The Generalized SLW Model

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Abstract. The Generalized SLW Method is presented, formulating the SLW method with the help of both the ALBDF and the Inverse ALBDF. The result is two equivalent symmetric models: the SLW Model and the Inverse SLW Model. The advantage of the unified dual formulation and of application of the ALBDF and the Inverse ALBDF is in more efficient implementation of the model and the elimination of the solution of the implicit equations for the absorption cross-sections in the construction of the spectral model in the case of non-isothermal media. The generalized approach explores all possibilities of the SLW method under both direct and inverse formulations including its limiting cases: the minimal one clear gas–one gray gas SLW-1 model, and the case when the number of gray gases approaches infinity termed the Exact SLW model. The present work outlines the steps in a unified construction of the generalized SLW model in isothermal and non-isothermal media, and compares different forms of the modelled radiative quantities in plane parallel media: directional total radiative flux, total emissivity, Planck mean and Rosseland mean absorption coefficients.

1. Introduction

The Spectral Line Weighted-sum-of-gray-gases (SLW) method is the first global approach for spectral modeling of radiative transfer in high temperature gases developed on the basis of the high resolution gas absorption spectrum [1-21]. Since that time the method has undergone a number of extensions, interpretations and variations, and other global models have been introduced: ADF, FSK, CW and MBWSGG methods, and hybrid forms of the SLW method with other spectral models such as EWBM, reordered wavenumber, correlated $k$-distributions, cumulative wavenumber, etc. [22,31-33,34-37].

The primary basis in accounting for spectral quantities in the SLW method is the distribution function $F(C,T_g,T_b,Y,p)$ of a continuous cross-section variable $C$ termed the Absorption Line Blackbody Distribution Function (ALBDF). The ALBDF is the characterization of the high resolution gas absorption cross-section $C_g(T_g)$ at gas temperature $T_g$, gas molar fraction $Y$, and total pressure $p$, regarding the Planck spectral distribution of the blackbody emissive power of radiation at the source temperature $T_b$. The present work is a unified formulation of the spectral line weighted-sum-of-gray-gases (SLW) approach to spectral modelling of gas radiation based on an alternative application of both the direct ALBDF and the inverse ALBDF. This yields two independent but completely equivalent symmetric gray gas spectral models which make the SLW method more universal, efficient, and
flexible. Both spectral models have a one-to-one correspondence between discretization of the cross-section variable $C$ (direct SLW model) and the ALBDF $F$ (inverse SLW model). Simultaneous synchronized discretization of both variables allows reduction of the number of gray gases to 3 - 6 gray gases without significant loss of accuracy, avoiding costly optimization. This approach is named here the compact SLW model. The resulting reordered absorption cross-section of the SLW spectral model allows clear definition and interpretation of the total characterization of the gas absorption coefficient such as total emissivity, and the Planck and the Rosseland mean absorption coefficients.

In the case of non-uniform media the SLW reference approach, based on assumption of correlation between absorption spectra at different states, alternates between applications of four direct and inverse ALBDFs calculated at the reference and the local states, providing more choices for better implementation of the SLW method. The advantage of this unified modelling approach is the elimination of the solution of implicit equations traditionally used in the reference approach. The Generalized SLW Method also includes two limiting cases: the minimal SLW model, which consists of a single clear gas – single gray gas model; and the Exact SLW model, in which the number of gray gases approaches infinity. These limiting cases also have alternative direct and inverse presentations.

This paper presents a unified treatment of the SLW direct and inverse methods in a single location, drawing some from previously published work as review to set the foundation for new material, and developing new explorations for other scenarios. The paper includes considerable detail, intending to serve as a reference for those seeking to implement one of the family of SLW models.

2. Absorption Line Blackbody Distribution Function (ALBDF) and Inverse ALBDF

2.1 Radiative Transfer Equation, gas absorption coefficient and the gas absorption cross-section

The spectral radiation intensity $I_{\eta}(s, \Omega)$ in absorbing and emitting media along a pathlength $s$ in a direction $\Omega$ is described by the Radiative Transfer Equation (RTE):

$$\frac{\partial I_{\eta}(s, \Omega)}{\partial s} = -\kappa_{\eta}(s)I_{\eta}(s, \Omega) + \kappa_{\eta}(s)I_{\eta} \left[ T(s) \right]$$

where $I_{\eta} \left[ T(s) \right]$ is the Planck blackbody intensity at the local gas temperature $T(s)$, and $\kappa_{\eta}(s)$ is the local spectral absorption coefficient. The construction of the SLW spectral model is typically based on the gas absorption cross-section $C_{\eta}(T_g, Y)$ rather than on the gas absorption coefficient. Consider the dependence of the gas absorption coefficient on its thermodynamic state (total pressure $p$, gas partial pressure $p_g$, molar fraction $Y$, gas temperature $T_g$, molar density $N$):

$$\kappa_{\eta} = \left( \frac{p}{RT_g} \right) \left( \frac{p_g}{p} \right) \cdot \frac{N_A}{mol} \cdot \frac{C_{\eta}(T_g, Y, p)}{mol} \cdot \frac{cm^2}{molecule} \cdot \frac{1}{10,000} \cdot \frac{m^2}{cm^2} \cdot \frac{[I/m]}{spectral\ absorption\ cross-section}$$

The gas absorption cross-section $C_{\eta}(T_g, Y, p)$, $[m^2/mol]$ defined by this relationship is used in the SLW method for characterization of the gas absorption spectrum.

The Line-by-Line (LBL) method solves the spectral RTE for each wavenumber $\eta$, after which the following spectral integration for obtaining the total intensity of radiation is used:

$$I(s, \Omega) = \int_\eta I_{\eta}(s, \Omega) d\eta$$
The SLW method is a particular case of the class of so-called global methods in gas radiation which, in contrast to the Line-by-Line method, starts first with the spectral integration of the RTE, and then solves it for the spectrally integrated intensity.

2.2. Definition of the ALBDF and the Inverse ALBDF

ALBDF $F(C, T_g, T_b, p, Y)$ is the distribution function which characterizes the high resolution line gas absorption spectrum $C_g(T_g, Y, p)$, $[m^2/mol]$ at a given thermodynamic state: gas temperature $T_g$, gas total pressure $p$, gas molar fraction $Y$. The function describes the fraction of the blackbody radiation power $E_b(T_b) = \sigma T_b^4$ emitted at temperature $T_b$ that is in the part of the spectrum where the gas line absorption cross-section $C_g(T_g)$ is below the prescribed value $C$ (see figure 1):

$$ F(C, T_g, T_b, p, Y) = \frac{1}{E_b(T_b)} \int_{C_g(T_g, p, Y) \leq C} E_b(T_b) \, d\eta $$

The ALBDF is an increasing monotonic function varying between 0 and 1 in variable $C$, and therefore, it is invertible.

Inverse ALBDF The inverse ALBDF $C(F, T_g, T_b)$ of the variable $F$ is defined such that

$$ F[C(F, T_g, T_b, p, Y), T_g, T_b, p, Y] = F, \text{ and } C[F(C, T_g, T_b, p, Y), T_g, T_b, p, Y] = C $$

The inverse ALBDF defines the value of the absorption cross-section $C$, for which the prescribed value $F$ is the fraction of the blackbody emissive power emitted at temperature $T_b$ in the part of the spectrum where the absorption cross-section at the prescribed gas thermodynamic state is below $C$. 

Figure 1. Definition of the direct ALBDF $F(C, T_g, T_b)$ and the inverse ALBDF $C(F, T_g, T_b)$
2.3 Tabulated LBL ALBDF
A finely resolved ALBDF look-up table has been developed previously [21] for H$_2$O, CO$_2$, and CO for a range of gas and blackbody source temperatures from 300K to 3000K (with increment 100K), and total pressures from 0.1 atm to 50 atm (at different mole fraction of H$_2$O for taking into account self and air-broadening). Calculations of the ALBDF were performed with the help of the HITEMP-2010 spectral database. An efficient algorithm has been used for fast multilinear interpolation and inversion for calculation of the direct and inverse ALBDF at the prescribed thermodynamic state. The accuracy of prediction based on the look-up table is nearly identical to LBL calculations, and can be used as a benchmark for verification of spectral models. The ALBDF look-up table is available for download at [http://albdf.byu.edu](http://albdf.byu.edu).

2.4 ALBDF Correlations
Because of simplicity and convenience in their applications, the most popular representations of the ALBDF are mathematical correlations of the form developed by Denison and Webb [1,3,6]. Despite the now-obsolete spectral database on which earlier correlations were based, these correlations are quite accurate compared to updated and more complete spectral data [27]. Coefficients of correlations are presented in [1,3].

\[
F_w(C,T_g,T_b,Y_w) = \frac{1}{2} \tanh \left[ P_w(T_g,T_b,\xi - \xi_{sb}) \right] + \frac{1}{2}
\]

\[
P_w(T_g,T_b,\xi - \xi_{sb}) = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} b_{lmn} \left( \frac{T_g}{2500} \right)^n \left( \frac{T_b}{2500} \right)^m (\xi - \xi_{sb})^l
\]

\[
\xi_{sb} = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} c_{lmn} \left( \frac{T_b}{2500} \right)^n \xi^n (Y_w)^{l+1}
\]

\[
\xi = \ln(C), \quad 1 \cdot 10^{-4} \leq C \leq 60 \left[ \frac{m^2}{mol} \right]
\]

\[
F_v(C,T_g,T_b) = \frac{1}{2} \tanh \left[ P_v(T_g,T_b,\xi) \right] + \frac{1}{2}
\]

\[
P_v(T_g,T_b,\xi) = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} d_{lmn} \left( \frac{T_g}{2500} \right)^n \left( \frac{T_b}{2500} \right)^m (\xi)^l
\]

\[
\xi = \ln(C), \quad 1 \cdot 10^{-4} \leq C \leq 60 \left[ \frac{m^2}{mol} \right]
\]

