Strong gravitational lensing with Gauss-Bonnet correction

J. Sadeghi\textsuperscript{a} and H. Vaez\textsuperscript{b}

\textsuperscript{a}Physics Department, Mazandaran University, P.O.Box 47416-95447, Babolsar, Iran
\textsuperscript{b}Young researchers club, Ayatollah Amoli branch, Islamic azad university, Amol, Iran

E-mail: pouriya@ipm.ir, h.vaez@umz.ac.ir

Received February 24, 2014
Revised May 6, 2014
Accepted May 11, 2014
Published June 12, 2014

Abstract. In this paper we investigate the strong gravitational lensing in a five dimensional background with Gauss-Bonnet gravity, so that in 4-dimensions the Gauss-Bonnet correction disappears. By considering the logarithmic term for deflection angle, we obtain the deflection angle $\hat{\alpha}$ and corresponding parameters $\bar{a}$ and $\bar{b}$. Finally, we estimate some properties of relativistic images such as $\theta_{\infty}$, $s$ and $r_m$.

Keywords: gravitational lensing, gravity

ArXiv ePrint: 1310.1701
1 Introduction

The deviation of the light rays in the gravitational fields is referred to gravitational lensing. The gravitational lensing (GL) in the weak limit has been used to test the General Relativity since its beginning [1, 2]. But, this theory in the weak limit was not able to describe the high bending and looping of the light rays. Hence, scientist community stated this phenomenon in the strong filed regime. In the strong field limit, the light rays pass very close to black hole and one set of infinitive relativistic “ghost” images would be produce on each side of black hole. These images are produced due to the light rays wind one or several times around the black hole before reaching to observer. At first, this phenomenon was proposed by Darwin [3]. Several studies of null geodesics in the strong gravitational field have been done in the past years [4]–[7]. In 2000, Virbhadra and Ellis showed that a source of light behind a schwarzschild black hole would produce an infinitive series of images on each side of the massive object [8]. Theses relativistic images are formed when the light rays travel very close to the black hole horizon, wind several times around the black hole before appearing at observer. By an alternative method, Frittelli et al. obtained an exact lens equation, integral expression for deflection and compared their results with Virbhadra et al. [9]. A new technic was proposed by Bozza et al. to find the position of the relativistic images and their magnification [10]. They used the first two terms of approximation to study schwarzschild black hole lensing. This method was applied to other works such as Eiroa, Romero and Torres studied a Reissner-Nordstrom black hole lensing [11, 12]; Petters calculated relativistic effects on microlensing events [14]. Afterward, the generalization of Bozza’s method for spherically symetric metric was developed in [15]. Bozza compared the image patterns for several interesting backgrounds and showed that by the separation of the first two relativistic images, we can distinguish two different collapsed objects. Further studies were developed for other black holes and metrics [16]–[32].

The gravitational lenses are important tools for probing the universe. In refs. [33, 34] Narasimha and Chitre predicted that the gravitational lensing of dark matter can give the useful data about the position of the dark matter in the universe. Also, the gravitational lens are used to detect the exotic objects in the universe, such as cosmic strings [35]–[37].

On the other hand, the gravitational theories in higher dimensions have attracted considerable attention. One of these higher dimension gravities is the supersymmetric string theory. Einstein-Gauss-Bonnet (EGB) theory, which emerges as the low-energy limit of this theory, can be considered as an effective model of gravity in higher dimensions. This theory yields a correction to Einstein-Hilbert action. The Gauss-Bonnet term involves up to second- order derivatives of the
metric with the same degrees of freedom as the Einstein theory [38, 39]. The variation of EGB action has different solutions and the spherically symmetric solution in the presence of Gauss-Bonnet gravity was obtained by Boulwar and Deser [40] and charged black hole one is found by Wiltshire [41]. The properties of the Gauss-Bonnet black holes have been studied in refs. [42]–[54]. In this paper we study the strong gravitational lensing and obtain the logarithmic deflection angle and corresponding coefficients. In the final we investigated some properties of relativistic images.

