Centralizers of irreducible subgroups in the projective special linear group

Clément Guérin*

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Abstract

In this paper, we classify conjugacy classes of centralizers of irreducible subgroups in $\text{PSL}(n, \mathbb{C})$ using alternate modules a.k.a. finite abelian groups with an alternate bilinear form. When $n$ is squarefree, we prove that these conjugacy classes are classified by their isomorphism classes. More generally, we define a finite graph related to this classification whose combinatorial properties are expected to help us describe the stratification of the singular (orbifold) locus in some character varieties.

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1 Definitions and results

A bad subgroup of a complex reductive Lie group $G$ is an irreducible subgroup $H$ of $G$ (see the definition below) whose centralizer strictly contains the center of $G$. Sikora in [Sik12] and Florentino-Lawton in [FL12] exhibited complex reductive Lie groups $G$.

*University of Luxembourg Campus Kirchberg, Mathematics Research Unit, BLG. 6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg, e-mail : clement.guerin@uni.lu
Lemma 1. We recall a simple lemma from a previous paper: \[
\text{commutator verifies } [B, Z] \in H \text{ and } [Z, B] \in Z 
\]

A group centralizer of \( \rho \) gives all the quotients of \( \text{SL}(n, \mathbb{C}) \). If \( P \) is a prime number) has been extensively studied in \([\text{Gue}16]\) from which we highlight:

**Theorem 1 in \([\text{Gue}16]\).** If \( \overline{H} \) is a bad subgroup of \( \text{PSL}(p, \mathbb{C}) \) then its centralizer \( Z_{\text{PSL}(p, \mathbb{C})}(\overline{H}) \) is either isomorphic to \( \mathbb{Z}/p \) or \( \mathbb{Z}/p \times \mathbb{Z}/p \), in the later case the irreducible subgroup is its own centralizer. Furthermore if two centralizers of irreducible subgroups of \( \text{PSL}(p, \mathbb{C}) \) are isomorphic then they are conjugate.

This result leads to a decomposition of the singular locus of the character variety (see loc. cit. for a definition and also paragraph 7 of \([\text{FLR}15]\) for Fuchsian groups into \( \text{PSL}(p, \mathbb{C}) \)). In this paper, we generalize the classification of centralizers of irreducible subgroups of \( \text{PSL}(p, \mathbb{C}) \) to \( \text{PSL}(n, \mathbb{C}) \). We recall some definitions for this paper.

A subgroup \( P \) of a reductive group \( G \) is said to be parabolic if \( G/P \) is a complete variety. When \( G = \text{SL}(n, \mathbb{C}) \) a subgroup is parabolic if and only if it is the stabilizer of a non-trivial flag in \( \mathbb{C}^n \) where \( \text{SL}(n, \mathbb{C}) \) acts canonically on \( \mathbb{C}^n \) (c.f. \([\text{Bor}91]\)).

A subgroup \( H \) of a reductive group \( G \) is said to be irreducible if for each parabolic subgroup \( P \) of \( G \) containing \( H \), we can find a Levi subgroup \( L \) of \( P \) such that \( H \subseteq L \). A representation \( \rho : \Gamma \rightarrow G \) is said to be irreducible (resp. completely reducible) if \( \rho(\Gamma) \) is irreducible (resp. completely reducible).

The centralizer \( Z_G(H) \) of a subgroup \( H \) of \( G \) is the set of elements \( g \in G \) commuting with any element of \( H \). The centralizer \( Z_G(\rho) \) of a representation \( \rho : \Gamma \rightarrow G \) is the centralizer of \( \rho(\Gamma) \).

Sikora gave a useful characterization of an irreducible group (corollary 17 in \([\text{Sik}12]\)). A group \( H \) in a reductive group \( G \) is irreducible if and only if it is completely reducible and \( [Z_G(H) : Z(G)] \) is finite.

Furthermore any finite group is a completely reducible subgroup and finite extensions of completely reducible subgroups are completely reducible subgroups.

The commutator \( [g, h] \) of \( g \) and \( h \) in a group \( G \) will classically be defined as \( ghg^{-1}h^{-1} \).

We recall a simple lemma from a previous paper:

**Lemma 1.** Let \( n \geq 1 \), \( A, B \) be two matrices in \( \text{GL}(n, \mathbb{C}) \) and \( \lambda \in \mathbb{C}^\ast \) such that their commutator verifies \( [A, B] = \lambda I_n \), then for all \( \mu \in \mathbb{C}^\ast \), \( A(E_\mu(B)) = E_{\lambda^{-1}\mu}(B) \) and \( B(E_\mu(A)) = E_{\lambda \mu}(A) \). In particular, \( B \) acts on \( \text{Sp}(A) \) by multiplying by \( \lambda \) and \( A \) acts on \( \text{Sp}(B) \) by multiplying by \( \lambda^{-1} \).

For \( n \geq 1 \), the center of \( \text{SL}(n, \mathbb{C}) \) is cyclic of order \( n \). For \( d \) dividing \( n \), denote \( \pi_d(\text{SL}(n, \mathbb{C})) \) the quotient of \( \text{SL}(n, \mathbb{C}) \) by the unique central subgroup of order \( d \). This gives all the quotients of \( \text{SL}(n, \mathbb{C}) \), in particular \( \pi_n(\text{SL}(n, \mathbb{C})) = \text{PSL}(n, \mathbb{C}) \).

The first result of this paper (see proposition \([\text{I}]\) and corollary \([\text{II}]\) in subsection \([\text{II}]\):

**Result 1.** Let \( n \geq 1 \) and \( \overline{H} \) be an irreducible subgroup of \( \text{PSL}(n, \mathbb{C}) \), then \( Z_{\text{PSL}(n, \mathbb{C})}(\overline{H}) \) is abelian, of exponent dividing \( n \) and of order dividing \( n^2 \).
In subsections 2.2 and 3.1 we classify conjugacy classes of centralizers of irreducible subgroups in $PSL(n, \mathbb{C})$ using alternate modules which are, by definition, abelian finite groups endowed with an alternate bilinear form, see \cite{TW15} or \cite{Wal64}. We show, in proposition 4, that we can associate to any centralizer of an irreducible subgroup in $PSL(n, \mathbb{C})$, a unique isometry class of alternate modules. According to this association, we get theorem 1:

**Result 2.** Let $n$ be a positive integer and $Z_1$, $Z_2$ be two centralizers of irreducible subgroups in $PSL(n, \mathbb{C})$, then $Z_1$ and $Z_2$ are conjugate if and only if their respective associated alternate modules are isometric.

Then, we give a necessary and sufficient condition for an alternate module to be associated to a centralizer of irreducible subgroup in $PSL(n, \mathbb{C})$ in theorem 2:

**Result 3.** Let $n \geq 1$ and $(A, \phi)$ be an alternate module then the following assertions are equivalent:

1. There exists an irreducible subgroup of $PSL(n, \mathbb{C})$ such that the alternate module associated to its centralizer is isometric to $(A, \phi)$.
2. The order of Lagrangians in $(A, \phi)$ divides $n$.
3. There exists an abelian group $B$ of order $n$ such that $(A, \phi)$ is isometrically embedded in the symplectic module $B \times B^*$.

This theorem is the generalization of theorem 1 in \cite{Gue16-1}. Alternate modules verifying the assertion 3 will be referred to as $n$-subsymplectic modules. When $n = p$ is prime, conjugacy classes of centralizers are classified by their isomorphism class. In the general case, we prove theorem 3 and corollary 6 in subsection 3.2:

**Result 4.** Conjugacy classes of centralizers of irreducible subgroups in $PSL(n, \mathbb{C})$ are classified by their isomorphism classes if and only if $n$ is squarefree. When $n$ is square-free, there are exactly $3^r$ conjugacy classes of centralizers of irreducible subgroups in $PSL(n, \mathbb{C})$ where $r$ is the number of distinct prime numbers dividing $n$.

After this, we define $(M_n, \leq)$ to be the set of isometry classes of $n$-subsymplectic modules ordered with the inclusion up to isometry. This allows us to draw a graph $G_n$ ruling those inclusions (see examples 2, 3 and 4). When $\Gamma$ is a finitely generated group, we show in proposition 3 that, by duality, this gives a stratification of the singular locus of the character variety $\chi_{Sing}(\Gamma, PSL(n, \mathbb{C}))$. In particular, one should be able to prove similar results to those that were proven in \cite{Gue16-1} when $\Gamma$ is Fuchsian.

Finally, in section 4, we deal with similar questions in partial quotients of $SL(n, \mathbb{C})$. We first get theorem 4 which generalizes result 1:

**Result 5.** Let $n$ be a positive integer, $d$ be a divisor of $n$ and $\overline{H}$ be an irreducible subgroup of $\pi_d(SL(n, \mathbb{C}))$. Then its centralizer in $\pi_d(SL(n, \mathbb{C}))$ is abelian of exponent dividing $\text{lcm}(n/d, d)$ and of order dividing $n^3/d$. If $\gcd(n/d, d) = 1$ then the order of its centralizer in $\pi_d(SL(n, \mathbb{C}))$ divides $nd$. 3
Likewise, propositions 13 and 14 respectively generalize results 2 and 3. From section 3, we deduce that classifying conjugacy classes of centralizers in $PSL(n, \mathbb{C})$ is equivalent to classifying conjugacy classes of centralizers in all quotients of $SL(n, \mathbb{C})$. In corollary 7, we characterize the isotropy group of the corresponding character variety when it is an orbifold (e.g. when $\Gamma$ is Fuchsian, see [Sik12]).

2 Properties of centralizers of irreducible in $PSL(n, \mathbb{C})$

In this section, we will demonstrate that any centralizer of an irreducible subgroup in $PSL(n, \mathbb{C})$ is abelian, of bounded exponent and of bounded order (first subsection).

In the second subsection, we shall see how to associate an alternate module to any centralizer of an irreducible subgroup in $PSL(n, \mathbb{C})$. The correspondence will be proven to be faithful.

Lemma 2. Let $G$ be a group and $N$ a subgroup of $Z(G)$, the center of $G$. Define $\pi : G \to G/N$ the quotient map. If $H$ is a subgroup of $G/N$, define $H := \pi^{-1}(H)$ and $U := \pi^{-1}(Z_{G/N}(H))$. The function

$\phi : U \to \text{Mor}(H, N)$

$\phi(u) := (h \mapsto [u, h])$

is well defined. It is a group morphism whose kernel is $Z_G(H)$.

Proof. Let $u \in U$ and $h \in H$ then $\pi([u, h]) = [\pi(u), \pi(h)] = N$ since $\pi(u)$ centralizes $\pi(h)$ by definition. Furthermore, if $h_1, h_2 \in H$ then:

$$[u, h_1 h_2] = uh_1h_2u^{-1}h_2^{-1}h_1^{-1} = uh_1(u^{-1}u)h_2u^{-1}h_2^{-1}h_1^{-1} = uh_1u^{-1}[u, h_2]h_1^{-1}$$

Since $[u, h_2]$ is central, $[u, h_1 h_2] = uh_1u^{-1}h_1^{-1}[u, h_2] = [u, h_1][u, h_2]$. As a result, $\phi_u : h \mapsto [u, h]$ is a morphism from $H$ to $N$ and $\phi$ is well defined. Using the exact same kind of argument, $\phi$ is easily proven to be itself a group morphism.

An element $u$ in $U$ belongs to $\text{Ker}(\phi)$ if and only if for all $h \in H$ we have $[u, h] = 1_G$, if and only if $u \in Z_G(H)$. \hfill $\square$

Once $n$ is given, $\xi$ will always denote a fixed primitive $n$-th root of the unity in the complex field. Let $n$ be a positive integer and $d$ a divisor of $n$. We define the natural projection $\pi_d : SL(n, \mathbb{C}) \to SL(n, \mathbb{C})/\langle \xi^d I_n \rangle$.

Quotients of $SL(n, \mathbb{C})$ are of the form $\pi_d(SL(n, \mathbb{C}))$, where $d$ divides $n$. We will study centralizers of their irreducible subgroups. Let $H$ be a subgroup of $SL(n, \mathbb{C})$, we define its $d$-centralizer $Z_d(H) := Z_{\pi_d(SL(n,\mathbb{C}))}(\pi_d(H)) \leq \pi_d(SL(n, \mathbb{C}))$.

Working in $SL(n, \mathbb{C})$ rather than in its quotients is a natural thing to do. Therefore, we define for $d$ dividing $n$ and a subgroup $H$ of $SL(n, \mathbb{C})$, the $d$-extended centralizer $U_d(H)$ of $H$ as the pull-back of $Z_d(H)$ by $\pi_d$. From the very definition of $U_d(H)$:
\[ U_d(H) = \{ g \in SL(n, \mathbb{C}) \mid \forall h \in H, \exists k \in \mathbb{Z}/d \text{ such that } [g, h] = \xi^{\frac{k}{d}} I_n \}. \]

Remark that if \( \mathcal{H} \) is an irreducible subgroup of \( \pi_d(SL(n, \mathbb{C})) \) then \( H := \pi_d^{-1}(\mathcal{H}) \) is an irreducible subgroup of \( SL(n, \mathbb{C}) \). Furthermore \( \pi_d(\pi_d^{-1}(\mathcal{H})) = \mathcal{H} \) and its centralizer \( Z_{\pi_d(SL(n, \mathbb{C}))}(\mathcal{H}) \) is equal to \( Z_d(H) \). As a result, rather than studying centralizers of irreducible subgroups of quotients of \( SL(n, \mathbb{C}) \), we can equivalently study \( d \)-extended centralizers of irreducible subgroups of \( SL(n, \mathbb{C}) \) for \( d \) dividing \( n \).

