General form of the deformation of the Poisson superbracket on (2,n)-dimensional superspace

S.E. Konstein* and I.V. Tyutin†‡

I.E. Tamm Department of Theoretical Physics,
P. N. Lebedev Physical Institute,
119991, Leninsky Prospect 53, Moscow, Russia.

Abstract

Continuous formal deformations of the Poisson superbracket defined on compactly supported smooth functions on \( \mathbb{R}^2 \) taking values in a Grassmann algebra \( \mathbb{G}^{n_-} \) are described up to an equivalence transformation for \( n_- \neq 2 \).

1 Introduction

In the present paper, we find the general form of the \( * \)-commutator in the case of a Poisson superalgebra of smooth compactly supported functions taking values in a Grassmann algebra \( \mathbb{G}^{n_-} \) for \( n_- \neq 2 \). It occurs that the case \( n_- = 2 \), where Poisson superalgebra has an additional deformation, needs separate investigation, which will be provided in [5]. The proposed analysis is essentially based on the results of the papers [4] by the authors, where the second cohomology space with coefficients in the adjoint representation of the Poisson superalgebra was found, and [3] where the general form of the \( * \)-commutator in the case of a Poisson superalgebra of smooth compactly supported functions on \( \mathbb{R}^{n_+} \) taking values in a Grassmann algebra was found for \( n_+ \geq 4 \).

1.1 Main definitions

Here we recall main definitions.

*E-mail: konstein@lpi.ru
†E-mail: tyutin@lpi.ru
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1.1.1 Deformations of topological Lie superalgebras

In this section, we recall some concepts concerning formal deformations of algebras (see, e.g., [1]), adapting them to the case of topological Lie superalgebras. Let $L$ be a topological Lie superalgebra over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) with Lie superbracket $\{\cdot,\cdot\}$. $\mathbb{K}[[h^2]]$ be the ring of formal power series in $h^2$ over $\mathbb{K}$, and $L[[h^2]]$ be the $\mathbb{K}[[h^2]]$-module of formal power series in $h^2$ with coefficients in $L$. We endow both $\mathbb{K}[[h^2]]$ and $L[[h^2]]$ by the direct-product topology. The grading of $L$ naturally determines a grading of $L[[h^2]]$: an element $f = f_0 + h^2 f_1 + \ldots$ has a definite parity $\varepsilon(f)$ if $\varepsilon(f) = \varepsilon(f_j)$ for all $j = 0, 1, \ldots$ Every $p$-linear separately continuous mapping from $L^p$ to $L$ (in particular, the bracket $\{\cdot,\cdot\}$) is uniquely extended by $\mathbb{K}[[h^2]]$-linearity to a $p$-linear separately continuous mapping over $\mathbb{K}[[h^2]]$ from $L[[h^2]]^p$ to $L[[h^2]]$. A (continuous) formal deformation of $L$ is by definition a $\mathbb{K}[[h^2]]$-bilinear separately continuous Lie superbracket $C(\cdot,\cdot)$ on $L[[h^2]]$ such that $C(f,g) = \{f,g\} \mod h^2$ for any $f,g \in L[[h^2]]$. Obviously, every formal deformation $C$ is expressible in the form

$$C(f,g) = \{f,g\} + h^2 C_1(f,g) + h^4 C_2(f,g) + \ldots, \quad f,g \in L, \quad (1)$$

where $C_j$ are separately continuous skew-symmetric bilinear mappings from $L \times L$ to $L$ ($2$-cochains with coefficients in the adjoint representation of $L$). Formal deformations $C^1$ and $C^2$ are called equivalent if there is a continuous $\mathbb{K}[[h^2]]$-linear operator $T = \text{id} + h^2 T_1 + h^4 T_2 + \ldots : L[[h^2]] \to L[[h^2]]$ such that $TC^1(f,g) = C^2(Tf,Tg)$, $f,g \in L[[h^2]]$. The problem of finding formal deformations of $L$ is closely related to the problem of computing Chevalle–Eilenberg cohomology of $L$ with coefficients in the adjoint representation of $L$. Let $C_p(L)$ denote the space of $p$-linear skew-symmetric separately continuous mappings from $L^p$ to $L$ (the space of $p$-cochains with coefficients in the adjoint representation of $L$). The space $C_p(L)$ possesses a natural $\mathbb{Z}_2$-grading: by definition, $M_p \in C_p(L)$ has the definite parity $\varepsilon(M_p)$ if the relation $\varepsilon(M_p(f_1, \ldots, f_p)) = \varepsilon(M_p) + \varepsilon(f_1) + \ldots + \varepsilon(f_1)$ holds for any $f_j \in L$ with definite parities $\varepsilon(f_j)$. We consider here only even Lie superbracket and only even deformation parameters. So, we consider that all $C_j$ in the expansion (1) are even $2$-cochains. The differential $d_p^\text{ad}$ is defined to be the linear operator from $C_p(L)$ to $C_{p+1}(L)$ such that

$$d_p^\text{ad}M_p(f_1, \ldots, f_{p+1}) = - \sum_{j=1}^{p+1} (-1)^{j+\varepsilon(f_j)} \varepsilon(f)_{i,j-1} + \varepsilon(f_j) \varepsilon_M p \{f_j, M_p(f_1, \ldots, \hat{f}_j, \ldots, f_{p+1})\} -$$

$$- \sum_{i<j} (-1)^{j+\varepsilon(f_j)} \varepsilon(f)_{i,j-1} M_p(f_1, \ldots f_{i-1}, \{f_i, f_j\}, f_{i+1}, \ldots, \hat{f}_j, \ldots, f_{p+1}), \quad (2)$$

for any $M_p \in C_p(L)$ and $f_1, \ldots, f_{p+1} \in L$ having definite parities. Here the hat means that the argument is omitted and the notation $|\varepsilon(f)|_{i,j} = \sum_{l=1}^j \varepsilon(f_l)$ has been used. Writing the Jacobi identity for a deformation $C$ of the form (1),

$$(-1)^{\varepsilon(f)\varepsilon(h)} C(f, C(g, h)) + \text{cycle}(f,g,h) = 0, \quad (3)$$
and taking the terms of the order $\hbar^2$, we find that
\[ d_2 \text{ad} C_1 = 0. \] (4)
Thus, the first order deformations of $L$ are described by 2-cocycles of the differential $d \text{ad}$.

1.1.2 Poisson superalgebra

Let $\mathcal{D}(\mathbb{R}^k)$ denote the space of smooth $\mathbb{K}$-valued functions with compact support on $\mathbb{R}^k$. This space is endowed by its standard topology. We set
\[ \mathbf{D}_{n+}^{n-} = \mathcal{D}(\mathbb{R}^{n+}) \otimes \mathbb{G}^n, \quad \mathbf{E}_{n+}^{n-} = C^\infty(\mathbb{R}^{n+}) \otimes \mathbb{G}^n, \]
where $\mathbb{G}^n$ is the Grassmann algebra with $n_-$ generators. The generators of the Grassmann algebra (resp., the coordinates of the space $\mathbb{R}^{n+}$) are denoted by $\xi^a$, $a = 1, \ldots, n_-$ (resp., $x^i$, $i = 1, \ldots, n_+$. We shall also use collective variables $z^A$ which are equal to $x^A$ for $A = 1, \ldots, n_+$ and are equal to $\xi^{A-n_+}$ for $A = n_+ + 1, \ldots, n_+ + n_-$. The spaces $\mathbf{D}_{n+}^{n-}$ and $\mathbf{E}_{n+}^{n-}$ possess a natural grading which is determined by that of the Grassmann algebra. The parity of an element $f$ of these spaces is denoted by $\varepsilon(f)$. We also set $\varepsilon_A = 0$ for $A = 1, \ldots, n_+$ and $\varepsilon_A = 1$ for $A = n_+ + 1, \ldots, n_+ + n_-.$

Let $\frac{\partial}{\partial z^A}$ and $\frac{\partial}{\partial \xi^A}$ be the operators of the left and right differentiation. The Poisson bracket is defined by the relation
\[ \{f, g\}(z) = f(z) \frac{\partial}{\partial z^B} \omega^{AB} \frac{\partial}{\partial z^B} g(z) = -(-1)^{\varepsilon(f)\varepsilon(g)} \{g, f\}(z), \] (5)
where the symplectic metric $\omega^{AB} = -(-1)^{\varepsilon_A \varepsilon_B} \omega^{BA}$ is a constant invertible matrix. For definiteness, we choose it in the form
\[ \omega^{AB} = \begin{pmatrix} \omega^{ij} & 0 \\ 0 & \lambda_\alpha \delta^{a\beta} \end{pmatrix}, \quad \lambda_\alpha = \pm 1, \quad i, j = 1, \ldots, n_+, \quad \alpha, \beta = 1, \ldots, n_- \]
where $\omega^{ij}$ is the canonical symplectic form (if $\mathbb{K} = \mathbb{C}$, then one can choose $\lambda_\alpha = 1$). The Poisson superbracket satisfies the Jacobi identity
\[ (-1)^{\varepsilon(f)\varepsilon(h)} \{f, \{g, h\}\}(z) + \text{cycle}(f, g, h) = 0, \quad f, g, h \in \mathbf{E}_{n+}^{n-}. \] (6)
By Poisson superalgebra $\mathcal{P}$, we mean the space $\mathbf{D}_{n+}^{n-}$ with the Poisson bracket (5) on it. The relations (4) and (6) show that this bracket indeed determines a Lie superalgebra structure on $\mathbf{D}_{n+}^{n-}$.

