Replication Error and Time Evolution of a Self-Replicating System

Dongsu Bak
Department of Physics, University of Seoul, Seoul 130-743, Korea

ABSTRACT
A self-replicating system where the elements belonging to a solution category can replicate themselves by copying their own informations, is considered. The information carried by each element is defined by an element of all the n multiple tensor product of a base space that consists of m different base elements. We assume that in the replication the processes of copying each base information are the same and independent from one another and that the copying error distribution in each process is characterized by a small variation with a quite small mean value. Concentrating on the number fluctuation of the informations in the copying process, we analyze the time evolution of the system. We illustrate the change of averaged number of informations carried by system objects and the variation of the number distribution as a function of time. Especially, it is shown that the averaged number of information grows in general after large number of generations.

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*email address: dsbak@mach.scu.ac.kr
1 Introduction

One of the essential features of the life system is that most of its element do possess an ability to replicate themselves by copying their own informations. The copying mechanisms in each element are mostly the same but its effectiveness is not perfect due to chemical and quantum fluctuations. If it were perfect, there should not be any change of the species of the system in its time evolution. The random fluctuation involved in this copying process is, hence, another important ingredient of the system in understanding its time evolution.

The natural consequences of the replication with random fluctuation are of great interest since they may explain main characteristics of our life system. Admitting the complication in treating the real life system, one may introduce a simplified version of the system that, however, comprises the essential features of our life system on the aspect of the replication and the distribution of the information numbers.

In this note, we shall consider the following simplified version. The system consists of objects that carry a characteristic information defined as an element belonging to all the $n$ multiple tensor products of a base space for all the nonnegative integer $n$. We shall then define the information number of an object by $N$ when it belongs to $N$-multiple tensor product space. The elements in the base space are fixed and their total number, $m$, is finite. In the real life system, for example, these base elements correspond to the four base acids. Namely, they are adenine, guanine, cytosine, and thymine. The information ensemble space is the place where the dynamics of interest occurs. A solution (moduli) space is defined as a subset of the information ensemble space with a restriction to the objects carrying the information element that may replicate themselves. We assume that the replication mechanism of a base element is mostly the same and the unit processes copying base information are independent events. Due to the chemical and quantum mechanical fluctuations, there might be copying errors that maps one specific element of the base space to an element in the ensemble space. Of course, the errors are quite small but still to be a main source of the dynamical flow of the system after generations. The errors include the change of the original base element into another base element, omission of the original base element and mapping the base element into more than a base element. The probability
of mapping the base element to more than two base units is quite small compared to the total probability of errors and, hence, the mean of the fluctuation is very close to the case of no errors. The mean of the errors is more or less inclined to the direction of growing of average numbers after one copy event because of the asymmetry reflected in the fact that there are no counterparts of mapping a base element to more than two base elements about the symmetry point of the mapping to just one element.

The unit time between subsequent generations will be set all the same and time flow will be measured by the number of generations. This implies that all the objects in the system at certain time have exactly the same numbers of ancestors born after an initial time. The next generation will be defined by the set of two identical descendents from each object of the original system which belongs to the solution category. The solution space is determined as a function of environment and the system itself. The properties of the solution space are not known mostly for realistic life systems. We shall introduce a minimal assumption that the density of solutions over the total ensemble space can be defined as a smooth function. This means that the solutions are densely distributed over the whole information configuration space.

Basic features of the time evolution of the system are as follows. Because of the copying errors, the consequence of all the replications of the objects of the original generation belonging to the solution space will be the next generation set whose elements may or may not belong to the solution space. After a unit time, the next generation again replicate its elements once they belong to the solution space. Repeating this process from generations to generations, the generations will flow in the information configuration space quite as the time flies, but the flow will be restricted to regions of near solution space. The errors propagate and the initial set diffuses into the new arena in the configuration space. What are the properties of the dynamics of the system as the number of generations grows?

