Radiating Collapse with Vanishing Weyl stresses

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In a recent approach in modelling a radiating relativistic star undergoing gravitational collapse the role of the Weyl stresses was emphasised. It is possible to generate a model which is physically reasonable by approximately solving the junction conditions at the boundary of the star. In this paper we demonstrate that it is possible to solve the Einstein field equations and the junction conditions exactly. This exact solution contains the Friedmann dust solution as a limiting case. We briefly consider the radiative transfer within the framework of extended irreversible thermodynamics and show that relaxational effects significantly alter the temperature profiles.

Keywords: gravitational collapse; heat flux; Weyl stresses

1. Introduction

Since the pioneering work of Oppenheimer and Snyder\(^1\) the final state of a star undergoing gravitational collapse has occupied a significant standing in relativistic astrophysics. An interesting extension of this problem is to include heat flow to a spherically symmetric model as a realistic model of a radiating star in gravitational collapse. Although this model is simplified, it does provide a good insight into the nature of collapse and serves as an excellent yardstick for more complicated radiating models. The available literature\(^2\)–\(^16\) on radiative gravitational collapse, and relativistic astrophysical applications of spherically symmetric gravitational fields with an imperfect matter distribution, is extensive. Models of relativistic radiating stars are useful in the investigation of the Cosmic Censorship hypothesis\(^17\)–\(^19\) which has attracted much attention in recent times amongst researchers in the field.

Since the derivation of the junction conditions\(^4\) for a radiating spherically symmetric star undergoing gravitational collapse and dissipating energy in the form of a radial heat flux, there have been many novel approaches to finding physically viable models. The model proposed by de Oliveira \textit{et al.}\(^5\) in which the star collapses from

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an initial static configuration has been studied extensively. Recently it has been shown that a similar approach can be adopted to look at the end state of collapse which results in a superdense star. Another model which has been useful in understanding the effects of dissipation is due to Kolassis in which the fluid trajectories are assumed to be geodesics. Recently Herrera et al. have imposed the condition of conformal flatness for a radiating, shear-free mass distribution undergoing gravitational collapse. This approach generalises earlier works by Patel and Tikekar and Grammenos in particular. In their paper Herrera et al. provided an approximate solution of the Einstein field equations. Their approximate solution highlighted the relaxational effects of the collapsing fluid on the temperature profile. These results were in agreement with earlier work carried out on the thermodynamics of radiating stars within the framework of extended irreversible thermodynamics. In this paper we demonstrate that it is possible to solve the field equations exactly with the assumption of vanishing Weyl stresses. Our exact solution provides a mathematical basis for the approximate approach of Herrera et al. and indicates that their model is physically significant.

This paper is organised as follows. In section two, we present the field equations governing the interior spacetime comprising a shear-free, spherically symmetric matter distribution with a radial heat flux. We also impose the condition for conformal flatness. In section three, the junction conditions required for the smooth matching of the interior spacetime to the Vaidya spacetime are outlined. In section four, an exact solution to the boundary condition is presented. We show that our exact solution has a Friedmann limit as in the case of the approximate analysis of Herrera et al. when the heat flux is absent. The stability of the model is also briefly discussed. Section five deals with the evolution of the temperature profiles within the framework of extended irreversible thermodynamics. We are in a position to calculate the temperature explicitly in both the causal and noncausal theories and it is further shown that relaxational effects lead to higher temperature within the stellar core.

2. Shear-free spacetimes

We investigate the gravitational collapse of a shear-free matter distribution with spherical symmetry. This is a reasonable assumption when modelling a relativistic radiating star. In this case there exists coordinates for which the line element may be expressed in a form that is simultaneously isotropic and comoving. With the coordinates \((x^a) = (t, r, \theta, \phi)\) the line element for the interior spacetime of the stellar model takes the form

\[
ds^2 = -A^2 dt^2 + B^2 \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\] (1)

where \(A = A(t, r)\) and \(B = B(t, r)\) are metric functions. In this paper we consider a model which represents a spherically symmetric, shear–free fluid configuration with heat conduction. For our model the energy-momentum tensor for the stellar fluid
becomes

\[ T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a. \]  

The fluid four–velocity \( u \) is comoving and is given by

\[ u^a = \frac{1}{A} \delta^a_0. \]  

The heat flow vector takes the form

\[ q^a = (0, q, 0, 0). \]  

since \( q^a u_a = 0 \) and the heat is assumed to flow in the radial direction on physical grounds because of spherical symmetry. The fluid collapse rate \( \Theta = u^a a \) of the stellar model is given by

\[ \Theta = 3 \frac{\dot{B}}{A B}. \]  

