Analytical solution of the cylindrical torsion problem for the relaxed micromorphic continuum and other generalized continua (including full derivations)

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Abstract
We solve the St. Venant torsion problem for an infinite cylindrical rod whose behaviour is described by a family of isotropic generalized continua, including the relaxed micromorphic and classical micromorphic model. The results can be used to determine the material parameters of these models. Special attention is given to the possible nonphysical stiffness singularity for a vanishing rod diameter, because slender specimens are, in general, described as stiffer.

Keywords
generalized continua, torsion, torsional stiffness, characteristic length, size effect, micromorphic continuum, Cosserat continuum, gradient elasticity, relaxed micromorphic model, bounded stiffness
I. Introduction

In this paper, we continue our endeavour to find analytical solutions to simple boundary value problems for families of generalized continua [1–3]. The focus is on non-homogeneous solutions that on the one hand activate the additional deformations modes offered by generalized continua (curvature terms) and which may be used, on the other hand, in calibrating the additional (many) material parameters. The renewed interest in models of generalized continua comes in part from the fact that for small specimens one may observe size effects, not accounted for by linear Cauchy elasticity. On the other hand, the description of man-made architecture materials/meta-materials need generalized continua to capture frequency band-gaps in the dynamic range, a prominent example being given by the relaxed micromorphic model [4–8].

Here, we consider the static St. Venant torsion problem. As we aim at identifying material parameters, let us first review what can be said for isotropic linear elasticity.

1.1. Material parameters in linear elasticity versus generalized continua

The determination of the two constitutive material parameters in isotropic linear elasticity can be achieved in several different ways. For example, Young’s modulus and Poisson’s ratio

\[ E_{\text{macro}} = \frac{9\kappa_{\text{macro}} \mu_{\text{macro}}}{3\kappa_{\text{macro}} + \mu_{\text{macro}}} , \]
\[ \nu_{\text{macro}} = \frac{3\kappa_{\text{macro}} - 2\mu_{\text{macro}}}{2(3\kappa_{\text{macro}} + \mu_{\text{macro}})} , \]
\[ \lambda_{\text{macro}} = \frac{3\kappa_{\text{macro}} - 2\mu_{\text{macro}}}{3} , \]
\[ \kappa_{\text{macro}} = \frac{2\mu_{\text{macro}} + 3\lambda_{\text{macro}}}{3} , \] (1)

can be uniquely determined from a homogeneous macroscopic tension–compression test. Moreover, the shear modulus \( \mu_{\text{macro}} \) and Young’s modulus \( E_{\text{macro}} \) can also be identified from the inhomogeneous torsion and bending test, respectively. Indeed, the classical torsional stiffness (per unit length) of a circular rod is given by

\[ T_{\text{macro}} = \mu_{\text{macro}} I_p = \mu_{\text{macro}} \frac{\pi R^4}{2} , \] (3)

and the bending stiffness (per unit length and per unit thickness) in cylindrical bending [2] is equivalent to

\[ D_{\text{macro}} = \frac{h^3}{12} \frac{E_{\text{macro}}}{(1 - \nu_{\text{macro}}^2)} = \frac{h^3}{12} \frac{4\mu_{\text{macro}} (3\kappa_{\text{macro}} + \mu_{\text{macro}})}{3\kappa_{\text{macro}} + 4\mu_{\text{macro}}} . \] (4)

A third independent identification can be achieved with dynamic measurements, determining the shear wave speed \( (c_s) \) and the pressure wave speed \( (c_p) \)

\[ c_s = \sqrt{\frac{\mu_{\text{macro}}}{\rho}} , \quad c_p = \sqrt{\frac{2\mu_{\text{macro}} + \lambda_{\text{macro}}}{\rho}} . \] (5)

In reality, all three methods may lead to slightly different values when used to fit real experiments due to the experimental setup. Nevertheless, they all are useful in complementing the identification procedure. We note that all mentioned tests convey a precise physical meaning to the appearing material parameters and this greatly helps in the mechanical application of linear elasticity to real-world structures.

The situation is much more involved when trying to determine material parameters for generalized continua. Even when restricting the attention to linear and isotropic response, the number of additional parameters increase significantly and it is also not clear a priori what the physical meaning of the additional parameters really is. Lakes [9, 10] has prominently investigated the fitting procedure for isotropic Cosserat solids. In the linear isotropic Cosserat model (Section 6) there appear already six independent parameters and a series of experiments with differently sized materials allows the Cosserat constants to be determined. A decisive tool for that purpose is the analytical solution for torsion and bending, which is already available in the literature [2, 11, 12]. The Cosserat model allows size effects to be described in the sense that more slender specimens have an increased apparent stiffness in bending and torsion. However, it is observed that the Cosserat model does have an unphysical stiffness singularity in bending [2] for a zero slenderness limit, the same appears in
general in torsion (Section 6) but can be avoided upon setting some curvature parameters to zero (Section 6.1). The mentioned stiffness singularity is not only an academic issue, but it concerns the stable identification of the material parameters [13]. Yet, in the Cosserat theory, Young’s modulus \( E_{\text{macro}} \) and Poisson’s ratio \( v_{\text{macro}} \) can still be determined in a size-independent manner with a homogeneous tension–compression test. In question are the so-called Cosserat couple modulus \( \mu_c \geq 0 \) and the three curvature parameters.

A first extension of the Cosserat model is the so-called micro-stretch model, which allows for infinitesimal rotation and volume stretch as independent kinematic fields. For the micro-stretch model we show that the additional kinematic degree of freedom is not activated in the torsion problem.

Another extension of the Cosserat model is given by the recently introduced relaxed micromorphic model \([14–16]\) (Section 4). In its static isotropic version it features only eight independent material parameters comparing favourably to the large number of constitutive parameters in the classical micromorphic model. Although the kinematics of the relaxed micromorphic model coincides with the classical micromorphic model (nine additional degrees of freedom: stretch, shrink, shear, rotations) the curvature term is a direct extension of the Cosserat curvature written in terms of \( \text{Curl} \mathbf{P} \). An important advantage of the model compared with the Cosserat model is that there is no stiffness singularity in whatsoever situation and four of the eight constants (\( \mu_{\text{macro}}, \lambda_{\text{macro}}, \mu_c \) and \( \lambda_c \)) can be determined \textit{ab initio} from size-independent homogeneous tests \([17]\). There remains to fit three curvature parameters and the Cosserat couple modulus \( \mu_c \geq 0 \) (which in some situations may be chosen to be zero because the model remains well-posed) \([18–22]\).

Another advantage of the isotropic relaxed micromorphic model is given by the fact that it can replace the isotropic Cosserat model in a straightforward manner without additional costs. Indeed, the Cosserat curvature parameters can be taken as such as well as the Cosserat couple modulus \( \mu_c \geq 0 \). The only new parameter set is \( \mu_{\text{micro}}, \lambda_{\text{micro}} \), an estimate of which can be inferred from the small-scale response \([17]\). Regularity and continuous dependence results for the relaxed micromorphic model have been obtained in \([14, 23–25]\) and first finite-element method (FEM) implementations in \( H(\text{Curl}) \)-space are presented in \([17]\).

Next, the micro-strain model \([26]\) is in a sense complementary to the Cosserat model: it assumes an additional strain-like symmetric field \( S \) as extra degree of freedom. Here, we recover a simplified micro-strain model without mixed terms and a choice for the curvature parameters, see also \([27]\) who considers a degenerate micro-strain model in disguise. We recover the analytical solution given by Hütter \([28]\) for the micro-strain torsion problem. It turns out that for bending \([2]\), simple shear \([1]\) and torsion (Section 10), the micro-strain solution does not show a stiffness singularity either. However, this is not a general feature of the micro-strain model, but only related to the restricted kinematic possibilities: bending and torsion activate prominently rotations, but these are ‘filtered out’ in the micro-strain model. Therefore, bounded stiffness in bending and torsion should come as no surprise. Next, we combine the Cosserat and the micro-strain ansatz in a novel \textit{ad hoc} model whose response is nevertheless governed by the Cosserat kinematics.

Lastly, we have the full micromorphic model \([29,30]\). The kinematics is augmented with a non-symmetric micro-distortion tensor \( \mathbf{P} \) (as for the relaxed micromorphic model, too) but the curvature energy depends on the full gradient \( D \mathbf{P} \) of the micro-distortion. For simplicity and for comparison, we consider a subclass without mixed terms and simplified curvature expression. In general, the bending and torsion responses show a stiffness singularity, which can be avoided in torsion by a very special ansatz for the curvature energy. However, nonphysical stiffness singularities cannot, in general, be avoided. Our investigation is complemented by considering the strain-gradient models and its couple-stress subclass. The reason for the singular stiffening behaviour in the other generalized continuum models (except the relaxed micromorphic one) can be connected to their \textit{nonredundant} formulation of the curvature measure \([31]\).