In this note, we shall concentrate on the information number fluctuation of the system objects as a result of the replication. Especially, we will consider the averaged information number and their variation as a function of time. In this way, we will demonstrate that there is a finite probability of the appearance of objects carrying a large number of information compared to that of an initial ancestor.
2 Self replicating system and its evolution

As mentioned in the introduction, the self-replicating system consists of objects that carry an information element belonging to the information configuration space. To define the configuration space, we first introduce a base space $B$ with a finite number of elements:

$$B = \{b_1, b_2, \ldots, b_m\}. \quad (2.1)$$

The information configuration space is then the direct sum of all n-multiple tensor product,

$$E = \bigoplus_{n=0}^{\infty} B^n, \quad (2.2)$$

where $B^n$ denotes the n multiple tensor product of the base space B. For each element of the information configuration space, we specify its number of information by the order of the tensor product where it belongs. As a subset of the ensemble space, the solution space $S_k$ is composed of the information elements, whose objects have an ability to replicate themselves from the $k$-th generation to the $(k+1)$-th generation. As remarked earlier, we will assume that the solution space elements are densely distributed over the information ensemble space, so that one may define the density of the solutions $\rho(n, k)$

$$\rho(n, k) = Z_\rho \frac{N(S_k \cap B^n)}{N(B^n)}. \quad (2.3)$$

as a smooth function of $n$ where $N(A)$ is the number of elements in a set $A$ and $Z_\rho$ is a normalization constant. The number fluctuation involved in the replication event of a base element takes the a distribution $d(1 + \mu_0, \sigma_0)$ with a mean $1 + \mu_0$ and a standard deviation $\sigma_0$. If one denotes the probability of $l$ base elements resulted from a unit base by $p_l$ for a copy, one may infer that $p_0 \sim p_2 = p$ ($p_2 - p_0 \ll p$), $p_3 \ll p$ and the contribution of all the higher mode may be ignored as explained earlier. By an explicit computation, one finds that $\sigma_0^2 \sim 2p$ and $\mu_0 \sim 2p_3 + p_2 - p_0 \ll \sigma_0^2$. Here, the variation $\sigma_0^2$ is much smaller than one, and we use an approximation that one is describing the distribution as if $n$ is a continuous parameter. When one replicates an object carrying $n$ information, the number fluctuation associated with this is described by the normal distribution

$$\delta n = z(n(1 + \mu_0), \sqrt{n} \sigma_0) \quad (2.4)$$
owing to the independence of all the replication events as well as the central limit theorem. The informations of the system is also given as a subset $A_k$ of the configuration space at the k-th generation. The distribution of the system informations over the ensemble space at the k-th generation is then described by

$$\phi(n, k) = Z_\phi N(A_k \cap B^n),$$

where $\phi(n, 0)$ represents the initial distribution of the informations and $Z_\phi$ denotes a normalization factor. With help of the distribution function in (2.4), one finds the density distribution of the system in the next generation is determined by the diffusion process from the set $A_k \cap S_k$ at the k-th generation. Namely, all the element in $A_k \cap S_k$ are doubled by replications and the number distribution of each descendent is described by (2.4). This is summarized in terms of the density description by

$$\phi(n, k + 1) = \int_0^\infty dn' G(n, n') \rho(n', k) \phi(n', k).$$

where $G(n, n')$ is the propagator of the distribution

$$G(n, n') = \frac{1}{\sqrt{2\pi n'\sigma_0}} \exp\left\{ -\frac{(n - n' - n'\mu_0)^2}{2n'\sigma_0^2} \right\}.$$ (2.7)

In (2.6), the factor two by the doubling as a result of replication is absorbed into the normalization factor of the density function. By the mathematical induction, the density of the system information at arbitrary generation is obtained from the initial data by

$$\phi(n, k) = \int_0^\infty dn' P(n, n'; k) \phi(n', 0),$$

where the propagator is defined as

$$P(n, n'; k) = \left( \prod_{i=1}^{k-1} \int_0^\infty dn_i \right) \left( \prod_{j=0}^{k-1} G(n_{j+1}, n_j) \rho(n_j, j) \right),$$

