Generalised Permutation Branes on a product of cosets $G_{k_1}/H \times G_{k_2}/H$

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Abstract

We study the modifications of the generalized permutation branes defined in [hep-th/0509153] which are required to give rise to the non-factorizable branes on a product of cosets $G_{k_1}/H \times G_{k_2}/H$. We find that for $k_1 \neq k_2$ there exists big variety of branes, which reduce to the usual permutation branes, when $k_1 = k_2$ and the permutation symmetry is restored.

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1 Introduction and summary

In the recent paper [1] were suggested the generalized permutation branes on a product of the WZW models $G_{k_1} \times G_{k_2}$ with the not necessarily equal levels $k_1$ and $k_2$. Geometrically the branes wrap the following submanifolds:

$$ (g_1, g_2) = \{(h_1 f h_2^{-1})^{k'_2}, \ (h_2 f h_1^{-1})^{k'_1} | h_1, h_2 \in G \} $$

(1)

where $k'_i = k_i / k$ and $k = \gcd(k_1, k_2)$. Obviously for $k_1 = k_2$ (1) reduces to the usual permutation branes [2–8]

$$ (g_1, g_2) = \{h_1 f h_2^{-1}, \ h_2 f h_1^{-1} \} $$

(2)

It is well known that for a submanifold to serve as D-brane in the WZW model the boundary two-form $\omega_C$ should exist trivializing the Wess-Zumino three-form on the brane [9–11]:

$$ \omega^{WZW}_{\text{brane}} = d\omega_C. $$

(3)

It was found in [1] that the restriction of the WZW form $k_1 \omega^{WZW}(g_1) + k_2 \omega^{WZW}(g_2)$ to the submanifold (1) indeed satisfies to this condition with $\omega_C^{(f)}$ given by the equation:

$$ \omega_C^{(f)}(h_1, h_2) = \frac{k_1 k_2}{k} \{\text{tr}(h_1^{-1}dh_1 f h_2^{-1}dh_2 f^{-1}) + \text{tr}(h_2^{-1}dh_2 f h_1^{-1}dh_1 f^{-1})\}

+ k_1 \sum_{j=1}^{k'_2 - 1} (k'_2 - j)\text{tr}(g^j(g^{-1}dg)g^{-j}g^{-1}dg)_{g = h_1 f h_2^{-1}}

+ k_2 \sum_{j=1}^{k'_1 - 1} (k'_1 - j)\text{tr}(g^j(g^{-1}dg)g^{-j}g^{-1}dg)_{g = h_2 f h_1^{-1}} $$

(4)

From the consideration of the global issues [9–11] it is deduced in [1] that $f = \exp \frac{\pi i \lambda}{\kappa}$, where $\lambda$ is an integral weight of G Lie algebra, and $\kappa = \text{lcm}(k_1, k_2)$.

Comparing the formulæ (1) and (2), describing generalized and usual permutation branes respectively one can deduce that the generalized branes preserve less symmetry. The usual permutation branes preserve two different twisted adjoint actions:

$$ (g_1, g_2) \to (mg_1, g_2 m^{-1}), \ (g_1, g_2) \to (g_1 m^{-1}, mg_2), $$

(5)

while the generalized branes preserve only the diagonal subgroup:

$$ (g_1, g_2) \to (mg_1 m^{-1}, mg_2 m^{-1}) $$

(6)
Using the form (4) boundary equations of motion have been analyzed in [1] and indeed found to be
\[ J_1 + J_2 = \bar{J}_1 + \bar{J}_2. \] (7)

Motivated by the recently established connection between the permutation D-branes in the minimal $N = 2$ supersymmetric models and the Landau-Ginzburg superpotential factorisation [12–14] it was suggested in [1] that the generalized permutation branes should exist also for cosets of the form $G_{k_1}/H \times G_{k_2}/H$ for the different levels $k_1$ and $k_2$. To produce such D-brane from the generalized D-branes (1) one should modify them in a way to preserve adjoint action of product $H \times H$:
\[ (g_1, g_2) \rightarrow (m_1 g_1 m_1^{-1}, m_2 g_2 m_2^{-1}) \] (8)
where $m_i \in H$. This problem is remained unsolved in [1].

Aim of this paper is to study the modifications of the ansatz (1) giving rise to the required symmetry. In the next section we show that one can propose two kinds of boundary conditions possessing with the necessary symmetries.

I. $(g_1, g_2) = \{(h_1 f h_2^{-1})^{k'_2} p t p^{-1}, \quad L^{-1}(h_2 f h_1^{-1})^{k'_1} n r n^{-1} L\}$ (9)
where $L, p, n \in H$ and run all the $H$ subgroup, $r$ and $t$ are fixed quantized elements of $H$. In other words we multiply both elements of the ansatz (1) by the (quantized) conjugacy classes of the subgroup $H$, and then smear the derived object along the adjoint action of $H$. The chain of the transformations:
\[ h_1 \rightarrow m_1 h_1, \quad h_2 \rightarrow m_1 h_2, \quad L \rightarrow m_1 L m_2^{-1}, \quad p \rightarrow m_1 p, \quad n \rightarrow m_1 n \] (10)
reproduces (8).

