Solving vibration characteristics of the space straight bridge based on the auxiliary system transfer matrix method

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Abstract. The transfer relationship between two ends of the bar was studied by using an auxiliary system with one end fixed and the other free. This transfer matrix method was the transfer matrix method of the auxiliary system. The method has high accuracy and was easy to calculate. In this paper, the vibration transfer matrix of spatial linear beam was derived based on discrete model and transfer matrix method of auxiliary system. Taking a span simply supported beam bridge as an example, the program was applied to calculate the natural vibration frequency and mode of spatial linear bridge. It demonstrates that this method was effective.

1. Introduction
The traditional transfer matrix method is a semi-numerical and semi-analytical method, which is based on the exact solution of the governing differential equation of the structure [1]. The auxiliary system transfer matrix method is an accurate solution to the field matrix by means of the auxiliary system [2]. The natural vibration frequency of the structure is the inherent vibration frequency of the vibration system. For the multi-particle system, the natural vibration frequency is related to its mass distribution (stiffness), boundary support conditions and vibration form [3-5]. The auxiliary system transfer matrix method is used to solve the natural vibration frequency and mode of space straight bridge, the method has guiding significance to the seismic performance of complex structures.

2. Calculation model of vibration characteristics of space straight bridge
The space straight bridge is equivalent to a massless straight segment (as shown in Figure 1.) and a concentrated mass and boundary.

The coordinate system is shown in Figure 2, x, y, z respectively represent axial direction, horizontal direction, vertical direction. u, v, w respectively represent x axial displacement, y horizontal displacement, z vertical displacement. \( \theta_x, \theta_y, \theta_z \) respectively represent the bending angle around the x axis, y axis, z axis. The cross section characteristics and material characteristics of the bridge remain unchanged along the x axis.

The vibration model of space straight bridge includes 13 state vectors, namely

\[ S = [N, Q_x, Q_y, M_x, M_y, M_z, u, v, w, \theta_x, \theta_y, \theta_z, \theta_z]^T. \]
Figure 1. Vibration characteristics calculation model of the space straight bridge

(a) Force and bending moment  
(b) Displacement and rotation angle

Figure 2. The coordinate system of transfer matrix of space straight beam

2.1. The space behavior transfer matrix of the massless straight beam segments

Based on the auxiliary system transfer matrix method, the space behavior transfer matrix of the straight beam is derived. Formula (1) is the state transfer matrix of the massless straight beam.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(1)

2.2. The space behavior transfer matrix of the concentrated mass

The inertial force of the structure free vibration acts only on the concentrated mass. The state vectors of the left and right sections of the i-th concentrated mass are expressed as:

\[
S_i = \begin{bmatrix} N_{si}^l & Q_{si}^l & M_{si}^l & M_{ci}^l & u_i^l & v_i^l & w_i^l & \theta_{si}^l & \theta_{yi}^l & \theta_{ci}^l \end{bmatrix}^T
\]

(2)

\[
S_i = \begin{bmatrix} N_{si}^r & Q_{si}^r & M_{si}^r & M_{ci}^r & u_i^r & v_i^r & w_i^r & \theta_{si}^r & \theta_{yi}^r & \theta_{ci}^r \end{bmatrix}^T
\]

(3)

Taking vertical action as an example, the inertial force and inertial torque are deduced. The inertial force of the concentrated mass is:

\[
F_i = ma = m\omega^2w
\]

(4)

Where \( F_i \) is the inertial force; \( m \) is mass; \( \omega \) is the natural frequency of the structure; \( a \) is accelerated speed.
The translational inertia of the concentrated mass is reflected by $m$, the rotational inertia of the concentrated mass is reflected by $J$. The rotational inertia moment of the concentrated mass is:

$$\vec{M} = J \beta = J \omega^2 \dot{\theta}$$  \hspace{1cm} (5)

Where $\vec{M}$ is rotational inertia moment of the concentrated mass; $J$ is rotational inertia of the concentrated mass; $\beta$ is rotational inertia of the concentrated mass.

$-m_i \omega^2 \dot{u}_i$ is the axial inertial force of the $i$-th concentrated mass, $-m_i \omega^2 \dot{v}_i$ is the horizontal inertial force, $-m_i \omega^2 \dot{w}_i$ is the vertical inertial force; $-J_{\alpha} \omega^2 \dot{\theta}_{\alpha}$ is the axial rotational inertia moment of the concentrated mass; $-J_{\beta} \omega^2 \dot{\theta}_{\beta}$ is the horizontal rotational inertia moment; $-J_{\gamma} \omega^2 \dot{\theta}_{\gamma}$ is the vertical rotational inertia moment.

