April 21, 2015

Embedding quantum systems with a non-conserved probability in classical environments

Alessandro Sergi

School of Chemistry and Physics, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, Pietermaritzburg, South Africa and KwaZulu-Natal Node, National Institute for Theoretical Physics (NITheP), South Africa

Quantum systems with a non-conserved probability can be described by means of non-Hermitian Hamiltonians and non-unitary dynamics. In this paper, the case in which the degrees of freedom can be partitioned in two subsets with light and heavy masses is treated. A classical limit over the heavy coordinates is taken in order to embed the non-unitary dynamics of the subsystem in a classical environment. Such a classical environment, in turn, acts as an additional source of dissipation (or noise), beyond that represented by the non-unitary evolution. The non-Hermitian dynamics of a Heisenberg two-spin chain, with the spins independently coupled to classical-like harmonic oscillators, is considered in order to illustrate the formalism.

I. INTRODUCTION

Historically, the development of non-Hermitian quantum mechanics started with the study of metastable states and tunneling by means of Hamiltonians with complex energy eigenvalues [1, 2]. Later a connection to $PT$-symmetry was found [3] and the concept of pseudo-hermiticity was also established [4, 5]. Nowadays, this field of research is constantly growing. A first general book on the topic has appeared [6]; applications of non-Hermitian quantum mechanics involve the study of scattering by complex potentials and quantum transport [7–17], description of metastable states [18–23], optical waveguides [24–26], multi-photon ionization [27–29], and nano-photonic and plasmonic waveguides [30]. The theoretical investigations are also undergoing rapid developments: non-Hermitian quantum mechanics has been investigated within a relativistic framework [31] and it has been adopted by various researchers as a means to describe open quantum systems [32–42]. Moreover, it seems that a few theoretical studies have been dedicated to the statistical mechanics

*Electronic address: sergi@ukzn.ac.za
and dynamics of systems with non-Hermitian Hamiltonians [43–54].

In the present work, the interest is focused on the development of a formalism to embed consistently the quantum dynamics of systems with a non-conserved probability in a classical environment, which is explicitly taken into account (i.e., it is not averaged-over) in the dynamics and which, in turn, acts as a source of disorder. Types of noise beyond those arising from Gaussian white noise [41] can then be treated. From a more general perspective, one goal of this work is to develop a numerical formalism (which is complementary to that based on master equations [55]) for studying the dissipative dynamics of, for example, quantum plasmonic metamaterials [56, 57] or processes of interest in quantum thermodynamics [58]. In order to obtain such a formalism, a composite system with heavy and light degrees of freedom is considered. First a classical limit over the heavy degrees of freedom is performed using the partial Wigner representation [59–67]. In such a way, a general quantum-classical approximation of non-Hermitian quantum mechanics is obtained. Finally, the limiting case in which the non-Hermitian part of the evolution does not affect the classical-like degrees of freedom (represented in Wigner phase space [68, 69]) is considered.

This paper is structured as follows. In Sec. II the quantum non-Hermitian equation of motion for the density matrix is taken as a starting point and the classical limit over the heavy mass coordinates is performed by using the partial Wigner transform and the linear expansion in the square root of the ratio between light and heavy masses. From this general case, one can easily derive the equation of motion valid for a decay operator depending only on the quantum degrees of freedom of the subsystem. In Sec. III the Hermitian part of the total Hamiltonian of the system is considered; an adiabatic Hamiltonian is extracted from this and its eigenstates are used to represent the quantum-classical non-Hermitian equation. In Sec. IV piecewise-deterministic algorithms (using the adiabatic basis) are presented explicitly in the case when the decay operator depends only on the quantum coordinates of the subsystem. The numerical approach is illustrated in Sec. V by studying (in the adiabatic approximation) the dynamics of a chain of two spins, coupled separately to an independent harmonic oscillator. Two different decay operators are explicitly treated. The evolution of the trace of the reduced density matrix of the spin chain and of relevant matrix elements is monitored. The results show that non-Hermitian quantum-classical dynamics (and the numerical algorithms developed in this work) can model efficiently the loss of probability and the damping expected in open quantum system. Conclusions are finally given in Sec. VI.
II. NON-HERMITIAN QUANTUM MECHANICS IN A CLASSICAL BATH

Consider a composite quantum system with quantum coordinates $\hat{r}, \hat{p}, \hat{R}, \hat{P} = (\hat{x}, \hat{X})$. A multidimensional notation will be adopted in the following so that, for example, $\hat{R}$ stands for $(\hat{R}_1, \hat{R}_2, ..., \hat{R}_N)$, where $N$ is the total number of degrees of freedom in configurational space of the subsystem represented by the operators $\hat{X}$. It is also assumed that the dynamics of the composite system is defined by the non-Hermitian Hamiltonian

$$\hat{H} = \hat{H} - i\hat{\Gamma},$$

where $\hat{H} = (\hat{H} + \hat{H}^\dagger)/2$ and $\hat{\Gamma} = i(\hat{H} - \hat{H}^\dagger)/2$ are Hermitian operators. In particular, in order to develop the formalism, one can assume that

$$\hat{H} = \frac{\hat{P}^2}{2M} + \frac{\hat{p}^2}{2m} + \hat{V}(\hat{r}, \hat{R}),$$

while $\hat{\Gamma}$ is left unspecified. In the following, it will also be assumed that $M$, the mass associated to the sub-system with coordinates $\hat{X}$, is much bigger than $m$, the mass associated to the subsystem with coordinates $\hat{x}$, i.e., $M \gg m$. This leads to the definition of the small parameter $\mu = (m/M)^{1/2}$. The non-normalized density matrix $\hat{\Omega}(t)$ of the composite system with non-Hermitian Hamiltonian $\hat{H}$ obeys the equation of motion

$$\frac{\partial}{\partial t} \hat{\Omega}(t) = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\Omega}(t) \right] - \frac{1}{\hbar} \left[ \hat{\Gamma}, \hat{\Omega}(t) \right] + ,$$

where $[..., ...]_-$ and $[..., ...]_+$ are the commutator and anticommutator, respectively.

In order to obtain the quantum-classical limit of Eq. (3) a rigorous procedure, which is based on the partial Wigner representation of the dynamics and the linear expansion in $\mu$, can be followed. Such a procedure was used for Hermitian Hamiltonians in Ref. [64]. Accordingly, one can introduce the partial Wigner transform of $\hat{\Omega}$ over only the coordinates of the heavy degrees of freedom:

$$\hat{\Omega}_W(X, t) = \frac{1}{(2\pi\hbar)^N} \int dZ e^{iP \cdot Z/\hbar} \langle R - Z/2|\hat{\Omega}(t)|R + Z/2 \rangle .$$

As a results $\hat{\Omega}_W(X, t)$ is an operator in terms of the quantum $\hat{x}$ variables and a function in terms of the $X$ variables (which are still quantum - they have only been represented in the Wigner quantum phase space). Analogously, an arbitrary quantum operator $\hat{\chi}$ of the composite system is partially transformed in Wigner phase space as

$$\hat{\chi}_W(X) = \int dZ e^{iP \cdot Z/\hbar} \langle R - Z/2|\hat{\chi}|R + Z/2 \rangle .$$
Moreover, the partial Wigner transform of a product of arbitrary operators \( \hat{\chi} \) and \( \hat{\xi} \) is given by

\[
(\hat{\chi}\hat{\xi})_W(X) \equiv \hat{\chi}_W(X)e^{\frac{i}{\hbar} \nabla B \nabla} \hat{\xi}_W(X),
\]

where \( \nabla = ((\partial/\partial R), (\partial/\partial P)) \) is the phase space gradient operator and

\[
B = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

is the symplectic matrix, so that \( \nabla B \) is basically the Poisson bracket, which it will also be denoted with the symbol \{...,\}.

