ON THE LANG–TROTTER CONJECTURE FOR SIEGEL MODULAR FORMS

ARVIND KUMAR, MONI KUMARI AND ARIEL WEISS

Abstract. Let \( f \) be a genus two cuspidal Siegel modular eigenform. We prove an adelic open image theorem for the compatible system of Galois representations associated to \( f \), generalising the results of Ribet and Momose for elliptic modular forms. Using this result, we investigate the distribution of the Hecke eigenvalues \( a_p \) of \( f \), and obtain upper bounds for the sizes of the sets \( \{ p \leq x : a_p = a \} \) for fixed \( a \in \mathbb{C} \), in the spirit of the Lang–Trotter conjecture for elliptic curves.

1. Introduction

If \( A \) is a non-CM elliptic curve over \( \mathbb{Q} \) of conductor \( N \) and if \( a \in \mathbb{Z} \), then the Lang–Trotter conjecture [LT76] states that

\[
\pi_A(x, a) := \# \{ p \leq x, \ p \nmid N : a_p = a \} \sim C(A, a)x^{1/2}\frac{\log x}{\log \log x},
\]

where \( a_p = p+1 - \#A(F_p) \) and \( C(A, a) \geq 0 \) is an explicit constant. The conjecture is formulated in terms of the two-dimensional compatible system of Galois representations attached to \( A \). More generally, if \( (\rho_\ell)_{\ell} \) is an arbitrary compatible system of Galois representations of conductor \( N \), whose Frobenius polynomials are defined over a number field \( E \), then it is natural to ask for the asymptotics of the set

\[
\{ p \leq x, \ p \nmid \ell N : \text{tr} \rho_\ell(\text{Frob}_p) = a \}
\]

for \( a \in \mathcal{O}_E \). When \( (\rho_\ell)_{\ell} \) is conjugate self-dual, a generalisation of the Lang–Trotter conjecture has been formulated by V. K. Murty [Mur99, Conj. 2.15].

The goal of this paper is to estimate the size of the set \( (1.1) \) when \( (\rho_\ell)_{\ell} \) is the compatible system of Galois representations attached to a genus two Siegel modular eigenform.

Let \( f \) be a cuspidal vector-valued genus two Siegel modular eigenform of weight \( (k_1, k_2) \)—i.e. weight \( \text{Sym}^{k_1-k_2} \det^{k_2} \)—with \( k_1 \geq k_2 \geq 2 \), level \( N \) and character \( \varepsilon \). Let \( E = \mathbb{Q}(\{a_p : p \nmid N\}, \varepsilon) \) be the number field generated by the image of \( \varepsilon \) and by the eigenvalues \( a_p \) of the Hecke operators \( T_p \). Let \( F \subseteq E \) be the field fixed by the inner twists of \( f \) (see Definition 1.3 and Section 2.1).

**Theorem 1.1.** Let \( \pi \) be the cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}_\mathbb{Q}) \) associated to \( f \) and assume that its functorial lift to \( \text{GL}_4(\mathbb{A}_\mathbb{Q}) \) exists, is cuspidal, and is neither an automorphic induction nor a symmetric cube lift. For \( a \in \mathcal{O}_F \), let

\[
\pi_f(x, a) := \# \{ p \leq x : a_p = a \}.
\]

Then, for any \( \epsilon > 0 \),

\[
\pi_f(x, a) \ll_{\epsilon, f} x^{\frac{1}{(\log x)^{1+\alpha-\epsilon}}} \quad \text{for} \quad \alpha = \frac{[F : \mathbb{Q}]}{10[F : \mathbb{Q}] + 1}.
\]

**2020 Mathematics Subject Classification.** 11F80, 11F46, 11R45.
**Key words and phrases.** Siegel modular forms, Images of Galois representations, Lang–Trotter conjecture.
If \( a = 0 \), then
\[
\pi_f(x, 0) \ll_{\epsilon, f} \frac{x}{(\log x)^{1+\alpha-\epsilon}}, \quad \alpha = \frac{[F : \mathbb{Q}]}{7[F : \mathbb{Q}]+1}.
\]

If we further assume the Generalised Riemann Hypothesis (GRH), we obtain the following strengthened result:

**Theorem 1.2.** Let \( f \) be as above, and assume GRH. If \( a \in \mathcal{O}_F \), then
\[
\pi_f(x, a) \ll_{f} \frac{x^{1-\alpha}}{(\log x)^{1-2\alpha}}, \quad \alpha = \frac{[F : \mathbb{Q}]}{11[F : \mathbb{Q}]+1}.
\]

If \( a = 0 \), then
\[
\pi_f(x, 0) \ll_{f} \frac{x^{1-\alpha}}{(\log x)^{1-2\alpha}}, \quad \alpha = \frac{[F : \mathbb{Q}]}{10[F : \mathbb{Q}]+1}.
\]

The assumption that the functorial lift of \( \pi \) is cuspidal and not an automorphic induction or a symmetric cube lift is equivalent to demanding that \( f \) is not CAP, endoscopic, CM, RM or a symmetric cube lift. The existence of this functorial lift follows from work of Weissauer [Wei08] and Asgari-Shahidi [AS06] when \( k_3 > 2 \), and from Arthur’s classification [Art13, GT19] when \( k_2 = 2 \). We refer the reader to [Wei18, Sec. 2.4] for further discussion.

The Lang–Trotter conjecture for elliptic curves was first investigated by Serre [Ser81], who showed that \( \pi_A(x, a) \ll \frac{x}{(\log x)^{3/4+\epsilon}} \) unconditionally, and that \( \pi_A(x, a) \ll x^{7/8}(\log x)^{1/2} \) under GRH. These bounds were subsequently improved by Wan [Wan90] and V. K. Murty [Mur97] unconditionally, and by Murty–Murty–Saradha under GRH [MMS88]. The best current estimates when \( a \neq 0 \) are \( \pi_A(x, a) \ll \frac{x(\log x)^{1/2}}{(\log x)^{2}} \) unconditionally [TZ18], and \( \pi_A(x, a) \ll \frac{x^{4/5}}{(\log x)^{2/5}} \) under GRH [Zyw15].

These results all apply verbatim to non-CM elliptic modular forms of weight \( \geq 2 \) with integer Hecke eigenvalues, and can easily be adapted to give bounds when the field \( E \) generated by the Hecke eigenvalues is arbitrary (see, for example, [Ser81, Sec. 7]). The bounds obtained in this way are independent of the field \( E \) and its subfield \( F \). A novel feature of our result is that our bounds improve as the degree \( [F : \mathbb{Q}] \) increases.

The higher dimensional case has been studied by Cojocaru–Davis–Silverberg–Stange [CDSS17] (see also [CJS20]), who formulate a precise conjecture for generic abelian varieties, and prove analogues of Theorems 1.1 and 1.2. If \( A \) is an abelian surface over \( \mathbb{Q} \) with \( \text{End}(A) = \mathbb{Z} \), then, conjecturally, for each prime \( \ell \), the \( \ell \)-adic Galois representation attached to \( A \) should be isomorphic to the \( \ell \)-adic Galois representation attached to a Siegel modular form \( f \) of weight \( (2, 2) \), paramodular level and integer Hecke eigenvalues. For such a Siegel modular form \( f \), our unconditional bound in Theorem 1.1 exactly matches that of [CDSS17, Thm. 1] (see also Remark 5.2). Our conditional bound of \( O(x^{11/12}(\log x)^{-5/6}) \) in Theorem 1.2 is slightly stronger than the bound \( O(x^{21/22+\epsilon}) \) of [CDSS17]. The authors of [CDSS17] also formulate a precise conjecture for the asymptotics of \( \pi_A(x, a) \), including the constant \( C(A, a) \). It would be interesting to formulate such a conjecture in the case of Siegel modular forms, particularly when \( F \neq \mathbb{Q} \), however, we do not pursue that here.

The proofs of the above results all use the strategy initiated by Serre [Ser81], which combines explicit versions of the Chebotarev density theorem with precise calculations of the images of Galois representations attached to modular forms and abelian varieties [Ser98, Rib75, Mom81, Rib85]. Our strategy is similar, however, in the case of Siegel modular forms, these image results are not available. The key technical input of this paper is a precise big image theorem for Galois representations attached to Siegel modular forms.
1.1. Images of Galois representations. There is a general philosophy that the image of an automorphic Galois representation should be as large as possible, unless there is an automorphic reason for it to be small. For example, let \( f \) be a Siegel modular eigenform as in the previous section. For each prime \( \lambda \) of \( E \), there is a \( \lambda \)-adic Galois representation

\[ \rho_{\lambda} : \operatorname{Gal}(\overline{Q}/Q) \to \operatorname{GSp}_4(E_{\lambda}) \]

associated to \( f \). If \( f \) is CAP or endoscopic, then \( \rho_{\lambda} \) is reducible for all \( \lambda \). Similarly, the image of \( \rho_{\lambda} \) is small if \( f \) has complex or real multiplication, or is a symmetric cube lift.

In [Wei18], building on previous work of Dieulefait and Dieulefait–Zenteno [Die02, DZ20], the third author showed that, if \( f \) is not in one of these exceptional cases, then the image of \( \rho_{\lambda} \) is large in the following sense: for all but finitely many primes \( \lambda \) (or for all \( \lambda \mid \ell \) for a set of primes \( \ell \) of density \( 1 \) if \( k_2 = 2 \)), the image of the residual representation \( \overline{\rho}_{\lambda} : \operatorname{Gal}(\overline{Q}/Q) \to \operatorname{GSp}_4(\mathcal{O}_{E/\lambda}) \) contains \( \operatorname{Sp}_4(F_{\ell}) \).

However, in order to prove Theorems 1.1 and 1.2, we need to pin down the exact image \( \overline{\rho}_{\lambda} \) for almost all primes. This study is complicated by an additional symmetry, that of inner twists, first described by Ribet [Rib77, Rib80] for elliptic modular forms. Fix an embedding \( E \hookrightarrow \mathbb{C} \).

**Definition 1.3.** An inner twist of \( f \) is a pair \((\sigma, \chi_{\sigma})\), where \( \sigma \in \operatorname{Hom}(E, \mathbb{C}) \) and \( \chi_{\sigma} \) is a Dirichlet character, such that \( \sigma(a_p) = \chi_{\sigma}(p)a_p \) for almost all primes \( p \).

We show in Section 2.1 that the set of such \( \sigma \in \operatorname{Hom}(E, \mathbb{C}) \) form an abelian subgroup \( \Gamma \) of \( \operatorname{Aut}(E/Q) \). Let \( F = E^K \) be its corresponding fixed field. For each \( \sigma \in \Gamma \), let \( K_{\sigma} \) be the number field cut out by \( \chi_{\sigma} \), and let \( K \) be the compositum of all the \( K_{\sigma} \)'s. Then, if \( p \) splits in \( K \), we have \( \sigma(a_p) = a_p \) for all \( \sigma \in \operatorname{Gal}(E/F) \), i.e. \( a_p \in F \subseteq E \). Thus, inner twists give a restriction on the image of \( \rho_{\lambda} \): the Frobenius elements associated to a positive density of primes \( p \) have trace contained in the proper subfield \( F \) of \( E \). Let \( \mathcal{G} = \mathcal{G}_f \) be the group scheme over \( \mathbb{Z} \) such that, for each \( \mathbb{Z} \)-algebra \( R \), we have

\[ \mathcal{G}(R) = \left\{ (g, \nu) \in \operatorname{GSp}_4(\mathcal{O}_F \otimes_{\mathbb{Z}} R) \times R^\times : \operatorname{sim}(g) = \nu^{k_1 + k_2 - 3} \right\}. \]

Here, \( \operatorname{sim} : \operatorname{GSp}_4 \to \mathbb{GL}_1 \) is the similitude character. Let \( \rho_{\ell} := \bigoplus_{\lambda \mid \ell} \rho_{\lambda} \). We show in Lemma 2.8 that, for almost all primes \( \ell \), the restriction

\[ \rho_{\ell \mid K} : \operatorname{Gal}(\overline{Q}/K) \to \operatorname{GSp}_4(E \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}) \]

factors through \( \operatorname{GSp}_4(F \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}) \), and extends to a representation

\[ R_{\ell} : \operatorname{Gal}(\overline{Q}/K) \to \mathcal{G}(\mathbb{Q}_{\ell}) \]

such that the projection to \( \operatorname{GSp}_4(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Q}_{\ell}) \) is \( \rho_{\ell \mid K} \) and the projection to \( \mathbb{Q}_{\ell}^\times \) is the cyclotomic character. Moreover, up to conjugation, we can assume that \( R_{\ell} \) takes values in \( \mathcal{G}(\mathbb{Z}_{\ell}) \). If \( \mathcal{L} \) is a set of rational primes, let \( \hat{Q}_{\mathcal{L}} = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{\ell \in \mathcal{L}} \mathbb{Z}_{\ell} \) and let

\[ R_{\mathcal{L}} := \bigoplus_{\ell \in \mathcal{L}} R_{\ell} : \operatorname{Gal}(\overline{Q}/K) \to \mathcal{G}(\hat{Q}_{\mathcal{L}}) \]

be the associated adelic Galois representation.

Our main technical result is the following determination of the images of these Galois representations, which generalises the results of Serre for elliptic curves [Ser98] and of Ribet, Momose and Loeffler for elliptic modular forms [Rib75, Rib77, Mom81, Rib85, Loe17].

**Theorem 1.4.** Let \( f \) be a cuspidal vector-valued Siegel modular eigenform of weight \((k_1, k_2)\), level \( N \) and character \( \varepsilon \). Define \( E, F \) and \( \mathcal{G} \) as above. Let \( \pi \) be the cuspidal automorphic
A representation of $\operatorname{GSp}_4(A_{\mathbb{Q}})$ associated to $\pi$ and assume that its functorial lift to $\operatorname{GL}_4(A_{\mathbb{Q}})$ exists, is cuspidal, and is neither an automorphic induction nor a symmetric cube lift.

Let $\mathcal{L}'$ be the set of rational primes $\ell$ such that $\ell \geq 5$, such that $\rho_\ell|_{\mathbb{Q}_\ell}$ is de Rham and such that $\rho_\ell|_{\mathbb{Q}_\ell}$ is crystalline if $\ell \nmid N$.\footnote{If $k_2 > 2$, then $\mathcal{L}'$ consists of all primes $\ell \geq 5$. If $k_2 = 2$, then $\mathcal{L}'$ has Dirichlet density 1 by [Wei18, Thm. 1.1].} Let $\mathcal{L} \subseteq \mathcal{L}'$ be the (cofinite) subset of primes such that $\rho_\ell|_{K}$ takes values in $\operatorname{GSp}_4(F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)$. Then:

(i) For each prime $\ell \in \mathcal{L}$, the image of $R_\ell$ is an open subgroup of $\mathcal{G}(\mathbb{Z}_\ell)$.

(ii) For all but finitely many primes $\ell \in \mathcal{L}$, the image of $R_\ell$ is exactly $\mathcal{G}(\mathbb{Z}_\ell)$.

(iii) The image of $R_{\mathcal{L}}$ is an open subgroup of $\mathcal{G}(\hat{\mathbb{Q}}_{\mathbb{C}})$.

### 1.2. Methods.

#### 1.2.1. The image of Galois.

To prove Theorem 1.4, in Section 2.1, we first generalise the notion of inner twists to the setting of Siegel modular forms, and show that they give a restriction on the image of $\rho_\ell$. Our key technical input is Lemma 2.11, which shows that the field $F$ cut out by the inner twists of $f$ is the trace field of the standard representation of $\rho_\ell$, obtained by composing $\rho_\ell$ with the maps $\operatorname{GSp}_4 \to \operatorname{PGSp}_4 \sim \operatorname{SO}_5 \to \operatorname{GL}_5$. As a result, we deduce that $F$ is generated over $\mathbb{Q}$ by $\{\frac{b_p}{c(p)} : p \nmid N\}$, where $b_p$ is the coefficient of $X^2$ in the characteristic polynomial of $\rho_\ell(\text{Frob}_p)$, which does not depend on $\ell$.