II. $(g_1, g_2) = \{(h_1 f h_2^{-1})^{k'_2} (s_1 l s_2^{-1})^{k'_2}, \quad L^{-1}(h_2 f h_1^{-1})^{k'_1} (s_2 l s_1^{-1})^{k'_1} L\}$ (11)
where $L, s_1, s_2 \in H$ run all the subgroup $H$ and $l$ is fixed quantized element. By words, we multiply the generalized permutation brane of $G_{k_1} \times G_{k_2}$ by the generalized permutation brane of $H_{k_3} \times H_{k_4}$ ($k_3 = x_e k_1$, $k_4 = x_e k_2$, $x_e$ is the embedding index of $H$ in $G$ [15]) and then again smear derived object along the adjoint action of $H$. We show in the next section that for $k_1 = k_2$ (9) and (11) are equivalent.
Using the Polyakov-Wiegmann identity

$$\omega^{WZW}(gh) = \omega^{WZW}(g) + \omega^{WZW}(h) - d(\text{tr}(g^{-1}dgdhh^{-1}))$$ \hspace{1cm} (12)

one obtains that the branes (9) satisfy to the condition (3) with the following boundary two-form:

$$\Omega(h_1, h_2, L, p, n) = \omega_C^{(f)}(h_1, h_2) + k_2\omega^{(2)}(C_2C_4, L) + k_1\omega^t(p) - k_1(\text{tr}(C_1^{-1}dC_1dC_3C_3^{-1})) - k_2(\text{tr}(C_2^{-1}dC_2dC_4C_4^{-1}))) + k_2\omega^t(n)$$ \hspace{1cm} (13)

where we denoted

- $$C_1 = (h_1fh_2^{-1})^{k_2}$$
- $$C_2 = (h_2fh_1^{-1})^{k_1}$$
- $$C_3 = ptp^{-1}$$
- $$C_4 = nrn^{-1}$$ \hspace{1cm} (14)

$$\omega_C^{(f)}$$ is the two-form given by the formula (4), $$\omega^t(p)$$ is the two-form found in [10,11]

$$\omega^t(p) = \text{tr}(p^{-1}dptp^{-1}dpt^{-1})$$ \hspace{1cm} (15)

and $$\omega^{(2)}$$ is the following useful two-argument two-form

$$\omega^{(2)}(g, U) = \text{tr}(dUU^{-1}(gdUU^{-1}g^{-1} + g^{-1}dg + dgg^{-1}))$$ \hspace{1cm} (16)

which we frequently encounter in this paper.

For an abelian subgroup $$H = U(1)$$, and equal levels $$k_1 = k_2$$ the brane (9) was studied in [16]. (In that paper this brane was written in the form $$(h_1fh_2^{-1}L^{-1}, h_2fh_1^{-1}L)$$, but after the redefinition $$L^{-1}h_2' = h_2$$ we derive (9).) It was checked for this case in [16], that the full Lagrangian with boundary term (13) enjoys with the symmetry (8). It was also shown in [16] that the geometry of this brane coincides with the shape corresponding to the permutation boundary state of the parafermions product. This serves for us as the hint that we found the correct solution.

One can also check that the branes (11) as well satisfy (3) with the two-form

$$\Omega(h_1, h_2, s_1, s_2, L) = \omega_C^{(f)}(h_1, h_2) + \omega_C^{(t)}(s_1, s_2) + k_2\omega^{(2)}(C_2C_4, L) - k_1(\text{tr}(C_1^{-1}dC_1dC_3C_3^{-1})) - k_2(\text{tr}(C_2^{-1}dC_2dC_4C_4^{-1})))$$ \hspace{1cm} (17)
where

\[C_5 = (s_1 s_2^{-1})^{k_2'}, \quad C_6 = (s_2 s_1^{-1})^{k_1'}. \quad (18)\]

The rest of the paper is organized as follows.

In the section 2 we obtain boundary conditions (9) and (11) by analyzing the action of the gauged WZW model on a world-sheet with boundary.

In the section 3 we consider these branes for product $SU(2)_{k_1}/U(1) \times SU(2)_{k_2}/U(1)$.

In the appendix A we deliver some algebraical calculations proving gauge invariance of action with boundary term given by (13).

In the appendix B we review different coordinate systems on $S^3$ sphere used in the section 3.

2 Lagrangian and symmetries

In this section we obtain the boundary conditions (9) and (11) by considering gauged WZW model action on a world-sheet with boundary along the way worked out in [17] and [18]. First of all we remind the bulk action of the gauged WZW model [19–23]:

\[S_{G/H}(g, A) = S^G + \frac{k_G}{2\pi} \int_{\Sigma} d^2 z \text{tr}\{A_z \partial_z g g^{-1} - A_z g^{-1} \partial_z g + A_z g A_z g^{-1} - A_z A_z\} \quad (19)\]

where

\[S^G = \frac{k_G}{4\pi} \left( \int_{\Sigma} d^2 z L^\text{kin} + \int_B \omega^\text{WZW}(g) \right) \quad (20)\]

is bulk WZW action, $L^\text{kin} = \text{tr}(\partial_z g \partial_z g^{-1})$, $\omega^\text{WZW} = \frac{1}{3} \text{tr}(d g g^{-1})^3$, $B$ is a three-dimensional manifold bounded by $\Sigma$, and $A$ is a gauge field taking values in the $H$ Lie algebra.

