Error sensitivity of a Quantum Simulator I: 
a first example

G. Strini

Dipartimento di Fisica dell'Università
Via Celoria 16 Milano - Italy

Abstract

As the first useful Quantum Computers presumably will be quantum simulators, here the minimum number of qubits necessary for the solution of Schrödinger’s equation in simple test problems is evaluated. From the present preliminary results it appears that it is possible to realize a useful quantum simulator with a register of only 10-15 qubits. The intrinsic sensitivity to some errors appears to be moderate without the need of error correcting methods. At present there is at least some indication that the amplitude errors are more dangerous than phase and decoherence errors. So corrections limited to amplitude errors may be very useful.

1. Introduction

After the pioneering work of Shor on the factorization [1] at present there is a growing interest in the study of quantum computers (QC) and the related problems of error control due to inaccuracies of the quantum gates, decoherence and so on (for a still useful review see [2]). As it is by now well known the QC may have an exponential advantage over the classical ones on many problems (for a very incomplete list see [3, 4, 5, 6, 7]), but due to implementation difficulties the first QC will not have many qubits. So it is almost obvious that the first useful QC will be quantum simulators and not true numerical computers.

This is the first of a series of papers devoted to the simulation of a quantum computer for the solution of a generic quantum mechanical problem. The point of view is that of an experimentalist wanting to know how many qubits are necessary and what amount of errors it is possible to tolerate. So the present problem is to find the minimum requirements for the setting-up of a useful simulator to solve simple problems.

The simulation of a quantum computer on a classical one was often done in the past [8], but for very specific problems. Here we study a system of general use to solve standard undergraduate problems in quantum mechanics as can be found in an exercise book [9]. Such a quantum simulator (QS) was simulated on a classical computer. Obviously the efficiency of such a simulation is exponentially slow, so nothing is gained in comparison with a direct solution of the various problems studied. From the experience gained so far, it is fair to say that a simulation of a QS on a
classical computer approximately requires a computational power similar to the one necessary for the direct solution on the classical computer, except for the simulation of some errors. The simulation of the various kind of errors is the most uncertain part of this work, because at present there are not sufficient experimental data on the sources of errors in the various implementations of real systems. It was then necessary to start with very simple hypotheses. Various problems were simulated: transmission and reflection of a Gaussian wavepacket on various barriers, unidimensional harmonic and anharmonic oscillators, the solution of the radial equation of the tridimensional harmonic oscillator etc.. Here we report only the results about the unidimensional harmonic oscillator as the results of this simpler case are very similar to those of the other problems studied.

2. Solution of the Schrödinger equation

The algorithm we will use has been described elsewhere \[10, 11\] so let us merely show the essential points. The Schrödinger equation:

$$i\hbar \frac{d}{dt} \psi(x; t) = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right] \psi(x; t) \equiv [H_0 + V(x)] \psi(x; t) \quad (1)$$

may be easily solved on a QC provided that it is possible to decompose the algorithm in a sequence of unitary operations enough simple and, what is more important, efficiently implementables on a simple quantum computer.

The first step is to decompose the region of interest of the space coordinate in $2^n$ intervals of length $\Delta$ and to represent these intervals in the Hilbert space of a $n$ qubits register. This space is the direct product of $n$ copies of the space of a single qubit:

$$|i_{n-1} \otimes i_{n-2} \otimes \cdots \otimes i_0\rangle = |i\rangle \quad (2)$$

where $(i_{n-1}, i_{n-2}, ..., i_0)$ is the binary decomposition of the index $i$.

