Harnack Inequality and Applications for SDEs Driven by G-Brownian Motion

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Abstract In this paper, Wang’s Harnack and shift Harnack inequality for a class of stochastic differential equations driven by G-Brownian motion are established. The results generalize the ones in the linear expectation setting. Moreover, some applications are also given.

Keywords Harnack inequality; shift Harnack inequality; stochastic differential equations; G-Brownian motion; G-expectation.

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1 Introduction

Since Wang [20] introduced dimensional-free Harnack inequality for diffusions on Riemannian manifold, this Harnack inequality has been extensively investigated. This type Harnack inequality acts as a powerful tool in the study of functional inequalities (see [1, 15, 16, 21, 22]), heat kernel estimates (see [6]), high order eigenvalues (see [9, 18]), transportation cost inequalities (see [4]), and short-time behavior of transition probabilities (see [2, 3, 9]). To establish Wang’s Harnack inequality, Wang and co-authors introduced the coupling by change of measures (see Wang [19] and references within for details).

On the other hand, for the potential applications in uncertainty problems, risk measures and the superhedging in finance, the theory of nonlinear expectation has been developed. Especially, Peng [12, 13] established the fundamental theory of G-expectation theory, G-Brownian motion and stochastic differential equations driven by G-Brownian motion (G-SDEs, in short).

To establish Wang’s Harnack inequality using coupling by change of measures in the linear probability setting, the Girsanov transform plays a crucial role. In [7, 11, 23], the Girsanov’s theorem has been extended to the G-framework, and Girsanov’s formula has been derived for G-Brownian motion. Song [17] firstly derived the gradient estimates for nonlinear diffusion semigroups by using the method of Wang’s coupling by change of measure. Recently, Hu et al. [8] studied the invariant and ergodic nonlinear expectations for G-diffusion processes.

In this paper, we investigate Wang’s Harnack and shift Harnack inequality and applications for the following G-SDE

\[ dX_t = b(X_t)dt + dB_t, \]  

where \( B_t \) is a G-Brownian motion.

The paper is organized as follows. In Section 2, we recall some preliminaries on G-Brownian motion, related stochastic calculus and transformation for G-expectation. In Section 3, Wang’s Harnack inequality and shift Harnack inequality are established for the nonlinear Markov operator associated with (1.1). In addition, we give some applications of Harnack inequality.

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2 Preliminaries

2.1 Sublinear Expectation Spaces

In this section, we propose some preliminaries and notations which appeared in Peng [12, 14]. Let \( \Omega \) be a given set and \( \mathcal{H} \) be a vector lattice of real valued functions defined on \( \Omega \), namely \( c \in \mathcal{H} \) for each constant \( c \) and \( |X| \in \mathcal{H} \) if \( X \in \mathcal{H} \). \( \mathcal{H} \) is considered as the space of random variables.

**Definition 2.1.** (Sublinear expectation space). A sublinear expectation \( \overline{E} \) on \( \mathcal{H} \) is a functional \( \overline{E} : \mathcal{H} \to \mathbb{R} \) satisfying the following properties: for all \( X, Y \in \mathcal{H} \), it holds that

- (a) **Monotonicity:** \( \overline{E}[X] \geq \overline{E}[Y] \), if \( X \geq Y \),
- (b) **Constant preservation:** \( \overline{E}[c] = c \),
- (c) **Sub-additivity:** \( \overline{E}[X + Y] \leq \overline{E}[X] + \overline{E}[Y] \),
- (d) **Positive homogeneity:** \( \overline{E}[\lambda X] = \lambda \overline{E}[X] \) for each \( \lambda \geq 0 \).

Then, \( (\Omega, \mathcal{H}, \overline{E}) \) is called a sublinear expectation space.

Let \( S^d \) be the collection of all \( d \times d \) symmetric matrices, \( X \) be a \( G \)-normal distributed random vector, and \( G : S^d \to \mathbb{R} \) is defined by

\[
G(A) := \frac{1}{2} \overline{E}[\langle AX, X \rangle] = \sup_{\gamma \in \Theta} \frac{1}{2} \text{tr}[\gamma A], \quad A \in S^d.
\]

Then the distribution of \( X \) is characterized by

\[
u(t, x) = \overline{E}[\varphi(x + \sqrt{t}X)], \quad \varphi \in C_{\text{t, lip}}(\mathbb{R}^d),
\]

where \( C_{\text{t, lip}}(\mathbb{R}^n) \) be the space of all real functions \( \varphi \) on \( \mathbb{R}^n \) satisfying

\[|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad x, y \in \mathbb{R}^n,\]

for some \( C > 0, m \in \mathbb{N} \) depending on \( \varphi \).

