DERIVATIVES OF ROTATION NUMBER OF ONE PARAMETER FAMILIES OF CIRCLE DIFFEOMORPHISMS

SHIGENORI MATSUMOTO

Dedicated to Professor Kazuo Masuda on his 65-th birthday

Abstract

We consider the rotation number \( r(t) \) of a diffeomorphism \( f_t = R_t \circ f \), where \( R_t \) is the rotation by \( t \) and \( f \) is an orientation preserving \( C^\infty \) diffeomorphism of the circle \( S^1 \). We shall show that if \( r(t) \) is irrational

\[
\limsup_{t \to t_0} (r(t') - r(t))/(t' - t) \geq 1.
\]

1. Introduction

Let \( f \) be an orientation preserving \( C^\infty \) diffeomorphism of the circle \( S^1 = \mathbb{R}/\mathbb{Z} \), and consider a one parameter family \( f_t, t \in J = [-1/2, 1/2] \), of diffeomorphisms defined by \( f_t = R_t \circ f \), where \( R_t \) denotes the rotation by \( t \). Fix once and for all a lift \( \tilde{f} : \mathbb{R} \to \mathbb{R} \) of \( f \) to the universal covering \( \mathbb{R} \) of \( S^1 \). Then a lift \( \tilde{f}_t \) of \( f_t \) is chosen as \( \tilde{f}_t = T_t \circ \tilde{f} \), where \( T_t \) is the translation by \( t \). The rotation number \( \rho(t) \in \mathbb{R} \) of \( \tilde{f}_t \) is a continuous and nondecreasing function of \( t \). Define a closed set \( C \) by

\[
C = J \cap \text{Int}(\rho^{-1}(\mathbb{Q})),
\]

and assume for simplicity that \( \rho(-1/2) = 0, \rho(1/2) = 1 \) and \( C \) is contained in the interior of \( J \).

V. I. Arnold [A] showed that \( m(C) > 0 \), where \( m \) denotes the Lebesgue measure. Denote by \( \mathcal{N} \) the set of non Liouville numbers and define a Borel subset \( N \) of \( C \) by

\[
N = \rho^{-1}(\mathcal{N}^c).
\]
M. R. Herman [H1] showed that $\rho$ is an absolutely continuous function and that $m(N) > 0$, (under a much less restrictive condition on the one parameter family).

A famous theorem of J.-C. Yoccoz says that if $t \in N$, then $f_t$ is $C^\infty$ conjugate to a rotation. Thus the result of M. R. Herman says that the set of the value $t$ such that $f_t$ is $C^\infty$ conjugate to a rotation has positive Lebesgue measure. On the other hand it is known ([H2] p. 170 and [KH] p. 412) that for a generic value $t$ in $C$ the conjugacy of $f_t$ to a rotation is a non absolutely continuous homeomorphism, provided that $f$ is a real analytic diffeomorphism and $f'$ is not constantly equal to 1. Nevertheless it is shown that $m(C \setminus N) = 0$ ([T]) and furthermore that $\dim_H(C \setminus N) = 0$ ([G]), where $\dim_H$ denotes the Hausdorff dimension. The purpose of this paper is to show a somewhat stronger result in this direction.

**Theorem 1.** If $\rho(t)$ is irrational, then we have

$$\limsup_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} \geq 1.$$  

Notice that the above theorem implies by the absolute continuity of $\rho$ that $\rho^{-1}(B)$ is null if $B$ is a null Borel set. As for the case $\rho(t)$ is rational, we have:

**Theorem 2.** Assume that $f$ is real analytic and $f'$ is not constantly equal to 1. For $t \in C$ such that $\rho(t) \in Q$, we have

$$\limsup_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} = \infty.$$  

These phenomena can be found in the computer graphics of the derivative $\rho'$ in [LV]. The plan of the paper is as follows. In Sect. 2, we prove a weaker version of Theorem 1 and apply it to a new proof of the result of [G]. In Sect. 3, we give an elaboration of the argument of Sect. 2, which yields a proof of Theorem 1 for $\rho(t)$ a Liouville number, while the non Liouville case is treated in Sect. 4. Finally Sect. 5 is devoted to the proof of Theorem 2. Also we shall remark that it is necessary to consider $\limsup$ instead of $\liminf$ in Theorem 1.

The author expresses his gratitude to the referee for valuable comments.