The paper is organized as follows: in section 2, we briefly present the Einstein-Gauss-Bonnet gravity. Section 3 is devoted to investigate the strong gravitational lensing in the presence of Gauss-Bonnet term. We consider the logarithmic term which was proposed by Bozza, and obtain its parameters \( \bar{a} \) and \( \bar{b} \). In section 4, some properties of relativistic images will be studied. Finally, in the last section we present summery.

2 Einstein-Gauss-Bonnet gravity

The action of Einstein-Gauss-Bonnet gravity in five dimensional is given by [40],

\[
I = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R + \frac{\alpha}{2} L_{GB} \right),
\]

(2.1)

where \( R \) and \( \alpha \) are Ricci scalar and Gauss-Bonnet constant respectively. \( G_5 \) is five-dimensional Newton’s constant and \( L_{GB} \) is the Gauss-Bonnet term as follows,

\[
L_{GB} = R^2 - 4 R_{ab} R^{ab} + R_{abcd} R^{abcd},
\]

(2.2)

here \( R_{ab} \) and \( R_{abcd} \) are Ricci tensor and Riemann tensor respectively. Note that the indexes run over the components of five dimensional space. The exact and spherically metric solution of the above action have been founded by Boulware and Deser [40],

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + c(r)d\Omega^2_3,
\]

(2.3)

where

\[
f(r) = 1 + \frac{r^2}{2a} \left( 1 - \sqrt{1 + \frac{8\alpha M}{r^4}} \right), \quad c(r) = r^2.
\]

(2.4)

Here \( M \) is related to ADM mass and note that we set \( G = c = 1 \). For simplicity, we introduce the dimensionless quantities as \( a = \frac{\alpha M}{r} \) and \( x = \frac{r}{\sqrt{2M}} \). So, we have,

\[
f(x) = 1 + \frac{x^2}{a} \left( 1 - \sqrt{1 + \frac{2a}{x^4}} \right).
\]

(2.5)

When \( a \) tends to zero the warp factor of the Myers-Perry metric is obtained [13]. The solution of \( f(x) = 0 \), \( x_h = \frac{1}{2} \sqrt{4 - 2a} \) is the horizon radius of the black hole. The variation of the horizon is plotted with respect to \( a/M \) in figure 1.
3 Lens equation, deflection angle with Gauss-Bonnet correction

The lens equation for a source of light and an observer situated at large distances from a lens(deflector) is given by \([18]\),

\[
D_{os} \tan \beta = \frac{D_{ol} \sin \theta - D_{ls} \sin(\hat{\alpha} - \theta)}{\cos(\hat{\alpha} - \theta)}.
\] (3.1)

Where, \(D_{ls}\) and \(D_{os}\) stand for the lens-source and observer-source diameter distance, respectively. The angular positions of source and images with respect to the optical axis (the line joining the observer and center of the lens) are represented by \(\beta\) and \(\theta\). The deflection of the light rays denotes by \(\hat{\alpha}\) which can be positive, \(\hat{\alpha} > 0\) (bending toward the lens) or be negative, \(\hat{\alpha} < 0\) (bending away from the lens). In the next section, we will obtain the deflection angle. The particular distance from the center of the lens to the null geodesic at the source position is called impact parameter, which is given by following expression,

\[
u = D_{ol} \sin \theta.
\] (3.2)

We can find the angular positions of images by the intersection of two functions \(\tan \theta - \tan \beta\) and \(\frac{D_{ol}}{D_{ls}}(\tan \theta + \tan(\hat{\alpha} - \theta))\) vs \(\theta\) for the same side and vs \(-\theta\) for opposite side. In addition to the primary and secondary image positions (due to the weak limit), there is a sequence of intersections that show the angular positions of the relativistic images. These points are very close to each other, so they are not distinguishable. For this reason, we call them relativistic images. These images are due to the bending of light rays more than \(3\pi/2\).
Now, we are going to investigate the deflection angle in the presence of Gauss-Bonnet correction gravity. By using the null geodesic equation for the following standard background metric,

