Topological symmetries of simply-connected four-manifolds and actions of automorphism groups of free groups

Shengkui Ye

June 27, 2022

Abstract

Let $M$ be a simply connected closed 4-manifold. It is proved that any (possibly finite) compact Lie group acting effectively and homologically trivially on $M$ by homeomorphisms is an abelian group of rank at most two, when $b_2(M) > 2$. As applications, let $\text{Aut}(F_n)$ be the automorphism group of the free group of rank $n$. We prove that any group action of $\text{Aut}(F_n)$ (and thus $\text{GL}_n(\mathbb{Z})$) $(n \geq 4)$ on $M \neq S^4$ by homologically trivial homeomorphisms factors through $\mathbb{Z}/2$.

1 Introduction

Let $F_n$ be a free group of rank $n$ and $\text{Aut}(F_n)$ the automorphism group, $\text{SAut}(F_n)$ its unique index-two subgroup. It is generally believed that any group action of $\text{Aut}(F_n)$ on a compact manifold $N^k$ factors a finite group when $k < n − 1$. This is a general version of the Zimmer program, studying actions of irreducible high-rank lattices on manifolds (cf. [7]). Bridson and Vogtmann [3] prove that any action of $\text{SAut}(F_n)$ on a sphere $S^k (k < n − 1)$ is trivial. Let $M$ be an orientable manifold of Euler characteristic not divisible by 6. The author [18] proves that any action of $\text{SAut}(F_n)$ on $M^k (k < n − 1)$ is trivial. In this article, we study actions of $\text{SAut}(F_n)$ on simply connected 4-manifolds. Note that the abelianization $F_n \rightarrow \mathbb{Z}^n$ induces epimorphisms $\text{Aut}(F_n) \rightarrow \text{GL}_n(\mathbb{Z})$ and $\text{SAut}(F_n) \rightarrow \text{SL}_n(\mathbb{Z})$. We prove the following.

Theorem 1.1 Let $\text{SAut}(F_n)$ be the unique index-two subgroup of the automorphism group of the free group $F_n$ and $M$ a closed 4-manifold with $H_1(M; \mathbb{Z}) = 0$. When $b_2(M) \geq 1$, any group action of $\text{SAut}(F_n)$ (and thus $\text{SL}_n(\mathbb{Z})$) $(n \geq 4)$ on $M$ by homologically trivial homeomorphisms is trivial.

Remark 1.2 (i) Combining Theorem 1.1 with a result of Bridson-Vogtmann [3], it is true that any group action of $\text{SAut}(F_n)$ $(n \geq 6)$ on any simply connected 4-manifold $M$ by homologically trivial homeomorphisms is trivial. Without restrictions of homological triviality, the group $\text{SAut}(F_n)$ could act non-trivially on some $M$ through a finite quotient group.

(ii) When $n = 3$, the group $\text{SAut}(F_3)$ could act through $\text{SL}_3(\mathbb{Z})$ on $\mathbb{C}P^2$ and $S^2$ (thus $S^2 \times S^2$) non-trivially. This means that the inequality $n \geq 4$ in Theorem 1.1 cannot
be improved. It is interesting to notice that the critical size \( n = 4 \) and the dimension of \( M \) (excluding \( S^4 \)) are the same, while it is generally required that \( n > \text{dim} \ M + 1 \) in Zimmer’s program.

The proof of Theorem 1.1 is based on the study of symmetries of 4-manifolds. Let \( M \) be a simply connected closed topological 4-manifold. Such a manifold was classified by Freedman in terms of the intersection pairing on second homology and the Kirby–Siebenmann invariant. As a first step, we study the topological transformation groups on \( M \). Since any finite group \( G \) acts freely on a simply connected closed 4-manifold \( M \) (the universal cover of a 4-manifold with fundamental group \( G \)) and acts effectively on the homology \( H_*(M; \mathbb{Z}) \) by the Lefschetz fixed-point theorem, we restrict our attention to the homologically trivial actions. It turns out that these actions are very restrictive. The locally linear version of the following result was firstly proved by McCooey [12].

**Theorem 1.3** Let \( G \) be a (possibly finite) compact Lie group acting effectively and homologically trivially on a topological closed 4-manifold \( M \) with \( H_1(M; \mathbb{Z}) = 0 \). When the second Betti number \( b_2(M) \geq 3 \), the group \( G \) is a subgroup of \( S^1 \times S^1 \).

**Remark 1.4** (i) The dihedral group \( D_n \) could act effectively and homologically trivially on \( S^2 \) and thus on \( S^2 \times S^2 \). This means that the bound of Betti numbers in Theorem 1.3 is sharp.

(ii) Let the cyclic group \( \mathbb{Z}/p \) of order prime \( p \) act on \( S^2 \) by rotations. The product \( \mathbb{Z}/p \times \mathbb{Z}/p \) acts on \( S^2 \times S^2 \) by rotating each factor with fixed points \( S^0 \times S^0 \). Taking connected sum along an invariant ball around a fixed point, the product \( \mathbb{Z}/p \times \mathbb{Z}/p \) could act homologically trivially on the connected sum \((S^2 \times S^2) \#(S^2 \times S^2)\). Therefore, the bound of ranks of the Lie group \( G \) in Theorem 1.3 is sharp.

The actions of finite groups on 4-manifolds are already studied by many people, eg. Edmonds [4, 5], Hambleton and Lee [8, 9], McCooey [12, 13] and Wilczyński [16], among others. For most of these works, the group actions are assumed to be locally linear (or smooth). The actions considered in this article are topological. Without locally linear assumptions, we cannot use the real representation of isotropy subgroups of a fixed point (see the first paragraph of Section 3 for an explicit list of difficulties). We will use the Smith theory of homology manifolds, equivariant cohomology, localization method and Borel formulas to deal with this difficulty.

The paper is organized as the following. In Section 2, we collect some basic facts and lemmas, most of which are already known in the literature. In Section 3, we study the group action of minimal non-abelian finite groups on simply connected 4-manifolds and Theorem 1.3 is proved. In the last section, we prove Theorem 1.1.

## 2 Preliminary

Let \( L = \mathbb{Z} \) or \( F_p \), the finite field with prime \( p \) elements. All homology groups are Borel-Moore homology groups with compact supports and coefficients in a sheaf \( \mathcal{A} \) of modules over \( L \). The homology groups of \( X \) are denoted by \( H_*(X; \mathcal{A}) \) and the cohomology groups (with coefficients in \( \mathcal{A} \) and compact supports) are denoted by \( H^*(X; \mathcal{A}) \). If \( \mathcal{A} \) is the constant sheaf, this is isomorphic to the Čech cohomology with compact supports. If \( F \) is a closed subset of \( X \), then sheaf cohomology satisfies \( H^k(X, F; \mathcal{A}) \cong H^k(X - F; \mathcal{A}) \). The
(co)homology $n$-manifold over $L$ considered in this article will be as in Borel [1]. Roughly speaking, a homology $n$-manifold over $L$ (denoted by $n$-$hm_L$) is a locally compact Hausdorff space that has a local cohomology structure (with coefficient group $L$) resembling that of the Euclidean $n$-space. Topological manifolds are (co)homology manifolds over $L$. Homology manifolds satisfy Poincaré duality between Borel-Moore homology and sheaf cohomology ([2], Thm 9.2, p.329).

We need several lemmas. The following result is generally called the Local Smith Theorem (cf. [2] Theorem 20.1, Prop 20.2, pp. 409-410).

**Lemma 2.1** Let $p$ be a prime and $L = F_p$. The fixed point set of any action of the cyclic group $\mathbb{Z}/p$ of order $p$ on an $n$-$hm_L$ is the disjoint union of (open and closed) components each of which is an $r$-$hm_L$ with $r \leq n$. If $p$ is odd then each component of the fixed point set has even codimension.

The following lemma is from Bredon [2], Theorem 2.5, p.79.

**Lemma 2.2** Let $G$ be a group of order 2 operating effectively on an $n$-cm over $\mathbb{Z}$, with non-empty fixed points. Let $F_0$ be a connected component of the fixed point set of $G$, and $r = \dim F_0$. Then $n - r$ is even (respectively odd) if and only if $G$ preserves (respectively reverses) the local orientation around some point of $F_0$.

The following lemma is from Bredon [2], Theorem 16.32, p.388.

**Lemma 2.3** If $X$ is a second countable $n$-$hm_L$, with or without boundary, and $n \leq 2$, then $X$ is a topological $n$-manifold.

Let a finite group $G$ act on a space (usually an $n$-$hm_L$) $X$. The group $G$ acts on the product $X \times EG$ diagonally, where $EG$ is the total space of a classifying space $BG$. The Borel construction is the quotient space of $X \times EG$, denoted by $X_G$. The Leray-Serre spectral sequence for the fibration $X \to X_G \to BG$ is

$$H^i(G; H^j(X)) \Rightarrow H^{i+j}(X_G),$$

which is also called the Borel spectral sequence. Let $R$ be a PID. When $G$ acts trivially on $H^*(X; R)$ (which is a finitely generated free $R$-module) and the Borel spectral sequence degenerates, we have $H^*(X_G; R) \cong H^*(BG; R) \otimes H^*(X; R)$ as graded modules (cf. [15], Proposition 1.18 of Chapter III, page 182). Let $\Sigma = \{x \in X \mid \text{the stabilizer } G_x \neq 1\}$ be the singular set. Let $S \supseteq \{1\}$ be a multiplicative set of $H^*(G)$ and $X^S = \{x \in X \mid S \cap \ker(H^*(G) \to H^*(G_x)) = \emptyset\}$. The localization theorem says that the restricted homomorphism

$$S^{-1}H^*(X_G) \to S^{-1}H^*(X^S_G)$$

is an isomorphism (cf. Hsiang [10], p.40).

We first prove the following collapsing of spectral sequences.