Defining

\[A_z = \partial_z \tilde{U} \tilde{U}^{-1}, \quad A_{\tilde{z}} = \partial_{\tilde{z}} U U^{-1} \quad (21)\]

we can write (19) in the form:

\[S^G_{G/H} (\tilde{g}, \tilde{h}) = S^G (\tilde{g}) - S^H (\tilde{h}) \quad (22)\]

where $\tilde{g} = U^{-1} g \tilde{U}$ and $\tilde{h} = U^{-1} \tilde{U}$. The level $k_H$ of the $S^H$ is related to $k_G$ through the embedding index $x_e$ of $H$ in $G$: $k_H = x_e k_G$ [15]. The expression (22)
is manifestly gauge invariant under the transformation:

\[ g \rightarrow mgm^{-1}, \quad U \rightarrow mU, \quad \tilde{U} \rightarrow m\tilde{U} \]  

(23)

For the case under consideration the bulk action is:

\[ S_{\text{bulk}}^{G/H}(\tilde{g}_1, \tilde{g}_2, \tilde{h}_1, \tilde{h}_2) = S^G(\tilde{g}_1) + S^G(\tilde{g}_2) - S^H(\tilde{h}_1) - S^H(\tilde{h}_2) \]  

(24)

where

\[ \tilde{g}_1 = U_1^{-1}g_1\tilde{U}_1 \]
\[ \tilde{g}_2 = U_2^{-1}g_2\tilde{U}_2 \]
\[ \tilde{h}_1 = U_1^{-1}\tilde{U}_1 \]
\[ \tilde{h}_2 = U_2^{-1}\tilde{U}_2 \]  

(25)

The levels \( k_3 \) and \( k_4 \) of the third and forth terms in (24), as explained above, are related to the levels \( k_1 \) and \( k_2 \) of the first and second terms respectively through the embedding index \( x_e \) of \( H \) in \( G \):

\[ k_3 = x_e k_1 \]
\[ k_4 = x_e k_2 . \]  

(26)

Consider the action (24) in the presence of boundary. We should specify boundary conditions. Let us choose for the sum of the first two actions the generalized permutation boundary conditions (1) and impose the arguments of the third and forth terms to take their values in the quantized conjugacy classes:

\[ \tilde{g}_1 = \left( (U_1^{-1}h_1)f(U_1^{-1}h_2)^{-1} \right)^{k_2} \]
\[ \tilde{h}_1 = \left( U_1^{-1}p \right)^{t^{-1}(U_1^{-1}p)^{-1}} \]
\[ \tilde{g}_2 = \left( (U_1^{-1}h_2)f(U_1^{-1}h_1)^{-1} \right)^{k_1} \]
\[ \tilde{h}_2 = \left( U_1^{-1}n \right)^{r^{-1}(U_1^{-1}n)^{-1}} \]  

(27)

It is easy to see that (27) brings to us conditions (9) for \( g_1 \) and \( g_2 \) with

\[ L = U_1U_2^{-1} \]  

(28)

Using the forms (1) and (15) one can write the full action with the boundary term:

\[ S = S_{\text{bulk}}^{G/H}(\tilde{g}_1, \tilde{g}_2, \tilde{h}_1, \tilde{h}_2) - \frac{1}{4\pi} \int_D \omega_C^{(f)}(U_1^{-1}h_1, U_1^{-1}h_2) - k_1\omega^{(t^{-1})}(U_1^{-1}p) - k_2\omega^{(r^{-1})}(U_1^{-1}n)\]  

(29)
where $\partial B = \Sigma + D$.

The action (29) is manifestly gauge invariant under the gauge transformation:

$$h_1 \rightarrow m_1 h_1, \quad h_2 \rightarrow m_1 h_2, \quad U_1 \rightarrow m_1 U_1, \quad U_2 \rightarrow m_2 U_2, \quad p \rightarrow m_1 p, \quad n \rightarrow m_1 n$$

implying, recalling the definition (28),

$$L \rightarrow m_1 L m_2^{-1}.$$  

(31)

We derived the transformation rules (10). Using the Polyakov-Wiegmann identities the action (29) can be written as:

$$S = S^G/H(g_1, A_1) + S^G/H(g_2, A_2) - \frac{1}{4\pi} \int_D \Omega$$  

(32)

where

$$\Omega = \omega_G^{(f)}(U_1^{-1} h_1, U_1^{-1} h_2) + k_1 \omega^{(t)}(U_1^{-1} p) + k_2 \omega^{(r)}(U_1^{-1} n) + k_1 \omega(g_1, U_1, \tilde{U}_1) + k_2 \omega(g_2, U_2, \tilde{U}_2)$$  

