An Exact Solution of BPS Domain Wall Junction

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Abstract

An exact solution of domain wall junction is obtained in a four-dimensional $\mathcal{N} = 1$ supersymmetric $U(1) \times U(1)'$ gauge theory with three pairs of chiral superfields which is motivated by the $\mathcal{N} = 2$ $SU(2)$ gauge theory with one flavor perturbed by an adjoint scalar mass. The solution allows us to evaluate various quantities including a new central charge $Y_k$ associated with the junction besides $Z_k$ which appears already in domain walls. We find that the new central charge $Y_k$ gives a negative contribution to the mass of the domain wall junction whereas the central charge $Z_k$ gives a dominant positive contribution. One has to be cautious to identify the central charge $Y_k$ alone as the mass of the junction.
Introduction

In recent years, there has been an intensive study of domain walls which appear in many areas of physics. These domain walls interpolate between degenerate discrete minima of a potential and spread over two spatial dimensions. This situation arises naturally in four-dimensional $\mathcal{N} = 1$ supersymmetric field theories \[1\]–\[3\] in addition to condensed matter physics. In supersymmetric unified models, domain walls can be formed during thermal evolution of our universe and often provide significant and interesting constraints on model building. On the other hand, it has been found that domain walls in supersymmetric theories can saturate the Bogomol’nyi bound \[4\]. Such a domain wall preserves half of the original supersymmetry and is called $1/2$ BPS state \[5\]. It has also been noted that these BPS states possess a topological charge which becomes a central charge $Z$ of the supersymmetry algebra \[6\].

Recently another interesting possibility for a BPS state has attracted much attention \[7\]–\[10\]. Domain walls occur in interpolating two discrete degenerate vacua in separate region of space. If three or more different discrete vacua occur in separate region of space, segments of domain walls separate each pair of the neighboring vacua. If the two spatial dimensions of all of these domain walls have one dimension in common, these domain walls meet at a one-dimensional junction. The solitonic configuration for the junction can preserve a quarter of supersymmetry. It has also been found that a new topological charge $Y$ can appear for such a $1/4$ BPS state \[7\] \[8\] \[10\].

There have been general considerations of junctions \[4\] \[8\] as well as more concrete numerical results \[9\]. In spite of these efforts, no exact or explicit solution has been obtained so far for the BPS junctions. In order to make progress in understanding these solitonic objects, it is quite useful to have exact solutions which allows us to investigate closely the behavior of these solitons and to evaluate explicitly the central charges $Z$ besides $Y$. In this respect, an exact solution offers informations complementary to general considerations and numerical studies.

The purpose of our paper is to present an exact solution of domain wall junction with three distinct vacua in a field theory model and explicitly work out various properties of the soliton including the new central charge $Y$ as well as the central charge $Z$. We believe that this is the first exact analytic solution of the BPS domain wall junction. The model is a simplified toy model simulating the $\mathcal{N} = 2$ supersymmetric $SU(2)$ Yang-Mills theory with one flavor which is explicitly broken to $\mathcal{N} = 1$ by giving a mass term to the adjoint chiral superfield. The central charge $Z$ is a two-dimensional complex vector which is determined by differences of superpotential at three distinct vacua. We give a formula which explicitly expresses the energy of the domain walls and junctions in terms of the central charges $Z$ and $Y$. We find in our model that the central charge $Y$ has a simple geometrical meaning of the $-2$ times the triangular area in field space which is enclosed by three domain walls connecting three distinct vacua at infinity. We also find that the main contribution to the mass of the domain wall junction configuration comes from the central charge $Z$ and the negative $Y$ is merely an additional small negative contribution. Our result gives a warning to a naive identification of the central charge $Y$ alone to be the mass of the junction.

Junctions and Central Charge
Using the convention of ref.[11], we denote the left-handed and right-handed supercharges of the $\mathcal{N} = 1$ supersymmetric four-dimensional field theory as $Q_\alpha, \bar{Q}_{\dot{\alpha}}$. If the translational invariance is broken as is the case for domain walls and/or junctions, the superalgebra in general receives contributions from central charges [1], [6]–[8], [10]. The anti-commutator between two left-handed supercharges has central charges $Z_k, k = 1, 2, 3$

$$\{Q_\alpha, Q_\beta\} = 2i(\sigma^k \bar{\sigma}^0)_\alpha^\gamma \epsilon_{\gamma\beta} Z_k. \quad (1)$$

The anti-commutator between left- and right-handed supercharges receives a contribution from central charges $Y_k, k = 1, 2, 3$

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^\mu P_\mu + \sigma^k Y_k), \quad (2)$$

where $P_\mu, \mu = 0, \cdots, 3$ are the energy-momentum four-vector of the system. Hermiticity of supercharges dictates that the central charges $Z_k$ are complex, and that $Y_k$ are real: $(Y_k)^* = Y_k$.