Upon taking the partial Wigner transform of Eq. (3), one obtains

\[
\frac{\partial}{\partial t} \hat{\Omega}_W(X, t) = -\frac{i}{\hbar} \left( \hat{H}_W(X)e^{\frac{i}{\hbar} \nabla B \nabla} \hat{\Omega}_W(X, t) - \hat{\Omega}_W(X, t)e^{\frac{i}{\hbar} \nabla B \nabla} \hat{H}_W(X) \right)
- \frac{1}{\hbar} \left( \hat{\Gamma}_W(X)e^{\frac{i}{\hbar} \nabla B \nabla} \hat{\Omega}_W(X, t) - \hat{\Omega}_W(X, t)e^{\frac{i}{\hbar} \nabla B \nabla} \hat{\Gamma}_W(X) \right).
\]

Equation (8) is still fully quantum in nature. Now, in order to take the quantum classical limit one can follow the procedure of Ref. [64], which was inspired by the theory of Brownian motion given in [73], and introduce scaled coordinates. Arbitrary units can be defined in the following way: one can introduce \( \epsilon_0 \) as unit of energy, \( t_0 = \hbar/\epsilon_0 \) as unit of time, and \( \lambda_0 = \hbar/(m\epsilon_0)^{1/2} \) as unit of length. One can also define \( p_0 = (m\epsilon_0)^{1/2} \) and \( P_0 = (M\epsilon_0)^{1/2} \) as the unit of the light and heavy momenta, respectively. As in [64], adimensional coordinates, making the momenta of the same order of magnitude, can be introduced:

\[
\hat{r}' = \frac{\hat{r}}{\lambda_0}, \quad \hat{R}' = \frac{\hat{R}}{\lambda_0},
\]

\[
\hat{p}' = \frac{\hat{p}}{p_0}, \quad \hat{P}' = \frac{\hat{P}}{P_0}.
\]

Accordingly, one has \( \hat{H}_W(\hat{x}, X) = \epsilon_0 \hat{H}'_W(\hat{x}, X) \) and \( \hat{\Gamma}_W(\hat{x}, X) = \epsilon_0 \hat{\Gamma}'_W(\hat{x}, X) \). In the scaled coordinates given in Eq. (9), Eq. (8) becomes

\[
\frac{\partial}{\partial t'} \hat{\Omega}'_W(X', t') = -\frac{i}{\hbar} \left( \hat{H}'_W(X')e^{\frac{i}{\hbar} \nabla' B \nabla'} \hat{\Omega}'_W(X', t') \right)
- \frac{1}{\hbar} \left( \hat{\Gamma}'_W(X')e^{\frac{i}{\hbar} \nabla' B \nabla'} \hat{\Omega}'_W(X', t') \right).
\]
Equation (10) is still fully quantum. However, now one can take advantage of the smallness of $\mu$ and expand the exponential operators retaining only the linear order. One obtains

$$\frac{\partial}{\partial t'} \hat{\Omega}_W(X', t') = -i \left[ \hat{H}'_W(X'), \hat{\Omega}_W(X', t') \right] - \left[ \hat{\Gamma}'_W(X'), \hat{\Omega}_W(X', t') \right] +$$

$$+ \frac{\mu}{2} B_{ab} \left( \nabla'_a \hat{H}'_W(X') \right) \nabla'_b \hat{\Omega}_W(X', t')$$

$$- \frac{\mu}{2} B_{ab} \left( \nabla'_a \hat{\Gamma}'_W(X') \right) \nabla'_b \hat{\Omega}_W(X', t')$$

$$- \frac{i\mu}{2} B_{ab} \left( \nabla'_a \hat{\Gamma}'_W(X') \right) \nabla'_b \hat{\Gamma}_W(X'),$$

(11)

where the sum over the repeated indices is implied. Equation (11) gives the quantum-classical approximation to Eqs. (3) and (8) in adimensional coordinates. Transforming back to fully dimensional variables, one finally obtains

$$\frac{\partial}{\partial t} \hat{\Omega}_W(X, t) = -\frac{i}{\hbar} \left[ \hat{H}_W(X), \hat{\Omega}_W(X, t) \right] - \frac{1}{\hbar} \left[ \hat{\Gamma}_W(X), \hat{\Omega}_W(X, t) \right] +$$

$$+ \frac{1}{2} \left\{ \left\{ \hat{H}_W(X), \hat{\Omega}_W(X, t) \right\} - \left\{ \hat{\Omega}_W(X, t), \hat{H}_W(X) \right\} \right\} -$$

$$- \frac{i}{2} \left\{ \left\{ \hat{\Gamma}_W(X), \hat{\Omega}_W(X, t) \right\} + \left\{ \hat{\Omega}_W(X, t), \hat{\Gamma}_W(X) \right\} \right\}. \quad (12)$$

Equation (12) provides the rigorous quantum-classical approximation for the non-Hermitian dynamics of composite systems. It is a valid approximation when the degrees of freedom of the system have two different De Broglie wavelengths, one short and one long. It is worth noting that, formally, Eq. (12) could have been obtained directly from Eq. (8) by taking an expansion to linear order in the limit $\hbar \to 0$, without going through the transformation to scaled coordinates given in Eq. (9). However, the $\mu$ expansion seems much more rigorous since $\hbar \to 0$ would imply a cancelation of quantum effects also on the $\hat{x}$ coordinates. Nevertheless, the practical agreement of the two limiting procedures indicates that the linear form in Eq. (12) can perhaps simply assumed as an ansatz for quantum-classical dynamics, as suggested in Ref. [74]. It is also very important to remark that Eq. (12) has a form that does not depend on any particular basis. Moreover, Eq. (12) is exact when the decay operator $\hat{\Gamma}_W(X)$ is linear in $X$, $\hat{H}_W(X)$ is at most quadratic in the $X$ coordinates and is linearly coupled through the $X$s with the quantum subsystem.

