A PERIOD DOUBLING ROUTE TO SPATIOTEMPORAL CHAOS IN A SYSTEM OF GINZBURG-LANDAU EQUATIONS FOR NEMATIC ELECTROCONVECTION

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(Communicated by Stephen Pankavich)

Abstract. In this paper we investigate the transition from periodic solutions to spatiotemporal chaos in a system of four globally coupled Ginzburg Landau equations describing the dynamics of instabilities in the electroconvection of nematic liquid crystals, in the weakly nonlinear regime. If spatial variations are ignored, these equations reduce to the normal form for a Hopf bifurcation with $O(2) \times O(2)$ symmetry. Both the amplitude system and the normal form are studied theoretically and numerically for values of the parameters including experimentally measured values of the nematic liquid crystal Merck I52. Coexistence of low dimensional and extensive spatiotemporal chaotic patterns, as well as a temporal period doubling route to spatiotemporal chaos, corresponding to a period doubling cascade towards a chaotic attractor in the normal form, and a kind of spatiotemporal intermittency that is characteristic for anisotropic systems are identified and characterized. A low-dimensional model for the intermittent dynamics is obtained by perturbing the eight-dimensional normal form by imperfection terms that break a continuous translation symmetry.

1. Introduction. In dissipative systems far from equilibrium, regular periodic states may undergo transitions to persistent spatiotemporal complex dynamics, a state known as spatiotemporal chaos (STC), which is unpredictable in both space and time and characterized by a fast decay of spatial and temporal correlations. It can be observed in physical systems when many spatial degrees of freedom contribute to the dynamics, like thermal convection [41, 22], nematic electroconvection [20], various pattern-forming chemical reactions, parametrically forced surface waves, and other large sustained nonequilibrium systems (see [9, 10, 50] for a review).

The problems of finding general approaches for the characterization of STC, as well as of identifying the instability mechanisms generating it, have been explored now for over two decades and are still challenging questions in nonlinear dynamics. While different scenarios for the transition to chaos in systems with few degrees of freedom have been established (see, e.g. [3, 45]), in spatially extended systems the transition from regular patterns to STC is still poorly understood. A possible
route to STC is via spatiotemporal intermittency (STI), a state of “weak turbulence” in which ordered (laminar) and disordered (turbulent) domains in space and time coexist \([6, 55, 52, 19]\).

In this paper we report on a study of the transition from periodic solutions to spatiotemporal chaos, via a temporal period doubling cascade, in a system of four globally coupled Ginzburg Landau equations describing the dynamics at the onset of a pattern forming instability in the electroconvection of nematic liquid crystals.

The period doubling cascade is one of the most intensively studied routes to chaos in low-dimensional dynamical systems. It has been rigorously proven to occur in parameter-dependent families of 1d unimodal iterated maps \([24, 7]\), the prototype of which is the logistic map \([39]\). In this scenario, a periodic state undergoes an infinite sequence of period doublings that accumulate at a parameter value beyond which chaos and periodic windows occur in alternation. This scenario is particularly intriguing owing to its universal scaling (and other, qualitative) properties in both parameter distance and amplitude ratio of consecutive period doublings. These scaling laws are quantified by two universal constants \([24, 7]\), referred to as Feigenbaum numbers, which have been confirmed experimentally in a variety of physical experiments including Rayleigh-Bénard convection \([34, 35]\), driven semiconductor oscillators \([56]\), optical bistability \([27]\), superfluids \([53]\), and Josephson junction analog devices \([63]\). As pointed out in \([36]\), measuring the Feigenbaum numbers and other, qualitative features of the period doubling scenario is challenging due to the noise sensitivity and the high reduction factor of the oscillation amplitudes at successive period doublings. According to \([36]\), such measurements require very good stability in fine-tuning and almost adiabatic changes of the parameters, see \([38]\) for a related discussion.

The above systems are all low dimensional, i.e. can be described by a few dynamical degrees of freedom. In particular, the convection experiments discussed in \([34, 35]\) were conducted in containers with small aspect ratios with up to six convective rolls present. Theoretically it is a common practice to approximate the governing PDE’s for small aspect ratio convection systems by a few-mode Galerkin-truncated system of ODE’s such as, for example, the Lorenz system \([37]\), which exhibits various period doubling transitions to chaos \([54]\). By contrast, the system of Ginzburg Landau equations studied in this paper describes instabilities in 2d anisotropic systems of much larger spatial extent than the intrinsic wavelengths of the patterns observed beyond the onset, which can lead to complexity in both space and time.

In attempts towards understanding spatiotemporal complexity, much effort has been devoted to the study of coupled map lattices as models for STC \([28, 29, 55]\). Coupled map lattices reside on 1d or 2d lattices, usually with periodic boundary conditions, where at each site the evolution of a continuous state space variable is governed by a local discrete-time map (often the logistic map) with additional couplings to the states of neighboring sites. In these systems, a variety of dynamical states including states characterized as defect turbulence and fully developed turbulence \([28]\), have been observed and studied. There are classes of coupled map lattices that are somewhat related to our Ginzburg Landau system for the selected parameter regime, but there are also important differences as will be discussed in Section 4.

A period doubling cascade leading to STC with one extended direction has been observed experimentally in “ramped” Taylor Couette vortex flow, where the inner
cylinder is deformed to an azimuthally varying cylinder with “hourglass geometry” [62]. These experiments were inspired by the theoretical observation that a slow variation of the external, patterning inducing parameter can lead to alternating phase slips and drifts from (Eckhaus-) stable to unstable wave numbers [31, 49]. This scenario depends, however, crucially on the non-homogeneity of the ramp and is not amenable to generic amplitude or phase equations.

Another important bifurcation that occurs in low-dimensional, parametrically forced systems such as the Mathieu equation [42], is the subharmonic bifurcation. This bifurcation is also a period doubling bifurcation, but it is usually not part of a period doubling cascade. In the context of spatiotemporal systems it occurs in parametrically forced fluid systems (Faraday experiment [23, 21]) and in vertically vibrated granular media. In the latter, the subharmonic bifurcation has been observed to occur in combination with a pattern forming instability leading to various kinds of ordered (squares and hexagons) patterns and more complicated and disordered patterns [40] as well as spatially localized oscillations [59]. As emphasized in [40], the observed subharmonic bifurcations in these experiments are not associated with a Feigenbaum period doubling sequence.

The system of Ginzburg Landau equations studied in this paper is calculated from the partial differential equations for electroconvection in nematic liquid crystals. Nematic liquid crystals differ from ordinary, isotropic liquids by the fact that their molecules are on average locally oriented along a preferred direction, called the director. For electroconvection, the nematic is sandwiched between two glass plates and an electric potential is applied across the electrode plates. At a critical value of the applied voltage, an electrohydrodynamic instability of the uniform state occurs that leads to the onset of electroconvection. Due to its anisotropy, nematic electroconvection provides a rich variety of pattern formation phenomena and has become a main paradigm for the study of anisotropic dissipative structures in spatially extended systems, ranging from periodic patterns of convection rolls to complex spatiotemporal states including worms, defects and STC [32, 5].