These central charges come from the total divergence and are non-vanishing when there are nontrivial differences in asymptotic behavior in different region of spatial infinity as is the case of domain walls and junctions. Therefore these charges are topological in the sense that they are determined completely by the boundary conditions at infinity. For instance, we can compute the anticommutators (1), (2) in the general Wess-Zumino models with arbitrary number of chiral superfields $\Phi^i$ and arbitrary superpotential $W$. The contributions to the central charges from bosonic components of chiral superfields are given by

$$Z_k = 2 \int d^3x \partial_k W^*(A^*), \quad (3)$$

$$Y_k = i\epsilon^{kmn} \int d^3x K_{ij} \partial_n (A^*^j \partial_m A^i), \quad \epsilon^{123} = 1, \quad (4)$$

where the scalar component of the $i$-th chiral superfield $\Phi^i$ is denoted as $A^i$ and the Kähler metric $K_{ij} = \partial^2 K(A^*, A)/\partial A^i \partial A^j$ is obtained from the Kähler potential $K$. We see that the central charge $Z_k$ is completely determined by the difference of values of the superpotential $W$ at spatial infinities where different discrete vacua are chosen for different directions. Since single domain wall has a field configuration which is nontrivial only in one dimension, one can see from eq.(4) that the central charge $Y_k$ vanishes whereas the central charge $Z_k$ is non-vanishing. The central charge $Y_k$ can be non-vanishing, if the field configuration at infinity is nontrivial in two-dimensions. This situation occurs when three or more different vacua occur at infinity as is the case for the domain wall junctions.

To examine the lower bound for the energy due to the hermiticity of the supercharges, we consider a hermitian linear combination of operators $Q$ and $\bar{Q}$ with an arbitrary complex two-vector $\beta^a$ and its complex conjugate $\bar{\beta}^\dot{a}$ as coefficients

$$K = \beta^a Q_a + \bar{\beta}^\dot{a} \bar{Q}_\dot{a}. \quad (5)$$
We treat $\beta^\alpha$ as c-numbers rather than the Grassmann numbers. Since $K$ is hermitian, the expectation value of the square of $K$ over any state is non-negative definite

$$\langle S|K^2|S\rangle \equiv (\beta^1,\beta^2,\bar{\beta}^1,\bar{\beta}^2)\hat{K}^2 \begin{pmatrix} \beta^1 \\ \beta^2 \\ \bar{\beta}_1 \\ \bar{\beta}_2 \end{pmatrix} \geq 0. \quad (6)$$

The equality holds if and only if the linear combination of supercharges $K$ is preserved by the state $|S\rangle$. Since we are interested in field configurations at rest, we obtain $P_k = 0, (k = 1, 2, 3)$ and the matrix $\hat{K}^2$ in terms of the central charges $Z_k, Y_k$ and the hamiltonian $H$ explicitly

$$\hat{K}^2 = \begin{pmatrix} \langle -Z_2 - iZ_1 \rangle & \langle iZ_3 \rangle & \langle H + Y_3 \rangle & \langle Y_1 - iY_2 \rangle \\ \langle iZ_3 \rangle & \langle -Z_2 + iZ_1 \rangle & \langle Y_1 + iY_2 \rangle & \langle H - Y_3 \rangle \\ \langle H + Y_3 \rangle & \langle Y_1 + iY_2 \rangle & \langle -Z_2^* + iZ_1^* \rangle & \langle -iZ_3^* \rangle \\ \langle Y_1 - iY_2 \rangle & \langle H - Y_3 \rangle & \langle -iZ_3^* \rangle & \langle -Z_2^* - iZ_1^* \rangle \end{pmatrix}. \quad (7)$$

For simplicity, let us assume that field configuration is two-dimensional, for instance, depends on $x_1, x_2$ only. Then we obtain $\langle Z_3 \rangle = \langle Y_1 \rangle = \langle Y_2 \rangle = 0$. The inequality (6) implies in this case that for any $\beta$ and any state

$$\langle H \rangle \geq \frac{-1}{|\beta|^2 + |\beta^2|^2} \left\{ (|\beta|^2 - |\beta^2|^2)\langle Y_3 \rangle + \text{Re} \left[ (\beta^1)^2 \langle -Z_2 - iZ_1 \rangle \right] \\ + \text{Re} \left[ (\beta^2)^2 \langle -Z_2 + iZ_1 \rangle \right] \right\}. \quad (8)$$

The minimum energy is achieved at the larger one of vanishing eigenvalues of the matrix $\hat{K}^2$

$$\det(\hat{K}^2) = (\langle H + Y_3 \rangle^2 - |\langle -iZ_1 - Z_2 \rangle|^2)(\langle H - Y_3 \rangle^2 - |\langle iZ_1 - Z_2 \rangle|^2) = 0. \quad (9)$$

Thus the BPS bound becomes $\langle H \rangle \geq \max \{H_1, H_{II} \}$ where $H_1$ and $H_{II}$ are two solutions of eq. (7)

$$H_1 \equiv |\langle -iZ_1 - Z_2 \rangle| - \langle Y_3 \rangle, \quad H_{II} \equiv |\langle iZ_1 - Z_2 \rangle| + \langle Y_3 \rangle. \quad (10)$$

The corresponding eigenvectors are given by $\bar{\beta}_1 = \beta^1 \langle iZ_1 + Z_2 \rangle / |\langle iZ_1 + Z_2 \rangle|$, $\beta_2 = \bar{\beta}_2 = 0$ for $\langle H \rangle = H_1$ and $\beta^1 = \bar{\beta}_1 = 0$, $\beta_2 = \beta^2 \langle -iZ_1 + Z_2 \rangle / |\langle -iZ_1 + Z_2 \rangle|$ for $\langle H \rangle = H_{II}$.

