Research Article

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The molecular characterization of anisotropic Herz-type Hardy spaces with two variable exponents

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Abstract: In this article, the authors establish the characterizations of a class of anisotropic Herz-type Hardy spaces with two variable exponents associated with a non-isotropic dilation on \( \mathbb{R}^n \) in terms of molecular decompositions. Using the molecular decompositions, the authors obtain the boundedness of the central \( \delta \)-Calderón-Zygmund operators on the anisotropic Herz-type Hardy space with two variable exponents.

Keywords: anisotropic Herz-type Hardy space, two variable exponents, molecular decomposition, central \( \delta \)-Calderón-Zygmund operator, boundedness

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1 Introduction

The theory of function spaces with variable exponents has rapidly made progress in the past 20 years since some elementary properties were established by Kováčik and Rákosník [1]. Lebesgue and Sobolev spaces with variable exponents have been extensively investigated, see [2] and the references therein. In 2012, Almeida and Drihem [3] introduced the Herz spaces with two variable exponents and obtain the boundedness of some sublinear operators on those spaces. In the same year, Wang and Liu [4] introduced the Herz-type Hardy spaces with variable exponents \( HK_p^{\alpha,q}(\mathbb{R}^n) \) and \( HK_p^{\alpha,q}(\mathbb{R}^n) \), which is a generalization of the classical Herz-type Hardy spaces. In 2015, Dong and Xu [5] introduced the Herz-type Hardy spaces with two variable exponents \( HK_p^{\alpha,q}(\mathbb{R}^n) \) and \( HK_p^{\alpha,q}(\mathbb{R}^n) \).

Recently, extending classical function spaces arising in harmonic analysis of Euclidean spaces to other domains and non-isotropic settings is an important topic. In 2003, Bownik [6] introduced the anisotropic Hardy spaces \( H^p_\delta(\mathbb{R}^n) \) associated with very general discrete groups of dilations. This formulation includes the classical isotropic Hardy space theory established by Fefferman and Stein [7] and the parabolic Hardy space theory established by Calderón and Torchinsky [8,9]. In 2008, Ding et al. [10] introduced the anisotropic Herz-type Hardy spaces \( HK_p^{\alpha,q}(A; \mathbb{R}^n) \) and \( HK_p^{\alpha,q}(A; \mathbb{R}^n) \) and established their atomic and molecular decompositions. In 2018, Zhao and Zhou [11] introduced the variable anisotropic Herz-type Hardy spaces \( HK_p^{\alpha,q}(A; \mathbb{R}^n) \) and \( HK_p^{\alpha,q}(A; \mathbb{R}^n) \) and established their atomic and molecular decompositions. Using these decompositions, they gave some applications. In 2019, Wang and Guo [12] introduced the variable anisotropic Herz-type Hardy spaces \( HK_p^{\alpha,q}(A; \mathbb{R}^n) \) and \( HK_p^{\alpha,q}(A; \mathbb{R}^n) \) and established their atomic decomposition and some applications.

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Inspired by the previous study, we would like to declare that the goal of this study is to establish the characterizations of a class of anisotropic Herz-type Hardy spaces with two variable exponents associated with a non-isotropic dilation on $\mathbb{R}^n$ in terms of molecular decompositions and obtain the boundedness of the central $\delta$-Calderón-Zygmund operators on the anisotropic Herz-type Hardy space with two variable exponents.

First, we recall some standard notations in variable function spaces. A measurable function $p(\cdot): \mathbb{R}^n \to (0, \infty)$ is called a variable exponent. Let $f$ be a measurable function on $\mathbb{R}^n$ and $p(\cdot) \in \mathcal{P}$. Then, the modular function (or, for simplicity, the modular) $\varrho_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting

$$
\varrho_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx
$$

and the Luxemburg (also called Luxemburg–Nakano) quasi-norm $\|f\|_{L^{p(\cdot)}}$ by

$$
\|f\|_{L^{p(\cdot)}} = \inf\{\lambda \in (0, \infty) : \varrho_{p(\cdot)}(f/\lambda) \leq 1\}.
$$

Moreover, the variable Lebesgue space $L^{p(\cdot)}$ is defined to the set of all measurable functions $f$ satisfying that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}}$. For any variable exponent $p(\cdot)$, let

$$
p_+ = \text{ess inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_- = \text{ess sup}_{x \in \mathbb{R}^n} p(x).
$$

Denote by $\mathcal{P}$ the set of all variable exponents $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$. We call $p'(\cdot)$ the conjugate exponent to $p(\cdot)$, that is, $p'(\cdot) = \frac{p_-}{p_-(p_--1)}$. Let $\mathcal{B}$ be the set of $p(\cdot) \in \mathcal{P}$, such that the Hardy–Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}$. It is well known that if $p(\cdot) \in \mathcal{P}$ and satisfies the following global log-Hölder continuous, then $p(\cdot) \in \mathcal{B}$.

**Definition 1.1.** Let $\alpha(\cdot)$ be a real function on $\mathbb{R}^n$.

(i) $\alpha(\cdot)$ is called log-Hölder continuous on $\mathbb{R}^n$ if there exists $C > 0$, such that

$$
|\alpha(x) - \alpha(y)| \leq \frac{C}{\log(e + 1/|x-y|)}
$$

for all $x, y \in \mathbb{R}^n$ and $|x-y| < \frac{1}{2}$.

(ii) $\alpha(\cdot)$ is called log-Hölder continuous at origin (or has a log decay at the origin), if there exists $C > 0$, such that

$$
|\alpha(x) - \alpha(0)| \leq \frac{C}{\log(e + 1/|x|)}
$$

for all $x \in \mathbb{R}^n$.

(iii) $\alpha(\cdot)$ is called log-Hölder continuous at infinity (or has a log decay at the infinity), if there exist some $\alpha_{\infty} \in \mathbb{R}^n$ and $C > 0$, such that

$$
|\alpha(x) - \alpha_{\infty}| \leq \frac{C}{\log(e + |x|)}
$$

for all $x \in \mathbb{R}^n$.