Then, neglecting the simbol $\otimes$, the wavefunction $\psi(x; t)$ is represented by the superposition:

$$\psi(x; t) = \sum_{i=0}^{2^n-1} C_i(t) |i\rangle = \sum_{i_{n-1}=0}^{1} \sum_{i_{n-2}=0}^{1} \cdots \sum_{i_0=0}^{1} C_{i_{n-1},i_{n-2},...,i_0}(t) |i_{n-1}i_{n-2}\cdots i_0\rangle =$$

$$= \frac{1}{N} \sum_{i=0}^{2^n-1} \psi(x_i = (i + 0.5) \Delta; t)|i\rangle \quad (3)$$

where $N$ is a normalization factor as the samples are not normalized. From this representation it is evident the exponential advantage to use a QS instead of a classical
computer: a register of \( n \) qubits can store \( 2^n - 1 \) complex numbers. This exponential advantage is completely lost in the simulation of the QS on a classical computer.

To implement the various operations to solve the (1) it is necessary to extend the Hilbert space of the QS given by the register \( |n\rangle \) with an ancilla register \( |m\rangle_a \) of \( m \) qubits. So the complete Hilbert space of the QS now reads:

\[
|m\rangle_a \otimes |n\rangle \quad (4)
\]

For the simulation of some errors of this QS on a classical computer it is necessary to add in the classical simulator the simulation of other qubits \( |l\rangle_l \) to describe unwanted quantum degrees of freedom or a rough representation of the environment. The complete Hilbert space of the classical simulation of the QS thus is:

\[
|l\rangle_l \otimes |m\rangle_a \otimes |n\rangle \quad (5)
\]

and for the classical simulator it is necessary to store \( 2^{(l+m+n)} \) complex numbers. The experience gained shows that the practical limits of the simulation appears to be due more to problems of storage than to computational power.

The Eq.(1) may be formally integrated by propagating the initial wavefunction every timestep \( \epsilon \):

\[
\psi(x; t + \epsilon) = e^{-\frac{i}{\hbar} [H_0 + V(x)] \epsilon} \cdot \psi(x; t) \quad (6)
\]

Using a time interval \( \epsilon \) small enough to neglect terms in \( \epsilon^2 \) it is possible to write:

\[
e^{-\frac{i}{\hbar} [H_0 + V(x)] \epsilon} \approx e^{-\frac{i}{\hbar} H_0 \epsilon} \cdot e^{-\frac{i}{\hbar} V(x) \epsilon} \quad (7)
\]

These operators are still unitary, simpler than the initial one and efficiently implementable on a QC. As the Fourier transform \( \mathcal{F} \) is also easily implemented [2]: by calling \( k \) the transformed variable, it is possible to write the first operator in the right hand side of (7) as:

\[
e^{+\frac{i}{\hbar} (\frac{\hbar^2}{2m} k^2) \epsilon} = U \quad (8)
\]

and so in the space representation:

\[
e^{-\frac{i}{\hbar} H_0 \epsilon} \equiv \mathcal{F}^{-1} \cdot U \cdot \mathcal{F} \quad (9)
\]

The problem is now reduced to the implementation of an operator of the form:

\[
e^{i f(x)} |x\rangle \quad (10)
\]

This is possible using an ancilla quantum register \( |m\rangle_a \) by means of the following steps:

\[
|0\rangle_a \otimes |x\rangle \rightarrow |f(x)\rangle_a \otimes |x\rangle \rightarrow e^{i f(x)} |f(x)\rangle_a \otimes |x\rangle \rightarrow \quad (11)
\]
\[ e^{i f(x)} |0\rangle_a \otimes |x\rangle \equiv |0\rangle_a \otimes e^{i f(x)} |x\rangle \quad (12) \]

The first step is a standard binary representation of the function \( f(x) \) on the ancilla register and may be implemented by means of c-not gates. The second step is the operation \( e^{im|m\rangle} \) done on the ancilla qubits and so efficiently implemented by means of standard techniques with phase gates giving a phase shift to every qubit proportional to the weight of the qubit. The third step is just the reverse of the first one and immediately implemented by the same array of gates as the first one but applied in the reversed order. It may be useful to note that, in the examples studied, the implementation required hundreds of c-not gates. As the ancilla qubits are returned to the standard configuration it is possible to use the same ancilla qubits for the kinetic energy and the potential energy computations. The potential energy part of the (7) is implemented in a similar way. In conclusion this algorithm can be implemented with no problems, except for some tricks necessary to minimize the number of the space and ancilla qubits.