If $H_1 > H_{II}$, then supersymmetry can only be preserved at $\langle H \rangle = H_1$ and the only one combination of supercharges is conserved

$$\left( Q_1 + \frac{\langle iZ_1 + Z_2 \rangle}{|\langle iZ_1 + Z_2 \rangle|} \hat{Q}_1 \right) \text{BPS} = 0. \quad (11)$$

If $H_{II} > H_1$, then supersymmetry can only be preserved at $\langle H \rangle = H_{II}$ and the only one combination of supercharges is conserved

$$\left( Q_2 + \frac{\langle -iZ_1 + Z_2 \rangle}{|\langle -iZ_1 + Z_2 \rangle|} \hat{Q}_2 \right) \text{BPS} = 0. \quad (12)$$
These cases correspond to the 1/4 BPS state. If two eigenvalues are degenerate \( H_1 = H_{II} \), we can have 1/2 BPS state at \( H = H_I = H_{II} \) where both two combinations of supercharges \((11)\) and \((12)\) are conserved.

The condition of supercharge conservation \((11)\) for \( H = H_I \) applied to chiral superfield \( \Phi^i = (A^i, \psi^i, F^i) \) gives after eliminating the auxiliary field \( F^i \)

\[
2iK_{ij} \frac{\langle iZ_1 + Z_2 \rangle}{\langle iZ_1 + Z_2 \rangle} D_z A^i = -\frac{\partial W^*}{\partial A^{*j}}, \tag{13}
\]

where complex coordinates \( z = x_1 + ix_2, \bar{z} = x_1 - ix_2 \), gauge covariant derivatives \( D_\mu, D_z = \frac{1}{2}(D_1 + iD_2) \) and \( D_\bar{z} = \frac{1}{2}(D_1 - iD_2) \) are introduced. The same BPS condition \((11)\) applied to \( U(1) \) vector superfield in the Wess-Zumino gauge \( V = (v_\mu, \lambda, D) \) gives after eliminating the auxiliary field \( D \)

\[
v_{12} = -D = \frac{1}{2} \sum_j A^i e_j A^j, \quad v_{03} = 0, \quad v_{01} = v_{31}, \quad v_{23} = -v_{02}, \tag{14}
\]

where \( v_{\mu\nu} \equiv \partial_\mu v_\nu - \partial_\nu v_\mu \) and \( e_j \) is the charge of the field \( A^j \). A similar condition holds in the case of non-Abelian gauge group.

Similarly the condition of supercharge conservation \((12)\) for \( H = H_{II} \) applied to chiral superfield gives after eliminating the auxiliary field

\[
2iK_{ij} \frac{\langle iZ_1 - Z_2 \rangle}{\langle iZ_1 - Z_2 \rangle} D_{\bar{z}} A^i = -\frac{\partial W^*}{\partial A^{*j}}, \tag{15}
\]

The BPS condition \((12)\) applied to \( U(1) \) vector superfield in the Wess-Zumino gauge gives

\[
v_{12} = D = -\frac{1}{2} \sum_j A^{*j} e_j A^j, \quad v_{03} = 0, \quad v_{01} = -v_{31}, \quad v_{23} = v_{02}. \tag{16}
\]

These BPS conditions \((13)\) and \((14)\) for \( H = H_I \) and \((15)\) and \((16)\) for \( H = H_{II} \) ensure that the configuration is BPS saturated.

Since \( \langle Z_k \rangle \) and \( \langle Y_k \rangle \) are given by total divergence as shown in eqs.\((3)\) and \((4)\), they are fixed by boundary condition at spatial infinity. Therefore the boundary condition determines which of the supercharges can be preserved \((11)\) and/or \((12)\).

Since the BPS states are the minimum energy solution for a given boundary condition at infinity, they are stable against any fluctuations preserving the boundary condition. The domain wall has the minimum energy and is stable as long as two different vacua occupy the order \( R \) region of boundary of large radius \( R \). The domain wall junction has also the minimum energy and is stable provided the three (or more) vacua remain in regions of order \( R \).

**The model**

There are many field theory models which have BPS domain wall or junction solutions. First example is the Wess-Zumino model of single chiral scalar field \( \Phi \) with a polynomial superpotential.
\[ W = \Lambda^2 \Phi - \frac{1}{n+1} \Lambda^{2-n} \Phi^{n+1}, \]
where \( n \) is an integer \( \geq 2 \) and \( \Lambda \) is a parameter with the dimension of mass. This model has \( n \) discrete supersymmetric vacua with vanishing vacuum energy. Therefore one can have domain wall solutions for \( n \geq 2 \) \cite{[1]}, and the junction solutions for \( n \geq 3 \) \cite{[1] - [3]}, interpolating among those vacua. Numerical studies have been performed for domain walls and junctions in these models \cite{[3]}. However, no explicit analytic solution has been found even for domain walls, apart from the simplest case of \( n = 2 \) where a kink solution has been known for sometime. No explicit solution has been found for more difficult problem of junctions.

