Witten-type topological field theory of self-organized criticality for stochastic neural networks

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Abstract. We study the Witten-type topological field theory (W-TFT) of self-organized criticality (SOC) for stochastic neural networks. The Parisi-Sourlas-Wu quantization of general stochastic differential equations (SDEs) for neural networks, the Becchi-Rouet-Stora-Tyutin (BRST)-symmetry of the diffusion system and the relation between spontaneous breaking and instantaneous connecting steady states of the SDEs, as well as the sufficient and necessary condition on pseudo-supersymmetric stochastic neural networks are obtained. Suppose neuronal avalanche is a mechanism of cortical information processing and storage [1][2][3] and the model of stochastic neural networks [7] is correct, as well as the SOC system can be looked up as a W-TFT with spontaneously broken BRST symmetry. Then we should recover the neuronal avalanches and spontaneously broken BRST symmetry from the model of stochastic neural networks. We find that, provided the divergence of drift coefficients is small and non-constant, the model of stochastic neural networks is BRST symmetric. That is, if the SOC of brain neural networks system can be looked upon as a W-TFT with spontaneously broken BRST symmetry, then the general model of stochastic neural networks which be extensively used in neuroscience [7] is not enough to describe the SOC. On the other hand, using the Fokker-Planck equation, we show the sufficient condition on diffusion so that there exists a steady state probability distribution for the stochastic neural networks. Rhythms of the firing rates of the neuronal networks arise from the process, meanwhile some biological laws are conserved.

Keywords: Witten-type topological field theory, self-organized criticality, stochastic neural networks, avalanche, Becchi-Rouet-Stora-Tyutin-symmetry
1 Introduction

It is well known that brain neural networks are organized based on self-organized criticality (SOC). The power-law of background neural activities had been discovered in EEG, ECoG and fMRI (see, for example, [8][12][14][1] etc.). Beggs and Plenz [1][2][3] studied spontaneous activity in an isolated slab of cortical tissue which followed up a much earlier series of experiments on isolated cortical slabs or slices carried out by Burns [4]. Their work revealed that an isolated cortical slab remains silent but excitable by brief current pulses. A strong enough pulse can trigger a sustained all-or-none response that propagates radially from the stimulation site, at a velocity of about 15 cm/s. Propagation of local field potentials (LFPs) in cortical circuits could be described by the same power-law that govern avalanches.

Can neuronal avalanches and SOC of brain arise from current often utilized general mathematical models for brain neural networks? Cowan, Neuman, Kiewiet and Drongelen [5] analyzed neural networks that exhibits self-organized criticality. Such criticality follows from the combination of a simple neural network with an excitatory feedback loop that generates bistability, in combination with an anti-Hebbian synapse in its input pathway. Using the methods of statistical field theory, they show that the network exhibits hysteresis in switching back and forward between its two stable states, each of which loses its stability at a saddlenode bifurcation. The Biological conservation law as an emerging functionality in dynamical neuronal networks was studied in [6]. However, so many different models were utilized in different papers. The lack of unifying mathematical theory and analyzing methods for the SOC impedes the effort to understand the neural information process arisen in our brain.

In [19], a scenario that a generic SOC system can be looked as a Witten-type topological field theory (W-TFT) with spontaneously broken Becchi-Rouet-Stora-Tyutin (BRST) symmetry, was proposed. But there are at least two unsolved problems: (1) From the quantum mechanical treatment, it is not clear why the distribution of avalanches must be a power-law; (2) Suppose we identify the instanton-induced Q-symmetry breakdown, when the spontaneous breakdown of Q symmetry occurs.

In this paper, we consider a general standard model of stochastic neural networks which was extensively used in neuroscience [7]. Suppose the neuronal avalanches is a mechanism of cortical information processing and storage [1][2][3] and the model of the stochastic neural networks [7] is correct, as well as the SOC system can be looked upon as a Witten-type topological field theory (W-TFT) with spontaneously broken Becchi-Rouet-Stora-Tyutin (BRST) symmetry. Then we should recover Beggs and Plenz etc.’s results and find power-law as well as spontaneously broken BRST symmetry from the model of stochastic neural networks. We utilize Parisi-Sourlas-Wu quantization to the stochastic differential equations
(SDEs) of the standard model of stochastic neural networks and find that, provided the divergence of drift coefficients is small and non-constant, the model of stochastic neural networks is BRST symmetric in the sense of W-TFT. That is, if the SOC of brain neural networks system can be looked upon as a W-TFT with spontaneously broken BRST symmetry, then the general model of the stochastic neural networks which be extensively used in neuroscience \[7\] is not enough to describe the SOC. At the same time, we solve the stochastic differential equations for the model of stochastic neural networks by numerical method, and try to recover Beggs and Plenz etc.’s results and find power-law. On the other hand, we study the sufficient condition on diffusion such that there exists a stationary probability distribution for the stochastic neural networks. The diffusion processes (solutions to the SDEs) take a long time inside some maximal equivalent class of the unperturbed diffusion process, and suddenly departing from one maximal equivalent class (up-state or down-state) and arriving to another (down-state or up-state). Rhythms of the firing rates of the neuronal network arise from the transitions between the maximal equivalent classes (tunneling effect), meanwhile some biological laws are conserved.