2.1 Abelianity, exponent and order

**Proposition 1.** Let \( n \geq 1 \) and \( H \) be an irreducible subgroup of \( SL(n, \mathbb{C}) \). Then, the \( n \)-centralizer \( Z_n(H) \) of \( H \) is abelian of exponent dividing \( n \).

**Proof.** According to lemma 2, define the group morphism :

\[
\phi_n : \frac{U_n(H)}{Ker(\phi_n)} \longrightarrow Mor(H, \langle \xi I_n \rangle)
\]

The kernel of \( \phi_n \) is the centralizer \( Z_{SL(n, \mathbb{C})}(H) \) of \( H \) in \( SL(n, \mathbb{C}) \). Since \( H \) is irreducible, Schur’s lemma implies that \( Z_{SL(n, \mathbb{C})}(H) = \langle \xi I_n \rangle \). Hence, the group \( U_n(H)/Ker(\phi_n) \) is isomorphic to a subgroup of \( Mor(H, \langle \xi I_n \rangle) \).

As a result, \( Z_n(H) \) is isomorphic to a subgroup of \( Mor(H, \langle \xi I_n \rangle) \). Since this group is abelian of exponent \( n \), it follows that \( Z_n(H) \) is also abelian of exponent dividing \( n \). \( \square \)

The next proposition justifies that the conjugacy class of any element in the \( n \)-extended centralizer of an irreducible subgroup in \( SL(n, \mathbb{C}) \) is well understood.

**Proposition 2.** Let \( n \geq 1 \) and \( H \) be an irreducible subgroup of \( SL(n, \mathbb{C}) \). If \( u \in U_n(H) \) and \( \pi_n(u) \) is of order \( d \) in \( Z_n(H) \) then there exists \( \lambda \in \mathbb{C}^* \) such that :

\[
\begin{pmatrix}
I_d \\
\lambda^{\frac{k}{d}} I_d \\
\vdots \\
\lambda^{\frac{k}{d}(d-1)} I_d
\end{pmatrix}
\]

where \( \lambda \in \left\{ \langle \xi I_n \rangle \text{ if } d \text{ is odd or } d \text{ even and } n/d \text{ even,} \right. \)
\[
\sqrt{\xi} \frac{\pi_n}{d} \langle \xi I_n \rangle \text{ if } d \text{ is even and } n/d \text{ odd.}
\]

**Proof.** First, \( \pi_n(H) \) is irreducible (since \( H \) is) so \( Z_n(H) \) is finite and \( U_n(H) \) is also finite. It follows that \( u \) is necessarily of finite order and, in particular, it is diagonalizable.
For all \( h \in H \), there exists \( s_h \in \mathbb{Z}/n \) such that \([h, u] = \xi^{s_h} I_n\). Applying lemma 2, the application \( s : H \rightarrow \mathbb{Z}/n \) is a group morphism. Let \( \xi^t \) be a generator of \( s(H) \).

Lemma 3 implies that \( H \) acts on the spectrum \( Sp(u) \) of \( u \) (i.e. the set of its eigenvalues) : if \( h \in H \) and \( \mu \in Sp(u) \) then \( h \cdot \mu := \xi^{s_h} \mu \). Let \( X \) be an orbit in \( Sp(u) \) for this action \( H \). The subspace \( \bigoplus \mu \in X E_\mu(u) \) of \( \mathbb{C}^{\mathbb{Z}/n} \) is stable by \( H \).

Since \( H \) is irreducible, this non-trivial subspace is the whole space \( \mathbb{C}^{\mathbb{Z}/n} \). In particular, the action of \( H \) on \( Sp(u) \) is transitive and all the eigenspaces have the same dimension \( v > 0 \). Remarking that \( H \) acts through the group morphism \( s \) whose image is generated by \( \xi^t \), we can say (if \( \lambda \) is some eigenvalue of \( u \)) that \( Sp(u) = \lambda \langle \xi^t \rangle \) and since \( u \) is diagonalizable:

\[
\begin{pmatrix}
I_v \\
\xi^t I_v \\
\vdots \\
\xi^{t(\frac{n}{2} - 1)} I_v
\end{pmatrix}
\]

\( u \) is conjugate to \( \lambda \begin{pmatrix}
I_v \\
\xi^t I_v \\
\vdots \\
\xi^{t(\frac{n}{2} - 1)} I_v
\end{pmatrix} \).

Using the dimensions, \( n = |Sp(u)| \times v = \frac{n}{2} v \), hence \( t = v \). Since \( \pi_n(u) \) is of order \( d \), this implies that \( t = \frac{n}{d} k \) with \( 0 < k < d \) prime to \( d \). Hence:

\[
\begin{pmatrix}
I_v \\
\xi^t I_v \\
\vdots \\
\xi^{t(d - 1)} I_v
\end{pmatrix}
\]

\( u \) is conjugate to \( \lambda \begin{pmatrix}
I_v \\
\xi^t I_v \\
\vdots \\
\xi^{t(d - 1)} I_v
\end{pmatrix} \).

The condition on \( \lambda \) is given by writing \( \det(u) = 1 \). Which leads to \( \lambda^n \xi^\frac{v n(d - 1)}{2} = 1 \).

If \( d \) is odd, then 2 divides \( d - 1 \) and then \( \lambda^n = 1 \).

If \( d \) is even and 2 divides \( \frac{n}{d} \), then \( \lambda^n = \xi^{-\frac{v n(d - 1)}{2}} = (\xi^n)^{\frac{2}{d}(d - 1)} = 1 \).

If \( d \) is even and 2 does not divide \( \frac{n}{d} \), then \( \lambda^n = \xi^{-\frac{v n(d - 1)}{2}} \) whence:

\[
\lambda \equiv \xi^{-\frac{v n(d - 1)}{2d}} \equiv \sqrt{\xi^{-\frac{n(d - 1)}{d}}} \mod \langle \xi I_n \rangle.
\]

Let \( n \geq 1 \) and \( H \) be an irreducible subgroup of \( SL(n, \mathbb{C}) \), we define the standard representation of \( U_n(H) \) as the natural inclusion \( \iota_H \) of \( U_n(H) \) in \( SL(n, \mathbb{C}) \). Its character \( Tr_{\iota_H} \) will be denoted \( \chi_H \). Computing this character appears to be easy.

**Proposition 3.** Let \( n \geq 1 \), \( H \) be an irreducible subgroup in \( SL(n, \mathbb{C}) \) and \( u \) in \( U_n(H) \),

then : \( \chi_H(u) = \begin{cases} 0 & \text{if } u \notin \langle \xi I_n \rangle \\ n \xi^k & \text{if } u = \xi^k I_n \text{ where } k \in \mathbb{Z}/n. \end{cases} \)
Proof. Let \( u \in U_n(H) \), if \( u = \xi^k I_n \in \langle \xi I_n \rangle \) then, \( \chi_H(u) = \text{Tr}(\iota_H(u)) = \text{Tr}(\xi I_n) = n \xi^k \).

If \( u \) is not central, then \( u \) is not trivial in \( U_n(H)/\langle \xi I_n \rangle = Z_n(H) \). Let \( d > 1 \) be its order in \( Z_n(H) \). By proposition 2 which explicitly gives the conjugacy class of \( u \), there exists \( \lambda \in \mathbb{C}^\times \) such that \( \chi_H(u) = \lambda \frac{d}{2}(1 + \xi^{\frac{d}{2}} + \cdots + \xi^{(d-1)}) \) and since \( \xi^{\frac{d}{2}} \) is a primitive \( d \)-th root of the unity for \( d > 1 \), the sum \( 1 + \xi^{\frac{d}{2}} + \cdots + \xi^{(d-1)} = 0 \), therefore \( \chi_H(u) = 0 \). □

Since \( U_n(H) \) is a finite group, we may use the theory of finite group representations in order to have some additional properties on \( Z_n(H) \). For instance :

**Corollary 1.** Let \( n \geq 1 \) and \( H \) be an irreducible subgroup of \( SL(n, \mathbb{C}) \) then \( |Z_n(H)| \) divides \( n^2 \). Furthermore, \( Z_n(H) \) is irreducible if and only if \( |Z_n(H)| = n^2 \).

**Proof.** First, we sum up classical results of the theory of finite groups representations, they can be found in [Ser77]. Say we are given a representation \( \rho : G \to GL(n, \mathbb{C}) \) of a finite group \( G \), then its character \( \chi := \text{Tr} \circ \rho \) has a norm defined by :

\[
\| \chi \|^2 := \frac{1}{|G|} \sum_{g \in G} \chi(g) \chi(g^{-1}).
\]

We know that \( \| \chi \|^2 \) is a natural number and \( \| \chi \|^2 = 1 \) if and only if the representation \( \rho \) is irreducible. Applying this to the standard representation \( \iota_H \) of \( U_n(H) \):

\[
\| \chi_H \|^2 = \frac{1}{|U_n(H)|} \sum_{u \in U_n(H)} \chi_H(u) \chi_H(u^{-1}) \quad \text{by proposition 3} \quad \text{\( \frac{n^3}{|U_n(H)|} \)}.
\]

Since \( \| \chi_H \|^2 \) must be an integer, \( |U_n(H)| \) divides \( n^3 \) and since \( |U_n(H)| \) is equal to \( n |Z_n(H)| \), \( |Z_n(H)| \) divides \( n^2 \). Furthermore, \( Z_n(H) \) is irreducible if and only if \( \pi_n^{-1}(Z_n(H)) = U_n(H) \) is irreducible if and only if \( \| \chi_H \|^2 = 1 \) if and only if \( |Z_n(H)| = n^2 \). □

By the following example, the bound is always reached :

**Example 1.** Let \( n \geq 1 \), define :

\[
u := \lambda \begin{pmatrix}
1 & \xi & \cdots & \\
\xi & \xi & \cdots & \\
\xi^{n-1} & \xi^{n-1} & \cdots & \\
& & \cdots & \end{pmatrix}
\text{ and } M := \lambda \begin{pmatrix}
0 & \cdots & 1 \\
1 & \cdots & \\
\cdots & \cdots & \\
& & 1 & 0
\end{pmatrix}
\]

where \( \lambda = 1 \) if \( n \) is odd and \( \sqrt{n} \) if \( n \) is even. Then \( H := \langle u, M \rangle \) is an irreducible subgroup of \( SL(n, \mathbb{C}) \) whose \( n \)-centralizer is \( \pi_n(H) \) which is of order \( n^2 \).

**Proof.** See the proof of proposition 3 □
The second corollary deals with conjugacy classes of centralizers of irreducible subgroups in $PSL(n, \mathbb{C})$.

**Corollary 2.** Let $H_1$ and $H_2$ be two irreducible subgroups in $SL(n, \mathbb{C})$. Then $Z_n(H_1)$ is conjugate to $Z_n(H_2)$ if and only if there exists an abstract group isomorphism $f$ from $U_n(H_1)$ to $U_n(H_2)$ such that for all $\xi^kI_n \in Z(SL(n, \mathbb{C}))$, $f(\xi^kI_n) = \xi^kI_n$.

Proof. We begin with the assumption that $Z_n(H_1)$ is conjugate to $Z_n(H_2)$ in $PSL(n, \mathbb{C})$. Then there exists $g \in SL(n, \mathbb{C})$ such that $\pi(g)Z_n(H_1)\pi(g)^{-1} = Z_n(H_2)$ and therefore, the element $g$ conjugates $U_n(H_1)$ to $U_n(H_2)$. As a result, the conjugation morphism by $g$ restricted to $U_n(H_1)$ will do the job.

Conversely, say such $f : U_n(H_1) \to U_n(H_2)$ exists. Recall that for $i = 1, 2$, $\iota_{H_i}$ defines the injection of $U_n(H_i)$ in $SL(n, \mathbb{C})$. We define the representation $\rho := \iota_{H_2} \circ f$ from $U_n(H_1)$ into $SL(n, \mathbb{C})$ and $\chi := tr \circ \rho$ its character. Remark that:

$$f(u) \begin{cases} \notin Z(SL(n, \mathbb{C})) \text{ if } u \notin Z(SL(n, \mathbb{C})) \\ = \xi^kI_n \text{ if } u = \xi^kI_n. \end{cases}$$

Applying proposition 3 to $\chi_{H_1}$ and $\chi_{H_2}$, we have that for $u = \xi^kI_n \in Z(SL(n, \mathbb{C}))$, $\chi_{H_1}(u) = n\xi^k = \chi(u)$ and for $u \notin Z(SL(n, \mathbb{C}))$, $\chi_{H_1}(u) = 0 = \chi(u)$. In particular, the representations $\iota_{H_1}$ and $\rho$ of $U_n(H_1)$ share the same character. As a result (see [Ser77]), they are conjugate. Since the respective images of $\iota_{H_1}$ and $\rho$ are $U_n(H_1)$ and $U_n(H_2)$, the groups $U_n(H_1)$ and $U_n(H_2)$ are conjugate.

Last consequence of proposition 3: any subgroup of $Z_n(H)$ remains the centralizer of an irreducible subgroup of $PSL(n, \mathbb{C})$.

