Bayesian Opponent Exploitation in Imperfect-Information Games

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Abstract

The two most fundamental problems in computational game theory are computing a Nash equilibrium and learning to exploit opponents given observations of their play (aka opponent exploitation). The latter is perhaps even more important than the former: Nash equilibrium does not have a compelling theoretical justification in game classes other than two-player zero-sum, and furthermore for all games one can potentially do better by exploiting perceived weaknesses of the opponent than by following a static equilibrium strategy throughout the match. The natural setting for opponent exploitation is the Bayesian setting where we have a prior model that is integrated with observations to create a posterior opponent model that we respond to. The most natural, and a well-studied prior distribution is the Dirichlet distribution. An exact polynomial-time algorithm is known for best-responding to the posterior distribution for an opponent assuming a Dirichlet prior with multinomial sampling in the case of normal-form games; however, for the case of imperfect-information games the best known algorithm is a sampling algorithm based on approximating an infinite integral without theoretical guarantees. The main result is the first exact algorithm for accomplishing this in imperfect-information games. We also present an algorithm for another natural setting where the prior is the uniform distribution over a polyhedron.

1 Introduction

Imagine you are playing a game repeatedly against one or more opponents. What algorithm should you use to maximize your performance? The classic “solution concept” in game theory is the Nash equilibrium. In a Nash equilibrium $\sigma$, each player is simultaneously maximizing his payoff assuming the opponents all follow their components of $\sigma$. So we should just find a Nash equilibrium strategy for ourselves and play it in all the game iterations, right?

Unfortunately, there are some complications. First, there can exist many Nash equilibria, and if the opponents are not following the same one that we have found (or are not following one at all), then our strategy would have no performance guarantees. Second, finding a Nash equilibrium is challenging computationally: it is PPAD-hard and is widely conjectured that no polynomial-time algorithms exist [2]. These challenges apply to both extensive-form games (of both perfect and imperfect information) and to normal-form games, for games with more than two players and two-player non-zero-sum games. While a particular Nash equilibrium may happen to perform well in practice,\(^1\) there is no theoretically compelling justification for why computing one and playing it repeatedly is a good approach. Two-player zero-sum games do not face these challenges: there exist polynomial-time algorithms for computing an equilibrium [11], and there exists a game value that is guaranteed in expectation in the worst case by all equilibrium strategies regardless of the strategy played by the opponent (and this value is the best worst-case guaranteed payoff for any of our strategies). However, even for this game class it would be desirable to deviate from equilibrium in order to learn and exploit perceived weaknesses of the opponent; for instance, if the opponent has played Rock in each of the first thousand iterations of rock-paper-scissors, it seems desirable to put additional probability mass on paper beyond the equilibrium value of $\frac{1}{3}$.

\(^1\)An agent for three player limit Texas hold 'em poker computed using the counterfactual regret minimization algorithm (which converges to Nash equilibrium in certain game classes) has been shown to perform well in practice despite a lack of theoretical justification [6].

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Thus, learning to exploit opponents’ weaknesses is desirable in all game classes. One approach would be to construct an opponent model consisting of a single mixed strategy that we believe the opponent is playing given our observations of his play and a prior distribution (perhaps computed from a database of historical play). This approach has been successfully applied to exploit weak agents in limit Texas hold ’em poker, a large imperfect-information game [4].\(^2\) A drawback of this approach is that it is potentially not robust. It is very unlikely that the opponent’s strategy matches this point estimate exactly, and we could perform poorly if our model is incorrect. A more robust approach, which is the natural one to use in this setting, is to use a Bayesian model, where the prior and posterior are full distributions over mixed strategies of the opponent, not single mixed strategies. A natural prior distribution, which has been studied and applied in this context previously, is the Dirichlet distribution. The pdf of the Dirichlet distribution returns the belief that the probabilities of \(K\) rival events are \(x_i\) given that each event has been observed \(\alpha_i - 1\) times: 
\[
    f(x, \alpha) = \frac{1}{B(\alpha)} \prod x_i^{\alpha_i - 1}.
\]
Some notable properties are that the mean is 
\[
    E[X_i] = \frac{\alpha_i}{\sum_k \alpha_k},
\]
and that, assuming multinomial sampling, the posterior distribution after including new observations is also a Dirichlet distribution with parameters updated based on the new observations.