In Pearson et al., [19], the HITEMP-2010 database was used to update the correlation coefficients $b_{lmn}$, $c_{lmn}$, $d_{lmn}$. Further, additional coefficients $a_{lmn}$ for correlations of the carbon monoxide ALBDF were reported:

\[
F_m(C,T_g,T_b) = \frac{1}{2} \tanh \left[ P_m(T_g,T_b,\xi) \right] + \frac{1}{2}
\]

\[
P_m(T_g,T_b,\xi) = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} d_{lmn} \left( \frac{T_g}{2500} \right)^n \left( \frac{T_b}{2500} \right)^m (\xi)^l
\]

\[
\xi = \ln(C), \quad 1 \cdot 10^{-4} \leq C \leq 60 \left[ \frac{m^2}{mol} \right]
\]
Note that the correlation for H\textsubscript{2}O includes the gas molar fraction $Y_w$ to account for the influence of self-broadening on the ALBDF (which is handled by a shift $\xi_{sb}$ of the variable $\xi$), whereas the ALBDF of CO\textsubscript{2} and of CO does not depend on gas molar fraction. The correlations of Denison and Webb \cite{1,3,6} were developed with the help of the originally generated spectral database and they are valid for the range of temperatures $400K \leq T_g, T_b \leq 2500K$. Extrapolation of these correlations to temperatures as low as room temperature has been found to be sufficiently accurate for engineering applications. The updated correlations of Pearson et al. \cite{19} were developed with the help of the updated HITEMP-2010 spectral database, and they are valid for a wider range of temperatures: $400K \leq T_g, T_b \leq 3000K$. In their most fundamental form, both sets of correlations are designated for application at standard atmospheric total pressure $p = 1 \text{ atm}$. 

Correlations for the ALBDF at variable total pressure in the range $0.1 \text{ atm} \leq p \leq 50 \text{ atm}$ were developed by Pearson et al. \cite{21}:

For H\textsubscript{2}O:

$$ F_w(C, T_g, T_b, Y_w, p) = \frac{1}{2} \tanh \left[ P_w(T_g, T_b, \xi - \xi_p) \right] + \frac{1}{2} $$

$$ P_w(T_g, T_b, \xi - \xi_p) = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} b_{lmn} \left( \frac{T_g}{2500} \right)^n \left( \frac{T_b}{2500} \right)^m \left( \xi - \xi_p \right)^l $$

$$ \xi_p = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} u_{lmn} \left( \frac{T_g T_b}{2500^2} \right)^n \xi^m \psi_{w}^{l+1} $$

$$ \psi_{w} = \frac{1}{10} \ln \left( 100 p_c \right) $$

$$ p_c = (1 + 8.17 \cdot Y_w) p $$

$$ \xi = \ln(C), \quad 1 \cdot 10^{-4} \leq C \leq 60 \left[ \frac{m^2}{mol} \right] $$

For CO\textsubscript{2}:

$$ F_c(C, T_g, T_b, p) = \frac{1}{2} \tanh \left[ P_c(T_g, T_b, \xi - \xi_p) \right] + \frac{1}{2} $$

$$ P_c(T_g, T_b, \xi) = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} d_{lmn} \left( \frac{T_g}{2500} \right)^n \left( \frac{T_b}{2500} \right)^m \left( \xi - \xi_p \right)^l $$

$$ \xi_p = \sum_{l=0}^{3} \sum_{m=0}^{3} \sum_{n=0}^{3} u_{lmn} \left( \frac{T_g T_b}{2500^2} \right)^n \xi^m \psi_{c}^{l+1} $$

$$ \psi_{c} = \frac{1}{10} \ln \left( 100 p \right) $$

$$ \xi = \ln(C), \quad 1 \cdot 10^{-4} \leq C \leq 600 \left[ \frac{m^2}{mol} \right] $$
The correlation coefficients at atmospheric pressure \( b_{\text{lum}}, c_{\text{lum}}, d_{\text{lum}}, a_{\text{lum}} \), and the coefficients for the correlations at variable total pressure \( u_{\text{lum}}, v_{\text{lum}} \), and \( w_{\text{lum}} \) are available for download at [http://albdf.byu.edu](http://albdf.byu.edu), together with the C++ code for calculation of the ALBDF using the correlations. The look-up table for the ALBDF generated from LBL spectral data used for development of correlations can also be found at this website. Other correlations for the ALBDF are developed in [22,26,27,29].

![Figure 2](image-url)  
**Figure 2.** Example of comparison of correlations and look-up table data [21]. \( Y_w=0 \) prescribes that in calculation of absorption coefficient only air broadening of spectral lines is taken into account (no self-broadening): a) \( \text{H}_2\text{O} \ (Y_w=0), \ T_g=1000\text{K}, \ p=1\text{atm} \); b) \( \text{H}_2\text{O} \ (Y_w=0), \ T_g=1000\text{K}, \ p=20\text{atm} \)

As seen in figure 2, the mathematical correlations of the ALBDF described previously reproduce the behavior of the ALBDF curves quite well and for most conditions. However, they deviate from the LBL data under some conditions. Note that dependence of the ALBDF on cross-section \( C \) is shown in logarithmic scale with six orders of magnitude variation. The steep growth of ALBDF with \( C \) prevents application of traditional approximations, for example, with the help of orthogonal polynomials which yield very strong oscillations of approximations. The problem of accurate polynomial representation of
ALBDF may be effectively solved with the help of the Method of Moments and Cutteridge-Devyatov Polynomials (CDP) in logarithmic scale.

2.5 Interpolation and inversion of the tabulated LBL ALBDF with the help of Method of Moments

The following approach provides the ALBDF as a continuous function of absorption cross-section $C$ utilizing the Method of Moments for polynomial representation of the functions. The basics of the method are developed and presented in [24,44], where solution of the Classical Moment Problem on a finite interval is formulated in the following form:

The Cutteridge-Devyatov polynomials (CDP) of order $n$, $\Phi^{(n)}_k(x)$, are defined as

$$\Phi^{(n)}_k(x) = \sum_{m=0}^{n-k} (-1)^{m+k} \frac{(n+k)!}{k!(k+1)!(n-k-1)!} \frac{(n+m)!}{m!(m+1)!(n-m-1)!} x^{k+n}$$

Let $f(x)$, $x \in [0,L]$ be a function such that $f(0) = 0$, and let the first $n$ generalized moments of $f(x)$ be defined by

$$J_k = \int_0^L x^k d[f(x)], \quad k = 0, 1, ..., n - 1$$

Let the function $f_n(x)$ be defined as a linear combination of $n$ polynomials $\Phi^{(n)}_k(x)$, $k = 0, 1, ..., n - 1$:

$$f_n(x) = \sum_{k=0}^{n-1} J_k \Phi^{(n)}_k(x)$$

where the coefficients $J_k$ are the generalized moments of $f(x)$. Then the first $n$ generalized moments of $f_n(x)$ and $f(x)$ coincide, and the sequence $f_n(x) \rightarrow f(x)$ approaches the continuous function. The outline of representation of the ALBDF with the help of CDP is the following:

One uses multilinear interpolation to read ALBDF data from the look-up table at a given thermodynamic state $T_g, T_p, p, Y$: $F_j = F(C, T_g, T_p, p, Y)$, $j = 0, 1, 2, ..., N$, where $N$ is the number of points in the table. Denote $C_{\min} = C_{g}, \ C_{\max} = C_{p}$ and $F_{\min} = F_g, \ F_{\max} = F_p$. Assume that $F(C, T_g, T_p, p, Y)$ is a linear function of the variable $C$ in the intervals $[C_{j-1}, C_j]$, $j = 1, ..., N$. The function $F(C, T_g, T_p, p, Y)$, by its definition, is strictly increasing on the entire interval $[C_{\min}, C_{\max}]$, and therefore, it is invertible.

Logarithmic change of variable and normalization is performed as follows:

$$c_j = \ln \frac{C_j}{C_{\min}} = \frac{\ln C_{\max}}{\ln C_{\min}}, \quad f_j = \frac{F_{j} - F_{\min}}{F_{\max} - F_{\min}}, \quad j = 0, 1, 2, ..., N$$

Assume the function $f(c)$, $c \in [0,1]$, $f(0) = 0$ and $f(1) = 1$ to be piecewise linear in the intervals $[C_{j-1}, C_j]$, $j = 1, ..., N$:

$$f(c) = \sum_{j=1}^N \left[ f_{j-1} - f_{j-1}^{(c)} \right] \left[ c - c_{j-1} \right] + f_{j-1}^{(c)} \left[ H(c - c_{j-1}) - H(c - c_j) \right], \quad c \in [0,1]$$
where $H(c)$ is the Heaviside unit step-function. The normalized function $f(c)$ is strictly increasing, and therefore, it is invertible. The inverse function is denoted as $c(f)$. An example of both the normalized direct and inverse functions for CO$_2$ at $T_g = T_b = 1000K$ are shown in figure 3.

![Figure 3. The normalized direct $f(c)$ and inverse $c(f)$ functions](image)

One may calculate the generalized moments of $f(c)$ and of the inverse function $c(f)$ accounting for the fact that both functions are piecewise linear:

$$J_k = \int_0^1 c^k d[f(c)] = \frac{1}{k+1} \sum_{j=1}^{N} \frac{c_j^{k+1} - c_{j-1}^{k+1}}{c_j - c_{j-1}} (f_j - f_{j-1}), \quad k=0,1,\ldots,n-1$$

$$M_k = \int_0^1 f^k d[c(f)] = \frac{1}{k+1} \sum_{j=1}^{N} \frac{f_j^{k+1} - f_{j-1}^{k+1}}{f_j - f_{j-1}} (c_j - c_{j-1}), \quad k=0,1,\ldots,n-1$$

Further, the function $f(c)$ and the inverse function $c(f)$ may be represented with the help of Cutteridge-Devyatov Polynomials:

$$f(c) = \sum_{k=0}^{n-1} J_k \Phi_k^{(n)}(c)$$

$$c(f) = \sum_{k=0}^{n-1} M_k \Phi_k^{(n)}(f)$$

Changing the variables $c$ and $f$ back to $C$ and $F$ to represent the ALBDF and the inverse ALBDF yields:

$$F(C,T_g,T_b,p,Y) = F_{min} + (F_{max} - F_{min}) \sum_{k=0}^{n-1} J_k \Phi_k^{(n)} \left( \ln \frac{C}{C_{min}} / \ln \frac{C_{max}}{C_{min}} \right)$$

$$C(F,T_g,T_b,p,Y) = C_{min} \cdot \left( \frac{C_{max}}{C_{min}} \right) \sum_{k=0}^{n-1} M_k \Phi_k^{(n)} \left( \frac{F - F_{min}}{F_{max} - F_{min}} \right)$$
With a sufficient number of moments (usually, 6 - 14), CDP provides very accurate and efficient representation of the ALBDF as shown in figures 4 and 5.