By $\mathcal{P}_0(\mathbb{R}^n)$ and $\mathcal{P}_\infty(\mathbb{R}^n)$, we denote the class of all exponents $p \in \mathcal{P}(\mathbb{R}^n)$, which are locally log-Hölder continuous at the origin and at the infinity, respectively.
Next, we will recall the notion of expansive dilations on $\mathbb{R}^n$; see [6, p. 5]. A real $n \times n$ matrix $A$ is called an expansive dilation, if all eigenvalues $\lambda$ of $A$ satisfy $|\lambda| > 1$. Suppose $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $A$ (taken according to the multiplicity), so that $1 < |\lambda_i| \leq \ldots \leq |\lambda_n|$. A set $\Delta \in \mathbb{R}^n$ is said to be an ellipsoid if $\Delta = \{x \in \mathbb{R}^n: |P| < 1\}$, for some nondegenerate $n \times n$ matrix $P$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. For a dilation $A$, there exists an ellipsoid $\Delta$ and $r > 1$, such that $\Delta \subset r\Delta \subset AD$, where $|\Delta|$, the Lebesgue measure of $\Delta$, equals 1. Let $B_k = A^k\Delta$ for $k \in \mathbb{Z}$, then we have $B_k = rB_k \subset B_{k+1}$, and $B_k = b^k$, where $b = |\det A| > 1$. Let $n$ be the smallest integer, so that $2B_0 \subset A^nB_0 = B_n$. A homogeneous quasi-norm associated with an expansive matrix $A$ is a measurable mapping $\rho_A: \mathbb{R}^n \to [0, \infty)$ satisfying

$$\rho_A(x) > 0 \text{ for } x \neq 0, \quad \rho_A(Ax) = |\det A|\rho_A(x) \text{ for } x \in \mathbb{R}^n,$$

where $C_A$ is a positive constant.

It was proved, in [6, p. 6, Lemma 2.4], that all homogeneous quasi-norms associated with a given dilation $A$ are equivalent. Define the step homogeneous quasi-norm $\rho$ on $\mathbb{R}^n$ induced by dilation $A$ as

$$\rho(x) = \begin{cases} b^i, & \text{if } x \in B_{i+1} \setminus B_i, \\ 0, & \text{if } x = 0. \end{cases}$$

Then, for any $x, y \in \mathbb{R}^n$, $\rho(x + y) \leq b^n(\rho(x) + \rho(y))$.

In the following we denote $C_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$. Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, denote $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, $\tilde{\chi}_k = \chi_k$ if $k \in \mathbb{Z}_+$, and $\tilde{\chi}_0 = \chi_{B_0}$, where $\chi_{C_k}$ is the characteristic function of $C_k$. Throughout this paper, we denote by $C$ a constant, which is independent of the main parameters and whose value may vary.

**Definition 1.2.** Let $a(\cdot): \mathbb{R}^n \to \mathbb{R}$ with $a(\cdot) \in L^\infty(\mathbb{R}^n)$, $0 < q \leq \infty$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous anisotropic Herz space $K^a_p (A; \mathbb{R}^n)$ associated with the dilation $A$ is defined by

$$K^a_p (A; \mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n) \setminus \{0\}: \|f\|_{K^a_p (\cdot)} < \infty \right\},$$

where

$$\|f\|_{K^a_p (\cdot)} \equiv \left\{ \sum_{k=-\infty}^{\infty} \|b^{kar(\cdot)} f_{\tilde{\chi}_k} \|_{L^p(\mathbb{R}^n)}^q \right\}^{1/q}.$$

The nonhomogeneous anisotropic Herz space $K^a_p (A; \mathbb{R}^n)$ associated with the dilation $A$ is defined by

$$K^a_p (A; \mathbb{R}^n) = \left\{ f \in L^p_{loc}(\mathbb{R}^n): \|f\|_{K^a_p (\cdot)} < \infty \right\},$$

where

$$\|f\|_{K^a_p (\cdot)} \equiv \left\{ \sum_{k=-\infty}^{\infty} \|b^{kar(\cdot)} f_{\tilde{\chi}_k} \|_{L^p(\mathbb{R}^n)}^q \right\}^{1/q}.$$

Here, the usual modifications are made when $q = \infty$.

In variable $L^p$ spaces, there are some important lemmas as follows.
Lemma 1.3. [1] Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). If \( f \in L^{p(\cdot)}(\mathbb{R}^n) \) and \( g \in L^{p(\cdot)}(\mathbb{R}^n) \), then \( fg \) is integrable on \( \mathbb{R}^n \) and
\[
\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)},
\]
where \( r_p = 1 + 1/p^- - 1/p^+ \).

The following lemmas are from [13].

Lemma 1.4. Suppose \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \). Then, there exists a constant \( C > 0 \), such that for all balls \( B \) in \( \mathbb{R}^n \),
\[
\frac{1}{|B|} \|\chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C.
\]

Lemma 1.5. Let \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \). Then, there exist \( 0 < \delta_1, \delta_2 < 1 \) depending only on \( p(\cdot) \) and \( n \), such that for all measurable subsets \( S \subset B \),
\[
\frac{\|\chi_S \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq \left( \frac{|S|}{|B|} \right)^{\delta_1},
\]
\[
\frac{\|\chi_S \|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_B \|_{L^{p(\cdot)}(\mathbb{R}^n)}} \leq \left( \frac{|S|}{|B|} \right)^{\delta_2}.
\]

Next, we introduce the definition of homogeneous anisotropic Herz-type Hardy space with two variable exponents \( HK_{p(\cdot)}^{a_1, q_1}(A; R^n) \) and the nonhomogeneous anisotropic Herz-type Hardy space with two variable exponents \( HK_{p(\cdot)}^{a_2, q_2}(A; R^n) \) and the atomic characterization of \( HK_{p(\cdot)}^{a_1, q_1}(A; R^n) \) and \( HK_{p(\cdot)}^{a_2, q_2}(A; R^n) \), which were obtained by Wang and Guo [12].

A \( C^\infty \) complex-valued function \( \varphi \) is said to belong to the Schwartz class \( S \), if for every integer \( \ell \in \mathbb{Z} \), and multi-index \( a, \|\varphi\|_{a, \ell} = \sup_{x \in R^n} |\partial^a \varphi(x)| < \infty \). The dual space of \( S \), namely, the space of all tempered distributions on \( R^n \) equipped with the weak-* topology, is denoted by \( S' \). For any \( N \in \mathbb{Z} \), let
\[
S_N = \{ \varphi \in S : \|\varphi\|_{a, \ell} \leq 1, \ |a| \leq N, \ell \leq N \}.
\]

For \( \varphi \in S \), \( k \in \mathbb{Z} \) and \( x \in R^n \), let \( \varphi_k(x) = b^{-k} \varphi(A^{-k}x) \).

