The Hoop Conjecture for Black Rings

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ABSTRACT

A precise formulation of the hoop conjecture for four-dimensional spacetimes proposes that the Birkhoff invariant \(\beta\) for an apparent horizon in a spacetime with mass \(M\) should satisfy \(\beta \leq 4\pi M\). The invariant \(\beta\) is the least maximal length of any sweepout of the 2-sphere apparent horizon by circles. An analogous conjecture in five spacetime dimensions was recently formulated, asserting that the Birkhoff invariant \(\beta\) for \(S^1 \times S^1\) sweepouts of the apparent horizon should satisfy \(\beta \leq \frac{16}{3}\pi M\). Although this hoop inequality was formulated for conventional five-dimensional black holes with 3-sphere horizons, we show here that it is also obeyed by a wide variety of black rings, where the horizon instead has \(S^2 \times S^1\) topology.
1 Introduction

Forty years ago, Thorne formulated the Hoop Conjecture, proposing that in four dimensions an horizon forms when and only when a mass $M$ is compacted into a region whose circumference $C$ in every direction satisfies the inequality $C \leq 4\pi M$ [1]. It is easily seen that the Schwarzschild black hole saturates this bound, and thus the conjecture would imply that other black holes of a given mass, such as those carrying charge or angular momentum, could not be “larger” than the Schwarzschild black hole. One of the problems with testing the validity of the hoop conjecture is to make rigorous the notion of the circumferences of the region in which the mass is confined. In an attempt to do this, Gibbons reformulated the hoop conjecture in terms of a quantity called the Birkhoff invariant [2]. He considered a Cauchy surface $\Sigma$ containing an outermost marginally trapped surface or apparent horizon $S$ with induced metric $g$, and constructed the Birkhoff invariant $\beta(g)$ for the pair $\{S, g\}$. This is defined by considering foliations of the topological 2-sphere $S$, with its metric $g$, as topological circles that grow from a point (a “north pole”) to a largest circumference (an “equator”) and then shrink to a point again (a “south pole”). The Birkhoff invariant $\beta(g)$ is then given by the smallest value amongst all the equitorial circumferences that can be achieved by foliating $S$ in all possible ways. The Gibbons reformulation of the hoop conjecture in four dimensions then asserts that

$$\beta(g) \leq 4\pi M,$$

with equality for the Schwarzschild black hole. The validity of this conjecture was demonstrated for various cases in [2], including the charged and rotating Kerr-Newman black hole.

Further tests of the inequality (1.1) were performed in [3], where it was shown to be satisfied by more general four-dimensional black holes, including those with a cosmological constant, and various rotating (multi)charged black holes in ungauged and gauged supergravities. Generalisations of the hoop conjecture to higher spacetime dimensions were also investigated in [3]. It becomes necessary then to find some appropriate definition of a higher-dimensional “hoop,” and then to give a precise definition of how its size is to be calculated. Since a conventional black hole in $D$ spacetime dimensions has an horizon with the topology of a $(D-2)$-sphere, one natural generalisation of a hoop would be to consider foliating the horizon with $(D-3)$-spheres, and define the analogous Birkhoff invariant as the volume of the smallest equitorial $(D-3)$-sphere amongst all the possible such foliations. There are, however, other natural possibilities. For example, in $D = 5$ spacetime dimensions, one could instead foliate a 3-sphere horizon with $S^1 \times S^1$ Clifford torii, and consider the Birkhoff invariant defined as the smallest achievable “equitorial” $S^1 \times S^1$ area amongst all possible such foliations. These, and other possible formulations of hoop inequalities were explored in [3], and tests of the conjectures were performed for a variety of rotating and charged black holes in higher dimensional gravities and supergravities. All the $D$-dimensional black hole examples studied in [3] had the “conventional” $S^{D-2}$ horizon topology.

In this paper, we shall study one of these hoop conjectures in five-dimensional spacetime, but applied now to the black holes known as black rings, which have horizon topology $S^2 \times S^1$, rather than $S^3$ [4]. The formulation that is particularly appropriate in this case uses the foliation by $S^1 \times S^1$ tori mentioned above. Now, however, instead of foliating 3-sphere horizons, the $S^1 \times S^1$ torii are to be thought of as foliating the $S^2 \times S^1$ horizons of the black rings. Thus one factor in the torus will be the circle of a foliation of $S^2$, while the other $S^1$ factor in the torus will be the $S^1$
factor in the horizon. Intriguingly, although the conjectured inequality for $S^1 \times S^1$ sweepouts in [3] was formulated only with five-dimensional black holes with $S^3$ horizon topologies in mind, we shall see that the identical inequality is satisfied also by all the black-ring solutions that we have investigated.