**2. Weaker version of Theorem 1**

The purpose of this section is to show the following proposition which is a weaker version of Theorem 1, and by applying it to prove that $\dim_H(C \setminus N) = 0$ ([G]).

A positive integer $q$ is called a closest return of an irrational number $x$ if for any $0 < j < q$, we have $|jx|_{S_1} > |qx|_{S_1}$, where $|x|_{S_1}$ is the distance of $x$ and 0 in $S^1$. If $|qx|_{S_1} = |qx - p|$ for some integer $p$, the rational number $p/q$ is called a convergent of $x$. 

\[ \text{shigenori matsumoto} \]
Proposition 2.1. Suppose $r(t)$ is irrational, $p/q$ a convergent of $r(t)$, $t' \in p^{-1}(p/q)$ the point nearest to $t$. Then we have

$$\frac{p(t') - p(t)}{t' - t} \geq e^{-M},$$

where $M = \|(\log f')'\|_{C^0}$.

To begin with let us prepare the following lemma.

Lemma 2.2. If $r(0)$ is irrational, then for any nonnegative integers $i, j$, we have $\mu((f^i)' \circ f^j) \geq 1$, where $\mu$ is the unique $f$-invariant probability measure on $S^1$.

Proof. By the downward concavity of log, it suffices to show

$$\mu(\log(f^i)' \circ f^j) = 0.$$ 

Since $\mu$ is $f$-invariant, this is equivalent to

$$\mu(\log(f^i)) = 0.$$

Again since $\mu$ is $f$-invariant and

$$\log(f^i)' = \sum_{v=0}^{i-1} \log f^v \circ f^v,$$

this follows from

$$\mu(\log f') = 0.$$

By the unique ergodicity of $f$, we have a uniform convergence

$$n^{-1} \log(f^n)' = n^{-1} \sum_{k=0}^{n-1} \log f^k \circ f^k \rightarrow \mu(\log f') = a.$$

But if $a > 0$ and if $n$ is sufficiently large we have

$$(f^n)' > \exp \frac{an}{2} > 1,$$

and if $a < 0$, then

$$(f^n)' < \exp \frac{an}{2} < 1.$$ 

In any case these contradict

$$\int_{S^1} (f^n)'(x) \, dx = 1.$$ 

$\square$
Proof of Proposition 2.1. Assume that $t$ in Proposition 2.1 is 0. The rotation number of $f$, $\rho(0) = x$, is irrational by the hypothesis.

Given $x, y \in S^1$, denote

$$[x, y] = \{ z \in S^1 \mid x \leq z \leq y \},$$

where $\leq$ is the positive cyclic order of $S^1$, and by $y - x$ the length of $[x, y]$.

Assume that $p/q$ is a convergent of $x$, and, to fix the idea, that $q - p < 0$. Thus we have $x < p/q$, and shall estimate the value of $t > 0$ such that $\rho(t) = p/q$.

Now since $q$ is a closest return, the intervals $R_j[0, -qx + p], 0 \leq j \leq q - 1$, are mutually disjoint, where $R_x$ denotes the rotation by $x$. The diffeomorphism $f$ is topologically conjugate to $R_x$ by an orientation preserving homeomorphism which maps a given point $x$ to 0. This implies that if we set $L(x) = [x, f^{-q}(x)]$, then

$$f^jL(x)$$

are mutually disjoint for $0 \leq j \leq q - 1$,

for any $x \in S^1$. Let $t$ be the smallest positive value such that $\rho(t) = p/q$. Our aim is to estimate the value of $(p/q - x)/t$ from below.

For $0 \leq s \leq t$ consider the point $f^q(x)$. For $s = 0$, this is just $f^q(x)$ and as $s \to t$ the point $f^q(x)$ increases from $f^q(x)$ towards $f^q(x)$ on the interval $f^qL(x) = [f^q(x), x]$. Thus for some $x$, $f^q_t(x) = x$, while for any $x f^q_t(x)$ lies on $f^qL(x)$, since $t$ is the smallest value such that $\rho(t) = p/q$. This shows that the point $f^q_t(x)$ lies on the interval $f^qL(x)$ for $1 \leq i \leq q$. See the figure.