Prior work has presented an efficient algorithm for optimally exploiting an opponent in normal-form games in the Bayesian setting with a Dirichlet prior [3]. The algorithm is essentially the fictitious play rule [1]. Given prior counts \(\alpha_i\) for each opponent action, the algorithm increments the counter for an action by one each time it is observed, and then best responds to a model for the opponent where he plays each strategy in proportion to the counters. This algorithm would also extend directly to sequential extensive-form games of perfect information, where we maintain independent counters at each of the opponent’s decision nodes; this would also work for games of imperfect information where the opponent’s private information is observed after each round of play (so that we would know exactly what information set he took the observed action from). For all of these game classes the algorithm would apply to both zero and general-sum games, for any number of players. However, it would not apply to imperfect-information games where the opponent’s private information is not observed after gameplay. An algorithm exists for approximating a Bayesian best response in imperfect-information games, which uses importance sampling to approximate the value of an infinite integral. This algorithm has been successfully applied to limit Texas hold ’em poker [15].\(^4\) However, it is only a heuristic approach with no theoretical performance guarantees. In this paper, we present the first algorithm that is provably optimal for this problem. The algorithm runs in time polynomial in the number of the opponent’s information states. We also present an algorithm for another natural prior distribution: the uniform distribution over a polyhedron.

## 2 Meta-Algorithm

At first glance the problem of developing efficient algorithms for optimizing against a posterior distribution, which is a full probability distribution over mixed strategies for the opponent (which are themselves probability distributions over his pure strategies) seems daunting. We need to be able to both compactly represent the posterior distribution and efficiently compute a best response to it. Fortunately, we show that our payoff of playing any strategy \(\sigma_i\) against a probability distribution over mixed strategies for the opponent equals our payoff of playing \(\sigma_i\) against the expectation of the distribution. Thus, we need only represent and respond to the single strategy that is the expectation of the distribution, and not to the full distribution. While this result was likely known previously, we have not seen it spelled out explicitly, and it is important enough to

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2This approach used an approximate Nash equilibrium strategy as the prior and is applicable even when historical data is not available, though if additional data were available a more informed prior that capitalizes on the data would be preferable.

3The normalizing constant \(B(\alpha)\) is the beta function \(B(\alpha) = \frac{\prod \Gamma(\alpha_i)}{\Gamma(\sum \alpha_i)}\), where \(\Gamma(n) = (n - 1)!\) is the gamma function.

4In addition to Bayesian Best Response, the paper also considers approaches for approximating Max A Posteriori Response and Thompson’s Response.
be highlighted so that it is on the radar of the AI community.

Suppose the opponent is playing mixed strategy $\sigma_{-i}$, where $\sigma_{-i}(s_{-j})$ is the probability that he plays pure strategy $s_{-j} \in S_{-j}$. Then, by definition of expected utility,

$$ u_i(\sigma_i, \sigma_{-i}) = \sum_{s_{-j} \in S_{-j}} [\sigma_{-i}(s_{-j}) \cdot u_i(\sigma_i, s_{-j})] $$

We can generalize this naturally to the case where the opponent is playing according to a probability distribution with pdf $f_{-i}$ over mixed strategies as follows:

$$ u_i(\sigma_i, f_{-i}) = \int_{\sigma_{-i} \in \Sigma_{-i}} [f_{-i}(\sigma_{-i}) \cdot u_i(\sigma_i, \sigma_{-i})] $$

Let $\overline{f_{-i}}$ denote the expectation of $f_{-i}$. That is, $\overline{f_{-i}}$ is the mixed strategy that selects action $s_{-j}$ with probability

$$ \int_{\sigma_{-i} \in \Sigma_{-i}} [\sigma_{-i}(s_{-j}) \cdot f_{-i}(\sigma_{-i})] . $$

Then we have the following result.

**Theorem 1.**

$$ u_i(\sigma_i, \overline{f_{-i}}) = u_i(\sigma_i, f_{-i}). $$

*That is, the payoff against the expectation of a strategy distribution equals the payoff against the full distribution.*

