Vector NLS hierarchy solitons revisited: dressing transformation and tau function approach

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Abstract

We discuss some algebraic aspects of the integrable vector non-linear Schrödinger hierarchies (GNLS\(_r\)). These are hierarchies of zero-curvature equations constructed from affine Kac-Moody algebras \(\hat{sl}_{r+1}\). Using the dressing transformation method and the tau-function formalism, we construct the N-soliton solutions of the GNLS\(_r\) systems. The explicit matrix elements in the case of GNLS\(_1\) are computed using level one vertex operator representations.
1 Introduction

It is well known that the 1 + 1-dimensional non-linear Schrödinger (NLS) equation is integrable and possesses exact soliton solutions \([1]\). It has been known that many soliton equations in 1 + 1 dimensions have integrable matrix generalizations, or more generally, integrable multi-component generalizations. The most well-known example for the multi-component case is the vector NLS equation, first studied by Manakov \([2]\). These type of equations and their higher-order generalizations find applications in non-linear optics (for a complete review of the most important references in the field see \([3]\)). The multi-soliton type solutions of these hierarchies can be obtained using diverse methods. For example, in \([4]\) the vector NLS equation has been studied in the framework of the inverse scattering method, the bright and dark multi-soliton solutions and their collisions have been studied.

One of the most fascinating applications of the Kac-Moody theory and its affine Lie algebras and their relevant groups is to exhibit hidden symmetries of soliton equations. According to the approach of \([5]\) a common feature of integrable hierarchies presenting soliton solutions is the existence of some special “vacuum solutions” such that the Lax operators evaluated on them lie in some abelian subalgebra of the associated Kac-Moody algebra. The soliton type solutions are constructed out of those “vacuum solutions” through the so called dressing transformation procedure. These developments lead to a quite general definition of tau functions associated to the hierarchies, in terms of the so called “integrable highest weight representations” of the relevant Kac-Moody algebra.

In this paper we obtain the multi-soliton solutions of the vector NLS equation using the dressing transformation method. We believe that the group theoretical point of view of finding the analytical results for the general case of \(N\)-soliton interactions could facilitate the study of their properties; for example, the asymptotic behaviour of trains of \(N\) solitonlike pulses with approximately equal amplitudes and velocities, as studied in \([6]\). The second point we should highlight relies upon the possible relevance of the NLS tau functions to its higher-order generalizations. We believe that the tau functions of the higher-order NLS generalization are related somehow to the basic tau functions of the usual (vector) NLS equations (this fact is observed for example in the case of the coupled NLS+DNLS system \([7, 8]\), in the second Ref. Hirota’s method has been used).

The plan is as follows. In section 2 we review the theory of the dressing transformations and the definition of the tau-function vectors. In section 3 we present the construction of the vector NLS equations (GNLS\(_r\)) associated to the homogeneous gradation of the Kac-Moody algebra \(\hat{sl}_{r+1}\), their relevant tau functions are defined and the construction of multiple-soliton solutions outlined. In section 4 we present a detailed study of the GNLS\(_1\) case; the first conserved charges are constructed in the context of this formalism and the explicit form of the \(N\)-soliton solutions are presented. Finally, for the sake of completeness, we have included Kac-Moody algebra notations and conventions, as well as, its well known theory of integrable highest weight representations, and level one homogeneous vertex operator representations (see appendices \([A, B, C]\)). Some of the details regarding the matrix elements appear in the appendices \([D, E, F]\).
2 Dressing Transformations

Consider a non linear system which can be formulated in terms of a system of first-order differential equations

\[ \mathcal{L}_N \Psi = 0, \quad (2.1) \]

where \( \mathcal{L}_N \) are Lax operators of the form

\[ \mathcal{L}_N = \frac{\partial}{\partial t_N} - A_N, \quad (2.2) \]

and the variables \( t_N \) are the various “times” of the hierarchy.

Then, the equations of the hierarchy are equivalent to the integrability or zero curvature conditions of (2.1),

\[ [\mathcal{L}_N, \mathcal{L}_M] = 0. \quad (2.3) \]

Therefore the Lax operators are of the form of a pure gauge

\[ A_N = \frac{\partial \Psi}{\partial t_N} \Psi^{-1}. \quad (2.4) \]

The type of integrable hierarchy considered here is based on a Kac-Moody algebra \( \hat{g} \) furnished with an integer gradation labelled by a vector \( s = (s_0, s_1, ..., s_r) \) of \( r + 1 \) non-negative co-prime integers such that

\[ \hat{g} = \bigoplus_{i \in \mathbb{Z}} \hat{g}_i(s) \quad \text{and} \quad [\hat{g}_i(s), \hat{g}_j(s)] \subseteq \hat{g}_{i+j}(s). \quad (2.5) \]

The connections we consider are of the form

\[ A_N = \sum_{i=0}^{t} A_{N,i} \quad \text{where} \quad A_{N,i} \in \hat{g}_i(s). \quad (2.6) \]

The “dressing transformations” are non local gauge transformations on \( A_N \) which maintain their form and gradation [5, 9]. Each of these gauge transformations is made with the help of two group elements \( \Theta_+ \) and \( \Theta_- \), such that

\[ A_N \rightarrow A_N^h \equiv \Theta_\pm A_N \Theta_{\pm}^{-1} + \partial_N \Theta_\pm \Theta_{\pm}^{-1} \quad (2.7) \]

or

\[ A_N^h = \frac{\partial (\Theta_\pm \Psi)}{\partial t_N} (\Theta_\pm \Psi)^{-1}. \quad (2.8) \]

We have a residual gauge transformation in (2.8)

\[ \Psi \rightarrow \Psi h, \quad (2.9) \]

where \( h \) is a constant group element. Therefore we can impose

\[ \Theta_+ \Psi = \Theta_- \Psi h \quad (2.10) \]
or equivalently
\[ \Theta_+ \Theta_+ = \Psi h \Psi^{-1} \] (2.11)

\[ \Theta_\Psi \] defines a new solution
\[ \Psi^h = \Theta_\Psi \] (2.12)

We admit a Gauss decomposition with respect to the gradation
\[ \Psi h \Psi^{-1} = \left( \Psi h \Psi^{-1} \right)_- \left( \Psi h \Psi^{-1} \right)_0 \left( \Psi h \Psi^{-1} \right)_+ \] (2.13)

We choose (see (2.11))
\[ \Theta_\Psi = \left( \left( \Psi h \Psi^{-1} \right)_- \right)^{-1} \] (2.14)

and therefore (2.12) can be written as
\[ \Psi^h = \left( \left( \Psi h \Psi^{-1} \right)_- \right)^{-1} \Theta_\Psi \Psi^{-1} = \left( \Psi h \Psi^{-1} \right)_0 \left( \Psi h \Psi^{-1} \right)_+ \Psi^{-1} \] (2.15)

where we used (2.10) and (2.11). \( \Psi^h \) in (2.15) is also a solution of the linear problem (2.2).

We shall consider solutions which belong to the orbits of the vacuum solutions.

Define
\[ \Theta^{-1}_- = \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right)_- \quad \text{and} \quad B^{-1} = \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right)_0 \] (2.16)

\[ N = \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right)_+ \quad \text{and} \quad \Omega = B^{-1} N. \] (2.17)

Therefore the dressing transformations generated by \( h \) becomes
\[ \Psi^{(\text{vac})} \rightarrow \Psi^h = \Theta_\Psi \Psi^{(\text{vac})^{-1}} = \Omega \Psi^{(\text{vac})} h^{-1}. \] (2.18)

Denote by \( | \lambda_i \rangle \) the state of highest weight of a fundamental representation such that \( s_i \neq 0 \).

Define the tau-function vector \[ [3] \]
\[ \tau_i(t^\pm) = \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right)_- | \lambda_i \rangle \]
\[ = \Theta^{-1}_- B^{-1} | \lambda_i \rangle \quad \text{for} \quad i = 0, 1, ..., r; \quad s_i \neq 0. \] (2.19)

Note that \( N | \lambda_i \rangle = \pm | \lambda_i \rangle \) and
\[ \tau_i(0)(t^\pm) = B^{-1} | \lambda_i \rangle = | \lambda_i \rangle \tilde{\tau}_i(0)(t^\pm), \] (2.20)

since \( | \lambda_i \rangle \) is an eigenstate of the sublagebra \( \hat{g}_o(s) \). Then
\[ \tilde{\tau}_i(0)(t^\pm) = \langle \lambda_0 | \left[ \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right]_+(0) | \lambda_0 \rangle, \] (2.21)

Using (2.19) we obtain
\[ \Theta^{-1}_- | \lambda_i \rangle = \frac{\tau_i(t^\pm)}{\tilde{\tau}_i(0)(t^\pm)} \] (2.22)
3 The \( \tau \) function and the N soliton solution for GNLS\(_r\)

The GNLS\(_r\) model is constructed for example in [10], where it was studied in the framework of the Zakharov-Shabat formalism and the context of hermitian symmetric spaces. In [11] an affine Lie algebraic foundation (loop algebra \( g \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) of \( g \)) for GNLS\(_r\) was given. Here instead we will consider the full Kac-Moody algebra \( \hat{sl}(r+1) \) with homogeneous gradation

\[
s = (1, 0, 0, ..., 0),
\]

and a semisimple element \( E^{(l)} \).

The connections are given by

\[
A_1 \equiv A = E^{(1)} + \sum_{i=1}^{r} \Psi_i^+ E^{(0)}_{\beta_i} + \sum_{i=1}^{r} \Psi_i^- E^{(0)}_{-\beta_i} + \nu_1 C,
\]

\[
A_2 \equiv B = E^{(2)} + \sum_{i=1}^{r} \Psi_i^+ E^{(1)}_{\beta_i} + \sum_{i=1}^{r} \Psi_i^- E^{(1)}_{-\beta_i} + \sum_{i=1}^{r} \partial_x \Psi_i^+ E^{(0)}_{\beta_i} - \sum_{i=1}^{r} \partial_x \Psi_i^- E^{(0)}_{-\beta_i} - \sum_{i=1}^{r} \Psi_i^+ \Psi_i^- \beta_i H^{(0)} - \sum_{i,j=1}^{r} \Psi_i^+ \Psi_i^- \epsilon(\beta_i, \beta_j) E^{(0)}_{\beta_i - \beta_j} + \nu_2 C,
\]

where \( \Psi_i^+, \Psi_i^-, \nu_1 \) and \( \nu_2 \) are the fields of the model.

We have

\[
E^{(l)} = 2\frac{\mu_r}{\alpha_r^2} H^{(l)}, \quad [D, E^{(l)}] = lE^{(l)}
\]

where \( \mu_r \) is a fundamental weight and \( \alpha_a \) are the simple roots of the \( sl(r+1) \) algebra. The \( \beta_i \) are the positive roots defined by

\[
\beta_i = \alpha_i + \alpha_{i+1} + ... + \alpha_r, \quad \text{with} \quad \alpha_i^2 = 2,
\]

and \( H_i \) being the generators of the cartan subalgebra in the Weyl-Cartan basis. We need also some relations in the Chevalley basis

\[
E = \frac{1}{r+1} \left( \sum_{a=1}^{r} a H^{(o)}_a \right), \quad \text{with} \quad H_a = \alpha_a H^{(o)}
\]

\[
\mu_r = \frac{1}{r+1} \left( \sum_{a=1}^{r} a \alpha_a \right), \quad \beta_a H = H_a + ... + H_r, \quad a = 1, ..., r
\]

\[
\left[ E^{(m)}_\beta, H^{(n)}_\alpha \right] = \frac{m}{r+1} \sum_{a=1}^{r} a \eta_{ab} C \delta_{m,-n},
\]

\[
\left[ E^{(m)}_\beta, E^{(n)}_{\pm \beta_i} \right] = \pm E^{(m+n)}_{\pm \beta_i},
\]

\[
\left[ H^{(m)}_a, E^{(n)}_{\pm \beta_i} \right] = \pm (\sum_{b=1}^{r} K_{ba}) E^{(m+n)}_{\pm \beta_i},
\]
where
\[
\eta_{ab} = \frac{2}{\alpha_b^2} K_{ab} = \eta_{ba} \quad \text{and} \quad K_{ab} = 2 \frac{\alpha_a \cdot \alpha_b}{\alpha_b^2}.
\] (3.7)

The potentials are in the subspaces
\[
A_1 \in \hat{g}_o(s) + \hat{g}_1(s), \quad A_2 \in \hat{g}_o(s) + \hat{g}_1(s) + \hat{g}_2(s)
\] (3.8)

The zero curvature condition \([\partial_t - B, \partial_x - A] = 0\) gives the following system of equations
\[
\begin{align*}
\partial_t \Psi^+ - \partial_x^2 \Psi^+ & = 2 \left( \sum_{j=1}^{r} \Psi^+_j \Psi^-_j \right) \Psi^+_i, \\
\partial_t \Psi^- - \partial_x^2 \Psi^- & = -2 \left( \sum_{j=1}^{r} \Psi^+_j \Psi^-_j \right) \Psi^-_i,
\end{align*}
\] (3.9)

\[
\partial_t \nu_1 - \partial_x \nu_2 = 0.
\]

The system of equations for the \(\Psi^\pm\) fields in (3.9), supplied with a convenient complexification of the time variable and the fields, are related to the well known integrable vector non-linear Schrödinger equation (vector NLS) [2, 4, 12].

Let us now study the vacuum solutions and dressing transformations. Let us note that \(\Psi^+ = \nu_1 = \nu_2 = 0\) is a solution of equations (3.9). Therefore from (3.3) we have the connections
\[
A_1^{(vac)} \equiv A^{(vac)} = E^{(1)}, \quad A_2^{(vac)} \equiv B^{(vac)} = E^{(2)},
\] (3.10)

These connections can be obtained from \(A_N^{(vac)} = \partial_t N \Psi \Psi^{-1}\) from the group element
\[
\Psi^{(vac)} = e^{x \hat{E}^{(1)} + t \hat{E}^{(2)} + X},
\] (3.11)

where
\[
X = \sum_{n=3}^{+\infty} t_n \hat{E}^{(n)}, \quad t_n \text{ are real parameters.}
\] (3.12)

