Online Matching Frameworks under Stochastic Rewards, Product Ranking, and Unknown Patience

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We study generalizations of online bipartite matching in which each arriving vertex (customer) views a ranked list of offline vertices (products) and matches to (purchases) the first one they deem acceptable. The number of products that the customer has patience to view can be stochastic and dependent on the products seen. We develop a framework that views the interaction with each customer as an abstract resource consumption process, and derive new results for these online matching problems under the adversarial, non-stationary, and IID arrival models, assuming we can (approximately) solve the product ranking problem for each single customer. To that end, we show new results for product ranking under two cascade-click models: an optimal algorithm when each item has its own hazard rate for making the customer depart, and a 1/2-approximate algorithm when the customer has a general item-independent patience distribution. We also present a constant-factor 0.027-approximate algorithm in a new model where items are not initially available and arrive over time. We complement these positive results by presenting three additional negative results relating to these problems.

1. Introduction

Online matching is a fundamental problem in e-commerce and online advertising, introduced in the seminal work of Karp et al. (1990). While offline matching has a long history in economics and computer science, online matching has exploded in popularity with the ubiquity of the internet and the emergence of online marketplaces. A common scenario in e-commerce is the online sale of unique goods due to the ability to reach niche markets via the internet (e.g., eBay); typical products include rare books, trading cards, art, crafts, and memorabilia. We will use this as a motivating
example to describe our setting. However, the settings we study can also model job search/hiring, crowdsourcing, online advertising, ride-sharing, and other online-matching problems.

In classical online bipartite matching, we start with a known set of offline vertices that may represent items for sale or ads to be allocated. Then, an unknown sequence of online vertices arrive, which may represent customers, users, or visitors to a webpage. These online vertices or customers arrive one-by-one, and the decision to match each customer or not (and if so, to which item) must be made irrevocably before the next customer is revealed. In the original formulation, the online vertices are chosen fully adversarially, although models that assume they are drawn from probability distributions have since been studied (Feldman et al. 2009, Alaei et al. 2012).

Many generalizations of online matching have also been proposed, including stochastic rewards and weighted graphs. Under stochastic rewards, there can be repeated interactions with a customer (recommending an item, and if they do not accept, recommending another item, etc.) before the next one arrives, as we describe subsequently. Our paper’s goal is to provide a framework that decouples the repeated-interaction problem for a single customer from the overall allocation problem over multiple customers, leading to new results and unification of old ones. Moreover, motivated by product ranking, we derive new results for the repeated-interaction problem with a single customer, including the extension in which the horizon for these interactions is unknown or stochastic.

**Description of stochastic rewards model, with patience.** In the stochastic rewards model, each edge exists independently according to a known probability; this probability is revealed upon the arrival of its incident online vertex. This is motivated by online platforms in which only a probabilistic prediction of whether a customer will buy an item is known at the time they arrive. The algorithm can “probe” edges incident to an online vertex, or equivalently recommend the customer an item, after which if they accept, then the item is sold committedly. If they otherwise reject, then under the basic stochastic rewards model (Mehta and Panigrahi 2012) there is no opportunity to offer another item; this is known as the customer having a patience of 1.

Other papers (Bansal et al. 2010, Adamczyk et al. 2015, Brubach et al. 2017) have more generally allowed the customer to have any deterministic patience \( \theta \). This can be interpreted as a
product ranking problem where \( \theta \) different items are listed on a page, and the customer will view them in order, stopping once they see an acceptable item, or reaching the end of the page. The product ranking problem where \( \theta \) is deterministic can be efficiently solved using dynamic programming (Purohit et al. 2019). More generally, if \( \theta \) is random, but drawn from a known distribution, then the customer may probabilistically depart after seeing any undesirable item; this is called the cascade-click model of product ranking. We will derive new results for the cascade-click model.

**Description of edge weights and stochastic arrival models.** In an orthogonal generalization of online bipartite matching, edges between items and customers may have a *weight*, which is the reward collected when that edge is matched. This can represent, e.g., the price at which that item is sold to the customer. When edges can take on different possible weights, parametric competitive ratios are known (Ma and Simchi-Levi 2020), but a competitive ratio that is an absolute constant is impossible in the original adversarial arrival model (see Mehta 2012). Therefore, many papers have focused on the relaxed models of *stochastic arrivals* or *vertex weights* instead, each of which circumvents this impossibility.

In the stochastic arrival models, the total number of online vertices \( T \) is known, and each online vertex \( t = 1, \ldots, T \) has a *type*\(^1\) drawn independently from a known distribution. Generally we allow distributions to be *non-stationary* and vary with \( t \), although we also consider the IID special case where these distributions are identical. Stochastic arrival models are motivated by settings with sufficient data to estimate the distribution over types. Meanwhile, in the model with vertex weights, all edges incident to any offline vertex \( u \) must have the same weight. This is motivated by each offline item having its own fixed price that is identical across customers.

### 1.1. Our Contributions

We develop a decoupling framework, which we describe in greater detail in Subsection 1.1.1 below, wherein we first study a simpler, single-customer version of various stochastic matching problems,

\(^1\)This includes everything that is known about the customer at the time of their arrival, including purchase probabilities, patience distribution, edge weights, etc.
and then use the algorithms for these problems to inform decisions during online customer arrivals. This approach allows us to derive new results for online bipartite matching with stochastic rewards and, in many cases, stochastic patience as well.

We consider both vertex weights and general edge weights in combination with the adversarial, non-stationary, and IID arrival models. Since our framework requires the repeated-interaction problems to be solvable for a single customer, we also make advancements on this front (see Subsection 1.1.2); namely, improving algorithms for the cascade-click model of product ranking, and deriving new algorithms in a model where items are arriving over time. Finally, we derive several negative results of interest (see Subsection 1.1.3).

1.1.1. Framework that decouples online matching from single-customer problems.

First we build a framework that takes as input a subroutine for solving the single-customer problem, and outputs an algorithm for the overall multi-customer online matching problem, under the aforementioned variants. The competitive ratios guaranteed by our framework are explained in Table 1 and we would like to highlight a key distinction in our approach. Existing analyses of stochastic rewards (Bansal et al. 2010, Mehta and Panigrahi 2012, Adameczyk et al. 2015, Brubach et al. 2017) all use an LP that is specific to the stochastic rewards matching process, which exhibits a stochasticity gap (see Subsection 6.1). By contrast, our framework uses an abstract LP, in which:

- There is a variable $x_v(\pi)$ for each of the (exponentially-many) policies $\pi$ that could be used for interacting with a single customer of type $v$;
- Each such policy $\pi$ ends up matching each available offline vertex $u$ with probability $p_{uv}(\pi)$;
- There is a single set of constraints tying together the customers over time, which enforce that each offline vertex $u$ is matched at most once in expectation.

Our framework abstracts away the details of the stochastic rewards matching process, deterministic vs. stochastic patience, etc., and holds as long as the policies represent different consumption processesootnote{Similar ideas have appeared in Cheung et al. (2022), who consider abstract “actions” that have different immediate rewards and different consumption distributions over resources. However, their focus is on learning these distributions.} that use up the offline vertices $u$ independently over time according to known probabilities.
Our results, however, are predicated on the existence of a subroutine that can (approximately) solve the repeated-interaction problems for each customer. We will call such a subroutine \( \kappa \)-approximate if given any weights \( w_{uv} \), it finds a policy \( \pi \) whose immediate expected reward \( \sum_u w_{uv} p_{uv}(\pi) \) is at least \( \kappa \) times the maximum possible immediate expected reward over all policies, for some \( \kappa \in [0,1] \). Equipped with a \( \kappa \)-approximate subroutine, our framework provides:

1. A \( \kappa/2 \)-competitive algorithm for vertex weights and adversarial arrivals (Subsection 4.1);
2. A \( \kappa/2 \)-competitive algorithm for edge weights and non-stationary arrivals (Subsection 4.2);
3. A \( (1 - 1/e)\kappa \)-competitive algorithm for edge weights and IID arrivals (Subsection 4.3); and
4. A \( (1 - 1/e)\kappa \)-competitive algorithm for vertex weights and non-stationary arrivals (Subsection 4.4).

The value \( \kappa = 1 \) is possible when \( \theta \) is deterministic (Purohit et al. 2019). We derive new results below showing that \( \kappa = 1 \) is also possible when \( \theta \) follows an (item-dependent) hazard rate model, and that \( \kappa = 1/2 \) is possible when \( \theta \) follows any (item-independent) distribution. This, in conjunction with our framework, justifies all of the results in Table 1.

1.1.2. New \( \kappa \)-approximate subroutines for single-customer problems. As discussed above, it is important for our framework to have \( \kappa \)-approximate subroutines for the repeated-interaction problems with a single customer. We make the following advancements on this front:

1. A 1-approximate (optimal) subroutine, in the model where each item \( i \) has a known hazard rate \( r_i \) and, if seen by the customer and undesired, causes the customer to depart with probability (w.p.) \( r_i \);
2. A 1/2-approximate subroutine, in the model where the customer has an arbitrary known patience distribution (and the probabilities of departing do not depend on the items seen);
3. A 0.027-approximate subroutine, in a new model where the customer has a deterministic patience, but the items are arriving over time according to Bernoulli processes.

The first two models can be motivated by product ranking in e-commerce. A special case of the first model (Subsection 5.1) is where \( r_i \) is equal to some \( r \) for all \( i \), which represents a
Table 1  Landscape of Online Matching Results

| Adversarial | Unweighted | Vertex-weighted | Edge-weighted |
|-------------|------------|-----------------|---------------|
| Non-stochastic | 0.632 (tight) | 0.632 (tight) | [must be weight-dependent] |
| Karp et al. (1990) | Aggarwal et al. (2011) | Ma and Simchi-Levi (2020) |
| Stochastic Rewards | 0.5 | ? → 0.5 | [must be weight-dependent] |
| Mehta and Panigrahi (2012) | | Ma and Simchi-Levi (2020) |
| Deterministic Patience/ Hazard Rate Model | ? → 0.5 | ? → 0.5 | – |
| Stochastic Patience | ? → 0.25 | ? → 0.25 | – |

| Non-stationary | Unweighted | Vertex-weighted | Edge-weighted |
|---------------|------------|-----------------|---------------|
| Non-stochastic | 0.632 | 0.632 | 0.5 |
| Alaei et al. (2012) | Alaei et al. (2012) | Alaei et al. (2012) |
| Deterministic Patience/ Hazard Rate Model | ? → 0.632 | ? → 0.632 | ? → 0.5 |
| Stochastic Patience | ? → 0.316 | ? → 0.316 | ? → 0.25 |

| Known IID | Unweighted | Vertex-weighted | Edge-weighted |
|-----------|------------|-----------------|---------------|
| Non-stochastic | 0.729 | 0.729 | 0.705 |
| Brubach et al. (2020) | Brubach et al. (2020) | Brubach et al. (2020) |
| Stochastic Rewards | 0.632 | 0.632 | 0.632 |
| Brubach et al. (2020) | Brubach et al. (2020) | Brubach et al. (2020) |
| Deterministic Patience/ Hazard Rate Model | 0.46 → 0.632 | 0.46 → 0.632 | 0.46 → 0.632 |
| Brubach et al. (2017) | Brubach et al. (2017) | Brubach et al. (2017) |
| Stochastic Patience | ? → 0.316 | ? → 0.316 | ? → 0.316 |

Landscape of online matching results grouped by arrival model, form of edge weights, and including the unknown patience models we introduce: the (item-dependent) hazard rate model, and the arbitrary (item-independent) stochastic patience model. **Bold** results with arrows show the improvements from this paper, with question marks denoting problems where no prior bound was known.

patience distribution with constant hazard rate $r$, i.e. a customer who departs w.p. $r$ after each position regardless of the item seen. Meanwhile, our $1/2$-approximation for the second model (**Subsection 5.2**) improves the state-of-the-art $1/e$-approximation from Chen et al. (2021) for this cascade-click model of product ranking. Their result also only holds in the special case of increasing hazard rate, while we extend it to general distributions by formulating and rounding a new LP relaxation for this single-customer problem. We note that general patience distributions are well-

\[^3\] However, we acknowledge that their $1/e$-approximation holds against a stronger benchmark that knows the patience in advance. This is only possible under some special cases of the patience distribution: as we show in **Subsection 6.3**
motivated in applications; see e.g. Aveklouris et al. (2021), who study a matching model where items are also arriving over time. On that note, our result for the third model (Subsection 5.3), when plugged into our frameworks, provides constant-factor guarantees in a related model where items (representing contractors in an online labor platform) may not be present at the beginning and need to arrive online after each customer (to acknowledge they can perform the customer’s task), and the customer has to then also accept that contractor. We contrast this new model with other online platform matching models in Section 2.

1.1.3. Negative results. Finally, our work presents three important negative results.

1. We formalize the notion of a stochasticity gap for LP-based approaches to these problems, and construct a stochastic bipartite graph in which even the offline maximum matching has expected size at most 0.544 times the value of the LP relaxation (Subsection 6.1). This means that the competitive ratio from the existing LP-based approaches cannot be better than 0.544, while our framework yields a $1 - 1/e \approx 0.632$-competitive algorithm.

2. We show that the simple family of greedy algorithms introduced in Mehta and Panigrahi (2012) cannot be better than 1/2-competitive (Subsection 6.2).

3. We show that when offering items to a single customer with random patience, if one compares to a benchmark that knows the realization of the patience in advance, then any constant-factor approximation is impossible (Subsection 6.3). Importantly, our counterexample holds even if the customer can be repeatedly offered the same item, which is identical to having an unknown number of opportunities to make a single sale (since the customer will buy at most one item). This is similar in spirit to the negative result derived in Alijani et al. (2020).

2. Further Related Work

Online matching with stochastic rewards. Online matching represents a large literature, which has been surveyed in Mehta (2012). We will describe the portion of this literature that such a result is impossible for the general patience distributions we consider, so our LP relaxation (necessarily) does not know the patience in advance.
focuses on stochastic rewards, where edges only match probabilistically upon being probed. This problem has been studied under both adversarial and stochastic arrival models, as well as different variants depending on the assumptions about edge weights/patience.

Online matching with stochastic edges was introduced in Bansal et al. (2010) as stochastic matching with timeouts (patience), where the authors showed a ratio of 0.12 for known IID arrivals and arbitrary edge weights. This was later improved to 0.46 in Brubach et al. (2017) and to 0.51 in Fata et al. (2019) for some cases. We improve these results by establishing a competitive ratio of 1/2 for non-stationary arrivals and $1 - 1/e$ for IID arrivals. We note that Borodin et al. (2022) concurrently prove these results, differing in three ways: i) they allow for more general constraints on which edges can be probed, beyond a simple patience constraint (although, they do not consider stochastic patience); ii) they compare against a more powerful offline benchmark that can switch back-and-forth between probing different online vertices; iii) they show that $1 - 1/e$ holds in the more general model of non-identical independent draws arriving in a uniformly random order. The same authors have also studied online matching with stochastic edges under the “secretary” model of random-order arrival (see Borodin et al. 2021).

For adversarial arrivals, most work has focused on the unweighted case, initially studied by Mehta and Panigrahi (2012) in the special case where patience $\theta_v$ equals 1 for all $v$. Under the further restriction of uniform vanishing edge probabilities, they showed that a competitive ratio of 0.53 is possible. This was extended to a ratio of 0.534 for unequal, but still vanishingly small probabilities (Mehta et al. 2015). These results were also recently improved to 0.576 and 0.572.

4 Their paper focuses on the offline matching with stochastic edges problem, which we do not consider in this literature review.

5 The techniques in Brubach et al. (2017) also involved solving a star graph problem with a black box. However, that work first solved an LP for a bipartite graph, and then used a black box probing algorithm to essentially round and probe the LP solution on the induced star graphs of arriving vertices. This differs from our work which uses algorithms for stochastic matching on star graphs as black boxes to solve a more sophisticated LP, then use that LP solution to guide the online algorithm.
respectively by Huang and Zhang (2020) and then to 0.596 for both models by Goyal and Udwani (2020); however, all these results focus on the case of vanishingly small probabilities, do not consider patience values greater than 1, and do not consider vertex weights. For arbitrary edge probabilities, general deterministic patience values, and vertex weights, our guarantee of 0.5 is the best-known. There is also a hardness result in Mehta and Panigrahi (2012) which shows that no algorithm for stochastic rewards with adversarial arrivals can achieve a competitive ratio greater than 0.62. This quantity is strictly less than $1 - \frac{1}{e}$, although we argue that this difference is artificially caused by the stochasticity gap, as we explain in Subsection 6.1.

Golrezaei et al. (2014) study another model of stochastic rewards, in which when a vertex $v$ (viewed as a customer) arrives online, an online algorithm chooses a set $S$ of potential matches for $v$ (viewed as an offering of products to the customer). Each customer (online vertex) has a general choice model which specifies the probability of the customer purchasing each item when offered each possible set of product assortments $S$. We contrast this model in more detail in Appendix B, but note that in this setting, a set of potential matches is chosen all at once rather than probed sequentially, with the outcome being determined by full set $S$ (the offered product assortment).

**Large starting capacities.** We do not study how our guarantees improve if there are at least $k$ copies of every offline vertex, although we believe our frameworks could be expanded to do so. The state-of-the-art for these $k$-dependent guarantees in online matching can be found for: adversarial arrivals (Ma and Simchi-Levi 2020), unweighted adversarial arrivals (Kalvanasundaram and Pruhs 2000), non-stationary arrivals with vertex weights (Alaei et al. 2012), general non-stationary arrivals (Jiang et al. 2022), and IID arrivals (Ma et al. 2021).