Fix a prime $\ell \in \mathcal{L}$. To prove that $R_\ell$ has open image, using the definition of the inner twists and an argument using Goursat’s lemma, we prove that it is enough to show that $\rho_\lambda|_K$ has open image inside

$$G_\lambda := \left\{ g \in \operatorname{GSp}_4(O_{F_\lambda}) : \text{sim}(g) \in \mathbb{Z}_\ell^{\times(k_1+k_2-3)} \right\}$$

for each $\lambda \mid \ell$. Let $H_\lambda$ be the image of $\rho_\lambda|_K$. Our starting point for that proving $\rho_\lambda|_K$ has open image is the result of [Wei18], that $H_\lambda$ is a Zariski-closed subgroup $\operatorname{GSp}_4(F_\lambda)$. In Section 3.1, we use this result in combination with a theorem of Pink [Pin98] to show that the projective image of $\rho_\lambda|_K$ is open in $\operatorname{PGSp}_4(F_\lambda)$, from which we can deduce the result.

Similarly, to prove that $R_\ell$ is surjective, again using Goursat’s lemma, we prove that it is enough to show that $\rho_\lambda$ surjects onto $G_\lambda$ for all $\lambda \mid \ell$. The surjectivity of $\rho_\lambda$ was proven by Dieulefait [Die02, Sec. 4.7] under the assumption that the field $F = \mathbb{Q}(\{\frac{b_p}{c(p)} : p \nmid N\})$ is generated over $\mathbb{Q}$ by $b_p$ for a single prime $p$. In Lemma 3.6, using our proof that the image of $R_\ell$ is open in $\mathcal{G}(\mathbb{Z}_\ell)$, we prove that Dieulefait’s assumption holds unconditionally.

Once we have proven parts (i) and (ii) of Theorem 1.4, part (iii) follows from a straightforward but technical generalisation of the group-theoretic results of Serre [Ser98], Ribet [Rib75] and Loeffler [Loe17].

#### 1.2.2. Lang–Trotter bounds.

Our proofs of Theorem 1.1 and Theorem 1.2 are very different in nature. To prove Theorem 1.1, we use the machinery of Serre [Ser81], which works by combining the explicit Chebotarev density theorem of [LO77] with the $\ell$-adic image of Galois for a single prime $\ell$. Rather than applying this machinery to the $\lambda$-adic Galois representation $\rho_\lambda$, as Serre does for elliptic modular forms in [Ser81, Sec. 7], the precision of Theorem 1.4 allows us to apply Serre’s machinery to the full $\ell$-adic Galois representation $\rho_\ell = \bigoplus_{\lambda \mid \ell} \rho_\lambda$. As a result, we obtain stronger bounds that improve with the size of $[F : \mathbb{Q}]$. For example, applying our method in
the case of an elliptic modular form \( f \) improves Serre’s bound of \( \pi_f(x, a) \ll_{f, \varepsilon} x/(\log x)^{5/4-\varepsilon} \) to the bound

\[
\pi_f(x, a) \ll_{f, \varepsilon} \frac{x}{(\log x)^{1+\alpha-\varepsilon}}, \quad \alpha = \frac{[F : \mathbb{Q}]}{3[F : \mathbb{Q}]+1}.
\]

The fact that our bounds improve with the size of \( [F : \mathbb{Q}] \) is in accordance with the generalised Lang–Trotter conjecture of Murty [Mur99, Conj. 3.1], which predicts that as soon as \( [F : \mathbb{Q}] \geq 3 \), we should actually have \( \pi_f(x, a) = O(1) \).

In contrast, our proof of Theorem 1.2 generalises the methods of Murty–Murty–Saradha [MMS88], which work by combining the explicit Chebotarev density theorem with the mod \( \ell \) image of Galois for infinitely many primes \( \ell \).

Assume, for the sake of exposition, that \( E = F \), so that \( f \) has no inner twists, and \( K = \mathbb{Q} \). Then, if \( a \in \mathcal{O}_F \), by the definition of the residual representation \( \overline{R}_\ell \), we have

\[
\pi_f(x, a) \leq \{ p \leq x : p \nmid \ell N, \ tr\overline{R}_\ell(Frob_p) \equiv a \pmod{\ell} \} + O(1),
\]

where, by \( tr\overline{R}_\ell \), we mean the trace of the \( \text{GSp}_4 \) component of \( G \). By Theorem 1.4, the image of the residual representation \( \overline{R}_\ell \) is \( G(F_\ell) \) for all but finitely many \( \ell \in \mathcal{L} \). Hence, for such \( \ell \), \( \overline{R}_\ell \) factors through a finite Galois extension \( L/K \), with Galois group \( G(F_\ell) \). Thus, it is sufficient to bound the number of primes \( p \leq x \) such that \( \overline{R}_\ell(Frob_p) \) is contained in the conjugation invariant subset \( \{(g, \nu) \in G(F_\ell) : tr(g) \equiv a \pmod{\ell} \} \subseteq G(F_\ell) \), which we can do using the Chebotarev density theorem. However, applying Chebotarev directly to these sets would not give the strongest possible bound.

The key idea is that, under GRH, we can bound the size of \( \pi_f(x, a) \) by bounding the size of the smaller set \( \{ p \leq x : a_p = a, \ \ell \text{ splits completely in } F(p) \} \) for almost all primes \( \ell \) that split completely in \( F \) (Lemma 6.1). Here, \( F(p) \) is the splitting field of the characteristic polynomial of \( \rho_\ell(Frob_p) \). If \( p \) is in this smaller set, then the eigenvalues of \( \overline{R}_\ell(Frob_p) \) are in \( F_\ell^\times \), so \( \overline{R}_\ell(Frob_p) \) is conjugate to an upper triangular matrix. We show in Lemma 6.4 that we can bound this set by applying the Chebotarev density theorem to an abelian extension, whose Galois group is isomorphic to the group of upper triangular matrices in \( G(F_\ell) \) modulo the subgroup of unipotent upper triangular matrices. In particular, since Artin’s holomorphic conjecture is known for abelian extensions, we can bound this set by using a stronger version of the Chebotarev density theorem due to Zywina [Zyw15].

As in the proof of Theorem 1.1, rather than working with the mod \( \lambda \) Galois representations \( \overline{\rho}_\lambda : \text{Gal}((\overline{Q}/Q) \to \text{GSp}_4(\mathcal{O}_E/\lambda) \), using Theorem 1.4, we can work with the mod \( \ell \) Galois representations \( \overline{\rho}_\ell : \text{Gal}((\overline{Q}/Q) \to \text{GSp}_4(\mathcal{O}_E \otimes \mathbb{F}_\ell)) \). Applying our method in the case of an elliptic modular form \( f \) gives the bound (under GRH)

\[
\pi_f(x, a) \ll_{f, \varepsilon} \frac{x^{1-\alpha}}{(\log x)^{1-2\alpha}}, \quad \alpha = \frac{[F : \mathbb{Q}]}{4[F : \mathbb{Q}]+1}.
\]

1.3. Outline of the paper. In Section 2, we recall key properties of the Galois representations attached to Siegel modular eigenforms, and generalise the notion of inner twists to Siegel modular forms. In Section 3, we prove Theorem 1.4, which is a key technical result. In Section 4, we recall explicit versions of the Chebotarev density theorem, a variant due to Serre [Ser81] and a refinement due to Zywina [Zyw15]. Using these inputs, we prove Theorems 1.1 and 1.2 in Sections 5 and 6.

2. Galois representations attached to Siegel modular forms

Let \( f \) be a cuspidal Siegel modular eigenform of weights \( (k_1, k_2) \), \( k_1 \geq k_2 \geq 2 \), level \( N \), and character \( \varepsilon \). We assume throughout this paper that, if \( \pi \) is the cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}_\mathbb{Q}) \) attached to \( f \), then the functorial lift of \( \pi \) to \( \text{GL}_4(\mathbb{A}_\mathbb{Q}) \) exists, is cuspidal,
and is neither an automorphic induction nor a symmetric cube lift. In particular, $f$ is not CAP, endoscopic, CM, RM or a symmetric cube lift. Let $E = \mathbb{Q}(\{a_p : p \nmid N\}, \varepsilon)$ be the subfield of $\mathbb{C}$ generated by the image of $\varepsilon$ and by the Hecke eigenvalues $a_p$ of the Hecke operators $T_p$. Then $E$ is a finite extension of $\mathbb{Q}$.

For a ring $R$, let

$$\text{GSp}_4(R) = \{g \in \text{GL}_4(R) : g^*Jg = \nu J, \ \nu \in R^\times\},$$

where $J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$. For $g \in \text{GSp}_4(R)$, the constant $\nu$ is called the similitude of $g$ and is denoted $\text{sim}(g)$. Let $\text{Sp}_4(R)$ be the subgroup of elements for which $\text{sim}(g) = 1$, let $\text{PSp}_4(R) = \text{Sp}_4(R)/Z(\text{Sp}_4(R))$, where $Z(\text{Sp}_4(R))$ is the centre of $\text{Sp}_4(R)$, and let $\text{PGSp}_4(R) = \text{GSp}_4(R)/Z(\text{GSp}_4(R))$.

By the work of Taylor, Laumon and Weissauer [Tay93, Lau05, Wei05, Wei08] when $k_2 \geq 2$, and Taylor [Tay91] when $k_2 = 2$ (see also [Mok14]), for each prime $\lambda$ of $E$, there exists a continuous semisimple symplectic Galois representation

$$\rho_\lambda : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(\overline{E}_\lambda)$$

that is unramified at all primes $p \nmid \ell N$, and is characterised by the property

$$\text{tr} \rho_\lambda(\text{Frob}_p) = a_p, \ \ \text{sim} \rho_\lambda(\text{Frob}_p) = \varepsilon(p)p^{k_1+k_2-3},$$

for all primes $p \nmid \ell N$. By [Ser18, Thm. 5.2.1], we can view $\rho_\lambda$ as a representation valued in $\text{GSp}_4(\mathcal{O}_{E_\lambda})$, and define the mod $\lambda$ representation

$$\overline{\rho}_\lambda : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(\mathcal{O}_E/\lambda)$$

to be the semisimplification of the reduction of $\rho_\lambda$ mod $\lambda$. This reduction is still symplectic by [BPP+19, Lemma 4.3.6]. The representation $\rho_\lambda$ should be defined over $E_\lambda$, but this seems not to be known in general. This ambiguity does not occur for $\overline{\rho}_\lambda$, since mod $\ell$ representations are always defined over their trace field.

By work of Ramakrishnan [Ram13], Dieulefait–Zenteno [DZ20] and the third author [Wei18], the image of $\rho_\lambda$ is generically large, in the following sense. Let $\mathcal{L}'$ be the set of rational primes $\ell$ such that $\ell \geq 5$, such that $\rho|_{\mathbb{Q}_\ell}$ is de Rham and such that $\rho|_{\mathbb{Q}}$ is crystalline if $\ell \nmid N$. Then $\mathcal{L}'$ is just the set of primes $\ell \geq 5$ if $k_2 > 2$, while if $k_2 = 2$, $\mathcal{L}'$ has Dirichlet density 1 [Wei18, Thm. 1.1].

**Theorem 2.1** ([Wei18, Thms. 1.1, 1.2]).

(i) If $\ell \in \mathcal{L}'$ and $\lambda | \ell$, then $\rho_\lambda$ is absolutely irreducible.

(ii) For all but finitely many $\ell \in \mathcal{L}'$, if $\lambda | \ell$, then the image of $\overline{\rho}_\lambda$ contains a subgroup conjugate to $\text{Sp}_4(\mathbb{F}_\ell)$.

Conjecturally, Theorem 2.1 should be true with $\mathcal{L}'$ the set of all primes, however, this question is open.

**Corollary 2.2.** For all but finitely many primes $\ell \in \mathcal{L}'$, for each prime $\lambda | \ell$, the Galois representation $\rho_\lambda$ descends to a representation

$$\rho_\lambda : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GSp}_4(E_\lambda).$$

*Proof.* By Theorem 2.1, $\overline{\rho}_\lambda$ is irreducible for all $\lambda | \ell$, for all but finitely many $\ell \in \mathcal{L}'$. Hence, for such primes $\lambda$, by [BPP+19, Lemma 4.3.8] and [Car94, Théorème 2], $\rho_\lambda$ is defined over its trace field. \qed

6
For each prime $\ell$, we form the $\ell$-adic representation
\[
\rho_\ell := \bigoplus_{\lambda \nmid \ell} \rho_\lambda: \text{Gal}(\overline{Q}/Q) \to \text{GSp}_4(E \otimes Q \overline{Q}_\ell),
\]
which, as before, we may view as taking values in $\text{GSp}_4(O_E \otimes \mathbb{Z} \overline{Q}_\ell)$, and the mod $\ell$ representation
\[
\overline{\rho}_\ell := \bigoplus_{\lambda \nmid \ell} \overline{\rho}_\lambda: \text{Gal}(\overline{Q}/Q) \to \text{GSp}_4(O_E \otimes \mathbb{Z} F_\ell).
\]
Finally, let $\mathcal{L} \subseteq \mathcal{L}'$ be a set of rational primes such that, for each $\ell \in \mathcal{L}$, $\rho_\ell$ takes values in $\text{GSp}_4(E \otimes Q \overline{Q}_\ell)$. If $Q_\mathcal{L} = Q \otimes \mathbb{Z} \prod_{\ell \in \mathcal{L}} \mathbb{Z}_\ell$, then we define
\[
\rho_\mathcal{L} := \bigoplus_{\ell \in \mathcal{L}} \rho_\ell: \text{Gal}(\overline{Q}/Q) \to \text{GSp}_4(E \otimes \hat{Q}_\mathcal{L}).
\]

2.1. Inner twists. In this section, we generalise the notion of inner twists to Siegel modular forms and discuss their key properties. The results in this section are well known in the case of elliptic modular forms [Mom81, Rib85]. Fix once and for all an embedding $\sigma_0: E \hookrightarrow C$. In particular, via this embedding, we may view the eigenvalues $a_p$ as elements of $C$.

**Definition 2.3.** A inner twist of $f$ is a pair $(\sigma, \chi)$, where $\sigma \in \text{Hom}(E, C)$ and $\chi$ is a Dirichlet character, such that $\sigma(a_p) = \chi(p)a_p$ for all primes $p \nmid N$.

We define $\Gamma$ to be the set of $\sigma \in \text{Hom}(E, C)$ for which such a $\chi$ exists. It is simple to show that $(\sigma_0, \chi)$ is a inner twist for a non-trivial character $\chi$ if and only if $f$ is CM or RM. In particular, since we have assumed that $f$ is not CM or RM, for each twist $(\sigma, \chi)$, the character $\chi$ is uniquely determined by $\sigma$, and we denote it by $\chi_\sigma$.

Fix an isomorphism $C \cong \overline{Q}_\ell$ for each $\ell$. Then each $\sigma \in \text{Hom}(E, C)$ induces a map $E \otimes Q_\ell \to \overline{Q}_\ell$. If the Galois representation $\rho_\ell$ takes values in $\text{GSp}_4(E \otimes Q_\ell)$, then, for each $\sigma \in \text{Hom}(E, C)$, we can define $^\sigma\rho_\ell$ to be the composition of $\rho_\ell$ with the map $\sigma: \text{GSp}_4(E \otimes Q_\ell) \to \text{GSp}_4(\overline{Q}_\ell)$. By the Chebotarev density theorem, a pair $(\sigma, \chi)$ is an inner twist if and only if $^\sigma\rho_\ell \cong ^\sigma\rho_\ell \otimes \chi$, where we view $\chi$ as a Galois character.

