Three families of grad div-conforming finite elements

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Abstract
Several smooth finite element de Rham complexes are constructed in three-dimensional space, which yield three families of grad div-conforming finite elements. The simplest element has only 8 degrees of freedom (DOFs) for a tetrahedron and 14 DOFs for a 3-rectangle. We show that these elements lead to conforming and convergent approximations to quad-div problems. As a by-product, we obtain some grad div-nonconforming elements. Numerical experiments validate the correctness and efficiency of the nonconforming elements for solving the quad-div problem.

Mathematics Subject Classification 65N30 · 65N15 · 65N12

1 Introduction

We are concerned in this paper with the grad div- or $H(\text{grad div})$-conforming finite elements used for discretizations of problems involving a quad-div operator (sometimes referred to as a fourth-order div operator). The quad-div operator appears in linear elasticity [1, 22, 23], where the integration of $(\nabla(\nabla \cdot u))^2$ represents the shear strain energy with the displacement of the elasticity body $u$. Moreover, the quad-div operator can be written as $((\nabla \cdot)^* \circ (\nabla \cdot))^* \circ (\nabla \cdot)$ which is one of the fourth-order operators of the formulation $(D^* \circ D)^* \circ D^* \circ D$. The biharmonic operator $\Delta^2$ and the quad-curl operator $(\nabla \times)^4$ are two well-known fundamental fourth-order operators of this formulation, which have been studied extensively. Many numerical
methods have been proposed for problems involving those two fourth-order operators. We refer to [3, 20, 21, 30–32] for conforming finite element methods and [8, 10, 11, 19, 25, 27–29, 33–35] for other methods. However, unlike the biharmonic operator and the quad-curl operator, very limited work has been done for problems involving the quad-div operator [16]. In this paper, we will propose grad div-conforming elements which can lead to conforming approximations of the quad-div problem.

We apply discrete de Rham complexes to construct three families of grad div-conforming finite elements in three space dimensions (3D). The discrete de Rham complex with appropriate smoothness has been an important and useful tool in designing finite elements and analyzing numerical schemes, see e.g., [4–6, 12, 18, 26]. Based on the de Rham complex with minimal smoothness, various well-known finite elements for computational electromagnetism or diffusion problems have been arranged in the finite element periodic table [7]. Motivated by problems in fluid and solid mechanics, there is a growing interest in constructing finite element de Rham complexes with enhanced smoothness, sometimes also referred to as Stokes complexes [13, 15, 20]. In this paper, for the conforming discretization of the quad-div problem, we will consider another variant of the de Rham complex, i.e.,

$$\mathbb{R} \subset \begin{array}{l} H^1(\Omega) \xrightarrow{\nabla} H(\text{curl}; \Omega) \xrightarrow{\nabla \times} H(\text{grad div}; \Omega) \xrightarrow{\nabla \cdot} H^1(\Omega) \rightarrow 0, \end{array} \quad (1)$$

where $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^3$. The precise definition of the notations used in the above complex is given in the subsequent sections. For simplicity of presentation, throughout this paper, we will assume that $\Omega$ is contractible. Then the exactness of (1) follows from standard results in, e.g., [4]. The two dimensional (2D) version of the complex (1) is

$$\mathbb{R} \subset \begin{array}{l} H^1(\Omega) \xrightarrow{\nabla \times} H(\text{grad div}; \Omega) \xrightarrow{\nabla \cdot} H^1(\Omega) \rightarrow 0, \end{array} \quad (2)$$

where $\nabla \times = (\partial_{x_2} u, -\partial_{x_1} u)^T$. If we rotate the complex (2) by $\frac{\pi}{2}$, we will get the following complex

$$\mathbb{R} \subset \begin{array}{l} H^1(\Omega) \xrightarrow{\nabla} H(\text{curl}^2; \Omega) \xrightarrow{\nabla \times} H^1(\Omega) \rightarrow 0 \end{array} \quad (3)$$

with $H(\text{curl}^2; \Omega) := \{u \in L^2(\Omega) : \nabla \times u \in L^2(\Omega) \text{ and } \nabla \times \nabla \times u \in L^2(\Omega)\}$, which has been studied in [20]. Therefore in this paper, we will only focus on the complex (1) to construct 3D grad div elements.

Our new finite elements fit into a subcomplex of (1):

$$\mathbb{R} \subset \Sigma_\Sigma \xrightarrow{\nabla} V_h \xrightarrow{\nabla \times} W_h \xrightarrow{\nabla \cdot} \Sigma_\Sigma^+ \rightarrow 0. \quad (4)$$

A starting point is to take $V_h$ as $C^0$ Lagrange finite element spaces. This leads to a natural choice of $V_h$, which is the first family of Nédélec elements. In addition, we choose Lagrange elements enriched with an interior bubble for $\Sigma_\Sigma^+$. The space $W_h \subset H(\text{grad div}; \Omega)$ is hence obtained as the curl of $V_h$ plus a complementary part,
mapped onto $\Sigma_h^+$ by div. Different orders of $\Sigma_h$ can yield different versions of $W_h$. Among the three versions of $W_h$ which we will construct in this paper, the simplest element has only 8 DOFs for a tetrahedron and 14 DOFs for a 3-rectangle. The space $W_h$ can be utilized as a conforming finite element space for solving the quad-div problem.

The tool of discrete complexes makes it possible to reach the goal of this paper, i.e., constructing grad div-conforming elements. Moreover, by this tool, we also fit the quad-div problem and its conforming finite element approximations into the framework of the finite element exterior calculus (FEEC) [4, 5] and hence enable the use of various tools from FEEC for the numerical analysis. For instance, we construct interpolation operators that commute with the differential operators. After that, the convergence result follows from a standard argument.

To validate the newly proposed elements, we fit conforming elements in the theoretical framework of the conforming finite element method and obtain the approximation property of the numerical solution. Moreover, we carry out a numerical experiment to validate the nonconforming elements.

The remaining part of the paper is organized as follows. In Sect. 2, we present notations and some basic facts from homological algebra. In Sect. 3, we define shape functions that form local exact sequences. In Sect. 4, we construct three families of grad div-conforming finite elements on tetrahedra and develop some theoretical results for them. In Sect. 5, we introduce the counterparts for 3-rectangles. In Sect. 6, we present a regularity estimate for the quad-div problem and apply the new elements to the problem. In Sect. 7, we provide numerical examples to verify the correctness and efficiency of our methods. Finally, concluding remarks and future work are given in Sect. 8.

2 Preliminaries

2.1 Notations

Unless otherwise specified, throughout the paper we assume $\Omega \subset \mathbb{R}^3$ is a contractible Lipschitz domain. We adopt conventional notations for Sobolev spaces such as $H^m(D)$ or $H^m_0(D)$ on a contractible sub-domain $D \subset \Omega$ furnished with the norm $\| \cdot \|_{m,D}$ and the semi-norm $|\cdot|_{m,D}$. In the case of $m = 0$, the space $H^0(D)$ coincides with $L^2(D)$ which is equipped with the inner product $\langle \cdot, \cdot \rangle_D$ and the norm $\| \cdot \|_D$. When $D = \Omega$, we drop the subscript $D$. We use $H^m(D)$ and $L^2(D)$ to denote the vector-valued Sobolev spaces $(H^m(D))^3$ and $(L^2(D))^3$.

In addition to the standard Sobolev spaces, we also define

$$H(\text{curl}; D) := \{ u \in L^2(D) : \nabla \times u \in L^2(D) \},$$

$$H(\text{div}; D) := \{ u \in L^2(D) : \nabla \cdot u \in L^2(D) \},$$

$$H(\text{grad div}; D) := \{ u \in L^2(D) : \nabla \cdot u \in H^1(D) \}.$$
For a subdomain $D$, a face $f$, or an edge $e$, we use $P_k$ to represent the space of polynomials with degree at most $k$, and $\tilde{P}_k$, the space of homogenous polynomials of degree $k$. The corresponding sets of vector polynomials are denoted as $P_k = (P_k(D))^3$ and $\tilde{P}_k = (\tilde{P}_k(D))^3$, respectively. We also define

$$R_k = P_{k-1} \oplus S_k$$

with $S_k = \{ p \in \tilde{P}_k \mid x \cdot p = 0 \}$, whose dimension is

$$\dim R_k = \frac{k(k + 2)(k + 3)}{2}.$$

We use $Q_{i,j,k}(D)$ to denote the polynomials with three variables ($x_1$, $x_2$, $x_3$) where the maximal degree is $i$ in $x_1$, $j$ in $x_2$, and $k$ in $x_3$. For simplicity, we drop the subscripts $i$ and $j$ when $i = j = k$. Similarly, we use $Q_{i,j}(f)$ to denote such polynomial spaces in 2D.