Since in anisotropic systems the minima of the neutral stability surfaces are isolated [17], they admit a unique reduced description through Ginzburg Landau type amplitude equations at the onset, whose form is mainly determined by the symmetries of the governing PDEs and the type of the instability [17, 13]. In particular, waves created in an instability can travel only in certain directions relative to the axis of the anisotropy. In this paper, we study the case of a Hopf bifurcation leading to the instability of oblique travelling rolls. For this instability, the reflection symmetries imply that the dynamics is governed by the interactions of waves travelling in four directions. For the parameter range considered, the chaotic dynamics observed in the experiments then arises immediately at the onset of convection and is characterized by defects in various wave components.

In [15, 43] we have used the system of four globally coupled complex Ginzburg Landau equations (GCCGLE), derived in [17], to study the weakly nonlinear evolution of the travelling wave envelopes in the weak electrolyte model (WEM) of Treiber and Kramer [57] for the nematic electroconvection. The WEM equations are extremely complicated, therefore a weakly nonlinear analysis near the onset is particularly useful.

One of the most important aspects of the WEM is the presence of a Hopf bifurcation that predicts the travelling wave patterns observed frequently near onset in experiments. In [43] we presented a systematic bifurcation analysis of the
WEM equations, for the nematic liquid crystal I52, when the primary instability is the Hopf bifurcation to oblique travelling rolls. Here we focus on the complex spatiotemporal dynamics occurring at the onset of this instability, including bistability of low-dimensional and extensive spatiotemporal chaotic solutions, as well as a temporal period doubling route to spatiotemporal chaos corresponding to a period doubling cascade towards a chaotic attractor in the normal form, to which the GCCGLE reduce if spatial modulations are ignored.

The paper is organized as follows. In Section 2 we introduce the GCCGLE and describe the basic properties of the solutions; in Section 3 we present a numerical study of the normal form dynamics and the spatiotemporal dynamics of the system of amplitude equations and identify a temporal period doubling route to spatiotemporal chaos. Simulations in the chaotic parameter regime show that bistability between low-dimensional chaos and extensive STC occurs. When parameters are varied further, a transition from this bistability to a switching-dynamics between symmetry-conjugated periodic or chaotic sets is observed, which we identify with a new kind of spatiotemporal intermittency characteristic for anisotropic systems. We show that an imperfect version of the normal form, obtained by adding additional small symmetry-breaking terms, captures the essential features of the switching-dynamics and hence proves to be a low-dimensional model for this kind of intermittency. In Section 4 we present concluding remarks and discuss our findings in relation to other studies of spatiotemporal systems.

2. Oblique Hopf bifurcation in the weak electrolyte model and the system of globally coupled Ginzburg Landau equations. To make the paper self-contained, we provide in this section some details about the derivation of the amplitude equations of the Ginzburg Landau type from the weak electrolyte model, along with the underlying oscillatory instability and its basic wave patterns. The Ginzburg-Landau formalism allows a complete classification of the basic wave patterns occurring near threshold and their stabilities in axially anisotropic, dissipative system with two extended dimensions. In such systems, the axial anisotropy induces reflection and translation invariance in both extended directions, thus the underlying symmetry group is $E(1) \times E(1)$ which compactifies to $O(2) \times O(2)$ if periodic boundary conditions are imposed.

2.1. The weak electrolyte model. In dimensionless units the evolution equations of the WEM are [57]

$$D_1 \sigma = -\alpha^2 \pi^2 \nabla \cdot (\mu \mathbf{E} \rho) - (r/2)(\sigma^2 - 1 - P_1 \pi^2 \alpha \rho^2),$$  

$$D_1 \mathbf{n} = \omega \times \mathbf{n} + d(\lambda \mathbf{A} \mathbf{n} - \mathbf{h}),$$  

$$P_1 D_1 \rho = -\nabla \cdot (\mu \mathbf{E} \sigma), \quad \rho = \nabla \cdot (\mathbf{E}),$$  

$$P_2 D_1 \mathbf{v} = -\nabla p - \nabla \cdot (\mathbf{T} + \mathbf{II}) + \pi^2 \rho \mathbf{E}, \quad \nabla \cdot \mathbf{v} = 0.$$  

where $D_1 = \partial_t + \mathbf{v} \cdot \nabla$, $\sigma$ is the local conductivity resulting from the ion dissociation-recombination dynamics, $\mathbf{n}$ is the director ($|\mathbf{n}| = 1$) giving the local orientation of the molecules, $\mathbf{v}$ is the fluid’s velocity, $\mathbf{E}$ is the electrical field, and $\rho$ is the charge distribution. The other fields and parameters occurring in (1)-(4) are defined below.

Electroconvection is driven by an external electrical field. We assume a constant electrical field in the $z$-direction of strength $\sqrt{2R/\pi}$,

$$\mathbf{E} = (\sqrt{2R/\pi}) \mathbf{e}_3 - \nabla \Phi,$$  

(5)
which is superimposed by an internally generated field \(-\nabla \Phi\) so that the first equation in (3) is actually an equation for the electric potential \(\Phi\).

The other derived fields in (1)-(4) are the vorticity, \(\omega = (1/2)(\nabla \times \mathbf{v})\), and the molecular field
\[
h = (\frac{\partial f}{\partial n} - \nabla \cdot \frac{\partial f}{\partial \nabla n}) - \varepsilon_a \pi^2 (\mathbf{n} \cdot \mathbf{E}) \mathbf{E},
\]
derived from the elastic energy density \(f\),
\[
2f = K_1 (\nabla \cdot \mathbf{n})^2 + K_2 [\mathbf{n} \cdot (\nabla \times \mathbf{n})]^2 + K_3 (\mathbf{n} \times (\nabla \times \mathbf{n}))^2,
\]
due to splay \((K_1)\), twist \((K_2)\), and bend \((K_3)\) deformations.

We use the scaling introduced in [57], in which the height of the layer is normalized to \(\pi\) and time is measured in terms of the director relaxation time, \(\tau_d\). This time unit as well as the other units are expressed in terms of various material parameters (in particular \(K_1\) is rescaled to unity), we refer to [57] for the details, see also [15, 43].

Assuming the common planar alignment of the director, the coordinate system is chosen such that \(\mathbf{n} = (1, 0, 0)\) at the upper and lower plates located at \(z = \pm \pi/2\). The rigid boundary conditions (blocking electrodes and no fluid motion) at the top and bottom of the layer then become
\[
\frac{\partial \sigma}{\partial z}, n_2, n_3, \Phi, v_i = 0 \text{ at } z = \pm \pi/2.
\]