3. Dimension of the registers

As long as at present qubits are a very expensive resource, the first important problem to be solved is to determine the minimum number of qubits. Here we devote a special care to this problem as it is usually totally neglected.

For the computation are needed \( n \) qubits to represent \( 2^n \) points of the coordinate space and \( m \) ancilla qubits are also needed for the computation of the potential and the kinetic energy with the resolution of \( 2^m \) steps. The most simple determination is the number \( n \) of the \( 2^n \) space points. This problem is common to every numerical representation of functions and is solved in a well known way by referring to the sampling theorem. As this theorem [12] is of central importance to keep the number of the qubits as small as possible it may be useful to remember some details. This theorem allows to compute the errors in reconstructing a function from its samples. It is enough to remember that the sampled function may be represented as the product of the original function times a train of Dirac delta functions. As long as the transform of a train of Dirac deltas is still a train of Dirac deltas, one gets replicas of the Fourier transform of the original function, shifted in frequency by the interval determined by the sampling frequency. So for a function whose spectrum is limited to a maximum frequency it is enough to sample at a frequency at least the double of this maximum frequency to avoid the superposition of the tails of the spectrum. At a lower sampling frequency there is an error in reconstruction due to tails superposition. This kind of error is referred to as aliasing error. To be concrete in the present problem it was assumed an initial wavefunction with a Gaussian shape. Since the Fourier transform is still a Gaussian so that the spectrum goes to zero very rapidly a surprisingly small number of samples was sufficient helping to keep the space register small.

The implementation of functions is relatively simple: it is enough to "write" the values of the function in the binary representation in an ancilla register of \( m \) qubits with a resolution of \( 2^m \) steps. This writing is realized by means of an array of c-not gates (the details will be given elsewhere). A hard task is the evaluation of the number
$m$ of the ancilla qubits necessary for the potential and kinetic energy computations. For this problem unfortunately there is not a theorem similar to the sampling theorem but only a less useful one on the characteristic function of the probability density of the function to be digitized (for a tutorial on these problems see [13] and cited references). So it was necessary to use some numerical tests.

An unexpected result was that usually the number of the ancilla qubits outweights the number of the qubits representing the space coordinate. The most sensitive part of this problem was the computation of the kinetic energy after the Fourier transform of the space wavefunction.

A first trick to keep this number as small as possible is to use wavefunctions smooth in the space coordinates, in order not to have strong components of short wavelength. To have a concrete idea of what is happening, in problems with a simple potential as the harmonic oscillator in the case of an initial Gaussian wavepacket with a spread about $1/50$ of the range of the space coordinate, and using only $n = 6$ qubits, i.e. $2^6 = 64$ points for the space coordinate, the components of the Fourier transform of the wavefunction are populated only in the very lower part of the spectrum. In this case, for the computation of the kinetic energy up to $m = 12$ ancilla qubits were necessary to reach a sufficient resolution. A second trick to keep this number small was to compute by means of the ancilla the kinetic energy limited to the most populated part of the spectrum and to disregard the rest, hence using the full resolution of $2^m$ steps only for the used part of the spectrum. Typical values used after various tests, were $n = 6$ space qubits and $m = 7$ ancilla qubits. A total of only $13$ qubits was sufficient for most tests.