**Corollary 3.** Let $n \geq 1$ and $H$ be an irreducible subgroup of $SL(n, \mathbb{C})$. For any subgroup $A$ of $Z_n(H)$, $Z_{PSL(n, \mathbb{C})}(Z_{PSL(n, \mathbb{C})}(A)) = A$. In particular, any subgroup of $Z_n(H)$ is itself the centralizer of an irreducible subgroup of $PSL(n, \mathbb{C})$.

Proof. We denote $B := Z_{PSL(n, \mathbb{C})}(Z_{PSL(n, \mathbb{C})}(A))$. The inclusion $A \subseteq B$ is obvious. Since $\pi_n(H) \leq Z_{PSL(n, \mathbb{C}))(A)$, we deduce that $B \leq Z_n(H)$. Hence $A \leq B \leq Z_n(H)$.

Let us first show that $Z_{PSL(n, \mathbb{C})}(B) = Z_{PSL(n, \mathbb{C})}(A)$. Since $A \leq B$, it is clear that $Z_{PSL(n, \mathbb{C})}(B) \leq Z_{PSL(n, \mathbb{C})}(A)$. Furthermore, if $z$ commutes with $A$ then any element $b \in B := Z_{PSL(n, \mathbb{C})}(Z_{PSL(n, \mathbb{C})}(A))$ will commute with $z$ by definition.

Hence $z \in Z_{PSL(n, \mathbb{C})}(B)$, so that $Z_{PSL(n, \mathbb{C})}(B) = Z_{PSL(n, \mathbb{C})}(A)$. (1)

Let $A_0$ (resp. $B_0$) be $\pi_n^{-1}(A)$ (resp. $\pi_n^{-1}(B)$) then $A_0 \leq B_0$, we denote $Z_1(A_0)$ (resp. $Z_1(B_0)$) the centralizer of $A_0$ (resp. $B_0$) in $SL(n, \mathbb{C})$. It follows that $Z_1(B_0) \leq Z_1(A_0)$. Let us show that the index of $Z_1(B_0)$ in $Z_1(A_0)$ is finite. Applying lemma 2 both indices $[U_n(A_0) : Z_1(A_0)]$ and $[U_n(B_0) : Z_1(B_0)]$ are finite. Since
\[ U_n(A_0) = \pi_n^{-1}(Z_{PSL(n, \mathbb{C})}(A)) \] by definition
\[ = \pi_n^{-1}(Z_{PSL(n, \mathbb{C})}(B)) \] by equation \[ \boxed{1} \]
\[ = U_n(B_0) \] by definition,
we get the following equality :

\[ [U_n(B_0) : Z_1(B_0)] = [U_n(A_0) : Z_1(B_0)] \]
\[ = [U_n(A_0) : Z_1(A_0)][Z_1(A_0) : Z_1(B_0)] \]
\[ \geq [Z_1(A_0) : Z_1(B_0)] \]

and end up with \[ [Z_1(A_0) : Z_1(B_0)] < \infty. \] \[ \Box \]

We make a proof by contradiction, say \( A \neq B \), then \( A_0 \neq B_0 \). Let us show that this contradicts the inequality \[ \Box \] We denote \( \rho_{A_0} \) (resp. \( \rho_{B_0} \)) the inclusion of \( A_0 \) (resp. \( B_0 \)) in \( SL(n, \mathbb{C}) \) whose character is \( \chi_{A_0} \) (resp. \( \chi_{B_0} \)). Since \( A_0 < B_0 \leq U_n(H) \), those representations are restrictions of the standard representation \( \iota_H \) of \( U_n(H) \) whose character has been computed in proposition \[ \boxed{3} \]

This leads to \( \| \chi_{A_0} \|^2 = \frac{n^3}{|A_0|} \) and \( \| \chi_{B_0} \|^2 = \frac{n^3}{|B_0|} \).

Since \( |A_0| < |B_0| \), we have \( \| \chi_{A_0} \| > \| \chi_{B_0} \| \). In terms of finite groups representations (cf. \[ \text{Ser77} \]), this means that there exists an irreducible \( B_0 \)-module of \( V := \mathbb{C}Z/n \) which is decomposed as a non-trivial sum of sub \( A_0 \)-modules. In particular the centralizer of the representation \( \rho_{B_0} \) is of infinite index in the centralizer of the representation \( \rho_{A_0} \). This contradicts the assertion \[ \Box \] As a result, \( A = Z_{PSL(n, \mathbb{C})}(Z_{PSL(n, \mathbb{C})}(A)) \).

In order to prove the last assertion of the corollary, it suffices to remark that \( Z_{PSL(n, \mathbb{C})}(A) \) contains \( \pi_n(H) \) and is, therefore, irreducible.

In the next subsection, we introduce a correspondence between \( n \)-centralizer of an irreducible subgroup and alternate modules.

### 2.2 The alternate module associated to a \( n \)-centralizer

Definitions and propositions concerning alternate modules used here can be found in \[ \text{TW15, Wal64} \] and \[ \text{Gue16-2} \]. We recall that an alternate module \((A, \phi)\) is an abelian group equipped with a bilinear map \( \phi : A \times A \to \mathbb{Q}/\mathbb{Z} \). We also remark that the group \( \mathbb{Q}/\mathbb{Z} \) contains a unique copy of \( \mathbb{Z}/n \) (namely, the subgroup generated by \( 1/n \)). In the next proposition, we construct an alternate module out of a finite group \( G \) containing a central cyclic subgroup \( C \) such that \( G/C \) is abelian.
Proposition 4. Let \( n \) be a positive integer, \( G \) be a finite group containing a central cyclic subgroup \( C \) of order \( n \) generated by \( c_0 \). Let \( A := G/C \) and \( \pi \) be the natural projection of \( G \) onto \( A \), assume that \( A \) is abelian. Taking for any \( a \in A \) an element \( \hat{a} \in G \) such that \( \pi(\hat{a}) = a \) (i.e. an arbitrary lift for \( a \)), we define :

\[
\phi_G : A \times A \rightarrow \mathbb{Z}/n
\]

\[
(a, b) \mapsto \phi_G(a, b) \text{ verifying } [\hat{a}, \hat{b}] = c_{0,a,b}^{\phi_G(a, b)}
\]

Then \( \phi_G \) is well defined (i.e. does not depend on the chosen lifts). Furthermore, \((A, \phi_G)\) is an alternate module whose kernel \( K_{\phi_G} \) is the image of the center of \( G \) by \( \pi \).

**Proof.** The fact that \( \phi_G \) has its values in \( \mathbb{Z}/n \) is clear since \( A \) is commutative. The fact that \( \phi_G \) does not depend on the chosen lifts \( \hat{a} \) is obvious because two different lifts of the same element are the same element up to a central element. Changing one for another does not change anything in the commutator \([\cdot, \cdot]\). Remark furthermore that if \( c_{0,a,b}^{k} \) is given, then \( k \) is well defined modulo \( n \) since \( c_0 \) is of order \( n \). The fact that \( \phi_G \) is bilinear is a consequence of lemma 2 and of the abelianity of \( A \). If \( a \in A \) then \([\hat{a}, \hat{a}] = I_n \). Whence \( \phi_G(a, a) = 0 \) modulo \( n \). Therefore \((A, \phi_G)\) is an alternate module.

Let \( a \in A \) then \( a \in K_{\phi_G} \) if and only if \([\hat{a}, \hat{b}] = I_n \) for all \( b \in A \), if and only if \( \hat{a} \in Z(G) \). Hence the radical of \((A, \phi_G)\) is the projection in \( A \) of the center of \( G \).

Let \( G_1 \) and \( G_2 \) be like in the proposition. Then \( f \) factors through \( C \leq G_1 \) and \( f(C) = C \leq G_2 \) and induces a group isomorphism \( \overline{f} \) between \( A_1 \) and \( A_2 \). If \( a, b \in A_1 \):

\[
\xi^{\phi_{G_2}}(\overline{f}(a), \overline{f}(b)) = \overline{f}(a, b) = f([\hat{a}, \hat{b}]) \text{ by definition of } \overline{f}
\]

\[
= f(\xi^{\phi_{G_1}}(a, b)) \text{ by definition of } \phi_G
\]

\[
= \xi^{\phi_{G_1}}(a, b) \text{ since } f \text{ fixes } C \text{ by assumption.}
\]

In particular the module \((A_1, \phi_{G_1})\) is isometric to the module \((A_2, \phi_{G_2})\) since \( \overline{f} \) is a group isomorphism. \(\square\)

In the sequel, given an irreducible subgroup \( H \) in \( SL(n, \mathbb{C}) \), we apply this proposition to \( G := U_n(H), \ C := Z(SL(n, \mathbb{C})), \ c_0 := \xi I_n \) and \( A := Z_n(H) \). The construction leads to an alternate module denoted \((Z_n(H), \phi_H)\) and called the associated alternate module to \( H \). Applying this proposition and corollary 2 the isometry class of \((Z_n(H), \phi_H)\) is invariant by conjugation of \( Z_n(H) \).
Since we want to classify conjugacy classes of centralizers of irreducible subgroups in $\text{PSL}(n, \mathbb{C})$, we will prove a lemma -a bit more general than we actually need it to be- that will eventually lead to a converse to the second statement in proposition [4].

Roughly speaking, it states that if we are given two groups with a central cyclic subgroup $C$ such that it leads to isometric alternate modules, then the two groups are isomorphic (provided they verify a condition on the order of the elements).

**Lemma 3.** Let $n \geq 1$, $C$ a cyclic group of order $n$ (denoted multiplicatively with a generator $c_0$) and $A$ a finite abelian group. Let $(d_1, \ldots, d_k)$ be the type of $A$, then $A$ is isomorphic to $(e_1) \times \cdots \times (e_1)$ where $e_i$ is an element of $A$ of order $d_i$.

Let $G$ and $H$ be two central extensions of $A$ by $C$, we denote $\pi_G$ (resp. $\pi_H$) the projection of $G$ (resp. $H$) on $A$. Assume that $\phi_G = \phi_H$ and for all $1 \leq i \leq r$, there exists $g_i \in G$ (resp. $h_i \in H$) such that $\pi_G(g_i) = e_i$ (resp. $\pi_H(h_i) = e_i$) and the order of $g_i$ is equal to the order of $h_i$ which is either $d_i$ or $2d_i$. Then, there exists an isomorphism $f$ between $G$ and $H$ sending $c_0 \in C \leq G$ to $c_0 \in C \leq H$.

**Proof.** We define a set theoretic section $u$ for $\pi_G$ and $v$ for $\pi_H$. By definition of $(e_i)$, for any element $a \in A$, there exists a unique $r$-uple $(\alpha_1, \ldots, \alpha_r)$ of integers such that :

$$a = \sum_{i=1}^{r} \alpha_i e_i \text{ and } 0 \leq \alpha_i < d_i.$$ 

We define $u(a) := g_0^{\alpha_r} \cdots g_1^{\alpha_1}$ and $v(a) := h_0^{\alpha_r} \cdots h_1^{\alpha_1}$. By definition of $g_i$ and $h_i$, $u$ and $v$ are respectively set-theoretic sections of $\pi_G$ and $\pi_H$. Any element of $G$ can be uniquely written as the product of an element $c$ of $C$ and $u(a)$ where $a \in A$ and we will define the isomorphism between $G$ and $H$ of the lemma as :

$$f : \begin{array}{rcl} G & \longrightarrow & H \\ cu(a) & \longmapsto & cv(a) \end{array} .$$

The application $f$ is onto, since any element $h$ in the group $H$ can be written as $cv(a)$ and $f(cv(a)) = cv(a) = h$, by definition. Furthermore, if $f(c_1 u(a_1)) = f(c_2 u(a_2))$, then $a_1 = \pi_H(f(c_1 u(a_1))) = \pi_H(f(c_2 u(a_2))) = a_2$. It follows that $c_1 = c_2$ and hence $c_1 u(a_1) = c_2 u(a_2)$ so that $f$ is a bijection. Furthermore, $f$ fixes point by point the elements of $C$ by definition. Remark that those properties of $f$ are easily deduced from the very definition of $f$. We shall need the assumptions of the lemma to show that $f$ is a group morphism. Let $k_1, k_2 \in G$, and write $k_1 = c_1 u(a_1)$, $k_2 = c_2 u(a_2)$ where $c_1, c_2 \in C$ and $a_1, a_2 \in A$. Then :

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\( f(k_1k_2) = f(k_1)f(k_2) \)
\( \iff \) \( f(c_1c_2u(a_1)u(a_2)) = c_1c_2u(a_1)u(a_2) \)
\( \iff \) \( f(c_1c_2u(a_1)u(a_2)u(a_1a_2)^{-1}u(a_1a_2)) = c_1c_2v(a_1)v(a_2) \)
\( \iff \) \( c_1c_2u(a_1)u(a_2)u(a_1a_2)^{-1}v(a_1a_2) = c_1c_2v(a_1)v(a_2) \)
\( \iff \) \( u(a_1)u(a_2)u(a_1a_2)^{-1} = v(a_1)v(a_2)v(a_1a_2)^{-1}. \)

We need to check that for all \( a_1, a_2 \in A \):
\[
 u(a_1)u(a_2)u(a_1a_2)^{-1} = v(a_1)v(a_2)v(a_1a_2)^{-1}. \tag{3}
\]

For \( i = 1, 2 \), we denote:
\[
a_i := \sum_{j=1}^{r} \gamma_j^i e_j \text{ where } 0 \leq \gamma_j^i < d_j.
\]

