On the Continuous Cohomology of a semi-direct product Lie group

Naoya Suzuki

Abstract

Let $G$ be a Lie group and $H$ be a subgroup of it. We can construct a bisimplicial manifold $N(G \rtimes H)$ and the de Rham complex $\Omega^*(N(G \rtimes H))$ on it. This complex is a triple complex and the cohomology of its total complex is isomorphic to $H^*(B(G \rtimes H))$. In this paper, we show that the total complex of the double complex $\Omega^p(N(G \rtimes H))$ is isomorphic to the continuous cohomology $H^c_\ast(G \rtimes H; S^Q G^* \otimes S^Q H^*)$ for any fixed $q$.

1 Introduction

Let $G$ be a Lie group. In the theory of simplicial manifold, there is a well-known simplicial manifold $N(G)$ called nerve of $G$. The de Rham complex $\Omega^*(N(G))$ on it is a double complex, and the cohomology of its total complex is isomorphic to $H^*(BG)$. In [2], Bott proved the cohomology of its horizontal complex $\Omega^p(N(G))$ is isomorphic to the continuous cohomology $H^c_\ast(G; S^Q G^*)$ for any fixed $q$.

On the other hand, for a subgroup $H$ of $G$ we can construct a bisimplicial manifold $N(G \rtimes H)$ and the de Rham complex $\Omega^*(N(G \rtimes H))$ on it. This complex is a triple complex and the cohomology of its total complex is isomorphic to $H^*(B(G \rtimes H))$ [10].

In this paper, we show that the total complex of the double complex $\Omega^p(N(G \rtimes H))$ is isomorphic to the continuous cohomology $H^c_\ast(G \rtimes H; S^Q G^* \otimes S^Q H^*)$ for any fixed $q$. 

1
2 Review of the simplicial de Rham complex

In this section we recall the relation between the simplicial manifold $NG$ and the classifying space $BG$. We also recall the notion of the equivariant version of the simplicial de Rham complex.

2.1 The double complex on simplicial manifold

For any Lie group $G$, we have simplicial manifolds $NG$, $PG$ and simplicial $G$-bundle $\gamma: PG \to NG$ as follows:

For any simplicial manifold $\{X_{\ast}\}$, we can associate a topological space $\|X_{\ast}\|$ called the fat realization defined as follows:

$$
}\left( g_{2}, \ldots, g_{q} \right)
}_i=0
$$
$$
\left( g_{1}, \ldots, g_{i+1}, \ldots, g_{q} \right)
}_i=1, \ldots, q-1
$$
$$
\left( g_{1}, \ldots, g_{q-1} \right)
}_i=q
$$

$$
\left( \bar{g}_{1}, \ldots, \bar{g}_{q+1} \right)
$$

We define $\gamma: PG \to NG$ as $\gamma(\bar{g}_{1}, \ldots, \bar{g}_{q+1}) = (\bar{g}_{1}\bar{g}_{2}^{-1}, \ldots, \bar{g}_{q}\bar{g}_{q+1}^{-1})$.

For any simplicial manifold $\{X_{\ast}\}$, we can associate a topological space $\|X_{\ast}\|$ called the fat realization defined as follows:

$$
\left( g_{2}, \ldots, g_{q} \right)
\left( g_{1}, \ldots, g_{i+1}, \ldots, g_{q} \right)
\left( g_{1}, \ldots, g_{q-1} \right)
$$

$$
\left( \bar{g}_{1}, \ldots, \bar{g}_{1}, \bar{g}_{i+2}, \ldots, \bar{g}_{q+1} \right)
}_i=0, 1, \ldots, q
$$

$$
\left( \bar{g}_{1}, \ldots, \bar{g}_{q} \right)
$$

Here $\Delta^n$ is the standard $n$-simplex and $\varepsilon^i$ is a face map of it. It is well-known that $\|\gamma\|:\|PG\|\to\|NG\|$ is the universal bundle $EG \to BG$ (see [5] [8] [9], for instance).

Now we introduce a double complex on a simplicial manifold.
Definition 2.1. For any simplicial manifold \( \{X_\ast\} \) with face operators \( \{\varepsilon_\ast\} \), we have a double complex \( \Omega^{p,q}(X) := \Omega^q(X_p) \) with derivatives as follows:

\[
\delta := \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \quad d' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).
\]

For \( NG \) and \( PG \) the following holds.