For each such $i$, consider the interval

$$I_i(x) = [f \circ f^{-i-1}_i(x), f^i(x)] \subset f^iL(x).$$

Since $f^i_t(x) = f \circ f^{-i-1}_i(x) + t$, these intervals have length $t$. Notice that the rightmost point of $I_i(x)$ is mapped by $f$ to the leftmost point of $I_{i+1}(x)$.

\[ f^qL(x) \]
\[ f^{q-1}I_1(x) \]
\[ f^{q-2}I_2(x) \]
\[ I_q(x) \]
\[ f^q_t(x) \]
\[ x \]

\[ f^{q-1}(x) \]
\[ I_{q-1}(x) \]

\[ f^q(x) \]
\[ f \circ f_t(x) I_2(x) \]
\[ f^t_t(x) \]
\[ f^t \]

\[ f(x) \]
\[ I_1(x) \]
\[ f_t(x) \]
\[ f \]

\[ x \]
\[ f \]

\[ L(x) \]
\[ f^{-q}(x) \]
The images
\[ f^{q-1}I_1(x), f^{q-2}I_2(x), \ldots, fI_{q-1}(x), I_q(x) \]
form a sequence of consecutive intervals towards the right. Their union
\[ \bigcup_{i=1}^{q} f^{q-i}I_i(x) = [f^q(x), f^q(x)] \]
is contained in \( f^qL(x) = [f^q(x), x] \) for any \( x \).

Let \( \tau_i(x) \) be the length of \( f^{q-i}I_i(x) \). By the Denjoy distortion lemma and (2.1), we get
\[ \tau_i(x) \geq e^{-M(f^{q-i})' \circ f^i(x)t}, \]
where \( M = \| (\log f')' \|_{C^0} \). Summing them up we get for any \( x \in S^1 \),
\[ x - f^q(x) \geq e^{-M}t \sum_{i=1}^{q} (f^{q-i})' \circ f^i(x). \]

Now let us evaluate the both hand sides by the invariant measure \( \mu \) of \( f \). It is well known that the evaluation of the left term yields \(-1\) times the rotation number of \( f^q \), i.e.
\[ \mu(\text{Id} - f^q) = p - qz. \]

Therefore Lemma 2.2 implies that
\[ (p/q - z)/t \geq e^{-M}, \]
as is desired. \( \square \)

Now let us start the proof of Graczyk’s Theorem ([G]). First we need the following easy lemma.

**Lemma 2.3.** Assume \( z \) is irrational, \( q > 1 \), and for some \( d > 3 \)
\[ |z - p/q| < 1/q^d. \]
Then \( p/q \) is a convergent of \( z \). \( \square \)

Let
\[ J_d(p/q) = \rho^{-1}((p/q - 1/q^d, p/q + 1/q^d)) - \rho^{-1}(p/q). \]
Also let
\[ D = \{ d \in (3, \infty) \mid q^d \notin \mathbb{Q}, \forall q \in \mathbb{N} \setminus \{1\} \}. \]
Since \( (3, \infty) \setminus D \) is countable, \( D \) is dense in \( (3, \infty) \).
Corollary 2.4. If \( q > 1 \) and \( d \in D \), then
\[
m(J_d(p/q)) \leq 2e^M q^{-d},
\]
where \( m \) denotes the Lebesgue measure.

Proof. Apply Proposition 2.1 to the irrational numbers \( z = p/q - 1/q^d \) and \( z = p/q + 1/q^d \).

Now the preimage \( L \) of the set of Liouville numbers by \( r \) can be described as
\[
L = \bigcap_{d \in D_{q0}} \bigcup_{q \geq q0} J_d(p/q),
\]
where in the union \( p \) runs over the integers \( 0 < p < q \), coprime to \( p \). Given any \( \alpha > 0 \), if \( d > 2/\alpha \), we have
\[
\sum_{q \geq q0} m(J_d(p/q))^{\alpha} \leq \sum_{q = q0}^{\infty} 2^{2M} q^{-\alpha d} \rightarrow 0 \quad (q0 \rightarrow \infty),
\]
which concludes that \( \dim_H(L) = 0 \).

3. Proof of Theorem 1 for Liouville \( \rho(t) \)

Let \( q_n \) be the \( n \)-th closest return of the irrational number \( z = \rho(0) \). Then the sequence \( \{q_n z\} \) converges to 0 in \( S^1 \), changing signs alternately. The closest returns satisfy
\[
q_{n+1} = a_{n+1}q_n + q_{n-1},
\]
where \( a_{n+1} \) is the \((n+1)\)-st denominator of the continued fraction of \( z \).