(33)

where

$$\omega(g_i, U_i, \tilde{U}_i) = \text{tr}(g_i^{-1} d g_i d \tilde{U}_i U_i^{-1} - d U_i U_i^{-1} d g_i g_i^{-1} - d U_i U_i^{-1} g_i d \tilde{U}_i U_i^{-1} g_i^{-1} + d U_i U_i^{-1} d \tilde{U}_i U_i^{-1})$$  

(34)

It is cumbersome but straightforward to check that the form (33) coincides with (13) with $L$ given by (28). The details of the calculations are delivered in the appendix A.

Boundary conditions (11) can be received in the same way, but now one should take the generalized boundary conditions as for the sum of the first two actions, as well for the sum of actions for the gauge groups:

$$\tilde{g}_1 = ((U_1^{-1} h_1) f(U_1^{-1} h_2)^{-1})^{k_2}$$

$$\tilde{h}_1 = ((U_1^{-1} s_2)^{-1}(U_1^{-1} s_1)^{-1})^{k_2}$$

$$\tilde{g}_2 = ((U_1^{-1} h_2) f(U_1^{-1} h_1)^{-1})^{k_1}$$

$$\tilde{h}_2 = ((U_1^{-1} s_1)^{-1}(U_1^{-1} s_2)^{-1})^{k_1}$$  

(35)

where we have taken into account that $\tilde{k} = \gcd(k_3, k_4) = x \gcd(k_1, k_2)$, and $k'_3 = k_3/\tilde{k} = k_1'$, $k'_4 = k_4/\tilde{k} = k_2'$.

One can easily check that these conditions are equivalent to (11) with again $L = U_1 U_2^{-1}$.
The full action with boundary term is:

\[ S = S_{\text{bulk}}^{G/H}(\bar{g} \bar{g}, \bar{h}, \bar{h}) - \frac{1}{4\pi} \int_D (\omega^{(f)}(U_1^{-1} h_1, U_1^{-1} h_2) - \omega^{(l)}(U_1^{-1} s_1, U_1^{-1} s_1)) \]  

where \( \partial B = \Sigma + D \). The action (36) is manifestly gauge invariant under the gauge transformation:

\[ h_1 \to m_1 h_1, \quad h_2 \to m_1 h_2, \quad U_1 \to m_1 U_1, \quad U_2 \to m_2 U_2, \quad s_1 \to m_1 s_1, \quad s_2 \to m_1 s_2 \]

implying, again,

\[ L \to m_1 L m_2^{-1}. \]

Using Polyakov-Wiegmann identities the action (36) can be written as:

\[ S = S^{G/H}(g_1, A_1) + S^{G/H}(g_2, A_2) - \frac{1}{4\pi} \int_D \Omega \]  

where

\[ \Omega = \omega^{(f)}(U_1^{-1} h_1, U_1^{-1} h_2) + \omega^{(l)}(U_1^{-1} s_1, U_1^{-1} s_2) + k_1 \omega(g_1, U_1, \tilde{U}_1) + k_2 \omega(g_2, U_2, \tilde{U}_2) \]

Repeating the same steps as outlined in the appendix A one can show that (40) coincides with (17).

Some comments:

Let us consider the branes (9) and (11) for equal levels \( k_1 = k_2 \), when permutation symmetry is restored:

\[ (g_1, g_2) = (C_1 C_3, L^{-1} C_2 C_4 L) = (h_1 f h_2^{-1} p t p^{-1}, L^{-1} h_1 f h_1^{-1} n r n^{-1} L) \]

\[ (g_1, g_2) = (C_1 C_5, L^{-1} C_2 C_6 L) = (h_1 f h_2^{-1} s_1 s_2^{-1}, L^{-1} h_2 f h_1^{-1} s_2 s_1^{-1} L) \]

By the redefinition

\[ h_2^{-1} C_5 = h_2', \quad L' = C_5^{-1} L \]

one can write the brane (42) in the form \((C_1, L^{-1} C_2 C_6 C_5 L)\). Taking into account that \( C_6 C_5 \) in this case is the usual conjugacy class, we see that at the point of the restored symmetry the family of the branes (42) coincides with the family (41).

Performing the same kind of redefinition in (41)

\[ h_2^{-1} C_3 = h_2', \quad L' = C_3^{-1} L \]
one can write all the branes (41) in the form

\[(g_1, g_2) = (C_1, L^{-1}C_2C_4C_3L)\]  \hspace{1cm} (45)

Presumably we can multiply both elements in (9) by the chain of conjugacy classes, as in [8], but for the case of \(k_1 = k_2\), when permutation symmetry is restored, we see, that one can cover all the family already multiplying just one of them. The conclusion is, that generically when \(k_1 \neq k_2\) we have two families of branes (9) and (11), which reduce to the branes of the form (45), when the permutation symmetry is restored.