The connections in the vacuum orbit are given by
\[
\begin{align*}
A & = \Theta_- E^{(1)} \Theta_-^{-1} + \partial_x \Theta_- \Theta_-^{-1} \\
& = M^{-1} N E^{(1)} N^{-1} M - M^{-1} \partial_x M + M^{-1} \partial_x N N^{-1} M,
\end{align*}
\] (3.13)

\[
\begin{align*}
B & = \Theta_- E^{(2)} \Theta_-^{-1} + \partial_t \Theta_- \Theta_-^{-1} \\
& = M^{-1} N E^{(2)} N^{-1} M - M^{-1} \partial_t M + M^{-1} \partial_t N N^{-1} M,
\end{align*}
\] (3.14)

with
\[
\Theta_- = \exp \left( \sum_{n>0} \sigma_{-n} \right), \quad M = \exp (\sigma_o), \quad N = \exp \left( \sum_{n>0} \sigma_n \right)
\] (3.15)
In the present case the gradation operator is \( Q_s = D \) with
\[
[D, \sigma_n] = n \sigma_n. \tag{3.16}
\]
We can relate the fields \( \Psi^\pm_i, \nu_1 \) and \( \nu_2 \) with some of the components of \( \sigma_n \). We have
\[
A = E^{(1)} + \left[ \sigma_{-1}, E^{(1)} \right] + \text{ terms of negative grade.} \tag{3.17}
\]
\[
= M^{-1} \left( E^{(1)} - \partial_x M. M^{-1} + \partial_x \sigma_1 \right) M + \text{ terms of grade > 1} \tag{3.18}
\]
\[
B = E^{(2)} + \left[ \sigma_{-1}, E^{(2)} \right] + \left[ \sigma_{-2}, E^{(2)} \right] + \frac{1}{2} \left[ \sigma_{-1}, \left[ \sigma_{-1}, E^{(2)} \right] \right] + \text{ terms of negative grade}
\]
\[
= M^{-1} \left( E^{(2)} - \partial_t M. M^{-1} + \partial_t \sigma_1 + \partial_t \sigma_2 + [\sigma_1, \partial_t \sigma_1] \right) M + \text{ terms of grade > 2} \tag{3.19}
\]
In (3.17) the term of degree \(-1\) must vanish and therefore
\[
\partial_x \sigma_{-1} + \left[ \sigma_{-2}, E^{(1)} \right] + \frac{1}{2} \left[ \sigma_{-1}, \left[ \sigma_{-1}, E^{(1)} \right] \right] = 0, \tag{3.20}
\]
Denote (consistently with (3.17))
\[
\sigma_{-1} = - \sum_{i=1}^{r} \Psi_i^+ E^{(-1)}_{i\beta_i} + \sum_{i=1}^{r} \Psi_i^- E^{(-1)}_{-i\beta_i} + \sum_{a=1}^{r} \sigma_a^1 H_a^{(-1)}; \tag{3.21}
\]
\[
\sigma_{-2} = \sum_{i=1}^{r} \sigma_{-1}^i E^{(-2)}_{i\beta_i} + \sum_{i=1}^{r} \sigma_{-2}^i E^{(-2)}_{-i\beta_i} + \sum_{a=1}^{r} \sigma_a^2 H_a^{(-2)},
\]
and therefore from (3.20) we obtain
\[
\partial_x \sigma_{-1}^a = \sum_{i=1}^{r} \Psi_i^+ \Psi_i^-, \quad a, i = 1, \ldots r. \tag{3.22}
\]
\[
\sigma_{-2}^{+i} = - \partial_x \Psi_i^+ + \frac{1}{2} \sum_{a=1}^{r} \sigma_a^1 \Psi_i^+ \left( \sum_{b=1}^{r} K_{ba} \right),
\]
\[
\sigma_{-2}^{-i} = - \partial_x \Psi_i^- + \frac{1}{2} \sum_{a=1}^{r} \sigma_a^1 \Psi_i^- \left( \sum_{b=1}^{r} K_{ba} \right),
\]
Substitution of \( \sigma_{-1}, \sigma_{-2} \) in (3.17) and (3.19) we obtain
\[
\nu_1 = - \frac{1}{r + 1} \sum_{a,b=1}^{r} a \eta_{ab} \sigma_{-1}^b,
\]
\[ \nu_2 = -\frac{2}{r+1} \sum_{a,b=1}^{r} a.\eta_{ab}\sigma_{2}. \] (3.23)

The \( \sigma_{-n} \)'s with the higher gradations are used to cancel out the undesired components. One of the tau-function vectors is given by

\[ \tau_o (x, t) = \left[ \Psi^{(0)} h \Psi^{(0)}\right] |\lambda_o\rangle \] (3.24)

\[ = \Theta^{-1} M^{-1} |\lambda_o\rangle, \] (3.25)

where \( h \) is a particular constant element of \( \hat{sl}(r+1) \).

Then we have

\[ \exp \left(- \sum_{n>0} \sigma_{-n} \right) \exp (-\sigma_o) |\lambda_o\rangle = \left[ \Psi^{(0)} h \Psi^{(0)}\right] |\lambda_o\rangle. \] (3.26)

We want to express the fields \( \Psi^\pm \) in terms of some tau functions, which are some matrix elements in an appropriate representation of \( \hat{sl}(r+1) \).

We can write down

\[ \sigma_o = \sum_{a=1}^{r} \sigma_{+a} E_{a_o}^{(0)} + \sum_{a=1}^{r} \sigma_{-a} E_{-a_o}^{(0)} + \sum_{a=1}^{r} \sigma^{a}_o H^{(0)} + \eta C \] (3.27)

or

\[ \sigma_o = \sum_{i=1}^{r} \sigma_{o}^{+i} e_i + \sum_{i=1}^{r} \sigma_{o}^{-i} f_i + \sum_{a=1}^{r} \sigma^{a}_o h_a + \eta C, \] (3.28)

where \( e_i \) and \( f_i \) (\( i = 0...r \)) are the generators in the Chevalley basis and \{ \( h_i \), \( D \) \} generates the Cartan subalgebra.

We have (see Appendix A)

\[ h_i |\lambda_o\rangle = 0, \quad f_i |\lambda_o\rangle = 0, \quad e_i |\lambda_o\rangle = 0 \quad \text{and} \quad C |\lambda_o\rangle = |\lambda_o\rangle, \quad i = 1, 2, ...r \] (3.29)

Therefore the zero gradation of (3.26) is

\[ \exp (-\sigma_o) |\lambda_o\rangle = \left[ \Psi^{(0)} h \Psi^{(0)}\right]_{(o)} |\lambda_o\rangle, \] (3.30)

and the left hand side can be written as

\[ \exp (-\sigma_o) |\lambda_o\rangle = |\lambda_o\rangle \hat{\tau}^{(o)} (x, t) \] (3.31)

with \( \hat{\tau}^{(o)} \) a real function given by the matrix element

\[ \hat{\tau}^{(o)} (x, t) = \langle \lambda_o \left[ \Psi^{(0)} h \Psi^{(0)}\right]_{(o)} |\lambda_o\rangle. \] (3.32)

Then the term with grade (-1) in (3.26) is

\[ -\sigma_{-1} |\lambda_o\rangle = \frac{\left[ \Psi^{(0)} h \Psi^{(0)}\right]_{(-1)} |\lambda_o\rangle}{\hat{\tau}^{(o)} (x, t)} \] (3.33)
or
\[
\left(-\sum_{i=1}^{r+} \Psi_i^+ E_{\beta_i}^{-(-1)} + \sum_{i=1}^{r-} \Psi_i^- E_{-\beta_i}^{-(-1)} + \sum_{a=1}^{\sigma} \sigma_a H_a^{-(-1)}\right) |\lambda_o\rangle =
\]
\[
- \left[\left[\Psi(0) h \Psi(0)^{-1}\right]_{(1)} \Psi \right]_{(-1)} |\lambda_o\rangle \frac{1}{\hat{\tau}(o)(x,t)},
\]
(3.34)

Using the commutation rules for the relevant Kac-Moody algebra elements we have

\[
\Psi_i^+ = \frac{\tau_i^+}{\hat{\tau}(o)} \quad \text{and} \quad \Psi_i^- = -\frac{\tau_i^-}{\hat{\tau}(o)},
\]
(3.35)

where the tau functions are defined by

\[
\tau_i^+ \equiv \langle \lambda_o | E^{(1)}_{-\beta_i} \left[\Psi(0) h \Psi(0)^{-1}\right]_{(-1)} |\lambda_o\rangle,
\]
(3.36)

\[
\tau_i^- \equiv \langle \lambda_o | E^{(1)}_{\beta_i} \left[\Psi(0) h \Psi(0)^{-1}\right]_{(-1)} |\lambda_o\rangle,
\]
(3.37)

and

\[
\hat{\tau}(o) \equiv \langle \lambda_o | \left[\Psi(0) h \Psi(0)^{-1}\right]_{(o)} |\lambda_o\rangle.
\]
(3.38)

In order to obtain the first non trivial solutions of (3.9) let us consider

\[
h = e^{aF_j}, \quad F_j = \sum_{n=-\infty}^{+\infty} \nu^n_j E_{-\beta_j}^{(-n)} , \quad \text{where} \ \nu_j \ \text{and} \ a \ \text{are real parameters.}
\]
(3.39)

with \( F_j \) being an eigenvector under the adjoint action of the generator \( E^{(n)} \), that is

\[
[xE^{(1)} + tE^{(2)} + X, F_j] = -[\nu_j (x + \nu_j t + \nu_j) F_j],
\]
(3.40)

where

\[
\nu_j = \sum_{n=3}^{+\infty} z_n \nu^n_j;
\]

denoting \( \varphi_j = \nu_j (x + \nu_j t + \nu_j) \) one can write

\[
\left[\Psi(0) h \Psi(0)^{-1}\right] = \exp \left(e^{-\varphi_j} aF_j\right)
\]
(3.41)

\[
= 1 + e^{-\varphi_j} aF_j,
\]

where we have used the fact that \( F_j^n = 0, \) for \( n \geq 2, \) that is, \( F_j \) are nilpotent (see Appendix B for more details).

The tau function \( \hat{\tau}(o) \) becomes

\[
\hat{\tau}(o) = \langle \lambda_o | \left(1 + e^{-\varphi_j} aF_j\right) |\lambda_o\rangle = 1,
\]
since

\[
\langle \lambda_o | E_{-\beta_i}^{(o)} |\lambda_o\rangle = 0.
\]
For the tau function $\tau_i^+$ we have

$$\tau_i^+ = \langle \lambda_o | \left[ E_{-\beta_i}^{(1)} \exp \left( e^{-\varphi_j} a \sum_{n=-\infty}^{+\infty} \nu_j^n E_{-\beta_i}^{(-n)} \right) \right] \rangle \langle \lambda_o |$$

$$= \langle \lambda_o | \left[ E_{-\beta_i}^{(1)} e^{-\varphi_j} a \sum_{n=-\infty}^{+\infty} \nu_j^n E_{-\beta_i}^{(-n)} \right] \rangle \langle \lambda_o |,$$

as $[E_{-\beta_i}^{(1)}, E_{-\beta_j}^{(-n)}] = 0$ and $E_{-\beta_i}^{(1)} |\lambda_o\rangle = 0$, it follows

$$\tau_i^+ = 0.$$

Now

$$\tau_i^- = \langle \lambda_o | \left[ E_{-\beta_i}^{(1)} e^{-\varphi_j} a \sum_{n=-\infty}^{+\infty} \nu_j^n E_{-\beta_i}^{(-n)} \right] \rangle \langle \lambda_o |,$$

if $i \neq j$ the last expression becomes

$$\tau_i^- = \langle \lambda_o | E_{-\beta_i}^{(1)} e^{-\varphi_j} a \nu_j |\lambda_o\rangle = 0,$$

then

$$\tau_i^- = a\delta_{ij} e^{-\varphi_j} \nu_j \langle \lambda_o | E_{-\beta_i}^{(1)} E_{-\beta_i}^{(-1)} |\lambda_o\rangle;$$

the matrix element can be written as

$$\langle \lambda_o | \left\{ \sum_{a=1}^{r} H_a^{(0)} + C + E_{-\beta_i}^{(-1)} E_{-\beta_i}^{(1)} \right\} |\lambda_o\rangle.$$

Then it follows

$$\tau_i^- = \delta_{ij} a \nu_j e^{-\varphi_j},$$

and therefore we obtain the solution

$$\Psi_i^+ = 0, \quad \Psi_i^- = -\delta_{ij} a \nu_j e^{-\varphi_j}. \quad (3.42)$$

We can also make the choice

$$h = e^{bG_j}, \text{ with } G_j = \sum_{n=-\infty}^{+\infty} \rho_j^n E_{-\beta_i}^{(-n)}, \text{ } \rho_j \text{ being a real parameter.}$$

with $G_j$ a new eigenvector of the adjoint action of $E^{(l)}$.

$$\left[ x E^{(1)} + t E^{(2)} + X, G_j \right] = \left( \rho_j (x + \rho_j t) + \overline{\rho_j} \right) G_j,$$

with

$$\overline{\rho_j} = \sum_{n=3}^{+\infty} \bar{z}_n \rho_j^n.$$

As in the previous case

$$[\Psi^{(0)} h \Psi^{(0)-1}] = \exp (e^n b G_j)$$
calculations as in the previous case, one gets
\[ G = 1 + e^{\eta} b G, \]  
(3.43)
where \( G^n = 0 \) for \( n \geq 2 \). Denoting \( \eta_j = \rho_j (x + \rho_j t) + \overline{\rho}_j \), and performing a similar calculations as in the previous case, one gets
\[ \hat{\tau}^{(o)} = 1, \quad \tau^- = 0 \quad \text{and} \quad \tau^+ = \delta_{ij} b \rho_j e^{\eta_j}, \]
and therefore we have a new solution of the form
\[ \Psi^- = 0, \quad \Psi^+ = \delta_{ij} b \rho_j e^{\eta_j}. \]  
(3.44)
Consider now the product
\[ h = e^{a_{j_1} F_{j_1} e^{b_{j_2} G_{j_2}}} j_1, j_2 = 1, 2, ..., \]  
(3.45)
where
\[ F_{j_1} = \sum_{n=-\infty}^{+\infty} \nu_{j_1, n} E^{(-n)} \beta_j, \quad G_{j_2} = \sum_{n=-\infty}^{+\infty} \rho_{j_2, n} E^{(-n)} \beta_j \quad a_{j_1}, b_{j_2} \text{ real parameters.} \]
Let us note that \( F_{j_1} \) and \( G_{j_2} \) are associated to positive and negative roots respectively. Therefore
\[ \left[ \Psi^{(0)}_h \Psi^{(0)-1} \right] = \left( 1 + e^{-\varphi_{j_1}} a_{j_1} F_{j_1} \right) \left( 1 + e^{\eta_{j_2}} b_{j_2} G_{j_2} \right) = 1 + e^{-\varphi_{j_1}} a_{j_1} F_{j_1} + e^{\eta_{j_2}} b_{j_2} G_{j_2} + a_{j_1} b_{j_2} e^{-\varphi_{j_1}} e^{\eta_{j_2}} F_{j_1} G_{j_2}, \]  
(3.46)
with \( \varphi_{j_1} = \nu_{j_1} (x + \nu t) + \overline{\nu}_j \quad \text{and} \quad \eta_{j_2} = \rho_{j_2} (x + \rho_j t) + \overline{\rho}_j \). The corresponding tau function are
\[ \hat{\tau}^{(o)} = 1 + \delta_{j_1, j_2} a_{j_1} b_{j_2} C_{j_1, j_2} e^{-\varphi_{j_1}} e^{\eta_{j_2}}, \quad C_{j_1, j_2} = \frac{\nu_{j_1} - \rho_{j_2}}{\nu_{j_1} \rho_{j_2}}, \]  
(3.47)
\[ \tau^+ = \langle \lambda_0 | E^{(1)}_{-\beta_j} b_{j_2} e^{\eta_{j_2}} G_{j_2} | \lambda_0 \rangle \]  
(3.48)
and
\[ \tau^- = \delta_{j_1, j_2} a_{j_1} \nu_{j_1} e^{-\varphi_{j_1}}. \]  
(3.49)
In Appendix \[ \Box \] we outline the form of the corresponding matrix elements. We therefore obtain
\[ \Psi^+ = \frac{\delta_{i, j_2} b_{j_2} \rho_{j_2} e^{\eta_{j_2}}}{1 + \delta_{j_1, j_2} a_{j_1} b_{j_2} C_{j_1, j_2} e^{-\varphi_{j_1}} e^{\eta_{j_2}}}, \quad \Psi^- = -\frac{\delta_{i, j_2} a_{j_1} \nu_{j_1} e^{-\varphi_{j_1}}}{1 + \delta_{j_1, j_2} a_{j_1} b_{j_2} C_{j_1, j_2} e^{-\varphi_{j_1}} e^{\eta_{j_2}}}. \]  
(3.50)
If \( j_1 \neq j_2 \) in (3.47) we recover the solution (3.44) and (3.44). Therefore in order to have one-soliton solutions we must have \( j_1 = j_2 = i \) and therefore a solution of the system of equations (3.9) is
\[ \Psi^+ = \frac{b_i \rho_i e^{\eta_i}}{1 + a_i b_i C_{i, i} e^{-\varphi_i} e^{\eta_i}}, \quad \Psi^- = -\frac{a_i \nu_i e^{-\varphi_i}}{1 + a_i b_i C_{i, i} e^{-\varphi_i} e^{\eta_i}}. \]  
(3.51)
In order to study the $N$-soliton solution consider the following group element

$$h = e^{a_1 F_{i_1} \ldots e^{a_N F_{i_N}} e^{b_1 G_{j_1} \ldots e^{b_N G_{j_N}}}},$$  \hspace{1cm} (3.52)

where

$$F_{jl} = \sum_{n=-\infty}^{+\infty} \nu_n E_{\beta j l}^{(-n)}, \quad G_{jl} = \sum_{n=-\infty}^{+\infty} \rho_n E_{\beta j l}^{(-n)}, \quad l = 1, 2, \ldots N; \quad i_l, j_l = 1, 2, \ldots r;$$

\[ a_l, b_l, \nu_l \text{ and } \rho_l \text{ are real parameters.} \]

where

$$\varphi_l = \nu_l (x + \nu_l t) + \overline{\nu_l}, \quad \eta_l = \rho_l (x + \rho_l t) + \overline{\rho_l}, \quad l = 1, 2, \ldots, N.$$  \hspace{1cm} (3.53)