We also write for \( 1 \leq j \leq r \),
\[
\delta_j := \begin{cases} 
\gamma_j^1 + \gamma_j^2 & \text{if } \gamma_j^1 + \gamma_j^2 < d_j \\
\gamma_j^1 + \gamma_j^2 - d_j & \text{else.}
\end{cases}
\]

Then:
\[
a_1 + a_2 = \sum_{j=1}^{r} \delta_j e_j \text{ with } 0 \leq \delta_j < d_j.
\]

By definition of \( u \) and the expressions of \( a_1 \) and \( a_2 \) given above:
\[
u(a_1)u(a_2) = g_r^{\gamma_1} \cdots g_1^{\gamma_1} g_r^{\gamma_2} \cdots g_1^{\gamma_2} \\
= g_r^{\gamma_1} \cdots g_2^{\gamma_1} g_r^{\gamma_2} g_1^{\gamma_1} c_0 \prod_{i=2}^{r} c_{\phi_{G_i}(e_i,e_j)} \gamma_i^{1} \gamma_i^{2}.
\]

We used the equality \( g_r g_1 = c_0^{\phi_{G_1}(e_1, e_1)} g_1 g_r \) and the fact \( c_0 \) is in the center of \( G \). By a straightforward induction:
\[
u(a_1)u(a_2) = g_r^{\gamma_1} + \gamma_2 g_{r-1}^{\gamma_1} \cdots g_1^{\gamma_1} g_{r-1}^{\gamma_2} \cdots g_1^{\gamma_2} \prod_{i=2}^{r-1} c_0 \phi_{G_i}(e_i,e_j) \gamma_i^{1} \gamma_i^{2}.
\]

We may do this again for \( g_{r-1}^{\gamma_2} \) up to \( g_1^{\gamma_2} \) and then we have:
\[
u(a_1)u(a_2) = g_r^{\gamma_1} + \gamma_2 \cdots g_1^{\gamma_1} + \gamma_2 \prod_{j=2}^{r-1} \prod_{i=1}^{j-1} c_0 \phi_{G_i}(e_i,e_j) \gamma_i^{1} \gamma_i^{2}.
\]

Finally remark that for \( 1 \leq i \leq r \), \( g_i^{\gamma_1 + \gamma_2 - d_i} \) is in \( C \) whence it is central. So that:
there exists at most one element of order 2 whence

\[ H(\delta) = g_r^{\gamma_1^1 + \gamma_2^2} \ldots g_1^{\gamma_1^1 + \gamma_2^2 - \delta_1} \]

in any case \( h \). Therefore,

\[ h \gamma_i = \gamma_i \delta_i = 1 \] \( i = 1 \) for \( a \in A \).

Finally : \( u(a_1)u(a_2)u(a_1 + a_2)^{-1} = \prod_{i=1}^{r} g_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} \prod_{j=1}^{r} \phi_G(\epsilon_j, e_j) \gamma_j^1 \gamma_j^2 \).

Using the same arguments applied to \( H \) :

\[ v(a_1)v(a_2)v(a_1 + a_2)^{-1} = \prod_{i=1}^{r} h_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} \prod_{j=1}^{r} \phi_H(\epsilon_j, e_j) \gamma_j^1 \gamma_j^2 \].

By assumption, \( \phi_G \) and \( \phi_H \) are equal, whence :

\[ \prod_{i=1}^{r} \phi_G(\epsilon_i, e_i) \gamma_i^1 \gamma_i^2 = \prod_{i=1}^{r} \phi_H(\epsilon_i, e_i) \gamma_i^1 \gamma_i^2 \].

Let \( 1 \leq i \leq r \), denote \( \lambda_i := g_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} \) and \( \mu_i := h_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} \). By assumption \( g_i \) and \( h_i \) have the same order which is either \( d_i \) or \( 2d_i \). If the order of \( g_i \) is \( d_i \) then, since \( \gamma_i^1 + \gamma_i^2 - \delta_i = 0 \) or \( d_i \), we have that \( \lambda_i = 1 \). If \( g_i \) and \( h_i \) are both of order \( 2d_i \), then, if \( \gamma_i^1 + \gamma_i^2 - \delta_i = 0 \), and we have \( \lambda_i = 1 \) or \( \mu_i = 1 \), otherwise \( \gamma_i^1 + \gamma_i^2 - \delta_i = d_i \), in which case \( \lambda_i \) and \( \mu_i \) are both elements of order 2 in a cyclic group of 2, but in a cyclic group, there exists at most one element of order 2 whence \( \lambda_i = \mu_i \).

In any case \( h_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} = g_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} \) for \( 1 \leq i \leq r \). It follows that :

\[ \prod_{i=1}^{r} g_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} = \prod_{i=1}^{r} h_i^{\gamma_i^1 + \gamma_i^2 - \delta_i} \].

Combining this equality to the equations [15] and [6] we get for \( a_1, a_2 \in A \) :

\[ u(a_1)u(a_2)u(a_1 + a_2)^{-1} = v(a_1)v(a_2)v(a_1 + a_2)^{-1} \].

Therefore, \( f \) is a group morphism and the lemma is proven.

We are now ready to state our first theorem.

**Theorem 1.** Let \( n \geq 1 \), \( \overline{H_1} \) and \( \overline{H_2} \) be two irreducible subgroups of \( PSL(n, \mathbb{C}) \). Denote \( H_i := \pi_n^{-1}(\overline{H_i}) \) the pull-back in \( SL(n, \mathbb{C}) \). Then \( Z_{PSL(n, \mathbb{C})(\overline{H_1})} \) is conjugate to \( Z_{PSL(n, \mathbb{C})(\overline{H_2})} \) if and only if \( (Z_{PSL(n, \mathbb{C})(\overline{H_1}), \phi_H_1}) \) is isometric to \( (Z_{PSL(n, \mathbb{C})(\overline{H_2}), \phi_H_2}) \).
Proof. First, remark that for $i = 1, 2$, $Z_{n}(H_{i}) \subseteq Z_{n}(H_{i})$. If $Z_{n}(H_{1})$ and $Z_{n}(H_{2})$, proposition 4 and corollary 2 imply that $(Z_{n}(H_{1}), \phi_{H_{1}})$ and $(Z_{n}(H_{2}), \phi_{H_{2}})$ are isometric.

Conversely, say $(Z_{n}(H_{1}), \phi_{H_{1}})$ and $(Z_{n}(H_{2}), \phi_{H_{2}})$ are isometric. Then, up to composing the projection $\pi_{1} : U_{n}(H_{1}) \to Z_{n}(H_{1})$ by the isometry between $(Z_{n}(H_{1}), \phi_{H_{1}})$ and $(Z_{n}(H_{2}), \phi_{H_{2}})$, we may assume that the alternate modules constructed from $U_{n}(H_{i})$ are both equal to $(A, \phi)$. Let $a$ be in $A$ and $d$ be its order, then, applying proposition 2, there exists $u_{i} \in U_{n}(H_{i})$ such that:

$$u_{i} \text{ is conjugate to } \lambda \begin{pmatrix} I_{\frac{n}{d}} & & & \\ & \xi I_{\frac{n}{d}} & & \\ & & \ddots & \\ & & & \xi^{n(d-1)} I_{\frac{n}{d}} \end{pmatrix}$$

with $\lambda \in \left\{ \begin{array}{ll} \langle \xi I_{n} \rangle & \text{if } d \text{ is odd or } d \text{ even and } n/d \text{ even} \\ \sqrt{\frac{n(d-1)}{d}} \langle \xi I_{n} \rangle & \text{if } d \text{ is even and } n/d \text{ odd.} \end{array} \right.$

Up to multiplying $u_{i}$ by $\xi^{k}$, we may take $\lambda = 1$ if $d$ is odd or $n/d$ is even (in which case the lift $u_{i}$ of $a$ is of order $d$) and $\lambda = \sqrt{\frac{n(d-1)}{d}}$ else (in which case the lift $u_{i}$ of $a$ is of order $2d$). We are, now, in the conditions of application of lemma 3 and there exists a group isomorphism between $U_{n}(H_{1})$ and $U_{n}(H_{2})$ fixing, point by point $\langle \xi I_{n} \rangle$. Hence $Z_{n}(H_{1})$ and $Z_{n}(H_{2})$ are conjugate by corollary 2.

In the next section, we shall see under which necessary and sufficient condition an alternate module $(A, \phi)$ is associated to a centralizer of an irreducible subgroup in $PSL(n, \mathbb{C})$.

3 Classification of centralizers in $PSL(n, \mathbb{C})$

In this section, we characterize the alternate modules associated to centralizers of irreducible subgroups in $PSL(n, \mathbb{C})$. Recall first that applying corollary 1 if an abelian group is isomorphic to such centralizer then its order divides $n^{2}$. A centralizer of an irreducible subgroup in $PSL(n, \mathbb{C})$ is full if its order is $n^{2}$.

In the first subsection, we give a necessary and sufficient condition for an alternate module to be associated to such centralizer. Full centralizers play a key role in the study. In the second section, we focus on the consequences of this result.

3.1 $n$-subsymplectic modules and associated centralizers

Once again, we begin with a result from representation theory. Let $G$ be a finite group and $\rho : G \to GL(n, \mathbb{C})$ be a representation of $G$ acting linearly on $V^{\mathbb{Z}/n}$ through $\rho$ then $V$ may be decomposed as a sum of maximal isotypic sub-modules $V_{1}, \ldots, V_{k}$ (i.e. which are linearly equivalent to $\lambda_{i} \cdot (W_{i}, \rho_{i})$ where $(W_{i}, \rho_{i})$ is an irreducible representation of...
G and $\lambda_i > 0$). Furthermore, up to the order of $(V_i)$, the decomposition happens to be unique. This leads to a technical lemma:

**Lemma 4.** Let $n \geq 1$, $G$ be a finite subgroup of $GL(n, \mathbb{C})$. Define $\iota_G$ to be the inclusion of $G$ into $GL(n, \mathbb{C})$. Let $V_1 \oplus \cdots \oplus V_k$ be the maximal isotypic decomposition of $(V, \iota_G)$. Then $N_{GL(n, \mathbb{C})}(G)$ acts on the set $\{V_1, \ldots, V_k\}$ of subspaces occurring in the maximal isotypic decomposition of $(V, \iota_G)$.

**Proof.** Let $n$ be in $N_{GL(n, \mathbb{C})}(G)$ and $1 \leq i \leq k$. For all $g \in G$ and $v \in V_i$:

$$g \cdot (n(v_i)) = (gn) \cdot v_i = n(n^{-1}gn) \cdot (v_i) = n n^{-1}gn \cdot (v_i).$$

As a result $nV_i$ is stable by $G$ and $nV_i$ is a submodule of $V$. Furthermore, if $(V_i, \rho'_i)$ is the induced representation on $V_i$ by $\iota_G$, then the representation on $nV_i$ induced by $\iota_G$ is $(nV_i, \rho'(n^{-1}n))$. In particular, $V_i$ being isotypic, $nV_i$ is also isotypic and is therefore included in a maximal isotypic submodule. This implies that there exists $1 \leq j \leq r$ such that $nV_i \leq V_j \Leftrightarrow V_i \leq n^{-1}V_j$.

Using the same argument, $n^{-1}V_j$ is also isotypic, but it contains $V_i$ which is maximal isotypic, it follows that $V_i = n^{-1}V_j$ whence $nV_i = V_j$. \qed

We recall notations and results from [Gue16-2]. Let $(A, \phi)$ be an alternate module, we say that $K$ included in $A$ is isotropic if $K$ is orthogonal to itself. We say that $L$ included in $A$ is Lagrangian if $L^\perp = L$. Remark that (proposition 2 in loc. cit.) Lagrangians are exactly the maximal isotropic subsets in $A$. Likewise, we have proven that any Lagrangian of the alternate module $(A, \phi)$ is of order $n_{A,\phi} := \sqrt{|A||K_\phi|}$ where $K_\phi$ denotes the radical (or kernel) of $(A, \phi)$.

**Proposition 5.** Let $n \geq 1$ and $H$ be an irreducible subgroup of $SL(n, \mathbb{C})$, then the order $n_{Z_n(H),\phi_H}$ of Lagrangians in $(Z_n(H), \phi_H)$ divides $n$.

**Proof.** Let $K$ be an isotropic subgroup in $(Z_n(H), \phi_H)$ and $K_0$ be its pull-back in $U_n(H)$. If $k_1, k_2 \in K_0$ then:

$$[k_1, k_2] = \xi^{\phi_H(\pi_n(k_1), \pi_n(k_2))} \text{ by definition of } \phi_H$$

$$= I_n \text{ since } (K, \phi_H|_{K \times K}) \text{ is trivial.}$$

Therefore, if $K$ is isotropic, then $\pi_n^{-1}(K)$ is an abelian subgroup in $U_n(H)$. In particular, if $L$ is a Lagrangian in $(Z_n(H), \phi_H)$, then $L_0 := \pi_n^{-1}(L)$ is an abelian subgroup of $U_n(H)$. Denote $\iota_0 : L_0 \to SL(n, \mathbb{C})$ the inclusion of $L_0$ in $SL(n, \mathbb{C})$. Let $h \in H$ and $x \in L_0$ then $hxh^{-1} \in \langle \xi I_n \rangle x$ since $x \in U_n(H)$. Because $\langle \xi I_n \rangle$ is included in $L_0$, it follows that $hxh^{-1}$ belongs to $L_0$ as well. Whence, $H$ normalizes $L_0$.