Let $T_h$ be a partition of the domain $\Omega$ consisting of tetrahedra or 3-rectangles. We denote $h_K$ as the diameter of an element $K \in T_h$ and $h$ as the mesh size of $T_h$. We adopt the following Piola mapping to relate the finite element function $u$ on a general element $K$ to a function $\hat{u}$ on the reference element $\hat{K}$ (the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ or the cube with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$):

$$u \circ F_K = \frac{B_K}{\det(B_K)} \hat{u}, \quad (5)$$

where the affine mapping

$$F_K(x) = B_K \hat{x} + b_K. \quad (6)$$

By a simple computation, we have

$$(\nabla \cdot u) \circ F_K = \frac{1}{\det(B_K)} \hat{\nabla} \cdot \hat{u}, \quad (7)$$

$$n \circ F_K = \frac{B_K^T \hat{n}}{\|B_K^T \hat{n}\|.} \quad (8)$$

We use $C$ to denote a generic positive $h$-independent constant.

### 2.2 Basic facts from homological algebra

We review some basic facts from homological algebra. For further details, we refer to, for example, [4]. A differential complex is a sequence of spaces $V^i$ and operators $d^i$:

$$0 \longrightarrow V^1 \xrightarrow{d^1} V^2 \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} V^n \xrightarrow{d^n} 0, \quad (9)$$
with \( d^{i+1} d^i = 0 \) for \( i = 1, 2, \ldots, n - 1 \). Denote \( \mathcal{N}(d^i) \) as the kernel space of the operator \( d^i \) in \( V^i \) and \( \mathcal{R}(d^i) \) as the image of the operator \( d^i \) in \( V^{i+1} \). Due to the definition of a complex, the inclusion \( \mathcal{N}(d^i) \subset \mathcal{R}(d^{i-1}) \) holds for each \( i \geq 2 \). Furthermore, if \( \mathcal{N}(d^i) = \mathcal{R}(d^{i-1}) \), we say that the complex (9) is exact at \( V^i \). At the two ends of the sequence, the complex is exact at \( V^1 \) if \( d^1 \) is injective (with trivial kernel), and is exact at \( V^n \) if \( d^{n-1} \) is surjective (with trivial cokernel). The complex (9) is referred to as exact if it is exact at all the spaces \( V^i \). If each space in (9) has finite dimensions, then a necessary (but not sufficient) condition for the exactness of (9) is the following dimension condition:

\[
\sum_{i=1}^{n} (-1)^i \dim(V^i) = 0.
\]

### 3 Local spaces and polynomial complexes

To define a finite element space, we must supply, for each element \( K \in T_h \),

- Shape functions;
- Degrees of freedom (DOFs) to guarantee appropriate continuity.

In this section, we will specify shape functions for each space involved in the complex (4). The local complex of function spaces on each \( K \in T_h \) for (4) is denoted as follows:

\[
\mathbb{R} \subset \Sigma_h^r(K) \xrightarrow{\nabla} V_h^r(K) \xrightarrow{\nabla \times} W_h^{r-1,k}(K) \xrightarrow{\nabla \cdot} \Sigma_h^{+,k-1}(K) \rightarrow 0. \tag{10}
\]

In addition to the complex (4), we introduce the complex (10) with two parameters \( r \) and \( k \) to specify degrees of spaces, which lead to several versions of complexes.

We let \( \Sigma_h^r(K) \) be \( P_r(K) \) for a tetrahedral element or \( Q_r(K) \) for a 3-rectangular element, and let \( V_h^r(K) \) be \( \mathcal{R}_r(K) \) for a tetrahedral element or \( \mathcal{Q}_{r-1, r, r}(K) \times \mathcal{Q}_{r, r-1, r}(K) \times \mathcal{Q}_{r, r, r}(K) \times \mathcal{Q}_{r, r, r, r}(K) \) for a 3-rectangular element. For a tetrahedral element \( K \), we set

\[
\Sigma_h^{+,k-1}(K) = \begin{cases} 
\Sigma_h^{k-1}(K), & k \geq 5, \\
\Sigma_h^{k-1}(K) \oplus \text{span}\{B_t\}, & k = 2, 3, 4,
\end{cases}
\]

where \( B_t = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \) with the barycentric coordinate \( \lambda_i \). For a 3-rectangular element \( K \), we set

\[
\Sigma_h^{+,k-1}(K) = \begin{cases} 
\Sigma_h^{k-1}(K), & k \geq 3, \\
\Sigma_h^{k-1}(K) \oplus \text{span}\{B_c\}, & k = 2,
\end{cases}
\]

where \( B_c = (x - x_l)(x - x_r)(y - y_f)(y - y_b)(z - z_d)(z - z_u) \) with the element \( K = (x_l, x_r) \times (y_f, y_b) \times (z_d, z_u) \).
To define $W^{r-1,k}(K)$, we introduce $p : C^\infty(\mathbb{R}^3) \mapsto \left[ C^\infty(\mathbb{R}^3) \right]^3$, which is defined by
\[
p u := \int_0^1 t^2 x(tx) \, dt,
\]
with $x := (x_1, x_2, x_3)^T$. As a special case of the Poincaré operators (see e.g., [13, 18]), $p$ has the following properties:

- the null-homotopy identity
  \[
  \nabla \cdot p u = u, \quad \forall u \in C^\infty(\mathbb{R}^3);
  \]
  \[\text{(11)}\]

- polynomial preserving property: if $u \in P_r(\mathbb{R}^3)$, then $p u \in P_{r+1}(\mathbb{R}^3)$.

To obtain grad div-conforming elements, we define a modified Poincaré operator $\tilde{p}$, whose normal component on each face in $\partial K$ is a constant. To this end, we first present the following lemmas.

**Lemma 1** A function $u \in V^k_h(K)$ is uniquely defined by

1. $\int_{e_i} u \cdot \tau_i \, ds$ for all $v \in P_{k-1}(e_i)$ at all edges $e_i$ of $K$,
2. $\int_{f_i} \nabla \times u \cdot n_i \, dA$ for all $v \in P_{k-1}(f_i)/\mathbb{R}$ or $Q_{k-1}(f_i)/\mathbb{R}$ at all faces $f_i$ of $K$,
3. $\int_{f_i} u_T \cdot v \, dA$ for all $v = \nabla_{f_i}(B_{f_i}q)$ with $q \in P_{k-3}(f_i)$ or $Q_{k-2}(f_i)$ at all faces $f_i$ of $K$,
4. $\int_K u \cdot v \, dV$ for all $v \in P_{k-3}(K)$ or $Q_{k-1,k-2,k-2}(K) \times Q_{k-2,k-1,k-2}(K) \times Q_{k-2,k-2,k-1}(K)$.

Here $\tau_i$ and $n_i$ are the unit tangential and the normal direction to the edge $e_i$ and the face $f_i$, respectively, $B_{f_i}$ is a bubble function on the face $f_i$, and $u_T = n_i \times u|_{f_i} \times n_i$.

**Proof** To prove the lemma, we show that $u = 0$ if all the above DOFs 1–4 vanish. We will prove only for a tetrahedral element $K$. We first show that $u_T = 0$ on $f \in \partial K$. By DOF 1 and integration by parts, we have
\[
\int_f \nabla \times u \cdot n_f \, dA = \int_{\partial f} u \cdot \tau_{\partial f} \, ds = 0,
\]
which, together with DOF 2 and the fact that $\nabla \times u \cdot n_f \in P_{k-1}(f)$, leads to
\[
\nabla \times u \cdot n_f = 0 \quad \text{on} \quad f.
\]

Therefore $u_T = \nabla_f p$ with $p \in P_k(f)$. By DOF 1 again, we have $u_T \cdot \tau_{\partial f} = 0$ on $\partial f$, i.e., the tangential derivative of $p$ along $\partial f$ is 0. Hence we can choose $p$ such that $p = 0$ on $\partial \Omega$, which implies $p = B_f q$ with $q \in P_{k-3}(f)$ and
\[
u_T = \nabla_f (B_f q).
\]

By DOF 3, we get $u_T = 0$ on $f$. The interior DOF 4 will then leads to $u = 0$ in $K$. \qed

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Lemma 2 For \( u \in \Sigma_h^k(K) \), there exist a function \( \varphi_u \in V_h^{k+1}(K) \) such that

\[
(pu - \nabla \times \varphi_u) \cdot n_i = \frac{1}{\text{area}(f_i)} \int_{f_i} pu \cdot n_i ds.
\]

Proof We define \( \varphi_u \in V_h^{k+1}(K) \) by setting the DOFs 1, 2, and 3 in Lemma 1 to be 0 and the DOF 2 as follows:

\[
\int_{f_i} \nabla \times \varphi_u \cdot n_i v dA = \int_{f_i} (pu - \nabla \times \varphi_u) \cdot n_i ds,
\]

for all \( v \in P_{k-1}(f_i)/\mathbb{R} \) or \( Q_{k-1}(f_i)/\mathbb{R} \) at all faces \( f_i \) of \( K \). According to the proof of Lemma 1, we have

\[
\nabla \times \varphi_u \cdot n_i = pu \cdot n_i - \frac{1}{\text{area}(f_i)} \int_{f_i} pu \cdot n_i ds,
\]

which finishes the proof.

