Leptonic Decays of Charged Pseudoscalar Mesons – 2012

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Abstract

We review the physics of purely leptonic decays of \( \pi^\pm \), \( K^\pm \), \( D^\pm \), \( D_s^\pm \), and \( B^\pm \) pseudoscalar mesons. The measured decay rates are related to the product of the relevant weak-interaction-based CKM matrix element of the constituent quarks and a strong interaction parameter related to the overlap of the quark and antiquark wave-functions in the meson, called the decay constant \( f_P \). The interplay between theory and experiment is different for each particle. Theoretical predictions of \( f_B \) that are needed in the \( B \) sector can be tested by measuring \( f_{D^+} \) and \( f_{D_s^+} \) in the charm sector. The lighter \( \pi^\pm \) and \( K^\pm \) mesons provide stringent comparisons between experiment and theory due to the accuracy of both the measurements and the theoretical predictions. This review was prepared for the Particle Data Group’s 2012 edition [1].
I. INTRODUCTION

Charged mesons formed from a quark and antiquark can decay to a charged lepton pair when these objects annihilate via a virtual $W$ boson. Fig. 1 illustrates this process for the purely leptonic decay of a $D^+$ meson.

\[
\Gamma(P \rightarrow \ell \nu) = \frac{G_F^2 f_P}{8\pi} m_\ell^2 M_P \left(1 - \frac{m_\ell^2}{M_P^2}\right)^2 |V_{q_1 q_2}|^2.
\] (1)

Here $M_P$ is the $P$ mass, $m_\ell$ is the $\ell$ mass, $V_{q_1 q_2}$ is the Cabibbo-Kobayashi-Maskawa (CKM) matrix element between the constituent quarks $q_1 \bar{q}_2$ in $P$, and $G_F$ is the Fermi coupling constant. The parameter $f_P$ is the decay constant, and is related to the wave-function overlap of the quark and antiquark.

The decay $P^\pm$ starts with a spin-0 meson, and ends up with a left-handed neutrino or right-handed antineutrino. By angular momentum conservation, the $\ell^\pm$ must then also be left-handed or right-handed, respectively. In the $m_\ell = 0$ limit, the decay is forbidden, and can only occur as a result of the finite $\ell$ mass. This helicity suppression is the origin of the $m_\ell^2$ dependence of the decay width.

There is a complication in measuring purely leptonic decay rates. The process $P \rightarrow \ell \nu \gamma$ is not simply a radiative correction, although radiative corrections contribute. The $P$ can make a transition to a virtual $P^*$, emitting a real photon, and the $P^*$ decays into $\ell \nu$, avoiding helicity suppression. The importance of this amplitude depends on the decaying particle and the detection technique. The $\ell \nu \gamma$ rate for a heavy particle such as $B$ decaying into a light particle such as a muon can be larger than the width without photon emission [2]. On the other hand, for decays into a $\tau^\pm$, the helicity suppression is mostly broken and these effects appear to be small.

Measurements of purely leptonic decay branching fractions and lifetimes allow an experimental determination of the product $|V_{q_1 q_2}| f_P$. If the CKM element is well known from other measurements, then $f_P$ can be well measured. If, on the other hand, the CKM element is not well measured, having theoretical input on $f_P$ can allow a determination of the CKM element. The importance of measuring $\Gamma(P \rightarrow \ell \nu)$ depends on the particle being considered. In the case of the $B^-$ the measurement of $\Gamma(B^- \rightarrow \tau^- \nu)$ provides an indirect determination of $|V_{ub}|$ provided that $f_B$ is provided by theory. In addition, $f_B$ is crucial for using measurements of $B^0 - \bar{B}^0$ mixing to extract information on the fundamental CKM elements.
parameters. Knowledge of $f_{B_s}$ is also needed, but it cannot be directly measured as the $B_s$ is neutral, so the violation of the SU(3) relation $f_{B_s} = f_B$ must be estimated theoretically. This difficulty does not occur for $D$ mesons as both the $D^+$ and $D^+_s$ are charged, allowing the direct measurement of SU(3) breaking and a direct comparison with theory.