2 stochastic neural networks

Consider a neural network of \(N\) neurons where each neuron receiving \(N\) synaptic inputs labeled by \(i = 1, 2, ..., N\). The firing rate of input \(i\) is denoted by \(v^i\) and the input rates are represented collectively by the \(N\)-component vector \(v\). The synaptic current \(I^i_s\) of \(i\)th neuron is generally modeled by (see \[7\] section 7.2 (7.6))

\[
\tau_s dI^i_s = (-I^i_s - E^i(u(t))) \, dt + \tau_s \alpha^i_c(I_s(t), t) d\eta^i(t),
\]

(1)

where \(\tau_s\) is the decay constant of synaptic conductance, \(E^i(u) := \partial_u E(u)\), \(\{\eta^i(t)\}_i\) are independent Wiener processes and \(\alpha^i_c\) are diffusion coefficients. For simplicity, assume \(\alpha^i_c(I_s(t), t) = \alpha^i_c(I_s(t))\).

For constant synaptic current, the firing rate of postsynaptic neuron \(i\) can be expressed as \(F(I^i_s)\) that will be used as the input firing rate at next time, where \(F\) is an increasing function and called an activation function. Assume that the time-dependent inputs are still given by this activation function

\[
u^i(t) = F(I^i_s(t)).
\]

(2)

\(F\) is sometimes taken to be a saturating function such as a sigmoid function. It is also bounded from above, which can be important in stabilizing a network against excessively high firing rates.
2.1 Stratonovich interpretation

By Ito formula, (1) and (2) imply

\[
\frac{du^i}{\tau_s} = \left(-F'(I_s^i) + \frac{E^i(u)}{\tau_s} + \frac{1}{2} F''(I_s^i) (\alpha_c^i(I_s))^2 \right) dt + F'(I_s^i) \alpha_c^i(I_s) \circ d\eta^i
\]

where, from the viewpoint of self-organized criticality (SOC) [19], we use the Stratonovich interpretation of solutions to SDE (3). Suppose \( u^i \) is the Stratonovich integration of SDE (3). The Ito’s equivalent SDE of \( u^i \)

\[
\frac{du^i}{\tau_s} = \left(-F'(I_s^i) + \frac{E^i(u)}{\tau_s} + \frac{1}{2} F''(I_s^i) (\alpha_c^i(I_s))^2 \right) dt + F'(I_s^i) \alpha_c^i(I_s) d\eta^i.
\]

2.2 sufficient condition on the existence of stationary probability distribution

Lemma 2.1 If \( \alpha_c^i(I_s(t)) := \lim_{c \rightarrow c_0} \alpha_c^i(I_s(t),t) = \sqrt{2T/f'(I_s^i(t))} \), then there is a stationary probability distribution of (4) with the following stationary density

\[
P_0(u) = \frac{1}{Z} \exp \left( -\frac{1}{T \tau_s} \tilde{E}(u) \right)
\]

where \( \tilde{E}(u) := \frac{1}{\tau_s} \left( \sum_i \int u^i \left( F^{-1}(r) - \frac{\tau_s f'(r)}{2f(r)} \right) dr + E(u) \right) \), \( f(r) = F'(F^{-1}(r)) \) and

\[
Z = \int [du] \exp \left( -\frac{1}{T \tau_s} \tilde{E}(u) \right).
\]

Proof. As \( c \rightarrow c_0 \), (4) is written as

\[
\frac{du^i}{\tau_s} = \left(-f(u^i)\tilde{E}(u^i) + T f'(u^i) \right) dt + \sqrt{2T f(u^i)} d\eta^i.
\]

Generally, suppose \( u = \{u_i\} \) satisfy following stochastic differential equations

\[
du_i(t) = b_i(u(t)) dt + \sigma_i(u(t)) d\eta^i(t).
\]

The transition density \( P(u,t|u(t_0),t_0) \) of \( u(t) \) must satisfy the Fokker-Planck equation

\[
\frac{\partial P}{\partial t} = \sum_i \frac{\partial}{\partial u^i} \left( \frac{1}{2} \frac{\partial \sigma_i^2 P}{\partial u^i} - b_i P \right).
\]

It follows that if

\[
b_i = \frac{1}{2} \frac{\sigma_i^2 - \tilde{E}'(u)}{T} + \frac{\partial}{\partial u^i} \frac{\sigma_i^2}{2}.
\]
then $P_0 = \frac{1}{Z} \exp(\frac{1}{T} \tilde{E}(u))$ is stationary to the Fokker-Planck equation. Note that
\[ \tilde{E}(u)(t) = \tilde{E}(u)(0), \quad \forall t \] is conserved.

Remark. Recall that the drift part of (5) is proportional to the gradient flow of the energy $\tilde{E}(u)$. As the parameter $T$ (temperature) decreases to zero, $P_0(u)$ will approach a stationary transition density on a set of minima of $\tilde{E}(u)$. This is the principle on which simulated annealing is based [18]. For the study of the behavior of the solutions to the SDE on large time intervals for small noise, an essential role is placed by the Markov chain on the set of boundaries of $\omega$-limit sets (see Fig.4). The stationary probability distribution of the diffusion process determined by SDE can be described by the stationary probability distribution of the Markov chain. The diffusion processes (solutions to the SDEs (5)) take a long time inside some maximal equivalent class of the unperturbed diffusion process, and suddenly departing from one maximal equivalent class (up-state / down-state) and arriving to another (down-state / up-state). Rhythms of the firing rates of the neuronal network arise from the transitions between the maximal equivalent classes of steady states, which is an important characteristic of brain neuronal networks.