Another example is the \( \mathcal{N} = 1 \) supersymmetric QCD with \( N_f \) flavor of quarks in the fundamental representation. For the case of \( SU(N_c) \) gauge group, it has \( N_c - N_f \) discrete supersymmetric vacua \cite{[12]}, and can have domain wall solutions \cite{[1] - [2]}. This model can also be obtained from the \( \mathcal{N} = 2 \) supersymmetric QCD by perturbing with a mass term for the adjoint chiral superfield. It reduces to the \( \mathcal{N} = 1 \) supersymmetric gauge theory in the infinite mass limit, whereas it ends up at the singular points of moduli space of the \( \mathcal{N} = 2 \) supersymmetric gauge theory in the limit of vanishing adjoint mass \cite{[13]}. The moduli space of the \( \mathcal{N} = 2 \) \( SU(2) \) supersymmetric Yang-Mills theory has two singularities where monopole or dyon becomes massless respectively \cite{[13]}. In order to discuss the model in a simpler setting, Kaplunovsky et. al. have proposed a toy model which can be treated as a local field theory \cite{[3]}. They introduced two pairs of chiral superfields \( \mathcal{M}, \tilde{\mathcal{M}} \) and \( \mathcal{D}, \tilde{\mathcal{D}} \) simulating the monopole, anti-monopole and the dyon, anti-dyon of the Seiberg-Witten theory respectively. Instead of the modulus \( u \) of the Seiberg-Witten theory, they introduced a linearized analogue \( T \) as a neutral chiral superfield. The gauge group was chosen as \( U(1) \times U(1)' \) simulating electric and magnetic gauge group and the quantum number of these chiral superfields are given by

\[
\begin{array}{cccccc}
\mathcal{M} & \tilde{\mathcal{M}} & \mathcal{D} & \tilde{\mathcal{D}} & T \\
U(1) & 0 & 0 & 1 & -1 & 0 \\
U(1)' & 1 & -1 & 1 & -1 & 0
\end{array}
\]

(17)

To mimic a massless monopole at \( T = \Lambda \) and a massless dyon at \( T = -\Lambda \), they consider a superpotential

\[
W = (T - \Lambda)\mathcal{M}\tilde{\mathcal{M}} + (T + \Lambda)\mathcal{D}\tilde{\mathcal{D}} - h^2 T,
\]

(18)

where the coupling parameter \( h^2 \) replaces the effect of the mass for the adjoint chiral superfield. Their model has two discrete \( \mathcal{N} = 1 \) supersymmetric vacua

\[
T = +\Lambda, \quad \mathcal{M}\tilde{\mathcal{M}} = h^2, \quad |\mathcal{M}| = |\tilde{\mathcal{M}}|, \quad \mathcal{D} = \tilde{\mathcal{D}} = 0,
\]

\[
T = -\Lambda, \quad \mathcal{D}\tilde{\mathcal{D}} = h^2, \quad |\mathcal{D}| = |\tilde{\mathcal{D}}|, \quad \mathcal{M} = \tilde{\mathcal{M}} = 0.
\]

(19)

For simplicity, they assumed that the Kähler metric of the model is flat and discussed the domain wall solution interpolating between the two vacua. For the special case of \( h^2 = 2\Lambda^2 \), they obtained an analytic solution of the domain wall which asymptotes to the vacuum at \( T = +\Lambda \) for \( x \to -\infty \) and to the other vacuum at \( T = -\Lambda \) for \( x \to +\infty \) of (19):

\[
\mathcal{M} = \tilde{\mathcal{M}} = \frac{h}{1 + e^{2\Lambda x}}, \quad \mathcal{D} = \tilde{\mathcal{D}} = \frac{h}{1 + e^{-2\Lambda x}}, \quad T = -\Lambda \tanh \Lambda x.
\]

(20)

They also studied the domain wall for general values of the coupling \( h^2 \neq 2\Lambda^2 \) numerically and found that the qualitative features are unchanged.
If we add a single flavor of quarks in the fundamental representation in the $\mathcal{N} = 2$ $SU(2)$ gauge theory, we obtain three singularities in the moduli space. For large bare mass of the quark, the additional singularity corresponds to the situation where the effective mass of quark vanishes, whereas the $Z_3$ symmetry among three singularities is realized in the limit of vanishing bare quark mass \[13\]. These three singularities become three discrete vacua of $\mathcal{N} = 1$ gauge theory when perturbed by the adjoint scalar mass \[14\]. In view of these features, we extend the $U(1) \times U(1)'$ model of ref.\[3\] by adding an additional pair of chiral superfields $Q, \tilde{Q}$ corresponding to the quark and anti-quark

$$
\begin{array}{c|cc}
Q & \tilde{Q} \\
\hline
U(1) & 1 & -1 \\
U(1) & 0 & 0 \\
\end{array}
$$

(21)