While Eq. (12) defines non-Hermitian quantum dynamics in a classical bath in the case of a general decay operator $\hat{\Gamma}_W(X)$, there is one interesting limiting situation that can be considered. It concerns the case in which the decay operator does not depend on the bath coordinates. When this happens the partial Wigner transform leaves $\hat{\Gamma}$ invariant, so that Eq. (12) reduces to

$$\frac{\partial}{\partial t} \hat{\Omega}_W(X, t) = -\frac{i}{\hbar} \left[ \hat{H}_W(X), \hat{\Omega}_W(X, t) \right] - \frac{1}{\hbar} \left[ \hat{\Gamma}, \hat{\Omega}_W(X, t) \right] +$$
\[ + \frac{1}{2} \left\{ \hat{H}_W(X), \hat{\Omega}_W(X,t) \right\} - \left\{ \hat{\Omega}_W(X,t), \hat{H}_W(X) \right\} . \]  

Equation (13) shows that, in this case, the effects on the subsystem dynamics arise from the anticommutator of \( \hat{\Gamma} \) and \( \hat{\Omega}_W(X) \) alone. Equation (13) might describe a situation in which two types of effect are present: the non-Hermitian dynamics, with its probability leakage or pumping, of a quantum subsystem embedded in a bath of classical degrees of freedom, whose influence is expressed through the Poisson bracket terms in the right hand side. If also the density matrix does not depend on the bath coordinates \( X, \hat{\Omega}_W(X,t) \to \hat{\Omega}(t) \), one obtains the purely quantum case, given by Eq. (3). Such a result constitutes a self-consistency check for the formalism.

### III. REPRESENTATION IN THE ADIABATIC BASIS

The partial Wigner transform of the Hamiltonian in Eq. (2) can be rewritten as

\[ \hat{H}_W(X) = \frac{P^2}{2M} + \hat{h}_W(R) . \]  

The adiabatic basis of \( \hat{H}_W(X) \) is defined by the eigenvalue problem

\[ \hat{h}_W|\alpha; R \rangle = E_{\alpha}(R)|\alpha; R \rangle . \]  

Such a basis can be used to represent the Eq. (12). Upon defining the quantities

\[ \Omega_{W}^{\alpha \alpha'}(X,t) = \langle \alpha; R|\hat{\Omega}_W(X,t)|\alpha'; R \rangle , \]  
\[ \omega_{\alpha \alpha'} = \frac{E_{\alpha}(R) - E_{\alpha'}(R)}{\hbar} , \]  
\[ F_{W}^{\alpha} = -\frac{\partial E_{\alpha}(R)}{\partial R} , \]  
\[ iL_{\alpha \alpha'} = \frac{P}{M} \cdot \frac{\partial }{\partial R} + \frac{1}{2}(F_{W}^{\alpha} + F_{W}^{\alpha'}) \cdot \frac{\partial }{\partial P} , \]  
\[ d_{\alpha \alpha'} = \langle \alpha; R|\frac{\partial }{\partial R}|\alpha'; R \rangle , \]  
\[ S_{\alpha \beta} = \left( \frac{P}{M} \cdot d_{\alpha \beta} \right)^{-1} \hbar \omega_{\alpha \beta} d_{\alpha \beta} , \]  
\[ T_{\alpha \alpha', \beta \beta'} = \delta_{\alpha \alpha'} \delta_{\beta \beta'} \frac{P}{M} \cdot d_{\alpha \beta} \left( 1 + \frac{1}{2} S_{\alpha \beta} \cdot \frac{\partial }{\partial P} \right) + \delta_{\alpha \beta} \delta_{\alpha' \beta'} \frac{P}{M} \cdot d_{\alpha' \beta'}^* \left( 1 + \frac{1}{2} S_{\alpha' \beta'}^* \cdot \frac{\partial }{\partial P} \right) , \]  
\[ iL_{0}^{\alpha \alpha', \beta \beta'} = (i \omega_{\alpha \alpha'} + iL_{\alpha \alpha'}) \delta_{\alpha \beta} \delta_{\alpha' \beta'} , \]  
\[ iL_{\alpha \alpha', \beta \beta'} = iL_{\alpha \alpha', \beta \beta'}^{(0)} + T_{\alpha \alpha', \beta \beta'} , \]
and using the results of Ref. [64], providing the representation of the first, third and fourth terms in the right hand side of Eq. (12), one obtains for Eq. (12):

\[
\frac{\partial}{\partial t} \Omega^\alpha_{\bar{W}} = - \sum_{\beta \gamma} iL_{\alpha \alpha', \beta \gamma} \Omega^\beta_{\bar{W}} - \frac{1}{\hbar} \langle \alpha; R | \left[ \hat{\Gamma}_W, \hat{\Omega}_W(X, t) \right] | \alpha'; R \rangle + \frac{i}{2} \langle \alpha; R | \left\{ \hat{\Gamma}_W, \hat{\Omega}_W(X, t) \right\} | \alpha'; R \rangle. \tag{25}
\]

In order to proceed with the derivation, one has to consider the term

\[
\langle \alpha; R | [\hat{\Gamma}_W, \hat{\Omega}_W(X, t)] | \alpha'; R \rangle = \sum_{\beta \gamma} \left( \Gamma^\alpha_{\bar{W}} \delta_{\alpha \beta} \Omega^\beta_{\bar{W}} (X, t) - \frac{i}{2} \langle \alpha; R | \left\{ \hat{\Gamma}_W, \hat{\Omega}_W(X, t) \right\} | \alpha'; R \rangle \right). \tag{26}
\]

Using Eq. (26), the equation of motion in (25) becomes

\[
\frac{\partial}{\partial t} \Omega^\alpha_{\bar{W}} = - \sum_{\beta \gamma} iL_{\alpha \alpha', \beta \gamma} \Omega^\beta_{\bar{W}} - \frac{1}{2\hbar} \sum_{\beta \gamma} \left( \Gamma^\alpha_{\bar{W}} \delta_{\alpha \beta} \Omega^\beta_{\bar{W}} (X, t) - \frac{i}{2} \langle \alpha; R | \left\{ \hat{\Gamma}_W, \hat{\Omega}_W(X, t) \right\} | \alpha'; R \rangle \right). \tag{27}
\]

One can then consider the third term in the right hand side of Eq. (27):

\[
\langle \alpha; R | \left\{ \hat{\Gamma}_W, \hat{\Omega}_W(X, t) \right\} | \alpha'; R \rangle = \sum_{\beta \gamma} \left( \langle \alpha; R | \partial \hat{\Gamma}_W / \partial R | \beta \gamma; R \rangle \partial \Omega^\alpha_{\bar{W}} (X, t) / \partial P - \frac{\partial \Gamma^\alpha_{\bar{W}}}{\partial P} (\gamma; R | \partial \hat{\Omega}_W(X, t) / \partial R | \alpha'; R) \right). \tag{28}
\]

In order to simplify further Eq. (28), one can consider the identities

\[
\langle \alpha; R | \partial \hat{\Gamma}_W / \partial R | \beta \gamma; R \rangle = \frac{\partial \Gamma^\alpha_{\bar{W}}}{\partial R} - \sum_{\sigma} \sigma^*_{\alpha \gamma} \Gamma^\alpha_{\bar{W}} - \sum_{\sigma} \Gamma^\sigma_{\bar{W}} d_{\sigma \gamma}, \tag{29}
\]

\[
\langle \gamma; R | \partial \hat{\Omega}_W / \partial R | \alpha'; R \rangle = \frac{\partial}{\partial R} \Omega^\alpha_{\bar{W}} - \sum_{\sigma} \sigma^*_{\sigma \alpha} \Omega^\sigma_{\bar{W}} - \sum_{\sigma} d_{\sigma \alpha} \Omega^\sigma_{\bar{W}}. \tag{30}
\]