In particular, \( \overline{E}[\varphi(X)] = u(1, 0) \), where \( u \) is the solution of the following parabolic PDE defined on \( [0, \infty) \times \mathbb{R}^d \):

\[
\begin{aligned}
\frac{\partial u}{\partial t} - G\left( \frac{\partial^2 u}{\partial x^2} \right) &= 0, \\
u(0, x) &= \varphi(x).
\end{aligned}
\]

This parabolic PDE is called a \( G \)-heat equation.

2.2 \( G \)-expectation and \( G \)-Brownian Motion

For any fixed \( T > 0 \), let \( \Omega = C_0([0, T], \mathbb{R}^d) \) be the space of all \( \mathbb{R}^d \)-valued continuous paths \( (\omega_t)_{t \in [0, T]} \) with \( \omega_0 = 0 \), equipped with the distance

\[
\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \left[ \max_{t \in [0,i]} |\omega^1_t - \omega^2_t| \right] \wedge 1.
\]

Consider the canonical process \( B_t(\omega) = \omega_t, \ t \in [0, \infty), \ \omega \in \Omega \). Let \( \Omega_T = \{\omega_{\cdot \wedge T} : \omega \in \Omega\} \), and set

\[
L_{\text{ip}}(\Omega_T) = \{\varphi(B_{t_1 \wedge T}, \ldots, B_{t_n \wedge T}) : n \in \mathbb{N}, \ t_1, \ldots, t_n \in [0, \infty), \ \varphi \in C_{\text{t, lip}}(\mathbb{R}^{d \times n})\}.
\]
Then $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T), \ t \leq T$. Set

$$L_{ip}(\Omega) = \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n).$$

Let $(\xi_i)_{i=1}^{\infty}$ be a sequence of $d$-dimensional random vectors on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ such that $\xi_i$ is $G$-normal distributed and $\xi_{i+1}$ is independent from $(\xi_1, \cdots, \xi_i)$ for each $i = 1, 2, \cdots$. We now introduce a sublinear expectation $\mathbb{E}$ defined on $L_{ip}(\Omega)$ via the following procedure: for each $X \in L_{ip}(\Omega)$ with

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}),$$

for some $\varphi \in C_{l,p}(\mathbb{R}^{d \times n})$, and $0 = t_0 < t_1 < \cdots < t_n < \infty$, we set

$$\mathbb{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})] = \mathbb{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \sqrt{t_2 - t_1}\xi_1, \cdots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

**Definition 2.2.** (G-expectation and G-Brownian motion). The sublinear expectation $\mathbb{E} : L_{ip}(\Omega) \to \mathbb{R}$ defined through the above procedure is called a G-expectation. The corresponding canonical process $(B_t)_{t \geq 0}$ on the sublinear expectation space $(\Omega, L_{ip}(\Omega), \mathbb{E})$ is called a G-Brownian motion.

**Remark 2.3.** Let $L_{ip}^G(\Omega_T)$ (respectively $L_{ip}^G(\Omega)$) be the completion of $L_{ip}(\Omega_T)$ (respectively $\Omega_T$) under the norm $(\mathbb{E}[|\cdot|^p])^{\frac{1}{p}}$. Then $\mathbb{E}[\cdot]$ can be continuously extends to a sublinear expectation on $(\Omega, L_{ip}^G(\Omega))$, which is still denoted by $\mathbb{E}$.

