On the determination of $\alpha_s$ from hadronic $\tau$ decays with contour-improved, fixed order and renormalon-chain perturbation theory

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Abstract

One of the largest theoretical uncertainties assigned to the strong coupling constant $\alpha_s$ as determined from hadronic tau decays stems from the differences in the results for Fixed Order Perturbation Theory (FOPT), Contour Improved Perturbation Theory (CIPT) and Renormalon Chain Perturbation Theory (RCPT). It is often argued that the three methods differ in the treatment of higher orders only and therefore the full difference should be treated as theoretical error. Recently other arguments either in favor of FOPT, CIPT or RCPT have been given, but none of those is able to combine all three to a single value in the strong coupling constant. In this note I will show that FOPT alone has a much larger uncertainty than previously assumed and therefore agrees within error with CIPT. Furthermore a more appropriate matching of the different schemes used in RCPT reduces the difference to the CIPT result by a factor of 6. Together with recently published results for the 4th order term $K_4$ this reduces the theoretical error on $\alpha_s$ by a factor of 2.5 compared to the previously assumed spread of the three perturbative approaches.

1 Introduction

Hadronic decays of the $\tau$ lepton are among the most actively studied fields in QCD. The unique situation of a small mass scale and still small non-perturbative contributions allow for a very precise determination of the strong coupling constant $\alpha_s$ [1–4] (For recent reviews see [5–10]). The ratio of the hadronic decay width of the $\tau$ and its leptonic width can be written as

$$R_{\tau} = 3 S_{EW} \left( |V_{ud}|^2 + |V_{us}|^2 \right) \left(1 + \delta_{EW} + \delta_{pert} + \delta_{non-pert}\right),$$

where $S_{EW} = 1.0198 \pm 0.0006$ [11, 12] and $\delta_{EW} = 0.0010 \pm 0.0010$ [13] are small electroweak corrections, $\delta_{non-pert}$ denotes a $O$(few%) non-perturbative correction and $\delta_{pert}$ is the perturbative prediction. Neglecting the masses of the quarks (as is a good approximation for
the non-strange decay width of the $\tau$-meson the perturbative part is given by

$$1 + \delta_{\text{pert}} = \sum_{n=0}^{4} \frac{K_n}{2\pi i} \oint_{|s|=m^2_\tau} \frac{ds}{s} \left(1 - 2 \frac{s}{m^2_\tau} + 2 \frac{s^3}{m^6_\tau} - \frac{s^4}{m^8_\tau}\right) \left(\frac{\alpha_s(-s)}{\pi}\right)^n + O(\alpha_s^5),$$

with the known coefficients [14–19]

$$K_0 = K_1 = 1,$$
$$K_2 = \frac{299}{24} - 9 \zeta(3),$$
$$K_3^{\text{MS}} = \frac{58057}{288} - \frac{779}{4} \zeta(3) + \frac{75}{2} \zeta(5),$$
$$K_4^{\text{MS}} = \frac{78631453}{20736} + \frac{4185}{8} \zeta(3)^2 - \frac{1704247}{432} \zeta(3) + \frac{34165}{96} \zeta(5) - \frac{1995}{16} \zeta(7),$$

where $K_3$ and $K_4$ are scheme dependent and given here in the MS-scheme and $\zeta(n)$ denotes the Riemann zeta function. The 4-th order term $K_4^{\text{MS}} \approx 49.0757$ deviates substantially from previous estimates and partial calculations of that coefficient $K_4^{\text{partial}} = 27 \pm 16$ [20, 21]. The fifth-order term has been estimated to $K_5 \approx 275$ in [19], but the large deviation of the exact $K_4$ from its prediction suggests that a 100% error on $K_5$ is realistic. For the purpose of evaluating differences stemming from the 5th and higher orders I'll use $K_5 = 400 \pm 400$ in this note.

The methods FOPT and CIPT [4] differ in the way (2) is calculated. In the CIPT approach the $\beta$-function is used to get numerical solutions for $\alpha_s(-s)$ in the complex $s$-plane by starting with $\alpha_s(m^2_\tau)$. The integrand is thus calculated in small steps on the circle $|s|=m^2_\tau$ and the sum of all pieces gives the total integral.

For the FOPT method the $\beta$-function and its derivatives are Taylor expanded in $s$ around $s_0 = m^2_\tau$ which leads to a power series representation of $\alpha_s(-s)$ in powers of $\alpha_s(m^2_\tau)$. The series is truncated at the desired order (here the 5th) in the strong coupling and inserted in the integral which becomes solvable now. The usual FOPT result reads:

$$\delta_{\text{pert}} = \frac{\alpha_s(m^2_\tau)}{\pi} + 5.2023 \frac{\alpha_s^2(m^2_\tau)}{\pi^2} + 26.366 \frac{\alpha_s^3(m^2_\tau)}{\pi^3} +$$
$$127.08 \frac{\alpha_s^4(m^2_\tau)}{\pi^4} + \left( K_5 + 307.78 \right) \frac{\alpha_s^5(m^2_\tau)}{\pi^5} + O(\alpha_s^6).$$

As is demonstrated in [19, 21] the fourth and fifth order terms contribute very little to the perturbative part and the difference between the FOPT and the CIPT result is much larger than the contributions from these terms even if generous errors are used for $K_5$. Taking $\alpha_s(m^2_\tau) = 0.35$ and $K_5 = 400$ as reference values we could first calculate $\delta_{\text{pert}}$ from the CIPT approach and extract $\alpha_s(m^2_\tau)$ again using FOPT:

$$\delta_{\text{pert}}^{\text{CIPT}}(\alpha_s(m^2_\tau) = 0.35) = 0.21179,$$
$$\alpha_s(m^2_\tau)^{\text{FOPT}}(\delta_{\text{pert}} = 0.21179) = 0.32543.$$

The deviation of either value from their mean is with $\Delta \alpha_s = \pm 0.012$ almost twice as large as the uncertainty due to higher orders $\Delta \alpha_s \Delta K_5 = 0.007$. The reason for this large

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difference is the choice of the point on the circle \( |s| = m_r^2 \) in the complex \( s \)-plane around which the \( \beta \)-function and its derivatives are Taylor expanded to approximate the strong coupling on the circle. In the following section the FOPT formalism will be generalized to allow for other choices.