(2.9)

with $n_k = n$ and $n_0 = n'$. The propersgator in (2.6) can be represented in a differential form for small $\mu_0$ and $\sigma_0^2$. To measure the smallness of the mean and the variation, we introduce new parameters $\mu$ and $\sigma$ by

$$\mu_0 = \mu\epsilon, \quad \sigma_0^2 = \sigma^2 \epsilon$$

(2.10)
such that the new parameter $\sigma^2$ is $O(1)$. To obtain the differential form of the propagation, we define the infinitesimal time evolution by

$$\psi(s, t + \epsilon) = \int_0^\infty ds G(s, s')\psi(s', t),$$

(2.11)

where one measures the time by $\epsilon$ multiplied by the number of generations $k$. Introducing a variable $\xi$ by

$$\frac{s - s'(1 + \mu \epsilon)}{\sigma \sqrt{s'}} = \xi \sqrt{\epsilon},$$

(2.12)

one may rewrite the above integral as

$$\psi(s, t + \epsilon) = \int_{-\infty}^{\infty} d\xi e^{-\xi^2/2} \left[ \frac{(1 + b \xi)}{(1 + \mu \epsilon)\sqrt{2\pi}} e^{-\xi^2/2} \psi \left( \frac{(1 + 2b \xi + 2b^2 \xi^2)s}{1 + \mu \epsilon}, t \right) + O(\epsilon^2) \right]$$

$$= \int_{-\infty}^{\infty} d\xi e^{-\xi^2/2} \frac{1 + b \xi}{\sqrt{2\pi}} \left[ (1 - \mu \epsilon)\psi + s(2b \xi + 2b^2 \xi^2 - \mu \epsilon)\psi' + 2b^2 \xi^2 s^2 \psi'' \right] + O(\epsilon^2)$$

(2.13)

with $b = \sigma \sqrt{\epsilon}/\sqrt{4(1 + \mu \epsilon)s}$. Integrating over the $\xi$-variable, one may obtain the following differential equation,

$$\frac{\partial}{\partial t} \psi(s, t) = \left[ \frac{\sigma^2}{2} \left( s^2 \frac{\partial^2}{\partial s^2} + 2 \frac{\partial}{\partial s} \right) - \mu s \frac{\partial}{\partial s} - \mu \right] \psi(s, t),$$

(2.14)

where the terms of $O(\epsilon^2)$ are ignored. Alternatively, the above equation may be presented by

$$\frac{\partial}{\partial t} \psi(r^2, t) = -H(p, x)\psi(r^2, t)$$

(2.15)

with the Hamiltonian

$$H(p, x) \equiv \frac{\sigma^2}{8}p^2 + \frac{i\mu}{2} x \cdot p + \mu = \frac{\sigma^2}{8} \left( p + \frac{2i\mu}{\sigma^2} x \right)^2 + \frac{\mu^2}{2\sigma^2} x^2$$

(2.16)

where the definition $r^2 = s = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and the four dimensional gradient $\nabla \equiv i\mathbf{p}$ are introduced. Although the Hamiltonian is not a Hermitian operator, the time evolution of the system is well defined.

Inclusion of the density contribution to the time evolution is a complicated problem. Some speculations on the dynamics with some generic density function will be relegated to the conclusion. Here, let us consider the case that it is possible to write the density function in a form

$$\rho(s, t) = \rho_0(t)(1 - \epsilon U(s, t) + O(\epsilon^2))$$

(2.17)
where the function $U(s,t)$ satisfies the requirement
\[ \epsilon \lim_{L \to \infty} \frac{1}{L} \int_0^L ds |U(s,t)| \ll 1. \] (2.18)

Note that the overall factor $\rho_0$ will be absorbed into the normalization of the distribution function $\phi$ without changing the probability amplitude. Thus the time evolution of the system function $\phi(s,t)$ for this case can be easily identified as
\[ \phi(r^2, t + \epsilon) = [1 - \epsilon H(p, x)][1 - \epsilon U(r^2, t)]\phi(r^2, t) + O(\epsilon^2), \] (2.19)
from the combination of (2.6) (2.14) and (2.17). Hence the differential equation describing the time evolution with the density function (2.17) is
\[ \frac{\partial}{\partial t} \phi(r^2, t) = -[H(p, x) + U(r^2, t)]\phi(r^2, t). \] (2.20)

As discussed in the introduction, the form of the potential $U(s,t)$ may depend upon the system function $\phi(s,t)$ because the system itself works as an environmental element. However, we won't further discuss this possibility in the following for the simplicity.