To make the quark massless at $T = m$ where $m$ is the bare mass parameter for the quark $Q$, the superpotential is extended as

$$
W = (T - \Lambda)M\tilde{M} + (T + \Lambda)D\tilde{D} + (T - m)Q\tilde{Q} - h^{2}T.
$$

(22)

This simple modification produces a model which possesses three distinct $\mathcal{N} = 1$ supersymmetric vacua and allows us to obtain an exact solution for junctions. Since the action is invariant under the three global $U(1)$ transformations

$$
\begin{align*}
\mathcal{M} &\to e^{i\delta_1} \mathcal{M}, & \tilde{\mathcal{M}} &\to e^{-i\delta_1} \tilde{\mathcal{M}}, \\
\mathcal{D} &\to e^{i\delta_2} \mathcal{D}, & \tilde{\mathcal{D}} &\to e^{-i\delta_2} \tilde{\mathcal{D}}, \\
Q &\to e^{i\delta_3} Q, & \tilde{Q} &\to e^{-i\delta_3} \tilde{Q}, \\
\end{align*}
$$

(23)

we can choose the vacuum configuration to be

Vac.1 : $T = m$, $Q = \tilde{Q} = h$, $\mathcal{M} = \tilde{\mathcal{M}} = \mathcal{D} = \tilde{\mathcal{D}} = 0$,

Vac.2 : $T = \Lambda$, $\mathcal{M} = \tilde{\mathcal{M}} = h$, $Q = \tilde{Q} = \mathcal{D} = \tilde{\mathcal{D}} = 0$,

Vac.3 : $T = -\Lambda$, $\mathcal{D} = \tilde{\mathcal{D}} = h$, $Q = \tilde{Q} = \mathcal{M} = \tilde{\mathcal{M}} = 0$.

(24)

We will consider a field configuration which is static and translationally invariant along $x_3$ direction. We assume that the three different vacua are realized in different directions at spatial infinity in $x_1, x_2$ plane.

**The solution**

The states which are saturated by the Bogomol’nyi bound obey the eqs. \[13\] and \[14\] or the eqs. \[15\] and \[16\]. For simplicity, we now choose the eqs. \[15\] and \[16\]. The BPS equation \[16\] for $U(1) \times U(1)'$ vector superfields can be satisfied trivially\[17\] by $v_{\mu} = 0$ and $D = 0$. The other BPS equation \[15\] becomes

$$
2K_{ij}\frac{\partial A^i}{\partial z} = \Omega \frac{\partial W^{*}}{\partial A^{*j}}, \quad \Omega = -i\frac{(iZ_1^{*} + Z_2^{*})}{|iZ_1^{*} + Z_2^{*}|}.
$$

(25)

* The $Z_3$ symmetric case of the vanishing bare quark mass in the Seiberg-Witten theory yields a different charge assignment for the third singularity $(n_m, n_e) = (1, 2)$ instead of $(n_m, n_e) = (0, 1)$ \[13\]. Even if we use this charge assignment for the $Q$ field, The vanishing $D$ term condition gives the same result.
We will look for a solution of this partial differential equation. We observe that the phase of $\Omega$ can be absorbed by a rotation of field configuration, since the BPS equation (25) is invariant under a phase rotation: $\Omega \to e^{i\delta} \Omega$, $z \to e^{-i\delta} z$. Later, we will check that the solution satisfies $H_{II} > H_I$.

Since we assume a canonical Kähler metric $K_{ij} = \delta_{ij}$, the BPS eqs. (25) become for our model
\[
2 \frac{\partial M}{\partial z} = \Omega \tilde{M}^*(T - \Lambda)^*,
2 \frac{\partial D}{\partial z} = \Omega \tilde{D}^*(T + \Lambda)^*,
2 \frac{\partial Q}{\partial z} = \Omega \tilde{Q}^*(T - m)^*,
2 \frac{\partial T}{\partial z} = \Omega \left( M \tilde{M} + D \tilde{D} + Q \tilde{Q} - h^2 \right)^*.
\]
(26)

Eqs. (24) and (26) are invariant under the following global phase changes of parameters, fields and complex coordinate $z$
\[
h \to e^{i\beta} h, \quad M \to e^{i\beta} M, \quad \tilde{M} \to e^{i\beta} \tilde{M},
\]
\[
D \to e^{i\beta} D, \quad \tilde{D} \to e^{i\beta} \tilde{D}, \quad Q \to e^{i\beta} Q, \quad \tilde{Q} \to e^{i\beta} \tilde{Q},
\]
\[
\Lambda \to e^{i\gamma} \Lambda, \quad T \to e^{i\gamma} T, \quad z \to e^{2i\beta + i\gamma} z.
\]
(27)

Arbitrary complex parameters $h$ and $\Lambda$ can be obtained from real-positive $h$ and $\Lambda$ by these phase changes. Therefore we shall take $h$ and $\Lambda$ to be real-positive in the following without loss of generality. The BPS condition $D = 0$ yields $|M(z, \bar{z})| = |\tilde{M}(z, \bar{z})|$, $|D(z, \bar{z})| = |\tilde{D}(z, \bar{z})|$, and $|Q(z, \bar{z})| = |\tilde{Q}(z, \bar{z})|$. Inspired by this condition, we wish to find a solution assuming
\[
M(z, \bar{z}) = \tilde{M}(z, \bar{z}), \quad D(z, \bar{z}) = \tilde{D}(z, \bar{z}), \quad Q(z, \bar{z}) = \tilde{Q}(z, \bar{z}).
\]
(28)

and that all of them are real-positive in the entire complex plane. We shall see that this Ansatz gives a consistent solution.