2 Generalized FOPT

The starting point is the perturbative expansion of the \( \beta \) function, which is given by

\[
\beta(a_s) = \frac{da_s(s)}{d\ln s} = -\beta_0 a_s^2(s) - \beta_1 a_s^3(s) - \beta_2 a_s^4(s) - \beta_3 a_s^5(s) - \ldots,
\]

where \( a_s(s) = \alpha_s(s)/(4\pi) \). The first two terms in the \( \beta \)-function [22–26] for \( n_f \) quark flavors,

\[
\beta_0 = 11 - \frac{2}{3} n_f, \\
\beta_1 = 102 - \frac{38}{3} n_f,
\]

are universal at leading twist whereas the higher order terms are scheme dependent. In the \( \overline{\text{MS}} \) scheme the first two scheme dependent coefficients are known [27–30]:

\[
\beta_2^{\overline{\text{MS}}} = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2, \\
\beta_3^{\overline{\text{MS}}} = \frac{149753}{6} + 3564\zeta(3) - \left( \frac{1078361}{162} + \frac{6508}{27}\zeta(3) \right) n_f + \\
\left( \frac{50065}{162} + \frac{6472}{81}\zeta(3) \right) n_f^2 + \frac{1093}{729} n_f^3.
\]

The Taylor expansion of the evolution equation (7) around \( s_0 \) reads up to the fifth order in \( \alpha_s \):

\[
\frac{\alpha_s(s)}{\pi} = \frac{\alpha_s(s_0)}{\pi} - \frac{1}{4} \beta_0 \ln \frac{s}{s_0} \left( \frac{\alpha_s(s_0)}{\pi} \right)^2 + \\
\frac{1}{16} \left( \beta_0^2 \ln^2 \frac{s}{s_0} - \beta_1 \ln \frac{s}{s_0} \right) \left( \frac{\alpha_s(s_0)}{\pi} \right)^3 - \\
\frac{1}{128} \left( 2 \beta_0^3 \ln^3 \frac{s}{s_0} - 5 \beta_0 \beta_1 \ln^2 \frac{s}{s_0} + 2 \beta_2 \ln \frac{s}{s_0} \right) \left( \frac{\alpha_s(s_0)}{\pi} \right)^4 + \\
\frac{1}{1536} \left( 6 \beta_0^4 \ln^4 \frac{s}{s_0} - 26 \beta_0^2 \beta_1 \ln^3 \frac{s}{s_0} + 9 \left( \beta_2^2 + 2 \beta_0 \beta_2 \right) \ln^2 \frac{s}{s_0} - 6 \beta_3 \ln \frac{s}{s_0} \right) \left( \frac{\alpha_s(s_0)}{\pi} \right)^5 + O(\alpha_s(s_0)^6).
\]

It should be noted that eq. (9) is strictly speaking not a Taylor approximation since the truncation occurs at a certain power of \( \alpha_s \) and not at a certain power in the expansion variable \( \ln(s/s_0) \).
Fig. 1: Quality of the Taylor expansion of $\alpha_s(m_\tau^2 \exp(i\varphi))$. The two plots show the absolute value $|\alpha_s|$ on the complex circle $s = m_\tau^2 \exp(i\varphi)$. The left plot shows with long-dashed, dash-dotted, narrow dotted and wide dotted lines the Taylor (FOPT) expansion up to 5th, 4th, 3rd, and 2nd order, respectively. The 4-loop result for the numerically solved $\beta$-function (CIPT) is drawn as a solid line for comparison. The right plot shows with dashed, dotted, and dash-dotted lines the numerical solutions for 3, 2 and 1 loop $\beta$-functions, respectively. Again the same 4-loop result as in the left plot is shown as a solid line. The reference value $\alpha_s(m_\tau^2) = 0.35$ was used for all curves.

Since both the integrand in eq. (2) and eq. (9) are power series in $\alpha_s$ it is interesting to compare the magnitudes of the coefficients in these series. The largest values in eq. (9) are obtained at $s = s_0 \exp(-i\pi)$ where the $|c_n|$ (the magnitude of the coefficient in front of $(\alpha_s/\pi)^n$) read 1, 7.07, 51.52, 390.8, 3023.85. At $s = s_0 \exp(-i\pi/2)$ (the average distance from $s_0$) the $|c_n|$ are 1, 3.53, 13.98, 62.33, 275.16. These numbers grow much (slightly) faster at $-\pi \ ( -\pi/2)$ than the $K_n$, where (starting with $K_1$) we have 1, 1.64, 6.37, 49.08, $\sim 275$. Therefore it is conceivable that the nature of the Taylor expansion of $\alpha_s$ dominates the uncertainty of the FOPT result and not missing higher order $K_n$ terms. To illustrate this the Taylor expansion of $\alpha_s$ is modified in the following.

Figure 1 shows that the deviation of the Taylor expanded $\alpha_s$ from the numerically solved $\alpha_s$ grows with the distance of $s = s_0 \exp(i\varphi)$ from the chosen development point $s_0 = m_\tau^2$. The CIPT results for 1 to 4-loop treatment on the circle are also shown. Starting at 2-loop level the CIPT results are almost indistinguishable while the FOPT deviations remain large even at 5th order! Thus the correct treatment of the logarithms and not higher orders in $\alpha_s$ are the dominant source of uncertainty.

