Identities of finitely generated graded algebras with involution.

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Abstract

We consider associative algebras with involution graded by a finite abelian group $G$ over a field of characteristic zero. Suppose that the involution is compatible with the grading. We represent conditions permitting PI-representability of such algebras. Particularly, it is proved that a finitely generated $(\mathbb{Z}/q\mathbb{Z})$-graded associative PI-algebra with involution satisfies exactly the same graded identities with involution as some finite dimensional $(\mathbb{Z}/q\mathbb{Z})$-graded algebra with involution for any prime $q$ or $q = 4$. This is an analogue of the theorem of A.Kemer for ordinary identities [31], and an extension of the result of the author for identities with involution [42]. The similar results were proved also for graded identities [1], [41].

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Introduction

One of the central problem of the theory of varieties and PI-theory is the Specht problem: the problem of existence of a finite base for any system of identities. The original Specht problem [10] was formulated for identities of associative algebras over a field of characteristic zero. It was solved positively by Alexander Kemer [31], [33], [35]. The principal and the most difficult part of the Kemer’s solution was the proof of PI-representability of finitely generated PI-superalgebras [31], [34]. An algebra (a superalgebra) is called $PI$-representable if it satisfies the same identities ($(\mathbb{Z}/2\mathbb{Z})$-graded identities) as some finite dimensional algebra (superalgebra). The phenomena of PI-representability of finitely generated algebras has also the proper interest. It is an intriguing question, what are the classes of algebras and identities such that their finitely generated algebras satisfy the same identities as finite dimensional algebras.

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The first results on PI-representability of associative algebras belong to A.Kemer. He proves that any finitely generated associative $(\mathbb{Z}/2\mathbb{Z})$-graded PI-algebra over a field of characteristic zero satisfies the same $(\mathbb{Z}/2\mathbb{Z})$-graded identities as some finite dimensional $(\mathbb{Z}/2\mathbb{Z})$-graded algebra over the same field \[31\], \[34\]. Later, he proves also that a finitely generated associative PI-algebra over an infinite field satisfies the same ordinary (non-graded) polynomial identities as some finite dimensional algebra \[32\]. PI-representability of finitely generated associative algebras over a commutative associative Noetherian ring with respect to ordinary polynomial identities was studied in \[14\]-\[22\].

The series of results were obtained also for graded identities and identities with involution of associative algebras over a field of characteristic zero. If \(G\) is a finite abelian group then a finitely generated \(G\)-graded associative PI-algebra over an algebraically closed field of characteristic zero satisfies the same graded identities as some finite dimensional \(G\)-graded algebra over the same field \[41\]. For more general case of a finite (not necessarily abelian) group \(G\) it was proved that a finitely generated \(G\)-graded associative PI-algebra over a field of characteristic zero satisfies the same graded identities as some finite dimensional \(G\)-graded algebra over some extension of the base field \[1\]. As the direct consequences of \[1\], \[41\] we have also the similar results for \(G\)-identities if \(G\) is a finite abelian group of automorphisms of an associative algebra.

Recently, PI-representability was proved also for identities with involution \[42\]. A finitely generated associative PI-algebra with involution over a field of characteristic zero satisfies the same identities with involution as some finite dimensional algebra with involution over the same field.

The special interest to graded identities in the case of characteristic zero is explained by the super-trick and relation with the Specht problem \[31\]. Therefore the problem of PI-representability of graded algebras with involution is of current interest also.

We consider associative algebras over a field \(F\) of characteristic zero. Further they will be called algebras.

An \(F\)-algebra \(A\) is graded by a group \(G\) (\(G\)-graded algebra) if \(A\) can be decomposed as a direct sum \(A = \bigoplus_{\theta \in G} A_\theta\) of its vector subspaces \(A_\theta\) \((\theta \in G)\), where \(A_\theta A_\xi \subseteq A_{\theta \xi}\) holds for all \(\theta, \xi \in G\). A subspace \(V \subseteq A\) is called graded if \(V = \bigoplus_{\theta \in G}(V \cap A_\theta)\). We consider only gradings by a finite abelian group.

Anti-automorphism \(*\) of the second order of an algebra \(A\) over \(F\) is called involution. Algebra with involution is also called \(*\)-algebra. An element \(a\) of a \(*\)-algebra \(A\) is called symmetric if \(a^* = a\), and skew-symmetric if \(a^* = -a\). Particularly, \(a + a^*\) is skew-symmetric and \(a - a^*\) is skew-symmetric for any element \(a \in A\). It is clear that \(A = A^+ \oplus A^\perp\), where \(A^+\) is the subspace formed by all symmetric elements (symmetric part), and \(A^\perp\) the subspace of all skew-symmetric elements of \(A\) (skew-symmetric part). We also use the notations \(a \circ b = ab + ba\), and \([a, b] = ab - ba\). It is clear that the symmetric part \(A^+\) of a \(*\)-algebra \(A\) with the operation \(\circ\) is a Jordan algebra, and the skew-symmetric part \(A^\perp\) with the operation \([,]\) is a Lie algebra.

Let \(G = \{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m\}\) be a finite abelian group of order \(m\) with the unit
Let us consider a $G$-graded algebra $A = \bigoplus_{\theta \in G} A_{\theta}$ with involution. We assume that involution $*$ is graded anti-automorphism of $A$, i.e. $A_{\theta}^* = A_{\theta}$ for any $\theta \in G$. This is equivalent to condition (see, e.g., [8]) that the subspaces $A^+$, $A^-$ are graded. Particularly, we have $A = \bigoplus_{\theta \in G}(A^+_{\theta} \oplus A^-_{\theta})$, where $A^\delta = \bigoplus_{\theta \in G} A^\delta_{\theta}$, $(\delta \in \{+,-\})$; and $A_{\theta} = A^+_{\theta} \oplus A^-_{\theta}$, $(\theta \in G)$. We say that an element $a \in A^\delta_{\theta}$ $(\delta \in \{+,-\}, \theta \in G)$ is homogeneous of complete degree $\deg_{\theta} a = (\delta, \theta)$ or simply $G$-homogeneous.

Note that if the base field $F$ contains a primitive root of unity $\sqrt[n]{1}$ of order $m = |G|$ then a $G$-grading on $A$ naturally corresponds to the action on $A$ of the group $\text{Irr}\{\mathbb{G}\} = \{\chi_k | k = 1, \ldots, |G|\} \cong G$ of irreducible characters of $G$. An irreducible character $\chi \in \text{Irr}\{\mathbb{G}\}$ acts on $A$ as an automorphism associating to any element $a = \sum_{\theta \in G} a_{\theta} \in A$ the element $\chi(a) = \sum_{\theta \in G} \chi(\theta) a_{\theta}$ ([29]). Then the involution $*$ of $A$ is graded iff it commutes with any $\chi \in \text{Irr}\{\mathbb{G}\}$. Thus in this case the group $\mathbb{G} = \text{Irr}\{\mathbb{G}\} \cong G \times \mathbb{Z}/2\mathbb{Z}$ of automorphisms and anti-automorphisms acts on $A$. We refer readers for more details about connection of gradings with automorphism group actions to [7], [8], [27], [29].

For two $G$-graded $*$-algebras $A$, $B$ their homomorphism is called graded $*$-homomorphism if it is graded, and commutes with involution. It happens iff it commutes with any element of $\mathbb{G}$. An ideal $I \subseteq A$ of a graded algebra with involution $A$ is a graded $*$-ideal if it is graded and invariant under the involution. For graded algebras with involution we consider only graded $*$-ideals and graded $*$-homomorphisms. In this case the quotient algebra $A/I$ is also a graded algebra with involution with the grading and involution induced from $A$. A $G$-graded $*$-algebra is called $*$-graded simple if it has no proper graded $*$-ideals.

We denote by $A_1 \times \cdots \times A_p$, the direct product of algebras $A_1, \ldots, A_p$, and by $A_1 \oplus \cdots \oplus A_p \subseteq A$ the direct sum of subspaces $A_i$ of an algebra $A$. It is clear that the direct product of graded algebras with involution is also a graded algebra with involution. Throughout the paper we denote by $J(A)$ the Jacobson radical of $A$, and by $\text{nd}(A)$ the degree of nilpotency of $J(A)$ if $A$ is finite dimensional. By default, all bases and dimensions of vector spaces are considered over the base field $F$.

We always consider the lexicographical order on the sets $\mathbb{N}_0^m$, $m$ is a positive integer number. Note that this order satisfies descending chain condition.

The concept of a graded identity with involution (graded $*$-identity) is the union of concepts of a graded identity (see [30], [31], [28], [29]) and identity with involution (see, e.g., [29]). It inherits the principal features of the notion of an ordinary polynomial identity. We refer the reader to the textbooks [25], [26], [29], and to [30], [31] on questions concerning ordinary polynomial identities.

Here we study graded $*$-identities of associative $G$-graded algebras with graded involution. We prove that a finitely generated $G$-graded PI-algebra with involution satisfies exactly the same graded $*$-identities as some finite dimensional graded algebra with involution under the condition of existence of some specific basis of a $*$-graded finite dimensional algebra (Theorem [5,2]). The required basis is defined in Lemma [3,2]. We also give a description of $*$-graded simple finite dimensional algebras over an algebraically closed field of characteristic zero for case of a grading by the
cyclic group $G$ of a prime order or of order 4 (Theorem 6.1). As a partial case we obtain

**Theorem 6.2** Let $q$ be a prime integer or $q = 4$. Assume that $F$ is a field of characteristic zero. Then for any $(\mathbb{Z}/q\mathbb{Z})$-graded finitely generated associative PI-algebra $A$ with graded involution over $F$ there exists a finite dimensional over $F$ $(\mathbb{Z}/q\mathbb{Z})$-graded associative algebra $C$ with graded involution such that the ideals of graded $*$-identities of $A$ and $C$ coincide.

Finally, we suppose that the next assumption can be true in general.

**Conjecture 0.1** Let $G$ be a finite abelian group, and $\overline{F}$ be an algebraically closed field of characteristic zero. Then any $G$-graded $*$-simple finite dimensional $\overline{F}$-algebra possesses a basis satisfying the claims of Lemma 3.2.

The problem of existence of the required basis is reduced to Assumption 2.1 concerning the classification of $*$-graded simple finite dimensional algebras over an algebraically closed field. The confirmation of Conjecture 6.1 will guarantee the existence of a basis defined in Lemma 3.2. In this case Theorem 5.2 immediately will imply PI-representability with respect to graded $*$-identities of any finitely generated $G$-graded PI-algebra with graded involution over a field $F$ of characteristic zero for any finite abelian group $G$.

**Conjecture 0.2** Let $F$ be a field of characteristic zero, and $G$ a finite abelian group. Then a proper $giT$-ideal of graded $*$-identities of a $G$-graded finitely generated associative PI-algebra with graded involution over $F$ coincides with the ideal of graded $*$-identities of some finite dimensional over $F$ $G$-graded associative algebra with graded involution.

It is worth to mention that the condition for a finitely generated algebra to be a PI-algebra in Theorems 5.1 5.2 6.2 is necessary. Since any finite dimensional algebra is a PI-algebra (an algebra satisfying non-trivial ordinary (non-graded) polynomial identity) then Theorems 5.1 5.2 6.2 can be applied only to $giT$-ideals containing some non-trivial T-ideal. We discuss briefly in Section 1 the conditions providing that a $giT$-ideal contains a non-trivial T-ideal.

First, we prove Theorem 5.1 about PI-representability with respect to graded $*$-identities for a field of characteristic zero which contains a primitive root of unity of order $m = |G|$. Afterwards, we extend this result for any field of characteristic zero (Theorem 5.2). In order to prove Theorem 5.1 we exploit the techniques created by A.R.Kemer [31] for the Specht problem solution modified for the case of graded identities with involution. Earlier these methods also were adopted by A.Belov e E.Aljadeff for group-graded identities [1], and by author for graded identities of algebras graded by a finite abelian group [41], and for non-graded identities with involution [42].
Here we follow the structure of the proof given in [42]. The majority of constructions, properties and arguments from [42] needs only slight adaptation for the graded case or even can be directly repeated. The main definitions are given for the completeness of the text, even if they directly repeat the non-graded versions. The proofs are repeated only if we need to point out some details or conditions which are peculiar in the graded case. In all other cases we refer the reader to the corresponding statements and arguments of [42] with the appropriate comments. We can refer the reader also to [41] for some technical details.

We introduce briefly in Section 1 the concept of a graded identity with involution and the concept of the free graded algebra with involution. In Section 2 we state the principal assumption (Assumption 2.1) concerning the classification of finite dimensional $*$-graded simple algebras over an algebraically closed field.

Section 3 is devoted to finite dimensional graded $*$-algebras. We consider their structure, define a specific basis, introduce the parameters $\text{par}_{gi}(A)$ of a finite dimensional graded $*$-algebra $A$, and the Kemer index $\text{ind}_{gi}(\Gamma)$ of the $giT$-ideal $\Gamma$ of graded identities with involution of a finitely generated graded PI-algebra with involution. We establish relations between the structural parameters $\text{par}_{gi}$ of finite dimensional graded $*$-algebras and indices $\text{ind}_{gi}$ of their $giT$-ideals. Lemmas 3.2, 3.15 in Section 3 are basic for the proof and represent the principal difference with the non-graded case [42]. Lemma 3.2 modulo Assumption 2.1 substitutes Lemma 4 of the non-graded case [42], and Lemma 3.15 takes place of Lemma 12 [42]. Lemmas 3.4, 3.14, 3.19, 3.20 are the graded versions of Lemmas 5, 9, 14, 15 in [42] respectively.

Section 4 is devoted to graded identities with forms, representable algebras, and to the technique of approximation of finitely generated algebras with involution by finite dimensional graded $*$-algebras. This section almost completely repeats the analogous section in [42]. Observe that the similar constructions (the free algebra with forms, identities with forms) can be found also in [32], [38], [44].

Section 5 contains the proof of the main theorems (Theorem 5.1 and Theorem 5.2). We consider algebras over a field of characteristic zero containing a primitive root of unity of order $m = |G|$ in Theorem 5.1. Its proof is also a slight modification of the proof in the non-graded case [42]. In Theorem 5.2 this result is extended for case of any field of characteristic zero.

In Section 6 we consider our problem in a partial case when the group $G$ is cyclic of a prime order $q$ or of the order $q = 4$. We give the classification of finite dimensional $*$-graded simple algebras over an algebraically closed field for a $(\mathbb{Z}/q\mathbb{Z})$-grading (Theorem 6.1), and obtain the PI-representability with respect to graded $*$-identities of finitely generated graded PI-algebras with graded involution in this case (Theorem 6.2).

Observe that in Section 1, in the definitions of free graded $*$-algebra with forms and graded $*$-identities with forms (in Section 4), and in principal Theorems 5.2 (Section 5), 6.2 (Section 6) we consider any field $F$ of characteristic zero. In Section 2 and in Theorem 6.1 we consider algebras over an algebraically closed field $\bar{F}$ of characteristic zero. In Section 3, in the major part of Section 4, and in Theorem 5.1
in Section 5 we assume that $F$ contains a primitive root of unity of order $m = |G|$.

1 Free graded algebra with involution.

Let $F$ be a field of characteristic zero, and $G$ a finite abelian group, $|G| = m$. Let us consider $Y = \{y_i \mid i \in \mathbb{N}, \theta \in G\}$, $Z = \{z_i \mid i \in \mathbb{N}, \theta \in G\}$ two countable sets of pairwise different indeterminants. We denote by $\deg_G y_i = \deg_G z_i = \theta$ the $G$-degree of the variables $Y \cup Z$ with respect to $G$-grading. Then $Y^0 = \{y_i \mid i \in \mathbb{N}\}$, $Z^0 = \{z_i \mid i \in \mathbb{N}\}$ are homogeneous variables of $G$-degree $\theta \in G$. We can define $*$-action on monomials over $Y \cup Z$ by equalities

$$w^* = (x_{i_1} \cdots x_{i_n})^* = x_{i_n}^* \cdots x_{i_1}^* = (-1)^{\delta(w)} x_{i_n} \cdots x_{i_1}, \quad \text{where}$$

$$y_j^* = y_j \theta, \quad z_j^* = -z_j \theta, \quad x_j \in Y \cup Z.$$

Here the sign is determined by the parity of the number $\delta(w)$ of variables of the set $Z$ in the monomial $w$. Linear extension of this action is an involution on the free associative algebra $\mathbb{F} = F(Y, Z)$ generated by the set $Y \cup Z$.

The algebra $\mathbb{F} = F(Y, Z)$ is $G$-graded with the grading $\mathbb{F} = \bigoplus_{\theta \in G} \mathbb{F}_\theta$ defined by $\mathbb{F}_\theta = \text{Span}_F \{x_{i_1} x_{i_2} \cdots x_{i_n} \mid \deg_G x_{i_1} \cdots \deg_G x_{i_n} = \theta, \ x_j \in Y \cup Z\}$. It is clear that the involution (1) is graded.

The algebra $\mathbb{F}$ is the free associative graded algebra with involution. Its elements are called graded $*$-polynomials. Note that the set $X^G = \{z_i \mid i \in \mathbb{N}, \theta \in G\}$ generates in $\mathbb{F}$ the $G$-graded subalgebra $\mathbb{F}(X^G)$ that is isomorphic to the free associative $G$-graded algebra $\mathbb{F}(X^G)$.