The integral on $\mathbf{D}_{n+}^{n-}$ is defined by the relation
\[ \bar{f} \text{ def } = \int dz f(z) = \int_{\mathbb{R}^{n+}} dx \int d\xi f(z), \]
where the integral on the Grassmann algebra is normed by the condition $\int d\xi \xi^1 \ldots \xi^{n_-} = 1$. 
1.2 Cohomology of \( P \)

Let

\[
N_1(x|f) = -2\Lambda(x_2) \int du\theta(x_1 - y_1)f(u),
\]

where \( \Lambda \in C^\infty(\mathbb{R}) \) be a function such that \( \frac{d}{dx}\Lambda \in \mathcal{D}(\mathbb{R}) \) and \( \Lambda(-\infty) = 0, \ \Lambda(+\infty) = 1 \),

\[
N_2^E(x|f,g) = \Theta(x|\partial_2 f g) - \Theta(x|f \partial_2 g) - 2(-1)^{n-\varepsilon(f)}\partial_2 f(z)\Theta(x|g) + 2\Theta(x|f)\partial_2 g(z)
\]

where

\[
\Theta(x|f) \overset{\text{def}}{=} \int du\delta(x_1 - u_1)\theta(x_2 - u_2)f(u),
\]

It is easily to prove that bilinear mapping \( N_2^E = N_2^D + d_1 N_1 \) maps \( (D_2^{n_-})^2 \) to \( D_2^{n_-} \).

Let \( Z_2^{n_-} = D_2^{n_-} \oplus C_{E_2^{n_-}}(D_2^{n_-}) \), where \( C_{E_2^{n_-}}(D_2^{n_-}) \) is a centralizer of \( D_2^{n_-} \) in \( E_2^{n_-} \). Evidently, \( C_{E_2^{n_-}}(D_2^{n_-}) \simeq \mathbb{K} \).

The following Theorem is proved in \([4]\):

**Theorem 1.1.**

Let the bilinear mappings \( m_1, m_3, m_5^0, m_5^1, m_5^2, \) and \( m_5^3 \) from \( (D_2^{n_-})^2 \) to \( D_2^{n_-} \) be defined by the relations

\[
m_1(z|f,g) = f(z) \left( \frac{\partial}{\partial z} \omega^{AB} \frac{\partial}{\partial z} \right)^3 g(z),
\]

\[
m_3(z|f,g) = \mathcal{E}_z f(z) g(z) - \left( -1 \right)^{\varepsilon(f)\varepsilon(g)} \mathcal{E}_Z g(z) f(z),
\]

\[
m_5^0(x|f,g) = N_2^D(x|f,g) + \frac{1}{2} \left( x^i \partial_i f(x) \right) g(x) - \frac{1}{2} f(x) \left( x^i \partial_i g(x) \right),
\]

\[
m_5^1(z|f,g) = N_2^D(z|f,g) - \Delta(x|f)g(z) + \left( -1 \right)^{\varepsilon(f)} f(z) \Delta(x|g)
\]

\[
- \frac{2}{3} \left( -1 \right)^{\varepsilon(f)} \left( \xi^1 \partial_1 f(z) \right) \Delta(x|g),
\]

\[
m_5^2(z|f,g) = N_2^D(z|f,g) - \Delta(x|f)g(z) + f(z) \Delta(x|g),
\]

\[
m_5^3(z|f,g) = N_2^D(z|f,g) - \Delta(x|f)g(z) + \left( -1 \right)^{\varepsilon(f)} f(z) \Delta(x|g) + \partial_{\xi^*} f(z) \Delta_{\alpha}(x|g) - \left( -1 \right)^{\varepsilon(f)} \Delta_{\alpha}(x|f) \partial_{\xi^*} g(z),
\]

where

\[
\mathcal{E}_z \overset{\text{def}}{=} 1 - \frac{1}{2} z \partial_z,
\]

\[
\Delta(x|f) \overset{\text{def}}{=} \int du\delta(x - y)f(u),
\]

\[
\Delta_{\alpha}(x|f) \overset{\text{def}}{=} \int du\eta_{\alpha}\delta(x - y)f(u),
\]

and \( z = (x_1, x_2, \xi_1, ..., \xi_{n_-}), u = (y_1, y_2, \eta_1, ..., \eta_{n_-}) \).

Let \( V_2^{n_-} \) be the subspace of \( C_2(D_2^{n_-}, D_2^{n_-}) \) generated by the cocycles \( m_1, m_3 \) and \( m_5^{n_-} \) for \( n_- = 0, 1, 2, 3 \), and by the cocycles \( m_1 \) and \( m_3 \) for \( n_- \geq 4 \).
Then there is a natural isomorphism $V_2^{n_-} \oplus (E_2^{n_-}/\mathbb{Z}_2^{n_-}) \simeq H_{\text{ad}}^2$ taking $(M_2, T) \in V_2^{n_-} \oplus (E_2^{n_-}/\mathbb{Z}_2^{n_-})$ to the cohomology class determined by the cocycle

$$M_2(z|f, g) + m_\zeta(f, g),$$

where

$$m_\zeta(f, g) = \{\zeta(z), f(z)\} \bar{g} - (-1)^{\varepsilon(f)\varepsilon(g)} \{\zeta(z), g(z)\} \bar{f},$$

and $\zeta \in E_2^{n_-}$ belongs to the equivalence class $T$.

2 Formulation of the results

For any $\kappa \in \mathbb{K}[[\hbar^2]]$, the Moyal-type superbracket

$$\mathcal{M}_\kappa(z|f, g) = \frac{1}{\hbar\kappa} f(z) \sinh \left( \hbar\kappa \frac{\partial}{\partial z} \omega^{AB} \frac{\partial}{\partial z^B} \right) g(z)$$

is skew-symmetric and satisfies the Jacobi identity and, therefore, gives a deformation of the initial Poisson algebra. For $\zeta \in E_{n_+}^{n_-}[[\hbar^2]]$, $\kappa, c \in \mathbb{K}[[\hbar^2]]$, we set

$$N_{\kappa, \zeta}(z|f, g) = \mathcal{M}_\kappa(z|f, g - \zeta \bar{f} - \zeta \bar{g}),$$

$$N_{\kappa, \zeta, c}(z|f, g) = \mathcal{M}_\kappa(z|f - \zeta \bar{f}, g - \zeta \bar{g}) + c \bar{f} \bar{g}.$$

Now we can formulate the main result of the present paper.

Theorem 2.1.

1. Let $n_- = 2k \neq 2$. Then every continuous formal deformation of the Poisson superalgebra $\mathcal{P}$ is equivalent either to the superbracket $N_{\kappa, \zeta}(z|f, g)$, where $\zeta \in \hbar^2 E_2^{n_-}[[\hbar^2]]$ is even and $\kappa \in \mathbb{K}[[\hbar^2]]$, or to the superbracket

$$C(z|f, g) = \{f(z), g(z)\} + m_\zeta(z|f, g) + cm_3(z|f, g),$$

where $\zeta \in \hbar^2 E_2^{n_-}[[\hbar^2]]$ is even and $c \in \hbar^2 \mathbb{K}[[\hbar^2]]$.

2. Let $n_- = 2k + 1$. Then every continuous formal deformation of the Poisson superalgebra $\mathcal{P}$ is equivalent to the superbracket $N_{\kappa, \zeta, c}(z|f, g)$, where $c, \kappa \in \mathbb{K}[[\hbar^2]]$ and $\zeta \in \hbar^2 E_2^{n_-}[[\hbar^2]]$ is an odd function such that $\mathcal{M}_\kappa(z|\zeta, \zeta) + c \in D_2^n[[\hbar^2]]$.

The rest of the paper consists of the proof of this Theorem.

3 The cases $n_- = 1$, $n_- = 3$ and $n_- \geq 4$

In these cases all even cohomologies are generated by $m_1$, $m_\zeta$ and $m_3$. We will not consider odd parameters of deformations, and thus even deformations of Poisson superalgebras for these values of $n_-$ can be considered literally in the same way, as for the case $n_+ \geq 4$ in \cite{3}.
4 Case \( n_- = 2 \).

There are additional deformation in this case, and it needs separate consideration, which will be provided elsewhere.

5 Case \( n_- = 0 \).

The rest of the paper is the proof of Theorem 2.1 for the case \( n_- = 0 \).

5.1 Notations

Introduce the following notation:

\[
N = x^i \partial_i, \\
\Theta(x|f) \equiv \int dy\delta(x^1 - y^1)\theta(x^2 - y^2)f(y), \quad \partial_2 \Theta(x|f) = f(x), \\
\Theta_{r_-}(x|f) = \int_{r_-}^{x^2} dy^2 f(x^1, y^2) \\
\tilde{\Delta}(x^1|f) \equiv \int dy\delta(x^1 - y^1)f(y), \\
\tilde{\Theta}(x^1|f) \equiv \int dy\theta(x^1 - y^1)f(y), \quad \partial_1 \tilde{\Theta}(x^1|f) = \tilde{\Delta}(x^1|f), \\
\Psi(x|f) = \tilde{\Theta}(x^1|f)\Lambda(x^2), \\
\Xi(x|f) = \Theta(x|f) - \tilde{\Delta}(x^1|f)\Lambda(x^2) \in D_2^0, \quad \Xi(|f|) = -\int dy[c(\Lambda) + y^2]f(y), \\
c(\Lambda) = \int_{-\infty}^{r_+} dy^2\Lambda(y^2) - r_+, \quad r_+ \supset \text{supp}(\partial_2 \Lambda), \\
\theta(x^1|f_1) = \int dy^1\theta(x^1 - y^1)f_1(y^1), \quad \theta(x^2|f_2) = \int dy^2\theta(x^2 - y^2)f_2(y^2).
\]

For shortness we will denote \( m_5^0 \) as \( m_5 \).