As $c \to c_0$, the Stratonovich interpretation of SDE (5) is rewritten as
\[ du^i = \left(-f(u^i)\tilde{E}^i(u) + T f'(u^i)\right) dt + \sqrt{2T f(u^i)} \circ d\eta^i, \quad \tilde{E}(u) := \frac{1}{\tau} \left(\sum_i \int u^i \left(F^{-1}(r)dr + E(u)\right)\right). \] (7)

Since the diffusion term $\sqrt{2T f(u^i)}$ in (7) depending on $u$, to avoid the quantization accounting the curvature of the target manifold, take the transformation
\[ \Phi(r) := \int^r_0 \frac{1}{\sqrt{f(s)}} ds \]
and $x^i = \Phi(u^i)$. Then
\[ dx^i = \left(-f(u^i)\tilde{E}^i(u) + T f'(u^i)\right) \frac{1}{\sqrt{f(u^i)}} dt + \sqrt{2T} \circ d\eta^i =: A^i(x) dt + \sqrt{2T} \circ d\eta^i \] (8)
is deduced from (7). Its Ito’s equivalent SDE
\[ dx^i = \left(-f(u^i)\tilde{E}^i(u) + T f'(u^i)\right) \frac{1}{\sqrt{f(u^i)}} dt + \sqrt{2T} d\eta^i. \] (9)

Remark that
\[ \partial_x A^i = \partial_{x^i} A^j, \] (10)
and there is a potential function $V(x)$ such that $A$ can be rewritten via $V$:

$$A^i(x) = \partial_x V(x). \quad (11)$$

### 3 quantization

Generally (3) is rewritten as an $N$-dimensional diffusion process

$$dv^i = A^i(v)dt + \sigma^i(v) \circ d\eta^i_t, \quad i = 1, 2, \ldots, N \quad (12)$$

where $\eta$ is the $N$-dimensional Wiener process, and $A(v) = \nabla_v V(v) + \tilde{A}(v)$. The nonpotential part $\tilde{A}$ is regarded as a magnetic field.

Following [19][20], where SOC was interpretation as Witten-type topological field theory with spontaneously broken BRST symmetry [26]. To complete this theory, there are at least two problems need to be fixed. First, the BRST-symmetry breakdown by instantons is only proved in one-dimension ([26]p.170). In multiple dimension case, even without the magnetic part $\tilde{A}$, the nonzero average $Q_{\alpha\beta}$ of a $Q$-exact operator only implies the perturbative ground state $\langle \alpha |$ doesn’t determine a supersymmetric ground state in the full theory ([25]p.223). It is unclear when a Hamiltonian related to $A$ is pseudo-Hermitian and when the hopping evolution of instanton and anti-instanton along the magnetic field $\tilde{A}$ induces pseudo-supersymmetry breakdown. Second, it is not clear why the BRST breakdown must be a power-law.

#### 3.1 Quantization by the De Witt-Faddeev-Popov method

We proceed with quantization [22] of equation (12) by the De Witt-Faddeev-Popov method. In light of the equation (9), the path integral representation of the Witten index is

$$Z = \int \left[ d\xi \right] e^{-\frac{1}{2} \int_0^\tau g^{jk} \xi^j \xi^k dt} \frac{det(\delta \xi)}{det(\delta v)} = \int \left[ dv \right] e^{-\frac{1}{2} \int_0^\tau g^{jk} (\partial_t v^j - A^j)(\partial_t v^k - A^k) dt} \frac{det(\delta \xi)}{det(\delta v)} \quad (13)$$

$g^{jk} = (g_{jk})^{-1}$ is the noise-noise correlator

$$\langle \xi^j(t)\xi^k(t') \rangle = g^{jk} \delta(t - t'), \quad d\xi^j(t) := \sigma^j(v) d\eta^j$$

which define a metric on the target manifold $M \ni v$ and which for now is assumed to be independent of the field $v$, and $det(\delta \xi)$ is the Jacobian of the map defined by the equation (3). Suppose $\sigma^j(v) \equiv 1$ (similar conclusions can be obtained for general case in the same spirit). By Gaussian multiple integral, the Jacobian can be represented as the path integral over the fermionic Faddeev-Popov ghosts

$$\tau det(\delta \xi) = \int [d\psi][d\bar{\psi}] e^{i \int_0^\tau \bar{\psi}_j (\delta^j_k \partial_t - \partial_v A^j) \psi^k dt} (\int [d\psi][d\bar{\psi}] e^{i \int_0^\tau \bar{\psi}_j \psi^j dt})^{-1}$$
and the path integral representation of the Witten index now is

$$Z = \int_{v, \bar{\psi}, \psi} e^{-\frac{1}{2}\int_0^\tau g_{jk}(\partial_tv^j - A^j)(\partial_tv^k - A^k) - i\bar{\psi}_j(\delta_k^l\partial_l - \partial_{v,k} A^l)\psi^k} dt$$

$$= (\det J)^{-1} \int_{v, B, \psi, \bar{\psi}} e^{-\int_0^\tau ig_{jk}(\partial_tv^j - A^j)B^k + \frac{1}{2}g_{jk}B^jB^k - i\bar{\psi}_j(\delta_k^l\partial_l - \partial_{v,k} A^l)\psi^k} dt$$

where the last equality derived by the Faddeev-Popov method.

### 3.2 BRST symmetry

The path integral representation of the Witten index is invariant under the nilpotent infinitesimal BRST transformation

$$\{Q, v^j\} = \psi^j, \quad \{Q, \psi^j\} = 0, \quad \{Q, \bar{\psi}_j\} = B_j, \quad \{Q, B_j\} = 0, \quad \{Q, \bar{\psi}\} = 0 \quad (15)$$

and the action is $Q$-exact

$$\int_0^\tau i g_{jk}(\partial_tv^j - A^j)B^k + \frac{1}{2}g_{jk}B^jB^k - i\bar{\psi}_j(\delta_k^l\partial_l - \partial_{v,k} A^l)\psi^k} dt = \int_0^\tau \{Q, \bar{\psi}_j(i(\partial_tv^j - A^j) + \frac{1}{2}B^j)\} dt \quad (16)$$

where $\{\cdot, \cdot\}$ is Poisson brackets (see [26](3.65)), $Q = -i\psi^j B^j$ is the BRST operator.