Now we define the modified Poincaré operator \( \tilde{p} : C^\infty(\mathbb{R}^3) \mapsto [C^\infty(\mathbb{R}^3)]^3 \) by

\[
\tilde{p}u = pu - \nabla \times \varphi_u.
\]

From the definition of \( \tilde{p} \), we can see that the null-homotopy identity (11) still holds.

Now we are ready to define \( W_r^{r-1,k}(K) \).

When \( r \geq k \) with \( k \geq 5 \) for a tetrahedral element and \( k \geq 3 \) for a 3-rectangular element, we define

\[
W^{r-1,k}_h(K) = \nabla \times V^r_h(K) \oplus \Sigma_h^{+,k-1}(K).
\]  (12)

When \( r \geq k \) with \( k = 2, 3, 4 \) for a tetrahedral element and \( k = 2 \) for a 3-rectangular element and \( r = k - 1 \) with \( k \geq 2 \), we define

\[
W^{r-1,k}_h(K) = \nabla \times V^r_h(K) \oplus \tilde{p} \Sigma_h^{+,k-1}(K).
\]  (13)

By (11), the right-hand side of (12) and (13) is a direct sum.

Lemma 3 The local sequence (10) is an exact complex.

Proof It’s obvious that \( \nabla \Sigma_h^r(K) \subset V^r_h(K) \). Because of the definition of \( W^{r-1,k}_h(K) \) and the null-homotopy identity (11), we have \( \nabla \times V^r_h(K) \subset W^{r-1,k}_h(K) \) and \( \nabla \cdot W^{r-1,k}_h(K) = \Sigma_h^{+,k-1}(K) \). This shows that (10) is a complex. It remains to show the exactness. It’s easy to check the exactness at \( \Sigma_h^r(K) \) and \( V^r_h(K) \). We now show that, for any \( \psi_h \in W^{r-1,k}_h(K) \) for which \( \nabla \cdot \psi_h = 0 \), there exists a \( u_h \in V^r_h(K) \) s.t. \( \psi_h = \nabla \times u_h \). Since \( \psi_h \in W^{r-1,k}_h(K) \), we have \( \psi_h = \nabla \times u_h + p \psi_h \) or \( \psi_h = \nabla \times u_h \) s.t. \( \nabla \cdot u_h = 0 \).
\( \nabla \times u_h + \tilde{p} w_h \) with \( u_h \in V_h^r(K) \) and \( w_h \in \Sigma_h^{+,k-1}(K) \). By the null-homotopy identity (11) again, \( 0 = \nabla \cdot v_h = w_h \). Thus \( \phi^{r_w} \) is also 0 when \( \tilde{p} \) is involved. Therefore, \( v_h = \nabla \times u_h \). To prove the exactness at \( \Sigma_h^{+,k-1}(K) \), we only need to show the div operator \( \nabla \cdot : W_h^{r-1,k}(K) \to \Sigma_h^{+,k-1}(K) \) is surjective. It is surjective since \( \nabla \times W_h^{r-1,k}(K) = \Sigma_h^{+,k-1}(K) \).

In the following lemma, we show that \( W_h^{r-1,k}(K) \) contains some polynomial subspaces. It plays an essential role in analyzing the approximation properties of the finite element space \( W_h^{r-1,k} \).

**Lemma 4** The inclusion \( P_{r-1}(K) \subseteq W_h^{r-1,k}(K) \) holds when \( r \leq k + 1 \).

**Proof** We claim that

\[
P_{r-1}(K) = \nabla \times P_r(K) \oplus pP_{r-2}(K).
\]

(14)

In fact, by the polynomial preserving property of \( p \), \( \nabla \times P_r(K) \oplus pP_{r-2}(K) \subseteq P_{r-1}(K) \). To show (14), it remains to show the two sides of (14) have the same dimension. By the null-homotopy identity (11), the right hand side is a direct sum. Therefore,

\[
\dim[\nabla \times P_r(K) \oplus pP_{r-2}(K)] = \dim \nabla \times P_r(K) + \dim P_{r-2}(K)
= \dim[\nabla \times P_r(K)/\nabla P_{r+1}(K)] + \dim P_{r-2}(K)
= \dim P_r(K) - \dim P_{r+1}(K) + 1 + \dim P_{r-2}(K),
\]

which is exactly the dimension of \( P_{r-1}(K) \). Combining (14) and the fact that \( P_{r-2}(K) \subseteq \Sigma_h^{+,k-1}(K) \), we get \( P_{r-1}(K) \subseteq \nabla \times P_r(K) \oplus p\Sigma_h^{+,k-1}(K) \subseteq W_h^{r-1,k}(K) \). Similarly, we can prove the lemma when \( \tilde{p} \) is involved. \( \square \)

We are now ready to construct grad div-conforming finite elements and complexes. We focus on analyzing tetrahedral elements and only present the definition of 3-rectangular elements in Sect. 5.

**4 Three families of grad div-conforming elements on tetrahedra**

The global discrete complex with specified degree for each space is given by

\[
\begin{aligned}
\mathbb{R} & \xleftarrow{\nabla} \Sigma_h^{r} & \xrightarrow{\nabla \times} & \quad V_h^{r} & \xrightarrow{\nabla \cdot} & \quad W_h^{r-1,k} & \xrightarrow{\nabla \cdot} & \quad \Sigma_h^{+,k-1} & \xrightarrow{0}.
\end{aligned}
\]

(15)

In this section, we construct grad div-conforming finite elements and complexes on tetrahedra. Assigning \( r = k - 1, k \), and \( k + 1 \) in (15) leads to three versions of grad div-conforming element spaces \( W_h^{k-2,k}, W_h^{k-1,k}, \) and \( W_h^{k,k} \), for which, Fig. 1 demonstrates the case \( k = 2 \).
4.1 Degrees of freedom and global finite element spaces

We define DOFs for the spaces in (15). The DOFs for the Lagrange element $\Sigma^r_h$ can be given as follows.

- **Vertex DOFs $M_v(u)$:**
  \[
  M_v(u) = \{ v_i(u) \text{ for all vertices } v_i \}.
  \]

- **Edge DOFs $M_e(u)$:**
  \[
  M_e(u) = \left\{ \frac{1}{\text{length}(e_i)} \int_{e_i} u v \, ds \text{ for all } v \in P_{r-2}(e_i) \text{ and for all edges } e_i \right\}.
  \]

- **Face DOFs $M_f(u)$:**
  \[
  M_f(u) = \left\{ \frac{1}{\text{area}(f_i)} \int_{f_i} u v \, dA \text{ for all } v \in P_{r-3}(f_i) \text{ and for all faces } f_i \right\}.
  \]

- **Interior DOFs $M_K(u)$:**
  \[
  M_K(u) = \left\{ \frac{1}{\text{vol}(K_i)} \int_{K_i} u v \, dV \text{ for all } v \in P_{r-4}(K_i) \text{ and all elements } K_i \right\}.
  \]

For $u \in H^{3/2+\delta}(\Omega)$ with $\delta > 0$, we can define an $H^1$ interpolation operator $\pi_h : H^{3/2+\delta}(\Omega) \rightarrow \Sigma^r_h$ by the above DOFs s.t.

\[
M_v(u - \pi_h u) = \{0\}, \quad M_e(u - \pi_h u) = \{0\}, \quad M_f(u - \pi_h u) = \{0\}, \quad \text{and } M_K(u - \pi_h u) = \{0\}.
\]

The DOFs for $\Sigma^+_{h,k-1}$ can be given similarly, with only one additional interior integration DOF on $K$ to deal with the interior bubble function. We denote $\tilde{\pi}_h$ as the $H^1$ interpolation operator to $\Sigma^+_{h,k-1}$ by these DOFs.

