Uniform asymptotic expansions for Laguerre polynomials and related confluent hypergeometric functions

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Abstract Uniform asymptotic expansions involving exponential and Airy functions are obtained for Laguerre polynomials $L_n^{(\alpha)}(x)$, as well as complementary confluent hypergeometric functions. The expansions are valid for $n$ large and $\alpha$ small or large, uniformly for unbounded real and complex values of $x$. The new expansions extend the range of computability of $L_n^{(\alpha)}(x)$ compared to previous expansions, in particular with respect to higher terms and large values of $\alpha$. Numerical evidence of their accuracy for real and complex values of $x$ is provided.

Keywords Asymptotic expansions · Laguerre polynomials · Confluent hypergeometric functions · Turning point theory · WKB methods · Numerical computation

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1 Introduction

In this paper we shall obtain computable asymptotic expansions for Laguerre polynomials $L^{(\alpha)}_n(x)$, for the case $n$ large and $\alpha$ small or large. These are defined by

$$L^{(\alpha)}_n(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}.$$  \hfill (1.1)

In terms of the confluent hypergeometric functions we have

$$L^{(\alpha)}_n(x) = \frac{\Gamma(n+\alpha+1)}{n!} M(-n, \alpha + 1, x) = \frac{(-1)^n}{n!} U(-n, \alpha + 1, x),$$  \hfill (1.2)

where $M$ denotes Olver’s scaled confluent hypergeometric function [15, Chap. 7, sec. 9].

We will use the techniques described in [8] for computing uniform asymptotic expansions of turning point problems. These involve Airy function expansions for solutions of differential equations having a simple turning point, but in a form that differs from the classical Airy function expansions of [15, Chap. 11]. Specifically, as described in [8], the coefficients in these new expansions are significantly easier to compute than previously, since the new method avoids nested integrations to evaluate them. Close to the turning point the method utilizes Cauchy’s integral formula to evaluate certain analytic coefficient functions.

We remark that the Cauchy integral method of [8] has potential applications to other forms of differential equations, and also the method was subsequently used in [9] to compute coefficients appearing in certain asymptotic expansions of integrals.

In this paper we shall also obtain Liouville-Green (L-G) expansions for Laguerre polynomials and confluent hypergeometric functions in domains that do not contain turning points, and these involve the exponential function. The form of these expansions differs from the standard ones (see [15, Chap. 10]), insomuch the coefficients in the expansions appear in the argument of the exponential. The advantage of this form is that the coefficients are again generally easier to compute. Indeed, L-G expansions of this form were used in [8] to obtain the new form of the Airy expansions described above.

Our results are valid in certain unbounded regions of the complex $x$ plane, which taken together include the positive real axis. These expansions hold for large $n$, uniformly for $0 < \delta \leq \alpha/n \leq \Delta < \infty$ and $-1 + \delta_1 \leq \alpha/n \leq -\delta_2 < 0$ ($\delta, \Delta, \delta_1, \delta_2$ fixed, with $\delta_1, \delta_2 \in (0, 1)$). If $nx$ is bounded away from 0 in a certain domain our $\alpha$ range of validity extends to $-1 + \delta_1 \leq \alpha/n \leq \Delta < \infty$.

A recent reference on the computation of Laguerre polynomials using asymptotics is [11], where three different asymptotic approximations are used: two expansions in terms of Bessel functions (from [10] and [19, Sect. 10.3.4]) and the Airy expansion of Frenzen and Wong [10]. These expansions, and in particular the Airy expansion, give
accurate approximations for large $n$ when $\alpha$ is not large. For bounded $\alpha$ see also [2] and [5] for certain expansions, as well as [4, 12] and [17] where the Riemann-Hilbert method was used.

For large $\alpha$ some expansions are available, like for instance those involving Whittaker functions of [6] and [14], or some in [18], but the coefficients of all these expansions are very hard to compute for the reasons outlined above. Also for large $\alpha$ the Riemann-Hilbert approach was used in [1] and [3], but these only provide leading order approximations.

In this paper expansions are derived that are readily computable and valid for large $|\alpha|$ as described above, and we provide numerical evidence of their high accuracy for real and complex values of $x$.

The Laguerre polynomials satisfy the following form of the confluent hypergeometric equation

$$x \frac{d^2 y}{dx^2} + (\alpha + 1 - x) \frac{dy}{dx} + ny = 0. \quad (1.3)$$

By the transformation $\tilde{y} = x^{(\alpha+1)/2} \exp\left(-\frac{1}{2}x\right) y$ we can remove the first derivative in the usual manner, and we get

$$\frac{d^2 \tilde{y}}{dx^2} = \frac{x^2 - 2(2n + \alpha + 1)x + \alpha^2 - 1}{4x^2} \tilde{y}. \quad (1.4)$$

On replacing $x$ by $uz$, where $u = n + \frac{1}{2}$ (a choice being explained after (2.9) below), this can then be re-written in the form

$$\frac{d^2 w}{dz^2} = \left\{ u^2 f(z) + g(z) \right\} w, \quad (1.5)$$

with a solution $w = z^{(\alpha+1)/2} \exp\left(-\frac{1}{2}uz\right) L_n^{(\alpha)}(uz)$. On defining $a$ by

$$\alpha = u \left( a^2 - 1 \right) \quad (a \geq 0), \quad (1.6)$$

we have in (1.5)

$$f(z) = \frac{(z - z_1)(z - z_2)}{4z^2}, \quad g(z) = -\frac{1}{4z^2}. \quad (1.7)$$

where

$$z_1 = (a-1)^2, \quad z_2 = (a+1)^2. \quad (1.8)$$

The latter are the turning points for large $u$, which is assumed here. These turning points coalesce with each other when $a = 0$ ($\alpha = -u$), and $z_1$ coalesces with the pole at $z = 0$ when $a = 1$ ($\alpha = 0$).

We shall consider the following cases separately.

**Case 1a** We obtain expansions for $z$ lying in domains containing the turning point $z_1$, which includes the interval $0 \leq z \leq z_2 - \delta$ (here and throughout $\delta > 0$). The turning point $z = z_2$ is excluded, and this is covered in Case 2 below. We assume $1 + \delta \leq a^2 \leq a_1 < \infty$ for fixed $a_1 \in (1, \infty)$. Thus $z_1$ cannot coalesce with the pole at $z = 0$ nor with the other turning point $z = z_2$, and $\alpha$ is positive and large, satisfying $u\delta \leq \alpha \leq u(a_1 - 1)$.
Case 1b This is the same as Case 1a, except now we consider \( \alpha \) negative. In particular, we assume \( 0 < a_0 \leq a^2 \leq 1 - \delta \) (for fixed \( a_0 \in (0, 1) \)), and in this case \( -u(1 - a_0) \leq \alpha \leq -u\delta < 0 \). Again \( z_1 \) cannot coalesce with the pole at \( z = 0 \) nor with the other turning point \( z = z_2 \).

Case 2 In this case expansions are derived for \( z \) lying in domains containing the turning point \( z_2 \), including the interval \( z_1 + \delta \leq z < \infty \), but not \( z_1 \) or 0. Here \( a \) lies in a larger interval than Cases 1a and 1b, namely \( 0 < a_0 \leq a^2 \leq a_1 < \infty \), and hence \( -u(1 - a_0) \leq \alpha \leq u(a_1 - 1) \). Note this can include large negative values of \( \alpha \) if \( a_0 < 1 \). The turning points again cannot coalesce (\( a = 0 \)) in this case, but \( z_1 \) can coalesce with \( z = 0 \) (since both points are excluded).

The more general case \( 0 \leq a < \infty \) with \( 0 \leq z < \infty \) will require an application of asymptotic expansions valid for a coalescing turning point and double pole, and also for two coalescing turning points. This will be studied in a subsequent paper.

The plan of this paper is as follows. In Section 2 we consider Case 1a and construct L-G expansions for the functions. In this section a detailed description of the Liouville transformation is provided. In Section 3 we obtain Airy function expansions for Case 1a. In Section 4 we obtain L-G and Airy expansions for Case 1b, and in Section 5 we likewise do this for Case 2. All of the expansions for Laguerre polynomials are summarized in Section 6. Finally, in Section 7 we present numerical results for the expansions of Cases 1a and 2 for Laguerre polynomials.

2 L-G expansions: Case 1a

We make the Liouville transformation

\[
\xi = \int_{z}^{z_1} f^{1/2}(t) dt. \tag{2.1}
\]

On explicit integration (and noting that \( \sqrt{z_1 z_2} = |a^2 - 1| \)) we obtain

\[
\xi = \frac{1}{2} (a^2 + 1) \ln \left\{ a^2 + 1 - z - S(z) \right\} - \frac{1}{2} S(z) - \max \left\{ a^2, 1 \right\} \ln(2a)
+ \frac{1}{2} |a^2 - 1| \ln \left\{ \frac{(a^2 - 1)^2 + |a^2 - 1| S(z) - (a^2 + 1) z}{z} \right\}. \tag{2.2}
\]

In this \( S(z) \) is given by

\[
S(z) = \left\{ (z_1 - z)(z_2 - z) \right\}^{1/2} = \left\{ z^2 - 2 \left( a^2 + 1 \right) z + \left( a^2 - 1 \right)^2 \right\}^{1/2}, \tag{2.3}
\]

where the principal logarithms are taken, and the branches of the square roots are chosen so that \( S(z) > 0 \) for \( -\infty < z < z_1 \), and is continuous elsewhere in the plane having a cut on the interval \([z_1, z_2]\). This means \( S(z) < 0 \) for \( z_2 < z < \infty \).

Thus \( \xi \) is real and positive when \( z \in (0, z_1) \), and varies continuously in the \( z \) plane having cuts along \((-\infty, 0]\) and \([z_1, \infty)\). Note that \( z = z_1 \) is mapped to
\[ \xi = 0, \text{ and } z = z_2 \pm i0 \text{ (respectively above and below the cut) is mapped to } \xi = \pm \min\{a^2, 1\}\pi i. \]

We find that \( \xi \to +\infty \) as \( z \to 0^+ \), such that

\[ \xi = \frac{1}{2} \left| a^2 - 1 \right| \left\{ 2 \ln \left| a^2 - 1 \right| - \ln(z) - 1 \right\} - a^2 |\ln(a)| + O(z). \]  

(2.4)

Furthermore, we have that

\[ \xi = \frac{1}{2} z - \frac{1}{2} \left( a^2 + 1 \right) \left\{ \ln(z) + 1 \right\} + a^2 \ln(a) + \min\{a^2, 1\}\pi i + O \left( z^{-1} \right). \]  

(2.5)

as \( z \to \infty \) in upper half plane.

