Transforming quantum states between reference frames

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Received 16 December 2019, revised 8 April 2020
Accepted for publication 22 April 2020
Published 8 June 2020

Abstract

In the 1970s, Fulling, Davis, and Unruh have shown that a quantum mechanical state must be described differently in different reference frames; otherwise, quantum mechanics would contain contradictions. We present a simple method for transforming any quantum state between the Minkowski and Rindler reference frames. We show that a Wigner-like distribution, commonly used in quantum optics, is useful for treating this problem. To illustrate our method, we transform the Minkowski vacuum and number states into Rindler space, and transform the Rindler vacuum into Minkowski space, as examples. Our method could be generalized to other cases as well.

Keywords: general relativity, quantum field theory, quantum phase space distributions, Wigner Weyl distribution, reference frames, Unruh acceleration radiation, coordinate transformation

1. Introduction

Professor Roy Glauber made many ground-breaking contributions to physics; his creative use of quasi-classical probability distributions, quantum optics being a prime example. In the present paper, we show how to use Wigner distributions to go between different reference frames in general relativity, for example, between the Minkowski and Rindler frames. It is a pleasure to dedicate this article to Roy Glauber, physicist-friend-familyman.

In the 1970s, a relativistic quantum mechanics effect was predicted [1–3], where the particle content of a single quantum state in two reference frames was shown to be different. Specifically, a vacuum in an inertial frame is a thermal state to an accelerating observer. This is called the ‘Unruh effect’ or ‘acceleration radiation’ in the literature. The underlying reason for the effect is that particles correspond to excitations of positive frequency modes, and since different reference frames have different notions of time, observers in different reference frames do not agree on each other’s definition of frequency.

Recently, quantum optics techniques were used for studying fundamental issues in quantum mechanics with relativity, quantum field theory in curved spacetime, and in particular, acceleration radiation and the dynamical Casimir effect [4–8].

In section 2, we introduce the basics of Rindler space [1, 9] as related to reference frames of observers with constant acceleration; in section 3, we discuss the Unruh–Minkowski (UM) modes, which are the natural Minkowski positive frequency modes for translating between inertial and constant accelerating frames; in section 4 we review the Wigner distribution and its property that enables us to transform between reference frames; in section 5, we present and apply our method to the problem of transforming the Minkowski vacuum to an accelerating frame; in section 6, we use our method and obtain the Rindler space vacuum in terms of UM modes, and in section 7, we find the Minkowski number states in terms of Rindler modes. In appendix A, we show how the Minkowski vacuum state could be transformed into Rindler space variables without our method; in appendix B
In this paper, we use the term ‘Wigner distribution’ to refer to a Wigner-like distribution in raising and lowering operators, rather than with the usual position-momentum phase-space. See appendix C.

All primed quantities are left-wedge quantities.

\[ \frac{c t}{\ell} = e^{-\frac{z}{c}} \sinh \left( \frac{c t'}{\ell} \right) \quad \frac{c t}{\ell} = \frac{1}{2} \ln \left( \frac{z + c t}{\ell} - z - c t \right) \]

or, in terms of the ‘null’ spacetime coordinates \( u = c t - z \), \( v = c t + z \), \( \mu = c t - \zeta \), and \( \nu = c t + \zeta \), the relationship between the spacetime coordinates is

\[ \frac{u}{\ell} = -e^{-\nu/\ell} \quad \frac{\nu}{\ell} = -\ln \left( \frac{u}{\ell} \right) \]

\[ \frac{v}{\ell} = e^{\mu/\ell} \quad \frac{\mu}{\ell} = \ln \left( \frac{v}{\ell} \right) \]

in the right wedge, where \( u < 0 \) and \( v > 0 \). The left wedge coordinates, where \( u > 0 \) and \( v < 0 \), could be obtained from equation (2) by taking \( \ell \rightarrow -\ell \)

\[ \frac{c t}{\ell} = e^{-\frac{\zeta}{c}} \sinh \left( \frac{c t'}{\ell} \right) \quad \frac{c t}{\ell} = \frac{1}{2} \ln \left( \frac{z + c t}{\ell} - z - c t \right) \]

or in terms of the left wedge null coordinates \( \mu' = c t' - \zeta' \) and \( \nu' = c t' + \zeta' \),

\[ \frac{u}{\ell} = e^{\nu'/\ell} \quad \frac{\nu'}{\ell} = -\ln \left( \frac{u}{\ell} \right) \]

\[ \frac{v}{\ell} = -e^{\mu'/\ell} \quad \frac{\mu'}{\ell} = \ln \left( \frac{v}{\ell} \right) \]

where \( \ell \) is the length-scale in the problem such that the 4-acceleration of a classical object at ‘rest’ at the Rindler coordinate \( \zeta = 0 \) is \( a = c^2/\ell \). To relate the coordinates in the right and left Rindler wedges, we take \( \ell \rightarrow -\ell \) because the constant acceleration trajectories of constant \( \zeta \) worldlines in the right and left wedges accelerate in opposite directions. Equations (2) and (4) are two separate Rindler coordinates, each covering separate parts of the Minkowski space. Refs. [3, 11] have shown that for each Rindler frequency \( \Omega = c|\zeta|/2\pi \ell \), there are two positive frequency, right-moving Unruh–Minkowski (UM) modes, \( \phi_\zeta(u) \) and \( \phi_{-\zeta}(u) \), and two positive-frequency left-moving UM modes, \( \phi_\zeta(v) \) and \( \phi_{-\zeta}(v) \), which are the subject of the following section.

3. The Unruh–Minkowski (UM) modes

Using the null coordinates \( u = c t - z \) and \( v = c t + z \) as above, the wave equation for massless fields is \( \partial_\mu \partial^\mu \phi = 0 \), the solutions to which are linear combinations of arbitrary right- and left-moving functions, \( f(u) \) and \( g(v) \). The right-moving Unruh–Minkowski (UM) modes \( \phi_\zeta(u) \) which correspond to the annihilation operator \( \hat{a}_\zeta \), are

\[ \phi_\zeta(u) = \frac{e^{-u\xi^4}}{\sqrt{4\pi \xi \sinh(\xi/2)}} \lim_{\lambda \rightarrow 0^+} (u/\ell - i\lambda)^{\xi^4/2\pi} \]

5 In this paper, we use the term ‘Wigner distribution’ to refer to a Wigner-like distribution in raising and lowering operators, rather than with the usual position-momentum phase-space. See appendix C.

6 All primed quantities are left-wedge quantities.

Figure 1. Minkowski spacetime diagram split into four parts (wedges) by a lightcone at the origin. The right and left wedges correspond to the right and left Rindler wedges. A constantly-accelerating observer with positive (negative) acceleration is confined to the right (left) Rindler wedge, where \( u \) is negative (positive). Since light signals have a 45° trajectory, no light signal can be sent between the right and left Rindler wedges—they are therefore said to be ‘causally-disconnected.’ The left and right Rindler wedges have independent particle excitations, corresponding to the annihilation operators \( \hat{b}_\xi \) and \( \hat{b}_\xi \), respectively.