**Proof.**

\[
\begin{align*}
    u_i(\sigma_i, \overline{f_{-i}}) &= \sum_{s_{-j} \in S_{-j}} \left[ u_i(\sigma_i, s_{-j}) \int_{\sigma_{-i} \in \Sigma_{-i}} [\sigma_{-i}(s_{-j}) \cdot f_{-i}(\sigma_{-i})] \right] \\
    &= \sum_{s_{-j} \in S_{-j}} \left[ \int_{\sigma_{-i} \in \Sigma_{-i}} [u_i(\sigma_i, s_{-j}) \cdot \sigma_{-i}(s_{-j}) \cdot f_{-i}(\sigma_{-i})] \right] \\
    &= \int_{\sigma_{-i} \in \Sigma_{-i}} \left[ \sum_{j \in S_{-j}} [u_i(\sigma_i, s_{-j}) \cdot \sigma_{-i}(s_{-j}) \cdot f_{-i}(\sigma_{-i})] \right] \\
    &= \int_{\sigma_{-i} \in \Sigma_{-i}} [u_i(\sigma_i, \sigma_{-i}) \cdot \overline{f_{-i}(\sigma_{-i})}] \\
    &= u_i(\sigma_i, \overline{f_{-i}}) \\
\end{align*}
\]

Theorem 1 applies to both normal-form and extensive-form games (with both perfect and imperfect information), for any number of players (we could let $\sigma_{-i}$ be a joint strategy profile for all agents besides ourselves).

Now suppose the opponent is playing according a prior distribution $p(\sigma_{-i})$, and let $p(\sigma_{-i}|x)$ denote the posterior probability given observations $x$. Let $\overline{p(\sigma_{-i}|x)}$ denote the expectation of $p(\sigma_{-i}|x)$. As an immediate consequence of Theorem 1, we have the following corollary:

**Corollary 1.** $u_i(\sigma_i, \overline{p(\sigma_{-i}|x)}) = u_i(\sigma_i, p(\sigma_{-i}|x))$.  

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Algorithm 1 Meta-algorithm for Bayesian opponent exploitation

Inputs: Prior distribution $p_0$, response functions $r_t$ for $0 \leq t \leq T$

\begin{align*}
M_0 &\leftarrow p_0(\sigma_{-i}) \\
R_0 &\leftarrow r_0(M_0)
\end{align*}

Play according to $R_0$ 

for $t = 1$ to $T$ do 

\begin{align*}
x_t &\leftarrow \text{observations of opponent’s play at time step } t \\
p_{t} &\leftarrow \text{posterior distribution of opponent’s strategy given prior } p_{t-1} \text{ and observations } x_t \\
M_t &\leftarrow \text{expectation of } p_t \\
R_t &\leftarrow r_t(M_t)
\end{align*}

Play according to $R_t$

Corollary 1 implies the meta-procedure for optimizing against an opponent who plays according to $p$ given by Algorithm 1.

There are several challenges for applying Algorithm 1. First, the algorithm assumes that we can compactly represent the prior and posterior distributions $p_t$, which have an infinite domain (the set of mixed strategy profiles for the opponents). Second, it requires a procedure to efficiently compute the posterior distributions given the prior and the observations, which will involve having to update potentially infinitely many strategies. Third, it requires an efficient procedure to compute the expectation of $p_t$. And fourth, it requires that the full posterior distribution from one round be compactly represented to be used as the prior distribution in the next round. We can address the fourth challenge by using the following modified $p_t$ update step:

\[ p_t \leftarrow \text{posterior distribution of opponent’s strategy} \]

\[ \text{given prior } p_0 \text{ and observations } x_1, \ldots, x_t. \]

We will be using this new rule in our main algorithm.

The response functions $r_t$ could be a standard best response, for which linear time algorithms exist in imperfect-information games, and a recent approach has enabled efficient computation in extremely large games [8]. It could also be a more robust form of a best response, e.g., one that places a limit on the exploitability of our own strategy, perhaps one that varies over time depending on estimators for how much we have won or lost [5, 7, 9]. In particular, the restricted Nash response has been demonstrated to outperform a full best response against agents in limit Texas hold ’em poker, whose actual strategy may differ from the exact model [9].

3 Exploitation Algorithm for Dirichlet Prior

As described in Section 1 the Dirichlet distribution is the conjugate prior for the multinomial distribution, and therefore the posterior is also a Dirichlet distribution, with the parameters $\alpha_i$ updated to reflect the new observations. Thus, the expectation of the posterior can be computed efficiently by computing the strategy for the opponent in which he plays each strategy in proportion to the updated weight, and Algorithm 1 yields an exact efficient algorithm for computing the Bayesian best response in normal-form games with a Dirichlet prior. However, the algorithm does not apply to games of imperfect information since we do not observe the private information held by the opponent, and therefore do not know which of his action counters we should increment. In this section we will present a new algorithm for this setting. We first present the algorithm in the context of a representative motivating game in Section 3.1, and present the algorithm for the general setting in Section 3.3.
### 3.1 Motivating game