Here we assume that \( z \) is a Liouville number. Thus the sequence \( \{a_{n+1}\} \) is unbounded. It is no loss of generality to assume that there is a subsequence \( n_i \) such that
\[
q_{n_i} z \uparrow 0 \text{ in } S^1 \quad \text{and} \quad a_{n_i} \rightarrow \infty.
\]
For simplicity we shall write \( n_i \) as \( n \) in what follows, and have in mind that
\[
q_n z < 0 < -q_n z < q_{n-1} z
\]
and that \( a_{n+1} \) is as large as desired. All the notations of the previous section are used by replacing \( p/q \) with \( p_n/q_n \).

Consider the first return map \( S \) of the rotation \( R^{-1}_z \) on the interval \([q_n z, q_{n-1} z]\). We have
\[
S = \begin{cases} 
R_{q_n}^{-q_0} \text{ on } [q_n z, (q_n + q_{n-1}) z] \text{ sending it to } [0, q_{n-1} z] \\
R_{q_{n-1}}^{-q_0} \text{ on } [(q_n + q_{n-1}) z, q_n z] \text{ sending it to } [q_n z, 0].
\end{cases}
\]
Since there are ordering
\[
0 < (1 - a_{n+1})q_n z < (q_n + q_{n-1}) z < -a_{n+1}q_n z < q_{n-1} z,
\]
the map $S = R^{-q_n}_a$ sends the interval $[0, (1 - a_{n+1})q_nz]$ onto $[-q_nz, -a_{n+1}q_nz]$. In particular $R^{-vq_n}_a[0, -q_nz]$, $0 \leq v < a_{n+1}$, form a consecutive sequence of intervals contained in $[0, q_{n-1}z]$. Translating into $f$ via the topological conjugacy, we have for any $x \in S^1$

\[(3.1) \quad \bigcup_{v=0}^{a_{n+1}-1} f^{-vq_n}L(x) \subset K(x),\]

where $L(x) = [x, f^{-q_n}(x)]$ as before and $K(x) = [x, f^{q_{n-1}}(x)]$.

As is well known, easy to show, (2.1) can be extended to:

\[(3.2) \quad f^jK(x) \text{ are disjoint for } 0 \leq j \leq q_n - 1.\]

So it looks plausible that the total length $l(x)$ of $\bigcup f^jL(x)$ is very small, since a large number of its iterates by $f^{-vq_n}$ are mutually disjoint (except the end points), by virtue of (3.1) and (3.2). On the other hand the Denjoy distortion lemma actually guarantees that the coefficient $e^{-M}$ in Proposition 2.1 can be replaced by $e^{-Mt}$ where $t$ is the maximum of $l(x)$, and hence if $l(x)$ were small enough, we should be able to prove Theorem 1. However we cannot do this for $L(x)$ itself and instead consider a subinterval $\hat{L}(x)$ defined by

\[
\hat{L}(x) = [x, f^{-q_n} \circ f^{q_n}_t(x)],
\]

where as before $t$ is the smallest value such that $\rho(t) = p_n/q_n$. We are going to show that the total length of the union of intervals

\[
\hat{I}(x) = m\left(\bigcup_{i=1}^{q_n} f^i\hat{L}(x)\right)
\]

is small, where $m$ denotes the Lebesgue measure as before. Notice that this is enough for our purpose of applying the Denjoy distortion lemma, that is, Proposition 2.1 can be improved as

\[(3.3) \quad \frac{\rho(t) - \rho(0)}{t} \geq e^{-Mt},\]

where $\hat{I} = \max\{\hat{I}(x) \mid x \in S^1\}$. Now we have

\[(3.4) \quad \hat{L}(x) = \bigcup_{i=1}^{q_n} f^{-i}I_i(x),\]

where as before

\[
I_i(x) = [f \circ f^{i-1}_t(x), f^i_t(x)] \subset f^iL(x).
\]

Put

\[
\hat{I}_i(x) = m\left(\bigcup_{j=1}^{q_n} f^{j-i}I_i(x)\right).
\]
Then we have by (3.4) and (2.1)

\[ \hat{I}(x) = \sum_{i=1}^{q_n} \hat{I}_i(x). \]

Also (3.1) and (3.2) implies that

\[ X^a_n + 1/C_0 \leq 0 \]

\[ \sum_{v=0}^{a_n+1} \hat{I}(f^{-vq_n}(x)) \leq 1. \]

Let us compare \( \hat{I}_i(x) \) with \( \hat{I}_i(f^{-vq_n}(x)) \) for \( 0 \leq v < a_{n+1} \). This is possible since the intervals \( I_i(x) \) and \( I_i(f^{-vq_n}(x)) \) are contained in \( f^K(x) \) and of length \( t \).