**Proposition 2.4 ([Rib80, Prop. 3.2]).** If $\sigma \in \Gamma$, then $\sigma(E) \subseteq E$.

**Proof.** Let $(\sigma, \chi)$ be an inner twist. Comparing the similitudes of $^\sigma\rho_\ell$ and $^\sigma\rho_\ell \otimes \chi$, we see that $\chi^2 = \sigma(\varepsilon) \cdot \varepsilon^{-1}$. Thus $\chi$ takes values in the field $Q(\varepsilon) \subseteq E$. But then
\[
\sigma(a_p) = \chi(p)a_p \in E
\]
for all $p \nmid \ell N$. Applying the same argument with a different prime $\ell$, it follows that $\sigma(a_p) \in E$ for all $p \nmid N$, and hence that $\sigma(E) \subseteq E$. \qed

In particular, $\Gamma$ is a subset of $\text{Aut}(E/Q)$. Moreover, the inner twists $(\sigma, \chi_\sigma)$ form a group under the multiplication
\[
(\sigma, \chi_\sigma) \cdot (\tau, \chi_\tau) = (\sigma\tau, \chi_\sigma \cdot \sigma(\chi_\tau)).
\]
Thus, $\Gamma$ is a subgroup of $\text{Aut}(E/Q)$.

**Definition 2.5.** Define $F = E^\Gamma$ to be the fixed field of $\Gamma$.

In fact, $E/F$ is an abelian Galois extension [Mom81, Prop. 1.7]. Each character $\chi_\sigma$ can be regarded as character of $\text{Gal}(\overline{Q}/Q)$. Its kernel is thus an open subgroup $H_\sigma$ of $\text{Gal}(\overline{Q}/Q)$.\]
Definition 2.6. Let \( H = \bigcap_{\sigma \in \Gamma} H_{\sigma} \), and let \( K \) be the corresponding Galois extension of \( \mathbb{Q} \).

Remark 2.7. If the Nebentypus character \( \varepsilon \) is non-trivial, then, using the Petersson inner product, \((c, \varepsilon^{-1})\) is always an inner twist, where \( c \) denotes complex conjugation. In particular, group \( H \) includes the kernel of \( \varepsilon \).

Recall from Theorem 2.1 that \( \mathcal{L}' \) is the set of primes \( \ell \geq 5 \) such that \( \rho_\ell|_{\mathbb{Q}} \) is de Rham, and crystalline if \( \ell \nmid N \).

Lemma 2.8. For all but finitely many primes \( \ell \in \mathcal{L}' \), \( \rho_\ell|_K \) descends to a representation

\[
\rho_\ell|_K : \text{Gal}(\overline{\mathbb{Q}}/K) \to \text{GSp}_4(F \otimes \mathbb{Q}_\ell)
\]

that is defined over \( F \otimes \mathbb{Q}_\ell \).

Proof. By part (ii) of Theorem 2.1, the residual representation \( \overline{\rho}_\ell|_K \) is irreducible for all but finitely many primes \( \ell \in \mathcal{L}' \). Hence, by the proof of Corollary 2.2, it is enough to show that \( \text{tr} \rho_\ell(\text{Gal}(\overline{\mathbb{Q}}/K)) \subseteq F \otimes \mathbb{Q}_\ell \). By the Chebotarev density theorem, it is enough to show that for all primes \( p \nmid \ell N \) that split completely in \( K \), we have \( \text{tr} \rho_\ell(\text{Frob}_p) \neq 0 \) if \( p \mid N \) such that \( \text{GSp}_4(\mathcal{O}_{\ell} \otimes \mathbb{Z}) \) is defined over \( K \), and the projection to \( \mathbb{Q}_\ell^\times \) is the cyclotomic character. Moreover, by [Ser18, Thm. 5.2.1], we can conjugate this representation to take values in \( \mathcal{G}(\mathbb{Z}) \).

We give a second interpretation of the number field \( F \). For each prime \( p \nmid N \), let \( b_p \) denote the coefficient of \( X^2 \) in the characteristic polynomial of \( \rho_\ell(\text{Frob}_p) \). Note that \( b_p = a_p^2 - a_{p^2} - p^{k_1+k_2-4} \) is independent of \( \ell \).

Definition 2.9. Define \( \mathcal{L} \subseteq \mathcal{L}' \) to be the set of primes in \( \mathcal{L}' \) such that \( \rho_\ell|_K \) is defined over \( F \otimes \mathbb{Q}_\ell \).

In particular, by Theorem 2.1 and Lemma 2.8, the set \( \mathcal{L} \) has density 1, and contains all but finitely many primes if \( k_2 > 2 \).

Definition 2.10. Let \( \mathcal{G} \) be the group scheme over \( \mathbb{Z} \) such that, for each \( \mathbb{Z} \)-algebra \( R \), we have

\[
\mathcal{G}(R) = \left\{(g, \nu) \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z} R) \times R^\times : \text{sim}(g) = \nu^{k_1+k_2-3}\right\}.
\]

By construction, if \( \ell \in \mathcal{L} \), then \( \rho_\ell|_K \) extends to a representation

\[
R_\ell : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathcal{G}(\mathbb{Q}_\ell)
\]

such that the projection to \( \text{GSp}_4(\mathcal{O}_{\ell} \otimes \mathbb{Z} \mathbb{Q}_\ell) \) is \( \rho_\ell|_K \), and the projection to \( \mathbb{Q}_\ell^\times \) is the cyclotomic character. Moreover, by [Ser18, Thm. 5.2.1], we can conjugate this representation to take values in \( \mathcal{G}(\mathbb{Z}) \).

Lemma 2.11. Let \( F_0 = \mathbb{Q}(\{b_p/p : p \nmid N\}) \). Then \( F = F_0 \).

Proof. Fix a prime \( \ell \in \mathcal{L}' \) such that \( \rho_\ell \) is defined over \( E \otimes \mathbb{Q}_\ell \). Now, \( F_0 \) is the field generated by \( \text{tr} \text{std} \rho_\ell(\text{Frob}_p) \), where \( \text{std} \rho_\ell \) is the standard representation obtained by composing \( \rho_\ell \) with the maps

\[
\text{GSp}_4 \to \text{PGSp}_4 \xrightarrow{\sim} \text{SO}_5 \to \text{GL}_5.
\]

Indeed, we have

\[
\wedge^2 \rho_\ell \otimes \text{sim}^{-1} \simeq \text{std} \rho_\ell \oplus 1,
\]
and
\[ \text{tr std } \rho_\ell(\text{Frob}_p) = \frac{b_p}{\text{sim } \rho_\ell(\text{Frob}_p)} - 1 = \frac{b_p}{p^{k_1 + k_2 - 3\varepsilon(p)}} - 1. \]

By Theorem 2.1, changing our choice of \( \ell \) if necessary, we can assume that the projection \( \sigma_\ell \) is irreducible for all but finitely many primes \( \ell \in \mathcal{L}' \). Indeed, for all but finitely many \( \ell \in \mathcal{L}' \), the image of the residual representation \( \sigma_\ell \) contains \( \text{Sp}_4(F_{\ell}) \). Hence, the image of \( \sigma_\ell \) contains \( \text{SO}_5(F_{\ell}) \), which is an irreducible subgroup of \( \text{GL}_5(F_{\ell}) \). Hence, \( \sigma_\ell \) is irreducible.

In general, if \( \rho_1, \rho_2 : G \to \text{GSp}_4(Q_{\ell}) \) are representations, then the projective representations
\[ \text{Proj } \rho_1, \text{Proj } \rho_2 : G \to \text{PGSp}_4(Q_{\ell}) \]
are isomorphic if and only if only if \( \rho_1 \) and \( \rho_2 \) are character twists of each other. Hence, \( \sigma \in \Gamma \) if and only if the two projective representations
\[ \text{Proj } \sigma_{\rho_{\ell}}, \text{Proj } \sigma_{\rho_{\ell}} : \text{Gal}(Q/Q) \to \text{PGSp}_4(Q_{\ell}) \]
are isomorphic. Via the exceptional isomorphism, it follows that \( \sigma \in \Gamma \) if and only if \( \sigma_{\rho_{\ell}} \) and \( \sigma_{\rho_{\ell}} \) are isomorphic as \( \text{SO}_5 \)-valued representations. Following the argument of [Ram00, p. 35], we see that \( \sigma_{\rho_{\ell}} \) and \( \sigma_{\rho_{\ell}} \) are isomorphic as \( \text{SO}_5 \) representations if and only if they are isomorphic as \( \text{GL}_5 \) representations. Indeed, both representations are irreducible, and the claim follows from the lemma on pp. 34 of [Ram00].

By the Brauer–Nesbitt theorem, we find that \( \sigma \in \text{Aut}(E/Q) \) is an element of \( \Gamma = \text{Aut}(E/F) \) if and only if \( \sigma_{\rho_{\ell}} \) and \( \sigma_{\rho_{\ell}} \) have the same trace, which, by the Chebotarev density theorem, is equivalent to having \( \sigma(\text{tr std } \rho_{\ell}(\text{Frob}_p)) = \sigma_0(\text{tr std } \rho_{\ell}(\text{Frob}_p)) \) for all \( p \not\mid N \). Running the argument again with a different choice of \( \ell \), we see that \( \sigma \in \Gamma \) if and only if \( \sigma \) fixes \( \frac{b_p}{\varepsilon(p)} \) for all \( p \not\mid N \). Thus \( \sigma \in \text{Aut}(E/Q) \) fixes \( F \) if and only if \( \sigma \) fixes \( F_0 \), so \( F = F_0 \). \( \square \)

3. The image of Galois

In this section, we prove Theorem 1.4. Let \( f \) be a cuspidal Siegel modular eigenform of weights \((k_1, k_2), k_1 \geq k_2 \geq 2, \) level \( N \) and character \( \varepsilon \). Assume, as always, that \( f \) is not CAP, endoscopic, CM, RM or a symmetric cube lift. Let \( E = Q(\{a_p : p \not\mid N\}, \varepsilon) \) be the coefficient field of \( f \), and let \( F, K \) be the number fields defined in Definition 2.5 and Definition 2.6. Let \( \mathcal{L} \) be the set of primes defined in Definition 2.9 and, for \( \ell \in \mathcal{L} \), write
\[ R_\ell : \text{Gal}(Q/K) \to \mathcal{G}(Z_\ell) \]
for the Galois representation defined just after Definition 2.10.

3.1. Open image. In this subsection, we prove part (i) of Theorem 1.4. Since the image of \( R_\ell \) surjects onto \( Z_\ell^+ \), to show that the image of \( R_\ell \) is open in \( \mathcal{G}(Z_\ell) \), it is equivalent to show that the image of \( \rho_\ell|_K \) is an open subgroup of
\[ G_\ell := \left\{ g \in \text{GSp}_4(O_F \otimes Z_{\ell}) : \text{sim}(g) \in Z_{\ell}^{\times (k_1 + k_2 - 3)} \right\}. \]

Fix a prime \( \ell \in \mathcal{L} \) and write \( F \otimes Q_\ell = \prod_{\lambda | \ell} F_{\lambda} \), where the product is over the primes \( \lambda \) of \( F \) above \( \ell \). Fix a prime \( \lambda | \ell \). Denote by \( \rho_{\lambda|K} \) the representation
\[ \rho_{\lambda|K} : \text{Gal}(Q/K) \to \text{GSp}_4(F_\lambda) \]
obtained via the projection \( F \otimes Q_\ell \to F_\lambda \), and let
\[ \text{Proj } \rho_{\lambda|K} : \text{Gal}(Q/K) \to \text{PGSp}_4(F_\lambda) \]
be the associated projective Galois representation. Let \( H_{\lambda}^{\text{ad}} \) be the image of \( \text{Proj } \rho_{\lambda|K} \).
Lemma 3.1. $H^\text{ad}_\lambda$ is Zariski dense in $\text{PGSp}_4(F_\lambda)$, where $\text{PGSp}_4$ is viewed as an algebraic group over $F_\lambda$.

Proof. By the assumption that $\ell \in \mathcal{L}$, $\rho_\lambda$ is irreducible, and hence $\rho_\lambda|_K$ is irreducible by [Wei18, Lem. 5.9]. Hence, by [Wei18, Cor. 5.6, Rem. 5.5], the Zariski closure of $H^\text{ad}_\lambda$ is $\text{PGSp}_4(F_\lambda)$. □

Lemma 3.2. $H^\text{ad}_\lambda$ is an open subgroup of $\text{PGSp}_4(F_\lambda)$.

Proof. We argue as in [Con19, Prop. 3.16]. $\text{PGSp}_4$ is an absolutely simple, connected adjoint group over $F_\lambda$ and the adjoint representation of $\text{PGSp}_4$ is irreducible. Since $H^\text{ad}_\lambda$ is Zariski dense in $\text{PGSp}_4(F_\lambda)$, by [Pin98, Thm. 0.7], there is a model $H$ of $\text{PGSp}_4$, defined over a closed subfield $L \subseteq F_\lambda$, such that $H^\text{ad}_\lambda$ is an open subgroup of $H(L)$. Moreover, by [Pin98, Prop. 0.6], the field $L$ is exactly the trace field of $H^\text{ad}_\lambda$, i.e., we have $L = F_\lambda$, and hence $H = \text{PGSp}_4$. It follows that $H^\text{ad}_\lambda$ is an open subgroup of $\text{PGSp}_4(F_\lambda)$. □

Lemma 3.3. The image of $\rho_\lambda|_K$ is conjugate to an open subgroup of

$$G_\lambda := \left\{ g \in \text{GSp}_4(O_{F_\lambda}) : \text{sim}(g) \in \mathbb{Z}_\ell^{(k_1+k_2-3)} \right\}.$$  

Proof. Up to conjugation, we may assume that the image $H_\lambda$ of $\rho_\lambda|_K$ is contained in $G_\lambda$. Arguing again as in [Con19, Prop. 3.16], we see that $H_\lambda$ must contain an open subgroup of $\text{Sp}_4(F_\lambda)$. Indeed, by Lemma 3.2, the projective image $H^\text{ad}_\lambda$ of $H_\lambda$ is an open subgroup of $\text{PGSp}_4(F_\lambda)$. Since the map $\text{Sp}_4(F_\lambda) \to \text{PGSp}_4(F_\lambda)$ has degree 2, and since $H_\lambda \cap \text{Sp}_4(F_\lambda)$ surjects onto $H^\text{ad}_\lambda \cap \text{PGSp}_4(F_\lambda)$, $H_\lambda$ must contain an open subgroup of $\text{Sp}_4(F_\lambda)$. In other words, $H_\lambda$ contains a principal congruence subgroup of $\text{Sp}_4(O_{F_\lambda})$. Thus, $H_\lambda \cap \text{Sp}_4(O_{F_\lambda})$ is open in $\text{Sp}_4(O_{F_\lambda})$. Since the similitude of $\rho_\lambda$ surjects onto $\mathbb{Z}_\ell^{(k_1+k_2-3)}$, it follows that $H_\lambda$ is open in $G_\lambda$. □

Proof of Theorem 1.4 (i). We argue as in [Mom81, Thm. 4.1]. We first show that, if $\lambda_1, \lambda_2$ are two distinct primes of $F$ above $\ell$, then the representations $\rho_{\lambda_1}|_K$ and $\rho_{\lambda_2}|_K$ are not isomorphic when restricted to any finite extension.

For each $i$, let $\tilde{\lambda}_i$ be a prime of $E$ above $\lambda_i$. Then, from the diagram

$$\text{Gal}(\overline{Q}/K) \xrightarrow{\rho_{\tilde{\lambda}_i}|_K} \text{GSp}_4(E_{\tilde{\lambda}_i}) \xrightarrow{\text{sim}} \text{GSp}_4(F_{\tilde{\lambda}_i})$$

it is clear that $\rho_{\lambda_1}|_K \cong \rho_{\lambda_2}|_K$ when viewed as representations valued in $E_{\tilde{\lambda}_i}$.