Let \( f \in S' \). The non-tangential maximal function \( M_\varphi(f) \) with respect to \( \varphi \) is defined by setting, for any \( x \in R^n \),
\[
M_\varphi(f)(x) = \sup \{|f \ast \varphi_k(y) : x - y \in B_k, k \in \mathbb{Z} \}.
\]

For any given \( N \in \mathbb{N} \), the non-tangential grand maximal function \( M_\varphi(f) \) of \( f \in S' \) is defined by setting, for any \( x \in R^n \),
\[
M_N(f)(x) = \sup_{\varphi \in S_N} M_\varphi(f)(x).
\]

For \( 0 < q < \infty \), we denote
\[
N_q = \left\lfloor \frac{(1/q - 1) \ln b / \ln \lambda - 2, \ 0 < q \leq 1,}{2, \ \ q > 1.}\right\rfloor
\]

Definition 1.6. Let \( a(\cdot) \in L^\infty \), \( 0 < q \leq \infty \), \( p(\cdot) \in \mathcal{P} \), and \( N > N_q \). The homogeneous anisotropic Herz-type Hardy space with variable exponents \( HK_{p(\cdot)}^{a(\cdot), q}(A; R^n) \) and the nonhomogeneous anisotropic Herz-type Hardy space with variable exponents \( HK_{p(\cdot)}^{a(\cdot), q}(A; R^n) \) are defined, respectively, by setting,
\[ H^a_{p(\cdot)}(A; \mathbb{R}^n) = \{ f \in S': M_{\mathcal{N}}(f) \in \mathring{K}^{a(\cdot),q}_{p(\cdot)}(A; \mathbb{R}^n) \} \]

and

\[ H^{\infty}_{p(\cdot)}(A; \mathbb{R}^n) = \{ f \in S': M_{\mathcal{N}}(f) \in K^{\infty}_{p(\cdot)}(A; \mathbb{R}^n) \}, \]

where

\[ \| f \|_{H^{\infty}_{p(\cdot)}(A; \mathbb{R}^n)} = \| M_{\mathcal{N}}(f) \|_{K^{\infty}_{p(\cdot)}(A; \mathbb{R}^n)} \quad \text{and} \quad \| f \|_{\mathring{H}^{\infty}_{p(\cdot)}(A; \mathbb{R}^n)} = \| M_{\mathcal{N}}(f) \|_{\mathring{K}^{\infty}_{p(\cdot)}(A; \mathbb{R}^n)}. \]

**Definition 1.7.** Let \( p(\cdot) \in \mathcal{P}, a(\cdot) \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_0^\log \), and nonnegative integer \( s \geq \max\{[(a(0) - \delta) \ln b / \ln \lambda], [(a_\infty - \delta) \ln b / \ln \lambda] \} \), where \( \delta \) as in Lemma 1.5. Here, \( a_0 = a_0, \) if \( l < 0, a_l = a_\infty, \) if \( l > 0. \)

1. An anisotropic central \((a(\cdot), p(\cdot), s)-\)atom is a measurable function \( a \) on \( \mathbb{R}^n \) satisfying
   
   (i) \( \operatorname{supp} a \subset B_l \), for some \( l \in \mathcal{Z} \);
   
   (ii) \( \| a \|_{L^p(B_l)} \leq |b|^{-ka}; \)
   
   (iii) \( \int_{\mathbb{R}^n} a(x) x^\beta dx = 0 \) for any \( \beta \in \mathbb{Z}_n^+ \) with \( |\beta| \leq s. \)

2. An anisotropic central \((a(\cdot), p(\cdot), s)-\)atom of restricted type is a measurable function \( a \) on \( \mathbb{R}^n \) satisfying
   
   (i) \( \operatorname{supp} a \subset B_l, \) for some \( l \in \mathcal{Z} \);
   
   (ii) \( \| a \|_{L^p(B_l)} \leq |b|^{-ka_\infty}; \)
   
   (iii) \( \int_{\mathbb{R}^n} a(x) x^\beta dx = 0 \) for any \( \beta \in \mathbb{Z}_n^+ \) with \( |\beta| \leq s. \)

## 2 Molecular decompositions of \( H^{\infty}_{p(\cdot)}(A; \mathbb{R}^n) \)

In this section, we first give the definitions of the molecules of the anisotropic Herz-type Hardy spaces with variable exponents. Before stating our results, we first give the notations of molecules.

**Definition 2.1.** Let \( 0 < q < \infty, p(\cdot) \in \mathcal{P}(\mathbb{R}^n), a(\cdot) \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_0^\log, \) and nonnegative integer \( s \geq \max\{[(a(0) - \delta) \ln b / \ln \lambda], [(a_\infty - \delta) \ln b / \ln \lambda] \} \), where \( \delta \) as in Lemma 1.5. Set \( \varepsilon > \max(s, (a(0) + \delta_1 - 1) \ln b / \ln \lambda, (a_\infty + \delta_1 - 1) \ln b / \ln \lambda) \) and \( d = 1 - \delta_1 + \varepsilon. \) Moreover, for any \( l \in \mathcal{Z}, \) when \( l < 0, a_l := a(0) \) and \( a := 1 - \delta_1 - a(0) + \varepsilon; \) when \( l \geq 0, a_l := a_\infty \) and \( a := 1 - \delta_1 - a_\infty + \varepsilon. \)

1. A function \( M_l \in L^p(l) \) with \( l \in \mathcal{Z} \) is said to be a dyadic central \((a(\cdot), p(\cdot); s, \varepsilon)_l\)-molecule if it satisfies
   
   (i) \( \| M_l \|_{L^p(l)} \leq b^{-ln}; \)
   
   (ii) \( \mathcal{R}_{p(\cdot)}(M_l) = \| M_l \|_{L^p(l)} \| (p(\cdot))_l M_l \|_{L^q(l)} < \infty; \)
   
   (iii) \( \int_{\mathbb{R}^n} M_l(x) x^\beta dx = 0, \) for any \( \beta \in \mathbb{Z}_n^+ \) with \( |\beta| \leq s. \)

2. A function \( M_l \in L^p(l) \) with \( l \in \mathbb{N}_0 \) is said to be a dyadic central \((a(\cdot), p(\cdot); s, \varepsilon)_l\)-molecule of restricted type if it satisfies (ii), (iii) and
   
   (i') \( \| M_l \|_{L^p(l)} \leq b^{-ln}. \)