Figures 1 and 2 depict the \( z-\xi \) map for \( 0 \leq \arg(z) \leq \pi \), with corresponding points labeled with the same letters. For the lower half \( z \) plane we can use from the Schwarz reflection principle that \( \xi(z) = \xi(\bar{z}) \).

With (2.1) and the new dependent variable given by

\[ V = f^{1/4}(z)w, \]  

(2.6)

the differential equation (1.5) is transformed to

\[ \frac{d^2V}{d\xi^2} = \left\{ u^2 + \phi(\xi) \right\} V, \]  

(2.7)

where

\[ \phi(\xi) = \frac{4f(z)f''(z) - 5f'^2(z)}{16f^3(z)} + \frac{g(z)}{f(z)} \]

\[ = \frac{z \left\{ 4z^3 - 4(3a^2 - 1)(a^2 - 3)z + 8(a^2 + 1)(a^2 - 1)^2 \right\}}{4(z - z_1)^3(z - z_2)^3}. \]  

(2.8)

The function \( \phi(\xi) \) is analytic in an unbounded domain \( \Delta \) (say) which excludes \( \xi = 0 \) and \( \xi = \pm \min\{a^2, 1\}\pi i \) (the singularities corresponding to the turning points). Then the part of \( \Delta \) corresponding to \( 0 \leq \arg(z) \leq \pi \) is the entire region.
Fig. 2 $\xi$ plane

depicted in Fig. 2, except with the points $\xi = 0$ and $\xi = \min \left\{ \alpha^2, 1 \right\} \pi i$ labeled E and F, respectively, excluded.

An important property is that

$$\phi(\xi) = O\left(\xi^{-2}\right),$$  \hspace{1cm} (2.9)

as $\xi \to \infty$ in $\Delta$. We remark that the choice of the large parameter in the form $u = n + \frac{1}{2}$, and the subsequent partitioning of (1.5), resulted in the desired behavior (2.9).

Asymptotic solutions of (2.7), accompanied by explicit error bounds, are given in [7] by

$$V_{n,1}(u, \xi) = \exp\left\{ u\xi + \sum_{s=1}^{n-1} \frac{E_s(\xi)}{u^s} \right\} + \epsilon_{n,1}(u, \xi),$$  \hspace{1cm} (2.10)

and

$$V_{n,2}(u, \xi) = \exp\left\{ -u\xi + \sum_{s=1}^{n-1} (-1)^s \frac{E_s(\xi)}{u^s} \right\} + \epsilon_{n,2}(u, \xi),$$  \hspace{1cm} (2.11)

where the coefficients $E_s(\xi)$ will be defined below.

In (2.10) $\epsilon_{n,1}(u, \xi) = e^{u\xi}O\left(\xi^{-n}\right)$ uniformly in a certain domain $\Xi_1$, with $e^{-u\xi} \epsilon_{n,1}(u, \xi) \to 0$ as Re$\xi \to -\infty$ in the domain, and likewise in (2.11) $\epsilon_{n,2}(u, \xi) = e^{-u\xi}O\left(\xi^{-n}\right)$ uniformly in a domain $\Xi_2$, with $e^{u\xi} \epsilon_{n,2}(u, \xi) \to 0$ as Re$\xi \to \infty$ in the domain. By virtue of (2.9), both $\Xi_1$ and $\Xi_2$ are unbounded, and are defined as follows.
In general the domain of validity $\Xi_1$ comprises the $\xi$ point subset of $\Delta$ for which there is a path $P_1$ (say) linking $\xi$ with $\alpha_1 = -\infty - \frac{1}{2} \left| a^2 - 1 \right| \pi i$ (corresponding to $z = \infty e^{\pi i}$) and having the properties (i) $P_1$ consists of a finite chain of $R_2$ arcs (as defined in [15, Chap. 5, sec. 3.3]), and (ii) as $t$ passes along $P_1$ from $\alpha_1$ to $\xi$, $\text{Re}(ut)$ is nondecreasing.

The domain $\Xi_2$ comprises the $\xi$ point subset of $\Delta$ for which there is a path $P_2$ (say) linking $\xi$ with $\alpha_2 = +\infty$ (corresponding to $z = 0$) and having the properties (i) $P_2$ consists of a finite chain of $R_2$ arcs, and (ii) as $t$ passes along $P_2$ from $\alpha_2$ to $\xi$, $\text{Re}(ut)$ is nonincreasing.

Let $\Xi_j^+ (j = 1, 2)$ denote the subsets of $\Xi_j$ corresponding to $0 \leq \text{arg}(z) \leq \pi$. Then $\Xi_1^+$ coincides with the domain shown in Fig. 2, but with the points $\xi = 0$ and $\xi = \min \left\{ a^2, 1 \right\} \pi i$ excluded. We denote by $D_1^+$ the $z$-domain corresponding to $\Xi_1^+$, and this is the entire upper half plane $\text{Im}(z) \geq 0$ excluding neighborhoods of the turning points $z = z_1, z_2$.

On the other hand, due to the monotonicity requirement (ii), the domain $\Xi_2^+$ must also exclude the unbounded region FGH of Fig. 2, as well as points on the segment EF. The corresponding $z$-domain $D_2^+$ (say) is the unshaded region depicted in Fig. 3, where the interval $[z_1, z_2]$ and the boundary curve FH must be excluded; the curve FH extends from $z = z_2$ to $z = \infty$ and is given parametrically by

$$\int_{z_2}^{\infty} f^{1/2} (t) \, dt = \tau i, \quad 0 \leq \tau < \infty. \quad (2.12)$$

We note that $z = +\infty$ is not contained in $D_2^+$ (but is contained in $D_1^+$).

Returning to the asymptotic expansions (2.10) and (2.11), the coefficients are given by

$$E_s(\xi) = \int F_s(\xi) d\xi \quad (s = 1, 2, 3, \ldots), \quad (2.13)$$
where
\[ F_1(\xi) = \frac{1}{2} \phi(\xi), \quad F_2(\xi) = -\frac{1}{4} \phi(\xi), \] (2.14)

and
\[ F_{s+1}(\xi) = -\frac{1}{2} F_s'(\xi) - \frac{1}{2} \sum_{j=1}^{s-1} F_j(\xi) F_{s-j}(\xi) \quad (s = 2, 3, \ldots). \] (2.15)

Primes are derivatives with respect to \( \xi \).

The choice of integration constants in (2.13) must be the same for both of the L-G solutions (2.10) and (2.11). As discussed in [8], the constants associated with the even terms \( (s = 2j, \ j = 1, 2, 3, \ldots) \) are arbitrary, but those for the odd terms \( (s = 2j + 1, \ j = 0, 1, 2, \ldots) \) must be precisely chosen, as described below.

It is preferable to work in terms of \( z \). Using
\[ \frac{d\xi}{dz} = f^{1/2}(z) = \frac{S(z)}{2z}, \] (2.16)

and writing \( \hat{E}_s(z) = E_s(\xi(z)) \) and \( \hat{F}_s(z) = F_s(\xi(z)) \), we have
\[ \hat{E}_s(z) = \int \frac{\hat{F}_s(z) S(z)}{2z} dz \quad (s = 1, 2, 3, \ldots), \] (2.17)

where
\[ \hat{F}_1(z) = \frac{1}{2} \phi(\xi(z)), \quad \hat{F}_2(z) = -\frac{z}{2S(z)} \frac{d\phi(\xi(z))}{dz}, \] (2.18)

and
\[ \hat{F}_{s+1}(z) = -\frac{z}{S(z)} \hat{F}_s'(z) - \frac{1}{2} \sum_{j=1}^{s-1} \hat{F}_j(z) \hat{F}_{s-j}(z) \quad (s = 2, 3, \ldots). \] (2.19)

We find by induction that
\[ \frac{\hat{F}_s(z) S(z)}{2z} = \frac{R_{2s+1}(z)}{S^{3s+2}(z)}, \] (2.20)

where \( R_n(z) \) is a polynomial of degree \( n \). From this we can show that
\[ \hat{E}_{2j}(z) = \frac{z T_{4j-1}(z)}{S^{6j}(z)}, \] (2.21)

and
\[ \hat{E}_{2j+1}(z) = \frac{T_{4j+3}(z)}{S^{6j+3}(z)}, \] (2.22)

where \( T_{n}^{(e)}(z) \) and \( T_{n}^{(o)}(z) \) are also polynomials of degree \( n \), provided the integration constants in (2.17) are chosen appropriately (which we assume).

For example, for the coefficient \( \hat{E}_1(z) \), we have
\[ T_{3}^{(o)}(z) = \frac{1}{48a^2} \left[ (a^2 - 1)^2 \left\{ (a^2 - 1)^2 - 4a^2 \right\} - 3(a^2 + 1)^3 z \right. \]
\[ \left. + 3(a^4 + 6a^2 + 1) z^2 - (a^2 + 1)^3 z^3 \right]. \] (2.23)
We note from (2.3) and (2.22) that each of \( S(z) \hat{E}_{2j+1}(z) \) \((j = 0, 1, 2, \cdots)\) is meromorphic at \( z = z_1 \) and at \( z = z_2 \). From [8] this is a requirement for the subsequent Airy function expansions to be valid at the turning points.