Let $f = f(x_1, \ldots, x_n) \in F(Y, Z)$ be a non-trivial graded $*$-polynomial ($x_i \in Y \cup Z$). We say that a graded $*$-algebra $A$ satisfies the graded $*$-identity (or graded identity with involution) $f = 0$ iff $f(a_1, \ldots, a_n) = 0$ for all homogeneous $a_i \in A^G_{\delta_i}$ of complete degree $\deg_G a_i = \deg_G x_i = (\delta_i, \theta_i)$, $\delta_i \in \{+, -\}$, $\theta_i \in G$ ($i = 1, \ldots, n$).

Then $\text{Id}^G(A) \leq F(Y, Z)$ is the ideal of all graded identities with involution of $A$. Similar to the case of graded identities and identities with involution ([31], [41], [42]) any ideal of graded identities with involution is two-sided graded $*$-ideal of the free graded algebra with involution $F(Y, Z)$, which is invariant under all its graded $*$-endomorphisms. We call such ideals $giT$-ideals. Also any $giT$-ideal $I$ of $F(Y, Z)$ is the ideal of graded $*$-identities of the graded algebra with involution $F(Y, Z)/I$. Given a set $S \subseteq F(Y, Z)$ the $giT$-ideal generated by $S$ is the minimal $giT$-ideal containing $S$. We denote it by $giT[S] \leq F(Y, Z)$. Two $G$-graded algebras with involution $A$ and $B$ are called $gi$-equivalent if they have the same $giT$-ideals of graded $*$-identities. We also say that $f = g \pmod{\Gamma}$ for a $giT$-ideal $\Gamma$ and graded $*$-polynomials $f, g \in F(Y, Z)$ if $f - g \in \Gamma$.

We assume that a $T$-ideal of ordinary polynomial identities $\Gamma_1$, a $GT$-ideal of $G$-graded identities $\Gamma_2$ or a $T$-ideal of non-graded identities with involution $\Gamma_3$ lies in a $giT$-ideal $\Gamma$ if the $giT$-ideal $\Gamma'$ generated by the corresponding ideal $\Gamma_i$ lies in $\Gamma$. Recall that a PI-algebra is an algebra satisfying an ordinary polynomial identity. It is clear that for a $G$-graded PI-algebra with involution $A$ the ideal of ordinary
polynomial identities \( \text{Id}(A) \), the ideal of graded identities \( \text{Id}^G(A) \), and the ideal of identities with involution \( \text{Id}^*(A) \) lie in \( \text{Id}^q(A) \). Note, that \( A \) is a PI-algebra iff the neutral component \( A_e \) satisfies a non-trivial \(*\)-identity, where \( e \) is the unit element of \( G \) (it follows from \[2\], \[3\], and \[1\], \[23\]). Also if \( F \) contains \( \sqrt{\text{T}} \) then a graded algebra with involution \( A \) is PI-algebra if it satisfies an essential \( \hat{G} \)-identity (\[5\], see also \[29\]).

We always can assume that a generating set of a finitely generated graded algebra with involution consists of homogeneous elements. Given a finitely generated graded \(*\)-algebra \( A \), and a finite homogeneous generating set \( K \) let us denote by \( \text{rkh}(K) \) the maximal number of generators of the same complete degree in \( K \). Then \( \text{rkh}(A) \) is the least \( \text{rkh}(K) \) for all finite homogeneous generating sets \( K \) of \( A \).

We also can consider the free \( G \)-graded algebra with involution \( F(Y_{(\nu)}, Z_{(\nu)}) \) of a finite rank \( \nu \) generated by the sets \( Y_{(\nu)} = \{ y_{i\theta} | i = 1, \ldots, \nu; \theta \in G \} \), and \( Z_{(\nu)} = \{ z_{i\theta} | i = 1, \ldots, \nu; \theta \in G \} \). Given a \( \text{giT} \)-ideal \( \Gamma \subseteq F(Y, Z) \) and a graded \(*\)-algebra \( B \), denote by \( \Gamma(B) = \{ f(b_1, \ldots, b_n) | f \in \Gamma, b_i \in B \} \subseteq B \) the verbal ideal of \( B \) corresponding to \( \Gamma \), here elements \( b_i \in B \) are homogeneous of appropriate complete degrees. Then Remark 1 \[42\] is also true in graded case.

The notions of homogeneous on polynomial degree identity, and linear identity are analogous to the case of ordinary identities (see \[23\], \[31\], \[29\]). Similarly it is enough to consider only multilinear graded \(*\)-identities in the case of characteristic zero. Let us denote for any \( \bar{n} = (n_{01}, n_{11}, \ldots, n_{0m}, n_{1m}) \in \mathbb{N}_0^{2m} \) (\( m = |G| \)) by \( P_{\bar{n}} \) the vector subspaces of \( \mathfrak{F} \) formed by all multilinear polynomials depending on \( y_{1\theta_1}, \ldots, y_{n_{01}\theta_1}, z_{1\theta_1}, \ldots, z_{n_{1\theta_1}}, \theta_i \in G \) (\( i = 1, \ldots, m \)). Then given a \( \text{giT} \)-ideal \( \Gamma \) the corresponding multilinear parts \( \Gamma_{\bar{n}} = \Gamma \cap P_{\bar{n}} \) of \( \Gamma \), and \( P_{\bar{n}}(\Gamma) = P_{\bar{n}}/\Gamma_{\bar{n}} \) of the relatively free graded algebra with involution \( \mathfrak{F}/\Gamma \) are \( (FS_{n_{01}} \otimes \cdots \otimes FS_{n_{1m}})\)-modules. Here \( S_{n_{ij}} \) acts on the corresponding set of \( \hat{G} \)-homogeneous variables independently.

**Lemma 1.1** Let \( G \) be a finite abelian group, and \( A \) a finitely generated associative \( G \)-graded PI-algebra with a graded involution. The \( \text{giT} \)-ideal of graded \(*\)-identities of \( A \) contains the ideal of graded \(*\)-identities of some finite dimensional \( G \)-graded algebra with involution.

**Proof.** Let \( \Gamma = \text{Id}^*(A) \) be the \(*\)-T-ideal of non-graded \(*\)-identities of \( A \). Then \( \Gamma \) is non-trivial and \( \Gamma \subseteq \text{Id}^q(A) \). By \[42\] \( \Gamma \) is the ideal of \(*\)-identities of some finite dimensional \(*\)-algebra \( B \). It is clear that the \( \text{giT} \)-ideal generated by \( \Gamma \) coincides with the ideal of graded \(*\)-identities of the finite dimensional \( G \)-graded algebra \( B \otimes F[G] \) with the involution induced from \( B \) (and trivial on the group algebra \( F[G] \)) \( (b \otimes \theta)^* = b^* \otimes \theta \), and the grading defined by \( \text{deg}_G b \otimes \theta = \theta \) for all \( b \in B, \theta \in G \). \( \Box \)

## 2 \(*\)-graded simple algebras. Assumption.

Let \( \bar{F} \) be an algebraically closed field of characteristic zero, and \( G \) a finite abelian group. Consider a finite dimensional \( G \)-graded \( \bar{F} \)-algebra \( C \) with involution. We call such an algebra \(*\)-graded simple if it does not contain a non-trivial graded \(*\)-ideal. It
is equivalent to the condition that an algebra has no non-trivial $\hat{G}$-invariant ideals, where $\hat{G} = \text{Irr} G \times \langle \ast \rangle$.

The Jacobson radical of a finite dimensional algebra is invariant under the action of the group of automorphisms and anti-automorphisms $\hat{G}$. Then over an algebraically closed field the radical is a $G$-graded $\ast$-ideal (see, e.g., [27]). Particularly, any finite dimensional $\ast$-graded simple algebra is semisimple.

For a finite dimensional $\ast$-graded simple $\hat{F}$-algebra $C$ there exist two possibilities. Either $C$ is a $G$-graded simple algebra with an involution compatible with the grading, or $C$ contains a proper graded simple ideal $B$. It is clear that in the second case $C = B \times \hat{B}^\ast$. Hence $C$ is isomorphic to the direct product $B \times \hat{B}^\ast$ of a graded simple algebra $B$ and its opposite algebra $\hat{B}^\ast$ with the exchange involution $(a,b)\ast = (b,a)$, $a \in B$, $b \in \hat{B}^\ast$. The description of $G$-graded simple algebras is given by the next lemma.

**Lemma 2.1 (Theorem 3, [6])** Let $\hat{F}$ be an algebraically closed field of characteristic zero. Then any finite dimensional $G$-graded simple algebra $C$ over $\hat{F}$ is isomorphic to $M_k(\hat{F}[H])$, a matrix algebra over the graded division algebra $\hat{F}[H]$, where $H$ is a subgroup of $G$ and $\zeta : H \times H \to \hat{F}^\ast$ is a 2-cocycle on $H$. The $G$-grading on $M_k(\hat{F}[H])$ is defined by a $k$-tuple $(\theta_1, \ldots, \theta_k) \in G^k$, so that $\deg_{\hat{G}}(E_{ij}\eta_k) = \theta_i^{-1}\xi\theta_j$ for any matrix unit $E_{ij}$ and any basic element $\eta_k$ of $\hat{F}[H]$, $\xi \in H$.

Here the graded division algebra $\hat{F}[H] = \text{Span}_\hat{F}\{\eta_k | \xi \in H\}$ is a twisted group algebra with the product on the basic elements $\eta_\theta \cdot \eta_\xi = \zeta(\theta, \xi) \eta_{\theta \xi}$ determined by a 2-cocycle $\zeta$ on a subgroup $H \leq G$ $(\theta, \xi \in H)$. It has the natural $H$-grading defined by $\deg_H \eta_\xi = \xi$ for any $\eta \in H$. Note that the set $\{E_{ij}\eta_k | 1 \leq i, j \leq k, \xi \in H\}$ forms a multiplicative basis of the $G$-graded simple algebra $M_k(\hat{F}[H])$.

**Definition 2.2** A graded involution on the $G$-graded simple algebra $M_k(\hat{F}[H])$ is called elementary if it satisfies the condition

$$
(E_{ij}\eta_k)^\ast = \alpha_{i,j,\xi} E_{i',j'}\eta_{\xi'}, \quad 1 \leq i', j' \leq k, \quad \xi' \in H, \quad \alpha_{i,j,\xi} \in \{1, -1\}
$$

for all $i, j = 1, \ldots, k$, $\xi \in H$.

Observe that $(i, j) = (i', j')$ in (2) implies that $\xi = \xi'$, because the involution is graded.

Let us give the principle assumption concerning finite dimensional $\ast$-graded simple algebras.

**Assumption 2.1** We suppose that any finite dimensional $\ast$-graded simple algebra is isomorphic as a graded $\ast$-algebra either to $G$-graded simple algebra $\hat{C}^{(1)} = M_k(\hat{F}[H])$ with an elementary involution, or to the direct product $\hat{C}^{(2)} = B \times B^\ast$ of a graded simple algebra $B = M_k(\hat{F}[H])$ and its opposite algebra $B^\ast$ with the exchange involution $\bar{\ast}$.

Moreover, in any case $H$ is a subgroup of $G$, and $\zeta : H \times H \to \mathbb{Q}[\sqrt{m}]^\ast$ is a 2-cocycle on $H$ with values in the algebraic extension of rational numbers $\mathbb{Q}$ by a primitive root $\sqrt{m}$ of one of degree $m = |G|$.
In chapters 3, 4, 5 we consider only such \( \ast \)-graded simple algebras.

## 3 Finite dimensional \( \ast \)-graded algebras.

Let \( F \) be a field of characteristic zero. Assume that \( F \) contains a primitive root \( \sqrt[m]{1} \) of one of degree \( m = |G| \). Suppose that \( A \) is a \( G \)-graded algebra with involution, finite dimensional over the base field \( F \). Repeating the proof of Lemma 2.2 [10] for the group \( \tilde{G} = \text{Irr} G \times \langle \ast \rangle \in \text{Aut}^\ast (A) \), and applying results of [13] we obtain the Wedderburn-Malcev decomposition for \( G \)-graded algebras with involution.

**Lemma 3.1** Let \( F \) be a field of characteristic zero containing a primitive root \( \sqrt[m]{1} \) of degree \( m = |G| \). Any finite dimensional \( G \)-graded \( F \)-algebra with involution \( A' \) is isomorphic as a graded \( \ast \)-algebra to a \( G \)-graded \( F \)-algebra with involution of the form

\[
A = C_1 \times \cdots \times C_p \oplus J.
\]

Where the Jacobson radical \( J = J(A) \) of \( A \) is a graded \( \ast \)-ideal, \( \tilde{B} = C_1 \times \cdots \times C_p \) is a maximal semisimple graded \( \ast \)-invariant subalgebra of \( A \), \( C_l \) are \( \ast \)-graded simple algebras \((p \in \mathbb{N} \cup \{0\})\).

Given a graded \( \ast \)-algebra \( B \) (not necessarily without unit) we denote by \( B^\# = B \oplus F \cdot 1_F \) the graded \( \ast \)-algebra with the adjoint unit \( 1_F \). We assume that \( 1_F \) has complete degree \((+,\cdot)\).

Similarly to [41, 42] we construct for an algebra \( A \) of the form [3] the graded algebra with involution with the free Jacobson radical. Given a graded \( \ast \)-subalgebra \( \tilde{B} \subseteq B \) we take the free product \( \tilde{B}^\ast \ast_F F\langle Y(q),Z(q)\rangle^\# \) of \( \tilde{B}^\# \) with the free unitary graded \( \ast \)-algebra \( F\langle Y(q),Z(q)\rangle^\# \) of rank \( q \). Consider its subalgebra \( \tilde{B}(Y(q),Z(q)) \) generated by the set \( \tilde{B} \cup F\langle Y(q),Z(q)\rangle \). It is clear that \( \tilde{B}(Y(q),Z(q)) = \tilde{B} \oplus \langle Y(q),Z(q) \rangle \) is a graded \( \ast \)-algebra, where \( Y(q),Z(q) \) is the two-sided graded \( \ast \)-ideal of \( \tilde{B}(Y(q),Z(q)) \) generated by the set of variables \( Y(q) \cup Z(q) \).

Given a \( \mathfrak{gt} \)-ideal \( \Gamma \) and a positive integer number \( s \) consider the quotient algebra

\[
\mathcal{R}_{q,s}(\tilde{B},\Gamma) = \tilde{B}(Y(q),Z(q))/(\Gamma(\tilde{B}(Y(q),Z(q)))) + (Y(q),Z(q))^s).
\]

Denote also \( \mathcal{R}_{q,s}(A) = \mathcal{R}_{q,s}(B,\text{Id}^\mathfrak{gt}(A)) \).

If \( F \) is algebraically closed then using Assumption 2.1 we obtain more detailed description of a finite dimensional graded \( \ast \)-algebra. Consider a sequence \( C_1,\ldots,C_p \) of \( p \) \( \ast \)-graded simple algebras of the types \( \tilde{C}^{(1)} \), \( \tilde{C}^{(2)} \). Suppose that the algebra \( C_l \) of this sequence is a \( G \)-graded simple algebra with an elementary involution (i.e. \( C_l \) is of the type \( \tilde{C}^{(1)} \)). Let us denote by

\[
e_{l,i,j}^{(\xi_l)} = E_{l,i,j} \eta_{\xi_l} \quad (1 \leq i,j \leq k_l, \quad \xi_l \in H_l)
\]

the basic element of \( C_l = M_{k_l}(F^{\langle i[H_l] \rangle}) \).
Consider the case when $C_l = \tilde{G}^{(2)} = \mathcal{H} \times \mathcal{H}^\circ$ is the direct product of a $G$-graded simple algebra $\mathcal{B} = M_{k_i}(\tilde{F}[H])$ and its opposite algebra $\mathcal{B}^\circ$ with the exchange involution ($C_l$ is of the type $\tilde{G}^{(2)}$). Then we denote

$$\epsilon_{l,(ij)}^{(\xi)} = \eta_i(\tilde{E}_{l,ij}, \tilde{E}_{l,ij}) = (\tilde{E}_{l,ij} \eta_i, \tilde{E}_{l,ij} \eta_i),$$

and

$$\tilde{\epsilon}_{l,(ij)}^{(\xi)} = \eta_i(\tilde{E}_{l,ij}, -\tilde{E}_{l,ij}) = (\tilde{E}_{l,ij} \eta_i, -\tilde{E}_{l,ij} \eta_i), \quad (1 \leq i, j \leq k_i),$$

$E_{l,ij}$ are the matrix units, $\eta_i$ is the basic element of $\tilde{F}[H]$ corresponding to $\xi_l \in H_l$. Note that in this case ($C_l = \tilde{G}^{(2)}$) all the elements $\epsilon_{l,(ij)}^{(\xi)}$ are symmetric, and $\tilde{\epsilon}_{l,(ij)}^{(\xi)}$ skew-symmetric with respect to involution.

For the second case let us consider also the elements

$$\epsilon_{l,(ij),i'}^{(\xi)} = \eta_i(\tilde{E}_{l,ij}, \tilde{E}_{l,i'j'}) = (\tilde{E}_{l,ij} \eta_i, \tilde{E}_{l,i'j'} \eta_i) = \frac{1}{2}(\epsilon_{l,(ij)}^{(\xi)} + \epsilon_{l,(i'j')}^{(\xi)}),$$

and

$$\tilde{\epsilon}_{l,(ij),i'}^{(\xi)} = \eta_i(\tilde{E}_{l,ij}, -\tilde{E}_{l,i'j'}) = (\tilde{E}_{l,ij} \eta_i, -\tilde{E}_{l,i'j'} \eta_i) = \frac{1}{2}(\epsilon_{l,(ij)}^{(\xi)} - \epsilon_{l,(i'j')}^{(\xi)}), \quad (1 \leq i, j, i', j' \leq k_i).$$

It is possible that these elements are not $G$-homogeneous, depending on the indices $i, j, i', j'$.