Furthermore the choice $s_0 = m_\tau^2$ for FOPT is completely arbitrary. This becomes even more obvious in the light of the usual procedure of comparing different $\alpha_s$ measurements by evolving them to the $Z^0$-mass with a numerically solved $\beta$-function. If we were to evolve the $\alpha_s$ from tau-decays to the $Z^0$-mass in a FOPT-like manner with equation (9) in just 3
steps (with the quark flavor transitions to $n_f = 4$ and $n_f = 5$ at $m_\tau$ and $m_b$, respectively) the numerical value of $\alpha_s(m^2)$ (for $\alpha_s(m^2) = 0.35$) would be 0.042, 0.206, 0.064, 0.160, for the 1, 2, 3, 4 loop beta function, respectively. Compared to the usual procedure of contour-improved evolution of the coupling (i.e. using eq. (9) in small steps) which yields 0.128, 0.123, 0.122, 0.122, for 1, 2, 3, 4 loop, respectively, the 3 step FOPT solutions converge very slowly, giving alternatingly lower and higher estimates of the coupling as more and more orders in $\alpha$ converge very slowly, giving alternatingly lower and higher estimates of the coupling as more and more orders in $\alpha$ are considered and still at 5th order resulting in a numerical value that is far below the exact result. The FOPT terms up to the 5th order do not compensate for the neglected large logarithms.

It is therefore natural to generalize this in the case of the $\tau$ and first evolve $\alpha_s(m^2)$ to $\alpha_s(m^2 \exp(i\varphi_0))$ with the numerically solved $\beta$-function and derive the Taylor series of $\delta_{\text{pert}}$ around this new point. The integral (2) can in fact be split in two pieces around $\varphi_0$ and $-\varphi_0$ since the strong coupling at $-\varphi_0$ is just the complex conjugate of the strong coupling at $\varphi_0$:

$$\alpha_s(m^2 \exp(-i\varphi_0)) = \alpha_s(m^2 \exp(i\varphi_0))^*$$

The resulting $\delta_{\text{pert}}$ up to the fifth order reads:

$$\delta_{\text{pert}} = a + \frac{8b}{3\pi} + a b \left( \frac{16 K_2}{3\pi} - \frac{\beta_0 \varphi_0}{2} + \left( \frac{16}{9\pi} - \frac{\pi}{4} \right) \beta_0 \right)$$

$$+ \left( a^2 - b^2 \right) \left( K_2 + \frac{19\beta_0}{48} + \frac{2\beta_0 \varphi_0}{3\pi} \right)$$

$$+ \left( a^3 - 3a b^2 \right) \left( -\frac{\beta_0^2 \varphi_0^2}{16} + \left( \frac{4}{9\pi} - \frac{\pi}{16} \right) \beta_0^2 \varphi_0 + \left( \frac{265}{1152} - \frac{\pi^2}{48} \right) \beta_0^2 \right)$$

$$\left( \frac{4 K_2 \beta_0 \varphi_0}{3\pi} + \frac{19 K_2 \beta_0}{24} + \frac{19 K_2 \beta_0}{6\pi} + K_3 + 19 \beta_1 \right)$$

$$- \left( b^3 - 3 a^2 b \right) \left( -\frac{\beta_0^2 \varphi_0^2}{6\pi} - \frac{19 \beta_0^2 \varphi_0}{96} + \left( \frac{13}{27\pi} - \frac{19\pi}{192} \right) \beta_0^2 - \frac{K_2 \beta_0 \varphi_0}{2} \right)$$

$$\left( \frac{16 K_2}{9\pi} - \frac{\pi K_2}{4} \right) \beta_0 - \frac{\beta_1 \varphi_0}{16} + \frac{8 K_3}{3\pi} + \left( \frac{2}{9\pi} - \frac{\pi}{32} \right) \beta_1 \right)$$

$$+ \left( a^4 - 6a^2 b^2 + b^4 \right) \left( K_4 + \frac{19 \beta_2}{768} - \frac{19 \beta_0^3 \varphi_0^2}{256} + \beta_0^2 \left( \frac{265 K_2}{384} - \frac{\pi^2 K_2}{16} \right) \right)$$

$$- \left( \frac{19 \pi^2}{768} - \frac{3355}{18432} \right) \beta_0^3 + \frac{19 K_2 \beta_1}{96} + \frac{19 K_3 \beta_0}{16} - \frac{3 K_2 \beta_0^2 \varphi_0^2}{16}$$

$$- \left( \frac{5 \pi^2}{384} - \frac{1325}{9216} \right) \beta_0 \beta_1 + \left( \frac{13}{36\pi} - \frac{19 \pi}{256} \right) \beta_0^3 \varphi_0$$

$$+ \beta_0^2 \varphi_0 \left( \frac{4 K_2}{3\pi} - \frac{3 \pi K_2}{16} \right) - \frac{\beta_0^3 \varphi_0^3}{24 \pi} + \frac{\beta_2 \varphi_0}{24 \pi} - \frac{5 \beta_0 \beta_1 \varphi_0^2}{128}$$

$$+ \frac{K_2 \beta_1 \varphi_0}{3\pi} + \frac{2 K_3 \beta_0 \varphi_0}{\pi} + \left( \frac{5}{18\pi} - \frac{5 \pi}{128} \right) \beta_0 \beta_1 \varphi_0$$