**Cascade-click models in product ranking.** We turn our literature review to papers that study the repeated-interaction/product ranking problems for a single customer. Our result from Subsection 5.1 shows how to optimally solve this problem under constant hazard rate, a special case of interest in Chen et al. (2021). Our result in Subsection 5.2 improves their guarantee and holds for general patience distributions. We should note that our results do not directly apply to
more general cascade-click models (see Kempe and Mahdian 2008) where the probability of the customer running out of patience depends on the specific item shown, but we believe that our simple LP-based technique in Subsection 5.2 could be useful for these generalized models. Other generalized ranking problems involving choice models are studied in Derakhshan et al. (2018).

In the related sequential assortment problem, multiple products can be shown to the customer at a time. The customer chooses between them according to a Multi-Nomial Logit (MNL) choice model (instead of independent click probabilities), and alternatively the customer could choose the option of viewing the next assortment, never to return. There is a constraint that the same product cannot be shown in different assortments. This problem is typically studied when the number of stages is deterministic and known (see Feldman and Segev (2019) and the references therein); however, the stage-dependent coefficients in Feldman and Segev (2019) can be used to capture our notion of a stochastic patience. Nonetheless, the PTAS derived in Feldman and Segev (2019) does not subsume our 1/2-approximation because: in the sequential assortment problem there is no constraint on the number of products offered at once, hence it does not capture our problem; also, in a PTAS, to get a $(1 - \varepsilon)$-approximation the runtime needs to be exponential in $1/\varepsilon$, whereas our LP-based technique has polynomial runtime independent of any error parameter.

Online matching where items arrive over time. Motivated by online platforms, many models where items arrive over time have been recently studied, with constant-factor approximations (Aouad and Sarataç 2022, Kessel et al. 2022) and optimal algorithms (Aveklouris et al. 2021, Kerimov et al. 2021) known under certain regimes. These papers focus on steady-state behavior, which is possible because items are arriving indefinitely. Our paper contrasts these models because there is still a finite supply of items; they merely need to “arrive” to acknowledge each customer, and we provide a constant-factor approximation for any finite time horizon and market size.

3. Problem Definition and Notation

We use $G = (U, V; E)$ to denote a bipartite graph with vertex set $U \cup V$ and edge set $E \subseteq U \times V$. Let $U = \{u_1, \ldots, u_m\}$ represent offline vertices and $V = \{v_1, \ldots, v_n\}$ represent online vertices. For
an edge $e = (u,v)$, we denote the \textit{weight} of edge $e$ by $w_e$ or $w_{u,v}$; for the special case of vertex weights, each offline vertex $u_i$ has a weight denoted by $w_i$, and $w_{u_i,v} = w_i$ for all $v \in V$. We will generally consider \textit{stochastic edges}, which means that for each edge $(u_i, v_j) \in U \times V$, there is a known probability $p_{i,j}$ with which that edge with independently exist when probed.

When considering the online matching problem for a single online vertex (customer) $v$, we will refer to it as a \textit{star graph}. In this case, we simplify notation and write $p_i$ to denote the probability of edge $(u_i, v)$. We also use $p_{u,v}$ for the given probability of edge $(u, v)$ when indices $i$ and $j$ are not required. Without loss of generality, we may assume that $p_{u,v}$ is defined even for $(u, v) \notin E$, since in this case we can simply let $p_{u,v} = 0$.

We are further given a \textit{patience} value $\theta_{v}$ for each online vertex in $V$ (we may also write $\theta_j$ for the patience of vertex $v_j \in V$) that signifies the number of times we are allowed to probe different edges incident on $v$ when it arrives. Each edge may be probed at most once and if it exists, we must match it and stop probing (probe-commit model).

We consider the online vertices arriving at positive integer times. In the adversarial arrival model, the vertices of $V = \{v_1, v_2, \ldots, v_n\}$ are fixed and the order of their arrival is set by an adversary so as to minimize the expected matching weight. We assume (wlog) that the vertices arrive in the order $v_1, v_2, \ldots, v_n$. When we consider the stochastic arrival models, $V$ instead specifies a set of vertex \textit{types}, and at time $t$, a vertex of an independently randomly chosen type from a known distribution arrives. Generally these distributions can vary across time, which we call the \textit{prophet arrival} model; we also consider the special case where these distributions are identical, which we refer to as \textit{IID arrivals}. In these models, we will let $T$ denote the length of the time horizon which is assumed to be known (otherwise the problem is impossible; see Subsection \[5.3\]).

When an online vertex $v$ arrives at time $t$, we attempt to match it to an available offline vertex. We are allowed to probe edges incident to $v_t$ one-by-one, stopping as soon as an edge $(u_i, v_t)$ is found to exist, at which point the edge is included in the matching and we receive a reward of $w_i$.

\[6\text{ This is because it was the arrival model of original focus in prophet inequality papers (Krengel and Sucheston 1977).}\]
We are allowed to probe a maximum of $\theta_t$ edges (in the stochastic patience models, $\theta_t$ is not known in advance and is discovered only after $\theta_t$ failed probes); if $\theta_t$ edges are probed and none of the edges exist, then vertex $v_t$ remains unmatched and we receive no reward. If we successfully match $v_t$ to $u_t$, we say that $w_i$ is the value or reward of $v_t$’s match; if $v_t$ remains unmatched, we say it has a value or reward of 0. The next online vertex $v_{t+1}$ does not arrive until we have finished attempting to match $v_t$ (either by exhausting the patience constraint, or by successfully matching $v_t$). Thus, there is only ever one online vertex available for matching at any one time.

We use $G$ to denote an instance, which includes the graph, weights, edge probabilities, and any arrival distributions including patience. Given any instance $G$ one can consider an optimal offline algorithm, which knows in advance the online vertices that will arrive. For the adversarial arrival model, this means knowing the sequence $v_1, v_2, \ldots, v_n$; for the stochastic arrival models, this means knowing the sequence of $T$ types that will be realized. We let $\text{OPT}(G)$ denote the expected reward collected by the best sequential probing algorithm on $G$ that has access to this offline information, noting that: i) $\text{OPT}(G)$ does not know the realizations of the stochastic edges in advance either; ii) computing $\text{OPT}(G)$ is difficult but unnecessary; iii) for stochastic arrival models, this expectation is also over the realizations of the $T$ types; and iv) we assume that the offline algorithm must finish the interactions with one online vertex before moving to the next. Meanwhile, we let $\text{ALG}(G)$ denote the expected reward collected by a fixed online algorithm on $G$, again taking a realization over types in the stochastic arrival models, and any potential randomness in the algorithm.

With this understanding, we say that a fixed (potentially randomized) online algorithm is $c$-competitive if $\text{ALG}(G)/\text{OPT}(G) \geq c$ for all instances $G$, where $c$ is a constant in $[0,1]$. We are interested in the maximum value of $c$ for which an algorithm can be $c$-competitive, which is referred to as the competitive ratio.

\footnote{We note that Borodin et al. (2022) derive results against a stronger benchmark, which can switch back-and-forth between online vertices.}
3.1. Outline for the Rest of the Paper

Our main algorithms and results for online matching with stochastic edges are presented in Section 4. In that section, we first present an algorithm for the vertex-weighted case, under adversarial arrivals, and show that it is $1/2$-competitive. To our knowledge, this is the first result for this setting. In addition, we provide an algorithm for the edge-weighted case, under prophet arrivals. Here too, we are able to show the algorithm is $1/2$-competitive; we further show that a slight modification can improve the competitive ratio to $1 - 1/e$ when either the edge-weighted assumption is relaxed to vertex weights or the non-stationary assumption is relaxed to known IID arrivals.

All of the algorithms of Section 4 rely on utilizing, as a black box, an algorithm for the simpler problem of a star graph, which corresponds to a single online customer. For this problem, when the patience of the customer is known, there is an optimal algorithm based on dynamic programming due to Purohit et al. (2019). However, the results in Section 4 are stated in an abstract general manner, which allows us to swap out the algorithm of Purohit et al. (2019) for algorithms solving the star graph problem under different settings of patience. In Section 5, we introduce new, stochastic, models for the patience of the customer, and give algorithms for these new settings. These new star graph algorithms can then be used as black boxes in the algorithms of Section 4, giving results for online matching under these new patience models.

Finally, in Section 6, we present our negative results following the order described earlier.

4. Algorithms for Online Matching with Stochastic Edges

In this section, we present our results for online matching with stochastic edges. Recall that in this problem, the items $U$ are known in advance while the customers arrive one by one in an online fashion. When a customer of type $v$ arrives, we learn the probability $p_{u,v}$, for each $u \in U$, that the customer will purchase item $u$ if offered; if purchased, we gain some reward specified by $w_{u,v}$, and if not, we may proceed to offer another item up to a total of $\theta_v$ offers. In the simplest setting, $\theta_v$ is known to the algorithm, although our framework can also handle settings where only a probability distribution over $\theta_v$ is known (see Section 5). An online algorithm must make offer items sequentially to the customer, and must make all its offers before the next one arrives. The goal is to maximize the expected total reward across all customers.
4.1. Vertex-Weighted, Adversarial Arrivals

We present a greedy algorithm AdvGreedy which achieves a 0.5-approximation for online matching with vertex weights, stochastic rewards, and patience constraints in the adversarial arrival model. Recall that in this setting, each offline item \( u_i \in U \) has a weight \( w_i \) such that \( w_{u_i, v} = w_i \) for all customers \( v \in V \), modeling the situation where items have a fixed price that is the same for all customers. If vertex \( v \) is the \( t^{th} \) vertex to arrive online, we say \( v \) arrives at time \( t \). Recall that the goal of the problem is to offer items to customer \( v_t \) one by one, until either the patience runs out or an item is successfully sold; and the entirety of this process is carried before the \((t+1)^{st}\) arrival, \( v_{t+1} \), arrives. The goal is to maximize the total revenue across all arrivals. Our algorithm makes use of a black box subroutine, StarBB, which takes as input a single online vertex, along with the probabilities and weights of its incident edges; StarBB(\( v, p, w \)) simply probes edges incident to online vertex \( v \) in some order, until either \( v \)'s patience is exhausted or a match is successful. Our results hold as long as StarBB is optimal, or within a constant factor \( \kappa \) of optimal. When the patience is deterministic and known to the algorithm, we can use the dynamic programming-based algorithm of Purohit et al. (2019) for StarBB; the dynamic program gives a sequence of \( \theta_v \) vertices in \( U \) to probe in this order. This is optimal for star graphs, so \( \kappa = 1 \). In Section 5, we extend this online stochastic matching result to settings where the patience is unknown and stochastic, by giving star graph algorithms for these settings and proving constant factor approximations for them.

Algorithm 1: Use a Star Graph Black Box to greedily match arriving vertices

Function AdvGreedy(\( U, V, p, w \)):

\[
\text{for Arriving vertex } v \in V \text{ do} \\
\quad \text{StarBB}(v, p, w)
\]

Let ALG(\( G \)) denote the expected size of the matching produced by this algorithm on the graph \( G \). Let OPT(\( G \)) denote the expected size of the matching produced by an optimal offline algorithm. Our main result here is:
Theorem 1. Given a $\kappa$-approximate black box for solving star graphs, Algorithm 1 achieves a competitive ratio of $0.5\kappa$; that is, for any bipartite graph $G$, $\frac{\text{ALG}(G)}{\text{OPT}(G)} \geq 0.5\kappa$.

To prove this, we first present an LP which provides an upper bound on the offline optimal.

4.1.1. An LP upper bound on $\text{OPT}(G)$. We formulate a new LP for our problem by adding a new constraint to, (1d), to the standard LP relaxation of the problem. This new LP gives a tighter upper bound on the offline optimal solution, $\text{OPT}(G)$. Note that our algorithm does not need to solve this new LP, as it is only used in our analysis.

$$\text{OPT}_{LP} := \max \sum_{u \in U} \sum_{v \in V} x_{u,v} p_{u,v} w_u$$ (1)

subject to

$$\sum_{v \in V} x_{u,v} p_{u,v} \leq 1 \quad \forall u \in U$$ (1a)

$$\sum_{u \in U} x_{u,v} p_{u,v} \leq 1 \quad \forall v \in V$$ (1b)

$$\sum_{u \in U} x_{u,v} \leq E[\theta_v] \quad \forall v \in V$$ (1c)

$$\sum_{u \in U'} x_{u,v} p_{u,v} w_u \leq \text{OPT}(U', v) \quad \forall U' \subseteq U, v \in V$$ (1d)

$$0 \leq x_{u,v} \leq 1 \quad \forall u \in U, v \in V$$

In this LP, we slightly abuse notation and write $\text{OPT}(U', v)$ to denote $\text{OPT}(G')$ for a star graph $G' = (U', \{v\}, U' \times \{v\})$. Recall that $\text{OPT}(U', v)$ can be computed by a black box.

We first show in Appendix A that our strengthened LP is a valid upper bound.

Lemma 1. For any bipartite graph $G$, $\text{OPT}_{LP}(G) \geq \text{OPT}(G)$.

4.1.2. Proof of 0.5-competitiveness. We can then bound the performance of our greedy algorithm relative to the solution of Linear Program (1).

Lemma 2. If StarBB is a $\kappa$-approximate algorithm for star graphs, then for any bipartite graph $G$, $\text{ALG}(G) \geq 0.5\kappa \text{OPT}_{LP}(G)$. 
By Lemmas 1 and 2, we have \( \text{ALG}(G) \geq 0.5\kappa\text{OPT}_\text{LP}(G) \geq 0.5\kappa\text{OPT}(G) \), which implies our main result of a \( \frac{\kappa}{2} \)-competitive algorithm. As previously mentioned, using the probing order given by the dynamic program of Purohit et al. (2019) as STARBB gives \( \kappa = 1 \), so we have a \( \frac{1}{2} \)-approximation. In Section 5, we present star graphs for stochastic patience settings, where \( \kappa \) is not necessarily 1.

4.2. Prophet Arrival Setting

If we wish to allow for arbitrary edge weights, we must consider a different arrival model. A popular arrival model in the literature is the known IID setting, as discussed in Section 2; here, we consider a generalization of the known IID setting, which we call the prophet arrival model. In this model, \( V \) specifies a set of possible arrival types; each arrival takes on one of these types randomly, according to a known distribution. The probabilities at each arrival are independent of previous arrivals, and the distribution over possible types can be different at each arrival.

For \( t = 1, 2, \ldots, T \) and \( v \in V \), denote by \( q_{tv} \) the probability that the vertex arriving at time \( t \) will be of type \( v \). For convenience, we denote by \( q_v = \sum_{t=1}^{T} q_{tv} \) the expected number of arrivals of a vertex of type \( v \).

We employ a new exponential-sized LP relaxation. In this LP, the variables correspond to policies for probing an arriving online vertex. A deterministic policy \( \pi \) for matching any online vertex type \( v \) is characterized by a permutation of some subset of \( U \). The policy specifies the strategy of attempting to match \( v \) to vertices of \( U \) in the order given by \( \pi \), until either a probe is successful, all vertices in \( \pi \) are attempted, or the patience of \( v \) is exhausted. Let \( \mathcal{P} \) denote the set of all deterministic policies.

We present our LP in (2) below. We let \( p_{uv}(\pi) \) denote the probability that an online arrival of type \( v \) is matched to offline vertex \( u \) when following policy \( \pi \), assuming that all vertices of \( \pi \) are still unmatched. The decision variables in the LP are given by \( x_v(\pi) \), and we let \( \text{OPT}_{\text{LP}}(G) \) denote the true optimal objective value of the exponential-sized LP.

\[
\text{OPT}_{\text{LP}} = \max \sum_{v \in V} \sum_{\pi \in \mathcal{P}} x_v(\pi) \sum_{u \in U} p_{uv}(\pi) w_{uv}
\]
subject to \[ \sum_{v \in V} \sum_{\pi \in P} x_v(\pi)p_{uv}(\pi) \leq 1 \quad \forall u \in U \] \hspace{1cm} (2a)

\[ \sum_{\pi \in P} x_v(\pi) = q_v \quad \forall v \in V \] \hspace{1cm} (2b)

\[ x_v(\pi) \geq 0 \quad \forall v \in V, \pi \in P \] \hspace{1cm} (2c)

We can interpret \( x_v(\pi) \) as the expected number of times policy \( \pi \) will be applied to an online vertex of type \( v \). Constraint (2a) says that in expectation each offline vertex \( u \) can be matched at most once. Constraint (2b) comes from the fact that exactly one policy (possibly the policy that makes zero probes) must be applied to each arriving vertex of type \( v \). It follows from standard techniques (see e.g. Bansal et al. 2010, Lemma 9) that even for a clairvoyant, who knows the realized types of arrivals in advance, if we let \( x_v(\pi) \) denote the expected number of times it applies policy \( \pi \) on an online vertex of type \( v \), then this forms a feasible solution to the LP with objective value equal to the clairvoyant’s expected weight matched. Therefore, if we can bound the algorithm’s expected reward relative to \( \text{OPT}_{\text{LPP}}(G) \), then this would yield a competitive ratio guarantee.

4.2.1. Solving the exponential-sized LP. The Linear Program (2) has a variable \( x_v(\pi) \) for every possible online vertex type \( v \) and policy \( \pi \in P \). If \( P \) has polynomial size, this LP therefore has polynomial size. This happens, for example, when all online vertices have a small constant patience. For instance, when the patience of all vertices is 1, a policy \( \pi \in P \) is characterized by a single offline vertex and so \( |P| = |U| \).