It is clear that all elements $\epsilon_{l,(ij)}^{(\xi)}$, $\tilde{\epsilon}_{l,(ij)}^{(\xi)}$ ($\xi_l \in H_l$, $1 \leq i, j \leq k_l$), admitted for $C_l$, form a $G$-homogeneous basis of $\tilde{C}_l$. Moreover, their symmetric and skew-symmetric parts with respect to the involution (eliminating proportional elements and zeros) form a $\tilde{G}$-homogeneous basis of $C_l$. This basis of $C_l$ will be considered as canonical.

More precisely, for simple algebras of the second type $\tilde{G}^{(2)}$ elements $\epsilon_{l,(ij)}^{(\xi)}$, $\tilde{\epsilon}_{l,(ij)}^{(\xi)}$ are $\tilde{G}$-homogeneous, and their symmetric and skew-symmetric parts coincide with them, $d_{l,(ij)}^{(\pm,\theta)} = \epsilon_{l,(ij)}^{(\xi)}$, $d_{l,(ij)}^{(-,\theta)} = \tilde{\epsilon}_{l,(ij)}^{(\xi)}$, $\theta = \deg_G \epsilon_{l,(ij)}^{(\xi)} = \deg_G \tilde{\epsilon}_{l,(ij)}^{(\xi)} = \theta^{-1} \xi_i \theta j_i$. For algebras of the type $\tilde{G}^{(1)}$ the next alternative follows from (2). If $(i, j) = (i', j')$ then $\epsilon_{l,(ij)}^{(\xi)}$ is a symmetric or skew-symmetric element. In this case $d_{l,(ij)}^{(\pm,\theta)} = \epsilon_{l,(ij)}^{(\xi)}$ for $(\delta, \theta) = \deg_G \epsilon_{l,(ij)}^{(\xi)}$. If $(i, j) \neq (i', j')$ then we have for the couple of indices $(i, j)$ and $(i', j')$, and elements $\xi_l, \xi_{l'} \in H_l$ the equalities $(\epsilon_{l,(ij)}^{(\xi)})^* = \alpha \epsilon_{l,(i'j')}^{(\xi)}$, $(\tilde{\epsilon}_{l,(ij)}^{(\xi)})^* = \alpha \tilde{\epsilon}_{l,(i'j')}^{(\xi)}$, where $\alpha \in \{1, -1\}$. Then we denote the symmetric part of $\epsilon_{l,(ij)}^{(\xi)}$ by $d_{l,(ij)}^{(+,\theta)} = 1/2(\epsilon_{l,(ij)}^{(\xi)} + \epsilon_{l,(i'j')}^{(\xi)})$, and the skew-symmetric part by $d_{l,(ij)}^{(-,\theta)} = 1/2(\epsilon_{l,(ij)}^{(\xi)} - \epsilon_{l,(i'j')}^{(\xi)})$, where $\theta = \deg_G \epsilon_{l,(ij)}^{(\xi)} = \theta^{-1} \xi_i \theta j_i = \deg_G \tilde{\epsilon}_{l,(ij)}^{(\xi)} = \theta^{+1} \xi_i \theta j_i$.

In any case the canonical basis of $C_l$ is formed by all non-zero elements $d_{l,(ij)}^{(\delta,\theta)}$. We denote by $\mathcal{I}_{l,(\delta,\theta)} = \{(i, j)|d_{l,(ij)}^{(\delta,\theta)} \neq 0\}$ the set of all couples of indices $(i, j)$ such that the corresponding basic element $d_{l,(ij)}^{(\delta,\theta)}$ has the complete degree $(\delta, \theta) \in \tilde{G}$.
Thus, Lemma 3.1 and Assumption 2.1 immediately imply the structure description of a finite dimensional $*$-graded simple algebra.

**Lemma 3.2** Let $F$ be an algebraically closed field of characteristic zero. Suppose that a finite abelian group $G$ admits the classification of finite dimensional $*$-graded simple algebras given in Assumption 2.7. Then any finite dimensional $G$-graded $F$-algebra with involution $A$ is isomorphic to a $G$-graded $F$-algebra with involution $A' = C_1 \times \cdots \times C_p \oplus J$. Where any $*$-graded simple subalgebra $C_l$ is isomorphic to an algebra $\bar{C}^{(1)}$ or $\bar{C}^{(2)}$ of Assumption 2.1 ($l = 1, \ldots, p$).

Moreover, $A'$ can be generated as a vector space by sets of its $\bar{G}$-homogeneous elements $D_{(\delta, \theta)}$, $U_{(\delta, \theta)} \subseteq A'$ ($\delta \in \{+,-\}, \theta \in G$) of the form

$$D_{(\delta, \theta)} = \{d_{i_1j_1}^{(\delta, \theta)} = \varepsilon_i d_{i_1j_1}^{(\delta, \theta)} e_l \mid (i_1, j_1) \in I_{(\delta, \theta)}; 1 \leq l \leq p\}, \quad (6)$$

$$U_{(+, \theta)} = \{(\varepsilon_i \varepsilon_r \varepsilon_{r'} - \varepsilon_r {r'} \varepsilon_{r'} \varepsilon_{r'})/2 \mid 1 \leq l' \leq l'' \leq p + 1; r \in J_0\} \quad (7)$$

Here $D = \bigcup_{(\delta, \theta) \in \bar{G}} D_{(\delta, \theta)}$ is the union of the canonical bases of $C_l$ ($l = 1, \ldots, p$), $U = \bigcup_{(\delta, \theta) \in \bar{G}} U_{(\delta, \theta)} \subseteq J$ is the set of $\bar{G}$-homogeneous radical elements.

Particularly, for any admitted element $e_{l(i_1j_1)}^{(\xi_l)}$ of $C_l$ ($\xi_l \in G$, $1 \leq i_1, j_1 \leq k_1$, $l = 1, \ldots, p$) there are two possibilities. In the first case $e_{l(i_1j_1)}^{(\xi_l)}$ is symmetric or skew-symmetric with respect to involution. Then it coincides with the corresponding element $d_{i_1j_1}^{(\delta, \theta)}$, where $\theta = \deg_G e_{l(i_1j_1)}^{(\xi_l)} = \theta_{i_1}^{-1} \xi_l \theta_{j_1}$. In the other case $e_{l(i_1j_1)}^{(\xi_l)}$ forms a pair with the uniquely defined element $e_{l(i_1j_1)}^{(\xi_l)} = \pm(e_{l(i_1j_1)}^{(\xi_l)})^*$. Any such pair bijectively corresponds to the pair $\{d_{i_1j_1}^{(+, \theta)}, d_{i_1j_1}^{(-, \theta)}\}$ of elements of $D$. Where $e_{l(i_1j_1)}^{(\xi_l)}$, $e_{l(i_1j_1)}^{(\xi_l)}$ are the sum, and the difference of $d_{i_1j_1}^{(+, \theta)}$, $d_{i_1j_1}^{(-, \theta)}$. And $d_{i_1j_1}^{(+, \theta)}$, $d_{i_1j_1}^{(-, \theta)}$ are the linear combinations of $e_{l(i_1j_1)}^{(\xi_l)}$, $e_{l(i_1j_1)}^{(\xi_l)}$ with coefficients $1/2, -1/2$. Here $\theta = \deg_G d_{i_1j_1}^{(\delta, \theta)} = \deg_G e_{l(i_1j_1)}^{(\xi_l)} = \theta_{i_1}^{-1} \xi_l \theta_{j_1}$. An element $e_{l(i_1j_1)}^{(\xi_l)}$ coincides with the corresponding $d_{i_1j_1}^{(-, \theta)}$, $\theta = \deg_G d_{i_1j_1}^{(\delta, \theta)} = \deg_G e_{l(i_1j_1)}^{(\xi_l)} = \theta_{i_1}^{-1} \xi_l \theta_{j_1} \quad (1 \leq i_1, j_1 \leq k_1, l = 1, \ldots, p)$.

The element $e_l = (1/\lambda_l) \sum_{i_1=1}^{k_l} e_{l(i_1j_1)}^{(\xi_l)}$ is the minimal orthogonal central idempotent of $B' = C_1 \times \cdots \times C_p$, corresponding to the unit element of the $l$-th $\bar{G}$-simple component $C_l$ of the algebra $A'$, $\lambda_l = \xi_l(e, e) \in Q[\sqrt{\epsilon}]$ (for any $l = 1, \ldots, p$).

In the definition (7) of the set $U = \bigcup_{(\delta, \theta) \in \bar{G}} U_{(\delta, \theta)}$ the element $r$ runs on a $G$-homogeneous set of elements of the Jacobson radical $J = \oplus_{r', r''=1}^{p+1}(\oplus_{\theta \in G} \varepsilon_r \varepsilon_{r'} \varepsilon_{r'} \varepsilon_{r'})$. $\varepsilon_{p+1} = 1 - (\varepsilon_1 + \cdots + \varepsilon_p)$ is the adjoint idempotent. Particularly, $\varepsilon_{p+1} = 0$ if $A$ is a unitary algebra. All idempotents $e_l$ are $\bar{G}$-homogeneous of degree $(+, \epsilon)$ ($l = 1, \ldots, p + 1$).

If we consider identities then the statement of Lemma 3.2 can be extended in some sense for the case of any field $F$ of characteristic zero containing a primitive root $\sqrt{\epsilon}$. 11
Definition 3.3 An $F$-algebra $A$ is called representable if $A$ can be embedded into some algebra $C$ that is finite dimensional over an extension $\bar{F} \supseteq F$ of the base field $F$.

Lemma 3.4 Let $F$ be a field of characteristic zero containing $\sqrt{\theta}$. Suppose that Assumption [2.1] is true for any algebraically closed extension $F \supseteq F$. Then any representable $G$-graded $F$-algebra with involution $A$ is $\vartheta$-equivalent to some $F$-finite dimensional $G$-graded algebra with involution $A'$ that satisfies all the claims of Lemma 3.2.

Proof. We always can assume that the extension $\bar{F} \supseteq F$ is algebraically closed. Suppose that $A$ is isomorphic to an $F$-subalgebra $B$ of a finite dimensional $\bar{F}$-algebra $\bar{B}$. It is clear that $B$ can be considered $G$-graded with involution induced from $A$. Consider a subalgebra $U = \{(b, b^*) | b \in B\}$ of the $F$-algebra $B \times B^\vartheta$. $U$ is $G$-graded with the grading $U_{\theta} = \{(b, b^*_\theta) | b_\theta \in B_\theta\}$, $\theta \in G$, and has the exchange involution $(b, b^*)^\vartheta = (b^\vartheta, b^\vartheta)$, $\theta \in B$. Then we consider $\bar{U} = \sum_{\theta \in G} \bar{U}_{\theta}$, where $\bar{U}_{\theta} = F \bar{U}_{\theta} \otimes_{\bar{F}} F\theta$ ($\theta \in G$). $\bar{U}$ is an $\bar{F}$-subalgebra of the algebra $\bar{B} \times B^\vartheta \otimes_{\bar{F}} F[G]$. Hence $\bar{U}$ is a finite dimensional $\bar{F}$-algebra. As an $F$-algebra $\bar{U}$ is $G$-graded with the graded involution $\vartheta$ defined by $(\alpha(b, b^*) \otimes \theta)^\vartheta = \alpha(b^\vartheta, b^\vartheta) \otimes \theta$, $\alpha \in \bar{F}$, $b \in B$, $\theta \in G$. If we consider all the algebras and graded $\vartheta$-identities over the base field $F$ then we have $\text{Id}^\vartheta(\bar{U}) = \text{Id}^\vartheta(B) = \text{Id}^\vartheta(A)$.

By Lemma 3.2 the graded $\bar{F}$-algebra with involution $\bar{U}$ has the decomposition (4), where the $\vartheta$-graded simple $\bar{F}$-algebras $\bar{C}_l$ are of the type $\bar{C}^{(1)}$ or $\bar{C}^{(2)}$. We can see from Assumption 2.1 that $\bar{C}_l = FC_l$ ($l = 1, \ldots, p$), where $C_l$ is the $\vartheta$-graded simple $F$-algebra generated as a vector space over the base field $F$ by the same canonical $G$-homogeneous basis as $C_l$ over $\bar{F}$. Let us take $B = C_1 \times \cdots \times C_p$, $\mathcal{R}$ is an $\bar{F}$-basis of $J(\bar{U})$, $\Gamma = \text{Id}^\vartheta(A)$, $q = \text{dim}_{\bar{F}}J(\bar{U}) = |\mathcal{R}|$, $s = \text{nd}(\bar{U})$. Then the $F$-algebra $A' = \mathcal{R}_{q,s}(B, \Gamma)$ defined by (1) is a graded algebra with involution which is finite dimensional over $F$. Note that $\bar{U}_\theta$ is $\bar{F}$-subspace of $\bar{U}$, and the involution $\vartheta$ of $\bar{U}$ is $\bar{F}$-linear. Hence, it is enough to verify multilinear graded $F$-identities with involution of $\bar{U}$ only on $G$-homogeneous elements $b \in B$, $r \in \mathcal{R}$. It follows from the graded version of Lemma 3.1 [2] that $A'$ satisfies all claims of Lemma 3.2 and $\text{Id}^\vartheta(A') = \text{Id}^\vartheta(\bar{U}) = \text{Id}^\vartheta(A)$.

Definition 3.5 We say that an $F$-finite dimensional $G$-graded $\vartheta$-algebra $A'$ has an elementary decomposition if it satisfies all the claims of Lemma 3.2.

It is clear that the direct product of algebras with elementary decomposition is the algebra with elementary decomposition. Also if $F$ is algebraically closed and admits for the group $G$ the classification of finite dimensional $\vartheta$-graded simple algebras given in Assumption 2.1 then any finite dimensional $G$-graded $F$-algebra with involution has an elementary decomposition.
Corollary 3.6 Let \( \bar{F} \supseteq F \) be an algebraically closed extension such that Assumption 2.7 is true over \( \bar{F} \). Suppose that \( F \) contains \( \sqrt[n]{1} \). Then any finite dimensional \( G \)-graded \( F \)-algebra with involution is \( \bar{g}_I \)-equivalent to a finite dimensional \( G \)-graded \( F \)-algebra with involution with elementary decomposition.

Proof. A finite dimensional graded \( F \)-algebra with involution \( A \) can be naturally embedded to the graded \( * \)-algebra \( \hat{A} = A \otimes_F \bar{F} \) preserving graded \( * \)-identities. We assume here \( (a \otimes \alpha)^* = a^* \otimes \alpha \), and \( \deg_G a \otimes \alpha = \deg_G a \), for all \( a \in A, \alpha \in \bar{F} \). The algebra \( \hat{A} \) is finite dimensional over \( \bar{F} \). By Lemma 3.4 there exists a finite dimensional \( G \)-graded \( * \)-algebra \( A' \) with elementary decomposition such that \( \Id^{g_I}(A') = \Id^{g_I}(A) = \Id^{g_I}(A) \). Where all identities are considered over the field \( F \). \( \square \)

Particularly, if Assumption 2.7 is true for the algebraic closure of \( F \) then for graded \( * \)-identities of finite dimensional algebras we can consider only algebras with elementary decomposition.

We assume further that the base field \( F \) contains a primitive root \( \sqrt[n]{1} \) of one of degree \( m = |G| \), and Assumption 2.7 is true for the group \( G \) over any algebraically closed extension \( \bar{F} \supseteq F \). It means that a basis of a finite dimensional graded \( F \)-algebra with involution \( A \) always can be chosen in the set \( D \cup U \). Where \( D \) is \( \bar{G} \)-homogeneous basis of the semisimple part \( B = C_1 \times \cdots \times C_p \) of \( A \). Particularly, we have \( \bigcup_{\delta, \theta} \mathcal{I}_{g_I, (\delta, \theta)} = \dim_F B_{\delta}^G \).

Therefore for a multilinear graded \( * \)-polynomial it is enough to consider only evaluations by elements of \( D \cup U \) of variables of corresponding \( \bar{G} \)-degree. Such evaluations are called elementary. Elements of the set \( D \) are called semisimple, and elements of \( U \) are radical.

Similarly to the case of group graded identities [41], and \( * \)-identities [42] we define the numeric parameters of a finite dimensional \( G \)-graded \( * \)-algebra, and of the ideal of graded \( * \)-identities of a finitely generated \( G \)-graded PI-algebra with involution. Assume that \( G = \{ \epsilon = \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m \} \), \( m = |G| \), \( \epsilon \) is the unit of \( G \). \( \bar{G} = \text{Irr} G \times <*> \subseteq \text{Aut}^*(A) \).

Definition 3.7 Let \( A = B \oplus J \) be a finite dimensional \( G \)-graded \( F \)-algebra with graded involution. Then \( B = \sum_{(\delta, \theta) \in \bar{G}} B_{\delta}^\theta \) is a maximal semisimple graded \( * \)-invariant subalgebra of \( A \), and \( J(A) = J \) the Jacobson radical of \( A \). We denote by \( \text{dim}_{g_I} A = (\dim B_{\hat{\theta}_1}^+, \dim B_{\hat{\theta}_1}^-, \ldots, \dim B_{\hat{\theta}_m}^+, \dim B_{\hat{\theta}_m}^-) \) the collection of dimensions of all \( \bar{G} \)-homogeneous parts of the semisimple subalgebra \( B \).

Recall also that we denote by \( \text{nd}(A) \) the degree of nilpotency of the radical \( J \). Then the parameter of \( A \) is \( \text{par}_{g_I}(A) = (\text{dim}_{g_I} A; \text{nd}(A)) \).