In fact again by the Denjoy distortion lemma and (3.2), we get

\[ m(\int f^{j-1}I_i(x)) / t \leq e^N m(\int f^{j-1}I_i(f^{-vq_n}(x))) / t \]

for any \( 0 < v < a_{n+1} \) and \( 1 \leq j \leq q_n \), where

\[ N = \max\{\|\log f'\|_{c^\alpha}, \|\log(f^{-1})'\|_{c^\alpha}\}. \]

Summing up (3.6) by \( j \), we get

\[ \hat{I}_i(x) \leq e^N \hat{I}_i(f^{-vq_n}x). \]

Again summing up by \( i \) we obtain

\[ \hat{I}(x) \leq e^N \hat{I}(f^{-vq_n}(x)). \]

Finally we get by (3.5)

\[ a_{n+1} \hat{I}(x) \leq e^N \sum_{v=0}^{a_n+1} \hat{I}(f^{-vq_n}(x)) \leq e^N. \]

Now \( N \) is a constant depending only on \( f \) and \( a_{n+1} \) can be chosen arbitrarily large. By virtue of (3.3), this completes the proof of Theorem 1 for Liouville \( \rho(t) \).

4. Proof of Theorem 1 for non-Liouville \( \rho(t) \)

Here we shall show that if \( \rho(t_0) \) is non Liouville, then \( \rho \) is differentiable at \( t_0 \) and \( \rho'(t_0) \geq 1 \).

In the first place we need the following theorem by P. Brunovský ([B]). For the proof see also [H1].

**Theorem 4.1.** Let \( g_t \) be a \( C^1 \)-path of \( C^1 \)-diffeomorphisms such that \( g_{t_0} \) is an irrational rotation for some \( t_0 \). Then we have

\[ \frac{d}{dt} \text{rot}(g_t)|_{t=t_0} = \int_{S^1} \frac{\partial}{\partial t} f(x)|_{t=t_0} dx, \]

where \( \text{rot}(g_t) \) denotes the rotation number of a lift of \( g_t \), chosen continuously on \( t \).
Since $\rho(t_0)$ is non Liouville, we have $f_0 = h \circ R_{\rho(t_0)} \circ h^{-1}$ for some $C^\infty$ diffeomorphism $h$ ([Y]). Applying the Brunovský theorem to the family $h^{-1} \circ f_t \circ h$, we get

$$
\rho'(t_0) = \int_{S^1} \frac{\partial (h^{-1} \circ f_t \circ h)}{\partial t}(x) \bigg|_{t=t_0} \, dx = \int_{S^1} (h^{-1})' \circ f_{t_0} \circ h(x) \cdot \frac{\partial f_t}{\partial t} \bigg|_{t=t_0} \circ h(x) \, dx
$$

$$
= \int_{S^1} (h^{-1})' \circ f_{t_0} \circ h(x) \, dx.
$$

Since $f_{t_0} \circ h = h \circ R_{\rho(t_0)}$ and the Lebesgue measure is invariant by the rotation we have

$$
\rho'(t_0) = \int_{S^1} (h^{-1})' \circ h(x) \, dx = \int_{S^1} h'(x)^{-1} \, dx.
$$

Now the Schwarz inequality concludes the proof of Theorem 1 for non-Liouville $\rho(t)$.

### 5. Proof of Theorem 2

By the assumption of Theorem 2, for any $p/q \in \mathbb{Q}$, the set $\rho^{-1}(p/q)$ is a nondegenerate interval and $C$ is a Cantor set. It is no loss of generality to assume that $0 \in J$ is the supremum of $\rho^{-1}(p/q)$ and to show

$$
\lim_{i \to 0} \frac{\rho(t) - (p/q)}{t} = \infty.
$$

The real analyticity of $f$ implies that the periodic points of $f_0 = f$ are finite in number, say $x_v$, $1 \leq v \leq ql$. Now since $0$ is the supremum of $\rho^{-1}(p/q)$, the graph of $f^q - p$ is above the diagonal and tangent to it just at the points $\pi^{-1}(x_v)$, where $\pi: \mathbb{R} \to S^1$ is the universal covering map. For $\epsilon > 0$, let $I_v = [a_v, b_v]$ be the $\epsilon$-neighbourhood of $x_v$. Choosing $\epsilon$ small enough, one can assume that the intervals $I_v$ are disjoint. Put $J_v = [b_v, a_{v+1}]$. Choose $N > 0$ big enough so that