The terms \(\mu, \epsilon, d, \Pi, T\) in (1)-(4) are the scaled mobility, dielectric, projection, Ericksen stress and viscous stress tensors, respectively, with components
\[
\mu_{ij} = \delta_{ij} + \sigma_a n_i n_j, \quad \epsilon_{ij} = \delta_{ij} + \epsilon_a n_i n_j, \quad d_{ij} = \delta_{ij} - n_i n_j, \quad \Pi_{ij} = \frac{\partial f}{\partial n_{k,i}} n_{k,i},
\]
\[-T_{ij} = \alpha_1 n_i n_j n_k n_l A_{kl} + \alpha_2 n_i N_i + \alpha_3 n_i N_j + \alpha_4 A_{ij} + \alpha_5 n_j n_k A_{kl} + \alpha_6 n_i n_k A_{kl},
\]
where \(\mathbf{n} = (n_1, n_2, n_3), \mathbf{N} = \mathbf{d}(\lambda A \mathbf{n} - \mathbf{h})\), and \(\varepsilon_a\) and \(\sigma_a\) are the anisotropies of the dielectric permittivity and the electric conductivity at the equilibrium, respectively. The (scaled) Leslie coefficients \(\alpha_1, \ldots, \alpha_6\) and the parameter \(\lambda\) in (2) are not independent and can be expressed in terms four independent parameters (rescaled Miesowicz coefficients) \(\eta_0, \eta_1, \eta_2, \eta_3\), as
\[
\lambda = \eta_1 - \eta_2, \quad \alpha_1 = \eta_0 - 2\eta_1 - 2\eta_2 + 2\eta_3 + 1, \quad \alpha_2 = -(1 + \lambda)/2, \quad \alpha_3 = (1 - \lambda)/2,
\]
\[
\alpha_4 = 2\eta_3, \quad \alpha_5 = 2\eta_1 - 2\eta_3 - (1 + \lambda)/2, \quad \alpha_6 = 2\eta_2 - 2\eta_3 - (1 - \lambda)/2.
\]

Lastly, \(P_1, P_2, r\) are Prandtl-type time scale ratios, \(P_1 = \tau_q/\tau_d, P_2 = \tau_{visc}/\tau_d, r = \tau_q/\tau_{rec}\), where \(\tau_q, \tau_{visc}, \tau_{rec}\) are the charge relaxation time, viscous relaxation time, and the ion recombination time. The parameter \(\alpha\) is proportional to \(\sqrt{\mu_+ \mu_-}\), where \(\mu_{\pm}\) are the mobilities of the charge carrying ions, and plays also the role of a Prandtl-type time-scale ratio (mobility parameter).

Overall the nondimensionalized WEM equations (1)-(4) depend on the main bifurcation parameter \(R\) (amplitude of the external electrical field), four Prandtl-type time-scale ratios \(P_1, P_2, r, \alpha\), the anisotropy coefficients \(\varepsilon_a\) and \(\sigma_a\), and the material parameters ratios \(K_2, K_3, \eta_0, \eta_1, \eta_2, \eta_3\). Typically, the Prandtl numbers \(P_1, P_2\) are very small compared to the other parameters (by factors \(10^{-3} - 10^{-6}\)), thus most studies including our previous work [15, 43] consider the limits \(P_1 = P_2 = 0\) (zero charge and viscous relaxation times).
The WEM equations are invariant under the reflections
\[(x, n_2, v_1) \rightarrow (-x, -n_2, -v_1),\]
\[(y, n_2, v_2) \rightarrow (-y, -n_2, -v_2),\]
(fields that preserve their signs are suppressed). Assuming an infinitely extended fluid layer in both \(x\) and \(y\), the equations are also invariant under the translations \((x, y) \rightarrow (x + \xi, y + \eta)\) but not rotationally invariant, thus the overall symmetry in \((x, y)\) is \(E_1 \times E_1, \ E_1\) being the symmetry group of translations and reflections on a line. The two reflection symmetries make the WEM a particular case of an axially anisotropic system, with a distinguished axis (the \(x\)-axis) and reflection invariance along and across this axis.

2.2. Oscillatory instabilities in the weak electrolyte model. For all values of the parameters, the basic state (no fluid motion) of (1)-(4) is \(\sigma = 1, \ u = 0, \ \Phi = 0, \ n = (1, 0, 0), \ p = \text{const.}\) When \(R\) is increased from zero, this solution becomes unstable at a critical value \(R_c\) that marks the onset of electroconvection. The value of \(R_c\) is found from the linearized equations of (1)-(4) for perturbational fields \((\delta \sigma, \delta n, \delta u, \delta v)\). To simplify the notation, we denote the vector of these variables by \(\delta \mathbf{u}\). Owing to the translation invariance in \((x, y)\) and \(t\), \(\delta \mathbf{u}\) can be represented by horizontal Fourier modes in the form
\[\delta \mathbf{u} = e^{\beta t} e^{i(px+qy)} \tilde{\mathbf{u}}(z, p, q),\]
with a linear growth rate \(\beta\) depending on \((p, q)\) and the parameters. Fixing all parameters except \(R\), the stability threshold \(R_c\) is the smallest value of \(R\) for which \(\text{Re}(\beta) = 0\) for some critical wave numbers \((p_c, q_c)\). If \(\beta = 0\) we have a stationary instability, referred to as steady state bifurcation, and if \(\beta = \pm i\omega_c \neq 0\) (in which case we fix the sign by \(\omega_c > 0\)) the instability is an oscillatory instability, referred to as Hopf bifurcation. In the case of a Hopf bifurcation, \(R_c\) is the minimum of an (oscillatory) neutral stability surface \(R = R_{ns}(p, q)\), defined by the condition of an imaginary growth rate \(\beta = \pm i\omega_{ns}(p, q)\), that is, \(R_c = R_{ns}(p_c, q_c)\) and \(\omega_c = \omega_{ns}(p_c, q_c)\).

Since the system is reflection invariant about both the \(x\)- and \(y\)-axes and has no rotational symmetry, the \(p\)- and \(q\)-axes are symmetry axes of the neutral stability surface. Generically, the minimum is either at the origin, or on a symmetry axis, or off both symmetry axes. Inspired by experiments, we consider here the most interesting case of a Hopf bifurcation in which the minimum is not on a symmetry axis. Due to the reflection symmetries there are then four minima on the neutral stability surface.

In such a case we have four critical wave numbers \((\pm p_c, \pm q_c)\) (we fix the signs by assuming \(p_c > 0, q_c > 0\)) that define two oblique (‘zig’ and ‘zag’) directions for two pairs of counter-propagating travelling wave solutions of the linearized WEM-equations at \(R = R_c\) of the form
\[u_j(t, x, y, z) = \text{Re}\{A_j e^{i(\omega_j t + p_{cj} x + q_{cj} y)} \tilde{u}_j(z)\}, \ 1 \leq j \leq 4,\]
where the \(A_j\) are arbitrary complex amplitudes, \(\tilde{u}_j(z) = \tilde{u}(z, p_{cj}, q_{cj})\), and the signs are specified as
\[p_{c1} = p_{c4} = p_c, \ p_{c2} = p_{c3} = -p_c, \ q_{c1} = q_{c2} = q_c, \ q_{c3} = q_{c4} = -q_c.\]
All other solutions of the linearized WEM equations at \(R = R_c\) are transient (decaying to zero).
2.3. The system of globally coupled complex Ginzburg Landau equations.

In our Ginzburg-Landau analysis of the WEM equations near an oblique Hopf bifurcation, the complex amplitudes $A_j$ are considered as slowly varying envelopes that modulate the travelling wave solutions of the linearized system at $R = R_c$. The dynamical equations for these envelopes follow from a weakly nonlinear, multiple scale analysis. We summarize here the basic procedure of this analysis and refer to [15, 16, 43] for details.