Then

$$\left[ \Psi^{(0)} h \Psi^{(0)-1} \right] = \left( 1 + e^{-\varphi_1} a_1 F_{i_1} \right) \ldots \left( 1 + e^{-\varphi_N} a_N F_{i_N} \right) \left( 1 + e^{\eta_1} b_1 G_{j_1} \right) \ldots \left( 1 + e^{\eta_N} b_N G_{j_N} \right),$$

Using (3.36), (3.37) and (3.38) we calculate the corresponding tau functions. Denoting

$$A_{i_l} \equiv a_l \nu_l e^{-\varphi_l}, \quad B_{j_l} \equiv b_l \rho_l e^{\eta_l},$$  \hspace{1cm} (3.54)

we have

$$\hat{\tau}^{(o)} = \langle \lambda_o \left| \left( \Psi^{(0)} h \Psi^{(0)-1} \right) \right| \lambda_o \rangle$$

$$\hat{\tau}^{(o)} = 1 +$$

$$\langle \lambda_o \left| \sum_{n=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N} \left( A_{i_1} \ldots A_{i_n} \right) \left( F_{i_1} \ldots F_{i_n} \right) \sum_{1 \leq k_1 < k_2 < \ldots < k_n \leq N} \left( B_{j_1} \ldots B_{j_n} \right) \right) \right| \lambda_o \rangle$$

$$= 1 + \sum_{n=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq k_1 < k_2 < \ldots < k_n \leq N} C_{i_1 \ldots i_n, j_1 \ldots j_n} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_n},$$  \hspace{1cm} (3.55)

where the coefficients are given by the matrix elements

$$C_{i_1 \ldots i_n, j_1 \ldots j_n} = \langle \lambda_o \left| F_{i_1} \ldots F_{i_n} G_{j_1} \ldots G_{j_n} \right| \lambda_o \rangle,$$  \hspace{1cm} (3.56)

Similarly

$$\tau^\pm_i = \langle \lambda_o \left| E_{\pm i}^{(1)} \Psi^{(0)} h \Psi^{(0)-1} \right| \lambda_o \rangle,$$
\[ \tau^+ = \sum_{n=0}^{N-1} \sum_{1 \leq l_1 < l_2 < \ldots < l_n \leq N, 1 \leq k_1 < k_2 < \ldots < k_{n+1} \leq N} C^+_{i_1 \ldots i_n j_1 \ldots j_{n+1}} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_{n+1}}, \]

(3.59)

\[ \tau^- = \sum_{n=0}^{N-1} \sum_{1 \leq l_1 < l_2 < \ldots < l_{n+1} \leq N, 1 \leq k_1 < k_2 < \ldots < k_n \leq N} C^-_{i_1 \ldots i_{n+1} i j_1 \ldots j_{k_{n+1}}} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_{n+1}}, \]

(3.60)

with the matrix elements given by

\[ C^+_{i_1 \ldots i_n j_1 \ldots j_{n+1}, i j_1 \ldots j_{k_n}} = \langle \lambda_0 | E^{(1)}_{-\beta_1} F_{i_1} \ldots F_{i_n} G_{j_1} \ldots G_{j_{n+1}} | \lambda_0 \rangle, \]

(3.61)

\[ C^-_{i_1 \ldots i_{n+1} i j_1 \ldots j_{k_n}} = \langle \lambda_0 | E^{(1)}_{\beta_1} F_{i_1} \ldots F_{i_n} G_{j_1} \ldots G_{j_k} | \lambda_0 \rangle. \]

(3.62)

The calculation of the corresponding matrix elements is outlined in Appendix D.

4 The example of GNLS

The hierarchy GNLS has a Lax operator

\[ L = \partial_x - E^{(1)} - \Psi^+ E^{(0)}_+ - \Psi^- E^{(0)}_- - \nu_1 C, \]

(4.1)

where \( \Psi^\pm \) and \( \nu_1 \) are the fields of the model.

The corresponding \( \hat{sl}(2) \) algebra in the Weyl Cartan basis is

\[
\begin{align*}
[H^{(m)}, H^{(n)}] &= \frac{n}{2} C \delta_{m+n,0}, \\
[H^{(n)}, E^{(m)}_\pm] &= \pm E^{(m+n)}_\pm, \\
[E^{(m)}_+, E^{(n)}_-] &= 2H^{(m+n)} + nC \delta_{m+n,0}.
\end{align*}
\]

(4.2)

The equations of the hierarchy are obtained as follows

\[ \frac{\partial L}{\partial t_N} = [B_N, L], \quad N > 0 \]

where

\[ B_N = \left( U H^{(N)} U^{-1} \right)_{\geq 0} \in C^\infty (\mathbb{R}, \mathfrak{g}_{\geq 0} (s)), \]

\[ B_N \subset \bigoplus_{i=0}^{N} \mathfrak{g}_i, \]

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with \( U \) being a group element obtained by exponentiating the negative degree elements

\[
U = \exp \left( \sum_{n > 1} T^{(-n)} \right), \quad [D, T^{(n)}] = nT^{(n)}.
\]

The first two \( B_N \) are

\[
B_1 = H^{(1)} + \Psi^+ E^{(0)}_+ + \Psi^- E^{(0)}_- + \nu_1 C, \quad (4.3)
\]

\[
B_2 = H^{(2)} + \Psi^+ E^{(1)}_+ + \Psi^- E^{(1)}_- - 2\Psi^+\Psi^- H^{(0)} + \partial_x \Psi^+ E^{(0)}_+ - \partial_x \Psi^- E^{(0)}_- + \nu_2 C, \quad (4.4)
\]

and the first equations of the hierarchy are

\[
\begin{align*}
\partial_t L &= [B_1, L] : \\
\partial_t \Psi^\pm &= \partial_x \Psi^\pm, \\
\partial_t \nu_1 &= \partial_x \nu_1.
\end{align*}
\]

\[
\partial_t L = [B_2, L] : \\
\partial_t \Psi^\pm &= \pm \partial_x^2 \Psi^\pm \mp 2 \left( \Psi^+ \Psi^- \right) \Psi^\pm, \\
\partial_t \nu_1 &= \partial_x \nu_2.
\]

The system of equations for the \( \Psi^\pm \) fields in (4.6), supplied with a convenient complexification of the time variable and the fields, are related to the well known non-linear Schrödinger equation (NLS) [13, 14]. The zero curvature condition for \( B_1 \) and \( B_N \) can be written as

\[
[\partial_t - B_N, \partial_x - B_1] = 0, \quad N = 1, 2, ...
\]

(4.7)

where \( B_N \) has the general form

\[
B_N = H^{(N)} + \sum_{n=0}^{N-1} B^{(n)}_N, \quad \text{with } B^{(n)}_N \in C^\infty (\mathfrak{g}, \mathfrak{g}_n (\text{hom})).
\]

(4.8)

As \( \Psi^\pm = \nu_N = 0 \) is a solution of each system of equations of the hierarchy, we have

\[
B_1^{(\text{vac})} = H^{(1)}, \quad B_N^{(\text{vac})} = H^{(N)},
\]

(4.9)

such connections can be obtained with the help of \( B_N = \partial_t \Psi \Psi^{-1} \) from the group element

\[
\Psi^{(\text{vac})} = \exp \left( xH^{(1)} + t_N H^{(N)} + \sum_{n=2,3,\ldots} t_n H^{(n)} \right) \equiv \exp \left( \sum_{n=1,2,\ldots} t_n H^{(n)} \right).
\]

(4.10)

Observe that according to (4.3) we have identified \( t_1 = x \).

The connections in the vacuum orbit are given by

\[
B_1 = \Theta H^{(1)} \Theta^{-1} + \partial_x \Theta \Theta^{-1},
\]

(4.11)
\[= M^{-1} \left( NH^{(1)} N^{-1} - \partial_x MM^{-1} + \partial_x NN^{-1} \right) M, \]

\[B_N = \Theta H^{(N)} \Theta^{-1} + \partial_N \Theta \Theta^{-1}, \]

\[= M^{-1} \left( NH^{(N)} N^{-1} - \partial_N MM^{-1} + \partial_N NN^{-1} \right) M. \tag{4.12} \]

Denote
\[\Theta = \exp \left( \sum_{n>0} \sigma_{-n} \right), \quad M = \exp \left( \sigma_o \right), \quad N = \exp \left( \sum \sigma_n \right), \tag{4.13} \]

\[[D, \sigma_n] = n \sigma_n, \]

Therefore we can relate \(\Psi^\pm\) to some \(\sigma_n\). For instance for \(N = 2\) and denoting \(t_2 = t\), we have
\[B_1 = H^{(1)} + [\sigma_{-1}, H^{(1)}] + \text{ terms of negative grade}, \tag{4.14} \]

\[= M^{-1} \left( H^{(1)} - \partial_x MM^{-1} + \partial_x \sigma_1 \right) M + \text{ terms of grade } > 1, \]

\[B_2 = H^{(2)} + [\sigma_{-1}, H^{(2)}] + [\sigma_{-2}, H^{(2)}] + \frac{1}{2} [\sigma_{-1}, [\sigma_{-1}, H^{(2)}]] + \]

\[+ \text{ terms of negative grade}. \tag{4.15} \]

\[= M^{-1} \left( H^{(2)} - \partial_t MM^{-1} + \partial_t \sigma_1 + \partial_t \sigma_2 + [\sigma_1, \partial_t \sigma_1] \right) M + \text{ terms of grade } > 2. \]

Let us observe that the next term (with degree \(-1\)) in (4.14) vanishes, and therefore we have
\[\partial_x \sigma_{-1} + [\sigma_{-2}, H^{(1)}] + \frac{1}{2} [\sigma_{-1}, [\sigma_{-1}, H^{(1)}]] = 0. \tag{4.16} \]

Denoting
\[\sigma_{-1} = -\Psi^+ E^{(-1)}_+ + \Psi^- E^{(-1)}_- + \sigma_{-1}^{o} H^{(-1)} \tag{4.17} \]

and
\[\sigma_{-2} = -\sigma_{-2}^+ E^{(-2)}_+ + \sigma_{-2}^- E^{(-2)}_- + \sigma_{-2}^o H^{(-2)}, \tag{4.18} \]

from (4.16) we obtain
\[\partial_x \sigma_{-1}^{o} = 2 \Psi^+ \Psi^-, \tag{4.19} \]

\[\sigma_{-2}^+ = -\partial_x \Psi^+ + \frac{1}{2} \sigma_{-1}^{o} \Psi^+, \tag{4.20} \]

\[\sigma_{-2}^- = -\partial_x \Psi^- + \frac{1}{2} \sigma_{-1}^{o} \Psi^- \tag{4.21} \]

Substituting these expressions for \(\sigma_{-1}\) and \(\sigma_{-2}\) in (4.14) and (4.15) we obtain (4.7) with
\[\nu_1 = -\frac{\sigma_{-1}^{o}}{2}, \quad \nu_1 = -\sigma_{-2}^o \tag{4.22} \]
The $\sigma_{-n}$'s with higher grades are to cancel the undesired components. The term of gradation $-2$ in (4.14) satisfies \[ \partial_x \sigma_{-2} + [\sigma_{-3}, H^{(1)}] + \frac{1}{2} [\sigma_{-2}, [\sigma_{-1}, H^{(1)}]] + \frac{1}{2} [\sigma_{-1}, [\sigma_{-2}, H^{(1)}]] = 0, \] (4.23)

where
\[ \sigma_{-3} = \sigma_{-3}^+ E_{+}^{(-3)} + \sigma_{-3}^- E_{-}^{(-3)} + \sigma_{-3}^0 H^{(-3)}. \]

From (4.23) we obtain \[ \partial_x \sigma_{-2}^0 = \Psi^- \partial_x \Psi^+ - \Psi^+ \partial_x \Psi^- . \] (4.24)

Equating to zero the terms of degree $(-1)$ and $(-2)$ in (4.15) we can obtain \[ \partial_t \sigma_{-1}^0 = 2 \left( \Psi^- \partial_x \Psi^+ - \Psi^+ \partial_x \Psi^- \right), \] (4.25)

and
\[ \partial_t \sigma_{-2}^0 = \frac{2}{3} \Psi^+ \Psi^- \left( \sigma_{-1}^0 \right)^2 - 2 \partial_x \Psi^+ \partial_x \Psi^- - \frac{2}{3} \left( \Psi^+ \Psi^- \right)^2 + \Psi^- \partial_x \Psi^+ + \Psi^+ \partial_x \Psi^- . \] (4.26)

From (4.19) and (4.25) we obtain \[ \partial_t \left( \Psi^+ \Psi^- \right) = \partial_x F[\Psi^+, \Psi^-], \] (4.27)

where $F$ is a functional of the fields and $\mathcal{H}_1$ is the first Hamiltonian given by
\[ \mathcal{H}_1 = \int_{-\infty}^{+\infty} dx. \Psi^+ \Psi^- . \]

In the same way (4.24) and (4.26) gives \[ \partial_t \left( \Psi^- \partial_x \Psi^+ - \Psi^+ \partial_x \Psi^- \right) = \partial_x G[\Psi^+, \Psi^-], \] (4.28)

where $G$ is a functional of the fields and $\mathcal{H}_2$ is the second Hamiltonian of GNLS$_1$ system given by
\[ \mathcal{H}_2 = \int_{-\infty}^{+\infty} dx. \left( \Psi^- \partial_x \Psi^+ - \Psi^+ \partial_x \Psi^- \right) . \] (4.29)

In this way one can construct the remaining Hamiltonians of higher order corresponding to every $\sigma_{n}^0 \ (n < -2)$.