Let

$$M_{ip}^G(\Omega_T) = \left\{ \eta = \sum_{j=0}^{N-1} \xi_j I_{[t_j, t_{j+1})]}; \xi_j \in L_{ip}^G(\Omega_{t_j}), N \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_N = T \right\}.$$

For $p \geq 1$, let $M_{ip}^G([0, T])$ be the completion of $M_{ip}^G([0, T])$ under the following norm

$$\|\eta\|_{M_{ip}^G([0, T])} = \left[ \mathbb{E} \left( \int_{0}^{T} |\eta_t|^p dt \right) \right]^{\frac{1}{p}}.$$

### 2.3 Capacity and Quasi-Sure Analysis for G-Brownian Paths

Denis et al.\cite{Denis} proved that there exists a weakly compact family $\{E_\theta : \theta \in \Theta\}$ of expectations introduced by probability measures $\{P_\theta : \theta \in \Theta\}$ defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\mathbb{E}[X] = \sup_{\theta \in \Theta} E_\theta[X], \ X \in L_{ip}(\Omega).$$

Then the associated Choquet capacity is given by

$$c(A) = \sup_{\theta \in \Theta} P_\theta(A), \ A \in \mathcal{B}(\Omega).$$

**Definition 2.4.** (Quasi-surely). A set $A \in \mathcal{B}(\Omega)$ is called polar if $c(A) = 0$ and a property holds quasi-surely (q.s.) if it holds outside a polar set.

**Remark 2.5.** Let $X$ and $Y$ be two random variables, we say that $X$ is a version of $Y$, if $X = Y$ q.s..
2.4 Transformation for \( G \)-expectation

From now on, we consider

\[
G(A) := \frac{1}{2} \sup_{\gamma \in [\sigma^2, \sigma^2]} \text{tr}(\gamma A), \quad A \in \mathbb{S}^d, 0 < \sigma^2 < \sigma^2
\]

are two matrices in \( \mathbb{S}^d \). \hspace{1cm} (2.1)

Hu et al.\textsuperscript{[7]} construct an auxiliary extended \( \tilde{G} \)-expectation space \((\tilde{\Omega}_T, L^1_{\tilde{G}}, \tilde{E})\) with \( \tilde{\Omega}_T = C_0([0, T], \mathbb{R}^{2d}) \) and

\[
\tilde{G}(A) = \frac{1}{2} \sup_{v \in [\sigma^2, \sigma^2]} \text{tr} \left[ A \begin{pmatrix} v & 1_d \\ 1_d & v^{-1} \end{pmatrix} \right], \quad A \in \mathbb{S}^{2d}.
\]

Let \((B_t, \overline{B}_t)\) be the canonical process in the extended space. Then \( \langle B_t, \overline{B}_t \rangle = t 1_d \), and

\[
\tilde{E}^{\tilde{G}}[\xi] = \tilde{E}[\xi], \quad \xi \in L^1_{\tilde{G}}(\Omega_T).
\]

The following lemma is taken from [7, Theorem 5.2].

**Lemma 2.6.** Let \((h_t)_{t \geq 0} \in M^2_{\tilde{G}}([0, T])\) and be a bounded process, then the process \( \tilde{B}_t := B_t + \int_0^T h_s ds \) is a \( \tilde{G} \)-Brownian motion under \( \tilde{E} \) for

\[
\tilde{E}[X] = \tilde{E}^{\tilde{G}}[X \exp \left( \int_0^T h_s d\overline{B}_s - \frac{1}{2} \int_0^T h_s^2 d\langle B \rangle_s \right)], \quad X \in L^1_{\tilde{G}}(\Omega_T).
\]

**Remark 2.7.** We remark that the \( \overline{B}_t \) is a \( \tilde{G} \)-Brownian motion under \( \tilde{E}^{\tilde{G}} \) with \( \tilde{G}(A) = \frac{1}{2} \sup_{\sigma^{-2} \leq v \leq \sigma^{-2}} \text{tr}[Av], \quad A \in \mathbb{S}^d. \)

3 Main Results

Denote by \( C^+_b(\mathbb{R}^d) \) the set of non-negative bounded continuous function on \( \mathbb{R}^d \). Let \( \overline{P} : C^+_b(\mathbb{R}^d) \to C^+_b(\mathbb{R}^d) \) be a nonlinear operator, we aim to establish the following Harnack-type inequality introduced by Feng-Yu Wang \textsuperscript{[19]}:

\[
\Phi(\overline{P} f(x)) \leq \overline{P} \Phi(f(y))e^{\Psi(x,y)}, \quad x, y \in \mathbb{R}^d, \quad f \in C^+_b(\mathbb{R}^d),
\]

where \( \Phi \) is a nonnegative convex function on \([0, \infty)\) and \( \Psi \) is a nonnegative function on \( \mathbb{R}^d \times \mathbb{R}^d \).