We note that the model acquires a $Z_3$ symmetry if we choose the bare mass $m$ of $Q$ as
\[
m = i\sqrt{3}\Lambda.
\]
(29)

In order to obtain the exact analytic solution of the domain wall junction, we specialize to this case, and shift the field $T$ as $T' = T - i\frac{1}{\sqrt{3}} \Lambda$ to make $T' = 0$ as the origin of the $Z_3$ rotation $T' \to e^{\pm \frac{2\pi}{3}} T'$. The three vacua (24) and BPS equations (26) take manifestly $Z_3$ symmetric forms
\[
\text{Vac.1 : } T' = \frac{2}{\sqrt{3}} e^{i\frac{\pi}{3}} \Lambda, \quad Q = \tilde{Q} = h, \quad M = \tilde{M} = D = \tilde{D} = 0,
\]
\[
\text{Vac.2 : } T' = \frac{2}{\sqrt{3}} e^{-i\frac{\pi}{3}} \Lambda, \quad M = \tilde{M} = h, \quad Q = \tilde{Q} = D = \tilde{D} = 0,
\]
\[
\text{Vac.3 : } T' = \frac{2}{\sqrt{3}} e^{-i\frac{\pi}{3}} \Lambda, \quad D = \tilde{D} = h, \quad Q = \tilde{Q} = M = \tilde{M} = 0.
\]
(30)
\[
\begin{align*}
2 \frac{\partial}{\partial z} \ln q_M &= \Omega \left( T'^* - \frac{2}{\sqrt{3}} e^{i \frac{1}{6} \pi} \Lambda \right), \\
2 \frac{\partial}{\partial z} \ln q_D &= \Omega \left( T'^* - \frac{2}{\sqrt{3}} e^{i \frac{5}{6} \pi} \Lambda \right), \\
2 \frac{\partial}{\partial z} \ln q &= \Omega \left( T'^* - \frac{2}{\sqrt{3}} e^{-i \frac{1}{2} \pi} \Lambda \right), \\
2 \frac{\partial}{\partial z} T' &= \Omega h^2 \left( q_M^2 + q_D^2 + q^2 - 1 \right), 
\end{align*}
\]

(31)

where we have normalized the scalar fields by the nonzero expectation value \( h \) at vacua

\[
\mathcal{M}(z, \bar{z}) = h q_M(z, \bar{z}), \quad \mathcal{D}(z, \bar{z}) = h q_D(z, \bar{z}), \quad \mathcal{Q}(z, \bar{z}) = h q(z, \bar{z}).
\]

(32)

The first of eq.(31) can be rewritten as

\[
q_M = C(\bar{z}) \exp \left( \frac{1}{2} \eta - \frac{1}{\sqrt{3}} \Omega e^{i \frac{1}{6} \pi} \Lambda z \right),
\]

(33)

\[
\frac{\partial}{\partial z} \eta(z, \bar{z}) = \Omega T'^*(z, \bar{z}),
\]

(34)

where the unknown function \( C(\bar{z}) \) is determined by the reality condition for \( q_M \) up to a constant which is absorbed into \( \eta \)

\[
C(\bar{z}) = \exp \left( -\frac{1}{\sqrt{3}} \Omega^* e^{-i \frac{1}{6} \pi} \Lambda \bar{z} \right).
\]

(35)

The remaining unknown function \( \eta(z, \bar{z}) \) should then be real. Consequently we obtain

\[
q_M = \exp \left( \frac{1}{2} \eta + \frac{2}{\sqrt{3}} \Lambda \text{Re} \left( -\Omega e^{i \frac{1}{6} \pi} z \right) \right), \quad \eta(z, \bar{z}) = (\eta(z, \bar{z}))^*.
\]

(36)

By an exactly similar procedure, we solve the second and third equations and obtain

\[
q_D = \exp \left( \frac{1}{2} \eta + \frac{2}{\sqrt{3}} \Lambda \text{Re} \left( -\Omega e^{i \frac{5}{6} \pi} z \right) + C_D \right), \quad C_D \in \mathbb{R},
\]

(37)

\[
q = \exp \left( \frac{1}{2} \eta + \frac{2}{\sqrt{3}} \Lambda \text{Re} \left( -\Omega e^{-i \frac{1}{2} \pi} z \right) + C \right), \quad C \in \mathbb{R},
\]

(38)

where \( C_D \) and \( C \) are integration constants. Let us assume that the origin \( z = 0 \) is the center of the domain wall junction and is \( Z_3 \) symmetric. Therefore, \( q_M = q_D = q \) at \( z = 0 \), which implies \( C_D = C = 0 \). Inserting eq.(34) to the complex conjugate of the last of eq.(31), we obtain

\[
2 \frac{\partial^2}{\partial z \partial \bar{z}} \eta = -h^2 \left[ 1 - e^{\eta} \left\{ \exp \left( \frac{4 \Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{i \frac{1}{6} \pi} z \right) \right) \\
+ \exp \left( \frac{4 \Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{i \frac{5}{6} \pi} z \right) \right) + \exp \left( \frac{4 \Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{-i \frac{1}{2} \pi} z \right) \right) \right\} \right].
\]