Setting $\varepsilon^2 = R - R_c$, the solution of (1)-(4) for small $\varepsilon > 0$ is represented as

$$u(x, y, z, t) = \text{Re}\{\sum_{j=1}^{4} \varepsilon A_j e^{i(\omega t + p_0 x + q_0 y)} \tilde{u}_j(z)\} + \varepsilon^2 u^{(2)} + \varepsilon^3 u^{(3)} + O(\varepsilon^4),$$

where the $A_j$ depend on the slow variables ($\varepsilon x, \varepsilon y, \varepsilon t$) and a “superslow” time $T = \varepsilon^2 t$. Substituting this solution ansatz into (1)-(4) shows at $O(\varepsilon^2)$ that $A_j = A_j(x_j, T)$, $1 \leq j \leq 4$, with $x_1 = (\xi_+, \eta_+)$, $x_2 = (\xi_-, \eta_+)$, $x_3 = (\xi_-, \eta_-)$, $x_4 = (\xi_+, \eta_-)$, where $\xi_{\pm} = x/v_{\pm}$ and $\eta_{\pm} = y/v_{\pm}$ are slow wave variables related to the critical group velocities $(v_x, v_y) = \nabla_{(p, q)} \omega_{ns}(p, q)|(p, q_c)$. At $O(\varepsilon^3)$ a solvability condition for a nonhomogeneous linear equation related to the linearized WEM equations results in a system of globally coupled complex Ginzburg-Landau equations (GCCGLE) [17] for the envelopes $A_j$. The equation for $A_1$ has the form

$$A_{1T}(\xi_+, \eta_+, T) = \{a_0 + D(\partial_{\xi_+}, \partial_{\eta_+}) + a_1 A_1(\xi_+, \eta_+, \tau)\}^2 + a_2 A_1(\eta_+, T)$$

$$+ a_3 A_2(\xi_+, \eta_+, s, T) + a_4 A_3(\xi_+, \eta_+, s, T) + a_5 A_4(\xi_+, \eta_+, s, T),$$

where $D$ is a complex second order differential operator,

$$D(\partial_{\xi_+}, \partial_{\eta_+}) = D_{11}\partial_{\xi_+}^2 + 2D_{12}\partial_{\xi_+}\partial_{\eta_+} + D_{22}\partial_{\eta_+}^2,$$

and the brackets denote averages over $s$. The equations for $A_2, A_3, A_4$ follow from equation (7) by the symmetry operations [17]

$$\begin{align*}
(x, A_1, A_2, A_3, A_4) & \rightarrow (-x, A_2, A_1, A_4, A_3), \\
(y, A_1, A_2, A_3, A_4) & \rightarrow (-y, A_3, A_1, A_2, A_4), \\
(y, A_1, A_2, A_3, A_4) & \rightarrow (-y, A_4, A_3, A_2, A_1).
\end{align*}$$

The linear and nonlinear coefficients occurring in (7) can be computed numerically from the underlying WEM equations for various choices of the parameters (see [15, 43] for details). The higher order terms $u^{(2)}, u^{(3)}$ in (6) depend on higher harmonics $e^{i(m_\omega t + kp_0 x + lq_0 y)}$ and products and derivatives of the $A_j, \tilde{A}_j$.

To illustrate the nature of the solutions (6), we consider fields for which the vertical modes are identical for all $1 \leq j \leq 4$ (e.g. the electric potential or the pressure). For such fields, the $O(\varepsilon)$-term in (6) is given by $\varepsilon \tilde{u}(z) U(x, y, t)$, where $\tilde{u}(z)$ is the vertical critical mode and $U$ is a superposition of oblique travelling waves modulated by the slowly varying envelopes, which we write more explicitly as

$$U(x, y, t) = \text{Re}\{[A_1(\xi_+, \eta_+, T)e^{i(p_0 x + q_0 y)} + A_2(\xi_-, \eta_+, T)e^{i(-p_0 x + q_0 y)}$$

$$+ A_3(\xi_-, \eta_-, T)e^{i(-p_0 x - q_0 y)} + A_4(\xi_+, \eta_-, T)e^{i(p_0 x - q_0 y)}]e^{i\omega t}\},$$

As is apparent from (9), the envelopes $(A_1, A_3)$ and $(A_2, A_4)$ modulate counterpropagating travelling waves in the directions $\pm (p_0, q_0)$ and $\pm (-p_0, q_0)$, respectively, thus we refer to $(A_1, A_3)$ as “zig”-amplitudes and to $(A_2, A_4)$ as “zag”-amplitudes. Given
2.4. The ODE normal form. The Ginzburg Landau system (7) can be viewed as an extension of an ODE normal form in the context of the equivariant bifurcation theory, derived using the center manifold theorem and normal form techniques [25] from the WEM under the restriction that the fields be periodic in \((x, y)\), with spatial periods \((2\pi/p_c, 2\pi/q_c)\). In general, while normal forms of equivariant bifurcation theory determine spatiotemporal patterns with fixed spatial periods, modulational equations of the Ginzburg Landau type allow to extend these patterns to families of varying periods and to study their modulational stability against periodic perturbations with other periods.

Under the above periodicity restriction, the center eigenspace of (1)-(4) at \(R = R_c\) is eight-dimensional and spanned by the real and imaginary parts of the basic wave solutions \(e^{i(\omega_c t + p_c x + q_c y)}\tilde{u}_j(z)\). The center manifold reduction and normal form transformation can be done using the same ansatz (6) as before, but with the \(A_j\) depending only on \(T\) (are spatially uniform). Thus the ODE normal form is obtained by restricting the GCCGLE to solutions \(A_j\) that are independent of the wave variables \(x_1 = (\xi, \eta)\) etc, resulting in the following ODE for \(A_1\),

\[
\frac{dA_1}{dT} = (a_0 + a_1|A_1|^2 + a_2|A_2|^2 + a_3|A_3|^2 + a_4|A_4|^2)A_1 + a_5A_2A_3A_4, \tag{10}
\]

and similar equations for \(A_2, A_3, A_4\) obtained by the application of the operations (8). This normal form plays the same role for the GCCGLE (7) as does the local nonlinear (e.g. logistic) map for coupled map lattices. Generally, normal form ODE’s associated with Ginzburg Landau-type amplitude equations and the nonlinear maps used in coupled map lattices govern spatially uniform solutions of the corresponding spatiotemporal systems. An important question is when the spatially uniform dynamics is stable in the spatiotemporal setting.

The normal form (10) has six basic periodic solutions corresponding to six basic wave patterns via (9). Each of these solutions is determined by a single complex amplitude \(A = |A|e^{i(\Omega T + \varphi)}\) with an arbitrary phase \(\varphi\). The modulus \(|A|\) and the frequency \(\Omega\) are determined by writing \((A_1, A_2, A_3, A_4)\) in terms of \(A\) as summarized in Table 2.4. Substituting this into (10) gives expressions of \(|A|\) and \(\Omega\) in terms of the nonlinear coefficients \(a_j\). The stabilities of the basic periodic solution in the setting of the ODE-system (10) are also classified through these coefficients (see [17, 51, 61]). Their patterns are obtained by substituting \((A_1, A_2, A_3, A_4)\) into (9) and are summarized for \(\varphi = 0\) in the last column of Table 2.4. The names of the basic periodic solutions are adapted to (and obvious from) the analytical forms of their patterns.