2. Rindler space basics

Minkowski space features two causally-independent wedges [1, 9] (and beforehand, in a footnote in [10]), called the right and left Rindler wedges. See figure 1. Each Rindler wedge has its own set of coordinates, \( (c t, \zeta) \) for the right wedge and \( (c t', \zeta') \) for the left wedge [6], where \( c \) is the speed of light. These are related to the Minkowski coordinates \( (c t, z) \) by...
and the left-moving modes are \( \phi_l(v) \). We will not treat the left-moving modes here because any results for the right-moving modes can be straight-forwardly generalized to the left-moving ones. We use the infinitesimal ‘\( \lambda \)’ to indicate that \( \phi_l(v) \) is defined with a branch cut in the top-half complex \( u \) plane (correspondingly, \( \phi_u(v) \) has a branch cut in the bottom-half complex \( u \) plane). To an accelerating observer, these modes appear to have a frequency \( \Omega = c|\xi|/2\pi \). That is, along the path \( \xi = 0 \), the modes go as \( \exp[-i\Omega t] \), where \( \Omega = c|\xi|/2\pi \).

For all (positive as well as negative) values of \( \xi \), the UM modes \( \phi_u(v) \) and \( \phi_l(v) \) have unit positive Klein–Gordon norm, and are linear combinations of the Minkowski positive frequency modes \( \exp(-i\omega u/\sqrt{2}\pi) \). In contrast, their complex-conjugates, \( \phi_u^*(v) \) and \( \phi_l^*(v) \), have negative norm. The mode \( \phi_u(v) \) has a constant amplitude for positive \( u \) and a different constant amplitude for negative \( u \); for positive \( \xi \) it has a larger amplitude in the right wedge than in the left wedge, and vice-versa for negative \( \xi \). The state defined by \( \hat{a}_\xi|0\rangle = 0 \) for all \( \xi \) is the Minkowski vacuum state. The UM modes form a complete and orthogonal set of functions like the complex exponentials, and therefore we can expand the quantum field \( \Phi \) in them

\[
\Phi(u, v) = \int_{-\infty}^{\infty} d\xi \left( \hat{a}_\xi \phi_u(\xi) + \hat{a}^\dagger_\xi \phi_l^*(\xi) \right) + \text{left-moving}. \tag{6}
\]

The field \( \hat{\Phi} \) could also be expanded in Rindler modes

\[
\hat{\Phi} = \int_{0}^{\infty} d\xi \left( \hat{b}_{R\xi} e^{-i(\xi/2\pi)t} + \hat{b}_{L\xi} e^{-i(\xi/2\pi)t} \right) + \text{H.a.} + [\mu \to v]. \tag{7}
\]

where the subscripts ‘\( R \)’ and ‘\( L \)’ correspond to the left and right Rindler wedges, and left-wedge coordinates are primed, as above. As can be seen from the coordinate transformation equations (2) and (4), the mode exp\( [-i\xi u/2\pi] \) \( / \sqrt{\xi/\ell} \), called ‘the Rindler mode in the right wedge,’ is only defined in the right Rindler wedge, and is a mode with a single Rindler frequency \( \Omega = c\xi/2\pi \). Since the right and left Rindler wedges are causally-disconnected, there is no coherence between a right and left Rindler mode, and therefore this mode is zero in the left Rindler wedge. Similarly, the Rindler mode in the left wedge, exp\( [-i\xi u/2\pi] \) \( / \sqrt{\xi/\ell} \), is zero in the right Rindler wedge. Comparing equations (6) and (7) for the field \( \hat{\Phi} \), using equations (2) and (4) to relate the Minkowski null coordinate \( u \) to the right and left wedge Rindler null coordinates \( \mu \) and \( \mu' \), one can see that for positive \( \xi \), the \( \hat{a} \) and \( \hat{b} \) operators are related by

\[
\hat{b}_{R\xi} = \frac{e^{\xi/2\ell} \hat{a} - e^{-\xi/2\ell} \hat{a}^\dagger}{\sqrt{2 \sinh (\xi/2)}} \quad \hat{a}_\xi = \frac{e^{\xi/2\ell} \hat{b}_{R\xi} - e^{-\xi/2\ell} \hat{b}_{L\xi}^\dagger}{\sqrt{2 \sinh (\xi/2)}},
\]

and similarly for their Hermitian adjoints. Thus, the Rindler creation and annihilation operators are expressible in terms of the UM mode annihilation and creation operators (and vice-versa). Analogously to the Minkowski vacuum state, the Rindler vacuum state is given by the state that satisfies \( \hat{b}_{R\xi}|0\rangle = 0 \) and \( \hat{b}_{L\xi}|0\rangle = 0 \) for all \( \xi \).

Throughout the paper, we focus on a single positive \( \xi \). That is, we focus on the positive-norm UM modes \( \phi_u \) and \( \phi_l \), and their corresponding annihilation operators \( \hat{a}_\xi \) and \( \hat{a}^\dagger_\xi \), respectively.

4. Wigner distribution of a quantum state

Any quantum state could be represented by its Wigner distribution \( w(q, p) = \int d^2q' \exp[iq'p]\hat{b}_{q'}|0\rangle\langle 0|\hat{b}^\dagger_{q'} - \pi/2 \), which is a phase-space representation of its density matrix, and therefore, could be used for describing the quantum state and for calculating expectation values [12–14]. The Wigner distribution is a pseudo-probability function in phase-space (the term ‘pseudo’ is used because the Wigner distribution can be negative). The Wigner distribution \( w(q, p) \), can be written as the expectation value of the displaced parity operator \( \hat{n}_{q,p}(\hat{q}, \hat{p}) \) [15], which is [16]

\[
w_{q,p}(\hat{q}, \hat{p}) = 2e^{2i\hat{q}p - \pi\hat{q}\hat{p}} - \pi/2, \tag{9}
\]

(see appendix B and F about the parity operator). That is, the Wigner distribution is the expectation value \( w(q, p) = \langle \hat{n}_{q,p} \rangle \), where the factor of \( 2\pi \) appears for normalization.

The c-numbers \( q \) and \( p \) correspond to the displacement of the configuration and the associated momentum operator, \( \hat{q} \) and \( \hat{p} \). Instead of the configuration-conjugate momentum basis, one can generalize the Wigner distribution to other bases having a continuous spectrum [17]. For example, one could use the Wigner distribution \( \hat{w}(\alpha, i\alpha^*) \) in the basis of coherent states \( |\alpha\rangle \), where \( \hat{a} |\alpha\rangle = \alpha |\alpha\rangle \). To get \( \hat{w} \), one should first get the proper seed operator, \( \hat{\nu} \), with the Wigner distribution \( \hat{w}(\alpha, i\alpha^*) = \langle \hat{n}_{\alpha,\alpha^*}(\hat{a}, i\alpha^*) \rangle /2\pi \), being its expectation value. As we discuss in appendix D, \( \hat{w} \) is obtained from the configuration-conjugate momentum Wigner seed operator \( \hat{w} \) in equation (9) by replacing the configuration and conjugate momentum operators, \( \hat{q} \) and \( \hat{p} \), and their corresponding c-functions, \( q \) and \( p \), by the ladder operators, \( \hat{a} \) and \( i\hat{a}^\dagger \),

\[
\hat{a} = (\hat{q} + i\hat{p})/\sqrt{2}, \quad i\hat{a}^\dagger = (\hat{q} - i\hat{p})/\sqrt{2}, \tag{10}
\]