Using Eqs. (29,30), Eq. (28) becomes

\[
\langle \alpha; R | \left\{ \hat{\Gamma}_W, \hat{\Omega}_W(X, t) \right\} | \alpha'; R \rangle = \sum_{\beta \gamma} \left( \frac{\partial \Gamma^\alpha_{\bar{W}}}{\partial R} \frac{\partial}{\partial P} \delta_{\alpha \beta} \Omega^\beta_{\bar{W}} (X, t) - \frac{\partial \Gamma^\alpha_{\bar{W}}}{\partial P} (\delta_{\alpha \beta} \Omega^\beta_{\bar{W}} (X, t) - \sum_{\sigma} \sigma^*_{\sigma \alpha} \frac{\partial}{\partial P} \Omega^\sigma_{\bar{W}} - \sum_{\sigma} d_{\sigma \alpha} \frac{\partial}{\partial P} \right). \tag{29}
\]
\[
\begin{align*}
&+ \Gamma_{W}^{\alpha \beta} d_{\sigma \beta} \delta_{\alpha', \beta'} \left( \frac{\partial}{\partial P} \right) \Omega_{W}^{\beta'} (X, t) \\
&+ \sum_{\beta'} \left( \sum_{\sigma} \frac{\partial \Gamma_{W}^{\alpha \sigma}}{\partial P} d_{\beta \sigma} \delta_{\alpha', \beta'} \right) \Omega_{W}^{\beta'} (X, t) \\
&+ \frac{\partial \Gamma_{W}^{\alpha \beta}}{\partial P} d_{\beta \alpha'} \right) \Omega_{W}^{\beta'} (X, t) . (31)
\end{align*}
\]

Using Eq. (31), Eq. (27) becomes

\[
\frac{\partial}{\partial t} \Omega_{W}^{\beta \alpha'} = - \sum_{\beta'} i \mathcal{L}_{\alpha \beta', \beta} \Omega_{W}^{\beta \beta'} - \frac{1}{\hbar} \sum_{\beta'} \left( \Gamma_{W}^{\alpha \beta} \delta_{\alpha', \beta'} + \Gamma_{W}^{\beta' \alpha'} \delta_{\alpha \beta} \right) \Omega_{W}^{\beta'} (X, t)
\]

\[
- \frac{i}{2} \sum_{\beta} \left( \frac{\partial \Gamma_{W}^{\alpha \beta}}{\partial R} \delta_{\alpha', \beta'} \left( \frac{\partial}{\partial P} \Omega_{W}^{\beta \beta'} \right) - \frac{\partial \Gamma_{W}^{\alpha \beta}}{\partial P} \delta_{\alpha', \beta'} \left( \frac{\partial}{\partial R} \Omega_{W}^{\beta \beta'} \right) \right)
\]

\[
- \sum_{\sigma} \left( d_{\sigma \alpha} \Gamma_{W}^{\alpha \beta} \delta_{\alpha', \beta'} \frac{\partial}{\partial P} + \Gamma_{W}^{\alpha \beta} d_{\sigma \beta} \delta_{\alpha', \beta'} \frac{\partial}{\partial P} \right)
\]

\[
+ \sum_{\beta} \left( \frac{\partial \Gamma_{W}^{\alpha \beta}}{\partial P} d_{\beta \alpha'} + \frac{\partial \Gamma_{W}^{\alpha \beta}}{\partial P} d_{\beta \alpha'} \right) \Omega_{W}^{\beta \beta'} (X, t)
\]

\[
- \frac{i}{2} \langle \alpha; R | \left\{ \hat{\Omega}_{W}(X, t), \hat{\Gamma}_{W} \right\} | \alpha'; R \rangle . (32)
\]

In order to complete the derivation, one must consider the last term in the right hand side of Eq. (32):

\[
\langle \alpha; R | \left\{ \hat{\Omega}_{W}(X, t), \hat{\Gamma}_{W} \right\} | \alpha'; R \rangle = \sum_{\gamma} \langle \alpha; R | \frac{\partial \hat{\Omega}_{W}(X, t)}{\partial R} | \gamma; R \rangle \frac{\partial \Gamma_{W}^{\gamma \alpha'}}{\partial P} - \sum_{\gamma} \frac{\partial \hat{\Omega}_{W}(X, t)}{\partial P} \delta_{\alpha \gamma} \langle \gamma; R | \frac{\partial \Gamma_{W}^{\gamma \alpha'}}{\partial R} | \alpha'; R \rangle . (33)
\]

To further simplify Eq. (33), one has to consider again (just with different indices) the identities given in Eqs. (29-30). Using Eqs. (29-30), Eq. (33) becomes

\[
\langle \alpha; R | \left\{ \hat{\Omega}_{W}(X, t), \hat{\Gamma}_{W} \right\} | \alpha'; R \rangle = \sum_{\beta \beta'} \left( \frac{\partial \Gamma_{W}^{\beta' \alpha'}}{\partial P} \delta_{\alpha \beta} \frac{\partial}{\partial R} \right) \hat{\Omega}_{W}^{\beta'} (X, t)
\]

\[
- \sum_{\beta \beta'} \left( \frac{d_{\beta \alpha}}{\partial \Gamma_{W}^{\beta' \alpha'}} \frac{\partial}{\partial P} \hat{\Omega}_{W}^{\beta'} (X, t) \right)
\]

\[
+ \sum_{\beta \beta'} \left( \frac{d_{\beta \alpha}}{\partial \Gamma_{W}^{\beta' \alpha'}} \delta_{\alpha \beta} \right) \hat{\Omega}_{W}^{\beta'} \Omega_{W}^{\beta' \beta'}
\]

\[
+ \sum_{\sigma} \left( \frac{d_{\sigma \beta}}{\partial \Gamma_{W}^{\beta' \alpha'}} \delta_{\alpha \beta} \right) \hat{\Omega}_{W}^{\beta'} \Omega_{W}^{\beta' \beta'}
\]
Using Eq. (34), Eq. (32) becomes