We denote $V_1, \ldots, V_k$ to be the maximal isotypic subspaces occuring in $(V, \iota_0)$. Using lemma 4, $H$ acts on $\{V_1, \ldots, V_k\}$. Furthermore, if the action were not transitive then $H$
would stabilize a non-trivial subspace of $V$ contradicting its irreducibility. Whence, the action of $H$ on $\{ V_1, \ldots, V_k \}$ is transitive and, in particular, $V_1, \ldots, V_k$ share the same dimension $\lambda > 0$.

Since $L_0$ is an abelian group, each $V_i$ can be decomposed as $\lambda$ copies of a 1-dimensional representation $(W_i, \rho_i)$ of $L_0$. We denote $\chi_0$ to be the character of $\iota_0$ and $\chi_i$ the character of $(V_i, \iota_0 | V_i)$. By considerations of finite groups representation theory (see [Ser77]), $\| \chi_i \|^2 = \lambda^2$ and by orthogonality of the characters $\chi_i$ for $i \neq j$:

$$\| \chi_0 \|^2 = k \lambda^2 = \lambda \dim(V).$$

On the other hand, $\chi_0 = \chi_H | L_0$ where $\chi_H$ is the character of the standard representation $\iota_H$ of $U_n(H)$ whose character has been computed in proposition 3. Hence:

$$\| \chi_0 \|^2 = \frac{n^3}{|L_0|}.$$ 

As a result, we have that $|L_0| = n^{\frac{3}{2}}$. Since $L_0 = \pi_n^{-1}(L)$ and the kernel of $\pi_n$ is of order $n$, we end up with $|L| = \frac{n^3}{3}$. In particular the order of $L$, which is $n_{Z_n(H), \phi_H}$, divides $n$.

We just found a necessary condition (on the cardinality of Lagrangians) for an alternate module to be associated to the centralizer of an irreducible subgroup of $PSL(n, \mathbb{C})$. It will be proven to be a sufficient condition as well in theorem 2. In order to prove this, we need to construct some particular examples of irreducible subgroups in $PSL(n, \mathbb{C})$.

**Proposition 6.** For any $n \geq 1$ and $B$ abelian group of order $n$, there exists a finite irreducible subgroup $H$ of $SL(n, \mathbb{C})$ of order $n^2$ such that $Z_n(H)$ is isomorphic to $B \times B$.

**Proof.** We prove it by induction on $n \geq 1$. If $n = 1$, then the property is obviously true. Assume $n > 1$. Let $B$ be an abelian group of order $n$ and $d$ be the exponent of $B$. Denote $B := B_1 \times \langle e \rangle$ where $e$ is of order $d$. Let $K$ be an irreducible subgroup of $SL(n/d, \mathbb{C})$ of order $(n/d)^2$ such that $Z_{n/d}(K)$ is isomorphic to $B_1$ (by induction hypothesis). We define a subgroup $K_0$ of $SL(n, \mathbb{C})$ by blocks of size $n/d$:

$$K_0 := \left\{ \begin{pmatrix} k & \cdots & \cdots \k \end{pmatrix} \mid k \in K \right\}.$$ 

We also denote:

$$M := \lambda \begin{pmatrix} 0 & I_{n/d} & \cdots & I_{n/d} \\ I_{n/d} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & I_{n/d} \\ I_{n/d} & 0 \end{pmatrix} \quad \text{and} \quad u := \lambda \begin{pmatrix} I_{n/d} \xi^n I_{n/d} \\ \xi^n I_{n/d} \\ \cdots \\ \xi^{n(d-1)} I_{n/d} \end{pmatrix}.$$
where $\lambda$ is defined as in proposition $\square$ (this implies that $\det(M) = \det(u) = 1$). Finally, let $H$ be the subgroup of $SL(n, \mathbb{C})$ generated by $K_0$, $M$, $u$ and $\xi I_n$.

Since $u$ is scalar by blocks of size $n/d$ with distinct eigenvalues, it follows that:

$$Z_{SL(n, \mathbb{C})}(u) = \left\{ \begin{pmatrix} g_0 \\
\vdots \\
g_{d-1} \end{pmatrix} \mid g_0, \ldots, g_{d-1} \in GL(n/d, \mathbb{C}) \text{ det}(g_0) \ldots \text{ det}(g_{d-1}) = 1 \right\}.$$

The commutator $[M, u]$ is of order dividing $d$. Since $\psi$ is surjective, it is of order dividing $d$. In the sequel, $U_n(H)$ is the subgroup of $U_n([u]) = \langle Z_{SL(n, \mathbb{C})}(u), M \rangle$. Let $g_0, \ldots, g_{d-1}$ be in $GL(n/d, \mathbb{C})$ and $b \geq 0$ such that:

$$x := \begin{pmatrix} g_0 \\
\vdots \\
g_{d-1} \end{pmatrix} M^b \in U_n(H).$$

Then $[M, x]$ is in $Z(SL(n, \mathbb{C}))$ and since $\pi_n(M)$ is of order $d$, it follows that $[M, x]$ is of order dividing $d$. Since $[M, u]$ is of order $d$, up to multiplying $x$ by some power of $u$, we may assume that $[M, x] = I_n$. But this implies (by a straightforward calculation) that $g_0 = \cdots = g_{d-1}$. Hence we just showed that:

$$U_n(H) \leq \left\{ \begin{pmatrix} g \\
\vdots \\
g \end{pmatrix} u^b M^b \mid g \in GL(n/d, \mathbb{C}), \text{ det}(g)^d = 1 \right\}.$$

In the sequel, $D(g)$ will be the matrix $\begin{pmatrix} g \\
\vdots \\
g \end{pmatrix}$ where $g \in GL(n/d, \mathbb{C})$.

Let $g \in GL(n/d, \mathbb{C})$ with $\det(g)^d = 1$ then $\det(g)$ is a $d$-th root of unity. Up to multiplying $D(g)$ by a $n/d$-th root of $\det(g)$ (which is a $n$-th root of the unity, i.e. $\xi^s$ for some integer $s$), we may assume that $\det(g) = 1$. whence:

$$U_n(H) \leq \left\{ \xi^s D(g) u^b M^b \mid g \in SL(n/d, \mathbb{C}) \text{ s, b, l } \geq 0 \right\}. \quad (7)$$

Let us define an application $\psi_0 : U_n(H) \rightarrow \langle \xi^{n/d} \rangle \times \langle \xi^{n/d} \rangle$, $v \mapsto (\xi^m, \xi^n)$. The application $\psi_0$ is a group morphism (consequence of lemma $\square$), it is surjective since $\psi_0(u)$ and $\psi_0(M)$ generate $\langle \xi^{n/d} \rangle \times \langle \xi^{n/d} \rangle$ and it factors through $Z_n(H)$. It follows that we have a surjective isomorphism $\psi$ from $Z_n(H)$ to $\mathbb{Z}/d \times \mathbb{Z}/d$. 

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From the inclusion \( \{ \pi_n(M), \pi_n(u) \} \) is of trivial intersection with \( \text{Ker}(\psi) \). Since it surjects onto \( \mathbb{Z}/d \times \mathbb{Z}/d \) through \( \psi \), it follows that:

\[
\langle \pi_n(M), \pi_n(u) \rangle \text{ is of trivial intersection with } \text{Ker}(\psi).
\]

Let \( k \in K \) and \( k_0 := D(k) \in K_0 \leq H \). For \( g \in SL(n/d, \mathbb{C}) \) : \( \langle k_0, D(g) \rangle = D([k,g]) \).

As a result, \( D(g) \in U_n(H) \) if and only if \( g \in U_{n/d}(K) \). In particular, this leads to a well defined group morphism modulo \( \langle \xi I_n \rangle \):

\[
\langle \pi_n(D(g)) \rangle \mapsto \pi_{n/d}(g).
\]

Because \( \varphi \) is injective, \( \text{Ker}(\psi) \) is isomorphic to \( \mathbb{Z}_{n/d}(K) \) which has been chosen to be isomorphic to \( B_1 \times B_1 \). As a result, \( \mathbb{Z}_n(H) \) is isomorphic to \( B_1 \times B_1 \times \mathbb{Z}/d \times \mathbb{Z}/d \).

Furthermore, if \( H_1 \) and \( H_2 \) are two irreducible subgroups verifying one of these assertions and \( \mathbb{Z}_n(H_1) \) is abstractly isomorphic to \( \mathbb{Z}_n(H_2) \) then \( H_1 \) and \( H_2 \) are conjugate.

**Proposition 7.** Let \( n \geq 1 \) and \( H \) be an irreducible subgroup of \( SL(n, \mathbb{C}) \) containing the center of \( SL(n, \mathbb{C}) \). Then the following assertions are equivalent:

1. \( |\mathbb{Z}_n(H)| = n^2 \)
2. \( \mathbb{Z}_n(H) \) is irreducible.
3. \( \pi_n(H) \) is abelian.
4. \( \pi_n(H) \) is its own centralizer.
5. \( (\mathbb{Z}_n(H), \phi_H) \) is isometric to \( B \times B^* \) where \( B \) is abelian of order \( n \) and \( B \times B^* \) is endowed with its canonical symplectic form.

Furthermore, if \( H_1 \) and \( H_2 \) are two irreducible subgroups verifying one of these assertions and \( \mathbb{Z}_n(H_1) \) is abstractly isomorphic to \( \mathbb{Z}_n(H_2) \) then \( H_1 \) and \( H_2 \) are conjugate.

**Proof.** The equivalence 1 \( \iff \) 2 is exactly the second assertion in corollary 1.

(2 \( \Rightarrow \) 3). Say \( \mathbb{Z}_n(H) \) is irreducible, then its centralizer is abelian by proposition 1, therefore, \( \pi_n(H) \) which commutes with \( \mathbb{Z}_n(H) \) is abelian as well.
(3 ⇒ 4). Let ρ be the inclusion of H into SL(n, C) and χ its character. Since H is included in U_n(H), ρ is the standard representation \(\iota_H\) of U_n(H) restricted to H whose character has been computed in proposition 3. It follows that \(\|\chi\|^2 = \frac{n^3}{|H|}\). On the other hand, since H is irreducible, ρ is irreducible and then \(\|\chi^2\| = 1\). Whence \(|H| = n^3\) and \(|\pi_n(H)| = n^2\). Since \(|Z_n(H)| \leq n^2\) and \(\pi_n(H)\) is included in \(Z_n(H)\), it follows that \(\pi_n(H) = Z_n(H)\) and \(\pi_n(H)\) is its own centralizer.

(4 ⇒ 2). If \(\pi_n(H)\) is its own centralizer then \(Z_n(H) = \pi_n(H)\) is irreducible.

(5 ⇒ 1) is obvious. (1 ⇒ 5). The alternate module \((Z_n(H), \phi_H)\) is of order \(n^2\). By proposition 3 we have that \(n_{Z_n(H), \phi_H} = \sqrt{|Z_n(H)||K_{\phi_H}|}\) divides \(n\). It follows that \(\sqrt{|K_{\phi_H}|}\) divides 1. Whence \(K_{\phi_H}\) is trivial and \((Z_n(H), \phi_H)\) is a symplectic module of order \(n^2\). By corollary 7.4 in [TW15], there exists an abelian group \(B\) of order \(n\) such that \((A, \phi)\) is isometric to \(B \times B^\ast\) with its canonical symplectic form.

Let \(H_1\) and \(H_2\) be two irreducible subgroups in \(SL(n, \mathbb{C})\) containing \(Z(SL(n, \mathbb{C}))\) such that \(Z_n(H_1)\) and \(Z_n(H_2)\) are isomorphic. Since they verify the assertion 5, \((Z_n(H_1), \phi_{H_1})\) and \((Z_n(H_2), \phi_{H_2})\) are both symplectic modules with isomorphic underlying groups. Since the isometry class of a symplectic module is simply given by the isomorphism class of its underlying group, it follows that \((Z_n(H_1), \phi_{H_1})\) and \((Z_n(H_2), \phi_{H_2})\) are isometric. As a result, applying theorem 4, the subgroups \(Z_n(H_1)\) and \(Z_n(H_2)\) are conjugate in \(PSL(n, \mathbb{C})\). Since for \(i = 1, 2\), \(\pi_n(H_i) = Z_n(H_i)\) and \(H_i = \pi_n^{-1}(\pi_n(H_i))\) it follows that \(H_1\) is conjugate to \(H_2\).

We end this subsection with a characterization theorem :

**Theorem 2.** Let \(n \geq 1\) and \((A, \phi)\) be an alternate module, then the following assertions are equivalent :

1. There exists an irreducible subgroup \(H\) in \(SL(n, \mathbb{C})\) such that \((Z_n(H), \phi_H)\) is isometric to \((A, \phi)\).

2. The order of Lagrangians in \((A, \phi)\) divides \(n\).

3. There exists an abelian group \(B\) of order \(n\) such that \((A, \phi)\) is isometrically embedded in \(B \times B^\ast\).

In particular, full centralizers of irreducible subgroups in \(SL(n, \mathbb{C})\) are the maximal elements among the centralizers of irreducible subgroups in \(SL(n, \mathbb{C})\).