This is a system of coupled partial differential equations in the variables \( A, B, \rho, p \) and \( q \). The nonvanishing components of the Weyl tensor \( C_{abcd} \) for the line element (1) are:

\[ C_{2323} = \frac{r^4}{3} B^2 \sin^2 \theta \left[ \left( \frac{A'}{A} - \frac{B'}{B} \right) \left( \frac{1}{r} + 2 \frac{B'}{B} \right) - \left( \frac{A''}{A} - \frac{B''}{B} \right) \right] \]

\[ C_{2323} = -r^4 \left( \frac{B}{A} \right)^2 \sin^2 \theta C_{0101} = 2r^2 \left( \frac{B}{A} \right)^2 \sin^2 \theta C_{0202} \]
If we now impose the condition of conformal flatness, then this would imply vanishing of all the Weyl tensor components. From the above relations we note that this condition is fulfilled if we demand

$$r \left( \frac{A''}{A} - \frac{B''}{B} \right) - \left( \frac{A'}{A} - \frac{B'}{B} \right) \left( 1 + 2r \frac{B'}{B} \right) = 0. \hspace{1cm} (10)$$

Equation (10) can be easily integrated to give

$$A(t, r) = \left[ C_1(t) r^2 + 1 \right] B \hspace{1cm} (11)$$

where $C_1(t)$ is an arbitrary function of $t$ yet to be determined.

The condition of pressure isotropy reduces to

$$\frac{B''}{B'} - 2 \frac{B'}{B} - \frac{1}{r} = 0 \hspace{1cm} (12)$$

which has the general solution

$$B = \frac{1}{C_2(t) r^2 + C_3(t)} \hspace{1cm} (13)$$

where $C_2(t)$ and $C_3(t)$ are arbitrary functions of integration.

3. Junction Conditions

Since the interior is radiating energy, the exterior spacetime is described by Vaidya’s outgoing solution given by

$$ds^2 = - \left( 1 - \frac{2m(v)}{R} \right) dv^2 - 2dvdR + R^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \hspace{1cm} (14)$$

The quantity $m(v)$ represents the Newtonian mass of the gravitating body as measured by an observer at infinity. The metric (14) is the unique spherically symmetric solution of the Einstein field equations for radiation in the form of a null fluid. The Einstein tensor for the line element (14) is given by

$$G_{ab} = -\frac{2}{R^2} \frac{dm}{dv} \delta_a^0 \delta_b^0. \hspace{1cm} (15)$$

The energy–momentum tensor for null radiation assumes the form

$$T_{ab} = \Phi w_a w_b \hspace{1cm} (16)$$

where the null four–vector is given by $w_a = (1, 0, 0, 0)$. Thus from (15) and (16) we have that

$$\Phi = -\frac{2}{R^2} \frac{dm}{dv} \hspace{1cm} (17)$$
for the energy density of the null radiation. Since the star is radiating energy to the exterior spacetime we must have \( \frac{dm}{dv} \leq 0 \).

The necessary conditions for the smooth matching of the interior spacetime to the exterior spacetime was first presented by Santos\(^4\) in his seminal paper. The junction conditions for the line elements (1) and (14) are given by

\[
\begin{align*}
(rB)_{\Sigma} &= R_{\Sigma} \\
\rho_{\Sigma} &= (qB)_{\Sigma} \\
m(v) &= \left\{ \frac{r^3}{2} \left( \frac{B^2 B'}{A^2} - \frac{B'^2}{B} \right) - r^2 B' \right\}_{\Sigma}.
\end{align*}
\]

Matching the interior spacetime (1) to the outgoing Vaidya solution (14) leads to the following ordinary differential equation

\[
\ddot{C}_1 b^2 + \ddot{C}_2 = -3 \frac{(\dot{C}_2 b^2 + \dot{C}_3)^2}{2 (C_2 b^2 + C_3)} - \frac{\dot{C}_1 b^2 (\dot{C}_2 b^2 + \dot{C}_3)}{C_1 b^2 + 1} - 2(\dot{C}_3 C_1 - \dot{C}_2) b
\]

\[
+ 2 \left( \frac{C_1 b^2 + 1}{C_2 b^2 + C_3} \right) \left[ C_2 (C_2 - C_1 C_3) b^2 + C_3 (C_1 C_3 - 2 C_2) \right] = 0. \tag{21}
\]

In the above \( C_1 = C_1(t), \) \( C_2 = C_2(t) \) and \( C_3 = C_3(t) \). Dots denote differentiation with respect to \( t \). Also \( r_{\Sigma} = b \) is a constant which defines the boundary of the star. To complete the gravitational description of this radiating star we must solve the junction condition (21).