We choose the space $V^r_h$ as the first family of Nédélec elements, which has the following DOFs:

- **Edge DOFs $M^e_e(u)$ (with a unit tangential vector $\tau_i$):**
  \[
  M^e_e(u) = \left\{ \int_{e_i} u \cdot \tau_i v \, ds \text{ for all } v \in P_{r-1}(e_i) \text{ and for all edges } e_i \right\}.
  \]

- **Face DOFs $M^f_f(u)$ (with a unit normal vector $n_i$):**
  \[
  M^f_f(u) = \left\{ \frac{1}{\text{area}(f_i)} \int_{f_i} u v \, dA \text{ for all } v = B_K \hat{v}, \hat{v} \in P_{r-2}(\hat{f}_i), \hat{v} \cdot \hat{n}_i = 0 \text{ and for all faces } f_i \right\}.
  \]
– Interior DOFs $M_K(u)$:

$$M_K(u) = \left\{ \int_{K_i} uv \, dV \text{ for all } v = \frac{B_{K_i}^T \hat{v}}{\det(B_{K_i})} \text{ with } \hat{v} \in P_{r-3}(\hat{K}_i) \right\}.$$ 

Assuming that $u \in H^{1/2+\delta}(\Omega)$ and $\nabla \times u \in L^p(\Omega)$ with $\delta \geq 0$ and $p > 2$. By the above DOFs, we define an $H(\text{curl})$ interpolation operator $r_h$ which maps to $V_h^r$ and satisfies

$$M_e(u - r_h u) = \{0\}, \quad M_f(u - r_h u) = \{0\}, \quad \text{and } M_K(u - r_h u) = \{0\}.$$ 

We now equip the space $W_h^{r-1,k}(K)$ with the following DOFs:

– Vertex DOFs $M_v(u)$ at all vertices $v_i$ of each $K$:

$$M_v(u) = \{(\nabla \cdot u)(v_i), \ i = 1, 2, \ldots, 4\}. \quad (16)$$

– Edge DOFs $M_e(u)$ at all edges $e_i$ of each $K$:

$$M_e(u) = \left\{ \frac{1}{\text{length}(e_i)} \int_{e_i} \nabla \cdot u q \, ds, \ \forall q \in P_{k-3}(e_i), i = 1, 2, \ldots, 6 \right\}. \quad (17)$$

– Face DOFs $M_f(u)$ at all faces $f_i$ of each $K$ (with the unit normal vector $n_i$):

$$M_f(u) = \left\{ \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \cdot u q \, dA, \ \forall q \in P_{k-4}(f_i), i = 1, 2, \ldots, 4 \right\} \cup \left\{ \int_{f_i} u \cdot n_i q \, dA, \ \forall q \in P_{r-1}(f_i), i = 1, 2, \ldots, 4 \right\}, \quad (18)$$

– Interior DOFs $M_K(u)$ for each element $K$:

$$M_K(u) = \left\{ \int_K u \cdot q \, dV, \ \forall q = B_K^{-T} \hat{q}, \ \hat{q} \in D \right\}, \quad (19)$$

where $D = \nabla P_{k-5}(\hat{K}) \oplus [P_{r-2}(\hat{K})/\nabla P_{r-1}(\hat{K})]$ when $k \geq 5; \ D = [P_{r-2}(\hat{K})/\nabla P_{r-1}(\hat{K})]$ when $k < 5$ and $r \geq 2; \ D = \emptyset$ when $k < 5$ and $r < 2$.

**Lemma 5** The DOFs (16)-(19) are well-defined for any $u \in H^{1/2+\delta}(K)$ and $\nabla \cdot u \in H^{3/2+\delta}(K)$ with $\delta > 0$.

**Proof** It follows from the Cauchy-Schwarz inequality that the face DOFs (18) and the interior DOFs (19) are well-defined since $u \in H^{1/2+\delta}(K)$ and $\nabla \cdot u \in H^{3/2+\delta}(K)$. By the embedding theorem, we have $\nabla \cdot u \in H^{3/2+\delta}(K) \hookrightarrow C^{0,\delta}(K)$, then the DOFs in (16) and (17) are well-defined. \(\Box\)
Lemma 6 The DOFs for \( W_{r-1}^{r-1,k}(K) \) are unisolvent.

Proof Since the decomposition (12) and (13) is a direct sum, \( \dim W_{r-1}^{r-1,k}(K) = \dim \nabla \times V_{r}^{r}(K) + \dim \Sigma_{h}^{r,k-1}(K) = \dim \nabla \times [R_{r}(K)/\nabla P_{r}(K)] + \dim P_{k-1}(K) = \dim[R_{r}(K)/\nabla P_{r}(K)] + \dim P_{k-1}(K) = (r+2)(r+3)(2r-1)/6 + k(k+1)(k+2)/6 + 1 \) when \( k \geq 5 \) and \( \dim W_{r-1}^{r-1,k}(K) = (r+2)(r+3)(2r-1)/6 + k(k+1)(k+2)/6 + 2 \) when \( k = 2, 3, 4 \). By counting the number of DOFs, we find the DOF set has the same dimension. Then it suffices to show that if all the DOFs vanish on a function \( u \), then \( u = 0 \). To see this, we first observe that \( \nabla \cdot u = 0 \) by the unisolvence of the DOFs of \( \Sigma_{h}^{r,k-1}(K) \). Then \( u = \nabla \times \phi \in P_{r-1}(K) \) for some \( \phi \in V_{h}^{r}(K) \). By the face DOFs of \( W_{r-1}^{r-1,k}(K) \), \( u \cdot n = 0 \) on faces. By integration by parts, we have, for any \( q \in P_{r-1}(K) \),

\[
(u, \nabla q)_K = (\nabla \times \phi, \nabla q)_K = (\nabla \times \phi \cdot n, q)_{\partial K} = (u \cdot n, q)_{\partial K} = 0,
\]

which, together with the interior DOFs, leads to

\[
(u, q)_K = 0 \text{ for any } q \in P_{r-2}(K).
\]
Since $u \cdot n = 0$ on faces, $\hat{u} = (\hat{x}_1\hat{\phi}_1, \hat{x}_2\hat{\phi}_2, \hat{x}_3\hat{\phi}_3)^T$. Note that $u$ and $\hat{u}$ are related by (5). Choosing $\varphi = B_K^{-T} \hat{\varphi} = B_K^{-T} (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3)^T \in P_{r-2}(K)$, we have by (20):

$$0 = (u, \varphi)_K = \det(B_K)^{-1}(\hat{u}, \hat{\varphi})_K = \det(B_K)^{-1}((\hat{x}_1\hat{\phi}_1, \hat{x}_2\hat{\phi}_2, \hat{x}_3\hat{\phi}_3)^T, \hat{\varphi})_K.$$ 

This implies that $\varphi = 0$ and hence $u = 0$. \hfill \Box

Provided $u \in H^{1/2+\delta}(\Omega)$, and $\nabla \cdot u \in H^{3/2+\delta}(\Omega)$ with $\delta > 0$ (see Lemma 5), we can define an $H(\text{grad div})$ interpolation operator $i_h$ whose restriction on $K$ is denoted as $i_K$ and defined by

$$M_v(u - i_K u) = \{0\}, \ M_e(u - i_K u) = \{0\}, \ M_f(u - i_K u) = \{0\}, \ M_K(u - i_K u) = \{0\},$$

where $M_v$, $M_e$, $M_f$ and $M_K$ are the sets of DOFs in (16)-(19).

Gluing the local spaces by the above DOFs, we obtain the global finite element space $W_h^{r-1,k}$.

**Lemma 7** The following conformity holds:

$$W_h^{r-1,k} \subset H(\text{grad div}; \Omega).$$

**Proof** It’s straightforward to have $\nabla \cdot W_h^{r-1,k} \subset \Sigma_h^{+,k-1} \subset H^1(\Omega)$. To show that $W_h^{r-1,k} \subset H(\text{grad div}; \Omega)$, it remains to prove that the normal component of functions in $W_h^{r-1,k}$ is single-valued on each face. From the definition of $W_h^{r-1,k}(K)$, the normal component of $u \in W_h^{r-1,k}(K)$ is a polynomial of order $r - 1$ on each face. The DOFs in (18) can uniquely determine the normal component of $u$. \hfill \Box

**Remark 1** If we use $p$ instead of $\hat{p}$ to define $W_h^{r-1,k}(K)$ in (13), then the normal component of functions $W_h^{r-1,k}(K)$ is a polynomial of order greater than $r - 1$. The resulting elements are nonconforming in $H(\text{grad div}; \Omega)$. The nonconforming elements can also be applied to solve the quad-div problem, see Sect. 7.

### 4.2 Global finite element complexes for the quad-div problem

With well-defined global finite element spaces, we now develop some properties of the complex (15) containing these spaces.