**Figure 4.** a) Comparison of the ALBDF look-up table data for H$_2$O at $T_g = T_b = 1000$ K with the CDP representation with the help of 14 moments; b) relative error.

**Figure 5.** Comparison of the ALBDF look-up table data (a) and the inverse ALBDF (b) of CO$_2$ at $T_g = T_b = 1000$ K with the CDP representation.
3. Construction of the Generalized SLW Spectral Model

3.1 Simultaneous subdivision of the variables C and F
The generalized SLW spectral model is based on simultaneous synchronized subdivision of the variable \( C \) into supplemental absorption cross sections \( \tilde{C}_0, \tilde{C}_1, ..., \tilde{C}_n \) and the subdivision of the variable \( F \) into corresponding values of \( \tilde{F}_0, \tilde{F}_1, ..., \tilde{F}_n \). The supplemental cross-sections define the wavenumber intervals of gray gases for spectral integration. That subdivision is followed by the corresponding subdivision into absorption cross-sections \( C_j, ..., C_n \) and associated values of \( F_j, ..., F_n \). The absorption cross-sections define the values of the gray gas absorption coefficients \( \kappa_j = NYC_j \).

**Figure 6.** Simultaneous partition of the variables \( C \) and \( F \): a) log-spaced absorption cross-sections \( \tilde{C}_j \) generate partition of \( \tilde{F}_j \); b) partition \( \tilde{F}_j \) based on Gauss-Legendre quadratures generates partition \( \tilde{C}_j \).

Construction of the standard SLW spectral model starts with subdivision of variable \( C \) into evenly log-spaced supplemental absorption cross sections \( \tilde{C}_0, \tilde{C}_1, ..., \tilde{C}_n \), as shown schematically in Fig. 6. Generally, one chooses the range \( C_{min}, C_{max} \) which effectively represents the absorption cross-section of gases. Then the supplemental cross-sections are defined by
\[
\tilde{C}_j = C_{min} \left( \frac{C_{max}}{C_{min}} \right)^{j/n}, \quad j = 0, 1, ..., n
\]
The absorption cross-sections for representation of the absorption coefficient as the geometrical mean of corresponding supplemental cross-sections are shown as:
\[
C_j = \sqrt[1/n]{\tilde{C}_j/C_j} \quad j = 1, ..., n
\]
The gray gas absorption coefficients are then calculated as
\[ \kappa_j = \text{NYC}_j \quad j = 1, \ldots, n \]
and the clear gas absorption coefficient is \( \kappa_0 = 0 \). Partition of the absorption cross-section variable \( C \) automatically generates the corresponding subdivision of variable \( F \):
\[ \tilde{F}_j = F\left(C_j, T_g, T_b\right) \quad j = 0, 1, \ldots, n \]
\[ F_j = F\left(C_j, T_g, T_b\right) \quad j = 1, \ldots, n \]

3.2 Outline of the compact SLW-Gauss model – construction of the SLW model in isothermal media

This approach is based on discretization of the ALBDF domain of the variable \( F \in \left[F_{\text{min}}, F_{\text{max}}\right] \) with the help of Gauss-Legendre quadratures, as shown in the right panel of figure 6. This approach is called compact, because, in general, without optimization, for satisfactory accuracy it requires fewer gray gases (3-5) than the traditional logarithmically spaced subdivision in the SLW method (8-10 gray gases). The procedure is summarized as follows:

1) Choose \( n \) = the number of gray gases in the model

2) Define \( \tilde{C}_0 = C_{\text{min}}, \quad \tilde{C}_n = C_{\text{max}} \)

3) Calculate
\[ F_{\text{min}} = \tilde{F}_0 = F\left(C_{\text{min}}, T_g = T, T_b = T\right) \]
\[ F_{\text{max}} = \tilde{F}_n = F\left(C_{\text{max}}, T_g = T, T_b = T\right) \]

4) Find positive abscissas \( x_j > 0 \) and weights \( w_j, \quad j = 1, 2, \ldots, n \) of the Gaussian-Legendre quadratures for integration over the interval \([-1, 1]\).

5) Discretize the interval \([F_{\text{min}}, F_{\text{max}}]\) by transformation of the Gaussian quadratures:
\[ \tilde{F}_0 = F_{\text{min}} \]
\[ \tilde{F}_j = F_{\text{min}} + \left(\sum_{k=1}^n w_k\right) \cdot \left(F_{\text{max}} - F_{\text{min}}\right) \quad j = 1, 2, \ldots, n \]
\[ F_j = F_{\text{min}} + x_j \cdot \left(F_{\text{max}} - F_{\text{min}}\right) \quad j = 1, 2, \ldots, n \]

6) Calculate the supplemental absorption cross-sections with the help of the inverse ALBDF:
\[ \tilde{C}_j = C\left(\tilde{F}_j, T_g = T, T_b = T\right) \quad j = 1, 2, \ldots, n \]

7) Calculate the gray gas absorption cross-sections with the help of the inverse ALBDF:
\[ C_j = C\left(F_j, T_g = T, T_b = T\right) \quad j = 1, 2, \ldots, n \]

8) Calculate the gray gas absorption coefficients:
\[ \kappa_0 = 0, \quad \kappa_j = \text{NYC}_j \quad j = 1, 2, \ldots, n \]

9) Calculate the gray gas weights:
\[ a_0 = \tilde{F}_0, \quad a_j = \tilde{F}_j - \tilde{F}_{j-1} \quad j = 1, 2, \ldots, n \]
10) Calculate the boundary gray gas weights:

\[ a_j^w = F\left(\bar{C}_j, T_g = T_b, T_b = T_w\right) - F\left(\bar{C}_0, T_g = T_b, T_b = T_w\right) \]

\[ a_0^w = F\left(\bar{C}_0, T_g = T_b, T_b = T_w\right) \]

11) The clear gas and the gray gas RTEs:

\[ \frac{\partial I_g(s, \Omega)}{\partial s} = 0 \]

\[ \frac{\partial I_j(s, \Omega)}{\partial s} = -\kappa_j I_j(s, \Omega) + a_j \kappa_j I_j(T) \]

subject to the corresponding boundary conditions.

**Figure 7.** The SLW reordered gray gas absorption coefficients and the SLW histogram spectrum.

The SLW method turns the continuous line gas absorption spectrum into a histogram spectrum with a few discrete values of the gray gas absorption coefficient \( \kappa_j = NYC_j \), as illustrated in the right panel of figure 7. Then these gray gas absorption coefficients are reordered according to their weights \( a_j \) (left panel of figure 7). With increase of the number of gray gases, the reordered distribution of \( \kappa_j \) is approaching multiplied by gas molar density and fraction the inverse ALBDF \( NYC\left(F, T_g, T_b\right) \) defined in the interval \( F \in \left[F_{min}, F_{max}\right] \).
Figure 8. Geometric Interpretation of the SLW Method and of the Inverse SLW Method
3.3 Summary of the SLW spectral model in isothermal medium

The SLW and the inverse SLW spectral models based on simultaneous discretisation of variables $C$ and $F$ are visualized in figure 8. There is a clear one-to-one correspondence between two models. Table 1 compiles all possible forms of the gray gas RTE in the generalized formulation with the help of direct and inverse ALBDF. The analytical solution for the directional radiative flux $q^+(x)$ in a plane parallel layer bounded by cold, black walls illustrates the differences and similarities in the different variations of the SLW method.

| Model     | RTE                                                                 | Total Intensity  | Directional Radiative Flux |
|-----------|----------------------------------------------------------------------|------------------|----------------------------|
|           |                                                                     | $I(s, \Omega)$   | $q^+(x)$                   |
| LBL       | $\frac{\partial I_s}{\partial s} = -\kappa_c I_s + \kappa_a I_{\theta}(T)$ | $\int_0^\pi I_s d\eta$ | $\pi I_b - 2\pi \int_0^\pi I_{\theta}(T) E_s(\text{NYC}, x) d\eta$ |
| SLW       | $\frac{\partial I_s}{\partial s} = -\kappa_c I + a_1 \kappa_a I_b$       | $\sum_{j=0}^n I_j$ | $\pi I_b - 2\pi \sum_{j=0}^n a_j E_s(\text{NYC}, x)$ |
| Inverse   | $\frac{\partial I_F}{\partial s} = -\kappa_c I(F) + \kappa_a I_b$          | $\sum_{j=0}^n I(F_j) dF_j$ | $\pi I_b - 2\pi \sum_{j=0}^n E_s \left( \text{NYC}(F_j) x \right)$ |
| SLW-1     | $\frac{\partial I_s}{\partial s} = -\kappa_c I + a_1 \kappa_a I_b$       | $I_0 + I_j$      | $\pi I_s a_1 - 2\pi I_s a_1 E_s \left( \text{NYC}, x \right)$ |
| Inverse   | $\frac{\partial I_F}{\partial s} = -\kappa_c I(F) + \kappa_a I_b$          | $I(F_0) dF_0 + I(F_j) dF_j$ | $\pi I_b - 2\pi \sum_{j=0}^n E_s \left( \text{NYC}(F_j) x \right)$ |
| Exact SLW | $\frac{\partial I(C)}{\partial s} = -\text{NYC} I(C) + \text{NYC} \frac{\partial F(C)}{\partial C} I_b$ | $\int_0^\pi I(C) dC$ | $\pi I_b - 2\pi \int_0^\pi E_s \left( \text{NYC}, x \right) \frac{\partial F(C)}{\partial C} dC$ |
| Inverse   | $\frac{\partial I(F)}{\partial s} = -\text{NYC}(F) I(F) + \text{NYC}(F) I_b$ | $\int_0^\pi I(F) dF$ | $\pi I_b - 2\pi \int_0^\pi E_s \left( \text{NYC}(F), x \right) dF$ |