(39)
For the special case of $h^2 = 2\Lambda^2$, eq. (39) can be solved analytically. Imposing the boundary conditions at infinity we obtain the solution

$$
\eta(z, \bar{z}) = -2 \ln \left[ \exp \left( \frac{2\Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{i\frac{1}{6}\pi} z \right) \right) + \exp \left( \frac{2\Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{i\frac{5}{6}\pi} z \right) \right) + \exp \left( \frac{2\Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{-i\frac{1}{2}\pi} z \right) \right) \right].
$$

(40)

Therefore we find solutions for scalar fields as

$$
\mathcal{M}(z, \bar{z}) = \tilde{\mathcal{M}}(z, \bar{z}) = \sqrt{2\Lambda s},
$$

$$
\mathcal{D}(z, \bar{z}) = \tilde{\mathcal{D}}(z, \bar{z}) = \sqrt{2\Lambda t},
$$

$$
\mathcal{Q}(z, \bar{z}) = \tilde{\mathcal{Q}}(z, \bar{z}) = \sqrt{2\Lambda u},
$$

$$
T'(z, \bar{z}) = \frac{2\Lambda}{\sqrt{3}} e^{-i\frac{1}{6}\pi} s + e^{-i\frac{5}{6}\pi} t + e^{i\frac{1}{2}\pi} u,
$$

(41)

$$
\begin{align*}
& s = \exp \left( \frac{2\Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{i\frac{1}{6}\pi} z \right) \right), \\
& t = \exp \left( \frac{2\Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{i\frac{5}{6}\pi} z \right) \right), \\
& u = \exp \left( \frac{2\Lambda}{\sqrt{3}} \text{Re} \left( -\Omega e^{-i\frac{1}{2}\pi} z \right) \right).
\end{align*}
$$

(42)

Now we will examine the solution more closely. The domain wall separating vacua $I$ and $J$ is characterized by a normal vector directing from $I$ to $J$ which is expressed as a complex number of unit modulus $\omega_{IJ}$. If the difference of the superpotential $W(Vac.I)$ at the vacuum $I$ and $W(Vac.J)$ at $J$ is denoted as $\Delta W_{IJ} = W(Vac.J) - W(Vac.I)$, the integral form of the BPS equation gives the condition on the direction of the domain walls as

$$
\Omega \frac{\Delta W_{IJ}}{|\Delta W_{IJ}|} \omega_{IJ} = 1.
$$

(43)

To orient the domain wall separating the vacuum 2 and 3 along the negative $x_2$ axis, we choose $\Omega = -1$. The modulus of the field $T'$ is plotted as a function of $x_1$ and $x_2$ in Fig. 1 where we can recognize three valleys corresponding to three domain walls.

Let us first examine the boundary conditions at spatial infinity $|z| \to \infty$. From eqs. (11) and (12), we find

$$
\begin{align*}
\text{when } & -\frac{1}{2}\pi < \arg(z) < \frac{1}{6}\pi, \text{ then } s \gg t, u, T' \to \frac{2}{\sqrt{3}} e^{-i\frac{5}{6}\pi} \Lambda, \text{ (vac.2)} \\
\text{when } & \frac{1}{6}\pi < \arg(z) < \frac{5}{6}\pi, \text{ then } u \gg s, t, T' \to \frac{2}{\sqrt{3}} e^{i\frac{1}{2}\pi} \Lambda, \text{ (vac.1)} \\
\text{when } & \frac{5}{6}\pi < \arg(z) < \frac{3}{2}\pi, \text{ then } t \gg s, u, T' \to \frac{2}{\sqrt{3}} e^{-i\frac{1}{2}\pi} \Lambda, \text{ (vac.3)}
\end{align*}
$$

(44)
Secondly, let us examine the asymptotic behavior along the region between two neighboring vacua. In the limit $x_2 \to -\infty$ with fixed $x_1$, eq. (41) reduces to

$$
\mathcal{M} \to \frac{\sqrt{2} \Lambda e^{\Lambda x_1}}{e^{\Lambda x_1} + e^{-\Lambda x_1}}, \quad \mathcal{D} \to \frac{\sqrt{2} \Lambda e^{-\Lambda x_1}}{e^{\Lambda x_1} + e^{-\Lambda x_1}}, \quad Q \to 0, \\
T' \to \Lambda \tanh \Lambda x_1 - \frac{\Lambda}{\sqrt{3}}i.
$$

Thus we recover the exact solution of domain wall (20) with $x$ replaced by $-x_1$. By the $Z_3$ symmetry, we also obtain respective exact domain wall solutions at the asymptotic region $x_1 = \pm \sqrt{3}x_2$ correctly.