For $B^-$ and $D^+_s$ decays, the existence of a charged Higgs boson (or any other charged object beyond the Standard Model) would modify the decay rates; however, this would not necessarily be true for the $D^+$ \[3, 4\]. More generally, the ratio of $\tau \nu$ to $\mu \nu$ decays can serve as one probe of lepton universality \[3, 5\].

As $|V_{ud}|$ has been quite accurately measured in super-allowed $\beta$ decays \[6\], with a value of 0.97425(22) \[7\], measurements of $\Gamma(\pi^+ \to \mu^+ \nu)$ yield a value for $f_\pi$. Similarly, $|V_{us}|$ has been well measured in semileptonic kaon decays, so a value for $f_K$ from $\Gamma(K^- \to \mu^- \bar{\nu})$ can be compared to theoretical calculations. Lattice gauge theory calculations, however, have been claimed to be very accurate in determining $f_K$, and these have been used to predict $|V_{us}|$ \[8\].

**II. CHARMED MESONS**

We review current measurements, starting with the charm system. The CLEO collaboration has performed the only measurement of the branching fraction for $D^+ \to \mu^+ \nu$ \[9\]. CLEO uses $e^+e^-$ collisions at the $\psi(3770)$ resonant energy where $D^-D^+$ pairs are copiously produced. They fully reconstruct one of the $D$’s, find a candidate muon track of opposite sign to the tag, and then use kinematical constraints to infer the existence of a missing neutrino and hence the $\mu\nu$ decay of the other $D$. They find $B(D^+ \to \mu^+ \nu) = (3.82\pm0.32\pm0.09) \times 10^{-4}$. We use the well-measured $D^+$ lifetime of 1.040(7) ps, and assuming $|V_{cd}|$ equals $|V_{us}| = 0.2246(12)$ \[10\], minus higher order correction terms \[11\], we find $|V_{cd}| = 0.2245(12)$. The CLEO branching fraction result then translates into a value of

$$f_{D^+} = (206.7 \pm 8.5 \pm 2.5) \text{ MeV}.$$  

This result includes a 1% correction (lowering) of the rate due to the presence of the radiative $\mu^+\nu\gamma$ final state based on the estimate by Dobrescu and Kronfeld \[11\].

Before we compare this result with theoretical predictions, we discuss the $D^+_s$. Measurements of $f_{D^+_s}$ have been made by several groups and are listed in Table \[12\] \[16\]. We exclude older values obtained by normalizing to $D^+_s$ decay modes that are not well defined. Many measurements, for example, used the $\phi\pi^+$ mode. This decay is a subset of the $D^+_s \to K^+K^-\pi^+$ channel which has interferences from other modes populating the $K^+K^-$ mass region near the $\phi$, the most prominent of which is the $f_0(980)$. Thus the extraction of effective $\phi\pi^+$ rate is sensitive to the mass resolution of the experiment and the cuts used to define the $\phi$ mass region \[17, 18\]. The CLEO, Babar and Belle $\mu^+\nu$ results rely on fully reconstructing all the final state particles except for the neutrino and using a missing-mass technique to infer the existence of the neutrino. CLEO uses $e^+e^- \to D_sD^+_s$ collisions at 4170 MeV, while Babar and Belle use $e^+e^- \to DKn\pi D^+_s$ collisions at energies near the $\Upsilon(4S)$.

When selecting the $\tau^+ \to \pi^+\nu$ and $\tau^+ \to \rho^+\bar{\nu}$ decay modes, CLEO uses both calculation of the missing-mass and the fact that there should be no extra energy in the event beyond that deposited by the measured tagged $D^+_s$ and the $\tau^+$ decay products. The $\tau^+ \to e^+\nu\bar{\nu}$ mode, however, uses only extra energy. BaBar measures $\Gamma(D^+_s \to \tau^+\nu)/\Gamma(D^+_s \to \overline{K^0}K^+)$ using the $\tau^+ \to e^+\nu\bar{\nu}$ mode.
Common systematic errors in the CLEO results have been taken into account. Numbers for \( f_{D^+_s} \) have been extracted using updated values for masses and \(|V_{cs}|\) (see text). Systematic uncertainties for errors on the \( D^+_s \) lifetime and mass are included; radiative corrections have been included. 

