UNIQUENESS SETS FOR FOURIER SERIES

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Abstract. The paper discusses some uniqueness sets for Fourier series.

1. Introduction

In this paper the following problem is considered: to find conditions on a set \( E \in [-\pi, \pi] \) such that, if a function \( f(x), -\pi < x < \pi \), belongs to some space and its Fourier series converges to zero at each point of the set \( E \), then \( f(x) \) is identically zero.

The first nontrivial result, for trigonometric series, was proved by G. Cantor and W. Young, see [1], p. 191.

**Theorem 1.** Let \( c_k \to 0 \) and for each point \( x \in [-\pi, \pi] \setminus F \) we have

\[
\lim_{n \to \infty} \sum_{k=-n}^{n} c_k e^{ikx} = 0,
\]

where \( F \) be a countable set. Then \( c_k = 0, \ k \in Z \).

D. Menshov, see [1], p. 806, construct a nonzero measure \( \mu \) which has support of zero Lebesgue measure, and its Fourier coefficients tend to zero. The partial summs, of that Fourier series converges to zeros out of \( \text{supp} (\mu) \).

2. Auxiliary definitions and results

More information, about the following quantities, related with Hausdorff’s measures and capacities, one can find in [3], pp. 13 - 46. For convenient of the reader, we give some definitions.

**Definition 2.** Let \( 0 \leq h(x), 0 \leq x \leq 1 \) be a nonnegative, increasing function and \( h(0) = 0 \). Let the subset \( E \subset \{ z; \ |z| = 1 \} \) be cover by the family of arcs \( \{ S_k \}_{k=1}^{\infty} \), i.e.

\[
E \subseteq \bigcup_{k=1}^{\infty} S_k.
\]
Then we put
\[ M_h(E) = \inf \left( \sum_{k=1}^{\infty} h(|S_k|) \right), \]
where \(|S|\) is the length of the arc \(S\) and the infimum is taken over all families of cover.

**Definition 3.** Let \(0 < \alpha < 1\) and \(E\) be bounded Borel set. The \(C_\alpha(E)\)-capacity of the set \(E\) is defined by formula
\[
C_\alpha(E) = \left( \inf_{\mu < E} \int_E \int_E \frac{d\mu(x)d\mu(y)}{|x - y|^{1+\alpha}} \right)^{-1},
\]
where \(\mu < E\) means, that \(\mu\) is probability measure with support in \(E\).

For each \(0 < \alpha < 1\) from Parseval’s equality follows that there is a constant \(M\) such that,
\[
\sum_{k=-\infty}^{\infty} |\hat{f}_k|^2 |k|^\alpha \leq M \int_{-\pi}^{\pi} |f(x)|^2 dx + M \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x - y|^{1+\alpha}} dxdy.
\]
The following statement is a fragment of A. Zygmund’s theorem, see [6], p.22. Let \(g(-\pi) = g(\pi)\) and \(g(x), -\pi \leq x \leq \pi\) have a bounded variation. If
\[
|g(x) - g(y)| \leq M \cdot h(|x - y|).
\]
Then, there is a constant \(B\) such that the Fourier coefficients of the function \(g(x)\) satisfy the inequalities
\[
\sum_{2^j \leq |k| < 2^{j+1}} |\hat{g}_k|^2 \leq B 2^{-j} h \left( \frac{\pi}{3} 2^{1-j} \right).
\]

**Definition 4.** Let us denote by \(\Lambda(n)\) the known Mangold’s function, equals
\[
\Lambda(p^n) = \ln p,
\]
where \(p\) is prime number and \(n\) is natural number. For other natural numbers \(m\)
\[
\Lambda(m) = 0.
\]

It is known, that for an arbitrary natural number \(n\) the equality
\[
\ln n = \sum_{d|n} \Lambda(d)
\]
holds, where the sum is taken over all positive divisors of \(n\).