Consider the following two-player game where we are player 2 and the opponent is player 1. Both players ante $1, and then player 1 is dealt a King (K) and Jack (J) with probability $\frac{1}{2}$, while player 2 is always dealt a Queen (Q). Player 1 is then allowed to bet $1 (b) or fold (f), and player 2 is allowed to call or fold vs. a bet. If player 1 folds, then player 2 wins the $2 pot; if player 1 bets and player 2 folds then player 1 wins the $2 pot; if player 1 bets and player 2 calls then the player with the higher card wins the $4 pot. (The full rules are presented for completeness and not needed for the analysis; all that is needed is the set of private states and actions available to player 1).

![Motivating game diagram](image)

Figure 1: Motivating game. Chance first deals player 1 a king or jack with probability $\frac{1}{2}$ each at the green node. Then player 1 selects bet or fold at a red node. Then player 2 chooses to call or fold facing a bet at a blue node.

If we observe player 1’s card after each hand, then we can apply the approach described above, where we maintain a counter for player 1 choosing each action with each card that is incremented for the selected action. However, if we do not observe player 1’s card after the hand (e.g., if he folds), then we would not know whether to increment the counter for the king or the jack.

To simplify analysis, we will assume that we never observe the opponent’s private card after the hand (which is not quite realistic since we would observe his card if he bets and we call); we can assume that we do not observe our payoff either until all game iterations are complete, since that could allow us to draw inferences about the opponent’s card. There are no known efficient algorithms even for the simplified case of fully unobservable opponent’s private information. We suspect that an algorithm for the case of partial observability (when the opponent’s private information is sometimes observed and sometimes not) can be constructed based on the algorithm we present here, and we plan to study this problem in future work.

Let $C$ denote player 1’s card, and $A$ denote his action. Then $P(C = K) = P(C = J) = \frac{1}{2}$. Let $q_{b|K}$ denote the probability that player 1 bets given a king: $q_{b|K} \equiv Pr(A = b|C = K)$. If we were using a Dirichlet prior with parameters $\alpha_1$ and $\alpha_2$ (where $\alpha_1 - 1$ is the number of times that action $b$ has been observed with a king, and $\alpha_2 - 1$ is the number of times $f$ has been observed with a king), then

$$Pr(q_{b|K} = \text{Dir}(q_{b|K};\alpha_1,\alpha_2) = \frac{(q_{b|K}^{\alpha_1})(1 - q_{b|K})^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)}$$

In general given observations $O$, Bayes’ rule gives the following, where $q$ is a mixed strategy that is given mass $p(q)$ under the prior, and $p(q|O)$ is the posterior:

$$p(q|O) = \frac{P(O|q)p(q)}{P(O)}$$
Now let us compute the expectation of the posterior distribution of the opponent’s probability of playing other parameters.

\[
E(A) = \int_a a P(A = a) da = \int_a \int_b P(A, B) db da = \int_a \int_b a P(A, B) db da
\]

Now let us compute the expectation of the posterior distribution of the opponent’s probability of playing \( b \) with a jack.

\[
P(A = b|O, C = J) = \int_{q_{b|J}} P(A = b|O, C = J, q_{b|J}) P(q_{b|J}|O) dq_{b|J}
\]

The general expression for the expectation of a continuous random variable is

\[
E[A] = \int_a a P(A = a) da = \int_a \int_b P(A, B) db da = \int_a \int_b a P(A, B) db da
\]
\[
\int q_{b|j}^{\alpha_K b}(1 - q_{b|j})^{\alpha_J f - 1} B(\alpha_K b + 1, \alpha_K f) \int q_{b|j}^{\alpha_K b}(1 - q_{b|j})^{\alpha_K f - 1} B(\alpha_K b + 1, \alpha_K f) dq_{b|K} dq_{b|J} = \int q_{b|j}^{\alpha_K b}(1 - q_{b|j})^{\alpha_J f - 1} B(\alpha_J b + 1, \alpha_J f) dq_{b|J} = 1,
\]

since the integrands are themselves Dirichlet distributions and all probability distributions integrate to 1. Similarly

\[
\int \int (q_{b|K}^{\alpha_K b-1}(1 - q_{b|j})^{\alpha_K f - 1} q_{b|j}^{\alpha_J b+1}(1 - q_{b|j})^{\alpha_J f - 1}) B(\alpha_K b, \alpha_K f) B(\alpha_J b + 2, \alpha_J f) = 1.
\]