We also note that

\[
\hat{E}_s(\infty) = \lambda_s, \tag{2.24}
\]

(say), where \( \lambda_{2j} = 0 \) \((j = 1, 2, 3, \cdots)\), whereas the odd terms are non-zero. The first three of the odd terms are found to be

\[
\lambda_1 = \frac{-1 + a^2}{48a^2}, \lambda_3 = \frac{7(1 + a^6)}{5760a^6}, \lambda_5 = \frac{-31(1 + a^{10})}{80640a^{10}}. \tag{2.25}
\]

We similarly find

\[
\hat{E}_s(0) = \mu_s, \tag{2.26}
\]

(say), where \( \mu_{2j} = 0 \) \((j = 1, 2, 3, \cdots)\), and again the odd terms are non-zero. The first two of these are found to be

\[
\mu_1 = \frac{a^4 - 6a^2 + 1}{48a^2 \left| a^2 - 1 \right|}, \mu_3 = -\frac{7a^{12} - 21a^{10} + 21a^8 - 30a^6 + 21a^4 - 21a^2 + 7}{5760a^6 \left| a^2 - 1 \right|^3}. \tag{2.27}
\]

Recall in this case we are assuming that \( 1 + \delta \leq a^2 \leq a_1 < \infty \), for fixed \( a_1 \in (1, \infty) \). Thus from (1.6) \( \alpha \) is positive, and so using

\[
L_n^{(\alpha)}(x) = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} \left\{ 1 + \mathcal{O}(x) \right\} \ (x \to 0), \tag{2.28}
\]

we therefore have the solution of (1.5) given by

\[
w_0(u, z) \equiv z^{(\alpha+1)/2} e^{-uz/2} L_n^{(\alpha)}(uz), \tag{2.29}
\]

which has the unique behavior as \( z \to 0 \)

\[
w_0(u, z) \equiv \frac{\Gamma(n + \alpha + 1)z^{(\alpha+1)/2}}{n! \Gamma(\alpha + 1)} \left\{ 1 + \mathcal{O}(z) \right\}. \tag{2.30}
\]

We now match solutions that are recessive at \( z = 0 \). From (2.6) we have by uniqueness (up to a multiplicative constant) of such solutions that

\[
w_0(u, z) \propto \frac{z^{1/2} V_{n,2}(u, \xi)}{((z_1 - z)(z_2 - z))^{1/4}}. \tag{2.31}
\]

Then, using (2.4), (2.11), (2.26), (2.29) and (2.30), we arrive at

\[
L_n^{(\alpha)}(uz) \sim \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} \left( \frac{\alpha}{u} \right)^{1/2} \left( \frac{u}{u + \alpha} \right)^{u/2} \left\{ \frac{\alpha^2}{(u + \alpha)uez} \right\}^{\alpha/2} \times \frac{1}{((z_1 - z)(z_2 - z))^{1/4}} \exp \left\{ \frac{1}{2} uz - u\xi + \sum_{s=1}^{\infty} (-1)^s \hat{E}_s(z - \mu_s) \right\}. \tag{2.32}
\]
This is uniformly valid for \( z \in D_2^+ \cup D_2^- \), where \( D_2^- \) is the conjugate region of \( D_2^+ \).

From [13, Eq. 13.2.24] a companion solution of (1.5) is given by

\[
    w_1(u, z) \equiv (uz)^{(\alpha + 1)/2} e^{uz/2} U(n + \alpha + 1, \alpha + 1, uze^{-\pi i}).
\]  
(2.33)

Now from [13, Eq. 13.7.3]

\[
    U(n + \alpha + 1, \alpha + 1, uze^{-\pi i}) = (uze^{-\pi i})^{-n-\alpha-1} \left\{ 1 + O\left( z^{-1} \right) \right\}
\]  
(2.34)

as \( z \to \infty \) for \( \left| \arg \left( z e^{-\pi i} \right) \right| \leq \frac{3}{2} \pi - \delta \), and we see that \( w_1(u, z) \) has the behavior

\[
    w_1(u, z) = (-1)^{n+1} e^{\alpha \pi i} (uz)^{-u-\alpha/2} e^{uz/2} \left\{ 1 + O\left( z^{-1} \right) \right\},
\]  
(2.35)

and in particular this is the unique solution that is recessive in the half-plane \( \left| \arg \left( z e^{-\pi i} \right) \right| \leq \frac{1}{2} \pi \).

Thus we similarly deduce by matching recessive solutions that

\[
    w_1(u, z) \propto \frac{z^{1/2} V_{n,1}(u, \xi)}{((z_1 - z)(z_2 - z))^{1/4}},
\]  
(2.36)

and hence using (2.5)

\[
    U(n + \alpha + 1, \alpha + 1, uze^{-\pi i}) \sim -\frac{i \exp \left\{ u + \frac{1}{2} \alpha + \max(\alpha, 0) \pi i \right\}}{u(u+1)^{1/2}(u + \alpha)^{(u+\alpha)/2}(uz)^{\alpha/2}}
\]

\[
    \times \frac{1}{((z_1 - z)(z_2 - z))^{1/4}} \exp \left\{ -\frac{1}{2} uz + u\xi + \sum_{s=1}^{\infty} \frac{\hat{E}_s(z) - \lambda_s}{u^s} \right\}.
\]  
(2.37)

This is uniformly valid in the upper half plane \( \text{Im}(z) \geq 0 \) excluding neighborhoods of the turning points \( z = z_1, z_2 \). For \( \text{Im}(z) < 0 \) a connection formula can be used.

We remark that this expansion also holds for negative \( \alpha \), and in particular for \( 0 < a_0 \leq a^2 \leq 1 - \delta \) (for fixed \( a_0 \in (0, 1) \)). Likewise for its Airy expansion (3.14) below.

### 3 Airy expansions: Case 1a

We now obtain asymptotic expansions which are valid at \( z = z_1 \). These involve Airy functions, and the standard Liouville transformation is given by

\[
    \xi = \left( \frac{3}{2} \zeta \right)^{2/3}, \quad W = \zeta^{-1/4} f^{1/4}(z) w,
\]  
(3.1)

where \( \xi \) is again given by (2.2). Here \( \zeta \) is defined to be analytic at \( z = z_1 \) (see [15, Chap. 11]), and moreover \( \zeta > 0 \) for \( 0 < z < z_1 \), and \( \zeta < 0 \) for \( z_1 < z < z_2 \).
With this transformation the differential equation (1.5) takes the form
\[
\frac{d^2 W}{d\xi^2} = \left[u^2 \xi + \psi(\xi)\right] W,
\]
where \(\psi(\xi)\) is analytic at \(\xi = 0\) (i.e. \(\xi = 0\) and \(z = z_1\)), and is given explicitly by
\[
\psi(\xi) = \frac{5}{16} \xi^{-2} + \xi \phi(\xi),
\]
where \(\phi(\xi)\) is given by (2.8).

From [8] the following three \textit{exact} solutions of (3.2) are given
\[
W_j(u, \zeta) = A_i \left(\frac{u^{2/3}}{\zeta}\right) A(u, z) + A_i^{'} \left(\frac{u^{2/3}}{\zeta}\right) B(u, z) (j = 0, \pm 1).
\]
Here \(A_i \left(\frac{u^{2/3}}{\zeta}\right) = \text{Ai}(u^{2/3} \zeta e^{-2\pi ij/3})\), which are the Airy functions that are recessive in the sectors \(S_j := \{\xi : |\arg(\xi e^{-2\pi ij/3})| \leq \pi/3\} \); see [16, §9.2(iii)]. In our case, as functions of \(z\), \(W_0(u, \zeta)\) is recessive at \(z = 0\), \(W_1(u, \zeta)\) is recessive at \(z = \infty e^{\pi i}\), and \(W_{-1}(u, \zeta)\) is recessive at \(z = \infty e^{-\pi i}\).

The coefficient functions \(A(u, z)\) and \(B(u, z)\) are analytic in a domain containing \(z = z_1\), and in [8, Theorem 2.1] it was shown that they possess the asymptotic expansions
\[
A(u, z) \sim \exp \left\{ \sum_{s=1}^{\infty} \frac{\hat{E}_{2s}(z) + a_{2s} \xi^{-2s}/(2s)}{u^{2s}} \right\} \cosh \left\{ \sum_{s=0}^{\infty} \frac{\hat{E}_{2s+1}(z) - a_{2s+1} \xi^{-2s-1}/(2s+1)}{u^{2s+1}} \right\},
\]
and
\[
B(u, z) \sim \frac{1}{u^{1/3} \xi^{1/2}} \exp \left\{ \sum_{s=1}^{\infty} \frac{\hat{E}_{2s}(z) + a_{2s} \xi^{-2s}/(2s)}{u^{2s}} \right\} \sinh \left\{ \sum_{s=0}^{\infty} \frac{\hat{E}_{2s+1}(z) - a_{2s+1} \xi^{-2s-1}/(2s+1)}{u^{2s+1}} \right\},
\]
where \(a_1 = a_2 = \frac{5}{72}, a_1' = a_2' = -\frac{7}{72}\), and for \(b = a\) and \(b = a'\)
\[
b_{s+1} = \frac{1}{2} (s+1)b_s + \frac{1}{2} \sum_{j=1}^{s-1} b_j b_{s-j} \quad (s = 2, 3, \cdots).
\]
it is straightforward to show that $D$ consists of the intersection of $D_1^+$ and $D_2^+$ (i.e. $D_2^+$ itself), along with the conjugate of $D_2^+$, and in addition all points on the interval $[z_1, z_2]$.

Thus $D$ is the unshaded region shown in Fig. 4. In this $|\arg(z)| \leq \pi$, the boundary curve $FH$ emanating from $z = z_2$ is defined by (2.12), and $FH'$ is the conjugate curve. Points on both curves (including $z = z_2$) are excluded from $D$, but the singularities $z = 0$ and $z = \infty e^{\pm \pi i}$ lie in $D$, as well as of course the turning point $z = z_1$.

For each of the three solutions the respective domain of validity extends beyond $D$. For example, for $W_0(u, \zeta)$ certain points accessible by crossing above or below the cut on the negative real axis can be included. For $W_1(u, \zeta)$ certain points crossing above this cut can be included, as well as points crossing the curve $FH$ (but not $FH'$). For our purposes the common domain of validity $D$ suffices, since an extension to a domain containing $z = z_2$ and $z = +\infty$ will be considered in the next section, and for the confluent hypergeometric function analytic continuation formulas can be used for values of $\arg(z)$ outside $[-\pi, \pi]$.

Fig. 4 Domain $D$ in $z$ plane
We now match the Airy function expansions with the Laguerre polynomial and confluent hypergeometric function. Firstly we have, on matching the solutions recessive at \( z = 0 \), namely \( W_0(u, \zeta) \) and \( \zeta^{-1/4} f^{1/4}(z) w_0(u, z) \) (see (2.29) and (3.4)), that

\[
W_0(u, \zeta) = C_0(u) \left\{ \frac{(z_1 - z)(z_2 - z)}{\zeta} \right\}^{1/4} z^{\alpha/2} e^{-uz/2} L_n^{(\alpha)}(uz),
\]

for some constant \( C_0(u) \).