| Experiment | Mode | \( \mathcal{B} \) | \( f_{D^+_s} \) (MeV) |
|------------|------|-----------------|-----------------|
| CLEO-c [12] | \( \mu^+\nu \) | \((5.65 \pm 0.45 \pm 0.17) \times 10^{-3}\) | \(257.6 \pm 10.3 \pm 4.3\) |
| BaBar [16] | \( \mu^+\nu \) | \((6.02 \pm 0.38 \pm 0.34) \times 10^{-3}\) | \(265.9 \pm 8.4 \pm 7.7\) |
| Belle [13] | \( \mu^+\nu \) | \((6.38 \pm 0.76 \pm 0.57) \times 10^{-3}\) | \(274 \pm 16 \pm 12\) |
| Average | \( \mu^+\nu \) | \((5.89 \pm 0.33) \times 10^{-3}\) | \(263.0 \pm 7.3\) |
| CLEO-c [12] | \( \tau^+\nu (\pi^+\pi^-) \) | \((6.42 \pm 0.81 \pm 0.18) \times 10^{-2}\) | \(278.0 \pm 17.5 \pm 4.4\) |
| CLEO-c [14] | \( \tau^+\nu (\rho^+\pi^-) \) | \((5.52 \pm 0.57 \pm 0.21) \times 10^{-2}\) | \(257.8 \pm 13.3 \pm 5.2\) |
| CLEO-c [15] | \( \tau^+\nu (e^+\nu\pi^-) \) | \((5.30 \pm 0.47 \pm 0.22) \times 10^{-2}\) | \(252.6 \pm 11.2 \pm 5.6\) |
| BaBar [16] | \( \tau^+\nu (e^+(\mu^+)\nu\pi^-) \) | \((5.00 \pm 0.35 \pm 0.49) \times 10^{-2}\) | \(245.4 \pm 8.6 \pm 12.2\) |
| Average | \( \tau^+\nu \) | \((5.43 \pm 0.31) \times 10^{-2}\) | \(255.7 \pm 7.2\) |

We extract the decay constant from the measured branching ratios using the \( D^+_s \) mass of 1.96847(33) GeV, the \( \tau^+ \) mass of 1.77682(16) GeV, and a lifetime of 0.500(7) ps. We use the first order correction \(|V_{cs}| = |V_{ud}| - |V_{cb}|^2/2\) \[10\]; taking \(|V_{ud}| = 0.97425(22)\) \[6\], and \(|V_{cb}| = 0.04\) from an average of exclusive and inclusive semileptonic \( B \) decay results as discussed in Ref. \[19\], we find \(|V_{cs}| = 0.97345(22)\). CLEO has included the radiative correction of 1% in the \( \mu^+\nu \) rate listed in the Table \[11\] (the \( \tau^+\nu \) rates need not be corrected). Other theoretical calculations show that the \( \gamma\mu^+\nu \) rate is a factor of 40–100 below the \( \mu^+\nu \) rate for charm \[20\]. As this is a small effect we do not attempt to correct the other measurements.

The average decay constant cannot simply be obtained by averaging the values in Table I since there are correlated errors between the \( \mu^+\nu \) and \( \tau^+\nu \) values. Table II gives the average values of \( f_{D_s} \) where the experiments have included the correlations.

**TABLE II:** Experimental results for \( f_{D^+_s} \) taking into account the common systematic errors in the \( \mu^+\nu \) and \( \tau^+\nu \) measurements.

| Experiment | \( f_{D^+_s} \) (MeV) |
|------------|----------------|
| CLEO-c     | 259.0 ± 6.2 ± 3.0 |
| BaBar      | 258.8 ± 6.4 ± 7.5 |
| Belle      | 273.8 ± 16.3 ± 12.2 |
| Average of \( \mu^+\nu + \tau^+\nu \) | 260.0 ± 5.4 |

Our experimental average is

\[
f_{D^+_s} = (260.0 \pm 5.4) \text{ MeV}.
\]
Furthermore, the ratio of branching fractions is found to be

$$R \equiv \frac{\mathcal{B}(D_s^+ \to \tau^+ \nu)}{\mathcal{B}(D_s^+ \to \mu^+ \nu)} = 9.2 \pm 0.7, \quad (2)$$

where a value of 9.76 is predicted in the Standard Model. Assuming lepton universality then we can derive improved values for the leptonic decay branching fractions of

$$\mathcal{B}(D_s^+ \to \mu^+ \nu) = (5.75 \pm 0.24) \times 10^{-3}, \quad \text{and}$$

$$\mathcal{B}(D_s^+ \to \tau^+ \nu) = (5.61 \pm 0.24) \times 10^{-2} \quad (3)$$

The experimentally determined ratio of decay constants is $f_{D^+_s}/f_{D^+} = 1.26 \pm 0.06$.