An alternative method to study the deformation of (finite) elastic cylinders is the semi-inverse method introduced by Ieşan in \([32,33]\), see also \([34]\). This method was also successfully used to study the deformation of elastic cylinders with microstructures, see \([35–37]\) and the book \([38]\), in which many of Ieşan’s results were unified. Regarding the semi-inverse method, all the results obtained in the classical micromorphic theory and all its subclasses (Cosserat, micro-stretch, micro-voids) are obtained by assuming that the internal energy is positive definite in terms of \( D \mathbf{P} \). In contrast, in the framework of the relaxed micromorphic model, the present results are valid also for internal energies which are not positive definite in terms of \( D \mathbf{P} \), but rather in terms of \( \text{Curl} \mathbf{P} \). We recall that an internal energy which is positive definite in terms of \( \text{Curl} \mathbf{P} \) is only semi-positive definite in terms of \( D \mathbf{P} \).

The paper is now structured as follows. After fixing our notation in Section 1.2 we shortly dwell on the formulation of the torsion problem in adapted variables, making it clear that we do not revert to express stresses...
and moments in cylindrical coordinate but we always use a Cartesian expression written in suitable variables. To set the stage we recall the linear isotropic torsion problem, which will then be suitably generalized.

1.2. Notation

For vectors \( a, b \in \mathbb{R}^n \), we define the scalar product \( \langle a, b \rangle := \sum_{i=1}^{n} a_i b_i \in \mathbb{R} \) and the dyadic product \( a \otimes b := (a_i b_j)_{i,j=1,\ldots,n} \in \mathbb{R}^{n \times n} \). In the same way, for tensors \( P, Q \in \mathbb{R}^{n \times n} \), we define the scalar product \( \langle P, Q \rangle := \sum_{i,j=1}^{n} P_{ij} Q_{ij} \in \mathbb{R} \) and the Frobenius-norm \( \|P\|^2 := \langle P, P \rangle \). Moreover, \( P^\top := (P_{ji})_{i,j=1,\ldots,n} \) denotes the transposition of the matrix \( P = (P_{ij})_{i,j=1,\ldots,n} \), which decomposes orthogonally into the skew-symmetric part \( \text{skw}\text{-sym} P := \frac{1}{2} (P - P^\top) \) and the symmetric part \( \text{sym} P := \frac{1}{2} (P + P^\top) \). The identity matrix is denoted by \( \mathbb{1} \), so that the trace of a matrix \( P \) is given by \( \text{tr} P := \langle P, \mathbb{1} \rangle \), whereas the deviatoric component of a matrix is given by \( \text{dev} P := P - \frac{\text{tr}(P)}{3} \mathbb{1} \). Given this, the orthogonal decomposition possible for a matrix is \( P = \text{dev} P + \text{skw} P + \frac{\text{tr}(P)}{3} \mathbb{1} \). The Lie algebra of skew-symmetric matrices is denoted by \( \mathfrak{so}(3) := \{ A \in \mathbb{R}^{3 \times 3} | A^\top = -A \} \), whereas the vector space of symmetric matrices \( \text{Sym}(3) := \{ S \in \mathbb{R}^{3 \times 3} | S^\top = S \} \). Using the one-to-one map \( \text{axl} : \mathfrak{so}(3) \rightarrow \mathbb{R}^3 \) we have

\[
A b = \text{axl}(A) \times b \quad \forall A \in \mathfrak{so}(3), \quad b \in \mathbb{R}^3.
\]  

(6)

where \( \times \) denotes the cross product in \( \mathbb{R}^3 \). The inverse of axl is denoted by \( \text{Anti} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3) \). The Jacobian matrix \( Du \) and the curl for a vector field \( u \) are defined as

\[
Du = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{pmatrix}, \quad \text{curl } u = \nabla \times u = \begin{pmatrix} u_{3,2} - u_{2,3} \\ u_{1,3} - u_{3,1} \\ u_{2,1} - u_{1,2} \end{pmatrix}.
\]

(7)

We also introduce the Curl and the Div operators of the \( 3 \times 3 \) matrix field \( P \) as

\[
\text{Curl } P = \begin{pmatrix} \langle \text{curl}(P_{11}, P_{12}, P_{13})^\top \rangle \\ \langle \text{curl}(P_{21}, P_{22}, P_{23})^\top \rangle \\ \langle \text{curl}(P_{31}, P_{32}, P_{33})^\top \rangle \end{pmatrix}, \quad \text{Div } P = \begin{pmatrix} \langle \text{div}(P_{11}, P_{12}, P_{13})^\top \rangle \\ \langle \text{div}(P_{21}, P_{22}, P_{23})^\top \rangle \\ \langle \text{div}(P_{31}, P_{32}, P_{33})^\top \rangle \end{pmatrix}.
\]

(8)

The cross product between a second-order tensor and a vector is also needed and is defined row-wise as follows

\[
m \times b = \begin{pmatrix} (b \times (m_{11}, m_{12}, m_{13}))^\top \\ (b \times (m_{21}, m_{22}, m_{23}))^\top \\ (b \times (m_{31}, m_{32}, m_{33}))^\top \end{pmatrix} = m \cdot \epsilon \cdot b = m_{ik} \epsilon_{kij} b_i,
\]

(9)

where \( m \in \mathbb{R}^{3 \times 3}, b \in \mathbb{R}^3 \) and \( \epsilon \) is the Levi-Civita tensor. The two indices contraction \( : \) is intended as

\[
B : \nabla m = B_{\nu\rho} m_{ij,\rho} = N_{ij}, \quad B : m = B_{ij} m_{ij} = b_i,
\]

(10)

where \( B \) and \( N \) are second-order tensors, \( m \) is a third-order tensor and \( b \) is a vector.

1.3. Cartesian variables expressed through cylindrical variables

To address the torsional problem in its natural environment but with the comfort of the classical Cartesian coordinate system, we introduce the cylindrical set of coordinates which allows us to express the classic Cartesian orthogonal set of coordinates \( x = \{x_1, x_2, x_3\} \) through a more suitable set of variables \( r = \{r, \varphi, z\} \), without switching completely to a cylindrical coordinate system, i.e., without expressing all the quantities (strains, stresses, etc.) in the basis corresponding to the cylindrical coordinates

\[
e_r = \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix}, \quad e_\varphi = \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

(11)
The quantities we want to obtain are

\[ x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z, \]

whereas the relations between the first and the second derivative of a generic vector field \( f \) are

\[
\begin{align*}
\frac{\partial f(r, \varphi, z)}{\partial r_j} &= \frac{\partial f(r, \varphi, z)}{\partial x_k} \frac{\partial x_k}{\partial r_j}, \\
\frac{\partial^2 f(r, \varphi, z)}{\partial r_j \partial r_k} &= \frac{\partial^2 f(r, \varphi, z)}{\partial x_m \partial x_n} \frac{\partial x_m}{\partial r_j} \frac{\partial x_n}{\partial r_k} + \frac{\partial f(r, \varphi, z)}{\partial x_m} \frac{\partial^2 x_m}{\partial r_j \partial r_k}.
\end{align*}
\]

The quantities we want to obtain are \( \frac{\partial f(r, \varphi, z)}{\partial x_k} \) and \( \frac{\partial^2 f(r, \varphi, z)}{\partial x_m \partial x_n} \), which are obtainable thanks to (13) (see Appendix A for full calculations).

It is emphasized again that we do not represent the torsional problem in cylindrical coordinates (namely all the differential operators, the equilibrium equation and the kinematic fields), but we use the classical Cartesian coordinates \( \{x_1, x_2, x_3\} \) parameterized in cylindrical variables \( \{r, \varphi, z\} \).

### 1.4. Structure of the higher-order ansatz

The ansatz for the displacement field for the cylindrical torsion problem, regardless of the treated model, is always given by

\[
\mathbf{u}(r, \varphi, z) = \vartheta \left( \begin{array}{c}
-x_2(r, \varphi)x_3(z) \\
-x_3(z) \\
0
\end{array} \right)
\]

(14)

Here, \( \vartheta \) is the rate of twist per unit length. It is highlighted that the displacement field has the third component equal to zero since we are studying a cylindrical rod, whose cross-section is not subjected to warping. The most general ansatz for the micro-distortion tensor, which will be used for the full and the relaxed micromorphic model, is

\[
\mathbf{P}(r, \varphi, z) = \vartheta \left( \begin{array}{ccc}
0 & -x_3(z) & -g_2(r)x_2(r, \varphi) \\
x_3(z) & 0 & g_2(r)x_1(r, \varphi) \\
g_1(r)x_2(r, \varphi) & -g_1(r)x_1(r, \varphi) & 0
\end{array} \right),
\]

(15)

where \( g_1, g_2 : [0, \infty) \rightarrow \mathbb{R} \). Starting from the form (15) of the ansatz for \( \mathbf{P} \), it is possible to obtain the ansatz for the micro-stretch model (\( \mathbf{A} = \text{skew} \mathbf{P} \) and \( \omega \mathbf{1} = \text{tr}(\mathbf{P}) \mathbf{1} \)), the Cosserat model (\( \mathbf{A} = \text{skew} \mathbf{P} \)), the micro-void model (\( \omega \mathbf{1} = \text{tr}(\mathbf{P}) \mathbf{1} \)), and the micro-strain model (\( \mathbf{S} = \text{sym} \mathbf{P} \)), by taking the skew-symmetric part, the trace of \( \mathbf{P} \), or the symmetric part depending on what is needed. Here we report the symmetric part, the skew-symmetric part and the trace of \( \mathbf{P} \).

\[
\mathbf{S}(r, \varphi, z) = \text{sym} \mathbf{P}(r, \varphi, z) = \vartheta \left( \begin{array}{ccc}
0 & 0 & g_m(r)x_2(r, \varphi) \\
0 & 0 & -g_m(r)x_1(r, \varphi) \\
g_m(r)x_2(r, \varphi) & -g_m(r)x_1(r, \varphi) & 0
\end{array} \right),
\]

(16)

\[
\mathbf{A}(r, \varphi, z) = \text{skew} \mathbf{P}(r, \varphi, z) = \vartheta \left( \begin{array}{ccc}
0 & -2x_3(z) & -g_p(r)x_2(r, \varphi) \\
x_3(z) & 0 & g_p(r)x_1(r, \varphi) \\
g_p(r)x_2(r, \varphi) & -g_p(r)x_1(r, \varphi) & 0
\end{array} \right),
\]

(17)

were \( g_p(r) = g_1(r) + g_2(r) \), \( g_m(r) = g_1(r) - g_2(r) \) and \( \omega \) is not reported because the ansatz (15) has a zero trace.