Finally let us evaluate the central charges $\langle Z_k \rangle$ and $\langle Y_k \rangle$ and check $H_{II} > H_I$ to confirm that this solution is indeed realized as a $1/4$ BPS state. Since these charges are determined solely by the boundary condition at spatial infinity (14), we evaluate them on a large cylindrical region with a disk of large radius $R$ ($R \gg \Lambda^{-1}$) centered at $z = 0$ and a height $\Delta x_3$. Field configurations on the surface of the large cylinder approaches a step-function across domain walls. We find

$$
\langle Z_1 \rangle = -12\Lambda^3 R \Delta x_3, \quad \langle Z_2 \rangle = i12\Lambda^3 R \Delta x_3, \quad \langle Y_3 \rangle = -2\sqrt{3}\Lambda^2 \Delta x_3,
$$

with corrections suppressed exponentially as $R \to \infty$. Therefore we obtain

$$
H_I = |\langle -iZ_1 - Z_2 \rangle| - \langle Y_3 \rangle = 2\sqrt{3}\Lambda^2 \Delta x_3, \\
H_{II} = |\langle iZ_1 - Z_2 \rangle| + \langle Y_3 \rangle = 24\Lambda^3 R \Delta x_3 - 2\sqrt{3}\Lambda^2 \Delta x_3.
$$

Figure 1: The modulus of the field $T'$ as a function of $x_1$ and $x_2$. We set $\Lambda = 1$ for simplicity.
We see that $H_{II} > H_I$ confirming the correctness of the choice of the BPS equation \(^{15}\). It is interesting to observe that $H_{II}$ is larger than $H_I$ primarily due to the different phases of $\langle Z_1 \rangle$ and $\langle Z_2 \rangle$ and not to the presence of $\langle Y_3 \rangle \neq 0$ term. This is in contrast to the case of a single domain wall where $H_I = H_{II}$ since $\langle Z_1 \rangle$ and $\langle Z_2 \rangle$ have the same phase factor and $\langle Y_3 \rangle = 0$. In fact we observe in eq.(47) that the contribution of $\langle Y_3 \rangle$ to the mass of the domain wall junction is actually negative\(^{\ddagger}\). To see this fact from another viewpoint, let us consider the central charge $\langle Y_3 \rangle$ further. The general formula \(^{3}\) for the case of many chiral superfields can be partially integrated as

$$Y_3 = \int dx_3 i \int d^2 x \left[ \partial_1 \left( K_i \partial_2 A^i \right) - \partial_2 \left( K_i \partial_1 A^i \right) \right] = \int dx_3 i \int K_i dA^i, \quad (48)$$

where $K_i \equiv \partial K / \partial A^i$ and the last integral in the field space should be done as a map from a counter clockwise contour in the $z$ plane. In our case, we have contributions to $\langle Y_3 \rangle$ from the field $T$ only, since eq.(4) clearly shows that fields with real values do not contribute. Moreover the Kähler metric in our case is trivial and the counter clockwise contour in $z$ is mapped to a counter clockwise contour in the field $T$. Therefore we obtain

$$Y_3 = \int dx_3 (-2) \int d(\text{Re}T)d(\text{Im}T) = \int dx_3 (-2) \sqrt{3} \Lambda^2, \quad (49)$$

where the integration region in the field space $T$ is the equilateral triangle whose vertices are the three vacuum field values. We see that the central charge $\langle Y_3 \rangle$ has a simple geometrical meaning of the $-2$ times the triangular area in field space which is enclosed by three domain walls connecting three distinct vacua at infinity. From this consideration, we again find that the central charge $\langle Y_3 \rangle$ should be negative and does not have a naive meaning of “junction mass”. Let us also note that the domain walls correspond to straight lines in field space in our simple model. For general models, it has been shown that lines corresponding to domain walls are not straight lines in field space $A^i$, but become straight lines if mapped to the complex plane of superpotential $W(A^i)$ \(^{3},^{4}\). Therefore the geometrical meaning of the central charge $\langle Y_3 \rangle$ in general situation (48) is that it is proportional to the area in field space spanned by the fields as measured by the Kähler potential \(^{3}\).

Let us emphasize that the central charge $\langle Y_3 \rangle$ has a simple geometrical meaning and is negative in our model. The main contribution to the mass of the domain wall junction configuration comes from the central charge $Z_k$ and the negative $\langle Y_3 \rangle$ is merely an additional small negative contribution. This result is not an artifact of our choice of the BPS equation \(^{14}\) rather than the other possibility \(^{13}\). If we choose the other BPS equation, we merely obtain the reflected domain wall junction solution $x_1 \rightarrow -x_1, x_2 \rightarrow x_2$. The solution in fact gives a positive $\langle Y_3 \rangle$, but it also accompanies a different formula for the mass of the configuration $\langle H \rangle = H_I \equiv |\langle -i Z_1 - Z_2 \rangle| - \langle Y_3 \rangle$ in \(^{14}\) where the central charge $\langle Y_3 \rangle$ contributes negatively to the mass. Therefore the final physical result is identical.

When we complete writing our paper, more new works appeared on domain walls and junctions \(^{15}\).

\(^{\ddagger}\) This fact seems to be against previous thoughts such as in ref.\(^{7}\).
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