According to the equilibrium conditions of the rigid body, the equilibrium equation of the $i$-th concentrated mass is obtained as follows:

$$\begin{align*}
N_{\alpha i}^i + m_i \omega^2 \dot{u}_i - N_{\alpha i}^i &= 0 \\
Q_{\beta i}^i + m_i \omega^2 \dot{v}_i - Q_{\beta i}^i &= 0 \\
Q_{\gamma i}^i + m_i \omega^2 \dot{w}_i - Q_{\gamma i}^i &= 0 \\
M_{\alpha i}^i + J_{\alpha} \omega^2 \dot{\theta}_{\alpha i} - M_{\alpha i}^i &= 0 \\
M_{\beta i}^i + J_{\beta} \omega^2 \dot{\theta}_{\beta i} - M_{\beta i}^i &= 0 \\
M_{\gamma i}^i + J_{\gamma} \omega^2 \dot{\theta}_{\gamma i} - M_{\gamma i}^i &= 0
\end{align*}$$  \hspace{1cm} (6)

According to the deformation coordination condition of the structure:

$$\begin{align*}
u_i^i &= u_i^i \\
v_i^i &= v_i^i \\
w_i^i &= w_i^i \\
\theta_i^i &= \theta_i^i \\
\theta_i^i &= \theta_i^i \\
\theta_i^i &= \theta_i^i
\end{align*}$$  \hspace{1cm} (7)

Therefore, the relationship between the state vector of the left section and the right section of the $i$-th concentrated mass can be expressed as:

$$S_i^l = \bar{T} S_i^r$$  \hspace{1cm} (8)

Where

$$\bar{T} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & m_i \omega^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & m_i \omega^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & m_i \omega^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & J_{\alpha} \omega^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & J_{\beta} \omega^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & J_{\gamma} \omega^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$  \hspace{1cm} (9)
Where $\mathbf{T}_i$ is the vibration point matrix of the i-th concentrated mass.

3. **The total transfer matrix of free vibration of space straight bridge**

   After all the field matrices and point matrices of the beam are determined, assuming that the state vector $S_0$ of the initial end, then the state vector $S_i$ of the i-th section can be expressed as:

   $$ S_i = \mathbf{T}_i \tilde{T}_{i-1} \mathbf{T}_{i-1} \cdots \tilde{T}_1 \mathbf{T}_1 S_0 $$  \hspace{1cm} (10)

   If the whole beam is divided into n sections, there is:

   $$ S_n = \mathbf{T}_n \tilde{T}_{n-1} \mathbf{T}_{n-1} \cdots \tilde{T}_1 \mathbf{T}_1 S_0 = \mathbf{T} S_0 $$  \hspace{1cm} (11)

   Where $\mathbf{T}$ is the total transfer matrix of space straight bridge.

4. **Solving the natural vibration frequency of the straight bridge**

   The boundary conditions at both ends of the structure are introduced into equation (11), the equation about the natural vibration frequency of the structure is obtained, namely:

   $$ f (\omega) = 0 $$  \hspace{1cm} (12)

   The natural vibration frequency of each order $\omega_j (j = 1 \sim n)$ is calculated by the frequency search method [6], and then the vibration characteristics of the system is determined. The solution steps of natural vibration frequency and vibration characteristics of the space straight bridge are shown in Figure 3.

![Figure 3. The solution steps of natural vibration frequency and vibration characteristics of the space straight bridge](image)

5. **Example calculation**

   The model calculation diagram of a single span straight bridge is shown in Figure 4. Considering the shear deformation of the structure, the basic parameters of the straight bridge are shown in Table 1. Based on the auxiliary system transfer matrix method, the natural vibration frequency of the example is solved by the Matlab calculation software and the 4-order vibration pattern diagrams are listed.

![Figure 4. The calculation model](image)
Table 1. The basic parameters of the straight bridge

| $l$ | $A$ | $E$ | $G$ | $I_x$ | $I_y$ | $I_z$ | $\mu$ | $m$ |
|-----|-----|-----|-----|-------|-------|-------|-------|-----|
| m   | m$^2$ | MPa | MPa | m$^4$ | m$^4$ | m$^4$ | kg   |
| 20  | 0.6  | $3 \times 10^4$ | $0.42E$ | 0.04  | 0.069 | 0.034 | 1.2   | 7500 |

The 4-order natural vibration frequency of the $x-z$ plane are listed in Table 2. The 4-order vibration modes of the structure are respectively shown in Figure 5.6.7.8.

Table 2. The natural vibration frequency of the straight bridge

| natural vibration frequency rad/s | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ |
|----------------------------------|------------|------------|------------|------------|
| $x-z$ plane                       | 25.89      | 102.90     | 255.91     | 367.31     |

Figure 5. The first vibration mode of $x-z$ plane

Figure 6. The second vibration mode of $x-z$ plane

Figure 7. The third vibration mode of $x-z$ plane

Figure 8. The fourth vibration mode of $x-z$ plane

6. Conclusion

The auxiliary system transfer matrix method is a method of approximating exact solutions, which can well solve the natural frequency and mode shape of structures, especially for complex structures.
auxiliary system transfer matrix method can also be extended to study the dynamic analysis of structure under seismic load and wind load.

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