Recall the definition of differentials \( d_1^{ad} \) and \( d_2^{ad} \):

\[
d_1^{ad} M_1(x|f, g) = \{f(x), M_1(x|g)\} - \{g(x), M_1(x|f)\} - M_1(x|\{f, g\}), \\
d_2^{ad} M_2(x|f, g, h) = \{f(x), M_2(x|g, h)\} + M_2(x|f, \{g, h\}) + \text{cycle}(f, g, h).
\]

The general solution of the equation \( d_2^{ad} M_2(x|f, g, h) = 0 \) has the form

\[
M_2(x|f, g) = c_1 m_1(x|f, g) + c_3 m_3(x|f, g) + c_5 m_5(x|f, g) + m_\zeta(f, g) + d_1^{ad} b^D(x|f, g),
\]

where \( b^D(f, g) \in D_2^0 \) and

\[
m_1(x|f, g) = f(x) \left( \frac{\partial}{\partial x^1} \right) g(x), \\
m_3(x|f, g) = [\mathcal{E}_x f(x)]g - [\mathcal{E}_x g(x)]f, \quad \mathcal{E}_x = 1 - \frac{1}{2} x^i \partial_i, \\
m_\zeta(x|f, g) = \{\zeta(x), f(x)\}g - \{\zeta(x), g(x)\}f, \quad \zeta \in E_2/\mathbb{Z}_2.
\]
Here, if $\zeta_1$ and $\zeta_2$ belong to the same equivalence class of $\mathbf{E}_2^0/\mathbf{Z}_2^0$ then $\zeta_1 - \zeta_2 = \text{const} + \zeta^D$, where $\zeta^D \in \mathbf{D}_2^0$. Then $m_{\zeta_1 - \zeta_2} = d_{1\zeta}^d m_{1\zeta^D}$ where $m_{1\zeta^D}(f) = \zeta^D(x)\tilde{f}$ can be included in $b^D$.

$$m_5(x|f,g) = m_{51}(x|f,g) + m_{52}(x|f,g) + m_{53}(x|f,g) + m_{54}(x|f,g),$$

$$m_{51}(x|f,g) = f(x)E_xg(x) - [E_x(x)]g(x),$$

$$m_{52}(x|f,g) = 2\Theta(x|f)\partial_2g(x) - 2\partial_2f(x)[\Theta(x|g)],$$

$$m_{53}(x|f,g) = -2\{f(x),\Psi(x|g)\} + 2\{g(x),\Psi(x|f)\},$$

$$m_{54}(x|f,g) = \Xi(x|\partial_2fg - f\partial_2g) = \tilde{m}_{254}(x|f,g) + f(x)g(x),$$

$\tilde{m}_{254}(x|f,g) = -2\Xi(x|f\partial_2g),$

$$m_5(|f,g) = \psi(f,g) = \psi^i(f\partial_i\gamma_1(f,g),$$

where

$$\psi^i(f) = \int dy[(-1)^iy^i + 2\delta^i_2c(\Lambda)]f(y),$$

$$\gamma_1(f) = \int dy[y^i + 2\Lambda(y)^iy^i]f(y).$$

The Moyal bracket is defined as

$$\mathcal{M}_\kappa(x|f,g) = \frac{1}{\hbar\kappa}f(x)\sinh\left(h\kappa\vec{\partial}_i\omega^{ij}\partial_j\right)g(x), \quad \kappa \in \mathbb{K}[h^2],$$

and shifted Moyal bracket depending on parameter $\zeta \in h^2\mathbf{E}_2^0[[h^2]]/\mathbf{Z}_2^0[[h^2]]$ (i.e. $\zeta = h^2\zeta_1 + h^4\zeta_2 + \ldots$, where $\zeta_i \in \mathbf{E}_2^0/\mathbf{Z}_2^0$) is defined as

$$\mathcal{N}_{\kappa,\zeta}(x|f,g) = \mathcal{M}_\kappa(x|f - \zeta\tilde{f},g - \zeta g).$$

It has the following obvious decompositions

$$\mathcal{N}_{\kappa,\zeta}(x|f,g) = \mathcal{M}_\kappa(x|f,g) + h^2m_{\zeta_1}(x|f,g) + O(h^4) =
\{f(x),g(x)\} + h^2\frac{\kappa^2}{6}f(x)\left(\vec{\partial}_i\omega^{ij}\partial_j\right)^3g(x) + h^2m_{\zeta_1}(x|f,g) + O(h^4),$$

$$\mathcal{N}_{\kappa,\zeta}(x|f,g) \in D, \mathcal{N}_{\kappa,\zeta}|(f,g) = 0.$$

Let $x = (x^1, x^2) \in \mathbb{R}^2$, $f, g, h \in \mathbf{D}_2^0$. We will consider various equations in the following domains in $\mathbb{R}^2 \times \mathbf{D}_2^0 \times \mathbf{D}_2^0 \times \mathbf{D}_2^0$:

**Definition.** Domain $\mathcal{U}^1$ consists of such $x \in \mathbb{R}^2$, $f, g, h \in \mathbf{D}_2^0$ that there exist vicinity $V_x$ of $x$ such that

$$[V_x \cup \text{supp}(h)] \cap [\text{supp}(f) \cup \text{supp}(g)] = \text{supp}(f) \cap \text{supp}(g) = \emptyset$$

**Definition.** Domain $\mathcal{U}^2$ consists of such $x \in \mathbb{R}^2$, $f, g, h \in \mathbf{D}_2^0$ that there exist vicinity $V_x$ of $x$ such that

$$[V_x \cup \text{supp}(h)] \cap [\text{supp}(f) \cup \text{supp}(g)] = \text{supp}(f) \cap \text{supp}(g) = \emptyset,$$

$$f(x) = f_1(x^1)f_2(x^2), \quad g(x) = g_1(x^1)g_2(x^2).$$
Definition. Domain $U^3$ consists of such $x \in \mathbb{R}^2$, $f, g, h \in D_2^0$ that there exist vicinity $V_x$ of $x$ such that

\[
[V_x \cup \text{supp}(h)] \cap [\text{supp}(f) \cup \text{supp}(g)] = \text{supp}(f) \cap \text{supp}(g) = \emptyset,
\]

\[
f(x) = f_1(x^1)f_2(x^2), \quad g(x) = g_1(x^1)g_2(x^2),
\]

\[
V_x \cap \text{supp}(\partial_2 \Lambda) = \emptyset.
\]

Definition. Domain $V$ consists of such $x \in \mathbb{R}^2$, $f, g, h \in D_2^0$ that for all $u, v \in \mathbb{R}$ there exist vicinities $V(x^1, u)$ and $V(v, x^2)$ of the points $(x^1, u) \in \mathbb{R}^2$ and $(v, x^2) \in \mathbb{R}^2$ correspondingly, such that

\[
[V(x^1, u) \cup V(v, x^2)] \cap [\text{supp}(f) \cup \text{supp}(g) \cup \text{supp}(\partial_2 \Lambda) \cup \text{supp}(C_2(|f, g|))] =
\]

\[
= \text{supp}(f) \cap \text{supp}(g) = \emptyset.
\]

Definition. Domain $W^2$ consists of such $x \in \mathbb{R}^2$, $f, g, h \in D_2^0$ that there exist vicinity $V_x$ of $x$ such that

\[
[V_x \cup \text{supp}(f) \cup \text{supp}(g)] \cap \text{supp}(h) = \emptyset.
\]

Definition. Domain $W^3$ consists of such $x \in \mathbb{R}^2$, $f, g, h \in D_2^0$ that there exist vicinity $V_x$ of $x$ such that

\[
V_x \cap [\text{supp}(f) \cup \text{supp}(g) \cup \text{supp}(h)] = \emptyset.
\]

Definition. Domain $W^4$ consists of such $x \in \mathbb{R}^2$, $f, g, h \in D_2^0$ that there exist vicinity $V_x$ of $x$ such that

\[
[V_x \cup \text{supp}(h)] \cap [\text{supp}(f) \cup \text{supp}(g)] = \emptyset.
\]

5.2 Jacobiators

Let $p(f, g)$ and $q(f, g)$ be two different 2-cochains taking values in $D_2^0$.

Jacobiators are defined as follows:

\[
J(p, q) \overset{\text{def}}{=} p(f, q(g, h)) + q(f, p(g, h)) + \text{cycle}(f, g, h),
\]

\[
J(p, p) \overset{\text{def}}{=} p(f, p(g, h)) + \text{cycle}(f, g, h).
\]

Evidently, $J(p, q)$ takes value in $D_2^0$.

If $m_0(f, g) = \{f, g\}$ then $J(p, m_0) = d_2^{\text{ad}} p$.