2 Hoop Inequality with $S^1 \times S^1$ Sweepouts

It was conjectured in [3] that the Birkhoff invariant $\beta(g)$ for $S^1 \times S^1$ sweepouts of an apparent horizon in a five-dimensional spacetime satisfies the inequality

$$\beta(g) \leq \frac{16\pi}{3} M,$$

(2.1)

where $M$ is the mass. The bound is exactly saturated by the five-dimensional Schwarzschild solution. It was shown in [3] that the bound is obeyed by the asymptotically flat rotating 3-charge black holes of ungauged supergravity [5], and by the asymptotically AdS rotating charged black holes of five-dimensional minimal gauged supergravity [6]. These, of course, all have horizons that are topologically $S^3$.

Here, we shall examine the inequality (2.1) in the case of the original uncharged black ring with a single rotation parameter [4], its charged generalisation [7], and the ring with two rotation parameters that was obtained in [8], together with its charged generalisation [9, 10]. In all cases, we shall find that the inequality is obeyed.

2.1 Single rotating uncharged black ring

The metric for the single rotating uncharged black ring was first obtained in [4]. It was recast in a somewhat more convenient form in [11, 12]:

$$ds^2 = \frac{F(y)}{F(x)} \left( dt + CR \frac{1+y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x-y)^2} F(x) \left[ -\frac{G(y)}{F(y)} dy^2 + G(x) + \frac{G(x)}{F(x)} d\phi^2 \right],$$

(2.2)

where $F(\zeta) = 1 + \lambda \zeta$, $G(\zeta) = (1 - \zeta^2)(1 + \nu \zeta)$ and $C(\nu, \lambda) = \sqrt{\lambda(\lambda - \nu)} \frac{1+\lambda}{1-\lambda}$. The apparent horizon is located at $y = -\nu^{-1}$. The coordinates are restricted to lie in the ranges $-1 \leq x \leq 1$ and $-\infty < y \leq -1$. The requirement that $G(\zeta)$ have real roots implies that the parameters should be such that $0 < \nu \leq \lambda < 1$.

To avoid conical singularities at $x = -1$ and at $y = -1$, one must respectively require that $\phi$ and $\psi$ be periodic with periods $\Delta \phi$ and $\Delta \psi$ given by:

$$\Delta \phi = \Delta \psi = \frac{2\pi \sqrt{1-\lambda}}{1-\nu}.$$

(2.3)

A similar conical singularity at $x = 1$ is avoided by requiring that $\phi$ be periodic with $\Delta \phi = 2\pi \sqrt{1+\lambda}/(1+\nu)$). Reconciling the two periodicities therefore implies that the parameters $\nu$ and $\lambda$ must be related by

$$\lambda = \frac{2\nu}{1+\nu^2}.$$

(2.4)
The apparent horizon, which has the topology $S^2 \times S^1$, is spanned by the coordinates $x$ and $\phi$ on the $S^2$, and $\psi$ on the $S^1$. The $S^2$ is foliated by circles parameterised by $\phi$, with singular orbits at $x = +1$ and $-1$, corresponding to the poles of the sphere. Thus we can obtain an upper bound on the value of the Birkhoff invariant by maximising the $S^1 \times S^1$ sweepout area

$$\mathcal{A}(x) = \Delta x \Delta \psi \sqrt{g_{\phi\phi} g_{\psi\psi} - g_{\phi\psi}^2} \bigg|_{y = -\frac{1}{\nu}}$$

as a function of $x$, for $-1 \leq x \leq 1$.

Substituting in the components of the metric and noting the simplification $G(y = -1/\nu) = 0$, we have

$$\mathcal{A}(x) = \frac{4\pi^2 R^2 (1 - \lambda)}{1 - \nu} \left[ \frac{\nu C(\nu, \lambda)^2}{(\lambda - \nu) (1 + \nu x)^2} \right]^{1/2}.$$  