It is highlighted that each section remains ‘rigid’ is not really correct, because the deformation of a cylinder section due to the displacement field (14) (which is a linear approximation of a rigid rotation) looks like

\[
\mathbf{x} + \mathbf{u}(\mathbf{x}) = \left( \begin{array}{c}
x_1 \\
x_2 \\
0
\end{array} \right) + \left( \begin{array}{ccc}
0 & -\vartheta x_3 & 0 \\
0 & 0 & 0 \\
-\vartheta x_3 & 0 & 0
\end{array} \right) \left( \begin{array}{c}
x_1 \\
x_2 \\
0
\end{array} \right) = \left( \begin{array}{c}
x_1 - x_2 x_3 \vartheta \\
x_2 + x_1 x_3 \vartheta \\
0
\end{array} \right).
\]

(18)

Of course this radial expansion does not contribute energetically under the small displacement hypothesis, and this can be seen from (23), in which it is clear that the symmetric strain tensor \( \mathbf{e} = \text{sym} \mathbf{D} \mathbf{u} \) does not depend on \( x_3 \equiv z \).
2. Overview of some generalized continuum models and their interconnections

Considering the following notation for the involved quantities:

\[ u : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \] displacement,

\[ P : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^{3 \times 3}, \] micro-distortion,

\[ A : \Omega \subset \mathbb{R}^3 \rightarrow \mathfrak{so}(3), \] micro-rotation,

\[ S : \Omega \subset \mathbb{R}^3 \rightarrow \text{Sym}(3), \] micro-strain,

\[ \omega : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}, \] micro-dilatation

and the orthogonal decomposition

\[ P = \text{dev} \text{sym} P + \text{skew} P + \frac{1}{3} \text{tr}(P) \mathbb{1} = \text{dev} S + A + \omega \mathbb{1} \tag{19} \]

we give the following genealogy tree of the generalized continuum models:

The strain gradient theory and second gradient theory are equivalent [3, 29] and contain, in addition, the couple stress theory as a special case. Using the Curl as primary differential operator for the curvature terms allows a neat unification of concepts.

3. Torsional problem for the isotropic Cauchy continuum

In order to set up a comparison with the models we present in the following sections, we start by presenting the solution of the classical cylindrical torsional problem. The strain energy for a linear elastic isotropic Cauchy continuum is

\[ W(Du) = \mu_{\text{macro}} \|\text{sym} Du\|^2 + \frac{\lambda_{\text{macro}}}{2} \text{tr}^2 (Du), \tag{20} \]

where \( \lambda_{\text{macro}} \) and \( \mu_{\text{macro}} \) are the macroscopic Lamé constants.
Figure 1. Geometry of the torsion problem: according to the St. Venant principle, we do not consider how the resultant end torque is applied. Furthermore, we assume that each cross-section (orthogonal to $x_3$) rotates as a rigid body with a constant rate of twist $\frac{\partial \Theta}{\partial x_3} = \vartheta$. As there is no warping, every cross-section remains in the same plane before and after the deformation. Note that the final solution for linear elasticity satisfies this ansatz only to within first order in the rate of twist $\vartheta$, see Figure 3.

In terms of the symmetric Cauchy stress tensor $\sigma = 2 \mu_{\text{macro}} \text{sym} \ Du + \lambda_{\text{macro}} \text{tr} (Du) \mathbb{1}$, where $\mathbb{1} = \text{sym} Du$ denotes the classical symmetric strain tensor, the equilibrium equation (in the absence of body forces) and the Neumann lateral boundary conditions (at the free surface) are

$$\text{Div} \sigma = 0, \quad t(r = R) = \sigma(r = R) \cdot e_r = 0.$$  \tag{21}$$

Our aim is to study a state of uniform torsion $\vartheta$ for an infinitely extended cylindrical rod. According to the cylindrical reference system shown in Figure 1, the ansatz for the displacement is

$$u(x_1, x_2, x_3) = u(r, \varphi, z) = \vartheta \begin{pmatrix} -x_2(r, \varphi) x_3(z) \\ x_1(r, \varphi) x_3(z) \\ 0 \end{pmatrix} = \begin{pmatrix} -z r \sin \varphi \\ z r \cos \varphi \\ 0 \end{pmatrix},$$  \tag{22}$$

where $\vartheta$ is the angle of twist per unit length. It is underlined that the third component of the displacement is chosen equal to zero because the cross-section is circular and therefore no warping is expected. The gradient of the displacement and its symmetric part are (the gradient is always taken with respect to the Cartesian coordinate system and then rewritten in the variables $\{r, \varphi, z\}$)

$$Du = \vartheta \begin{pmatrix} 0 & -z & -r \sin \varphi \\ z & 0 & r \cos \varphi \\ 0 & 0 & 0 \end{pmatrix}, \quad e = \text{sym} Du = \frac{\vartheta}{2} \begin{pmatrix} 0 & 0 & -r \sin \varphi \\ 0 & 0 & r \cos \varphi \\ -r \sin \varphi & r \cos \varphi & 0 \end{pmatrix}.$$  \tag{23}$$

Substituting the ansatz (23) into the equilibrium equation (21), it is easy to verify that they are identically satisfied.

In order to help the geometric interpretation of the torque, see Figure 2, we present its expression in Cartesian coordinates along with its representation in the cylindrical variables

$$M_c(\vartheta) := \int \int \left[ \frac{\text{traction}}{\text{per unit area}} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{x_1^2 + x_2^2} \sqrt{x_1^2 + x_2^2} \end{pmatrix} \right] dx_1 dx_2$$

$$= \int \int \begin{pmatrix} x_1 \sigma_{23} - x_2 \sigma_{13} \end{pmatrix} dx_1 dx_2 = \int_0^{2\pi} \int_0^R \begin{pmatrix} \sigma \cdot e_\varphi \end{pmatrix} r dr d\varphi,$$  \tag{24}$$

where $e_3 = e_z = (0, 0, 1)$ is the unit vector aligned with the mid-axis of the cylindrical rod.
The torque (or moment of torsion) about the $x_3$-axis and energy (per unit length $dx_3$) expressions are

$$M_c(\vartheta) := \int_0^{2\pi} \int_0^R \left[ \langle \sigma \cdot e_z, e_\varphi \rangle r \right] r \, dr \, d\varphi = \mu_{\text{macro}} \frac{\pi R^4}{2} \vartheta = \mu_{\text{macro}} I_p \vartheta = T_{\text{macro}} \vartheta, \quad (26)$$

$$W_{\text{tot}}(\vartheta) := \int_0^{2\pi} \int_0^R W(Du) \, r \, dr \, d\varphi = \frac{1}{2} \mu_{\text{macro}} \frac{\pi R^4}{2} \vartheta^2 = \frac{1}{2} \mu_{\text{macro}} I_p \vartheta^2 = \frac{1}{2} T_{\text{macro}} \vartheta^2,$$

where $\mu_{\text{macro}}$ is the macroscopic shear modulus, $I_p = \frac{\pi R^4}{2}$ is the polar moment of inertia and $T_{\text{macro}} = \mu_{\text{macro}} I_p$ is the torsional stiffness. It is also highlighted that

$$\frac{d}{d\vartheta} W_{\text{tot}}(\vartheta) = M_c(\vartheta) = T_{\text{macro}} \vartheta, \quad \frac{d^2}{d\vartheta^2} W_{\text{tot}}(\vartheta) = T_{\text{macro}}. \quad (27)$$

Here and in the remainder of this work, the elastic coefficients $\mu_i, \lambda_i, \kappa_i$ are expressed in MPa, the coefficients $a_i$ are dimensionless, the characteristic lengths $L_c$ and the radius $R$ in meters and the rate of twist $\vartheta$ in m$^{-1}$ (see Figure 3).