3 Generalized permutation branes on \(SU(2)\) cosets

In this section we consider permutation branes on product of \(SU(2)/U(1)\) cosets. At the beginning we consider usual permutation brane for \(k_1 = k_2\) and show that the geometrical description given above coincide with the permutation boundary state [3] overlap with the graviton wave packet. Actually this calculation was performed in [16], but for the completeness and the reader’s convenience we repeat it (slightly generalized and with corrected typos) here. Then we elaborate the geometry of the simplest generalized permutation brane.

With the \(U(1)\) subgroup generated by \(\sigma_3\) the brane (9) for \(k_1 = k_2 = k\) takes the form:

\[\left. (g_1, g_2) \right|_{\text{brane}} = (fh_1^{-1}e^{i\alpha f_{\frac{k}{2}}}, e^{-i\alpha f_{\frac{k}{2}}}e^{i\frac{\pi M}{k}f_{\frac{k}{2}}}). \]  \hspace{1cm} (46)

where \(f = e^{i\hat{\psi} f_{\frac{k}{2}}}\), \(\hat{\psi} = \frac{2j\pi}{k}\), \(j = 0, \ldots, \frac{k}{2}\), and \(M\) is an integer. The factor \(e^{i\frac{\pi M}{k}f_{\frac{k}{2}}}\) reflects \(\mathbb{Z}_k\) symmetry of an abelian coset [24]. One can multiply with this factor also the first element in (46), but as explained above, performing the redefinition (44), one gets again (46). We see that all the branes are labelled by two indices \(\hat{\psi}\) and \(M\), exactly as the permutation states of the parafermions product. The elements \(g_1\) and \(g_2\) belong to the brane surface if the following equation admits a solution for the parameter \(\alpha\),

\[\text{tr} \left( g_1 e^{-i\alpha f_{\frac{k}{2}}} g_2 e^{i\alpha f_{\frac{k}{2}}} e^{-i\frac{\pi M}{k}f_{\frac{k}{2}}} \right) = 2 \cos \hat{\psi}. \]  \hspace{1cm} (47)

This equation can be further elaborated in the Euler coordinates, reviewed in the appendix B. The formulae for the Euler angles of a product of two elements \(\hat{g} = g_1g_2\) are given in [25].
\[
\begin{align*}
\cos \hat{\theta} &= \cos \tilde{\theta}_1 \cos \tilde{\theta}_2 - \sin \tilde{\theta}_1 \sin \tilde{\theta}_2 \cos(\chi_2 + \varphi_1), \quad (48) \\
e^{i\hat{\phi}} &= e^{\frac{i\chi_1 + \phi_2}{2}} \left( \cos \frac{\tilde{\theta}_1}{2} \cos \frac{\tilde{\theta}_2}{2} e^{i\chi_1 + \phi_2} - \sin \frac{\tilde{\theta}_1}{2} \sin \frac{\tilde{\theta}_2}{2} e^{-i\chi_2 + \phi_1} \right). \quad (49)
\end{align*}
\]

where the hatted variables refer to the product \( \hat{g} \).

Denoting by \( \hat{\Theta}, \hat{\Phi} \) Euler angles \( \hat{\theta} \) and \( \hat{\phi} \) of the product \( g_1 e^{-i\alpha} g_2 \) and using (48) and (49) we can rewrite (47) as

\[
\cos \frac{\hat{\Theta}}{2} \cos(\gamma/2 - \xi/2 - \hat{\phi}_1 - \hat{\phi}_2 + \frac{\pi M}{2k}) = \cos \hat{\psi}, \quad (50)
\]

where

\[
\cos \hat{\Theta} = \cos \tilde{\theta}_1 \cos \tilde{\theta}_2 - \sin \tilde{\theta}_1 \sin \tilde{\theta}_2 \cos \gamma, \quad (51)
\]

and we have introduced new labels \( \gamma = \chi_2 + \varphi_1 - \alpha \) and \( \xi/2 = \hat{\Phi} - \frac{\chi_1 + \phi_2}{2} \). The variables \( \xi \) and \( \gamma \) are related to each other by the equation

\[
e^{i\frac{\xi}{2}} = \frac{1}{\cos \frac{\phi}{2}} \left( \cos \frac{\tilde{\theta}_1}{2} \cos \frac{\tilde{\theta}_2}{2} e^{i\frac{\xi}{2}} - \sin \frac{\tilde{\theta}_1}{2} \sin \frac{\tilde{\theta}_2}{2} e^{-i\frac{\xi}{2}} \right). \quad (52)
\]