The ADM mass is given by \cite{11, 12}

$$M = \frac{3\pi R^2}{4} \frac{\lambda}{(1 - \nu)},$$

and so verifying the hoop conjecture \eqref{2.1} amounts to showing that

$$Z \equiv 1 - \left[ \frac{3\mathcal{A}(x)}{16\pi M} \right]^2 = 1 - \frac{\nu (1 - \lambda^2) (1 - x^2)}{\lambda} (1 + \nu x)^2 \geq 0$$

when $\lambda$ is given by \eqref{2.4} and

$$-1 \leq x \leq 1, \quad \text{and} \quad 0 < \nu \leq 1.$$  

Since $\nu \leq \lambda$, the inequality \eqref{2.8} will be established if we can show that

$$P(x) \equiv (1 - \nu^2) (x^2 - 1) + 1 + \nu x \geq 0.$$  

Writing this as

$$P(x) = \frac{1}{4} (3 - \nu x)(x + \nu)^2 + \frac{1}{4} (1 + \nu x)(x - \nu)^2,$$

and noting that $(3 - \nu x) > 0$ and $(1 - \nu x) \geq 0$ when the conditions in \eqref{2.9} are satisfied, we see that $P(x)$ is manifestly non-negative. The apparent horizon of the uncharged black ring \eqref{2.2} with a single rotation parameter therefore obeys the hoop inequality \eqref{2.1}.

An alternative proof, which we shall present here to illustrate an analogous one that we shall use later for the more complicated example of the charged black ring with two rotation parameters, is to take the expression for $Z$ in \eqref{2.8}, with $\lambda$ given by \eqref{2.4}, and parameterise $x$ and $\nu$ by

$$x = -1 + \frac{2}{1 + u}, \quad \nu = \frac{1}{1 + v},$$

where $u \geq 0$ and $v \geq 0$. It can then be seen that $Z$ is given by

$$Z = \alpha (4 + 4u + 10v + 12uv + 2u^2v + 10v^2 + 6u^2v^2 + 4u^4v^2 + 5v^3 + 3u^2v^3 + v^4 + u^2v^4),$$

where $\alpha$ is the non-negative quantity given by

$$\alpha^{-1} = (1 + \nu^2)(1 + \nu x)(1 + u)^2 (1 + v)^4.$$  

Since the coefficient of every term in \eqref{2.13} is positive, it follows that $Z \geq 0$ and the hoop bound is again verified.
2.2 Single rotating charged black ring

This solution was first obtained in [7]. For our purposes, it will be convenient to parameterise it somewhat differently, in a form that reduces to (2.2) in the case that the charge parameter is set to zero. A simple way to obtain the solution in this form is in fact to set to zero one of the rotation parameters in the more general doubly-rotating charged ring solution obtained in [10], and then to make appropriate parameter and coordinate redefinitions to match those in (2.2). This is achieved by making the replacements

\[ \nu = 0, \quad \lambda \to \nu, \quad k \to \frac{R}{\sqrt{2(1 + \nu^2)}}, \quad \phi \to -\sqrt{1 + \nu^2} \psi, \quad \psi \to \sqrt{1 + \nu^2} \phi \]  

(2.15)

in the metric in [10] (which appears in (2.24) below). This gives the metric

\[
d s^2 = -D^{-2/3} F(y) \left( dt + C \frac{(1 + y) R c d \psi}{F(y)} \right)^2 + D^{1/3} \frac{R^2}{(x - y)^2} F(x) \left[ -\frac{G(y)}{F(y)} d \psi^2 + \frac{G(x)}{F(x)} d \phi^2 + \frac{d x^2}{G(x)} - \frac{d y^2}{G(y)} \right],
\]

(2.16)

\[
D = 1 + s^2 \left( \frac{2 \nu(x - y)}{(1 + \nu^2) F(x)} \right),
\]

(2.17)

where \( c \equiv \cosh \delta \) and \( s \equiv \sinh \delta \), with \( \delta \) parameterising the charge. The functions \( F \) and \( G \) are precisely those defined under (2.2), and the constant \( C \) is defined under (2.2), where in all cases \( \lambda \) is given by (2.4). The mass and conserved electric charge for this solution are given by

\[
M = (1 + \frac{2}{3} s^2) M_0, \quad Q = \frac{\pi R^2 \nu s c}{(1 - \nu)(1 + \nu^2)},
\]

(2.19)

where \( M_0 \) is the mass of the uncharged black ring, given in (2.7).

The apparent horizon is again located at \( y = -\nu^{-1} \), and the area of the \( S^1 \times S^1 \) sweepout on an \( x = \) constant latitude of the \( S^2 \) is given by

\[
A(x) = \frac{c}{D^{1/6}} A_0(x),
\]

(2.20)

where \( A_0(x) \) is the area of the sweepout in the uncharged case \( (\delta \to 0) \) and therefore is given by the expression (2.6). Thus

\[
\frac{3}{16\pi} = \frac{c}{D^{1/6}(1 + \frac{2}{3} s^2)} \left[ \frac{3 A_0(x)}{16\pi M_0} \right],
\]

(2.21)

and since the hoop inequality (2.1) has already been verified for the uncharged case, we need only show that

\[
\frac{c}{D^{1/6}(1 + \frac{2}{3} s^2)} \leq 1
\]

(2.22)

for \(-1 \leq x \leq 1, 0 < \nu \leq 1\) and all \( \delta \) in order to verify it for the charged case too.