4. Sensitivity to memory errors

A further problem is the evaluation of the sensitivity of the simulator to errors. In the past various studies of the sensitivity to errors were done on numerical QCs [14, 15, 16, 17] and quantum computers dedicated to special problems [7], but here we concentrate on simulators able to solve a generic quantum mechanical problem. It is necessary to define exactly which kind of errors are to be studied and then to give an evaluation of the effect of these errors. The effect of the errors depends on the observable of interest. Here the effect was estimated by computing the fidelity defined by the comparison of the erroneous wavefunction $|\psi\rangle_{err}$ with the correct one $|\psi\rangle$ computed using the same system and zero errors.

$$f = |\langle\psi|\psi\rangle_{err}|^2$$

(13)

This quantity gives the probability that the erroneous wavefunction will pass a test for the correct one. The first kind of errors studied are the memory errors. These errors are simulated by applying a generic unitary matrix with random parameters to one qubit at a time of the register representing the space coordinate. This matrix:

$$U = \begin{bmatrix} \cos\theta & \exp(i\alpha) \cdot \sin\theta \\ -\exp(i\beta) \cdot \sin\theta & \exp(i(\alpha + \beta) \cdot \cos\theta) \end{bmatrix}$$

(14)

is applied at every timestep with random parameters with uniform distribution be-
tween plus and minus a given maximum (only a single parameter at a time is different from zero: as the errors are assumed to be small the probability of two or more simultaneous errors is taken to be negligible). To ease the comparison of the effect of different errors the same string of random numbers was used in all the tests. To be more precise, the integration of Schrödinger’s equation was done for 40 timesteps and the error was applied every step between steps 10 and 30.

The dependence of the fidelity on the error parameter, the maximum amplitude of the error and the qubit in error have been explored. In the parametrization used the parameters $\alpha$ and $\beta$ have an analogous effect so it is enough to study only the effect of the errors described by the $\alpha$ and $\theta$ parameters. For brevity we refer as $\alpha$, $\beta$ and $\theta$ errors the errors due to the $U$ matrix with only the specified parameter different from zero. Also for error $\theta = 0.1$ radians we mean a series of $U$ matrices with only the parameter $\theta$ different from zero and random with uniform distribution between $\pm 0.1$ radians. Similarly for the other parameters. The results show a moderate effect of errors $\alpha$ (and so also $\beta$), and a strong effect of the $\theta$ errors.

The errors are applied from the steps 10 to 30 of the 40 integration, so that the fidelity starts with the value 1, remains in this value until the first application of the error matrix and then decreases randomly to the minimum when the last error is applied. In Table 1 we report this final value of the fidelity resulting from the error matrix applied to the various qubits of the register representing the space coordinate.

| qubit | $\alpha = 0.30$ | $\theta = 0.05$ | $\theta = 0.10$ | $\theta = 0.20$ | $\theta = 0.30$ |
|-------|----------------|----------------|----------------|----------------|----------------|
| 0     | 0.90           | 0.96           | 0.96           | 0.83           | 0.66           |
| 1     | 0.90           | 0.99           | 0.96           | 0.85           | 0.70           |
| 2     | 0.90           | 0.99           | 0.97           | 0.89           | 0.76           |
| 3     | 0.83           | 0.99           | 0.96           | 0.86           | 0.71           |
| 4     | 0.96           | 0.98           | 0.91           | 0.69           | 0.43           |
| 5     | 0.98           | 0.98           | 0.93           | 0.75           | 0.53           |

The first column indicates the qubit acted-on by the error matrix (14). The second column labeled ($\alpha = 0.30$) gives the final fidelity obtained with errors of maximum 0.30 radians on the parameter $\alpha$ of the matrix (14). The following 4 columns give the final fidelity for errors with the parameter $\theta$ respectively of maximum 0.05, 0.10, 0.20 and 0.30 radians. By comparing the second and the last column it appears that the average effect of the errors in parameter $\alpha$ are almost negligible in comparison with the error in parameter $\theta$. It is possible to note an unexpected low sensitivity to $\theta$ errors as high as 0.10 radians, as shown in the 4th column. Another unexpected result is the low correlation of the fidelity with the number $n$ of the qubit in error (of weight $2^n$) indicated in the first column.