**Lemma 2.4** Let $G$ be a finite group acting on a manifold $M^{2n}$ with vanishing odd-dimensional integral cohomology groups and torsion-free even dimensional integral cohomology groups. Suppose that the group action is homologically trivial and $H^i(G; \mathbb{Z}) = 0$ for each odd $i$. Then the Borel spectral sequence $H^i(BG; H^j(M; \mathbb{Z})) \Rightarrow H^{i+j}(M_G; \mathbb{Z})$ collapses.
Proof. The differential map
\[ d_s : H^i(BG; H^j(M; \mathbb{Z})) \to H^{i+s+1}(BG; H^{j-s}(M; \mathbb{Z})) \]
is trivial when \( s \) is even, since the odd dimensional cohomology group of \( G \) is trivial. When \( s \) is odd, the map \( d_s \) is still trivial since either \( H^j(M; \mathbb{Z}) \) or \( H^{j-s}(M; \mathbb{Z}) \) is trivial.

The following is essentially Cor. 9.3 of Bredon [2] (p.249).

Lemma 2.5 The inclusion \( \Sigma \to M \) induces isomorphism \( H^i(M_G; A) \to H^i(\Sigma_G; A) \) for any coefficients \( A \) and \( i > n \).

We will need the following lemma, part of whose proof is already contained in Prop. 2.4 of Edmonds [4].

Lemma 2.6 Let \( G = \mathbb{Z}/p \) be a cyclic group of a prime order \( p \) acting effectively and homologically trivially on a simply connected closed topological 4-manifold \( M \). The fixed point set \( M^G \) is a disjoint union of spheres \( S^2 \) and discrete points.

Proof. By the Smith theory, the fixed point set \( M^G \) is a homological manifold over \( \mathbb{Z}/p \) of codimension even (cf. Lemma 2.1 and Lemma 2.2). Lemma 2.3 implies that \( M^G \) is a topological manifold. It is enough to prove the first Betti number of \( M^G \) is zero. But this is already known by Edmonds [4]. For convenience, we repeat the proof.

By Lemma 2.5 \( H^5(M_G; \mathbb{Z}) \cong H^5(\Sigma_G; \mathbb{Z}) \). When \( p \) is prime, the action of \( G = \mathbb{Z}/p \) is semi-free and \( \Sigma \) is the fixed point set. Therefore, we have the following isomorphisms
\[
H^5(M_G; \mathbb{Z}) \cong H^5(\Sigma_G; \mathbb{Z}) = H^5(\Sigma \times BG; \mathbb{Z}) \cong \bigoplus_{i+j=5} H^i(G; H^j(F; \mathbb{Z}))
\]
\[
\cong H^4(G; H^1(\Sigma; \mathbb{Z})).
\]

Note that \( H^i(G; \mathbb{Z}) = 0 \) when \( i \) is odd and \( \mathbb{Z}/p \) when \( i > 0 \) is even. This implies that the Borel spectral sequence \( H^i(G; H^j(M; \mathbb{Z})) \Rightarrow H^{i+j}(M_G; \mathbb{Z}) \) collapses by Lemma 2.3. Therefore, the \( p \)-rank of \( H^5(M_G; \mathbb{Z}) \) is the same as that of \( \bigoplus_{i+j=5} H^i(G; H^j(M; \mathbb{Z})) \cong 0 \). Thus the first Betti number \( b_1(\Sigma) = 0 \), which gives that \( \Sigma \) is a disjoint union of spheres \( S^2 \) and discrete points.

We need two more lemmas from Edmonds [4] (Prop. 1.2 and Cor. 2.6).

Lemma 2.7 Let \( g \) be a periodic map of prime order \( p \) which acts orientation-preservingly on a closed four-manifold \( M \) with \( H_1(M; \mathbb{Z}) = 0 \). Then the Euler characteristic \( \chi(\text{Fix}(g)) = \chi(M) = 2 + b_2(M) \).

Lemma 2.8 Let \( g \) be a periodic map of prime order \( p \) which acts orientation-preservingly on a closed four-manifold \( M \) with \( H_1(M; \mathbb{Z}) = 0 \). If \( \text{Fix}(g) \) is not purely 2-dimensional; then the 2-dimensional components of \( \text{Fix}(g) \) represent independent elements of \( H_2(M; F_p) \). If it is purely two-dimensional; and has \( k \) two-dimensional components, then the two-dimensional components span a subspace of \( H_2(M; F_p) \) of dimension at least \( k - 1 \); with any \( k - 1 \) components representing independent elements.

Corollary 2.9 Let \( M \) be a closed four-manifold \( M \) with \( H_1(M; \mathbb{Z}) = 0 \) and \( G \) a finite group acting on \( M \) effectively and homologically trivially. Suppose that \( s \in G \) is of order \( p \) and \( t \) is a normalizer of the cyclic subgroup \( \langle s \rangle \). When \( b_2(M) \geq 3 \), the element \( t \) acts invariantly and homologically trivially on each 2-sphere in the fixed point set \( \text{Fix}(s) \).
Proof. When $b_2(M) \geq 3$, the Euler characteristic $\chi(M^s) = 2 + b_2(M) \geq 5$ by Lemma 2.7. Any two spheres in $\text{Fix}(s)$ represent linearly independent elements of $H_2(M; F_p)$ by Lemma 2.8. Therefore, the element $t$ acts trivially on the homology classes and preserves each sphere component.

Corollary 2.9 implies that when $b_2(M) \geq 3$, the fixed point set of an elementary abelian $p$-group acting effectively and homologically trivially on a simply connected 4-manifold $M$ is a union of discrete points and pairwise disjoint 2-spheres.

3 Finite group actions on 4-manifolds

In this section, we will prove Theorem 1.3. We will follow the strategy of McCooey [12]. However, there are several places where McCooey's argument does not directly generalize. When the group action of $G$ on $M$ is locally linear, the group $G$ acts linearly on the ‘tangent space’ $T_p M$ for any fixed point $p \in \text{Fix}(G)$. This fact is used repeatedly in the proof of Proposition 4, the applications of Lemma 10 to restrict the types of fixed points, the local representation on page 845 of [12] and so on. Without the local linearity, the singular set of the group action could be very complicated.

We will prove that no non-abelian finite group can act effectively, homologically trivially on a simply connected closed 4-manifold when the second Betti number is bigger than 2. The proof is by contradiction. Suppose that there is such an action. There would be an action of a minimal non-abelian group $D$ (i.e. any proper subgroup of $D$ is abelian). Let $\Sigma$ be the singular set. We will prove that this is impossible by computing the cohomology groups $H^*(M_D)$ and $H^*(\Sigma_D)$, which are isomorphic by Lemma 2.5 in high dimensions.

First, let us recall the knowledge of minimal non-abelian finite groups. A non-abelian finite group is minimal if any proper subgroup is abelian. The minimal non-abelian finite groups are classified as follows according to the rank of their elementary abelian subgroups ($p, q$ are prime numbers, cf. [14, 17], Section 4 of [12]):

- In rank 1, we have the groups
  $$\mathbb{Z}/p \rtimes \mathbb{Z}/q^n,$$
  (where the semi-direct product automorphism has order $q$) and the quaternion group
  $$D_2^a = \langle a, b \mid a^4 = 1, a^2 = b^2, [a, b] = a^2 \rangle;$$

- In rank 2, we have the groups
  $$(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/q^n,$$
  (where the semi-direct product automorphism $\sigma$ has order $q$ and $\sigma$ acts trivially on any proper invariant submodule of $\mathbb{Z}/p \times \mathbb{Z}/p$),
  $$\mathbb{Z}/p^m \rtimes \mathbb{Z}/p^n = \langle a, b \mid a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^{m-1}}, m \geq 2 \rangle$$
  and $(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$. In this case, there is a normal rank-two subgroup.

- In rank 3 and higher, we will not need the structure.
The case of rank-1 groups will be proved in Theorem 3.5 (for \(\mathbb{Z}/p \times \mathbb{Z}/q^n, n = 1\)), Theorem 3.8 (for \(\mathbb{Z}/p \times \mathbb{Z}/q^n, n > 1\)) and Theorem 3.9 (for the quaternion group \(D_2\)). The case of high-rank groups will be proved in Theorem 3.15.

In lots of cases, we will need the following information on cohomology groups.

**Lemma 3.1** The cohomology groups

\[
H^i(\mathbb{Z}/p \times \mathbb{Z}/q^n; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } i = 0 \\
\mathbb{Z}/q^n, & \text{if } i \text{ is even but } 2q \nmid i \\
\mathbb{Z}/p \bigoplus \mathbb{Z}/q^n, & \text{if } 2q| i \\
0, & \text{otherwise.}
\end{cases}
\]

\[
H^i(D_2; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } i = 0 \\
\mathbb{Z}/2 \bigoplus \mathbb{Z}/2, & \text{if } i \equiv 2 \mod 4 \\
\mathbb{Z}/8, & \text{if } i \equiv 0 \mod 4, i > 0 \\
0, & \text{otherwise.}
\end{cases}
\]

**3.1 Singular sets**

In this subsection, we will study the singular set \(\Sigma\) for the action of the minimal non-abelian group \(D = \mathbb{Z}/p \times \mathbb{Z}/q^n\) \((n \geq 1)\) on a 4-manifold \(M\). Let \(\mathbb{Z}/p = \langle s \rangle\) be a cyclic group of prime order \(p\) and \(\mathbb{Z}/q^n = \langle t \rangle\) a cyclic group of prime order \(q^n\) acting on \(\mathbb{Z}/p\).

Note that \(q \mid p - 1\) and \(tst^{-1} = s^k\) for some \(1 < k < p\). For example, when \(q = 2, n = 1\), we have that

\[D = D_p = \mathbb{Z}/p \times \mathbb{Z}/2 = \langle s, t : s^p = t^2 = 1, tst^{-1} = s^{-1} \rangle\]

is the dihedral group.