\[ ds^2 = -A(r)dt^2 + A^{-1}(r)dr^2 + C(r)\, d\phi^2 + D(r)\, d\psi^2, \quad (3.3) \]

one can find the following equations,

\[ \dot{t} = \frac{E}{A(r)}, \]
\[ \dot{\phi} = \frac{L_\phi}{C(r)}, \]
\[ \dot{\psi} = \frac{L_\psi}{D(r)}, \quad (3.4) \]
\[ (\dot{r})^2 = \frac{1}{B(r)} \left[ \frac{D(r)E - A(r)\, L_\psi^2}{A(r)D(r)} - \frac{L_\phi^2}{C(r)} \right]. \quad (3.5) \]

where \( E \) is the energy of photon and \( L_\phi \) and \( L_\psi \) are angular momentums in \( \phi \) and \( \psi \) directions. Here a dot denotes derivation with respect to affine parameter. If we consider the \( \theta \) component of geodesic equations in the equatorial plane (\( \theta = \pi/2 \)), we have

\[ \dot{\phi} \left[ D(r)\dot{\psi} \right] = \dot{\phi}L_\psi = 0. \quad (3.6) \]

Here, if we consider \( \dot{\phi} = 0 \), the deflection angle of light ray becomes zero and this is illegal, therefore we set \( L_\psi = 0 \). For a light ray coming from infinity the deflection angle in the directions \( \phi \) is given by \([26, 27]\),

\[ \hat{\alpha}_\phi = I_\phi(x_0) - \pi, \quad (3.7) \]

where

\[ I(x_0) = 2 \int_{x_0}^{\infty} \left[ \frac{C(x)}{C(x_0)} A(x_0) - A(x) \right]^{-\frac{1}{2}} \frac{dx}{x}, \quad (3.8) \]

where \( x_0 \) is the closet approach distance for the light ray when it passes near the lens. The impact parameter for the closet approach is expressed by,

\[ u(x_0) = \sqrt{\frac{C(x_0)}{A(x_0)}} = x \sqrt{\frac{1}{x^2 -\left(1 - \frac{2a}{x} \right)^{\frac{1}{2}}}}, \quad (3.9) \]

The above relation is obtained from the null geodesic equation (3.5) with setting \( dr/d\phi = 0 \). By using (3.2) and (3.9) one can relate the image position to the closet approach and this relation allows us to write the deflection angle as a function of image position. The image position is plotted as a function of the closest approach in figure 2. We see that the image positions and distances between relativistic images reduce by increasing the Gauss-Bonnet parameter. For the large values of \( x_0 \) the curves coincide for any value of Gauss-Bonnet parameter and this means that primary and secondary image position remain unchanged. There is a minimum value for the closest approach that is called the photon sphere radius and is a \( r = \text{const} \) null geodesic. The photon sphere is the root of derivative of the impact parameter with respect to \( x_0 \) which is given by,

\[ x_{ps} = (4 - 2a)^{\frac{1}{4}}. \quad (3.10) \]
The dashed curve in the figure 1 shows the variation of photon sphere radius. It decreases with increasing the Gauss-Bonnet parameter and tends to zero at $\alpha = 2$. When $x_0$ asymptotically approaches the photon sphere radius, the photon reveals around the lens more times and the deflection angle diverges as $x_0$ tends to photon sphere. We can rewrite the equation (3.8) as,

$$I(x_0) = 2 \int_{0}^{1} F(z, x_0) \, dz,$$  \hspace{1cm} (3.11)

and

$$F(z, x_0) = \frac{1}{\sqrt{A(x_0) - A(x)\frac{c(z_0)}{c(x)}}},$$  \hspace{1cm} (3.12)

where $z = 1 - \frac{x}{x_0}$.