Letting \(Z\) denote the denominator, we have

\[
P(b|O, J) = \frac{B(\alpha_K b + 1, \alpha_K f) B(\alpha_J b + 1, \alpha_J f) + B(\alpha_K b, \alpha_K f) B(\alpha_J b + 2, \alpha_J f)}{Z},
\]

where the normalization term is

\[
Z = B(\alpha_K b + 1, \alpha_K f) B(\alpha_J b + 1, \alpha_J f) + B(\alpha_K b, \alpha_K f) B(\alpha_J b + 2, \alpha_J f) + B(\alpha_K b, \alpha_K f + 1) B(\alpha_J b, \alpha_J f + 1) + B(\alpha_K b, \alpha_K f) B(\alpha_J b, \alpha_J f + 2)
\]

\(P(f|O, J), P(b|O, K), \) and \(P(f|O, K)\) can be computed analogously. As stated earlier, \(B(\alpha) = \frac{\prod \Gamma(\alpha_i)}{\Gamma(\sum \alpha_i)}\), where \(\Gamma(n) = (n - 1)\), which can be computed efficiently.

Note that the algorithm we have presented applies for the case where we play one more game iteration and collect one additional observation. However, it is problematic for the general case we are interested in where we play many game iterations, since the posterior expression in Equation 1 is not Dirichlet, and therefore we cannot just apply the same procedure in the next iteration using the computed posterior as the new prior. We will need to derive a new expression for \(P(b|O, J)\) for this setting. Suppose that we have observed the opponent play action \(b\ \theta_b\) times and \(f\ \theta_f\) times (in addition to the number of fictitious observations reflected in the prior \(\alpha\)), though we do not observe his card. Note that there are \(\theta_b + 1\) possible ways that he could have played \(b\ \theta_b\) times: 0 times with \(K\) and \(\theta_b\) times with \(J\), 1 time with \(K\) and \(\theta_b - 1\) times with \(J\), etc. Thus the expression for \(p(q|O)\) will have \(\theta_b + 1\) terms in it instead of two (we can view \(\theta_b + 1\) as being a constant if we assume that the number of game iterations is a constant, but in any case this is linear in the number of iterations).

We have the new equation

\[
P(q|O) = \frac{1}{2B(\alpha_K b, \alpha_K f) B(\alpha_J b, \alpha_J f) p(O)} \cdot \sum_{i=0}^{\theta_b} \sum_{j=0}^{\theta_f} q_{b|K}^{\alpha_K b-1+i} (1 - q_{b|K})^{\alpha_K f - 1+j} q_{b|J}^{\alpha_J b-1+i+\theta_f} (1 - q_{b|J})^{\alpha_J f - 1+(\theta_f-j)}
\]

Using similar reasoning as above, this gives

\[
P(b|O, J) = \frac{\sum_i \sum_j [B(\alpha_K b + i, \alpha_K f + j) B(\alpha_J b + \theta_b - i + 1, \alpha_J f + \theta_f - j)]}{Z}
\]

The normalization term is

\[
Z = \sum_i \sum_j [B(\alpha_K b + i, \alpha_K f + j) B(\alpha_J b + \theta_b - i + 1, \alpha_J f + \theta_f - j)]
\]
Thus the algorithm for responding to the opponent is the following. We start with the prior counters on each private information and action combination, $\alpha_{Kb}, \alpha_{Kf}, \alpha_{Jb}, \alpha_{Jf}$. We keep separate counters $\theta_b, \theta_f$ for the number of times we have observed the actions $b$ and $f$ during the course of play. Then we combine these counters according to Equation 2 in order to compute the strategy for the opponent that is the expectation of the posterior given the prior and observations, and we best respond to this strategy, which gives us the same payoff as best responding to the full posterior distribution according to Theorem 1.