From the definition that \(\Sigma = \{x \in M \mid \text{the stabilizer } D_x \neq \{e\}\}\), we know that \(\Sigma = \bigcup_{g \in G \setminus \{e\}} \text{Fix}(g)\).

We consider the case when \(n = 1\) first. For any integer \(i\), we have \(tst^{-1} = s^i\) for some integer \(j\). For any integers \(k, 0 < l < q\), we have \(\langle s^k t^l \rangle = \langle s^k t \rangle\) for some \(k'\). Therefore, the singular set \(\Sigma = \bigcup \text{Fix}(s) \cup \bigcup_{i=0}^{p-1} \text{Fix}(s^i t)\). The following lemma gives the global picture of the singular set.

**Lemma 3.2** Let the group \(D = \mathbb{Z}/p \times \mathbb{Z}/q\) act effectively homologically trivially on a closed 4-manifold \(M\) with \(H_1(M; \mathbb{Z}) = 0\) by homeomorphisms. The singular set \(\Sigma = \bigcup \text{Fix}(s) \cup \bigcup_{i=0}^{p-1} \text{Fix}(s^i t)\) satisfies the following:

1. Each fixed point set \(\text{Fix}(s^i t)\) or \(\text{Fix}(s)\) is a disjoint union of 2-spheres and discrete points. Moreover, the Euler characteristic \(\chi(M) = \chi(\text{Fix}(s)) = \chi(\text{Fix}(s^i t))\) for each \(i = 0, ..., p - 1\);

2. The intersection \(\text{Fix}(s) \cap \text{Fix}(s^i t)\) (or \(\text{Fix}(s^i t) \cap \text{Fix}(s^j t), i \neq j\)) is a part of the global fixed point set \(\text{Fix}(D) = M^D\);

3. The global fixed point set \(M^D\) is a disjoint union of (possibly empty) discrete points, 1-spheres and 2-spheres.

**Proof.** Since each element of \(\{s^i t \mid i = 0, ..., p - 1\}\) is of prime order \(q\) and the group action is orientation-preserving, it follows Lemma 2.6 that the fixed point set is a disjoint
union of 2-spheres and discrete points. The equalities of Euler characteristics are classical (cf. Lemma 2.7).

Since any two distinct elements in \{s, st \mid i = 0, \ldots, p - 1\} generate the whole group \(D\), the claim (2) is proved. Each component of \(M^D\) comes from an action of \(t\) on \(\text{Fix}(s)\), which is discrete or an \(r\)-sphere \((r = 0, 1, \text{or } 2)\). This proves (3). \(\square\)

From (1) of the previous lemma, a connected component in \(\text{Fix}(t)\) or \(\text{Fix}(s)\) is either a discrete point or a 2-sphere. A connected component in the intersection \(\text{Fix}(t) \cap \text{Fix}(s)\) can be either a discrete point or a whole 2-sphere in the singular set \(\Sigma\). The following result implies that two distinct 2-spheres in \(\text{Fix}(t) \cup \text{Fix}(s)\) cannot have a non-trivial intersection.

**Theorem 3.3** Let the group \(D = \mathbb{Z}/p \rtimes \mathbb{Z}/q\) act effectively homologically trivially on a closed 4-manifold \(M\) with \(H_1(M; \mathbb{Z}) = 0\) by homeomorphisms. Suppose that a discrete point \(x \in \text{Fix}(D)\) is contained in a non-discrete component of \(\text{Fix}(\mathbb{Z}/q)\). Then \(x\) is a discrete point in \(\text{Fix}(\mathbb{Z}/p)\).

**Proof.** It’s already known that \(\text{Fix}(\mathbb{Z}/p)\) is a union of 2-spheres and discrete points. Suppose that \(x \in S^2 \subset \text{Fix}(\mathbb{Z}/q)\). Suppose that \(tst^{-1} = s^k\) for some \(1 < k < p\). Then \(s^iS^2 = \text{Fix}(s^k)\) for each \(1 \leq i \leq p - 1\). When \(x\) is discrete in \(\text{Fix}(D)\), we know that \(\cap_{i=1}^{p-1} s^iS^2 = \{x\}\). Let \(U\) be a \(D\)-invariant open small ball containing \(x\) and \(\Sigma' = \Sigma \cap U\). Then we have \(H_c^k(U_D) = H_c^k(\Sigma'_G)\) when \(k > 4\) by Lemma 2.3 (where \(H_c^k(\Sigma')\) is the cohomology group with compact supports). Note that \(H_c^k(U) = \mathbb{Z}\) when \(k = 4\) and 0, when \(k \neq 4\). Moreover, \(H_c^k(\Sigma') = \lim_{\longrightarrow} H^k(\Sigma', \Sigma' - K)\) (the direct limit of the relative singular cohomology group with \(K\) ranges over compact subsets).

Suppose that \(x\) belongs to a 2-sphere \(S^2_p\) component of \(\text{Fix}(\mathbb{Z}/p)\). Then \(\Sigma'\) is a union of \(p+1\) 2-discs with a common intersection point \(x\). Note that \(\mathbb{Z}/q\) acts invariantly on the 2-sphere \(S^2_p\). Therefore, \(H_c^2(\Sigma') = \mathbb{Z}^{p+1}\) and \(H_c^1(\Sigma') = \mathbb{Z}^p\) (viewed as the \(D\)-module \(\mathbb{Z}[D] \otimes_{\mathbb{Z}/q} \mathbb{Z}\)) and \(H_c^0(\Sigma') = 0\).

When \(q = 2\), we have the following

\[
H^6(D; H_c^2(\Sigma')) \cong H^6(D; \mathbb{Z}) \bigoplus H^6(\mathbb{Z}/q; \mathbb{Z}) \\
\cong \mathbb{Z}/q \bigoplus \mathbb{Z}/q,
\]

\[
H^7(D; H_c^1(\Sigma')) \cong H^7(D; \mathbb{Z}[D] \otimes_{\mathbb{Z}/q} \mathbb{Z}) \\
\cong H^7(\mathbb{Z}/q; \mathbb{Z}) = 0.
\]

Therefore, the Borel spectral sequence \(H^i(D; H^j_l(\Sigma')) \implies H^i_l(\Sigma'_G)\) (cf. [2], Theorem 9.5, page 251) implies that \(H^8(\Sigma'_G)\) is isomorphic to a quotient group of \(H^6(D; H_c^2(\Sigma')) \cong \mathbb{Z}/q \bigoplus \mathbb{Z}/q\). However,

\[
H^8(U_G) \cong H^4(D; H_c^4(U)) \cong \mathbb{Z}/p \bigoplus \mathbb{Z}/2
\]

when \(q = 2\), which is not isomorphic to \(H^8(\Sigma'_G)\) considering the \(p\)-part.

When \(q > 2\), we have

\[
H^{2q+2}(D; H_c^2(\Sigma')) \cong H^{2q+2}(D; \mathbb{Z}) \bigoplus H^{2q+2}(\mathbb{Z}/q; \mathbb{Z}) \\
\cong \mathbb{Z}/q \bigoplus \mathbb{Z}/q.
\]
and the Borel spectral sequence gives that $H^{2q+4}_{c}(\Sigma_{G}')$ is isomorphic to a quotient group of $\mathbb{Z}/q \bigoplus \mathbb{Z}/q$. However,

$$H^{2q+4}_{c}(U_{G}) \cong H^{2q}(D; H^{4}_{c}(U)) \cong \mathbb{Z}/p \bigoplus \mathbb{Z}/q,$$

which is not isomorphic to $H^{2q+4}_{c}(\Sigma_{G}')$ considering the $p$-part. This proves that $x$ is a discrete component in $\text{Fix}(\mathbb{Z}/p)$. ■

We now consider the case when $n > 1$. Since the $p$-th power $(s^{i}t^{q})^{p} = t^{pq}$ for each $i$ and $j > 0$ and $(s^{i})^{q} = t^{q}$, we see that the fixed point set $\text{Fix}(s^{i}t^{q}) \subset \text{Fix}(t^{q-1})$, which is also a disjoint union of discrete points and 2-dimensional spheres. Therefore, the singular $\Sigma = \text{Fix}(s) \cup \text{Fix}(t^{q-1})$ is a union of spheres and discrete points.

**Theorem 3.4** Let the group $D = \mathbb{Z}/p \times \mathbb{Z}/q^n$ ($n > 1$) act effectively homologically trivially on a closed 4-manifold $M$ with $H_1(M;\mathbb{Z}) = 0$ by homeomorphisms. Suppose that a discrete point $x \in \text{Fix}(D)$ is contained in a non-discrete component of $\text{Fix}(t^{q-1})$, then $x$ is a discrete point in $\text{Fix}(\mathbb{Z}/p)$.

**Proof.** Suppose that $x \in S^{2}_{1} \cap S^{2}_{2}$ for a 2-sphere $S^{2}_{1} \subset \text{Fix}(t^{q-1})$ and another 2-sphere $S^{2}_{2} \subset \text{Fix}(\mathbb{Z}/p)$. Since $s$ commutes with $t^{q-1}$, we have $sS^{2}_{2} = S^{2}_{2}$ as a set. The quotient group $D/\langle t^{q-1} \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/q$ acts invariantly on $\text{Fix}(t^{q-1})$. Let $U$ be an $D$-invariant open small ball containing $x$ and $\Sigma' = \Sigma \cap U$. Note that $H^{4}_{c}(U) = \mathbb{Z}$, $H^{2}_{c}(\Sigma') = \mathbb{Z}^{2}$ and $H^{4}_{c}(\Sigma') = \mathbb{Z}$, $H^{6}_{c}(\Sigma') = 0$, with trivial $D$-actions.