and their corresponding c-functions, \( \alpha \) and \( i\alpha^* \), respectively. That is,

\[
\hat{w}_{\alpha,\alpha^*}(\hat{\nu}, i\hat{\nu}^\dagger) = 2e^{2(i\hat{\nu}^\dagger - i\alpha^*)\hat{a} - \alpha} = 2e^{-2i\hat{a}^\dagger - i\alpha^*}\hat{a} - \alpha, \tag{11}
\]

where the c-number \( \alpha \) corresponds to the displacement of the coherent state \( |\alpha\rangle \), and the factor of \( i \) multiplying the \( \alpha^* \) is included so that the commutation relation of \( \hat{\nu} \) with \( i\hat{a}^\dagger \) be the same as of \( \hat{q} \) with \( \hat{p} \) [18, 19, 16]. In appendix D we discuss that the Wigner distributions \( w(q, p) \) and \( \hat{w}(\alpha, i\alpha^*) \) are related by

\[
\text{7 The semicolon ‘\( ; \)’ in the exponent denotes Schwinger operator ordering, where (in this case) all factors of \( (\hat{q} - p) \) are to the left of any factor of \( (\hat{q} - q) \), as shown explicitly in equation (C3).}
direct substitution in their arguments,
\[ \tilde{w}(\alpha, i\alpha^\theta) = w \left( \frac{\alpha + \alpha^\theta}{\sqrt{2}}, \frac{\alpha - \alpha^\theta}{i\sqrt{2}} \right), \]
where the transformation
\[ q \rightarrow \frac{\alpha + \alpha^\theta}{\sqrt{2}}, \quad p \rightarrow \frac{\alpha - \alpha^\theta}{i\sqrt{2}}. \]
was obtained from equation (10).

Instead of treating the configuration and conjugate momentum as primary, we will treat the coherent-state basis as primary, and write the argument of the Wigner distribution as \( W(\alpha) \), which stands for \( W(\alpha, \alpha^\theta) \), which really stands for \( \tilde{w}(\alpha, i\alpha^\theta) \). The Wigner seed operator \( \hat{W}_0 \) for a single mode of the field is the same as in equation (12),
\[ \hat{W}_0 = 2e^{-2\alpha^\theta - \alpha^* \theta}(\hat{a}^\theta - \alpha), \]
and the Wigner distribution is its expectation value, \( W(\alpha) = \langle \hat{W}_0 \rangle / 2\pi \).

As we mentioned in the end of the previous section, we focus on a single positive \( \xi \) for the Rindler and UM modes. The Wigner distribution\(^9\) for the two modes of \( \phi_\xi \) and \( \phi_{-\xi} \) is obtained from the density matrix \( \hat{\rho} \) via\(^8\) \( W(\alpha_\xi, \alpha_{-\xi}) = \langle \hat{W}_m \hat{W}_{-m} \rangle / (2\pi)^2 \), where \( \hat{W}_m \) is the Wigner seed operator of the \( j \)-th mode with parameter \( \alpha_j \) and corresponding creation and annihilation operators
\[ \hat{W}_j = 2e^{-2\alpha^\theta_j - \alpha^* \theta_j}(\hat{a}^\theta_j - \alpha_j). \]

We show that describing a quantum state in terms of its Wigner distribution is very useful because it allows us to transform quantum states between different reference frames in a simple-minded way. In particular, our method entails only direct substitution in the arguments. The form of the transformation [see equations (8) and (19)] is obtained from the spacetime structure of the modes [11], whereas the transformation is performed in the coherent state basis.

5. The Minkowski vacuum

An important example is the Wigner distribution of the Minkowski vacuum state \( \hat{\rho} = \left| \Omega_M \right\rangle \left\langle \Omega_M \right| \), which is
\[ W_{\text{Mink}}(\{\alpha_\xi, \alpha_{-\xi}\}) = \prod_\xi W(\alpha_\xi, \alpha_{-\xi}) = \left( \prod_\xi \frac{\hat{W}_m(\hat{W}_{-m})}{(2\pi)^2} \right), \]
where \( \alpha_\xi \) is the phase-space displacement (in complex \( \alpha \)-space) of the annihilation operator \( \hat{a}_\xi \) or of its eigenfunction \( |\alpha_\xi\rangle \). Throughout the paper, we focus on a single positive \( \xi \).

\(^8\) See appendix C for more information about Wigner distributions and how they could be obtained from the density matrix.
\(^9\) We often write expressions like \( W(\alpha_\xi, \alpha_{-\xi}) \), and the dependence on the conjugate variables, \( \alpha^* \) and \( \alpha^* \theta \), is to be understood. That is, \( W(\alpha_\xi, \alpha_{-\xi}) \) stands for \( W(\alpha_\xi, \alpha^* \theta, \alpha_{-\xi}, \alpha^* \theta) \).

keeping in mind that the full state contains all frequencies. For a single UM frequency\(^10\) (that is, the \( \xi \) term in the product (17)), the Wigner distribution of the Minkowski vacuum state is
\[ W(\alpha_\xi, \alpha_{-\xi}) = \frac{1}{\pi} e^{-2|\alpha_\xi|^2 + |\alpha_{-\xi}|^2}, \]
see equation (C12). Since the coherent state \( |\alpha\rangle \) has an average number of excitations \( |\alpha|^2 \), the value of \( W(\alpha_\xi, \alpha_{-\xi}) \), after integrating over the phases of \( \alpha_\xi \) and \( \alpha_{-\xi} \), heuristically corresponds to the probability of having \( |\alpha_\xi|^2 \) excitations of the mode \( \phi_\xi \) and \( |\alpha_{-\xi}|^2 \) excitations of the mode \( \phi_{-\xi} \). This state is given in terms of the coherent state \( \alpha_\xi \) and \( \alpha_{-\xi} \) of the UM modes, \( \phi_\xi \) and \( \phi_{-\xi} \), respectively. As we discuss in appendix E, we can express the same quantum state (the Minkowski vacuum) in terms of coherent states of Rindler modes, \( \beta_\xi \) and \( \beta_{-\xi} \), of the right and left wedges, respectively; to do so, we make the canonical transformation in the Wigner distribution [14, 16, 19–22] in equation (18),
\[ \alpha_\xi \rightarrow \frac{1}{\sqrt{2\sinh(\xi/2)}} \left( e^{i\xi/4} \beta_\xi - e^{-i\xi/4} \beta^*_\xi \right), \]
\[ \alpha_{-\xi} \rightarrow \frac{1}{\sqrt{2\sinh(\xi/2)}} \left( e^{i\xi/4} \beta_{-\xi} - e^{-i\xi/4} \beta^*_{-\xi} \right), \]
which we obtained by taking equations (8) and converting them into c-functions. The expressions for \( \alpha_\xi \) and \( \alpha_{-\xi} \) are obtained by complex-conjugation of equations (19). Doing so, we obtain the Wigner distribution for the Minkowski vacuum state in terms of the Rindler modes,
\[ \pi^2 W(\beta_\xi, \beta_{-\xi}) = \exp \left[ \frac{2\beta_\xi \beta_{-\xi} + \beta^*_\xi \beta^*_{-\xi}}{\sinh(\xi/2)} \right] - 2(|\beta_\xi|^2 + |\beta_{-\xi}|^2) \coth(\xi/2). \]
Thus, we started with a quantum state (the Minkowski vacuum) for the field [equation (18)], and through direct substitution [equation (19)], were able to express the same state in terms of Rindler modes [equation (20)]. Besides making the transformation between reference frames easy (direct substitution), the Wigner representation gives a simpler expression [equation (20)] than its Fock space representation [equation (21)] [11],
\[ \hat{\rho} = \prod_\xi \prod_{\xi'} \sqrt{1 - \exp[-\xi]} \sqrt{1 - \exp[-\xi']} \sum_{n_\xi} \sum_{n_{\xi'}} \exp \left[ \left( \frac{n_\xi \xi}{2} + \frac{n_{\xi'} \xi'}{2} \right) \right] |n_\xi, n_{\xi'}\rangle \langle n_\xi, n_{\xi'}|, \]
(see appendix A) where \( |n_\xi, n_{\xi'}\rangle \) is the state of \( n_\xi \) excitations in the right wedge and \( n_{\xi'} \) excitations in the left wedge.