**Theorem 1** The complex (15) is exact on contractible domains.

**Proof** We first show the exactness at $V_h^r$ and $W_h^{r-1,k}$. To this end, we show that for any $v_h \in V_h^r \subset H(\text{curl}; \Omega)$ and $u_h \in W_h^{r-1,k} \subset H(\text{grad div}; \Omega) \subset H(\text{div}; \Omega)$ satisfying $\nabla \times v_h = 0$ and $\nabla \cdot u_h = 0$, there exists $p_h \in \Sigma_h^r$ and $\phi_h \in \Sigma_h^{+,k-1}$ such that $v_h = \nabla p_h$ and $u_h = \nabla \times \phi_h$. Actually, this follows from the exactness of the standard finite element differential forms (e.g., [4]). To prove the exactness at $\Sigma_h^{+,k-1}$, that is to
prove the operator $\nabla \cdot$ from $W^{r-1,k}_h$ to $\Sigma^{+,k-1}_h$ is surjective, we count the dimensions. The dimension count of the Lagrange elements reads:

$$\dim \Sigma^r_h = V + (r - 1)E + \frac{1}{2}(r - 2)(r - 1)F + \frac{1}{6}(r - 3)(r - 2)(r - 1)K,$$

where $V$, $E$, $F$, and $K$ denote the number of vertices, edges, faces, and 3D cells, respectively. The dimension count of the space $V^r_h$ reads:

$$\dim V^r_h = E + (r - 1)F + \frac{1}{2}r(r - 1)(r - 2)K.$$

From the DOFs (16)-(19),

$$\dim W^{r-1,k}_h - \dim \Sigma^{+,k-1}_h = \frac{1}{2}r(r + 1)F + \frac{1}{6}r(r + 1)(2r - 5)K.$$

From the above dimension count, we have

$$-1 + \dim \Sigma^r_h - \dim V^r_h + \dim W^{r-1,k}_h - \dim \Sigma^{+,k-1}_h = 0,$$

where we have used Euler’s formula $V - E + F - K = 1$. This completes the proof. □

We summarize the interpolations defined in Sect. 4.1 in the following diagram:

$$\begin{align*}
\mathbb{R} \xleftarrow{\pi_h} H^1(\Omega) & \xrightarrow{\nabla} H(\text{curl}; \Omega) \xrightarrow{\nabla \times} H(\text{grad div}; \Omega) \xrightarrow{\nabla \cdot} H^1(\Omega) \rightarrow 0 \\
\mathbb{R} \xleftarrow{\Sigma} V & \xrightarrow{\nabla} V \xrightarrow{\nabla \times} W \xrightarrow{\nabla \cdot} \Sigma \rightarrow 0 \quad (21) \\
\mathbb{R} \xleftarrow{\Sigma^r_h} V^r_h & \xrightarrow{\nabla \times} W^{r-1,k}_h \xrightarrow{\nabla \cdot} \Sigma^{+,k-1}_h \rightarrow 0.
\end{align*}$$

Here $\Sigma$, $V$, $W$ are three subspaces of $H^1(\Omega)$, $H(\text{curl}; \Omega)$, and $H(\text{grad div}; \Omega)$ in which $\pi_h$ (or $\tilde{\pi}_h$), $r_h$, and $i_h$ are well-defined.

Now we show that the interpolations in (21) commute with the differential operators. In addition to Lemma 4, this result also plays a key role in the error analysis below for the interpolations.

**Lemma 8** The last two rows of the complex (21) are a commuting diagram, i.e.,

$$\begin{align*}
\nabla \pi_h u &= r_h \nabla u \text{ for all } u \in \Sigma, \quad (22) \\
\nabla \times r_h u &= i_h \nabla \times u \text{ for all } u \in V, \quad (23) \\
\nabla \cdot i_h u &= \tilde{\pi}_h \nabla \cdot u \text{ for all } u \in W. \quad (24)
\end{align*}$$
The following lemma relates the interpolation on \( K \) to that on \( \hat{K} \).

**Lemma 9** For \( u \in W \), under the transformation (5), we have \( \hat{i}_K u = i_{\hat{K}} \hat{u} \).

**Proof** By the transformations (5), (7), and (8), we have all the DOFs in (19) and the second set of (18) are equivalent with those for \( \hat{u} \) on \( \hat{K} \). In addition, all the DOFs in (16), (17), and the first set of (18) are differed from those for \( \hat{u} \) on \( \hat{K} \) by a factor \( \frac{1}{\det(B_K)} \). This means the DOFs for defining \( \hat{i}_K u \) and those for defining \( i_{\hat{K}} \hat{u} \) are differed by a constant factor. According to Proposition 3.4.7 in [9], we complete the proof. □

Next, we establish the approximation property of the interpolation operator.

**Theorem 2** For \( r = k - 1, k \), or \( k + 1 \), if \( u \in H^{s+(r-k)}(\Omega) \) and \( \nabla \cdot u \in H^s(\Omega) \), \( 3/2 + \delta \leq s \leq k \) with \( \delta > 0 \), then we have the following error estimates for the interpolation \( i_h \),

\[
\| u - i_h u \|_K \leq C h^{s+(r-k)} (\| u \|_{s+(r-k)} + \| \nabla \cdot u \|_s),
\]

\[
\| \nabla \cdot (u - i_h u) \| \leq C h^{s} \| \nabla \cdot u \|_s.
\]

\[
| \nabla \cdot (u - i_h u) |_1 \leq C h^{s-1} \| \nabla \cdot u \|_s.
\]

**Proof** We only prove the results for integer \( s \) to avoid the technical complications. To prove (25), we first apply the transformation (5) and Lemma 9 to derive, for a general element \( K \in \mathcal{T}_h \),

\[
\| u - i_K u \|_K = \left( \int_{\hat{K}} \det(B_K)^{-1} \left| B_K (\hat{u} - i_{\hat{K}} \hat{u}) \right|^2 d\hat{V} \right)^{\frac{1}{2}}
\]

\[
\leq \left| \det(B_K) \right|^{-\frac{1}{2}} B_K \| \hat{u} - i_{\hat{K}} \hat{u} \|_{\hat{K}}.
\]

From Lemma 4, \( P_{r-1}(K) \subseteq W^{r-1,k}_h(K) \). Therefore, we have \( i_{\hat{K}} \hat{p} = \hat{p} \) when \( \hat{p} \in P_{r-1}(\hat{K}) \). We obtain, with the help of Lemma 5 and Theorem 5.5 in [24],

\[
\| \hat{u} - i_{\hat{K}} \hat{u} \|_{\hat{K}} = \left\| (I - i_{\hat{K}}) (\hat{u} + \hat{p}) \right\|_{\hat{K}}
\]

\[
\leq \inf_{\hat{p} \in P_{r-1}(\hat{K})} C \left( \| \hat{u} + \hat{p} \|_{s+(r-k),\hat{K}} + \| \hat{\nabla} \cdot (\hat{u} + \hat{p}) \|_{s,\hat{K}} \right)
\]

\[
\leq C \left( \| \hat{u} \|_{s+(r-k),\hat{K}} + \| \hat{\nabla} \cdot \hat{u} \|_{s,\hat{K}} \right).
\]

Mapping back to the general element \( K \) leads to

\[
\| u - i_K u \|_K \leq C h^{s+(r-k)} (\| u \|_{s+(r-k),K} + \| \nabla \cdot u \|_{s,K}).
\]

Summing the above inequality (28) over \( K \in \mathcal{T}_h \), we obtain (25).
As for (26) and (27), we use Lemma 8 and Lemma 4 to obtain, for $i = 0, 1$,

$$\|\nabla \cdot (u - i_h u)\|_i = \|\nabla \cdot u - \tilde{\pi}_h \nabla \cdot u\|_i \leq C h^{s-i} \|\nabla \cdot u\|_s,$$

where we have used the standard approximation property of the Lagrange finite element.

\[\square\]

5 Three families of grad div-conforming elements on 3-rectangles

In this section, we will construct three families of grad div-conforming finite element spaces $W_{r-1,k}$ for a 3-rectangular mesh by taking $r = k - 1, k, \text{ and } k + 1$ in (15).

Proceeding as in the tetrahedral case, we choose the Lagrange element space of order $r$ on 3-rectangles for $\Sigma^r_h$. The space $\Sigma^{+,k-1}_h$ is the $(k-1)$-th order Lagrange element space, which is enriched with an interior bubble function when $k = 2$. In addition, we choose the first Nédélec element space of order $r$ for $V^r_h$. The DOFs for these spaces can be found in [24, Chapter 6]. In the following, we will define DOFs for $W_{r-1,k}$. The three grad div elements with $r = k - 1, r = k, \text{ and } r = k + 1$ with $k = 2$ for a 3-rectangular element are shown in Fig. 2.