We will use notations $J_{ab} \overset{\text{def}}{=} J(m_a, m_b)$ for Jacobiators of coboundaries.

According to [3],

\[
J_{\zeta, 3} = J_{3, 3} = 0.
\]

Further, one can easily check, that

\[
J_{1, 3}(x|f, g, h) = -2m_1(x|f, g)\bar{h} + \text{cycle}(f, g, h), \tag{15}
\]

\[
J_{\zeta, 5} = d_2^{\text{ad}} \sigma_\zeta, \tag{16}
\]
where \( \sigma_\zeta(x|f, g) = \{ f(x), \zeta(x) \} \gamma_1(g) + \{ f(x)E_x \zeta(x) + 2\Theta(x|f) \partial_\zeta(x) + \)
\[ + 2\{ \zeta(x), \Psi(x|f) \} - 2\Theta(x|f \partial_\zeta) + 2\Delta(x|f \partial_\zeta) \Lambda(x^2) - \]
\[ - \Theta_{r_\zeta}(x|E_x \zeta) \partial_2 f(x)|g - (f \leftrightarrow g), \]
(17)

and
\[ J(m_\zeta, \sigma_\zeta) = 0. \]
(18)

One can decompose \( J_{5,5} \) in the form
\[ J_{5,5} = \tilde{J}_{5,5} + d_{2}^{ad} \sigma_1, \]
(19)

where
\[ \sigma_4(x|f, g) = 4[\Theta(x|f) \partial_2 \Lambda(x^2) - \Theta(x|f \partial_2 \Lambda) + \tilde{\Delta}(x^1|f \partial_2 \Lambda) \Lambda(x^2) - \]
\[ - \tilde{\Delta}(x^1|f) \Lambda(x^2) \partial_2 \Lambda(x^2)]\tilde{\Theta}(x^1|g) - (f \leftrightarrow g), \]
\[ \sigma_4(x|f, g) \in D_2. \]

We will need the expressions for \( J_{ab} \) in different domains:

**Domain \( U^2 \)** In this domain
\[ \hat{J}_{1,3}(x|f, g, h) = \hat{J}_{1,5}(x|f, g, h) = 0, \]
\[ \hat{J}_{3,51}(x|f, g, h) = \hat{J}_{3,54}(x|f, g, h) = 0, \]
\[ \hat{J}_{3,52}(x|f, g, h) = 2[\bar{f}_1 \bar{f}_2 g_1(x^1)|x^2|g_2) - (f \leftrightarrow g)] \partial_2 h(x), \]
\[ \hat{J}_{3,53}(x|f, g, h) = \{ h(x), \sigma_1(x|f, g) \}, \]
\[ \sigma_1(x|f, g) = [\bar{f}_1 \theta(x^1|g_1) - \theta(x^1|f_1) \bar{g}_1] \bar{f}_2 \bar{g}_2 [2\Lambda_2(x^2) + x^2 \partial_2 \Lambda_2(x^2)]. \]
(20)

\( \sigma_1 \) depends only on \( x^1 \) and \( \sigma_1 \in D(\mathbb{R}) \) for fixed \( f, g. \)

\[ \hat{J}_{51,5k}(x|f, g, h) = \hat{J}_{54,5k}(x|f, g, h) = 0, \]
\[ k = 1, ..., 4, \]
\[ \hat{J}_{5,5}(x|f, g, h) = 0, \]
\[ d_{2}^{ad} \hat{\sigma}_4(x|f, g, h) = \{ h(x), \sigma_4(x|f, g) \}. \]
(21)
(22)

Here and below the sign \( \hat{\ } \) over form means that we consider the restriction of the form on the domain under consideration. We use also the notation \( d_{2}^{ad} \hat{P} \) instead of \( d_{2}^{ad} P. \)

\(^1\text{We suppose in each formula, containing the expression } \Theta_{r_\zeta}, \text{ that supports of the functions } f, g \text{ and } h \)
\text{are above the line } x^2 = r_- \text{ in } \mathbb{R}^2. \text{ This restriction is used for the purpose of finding some constants, and occurs to be correct.}
Domain $\mathcal{W}^2$ In this domain
\[
\hat{J}_{1.5}(x|f, g, h) = \{f(x), \hat{n}^{(3)}(x|g, h)\} - \{g(x), \hat{n}^{(3)}(x|f, h)\} - \hat{n}^{(3)}(x|\{f(x), g(x)\}, h),
\]
\[
n^{(3)}(x|f, h) = n^{(3)}_1(x|f, h) - n^{(3)}_1(x|h, f)
\]
\[
n^{(3)}_1(x|f, h) = 2[\partial_1^3 f(x)\partial_2^3 \Lambda(x^2)\tilde{\Theta}(x^1|h) - 3\partial_1^2 \partial_2 f(x)\partial_2^2 \Lambda(x^2)\partial_1 \tilde{\Theta}(x^1|h) +
+3\partial_1 \partial_2^2 f(x)\partial_2 \Lambda(x^2)\partial_1 \tilde{\Theta}(x^1|h) - \partial_2^2 f(x)\Lambda(x^2)\partial_1^2 \tilde{\Theta}(x^1|h) +
+\partial_2^3 f(x)\partial_1^3 \Theta(x|h)].
\]
\[
\hat{J}_{5.5}(x|f, g, h) = [\partial_1 f(x)g(x) - f(x)\partial_1 g(x)]\gamma_2^i(x|h),
\]
\[
\gamma_2^i(x|h) = \omega^{ij}\partial_j \sigma_2(x|h) + \delta_2^i 2\Theta(x|h),
\]
\[
\sigma_2(x|h) = 2\mathcal{E}_2 \Psi(x|h) + x^1 \Theta(x|h),
\]
\[
\hat{J}_{1.3}(x|f, g, h) = 0,
\]
\[
\hat{J}_{1.5}(x|f, g, h) = 2\Xi(x|\partial_2 f g \left(\tilde{T}_{ij} \omega^{ij}\partial_j\right)^3 h) +
+cyclic(f, g, h).
\]

Domain $\mathcal{W}^3$ In this domain
\[
\hat{J}_{1.3}(x|f, g, h) = 0,
\]
\[
\hat{J}_{1.5} = 2\partial_3^3 h(x)\partial^3 \Theta(x|f\partial_2 g) + 2h(x) \left(\tilde{T}_{ij} \omega^{ij}\partial_j\right)^3 \left[\tilde{\Lambda}(x^1|f\partial_2 g)\Lambda(x^2)\right] - 2\partial_2 h(x)\Theta(x|f \left(\tilde{T}_{ij} \omega^{ij}\partial_j\right)^3 g) - 2\{h(x), \Psi(x|f \left(\tilde{T}_{ij} \omega^{ij}\partial_j\right)^3 g)\},
\]

5.3 $\hbar^2$-order equation for $C(x|f, g)$

Using decomposition
\[
C(x|f, g) = \{f(x), g(x)\} + \hbar^2 C_1(x|f, g) + O(\hbar^4), \quad C_1(x|f, g) \in \mathbf{D}_2^0,
\]

one obtains from Jacobi identity
\[
J(C, C) = 0
\]
the following equation
\[
d^{\text{ad}}_2 C_1(x|f, g, h) = 0.
\]

The first order deformation has the form
\[
C_1(x|f, g) = \frac{1}{6} \kappa_1^2 m_1(x|f, g) + c_{31} m_3(x|f, g) + c_{51} m_5(x|f, g) +
+m_{c_1}(x|f, g) + d^{\text{ad}}_1 b^D_1(x|f, g)
\]
or, after similarity transformation with $T = \text{id} + \hbar^2 T_1 + ...$, where $T_1(f) = -b^3(f)$.

Below, we will mean the similarity transformation of such form, writing "up to similarity transformation".

$$C_1(x|f, g) = \frac{1}{6} \kappa_1^2 m_1(x|f, g) + c_{31} m_3(x|f, g) + c_{51} m_5(x|f, g) + m_{\zeta_1}(x|f, g).$$

### 5.4 $\hbar^4$-order equation for $C(f, g)$

Represent $C(x|f, g)$ in the form

$$C(x|f, g) = N_{\kappa_1, \zeta_1}(x|f, g) + \hbar^2 c_{31} m_3(x|f, g) + \hbar^2 c_{51} m_5(x|f, g) + \hbar^4 C_2(x|f, g) + O(\hbar^6),$$

$$C_2(x|f, g) \in D_2^0.$$

The Jacobi identity (25) for $C(x|f, g)$ gives

$$d^\text{ad} D_2(x|f, g, h) + \frac{\kappa_1^2 c_{31}}{6} J_{1,3}(x|f, g, h) + \frac{\kappa_1^2 c_{51}}{6} J_{1,5}(x|f, g, h) + c_{51} c_{31} J_{3,5}(x|f, g, h) + c_{51} J_{5,5}(x|f, g, h) = 0,$$

where

$$D_2 = C_2 + c_{51} \sigma_{\zeta_1} \in D_2^0.$$

The following proposition follows from (26)

**Proposition 5.1.** $c_{51} c_{31} = 0$.

This proposition is proved in Appendix 1.

Further, we have to consider 2 cases: $c_{31} \neq 0$ and $c_{31} = 0$.