Table 1. SLW spectral models in isothermal media
4. The Generalized SLW Spectral Model in Non-Isothermal Media

4.1 The SLW Reference Approach

The argument of correlated spectra (sometimes called the “ideal spectrum” assumption) is an assumption of relationship between the absorption cross-section at some chosen reference state and at the local state. The reference approach consists in the following. One chooses the reference supplemental cross-sections \( \tilde{C}_0^\text{ref}, \tilde{C}_1^\text{ref}, \ldots, \tilde{C}_n^\text{ref} \). Their intersection with the gas absorption cross-section at the reference state \( C^\eta(T_g = T_{\text{ref}}, Y = Y_{\text{ref}}) \) generate the spectral intervals \( \Delta \eta_{i,j} \). Then there exist the local supplemental cross-sections \( \tilde{C}_0^\text{loc}, \tilde{C}_1^\text{loc}, \ldots, \tilde{C}_n^\text{loc} \) such that their intersection with the absorption cross-section at local state \( C^\eta(T_g = T_{\text{loc}}, Y = Y_{\text{loc}}) \) generate the same spectral intervals, as shown in Fig. 9 and as described mathematically as:

\[
\Delta \eta_{i,j} = \left\{ \eta : \tilde{C}_i^\text{ref} \leq C^\eta(T_g = T_{\text{ref}}, Y = Y_{\text{ref}}) \leq \tilde{C}_j^\text{ref} \right\} = \left\{ \eta : \tilde{C}_i^\text{loc} \leq C^\eta(T_g = T_{\text{loc}}, Y = Y_{\text{loc}}) \leq \tilde{C}_j^\text{loc} \right\}
\]

It follows from the equality of the intervals of integration that for the ALBDF at the reference source temperature

\[
F\left( \tilde{C}_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}} \right) = F\left( \tilde{C}_j^\text{loc}, T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right), \quad j = 0, 1, \ldots, n
\]

\[
F\left( C_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}} \right) = F\left( C_j^\text{loc}, T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right), \quad j = 1, 2, \ldots, n
\]

or the equivalent relationships should hold for the ALBDF at the local source temperature

\[
F\left( \tilde{C}_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{loc}} \right) = F\left( \tilde{C}_j^\text{loc}, T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right), \quad j = 0, 1, \ldots, n
\]

\[
F\left( C_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{loc}} \right) = F\left( C_j^\text{loc}, T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right), \quad j = 1, 2, \ldots, n
\]

Then the set of local supplemental cross-sections \( \tilde{C}_0^\text{loc}, \tilde{C}_1^\text{loc}, \ldots, \tilde{C}_n^\text{loc} \) and local absorption cross-sections \( C_0^\text{loc}, C_1^\text{loc}, \ldots, C_n^\text{loc} \) can be found by solution of these implicit equations. In the case of the Generalized SLW spectral model they can be found explicitly (rather than by inversion of the ALBDF) with the help of the inverse ALBDF \( C(F) \) at the reference source temperature:

\[
\tilde{C}_j^\text{loc} = C\left[ F\left( \tilde{C}_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}} \right), T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right], \quad j = 0, 1, \ldots, n
\]

\[
C_j^\text{loc} = C\left[ F\left( C_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}} \right), T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right], \quad j = 1, 2, \ldots, n
\]

or by the alternative equations at the local source temperature:

\[
\tilde{C}_j^\text{loc} = C\left[ F\left( \tilde{C}_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{loc}} \right), T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right], \quad j = 0, 1, 2, \ldots, n
\]

\[
C_j^\text{loc} = C\left[ F\left( C_j^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{loc}} \right), T_g = T_{\text{loc}}, T_b = T_{\text{loc}} \right], \quad j = 1, 2, \ldots, n
\]

The alternative approaches for construction of the local cross-sections \( \tilde{C}_j^\text{loc} \) and \( C_j^\text{loc} \) are visualized in figure 9. One may define the gray gas reference correlation function, which relates the local gray gas absorption cross-sections with those at the reference state:
\[ B_j^{\text{loc}} = \frac{C_j^{\text{loc}}}{C_j^{\text{ref}}} \]

Then the local absorption cross-sections can be found from

\[ C_j^{\text{loc}} = B_j^{\text{loc}} C_j^{\text{ref}} \]

The gray gas absorption coefficients in the RTE are local variables:

\[ \kappa_g(s) = \kappa_g^{\text{loc}} = N^{\text{loc}} Y^{\text{loc}} C_j^{\text{loc}} = N^{\text{loc}} Y^{\text{loc}} B_j^{\text{loc}} C_j^{\text{ref}}; \quad \kappa_0^{\text{loc}} = 0 \]

**Figure 9.** Correlated SLW model in non-isothermal medium – the alternative reference approaches

**4.2 Outline of construction of the compact SLW model at the Reference State**

The compact SLW spectral model based on Gauss-Legendre quadrature (as referenced in Section 3.2) at the reference state is constructed via the following steps.

1) Choose \( T_{\text{ref}}, Y_{\text{ref}}, P_{\text{ref}} \) to define the Reference State

2) Choose \( n \) = the number of gray gases in the model
3) Define
\[ \tilde{C}_0^{\text{ref}} = C_{\min}, \quad \tilde{C}_n^{\text{ref}} = C_{\max} \]

4) Calculate
\[ F_{\min}^{\text{ref}} = \tilde{F}_{\min}^{\text{ref}} = F\left(C_{\min}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}}\right) \]
\[ F_{\max}^{\text{ref}} = \tilde{F}_{\max}^{\text{ref}} = F\left(C_{\max}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}}\right) \]

5) Calculate positive abscissas \( x_j > 0 \) and weights \( w_j, \quad j = 1, 2, ..., n \) of the Gaussian-Legendre quadratures for integration over interval \([-I, I]\).

6) Discretize the interval \([F_{\min}^{\text{ref}}, F_{\max}^{\text{ref}}]\) by transformation of the Gaussian quadratures:
\[ \tilde{F}_j^{\text{ref}} = F_{\min}^{\text{ref}} + \left( \sum_{k=1}^{J} w_k \right) \left( F_{\max}^{\text{ref}} - F_{\min}^{\text{ref}} \right) \quad j = 1, 2, ..., n \]
\[ F_j^{\text{ref}} = F_{\min}^{\text{ref}} + x_j \left( F_{\max}^{\text{ref}} - F_{\min}^{\text{ref}} \right) \quad j = 1, 2, ..., n \]

7) Calculate the reference supplemental absorption cross-sections with the help of inverse ALBDF:
\[ \tilde{C}_j^{\text{ref}} = C\left( \tilde{F}_j^{\text{ref}}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}}\right), \quad j = 1, 2, ..., n \]

8) Calculate the reference gray gas absorption cross-sections with the help of inverse ALBDF:
\[ C_j^{\text{ref}} = C\left( F_j^{\text{ref}}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}}\right), \quad j = 1, 2, ..., n \]

9) The finite increments of discretization become:
\[ \Delta C_j^{\text{ref}} = \tilde{C}_j^{\text{ref}} - \tilde{C}_{j-1}^{\text{ref}}, \quad j = 1, 2, ..., n \]
\[ \Delta C_0^{\text{ref}} = \tilde{C}_0^{\text{ref}} \]
\[ \Delta F_j^{\text{ref}} = \tilde{F}_j^{\text{ref}} - \tilde{F}_{j-1}^{\text{ref}}, \quad j = 1, 2, ..., n \]
\[ \Delta F_0^{\text{ref}} = \tilde{F}_0^{\text{ref}} \]

To determine the reference gray gas spectral intervals of integration, the entire set of wavenumbers \((0, \infty)\) is portioned into subintervals \( \Delta \eta_{i,j} \) according to discretization of the continuous absorption cross-section \( C_\eta\left(T_g = T_{\text{ref}}\right) \) into reference supplemental cross-sections \( \tilde{C}_0^{\text{ref}}, \tilde{C}_j^{\text{ref}}, ..., \tilde{C}_n^{\text{ref}} \)

\[ \Delta \eta_{i,0} = \{ \eta : \quad C_\eta\left(T_g = T_{\text{ref}}\right) \leq \tilde{C}_0^{\text{ref}} \}, \quad \text{clear gas} \]
\[ \Delta \eta_{i,j} = \{ \eta : \quad \tilde{C}_{j-1}^{\text{ref}} \leq C_\eta\left(T_g = T_{\text{ref}}\right) \leq \tilde{C}_j^{\text{ref}} \}, \quad j = 1, 2, ..., n \quad \text{gray gases} \]

The domain of spectral integration corresponding to gray gas \( j \) becomes
\[ \Delta \eta_j = \bigcup \Delta \eta_{i,j}, \quad j = 0, 1, 2, ..., n \]
The reference spectral model is set, following which the SLW Reference Approach can be applied.

4.3 Outline of the Generalized Correlated SLW Model in Non-Isothermal Medium

1) Define the reference state by choosing \( T_{\text{ref}}, Y_{\text{ref}}, p_{\text{ref}} \).