3 Solution and statistics of self-replicating system

In the preceding section, we discuss the time evolution process and the differential equation governing the system dynamics. For the case that one may characterize the system by the potential $U$ for the solution density contribution, the time evolution of the system follows the differential equation (2.20). Let us first consider the case $U = 0$, which implies that the density function is constant in its argument. Namely, the solution space distribution is uniform over the information configuration space. To find the dynamics, it is convenient to construct the kernel $K(r^2, r'^2; t)$ that is defined as a solution of the equation (2.20) with the initial condition
\[ \phi_0(r^2) = \delta(r^2 - r'^2). \] (3.1)

With help of the kernel, the solution for a general initial condition $\phi(r^2, 0)$ is obtained with
\[ \phi(r^2, t) = \int_0^\infty d(r'^2)K(r^2, r'^2; t)\phi(r'^2, 0). \] (3.2)
In order to find the kernel, we first introduce the function \( \tilde{\phi}(r^2, t) \) by
\[
\phi(r^2, t) = e^{\mu r^2/\sigma^2} \tilde{\phi}(r^2, t),
\]
and insert this into the equation (2.20). One finds that the equation for \( \tilde{\phi} \) now reads
\[
\frac{\partial}{\partial t} \tilde{\phi}(r^2, t) = -\tilde{H}(p, x) \tilde{\phi}(r^2, t)
\]
with a new Hamiltonian
\[
\tilde{H}(p, x) = \frac{\sigma^2}{8} p^2 + \frac{\mu^2}{2\sigma^2} x^2.
\]
It is interesting to note that this is the Hamiltonian that describes a simple harmonic oscillator in four dimensional flat Euclidean space. The kernel for the equation (3.4) is simply a product of the propagator for one dimensional simple harmonic oscillator [1]:
\[
\tilde{K}(x, x'; t) = \left( \frac{\mu}{2\pi \sinh \mu t} \right)^2 \exp \left\{ -\frac{2\mu}{\sigma^2 \sinh \mu t} [(x^2 + x'^2) \cosh \mu t - 2x \cdot x'] \right\},
\]
with the solution of the differential equation (3.4)
\[
g(x, t) = \int d^4x' \tilde{K}(x, x'; t) g(x', 0).
\]
for an arbitrary initial condition \( g(x, 0) \). The kernel for our system is then obtained by taking \( g(x', 0) = e^{-\mu r'^2/\sigma^2} \delta(r'^2 - r^2) \) in (3.7) and multiplying the factor \( e^{\mu r^2/\sigma^2} \). Using the integral representation of the Bessel function [2], one finds that the expression for the kernel reads
\[
K(r^2, r'^2; t) = e^{\mu r^2/\sigma^2} \int d^4x' \tilde{K}(x, x'; t) e^{-\mu r'^2/\sigma^2} \delta(r'^2 - r^2) = \frac{\mu r_0}{2\pi \sinh \mu t} \exp \left\{ \frac{2\mu}{\sigma^2} [(r^2 - r_0^2) - (r^2 + r_0^2) \coth \mu t] \right\} I_1\left( \frac{2\mu r_0}{\sigma^2 \sinh \mu t} \right).
\]
where \( I_1(x) \) is the first-kind Bessel function of imaginary argument. For the consistency of the equation (3.2), the kernel must satisfy the relation
\[
K(r^2, r'^2; t + t') = \int_0^\infty d(z^2) K(r^2, z^2; t) K(z^2, r'^2; t'),
\]
which may obtained by applying the equation (3.2) twice for time intervals \([0, t']\) and \([t', t + t']\). Upon using the expression (3.8), this composition rule may be checked
explicitly by a straightforward computation with help of the formulas for the definite integrals involving the Bessel function.\footnote{4}