The figures 1 and 2 give a plot of $|\psi(x; t)|^2$ for $\theta$ errors in the qubits ”0” and ”5”. It is useful to give a physical interpretation of these errors. This error gives a probability of reversal of the value of the qubit acted-on by the error. Upon indicating
with \( \Delta \) the step in the space coordinate, the \( \theta \) error in the qubit "0" gives a reshuffling of the wavefunction values in adjacent intervals. A \( \theta \) error in the qubit "1" gives an analogous reshuffling for intervals \( 2\Delta \), and so on for the \( \theta \) errors in the other qubits of higher weight.

As the wavefunction used is a Gaussian wavepacket there are at least two different regimes depending on the width of the reshuffling interval, namely if it is smaller or greater than the width of the wavepacket. This effect is clearly evident in figure 1 and 2, referring respectively to \( \theta \) errors in the qubits "0" and "5" and so referring to space intervals of length \( \Delta \) and \( 32\Delta \) respectively.

5. Effect of the leak errors

Other sources of errors are quantum leaks, gate errors and decoherence. Here we limit ourselves to leak errors. The quantum leaks are due to unwanted interactions of the qubits of the working register of the QS with quantum degrees of freedom not involved in the computation. To simulate these errors it is necessary to add some auxiliary qubits \( |l \rangle_l \) to the simulator Hilbert space. So this space now is given by (5).

Here we study the effect of the leak errors due to the interaction of a single qubit of the space register with a single extra quantum degree of freedom described by a single qubit starting in a pure state. So it is possible to represent this interaction by means of a unitary 4x4 matrix applied to a two qubit space given by the qubit supposed to be acted by the error and the leak qubit at the specified times and tracing away the ancilla leak qubit at the end of the computation. Also in this case the error matrix is applied every timestep between steps 10 and 30. The error matrix is now 4x4 and so it is in general described by too many parameters to be useful. So in the absence of sufficient experimental data we did select two particular very interesting matrices \( U_1 \) and \( U_2 \) depending from a single parameter given by [18]:

\[
U_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & -\sin \theta \\
0 & 0 & 1 & 0 \\
0 & \sin \theta & 0 & \cos \theta
\end{bmatrix} \quad U_2 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
0 & \sin \theta & 0 & -\cos \theta \\
0 & 0 & 1 & 0
\end{bmatrix}
\]

(15)

To have an idea of the reason of the choice of these matrices, it is useful to note that the matrix \( U_1 \), when applied to a system composed of a single qubit and a single ancilla qubit gives a pure decoherence. In fact denoting with \( X, Y, Z \) the parameters of the Poincaré sphere of the density matrix of a single qubit:

\[
\rho = \frac{1}{2} \begin{bmatrix}
1 + Z & X - iY \\
X + iY & 1 - Z
\end{bmatrix}
\]

(16)

the matrix \( U_1 \) applied to a system composed of a single qubit with this density matrix and an ancilla qubit in the pure state \( |0 \rangle \) after the trace of the ancilla qubit gives for the system reduced density matrix the transformation:

\[
X' = \cos \theta X \quad Y' = \cos \theta Y \quad Z' = Z
\]

(17)
and so the coherence \((X - iY)\) is reduced. The analogous computation with the matrix \(U_2\) gives:

\[
X' = \cos \theta X \quad Y' = \cos \theta Y \quad Z' = \sin^2 \theta + \cos^2 \theta Z \quad (18)
\]

and so the coherence is reduced and there is also a shift and a factor lower than 1 of the \(Z\) component of the reduced density matrix representing the population. The fidelity resulting from errors due to the application of the \(U_1\) and the \(U_2\) matrices to the various qubits of the space register was computed in a way similar to the preceding case but with the reduced matrix obtained by tracing away the ancilla qubit \(|l\rangle\).

