The integrability of the one-dimensional (1D) fermion chain model is investigated in the framework of the Quantum Inverse Scattering Method (QISM). We introduce a new $R$-operator for the fermion chain model, which is expressed in terms of the fermion operators. The $R$-operator satisfies a new type of the Yang-Baxter relation with fermionic $L$-operator. We derive the fermionic Sutherland equation from the relation, which is equivalent to the fermionic Lax equation. It also provides a mathematical foundation of the boost operator approach for the fermion model. In fact, we obtain some higher conserved quantities of the fermion model using the boost operator.

KEYWORDS:
quantum inverse scattering method, fermionic formulation, fermion chain model, Sutherland equation, boost operator

1 Introduction

In the last decades, many integrable spin chain models have been investigated by means of Quantum Inverse Scattering Method (QISM for brevity). In the QISM, the $R$-matrix, which satisfies the Yang-Baxter relation with the $L$-operator

$$ R_{12}(u_1, u_2) L_j(u_1) L_j(u_2) = L_j(u_2) L_j(u_1) R_{12}(u_1, u_2), \quad (1) $$

plays the essential role. For the spin chain models, the $R$-matrix $R_{12}(u_1, u_2)$ is a $c$-number matrix and satisfies the Yang-Baxter equation

$$ R_{12}(u_1, u_2) R_{13}(u_1, u_3) R_{23}(u_2, u_3) = R_{23}(u_2, u_3) R_{13}(u_1, u_3) R_{12}(u_1, u_2). \quad (2) $$
From the Yang-Baxter relation (1), it follows that the transfer matrix defined by

\[ t(u) = \text{tr}_a(\hat{L}_N(u) \ldots \hat{L}_1(u)) \]  

constitutes a commuting family

\[ [t(u), t(v)] = 0. \]  

Expansion of \( t(u) \) in terms of spectral parameter \( u \) gives the conserved quantities. Therefore, this proves the existence of the conserved operators \( I^{(j)} \) which are involutive,

\[ [I^{(j)}, I^{(k)}] = 0. \]  

We now know that there also exist many solvable fermion chain models which are important in condensed matter physics. They are usually related to the spin models through the Jordan-Wigner transformation. For example, the fermion chain model related to the \( XXZ \) model is given by [7, 8]

\[ \mathcal{H} = \sum_{j=1}^{N} \left\{ \hat{c}_j^\dagger \hat{c}_{j+1} + \hat{c}_{j+1}^\dagger \hat{c}_j + \frac{\Delta}{2} (2n_j - 1)(2n_{j+1} - 1) \right\}. \]  

Here \( \hat{c}_j^\dagger \) and \( \hat{c}_j \) are the creation and the annihilation operators at site \( j \) which satisfy the canonical anti-commutation relations

\[ \{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0, \quad \{c_j, c_k^\dagger\} = \delta_{jk}, \]  

and \( n_j \) is the number operator

\[ n_j = \hat{c}_j^\dagger \hat{c}_j. \]  

The periodic boundary condition (PBC) is assumed in (6),

\[ c_{N+1} = c_1, \quad \hat{c}_{N+1}^\dagger = \hat{c}_1^\dagger. \]  

We refer to this model (6) as the \( XXZ \) fermion model.

Since the Jordan-Wigner transformation connects the spin model and the fermion model, physical properties of the fermionic chain models including the exact integrability are often discussed using the corresponding spin chain model. However, there are some important differences between the fermion models and the spin models. In particular, the PBC for the fermion model does not correspond to the PBC for the spin model due to the non-locality of the Jordan-Wigner transformation. Therefore it is more appropriate to treat the fermion models keeping their fermionic nature as much as possible (c.f. refs. 9 and 10).

The QISM for the fermion chain model (6) has been developed by several authors, [7, 8, 11, 12] where the fermionic \( L \)-operator is introduced. The fermionic \( R \)-matrix which
satisfies the graded Yang-Baxter relation was also found. The existence of the fermionic \( R \)-matrix can be used to establish the integrability of the fermion model under the PBC (1) or other twisted boundary conditions. The most intriguing point of this method is that the matrix elements of the fermionic \( L \)-operator consist of the fermion operators, which means that the quantum space is fermionic. On the other hand, the auxiliary space remains to be a usual spin space and correspondingly the \( R \)-matrix has the \( c \)-number elements. The fact suggests that there still requires further investigation on the fermionic formulation of the QISM.