The function $F(z, x_0)$ diverges as $z$ approaches to zero. Therefore, we can split the integral (3.11) in two parts, the divergent part $I_D(x_0)$ and the regular one $I_R(x_0)$, as follows [15]

$$I_D(x_0) = 2 \int_{0}^{1} F_0(z, x_0) \, dz,$$  \hspace{1cm} (3.13)

$$I_R(x_0) = 2 \int_{0}^{1} [F(z, x_0) - F_0(z, x_0)] \, dz.$$  \hspace{1cm} (3.14)

Here we expand the argument of the square root in $F(z, x_0)$ up to the second order in $z$

$$F_0(z, x_0) = \frac{1}{\sqrt{p(x_0)z + q(x_0)z^2}},$$  \hspace{1cm} (3.15)
where

\[
p(x_0) = \frac{x_0}{c(x_0)} \left[ c'(x_0)f(x_0) - c(x_0)f'(x_0) \right] \\
= -\frac{2}{x_0^4 + 2a} \left( -x_0^4 - 2a + 2x_0^2 \sqrt{\frac{x_0^4 + 2a}{x_0}} \right)
\]

\[
q(x_0) = \frac{x_0^2}{2c(x_0)} \left[ 2c'(x_0)c(x_0)f'(x_0) - 2c(x_0)^2 f(x_0) + f(x_0)c(x_0)c''(x_0) - c^2(x_0)f''(x_0) \right] \\
= -\frac{x_0^8}{x_0^6 - 4x_0^4 a - 4a^2 + 6x_0^6 \sqrt{\frac{x_0^4 + 2a}{x_0}} + 4x_0^2 \sqrt{\frac{x_0^4 + 2a}{x_0}} a}{(x_0^4 + 2a)^2}
\]

(3.16)

(3.17)

As \( a \) goes to zero, \( p \) and \( q \) tend to five dimensional schwarzschild ones, \( p = -\frac{4}{x_0^2} + 2 \) and \( q = \frac{6}{x_0^2} - 1 \). For \( x_0 > x_{ps} \), \( p(x_0) \) is nonzero and the leading order of the divergence in \( F_0 \) is \( z^{-1/2} \), which have a finite result. As \( x_0 \to x_{ps} \), \( p(x_0) \) approaches zero and the divergence is of order \( z^{-1} \), that makes the integral divergent logarithmically. Therefore, the deflection angle can be approximated in the following form [15]

\[
\hat{\alpha} = -\bar{a} \log \left( \frac{u}{u_{ps}} - 1 \right) + \bar{b} + O(u - u_{ps}),
\]

(3.18)

where

\[
\bar{a} = \frac{1}{\sqrt{q(x_{ps})}} \approx \frac{\sqrt{2}}{2} + 0.128 a + 0.161 a^2
\]

\[
\bar{b} = -\pi + b_R + \bar{a} \log \frac{x_{ps}^2 [c''(x_{ps})C(x_{ps})F(x_{ps}) - C(x_{ps})F''(x_{ps})]}{u_{ps} \sqrt{F'(x_{ps})C(x_{ps})}} \approx 0.6902 + 0.154 a + 0.373 a^2,
\]

\[
b_R = I_R(x_{ps}), \quad u_{ps} = \sqrt{\frac{C(x_{ps})}{F(x_{ps})}}.
\]

(3.19)

When \( a \) tends to zero, we have \( a = \frac{\sqrt{2}}{2} \) and \( b = 0.6902 \), that these values belong to Myers-Perry metric [13]. Using (3.18) and (3.19), we can investigate the properties of strong gravitational lensing in the presence of Gauss-Bonnet correction. The variations of the \( u_{ps} \) is shown in figure 1. Also, coefficients \( \bar{a}, \bar{b} \), and the deflection angle \( \hat{\alpha} \) have been plotted with respect to the Gauss-Bonnet correction in figures 3-4. We see that by increasing \( \alpha \), the deflection angle \( \hat{\alpha} \) and \( \bar{a} \) increase and \( \bar{b} \) decreases. The deflection angle becomes diverge as \( \alpha \to 2 \).