4. Torsional problem for the isotropic relaxed micromorphic model

The relaxed micromorphic model, in contrast to all the other proposals for generalized continua in the literature, lives on two well-defined and separated scales, each describing linear elastic response: the classical macroscopic...
Figure 4. Macro- and micro-scale stiffness governed by two springs in series. If \( \mu_{\text{micro}} \to \infty \), this implies that \( \mu_{\text{macro}} = \mu_e \). In all suitable cases for our family of considered generalized continua (depending on the kinematics), we use the same/similar lower-order energy expression (the energy without curvature).

Response (characteristic length \( L_c \to 0 \), available for experiments with large specimens) is described as usual by

\[
E_{\text{macro}} = \frac{9 \kappa_{\text{macro}} \mu_{\text{macro}}}{3 \kappa_{\text{macro}} + \mu_{\text{macro}}}, \quad \nu_{\text{macro}} = \frac{3 \kappa_{\text{macro}} - 2 \mu_{\text{macro}}}{2(3 \kappa_{\text{macro}} + \mu_{\text{macro}})}, \quad \lambda_{\text{macro}} = \frac{3 \kappa_{\text{macro}} - 2 \mu_{\text{macro}}}{3},
\]

(28)

The macroscopic parameters can be uniquely determined from a homogeneous macroscopic tension–compression test. However, the shear modulus \( \mu_{\text{macro}} \) and Young’s modulus \( E_{\text{macro}} \) can also be identified from the inhomogeneous torsion and bending test, respectively. Indeed, the classical torsional stiffness of a circular rod is given by

\[
T_{\text{macro}} = \frac{\mu_{\text{macro}} I_p}{\pi R^4} = \mu_{\text{macro}} \frac{\pi R^4}{2}.
\]

(30)

The microscopic scale (appearing for \( L_c \to \infty \)), representing a surrogate stiffness connected to the smallest meaningful scale of the material is described by the parameters

\[
E_{\text{micro}} = \frac{9 \kappa_{\text{micro}} \mu_{\text{micro}}}{3 \kappa_{\text{micro}} + \mu_{\text{micro}}}, \quad \nu_{\text{micro}} = \frac{3 \kappa_{\text{micro}} - 2 \mu_{\text{micro}}}{2(3 \kappa_{\text{micro}} + \mu_{\text{micro}})}, \quad \lambda_{\text{micro}} = \frac{3 \kappa_{\text{micro}} - 2 \mu_{\text{micro}}}{3},
\]

(31)

The macroscopic parameters \( \mu_{\text{macro}} \) and \( \lambda_{\text{macro}} \) (Figure 4) do not directly intervene in the formulation of the relaxed micromorphic model (34), but the connection is necessarily given by the Reuss-like homogenization formula [39]

\[
\mu_{\text{macro}} = \frac{\mu_e \mu_{\text{micro}}}{\mu_e + \mu_{\text{micro}}}, \quad \mu_e = \frac{\mu_{\text{macro}} \mu_{\text{micro}}}{\mu_{\text{micro}} - \mu_{\text{macro}}}, \quad \kappa_{\text{macro}} = \frac{\kappa_e \kappa_{\text{micro}}}{\kappa_e + \kappa_{\text{micro}}}, \quad \kappa_e = \frac{\kappa_{\text{macro}} \kappa_{\text{micro}}}{\kappa_{\text{micro}} - \kappa_{\text{macro}}}.
\]

(33)

Note that the Cosserat couple modulus \( \mu_c \geq 0 \) is not appearing in the homogenization formulae (33). As a consequence, both parameter sets (29)–(33) can be identified independently of the scale consideration (being particularly careful with the techniques for the micro-parameters identification) and they uniquely determine the meso-scale parameter set \( \mu_e, \lambda_e \) appearing in (33)2.

The general expression of the strain energy for the isotropic relaxed micromorphic continuum is

\[
W(Du, P, \text{Curl} P) = \mu_e \| \text{sym} (Du - P) \|^2 + \frac{\lambda_e}{2} \text{tr}^2 (Du - P) + \mu_c \| \text{skew} (Du - P) \|^2
+ \mu_{\text{micro}} \| \text{sym} P \|^2 + \frac{\lambda_{\text{micro}}}{2} \text{tr}^2 (P)
+ \frac{\mu L_c^2}{2} \left( a_1 \| \text{dev sym Curl} P \|^2 + a_2 \| \text{skew Curl} P \|^2 + \frac{a_3}{3} \text{tr}^2 (\text{Curl} P) \right),
\]

(34)
where \((\mu_e, \lambda_e), (\mu_{\text{micro}}, \lambda_{\text{micro}}), \mu_e, L_e > 0\) and \((a_1, a_2, a_3)\) are the parameters related to the meso-scale, the parameters related to the micro-scale, the Cosserat couple modulus, the characteristic length and the three general isotropic curvature parameters, respectively. This energy expression represents the most general isotropic form possible for the relaxed micromorphic model. It is important to underline that, given the subsequent ansatz (38), it holds that \(\text{skew} \, \text{Curl} \, P = 0\). This reduces immediately the number of curvature parameters appearing in the torsion solution. In the absence of body forces, the equilibrium equations are then

\[
\begin{align*}
\text{Div} \left[ 2\mu_e \text{sym} (Du - P) + \lambda_e \text{tr} (Du - P) \mathbb{1} + 2\mu_e \text{skew} (Du - P) \right] = 0, \\
\tilde{\sigma} - 2\mu_{\text{micro}} \text{sym} P - \lambda_{\text{micro}} \text{tr} (P) \mathbb{1} - \mu L_e^2 \text{Curl} \left( a_1 \text{dev} \, \text{sym} \, \text{Curl} \, P + \frac{a_3}{3} \text{tr} (\text{Curl} \, P) \mathbb{1} \right) = 0.
\end{align*}
\]

(35)

The boundary conditions at the lateral free surface are

\[
\begin{align*}
\tilde{\tau}(r = R) &= \tilde{\sigma}(r) \cdot e_r = 0_{\mathbb{R}^3}, & \text{(traction free),} \\
\eta(r = R) &= m(r) \cdot \epsilon \cdot e_r = m(r) \times e_r = 0_{\mathbb{R}^{3 \times 3}}, & \text{(moment free),}
\end{align*}
\]

(36)

where

\[
m = \mu L_e^2 \left( a_1 \text{dev} \, \text{sym} \, \text{Curl} \, P + \frac{a_3}{3} \text{tr} (\text{Curl} \, P) \mathbb{1} \right)
\]

(37)

is a generalized nonsymmetric second-order moment tensor, the (nonsymmetric) force-stress tensor \(\tilde{\sigma}\) is given in (35), \(e_r\) is the radial unit vector and \(\epsilon\) is the Levi-Civita tensor. The vector \(\tilde{\tau}(r) \in \mathbb{R}^3\) is the generalized traction and the tensor \(\eta(r) \in \mathbb{R}^{3 \times 3}\) is called the generalized double traction tensor. According to the cylindrical reference system shown in Figure 1, the ansatz for the displacement and for the micro-distortion \(P\) is

\[
\begin{align*}
u(x_1, x_2, x_3) &= u(r, \varphi, z) = \vartheta \begin{pmatrix} -x_2(r, \varphi) x_3(z) \\ x_1(r, \varphi) x_3(z) \\ 0 \end{pmatrix}, \\
\mathbf{P}(x_1, x_2, x_3) &= \mathbf{P}(r, \varphi, z) = \vartheta \begin{pmatrix} 0 & -x_3(z) & -g_2(r) x_2(r, \varphi) \\ x_3(z) & 0 & g_2(r) x_1(r, \varphi) \\ g_1(r) x_2(r, \varphi) & -g_1(r) x_1(r, \varphi) & 0 \end{pmatrix},
\end{align*}
\]

(38)

where \(x_1(r, \varphi) = r \cos \varphi, x_2(r, \varphi) = r \sin \varphi\) and \(x_3(z) = z\). The Cartesian \(Du\) and the Cartesian \(\text{Curl} \, P\) expressed in the cylindrical variables \((r, \varphi, z)\) are

\[
\begin{align*}
Du(r, \varphi, z) &= \vartheta \begin{pmatrix} 0 & -z & -r \sin \varphi \\ z & 0 & r \cos \varphi \\ 0 & 0 & 0 \end{pmatrix}, \\
\text{Curl} \, \mathbf{P}(r, \varphi, z) &= \vartheta \begin{pmatrix} 1 - g_2(r) - r g'_2(r) \sin^2 \varphi & r g'_2(r) \sin \varphi \cos \varphi & 0 \\ r g'_2(r) \sin \varphi \cos \varphi & 1 - g_2(r) - r g'_2(r) \cos^2 \varphi & 0 \\ 0 & 0 & -(2 g_1(r) + r g'_1(r)) \end{pmatrix}.
\end{align*}
\]

(39)