### 4. An exact model

Herrera \textit{et al.}\(^8\) have presented a simple approximate solution of (21) which reduces to the Friedmann dust sphere in the appropriate limit. They assumed the following forms for the temporal functions

\[
C_1 = \epsilon C_1(t), \quad C_2 = 0, \quad C_3 = \frac{a}{t^2}. \tag{22}
\]

where \( 0 < \epsilon << 1 \) and \( a \) is a positive constant. With these assumptions, the junction condition (21) reduces to

\[
\dot{C}_1 + \left( \frac{t}{b^2} + \frac{2}{b} \right) C_1 \approx 0 \tag{23}
\]

which has the approximate solution

\[
C_1 \approx C_1(0) \exp \left( \frac{-t^2}{2b^2} - \frac{2t}{b} \right). \tag{24}
\]

The approximate solution (24) makes it possible to perform a qualitative analysis of the physical features of the model.
However it is possible to solve (21) exactly for particular choices of the arbitrary temporal functions. We now present a simple exact solution of the junction condition (21). Guided by the approximate solution of Herrera et al.\textsuperscript{8} we take

\begin{equation}
C_2(t) = 0.
\end{equation}

Substituting (25) into the boundary condition (21) we obtain

\begin{equation}
C_3 \dddot{C}_3 - \frac{3}{2} \ddot{C}_3 - \left[ \frac{\dot{C}_1 b^2}{C_1 b^2 + 1} + 2C_1 b \right] C_3 \ddot{C}_3 + 2(C_1 b^2 + 1) C_1 C_2 = 0.
\end{equation}

The transformation

\begin{equation}
C_3(t) = u^{-2}
\end{equation}

enables us to write (26) in the equivalent form

\begin{equation}
\dddot{u} - \ddot{u} \left[ \frac{\dot{C}_1 b^2}{C_1 b^2 + 1} + 2C_1 b \right] - (C_1 b^2 + 1) C_1 u = 0.
\end{equation}

Equation (28) is a second order differential equation which is linear in $u(t)$ if the function $C_1$ is specified. A simple choice for $C_1$ is

\begin{equation}
C_1 = \mathcal{C}
\end{equation}

where $\mathcal{C}$ is a constant. It follows that (28) becomes

\begin{equation}
\dddot{u} - 2C_b \ddot{u} - (C b^2 + 1) \mathcal{C} u = 0.
\end{equation}

This differential equation has three categories of solutions.

In terms of the original function $C_3$ we can present the solutions as

**Case I:** \quad $2C^2b^2 + \mathcal{C} > 0$

\begin{equation}
C_3(t) = \left[ \beta_1 e^{(C b + \sqrt{2C^2b^2 + \mathcal{C})} t} + \beta_2 e^{(C b - \sqrt{2C^2b^2 + \mathcal{C})} t} \right]^{-2},
\end{equation}

**Case II:** \quad $2C^2b^2 + \mathcal{C} < 0$

\begin{equation}
C_3(t) = \left[ \beta_1 e^{2Cbt} \cos(t \sqrt{2C^2b^2 + \mathcal{C})} + \beta_2 e^{Cbt} \sin(t \sqrt{2C^2b^2 + \mathcal{C})} \right]^{-2},
\end{equation}

**Case III:** \quad $2C^2b^2 + \mathcal{C} = 0$

\begin{equation}
C_3(t) = (\beta_1 + \beta_2 t)^{-2} e^{-2Cbt},
\end{equation}

where $\beta_1$ and $\beta_2$ are constants of integration. We have therefore exhibited an exact solution to the boundary condition (21) where

\begin{equation}
C_1(t) = \mathcal{C}, \quad C_2(t) = 0,
\end{equation}

\begin{equation}
C_3(t) = 0.
\end{equation}
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and the three analytic forms for $C_3$ are given by (31)-(33). This exact solution provides a formal mathematical basis for the approximate model of Herrera et al.\(^8\) for a radiating star.