**Definition 2.2.** Let \( 0 < q < \infty, p(\cdot) \in \mathcal{P}(\mathbb{R}^n), a(\cdot) \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_0^\log, \) and nonnegative integer \( s \geq \max\{[(a(0) - \delta) \ln b / \ln \lambda], [(a_\infty - \delta) \ln b / \ln \lambda] \} \), where \( \delta \) as in Lemma 1.5. Set \( \varepsilon > \max(s, (a(0) + \delta_1 - 1) \ln b / \ln \lambda, (a_\infty + \delta_1 - 1) \ln b / \ln \lambda) \) and \( d = 1 - \delta_1 + \varepsilon. \) Moreover, for any function \( M \in L^p(l) \), when \( \| M \|_{L^p(l)} > 1, a := 1 - \delta_1 - a(0) + \varepsilon; \) when \( \| M \|_{L^p(l)} \leq 1, a := 1 - \delta_1 - a_\infty + \varepsilon. \)
(1) A function \( M \in L^p \) is said to be a central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule if it satisfies

(i) \( \mathcal{R}_{p(\cdot)}(M) = \|M^{\alpha/d}\|_p \|\rho(\cdot)^d M\|_{L^p}^{1-a/d} < \infty \);

(ii) \( \int_{\mathbb{R}^n} M(x) x^\beta \, dx = 0 \), for any \( \beta \) with \( |\beta| \leq s \).

(2) A function \( M \in L^p \) is said to be a central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule of restricted type if it satisfies (i), (ii) and

(i') \( \|M\|_{L^p} \leq 1 \).

The following lemma shows that a central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule is a generalization of the central \((\alpha(\cdot), p(\cdot), s)\)-atom.

**Lemma 2.3.** Let \( 0 < q < \infty \), \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), \( \alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty \), and nonnegative integer \( s \geq \max\{[(\alpha(0) - \delta_1) \ln b/\ln \lambda_1], [(\alpha_\infty - \delta_1) \ln b/\ln \lambda_1], [(\alpha(0) - \delta_2) \ln b/\ln \lambda_2], [(\alpha_\infty - \delta_2) \ln b/\ln \lambda_2]\} \), where \( \max\{|\delta_1, \delta_2| \leq \alpha(0), \alpha_\infty < \infty \) and \( \delta_1, \delta_2 \) as in Lemma 1.5. Set \( \varepsilon > \max\{\alpha(0), \alpha_\infty - (\delta_1 - 1) \ln b/\ln \lambda_1, (\alpha_\infty + \delta_1 - 1) \ln b/\ln \lambda_1\} \) and \( d = 1 - \delta_1 + \varepsilon \). Moreover, for any function \( M \in L^p \), when \( \|M\|_{L^p} > 1 \), \( a := 1 - \delta_1 - \alpha(0) + \varepsilon \); when \( \|M\|_{L^p} \leq 1 \), \( a := 1 - \delta_1 - \alpha_\infty + \varepsilon \).

(i) If \( M \) is a central \((\alpha(\cdot), p(\cdot), s)\)-atom, then \( M \) is a central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule, such that \( \mathcal{R}_{p(\cdot)}(M) < C \) with \( C \) independent of \( M \).

(ii) If \( M \) is a central \((\alpha(\cdot), p(\cdot), s)\)-atom of restricted type, then \( M \) is a central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule of restricted type, such that \( \mathcal{R}_{p(\cdot)}(M) < C \) with \( C \) independent of \( M \).

**Proof.** We only prove (i). (ii) can be proved in the similar way.

Let \( M \) be a \((\alpha(\cdot), p(\cdot), s)\)-atom with support on a ball \( B_k \), then we get

\[
\|M\|_{L^p}^{\alpha/d} \|\rho(\cdot)^d M\|_{L^p}^{1-a/d} \leq L \|M\|_{L^p} \leq C.
\]

Now, we give the molecular decompositions of anisotropic Herz-type Hardy spaces with two variable exponents.

**Theorem 2.4.** Let \( 0 < q < \infty \), \( p(\cdot) \in \mathcal{B}(\mathbb{R}^n) \), \( \alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty \), and nonnegative integer \( s \geq \max\{[(\alpha(0) - \delta_1) \ln b/\ln \lambda_1], [(\alpha_\infty - \delta_1) \ln b/\ln \lambda_1], [(\alpha(0) - \delta_2) \ln b/\ln \lambda_2], [(\alpha_\infty - \delta_2) \ln b/\ln \lambda_2]\} \), where \( \max\{|\delta_1, \delta_2| \leq \alpha(0), \alpha_\infty < \infty \) and \( \delta_1, \delta_2 \) as in Lemma 1.5. Set \( \varepsilon > \max\{\alpha(0), \alpha_\infty - (\delta_1 - 1) \ln b/\ln \lambda_1, (\alpha_\infty + \delta_1 - 1) \ln b/\ln \lambda_1\} \) and \( d = 1 - \delta_1 + \varepsilon \). Moreover, for any function \( M \in L^p \), when \( \|M\|_{L^p} > 1 \), \( a := 1 - \delta_1 - \alpha(0) + \varepsilon \); when \( \|M\|_{L^p} \leq 1 \), \( a := 1 - \delta_1 - \alpha_\infty + \varepsilon \).

(i) \( f \in HK_{p(\cdot), q}^{\alpha(\cdot), \beta}(A; \mathbb{R}^n) \) if and only if \( f \) can be represented as

\[
f = \sum_{k=-\infty}^{\infty} \lambda_k M_k, \text{ in } S',
\]

where each \( M_k \) is a dyadic central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule and \( \sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty \). Moreover,

\[
\|f\|_{HK_{p(\cdot), q}^{\alpha(\cdot), \beta}(A; \mathbb{R}^n)} \sim \inf \left( \sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p},
\]

where the infimum is taken over all above decompositions of \( f \).

(ii) \( f \in HK_{p(\cdot), q}^{\alpha(\cdot), \beta}(A; \mathbb{R}^n) \) if and only if \( f \) can be represented as

\[
f = \sum_{k=0}^{\infty} \lambda_k M_k, \text{ in } S',
\]

where each \( M_k \) is a dyadic central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule of restricted type and \( \sum_{k=0}^{\infty} |\lambda_k|^p < \infty \). Moreover,
where the infimum is taken over all above decompositions of \( f \).