In this paper, we introduce an \( R \)-operator for the \( XXZ \) fermion model. It consists of the fermion operators and satisfies a new type of the Yang-Baxter relation with the fermionic \( L \)-operator. The role of the auxiliary space and the quantum space for the fermionic \( L \)-operator is exchanged in the Yang-Baxter relation. We derive the fermionic Sutherland equation from the Yang-Baxter relation, which is shown to be equivalent to the fermionic Lax equation. \([12, 13]\) To obtain the higher conserved operators for the lattice spin models, it is in general very useful to introduce the boost operator. \([14, 15, 16]\) The fermionic Sutherland equation provides the basis for the application of the boost operator to the fermion models. We derive some higher conserved operators for the \( XXZ \) fermion model using the boost operator.

The paper is organized as follows. In §2, we summarize salient points of the QISM for the \( XXZ \) fermion model. In §3, we consider the exchange of the auxiliary and quantum spaces of the fermionic \( L \)-operator and introduce an \( R \)-operator. We also discuss the fundamental properties of the \( R \)-operator. In particular, we find that the \( R \)-operator satisfies the Yang-Baxter relation. In §4, we further discuss some applications of the fermionic \( R \)-operator and the Yang-Baxter relation. We derive the Sutherland equation and introduce the boost operator. The last section is devoted to concluding remarks.

## 2 Graded Yang-Baxter Relation for the \( XXZ \) Fermion Model

In this section we briefly summarize the QISM for the \( XXZ \) fermion model (6). The fermionic \( L \)-operator is given by

\[
\mathcal{L}_j(u) = \begin{pmatrix}
\alpha(u)n_j + \gamma(u)(1 - n_j) & 2\beta(u)c_j \\
-2i\beta(u)c_j^\dagger & \alpha(u)(1 - n_j) - i\gamma(u)n_j
\end{pmatrix}, \tag{10}
\]

where

\[
\alpha(u) = \sin(u + 2\eta), \\
\gamma(u) = \sin u, \\
2\beta(u) = \sin 2\eta. \tag{11}
\]
We express by $\otimes_s$ the Grassmann (graded) direct product

$$[A \otimes_s B]_{\alpha\gamma,\beta\delta} = (-1)^{P(\alpha)+P(\beta)}P(\gamma) A_{\alpha\beta} B_{\gamma\delta}$$

$$P(1) = 0, \quad P(2) = 1.$$  \hspace{1cm} (12)

Then there exists the fermionic $R$-matrix which satisfies the graded Yang-Baxter relation (Fig. 1),

$$R_{12}(u_1-u_2)\mathcal{L}(u_1)\mathcal{L}(u_2) = \mathcal{L}(u_2)\mathcal{L}(u_1)R_{12}(u_1-u_2)$$ \hspace{1cm} (13)

where, with $I$ being $2 \times 2$ identity matrix,

$$\mathcal{L}(u_1) = \mathcal{L}_j(u_1) \otimes_s I, \quad \mathcal{L}(u_2) = I \otimes_s \mathcal{L}_j(u_2).$$ \hspace{1cm} (14)

The explicit form of the fermionic $R$-matrix is

$$R_{12}(u) = \begin{pmatrix}
 \sin(u+2\eta) & 0 & 0 & 0 \\
 0 & -i\sin u & \sin 2\eta & 0 \\
 0 & \sin 2\eta & i\sin u & 0 \\
 0 & 0 & 0 & -\sin(u+2\eta)
\end{pmatrix}.$$ \hspace{1cm} (15)

It fulfills the graded Yang-Baxter equation (Fig. 2)

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)$$ \hspace{1cm} (16)

Note that non-zero elements of (15) are even with respect to the parity $P(\alpha)$, i.e.,

$$P(\alpha) + P(\beta) + P(\alpha') + P(\beta') = 0 \pmod{2} \quad \text{for} \quad R_{\alpha\beta,\alpha'\beta'} \neq 0.$$ \hspace{1cm} (17)

The monodromy matrix $T(u)$ is defined as an ordered product of the fermionic $L$-operators

$$T(u) = \mathcal{L}_N(u) \ldots \mathcal{L}_1(u).$$ \hspace{1cm} (18)

From the local relation (13), we have the global relation for the monodromy matrix

$$R_{12}(u_1-u_2)T_j(u_1)T_j(u_2) = T_j(u_2)T_j(u_1)R_{12}(u_1-u_2).$$ \hspace{1cm} (19)