3.1 Harnack and Shift Harnack Inequality

In the setting of \( G \)-SDEs, we establish this type inequality for the associated Markov operator \( \overline{P}_T \) which will be defined below. For simplicity, we consider the 1-dimensional \( G \)-Brownian motion case, but our results and methods still hold for the case \( d > 1 \). More precisely, consider the following \( G \)-SDE

\[
dX_t = b(X_t)dt + dB_t, \quad (3.1)
\]

where \( B_t \) is a \( G \)-Brownian motion for \( \sigma^2 = \overline{E}[B_t^2] \geq -\overline{E}[-B_t^2] = \sigma^2 > 0 \), and \( b : \mathbb{R} \to \mathbb{R} \) satisfies

**(H1)** \(|b(x) - b(y)| \leq K|x - y|, \quad x, y \in \mathbb{R} \).
for some constant $K > 0$. From [13], under (H1) the $G$-SDE (3.1) has a unique solution $X \in M^2_G([0, T])$ for any initial value. In what follows, for $T > 0$, we define
\[
\mathcal{P}_T f(x) = \mathbb{E} f(X_T^x), \quad f \in C^+_b(\mathbb{R}),
\]
where $X_T^x$ solves (3.1) with initial value $x$.

**Remark 3.1.** In order to ensure the term $f(X_T^x) \in L^2_G(\Omega_T)$, we always assume $f \in C^+_b(\mathbb{R})$.

In view of the definition of $\mathcal{E}^\hat{G}$ (in section 2.4), it holds that
\[
\mathbb{E}[f(X_T^x)] = \mathcal{E}^\hat{G}[f(X_T^x)] =: \mathcal{P}_T^\hat{G} f(x).
\]
We have the following result.

**Theorem 3.2.** Under (H1), for any nonnegative $f \in C^+_b(\mathbb{R})$ and $T > 0, x, y \in \mathbb{R}$, it holds that
\[
(\mathcal{P}_T f)^p(y) \leq \mathcal{P}_T f^p(x) \exp \left\{ \frac{p|x - y|^2}{2\sigma^2(p - 1)} \left( \frac{1}{T} + K + \frac{K^2 T}{3} \right) \right\}. \tag{3.3}
\]

**Proof.** We use the coupling by change of measures as explained in [19]. Consider the following coupled stochastic differential equations
\[
dY_t = b(Y_t)dt + dB_t + \left( b(X_t) - b(Y_t) - \frac{v}{T} \right)dt, \quad Y_0 = y + v,
\]
\[
dX_t = b(X_t)dt + dB_t, \quad X_0 = x.
\]
It’s easy to see that $Y_t = X_t + \frac{T - t}{T}v$. In particular, $Y_T = X_T$. It follows from (H1) that
\[
\left| \left( b(X_t) - b(Y_t) - \frac{v}{T} \right) \right| \leq \frac{1 + K(T - t)}{T}|v|. \tag{3.4}
\]
Let
\[
M_T = \exp \left\{ -\int_0^T u_s dB_s - \frac{1}{2} \int_0^T |u_s|^2 d\langle B \rangle_s \right\},
\]
where $u_t = b(X_t) - b(Y_t) - \frac{v}{T}$, and $\overline{B}_t$ is a $\hat{G}$-Brownian motion under $\mathcal{E}^\hat{G}$ in Remark 2.7.

Define a sublinear expectation $\hat{E}$ by $\hat{E}[\cdot] := \mathcal{E}^\hat{G}[\cdot; M_T]$. By Lemma 2.6, the process
\[
\hat{B}_t := B_t + \int_0^t u_s ds, \quad t \geq 0
\]
is a $G$-Brownian motion under $\hat{E}$.

Then, $Y_t$ can be reformulated by
\[
dY_t = b(Y_t)dt + d\hat{B}_t.
\]
Therefore, $\mathcal{P}_T f(y) = \mathbb{E} f(X_T^y) = \hat{E} f(Y_T^y) = \mathcal{E}^\hat{G} f(X_T^y) = \mathcal{E}^\hat{G} (M_T f(X_T^y))$.