Theorem 2.1 ([3] [5] [8]). There exist ring isomorphisms

\[
H^*(\Omega^*(NG)) \cong H^*(BG), \quad H^*(\Omega^*(PG)) \cong H^*(EG).
\]

Here \( \Omega^*(NG) \) and \( \Omega^*(PG) \) mean the total complexes.

2.2 Equivariant version

When a Lie group \( H \) acts on a manifold \( M \), there is the complex of equivariant differential forms \( \Omega^*_H(M) := (\Omega^*(M) \otimes \mathcal{SH}^*)^H \) with suitable differential \( d_H \) ([1] [4]). Here \( \mathcal{H} \) is the Lie algebra of \( H \) and \( \mathcal{SH}^* \) is the algebra of polynomial functions on \( \mathcal{H} \). This is called the Cartan Model. When \( M \) is a Lie group \( G \), we can define a double complex \( \Omega^*_H(NG(*)) \) below in the same way as in Definition 2.1.

\[
\begin{array}{ccc}
\Omega^p_H(G) & \xrightarrow{\delta} & \Omega^p_H(NG(2)) \\
\uparrow_{-d_H} & & \uparrow_{d_H} \\
\Omega^{p-1}_H(G) & \xrightarrow{\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*} & \Omega^{p-1}_H(NG(2)) \\
\uparrow_{d_H} & & \\
\Omega^{p-2}_H(NG(2)) & & \\
\uparrow_{d_H} & & \\
\vdots & & \\
\Omega^1_H(NG(p)) & \xrightarrow{(-1)^p \delta} & \Omega^1_H(NG(p+1)) \\
\uparrow_{(-1)^p d_H} & & \\
\Omega^0_H(NG(p)) & \xrightarrow{\sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*} & \Omega^0_H(NG(p+1)) \\
\end{array}
\]
3 The cohomology of the horizontal complex

At first, we recall the description of the cohomology of groups in terms of resolutions due to Hochschild and Mostow [7].

**Theorem 3.1** ([7]). If $G$ is a topological group and $M$ is a topological $G$-module, then the continuous cohomology $H^c(G; M)$ is isomorphic to the cohomology of the invariant complex

$$\text{Inv}_GM \rightarrow \text{Inv}_GX_0 \rightarrow \text{Inv}_GX_1 \rightarrow \cdots$$

for any continuously injective resolution $M \rightarrow X_0 \rightarrow X_1 \rightarrow \cdots$ of $M$.

Now we recall the result of Bott in [2], which gives the cohomology of the horizontal complex of $\Omega^*(NG)$.

**Theorem 3.2** (Bott,[2]). For any fixed $q$,

$$H^{p+q}_d(\Omega^q(NG)) \cong H^p_c(G; S^qG^*).$$

Here $G^*$ is a $\mathbb{R}$-module of left-invariant 1-forms on $G$.

**Proof.** Let $n$ denote the ordered set $\{0 < 1 < \cdots < n\}$ and $n^\sharp$ the underlying set of it. We define $CZ(n) := \mathbb{Z}(n^\sharp)$ as a free group generated by $n^\sharp$ then we have a natural arrow

$$r : CZ \rightarrow \mathbb{Z}$$

defined by

$$r(n) \left( \sum_{a = 0, \ldots, n} a_\alpha \alpha \right) = \sum a_\alpha, \quad a_\alpha \in \mathbb{Z}.$$  

Bott called the kernel of $r$ the suspension of $\mathbb{Z}$ and denote it $\Sigma \mathbb{Z}$.

We define the suspension of $G^*$ as $\Sigma G^* := CZ \otimes G^*$. Then there exists the following isomorphism:

$$\Omega^q(NG(n)) \cong \text{Inv}_G[k\{\Omega^0(PG(n)) \otimes \Lambda^q\Sigma G^*(n)\}].$$

Before we consider the cohomology of the horizontal complex $H^*_d(\text{Inv}_G[k\{\Omega^0(PG) \times \Lambda^q\Sigma G^*\}])$, we observe the complex $\mathfrak{F}^*_dG := k\{\Omega^0(PG(\ast)) \times \Lambda^q\Sigma G^*(\ast)\}$. 