Table III: Theoretical predictions of $f_{D^+_s}$, $f_{D^+}$, and $f_{D^+_s}/f_{D^+}$. Quenched lattice calculations are omitted, while PQL indicates a partially-quenched lattice calculation. (Only selected results having errors are included.)

| Model                      | $f_{D^+_s}$ (MeV) | $f_{D^+}$ (MeV) | $f_{D^+_s}/f_{D^+}$ |
|----------------------------|-------------------|----------------|---------------------|
| Experiment (our averages)  | 260.0 ± 5.4       | 206.7 ± 8.9    | 1.26 ± 0.06         |
| Lattice (HPQCD) [21]       | 248.0 ± 2.5       | 213 ± 4        | 1.164 ± 0.018       |
| Lattice (FNAL+MILC) [22]   | 260.1 ± 10.8      | 218.9 ± 11.3   | 1.188 ± 0.025       |
| PQL [23]                   | 244 ± 8           | 197 ± 9        | 1.24 ± 0.03         |
| QCD sum rules [24]         | 205 ± 22          | 177 ± 21       | 1.16 ± 0.01 ± 0.03  |
| QCD sum rules [25]         | 245.3 ± 15.7 ± 4.5| 206.2 ± 7.3 ± 5.1| 1.193 ± 0.025 ± 0.007|
| Field correlators [26]     | 260 ± 10          | 210 ± 10       | 1.24 ± 0.03         |
| Light front [27]           | 268.3 ± 19.1      | 206 (fixed)    | 1.30 ± 0.04         |

Table III compares the experimental $f_{D^+_s}$ with theoretical calculations [21–27]. While most theories give values lower than the $f_{D^+_s}$ measurement, the errors are sufficiently large, in most cases, to declare success. The largest discrepancy (2.0 standard deviations) is with an unquenched lattice calculation [21].

Upper limits on $f_{D^+}$ and $f_{D_s}$ of 230 and 270 MeV, respectively, have been determined using two-point correlation functions by Khodjamirian [28]. The $D^+$ result is safely below this limit, while the average $D_s$ result is also, but older results [1] not used in our average are often above the limit.

Akeroyd and Chen [29] pointed out that leptonic decay widths are modified in two-Higgs-doublet models (2HDM). Specifically, for the $D^+$ and $D^+_s$, Eq. (1) is modified by a factor $r_q$ multiplying the right-hand side [30]:

$$r_q = \left[ 1 + \left( \frac{1}{m_c + m_q} \right) \left( \frac{M_{D_q}}{M_{H^+}} \right)^2 \left( m_c - \frac{m_q \tan^2 \beta}{1 + \epsilon_0 \tan \beta} \right) \right]^2,$$

where $m_{H^+}$ is the charged Higgs mass, $M_{D_q}$ is the mass of the $D$ meson (containing the light quark $q$), $m_c$ is the charm quark mass, $m_q$ is the light-quark mass, and $\tan \beta$ is the ratio of the vacuum expectation values of the two Higgs doublets. In models where the fermion
mass arises from coupling to more than one vacuum expectation value $\epsilon_0$ can be non-zero, perhaps as large as 0.01. For the $D^+$, $m_d \ll m_c$, and the change due to the $H^+$ is very small. For the $D_s^+$, however, the effect can be substantial.