2) Construct the generalized spectral model at the Reference State:

\[
C_{0}^{\text{ref}}, C_{1}^{\text{ref}}, ..., C_{n}^{\text{ref}}
\]

\[
\tilde{F}_{j}^{\text{ref}} = F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{ref}} \right), \quad j = 0, 1, 2, ..., n
\]

\[
F_{j}^{\text{ref}} = F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{ref}} \right), \quad j = 1, 2, ..., n
\]

\[
\tilde{F}_{j}^{\text{loc}} = F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{loc}} \right), \quad j = 0, 1, 2, ..., n
\]

\[
F_{j}^{\text{loc}} = F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{loc}} \right), \quad j = 1, 2, ..., n
\]

The increments of discretization are defined by:

\[
\Delta C_{j}^{\text{ref}} = C_{j}^{\text{ref}} - C_{j-1}^{\text{ref}}, \quad j = 1, 2, ..., n
\]

\[
\Delta C_{0}^{\text{ref}} = C_{0}^{\text{ref}}
\]

\[
\Delta F_{j}^{\text{ref}} = F_{j}^{\text{ref}} - F_{j-1}^{\text{ref}}, \quad j = 1, 2, ..., n
\]

\[
\Delta F_{0}^{\text{ref}} = F_{0}^{\text{ref}}
\]

\[
\Delta F_{j}^{\text{loc}} = F_{j}^{\text{loc}} - F_{j-1}^{\text{loc}}, \quad j = 1, 2, ..., n
\]

\[
\Delta F_{0}^{\text{loc}} = F_{0}^{\text{loc}}
\]

3) Calculate the local absorption cross-sections and the local absorption coefficients with the help of the inverse ALBDF at the reference source temperature:

\[
\tilde{C}_{j}^{\text{loc}} = C \left[ F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{loc}} \right), T_{g} = T_{\text{loc}}, T_{b} = T_{\text{ref}} \right], \quad j = 0, 1, ..., n
\]

\[
C_{j}^{\text{loc}} = C \left[ F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{loc}} \right), T_{g} = T_{\text{loc}}, T_{b} = T_{\text{loc}} \right], \quad j = 1, 2, ..., n
\]

or by alternative equations with the help of the inverse ALBDF at the local source temperature:

\[
\tilde{C}_{j}^{\text{loc}} = C \left[ F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{loc}} \right), T_{g} = T_{\text{loc}}, T_{b} = T_{\text{loc}} \right], \quad j = 0, 1, 2, ..., n
\]

\[
C_{j}^{\text{loc}} = C \left[ F\left( C_{j}^{\text{ref}}, T_{g} = T_{\text{ref}}, T_{b} = T_{\text{loc}} \right), T_{g} = T_{\text{loc}}, T_{b} = T_{\text{loc}} \right], \quad j = 1, 2, ..., n
\]

Then the local gray gas absorption coefficients are

\[
\kappa_{j}^{\text{loc}} = \kappa_{j}^{\text{loc}} = N^{\text{loc}} Y^{\text{loc}} C_{j}^{\text{loc}}, \quad j = 1, 2, ..., n
\]
\( k_{0}^{loc} = 0 \)

4) Calculate the local gray gas weights by one of the following alternative approaches:

\[
\Delta F_{j}^{loc} = a_{j}^{loc} = F\left( \tilde{C}_{j}^{ref}, T_{g} = T_{ref}, T_{b} = T_{loc} \right) - F\left( \tilde{C}_{j-1}^{ref}, T_{g} = T_{ref}, T_{b} = T_{loc} \right)
\]

\[
\Delta F_{0}^{loc} = a_{0}^{loc} = F\left( \tilde{C}_{0}^{loc}, T_{g} = T_{loc}, T_{b} = T_{loc} \right)
\]

5) The gray gas RTE:

\[
\frac{\partial}{\partial s} I_{j}(s, \Omega) = -k_{j}^{loc} I_{j}(s, \Omega) + a_{j}^{loc} k_{j}^{loc} I_{b}(T_{loc})
\]

The alternative approaches in construction of the Generalized SLW model are visualized in Fig. 9.

4.4 Additional new versions of the Reference Approach of the Generalized SLW Model

The presented alternative SLW reference approaches correspond to original SLW reference approach of Denison and Webb [5] and to the reference approach applied in the ADF method of Riviére et al. [35]. The Generalized SLW Model allows exploration of more possibilities for the reference approach: depending on starting partition into gray gases rather with the help of the supplemental cross-sections \( \tilde{C}_{j}^{ref} \) or with the help of \( \tilde{F}_{j}^{ref} \); depending on the choice of the blackbody source temperature rather \( T_{b} = T_{ref} \) or \( T_{b} = T_{loc} \) to establish gas absorption spectra correlation at different thermodynamical states; depending on the choice of calculation of the local gray gas weights \( a_{j}^{loc} \) with the help rather of the reference supplemental cross-sections \( \tilde{C}_{j}^{ref} \) or with the help of the local supplemental cross-sections \( \tilde{C}_{j}^{loc} \), totally, 8 different versions of the SLW reference approach are possible. All these possible versions are the particular cases of the visualization of the Generalized SLW model reference approach shown in Fig. 9. The full investigation and comparison of all possibilities might help to overcome the main challenge for the global methods dealing with prediction of radiative transfer in non-isothermal gaseous medium.

4.5 The finite increment form of the gray gas RTEs

One may define the following forms of the gray gas intensity

\[
I_{j}(s, \Omega) = I\left( \tilde{C}_{j}^{ref}, s, \Omega \right) \cdot \Delta C_{j}^{ref}
\]

\[ \text{The SLW gray gas increment intensity} \]

\[
I_{j}(s, \Omega) = I\left( \tilde{F}_{j}^{ref}, s, \Omega \right) \cdot \Delta F_{j}^{ref}
\]

\[ \text{The Inverse SLW gray gas increment intensity} \]

The gray gas RTEs may then be written as (with two alternative forms of the gray gas RTEs which are equivalent provided that the gas spectra are correlated):

\[
\frac{\partial}{\partial s} I\left( \tilde{C}_{j}^{ref}, s, \Omega \right) = -k_{j}^{loc} I\left( \tilde{C}_{j}^{ref}, s, \Omega \right) + \frac{\Delta F_{j}^{loc}}{\Delta C_{j}^{ref}} k_{j}^{loc} I_{b}(T_{loc})
\]

\[ \text{The finite increment form of the gray gas RTEs} \]
\[
\frac{\partial}{\partial s} I\left(\tilde{F}_{j}^{ref}, s, \Omega\right) = -\kappa_{j}^{loc} I\left(\tilde{F}_{j}^{ref}, s, \Omega\right) + \frac{\Delta F_{j}^{loc}}{\Delta F_{j}^{ref}} \kappa_{j}^{loc} I_{b}(T_{loc})
\]

where the gray gas weights are calculated in one of two ways:

\[
\Delta F_{j}^{loc} \equiv \Delta f_{j}^{loc} = F\left(\tilde{C}_{j}, T_{g} = T_{ref}, T_{b} = T_{loc}\right) - F\left(\tilde{C}_{j-1}, T_{g} = T_{ref}, T_{b} = T_{loc}\right)
\]

\[
\Delta F_{j}^{ref} \equiv \Delta f_{j}^{ref} = F\left(\tilde{C}_{j}, T_{g} = T_{ref}, T_{b} = T_{ref}\right) - F\left(\tilde{C}_{j-1}, T_{g} = T_{ref}, T_{b} = T_{ref}\right)
\]

The clear gas RTEs are described as

\[
\frac{\partial}{\partial s} I_{b}(s, \Omega) = 0
\]

\[
\frac{\partial}{\partial s} I\left(\tilde{C}_{0}^{ref}, s, \Omega\right) = 0
\]

\[
\frac{\partial}{\partial s} I\left(\tilde{F}_{0}^{ref}, s, \Omega\right) = 0
\]

One solves the gray gas and clear gas RTEs subject to the boundary conditions, following which the total emissivity is calculated by summation

\[
I(s, \Omega) = \sum_{j=0}^{N} I_{j}(s, \Omega)
\]

\[
I(s, \Omega) = \sum_{j=0}^{N} I\left(\tilde{C}_{j}^{ref}, s, \Omega\right) \cdot \Delta C_{j}^{ref}
\]

\[
I(s, \Omega) = \sum_{j=0}^{N} I\left(\tilde{F}_{j}^{ref}, s, \Omega\right) \cdot \Delta F_{j}^{ref}
\]

### 4.6 Exact Limit of the Generalized SLW Model

In the limit when the number of gray gases approaches infinity, the finite increment RTEs become equations in terms of intensity as a continuous functions of the variables \(C\) and \(F\)

\[
\frac{\partial}{\partial s} I\left(C, s, \Omega\right) = -N_{loc}^{loc} Y_{loc}^{loc} B_{loc}^{loc}(C) CI\left(C, s, \Omega\right) + \frac{\partial F\left(C, T_{g} = T_{ref}, T_{b} = T_{loc}\right)}{\partial C} N_{loc}^{loc} Y_{loc}^{loc} B_{loc}^{loc}(C) CI_{b}(T_{loc})
\]

\[
\frac{\partial}{\partial s} I\left(F, s, \Omega\right) = -N_{loc}^{loc} Y_{loc}^{loc} C\left[F, T_{g} = T_{ref}, T_{b} = T_{loc}\right] I\left(F, s, \Omega\right) + \frac{\partial F\left[F, T_{g} = T_{ref}, T_{b} = T_{ref}, T_{s} = T_{loc}, T_{b} = T_{loc}\right]}{\partial F} N_{loc}^{loc} Y_{loc}^{loc} C\left[F, T_{g} = T_{ref}, T_{b} = T_{loc}\right] I_{b}(T_{loc})
\]

The total intensity is obtained by integration

\[
I(s, \Omega) = \int_{0}^{\infty} I\left(C, s, \Omega\right)dC
\]
\[ I(s, \Omega) = \int_o^1 I(F, s, \Omega) dF \]

4.7 Exact SLW model – Analytical Solutions based on tabulated LBL ALBDF

Calculations in the Exact SLW model can be effectively performed with the help of piecewise linear ALBDF defined by an ALBDF look-up table generated from line-by-line data. The tabulated LBL ALBDF is expressed as

\[ F_j = F(C_j, T_g, T_b, Y, p) \]

interpolated to state \((T_g, T_b, Y, p), \quad j = 1, 2, ..., N\)

Assuming that the ALBDF function is a piecewise linear function between the discrete entries of the look-up table

\[ F(C, T_g, T_b, Y, p) = \sum_{j=1}^N \left[ \frac{F_j - F_{j-1}}{C_j - C_{j-1}} \left( C - C_{j-1} \right) + F_{j-1} \right] \left[ H(C - C_{j-1}) - H(C - C_j) \right] \]

the derivative of ALBDF in the intervals \((C_{j-1}, C_j)\) becomes

\[ \frac{\partial}{\partial C} F(C, T_g, T_b, Y, p) = \sum_{j=1}^N \frac{F_j - F_{j-1}}{C_j - C_{j-1}} \left[ H(C - C_{j-1}) - H(C - C_j) \right] \]

Note that the derivative of ALBDF is mathematically a piecewise constant function.