When $q = 2$, we have

$$H^{8}(U_{G}) \cong H^{4}(D; H^{4}_{c}(U)) \cong \mathbb{Z}/p \bigoplus \mathbb{Z}/2,$$

$$H^{8}(D; H^{2}_{c}(\Sigma')) \cong H^{6}(D; \mathbb{Z}^{2}) \cong (\mathbb{Z}/q^{n})^{2},$$

$$H^{7}(D; H^{4}_{c}(\Sigma')) = 0.$$

The Borel spectral sequence $H^{i}(D; H^{j}_{c}(\Sigma')) \Rightarrow H^{i+j}_{c}(\Sigma_{G}')$ implies that $H^{8}_{c}(\Sigma_{G}')$ is isomorphic to a quotient group of $(\mathbb{Z}/q^{n})^{2}$, which has a trivial $p$-part. This is a contradiction to that $H^{8}(\Sigma_{G}') \cong H^{8}(U_{G})$.

When $q > 2$, we have

$$H^{2q+4}_{c}(U_{G}) \cong H^{2q}(D; H^{4}_{c}(U)) \cong \mathbb{Z}/p \bigoplus \mathbb{Z}/q^{n},$$

$$H^{2q+2}(D; H^{2}_{c}(\Sigma')) \cong H^{2q+2}(D; \mathbb{Z}^{2}) \cong (\mathbb{Z}/q^{n})^{2},$$

$$H^{2q+3}(D; H^{4}_{c}(\Sigma')) = 0.$$

The Borel spectral sequence again implies that $H^{2q+4}_{c}(\Sigma_{G}')$ is isomorphic to a quotient of $(\mathbb{Z}/q^{n})^{2}$ and thus not isomorphic to $H^{2q+4}_{c}(U_{G})$, considering the $p$-part. This is a contradiction. ■

### 3.2 Group actions of $\mathbb{Z}/p \rtimes \mathbb{Z}/q$

In this subsection, we will prove the following.

**Theorem 3.5** Let $p, q$ be two primes and $D = \mathbb{Z}/p \rtimes \mathbb{Z}/q$ a nontrivial semi-direct product. Suppose that $M$ is a 4-dimensional manifold with $H_{1}(M;\mathbb{Z}) = 0$ and the second Betti number $b_{2}(M) \geq 3$. The group $D$ cannot act effectively homologically trivially on $M$ by homeomorphisms.
The proof of this theorem is based on a detailed study of singular sets and applications of Borel spectral sequences.

**Lemma 3.6** Two distinct 2-spheres \( S_1^2, S_2^2 \subset \text{Fix}(t) \) have disjoint orbits under the action of \( \mathbb{Z}/p = \langle s \rangle \).

**Proof.** Suppose that \( s^i S_1^2 \cap s^j S_2^2 \neq \emptyset \) for some \( 0 \leq i \neq j < p \). Then \( S_1^2 \cap s^{-i} S_2^2 \neq \emptyset \) and there is a point \( x \in S_1^2 \cap s^{-i} S_2^2 \). Note that \( tx = x \) and \( s^{-j} ts^{-j} x = x \). Therefore, \( x \in \text{Fix}(D) \), a global fixed point. However, \( x = s^{-j} x \in S_2^2 \) implies \( S_1^2 \cap S_2^2 \neq \emptyset \), which is a contradiction to the fact that each component of \( \text{Fix}(t) \) is a manifold (see Lemma 3.2).

From the previous subsection, we know that \( \Sigma = \cup \text{Fix}(s) \cup \bigcup_{i=0}^{p-1} \text{Fix}(s^i t) \). Moreover, the 2-spheres in \( \text{Fix}(s) \) and \( \text{Fix}(t) \) can only be disjoint or identical by Theorem 3.3 (the intersection of the two spheres cannot be a circle, since \( t \) acts homologically trivial on \( \text{Fix}(s) \)). By Lemma 3.6, two 2-spheres in \( \bigcup_{i=0}^{p-1} \text{Fix}(s^i t) \) from distinct \( \langle s \rangle \)-orbits are disjoint. Therefore, a connected component \( X \) of \( \Sigma \) is either a isolated point, or a single 2-sphere, or a union of \( p \) 2-spheres (from \( \bigcup_{i=0}^{p-1} \text{Fix}(s^i t) \)) having non-empty intersections. The remaining part of the proof of Theorem 3.5 will be the similar to that of Proposition 13 in McCooey [12].

Suppose that the singular set \( \Sigma \) has \( k \) isolated global fixed points, \( l \) connected components of single 2-spheres in \( \text{Fix}(D) \) and \( n_1 \) connected components of unions of \( p \) 2-spheres with non-empty intersections. Suppose that the rest of \( \text{Fix}(s) \) has \( n_2 \) free \( \langle t \rangle \)-orbits of 2-spheres, and \( n_3 \) free \( \langle t \rangle \)-orbits of isolated points. Moreover, the rest of \( \text{Fix}(t) \) contains \( n_4 \) free \( \langle s \rangle \)-orbits of 2-spheres and \( n_5 \) free \( \langle s \rangle \)-orbits of isolated points.

The following was first essentially obtained by McCooey [12] (page 847).

**Lemma 3.7** Let \( \Sigma' \subset \Sigma \) be the union of the connected components which are not isolated points or single 2-spheres in \( \text{Fix}(D) \). Suppose that \( \Sigma' \) has \( n_1 \) connected components consisting of unions of \( p \) 2-spheres intersecting at \( m_i \geq 1 \) points. The Borel spectral sequence for \( H^*(\Sigma'_D; \mathbb{Z}) \) collapses. Actually, we have

\[
H^n(D; H^0(\Sigma')) \cong \begin{cases} 
0, & \text{if } n \text{ is odd}, \\
(\mathbb{Z}/q)^{n_1+n_4+n_5} \bigoplus (\mathbb{Z}/p)^{n_2+n_3}, & \text{if } i \text{ is even but } 2q \mid n \\
(\mathbb{Z}/q)^{n_1+n_4+n_5} \bigoplus (\mathbb{Z}/p)^{n_1+n_2+n_3}, & \text{if } 2q \nmid n.
\end{cases}
\]

\[
H^n(D; H^1(\Sigma')) \cong \begin{cases} 
\bigoplus_{i=1}^{n_1} (\mathbb{Z}/p)^{m_{i}-1}, & \text{if } n = 2q - 1 \text{ mod } 2q, \\
0, & \text{otherwise}.
\end{cases}
\]

\[
H^n(D; H^2(\Sigma')) \cong \begin{cases} 
0, & \text{if } n \text{ is odd}, \\
(\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2}, & \text{if } n \text{ is even}.
\end{cases}
\]

**Proof.** When \( q = 2 \), this is proved by McCooey [12]. For \( q > 2 \), the proof is similar. For completeness, we repeat it here. Let \( X \) be a union of \( p \) 2-spheres intersecting at \( m_i \geq 1 \) points \( x_1, x_2, \ldots, x_{m_i} \). The \( D \)-equivariant chain complex for \( X \) is of the form

\[
0 \to \mathbb{Z}D \bigotimes_{\langle t \rangle} \mathbb{Z} \to (\mathbb{Z}D \bigotimes_{\langle t \rangle} \mathbb{Z})^{m_i-1} \to \mathbb{Z}^{m_i} \to 0.
\]

Let \( \text{Ck}(\mathbb{Z}) = \text{coker}(\mathbb{Z} \to \text{Hom}_{\mathbb{Z}p}(D, \mathbb{Z})) \), i.e.

\[
0 \to \mathbb{Z} \to \text{Hom}_{\mathbb{Z}p}(\mathbb{Z}, \mathbb{Z}) \to \text{Ck}(\mathbb{Z}) \to 0.
\]
The Shapiro’s Lemma implies that $H^i(D; \text{Hom}_{\mathbb{Z}[\mathbb{Z}/q]}(\mathbb{Z}[D], \mathbb{Z})) \cong H^i(\mathbb{Z}/q; \mathbb{Z})$ and the long exact sequence gives

$$\cdots \rightarrow H^i(D; \mathbb{Z}) \rightarrow H^i(\mathbb{Z}/q; \mathbb{Z}) \rightarrow H^i(D; Ck(\mathbb{Z})) \rightarrow H^{i+1}(D; \mathbb{Z}) \rightarrow \cdots.$$ 

Therefore,

$$H^n(D; Ck(\mathbb{Z})) \cong \begin{cases} \mathbb{Z}/p, & \text{if } n \equiv 2q - 1 \text{ mod } 2q, \\ 0, & \text{otherwise.} \end{cases}$$

The $D$-equivariant chain complex for $X$ gives

$$H^0(X; \mathbb{Z}) = \mathbb{Z}, \quad H^1(X; \mathbb{Z}) = (Ck(\mathbb{Z}))^{m_1-1}, \quad H^2(X; \mathbb{Z}) = \text{Hom}_{\mathbb{Z}[\mathbb{Z}/q]}(\mathbb{Z}[D], \mathbb{Z}).$$

The cohomology groups of $\Sigma'$ are

$$H^0(\Sigma'; \mathbb{Z}) \cong \mathbb{Z}^{n_1} \bigoplus \text{Hom}_{\mathbb{Z}[\mathbb{Z}/p]}(\mathbb{Z}[D], \mathbb{Z})^{n_2+n_3} \bigoplus \text{Hom}_{\mathbb{Z}[\mathbb{Z}/q]}(\mathbb{Z}[D], \mathbb{Z})^{n_4+n_5},$$

$$H^1(\Sigma'; \mathbb{Z}) \cong \bigoplus_{i=1}^{n} (Ck(\mathbb{Z}))^{m_i-1},$$

$$H^2(\Sigma'; \mathbb{Z}) \cong \text{Hom}_{\mathbb{Z}[\mathbb{Z}/p]}(\mathbb{Z}[D], \mathbb{Z})^{n_2} \bigoplus \text{Hom}_{\mathbb{Z}[\mathbb{Z}/q]}(\mathbb{Z}[D], \mathbb{Z})^{n_1+n_4}.$$ 

The Shapiro’s lemma gives the required isomorphisms of the lemma. Since the $E_2^{i,2} = H^i(D; H^2(\Sigma'; \mathbb{Z}))$ and $E_2^{i+2,0}$ vanish in odd dimensions and $E_2^{i,1}$ vanishes in even dimensions, the Borel spectral sequence collapses. ■

**Proof of Theorem 3.5** Suppose that $D$ acts effectively homologically trivially on $M$. Since the odd-dimensional cohomology groups of $D$ vanish (cf. Lemma 3.1), the Borel spectral sequence collapses by Lemma 2.4. Therefore, the graded module (cf. [15], Proposition 1.18 of Chapter III, page 182)

$$\text{GR}(H^i(M_D)) \cong H^i(D; \mathbb{Z}) \bigoplus H^{i-2}(D; H^2(M; \mathbb{Z})) \bigoplus H^{i-4}(D; \mathbb{Z}).$$

which is isomorphic to $\text{GR}(H^i(\Sigma_D))$ when $i > 4$ by Lemma 2.2. We will prove that this is impossible by comparing $H^{4q}(M_D)$ and $H^{4q}(\Sigma_D)$.