Using Hölder’s inequality, we have
\[
(\mathcal{P}_T f)^p(y) = (\mathcal{E}^\hat{G} [M_T f(X_T^y)])^p \leq (\mathcal{E}^\hat{G} [f^p(X_T^y)]) (\mathcal{E}^\hat{G} [M_T^p])^{p-1}. \tag{3.5}
\]
Now we estimate the moment of $M_T$. Indeed,

$$
\mathbb{E}^\mathcal{G} \left[ M_T^{\frac{p}{2}} \right] = \mathbb{E}^\mathcal{G} \exp \left\{ - \frac{p}{p - 1} \int_0^T u_s dB_s - \frac{p}{2(p - 1)} \int_0^T |u_s|^2 dB_s \right\} \\
= \mathbb{E}^\mathcal{G} \exp \left\{ - \frac{p}{p - 1} \int_0^T u_s dB_s - \frac{p^2}{2(p - 1)^2} \int_0^T |u_s|^2 dB_s \right\} \\
+ \frac{p}{2(p - 1)^2} \int_0^T |u_s|^2 dB_s \right\} \\
= \mathbb{E}^\mathcal{G} \exp \left\{ \frac{p^2 - 2v^2}{2(p - 1)^2} \left( \frac{1}{T} + K + \frac{K^2T}{3} \right) \right\}. 
$$  \hspace{1cm} (3.6)

According to Remark 2.7 and inequality (3.4), it holds that

$$
\exp \left\{ \frac{p}{2(p - 1)^2} \int_0^T |u_s|^2 dB_s \right\} \leq \exp \left\{ \frac{p^2 - 2v^2}{2(p - 1)^2} \int_0^T |u_s|^2 ds \right\} \\
\leq \exp \left\{ \frac{p^2 - 2v^2}{2(p - 1)^2} \left( \frac{1}{T} + K + \frac{K^2T}{3} \right) \right\}. 
$$

Substituting this into (3.6), we have

$$
\mathbb{E}^\mathcal{G} \left[ M_T^{\frac{p}{2}} \right] \leq \exp \left\{ \frac{p^2 - 2|y|^2}{2(p - 1)^2} \left( \frac{1}{T} + K + \frac{K^2T}{3} \right) \right\}. \hspace{1cm} (3.7)
$$

Combining (3.5) and (3.7), we prove (3.3). \hfill \square

**Theorem 3.3.** Under (H1), for any nonnegative $f \in C_b^+ (\mathbb{R})$ and $T > 0$, $x, y \in \mathbb{R}$, the following shift Harnack inequality holds

$$
(\mathcal{P}_T f(x))^p \leq (\mathcal{P}_T f^p(y + \cdot))(y) \exp \left\{ \frac{pm^2}{2v^2(p - 1)} \left( \frac{1}{T} + K + \frac{K^2T}{3} \right) \right\}. \hspace{1cm} (3.8)
$$

**Proof.** Let $Y_t = X_t + \frac{t}{T}v$ with $Y_0 = X_0 = x$, then $Y_T = X_T + v$. Let

$$
R_t = \exp \left\{ - \int_0^t \frac{v}{T} b(X_s) - b(Y_s) dB_s - \frac{1}{2} \int_0^t \left( \frac{v}{T} b(X_s) - b(Y_s) \right)^2 dB_s \right\},
$$

where $\mathcal{B}_t$ is $G$-Brownian motion under $\mathbb{E}^\mathcal{G}$, which is an auxiliary process. It follows from (H1) that

$$
\left| \frac{v}{T} b(X_s) - b(Y_s) \right| \leq \frac{1 + Ks}{T} |v|. \hspace{1cm} (3.9)
$$

Due to Lemma 2.6,

$$
\tilde{B}_t := B_t + \int_0^t \left( \frac{v}{T} b(X_s) - b(Y_s) \right) ds
$$

is a $G$-Brownian motion under $\mathbb{E}$ with

$$
\mathbb{E}[\xi] = \mathbb{E}^\mathcal{G}[\xi R_T], \quad \xi \in L^1_G(\Omega_T).
$$