$$+ \left( a^3 b - a b^3 \right) \left( \frac{\beta_0^3 \varphi_0^3}{16} + \left( \frac{40}{27\pi} - \frac{265 \pi}{768} + \frac{\pi^3}{64} \right) \beta_0^3 - \frac{\beta_2 \varphi_0}{16} \right)$$
\[
+ \beta_0^2 \left( \frac{52 K_2}{9 \pi} - \frac{19 \pi K_2}{16} \right) + \frac{32 K_4}{3 \pi} + \left( \frac{2}{9 \pi} - \frac{\pi}{32} \right) \beta_2 \\
+ \beta_1 \left( \frac{16 K_2}{9 \pi} - \frac{\pi K_2}{4} \right) + \beta_0 \left( \frac{32 K_3}{3 \pi} - \frac{3 \pi K_3}{2} \right) - \frac{K_2 \beta_1 \varphi_0}{2} - 3 K_3 \beta_0 \varphi_0 \\
- \frac{95 \beta_0 \beta_1 \varphi_0}{192} - \frac{19 K_2 \beta_0^2 \varphi_0}{8} + \left( \frac{65}{54 \pi} - \frac{95 \pi}{384} \right) \beta_0 \beta_1 - \left( \frac{2}{3 \pi} - \frac{3 \pi}{32} \right) \beta_0^3 \varphi_0^2 \\
+ \left( \frac{\pi^2}{16} - \frac{265}{384} \right) \beta_0^3 \varphi_0 - \left( \frac{5 \beta_0 \beta_1 \varphi_0^2}{12 \pi} - \frac{2 K_2 \beta_0^2 \varphi_0^2}{\pi} \right) \\
+ \left( a^5 - 10 a^3 b^2 + 5 a b^4 \right) \left( K_5 + \frac{\beta_0^4 \varphi_0^4}{256} - \frac{3 \beta_1^2 \varphi_0^2}{512} + \beta_0^2 \left( \frac{265 K_3}{192} - \frac{\pi^2 K_3}{8} \right) \right) \\
+ \frac{19 \beta_3}{3072} + \beta_0^3 \left( \frac{3355 K_2}{4608} - \frac{19 \pi^2 K_2}{192} \right) + \left( \frac{\pi^4}{1280} - \frac{265 \pi^2}{9216} + \frac{41041}{221184} \right) \beta_0^4 \\
- \left( \frac{\pi^2}{512} - \frac{265}{12288} \right) \beta_1^2 + \frac{19 K_2 \beta_2}{384} + \frac{19 K_3 \beta_1}{64} + \frac{19 K_4 \beta_0}{12} - \frac{19 K_2 \beta_0^3 \varphi_0^2}{64} \\
- \frac{3 K_3 \beta_0^2 \varphi_0^2}{8} - \frac{247 \beta_0^2 \beta_1 \varphi_0^2}{3072} + \left( \frac{10}{27 \pi} - \frac{265 \pi}{3072} + \frac{\pi^3}{256} \right) \beta_0^4 \varphi_0 \\
- \left( \frac{\pi^2}{256} - \frac{265}{6144} \right) \beta_0 \beta_2 + \left( \frac{1}{24 \pi} - \frac{\pi}{32 \pi} \right) \beta_1^2 \varphi_0 + \frac{\beta_0^3 \varphi_0}{96 \pi} - \frac{3 \beta_0 \beta_2 \varphi_0^2}{256} \\
+ \beta_0^2 \varphi_0 \left( \frac{8 K_3}{3 \pi} - \frac{3 \pi K_3}{8} \right) + \left( \frac{\pi^2}{128} - \frac{265}{3072} \right) \beta_0^4 \varphi_0^2 + \frac{\beta_3 \varphi_0}{96 \pi} - \frac{3 \beta_0 \beta_2 \varphi_0^2}{256} \\
+ \beta_0 \beta_1 \left( \frac{1855 K_2}{4608} - \frac{7 \pi^2 K_2}{192} \right) - \frac{247 \pi^2}{9216} + \frac{43615}{221184} \beta_0^2 \beta_1 \\
- \left( \frac{1}{18 \pi} - \frac{\pi}{128} \right) \beta_0^3 \varphi_0^3 + \frac{169}{432 \pi} - \frac{247 \pi}{3072} \beta_0^2 \beta_1 \varphi_0 - \frac{K_2 \beta_0^3 \varphi_0^3}{6 \pi} \\
- \frac{13 \beta_0^2 \beta_1 \varphi_0^3}{12 \pi} + \frac{K_2 \beta_2 \varphi_0}{2 \pi} + \frac{K_3 \beta_1 \varphi_0}{8 \pi} + \frac{8 K_4 \beta_0 \varphi_0}{3 \pi} - \frac{7 K_2 \beta_0 \beta_1 \varphi_0^2}{64} \\
+ \left( \frac{1}{12 \pi} - \frac{3 \pi}{256} \right) \beta_0 \beta_2 \varphi_0 + \beta_0 \beta_1 \varphi_0 \left( \frac{7 K_2}{9 \pi} - \frac{\pi K_2}{64} \right) \\
+ \left( b^5 + 5 a^4 b - 10 a^2 b^3 \right) \left( \frac{19 \beta_0^4 \varphi_0^3}{768} + \frac{121}{324 \pi} - \frac{3355 \pi}{36864} + \frac{19 \pi^3}{3072} \right) \beta_0^4 \\
+ \beta_0^3 \left( \frac{40 K_2}{27 \pi} + \frac{\pi^3 K_2}{64} - \frac{265 \pi K_2}{768} \right) - \frac{\beta_3 \varphi_0}{256} + \left( \frac{13}{288 \pi} - \frac{\pi}{2048} \right) \beta_1^2 \\
+ \beta_0^2 \left( \frac{26 K_3}{9 \pi} - \frac{19 \pi K_3}{32} \right) + \frac{8 K_5}{3 \pi} - \frac{19 \beta_1^2 \varphi_0}{1024} + \left( \frac{1}{72 \pi} - \frac{\pi}{512} \right) \beta_3 \\
+ \beta_2 \left( \frac{K_2}{9 \pi} - \frac{\pi K_2}{64} \right) + \beta_1 \left( \frac{2 K_3}{3 \pi} - \frac{3 \pi K_3}{32} \right) + \beta_0 \left( \frac{32 K_1}{9 \pi} - \frac{\pi K_4}{2} \right) + \frac{K_2 \beta_0^3 \varphi_0^3}{16} \\
+ \frac{13 \beta_0^2 \beta_1 \varphi_0^3}{768} + \frac{65}{162 \pi} - \frac{3445 \pi}{36864} + \frac{13 \pi^3}{3072} \beta_0^2 \beta_1 - \frac{K_2 \beta_3 \varphi_0}{32} - \frac{3 K_3 \beta_1 \varphi_0}{16} \\
-K_4 \beta_0 \varphi_0 - \frac{19 \beta_0 \beta_2 \varphi_0}{512} - \frac{\beta_0^3 \varphi_0}{16} \left( \frac{265 K_2}{384} - \frac{\pi^2 K_2}{16} \right) + \frac{\beta_0^4 \varphi_0^4}{96 \pi} - \frac{\beta_1^2 \varphi_0^2}{64 \pi}
\]
\(-\frac{19}{16} K_3 \beta_0^2 \varphi_0 + \left( \frac{13}{1024} - \frac{19 \pi}{108} \right) \beta_0 \beta_2 + \beta_0 \beta_1 \left( \frac{91 K_2}{- \frac{3 \pi}{768}} - \frac{133 \pi K_2}{768} \right) \)
\(\left( \frac{13}{72 \pi} - \frac{19 \pi}{512} \right) \beta_0^4 \varphi_0^2 - \beta_0^3 \varphi_0^2 \left( \frac{2 K_2}{3 \pi} - \frac{3 \pi K_2}{32} \right) \)
\(\left( \frac{19 \pi^2}{768} - \frac{3355}{18432} \right) \beta_0^4 \varphi_0 - \beta_0^2 \varphi_0^2 \left( \frac{32 \pi}{384} - \frac{133 K_2 \beta_0 \beta_1 \varphi_0}{384} - \frac{K_2 \beta_0^2 \varphi_0^2}{\pi} \right) \)
\(\left( \frac{13}{72 \pi} - \frac{13 \pi}{512} \right) \beta_0^2 \beta_1 \varphi_0^2 + \left( \frac{13 \pi^2}{768} - \frac{3445}{18432} \right) \beta_0^2 \beta_1 \varphi_0 - \frac{7 K_2 \beta_0 \beta_1 \varphi_0^2}{24 \pi} \),