However, for general patience values the LP has exponentially many variables. Nonetheless, since there are only a polynomial number of constraints, a sparse solution which is polynomially-sized still exists. To help in solving the LP, we consider the dual.

\[
\begin{align*}
\text{minimize} & \quad \sum_{u \in U} \alpha_u + \sum_{v \in V} q_v \beta_v \\
\text{s.t.} & \quad \sum_{u \in U} p_{uv}(\pi)\alpha_u + \beta_v \geq \sum_{u \in U} p_{uv}(\pi)w_{uv} \quad \forall v \in V, \pi \in P \\
& \quad \alpha_u \geq 0 \quad \forall u \in U
\end{align*}
\] \hspace{1cm} (3)

Note that the exponential family of constraints (3a) can be re-written as

$$\max_{\pi \in \mathcal{P}} \sum_{u \in U} p_{uv}(\pi)(w_{uv} - \alpha_u) \leq \beta_v, \quad \forall v \in V$$ (4)

which is equivalent to, for every online type $v \in V$, solving the probing problem for a star graph consisting of a single online vertex of type $v$ and adjusted edge weights $w'_{uv} = w_{uv} - \alpha_u$.

Recall that if for every online type $v \in V$, we have a black box which can solve the maximization problem in (4) to optimality, then we have a separation oracle for the dual LP which can either establish dual feasibility or find a violating constraint given by $v$ and $\pi$. By the equivalence of separation and optimization, this allows us to solve both the dual LP (2) and in turn the primal LP (3) using the ellipsoid method, as formalized in the proposition below.

**Proposition 1.** Given black box star graph algorithms for every online vertex type which can verify (4) in polynomial time, the exponential-sized LP (2) can be solved in polynomial time.

An explicit statement which directly establishes Proposition 1 can be found in Chapter 14 of Schrijver (1998). We should note that the ellipsoid method is only used to establish poly-time solvability in theory, and that a column generation method which adds primal variables $x_v(\pi)$ as needed on-the-fly is usually more efficient in practice.

In either case, while the dynamic programming approach of Purohit et al. (2019) gives us an optimal algorithm for verifying (4) in polynomial time, for the setting of general stochastic patience described in Subsection 5.2 the algorithm we present is only a 1/2-approximation for the star graph problem. Nonetheless, one can still use the structure of the dual LP, along with this approximate separation, to compute a $(1/2 - \epsilon)$-approximate solution to the primal LP. This is formalized in the proposition below.

**Proposition 2.** Suppose for every type $v \in V$, we are given a black box which is a $\kappa$-approximation algorithm for the maximization problem in (4). Then a solution to LP (2) with objective value at least $(\kappa - \epsilon)\text{OPT}_{LPP}(G)$ can be computed, in time polynomial in the problem parameters and $1/\epsilon$. 
Proposition 2 follows from the potential-based framework of Cheung and Simchi-Levi (2016), which is elaborated on in Appendix C. Armed with Propositions 1 and 2, we can use any optimal or approximately optimal algorithm for star graphs to obtain an (approximately) optimal solution to LP (2), which we can then use to solve the problem of online matching under prophet arrivals.

4.2.2. Algorithm and analysis based on exponential-sized LP. We now show how to use the LP (2) to design a $\kappa/2$-competitive online algorithm, given a feasible LP solution $x^*_v(\pi)$ which is at least $\kappa \cdot \text{OPT}_{LPP}(G)$, for some $\kappa \leq 1$. For each vertex $u \in U$, let $w_u^* = \sum_{v \in V} w_{uv} \sum_{\pi \in P} p_{uv}(\pi) x^*_v(\pi)$ denote the expected reward of matching $u$ according to the assignment $x^*$. Notice that the objective value of the given solution is $\sum_{u \in U} w_u^*$, which is at least $\kappa \cdot \text{OPT}_{LPP}(G)$. Using this notation, our algorithm is given in Algorithm 2. By LP constraint (2b), we have $\sum_{\pi \in P} x^*_v(\pi)/q_v = 1$, so the probability distribution over policies defined in Algorithm 2 is proper. We also note that since at most polynomially many variables $x^*_v(\pi)$ will be nonzero, this distribution has polynomial-sized support and can be sampled from in polynomial time. The algorithm itself is fairly simple, selecting a policy $\pi$ at random with probability proportional to LP solution $x^*_v(\pi)$ upon the arrival of a vertex $v_t$ of type $v$. Then, policy $\pi$ is followed in order to attempt to match the online vertex, but with two quirks: first, we skip probing any edge $(\pi_i, v_t)$ for which the weight of the edge is too small (specifically, if $w_{\pi_i, v} < w_{\pi_i}^*/2$), and second, if $\pi$ tells us to probe an edge $(\pi_i, v_t)$ for which offline vertex $\pi_i$ is unavailable, we “simulate” the probing instead, terminating with no reward if the simulated probe is successful. This simulation technique was used in Brubach et al. (2017) and is also used by our star graph algorithm in Subsection 5.2.

**Theorem 2.** Under general edge weights and known arrival distributions, Algorithm 2 is $\kappa/2$-competitive for the online bipartite matching with patience problem, assuming we are given a solution to LP (2) with objective value at least $\kappa \cdot \text{OPT}_{LPP}(G)$.

The proof of Theorem 2 is deferred to Appendix A. In the case of deterministic, known patience, the algorithm of Purohit et al. (2019) can be used as the black box star graph algorithms to allow verifying (4), and thus solving LP (2) exactly. In this case, $\kappa = 1$ in Theorem 2, and we have a
1/2-competitive algorithm for online matching with patience, under prophet arrivals and arbitrary edge weights. In Section 5 we extend the classical online stochastic matching problem to consider stochastic patience, and give star graph algorithms with constant factor approximation guarantees, allowing us to apply Theorem 2 to these settings as well.

4.3. Improvement for IID Arrivals

In the case of IID arrivals, i.e., where $q_{1v} = q_{2v} = \cdots = q_{Tv}$ for all vertex types $v \in V$, a slight modification to Algorithm 2 yields a competitive ratio of $(1 - 1/e) \kappa$ when given a feasible solution to LP 2 that is at least $\kappa \cdot \text{OPT}_{LPP}(G)$ of optimal.

The only change to the algorithm from the previous non-IID setting is that for IID arrivals, we do not skip probing vertices $u \in U$ when $w_{uv} < w_{uv}^*/2$. The full pseudocode is given in Algorithm 3.
**Algorithm 3:** Random Arrivals from Known Identical Distributions (IID Arrivals)

```plaintext
Function OnlineMatch(U, V, p, w):
    for t = 1 to T do
        Online vertex v_t, of type v ∈ V, arrives
        Choose a policy π = (π_1, π_2, ..., π_ℓ) with probability x_v^*(π)/q_v
        for i := 1 to ℓ do
            if π_i is unmatched then
                Probe edge (π_i, v_t) and match if successful for reward w_{π_i,v_t}
            else
                Simulate probing (π_i, v_t). If successful, move to next arrival without matching v_t.
```

**Theorem 3.** Under general edge weights and known IID arrivals, Algorithm 3 is κ(1 − 1/e)-competitive for the online bipartite matching with patience problem, assuming we are given a solution to LP (2) with objective value at least κ·OPT_{LPP}(G).

The proof of Theorem 3 is deferred to Appendix A. As with Theorem 2, the dynamic program of Purohit et al. (2019) can be used to get an optimal LP solution, so that Theorem 3 gives a competitiveness of 1 − 1/e for Algorithm 3 under the classical deterministic patience setting. Section 5 extends the result to online stochastic matching with stochastic patience, by providing black box algorithms for those settings which can then be used to find approximately optimal solutions to LP (2) as per Proposition 2.

**4.4. Improvement for Vertex Weights**

With a new analysis, we can show that Algorithm 3 still achieves a competitive ratio of 1 − 1/e in the prophet (non-stationary) setting in the case of vertex weights.
Theorem 4. Under vertex-weights and known arrival distributions, Algorithm \( \mathcal{A} \) is \( \kappa(1 - 1/e) \)-competitive for the online bipartite matching with patience problem, assuming we are given a solution to LP \( (2) \) with objective value at least \( \kappa \cdot \text{OPT}_{LPP}(G) \).

The proof of Theorem 4 is deferred to Appendix A. It has identical implications as Theorem 3, with the improvement beyond the \( 1/2 \)-guarantee of Theorem 2 coming from the vertex-weighted assumption instead of the IID assumption.

5. Algorithms for Star Graphs

All of the algorithms in Section 4 make use of a black box algorithm for solving the special case of a single customer (called the “star graph” problem, since it corresponds to a star graph with the customer as the center vertex): Under adversarial arrivals, the black box is used for each arriving vertex, while for prophet arrivals it is used as a separation oracle for the dual LP. For the classic problem of online matching with stochastic edges and finite patience, we may use the dynamic program of Purohit et al. (2019) as this black box. Since this dynamic program is optimal for star graphs, \( \kappa = 1 \) in all of the results of Section 4. In this section, we consider the problem where the patience of the customer is not known, but is instead determined by some stochastic process. These algorithms can be used as black boxes for the algorithms in Section 4 giving competitive ratios for online matching with stochastic edges (with multiple customers) under these new stochastic patience settings. Finally, we also give an algorithm for an alternate setting where the customer has a deterministic patience, but items arrive over time according to Bernoulli processes. Throughout, we use \( m \) to denote the number of items, as in section 3.

5.1. Constant Hazard Rate

In this setting, the patience is random and unknown, with a constant “hazard rate” \( r_i \) for each \( i \in [m] \). When an attempted match with \( u_i \) is unsuccessful, the customer \( v \) runs out of patience with probability \( r_i \) and remains available for another match attempt with probability \( 1 - r_i \). When

\(^8\) Throughout this paper, we use the notation \([m] := \{1, 2, \ldots, m\}\)
the hazard rate $r_i$ is the same for all $i \in [m]$ (i.e., $r_i = r$ for some $r$ and every $i$), this is equivalent to the patience having a constant hazard rate of $r$.

**Theorem 5.** For maximizing the expected weight of the matched item in the (item-dependent) Constant Hazard Rate patience model, it is optimal to probe items in decreasing order of

\[
\frac{w_ip_i}{p_i + (1 - p_i)r_i}.
\]

Theorem 5 tells us that, in the special case where the patience distribution can be modeled with individual per-item hazard rates, we can achieve an optimal star graph probing strategy. As such, we can apply all of the algorithms and results from Section 4 with $\kappa = 1$.

### 5.2. Arbitrary Patience Distributions

We address the case where the patience is stochastic (and unknown) and can follow an arbitrary known distribution. Without loss of generality, we can assume the patience distribution has finite support (more specifically, that the patience $\theta \in [m]$); this is because a patience greater than $m$ is equivalent to a patience of $m$ (additionally, a patience of 0 indicates a vertex which can never be matched by any algorithm, since no probes can be made, so we can simply ignore such vertices).

We use an LP-based approach for this problem. We denote by $q_\theta$ the probability that the patience of the online vertex $v$ is at least $\theta$. Notice that $1 = q_1 \geq q_2 \geq \ldots \geq q_n$. Our approach utilizes the LP (5) described in the next paragraph. The variables are $x_{j\theta}$, which correspond to the probability of attempting to match with $j$ on the $\theta^{th}$ attempt. The value $s_\theta$ represents the probability that the online vertex is available for a $\theta^{th}$ match attempt, meaning its patience is at least $\theta$ and all previous match attempts were unsuccessful. This can be calculated from the $x_{j\theta}$, $q_\theta$, and $p_j$ values.

\[
\begin{align*}
\max & \sum_{j=1}^{m} w_j p_j \sum_{\theta=1}^{m} x_{j\theta} \\
\text{subject to} & \sum_{\theta' = \theta}^{m} x_{j\theta'} \leq s_\theta, \quad \forall j \in \{1, 2, \ldots, m\}, \theta \in \{1, 2, \ldots, m\}
\end{align*}
\]
\[
\sum_{j=1}^{m} x_{j\theta} \leq s_{\theta}, \quad \forall \theta \in \{1, 2, \ldots, m\} \tag{5b}
\]

\[
x_{j\theta} \geq 0, \quad \forall j \in \{1, 2, \ldots, m\}, \theta \in \{1, 2, \ldots, m\}
\]

\[
s_1 = 1, \tag{5c}
\]

\[
s_{\theta} = \frac{q_{\theta}}{q_{\theta-1}} \left( s_{\theta-1} - \sum_{j=1}^{m} p_j x_{j,\theta-1} \right), \quad \forall \theta \in \{2, \ldots, m\} \tag{5d}
\]

As an example, consider the e-commerce application, where we wish to offer items to a customer one by one. The variable \( x_{j\theta} \) indicates the probability that the customer is offered item \( j \) after \( \theta - 1 \) other items have already been offered (and rejected). This can only happen if the customer’s patience is at least \( \theta \) and the customer has not already purchased an item (prior to the \( \theta \)th offer), since otherwise no offers can be made at this point. The value \( s_{\theta} \) denotes precisely this probability, that is the probability that the customer’s patience is at least \( \theta \) and the customer has rejected all previous offers. This suggests a simple constraint: \( x_{j\theta} \leq s_{\theta} \). However, two stronger conditions can be given: first, no valid strategy can offer item \( j \) at any point on or after the \( \theta \)th attempt, if the customer has patience less than \( \theta \) or purchases an item offered before the \( \theta \)th. Summing over all offers after \( \theta - 1 \), \( \sum_{\theta' = \theta} x_{j,\theta'} \) is the probability that item \( j \) is offered at or after the \( \theta \)th step, and the reasoning above gives us the constraints (5a). Further, if the customer is unavailable (due to having purchased an item or running out of patience) for the \( \theta \)th attempt, then no item can be offered on the \( \theta \)th attempt; thus, the probability of offering an item on attempt number \( \theta \) cannot exceed the probability that the customer is available for the \( \theta \)th attempt. This gives constraints (5b). We note that the family of constraints (5a) differs from similar time-indexed LP’s in the literature (Ma 2014), in that there is a constraint for the sum starting at every attempt \( \theta \) instead of a single constraint where \( \theta = 1 \).

Constraints (5a) and (5d) simply give a closed form expression for the quantities \( s_{\theta} \), where \( \frac{q_{\theta}}{q_{\theta-1}} \) is understood to be 0 if both \( q_{\theta} \) and \( q_{\theta-1} \) are 0. Finally, since \( x_{j\theta} \) corresponds to a probability, we require \( x_{j\theta} \in [0, 1] \) (we do not explicitly write the constraint \( x_{j\theta} \leq 1 \) in the LP above, since it is redundant, being implied by (5b) for \( \theta = 1 \), since \( s_1 = 1 \)). We can see that this LP upper bounds
the optimal algorithm, since taking $x_{j\theta}$ to be the probability of the algorithm probing $j$ on attempt $\theta$ for all $j$ and $\theta$, we get a feasible solution to the LP with objective value equal to the algorithm’s expected weight matched.

Our algorithm is simple: we solve LP (5) to get an optimal solution $x^*$, along with the values $s^*$. Then, when making the $\theta$th probe, choose each offline vertex $j = 1, \ldots, n$ with probability $x^*_{j\theta}/s^*_\theta$ (note that if $s^*_\theta = 0$ then $x^*_{j\theta} = 0$), which defines a proper probability distribution by (5b). If a vertex $j$ is chosen to be probed, but has already been unsuccessfully probed in a previous attempt, we “simulate” probing $j$ instead, and terminate with no reward if the simulated probe is successful. This simulation technique is important in ensuring that the probability of surviving to the $\theta$’th attempt is consistent with the LP value $s^*_\theta$.

**Example 1.** We provide an example to illustrate our LP and algorithm. Let $m = 2$, and suppose there are two items with weights $w_1 = 1, w_2 = 2$ and probabilities $p_1 = 3/4, p_2 = 1/4$.

First suppose $q_1 = q_2 = 1$, i.e. the patience is deterministically 2. Then the non-zero $x$-values in the optimal LP solution are $x_{2,1} = 1, x_{1,2} = 0.75$. This corresponds to probing item 2 (with the higher reward if it succeeds) first, and if the probe fails (occurring w.p. 3/4), probing item 1 afterward. The expected reward is $w_2p_2 + (1-p_2)w_1p_1 = 2/4 + 3/4 \cdot 3/4 = 17/16 = 1.0625$, which is also the optimal objective value of the LP.

Now consider a more interesting example where $q_1 = 1, q_2 = 1/3$. Then the non-zero $x$-values in the optimal LP solution are $x_{1,1} = 0.9, x_{1,2} = 0.1, x_{2,1} = 0.1$. Our algorithm in this case will probe item 1 first w.p. 0.9, and otherwise (w.p. 0.1) probe item 2 first. If it survives to the 2nd probe, which occurs w.p. $s_2 = 0.1$, it will always probe item 1. However, note that this will be a “simulated” probe (which generates no reward) unless item 2 was probed on the 1st probe, making item 1 still available for the 2nd probe. The probability of this occurring is $0.1 \cdot 3/4 \cdot 1/3 = 0.025$. Therefore, the expected reward of our algorithm is $(0.9 + 0.025) \cdot 3/4 + 0.1 \cdot 2/4 \approx 0.743$. Meanwhile, the LP optimal value is 0.8.