Note that the singular set $\Sigma = \Sigma^1 \cup \Sigma^2 \cup \Sigma'$, is a disjoint union of $\Sigma^1$ consisting of $k$ isolated global fixed points, $\Sigma^2$ consisting of $l$ components of single 2-spheres in $\text{Fix}(D)$, and $\Sigma'$ the remaining part. Therefore,

$$H^n(\Sigma_D; \mathbb{Z}) \cong H^n(\Sigma'_D; \mathbb{Z}) \bigoplus H^n(\Sigma^1_D; \mathbb{Z}) \bigoplus H^n(\Sigma^2_D; \mathbb{Z}).$$

Lemma 2.4 implies that the Borel spectral sequences for $\Sigma^1_D \rightarrow BD$ and $\Sigma^2_D \rightarrow BD$ collapse. By Lemma 3.7 we have

$$H^{4q}(G; H^0(\Sigma; \mathbb{Z})) \bigoplus H^{4q-1}(G; H^1(\Sigma; \mathbb{Z})) \bigoplus H^{4q-2}(G; H^2(\Sigma; \mathbb{Z})) \cong (\mathbb{Z}/pq)^{k+l+n_1} \bigoplus (\mathbb{Z}/q)^{n_2+n_3} \bigoplus (\mathbb{Z}/q)^{n_4+n_5} \bigoplus (\mathbb{Z}/p)^{\sum_{i=1}^{n_1}} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4} \bigoplus (\mathbb{Z}/p)^{n_2} \bigoplus (\mathbb{Z}/q)^{n_1+n_4}.$$ 

Since

$$H^{4q}(M_G) \cong \mathbb{Z}/pq \bigoplus (\mathbb{Z}/q)^{b_2(M)} \bigoplus \mathbb{Z}/pq.$$
when \( q = 2 \) (or \( \mathbb{Z}/pq \bigoplus (\mathbb{Z}/q)^{b_2(M)} \) when \( q > 2 \)), we have that
\[
k + l + n_1 + 2n_2 + n_3 + \sum (m_i - 1) = 2
\]
by considering the \( p \)-part. When \( n_1 = 0 \), we have that \( \Sigma \) is empty and \( k + l + 2n_2 + n_3 = 2 \). Since \( 2 + b_2(M) = \chi(Fix(s)) = k + 2l + 4n_2 + 2n_3 \leq 4 \), we get that \( b_2(M) \leq 2 \). When \( n_1 = 1 \), we have that \( n_2 = 0 \) and \( k + l + n_3 + (m_1 - 1) \leq 1 \). Once again, \( 2 + b_2(M) = \chi(Fix(s)) = k + 2l + 2n_3 + m_1 \leq 4 \) and thus \( b_2(M) \leq 2 \). When \( n_1 = 2 \), we have \( k + l + 2n_2 + n_3 + \sum (m_i - 1) = 0 \) and thus \( 2 + b_2(M) = \chi(Fix(s)) \leq 4 \), implying \( b_2(M) \leq 2 \). These are contradictions to \( b_2(M) \geq 3 \).

When \( q > 2 \), we have that
\[
k + l + n_1 + 2n_2 + n_3 + \sum (m_i - 1) = 1
\]
by considering the \( p \)-part. Similar argument shows that \( b_2(M) \leq 2 \), a contradiction. ■

### 3.3 Group actions of \( \mathbb{Z}/p \times \mathbb{Z}/q^n \), \( n > 1 \)

In this subsection, we study the group action of \( \mathbb{Z}/p \times \mathbb{Z}/q^n \), \( n > 1 \) and prove the following result. We still assume \( s \) is a generator of \( \mathbb{Z}/p \) and \( t \) is a generator of \( \mathbb{Z}/q^n \).

**Theorem 3.8** Let \( p, q \) be two primes and \( D = \mathbb{Z}/p \times \mathbb{Z}/q^n \) \( (n > 1) \) a minimal non-abelian finite group. Suppose that \( M \) is a four-dimensional closed manifold with \( H_1(M; \mathbb{Z}) = 0 \) and the second Betti number \( b_2(M) \geq 3 \). The group \( D \) cannot act effectively homologically trivially on \( M \) by homeomorphisms.

**Proof.** Suppose that \( D \) acts effectively homologically trivially on \( M \). The singular set \( \Sigma \) is described as following. Note that the fixed point set \( Fix(s) \) is a disjoint union of discrete points and 2-dimensional spheres. Since the \( p \)-th power \((s^it^j)^p = t^{pq^j} \) for each \( i \) and \( j > 0 \) and \((s^it^j)t = t^n \), we see that the fixed point set \( Fix(s^it^j) \subset Fix(t^{pq^j}) \), which is also a disjoint union of discrete points and 2-dimensional spheres. Therefore, the singular set \( \Sigma = Fix(s) \cup Fix(t^{pq^j}) \) is an union of spheres and discrete points. By Theorem 3.4 distinct 2-spheres in \( Fix(s) \cup Fix(t^{pq^j}) \) cannot have a discrete intersection. Therefore, the singular set \( \Sigma \) consists of (say \( n_1 \)) isolated fixed points and (say \( n_2 \)) 2-spheres. When \( b_2(M) \geq 3 \), the fixed point set \( Fix(g) \) of any order-\( p \) element \( g \in D \) cannot be purely 2-dimensional consisting of two 2-spheres (otherwise, the \( \chi(Fix(g)) = 4 = 2 + b_2(M) \), a contradiction). Lemma 2.8 implies arbitrary two 2-spheres in \( Fix(g) \) are independent in \( H_2(M; F_p) \). This implies that any element \( h \in N_D(S_g) \) (a normalizer of \( \langle g \rangle \) ) preserves each 2-sphere component of \( Fix(g) \) (see Corollary 2.9). Therefore, each point and each sphere in \( \Sigma \) is invariant under the action of \( D \).

Since the odd-dimensional cohomology groups of \( D \) are vanishing (cf. Lemma 3.1), the Borel spectral sequence collapses for both \( M_D \to BD \) and \( \Sigma_D \to BD \) by Lemma 2.4. We have that
\[
H^q(\Sigma_D; \mathbb{Z}) \cong H^q(D; H^0(\Sigma; \mathbb{Z})) \bigoplus H^{q-2}(D; H^2(\Sigma; \mathbb{Z}))
\]
\[
\cong (\mathbb{Z}/pq^n)^{n_1+n_2} \bigoplus (\mathbb{Z}/q^n)^{n_2},
\]
\[
H^q(M_D; \mathbb{Z}) \cong H^q(D; H^0(M; \mathbb{Z})) \bigoplus H^{q-2}(D; H^2(M; \mathbb{Z})) \bigoplus H^{q-4}(D; H^4(M; \mathbb{Z}))
\]
\[
\cong (\mathbb{Z}/pq^n)^2 \bigoplus (\mathbb{Z}/q^n)^{b_2(M)}
\]
when \( q = 2 \) (or \( \mathbb{Z}/pq^n \bigoplus (\mathbb{Z}/q^n)^{b_2+1} \) when \( q > 2 \)). By Lemma 2.7, \( H^q(\Sigma_D; \mathbb{Z}) \cong H^q(M_D; \mathbb{Z}) \), which gives that \( n_1 + n_2 = 2 \) (or \( n_1 + n_2 = 1 \) when \( q > 2 \)) and \( n_1 + 2n_2 = 2 + b_2 \leq 4 \), which is a contradiction to that \( b_2(M) \geq 3 \). ■
3.4 Group actions of the quaternion group

In this subsection, we study the group action of the quaternion group $D_2 = \langle a, b \mid a^4 = 1, a^2 = b^2, [a, b] = a^2 \rangle$. We prove the following.

**Theorem 3.9** Suppose that $M$ is a four-dimensional closed manifold with $H_1(M; \mathbb{Z}) = 0$ and the second Betti number $b_2(M) \geq 3$. The group $D_2^*$ cannot act effectively homologically trivially on $M$ by homeomorphisms.