If our observable is confined to the right Rindler wedge, we may integrate the Minkowski vacuum state, equation (20), over the \( \beta_{-\xi} \)-phase space (because \( \beta_{-\xi} \) refers to the left

\(^{10}\) Note that there are four modes with the ‘same’ frequency \( \xi \): the left- and right-moving UM modes, and positive and negative \( \xi \).
Rindler wedge modes). This would give the Minkowski vacuum state as it pertains to any observer or observable confined to the right Rindler wedge (in density matrix language, this amounts to performing a partial-trace over the left wedge variables). Doing so, we obtain the state

$$W_{\text{reduced}}(\beta_{\text{LR}}) = \frac{1}{\pi} \tanh(\xi/2) e^{-2/\beta_{\text{LR}}^2 \tanh(\xi/2)},$$

(22)

which is a thermal state [20, 21] with temperature $k_B T_U = \Omega/\xi = e/2\pi\ell$, which is the Unruh result [3], where $k_B$ is Boltzmann’s constant.

6. Rindler vacuum in Minkowski phase-space

One may also wish to transform a quantum state that is known in terms of Rindler excitations into Minkowski space. Whereas in the previous section, we showed how to transform a state known in Minkowski space into a Rindler space, in terms of Rindler excitations into Minkowski space.

In terms of Rindler modes, the Rindler vacuum state $\hat{\rho} = |0\rangle_\text{LR} \langle 0|$ is

$$W(\beta_{\text{LR}}, \beta_{\text{LL}}) = \frac{1}{\pi^2} e^{-2|\beta_{\text{LR}}|^2 + |\beta_{\text{LL}}|^2},$$

(23)

From equation (8), we have that the annihilation operators for the Rindler modes in the right and left wedges are related to the UM modes by [11]

$$\hat{b}_{\text{LR}} = \frac{1}{\sqrt{2\sinh(\xi/2)}} \{ e^{\xi/4} \hat{a}_\xi + e^{-\xi/4} \hat{a}^\dagger_\xi \},$$

$$\hat{b}_{\text{LL}} = \frac{1}{\sqrt{2\sinh(\xi/2)}} \{ e^{\xi/4} \hat{a}_{-\xi} + e^{-\xi/4} \hat{a}^\dagger_{-\xi} \},$$

(24)

and using their c-number analogs,

$$\beta_{\text{LR}} \longrightarrow \frac{1}{\sqrt{2\sinh(\xi/2)}} \{ e^{\xi/4} \alpha_\xi + e^{-\xi/4} \alpha^*_\xi \},$$

$$\beta_{\text{LL}} \longrightarrow \frac{1}{\sqrt{2\sinh(\xi/2)}} \{ e^{\xi/4} \alpha_{-\xi} + e^{-\xi/4} \alpha^*_\xi \},$$

(25)

in equation (23), the Wigner distribution of the Rindler vacuum is (see appendix E)

$$W(\alpha_\xi, \alpha_{-\xi}) = \frac{1}{\pi^2} \exp \left[ -2 \frac{\alpha_\xi \alpha_{-\xi} + \alpha^*_\xi \alpha^*_{-\xi}}{\sinh(\xi/2)} \right]$$

$$\times \exp \left[ -2(|\alpha_\xi|^2 + |\alpha_{-\xi}|^2) \coth(\xi/2) \right],$$

(26)

in terms of Unruh–Minkowski modes. Thus, we see that our method can transform a quantum state’s description between the reference frames in both directions (Minkowski $\leftrightarrow$ Rindler).

7. Minkowski number states

It is common perception that the Minkowski number states are tedious to treat, and indeed, we agree that they are laborious in the Rindler Fock space representation. However, in the Wigner phase-space language, a simplification occurs.

In terms of Unruh–Minkowski modes, a Minkowski mode number state is

$$W^{(0)}(\alpha_\xi, \alpha_{-\xi}) = \frac{1}{\pi^2} e^{-2(|\alpha_\xi|^2 + |\alpha_{-\xi}|^2) L_n(4 |\alpha_\xi|^2)},$$

(27)

where $L_n$ is the Laguerre polynomial of order $n$, see appendix G. Using the transformation equations (19), we get that the $n$-Minkowski particle state is

$$W^{(0)}(\beta_{\text{LR}}, \beta_{\text{LL}}) = W^{(0)}(\beta_{\text{LR}}, \beta_{\text{LL}}) L_n(\frac{2}{\sinh(\xi/2)})$$

$$\times e^{\xi/2 |\beta_{\text{LR}}|^2 - \beta_{\text{LR}}^2 - \beta_{\text{LL}}^2} + e^{-\xi/2 |\beta_{\text{LL}}|^2 - \beta_{\text{LR}}^2}).$$

(28)

in terms of Rindler modes, where $W^{(0)}$ is the Minkowski vacuum state in terms of Rindler modes, equation (20).

8. Conclusions

We have presented a method for transforming any arbitrary quantum state between Unruh–Minkowski and Rindler modes, and vice-versa. This method could be applied to more general transformation cases. We have shown that translating the Minkowski vacuums and number states into the accelerating frame and the Rindler vacuum into the Minkowski frame are all easy to do using our method.

Acknowledgments

JSB and MOS would like to thank the Robert A. Welch Foundation (Grant No. A-1261), the Office of Naval Research Award No. N00014-16-1-3054, the Air Force Office of Scientific Research (FA9550-18-1-0141), and the King Abdullah City for Science and Technology (KACST) grant for their the support. WGU would like to thank the Hagler IAS at Texas A&M, NSERC, and CIFAR for their support.