The primes $\tilde{\lambda}_1, \tilde{\lambda}_2$ correspond to two embeddings $\sigma_1, \sigma_2 \in \text{Hom}(E, \overline{Q}_\ell)$. Moreover, since $\lambda_1 \neq \lambda_2$, $\sigma_1|_F \neq \sigma_2|_F$. For each $i$, $\sigma_i|_\ell \simeq \rho_{\tilde{\lambda}_i}$. Suppose that $\sigma_i|_L \simeq \sigma_i|_L$ for some finite extension $L/Q$. It follows that $\sigma_i|_\ell \simeq \chi \simeq \sigma_i|_\ell$ for some Dirichlet character $\chi$. Hence, by the definition of $F$ as the field fixed by the inner twists, we must have $\sigma_1|_F = \sigma_2|_F$, a contradiction.

We can now apply the analysis of [Rib76, Ch. IV, §4]. Let $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \overline{Q}_\ell$, where $\mathfrak{g}$ is the Lie algebra of the image of $\rho_\ell$. Let $\overline{\mathfrak{g}} = \mathfrak{g} \otimes \overline{Q}_\ell$, where $\mathfrak{g}$ is the Lie algebra of $G_\ell$.

By Lemma 3.3, the projection of $\overline{\mathfrak{g}}$ to $\mathfrak{gsp}_4(V_\sigma)$ is surjective. By the above argument, $V_{\sigma_1}$ and $V_{\sigma_2}$ are not isomorphic as $\mathfrak{g}$-modules for any distinct $\sigma_1, \sigma_2 \in \text{Hom}(F, \mathcal{C})$. It follows that the projections of $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{g}}$ onto $\mathfrak{gsp}_4(V_{\sigma_1}) \times \mathfrak{gsp}_4(V_{\sigma_2})$ are the same. By the Lie algebra version of Goursat’s lemma [Rib76, Lemma, p.790], it follows that $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}$. Since $\mathfrak{g} \subseteq \mathfrak{g}$, it follows that

10
$\frak{h} = \frak{g}$, i.e. that the image of $\rho_\ell|_K$ is an open subgroup of $G_\ell$. Hence, the image of $R_\ell$ is an open subgroup of $\mathcal{G}(\mathbb{Z}_\ell)$. \hfill \Box

3.2. The precise image for almost all primes. In this section, we prove part (ii) of Theorem 1.4. As in the previous section, to show that $R_\ell$ surjects onto $\mathcal{G}(\mathbb{Z}_\ell)$, it is sufficient to show that $\rho_\ell|_K$ surjects onto $G_\ell$. Moreover, as the following lemma shows, it is sufficient to show that the residual representation $\overline{\rho}|_K$ surjects onto

$$\overline{\mathcal{G}}_\ell := \left\{ g \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell) : \text{sim}(g) \in \mathbb{F}_\ell^{\times(k_1+k_2-3)} \right\}.$$ 

Lemma 3.4. Let $\ell \geq 5$ be prime, and let $F_1, \ldots, F_\ell$ be finite unramified extensions of $\mathbb{Q}_\ell$ with rings of integers $\mathcal{O}_1, \ldots, \mathcal{O}_\ell$ and residue fields $F_1, \ldots, F_\ell$. Let $H$ be a closed subgroup of $\text{Sp}_4(\mathcal{O}_1) \times \cdots \times \text{Sp}_4(\mathcal{O}_\ell)$, which surjects onto $\text{PSp}_4(F_1) \times \cdots \times \text{PSp}_4(F_\ell)$. Then $H = \text{Sp}_4(\mathcal{O}_1) \times \cdots \times \text{Sp}_4(\mathcal{O}_\ell)$. The result now follows from [DKR01, Lem. 2]. \hfill \Box

Corollary 3.5. Let $\ell \geq 5$ be prime that is unramified in $F$, and suppose that $H$ is a closed subgroup of $G_\ell$ that surjects onto $\overline{\mathcal{G}}_\ell$. Then $H = G_\ell$.

Proof. Let $H'$ be the commutator subgroup of $H$ and let $H_0 = H \cap \text{Sp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$. Clearly, $H_0 \supseteq H'$. Since $\text{Sp}_4(F)$ is a perfect group (i.e. a group that is equal to its own commutator subgroup) whenever $F$ is a field order at least 5, we see that $H_0$ surjects onto $\text{Sp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$. Therefore, by Lemma 3.4, $H_0 = \text{Sp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$. Hence, $H$ is a closed subgroup of $G_\ell$ that contains $\text{Sp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$ and surjects onto $\mathbb{Z}_\ell^{\times(k_1+k_2-3)}$. Thus, $H = G_\ell$. \hfill \Box

Recall Lemma 2.11, that the field $F$ is equal to $\mathbb{Q}(\{\frac{b_p}{\varepsilon(p)} : p \nmid N\})$, where $b_p$ is the coefficient of $X^2$ in the characteristic polynomial of $\rho_p(\text{Frob}_p)$.

Lemma 3.6. There exists a prime $q \nmid N$ such that $F = \mathbb{Q}(b_q)$.

Proof. We argue as in [Rib85, Thm. 3.1]. Let $H_\ell$ be the image of $\rho_\ell|_K$ and, for an element $g \in H_\ell$, let $b(g)$ denote the coefficient of $X^2$ in its characteristic polynomial. Consider the set

$$U = \left\{ g \in H_\ell : \frac{b(g)}{\varepsilon(g)} \text{ generates } F \otimes \mathbb{Q}_\ell \text{ as a } \mathbb{Q}_\ell\text{-algebra} \right\}.$$

Then $U$ is an open subset of $H_\ell$, which is in turn an open subgroup of $G_\ell$ by part (i) of Theorem 1.4. Since $G_\ell$ contains elements $g$ such that $\frac{b(g)}{\varepsilon(g)}$ generates $F \otimes \mathbb{Q}_\ell$ as a $\mathbb{Q}_\ell$-algebra, we see that $U$ is the intersection of two non-empty open subgroups of $G_\ell$, so is itself non-empty. Since $U$ is closed under conjugation, by the Chebotarev density theorem, there exists a rational prime $q \nmid \ell N$ that splits completely in $K$ such that $\rho_q(\text{Frob}_q) \in U$. Hence, $\frac{b_q}{\varepsilon(q)}$ generates $F \otimes \mathbb{Q}_\ell$ as a $\mathbb{Q}_\ell$-algebra, so $F = \mathbb{Q}(b_q)$. By Remark 2.7, $\varepsilon(q) = 1$, so $F = \mathbb{Q}(b_q)$. \hfill \Box

Lemma 3.7. For all but finitely many primes $\ell \in \mathcal{L}$, for all primes $\lambda | \ell$ of $F$, the image of $\overline{\mathcal{G}}_\lambda|_K$ is exactly

$$\overline{G}_\lambda := \left\{ g \in \text{GSp}_4(F_\lambda) : \text{sim}(g) \in \mathbb{F}_\ell^{\times(k_1+k_2-3)} \right\}.$$
Proof. We follow a similar argument to [Die02, Section 4.7]. Since the similitude of $\overline{\rho}_\lambda|K$ surjects onto $F_\ell \times (k_1+k_2-3)$, it is enough to show that the image of $\overline{\rho}_\lambda|K$ contains $\text{Sp}_4(F_\ell)$ for all $\lambda \mid \ell$, for all but finitely many primes $\ell \in \mathcal{L}$.

Let $\ell \in \mathcal{L}$ and, for each prime $\lambda \mid \ell$, let $\overline{\rho}_\lambda$ be the image of $\overline{\rho}_\lambda|K$ and let $\overline{\rho}_\lambda^{\text{ad}} \subseteq \text{PGSp}_4(F_\ell)$ be the projective image of $\overline{\rho}_\lambda|K$. By Theorem 2.1, we may assume that $\overline{\rho}_\lambda^{\text{ad}}$ contains $\text{PSp}_4(F_\ell)$.

Let $F$ be the largest subfield of $F_\ell$ such that $\overline{\rho}_\lambda^{\text{ad}}$ contains $\text{PSp}_4(F_\ell)$. We will show that $F = F_\ell$.

Let $q$ be the prime from Lemma 3.6. Suppose further that $\ell \neq q$, that $\ell$ does not divide the discriminant of $b_q$ and that $a_q = \text{tr} \rho_\lambda(\text{Frob}_q)$ is invertible modulo $\ell$; these conditions exclude only finitely many primes $\ell \in \mathcal{L}$. Then $F_\lambda = F_\ell(b_q)$. Suppose that $F \subseteq F_\lambda$. Then every element of $\overline{\rho}_\lambda^{\text{ad}}$ has a lift to $\overline{\rho}_\lambda$ with characteristic polynomial in $F[X]$. Since the characteristic polynomial of $\overline{\rho}_\lambda(\text{Frob}_q)$ is $X^4 - a q X^3 + b_q X^2 - q a q^{k_1+k_2-3} X + q^{2(k_1+k_2-3)} (\mod \lambda)$, we see that there exists an element $\ell \in F_\lambda^\times$ such that $\ell a q, \ell^2 b_q, \ell^4 a_q \in F$. Since $a_q$ is invertible modulo $\ell$, it follows that $\ell^2 = \frac{\ell^4 a_q}{a_q} \in \tilde{F}^\times$, and hence that $b_q \in F$, contradicting the fact that $F_\lambda = F_\ell(b_q)$.

It follows that $\overline{H}_\lambda^{\text{ad}}$ contains $\text{PSp}_4(F_\ell)$ and hence that $\overline{H}_\lambda$ contains $\text{Sp}_4(F_\lambda)$. Thus, $\overline{H}_\lambda = \overline{G}_\lambda$. □

Part (ii) of Theorem 1.4 now follows inductively from Goursat’s Lemma:

Lemma 3.8 (Goursat’s Lemma, [Rib75, Lem. 3.2]). Let $G_1, G_2$ be groups and let $H$ be a subgroup of $G_1 \times G_2$ for which the two projections $p_i: H \to G_i$ are surjective. Let $N_1$ be the kernel of $p_2$ and let $N_2$ be the kernel of $p_1$. Then the image of $H$ in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism $G_1/N_1 \sim \to G_2/N_2$.

Proof of Theorem 1.4 (ii). We argue as in the proof of (3.1) of [Rib75]. If $\lambda_1, \lambda_2$ are two primes of $F$ above $\ell$, then the image $\overline{H}$ of $\overline{\rho}_{\lambda_1}|K \times \overline{\rho}_{\lambda_2}|K$ is a subgroup of $\overline{G}_{\lambda_1} \times \overline{G}_{\lambda_2}$ and, by Lemma 3.7, we can assume that the image of each of the two projections to $\overline{G}_{\lambda_i}$ is surjective.

Let $N_2$ and $N_1$ be the kernels of the projections of $\overline{H}$ onto $\overline{G}_{\lambda_1}$ and $\overline{G}_{\lambda_2}$. Then, by Goursat’s lemma, the image of $\overline{H}$ in $\overline{G}_{\lambda_1}/N_1 \times \overline{G}_{\lambda_2}/N_2$ is the graph of an isomorphism $\overline{G}_{\lambda_1}/N_1 \sim \to \overline{G}_{\lambda_2}/N_2$.

Let $\overline{G}$ be the projection of $\overline{G}_\ell$ onto $\overline{G}_{\lambda_1} \times \overline{G}_{\lambda_2}$. Since the kernel of $\overline{G} \to \overline{G}_{\lambda_1}$ is $\text{Sp}_4(F_{\lambda_2})$ we have $N_2 \leq \text{Sp}_4(F_{\lambda_2})$ and similarly $N_1 \leq \text{Sp}_4(F_{\lambda_1})$. By the isomorphism $\overline{G}_{\lambda_1}/N_1 \sim \to \overline{G}_{\lambda_2}/N_2$ we have $N_2 = \text{Sp}_4(F_{\lambda_2})$ if and only if $N_1 = \text{Sp}_4(F_{\lambda_1})$, in which case $\overline{H} = \overline{G}$.

Thus, if $\overline{H}$ is a proper subgroup of $\overline{G}$, then, for each $i$, $N_i$ is a proper normal subgroup of $\text{Sp}_4(F_{\lambda_i})$, so $N_i \leq \{ \pm I \}$. The isomorphism $\overline{G}_{\lambda_1}/N_1 \sim \to \overline{G}_{\lambda_2}/N_2$ now implies that $F_{\lambda_1} = F_{\lambda_2}$ and that there are elements $\sigma \in \text{Gal}(F_{\lambda_1}/F_\ell)$ and $S \in \text{GSp}_4(F_{\lambda_1})$ such that, for each $(g_1, g_2) \in \overline{G}$, there is a scalar $\lambda(g_1, g_2)$ such that $g_2 = \lambda(g_1, g_2) \cdot (S g_1 S^{-1})$. Since $\sim \overline{\rho}_{\lambda_1}|K = \sim \overline{\rho}_{\lambda_2}|K \in F_\ell^\times$, it follows that $\lambda(g_1, g_2) = 1$ for all $(g_1, g_2) \in \overline{G}$. Hence, the characteristic polynomials of $g_2$ and $\lambda(g_1, g_2) \cdot (S g_1 S^{-1})$ and equating the coefficient of $X^2$, we find that

$$b(g_2) = \sigma(b(g_1))$$

for all $g_1, g_2 \in \overline{G}$, where $b(g_i)$ denotes the coefficient of $X^2$ in the characteristic polynomial of $g_i$. This contradicts Lemma 3.6, which states there exists a prime $q$ such that $F = \mathbb{Q}(b_q)$, i.e. if $\ell$ is large enough, the element $\text{tr} \overline{\rho}_{\ell}(\text{Frob}_q) \in \mathcal{O}_F \otimes \mathbb{Z} F_\ell$ generates $\mathcal{O}_F \otimes \mathbb{Z} F_\ell$ as an $F_\ell$-algebra.

It follows that $\overline{H}$ of $\overline{\rho}_{\lambda_1}|K \times \overline{\rho}_{\lambda_2}|K$ surjects onto $\overline{G}$. Hence, by induction, the image of $\overline{\rho}_{\ell}|K$ is $\overline{G}_{\ell}$, and the result follows from Corollary 3.5. □
3.3. Adelic large image. Finally, we prove part (iii) of Theorem 1.4. We begin with some group theoretic results, which are mostly generalisations of [Ser98, Ch. IV] and [Loc17, Sec. 1].

Definition 3.9 ([Ser98, IV-25]). Let \( Y \) be a profinite group, and let \( \Sigma \) be a non-abelian finite simple group. We say that \( \Sigma \) occurs in \( Y \) if there exist closed subgroups \( Y_1, Y_2 \) of \( Y \) such that \( Y_1 \triangleleft Y_2 \) and \( Y_2/Y_1 \simeq \Sigma \). We write \( \text{Occ}(Y) \) for the set of non-abelian finite simple groups occurring in \( Y \).

Lemma 3.10 ([Ser98, IV-25]). If \( Y = \lim Y_a \) and each \( Y \to Y_a \) is surjective, then \( \text{Occ}(Y) = \bigcup \text{Occ}(Y_a) \). If \( Y \) is an extension of \( Y' \) and \( Y'' \), then \( \text{Occ}(Y) = \text{Occ}(Y') \cup \text{Occ}(Y'') \).

Lemma 3.11. Let \( L \) be a finite extension of \( \mathbb{Q}_\ell \) for some prime \( \ell \), with ring of integers \( \mathcal{O} \), uniformiser \( \varpi \) and residue field \( \mathbf{F} \). We have \( \text{Occ}(\text{GSp}_4(\mathcal{O})) = \text{Occ}(\text{Sp}_4(\mathcal{O})) = \text{Occ}(\text{PSp}_4(\mathbf{F})) \).