4.7.1 Exact SLW modelling of Planck Mean Absorption Coefficient \(\kappa_p\)

Exact SLW model

\[ \kappa_p = NY \int_0^\infty C \frac{\partial F}{\partial C} dC = NY \sum_{j=1}^N \left( F_j - F_{j-1} \right) \left( \frac{C_j + C_{j-1}}{2} \right) \]

SLW-n model

\[ \kappa_p = NY \sum_{j=1}^N \left( F_j - F_{j-1} \right) \sqrt{C_j C_{j-1}} \]

4.7.2 Exact SLW modelling of the Total Emissivity \(\varepsilon\)

The Exact SLW model form of the emissivity may be stated as

\[ \varepsilon = (F_{max} - F_{min}) \frac{1}{NYL} \sum_{j=1}^\infty \left( \frac{F_j - F_{j-1}}{C_j - C_{j-1}} \right) \left( e^{NYLC_{j-1}} - e^{-NYLC_j} \right) \]

which is compared to the SLW-n model of total emissivity as

\[ \varepsilon = \sum_{j=1}^N \left( F_j - F_{j-1} \right) \left( 1 - e^{-NYLC_j} \right) \]

4.7.3 Exact SLW modelling of the net radiative flux \(q(x)\)

The analytical solution for the total directional radiative fluxes in a non-scattering plane layer bounded by cold, black walls may be stated as:

\[ q^+(x) = \pi I_0 - 2\pi I_0 \int_0^{NYC} E_j(NYC) \frac{\partial F}{\partial C} dC \]
Calculation with the piecewise linearly defined ALBDF is as follows:

\[ q^+ (x) = \pi I_b - 2\pi I_b \sum_{j=1}^{N} \left( \frac{F_j - F_{j-1}}{C_j - C_{j-1}} \right) \left[ -E_j \left( NYC_j, x \right) + E_x \left( NYC_{j-1}, x \right) \right] \]

\[ q^- (x) = \pi I_b - 2\pi I_b \sum_{j=1}^{N} \left( \frac{F_j - F_{j-1}}{C_j - C_{j-1}} \right) \left[ -E_j \left[ NYC_j \left( L - x \right) \right] + E_x \left[ NYC_{j-1} \left( L - x \right) \right] \right] \]

Note that the 4\textsuperscript{th} order integral-exponential function appears in these equations. The exact solution may be compared to the SLW-n model result:

\[ q^+ (x) = \pi I_b - 2\pi I_b \sum_{j=1}^{N} \left( F_j - F_{j-1} \right) E_j \left( NYC_j, x \right) \]

\[ q^- (x) = \pi I_b - 2\pi I_b \sum_{j=1}^{N} \left( F_j - F_{j-1} \right) E_x \left[ NYC_j \left( L - x \right) \right] \]

The total net radiative flux becomes

\[ q(x) = q^+ (x) - q^- (x) \]

\[ = 2\pi I_b \sum_{j=1}^{N} \left( F_j - F_{j-1} \right) \left[ E_j \left[ NYC_j \left( L - x \right) \right] - E_x \left( NYC_j, x \right) \right] \]

4.7.4 Exact SLW Model analytical solution for the total divergence of the net radiative flux

The solution for the divergence of the net radiative flux in a plane layer bounded by cold black walls may be stated as:

\[ Q(x) = -2\pi I_b (T) \int_{0}^{N} NCx \left( NCx \right) \frac{\partial F}{\partial C} dC - 2\pi I_b (T) \int_{0}^{N} NC \left( NC \right) \left[ NC \left( L - x \right) \right] \frac{\partial F}{\partial C} dC \]

The piecewise linear LBL ALBDF solution may likewise be stated as:

\[ Q(x) = -2\pi I_b \sum_{j=1}^{N} \left( F_j - F_{j-1} \right) \left[ E_j \left( k_j, x \right) + E_x \left( k_{j-1}, x \right) \right] \]

\[ + 2\pi I_b \sum_{j=1}^{N} \left( F_j - F_{j-1} \right) \left[ k_j \cdot E_j \left( k_j, x \right) - k_{j-1} \cdot E_x \left( k_{j-1}, x \right) \right] \]

\[ - 2\pi I_b \sum_{j=1}^{N} \left( F_j - F_{j-1} \right) \left[ E_x \left( \left( L - x \right) k_j \right) + E_x \left( \left( L - x \right) k_{j-1} \right) \right] \]

\[ + 2\pi I_b \sum_{j=1}^{N} \left( F_j - F_{j-1} \right) \left[ k_j \cdot E_j \left( \left( L - x \right) k_j \right) - k_{j-1} \cdot E_x \left( \left( L - x \right) k_{j-1} \right) \right] \]
The analytical Exact SLW Model solutions with piecewise linear interpolation of the tabulated LBL ALBDF provides the benchmark solution for validation of the SLW-\(n\) model.

5 The SLW-1 model

5.1 Construction of the least-squares optimized SLW-1 Model

The SLW-1 model is a minimal SLW spectral model which consists of one gray gas \((\kappa_j, a_j)\), and one clear gas \((\kappa_0, a_0)\), for which \(\kappa_0 = 0\) and \(a_0 = 1 - a_j\). Therefore, only two gray gas parameters \((\kappa_j, a_j)\) are needed to define the SLW-1 model. The histogram absorption spectrum of the SLW-1 model is defined by the supplemental cross-section \(\tilde{C}_0\), which separates the clear gas and the gray gas, and by absorption cross-section \(C_j\), which defines the magnitude of the gray gas (see figure 10). The SLW-1 spectral model is not just a simple reduction of the SLW-\(n\) model to the case of one gray gas, \(n=1\). Rather, its construction has some unique characteristics. An arbitrary SLW-1 model is not intended to be extremely accurate, although certainly not as accurate as the SLW-\(n\) model with sufficient gray gases. Rather, optimization is required for implementation of the SLW-1 model. The efficient SLW-1 model has been introduced previously [15,16]. These previously reported methods for construction of the SLW-1 model are simple methods which are based on exact matching of two accurately calculated radiative quantities such as the total emissivity or total flux divergence with the help of the SLW-\(n\) model using a large number of gray gases. These approaches are straightforward, but not robust and universal since the spectral model constructed might be sensitive to the choice of the matching parameters. The least-squares optimized SLW-1 spectral model in closed form was proposed in [17]. The method is universal, but retains its simplicity because the spectral parameters are found from explicit analytical relations. It is based on exponential approximation of the integro-exponential function widely used in radiation theory. In this approach, the objective function was defined on the entire layer for \(0 \leq x \leq L\). In recent research [46], it has been observed that minimization of the objective function on the equivalent plane layer \(0 \leq x \leq R\), where the reduced layer thickness \(R = c \cdot L\) with some coefficient \(c<1\), might yield better results for prediction of radiative transfer in the entire layer.

Construction of the “true” least-squares optimized SLW-1 model consists in the following: consider the normalized total net radiative flux in the isothermal layer at temperature \(T\) bounded by cold black walls. Expressions for the dimensionless radiative flux for the SLW-\(n\) and SLW-1 models may be stated as

\[
\frac{q_{SLW-n}(x)}{2\pi I_b(T)} = -\sum_{j=0}^{\kappa} a_j \left[ E_j(\kappa_j, x) - E_j(\kappa_j, (L-x)) \right]
\]
If one discretizes the spatial interval $0 \leq x \leq R$: $x_m = m \cdot R/M, m = 0,1,2,\ldots,M$ (i.e., uniform partition) Then one may find the parameters $\kappa_i$ and $a_i$ of the SLW-1 model determined from minimizing the least-square error

$$P(a_i, \kappa_i) = \frac{1}{2\pi I_b(T)} \sum_{m=1}^{M} \left[q_{\text{SLW-1}}(x_m) - q_{\text{SLW-1}}(x_m)\right]^2 \rightarrow \text{min}$$

The optimal solution should be at the points $\kappa_i$ and $a_i$ where the partial derivatives of the objective function $P(a_i, \kappa_i)$ with respect to $\kappa_i$ and $a_i$ vanish (i.e., at the critical points):

$$\frac{\partial}{\partial a_i} P(a_i, \kappa_i) = \frac{\partial}{\partial a_i} \left[ q_{\text{SLW-1}}(x_m) - q_{\text{SLW-1}}(x_m) \right] = 0$$
One may now differentiate and solve this equation for $a_i$ in terms of $\kappa_i$:

$$a_i = \frac{2 \sum_{m=1}^{M} \left[ \frac{q_{SLW-n}(x_m)}{2\pi I_b(T)} \left( E_i(\kappa_i,x_m) - \kappa_i \left( L - x_m \right) \right) \right]}{\sum_{m=1}^{M} \left[ E_i(\kappa_i,x_m) - \kappa_i \left( L - x_m \right) \right]^2}$$

Then the differentiation of the objective function with respect to $\kappa_i$ yields

$$\frac{\partial}{\partial \kappa_i} P(a_i, \kappa_i) = 2a_i \sum_{m=0}^{M} \left[ \frac{q_{SLW-n}(x_m)}{2\pi I_b(T)} \right] - a_i \left\{ E_i \left[ \kappa_i \left( L - x \right) \right] - E_i \left( \kappa_i \right) \right\} \left\{ (L - x_m) E_i \left[ \kappa_i \left( L - x_m \right) \right] - x_m E_i \left( \kappa_i x_m \right) \right\} = 0$$

One may then replace $a_i$ and obtain the expression for the unknown $\kappa_i$:

$$\sum_{m=0}^{M} \left[ \frac{q_{SLW-n}(x_m)}{2\pi I_b(T)} \right] = \frac{2 \sum_{m=0}^{M} \left[ \frac{q_{SLW-n}(x_m)}{2\pi I_b(T)} \left( E_i(\kappa_i,x_m) - \kappa_i \left( L - x_m \right) \right) \right]}{\sum_{m=1}^{M} \left[ E_i(\kappa_i,x_m) - \kappa_i \left( L - x_m \right) \right]^2} \times \left\{ x_m E_i(\kappa_i,x_m) - (L - x_m) E_i \left[ \kappa_i \left( L - x_m \right) \right] \right\} = 0$$

Finally, this equation is solved numerically for $\kappa_i$ (it has a unique solution for $\kappa_i > 0$), and a calculation of $a_i$ follows. An example of the SLW-1 model is shown in figure 11, where the gray gas absorption coefficient $\kappa_i$ is found for a layer of water vapor of thickness $L = 0.75m$ and temperature of $750K$. For the case of black cold boundaries, the total net radiative flux $q(x)$, kW/m$^2$ is calculated with the help of the SLW-10, and with the optimized SLW-1 model (see table 1).