\( \text{cpar}_{g_I}(A) = (\text{par}_{g_I}(A); \dim J(A)) \) is called the complex parameter of \( A \).

Note that for any nonzero two-sided graded \( * \)-ideal \( I \leq A \) of \( A \) it holds \( \text{cpar}_{g_I}(A/I) < \text{cpar}_{g_I}(A) \).

Let \( f = f(s_1, \ldots, s_k, x_1, \ldots, x_n) \in F[Y, Z] \) be a polynomial linear on a set \( S = \{s_1, \ldots, s_k\} \) of homogeneous variables \( (S \subseteq Y^\theta \text{ or } S \subseteq Z^\theta, \theta \in G) \). Then \( f \)
is alternating on $S$, if $f(s_{\sigma(1)}, \ldots, s_{\sigma(k)}, x_1, \ldots, x_n) = (-1)^{\sigma} f(s_1, \ldots, s_k, x_1, \ldots, x_n)$ holds for any permutation $\sigma \in S_k$.

It is clear that a polynomial $f(s_1, \ldots, s_k, x_1, \ldots, x_n)$ is alternating on the set $S = \{s_1, \ldots, s_k\}$ if and only if

$$f(s_1, \ldots, s_k, x_1, \ldots, x_n) = A_S(g) = \sum_{\sigma \in S_k} (-1)^{\sigma} g(s_{\sigma(1)}, \ldots, s_{\sigma(k)}, x_1, \ldots, x_n)$$

for some graded *-polynomial $g(s_1, \ldots, s_k, x_1, \ldots, x_n)$ which is linear on the set $S$. The mapping $A_S$ is a graded linear transformation and is called alternator. The properties of alternating graded *-polynomials are similar to the case of ordinary polynomials (see, e.g. [25], [31], [29]). Note also that an alternator commutes with involution.

Given a 2m-tuple $\mathbf{t} = (t_1, \ldots, t_{2m}) \in \mathbb{N}_0^{2m}$ we say that a graded *-polynomial $f \in F(Y, Z)$ has the collection of $\mathbf{t}$-alternating homogeneous variables ($f$ is $\mathbf{t}$-alternating) if $f(Y_1, Z_1, \ldots, Y_m, Z_m, X)$ is linear on $\bigcup_{j=1}^{m} (Y_j \cup Z_j)$, and $f$ is alternating on each of the sets $Y_j \subseteq Y^{\hat{\theta}_j}$, $Z_j \subseteq Z^{\hat{\theta}_j}$, where $|Y_j| = t_{2j-1}$, $|Z_j| = t_{2j}$, $j = 1, \ldots, m$.

Recall that we order 2m-tuples lexicographically. The definitions of the type of a multihomogeneous on degrees polynomial $f \in F(Y, Z)$, and the Kemer index of giT-ideal of a finitely generated graded PI-algebra with graded involution repeat the corresponding definitions for the case of graded polynomials [41]. We will consider them for the completeness of the text.

**Definition 3.8** Given a 2m-tuple $\mathbf{t} = (t_1, \ldots, t_{2m}) \in \mathbb{N}_0^{2m}$ consider some (possibly different) collections $\tau_1, \ldots, \tau_s \in \mathbb{N}^{2m}_0$ satisfying the conditions $\tau_j > \mathbf{t}$ for any $j = 1, \ldots, s$. Let $f \in F(Y, Z)$ be a multihomogeneous graded *-polynomial. Suppose that $f = f(S_1, \ldots, S_{s+\mu}, X)$ is simultaneously $\tau_j$-alternating on $S_j = \bigcup_{G \in G} (Y_j^G \cup Z_j^G)$ for any $j = 1, \ldots, s$, and $\mathbf{t}$-alternating on any $S_j = \bigcup_{G \in G} (Y_j^G \cup Z_j^G)$, $j = s + 1, \ldots, s + \mu$. All the collections $S_j$ are pairwise disjoint. Then we say that $f$ has the type $(\mathbf{t}; s; \mu)$.

Here $|Y_j^G| = \tau_{2i-1}^{\mathbf{t}}$, $|Z_j^G| = \tau_{2i}^{\mathbf{t}}$ for any $j = 1, \ldots, s$ or $|Y_j^G| = t_{2i-1}^{\mathbf{t}}$, $|Z_j^G| = t_{2i}^{\mathbf{t}}$ for all $j = s + 1, \ldots, s + \mu$ ($i = 1, \ldots, m$).

Note that multihomogeneous polynomials $f$ and $f^*$ always have the same type.

**Definition 3.9** Given a giT-ideal $\Gamma \subseteq F(Y, Z)$ the parameter $\beta(\Gamma) = \mathbf{t}$ is the greatest lexicographic 2m-tuple $\mathbf{t} \in \mathbb{N}_0^{2m}$ such that for any $\mu \in \mathbb{N}$ there exists a graded *-polynomial $f \notin \Gamma$ of the type $(\mathbf{t}; 0; \mu)$.

**Definition 3.10** Given a nonnegative integer $\mu$ let $\gamma(\Gamma; \mu) = s \in \mathbb{N}$ be the smallest natural $s$ such that any graded *-polynomial of the type $(\beta(\Gamma); s; \mu)$ belongs to $\Gamma$.

$\gamma(\Gamma; \mu)$ is a positive non-increasing function on $\mu$. Let us denote the limit of this function $\lim_{\mu \to \infty} \gamma(\Gamma; \mu) 

**Definition 3.11** The $(2m + 1)$-tuple $\text{ind}_{gi}(\Gamma) = (\beta(\Gamma); \gamma(\Gamma))$ is called by Kemer index of a giT-ideal $\Gamma$. 

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A finitely generated PI-algebra satisfies a non-graded Capelli identity \[36\]. Similarly to the case of GT-ideals [11] and *T-ideals [42] the Kemer index is well defined for the giT-ideal of graded *-identities of a finitely generated G-graded PI-algebra with involution. We denote \( \text{ind}_{\text{gi}}(A) = \text{ind}_{\text{gi}}(\text{Id}_{\text{gi}}(A)) \) for a finitely generated graded PI-algebra \( A \) with involution. We denote \( \text{ind}_{\text{gi}}(A) = \text{par}_{\text{gi}}(A) = (0, \ldots, 0; s) \).

We obtain automatically the notion of \( \mu \)-boundary polynomials for a giT-ideal.

**Definition 3.12 (Definition 7, [42])**
Given a nonnegative integer \( \mu \) a nontrivial multihomogeneous polynomial \( f \in F(Y, Z) \) is called \( \mu \)-boundary polynomial for a giT-ideal \( \Gamma \) if \( f \notin \Gamma \), and \( f \) has the type \( (\beta(\Gamma); \gamma(\Gamma) - 1; \mu) \).

Denote by \( S_{\mu}(\Gamma) \) the set of all \( \mu \)-boundary polynomials for \( \Gamma \). Then \( S_{\mu}(\Gamma) = S_{\mu}(\text{Id}_{\text{gi}}(\Gamma)), \ K_{\mu}(\Gamma) = \text{giT}[S_{\mu}(\Gamma)], \ K_{\mu, A} = \text{giT}[S_{\mu}(A)] \).

The set \( S_{\mu}(\Gamma) \) of all \( \mu \)-boundary polynomials of a giT-ideal \( \Gamma \), and the Kemer index satisfy the same basic properties as in the case of GT-ideals and *T-ideals (see Lemmas 4-10 [11], and Lemmas 6-8 [42]). Observe that these properties do not depend on the type of identities. They are completely determined by Definitions 3.9-3.12 (see the arguments in [11]).

We can consider also the graded version of a *PI-reduced algebra. We call it gi-reduced algebra.

**Definition 3.13**
A finite dimensional G-graded *-algebra \( A \) with elementary decomposition is gi-reduced if there do not exist finite dimensional G-graded *-algebras with elementary decomposition \( A_1, \ldots, A_\varnothing (\varnothing \in \mathbb{N}) \) such that \( \bigcap_{i=1}^{\varnothing} \text{Id}_{\text{gi}}(A_i) = \text{Id}_{\text{gi}}(A) \), and \( \text{cpar}_{\text{gi}}(A_i) < \text{cpar}_{\text{gi}}(A) \) for all \( i = 1, \ldots, \varnothing \).

The next graded modification of Lemma 9 [42] holds.

**Lemma 3.14**
Given a gi-reduced algebra \( A \) with the Wedderburn-Malcev decomposition \( A = (C_1 \times \cdots \times C_p) \oplus J \), we have \( C_{\sigma(1)}JC_{\sigma(2)}J \cdots JC_{\sigma(p)} \neq 0 \) for some \( \sigma \in \text{Sym}_p \).

**Proof.** Suppose that \( C_{\sigma(1)}JC_{\sigma(2)}J \cdots JC_{\sigma(p)} = 0 \) for any \( \sigma \in \text{Sym}_p \). Consider G-graded *-algebras with elementary decomposition \( A_i = ( \bigcap_{1 \leq j \leq p, j \neq i} C_j ) \oplus J(A) \) (\( i = 1, \ldots, p \)). We have for them \( \text{Id}_{\text{gi}}(A) = \bigcap_{i=1}^{p} \text{Id}_{\text{gi}}(A_i) \), and \( \text{dims}_{\text{gi}}(A_i) < \text{dims}_{\text{gi}}(A) \) for any \( i = 1, \ldots, p \). This contradicts to the definition of gi-reduced algebra. \( \square \)

Particularly, we have \( \text{nd}(A) \geq p \) for a gi-reduced algebra \( A \). Then Corollary 3.6 along with the properties of \( \mu \)-boundary polynomials implies also the graded versions of Lemmas 10, 11 [42] (see the proofs in [11]).

The Kemer index and parameters of gi-reduced algebras are related in a similar way as in case of non-graded involution. It is the crucial point of our proof.
Lemma 3.15 Given a $g_l$-reduced algebra $A$ we have $\beta(A) = \text{dims}_{g_l} A$. Any $*$-graded simple finite dimensional algebra $C$ with elementary decomposition is $g_l$-reduced, and $\text{ind}_{g_l}(C) = \text{par}_{g_l}(C) = (t_1, \ldots, t_{2m};1)$.

Proof. If $A$ is nilpotent then the assertion of Lemma is trivial. Suppose that $A$ is a non-nilpotent $g_l$-reduced algebra. By the graded version of Lemma 6 \cite{12} we have $\beta(A) \leq \text{dims}_{g_l} A$. Thus it is enough to find a graded $*$-polynomial of the type $(\text{dims}_{g_l} A; 0; \hat{s})$ which is not a graded identity with involution of $A$ for any $\hat{s} \in \mathbb{N}$.

Consider the elementary decomposition \cite{13} of $A$. Similarly to Lemma 12 \cite{12} for any $*$-graded simple component $C_l$ ($l = 1, \ldots, p$) we take $\hat{s}$ sets of distinct $\hat{G}$-homogeneous variables $Y^{l(\delta, \theta)}_{l,m} = \{y_{l,(i,j),m}^{(\delta, \theta)} | (i,j) \in I_{l(\delta, \theta)}\}$ corresponding to the canonical basic elements $q^{l(\delta, \theta)}_{i,j,m} \in D^{l(\delta, \theta)}$ \cite{13}. Here $Y_{l,m}^{(+, \theta)} \subseteq Y^{\theta}$, and $Y_{l,m}^{(-, \theta)} \subseteq Z^{\theta}$.

Suppose that $C_l = \hat{G}^{(q)}$ with $q = 1, 2$. Then consider the polynomial $w_{l,m}(Y_{l,m}, X_l)$ which is the product of all variables of the set $Y_{l,m} = \bigcup_{(\delta, \theta) \in \hat{G}} Y^{l(\delta, \theta)}_{l,m}$ connected by $x_{l,(ij)}$ if $q = 1$ or by $y_{l,(ij,j')}$ if $q = 2$. Here we take $x_{l,(ij)} = \frac{1}{2}(\pi_3 \tilde{y}_{l,(ij)} + \pi_4 \tilde{z}_{l,(ij)})$, and $y_{l,(ij,j')} = \frac{1}{2}(\pi_3 \tilde{y}_{l,(ij,j')})$. Where $\pi_3 \in \{0, 1\}$, and $\tilde{y}_{l,(ij)} \in Y_{l,m}^{(1, \theta)}$. $\tilde{z}_{l,(ij)} \in Z$ with $\deg_{\hat{G}} y_{l,(ij)} = \deg_{\hat{G}} \tilde{z}_{l,(ij)} = \deg_{\hat{G}} e_{l(\delta, \theta)}^{(l(l,j))}$. We also say here that $x_{l,(ij,j')}$, and $y_{l,(ij,j'),m}$ connect the variable $y_{l,(i_1,j_1),m}$ with $y_{l,(i_2,j_2),m}$, and $x_{l,(i_1,j_1),m}$ $x_{l,(i_2,j_2),m}$. Consider the corresponding product of these variables. Then we denote $X_l = \{\tilde{y}_{l,(ij)}, \tilde{z}_{l,(ij)} | 1 \leq i, j \leq k_l\}$.

By Lemma 3.14 we can assume without lost of generality that $A$ contains an element

$$a = e_{l_1(1,s_1,s_1)}^{(l_1(1,s_1,s_1))} e_{l_2(2,s_2,s_2)}^{(l_2(2,s_2,s_2))} \cdots e_{l_{p-1}(p-1,s_{p-1},s_{p-1})}^{(l_{p-1}(p-1,s_{p-1},s_{p-1}))} e_{l_p(1,s_p,s_p)}^{(l_p(1,s_p,s_p))} \neq 0,$$

where $r_l \in U$ are some $\hat{G}$-homogeneous basic radical elements chosen in the set $U$ (Lemma 3.2).

Then in the case $q = 1$ let us consider the graded $*$-polynomial $W_{l,s_i} = x_{l,(s_t,t_i)} \cdot (\prod_{m=1}^{l} w_{l,(t_l,t_1) \cdot w_{l,m})} \cdot x_{l,(t_s,t_i)}^{l(1)}$. And for $q = 2$ let us take $W_{l,s_i} = x_{l,(s_t,t_i)} \cdot (\prod_{m=1}^{l} w_{l,(t_l,t_1) \cdot w_{l,m})} \cdot x_{l,(t_s,t_i)}^{l(1)}$. Where $s_i$ are given by \cite{13}, and $t_l$, $t_i$ are chosen to connect the word $w_{l,m}$ with $w_{l,m+1}$. The variables $x_{l,(i,j)}^{l(1)}$ and $y_{l,(i,j)}^{l(1)}$ are defined as linear combinations of $\hat{G}$-homogeneous variables of the set $X_l = \{\tilde{y}_{l,(ij)}, \tilde{z}_{l,(ij)} | 1 \leq i, j \leq k_l\}$ similarly to $x_{l,(ij)}$, and $x_{l,(i,j,j')}$. Denote $Y^{l(\delta, \theta)}_{m} = \bigcup_{l=1}^{p} Y_{l,m}^{l(\delta, \theta)}$ \cite{13}. The polynomial

$$f = (\prod_{m=1}^{\hat{s}} \prod_{(\delta, \theta) \in \hat{G}} A_{l(\delta, \theta)}^{l(\delta, \theta)} W_{l,s_1} \hat{x}_1 W_{l,s_2} \hat{x}_2 \cdots \hat{x}_p W_{l,s_p}$$

is $(\text{dims}_{g_l} A)$-alternating on any set $Y^{l}_{m}$ for all $m = 1, \ldots, \hat{s}$. Here $\hat{x}_q \in Y^{\theta}$ if $r_q \in U_{(+, \theta)}$, and $\hat{x}_q \in Z^{\theta}$ if $r_q \in U_{(-, \theta)}$ in \cite{13}, $q = 1, \ldots, p-1$. 16
Then the evaluation
\[ y_{l,(i,j);m}^{(\delta,\theta)} = a_{l(i,j);m}^{(\delta,\theta)}, \quad (i_l,j_l) \in \mathcal{I}_{l,(\delta,\theta)}; \]
\[ \hat{x}_q = r_q; \]
\[ l = 1, \ldots, p; \quad q = 1, \ldots, p - 1; \quad m = 1, \ldots, \hat{s}; \quad (\delta, \theta) \in \hat{G}; \quad (10) \]
of the polynomial \( f \) is equal to \( \alpha a \neq 0 \), where the element \( \alpha \in A \) is defined in \( \mathfrak{S} \), and the non-zero coefficient \( \alpha \in F \) is the product of the corresponding values of the 2-cocycles \( \zeta \) divided by \( 2^q \cdot c_t \in \mathbb{N} \) (see the proof of Lemma 12 [42], and Lemmas 11, 15 [41]). Here the elementary substitution of variables of the sets \( X_l, X'_l \), and the coefficients \( \pi_s \in \{0, 1, -1\} \) for any collection \( (l, (i, j)) \) are chosen to guarantee \( x_{l,(i,j);l} = e_{l,(i,j)}^{(\epsilon)} \), \( x_{l,(i,j);l} = e_{l,(i,j);l}^{(\epsilon)} \) (see Lemma 3.2 and 5); and
\[ x_{l,(i,j);l}^{(\delta)} = e_{l,(i,j)}^{(\theta)}, \quad x_{l,(i,j);l}^{(\delta)} = \eta_{l}(E_{l(i,j)}; (-1)^{k_l}H_l|E_{l(j)};p) \]
for the suitable element
\[ \theta_l = H_l (1 \leq i_l, j_l, l \leq l_1, 1 \leq l \leq p). \]
Therefore we have \( f \notin \text{Rad}^{(l)}(A) \). Hence at least one multihomogeneous component \( \tilde{f} \) of \( f \) is not a graded \( \ast \)-identity of \( A \) also, and it is \((\dim_{\mathfrak{g}} A)\)-alternating on any set \( Y_{(m)}, m = 1, \ldots, \hat{s} \). Thus \( \tilde{f} \) is the required polynomial.