Now as \( z \to 0^+ \) (\( \zeta \to +\infty \)) we find from (3.1), (3.5), (3.6), and the behavior of the Airy function and its derivative for large argument (see [16, §9.7(ii)]), that the LHS behaves as

\[
W_0(u, \zeta) \sim \frac{1}{2\pi^{1/2} u^{1/6} \zeta^{1/4}} \exp \left\{ -u\xi - \sum_{s=0}^{\infty} \frac{\mu_{2s+1}}{u^{2s+1}} \right\}.
\]

On the other hand, from (2.28), the RHS of (3.8) has the behavior

\[
C_0(u) \left\{ \frac{(z_1 - z)(z_2 - z)}{\zeta} \right\}^{1/4} z^{\alpha/2} e^{-uz/2} L_n^{(\alpha)}(uz) \\
\sim \frac{C_0(u)}{n! \Gamma(\alpha + 1)} \frac{(a^2 - 1)^{1/2} \zeta^{1/4}}{\zeta^{1/4}}.
\]

Solving for \( C_0(u) \) from (3.8)–(3.10), and on referring to (1.6) and (2.5), we then arrive at

\[
L_n^{(\alpha)}(uz) \sim \frac{2\pi^{1/2} u^{1/6} \Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} \left( \frac{\alpha}{u} \right)^{1/2} \left( \frac{u}{u + \alpha} \right)^{u/2} \\
\times \left( \frac{\alpha^2}{(u + \alpha) uz} \right)^{\alpha/2} \left\{ \frac{\zeta}{(z_1 - z)(z_2 - z)} \right\}^{1/4} \exp \left\{ \frac{1}{2} uz + \sum_{s=0}^{\infty} \frac{\mu_{2s+1}}{u^{2s+1}} \right\} \\
\times \{ Ai(u^{2/3}\zeta) A(u, z) + Ai'(u^{2/3}\zeta) B(u, z) \}.
\]

Similarly, on matching solutions recessive at \( z = \infty e^{\pi i} \) we assert the existence of a constant \( C_1(u) \) such that

\[
W_1(u, \zeta) = C_1(u) \left\{ \frac{(z_1 - z)(z_2 - z)}{\zeta} \right\}^{1/4} z^{\alpha/2} e^{\alpha/2} U(n + \alpha + 1, \alpha + 1, uz e^{-\pi i}).
\]

The constant is found by comparing both sides as \( z \to \infty e^{\pi i} \), and as a result, on using

\[
Ai_1(u^{2/3}\zeta) A(u, z) + Ai'_1(u^{2/3}\zeta) B(u, z) \\
\sim \frac{e^{\pi i/6}}{2\pi^{1/2} u^{1/6} \zeta^{1/4}} \exp \left\{ u\xi + \sum_{s=0}^{\infty} \frac{\lambda_{2s+1}}{u^{2s+1}} \right\},
\]

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we find that

\[ U \left( n + \alpha + 1, \alpha + 1, u e^{-\pi i} \right) \]
\[ \sim - \frac{2\pi^{1/2} e^{\pi i/3}}{u^{1/3} \{ u (u + \alpha) \}^{(u+\alpha)/2} z^{\alpha/2}} \left\{ \frac{\zeta}{(z_1 - z) (z_2 - z)} \right\}^{1/4} \]
\[ \times \exp \left\{ - \frac{1}{2} u z + u + \frac{1}{2} \alpha + \max (\alpha, 0) \pi i - \sum_{s=0}^{\infty} \frac{\lambda_{2s+1}}{u^{2s+1}} \right\} \]
\[ \times \left\{ \text{Ai}_1 \left( u^{2/3} \zeta \right) A (u, z) + \text{Ai}'_1 \left( u^{2/3} \zeta \right) B (u, z) \right\}. \tag{3.14} \]

By the matching of solutions that are recessive as \( z \to \infty e^{-\pi i} \), we obtain corresponding L-G and Airy asymptotic expansions which are given by (2.37) and (3.14) respectively, with \( i \) replaced by \( -i \) in both, and \( \text{Ai}_1 \) replaced by \( \text{Ai}_{-1} \) in the latter. Likewise for (5.18) and (5.20) below.

4 L-G and Airy expansions: Case 1b

Now let us consider the case \( 0 < a_0 \leq a^2 \leq 1 - \delta \ ( -u (1 - a_0) \leq \alpha \leq -u \delta < 0 ) \). As mentioned earlier, in this case the expansions (2.37) and (3.14) remain valid for \( U \left( n + \alpha + 1, \alpha + 1, u e^{-\pi i} \right) \). However, the same is not true for the corresponding expansions (2.32) and (3.11) of \( L_n^{(\alpha)} (u z) \). This is because \( w_0 (u, z) \) (defined by (2.29)) is in general not recessive at \( z = 0 \) when \( \alpha < 0 \) (except when \( \alpha \) is an integer). Indeed, this time the appropriate solution recessive at the origin is given by

\[ w_2 (u, z) \equiv z^{(\alpha+1)/2} e^{-uz/2} N \left( -n, \alpha + 1, u z \right), \tag{4.1} \]

where \( N (a, c, x) \) is Olver’s scaled confluent hypergeometric function defined in [15, Chap. 7, sec. 9]. In particular we have for \( \alpha \neq 1, 2, 3, \cdots \)

\[ N (n, \alpha + 1, x) = \frac{x^{-\alpha}}{\Gamma (1 - \alpha)} \left\{ 1 + \mathcal{O} (x) \right\} (x \to 0), \tag{4.2} \]

and hence

\[ w_2 (u, z) \equiv \frac{z^{(1-\alpha)/2}}{\Gamma (1 - \alpha) u^\alpha} \left\{ 1 + \mathcal{O} (z) \right\} (z \to 0). \tag{4.3} \]

Comparing this to (2.30) we see that \( w_2 (u, z) \) has the desired recessive behavior at \( z = 0 \) when \( \alpha < 0 \). We remark that when \( \alpha \) is a negative integer \( w_0 (u, z) \) and \( w_2 (u, z) \) are multiples of one another, since if \( \alpha = -p \) for any integer \( p \in [1, n] \) we have from (1.1) and (2.29)

\[ w_0 (u, z) = \frac{(-u)^p z^{(p+1)/2}}{p!} \left\{ 1 + \mathcal{O} (z) \right\} (z \to 0). \tag{4.4} \]
Uniform asymptotic expansions for Laguerre polynomials...

By identifying \( w_2(u, z) \) and \( f^{-1/4}(z) V_{n,2}(u, \xi) \) in a similar manner to (2.31), we find for \( -u (1 - a_0) \leq \alpha \leq -u \delta < 0 \) that

\[
N(-n, \alpha + 1, uz) \sim \frac{1}{\Gamma(1 - \alpha)} \left( \frac{|\alpha|}{u} \right)^{1/2} \left( \frac{u + \alpha}{u} \right)^{u/2} \left\{ \frac{(u + \alpha)e^{1/2}}{\alpha^2 uz^2} \right\}^{\alpha/2} \\
\times \frac{1}{\{(z_1 - z)(z_2 - z)\}^{1/4}} \exp \left\{ \frac{1}{2} uz - \frac{1}{2} u\xi + \sum_{s=1}^{\infty} (-1)^s \frac{\hat{E}_s(z) - \mu_s}{u^s} \right\},
\]

(4.5)

uniformly for \( z \in D_2^+ \cup D_2^- \).

Likewise, analogously to (3.11)

\[
N(-n, \alpha + 1, uz) \sim \frac{2\pi^{1/2} u^{1/6}}{\Gamma(1 - \alpha)} \left( \frac{|\alpha|}{u} \right)^{1/2} \left( \frac{u + \alpha}{u} \right)^{u/2} \\
\times \left\{ \frac{(u + \alpha)e^{1/2}}{\alpha^2 uz^2} \right\}^{\alpha/2} \left\{ \frac{\zeta}{(z_1 - z)(z_2 - z)} \right\}^{1/4} \exp \left\{ \frac{1}{2} uz + \sum_{s=0}^{\infty} \frac{\mu_{2s+1}}{u^{2s+1}} \right\} \\
\times \left\{ \text{Ai}(u^{2/3}\zeta) A(u, z) + \text{Ai}'(u^{2/3}\zeta) B(u, z) \right\},
\]

(4.6)

uniformly for \( z \in \mathbb{D} \).

To obtain the desired expansions for the Laguerre polynomials we use (1.2) and the connection formula

\[
M(-n, \alpha + 1, z) = \frac{e^{\alpha \pi i n!}}{\Gamma(n + \alpha + 1)} N(-n, \alpha + 1, z) \\
- \frac{\sin(\pi \alpha)}{\pi} e^{\alpha \pi i} \Gamma(n + \alpha + 1) e^{\alpha \pi i} U(n + \alpha + 1, \alpha + 1, ze^{-\pi i}),
\]

(4.7)

to obtain

\[
L_n^{(\alpha)}(uz) = e^{\alpha \pi i} N(-n, \alpha + 1, z) \\
- \pi^{-1} \sin(\pi \alpha) \Gamma(n + \alpha + 1) e^{\alpha \pi i} U(n + \alpha + 1, \alpha + 1, ze^{-\pi i}).
\]

(4.8)

On inserting the expansions (2.37) and (4.5) into (4.8) yields the desired L-G expansion for \( L_n^{(\alpha)}(uz) \) for \( -u (1 - a_0) \leq \alpha \leq -u \delta < 0 \). Likewise, the corresponding Airy expansion comes from (3.14), (4.6), and (4.8).

5 L-G and Airy expansions: Case 2

We note that \( S(z) < 0 \) for \( z \in (z_2, \infty) \), where \( S(z) \) is given by (2.3). Hence the same is true for \( f^{1/2}(z) = S(z)/(2z) \), and with this in mind we now define the L-G variable by

\[
\tilde{\xi} = -\int_{z_2}^z f^{1/2}(t) \, dt,
\]

(5.1)
so that \( \tilde{\xi} \geq 0 \) for \( z \geq z_2 \). Explicit integration yields

\[
\tilde{\xi} = -\frac{1}{2} (a^2 + 1) \ln \left\{ z - S(z) - a^2 - 1 \right\} - \frac{1}{2} S(z) + \max \{a^2, 1\} \ln (2a) - \frac{1}{2} |a^2 - 1| \ln \left\{ \frac{(a^2 + 1) z - (a^2 - 1)^2 + |a^2 - 1| S(z)}{z} \right\}.
\]

We observe that \( \tilde{\xi} = \xi - \min \{a^2, 1\} \pi i \), where \( \xi \) is the corresponding L-G variable from case 1.

The L-G asymptotic solution that we employ this time is given by the expansion

\[
\tilde{V}_{n,2} (u, \tilde{\xi}) = \exp \left\{ -u \tilde{\xi} + \sum_{s=1}^{n-1} (-1)^s \frac{\hat{E}_s(z)}{u^s} \right\} + \tilde{\varepsilon}_{n,2} (u, \tilde{\xi}),
\]

where the coefficients \( \hat{E}_s(z) \) are the same as in Cases 1a and 1b.