![Figure 11](image_url)

**Figure 11.** a) The optimized parameter $\kappa_i$; b) net radiative flux by SLW-10 and SLW-1 models
5.2 Outline of the SLW-1 Model in Non-Isothermal Medium – the Reference Approach

The development of the SLW-1 model in non-isothermal media using the Reference Approach uses the following sequence (see figure 12 for visualization):

1. Choose the reference state \( T_{\text{ref}}, Y_{\text{ref}}, p_{\text{ref}} \).
   
   Determine the parameters of the efficient SLW-1 model at the reference state:
   
   \[
   \kappa_{0\text{ref}}^\text{ref}, a_{1\text{ref}}^\text{ref}, \quad \text{then calculate} \quad \kappa_0^\text{ref} = 0, \quad a_0^\text{ref} = 1 - a_1^\text{ref}
   \]

2. Determine the cross-sections of the SLW-1 model at the reference state using inverse ALBDF:
   
   \[
   C_{0\text{ref}}^\text{ref} = C \left( a_0^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}} \right)
   \]
   
   \[
   C_{1\text{ref}}^\text{ref} = \frac{\kappa_1^\text{ref}}{N_{\text{ref}}^\text{ref} Y_{\text{ref}}}
   \]

3. Determine the absorption cross-section \( C_{i\text{loc}}^\text{loc} \) at the local state using inverse ALBDF:
   
   \[
   C_{i\text{loc}}^\text{loc} = C \left[ F \left( C_{i\text{ref}}^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{ref}} \right), T_g = T_{\text{loc}}, T_b = T_{\text{ref}} \right]
   \]

4. Calculate the SLW-1 model parameters at the local state:
   
   \[
   a_0^\text{loc} = F \left( C_{0\text{ref}}^\text{ref}, T_g = T_{\text{ref}}, T_b = T_{\text{loc}} \right), \quad a_1^\text{loc} = 1 - a_0^\text{loc}
   \]
   
   \[
   a_0 \left( T_o \right) = F \left( C_{0\text{ref}}^\text{ref}, T_g = T_{\text{ref}}, T_b = T_o \right), \quad a_1 \left( T_o \right) = 1 - a_0 \left( T_o \right)
   \]
   
   \[
   a_0 \left( T_L \right) = F \left( C_{0\text{ref}}^\text{ref}, T_g = T_{\text{ref}}, T_b = T_L \right), \quad a_1 \left( T_L \right) = 1 - a_0 \left( T_L \right)
   \]
   
   \[
   \kappa_0^\text{loc} = 0, \quad \kappa_i^\text{loc} = N_{\text{loc}}^\text{loc} Y_{\text{loc}} C_{i\text{loc}}^\text{loc}
   \]

5. RTEs for clear and gray gas:
   
   \[
   \frac{\partial I_j^i \left( x, \mu \right)}{\partial x} = 0
   \]
   
   \[
   \frac{\partial I_o^i \left( x, \mu \right)}{\partial x} = 0
   \]
   
   \[
   \mu \frac{\partial I_j^i \left( x, \mu \right)}{\partial x} = -\kappa_i \left( x \right) I_j^i \left( x, \mu \right) + \kappa_i \left( x \right) a_i \left( x \right) I_o^i \left( T \right)
   \]
   
   \[
   -\mu \frac{\partial I_j^i \left( x, \mu \right)}{\partial x} = -\kappa_i \left( x \right) I_j^i \left( x, \mu \right) + \kappa_i \left( x \right) a_i \left( x \right) I_o^i \left( T \right)
   \]

6. Boundary conditions in the case of the gray diffusely reflective boundaries are:
   
   \[
   I_j^i \left( 0, \mu \right) = \varepsilon_i a_j \left( T_o \right) I_o^i \left( T_o \right) + 2 \rho_j \int_0^1 I_j^i \left( 0, \mu \right) \mu d\mu \quad j = 0, 1
   \]
   
   \[
   I_j^i \left( L, \mu \right) = \varepsilon_i a_j \left( T_L \right) I_o^i \left( T_L \right) + 2 \rho_j \int_0^1 I_j^i \left( L, \mu \right) \mu d\mu \quad j = 0, 1
   \]
5.3 The SLW-1 Model in 2-D and 3-D geometries
Extensive investigation of different approaches to the construction of the SLW-1 model in two-dimensional rectangular and cylindrical geometries; and three-dimensional rectangular, cylindrical and spherical geometries for radiative transfer of mixtures without soot and with soot particles has been performed recently [47]. The approach taken there is identifying an equivalent plane layer in which the SLW-1 model is built with the help of least-squares optimization and then applied to higher spatial dimension. It has been observed that for most cases considered a satisfactory SLW-1 model can be constructed. However, the research is still far from a definitive conclusion and an appropriate recommendation regarding the implementation of the SLW-1 model in multidimensional situations.

6. The SLW modeling of the total emissivity, and the Planck and the Rosseland mean absorption coefficients
Total radiation quantities such as gas total emissivity or different types of averaged absorption coefficients (e.g., Planck or the Rosseland mean absorption coefficients) are different characterizations of the gas absorption spectrum. They are used in some spectral models for prediction of radiative transfer, for example, in the optically thin or in the optically thick approximations. However, they can also be used for construction of spectral models which preserve these total characteristics. For
example, the parameters of the SLW-n or SLW-1 models can be found by fitting the parameters \( \varepsilon \), \( \kappa_p \), or \( \kappa_R \), evaluated by one of several methods in optically thin or optically thick media.

One must emphasize that in the Generalized SLW model, \( \varepsilon \), \( \kappa_p \), and \( \kappa_R \) are not total characteristics of the actual high resolution absorption spectrum but rather, the total characteristics of the gray gas absorption coefficient reordered with the help of the direct and the inverse ALBDF. They have very clear mathematical interpretations and geometrical visualization. The generalized SLW model yields a variety of the representations of the total characteristics summarized in the table 2. Note that the SLW model calculation of \( \varepsilon \), \( \kappa_p \), or \( \kappa_R \) involves summation only over gray gases \( j = 1,2,...,n \). Therefore, these parameters are treated as characterization of the reordered gray gas absorption coefficients of the SLW model. The approach to and implications of calculating these parameters is considered in this section.

### 6.1 SLW Modelling of Total Emissivity

Spectral integration of the spectral emissivity with the SLW model yields the total emissivity of a gas layer [1-3]

\[
\varepsilon = \frac{\int_0^\infty \varepsilon I_m(T) d\eta}{I_b(T)} = \sum_{j=1}^n a_j \left( 1 - e^{-\varepsilon_j} \right)
\]

where the gray gas weights are calculated with the help of the ALBDF:

\[
a_j = F(C_j, T_g = T, T_b = T) - F(C_{j-1}, T_g = T, T_b = T)
\]

Note that the clear gas does not contribute to the emissivity. All possible SLW forms of modeling of the emissivity \( \varepsilon \) are shown in table 2.

### 6.2 SLW Modelling of the Planck Mean Absorption Coefficient \( \kappa_p \)

The SLW modeling of the Planck mean absorption coefficient yields [8,9]:

\[
\kappa_p(T) = \frac{\int_0^\infty \kappa p I_m(T) d\eta}{I_b(T)} = \sum_{j=1}^n \kappa_p a_j = \sum_{j=1}^n \kappa_j \left[ F(C_j, T_g = T, T_b = T) - F(C_{j-1}, T_g = T, T_b = T) \right]
\]

The geometric meaning of \( \kappa_p \) is the area under the graph of the \( F(C) \) curve. The modified Planck mean absorption coefficient is defined by distinguishing the gas and source temperatures

\[
\kappa_p(T_g, T_b) = \sum_{j=1}^n \kappa_j a_j = \sum_{j=1}^n \kappa_j \left[ F(C_j, T_g, T_b) - F(C_{j-1}, T_g, T_b) \right]
\]

Note that the clear gas does not contribute to \( \kappa_p \). All possible SLW forms of \( \kappa_p \) are listed in table 2.

### 6.3 SLW Modelling of the Rosseland Mean Absorption Coefficient \( \kappa_R \)

The Rosseland mean absorption coefficient appears in the diffusion approximation of radiation transfer through optically thick media, and is defined as

\[
\frac{1}{\kappa_R(T_g, T_b)} = \frac{1}{\frac{\partial}{\partial T_b} [I_b(T_b)]} \int_0^\infty \frac{1}{\kappa_p(T_g)} \frac{\partial}{\partial T_b} [I_m(T_g)] d\eta = \frac{\pi}{4 \sigma_T} \int_0^\infty \frac{1}{\kappa_p(T_g)} \frac{\partial}{\partial T_b} [I_m(T_g)] d\eta
\]
The term \( \frac{l}{\kappa_R} \) may be treated as a mean free path in the absorbing medium. If the medium is optically thick over the entire wavenumber region, the Rosseland mean absorption coefficient \( \kappa_R(T_e, T_b) \) exists and is calculated in a straightforward way. The Rosseland mean of the absorption coefficient of a single Lorentz line without truncation which is always non-zero exists and it can be calculated directly. Therefore, the Rosseland mean of the absorption coefficient of a gas spectrum constructed as a superposition of the Lorentz lines without truncation exists also. The problem of calculating \( \frac{l}{\kappa_R} \) for gases appears when the absorption spectrum due to truncation of spectral lines includes completely transparent bands, i.e., where \( \frac{l}{\kappa_R} \) is not defined. Therefore, the Rosseland mean for gases which include spectral windows does not exist. Moreover, for any gas, there are always spectral intervals where the gas is optically thin. Therefore, \( \frac{l}{\kappa_R} \) is defined for strong bands only, where the Rosseland mean absorption coefficient refers to the mean over the optically thick bands [38,39].

