ONCE MORE ON POSITIVE COMMUTATORS

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Abstract. Let $A$ and $B$ be bounded operators on a Banach lattice $E$ such that the commutator $C = AB - BA$ and the product $BA$ are positive operators. If the product $AB$ is a power-compact operator, then $C$ is a quasi-nilpotent operator having a triangularizing chain of closed ideals of $E$. This theorem answers an open question posed in [3], where the study of positive commutators of positive operators has been initiated.

1. Introduction

Let $X$ be a Banach space. The spectrum and the spectral radius of a bounded operator $T$ on $X$ are denoted by $\sigma(T)$ and $r(T)$, respectively. A bounded operator $T$ on $X$ is said to be power-compact if $T^n$ is a compact operator for some $n \in \mathbb{N}$. A chain $\mathcal{C}$ is a family of closed subspaces of $X$ that is totally ordered by inclusion. We say that $\mathcal{C}$ is a complete chain if it is closed under arbitrary intersections and closed linear spans. If $\mathcal{M}$ is in a complete chain $\mathcal{C}$, then the predecessor $\mathcal{M}_-$ of $\mathcal{M}$ in $\mathcal{C}$ is defined as the closed linear span of all proper subspaces of $\mathcal{M}$ belonging to $\mathcal{C}$.

Let $E$ be a Banach lattice. An operator $T$ on $E$ is called positive if the positive cone $E^+$ is invariant under $T$. It is well-known that every positive operator $T$ is bounded and that $r(T)$ belongs to $\sigma(T)$. A bounded operator $T$ on $E$ is said to be ideal-reducible if there exists a non-trivial closed ideal of $E$ invariant under $T$. Otherwise, it is ideal-irreducible. If the chain $\mathcal{C}$ of closed ideals of $E$ is maximal in the lattice of all closed ideals of $E$ and if every one of its members is invariant under an operator $T$ on $E$, then $\mathcal{C}$ is called a triangularizing chain for $T$, and $T$ is said to be ideal-triangularizable. Note that such a chain is also maximal in the lattice of all closed subspaces of $E$ (see e.g. [1, Proposition 1.2]).

In [3] positive commutators of positive operators on Banach lattices are studied. The main result [3, Theorem 2.2] is the following
**Theorem 1.1.** Let $A$ and $B$ be positive compact operators on a Banach lattice $E$ such that the commutator $C = AB - BA$ is also positive. Then $C$ is an ideal-triangularizable quasi-nilpotent operator.

Examples in [3] show that the compactness assumption of Theorem 1.1 cannot be omitted. They are based on a simple example that can be obtained by setting $A = S^*$ and $B = S$, where $S$ is the unilateral shift on the Banach lattice $l^2$.

Theorem 1.1 has been further extended in [5, Theorem 3.4]. Recall that a bounded operator $T$ on a Banach space is called a Riesz operator or an essentially quasi-nilpotent operator if $\{0\}$ is the essential spectrum of $T$.

**Theorem 1.2.** Let $A$ and $B$ be positive operators on a Banach lattice $E$ such that the sum $A + B$ is a Riesz operator. If the commutator $C = AB - BA$ is a power-compact positive operator, then it is an ideal-triangularizable quasi-nilpotent operator.

In this note we answer affirmatively the open question posed in [3, Open questions 3.7 (1)] whether is it enough to assume in Theorem 1.1 that only one of the operators $A$ and $B$ is compact.

**2. Preliminaries**

If $T$ is a power-compact operator on a Banach space $X$, then, by the classical spectral theory, for each $\lambda \in \mathbb{C} \setminus \{0\}$ the operator $\lambda - T$ has finite ascent $k$, i.e., $k$ is the smallest natural number such that $\ker ((\lambda - T)^k) = \ker ((\lambda - T)^{k+1})$. In this case the (algebraic) multiplicity $m(T, \lambda)$ of $\lambda$ is the dimension of the subspace $\ker ((\lambda - T)^k)$.

We will make use of the following extension of Ringrose’s Theorem.

**Theorem 2.1.** Let $T$ be a power-compact operator on a Banach space $X$, and let $\mathcal{C}$ be a complete chain of closed subspaces invariant under $T$. Let $\mathcal{C}'$ be a subchain of $\mathcal{C}$ of all subspaces $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M}_- \neq \mathcal{M}$. For each $\mathcal{M} \in \mathcal{C}'$, define $T_\mathcal{M}$ to be the quotient operator on $\mathcal{M}/\mathcal{M}_-$ induced by $T$. Then

$$\sigma(T) \setminus \{0\} = \bigcup_{\mathcal{M} \in \mathcal{C}'} \sigma(T_\mathcal{M}) \setminus \{0\}.$$

Moreover, for each $\lambda \in \mathbb{C} \setminus \{0\}$ we have

$$m(T, \lambda) = \sum_{\mathcal{M} \in \mathcal{C}'} m(T_\mathcal{M}, \lambda).$$

**Proof.** In the case of a compact operator $T$ the first equality is proved in [12, Theorem 7.2.7], while the second equality follows from the theorem
Theorem 2.2. Let $A$ and $B$ be bounded operators on a Banach space. If $AB$ is power-compact, then $BA$ is power-compact and

$$m(AB, \lambda) = m(BA, \lambda)$$

for each $\lambda \in \mathbb{C} \setminus \{0\}$.