The transfer matrix $\tau(u)$ is defined by the supertrace of the monodromy matrix $T(u)$

$$\tau(u) = \text{str} T(u) \equiv \text{tr}\{\sigma^z T(u)\}.$$ \hspace{1cm} (20)

Then the global graded Yang-Baxter relation (19) leads to the commutativity of the transfer matrix,

$$[\tau(u_1), \tau(u_2)] = 0.$$ \hspace{1cm} (21)
The last equation establishes the exact integrability of the model (3) under the PBC. In fact, we have

$$\tau(u) = \tau(0)\{1 + u\mathcal{H} + \ldots\}, \quad (22)$$

with the identification

$$\Delta = \cos 2\eta. \quad (23)$$

For later use, we introduce a notation

$$\mathcal{H} = \sum_{j=1}^{N} \mathcal{H}_{j,j+1}, \quad (24)$$

where

$$\mathcal{H}_{j,k} = \frac{1}{\sin 2\eta}\left\{ (c^\dagger_j c_k + c^\dagger_k c_j) + \frac{1}{2} \cos 2\eta (2n_j - 1)(2n_k - 1) + \frac{1}{2} \cos 2\eta \right\}. \quad (25)$$

We call $\mathcal{H}_{j,k}$ the two-point Hamiltonian density.

![Fig. 1 Graded Yang-Baxter relation, eq. (13)](image1)

![Fig. 2 Graded Yang-Baxter equation, eq. (16)](image2)
3 Fermionic $R$-Operator

One of the characteristic features of the fermionic $L$-operator is that its matrix elements consist of the fermion operators, that is, quantum space of the fermionic $L$-operator is fermion Fock space. However, the auxiliary space remains to be a usual spin space. The fermionic $L$-operator is therefore asymmetric with respect to the exchange of the quantum space and the auxiliary space. The reader may recall the similar situation when we construct the spin-$S$ XXZ model from the spin-$\frac{1}{2}$ XXZ model. In this case, we first consider the intermediate $L$-operator, whose quantum space consists of spin operators of magnitude $S$. Then we exchange the auxiliary and quantum spaces of the $L$-operator and consider a new Yang-Baxter relation. The $R$-matrix which satisfies the Yang-Baxter relation should be in a $(2S+1) \times (2S+1)$ matrix form. From the $R$-matrix, we can construct a new transfer matrix which generates the spin-$S$ XXZ model. [6, 17, 18]

We apply the similar procedure to the fermion chain model. Consider the following new Yang-Baxter relation (Fig. 3)

$$R^f_{12}(u_1 - u_2) \mathcal{L}_1(u_1) \mathcal{L}_2(u_2) = \mathcal{L}_2(u_2) \mathcal{L}_1(u_1) R^f_{12}(u_1 - u_2)$$  (26)

Notice that the role of the auxiliary and quantum spaces of the fermionic $L$-operator is exchanged in (13) and (26). Inevitably, the $R$-operator $R^f(u)$ is no longer $c$-number matrix and consists of fermion operators. Of course, there is no assurance for the existence of such $R$-operator at this stage. We assume a form of the $R$-operator as

$$R^f_{12}(u_1 - u_2) = g_1 n_1 n_2 + g_2 n_1 (1 - n_2) + g_3 (1 - n_1) n_2 + g_4 (1 - n_1)(1 - n_2) + g_5 c_1^\dagger c_2 + g_6 c_2^\dagger c_1,$$  (27)

where $g_i, i = 1, \ldots, 6$ are scalar functions of $u_1$ and $u_2$. Substituting the expression (27) in (26), we have the following equations,

$$g_1 = -g_4, \quad g_3 = g_2, \quad g_6 = g_5,$$

$$\alpha(u_2) g_1 = -\gamma(u_2) g_2 - \alpha(u_1) g_5,$$

$$\gamma(u_1) g_1 = -\alpha(u_1) g_2 - \gamma(u_2) g_5,$$

$$\beta(u_1) \beta(u_2) g_2 = \{ \gamma(u_1) \alpha(u_2) - \alpha(u_1) \gamma(u_2) \} g_5.$$  (28)

A set of functional equations (28) is solved to give (up to an overall factor)

$$g_1 = -g_4 = -\frac{\sin(u_1 - u_2 + 2\eta)}{\sin 2\eta},$$

$$g_2 = g_3 = \frac{\sin(u_1 - u_2)}{\sin 2\eta},$$

$$g_5 = g_6 = 1.$$  (29)
Thus obtained $R$-operator, which we call fermionic $R$-operator,
\[
R_{12}^f(u_1 - u_2) = \frac{\sin(u_1 - u_2 + 2\eta)}{\sin 2\eta} (1 - n_1 - n_2) + \frac{\sin(u_1 - u_2)}{\sin 2\eta} (n_1 + n_2 - 2n_1n_2) + c_1^\dagger c_2 + c_2^\dagger c_1,
\]
(30)
satisfies (29).