4 Relativistic images properties

In the previous section, we investigated the strong gravitational lensing by using a simple and reliable logarithmic formula for deflection angle that was obtained by Bozza et al. and obtained corresponding parameters \( \bar{a} \) and \( \bar{b} \). Now we study some properties of relativistic images in the presence of Gauss-Bonnet gravity. When a source, lens, and observer are highly aligned, we can write the lens equation in strong gravitational lensing, as following [15]

\[
\beta = \theta - \frac{D_{ls}}{D_{os}} \Delta \alpha_n,
\]

(4.1)
where $\Delta \alpha_n = \alpha - 2n\pi$ is the offset of deflection angle in which all the loops are subtracted, and the integer $n$ indicates the $n$-th image. The image position $\theta_n$ and the image magnification $\mu_n$ can be approximated as obtained in ref. [10],

\begin{align}
\theta_n &= \theta^0_n + \frac{u_{ps}(\beta - \theta^0_n)}{\bar{a}D_{ls}D_{ol}} \left( 1 + e^{\frac{b-2n\pi}{a}} \right) \left( 1 + e^{\frac{b-2n\pi}{a}} \right) D_{os}, \\
\mu_n &= \frac{u_{ps} \left( 1 + e^{\frac{b-2n\pi}{a}} \right) \left( 1 + e^{\frac{b-2n\pi}{a}} \right) D_{os}}{\bar{a}D_{ls}D_{ol}^2}.
\end{align}

where

$$\theta^0_n = \theta_{ps} \left( 1 + e^{(b - m\pi)/a} \right),$$

$\theta^0_n$ is the angular position of $\alpha = 2n\pi$. They separate the outer most image $\theta_1$ from the others images which are packed together at $\theta_\infty$. Therefore, the separation between $\theta_1$ and $\theta_\infty$ and ratio of their magnification can be considered by,

\begin{align}
s &= \theta_1 - \theta_\infty, \\
\mathcal{R} &= \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n}.
\end{align}
The asymptotic position of the set of images $\theta_\infty$ can be obtained from the minimum of the impact parameter as,

$$\theta_\infty = \frac{\upsilon_{ps}}{D_{ol}}. \tag{4.7}$$

By considering equation (4.7), we can approximate equations (4.6) as,

$$s = \theta_\infty e^{\frac{\bar{b}}{2\pi}} - \frac{2\pi}{\bar{a}},$$

$$\mathcal{R} = e^{\frac{2\pi}{\bar{a}}}. \tag{4.8}$$

Another property that can be defined for relativistic images is the relative magnification of the outermost relativistic image with the other ones. This is shown by $r_m$ which is related to $\mathcal{R}$ as,

$$r_m = 2.5 \log \mathcal{R}. \tag{4.9}$$

If we suppose a five dimensional black hole with mass $4.31 \times 10^6 M_\odot$ (Galaxy center mass) and the distance between the observer and black hole is $D_{OL} = 8.5 kpc$ (The distance between the sun and galaxy center) [55], we can study the effect of the Gauss-Bonnet parameter on these quantities. Our results are presented in figures 4–6 and table 1.

5 Summary

The light rays can be deviated from a straight way in the gravitational field as predicted by General Relativity in which this deflection of light rays is known as gravitational lensing. In the strong field limit, the deflection angle of the light rays passing very close to the black hole, becomes so large
that, the light rays wind several times around the black hole before appearing at the observer. Therefore the observer would detect two infinite set of faint relativistic images produced on each side of the black hole. On the other hand, the gravitational theories in higher dimensions have been
Table 1. Numerical estimations for the coefficients and observables of strong gravitational lensing with Gauss-Bonnet correction. (Not that the numerical values for $\theta_\infty$ and $s$ are of order microarcsec).