Similar to the previous case, the fidelity starts with the value 1 and then randomly decreases to the final value when the last error matrix is applied. The parameter \(\theta\) is supposed to be random with uniform distribution between plus and minus the maximum value indicated and the error matrices are applied from the timestep 10 to the timestep 30.

| qubit | \(\theta = 0.05\) | \(\theta = 0.10\) | \(\theta = 0.30\) |
|-------|-----------------|-----------------|-----------------|
| 0     | 1.0             | 0.99            | 0.82            |
| 1     | 1.0             | 0.98            | 0.89            |
| 2     | 1.0             | 0.99            | 0.79            |
| 3     | 1.0             | 0.99            | 0.77            |
| 4     | 1.0             | 1.0             | 0.95            |
| 5     | 1.0             | 1.0             | 0.93            |

| qubit | \(\theta = 0.05\) | \(\theta = 0.10\) | \(\theta = 0.30\) |
|-------|-----------------|-----------------|-----------------|
| 0     | 0.99            | 0.76            |
| 1     | 1.0             | 0.98            | 0.83            |
| 2     | 1.0             | 0.97            | 0.82            |
| 3     | 1.0             | 0.94            | 0.78            |
| 4     | 0.99            | 0.97            | 0.79            |
| 5     | 1.0             | 0.97            | 0.66            |

In Table 2 we report the final fidelity for leak errors on the various qubits of the register of space coordinate. The first column indicates the qubit acted-on by the leak.
The following three columns give the final fidelity for various values of the maximum of the error parameter $\theta$ in the matrix $U_1$. The Table 3 give the analogous quantity for the matrix $U_2$. In both cases there is a low correlation of the fidelity with the weight of the qubit in error.

The leak error is due to the transfer of some of the quantum information on the system to a further quantum degree of freedom. But as the computation of the fidelity is done by neglecting this degree of freedom and calculating the reduced density matrix, part of the quantum information is regained. This effect may explain the (lower than expected) effect of the quantum leak, even with an error as high as the 0.30 radians of the parameter used in the matrices $U_1$ and $U_2$.

In figure 3 and 4 we report a plot of $|\psi(x; t)|^2$ for the errors due to the matrix $U_1$ applied to the space qubit ”0” and ”5” for the error $\theta = 0.1$ radians, corresponding to the third column of Table 2. It is possible to note a very small effect of this kind of error. In figure 5 and 6 are reported the analogous results for the matrix $U_2$. The matrix $U_2$ generates, along with decoherence errors, also population errors. In this case it is possible to distinguish at least two different regimes. This last kind of error gives an effect similar to the $\theta$ error from the matrix $U$ shown in Table 1 and figure 1 and 2. As in the case of the figure 1, in figure 5 it is evident a granularity due to the reshuffling of the wavefunction values in adjacent space intervals. This granularity changes to a random noise in figure 6 referring to errors in qubit ”5” as in the corresponding $U$ case reported in figure 2.

By comparing the results obtained from the matrix $U_1$ (decoherence) and those from the matrix $U_2$ (decoherence and population), it appears that the last kind of errors is more dangerous.

6. Conclusions and future developments

The first aim of this work was to have reliable data on how many qubits are necessary to implement a QS for the solution of a generic and simple quantum mechanical problem. Surprisingly the number of ancilla qubits necessary for the implementation of the various operators outweighs the number of the qubits representing the space degree of freedom of the system. But for the simple problems studied here, with some care a total of only 13 qubits proved to be sufficient. Another useful result is that the QS appears to be quite robust against the explored errors: an error of 0.1 radians on all the parameters indicated gives a fidelity higher than about 0.9. Obviously for other observables and for a different measure of the error this conclusion may be not valid. The fidelity is largely independent from the weight of the qubit in error.

To ease the comparison of the memory errors as defined here with other definitions, it is sufficient to say that the $\alpha$ and $\beta$ errors are similar to the phase errors and the $\theta$ error is similar to the amplitude error. So it is possible to say that in this QS the amplitude errors are more dangerous than the phase errors.

The most important development is to enlarge the single leak qubit to a larger space in order to model the environment. The interaction of a single qubit with the environment has been modeled by various authors[19, 20]; it is known that two ancilla qubits are sufficient, but one needs to trace away these qubits at every integration step in order to wash-out any memory of the system taken by the environment. Hence
it is necessary to describe the system by means of the density matrix. This does not create any basic problem, but only problems of storage and computational power of the classical computer.