Notice that the condition of \( gi \)-reducibility of \( A \) is necessary only to find in \( A \) a non-zero element \( \mathfrak{S} \). If \( A \) is a \( \ast \)-graded simple algebra with elementary decomposition \( (p = 1) \) then instead of \( a \) in \( \mathfrak{S} \) we take \( e_{l,(11)}^{(\epsilon)} \), and will also obtain \( \beta(A) = \dim_{\mathfrak{g}} A \). Since \( \text{ind}_{g_l}(A) \leq \text{par}_{g_l}(A) = (\beta(A); 1) \), and \( \gamma(A) > 0 \) then \( \text{ind}_{g_l}(A) = \text{par}_{g_l}(A) \). By the graded version of Lemma 6 [42] the conditions \( \dim J(A) = 0 \), and \( \text{ind}_{g_l}(A) = \text{par}_{g_l}(A) \) imply that \( A \) is \( gi \)-reduced. \( \square \)

Assume that a finite dimensional graded algebra with involution \( R \) has an elementary decomposition. Similarly to [41], [42] we define special types of evaluations of variables of a graded \( \ast \)-polynomial in \( A \): elementary complete and elementary thin evaluations. We also define the notion of an exact polynomial related with these evaluations.

**Definition 3.16** An elementary evaluation \( (a_1, \ldots, a_n) \) of \( G \)-homogeneous elements of \( A \) (namely, \( a_i \in D \cup U \subseteq A \) [29], [7]) is called incomplete if there exists \( j = 1, \ldots, p \) such that
\[ \{a_1, \ldots, a_n\} \cap (C_j \oplus \oplus_{l=1}^{p+1} (e_j J e_l + e_l J e_j)) = \emptyset. \]
Otherwise the evaluation \( (a_1, \ldots, a_n) \) is called complete.

**Definition 3.17** An elementary evaluation \( (a_1, \ldots, a_n) \in A^n \) is called thin if it contains strictly less than \( \text{nd}(A) - 1 \) radical elements (not necessarily distinct).

**Definition 3.18** We say that a multilinear graded \( \ast \)-polynomial \( f(x_1, \ldots, x_n) \in F(Y, Z) \) is exact for a finite dimensional graded \( \ast \)-algebra \( A \) with elementary decomposition if \( f(a_1, \ldots, a_n) = 0 \) holds in \( A \) for any thin or incomplete evaluation \( (a_1, \ldots, a_n) \in A^n \).
By Lemma 3.2 we have that \( \dim_F(C_l)^+ > 0 \) for any \( l = 1, \ldots, p \). Hence the arguments similar to original ones prove the graded version of Lemma 13 [42]. The next two statements are also true for graded \(*\)-identities.

**Lemma 3.19** Any nonzero \( \tilde{g} \)-reduced algebra \( A \) has an exact polynomial, that is not a graded \(*\)-identity of \( A \).

**Proof.** If \( A \) is nilpotent then the assertion follows from the graded version of Lemma 13 [42]. Suppose that \( A \) is a non-nilpotent graded \(*\)-algebra satisfying the claims of Lemma 3.2. Consider its \(*\)-invariant graded subalgebras \( A_i = ( \prod_i C_j ) \oplus J(A) \)

for all \( i = 1, \ldots, p \). Take \( q = \dim_F J(A), \ s = \text{nd}(A) - 1 \). By the graded version of Lemma 3 [42] the graded \(*\)-algebra \( R_{q,s}(A) \) has an elementary decomposition, and \( \text{Id}^q(A) \subseteq \text{Id}^q(\tilde{A}) \), where \( \tilde{A} = A_1 \times \cdots \times A_p \times R_{q,s}(A) \). Consider any map of the set \( Y(q) \cup Z(q) \) onto a \( \tilde{G} \)-homogeneous basis of \( J(A) \) of the form (7) which preserves \( \tilde{G} \)-degrees of variables. Such map can be extended to a surjective graded \(*\)-homomorphism \( \varphi : B(Y(q), Z(q)) \rightarrow A \), assuming \( \varphi(b) = b \) for any \( b \in B \).

Therefore similarly to Lemma 14 [42] any multilinear graded \(*\)-identity of the algebra \( \tilde{A} \) is exact for \( A \). It is clear that all the algebras \( A_i \), and the algebra \( R_{q,s}(A) \) have the complex parameter less than \( A \). Since \( A \) is \( \tilde{g} \)-reduced then we have \( \text{Id}^q(A) \subseteq \text{Id}^q(\tilde{A}) \). Any multilinear polynomial \( f \) such that \( f \in \text{Id}^q(\tilde{A}) \), and \( f \notin \text{Id}^q(A) \) satisfies the assertion of the lemma. \( \square \)

**Lemma 3.20** Let \( A \) be a finite dimensional graded \(*\)-algebra with an elementary decomposition, \( h \) an exact polynomial for \( A \), and \( \bar{a} \in A^n \) a complete evaluation of \( h \) containing exactly \( \bar{s} = \text{nd}(A) - 1 \) radical elements. Then for any \( \mu \in N_0 \) there exist a graded \(*\)-polynomial \( h_\mu \in \tilde{g}T[h] \), and an elementary evaluation \( \bar{u} \) of \( h_\mu \) in \( A \) such that:

1. \( h_\mu(Z_1, \ldots, Z_{\tilde{s}+\mu}, X) \) is \( \tau_j \)-alternating on any set \( Z_j \) with \( \tau_j > \beta = \text{dim}_g(A) \) for all \( j = 1, \ldots, \tilde{s} \), and is \( \beta \)-alternating on any \( Z_j \) for \( j = \tilde{s} + 1, \ldots, \tilde{s} + \mu \) (all the sets \( Z_j, X \subseteq (Y \cup Z) \) are disjoint),

2. \( h_\mu(\bar{u}) = h(\bar{a}) \),

3. all the variables from \( X \) are replaced by semisimple elements.

**Proof.** Consider the decomposition (3) of \( A \). Take any \( l = 1, \ldots, p \). Let \( W_{l,s_l}(\bar{Y}_l, \bar{X}_l) \) be defined as in Lemma 3.15 for \( \bar{s} = \tilde{s} + \mu \). Here \( \bar{Y}_l = \bigcup_{m=1}^{\tilde{s}+\mu} Y_{l,m}, \ \bar{X}_l = X_l \cup X'_l \). Suppose that the evaluation \( \alpha_{l,s_l}^{e_l(s_l,s_l)} \) of the polynomial \( W_{l,s_l} \) is equal to \( \alpha_{l,s_l}^{e_l(s_l,s_l)} \) (Lemma 3.15 see also Lemma 12 [42], and Lemmas 11, 15 [11]), where \( \alpha_{l,s_l} \in F \) is the non-zero coefficient. Consider the polynomial \( \tilde{f}_l(\bar{Y}_l, \bar{X}_l) = (1/\lambda_l) \sum_{s_l=1}^{k_l} 1/\alpha_{l,s_l} W_{l,s_l} \), where \( \lambda_l = \zeta_l(\epsilon, \epsilon) \) (see Lemma 3.2).

Notice that the polynomial \( \tilde{f}_l \) is not necessary \( G \)-homogeneous due to terms \( x_{l,(ij,ij')} \) that can be non-homogeneous. Denote by \( \tilde{f}_l \) the \( e \)-component of \( \tilde{f}_l \) in \( G \)-grading, and by \( \tilde{f}_l^e = (\tilde{f}_l + \tilde{f}_l^*)/2 \) its symmetric part.
From the proof of Lemma \[3.15\] it is clear that the evaluation \([10]\) of the polynomial \(f'_l = (\prod_{m=1}^{\delta+\mu} Y_{l,m}^{(\delta,\theta)})f_l\) is equal to \(\varepsilon_l\). Since \(\varepsilon_l\) is a \(G\)-homogeneous element of degree \((+\mu, \varepsilon)\) then the result of this evaluation of the polynomial \(f'_l = (\prod_{m=1}^{\delta+\mu} Y_{l,m}^{(\delta,\theta)})f_l\) is the same. Recall that any alternator is graded and commutes with involution.

Assume that \(\zeta_1, \ldots, \zeta_s\) are the variables of \(h\) evaluated by radical elements of \(\tilde{a}\). Let us denote \(Z_m^{(\delta,\theta)} = \bigcup_{l=1}^p Y_{l,m}^{(\delta,\theta)} \cup \{\zeta_m\}\) if \(m = 1, \ldots, s\), and \(\deg_{G} \zeta_m = (\delta, \theta)\) or \(Z_m^{(\delta,\theta)} = \bigcup_{l=1}^p Y_{l,m}^{(\delta,\theta)}\) otherwise. Then \(Z_m = \bigcup_{(\delta,\theta) \in \tilde{G}} Z_m^{(\delta,\theta)}\). Let us denote by \(\hat{f}_1\) the polynomial \(\frac{1}{2}f_1\) in the case \(p = 1\), and \(h(\tilde{a}) \in \varepsilon_1 \varepsilon_1\). The polynomials \(\hat{f}_l = f'_l\) \((l = 1, \ldots, p)\) must be taken in all other cases. We obtain the proof of our lemma in graded case if we replace the polynomials \(f_l\) by \(\hat{f}_l\) in the proof of Lemma \[15\] and apply to the polynomial \(h'\) the product of the alternators acting on \(Z_m^{(\delta,\theta)}\) (for all \((\delta, \theta) \in \tilde{G}, m = 1, \ldots, \tilde{s} + \mu)\). Note that all other elements considered in Lemma \[15\] are \(G\)-homogeneous. Remark also that we obtain the evaluation \(\tilde{u}\) replacing the variables of the polynomials \(\hat{f}_1\) as in Lemma \[3.15\] (see \([10]\)) and the variables \(\zeta_m, \tilde{x}_{nt}\) by the corresponding elements \(a_s\) as in Lemma \[15\] (see \([13]\)).

Similarly to the case of non-graded \(*\)-polynomials \([42]\), and polynomials graded by an abelian group \([41]\), Lemmas \[3.15\], \[8.19\] imply the graded versions of Lemmas 16-19 \([42]\).

### 4 Representable algebras.

Consider a graded version of \(*\)-identities with forms introduced in \([42]\). Let \(F\) be a field, and \(R\) a commutative associative \(F\)-algebra. Suppose that a \(G\)-graded \(F\)-algebra \(A\) with involution has a structure of \(R\)-algebra satisfying \(RA_{\theta} \subseteq A_{\theta}\), \(\forall \theta \in \tilde{G}\), and the involution of \(A\) is \(R\)-linear, i.e. \(\deg_{G} ra = \deg_{G} a, \ ra = ar, \ (ra)^* = ra^*\) for all \(r \in R, \ a \in A\). Particularly, it happens if \(R = F\) or if \(R \subseteq Z(A) \cap A_{\tilde{\varepsilon}}^*\), where \(Z(A)\) is the center of \(A\).

**Definition 4.1 (Definition 13 \([42]\))** Let \(A\) be an \(R\)-algebra with involution. Any \(R\)-multilinear mapping \(\hat{\cdot} : A^n \to R\) is called \(n\)-linear form on \(A\).

**Definition 4.2** Suppose that \(A, B\) are \(F\)-algebras with an \(n\)-linear form \(\hat{\cdot} : A^n \to R\). A homomorphism of \(F\)-algebras \(\varphi : A \to B\) preserves the form \(\hat{\cdot}\) if

\[
\varphi(a_0 \hat{\cdot} (a_1, \ldots, a_n)) = \varphi(a_0) \hat{\cdot} (\varphi(a_1), \ldots, \varphi(a_n)), \quad \forall a_i \in A.
\]

Let us consider the free \(*\)-algebra with forms \(FS(Y, Z) = F(Y, Z) \otimes F S\) defined for the \(G\)-graded free algebra \(F(Y, Z)\), a bilinear form \(f_2\), and a linear form \(f_1\) (see \([42]\)). Here the algebra of graded pure form \(*\)-polynomials \(S\) is the free associative commutative algebra with unit generated by \(f_2(u_1, u_2), f_1(u_3)\) for all nonempty graded \(*\)-monomials \(u_1, u_2, u_3 \in F(Y, Z)\). Then \(FS(Y, Z)\) is a \(G\)-graded algebra with
the grading induced from $F(Y, Z)$ assuming $\deg_G f \otimes s = \deg_G f$, for all $f \in F(Y, Z)$, $s \in S$. The algebra $FS(Y, Z)$ is called free graded $*$-algebra with forms.

The concept of graded $*$-identities with forms is introduced as usual with regard to the grading. Let $A$ be a graded $R$-algebra with involution and forms, $f(x_1, \ldots, x_n) \in FS(Y, Z)$ be a graded $*$-polynomial with forms. $A$ satisfies the graded $*$-identity with forms $f = 0$ if $f(a_1, \ldots, a_n) = 0$ holds in $A$ for any $a_i \in A$ with $\deg_G x_i = \deg_G a_i$. The ideal of graded $*$-identities with forms of an algebra $A$ $S\text{Id}^g(A) = \{ f \in FS(Y, Z) | A$ satisfies $f = 0 \}$ is a graded $S$-ideal of $FS(Y, Z)$ invariant with respect to involution and closed under all graded $*$-endomorphisms $\varphi$ of $FS(Y, Z)$ which preserve the forms. $S\text{Id}^g(A)$ also has the property that $g_1 \cdot f_2(f, g_2), g_1 \cdot f_2(g_2, f), g_1 \cdot f_1(f) \in S\text{Id}^g(A)$ for any $g_1, g_2 \in FS(Y, Z)$, $f \in S\text{Id}^g(A)$. Ideals of $FS(Y, Z)$ with all mentioned properties are called $g|$TS-ideals. Given a $g|$TS-ideal $\Gamma$ we define the relatively free graded $*$-algebra with forms of infinite rank $\overline{FS}(Y, Z) = FS(Y, Z)/\Gamma$, and of a rank $\nu$ $\overline{FS}(Y(\nu), Z(\nu)) = FS(Y(\nu), Z(\nu))/\overline{\Gamma}$ $\cap$ $FS(Y(\nu), Z(\nu))$. The equality of graded $*$-polynomials with forms modulo $\overline{\Gamma}$ is defined similarly to non-graded case [32]. We denote also by $g|$TS$[\nu]$ the $g|$TS-ideal generated by a set $\mathcal{V} \subseteq FS(Y, Z)$.

Assume now that $F$ is a field of characteristic zero, and $\overline{\sqrt{\Gamma}} \in F$. Let us define forms on a finite dimensional $G$-graded $F$-algebra with involution $A = B \oplus J$ with the Jacobson radical $J$, and the semisimple part $B$. Consider for any element $b \in B$, the linear operator $\overline{\mathcal{T}}_b : B \rightarrow B$ on the graded $*$-subalgebra $B$ defined by

$$\overline{\mathcal{T}}_b(c) = b \circ c, \quad c \in B. \quad (11)$$

It is clear that $\overline{\mathcal{T}}_{\alpha_1 b_1 + \alpha_2 b_2} = \alpha_1 \overline{\mathcal{T}}_b + \alpha_2 \overline{\mathcal{T}}_b$ for all $\alpha_i \in F$, $b_i \in B$. If $b \in B^+$ is symmetric element then the subspaces $B^g_\delta$ are stable under $\overline{\mathcal{T}}_b$ for all $(\delta, \theta) \in G$. If $b \in B^-$ is skew-symmetric then $\overline{\mathcal{T}}_b(B^g_\delta) \subseteq B^g_{\delta - \theta}$, and $\overline{\mathcal{T}}_b(B^g_\delta) \subseteq B^g_{\delta + \theta}$, $\theta \in G$. Particularly, the trace of the operator $\overline{\mathcal{T}}_b$ is zero for any $b \in B^-$. Then the bilinear form $f_2 : A^2 \rightarrow F$, and the linear form $f_1 : A \rightarrow F$ are defined on $A$ by the rules

$$f_2(a_1, a_2) = f_2(b_1^{(\xi)}, b_2^{(\xi)}) = \text{Tr}(\overline{\mathcal{T}}_{b_1^{(\xi)}} \cdot \overline{\mathcal{T}}_{b_2^{(\xi)}}),$$

$$f_1(a_1) = f_1(b_1^{(\xi)}) = \text{Tr}(\overline{\mathcal{T}}_{b_1^{(\xi)}}), \quad (12)$$

$$a_i = b_i + r_i \in A, \quad b_i = \sum_{\theta \in G} b_i^{(\theta)} \in B, \quad r_j \in J, \quad b_i^{(\theta)} \in B_\theta, \quad \theta \in G,$$

where $\overline{\mathcal{T}}_2 \cdot \overline{\mathcal{T}}_2$ is the product of linear operators, and $\text{Tr}$ is the usual trace function of linear operator. Suppose that $A = A_1 \times \cdots \times A_p$. Observe that in this case the restrictions on $A_i$ of the forms $f_1$, $f_2$ of $A$ coincide with the forms defined by (12) on $A_i$ directly. It is clear that $f_2$ is a symmetric form satisfying $f_2(r, a) = 0$ for any $r \in J$, $a \in A$, $f_2(a_1, a_2) = 0$ for any $a_1 \in A^-$, $a_2 \in A^+$, and $f_2(a_1, a_2) = 0$ for any $a_1 \in A_\theta$, $\theta \neq \epsilon$, $a_2 \in A$. The linear form $f_1$ also satisfies $f_1(r) = 0$ for any $r \in J$, $f_1(a) = 0$ for any $a \in A^-$, and $f_1(a) = 0$ for any $a \in A_\theta$, $\theta \neq \epsilon$. Particularly, the next lemma holds.
Lemma 4.3 Let $A$ be a finite dimensional graded $F$-algebra with involution and with the forms (12). Given a graded form $\ast$-polynomial $h \in \mathcal{F}(Y, Z)$, and variables $x_1, x_2, x_3 \in Y \cup Z$ with the exception of three cases $x_1, x_2 \in Y^\ast$, or $x_1, x_2 \in Z^\ast$, or $x_3 \in Y^\ast$, $A$ satisfies graded $\ast$-identities with forms
\[ f_1(x_3) \cdot h = 0, \quad f_2(x_1, x_2) \cdot h = 0. \]

Applying the arguments of Lemma 4.4 [12] and considering restrictions of the corresponding operators on $B_\delta^\circ ((\delta, \theta) \in \mathcal{G})$ we obtain the following lemma in graded case. Observe that here it is enough to consider semisimple or radical evaluations of variables (not necessary elementary ones).