Then

$$
dY_t = b(Y_t) dt + d\tilde{B}_t, \quad Y_0 = x.
Thus, for \( f \in C_b^+(\mathbb{R}) \), \( p \geq 1 \), by Hölder inequality, it holds that
\[
(\mathcal{P}_T f)^p(x) = (\mathbb{E}[f(Y_T^x)])^p = (\mathbb{E}^\mathcal{G}[R_T f(X_T^x + v)])^p \leq (\mathcal{P}_T f^p(x) \mathbb{E}^\mathcal{G}[R_T f^p(X_T^x + v)])^p \leq (\mathcal{P}_T f^p(x) \mathbb{E}^\mathcal{G}[R_T f^p(X_T^x + v)])^p \leq \left( \mathcal{P}_T f^p(x) \left( \mathbb{E}^\mathcal{G}[R_T f^p(X_T^x + v)] \right)^p \right).
\]

Letting \( h_s = \frac{x}{T} + b(X_s) - b(Y_s) \), by Remark 2.7 and (3.9), we conclude that
\[
\exp \left\{ \frac{p}{2(p-1)^2} \int_0^T |h_s|^2 d(\mathcal{B})_s \right\} \leq \exp \left\{ \frac{p \sigma^2 - 2}{2(p-1)^2} \int_0^T |h_s|^2 ds \right\} = \exp \left\{ \frac{p \sigma^2 - 2}{2(p-1)^2} \left( \frac{1}{T} + K + \frac{K^2 T}{3} \right) \right\}.
\]

Similar to the arguments in Theorem 3.2, we have
\[
\mathbb{E}^\mathcal{G}[R_T f^p] \leq \exp \left\{ \frac{p \sigma^2 - 2}{2(p-1)^2} \left( \frac{1}{T} + K + \frac{K^2 T}{3} \right) \right\}.
\]

It follows from (3.10) and (3.11) that (3.8) holds.

\[\square\]

3.2 Applications of Harnack Inequality

In this subsection, we give some applications of Harnack inequality. Before that, we give some definitions on quasi-invariant linear (nonlinear) expectation as follows.

**Definition 3.4.** Let \( E \) be a linear (nonlinear) expectation, and \( \mathcal{P} \) be a nonlinear operator defined on \( C_b^+(\mathbb{R}^d) \). \( E \) is called a quasi-invariant linear (nonlinear) expectation of \( \mathcal{P} \), if there exists a function \( g \in C_b^+(\mathbb{R}^d) \), such that
\[
E[\mathcal{P} f] \leq E[g f], \quad f \in C_b^+(\mathbb{R}^d).
\]

Moreover, if
\[
E[(\mathcal{P} f)] = E[f], \quad f \in C_b^+(\mathbb{R}^d),
\]
then \( E \) is called an invariant linear (nonlinear) expectation of \( \mathcal{P} \).

To illustrate the above definition, we consider two examples as follows.

**Example 1.** Consider the following Ornstein-Uhlenbeck process driven by \( G \)-Brownian motion: for each \( x \in \mathbb{R}^d \),
\[
Y_t^x = x - \alpha \int_0^t Y_s^x ds + B_t, \quad t \geq 0,
\]
where \( \alpha > 0 \), \( B_t \) is a \( d \)-dimensional \( G \)-Brownian motion. Hu et al. [8, Lemma 3.15] proved that the unique invariant nonlinear expectation for \( G \)-Ornstein-Uhlenbeck process \( Y \) is the \( G \)-normal distribution of \( \sqrt{\frac{1}{2\alpha}} B_t \).

**Example 2.** Besides (H1), assume \( b(0) = b(0) = \sigma(0) = 0 \), then it is easy to check that SDE (1.1) has a unique solution with initial value 0, i.e. \( X_0 = 0 \). Again, Hu et al. [8, Example 3.14] showed that the invariant expectation for \( X_t \) is \( \delta_0 \), which is a invariant linear expectation. However, in this case, the Harnack inequality may not hold since \( \sigma(0) = 0 \).
Now, we consider Harnack-type inequality for
\[ \Phi(\mathcal{P}f(x)) \leq \mathcal{P}\Phi(f)(y) e^{\Psi(x,y)}, \quad f \in C_b^+(\mathbb{R}^d), \] (3.13)
where \( \Phi \) is a nonnegative continuous convex function on \([0, \infty)\) and \( \Psi \) is a nonnegative continuous function on \( \mathbb{R}^d \times \mathbb{R}^d \).