In (5.3) \( \tilde{\varepsilon}_{n,2} (u, \tilde{\xi}) = e^{-u \tilde{\xi}} O \left( u^{-n} \right) \) uniformly in a certain unbounded domain \( \tilde{\Xi}_2 \), with \( e^{u \tilde{\xi}} \tilde{\varepsilon}_{n,2} (u, \tilde{\xi}) \to 0 \) as \( \text{Re} \tilde{\xi} \to \infty \) in the domain. Let \( \tilde{\Delta} \) denote the domain in the \( \tilde{\xi} \) plane corresponding to \( \Delta \) in the \( \xi \) plane: thus it is \( \Delta \) shifted by the factor \( -\min \{a^2, 1\} \pi i \). Then the domain \( \tilde{\Xi}_2 \) comprises the \( \tilde{\xi} \) point subset of \( \tilde{\Delta} \) for which there is a path \( \tilde{\mathcal{P}}_2 \) (say) linking \( \tilde{\xi} \) with \( \tilde{\alpha}_2 = +\infty \) (corresponding to \( z = +\infty + i0 \)) and having the properties (i) \( \tilde{\mathcal{P}}_2 \) consists of a finite chain of \( R_2 \) arcs, and (ii) as \( t \) passes along \( \tilde{\mathcal{P}}_2 \) from \( \tilde{\alpha}_2 \) to \( \xi \), \( \text{Re} (ut) \) is nonincreasing.

If \( \tilde{\Xi}_2^+ \) denotes the subset of \( \tilde{\Xi}_2 \) corresponding to \( 0 \leq \text{arg} (z) \leq \pi \), then due to condition (ii) above \( \tilde{\Xi}_2^+ \) must also exclude all points in the fourth quadrant of the \( \tilde{\xi} \) plane. The corresponding \( z \)-domain \( \tilde{D}_2^+ \) (say) is thus the unshaded region depicted in Fig. 5, where the interval \( [z_1, z_2] \) and the boundary curve EB must be excluded; the curve EB emanates from \( z = z_1 \) to a point on the upper part of the cut along \( (-\infty, 0) \), and is given parametrically by

\[
\int_{z_1}^{z} f^{1/2}(t) \, dt = -\tau i, \quad 0 \leq \tau \leq \frac{1}{2} |a^2 - 1|.
\]

We remark that \( z = 0 \) is not contained in \( \tilde{D}_2^+ \).

We now use the behavior of the Laguerre polynomials at infinity to match it with the L-G expansion (5.3). Specifically, we know that

\[
L_n^{(\alpha)}(uz) = \frac{(-1)^n}{n!} \left( -n, \alpha + 1, uz \right),
\]

and hence the following solution of (1.5)

\[
\tilde{w}_0 (u, z) \equiv (uz)^{(\alpha+1)/2} e^{-uz/2} L_n^{(\alpha)}(uz),
\]

has, from [13, Eq. 13.7.3], the behavior as \( z \to \infty \)

\[
\tilde{w}_0 (u, z) = \frac{(-1)^n}{n!} (uz)^{(2u+\alpha)/2} e^{-uz/2} \left\{ 1 + O \left( \frac{1}{z} \right) \right\}.
\]
On comparing this to (2.35) we see that \( \tilde{w}_0(u, z) \) is the unique solution that is recessive in right half plane. On matching, we therefore ascertain that there exits a constant \( \tilde{c}_0(u) \) such that
\[
\tilde{w}_0(u, z) = \tilde{c}_0(u) f^{-1/4}(z) \tilde{V}_{n,2}(u, \tilde{\xi}),
\]
where \( \tilde{V}_{n,2}(u, \tilde{\xi}) \) is given by (5.3).

Using
\[
\tilde{\xi} = \frac{1}{2z} - \frac{1}{2} \left( a^2 + 1 \right) \left( \ln(z) + 1 \right) + a^2 \ln(a) + \mathcal{O}\left( z^{-1} \right) \quad (z \to \infty),
\]
we have from (5.3), (2.24) and (5.7)
\[
\tilde{c}_0(u) = \lim_{z \to \infty} \left\{ \frac{f^{1/4}(z) \tilde{w}_0(u, z)}{\tilde{V}_{n,2}(u, \tilde{\xi})} \right\} \sim \frac{(-1)^n u^{a/2} (u + \alpha)^{(u+\alpha)/2}}{\sqrt{2e^{u+(\alpha/2)n!}}} \exp \left\{ -\sum_{s=1}^{\infty} (-1)^s \frac{\hat{\lambda}_s}{u^s} \right\}. \tag{5.10}
\]

Hence from (5.6), (5.8) and (5.10)
\[
L_n^{(\alpha)}(uz) \sim \frac{(-1)^n u^{(u-1)/2} (u + \alpha)^{(u+\alpha)/2}}{e^{u+(\alpha/2)n!} (uz)^{a/2} ((z - z_1)(z - z_2))^{1/4}} \\
\times \exp \left\{ \frac{1}{2} uz - u\tilde{\xi} + \sum_{s=1}^{\infty} (-1)^s \frac{\hat{E}_s(z) - \lambda_s}{u^s} \right\}, \tag{5.11}
\]
uniformly for \( z \in \tilde{D}_2^+ \cup \tilde{D}_2^- \), where \( \tilde{D}_2^- \) is the conjugate region of \( \tilde{D}_2^+ \). We emphasize that this expansion is not valid on the interval \([z_1, z_2]\).
We next construct an Airy function expansion, which is valid at \( z = z_2 \), similarly to (3.11). We have, again by matching solutions (3.4) with \( j = 0 \) (all recessive at \( z = +\infty \)) with (5.6), that

\[
\tilde{C}_0 (u) \left\{ \frac{(z - z_1) (z - z_2)}{\zeta} \right\}^{1/4} \frac{z^{\alpha/2}}{e^{-uz/2}} L_n^{(\alpha)} (uz) = \text{Ai} \left( u^{2/3} \tilde{\zeta} \right) \tilde{A} (u, z) + \text{Ai}' \left( u^{2/3} \tilde{\zeta} \right) \tilde{B} (u, z),
\]

for some constant \( \tilde{C}_0 (u) \). Here \( \tilde{\zeta} = \left( \frac{3}{2} \tilde{\xi} \right)^{2/3} \), where \( \tilde{\xi} \) is given by (5.2), and the coefficient functions \( \tilde{A} (u, z) \) and \( \tilde{B} (u, z) \) are analytic at \( z = z_2 \) (\( \tilde{\zeta} = 0 \)). These have the expansions (c.f. (3.5) and (3.6))

\[
\tilde{A} (u, z) \sim \exp \left\{ \sum_{s=1}^{\infty} \frac{\hat{E}_{2s}(z) + \tilde{a}_{2s} \tilde{\zeta}^{-2s}}{u^{2s}} \right\} \times \cosh \left\{ \sum_{s=0}^{\infty} \frac{\hat{E}_{2s+1}(z) - \tilde{a}_{2s+1} \tilde{\zeta}^{-2s-1}}{u^{2s+1}} \right\},
\]

and

\[
\tilde{B} (u, z) \sim \frac{1}{u^{1/3} \tilde{\zeta}^{1/2}} \exp \left\{ \sum_{s=1}^{\infty} \frac{\hat{E}_{2s}(z) + a_{2s} \tilde{\zeta}^{-2s}}{u^{2s}} \right\} \times \sinh \left\{ \sum_{s=0}^{\infty} \frac{\hat{E}_{2s+1}(z) - a_{2s+1} \tilde{\zeta}^{-2s-1}}{u^{2s+1}} \right\}.
\]

uniformly for \( z \in \tilde{D} \); this domain is the unshaded region depicted in Fig. 6, where \( EB' \) is the conjugate curve of \( EB \). All points on the boundary curve \( B'EB \) are excluded from \( \tilde{D} \).

Now as \( \tilde{\zeta} \to \infty \) we find from (5.13), (5.14), and the behavior of the Airy function and its derivative for large argument [16, §9.7(ii)], that the RHS behaves as

\[
\text{Ai} \left( u^{2/3} \tilde{\zeta} \right) \tilde{A} (u, z) + \text{Ai}' \left( u^{2/3} \tilde{\zeta} \right) \tilde{B} (u, z) \sim \frac{1}{2\pi^{1/2} u^{1/6} \tilde{\zeta}^{1/4}} \exp \left\{ -u\tilde{\zeta} - \sum_{s=0}^{\infty} \frac{\lambda_{2s+1}}{u^{2s+1}} \right\}.
\]

On the other hand, from (5.7), the LHS has the behavior

\[
\tilde{C}_0 (u) \left\{ \frac{(z - z_1) (z - z_2)}{\zeta} \right\}^{1/4} \frac{z^{\alpha/2}}{e^{-uz/2}} L_n^{(\alpha)} (uz) \sim (-1)^{n} \tilde{C}_0 (u) u^{n+\zeta (2n+\alpha+1)/2} e^{-uz/2} / n! \tilde{\zeta}^{1/4}.
\]
Thus from (1.6), (5.9), (5.12)–(5.16) we can solve for $\tilde{C}_0 (u)$, and from this deduce that

$$L_n^{(\alpha)} (uz) \sim \frac{(-1)^n 2 \pi^{1/2} u^{(u/2)-(1/3)} (u + \alpha)^{(u+\alpha)/2}}{n! (uz)^{\alpha/2}} \times \left\{ \frac{\tilde{\zeta}}{(z - z_1) (z - z_2)} \right\}^{1/4} \exp \left\{ \frac{1}{2} uz - u - \frac{1}{2} \alpha + \sum_{s=0}^{\infty} \frac{\lambda_{2s+1}}{u^{2s+1}} \right\} \times \left\{ \text{Ai} \left( u^{2/3} \tilde{\zeta} \right) \tilde{A} (u, z) + \text{Ai}' \left( u^{2/3} \tilde{\zeta} \right) \tilde{B} (u, z) \right\}. \quad (5.17)$$
For the complementary solution we have, equivalent to (2.37), the asymptotic expansion

\[
U(n + \alpha + 1, \alpha + 1, uze^{-\pi i}) \sim \frac{(-1)^{n+1} e^{\alpha \pi i + u + (\alpha / 2)}}{u^{(u+1)/2} (u + \alpha)^{(u+\alpha)/2}} \times \frac{1}{(uz)^{\alpha/2} [(z - z_1)(z - z_2)]^{1/4}} \times \exp \left\{ - \frac{1}{2} uz + u\tilde{\xi} + \sum_{s=1}^{\infty} \frac{\hat{E}_s(z) - \lambda_s}{u^s} \right\}.
\]

Similarly to the derivation of (5.17), using

\[
\text{Ai}_1 \left( \frac{u^{2/3}}{\zeta} \right) \tilde{A}(u, z) + \text{Ai}'_1 \left( \frac{u^{2/3}}{\zeta} \right) \tilde{B}(u, z) \sim \frac{e^{\pi i / 6}}{2\pi^{1/2}u^{1/6}\tilde{\zeta}^{1/4}} \exp \left\{ u\tilde{\xi} + \sum_{s=0}^{\infty} \frac{\lambda_{2s+1}}{u^{2s+1}} \right\},
\]

we arrive at

\[
U(n + \alpha + 1, \alpha + 1, uze^{-\pi i}) \sim \frac{(-1)^{n+1} 2\pi^{1/2} e^{-\pi i / 6} e^{\alpha \pi i}}{u^{1/3} (u + \alpha)^{(u+\alpha)/2} \zeta^{\alpha/2}} \times \left\{ \frac{\tilde{\zeta}}{(z - z_1)(z - z_2)} \right\}^{1/4} \exp \left\{ - \frac{1}{2} uz + u + \frac{1}{2} \alpha - \sum_{s=0}^{\infty} \frac{\lambda_{2s+1}}{u^{2s+1}} \right\} \times \left\{ \text{Ai}_1 \left( \frac{u^{2/3}}{\zeta} \right) \tilde{A}(u, z) + \text{Ai}'_1 \left( \frac{u^{2/3}}{\zeta} \right) \tilde{B}(u, z) \right\}.
\]