or numerically:

\[
\delta_{\text{pert}} = a + 0.8488 b + (-4.5 \varphi_0 + 0.8082) a b + \\
(1.9099 \varphi_0 + 5.2023) (a^2 - b^2) + \\
(-5.0625 \varphi_0^2 + 5.2138 \varphi_0 + 26.366) (a^3 - 3 a b^2) + \\
(4.2972 \varphi_0^2 + 27.410 \varphi_0 + 12.356) (b^3 - 3 a^2 b) + \\
(-9.6687 \varphi_0^3 - 101.51 \varphi_0^2 - 71.629 \varphi_0 + 127.08) (a^4 - 6 a^2 b^2 + b^4) + \\
(45.563 \varphi_0^3 - 100.94 \varphi_0^2 - 918.59 \varphi_0 - 521.11) (a^3 b - a b^3) + \\
(25.629 \varphi_0^4 - 92.897 \varphi_0^3 - 1220.5 \varphi_0^2 - 1272.5 \varphi_0 + \\
K_5 + 307.78) (a^5 - 10 a^3 b^2 + 5 a b^4) + \\
(21.755 \varphi_0^4 + 324.78 \varphi_0^3 + 271.83 \varphi_0^2 - 1612.0 \varphi_0 + \\
0.8488 K_5 - 1413.5) (b^5 + 5 a^4 b - 10 a^2 b^3),
\]

with \(\varphi_0 \in [-\pi, 0]\), and \(\alpha_s(m_c^2 \exp(i \varphi_0))/\pi = a + ib\). Three points should be noted about equation (11):

1. it resembles the usual FOPT result for \(\varphi_0 = 0\) and \(b = 0\);

2. \(\delta_{\text{pert}}\) remains real for all choices of \(\alpha_s\) and \(\varphi_0\);

3. inserting the Taylor expanded \(\alpha_s(\varphi_0)\) in eq. (11) and Taylor expanding the resulting \(\delta_{\text{pert}}\) again around \(\alpha_s(\varphi_0 = 0)\) leads also to the usual FOPT result.

The last point demonstrates that FOPT can be generalized only if the ‘exact’ value for \(\alpha_s(\varphi_0)\) is used in the expansion.

Figure 2 shows \(\delta_{\text{pert}}\) as a function of \(\varphi_0\) with \(\alpha_s(m_c^2) = 0.35\) and \(K_5 = 400, 0.800\) as reference values. The consequences of the generalized FOPT solution are discussed in the following section.

3 Discussion of the generalized FOPT solution

As can be seen from figure 2 the FOPT result depends largely on the choice of \(\varphi_0\). The FOPT curves intersect with the CIPT curves around \(\varphi_0 \simeq -1\) but span over a much larger range of \(\delta_{\text{pert}}\) values. Compared to the uncertainty from the neglected higher orders this intrinsic error is 4 times larger as none of the choices for \(\varphi_0\) should be excluded. The default choice of \(\varphi_0 = 0\) leads to the largest possible value of \(\delta_{\text{pert}}\) and therefore \(\alpha_s\) from
Fig. 2: $\delta_{\text{pert}}$ as function of the development point $\varphi_0$. The solid, medium-dashed and short-dashed lines show the CIPT result to 5th order for $K_5 = 400$, $0$, and $800$, respectively. The long-dashed, dash-dotted and dotted lines show the generalized FOPT result to 5th order for $K_5 = 400$, $0$, and $800$, respectively. The reference value $\alpha_s(m_Z^2) = 0.35$ was used for all curves.