A major concern is the need for the Standard Model (SM) value of $f_{D^+}$. We can take that from a theoretical model. Our most aggressive choice is that of the unquenched lattice calculation [21], because it claims the smallest error. Since the charged Higgs would lower the rate compared to the SM, in principle, experiment gives a lower limit on the charged Higgs mass. However, the value for the predicted decay constant using this model is 2.0 standard deviations below the measurement. If this small discrepancy is to be taken seriously, either (a) the model of Ref. [21] is not representative; (b) no value of $m_{H^+}$ in the two-Higgs doublet model will satisfy the constraint at 99% confidence level; or (c) there is new physics, different from the 2HDM, that interferes constructively with the SM amplitude such as in the R-parity-violating model of Akeroyd and Recksiegel [31].

To sum up, the situation is not clear. To set limits on new physics we need an independent calculation of $f_{D_s}$ with comparable accuracy, and more precise measurements would also be useful.

III. THE $B$ MESON

The Belle and BaBar collaborations have found evidence for $B^- \rightarrow \tau^- \bar{\tau}$ decay in $e^+e^- \rightarrow B^-B^+$ collisions at the $\Upsilon(4S)$ energy. The analysis relies on reconstructing a hadronic or semi-leptonic $B$ decay tag, finding a $\tau$ candidate in the remaining track and or photon candidates, and examining the extra energy in the event which should be close to zero for a real $\tau^-$ decay to $e^-\nu\bar{\nu}$ or $\mu^-\nu\bar{\nu}$ opposite a $B^+$ tag. The results are listed in Table IV.

| Experiment Tag | $B$ (units of $10^{-4}$) |
|----------------|--------------------------|
| Belle [32] Hadronic | $1.79^{+0.36}_{-0.49} +0.36$ |
| Belle [33] Semileptonic | $1.54^{+0.38}_{-0.37} +0.29$ |
| Belle Our average | $1.62 \pm 0.40$ |
| BaBar [34] Hadronic | $1.80^{+0.57}_{-0.54} +0.26$ |
| BaBar [35] Semileptonic | $1.7 \pm 0.8 \pm 0.2$ |
| BaBar Average [34] | $1.76 \pm 0.49$ |
| Our average | $1.68 \pm 0.31$ |

There are large backgrounds under the signals in all cases. The systematic errors are also quite large, on the order of 20%. Thus, the significances are not that large. Belle quotes $3.5\sigma$ and $3.6\sigma$ for their hadronic and semileptonic tags, respectively, while BaBar quotes $3.3\sigma$ and $2.3\sigma$, again for hadronic and semileptonic tags. We note that the four central values are remarkably close to the average considering the large errors on all the measurements. More accuracy would be useful to investigate the effects of new physics.
We extract a SM value using Eq. (1). Here theory provides a value of $f_B = (194 \pm 9)$ MeV. We also need a value for $|V_{ub}|$. Here significant differences arise between using inclusive charmless semileptonic decays and the exclusive decay $B \to \pi \ell^+ \nu$. The inclusive decays give rise to a value of $|V_{ub}| = (4.27 \pm 0.38) \times 10^{-3}$ while the exclusive measurements yield $|V_{ub}| = (3.38 \pm 0.36) \times 10^{-3}$, where the errors are dominantly theoretical. Their average, enlarging the error in the standard manner because the results differ, is $|V_{ub}| = (3.80 \pm 0.44) \times 10^{-3}$. Using these values and the PDG values for the $B^+$ mass and lifetime, we arrive at the SM prediction for the $\tau^- \bar{\nu}$ branching fraction of $0.96 \pm 0.24 \times 10^{-4}$. This value is about a factor of two smaller than the measurements. There is a 6.6% probability that the data and the SM prediction are consistent. This difference is more clearly seen by examining the correlation between the CKM angle $\beta$ and $B(B^- \to \tau^- \bar{\nu})$. The CKM fitter group provides a fit to a large number of measurements involving heavy quark transitions. The point in Fig. 2 shows the directly measured values, while the predictions from their fit without the direct measurements are also shown. There is about a factor of two discrepancy between the measured value of $B(B^- \to \tau^- \bar{\nu})$ and the fit prediction.