**Proof.** Since the order of $a^2$ is 2, the fixed point set $\text{Fix}(a^2)$ is a disjoint union of 2-spheres and discrete points by Lemma 2.6. Note that any non-trivial element $g \neq a^2$ has $g^2 = a^2$. This implies that $\text{Fix}(g) \subset \text{Fix}(a^2)$ and the singular set $\Sigma = \text{Fix}(a^2)$. Suppose that $\text{Fix}(a^2)$ consists of $n_1$ discrete points and $n_2$ 2-spheres. The quotient group $D_2^*/\langle a^2 \rangle$ acts invariantly on the fixed point set $\text{Fix}(a^2)$, preserving the $n_2$ individual 2-spheres, while the action could permute the $n_1$ discrete points. Since each 2-sphere in $\text{Fix}(a^2)$ represents a non-trivial homology class, $D_2^*/\langle a^2 \rangle$ preserves each such a 2-sphere. Let $n_{1i} (i = 0, 1, 2)$ denote the number of $D_2^*/\langle a^2 \rangle$-orbits of discrete points in $\text{Fix}(a^2)$ with stabilizers of 2-rank $i$. Since the odd-dimensional cohomology groups of $G := D_2^*$ vanish (cf. Lemma 3.1), the Borel spectral sequence collapses by Lemma 2.4. Note that

$$H^0(\Sigma; \mathbb{Z}) \cong (\mathbb{Z}[G] \otimes_{\mathbb{Z}(a^2)} \mathbb{Z}) ^{n_{10}} \oplus (\mathbb{Z}[G] \otimes_{\mathbb{Z}[\mathbb{Z}/4]} \mathbb{Z}) ^{n_{11}} \oplus \mathbb{Z} ^{n_{12} + n_2}.$$

Therefore, we have the isomorphisms of graded modules

$$\text{GR}(H^8(M_G; \mathbb{Z})) \cong H^8(G; \mathbb{Z}) \oplus H^6(G; H^2(M; \mathbb{Z})) \oplus H^4(G; H^4(M; \mathbb{Z}))$$

$$\cong (\mathbb{Z}/8)^2 \oplus (\mathbb{Z}/2) ^{2b_2(M)},$$

$$\text{GR}(H^8(\Sigma_G; \mathbb{Z})) \cong H^8(G; H^0(\Sigma; \mathbb{Z})) \oplus H^6(G; H^2(\Sigma; \mathbb{Z})),$$

$$\cong (\mathbb{Z}/8)^{n_{12} + n_2} \oplus (\mathbb{Z}/4) ^{n_{11}} \oplus (\mathbb{Z}/2) ^{n_{10}} \oplus (\mathbb{Z}/2) ^{2n_2}$$

and

$$\text{GR}(H^6(M_G; \mathbb{Z})) \cong H^6(G; \mathbb{Z}) \oplus H^4(G; H^2(M; \mathbb{Z})) \oplus H^2(G; H^4(M; \mathbb{Z}))$$

$$\cong (\mathbb{Z}/8)^{2b_2(M)} \oplus (\mathbb{Z}/2)^4,$$

$$\text{GR}(H^6(\Sigma_G; \mathbb{Z})) \cong H^6(G; H^0(\Sigma; \mathbb{Z})) \oplus H^4(G; H^2(\Sigma; \mathbb{Z})),$$

$$\cong (\mathbb{Z}/2) ^{2(n_{12} + n_2)} \oplus (\mathbb{Z}/4) ^{n_{11}} \oplus (\mathbb{Z}/2) ^{n_{10}} \oplus (\mathbb{Z}/8) ^{n_2}.$$

Considering Lemma 2.2, the cardinalities of modules give that $6 + 2b_2(M) = 3(n_{12} + n_2) + 2n_{11} + n_{10} + 2n_2$ and $3b_2(M) + 4 = 2(n_{12} + n_2) + 2n_{11} + n_{10} + 3n_2$. Thus, $b_2(M) = 2 - n_{12} \leq 2$, a contradiction. \[\square\]

3.5 High rank case

In this subsection, we will prove that a high-rank finite group acting effectively homologically trivially on a simply connected 4-manifold $M$ with large second Betti number has to be $\mathbb{Z}/p \times \mathbb{Z}/p$.

**Lemma 3.10** (Borel [1], Theorem 4.3, p.182) Let $G$ be an elementary $p$-group operating on a first countable cohomology $n$-manifold $X \mod p$. Let $x \in X$ be a fixed point of
G on X and let n(H) be the cohomology dimension mod p of the component of x in the fixed point set of a subgroup H of G. If r = n(G), we have

\[ n - r = \sum_H (n(H) - r) \]

where H runs through the subgroups of G of index p.

The following is a special case of Lemma 2.1 in [13].

**Lemma 3.11** Let M be a closed 4-manifold with \( H_1(M; F_p) = 0 \) and G a finite group acting homologically trivially on M. When \( b_2(M) \geq 3 \), the Borel spectral sequence

\[ H^i(G; H^j(M; F_p)) \Rightarrow H^{i+j}(M_G; F_p) \]

collapses.

**Lemma 3.12** Let \( G = (\mathbb{Z}/p)^k \) act effectively homologically trivially on a 4-manifold M with \( H_1(M; \mathbb{Z}) = 0 \).

(i) If \( b_2(M) \geq 3 \), we have \( k \leq 2 \). When \( k = 2 \), the singular set \( \Sigma \) consists of chains of 2-spheres arranged in closed loops. Moreover, the global fixed point \( \text{Fix}(G) \) consists of \( b_2(M) + 2 \) points.

(ii) If \( b_2(M) = 1, p = 2 \) and \( k = 2 \), the singular set \( \Sigma \) consists of chains of 3 2-spheres arranged in a closed loop. The global fixed point \( \text{Fix}(G) \) consists of the 3 intersection points.

(iii) If \( b_2(M) = 2, p > 2 \) and \( k = 2 \), the singular set \( \Sigma \) consists of either two loops of two 2-spheres or a single loop of four 2-spheres, linking together at global fixed points.

**Proof.** When \( b_2(M) \geq 3 \), the Borel spectral sequence collapses (cf. Lemma 3.11). Therefore, the module \( H^*(M_G; F_p) \) is isomorphic to \( H^*(G; F_p) \otimes H^*(M; F_p) \). Recall that \( H^*(G; F_p) \) is isomorphic to \( F_p[x_1, \ldots, x_k] \) (with deg \( x_i = 1 \) for each \( i \)) when \( p = 2 \), or \( F_p[y_1, \ldots, y_k] \otimes \Lambda[x_1, \ldots, x_k] \) (with deg \( x_i = 1 \) and deg \( y_j = 2 \) for any \( i, j \)) when \( p > 2 \) (see [10], p.45). Note that we cannot apply Lemma 2.4 to get the collapse of the Borel spectral sequence, since the odd dimensional \( F_p \)-coefficient cohomology groups may not vanish.

Let S be the multiplicative set of non-trivial elements in the polynomial ring \( F_p[x_1, \ldots, x_k] \) (or \( F_p[y_1, \ldots, y_k] \)). Then the set

\[ M^S = \{ x \in M : S \cap \ker(H^*(G; F_p) \rightarrow H^*(G_x; F_p)) = \emptyset \} \]

would be the global fixed point set \( M^G \). The localization theory gives an isomorphism

\[ S^{-1}H^*_G(M; F_p) \cong S^{-1}H^*_G(M^G; F_p) \cong S^{-1}H^*(G; F_p) \otimes H^*(M^G; F_p) \]

This clearly implies that \( M^G \) is non-empty. From Lemma 2.6, we know that the fixed point set \( \text{Fix}(H) \) is a union of (possibly empty) 2-spheres \( S^2 \) and discrete points for any subgroup \( H < G \) of index p. From the Borel’s formula (cf. Lemma 3.10)

\[ 4 - n_0 = \sum n(H) - n_0, \]
where \( n(H) \) is the dimension of \( \text{Fix}(H) \) and \( n_0 \) is the dimension of \( \text{Fix}(G) \), we know that there are must be some \( H \) such that \( n(H) = 2 \) and \( n_0 = 0 \). We may assume that for some nontrivial element \( a \in G \), the fixed point set \( \text{Fix}(a) \) contains \( S^2 \), which contains a global fixed point in \( M^G \). Fix a decomposition \( G = \langle a \rangle \times (\mathbb{Z}/p)^{k-1} \). Then the complement \( (\mathbb{Z}/p)^{k-1} \) acts invariantly on \( S^2 \), since two different \( S^2 \) components in \( \text{Fix}(a) \) represent different homology classes in \( H_2M \) (cf. Lemma 2.3).

Suppose that \( k \geq 3 \). When \( p > 2 \), the subgroup \( (\mathbb{Z}/p)^2 \) cannot act effectively on \( S^2 \) as seen from the Borel’s formula. Thus there is an element \( g \in (\mathbb{Z}/p)^{k-1} \) acting trivially on \( S^2 \). Then the subgroup \( \langle a, g \rangle \) acts effectively on \( M \) with a global fixed point set \( S^2 \). However, this is impossible by Borel’s formula again. When \( p = 2 \), each element in \( (\mathbb{Z}/p)^2 \) acts on \( S^2 \) by orientation-preserving since the codimension of the global fixed point set of each element is even. Once again, the action of \( (\mathbb{Z}/2)^2 \) cannot be effective, since there is a global fixed point in \( S^2 \). Using Borel’s formula again, this is impossible.

The only admitted case is that \( k = 2 \) and each generator \( a, b \) of \( G = \langle a \rangle \times \langle b \rangle \cong (\mathbb{Z}/p)^2 \) fixes several copies of \( S^2 \). Moreover, one generator rotates the spheres fixed by the other generator. Therefore, the singular set is some chains of \( 2 \)-spheres \( S_1, S_2, \ldots, S_k \) arranged in closed loops with \( S_i \) intersects once with \( S_{i-1} \) and \( S_{i+1} \) respectively. Furthermore, the \( 2 \)-spheres \( S_{2i-1} \) (resp. \( S_{2i} \)) are fixed by the generator \( a \) (resp. \( b \)). The fixed point \( \text{Fix}(G) \) consists of \( b(M) + 2 \) points considering the Euler characteristics.

