A uniqueness theorem for entire functions

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Abstract

Let $G(k) = \int_0^1 g(x)e^{kx}dx$, $g \in L^1(0,1)$. The main result of this paper is the following theorem.

**Theorem.** If $\limsup_{k \to +\infty} |G(k)| < \infty$, then \(g = 0\). There exists \(g \not\equiv 0, g \in L^1(0,1)\), such that $G(k_j) = 0$, $k_j < k_{j+1}$, $\lim_{j \to \infty} k_j = \infty$, $\lim_{k \to \infty} |G(k)|$ does not exist, $\limsup_{k \to +\infty} |G(k)| = \infty$. This $g$ oscillates infinitely often in any interval $[1-\delta,1]$, however small $\delta > 0$ is.

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1 Introduction

Let $g \in L^1(0,1)$ and $G(k) = \int_0^1 g(x)e^{kx}dx$. The function $G$ is an entire function of the complex variable $k$. It satisfies for all $k \in \mathbb{C}$ the estimate

$$|G(k)| \leq c_0e^{|k|}, \quad c_0 := \int_0^1 |g(x)|dx,$$

(1)

it is bounded for $\Re k \leq 0$:

$$|G(k)| \leq c_0, \quad \Re k \leq 0,$$

(2)

and it is bounded on the imaginary axis:

$$\sup_{\tau \in \mathbb{R}} |G(i\tau)| \leq c_0,$$

(3)

where $\mathbb{R}$ denotes the real axis.
It is not difficult to prove that if $g$ keeps sign in any interval $[1 - \delta, 1]$, and $\int_{1-\delta}^{1} |g(x)|\,dx > 0$ then $\lim_{k \to \infty} |G(k)| = \infty$. By $\infty$ we mean $+\infty$ in this paper.

However, there exists a $g \neq 0$, $g \in L^1(0, 1)$, such that $\lim_{k \to \infty} |G(k)|$ does not exist, but $\limsup_{k \to \infty} |G(k)| = \infty$. This $g$ changes sign (oscillates) infinitely often in any interval $[1 - \delta, 1]$, however small $\delta > 0$ is, and the corresponding $G(k)$ changes sign infinitely often in any neighborhood of $+\infty$, i.e., it has infinitely many isolated positive zeros $k_j$, which tend to $\infty$.

Our main result is:

**Theorem 1.1.** If $c_1 := \limsup_{k \to \infty} |G(k)| < \infty$, then $g = 0$. There exists $g \neq 0$, $g \in L^1(0, 1)$, such that $\limsup_{k \to \infty} |G(k)| = \infty$, $\lim_{k \to \infty} |G(k)|$ does not exist, $G(k_j) = 0$, where $k_j < k_{j+1}$, $\lim_{j \to \infty} k_j = \infty$, and $g$ oscillates infinitely often in any interval $[1 - \delta, 1]$, however small $\delta > 0$ is.

In Section 2 proofs are given. The main tools in the proofs are a Phragmen-Lindelöf (PL) theorem (see, e.g.,[1], [3]), and a construction, developed in [2] for a study of resonances in quantum scattering on a half line.

For convenience of the reader we formulate the Phragmen-Lindelöf theorem, used in Section 2. A proof of this theorem can be found, e.g., in [1].

By $\partial Q$ the boundary of the set $Q$ is denoted, $M > 0$ is a constant, and $0 < \alpha < 2\pi$.

**Theorem (PL).** Let $G(k)$ be holomorphic in an angle $Q$ of opening $\frac{\pi}{\alpha}$ and continuous up to its boundary. If $\sup_{k \in \partial Q} |G(k)| \leq M$, and the order $\rho$ of $G(k)$ is less than $\alpha$, i.e.,

$$\rho := \limsup_{r \to \infty} \frac{\ln \ln \max_{|k|=r} |G(k)|}{\ln r} < \alpha,$$

then $\sup_{k \in Q} |G(k)| \leq M$.

In Section 2 this Theorem will be used for $\alpha = \frac{\pi}{2}$ and $\rho = 1 < \alpha$.

### 2 Proofs

The proof of Theorem 1.1 is based on two lemmas, and the conclusion of Theorem 1.1 follows immediately from these lemmas.

**Lemma 2.1.** If

$$\limsup_{k \to \infty} |G(k)| < \infty,$$

(4)
then
\[ g = 0. \]  

**Proof.** The entire function \( G \) in the first quadrant \( Q_1 \) of the complex plane \( k \), that is, in the region \( 0 \leq \arg k \leq \frac{\pi}{2}, |k| \in [0, \infty) \), satisfies estimate (1) and is bounded on the boundary of \( Q_1 \):
\[
\sup_{k \in \partial Q_1} |G(k)| \leq \max(c_0, c_1) := c_2. \tag{6}
\]

By a Phragmen-Lindelöf theorem (see, e.g., [3], p.276) one concludes that
\[
\sup_{k \in Q_1} |G(k)| \leq c_2. \tag{7}
\]

A similar argument shows that
\[
\sup_{k \in Q_m} |G(k)| \leq c_2, \quad m = 1, 2, 3, 4, \tag{8}
\]
where \( Q_m \) are quadrants of the complex \( k \)-plane. Consequently, \( |G(k)| \leq c_2 \) for all \( k \in \mathbb{C} \), and, by the Liouville theorem, \( G(k) = \text{const} \). This \text{const} is equal to zero, because, by the Riemann-Lebesgue lemma, \( \lim_{r \to \infty} G(i\tau) = 0 \).