Let us define the tau-function vector
\[ \tau (x, t_2, t_3, ...) = \Psi^{(\text{vac})} h \Psi^{(\text{vac})-1} |\lambda_o\rangle, \] (4.30)
\[ = \Theta^{-1} M^{-1} |\lambda_o\rangle, \] (4.31)

where $h$ is a particular element of the group $\hat{sl}(2)$ which generates a dressing transformation. \footnote{We use $\partial e^\sigma e^{-\sigma} = \partial \sigma + \frac{1}{2} [\sigma, \partial \sigma] + \frac{1}{4!} [\sigma, [\sigma, \partial \sigma]] + \cdots$}
Therefore we have
\[
\exp \left( - \sum_{n>0} \sigma_{-n} \right) \exp ( -\sigma_o ) |\lambda_o\rangle = \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} |\lambda_o\rangle ,
\]
and then one can write
\[
\sigma_o = \sigma_o^0 H + \sigma_o^+ E_+^{(o)} + \sigma_o^- E_-^{(o)} + \eta C ,
\]
or
\[
\sigma_o = \sigma_o^0 h_1 + \sigma_o^+ e_1 + \sigma_o^- f_1 + \eta C ,
\]
where we have used \( h_1 |\lambda_o\rangle = 0 , \quad f_1 |\lambda_o\rangle = 0 \) and \( C |\lambda_o\rangle = |\lambda_o\rangle \). In this way the zero gradation of expression (4.32) is
\[
\exp ( -\sigma_o ) |\lambda_o\rangle = \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) (o) |\lambda_o\rangle ,
\]
which can be rewritten as
\[
\exp ( -\sigma_o ) |\lambda_o\rangle = |\lambda_o\rangle \tilde{\tau}^{(o)} (x,t)
\]
where \( \tilde{\tau}^{(o)} (x,t) \) is a function of \( x \) and the times \( t_n \) given by the following matrix element
\[
\tilde{\tau}^{(o)} (x,t) = \langle \lambda_o | \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) (o) |\lambda_o\rangle .
\]
The term with degree \((-1)\) in (4.32) becomes
\[
(-\sigma_{-1}) |\lambda_o\rangle = \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) (-1) |\lambda_o\rangle / \tilde{\tau}^{(o)} (x,t),
\]
or
\[
\left( -\Psi^+ E_+^{(-1)} + \Psi^- E_-^{(-1)} + \sigma_{-1}^0 H^{(-1)} \right) |\lambda_o\rangle = - \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) (-1) |\lambda_o\rangle / \tilde{\tau}^{(o)} (x,t),
\]
As \( H^{(0)} |\lambda_o\rangle = h_1 |\lambda_o\rangle = 0 , \quad E_\pm^{(1)} |\lambda_o\rangle = 0 \) we can write
\[
\Psi^+ = \frac{\tau^+}{\tilde{\tau}^{(o)}} \quad \text{and} \quad \Psi^- = - \frac{\tau^-}{\tilde{\tau}^{(o)}},
\]
where
\[
\tau^+ \equiv \langle \lambda_o | E_-^{(1)} \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) (-1) |\lambda_o\rangle ,
\]
\[
\tau^- \equiv \langle \lambda_o | E_+^{(1)} \left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) (-1) |\lambda_o\rangle .
\]
The relations (4.35), (4.38) and (4.39) define the tau-functions of the GNLS\(_1\) system of equations (4.6).
In order to obtain the first non trivial solution we choose
\[
h = e^F , \quad \text{with} \quad F = \sum_{n=-\infty}^{+\infty} \nu_1^n E_-^{(-n)}. \quad (4.40)
\]
Since
\[
\left[ \sum_{n=1}^{+\infty} t_n H^{(n)}, F \right] = - \left( \sum_{n=1}^{+\infty} t_n \nu_1^n \right) F, \quad \text{with } \nu_1 \text{ a real parameter},
\]
we may obtain
\[
\left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) = \exp \left( e^{\varphi_1} F \right) = 1 + e^{\varphi_1} F,
\]
with
\[
\varphi_1 = \sum_{n=1}^{+\infty} t_n \nu_1^n.
\]
Where we have used the property \( F^n = 0 \), for \( n \geq 2 \). The tau functions become
\[
\tau^{(o)} = \langle \lambda_o | \left( 1 + e^{-\varphi_1} E_{-}^{(o)} \right) | \lambda_o \rangle = 1,
\]
\[
\tau^{+} = \langle \lambda_o | E_{-}^{(1)} e^{-\varphi_1} \nu_1 E_{-}^{(-1)} | \lambda_o \rangle = 0,
\]
\[
\tau^{-} = \langle \lambda_o | E_{+}^{(1)} e^{-\varphi_1} \nu_1 E_{-}^{(-1)} | \lambda_o \rangle = \nu_1 e^{-\varphi_1}.
\]
Using (4.37) we obtain the following solution of the Eqs. (4.6)
\[
\Psi^{+} = 0 \quad \text{and} \quad \Psi^{-} = - \nu_1 e^{-\varphi_1}.
\]
Now let us choose
\[
h = e^{G}, \quad G = \sum_{n=-\infty}^{+\infty} \rho_1^n E_{+}^{(-n)}, \quad \rho_1 \text{ is a real parameter.}
\]
Since
\[
\left[ \sum_{n=1}^{+\infty} t_n H^{(n)}, G \right] = \left( \sum_{n=1}^{+\infty} t_n \rho_1^n \right) G,
\]
we obtain
\[
\left( \Psi^{(\text{vac})} h \Psi^{(\text{vac})^{-1}} \right) = \exp \left( e^{\eta_1} G \right) = 1 + e^{\eta_1} G, \quad \text{with } \eta_1 = \sum_{n=1}^{+\infty} t_n \rho_1^n,
\]
where we used \( G^n = 0, \ n \geq 2 \). Therefore the tau functions become
\[
\hat{\tau}^{(o)} = \langle \lambda_o | \left( 1 + e^{\eta_1} E_{+}^{(o)} \right) | \lambda_o \rangle = 1,
\]
\[ \tau^- = \langle \lambda_o | \left( E_+^{(1)} e^n \rho_1 E_+^{(-1)} \right)_{(o)} | \lambda_o \rangle = 0, \]
\[ \tau^+ = \langle \lambda_o | E_-^{(1)} e^n \rho_1 E_+^{(-1)} | \lambda_o \rangle = \rho_1 e^n, \]
and the corresponding solutions
\[ \Psi^- = 0 \quad \text{and} \quad \Psi^+ = \rho_1 e^n. \tag{4.51} \]

In order to obtain one-soliton solutions, let us choose
\[ h = e^{aF} e^{bG}, \quad \text{where} \quad F \text{ and } G \text{ are given in (4.40) and (4.48)} \tag{4.52} \]
with \( a \) and \( b \) real parameters.

Then
\[ \Psi^{(\text{vac})} h \Psi^{(\text{vac})}^{-1} = \exp \left( e^{-\varphi} aF \right) \exp \left( e^{\eta} bG \right) \tag{4.53} \]
\[ = 1 + e^{-\varphi} aF + e^{\eta} bG + e^{-\varphi} e^\eta aFbG, \tag{4.54} \]
with \( \varphi = \sum_{n=1}^{+\infty} t_n \nu^n \) and \( \eta = \sum_{n=1}^{+\infty} t_n \rho^n \). Let us compute the relevant tau functions. The expression for \( \tilde{\tau}^{(o)} \) becomes
\[ \tilde{\tau}^{(o)} = 1 + a b c e^{-\varphi} e^\eta, \tag{4.55} \]
where \( c \) is a matrix element of the following form
\[ c = \langle \lambda_o | (FG)_{(o)} | \lambda_o \rangle \]
\[ = \langle \lambda_o \left| \sum_{n,m>0}^{+\infty} \nu_1^{-n} \rho_1^{-m} E_-^{(n)} E_+^{(-m)} \right| \lambda_o \rangle \]
\[ = \langle \lambda_o \sum_{n,m>0}^{+\infty} \nu_1^{-n} \rho_1^{-m} \left( -2H^{(n-m)} + m\delta_{n-m,0} C \right)_{(o)} | \lambda_o \rangle \]
\[ = \sum_{n=0}^{+\infty} n \left( \frac{\rho_1}{\nu_1} \right)^n \]
\[ = \frac{\nu_1 \rho_1}{(\rho_1 - \nu_1)^2}. \tag{4.56} \]

where, for pedagogical reasons, we have used step by step, the properties of the integrable highest weight representation of the algebra \( \hat{sl}(2) \), see Appendix A (Eqs. (A.31)-(A.37)). The computation of higher order matrix elements is performed very quickly using the vertex operator formalism, see Appendix B.

The remaining tau functions are given by
\[ \tau^+ = \langle \lambda_o | E_-^{(1)} \left( be^{\eta} \rho_1 E_+^{(-1)} \right) | \lambda_o \rangle \]
\[ = be^{\eta} \rho_1 \langle \lambda_o | E_+^{(1)} \left( -2H^{(0)} + C \right) | \lambda_o \rangle \]
\[ = b \rho_1 e^{\eta} \tag{4.57} \]

18
\[ \tau^- = \langle \lambda_o | E_+^{(1)} \left( a.e^{-\varphi} \nu_1 E_-^{(-1)} \right) | \lambda_o \rangle = a.\nu_1 e^{-\varphi}. \] (4.58)

Thus, a one-soliton solution is
\[ \Psi^+ = \frac{b \nu_1 e^n}{1 + a b c e^{-\varphi} e^n}, \] (4.59)
and
\[ \Psi^- = -\frac{a \nu_1 e^{-\varphi}}{1 + a b c e^{-\varphi} e^n}. \] (4.60)

Let us write down the explicit form of this one-soliton solution. As a particular case we set the following relations
\[ \rho_1 = -\nu_1, \quad b = -a = -2 \quad \text{and} \quad t_{2n+1} = 0 \quad (n \geq 0), \] (4.61)
then the relations (4.59) and (4.60) become
\[ \Psi^+ = \nu_1 \exp \left( \nu_1^2 t + \varphi_1 \right) \sech (\nu_1 x) \] (4.62)
and
\[ \Psi^- = -\nu_1 \exp \left( -\nu_1^2 t - \varphi_1 \right) \sech (\nu_1 x), \] (4.63)
where \( \varphi_1 = \sum_{n=2}^{+\infty} \nu_1^{2n} t_{2n} \) is a phase parameter as far as only the equation (4.60) of the whole hierarchy is considered. The solutions of types (4.62) and (4.63), are known as an ‘envelope soliton’ or ‘bright soliton’ solutions in the context of the non-linear Schrödinger equation [15].

Next let us choose
\[ h = e^{a_1 F_1} e^{a_2 F_2} e^{b_1 G_1} e^{b_2 G_2}, \quad a_i, b_i, \nu_i, \rho_i \text{ real parameters}; \]
with \( F_i = \sum_{n=-\infty}^{+\infty} \nu_i^n E_+^{(-n)} \) and \( G_i = \sum_{n=-\infty}^{+\infty} \rho_i^n E_-^{(-n)}, \ i = 1, 2. \) (4.64)

Then
\[ \Psi^{(\text{vac})}_i \Psi^{(\text{vac})-1}_i = 1 + e^{-\varphi_1} a_1 F_1 + e^{-\varphi_2} a_2 F_2 + e^{\eta_1} b_1 G_1 + e^{\eta_2} b_2 G_2 + e^{\nu_1} e^{\eta_2} b_1 G_1 b_2 G_2 + e^{-\varphi_1} e^{-\varphi_2} a_1 F_1 a_2 F_2 + e^{-\varphi_1} e^{\eta_1} a_1 F_1 b_1 G_1 + e^{-\varphi_1} e^{\eta_2} a_1 F_1 b_2 G_2 + e^{-\varphi_2} e^{\eta_1} a_2 F_2 b_1 G_1 + e^{-\varphi_2} e^{\eta_2} a_2 F_2 b_2 G_2 + e^{-\eta_1} e^{-\eta_2} a_1 F_1 a_2 F_2 b_1 G_1 + e^{-\eta_1} e^{-\eta_2} a_1 F_1 a_2 F_2 b_2 G_2 + e^{-\eta_1} e^{\varphi_2} e^{\eta_2} a_1 F_1 a_2 F_2 b_1 G_1 + e^{-\eta_1} e^{\varphi_2} e^{\eta_2} a_1 F_1 a_2 F_2 b_2 G_2, \] (4.65)
with
\[ \varphi_i = \sum_{n=1}^{+\infty} \nu_i^n t_n, \quad \eta_i = \sum_{n=1}^{+\infty} \rho_i^n t_n, \quad i = 1, 2. \]
The corresponding tau functions are

\[
\tilde{\tau}^{(o)} = 1 + e^{-\varphi_1}e^{m_1}a_1b_1 \langle \lambda_o | F_1G_1 | \lambda_o \rangle + e^{-\varphi_1}e^{m_2}a_1b_2 \langle \lambda_o | F_1G_2 | \lambda_o \rangle + e^{-\varphi_2}e^{m_2}a_2b_2 \langle \lambda_o | F_2G_2 | \lambda_o \rangle + e^{m_1}e^{-\varphi_2}b_1a_2 \langle \lambda_o | G_1F_2 | \lambda_o \rangle + e^{-\varphi_1}e^{m_1}e^{-\varphi_2}e^{m_2}a_1b_1a_2b_2 \langle \lambda_o | F_1G_1F_2G_2 | \lambda_o \rangle, \tag{4.66}
\]

\[
\tau^+ = e^{m_1}b_1 \langle \lambda_o | E^{(1)}_+ G_1 | \lambda_o \rangle + e^{m_2}b_2 \langle \lambda_o | E^{(1)}_- G_2 | \lambda_o \rangle + e^{-\varphi_1}e^{m_1}e^{m_2}a_1b_1b_2 \langle \lambda_o | E^{(1)}_+ F_1G_1G_2 | \lambda_o \rangle + e^{-\varphi_2}e^{m_1}e^{m_2}a_2b_1b_2 \langle \lambda_o | E^{(1)}_- F_1F_2G_2 | \lambda_o \rangle, \tag{4.67}
\]

and

\[
\tau^- = e^{-\varphi_1}a_1 \langle \lambda_o | E^{(1)}_+ F_1 | \lambda_o \rangle + e^{-\varphi_2}a_2 \langle \lambda_o | E^{(1)}_- F_2 | \lambda_o \rangle + e^{-\varphi_1}e^{-\varphi_2}e^{m_1}e^{m_2}a_1a_2b_1b_2 \langle \lambda_o | E^{(1)}_+ F_1F_2G_2 | \lambda_o \rangle + e^{-\varphi_1}e^{-\varphi_2}e^{m_1}e^{m_2}a_1a_2b_1b_2 \langle \lambda_o | E^{(1)}_- F_1F_1G_2 | \lambda_o \rangle. \tag{4.68}
\]

Notice that only some terms of the expansion \( \Psi^{(vac)h} \Psi^{(vac)-1} \) contribute to the tau functions. For example, the terms \( \langle \lambda_o | F_1 | \lambda_o \rangle \), \( \langle \lambda_o | G_1 | \lambda_o \rangle \), \( \langle \lambda_o | F_2G_1 | \lambda_o \rangle \) and \( \langle \lambda_o | G_1F_2 | \lambda_o \rangle \) do not contribute to the computation of \( \tilde{\tau}^{(o)} \), since these matrix elements vanish.

In particular let us consider \( \nu_i = -\nu_i \), \( b_i = -a_i = -2 \) and \( t_{2n+1} = 0 \), then

\[
\tau^+ = a_1\nu_1\hat{\varphi}_1 + a_2\nu_2\hat{\varphi}_2 + a_1a_2\Delta_1e^{\varphi_1}e^{-\varphi_2} + a_1a_2\Delta_2e^{-\varphi_1}e^{\varphi_2}, \tag{4.69}
\]

\[
\tau^- = a_1\nu_1e^{-\varphi_1} + a_2\nu_2e^{-\varphi_2} + a_1a_2\Delta_1e^{-\varphi_1}e^{-\varphi_2} + a_1a_2\Delta_2e^{-\varphi_1}e^{\varphi_2}, \tag{4.70}
\]

where

\[
\hat{\varphi}_i = -\nu_i(x + \nu_it) + \eta_i, \tag{4.71}
\]

\[
\varphi_i = \nu_i(x + \nu_it) + \eta_i, \tag{4.72}
\]

\[
\Delta_i = \frac{\nu_i}{4} \left( \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \right)^2, \quad i = 1, 2;
\]

now let us choose \( a_1 \) and \( a_2 \) such that

\[
\nu_i = a_i^2\Delta_i, \quad (i \neq j),
\]

then

\[
a_1 = a_2 \equiv a = 2 \left( \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2} \right).
\]

Let us remark that the parameters \( \eta_i \) are some phase parameters if only the equations \( \{4.6\} \) of the hierarchy are to be considered. With this choice of parameters the \( \tilde{\tau}^{(o)} \) function turns out to be

\[
\tilde{\tau}^{(o)} = e^{-(\nu_1+\nu_2)x} \left\{ \frac{4}{a^2} (e^{(\nu_1-\nu_2)x} + e^{-(\nu_1-\nu_2)x}) + e^{(\nu_1+\nu_2)x} + e^{-(\nu_1+\nu_2)x} + 4\frac{\nu_1\nu_2}{(\nu_1-\nu_2)^2} (e^{(\nu_1^2-\nu_2^2) t+(\eta_1-\eta_2)} + e^{-(\nu_1^2-\nu_2^2) t-(\eta_1-\eta_2)}) \right\}. \tag{4.73}
\]
Therefore the fields $\Psi^+$ and $\Psi^-$ become

$$
\Psi^\pm = \pm ae^\pm [(\nu_1^2 + \nu_2^2) (t + (\mathfrak{v}_1 + \mathfrak{v}_2))],
$$
which are the two-soliton solutions of \(4.6\) or the two-soliton solutions (after relevant complexification) of the corresponding non-linear Schrödinger equation \(16\).