The condition $c_{31} \neq 0$ gives $c_{51} = 0$ and, according to [3], $\kappa_1 = 0$, and up to similarity transformation

$$C_2(x|f, g) = c_{12} m_1(x|f, g) + c_{32} m_3(x|f, g) + c_{52} m_5(x|f, g) + m_{\zeta_1}(x|f, g).$$

**Proposition 5.2.** If $c_{31} \neq 0$ then $c_{52} = 0$.

This proposition is proved in Appendix 2.

Analogously one proves that if $c_{31} \neq 0$ then $c_{5i} = 0$ for all $i$, where $c_{5i}$ are coefficient at $m_5$ in the decomposition of $C(f, g)$.

Thus, in the case under consideration the coboundary $m_5$ plays no role in deformation. Such situation was investigated in [3] and $C(f, g)$ is described by item 1 of Theorem 2.1.

### 5.5 $c_{31} = 0$

Now consider the case $c_{31} = 0$, where proposition 5.1 gives no information about $c_{51}$. In Appendix 3 the following proposition is proved:

**Proposition 5.3.** $c_{51} = 0$.

Further consider the case $c_{31} = c_{51} = 0$, $\kappa_1 \neq 0$.

Using representation

$$C(x|f, g) = N_{\kappa_1, \zeta_1}(x|f, g) + \hbar^4 C_2(x|f, g) + O(\hbar^6),$$

$$C_2(x|f, g) \in D_2^0.$$
we have
\[ d^{ad}_2 C_2 = 0, \]
and after similarity transformation
\[ C_2(x|f, g) = \frac{\kappa_1 \kappa_2}{3} m_1(x|f, g) + c_{32} m_3(x|f, g) + c_{52} m_5(x|f, g) + m_2(x|f, g). \]

**Proposition 5.4.** If \( c_{31} = c_{51} = 0 \) and \( \kappa_1 \neq 0 \) then \( c_{5i} = 0 \) for all \( i \).

This proposition is proved in Appendix 4.

Thus, if \( c_{31} = c_{51} = 0 \) and \( \kappa_1 \neq 0 \) then \( C(f, g) = N_{\kappa, \zeta}(z|f, g) \) in this case.

Now, consider the case \( \kappa_1 = c_{31} = c_{51} = 0 \).

Represent \( C(x|f, g) \) in the form
\[ C(x|f, g) = N_0, \zeta_1(x|f, g) + \hbar^2 m_1(x|f, g), \]
where
\[ N_0, \zeta_1(x|f, g) = \{ f(x), g(x) \} + \hbar^2 m_1(x|f, g), \]
This implies
\[ d^{ad}_2 C_2 = 0, \]
and (after some similarity transformation)
\[ C_2(x|f, g) = \frac{\kappa_2}{6} m_1(x|f, g) + c_{32} m_3(x|f, g) + c_{52} m_5(x|f, g) + m_2(x|f, g). \]

Represent \( C(x|f, g) \) in the form
\[ C(x|f, g) = N_{\hbar^2, \zeta_1}(x|f, g) + \hbar^4 c_{32} m_3(x|f, g) + \hbar^4 c_{52} m_5(x|f, g) + \hbar^6 C_3(x|f, g) + O(\hbar^8), \]
\[ C_3(x|f, g) \in D^0_2. \]

The Jacobi identity \( \text{(25)} \) for \( C(x|f, g) \) gives
\[ d^{ad}_2 D_3(x|f, g, h) = 0, \quad \text{(27)} \]
where
\[ D_3 = C_3 + c_{52} \sigma_1. \]

Solution of Eq. \( \text{(27)} \) described by Eq. \( \text{(14)} \). As a result, we obtain (after similarity transformation)
\[ C_3(x|f, g) = -c_{52} \sigma_1(x|f, g) + \frac{\kappa_2 \kappa_3}{3} m_1(z|f, g) + c_{33} m_3(z|f, g) + c_{53} m_5(z|f, g) + m_3(z|f, g), \]
and we can represent \( C(x|f, g) \) in the form
\[ C(x|f, g) = N_{\hbar^2, \zeta_1}(x|f, g) - \hbar^6 c_{52} \sigma_1(x|f, g) + c_{33} m_3(x|f, g) + c_{53} m_5(x|f, g) + \hbar^8 C_4(x|f, g) + O(\hbar^{10}), \]
\[ C_4(x|f, g) \in D^0_2. \]

\[ C_4(x|f, g) \in D^0_2. \]
Here
\[ \kappa[n] = \sum_{k=2}^{n} \hbar^{2k-2} \kappa_k, \quad c_5[n] = \sum_{k=2}^{n} \hbar^{2k} c_{5k}. \]

**Proposition 5.5.** If \( c_{31} = c_{51} = \kappa_1 = 0 \) then \( c_{32} c_{52} = 0 \).
This proposition is proved in Appendix 3

**Proposition 5.6.** If \( c_{31} = c_{51} = \kappa_1 = 0 \) and \( c_{32} \neq 0 \) then \( c_{5i} = \kappa_i = 0 \) for all \( i \).
This proposition is proved in Appendix 3

**Proposition 5.7.** Let \( c_{31} = c_{51} = \kappa_1 = 0 \) and \( c_{32} = 0 \). Then \( c_{52} = 0 \).
This proposition is proved in Appendix 4

So (after similarity transformation)
\[ C_4 = -c_{33} \sigma \zeta_1 + \left( \frac{\kappa_2^2}{6} + \frac{\kappa_3 \kappa_4}{3} \right) m_1 + c_{34} m_3 + c_{54} m_5 + m \zeta_4. \]

**Proposition 5.8.** Let \( c_{31} = c_{51} = \kappa_1 = 0 \), \( c_{32} = c_{52} = 0 \), and \( \kappa_2 \neq 0 \). Then \( c_{3i} = c_{5i} = 0 \) for all \( i \).

**Proof.**
Consider \( \hbar^{10} \)-order. Represent \( C(x|f,g) \) in the form
\[
C(x|f,g) = N_{h_{[3]}^{[3]} \zeta_4} (x|f,g) - h^8 c_{33} \sigma \zeta_1 (x|f,g) + c_{3[4]} m_3 (x|f,g) +
+c_{5[4]} m_5 (x|f,g) + h^{10} C_5 (x|f,g) + O(h^{12}),
\]
\[ C_5 (x|f,g) \in D_0^2. \]

The Jacobi identity (25) for \( C(x|f,g) \) gives
\[
d^a d^a D_5 (x|f,g,h) + \frac{\kappa_2^2 c_{33}}{6} J_{1,3} (x|f,g,h) + \frac{\kappa_2^2 c_{53}}{6} J_{1,5} (x|f,g,h) = 0, \quad (29)
\]
\[ D_5 = C_5 + c_{54} \sigma \zeta_1 + c_{53} \sigma \zeta_2 \in D_0^2. \]

Eq. (29) coincides exactly with Eq. (A4.1) and consideration of Eq. (29) in Domains \( U^2, V, W^3, W^2, W^4 \) gives
\[ c_{33} = c_{53} = 0 \]
and so on:
\[ c_{3k} = c_{5k} = 0 \quad \text{for all } k. \]

At last, consider the case \( c_{31} = c_{51} = \kappa_1 = c_{32} = c_{52} = \kappa_2 = 0 \), and show, that \( c_{5k} = 0 \) for all \( k \).

Eq. (29) reduces to
\[ d^a d^a D_5 (x|f,g,h) = 0 \]
and we find (after similarity transformation and some renaming)
\[ C = N_{h_{[3]}^{[3]} \zeta_4} (x|f,g) - h^{10} c_{54} \sigma \zeta_1 - h^{10} c_{53} \sigma \zeta_2 + c_{3[5]} m_3 (x|f,g) +
+c_{3[5]} m_5 (x|f,g) + h^{12} C_6 (x|f,g) + O(h^{14}),
\]
\[ C_6 (x|f,g) \in D_0^2. \]
The Jacobi identity (25) for $C(x|f,g)$ gives

$$d_2^{\text{ad}} D_b(x|f,g,h) + \frac{\kappa_3^2 c_{33}}{6} J_{1,3}(x|f,g,h) + \frac{\kappa_3^2 c_{53}}{6} J_{1,5}(x|f,g,h) = 0,$$

(30)

$$D_5 = C_5 + c_{55} \sigma_1 + c_{54} \sigma_2 + c_{53} \sigma_3 \in D^0_2.$$

Eq. (30) coincides with Eq. (A4.1) and further decomposition $C(x|f,g)$ on $\hbar^2$ leads to the same equation which implies

$$c_{5k} = 0 \ \forall k.$$

Thus, the coboundary $m_5$ plays no role in deformation, and $C(f,g)$ is described by item 4 of Theorem 2.1.