The application of one of the operations in (8) to a basic periodic solution creates a symmetry-conjugated solution corresponding to a wave pattern with a different orientation. For example, the first operation in (8) changes the TW-solution \((A, 0, 0, 0)\) and its associated zig-wave pattern to \((0, A, 0, 0)\) corresponding to a zag-wave. Note that the wave pattern of AW is seen as a periodic alternation between standing waves in the zig- and zag-directions.

For the WEM, the patterns corresponding to the basic periodic solutions are created via (6). Generally for convection systems, a TW-solution induces a travelling roll pattern and a SW-solution a non-moving roll pattern with periodic reversals of the fluid circulation. The two TR-solutions give rise to travelling rectangular
convection cells and the SR-solution to non-moving convection cells with periodic reversals of the fluid motion. An AW-solution manifests itself as periodic alternation between rolls in the zig- and zag-directions.

In addition to the basic periodic solutions, (10) can show more complicated non-transient solutions as attractors, including quasiperiodic solutions and heteroclinic cycles (characterized by cyclic transitions between three or four basic periodic solutions) [51, 61] as well as chaotic attractors [43, 11]. If periodic boundary conditions in \((x, y)\) with periods \((2\pi/p_c, 2\pi/q_c)\) are imposed, the non-transient solutions of the WEM slightly above the convection threshold \(R_c\) are in one-to-one correspondence to the non-transient solutions of (10). Some examples of possible scenarios for WEM parameters of the nematic I52, including bistability, quasiperiodic solutions, chaotic attractors and heteroclinic cycles are presented in [43].

For PDE-systems posed in an infinitely extended \((x, y)\)-plane without periodicity restriction (solutions only required to be bounded), the solutions created by (10) still induce (spatially periodic) solutions, but attractors of (10) may not correspond to stable solutions of the PDEs anymore. Specifically, the basic periodic solutions of the normal form (10) induce wave solutions with critical wave numbers \((p_c, q_c)\). The globally coupled system of complex Ginzburg Landau equations (7) allows to extend these solutions to families of wave solutions with nearby critical wave numbers \((p_c + \varepsilon p, q_c + \varepsilon q)\) and to analyze their (modulational) stability.

In the next section we report on numerical results showing chaotic dynamics in the normal form (10) in relation to spatiotemporal complex dynamics of the associated pattern \(U(x, y, t)\), for parameters calculated for the nematic I52.

### 3. Chaotic normal form dynamics and spatiotemporal chaotic patterns

The rescaled WEM equations depend on the bifurcation parameter \(R\) (the amplitude of the external electric field) and ten dimensionless material parameters usually known experimentally, excepting \(\alpha\) (the mobility parameter) and \(r\) (the recombination parameter) [57]. In our parameter study we fixed eight of the known parameters to the following measured values of the nematic I52,

\[
\begin{array}{cccccccc}
\eta_0 & \eta_1 & \eta_2 & \eta_3 & \epsilon_a & \sigma_a & K_2 & K_3 \\
0.8038 & 0.8769 & 0.0769 & 0.1019 & -0.0089 & 0.38 & 0.875 & 1.25
\end{array}
\]

and varied \(r\) and \(\alpha\). The details of the linear stability analysis and the computation of the critical data \((p_c, q_c, R_c)\) are given in [43]. We identified basic wave patterns occurring stably as solutions of (10), in different regions of the parameter space. The Eckhaus stability of the travelling waves has been investigated in [16]. In regions where no stable basic solutions are predicted, we complemented our analysis with the numerical study of the normal form (10) dynamics and compared it with
Figure 1. Time series $|A_j(T)|$, $1 \leq j \leq 4$, showing the period doubling sequence, obtained from numerical simulations of (10) for (a) $\alpha = 0.024$, (b) $\alpha = 0.025$, (c) $\alpha = 0.02515$, and (d) $\alpha = 0.0252$. In (c) the upper time series in the two plots are for $|A_1|$, $|A_4|$, and the lower time series for $|A_2|$, $|A_3|$.

In the numerical simulation of the spatiotemporal dynamics of the Ginzburg Landau equations (7). In the next subsections we present the results of the parameter study in the case of spatiotemporal complex dynamics of (7) induced by chaotic dynamics in the normal form (10).

3.1. Normal form dynamics: Period doubling cascade to a chaotic attractor. Here we report on a specific sequence of bifurcations observed in numerical simulations of (10) for values of the coefficients $a_0, \ldots, a_5$ calculated from the WEM. The overall bifurcation is a transition from a quasiperiodic attractor to a travelling rectangle via a cascade of period doubling bifurcations and a reverse cascade of period doubling. This scenario occurs for $r = 0.03$ and $\alpha$ varying from 0.024 to 0.026. In Figure 1 we show four time series of the absolute values of the $A_j(T)$, $1 \leq j \leq 4$, for $\alpha = 0.024$, 0.025, 0.02515 and 0.0252.
At $\alpha = 0.024$, the linear stability analysis predicts no stable waves, but the possibility of an attracting quasiperiodic solution $QP_1$ residing in the 4d fixed point subspace $(A_1, A_2, A_3, A_4) = (A, B, A, B)$. This prediction has been confirmed by numerical simulations for several initial conditions. In Figure 1(a) we show the time series of the $|A_j(T)|$ for $QP_1$ resulting from a simulation with initial condition close to a travelling rectangle $(A, 0, 0, A)$.

Slightly above $\alpha = 0.024$ the $QP_1$ becomes unstable, and we observe attraction to another quasiperiodic solution $QP_2$ (not shown) in the 4d subspace $(A, B, B, A)$. When $\alpha$ increases towards 0.025, $QP_2$ also becomes unstable. For this value of $\alpha$ the necessary conditions for the existence of an attracting heteroclinic cycle are satisfied, but we could not find this cycle in our simulations. Instead, we found an attracting orbit that is periodic in the absolute values (presumably also quasiperiodic in the full $(A_1, A_2, A_3, A_4)$-space) with a long period. The time series of this orbit is shown in Figure 1(b). The choice of the initial condition (close to $AW$, which resides in the same 4d subspace as $QP_1$) leads first to $QP_1$ for a certain transient time before the attracting orbit with large period is approached — this is not the case for initial conditions close to $TR_x$. This attracting orbit does not reside in a 4d fixed point subspace and is not a structurally stable heteroclinic cycle (confirmed through numerical simulation up to $T = 5 \times 10^8$). The orbit comes periodically close to $TR_x$ (at the peaks where $|A_2|$ and $|A_3|$ are close to zero). The other peaks do not admit an interpretation as a basic wave. We observe that $|A_1(T)|$ and $|A_4(T)|$ as well as $|A_2(T)|$ and $|A_3(T)|$ have identical wave forms and a phase shift of $\pi$ relative to each other.