We note that in this case, the best algorithm is to probe item 1 first (which is the “safe bet”, given that the customer has a 2/3 chance of departing after the 1st probe) followed by item 2, which
would have expected reward $19/24 \approx 0.791$, worse than the LP value. It would have been better than our algorithm though. However, note that the optimal ordering given many items and an arbitrary patience distribution is generally non-trivial to solve (even in our examples, the optimal ordering switched from 2,1 to 1,2 depending on the patience distribution), and to our knowledge the best approximation algorithm is the 1/2-approximation provided by our randomized algorithm.

**Theorem 6.** The online algorithm based on LP (5) is a 1/2-approximation for the star graph probing problem, for an arbitrary patience distribution which is given explicitly.

The proof of Theorem 6 is in Appendix A; we note that the analysis in the proof is tight.

We further note that the result of Theorem 6 compares to a benchmark (LP (5)) that does not know the full realization of the patience values in advance. This is necessary, since Theorem 10 states that comparing to a benchmark which knows the patience in advance leads to arbitrarily bad competitive ratios.

Theorem 6 allows for us to solve online matching problems when the patience of each customer is stochastic and follows an arbitrary distribution that is known to the algorithm. We simply use our star graph algorithm as a black box for our algorithms in Section 4 with $\kappa = 1/2$. This gives us $1/4$-competitive algorithms for vertex-weighted adversarial arrivals and edge-weighted prophet arrivals; as per Theorems 3 and 4 we have improved competitive ratios of $1/2 (1 - 1/e)$ under known IID arrivals (even with arbitrary edge weights) and vertex-weighted prophet arrivals (even when the distributions are not identical).

### 5.3. Item Arrivals

Next we consider a different setting in which after a customer arrives, the “items” (interpreted as contractors in an online platform) are initially unavailable, and only show up (to acknowledge that they can do the customer’s job) following Bernoulli processes. More specifically, each item $i \in [m]$ has two given probabilities: the matching probability $p_i$ and an arrival probability $q_i$. The customer has a known deterministic patience $\theta$, and the process unfolds in discrete time steps; at
time \( t \in \{1, \ldots, \theta\} \), the algorithm must choose at most one item that has arrived to offer to the customer. After time \( t = \theta \), if no item has been purchased, the customer runs out of patience and becomes permanently unavailable for matches. In contrast to the other patience settings, in this setting the patience corresponds to the *amount of time* the customer is willing to wait, rather than the *number of items* they may be offered. Thus, if the algorithm makes no offer at time \( t \) (because it is waiting for items to become available), we still move one step closer to exhausting the customer’s patience.

When the customer first arrives (immediately prior to time \( t = 1 \)), no items are available to be offered. However, at each time step, each item \( i \) becomes available independently with probability \( q_i \). Once an item \( i \) becomes available to the customer, say initially at time \( t \), then it can be offered to the customer at most once, at any time step \( t' \geq t \). When item \( i \) is offered, the customer purchases it with probability \( p_i \), in which case a weight of \( w_i \) is achieved and the process terminates; with probability \( 1 - p_i \), the item is not purchased and the process immediately proceeds to the next time step. As with all star graph problems, our goal is to develop an algorithm which maximizes the expected weight of the item sold to the customer (achieving a weight of 0 if no item is sold).

We begin with a linear programming relaxation of the problem.

\[
\text{LP} := \max \sum_{i=1}^{m} w_i x_i p_i \tag{6}
\]

subject to

\[
\sum_{i=1}^{m} x_i p_i \leq 1 \tag{6a}
\]
\[
\sum_{i=1}^{m} x_i \leq \theta \tag{6b}
\]
\[
x_i \leq 1 - (1 - q_i)^{\theta} \quad \forall i \in \{1, 2, \ldots, m\} \tag{6c}
\]
\[
x_i \geq 0 \quad \forall i \in \{1, 2, \ldots, m\} \tag{6d}
\]

We call an item “large” if \( q_i \geq c/\theta \). Otherwise, if \( q_i < c/\theta \), we say that item \( u_i \) is “small.” Let \( I_{\text{LARGE}} = \{i \in [m] \mid x_i \geq c/\theta\} \) denote the set of large items, and \( I_{\text{SMALL}} = \{i \in [m] \mid x_i < c/\theta\} \) denote the set of small items. Our algorithm first solves the LP \( \text{LP} \) to obtain an optimal solution \((x_i)_{i \in [m]}\);
then, it makes use of one of two different strategies, choosing between the two depending on relative contribution of large vs small items to the LP objective. The motivation here is as follows: intuitively, we wish to choose to offer an item $i$ with some probability proportional to $x_i$. However, if most of the contribution to the objective value in the LP comes from small items, there may be a high probability of no items arriving in any one time step; in this case, we may be better off simply offering any item that arrives in a time step where we are lucky enough to have an arrival.

**The LARGE strategy.** First, let $\rho \in (0, \frac{1}{2})$ be a fixed parameter. Our strategy $\pi_{\text{LARGE}}$ does the following: at each time step, we select an item at random. When selecting a random item, we choose item $i$ with probability $x_i/\theta$. With probability $1 - \sum_{i=1}^{m} x_i/\theta$, we select no item. It follows from constraint (6b) that this forms a valid distribution.

The algorithm selects this item at random, and if the item has arrived and has not yet been offered, we offer it to the buyer with probability $\rho$ (and with probability $1 - \rho$ we make no offer).

**The SMALL strategy.** Our strategy $\pi_{\text{SMALL}}$ does the following: At each time step, if at least one small item arrives in that step, choose one of the small arrivals at random and offer it to the buyer. We ignore large items. Any small item which arrives and is not chosen is permanently discarded (i.e., it will never be offered to the buyer).

**The full algorithm.** We fix a parameter $\varphi \in (0, 1)$. First, if $\theta = 1$, we simply take the optimal choice, offering the item with the highest expected reward among items that arrived. Then, for $\theta \geq 2$, if $\sum_{i \in I_{\text{LARGE}}} w_i x_i p_i \geq (1 - \varphi)\text{LP}$, we use strategy $\pi_{\text{LARGE}}$ at every time step. Otherwise, $\sum_{i \in I_{\text{SMALL}}} w_i x_i p_i > \varphi\text{LP}$, and we use $\pi_{\text{SMALL}}$ at every time step.

We show that this algorithm is a 0.027-approximation for the problem. This is done by considering the two cases (corresponding to using the LARGE and SMALL strategies) separately.

**Lemma 3.** If $\sum_{i \in I_{\text{LARGE}}} w_i x_i p_i \geq (1 - \varphi)\text{LP}$, then $\pi_{\text{LARGE}}$ achieves an expected matching weight of $(1 - \varphi)\rho(1 - 2\rho)\text{LP}$.

**Lemma 4.** If $\sum_{i \in I_{\text{LARGE}}} w_i x_i p_i < (1 - \varphi)\text{LP}$, then $\pi_{\text{SMALL}}$ achieves an expected matching weight of at least

$$\varphi \left(1 - \frac{c}{2}\right) \frac{2}{1 - (1 - c/2)^2} \left(\frac{1}{c} - \frac{e^{-c}}{1 - e^{-c}}\right)\text{LP} \quad (7)$$
Using the bounds for both the LARGE and SMALL strategies, we can now give our final result.

**Theorem 7.** For an appropriate choice of parameters $\varphi, \rho, c$, our algorithm is a $0.027$-approximation.

Lemmas 3 and 4 and Theorem 7 are proved in Appendix A. Our result for this setting can be used as a black box for a new kind of online stochastic matching problem with two-sided arrivals, where after the arrival of each customer, all items (contractors in an online labor platform) are initially unavailable, and arrive over time following Bernoulli processes (when they “discover” the customer’s task). These acknowledgments “reset” after each customer, who presents a new job, and we note that that each (contractor, customer)-pair can have a different rate for its Bernoulli process of the contractor arriving, as well as a different probability for the customer accepting that contractor. A contractor, once matched, spends the time horizon (e.g. one day) doing that task and hence never returns. For such an online matching problem, we may use our strategy as a black box for the algorithms of Section 4, where we have $\kappa = 0.027$, to get a constant-factor competitive ratio for any finite market size and time horizon.

6. **Negative Results**

6.1. **Stochasticity Gap**

The *stochasticity gap* is a fundamental gap in Linear Programming relaxations for stochastic problems which replace probabilities with deterministic fractional weights. The notion was first discussed informally in Brubach et al. (2017), and was later also observed by Purohit et al. (2019) (where they referred to it as a “probing gap”). When these LP relaxations are used as upper bounds on the offline optimal solution, or as a benchmark for the competitive ratio, the stochasticity gap represents a barrier to the best achievable competitive ratio. One interpretation of such a result is that better competitive ratios are not possible. However, one may alternatively view it as a result showing the limitations of using a particular LP as a benchmark for competitive ratios.

We present a stochasticity gap for a common LP relaxation of the online matching problem with stochastic rewards. Recall from Subsection 4.1.1 that LP (1) with the constraints (1a)–(1c)
(and excluding our additional family of constraints \(1d\)) is a standard LP relaxation for bipartite matching with (known) patience constraints and adversarial arrivals. This is essentially an extension of the “Budgeted Allocation” LP from Mehta and Panigrahi (2012) to include the patience constraints. For convenience, we reproduce this standard LP below in the vertex-weighted setting.

$$\begin{align*}
\text{max} & \sum_{u \in U} \sum_{v \in V} x_{u,v} p_{u,v} w_u \\
\text{subject to} & \sum_{v \in V} x_{u,v} p_{u,v} \leq 1 \quad \forall u \in U \\
& \sum_{u \in U} x_{u,v} p_{u,v} \leq 1 \quad \forall v \in V \\
& \sum_{u \in U} x_{u,v} \leq \mathbb{E}[\theta_v] \quad \forall v \in V \\
& 0 \leq x_{u,v} \leq 1 \quad \forall u \in U, v \in V
\end{align*}$$

Simple LP formulations like this, while useful, can give too large of an upper bound on the performance of any offline algorithm, and thus make it difficult to get larger competitive ratios. As such, more complex (and, often, exponentially-sized) LPs have been used in recent work (see, e.g., Gamlath et al. (2019)) to achieve better results. Our LP-based techniques in Section 4 use different exponential-sized LPs to overcome the limitations of stochasticity gaps.

We start with a simple example demonstrating the notion of stochasticity gap, where the bipartite graph has a single offline vertex \(u\), and \(n\) online vertices arriving in any order. Suppose \(p_{uv} = 1/n\) and \(w_{uv} = 1\) for all online vertices \(v \in V\). The LP given by \((1a)-(1c)\) can assign \(x_{u,v} = 1\) for all edges, achieving an objective value of 1. However, the best any online algorithm can do is probe the single edge \((u,v)\) whenever vertex \(v\) arrives online, which matches the single offline vertex \(u\) with probability \(1 - 1/e\). Thus, it is impossible for any online algorithm to guarantee a matching of expected weight better than \((1 - 1/e)\) times the LP value. This establishes a stochasticity gap of \(1 - 1/e\) for this formulation, and suggests that if we wish to beat the \(1 - 1/e\) barrier, we must use a different LP benchmark. However, the stochasticity gap for the LP of \((1a)-(1c)\) is even worse. To establish this, we consider a complete bipartite graph with \(n\) vertices on each side, and edge probabilities \(1/n\); a result on random graphs then implies Theorem 8, whose proof is in Appendix A.
Theorem 8. The LP given by the objective function (1) and constraints (1a)–(1c) has a stochasticity gap of at most $\approx 0.544$.

We should note that Fata et al. (2019) establishes a smaller upper bound of $1 - \ln(2 - 1/e) \approx 0.51$ relative to this LP, but they restrict to online probing algorithms. Our higher upper bound holds even for the offline optimal matching, hence reflecting a true “stochasticity gap”.

6.2. 0.5 Upper Bound for SimpleGreedy

As defined in Mehta and Panigrahi (2012), an opportunistic algorithm for the Stochastic Rewards setting is one which always attempts to probe an edge incident to an online arriving vertex $v \in V$ if one exists. The work of Mehta and Panigrahi (2012) showed that in the unweighted Stochastic Rewards ($\theta_v = 1$ for all online vertices $v \in V$) problem, any opportunistic algorithm achieves a competitive ratio of $1/2$. The simplest opportunistic algorithm is the one which, when $v \in V$ arrives online, chooses a neighbor $u \in U$ of $v$ arbitrarily and probes the edge $(u, v)$. We call this algorithm “SimpleGreedy”. Since SimpleGreedy is opportunistic, the result of Mehta and Panigrahi (2012) shows that SimpleGreedy achieves a competitive ratio of at least $1/2$; Theorem 9 shows that this is tight even when restricted to small, uniform $p$.

Theorem 9. There exists a family of unweighted graphs under stochastic rewards and adversarial arrivals for which SimpleGreedy achieves a competitive ratio of at most $1/2$ even when all edges have uniform probability $p = O(1/n)$.

We present our construction here. Let $k$ be a fixed positive integer constant. Let $U = U_0 \cup U_n$, where $U_0$ and $U_n = \{u_1, \ldots, u_n\}$ are disjoint, and $|U_0| = k$. Let $V = V_0 \cup V_n$ where $V_0$ and $V_n = \{v_1, \ldots, v_n\}$ are disjoint, and $|V_0| = kn^2$. Let $E = E_0 \cup E_n$ where $E_0 = U_0 \times V$ and $E_n = \{(u_i, v_i) \mid i = 1, \ldots, n\}$. Let $p = k/n$.

For the bipartite graph $G(U, V; E)$, an offline algorithm can achieve a matching of expected size at least $2k$ by first probing edges $(u, v) \in U_0 \times V_0$ until all edges are probed or the maximum possible successful matches, $k$, is achieved. This strategy achieves $k$ successful matches among these edges.
in expectation. Then, the offline optimal will probe all edges of \( E_n \) in any order, achieving an expected number of successful matches of \( k \). The total expected size of the achieved matching is then \( 2k \). We complete the proof of Theorem 9 by showing that an online algorithm cannot earn more than \( k + o(k) \), in Appendix A.

6.3. Hardness of Unknown Patience

We now show that when offering items to a single customer with random patience, one should not be comparing to a benchmark that knows the realization of the patience in advance, or else the competitive ratio will be 0. The same counterexample shows for single-item IID-valued online accept/reject problems that the competitive ratio will be 0 if the number of arrivals is unknown, recovering the result of Alijani et al. (2020).

Theorem 10. For the star graph probing problem with patience \( \theta \) drawn from an arbitrary distribution \( \theta \), the reward of an online algorithm relative to a clairvoyant who sees the realization of \( \theta \) in advance must be 0, even if there are infinite copies of every offline vertex.

Theorem 10 is proved in Appendix A and its construction is presented below. The significance of allowing infinite copies of offline vertices is the following. Essentially, we are left with a pricing problem where there are an unknown \( \theta \) number of opportunities to make a single sale to a customer; the different offline vertices’ weights correspond to different prices that can be tried, and after each trial we get an independent realization (due to the infinite copies) whose probability depend on the price. One can further transform such an instance into an online accept/reject problem facing a stream of \( \theta \) IID draws, where the pricing decisions correspond to acceptance thresholds. Therefore, our hardness result implies the following. Although already known to Alijani et al. (2020, Appendix A.1), we re-derive it using our construction to articulate the connection, which we believe is instructive.

\(^9\) See Subsection 5.2 for the exact problem statement. There we compared to an LP that also did not know the realization of the patience in advance, the importance of which is fully justified by the present theorem.
Corollary 1. Consider the simple optimal stopping problem where an online algorithm can accept at most one of $\theta$ values that are drawn IID from a known distribution and presented one-by-one. If $\theta$ is unknown, then the competitive ratio is 0.

Our construction. Fix a positive integer $k$ and let $m$ be another positive integer that we will drive to $\infty$. Consider a star graph, i.e. a bipartite graph with many offline vertices $u \in U$ and a single online vertex $v$. Consider the following distribution over the patience of $v$:

$$\tilde{\theta}_v = \begin{cases} 
m^{2i} & \text{w.p. } m^{-i} - m^{-i-1} \quad \forall i = 0, \ldots, k-1; \\
m^{2k} & \text{w.p. } m^{-k}. 
\end{cases}$$

(8)

In our construction, there are $m^{2k}$ identical offline vertices for each $i = 0, \ldots, k$, with weight $m^i$ and probability $m^{-2i}$. We note that $m^{2k}$ is greater than the largest possible realization of $\tilde{\theta}_v$, so the constraints on the availability of offline vertices are never binding. That is, our construction applies even in the more restrictive setting where there are infinite copies of every offline vertex.