It can be remarked that \(\text{Curl} \, \mathbf{P}\) is symmetric.
Inserting the ansatz (38)–(39) into (35), the 12 equilibrium equations are reduced to the following 4 ordinary differential equilibrium equations

\[
\frac{1}{3} \vartheta \sin \varphi \left( r \mu L_c^2 (3 a_{1} + a_{3}) g''_{1}(r) + 2 a_{1} g''_{2}(r) + 3 \mu_{c} (g_{1}(r) + g_{2}(r) - 1) \right)
\]

\[
- 3 (\mu_{c} + \mu_{micro}) (g_{1}(r) - g_{2}(r)) - 3 \mu_{c} + 3 \mu_{e} L_c^2 ((a_{1} - a_{3}) g_{1}(r) - (2a_{1} + a_{3}) g_{2}(r)) = 0,
\]

\[
\frac{1}{3} \vartheta \cos \varphi \left( r \mu L_c^2 ((a_{1} - a_{3}) g''_{1}(r) + (2a_{1} + a_{3}) g''_{2}(r)) - 3 \mu_{e} (g_{1}(r) + g_{2}(r) - 1) \right)
\]

\[
+ 3 (\mu_{e} + \mu_{micro}) (g_{1}(r) - g_{2}(r)) + 3 \mu_{e} L_c^2 ((a_{1} - a_{3}) g_{1}(r) + (2a_{1} + a_{3}) g_{2}(r)) = 0,
\]

\[
\frac{1}{3} \vartheta \sin \varphi \left( r \mu L_c^2 ((2a_{1} + a_{3}) g''_{1}(r) + (a_{3} - a_{1}) g''_{2}(r)) - 3 \mu_{e} (g_{1}(r) + g_{2}(r) - 1) \right)
\]

\[
+ (\mu_{e} + \mu_{micro}) (g_{1}(r) - g_{2}(r)) + 3 \mu_{e} L_c^2 ((2a_{1} + a_{3}) g_{1}(r) + (a_{3} - a_{1}) g_{2}(r)) = 0,
\]

\[
\frac{1}{3} \vartheta \cos \varphi \left( r \mu L_c^2 ((a_{1} - a_{3}) g''_{2}(r) - (2a_{1} + a_{3}) g''_{1}(r)) + 3 \mu_{e} (g_{1}(r) + g_{2}(r) - 1) \right)
\]

\[
+ (\mu_{e} + \mu_{micro}) (g_{1}(r) - g_{2}(r)) + 3 \mu_{e} L_c^2 ((a_{1} - a_{3}) g_{2}(r) - (2a_{1} + a_{3}) g_{1}(r)) = 0.
\]

It is important to underline that (35)1 is identically satisfied, and that from the entire set of four equilibrium equations (40) only two are not redundant because (40)1 = \tan \varphi (40)2 and (40)3 = \tan \varphi (40)4.

It is also pointed out that the two remaining linearly independent equations (40)1,3 can be uncoupled and are of the Bessel ordinary differential equation (ODE) type (see Appendix B). Indeed, if we take their sum and difference, while being careful of substituting \( g_{p}(r) = g_{1}(r) + g_{2}(r) \) and \( g_{m}(r) = g_{1}(r) - g_{2}(r) \) along with their derivatives, we deduce

\[
\vartheta \sin \varphi \left( a_{1} \mu L_c^2 (3 g''_{m}(r) + r g''_{m}(r)) - 2 r \mu_{c} (g_{m}(r) + 1) - 2 r g_{m}(r) \mu_{micro} \right) = 0,
\]

\[
\frac{1}{3} \vartheta \sin \varphi \left( 6 r \mu_{c} (g_{p}(r) - 1) - \mu L_c^2 (a_{1} + 2a_{3}) (3 g_{p}(r) + r g''_{p}(r)) \right) = 0.
\]

As \( g_{1}(r) := \frac{g_{p}(r)+g_{m}(r)}{2} \) and \( g_{2}(r) := \frac{g_{p}(r)-g_{m}(r)}{2} \), the solution in terms of \( g_{1}(r) \) and \( g_{2}(r) \) of (41) is

\[
g_{1}(r) = \frac{1}{2} \left( 1 - \frac{ia_{1} L_{1} \left( \frac{r \mu L_{c}}{L_{c}} \right) - A_{2} Y_{1} \left( -i \frac{r \mu L_{c}}{L_{c}} \right) + iB_{1} I_{1} \left( \frac{r \mu L_{c}}{L_{c}} \right) - B_{2} Y_{1} \left( -i \frac{r \mu L_{c}}{L_{c}} \right) - \mu_{e} \mu_{micro}}{\mu_{e} + \mu_{micro}} \right),
\]

\[
g_{2}(r) = \frac{1}{2} \left( 1 + \frac{ia_{1} L_{1} \left( \frac{r \mu L_{c}}{L_{c}} \right) - A_{2} Y_{1} \left( -i \frac{r \mu L_{c}}{L_{c}} \right) + iB_{1} I_{1} \left( \frac{r \mu L_{c}}{L_{c}} \right) + B_{2} Y_{1} \left( -i \frac{r \mu L_{c}}{L_{c}} \right) + \mu_{e} \mu_{micro}}{\mu_{e} + \mu_{micro}} \right),
\]

\[
f_{1} := \sqrt{\frac{6 \mu_{e}}{(a_{1} + 2a_{3}) \mu}}, \quad f_{2} := \sqrt{\frac{2(\mu_{e} + \mu_{micro})}{a_{1} \mu}},
\]

where \( I_{n} (\cdot) \) is the modified Bessel function of the first kind, \( Y_{n} (\cdot) \) is the Bessel function of the second kind (see Appendix B for the formal definitions), and \( A_{1}, B_{1}, A_{2}, B_{2} \) are integration constants.

The values of \( A_{1}, B_{1} \) are determined from the boundary conditions (36), whereas, owing to the divergent behaviour of the Bessel function of the second kind at \( r = 0 \), we have to set \( A_{2} = 0 \) and \( B_{2} = 0 \) in order to have a continuous solution. The fulfilment of the boundary conditions (36) allows us to find the expressions of the
The classical torque, the higher-order torque and energy (per unit length d\(z\)) expressions are

\[
M_e(\theta) := \int_0^{2\pi} \int_0^R \left[ (\sigma \cdot \varepsilon_z, e_y) \right] r \, dr \, d\theta
= \int_0^{2\pi} \int_0^R \left[ \left( (\mathbf{m} \times \mathbf{e}_z) e_y, - (\mathbf{m} \times \mathbf{e}_z) e_y \right) \right] r \, dr \, d\theta
= T_e(\theta),
\]

\[
M_m(\theta) := \int_0^{2\pi} \int_0^R \left[ (\mathbf{W}, \mathbf{P}, \text{CurlP}) \right] r \, dr \, d\theta
= \int_0^{2\pi} \int_0^R \left[ \frac{4\mu_e f_2 f_3 (\frac{R}{c})}{c} \left( \frac{R}{c} \right)^2 \left( \frac{R}{c} \right)^3 \right] r \, dr \, d\theta
= \frac{1}{2} T_m(\theta),
\]

\[
W_m(\theta) := \int_0^{2\pi} \int_0^R \mathbf{W} \, d\mathbf{u} \cdot d\mathbf{r} \, d\theta
= \int_0^{2\pi} \int_0^R \left[ \left( \frac{4\mu_e f_2 f_3 (\frac{R}{c})}{c} \left( \frac{R}{c} \right)^2 \left( \frac{R}{c} \right)^3 \right) \right] r \, dr \, d\theta
= \frac{1}{2} T_m(\theta)^2,
\]

where

\[ q_1 := 3a_1 f_1 f_2 z_1, \quad q_2 := 2f_2(a_1 - a_3). \]
Both quantities \( \tau_e \) and \( \tau_w \) for which can be used to define the following torsional stiffnesses \( \kappa_{\text{macro}} = \frac{\kappa_e \kappa_{\text{micro}}}{\kappa_e + \kappa_{\text{micro}}} \) and \( \kappa_{\text{micro}} = \frac{\kappa_e \kappa_{\text{micro}}}{\kappa_e + \kappa_{\text{micro}}} \), with \( \kappa_i = \frac{2(\kappa_e + 3\kappa_i)}{3} \), \( i \in \{e, \text{micro}, \text{macro}\} \), \( \mu_{\text{macro}} = \frac{\mu_e \mu_{\text{micro}}}{\mu_e + \mu_{\text{micro}}} \), \( \mu_{\text{micro}} = \frac{\mu_e \mu_{\text{micro}}}{\mu_e + \mu_{\text{micro}}} \), with \( \kappa_i = \frac{2(\kappa_e + 3\kappa_i)}{3} \), \( i \in \{e, \text{micro}, \text{macro}\} \), which can be used to define the following torsional stiffnesses

\[
T_{\text{macro}} = \mu_{\text{macro}} I_p = \frac{\mu_{\text{micro}} \mu_e}{\mu_{\text{micro}} + \mu_e} I_p, \quad T_{\text{micro}} = \mu_{\text{micro}} I_p, \quad T_e = \mu_e I_p.
\]

The plots of the torsional stiffness for the classical torque (light blue), the higher-order torque (red) and the torque energy (green) for \( \mu_e = 0, 1/2, \infty \) while varying \( L_e \) is shown in Figure 5.