Proof. We argue as in [Kan05, Lem. 10]. Since \( \text{GSp}_4(\mathcal{O})/\text{Sp}_4(\mathcal{O}) \) is abelian, the first equality follows from Lemma 3.10. Similarly, \( \text{Occ}(\text{PSp}_4(\mathbf{F})) = \text{Occ}(\text{Sp}_4(\mathbf{F})) \). Since \( \text{Sp}_4(\mathcal{O}) = \varprojlim \text{Sp}_4(\mathcal{O}/\varpi^n) \), by Lemma 3.10 again, we have \( \text{Occ}(\text{Sp}_4(\mathcal{O})) = \bigcup_n \text{Occ}(\text{Sp}_4(\mathcal{O}/\varpi^n)) \). It remains to show that \( \text{Occ}(\text{Sp}_4(\mathcal{O}/\varpi^n)) = \text{Occ}(\text{Sp}_4(\mathbf{F})) \) for each \( n \).

Observe that the kernel \( X \) of \( \text{Sp}_4(\mathcal{O}/\varpi^n) \to \text{Sp}_4(\mathbf{F}) \) is an \( \ell \)-group. Indeed, any matrix in the kernel can be written as \( I + \varpi A \) where \( A \in M_4(\mathcal{O}/\varpi^n) \). Hence, in \( M_4(\mathcal{O}/\varpi^n) \), we have \( (I + \varpi A)^\ell = I \), so every element of \( X \) has order a power of \( \ell \).

Since every \( \ell \)-group is solvable, it follows that \( \text{Occ}(X) = \emptyset \), so \( \text{Occ}(\text{Sp}_4(\mathcal{O}/\varpi^n)) = \text{Occ}(\text{Sp}_4(\mathbf{F})) \) by Lemma 3.10.

\( \square \)

Corollary 3.12. If \( p \neq 2, \ell \) and \( q = p^r \) for some \( r \), then \( \text{PSp}_4(\mathbf{F}_q) \notin \text{Occ}(\text{Sp}_4(\mathcal{O})) \).

Proof. By Lemma 3.11, \( \text{Occ}(\text{GSp}_4(\mathcal{O})) = \text{Occ}(\text{PSp}_4(\mathbf{F})) \). The result is now immediate from the classification of the maximal subgroups of \( \text{PSp}_4(\mathbf{F}) \) [Mit14].

\( \square \)

Recall that \( \mathcal{G} \) is the group scheme whose \( R \) points are

\[ \mathcal{G}(R) = \left\{ (g, \nu) \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbf{Z} R) \times R^\times : \text{sim}(g) = \nu^{k_1+k_2-3} \right\}. \]

Let

\[ \mathcal{G}^0(R) = \text{Sp}_4(\mathcal{O}_F \otimes \mathbf{Z} R). \]

In particular, we have \( \mathcal{G}^0(\mathbf{Z}_\ell) = \prod_{\lambda \nmid \ell} \text{Sp}_4(\mathcal{O}_{F_{\lambda}}) \), where the product is over primes \( \lambda \) of \( F \) above \( \ell \), and \( \mathcal{O}_{F_{\lambda}} \) is the ring of integers of the completion \( F_{\lambda} \) of \( F \).

Theorem 3.13. Fix a set of primes \( \mathcal{L} \). Let \( U^0 \) be a closed compact subgroup of \( \mathcal{G}^0(\hat{\mathbf{Q}}_\mathcal{L}) \) such that:

- for every prime \( \ell \in \mathcal{L} \), the projection of \( U^0 \) to \( \mathcal{G}^0(\mathbf{Q}_\ell) \) is open in \( \mathcal{G}^0(\mathbf{Q}_\ell) \);
- for all but finitely many primes \( \ell \in \mathcal{L} \), the projection of \( U^0 \) to \( \mathcal{G}^0(\mathbf{Q}_\ell) \) is \( \mathcal{G}^0(\mathbf{Z}_\ell) \).

Then \( U^0 \) is open in \( \mathcal{G}^0(\hat{\mathbf{Q}}_\mathcal{L}) \).
Proof. This is a generalisation of [Ser98, Main Lemma, IV-19] and [Loe17, Thm. 1.2.2]. Let $S \subseteq \mathcal{L}$ be a finite set of primes containing $\{2, 3, 5\} \cap \mathcal{L}$ and the finitely many primes such that the projection $U^o$ to $\mathcal{G}^o(Q_\ell)$ is not $\mathcal{G}^o(Z_\ell)$.

First, note that the projection $U^o_S$ of $U^o$ to $\prod_{\ell \in S} \mathcal{G}^o(Q_\ell)$ is open. We argue as in [Ser98, Lem. 4 pp. IV-24]. Replacing $U^o$ with a finite index subgroup, we can assume that, for each $\ell \in S$, the projection of $U^o$ to $\mathcal{G}^o(Q_\ell)$ is contained in the group of elements congruent to $1 \mod \ell$, i.e. is pro-$\ell$. It follows that $U^o_S$ is pro-nilpotent, and hence is a product of its Sylow subgroups. Thus, $U^o_S \cong \prod_{\ell \in S} U^o_{\ell}$, where $U^o_{\ell}$ is the projection of $U^o$ to $\mathcal{G}^o(Q_\ell)$. Since each $U^o_{\ell}$ is open in $\mathcal{G}^o(Q_\ell)$ by assumption, it follows that $U^o_S$ is open in $\prod_{\ell \in S} \mathcal{G}^o(Q_\ell)$.

To conclude, it is sufficient to show that $U^o$ contains an open subgroup of $\mathcal{G}^o(Q_\ell)$. Since $U^o_S$ is open in $\prod_{\ell \in S} \mathcal{G}^o(Q_\ell)$, it is enough to show that $\prod_{\ell \in \mathcal{L}\setminus S} \mathcal{G}^o(Z_\ell) \subseteq U^o$. To show this, it is enough to show that $U^o$ contains $\mathcal{G}^o(Z_\ell) = \text{Sp}_4(O_F \otimes Z_\ell)$ for every $\ell \in \mathcal{L}\setminus S$.

For each $\ell \in \mathcal{L}\setminus S$, let $H_\ell = U^o \cap \mathcal{G}^o(Z_\ell)$. By assumption, the projection of $U^o$ to $\mathcal{G}^o(Q_\ell)$ is $\mathcal{G}^o(Z_\ell)$, which in turn surjects onto $\text{PSp}_4(F_\lambda)$ for each $\lambda \mid \ell$. Hence, $\text{PSp}_4(F_\lambda) \in \text{Occ}(U^o)$. On the other hand, $U^o/H_\ell$ is isomorphic to a closed subgroup of $\prod_{q \in \mathcal{L}\setminus \{\ell\}} \mathcal{G}^o(Q_q)$, so by Corollary 3.12, $\text{PSp}_4(F_\lambda) \notin \text{Occ}(U^o/H_\ell)$. It follows from Lemma 3.10 that $\text{PSp}_4(F_\lambda) \in \text{Occ}(H_\ell)$. Hence, $H_\ell$ is a subgroup of $\prod_{\lambda \mid \ell} \text{Sp}_4(O_{F_\lambda})$ whose projection to $\prod_{\lambda \mid \ell} \text{PSp}_4(F_\lambda)$ is surjective. It follows from Lemma 3.4 that $H_\ell = \prod_{\lambda \mid \ell} \text{Sp}_4(O_{F_\lambda}) = \mathcal{G}^o(Z_\ell)$. The result follows.

**Theorem 3.14.** Fix a set of primes $\mathcal{L}$. Let $U$ be a closed compact subgroup of $\mathcal{G}(\hat{Q}_\mathcal{L})$ such that:

- for every prime $\ell \in \mathcal{L}$, the projection of $U$ to $\mathcal{G}(Q_\ell)$ is open in $\mathcal{G}(Q_\ell)$;
- for all but finitely many primes $\ell \in \mathcal{L}$, the projection of $U$ to $\mathcal{G}(Q_\ell)$ is $\mathcal{G}(Z_\ell)$;
- the image of $U$ in $\hat{Q}_\mathcal{L}^x$ is open.

Then $U$ is open in $\mathcal{G}(\hat{Q}_\mathcal{L})$.

**Proof.** We follow [Loe17, Thm. 1.2.3]. Let $U^o = U \cap \mathcal{G}^o(\hat{Q}_\mathcal{L})$. We claim that $U^o$ satisfies the hypotheses of Theorem 3.13. Since $\mathcal{G}(Q_\mathcal{L})/\mathcal{G}^o(\hat{Q}_\mathcal{L}) \cong \hat{Q}_\mathcal{L}^x$ is abelian, the group $U^o$ contains the closure of the commutator subgroup of $U$. When $\ell \geq 3$, $\text{Sp}_4(O_F \otimes Z_\ell)$ is the closure of its commutator subgroup. Hence, if $\ell \geq 3$ and $U$ surjects onto $\mathcal{G}(Z_\ell)$, then $U^o$ surjects onto $\mathcal{G}^o(Z_\ell) = \text{Sp}_4(O_F \otimes Z_\ell)$. When $\ell = 2$ the commutator subgroup of $\text{Sp}_4(O_F \otimes Z_\ell)$ still has finite index. Hence, $U^o$ satisfies the hypotheses of Theorem 3.13. Thus, $U$ contains an open subgroup of $\mathcal{G}^o(Q_\mathcal{L})$. Since the image of $U$ in $\hat{Q}_\mathcal{L}^x \cong \mathcal{G}(\hat{Q}_\mathcal{L})/\mathcal{G}^o(\hat{Q}_\mathcal{L})$ is open, it follows that $U$ is open in $\mathcal{G}^o(Q_\mathcal{L})$.

**Proof of Theorem 1.4 (iii).** The result now follows immediately from Theorem 1.4 parts (i) and (ii) and Theorem 3.14.

3.4. The image of $\text{Gal}(\overline{Q}/Q)$. In Theorem 1.4, we computed the image of $R_\ell$ for all but finitely many primes $\ell \in \mathcal{L}$. Suppose that $\ell \in \mathcal{L}$ is such that the image of $R_\ell$ is surjective. In this section, we use methods of E. Papier (see [Rib85, Thm. 4.1]) to compute the image of

$$\rho_\ell: \text{Gal}(\overline{Q}/Q) \to \text{GSp}_4(O_E \otimes Z_\ell).$$

Suppose that $(\sigma, \chi_\sigma)$ is an inner twist, and let $\sigma(\rho_\ell)$ denote the representation obtained by composing $\rho_\ell$ with $\sigma: O_E \otimes Z_\ell \to O_E \otimes Z_\ell$. Then the representations $\sigma(\rho_\ell)$ and $\rho_\ell \otimes \chi_\sigma$ have
the same trace. Since both are semisimple, it follows that they are isomorphic, so there is a matrix \( X \in \text{GSp}_4(E \otimes \mathbb{Q}) \) such that
\[
X \sigma(\rho_\ell) X^{-1} = \rho_\ell \otimes \chi_\sigma.
\]
By definition, \( \chi_\sigma|_K \) is trivial and \( \sigma(\rho_\ell|_K) = \rho_\ell|_K \). It follows that \( X \) commutes with the image of \( \rho_\ell|_K \). Since, for example, the image of \( \rho_\ell|_K \) contains \( \text{Sp}_4(\mathcal{O}_F \otimes \mathbb{Z} \mathbb{Z}_\ell) \), we see that \( X \) is a scalar matrix, so there is an equality of matrices
\[
\sigma(\rho_\ell(\gamma)) = \rho_\ell(\gamma) \chi_\sigma(\gamma)
\]
for all \( \gamma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) and for all inner twists \((\sigma, \chi_\sigma)\).

Recall that the group structure on the inner twists is given by
\[
(\sigma, \chi_\sigma) \cdot (\tau, \chi_\tau) = (\sigma\tau, \chi_\sigma \cdot \sigma(\chi_\tau)).
\]
Hence, the map \( \sigma \mapsto \chi_\sigma \) defines an element of \( H^1(\mathbb{Q}, \overline{\mathbb{Q}}^\times) \). By Hilbert’s theorem 90, this cohomology group is trivial, so, for each \( \gamma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), we can choose \( \alpha(\gamma) \in \mathbb{E}^\times \) such that
\[
\chi_\sigma(\gamma) = \frac{\sigma(\alpha(\gamma))}{\alpha(\gamma)}
\]
for all \( \sigma \in \Gamma \). Moreover, we can choose the elements \( \alpha(\gamma) \) to be independent of \( \ell \), and so that \( \alpha(\gamma) \) only depends on the image of \( \gamma \) in \( \text{Gal}(K/\mathbb{Q}) \). Thus, there are exactly \([K : \mathbb{Q}]\) numbers \( \alpha(\gamma) \) and, when \( \ell \) is large enough, we will have \( \alpha(\gamma) \in \mathcal{O}_E \otimes \mathbb{Z} \mathbb{Z}_\ell \) for all \( \gamma \). We deduce the following generalisation of [Rib85, Thm. 4.1]:

**Theorem 3.15.** For all but finitely many primes \( \ell \in \mathcal{L} \), the image of \( \rho_\ell \) is generated by
\[
G_\ell = \left\{ g \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z} \mathbb{Z}_\ell) : \text{sim}(g) \in \mathbb{Z}_\ell^{\times}(k_1+k_2-3) \right\}
\]

**Proof.** By definition, for all \( \gamma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \),
\[
\rho_\ell(\gamma) \alpha(\gamma)^{-1} = \rho_\ell(\gamma) \chi_\sigma(\gamma) \sigma(\alpha(\gamma))^{-1} = \sigma(\rho_\ell(\gamma) \alpha(\gamma)^{-1}).
\]
Hence, \( \rho_\ell(\gamma) \alpha(\gamma)^{-1} \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z} \mathbb{Z}_\ell) \). Taking similitudes, we see that \( \varepsilon(\gamma) \alpha(\gamma)^{-2} \in (\mathcal{O}_F \otimes \mathbb{Z} \mathbb{Z}_\ell)^{\times} \).

Moreover, we have an equality
\[
\rho_\ell(\gamma) = \left( \begin{array}{cc}
\alpha(\gamma) & \varepsilon(\gamma)/\alpha(\gamma) \\
\alpha(\gamma) & \varepsilon(\gamma)/\alpha(\gamma)
\end{array} \right) \left( \begin{array}{cc}
1 & \alpha(\gamma)^2/\varepsilon(\gamma) \\
\alpha(\gamma)^2/\varepsilon(\gamma) & 1
\end{array} \right) \alpha(\gamma)^{-1} \rho_\ell(\gamma),
\]
and the product in the second set of brackets belongs to \( G_\ell \). The result now follows from Theorem 1.4. \( \square \)

**Corollary 3.16.** For all but finitely many primes \( \ell \in \mathcal{L} \), the image of \( \rho_\ell \) is the disjoint union of at most \([K : \mathbb{Q}]\) cosets
\[
\prod \left( \begin{array}{cc}
\alpha(\gamma) & \varepsilon(\gamma)/\alpha(\gamma) \\
\alpha(\gamma) & \varepsilon(\gamma)/\alpha(\gamma)
\end{array} \right) G_\ell,
\]
where \( \gamma \) ranges over some subset of \( \text{Gal}(K/\mathbb{Q}) \).
4. The Chebotarev density theorem

Let $L/K$ be a Galois extension of number fields, with Galois group $G$. Let $M_K$ denote the set of primes of $K$ and, for each prime $p \in M_K$ that is unramified in $L$, let $\text{Frob}_p \in G$ be a choice of Frobenius element. Let $C$ be a non-empty subset of $G$ that is stable under conjugation.

**Definition 4.1.** For any $x > 0$, define

$$\pi_C(x, L/K) := \# \{ p \in M_K, \text{unramified in } L : N_{K \mathbb{Q}}(p) \leq x, \text{Frob}_p \in C \}.$$  

The Chebotarev density theorem states that

$$\pi_C(x, L/K) \sim \frac{|C|}{|G|} \pi(x).$$

To obtain explicit bounds on the size of $\pi_C(x, L/K)$, we will require an effective version of the Chebotarev density theorem.