The fermionic $R$-operator enjoys the following properties.

1. Regularity (Initial condition)
\[
R_{12}^f(u = 0) = K_{12}.
\]
(31)

Here
\[
K_{ij} = 1 - (c_i^\dagger - c_j^\dagger)(c_i - c_j)
\]
is the permutation operator for the fermion operators:
\[
K_{ij} = K_{ji}, \quad K_{jj} = 1,
\]
\[
K_{ij}c_i = c_jK_{ij}, \quad K_{ij}c_i^\dagger = c_j^\dagger K_{ij},
\]
\[
K_{ij}K_{ij} = 1.
\]
(32)

2. Local Hamiltonian:
\[
\left. \frac{dR_{12}^f(u)}{du} \right|_{u=0} = K_{12}H_{12}.
\]
(34)

Here $H_{12}$ is the 2-point Hamiltonian density given in (25).

3. Yang-Baxter equation:
The fermionic $R$-operator satisfies the Yang-Baxter equation (Fig. 4) as
\[
R_{12}^f(u_1 - u_2)R_{13}^f(u_1)R_{23}^f(u_2) = R_{23}^f(u_2)R_{13}^f(u_1)R_{12}^f(u_1 - u_2).
\]
(35)

4. Unitarity:
\[
R_{12}^f(u)R_{12}^f(-u) = \left[ 1 - \frac{\sin^2 u}{\sin^2 2\eta} \right] 1.
\]
(36)

Now we consider a product of the fermionic $R$-operators
\[
T_a^f(u) = R_{aN}^f(u) \ldots R_{a2}^f(u)R_{a1}^f(u).
\]
(37)

Then from (35), we see that $T_a^f(u)$ satisfies the Yang-Baxter relation,
\[
R_{12}^f(u_1 - u_2)T_1^f(u_1)T_2^f(u_2) = T_2^f(u_2)T_1^f(u_1)R_{12}^f(u_1 - u_2).
\]
(38)
Here we note the commutativity,
\[ R_{1j}(u_1) R_{2k}(u_2) = R_{2k}(u_2) R_{1j}(u_1), \hspace{1cm} (j \neq k). \] (39)

We refer to \( T^f_j(u) \) as monodromy operator. We can also introduce an analog of the transfer matrix,
\[ \tau^f(u) = \text{Str} T_a^f(u) \]
\[ \equiv a \langle 0 | T_a^f(u) | 0 \rangle_a - a \langle 1 | T_a^f(u) | 1 \rangle_a, \] (40)
where \(|0\rangle_a\) and \(|1\rangle_a\) are defined by
\[ c_a |0\rangle_a = 0, \quad |1\rangle_a = c^\dagger_a |0\rangle_a. \] (41)

The fermionic transfer operator \( \tau^f(u) \) constitutes a commuting family due to (38)
\[ [\tau^f(u), \tau^f(v)] = 0, \] (42)
which also gives Hamiltonian (6),
\[ H = \sum_{j=1}^{N} \mathcal{H}_{jj+1} = \frac{d \log \tau^f(u)}{du} \bigg|_{u=0}. \] (43)
4 Sutherland Equation and Boost Operator

In this section we discuss further some applications of the fermionic $R$-operator and the Yang-Baxter relation.