**Appendix 1. Proof of Proposition 5.1**

**Proposition.** $c_{51} c_{31} = 0$.

In the Domain $\mathcal{U}^2$, Eq. (26) takes the form

$$\{h, \hat{D}_2\} + c_{51} c_{31} (\hat{J}_{3,52} + \hat{J}_{3,53}) + c_5^2 (J_{52,53} + \hat{J}_{53,53}) = 0,$$

or

$$\{h, \hat{D}_2 + \hat{\sigma}^{(2)}\} = 2 c_{51} c_{31} [f_1(x^1) \theta(x^2|f_2) \bar{g}_1 \bar{g}_2 - (f \leftrightarrow g)] \partial_2 h(x),$$

(A1.1)

where

$$\sigma^{(2)}(x|f,g) = c_{51} c_{31} \sigma_1 (x|f,g) + c_5^2 \sigma_4 (x|f,g) \in D^0_2.$$

It follows from Eq. (A1.1) that

$$-\partial_1 (\hat{D}_2 + \hat{\sigma}^{(2)})(x|f,g) = 2 c_{51} c_{31} [f_1(x^1) \theta(x^2|f_2) \bar{g}_1 \bar{g}_2 - (f \leftrightarrow g)],$$

which implies after integrating over $x^1$

$$0 = c_{51} c_{31} [\theta(x^2|f_2) \bar{g}_2 - \theta(x^2|g_2) \bar{f}_2] \bar{f}_1 \bar{g}_1$$

giving the result

$$c_{51} c_{31} = 0.$$

**Appendix 2. Proof of Proposition 5.2**

**Proposition.** If $c_{31} \neq 0$ then $c_{5i} = 0$ for all $i$.

To prove this proposition, consider next, $\hbar^6$-order of decomposition.

Represent $C(x|f,g)$ in the form

$$C(x|f,g) = \mathcal{N}_{h^2 \zeta_2}(x|f,g) + c_{3[2]} m_3(x|f,g) + h^4 c_{52} m_5(x|f,g) + h^6 C_3(x|f,g) + O(h^8),$$

$$C_3(x|f,g) \in D^0_2,$$

where

$$c_{3[n]} = \sum_{k=1}^n h^{2k} c_{3k}, \quad \zeta_{[n]} = \sum_{k=1}^n h^{2k} \zeta_n.$$
The Jacobi identity (25) for $C(x|f,g)$ gives

$$d^a_d D_3(x|f,g,h) + \frac{\kappa^2 c_{31}}{6} J_{1,3}(x|f,g,h) + c_{52} c_{31} J_{3,5}(x|f,g,h) = 0,$$  \hspace{1cm} (A2.1)

$$D_3 = C_3 + c_{52} \sigma_{\zeta_1} \in D^0_2.$$

which implies

$$c_{52} = 0.$$  \hspace{1cm} (A2.2)

Indeed, consider (A2.1) in the Domain $U^3$. Then we have from Eq. (A2.1)

$$\{ h, \hat{D}_3 + c_{52} c_{31} \hat{\sigma}_1 \} = 2 c_{52} c_{31} [f_1(x^1) \theta(x^2|f_2) \bar{g}_1 \bar{g}_2 - (f \leftrightarrow g)] \partial_2 h(x),$$

which implies

$$\partial_1 (\hat{D}_3 + c_{52} c_{31} \hat{\sigma}_1) = 2 c_{52} c_{31} [f_1(x^1) \theta(x^2|f_2) \bar{g}_1 \bar{g}_2 - (f \leftrightarrow g)]$$  \hspace{1cm} (A2.3)

Because $\sigma_1 \in D^0_2$, one can conclude

$$c_{52} c_{31} = 0,$$

and so $c_{52} = 0$.

In the same way it is possible to prove, that if $c_{31} \neq 0$, then $c_{5k} = 0$ for all $k$.

**Appendix 3. Proof of Proposition 5.3.**

**Proposition.** If $c_{31} = 0$ then $c_{51} = 0$.

Consider the case $c_{31} = 0$ starting from $h^4$-order of deformation decomposition.

**A3.1. $h^4$-order**

Represent $C(x|f,g)$ in the form

$$C(x|f,g) = N_{c_{31}, \zeta_1}(x|f,g) + h^2 c_{51} m_5(x|f,g) + h^4 C_2(x|f,g) + O(h^6),$$

$$C_2(x|f,g) \in D^0_2.$$

The Jacobi identity (25) for $C(x|f,g)$ gives

$$d^a_d (C_2 + c_{51} \sigma_{\zeta_1}) + \frac{\kappa^2 c_{51}}{6} J_{1,5} + c_{51}^2 J_{5,5,5} = 0,$$  \hspace{1cm} (A3.1)

Then Eq. (A3.1) transforms to the form

$$d^a_d D_2 + \frac{\kappa^2 c_{51}}{6} J_{1,5} + c_{51}^2 \tilde{J}_{5,5,5} = 0,$$  \hspace{1cm} (A3.2)

$$D_2 = C_2 + c_{51} \sigma_{\zeta_1} + c_{51}^2 \sigma_4 \in D,$$

The forms $\tilde{J}_{5,5,5}$ and $\sigma_4$ are defined in Section 5.2.

In the Domain $U^1$ we have

$$d^a_d \hat{D}_2(x|f,g,h) = 0$$
and so, as it was proved in \cite{2}

\[ D_2(x|f, g) = D_{2|1}(x|f, g) + D_{2|2}(x|f, g), \tag{A3.3} \]

where \( D_{2|1}(x|f, g) \) and \( D_{2|2}(x|f, g) \) have the form

\[
D_{2|1}(x|f, g) = \sum_{q=0}^{Q} [(\partial_i^q f(x)m^{1(i)_q}(x|g) - m^{1(i)_q}(x|f)(\partial_i^q g(x)],
\]

\[
D_{2|2}(x|f, g) = \sum_{q=0}^{Q} m^{2(i)_q}(x)[(\partial_i^q f(x) - f(\partial_i^q g(x) + m^3(f, g).
\]

To specify \( D_{2|1}(x|f, g) \) and \( D_{2|2}(x|f, g) \), consider Jacobi identity in the following 2 domains.

**A3.2. Domain \( V \)**

In the Domain \( V \) we have

\[
\sigma_\zeta(x|f, g) = \sigma_4(x|f, g) = C_2(x|f, g) = 0
\]

and thus \( \hat{m}^3(f, g) = 0 \). This implies \( m^3(f, g) = m^3_{\text{loc}}(f, g) \). So

\[
D_{2|2}(x|f, g) = \sum_{q=1}^{Q} m^{2(i)_q}(x)[(\partial_i^q f(x) - f(\partial_i^q g(x)), q = 2l + 1.
\]

**A3.3. Domain \( W^2 \)**

In the Domain \( W^2 \), Eq. \( \text{(A3.1)} \) reduces to

\[
d_2^d D_{2|1} + c_5^2 \hat{J}_{5,5} = 0, \tag{A3.4}
\]

where

\[
D_{2|1}' = C_2 + c_{51}\sigma_\zeta + c_{2}^2\sigma_4 + \frac{\kappa_1^2 c_{51}}{6} n^{(3)} = \]

\[
= \sum_{q=0}^{Q} [(\partial_i^q f(x)m^{1(i)_q}(x|g) - m^{1(i)_q}(x|f)(\partial_i^q g(x)] \in D_2^0,
\]

and

\[
m^{n(i)_q}(x|h)(\partial_i^q f(x) = m^{1(i)_q}(x|h)(\partial_i^q f(x) + \frac{\kappa_1^2 c_{51}}{6} \delta_{q,3} n^{(3)}(x|f, h),
\]

Let \( f(x) = e^{px}, g(x) = e^{kx} \) in some vicinity of \( x \). Then \( \text{(A3.4)} \) takes the form

\[
(p_1 k_2 - p_2 k_1) \sum_{q=0}^{Q} [(p_i)^q + (k_i)^q - (p_i + k_i)^q] m^{n(i)_q}(x|h) +
\]

\[
+ \sum_{q=0}^{Q} [(k_i)^q\{px, \hat{m}^{n(i)_q}(x|h)] - (p_i)^q\{kx, \hat{m}^{n(i)_q}(x|h)]} +
\]

\[
+ c_{51}^2 (p_i - k_i) \gamma^2_1 = 0,
\]
or, equivalently

\[
(p_1 k_2 - p_2 k_1) \sum_{q=0}^{Q} [(p_i)^q + (k_i)^q - (p_i + k_i)^q] \hat{\phi}^{(i)q} (x|h) + \\
+ \sum_{q=0}^{Q} [(k_i)^q \{p x, \hat{\phi}^{(i)q} (x|h)\} - (p_i)^q \{k x, \hat{\phi}^{(i)q} (x|h)\}] + \\
c_{51}^2 \{(p - k) x, \sigma_2 (x|h)\} + 2c_{51}^2 (p_2 - k_2) \Theta (x|h) = 0. \tag{A3.5}
\]

**Proposition.** \( Q \leq 1 \)

Indeed, let \( Q \geq 2 \). It follows from Eq. \( (A3.3) \) \([(p_i)^Q + (k_i)^Q - (p_i + k_i)^Q] \hat{\phi}^{(i)Q} (x|h) = 0, \) and so \( \hat{\phi}^{(i)Q} (x|h) = 0 \) if \( Q \geq 2 \).