| $\alpha/M$ | $\theta_\infty$ | $s$     | $r_m$       | $a$          | $b$          |
|-----------|----------------|--------|-------------|--------------|--------------|
| 0         | 20.0202002     | 0.001043418 | 9.647597725 | 0.707106781 | -0.690292235 |
| 0.3       | 19.6255146     | 0.001970453 | 8.894645615 | 0.766964988 | -0.777741402 |
| 0.6       | 19.18509246    | 0.00377615  | 8.071759368 | 0.845154255 | -0.92871426  |
| 0.9       | 18.68212151    | 0.008402205 | 7.15484996  | 0.953462589 | -1.06498697  |
| 1.2       | 18.08715205    | 0.020603386 | 6.10167655  | 1.118033989 | -1.29429156  |
| 1.5       | 17.33784593    | 0.058672906 | 4.823798862 | 1.414213562 | -1.761807496 |
| 1.8       | 16.1916094     | 0.210208553 | 2.973589325 | 2.294157338 | -3.6829744   |
| 1.95      | 15.12420331    | 0.480609627 | 1.364376354 | 5.00123     | -10.96179596 |

Figure 7. The relative magnification $r_m$ versus $\alpha$.

attracting considerable attention in recent decades. Einstein-Gauss-Bonnet theory that emerges as the low-energy limit of supersymmetric string theory, is one of the candidates for higher dimension theory. We considered five dimensional metric with Gauss-Bonnet correction and studied the strong gravitational lensing and obtained the deflection angle and corresponding parameters $\bar{a}$ and $\bar{b}$. We saw that by increasing $\alpha$, the deflection angle $\hat{\alpha}$ and $\bar{a}$ increase and $\bar{b}$ decreases. The deflection angle became diverge as $\alpha \rightarrow 2$.

Finally, we estimated some properties of relativistic images which can be detected by astronomical instruments. Our results have been presented in figures 5–7. In figures 5 and 7, the variations of $\theta_\infty$ and $r_m$, shown that the position of compacted images and relative magnification reduce with increasing $\alpha$. Also the angular separation is an increasing function and diverges, as $\alpha$ tends to two (figure 6). Furthermore, we saw that the position of images reduces with $\alpha$ (see figure 2). But you should note the decreasing rate of $\theta_\infty$ is more than $\theta_1$, therefore $s$ will be increasing function.
References

[1] A. Einstein, *Lens-Like Action of a Star by the Deviation of Light in the Gravitational Field*, Science 84 (1936) 506.

[2] P. Schneider, J. Ehlers and E.E. Falco, *Gravitational Lenses*, Springer-Verlag, Berlin (1992).

[3] C. Darwin, *The gravity field of a particle. I*, Proc. Roy. Soc. Lond. 249 (1959) 180.

[4] J.M. Bardeen, *Black Holes*, C. DeWitt and B.S. deWitt eds, Gordon and Breach 215 (1973).

[5] S.U. Viergutz, *Image generation in Kerr geometry. I. Analytical investigations on the stationary emitter-observer problem*, Astron. Astrophys. 272 (1993) 355.

[6] R.J. Nemiroff, *Visual distortions near a neutron star and black hole*, Am. J. Phys. 61 (1993) 619 [astro-ph/9312003] [inSPIRE].

[7] H. Falcke, F. Melia and E. Agol, *Viewing the Shadow of the Black Hole at the Galactic Center*, Phys. Rev. Lett. 528 (2000) 0113.

[8] K.S. Virbhadra and G.F.R. Ellis, *Schwarzschild black hole lensing*, Phys. Rev. D 62 (2000) 084003 [astro-ph/9904193] [inSPIRE].

[9] S. Frittelli, T.P. Kling and E.T. Newman, *Space-time perspective of Schwarzschild lensing*, Phys. Rev. D 61 (2000) 064021 [gr-qc/0001037] [inSPIRE].

[10] V. Bozza, S. Capozziello, G. Iovane and G. Scarpetta, *Strong field limit of black hole gravitational lensing*, Gen. Rel. Grav. 33 (2001) 1535 [gr-qc/0102068] [inSPIRE].