The Rosseland mean absorption coefficient is defined here as the mean of the reordered absorption coefficient \( \frac{l}{\kappa_R(T_e, T_b)} = \frac{1}{4\sigma T_b^4} \sum_j \left[ \frac{1}{\kappa_j(T_b)} \frac{\partial}{\partial T_b} E_{b0}(T_b) d\eta \right] \) defined in \( F \in [F_{\min}, F_{\max}] \) without inclusion of clear gas. Therefore, it is the Rosseland mean of the absorbing gray gases only and it characterizes the SLW spectral model rather than the actual high resolution gas absorption coefficient (which might include spectral windows). Its application has a sense only for the current SLW spectral model. The derivation of the SLW model for the term \( \frac{l}{\kappa_R} \) is the following:

\[
\frac{l}{\kappa_R(T_e, T_b)} = \frac{1}{4\sigma T_b^4} \sum_j \left[ \frac{1}{\kappa_j(T_b)} \frac{\partial}{\partial T_b} E_{b0}(T_b) d\eta \right]
\]

\[
= \frac{1}{4\sigma T_b^4} \sum_j \left[ \frac{1}{\kappa_j(T_b)} \frac{\partial}{\partial T_b} \left[ E_b(T_b) \left( F(C_j, T_e, T_b) - F(C_{j-1}, T_e, T_b) \right) \right] \right]
\]

\[
= \frac{1}{4\sigma T_b^4} \sum_j \left[ \frac{1}{\kappa_j(T_b)} \frac{\partial}{\partial T_b} \left[ E_b(T_b) \cdot a_j \right] \right]
\]

\[
= \frac{1}{4\sigma T_b^4} \sum_j \left[ \frac{1}{\kappa_j(T_b)} \frac{\partial}{\partial T_b} \left[ a_j + \frac{T_b}{4} \frac{\partial}{\partial T_b} a_j \right] \right]
\]

Therefore, the SLW model representation of the Rosseland mean absorption coefficient of gray gases is given by

\[
\frac{l}{\kappa_R(T_e, T_b)} = \sum_j \frac{l}{\kappa_j(T_b)} \left[ a_j + \frac{T_b}{4} \frac{\partial}{\partial T_b} a_j \right], \quad a_j = F(C_j, T_e, T_b) - F(C_{j-1}, T_e, T_b)
\]

Under the SLW-1 modeling approach this reduces to

\[
\frac{l}{\kappa_R(T_e, T_b)} = \frac{1}{\kappa_j(T_b)} \left[ a_j + \frac{T_b}{4} \frac{\partial}{\partial T_b} a_j \right]
\]
or it can also be expressed in terms of the weight of the clear gas

\[
\frac{I}{\kappa_a(T_s, T_b)} = \frac{I}{\kappa_1} \left[ 1 - a_0 - \frac{T_b}{4} \frac{\partial}{\partial T_b} a_0 \right]
\]

The Exact SLW Model of the Rosseland mean absorption coefficient may be derived as follows:

\[
\frac{I}{\kappa_b(T_s, T_b)} = \sum_{j=1}^{NYC} \frac{I}{\kappa_j} \frac{\partial}{\partial T_b} \left[ E_b(T_b) \left( F \left( C_j, T_s, T_b \right) - F \left( C_{j-1}, T_s, T_b \right) \right) \right]
\]

Application of the LBL look-up table of the ALBDF for calculation of the Rosseland mean absorption coefficient becomes

\[
\frac{I}{\kappa_a(T_s, T_b)} = \frac{I}{NY} \sum_{i=1}^{NY} \frac{C_i}{C_{i-1}} \left[ F \left( C_i, T_s, T_b \right) - F \left( C_{i-1}, T_s, T_b \right) + \frac{T_b}{4} \frac{\partial}{\partial T_b} \frac{\partial}{\partial C} F \left( C_i, T_s, T_b \right) \right] dC
\]

Geometrical interpretation of the SLW and the Exact SLW modelling of mean coefficients \( \kappa_a \) and \( \kappa_b \) are shown in figure 13.
**Figure 13.** Reordered absorption coefficient of the Exact SLW model and geometrical interpretation of SLW modelling of $\kappa_p$ and $\kappa_R$

**Figure 14.** Dependence of the Rosseland mean absorption coefficient $\kappa_R (T_g, T_p)$ on gas temperature
a) H$_2$O, $Y_w = 1.0$ (self-broadening); b) CO$_2$, $Y_c = 1.0$ (air-broadening); c) CO, $Y_m = 1.0$ (air-broadening)
| Model     | Total Emissivity $\epsilon (L)$ | Planck Mean $\kappa_p$ | Rosseland Mean $I/\kappa_R$ |
|-----------|---------------------------------|------------------------|----------------------------|
| LBL       | $\frac{I}{I_b} \int_0^\infty (1-e^{-\kappa_p L}) \epsilon \eta d\eta$ | $\frac{I}{I_b} \int_0^\infty \kappa_p \eta \epsilon \eta d\eta$ | $\frac{I}{\partial T_b} \int_0^\infty \frac{I}{I_b(T_b)} \kappa_p \epsilon \eta \eta d\eta$ |
| SLW       | $\sum_{j=1}^{n} a_j (1-e^{-NYC_j L})$ | $\sum_{j=1}^{n} \kappa_j a_j$ | $\sum_{j=1}^{n} \frac{I}{NYC_j} \left[ a_j + T_b \frac{\partial a_j}{\partial T} \right]$ |
| Inverse SLW | $\sum_{j=1}^{n} \Delta F_j \left[ 1-e^{-NYC_j(F)_j} \right]$ | $NY \sum_{j=1}^{n} \Delta F_j C(F_j)$ | $\sum_{j=1}^{n} \frac{I}{NYC(F_j)} \left[ \Delta F_j + T_b \frac{\partial \Delta F_j}{\partial T} \right]$ |
| SLW-1     | $a_j (1-e^{-\kappa_j L})$ | $\kappa_j$ | $\frac{I}{\kappa_j} \left[ a_j - \frac{T_b}{4} \frac{\partial a_j}{\partial T} \right]$ |
| Inverse SLW-1 | $\Delta F_j \left[ 1-e^{-NYC_j(F)_j} \right]$ | $NY \Delta F_j C(F_j)$ | $\frac{I}{\kappa_j} \left[ \Delta F_j + \frac{T_b}{4} \frac{\partial \Delta F_j}{\partial T} \right]$ |
| Exact SLW | $\int_0^\infty (1-e^{-NYC(C)}) \frac{\partial F(C)}{\partial C} dC$ | $NY \int_0^\infty C \frac{\partial F(C)}{\partial C} dC$ | $\frac{I}{NY} \int_0^\infty C \left[ \frac{\partial F(C)}{\partial C} + \frac{T_b}{4} \frac{\partial^2 F(C,T,T_b)}{\partial T \partial C} \right] dC$ |
| Inverse Exact SLW | $1 - NYL \int_0^\infty e^{-NYC(C)} dC$ | $NY \int_0^\infty \left[ 1 - F(C) \right] dC$ | $\approx \frac{I}{NY} \int_0^\infty dF(C(F))$ |
Comment regarding application of the $\kappa_P$ and $\kappa_R$ for construction of the semi-gray model

If the term with the derivative of weight with respect to source temperature is neglected, then the simplified SLW-1 model representation of $\kappa_k$ is given by

$$\frac{1}{\kappa_k(T_a, T_b)} = \frac{1}{\kappa_i} \left[ a_j + \frac{T_a}{4} \frac{\partial}{\partial T_b} a_j \right] \approx \frac{a_j}{\kappa_i}$$

Recall also the SLW-1 model representation of the Planck mean absorption coefficient is

$$\kappa_p = a_j \kappa_i$$

These two equations can be used for determining the parameters of the SLW-1 model in terms of $\kappa_p$ and $\kappa_k$:

$$\kappa_i = \sqrt{\kappa_p \kappa_R} \quad \text{and} \quad a_j = \sqrt{\kappa_p / \kappa_R}$$

These quantities which are the parameters of the SLW-1 model based on $\kappa_p$ and $\kappa_k$ are also known, $\kappa_j = \sqrt{\kappa_p \kappa_R}$ as the mean absorption coefficient, and $a_j = \sqrt{\kappa_p / \kappa_R}$ as the weighted nongrayness [45]. It is interesting to note that application of the SLW-1 model for $\kappa_p$ and of the simplified SLW-1 model for $\kappa_k$ yields also the popular semi-gray model RTE previously published in [40-42]:

$$\frac{\partial}{\partial S} I = -\sqrt{\kappa_p \kappa_R} \cdot I + \kappa_p \cdot I_b$$

However, it should be mentioned that the main difference between the SLW-1 model and the semi-gray model is that the SLW method also includes an additional equation for the clear gas.

7. Conclusions

A unified approach explores and reveals more possibilities of the SLW method, exploring both the direct and inverse implementations of the model. The present work outlines the detailed steps in the construction of the generalized SLW model in isothermal and non-isothermal media, and compares different forms of the modelled radiative quantities: directional total radiative flux, total emissivity, Planck mean and Rosseland mean absorption coefficients. Unification of both spectral models allows the resolution of some difficulties which appear when the methods are applied individually, such as inversion of the distribution function or solution of the implicit equations in applications to non-isothermal gas media. The present work can also serve as a handbook in different approaches of the SLW method.

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