4.1. Unconditional effective Chebotarev. The following theorem is unconditional:

**Theorem 4.2** ([LMO79, Thm. 1.4]). Assume that $L/K$ is finite. There exist constants $c_1, c_2$ such that, if

$$\log x > c_1 (\log |\text{disc}(L/\mathbb{Q})|)(\log \log |\text{disc}(L/\mathbb{Q})|)(\log \log \log |6 \text{disc}(L/\mathbb{Q})|),$$

then

$$\pi_C(x, L/K) \leq c_2 \frac{|C|}{|G|} \text{Li}(x),$$

where $\text{Li}(x) = \int_2^x \frac{dt}{\log t}$ is the logarithmic integral function.

Using this theorem, Serre [Ser81] shows how to deduce upper bounds for $\pi_C(x, L/K)$ assuming that $G = \text{Gal}(L/K)$ is an $\ell$-adic Lie group. Indeed, let $G$ be a compact $\ell$-adic Lie group of dimension $D$ and let $C \subseteq G$ be a non-empty closed subset that is stable under conjugation.

**Definition 4.3** ([Ser81, Sec. 3]). Let $C_n$ denote the image of $C$ in $G/\ell^n G$. We say that $C$ has *Minkowski dimension* $\dim_M(C) \leq d$ if $|C_n| = O(\ell^{nd})$ as $n \to \infty$.

Now, if $s \in C$, then the centraliser $Z_G(s)$ of $s$ is a closed Lie subgroup of $G$, so has a well-defined dimension.

**Theorem 4.4** ([Ser81, Thm. 12]). Suppose that $\dim_M(C) \leq d$, and set

$$r_C = \inf_{s \in C} \dim \frac{G}{Z_G(s)}.$$  

Then, for any $\epsilon > 0$,

$$\pi_C(x, L/K) \ll \frac{x}{\log(x)^{1+\alpha-\epsilon}},$$

where $\alpha = (D - d)/(D - r_C/2)$. 

---

16
4.2. Chebotarev density under the Generalised Riemann Hypothesis. Assuming that the Generalised Riemann Hypothesis holds for $L$, there are stronger effective versions of the Chebotarev density theorem. The following version is due to Lagarias–Odlyzko [LO77].

**Theorem 4.5** ([Ser81, Thm. 4]). Assume that the Dedekind zeta function $\zeta_L(s)$ satisfies the Riemann Hypothesis. Then

$$\pi_C(x, L/K) = \frac{|C|}{|G|} \log x + O\left( \frac{|C|}{|G|} x^{1/2} (\log |\text{disc}(L/Q)| + [L : Q] \log x) \right).$$

Finally, if we further assume that $L/K$ is abelian, then Artin’s holomorphicity conjecture is known to hold for $L/K$, and we obtain the following stronger result due to Zywina:

**Theorem 4.6** ([Zyw15, Thm. 2.3]). Suppose that $L/K$ is finite and abelian. Assume that the Dedekind zeta function $\zeta_L(s)$ satisfies the Riemann Hypothesis. Then

$$\pi_C(x, L/K) \ll \frac{|C|}{|G|} x \log x + [K : Q] \log x + [K : Q] \frac{1}{2} \log(M(L/K)),$$

where

$$M(L/K) := 2[L : K] \log(L/K) \prod_{p \in \mathcal{P}(L/K)} p.$$

Here, we define $\mathcal{P}(L/K)$ to be the set of rational primes $p$ that are divisible by some prime $p$ of $K$ that ramifies in $L$.

In order to apply these theorems, it is helpful to have a bound on $\log |\text{disc}(L/Q)|$. We will use the following result:

**Proposition 4.7** ([Ser81, Prop. 5]). Let $L/K$ be a finite extension of number fields. Then

$$\log |N_{K/Q}(\text{disc}(L/K))| \leq \left( [L : Q] - [K : Q] \right) \sum_{p \in \mathcal{P}(L/K)} \log p + [L : Q] \log[L : K].$$

In order to estimate the size of $\pi_C(x, L/K)$, it is often more convenient to study the following weighted version:

**Definition 4.8.** For any $x > 0$, define

$$\tilde{\pi}_C(x, L/K) := \sum_{p \in \mathcal{P}(L/K), m \geq 1} \frac{1}{m} \delta_C(\text{Frob}_p^m),$$

where $\delta_C : G \to \{0, 1\}$ is such that $\delta_C(g) = 1$ if and only if $g \in C$.

This weighted sum $\tilde{\pi}_C(x, L/K)$ is a good approximation of $\pi_C(x, L/K)$:

**Lemma 4.9** ([Zyw15, Lem. 2.7]). We have

$$\tilde{\pi}_C(x, L/K) = \pi_C(x, L/K) + O\left( [K : Q] \left( \frac{x^{1/2}}{\log x} + \log M(L/K) \right) \right),$$

where $M(L/K)$ is the constant defined in Theorem 4.6.

We end this section by recalling the following result of Zywina [Zyw15, Lemma 2.6].
Lemma 4.10. (i) Let $H$ be a subgroup of $G$ and suppose that every element of $C$ is conjugate to some element of $H$. Then

$$\overline{\pi}_C(x, L/K) \leq \overline{\pi}_{C \cap H}(x, L/L^H).$$

(ii) Let $N$ be a normal subgroup of $G$ and suppose that $NC \subset C$. Then

$$\overline{\pi}_C(x, L/K) = \overline{\pi}_N(x, L^N/K),$$

where $\overline{\pi}$ is the image of $C$ in $G/N = \text{Gal}(L^N/K)$.

5. UNCONDITIONAL BOUNDS ON $\pi_f(x, a)$

Recall that $f$ is a cuspidal Siegel modular eigenform of weights $(k_1, k_2)$, $k_1 \geq k_2 \geq 2$, level $N$ and character $\varepsilon$. Assume that $f$ is not CAP, endoscopic, CM, RM or a symmetric cube lift. Let $E = \mathbb{Q}(\{a_p : p \nmid N\}, \varepsilon)$ be the coefficient field of $f$, and let $F, K$ be the number fields defined in Definition 2.5 and Definition 2.6. Let $\mathcal{L}$ be the set of primes defined in Definition 2.9 and, for $\ell \in \mathcal{L}$, write

$$R_\ell : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathcal{G}(\mathbb{Z}_\ell)$$

for the Galois representation defined just after Definition 2.10. Here,

$$\mathcal{G}(\mathbb{Z}_\ell) = \left\{(g, \nu) \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell) \times \mathbb{Z}_\ell^\times : \text{sim}(g) = \nu^{k_1+k_2-3}\right\}.$$

In particular, the projection of $R_\ell$ onto $\text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$ is exactly $\rho|_K$, and its projection to $\mathbb{Z}_\ell^\times$ is the $\ell$-adic cyclotomic character. By Theorem 1.4, $R_\ell$ has open image in $\mathcal{G}(\mathbb{Z}_\ell)$ for all $\ell \in \mathcal{L}$ and is surjective for all but finitely many $\ell \in \mathcal{L}$. Using the fact that the dimension of the $\ell$-adic Lie group $\text{Sp}_4(\mathcal{O}_F)$ is $10|F_\chi : \mathbb{Q}_\ell|$ for each prime $\chi \mid \ell$, we see that

$$\dim \mathcal{G}(\mathbb{Z}_\ell) = 10|F : \mathbb{Q}| + 1.$$  \hspace{1cm} (5.1)

5.1. The case $a \neq 0$. Fix a non-zero algebraic integer $a \in \mathcal{O}_F$. Our goal is to bound the size of the set $\pi_f(x, a) := \{p \leq x : a_p = a\}$.

Proposition 5.1. Assume that $a \neq 0$. Then

$$\pi_f(x, a) = \frac{1}{[K : \mathbb{Q}]} \# \{p \in M_K, N(p) = p \leq x : a_p = a\}.$$  \hspace{1cm} (5.2)

Proof. Recall from Section 2.1 that, by definition, $\Gamma = \text{Gal}(E/F)$ is the group of $\sigma \in \text{Aut}(E/\mathbb{Q})$ such that $(\sigma, \chi_\sigma)$ is an inner twist. Hence, if $a_p = a$ for some non-zero $a \in \mathcal{O}_F$, then, for every inner twist $(\sigma, \chi_\sigma)$, we have

$$a_p = \sigma(a_p) = \chi_\sigma(p) a_p,$$

from which it follows that $\chi_\sigma(p) = 1$. Since $K$ is, by definition, the field cut out by all the $\chi_\sigma$’s, we see that, if $a_p = a$ for a prime $p$, then $p$ splits completely in $K$. \hfill \square

It follows from Proposition 5.1 that bounding the size of $\pi_f(x, a)$ is exactly the same as bounding the size of $\# \{p \in M_K, N(p) = p \leq x : a_p = a\}$, up to the constant $[K : \mathbb{Q}]$.

Let

$$C_\ell(a) = \{(g, \nu) \in \text{Im} R_\ell \subseteq \mathcal{G}(\mathbb{Z}_\ell) : \text{tr}(g) = a\}.$$  \hspace{1cm} (5.2)

Then, for any prime $\ell \in \mathcal{L}$, we have

$$\# \{p \in M_K, N(p) = p \leq x : a_p = a\} = \pi_{C_\ell(a)}(x, L/K) + O(1),$$

where $L$ is the fixed field of the kernel of $R_\ell$, and the $O(1)$ is to account for the finitely many primes $p \mid \ell N$. In order to prove Theorem 1.1, we use Theorem 4.4 to estimate the size of $\pi_{C_\ell(a)}(x, L/K)$.  

18
Proof of Theorem 1.1 (i). We show that the set $C(a)$ has Minkowski dimension $9[F : Q] + 1$. Write $a = (a_\alpha)_\lambda$ and $\text{tr}(g) = (\text{tr}(g)_\lambda)_\alpha$ via the isomorphism $F \otimes Q \isom \prod_{\ell \not \mid \infty} F_\ell$. Then $C(a)$ is the closed subset of $\text{Im} R_\ell$ cut out by the $[F : Q]$ equations $\text{tr}(g)_\lambda = a_\lambda$. By Theorem 1.4, $\text{Im} R_\ell$ is an open subgroup of $G(Z_\ell)$. Hence, rescaling by $s$, it has dimension $10[F : Q] + 1$ as an $\ell$-adic Lie group. By [Ser81, Sec. 3.2], it follows that $\dim_M C(a) \leq (10[F : Q] + 1) - [F : Q] = 9[F : Q] + 1$. It follows from Theorem 4.4 that

$$\pi_{C_\ell(a)}(x, L/K) \ll \frac{x}{\log(x)^{1 + \alpha - \epsilon}},$$

where $\alpha = \frac{[F : Q]}{10[F : Q] + 1}$. The result follows from Proposition 5.1 and (5.2).

5.2. The case $a = 0$. Let $PG_\ell$ denote the image of the set

$$G_\ell = \left\{ g \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell) : \text{sim}(g) \in \mathbb{Z}_\ell^{(k_1 + k_2 - 3)} \right\}$$

in $\text{PGSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$, and let $H_\ell$ denote the image of $\text{Proj} \rho_\ell : \text{Gal}(\overline{Q}/Q) \to \text{PGSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$. Let $L = \overline{Q}^{H_\ell}$ be the field cut out by the kernel of $\text{Proj} \rho_\ell$. By Theorem 3.15, there is a subgroup of $H_\ell$ of index at most $[K : Q]$ that is open in $PG_\ell$. In particular, $H_\ell$ and $PG_\ell$ have the same dimension as $\ell$-adic Lie groups, i.e. both have dimension $9[F : Q] + 1$.

We can write $PG_\ell = \prod_{\lambda \not \mid \ell} PG_\lambda$, where $PG_\lambda$ is the image of the set

$$G_\lambda = \left\{ g \in \text{GSp}_4(\mathcal{O}_{F_\lambda}) : \text{sim}(g) \in \mathbb{Z}_\ell^{(k_1 + k_2 - 3)} \right\}$$

in $\text{PGSp}_4(\mathcal{O}_{F_\lambda})$. Since $H_\ell$ is contained in a union of cosets of $G_\ell$, we can similarly define $H_\lambda$ to be the projection of $H_\ell$ onto the corresponding coset of $PG_\lambda$.

Proof of Theorem 1.1 (ii). Set

$$C(0) = \left\{ g \in H_\ell : \text{tr}(g) = 0 \right\}.$$

Note that having trace 0 is well defined on $\text{PGSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$, so this definition makes sense. As in the previous section, we have $\dim H_\ell - \dim_M C(0) = [F : Q]$.

We compute the quantity

$$r = r_{C(0)} = \inf_{s \in C(0)} \dim \frac{H_\ell}{Z_{H_\ell}(s)}.$$

If $s \in C(0)$, then, by Corollary 3.16, we can write

$$s = \left( \frac{\alpha}{\varepsilon / \alpha} \right) h,$$

where $h \in PG_\ell$ and $\alpha, \varepsilon \in \mathcal{O}_E \otimes \mathbb{Z}_\ell$. Moreover, as in Theorem 3.15, we have $\varepsilon / \alpha^2 \in \mathcal{O}_F \otimes \mathbb{Z}_\ell$. Hence, rescaling by $\alpha^{-1}$, we can assume that $s \in \text{PGSp}_4(\mathcal{O}_F \otimes \mathbb{Z}_\ell)$.

Write $s = (s_\lambda)_\lambda$ via the isomorphism $F \otimes Q = \prod_{\lambda \not \mid \ell} F_\lambda$. Then we have

$$\dim \frac{H_\ell}{Z_{H_\ell}(s)} = \sum_{\lambda \not \mid \ell} \dim H_\lambda - \dim Z_{H_\lambda}(s_\lambda).$$

We now argue as in [CDSS17, Thm. 1]. By [CDSS17, Thm. A.1], we have $\dim H_\lambda - \dim Z_{H_\lambda}(s_\lambda) \geq 4[F_\lambda : Q_\ell]$ for all $s \in C(0)$. It follows that

$$\dim \frac{H_\ell}{Z_{H_\ell}(s)} \geq 4[F : Q].$$
Hence, by Theorem 4.4, we have
\[ \pi_{C(x)}(x, L/Q) \ll \frac{x}{(\log x)^{1+\alpha-\epsilon}}, \]
where
\[ \alpha = \frac{[F : Q]}{9[F : Q] + 1 - 4[F : Q]^{1/2}} = \frac{[F : Q]}{\ell[F : Q] + 1}. \]

**Remark 5.2.** Suppose that \( a \in \mathcal{O}_F \) is such that there exists a prime \( \ell \in \mathcal{L} \) such that \( \ell \mid \frac{a}{2} \). Then, arguing as in the above proof (and as in [CDSS17, Thm. 1]), it follows that \( r_{C(x)} \geq 4[F : Q] \).

Applying Theorem 4.4, it follows that
\[ \{ p \leq x : a_p = a \} \ll \frac{x}{(\log x)^{1+\alpha-\epsilon}}, \]
where
\[ \alpha = \frac{[F : Q]}{8[F : Q] + 1}, \]
a stronger bound than Theorem 1.1. In particular, if \( \mathcal{L} \) is the set of all primes, which is conjecturally the case when \( E = F \), then, as in [CDSS17, Thm. 1], we obtain this stronger bound whenever \( a \neq \pm 4 \).