- **Vertex DOFs** $M_v(u)$ at all vertices $v_i$ of $K$:

$$M_v(u) = \{(\nabla \cdot u)(v_i), \ i = 1, 2, \ldots, 8\}.$$

- **Edge DOFs** $M_e(u)$ at all edges $e_i$ of $K$:

$$M_e(u) = \left\{ \frac{1}{\text{length}(e_i)} \int_{e_i} \nabla \cdot u q ds, \ \forall q \in P_{k-3}(e_i), \ i = 1, 2, \ldots, 12 \right\}.$$

- **Face DOFs** $M_f(u)$ at all faces $f_i$ of $K$ (with the unit normal vector $n_i$):
\[
M_f(u) = \left\{ \frac{1}{\text{area}(f_i)} \int_{f_i} \nabla \cdot u q dA, \ \forall q \in Q_{k-3,k-3}(f_i), i = 1, 2, \ldots, 6 \right\} \\
\cup \left\{ \int_{f_i} u \cdot n_i q dA, \ \forall q \in Q_{r-1,r-1}(f_i), i = 1, 2, \ldots, 6 \right\},
\]

- Interior DOFs \( M_K(u) \):

\[
M_K(u) = \left\{ \int_K u \cdot q dV, \ \forall q = B_K^{-T} \hat{q}, \ \hat{q} \in D \right\},
\]

where \( D = \emptyset \) when \( k = 2, r < 2 \); \( D = \{ Q_{r-2,r-1,r-1}(K) \times Q_{r-1,r-2,r-1}(K) \times Q_{r-1,r-1,r-2}(K)/\nabla Q_{r-1}(K) \} \) when \( k = 2, r \geq 2 \); \( D = \{ Q_{r-2,r-1,r-1}(K) \times Q_{r-1,r-2,r-1}(K) \times Q_{r-1,r-1,r-2}(K)/\nabla Q_{r-1}(K) \} \oplus \nabla Q_{k-3}(K) \) when \( k \geq 3 \).

The same theoretical results as the tetrahedral elements can be obtained by a similar argument. We omit them for this case.

### 6 The application to quad-div problems

In this section, we use the \( H(\text{grad div}) \)-conforming finite elements to solve the quad-div problem which is stated as follows:

For \( f \in H(\text{curl}; \ \Omega) \) and \( g \in L^2(\Omega) \), find \( u \), such that

\[
(\nabla \nabla) \cdot u + u = f \text{ in } \Omega, \\
\nabla \times u = g \text{ in } \Omega, \\
\quad u \cdot n = 0 \text{ on } \partial \Omega, \\
\quad \nabla \cdot u = 0 \text{ on } \partial \Omega. \tag{29}
\]

Here, to make the problem consistent, \( g = \nabla \times f \). By taking curl on both sides of the first equation of (29), we see that the condition \( \nabla \times u = g \) holds automatically.

We define \( H_0(\text{grad div}; \ \Omega) \) and \( H_0(\text{div}; \ \Omega) \) with vanishing boundary conditions:

\[
H_0(\text{grad div}; \ \Omega) := \{ u \in H(\text{grad div}; \ \Omega) : u \cdot n = 0 \text{ and } \nabla \cdot u = 0 \text{ on } \partial \Omega \}, \\
H_0(\text{div}; \ \Omega) := \{ u \in H(\text{div}; \ \Omega) : u \cdot n = 0 \text{ on } \partial \Omega \}.
\]

The variational formulation is to seek \( u \in H_0(\text{grad div}; \ \Omega) \) such that

\[
a(u, v) = (f, v) \quad \forall v \in H_0(\text{grad div}; \ \Omega), \tag{30}
\]

with \( a(u, v) := (\nabla(\nabla \cdot u), \nabla(\nabla \cdot v)) + (u, v) \).

It follows from Lax-Milgram Lemma that (30) is well-posedness. Taking \( v = \nabla \times \psi \) with \( \psi \in H_0(\text{curl}; \ \Omega) \) in (30) leads to

\[
(u, \nabla \times \psi) = (f, \nabla \times \psi) = (g, \psi),
\]
which implies $\nabla \times \mathbf{u} = \mathbf{g}$ holds in the sense of $L^2(\Omega)$. Since the regularity of the solution plays a crucial role in the error analysis, we will first derive a regularity result for the quad-div problem before proceeding further.

**Lemma 10** Suppose $\Omega$ is a polyhedron. Consider the problem $-\Delta \mathbf{u} = \mathbf{f}$ in $\Omega$ with $\mathbf{u} = 0$ on $\partial \Omega$. There exists a constant $\alpha_0 > 1/2$ satisfying the same conditions as in [14, Theorem 2.2.1] such that

$$u \in H^{1+\alpha_0}(\Omega) \text{ when } f \in H^{\alpha_0-1}(\Omega).$$

In particular, when $\Omega$ is convex, $\alpha_0$ is at least 1.

**Theorem 3** Under the assumption of $\Omega$, we assume further $\Omega$ is polyhedron. For $1/2 < \alpha \leq \min\{\alpha_0, 1\}$ with $\alpha_0$ defined in Lemma 10, the solution $\mathbf{u}$ of (29) satisfies $\mathbf{u} \in H^{\alpha}(\Omega)$ and $\nabla \cdot \mathbf{u} \in H^{1+\alpha}(\Omega)$. Moreover, it admits the following decomposition:

$$\mathbf{u} = \nabla \times \mathbf{w}^{\text{sing}} + \mathbf{u}^{\text{reg}}$$

with $\mathbf{u}^{\text{reg}} \in H^{2+\alpha}(\Omega)$ and $\mathbf{w}^{\text{sing}} \in H^{1+\alpha}(\Omega)$.

**Proof** Let $w = \nabla \cdot \mathbf{u}$, then, from the first equation of (29), we have

$$\nabla (\nabla \cdot \nabla w) = \nabla (\Delta w) = \mathbf{f} - \mathbf{u} \in L^2(\Omega),$$

which implies $\Delta w \in H^1(\Omega)$. Note that $w = \nabla \cdot \mathbf{u} = 0$ on $\partial \Omega$. Applying Lemma 10, we obtain $w = \nabla \cdot \mathbf{u} \in H^{1+\alpha_1}(\Omega)$ with $1/2 < \alpha_1 \leq \alpha_0$.

Since $\mathbf{u} \in H_0(\text{div}; \Omega) \cap H(\text{curl}; \Omega) \hookrightarrow H^{\alpha_2}(\Omega)$ with $1/2 < \alpha_2 \leq \min\{1, \alpha_0\}$ [2, Proposition 3.7], we have $\mathbf{u} \in H^{\alpha_2}(\Omega)$. Taking $\alpha = \min\{\alpha_1, \alpha_2\}$ yields $\mathbf{u} \in H^{\alpha}(\Omega)$ and $\nabla \cdot \mathbf{u} \in H^{1+\alpha}(\Omega)$.

Let $\Omega$ be a bounded, smooth, contractible open set with $\overline{\Omega} \subset \Omega$. For the solution $\mathbf{u}$, we can extend $\mathbf{u}$ in the following way:

$$\tilde{\mathbf{u}} = \begin{cases} \mathbf{u}, & \Omega, \\ 0, & \Omega - \overline{\Omega}. \end{cases}$$

It follows from $\mathbf{u} \in H_0(\text{div}; \Omega)$ and $\nabla \cdot \mathbf{u} \in H^{1+\alpha}(\Omega) \cap H_0^1(\Omega)$ that $\tilde{\mathbf{u}} \in H_0(\text{div}; \Omega)$ and $\nabla \cdot \tilde{\mathbf{u}} \in H_0^1(\Omega) \cap H^{1+\alpha}(\Omega)$. We consider the problem of finding $\psi$ defined in $\Omega$ such that

$$-\Delta \psi = -\nabla \cdot \tilde{\mathbf{u}} \in H^{1+\alpha}(\Omega) \text{ in } \Omega,$$

$$\psi = 0 \text{ on } \partial \Omega. \quad (31)$$

By the regularity result of the Laplace problem [17, Theorem 1.8], there exists a function $\psi \in H^{3+\alpha}(\Omega)$ satisfying (31) and (32). Rewriting (31) and restricting on $\Omega$, we have

$$\nabla \cdot (\mathbf{u} - \nabla \psi) = 0 \text{ in } \Omega.$$
with \( \mathbf{u} - \nabla \psi \in H^\alpha(\Omega) \). According to [17, Remark 3.12], there exists \( \mathbf{w} \in H^{1+\alpha}(\Omega) \) such that

\[
\mathbf{u} - \nabla \psi = \nabla \times \mathbf{w} \quad \text{and} \quad \nabla \cdot \mathbf{w} = 0 \quad \text{in} \quad \Omega.
\] (33)

Denoting \( \mathbf{u}^{\text{reg}} = \nabla \psi \in H^{2+\alpha}(\Omega) \) and \( \mathbf{w}^{\text{sing}} = \mathbf{w} \in H^{1+\alpha}(\Omega) \), we have, from (33), that \( \mathbf{u} = \nabla \times \mathbf{w}^{\text{sing}} + \mathbf{u}^{\text{reg}} \).

\[ \blacksquare \]

Remark 2 With Theorem 3, the interpolation \( i_h \mathbf{u} \) is well-defined.