### 3.2 Example

Suppose the prior is that the opponent played $b$ with $K$ 10 times, played $f$ with $K$ 3 times, played $b$ with $J$ 4 times, and played $f$ with $J$ 9 times. Thus $\alpha_{Kb} = 10, \alpha_{Kf} = 3, \alpha_{Jb} = 4, \alpha_{Jf} = 9$. Now suppose we observe him play $b$ at the next iteration. Applying our algorithm using Equation 1 gives us

$$p(b|O,J) = \frac{B(11, 3)B(5, 9) + B(10, 3)(6, 9)}{Z} = \frac{0.0011655 \cdot 0.00015540 + 0.00151515 \cdot 0.00005550 = \frac{2.65209525 e^{-7}}{Z}}{Z}$$

$$p(f|O,J) = \frac{B(10, 4)B(4, 10) + B(10, 3)(4, 11)}{Z} = \frac{0.00034965 \cdot 0.00034965 + 0.00024975 \cdot 0.00151515 = \frac{5.00663835 e^{-7}}{Z}}{Z}$$

Therefore

$$p(b|O,J) = \frac{2.65209525 e^{-7}}{2.65209525 e^{-7} + 5.00663835 e^{-7}} = 0.3462837838.$$

So we think that with a jack he is playing a strategy that bets with probability 0.346 and folds with probability 0.654. Notice that previously we thought his probability of betting with a jack was $\frac{4}{13} = 0.308$, and had we been in the setting where we always observe his card after gameplay and observed that he had a jack, the posterior probability would be $\frac{5}{14} = 0.357$.

An alternative “naïve” (and incorrect) approach would have been to increment the counter for $\alpha_{Jb}$ by $\alpha_{Jb}/\alpha_{Jb} + \alpha_{Kb}$, the ratio of the prior probability that he bets given $J$ to the total prior probability that he bets. This would give a posterior probability of him betting with $J$ of $\frac{4 + \frac{14}{14}}{14} = 0.306$, which differs significantly from the correct value.

Similarly we compute his strategy with a king:

$$p(b|O,K) = \frac{B(5, 9)B(11, 3) + B(4, 9)(12, 3)}{Z} = \frac{0.0011655 \cdot 0.00116550 + 0.00050505 \cdot 0.00091575 = \frac{6.43618238 e^{-7}}{Z}}{Z}$$

$$p(f|O,K) = \frac{B(4, 10)B(10, 4) + B(4, 9)(10, 5)}{Z} = \frac{0.00034965 \cdot 0.00034965 + 0.00050505 \cdot 0.00009990 = \frac{1.72709618 e^{-7}}{Z}}{Z}$$

$$p(b|O,K) = \frac{6.43618238 e^{-7}}{6.43618238 e^{-7} + 1.72709618 e^{-7}} = 0.7884284676.$$
So we think he will bet with king with probability 0.788 and fold with probability 0.212. By comparison, 
\( \frac{10}{13} = 0.769, \frac{11}{14} = 0.786, \) and \( \frac{10+12}{14} = 0.765 \) (so the “naïve” incorrect approach would actually have reduced the probability despite observing a bet). Interestingly, in this case the posterior probability is actually higher than if we were in the fully observable setting and saw that he had a king and bet which would correspond to the Dirichlet posterior.

### 3.3 General setting

We now consider the general setting where the opponent can have \( n \) different states of private information according to an arbitrary distribution \( \pi \) and can take \( m \) different actions. Assume he is given private information \( x_i \) with probability \( \pi_i \), for \( i = 1, \ldots, n \), and can take action \( k_i \), for \( i = 1, \ldots, m \). Assume the prior is Dirichlet with parameters \( \alpha_{ij} \) for the number of times action \( j \) was played with private information \( i \) (so the expectation of the prior has the player selecting action \( k_j \) at information state \( x_i \) with probability \( \frac{\alpha_{ij}}{\sum_j \alpha_{ij}} \)).

\[
\begin{align*}
Pr(C = x_i) &= \pi_i \\
Pr(A = k_j | C = x_i) &= q_{kj|x_i} \\
Pr(q_{kj|x_i}) &= \text{Dir}(q_{kj|x_i}; \alpha_{i1}, \ldots, \alpha_{im}) = \prod_j q_{kj|x_i}^{\alpha_{ij}-1} B(\alpha_{i1}, \ldots, \alpha_{im})
\end{align*}
\]

As before, using Bayes’ rule we have

\[
\begin{align*}
p(q|O) &= \frac{P(O|q)p(q)}{P(O)} \\
&= \frac{\sum_i P(O, C = x_i | q)p(q)}{p(O)} \\
&= \frac{\sum_i P(O|C = x_i, q)\pi_i p(q)}{p(O)} \\
&= \frac{p(q) \sum_i P(O|x_i, q)\pi_i}{p(O)} \\
&= \frac{\sum_i [P(O|x_i)p(q)\pi_i]}{p(O)}
\end{align*}
\]