### 6. Conditional bounds on \( \pi_f(x, a) \)

Recall that \( f \) is a cuspidal Siegel modular eigenform of weights \((k_1, k_2), k_1 \geq k_2 \geq 2\), level \( N \) and character \( \epsilon \). Assume that \( f \) is not CAP, endoscopic, CM, RM or a symmetric cube lift. Let \( E = \mathbb{Q}((a_p : p \nmid N, \epsilon)) \) be the coefficient field of \( f \), and let \( F, K \) be the number fields defined in Definition 2.5 and Definition 2.6. Let \( \mathcal{L} \) be the set of primes defined in Definition 2.9 and, for \( \ell \in \mathcal{L} \), write
\[ \overline{R}_\ell : \text{Gal}(\overline{\mathbb{Q}}/K) \to \mathcal{G}(F_\ell) \]
for the reduction of \( R_\ell \) modulo \( \ell \). Here,
\[ \mathcal{G}(F_\ell) = \left\{ (g, \nu) \in \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z} F_\ell) \times F_\ell^\times : \text{sim}(g) = \nu^{k_1+k_2-3} \right\}. \]

In particular, the projection of \( \overline{R}_\ell \) onto \( \text{GSp}_4(\mathcal{O}_F \otimes \mathbb{Z} F_\ell) \) is exactly \( \overline{\mathcal{G}}_{|K} \), and its projection to \( F_\ell^\times \) is the mod \( \ell \) cyclotomic character. By Theorem 1.4, \( \overline{R}_\ell \) is surjective for all but finitely many primes \( \ell \in \mathcal{L} \).

#### 6.1. The case \( a \neq 0 \)

Fix a non-zero algebraic integer \( a \in \mathcal{O}_F \). Our goal is to bound the size of \( \pi_f(x, a) = \# \{ p \leq x : a_p = a \} \). As in the previous section, by Proposition 5.1, bounding the size of \( \pi_f(x, a) \) is exactly the same as bounding the size of \( \# \{ p \in M_K, N(p) = p \leq x : a_p = a \} \), up to the constant \([K : \mathbb{Q}]\). We will bound the size of this set by generalising the strategy of [MMS88].

For a prime \( p \) with \( p \nmid N \) such that \( p \) splits completely in \( K \), let \( F(p) \) be the splitting field of the characteristic polynomial of \( \rho_\ell(\text{Frob}_p) \). By definition, this characteristic polynomial does not depend on \( \ell \). Define
\[ \pi(x, a; \ell) := \# \{ p \in M_K, N(p) = p \leq x : a_p = a, \ \ell \text{ splits completely in } F(p) \}. \]

The following lemma will allow us to use \( \pi(x, a; \ell) \) to bound \( \pi_f(x, a) \):

**Lemma 6.1** (c.f. [MMS88, Lem. 4.4]). Let \( I \) be the interval \([y, y+u]\), where \( y, u \) are chosen so that \( x \geq y \geq u \geq y^{1/2}(\log y)^{1+\epsilon}(\log xy) \) for some \( \epsilon \geq 0 \). Then, assuming GRH, we have
\[ \# \{ p \in M_K, N(p) = p \leq x : a_p = a \} \ll \max_{\ell \in I} \pi(x, a; \ell). \]
To prove Lemma 6.1, we will require the following bound on the discriminant of \( F(p) \):

**Lemma 6.2.** We have \( \log |\text{disc}(F(p)/\mathbb{Q})| = O(\log p) \).

**Proof.** Let \( g(x) \in \mathcal{O}_F[x] \) be the characteristic polynomial of \( \rho_1(\text{Frob}_p) \). Let \( \pi \) be the unitary cuspidal automorphic representation of \( \text{GSp}_4(\mathbb{A}_F) \) associated to \( f \). By assumption, \( \pi \) lifts to a cuspidal automorphic representation \( \Pi \) of \( \text{GL}_4(\mathbb{A}_F) \). Let \( \alpha_1, \ldots, \alpha_4 \) be the Satake parameters of the local representation \( \Pi_p \). By [JS81, Cor. 2.5], we have

\[
p^{-\frac{1}{2}} < |\alpha_i| < p^{\frac{1}{2}}
\]

for each \( i \). In fact, if the weight \( k_2 > 2 \), then the Ramanujan conjecture is known, and we have \( |\alpha_i| = 1 \).

By definition, the four roots of \( g(x) \) are \( \alpha_1 p^{\frac{1}{2}}((k_1+k_2-3)), \ldots, \alpha_4 p^{\frac{1}{2}}((k_1+k_2-3)) \). It follows that the coefficients of \( g(x) \) are \( O(p^{2(k_1+k_2-2)}) \). Since the discriminant of \( g(x) \) is a polynomial in its coefficients, we see that \( \log |\text{disc}(g(x))| = O(\log p) \).

Let \( L_i = F(\alpha_i p^{\frac{1}{2}}((k_1+k_2-3))) \). Then

\[
\text{disc}(L_i/\mathbb{Q}) = N_{F/\mathbb{Q}}(\text{disc}(L_i/F)) \cdot \text{disc}(F/\mathbb{Q})[L_i:F].
\]

Since \( \text{disc}(F/\mathbb{Q})[L_i:F] = O(1) \), it follows from the above reasoning that \( \log |\text{disc}(L_i/\mathbb{Q})| = O(\log p) \). But \( F(p) \) is the compositum of the \( L_i \)'s and, in general, \( \mathfrak{d}(K_1 \cdot K_2/\mathbb{Q}) \cdot \mathfrak{d}(K_1/\mathbb{Q}) \cdot \mathfrak{d}(K_2/\mathbb{Q}) \) for arbitrary fields \( K_1, K_2 \), where \( \mathfrak{d} \) denotes the relative different ideal. It follows that \( \log |\text{disc}(F(p)/\mathbb{Q})| = O(\log p) \). \( \square \)

**Proof of Lemma 6.1.** Observe that

\[
\sum_{\ell \in I} \pi(x, a; \ell) = \sum_{\substack{\ell \in I \colon \ell \text{ splits completely in } F(p) \atop N(p) = p \leq x}} \# \{\ell \in I : \ell \text{ splits completely in } F(p)\}.
\]

By taking the trivial conjugacy class \( C = \{1\} \) of the Galois group of \( F(p) \) over \( \mathbb{Q} \) and applying Theorem 4.5, under GRH, the size of the set \( \{\ell \leq z : \ell \text{ splits completely in } F(p)\} \) is equal to

\[
\frac{1}{[F(p) : \mathbb{Q}]} \pi(z) + O\left( \frac{1}{[F(p) : \mathbb{Q}]} z^{1/2} (\log p + [F(p) : \mathbb{Q}] \log z) \right),
\]

for any real number \( z \gg 0 \). Here, we have used Lemma 6.2, that \( \log |\text{disc}(F(p)/\mathbb{Q})| \ll \log p \).

Since \( u \geq y^{1/2}(\log y)^{2+\epsilon} \), under GRH, we have

\[
\pi(y + u) - \pi(y) \gg \frac{u}{\log u}.
\]

It follows that

\[
\# \{\ell \in I : \ell \text{ splits completely in } F(p)\} \gg \pi(y + u) - \pi(y),
\]

where the implied constant in the above estimate is uniform in \( p \). Using this estimate in (6.1) yields

\[
\sum_{\substack{\ell \in I \colon \ell \text{ splits completely in } F(p) \atop N(p) = p \leq x}} \frac{1}{\pi(y + u) - \pi(y)} \sum_{\ell \in I} \pi(x, a; \ell) \ll \max_{\ell \in I} \pi(x, a; \ell).
\]

\( \square \)
Remark 6.3. In [Mur97, pp. 304], Murty proves a two-dimensional analogue of Lemma 6.1 without assuming GRH. Murty’s method makes essential use of the fact that, in the elliptic modular forms case, the field $F(p)$ is a quadratic extension of $F$. Hence, the quantity $\# \{ \ell \in I : \ell \text{ splits completely in } F(p) \}$ can be estimated via a character sum. In our case, $F(p)$ need not even be an abelian extension of $F$, so Murty’s method does not apply. It would be interesting to see if a version of Lemma 6.1 can be proven without assuming GRH. By combining the methods of this section with the unconditional Chebotarev density theorem of [TZ18], such a result would lead to an improved unconditional bound in Theorem 1.1.

Let $\ell \in \mathcal{L}$ be a prime such that $\overline{\mathcal{H}}_{\ell}$ is surjective, and let $L$ be the field cut out by the kernel of $\overline{\mathcal{H}}_{\ell}$. Then $L$ is a finite Galois extension of $K$ with Galois group $\mathcal{G}(\mathbb{F}_\ell)$. Define:

- $\mathcal{C}(a, \ell) = \{(g, \nu) \in \mathcal{G}(\mathbb{F}_\ell) : \text{tr}(g) = a \pmod{\ell}, \text{ and all the eigenvalues of } g \text{ are in } \mathbb{F}_\ell^x \}$,
- $\mathcal{B}_\ell = \{(g, \nu) \in \mathcal{G}(\mathbb{F}_\ell) : g \text{ upper triangular} \}$,
- $\mathcal{U}_\ell = \{(g, \nu) \in \mathcal{G}(\mathbb{F}_\ell) : g \text{ unipotent upper triangular} \}$,
- $\mathcal{C}(a, \ell)\overline{\mathcal{H}} = \text{the image of } \mathcal{C}(a, \ell) \cap \mathcal{B}_\ell \text{ in } \mathcal{B}_\ell/\mathcal{U}_\ell$.

Then $\mathcal{C}(a, \ell)$ is a subset of $\mathcal{G}(\mathbb{F}_\ell)$ that is closed under conjugation. Note that $\mathcal{U}_\ell$ is normal in $\mathcal{B}_\ell$ and that $\mathcal{B}_\ell/\mathcal{U}_\ell$ is abelian with Galois group $\text{Gal}(L^{un}/L^{B\ell})$.

Lemma 6.4. Let $\ell \in \mathcal{L}$ be a prime which splits completely in $F$. Then $\pi(x, a; \ell) \ll \pi(\mathcal{C}(a, \ell))(x, L^{B\ell}/L^{B\ell})$, where $\tilde{\pi}$ is as in Definition 4.8.

Proof. If $\ell$ splits in $F(p)$, then all the roots of the characteristic polynomial of $\rho(\text{Frob}_p)$ are in $\mathbb{F}_\ell^x$. It follows that $\pi(x, a; \ell) \ll \pi_{\mathcal{C}(a, \ell)}(x, L/K)$.

Now, $\mathcal{C}(a, \ell)$ is a union of conjugacy classes of $\mathcal{G}(\mathbb{F}_\ell)$, and each conjugacy class contains an element of $\mathcal{B}_\ell$. Hence, by Lemma 4.10 (i),

$$\pi_{\mathcal{C}(a, \ell)}(x, L/K) \leq \pi_{\mathcal{C}(a, \ell)\cap B_\ell}(x, L/L^{B\ell}).$$

Since multiplication by elements of $\mathcal{U}_\ell$ preserves the set $\mathcal{B}_\ell$, by Lemma 4.10 (ii), we have

$$\pi_{\mathcal{C}(a, \ell)\cap B_\ell}(x, L/L^{B\ell}) = \pi_{\mathcal{C}(a, \ell)}(x, L^{B\ell}/L^{B\ell}).$$

Combining the above estimates gives the desired result.

Lemma 6.5. Let $[F : Q] = n$, and suppose that $\ell$ is unramified in $F$. Then we have

(i) $|\mathcal{B}_\ell| \asymp \ell^{6n+1}, |\mathcal{U}_\ell| \asymp \ell^{4n}$.

(ii) $|\mathcal{C}(a, \ell)| \asymp \ell^{n+1}$.

(iii) $[L^{B\ell} : K] \ll \ell^{4n}$ and $\log M(L^{B\ell}/L^{B\ell}) \ll \log \ell$, where $M(L^{B\ell}/L^{B\ell})$ is as in Theorem 4.6.

Proof. If $F$ is a finite field of cardinality $q$, then the set of upper triangular matrices in $\text{GSp}_4(F)$ is

$$\left\{ \begin{pmatrix} a & b & c & 0 \\ \ell & d & e & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} : a, b, c, d, e, f, s, t \in F, n, r, s, t \in F \}.$$

Therefore, for any $\nu \in \mathbb{F}_\ell^x$, it follows that

$$\# \{ g \in \text{GSp}_4(F) : \text{sim}(g) = \nu, \ g \text{ upper triangular} \} = q^6 + O(q^5) \asymp q^6$$

and

$$\# \{ g \in \text{GSp}_4(F) : g \text{ unipotent upper triangular} \} = q^4 + O(q^3) \asymp q^4.$$
(i) Since $\ell$ is unramified in $F$, we have $\mathcal{O}_F \otimes \mathbb{Z} F_\ell \simeq \prod_{\lambda \mid \ell} F_\lambda$, where the product runs over the primes $\lambda \mid \ell$ of $F$. From (6.2), via this isomorphism, we have

\[
|B| = \sum_{\nu \in F_\ell^c} \left( \prod_{\lambda \mid \ell} \# \{ g \in \text{GSp}_4(F_\lambda) : \text{sim}(g) = \nu, \ g \text{ upper triangular} \} \right) \\
\asymp \sum_{\nu \in F_\ell^c} \prod_{\lambda \mid \ell} N(\lambda)^6 \asymp \ell^{6n+1}.
\]

Similarly, from (6.3), we have

\[
|U| = \left( \prod_{\lambda \mid \ell} \# \{ g \in \text{GSp}_4(F_\lambda) : g \text{ unipotent upper triangular} \} \right) \\
\asymp \prod_{\lambda \mid \ell} N(\lambda)^4 \asymp \ell^{4n}.
\]

(ii) From the definition of $\overline{C}(a, \ell)$, we observe that its elements are in bijection with

\[
\{ (g, \nu) \in G(F_\ell) : g \text{ diagonal}, \text{tr}(g) = a \}.
\]

Writing $a = (a_\lambda)_\lambda$ via the isomorphism $\mathcal{O}_F \otimes \mathbb{Z} F_\ell \simeq \prod_{\lambda \mid \ell} F_\lambda$ and proceeding as before, we obtain

\[
|\overline{C}(a, \ell)| = \sum_{\nu \in F_\ell^c} \left( \prod_{\lambda \mid \ell} \# \{ g \in \text{GSp}_4(F_\lambda) : \text{sim}(g) = \nu, \ g \text{ diagonal, tr}(g) = a_\lambda \} \right) \\
\asymp \sum_{\nu \in F_\ell^c} \prod_{\lambda \mid \ell} N(\lambda) \asymp \ell^{n+1}.
\]

(iii) Using formulae for the size of $\text{GSp}_4(F)$ over finite fields $F$, it is easy to check that $|G(F_\ell)| \asymp \ell^{10n+1}$. Since $[L^{B_\ell} : K] = |G(F_\ell) : B_\ell|$, it follows from (i) that $[L^{B_\ell} : K] \ll \ell^{4n}$. The second bound follows from part (i), Proposition 4.7 and the fact that $[L^{U_\ell} : L^{B_\ell}] = [B_\ell : U_\ell]$.

\[\square\]

**Proof of Theorem 1.2** (i). First observe that the group $B_\ell/U_\ell$ is abelian, which is the Galois group of the extension $L^{B_\ell}/L^{U_\ell}$. Hence, under GRH, applying Theorem 4.6, Lemma 4.9 and Lemma 6.5 yields

\[
\pi_{\overline{C}(a, \ell)}(x, L^{U_\ell}/L^{B_\ell}) \ll \frac{\overline{C}(a, \ell)}{|B_\ell|/|U_\ell| \log x} + [\overline{C}(a, \ell)]^{1/2}[L^{B_\ell} : Q]^{1/2} \log M(L^{U_\ell}/L^{B_\ell}) \\
\ll \frac{1}{\ell^n} \frac{x}{\log x} + \ell^{1/2} \log x^{1/2} \log x.
\]

Let $y \asymp \frac{x^{\alpha/n}}{(\log x)^{2n+1}}$, where $\alpha = \frac{n}{11n+1}$. By Theorem 1.4, the set of primes such that $\overline{R}_\ell$ is surjective has density 1. Hence, for $y$ sufficiently large, we can choose $\ell \in [y, 2y]$ such that $\ell$ that splits completely in $F$ and such that $\overline{R}_\ell$ is surjective. By Lemma 6.4, we have

\[
\pi(x, a; \ell) \ll \frac{1}{y^n} \frac{x}{\log x} + y^{1/2} \log y \frac{x^{1/2}}{\log x} \\
\ll \frac{x^{1-\alpha}}{(\log x)^{1-2\alpha}}.
\]

The result now follows from Lemma 6.1. \[\square\]
6.2. The case \(a = 0\). Let
\[
\mathcal{G}_\ell := \left\{ g \in \mathrm{GSp}_4(O_F \otimes \mathbb{Z} F_\ell) : \mathrm{sim}(g) \in F_\ell^{x(k_1+k_2-3)} \right\}.
\]
Then, by Corollary 3.16, for all but finitely many primes \(\ell \in \mathcal{L}\) the image of \(\pi_\ell^* : \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q}) \to \mathrm{GSp}_4(O_F \otimes \mathbb{Z} F_\ell)\) is a disjoint union of at most \([K : \mathbb{Q}]\) cosets
\[
\bigotimes \left( \frac{\alpha(\gamma)}{\alpha(\gamma)} \right) \mathcal{G}_\ell,
\]
where \(\gamma\) ranges over some subset of \(\mathrm{Gal}(K / \mathbb{Q})\). Moreover, taking \(\ell\) sufficiently large, the quantity \(\varepsilon(\gamma)\alpha(\gamma)^{-2}\) is an element of \((O_F \otimes \mathbb{Z} F_\ell)^{\times}\). Write \(\mathcal{G}_\ell^\gamma\) for the coset indexed by \(\gamma\) and, for each \(\gamma\), let \(\mathcal{C}_\gamma(0, \ell)\) denote the set of trace 0 elements in \(\mathcal{G}_\ell^\gamma\).