We now present the finite element scheme. We define the finite element space with vanishing boundary conditions

\[
W_h^0 = \{ \mathbf{v}_h \in W_h^{r-1,k}, \quad \mathbf{n} \cdot \mathbf{v}_h = 0 \quad \text{and} \quad \nabla \cdot \mathbf{v}_h = 0 \quad \text{on} \quad \partial \Omega \}.
\]

The \( H(\text{grad div}) \)-conforming finite element method reads: seek \( u_h \in W_h^0 \), such that

[34]

\[
a(u_h, v_h) = (f, v_h) \quad \forall v_h \in W_h^0.
\]

Then we have the following approximation property of \( u_h \).

Theorem 4 For \( r = k-1 \), \( r = k \), or \( r = k+1 \), if \( \mathbf{u} \in H^{s+(r-k)}(\Omega) \) and \( \nabla \cdot \mathbf{u} \in H^s(\Omega) \), \( 3/2 + \delta \leq s \leq k \) with \( \delta > 0 \), then we have the following error estimates for the numerical solution \( u_h \),

\[
\| \mathbf{u} - u_h \| + \| \nabla \nabla \cdot (\mathbf{u} - u_h) \| \leq C h^{s-1} (\| \nabla \cdot \mathbf{u} \|_s + \| \mathbf{u} \|_{s-1}).
\] (35)

\[
\| \nabla \cdot (\mathbf{u} - u_h) \| \leq C h^{\min\{s,2(s-1)\}} (\| \nabla \cdot \mathbf{u} \|_s + \| \mathbf{u} \|_{s-1}),
\] (36)

\[
\| \mathbf{u} - u_h \| \leq C h^{\min\{s,2(s-1)\}} (\| \mathbf{u} \|_s + \| \nabla \cdot \mathbf{u} \|_s), \quad \text{when} \quad r = k, k + 1.
\] (37)

Proof The estimates (35) and (36) follow immediately from Céa’s lemma and the duality argument. By a similar idea to the proof of [27, Theorem 6], we can obtain (37).

\[ \blacksquare \]

Remark 3 When \( \Omega \) is a Lipschitz polyhedron, even with the lowest regularity, the scheme still has a convergence order \( 1/2 \) in \( H(\text{grad div}) \) norm.

Remark 4 The estimate for \( \| \mathbf{u} - u_h \| \) is not optimal when \( r = k + 1 \).

7 Numerical experiments

The validity of the conforming elements can be guaranteed by the theoretical analysis. We now use several numerical tests to validate the nonconforming elements without the modification of the Poincaré operator. We consider the problem (29) on a unit cube \( \Omega = (0, 1) \times (0, 1) \times (0, 1) \) with an exact solution

\[
\mathbf{u} = \nabla \left( x^3 y^3 z^3 (x - 1)^3 (y - 1)^3 (z - 1)^3 \right).
\] (38)
Three families of grad div-conforming finite elements

Table 1 Example 1: numerical results by the lowest-order \((k = 2)\) tetrahedral element in the family \((r = k - 1)\) of \(H(\text{grad div})\)-nonconforming elements

| \(N\) | \(\|e_h\|\) Rates | \(\|\nabla \cdot e_h\|\) Rates | \(\|\nabla (\nabla \cdot e_h)\|\) Rates |
|---|---|---|---|
| 16 | 7.338806e-07 | 3.773907e-06 | 1.261805e-04 |
| 20 | 5.585337e-07 | 1.2236 | 2.462834e-06 | 1.9127 |
| 24 | 4.511530e-07 | 1.1711 | 1.728736e-06 | 1.9412 |
| 28 | 3.788654e-07 | 1.1328 | 1.278389e-06 | 1.9578 |
| 32 | 3.268841e-07 | 1.1052 | 9.829309e-07 | 1.9682 |

Table 2 Example 1: numerical results by the lowest-order \((k = 2)\) tetrahedral element in the family \((r = k)\) of \(H(\text{grad div})\)-nonconforming elements

| \(N\) | \(\|e_h\|\) Rates | \(\|\nabla \cdot e_h\|\) Rates | \(\|\nabla (\nabla \cdot e_h)\|\) Rates |
|---|---|---|---|
| 8 | 1.232033e-06 | 1.150197e-05 | 3.902786e-04 |
| 12 | 5.905553e-07 | 1.8136 | 5.614381e-06 | 1.7688 |
| 16 | 3.416300e-07 | 1.9026 | 3.269987e-06 | 1.8790 |
| 20 | 2.215699e-07 | 1.9404 | 2.127621e-06 | 1.9260 |
| 24 | 1.549977e-07 | 1.9599 | 1.490982e-06 | 1.9502 |

Then, by a simple calculation, the source term \(f\) can be derived. Note that in this case \(g = 0\). We denote the finite element solution as \(u_h\). To measure the error between the exact solution and the finite element solution, we also denote

\[ e_h = u - u_h. \]

**Example 1** In this example, we test the tetrahedral elements. To this end, we partition the unit cube into \(N^3\) small cubes and then partition each small cube into 6 congruent tetrahedra. We use the lowest-order elements in three families to solve the problem (29) on the uniform tetrahedral mesh.

Tables 1, 2, and 3 illustrate various errors and convergence rates for three families. We observe from the tables that the numerical solution converges to the exact solution with a convergence order 1 for the family \(r = k - 1\), 2 for the family \(r = k\), and 2 for the family \(r = k + 1\) in the sense of \(L^2\)-norm. In addition, the three families have the same convergence order 2 in the \(H(\text{div})\)-norm and 1 in the \(H(\text{grad div})\)-norm, respectively. All the results confirm the correctness of the nonconforming elements.

**Example 2** In this example, we test the 3-rectangular grad div-conforming elements. We use uniform cubical meshes with the mesh size \(h\) varying from 1/12 to 1/20. Unlike tetrahedral elements, in this test, we also explore superconvergence of the 3-rectangular elements. To this end, we denote \(\{w_n\}_{n=1}^p\) and \(\{g_n\}_{n=1}^p\) as the weights and nodes of Legendre-Gauss quadrature rule of an order \(p\). We also denote \(\{w_n\}_{n=1}^p\) and \(\{l_n\}_{n=1}^p\) as the weights and nodes of Legendre-Gauss-Lobbato quadrature rule of an
Table 3 Example 1: numerical results by the lowest-order \((k = 2)\) tetrahedral element in the family \((r = k + 1)\) of \(H(\text{grad} \ \text{div})\)-nonconforming elements

| \(N\) | \(\|e_h\|\) | Rates | \(\|\nabla \cdot e_h\|\) | Rates | \(\|\nabla(\nabla \cdot e_h)\|\) | Rates |
|-------|----------------|-------|----------------|-------|----------------|-------|
| 8     | 1.224295e-06  |       | 1.149723e-05  |       | 2.377994e-04  |       |
| 10    | 8.220074e-07  | 1.7853| 7.812974e-06  | 1.7313| 1.954954e-04  | 0.8779|
| 12    | 5.864916e-07  | 1.8516| 5.613355e-06  | 1.8135| 1.654135e-04  | 0.9165|
| 14    | 4.381462e-07  | 1.8917| 4.211742e-06  | 1.8636| 1.431136e-04  | 0.9394|
| 16    | 3.391664e-07  | 1.9176| 3.269652e-06  | 1.8961| 1.259941e-04  | 0.9541|

Table 4 Example 2: numerical results by the lowest-order \((k = 2)\) 3-rectangular element in the family \((r = k - 1)\) of \(H(\text{grad} \ \text{div})\)-nonconforming elements

| \(h\)  | \(\|e_h\|\) | \(\|e_h\|_V\) | \(\|\nabla \cdot e_h\|_U\) | \(\|\nabla \cdot e_h\|_W\) | \(\|\nabla \cdot e_h\|_W\) |
|-------|----------------|----------------|----------------|----------------|----------------|
| 1/8   | 1.2939e-06     | 8.2349e-07     | 6.6566e-06     | 2.7601e-06     | 1.5795e-04     |
| 1/16  | 5.6099e-07     | 2.1371e-07     | 1.7020e-06     | 6.6734e-07     | 7.6700e-05     |
| 1/24  | 3.6063e-07     | 9.5663e-08     | 7.5957e-07     | 2.9471e-07     | 5.0814e-05     |
| 1/32  | 2.6677e-07     | 5.3946e-08     | 4.2787e-07     | 1.6541e-07     | 3.8025e-05     |
| 1/40  | 2.1201e-07     | 3.4566e-08     | 2.7402e-07     | 1.0575e-07     | 3.0388e-05     |

order \(p\). For \(u = (u_1, u_2, u_3)^T\), we define three discrete norms.