**Proof.** The implication 1 ⇒ 2 is exactly the statement of proposition 3.

Let us show that 2 ⇒ 3. Let \((A, \phi)\) be an alternate module and \(d = n_{A, \phi}\) be the order of one of its Lagrangians which divides \(n\) by assumption. Using the theorem proven in [Gue16] (alternate modules are subsymplectic), there exists an abelian group \(B_0\) of order \(d\) such that \((A, \phi)\) is isometrically embedded in \(B_0 \times B_0^\ast\). Defining the abelian group \(B := B_0 \times \mathbb{Z}/(n/d)\), \(B_0 \times B_0^\ast\) is isometrically embedded in \(B \times B^\ast\), so that \((A, \phi)\) is also isometrically embedded in \(B \times B^\ast\). Since \(|B| = |B_0|n/d = n\), we have assertion 3.
Let us show that $3 \Rightarrow 1$. Assume that $(A, \phi)$ is isometrically embedded in $B \times B^*$. Let $H$ be an irreducible subgroup such that $Z_n(H)$ is isomorphic (as a group) to $B \times B$ (by proposition 3). Since $|Z_n(H)| = |B|^2 = n^2$, it follows, by proposition 4, that $(Z_n(H), \phi_H)$ is isometric to $B \times B^*$. We denote $f$ a group isomorphism from $Z_n(H)$ to $B \times B^*$ which is an isometry.

Denote $(A_0, \phi_0)$ the submodule of $(Z_n(H), \phi_H)$ which is sent to $(A, \phi)$ by $f$. By corollary 3, $A_0$ is the $n$-centralizer of an irreducible subgroup $K$ of $SL(n, \mathbb{C})$, that is to say $Z_n(K) = A_0$. Furthermore, from the definition of $(A_0, \phi_K)$ and $(Z_n(H), \phi_H)$, $\phi_{K|A_0 \times A_0} = \phi_K$. Since $\phi_0 := \phi_{H|A_0 \times A_0}$, the alternate module $(A_0, \phi_0)$ is of the form $(Z_n(K), \phi_K)$. Since $(A, \phi)$ is isometric to $(A_0, \phi_0)$ we have the assertion 1.

By corollary 4, full centralizers reach the maximal order of centralizers of irreducible subgroups in $SL(n, \mathbb{C})$. Whence, full centralizers are maximal among the centralizers of irreducible subgroups in $SL(n, \mathbb{C})$.

Conversely, if $H$ is an irreducible subgroup in $SL(n, \mathbb{C})$, then $(Z_n(H), \phi_H)$ is isometrically embedded in $B \times B^*$ (since $1 \iff 3$) with $B$ abelian group of order $n$. Using propositions 6 and 7, there exists an irreducible subgroup $S$ of $PSL(n, \mathbb{C})$ such that $S$ is isomorphic to $B \times B^*$. If $A$ is the image of $Z_n(H)$ into $B \times B^*$ then $A = Z_n(K)$ by corollary 3. It follows that $(Z_n(H), \phi_H)$ is isometric to $(Z_n(K), \phi_K)$. Hence $Z_n(H)$ and $Z_n(K)$ are conjugate (by theorem 4), since $Z_n(K)$ is included in $S = Z_{PSL(n,\mathbb{C})}(S)$ which is a full centralizer, we have that any centralizer of irreducible subgroup in $PSL(n, \mathbb{C})$ is contained in a full centralizer. This implies that all maximal elements among the centralizers of irreducible subgroups in $SL(n, \mathbb{C})$ must be full centralizers.

This theorem will have some consequences that we are going to sum up in the next subsection. In the sequel, if $(A, \phi)$ is an alternate module, we will say that $(A, \phi)$ is $n$-subsymplectic if there exists an abelian group $B$ of order $n$ such that $(A, \phi)$ is isometrically embedded in $B \times B^*$.

### 3.2 Consequences of the classification

Something that may be the most obvious consequence is the following:

**Corollary 4.** Let $n \geq 1$, then the number of conjugacy classes of centralizers of irreducible subgroups in $PSL(n, \mathbb{C})$ is finite.

**Proof.** Using theorem 4, we can equivalently bound the number of isometry classes of associated alternate modules. Using theorem 2, it suffices to show that there are only a finite number of isometry classes of $n$-subsymplectic modules.

When we fix $B$, the module $B \times B^*$ has only a finite number of submodules, since there exist only a finite number of abelian groups $B$ of order $n$, there are only a finite number of symplectic modules of order $n^2$, up to isometry, and hence, a finite number of $n$-subsymplectic modules up to isometry. \hfill \square
Lemma 5. Let $n \geq 1$ and $A$ be an abelian group. Then there exists an irreducible subgroup $H$ of $SL(n, \mathbb{C})$ such that $Z_n(H)$ is isomorphic to $A$ if and only if there exists an abelian group $B$ of order $n$ such that $A$ is isomorphic to a subgroup of $B \times B$.

In particular, for any abelian group $A$, there exists an integer $n \geq 1$ and an irreducible subgroup $H$ of $SL(n, \mathbb{C})$ such that $Z_n(H)$ is isomorphic to $A$.

Proof. If $A$ is isomorphic to $Z_n(H)$ where $H$ is an irreducible subgroup of $SL(n, \mathbb{C})$ then, by theorem 2, there exists an abelian group $B$ of order $n$ such that $(Z_n(H), \phi_H)$ is isometrically embedded in $B \times B^*$. In particular $Z_n(H)$ is isomorphic to a subgroup of $B \times B^*$ which is (non-canonically) isomorphic to $B \times B$. Hence $A$ is isomorphic to a subgroup of $B \times B$.

Let $B$ be an abelian group $B$ of order $n$ such that $A$ is isomorphic to a subgroup of $B \times B$. By proposition 6 there exists an irreducible subgroup $H$ of $SL(n, \mathbb{C})$ such that $Z_n(H) = B \times B$. Let $A_0$ be the image of $A \leq B \times B$ in $Z_n(H)$, by corollary $A$ is the $n$-centralizer of an irreducible subgroup of $SL(n, \mathbb{C})$.

Remark that $A$ is always included in $A \times A$. Hence, if $n := |A|$, applying what we have already done in this corollary, there exists an irreducible subgroup $H$ of $SL(n, \mathbb{C})$ such that $A$ is isomorphic to $Z_n(H)$. $\square$

If $p$ is a prime number, we have recalled in the introduction that conjugacy classes of centralizers of irreducible subgroups in $PSL(p, \mathbb{C})$ are classified by their isomorphism classes. We would like to know exactly when this convenient property holds. We recall that an integer $n \geq 2$ is squarefree if it cannot be divided by a non-trivial square.

Corollary 5. Let $n \geq 1$ and $A$ be an abelian group. Then there exists an irreducible subgroup $H$ of $SL(n, \mathbb{C})$ such that $Z_n(H)$ is isomorphic to $A$ if and only if there exists an abelian group $B$ of order $n$ such that $A$ is isomorphic to a subgroup of $B \times B$.

Proof. If $A$ is isomorphic to $Z_n(H)$ where $H$ is an irreducible subgroup of $SL(n, \mathbb{C})$ then, by theorem 2, there exists an abelian group $B$ of order $n$ such that $(Z_n(H), \phi_H)$ is isometrically embedded in $B \times B^*$. In particular $Z_n(H)$ is isomorphic to a subgroup of $B \times B^*$ which is (non-canonically) isomorphic to $B \times B$. Hence $A$ is isomorphic to a subgroup of $B \times B$.

Let $B$ be an abelian group $B$ of order $n$ such that $A$ is isomorphic to a subgroup of $B \times B$. By proposition 6 there exists an irreducible subgroup $H$ of $SL(n, \mathbb{C})$ such that $Z_n(H) = B \times B$. Let $A_0$ be the image of $A \leq B \times B$ in $Z_n(H)$, by corollary $A$ is the $n$-centralizer of an irreducible subgroup of $SL(n, \mathbb{C})$.

Remark that $A$ is always included in $A \times A$. Hence, if $n := |A|$, applying what we have already done in this corollary, there exists an irreducible subgroup $H$ of $SL(n, \mathbb{C})$ such that $A$ is isomorphic to $Z_n(H)$. $\square$

Lemma 5. Let $n$ be a squarefree integer and $(A, \phi)$ be an alternate module of rank 2, of exponent dividing $n$ such that $n_{A,\phi}$ divides $n$. Let $A$ be isomorphic to $(b) \times (a)$ with $e$ (the order of $b$) dividing $d$ (the order of $a$) dividing $n$. Then :

$(A, \phi)$ is isometric to $\left( \frac{\mathbb{Z}}{e/e} \times \frac{\mathbb{Z}}{d}, \begin{pmatrix} 0 & -1/e \\ 1/e & 0 \end{pmatrix} \right)$.

Proof. Let $e'$ be the order of $\phi(b,a)$. Then $K_{\phi}$ contains $\langle b^{e'} \rangle$ of order $e/e'$ and $\langle a^{e'} \rangle$ of order $d/e'$. Since those two groups are in trivial intersection, $K_{\phi}$ contains their direct product and $|K_{\phi}|$ is divided by $\frac{d}{(e')^2}$. Remark that $|A| = de$. It follows that

$$n_{A,\phi} := \sqrt{|A||K_{\phi}|} = de/e'.$$

Since $n_{A,\phi}$ divides $n$ by assumption, $d/e'$ divides $n$. Assume that $e'$ is a strict divisor of $e$ and take $p$ a prime number dividing $\frac{e}{e'}$ then $p$ divides $d$ and, hence, $p^2$ divides $n$ which contradicts the fact that $n$ is squarefree. Hence $\phi(b,a)$ is of order $e$. 21
Denote $\phi(b, a) = \frac{b}{a} \in \mathbb{Q}/\mathbb{Z}$. Since $\phi(b, a)$ is of order $e$, $\lambda$ is prime to $e$. Let $\mu$ be an integer such that $\lambda\mu = 1 \mod e$. Then the automorphism $f : A \to A$ sending $b$ to $b^\mu$ and $a$ to $a$ is an isometry between $(A, \phi)$ and \( \left( \mathbb{Z}/e \times \mathbb{Z}/d, \begin{pmatrix} 0 & -1/e \\ 1/e & 0 \end{pmatrix} \right) \). \hfill \square

This lemma leads to the following theorem:

**Theorem 3.** Let $n \geq 2$. Then conjugacy classes of centralizers of irreducible subgroups of $\text{PSL}(n, \mathbb{C})$ are classified by their isomorphism classes if and only if $n$ is squarefree.

**Proof.** Assume that $n$ is squarefree. Let $H_1$ and $H_2$ be two irreducible subgroups in $\text{SL}(n, \mathbb{C})$ such that $Z_n(H_1)$ and $Z_n(H_2)$ are isomorphic. First remark that, by corollary 5, $Z_n(H_1)$ is necessarily isomorphic to a subgroup of $B \times B$ where $B$ is abelian of order $n$. Since $n$ is squarefree, $B$ is actually cyclic. It follows that the common rank of $Z_n(H_1)$ and $Z_n(H_2)$ is lesser or equal to 2.

If the common rank of $Z_n(H_1)$ and $Z_n(H_2)$ is 0 or 1, then for $i = 1, 2$, $(Z_n(H_i), \phi_{H_i})$ is necessarily the trivial module (i.e. $\phi_{H_i} = 0$). Since $Z_n(H_1)$ and $Z_n(H_2)$ are isomorphic, it is clear that $(Z_n(H_1), \phi_{H_1})$ and $(Z_n(H_2), \phi_{H_2})$ are isometric. Whence, using theorem 1, $Z_n(H_1)$ and $Z_n(H_2)$ are conjugate.

Else, the common rank of $Z_n(H_1)$ and $Z_n(H_2)$ is 2. Let $\mathbb{Z}/e \times \mathbb{Z}/d$ be the isomorphism class of both $Z_n(H_1)$ and $Z_n(H_2)$. Since $a_{Z_n(H_1), \phi_{H_1}}$ and $a_{Z_n(H_2), \phi_{H_2}}$ both divide $n$, we are in the condition of application of lemma 5. In particular $(Z_n(H_1), \phi_{H_1})$ and $(Z_n(H_2), \phi_{H_2})$ are isometric and, by theorem 1, $Z_n(H_1)$ and $Z_n(H_2)$ are conjugate.

Assume that $n$ is not squarefree. Let $p$ be a prime number such that $p^2$ divides $n$. We define two structures of alternate module on $\mathbb{Z}/p \times \mathbb{Z}/p$:

\[
M_1 := (\mathbb{Z}/p \times \mathbb{Z}/p, 0) \text{ where the form is trivial.} \\
M_2 := \mathbb{Z}/p \times (\mathbb{Z}/p)^* \text{ where the form is symplectic.}
\]

Remark that $M_1$ is Lagrangian in $M_1$ so the order of its Lagrangian is $p^2$ which divides $n$. By theorem 2 there exists an irreducible subgroup $H_1$ of $\text{SL}(n, \mathbb{C})$ such that $(Z_n(H_1), \phi_{H_1})$ is isometric to $M_1$. On the other hand the radical of $M_2$ is trivial, so the order of its Lagrangian is $p$ which divides $n$. By theorem 2 there exists an irreducible subgroup $H_2$ of $\text{SL}(n, \mathbb{C})$ such that $(Z_n(H_2), \phi_{H_2})$ is isometric to $M_2$.