When \( b_2(M) = 1 \) and \( a \neq b \), the singular set \( \Sigma = \text{Fix}(a) \cup \text{Fix}(b) \cup \text{Fix}(ab) \). If \( \Sigma \) is of dimension 0, the element \( b \) acts invariantly on \( \text{Fix}(a) \) consisting of 3 points (by Lemma 2.7) and thus has a fixed point. But this is impossible by the Borel’s formula. After changing of basis, we may assume that \( \text{Fix}(a) \) consists of a \( 2 \)-sphere \( S^2_a \) and a discrete point \( x \). Since \( b \) acts trivially on \( x \), there is a sphere \( S^2_b \subset \text{Fix}(b) \) containing \( x \) by the Borel formula. Note that \( a \) acts invariantly and non-trivially on \( S^2_b \), the sphere \( S^2_a \) intersects \( S^2_b \) at a point \( y \). The Borel’s formula for \( y \) shows that \( \text{Fix}(ab) \) contains another sphere \( S^2_{ab} \) connecting \( S^2_a \) and \( x \).

We consider the case when \( b_2(M) = 2 \) and \( p > 2 \). The singular set \( \Sigma = \text{Fix}(a) \cup \bigcup_{i=0}^{p-1} \text{Fix}(a^ib) \), a union of fixed point sets. Note that \( \text{Fix}(a) \) is a union of (possibly empty) discrete points and \( 2 \)-spheres. If \( \Sigma \) is of dimension 0, the element \( b \) acts invariantly on \( \text{Fix}(a) \) consisting of 4 points and thus has a fixed point. But this is impossible by the Borel’s formula. If \( \Sigma \) is of dimension two, after changing of basis we may assume that \( \text{Fix}(a) \) contains a \( 2 \)-sphere. Note that the Borel’s formula implies that at each global fixed point \( z \in \text{Fix}(G) \), there are two elements \( g, h \in G \) generating different subgroups with two distinct spheres \( S^2_g \subset \text{Fix}(g) \) and \( S^2_h \subset \text{Fix}(h) \) such that \( z \in S^2_g \cap S^2_h \). Since any element in \( G \) acts invariantly on these spheres, each sphere \( S^2_g \) or \( S^2_h \) contains two global fixed points in \( \text{Fix}(G) \). These spheres form either two loops of \( 2 \)-spheres or a single loop consisting of four spheres, linking together at global fixed points.

**Remark 3.13** There are actions of \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) on \( S^2 \times S^2 \) and actions of \( \mathbb{Z}/3 \times \mathbb{Z}/3 \) on \( \mathbb{C}P^2 \) without global fixed points. The isometry group \( (\text{SO}(3) \times \text{SO}(3)) \times \mathbb{Z}/2 \) of \( S^2 \times S^2 \) and the isometry group \( PU(3) \) of \( \mathbb{C}P^2 \) both contain elementary subgroup of ranks larger 2. This means that the restrictions of Betti numbers and prime \( p \) in Lemma 2.12 cannot be dropped. More information about group actions on \( \mathbb{C}P^2 \) can be found in [8].

It is proved in [13] (Lemma 2.6) that the singular set \( \Sigma \) in Lemma 3.12 is actually connected and there is only one such a loop. For convenience, we repeat the proof here (note that there are some typos in the proof of Corollary 2.5 and Lemma 2.6 in [13]).
Lemma 3.14 Let $G = (\mathbb{Z}/p)^2$ act effectively homologically trivially on a closed 4-manifold $M$ with $H_1(M; \mathbb{Z}) = 0$. Suppose that one of the following conditions holds:

1. $b_2(M) \geq 3$,
2. $b_2(M) = 1$ and $p = 2$,
3. $b_2(M) = 2$ and $p > 2$.

Then the singular set $\Sigma$ is a chain of 2-spheres arranged in a closed loop. Moreover, each sphere represents a primitive class in $H_2(M; \mathbb{Z})$ and together these classes generate $H_3(M; \mathbb{Z})$. The number of fixed spheres is $b_2(M) + 2$.

Proof. Let $N$ be the number of connected components of $\Sigma$. Consider the long exact sequence

$$
\cdots \to H^1(M, \Sigma) \to H^1(M) \to H^1(\Sigma) \to H^2(M, \Sigma) \to H^2(M) \to H^2(\Sigma) \to H^3(M, \Sigma) \to \cdots
$$

for the pair $(M, \Sigma)$, we have that $H^1(M, \Sigma) \cong \mathbb{Z}^{N-1}$, $H^2(M, \Sigma) \cong \mathbb{Z}^{N+L}$, where $L$ is the rank of the cokernel $H^1(\Sigma) \to H^2(M, \Sigma)$. By Lemma 3.12, the singular $\Sigma$ consists of loops of spheres. If a component of $\Sigma$ contains at least three spheres, then each sphere intersects its neighbor geometrically once and thus represents primitive element in $H_2(M)$. If a component of $\Sigma$ contains only two spheres, the homology class represented by the sphere may be a multiple of 2. But this case only happen when $p > 2$, since the homology class is non-trivial in $H_2(M; \mathbb{Z}/2)$ when $p = 2$ (cf. Lemma 2.3). Therefore, the relative cohomology $H^i(M, \Sigma)$ as the cokernel of $H^2(M) \to H^2(\Sigma)$ is isomorphic to $\mathbb{Z}^{L+2} \oplus T$ by counting ranks, where $T = 0$ when $p = 2$ and $2T = 0$ when $p > 2$.

Let $\pi : M \to M/G \to \Sigma/G$ be the projection and denote by $M^* = M/G, \Sigma^* = \Sigma/G$. Since $M - \Sigma$ is a manifold, there is a diagram (not commutative in the ordinary sense) given by the Poincaré duality:

$$
\begin{align*}
H^3(M, \Sigma) &\cong H_1(M - \Sigma) \\
\uparrow \pi^* &\quad \downarrow \pi_* \\
H^3(M^*, \Sigma^*) &\cong H_1(M^* - \Sigma^*).
\end{align*}
$$

Note that $H_1(M - \Sigma)$ is generated by meridians to the spheres in $\Sigma$ and each of these is a $p$-fold cover of its image in $H_1(M^* - \Sigma^*)$.

Consider the Borel spectral sequence $E_{i,j}^2 = H^i(G; H^j(M, \Sigma)) \Longrightarrow H^{i+j}(M^*, \Sigma^*)$. Since $\pi^*$ factors through $H^3(M^*, \Sigma^*) \to E_{0,3}^0 \to E_{0,3}^3 = H^3(M, \Sigma)$, we see that the cokernel($E_{0,3}^3 \to E_{2,3}^3$) is of exponent at most $p$. Since $H^i(M^*, \Sigma^*) \cong \mathbb{Z}$, the stable term $E_{i,j}^2 = 0$ if $i + j \geq 4, i > 0$. We consider $E_{3,1}^2$ and $E_{2,2}^2$. Note that $E_{3,1}^2$ is killed by $E_{3,3}^0$ and $E_{2,2}^2$ is killed by $E_{0,3}^0$ and $E_{2,1}^4$. Therefore,

$$
\text{rk} E_{0,3}^3 + \text{rk} E_{4,1}^2 \geq \text{rk} E_{2,2}^{2,2} + \text{rk} E_{2,1}^{3,1},
$$

which gives that $(2 + L) + (3N - 3) \geq 2N + 2L + (N - 1) + L = 0$. This shows that $H^2(M) \to H^2(\Sigma)$ is injective and thus $\Sigma$ represents all of $H_2(M)$.

Note that $\pi^*$ is injective and thus $H^3(M^*, \Sigma^*) \to E_{0,3}^3$ is isomorphic. Therefore, $E_{2,1}^2 = 0$ and $E_{2,1}^2$ is killed by $E_{0,2}^3 = H^2(M, \Sigma)$, thus $N \geq 2(N - 1), N \leq 2$. Suppose that $N = 2$. We will prove this is impossible to get $N = 1$. Denote by $b, c$ some two generators on $H^2(M, \Sigma)$ and $a$ a generator of $H^1(M, \Sigma)$. The surjective map $d_2 : E_{0,2}^2 \to E_{1,1}^2$ gives that $d_2(b) = \alpha a, d_2(c) = \beta a$ for some integers $\alpha, \beta$. The multiplicative properties of the
spectral sequence implies that \( \ker(d_{2,2}^{2,2} : E_2^{2,2} \to E_2^{4,1}) = \langle \beta b - \alpha c \rangle \) and thus there is some \( e \in H^3(M, \Sigma) \) such that \( d_2(e) = \beta b - \alpha c \). Since \( E_3^{3,1} = E_3^{3,1} \cong \mathbb{Z}/p \cong \langle \mu \rangle \), we have \( d_3(f) = \mu a \) for some \( f \in H^3(M, \Sigma) \), independent of \( e \). But \( d_3(\alpha f) = d_3(\beta f) = 0 \), since \( \mu a, \mu b \) are in the image of \( E_2^{4,2} \). Therefore, \( \ker(d_{2,3}^{2,3} : E_2^{2,3} \to E_2^{4,2}) \) is of \( \mathbb{Z}/p \)-rank at least two. Since \( \text{Im}(E_2^{0,4} \to E_2^{2,3}) \) is of \( \mathbb{Z}/p \)-rank at most 1, the stable term \( E_\infty^{2,3} \) has \( \mathbb{Z}/p \)-rank at least 1. This is a contradiction and thus \( N = 1 \).

**Theorem 3.15** Suppose that \( M \) is a 4-manifold with \( H_1(M; \mathbb{Z}) = 0 \) and the second Betti number \( b_2(M) \geq 3 \). Let \( G \) be a minimal non-abelian finite group of rank at least 2. Then \( G \) cannot act effectively homologically trivially on \( M \) by homeomorphisms.