Acknowledgments

This work was supported by M.P.I. I would like to thank prof. V.G. Benza for his careful reading of the manuscript and the many useful suggestions.

(* ) E-mail giuliano.strini@mi.infn.it

References
[1] P. W. Shor. Proc. 35th Ann. Symp. Found. Comp. Sci. IEEE Comp. Soc. Pr. 124 (1994).
[2] A. Eckert and R. Jozsa. Reviews of Modern Physics 68, 733 (1996).
[3] S. Lloyd. Science 273, 1073 (1996).
[4] D. S. Abrams and S. Lloyd. Phys. Rev. Lett. 79, 2586 (1997).
[5] D. A. Lidar and H. Wang. Phys. Rev. E59 2429 (1999).
[6] D. A. Lidar and O. Biham. Phys. Rev. E56 3661 (1997).
[7] G. Benenti, G. Casati, S. Montangero and D. L. Shepelyansky. arXiv:quant-ph/0107036.
[8] J. Wallace Quantum Computer Simulators. A Review. http://www.dcs.ex.ac.uk/~jwallac.
[9] I. I. Gol’dman et al.. Problems in Quantum Mechanics. Infosearch Limited. London 1960.
[10] C. Zalka. Fortschr. Phys. 46, 877 (1988).
[11] S. Wiesner. arXiv:quant-ph/9603028.
[12] F. M. Reza. An introduction to information theory. McGraw-Hill 1961.
[13] A. Grassi and G. Strini. Alta frequenza XXXIII E547 (1964).
[14] K. M. Obenland and A. M. Despain. Proc. of High-Performance Computing HPC’98 228 (1998).
[15] C. Miquel, J. P. Paz and R. Perazzo. Phys. Rev. A54 2605 (1996).
[16] A. Barenco and A. Ekert. Phys. Rev. A54 139 (1996).
[17] C. Miquel, J. P. Paz and W. H. Zurek. Phys. Rev. Lett. 78 3971 (1997).
[18] Chi-Sheng Niu and R. Griffiths. Phys. Rev. A60, 2764 (1999).
[19] B. Schumacher. Phys. Rev. A54, 2614 (1966).
[20] G. Strini. Unpublished Lecture Notes (2000).
Figure captions

Figure 1. Plot of $|\psi(x; t)|^2$ obtained with the memory error matrix $U$ applied to the qubit "0" of weight $\Delta$. The granulosity due to the reshuffling of the wavefunction values at a distance $\Delta$ is clearly visible.

Figure 2. Plot of $|\psi(x; t)|^2$ obtained with the memory error matrix $U$ applied to the qubit "5" of weight $32\Delta$. The reshuffling of the wavefunction values at a distance $32\Delta$ originates a random noise.

Figure 3. Plot of $|\psi(x; t)|^2$ obtained with the leak error matrix $U_1$ applied to the qubit "0" of weight $\Delta$. This leak originates a decoherence error and a very low noise.

Figure 4. Plot of $|\psi(x; t)|^2$ obtained with the leak error matrix $U_1$ applied to the qubit "5" of weight $32\Delta$. As in the preceding figure 3 this leak originates a decoherence error and a very low noise.

Figure 5. Plot of $|\psi(x; t)|^2$ obtained with the leak error matrix $U_2$ applied to the qubit "0" of weight $\Delta$. This leak originates a decoherence error and an amplitude error. The amplitude error in the qubit "0" originates a reshuffling of the wavefunction values at a distance $\Delta$ and so a granulosity very similar to the one due to memory errors reported in figure 1.

Figure 6. Plot of $|\psi(x; t)|^2$ obtained with the leak error matrix $U_2$ applied to the qubit "5" of weight $32\Delta$. This leak originates a decoherence error and an amplitude error. The amplitude error in the qubit "5" originates a reshuffling of the wavefunction values at a distance $32\Delta$ and so originates a random noise similar to the one due to memory errors reported in figure 2.
