta \) and \( \epsilon \) are consistent

\[
(\Delta \otimes id) \circ \varphi = (id \otimes \varphi) \circ \varphi, \quad (\epsilon \otimes id) \circ \varphi = id \quad (17)
\]

Provided the elements \( x, y \) commute with the generators \( a, b, c, d \) of \( F_q \) one can show that the relation \( xy = qyx \) (the quantum plane [18]) is preserved under the coaction: \( \varphi(x)\varphi(y) = q\varphi(y)\varphi(x) \). Thus \( V \) can be characterized as an \( F_q \)-comodule algebra. As in the previous example of the \( q \)-oscillator and \( q \)-spin one can interpret corepresentations as addition of two physical systems, considering \( F_q \) and \( V \) as some algebras of observables, such that in the combined system \( F_q \otimes V \) one can reproduce by the coaction map \( \varphi \) one of the original algebras i.e. \( V \).

By analogy with representations we can take few copies of corepresentations e.g. \( V_1, V_2 \). However, to have a corepresentation in the tensor product \( V_1 \otimes V_2 \) the generators \( x_1, y_1 \) of \( V_1 \) must not commute with those \( x_2, y_2 \) of \( V_2 \). In order to define corepresentation they should satisfy the relations (\( \omega = q - q^{-1} \))

\[
\begin{align*}
    x_i y_i = q y_i x_i, & \quad x_1 y_2 = y_2 x_1 + \omega x_2 y_1 \\
    x_2 y_1 = y_1 x_2, & \quad x_1 x_2 = q x_2 x_1, \quad y_1 y_2 = q y_2 y_1
\end{align*}
\quad (18)
\]

The simplest way to explain the covariance of the relations (18) is the \( R \)-matrix formalism [1] and the exchange or Zamolodchikov - Faddeev algebra [13b]. Using the \( R \)-matrix (3) and two component spinors \( v_i^t = (x_i, y_i), i = 1, 2 \) one can write (18) in the
compact form
\[ \hat{R}v_i \otimes v_i = q v_i \otimes v_i, \quad \hat{R}v_2 \otimes v_1 = v_1 \otimes v_2, \quad \hat{R} = \mathcal{P}R. \quad (19) \]

It is obvious that due to (3) these relations are invariant w.r.t. the coaction \( v_i \rightarrow \varphi(v_i) = T v_i \). The element \( x_1 y_2 - q y_1 x_2 = y_2 x_1 - q^{-1} x_2 y_1 \) is central. In fact, relations (18) coincide with (1) and the central element with \( det_q \). However, one can extend this composition rule to any number of corepresentations \( V_j; j = 1, 2, 3, \ldots \) with relations (19) for any ordered pair of indices \( i > j \).

This extension procedure can be applied also to the covariant algebra \( \mathcal{K} \) (8). However we shall do it using the equivalent algebra \( \mathcal{B} \) [2,10,16], directly related to the braid group. The algebra \( \mathcal{B} \) has a \( 2 \times 2 \) matrix of generators \( M \) satisfying the following reflection equation
\[ M_1 R^{-1} M_2 \tilde{R}^{-1} = R^{-1} M_2 \tilde{R}^{-1} M_1. \quad (20) \]

where \( \tilde{R} = \mathcal{P}R\mathcal{P} \) and \( \mathcal{P} \) is the permutation (flip) operator of two spaces \( C^2 \otimes C^2 \). The corresponding coaction of the quantum group \( F_q(GL(2)) \) is
\[ M \rightarrow M_T = TMT^{-1} \quad (21) \]

Let us consider two copies of this algebra \( \mathcal{B}_i; i = 1, 2 \), with matrices of generators \( M(i) \) satisfying (20). In order for the product of these matrices \( M(1)M(2) = M^{(2)} \) to satisfy (20) it is sufficient to require the following relations among the generators of \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \)
\[ M_1(2)R^{-1}M_2(1)R = R^{-1}M_2(1)RM_1(2). \quad (22) \]

The proof is straightforward, however, we should point out probably more obvious graphical proof which follows from the connection of the reflection equation (20) with the braid group in a solid handlebody [6]. Multiplying (20) by \( \mathcal{P} \) from left and right, identifying \( M_2 = \tau, \mathcal{P}R^{-1} = \sigma_1 \) one obtains the additional relation [6,19] between generators \( \tau \) and \( \sigma_1 \) of \( B_n^{(1)} \) (cf. Fig 1)
\[ \tau \sigma_1 \tau \sigma_1 = \sigma_1 \tau \sigma_1 \tau. \quad (23) \]