Since \( x \geq -1 \) and \( y \leq -\nu^{-1} < -1 \), it follows that \( (x - y) > 0 \). Also, \( F(x) = 1 + 2\nu x / (1 + \nu^2) \) and so with \( 0 < \nu \leq 1 \) it follows that \( F(x) \geq 0 \). Hence, from (2.18), we see that \( D > 1 \), and so it remains only to show that \( c/(1 + \frac{2}{3} s^2) \leq 1 \). This is clear, since

\[
\frac{c}{1 + \frac{2}{3} s^2} = \frac{3c}{1 + 2c^2} = 1 - \frac{(c - 1)(2c - 1)}{1 + 2c^2},
\]

(2.23)

and \( c = \cosh \delta \geq 1 \). Thus the hoop inequality (2.1) is satisfied by the single rotating charged black ring.
2.3 Doubly rotating charged black ring

The solution for an uncharged doubly rotating black ring was obtained in [8]. This was generalised [9] to a 2-charged doubly rotating black ring, and further in [10] to the more general 3-charge doubly-rotating black ring solution in the $\mathcal{N} = 2$ STU supergravity theory. It was shown in [10] that in order to obtain a solution with no conical singularities at the poles of the $S^2$ surfaces, two of the three charges must be set to zero. The metric is then given by

\[
\begin{align*}
\text{ds}^2 &= -D^{-2/3} \frac{H(y,x)}{H(x,y)} (dt + c \Omega)^2 + D^{1/3} \left( - \frac{F(x,y)}{H(y,x)} \frac{d\phi^2}{H(y,x)} - 2 \frac{J(x,y)}{H(y,x)} \frac{d\phi dy}{H(y,x)} \right) \\
&\quad + \frac{F(y,x)}{H(y,x)} \frac{d\psi^2}{(x-y)^2} + \frac{2k^2 H(x,y)}{(x-y)^2(1-\nu)^2} \left( \frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} \right) \tag{2.24}
\end{align*}
\]

where

\[
\begin{align*}
G(x) &= (1-x^2)(1+\lambda x + \nu x^2), \\
H(x,y) &= 1 + \lambda^2 - \nu^2 + 2\lambda \nu(1-x^2) + 2x^2 y(1-y^2 \nu^2) + x^2 y^2 \nu (1-\lambda^2 - \nu^2), \\
J(x,y) &= \frac{2k^2 (1-x^2)(1-y^2) \lambda \nu}{(x-y)(1-\nu)^2} [1 + \lambda^2 - \nu^2 + 2(x+y) \lambda \nu - xy \nu (1-\lambda^2 - \nu^2)], \\
F(x,y) &= \frac{2k^2}{(x-y)^2(1-\nu)^2} \left\{ G(x)(1-y^2) \left[ (1-\nu^2 - \lambda^2)(1+\nu) + y\lambda(1-\lambda^2 + 2\nu - 3\nu^2) \right] \\
&\quad + G(y) \left[ 2\lambda^2 + x\lambda(1-\nu^2 + \lambda^2) + x^2 [(1-\nu^2 - \lambda^2)(1+\nu) + x^3 \lambda(1-\lambda^2 - 3\nu^2 + 2\nu^3) - x^4 (1-\nu) \nu (-1 + \lambda^2 + \nu^2) \right] \right\}, \tag{2.25}
\end{align*}
\]

together with

\[
\begin{align*}
\Omega &= \frac{2k\lambda \sqrt{(1+\nu)^2 - \lambda^2}}{H(y,x)} \left[ \frac{1+y}{1-\lambda + \nu} \right] (1+\lambda - \nu + x^2 \nu (1-\lambda - \nu) + 2\nu x (1-y)) d\phi \\
&\quad + (1-x^2)y \sqrt{\nu} d\psi \right] \right], \\
D &= 1 + \frac{2\lambda s^2 (1-\nu) (x-y)(1-\nu xy)}{H(x,y)}, \tag{2.26}
\end{align*}
\]

where $-1 \leq x \leq 1$, $y \leq -1$ and $\phi$ and $\psi$ each have period $2\pi$. The parameters are restricted by the requirements $0 \leq \nu < 1$ and $2\sqrt{\nu} \leq \lambda < 1 + \nu$. The apparent horizon is located at

\[
y_h = \frac{-\lambda + \sqrt{\lambda^2 - 4\nu}}{2\nu}, \tag{2.27}
\]

and the ADM mass is given by

\[
M = \frac{(3 + 2s^2) k^2 \pi \lambda}{(1-\lambda + \nu)}. \tag{2.28}
\]