FOPT used to be smaller than from CIPT. The deviation can however not be attributed to higher order terms in the series of $\delta_{\text{pert}}$. Instead the extraction of $\alpha_s$ with FOPT should use the average of the two extremes $\alpha_s(\varphi_0 = 0)$ and $\alpha_s(\varphi_0 = -\pi)$ and half of their difference as additional theoretical error. Consequently the most accurate way for the determination of $\alpha_s$ from $\tau$ decays is the CIPT approach. There is no reason to add the same error to the CIPT result as it does not depend on the choice of $\varphi_0$. Also since the FOPT result agrees within its own error with CIPT there is no discrepancy anymore between results with these two approaches. In fact the CIPT result is what FOPT would converge to for $n \to \infty$ if $n$ equidistant points on the circle $s = s_0 \exp(i\varphi)$ would be used in the expansion. The case $n = 1$ could therefore be regarded as an approximation for CIPT and the choice $\varphi_0 = 0$ is just one of the many possible choices for $n = 1$. 
4 Renormalon Chains

The third theory often used in evaluating $\alpha_s$ from $\tau$ decays uses so called ‘Renormalon Chains’ \cite{31–34} and re-sums the $\beta^0$ parts of $\delta_{pert}$ to all orders in $\alpha_s$. In \cite{32} the result obtained from this re-summation is corrected by the known FOPT terms by first subtracting the large-$\beta^0$ part up to the desired order of FOPT and then adding the FOPT part. Thus $\delta_{pert}$ for the Renormalon Chain Perturbation Theory (RCPT) can be written as

$$\delta_{pert}^{RCPT} = \delta_{renormalon} - \delta_{large-\beta^0}^{FOPT} + \delta_{pert}^{FOPT},$$

where the three terms in the sum refer to the renormalon chain result, the large-$\beta^0$ re-summed result up to the order used in FOPT, and the FOPT result, respectively. As is pointed out in \cite{32} the renormalon chain re-summation includes parts of the CIPT re-summation, namely the terms $(-\beta^0/4\ln(s/s_0))^n$ which are part of the coefficient in front of $(\alpha_s/\pi)^n$ in eq. (9). Therefore the CIPT result can not be used instead of the FOPT result in eq.(13). Still, for the FOPT correction and the fixed order large-$\beta^0$ correction the same arbitrariness of the choice of $\varphi_0$ as discussed in the first part of this note exists, as long as both the FOPT term and the fixed order large-$\beta^0$ term are expanded around the same $\varphi_0$. Therefore the variation of $\delta_{pert}^{FOPT} - \delta_{large-\beta^0}^{FOPT}$ with $\varphi_0$ is a source of uncertainty in the RCPT approach. The generalized $\delta_{large-\beta^0}^{FOPT}$ can be derived from eq. (11) by setting $\beta_n = 0$ for $n > 0$ and replacing the $K_n$ with $\beta_n^{(n-1)}K_n$, which are given up to $n = 4$ in \cite{32} and up to $n = 12$ in \cite{10}. Numerically $\delta_{large-\beta^0}^{FOPT}$ up to the fifth order in $\alpha_s$ is given by:

$$\begin{align*}
\delta_{\beta^0}^{FOPT} &= a + 0.8488b + (-4.5\varphi_0 + 0.6668)ab + \\
& ((1.9099\varphi_0 + 5.1190)(a^2 - b^2) + \\
& (-5.0625\varphi_0^2 + 1.5002\varphi_0 + 28.779)(a^3 - 3ab^2) + \\
& (4.2972\varphi_0^2 + 23.035\varphi_0 + 2.5067)(b^3 - 3ab) + \\
& (-9.6687\varphi_0^3 - 77.745\varphi_0^2 - 16.920\varphi_0 + 156.67)(a^4 - 6a^2b^2 + b^4) + \\
& (45.563\varphi_0^3 - 20.253\varphi_0^2 - 777.03\varphi_0 - 433.69)(a^3b - ab^3) + \\
& (25.629\varphi_0^4 - 15.189\varphi_0^3 - 874.16\varphi_0^2 - 975.80\varphi_0 + \\
& 900.78)(a^5 - 10a^3b^2 + 5ab^4) + \\
& (21.755\varphi_0^5 + 233.23\varphi_0^4 + 76.141\varphi_0^3 - 1410.1\varphi_0 - \\
& 615.93)(b^5 + 5a^4b - 10a^2b^3),
\end{align*}$$

with $\varphi_0 \in [-\pi, 0]$, and $\alpha_s(m^2_{\tau}\exp(i\varphi_0))/\pi = a + ib$ as in eq. (12). The difference $\delta_{pert}^{FOPT} - \delta_{large-\beta^0}$ is not as sensitive to the choice of $\varphi_0$ as $\delta_{pert}^{FOPT}$ alone and roughly halves the associated uncertainty in $\alpha_s$.

Figure 3 shows the RCPT result using the $\delta_{renormalon}$ as in \cite{32} but the modified $\delta_{pert}^{FOPT} - \delta_{large-\beta^0}$ from eqs. (12,14) to correct the result up to the fifth order in $\alpha_s$. The reference value of $\alpha_s(m^2_{\tau}) = 0.35$ is used again for all curves. It is clear from the figure that RCPT would still require a much smaller $\alpha_s \simeq 0.31$ compared to CIPT even with the modified corrections in the fixed order parts.

This observation relies however on the fact that $\alpha_s$ in the renormalon part and the fixed order part of eq. (13) refers to the same quantity. This is probably not the case. The renormalon part in \cite{32} is derived from the one-loop coupling in the so-called V scheme,
Fig. 3: $\delta_{\text{pert}}$ as function of the development point $\varphi_0$. The solid, medium-dashed and short-dashed lines show the CIPT result to 5th order for $K_5 = 400$, $0$, and $800$, respectively. The long-dashed, dash-dotted and dotted lines show the RCPT result with 0-loop matching and generalized FOPT correction to 5th order for $K_5 = 400$, $0$, and $800$, respectively. The reference value $\alpha_s(m_t^2) = 0.35$ was used for all curves.