Intuition behind hardness. In our construction, there are essentially an unknown number of opportunities to sell a single item. During each opportunity, one must choose a consumption option $i = 0, \ldots, k$, which has a $m^{-2i}$ probability of selling the item at price $m^i$. The immediate reward from consumption option $i$ is $m^i \cdot m^{-2i} = m^{-i}$, which is decreasing in $i$, but smaller indices of $i$ also has a higher chance of closing the sale and eliminating future opportunities. Therefore, there is a tradeoff between offering “longshot” prices with a high index of $i$ (desirable if a large number of opportunities remain), vs. the “safe” option $i = 0$ which makes a sale w.p. $m^{-0} = 1$ (desirable on the final opportunity). Our proof of Theorem 10 in Appendix A shows that a clairvoyant who tries only option $i$ when they know the patience will be $m^{2i}$, for all $i = 0, \ldots, k$, can earn $\approx (1 - 1/e)(k + 1)$. Meanwhile, any online algorithm is best off using the “safe” option $i = 0$ on the first try and finishing, since the customer only has a small chance of having patience greater than 1 (we prove this through backwards induction on the optimal dynamic program). This establishes an unbounded separation when $k$ is taken to be large in our construction.
Transformed hard instance to establish Corollary \([\square]\). For concreteness, we show how to transform our construction to the accept/reject problem. The IID draws from the distribution should take one of \(k + 1\) possible values, indexed by \(i = 0, \ldots, k\). The decision each period is to set a threshold on the minimum acceptable value, where each option \(i = 0, \ldots, k\) should correspond to a “consumption option” that has probability \(m^{-2i}\) of accepting. Therefore, the probability of the IID draw taking value index \(i\) for all \(i = 0, \ldots, k - 1\) should be \(m^{-2i} - m^{-2(i+1)}\) and the probability of value index \(i = k\) should be \(m^{-2k}\), so that accepting all levels with index at least \(i\) has probability

\[
(m^{-2i} - m^{-2(i+1)}) + (m^{-2(i+1)} - m^{-2(i+2)}) + \cdots + m^{-2k} = m^{-2i}
\]

of making an acceptance. Now, the exact values for each index \(i\) must be calibrated so that the immediate reward from each consumption option \(i\) is \(m^{-i}\). Using backwards induction over \(i = k, \ldots, 1\), we can solve that the value for index \(i = k\) should be \(m^k\) and that the value for each index \(i = 0, \ldots, k - 1\) should be

\[
\frac{m^{-i} - m^{-(i+1)}}{m^{-2i} - m^{-2(i+1)}} = \frac{m^{i+2} - m^{i+1}}{m^2 - 1} = \frac{m^{i+1}}{m + 1}.
\]

This completes the construction of our transformed instance for the accept/reject problem.

Applying Yao’s minimax principle. Finally, we explain why in Corollary \([\square]\) there is no difference between \(\theta\) being completely unknown vs. \(\theta\) being drawn from a known distribution (but competing against a clairvoyant who knows its realization in advance). Formally, for any patience \(\theta\) and any (deterministic) non-clairvoyant algorithm \(\psi\), let \(\text{ALG}(\psi, \theta)\) denote the algorithm’s expected reward when the patience realizes to \(\theta\). Meanwhile, let \(\text{OPT}(\theta)\) denote the clairvoyant’s expected reward when the patience is known to be \(\theta\). Let \(\text{D}\) denote a distribution over patiences \(\theta\), and \(\Psi\) denote a distribution over algorithms \(\psi\). Yao’s minimax principle says that

\[
sup_{\Psi} \inf_{\theta} \frac{\mathbb{E}_{\psi \sim \Psi}[\text{ALG}(\psi, \theta)]}{\mathbb{E}_{\theta \sim \text{OPT}}[\text{OPT}(\theta)]} = \inf_{\text{D}} \sup_{\psi} \frac{\mathbb{E}_{\theta \sim \text{D}}[\text{ALG}(\psi, \theta)]}{\mathbb{E}_{\theta \sim \text{D}}[\text{OPT}(\theta)]}
\]

(where the second equality holds via rescaling worst-case distributions \(\text{D}\) by \(\text{OPT}(\theta))\). The existence of our family of distributions in \([\square]\) shows that the RHS expression, and hence all of these
expressions, equal 0. The LHS expression equaling 0 implies that for any fixed (randomized) online algorithm that does not know the value of $\theta$ in advance, an adversary can always set a horizon length $\theta$ for which the algorithm performs unboundedly worse relative to $\text{OPT}(\theta)$.

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Appendix A: Deferred Proofs

Proof of Lemma 1. Consider an adaptive offline algorithm which is optimal. Let \( x_{u,v} \) be the probability that this strategy probes edge \((u,v)\). For any vertex \( u \in U \), the probability that \( u \) is successfully matched is at most \( \sum_{v \in V} x_{u,v} p_{u,v} \leq 1 \), and similarly for the probability of successfully matching any online vertex \( v \in V \). Thus, this assignment satisfies constraints (1a) and (1b). By the definition of \( \text{OPT}(G) \), we cannot probe more than \( \theta_v \) edges incident on an online vertex \( v \). So constraint (1c) is satisfied.

Finally, we argue that the new constraint (1d) is satisfied by this assignment. Suppose instead there is some vertex \( v' \in V \) and some \( U' \subseteq U \) for which \( \sum_{u \in U'} x_{u,v'} p_{u,v'} w_u > \text{OPT}(U', v') \). Then, we can define a new offline probing strategy on the star graph \((U', \{v'\}, U' \times \{v'\})\) which simply simulates our original algorithm on \( G \) and probes only those edges which are in \( U' \times \{v'\} \). This achieves an expected matching weight on the star graph of at least \( \sum_{u \in U'} x_{u,v'} p_{u,v'} w_u > \text{OPT}(U', v') \), but this contradicts the fact that \( \text{OPT}(U', v') \) is the optimal expected matching weight for the star graph. Thus, this assignment must satisfy constraint (1d). It follows that LP (1) must have objective value at least as large as the expected matching weight of the optimal offline algorithm.

□

Proof of Lemma 2. For the sake of analysis, suppose we have solved LP (1) on the graph \( G \). Let \( c(x) = \sum_{u \in U} \sum_{v \in V} x_{u,v} p_{u,v} w_u \) be the value of the objective function for \( x \) and let \( c_v(x) = \sum_{u \in U} x_{u,v} p_{u,v} w_u \) be the value “achieved” by a given online vertex \( v \) with \( c(x) = \sum_{v \in V} c_v(x) \). Let \( x^* \) be the optimal assignment given by LP (1), for the graph \( G \). So \( c(x^*) = \sum_{v \in V} c_v(x^*) = \text{OPT}_{LP}(G) \).

We will make the following charging argument. Imagine that when a vertex \( v \) is matched to some \( u \in U \), we assign 0.5\( w_u \) to \( v \) and for all \( v' \in V \) (including \( v \) itself) we assign 0.5\( x_{u,v'} p_{u,v'} w_u \) to \( v' \). Note we have assigned at most \( w_u \) weight in total since \( \sum_{v' \in V} 0.5 x_{u,v'} p_{u,v'} w_u \leq 0.5 w_u \) due to LP constraint (1a).

Let \( w_v \) for online vertex \( v \in V \) be equal to the weight \( w_u \) of the offline vertex \( u \) which is matched to \( v \) or 0 if \( v \) is unmatched at the end of the arrivals. Let \( U_m \subseteq U \) be the set of offline vertices which are matched at the end of the arrivals. We define

\[
c_v(\text{ALG}) = 0.5 w_v + 0.5 \sum_{u \in U_m} x_{u,v} p_{u,v} w_u
\]
as the weight assigned to \( v \) in our imaginary assignment. By the linearity of expectation

\[
\text{ALG}(G) = \mathbb{E} \left[ \sum_{v \in V} c_v(\text{ALG}) \right] = \sum_{v \in V} \mathbb{E}[c_v(\text{ALG})]
\]

Thus, to complete the proof, we must show that

\[
\sum_{v \in V} \mathbb{E}[c_v(\text{ALG})] \geq \frac{1}{2} \text{OPT}_{\text{LP}}(G)
\]

Consider an online vertex \( v \) arriving at time \( k \). Let \( U_v \subseteq U \) be the set of vertices available (unmatched) when \( v \) arrives and \( U_{-v} = U \setminus U_v \) be the set of vertices which are already matched when \( v \) arrives. Note that when \( v \) arrives, it has already been assigned a value of \( 0.5 \sum_{u \in U_{-v}} x_{u,v}p_{u,v}w_u \).

After using a \( \kappa \)-approximate (\( \kappa \leq 1 \)) black box for matching \( v \) to \( U_v \), we have assigned an expected value to \( v \) of at least \( 0.5 \kappa \text{OPT}(U_v,v) + 0.5 \sum_{u \in U_{-v}} x_{u,v}p_{u,v}w_u \).

Thus, we have

\[
\sum_{v \in V} \mathbb{E}[c_v(\text{ALG})] \geq \sum_{v \in V} \sum_{U_v \subseteq U} \Pr[U_v] \left( 0.5 \kappa \text{OPT}(U_v,v) + 0.5 \sum_{u \in U_{-v}} x_{u,v}p_{u,v}w_u \right)
\]

\[
\geq \sum_{v \in V} \sum_{U_v \subseteq U} \Pr[U_v] \left( 0.5 \kappa \sum_{u \in U_v} x_{u,v}p_{u,v}w_u + 0.5 \kappa \sum_{u \in U_{-v}} x_{u,v}p_{u,v}w_u \right) : \text{(1d)}
\]

\[
\geq 0.5\kappa \sum_{v \in V} \sum_{u \in U_v} x_{u,v}p_{u,v}w_u
\]

\[
= \kappa \frac{1}{2} \text{OPT}_{\text{LP}}(G)
\]

which completes the proof. \( \square \)

**Proof of Theorem 2.** Given a feasible solution \( x^*_v(\pi) \) to LP 2 whose objective value is at least \( \kappa \cdot \text{OPT}_{\text{LP}}(G) \), we show that an online algorithm can form a matching whose weight is in expectation at least \( \frac{\kappa}{2} \cdot \text{OPT}_{\text{LP}}(G) \).

First, we establish some notation. Let \( Q^t_v \in \{0,1\} \) be the indicator random variable for the event that the arrival at time \( t \) has type \( v \in V \). Let \( Y^t_{uv} \in \{0,1\} \) be the indicator random variable for \( u \) being matched to a vertex of type \( v \) at time \( t \). Let \( N^t_u \) be the indicator random variable for offline vertex \( u \) becoming matched by the end of time \( t \). And, let \( \Pi_t \) be denote the policy chosen by Algorithm 2 for time \( t \). We may now write the online algorithm’s weight matched as

\[
\sum_{t,u,v} w_{uv}Y^t_{uv} = \sum_{t,u,v} \left( w_{uv} - \frac{w_u^*}{2} \right) Y^t_{uv} + \sum_{t,u} \frac{w_u^*}{2} Y^t_{uv}
\]

\[
= \sum_{u,v:w_{uv} \geq w_u^*/2} \left( w_{uv} - \frac{w_u^*}{2} \right) Y^t_{uv} + \sum_{u} \frac{w_u^*}{2} N^T_u
\]

(9)
Now we wish to analyze the value of $\mathbb{E}[Y_{uv}^t]$. For a pair $u,v$ such that $w_{uv} \geq w_u^*/2$ (i.e. edge $uv$ is not skipped over when executing policies $\pi \in \mathcal{P}$ during the execution of Algorithm 2), we have

$$\mathbb{E}[Y_{uv}^t] = \mathbb{E}[Y_{uv}^t | (Q_t^v = 1) \cap (N_{u}^{t-1} = 0)] \Pr[(Q_t^v = 1) \cap (N_{u}^{t-1} = 0)]$$

$$= q_{uv} (1 - \mathbb{E}[N_{u}^{t-1}]) \sum_{\pi \in \mathcal{P}} \mathbb{E}[Y_{uv}^t | (Q_t^v = 1) \cap (N_{u}^{t-1} = 0) \cap (\Pi_t = \pi)] \Pr[\Pi_t = \pi | Q_t^v = 1]$$

(10)

Next, notice that $\mathbb{E}[N_{u}^{t-1}] \leq \mathbb{E}[N_u^T]$ since the probability of offline vertices becoming matched only increases over time. Notice also that since $w_{uv} \geq w_u^*/2$, no policy for a vertex of type $v$ will skip over probing $u$. Further, skipping over other vertices can only increase the probability that $u$ is probed. Hence, we can conclude that $\mathbb{E}[Y_{uv}^t | (Q_t^v = 1) \cap (N_{u}^{t-1} = 0) \cap (\Pi_t = \pi)] \geq p_{uv}(\pi)$. Finally, we observe that $\Pr[\Pi_t = \pi | Q_t^v = 1] = x_v^*(\pi)/q_v$.

Applying the above to (10), we have

$$\mathbb{E}[Y_{uv}^t] \geq q_{uv} (1 - \mathbb{E}[N_{u}^{T}]) \sum_{\pi \in \mathcal{P}} p_{uv}(\pi) \frac{x_v^*(\pi)}{q_v}$$

and combining with (9) we get can bound the algorithm’s expected weight as

$$\mathbb{E} \left[ \sum_{t,u,v} w_{uv} Y_{uv}^t \right] \geq \sum_{u,v:w_{uv} \geq w_u^*/2} (w_{uv} - \frac{w_u^*}{2}) q_{uv} \sum_{\pi \in \mathcal{P}} p_{uv}(\pi) \frac{x_v^*(\pi)}{q_v} + \sum_{u} \frac{w_u^*}{2} \mathbb{E}[N_u^T]$$

$$\geq \sum_{u} (1 - \mathbb{E}[N_u^T]) \sum_{v} (w_{uv} - \frac{w_u^*}{2}) \sum_{\pi \in \mathcal{P}} p_{uv}(\pi) x_v^*(\pi) + \sum_{u} \frac{w_u^*}{2} \mathbb{E}[N_u^T]$$

$$= \sum_{u} (1 - \mathbb{E}[N_u^T]) \left( w_u^* - \frac{w_u^*}{2} \sum_{v} \sum_{\pi \in \mathcal{P}} p_{uv}(\pi) x_v^*(\pi) \right) + \sum_{u} \frac{w_u^*}{2} \mathbb{E}[N_u^T]$$

which is at least $\sum_{u} w_u^*/2$. By assumption on the LP solution, we have $\sum_{u} w_u^*/2 \geq \kappa \cdot \text{OPT}_{LPP}(G)/2$, completing the proof. □

**Proof of Theorem 3.** Given a feasible solution $x_v^*(\pi)$ to LP 2 whose objective value is at least $\kappa \cdot \text{OPT}_{LPP}(G)$, we show that an online algorithm can form a matching whose weight is in expectation at least $\kappa(1-1/e) \cdot \text{OPT}_{LPP}(G)$.

We let $Q_v^t$, $Y_v^t$, $N_v^t$, and $\Pi_t$ be defined as in the proof of Theorem 2. Further, let $Z_{uv}^t \in \{0,1\}$ be the indicator random variable for $(u,v)$ being successfully probed (real or simulated) at time $t$. Then, let $Z^t := \sum_{v \in V} Z_{uv}^t$. Note that $Z^t \in \{0,1\}$ since $Z_{uv}^t$ can only be 1 for the single $v \in V$ that corresponds to the vertex type arriving at time $t$.

Next, since $u$ is available to be matched at time $t$ if and only if $Z_1^t = Z_2^t = \cdots = Z_u^{t-1} = 0$, we can write $Y_{uv}^t = Z_{uv}^t \cdot 1[Z_1^t = Z_2^t = \cdots = Z_u^{t-1} = 0]$. 
Finally, substituting back into (14), we get that the algorithm's expected weight is at least

\[
E \left[ \sum_{t,u,v} w_{uv} Y_{uv}^t \right] = \sum_{u,t} E \left[ \sum_v w_{uv} Z_{uv}^t \right] E[1(Z_u^1 = \cdots = Z_u^{t-1} = 0)] \tag{11}
\]

Under IID assumption, we know \( q_{tv} = q_v/T \) for all \( t \), and we can derive the following:

\[
E[Z_{uv}^t] = \Pr[Q_v^t = 1] \sum_{\pi \in P} E[Z_{uv}^t | (Q_v^t = 1) \cap (\Pi_t = \pi)] \Pr[\Pi_t = \pi | Q_v^t = 1] = \frac{q_v}{T} \sum_{\pi \in P} \frac{p_{uv}(\pi) x_v^*(\pi)}{q_v} = \frac{1}{T} \sum_{\pi \in P} p_{uv}(\pi) x_v^*(\pi) \tag{12}
\]

Using (12), we can derive the following for any offline vertex \( u \):

\[
E\left[ \sum_v w_{uv} Z_{uv}^t | Z_u^t = 1 \right] = \frac{\sum_v w_{uv} E[Z_{uv}^t]}{\Pr[Z_u^t = 1]} = \frac{\sum_v w_{uv} E[Z_{uv}^t]}{\sum_{v} E[Z_{uv}^t]} = \frac{\sum_{v,\pi} w_{uv} p_{uv}(\pi) x_v^*(\pi)}{\sum_{v,\pi} p_{uv}(\pi) x_v^*(\pi)} \tag{13}
\]

Substituting (13) back into (11), we get

\[
E \left[ \sum_{t,u,v} w_{uv} Y_{uv}^t \right] = \sum_{u,t} \left( \frac{\sum_{v,\pi} w_{uv} p_{uv}(\pi) x_v^*(\pi)}{\sum_{v,\pi} p_{uv}(\pi) x_v^*(\pi)} \right) \Pr[\Pi_t = \pi | Q_v^t = 1] E[1(Z_u^1 = \cdots = Z_u^{t-1} = 0)]
\]

\[
= \sum_{u} \left( \frac{\sum_{v,\pi} w_{uv} p_{uv}(\pi) x_v^*(\pi)}{\sum_{v,\pi} p_{uv}(\pi) x_v^*(\pi)} \right) \sum_{t} E[1(Z_u^t = 1) 1(Z_u^1 = \cdots = Z_u^{t-1} = 0)] \tag{14}
\]

For any \( u \), let’s analyze \( E[\sum_t 1(Z_u^t = 1) 1(Z_u^1 = \cdots = Z_u^{t-1} = 0)] \). The expression inside the \( E \) is 0 if all of the \( Z_u^t \) are 0, and 1 otherwise. Therefore, by independence, the expectation equals

\[
1 - \prod_t \Pr[Z_u^t = 0] = 1 - \prod_t (1 - \Pr[\sum_v Z_{uv}^t]).
\]

Substituting in (12), we get this is equal to

\[
1 - \prod_t (1 - \frac{1}{T} \sum_v \Pr[p_{uv}(\pi) x_v^*(\pi)]) \geq 1 - \exp(- \sum_t \frac{1}{T} \sum_{v,\pi} p_{uv}(\pi) x_v^*(\pi))
\]

\[
= 1 - \exp(- \sum_{v,\pi} p_{uv}(\pi) x_v^*(\pi))
\]

Finally, substituting back into (14), we get that the algorithm’s expected weight is at least

\[
\sum_u \sum_{v,\pi} w_{uv} p_{uv}(\pi) x_v^*(\pi) \frac{1 - \exp(- \sum_{v,\pi} p_{uv}(\pi) x_v^*(\pi))}{\sum_{v,\pi} p_{uv}(\pi) x_v^*(\pi)}
\]
The function \( \frac{1 - \exp(-z)}{z} \) is decreasing in \( z \) and the expression \( \sum_{v, \pi} p_{uv}(\pi)x^*_v(\pi) \) is at most 1 by LP constraint (2a), hence the fraction in the expression above is uniformly lower-bounded by \( 1 - 1/e \). Therefore, the algorithm’s expected weight is at least \( (1 - 1/e) \sum_u \sum_{v, \pi} w_{uv}p_{uv}(\pi)x^*_v(\pi) \), which in turn is at least \( (1 - 1/e)\kappa \cdot \text{OPT}(G) \) by presumption on the given LP solution \( x^*_v(\pi) \), completing the proof. \( \square \)

Proof of Theorem 4. Given a feasible solution \( x^*_v(\pi) \) to LP (2) whose objective value is at least \( \kappa \cdot \text{OPT}_{LPP}(G) \), we show that an online algorithm can form a matching whose weight is in expectation at least \( \kappa(1 - 1/e) \cdot \text{OPT}_{LPP}(G) \).