Let us recall that the vectorial gauging of \( U(1) \) symmetry is corresponding to the translation of \( \phi \) and the resulting target space of the \( SU(2)_k/U(1) \) model, derived after the gauge fixing \( \phi = 0 \) and integrating out of the gauge field, is the two-dimensional disc, parameterized by \( \theta \) and \( \bar{\phi} \). In the case of product the target space is parameterized by \( \theta_1, \theta_2, \bar{\phi}_1, \bar{\phi}_2 \). Hence the brane consists of those points for which equation (50) admits a solution for \( \gamma \). \( \Theta \) and \( \xi \) are considered here as the complicated functions of \( \hat{\theta}_1, \hat{\theta}_2 \) and \( \gamma \) given by (51) and (52) respectively. For \( \hat{\psi} = 0 \) there are additional constraints, which imply that in this case the brane is two dimensional and given by the equations

\[
\hat{\theta}_1 = -\hat{\theta}_2, \quad \bar{\phi}_1 = -\bar{\phi}_2 + \frac{\pi M}{2k}. \quad (53)
\]

Now we calculate the effective geometry corresponding to the permutation boundary state [3]:

\[
|L, M\rangle = \sum_{j,m} S_{Lj} e^{i\pi Mm/k} \sum_{N_1, N_2} |j, m, N_1\rangle_1 \otimes |j, m, N_1\rangle_2 \otimes |j, m, N_2\rangle_2 \otimes |j, m, N_2\rangle_1
\]

(54)
where $S_{Lj}$ is matrix of the modular transformation of $SU(2)_k$

$$S_{Lj} = \sqrt{\frac{2}{k+2}} \sin \left( \frac{(2L+1)(2j+1)\pi}{k+2} \right). \quad (55)$$

To obtain the effective geometry, one should compute the overlap $\langle \theta_1, \tilde{\phi}_1, \theta_2, \tilde{\phi}_2 | L, M \rangle$. At the beginning we should find the wave-functions of the parafermion disc theory [24]:

$$\Psi_{j,m}(\theta, \tilde{\phi}) = \langle \theta, \tilde{\phi} | j,m \rangle \quad (56)$$

The wave-functions of the disc are the $SU(2)$ wave-functions that are invariant under translation of $\phi$. (Note that in [24] axial gauging is considered, and as a consequence the roles of $\phi$ and $\tilde{\phi}$ are interchanged). Recalling that the $SU(2)$ wave-functions are the normalized Wigner functions

$$\sqrt{2j+1} D_{jnm}(g(\vec{\theta})) = \sqrt{2j+1} e^{-i(n\chi + m\phi)} d_{jnm}(\cos \tilde{\Theta}), \quad (57)$$

we see that the function on disc are those of them with $m = n$. Using that for the large $k$

$$\frac{S_{Lj}}{S_{0j}} \sim \frac{(k+2)}{\pi(2j+1)} \sin[(2j+1)\hat{\psi}], \quad (58)$$

where $\hat{\psi} = \frac{(2L+1)\pi}{k+2}$, one obtains that in the large-$k$ limit the overlap reduces to

$$\langle \tilde{\theta}_1, \tilde{\theta}_2 | L, M \rangle \sim \sum_j \sum_m \sin[(2j+1)\hat{\psi}] e^{i\pi Mm/k} D_{jmm}(g_1(\tilde{\theta}_1)) D_{jmm}(g_2(\tilde{\theta}_2)). \quad (59)$$

It is known [25] that $d_{jnm}$ are satisfying the relation (note that there is no summation assumed for the repeated indices)

$$d_{jmm}(\cos \tilde{\Theta})d_{jmm}(\cos \tilde{\Theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi(\gamma - \xi)} d_{jmm}(\cos \tilde{\Theta}) d\gamma, \quad (60)$$

The functions $\Theta$ and $\xi$ are functions of $\tilde{\theta}_1, \tilde{\theta}_2$ and $\gamma$ defined in equations (51) and (52). Using (60) the overlap of the boundary state with the bulk probe can be written as

$$\langle \tilde{\theta}_1, \tilde{\theta}_2 | L, M \rangle \sim \sum_j \sum_m \int_{-\pi}^{\pi} \sin[(2j+1)\hat{\psi}] e^{i\xi(\gamma - \xi - 2\tilde{\phi}_1 - 2\tilde{\phi}_2 + \pi M/k)} d_{jmm}(\cos \tilde{\Theta}) d\gamma. \quad (61)$$

Now using that $\sum_m D_{jmm}(g) = \frac{\sin(2j+1)\psi}{\sin \psi}$, where $\psi$ is the angle of the standard metric and defined by the relation $Tr g = 2 \cos \psi$, and the completeness of $\sin[(2j+1)\psi]$ on the interval $[0, \pi]$ we get

$$\langle \tilde{\theta}_1, \tilde{\theta}_2 | L, M \rangle \sim \int_{-\pi}^{\pi} \frac{\delta(\psi - \hat{\psi})}{\sin \psi} d\gamma, \quad (62)$$
where

$$\cos \psi = \cos \frac{\tilde{\Theta}}{2} \cos (\gamma/2 - \xi/2 - \tilde{\phi}_1 - \tilde{\phi}_2 + \frac{\pi M}{2k})$$  \hspace{1cm} (63)$$

From this equation it follows that the brane consist of all those points for which the expression in the argument of the $\delta$ function has a root for $\gamma$. This is the same condition as the one coming from equation (50), obtained in the Langrangian approach.