$\alpha_s^V(\mu^2)$ which is matched on the 0-loop level to $\alpha_s^\text{MS}(\exp(-5/3)\mu^2)$. The problem therefore is that we have a coupling constant on the 3-loop level\footnote{Since $\beta_3$ enters only in the 5th order in $R_\tau$ $\alpha_s$ is effectively used as a 3-loop coupling constant in FOPT which goes up to the 4th (plus estimated 5th) order in $\alpha_s$.} in the FOPT parts, but treat it as a one-loop coupling in the renormalon parts. A possible solution would be to use 2-loop matching to go from the $\overline{\text{MS}}$-scheme to the V scheme which is given by [35, 36]

$$
\frac{\alpha_s^V(\mu^2)}{\pi} = \frac{\alpha_s^\text{MS}(e^{-5/3}\mu^2)}{\pi} - 2 \left( \frac{\alpha_s^\text{MS}(e^{0.4221}\mu^2)}{\pi} \right)^2 - 7.72816 \left( \frac{\alpha_s^\text{MS}(e^{0.4221}\mu^2)}{\pi} \right)^3.
$$

Figure 4 shows again the RCPT result as before but with the 2-loop matching for $\alpha_s^V$. The reference value of $\alpha_s(m_t^2) = 0.35$ is used for all RCPT and CIPT curves. The large
Fig. 4: $\delta_{\text{pert}}$ as function of the development point $\varphi_0$. The solid, medium-dashed and short-dashed lines show the CIPT result to 5th order for $K_5 = 400$, 0, and 800, respectively. The long-dashed, dash-dotted and dotted lines show the RCPT result with 2-loop matching and generalized FOPT correction to 5th order for $K_5 = 400$, 0, and 800, respectively. The reference value $\alpha_s(m_\tau^2) = 0.35$ was used for all curves.

overlap of the CIPT and RCPT curves shows that the differences in the deduced strong couplings from both theories are much smaller than previously assumed.

5 Discussion of the modified RCPT solution

Unlike in the case of the generalized FOPT the RCPT solution can not be regarded as the first iteration of a contour improved result as this would result in inconsistent definitions of $\alpha_s$ on the circle $s = |m_\tau^2|$. Therefore the magnitude of the spread of $\alpha_s$ values obtained from a fixed $\delta_{\text{pert}}$ and a fixed $K_5$ could be regarded as induced by higher order terms. This spread has furthermore the same magnitude as the difference between CIPT and RCPT at $\varphi_0 = 0$, which at the same time shrunk by a factor of six by using the 2-loop matching instead of 0-loop matching for the transition from the V scheme to the MS
Fig. 5: $\alpha_s(m_\tau^2)$ as function of the development point $\varphi_0$. The solid, medium-dashed and short-dashed lines show the CIPT result to 5th order for $K_5 = 400$, 0, and 800, respectively. The long-dashed, dash-dotted and dotted lines show the RCPT result with 2-loop matching and generalized FOPT correction to 5th order for $K_5 = 400$, 0, and 800, respectively. All curves are obtained with $\delta_{\text{pert}} = 0.21179$ as reference value.

scheme. Averaging over all $\varphi_0 = 0$ values for RCPT leads to the same numerical value for $\alpha_s$ as CIPT at the central choice for $K_5$, while the RCPT results for $K_5 = 0, 800$ stay much closer to the central $\alpha_s$ compared to the corresponding CIPT values, showing that the large $\beta_0$ re-summation reduces the impact of higher order terms.

Recently the situation about the influence of higher-order corrections has been revisited in [10], where the authors study the renormalon structure in different models and fix the Borel transform of the Adler function such that the known terms up to 4th order are reproduced. Using this matched Adler function for $R_\tau$ and comparing the full re-summed result with standard FOPT shows again good agreement, while the distance to CIPT is large. But as discussed in section 2, the generalized FOPT solution would on average reproduce the CIPT result and therefore the model of [10] would also deviate from the approach advocated in section 2.
The power corrections to $R_\tau$ have been re-examined in [37] where duality violation parts [38, 39] of the order of 0.01 (but found to be negligible in [9]) and tachyonic mass corrections from the gluon are considered. It is argued in [37] that the difference between the Borel-sum and the truncated series at 4th order in the large-$\beta_0$ limit can be regarded as non-standard dimension 2 power corrections to $R_\tau$. Since these corrections are of the order $0.04$ for CIPT and $0.02$ for FOPT, respectively, they dominate over the duality violation effects and if taken at face value eliminate the difference between FOPT and CIPT. However, it should be noted that the Borel transform $\hat{D}(b)$ of the Adler function does not have a pole at $b = 1$ and therefore no renormalon ambiguity of dimension 2, which makes it difficult to associate a dimension 2 correction to the observed difference.

Here I'll concentrate on the perturbative parts only and neglect any non-standard power correction. Figures 2 and 4 show that there is no large cancellation mechanism which would prevent the FOPT or the RCPT result from depending strongly on the arbitrary choice of the development point $\varphi_0$. Including this arbitrary choice in the uncertainty estimate shows that CIPT provides the most accurate estimate. It is however re-assuring that correcting the large-$\beta_0$ re-summed result with the known fixed order terms up to 5th order reduces this dependency by 50% leaving the remaining 50% to the exact logarithms for $\beta_n$ with $n > 0$.