![FIG. 2: Measured versus predicted values of $B(B^- \to \tau^- \bar{\nu})$ versus $\sin 2\beta$ from the CKM fitter group. The point with error bars shows the measured values, while the predictions are in colors, with the color being related to the confidence level. (From the CKM Fitter group.)](image)

IV. CHARGED PIONS AND KAONS

We now discuss the determination of charged pion and kaon decay constants. The sum of branching fractions for $\pi^- \rightarrow \mu^- \bar{\nu}$ and $\pi^- \rightarrow \mu^- \bar{\nu} \gamma$ is 99.98770(4)%%. The two modes are difficult to separate experimentally, so we use this sum, with Eq. (1) modified to include photon emission and radiative corrections. The branching fraction together with the lifetime 26.033(5) ns gives

$$f_{\pi^-} = (130.41 \pm 0.03 \pm 0.20) \text{ MeV}.$$  

The first error is due to the error on $|V_{ud}|$, 0.97425(22); the second is due to the higher-order corrections, and is much larger.
Similarly, the sum of branching fractions for \( K^- \to \mu^- \bar{\nu} \) and \( K^- \to \mu^- \bar{\nu}\gamma \) is 63.55(11)\%, and the lifetime is 12.3840(193) ns \[41\]. Measurements of semileptonic kaon decays provide a value for the product \( f_K(0)|V_{us}| \), where \( f_K(0) \) is the form-factor at zero four-momentum transfer between the initial state kaon and the final state pion. We use a value for \( f_K(0)|V_{us}| \) of 0.21664(48) \[41\]. The \( f_K(0) \) must be determined theoretically. We follow Blucher and Marciano \[7\] in using the lattice calculation \( f_K(0) = 0.9644 \pm 0.0049 \) \[42\], since it appears to be more precise than the classic Leutwyler-Roos calculation \( f_K(0) = 0.961 \pm 0.008 \) \[43\]. [Other recent averages are \( 0.956 \pm 0.008 \) \[49\], and \( 0.9588 \pm 0.0044 \) \[44\].] Using the value from Ref. \[42\], the result is \( |V_{us}| = 0.2246 \pm 0.0012 \), consistent with the hyperon decay value of 0.2250 \pm 0.0027 \[45\]. We derive

\[
f_K^- = (156.1 \pm 0.2 \pm 0.8 \pm 0.2) \text{ MeV}.
\]

The first error is due to the error on \( \Gamma \); the second is due to the CKM factor \( |V_{us}| \), and the third is due to the higher-order corrections. The largest source of error in these corrections depends on the QCD part, which is based on one calculation in the large \( N_c \) framework. We have doubled the quoted error here; this would probably be unnecessary if other calculations were to come to similar conclusions. A large part of the additional uncertainty vanishes in the ratio of the \( K^- \) and \( \pi^- \) decay constants, which is

\[
f_K^-/f_{\pi^-} = 1.197 \pm 0.002 \pm 0.006 \pm 0.001.
\]

The first error is due to the measured decay rates; the second is due to the uncertainties on the CKM factors; the third is due to the uncertainties in the radiative correction ratio.

These measurements have been used in conjunction with calculations of \( f_K/f_{\pi} \) in order to find a value for \( |V_{us}|/|V_{ud}| \). Three recent lattice predictions of \( f_K/f_{\pi} \) are 1.189 \pm 0.007 \[46\], 1.192 \pm 0.007 \pm 0.006 \[47\], and 1.197 \pm 0.002 \pm 0.003 \[48\], yielding an average by the FLAG group of 1.195 \pm 0.005 \[49\]. [There is also a new value 1.1872 \pm 0.0041 (statistical errors only) \[50\].] Together with the precisely measured \( |V_{ud}| \), this gives an independent measure of \( |V_{us}| \) \[8, 41\].

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We have not included a BaBar result: $B(D_s^+ \to \mu^+\nu) = (6.67 \pm 0.83 \pm 0.26 \pm 0.66) \times 10^{-3}$ and $f_{D_s^+} = (281 \pm 17 \pm 7 \pm 14)$ MeV based on $B(D_s^+ \to \phi\pi^+) = (4.71 \pm 0.46)\%$. These measurements determined the ratio of the leptonic decay to a hadronic decay, usually $\Gamma(D_s^+ \to \ell^+\nu)/\Gamma(D_s^+ \to \phi\pi^+)$. See B. Aubert \textit{et al.} (BABAR Collaboration), Phys. Rev. Lett. 98, 141801 (2007).

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