Suppose $\partial_v A^k = \partial_{\psi, k} A^j$. As [26](3.62), the action

$$\int_0^\tau i g_{jk}(\partial_tv^j - \sigma A^j)B^k + \frac{1}{2}g_{jk}B^jB^k - i\bar{\psi}_j(\delta_k^l\partial_l - \sigma\partial_{v,k} A^l)\psi^k} dt$$

is invariant under the discrete transformation

$$\sigma \rightarrow -\sigma, \quad \psi \rightarrow \bar{\psi}, \quad \bar{\psi} \rightarrow \psi, \quad B \rightarrow B - 2i\sigma A \quad (17)$$

and there is a second BRST operator $\bar{Q} := -i\bar{\psi}^k(B^k - 2iA^k)$ with $\sigma = 1$. It is remarkable that, even if $\partial_v A^k \neq \partial_{\psi, k} A^j$, utilizing the Poisson brackets we still can prove that $\bar{Q}$ is nilpotent and

$$\{Q, \bar{Q}\} = 2i(\frac{1}{2}(\partial_tv^j - A^j)^2 + (\partial_tv^j - A^j) A^j - i\bar{\psi}^j \partial_{v,k} A^j\psi^k) = 2iH. \quad (18)$$

The interaction representations of the BRST operators

$$Q = -i\psi^j B^j = -\psi^j(\partial_tv^j - A^j) \rightarrow Q = -\psi^j \partial_v,$$

$$\bar{Q} = -i\bar{\psi}^k(B^k - 2iA^k) = -\bar{\psi}^k(\partial_tv^k + A^k) \rightarrow \bar{Q} = -\bar{\psi}^k(\partial_{v,k} + 2A^k) \quad (19)$$

and the Hamiltonian

$$H = -\frac{1}{2}\Delta_v - A \cdot \nabla_v - div_v(A), \quad (20)$$
are derived by an appropriate quantum

\[
\Pi^j := \frac{\delta^R L}{\delta v^j} = \partial_v v^j - A^j \rightarrow \partial_v v^j,
\]

\[
\Pi^j A^j \rightarrow \frac{1}{2}[\partial_v A^j, A^j]_+ = A \cdot \nabla_v + \frac{1}{2} \text{div}_v A,
\]

\[-i\bar{\psi}^j \psi^k \rightarrow \frac{1}{2}[\partial_v \psi^k, \psi^j]_- = \frac{1}{2} \delta^{jk}.
\]

Notice that

\[
Q\bar{Q} = i(\frac{-1}{2} \Delta_v - A \cdot \nabla_v - \text{div}_v(A)) = iH.
\]

The natural representation of the above algebra of observable operators is on the space of differential forms (see [25](10.214)(10.225))

\[
\mathcal{H} = \Omega(M) \otimes \mathbb{C}
\]

equipped with the Hermitian inner product

\[
(\omega_1, \omega_2) = \int_M \bar{\omega}_1 \wedge \ast \omega_2.
\]

3.3 pseudo-supersymmetry

Generally [27], \( H \) is pseudo-Hermitian if there is a linear invertible Hermitian \( \Lambda \) and the adjoint operator of \( H \) satisfies

\[
H^\dagger = -\Lambda H \Lambda^{-1}.
\]

If \( Q^2 := \Lambda^{-1} Q^\dagger \Lambda \) is also nilpotent and \( 2iH = \{Q, Q^2\} \), we call the system 2-pseudo-s supersymmetric [27]. Here the notations are slightly different from [27] where the Hamiltonian \( H_M \) utilized in [27] satisfies \( 2H_M = \{Q, Q^2\} \). So we have to use \( H_M := iH \). Furthermore, if \( Q \) is BRST and there exists at least one physical state in the sense of gauge theory or W-TFT [26], the system is called BRST-supersymmetric in the sense of gauge theory or W-TFT respectively. Otherwise, the supersymmetries are broken. The BRST-supersymmetry breaking is spontaneous.

In light of (22), \( H \) is pseudo-Hermitian with respect to \( \Lambda \) if and only if

\[
-iQ\bar{Q} = H = -\Lambda^{-1} H^\dagger \Lambda = -i\Lambda^{-1} \bar{Q}\Lambda \Lambda^{-1} Q^\dagger \Lambda.
\]

**Proposition 3.1** The system (12) is 2-pseudo-supersymmetric if there is a linear invertible Hermitian \( \Lambda \) s.t.

\[
\bar{Q} = \Lambda^{-1} Q^\dagger \Lambda.
\]
Corollary 3.2 If $A$ is given via a potential $V$: $A = \nabla_v V$, then the system (12) is 2-pseudo-supersymmetric.

Proof. Take $\Lambda = \exp\{2V\}$ and note that
$$Q^\dagger \Lambda = \Lambda(-\bar{\psi})(\nabla_v + 2A) = \Lambda \bar{Q}.$$ 

3.4 spectrum

It was proved in [27] that the Hamiltonian $H$ is pseudo-Hermitian if the spectrum of $H$ consists of real and the pairs of complex-conjugate energies. It was well known that, when the Hamiltonian is self-adjoint (i.e. Hermitian), all eigenvalues must be real. So non-self-adjoint is a necessary condition of pseudo-supersymmetric breaking. It is easy to see that

Lemma 3.3 $H$ is self-adjoint (i.e. Hermitian) in $L^2$ if and only if $A(v) \equiv 0$.

Moreover $A(v) = 0$ implies that $v$ is a steady state of the unperturbed dynamical system of the equation (12).

Consider the spectrum of the Hamilton $H$ on $M$ with periodic boundary condition (P.B.C.) or Dirichlet boundary condition (D.B.C.) or Neuman boundary condition (N.B.C.).