Even if $Z_n(H_1)$ and $Z_n(H_2)$ are both isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$, $Z_n(H_1)$ and $Z_n(H_2)$ are not conjugate because $M_1$ and $M_2$ are not isometric (see the last assertion of proposition 4). \hfill \square

As a result, when $n$ is squarefree, we can compute the number of conjugacy classes of centralizers of irreducible subgroups in $\text{PSL}(n, \mathbb{C})$.

**Corollary 6.** Let $n = p_1 \ldots p_r$ be a squarefree integer (i.e. $p_1, \ldots, p_r$ are two-by-two different prime numbers). There are exactly $3^r$ conjugacy classes of centralizers of irreducible subgroups in $\text{PSL}(n, \mathbb{C})$. 

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Proof. From theorem 3 it suffices to compute the number of different isomorphism classes. When $n$ is squarefree, the cyclic group of order $n$ is the only abelian group of order $n$. Because of corollary 5, it suffices to compute the number of subgroups in $\mathbb{Z}/n \times \mathbb{Z}/n$. In order to do this, we associate to any subgroup $A$ of $\mathbb{Z}/n \times \mathbb{Z}/n$ the $r$-uple $(A_{p_1}, \ldots, A_{p_r})$ of its $p_i$-Sylows. Since the isomorphism class of $A$ only depends on $(A_{p_1}, \ldots, A_{p_r})$, it suffices to compute the number of possible choices for the $p_i$-Sylows.

Since $A_{p_i}$ is clearly included in the $p_i$-Sylow of $\mathbb{Z}/n \times \mathbb{Z}/n$ which is isomorphic to $\mathbb{Z}/p_i \times \mathbb{Z}/p_i$, it follows that we have exactly three choices for $A_{p_i}$: $\{0\}$, $\mathbb{Z}/p_i$ and $\mathbb{Z}/p_i \times \mathbb{Z}/p_i$. As a result we have $3^r$ different choices for the isomorphism class of $A$. □

Now we would like to highlight a last consequence that might be the most fruitful. Let $\mathcal{M}_n$ be the set of isometry classes of alternate modules which are $n$-subsymplectic. We see $(\mathcal{M}_n, \leq)$ as an ordered set, where $\leq$ is the usual relation of inclusion (up to isometry). For any $n \geq 1$, we define a graph structure $G_n$, by taking for the set of vertices, the set $\mathcal{M}_n$ and any two classes of modules $M_1$ and $M_2$ are linked by an edge if $M_1 \leq M_2$ and $|M_2|/|M_1|$ is a prime number or $M_2 \leq M_1$ and $|M_1|/|M_2|$ is a prime number.

Before giving some examples we give a notation for some alternate modules. For any finite abelian group $B$, the symplectic module $B \times B^*$ will be denoted $S(B)$. For any integer $k \geq 1$, the trivial module on $\mathbb{Z}/k$ will be denoted by $C_k$. For any triple $(e', e, d)$ of integers such that $e'$ divides $e$ and $e$ divides $d$ :

$$M_{e', e, d} := \left( \mathbb{Z}/e \times \mathbb{Z}/d, \begin{pmatrix} 0 & -1/e' \\ 1/e' & 0 \end{pmatrix} \right).$$

Furthermore, here, $\oplus$ will denote the orthogonal sum.

Example 2. If $p$ is a prime number, there are exactly 3 conjugacy classes of centralizers of irreducible subgroups in $\text{PSL}(p, \mathbb{C})$.

Proof. Indeed, it suffices to compute the cardinal of $\mathcal{M}_p$ by theorem 2. In this case, the graph $G_p$ can easily seen to be like in figure 1.

Figure 1: The graph $G_p$ □
Example 3. If $p$ is a prime number, there are exactly 9 conjugacy classes of centralizers of irreducible subgroups in $PSL(p^2, \mathbb{C})$.

Proof. Indeed, it suffices to compute the cardinal of $\mathcal{M}_{p^2}$ by theorem 2. In this case, the graph $G_{p^2}$ can be computed to be like in figure 2.

Example 4. If $p$ is a prime number, there are exactly 24 conjugacy classes of centralizers of irreducible subgroups in $PSL(p^3, \mathbb{C})$.

Proof. Indeed, it suffices to compute the cardinal of $\mathcal{M}_{p^3}$ by theorem 2. In this case, the graph $G_{p^3}$ can be computed like in figure 3.

In [Gue16-1], we defined for $G$ a complex semi-simple Lie group and $\Gamma$ a finitely generated group a bad representation to be an irreducible representation $\rho : \Gamma \to G$ with a non-trivial centralizer. We also defined the singular locus of the character variety to be the set of conjugacy classes of bad representations. This set is denoted $\chi_{Sing}^i(\Gamma, G)$. In the same paper, we used the theorem classifying centralizers of irreducible subgroups in $PSL(p, \mathbb{C})$ ($p$ is a prime number) to study, for any finitely generated group $\Gamma$, the singular locus of the character variety $\chi_{Sing}^i(\Gamma, PSL(p, \mathbb{C}))$. 

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Somehow, we hope that the same thing is possible when the prime number is replaced by any integer $n$. Basically, the idea is the following. Let $M$ be the isometry class of

Figure 3: The graph $G_p^3$
an alternate module in $\mathcal{M}_n$. Because of theorem \ref{thm:centralizer}, there exists a centralizer $Z_M$ of an irreducible subgroup in $\text{PSL}(n, \mathbb{C})$ whose associated alternate module is $M$. For any $M \in \mathcal{M}_n$, we define a subset $\chi_M$ of the character variety:

$$\chi_M := \left\{ \text{PSL}(n, \mathbb{C}) \cdot \rho \left| \rho \in \text{Hom}(\Gamma, \text{PSL}(n, \mathbb{C})(Z_M)) \land \rho \text{ is irreducible} \right. \right\}.$$ 

Using the fact that $Z_M$ is well defined up to conjugation, $\chi_M$ does not depend on the conjugate of $Z_M$ we chose. For any $\text{PSL}(n, \mathbb{C}) \cdot \rho$ in $\chi_M$, $\rho$ is centralized by some conjugate of $Z_M$. Furthermore, using again the unicity, up to conjugation of $Z_M$, if $\rho$ is an irreducible representation of $\Gamma$ into $\text{PSL}(n, \mathbb{C})$ such that the alternate module $M_\rho$ associated to its centralizer contains an isometric copy of $M$ then the orbit of $\rho$ is contained in $\chi_M$. As a result, if we consider $(\mathcal{L}S_n, \leq)$ to be the set of $\chi_M$ for $M$ being an element of $\mathcal{M}_n$ and $\leq$ is the usual inclusion then $(\mathcal{L}S_n, \leq)$ is a partially ordered set which verifies:

**Proposition 8.** For $n \geq 2$ and $\Gamma$ a finitely generated group, the application $\chi : \mathcal{M}_n \to \mathcal{L}S_n$ sending $M$ to $\chi_M$ is decreasing. Furthermore:

$$\chi^\text{Sing}(\Gamma, \text{PSL}(n, \mathbb{C})) = \bigcup_{M \in \mathcal{M}_n \mid |M| > 1} \chi_M.$$ 

Finally, if $M \in \mathcal{M}_n$ is of order $n^2$ then $\chi_M$ is finite.

**Proof.** Assume that $M \leq N$ with $M, N \in \mathcal{M}_n$. Then, we may choose $Z_M \leq Z_N$. Hence, we have $Z_{\text{PSL}(n, \mathbb{C})(Z_N)} \leq Z_{\text{PSL}(n, \mathbb{C})(Z_M)}$ which leads to $\chi_N \leq \chi_M$.

Let $\rho$ be a representation such that $\text{PSL}(n, \mathbb{C}) \cdot \rho$ belongs to $\chi_M$ with $|M| > 1$ then $\rho$ is irreducible by definition and $\rho$ is centralized by one conjugate of $Z_M$ which is not trivial. It follows that $Z(\rho)$ cannot be trivial and $\rho$ is bad representation. It follows that its conjugacy class belongs to the singular locus of the character variety.

Conversely, assume that $\rho$ is a bad representation, then define $M$ to be the isometry class of the alternate module associated to $Z(\rho)$. Applying theorem \ref{thm:centralizer} $M \in \mathcal{M}_n$ and since $Z(\rho)$ is not trivial, $|M| > 1$. By theorem \ref{thm:irreducible} it follows that $Z(\rho)$ is conjugate to $Z_M$. Since $\rho$ is irreducible, we do get that its conjugacy class belongs to $\chi_M$ where $|M| > 1$. Hence we get:

$$\chi^\text{Sing}(\Gamma, \text{PSL}(n, \mathbb{C})) = \bigcup_{M \in \mathcal{M}_n \mid |M| > 1} \chi_M.$$ 

For the last assertion, remark that if $M$ is of order $n^2$ then $Z_M$ is a full centralizer. By proposition \ref{prop:full-centralizer} $Z_M$ is irreducible and self-centralizing. In particular $Z_{\text{PSL}(n, \mathbb{C})(Z_M)} = Z_M$. It follows that for any representation $\rho$ of $\Gamma$ such that its conjugacy class belongs to $\chi_M$, the image of $\rho$ must be equal to $Z_M$ (taking $H := \pi_n^{-1}(\rho(\Gamma))$ in proposition \ref{prop:irreducible}).
Consider, in $\Gamma$, the set $X$ of normal subgroups $N$ such that $\Gamma/N$ is isomorphic to $Z_M$. Since $Z_M$ is abelian, $X$ is in bijection with the set $X'$ of subgroups $A$ in $\Gamma^{Ab}$ such that $\Gamma^{Ab}/A$ is isomorphic to $Z_M$. Since $\Gamma$ is finitely generated, $\Gamma^{Ab}$ is finitely generated. This implies that the set $X'$ is necessarily finite. For any representation $\rho$ whose conjugacy class belongs to $\chi_M$, we get that $Ker(\rho) \in X$.

If we denote $\chi_{M,N}$ to be the conjugacy classes of $\rho$ whose kernel is equal to $N$, then $\chi_{M,N}$ is exactly the set of conjugacy classes of representations from $\Gamma/N$ onto $Z_M$. Since $\chi_M$ is a union of $\chi_{M,N}$ for $N \in X$ and $X$ is finite, it follows that $\chi_M$ is finite as well.

This leads to a decomposition of the singular locus $\chi_{Sing}(\Gamma, PSL(n, \mathbb{C}))$ governed by the graph $G_n$ we defined earlier. Each vertex $M$ of $G_n$ leads to a stratum of the singular locus, and the minimal strata of the singular locus are simply a finite union of points, namely the $\chi_M$ where $M$ is a symplectic module of order $n^2$.

### 4 Centralizers in quotients of $SL(n, \mathbb{C})$

In this section, we gather some generalizations of the results obtained for centralizers in $PSL(n, \mathbb{C})$ to centralizers of irreducible subgroups in $\pi_d(SL(n, \mathbb{C}))$ where $d$ divides $n$. Roughly speaking, we shall see that the results are almost the same.

#### 4.1 Abelianity, exponent and order

First, we handle the abelianity of the $d$-centralizers.

**Proposition 9.** Let $n \geq 1$, $d$ be a divisor of $n$ and $H$ be an irreducible subgroup of $SL(n, \mathbb{C})$, then $Z_d(H)$ is abelian.

**Proof.** Let $u, v$ be two elements in $U_d(H)$ (we recall that $U_d(H) := \pi_d^{-1}(Z_d(H))$). Since we have $U_d(H) \leq U_n(H)$ and $Z_n(H)$ abelian (by proposition 1), it follows that $[u, v]$ belongs to $\langle \xi I_n \rangle$. In particular, lemma 2 implies:

$$[u, v]^d = [u^d, v].$$  \hfill (9)

On the other hand, since $u \in U_d(H)$, we also have that for all $h \in H$ $[u, h]$ is in $\langle \xi I_n \rangle$ of order dividing $d$. It follows (by lemma 2 again) that $[u^d, h] = I_n$ for all $h \in H$ i.e. $u^d$ centralizes $H$. Since $H$ is irreducible, Schur’s lemma implies that $u^d \in \langle \xi I_n \rangle$. It follows, from equation 9 that $[u, v]$ is of order dividing $d$. Since it is central, we get, by definition, that $[u, v] \in ker(\pi_d)$. As a result $Z_d(H) = \pi_d(U_d(H))$ is abelian. \hfill $\square$

The second proposition deals with the exponent of $Z_d(H)$.