Appendix A. Fock space derivation of the Minkowski vacuum in Rindler spacetime

The Minkowski vacuum is defined as the state which is annihilated by all annihilation operators, for example

$$\hat{a}_\xi |0_M\rangle = \frac{1}{\sqrt{2 \sinh(\xi/2)}} (e^{\xi/2} \hat{b}_{\text{LR}} - e^{-\xi/2} \hat{b}^\dagger_{\text{LL}}) |0_M\rangle = 0.$$  (A1)
Since \(|0_M\rangle\) must satisfy equation (A1) for every \(\xi\) independently, we see that the Minkowski vacuum is

\[
|0_M\rangle = \prod_\xi \hat{f}_\xi (\hat{b}_\xi^\dagger, \hat{b}_\xi^\dagger)|0_R\rangle,
\]  
(A2)

in terms of Rindler modes. Focusing first on a single \(\xi\), we have that

\[
|0_{M\xi}\rangle = \hat{f}_\xi (\hat{b}_\xi^\dagger, \hat{b}_\xi^\dagger)|0_R\rangle.
\]  
(A3)

Then from equations (A1) and (A3), we have that

\[
(e^{\xi/4} \hat{b}_{\xi R} - e^{-\xi/4} \hat{b}_{\xi L}^\dagger)|0_R\rangle = 0,
\]  
(A4)

and noticing that

\[
\frac{\partial \hat{f}_\xi}{\partial \hat{b}_{\xi R}^\dagger} = e^{-\xi/2} \hat{b}_{\xi L}^\dagger \hat{f}_\xi,
\]  
(A6)

giving that \(\hat{f}_\xi\) is

\[
\hat{f}_\xi = f_\xi \exp[-\xi/2 \hat{b}_{\xi L}^\dagger \hat{b}_{\xi R}].
\]  
(A7)

Thus, in terms of Rindler modes, the Minkowski vacuum \(|0_M\rangle\) is

\[
|0_M\rangle = \hat{f}_\xi (\hat{b}_\xi^\dagger, \hat{b}_\xi^\dagger)|0_R\rangle = f_\xi \prod_\xi \prod_{\xi} \frac{e^{-\xi/2}}{n_\xi !} (\hat{b}_{\xi L}^\dagger \hat{b}_{\xi R}^\dagger)|0_R\rangle,
\]  
(A8)

where \(f_\xi = \sqrt{1 - \exp[-\xi]}\) is a normalization constant. Equation (A8) is just for a single \(\xi\) (notice that the normalization constant is \(\xi\)-dependent). For all \(\xi\)’s, we have in terms of Rindler Fock quanta, the Minkowski vacuum state is

\[
|0_M\rangle = \prod_\xi \sqrt{1 - \exp[-\xi]} \prod_{\xi} \frac{e^{-\xi/2}}{n_\xi !} |n_R\rangle, \quad |n_L\rangle,
\]  
(A9)

where \(|n_R\rangle, |n_L\rangle\) is the state of \(n_R\) excitations in the right wedge and \(n_L\) excitations in the left wedge. To get the state that an observer accelerating to the right would interact with, we need to trace-out the left-wedge (primed) quanta in the Minkowski vacuum density matrix

\[
|0_M\rangle \langle 0_M| = \prod_\xi \prod_{\xi} \sqrt{1 - \exp[-\xi]} \sqrt{1 - \exp[-\xi]} \prod_{n_\xi, n_\xi} \frac{\exp\left(-\frac{n_\xi \xi}{2} + \frac{n_\xi^2 \xi}{2}\right)}{n_\xi !} |n_\xi, n_\xi \rangle \langle n_\xi, n_\xi|.
\]  
(A10)

Tracing over the left-wedge quanta, the density matrix is thermal

\[
\hat{\rho}_{\text{thermal}} = \prod_\xi \left(1 - e^{-\xi}\right) \sum_n e^{-n_\xi \xi/2} |n_\xi \rangle \langle n_\xi|,
\]  
(A11)

with temperature \(k_B T_0 = \Omega/\xi = \hbar c/2\pi \ell\).

**Appendix B. Position-momentum Wigner distributions**

Here we discuss the fact that the position-momentum Wigner distribution is the expectation value of the seed operator [15, 16]

\[
\hat{w}_{q, p}(\hat{q}, \hat{p}) = e^{2i(\hat{q} \hat{p} - \hat{p} \hat{q})},
\]  
(B1)

Starting with the more familiar formula for the Wigner distribution

\[
w(q, p) = \frac{1}{2\pi} \int dq' \left\langle q + \frac{q'}{2}|\hat{p}| q - \frac{q'}{2}\right\rangle e^{-i q' p},
\]  
(B2)

and changing variables according to \(q' \to 2(q' - q)\),

\[
w(q, p) = \frac{1}{2\pi} \int dq' \langle q'|\hat{p}|2q - q'\rangle e^{-2i(q' - q)p},
\]  
(B3)

and noticing that

\[
[2q - q'] = |q' - 2(q' - q)| = e^{2i(q' - q)p}|q'angle,
\]  
(B4)

the Wigner distribution, equation (B3), becomes

\[
w(q, p) = \frac{1}{2\pi} \int dq' \langle q'|\hat{p}|2q - q'\rangle e^{-2i(q' - q)p}
\]  
(B6)

\[
= \frac{1}{2\pi} \int dq' \langle q'|\hat{p}|2q - q'\rangle e^{-2i(q' - q)p}
\]  
(B7)

which is the expectation value of the Wigner seed operator, equation (B1).

Integrating the Wigner distribution over the configuration variable gives the probability of the associated momentum, and integrating over the momentum variable gives the probability of the configuration variable

\[
|\varphi(p)|^2 = \int dq \, w(q, p),
\]  
(B9)

\[
|\psi(q)|^2 = \int dp \, w(q, p).
\]  
(B10)

Therefore, the Wigner distribution satisfies the marginals (this is a property of all two-variable probability distributions). As implied by equation (B10), the Wigner distribution is normalized

\[
\int dq dp \, w(q, p) = 1,
\]  
(B11)

which is another property of probability distributions. However, the Wigner distribution can (and usually does) have negative values—so it is not a proper probability distribution. For these reasons, it is often referred-to as a quasi-probability distribution.
Appendix C. Wigner-like functions

Here, we give a quick introduction on how to obtain a Wigner distribution for a given quantum state \( |\beta\rangle \).

The Wigner distribution \( W(\alpha) \) of a quantum state is the expectation value \([16, 23] \)
\[
W(\alpha) = \frac{1}{2\pi} \text{Tr} \left[ \hat{W}_\alpha (\hat{a}, \hat{a}^\dagger) \right],
\]
(C1)
of the ‘displaced parity operator’ \([12, 15] \), also known as the ‘Wigner seed operator’ \([16] \)
\[
\hat{W}_\alpha (\hat{a}, \hat{a}^\dagger) = 2e^{-2i\alpha^*c(\hat{a}^\dagger - \hat{a})},
\]
(C2)
where the semicolon ‘;’ in the exponent signifies the Schwinger ordered-exponential
\[
e^{\hat{A}\hat{B}} = \sum_{n=0}^{\infty} \frac{1}{n!} (\hat{A}\hat{B})^n,
\]
(C3)
and the parameter \( \alpha \) is the argument of the Wigner distribution coming from it, see equation (C1). There are many ways to express the Wigner seed operator \([23] \), but we find equation (C2) to be the simplest. For the two modes, \( \varphi_\zeta \) and \( \varphi_{-\zeta} \), the Wigner distribution is
\[
W(\alpha_\zeta, \alpha_{-\zeta}) = \left\langle \hat{W}_{\alpha_\zeta} \hat{W}_{\alpha_{-\zeta}} \right\rangle, W(0, \alpha_{-\zeta}) = \left\langle \hat{W}_{\alpha_{-\zeta}} \right\rangle,
\]
(C4)
and similarly for the Wigner seed operator of the second mode, \( \hat{W}_{\alpha_{-\zeta}} \).