First, we differentiate the Yang-Baxter relation (26) with respect to the spectral parameter $u_2$ at $u_2 = u_1$. Then we get

$$
-\mathcal{K}_{12} \mathcal{L}_1(u_1) \mathcal{L}_2(u_1) + \mathcal{K}_{12} \mathcal{L}_1(u_1) \frac{d \mathcal{L}_2(u_2)}{du_2} \bigg|_{u_2 = u_1}
= \frac{d \mathcal{L}_2(u_2)}{du_2} \bigg|_{u_2 = u_1} \mathcal{L}_1(u_1) \mathcal{K}_{12} - \mathcal{L}_2(u_1) \mathcal{L}_1(u_1) \mathcal{K}_{12} \mathcal{K}_{12}.
$$

(44)

For simplicity, we do not write the auxiliary space dependence of the $L$-operator here and hereafter. By use of the relations

$$
\mathcal{L}_1(u) \mathcal{K}_{12} = \mathcal{K}_{12} \mathcal{L}_2(u), \quad \mathcal{L}_2(u) \mathcal{K}_{12} = \mathcal{K}_{12} \mathcal{L}_1(u),
$$

(45)

eq (44) can be rewritten as

$$
[\mathcal{H}_{12}, \mathcal{L}_1(u) \mathcal{L}_2(u)] = \mathcal{L}_1(u) \frac{d \mathcal{L}_2}{du}(u) - \frac{d \mathcal{L}_1}{du}(u) \mathcal{L}_2(u),
$$

(46)

This is the fermionic version of the Sutherland equation, which enables us to show the commutativity of the Hamiltonian $\mathcal{H}$ and the transfer matrix $\tau(u)$,

$$
[\mathcal{H}, \tau(u)] = 0.
$$

(47)

The fermionic Sutherland equation (46) is equivalent to the Lax equation,

$$
\frac{d \mathcal{L}_j(u)}{dt} \equiv i[\mathcal{L}_j(u), \mathcal{H}]
= \mathcal{M}_{j+1}(u) \mathcal{L}_j(u) - \mathcal{L}_j(u) \mathcal{M}_j(u).
$$

(48)

Here $\mathcal{M}_j$ is given by the formula,

$$
\mathcal{M}_j(u) = -i \mathcal{L}_j^{-1}(u) \left\{ \frac{d \mathcal{L}_j(u)}{du} + [\mathcal{H}_{j-1}, \mathcal{L}_j(u)] \right\}.
$$

(49)

We remark that the formula (49) for the usual spin models is well-known. However, to our knowledge, the formula (49) for the fermion models is not known before and is proved here for the first time. We can conclude that the fermionic Lax equation follows from the Yang-Baxter relation (26).

Using (10) and (25) we can obtain the Lax operator $\mathcal{M}_j(u)$ for the fermion $XXZ$ model as

$$
\mathcal{M}_j(u) = \begin{pmatrix} M_{11}(u) & M_{12}(u) \\ M_{21}(u) & M_{22}(u) \end{pmatrix},
$$

(50)
where
\[
\mathcal{M}_{11}(u) = \frac{1}{\sin 2\eta} \left\{ i - \frac{\sin u}{\sin(u + 2\eta)} \right\} c_j^\dagger c_{j-1} + \frac{1}{\sin 2\eta} \left\{ i + \frac{\sin u}{\sin(u - 2\eta)} \right\} c_{j-1}^\dagger c_j \\
- \frac{i \sin 4\eta}{\sin(u + 2\eta) \sin(u - 2\eta)} (1 - n_j)(1 - n_{j-1}) - i \cot(u + 2\eta),
\]
\[
\mathcal{M}_{12}(u) = \frac{1}{\sin(u - 2\eta)} \left\{ i n_{j-1} c_j - (1 - n_j)c_{j-1} \right\} - \frac{1}{\sin(u + 2\eta)} \left\{ i(1 - n_{j-1})c_j - n_jc_{j-1} \right\},
\]
\[
\mathcal{M}_{21}(u) = - \frac{1}{\sin(u + 2\eta)} \left\{ i n_{j-1} c_j^\dagger + (1 - n_j)c_{j-1}^\dagger \right\} + \frac{1}{\sin(u - 2\eta)} \left\{ i(1 - n_{j-1})c_j^\dagger + n_jc_{j-1}^\dagger \right\},
\]
\[
\mathcal{M}_{22}(u) = \frac{1}{\sin 2\eta} \left\{ i - \frac{\sin u}{\sin(u + 2\eta)} \right\} c_{j-1}^\dagger c_j + \frac{1}{\sin 2\eta} \left\{ i + \frac{\sin u}{\sin(u + 2\eta)} \right\} c_j^\dagger c_{j-1} \\
- \frac{i \sin 4\eta}{\sin(u + 2\eta) \sin(u - 2\eta)} n_j n_{j-1} - i \cot(u + 2\eta).
\]

This result coincides with the known one [12] up to the terms proportional to the identity matrix.