**Theorem 3.5.** Let \( E \) be a quasi-invariant nonlinear expectation of \( \mathcal{P} \) and \( \Phi \) be a strictly increasing function with \( \Phi(\infty) := \lim_{r \to \infty} \Phi(r) = \infty \) such that (3.13) holds.

(1) For any \( H \in C_b^+(\mathbb{R}^d) \), and \( n \geq 1 \), we have
\[ \mathcal{P}H(x) \leq \frac{\Phi^{-1}\left(\frac{E[\Phi(1+nH)g]}{E[e^{-\Psi(x,\cdot)}]}\right)}{n} - 1, \quad x \in \mathbb{R}^d. \]

(2) If \( E \) is a invariant nonlinear expectation of \( \mathcal{P} \), it holds that
\[ \sup_{f \in C_b^+(\mathbb{R}^d), E[\Phi(f)] \leq 1} \Phi(\mathcal{P}f(x)) \leq \frac{1}{E[e^{-\Psi(x,\cdot)}]}. \]

**Proof.**

(1) Applying (3.13) to \( f = 1 + nH \), we have
\[ \Phi(1 + n\mathcal{P}H(x)) \leq \mathcal{P}\Phi(1 + nH)(y) e^{\Psi(x,y)}, \quad x, y \in \mathbb{R}^d. \]
This yields
\[ \Phi(1 + n\mathcal{P}H(x)) e^{-\Psi(x,y)} \leq \mathcal{P}\Phi(1 + nH)(y), \quad x, y \in \mathbb{R}^d. \]
Since \( E \) is a quasi-invariant expectation of \( \mathcal{P} \), it follows from (3.12) that
\[ \Phi(1 + n\mathcal{P}H(x)) \leq \frac{E[\mathcal{P}\Phi(1 + nH)g]}{E[e^{-\Psi(x,\cdot)}]} = \frac{E[\Phi(1 + nH)g]}{E[\Phi(1 + nH)]}. \]
Therefore,
\[ \mathcal{P}H(x) \leq \frac{\Phi^{-1}\left(\frac{E[\Phi(1+nH)g]}{E[e^{-\Psi(x,\cdot)}]}\right)}{n} - 1, \]
where \( \Phi^{-1} \) is the inverse of \( \Phi \).

(2) From (3.13), for any \( f \in C_b^+(\mathbb{R}^d) \), it holds that
\[ e^{-\Psi(x,y)} \Phi(\mathcal{P}f(x)) \leq \mathcal{P}\Phi(f)(y). \]
Taking expectation \( E \) with respect to \( y \) in the above inequality and using the fact that \( E \) is a invariant nonlinear expectation of \( \mathcal{P} \), we obtain, for any \( E[\Phi(f)] \leq 1 \),
\[ \Phi(\mathcal{P}f(x)) \leq \frac{E[\mathcal{P}\Phi(f)]}{E[e^{-\Psi(x,\cdot)}]} = \frac{E[\Phi(f)]}{E[e^{-\Psi(x,\cdot)}]} \leq \frac{1}{E[e^{-\Psi(x,\cdot)}]}, \]
which is the desired result. \( \square \)

**Remark 3.6.** Let us compare the above result with the known ones in the linear expectation setting. In the linear expectation space, \( \mathcal{P} \) can be defined form \( B_b^+(\mathbb{R}^d) \) to \( B_b^+(\mathbb{R}^d) \). Thus, taking \( H = 1_A \) with \( E1_A = 0 \) in Theorem 3.5 (1), we obtain \( \mathcal{P}1_A = 0 \), which means \( \mathcal{P} \) is absolutely continuous with respect to \( E \), i.e., there exists a a function \( 0 \leq g \in B_b(\mathbb{R}^d) \) such that
\[ \mathcal{P}f = E[gf], \quad 0 \leq f \in B_b(\mathbb{R}^d). \]

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