**Theorem 2.5.** Let \( 0 < q < 1, \, p(\cdot) \in B(\mathbb{R}^n), \, \alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_{\infty}, \) and nonnegative integer \( s \geq \max([((\alpha(0) - \delta_1) \ln b/\ln \lambda.), \, ((\alpha_\infty - \delta_1) \ln b/\ln \lambda.), \, ((\alpha(0) - \delta_2) \ln b/\ln \lambda.), \, ((\alpha_\infty - \delta_2) \ln b/\ln \lambda.)], \) where \( \max(\delta_1, \delta_2) \leq \alpha(0), \, \alpha_\infty < \infty \) and \( \alpha_\infty, \, \delta_1, \) \( \alpha_\infty, \, \delta_2 \) as in Lemma 1.5. Set \( \varepsilon > \max(s, (\alpha(0) + \delta_1 - 1) \ln b/\ln \lambda., \, (\alpha_\infty + \delta_1 - 1) \ln b/\ln \lambda.) \) and \( d = 1 - \delta_1 + \varepsilon. \) Moreover, for any function \( M \in L^{p(\cdot)}, \) when \( \|M\|_{L^{p(\cdot)}} > 1, \, a := 1 - \delta_1 - \alpha(0) + \varepsilon; \) when \( \|M\|_{L^{p(\cdot)}} \leq 1, \, a := 1 - \delta_1 - \alpha_\infty + \varepsilon. \)

(i) \( f \in H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n) \) if and only if can be represented as

\[
f = \sum_{k=0}^{\infty} \lambda_k M_k, \quad \text{in } S',
\]

where each \( M_k \) is a central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule and \( \sum_{k=0}^{\infty} |\lambda_k|^q < \infty. \) Moreover,

\[
\|f\|_{H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \sim \inf \left( \sum_{k=0}^{\infty} |\lambda_k|^q \right)^{1/q},
\]

where the infimum is taken over all above decompositions of \( f. \)

(ii) \( f \in H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n) \) if and only if can be represented as

\[
f = \sum_{k=0}^{\infty} \lambda_k M_k, \quad \text{in } S',
\]

where each \( M_k \) is a central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule of restricted type and \( \sum_{k=0}^{\infty} |\lambda_k|^q < \infty. \) Moreover,

\[
\|f\|_{H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \sim \inf \left( \sum_{k=0}^{\infty} |\lambda_k|^q \right)^{1/q},
\]

where the infimum is taken over all above decompositions of \( f. \)

By theorem 3.2 of [12] and Lemma 2.3, we see that Theorems 2.4 and 2.5 can be obtained from the following lemma.

**Lemma 2.6.** Let \( 0 < q < \infty, \, p(\cdot) \in B(\mathbb{R}^n), \, \alpha \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_{\infty}, \) and nonnegative integer \( s \geq \max([((\alpha(0) - \delta_1) \ln b/\ln \lambda.), \, ((\alpha_\infty - \delta_1) \ln b/\ln \lambda.), \, ((\alpha(0) - \delta_2) \ln b/\ln \lambda.), \, ((\alpha_\infty - \delta_2) \ln b/\ln \lambda.)], \) where \( \max(\delta_1, \delta_2) \leq \alpha(0), \, \alpha_\infty < \infty \) and \( \alpha_\infty, \, \delta_1, \) \( \alpha_\infty, \, \delta_2 \) as in Lemma 1.5. Set \( \varepsilon > \max(s, (\alpha(0) + \delta_1 - 1) \ln b/\ln \lambda., \, (\alpha_\infty + \delta_1 - 1) \ln b/\ln \lambda.) \) and \( d = 1 - \delta_1 + \varepsilon. \) Moreover, for any function \( M \in L^{p(\cdot)} \) and \( l \in \mathbb{Z}, \) when \( \|M\|_{L^{p(\cdot)}} > 1 \) or \( l \geq 0, \) let \( a_l := \alpha(0) \) and \( a := 1 - \delta_1 - \alpha(0) + \varepsilon; \) when \( \|M\|_{L^{p(\cdot)}} \leq 1 \) or \( l \geq 0, \) let \( a_l := \alpha_\infty \) and \( a := 1 - \delta_1 - \alpha_\infty + \varepsilon. \)

(i) If \( 0 < q \leq 1, \) there exists a constant \( C, \) such that for any central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule \( M \) and any central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule of restricted type \( M, \)

\[
\|M\|_{H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \leq C \quad \text{and} \quad \|M\|_{H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \leq C,
\]

respectively.

(ii) There exists a constant \( C, \) such that for any dyadic central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule \( M_l, \) \( l \in \mathbb{Z}, \) and any dyadic central \((\alpha(\cdot), p(\cdot); s, \varepsilon)\)-molecule of restricted type \( M_l, \) \( l \in \mathbb{N}_0, \)

\[
\|M_l\|_{H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \leq C \quad \text{and} \quad \|M_l\|_{H_{p(\cdot)}^{\alpha(\cdot), q}(A; \mathbb{R}^n)} \leq C,
\]

respectively.
**Proof.** We only prove (i) for the homogeneous case, the proof of the nonhomogeneous case and (ii) are similar.

Suppose that $M$ is a central $(\alpha(\cdot), p(\cdot); s, \varepsilon)$-molecule. Taking

$$r = \begin{cases} \|M\|_{L^{p(\cdot)}}^{1/\alpha(0)}, & \|M\|_{L^{p(\cdot)}} > 1, \\ \|M\|_{L^{p(\cdot)}}^{1/\alpha(\infty)}, & \|M\|_{L^{p(\cdot)}} \leq 1, \end{cases}$$

and denote by $\sigma_n$, the unique integer satisfying $b^{\sigma_n} < r \leq b^{\sigma_n+1}$. Denote $E_0 = B_0$, and $E_k = B_{b^r \cdot k} \setminus B_{b^r \cdot k - 1}$ for $k \in \mathbb{N}$. Set

$$M(x)\chi_{E_k}(x) = \frac{\chi_{E_k}(x)}{|E_k|} \int_{\mathbb{R}^n} M(y)\chi_{E_k}(y) dy = H_k(x) - F_k(x).$$

It follows that

$$M(x) = \sum_{k=0}^{\infty} (H_k(x) - F_k(x)) + \sum_{k=0}^{\infty} \frac{\chi_{E_k}(x)}{|E_k|} \int_{\mathbb{R}^n} M(y)\chi_{E_k}(y) dy.$$  

Obviously, $supp (H_k(x) - F_k(x)) \subset B_{b^r \cdot k}$ and $\int_{\mathbb{R}^n} (H_k(x) - F_k(x)) dx = 0$. We claim that

(a) There is a positive constant $C$ and a sequence of numbers $\{\lambda_k\}$, such that

$$\sum_{k=0}^{\infty} |\lambda_k|^N < \infty, \quad H_k - F_k = \lambda_k a_k,$$

where each $a_k$ is a $(\alpha(\cdot), p(\cdot), 0)$-atom;

(b) $\sum_{k=0}^{\infty} F_k$ has a $(\alpha(\cdot), p(\cdot), 0)$-atom decomposition,

then our desired conclusion can be deduced directly.