We let \( Q_t, Y_{uv}^t, Z_{uv}^t, Z_u^t \), and \( N_u^t \) be as defined in the proof of Theorem 2. The LP objective value can be written as

\[
\sum_{u \in U} w_u \left( \sum_{v \in V} \sum_{\pi \in \mathcal{P}} x^*_v(\pi)p_{uv}(\pi) \right).
\]

We will show that Algorithm 3 in this case matches each offline vertex \( u \) with probability at least \( (1 - 1/e) \sum_{v \in V} \sum_{\pi \in \mathcal{P}} x^*_v(\pi)p_{uv}(\pi) \). Notice that a vertex \( u \) is left unmatched only if \( Z_u^1 = \cdots = Z_u^T = 0 \). Since \( Z_u^1, \ldots, Z_u^T \) are independent, the probability of this is

\[
\prod_{t=1}^T \left( 1 - \Pr[Z_u^t = 1] \right) = \prod_{t=1}^T \left( 1 - \sum_{v \in V} \mathbb{E}[Z_{uv}^t] \right)
\]

\[
= \prod_{t=1}^T \left( 1 - \sum_{v \in V} \sum_{\pi \in \mathcal{P}} q_{tv} \frac{p_{uv}(\pi)x^*_v(\pi)}{q_v} \right)
\]

\[
\leq \prod_{t=1}^T \exp \left( - \sum_{v \in V} \sum_{\pi \in \mathcal{P}} q_{tv} \frac{p_{uv}(\pi)x^*_v(\pi)}{q_v} \right)
\]

\[
= \exp \left( - \sum_{v \in V} \sum_{\pi \in \mathcal{P}} p_{uv}(\pi)x^*_v(\pi) \sum_{t=1}^T \frac{q_{tv}}{q_v} \right)
\]

Therefore, the probability of Algorithm 3 matching an offline vertex \( u \) is at least

\[
1 - \exp\left( - \sum_{v \in V} \sum_{\pi \in \mathcal{P}} p_{uv}(\pi)x^*_v(\pi) \right)
\]

which we know is at least \( (1 - 1/e) \sum_{v \in V} \sum_{\pi \in \mathcal{P}} x^*_v(\pi)p_{uv}(\pi) \) by the same analysis as at the end of the proof of Theorem 3. Therefore, the total expected weight accrued by the algorithm across all offline vertices \( u \in U \) is at least \( (1 - 1/e) \sum_{u \in U} \sum_{v \in V} \sum_{\pi \in \mathcal{P}} x^*_v(\pi)p_{uv}(\pi) \), which in turn is at least

\[
(1 - 1/e)\kappa \cdot \text{OPT}_{LPP}(G)
\]

by presumption on the given LP solution \( x^*_v(\pi) \), completing the proof. \( \square \)

Proof of Theorem 6. In our analysis, we define \( \theta_0 \) to be the random indicator variable corresponding to the event that the probing process has continued to the \( \theta \)-th attempt. Additionally,
$X_{j\theta}$ is the indicator variable for the event that vertex $j$ is probed in the $\theta$th attempt, including simulated probes.

We claim that $\mathbb{E}[S_\theta] = s^*_\theta$ for all $\theta$. This can be seen to be true by definition in (5c)–(5d) and the fact that we simulate probes even when a vertex $j$ is unavailable. Further, we can immediately see that $\mathbb{E}[X_{j\theta}] = \mathbb{E}[X_{j\theta}|S_\theta = 1] \Pr[S_\theta = 1] = \frac{x^*_{j\theta}}{s^*_\theta}$ as long as $s^*_\theta > 0$, and that both $\mathbb{E}[X_{j\theta}]$ and $x^*_{j\theta}$ must be 0 if $s^*_\theta = 0$. Therefore, $\mathbb{E}[X_{j\theta}] = x^*_{j\theta}$ for all $j$ and $\theta$.

The algorithm’s expected weight matched only counts non-simulated probes, and can be written

$$\sum_{j=1}^{n} w_j p_j \sum_{\theta=1}^{n} \mathbb{E} \left[ X_{j\theta} \prod_{\theta'<\theta} \left(1 - X_{j\theta'}\right) \right].$$

(15)

We will analyze the expectation in (15) for an arbitrary $j$ and $\theta$. We assume $s^*_\theta > 0$, since otherwise $X_{j\theta} = 0$ w.p. 1 and the expectation must be 0.

**Step 1: Using conditional independence.** The terms inside the expectation in (15) are generally not independent, since $X_{j\theta} = 1$ (i.e. probing $j$ on attempt $\theta$) is only possible if $S_\theta = 1$ (i.e. the online vertex is still available on attempt $\theta$), which is affected by whether $X_{j\theta'} = 1$ for $\theta' < \theta$.

Nonetheless, we can decompose the expectation as follows:

$$\mathbb{E} \left[ X_{j\theta} \prod_{\theta'=1}^{\theta-1} \left(1 - X_{j\theta'}\right) \right] = \mathbb{E}[X_{j\theta}] \mathbb{E} \left[ \prod_{\theta'=1}^{\theta-1} \left(1 - X_{j\theta'}\right) | X_{j\theta} = 1 \right]$$

$$= x^*_{j\theta} \prod_{\theta'=1}^{\theta-1} \mathbb{E}[(1 - X_{j\theta'}) | X_{j\theta} = 1]$$

$$= x^*_{j\theta} \prod_{\theta'=1}^{\theta-1} \mathbb{E}[(1 - X_{j\theta'}) | S_{\theta'+1} = 1].$$

The second equality holds because $\{X_{j\theta'} : \theta' < \theta\}$ were only correlated through their dependence on the availability of the online vertex, but conditional on $X_{j\theta} = 1$ which reveals that the online vertex is available for all attempts $\theta' < \theta$, whether $j$ is probed on any attempt is determined by an independent coin flip. The third equality holds because from the perspective of $X_{j\theta'}$, conditioning on $X_{j\theta} = 1$ is equivalent to conditioning on $S_{\theta'+1} = 1$. Indeed, the only future information which distorts the likelihood of having probed $j$ on attempt $\theta'$ is whether the online vertex survived to attempt $\theta'+1$; any events after that have no interdependence with $X_{j\theta'}$.

For an arbitrary $\theta'$, noting that $S_{\theta'+1} = 1$ implies $S_{\theta'} = 1$, we can evaluate

$$\mathbb{E}[(1 - X_{j\theta'}) | S_{\theta'+1} = 1] = \frac{\Pr[X_{j\theta'} = 0 \cap S_{\theta'+1} = 1 | S_{\theta'} = 1]}{\Pr[S_{\theta'+1} = 1 | S_{\theta'} = 1]}.$$

(16)

Now, since $X_{j\theta'} = 0$ and $X_{j\theta'} = 1$ are mutually exclusive and collectively exhaustive events,

$$\Pr[S_{\theta'+1} = 1 | S_{\theta'} = 1] = \Pr[X_{j\theta'} = 0 \cap S_{\theta'+1} = 1 | S_{\theta'} = 1] + \Pr[X_{j\theta'} = 1 \cap S_{\theta'+1} = 1 | S_{\theta'} = 1]$$

$$\leq \Pr[X_{j\theta'} = 0 \cap S_{\theta'+1} = 1 | S_{\theta'} = 1] + \frac{x^*_{j\theta'}}{s^*_{\theta'}}.$$
where the inequality uses the fact that \( \Pr[X_{j\theta'} = 1 | S_{\theta'} = 1] = x_{j\theta'/s_{\theta'}}. \) Substituting into (16),

\[
\mathbb{E}[(1 - X_{j\theta'}) | S_{\theta'+1} = 1] \geq 1 - \frac{x_{j\theta'/s_{\theta'}}}{\Pr[S_{\theta'+1} = 1 | S_{\theta'} = 1]}
\]

\[
= 1 - \frac{x_{j\theta'}}{s_{\theta'+1}}.
\]

In combination with the fact that expectations are non-negative, we can substitute back into expression (15) to conclude that for any node \( j \),

\[
\sum_{\theta=1}^{n} \mathbb{E} \left[ X_{j\theta} \prod_{\theta' = 1}^{\theta-1} (1 - X_{j\theta'}) \right] \geq \sum_{\theta: s_{\theta'} > 0} \sum_{\theta: s_{\theta'} > 0} x_{j\theta} \prod_{\theta' = 1}^{\theta-1} \max \{1 - \frac{x_{j\theta'}}{s_{\theta'+1}}, 0\}. \tag{17}
\]

**Step 2: Using the improved constraints.** Now that in (17) we have bounded the algorithm’s expected weight matched as an expression of variables from the LP’s optimal solution, we can use the improved constraints (5a) to derive a bound relative to the LP’s optimal objective value.

For this part of the proof, fix an arbitrary \( j \) and we will omit the subscript \( j \). To ensure that we don’t divide by 0 on the RHS of (17), let \( \bar{\theta} \) denote the maximum index \( \theta \) for which \( x_{\bar{\theta}} > 0 \); note it must be the case that \( s_{\bar{\theta}} > 0 \). We can then apply (5a) on the values of \( s_{\theta'+1} \) to see that the RHS of (17) is lower-bounded by

\[
\sum_{\theta=1}^{\bar{\theta}} x_{\theta} \prod_{\theta' = 1}^{\theta-1} \max \{1 - \frac{x_{j\theta'}}{s_{\theta'+1}}, 0\}. \tag{18}
\]

**Lemma 5.** The algorithm’s probability of probing any offline node in a non-simulated fashion is at least half the total from the LP optimal solution, when both sums are truncated at the index \( \bar{\theta} \) defined above. That is, \( \sum_{\theta=1}^{\bar{\theta}} x_{\theta} \prod_{\theta' = 1}^{\theta-1} \max \{1 - \frac{x_{j\theta'}}{s_{\theta'+1}}, 0\} \geq \frac{1}{2} \sum_{\theta=1}^{\bar{\theta}} x_{\theta} \).

The proof of Lemma 5 is presented next. It directly completes the present proof from expression (18). \( \square \)

The analysis in Theorem 6 is tight. Indeed, consider an example with \( n = 2 \) offline vertices, where the first has \( w_1 = 1 \) and \( p_1 = \epsilon \) for some small \( \epsilon > 0 \), while the second has \( w_2 = 0 \) and \( p_2 = 1 \). The patience is deterministically 2, i.e. \( q_1 = q_2 = 1 \). The following defines an optimal LP solution: \( s_1 = 1, x_{11} = \frac{1}{2(1-\epsilon)}, x_{21} = \frac{1-\epsilon}{2(1-\epsilon)}, s_2 = 1/2, x_{12} = \frac{1-\epsilon}{2(1-\epsilon)}, x_{22} = 0 \), which has objective value \( \epsilon \) (all coming from the first vertex). However, the algorithm can only probe the first vertex in a non-simulated fashion on attempt 1, since conditional on it surviving to attempt 2, the first vertex must have already been probed on attempt 1. As a result, the algorithm probes the first vertex with a total probability of \( \frac{1}{2(1-\epsilon)} \), matching expected weight \( \frac{\epsilon}{2(1-\epsilon)} \) which approaches half of the LP’s objective value as \( \epsilon \to 0 \).
Proof of Lemma 3. To simplify notation in the proof of (18), we let $x_{\theta'}^*$ denote $x_{\theta'+1}^* + \cdots + x_{\theta}^*$ for any $\theta'$; we also let $[\cdot]^+$ denote $\max\{\cdot, 0\}$. If $\bar{\theta} = 1$ then the desired result is trivial; if $\bar{\theta} = 2$ then (18) equals $x_1^* + x_2^*[1 - x_1^*/x_2^*]^+ = \max\{x_1^*, x_2^*\}$ which also leads to the desired result that it is at least $\frac{1}{2} \sum_{\theta=1}^{\bar{\theta}} x_\theta^*$. Hereafter, we assume $\bar{\theta} \geq 3$. The following can then be derived:

$$
\sum_{\theta=1}^{\bar{\theta}} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ = \sum_{\theta=1}^{\bar{\theta}-2} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \left( x_\bar{\theta}^* \prod_{\theta' = 1}^{\bar{\theta}-2} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ \right) \prod_{\theta' = 1}^{\bar{\theta}-2} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+
$$

$$
= \sum_{\theta=1}^{\bar{\theta}-2} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \sum_{\theta=1}^{\bar{\theta}-3} x_{\bar{\theta}-2}^* \prod_{\theta' = 1}^{\bar{\theta}-3} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ \prod_{\theta' = 1}^{\bar{\theta}-2} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+
$$

$$
= \sum_{\theta=1}^{\bar{\theta}-2} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \sum_{\theta=1}^{\bar{\theta}-3} x_{\bar{\theta}-2}^* \prod_{\theta' = 1}^{\bar{\theta}-3} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ \prod_{\theta' = 1}^{\bar{\theta}-2} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+
$$

$$
\geq \sum_{\theta=1}^{\bar{\theta}-2} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \frac{x_{\bar{\theta}-2}^*}{2} \prod_{\theta' = 1}^{\bar{\theta}-2} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ \prod_{\theta' = 1}^{\bar{\theta}-2} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ \tag{19}
$$

where the inequality applies the facts that $\frac{\max(x_{\bar{\theta}-2}^*, x_{\bar{\theta}-2}^*)}{x_{\bar{\theta}-2}^*} \geq 1/2$ and $[x_{\bar{\theta}-2}^* - x_{\bar{\theta}-2}^*]^+ \geq x_{\bar{\theta}-2}^* - x_{\bar{\theta}-2}^*$, after acknowledging that all terms in the product are initially non-negative.

Now, for any $k = \bar{\theta} - 2, \bar{\theta} - 1, \ldots, 2$, the following can be derived:

$$
\sum_{\theta=1}^{k-1} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \frac{x_{k-1}^* - x_k^*}{2} \prod_{\theta' = 1}^{k-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+
$$

$$
= \sum_{\theta=1}^{k-1} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \frac{x_{k-1}^* - x_k^*}{2} \prod_{\theta' = 1}^{k-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+
$$

$$
= \sum_{\theta=1}^{k-1} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \frac{x_{k-1}^* - x_k^*}{2} \prod_{\theta' = 1}^{k-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+
$$

$$
\geq \sum_{\theta=1}^{k-1} x_\theta^* \prod_{\theta' = 1}^{\theta-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+ + \frac{x_{k-1}^* - x_k^*}{2} \prod_{\theta' = 1}^{k-1} \left[ 1 - \frac{x_{\theta'}^*}{x_{\theta'}^*} \right]^+. \tag{20}
$$

When $k = \bar{\theta} - 2$, expression (20) equals (19). We have inductively established that this is at least the value of expression (21) when $k = 2$, which is $x_1^* + \frac{x_{2}^* - x_1^*}{2} = \frac{1}{2}(x_1^* + \cdots + x_{\bar{\theta}}^*)$, completing the proof. □

Proof of Theorem 3. Let $q_i = p_i + (1 - p_i)r_i$ be the probability that probing terminates after attempting to match $i$ (which occurs either when the match is successful, or when the match fails and the patience is subsequently exhausted); the probability that another match can be attempted is then $1 - q_i$.