Lemma 4.4 Given a finite dimensional $G$-graded $\ast$-algebra $A$ with the forms (12) over a field $F$, and a graded $\ast$-polynomial $f \in \mathcal{F}(Y, Z)$ of type $(\dim_{g_{\ast}} A, \text{nd}(A) - 1, 1)$ suppose that $\{x_1, \ldots, x_t\} \in Y \cup Z$ are variables on which $f$ is $(\dim_{g_{\ast}} A)$-alternating $(t = \dim B)$. Then $A$ satisfies the graded $\ast$-identities with forms
\[ f_2(y_1, y_2) = \sum_{i=1}^{t} f|_{x_i = y_1 y_2}, \quad y_1, y_2 \in Y^\ast, \]
\[ f_2(z_1, z_2) = \sum_{i=1}^{t} f|_{x_i = z_1 z_2}, \quad z_1, z_2 \in Z^\ast, \]
\[ f_1(y) = \sum_{i=1}^{t} f|_{x_i = y_{0x_i}}, \quad y \in Y^\ast. \]

Lemma 4.5 Let $f(\bar{x}_1, \ldots, \bar{x}_k) \in \mathcal{F}(Y, Z)$ be a graded $\ast$-polynomial of a type $(\beta; \gamma - 1; 1)$ (for some $\beta \in \mathbb{N}_0^m$, $\gamma \in \mathbb{N}$), and $s(\zeta_1, \ldots, \zeta_d) \in \mathcal{S}$ a graded pure form $\ast$-polynomial $(\{\zeta_1, \ldots, \zeta_d\} \subseteq Y \cup Z)$. Then there exists a graded $\ast$-polynomial $g_s(\bar{x}_1, \ldots, \bar{x}_k, \zeta_1, \ldots, \zeta_d) \in \mathcal{G}T[f]$ such that any finite dimensional $G$-graded $\ast$-algebra $A$ with forms (12) having parameter $\text{par}_{g_{\ast}}(A) = (\beta; \gamma)$ satisfies the graded $\ast$-identity with forms
\[ s(\zeta_1, \ldots, \zeta_d) \cdot f(\bar{x}_1, \ldots, \bar{x}_k) - g_s(\bar{x}_1, \ldots, \bar{x}_k, \zeta_1, \ldots, \zeta_d) = 0. \]

Proof. Assume that $f$ is $(\dim_{g_{\ast}} A)$-alternating on $\{\bar{x}_1, \ldots, \bar{x}_t\}$, $t = \dim B$. Suppose that $w_i$ are $\mathcal{G}$-homogeneous polynomials of $\mathcal{G}$-degree $\deg_{\mathcal{G}} \bar{x}_i$ ($i = 1, \ldots, t$), and $\tilde{\zeta}_j \in Y^\ast \cup Z^\ast$ are variables satisfying the claims of Lemma 4.4. Applying consequently Lemma 4.4 we obtain that $f_2(\tilde{\zeta}_1, \tilde{\zeta}_2) \cdot f_2(\tilde{\zeta}_2^{n_1-1}, \tilde{\zeta}_2^{n_2}) \cdot f_1(\tilde{\zeta}_2^{n_2+1}) \cdot f(\tilde{\zeta}_j^{n_1} \circ (\tilde{\zeta}_j \circ \bar{x}_i)))$. Therefore $\tilde{w}_i$ are right normed jordan monomials of the same type. Replacing $\tilde{\zeta}_j$ by homogeneous elements of $\mathcal{F}(Y, Z)$ of the corresponding $\mathcal{G}$-degrees, and applying Lemma 4.3 as in the proof of Lemma 22 [12] we obtain that $s(\zeta_1, \ldots, \zeta_d) \cdot f(\bar{x}_1, \ldots, \bar{x}_k) = 0 \mod \text{SId}_g^\ast(A)$. Observe that the graded $\ast$-polynomial $g_s$ does not depend on $A$ and $g_s \in \mathcal{G}T[f]$. \[ \square \]

Assume that $A$ is a $G$-graded finite dimensional $\ast$-algebra with the Jacobson radical $J$, and the semisimple part $B$. Let us denote $t_\theta = \dim B^\circ_{\theta}$, $q_\theta = \dim J^\circ_{\theta}$ for any $\theta = (\delta, \theta) \in \mathcal{G}$, and $t = \sum_{\theta \in \mathcal{G}} t_\theta = \dim B$. Given a positive integer $\nu$ take $\Lambda_\nu = \{\lambda_{\theta j} \mid \theta \in \mathcal{G}; 1 \leq i \leq \nu; 1 \leq j \leq t_\theta + q_\theta\}$, and the free commutative
associative unitary algebra $F[\Lambda_\nu]^\#$ generated by the set $\Lambda_\nu$. Let us consider the extension $P_\nu(A) = F[\Lambda_\nu]^\# \otimes_F A$ of $A$ by $F[\Lambda_\nu]^\#$.

$P_\nu(A)$ is a graded algebra with the involution defined by $(f \otimes a)^* = f \otimes a^*$ ($f \in F[\Lambda_\nu]^\#$, $a \in A$), and the grading $(P_\nu(A))_\theta = F[\Lambda_\nu]^\# \otimes_F A_\theta$, $\theta \in G$. The forms $f_2, f_1$ of $A$ defined by (12) can be naturally extended to the $F[\Lambda_\nu]^\#$-bilinear form $f_2 : P_\nu(A)^2 \to F[\Lambda_\nu]^\#$, and $F[\Lambda_\nu]^\#$-linear form $f_1 : P_\nu(A) \to F[\Lambda_\nu]^\#$ respectively.

We call by a Cayley-Hamilton type graded $*$-polynomial a degree homogeneous $*$-polynomial with forms of the following type

$$x^n + \sum_{i_0 + i_1 + \cdots + i_k = n, \ 0 < i_0 < n, \ 1 \leq k \leq k_1} \alpha_{(i),(j)} f_2(x^{i_1}, x^{j_1}) \cdots f_2(x^{i_k}, x^{j_k}) f_1(x^{i_{k+1}}) \cdots f_1(x^{i_{k_1+1}}),$$

where $\alpha_{(i),(j)} \in F$, $x = y + z$, $y \in Y^\nu$, $z \in Z^\nu$. Note that here $i_l, j_l > 0$ ($l \geq 0$). A Cayley-Hamilton type polynomial is not $\tilde{G}$-homogeneous, but it is $G$-homogeneous of the neutral degree.

**Lemma 4.6** $P_\nu(A)$ satisfies a Cayley-Hamilton type graded $*$-identity $\mathcal{K}^{nd(A)}_{3t+1}(x) = 0$ for some Cayley-Hamilton type graded $*$-polynomial $\mathcal{K}_{3t+1}(x)$ of degree $3t + 1$, $t = \dim B$, $x = y + z$, $y \in Y^\nu$, $z \in Z^\nu$.

**Proof.** By Lemma 23 [42] the algebra $P_\nu(A)$ satisfies the non-graded $*$-identity $\mathcal{K}^{nd(A)}_{3t+1}(x) = 0$, where $\mathcal{K}_{3t+1}(x)$ is a Cayley-Hamilton type non-graded $*$-polynomial of degree $3t + 1$ with the forms defined by (15), (16) in [42]. Particularly, $\mathcal{K}^{nd(A)}_{3t+1}(x) = 0$ holds for any $x \in (P_\nu(A))_\xi$. Observe that for all powers of an element $x \in (P_\nu(A))_\xi$ the definition (15) of the forms $f_1, f_2$ in non-graded case given in [42] coincides with the corresponding definition (12) in the $G$-graded case. \hfill \square

Let $\{\hat{b}_{\delta_1}, \ldots, \hat{b}_{\delta_\nu}\}$ be a basis of the $\tilde{G}$-homogeneous part $B_\delta^\theta$ of a semisimple part $B$ of $A$, and $\{\hat{r}_{\theta_1}, \ldots, \hat{r}_{\theta_\nu}\}$ a basis of the $\hat{G}$-homogeneous part $J_\delta^\theta$ of the Jacobson radical $J = J(A)$ of $A$, $\theta = (\delta, \theta) \in \tilde{G}$. If $A$ has an elementary decomposition then all these bases can be chosen in the set $\bigcup_{\delta \in \hat{G}} (D_\delta \cup U_\delta)$ (14), (7), Lemma 3.2. Let us take the elements

$$\eta_{\delta i} = \sum_{j=1}^{t_\delta} \lambda_{\delta ij} \otimes \hat{b}_{\delta j} + \sum_{j=1}^{q_\delta} \lambda_{\delta ij+t_\delta} \otimes \hat{r}_{\delta j} \in P_\nu(A), \quad \delta \in \tilde{G}, \ 1 \leq i \leq \nu. \quad (13)$$

All elements $\eta_{\delta i}$ are $\tilde{G}$-homogeneous of $\tilde{G}$-degree $\delta$. Denote by $\mathcal{G}_\nu = \{\eta_{\delta i} | \delta \in \tilde{G}; \ 1 \leq i \leq \nu\}$ the set of these elements. Consider the $G$-graded $*$-invariant $F$-subalgebra $F_\nu(A)$ of $P_\nu(A)$ generated by $\mathcal{G}_\nu$. Consider any map $\varphi$ of the generators $\mathcal{G}_\nu$ to arbitrary $\tilde{G}$-homogeneous elements $\tilde{a}_{\delta i} \in A$ of the corresponding $\tilde{G}$-degrees

$$\varphi : \eta_{\delta i} \mapsto \tilde{a}_{\delta i} = \sum_{j=1}^{t_\delta} \tilde{\alpha}_{\delta ij} \hat{b}_{\delta j} + \sum_{j=1}^{q_\delta} \tilde{\alpha}_{\delta ij+t_\delta} \hat{r}_{\delta j} \quad (\delta \in \tilde{G}; \ i = 1, \ldots, \nu), \quad (14)$$
here $\tilde{\alpha}_{\delta ij} \in F$. It is clear that $\varphi$ can be extended to the graded $*$-homomorphism of $F$-algebras $\varphi : F_\nu(A) \to A$, also inducing the graded $*$-homomorphism $\tilde{\varphi} : P_\nu(A) \to A$ by the equalities

$$\tilde{\varphi}((\lambda_{\delta_{ijj1}} \cdots \lambda_{\delta_{ikek}}) \otimes a) = (\tilde{\alpha}_{\delta_{ijj1}} \cdots \tilde{\alpha}_{\delta_{ikek}}) \cdot a \quad \forall a \in A. \quad (15)$$

The graded $*$-homomorphism $\tilde{\varphi}$ preserves the forms on $P_\nu(A)$ and $A$ defined by $[12]$. Elements of $F_\nu(A)$ are called quasi-polynomials on the variables $\mathcal{Y}_\nu$. Products of the generators $\eta_{\alpha_\nu} \in \mathcal{Y}_\nu$ of the algebra $F_\nu(A)$ are called quasi-monomials. We have also $\text{Id}^g(F_\nu(A)) \supseteq \text{Id}^g(P_\nu(A)) = \text{Id}^g(A)$ for any $\nu \in \mathbb{N}$.

Now we can state the relation between graded $*$-identities of finitely generated PI-algebras and graded $*$-identities of finite dimensional algebras assuming that Assumption 2.1 is true over any $G$-graded PI-algebra with involution, i.e. we assume that $\Gamma$ contains a non-trivial $T$-ideal.

5 Graded $*$-identities of finitely generated algebras.

Now we can state the relation between graded $*$-identities of finitely generated PI-algebras and graded $*$-identities of finite dimensional algebras assuming that As-
The decomposition \( G \) implies that ind\(_{gi}\) (

\[ \text{by (4) for } A \]

\[ \nu \]

\( \ast \)-identities of some finite dimensional \( F \)-graded associative algebra with graded involution.

\[ \ast \]-homogeneous \( D \) coincides with the \( gi \)-ideal of graded \( \ast \)-identities of some finite dimensional \( F \)-graded associative algebra with graded involution.

**Proof.** Let \( \Gamma = \text{Id}^{gi}(D) \). We use the induction on the Kemer index \( \text{ind}_{gi}(\Gamma) = \kappa = (\beta; \gamma) \) of \( \Gamma \).

The base of induction. Let \( \text{ind}_{gi}(\Gamma) = (\beta; \gamma) \) with \( \beta = (0, \ldots, 0) \). Then \( D \) is a nilpotent finitely generated algebra. Hence, \( D \) is finite dimensional.

The inductive step. The \( \Gamma \)-graded versions of Lemmas 11, 26, 27 \[42\] imply that \( \Gamma \geq \text{Id}^{gi}(A) \), where \( A = \mathcal{O}(A) \times \mathcal{Y}(A) \) is a finite dimensional \( \Gamma \)-graded algebra with involution, such that \( \text{ind}_{gi}(\Gamma) = \text{ind}_{gi}(A) = \kappa \). Moreover, \( S_{\tilde{\mu}}(\Gamma) = S_{\tilde{\mu}}(\mathcal{O}(A)) = S_{\tilde{\mu}}(A) \leq \text{Id}^{gi}(\mathcal{Y}(A)) \) for some \( \tilde{\mu} \in \mathbb{N} \). Here \( \mathcal{O}(A), \mathcal{Y}(A) \) are finite dimensional \( \Gamma \)-graded \( \ast \)-algebras with elementary decomposition. \( \mathcal{O}(A) = A_1 \times \cdots \times A_{\rho} \), where \( A_i \) are \( gi \)-reduced algebras, \( \text{ind}_{gi}(\mathcal{O}(A)) = \text{ind}_{gi}(A_i) = \kappa \forall i = 1, \ldots, \rho \), and \( \text{ind}_{gi}(\mathcal{Y}(A)) < \kappa \) (see the graded version of Lemma 11 \[42\]).

Denote \( (t_1, \ldots, t_{2m}) = \beta(\Gamma) = \text{dim}_{gi}A_i, \ t = \sum_{j=1}^{2m} t_j; \ \gamma = \gamma(\Gamma) = \text{nd}(A_i) \) (for all \( i = 1, \ldots, \rho \)). Let us take for any \( i = 1, \ldots, \rho \) the algebra \( \tilde{A}_i = \mathcal{R}_{q_i,s}(A_i) \) defined by \( \{4\} \) for \( A_i \) with \( q_i = \dim F A_i, \ s = (t+1)(\gamma + \tilde{\mu}). \) \( \tilde{A}_i \) is a finite dimensional \( \gamma \)-graded \( \ast \)-algebra with elementary decomposition. We have also \( \Gamma_i = \text{Id}^{gi}(\tilde{A}_i) = \text{Id}^{gi}(A_i) \), and \( \text{dim}_{gi}\tilde{A}_i = \text{dim}_{gi}A_i = \beta \). The Jacobson radical \( J(\tilde{A}_i) = (Y(q_i), Z(q_i))/I \) is nilpotent of the degree at most \( s = (t+1)(\gamma + \tilde{\mu}), \) where \( I = \Gamma_i(B_i\langle Y(q_i), Z(q_i) \rangle) + (Y(q_i), Z(q_i))^s \). Here the algebra \( B_i \) can be considered as the semisimple part of \( A_i \) and of \( \tilde{A}_i \) simultaneously (the graded version of Lemma 3 \[42\]). Take \( \tilde{A} = \times_{i=1}^{\rho} \tilde{A}_i \), and \( \nu = \text{rk}(D) \). By the graded version of Remark 2 \[42\], and Lemma 3.4 there exists an \( F \)-finite dimensional \( \Gamma \)-graded algebra with graded involution and elementary decomposition \( C \) such that \( \text{Id}^{gi}(C) = \text{Id}^{gi}(\mathcal{T}_\nu(\tilde{A})/\Gamma(\mathcal{T}_\nu(\tilde{A}))) \).

Let us denote \( \tilde{D}_\nu = F(Y(\nu), Z(\nu))/((\Gamma + K_{\tilde{\mu}}(\Gamma)) \cap F(Y(\nu), Z(\nu))) \). The graded versions of Lemmas 6, 8 \[42\] imply that \( \text{ind}_{gi}(\tilde{D}_\nu) \leq \text{ind}_{gi}(\Gamma + K_{\tilde{\mu}}(\Gamma)) < \text{ind}_{gi}(\Gamma) \). By the inductive step we obtain \( \text{Id}^{gi}(\tilde{D}_\nu) = \text{Id}^{gi}(\tilde{U}), \) where \( \tilde{U} \) is a finite dimensional over \( F \) \( \Gamma \)-graded \( \ast \)-algebra. The graded version of Remark 2 \[42\] yields \( \Gamma \subseteq \text{Id}^{gi}(C \times \tilde{U}) \).