Figure 5 shows a numerical example for $\delta_{\text{pert}} = 0.21179$ comparing the values of $\alpha_s$ for $K_5 = 400, 0, 800$ from fits to CIPT and RCPT as a function of $\varphi_0$. The CIPT numbers are:

$$
\alpha_s(m_\tau^2, \delta_{\text{pert}} = 0.21179, K_5 = 0)_{\text{CIPT}} = 0.3572,
\alpha_s(m_\tau^2, \delta_{\text{pert}} = 0.21179, K_5 = 400)_{\text{CIPT}} = 0.35,
\alpha_s(m_\tau^2, \delta_{\text{pert}} = 0.21179, K_5 = 800)_{\text{CIPT}} = 0.3432,
\tag{16}
$$

and for RCPT the result is:

$$
\alpha_s(m_\tau^2, \delta_{\text{pert}} = 0.21179, K_5 = 0)_{\text{RCPT}} = 0.3519 \pm 0.0097,
\alpha_s(m_\tau^2, \delta_{\text{pert}} = 0.21179, K_5 = 400)_{\text{RCPT}} = 0.3498 \pm 0.0063,
\alpha_s(m_\tau^2, \delta_{\text{pert}} = 0.21179, K_5 = 800)_{\text{RCPT}} = 0.3480 \pm 0.0034,
\tag{17}
$$

where the errors are given by the RMS of the $\alpha_s$ values over the range $-\pi \leq \varphi_0 \leq 0$. Now RCPT and CIPT agree on the central value of $\alpha_s$ and both give similar estimates for the uncertainty due to (different) neglected higher order terms:

$$
\alpha_s(m_\tau^2, \delta_{\text{pert}} = 0.21179) = 0.3499 \pm 0.0072^{+0.0052}_{-0.0007\mu}.
\tag{18}
$$

where the first error is due to $K_5 \approx 400 \pm 400$ and the second due to the variation of the renormalization scale $0.4 \leq \mu^2/m_\tau^2 \leq 1.6$.

6 Numerical analysis

Using the same numerical value for $\delta_{\text{pert}} = 0.2042 \pm 0.0038_{\text{exp}} \pm 0.0033_{\text{non-pert}}$ as obtained in [9] and used in [10], where the first error is the experimental one, dominated by the
non-strange hadronic decay ratio of the $\tau$, $R_{\tau,V+A}$ and the second is due to the non-perturbative and quark-mass corrections, the results for CIPT and generalized FOPT and RCPT read:

\[
\begin{align*}
\alpha_s^{\text{CIPT}}(m_\tau^2) & = 0.3406 \pm 0.0047_{\text{exp}} \pm 0.0041_{\text{non-pert}} \pm 0.0066_{K_5}, \\
\alpha_s^{\text{FOPT}}(m_\tau^2) & = 0.3535 \pm 0.0061_{\text{exp}} \pm 0.0053_{\text{non-pert}} \pm 0.0208_{\varphi_0-0.0001_{K_5}}, \\
\alpha_s^{\text{RCPT}}(m_\tau^2) & = 0.3440 \pm 0.0030_{\text{exp}} \pm 0.0026_{\text{non-pert}} \pm 0.0061_{\varphi_0} \pm 0.0019_{K_5}.
\end{align*}
\]  

(19)

All three results agree within the error due to $\varphi_0$ which is very large for FOPT but moderate in case of RCPT. The difference between CIPT and RCPT is of the same size as the error due to $\varphi_0$ for RCPT and the average between both values (and conservatively assigning the larger of the two results errors to the average) leads to:

\[
\alpha_s(m_\tau^2) = 0.3423 \pm 0.005_{\text{exp}} \pm 0.007_{\Delta K_5} \pm 0.004_{\text{non-pert}}^{+0.005}_{-0.001\mu},
\]

(20)

where the fourth error is due to the variation of the renormalization scale. The total theoretical error (including the non-perturbative part) is with $\pm 0.008$ only marginally larger than the experimental error.

Reducing $\alpha_s$ given by eq. (20) from $m_\tau = 1.7768$ GeV to $m_{Z^0} = 91.1876$ with $m_c(m_c) = 1.27^{+0.07}_{-0.11}$ GeV and $m_b(m_b) = 4.20^{+0.17}_{-0.07}$ GeV [40] with the flavor thresholds to $n_f = 4$ and $n_f = 5$ at $m_\tau$ and $m_b$, respectively, gives:

\[
\alpha_s(m_{Z^0}^2) = 0.1213 \pm 0.0006_{\text{exp}} \pm 0.0008_{\Delta K_5} \pm 0.0004_{\text{non-pert}}^{+0.0005}_{-0.0001\mu} \pm 0.0002_{\text{ev}},
\]

(21)

where the last error is the evolution uncertainty due to the variation of the thresholds $m_q < m_{\text{thresh}} < 2m_q$ and the quark masses itself within their respective errors.

### 7 Conclusions

Modifying the usual Taylor expansion in fixed order perturbation theory by allowing starting points other than $\alpha_s(m_c^2)$ on the complex circle $|s| = m_c^2$ reveals a larger intrinsic uncertainty of FOPT than previously assumed. Giving equal weight to all possible choices and averaging over the different values of $\alpha_s^{\text{FOPT}}$ so obtained brings the FOPT result in agreement with CIPT. Since CIPT does not bear this additional intrinsic uncertainty the CIPT solution should be preferred. The large-$\beta_0$ re-summed result can be modified in its fixed order parts with a similar approach. Here the variation due to the starting point of the strong coupling on the complex circle alone does no account for the difference to CIPT. But applying 2-loop matching instead of 0-loop matching to combine the large-$\beta_0$ parts with the known fixed order parts up to $\beta_2$ cancels the difference of RCPT and CIPT. In this way all three perturbative approaches finally agree on the central value of $\alpha_s$ from the $\tau$. The final result from eq. (21) is

\[
\alpha_s(m_{Z^0}^2) = 0.1213 \pm 0.0006_{\text{exp}} \pm 0.0010_{\text{theo}},
\]

(22)

where the first error is experimental and the second theoretical including power corrections. This confirms with different theoretical arguments the large $\alpha_s$ obtained in [9] and is not compatible with the numerically lower values in [10,37,41,42].
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