As in the previous examples, we consider the family of $S^1 \times S^1$ sweepouts of the horizon that are parameterised by the coordinate $x$. The area of the sweepout is given by

\[
A(x) = (2\pi)^2 \sqrt{g_{\phi\phi} g_{\psi\psi} - g_{\phi\psi}^2} \bigg|_{y=y_h}. \tag{2.29}
\]
Writing
\[
\left[ \frac{3A(x)}{16\pi M} \right]^2 \equiv D^{-1/3} Y, \quad (2.30)
\]
the verification of the hoop conjecture (2.1) amounts to showing that \( D^{-1/3} Y \leq 1 \).

The algebra in this example is rather too complicated to present. However, the verification of the hoop conjecture may be demonstrated as follows. The strategy is to reparameterise the constants \( \nu, \lambda \) and \( \delta \), and the latitude coordinate \( x \) on the 2-spheres, so that each parameter ranges, unrestricted, over the non-negative range 0 to \( \infty \). We do this by first defining
\[
\nu = \tanh^2 \beta, \quad \lambda = 2 \tanh \beta \cosh \alpha. \quad (2.31)
\]
The parameter \( \beta \) lies in the range \( 0 \leq \beta \leq \infty \), while \( \alpha \) should range over \( 0 \leq \alpha \leq \alpha_{\text{max}} \), where
\[
\alpha_{\text{max}} = \coth \beta. \quad (2.32)
\]
Note that the horizon will be located at \( y_h = -e^{-\alpha} \coth \beta \). Finally, we introduce new parameters \((u, v, w, z)\), each lying in the range from 0 to \( \infty \), such that
\[
e^\beta = 1 + u, \quad e^\alpha = 1 + \frac{2}{(1 + v)((1 + u)^2 - 1)}, \quad x = -1 + \frac{2}{1 + w}, \quad e^\delta = 1 + z. \quad (2.33)
\]
This gives a complete covering of the parameter space, and the \( x \) coordinate range on the 2-sphere, in terms of independent and unrestricted non-negative quantities. (We can, without loss of generality, assume that \( \delta \geq 0 \) since the metric is insensitive to the sign of the electric charge.)

We then find that \( 1 - Y \) is a rational function of \( u, v, w, z \), in which every term in the numerator and denominator multinomials has a positive coefficient. (For example, the numerator is a multinomial with 3350 terms, all having positive coefficients.) This establishes that \( Y \leq 1 \).

Likewise, we find that \( D - 1 \) is a rational function with all positive coefficients, and so \( D \geq 1 \). Therefore, from (2.30), we see that the hoop conjecture is verified for the single-charged doubly rotating black ring.

3 Conclusions

A formulation of a hoop conjecture for apparent horizons in five-dimensional spacetimes (2.1) was proposed in [3], based on a definition of a “hoop” in terms of a Birkhoff invariant for a least maximal sweepout by \( S^1 \times S^1 \) foliations of the three-dimensional horizon. This extended an earlier reformulation of Thorne’s original [1] four-dimensional hoop conjecture by Gibbons, in which the hoop was characterised by the Birkhoff invariant for \( S^1 \) foliations of the \( S^2 \) horizon [2].

The conjecture in five spacetime dimensions was shown to be valid for various known black hole solutions in [3], but only those with \( S^3 \) horizon topology were investigated there. In the present paper, we have examined the conjecture also for black ring metrics, where the horizon topology is instead \( S^2 \times S^1 \). Intriguingly, we find that the identical inequality is obeyed in these examples also.

In closing, we note that although the Gibbons reformulation [2] of the original Thorne conjecture [1] has the great merit of replacing the rather heuristic and loosely-defined notion of bounding the size of an apparent horizon “in all directions” by the precise notion of a Birkhoff invariant for the horizon, it does come at the price of considerably weakening the original concept of a hoop
bound. Thus, instead of asserting that the black hole could be “passed through the hoop” with any orientation, it asserts that there exists some orientation for which it would pass through the hoop. If, for example, in four dimensions the horizon had the shape of a prolate ellipsoid, then the Birkhoff invariant would be given by the more slender circumference around the equator of the ellipsoid, leaving open the possibility that the dimension of a circumference passing through the poles might exceed the conjectured hoop bound. It would be very interesting to investigate whether well-defined conjectures that encompassed the possibility of more powerful bounds in the spirit of the original hoop conjecture might be formulated.

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