\[
\frac{\partial}{\partial t} \Omega_{W}^{\alpha\alpha'} = - \sum_{\beta\beta'} i \mathcal{L}_{\alpha\alpha',\beta\beta'} \Omega_{W}^{\beta\beta'} - \frac{1}{\hbar} \sum_{\beta\beta'} \left( \Gamma_{W}^{\alpha\beta} \delta_{\alpha'\beta'} + \Gamma_{W}^{\beta\alpha'} \delta_{\alpha\beta} \right) \Omega_{W}^{\beta\beta'}(X, t) \\
- \frac{i}{2} \sum_{\beta\beta'} \left[ \left( \frac{\partial \Gamma_{W}^{\alpha\beta}}{\partial R} \delta_{\alpha'\beta'} - \frac{\partial \Gamma_{W}^{\beta\alpha'}}{\partial R} \delta_{\alpha\beta} \right) \frac{\partial}{\partial P} \right] \\
+ \left( \frac{\partial \hat{\Gamma}_{W}^{\beta\alpha'}}{\partial P} \delta_{\alpha\beta} - \frac{\partial \hat{\Gamma}_{W}^{\alpha\beta}}{\partial P} \delta_{\alpha'\beta'} \right) \frac{\partial}{\partial R} \\
- \sum_{\sigma} \left( d_{\sigma\alpha}^{*} \Gamma_{W}^{\sigma\beta} \delta_{\alpha'\beta'} + \Gamma_{W}^{\sigma\alpha'} d_{\sigma\beta} \delta_{\alpha\beta} \right) \frac{\partial}{\partial P} \\
+ d_{\sigma\beta}^{*} \Gamma_{W}^{\sigma\alpha} \delta_{\alpha\beta} + \Gamma_{W}^{\beta\sigma} d_{\sigma\alpha} \delta_{\alpha'\beta'} \right) \frac{\partial}{\partial P} \\
+ \sum_{\sigma} \left( \frac{\partial \Gamma_{W}^{\sigma\alpha}}{\partial P} d_{\sigma\alpha}^{*} \delta_{\alpha'\beta'} - d_{\sigma\beta}^{*} \frac{\partial \hat{\Gamma}_{W}^{\sigma\alpha'}}{\partial P} \delta_{\alpha\beta} \right) \\
+ \frac{\partial \Gamma_{W}^{\beta\alpha}}{\partial P} d_{\beta\alpha} - d_{\beta\alpha}^{*} \frac{\partial \hat{\Gamma}_{W}^{\beta\alpha'}}{\partial P} \right] \hat{\Omega}_{W}^{\beta\beta'}.
\]  

Equation (35) provides the searched result: it is the representation of the quantum-classical non-Hermitian dynamics, given by Eq. (12), in the adiabatic basis of \( \hat{H}_{W}(X) \). As it stands, it looks quite formidable and not amenable of being numerically integrated in a simple way.

When the decay operator depends only on the quantum coordinates of the subsystem, one has to consider the representation of Eq. (13) in the adiabatic basis. It can be verified that Eq. (35) reduces to

\[
\frac{\partial}{\partial t} \Omega_{W}^{\alpha\alpha'}(X, t) = - \sum_{\beta\beta'} i \mathcal{L}_{\alpha\alpha',\beta\beta'} \Omega_{W}^{\beta\beta'}(X, t) \\
- \frac{1}{\hbar} \sum_{\beta\beta'} \left( \Gamma_{W}^{\alpha\beta} \delta_{\alpha'\beta'} + \Gamma_{W}^{\beta\alpha'} \delta_{\alpha\beta} \right) \Omega_{W}^{\beta\beta'}(X, t).
\]  

Equation (36) is much simpler than the general Eq. (35); it can be used to model dissipative effects on a quantum subsystems arising both from sources and sinks of probability and disorder because of the coupling to a classical environment. Equation (36) can be integrated by means of piecewise-deterministic algorithms. These will be sketched in Sec. IV.

IV. PIECEWISE-DETERMINISTIC ALGORITHMS

Equation (34), which is the adiabatic representation of the abstract Eq. (12), can be numerically integrated by means of piecewise-deterministic algorithms. However, given the complexity of
Eq. (35), only the limiting case given by Eq. (36) will be explicitly considered here.

Consider first Eq. (36). In such a case, it is convenient to decompose the representation of the decay operator in the adiabatic basis in terms of a diagonal, $\Gamma_{d}^{\alpha\alpha}$, and an off-diagonal part, $\Gamma_{o}^{\alpha\beta}$:

$$\Gamma^{\alpha\beta} = \Gamma_{d}^{\alpha\alpha} + \Gamma_{o}^{\alpha\beta}.$$

Equation (36) becomes

$$\frac{\partial}{\partial t}\Omega_{W}^{\alpha',\beta'} = -\sum_{\beta''} (iL_{\alpha',\beta'}^{(\gamma)} + T_{\alpha',\beta'}^{\Gamma} + T_{\alpha',\beta'}^{\Gamma}) \delta_{\alpha'\beta'} \Delta t \omega_{\alpha'\alpha'},$$

where one has defined

$$iL_{\alpha',\beta'}^{(\gamma)} = i\omega_{\alpha'\alpha'} + \gamma_{\alpha'\alpha'} + iL_{\alpha'\alpha'}.$$

A piecewise-deterministic algorithm for the integration of Eq. (41) can be found by using the sequential short-time propagation (SSTP) scheme \cite{66, 75}. A trajectory can be seen as the concatenation of small finite time steps $\Delta t$. Accordingly, for a single step, the propagator associated with Eq. (41) can be written as

$$e^{-i\Delta t(L^{(\gamma)} + T^{\Gamma})}_{\alpha',\beta'} \approx e^{-i\Delta tL^{(\gamma)}_{\alpha',\alpha'} + \Delta tT_{\alpha',\beta'}^{\Gamma}}.$$

The propagator decomposition in Eq. (43) can be used as the basis for a SSTP algorithm for integrating Eq. (41). The action of $T_{\alpha\beta}^{\alpha',\beta'}$ and $T_{\alpha\beta}^{\Gamma}$ must be sampled using either basic \cite{66, 75} or more advanced schemes \cite{76, 78} for long-time stable integration. The momentum-jump approximation can be adopted in the expression of $T_{\alpha\beta}^{\alpha',\beta'}$ and, moreover, the damping (or enhancing) frequency $\gamma_{\alpha'\alpha'}$ must be considered in the action of $iL_{\alpha',\alpha'}^{(\gamma)}$. 
V. NON-HERMITIAN SPIN CHAIN IN HARMONIC BATHS

Consider a subsystem described by the Hermitian Hamiltonian

\[ \hat{H}_S = -j_x \hat{\sigma}_x^{(s_1)} \hat{\sigma}_x^{(s_2)} - j_y \hat{\sigma}_y^{(s_1)} \hat{\sigma}_y^{(s_2)} - j_z \hat{\sigma}_z^{(s_1)} \hat{\sigma}_z^{(s_2)}, \]  

which represents a chain of two coupled quantum spins, \( s_k, k = 1, 2 \). The constants \( j_\ell \), with \( \ell = x, y, z \), determine the spin coupling strength. The operators \( \hat{\sigma}_\ell^{(s_k)} \) are given by the Pauli matrices for spin \( k \).