If there are two handles in the body the braid group \( B_n^{(2)} \) has two additional generators
\[ \tau_i, i = 1, 2 \text{ which satisfy (23) and relation between them (cf. Fig.2)} \]

\[ \tau_2 \sigma_1 \tau_1 \sigma_1^{-1} = \sigma_1 \tau_1 \sigma_1^{-1} \tau_2 . \]  

which coincides with (22). Composition of the generators \( \tau_1 \tau_2 \) corresponds to \( M(1)M(2) \) and satisfies the same relation (23), (20) as one generator (Fig.3). One can develop this for any number of handles \( g, B_n^{(g)} \). The algebraic properties of constructed set of algebras \( B_i, i = 1, 2, \ldots, g \) (central elements, abelian subalgebras, representations) are discussed in [22].

One can also interpret some quantum algebras as spaces of quantum states of some physical model. The covariant algebra \( K \) which due to its transformation property

\[ k_{ij} \rightarrow \sum_{m,n} T_{im} T_{jn} k_{mn} \]  

may be called a \( q \)-tensor of the second rank (generalizations to higher ranks were proposed in [10]), has two central elements (10). These central elements enter into the characteristic equation for the matrix \( K \) [10]:

\[ K \epsilon_q K = z_2 \epsilon_q - qz_1 K , \]  

where \( \epsilon_q \) is the \( 2 \times 2 \) off diagonal matrix of the quantum metric for \( F_q(GL(2)) \). The algebra \( K \) is reduced by fixing one of central elements e.g. \( z_1 = \beta - q \gamma = 0 \). The reduced or factor algebra \( L_q \) is generated by three elements \( \alpha, \gamma, \delta \) \( (\beta = q \gamma) \) satisfying relations

\[ \alpha \gamma = q^2 \gamma \alpha, \quad \gamma \delta = q^2 \delta \gamma, \quad [\alpha, \delta] = q(q^2 - q^{-2}) \gamma^2 . \]  

These are the relations of the generators of the quantum two dimensional sphere [20]. The second central element \( z_2 = \alpha \delta - q^3 \gamma^2 \) corresponds to the radius of the sphere and the matrix \( K \epsilon_q \) corresponds to the classical matrix \( S = \Sigma x_i \sigma_i \) with the constraint \( S^2 = \Sigma x_i^2 \) (cf.(26)). Like \( L_2(S^2) \) this algebra \( L_q \) being covariant w.r.t. the quantum group \( F_q(SU(2)) \) coaction \( e_i \rightarrow \Sigma t_{ij}^{(1)} \otimes e_j \) is decomposed as a linear space into a direct sum of irreducible (co)representations \( L_q = \Sigma V_n, n = 0, 1, 2, \ldots \). There exists an invariant integral \( \nu \) on \( L_q \) as a linear functional \( \nu(\eta) \in C, \eta \in L_q \). Using this invariant integral one
can think on $\mathcal{L}_q$ as a space of states of the quantum particle on the $q$-deformed sphere. The $q$-deformed statistics of these particles is discussed in [21]. Being dual to $F_q(SU(2))$ the quantum algebra $su_q(2)$ acts on $\mathcal{L}_q$ by “differential” or difference operators and its central element $c_2$ (10) is the $q$-Casimir operator in $\mathcal{L}_q$, which has eigenvalue $[n]_q[n+1]_q$ over each irreducible component $V_n$.

By another identification of the reduced algebra $\mathcal{K}$ generators

$$\alpha \rightarrow A^+, \quad \delta \rightarrow -A, \quad q^3(1-q^{-4})\gamma^2 \rightarrow q^{-2N}, \quad q \rightarrow q^{1/2}$$

one gets the $q$-oscillator algebra (7).

## 3. Conclusion

The preceding considerations give quite different physical interpretations of the $q$-deformed algebras: as the symmetry algebra of a model, as an algebra of observables and even as a space of physical states. After appropriate quasiclassical limits all these interpretations have corresponding classical counterparts. Hence the $q$-deformed objects are analogous with respect to many properties to the usual Lie groups and Lie algebras though their use and interpretation is sometimes rather elaborated.

In attempting to understand these ideas more completely we shall sadly miss the contributions of the late Professor M.C. Polivanov.

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