Theorem 3.4 $(\sigma I + H)^{-1}$ for some $\sigma > 0$ is compact, positive from $W^{1,2}(M)$ to itself. Moreover, suppose any two eigenvectors are orthogonal. Then all the eigenvalues of $H$ are real if and only if $A(v) \equiv 0$.

Proof. The spectrum problem of the Hamilton $H$ is equivalent to $H_\sigma \Psi := (\sigma I + H)\Psi = (\lambda + \sigma)\Psi$ that can be extended to a weak form in $W^{1,2}(M)$
$$\mathcal{H}_\sigma v - \sigma E_1 E\Psi = \lambda E_1 E\Psi,$$
where $\mathcal{H}_\sigma$ is the extension of $H_\sigma$ on $W^{1,2}(M)$, $E : W^{1,2} \to L^2$ and $E_1 : L^2 \to (W^{1,2})^*$ are two embedding maps. From the Sobolev theorem, the embedding map $E : W^{1,2} \to L^2$ is compact. while $E_1 : L^2 \to (W^{1,2})^*$ is continuous. So $\mathcal{H}_\sigma^{-1} E_1 E : W^{1,2} \to W^{1,2}$ is compact. Similar proof can be applied to D.B.C. or N.B.C..

To study complex eigenvalue of $H$, we have to consider the Hilbert space $W^{1,2}(M)$ of complex valued functions. Then $H$ is self-adjoint (i.e. Hermitian) if and only if all the eigenvalues are real. By Lemma 3.3 all the eigenvalues of $H$ are real if and only if $A(v) \equiv 0$. 

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3.5 BRST supersymmetric breaking

It is easy to see that $H$ is pseudo-Hermitian. First we are looking for eigenvalues

$$H_0 \Phi := -\frac{1}{2} \Delta x \Phi - A^j(x) \nabla x, \Phi = \lambda \Phi, \quad \forall x \in M$$

(23)

with P.B.C or D.B.C or N.B.C. It is easy to see that, in case of P.B.C. or N.B.C., any non-zero constant is an eigenvector with zero eigenvalue. Generally we have

**Lemma 3.5** Suppose

either (1) $\text{div}_x A(x) = 0$ and $\Phi$ satisfies D.B.C or $\text{div}_x A(x) = 0$ and $\Phi$ as well as $A$ satisfy P.B.C.;

or (2) $\Phi$ satisfies D.B.C., $M$ is star shape, $\max_{x \in M} |x||A(x)| < \frac{\sqrt{2}}{3}$ and $|\text{div}_x A(x)| < C_M$

where $C_M$ is the Poincaré constant for D.B.C.;

or (3) $\Phi$ and $A$ satisfy P.B.C., $|\nabla_x \Phi(x)|^2 \leq 2|\partial_x \Phi(x)|^2 (\forall x \in \partial M)$, $M$ is symmetric under the transformation $x_j \to -x_j$ ($\forall j = 1, 2, ..., N$), $\max_{x \in M} |x||A(x)| < \frac{\sqrt{2}}{3}$ and $|\text{div}_x A(x)| < C_M$

where $C_M$ is the Poincaré constant for P.B.C.;

or (4) there is $V$ such that $A = \nabla_x V$, and $\Phi$ satisfies D.B.C. or N.B.C. or $\Phi$ and $V$ satisfy P.B.C..

Then, in case of P.B.C. or N.B.C. the only lowest energy state of $H_0$ is a constant with zero eigenvalue, and in case of D.B.C. the lowest energy state of $H_0$ is positive.

**Proof.** (1) Note that

$$\frac{1}{2} \int_M |\nabla x \Phi(x)|^2 dx = \text{Re}(\lambda) \int_M |\Phi(x)|^2 dx$$

(24)

which implies that the only lowest energy state is a constant with zero eigenvalue.

(2)(3) First note that the symmetry of $M$ under the transformation $x_j \to -x_j$ ($\forall j = 1, 2, ..., N$) implies that $M$ is star shape. Multiply $H_0 \Phi = \lambda \Phi$ by $\Phi - \frac{1}{|M|} \int_M \Phi dx$ and integrate over $M$

$$\frac{1}{2} \int_M |\nabla x (\Phi(x) - \frac{1}{|M|} \int_M \Phi)|^2 dx + \frac{1}{2} \int_M \text{div} A(x) |\Phi(x) - \frac{1}{|M|} \int_M \Phi|^2 dx$$

$$= \text{Re}(\lambda)(\int_M |\Phi(x)|^2 dx - \frac{1}{|M|} \int_M |\Phi(x)|^2 dx), \quad \forall \Phi \in W^{1,2}(M) \quad \text{with P.B.C.}$$

(25)

and recall the Schwartz inequality and Poincaré inequality

$$\int_M |\Phi(x)|^2 dx - \frac{1}{|M|} \int_M |\Phi(x)|^2 dx \geq 0,$$

$$\int_M |\nabla x (\Phi(x) - \frac{1}{|M|} \int_M \Phi)|^2 dx \geq C_M \int_M |\Phi(x) - \frac{1}{|M|} \int_M \Phi|^2 dx > \int_M \text{div} A(x) |\Phi(x) - \frac{1}{|M|} \int_M \Phi|^2 dx.$$
We discover that the eigenvalues of $H_0$ are non-negative. Furthermore, multiply $H_0\Phi = 0$
by $x \cdot \nabla_x \Phi$ and integrate over $M$

$$\frac{N - 2}{4} \int_M |\nabla_x \Phi(x)|^2 dx + \int_{\partial M} \left( \frac{1}{2} \text{Re} \left( \partial_x \Phi(x) \cdot \nabla_x \Phi(x) \right) - \frac{1}{4} (x \cdot \nu(x)) |\nabla_x \Phi(x)|^2 \right) d\mathcal{H}^{N-1}(x)$$

$$= -\text{Re} \int_M (A(x) \cdot \nabla_x \Phi(x))(x \cdot \nabla_x \Phi(x)) dx$$

$$\leq (\max_{x \in M} |x||A(x)|) \int_M |\nabla_x \Phi(x)|^2 dx, \quad \forall \Phi \in W^{1,2}(M)$$