As another application of the fermionic Sutherland equation, we consider the boost operator for the fermion chain model as
\[
\mathcal{B} = \sum_{j=1}^{N} j \mathcal{H}_{j,j+1}.
\]

By making use of the fermionic Sutherland equation, we can show the following relations,
\[
[\mathcal{B}, \tau(u)] = \frac{d\tau(u)}{du} - N \text{str} \left( \mathcal{L}_N \ldots \mathcal{L}_2 \frac{d\mathcal{L}_1}{du} \right).
\]

We define \( I^{(n)} \) as \( n \)-th local conserved operator and \( I^{(1)} = \mathcal{H} \). Then, from \((53)\), a recursion relation for the conserved operator is
\[
I^{(n+1)} = [\mathcal{B}, I^{(n)}] + \text{(some other terms)}.
\]

By use of the boost operator \((52)\) and the recursion relation \((54)\), we immediately obtain some higher conserved operators for the fermion XXZ model. Notice that the second term in the r.h.s. of eq. \((54)\) originates from the second term in the r.h.s. of eq. \((53)\), and we can estimate its contribution. Then we get
\[
I^{(2)} = \frac{1}{\sin^2 2\eta} \sum_{j=1}^{N} \left[ \left( c_{j+2}^\dagger c_j - c_j^\dagger c_{j+2} \right) - 2 \cos 2\eta \left( c_{j+1}^\dagger c_j - c_j^\dagger c_{j+1} \right) \\
- n_{j+2}(c_{j+1}^\dagger c_j - c_j^\dagger c_{j+1}) - (c_{j+2}^\dagger c_{j+1} - c_{j+1}^\dagger c_{j+2})n_j \right] \]
\]
\[ I^{(3)} = \frac{2}{\sin^3 2\eta} \sum_{j=1}^{N} \left[ (c_{j+3}^\dagger c_j + c_j^\dagger c_{j+3} + c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) 
+ \cos 2\eta \left\{ -3(c_{j+2}^\dagger c_j + c_j^\dagger c_{j+2}) + 2n_{j+3}(c_{j+2}^\dagger c_j + c_j^\dagger c_{j+2}) 
+ 2n_{j+1}(c_{j+2}^\dagger c_j + c_j^\dagger c_{j+2}) + 2n_{j-1}(c_{j+2}^\dagger c_j + c_j^\dagger c_{j+2}) 
+ 2(c_{j+3}^\dagger c_{j+2} - c_{j+2}^\dagger c_{j+3})(c_{j+1}^\dagger c_j - c_j^\dagger c_{j+1}) - 2n_j + 4n_{j+1}n_j - 2n_{j+2}n_j \right\} 
+ 2\cos^2 2\eta \left\{ (c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) + 2n_{j+2}(c_{j+2}^\dagger c_{j+1} + c_{j+1}^\dagger c_{j+2})n_j 
- n_{j+2}(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}) - n_j(c_{j+2}^\dagger c_{j+1} + c_{j+1}^\dagger c_{j+2}) \right\} \right] . \] 

(56)

We remark that these results are consistent with the known ones. \[11, 12, 19\]

5 Concluding Remarks

In this paper, we have studied the quantum inverse scattering method (QISM) for the XXZ fermion model. We have introduced a fermionic \( R \)-operator for the fermion chain model. It consists of the fermion operators and satisfies the Yang-Baxter equation. The \( R \)-operator intertwines the fermionic \( L \)-operator in a different way from the usual one. The auxiliary space and the quantum space of the fermionic \( L \)-operator are exchanged. We have shown that the new relation plays a complementary role to the usual graded Yang-Baxter relation. In particular, we can derive the fermionic Sutherland equation from the relation. It is shown that the fermionic Lax equation is equivalent to the fermionic Sutherland equation. The Sutherland equation also gives a mathematical foundation of the boost operator approach for the fermion chain model.

We have also defined a fermionic monodromy operator by a product of the fermionic \( R \)-operators. The transfer operator, which is an analog of the transfer matrix, can be constructed by taking the expectation values in the auxiliary space, \( \text{Str} \), of the monodromy operator. The transfer operator constitutes a commuting family, which proves the integrability of the fermion chain model. The diagonalization of the fermionic transfer operator will be discussed in a separate paper.

Our approach is applicable to other integrable fermion models. Among them, we shall consider the Hubbard model in subsequent papers.

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