**Proof.** Note that \( G \) contains a normal subgroup isomorphic to \( (\mathbb{Z}/p)^k, k \geq 2 \). Lemma 3.12 implies that \( k = 2 \) and the singular set \( \Sigma_0 \) of \( (\mathbb{Z}/p)^k \) is a chain of 2-spheres \( S_1, S_2, \ldots, S_{b_2(M)/2+1} \) arranged in a closed loop when \( k = 2 \). Moreover, the 2-spheres \( S_{2i-1} \) (resp. \( S_{2i} \)) are fixed by the generator \( a \) (resp. \( b \)) of \( \langle a \rangle \times \langle b \rangle \cong (\mathbb{Z}/p)^2 \). Choose an element \( g \in G \) normalizing \( (\mathbb{Z}/p)^k \) non-trivially. Suppose that \( gag^{-1} = a^ib^j \) for some integers \( i,j \). Since \( G \) is minimal non-abelian, the integer \( j \neq 0 \). Note that adjacent spheres \( S_i \) and \( S_{i+1} \) represent different homology classes in \( H_2(M) \) when \( b_2(M) \geq 3 \) by Lemma 3.14. Since the action of \( g \) on \( M \) is homologically trivial and the \( b_2(M) + 2 \) spheres in \( \Sigma_0 \) represent at least \( b_2(M) \) homology classes in \( H_2(M) \), there is a fixed point \( x \in \Sigma_0 \cap \text{Fix}(G) \) and the element \( g \) acts invariantly on each sphere \( S_i \). However, we have that \( g\text{Fix}(a) = \text{Fix}(gag^{-1}) = \text{Fix}(a^ib^j) \neq \text{Fix}(a), \) a contradiction.

**Proof of Theorem 1.3.** Since every non-abelian compact Lie group contains a non-abelian finite subgroup (cf. [12] Lemma 15), Theorems 3.5 3.8 3.9 3.15 imply that the Lie group \( G \) is abelian considering the classification of minimal non-abelian finite groups. The rank of \( G \) is at most 2 by Lemma 3.12.

## 4 Actions of automorphism groups of free groups

Fixing a basis \( \{a_1, \ldots, a_n\} \) for the free group \( F_n \), we define several elements in \( \text{Aut}(F_n) \) as follows. The inversions are defined as

\[
e_i : a_i \mapsto a_i^{-1}, a_j \mapsto a_j \quad (j \neq i);
\]

while the permutations are

\[
(ij) : a_i \mapsto a_j, a_j \mapsto a_i, a_k \mapsto a_k \quad (k \neq i, j).
\]

The subgroup \( N < \text{Aut}(F_n) \) generated by all \( e_i \) \( (i = 1, \ldots, n) \) is isomorphic to \( (\mathbb{Z}/2)^n \). The subgroup \( W_n < \text{Aut}(F_n) \) is generated by \( N \) and all \( (ij) \) \( (1 \leq i \neq j \leq n) \). Denote \( SW_n = W_n \cap \text{SAut}(F_n) \) and \( SN = N \cap \text{SAut}(F_n) \). The element \( \Delta = e_1e_2\cdots e_n \) is central in \( W_n \) and lies in \( \text{SAut}(F_n) \) precisely when \( n \) is even.

The following result is Proposition 3.1 of [3].

**Lemma 4.1** Suppose \( n \geq 3 \) and let \( f \) be a homomorphism from \( \text{SAut}(F_n) \) to a group \( G \). If \( f|_{SW_n} \) has non-trivial kernel \( K \), then one of the following holds:

1. \( n \) is even, \( K = \langle \Delta \rangle \) and \( f \) factors through \( \text{PSL}(n, \mathbb{Z}) \),
2. \( K = SN \) and the image of \( f \) is isomorphic to \( \text{SL}(n, \mathbb{Z}/2) \), or
3. \( f \) is the trivial map.
Proof of Theorem 1.1. Let $G$ be the group of homologically trivial homeomorphisms of $M$ and $f: \text{SAut}(F_n) \to G$ a group homomorphism. Let $SW_3 < \text{SAut}(F_3)$ (viewed as a subgroup of $\text{SAut}(F_n)$) fixing all $a_i, i > 3$ be the finite group defined as above.

When $b_2(M) \geq 3$, any finite subgroup of $G$ is abelian by Theorem 1.3. This implies that the kernel $K$ of $f|_{SW_3}$ is non-trivial. If $K = SN$, the image of $f|_{SAut(F_3)}$ is isomorphic to the non-abelian group $\text{SL}(3, Z/2)$ by Lemma 4.1 and this is impossible by Theorem 3.15. Therefore, the action of $\text{SAut}(F_3)$, and thus that $\text{SAut}(F_n)$, is trivial.

When $b_2(M) = 2$, we construct a subgroup $H = (Z/3)^2 \rtimes Z/2 < \text{SAut}(F_4)$ (viewed as a subgroup of $\text{SAut}(F_n)$ fixing all $a_i, i > 4$), where the generator $b$ of $Z/2$ permutes the two generators of $(Z/3)^2$. When this is done, Lemma 3.14 implies that the singular set $\Sigma$ of $(Z/3)^2$ is a chain of 2-spheres arranged in a closed loop. Moreover, each sphere represents a primitive class in $H_2(M; Z)$ and together these classes generate $H_2(M; Z)$. However, the element $b$ permutes spheres in the singular set $\Sigma$, which is impossible since the group action of $b$ is homologically trivial. This shows that the action of $H$ is not effective and the theorem is proved by Lemma 4.1 and Theorem 1.3. Actually, for $i = 1, 2$, we define $R_i : F_n \to F_n$ as

$$a_{2i−1} \mapsto a_{2i}^{−1}, a_{2i} \mapsto a_{2i}^{−1}, a_{2i−1}, a_j \mapsto a_j(j \neq 2i, 2i−1).$$

The subgroup generated by $R_1, R_2$ is isomorphic to $(Z/3)^2$ (cf. Bridson-Vogtmann 3, Lemma 3.2). The generator $R_1$ is conjugate to $R_2$ by the involution $(13)(24)$, giving the generator of $Z/2$.

When $b_2(M) = 1$, we consider the subgroup $N \cap \text{SAut}(F_4)$, which is isomorphic to $(Z/2)^3$ whose generators will be denoted by $a, b, c$. Suppose that $(Z/2)^3$ acts effectively. Lemma 3.14 implies that the singular set $\Sigma$ of $a, b$ is a loop of $b_2(M) + 2 = 3$ 2-spheres, intersecting at three global fixed points $x, y, z \in \text{Fix}(a, b)$. But $c$ acts invariantly on each sphere in $\Sigma$. The Borel formula 3.10 gives that the action of $c$ is non-trivial. Therefore, the fixed point set of any index-2 subgroup of $(Z/2)^3$ is discrete, which is impossible by Borel’s formula again. This implies that some non-trivial element $g \in (Z/2)^3$ acts trivially. Lemma 4.1 implies that $\text{Im } f$ is either trivial or contains a non-abelian subgroup. Considering Theorem 1.3, the image $\text{Im } f$ has to be trivial.

Acknowledgements

The author is grateful to the referee for detailed comments on a previous version of this article. This work is supported by NSFC (No. 11971389).

References

[1] A. Borel, Seminar on transformation groups. Annals of Mathematics Studies, No. 46, Princeton University Press, Princeton, N.J. 1960.
[2] G.E. Bredon, Sheaf Theory, second edition. Graduate Texts in Mathematics, 170. Springer-Verlag, New York, 1997.
[3] M. Bridson and K. Vogtmann, Actions of automorphism groups of free groups on homology spheres and acyclic manifolds, Commentarii Mathematici Helvetici 86(2011), 73-90.
[4] A.L. Edmonds, Aspects of group actions on four-manifolds, Topol. Appl. 31 (2) (1989) 109-124.
[5] A.L. Edmonds, *Homologically trivial group actions on 4-manifolds*, arXiv:math/9809055.

[6] A. Edmonds, *Construction of group actions on four-manifolds*, Trans. Amer. Math. Soc. 299 (1987), 155-170.

[7] D. Fisher, *Groups acting on manifolds: Around the Zimmer program*, In Geometry, Rigidity, and Group Actions 72-157. Univ. Chicago Press, Chicago, 2011.

[8] I. Hambleton, R. Lee, *Finite group actions on $\mathbb{P}^2(\mathbb{C})$*, J. Algebra 116 (1) (1988) 227-242.

[9] I. Hambleton, R. Lee, *Smooth group actions on definite 4-manifolds and moduli spaces*, Duke Math. J. 78 (3) (1995) 715–732.

[10] W. Y. Hsiang, *Cohomology Theory of Topological Transformation Groups*, Springer, New York 1975.

[11] S. Kwasik, R. Schultz, *Homological properties of periodic homeomorphisms of 4-manifolds*, Duke Math. J. 58 (1989), no. 1, 241–250

[12] M. P. McCooey, *Symmetry groups of four manifolds*, Topology, 41(2002), 835-851

[13] M. P. McCooey, *Four manifolds which admit $\mathbb{Z}/p \times \mathbb{Z}/p$ actions*, Forum Math., 41(2002), 835-851

[14] L. Redei, *Das scheife Produkt in der Gruppentheorie*, Commentarii Mathematici Helvetici, 20 (1947) 225–264 (German).

[15] T. tom Dieck, *Transformation Groups*, Walter de Gruyter & Co., Berlin, 1987.

[16] D. Wilczyński, *Group actions on the complex projective plane*, Trans. Amer. Math. Soc. 303 (2) (1987) 707–731.

[17] N. Yagita, *On the dimension of spheres whose product admits a free action by a non-abelian group*, Quart. J. Math. Oxford. Second Series. 36 (141) (1985) 117–127.

[18] S. Ye, *Euler characteristics and actions of automorphism groups of free group*, Algebraic and Geometric Topology 18 (2018) 1195-1204.

NYU Shanghai, 1555 Century Avenue, Shanghai, 200122, China.
NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 3663 Zhongshan Road North, Shanghai, 200062, China
E-mail: sy55@nyu.edu