It is highlighted here that the torsional stiffness obtainable from the energy \( T_w \) is the only stiffness available experimentally.

### 4.1. Limits

#### 4.1.1. The relaxed micromorphic model with symmetric force stresses \( (\mu_e \to 0) \)

The relaxed micromorphic model (Figure 6) is

\[
W_{\text{tot}}(\theta) := \int_0^{2\pi} \int_0^R W(Du, P, \text{Curl}P) r \, dr \, d\varphi
\]

\[
= \frac{1}{2} \left[ f_2 I_0 \left( \frac{R^2}{L_c} \right) \left( v_2 \mu_{\text{micro}} \frac{L_c^2}{R^2} + \mu_e (\mu_e + \mu_{\text{micro}}) \right) / f_2 I_0 \left( \frac{R^2}{L_c} \right) - 2v_1 I_1 \left( \frac{R^2}{L_c} \right) \frac{L_c}{R} \right.
\]

\[
- \frac{2I_1 \left( \frac{R^2}{L_c} \right) \left( v_2 \mu_{\text{micro}} \frac{L_c^2}{R^2} + v_1 \mu_e (\mu_e + \mu_{\text{micro}}) \right) \frac{L_c}{R}}{f_2 I_0 \left( \frac{R^2}{L_c} \right) - 2v_1 I_1 \left( \frac{R^2}{L_c} \right) \frac{L_c}{R}} \left. \right] \frac{\mu_{\text{micro}}}{(\mu_e + \mu_{\text{micro}})^2 I_p} \theta^2,
\]

\[
v_1 := \frac{a_1 + 2a_3}{a_1 + 8a_3}, \quad v_2 := \frac{24a_1 a_3 \mu}{a_1 + 8a_3}.
\]
Figure 6. Relaxed micromorphic model. Torsional stiffness for the torque energy while varying $L_c$, for different values of $\mu_c = \{0, 1/30, 1/10, 1/5, 1, \infty\}$. The torsional stiffness remains bounded as $L_c \to \infty$ ($R \to 0$) and the model does not collapse into a linear elastic model. The values of the other parameters used are $\mu = 1$, $\mu_c = 1/3$, $\mu_{\text{micro}} = 1/4$, $a_1 = 10$, $a_3 = 1/50$ and $R = 1$. Here, varying $\mu_c$ does not intervene with $T_{\text{macro}}$ and $T_{\text{micro}}$.

Note that the torsional stiffness at the micro-scale $T_{\text{micro}}$ is here independent of the Cosserat couple modulus $\mu_c$, see (47).

4.1.2. The relaxed micromorphic model with conformal curvature energy ($a_3 = 0$) while varying the Cosserat couple modulus $\mu_c$.

In the particular case for which the parameter $a_3$ is equal to zero (Figure 7) the elastic energy turns into

$$W(Du, P, \text{Curl} P) = \mu_c \|\text{sym} (Du - P)\|^2 + \frac{\lambda_c}{2} \text{tr}^2 (Du - P) + \mu_c \|\text{skew} (Du - P)\|^2 + \mu_{\text{micro}} \|\text{sym} P\|^2 + \frac{\lambda_{\text{micro}}}{2} \text{tr}^2 (P) + \frac{\mu L_c^2}{2} a_1 \|\text{dev sym Curl} P\|^2.$$ (49)

In this case, the torsional stiffness at the micro-scale, namely for $L_c \to \infty$ ($R \to 0$), depends also on $\mu_c$

$$\tilde{T} := \lim_{L_c \to \infty} T_w = \frac{\mu_{\text{micro}} (9 \mu_c + \mu_e)}{(9 \mu_c + \mu_e) + \mu_{\text{macro}}} I_p.$$ (50)

For $\mu_c \to 0$ we obtain a linear elastic model with stiffness $T_{\text{macro}}$, for $\mu_c \to \infty$ a model is recovered that has $T_{\text{micro}}$ at the micro-scale, whereas for intermediate values of $0 < \mu_c < \infty$ a torsional stiffness between $T_{\text{macro}}$ and $T_{\text{micro}}$ appears.

We may consider a further limit in (50). It holds

$$\bar{T} := \lim_{\mu_{\text{micro}} \to \infty} \tilde{T} = (9 \mu_c + \mu_e) I_p = (9 \mu_c + \mu_{\text{macro}}) I_p,$$ (51)

where the last relation for which we have $\mu_c = \mu_{\text{macro}}$ is obtained from (33) taking $\mu_{\text{micro}} \to \infty$.

4.1.3. The Cosserat model as a limit of the relaxed micromorphic model ($\mu_{\text{micro}} \to \infty$).

The Cosserat model (Figure 8) can be obtained from the relaxed micromorphic model by formally letting $\mu_{\text{micro}} \to \infty$ and $\kappa_{\text{micro}} \to \infty$. From the homogenization formula (46) it is possible to see that for $\mu_{\text{micro}} \to \mu_{\text{macro}}$ we have $\mu_c \to \infty$, whereas $\mu_{\text{macro}} = \mu_c$ for $\mu_{\text{micro}} \to \infty$, which is the stiffness at the macro-scale for the Cosserat model.

4.1.4. Sensitivity of the relaxed micromorphic model with respect to the curvature parameters $a_1$ and $a_3$.

Sensitivity study for the relaxed micromorphic model while varying $a_1$ and $a_3$ independently.

The parametric study represented in Figure 9 has not been carried out for the limit $a_1 \to 0$ and $a_3 \to 0$ because we would have had an indeterminate form for $L_c \to \infty$, and that is why we used the symbol $\sim 0$. The
Figure 7. Relaxed micromorphic model with conformal curvature energy. Torsional stiffness for the torque energy while varying $L_c$, for different values of $\mu_c = \{0, 1/30, 1/10, 1/5, 1, \infty\}$. The torsional stiffness remains bounded as $L_c \to \infty (R \to 0)$ and the model does not collapse in a linear elastic model except for the case $\mu_c = 0$. The values of the other parameters used are $\mu = 1, \mu_e = 1/3, \mu_{micro} = 1/4, a_1 = 2$ and $R = 1$. In this case, varying $\mu_c$ influences the torsional stiffness also for small specimen size.

Figure 8. (a) Relaxed micromorphic model with full curvature. Torsional stiffness for the torque energy while varying $L_c$, for different values of $\mu_{micro} = \{0, 1/20, 1/7, 1/4, 1/2, \infty\}$. The torsional stiffness becomes unbounded as $L_c \to \infty (R \to 0)$ when $\mu_{micro} \to \infty$. The values of the other parameters used are $\mu = 1, \mu_{macro} = 1/10, \mu_c = 1/2, a_1 = 1/5, a_3 = 1/7$ and $R = 1$. The Cosserat solution appears for $\mu_{micro} \to \infty$. (b) Cosserat model and relaxed micromorphic model with conformal curvature. Torsional stiffness for the torque energy while varying $L_c$. The torsional stiffness is bounded as $L_c \to \infty (R \to 0)$. For the Cosserat model we chose $\mu_c = 1/9$ whereas for the relaxed micromorphic model $\mu_c = 1/2$ and $\mu_{micro} = 3$ in order to have the same upper bound $T = \tilde{T}$. The values of the other parameters used are $\mu = 1, \mu_{macro} = 1, a_1 = 5$ and $R = 1$.

solution of the problem while having $a_3 = 0$ a priori is analysed carefully in Section 4.1.2, and the solution of the problem while having $a_1 = 0$ a priori makes the relaxed model collapse into a classical linear elastic model with torsional stiffness $T_{macro}$.

4.2. Finite-element simulations
Using finite-element analysis as a tool of comparison, in this section we

(i) test the validity of the solution in terms of the hypothesis of small deformations (i.e., small twist rate);
(ii) discuss the validity of the St. Venant principle for the relaxed micromorphic model.
Figure 9. Relaxed micromorphic model. Response of the relaxed micromorphic model while varying (a) the curvature parameter $a_1$ having $a_3 = 20$ and (b) the curvature parameter $a_3$ having $a_1 = 20$. The values of the other parameters are $\mu = 1$, $\mu_c = 1/5$, $\mu_e = 1$, $\mu_{micro} = 1/9$ and $R = 1$.

In this analysis we take a finite-size cylindrical rod and we apply opposite and equal finite rotation at both of its ends ($z = \pm L/2$; Figure 10). Accordingly, the boundary conditions are

$$u(z = \pm L/2) = \begin{pmatrix} \cos \pm \Theta & \sin \pm \Theta & 0 \\ -\sin \pm \Theta & \cos \pm \Theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_2/L \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_2/L \end{pmatrix},$$

$$P(z = \pm L/2) \times e_1 = \begin{pmatrix} -P_{12} & P_{11} & 0 \\ -P_{22} & P_{21} & 0 \\ -P_{32} & P_{31} & 0 \end{pmatrix} = \begin{pmatrix} \sin \pm \Theta & \cos \pm \Theta - 1 & 0 \\ 1 - \cos \pm \Theta & \sin \pm \Theta & 0 \\ 0 & 0 & 0 \end{pmatrix} = D u(z = \pm L/2) \times e_1,$$

where $P(z) \times e_1 = D u(z) \times e_1$ are the consistent boundary conditions on the tangential part for the micro-distortion tensor $P$.