We first show (a). Without loss of generality, we can suppose that $\mathcal{R}_{p(\cdot)}(M) = 1$, which implies that

$$\|\rho(\cdot)^d M\|_{L^{p(\cdot)}} = \|M\|_{L^{p(\cdot)}}^{a/d - a} = r^a.$$  

For $k = 0$, we have

$$\|H_0(x) - F_0(x)\|_{L^{p(\cdot)}} \leq \|M\|_{L^{p(\cdot)}} + \frac{\|X_{B_{b^r}}\|_{L^{p(\cdot)}}}{|B_{b^r}|} \int_{\mathbb{R}^n} \left| M(y)\chi_{B_{b^r}} \right| dy$$

$$\leq C \|M\|_{L^{p(\cdot)}} = C \cdot |B_{b^r}|^{-a},$$

and for $k \in \mathbb{N}$,

$$\|H_k(x) - F_k(x)\|_{L^{p(\cdot)}} \leq \|H_k(x)\|_{L^{p(\cdot)}} + \|F_k(x)\|_{L^{p(\cdot)}}$$

$$\leq \|H_k(x)\|_{L^{p(\cdot)}} + \frac{C}{|E_k|} \|H_k(x)\|_{L^{p(\cdot)}} \|\chi_{E_k}\|_{L^{p(\cdot)}} \|\chi_{E_k}\|_{L^{p(\cdot)}}$$

$$\leq C \|H_k(x)\|_{L^{p(\cdot)}}$$

$$\leq C \|\rho(\cdot)^d M\|_{L^{p(\cdot)}} \cdot (b^{a \cdot k})$$

$$= Cr \cdot (b^{a \cdot k})^{-d} \leq Cr^{-d} \cdot |b^r \cdot k|^{-a}.$$  

Thus, for any $k \in \mathbb{N} \cup \{0\}$, there is a constant $C$ independent of $k$, such that
\[ \|H_k(x) - F_k(x)\|_{L^p} \leq C b^{-ka} |B_{a+k}|^{-a_0}. \]

If we denote \( \lambda_{i,k} = C b^{-ka} \) and \( a_{i,k} = (H_k(x) - F_k(x))/\lambda_{i,k} \), then the \( a_{i,k} \) are central \((c(\cdot), p(\cdot), 0)\)-atoms and \( \sum_{k=0}^{\infty} (H_k(x) - F_k(x)) = \sum_{k=0}^{\infty} \lambda_{i,k} a_{i,k}(x) \). Moreover,

\[
\sum_{k=0}^{\infty} |\lambda_{i,k}|^{p} \leq C \sum_{k=0}^{\infty} b^{-ka} \leq C,
\]

where \( C \) is independent of \( M \).

Next, we will show (b). Set

\[
m_k = \sum_{i=k}^{\infty} \int_{\mathbb{R}^n} M(x) \chi_\varepsilon(x) \, dx, \quad \varphi_k(x) = \frac{\chi_\varepsilon(x)}{|E_k|}.
\]

Noting that \( m_0 = 0 \), summing by parts, we have

\[
\sum_{i=k}^{\infty} F_k(x) = \sum_{i=k}^{\infty} (m_k - m_{k+1}) \varphi_k(x) = \sum_{i=k}^{\infty} m_{k+1} (\varphi_{k+1}(x) - \varphi_k(x)).
\]

Clearly,

\[
\int_{\mathbb{R}^n} m_{k+1}(\varphi_{k+1}(x) - \varphi_k(x)) \, dx = 0, \quad \text{supp}(m_{k+1}(\varphi_{k+1} - \varphi_k)) \subset B_{a+k+1},
\]

and

\[
m_k = m_0 - \int_{B_{a+k-1}} M(x) \, dx.
\]

Hence, we obtain

\[
\|m_{k+1}(\varphi_{k+1} - \varphi_k)\|_{L^p} = \left\| \int_{B_{a+k-1}} M(x) \chi_\varepsilon(x) \, dx (\varphi_{k+1} - \varphi_k) \right\|_{L^p} \leq \left\| \sum_{i=k}^{\infty} \int_{B_{a+k-1}} M(x) \chi_\varepsilon(x) \, dx (\varphi_{k+1} - \varphi_k) \right\|_{L^p}.
\]

where \( C \) is independent of \( k \). Setting
\[
\lambda_{2,k} = Cb^{-ka} \quad \text{and} \quad a_{2,k} = m_{k+1}(\varphi_{k+1} - \varphi_k)/\lambda_{2,k},
\]

we have
\[
\sum_{k=0}^{\infty} \frac{X_{E_k(x)}}{|E_k|} \int_{\mathbb{R}^n} M(y) \chi_{E_k}(y) dy = \sum_{k=0}^{\infty} \lambda_{2,k} a_{2,k},
\]

where the \(a_{2,k}\) are central \((a(\cdot), p(\cdot), 0)\)-atoms. Furthermore,
\[
\sum_{k=0}^{\infty} |\lambda_{2,k}|^p \leq C \sum_{k=0}^{\infty} b^{kaq} \leq C,
\]

where \(C\) is independent of \(M\). The conclusion (b) then holds. Hence, the proof of Lemma 2.6 is completed. \(\square\)

3 Applications

In this section, we give an application of the molecular decomposition theory established in Section 2. We study the boundedness of the central \(\delta\)-Calderón-Zygmund operators from \(HK^{a(\cdot), q}_{\rho(\cdot)}(A; \mathbb{R}^n)\) to \(HK^{a(\cdot), q}_{\rho(\cdot)}(A; \mathbb{R}^n)\). The following condition is necessary for our discussion on the central \(\delta\)-Calderón-Zygmund operators on the \(HK^{a(\cdot), q}_{\rho(\cdot)}(A; \mathbb{R}^n)\) spaces:

\[
f = \sum_{i \in \mathbb{N}} \lambda_i a_i \text{ in } S' \implies Tf = \sum_{i \in \mathbb{N}} \lambda_i Ta_i \text{ in } S'.
\]

The central \(\delta\)-Calderón-Zygmund operators, which are more general than the classical Calderón-Zygmund operators, were introduced by Lu and Yang [14] in the isotropic setting of \(\mathbb{R}^n\). Moreover, Ding et al. [10] extended them to the following non-isotropic setting of \(\mathbb{R}^n\) associated with the dilation \(A\).