Regarding the solutions of the equations of higher order of the hierarchy \(4.7\), we may argue that the same solutions, 1-soliton \(4.59\)-\(4.60\) and 2-soliton constructed with the tau functions \(4.66\), \(4.67\) and \(4.68\) satisfying \(4.3\) should satisfy the higher order equations of the hierarchy GNLS$_1$, each equation with its corresponding time scale $t_n$. This behaviour is also observed in the study of the KdV system using the perturbative reduction and multiple time scaling approach \(17\).

### 4.1 N-soliton solutions

The generalization for a N-soliton solution can be made choosing

$$h = e^{a_1 F_1} ... e^{a_N F_N} e^{b_1 G_1} ... e^{b_N G_N},$$

where

$$F_i = \sum_{n=-\infty}^{+\infty} \nu_i^n E_{-n}^i, \quad G_i = \sum_{n=-\infty}^{+\infty} \rho_i^n E_{n}^i, \quad i = 1, 2, ..., N$$

with $a_i, b_i, \nu_i$ and $\rho_i$ are real parameters.

The following expression plays an important role in the construction of the tau functions

$$\Psi^{(vac)} h \Psi^{(vac)}^{-1} = \left(1 + a_1 e^{-\varphi_1} F_1\right) \left(1 + b_1 e^{\eta_1} G_1\right) ... \left(1 + a_N e^{-\varphi_N} F_N\right) \left(1 + b_N e^{\eta_N} G_N\right),$$

where

$$\varphi_i = \sum_{n=1}^{+\infty} \nu_i^n t_n, \quad \eta_i = \sum_{n=1}^{+\infty} \rho_i^n t_n, \quad i = 1, 2, ..., N;$$

the parameters $\nu_i$ and $\rho_i$ will characterize each soliton.

It will be convenient to write the various $F_i$ and $G_i$ in terms of the vertex operators (see Appendix \(C\))

$$F_i \rightarrow \nu_i \Gamma_- (\nu_i), \quad G_i \rightarrow \rho_i \Gamma_+ (\rho_i),$$

then

$$\Psi^{(vac)} h \Psi^{(vac)}^{-1} = \left(1 + a_1 \nu_1 e^{-\varphi_1} \Gamma_- (\nu_1)\right) \left(1 + b_1 \rho_1 e^{\eta_1} \Gamma_+ (\rho_1)\right) ... \left(1 + a_N \nu_N e^{-\varphi_N} \Gamma_- (\nu_N)\right) \left(1 + b_N \rho_N e^{\eta_N} \Gamma_+ (\rho_N)\right).$$

Denoting

$$A_n \equiv a_n \nu_n e^{-\varphi_n}, \quad B_n \equiv b_n \rho_n e^{\eta_n},$$
we may compute the relevant tau functions

$$\hat{\tau}^{(o)} = 1 + \langle \lambda_o \rangle \sum_{n=1}^{N}$$

\[ \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N} A_{i_1} \ldots A_{i_n} \Gamma_+ (\nu_{i_1}) \ldots \Gamma_+ (\nu_{i_n}) \sum_{1 \leq j_1 < j_2 < \ldots < j_n \leq N} B_{j_1} \ldots B_{j_n} \Gamma_+ (\rho_{j_1}) \ldots \Gamma_+ (\rho_{j_n}) |\lambda_o\rangle. \]

We are using some properties of the product of vertex operators acting on $|\lambda_o\rangle$ and its dual $\langle \lambda_o |$ (see Appendix B and C). Then

$$\hat{\tau}^{(o)} = 1 + \sum_{n=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_n \leq N} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_n} \cdot (4.80)$$

\[ \prod_{1 \leq l \leq m \leq n} (\nu_{i_l} - \nu_{i_m})^2 (\rho_{j_l} - \rho_{j_m})^2 \epsilon (\alpha_{i_l}, \alpha_{i_m}) \epsilon (\alpha_{j_l}, \alpha_{j_m}) \] (4.81)

\[ \prod_{1 \leq l \leq m \leq n} (\nu_{i_l} - \rho_{j_m})^2 (\rho_{j_l} - \rho_{j_m})^2 \epsilon (\alpha_{i_l}, \alpha_{j_m})^{-1}. \] (4.82)

Denoting

$$\epsilon (\alpha_{i_l}, \alpha_{i_m}) = \epsilon (-\alpha, -\alpha) \equiv \epsilon (-, -),$$

$$\epsilon (\alpha_{i_l}, \alpha_{j_m}) = \epsilon (\alpha, \alpha) \equiv \epsilon (+, +),$$

$$\epsilon (\alpha_{i_l}, \alpha_{j_m}) = \epsilon (-\alpha, \alpha) \equiv \epsilon (-, +),$$

and using the properties of the cocycles in the case of $\widehat{sl}(2)$ [18]

$$\epsilon (+, +) = \epsilon (-, -) = -1,$$ (4.83)

$$\epsilon (+, -) = \epsilon (-, +) = 1,$$ (4.84)

we can write

$$\hat{\tau}^{(o)} = 1 + \sum_{n=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_n \leq N} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_n} \cdot (4.85)$$

\[ \prod_{1 \leq l \leq m \leq n} (\nu_{i_l} - \nu_{i_m})^2 (\rho_{j_l} - \rho_{j_m})^2 \prod_{1 \leq l \leq m \leq n} (\nu_{i_l} - \rho_{j_m})^2 \] (4.86)

and

$$\tau^\pm = \frac{1}{2\pi i} \int dz \cdot \langle \lambda_o | \Gamma_+ (z) \left( 1 + a_1 \nu_1 e^{-\varphi_1} \Gamma_+ (\nu_1) \right) (1 + b_1 \rho_1 e^{\eta_1} \Gamma_+ (\rho_1)) \cdots \right.$$ (4.87)

\[ \left. (1 + a_N \nu_N e^{-\varphi_N} \Gamma_+ (\nu_N)) (1 + b_N \rho_N e^{\eta_N} \Gamma_+ (\rho_N)) \right) |\lambda_o\rangle. \] (4.88)

According to Eq [C.15], in order to have non vanishing terms, we must have equal number of operators $\Gamma_+$ and $\Gamma_-$ inside the states $\langle \lambda_o |$ and $|\lambda_o \rangle$, then

$$\tau^\pm = \frac{1}{2\pi i} \int dz \sum_{n=1}^{N} A_1^{m_1} \ldots A_N^{m_N} B_1^{n_1} \ldots B_N^{n_N} \langle \lambda_o | \Gamma_+ (z) \Gamma_+^{m_1} (\nu_1) \ldots \Gamma_+^{m_N} (\nu_N) \right.$$ (4.89)

\[ \Gamma_+^{n_1} (\rho_1) \ldots \Gamma_+^{n_N} (\rho_N) |\lambda_o\rangle, \] (4.90)
where the exponents satisfy the following relations

\[ \sum_{i=1}^{N} m_i \pm 1 = \sum_{i=1}^{N} n_i = n \]  
(4.91)

\[ m_i, n_i = 0, 1. \]  
(4.92)

Reordering the operators we may write

\[ \tau^+ = \frac{1}{2\pi i} \oint d\nu.n \sum_{n=0}^{N-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq N} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_{n+1}} \]  
(4.93)

\[ \langle \lambda_0 | \Gamma_+ (\nu) \Gamma_+ (\nu_{i_1}) \ldots \Gamma_+ (\nu_{i_n}) \Gamma_+ (\rho_{j_1}) \ldots \Gamma_+ (\rho_{j_{n+1}}) | \lambda_0 \rangle, \]  
(4.94)

\[ \tau^+ = \frac{1}{2\pi i} \sum_{n=0}^{N-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq N} \epsilon (-\alpha, \alpha_{i_l}) (\nu - \nu_{i_l})^2 \]  
(4.95)

\[ \left( \prod_{0 < l < n} \epsilon (-\alpha, \alpha_{j_m}) (\nu - \rho_{j_m})^2 \right)^{-1} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_{n+1}} \]  
(4.96)

\[ \left( \prod_{0 \leq l < m \leq n} \epsilon (\alpha_{i_l}, \alpha_{i_m}) (\nu_{i_l} - \nu_{i_m})^2 \right) \cdot \left( \prod_{0 \leq l < m \leq n} \epsilon (\alpha_{j_l}, \alpha_{j_m}) (\rho_{j_l} - \rho_{j_m})^2 \right) \]  
(4.97)

\[ \left( \prod_{1 \leq l \leq n \leq n+1, l \neq n+1} \epsilon (\alpha_{i_l}, \alpha_{j_m}) (\nu_{i_l} - \rho_{j_m})^2 \right)^{-1} \]  
(4.98)

In Appendix\[\] we show that the contour integration in the variable \( \nu \) is equal to \( 2\pi i \) for any value of \( n \). Then,

\[ \tau^+ = \sum_{n=0}^{N-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq N} A_{i_1} \ldots A_{i_n} B_{j_1} \ldots B_{j_{n+1}} \left( \prod_{0 \leq l < m \leq n} \epsilon (\alpha_{i_l}, \alpha_{i_m}) (\nu_{i_l} - \nu_{i_m})^2 \right) \cdot \]

\[ \left( \prod_{1 \leq l \leq n \leq n+1} \epsilon (\alpha_{j_l}, \alpha_{j_m}) (\rho_{j_l} - \rho_{j_m})^2 \right) \cdot \left( \prod_{0 \leq l < n} \epsilon (-\alpha, \alpha_{i_l}) \right) \cdot \]

\[ \left( \prod_{1 \leq l \leq n \leq n+1} \epsilon (\alpha_{i_l}, \alpha_{j_m}) (\nu_{i_l} - \rho_{j_m})^2 \right)^{-1} \left( \prod_{0 \leq m \leq n+1} \epsilon (-\alpha, \alpha_{j_m}) \right)^{-1}. \]

Likewise we have

\[ \tau^- = \frac{1}{2\pi i} \oint d\rho.n \sum_{n=0}^{N-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq N} B_{i_1} \ldots B_{i_n} A_{j_1} \ldots A_{j_{n+1}}. \]

\[ \langle \lambda_0 | \Gamma_+ (\rho) \Gamma_+ (\rho_{i_1}) \ldots \Gamma_+ (\rho_{i_n}) \Gamma_+ (\nu_{j_1}) \ldots \Gamma_+ (\nu_{j_{n+1}}) | \lambda_0 \rangle \]

\[ = \frac{1}{2\pi i} \sum_{n=0}^{N-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq N} \oint d\rho.n \left( \prod_{0 \leq l < n} \epsilon (\alpha, \alpha_{i_l}) (\rho - \rho_{i_l})^2 \right) \]
\[
\left( \prod_{0<m \leq n+1} \epsilon (\alpha, \alpha_m) (\rho - \nu_m)^2 \right)^{-1} B_i B_{i} A_{j_1} A_{j_{n+1}} \\
\left( \prod_{1 \leq l < m \leq n} \epsilon (\alpha_{i_l}, \alpha_{i_m}) (\rho_{i_l} - \rho_{i_m})^2 \cdot \left( \prod_{1 \leq l < m \leq n+1} \epsilon (\alpha_{j_l}, \alpha_{j_m}) (\nu_{j_l} - \nu_{j_m})^2 \right) \right)^{-1},
\]

the contour integration in \( \rho \) is also equal to \( 2\pi i \) (see Appendix F) for any value of \( n \). Therefore

\[
\tau^{-} = \sum_{n=0}^{N-1} \sum_{1 \leq i_1 < i_2 < \ldots < i_n \leq N, 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq N} B_{i_1} B_{i_2} A_{j_1} A_{j_{n+1}} \left( \prod_{1 \leq l < m \leq n} \epsilon (\alpha_{i_l}, \alpha_{i_m}) (\rho_{i_l} - \rho_{i_m})^2 \right) \cdot \\
\left( \prod_{1 \leq l < m \leq n+1} \epsilon (\alpha_{j_l}, \alpha_{j_m}) (\nu_{j_l} - \nu_{j_m})^2 \right) \left( \prod_{0 < l \leq n} \epsilon (\alpha, \alpha_{i_l}) \right)^{-1} \\
\left( \prod_{0 < m \leq n+1} \epsilon (\alpha, \alpha_{j_m}) \right)^{-1}.
\]

Similar expressions were found by Kac and Wakimoto in the context of their generalized Hirota equations approach.

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A Appendix: The “untwisted” Kac-Moody algebras and their integrable highest-weight representations

We present the necessary Kac-Moody algebra notations and conventions used to construct integrable models, as well as, the theory of the so called “highest weight integrable representations” which are useful in the construction of their soliton solutions. A complete treatment can be found in [13, 20].

An “untwisted” Kac-Moody algebra \( \hat{g} \) affine to a finite Lie algebra \( g \) can be realized as an extension of the “loop algebra” of \( g \):

\[
\hat{g} = \left( g \otimes \mathbb{C}[z,z^{-1}] \right) \oplus \mathbb{C}C \oplus CD, \quad (A.1)
\]
where \( C[z, z^{-1}] \) is the algebra of Laurent Polynomials in \( z \) and \( \mathbb{C}x (x = C, D) \) is a 1 dimensional subspace. Writing an element of the loop algebra as \( a_n \equiv (a \otimes z^m) \), where \( a \in g \) and \( n \in \mathbb{Z} \), then the algebra can be written as

\[
[a_n, b_m] = [a, b]_{n+m} + \delta_{m+n,0} (a, b) mC,
\]

\[
[D, a_n] = na_n
\]

\[
[D, D] = [C, D] = [C, C] = [C, a_n] = 0
\]  

where \((a, b)\) is the Killing form of \( g \) and \([a, b]\) is the Lie bracket in \( g \). \( C \) is the central element of \( \hat{g} \), and \( D \) is a derivative operator which induces a natural integer gradation of \( \hat{g} \)

\[
\hat{g} = \bigoplus_{i \in \mathbb{Z}} \hat{g}_i,
\]

where \([D, \hat{g}_i] = i\hat{g}_i\). The operator \( D \) defines the so called “homogeneous gradation”.

Here let us point out that there is a natural Heisenberg subalgebra of \( \hat{g} \). Introducing a triangular decomposition of the finite algebra \( g = n_- \oplus h \oplus n_+ \), one can define an homogeneous Heisenberg algebra as an algebra composed of the elements \( \{h \otimes z^n, C\} \) such that

\[
[a_n, b_m] = \delta_{m+n,0} (a, b) mC.
\]

In the Cartan-Weyl basis the commutation relations are given as

\[
[H_i^m, H_j^n] = mC \delta_{ij} \delta_{m,-n},
\]

\[
[H_i^m, E_\alpha^n] = \alpha_i E_{\alpha}^{m+n},
\]

\[
[E_\alpha^m, E_\beta^n] = \begin{cases} 
\varepsilon(\alpha, \beta) E^{m+n}_{\alpha+\beta}, & \text{if } \alpha + \beta \text{ is a root} \\
\frac{2}{\alpha} \alpha \cdot H^{m+n} + Cm \delta_{m+n,0}, & \text{if } \alpha + \beta = 0 \\
0, & \text{in other case}
\end{cases}
\]

\[
[C, E_\alpha^m] = [C, H_\alpha^m] = 0
\]

\[
[D, E_\alpha^n] = nE_\alpha^n, \quad [D, H_\alpha^n] = nH_\alpha^n.
\]

In this case the Cartan subalgebra is formed by the generators \( \{H_i^{(0)}, C, D\} \) and the step operators are:

- \( E_\alpha^n \) associated to the roots \( a = (\alpha, 0, n) \), where \( \alpha \) belongs to the set of roots of \( g \) and \( n \) are integers,
- \( H_i^n \) associated to the roots \( n\delta = (0, 0, n) \) with \( n \neq 0 \).