(27)

where $\nu(x)$ is the unit outward normal vector of $\partial M$, and for star shape domain $M$,
$x \cdot \nu(x) \geq 0$ ($\forall x \in \partial M$). Particularly, P.B.C. and $|\nabla_x \Phi(x)|^2 \leq 2|\partial_x \Phi(x)|^2(\forall x \in \partial M)$
as well as the symmetry of $M$ under the transformation $x_j \to -x_j$ ($\forall j = 1, 2, ..., N$) implies
the positivity of the second integration in the first line of (27). From (27) we find that
the only lowest energy state is a constant with zero eigenvalue in case (3). The proof of (2)
is similar.

(4) Multiplying $H_0\Phi = \lambda \Phi$ by exp($2V$)$\Phi$ and integrating over $M$, in light of
$A = \nabla_x V$, we find

$$\frac{1}{2} \int_M \exp(2V)|\nabla_x \Phi(x)|^2 dx = \lambda \int_M \exp(2V)|\Phi(x)|^2 dx.$$   (28)

Then the eigenvalues must be real and the only lowest energy state is a constant with zero
eigenvalue. Particularly, for D.B.C., the lowest energy is positive.

Suppose $\lambda_0^{H_0}$ and $\lambda_1^{H_0}$ are the 1st and 2nd eigenvalues of $H_0$. Then from Lemma 3.5,

$$\lambda_0^{H_0} = 0, \quad \lambda_1^{H_0} > 0.$$  

We have

Theorem 3.6 Suppose $\text{div}_x A(x) \neq \text{constant}$ and either (1)

$$\sup_{x \in M} |\text{div}A(x)| < \min_{\zeta \in C: |\zeta| = \frac{-\lambda_0^{H_0}}{2}} \frac{1}{2(1 + |\zeta|^2)(1 + \|H_0 - \zeta^{-1}\|^2)^{\frac{1}{2}}}$$

(29)

as well as Lemma 3.5(3);
or (2) there is $V$ such that $A = \nabla_x V$, $\Phi$ and $V$ satisfy P.B.C. or $\Phi$ and $V$ satisfy N.B.C..

Then the corresponding ground state of $H$ must be non-constant and $H$ is BRST-
supersymmetric breaking in gauge theory provided $H$ is 2-pseudo-supersymmetric. Moreover
$H$ is BRST-supersymmetric in W-TFT in case (2).
Proof. Step 1. Suppose (29), in light of Lemma 3.5 and (28) (Th.3.18), the first eigenvalue (lowest energy) $\lambda^H_0$ of $H$ is in a small neighborhood of $\lambda^{H_0}_0$ which has the same multiplicity as $\lambda^{H_0}_0$, that is, $\lambda^H_0$ is also simple. Furthermore the corresponding ground state $\Phi^H_0$ must be non-constant. If not, $\Phi^H_0 \equiv C (\neq 0)$, then

$$H\Phi^H_0 = -C \text{div}_x A(x) = \lambda^H_0 C \Rightarrow \text{div}_x A(x) \equiv \lambda^H_0$$

which is contradiction with the assumption $\text{div}_x A(x) \neq \text{constant}$.

Step 2. Notice that $Q^H\Phi^H_0 \neq 0$ for any non-constant state $\Phi^H_0$. Thus there does not exist any physical state in the sense of gauge theory (see [26] section 3.6). Then $H$ is BRST-supersymmetric breaking in gauge theory provided $H$ is 2-pseudo-supersymmetric.

Step 3. Suppose there is $V$ such that $A = \nabla_x V$. Multiplying $H\Phi = \lambda\Phi$ by $\exp(2V)\bar{\Phi}$ and integrating over $M$, in light of $A = \nabla_x V$ and $\int_{\partial M} e^{2V} \nu(x) \cdot \nabla_x V(x)|\Phi(x)|^2d\mathcal{H}^{N-1} \leq 0$, we find

$$0 \leq \int_M \exp(2V) \left( \frac{1}{2}|\nabla_x \Phi(x)|^2 + 2|\nabla_x V|^2|\Phi|^2 + 2\text{Re}\bar{\Phi} \nabla_x V \cdot \nabla_x \Phi \right) dx = \lambda \int_M \exp(2V)|\Phi(x)|^2 dx. \tag{30}$$

On the other hand, it is easy to check that $Q\exp(-2V(x))\prod_{j=1}^N \psi^j = 0$ and $Q\exp(-2V(x))\prod_{j=1}^N \psi^j = 0$. So the lowest energy of $H$ is zero, and $H$ is BRST-supersymmetric in W-TFT.

Remark. 1. Since any $A \in L^2$ can be decomposed as $A(x) = \nabla_x V(x) + \tilde{A}(x)$ with $\text{div}_x \tilde{A}(x) = 0$, the condition for $\text{div}_x A$ is replaced for $\Delta_x V$.

Remark. 2. Generally, $H$ may be non-Hermitian. The eigenvalues of $H$ could be negative and complex. To extend the Witten index to these $H$ (see, for example, [27]), the existence and multiplicity of zero eigenvalue to $H$ is important. We have to investigate the condition when the eigenvalues and their multiplicity of $H$ deviate from the zero eigenvalue of $H_0$. The results in this section mentioned several necessary conditions on the variation of negativity and multiplicity of eigenvalues from $H_0$ to $H$.