Let us turn now to the generalized permutation brane on a product $SU(2)_{k_1}/U(1) \times SU(2)_{k_2}/U(1)$. The branes (9) and (11) for abelian case take forms

$$H = \{(h_1 f h_2^{-1})^{k_2} e^{\frac{n M_1}{k_1} \sigma_3^2}, \quad e^{i \alpha \sigma_3^2} (h_2 f h_1^{-1})^{k_1} e^{-i \alpha \sigma_3^2} e^{i \frac{n M_2}{k_2} \sigma_3^2}\}$$  \hspace{1cm} (64)$$

$$H = \{(h_1 f h_2^{-1})^{k_2} e^{i \beta \sigma_3^2} e^{i \frac{n M_1}{k_1} \sigma_3^2}, \quad e^{i \alpha \sigma_3^2} (h_2 f h_1^{-1})^{k_1} e^{-i \beta \sigma_3^2} e^{-i \frac{n M_2}{k_2} \sigma_3^2}\}$$  \hspace{1cm} (65)$$

We see that for $k_1 \neq k_2$ we have much bigger variety of branes, which all degenerate to (46) when $k_1 = k_2$.

Now we describe geometry of the generalized permutation branes (64) and (65) for the simplest case $f = e$. For this case the branes are:

$$(g_1, g_2) = (g^{k_2} e^{i \frac{n M_1}{k_1} \sigma_3^2}, \quad e^{i \alpha \sigma_3^2} g^{k_1} e^{-i \frac{n M_2}{k_2} \sigma_3^2})$$  \hspace{1cm} (66)$$

$$(g_1, g_2) = (g^{k_2} e^{i \beta \sigma_3^2} e^{i \frac{n M_1}{k_1} \sigma_3^2}, \quad e^{i \alpha \sigma_3^2} g^{k_1} e^{-i \beta \sigma_3^2} e^{-i \frac{n M_2}{k_2} \sigma_3^2})$$  \hspace{1cm} (67)$$

To elaborate the geometry of (66) and (67) we first recall the useful fact mentioned in [1] that the element $g^n$ in the coordinates (88) has the same angles $\xi$ and $\eta$ as $g$ but $\psi$ has to be replaced by $n \psi$. Then we need the formulae of transformation from the coordinates (88) to the Euler coordinates:

$$\tan \tilde{\phi} = \tan \psi \cos \xi$$

$$\sin \theta = \sin \psi \sin \xi$$  \hspace{1cm} (68)$$

Taking finally into account that the adjoint action of $U(1)$ does not change $\theta$ and $\tilde{\phi}$ angles of an element one can describe the geometry of (66) and (67) by the following equations of embedding respectively:

$$\tan \left(\frac{\phi_1 - \pi M_1}{2k_1}\right) = \tan k_2' \psi \cos \xi$$

$$\sin \theta_1 = \sin k_2' \psi \sin \xi$$

$$\tan \left(\frac{\phi_2 - \pi M_2}{2k_2}\right) = -\tan k_1' \psi \cos \xi$$

$$\sin \theta_2 = -\sin k_1' \psi \sin \xi$$  \hspace{1cm} (69)$$
\[
\tan \left( \tilde{\phi}_1 - k'_2 \beta - \frac{\pi M_1}{2k_1} \right) = \tan k'_2 \psi \cos \xi \\
\sin \theta_1 = \sin k'_2 \psi \sin \xi \\
\tan \left( \tilde{\phi}_2 + k'_1 \beta - \frac{\pi M_2}{2k_2} \right) = - \tan k'_1 \psi \cos \xi \\
\sin \theta_2 = - \sin k'_1 \psi \sin \xi
\] (70)

The brane (66) is two-dimensional with the world-volume coordinates \((\psi, \xi)\), whereas the brane (67) is three-dimensional with the world-volume coordinates \((\psi, \xi, \beta)\). It is easy to check that when \(k_1 = k_2\), and \(k'_1 = k'_2 = 1\), (69) and (70) reduce to (53). To find various other properties of the branes like mass, spectrum \textit{et c.}, is left for the future work.
A Details of calculations

To show that (33) coincides with (13) at the beginning we solve (27): 

\[ \tilde{U}_1 = C_3^{-1}U_1, \quad (71) \]
\[ \tilde{U}_2 = U_2U_1^{-1}C_4^{-1}U_1, \quad (72) \]
\[ g_1 = C_1C_3 \quad (73) \]
\[ g_2 = L^{-1}C_2C_4L = (U_1U_2^{-1})^{-1}C_2C_4(U_1U_2^{-1}) \quad (74) \]

where \( C_1, C_2, C_3, C_4 \) are defined in (14).