The positive roots are \((\alpha, 0, n)\) for \( n > 0 \) or \( n = 0 \) and \( \alpha > 0 \), and among them the simple roots being \( a_i = (\alpha_i, 0, 0), \quad i = 1, \ldots, r \), and \( \alpha_o = (-\psi, 0, 1) \) with \( \psi \) the maximal root of \( g \).
The representation theory of the Kac-Moody algebra in terms of vertex operators usually make use of the Cartan-Weyl basis \([18]\), see Appendix B. Instead, to construct the so-called “integrable highest weight representations” \([13, 20]\) of the “untwisted” affine Kac-Moody algebra we will need the Chevalley basis commutation relations (the notations and presentation here follow closely the Appendix of \([21]\))

\[
\begin{align*}
[H^m_a, H^b_n] &= mC\eta_{ab}\delta_{m+n,0} \quad (A.9) \\
[H^m_a, E^n_\alpha] &= \sum_{b=1}^r m^\alpha_b K_{ba} E^{m+n}_\alpha \quad (A.10) \\
[E^m_\alpha, E^{-n}_{\alpha-}\beta] &= \frac{q}{\alpha^2} \sum_{a=1}^r \alpha_a^2 \delta_{\alpha+\beta,0} \quad (A.11) \\
[C, E^m_\alpha] &= [C, H^m_0] = 0 \quad (A.12) \\
[D, E^m_n] &= nE^m_n, \quad [D, H^m_0] = nH^m_0. \quad (A.13)
\end{align*}
\]

where \(K_{ab} = 2\alpha_a \cdot \alpha_b / \alpha_2^b\) is the Cartan matrix of the finite simple Lie algebra \(g\) associated to \(\hat{g}\) and generated by \(\{H^0_0, E^0_\alpha\}\). \(\eta_{ab} = \frac{2}{\alpha_a^2} K_{ab} = \eta_{ba}\), \(q\) is the highest positive integer such that \(\beta - q\alpha\) is a root, \(\varepsilon(\alpha, \beta)\) are \((\pm)\) signs determined by the Jacobi identities, \(l^\alpha_a\) and \(m^\alpha_a\) are the integers in the expansion \(\alpha / \alpha^2 = \sum_{r=1}^r l^\alpha_a \alpha_a / \alpha_2^a\) and \(\alpha = \sum_{r=1}^r m^\alpha_a \alpha_a\) respectively, where \(\alpha_1, ..., \alpha_r\) are the simple roots of \(g\) \((r \equiv \text{rank of } g)\). \(\hat{g}\) has a symmetric non-degenerate bilinear form which can be normalized as

\[
\begin{align*}
\text{Tr } (H^m_a H^b_n) &= \eta_{ab}\delta_{m+n,0} \\
\text{Tr } (E^m_\alpha E^{-n}_{\alpha-}\beta) &= \frac{2}{\alpha^2} \delta_{\alpha+\beta,0}\delta_{m+n,0} \\
\text{Tr } (CD) &= 1.
\end{align*}
\]

The integer gradations of \(\hat{g}\)

\(\hat{g} = \bigoplus_{n \in \mathbb{Z}} \hat{g}_n\)

have been presented in \([13]\). The gradation operator \(Q_s\) satisfying

\[ [Q_s, \hat{g}_n] = n\hat{g}_n; \quad n \in \mathbb{Z}, \quad (A.15) \]

is defined by

\[ Q_s = H_s + N_s D + \sigma C, \quad H_s = \sum_{a=1}^r s_a \lambda_a^\psi \cdot H^0, \quad H^0 = (H^0_1, ..., H^0_r) \quad (A.16) \]

where \((s_0, s_1, ..., s_r)\) is a \(n\)-tuple of non-negative co-prime integers, and \(\lambda_a^\psi \equiv 2\lambda_a / \alpha_2^a\) with \(\lambda_a\) and \(\alpha_a\) being the fundamental weights and the simple roots of \(g\) respectively. Moreover,

\[ N_s = \sum_{i=0}^r s_i m^\psi_i, \quad \psi = \sum_{a=1}^r m^\psi_a \alpha_a, \quad m^\psi_0 \equiv 1, \quad (A.17) \]
with $\psi$ being the maximal root of $g$. The value of $\sigma$ is arbitrary. Therefore

$$[Q_s, H^n_\alpha] = nN_s H^n_\alpha$$
$$[Q_s, E^n_\alpha] = \left( \sum_{a=1}^r m^\alpha_a s_a + nN_s \right) E^n_\alpha.$$  

The positive and negative “step operators” of $\hat{g}$ associated to the simple roots are:

$$e_a \equiv E^0_{\alpha_a}, \quad e_0 \equiv E^1_{-\psi}, \quad f_a \equiv E^{-\alpha_a} \quad e \quad f_0 \equiv E^{-1}_{-\psi},$$  

and its Cartan subalgebra generated by

$$h_a \equiv H^0_\alpha, \quad h_0 \equiv -\sum_{a=1}^r l^\psi_a H^0_\alpha + \frac{2}{\psi^2} C \quad e \quad D,$$

with $l^\psi_a$ given above; then they satisfy

$$[Q_s, h_i] = [Q_s, D] = 0; \quad [Q_s, e_i] = s_i e_i; \quad [Q_s, f_i] = -s_i f_i; \quad i = 0, 1, ..., r.$$  

An important class of representations of the Kac-Moody algebra are the so called “integrable highest-weight representations” [13]. They are defined in terms of a highest weight $|\lambda_s\rangle$ labelled by the gradation $s$ of $\hat{g}$. That state is annihilated by the positive grade generators

$$\hat{g}_+ \mid \lambda_s \rangle = 0,$$

and it is an eigenstate of the generators of the subalgebra $\hat{g}_0$

$$h_i \mid \lambda_s \rangle = s_i \mid \lambda_s \rangle$$

$$f_i \mid \lambda_s \rangle = 0; \quad \text{for any } i \text{ with } s_i = 0$$

$$Q_s \mid \lambda_s \rangle = \eta_s \mid \lambda_s \rangle$$

$$C \mid \lambda_s \rangle = \frac{\psi^2}{2} \left( \sum_{i=0}^r l^\psi_i s_i \right) \mid \lambda_s \rangle,$$

where $l^\psi_i$ is given by

$$\frac{\psi}{\psi^2} = \sum_{a=1}^r l^\psi_a \frac{\alpha_a}{\alpha^2_a}; \quad l^\psi_0 = 1.$$  

The eigenvalue of the central element $C$ in the representation $|\lambda_s\rangle$, is known as the level of the representation

$$c = \frac{\psi^2}{2} \left( \sum_{i=0}^r l^\psi_i s_i \right);$$  

in particular, the “highest-weight integrable representations” with $c = 1$ are known as “basic representations”.

The states of highest weight $|\lambda_s\rangle$ can be realized as
$|\lambda_s\rangle \equiv \bigotimes_{i=0}^{r} |\hat{\lambda}_i\rangle \otimes s_i$, \hspace{1cm} (A.28)

where $|\hat{\lambda}_i\rangle$ are the highest states of the fundamental representations of $\hat{g}$, and $\hat{\lambda}_i$ are the relevant fundamental weights of $\hat{g}$. They are given by \[18\]

\[
\hat{\lambda}_0 = \left(0, \frac{\psi^2}{2}, 0\right) \tag{A.29}
\]

\[
\hat{\lambda}_a = \left(\lambda_a, \frac{l_a^0 \psi^2}{2}, 0\right), \tag{A.30}
\]

where $\lambda_a$, $a = 1, 2, \ldots, r$ are the fundamental weights of the finite Lie algebra $g$ associated to $\hat{g}$, $l_a^0$ is defined in \[26\], and the corresponding components are the eigenvalues of $H_a^0$, $C$ and $D$ respectively, viz.

\[
H_a^0 |\hat{\lambda}_0\rangle = 0; \hspace{1cm} C |\hat{\lambda}_0\rangle = \frac{\psi^2}{2} |\hat{\lambda}_0\rangle \tag{A.31}
\]

\[
H_a^0 |\hat{\lambda}_a\rangle = \delta_{a,b} |\hat{\lambda}_a\rangle; \hspace{1cm} C |\hat{\lambda}_a\rangle = \frac{\psi^2}{2} l_a^0 |\hat{\lambda}_0\rangle \tag{A.32}
\]

and

\[
D |\hat{\lambda}_i\rangle = 0. \tag{A.33}
\]

Let us notice that in each of the $r + 1$ fundamental representations of $\hat{g}$, the (unique) highest weight state satisfies

\[
h_j |\hat{\lambda}_i\rangle = \delta_{ij} |\hat{\lambda}_i\rangle \tag{A.34}
\]

\[
e_j |\hat{\lambda}_i\rangle = 0, \hspace{1cm} \forall j \tag{A.35}
\]

\[
f_j |\hat{\lambda}_i\rangle = 0, \hspace{1cm} \text{for } j \neq i \tag{A.36}
\]

\[
f_j^2 |\hat{\lambda}_i\rangle = 0. \tag{A.37}
\]

Then the generators $e_i$ and $f_i$ are nilpotent when acting on $|\lambda_s\rangle$, and these are indeed, integrable representations.

**B** The construction of the homogeneous vertex operators

The construction of vertex operator representations of Kac-Moody algebras can be found in \[13, 22\]. The construction of the homogeneous vertex operators is based on the homogeneous Heisenberg subalgebra $h$. Here we follow the construction developed in \[13\].

Consider

\[
g = h \bigoplus \left( \bigoplus_{\alpha \in \Delta} CE_{\alpha} \right), \tag{B.1}
\]
$g$ being a simple finite Lie algebra of type $A_l, D_l$ or $E_l$, whose commutation relations are given by

$$[h, h'] = 0, \quad \text{if } h, h' \in h$$  \hfill (B.2)

$$[h, E_\alpha] = (h | \alpha) E_\alpha, \quad \text{if } h \in h, \ \alpha \in \Delta$$  \hfill (B.3)

$$[E_\alpha, E_{-\alpha}] = -\alpha, \quad \text{if } \alpha \in \Delta$$

$$[E_\alpha, E_\beta] = 0, \quad \text{if } \alpha, \beta \in \Delta, \quad \alpha + \beta \notin \Delta \cup \{0\}$$

$$[E_\alpha, E_{\beta'}] = \varepsilon(\alpha, \beta) E_{\alpha + \beta'}, \quad \text{if } \alpha, \beta, \alpha + \beta \in \Delta,$$

where $\Delta = \{\alpha \in Q/(\alpha | \alpha) = 2\}$ and $Q$ is the root lattice; $(\quad | \quad)$ is the invariant symmetric form of $g$, normalized as follows:

$$(h | E_\alpha) = 0, \quad \text{if } h \in h, \ \alpha \in \Delta; \quad (E_\alpha, E_\beta) = -\delta_{\alpha, -\beta}, \quad \text{if } \alpha, \beta \in \Delta.$$

Let

$$\hat{g} = C [t, t^{-1}] \otimes_C g + CC + CD,$$  \hfill (B.4)

be an affine algebra of type $A_n^{(1)}, D_n^{(1)}$ or $E_n^{(1)}$, respectively.

Consider the complex commutative associative algebra

$$V = S \left( \bigoplus_{j < 0} (t^j \otimes h) \right) \otimes_C C [Q],$$  \hfill (B.5)

where $S$ stands for the symmetric algebra and $C [Q]$ stands for the group algebra of the root lattice $Q \subset h$ of $g$. Let $\alpha \rightarrow e^\alpha$ denote the inclusion $Q \subset C [Q]$ (a base of the vector space $C [Q]$ is given by the elements $e^\alpha, \alpha \in Q$, and, it is defined the twisted product of the group algebra elements, $e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha + \beta}$ )

$u^{(n)}$ will stand for $t^n \otimes u (n \in \mathbb{Z}, u \in g)$. For $n > 0, u \in h$, denote by $u(-n)$ the operator on $V$ of multiplication by $u^{(-n)}$. For $n \geq 0$, $u \in h$, denote by $u(n)$ the derivation of the algebra $V$ defined by the formula

$$u(n) \left( v^{(-m)} \otimes e^\alpha \right) = n\delta_{n,-m} \left( u | v \right) \otimes e^\alpha + \delta_{n,0} \left( \alpha | u \right) v^{(-m)} \otimes e^\alpha$$  \hfill (B.6)

Choosing dual bases $u_i$ and $u^i$ of $h$, define the operator $D_o$ on $V$ by the formula

$$D_o = \sum_{i=1}^l \left( \frac{1}{2} u_i(0) u^i(0) + \sum_{n \geq 1} u_i(-n) u^i(n) \right).$$  \hfill (B.7)

Furthermore, for $\alpha \in Q$, define the sign operator $c_\alpha$:

$$c_\alpha \left( f \otimes e^\beta \right) = \varepsilon(\alpha, \beta) f \otimes e^\beta.$$  \hfill (B.8)

Finaly, for $\alpha \in \Delta \subset Q$ introduce the vertex operator

$$\Gamma_\alpha(z) = \exp \left( \sum_{j \geq 1} \frac{z^j}{j} \alpha(-j) \right) \exp \left( - \sum_{j \geq 1} \frac{z^{-j}}{j} \alpha(j) \right) e^\alpha z^{\alpha(0)} c_\alpha,$$  \hfill (B.9)
here $z$ is viewed as an indeterminate. Expanding in powers of $z$:

$$\Gamma_\alpha(z) = \sum_{j \in \mathbb{Z}} \Gamma_\alpha^{(j)} z^{-j-1}, \quad (B.10)$$

we obtain a sequence of operators $\Gamma_\alpha^{(j)}$ on $V$. Now we can state the result.

**Theorem:** The map $\sigma : \hat{g} \rightarrow \text{End}(V)$ defined by

$$
\begin{align*}
C & \rightarrow 1, \\
u^{(n)} & \rightarrow u^{(n)}, \quad \text{for } u \in \mathfrak{h}, \quad n \in \mathbb{Z}, \\
E_\alpha^{(n)} & \rightarrow \Gamma_\alpha^{(n)}, \quad \text{for } \alpha \in \Delta, \quad n \in \mathbb{Z}, \\
D & \rightarrow -D_o,
\end{align*}
$$

defines the basic representation of the affine algebra $\hat{g}$ on $V$. $\text{End}(V)$ denotes the space of the linear maps of an vector space $V$ on itself.

The proof of this theorem is presented in [13].

### C Homogeneous vertex operator calculus

Defining

$$
\Gamma_\alpha^\pm(z) = \exp \sum_{j \geq 1} \frac{\alpha(\pm j)}{\mp j} z^{\pm j}, \quad \Gamma_\alpha^0(z) = e^{\alpha z^{(0)}} c_\alpha, \quad (C.1)
$$

the following relations can be obtained:

$$
\Gamma_\alpha^-(z_1)\Gamma_\beta^+(z_2) = \Gamma_\beta^- (z_2)\Gamma_\alpha^-(z_1) \left( 1 - \frac{z_2}{z_1} \right)^{\alpha|\beta}
$$

and

$$
\Gamma_\alpha^0(z_1)\Gamma_\beta^0(z_2) = e^{\alpha + \beta} z_1^{\alpha(0)} z_2^{\beta(0)} c_\alpha c_\beta \varepsilon(\alpha, \beta), \quad (C.3)
$$

where $\left( 1 - \frac{z_2}{z_1} \right)^m$, $m \in \mathbb{Z}$, with $\left| \frac{z_2}{z_1} \right| \leq 1$.