Consider a multilinear polynomial \( f(\tilde{x}_1, \ldots, \tilde{x}_m) \in \text{Id}^{gi}(C \times \tilde{U}) \) in variables \( \tilde{x}_j \in Y \cup Z \). Let us take any multihomogeneous with respect to degrees of variables and \( \tilde{G} \)-homogeneous \( \ast \)-polynomials \( w_1, \ldots, w_m \in F(Y(\nu), Z(\nu)) \) (\( \deg_{\tilde{G}} w_j = \deg_{\tilde{G}} \tilde{x}_j, \ j = 1, \ldots, m \)). We have \( f(w_1, \ldots, w_m) = g+h \) for some multihomogeneous graded \( \ast \)-polynomials \( g \in \Gamma, \ h \in K_{\tilde{\mu}}(\Gamma) \) also depending on \( Y(\nu) \cup Z(\nu) \). Then by the graded version of Lemma 24 \[42\] we obtain \( h = f(w_1, \ldots, w_m) - g \in \mathcal{S}\Gamma + \text{Id}^{gi}(\tilde{A}_i) \) for any \( i = 1, \ldots, \rho \). Hence \( \tilde{h}(\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathcal{S}\Gamma + \text{Id}^{gi}(\tilde{A}_i) \) holds also for the full linearization

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\( \tilde{h} \) of \( h \). The graded version of Lemma 18 \cite{12} implies that \( \tilde{h} \) is exact for \( A_i \) (\( i = 1, \ldots, \rho \)).

Fix any \( i = 1, \ldots, \rho \). Assume that \( (c_1, \ldots, c_{q_i}) \) is a basis of \( A_i \) consisting of \( \tilde{G} \)-homogeneous elements chosen in \( D \cup U \) (Lemma 3.2), and fix the order of the basic elements. Suppose that \( \tilde{a} \) is an elementary complete evaluation of \( \tilde{h} \) in the algebra \( A_i \) with \( \gamma - 1 \) radical elements. By Lemma 3.20 there exists a polynomial \( h_{\tilde{\mu}}(\tilde{Z}_1, \ldots, \tilde{Z}_{\gamma - 1 + \tilde{\mu}}, \chi) \in S \Gamma + \text{SId}^g(A_i) \) of type \((\beta, \gamma - 1, \tilde{\mu})\), and an elementary evaluation \( \tilde{u} \) of \( h_{\tilde{\mu}} \) in \( A_i \) such that \( h_{\tilde{\mu}}(\tilde{u}) = \tilde{h}(\tilde{a}) \).

Moreover, \( h_{\tilde{\mu}} \) is alternating in any \( \tilde{Z}_j \) (\( j = 1, \ldots, \gamma - 1 + \tilde{\mu} \)), and all variables from \( \chi \) are replaced by semisimple elements. Then we have

\[
\alpha_2 h_{\tilde{\mu}} = \left( \prod_{m=1}^{\gamma - 1 + \tilde{\mu}} \prod_{\vartheta \in \tilde{G}} A_{Z_m}^{\vartheta} \right) h_{\tilde{\mu}} = \sum_j \left( \prod_{m=1}^{\gamma - 1 + \tilde{\mu}} \prod_{\vartheta \in \tilde{G}} A_{Z_m}^{\vartheta} \right) (\tilde{s}_j \tilde{g}_j) \pmod{\text{SId}^g(A_i)},
\]

where \( Z_m^\vartheta \) is the subset of \( \tilde{G} \)-homogeneous variables of \( Z_m \) of complete degree \( \vartheta = (\delta, \theta) \in \tilde{G} \); \( \alpha_2 \in F, \alpha_2 \neq 0 \); \( \tilde{g}_j \in \Gamma \), \( \tilde{s}_j \in S \). Denote by \( \{\xi_1, \ldots, \xi_n\} \) the variables \( Z \cup \chi \) of \( h_{\tilde{\mu}} \) (the first \( (t + 1)(\gamma - 1) + t\tilde{\mu} \) variables are from \( Z = \bigcup_{m=1}^{\gamma - 1 + \mu} \tilde{Z}_m \)), and by \( Z^\vartheta = \bigcup_{m=1}^{\gamma - 1 + \mu} Z_m^\vartheta \) the \( \tilde{G} \)-homogeneous part of variables \( Z \) of complete degree \( \vartheta \in \tilde{G} \).

Let \( u_k \in A_i \) be an element of the mentioned evaluation \( \tilde{u} = (u_1, \ldots, u_n) \) of \( h_{\tilde{\mu}} \).

We take in the algebra \( \tilde{A}_i \), the elements \( \tilde{y}_\pi(k)^\vartheta = y_\pi(k)^\vartheta + I \), \( \tilde{z}_\pi(k)^\vartheta = z_\pi(k)^\vartheta + I \), where \( y_\pi(k)^\vartheta \in Y_{(q_i)}^\vartheta \), \( z_\pi(k)^\vartheta \in Z_{(q_i)}^\vartheta \) are variables, and \( u_k = c_{\pi(k)}(\pi(k)) \) is the ordinal number of the element \( u_k \) in our basis of \( A_i \), \( 1 \leq \pi(k) \leq q_i \), \( \deg_G u_k = \theta \).

Consider in the algebra \( \tilde{A}_i \) the following evaluation of \( h_{\tilde{\mu}}(\xi_1, \ldots, \xi_n) \)

\[
\zeta_k = \tilde{y}_\pi(k)^{\vartheta} \in J(\tilde{A}_i) \quad \text{if} \quad \zeta_k \in Z^{(+, \theta)}(\theta \in G),
\]

\[
\zeta_k = \tilde{z}_\pi(k)^{\vartheta} \in J(\tilde{A}_i) \quad \text{if} \quad \zeta_k \in Z^{(-, \theta)}(\theta \in G),
\]

\[
\zeta_k = u_k \quad \text{if} \quad \zeta_k \in \chi.
\]

If a graded pure form polynomial \( \tilde{s}_j \) in (16) depends essentially on \( Z \) then \( (\prod_{m=1}^{\gamma - 1 + \mu} \prod_{\vartheta \in \tilde{G}} A_{Z_m}^{\vartheta} ) (\tilde{s}_j \tilde{g}_j) \neq 0 \), since the forms are zero on radical elements (see \cite{12}). If \( \tilde{s}_j \) does not depend on \( Z \) then \( (\prod_{m=1}^{\gamma - 1 + \mu} \prod_{\vartheta \in \tilde{G}} A_{Z_m}^{\vartheta} ) (\tilde{s}_j \tilde{g}_j) = \tilde{s}_j \tilde{g}_j \), where \( \tilde{g}_j = (\prod_{m=1}^{\gamma - 1 + \mu} \prod_{\vartheta \in \tilde{G}} A_{Z_m}^{\vartheta} ) \tilde{g}_j \in \Gamma \). If \( \tilde{g}_j \neq 0 \) in \( \tilde{A}_i \) then one of the multi-homogeneous on degrees components of \( \tilde{g}_j \) is a \( \tilde{\mu} \)-boundary polynomial for \( \tilde{A}_i \). And it is not a \( \tilde{\mu} \)-boundary polynomial for \( \Gamma \), because it belongs to \( \Gamma \). It implies \( \text{SId}_{\tilde{\mu}}(A) \neq \text{SId}_{\tilde{\mu}}(\Gamma) \), that contradicts to the properties of \( A \). Therefore \( \tilde{g}_j \neq 0 \). Thus in any case \( h_{\tilde{\mu}} \neq 0 \) holds in the algebra \( \tilde{A}_i \). Consider in the algebra \( B_i(Y_{(q_i)}, Z_{(q_i)}) \) the elements

\[
v_k = y_\pi(k)^{\vartheta} \quad \text{if} \quad \zeta_k \in Z^{(+, \theta)}(\theta \in G),
\]

\[
v_k = z_\pi(k)^{\vartheta} \quad \text{if} \quad \zeta_k \in Z^{(-, \theta)}(\theta \in G),
\]

\[
v_k = u_k \quad \text{if} \quad \zeta_k \in \chi.
\]
Hence the evaluation $\zeta_k = v_k$ ($k = 1, \ldots, \tilde{n}$) of the polynomial $h_\tilde{\mu}$ is equal to $h_\tilde{\mu}(v_1, \ldots, v_n) \in I = \Gamma_i(B_i(Y(q_i), Z(q_i))) + (Y(q_i), Z(q_i))^\ast$ in the algebra $B_i(Y(q_i), Z(q_i))$. Since $|Z| < s$, the polynomial $h_\tilde{\mu}$ is linear on variables $Z$, and variables of $\mathcal{A}$ are replaced by semisimple elements then we obtain $h_\tilde{\mu}(v_1, \ldots, v_n) \in \Gamma_i(B_i(Y(q_i), Z(q_i)))$.

Consider the map $\varphi : y_{j\theta} \mapsto c_j$ if $\deg \hat{\mathcal{G}} c_j = (+, \theta)$, and $\varphi : z_{j\theta} \mapsto c_j$ if $\deg \hat{\mathcal{G}} c_j = (-, \theta)$, $j = 1, \ldots, q_i$. It is clear that $\varphi$ can be extended to a graded $*$-homomorphism $\varphi : B_i(Y(q_i), Z(q_i)) \to A_i$ assuming $\varphi(b) = b$ for any $b \in B_i$. Then $\varphi(h_\tilde{\mu}(v_1, \ldots, v_n)) = h_\tilde{\mu}(\varphi(v_1), \ldots, \varphi(v_n)) = h_\tilde{\mu}(\tilde{u}) \in \varphi(\Gamma_i(B_i(Y(q_i), Z(q_i)))) \subseteq \Gamma_i(A_i) = (0)$.

Therefore $h(\tilde{a}) = h_\tilde{\mu}(\tilde{u}) = 0$ holds in $A_i$ for any elementary complete evaluation $\tilde{a} \in A_i^n$ containing $\gamma - 1$ radical elements. Since $\tilde{h}$ is a multilinear exact polynomial for $A_i$, and $\gamma = \text{rd}(A_i)$ then $\tilde{h} \in \text{Id}^{gi}(A_i)$. Hence $\tilde{h} \in \bigcap_{i=1}^{\tilde{h}} \text{Id}^{gi}(A_i)$, and $h \in \text{Id}^{gi}(\mathcal{O}(A) \times \mathcal{Y}(A)) \subseteq \text{Id}^{gi}(A) \subseteq \Gamma$. Thus we have $f(w_1, \ldots, w_m) = g + h \in \Gamma$ for all multihomogeneous $\mathcal{G}$-homogeneous graded $*$-polynomials $w_1, \ldots, w_m \in F(\mathcal{Y}(\nu), Z(\nu))$ of corresponding $\tilde{\mathcal{G}}$-degrees. By the graded version of Remark 1 [42] it implies $\text{Id}^{gi}(C \times \tilde{U}) \subseteq \Gamma$.

Therefore $\Gamma = \text{Id}^{gi}(C \times \tilde{U})$. Theorem is proved. \qed

**Theorem 5.2** Let $G$ be a finite abelian group, and $F$ a field of characteristic zero. Suppose that Assumption [23] holds for the group $G$ over any algebraically closed extension $\tilde{F}$ of $F$. Let $D$ be a finitely generated $G$-graded associative PI-algebra over $F$ with graded involution. Then the $g\mathcal{T}$-ideal of graded $*$-identities of $D$ coincides with the $g\mathcal{T}$-ideal of graded $*$-identities of some finite dimensional over $F G$-graded associative algebra with graded involution.

**Proof.** Assume that $F$ does not contain $j = \sqrt{1}$. Consider the extension $K = F[j]$ of $F$ by $j$. It is clear that any algebraically closed extension $\tilde{F}$ of $F$ contains also $K$. Since Assumption [23] holds for the group $G$ over any algebraically closed extension $\tilde{F}$ of $F$ then it is true also over any algebraically closed extension $\tilde{K}$ of $K$. Consider the algebra $\tilde{D} = D \otimes F K$. $\tilde{D}$ is a finitely generated $K$-algebra with the $G$-grading $\tilde{D}_g = D_\theta \otimes F K$, $\theta \in G$. The graded involution on $\tilde{D}$ is naturally induced from $D$ by equalities $(a \otimes \alpha)^* = a^* \otimes \alpha$, for any $a \in D$, $\alpha \in K$. It is clear that $D$ can be considered as an $F$-subalgebra of $\tilde{D}$, and the graded $F$-identities with involution of $D$ and $\tilde{D}$ coincide $\text{Id}_{\tilde{D}}^{gi}(D) = \text{Id}_{\tilde{D}}^{gi}(\tilde{D})$. Particularly, $\tilde{D}$ is a PI-algebra ($\text{Id}_{\tilde{D}}^{gi}(D) \subseteq \text{Id}_{\tilde{K}}^{gi}(\tilde{D})$). By Theorem [5.1] we obtain that $\text{Id}_{\tilde{K}}^{gi}(\tilde{D}) = \text{Id}_{\tilde{K}}^{gi}(C)$ over the field $K$ for some $G$-graded algebra $C$ with graded involution, finite dimensional over $K$. $C$ can be considered also as an $F$-algebra. And as an $F$-algebra $C$ preserves the same $G$-grading and involution. Since $K = F[j]$ is the finite extension of $F$ then $C$ is also finite dimensional over $F$. It is clear that $\text{Id}_{\tilde{F}}^{gi}(C) = \text{Id}_{\tilde{K}}^{gi}(C) \cap F(\mathcal{Y}, Z)$. Therefore we have $\text{Id}_{\tilde{F}}^{gi}(C) = \text{Id}_{\tilde{F}}^{gi}(\tilde{D}) = \text{Id}_{\tilde{F}}^{gi}(D)$. And the $F$-algebra $C$ is the required finite dimensional algebra. \qed

Observe that the final result is obtained for any base field of characteristic zero. The unique restrictions that we have are Assumption [23] and the requirement for a $g\mathcal{T}$-ideal to contain a non-trivial $T$-ideal. The second condition is necessary. An
ideal of group-graded identities of a finitely generated algebra can not contain a non-trivial ordinary non-graded identity (see, e.g., the comment after Theorem 1 [11]). And a finite dimensional algebra is always a PI-algebra. We have discussed in Section 1 the conditions which provide this property for a giT-ideal.

6 PI-representability of \((\mathbb{Z}/q\mathbb{Z})\)-graded algebras.

Suppose that \(G\) is a cyclic group of order \(q\), where \(q\) is a prime number or \(q = 4\). We use the additive notation for the group \(G\) in this case.

Consider the function \(\chi : \mathbb{Z}/4\mathbb{Z} \to \{0, 1\}\) defined on the group \(\mathbb{Z}/4\mathbb{Z}\) by the rules \(\chi(0) = \chi(1) = 0\), \(\chi(2) = \chi(3) = 1\). The next properties of \(\chi\) can be checked directly.

**Lemma 6.1** \(\chi(x) + \chi(y) = \chi(x + y) + 1 \mod 2\) if \(x, y \in \{1, 3\}\), and \(\chi(x) + \chi(y) = \chi(x + y) \mod 2\) if \(x\) or \(y\) is even.

Recall that an elementary grading on the matrix algebra \(M_k(\bar{F})\) is the \(G\)-grading defined by a \(k\)-tuple \((\theta_1, \ldots, \theta_k) \in G^k\), so that \(\deg_G(E_{ij}) = -\theta_i + \theta_j\) for any matrix unit \(E_{ij}\) (see, e.g., [6], [7], [8], [11]).

We obtain the description of \(*\)-graded simple finite dimensional algebras over an algebraically closed field \(\bar{F}\) for the group \(G\). It is based on the classification of simple \(G\)-graded algebras given in Lemma 2.1 (Theorem 3 [6]).

**Theorem 6.1** Let \(q\) be a prime number or \(q = 4\), and \(G\) a cyclic group of order \(q\). Suppose that \(\bar{F}\) is an algebraically closed field of characteristic zero, and \(C\) is a \(G\)-graded finite dimensional \(\bar{F}\)-algebra with graded involution. Then \(C\) is \(*\)-graded simple if and only if \(C\) is isomorphic as a graded \(*\)-algebra to one of the algebras of the list:

1. the direct product \(B \times B^{op}\) of a graded simple algebra \(B = M_k(\bar{F}[H])\), and its opposite algebra \(B^{op}\) with the exchange involution \(\bar{+}\), where \(\bar{F}[H]\) is the group algebra of the group \(H\), and \(H\) is the trivial group, \(G\), or \(H = \{0, \bar{2}\} \leq \mathbb{Z}/4\mathbb{Z}\);

2. the full matrix algebra \(M_k(\bar{F})\) with an elementary grading and an elementary involution;

3. the full matrix algebra \(M_k(\bar{F}[H])\) over the group algebra \(\bar{F}[H]\) with the grading induced by the natural grading of \(\bar{F}[H]\) (\(\deg_G(X_\theta \eta_\theta = \theta\)), and involution \((\sum_{\theta \in H} X_\theta \eta_\theta)^\ast = \sum_{\theta \in H} X_{\theta}^\ast \eta_\theta\), where \(t\) is the transpose or symplectic involution on the matrix algebra \(M_k(\bar{F})\), \(X_\theta \in M_k(\bar{F})\), \(\theta \in H\), \(H\) is a cyclic group;

4. the full matrix algebra \(M_k(\bar{F}[H])\) over the group algebra \(\bar{F}[H]\) with the grading induced by the natural grading of \(\bar{F}[H]\) and involution \((\sum_{\theta \in H} X_\theta \eta_\theta)^\ast = \sum_{\theta \in H} (-1)^\theta X_{\theta}^\ast \eta_\theta\), where \(t\) is the transpose or symplectic involution on the matrix algebra \(M_k(\bar{F})\), and \(H \cong \mathbb{Z}/2\mathbb{Z}\), or \(H \cong \mathbb{Z}/4\mathbb{Z}\);
5. the full matrix algebra $M_k(\tilde{F}[H])$ over the group algebra of $H = \{0, 2\}$ with $(\mathbb{Z}/4\mathbb{Z})$-grading defined as in Lemma 2.1 by a $k$-tuple $(\theta_1, \ldots, \theta_k) \in \{0, 1\}^k$ and an elementary involution.