Inserting (71) and (73) in (34) for \( i = 1 \) one can show that 

\[ \omega(g_1, U_1, \tilde{U}_1) = -\text{tr}(C_1^{-1}dC_1dC_3C_3^{-1}) - \omega(2)(C_1, U_1) - \omega(2)(C_3, U_1) \quad (75) \]

Inserting (72) and (74) in (34) for \( i = 2 \) one obtains

\[ \omega(2)(g_2, U_2, \tilde{U}_2) = -\omega(2)(C_2, U_1) - \omega(2)(C_4, U_1) + \omega(2)(C_2C_4, U_1U_2^{-1}) - \text{tr}(C_2^{-1}dC_2dC_4C_4^{-1}) \quad (76) \]

To deal with the second and third terms in (33) we recall the following useful identity derived in [17]:

\[ \omega^{(l)}(U_1^{-1}p) = \omega^{(l)}(p) + \omega^{(2)}(C_3, U_1) \quad (77) \]

It was shown in [17] that (77) guarantees that the full WZW Lagrangian on a world-sheet with boundary, with boundary conditions specified by the conjugacy class \( C_3 \), enjoys with the full diagonal subalgebra:

\[ g(z, \bar{z}) \rightarrow k_L(z)g(z, \bar{z})k_R^{-1}(\bar{z}), \quad k_L|_{\text{boundary}} = k_R|_{\text{boundary}} \quad (78) \]

The last ingredient which we need is the formula giving transformation properties of \( \omega_C \):

\[ \omega_C^{(f)}(U_1^{-1}h_1, U_1^{-1}h_2) = \omega_C^{(f)}(h_1, h_2) + k_1\omega^{(2)}(C_1, U_1) + k_2\omega^{(2)}(C_2, U_1) \quad (79) \]

It is possible to derive this formula using the definition (4). But this formula is nothing else as the global form of the equation (14), reflecting symmetry properties of the generalized permutation brane. It is straightforward to check that (79) guarantees that the WZW Lagrangian on a world-sheet with boundary with the
boundary conditions given by the generalized permutation brane enjoys with the symmetry
\[
g_1(z, \bar{z}) \rightarrow k_L(z) g_1(z, \bar{z}) k^{-1}_R(\bar{z}), \quad g_2(z, \bar{z}) \rightarrow h_L(z) g_2(z, \bar{z}) h^{-1}_R(\bar{z}),
\]

\[
k_L|_{\text{boundary}} = k_R|_{\text{boundary}} = h_L|_{\text{boundary}} = h_R|_{\text{boundary}} \quad (80)
\]

Inserting (75), (76), (77) and (79) and in (33) one ends up with (13).

**B Various coordinate systems for the sphere and relations between them**

A three-sphere $S^3$ is a group manifold of the $SU(2)$ group. A generic element in this group can be written as

\[
g = X_0\sigma_0 + i(X_1\sigma_1 + X_2\sigma_2 + X_3\sigma_3) = \begin{pmatrix} X_0 + iX_3 & X_2 + iX_1 \\ -(X_2 - iX_1) & X_0 - iX_3 \end{pmatrix} \quad (81)
\]

subject to condition that the determinant is equal to one

\[
X_0^2 + X_1^2 + X_2^2 + X_3^2 = 1. \quad (82)
\]

The metric on $S^3$ can be written in the following three ways, which will be used in the main text. Firstly, using the Euler parametrization of the group element we have

\[
g = e^{i\chi \sigma_3^3} e^{i\tilde{\theta} \sigma_3^1} e^{i\phi \sigma_3^2}
\]

\[
ds^2 = \frac{1}{4} ((d\chi + \cos \tilde{\theta} d\phi)^2 + d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\phi^2) \quad (83)
\]

The ranges of coordinates are $0 \leq \tilde{\theta} \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \chi \leq 4\pi$.

Secondly, we can use coordinates that are analogue to the global coordinate for $AdS_3$

\[
X_0 + iX_3 = \cos \theta e^{i\phi}, \quad X_2 + iX_1 = \sin \theta e^{i\phi} \quad (85)
\]

\[
ds^2 = d\theta^2 + \cos^2 \theta d\tilde{\phi}^2 + \sin^2 \theta d\phi^2. \quad (86)
\]

The relation between the metrics (83) and (85) is given by

\[
\chi = \tilde{\phi} + \phi, \quad \varphi = \tilde{\phi} - \phi, \quad \theta = \frac{\tilde{\theta}}{2}. \quad (87)
\]
The ranges of coordinates are \(-\pi \leq \tilde{\phi}, \phi \leq \pi\) and \(0 \leq \theta \leq \frac{\pi}{2}\).

Thirdly, the standard metric on \(S^3\) is given by (\(\vec{n}\) is a unit vector on \(S^2\))

\[
g = e^{2i\psi \frac{\vec{n} \cdot \vec{\sigma}}{2}}, \quad ds^2 = d\psi^2 + \sin^2 \psi (d\xi^2 + \sin^2 \xi d\eta^2) \tag{88}
\]

\[
X_0 + iX_3 = \cos \psi + i \sin \psi \cos \xi, \quad X_2 + iX_1 = \sin \psi \sin \xi e^{i\eta}. \tag{89}
\]

The ranges of the coordinates are \(0 \leq \psi, \xi \leq \pi\) and \(0 \leq \eta \leq 2\pi\).
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