From (C.2) and (C.3) it follows

$$
\begin{align*}
\Gamma_\alpha(z_1)\Gamma_\beta(z_2) &= \left( 1 - \frac{z_2}{z_1} \right)^{\alpha|\beta} z_1^{\alpha(\beta)} \varepsilon(\alpha, \beta) \exp \left[ \sum_{j \geq 1} \frac{1}{j} \left( z_1^{j} \alpha(-j) + z_2^{j} \beta(-j) \right) \right] \\
&\exp \left[ -\sum_{j \geq 1} \frac{1}{j} \left( z_1^{-j} \alpha(j) + z_2^{-j} \beta(j) \right) \right] e^{\alpha + \beta} z_1^{\alpha(0)} z_2^{\beta(0)} c_\alpha c_\beta. \quad (C.4)
\end{align*}
$$

If $z \equiv z_1 = z_2$ and $\alpha = \beta$, from (C.4) we can obtain

$$
\Gamma_\alpha(z)\Gamma_\alpha(z) = 0,
$$

or

$$
[\Gamma_\alpha(z)]^n = 0, \quad \text{for } n \geq 2. \quad (C.5)
$$
A useful formula

\[ \Gamma_{\alpha_N}(z_N)\Gamma_{\alpha_{N-1}}(z_{N-1})...\Gamma_{\alpha_1}(z_1) (1 \otimes 1) = \prod_{1 \leq i < j \leq N} \varepsilon(\alpha_i, \alpha_j)(z_i - z_j)^{(\alpha_i|\alpha_j)}. \]

\[ \left( \prod_{i=1}^{N} \exp \sum_{j \in \mathbb{N}} \frac{z_j^i}{j} \alpha_i(-j) \right) \otimes \exp \left( \sum_{i=1}^{N} \alpha_i \right) \] (C.6)

**Proof.** We will prove by induction for \( N = 1 \)

\[ \Gamma_{\alpha}(z_1) (1 \otimes 1) = \exp \sum_{j \geq 1} \frac{z_j^1}{j} \alpha(-j) \exp - \sum_{j \geq 1} \frac{z_j^1}{j} \alpha(j)e^{\alpha}z_1^{\alpha(0)}c_\alpha ; \] (C.7)

we have the following relations:

\[ c_\alpha(1 \otimes 1) = \varepsilon(\alpha, 0)1 \otimes 1, \quad \text{consider the convention: } \varepsilon(\alpha, 0) \equiv 1 \]
\[ = 1 \otimes 1 \] (C.8)

\[ z_1^{\alpha(0)}(1 \otimes 1) = 1 \otimes \left(e^{\ln z_1^{\alpha(0)}}\right) 1 \]
\[ = 1 \otimes 1. \] (C.9)

where we used

\[ e^{\alpha} (1 \otimes 1) \equiv e^{\alpha} (1 \otimes e^{\alpha}) = 1 \otimes e^{\alpha} \] (C.10)

From (B.6) we have

\[ u(n) \left( v^{(0)} \otimes e^{\alpha} \right) \equiv u(n) (1 \otimes e^{\alpha}) = n \delta_{n,0} (u | v) \otimes e^{\alpha} + \delta_{n,0} (\alpha | u) v^{(0)} \otimes e^{\alpha} \]

then for \((n > 0)\)

\[ u(n) (1 \otimes 1) = 0. \] (C.11)

Therefore

\[ \exp - \sum_{j \geq 1} \frac{z_j^1}{j} \alpha(j)(1 \otimes e^{\alpha}) = \left[ \left( 1 - \sum_{j \geq 1} \frac{z_j^1}{j} \alpha(j) + ... \right) 1 \right] \otimes e^{\alpha} \]
\[ = 1 \otimes e^{\alpha}. \] (C.12)

Besides we can write

\[ \exp \sum_{j \geq 1} \frac{z_j^1}{j} \alpha(-j)(1 \otimes e^{\alpha}) = \sum_{j \geq 1} \frac{z_j^1}{j} \alpha(-j) \otimes e^{\alpha}, \]

thus

\[ \Gamma_{\alpha_1}(z_1) (1 \otimes 1) = \left( \exp \sum_{j \geq 1} \frac{z_j^1}{j} \alpha_1(-j) \right) \otimes e^{\alpha_1}, \] (C.13)
which is equal to the equation (C.6) for \( N = 1 \).

Now, let us assume that (C.6) is true for a given \( N \), and we will write (C.6) as
\[
\Gamma_{\alpha N}(z_N) \Gamma_{\alpha N-1}(z_{N-1}) \ldots \Gamma_{\alpha 1}(z_1) (1 \otimes 1) = \prod_{1 \leq i < j \leq N} \varepsilon(\alpha_i, \alpha_j) (z_i - z_j)^{\langle \alpha | \alpha \rangle}.
\]

\[
\left( \prod_{i=1}^{N} \exp \sum_{j \in \mathbb{N}} \frac{z_j}{j} \alpha_i \right) \exp \left( \sum_{i=1}^{N} \alpha_i \right) (1 \otimes 1). 
\tag{C.14}
\]

Multiplying (C.14) by \( \Gamma_{\alpha N+1}(z_{N+1}) \) and using the relations (C.2), (C.3) and (C.7)-(C.12) many times, we can write
\[
\Gamma_{\alpha N+1}(z_{N+1}) \Gamma_{\alpha N}(z_N) \ldots \Gamma_{\alpha 1}(z_1) (1 \otimes 1) = \prod_{1 \leq i < j \leq N} \varepsilon(\alpha_i, \alpha_j) (z_i - z_j)^{\langle \alpha | \alpha \rangle}.
\]

\[
\prod_{1 \leq k < N+1} \varepsilon(\alpha_k, \alpha_{N+1}) (z_k - z_{N+1})^{\langle \alpha_k | \alpha_{N+1} \rangle} \cdot \left( \prod_{j \in \mathbb{N}} \frac{z_j}{j} \alpha_{N+1} \right)^{\langle \alpha_{N+1} | \alpha \rangle} e^{\alpha_{N+1}}.
\]

\[
\left( \prod_{i=1}^{N} \exp \sum_{j \in \mathbb{N}} \frac{z_j}{j} \alpha_i \right) \exp \left( \sum_{i=1}^{N} \alpha_i \right) (1 \otimes 1)
\]

\[
= \prod_{1 \leq i < j \leq N+1} \varepsilon(\alpha_i, \alpha_j) (z_i - z_j)^{\langle \alpha_i | \alpha_j \rangle} \left( \prod_{i=1}^{N+1} \exp \sum_{j \in \mathbb{N}} \frac{z_j}{j} \alpha_i \right)^{\langle \alpha | \alpha \rangle}.
\]

\[
\exp \left( \sum_{i=1}^{N+1} \alpha_i \right) (1 \otimes 1)
\]

\[
= \prod_{1 \leq i < j \leq N+1} \varepsilon(\alpha_i, \alpha_j) (z_i - z_j)^{\langle \alpha_i | \alpha_j \rangle} \left( \prod_{i=1}^{N+1} \exp \sum_{j \in \mathbb{N}} \frac{z_j}{j} \alpha_i \right)^{\langle \alpha | \alpha \rangle} \otimes \exp \left( \sum_{i=1}^{N+1} \alpha_i \right),
\]

which is exactly the expression (C.6) written for \( N + 1 \). This way the equation (C.6) is proved.

Moreover, making the correspondence
\[
|\lambda_o\rangle \leftrightarrow 1 \otimes 1,
\]
where \( |\lambda_o\rangle \) denotes a highest weight state of a fundamental representation, we can write
\[
\langle \lambda_o | \Gamma_{\alpha N}(z_N) \Gamma_{\alpha N-1}(z_{N-1}) \ldots \Gamma_{\alpha 1}(z_1) | \lambda_o \rangle =
\]

\[
\begin{cases}
0, & \text{if } \sum_{i=1}^{N} \alpha_i \neq 0 \\
\prod_{1 \leq i < j \leq N} \varepsilon(\alpha_i, \alpha_j) (z_i - z_j)^{\langle \alpha_i | \alpha_j \rangle} & \text{if } \sum_{i=1}^{N} \alpha_i = 0
\end{cases}
\tag{C.15}
\]
where we have used the fact that

\[
\langle \lambda_0 | \exp \sum_{j \in \mathbb{N}} \frac{z_j^j}{j} \alpha_i(-j) = \langle \lambda_0 | \left( 1 + \sum_{j \in \mathbb{N}} \frac{z_j^j}{j} \alpha_i(-j) + \ldots \right) = \langle \lambda_0 |
\]

and

\[
\langle \lambda_0 | \exp \left( \sum_{i=1}^{N} \alpha_i \right) = \begin{cases} 
0, & \text{if } \sum_{i=1}^{N} \alpha_i \neq 0 \\
\langle \lambda_0 |, & \text{if } \sum_{i=1}^{N} \alpha_i = 0 
\end{cases}
\]

## D Matrix elements using a vertex operator representation of the algebra \( \widehat{sl}_{r+1} \)

Consider the correspondence

\[
F_i \rightarrow \nu_i \Gamma_{-\alpha_i} (\nu_i),
\]

\[
G_i \rightarrow \rho_i \Gamma_{\alpha_i} (\rho_i).
\]

then we can write

\[
C_{i_1 \ldots i_n, j_1 \ldots j_n} = \langle \lambda_0 | F_{i_1} \ldots F_{i_n} G_{j_1} \ldots G_{j_n} | \lambda_0 \rangle = \nu_{i_1} \ldots \nu_{i_n} \rho_{j_1} \ldots \rho_{j_n} \langle \lambda_0 | \Gamma_{-\alpha_{i_1}} (\nu_{i_1}) \ldots \Gamma_{-\alpha_{i_n}} (\nu_{i_n}) \Gamma_{\beta_{j_1}} (\rho_{j_1}) \ldots \Gamma_{\beta_{j_n}} (\rho_{j_n}) | \lambda_0 \rangle \]

\[
= \{ \delta \rightarrow_{0, \beta_{j_1}, \ldots, \beta_{j_n}} \nu_{i_1} \ldots \nu_{i_n} \rho_{j_1} \ldots \rho_{j_n} \cdot \prod_{1 \leq a < b \leq n} \epsilon (\beta_{j_a}, \beta_{j_b}) \epsilon (-\beta_{i_a}, -\beta_{i_b}) (\rho_{j_a} - \rho_{j_b}) (\beta_{j_a} | \beta_{j_b}) \}. \]

\[
(\nu_{i_a} - \nu_{i_b}) (\beta_{i_a} | \beta_{i_b}) \} / \{ \prod_{1 \leq a < b \leq n} \epsilon (-\beta_{i_a}, \beta_{i_b}) (\nu_{i_a} - \rho_{i_b}) (\beta_{i_a} | \beta_{i_b}) \}, \quad (D.1)
\]

where we have used equation (C.15).

\[
C_{i_1 \ldots i_n, j_1 \ldots j_{n+1}}^{+} = \langle \lambda_0 | E_{-\alpha_{i_1}}^{(1)} F_{i_1} \ldots F_{i_n} G_{j_1} \ldots G_{j_{n+1}} | \lambda_0 \rangle = \frac{1}{2 \pi i} \oint d\nu. \nu \nu_{i_1} \ldots \nu_{i_n} \rho_{j_1} \ldots \rho_{j_{n+1}} \cdot
\]

\[
\langle \lambda_0 | \Gamma_{-\alpha_i} (\nu) \Gamma_{-\alpha_{i_1}} (\nu_{i_1}) \ldots \Gamma_{-\alpha_{i_n}} (\nu_{i_n}) \Gamma_{\beta_{j_1}} (\rho_{j_1}) \ldots \Gamma_{\beta_{j_{n+1}}} (\rho_{j_{n+1}}) | \lambda_0 \rangle
\]
In the last relation we have written the following contour integration:

\[
\left( \rho_{jka} - \rho_{jkb} \right)^{-\left( \beta_{jka} | \beta_{jkb} \right)} \} / \{ \prod_{0 \leq a < b \leq n, a \neq n+1} e^{-\left( \beta_{ia} , \beta_{ib} \right) \left( \nu_{ia} - \nu_{ib} \right) \left( \beta_{ia} | \beta_{ib} \right)} \}.
\]

(D.2)

In the last relation we have written the following contour integration:

\[
I_1 = \frac{1}{2\pi i} \oint d\nu \nu_{i_1} \cdots \nu_{i_n} \rho_{j_{k_1}} \cdots \rho_{j_{k_n+1}} \delta \rho_{\beta_{j_{k_1}} + \cdots + \beta_{j_{k_{n+1}}} - \beta_{i_1} - \cdots - \beta_{i_n}} \cdot
\]

\[
\prod_{0 < a \leq n} e^{-\left( \beta_{i_a} , -\beta_{i_a} \right) \left( \nu - \nu_{i_a} \right) \left( \beta_{i_a} | \beta_{i_a} \right)} \prod_{0 < b \leq n+1} e^{-\left( \beta_{j_{k_b}}, \beta_{j_{k_b}} \right) \left( \nu - \rho_{j_{k_b}} \right) \left( \beta_{j_{k_b}} | \beta_{j_{k_b}} \right)}.
\]

\[
I_1 = \frac{1}{2\pi i} \oint d\nu \nu_{i_1} \cdots \nu_{i_n} \rho_{j_{k_1}} \cdots \rho_{j_{k_n}} \cdot
\]

Likewise we have

\[
C_{i_{n+1} \cdots i_k}^{\ensuremath{-1}} \cdots \Delta_{j_{k_1}} G_{j_{k_n}} = \langle \lambda_{\rho} | G_{j_{k_1}} \cdots G_{j_{k_n}} | \lambda_{\omega} \rangle = \frac{1}{2\pi i} \oint dp \rho_{\nu_{i_1}} \cdots \rho_{\nu_{i_n+1}} \rho_{\rho_{j_{k_1}}} \cdots \rho_{\rho_{j_{k_n}}} \cdot
\]

\[
\langle \lambda_{\rho} | \Gamma_{\beta_{j_{k_1}}} (\rho) \Gamma_{\beta_{j_{k_1}}} (\nu_{i_1}) \cdots \Gamma_{\beta_{j_{k_{n+1}}}} (\nu_{i_{n+1}}) \Gamma_{\beta_{j_{k_1}}} (\rho_{j_{k_1}}) \cdots \Gamma_{\beta_{j_{k_n}}} (\rho_{j_{k_n}}) | \lambda_{\omega} \rangle
\]

\[
= \frac{1}{2\pi i} \oint dp \rho_{\nu_{i_1}} \cdots \rho_{\nu_{i_n+1}} \rho_{j_{k_1}} \cdots \rho_{j_{k_n}} \delta \rho_{\beta_{j_{k_1}} + \cdots + \beta_{j_{k_{n+1}}} - \beta_{i_1} - \cdots - \beta_{i_{n+1}}} \cdot
\]

\[
\prod_{0 < a \leq n} e^{-\left( \beta_{i_a} , \beta_{j_{k_a}} \right) \left( \rho - \rho_{j_{k_a}} \right) \left( \beta_{i_a} | \beta_{j_{k_a}} \right)} \prod_{0 < b \leq n+1} e^{-\left( \beta_{i_b} , \beta_{j_{k_b}} \right) \left( \rho - \nu_{i_b} \right) \left( \beta_{i_b} | \beta_{j_{k_b}} \right)}.
\]

\[
\prod_{0 \leq a < b \leq n} e^{-\left( \beta_{j_{k_a}} , \beta_{j_{k_b}} \right) \left( \rho_{j_{k_a}} - \rho_{j_{k_b}} \right) \left( \beta_{j_{k_a}} | \beta_{j_{k_b}} \right)} \prod_{0 \leq a < b \leq n+1} e^{-\left( \beta_{i_a} , -\beta_{i_b} \right)}.
\]

\[
\left( \nu_{i_a} - \nu_{i_b} \right)^{-\left( \beta_{i_a} | \beta_{i_b} \right)} \} / \{ \prod_{0 \leq a < b \leq n, b \neq n+1} e^{-\left( \beta_{i_a} , \beta_{j_{k_b}} \right) \left( \nu_{i_a} - \rho_{j_{k_b}} \right) \left( \beta_{i_a} | \beta_{j_{k_b}} \right)} \}.
\]

(D.4)

This time the relevant contour integration becomes

\[
I_2 = \frac{1}{2\pi i} \oint dp \rho \prod_{0 < a \leq n} e^{-\left( \beta_{i_a} - \beta_{j_{k_a}} \right) \left( \rho - \rho_{j_{k_a}} \right) \left( \beta_{i_a} | \beta_{j_{k_a}} \right)} \prod_{0 < b \leq n+1} e^{-\left( \beta_{i_b} \right) \left( \rho - \nu_{i_b} \right) \left( \beta_{i_b} | \beta_{j_{k_b}} \right)}.
\]

(D.5)

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E Matrix elements using the homogeneous vertex operator representation of the Kac-Moody algebra $\hat{sl}_2$

In the case of the affine algebra $\hat{sl}_2$, we have the generators

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and let us choose the following dual bases of $\hat{sl}_2$

$$\{t^n \alpha, t^n e, t^n f, C, D\}, \quad \text{and} \quad \left\{ \frac{1}{2} t^{-n} \alpha, t^{-n} f, t^{-n} e, D, C \right\}.$$

We have

$$Q = Z \alpha, \quad (\alpha | \alpha) = 2, \quad \varepsilon(\alpha, \alpha) = \varepsilon(-\alpha, -\alpha) = -1, \quad \varepsilon(\alpha, \alpha) = \varepsilon(-\alpha, -\alpha) = 1$$

(we are using the “gauge fixing” of [18, 24] for $\varepsilon(\alpha, \beta)$).