Proof. It is clear that all alternatives are $*$-graded simple algebras. Suppose that $C$ is a $*$-graded simple finite dimensional $\tilde{F}$-algebra. Then $C$ is a $G$-graded semisimple algebra (Lemma 3.1), and it contains a $G$-graded simple ideal $B$. $B$ is isomorphic as a $G$-graded algebra to $M_k(\tilde{F}^c[H])$ by Lemma 2.1 where $H$ is a subgroup of $G$, and $\zeta : H \times H \to \tilde{F}^*$ is a 2-cocycle on $H$. The canonical grading of $M_k(\tilde{F}^c[H])$ is defined by a $k$-tuple $(\theta_1, \ldots, \theta_k) \in G^k$, so that $\deg_C(E_{ij}\eta_k) = -\theta_i + \xi + \theta_j$. It is well-known (see, e.g., [24]) that the second cohomologies of a cyclic group in this case are trivial. Thus $\tilde{F}^c[H]$ is isomorphic as a graded algebra to the group algebra $\tilde{F}[H]$ of $H$, where $H$ is one of the group of the list: $\{e\}, G, \{0, 2\} \leq \mathbb{Z}/4\mathbb{Z}$. Then either $C = B$ or $C = B \times B^\ast$. In the last case $C$ is isomorphic to $B \times B^{op}$ with the exchange involution. The isomorphism is given by $\varphi : a + b \mapsto (a, b^\ast)$, where $a \in B$, $b \in B^\ast$. Hence we obtain the first alternative.

Suppose that $C = B$ is a $G$-graded simple algebra with graded involution. Thus $C \cong M_k(\tilde{F}[G])$, where $H \in \{\{e\}, \{0, 2\}, G\}$. If $H = \{e\}$ then $C \cong M_k(\tilde{F})$ is the full matrix algebra with an elementary grading and graded involution. The results of Y.A. Bakhurin, I.P. Shestakov, and M.V. Zaicev ([8], [11]) yields in this case that $C$ is isomorphic as a $*$-graded algebra to $M_k(\tilde{F})$ with an elementary grading and an elementary involution.

Suppose that $H = G$, and $C = M_k(\tilde{F}[G])$ with the grading defined in Lemma 2.1 and graded involution. Consider the algebra $C' = M_k(\tilde{F}[G]) = \text{Span}_F\{E_{ij}\tilde{\eta}_k|i, j = 1, \ldots, k; \xi \in G\}$ with the $G$-grading induced by the natural grading of $\tilde{F}[G]$. The $\tilde{F}$-linear map $\varphi : E_{ij}\tilde{\eta}_k \mapsto E_{ij}\tilde{\eta}_k - \theta_i + \theta_j$ is a $G$-graded isomorphism of the algebras $C$ and $C'$. The involution in $C'$ is induced from $C$ by $\varphi$.

A $G$-homogeneous element of $C'$ of degree $\theta \in G$ has the form $X_\theta \tilde{\eta}_\theta = (X_\theta \tilde{\eta}_k) \cdot (I \tilde{\eta}_\theta)$, where $X_\theta \in M_k(\tilde{F})$ is a matrix, $I$ is the identity matrix of order $k$, $\xi$ is the unit of the group $G$. Observe that the element $X_\theta \tilde{\eta}_k$ belongs to the neutral component $C'_\xi$ of $C'$. $C'_\xi \cong M_k(\tilde{F})$, and it is a $*$-invariant subalgebra of $C'$. By Theorem 4.6.12 [15] (see also the proof of Theorem 3.6.8 [29]) the restriction of the involution on $C'_\xi$ can be taken as the transpose or symplectic involution up to an inner automorphism of $C'_\xi$.

Consider a generator $\xi$ of the group $G$. Observe that $I\tilde{\eta}_k$ is a central element of $C'$ of $G$-degree $\xi$. Then $(I\tilde{\eta}_k)^\ast$ has the same $G$-degree, and also belongs to the center of $C'$. Thus $(I\tilde{\eta}_k)^\ast = \alpha I\tilde{\eta}_k$ for some $\alpha \in \tilde{F}$. Since $(I\tilde{\eta}_k)^q = I\tilde{\eta}_k$, and $(I\tilde{\eta}_k)^\ast = I\tilde{\eta}_k$ $(I\tilde{\eta}_k$ is the unit of $C'$) then we obtain $\alpha^q = 1$. We also deduce $\alpha^2 = 1$ from $I\tilde{\eta}_k = ((I\tilde{\eta}_k)^q)^\ast = \alpha^2 I\tilde{\eta}_k$. If $q$ is an odd prime number then $\alpha = 1$. If $q = 2$ or $q = 4$ then $\alpha \in \{-1, 1\}$.

Then for any $\theta \in G$ we have $\theta = \xi^m$ for some integer $m$. Hence, we can assume that $(X_\theta \tilde{\eta}_\theta)^\ast = (I \tilde{\eta}_m)^\ast \cdot (X_\theta \tilde{\eta}_\theta)^\ast = ((I \tilde{\eta}_k)^q)^\ast \cdot (X_\theta \tilde{\eta}_k)^\ast = \alpha^m X_\theta \tilde{\eta}_k$, where $t$ denotes the transpose or symplectic involution on the matrix algebra. Therefore we obtain the alternative 3 or 4.
Similarly to the previous case we obtain that

\[ a = \sum_{i=0}^{3} a_i = (a_0 + a_1) + (a'_0 + a'_1)(I\eta_2), \]  

where \( a_i \in C_i \) are \( G \)-homogeneous components of \( a \), \( a'_0 = a_2\eta_2 \in C_0 \), \( a'_1 = a_3\eta_2 \in C_1 \).

Similarly to the previous case we obtain that \( a^* = (a_0 + a_1)^* + (I\eta_2)^*(a'_0 + a'_1)^* = (a_0 + a_1)^* + \alpha(a'_0 + a'_1)^* (I\eta_2) \), where \( \alpha \in \{-1, 1\} \), \( I \) is the identity matrix. Hence it is enough to describe the restriction of our involution on the graded subspace \( C_0 \oplus C_1 \) of \( C \).

Let us take the vector space \( A = C_0 \oplus C_1 \). Define in \( A \) the multiplication by the rule \( (a_0 + a_1) \odot (b_0 + b_1) = a_0b_0 + a_0b_1 + a_1b_0 + a_1b_1\eta_2 \) in \( C \), where \( a_i, b_i \in C_i \). It is clear that \( A \) is a superalgebra \((\mathbb{Z}/2\mathbb{Z})\)-graded algebra with the \((\mathbb{Z}/2\mathbb{Z})\)-grading \( A = C_0 \oplus C_1 \). The map \( \bar{\star} \) is naturally defined in \( A \) by \( (a_0 + a_1)^\bar{\star} = (a_0 + a_1)^* = a'_0 + a'_1 \), \( a_i \in C_i \). Since the involution \( \bar{\star} \) in \( C \) is graded then \( \bar{\star} \) is a \((\mathbb{Z}/2\mathbb{Z})\)-graded linear operator of the second order which satisfies \( (a_i \odot b_1)^\bar{\star} = a^i\bar{\star} (b_j^i \odot a_k^i) \), where \( a_i, b_i \in C_i \). A linear operator with all mentioned properties is called \( \alpha \)-involution. It is clear that a \((1)\)-involution is a graded involution on the superalgebra \( A \), and a \((-1)\)-involution is a superinvolution on the superalgebra \( A \). We denote by \( \Phi(C) \) the superalgebra \( A \) with the multiplication \( \odot \) and the \( \alpha \)-involution \( \bar{\star} \) obtained of the \((\mathbb{Z}/4\mathbb{Z})\)-graded algebra \( C \) with a graded involution \( \ast \) by the represented procedure.

The superalgebra \( \Phi(C) \) is isomorphic as a superalgebra to the full matrix algebra \( \mathcal{A} = M_k(\bar{F}) \) with the elementary \((\mathbb{Z}/2\mathbb{Z})\)-grading defined by the \( k \)-tuple \((\bar{\theta}_1, \ldots, \bar{\theta}_k) \). Where \( \theta_i = \theta_i + H \in (\mathbb{Z}/4\mathbb{Z})/H \) if \((\theta_1, \ldots, \theta_k) \) is the \( k \)-tuple defining the \((\mathbb{Z}/4\mathbb{Z})\)-grading of \( C \). The graded isomorphism \( \varphi : \Phi(C) \to M_k(F) \) is given by the rule \( \varphi(E_{ij}^\eta\xi_{ij}) = E_{ij} \). Denote by \( \bar{\star} \) the \( \alpha \)-involution on the superalgebra \( A \) induced by the \( \alpha \)-involution \( \bar{\star} \) with respect to the isomorphism \( \varphi \). Hence we have \( \varphi(a^\bar{\star}) = \varphi(a)\bar{\star} \) for any \( a \in \Phi(C) \), and \( \varphi \) is the isomorphism of superalgebras with \( \alpha \)-involution.

If \( \alpha = 1 \) then \((\mathcal{A}, \bar{\star})\) is isomorphic as a \( \ast \)-graded algebra to the matrix algebra \( M_k(\bar{F}) \) with an elementary grading and an elementary involution by [8], [11]. For \( \alpha = -1 \) superinvolutions on \((\mathbb{Z}/2\mathbb{Z})\)-graded matrix algebras are described by M.L. Racine [37] (Proposition 13, 14) (see also [9], Proposition 1). In both of the cases \((\mathcal{A}, \bar{\star})\) is isomorphic as a superalgebra with \( \alpha \)-involution to the algebra \( \bar{\mathcal{A}} = M_k(\bar{F}) \) with an elementary grading and an elementary \( \alpha \)-involution \( \bar{\star} \). An \( \alpha \)-involution \( \bar{\star} \) on a matrix superalgebra \( M_k(\bar{F}) \) is called elementary if \( (E_{ij})^{\bar{\star}} = \pm E_{st} \) for all \( i, j = 1, \ldots, k \) and some \( s, t \).

Consider the \( k \)-tuple \((\bar{\vartheta}_1, \ldots, \bar{\vartheta}_k) \in \mathbb{Z}/2\mathbb{Z} \) defining the elementary grading of \( \bar{\mathcal{A}} \). Suppose that \( \bar{\vartheta}_i = \bar{\vartheta}_i + 2\mathbb{Z} \), \( \vartheta_i \in \{0, 1\} \), \( i = 1, \ldots, k \). Let us take the \( k \)-tuple \((\bar{\vartheta}_1, \ldots, \bar{\vartheta}_k) \), where \( \bar{\vartheta}_i = \theta_i + 4\mathbb{Z} \in \mathbb{Z}/4\mathbb{Z} \). Denote by \( \bar{C} \) the algebra \( M_k(F[H]) \) with the canonical \((\mathbb{Z}/4\mathbb{Z})\)-grading defined by the \( k \)-tuple \((\bar{\vartheta}_1, \ldots, \bar{\vartheta}_k) \) (see Lemma 21). Observe that \( \Phi(C) \) is isomorphic as a superalgebra to \( \bar{\mathcal{A}} \) by the arguments above. Suppose that \( \bar{\varphi} : \Phi(C) \to \bar{\mathcal{A}} \) is the \((\mathbb{Z}/2\mathbb{Z})\)-graded isomorphism given by the rules \( \bar{\varphi}(E_{ij}^\eta\xi_{ij}) = E_{ij} \), \( i, j = 1, \ldots, k \).
Define a map $*$ on the algebra $\bar{C}$ by the rule:

$((c_0 + c_1) + (c_0' + c_1')(I\eta_2))^* = \tilde{\phi}^{-1}((\tilde{\phi}(c_0 + c_1))^*) + \alpha\tilde{\phi}^{-1}((\tilde{\phi}(c_0' + c_1'))^*)(I\eta_2).$ \hfill (19)

Where $c = (c_0 + c_1) + (c_0' + c_1')(I\eta_2)$ is the form (18) for an element $c \in \bar{C}$, $c_i, c_i' \in \bar{C}$. It is clear that $*$ is a $(\mathbb{Z}/4\mathbb{Z})$-graded linear operator of $\bar{C}$ satisfying $(I\eta_2)^* = \alpha(I\eta_2)$. The restriction of $*$ on $\Phi(\bar{C})$ is $\tilde{\phi}^{-1}\tilde{\phi} \tilde{\phi}$. Hence $\tilde{\phi}$ is an $\alpha$-involution, and $\tilde{\phi}$ is an isomorphism of superalgebras with $\alpha$-involution $(\Phi(\bar{C}), \tilde{\phi})$ and $(\bar{A}, \tilde{\phi})$. Particularly, $(\Phi(\bar{C}), \tilde{\phi})$ is isomorphic as a superalgebra with $\alpha$-involution to $(\Phi(C), \tilde{\phi})$. Moreover, for a basic element $b = E_{ij}\eta_\xi$ of the canonical basis of $\bar{C}$ we have

$b^* = (E_{ij}\eta_\xi)^* = \alpha^* E_{ij}^{\bar{\xi}} \eta_\xi,$

$s = \chi(\deg_{(\mathbb{Z}/4\mathbb{Z})} b), \quad \bar{\xi} = \xi + \deg_{(\mathbb{Z}/4\mathbb{Z})} E_{ij} - \deg_{(\mathbb{Z}/4\mathbb{Z})} E_{ij}^{\bar{\xi}}.$ \hfill (20)

Using (19) or (20) and Lemma 6.1 it can be directly checked that $*$ is a $(\mathbb{Z}/4\mathbb{Z})$-graded involution of $\bar{C}$. The isomorphism of superalgebras with $\alpha$-involution $\psi : (\Phi(\bar{C}), \tilde{\phi}) \to (\Phi(C), \tilde{\phi})$ induces the isomorphism of $(\mathbb{Z}/4\mathbb{Z})$-graded algebras with involution $\Psi : (\bar{C}, \tilde{\phi}) \to (C, \tilde{\phi})$ by the rule based on the representation (18) of elements $c \in \bar{C}$:

$\Psi(c) = \Psi((c_0 + c_1) + (c_0' + c_1')(I\eta_2)) = \psi(c_0 + c_1) + \psi(c_0' + c_1')(I\eta_2).$

Since $*$ is an elementary involution (by (20)) then we obtain the last alternative of the theorem. It completes the proof. \hfill \Box

The direct consequence of Theorem 6.1 is the next corollary.

**Corollary 6.2** Let $\bar{F}$ be an algebraically closed field of characteristic zero. Assumption 2.1 is true over $\bar{F}$ for a cyclic group $G$ of a prime order or of the order 4.

Hence Theorems 6.1, 5.2 immediately imply Theorem 6.2

**Theorem 6.2** Let $q$ be a prime number or $q = 4$, $F$ a field of characteristic zero. Then for any $(\mathbb{Z}/q\mathbb{Z})$-graded finitely generated associative PI-algebra $A$ with graded involution over $F$ there exists a finite dimensional over $F$ $(\mathbb{Z}/q\mathbb{Z})$-graded associative algebra $C$ with graded involution such that the ideals of graded identities with involution of $A$ and $C$ coincide.

It is an interesting problem to describe all groups $G$ such that Assumption 2.1 is true for $G$-graded $*$-algebras over an algebraically closed field. We suppose that Assumption 2.1 should be true for any finite abelian group.

**Conjecture 6.1** Let $G$ be a finite abelian group, and $\bar{F}$ an algebraically closed field of characteristic zero. Given a $G$-graded finite dimensional algebra $A$ with graded involution $A$ is $*$-graded simple if and only if $A$ is isomorphic as a graded $*$-algebra either to $G$-graded simple algebra $\bar{C}^{(1)} = M_k(\bar{F}^\mathbb{C}[H])$ with an elementary involution, or to the direct product $\bar{C}^{(2)} = \mathcal{B} \times \mathcal{B}^{op}$ of a graded simple algebra $\mathcal{B} = M_k(\bar{F}^\mathbb{C}[H])$.
and its opposite algebra $B^{op}$ with the exchange involution $\bar{\cdot}$. Where $H$ is a subgroup of $G$, and $\zeta: H \times H \to \mathbb{Q}[\sqrt[m]{1}]^*$ is a 2-cocycle on $H$ with values in the algebraic extension of rational numbers $\mathbb{Q}$ by a primitive root $\sqrt[m]{1}$ of 1 of degree $m = |G|$.

If it is true then any $G$-graded finitely generated PI-algebra with graded involution over a field of characteristic zero should be PI-representable with respect to graded $*$-identities.

Both of the questions (the classification of finite dimensional $*$-graded simple algebras, and PI-representability of finitely generated algebras) are also interesting in case of a finite (not necessary abelian) group.

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