Suppose we have an optimal strategy which probes in the order $u_1, u_2, \ldots, u_m$ and achieves the maximum expected reward. We denote this reward by $R^*$, and note that

$$
R^* = \sum_{i=1}^{m} \left[ \prod_{j=1}^{i-1} (1 - q_j)w_ip_i \right] \tag{22}
$$
If the permutation satisfies $w_ip_i/q_i \geq w_{i+1}p_{i+1}/q_{i+1}$ for all $i \in [m-1]$, then we are done. Suppose to the contrary that there is some $k \in [m-1]$ for which $w_kp_k/q_k < w_{k+1}p_{k+1}/q_{k+1}$. Consider the alternate strategy where we swap the order of probing $k$ and $k+1$. That is, we probe in the order $u_1, u_2, \ldots, u_{k-1}, u_k, u_{k+2}, \ldots, u_m$. Denote the expected reward of this new strategy by $R'$. We have

$$R' = \sum_{i=1}^{k-1} \prod_{j=1}^{i-1} (1 - q_j)w_ip_i + \prod_{j=1}^{k-1} (1 - q_j)(w_{k+1}p_{k+1} + (1 - q_{k+1})w_kp_k) + \sum_{i=k+1}^m \prod_{j=1}^{i-1} (1 - q_j)w_ip_i$$

(23)

Subtracting (23) from (22) yields

$$R^* - R' = \prod_{j=1}^{k-1} (1 - q_j)(w_kp_k + (1 - q_k)w_{k+1}p_{k+1}) - \prod_{j=1}^{k-1} (1 - q_j)(w_{k+1}p_{k+1} - (1 - q_{k+1})w_kp_k)$$

which is equivalent to

$$R^* - R' = \prod_{j=1}^{k-1} (1 - q_j) \left( w_kp_k + (1 - q_k)w_{k+1}p_{k+1} - w_{k+1}p_{k+1} - (1 - q_{k+1})w_kp_k \right)$$

$$= \prod_{j=1}^{k-1} (1 - q_j) \left( q_{k+1}w_kp_k - q_kw_{k+1}p_{k+1} \right) < 0$$

where the inequality holds since, by assumption, $w_kp_k/q_k < w_{k+1}p_{k+1}/q_{k+1}$, and thus $w_kp_kq_{k+1} < w_{k+1}p_{k+1}q_k$.

However, this means that $R^* < R'$, contradicting the assumption that $R^*$ is optimal.

**Proof of Lemma**

For a large item $i$, let $A_{i,t}$ be the event that $i$ arrives before or at time $t$. Let $B_{i,t}$ be the event that $i$ is offered before time $t$. Let $C_{i,t}$ be the event that we have successfully sold some item in $I_{\text{LARGE}} - \{i\}$ before time $t$.

We note that for an item $i$ and time step $t$:

$$\Pr(i \text{ is probed at time } t) = \Pr(A_{i,t} \wedge \bar{B}_{i,t} \wedge \bar{C}_{i,t})\frac{px_i}{\theta}$$

(24)

Next, we observe that the probability that $i$ has arrived by time $t$ is given by

$$\Pr(A_{i,t}) = 1 - (1 - q_i)^t.$$  

(25)

Next, we consider $\Pr(\bar{B}_{i,t} \cap \bar{C}_{i,t})$, which is given by:

$$\Pr(\bar{B}_{i,t} \cap \bar{C}_{i,t}) = \prod_{t' = 1}^{t-1} \Pr(\bar{B}_{i,t'} \cap \bar{C}_{i,t'})$$

(26)

where $B_{i,t'}$ is the event that $i$ is offered at time $t'$ and $C_{i,t'}$ is the event that some item in $I_{\text{LARGE}} - \{i\}$ is offered at time $t'$. Next, we note that

$$\Pr(\bar{B}_{i,t'} \cap \bar{C}_{i,t'}) = 1 - \Pr(B_{i,t'} \vee C_{i,t'}) \geq 1 - \Pr(B_{i,t'}) - \Pr(C_{i,t'}).$$

Then,

$$\Pr(B_{i,t'}) \leq \frac{\rho x_i}{\theta} \leq \frac{\rho}{\theta}.$$
and
\[ \Pr(C'_{i,t}) \leq \sum_j \frac{\rho x_j}{\Theta} p_j = \frac{\rho}{\Theta} \sum_j x_j p_j \leq \frac{\rho}{\Theta} \]
where in the final inequality, we use constraint (6a). These together give
\[ \Pr(B'_{i,t} \cap C'_{i,t}) \geq 1 - \frac{2\rho}{\Theta} \]
which combined with (26) gives (27).

Combining (25) and (27) with (24) gives us (28).

Finally, we note that
\[ \sum_{t=1}^{\theta} (1 - q_i)^t \leq (1 - 1 - q_i)^{\theta} \]
For a large item \(i\), we have
\[ (1 - q_i) \frac{1 - (1 - q_i)^\theta}{q_i} \leq \frac{1 - c/\theta}{c/\theta} \left( 1 - (1 - c/\theta)^\theta \right) = \frac{\theta}{c} \left( 1 - \frac{c}{\theta} \right) (1 - (1 - c/\theta)^\theta) \]
where the inequality follows from the fact that the first expression is decreasing in \(q_i\) (and, for large items, \(q_i \geq c/\theta\)).

This finally gives us a bound for large \(i\):
\[ \Pr(i \text{ gets probed}) \geq \rho(1 - 2\rho) \left( 1 - \frac{1}{c} (1 - c/\theta) (1 - (1 - c/\theta)^\theta) \right) x_i \]
We argue that the expression on the RHS is decreasing in \(\theta\). To see this, let
\[ f_c(\theta) = \left( 1 - \frac{c}{\theta} \right) \left( 1 - \left( 1 - \frac{c}{\theta} \right)^\theta \right) \]
so that we can rewrite equation (29) as

\[ \Pr(\text{i gets probed}) \geq \rho(1 - 2\rho) \left(1 - \frac{1}{c} f_c(\theta)\right) x_i \]

To show that the RHS of (29) is decreasing in \( \theta \), it suffices to show that \( f_c(\theta) \) is increasing. Taking the derivative, we get:

\[
\frac{df}{d\theta} = \frac{c}{\theta^2} \left(1 - \left(1 - \frac{c}{\theta}\right)^\theta\right) + \left(1 - \frac{c}{\theta}\right)^\theta \ln\left(1 - \frac{c}{\theta}\right) \frac{c}{\theta^2} \]

\[
= \frac{c}{\theta^2} \left(1 - \left(1 - \frac{c}{\theta}\right)^\theta\right) + \left(1 - \frac{c}{\theta}\right)^{\theta+1} \ln\left(1 - \frac{c}{\theta}\right) \]

\[
= \frac{c}{\theta^2} \left(1 - \left(1 - \frac{c}{\theta}\right)^\theta\right) \left(1 - \left(1 - \frac{c}{\theta}\right) \ln\left(1 - \frac{c}{\theta}\right)\right) \tag{30}
\]

We wish to show that (30) is nonnegative. First, we observe that

\[
1 - \left(1 - \frac{c}{\theta}\right) \ln\left(1 - \frac{c}{\theta}\right) \leq 1 - \ln\left(1 - \frac{c}{\theta}\right) < 1 + \frac{\frac{c}{\theta}}{1 - \frac{c}{\theta}} = \frac{1}{1 - \frac{c}{\theta}} \tag{31}
\]

where the second inequality follows from the fact that \( \ln(1 + x) > x/(1 + x) \) when \( x > -1 \) (and taking \( x = -c/\theta \)). Combining (31) with (30), we get:

\[
\frac{df}{d\theta} \geq \frac{c}{\theta^2} \left(1 - \left(1 - \frac{c}{\theta}\right)^\theta\right) \left(1 - \left(1 - \frac{c}{\theta}\right)^{\theta-1}\right) > 0
\]

Therefore, \( f_c(\theta) \) is increasing in \( \theta \), and the RHS of (29) is decreasing in \( \theta \). As a result, the expression is minimized when \( \theta \to \infty \), so we can simplify our bound further.

\[ \Pr(\text{i gets probed}) \geq \rho(1 - 2\rho) \left(1 - \frac{1}{c} (1 - e^{-c})\right) x_i \tag{32} \]

Thus, the total expected weight produced by \( \pi_{\text{LARGE}} \) is at least

\[ \sum_{i \in \text{LARGE}} w_i p_i \Pr(\text{i gets probed}) \geq \sum_{i \in \text{LARGE}} w_i p_i \left(1 - \frac{1}{c} (1 - e^{-c})\right) \rho(1 - 2\rho) \]

\[
\geq \left(1 - \frac{1}{c} (1 - e^{-c})\right) \rho(1 - 2\rho)(1 - \varphi)\text{LP} \tag{33}
\]

where the first inequality comes from (32) and the second comes from the assumption. \( \square \)

**Proof of Lemma 4.** Since \( \sum_{i \in \text{LARGE}} w_i x_i p_i < (1 - \varphi)\text{LP} \), we must have that \( \sum_{i \in \text{SMALL}} w_i x_i p_i \geq \varphi\text{LP} \).

As before, we consider the probability that an item \( i \) is probed at time step \( t \). Let \( A_{i,t} \) be the event that \( i \) arrives at time \( t \), \( B_t \) be the event that an item is successfully sold before time \( t \), and \( C_{i,t} \) be the event that some other small item \( j \neq i \) arrives at time \( t \) and wants to be probed.

\[ \Pr(\text{i probed at time } t) = \Pr(A_{i,t} \cap B_t \cap \overline{C_{i,t}}) \frac{x_i}{1 - (1 - q_i)^\theta} \]
\[
= \Pr(A_{i,t}) \Pr(B_t \cap \bar{C}_{i,t} | A_{i,t}) \frac{x_i}{1 - (1 - q_i)\theta} \\
= \Pr(A_{i,t}) \Pr(B_t | A_{i,t}) \Pr(\bar{C}_{i,t} | A_{i,t} \cap \bar{B}_t) \frac{x_i}{1 - (1 - q_i)\theta} \\
= \Pr(A_{i,t}) (1 - \Pr(B_t | A_{i,t})) \Pr(\bar{C}_{i,t} | A_{i,t} \cap \bar{B}_t) \frac{x_i}{1 - (1 - q_i)\theta}
\]

The probability of \( i \) arriving precisely at time \( t \) is

\[
\Pr(A_{i,t}) = q_i (1 - q_i)^{t-1}.
\]

For the probability that an item is successfully sold before time \( t \), since we condition on \( i \) arriving at time \( t \) we only need to consider the probability of successfully selling some item \( j \neq i \):

\[
\Pr(B_t | A_{i,t}) \leq \sum_{j \in I_{\text{SMALL}} - \{i\}} \left( \sum_{t'=1}^{t-1} q_j (1 - q_j)^{t'-1} \frac{x_j}{1 - (1 - q_j)\theta} p_j \right)
\]

\[
= \sum_{j \in I_{\text{SMALL}} - \{i\}} (1 - (1 - q_j)^{t-1}) \frac{x_j}{1 - (1 - q_j)\theta} p_j
\]

\[
= \sum_{j \neq I_{\text{SMALL}} - \{i\}} 1 - (1 - q_j)^{t-1} \frac{x_j}{1 - (1 - q_j)\theta} p_j
\]

Next, we note that the expression

\[
\frac{1 - (1 - q_j)^{t-1}}{1 - (1 - q_j)\theta}
\]

is increasing in \( q_j \) and so we have

\[
\Pr(B_t | A_{i,t}) \leq \sum_{j \neq I_{\text{SMALL}} - \{i\}} 1 - (1 - c/\theta)^{t-1} \frac{x_j}{1 - (1 - q_j)\theta} p_j
\]

since \( q_j \leq c/\theta \) for small \( j \). Using LP constraint (6a) then gives (34).

\[
\Pr(B_t | A_{i,t}) \leq \frac{1 - (1 - c/\theta)^{t-1}}{1 - (1 - c/\theta)\theta}
\]

For the probability that another item \( j \neq i \) arrives and wants to be probed, we have:

\[
\Pr(\bar{C}_{i,t}) \geq \prod_{j \in I_{\text{SMALL}} - \{i\}} \left( 1 - (1 - q_j)^{t-1} q_j \frac{x_j}{1 - (1 - q_j)\theta} \right) \geq \prod_{j \in I_{\text{SMALL}} - \{i\}} \left( 1 - q_j \frac{x_j}{1 - (1 - q_j)\theta} \right)
\]

We wish to lower bound this product. Let \( z_{i,j} := q_j \frac{x_j}{1 - (1 - q_j)\theta} \). Notice that since \( q_j \leq c/\theta \) (because \( j \) is small) and \( x_j / (1 - (1 - q_j)^\theta) \leq 1 \) (by constraint (3b)), we have that \( z_j \leq c/\theta \). Further:

\[
\sum_{j \in I_{\text{SMALL}} - \{i\}} z_j = \sum_{j \in I_{\text{SMALL}} - \{i\}} q_j \frac{x_j}{1 - (1 - q_j)\theta} \leq \sum_{j \in I_{\text{SMALL}} - \{i\}} \frac{c}{\theta} \frac{x_j}{1 - (1 - q_j)^\theta} \leq \frac{c}{1 - (1 - c/\theta)\theta}
\]

A lower bound is therefore given by the solution to the following minimization problem:

\[
\begin{align*}
\min_{j \in I_{\text{SMALL}} - \{i\}} & \quad (1 - z_j) \\
\text{s.t.} & \quad z_j \leq c/\theta, \quad \forall j \in I_{\text{SMALL}} - \{i\} \\
& \quad \sum_{j \in I_{\text{SMALL}} - \{i\}} z_j \leq \frac{c}{1 - (1 - c/\theta)\theta}
\end{align*}
\]
Lemma 4 (the precise statement and proof of which is presented after the present proof) gives us a lower bound on the solution to this minimization problem. Using $b = c/\theta$ and $S = c/(1 - (c/\theta)^{\theta})$, Lemma 4 tells us that

$$\Pr(C_{i,t} \mid A_{i,t} \cap B_t) \geq \left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}}$$

We can now put everything together to get

$$\Pr(i \text{ probed at time } t) \geq q_i (1 - q_i)^{t-1} \left(1 - \frac{1 - (1 - \frac{c}{\theta})^{t-1}}{1 - (1 - \frac{c}{\theta})^{\theta}}\right) \left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}}$$

By summing over all time steps $t$, we can get a bound on the probability that a small item $i$ gets probed:

$$\Pr(i \text{ gets probed}) \geq \sum_{t=1}^{\theta} q_i (1 - q_i)^{t-1} \frac{x_i}{1 - (1 - q_i)^{\theta}} \left(1 - \frac{1 - (1 - \frac{c}{\theta})^{t-1}}{1 - (1 - \frac{c}{\theta})^{\theta}}\right) \left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}}$$

$$\geq \sum_{t=1}^{\theta} q_i (1 - q_i)^{t-1} \frac{x_i}{1 - (1 - q_i)^{\theta}} \left(1 - \frac{1 - (1 - \frac{c}{\theta})^{t-1}}{1 - (1 - \frac{c}{\theta})^{\theta}}\right) \left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}} \sum_{t=1}^{\theta} q_i (1 - q_i)^{t-1}$$

$$= \frac{1}{\theta} \left(1 - \left(1 - \frac{c}{\theta}\right)^{\theta}\right) \sum_{t=1}^{\theta} \left(1 - \left(1 - \frac{c}{\theta}\right)^{t-1}\right) \left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}}$$

Now, note that

$$\sum_{t=1}^{\theta} \left(1 - \left(1 - \frac{c}{\theta}\right)^{t-1}\right) \left(1 - \left(1 - \frac{c}{\theta}\right)^{t-1}\right)$$

$$= \sum_{t=1}^{\theta} \left(1 - \left(1 - \frac{c}{\theta}\right)^{t-1}\right)$$

$$= -\theta(1 - \frac{c}{\theta}) + \sum_{t=1}^{\theta} (1 - \frac{c}{\theta})^{t-1}$$

$$= -\theta(1 - \frac{c}{\theta}) + \frac{\theta}{c}, \left(1 - \frac{c}{\theta}\right)$$

which gives

$$\Pr(i \text{ gets probed}) \geq \left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}} \left(1 - \left(1 - \frac{c}{\theta}\right)^{\theta}\right) \left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}}$$

since $(1 - c/\theta)^{\theta}$ is maximized when $\theta \to \infty$ so that $(1 - c/\theta)^{\theta} \leq e^{-c}$. Further, we note that

$$\left(1 - \frac{c}{\theta}\right)^{1-(1-c/\theta)^{\theta}} \leq e^{-c}$$
is increasing in $\theta \geq 2$ when $c \in (0,1)$, and so we have
\[
(1 - \frac{c}{\theta})^{\frac{\theta}{1-(1-c/\theta)^2}} \geq (1 - \frac{c}{2})^{\frac{2}{1-(1-c/2)^2}}
\]
which allows us to get our final bound:
\[
\sum_{i \in I_{\text{SMALL}}} w_i p_i \Pr(i \text{ gets probed}) \geq \sum_{i \in I_{\text{SMALL}}} w_i p_i x_i \left(1 - \frac{c}{1 - e^{-c}}\right) \left(1 - \frac{c}{1 - e^{-c}}\right) \left(1 - \frac{c}{2}\right)^{\frac{2}{1-(1-c/2)^2}} \geq \varphi \left(1 - \frac{c}{2}\right)^{\frac{2}{1-(1-c/2)^2}} \left(1 - \frac{c}{1 - e^{-c}}\right) LP
\]
as desired. □

**Lemma 6.** Consider the following optimization problem over variables $z_j$ for $j \in J$, with fixed positive constants $b, S$:
\[
F = \min \prod_{j \in J} (1 - z_j) \tag{36}
\]
\[
\text{s.t. } z_j \leq b, \quad \forall j \in J \tag{37}
\]
\[
\sum_{j \in J} z_j \leq S \tag{38}
\]
Then, the solution satisfies
\[
F \geq (1 - b)^{S/b}
\]

**Proof of Lemma** First, we claim that in the optimal solution to the above program, at most one $j$ has $z_j \in (0,b)$; the rest must all be equal to either 0 or $c/\theta$. To see this, note that if we have two items $j,k$ for which both $z_j \in (0,b)$ and $z_k \in (0,b)$, then the objective value is $Z \cdot (1 - z_j)(1 - z_k)$, where $Z = \prod_{l \in I_{\text{SMALL}} - \{i,j,k\}} (1 - z_l)$. Assume wlog that $z_j \geq z_k$. Then, we consider the alternate assignment $z'_j = z_j + \epsilon$, $z'_k = z_k - \epsilon$, where $\epsilon = \min \{b - z_j, z_k\}$. Clearly, since $0 < z_k \leq z_j < b$, we have $0 < \epsilon < b$. We can then see that
\[
(1 - z'_j)(1 - z'_k) = (1 - z_j - \epsilon)(1 - z_k + \epsilon) = 1 - z_j - z_k + z_j z_k + \epsilon z_j + \epsilon^2 = (1 - z_j)(1 - z_k) - \epsilon(z_j - z_k + \epsilon) < (1 - z_j)(1 - z_k)
\]
since $\epsilon > 0$ and $z_j \geq z_k$. This alternate assignment would therefore have an objective value of $Z(1 - z'_j)(1 - z'_k) < Z(1 - z_j)(1 - z_k)$.