**Proposition 10.** Let $n \geq 1$, $d$ be a divisor of $n$ and $H$ be an irreducible subgroup of $SL(n, \mathbb{C})$, then the exponent of $Z_d(H)$ divides $\text{lcm}(n/d, d)$.
Proof. Let $u$ be an element of $U_d(H)$ and $d_0$ be the order of $\pi_n(u)$. Let $h$ be in $H$ such that $[h, u]$ is of order $d_0$. Since $u \in U_d(H)$, it follows that $[h, u]$ must also be of order dividing $d$. It follows that $d_0$ divides $d$. Using proposition 2:

$$u \text{ is conjugate to } \lambda \begin{pmatrix}
I_{\frac{n}{d_0}} & \xi^{\frac{m}{d_0}} I_{\frac{n}{d_0}} \\
& \ddots \\
& & \xi^{\frac{m}{d_0}(d_0-1)} I_{\frac{n}{d_0}}
\end{pmatrix}$$

where $\lambda \in \left\{ \begin{aligned}
&\langle \xi I_n \rangle & \text{if } d_0 \text{ is odd or } d_0 \text{ is even and } n/d_0 \text{ even,} \\
&\sqrt{\xi^{\frac{n}{d_0}(d_0-1)}} \langle \xi I_n \rangle & \text{if } d_0 \text{ is even and } n/d_0 \text{ odd.}
\end{aligned} \right.$

Let $m$ be a common multiplicator of $n/d$ and $d$. Then :

$$u^m \text{ is conjugate to } \lambda^m \begin{pmatrix}
I_{\frac{n}{d_0}} & \xi^{\frac{m}{d_0}} I_{\frac{n}{d_0}} \\
& \ddots \\
& & \xi^{\frac{m}{d_0}(d_0-1)} I_{\frac{n}{d_0}}
\end{pmatrix}$$

Since $d_0$ divides $d$ which divides $m$, $u^m$ is conjugate to $\lambda^m I_n$. On one hand, if $d_0$ is odd, or $d_0$ is even and $n/d_0$ even, then $\lambda^m \in \langle \xi^m \rangle \leq \langle \xi^{n/d} \rangle$. In particular, $u^m$ is central of order dividing $d$, whence $u^m \in \ker(\pi_d)$ and $\pi_d(u)$ is of order dividing $m$. On the other hand, if $d_0$ is even and $n/d_0$ is odd then $\lambda = \sqrt{\xi^{\frac{n}{d_0}(d_0-1)}} \xi^k$ where $0 \leq k \leq n-1$. Remark that $2$ divides $m$ since it divides $d_0$. Hence :

$$\sqrt{\xi^{\frac{n}{d_0}(d_0-1)}} m = (\xi^{\frac{n}{d_0}})^{-\frac{m}{d_0}(d_0-1)} \text{ belongs to } \langle \xi^{n/d} \rangle.$$ 

Since $\langle \xi^k \rangle^m$ is also in $\langle \xi^{n/d} \rangle$, $\lambda^m$ belongs to $\langle \xi^{n/d} \rangle$. Whence $u^m$ is central of order dividing $d$ and $u^m \in \ker(\pi_d)$, so that $\pi_d(u)$ is of order dividing $m$. In any case, all elements in $Z_d(H)$ are of order dividing $m$. This is, in particular, true for $m = \text{lcm}(n/d, d)$. \qed

We finally handle the order :

**Proposition 11.** Let $n \geq 1$, $d$ be a divisor of $n$ and $H$ be an irreducible subgroup of $SL(n, \mathbb{C})$ then $Z_d(H)$ is of order dividing $n^3/d$.

**Proof.** We know that $U_d(H) \leq U_n(H)$. Furthermore, $\pi_d(U_d(H)) = Z_d(H)$. Hence :
\[ |Z_d(H)| = \frac{|U_d(H)|}{d} \]
divides \( \frac{|U_n(H)|}{d} \) since \( U_d(H) \leq U_n(H) \)

\[ \text{divides} \quad \frac{n|Z_n(H)|}{d} \quad \text{since} \quad \pi_n(U_n(H)) = Z_n(H) \]

\[ \text{divides} \quad \frac{n^3}{d} \quad \text{by corollary} \, \text{I} \]

\[ \square \]

In this proof, we used an inclusion which is, a priori, very weak: \( U_d(H) \leq U_n(H) \).

The next example shows that the bound of proposition I cannot be improved in general (computations are left to the reader).

**Example 5.** We study the case \( n := 4 \) and \( d := 2 \). There exists an exceptional isomorphism between \( SL(4, \mathbb{C}) \) and \( Spin(6, \mathbb{C}) \). This makes of \( SL(4, \mathbb{C}) \) the universal cover of \( SO(6, \mathbb{C}) \). In particular, \( \pi_2(SL(4, \mathbb{C})) \) is identified to \( SO(6, \mathbb{C}) \). We denote \( \pi_2 : SL(4, \mathbb{C}) \to SO(6, \mathbb{C}) \) the induced projection. We denote \( \overline{H} \leq SO(6, \mathbb{C}) \), the subgroup of diagonal matrices in \( SO(6, \mathbb{C}) \). Then \( \overline{H} \) is a finite group of order 32, which is its own centralizer in \( SO(6, \mathbb{C}) \). Let \( H := \pi_2^{-1}(\overline{H}) \) then \( |Z_2(H)| = 32 = \frac{4^3}{2} = \frac{n^3}{d} \).

In order to improve the bound in some particular cases, we prove a lemma:

**Lemma 6.** Let \( n \geq 1 \), \( d \) be a divisor of \( n \) and \( H \) be an irreducible subgroup of \( SL(n, \mathbb{C}) \). Recall that we have \( U_d(H) \leq U_n(H) \) and \( U_n(H)/\langle \xi I_n \rangle \) is equal to \( Z_n(H) \) then the quotient group \( U_d(H)/\langle \xi I_n \rangle \), included in \( Z_n(H) \), is equal to \((Z_n(H))_{(d)}\), the \( d \)-torsion of the abelian group \( Z_n(H) \).

**Proof.** Let \( u \) be in \( U_d(H) \) then we have that for all \( h \in H \) \( [u, h] \) is in \( \langle \xi I_n \rangle \) of order dividing \( d \). In particular \( u^d \) (by lemma \( 2 \)) commutes with \( H \). It follows that \( u^d \) is central by Schur’s lemma, whence \( u^d \in \langle \xi I_n \rangle \). As a result, \( \pi_n(u) \) is of order dividing \( d \). Finally this leads to \( U_d(H)/\langle \xi I_n \rangle \leq (Z_n(H))_{(d)} \).

Let \( u \in U_n(H) \) be such that \( u \in \pi_n^{-1}((Z_n(H))_{(d)}) \). It follows that \( u^d \) is central. Let \( h \in H \), since \( u \in U_n(H) \), we know that \([u, h]\) is central. By lemma \( 2 \) \([u, h]^d = [u^d, h]\) and since \( u^d \) is central, \([u, h]\) is of order dividing \( d \). By definition of \( U_d(H) \), this implies that \( u \in U_d(H) \). With the first inclusion, we have \( U_d(H)/\langle \xi I_n \rangle = (Z_n(H))_{(d)} \).

\[ \square \]

This leads to the following proposition:

**Proposition 12.** Let \( n \geq 1 \), \( d \) dividing \( n \) and \( H \) an irreducible subgroup of \( SL(n, \mathbb{C}) \). If \( \gcd(d, n/d) = 1 \) then the order of \( Z_d(H) \) divides \( nd \).
Proof. The order of $Z_n(H)$ divides $n^2$ by corollary 1. We decompose $n$ as a product of prime numbers $n = p_1^{a_1} \cdots p_r^{a_r}$. Since $gcd(d, n/d) = 1$ and the order of the $p_i$’s does not matter, we may as well assume that $d = p_1^{a_1} \cdots p_s^{a_s}$ and $n/d = p_{s+1}^{a_{s+1}} \cdots p_r^{a_r}$.

Since $Z_n(H)$ is abelian by proposition 1 of order dividing $n^2$ (by corollary 1), there exists, for any $1 \leq i \leq r$ a unique subgroup $S_i \leq Z_n(H)$ whose order is a power of $p_i$ such that: $Z_n(H) = S_1 \times \cdots \times S_r$. Clearly, $(Z_n(H))_{(d)} = S_1 \times \cdots \times S_s$ and since $|Z_n(H)|$ divides $n^2$, we deduce from this, that $|(Z_n(H))_{(d)}|$ divides $d^2$. Lemma 6 states that $U_d(H) = \pi_n^{-1}((Z_n(H))_{(d)})$. Hence:

$$|Z_d(H)| = |\pi_d(U_d(H))| = |U_d(H)|/d = |\pi_n^{-1}((Z_n(H))_{(d)})| = n|(Z_n(H))_{(d)}|/d$$

Since $|(Z_n(H))_{(d)}|$ divides $d^2$, it follows that $|Z_d(H)|$ divides $nd$. \qed

We gather the results of this subsection in one single theorem:

**Theorem 4.** Let $n \geq 1$, $d$ be a divisor of $n$ and $H$ be an irreducible subgroup of $SL(n, \mathbb{C})$ then $Z_d(H)$ is abelian of exponent dividing lcm$(n/d, d)$ and of order dividing $n^3/d$. If $gcd(n/d, d) = 1$ then $Z_d(H)$ is of order dividing $nd$.

In the next subsection, we shall study the conjugacy classes of $d$-centralizer of irreducible subgroups in $SL(n, \mathbb{C})$ and briefly apply this to the study of the corresponding character variety.

### 4.2 Singularities in the associated character variety

First, we remark that we can always project $Z_d(H)$ onto a subgroup $A_d(H)$ of $PSL(n, \mathbb{C})$. Since it is a subgroup of $Z_n(H)$ (by lemma 6) it follows, by corollary 10, that the group $A_d(H)$ is the centralizer of an irreducible subgroup of $PSL(n, \mathbb{C})$. In particular we can associate to $A_d(H)$ a structure of alternate module. Denote $M_d(H) := (A_d(H), \phi_{H,d})$ the alternate module that we get with this construction. Now we easily generalize theorem 11.

**Proposition 13.** Let $n \geq 1$, $d$ be a divisor of $n$ and $H_1, H_2$ be two irreducible subgroups of $SL(n, \mathbb{C})$ then $Z_d(H_1)$ and $Z_d(H_2)$ are conjugate if and only if $M_d(H_1)$ and $M_d(H_2)$ are isometric.

**Proof.** First remark that $Z_d(H_1)$ is conjugate to $Z_d(H_2)$ if and only if their respective projections $A_d(H_1)$ and $A_d(H_2)$ are conjugate in $PSL(n, \mathbb{C})$. Since $A_d(H_1)$ and $A_d(H_2)$ are centralizer of irreducible subgroups in $PSL(n, \mathbb{C})$, $A_d(H_1)$ and $A_d(H_2)$ are conjugate if and only if $M_d(H_1)$ and $M_d(H_2)$ are isometric applying theorem 11. \qed

Likewise we may generalize theorem 2.
**Proposition 14.** Let $n \geq 1$, $d$ be a divisor of $n$ and $(A, \phi)$ be an alternate module then there exists an irreducible subgroup $H$ in $SL(n, \mathbb{C})$ such that $(A, \phi)$ is isometric to $M_d(H)$ if and only if $n_{A,\phi}$ (the order of Lagrangians in $(A, \phi)$) divides $n$ and $A$ is of exponent dividing $d$.

**Proof.** Let $H$ be an irreducible subgroup in $SL(n, \mathbb{C})$ then, by definition $A_d(H) = \pi_n(U_d(H))$ is of exponent dividing $d$ by lemma 8. Whence $M_d(H)$ is an alternate module of $d$-torsion. Furthermore $M_d(H)$ is isometrically embedded in $(Z_n(H), \phi_H)$. Let $L$ be Lagrangian of $M_d(H)$. It follows that $L$ is isotropic in $(Z_n(H), \phi_H)$, in particular, $L \leq L_0$ where $L_0$ is a Lagrangian in $(Z_n(H), \phi_H)$. It follows that $n_{M_d(H)}$ divides $n_{Z_n(H),\phi_H}$ which divides $n$ (by theorem 2).

Conversely, let $(A, \phi)$ be an alternate module of exponent dividing $d$ and such that $n_{A,\phi}$ divides $n$. By theorem 2 there exists an irreducible subgroup $H$ in $SL(n, \mathbb{C})$ such that $(Z_n(H), \phi_H)$ is isometric to $(A, \phi)$. Since $Z_n(H)$ is of exponent dividing $d$, it follows that $\pi_n(U_d(H)) = Z_n(H)$ by lemma 6. In particular $A_d(H) = Z_n(H)$ and hence $M_n(H) = (Z_n(H), \phi_H)$ by definition. Hence $M_d(H)$ is isometric to $(A, \phi)$. \hfill $\square$

Before stating a straightforward corollary we recall a few facts about the character variety. Let $\Gamma$ be a Fuchsian group and $G$ be a semi-simple complex Lie group. It is known that in such case, $\chi^i(\Gamma, G)$ admits a structure of orbifold (see [Sik12]). Furthermore the local isotropy group of $G \cdot \rho$ is, up to conjugation, $Z_G(\rho)/Z(G)$.

**Corollary 7.** Let $n \geq 1$, $d$ be a divisor of $n$ and $\rho$ be an irreducible representation of $\Gamma$ into $G := \pi_d(SL(n, \mathbb{C}))$. Then the isotropy group of $G \cdot \rho$ in the orbifold $\chi^i(\Gamma, G)$ is an abelian group $A$ of exponent dividing $d$ such that there exists an abelian group $B$ of order $n$ with $A$ included in $B \times B$.

**Proof.** Let $H := \pi_d^{-1}(\rho(\Gamma))$. Then $H$ is irreducible and $Z_G(\rho) = Z_d(H)$ by definition. By proposition 14 $M_d(H)$ is an alternate module of exponent dividing $d$ and with its Lagrangians of order dividing $n$. It follows that $A_d(H) = Z_n(H)$, the underlying group of $M_d(H)$, is of exponent dividing $d$. By corollary 5 there exists an abelian group $B$ of order $n$ such that $A_d(H) = Z_n(H)$ is included in $B \times B$. The result follows since $Z_G(\rho)/Z(G) = A_d(H)$. \hfill $\square$

In conclusion of this short section, we can say that studying the case of $PSL(n, \mathbb{C})$ is equivalent to studying the case of quotients of $SL(n, \mathbb{C})$ in general.

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