Example: Vacuum state. As an example of how to obtain the Wigner distribution from the Fock representation, we show how to obtain the Wigner distribution corresponding to the vacuum state \( |0\rangle \)
\[
W(\alpha) = \frac{1}{2\pi} \text{Tr} \left[ 2e^{-2i\alpha^*c(\hat{a}^\dagger - \hat{a})}|0\rangle \langle 0| \right],
\]
(C6)
\[
= \frac{1}{2\pi} \langle 0|2e^{-2i\alpha^*c[(\hat{a}^\dagger - \hat{a})]|0\rangle,
\]
(C7)
Since we have Schwinger ordering of the operator, we have
\[
\langle 0|e^{-2i\alpha^*c[(\hat{a}^\dagger - \hat{a})]}|0\rangle
\]
\[
= \sum_{n=0}^{\infty} \frac{(-2\alpha)^n}{n!} \langle 0| (\hat{a}^\dagger - \hat{a})^n |0\rangle
\]
\[
= \sum_{n=0}^{\infty} \frac{(-2\alpha)^n}{n!} \langle 0| (\hat{a}^\dagger - \hat{a})^n |0\rangle
\]
\[
= e^{-2|\alpha|^2},
\]
(C10)
\[ \text{13} \text{ We sometimes express the operator’s dependence on } \hat{a} \text{ and } \hat{a}^\dagger \text{ explicitly —as in equation (C5)—and sometimes not—like in equation (C4). Notice that the Wigner seed operator } \hat{W}_\alpha (\hat{a}, \hat{a}^\dagger) \text{ is a function of the field mode creation and annihilation operators } \hat{a} \text{ and } \hat{a}^\dagger, \text{ and is parameterized by a phase-space displacement } \alpha, \text{ which we include as a subscript. At } \alpha = 0, \text{ for instance, } \hat{W}_0 \text{ is the displaced parity operator with zero displacement—that is, it is the parity operator—see appendix F. The Wigner distribution } W(\alpha) \text{ is a function of the phase-space displacement parameter } \alpha. \]
Once we are familiar with Schwinger-ordered operators, we realize that we could have just written
\[
\langle 0|e^{-2i\alpha^*c[(\hat{a}^\dagger - \hat{a})]}|0\rangle = e^{-2|\alpha|^2(|0\rangle \langle 0|).}
\]
(C11)
Thus, we see that the vacuum state is
\[
W(\alpha) = \frac{1}{\pi} e^{-2|\alpha|^2}.
\]
(C12)
We give another example in appendix G, where we find the Wigner distribution of a number state.

Appendix D. Canonical transformations

Here we show that if \( [\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] \), then the Wigner seed operators in the \( \hat{a} \) and \( \hat{b} \) bases are
\[
\hat{W}_\alpha (\hat{a}, \hat{a}^\dagger) = 2e^{-2i\alpha^*c(\hat{a}^\dagger - \hat{a})}
\]
\[
\hat{W}_\beta (\hat{b}, \hat{b}^\dagger) = 2e^{2i\beta^*c(\hat{b}^\dagger - \hat{b})},
\]
(D1)
leading to Wigner distributions \( \hat{W}_\alpha (\alpha) \) and \( \hat{W}_\beta (\beta) \). Furthermore, in equation (D15) we show that these are related via direct substitution.

Our discussion is based on \([20] \), where a more general case is done. An even more general case is done in \([23] \). As we learn from Royer \([15] \), the Wigner seed operator is the ‘displaced parity operator’ (see appendix F)
\[
\hat{W}_\alpha (\hat{a}, \hat{a}^\dagger) = 2e^{2i\alpha^*c(\hat{a}^\dagger - \hat{a})} e^{-2i\alpha^*c(\hat{a}^\dagger - \hat{a})}
\]
\[
= \hat{D}_\alpha (\hat{a}, \hat{a}^\dagger) \hat{W}_0 (\hat{a}, \hat{a}^\dagger) \hat{D}_\alpha^{-1} (\hat{a}, \hat{a}^\dagger),
\]
(D2)
where we have used equation (C2) for \( \hat{W}_\alpha \), where \( \hat{W}_0 \) is \( \hat{W}_0 \) with \( \alpha = 0 \) (also known as the parity operator—see appendix F) and where \( \hat{D}_\alpha (\hat{a}, \hat{a}^\dagger) \) is the displacement operator \( \exp[i\alpha \hat{a}^\dagger - \alpha^*\hat{a}] \). Suppose the transformation between \( (\hat{a}, \hat{a}^\dagger) \) and \( (\hat{b}, \hat{b}^\dagger) \) operators has the form
\[
\hat{b} = \hat{S} \hat{a} \hat{S}^{-1} = \alpha \hat{a} + \beta \hat{a}^\dagger
\]
\[
\hat{b}^\dagger = \hat{S} \hat{a}^\dagger \hat{S}^{-1} = \gamma \hat{a} + \delta \hat{a}^\dagger,
\]
(D4)
This implies that the commutators remain unchanged
\[
[\hat{b}, \hat{b}^\dagger] = [\hat{S} \hat{a}, \hat{S} \hat{a}^\dagger] \hat{S}^{-1} = [\hat{a}, \hat{a}^\dagger],
\]
(D5)
giving that \( \alpha \delta - \beta \gamma = 1 \). The transformation is also invertible,
\[
\hat{a} = \delta \hat{b} - \beta \hat{b}^\dagger
\]
\[
\hat{a}^\dagger = \alpha \hat{b}^\dagger - \gamma \hat{b},
\]
(D6)
Importantly, the displacement operator transforms like
\[
e^{\mathbf{i}\beta \cdot \mathbf{b}^\dagger} \hat{D}_\beta (\hat{b}, \hat{b}^\dagger) = \hat{S} \hat{D}_\alpha (\hat{a}, \hat{a}^\dagger) \hat{S}^{-1}
\]
\[
= \hat{D}_\alpha (\hat{a}, \hat{a}^\dagger),
\]
(D7)
where we recognized \( \alpha (\beta, \beta^*) = (\delta \beta - \beta \beta^*) \) and \( \alpha^* (\beta, \beta^*) = (\alpha \beta^* - \gamma \beta) \) from equation (D6) (having doffed the hats).
A particular realization of \( \hat{S} \) is
\[
\hat{S} = \exp \left\{ [\hat{a}^\dagger]_2 \alpha + \hat{a}^\dagger \hat{a} \beta + (\hat{a} \hat{a}^\dagger)_2 \gamma \right\},
\]
(D8)
giving that equation (D4) is
\[ \tilde{S}^{-1} = a^\dagger \left( \cosh \Delta - \beta \frac{\sinh \Delta}{\Delta} \right) - \alpha \left( 2 \alpha \frac{\sinh \Delta}{\Delta} \right), \]
and therefore, operating from the right with \( S^{-1} \),
\[ e^{-2b^\dagger \cdot b} \tilde{S} e^{-2a^\dagger \cdot a} = e^{-2a^\dagger \cdot a}, \]
which is a surprising result, and is true because \([\tilde{a}, \tilde{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] \).