[11] E.F. Eiroa, G.E. Romero and D.F. Torres, *Reissner-Nordstrom black hole lensing*, Phys. Rev. D 66 (2002) 024010 [gr-qc/0203049] [inSPIRE].

[12] E.F. Eiroa, *Gravitational lensing by Einstein- Born-Infeld black holes*, Phys. Rev. D 73 (2006) 043002 [gr-qc/0511065] [inSPIRE].

[13] E.F. Eiroa, *A Braneworld black hole gravitational lens: Strong field limit analysis*, Phys. Rev. D 71 (2005) 083010 [gr-qc/0410128] [inSPIRE].

[14] A. Bhadra, *Gravitational lensing by a charged black hole of string theory*, Phys. Rev. D 67 (2003) 103006 [gr-qc/0210109] [inSPIRE].

[15] V. Bozza, *Quasiequatorial gravitational lensing by spinning black holes in the strong field limit*, Phys. Rev. D 78 (2008) 103005 [arXiv:0807.3872] [inSPIRE].

[16] A.N. Aliev and P. Talazan, *Gravitational Effects of Rotating Braneworld Black Holes*, Phys. Rev. D 80 (2009) 044023 [arXiv:0906.1465] [inSPIRE].
[23] K.S. Virbhadra and G.F.R. Ellis, *Gravitational lensing by naked singularities*, *Phys. Rev. D* 65 (2002) 103004 [arXiv:SPIRE].

[24] K.S. Virbhadra, *Relativistic images of Schwarzschild black hole lensing*, *Phys. Rev. D* 79 (2009) 083004 [arXiv:0810.2109] [arXiv:SPIRE].

[25] K.S. Virbhadra and C.R. Keeton, *Time delay and magnification centroid due to gravitational lensing by black holes and naked singularities*, *Phys. Rev. D* 77 (2008) 124014 [arXiv:0710.2333] [arXiv:SPIRE].

[26] K.S. Virbhadra and C.R. Keeton, *Role of the scalar field in gravitational lensing by black holes and naked singularities*, *Phys. Rev. D* 77 (2008) 124014 [arXiv:0710.2333] [arXiv:SPIRE].

[27] K.S. Virbhadra and C.R. Keeton, *Time delay and magnification centroid due to gravitational lensing by black holes and naked singularities*, *Phys. Rev. D* 77 (2008) 124014 [arXiv:0710.2333] [arXiv:SPIRE].

[28] G.N. Gyulchev and I.Zh. Stefanov, *Gravitational Lensing by Phantom Black holes*, *Phys. Rev. D* 87 (2013) 063005.

[29] Y. Liu, S. Chen and J. Jing, *Strong gravitational lensing in a squashed Kaluza-Klein black hole spacetime*, *Phys. Rev. D* 81 (2010) 124019 [arXiv:1002.0086] [arXiv:SPIRE].

[30] J. Sadeghi, A. Banijamali and H. Vaez, *Strong Gravitational Lensing in a Charged Squashed Kaluza-Klein Black hole*, *Astrophys. Space Sci.* 343 (2013) 559 [arXiv:1205.0805] [arXiv:SPIRE].

[31] J. Sadeghi, J. Naji and H. Vaez, *Strong gravitational lensing in a charged squashed Kaluza-Klein Gödel black hole*, *Phys. Lett. B* 728 (2014) 170 [arXiv:1310.4486] [arXiv:SPIRE].

[32] H. Saadat, *Gravitational lensing of STU black holes*, *Int. J. Theor. Phys.* 52 (2013) 4569 [arXiv:1306.0601] [arXiv:SPIRE].

[33] D. Narasimha and S.M. Chitre, *Lensing of extended sources by dark galactic haloes*, *Astron. J.* 97 (1989) 327.

[34] D. Narasimha, *Gravitational lens as a probe of dark matter in the universe, ICNAPP*, R. Cowsik eds., (1994), pg. 251.

[35] C.J. Hogan, R. Narayan, *Gravitational lensing by cosmic strings*, *Mon. Not. Roy. Astron. Soc.* 211 (1984) 575.