The excited and ground state of the spins are denoted by \( |e^{(s_k)}> \) and \( |g^{(s_k)}> \) (\( k = 1, 2 \)), respectively. As in Ref. [79], the subsystem basis is defined by the following vectors:

\[ |1> = |e^{(s_1)}, e^{(s_2)}>, |2> = |e^{(s_1)}, g^{(s_2)}>, |3> = |g^{(s_1)}, e^{(s_2)}>, |4> = |g^{(s_1)}, g^{(s_2)}>. \]

The bath is composed by two harmonic oscillators and has the following partially Wigner-transformed Hamiltonian:

\[ H_{B,W} = \sum_{k=1}^{2} \left( \frac{P_{(s_k)}^2}{2M} + \frac{M \omega^2}{2} R_{(s_k)}^2 \right). \]  

Equation (45) provides the partially Wigner-transformed Hamiltonian of two independent harmonic oscillators with mass \( M \) and frequency \( \omega \). Oscillator 1 is coupled to spin 1 while oscillator 2 is coupled to spin 2. The corresponding Hamiltonian (in the partial Wigner representation) is

\[ \hat{H}_{SB,W} = -2 \sum_{k=1}^{2} cR_{(s_k)} \hat{\sigma}_z^{(s_k)}. \]  

The total partially Wigner-transformed Hermitian Hamiltonian of the system is (of course) given by \( \hat{H}_W(X) = \hat{H}_S + \hat{H}_{SB,W} + H_{B,W} \). Since the total bath is harmonic and the coupling with the spin chain is bilinear, the linear approximation of the partially Wigner represented dynamics is exact. This means that the classical-like representation of the bath in Wigner space is, in fact, fully quantum in nature.

In order to illustrate the numerical implementation of the formalism, two decay operators are considered:

\[ \hat{\Gamma}^{(1)} = \gamma_1 \hat{I}, \]  
\[ \hat{\Gamma}^{(2)} = \gamma_2 |e^{(s_1)}, e^{(s_2)}><e^{(s_1)}, e^{(s_2)}|. \]

The symbol \( \hat{I} \) denotes the identity operator in the Hilbert space of the spin chain while \( \gamma_j, j = 1, 2 \), are constants. The dynamics of the density matrix is determined by substituting \( \hat{\Gamma}^{(j)}, j = 1, 2 \), into Eq. (13). The adiabatic basis representation of Eqs. (13) has been given in Sec. III. Non-adiabatic corrections to the dynamics can be disregarded upon assuming a weak coupling to the
FIG. 1: Adiabatic time evolution of the trace of the reduced density matrix, $\text{Tr}[\Omega_S]$, for $\beta = 0.1$, $j_x = j_y = -1$, $j_z = 0.5$, $c = 0.24$. The numerical time step of integration is $\Delta t = 0.01$. Adimensional parameters are used. Results for non-Hermitian dynamics with decay operator $\hat{\Gamma}^{(1)} = \gamma_1 \hat{1}$ and initial reduced density matrix $\hat{\Omega}_S(t_0) = |\Phi><\Phi|$, with $|\Phi| = |e^{(s_1)}, g^{(s_2)}>$. The upper curve shows the results for $\gamma_1 = 0$ (Hermitian dynamics). Then, from top to bottom, the curves for $\gamma_1 = 0.1$, $0.5$, $1$ are displayed. All curves are drawn with statistical error bars.

environment: This means that the transition operator $T_{\alpha\alpha';\beta\beta'}$ in Eq. (41) can be neglected. If $\Delta t$ is the numerical integration step, a single-step SSTP propagator in the adiabatic approximation is written as

$$e^{-i\Delta t L^{(\gamma)}_{\alpha\alpha'}} \delta_{\alpha\beta} \delta_{\alpha'\beta'} = e^{-i \int_0^{\Delta t} d\tau \omega_{\alpha\alpha'}(\tau)} e^{-\frac{1}{\hbar} \int_0^{\Delta t} d\tau \gamma_{\alpha\alpha'}(\tau)} e^{-i \Delta t L_{\alpha\alpha'}} \delta_{\alpha\beta} \delta_{\alpha'\beta'} .$$

(49)

where $\gamma_{\alpha\alpha'}$ is defined in Eq. (39). The right hand side of Eq. (49) can be derived by means of the Dyson identity, as explained in Ref. [64]. In the calculations reported either $\hat{\Gamma}^{(1)}$ or $\hat{\Gamma}^{(2)}$ have been used to obtain $\gamma_{\alpha\alpha'}$, depending on the case.

In order to perform the numerical study, the density matrices of the subsystem $\hat{\Omega}_S(t)$ and of the oscillators $\Omega_{B,W}(X, t)$ are considered initially uncorrelated:

$$\hat{\Omega}_W(X, t_0) = \hat{\Omega}_S(t_0) \otimes \Omega_{B,W}(X, t_0) ,$$

(50)

where

$$\Omega_{B,W}(X, t_0) = \prod_{k=1}^{2} \frac{\tanh(\beta \omega/2)}{\pi} \exp \left[ -2 \frac{\tanh(\beta \omega/2)}{\omega} H_{B,W}(X) \right] ,$$

(51)

with $\beta = 1/k_B T$ inverse thermodynamics temperature ($k_B$ denotes the Boltzmann’s constants) and $H_{B,W}(X)$ is defined in Eq. (45). The initial condition for the reduced density matrix of the spin chain has been chosen as $\hat{\Omega}_S(t_0) = |\Phi><\Phi|$ with $|\Phi| = |e^{(s_1)}, g^{(s_2)}>$ when using $\hat{\Gamma}^{(1)}$ and as $\hat{\Omega}_S(0) = |\Psi><\Psi|$, with $|\Psi| = (1/\sqrt{2}) (|e^{(s_1)}, e^{(s_2)}> - |e^{(s_1)}, g^{(s_2)}>)$ when using $\hat{\Gamma}^{(2)}$. Upon
FIG. 2: Adiabatic time evolution of matrix element $\Omega_{22}^S$, in the subsystem basis, for $\beta = 0.1$, $j_x = j_y = -1$, $j_z = 0.5$, $c = 0.24$. The numerical time step of integration is $\Delta t = 0.01$. Adimensional parameters are used. Results for non-Hermitian dynamics with decay operator $\Gamma^{(1)}_W = \gamma_1 \hat{1}$ and initial reduced density matrix $\hat{\Omega}_S(t_0) = |\Phi><\Phi|$, with $|\Phi> = |e(s_1), g(s_2)>$. The upper curve displays the results for $\gamma_1 = 0$ (Hermitian Dynamics). Then, from top to bottom, the curves for $\gamma_1 = 0.1, 0.5, 1$ are displayed. Starting from the top curve, a constant shift of 1.5 in the negative $y$ direction has been applied for visualization purposes. All curves are drawn with statistical error bars.

choosing $\omega$ as reference frequency, one can introduce a unit of energy, $\hbar \omega$, a dimensionless time, $t \rightarrow t/\omega$, and an inverse thermodynamical temperature, $\beta \rightarrow \beta/\hbar \omega$. One can also introduce the adimensional coordinates $R \rightarrow (\hbar/M\omega)^{1/2} R$ and $P \rightarrow (\hbar\omega M)^{1/2} P$. Correspondingly, one can use the following adimensional constants $j_\ell \rightarrow j_\ell/\hbar \omega (\ell = x, y, z)$, $c \rightarrow c/(\hbar M \omega)^2$, $\gamma_1 \rightarrow \gamma_1/\hbar \omega$, $\gamma_2 \rightarrow \gamma_2/\omega (\hbar M \omega)^{1/2}$ and $\gamma_3 \rightarrow \gamma_3/\hbar M \omega^2$. The values adopted in the calculations have been $\beta = 0.1$, $j_x = j_y = -1$, $j_z = 0.5$, and $c = 0.24$, and $\gamma_2 = 5$. Phase space averages with negligible statistical errors have been calculated using $5 \times 10^4$ points.