In Figure 11 it is possible to see how the nonidentically zero components of the micro-distortion $P$ vary across the diameter aligned to the $x_1$-axis ($\varphi = \pi/2$) of the cross-section placed in the middle of the cylindrical rod ($z = 0$). We chose the middle section in order to study the solution far away enough from the disturbance region on which the boundary conditions have been applied. The values of the components of $P$ of Figure 11 (twist rate $\vartheta = \pi/50$) are in perfect agreement with the analytical solution, confirming the validity of the small-deformation solution obtained in Section 4.
Figure 11. Plots of the components (a) $P_{12}$, (b) $P_{13}$, (c) $P_{21}$ and (d) $P_{31}$ of the micro-distortion tensor $P$ at the cross-section $z = 0$. The purple line corresponds to $L_c = 0$, the grey line to $L_c = 1$ and the green line to $L_c = \infty$. The values of the other parameters used are $\mu = 1$, $\mu_c = 1$, $\mu_e = 1$, $\mu_{\text{micro}} = 1$, $a_1 = 1$, $a_3 = 1$, $R = 1$, $\theta = \pi/50$ and $L = 10$.

Figure 12. Plots of how the components (a) $P_{13}$ and (b) $P_{31}$ vary along a line on the external boundary ($\phi = \pi/2$ and $r = R$): the solid lines are the analytical solution whereas the markers are the numerical values obtained thanks to a finite-element analysis. The purple line has been obtained for $L_c = 0$, the grey line for $L_c = 1$ and the green line for $L_c = \infty$. The values of the other parameters used are $\mu = 1$, $\mu_c = 1$, $\mu_e = 1$, $\mu_{\text{micro}} = 1$, $a_1 = 1$, $a_3 = 1$, $R = 1$, $\phi = \pi/2$. As can be seen, the solution does not converge stably and not perfectly symmetrically (the mesh is not symmetric) to the analytical solution, but nevertheless it converges rapidly.

Furthermore, in Figure 12 we show how the solution obtained while applying consistent boundary conditions converges to that obtained analytically in a distance from the boundary which is more or less between one radius and one diameter. This is the pinnacle expression of the St. Venant principle: we have applied not only a finite-rotation instead of a linearized one, but we have also used consistent boundary conditions for $P$ which we know are different from the correct values that the tangential part of $P$ should have, and we obtained nevertheless the analytical linearized solution after a rather small boundary layer.
Figure 13. Plots of the component $P_{13}$ across the section placed at $z = 0$ obtained (a) analytically and (b) with the finite-element analysis. The two results are in perfect agreement. The values of the other parameters used are $\mu = 1$, $\mu_c = 1$, $\mu_e = 1$, $\mu_{\text{micro}} = 1$, $a_1 = 1$, $a_3 = 1$, $R = 1$, $\varphi = \pi/2$ and $L_c = 1$.

We describe, in particular, the component $P_{31}$ (rather than the component $P_{13}$) because, owing to the consistent boundary conditions, it is forced to start from zero at the lateral boundary. In Figure 12 we plot this component which is evaluated for the length of the rod on the external surface ($\varphi = \pi/2$ and $r = R$).

In Figure 13 is shown how the component $P_{13}$ vary on the cross-section centered in the origin of the reference system ($z = 0$).

The implications of this results are of great value in the context of the identification of the elastic material parameters: it is clear how to apply consistent boundary conditions on a real sample in a laboratory (Dirichlet hard), and thanks to these results, we now know that our analytical solution is taking place far away enough from the boundary layer.

5. Torsional problem for the isotropic micro-stretch model in dislocation format

In the micro-stretch model in dislocation format [41–45], in contrast to the relaxed micromorphic model, the micro-distortion tensor is devoid of the deviatoric component $\text{dev} \, \text{sym} \, P = 0 \Leftrightarrow P = A + \omega \mathbb{1}$, $A \in \mathfrak{so}(3)$ and $\omega \in \mathbb{R}$. The expression of the strain energy for this model in dislocation format can be written as [41]

$$W(Du, A, \omega, \text{Curl} (A - \omega \mathbb{1})) = \mu_{\text{macro}} \|\text{dev} \, \text{sym} \, Du\|^2 + \kappa_e \frac{1}{2} \|Du - \omega \mathbb{1}\|^2 + \mu_c \|\text{skew} (Du - A)\|^2 + \frac{9}{2} \kappa_{\text{micro}} \omega^2$$

$$+ \frac{\mu L_c^2}{2} \left( a_1 \|\text{dev} \, \text{sym} \, \text{Curl} A\|^2 + a_2 \|\text{skew} \, \text{Curl} (A + \omega \mathbb{1})\|^2 + \frac{a_3}{3} \text{tr}^2 (\text{Curl} A) \right),$$

because $\text{Curl} (\omega \mathbb{1}) \in \mathfrak{so}(3)$. The equilibrium equations, in the absence of body forces, are then

$$\text{Div} \left[ 2\mu_{\text{macro}} \, \text{dev} \, \text{sym} \, Du + \kappa_e \text{tr} (Du - \omega \mathbb{1}) \mathbb{1} + 2\mu_c \, \text{skew} (Du - A) \right] = 0,$$

$$2\mu_c \, \text{skew} (Du - A)$$

$$- \mu L_c^2 \, \text{skew} \, \text{Curl} \left( a_1 \, \text{dev} \, \text{sym} \, \text{Curl} A + a_2 \, \text{skew} \, \text{Curl} (A + \omega \mathbb{1}) + \frac{a_3}{3} \text{tr} (\text{Curl} A) \mathbb{1} \right) = 0,$$

$$\text{tr} \left[ 2\mu_{\text{macro}} \, \text{dev} \, \text{sym} \, Du \right.$$

$$+ \kappa_e \text{tr} (Du - \omega \mathbb{1}) \mathbb{1} - \kappa_{\text{micro}} \text{tr} (\omega \mathbb{1}) \mathbb{1} - \mu L_c^2 \, a_2 \, \text{Curl} \, \text{skew} \, \text{Curl} (\omega \mathbb{1} + A) \bigg] = 0.$$
The boundary conditions at the free surface are
\begin{align*}
\tilde{\eta}(r = R) &= \tilde{\sigma}(r) \cdot e_r = 0_{\mathbb{R}^3}, \\
\eta(r = R) &= \text{skew} (m(r) \cdot \epsilon \cdot e_r) = \text{skew} (m(r) \times e_r) = 0_{\mathbb{R}^{3 \times 3}}, \\
\gamma(r = R) &= \frac{1}{3} \text{tr} (m(r) \cdot \epsilon \cdot e_r) = \frac{1}{3} \text{tr} (m(r) \times e_r) = 0.
\end{align*}
(55)

According to the reference system shown in Figure 1, the ansatz for the displacement and micro-distortion fields is
\begin{align*}
u(x_1, x_2, x_3) &= u(r, \varphi, z) = \vartheta \begin{pmatrix} -x_2(r, \varphi) x_3(z) \\ x_1(r, \varphi) x_3(z) \\ 0 \end{pmatrix}, \quad \omega = 0, \\
A(x_1, x_2, x_3) &= A(r, \varphi, z) = \frac{\vartheta}{2} \begin{pmatrix} 0 & -2x_3(z) & -g_p(r)x_2(r, \varphi) \\ 2x_3(z) & 0 & g_p(r)x_1(r, \varphi) \\ g_p(r)x_2(r, \varphi) & -g_p(r)x_1(r, \varphi) & 0 \end{pmatrix}. \quad (56)
\end{align*}

As the ansatz requires \(\omega = 0\), the micro-stretch model coincides with the Cosserat model as presented in the next section.