Following the standard rule, the mean $M(t)$ and $V(t)$ variation of the number of information are defined by

\begin{align}
M(t) & \equiv \frac{\int_0^\infty ds \, s \phi(s,t)}{\int_0^\infty ds \, \phi(s,t)} \quad (3.10) \\
V(t) & \equiv \frac{\int_0^\infty ds \, s^2 \phi(s,t)}{\int_0^\infty ds \, \phi(s,t)} - M^2(t). \quad (3.11)
\end{align}

For the case of the initial condition $\phi_0(s) = \delta(s - s_0)$, one may compute explicitly the mean and variation noting that $\phi(s,t)$ is given by $K(s,s;0)$. For this initial condition, we first compute the total amplitude $P(t)$; the resulting expression reads

\begin{equation}
P(t) \equiv \int_0^\infty ds \, \phi(s,t) = 1 - \text{Exp}\left\{-\frac{2\mu s_0 e^{\mu t}}{\sigma^2 \sinh \mu t}\right\} \quad (3.12)
\end{equation}

As a function of $t$, $\mu$ and $\sigma$, the mean and variation for the initial condition are explicitly

\begin{align}
M(t) & = s_0 e^{2\mu t} P^{-1}(t) \quad (3.13) \\
V(t) & = \left[\sigma^2 s_0 P(t) e^{3\mu t} \mu^{-1} \sinh \mu t - s_0^2 \text{Exp}\{4\mu t - \frac{2\mu s_0 e^{\mu t}}{\sigma^2 \sinh \mu t}\}\right] P^{-2}(t). \quad (3.14)
\end{align}

By taking the limit that $\mu$ goes to zero from these expressions, one may obtain the mean and variation of the system with $\mu = 0$ and they are

\begin{align}
M_0(t) & = \frac{s_0}{1 - e^{-\lambda t}} \quad (3.15) \\
V_0(t) & = \frac{s_0^2 e^{\frac{\lambda t}{2}}}{(e^{\frac{\lambda t}{2}} - 1)^2} \left[\frac{2t}{\lambda} (e^{\frac{\lambda t}{2}} - 1) - 1\right]. \quad (3.16)
\end{align}

where $\lambda = 2s_0/\sigma^2$. For $t \ll \lambda$, one finds that the mean and variation are

\begin{align}
M_0(t) \sim s_0, \quad V_0(t) \sim 2s_0^2 t/\lambda, \quad (3.17)
\end{align}

which agree with those computed from the initial condition $\phi(s,0) = \delta(s - s_0)$. On the other hand, for $t \gg \lambda$, we have

\begin{align}
M_0(t) \sim \frac{\sigma^2 t}{2}, \quad V_0(t) \sim \left(\frac{\sigma^2 t}{2}\right)^2. \quad (3.18)
\end{align}
Hence the mean and variation are independent of the initial parameter $s_0$ and grow in a power law of time as the time gets larger. With the nonvanishing $\mu$, one finds that as far as $|\mu t| \ll 1$, the mean and the variation behave in the same way as the case with $\mu = 0$. On the other hand, for the case $\mu t \gg 1$, the mean and variation are approximated by

$$M(t) \sim \frac{s_0 e^{2\mu(t+\lambda)}}{e^{2\mu\lambda} - 1}, \quad (3.19)$$

$$V(t) \sim \frac{s_0^2 e^{2\mu(2t+\lambda)}}{(e^{2\mu\lambda} - 1)^2} \left[ \frac{1}{\mu\lambda} (e^{2\mu\lambda} - 1) - 1 \right]. \quad (3.20)$$

For the negative $\mu$, the large time (i.e. $t \gg -\mu^{-1}$) behaviors are simply $M(t) \sim \sigma^2/(4|\mu|)$ and $V(t) \sim \sigma^4/(4\mu)^2$ and hence only in this case the mean and variation of the system are bounded from above all the time.