Further, note from the definition of $\epsilon$ that we have $z'_j \leq b$ and $z'_k \geq 0$, with at least one of these being equality. Certainly, we also have $z_j > 0$ and $z_k < b$, and it is clear that $z'_j + z'_k = z_j + z_k$; thus, the new assignment is still feasible. This means that if we have two items $j,k$ with $0 < z_k \leq z_j < b$, then this is not an optimal assignment since we can decrease the objective value by either increasing $z_j$ to $b$ or decreasing $z_k$ to 0.
We can therefore see that the objective is minimized when $z_j = b$ for as many items as possible. Let $\nu = \lfloor S/b \rfloor$, and $\eta = S - \nu b$. The optimal solution sets $z_j = b$ for $\nu$ items, $z_j = \eta$ for a single item, and $z_j = 0$ for all other items. This gives an objective value of

$$F = (1 - b)^\nu (1 - \eta)$$

and we wish to show that this is lower bounded by $(1 - b)^{S/b}$. Noting that $S/b = \nu + \eta b$, we wish to show that

$$(1 - b)^\nu (1 - \eta) \geq (1 - b)^{\nu + \eta b}$$

Simplifying this expression, we find it is equivalent to

$$(1 - \eta)^{1/\eta} \geq (1 - b)^{1/b}$$ \hspace{1cm} (39)

The Taylor series expansion of the natural log gives that

$$\frac{1}{x} \ln(1 - x) = -\frac{1}{x} \sum_{r=1}^{\infty} \frac{x^r}{r!} = -\sum_{r=1}^{\infty} \frac{x^{r-1}}{r!}$$

From this, it is clear that since $\eta \leq b$, we must have

$$\sum_{r=1}^{\infty} \frac{\eta^{r-1}}{r!} \leq \sum_{r=1}^{\infty} \frac{b^{r-1}}{r!}$$

and therefore $\frac{1}{\eta} \ln(1 - \eta) \geq \frac{1}{b} \ln \left(1 - \frac{1}{b}\right)$, which implies (39) as desired. \hfill \Box

Proof of Theorem 7 We use $\rho = 1/4$ and $\phi = 0.37$. For this $\rho, \phi$, strategy $\pi_{\text{LARGE}}$ achieves a ratio of

$$(1 - 0.37) \left( \frac{1}{8} \right) \left( 1 - \frac{1}{c} (1 - e^{-c}) \right)$$

and strategy $\pi_{\text{SMALL}}$ achieves a ratio of

$$0.37 \left( 1 - \frac{c}{2} \right)^{1 - \left( 1 - c/2 \right)^2} \left( \frac{1}{c} - \frac{e^{-c}}{1 - e^{-c}} \right)$$

The equation

$$(1 - 0.37) \left( \frac{1}{8} \right) \left( 1 - \frac{1}{c} (1 - e^{-c}) \right) = 0.37 \left( 1 - \frac{c}{2} \right)^{1 - \left( 1 - c/2 \right)^2} \left( \frac{1}{c} - \frac{e^{-c}}{1 - e^{-c}} \right)$$

has a solution at $c \approx 0.9249$, where both $\pi_{\text{LARGE}}$ and $\pi_{\text{SMALL}}$ achieve an approximation ratio of approximately 0.0273714 > 0.027. \hfill \Box

Proof of Theorem 8 Consider the complete bipartite graph $G = (U, V; E)$ with $|U| = |V| = n$ and $p_{uv} = 1/n$ for all $(u, v) \in U \times V$. We state the following result that is implied by Karp and Sipser (1981).
**Lemma 7** (Theorem 14 of Bollobas and Brightwell (1995); implied by Karp and Sipser (1981)). Let $G$ be a random bipartite graph with both partitions of size $n$ and where each edge exists independently with probability $p = 1/n$. Let $\gamma$ be the solution to the equation $\gamma = e^{-\gamma}$. Then, the largest independent set of $G$ has size $n(2\gamma + \gamma^2))[1 + o(1)]$ with probability $1 - o(1)$.

Lemma 7 implies that, almost surely as $n \to \infty$, there exists an independent set of size $n(2\gamma + \gamma^2)$, and hence there exists a vertex cover of size $2n - n(2\gamma + \gamma^2) \approx 0.544n$ (found by taking all the vertices not in the independent set). Therefore almost surely no matching can have size greater than $0.544n$. It follows that no online or offline algorithm can achieve an expected matching size greater than

$$\frac{(1 - o(1))0.544n + o(1)n}{n} = 0.544 + o(1),$$

completing the proof that the stochasticity gap is at least $0.544$. □

**Proof of Theorem 9.** We complete the proof of Theorem 9 by showing that an online algorithm cannot earn more than $k + o(k)$.

An adversary in the online setting may expose all vertices of $V_n$ before any of the vertices of $V_0$ to the greedy algorithm. SimpleGreedy choosing arbitrarily may in the worst case choose to probe edges of $E_0$ first, preventing some vertices of $V_0$ from being matched later. We consider the case where SimpleGreedy chooses an edge $(u, v_i) \in E_0$ for each online vertex $v_i \in V_n$ if any $u \in U_0$ is available at $v_i$’s arrival. We calculate the expected size of the matching produced by this strategy.

Let $M$ be a random variable corresponding to the size of the final matching, and let $M_0$ and $M_n$ be random variables corresponding to the number of matched vertices in $V_0$ and $V_n$, respectively. Then, the expected size of the matching is

$$\mathbb{E}[M] = \mathbb{E}[M_n] + \mathbb{E}[M_0] = k + \mathbb{E}[M_0].$$

We now consider $\mathbb{E}[M_0]$. If $l < k$ vertices of $V_n$ are matched successfully, then when the vertices of $V_0$ arrive online, there will only be $k - l$ vertices of $U_0$ remaining to be matched. Since $|V_0| = kn^2$, greedy will almost surely match all of them successfully. Thus, we get (as $n \to \infty$)

$$\mathbb{E}[M_0] = \sum_{l=0}^{k-1} \binom{n}{l} (1 - k/n)^{n-l} (k/n)^l (k-l) / l! \approx \sum_{l=0}^{k-1} \frac{k^l}{l!} \frac{e^{-k}k^l}{n^l} \frac{1}{l!} = \sum_{l=0}^{k-1} \frac{e^{-k}k^l}{(l-1)!} \frac{1}{l!}$$

$$= k \left[ \sum_{l=0}^{k-1} \frac{e^{-k}k^l}{l!} - \sum_{i=0}^{k-2} \frac{e^{-k}k^l}{l!} \right] = k \cdot e^{-k}k^{k-1} = k \cdot e^{-k}k^k / (k-1)! = k \cdot e^{-k}k^k / k!.$$

Finally, we observe that for large $k$, $e^{-k}k^k / k! \sim (2\pi k)^{-1/2}$, due to Stirling’s formula. Therefore, $\mathbb{E}[M] = k + o(k)$. □
Proof of Theorem 10. Fix a positive integer \( k \) and let \( m = \omega(k) \) be large. An algorithm which knows the patience in advance may adopt the following strategy. For any \( i = 0, \ldots, k \), if the patience of \( v \) is \( m^{2i} \), then repeatedly probe edges for offline vertices of type \( i \), which have weight \( m^i \) and probability \( m^{-2i} \), until a success or until patience expires. If any of these edges is successfully matched, a reward of \( m^i \) is achieved, so the expected reward in this case is \([1 - (1 - m^{-2i})m^{2i}]m^i \geq (1 - 1/e)m^i\). Thus, the expected reward of this strategy is at least

\[
\sum_{i=0}^{k-1} (m^{-i} - m^{i-1})(1 - 1/e)m^i + m^{-k}(1 - 1/e)m^k
\]

\[
= \sum_{i=0}^{k-1} (1 - 1/m)(1 - 1/e) + (1 - 1/e)
\]

\[
= (k+1)(1 - 1/e) - k/m(1 - 1/e)
\]

\[
= (1 - 1/e)(k + 1 - k/m).
\]

We now show that an online algorithm cannot achieve an expected reward greater than 1. We split the online probing strategy into stages, where in stage \( i \) it is known that the patience is at least \( m^{2i} \), for all \( i = 0, \ldots, k \). That is, stage 0 begins on the first probe, and stage \( i \) begins with the \( m^{2(i-1)} + 1 \)'th probe for all \( i = 1, \ldots, k \). We will let \( V(i) \) denote the expected reward received from stage \( i \) onward, conditioned on reaching stage \( i \) (with the online vertex still unmatched). Note that the expected reward of the offline algorithm is then given by \( V(0) \).

We proceed by backward induction. First, consider the case where the algorithm has reached the final stage \( i = k \). This occurs after \( m^{2(k-1)} \) unsuccessful probes, so that there are \( m^{2(k-1)}(m^2 - 1) \) remaining probes before the patience is exhausted. In this case, the expected reward is at most \( m^k \), since the maximum weight of any edge is \( m^k \).

Our inductive hypothesis is \( V(i) \leq m^i + (k - i)m^{i-1} \), which we already demonstrated holds in the base case \( i = k \). Now, consider an arbitrary stage \( i < k \). At the beginning of stage \( i \), the algorithm has made \( m^{2(i-1)} \) probes, and the patience is known to be at least \( m^{2i} \). The stage ends after \( m^{2i} - m^{2(i-1)} \) unsuccessful probes (it may end sooner if any probe is successful), at which point the algorithm either learns that the patience is exactly \( m^{2i} \) (and can make no more probes), or learns that the patience is at least \( m^{2(i+1)} \) (and enters stage \( i + 1 \)).

We make the following observations about the optimal online algorithm at stage \( i \). The inductive hypothesis implies that \( V(i) \leq m^{i+1} + (k - (i + 1))m^i \). Moreover, conditional on the patience being at least \( m^{2i} \), the probability of the patience being at least \( m^{2(i+1)} \) (i.e., the next stage existing) is \( 1/m \). Therefore, \( V(i) \) cannot be greater than in a hypothetical universe where the online algorithm can make \( m^{2i} - m^{2(i-1)} \) probes, and if none of them are successful, then collects a deterministic reward of \( \frac{1}{m}(m^{i+1} + (k - (i + 1))m^i) = m^i + (k - (i + 1))m^{i-1} \).
In this hypothetical universe, the online algorithm should never probe an edge with weight \( m' \) with \( i' \leq i \), since it would collect a better reward from waiting for stage \( i+1 \). On the other hand, if it probes only edges with weight \( m' \) with \( i' > i \) during its \( m^2i - m^2(i-1) \) probes, then \( V(i) \) can be no greater than

\[
m^2m^{-i+1} + m^i + (k - (i + 1))m^{i-1}
\]  

(40)
since the online algorithm has \( m^2i - m^2(i-1) \leq m^{2i} \) probes and the expected value of each probe can be at most \( m^{-2i'} m' = m^{-i'} \leq m^{-(i+1)} \), with the final term \( m^i + (k - (i + 1))m^{i-1} \) representing the online algorithm’s reward if none of the probes are successful (expression (40) assumes this reward is always collected, which is clearly a valid upper bound). Expression (40) equals \( m^i + (k - i)m^{i-1} \), completing the induction.

Therefore, we have established that \( V(0) \leq 1 + k/m \). The ratio between the expected earnings of the best online algorithm relative to the offline (which knows the patience realization in advance) is at most

\[
\frac{1 + k/m}{(1 - 1/e)(k + 1 - k/m)}
\]

which approaches \( \frac{1}{(1-1/e)(k+1)} \) as \( m \to \infty \), completing the proof. \( \square \)

Appendix B: Relation to the Online Assortment Problem

The work of Golrezaei et al. (2014) considers a model in which online vertices represent customers and offline vertices represent products, and a merchant wishes to offer products to consumers so as to maximize profit. This setting differs from ours in that the merchant offers a collection of several products all at once. Then, the customer chooses to either purchase some product (or multiple products at once) based on products offered or purchase nothing. By contrast, in our model the algorithm (the “merchant” in our setting) attempts one match at a time, stopping when a successful match occurs or the number of unsuccessful attempts equals the patience constraint.

In the setting of Golrezaei et al. (2014), each customer \( v \) has a “general choice model” \( \phi_v(S, u) \) that specifies the probability that customer \( v \) purchases item \( u \) when offered the set \( S \) of items. More generally, since the model considers that \( v \) may purchase more than one item, \( \phi_v(S, S') \) is used to denote the probability that \( v \) will purchase exactly the items \( S' \) when offered \( S \) (and then \( \phi_v(S, u) \) is defined to be \( \sum_{S':u \in S'} \phi_v(S, S') \)). It is assumed that the customer will only purchase products that were offered as part of the assortment \( S \) (that is, \( \phi_v(S, S') = 0 \) if \( S' \not\subseteq S \)).

The algorithm they propose for their model can be viewed as a greedy algorithm which presents an online-arriving customer \( v \) with the set \( S \) that maximizes the expected profit of the items \( v \) purchases. Doing so would guarantee a competitive ratio of at least 0.5, though this maximization step is not necessarily solvable in polynomial time for arbitrary choice models.
Their results do not immediately extend to our setting, as their stochastic model is somewhat different. Extending their results to our setting requires a reduction from our sequential probing with the probe-commit model to this all-at-once model by construction of appropriate choice models \( \phi \). Further, such a reduction would not necessarily yield a polynomial-time result without also designing an algorithm for solving the aforementioned maximization in polynomial time.

One contribution of the present work is Algorithm 1 which indeed can be viewed as greedily maximizing the expected weight (or profit) of \( v \)'s match (or purchase). However, without also constructing a reduction from our sequential probing model to this all-at-once model, the result of Golrezaei et al. (2014) does not extend to give a competitive ratio of 0.5 for our problem. Rather, in the present work, we present a clean, self-contained analysis of Algorithm 1 to achieve a competitive ratio of 0.5 for our problem without relying on the results of Golrezaei et al. (2014) or the similar framework of Cheung et al. (2022) based on “actions”.

**Appendix C: Proof of Proposition 2 from Cheung and Simchi-Levi (2016)**

The potential-based framework of Cheung and Simchi-Levi (2016) finds for each \( v \in V \) a policy \( \pi \) that is \( \kappa \)-approximate to the problem of \( \max_{\pi \in \mathcal{P}} \sum_{u \in U} (w_{uv} - \alpha_u) p_{uv}(\pi) \), updates the dual variables \( \alpha_u \) accordingly, and then repeats iteratively. The analysis uses a potential function motivated by the multiplicative weights method Arora et al. (2012). The final solution returned is the average of all the policies found over the iterations, and the total number of iterations needed to average into a feasible and \((\kappa - \epsilon)\)-approximate LP solution is polynomial in \( 1/\epsilon \).

Although it is stated in the context of assortment optimization, Theorem 3.4 from Cheung and Simchi-Levi (2016) (and its generalization in Appendix G of Cheung and Simchi-Levi (2016)) directly implies our Proposition 2 after the following observations about the correspondence with our context of online matching with patience. In assortment optimization, there are an exponential number of “assortments” \( S \) (unordered subsets) of offline vertices that could be offered to an online vertex of type \( v \), after which there is a probability \( p_{uv}(S) \) of each \( u \in S \) being matched. The only ingredient needed in Cheung and Simchi-Levi (2016) is that given any set of (possibly negative) adjusted weights \( w'_{uv} \) for an online type \( v \), an assortment which solves \( \max_S \sum_{u \in S} w'_{uv} p_{uv}(S) \) within a factor of \( \kappa \) can be found in polynomial time. The framework is unchanged if in our setting, we consider policies \( \pi \) which are ordered subsets of offline vertices instead. Moreover, our black boxes for finding optimal or approximately optimal policies \( \pi \) can ignore any negative adjusted weights, since it is never beneficial to probe a negative edge. Finally, the general packing constraints and flexible products allowed in Cheung and Simchi-Levi (2016) are generalizations which can be ignored, and as a result the statement of Theorem 3.4 from Cheung and Simchi-Levi (2016) implies our Proposition 2.