The following theorem is a consequence of [9, Theorem 4.3]; see a recent paper [7, Theorem 0.1] which also contains the easily proved proposition [7, Proposition 0.2] that a positive operator is ideal-irreducible if and only if it is semi non-supporting (the notion used in [9]).

Theorem 2.3. Let $S$ and $T$ be positive operators on a Banach lattice $E$ such that $S \leq T$ and $r(S) = r(T)$. If $T$ is an ideal-irreducible power-compact operator, then $S = T$.

3. Results

The main result of this note is the following extension of Theorem 1.1 (and [3, Theorem 2.4] as well).

Theorem 3.1. Let $A$ and $B$ be bounded operators on a Banach lattice $E$ such that $AB \geq BA \geq 0$ and $AB$ is a power-compact operator. Then the commutator $C = AB - BA$ is an ideal-triangularizable quasi-nilpotent operator.

Proof. Let $\mathcal{C}$ be a chain (of closed ideals) that is maximal in the lattice of all closed ideals invariant under $AB$. By maximality, this chain is complete. Let $\mathcal{C}'$ be a subchain of all subspaces $\mathcal{M} \in \mathcal{C}$ such that $\mathcal{M}_\perp \neq \mathcal{M}$. Since $AB \geq BA \geq 0$ and $AB \geq C \geq 0$, every member of $\mathcal{C}$ is also invariant under the operators $BA$ and $C$, and these operators are power-compact operators by the Aliprantis-Burkinshaw theorem [2, Theorem 5.14]. For any ideal $\mathcal{M} \in \mathcal{C}'$, $r((AB)_{\mathcal{M}}) \geq r((BA)_{\mathcal{M}})$, since $(AB)_{\mathcal{M}} \geq (BA)_{\mathcal{M}} \geq 0$. We will prove that $r((AB)_{\mathcal{M}}) = r((BA)_{\mathcal{M}})$ for every ideal $\mathcal{M} \in \mathcal{C}'$, and so $(AB)_{\mathcal{M}} = (BA)_{\mathcal{M}}$ by Theorem 2.3.
Assume there are ideals $M \in C'$ such that $r((AB)_M) > r((BA)_M)$. Among them choose $M_0 \in C'$ for which $\lambda_0 := r((AB)_M)$ is maximal. Such an ideal exists, because for each $\epsilon > 0$ there are only finitely many eigenvalues of $AB$ with the absolute value at least $\epsilon$. For each ideal $M \in C'$ with $r((AB)_M) > \lambda_0$, we must have $r((AB)_M) = r((BA)_M)$, and so $(AB)_M = (BA)_M$ by Theorem 2.3. The same conclusion holds in the case when $r((AB)_M) = r((BA)_M) = \lambda_0$. If $\lambda_0 = r((AB)_M) > r((BA)_M)$, then 

$$m((AB)_M, \lambda_0) > 0 = m((BA)_M, \lambda_0).$$

If $r((AB)_M) < \lambda_0$, then 

$$m((AB)_M, \lambda_0) = 0 = m((BA)_M, \lambda_0).$$

In view of Theorem 2.1 we now conclude that $m(AB, \lambda_0) > m(BA, \lambda_0)$. However, by Theorem 2.2 we have $m(AB, \lambda_0) = m(BA, \lambda_0)$. This contradiction shows that, for each $M \in C'$, $(AB)_M = (BA)_M$ and so $C_M = (AB)_M - (BA)_M = 0$. By Theorem 2.1, we conclude that $C$ is quasi-nilpotent.

Finally, it is a simple consequence (see e.g. [5, Theorem 1.3]) of the well-known de Pagter’s theorem (see [1, Theorem 9.19] or [10]) that $C$ has a triangularizing chain of closed ideals of $E$. In fact, we can simply complete the chain $C$ to a triangularizing chain of closed ideals for the operator $C$. □

As a corollary we obtain the answer to an open question posed in [3, Open questions 3.7 (1)].

**Corollary 3.2.** Let $A$ and $B$ be positive operators on a Banach lattice $E$ such that the commutator $C = AB - BA$ is a positive operator. If one of the operators $A$ and $B$ is power-compact (in particular, compact), then the commutator $C$ is an ideal-triangularizable quasi-nilpotent operator.

**Proof.** By a simple induction, we have $0 \leq (AB)^n \leq A^n B^n$ for every $n \in \mathbb{N}$. Assume now that for $n \in \mathbb{N}$ one of the operators $A^n$ and $B^n$ is compact, so that the operator $A^n B^n$ is compact. Then the operator $(AB)^m$ is also compact by the Aliprantis-Burkinshaw theorem [2, Theorem 5.14]. Therefore, Theorem 3.1 can be applied. □

It should be noted that a recent preprint [6, Theorem 4.5] gives an independent proof of Corollary 3.2 in the case when one of the operators $A$ and $B$ is compact.

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