In the following theorem A. Broman, see [2], p. 851, gived the characterization of close exceptional sets.
Theorem 5. Let $0 < \alpha < 1$ and 
\[ \sum_{n=-\infty}^{\infty} \frac{|c_n|^2}{|n|^\alpha + 1} < \infty. \]

Let $F$ be close set and 
\[ \lim_{r \to 1-0} \sum_{k=-\infty}^{\infty} r^{|k|} c_k e^{ikx} = 0, \]
for arbitrary $x \in [-\pi, \pi] \setminus F$. Then $c_k = 0$, $k \in \mathbb{Z}$, if and only if 
\[ C_{1-\alpha}(F) = 0. \]

A. Zygmund, see [7], proved the following nontrivial result.

Theorem 6. Let $\epsilon > 0$ and $\epsilon_n > 0$, $n = 1, 2, \ldots$ be an arbitrary decreasing sequence, tending to zero. Let $|c_n| \leq \epsilon_n$, $n = 1, 2, \ldots$. There is a subset $E \subseteq [-\pi, \pi]$ with measure, i.e. $m(E) > 2\pi - \epsilon$, such that, if for each point $x \in [-\pi, \pi] \setminus E$ we have 
\[ \lim_{n \to \infty} \sum_{k=-n}^{n} c_k e^{ikx} = 0, \]
then $c_k = 0$, $k \in \mathbb{Z}$.

The proof of the following theorem one can find in [5].

Theorem 7. Let $0 \leq \alpha < 1$ and 
\[ \int_{-\pi}^{\pi} |f(x)|dx + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x)-f(y)|}{|x-y|^{1+\alpha}}dx < \infty \]
Let $E \subset [-\pi, \pi]$ be a subset such that for almost all points $x_0 \in [-\pi, \pi]$ we have 
\[ \sum_{n=1}^{\infty} 2^n (1-\alpha) C_{1-\alpha}(E_n(x_0)) = \infty, \]
where 
\[ E_n(x_0) = \left\{ x \in E; \frac{1}{2^{n+1}} \leq |x-y| < \frac{1}{2^n} \right\}. \]
If 
\[ \lim_{n \to \infty} \sum_{k=-n}^{n} \hat{f}_k e^{ikx} = 0, \quad x \in E \]
then $f(x) = 0$, $x \in [-\pi, \pi]$.

In this paper we prove a new result of this type for other classes of functions.
3. New uniqueness result

The following result, in different form, one can find in the paper, see [4].

**Theorem 8.** Let \( f(-\pi) = f(\pi) \) be differentiable function. Then

\[
\frac{1}{\pi} \sum_{p \in P} \ln p \left( \sum_{n=1}^{\infty} \frac{1}{p^n} \sum_{k=1}^{p^n} f \left( \frac{2\pi k}{p^n} \right) - \hat{f}_0 \right) = \sum_{n \neq 0, n = -\infty}^{\infty} \hat{f}_n \ln |n|.
\]

**Proof.** We have

\[
\sum_{n=1}^{\infty} \Lambda(n) \left( \sum_{k=1}^{n} \frac{1}{|1 - z \exp \left( -\frac{2\pi ik}{n} \right)|^2} - 1 \right) =
\]

\[
= \sum_{n=1}^{\infty} \Lambda(n) \sum_{k=1}^{n} \left( \sum_{j=-\infty}^{\infty} r^{|j|} \exp \left\{ ixj - \frac{2\pi ikj}{n} \right\} - 1 \right) =
\]

\[
= \sum_{j=1}^{\infty} r^{|j|} \left( \sum_{j/n} \Lambda(n) \right) \cos(jx) = \sum_{j=1}^{\infty} r^{|j|} \cos(jx) \ln j.
\]

where \( z = re^{ix}, \quad 0 \leq r < 1. \)

Multiplying by the function \( f(x) \) and after integrating we get

\[
\sum_{n=1}^{\infty} \Lambda(n) \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{\left| 1 - r \exp \left( ix - \frac{2\pi ik}{n} \right) \right|^2} f(x) dx - \hat{f}_0 \right) =
\]

\[
= \frac{1}{2} \sum_{j \neq 0, j = -\infty}^{\infty} r^{|j|} \hat{f}_j \ln |j|.
\]

Passing to the limit if \( r \to 1 - 0 \) we get the required equality. \( \square \)

**Remark.** The getting result we can write in the form

\[
\frac{1}{\pi} \sum_{p \in P} \ln p \left( \sum_{n=1}^{\infty} \frac{1}{p^n} \sum_{k=1}^{p^n} \delta \left( x - \frac{2\pi k}{p^n} \right) - 1 \right) = \sum_{n \neq 0, n = -\infty}^{\infty} e^{inx} \ln n.
\]

This formula is a generalization of Poisson’s well known formula:

\[
\sum_{n=-\infty}^{\infty} \delta(x - 2\pi n) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{inx}, \quad -\infty < x < \infty.
\]