From equations (D7) and (D11), we see that
\[ \hat{W}_{a,b}(\tilde{a}, \tilde{a}^\dagger) = \hat{W}_b(\hat{b}, \hat{b}^\dagger) \]
and therefore, the Wigner distribution corresponding to the operator \( \hat{A}(\hat{b}, \hat{b}^\dagger) \) is
\[ A_{\hat{b}}(\beta, \beta^*) = \text{Tr} \{ \hat{A} \hat{W}_b(\hat{b}, \hat{b}^\dagger) \} = \text{Tr} [ \hat{A} \hat{W}_{a,b}(\tilde{a}, \tilde{a}^\dagger) ] = \alpha_{A}(\beta, \beta^*), \]
where the subscripts on the c-number \( A \) denote the operator basis of the Wigner operator used.

For example, if we were given the Wigner distribution \( W_{a,b}(\alpha, \alpha^*) \) in terms of the coherent state basis \([\alpha]\), but we are interested in the Wigner distribution in the basis of the field variables \( \hat{X} = (a + \alpha)/(\sqrt{2}) \) and \( \hat{K} = (a - \alpha)/(\sqrt{2}) \), which have the commutation relation \([\hat{X}, \hat{K}] = 1 \) (making the transformation a canonical one), then we can use the transformation
\[ \alpha \rightarrow \frac{x - k}{\sqrt{2}}, \quad \alpha^* \rightarrow \frac{x + k}{\sqrt{2}}, \]
in the given Wigner distribution \( W_{\hat{X}, \hat{K}}(x, k) \) to obtain
\[ W_{\hat{X}, \hat{K}}(x, k) = \hat{W}_{a,b}(\alpha, \alpha^*) \frac{(x - k) \sqrt{2}}{(x + k) \sqrt{2}}, \]
where \( x \) and \( k \) are the c-numbers corresponding to eigenvalues of \( \hat{X} \) and \( \hat{K} \), respectively. Alternatively, given \( W_{\hat{X}, \hat{K}}(x, k) \), we can obtain \( W_{a,b}(\alpha, \alpha^*) \) by the inverse of the transformation given in equation (D16).

Appendix E. Two-mode Wigner identities

Here, we generalize the results of appendix D to the two-mode case, which is interesting for us, because of
\[ \text{equations (8) and (24). Generalizing equation (8), we have the transformation} \]
\[ \hat{b}_1 = \tilde{S} \hat{a}_1 \tilde{S}^{-1} = \alpha \hat{a}_1 + \beta \hat{a}_1^\dagger \]
\[ \hat{b}_2 = \tilde{S} \hat{a}_2 \tilde{S}^{-1} = \alpha \hat{a}_2 + \beta \hat{a}_2^\dagger \]
\[ \hat{b}_3 = \tilde{S} \hat{a}_3 \tilde{S}^{-1} = \alpha \hat{a}_3 + \beta \hat{a}_3^\dagger, \]
\[ \text{E1} \]

Our case (which is less general), could be obtained from \( S = \exp[\xi \hat{a}_2 - \xi \hat{a}_1^\dagger \hat{a}_2^\dagger] \) [24], which gives
\[ \hat{b}_1 = \hat{a}_1 \cosh \xi + a_1^\dagger e^{i\theta} \sinh \xi \]
\[ \hat{b}_2 = \hat{a}_2 \cosh \xi + a_2^\dagger e^{i\theta} \sinh \xi \]
\[ \hat{b}_3 = \hat{a}_3 \cosh \xi + a_3^\dagger e^{i\theta} \sinh \xi, \]
\[ \text{E2} \]

In terms of the definition in appendix D, the displacement operator \( \hat{D} \) is
\[ \hat{D}_{\alpha, \alpha^*}(\alpha, \alpha^*) = \hat{D}_{\alpha, \alpha^*}(\alpha, \alpha^*) \hat{D}_{\alpha, \alpha^*}(\alpha, \alpha^*) \]
\[ = \exp[\alpha \hat{a}_1 + \hat{a}_1^\dagger \alpha^* + \hat{a}_2 \alpha^* - \hat{a}_2^\dagger \alpha^*], \]
\[ \text{E5} \]

which satisfies
\[ \hat{D}_{\beta, \beta^*}(\beta, \beta^*) \hat{b}_1 = \hat{b}_2 \hat{D}_{\beta, \beta^*}(\beta, \beta^*) \tilde{S} \]
\[ = \hat{D}_{\beta, \beta^*}(\beta, \beta^*) \hat{a}_1 \hat{a}_1^\dagger \alpha \alpha^* - \hat{a}_2^\dagger \alpha \alpha^* - \hat{a}_2 \alpha^* \alpha^*, \]
\[ \text{E8} \]

where the \( \alpha \)’s as a function of the \( \beta \)’s are obtained from equation (E4) after replacing the operators by their c-numbers. We see that since \( S \) has dependence only on products of even powers of creation and annihilation operators,
\[ (e^{-2\beta_1 \hat{a}_1^\dagger \hat{a}_2} e^{-2\beta_2^\dagger \hat{a}_2^\dagger \hat{a}_2}) \tilde{S} (e^{-2\beta_1^\dagger \hat{a}_1^\dagger \hat{a}_2^\dagger} e^{-2\beta_2^\dagger \hat{a}_2^\dagger \hat{a}_2}) = \tilde{S}, \]
\[ \text{E9} \]

and therefore,
\[ e^{-2\beta_1 \hat{a}_1^\dagger \hat{a}_2} e^{-2\beta_2^\dagger \hat{a}_2^\dagger \hat{a}_2} = \tilde{S} e^{-2\beta_1 \hat{a}_1^\dagger \hat{a}_2} e^{-2\beta_2^\dagger \hat{a}_2^\dagger \hat{a}_2} \tilde{S}^{-1} \]
\[ = e^{-2\beta_1 \hat{a}_1^\dagger \hat{a}_2} e^{-2\beta_2^\dagger \hat{a}_2^\dagger \hat{a}_2}. \]
\[ \text{E10} \]

In terms of the definition in equation (D3), the Wigner seed operator is
\[ W_{\beta, \beta^*}(\beta, \beta^*) \tilde{S} \]
\[ = W_{\beta, \beta^*}(\beta, \beta^*) \tilde{S} \]
\[ = W_{\beta, \beta^*}(\beta, \beta^*). \]
\[ \text{E12} \]
\[ W_{\beta,\alpha}(\hat{b}_1, \hat{b}_1^{\dagger}, \hat{b}_2, \hat{b}_2^{\dagger}) = \hat{D}_{\beta,\alpha}(\hat{b}_1, \hat{b}_1^{\dagger}, \hat{b}_2, \hat{b}_2^{\dagger}) e^{-2\hat{b}_1^{\dagger}\hat{b}_1 - 2\hat{b}_2^{\dagger}\hat{b}_2} \times \hat{D}_{\alpha,\beta}(\hat{b}_1, \hat{b}_1^{\dagger}, \hat{b}_2, \hat{b}_2^{\dagger}) \]  
(E13)

\[ = \hat{D}_{\alpha,\beta}(\hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2, \hat{a}_2^{\dagger}) e^{-2\hat{a}_1^{\dagger}\hat{a}_1 - 2\hat{a}_2^{\dagger}\hat{a}_2} \times \hat{D}_{\alpha,\beta}(\hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2, \hat{a}_2^{\dagger}) \]  
(E14)

\[ = W_{\alpha,\beta}(\hat{a}_1, \hat{a}_1^{\dagger}, \hat{a}_2, \hat{a}_2^{\dagger}). \]  
(E15)