Now assume that action \( k_{j^*} \) was observed in a new time step, while the opponent’s private information was not observed.

\[
P(q|O) = \frac{\sum_{i=1}^n \prod_j \gamma_j^{\sum_{i=1}^n \left[ \pi_i q_{kj^* | x_i} \prod_{h=1}^n \prod_{j=1}^n q_{kj|h|x_j}^{\alpha_{ijh}-1} \right]}}{p(O) \prod_{i=1}^n B(\alpha_{i1}, \ldots, \alpha_{im})}
\]

We now compute the expectation for the posterior probability that the opponent plays \( k_{j^*} \) with private information \( x_{i^*} \) as done in Section 3.1.

\[
P(A = k_{j^*} | O, C = x_{i^*}) = \frac{\int \left[ \sum_{i=1}^n \left[ \pi_i q_{kj^* | x_i} \prod_{h=1}^n \prod_{j=1}^n q_{kj|h|x_j}^{\alpha_{ijh}-1} \right] \right]}{p(O) \prod_{i=1}^n B(\alpha_{i1}, \ldots, \alpha_{im})}
\]

\[
= \frac{\sum_i \left[ \pi_i \prod_j B(\gamma_{1j}, \ldots, \gamma_{nj}) \right]}{Z},
\]
For each

\[ \gamma_{ij} = \alpha_{ij} + 2 \text{ if } i = i^* \text{ and } j = j^*; \gamma_{ij} = \alpha_{ij} + 1 \text{ if } j = j^* \text{ and } i \neq i^*, \text{ and } \gamma_{ij} = \alpha_{ij} \text{ otherwise.} \]

If we denote the numerator by \( \tau_{i^*j^*} \) then \( Z = \sum_{i^*} \tau_{i^*j^*}. \)

As in Section 3.1, we will need to generalize this to the case of multiple observed actions in order to obtain an efficient algorithm, because the posterior is not Dirichlet and can not be used directly as the prior for the next iteration. Suppose we have observed action \( k_j \theta_j \) times (in addition to the number of fictitious times indicated by the prior counts \( \alpha_{ij} \)). We compute \( P(q|O) \) analogously as

\[
P(q|O) = \frac{\sum_{i=1}^{n} \left[ \pi_i \sum_{\{\rho_{ab}\}} \prod_{h=1}^{m} \prod_{j=1}^{n} q_{kh|j}^{\rho_{ab}-1+\rho_{hb}} \right]}{p(O) \prod_{i=1}^{n} B(\alpha_{i1}, \ldots, \alpha_{im})},
\]

where the \( \sum_{\{\rho_{ab}\}} \) is over all values \( 0 \leq \rho_{ab} \leq \theta_{b} \) with \( \sum_{a} \rho_{ab} = \theta_{b} \) for each \( b \), for \( 1 \leq a \leq n, 1 \leq b \leq m \).

We can write this as \( \sum_{1} \ldots \sum_{m} \) where

\[
\Sigma_{b} = \sum_{\rho_{1b}=0}^{\theta_{b}} \sum_{\rho_{2b}=0}^{\theta_{b}-\rho_{1b}} \ldots \sum_{\rho_{nb}=0}^{\theta_{b}-\sum_{r=0}^{n-1} \rho_{rb}}.
\]

For each \( b \), the number of terms equals the number of ways of distributing the \( \theta_{b} \) observations amongst the \( n \) possible private information states, which equals

\[
C(\theta_{b} + n - 1, n - 1) = \frac{(\theta_{b} + n - 1) \ldots n}{\theta_{b}!} = O(n^{\theta_{b}}).
\]

Since we must do this for each of the \( m \) actions, the total number of terms is \( O(n^{Tm}) \), where \( T \) is the total number of game iterations which upper bounds the \( \theta_{b} \)'s. So the number of terms is exponential in the number of game iterations and the number of actions, but polynomial in the number of private information states. The final expression for \( P(A = k_{j^*}|O, C = x_{i^*}) \) is

\[
P(A = k_{j^*}|O, C = x_{i^*}) = \frac{\sum_{i} \left[ \pi_i \sum_{\{\rho_{ab}\}} \prod_{h} B(\alpha_{1h} + \rho_{1h}, \ldots, \alpha_{nh} + \rho_{nh}) \right]}{Z}.
\]

### 4 Algorithm for Uniform Prior Distribution Over Polyhedron

Another natural prior that has been studied previously is the uniform distribution over a polyhedron. This can model the situation when we think the opponent is playing uniformly at random within some region of a fixed strategy, such as a specific Nash equilibrium or a “population average” strategy based on historical data. Prior work has used this model to generate a class of opponents who are significantly more sophisticated than opponents who play uniformly at random over the entire space [5]). For example, in rock-paper-scissors, we may think the opponent is playing a strategy uniformly at random out of strategies that play each action with probability within \([0.31,0.35]\), as opposed to completely random over \([0,1]\).