**Theorem 9.** Let \( 0 < \alpha < 1 \) and a nonnegative function \( x \leq h(x), \quad 0 \leq x < 1 \) satisfy the condition

\[
\int_{0}^{1} \frac{h(x)}{x^{2-\alpha}} \ln^2 \frac{e}{x} dx < \infty.
\]
Let for the function \( f(x) \) we have
\[
\int_{-\pi}^{\pi} |f(x)|^2 \, dx + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} \, dx \, dy < \infty.
\]

Let there is a subset \( E \subset [-\pi, \pi] \) such that:

1. \( M_h(E) > 0 \),
2. if \( x \in E \) then an arbitrary point \( x + \frac{2\pi k}{p^n} \),

where \( k \in \mathbb{Z}, n \in \mathbb{N} \) and \( p \) is prime number, by \( \text{mod}(2\pi) \), belongs to the set \( E \).

If for every \( x \in E \) we have
\[
\lim_{r \to 1-0} \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}_k e^{ikx} = 0,
\]
then the function \( f(x) \) is identically zero.

Proof. By O. Frostman’s theorem, see [3], p. 14, there is a probability measure \( d\mu \) such that \( \text{supp}(d\mu) \subseteq E \), \( M_h(\text{supp}(\mu)) > 0 \).

and for each \( 0 < \delta \) the inequality
\[
\int_{[x, x+\delta]} d\mu \leq Ah(\delta)
\]
hold. Let us assume at the points 0 and \( 2\pi \) the function \( \mu \) is continuous and \( \mu(0) + 1 = \mu(2\pi) \). Denote
\[
f(re^{ix}) = \sum_{n=-\infty}^{\infty} r^{|n|} \hat{f}_n e^{inx}.
\]

Then we have
\[
\frac{1}{\pi} \sum_{p \in P} \ln p \left( \sum_{n=1}^{\infty} \left[ \frac{1}{p^n} \sum_{k=1}^{p^n} \int_E f \left( r \exp \left( \frac{2\pi ik}{p^n} + ix \right) \right) \, d\mu(x) - \hat{f}_0 \right] \right) =
\]
\[
= \sum_{n=2}^{\infty} r^n \left[ \hat{f}_n \int_E e^{inx} \, d\mu(x) + \hat{f}_{-n} \int_E e^{-inx} \, d\mu(x) \right] \ln n =
\]
\[
= 2\pi i \sum_{n=2}^{\infty} \left( \hat{f}_n \hat{g}_{-n} - \hat{f}_{-n} \hat{g}_n \right) r^a n \ln n.
\]

where
\[
g(x) = \mu(x) - \frac{x}{2\pi}.
\]
Since the function $f(e^{ix})$ vanish on the set $E$ so, we have
\[ \left| \frac{1}{\pi} \sum_{p \in P} \hat{f}_0 \ln p \right| \leq 2 \sum_{n=2}^{\infty} (|\hat{f}_n||\hat{\gamma}_n| + |\hat{f}_n||\hat{\gamma}_{-n}|) n \ln n. \]

Let us note
\[ \sum_{n=2}^{\infty} |\hat{f}_n| |\hat{\gamma}_{-n}| n \ln n \leq \left( \sum_{n=1}^{\infty} |\hat{f}_n|^2 n^\alpha \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} |\hat{\gamma}_{-n}|^2 n^{2-\alpha} \ln^2 n \right)^{\frac{1}{2}} \leq \left( \sum_{n=1}^{\infty} |\hat{f}_n|^2 n^\alpha \right)^{\frac{1}{2}} \left( \sum_{j=1}^{\infty} j^2 2^{(2-\alpha)j} \sum_{n=2^j}^{2^{j+1}-1} |\hat{\gamma}_{-n}|^2 \right)^{\frac{1}{2}} \leq M \left( \sum_{j=1}^{\infty} j^2 2^{(1-\alpha)j} h(2^{-j}) \right)^{\frac{1}{2}} < \infty. \]

The inequality
\[ \left| \frac{1}{\pi} \sum_{p \in P} \hat{f}_0 \ln p \right| < \infty \]
valid only if
\[ \hat{f}_0 = 0. \]

Considering the functions $e^{inx} f(x)$, $n \in Z$ we prove that $\hat{f}(n) = 0$, $n \in Z$. □

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