### Appendix F. The parity operator

Here we show that \( \hat{P} = \hat{W}_0 / 2 = \exp[-2\hat{a}^{\dagger}; \hat{a}] \) is the parity operator [16]. Here, the semicolon (;) denotes Schwinger operator ordering, so \( \hat{P} \) is

\[ \hat{P} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!} (\hat{a}^{\dagger})^n (\hat{a})^n, \]  
(F1)

which is normal ordered. In equation (F1) we used the shorthand \( \hat{W}_0 \) to mean \( \hat{W}_0(\hat{a}, \hat{a}^{\dagger}) \) (the longer, more explicit expression is useful in appendix D and E). We accomplish this by examining its matrix elements through its action on a complete set of states. This operator acting on the coherent state \( |\alpha\rangle \) gives

\[ e^{-2\hat{a}^{\dagger}\hat{a}} |\alpha\rangle = e^{-2\alpha^2} |\alpha\rangle, \]  
(F2)

since all \( \hat{a}^{\dagger} \)'s are to the right. Noticing that \( e^{-2\alpha^2} \) shifts a coherent state by \(-2\alpha \), we have that

\[ \hat{P} |\alpha\rangle = | - \alpha \rangle, \]  
(F3)

thus, we have reflected the state \( \alpha \) about the origin \( \alpha = 0 \). Also, \( \langle \hat{P}^2 |\alpha\rangle = |\alpha\rangle \) for any \( \alpha \), which shows that the square of \( \hat{W}_0 \) is the identity operator

\[ \hat{P}^2 = \hat{1}. \]  
(F4)

Since

\[ \langle n | - \alpha \rangle = \langle n | \hat{P} |\alpha\rangle = \langle \hat{P}^\dagger n |\alpha\rangle \]  
(F5)

and since \( \hat{W}_0 \) is Hermitian,

\[ \hat{P} = \hat{P}^\dagger, \]  
(F6)

and since

\[ \langle n | - \alpha \rangle = (-)^n \langle n |\alpha\rangle, \]  
(F7)

we see that

\[ \hat{W}_0(n) = (-)^n \langle n \rangle, \]  
(F8)

the sign indicating whether \( n \) is even or odd. From equation (F1), we see that

\[ \hat{a} \hat{P} = -\hat{P} \hat{a} \quad \hat{a}^{\dagger} \hat{P} = -\hat{P} \hat{a}^{\dagger}, \]  
(F9)

and using equations (F4) and (F6), we see that equation (F9) gives

\[ \hat{P} \hat{a} \hat{P} = -\hat{a} \quad \hat{P} \hat{a}^{\dagger} \hat{P} = -\hat{a}^{\dagger}. \]  
(F10)

### Appendix G. The Wigner distribution of number states

Here we present a derivation of the Wigner distribution of a number state. The Wigner seed operator, equation (C2),

\[ \hat{W}_0 = 2 e^{-2\alpha^{\dagger} - \alpha^2}, \]  
(G1)

anticommutes with the operators

\[ (\hat{a} - \alpha) \hat{W}_0 = -\hat{W}_0(\hat{a} - \alpha), \]  
(G2)

\[ (\hat{a}^{\dagger} - \alpha^2) \hat{W}_0 = -\hat{W}_0(\hat{a}^{\dagger} - \alpha^2), \]  
(G3)

equation (F9) is a special case for \( \alpha = 0 \). From equations (G1) and (G2) we can obtain differential operators on the Wigner seed operator \( \hat{W} \) which bring down an \( \hat{a} \) to its left or an \( \hat{a}^{\dagger} \) to its right

\[ \hat{a} \hat{W}_0 = (\alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*}) \hat{W}_0, \]  
(G4)

\[ \hat{W}_0 \hat{a}^{\dagger} = (e^{\alpha^2} - \frac{1}{2} \frac{\partial}{\partial \alpha}) \hat{W}_0, \]  
(G5)

which we will use to raise the number of excitations. Thus, the Wigner distribution of a number state is

\[ W^{(n)} = \langle n | \hat{W} |n\rangle \]  
(G6)

\[ = \langle 0 | (\hat{a}^{\dagger})^n \hat{W}_0 (\hat{a})^n | 0 \rangle \]  
(G7)

\[ = (\alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*})^n (e^{\alpha^2} - \frac{1}{2} \frac{\partial}{\partial \alpha})^n \hat{W}^{(0)}. \]  
(G8)

Using that

\[ \frac{W^{(n)}}{W^{(0)}} = \left( 2 \alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*} \right)^n (e^{\alpha^2} - \frac{1}{2} \frac{\partial}{\partial \alpha})^n \]  
(G9)

\[ = \left( \frac{\partial}{\partial \alpha} \right)^n e^{\beta(2\alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*})} e^{\gamma(2\alpha - \frac{1}{2} \frac{\partial}{\partial \alpha})} \]  
(G10)

\[ = \left( \frac{\partial}{\partial \alpha} \right)^n e^{\beta(2\alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*})} \]  
(G11)

Taking the derivatives with respect to \( \gamma \),
Moving the exponential to the left by displacing the derivatives, and setting \( \beta = \gamma = 0 \) in the exponent, we get

\[
\frac{W^{(n)}}{W^{(0)}} = \left( \frac{\partial}{\partial \beta} + 2\alpha - \gamma \right)^n (2\alpha^n - \beta)^n \bigg|_{\beta=0, \gamma=0} \tag{G12}
\]

\[
= \left( \frac{\partial}{\partial \beta} + 2\alpha \right)^n (2\alpha^n - \beta)^n \bigg|_{\beta=0} \tag{G13}
\]

which can be written as a Schwinger ordered-exponential

\[
\frac{W^{(n)}}{W^{(0)}} = \frac{\partial^n}{\partial \eta^n} \exp \left[ i n \left( \frac{1}{2} \frac{\partial}{\partial \beta} - 2\alpha \right) (\beta - 2\alpha \eta) \right] \bigg|_{\beta=0, \eta=0} \tag{G14}
\]

\[
= \frac{\partial^n}{\partial \eta^n} \frac{1}{\eta - 1} e^{-\frac{\eta}{2}(\beta - 2\alpha \eta) \left( \frac{1}{2} \frac{\partial}{\partial \beta} - 2\alpha \right)} \bigg|_{\beta=0, \eta=0} \tag{G15}
\]

where we used properties of Schwinger ordered-exponentials to switch their order. The derivatives to the right give zero, and we can set giving

\[
W^{(n)} = \frac{W^{(0)}}{\eta^n} \frac{1}{\eta - 1} e^{4\alpha |\eta|^{-2}} \bigg|_{\eta=0} \tag{G16}
\]

which we recognize as the generating function for the Laguerre polynomials

\[
\sum_{n=0}^{\infty} \eta^n L_n(\alpha) = \frac{1}{\eta - 1} e^{4\alpha |\eta|^{-2}}. \tag{G17}
\]

Thus,

\[
W^{(n)} = \frac{1}{\pi} e^{-2|\alpha|^2} L_n(4|\alpha|^2) = W^{(0)} L_n(4|\alpha|^2), \tag{G18}
\]

which is equation (27).

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