**Definition 3.1.** Let \(0 < \delta < 1\) and \(1 < p < \infty\). Let \(T: S'(\mathbb{R}^n) \to S'(\mathbb{R}^n)\) be a linear continuous operator. If there exists \(K(x, y) \in S'(\mathbb{R}^n \times \mathbb{R}^n)\), being continuous away from the diagonal in \(\mathbb{R}^{2n}\) and satisfying:

(i) \(K(x, 0) + |K(0, x)| \leq C (\rho(x))^{-1}, \text{ for all } x \neq 0;\)
(ii) \(K(x, 0) - K(0, x) + |K(0, x) - K(y, x)| \leq C (\rho(y))^{\delta} (\rho(x))^{-1-\delta}, \text{ when } \rho(x) \geq b^{2p} \rho(y);\)
(iii) \(\langle T(f), g \rangle = \int_{\mathbb{R}^n} K(x, y) f(y) g(x) dx, \text{ for } f, g \in S(\mathbb{R}^n)\) with disjoint supports, and if \(T\) can be extended to a bounded operator on \(L^p(\mathbb{R}^n)\), then we say \(T\) is a central \(\delta\)-Calderón-Zygmund operator in \(L^p(\mathbb{R}^n)\).

Using the molecular theory of \(HK^{a(\cdot), q}_{\rho(\cdot)}(A; \mathbb{R}^n)\), we can prove the following theorem:

**Theorem 3.2.** Let \(0 < \delta < 1\), \(0 < q < \infty\), \(p(\cdot) \in \mathcal{B}(\mathbb{R}^n)\), \(a(\cdot) \in L^\infty \cap \mathcal{P}_0 \cap \mathcal{P}_\infty\) and \(\max \{\delta_2, \delta_1\} \leq \delta_1 < \min \{(\delta_1 + \delta) \ln \lambda / \ln b, (\delta_2 + \delta) \ln \lambda / \ln b\}\) with \(\alpha_i\) as in Definition 1.7. Suppose that \(T\) is a central \(\delta\)-Calderón-Zygmund operator and is bounded on \(L^p(\mathbb{R}^n)\). If \(T\) satisfies (3.1) for every central atomic decomposition and \(\int_{\mathbb{R}^n} Ta(x) dx = 0\) for each central \((a(\cdot), p(\cdot), 0)\)-atom \(a(x)\), then \(T\) can be extended to a bounded operator from \(HK^{a(\cdot), q}_{\rho(\cdot)}(A; \mathbb{R}^n)\) to \(HK^{a(\cdot), q}_{\rho(\cdot)}(A; \mathbb{R}^n)\).
Remark 3.3. The boundedness of the central $\delta$-Calderón-Zygmund operators on the homogeneous anisotropic Herz-type Hardy space $HK^{a,q}_{\rho}(A; \mathbb{R}^n)$ is contained by Theorem 3.2, and the results also hold for the nonhomogeneous anisotropic Herz-type Hardy space $HK^{a,q}_{\rho}(A; \mathbb{R}^n)$.

Proof. Case 1. For $0 < q < 1$. Let $a$ be a central $(\alpha(), p(\cdot), 0)$-atom with its support in $B_k$ for some $k \in \mathbb{Z}$. If $Ta$ is a central $(\alpha(), p(\cdot); 0, \varepsilon)$-molecule for some $\delta + \delta_1 > \delta + \alpha_1 - 1$ and by condition (3.1) and Lemma 2.6, we have $\|Ta\|_{HK^{a,q}_{\rho}(A; \mathbb{R}^n)} \leq C$. Then, for homogeneous case

$$\|Ta\|_{HK^{a,q}_{\rho}(A; \mathbb{R}^n)} \leq \sum_{k=-\infty}^{\infty} |\lambda_k| \|Ta\|_{HK^{a,q}_{\rho}(A; \mathbb{R}^n)} \leq \infty.$$  

Thus, $T$ is a bounded operator on $HK^{a,q}_{\rho}(A; \mathbb{R}^n)$ by taking supremum of the above formula. It suffices to show $Ta$ is a central $(\alpha(), p(\cdot), 0, \varepsilon)$-molecule for some $\delta + \delta_1 > \delta + \alpha_1 - 1$. To this aim, let $a = 1 - \delta_1 - \alpha_1 + \varepsilon, d = 1 - \delta_1 + \varepsilon$. Obviously, we only need to verify the size condition of molecules, that is,

$$R_{p(\cdot)}(Ta) = \|Ta\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\alpha(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C,$$  

with $C$ independent of $a$. From the hypothesis of the theorem, we need to show only (3.2). To do this, we first estimate $\|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$. By the boundedness of $T$ on the $L^{p(\cdot)}$, we have

$$\|p(\cdot)\|_{L^{p(\cdot)}(B_{2^k,2^n})} = Cb^{n\delta} \|Ta\|_{L^{p(\cdot)}(B_{2^k,2^n})} \leq Cb^{n\delta} \|a\|_{L^{p(\cdot)}(B_{2^k,2^n})} \leq Cb^{n(d-a)}.$$  

On the other hand, if $x \in B_{2^k,2^n}$, from the condition(ii) of definition 3.1 and $\int_{B_k} \alpha(x) dx = 0$, we can get

$$|T(a)| = \left| \int_{B_k} (K(x, y) - K(x, 0)) a(y) dy \right| \leq C \int_{B_k} \frac{\rho(y)^\delta}{\rho(x)^\delta} \alpha(y) dy \leq Cb^{k(1+\delta)} \rho(x)^{1-\delta} \alpha(x) \int_{B_k} \alpha(x) dx \leq Cb^{k(1+\delta)} \alpha(x)^{1-\delta} \frac{1}{|B_k|} \int_{B_k} \alpha(x) dx \leq Cb^{k(1+\delta)} \alpha(x)^{1-\delta} M(\alpha)(x).$$  

Therefore, noting that $\varepsilon < \delta + \delta_1$, we have

$$\|p(\cdot)\|_{L^{p(\cdot)}(B_{2^k,2^n})} \leq Cb^{k(1+\delta)} \|p(\cdot)\|_{L^{p(\cdot)}(B_{2^k,2^n})} \leq Cb^{k(1+\delta)} \|Ma(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{k(1+\delta)} \|a(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{k(1+\delta)} \alpha(x)^{1-\delta} M(\alpha)(x).$$  

That is, $\|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{k(1+\delta)}$. Thus,

$$R_{p(\cdot)}(Ta) = \|Ta\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\alpha(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{n\delta} \|Ta\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\alpha(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$  

$$\leq Cb^{n\delta} \|Ta\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\alpha(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{n\delta} \|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\alpha(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{n(d-a)} \|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\alpha(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq Cb^{n(d-a)} \|a\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|p(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\alpha(\cdot)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C < \infty,$$

where $C$ is independent of $a$.