For $0.025 < \alpha < 0.0251$ the orbit shown in Figure 1(b) undergoes a period doubling cascade towards a chaotic attractor which persists for $0.0251 \leq \alpha \leq 0.02515$. The time series of the chaotic attractor are shown in Figure 1(c) for $\alpha = 0.02515$. The normal form parameters for this value of $\alpha$ are

$$a_1 = -1 - 1.1896i, \quad a_2 = -0.9501 - 0.5541i, \quad a_3 = -0.7135 + 1.4141i,$$
$$a_4 = -0.2097 - 3.3502i, \quad a_5 = -0.4867 - 0.2386i.$$  \quad (11)

In Figure 2(a), (b) and (c) we present the phase space plots of $|A_4|$ versus $|A_1|$ and $|A_3|$ versus $|A_2|$ for $\alpha = 0.025, 0.025025$ and 0.02515, respectively. When $\alpha$ increases further, we observe for $0.02515 < \alpha < 0.0252$ first a reverse period doubling cascade, then $QP_2$ again, and then, very close to 0.0252, $QP_1$. At $\alpha = 0.0252$, $QP_1$ is unstable again, and the travelling rectangle $TR_x$ is the only attractor which remains so up to $\alpha = 0.026$. A time series demonstrating the approach to $TR_x$ for $\alpha = 0.0252$ (for this value the approach is slow) is shown in Figure 1(d).

Similar calculations for nearby values of $r$, $0.029 < r < 0.031$, show the same scenario, suggesting that the chaotic attractor occurs in a small “island” in the $(r, \alpha)$-plane, surrounded by small concentric regions with quasiperiodic solutions that double their period when moving from the border to the center. Although the scan has been done in a parameter range accessible to experiments, we believe that the experimental detection of the above scenario is challenging, due to the very narrow window in parameters space where it occurs.

The presence of the period doubling cascade in the ODE normal form (10) makes the GCCGLE (7) similar (but with continuous time and space) to 2d coupled map lattices when the basic nonlinear map is the logistic map and the coupling is diffusive. Accordingly, we identify the period doubling cascade shown by (10) with a period doubling route to STC, as is done in studies of coupled map lattices [1, 55].
We next present simulations of the GCCGLE demonstrating bistability of different types of STC as well as intermittency.

3.2. Numerical simulations of the GCCGLE: Bistability of low and high dimensional STC and intermittency. We first present numerical simulations of the Ginzburg Landau system (7), with the data fixed by the values calculated for the WEM parameters for which the normal form (10) shows the chaotic attractor.
described in 3.1, and none of the basic waves is stable. For these parameters (7) shows bistability between a state of extensive STC, with only two envelopes active giving rise to patches of travelling rectangles, and a state of low-dimensional STC, spatially ordered, in which two of the envelopes behave as in the normal form, whereas the other two envelopes show one-dimensional spatial variation. In what follows we describe the main features of these two spatiotemporal chaotic patterns.

In our simulations we use a Fourier–Galerkin approximation of the $A_j$, viewed as functions of $(\xi, \eta, T)$ $(\xi = \xi \pm, \eta = \eta \pm)$,

$$ A_j(\xi, \eta, T) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_j(m, n, T)e^{i(m\xi + n\eta)}, \quad 1 \leq j \leq 4, \quad (12) $$

(see [43] for details concerning the numerical method). Given a numerical solution of (7) at time $T$, we can visualize the associated pattern by plotting $U$ defined in (9). Here we choose $\varepsilon = 0.2$.

We refer to the modes with $m = n = 0$ as central modes. If spatial variations are ignored, these modes represent the $A_j$ and satisfy the system of normal form ODEs (10) along with the corresponding equations for $A_2$, $A_3$, $A_4$.

**Simulation 1: Low dimensional spatiotemporal chaos.** In our first simulation we have chosen random initial values $|a_j(0, 0, 0)| < 1$ for the central modes, and small random perturbations $|a_j(m, n, 0)| < 0.1 \ ((m, n) \neq (0, 0))$ for the non-central modes. To see which modes remain active in the long run, we computed temporal averages $<|a_j(m, n)|>$ over $0 \leq T \leq 1200$. For $j = 2, 3$ only the central modes remain active (average 0.3259 for both $j = 2$ and 3, the averages of all other modes are below 0.0087). For $j = 1, 4$ the central modes are also dominant, but in addition about eight other modes distributed to the left and right of the central modes are active (the averages of the neighboring modes are $\approx 0.24$). Accordingly, $A_2$ and $A_3$ are approximately spatially uniform, whereas $A_1$ and $A_4$ are approximately translation invariant in the $y$-direction.

In Figure 3 a snapshot of the pattern $U$, equation (9), is shown. (In this and the other color-coded plots red is mapped to high and blue to low values.) Here we can recognize regions dominated by approximative stripe patterns and regions dominated by approximative rectangular patterns. Generally, the spatial structure of the patterns observed in this simulation does not appear complex. We observe, however, temporal complexity in the time series of all active mode amplitudes. In Figure 4(a) and (b) we show the time series of the absolute values of all central modes and of the next-to-central modes of $A_1$, $A_4$, respectively, for $0 \leq T \leq 20,000$. The dominant feature of the central mode time series are sharp “downwards” (close to zero) peaks occurring at apparently irregular times. In Figure 4(c) and (d) two different phase plane plots are shown.

It is interesting to note that chaotic dynamics represented in the plot of $|a_3(0, 0)|$ versus $|a_2(0, 0)|$ in Figure 4(d) resembles the (right) plot of $|A_3|$ versus $|A_2|$ in Figure 2(c), however, the plot of $|a_4(0, 0)|$ versus $|a_1(0, 0)|$ in Figure 4(c) appears rather different from, although reminiscent in shape to the (left) plot of $|A_4|$ versus $|A_1|$ in Figure 2(c). Overall, the temporal dynamics resulting from Simulation 1 is a low-dimensional temporal chaos, reminiscent of the chaotic attractor of the normal form for the same values of the coefficients, but with a few additional modes being activated. We note that a similar type of low-dimensional spatiotemporal chaotic
dynamics shown by (7), referred to as “zigzag-chaos”, has been identified and analyzed in [44].

**Simulation 2: Extensive spatiotemporal chaos.** In our second simulation we chose fully random initial conditions. The resulting dynamics turned out to be radically different from that of Simulation 1, as is apparent in the temporal averages of the mode amplitudes shown in Figure 5(a). Now $A_1$ and $A_4$ decay, whereas all modes of $A_2$ and $A_3$ remain active in the long run, indicating spatiotemporal complexity. The temporal complexity can be observed in the time series of the averages of $|a_2(m, n, T)|$ and $|a_3(m, n, T)|$ over $(m, n)$ shown in Figure 5(b). The spatial complexity is revealed in the snapshots of $|A_j|$, $j = 2, 3$, shown in Figure 6(a), as well as in the pattern snapshot displayed in Figure 6(b). Note the steep troughs of $|A_2|$ and $|A_3|$, where the moduli of $A_2$ and $A_3$ are approximately zero, corresponding to the “holes” observed already in simulations of a single 2d complex Ginzburg Landau equation (see [33] for a review). The pattern in Figure 6(b) is dominated by rectangles (for the normal form (10) the $TR_x$-solutions have the form $(A, 0, 0, A)$ or $(0, A, A, 0)$), interlaced with dislocations leading to phase slips and grain boundaries.