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Lemma 3.1.

\[ H_\delta(\Omega^0(PG(n))) \cong \begin{cases} \mathbb{R} & (n = 0) \\ 0 & \text{otherwise} \end{cases}, \quad H_\delta(\Lambda^qSG^*(n)) \cong \begin{cases} S^qG^* & (n = q) \\ 0 & \text{otherwise} \end{cases} \]

So

\[ H^n_\delta(\mathcal{P}G) \cong \begin{cases} S^qG^* & (n = q) \\ 0 & \text{otherwise} \end{cases} \]

Since the cochain complex

\[ \mathcal{P}G : \Omega^0(PG(0)) \otimes \Lambda^qSG^*(0) \xrightarrow{\delta_q} \Omega^0(PG(1)) \otimes \Lambda^qSG^*(1) \xrightarrow{\delta_{q+1}} \cdots \]

is continuously injective, we obtain the following continuously injective resolution of \( S^qG^* \) from Lemma 3.1.

\[ S^qG^* = \ker\delta_q / \text{Im}\delta_q \xrightarrow{\delta_q} (\Omega^0(PG(q + 1)) \otimes \Lambda^qSG^*(q + 1)) / \text{Im}\delta_q \]

\[ \xrightarrow{\delta_{q+1}} \Omega^0(PG(q + 2)) \otimes \Lambda^qSG^*(q + 2) \xrightarrow{\delta_{q+2}} \cdots \quad (\text{exact}). \]

Therefore \( H^p_c(G; S^qG^*) \) is equal to the \( p \)-th cohomology of the complex below.

\[ \text{Inv}_G S^qG^* \xrightarrow{\delta_q} \text{Inv}_G[\Omega^0(PG(q + 1)) \otimes \Lambda^qSG^*(q + 1)] / \text{Im}\delta_q \]

\[ \xrightarrow{\delta_{q+1}} \text{Inv}_G[\Omega^0(PG(q + 2)) \otimes \Lambda^qSG^*(q + 2)] \xrightarrow{\delta_{q+2}} \cdots \]

So we obtain the following isomorphism.

\[ H^p_c(G; S^qG^*) \cong H^{p+q}_\delta(\text{Inv}_G[k\{\Omega^0(PG) \times \Lambda^qSG^*\}]). \]

\[ \square \]

Corollary 3.1 (Bott \([2]\)). If \( G \) is compact,

\[ H^p_\delta(\Omega^q(NG)) \cong \begin{cases} \text{Inv}_G S^qG^* & (p = q) \\ 0 & \text{otherwise} \end{cases} \]
4 The triple complex on bisimplicial manifold

In this section we construct a triple complex on a bisimplicial manifold.

A bisimplicial manifold is a sequence of manifolds with horizontal and vertical face and degeneracy operators which commute with each other. A bisimplicial map is a sequence of maps commuting with horizontal and vertical face and degeneracy operators. For a subgroup $H$ of $G$, we define a bisimplicial manifold $NG(*) \times NH(*)$ as follows:

$$NG(p) \times NH(q) := \overbrace{G \times \cdots \times G}^{p-times} \times \overbrace{H \times \cdots \times H}^{q-times}.$$

Horizontal face operators $\varepsilon^G_i : NG(p) \times NH(q) \to NG(p-1) \times NH(q)$ are the same as the face operators of $NG(p)$. Vertical face operators $\varepsilon^H_i : NG(p) \times NH(q) \to NG(p) \times NH(q-1)$ are

$$\varepsilon^H_i (\bar{g}, h_1, \cdots, h_q) = \begin{cases} (\bar{g}, h_2, \cdots, h_q) & i = 0 \\ (\bar{g}, h_1, \cdots, h_i h_{i+1}, \cdots, h_q) & i = 1, \cdots, q-1 \\ (h_q g h^{-1}_q, h_1, \cdots, h_{q-1}) & i = q. \end{cases}$$

Here $\bar{g} = (g_1, \cdots, g_p)$.

We define a bisimplicial map $\gamma_\times : PG(p) \times PH(q) \to NG(p) \times NH(q) as \gamma_\times (\bar{g}, \bar{h}_1, \cdots, \bar{h}_{q+1}) = (\bar{h}_{q+1}, \gamma_\times (\bar{g}) \bar{h}_q^{-1}, \gamma_\times (h_1, \cdots, h_{q+1}))$. Now we fix a semi-direct product operator $\cdot_\times$ of $G \times H$ as $(g, h) \cdot_\times (g', h') := (ghg'h^{-1}, hh')$, then $G \times H$ acts $PG(p) \times PH(q)$ by right as $(\bar{g}, \bar{h}) \cdot (g, h) = (h^{-1} \bar{g}gh, \bar{h}h)$ and $\| \gamma_\times \|$ is a model of $E(G \times H) \to B(G \times H)$.