### 3.6 topological invariant for general pseudo-Hermitian Hamiltonians

As the eigenspaces of $H$ that are associated with non-zero eigenvalues consist of pseudo-superpartner pairs of state vectors, under continuous pseudo-supersymmetry preserving deformations of $H$ or $H$ the Witten index is left invariant. Hence, it is a topological invariant (see [27]). The theory of the topological invariant conditions of the deformations in previous section can be extended to more general pseudo-Hermitian Hamiltonians. Suppose
$H_0$ and $H$ are general pseudo-Hermitian and have pseudo-supersymmetry generated by $Q_{H_0}$ and $Q_H$ respectively. Suppose $\lambda_{0}^{H_0}$ and $\lambda_{1}^{H_0}$ are the 1st and 2nd eigenvalues of $H_0$, and $\lambda_{0}^{H_0} = 0$, $\text{Multiplicity of } \lambda_{0}^{H_0} = 1$, $\lambda_{1}^{H_0} > 0$.

Utilizing operator perturbation theory \[28\] (chap. 3, Th.3.18) we have

**Theorem 3.7** Suppose the domain $D(H) \supset D(H_0)$, $\lambda_{0}^{H_0}$ is simple and

$$\|H - H_0\| < \frac{1}{\min_{\zeta \in \mathbb{C}; |\zeta| = |\lambda_{0}^{H_0}|} \frac{1}{2(1 + |\zeta|^2)(1 + \|H - \zeta\|^{-1})^{\frac{1}{2}}}}. \quad (31)$$

Then the first eigenvalue $\lambda_{0}^{H}$ of $H$ is inside the circle $\{\zeta \in \mathbb{C} : |\zeta| < \frac{|\lambda_{0}^{H_0}|}{2}\}$, and the multiplicity of $\lambda_{0}^{H}$ is also simple. That is, if $H$ has zero state then the Witten index is invariant from $H_0$ to $H$ and if $H$ has not zero state then the $Q$-symmetry of $H$ is breaking.

As another example, consider an even pseudo-Hermitian Hamiltonian $H$ on Hilbert space $\mathcal{H} = \mathcal{H}_+ \bigoplus \mathcal{H}_-$ which has a pseudo-supersymmetry generated by $Q : \mathcal{H}_+ \to \mathcal{H}_-$. The extended Witten index of pseudo-supersymmetry of $H$ is defined by $\dim(\ker(H_+)) - \dim(\ker(H_-))$ where the pseudo-superpartner Hamiltonian $H_\pm := H|_{\mathcal{H}_\pm}$. Applying Th.3.7 to $H_0 = H_+$ and $H = H_-$, we discover that if $H_-$ has zero state then the Witten index is zero and if $H_-$ has not zero state then there exists a supersymmetric ground state and $H$ is BRST-symmetric.

## 4 bio-conservation law and rhythms arisen in the stochastic neural networks

In the remaining of this section, we consider \[8\] with the function

$$E(u) = -\frac{1}{2} \sum_{i,j=1}^{N} w_{ij} u^i u^j - \sum_{i=1}^{N} \theta_i u^i \quad (32)$$

that results in

$$E^i(u) = -\sum_{j=1}^{N} \frac{w_{ij} + w_{ji}}{2} u^j - \theta_i, \quad (33)$$

and the activation function

$$u^i(t) = F(T^i_a(t)), \quad F(r) = \begin{cases} 
1 - e^{-\beta(r_1-r_0)}, & \forall r \geq r_1 \\
1 - e^{-\beta(r-r_0)}, & \forall r \in [r_0, r_1] \\
0, & \forall r < r_0, 
\end{cases} \quad (34)$$
where $\beta > 0$, $r_0 \in [0,1)$ and $r_1 > r_0$. Comparing with the half-wave rectification $[I^*_s(t) - r_0]_+$ (see [7] section 7.2 where $r_0$ is a threshold and the notation $[\cdot]_+$ denotes half-wave rectification), as $\beta \to \infty$, this activation function converges to Heaviside function. Remark that the conclusions obtained in this section can be easily extended to more general functions $E$ and $F$.

For (34), we have $F''/(F')^2 = -1/(1-u^i)$, $F^{-1}(u^i) = 1/\beta \ln 1/(1-u^i) + r_0$ and $f(u^i) = \beta(1-u^i)$.

4.1 the quantization of the stochastic neural networks are BRST supersymmetric

In light of (11) and the topological field theory obtained in the last section, we have

Theorem 4.1 There is $V(x)$ such that $A(x) = \nabla_x V(x)$, $Q$ and $\bar{Q}$ of (19) are the BRST operators of the quantization of (8), and the BRST operators of the quantization of (8) is 2-pseudo-supersymmetric (see Corollary 3.2). The quantization of (8) is BRST supersymmetry in W-TFT. That is, if the self-organized criticality of the neural networks system is looked upon as a W-TFT with spontaneously broken BRST symmetry, then the stochastic neural networks (8) which are extensively used in neuroscience [7] are not enough to simulate the SOC.

4.2 log spike number vs log window number

We solve the stochastic differential equations by numerical method (SDE TooLBOX [29]), and try to recover Beggs and Plenz etc.’s results and find power-law. Fig 1 shows the simulating results of the size distribution of neural activities (the numbers of synchronized firing neurons in a window (1 sec) v.s. the numbers of the window) for seven different weighted connecting matrices $W$ from the stochastic neural networks with $\beta = 0.1$, $\tau_s = 0.001(= 0.01$ sec), $T = 100$, $N = 20$ (4 inhibited neurons). The weighted connecting matrices $W$ with at least two stable steady states are selected. The distributing curves of log spike number vs log window number is more likely nonlinear than linear.