If \( G = 0 \), then \( g = 0 \) by the injectivity of the Fourier transform. Lemma 2.1 is proved.

**Lemma 2.2.** There exists \( g \neq 0, g \in C^\infty(0, 1) \), such that
\[
\lim_{k \to \infty} \sup_{k \to \infty} |G(k)| = \infty,
\]
\[
\lim_{k \to \infty} |G(k)| \text{ does not exist, } G(k_j) = 0, \text{ where } k_j < k_{j+1}, \lim_{j \to \infty} k_j = \infty,
\]
and \( g \) oscillates infinitely often in any interval \([1-\delta, 1]\), however small \( \delta > 0 \) is.

**Proof.** Let \( \Delta_j = [x_j, x_{j+1}], 0 < x_j < x_{j+1} < 1, \lim_{j \to \infty} x_j = 1, j = 2, 3, ..., \), \( f_j(x) \geq 0, f_j \in C_0^\infty(\Delta_j), \int_{\Delta_j} f_j(x)dx = 1, \) and \( \epsilon_j > 0, \epsilon_{j+1} < \epsilon_j \), be a sequence of numbers such that \( f(x) \in C^\infty(0, 1) \), where
\[
f(x) := \sum_{j=2}^{\infty} \epsilon_j f_j(x). \tag{9}
\]

Define
\[
g(x) := \sum_{j=2}^{\infty} (-1)^j \epsilon_j f_j(x), \quad g \neq 0. \tag{10}
\]
The function \( g \) oscillates infinitely often in any interval \([1 - \delta, 1]\), however small \( \delta > 0 \) is.

Let

\[
G(k) := \sum_{j=2}^{\infty} (-1)^j \epsilon_j G_j(k), \quad G_j(k) := \int_{\Delta_j} f_j(x) e^{kx} dx.
\]  

(11)

Note that \( G_j(k) > 0 \) for all \( j = 2, 3, \ldots \) and all \( k > 0 \), and

\[
\lim_{k \to \infty} \frac{G_m(k)}{G_j(k)} = \infty, \quad m > j,
\]  

(12)

because intervals \( \Delta_j \) and \( \Delta_m \) do not intersect if \( m > j \).

Fix an arbitrary small number \( \omega > 0 \).

Let us construct a sequence of numbers

\[
b_j > 0, \quad b_{j+1} > b_j, \quad \lim_{j \to \infty} b_j = \infty,
\]  

such that

\[
G(b_{2m}) \geq \omega, \quad G(b_{2m+1}) \leq -\omega, \quad m = 1, 2, \ldots
\]  

(13)

This implies the existence of \( k_j \in (b_{2m}, b_{2m+1}) \), such that

\[
G(k_j) = 0, \quad k_j < k_{j+1}, \quad \lim_{j \to \infty} k_j = \infty.
\]

Moreover,

\[
\limsup_{k \to \infty} |G(k)| = \infty,
\]  

(14)

because otherwise \( G = 0 \) by Lemma 1, so \( g = 0 \) contrary to the construction, see (10).

Let us choose \( b_2 > 0 \) arbitrary. Then \( G_2(b_2) > 0 \). Making \( \epsilon_3 > 0 \) smaller, if necessary, one gets

\[
\epsilon_2 G_2(b_2) - \epsilon_3 G_3(b_2) \geq \omega.
\]  

(15)

Choosing \( b_3 > b_2 \) sufficiently large, one gets

\[
\epsilon_2 G_2(b_3) - \epsilon_3 G_3(b_3) \leq -\omega.
\]  

(16)

This is possible because of (12).
Suppose that $b_2, \ldots, b_{2m+1}$ are constructed, so that

$$
\sum_{j=2}^{2m} (-1)^j \epsilon_j G_j(b_{2p}) \geq \omega, \quad p = 1, 2, \ldots, m,
$$

(17)

and

$$
\sum_{j=2}^{2m+1} (-1)^j \epsilon_j G_j(b_{2p+1}) \leq -\omega, \quad p = 1, \ldots, m.
$$

(18)

Let $m \to \infty$. Then

$$
\lim_{m \to \infty} \sum_{j=2}^{2m} (-1)^j \epsilon_j G_j(k) = \lim_{m \to \infty} \sum_{j=2}^{2m+1} (-1)^j \epsilon_j G_j(k) = G(k),
$$

(19)

where $G(k)$ is defined in (11), and the convergence in (19) is uniform on compact subsets of $[0, \infty)$. Therefore inequalities (13) hold. Lemma 2.2 is proved. □

Theorem 1 follows from Lemmas 1 and 2. □

References

[1] G.Polya, G.Szegö, Problems and theorems in analysis, Springer-Verlag, Berlin, 1983.

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[3] W.Rudin, Real and complex analysis, McGraw Hill, New York, 1974.