In summary, the dynamics observed in this simulation appears temporally and spatially complex and may be identified with ‘extensive spatiotemporal chaos’. Some of the numerically computed mode time series resemble qualitatively those extracted from an experimental pattern that was diagnosed in [12] using a demodulation analysis. Another similarity between the pattern presented in this paper and the experimental pattern is the dominance of the envelopes $A_2$ and $A_3$, although in the experimental pattern the other two envelopes did not decay away but persisted with smaller amplitudes.
Figure 4. Time series and phase space plots of dominant and secondary mode amplitudes plots for Simulation 1.

Figure 5. Averages of the mode amplitudes \(|a_j(m,n,T)|\) for Simulation 2. (a): Time averages. (b) Time series of the \((m,n)\)-averages for \(j = 2, 3\).
Moreover, the extensive STC coexists with the low-dimensional temporally chaotic but spatially ordered pattern described in the previous paragraph, for the same values of the parameters for which the normal form has a chaotic attractor and none of the basic waves is stable. A similar competition between ordered undulations and spatiotemporal undulation chaos has been observed in another anisotropic extended pattern forming system exhibiting spatiotemporal complexity, see [18] and references therein.

Overall, these numerical simulations indicate the existence of a temporal period doubling route towards the low dimensional spatiotemporal chaos, induced by the
period doubling cascade to the chaotic attractor observed in the normal form dynamics, which coexists with an extensive STC-state dominated by holes.

**Simulation 3: Switching dynamics.** The two envelope pairs in the low dimensional STC would (ideally) correspond to rectangles travelling in opposite directions. However, for the nearby parameter values

\[
\begin{align*}
    a_1 &= -1 - 1.1806i, \quad a_2 = -0.923 - 0.5538i, \quad a_3 = -0.6442 + 1.3613i, \\
    a_4 &= -0.2223 - 3.3025i, \quad a_5 = -0.4647 - 0.2472i,
\end{align*}
\]

for which similar chaotic normal form dynamics as shown in Figure 1(c) is observed [11], we find different spatiotemporal dynamics in the solutions of the Ginzburg Landau system (7): the extensively chaotic state has disappeared, and irregular rapid transitions (“switches”) between symmetry-conjugated copies of the low-dimensional STC-state occur. The switches are clearly apparent in the dynamics of the central modes shown in Figure 7(a), and are accompanied by large excursions of the secondary modes as illustrated in Figure 7(b). These switches are likely created through a crisis of two attractors or through a heteroclinic orbit between symmetry-conjugated chaotic saddles.

Overall, four symmetry-conjugated (via the reflection operations in (8)) chaotic attractors of the normal form (10) identified in [11] become two symmetrized (with respect to the \(y\)-reflection) pairs of chaotic sets with connecting, switching inducing orbits between them. The bursts in the time series of the secondary, harmonic \((1,0)\)-modes of the envelopes \((A_1, A_4)\) accompanied by quiescent phases of the same modes for \((A_2, A_3)\) and vice versa seen in Figure 7(b) (similar bursts and quiescent phases occur in the higher \((m,0)\)-modes) show spatiotemporal bursts alternating with spatially homogeneous but temporally chaotic dynamics alternating irregularly between two pairs of envelopes. This is clearly a case of spatiotemporal intermittency for the envelope dynamics that is indicative of in-out intermittency [4]. Since the invariant subspaces underlying this switching dynamics originate from the anisotropy of the spatiotemporal systems (including the WEM) from which the GCCGLE (7) can be derived, we interpret this dynamics as a kind of spatiotemporal intermittency that is characteristic for anisotropic systems.
To gain further insight into the nature of the switching dynamics, we construct a simple, symmetry-broken model, in which the coupling of the first $X$-modes (in the full Ginzburg Landau simulations we observe that the $Y$-modes decay to zero), to the central modes is replaced by constant terms. Denoting the central modes by $z_j = a_{0,0}^{(j)}$, $j = 1, \ldots, 4$, and replacing the first mode amplitudes by a constant $b$, we obtain the following system for the $z_j$,

\begin{align*}
\dot{z}_1 &= a_0 z_1 + \theta z_1 + (a_1|z_1|^2 + a_2|z_2|^2 + a_3|z_3|^2 + a_4|z_4|^2)z_1 + a_5 z_2 z_3 z_4, \\
\dot{z}_2 &= a_0 z_2 + \theta z_2 + (a_1|z_2|^2 + a_2|z_1|^2 + a_3|z_4|^2 + a_4|z_3|^2)z_2 + a_5 z_1 z_4 z_3, \\
\dot{z}_3 &= a_0 z_3 + \theta z_3 + (a_1|z_3|^2 + a_2|z_4|^2 + a_3|z_1|^2 + a_4|z_2|^2)z_3 + a_5 z_4 z_1 z_2, \\
\dot{z}_4 &= a_0 z_4 + \theta z_4 + (a_1|z_4|^2 + a_2|z_3|^2 + a_3|z_2|^2 + a_4|z_1|^2)z_4 + a_5 z_3 z_2 z_1, \\
\end{align*}

(14) where $\theta = (a_4 + a_5)b^2$. The effect of $b \neq 0$ is to break the $y$-translation invariance. Thus the perturbed normal form (14) can also be considered as an imperfect version of the normal form for a Hopf bifurcation with one broken translation (or circular) symmetry. Investigating the effect of breaking a translation symmetry in (10) is another motivation for studying (14), which extends a similar study pursued in [14], motivated by setting up a low-dimensional model describing the effect of distant sidewalls on travelling and standing waves in 1d systems, to the 2d anisotropic case.

The results of simulations of (14) for $b = 0.2$ are shown in Figure 8; we refer to [64] for further properties of this system, including various period doubling cascades, intermittencies and crises. In Figure 8 we observe similar switching dynamics as in our simulations of the Ginzburg Landau system, Figure 7(a), confirming that (14) provides a low-dimensional model for this kind of dynamics characterized by switches between symmetry-conjugated chaotic sets.

4. Conclusions and discussion. We have presented the complex spatiotemporal dynamics shown by a system of four globally coupled complex Ginzburg Landau equations, which govern the dynamics of wave instabilities in two-dimensional anisotropic systems with two translational and two reflectional symmetries. We have found a rich variety of patterns at onset, including complex structures like low-dimensional STC, extensive STC dominated by holes, phase slips and grain boundaries, and intermittent switching between conjugated chaotic sets in different
parameter regimes, some of which were computed from the Weak Electrolyte Model for the material parameters of the nematic liquid crystal Merck I52. In the following we discuss these findings in relation to the dynamics observed and studied in other spatiotemporal systems.