Figure 1 displays the adiabatic time evolution of the trace of the reduced density matrix of the spin chain, $\text{Tr}[\hat{\Omega}_S]$, when the decay operator is $\Gamma^{(1)}_W = \gamma_1 \hat{1}$. and initial reduced density matrix $\hat{\Omega}_S(0) = |\Phi><\Phi|$, with $|\Phi> = |e(s_1), g(s_2)>$. The upper curve shows the results for $\gamma_1 = 0$ (Hermitian dynamics). Then, from top to bottom, the curves for $\gamma_1 = 0.1, 0.5, 1$ are displayed. All curves are drawn with statistical error bars (which are already negligible by using just 50000 phase space points). As expected the “loss of probability” (given by the fact that non-Hermitian dynamics represents in an effective way the effect of additional states, which do not appear in the Hamiltonian, whose occupation can grow at the expense of the occupation of the explicitly described states; such is the case when escaping from a well toward infinity or decaying of a metastable state) increases upon increasing $\gamma_1$. Figure 2 displays the corresponding damping phenomenon for the reduced matrix element $\hat{\Omega}_{22}^S$ of the spin chain.
FIG. 3: Adiabatic time evolution of the trace of the reduced density matrix, Tr[\Omega_S], for \( \beta = 0.1, j_x = j_y = -1, j_z = 0.5, c = 0.24 \). The numerical time step of integration is \( \Delta t = 0.01 \). Adimensional parameters are used. Results for non-Hermitian dynamics with decay operator \( \hat{\Gamma}^{(2)} = \gamma_2 |e^{(s_1)}, e^{(s_2)} > < e^{(s_1)}, e^{(s_2)}| \) and initial reduced density matrix \( \hat{\Omega}_S(t_0) = |\Psi > < \Psi| \), with \( |\Psi > = (1/\sqrt{2}) (|e^{(s_1)}, e^{(s_2)} > -|e^{(s_1)}, g^{(s_2)} >) \). The curves, from top to bottom, show the results for \( \gamma_2 = 0.001, 0.01, 0.1 \). All curves are drawn with statistical error bars.

Figure 3 displays the adiabatic time evolution of the trace of the reduced density matrix of the spin chain, Tr[\Omega_S], when the decay operator is \( \hat{\Gamma}^{(2)} = \gamma_2 |e^{(s_1)}, e^{(s_2)} > < e^{(s_1)}, e^{(s_2)}| \) and the initial reduced density matrix \( \hat{\Omega}_S(0) = |\Psi > < \Psi| \), with \( |\Psi > = (1/\sqrt{2}) (|e^{(s_1)}, e^{(s_2)} > -|e^{(s_1)}, g^{(s_2)} >) \). The upper curve shows the results for \( \gamma_2 = 0.001 \). Then, from top to bottom, the curves for \( \gamma_2 = 0.01 \) and \( \gamma_2 = 0.1 \) are shown. Figure 4 displays the corresponding damping phenomenon for the reduced matrix element \( \hat{\Omega}_S^{11} \) of the spin chain. In this case, the trace decays because of the depletion of the state \( |e^{(s_1)}, e^{(s_2)} > \).

VI. CONCLUSIONS

In this work, a formalism to embed non-Hermitian quantum dynamics in a classical bath has been provided. In order to achieve this, a quantum-classical approximation for the non-Hermitian equations of motion of composite systems (with degrees of freedom having light and heavy masses, \( m \) and \( M \), respectively) has been first been considered, using a partial Wigner representation. Then, the limiting case when the non-Hermitian part of the evolution does not involve the classical-like degrees of freedom has been taken into account. The classical bath embedding the quantum system with non-conserved probability can be used as a source of realistic noise.

When the adiabatic part of the Hermitian Hamiltonian is considered, its eigenstates (defining the adiabatic basis) can be used to represent the non-Hermitian quantum-classical equation of motion.
FIG. 4: Adiabatic time evolution of matrix element $\Omega_{S}^{22}$, in the subsystem basis, for $\beta = 0.1, j_x = j_y = -1, j_z = 0.5, \epsilon = 0.24$. The numerical time step of integration is $\Delta t = 0.01$. Adimensional parameters are used. Results for non-Hermitian dynamics with decay operator $\hat{\Gamma}^{(2)} = \gamma_2 |e^{(s_1)}, e^{(s_2)}><e^{(s_1)}, e^{(s_2)}|$ and initial reduced density matrix $\hat{\Omega}_S(t_0) = |\Psi><\Psi|$, with $|\Psi> = (1/\sqrt{2}) (|e^{(s_1)}, e^{(s_2)}>-|e^{(s_1)}, g^{(s_2)}>)$. The curves, from top to bottom, show the results for $\gamma_2 = 0.001, 0.01, 0.1$. All curves are drawn with statistical error bars.

Once the equations of motion are represented in this adiabatic basis, algorithms can be developed using a sequential short-time propagation scheme. For the sake of illustrating the formalism, a Heisenberg chain with two spins, each weakly coupled to a separate harmonic oscillator has been studied. Two different decay operators have been explicitly considered showing that the algorithms lead to a stable and efficient numerical approach.

Future applications will be devoted to the modeling of nano-scale solid state devices in dissipative environments.

Acknowledgements

This work is based upon research supported by the National Research Foundation of South Africa.