Let \( v_{i,j} \) denote the \( j \)th vertex for player \( i \), where vertices correspond to mixed strategies. Let \( p^0 \) denote the prior distribution over vertices, where \( p^0_{i,j} \) denotes the probability that player \( i \) plays the strategy corresponding to vertex \( v_{i,j} \). Let \( V_i \) denote the number of vertices for player \( i \). Algorithm 2 gives an algorithm for computing the Bayesian best response in this setting. Correctness follows straightforwardly by applying Corollary 1 with the formula for the expectation of the uniform distribution.

### 5 Conclusion

One of the most fundamental problems in game theory is that of learning to play optimally against opponents who may make mistakes. We have presented the first exact algorithm for performing exploitation in
Algorithm 2 Algorithm for opponent exploitation with uniform prior distribution over polyhedron

Inputs: Prior distribution over vertices $p^0$, response functions $r_t$ for $0 \leq t \leq T$

\begin{align*}
M_0 &\leftarrow \text{strategy profile assuming opponent } i \text{ plays each vertex } v_{i,j} \text{ with probability } p^0_{i,j} = \frac{1}{V_i} \\
R_0 &\leftarrow r_0(M_0) \\
\text{Play according to } R_0 \\
\text{for } t = 1 \text{ to } T \text{ do} \\
\quad \text{for } i = 1 \text{ to } N \text{ do} \\
\quad \quad a_i &\leftarrow \text{action taken by player } i \text{ at time step } t \\
\quad \quad \text{for } j = 1 \text{ to } V_i \text{ do} \\
\quad \quad \quad p^t_{i,j} &\leftarrow p^{t-1}_{i,j} \cdot v_{i,j}(a_i) \\
\quad \quad \text{Normalize the } p^t_{i,j} \text{'s so they sum to 1} \\
M_t &\leftarrow \text{strategy profile assuming opponent } i \text{ plays each vertex } v_{i,j} \text{ with probability } p^t_{i,j} \\
R_t &\leftarrow r_t(M_t) \\
\text{Play according to } R_t
\end{align*}

imperfect-information games in the Bayesian setting using a very natural and the most well-studied prior distribution for this problem, the Dirichlet distribution. The algorithm runs in time polynomial in the number of information states for the opponent. Previously an exact algorithm had only been presented for normal-form games, and the best previous algorithm was a heuristic with no guarantees. We focused on the setting where the opponent’s private information was not observed after each iteration. We observed interesting counter-intuitive phenomena in a natural game. Future work can extend our analysis to many important settings. For example, we would like to study the setting when the opponent’s private information is observed after some iterations and not after others and general sequential games where the agents can take multiple actions (we focused on the case where each player took one action). We would also like to consider algorithms that could have an improved dependence on the number of actions and game iterations. Perhaps some of these extensions can be developed straightforwardly from our results. Relatedly, we would like to extend analysis for any number of agents. Our algorithm is not specialized for two-player zero-sum games (it applies to general-sum games); if we are able to compute the expectation of the posterior strategy against multiple opponent agents, then best responding to this strategy profile is just a single agent optimization and can be done in time linear in the size of the game regardless of the number of opponents. While the Dirichlet is the most natural and well-studied prior for this problem, we would also like to study other important distributions. We have also presented an algorithm for a natural prior that is the uniform distribution over a polyhedron, which could model the situation where we think the opponent is playing a strategy from a uniform distribution in a region around a particular strategy, such as a specific Nash equilibrium or a “population average” strategy based on historical data.

While our main algorithm is likely impractical for a large number of game iterations or actions due to its exponential dependence, it would be interesting to compare it experimentally for a small number of iterations and actions to the approximation heuristic that has been previously successful in limit Texas hold 'em poker [15]. Opponent exploitation is a fundamental problem, and our algorithm and extensions could be applicable to any domain modeled as an imperfect-information game. For example, many security game models have imperfect information, e.g., [10, 12], and opponent exploitation in security games has been a very active area of recent study, e.g., [13, 14].
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