One may further consider the case with nonvanishing potential $U(s,t)$. Of course, it is not possible to solve the system explicitly without an explicit form of the potential. However, the influence of the potential to the time evolution of the system is not so complicated to understand. Since the role of the potential $U$ appears as a weighting factor $e^{-U(s,t)}$ on the amplitude in each propagating interval $[t, t + \epsilon]$, the larger positive value of $U(s,t)$ at a certain position results in the more suppression of the amplitude, while the larger negative value leads to the more amplification of the probability amplitude. Especially, for the case that the harmonic potential term in (3.5) dominates the potential $U$ at large $s$, e.g.

$$|U(s,t)| \ll \mu^2 s/2\sigma^2 \quad \text{as } s \to \text{large}, \quad (3.21)$$

the large time features of the system mostly agrees with those in (3.19) and (3.20). For the case that $U$ is much larger than $U_0 = \mu^2 s/2\sigma^2$, the growth of the mean and variation are relatively suppressed. The other limiting case, $-U/U_0 \gg 1$, the mean and variation grow much faster in time compared to the case $U = 0$.

What happens to the case that the influence of the density cannot be presented in terms of the potential description? In this case, the amplification by the density multiplication in each generation are $O(1)$ and become huge after a large number of generations are passed. The distribution function of the system will be sharply
peaked at the maximum of the density after a large number of generations. This is characterized by the mean $M(t) \to \text{Max}(\rho(s,t))$ with $V(t) \to 0$. Thus, in this case one may conclude that there is no evolution of the system at all after a large time as far as the probability distribution of the system is concerned.

One may speculate that the real life system presides in between the two limiting cases of the $O(1)$ variation of the density function and $U = 0$.

4 Outlook

As a simplified version of the real life system, we have modeled the self-replicating system whose dynamics can be projected onto the space of information configurations. Each object in the system carries an information element in the configuration space. The unit copying process for the replication of the base information is characterized by the number fluctuation with the mean $\mu_0$ and the standard deviation $\sigma_0$, and each of unit copying is assumed to be an independent event. We define the solution space by the condition that its element has an enough information to replicate itself to the next generation. To characterize the properties of the solution space, we define the density function of the solution space over the information configuration space. We have shown that the dynamics of the system is described with help of the time evolution kernel and may be mapped into the evolution of the four dimensional harmonic oscillator for the case of uniform density of the solution space. In addition, we have discussed the system dynamics for the various case of nonuniform density. Especially, we have proved that the time evolution of the mean and variation of the information numbers over the system configuration are growing with time with a few exceptions. This exception compiles the case of the negative $\mu$, which is improbable in the realistic system.

In our model, some details of the real life system are not included for simplicity. For example, the unit time intervals between generations are not all the same but depend upon the generations and the information number variable in the real life system. The number of descendents from a system object after a generation is also dependent on the time and the number variable. Moreover, there are two kinds of
the replication processes, which are nothing but asexual and sexual reproductions. There may be also a probability to fail to produce something else rather than base elements in the unit copying process, though it is expected to be extremely small. As mentioned earlier, it may be the case that the density of the solution space is a functional of the system distribution $\phi(s,t)$. In this case, the equation governing the time evolution inevitably becomes nonlinear in $\phi(s,t)$ and this nonlinear effect may be important in understanding the decelerating force of the population growth due to the limitation of resources.

These kind of fluctuations may be important in understanding the local dynamical evolution at a certain local time interval $[t, t + \Delta t]$. However, if the fluctuations are averaged over the long time, the effects of these deviations may be effectively ignored without introducing any serious change in global pictures. Furthermore, the detailed description of these variations may be incorporated by the slight modifications of the model presented.

Nevertheless, the effect of these variation may be crucial in comparing the theory to the characteristics of the real life system because observations on the real life system is confined to a present short time interval. In this sense, the experimental implication of our model need more investigations because how to extract essential features of the real system dynamics from the present data depends considerably on the detailed form of local fluctuations.

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