Case 2. For $1 < q < \infty$. By a proof similar to that of [3, Proposition 3.8], we easily obtain an important lemma as follows. \hfill \Box
Lemma 3.4. Let \( \alpha(\cdot) \in L^\infty \cap \mathcal{P}_{0} \cap \mathcal{P}_{s}, \ p(\cdot) \in \mathcal{P} \) and \( q \in (0, \infty) \), then

\[
\| \mathcal{A} \|_{p,q}^{\mathcal{A}(\mathbb{R}^n)} \approx \left( \sum_{k=0}^{\infty} b^{\alpha_k q} \| f_k \|_{L^p}^q \right)^{1/q} + \left( \sum_{k=0}^{\infty} b^{\alpha_k q} \| f_k \|_{L^p}^q \right)^{1/q}.
\]

We now proceed with the proof of Theorem 3.2. Let \( a \) be a central \((\alpha(\cdot), p(\cdot), 0)\)-atom with its support in \( B_k \) for some \( k \in \mathbb{Z} \). We write

\[
\| M_N T(f) \|_{p(1)}^{q} \approx \left| \sum_{k=0}^{\infty} b^{\alpha_k q} \| M_N T(f) \|_{L^p}^q \right| + \left| \sum_{k=0}^{\infty} b^{\alpha_k q} \| M_N T(f) \|_{L^p}^q \right|
\]

For \( I_3 \), it is easy to see that

\[
I_3 \leq C \left| \sum_{k=0}^{\infty} b^{\alpha_k q} \left( \sum_{j=0}^{\infty} |\lambda_j| \| M_N T(a_j) \|_{L^p}^q \right) \right|^q + C \left| \sum_{k=0}^{\infty} b^{\alpha_k q} \left( \sum_{j=k-\sigma}^{\infty} |\lambda_j| \| M_N T(a_j) \|_{L^p}^q \right) \right|^q
\]

We first estimate \( M_N T(a) \) on \( C_k \) for \( k \geq k_0 + \sigma + 1 \). For any \( x \in C_k, \varphi \in S_N, j \in \mathbb{Z} \) and a polynomial \( P_s \) of degree \( s \), by a proof similar to those of [10, p. 1454], we have

\[
M_N T(a)(x) \leq C b^{\alpha_0 \cdot \delta_0} \| x \|_{L^p} \| x \|_{L^p} \leq b^{\alpha_s \cdot \delta_s} \| x \|_{L^p} \leq b^{\alpha_s \cdot \delta_s} \| x \|_{L^p}
\]

where \( m = k - k_0 - 1 - \sigma \). By the Hölder inequality, \( |b^{\alpha_0 \cdot \delta_0} \| x \|_{L^p} < 1 \) and (3.3), we obtain

\[
I_{31} \leq C \left| \sum_{k=0}^{\infty} b^{\alpha_k q} \left( \sum_{j=0}^{\infty} |\lambda_j| \| M_N T(a_j) \|_{L^p}^q \right) \right|^q
\]

\[
\leq C \left| \sum_{k=0}^{\infty} b^{\alpha_k q} \left( \sum_{j=k-\sigma}^{\infty} |\lambda_j| \| M_N T(a_j) \|_{L^p}^q \right) \right|^q
\]

\[
\leq C \left| \sum_{k=0}^{\infty} \sum_{j=k-\sigma}^{\infty} |\lambda_j|^q \| b^{\alpha_0 \cdot \delta_0} \| x \|_{L^p}^q \right|^q
\]

\[
\leq C \left| \sum_{k=0}^{\infty} \sum_{j=k-\sigma}^{\infty} |\lambda_j|^q \| b^{\alpha_0 \cdot \delta_0} \| x \|_{L^p}^q \right|^q
\]

\[
\leq C \sum_{j=k-\sigma+1}^{\infty} |\lambda_j|^q \| b^{\alpha_0 \cdot \delta_0} \| x \|_{L^p}^q
\]

\[
\leq C \sum_{j=k-\sigma+1}^{\infty} |\lambda_j|^q.
\]
From the $L^{p(\cdot)}$ boundedness of $M_N$, the size condition of $a$, and the Hölder inequality, we conclude that

$$ I_{32} \leq C \sum_{k=-\infty}^{-1} b_{n,k}^{a_{q,k}} \left( \sum_{j=k-\theta}^{+\infty} |\lambda_j|q T(a) \chi_{\lambda_k} \|L^{p(\cdot)}\right)^q $$

$$ \leq C \sum_{k=-\infty}^{-1} b_{n,k}^{a_{q,k}} \left( \sum_{j=k-\theta}^{+\infty} |\lambda_j||B_j|^{-a_{q,j}} \right)^q $$

$$ \leq C \sum_{k=-\infty}^{-1} b_{n,k}^{a_{q,k}} \left( \sum_{j=k-\theta}^{-1} |\lambda_j||B_j|^{-a_{q,j}} \right)^q + C \sum_{k=-\infty}^{-1} b_{n,k}^{a_{q,k}} \left( \sum_{j=0}^{+\infty} |\lambda_j||B_j|^{-a_{q,j}} \right)^q $$

$$ \leq C \sum_{k=-\infty}^{-1} b_{n,k}^{a_{q,k}} \left( \sum_{j=k-\theta}^{-1} |\lambda_j||B_j|^q |B_j|^{-a_{q,j}/2} \right)^q $$

$$ \leq C \sum_{j=0}^{+\infty} \sum_{k=-\infty}^{-1} |\lambda_j|^q b_{n,k}^{a_{q,k} |B_j|^q |B_j|^{-a_{q,j}/2}} + C \sum_{j=0}^{+\infty} |\lambda_j|^q $$

$$ \leq C \sum_{j=0}^{+\infty} |\lambda_j|^q $$

Therefore, we finish the proof of Theorem 3.2.

The proof of $I_4$ is similar to $I_3$, we are omitting it there. From the $I_3$, $I_4$, we can get

$$ \|M_N T(f)\|_{\mathcal{L}^{p(\cdot),q}(A;R)} \leq C \sum_{j \in \mathcal{Z}} |\lambda_j|^q. $$

Thus,

$$ \|T(f)\|_{\mathcal{L}^{p(\cdot),q}(A;R)} \leq C \|f\|_{\mathcal{L}^{p(\cdot),q}(A;R)}. $$

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