4.3 rhythms arising from transitions between up-steady states and down-steady states of the stochastic neural networks

Even though the stochastic neural networks are supersymmetric in W-TFT and the distributing curves of log spike number vs log window number are more likely non-power-law than power-law, similar rhythms discovered in cortical EEG and ECoG can be simulated by the stochastic neural networks.
Fig 2 is an example for $N = 20$. Rhythms of the firing rates of the neuronal network arise from the solution transitioning between up-steady state $\pi$ and down-steady state $\mu$ of the diffusion equation (7) (tunneling effect, see Fig 2(A)(B)), meanwhile the biological laws (6)
is conserved.

Assume the target manifold $M$ is a domain with smooth and compact closure in the state space of the diffusion process (7). Suppose $K$ be the maximal equivalent class of a $\omega$-limit set of the unperturbed diffusion process $X(t)$. That is, the set containing at least one $\omega$-limit set, all whole trajectories of the dynamical system (7) with $T = 0$ starting from any $x_0 \in K$. The attractable basin $D(K)$ of $K$ is the set of initial states from which the unperturbed
diffusion process $X(t)$ converge to $K$ with probability one:

$$
D(K) := \{ x \in M : P\{ \exists \tau > 0 \ s.t \ X_t \in K, \ \forall t > \tau | X_0 = x \} = 1 \}.
$$

Obviously, $K \subset D(K)$ holds (see Fig 3).

Assume in $M$ there are a finite number of maximal equivalent classes of the unperturbed diffusion process. Denote by $\partial M$ the boundary of $M$. Let $\mathcal{K} := \{ K_n \}_{n=1}^{n_0}$ be all possible maximal equivalent sets of the unperturbed diffusion process in $M$. Let $\rho_0$ be a positive number smaller than half of the minimum of the distances between $K_i$ and $K_j$ and between $K_i$ and $\partial M$. Let $0 < \rho_1 < \rho_0$. We denote by $O_\rho(K)$ the $\rho$-neighborhood of $K$, by $\partial O_\rho(K)$
the boundary of $O_\rho(K)$. We introduce the random times

$$
\tau_0 = 0, \quad \tau_n = \inf\{t \geq \tau_{n-1}: X^\epsilon(t) \in M \setminus \bigcup_{j=1}^{n_0} O_\rho_0(K_j)\},
$$

$$
\tau_{n+1} = \inf\{t \geq \tau_n: X^\epsilon(t) \in \partial M \cup \left(\bigcup_{j=1}^{n_0} \partial O_\rho_1(K_j)\right)\}
$$

and consider the Markov chain

$$X^\epsilon(\tau_n), \quad \forall n = 0, 1, 2, \ldots$$

From $n = 1$ on, $X^\epsilon(\tau_n)$ belongs to $\partial M \cup \left(\bigcup_{j=1}^{n_0} \partial O_\rho_1(K_j)\right)$. As far as the times $\tau_n$ are concerned, $X^\epsilon(\tau_0)$ can be any point of $M \setminus \bigcup_{j=1}^{n_0} O_\rho_0(K_j)$; all the following $X^\epsilon(\tau_n) \in M \setminus \bigcup_{j=1}^{n_0} O_\rho_1(K_j)$, until the time of exit of $X^\epsilon(t)$ to $\partial M$, belong to one of the surfaces $\partial O_\rho_0(K_j)$. After exit to the boundary $\partial M$, we have

$$\tau_n = \tau_{n-1} = \tau_{n+1} = \tau_{n+2} = \ldots$$

and the chain $\{X^\epsilon(\tau_n)\}_n$ stops (see Fig 4). The diffusion process (7) takes many times inside some maximal equivalent class of the unperturbed diffusion process, while suddenly transitioning at some $\bar{\tau}_n$ from one maximal equivalent class (up-state or down-state) to another (down-state or up-state). Rhythms of the firing rates of the neuronal network arise from the solution transitioning between the maximal equivalent classes meanwhile the biological law (6) is conserved.

5 conclusion

In this paper the quantum field theory is applied to study the phase transitions in large-scale brain activity, and the associated phenomena associated with critical behavior.

Suppose SOC can be interpreted as Witten-type topological field theory with spontaneously broken BRST symmetry [26][19]. Then the stochastic neural networks (8) which be extensively used in neuroscience [7] are not enough to simulate the SOC. The BRST-symmetry breakdown by instantons is proved in one-dimension (26, p.170). In multiple dimension case, the sufficient and (or) necessary conditions when a Hamiltonian related to the diffusion process (12) is pseudo-Hermitian and pseudo-supersymmetry, as well as the relation between the interaction representations of the BRST operators and the Hamiltonian are discovered. Some examples on the hopping evolution of instanton and anti-instanton along the magnetic field $\tilde{A}$ inducing pseudo-supersymmetry and (or) BRST breakdown are given.

We find that the distributing curves of log spike number vs log window number are more likely non-power-law than power-law, and the stochastic neural networks do not break BRST supersymmetry in W-TFT. Furthermore, the rhythms discovered in cortical EEG and ECoG
can be simulated by the stochastic neural networks. The sufficient condition on diffusion such that there exists a stationary probability distribution for the stochastic neural networks is obtained. The diffusion process (7) takes a long time inside some maximal equivalent class of the unperturbed diffusion process (7) without noise, and suddenly departing from one maximal equivalent class (up-state / down-state) and arriving to another (down-state / up-state). Rhythms of the firing rates of the stochastic neuronal networks arise from the transitions between the maximal equivalent classes (tunneling effect), meanwhile the biological law (6) is conserved.

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