6 Summary of main results for Laguerre polynomials

Let \( u = n + \frac{1}{2} \), \( a = ((\alpha / u) + 1)^{1/2} \geq 0 \), \( z_1 = (a - 1)^2 \), \( z_2 = (a + 1)^2 \), and define \( \xi = \xi(z) \) by (2.2) and (2.3). Figures 1 and 2 depict the \( z - \xi \) map for \( 0 \leq \arg(z) \leq \pi \), with corresponding points labeled with the same letters. For the lower half \( z \) plane we use \( \xi(z) = \xi(\bar{z}) \). Define coefficients \( \hat{E}_s(z) \) by (2.8), (2.17)–(2.19). The integration constants in (2.17) are chosen so that for the even terms \( \hat{E}_{2j}(0) = \hat{E}_{2j}(\infty) = 0 \) (\( j = 1, 2, 3, \ldots \)), and for the odd terms \( S(z)\hat{E}_{2j+1}(z) \) (\( j = 0, 1, 2, \ldots \)) are meromorphic at both turning points \( z = z_1 \) and \( z = z_2 \) (see (1.8), (2.3) and (2.22)).
Case 1a In this assume $1 + \delta \leq a_1^2 \leq a_1 < \infty$ for fixed $a_1 \in (1, \infty)$. The relevant L-G expansion is given by (2.32), and after truncating after $N \geq 1$ terms we have

$$L_n^{(\alpha)}(uz) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \left(\frac{\alpha}{u}\right)^{1/2} \left(\frac{u}{u + \alpha}\right)^{u/2} \times \left\{ \frac{a^2}{(u + \alpha)uez} \right\}^{\alpha/2} \left\{ \frac{1}{(z_1 - z)(z_2 - z)} \right\}^{1/4} \times \exp \left\{ \frac{1}{2}uz - u\xi + \sum_{s=1}^{N} \frac{1}{u^{2s}} \right\} \left\{ 1 + O\left( \frac{1}{u^{N+1}} \right) \right\}.$$  

(6.1)

The order term is uniformly valid for $z \in D_2^+ \cup D_2^-$, where $D_2^+$ is shown in Fig. 3, and $D_2^-$ is the conjugate region of $D_2^+$.  

Let $\xi = \left(\frac{3}{2}\xi\right)^{2/3}$. The Airy expansion is given by (3.11), and (3.5) and (3.6) are used to approximate the coefficient functions. Therefore uniformly for $z \in D$ (the domain of validity depicted in Fig. 4) we have

$$L_n^{(\alpha)}(uz) = \frac{2\pi^{1/2}u^{1/6}\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \left(\frac{\alpha}{u}\right)^{1/2} \left(\frac{u}{u + \alpha}\right)^{u/2} \times \left\{ \frac{a^2}{(u + \alpha)uez} \right\}^{\alpha/2} \exp \left\{ \frac{1}{2}uz + \sum_{s=0}^{m-1} \frac{1}{u^{2s+1}} \right\} \times \left[ \text{Ai}\left(u^{2/3}\xi\right) \left\{ A_m(u, z) + O\left( \frac{1}{u^{2m+1}} \right) \right\} + \text{Ai}'\left(u^{2/3}\xi\right) \left\{ B_m(u, z) + O\left( \frac{1}{u^{2m+(4/3)}} \right) \right\} \right].$$  

(6.2)

In this, for any positive integer $m$, we have introduced the truncated expansions

$$A_m(u, z) = \chi(u, z) \exp \left\{ \sum_{s=1}^{m} \frac{\hat{E}_{2s}(z) + a_{2s}\xi^{-2s}/(2s)}{u^{2s}} \right\} \times \cosh \left\{ \sum_{s=0}^{m-1} \frac{\hat{E}_{2s+1}(z) - a_{2s+1}\xi^{-2s-1}/(2s + 1)}{u^{2s+1}} \right\},$$  

(6.3)

and

$$B_m(u, z) = \frac{\chi(u, z)}{u^{1/3}\xi^{1/2}} \exp \left\{ \sum_{s=1}^{m} \frac{\hat{E}_{2s}(z) + a_{2s}\xi^{-2s}/(2s)}{u^{2s}} \right\} \times \sinh \left\{ \sum_{s=0}^{m-1} \frac{\hat{E}_{2s+1}(z) - a_{2s+1}\xi^{-2s-1}/(2s + 1)}{u^{2s+1}} \right\},$$  

(6.4)

where

$$\chi(u, z) = \left\{ \frac{\xi}{(z_1 - z)(z_2 - z)} \right\}^{1/4}.$$  

(6.5)
In (6.3) and (6.4) \( a_1 = a_2 = \frac{5}{72}, \) \( \bar{a}_1 = \bar{a}_2 = -\frac{7}{72}, \) and subsequent terms given by (3.7) with \( b = a \) or \( b = \bar{a}. \)

We remark that the term \( \chi(u, z) \) has been absorbed into the scaled functions \( A_m(u, z) \) and \( B_m(u, z) \) because it has a removable singularity at \( z = z_1 (\zeta = 0), \) and hence it is easier to compute via Cauchy’s integral formula when \( z \) is close to \( z_1, \) as described in the next section.

**Case 1b** We assume \( 0 < a_0 \leq a^2 \leq 1 - \delta \) (for fixed \( a_0 \in (0, 1) \)), so that \( \alpha \) is negative. We use the connection formula

\[
L_n^{(\alpha)}(uz) = e^{a\pi i} N(-n, \alpha + 1, z) \left(-\pi^{-1} \sin(\pi\alpha) \Gamma(n + \alpha + 1) e^{\pi i} U(n + \alpha + 1, \alpha + 1, ze^{-\pi i})\right). \tag{6.6}
\]

On inserting the expansions (2.37) and (4.5) into (6.6) (where \( \mu_s = \hat{E}_s(0) \) and \( \lambda_s = \hat{E}_s(\infty) \)) yields the desired L-G expansion for \( L_n^{(\alpha)}(uz). \) As for (6.1), this compound expansion is uniformly valid for \( z \in D^+ + 2 \cup D^- - 2. \)

Likewise, the corresponding Airy expansion comes from inserting the expansions (3.14) and (4.6) into (6.6), and this is uniformly valid for \( z \in \hat{D}. \)

**Case 2** In this case we assume that \( 0 < a_0 \leq a^2 \leq a_1 < \infty. \) Define \( \bar{\zeta} \) by (5.2), and from (5.11) the L-G approximation with \( N \) terms is now given by

\[
L_n^{(\alpha)}(uz) = \frac{(-1)^n u^{(u-1)/2} (u + \alpha)^{(u+\alpha)/2}}{e^{u+(\alpha/2)} n! (uz)^{u/2} \left\{ (z - z_1) (z - z_2) \right\}^{1/4}}
\times \exp \left\{ \frac{1}{2} uz - u \bar{\zeta} + \sum_{s=1}^{N} (-1)^s \hat{E}_s(z) - \hat{E}_s(\infty) \frac{u^s}{u} \right\} \left\{ 1 + \mathcal{O}\left(\frac{1}{u^{N+1}}\right)\right\}. \tag{6.7}
\]

The order term is uniformly valid for \( z \in \hat{D}^+ + \hat{D}^- \), where \( \hat{D}^+ \) is shown in Fig. 5, and \( \hat{D}^- \) is the conjugate region of \( \hat{D}^+. \)

Let \( \bar{\zeta} = \left(\frac{3}{2} \bar{\xi}\right)^{2/3}. \) The Airy expansion is given by (3.11), (5.13) and (5.14), uniformly for \( z \in \hat{D} \) (the domain of validity depicted in Fig. 6). Hence truncating similarly to (6.2)–(6.4) we have

\[
L_n^{(\alpha)}(uz) = \frac{(-1)^n 2\pi^{1/2} u^{(u/2)-(1/3)} (u + \alpha)^{(u+\alpha)/2}}{n! (uz)^{u/2}}
\times \exp \left\{ \frac{1}{2} uz - u - \frac{1}{2} \alpha + \sum_{s=0}^{m-1} \hat{E}_{2s+1}(\infty) \frac{u^{2s+1}}{u^{2s+1}} \right\}
\times \left[ \text{Ai}\left(uz^{2/3} \bar{\zeta}\right) \left\{ \hat{A}_m(u, z) + \mathcal{O}\left(\frac{1}{u^{2m+1}}\right)\right\} \right]
+ \text{Ai}'\left(uz^{2/3} \bar{\zeta}\right) \left[ \hat{B}_m(u, z) + \mathcal{O}\left(\frac{1}{u^{2m+(4/3)}}\right)\right]. \tag{6.8}
\]
where, for any positive integer $m$,
\[
\tilde{A}_m(u, z) = \tilde{\chi}(u, z) \exp \left\{ \sum_{s=1}^{m} \frac{\hat{E}_{2s}(z) + \hat{a}_{2s}\hat{\xi}^{2s-2s}/(2s)}{u^{2s}} \right\} \times \cosh \left\{ \sum_{s=0}^{m-1} \frac{\hat{E}_{2s+1}(z) - \hat{a}_{2s+1}\hat{\xi}^{2s+1-2s-1}/(2s+1)}{u^{2s+1}} \right\},
\]
(6.9)
and
\[
\tilde{B}_m(u, z) = \tilde{\chi}(u, z) \frac{u^{1/3}\hat{\zeta}^{1/2}}{\tilde{\chi}(u, z)} \exp \left\{ \sum_{s=1}^{m} \frac{\hat{E}_{2s}(z) + \hat{a}_{2s}\hat{\xi}^{2s-2s}/(2s)}{u^{2s}} \right\} \times \sinh \left\{ \sum_{s=0}^{m-1} \frac{\hat{E}_{2s+1}(z) - \hat{a}_{2s+1}\hat{\xi}^{2s+1-2s-1}/(2s+1)}{u^{2s+1}} \right\},
\]
(6.10)
in which
\[
\tilde{\chi}(u, z) = \left\{ \frac{\hat{\zeta}}{(z - z_1)(z - z_2)} \right\}^{1/4}.
\]
(6.11)

7 Numerical results

Here we illustrate the accuracy of the new expansions for Cases 1a and 2. We concentrate on Laguerre polynomials with $n$ large and $\alpha$ non-negative, but analogous results are available for negative $\alpha$, as well as for the complementary confluent hypergeometric functions.