[1] G. A. Gamow, Zeitschrift für Physik 51, 204 (1928).
[2] L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Pergamon Press, New York, 1977).
[3] C M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[4] A. Mostafazadeh, J. Math. Phys. 43, 205 (2002).
[5] A. Mostafazadeh, J. Phys. A 36, 7081 (2003).
[6] N. Moiseyev, Non-Hermitian Quantum Mechanics (Cambridge University Press, Cambridge, 2011).
[7] H. Suura, Prog. Theor. Phys. 12, 49 (1954).
[8] F. Coester and H. Kümmel, Nucl. Phys. 9, 225 (1958).
[9] A. J. Layzer, Phys. Rev. 129, 908 (1963).
[10] C. M. Bender, Rep. Prog. Phys. 70, 947 (2007).
[11] B. D. Wibking and K. Varga, Phys. Lett. A 376, 365 (2012).
[12] K.-F. Berggren, I. I. Yakimenko, and J. Hakanen, New. J. Phys. 12, 073005 (2010).
[13] M. Znojil, Phys. Rev. D 80, 045009 (2009).
[14] K. Varga and S. T. Pantelides, Phys. Rev. Lett. 98, 076804 (2007).
[15] J.G. Muga, J.P. Palao, B. Navarro, and I. L. Egusquiza, Phys. Rep. 395, 357 (2004).
[16] A. Thilagam, J. Chem. Phys. 136, 065104 (2011).
[17] B. Wahlstrand, I. I. Yakimenko, and K.-F. Berggren, Physical Review E 89, 062910 (2014).
[18] N. Moiseyev, Phys. Rep. 302, 211 (1998).
[19] W. John, B. Milek, H. Schanz, and P. Seba, Phys. Rev. Lett. 67, 1949 (1991).
[20] C. A. Nicolaides and S. I. Themelis Phys. Rev. A 45, 349 (1992).
[21] H. Feshbach, Ann. Phys. 5, 357 (1958).
[22] H. Feshbach, H. Feshbach, Ann. Phys. 19, 287 (1962).
[23] E. C. G. Sudarshan, Phys. Rev. D 18, 2914 (1978).
[24] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nature Physics 6, 192 (2010).
[25] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. 103, 093902 (2009).
[26] H. Schomerus and J. Wiersig, Physical Review A 90, 053819 (2014).
[27] S. Selstø, T. Birkeland, S. Kvaal, R. Nepstad, and M. Ferre, J. Phys. B: At. Mol. Opt. Phys. 44, 215003 (2011).
[28] H. C. Baker, Phys. Rev. A 30, 773 (1984).
[29] S.-I. Chu and W. P. Reinhardt, Phys. Rev. Lett. 39, 1195 (1977).
[30] H. Alaeian and J. A. Dionne, Physical Review B 89, 075136 (2014).
[31] K. Jones-Smith and H. Mathur, Phys. Rev. D 89, 125014 (2014).
[32] J. Korringa, Phys. Rev. 133, 1228 (1964).
[33] J. Wong, J. Math. Phys. 8, 2039 (1967).
[34] G. C. Hegerfeldt, Phys. Rev. A 47, 449 (1993).
[35] S. Baskoutas, A. Jannussis, R. Mignani, and V. Papatheou, J. Phys. A: Math. Gen. 26, L819 (1993).
[36] P. Angelopoulos, S. Baskoutas, A. Jannussis, R. Mignani, and V. Papatheou, Int. J. Mod. Phys. B 9, 2083 (1995).
[37] I. Rotter, J. Phys. A 42, 153001 (2009).
[38] R. Lo Franco, B. Bellomo, S. Maniscalco, and G. Compagno, Int. J. Mod. Phys. B 27, 1345053 (2013).
[39] S. Banerjee and R. Srikanth, Mod. Phys. Lett. B 24, 2485 (2010).
[40] F. Reiter and A. S. Sørensen, Phys. Rev. A 85, 032111 (2012).
[41] D. C. Brody and E. M. Graefe, Phys. Rev. Lett. 109, 230405 (2012).
[42] K. G. Zloshchastiev and A. Sergi, J. Mod. Optics 61, 1298 (2014).
[43] H. C. Baker and R. L. Singleton, Phys. Rev. A 42, 10 (1990).
[44] G. Dattoli, A. Torre, and R. Mignani, Phys. Rev. A 42, 1467 (1990).
[45] W. H. Hu, L. Jin, Y. Li, and Z. Song, Phys. Rev. A 86, 042110 (2012).
[46] E. M. Graefe, H. J. Korsch, and A. E. Niederle, Phys. Rev. Lett. 101, 150408 (2008).
[47] E. M. Graefe and R. Schubert, Phys. Rev. A 83, 060101(R) (2011).
[48] H. F. Jones and E. S. Moreira, J. Phys. A 43, 055307 (2010).
[49] A. Sergi, Comm. Theor. Phys. 56, 96 (2011).
[50] A. Sergi and K. Zloshchastiev, Int. J. Mod. Phys. B 27, 1350163 (2013).
[51] A. Sergi and K. G. Zloshchastiev, quant-ph arXiv:1412.5782 (2014).
[52] T. Kawamoto and N. Hatano, JPS Conf. Proc. 1, 012126 (2014).
[53] X. Lian, H. Zhong, Q. Xie, X. Zhou, Y. Wu, W. Liao, European Physical Journal D 68, 1 (2014).
[54] E. Karakaya, F. Altintas, and K. Gümüş, and Ö. Müstecaplıoğlu, Europhys. Lett. 105, 40001 (2014).
[55] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, Oxford, 2007).
[56] M. S. Tame, K. R. McEnery, S. K. Özdemir, J. Lee, S. A. Maier, and M. S. Kim, Nature Phys. 9, 329 (2013).
[57] K. R. McEnery, M. S. Tame, S. A. Maier, and M. S. Kim, Phys. Rev. A 89, 013822 (2014).
[58] Günter Mahler, Quantum Thermodynamic Processes (CRC Press, Boca Raton, 2015).
[59] T. A. Osborn, M. F. Kondrateva, G. C. Tabisz, and B. R. McQuarrie, J. Phys. A 32, 4149 (1999).
[60] W. Y. Zhang and R. Balescu, J. Plasma. Phys. 40, 199 (1988).
[61] W. Boucher and J. Traschen, Phys. Rev. D 37, 3522 (1988).
[62] G. Stock and M. Thoss, Phys. Rev. Lett. 78, 578 (1997).
[63] C. C. Martens and J.-Y. Fanga, J. Chem. Phys. 106, 4918 (1997).
[64] R. Kapral and G. Ciccotti J. Chem. Phys. 110, 8919 (1999).
[65] A. Sergi and R. Kapral, J. Chem. Phys. 118, 8566 (2003).
[66] A. Sergi, D. MacKernan, G. Ciccotti and R. Kapral, Theor. Chem. Acc. 110, 49 (2003).
[67] A. Sergi and R. Kapral, J. Chem. Phys. 121, 7565 (2004).
[68] E. Wigner, Phys. Rev. 40, 749 (1932).
[69] M. Hillery, R. F. O’Connell, M. O. Scully, E. P. Wigner, Phys. Rep. 106, 121 (1984).
[70] E. M. Graefe, M. Hönig, and H. J. Korsch, J. Phys. A Theor. 43, 075306 (2010).
[71] E. M. Graefe, H. J. Korsch, A. E. Niederle, Phys. Rev. A 82, 013629 (2010).
[72] E. M. Graefe, H. J. Korsch, A. Rush, R. Schubert, Journal of Physics A 48, 055301 (2015).
[73] P. Mazur and I. Oppenheim, Physica 50, 241 (1970).
[74] A. Sergi, Phys. Rev. E 72, 066125 (2005).
[75] D. MacKernan, R. Kapral, and G. Ciccotti, J. Phys. Condens. Matter 14, 9069 (2002).
[76] A. Sergi and F. Petruccione, Phys. Rev. E 81, 032101 (2010).
[77] D. A. Uken, A. Sergi, and F. Petruccione, Physica Scripta T143, 014024 (2011).
[78] D. A. Uken, A. Sergi, and F. Petruccione, Phys. Rev. E 88, 033301 (2013).
[79] A. Sergi, I. Sinayskiy, and F. Petruccione, Phys. Rev. A 80, 012108 (2009).