Further \( Q \leq 1 \), and so \( m' = m \). Introduce \( m^{(i)0}, \hat{\phi}^{(i)0} (x|h) = m^{(i)0} (x|h) + c_{51}^2 \sigma_2 (x|h), \) where \( m^{(i)0} (x|h) \equiv \hat{\phi}^{(i)0} (x|h) \). We obtain from Eq. \( (A3.5) \)

\[
\begin{align*}
\partial_2 \hat{\phi}^{(i)0} (x|h) &= 0, \quad \tag{A3.6} \\
\partial_1 \hat{\phi}^{(i)0} (x|h) &= 2c_{51}^2 \Theta (x|h), \quad \tag{A3.7}
\end{align*}
\]

which implies for the kernel of this form

\[
\begin{align*}
\partial_2 \hat{\phi}^{(i)0} (x|y) &= 0, \\
\partial_1 \hat{\phi}^{(i)0} (x|y) &= 2c_{51}^2 \delta (x^1 - y^1) \theta (x^2 - y^2)
\end{align*}
\]

and so

\[
\begin{align*}
\partial_2 m^{(i)0} (x|y) &= \partial_2 \sum_{p,q=0} \partial_1^p \partial_2^q \delta (x - y) U^{pq} (y) + \sum_{p=0} \partial_1^p \delta (x - y) V^p (y) \\
m^{(i)0} (x|y) &= \sum_{p,q=0} \partial_1^p \partial_2^q \delta (x - y) U^{pq} (y) + \\
&+ \sum_{p=0} \partial_1^p \delta (x^1 - y^1) \theta (x^2 - y^2) V^p (y) + u (x^1|y),
\end{align*}
\]

which results as

\[
\sum_{p=0} \partial_1^{p+1} \delta (x^1 - y^1) \theta (x^2 - y^2) V^p (y) + \partial_1 \hat{u} (x^1|y) = \\
= 2c_{51}^2 \delta (x^1 - y^1) \theta (x^2 - y^2) \quad \text{for} \quad (x^i) \neq (y^i).
\]

Considering the case \( y^2 > x^2 \) gives \( \partial_1 \hat{u} (x^1|y) = 0 \), and then the case \( y^2 < x^2 \) gives

\[
\sum_{p=0} \partial_1^{p+1} \delta (x^1 - y^1) V^p (y) = 2c_{51}^2 \delta (x^1 - y^1),
\]

which implies

\( c_{51} = 0. \)

**Appendix 4. Proof of Proposition 5.4.**
Proposition. If \( c_31 = c_51 = 0 \) and \( \kappa_1 \neq 0 \) then \( c_5i = 0 \) for all \( i \).
To prove this proposition consider successive terms in the decomposition on of \( h^2 \).

**A4.1. \( h^0 \)-order**

Represent \( C(x|f, g) \) in the form

\[
C(x|f, g) = N_{k[3], \xi[3]}(x|f, g) + \hbar^3 c_{32} m_3(x|f, g) + \hbar^4 c_{52} m_5(x|f, g) + \hbar^6 C_3(x|f, g) + O(h^8),
\]

\[C_3(x|f, g) \in D^0_2. \]

The Jacobi identity \((25)\) for \( C(x|f, g) \) gives

\[
d^{3d}_{\bar{2}} D_3 + \frac{\kappa_1^2 c_{32}}{6} J_{1,3} + \frac{\kappa_1^2 c_{52}}{6} J_{1,5} = 0, \tag{A4.1}
\]

where

\[D_3 = C_3 + c_{32} \sigma_{i} \in D. \]

The consideration of Eq. \((A4.1)\) in Domain \( U^2 \) and Domain \( V \) gives (according to Subsec. \( A3.1 \) and taking into account that \( J_{1,3}(x|f, g, h) = 0 \) in these Domains)

\[
D_3(x|f, g) = D_3[1](x|f, g) + D_3[2](x|f, g),
\]

\[
D_3[1](x|f, g) = \sum_{q=0}^{Q} [(\partial^2)^q f(x)m^{1/(i)}_q(x|g) - m^{1/(i)}_q(x|f)(\partial^2)^q g(x)],
\]

\[
D_3[2](x|f, g) = \sum_{q=0}^{Q} m^{2/(i)}_q(x|f)[(\partial^2)^q f - f(\partial^2)^q g], q = 2l + 1.
\]

Consider Eq. \((A4.1)\) in the Domain \( W^3 \):

In this Domain Eq. \((A4.1)\) takes the form

\[
\sum_{q=0}^{Q} \hat{m}^{2/(i)}_q (x|[(\partial^2)^q f]\{g, h\} - f_1(\partial^2)^q \{g, h\}) + \text{cycle}(f, g, h) =
\]

\[
= -\frac{\kappa_1^2 c_{52}}{3} \Xi(x|\partial_2 fm_1(x|g, h)) + \text{cycle}(f, g, h). \tag{A4.2}
\]

Let \( f(x) = e^{px} \), \( g(x) = e^{kx} \) in some vicinity of \( \text{supp}(h) \) and let \( h(x) \rightarrow e^{-(p+k)x} h(x) \). R.h.s. of Eq. \((A4.2)\) takes the form

\[-\kappa_1^2 c_{52}(p_1 k_2 - p_2 k_1)(p_2 k_2 + p_2 k_2)\partial^2_1 \Xi(x|h) -
\]

\[-\frac{\kappa_1^2 c_{52}}{3}(p_2 k_2 - p_2 k_2)\partial^3_1 \Xi(x|h).
\]

Let \( Q \geq 5 \). Then we have from \((A4.2)\)

\[(p_1 k_2 - p_2 k_1)[F_Q(p) + F_Q(k) - F_Q(p + k)] = 0 \implies
\]

\[F_q(p) = 0 \implies \hat{m}^{2/(i)}_q(x|h) = 0, q \geq 5,
\]
where $F_q(p) = (p_i)^q \Xi^{2(i)q}(x|h)$. For the terms of the 5-th order in $p, k$ in Eq. (A4.2) ($Q = 3$) we find

\[ 6(p, p_j k_l + p_j k_j k_l) \Xi^{2(i)k}(x|h) = \kappa^2_{c_{32}}(p^2_2 k_2 + p_2 k'^2_2) \Xi^{2}(x|h) \implies \]

\[ \Xi^{2(1)j}(x|h) = 0, \quad \Xi^{2(222)}(x|h) = \frac{\kappa^2_{c_{32}}}{6} \Xi^{2}(x|h). \]

The terms of the 4-th order in $p, k$ (which include $\Xi^{2(1)j}(x|h)$ only) are canceled identically and we obtain

\[ \Xi^{2(1)}(x[\partial, f]\{g, h\} - f \partial, \{g, h\}) + \text{cycle}(f, g, h) = 0. \quad \text{(A4.3)} \]

In the Domain $W^2$, we find

\[ d^a \hat{D}_3 = d^a \hat{D}_3^a(x|f, g), \]

so

\[ \sum_{q=0}^{Q} \left[ \{f(x), [\partial, f]g(x)] \Xi^{2(1)q}(x|h) \} - \{g(x), [\partial, f]f(x)] \Xi^{2(1)q}(x|h) \} - \right. \]

\[ - \left. \{\partial, f\{f(x), f(x)\]} \Xi^{2(1)q}(x|h) \right] + \frac{\kappa^2_{c_{32}}}{6} J_{1,3}(x|f, g, h) = 0, \quad \text{(A4.4)} \]

Here

\[ \Xi^{2(1)q}(x|h)(\partial, f) f(x) = \Xi^{2(1)q}(x|h)(\partial, f) f(x) + \frac{\kappa^2_{c_{32}}}{6} \delta_{q,3} n^{(3)}(x|f, h), \]

and $J_{1,3}(x|f, g, h) = -2m_1(x|f, g)$. Let $f(x) = e^{px}$, $g(x) = e^{kx}$ in some vicinity of $x$. Then Eq. (A4.4) reduces to

\[ (p_1 k_2 - p_2 k_1) \sum_{q=0}^{Q} \left[ (p_i)^q(k_i)^q - (p_i + k_i)^q \Xi^{2(1)q}(x|h) \right] + \]

\[ + \sum_{q=0}^{Q} \left[ (k_i)^q \{p x, \Xi^{2(1)q}(x|h)\} - (p_i)^q \{k x, \Xi^{2(1)q}(x|h)\} \right] - \frac{\kappa^2_{c_{32}}}{3} (p_1 k_2 - p_2 k_1)^3 h = 0. \quad \text{(A4.5)} \]

Let $Q \geq 5$. It follows from Eq. (A4.3)

\[ [(p_i)^Q + (k_i)^Q - (p_i + k_i)^Q] \Xi^{2(1)q}(x|h) = 0 \implies \Xi^{2(1)q}(x|h) = 0, \quad q \geq 4. \]

Let $Q = 4$. It follows from Eq. (A4.5)

\[ [(p_i)^4 + (k_i)^4 - (p_i + k_i)^4] \Xi^{2(1)q}(x|h) = \frac{\kappa^2_{c_{32}}}{3} (p_1 k_2 - p_2 k_1)^2 h. \]
Setting \( p = k \) in this equation, we obtain
\[
\hat{m}^{(i)}(x|h) = 0 \implies c_{32} = 0.
\]

Then we obtain \( \hat{m}^{(i)}(x|h) = \hat{m}^{(i)}(x|h) = 0 \) and
\[
\partial_t \hat{m}^{(i)}(x|h) = -\hat{m}^{(i)}(x|h), \quad \partial_t \hat{m}^{(i)}(x|h) = 0
\] (A4.6)

Eqs. (A4.3) and (A4.6) was solved in \( \mathcal{W} \) and the solution gives the following expression for \( C_3 \):
\[
C_3(x|f, g) = C_{3loc}(x|f, g) - c_{52}\sigma_1(x|f, g) - \frac{\kappa^2 c_{52}}{6} [n^{(3)}(x|f, g) - n^{(3)}(x|g, f)] + \]
\[+ \frac{\kappa^2 c_{52}}{6} \partial^2 \Xi(x|\partial^3 f g - f \partial^3 g) + T_3(x|f, g), \]
\[T_3(x|f, g) = \mathcal{E}_x f(x)a(g) - \mathcal{E}_x g(x)a(f) + \partial_2 f(x)\Theta(x|V_1 g) - \partial_2 g(x)\Theta(x|V_1 f) + \]
\[+ \{f(x), \nu(x|g)\} - \{g(x), \nu(x|f)\} + V_2(x)\Theta(x|\partial_2 f g - f \partial_2 g) + \mathcal{d}_2 \mathcal{E}_3(x|f, g), \]

where \( a(f), \nu(x|f), \mathcal{E}_3(x|f) \) are some functionals, \( V_1(x), V_2(x) \) are some distributions.