We now consider Case III more closely. The line element in this case becomes

$$ds^2 = (\beta_1 + \beta_2 t)^4 e^{4Cb t} \left[ - (C r^2 + 1)^2 dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (35)$$

If we set $\beta_1 = C = 0, \beta_2 = 1$ in (35) we obtain

$$ds^2 = t^4 \left[ -dt^2 + dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (36)$$

which is the Friedmann dust solution. Herrera et al.\(^8\) also regain (36) for their approximate solution. However our limit arises from an exact model.

From (6)-(8) we obtain the expressions

$$\rho = \frac{12}{z^6} \frac{[\beta_2 + bC z] e^{-4Cbt}}{(C r^2 + 1)^2} \quad (37)$$

$$p = \frac{4C}{z^5} \frac{[(1 - b^2 C + C r^2)\beta_1 + (-2 b + t + (r^2 - b^2) C t)\beta_2]}{(C r^2 + 1)^2} \quad (38)$$

$$qB = -\frac{8C r [\beta_2 + bCb z] e^{-4Cbt}}{(C r^2 + 1)^2} \quad (39)$$

for the line element (35) where $z(t) = \beta_1 + \beta_2 t$. The luminosity radius is given by

$$(rB)\Sigma = (\beta_1 + \beta_2 t)^2 e^{2Cbt} \quad (40)$$

which starts off at arbitrary large values at $t = -\infty$ and evolves towards $t = 0$. The four-acceleration of this model is nonvanishing as

$$a_a = u^b \nabla_b u_a = \frac{A'}{A} \delta_a^1 = \frac{2Cr}{1 + C r^2} \quad (41)$$

which is also independent of time. The measure of the dynamical instability of the stellar configuration at any given instant in time is given by

$$\Gamma = \frac{d\ln \rho}{d\ln \rho} \quad (42)$$

For our model we obtain

$$\Gamma_{\text{centre}} - \Gamma_{\text{surface}} = -\frac{b^2 C^2 (\beta_1 + \beta_2 t)^2}{6(\beta_2 + bC (\beta_1 + \beta_2 t)^2} < 0. \quad (43)$$

This implies that $\Gamma_{\text{centre}} < \Gamma_{\text{surface}}$ for all times. This physical result indicates that the centre of the collapsing star is more unstable than the outer regions, reinforcing the approximate results of Herrera et al.\(^8\) as well as the earlier work by Denmat et al.\(^20\) which is also related.
5. Thermodynamics

In this section we investigate the evolution of the temperature profile of our model within the context of extended irreversible thermodynamics. The causal transport equation in the absence of rotation and viscous stress is

$$\tau h_{ab} q_b + q_a = -\kappa (h_{ab} \nabla_b T + T \dot{u}_a)$$  \hspace{1cm} (44)

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, $T$ is the local equilibrium temperature, $\kappa \geq 0$ is the thermal conductivity, and $\tau \geq 0$ is the relaxational time-scale which gives rise to the causal and stable behavior of the theory. To obtain the noncausal Fourier heat transport equation we set $\tau = 0$ in (44). For the metric (1), equation (44) becomes

$$\tau(qB) \dot{T} + AqB = -\frac{\kappa(AT)'}{B}.$$  \hspace{1cm} (45)

In order to obtain a physically reasonable stellar model we will adopt the thermodynamic coefficients for radiative transfer. Hence we are considering the situation where energy is transported away from the stellar interior by massless particles, moving with long mean free path through matter that is effectively in hydrodynamic equilibrium, and that is dynamically dominant. Govender et al\textsuperscript{14} have shown that the choice

$$\kappa = \gamma T^3 \tau_c, \quad \tau_c = \left(\frac{\alpha}{\gamma}\right)^{-\sigma}, \quad \tau = \left(\frac{\beta \gamma}{\alpha}\right) \tau_c,$$

is a physically reasonable choice for the thermal conductivity $\kappa$, the mean collision time between massive and massless particles $\tau_c$ and the relaxation time $\tau$. The quantities $\alpha \geq 0$, $\beta \geq 0$ and $\sigma \geq 0$ are constants. Note that the mean collision time decreases with growing temperature as expected except for the special case $\sigma = 0$, when it is constant. With these assumptions the causal heat transport equation (45) becomes

$$\beta(qB) T^{-\sigma} + A(qB) = -\alpha \frac{T^{3-\sigma}(AT)'}{B}.$$  \hspace{1cm} (47)