$$
f^q_{f_i(v)}(b_v) \notin J_v, \quad 1 \leq j(v) \leq N + 1.
$$

This means that any orbit by $f^q$ stays consecutively in $J_v$ for at most $N$ times. Then since $f^q_{f_i} > f^q$ for $t > 0$ and the speed for $f^q_{f_i}$ is bigger than that for $f^q$, any orbit by $f^q_{f_i}$ stays consecutively in $J_i$ for at most $N$ times.

On the other hand straightforward computation shows that $\partial f^q_{f_i}/\partial t \geq 1$, and therefore $f^q_{f_i} - p \geq t$. This shows that any orbit by $f^q_{f_i}$ stays consecutively in $I_i$ for at most $2\epsilon/t$ times, $2\epsilon$ being the length of $I_v$.

Let us estimates the times of iterations of $f^q_{f_i}$ needed for some point to go around $S^1$ once. The above observation shows that the times needed for a round trip does not exceed $(N + 2\epsilon/t)ql$. Translated into the language of rotation number we have

$$
qp(t) - p \geq ((N + 2\epsilon/t)ql)^{-1}.
$$
Therefore if \( t < \frac{\varepsilon}{N} \), we have

\[
\frac{\rho(t) - (p/q)}{t} \geq (3\delta q^2 t)^{-1},
\]

completing the proof of Theorem 2.

Finally let us remark that taking \( \limsup \) instead of \( \liminf \) is necessary for Theorem 1.

**Proposition 5.1.** Assume that \( f \) is real analytic and \( f' \) is not constantly equal to 1. There is a residual subset \( R \) in \( C \) such that for any \( t \in R \)

\[
\liminf_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} = 0, \quad \text{and} \quad \limsup_{t' \to t} \frac{\rho(t') - \rho(t)}{t' - t} = \infty.
\]

**Proof.** Let \( \rho(J) \cap \mathbb{Q} = \{ x_n \mid n \in \mathbb{N} \} \) and set \( [a_n, b_n] = \rho^{-1}(x_n) \). Then there is \( c_n > b_n \) very near \( b_n \) such that if \( t \in (b_n, c_n) \)

\[
\frac{\rho(t) - x_n}{t - a_n} < \frac{1}{n} \quad \text{and} \quad \frac{\rho(t) - x_n}{t - b_n} > n.
\]

We used Theorem 2 for the second inequality. Now the set

\[
R = \bigcap_{p, n > p} (b_n, c_n) \cap C
\]

is residual in \( C \) and satisfies the condition of the theorem. \( \square \)

**References**

[A] V. I. Arnold, Small denominator I: on the mapping of a circle into itself, Izvestija Akad. Nauk. ser. Math. 25 (1961), 21–86, Translations A. M. S., 2nd Series 46, 213–284.

[B] P. Brunovsky, Generic properties of the rotation number of one-parameter diffeomorphisms of the circle, Czech. Math. J. 24 (1974), 74–90.

[G] J. Graczyk, Linearizable circle diffeomorphisms in one-parameter families, Bol. Soc. Bras. Mat. 24 (1993), 201–210.

[H1] M. R. Herman, Mesure de Lebesgue et nombre de rotation, Lecture notes in math. 597, Springer, 1977, 271–293.

[H2] M. R. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle a des rotations, Publ. I. H. E. S. 49 (1979), 5–233.

[KH] A. Katok and B. Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge University Press, 1995.

[LV] A. Luque and J. Villanueva, Computation of derivatives of the rotation number for parametric families of circle diffeomorphisms, Physica D 237 (2008), 2599–2615.

[T] M. Tsujii, Rotation number and one-parameter families of circle diffeomorphisms, Erg. Th. Dyn. Sys. 12 (1992), 359–363.
[Y] J.-C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, Ann. Sci. Ecole Norm. Sup. 17 (1984), 333–359.

Shigenori Matsumoto
Department of Mathematics
College of Science and Technology
Nihon University
1-8-14 Kanda, Surugadai
Chiyoda-ku, Tokyo, 101-8308
Japan
E-mail: matsumo@math.cst.nihon-u.ac.jp