**Lemma 6.6.** For any \(\gamma \in \mathrm{Gal}(K / \mathbb{Q})\), we have \(|\mathcal{C}_\gamma(0, \ell)| \asymp |\mathcal{C}_1(0, \ell)|\).

**Proof.** First note that, since \(\varepsilon(\gamma)\alpha(\gamma)^{-2} \in (O_F \otimes \mathbb{Z} F_\ell)^{\times}\), we have \(\alpha(\gamma)^{-1} \mathcal{G}_\ell^\gamma \subseteq \mathrm{GSp}_4(O_F \otimes \mathbb{Z} F_\ell)^\gamma\). Set \(b = \varepsilon(\gamma)\alpha(\gamma)^{-2}\). Then \(\mathcal{C}_\gamma(0, \ell)\) is precisely the set of trace 0 elements in the coset
\[
\left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \mathcal{G}_\ell.
\]
Hence,
\[
|\mathcal{C}_\gamma(0, \ell)| = \# \left\{ g \in \mathcal{G}_\ell^\gamma : \mathrm{tr}(g) = 0 \right\}
= \# \left\{ g \in \mathcal{G}_\ell : \mathrm{tr}(g) = 0 \right\}
= \sum_{\nu \in F_\ell^{x(k_1+k_2-3)}} \prod_{\lambda | \ell} \# \left\{ g = (g_{ij}) \in \mathrm{GSp}_4(F_\lambda) : \mathrm{sim}(g) = \nu b^{-2}, g_{11} + g_{22} + b g_{33} + b g_{4} = 0 \right\}
\asymp \sum_{\nu \in F_\ell^{x(k_1+k_2-3)}} \prod_{\lambda | \ell} N(\lambda)^{9/2}.
\]
Since the above calculation did not depend on \(\gamma\), the result follows. \(\square\)

**Corollary 6.7.** We have
\[
\pi_f(x, 0) \ll \# \left\{ p \in M_K : N(p) = p \leq x : \mathrm{tr}\pi_\ell(p)(\mathrm{Frob}_p) \equiv 0 \pmod{\ell} \right\}.
\]

**Proof.** We have
\[
\pi_f(x, 0) \ll \# \left\{ p \leq x : \mathrm{tr}\pi_\ell(p)(\mathrm{Frob}_p) \equiv 0 \pmod{\ell} \right\}.
\]
By the previous lemma, the set on the right hand side splits into at most \([K : \mathbb{Q}]\) subsets, each of size
\[
\# \left\{ p \leq x : p \text{ splits completely in } K, \mathrm{tr}\pi_\ell(p)(\mathrm{Frob}_p) \equiv 0 \pmod{\ell} \right\}.
\]
Indeed, as in Proposition 5.1, this set is exactly \(\mathcal{C}_1(0, \ell)\). The result follows from the fact that this set has size
\[
\frac{1}{[K : \mathbb{Q}]} \# \left\{ p \in M_K : N(p) = p \leq x : \mathrm{tr}\pi_\ell(p)(\mathrm{Frob}_p) \equiv 0 \pmod{\ell} \right\}.
\]
\(\square\)

Now, as in the previous section, for a prime \(p\) with \(p \nmid N\) such that \(p\) splits completely in \(K\), let \(F(p)\) be the splitting field of the characteristic polynomial of \(p_\ell(\mathrm{Frob}_p)\), and let
\[
\pi(x, 0; \ell) := \# \left\{ p \in M_K : N(p) = p \leq x : a_p = 0, \ell \text{ splits completely in } F(p) \right\}.
\]
Then, by Lemma 6.1 and Corollary 6.7, we can use \(\pi(x, 0; \ell)\) to bound \(\pi_f(x, 0)\).
Fix a prime $\ell \in \mathcal{L}$ such that $\overline{\mathbf{R}}_\ell$ is surjective, and let $L$ be the field cut out by the kernel of $\overline{\mathbf{R}}_\ell$. Let

\[
\mathcal{C}(0, \ell) = \{(g, \nu) \in \mathcal{G}(\mathbf{F}_\ell) : \text{tr}(g) = 0 \pmod{\ell}\}, \quad \mathcal{B}_\ell = \{(g, \nu) \in \mathcal{G}(\mathbf{F}_\ell) : g \text{ upper triangular}\},
\]

\[
\mathcal{H}_\ell = \{(g, \nu) \in \mathcal{G}(\mathbf{F}_\ell) : g \text{ upper triangular with } 4 \text{ equal eigenvalues}\},
\]

\[
\overline{\mathcal{C}}(0, \ell) = \text{the image of } \mathcal{C}(0, \ell) \cap \mathcal{B}_\ell \text{ in } \mathcal{B}_\ell/\mathcal{H}_\ell.
\]

The proof of the following lemma is essentially identical to that of Lemma 6.5.

**Lemma 6.8.** Let $[F : \mathbb{Q}] = n$. Then we have

(i) $|\mathcal{B}_\ell| \asymp \ell^{6n+1}, |\mathcal{H}_\ell| \asymp \ell^{5n}$.  

(ii) $|\overline{\mathcal{C}}(0, \ell)| \asymp \ell^n$.  

(iii) $[L^B_\ell : K] \ll \ell^{4n}$ and $\log M(L^{H_\ell}/L^{B_\ell}) \ll \log \ell$, where $M(L^{H_\ell}/L^{B_\ell})$ is as in Theorem 4.6.

**Proof of Theorem 1.2 (ii).** Since the product of a scalar matrix and a trace zero matrix has trace zero, we have

\[
\mathcal{H}_\ell \cdot (\mathcal{C}(0, \ell) \cap \mathcal{B}_\ell) = \mathcal{C}(0, \ell) \cap \mathcal{B}_\ell.
\]

Hence, from Lemma 4.10, we have

\[
\pi_{\mathcal{C}(0, \ell)}(x, L/K) \leq \pi_{\mathcal{C}(0, \ell) \cap \mathcal{B}_\ell}(x, L/L^{B_\ell}) \leq \pi_{\overline{\mathcal{C}}(0, \ell)}(x, L^{H_\ell}/L^{B_\ell}).
\]

Under GRH, by Theorem 4.6, Lemma 4.9 and Lemma 6.8, we have

\[
\pi_{\overline{\mathcal{C}}(0, \ell)}(x, L^{H_\ell}/L^{B_\ell}) \ll \frac{\overline{\mathcal{C}}(0, \ell)}{|\mathcal{B}_\ell|/|\mathcal{H}_\ell| \log x} + |\overline{\mathcal{C}}(0, \ell)|^{1/2}[L^B_\ell : \mathbb{Q}]^{x/2} \log M(L^{H_\ell}/L^{B_\ell})
\]

\[
\ll \frac{1}{\ell^n} \frac{x}{\log x} + \ell^{4(8n+1)} \log \ell \frac{x^{1/2}}{\log x}.
\]

Let $y \asymp \frac{x^{\alpha/n}}{(\log x)^{2n+1}}$, where $\alpha = \frac{n}{10n+1}$. By Theorem 1.4, the set of primes such that $\overline{\mathbf{R}}_\ell$ is surjective has density 1. Hence, for $y$ sufficiently large, we can choose $\ell \in [y, 2y]$ such that $\ell$ that splits completely in $F$ and such that $\overline{\mathbf{R}}_\ell$ is surjective. By the same argument as Lemma 6.4,

\[
\pi(x, 0; \ell) \ll \frac{x^{1-\alpha}}{(\log x)^{1-2\alpha}}.
\]

The result now follows from Lemma 6.1. \qed

**Acknowledgements**

The authors are grateful to Tobias Berger, Andrea Conti and Tian Wang for helpful correspondences and comments. The first author was supported by grant no. 692854 provided by the European Research Council (ERC). The second author was supported by Israeli Science Foundation grant 1400/19. The third author was supported by BSF grant 2018250.

**References**

[Art13] James Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups. MR 3135650

[AS06] Mahdi Asgari and Freydoon Shahidi, *Generic transfer from $\text{GSp}(4)$ to $\text{GL}(4)$*, Compos. Math. 142 (2006), no. 3, 541–550. MR 2231191

[BPP+19] Armand Brumer, Ariel Pacetti, Cris Poor, Gonzalo Tornaría, John Voight, and David S. Yuen, *On the paramodularity of typical abelian surfaces*, Algebra Number Theory 13 (2019), no. 5, 1145–1195. MR 3981316
On [Rib75] Kenneth A. Ribet, [Ram13] Dinakar Ramakrishnan, Recovering modular forms from squares. Compact subgroups of linear algebraic groups [Mur99] V. Kumar Murty, [Mom81] Fumiyuki Momose, On the [Mok14] Chung Pang Mok, Modular forms and the Chebotarev density theorem [MMS88] M. Ram Murty, V. Kumar Murty, and N. Saradha, [Mit14] Howard H. Mitchell, The subgroups of the quaternary abelian linear group, Trans. Amer. Math. Soc. 15 (1914), no. 4, 379–396. MR1500986 †3, 13 [MMSS88] M. Ram Murty, V. Kumar Murty, and N. Saradha, Modular forms and the Chebotarev density theorem, Amer. J. Math. 110 (1988), no. 2, 253–281. MR935007 †2, 5, 20 [Mok14] Chung Pang Mok, Galois representations attached to automorphic forms on GL2 over CM fields, Compos. Math. 150 (2014), no. 4, 523–567. MR3200667 †6 [Mom81] Fumiyuki Momose, On the l-adic representations attached to modular forms, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 1, 89–109. MR617807 †2, 3, 7, 10 [Mur97] V. Kumar Murty, Modular forms and the Chebotarev density theorem. II, Analytic number theory (Kyoto, 1996), 1997, pp. 287–308. MR1694997 †2, 22 [Mur99], Frobenius distributions and Galois representations, Automorphic forms, automorphic representations, and arithmetic (Fort Worth, TX, 1996), 1999, pp. 193–211. MR1703751 †1, 5 [Pin98] Richard Pink, Compact subgroups of linear algebraic groups, J. Algebra 206 (1998), no. 2, 438–504. MR1637068 †4, 10 [Ram00] Dinakar Ramakrishnan, Recovering modular forms from squares. Appendix to a problem of Linnik for elliptic curves and mean-value estimates for automorphic representations (by W. Duke and E. Kowalski), Invent. Math. 139 (2000), no. 1, 1–39. MR1728875 †9 [Ram13], Decomposition and parity of Galois representations attached to GL(4), Automorphic representations and L-functions, 2013, pp. 427–454. MR3156860 †6 [Rib75] Kenneth A. Ribet, On l-adic representations attached to modular forms, Invent. Math. 28 (1975), 245–275. MR0419358 †2, 3, 4, 11, 12 [Rib76] Galois action on division points of Abelian varieties with real multiplications, Amer. J. Math. 98 (1976), no. 3, 751–804. MR457455 †10
[Rib77] ———, *Galois representations attached to eigenforms with Nebentypus*, Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), 1977, pp. 17–51. Lecture Notes in Math., Vol. 601. MR0453647 ↑3

[Rib80] ———, *Twists of modular forms and endomorphisms of abelian varieties*, Math. Ann. 253 (1980), no. 1, 43–62. MR594532 ↑3, 7

[Rib85] ———, *On l-adic representations attached to modular forms. II*, Glasgow Math. J. 27 (1985), 185–194. MR819838 ↑2, 3, 7, 11, 14, 15

[Ser18] Jean-Pierre Serre, *On the mod p reduction of orthogonal representations*, Lie groups, geometry, and representation theory, 2018, pp. 527–540. MR3890220 ↑3

[Ser81] ———, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. 54 (1981), 323–401. MR644559 ↑2, 4, 5, 16, 17, 19

[Ser98] ———, *Abelian l-adic representations and elliptic curves*, Research Notes in Mathematics, vol. 7, A K Peters, Ltd., Wellesley, MA, 1998. With the collaboration of Willem Kuyk and John Labute, Revised reprint of the 1968 original. MR1484415 ↑2, 3, 4, 13, 14

[Tay91] Richard Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. 63 (1991), no. 2, 281–332. MR1115109 ↑6

[Tay93] ———, *On the l-adic cohomology of Siegel threefolds*, Invent. Math. 114 (1993), no. 2, 289–310. MR1240640 ↑6

[TZ18] Jesse Thorner and Asif Zaman, *A Chebotarev variant of the Brun-Titchmarsh theorem and bounds for the Lang-Trotter conjectures*, Int. Math. Res. Not. IMRN 16 (2018), 4991–5027. MR3848226 ↑2, 22

[Wan90] Da Qing Wan, *On the Lang-Trotter conjecture*, J. Number Theory 35 (1990), no. 3, 247–268. MR1062334 ↑2

[Wei05] Rainer Weissauer, *Four dimensional Galois representations*, Astérisque 302 (2005), 67–150. Formes automorphes. II. Le cas du groupe GSp(4). MR2234860 ↑6

[Wei08] ———, *Existence of Whittaker models related to four dimensional symplectic Galois representations*, Modular forms on Schiermonnikoog, 2008, pp. 285–310. MR2530981 ↑2, 6

[Wei18] Ariel Weiss, *On the images of Galois representations attached to low weight Siegel modular forms* (2018), available at arXiv:1802.08537. To appear in Journal of the LMS. ↑2, 3, 4, 6, 10

[Zyw15] David Zywina, *Bounds for the Lang-Trotter conjectures*, SCHOLAR—a scientific celebration highlighting open lines of arithmetic research, 2015, pp. 235–256. MR3453123 ↑2, 5, 17

Arvind Kumar, Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmund J. Safra Campus, Jerusalem 91904-01, Israel.

Email address: arvind.kumar@mail.huji.ac.il

Moni Kumari, Department of Mathematics, Bar-Ilan University, Ramat Gan 52900-02, Israel.

Email address: moni.kumari@biu.ac.il

Ariel Weiss, Department of Mathematics, Ben-Gurion University of the Negev, Be’er Sheva 841050, Israel.

Email address: arielweiss@post.bgu.ac.il