Solutions are easily obtainable when in the noncausal case ($\beta = 0$) as was shown by Govinder and Govender\textsuperscript{16} in their general treatment. All noncausal solutions of (47) are

$$(AT)^{4-\sigma} = \frac{\sigma - 4}{\alpha} \int A^{4-\sigma} qB^2 \, dr + F(t), \quad \sigma \neq 0$$  \hspace{1cm} (48)

$$\ln (AT) = -\frac{1}{\alpha} \int qB^2 \, dr + F(t), \quad \sigma = 4.$$  \hspace{1cm} (49)

where $F(t)$ is a function of integration which is fixed by the surface temperature of the star. Note that $T$ corresponds to the noncausal temperature when $\beta = 0$. For
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a constant mean collision time \( (\sigma = 0) \), (47) can be integrated to give the causal temperature, i.e.,

\[
(\mathcal{T}^4)' = -\frac{4}{\alpha} \left[ \beta \int A^3 B \langle qB \rangle dr + \int A^4 qB^2 dr \right] + F(t). \tag{50}
\]

In (46) we can think of \( \beta \) as the ‘causality’ index, measuring the strength of relaxational effects, with \( \beta = 0 \) giving the noncausal case.

The effective surface temperature of a star is given by

\[
(\bar{T}^4)_\Sigma = \left( \frac{1}{r^2 B^2} \right) \left( \frac{L}{4\pi \delta} \right) \tag{51}
\]

where \( L \) is the luminosity at infinity and \( \delta (> 0) \) is a constant. The luminosity at infinity can be calculated from

\[
L_\infty = -\frac{dm}{dv} \tag{52}
\]

where

\[
m(v) = \left[ \frac{r^3 B B^2}{2A^2} - r^2 B' - \frac{r^3 B'^2}{2B} \right]_\Sigma. \tag{53}
\]

For our model we can calculate the temperature in both the causal and noncausal theories explicitly. The noncausal temperature for constant collision time is given by

\[
\tilde{T}^4 = \left[ \frac{1 + Cb^2}{1 + Cr^2} \right]^4 T^4_\Sigma + \frac{16(\beta z^{-1} + Cb)}{3\alpha z^2} \left[ \frac{(1 + Cr^2)^3 - (1 + Cb^2)^3}{(1 + Cr^2)^4} \right]. \tag{54}
\]

where \( z(t) = \beta_1 + \beta_2 t \). The causal temperature for constant collision time is given by

\[
T^4 = \left[ \frac{1 + Cb^2}{1 + Cr^2} \right]^4 T^4_\Sigma + \frac{8\beta}{\alpha} \left[ \frac{(\beta z^{-1} + Cb)}{z^4} e^{-2C\ell t} \right] \left[ \frac{(1 + Cr^2)^2 - (1 + Cb^2)^2}{(1 + Cr^2)^4} \right] + \frac{16(\beta z^{-1} + Cb)}{3\alpha} \left[ \frac{(1 + Cr^2)^3 - (1 + Cb^2)^3}{(1 + Cr^2)^4} \right] e^{-2C\ell t}. \tag{55}
\]

We note that at the boundary of the star, the causal and noncausal temperatures, \( T \) and \( \tilde{T} \) respectively, are equal. In fact the causal temperature is everywhere greater than its noncausal counterpart within the interior of the star. From (45) we can write

\[
\kappa(T)(\mathcal{T})' - \kappa(\tilde{T})(\mathcal{\tilde{T}})' = -(B\tau)(qB)' \tag{56}
\]

which takes the form

\[
(\mathcal{T}^{4-\sigma})' - (\mathcal{\tilde{T}}^{4-\sigma})' = -\frac{4-\sigma}{\alpha} (A^{3-\sigma} B \tau)(qB)' \tag{57}
\]

for the assumptions (46) where we have defined \( \mathcal{T} = AT \) and \( \mathcal{\tilde{T}} = A\tilde{T} \). We can immediately see that the relative spatial gradient of \( \mathcal{T} \) is greater than that of \( \mathcal{\tilde{T}} \),
since $(qB) > 0$ and $4 - \sigma > 0$. This result confirms earlier findings\cite{14} in which the acceleration was vanishing and the model had a Friedmann limit.

The evolution of the temperature has been studied within the framework of extended irreversible thermodynamics and are in keeping with earlier works on radiating collapse. This model has shown that relaxational effects can significantly alter the temperature of the star, especially during the latter stages of collapse. We have further shown that the model is more unstable closer to the centre as opposed to the outer regions of the star. Such instabilities have been shown to result in peeling or cracking up\cite{21} of the stellar configuration.

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