7.1 Case 1a

Let $z \in D$: if this point is not too close to the turning point $z_1$ we can use (6.3) and (6.4) directly for their numerically stable computation in (6.2). On the other hand, if $z$ is close to $z_1$ these expansions are not stable, since each $\hat{E}_s(z)$ is unbounded at this turning point. Instead we follow [8] to compute these functions via Cauchy’s integral formula. Now, neither $A_m(u, z)$ nor $B_m(u, z)$ are analytic at $z = z_1$, but $\chi(u, z) A(u, z)$ and $\chi(u, z) B(u, z)$ are. Thus, we have
\[
\chi(u, z) A(u, z) = \frac{1}{2\pi i} \oint_{L_1} \frac{\chi(u, t) A(u, t)}{t - z} dt
\]
\[
= \frac{1}{2\pi i} \oint_{L_1} \frac{A_m(u, t) + O(u^{-2m-1})}{t - z} dt
\]
(7.1)
\[
= \frac{1}{2\pi i} \oint_{L_1} \frac{A_m(u, t)}{t - z} dt + O\left(\frac{1}{u^{2m+1}}\right),
\]
for some suitably-chosen bounded simple loop $L_1$ in the $t$-plane that is positively orientated, lies in an equivalent domain to $D$, and surrounds $t = z$ and $t = z_1$. Hence in (6.2) it is legitimate (if necessary) to replace $A_m(u, z)$ by
\[(2\pi i)^{-1} \oint_{\mathcal{L}_1} A_m(u, t) (t - z)^{-1} dt, \text{ and similarly for } B_m(u, z). \] We then compute these integrals using the trapezoidal rule, which has exponential convergence.

In our computations we shall take \(N\) even and \(m = N/2\) so that the order terms in (6.1) and (6.2) are of comparable magnitude (and likewise for Case 2 considered later). Note that for fixed \(\zeta\)

\[u^{1/6} \text{Ai}(u^{2/3} \xi) = \exp(- u \xi) O(1), \]

\[u^{1/6} \text{Ai}'(u^{2/3} \xi) = \exp(- u \xi) O(u^{1/3}), \]

as \(u \to \infty\) (see [16, §9.7(ii)]).

If we consider the natural choice of \(\mathcal{L}_1\) being a circular path centered on \(z_1\), it should have a radius smaller than

\[r_m = \min\{z_1, z_2 - z_1\}\]

so that we can use Eq. (7.1). We have that \(r_m = z_1\) when \(a \leq 3 + 2\sqrt{2}\) (that is, \(\alpha/(n + 1/2) \leq 4(4 + 3\sqrt{2})\)) and \(r_m = z_2 - z_1\) if \(a \geq 3 + 2\sqrt{2}\). When we have to use Cauchy’s integral formula (for \(z\) close to \(z_1\)) it is clear that \(\text{Re } z > 0\); however, we have shown that the validity of both the L-G approximation and the Airy expansion away from the turning point \(z_1\) extends to \(\text{Re } z < 0\). We first consider circular \(\mathcal{L}_1\) as described above with a radius smaller that \(r_m\), and later we give details for \(\text{Re } z < 0\) where we can use (6.3) and (6.4) directly (without resorting to Cauchy integrals).

Let the relative error of an approximation \(f^*\) of a function \(f\) be defined in the usual manner by

\[\epsilon_{\text{rel}} = \left|\frac{f - f^*}{f}\right|.\]

Then in Fig. 7 we show the relative errors of the L-G expansion (6.1) and the Airy expansion (6.2) for the Laguerre polynomial. In both cases \(\epsilon_{\text{rel}}\) is shown as a function of the angle when the upper half circle centered at \(z_1\) is followed clockwise. As expected, we observe a step increase in the relative error for the L-G approximation as we cross a Stokes line; contrarily, the Airy expansion works well for all angles. In addition, we observe that the relative error for the Airy expansion is smaller than for the L-G expansion for all \(\theta\). Figure 8 also shows that this is true for real values \(0 < z < z_1\), while for negative \(z\) both expansions give similar accuracies.

In these and in most of the figures, we consider the case of Laguerre polynomials of degree \(n = 100\) and we use (6.1) with \(N = 16\) and (6.3) and (6.4) with \(m = 8\). If a higher degree \(n\) or a larger number of coefficients is considered, the accuracy will improve (except, of course, if \(N\) or \(m\) is too large, because the series are asymptotic but not convergent).

Figure 9 shows the maximum relative error over circles centered at \(z_1\) of different radii and with \(\text{Re } z < z_2\). This error is shown as a function of \(|\tau|\), with \(\tau = (z - z_1)/(z_2 - z_1)\). For the two smaller values of \(\alpha\), some of the circles include negative values of \(\text{Re } z\) for the two smaller values of \(\alpha\), specifically when \(|\tau| > |\tau(0)|\), with \(\tau(0) \approx -0.0301\) when \(\alpha = 100\) and \(< \tau(0) \approx -0.4028\) when \(\alpha = 1000\).
We observe that for computing the coefficients of the Airy expansion by Cauchy integrals only circles with $|\tau| < \min(1, |\tau(0)|)$ can be considered (that is, circles of radius smaller than $r_m = \min\{z_1, z_2 - z_1\}$). Indeed, we can not consider

![Fig. 7](image)

**Fig. 7** Relative errors $\epsilon_{rel}$ for the L-G expansion (6.1) and the Airy expansion (6.2), with coefficients computed from (6.3) and (6.4), as a function of $\theta$, where $z = z_1 - Re^{-i\theta}$, $\theta \in [0, \pi]$ and $R = 0.5r_m = 0.5z_1$.

![Fig. 8](image)

**Fig. 8** Relative errors $\epsilon_{rel}$ for the L-G expansion (6.1) and the Airy expansion (6.2), with coefficients computed from (6.3) and (6.4), as a function of $z/z_1$ for real values of $z$, $z < z_1$. 


$z > z_2$ because the expansions are meaningless there, and we can not consider $\text{Re } z < 0$ because of the discontinuity branch at the negative real axis (which implies a discontinuity in the coefficients $A(u, z)$ and $B(u, z)$).

However, as commented earlier and shown in Figs. 8 and 9, the Airy expansion (with coefficients computed by L-G asymptotics) is valid for $\text{Re } z < 0$. In particular it is also valid for negative $z$, as Fig. 10 also shows. In this figure, we plot the relative error of the Airy expansion (with coefficients from asymptotics) for real $z$ as a function of $\tau = (z - z_1) / (z_2 - z_1)$. Again this plot includes negative values of $z$ for the two smaller values of $\alpha$, for $\tau < \tau (0) \approx -0.0301$ when $\alpha = 100$ and for $\tau < \tau (0) \approx -0.4028$ when $\alpha = 1000$ (for the other case $\tau (0) < -1$ and therefore $z > 0$ in the figure). The expansion is also accurate for negative $z$ and, as expected, it fail close to the turning points ($\tau = 0$, $\tau = 1$).

As commented before, even when the expansions are valid for $\text{Re } z < 0$, these values can not be used for computing the coefficients by means of Cauchy integrals, due to the discontinuity at the branch cut. Figure 11 shows this: we plot the imaginary part of the approximations $A_m(u, z)$ and $B_m(u, z)$ as functions of the angle when the radius is such that the circle cuts the negative real axis ($r = 1.1$), and when it does not ($r = 0.9$). For the second case, both coefficients are real over the real line, but not in the first case, when they have a non-zero imaginary part when $z$ is real and negative, that is, when $\theta = 0$ (we move along the circles clockwise).

Finally, we notice that the relative error close to the turning point becomes large, as shown in Fig. 10; this indicates that, as expected, the L-G expansions for the coefficients of the Airy expansion are not accurate close to the turning point. This loss of accuracy is repaired by computing these coefficients by Cauchy integrals over
Fig. 10 Relative error $\epsilon_{rel}$ of the Airy expansion (6.2), with coefficients computed from (6.3) and (6.4), for real values of $z$ as a function of $\tau = (z - z_1)/(z_2 - z_1)$, a contour encircling the turning point (but sufficiently away from it) and contained in the half-plane $\text{Re } z > 0$. This is illustrated in Fig. 12. The Cauchy integrals are computed using the trapezoidal rule with a discretization of 150 points in the upper half of the Cauchy contour (in the lower half we employ complex conjugation).

We notice that, as is well known, the Laguerre polynomials $L_n^{(\alpha)}(uz)$ have $n$ positive real zeros when $\alpha > -1$ and most of them in the interval $(z_1, z_2)$. Of course, the relative error (7.4) at the zeros is meaningless, and loss of relative accuracy is unavoidable very close to these zeros. In the previous figures, this loss of accuracy is not clearly revealed because the function is sampled at values of $z$ which are not too

Fig. 11 Imaginary parts of the coefficient function approximations $A_{N/2}(u, z)$ and $B_{N/2}(u, z)$ as a function of $\theta$, where $z = z_1 - rz_1 e^{-i\theta}$ for two values of $r$. 

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Fig. 12 Relative error $\epsilon_{rel}$ of the Airy expansion (6.2) for real values of $z$ as a function of $(z - z_1)/r_m$, with the coefficient approximations $A_m(u, z)$ and $B_m(u, z)$ computed by Cauchy integrals. The Cauchy contour is a circle of radius $R = 0.7r_m$, $r_m = \min\{z_1, z_2 - z_1\}$, centered at $z_1$.

close to the zeros (the sample values are not selected to avoid the zeros, simply happen to be not too close). Values of $z$ very close to the zeros are needed to observe a significant accuracy loss; however, the previous plots show some effect of the zeros because the relative error gives a relatively noisy plot for values of $z$ where zeros

Fig. 13 Relative errors $\epsilon_{rel}$ for the L-G expansion (6.7) and the Airy (6.8) expansion, with coefficients computed from (6.9) and (6.10), as a function of $\theta$, where $z = z_2 + Re^{i\theta}$, $\theta \in [0, \pi]$ and $R = 0.25(z_2 - z_1)$.
occur when compared to the cases without zeros. Compare, for example, the positive values of the abscissa (for which there are zeros) with the negative values (no zeros) in Figs. 10 and 12. When we discuss Case 2 in the next section, we will show a detailed example of computation close to the zeros.

7.2 Case 2

Now we provide numerical evidence of the accuracy of the expansions for Laguerre polynomials in domains containing the turning point $z_2$.