To prove that \( c_{52} = 0 \) consider Eq. (A4.11) in the Domain \( \mathcal{W}^4 \).

In Domain \( \mathcal{W}^4 \), Eq. (A4.1) gives
\[
d_2^{ad} \hat{T}_3(x|f, g, h) = \frac{\kappa^2 c_{52}}{3} \left[ \partial_2 h(x)\Theta(x|f) \left( \partial_i \omega^{ij} \partial_j \right)^3 g \right] + \]
\[+ \{h(x), \psi(x|f) \left( \partial_i \omega^{ij} \partial_j \right)^3 g \} \] (A4.7)

The expression for \( d_2^{ad} \hat{T}_3(x|f, g, h) \) was calculated in \( \mathcal{W} \):
\[
d_2^{ad} \hat{T}_3(x|f, g, h) = \{h(x), V_2(x)\Theta(x|\partial_2 f g - f \partial_2 g)\} + \mathcal{E}_x h(x)a(\{f, g\}) + \]
\[+ \partial_2 h(x)\Theta(x|V_1 \{f, g\}) + \{h(x), \nu(\{f, g\})\}. \]

We have
\[
-2\{h(x), V_2(x)\Theta(x|f \partial_2 g)\} + \mathcal{E}_x h(x)a(\{f, g\}) + \partial_2 h(x)\Theta(x|V_1 \{f, g\}) \]
\[+ \{h(x), \nu(\{f, g\})\} = \frac{\kappa^2 c_{52}}{3} [\partial_2 h(x)\partial_i^3 \Theta(x|f \partial^3 g) + \]
\[+ \{h(x), \partial_i^3 \hat{\Lambda}(x^1|f \partial^3 g) \Lambda(x^2)\}] \] (A4.8)

Let \( f(x) = e^{-px} \) in some vicinity of \( \text{supp}(g) \), and replace \( g(x) \) by \( e^{px} g(x) \). Consider the terms proportional to \( \partial_1 h(x)p_2^3 \) in Eq. (A4.3), :
\[
\frac{\kappa^2 c_{52}}{3} \partial_1 h(x)p_2^3 \partial_1^2 \hat{\Lambda}(x^1|g) \partial_2 \Lambda(x^2) = 0
\]

which implies
\[
c_{52} = 0,
\]
Analogously, if $\kappa_1 \neq 0$ then $c_{5k} = c_{3k} = 0$.

**Appendix 5. Proof of Proposition 5.5**

**Proposition.** If $c_{31} = c_{51} = \kappa_1 = 0$ then $c_{32} c_{52} = 0$.

For the proof, consider the 8th order terms in the decomposition $C(x|f, g)$ on $\hbar^2$.

The Jacobi identity (25) for (28) gives

$$
d_{2}^{ad}D_{4}(x|f, g, h) + \frac{\kappa_2^{2}c_{32}}{6}J_{1,3}(x|f, g, h) + \frac{\kappa_2^{2}c_{52}}{6}J_{1,5}(x|f, g, h) + c_{32}c_{52}J_{3,5}(x|f, g, h) + c_{52}^{2}J_{5,5}(x|f, g, h) = 0, \tag{A5.1}
$$

where

$$D_{4} = C_{4} + c_{53}\sigma_{\zeta_{1}} + c_{52}\sigma_{\zeta_{2}} \in D_{0}^{2}.$$  

Consider Eq. (A5.1) in the Domain $U^{2}$ In the case under consideration Eq. (A5.1) takes the form

$$\{h, \dot{D}_{4}\} + c_{32}c_{52}J_{3,52} + c_{52}^{2}(\dot{J}_{52,53} + J_{53,54}) = 0,$$

which implies

$$\omega^{ij}\partial_{j}(\dot{D}_{4} + \dot{\sigma}^{(4)})(x|f, g) = 2c_{32}c_{52}|f_{1}(x^{1})\theta(x^{2}|f_{2})g_{1}g_{2} - (f \leftrightarrow g)|\delta_{i}^{j}, \tag{A5.2}
$$

where

$$\sigma^{(4)}(x|f, g) = c_{32}c_{52}\sigma_{1}(x|f, g) + c_{52}^{2}\sigma_{4}(x|f, g).$$

Analogously to Appendix 1, we obtain from (A5.2)

$$c_{52}c_{32} = 0.$$

**Appendix 6. Proof of Proposition 5.6**

**Proposition.** If $c_{31} = c_{51} = \kappa_1 = 0$ and $c_{32} \neq 0$ then $c_{5i} = \kappa_{i} = 0$ for all $i$.

Indeed, if $c_{32} \neq 0$, then $c_{52} = 0$ and Eq. (A5.1) takes the form

$$d_{2}^{ad}D_{4}(x|f, g, h) + \frac{\kappa_2^{2}c_{32}}{6}J_{1,3}(x|f, g, h) = 0, \tag{A6.1}
$$

$$D_{4} = C_{4} + c_{53}\sigma_{\zeta_{1}} \in D_{0}^{2}.$$  

It follows from Eq. (A6.1) $\kappa_{2} = 0$.

Further, after some renaming, we find (up to similarity transformation)

$$C(x|f, g) = N_{\kappa_{2}^{2}\kappa_{i}^{2}\zeta_{4}}(x|f, g) + c_{3[4]}m_{3}(x|f, g) + c_{5[4]}m_{5}(x|f, g) - h^{8}c_{53}\sigma_{\zeta_{1}}(x|f, g) + h^{10}C_{5}(x|f, g) + O(h^{12}),$$

$C_{5}(x|f, g) \in D_{0}^{2}.$
Here

\[
\kappa_{[n]} = \hbar^2 \sum_{k=3}^n \hbar^{2(k-3)} \kappa_k, \quad \zeta_{[n]} = \sum_{k=1}^n \hbar^{2k} \zeta_n,
\]

\[
c_{3[n]} = \sum_{k=2}^n \hbar^{2k} c_{3k}, \quad c_{5[n]} = \sum_{k=3}^n \hbar^{2k} c_{5k}.
\]

The Jacobi identity \[26\] for \(C(x|f,g)\) gives

\[
d_2^\text{ad} D_5(x|f,g,h) + \frac{\kappa_2^2 c_{32}}{6} J_{1,3}(x|f,g,h) + c_{32} c_{53} J_{3,5}(x|f,g,h) = 0, \]

\[\text{(A6.2)}\]

\[
D_4 = C_4 + c_{54} \sigma_{\zeta_1} + c_{53} \sigma_{\zeta_2} \in D.
\]

Considering Eq. \[\text{(A6.2)}\] in Domain \(U^2\) we obtain

\[
c_{53} = 0
\]

and successively

\[
\kappa_k = c_{5k} = 0, \quad \forall k.
\]

**Appendix 7. Proof of Proposition 5.7**

**Proposition.** Let \(c_{31} = c_{51} = \kappa_1 = 0\) and \(c_{32} = 0\). Then \(c_{52} = 0\).

In this case, Eq. \[\text{(A5.1)}\] takes the form

\[
d_2^\text{ad} D_4(x|f,g,h) + \frac{\kappa_2^2 c_{52}}{6} J_{1,5}(x|f,g,h) + c_{52}^2 \tilde{J}_{5,5}(x|f,g,h) = 0,
\]

\[\text{(A7.1)}\]

\[
D_4 = C_4 + c_{53} \sigma_{\zeta_1} + c_{52} \sigma_{\zeta_2} + c_{52}^2 \sigma_4 \in D_0^2.
\]

Eq. \[\text{(A7.1)}\] coincides exactly with Eq. \[\text{(A3.2)}\] and consideration of Eq. \[\text{(A7.1)}\] in Domains \(U^2, V, W^2\) gives (analogously to Appendix 4)

\[
c_{52} = 0.
\]

**References**

[1] M. Gerstenhaber, Ann. Math. 79 (1964), 59–103; ibid. 99 (1974), 257–276.

[2] S. E. Konstein, A. G. Smirnov and I. V. Tyutin, Cohomologies of the Poisson superalgebra, Teor. Mat. Fiz., 143 (2005), 625; hep-th/0312109

[3] S. E. Konstein, A. G. Smirnov and I. V. Tyutin, General form of the deformation of the Poisson superbracket, Teor. Mat. Fiz., 148 (2006), 1011; hep-th/0401023

[4] S. E. Konstein and I. V. Tyutin, Cohomology of the Poisson superalgebra on spaces of superdimension \((2, n-1)\), Teor. Mat. Fiz., 145 (2005), 1619; hep-th/0411235

[5] S. E. Konstein and I. V. Tyutin, General form of the deformation of the Poisson superalgebra on \((2,2)\)-dimensional superspace, (in preparation).