\[
\|u\|_V^2 = \sum_{K \in T_h} \sum_{r, s, t = 1}^k \omega_r \omega_s \omega_t \left( h_x^K h_y^K h_z^K \|u(x^K_c + h_x^K l_r, y^K_c + h_y^K l_s, z^K_c + h_z^K l_t)\|^2 \right),
\]

\[
\|u\|_W^2 = \sum_{K \in T_h} \omega_m \omega_n \left( h_x^K h_y^K \|u_1(\cdot, y^K_c + h_y^K g_m, z^K_c + h_z^K g_n)\|^2 + h_y^K h_z^K \|u_2(x^K_c + h_x^K g_m, \cdot, z^K_c + h_z^K g_n)\|^2 + h_z^K h_x^K \|u_3(x^K_c + h_x^K g_m, y^K_c + h_y^K g_n, \cdot)\|^2 \right),
\]

and

\[
\|u\|_W^2 = \sum_{K \in T_h} \sum_{n = 1}^{k-1} \omega_l \left( h_x^K \|u_1(x^K_c + h_x^K g_n, \cdot, \cdot)\|^2 + h_y^K \|u_2(\cdot, y^K_c + h_y^K g_n, \cdot)\|^2 + h_z^K \|u_3(\cdot, \cdot, z^K_c + h_z^K g_n)\|^2 \right),
\]

where \((x^K_c, y^K_c, z^K_c)\) is the center of element \(K\) and \(2h_x^K, 2h_y^K, 2h_z^K\) are the lengths of edges parallel to \(x, y, z\) axes, respectively.

Tables 4, 5, and 6 shows errors measured in various norms for the lowest-order 3-rectangular elements in the three families. We also depict error curves with a log-log scale in Fig. 3. From Fig. 3A, we can observe superconvergence phenomena that
Table 5 Example 2: numerical results by the lowest-order \((k = 2)\) 3-rectangular element in the family \((r = k)\) of \(H(\text{grad div})\)-nonconforming elements

| \(h\) | \(\|e_h\|\) | \(\|e_h\|_V\) | \(\|\nabla \cdot e_h\|\) | \(\|\nabla \cdot e_h\|_U\) | \(\|\nabla (\nabla \cdot e_h)\|\) | \(\|\nabla (\nabla \cdot e_h)\|_W\) |
|-------|-------------|-------------|---------------|----------------|----------------|----------------|
| 1/4   | 2.2275e-06  | 2.1791e-06  | 2.0877e-05    | 1.4226e-05     | 3.2323e-04     | 2.0116e-04     |
| 1/10  | 3.2909e-07  | 3.2124e-07  | 3.2554e-06    | 2.4023e-06     | 1.2317e-04     | 3.1825e-05     |
| 1/16  | 1.2730e-07  | 1.2419e-07  | 1.2547e-06    | 9.2031e-07     | 7.6282e-05     | 1.2340e-05     |
| 1/22  | 6.7137e-08  | 6.5485e-08  | 6.6217e-06    | 9.2031e-07     | 7.6282e-05     | 1.2340e-05     |
| 1/28  | 4.1395e-08  | 4.0373e-08  | 4.0839e-07    | 2.9756e-07     | 4.3414e-05     | 4.0157e-06     |

Table 6 Example 2: numerical results by the lowest-order \((k = 2)\) 3-rectangular element in the family \((r = k + 1)\) of \(H(\text{grad div})\)-nonconforming elements

| \(h\) | \(\|e_h\|\) | \(\|e_h\|_V\) | \(\|\nabla \cdot e_h\|\) | \(\|\nabla \cdot e_h\|_U\) | \(\|\nabla (\nabla \cdot e_h)\|\) | \(\|\nabla (\nabla \cdot e_h)\|_W\) |
|-------|-------------|-------------|---------------|----------------|----------------|----------------|
| 1/4   | 2.5839e-06  | 2.5818e-06  | 2.0796e-05    | 1.4263e-05     | 3.2318e-04     | 2.0132e-04     |
| 1/10  | 3.2119e-07  | 3.2115e-07  | 3.2315e-06    | 2.4030e-06     | 1.2317e-04     | 3.1829e-05     |
| 1/16  | 1.2417e-07  | 1.2416e-07  | 1.2541e-06    | 9.2042e-07     | 7.6282e-05     | 1.2340e-05     |
| 1/22  | 6.5478e-08  | 6.5476e-08  | 6.6200e-07    | 4.8351e-07     | 5.5322e-05     | 6.5117e-06     |

\(\|\nabla (\nabla \cdot e_h)\|_W\) and \(\|e_h\|_V\) converge to 0 with one order higher than \(\|\nabla (\nabla \cdot e_h)\|\) and \(\|e_h\|\). In addition, from Fig. 3B, C, we can observe superconvergence of \(\|\nabla (\nabla \cdot e_h)\|_W\).

We can not observe any superconvergence of \(\|e_h\|_V\) for \(r = 2, 3\), and \(\|\nabla \cdot e_h\|_U\) for all the 3 families when \(k = 2\). To further investigate the superconvergence of \(\nabla \cdot e_h\), we test the third-order \((k = 3)\) element in the family of \(r = k - 1\). The results are shown in Table 7 and Fig. 3D. In this case, we can observe superconvergence of \(\nabla \cdot e_h\).

Using these superconvergent results, together with some recovery techniques, we can construct a solution with higher accuracy if needed, which is one of the reasons that we explore the superconvergence of 3-rectangular elements.

We conclude this section by pointing out that the three families of elements bear their own advantages. The family of \(r = k - 1\) can be the best choice if we pursue a low computational cost, while the family with \(r = k + 1\) stands out for its higher accuracy in the \(L^2\)-norm without any recovery techniques.

### 8 Conclusion

In this paper, we constructed conforming finite element de Rham complexes with enhanced smoothness. This leads to grad div-conforming elements in 3D which can be utilized to solve the quad-div problem. The simplest elements in our construction have only 8 and 14 DOFs for a tetrahedron and a 3-rectangle respectively, which makes commercial adoption of the elements feasible.

Since the quad-div problem has not been extensively studied both in mathematical theory and numerical methods, there are still some mysteries about the regularity and the discretization of this problem. However, by the de Rham complex, we relate the
Example 2: numerical results by the third-order \((k = 3)\) 3-rectangular element in the family \((r = k - 1)\) of \(H(\text{grad div})\)-nonconforming elements

\[
\begin{array}{ccccccc}
 h & \|e_h\| & \|e_h\|_V & \|\nabla \cdot e_h\|_U & \|\nabla \cdot e_h\|_W & \|\nabla (\nabla \cdot e_h)\|_W & \|\nabla (\nabla \cdot e_h)\|_W \\
 1/4 & 6.3209e-07 & 2.2806e-07 & 2.9623e-06 & 8.8580e-07 & 7.8540e-05 & 2.5723e-05 \\
 1/10 & 7.6991e-08 & 1.1031e-08 & 1.8971e-07 & 2.8011e-08 & 1.2378e-05 & 1.9063e-06 \\
 1/16 & 2.8833e-08 & 2.5640e-09 & 4.6416e-08 & 4.3703e-09 & 4.8272e-06 & 4.7467e-07 \\
 1/22 & 1.5050e-08 & 9.7038e-10 & 1.7869e-08 & 1.2319e-09 & 2.5519e-06 & 1.8377e-07 \\
\end{array}
\]

grad div-conforming elements to the FEEC. This allows further systematic developments of the new elements and the quad-div problem. We believe that the framework may shed new light into studying this problem.

In the future, we will apply the newly proposed elements to solve practical problems and further investigate the superconvergence phenomena. Moreover, we will construct finite element subcomplexes for the following variant of de Rham complex:
\[ \mathbb{R} \subset H^1(\Omega) \xrightarrow{\nabla} \Phi(\Omega) \xrightarrow{\nabla \times} H^1(\Omega) \xrightarrow{\nabla \cdot} L^2(\Omega) \rightarrow 0, \]

where \( \Phi(\Omega) := \{ v \in L^2(\Omega), \nabla \times v \in H^1(\Omega) \} \).

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