**Definition 4.1.** For a bisimplicial manifold $NG(*) \times NH(*)$, we have a triple complex as follows:

$$\Omega^{p,q,r}(NG(*) \times NH(*)) := \Omega^*(NG(p) \times NH(q))$$

Derivatives are:

$$\delta_G := \sum_{i=0}^{p+1} (-1)^i (\varepsilon^G_i)^*, \quad \delta_H := \sum_{i=0}^{q+1} (-1)^i (\varepsilon^H_i)^* \times (-1)^p$$

$$d' := (-1)^{p+q} \times \text{the exterior differential on } \Omega^*(NG(p) \times NH(q)).$$
Theorem 4.1 ([10]). If $H$ is compact, there exist isomorphisms
\[ H(Ω^*_H(NG)) \cong H(Ω^*(NG \rtimes NH)) \cong H^*(B(G \rtimes H)). \]

Here $Ω^*_H(NG)$ means the total complex in subsection 2.2 and $Ω^*(NG \rtimes NH)$ means the total complex of the triple complex.

5 Main theorem

Theorem 5.1. For any fixed $q$,
\[ H^{δ+q}_δ(Ω^q(NG \rtimes NH)) \cong H^p_c(G \rtimes H; S^qG^* \otimes S^qH^*). \]
Here $δ := δ_G + δ_H$.

Proof. We identify $Ω^q(NG(n) \rtimes NH(m))$ with $\text{Inv}_{G \rtimes H}[Ω^0(\Omega^G(n)) \otimes Λ^qΣG^*(n) \otimes Ω^0(\Omega^H(m)) \otimes Λ^qΣH^*(m)]$.

Before we deal with the cohomology $H^*_δ(\text{Inv}_{G \rtimes H}[k\{Ω^0(\Omega^G) \times Λ^qΣG^* \times Ω^0(\Omega^H) \times Λ^qΣH^*\}])$, we observe the total complex of the double complex
\[ Ω^q_{δ_G} G \otimes Ω^q_{δ_H} H = k\{Ω^0(Ω^G) \times Λ^qΣG^* \times Ω^0(Ω^H) \times Λ^qΣH^*\}. \]

From Lemma 3.1, we obtain:
\[ H^n(Ω^q_G \otimes Ω^q_H) \cong \begin{cases} S^qG^* \otimes S^qH^* & (n = q) \\ 0 & \text{otherwise.} \end{cases} \]

Since the total complex
\[ k_δ(Ω^qG \times Ω^qH)(0) \xrightarrow{δ_0} k_δ(Ω^qG \times Ω^qH)(1) \xrightarrow{δ_1} \cdots \]
is continuously injective, we obtain the following continuously injective resolution of $S^qG^* \otimes S^qH^*$.
\[ S^qG^* \otimes S^qH^* = \text{Ker}δ_q/\text{Im}δ_{q-1} \xrightarrow{δ_q} k_δ(Ω^qG \times Ω^qH)(q + 1)/\text{Im}δ_q \xrightarrow{δ_{q+1}} k_δ(Ω^qG \times Ω^qH)(q + 2)/\text{Im}δ_{q+1} \cdots \text{(exact)}. \]
Therefore $H^p_c(G \rtimes H; S^qG^* \otimes S^qH^*)$ is equal to the $p$-th cohomology of the complex below.

$$\text{Inv}_{G \rtimes H}(S^qG^* \otimes S^qH^*) \xrightarrow{\delta_q} \text{Inv}_{G \rtimes H}[k_\delta(\mathfrak{P}^qG \times \mathfrak{P}^qH)(q + 1)/\text{Im}\delta_q]$$

$$\xrightarrow{\delta_{q+1}} \text{Inv}_{G \rtimes H}[k_\delta(\mathfrak{P}^qG \times \mathfrak{P}^qH)(q + 2)] \xrightarrow{\delta_{q+2}} \cdots$$

So we obtain the following isomorphisms.

$$H^p_c(G \rtimes H; S^qG^* \otimes S^qH^*) \cong H^{p+q}_\delta(\text{Inv}_{G \rtimes H}[k_\delta(\mathfrak{P}^qG \times \mathfrak{P}^qH)])$$

$$\cong H^{p+q}_\delta(\Omega^q(NG \rtimes NH)).$$

\begin{flushright}
$\square$
\end{flushright}

**Corollary 5.1.** If $G$ is compact,

$$H^p_\delta(\Omega^q(NG \rtimes NG)) \cong \begin{cases} 
\text{Inv}_{G \times G}(S^qG^* \otimes S^qG^*) & (p = q) \\
0 & \text{otherwise.}
\end{cases}$$

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