[36] A. Vilenkin, *Cosmic strings as gravitational lenses*, *Astrophys. J. Lett.* 282 (1984) L52.

[37] A. Vilenkin, *Looking for cosmic strings*, *Nature* 322 (1986) 613.

[38] B. Zwiebach, *Curvature Squared Terms and String Theories*, *Phys. Lett. B* 156 (1985) 315 [SPIRE].

[39] B. Zumino, *Gravity Theories in More Than Four-Dimensions*, *Phys. Rept.* 137 (1986) 109 [SPIRE].

[40] D.G. Boulware and S. Deser, *String Generated Gravity Models*, *Phys. Rev. Lett.* 55 (1985) 2656 [arXiv:SPIRE].

[41] D.L. Wiltshire, *Black Holes in String Generated Gravity Models*, *Phys. Rev. D* 38 (1988) 2445 [arXiv:SPIRE].

[42] B.-G. Cai and Q. Guo, *Gauss-Bonnet black holes in dS spaces*, *Phys. Rev. D* 69 (2004) 104025 [hep-th/031102] [arXiv:SPIRE].

[43] B.-G. Cai, *Gauss-Bonnet black holes in AdS spaces*, *Phys. Rev. D* 65 (2002) 084014 [hep-th/0109133] [arXiv:SPIRE].

[44] J. Crisostomo, R. Troncoso and J. Zanelli, *Black hole scan*, *Phys. Rev. D* 62 (2000) 084013 [hep-th/0003271] [arXiv:SPIRE].

[45] S. Ogushi and M. Sasaki, *Holography in Einstein Gauss-Bonnet gravity*, *Prog. Theor. Phys.* 113 (2005) 979 [hep-th/0407083] [arXiv:SPIRE].
[46] T. Clunan, S.F. Ross and D.J. Smith, On Gauss-Bonnet black hole entropy, 
*Class. Quant. Grav.* **21** (2004) 3447 [gr-qc/0402044] [nSPIRE].

[47] A. Barrau, J. Grain and S.O. Alexeyev, Gauss-Bonnet black holes at the LHC: Beyond the dimensionality of space, *Phys. Lett.* B **584** (2004) 114 [hep-ph/0311238] [nSPIRE].

[48] M. Bañados, Charged black holes in Gauss-Bonnet extended gravity, *Phys. Lett.* B **579** (2004) 13 [hep-th/0310160] [nSPIRE].

[49] M. Cvetić, S. Nojiri and S.D. Odintsov, Black hole thermodynamics and negative entropy in de Sitter and anti-de Sitter Einstein-Gauss-Bonnet gravity, *Nucl. Phys.* B **628** (2002) 295 [hep-th/0112045] [nSPIRE].

[50] S. Nojiri and S.D. Odintsov, de Sitter space versus Nariai black hole: Stability in d5 higher derivative gravity, *Phys. Lett.* B **523** (2001) 165 [hep-th/0110064] [nSPIRE].

[51] S. Nojiri, S.D. Odintsov and S. Ogushi, Cosmological and black hole braneworld universes in higher derivative gravity, *Phys. Rev.* D **65** (2002) 023521.

[52] G. Dotti and R.J. Gleiser, Gravitational instability of Einstein-Gauss-Bonnet black holes under tensor mode perturbations, *Class. Quant. Grav.* **22** (2005) L1 [gr-qc/0409005] [nSPIRE].

[53] I.P. Neupane, Thermodynamic and gravitational instability on hyperbolic spaces, *Phys. Rev.* D **69** (2004) 084011.

[54] I.P. Neupane, Black hole entropy in string-generated gravity models, *Phys. Rev.* D **67** (2003) 061501.

[55] S. Gillessen, F. Eisenhauer, S. Trippe, T. Alexander, R. Genzel et al., Monitoring stellar orbits around the Massive Black Hole in the Galactic Center, *Astrophys. J.* **692** (2009) 1075 [arXiv:0810.4674] [nSPIRE].