Considering $q = e^\alpha$, we identify $C[Q]$ with $C[q, q^{-1}]$. Thus the homogeneous vertex operator construction can be described as follows

$$L(\lambda_0) = C[x_1, x_2, ...; q, q^{-1}];$$

$$\alpha^{(n)} \mapsto 2 \frac{\partial}{\partial x_n} \quad \text{and} \quad \alpha^{(-n)} \mapsto nx_n \quad \text{for} \quad n > 0, \quad \alpha^{(0)} \mapsto 2q \frac{\partial}{\partial q};$$

$$C \mapsto 1, \quad D \mapsto -q \left( \frac{\partial}{\partial q} \right)^2 - \sum_{n \geq 1} nx_n \frac{\partial}{\partial x_n};$$

$$E(z) := \sum_{n \in \mathbb{Z}} E^{(n)}_\pm z^{-n-1} \mapsto \Gamma_\pm(z)$$

where

$$\Gamma_\pm(z) = \exp \left( \pm \sum_{j \geq 1} z^j x_j \right) \exp \left( \mp 2 \sum_{j \geq 1} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} \right) q^{\pm 1} z^{\pm 2q \frac{\partial}{\partial q}} c_\pm \alpha,$$  \hspace{1cm} (E.1)

(note that $z^{\pm 2q \frac{\partial}{\partial q}} (q^n) = z^{\pm 2n q^n}$).

Then, we can make use of the Eqn. (C.15) to compute the matrix elements, viz.,

$$\langle \lambda_0 | \Gamma_{\alpha_N}(z_N) \Gamma_{\alpha_{N-1}}(z_{N-1}) ... \Gamma_{\alpha_1}(z_1) | \lambda_0 \rangle =$$

$$\begin{cases} 0, & \text{if } \sum_{i=1}^N \alpha_i \neq 0 \\ \Pi_{1 \leq i < j \leq N} \varepsilon(\alpha_i, \alpha_j) (z_i - z_j)^{(\alpha_i | \alpha_j)} & \text{if } \sum_{i=1}^N \alpha_i = 0 \end{cases} \quad (E.2)$$

(E.3)

In this way we have two types of operators, $\Gamma_\pm(z)$, associated to $\alpha$ and $-\alpha$ respectively. We should have an even number of $\Gamma$’s in Eqn. (C.15) in order to have a non zero value.
for \( \langle \lambda_0 | \Gamma_{\alpha N}(z_N) \Gamma_{\alpha N-1}(z_{N-1}) \ldots \Gamma_{\alpha 1}(z_1) | \lambda_0 \rangle \); thus we may choose 2N operators, such that N operators correspond to \( \alpha \) and the remaining \( N \) of them correspond to \( -\alpha \).

We provide some of the matrix components we used in the construction of one-soliton and two-soliton solutions of the system NLS. Defining

\[
G_i = \sum_{n=-\infty}^{+\infty} \nu_i^{-n} E^{(n)}_+, \quad F_i = \sum_{n=-\infty}^{+\infty} \nu_i^{-n} E^{(n)}_-. \tag{E.4}
\]

we can make the correspondence

\[
G_i \rightarrow \rho_i \Gamma_+ (\rho_i), \quad F_i \rightarrow \nu_i \Gamma_- (\nu_i). \tag{E.5}
\]

Then

\[
\langle \lambda_0 | G_i F_j | \lambda_0 \rangle = \langle \lambda_0 | F_j G_i | \lambda_0 \rangle = \rho_i \nu_j \langle \lambda_0 | \Gamma_+ (\rho_i) \Gamma_- (\nu_j) | \lambda_0 \rangle = \frac{\rho_i \nu_j \varepsilon(\nu_j, \rho_i)}{(\rho_i - \nu_j)^2}.
\]

As a special case we compute the following expression

\[
\langle \lambda_0 | E_1^1 G_1 | \lambda_0 \rangle = \frac{1}{2\pi i} \oint d\rho_i (z - \rho_i)^2 d\rho_i = \rho_i,
\]

where we have used

\[
E_1^1 = \oint d\rho_i \Gamma_-(\rho_i),
\]

where integration is over some curve encircling the origin.

The same method can be used to compute the following matrix element

\[
\langle \lambda_0 | E_1^1 G_1 F_2 G_2 | \lambda_0 \rangle = \frac{1}{2\pi i} \oint d\rho_1 \int d\rho_2 \rho_1 \nu_2 \rho_2 \langle \lambda_0 | \Gamma_- (z) \Gamma_+ (\rho_1) \Gamma_- (\nu_2) \Gamma_+ (\rho_2) | \lambda_0 \rangle = \frac{1}{2\pi i} \oint d\rho_1 \int d\rho_2 \rho_1 \nu_2 \rho_2 \varepsilon(\rho_1, \rho_2) \varepsilon(\nu_2, \nu_1) \varepsilon(\rho_1, \nu_2) \varepsilon(\nu_1, \rho_2) \varepsilon(\nu_1, \nu_2) (\rho_2 - \rho_1)^2 (\nu_1 - \nu_2)^2 (\rho_1 - \rho_2)^2 (\nu_2 - \nu_1)^2 (\rho_1 - \nu_1)^2 (\nu_2 - \nu_2)^2 (\rho_2 - \nu_2)^2 (\nu_1 - \nu_1)^2
\]

\[
= \rho_1 \nu_2 \rho_2 (\rho_2 - \rho_1)^2 (\nu_2 - \nu_1)^2 (\rho_2 - \nu_2)^2 (\nu_2 - \rho_2)^2 (\rho_1 - \rho_1)^2 (\nu_1 - \nu_1)^2 (\rho_1 - \nu_1)^2 (\nu_2 - \nu_2)^2 (\rho_2 - \nu_2)^2 (\nu_1 - \nu_1)^2.
\]

The remaining matrix elements can be computed in the same way. We give some of them

\[
\langle \lambda_0 | E_1^1 G_1 | \lambda_0 \rangle = \frac{1}{2\pi i} \oint d\rho_i (z - \rho_i)^2 d\rho_i = \rho_i,
\]

...
\[ \langle \lambda_o | F_i G_i F_j G_j | \lambda_o \rangle = \frac{\rho_i \nu_j \rho_j (\rho_j - \rho_i)^2 (\nu_j - \nu_i)^2}{(\rho_j - \nu_j)^2 (\rho_j - \nu_i)^2 (\rho_i - \nu_j)^2 (\rho_i - \nu_i)^2}, \quad (E.6) \]

\[ \langle \lambda_o | E^1_- G_i F_j G_j | \lambda_o \rangle = \frac{\rho_i \nu_j \rho_j (\nu_j - \rho_i)^2}{(\rho_j - \nu_j)^2 (\nu_j - \rho_i)^2}, \quad (E.7) \]

\[ \langle \lambda_o | E^1_+ F_i | \lambda_o \rangle = \nu_i, \quad (E.8) \]

\[ \langle \lambda_o | E^1_+ F_i G_i F_j | \lambda_o \rangle = \frac{\nu_i \nu_j (\nu_j - \nu_i)^2}{(\rho_i - \nu_j)^2 (\nu_i - \rho_i)^2}, \quad (E.9) \]

These results suggest that a general matrix element could be expressed in terms of Vandermonde-like determinants. In fact, recently in [25] there was derived a natural relationship with the Vandermonde-like determinants. The resulting framework in our case may be well-suited to achieve a compactness and transparency in \( N \)-soliton formulas.

F The contour integration of the matrix elements in the case of \( \widehat{sl}_2 \)

Let us show that the integral

\[ I_n = \frac{1}{2\pi i} \oint dz \frac{(z - \nu_1)^2 \cdots (z - \nu_{n-1})^2}{(z - \rho_1)^2 \cdots (z - \rho_n)^2}, \quad n \geq 1 \quad (F.1) \]

is equal to unity. We will prove by induction.

For \( n = 1 \) :

\[ I_1 = \frac{1}{2\pi i} \oint dz \frac{z}{(z - \rho_1)^2} = 1. \]

Assuming that \( I_n = 1 \) for some \( n > 1 \) let us determine the form of \( I_{n+1} \).

The expression for \( I_{n+1} \) can be written as

\[ I_{n+1} = I_n + C_n, \quad (F.2) \]

where

\[ C_n = \frac{1}{2\pi i} \frac{\partial}{\partial \rho_1} \cdots \frac{\partial}{\partial \rho_{n+1}} \oint dz \frac{(z - \nu_1)^2 \cdots (z - \nu_{n-1})^2}{(z - \rho_1) \cdots (z - \rho_{n+1})} (\rho_{n+1} - \nu_n)^2 \]

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\[
\frac{1}{2\pi i} \left( \frac{\partial}{\partial \rho_1} \cdots \frac{\partial}{\partial \rho_{n+1}} \right)_z \int dz \, (\rho_2 - \nu_1)^2 (\rho_3 - \nu_2)^2 \cdots (\rho_{n+1} - \nu_n)^2.
\]

\[
\frac{(z - \rho_1) \cdots (z - \rho_{n+1})}{\partial \rho_1} + \frac{\partial}{\partial \rho_1} \frac{\partial}{\partial \rho_2} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 \cdots (\rho_{n+1} - \nu_n)^2.
\]

\[
\frac{(z - \rho_1)(z - \rho_3) \cdots (z - \rho_{n+1})}{\partial \rho_1 \partial \rho_2} \frac{\partial}{\partial \rho_4} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_2 - \nu_1)^2 (\rho_4 - \nu_3)^2 \cdots (\rho_{n+1} - \nu_n)^2.
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_2 \partial \rho_4} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_2 - \nu_1)^2 (\rho_3 - \nu_2)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]

\[
\dotfill
\]

\[
\frac{(z - \rho_1)(z - \rho_3)(z - \rho_{n-1})(z - \rho_{n+1})}{\partial \rho_1 \partial \rho_3 \partial \rho_5} \cdots \frac{\partial}{\partial \rho_{n+1}} \int dz \, (\rho_3 - \nu_2)^2 (\rho_5 - \nu_4)^2 (\rho_{n+1} - \nu_n)^2.
\]
\[
\frac{\partial}{\partial \rho_1} \ldots \frac{\partial}{\partial \rho_{n-3}} \frac{\partial}{\partial \rho_{n+1}} \int dz (\rho_2 - \nu_1)^2 \cdots (\rho_{n-3} - \nu_{n-4})^2 (\rho_{n+1} - \nu_n)^2. \\
\frac{\partial}{\partial \rho_1} \int dz (\rho_{n+1} - \nu_n)^2 \frac{z}{(z - \rho_1) (z - \rho_{n+1})}. 
\]

\[
\text{(F.3)}
\]

It is easy to show the following:

i) \[
\frac{1}{2\pi i} \int dz \frac{z}{(z - \rho_1) (z - \rho_2)} = 1, 
\]
and ii) \[
\int dz \frac{z}{(z - \rho_1) \cdots (z - \rho_n)} = 0 \quad \text{for} \quad n > 2. 
\]

writing \[
\frac{1}{(z - \rho_1) \cdots (z - \rho_n)} = \frac{1}{\det \Delta} \left[ \frac{A_{n,1}}{z - \rho_1} + \frac{A_{n,2}}{z - \rho_2} + \cdots + \frac{A_{n,n}}{z - \rho_n} \right], 
\]
with \[
\Delta = 
\begin{pmatrix}
- \sum_{i=1, i \neq 1}^{n} \rho_i & - \sum_{i=1, i \neq 2}^{n} \rho_i & \cdots & - \sum_{i=1, i \neq n}^{n} \rho_i \\
- \sum_{1 \leq i < j \leq n, i,j \neq 1}^{n} \rho_i \rho_j & - \sum_{1 \leq i < j \leq n, i,j \neq 2}^{n} \rho_i \rho_j & \cdots & - \sum_{1 \leq i < j \leq n, i,j \neq n}^{n} \rho_i \rho_j \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{n-1} \rho_2 \rho_3 \cdots \rho_n & (-1)^{n-1} \rho_1 \rho_3 \rho_4 \cdots \rho_n & \cdots & (-1)^{n-1} \rho_1 \rho_2 \cdots \rho_{n-1}
\end{pmatrix} \tag{F.7}
\]

\(A_{i,j}\) denotes the cofactor of the element \(\Delta_{i,j}\). Let us note that, multiplying by \(\sum_{i=1}^{n} \rho_i\) the first row and adding to the second row of the matrix \(\Delta\), the \(\det \Delta\) and the cofactors \(A_{n,i}\) do not change; then (F.6), can be written as

\[
\frac{1}{(z - \rho_1) \cdots (z - \rho_n)} = \frac{1}{\det d} \left[ \frac{B_{n,1}}{z - \rho_1} + \frac{B_{n,2}}{z - \rho_2} + \cdots + \frac{B_{n,n}}{z - \rho_n} \right] \tag{F.8}
\]

where \(\det \Delta = \det d\) and \(A_{i,j} = B_{i,j}\). \(B_{i,j}\) are the cofactors of the matrix elements \(d_{i,j}\). It is easy to realize that the matrix \(d\) has \(d_{2,1} = \rho_1\) as the elements of the second row. Then the contour integration of the relevant terms of (F.8) is

\[
\frac{1}{2\pi i} \int dz \frac{z}{(z - \rho_1) \cdots (z - \rho_n)} = \frac{1}{\det d} [\rho_1 B_{n,1} + \rho_2 B_{n,2} + \cdots + \rho_n B_{n,n}].
\]

We know that

\[
\sum_{i=1}^{n} d_{i,j} B_{k,j} = 0, \quad l \neq k
\]

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and therefore, we can conclude

\[ \oint \frac{dz}{(z - \rho_1) \cdots (z - \rho_n)} = 0, \quad \text{for } n > 2 \]  

(F.9)

Thus, the last term of \( C_n \) vanishes due to (F.4) and the remaining terms vanish because of (F.5). Then \( C_n = 0 \) for \( n > 1 \), and the Eqn.(F.2) becomes

\[ I_{n+1} = I_n, \quad n > 1, \]

this shows that

\[ I_n = 1, \quad n \geq 1. \]

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