As in Case 1a, if $z$ is close to the turning point (this time $z_2$) we can replace $\tilde{A}_m(u, z)$ by $\frac{1}{2\pi i} \oint_{L_2} \tilde{A}_m(u, t) (t - z)^{-1} \, dt$ in (6.8), and similarly for $\tilde{B}_m(u, z)$. Here $L_2$ is a bounded contour in the $t$-plane that is positively orientated, lies in an equivalent domain to $\tilde{D}$, and surrounds $t = z$ and $t = z_2$. As in Case 1a we typically take this to be a circle.

Figures 13 and 14 compare the accuracy attainable for the L-G expansion (6.7) and the Airy expansion (6.8)–(6.10), both for real and complex variables. Figures 15 and 16 illustrate the accuracy of this same Airy expansion but for different values of $\alpha$. And, finally, Fig. 17 shows the accuracy of the Airy expansion with the coefficients computed by Cauchy integrals. In all of these cases, the coefficients of the Airy expansion are computed using $N = 16$ ($m = 8$), and the degree of Laguerre polynomials is set to $n = 100$.

In Fig. 13, we compare the L-G expansion with the Airy expansion away from the turning point (with coefficient functions still computed with (6.9) and (6.10)). The
relative accuracy over a circle of radius $R = 0.25(z_2 - z_1)$ centered at $z_2$ is shown as a function of the angle $\theta \in [0, \pi]$; this half-circle is circulated counterclockwise. We observe that the L-G expansion (6.7) tends to fail for large $\theta$ in this interval, as the expansion loses meaning when we cross a Stokes line. Contrarily, the Airy type expansion (6.8)–(6.10) is valid in all of the interval. In addition, we observe that the relative error for the Airy expansion is smaller than for the L-G expansion for all $\theta$. We further explore this fact for real variables $z > z_2$ in Fig. 14.

Figure 15 shows the maximum relative error of the Airy expansion (6.8)–(6.10), over circles centered at $z_2$. As expected, the relative error increases both when the radius is small (because we are too close to the turning point $z_2$) as well as when part of the circle becomes too close to $z_1$. Figure 16 shows again the relative error of the Airy expansion (6.8)–(6.10) with coefficients computed with asymptotic series, but for real variable. Again, the relative error increases close to the turning points. For $z > z_2$ the relative error decreases as $z$ increases, as can be expected. This figure shows that our alternative Cauchy integral method for the computation of the coefficients $\tilde{A}_m(u,z)$ and $\tilde{B}_m(u,z)$ is needed around the turning point.

Figure 17 provides this computation using the trapezoidal rule with a discretization of 100 points in the upper half of the Cauchy contour (in the lower half we consider complex conjugation). Combining this computation with the results in Fig. 16 we observe that it is possible to compute accurately the Laguerre polynomials for $z > z_1$, but not too close to $z_1$.

Observe that, according to Fig. 15 (see also Fig. 16), over a circle of radius $0.7(z_2 - z_1)$ centered at $z_2$, the relative errors are of the order of $10^{-30}$, and we observe that the application of Cauchy integrals permits us to maintain this accuracy inside
Fig. 16 Relative error $\epsilon_{rel}$ of the Airy expansion (6.8), with coefficients computed from (6.9) and (6.10), for real values of $z$ as a function of $\rho = (z - z_2) / (z_2 - z_1)$

the circle (but not too close to the circle). As we see in Fig. 17, the relative error has little variation and it is of the order of $10^{-30}$ inside the circle $|z - z_2| < 0.5 (z_2 - z_1)$ (the figure is only for real $z$ but the same is true for complex $z$).

Fig. 17 Relative error $\epsilon_{rel}$ of the Airy expansion (6.8) for real values of $z$ as a function of $\rho = (z - z_2) / (z_2 - z_1)$, with the coefficient approximations $\tilde{A}_m(u, z)$ and $\tilde{B}_m(u, z)$ computed by Cauchy integrals. The Cauchy contour is a circle of radius $R = 0.7 (z_2 - z_1)$ centered at $z_2$.
As we commented in the previous subsection, the relative accuracy unavoidably degrades very close to the zeros and this degradation is, for example, responsible for the different appearance of the graphs for $\rho > 0$ and $\rho < 0$ in Figs. 16 and 17.

To illustrate the uniform accuracy of our approximations near the zeros we need to replace the denominator of (7.4) with an “envelope” of the Laguerre polynomial, which mimics the amplitude of $L_n^{(\alpha)}(x)$ but does not vanish at its zeros. For polynomials having simple zeros, the envelope function $\text{env} f(x) = \{ f^2(x) + f'^2(x) \}^{1/2}$ serves this purpose. Then we define the modified relative error of an approximation $f^*$ to a function $f$ as

$$\hat{\epsilon}_{rel} = \frac{|f - f^*|}{\text{env} f},$$

(7.5)

where in our case $f(x) = L_n^{(\alpha)}(x)$.

In Fig. 18 we then show similar results for the Airy expansion in a more restricted interval containing two zeros. In this both the relative error $\epsilon_{rel}$ (solid line) and the modified relative error $\hat{\epsilon}_{rel}$ (dashed line) are shown. In the figure we used many more sample points than in Fig. 17, so that the (unavoidable) degradation of the relative error $\epsilon_{rel}$ becomes more apparent, whereas in contrast the modified relative error $\hat{\epsilon}_{rel}$ remains bounded. As can be seen, the degradation of $\epsilon_{rel}$ only takes place very close to the zeros. Of course, this relative error degradation is unavoidable and common to any method of numerical computation. On the other hand, $\hat{\epsilon}_{rel}$ is small throughout the interval which illustrates the uniform absolute accuracy of our Airy expansion in the whole interval.

Finally, we give some additional results for other values of $n$, $\alpha$ and $N$ as a further illustration of the accuracy of the computation using Cauchy integrals. We pick a

![Figure 18](image)
Table 1  Relative error $\epsilon_{rel}$ of the Airy expansion (6.8) with $z = z_2 + 0.1(z_2 - z_1)$, where the coefficient approximations $\tilde{A}_m(u, z)$ and $\tilde{B}_m(u, z)$ are computed by Cauchy integrals over the contour $|z - z_2| = 0.7|z_2 - z_1|$; different selections of the degree $n$, the parameter $\alpha$ and the number of coefficients in the asymptotic expansion $N = 2n$ are considered

| $N \Rightarrow$ | 2   | 4   | 8   | 12  | 16  |
|----------------|-----|-----|-----|-----|-----|
| $\alpha, n$   |     |     |     |     |     |
| 0, 10          | $1.15 \times 10^{-6}$ | $2.88 \times 10^{-9}$ | $2.20 \times 10^{-13}$ | $1.28 \times 10^{-16}$ | $2.86 \times 10^{-19}$ |
| 0, 100         | $1.46 \times 10^{-9}$ | $4.03 \times 10^{-14}$ | $3.65 \times 10^{-22}$ | $2.45 \times 10^{-29}$ | $6.24 \times 10^{-36}$ |
| 0, 1000        | $1.50 \times 10^{-12}$ | $4.21 \times 10^{-19}$ | $3.89 \times 10^{-31}$ | $2.66 \times 10^{-42}$ | $6.85 \times 10^{-53}$ |
| 100, 10        | $4.90 \times 10^{-7}$ | $1.22 \times 10^{-9}$ | $9.53 \times 10^{-13}$ | $5.68 \times 10^{-17}$ | $1.30 \times 10^{-19}$ |
| 100, 100       | $7.79 \times 10^{-10}$ | $1.94 \times 10^{-14}$ | $1.69 \times 10^{-22}$ | $1.13 \times 10^{-29}$ | $2.88 \times 10^{-36}$ |
| 100, 1000      | $1.31 \times 10^{-12}$ | $3.41 \times 10^{-19}$ | $2.76 \times 10^{-31}$ | $1.70 \times 10^{-42}$ | $4.06 \times 10^{-53}$ |

value of $z$ in the disk $|z - z_2| < 0.5 (z_2 - z_1)$. In particular, we fix $z = z_1 + 0.1(z_2 - z_1)$. We show the corresponding relative errors in Table 1. In the table, we take 300 points in the upper half of the Cauchy contour because with the previous selection (100 points) the discretization error is not small enough in some cases; in particular for $n = 1000$ and $N = 16$ ($m = N/2 = 8$), when the relative error becomes of the order of $10^{-50}$.

We observe in Table 1 that the relative error shows little variation with the value of $\alpha$, as the previous figures also showed, and that the dependence on $n$ and $N$ is as expected for asymptotic approximations with an error of $O(n^{-N-1})$.

This is shown more explicitly in Table 2, where we give the values of the computational asymptotic error constants $n^{N+1}\epsilon_{rel}$, with $\epsilon_{rel}$ the relative errors in Table 1; we observe these experimental constants have a slow variation as a function of $n$ and $\alpha$. While we also expect these constants to be $O(1)$ (at least for moderate values of $N$) they are in fact quite small for all the values of $N$ under consideration, which illustrates the uniform high accuracy of our Airy expansions.

Table 2  Computational asymptotic error constants estimated from the errors of Table 1

| $N \Rightarrow$ | 2   | 4   | 8   | 12  | 16  |
|----------------|-----|-----|-----|-----|-----|
| $\alpha, n$   |     |     |     |     |     |
| 0, 10          | $1.15 \times 10^{-3}$ | $2.88 \times 10^{-4}$ | $2.20 \times 10^{-4}$ | $1.28 \times 10^{-3}$ | $2.86 \times 10^{-2}$ |
| 0, 100         | $1.46 \times 10^{-3}$ | $4.03 \times 10^{-4}$ | $3.65 \times 10^{-4}$ | $2.45 \times 10^{-3}$ | $6.24 \times 10^{-2}$ |
| 0, 1000        | $1.50 \times 10^{-3}$ | $4.21 \times 10^{-4}$ | $3.89 \times 10^{-4}$ | $2.66 \times 10^{-3}$ | $6.85 \times 10^{-2}$ |
| 100, 10        | $4.90 \times 10^{-4}$ | $1.22 \times 10^{-4}$ | $9.53 \times 10^{-5}$ | $5.68 \times 10^{-4}$ | $1.30 \times 10^{-2}$ |
| 100, 100       | $7.79 \times 10^{-4}$ | $1.94 \times 10^{-4}$ | $1.69 \times 10^{-4}$ | $1.13 \times 10^{-3}$ | $2.88 \times 10^{-2}$ |
| 100, 1000      | $1.31 \times 10^{-3}$ | $3.41 \times 10^{-4}$ | $2.76 \times 10^{-4}$ | $1.70 \times 10^{-3}$ | $4.06 \times 10^{-2}$ |
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