4.1. Coupled map lattices. We have identified the mechanism leading to spatiotemporal complex dynamics with a period doubling route to STC, which is induced by a period doubling cascade to a chaotic attractor in the underlying normal form. This is similar to the class of coupled map lattices (CML’s) whose local dynamics is governed by a logistic map [28, 29, 55]. Regarding symmetries, CML’s also don’t have a continuous spatial rotation invariance due to the discrete space, but in most studies a maximal (square, or almost square) symmetry is retained. By contrast, the GCCGLE have different couplings in the two directions as well as mixed couplings and global couplings. Since most CML’s depend only on two parameters and have a much simpler structure than our globally coupled system of PDE’s, periodic and chaotic solutions are amenable to a far more detailed parameter study numerically as well as theoretically. In particular, the scaling in the bifurcation structure in the 2d parameter plane can be quantified by extending the renormalization approach for 1d unimodal maps [24, 7] to coupled maps to extract additional scaling exponents for the coupling parameter and the coherence length (with different values for linear coupling and feedforward coupling) [1, 30]. The presence of several complex coupling parameters in the GCCGLE and the numerical effort required for determining period doublings of periodic solutions of differential equations prevented us so far from a more systematic parameter study.

There are important qualitative differences of our GCCGLE compared to the 2d CML’s that have been studied, e.g. in [28, 29, 55]. Firstly, the GCCGLE have a high degree of anisotropy revealed in different coupling strengths in the two directions, whereas in 2d CML’s there is usually only one coupling parameter measuring the coupling strength in both directions. Accordingly, the preferred state of 2d CML’s, when the spatially uniform state becomes unstable due to a short wavelength instability, is the so called checkerboard state [28, 29], which is a consequence of the square symmetry. Secondly, the GCCGLE have three continuous translation invariances (spatial translations in $x$ and $y$ and time translations) revealed as phase shift symmetries for the $A_j$, which are not present in the CML’s. Since the period doubling cascade in the normal form occurs in the space of amplitudes, application of these symmetry operations allows to translate any periodic or chaotic attractor orbit along circles in the complex $A_j$-planes. Moreover, the linear couplings in the GCCGLE contain dissipative (real parts of the diffusion coefficients) and dispersive (imaginary parts) components.

A CML-model incorporating these properties to some extent would be the following:

$$u_{t+1}(m,n) = f_\lambda(u_t(m,n)) + c(1+i\alpha) \sum_{\nu=\pm 1} [f_\lambda(u_t(m+\nu,n)) - f_\lambda(u_t(m,n))] + d(1+i\beta) \sum_{\nu=\pm 1} [f_\lambda(u_t(m,n+\nu)) - f_\lambda(u_t(m,n))],$$

where $1 \leq m \leq M$, $1 \leq n \leq N$ are lattice labels (periodically repeated) and $u_t(m,n) \in \mathbb{C}$ is the state at site $(m,n)$ and time $t$. The basic nonlinear map is

$$f_\lambda(u) = u(\lambda - |u|^2), \quad u \in \mathbb{C},$$

where $1 \leq m \leq M$, $1 \leq n \leq N$ are lattice labels (periodically repeated) and $u_t(m,n) \in \mathbb{C}$ is the state at site $(m,n)$ and time $t$. The basic nonlinear map is

$$f_\lambda(u) = u(\lambda - |u|^2), \quad u \in \mathbb{C}, \quad (16)$$
where for simplicity we assume that it depends only on a real parameter, $\lambda > 0$, so that we have rotation invariance in the complex plane. Accordingly, the map leaves each straight line through the origin invariant and shows separate (reflection-related) period doubling cascades on each of the associated two half lines when $\lambda$ is increased, with the first period doubling at $\lambda = 2$, the second at $\lambda = \sqrt{5}$, and the accumulation point at $\lambda_c \approx 2.305$. For larger $\lambda$ there is a symmetry-increasing bifurcation (the orbits on the two half lines merge into one), thus for studying the effect of the standard period doubling cascade on the coupled system we have to stay away from this regime. As coupling we prefer to choose feedforward coupling (using $f_\lambda$) which, although not as universal as linear coupling, appears more robust for the period doubling route in CML’s, see [1].

A more detailed study of this map is pending, but preliminary simulations show that in certain parameter regimes we find qualitative behavior that is similar to the low and high dimensional chaos observed in Simulations 1 and 2 for the GCCGLE. Snapshots showing the spatial states after transients have died out for initial conditions close to the uniform state and fully random initial conditions are shown in Figure 9 (a) and (b), respectively. Note that, due to the anisotropy, the pattern in Figure 9(a) is a stripe pattern instead of a checkerboard pattern.

4.2. Period doubling cascade induced by spatial ramps. As mentioned in the introduction, the period doubling cascade to STC observed in Taylor Couette vortex flow with the inner cylinder deformed to hourglass geometry [62], is theoretically explained by the non-homogeneity (ramp) of the apparatus, which induces a slow variation of the external parameter driving the pattern formation [31, 49]. The ramp can induce a drift in the selected wave number from the Eckhaus-stable band to the Eckhaus-unstable band leading to phase slips, which is repeated in a periodic fashion. As demonstrated in [49] in numerical simulations of a 1d reaction diffusion system, the resulting oscillations may undergo a period doubling cascade leading to STC in systems with one extended direction ($z$-direction in the case of the Taylor
Couette system). This scenario is, however, not amenable to generic amplitude or phase equations due to the non-homogeneity of the ramped system, and we are not aware of attempts to explain this behavior through 1d CML’s. Generally (whether isotropic or anisotropic), phase slips in 1d systems are replaced by defects (or holes) in 2d systems. We refer to [47] for a study of double phase slips in 1d systems and bound defect pairs in 2d systems, and to [26] for the modelling of defect chaos in parametrically driven systems by two locally coupled complex Ginburg Landau equations. Clearly, the high dimensional chaos of Simulation 2 is also dominated by defects and grain boundaries as is the defect-mediated turbulence shown by a single complex Ginzburg Landau equation [33]. We are not aware of a period doubling scenario analogous to the ramp-induced cascade in the case of 2d extended systems.

4.3. Vertically vibrated granular media. Due to the lack of reliable macroscopic equations, a modified complex Ginzburg Landau equation coupled to a conservation of mass law [2, 58], a generalized Swift Hohenberg equation [8], and an iterated map residing in a continuous space with local couplings [60] have been successfully used as models for parametrically forced systems to generate the rich variety of patterns observed in vibrated granular media [40, 59] theoretically. All of these systems are 2d-isotropic, so the ordered patterns shown by them are squares and hexagons consistent with the experiments. By contrast, truly anisotropic systems don’t exhibit square or hexagonal spatial patterns. For example, the normal form (10) underlying the GCCGLE (7) shows standing and travelling rectangles as basic solutions with different stability coefficients for rectangles travelling in the $x$- and $y$-directions. There are generically no travelling squares except when the $O(2) \times O(2)$ (or $D_2 \times T^2$ where $T^2$ is the 2-torus and $D_n$ the dihedral group of order $n$) symmetry degenerates to $D_4 \times T^2$. However, the chaotic normal form dynamics occurs significantly away from this degeneracy, thus the period doubling cascade disappears in the $D_4 \times T^2$-limit. In addition, the subharmonic bifurcation identified in the experiments is not part of a period doubling sequence. It would be interesting though to modify the spatial map from [60] to become anisotropic and perform a weakly nonlinear analysis near strong resonances combined with generic anisotropic pattern formation, to extract canonical Ginzburg Landau-type iterated maps.

Acknowledgment. This research has been supported by the National Science Foundation under Grant No. DMS-1615909. We thank the referees for their useful comments that helped to improve the presentation of the paper.

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Received December 2016; revised October 2017.

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