6. Torsional problem for the isotropic Cosserat continuum

The strain energy for the isotropic Cosserat continuum in dislocation tensor format (curvature expressed in term of \(\text{Curl}A\)) can be written as [1, 2, 46–51, 71, 72]
\begin{align*}
W(Du, A, \text{Curl}A) &= \mu_{\text{macro}} \|\text{sym} Du\|^2 + \frac{\lambda_{\text{macro}}}{2} \text{tr}^2 (Du) + \mu_c \|\text{skew} (Du - A)\|^2 \\
&\quad + \frac{\mu L_c^2}{2} \left( a_1 \|\text{dev} \text{sym} \text{Curl}A\|^2 + a_2 \|\text{skew} \text{Curl}A\|^2 + \frac{a_3}{3} \text{tr}^2 (\text{Curl}A) \right),
\end{align*}
(57)
where \(A \in so(3)\). It is underlined that for the ansatz (61), which is presented later in this section, it holds that \(\text{skew} (\text{Curl}A) = 0\) (see calculation (39)_2). The equilibrium equations, in the absence of body forces, are therefore the following
\begin{align*}
\tilde{\sigma} := \text{Div} \left[ 2 \mu_{\text{macro}} \text{sym} Du + \lambda_{\text{macro}} \text{tr} (Du) I + 2 \mu_c \text{skew} (Du - A) \right] = 0, \\
2 \mu_c \text{skew} (Du - A) - \mu L_c^2 \text{skew} \text{Curl} \left( a_1 \text{dev} \text{sym} \text{Curl}A + \frac{a_3}{3} \text{tr} (\text{Curl}A) I \right) = 0.
\end{align*}
(58)

The boundary conditions at the free surface are
\begin{align*}
\tilde{\eta}(r = R) &= \tilde{\sigma}(r) \cdot e_r = 0_{\mathbb{R}^3}, \\
\eta(r = R) &= \text{skew} (m(r) \cdot \epsilon \cdot e_r) = \text{skew} (m(r) \times e_r) = 0_{\mathbb{R}^{3 \times 3}}, \\
\gamma(r = R) &= \frac{1}{3} \text{tr} (m(r) \cdot \epsilon \cdot e_r) = \frac{1}{3} \text{tr} (m(r) \times e_r) = 0.
\end{align*}
(59)

where the second-order moment stress tensor is now given by
\begin{align*}
m = \mu L_c^2 \left( a_1 \text{dev} \text{sym} \text{Curl}A + \frac{a_3}{3} \text{tr} (\text{Curl}A) I \right), \quad (60)
\end{align*}
the expression of \(\tilde{\sigma}\) is in (58), \(e_r\) is the radial unit vector and \(\epsilon\) is the Levi-Civita tensor.

According to the reference system shown in Figure 1, the ansatz for the displacement field and the micro-rotation is
\begin{align*}
u(x_1, x_2, x_3) &= u(r, \varphi, z) = \vartheta \begin{pmatrix} -x_2(r, \varphi) x_3(z) \\ x_1(r, \varphi) x_3(z) \\ 0 \end{pmatrix}, \quad (61)
\end{align*}
\begin{align*}
A(x_1, x_2, x_3) &= A(r, \varphi, z) = \frac{\vartheta}{2} \begin{pmatrix} 0 & -2x_3(z) & -g_p(r)x_2(r, \varphi) \\ 2x_3(z) & 0 & g_p(r)x_1(r, \varphi) \\ g_p(r)x_2(r, \varphi) & -g_p(r)x_1(r, \varphi) & 0 \end{pmatrix},
\end{align*}
where, in relation to the ansatz (38), we define $g_p(r) := g_1(r) + g_2(r)$, so that there is only one unknown function to be determined. Substituting the ansatz (61) in (58) the six equilibrium equations are equivalent to

\[
\begin{align*}
\frac{1}{6} \psi \sin \varphi \left( 6r \mu_c (g_p(r) - 1) - \mu L_c^2 (a_1 + 2a_3) \left( 3g_p'(r) + r g_p''(r) \right) \right) &= 0, \\
\frac{1}{6} \psi \cos \varphi \left( 6r \mu_c (g_p(r) - 1) - \mu L_c^2 (a_1 + 2a_3) \left( 3g_p'(r) + r g_p''(r) \right) \right) &= 0.
\end{align*}
\]

(62)

It is important to underline that (58)1 is identically satisfied, and that between the two equilibrium equations (62) there is only one independent equation because (62)1 = \tan \varphi (62)2. The solution of (62) is

\[
g_p(r) = 1 - \frac{i A_1 I_1 \left( \frac{rf_1}{L_c} \right)}{r} + \frac{A_2 Y_1 \left( -\frac{rf_1}{L_c} \right)}{r}, \quad f_1 := \frac{6 \mu_c}{\sqrt{(a_1 + 2a_3) \mu}},
\]

(63)

where $I_n(\cdot)$ is the modified Bessel function of the first kind, $Y_n(\cdot)$ is the Bessel function of the second kind (see Appendix B for the formal definitions), and $A_1, A_2$ are integration constants.

The value of $A_1$ is determined from the boundary conditions (59), where, owing to the divergent behaviour of the Bessel function of the second kind at $r = 0$, we have to set $A_2 = 0$ in order to have a continuous solution. The fulfilment of the boundary conditions (59) allows us to find the expressions of the integration constants

\[
A_1 = -\frac{i RL_c}{f_1 R z_1 \left( I_0 \left( \frac{rf_1}{L_c} \right) + I_2 \left( \frac{rf_1}{L_c} \right) \right) + z_2 L_c I_1 \left( \frac{rf_1}{L_c} \right)}, \quad z_1 := \frac{a_1 + 2a_3}{3a_1}, \quad z_2 := \frac{4a_3 - a_1}{3a_1}.
\]

(64)

The classical torque, the higher-order torque and energy (per unit length $dz$) expressions are

\[
\begin{align*}
M_c(\psi) &:= \int_0^{2\pi} \int_0^R \left[ \langle \hat{\mathbf{e}} \cdot \mathbf{e} \rangle r \right] r \, dr \, d\varphi \\
&= \int_0^{2\pi} \int_0^R \left[ \mu_{\text{macro}} + \frac{4 \mu_c I_2 \left( \frac{rf_1}{L_c} \right) \frac{L_c^2}{R^2}}{f_1 \left( 2f_1 z_1 I_0 \left( \frac{rf_1}{L_c} \right) + (z_2 - 2z_1) I_1 \left( \frac{rf_1}{L_c} \right) \frac{L_c}{R} \right)} \right] I_p \psi = T_c \psi, \\
M_m(\psi) &:= \int_0^{2\pi} \int_0^R \left[ \langle \text{skew} \left( \mathbf{m} \times \mathbf{e} \right) \rangle \mathbf{e}_r, \mathbf{e}_r \right] r \, dr \, d\varphi \\
&= \int_0^{2\pi} \int_0^R \left[ \frac{2 \mu \left( 3a_1 f_1 z_1 I_0 \left( \frac{rf_1}{L_c} \right) \frac{L_c^2}{R^2} - 2(a_1 - a_3) I_1 \left( \frac{rf_1}{L_c} \right) \frac{L_c}{R} \right)}{6f_1 z_1 I_0 \left( \frac{rf_1}{L_c} \right) - 3I_1 \left( \frac{rf_1}{L_c} \right) \frac{L_c}{R}} \right] I_p \psi = T_m \psi, \\
W_{\text{tot}}(\psi) &:= \int_0^{2\pi} \int_0^R W(\mathbf{D}u, \mathbf{A}, \text{CurlA}) r \, dr \, d\varphi \\
&= \frac{1}{2} \left[ \mu_{\text{macro}} + \frac{4 \mu_c I_2 \left( \frac{rf_1}{L_c} \right) \frac{L_c^2}{R^2}}{f_1 \left( 2f_1 z_1 I_0 \left( \frac{rf_1}{L_c} \right) + (z_2 - 2z_1) I_1 \left( \frac{rf_1}{L_c} \right) \frac{L_c}{R} \right)} \right. \\
&\quad + \left. \frac{2 \mu \left( 3a_1 f_1 z_1 I_0 \left( \frac{rf_1}{L_c} \right) \frac{L_c^2}{R^2} - 2(a_1 - a_3) I_1 \left( \frac{rf_1}{L_c} \right) \frac{L_c}{R} \right)}{6f_1 z_1 I_0 \left( \frac{rf_1}{L_c} \right) - 3I_1 \left( \frac{rf_1}{L_c} \right) \frac{L_c}{R}} \right] I_p \psi^2
\]
\]

(65)

\[
= \frac{1}{2} T_w \psi^2.
\]

The validity of (65)3 for $M_m$ is shown in Appendix C. The plot of the torsional stiffness for the classical torque (light blue), the higher-order torque (red) and the torque energy (green) while varying $L_c$ is shown in Figure 14(a).
6.1. Cosserat conformal curvature case: bounded stiffness in torsion

In the particular case for which the parameter $a_3$ is equal to zero the elastic energy turns into

$$W(Du, A, \text{Curl} A) = \mu_{\text{macro}} \|\text{sym} Du\|^2 + \frac{\lambda_{\text{macro}}}{2} tr^2(Du) + \mu_c \|\text{skew}(Du - A)\|^2 + \frac{\mu L_c^2}{2} a_1 \|\text{dev sym Curl} A\|^2. \quad (66)$$

In terms of $\phi = a\text{xl}(A)$, the curvature energy can be written as $\frac{\mu L_c^2}{2} a_1 \|\text{dev sym Daxl}(A)\|^2$ which is the conformal curvature case [52] (Figure 15). In this special case, the torsional stiffness remains bounded as $L_c \to \infty$ ($R \to 0$), namely $T := (9 \mu_c + \mu_{\text{macro}}) I_p$, which is consistent with the results in (51).
6.2. Cosserat limit case \( \mu_c \to \infty \) (indeterminate couple stress model)

We have (Figure 16)

\[
\lim_{\mu_c \to \infty} M_c(\varphi) = \left[ \mu_{\text{macro}} + a_1 \mu \frac{L_c^2}{R^2} \right] I_p \varphi = T_c \varphi, \quad \lim_{\mu_c \to \infty} M_m(\varphi) = 2a_1 \mu \frac{L_c^2}{R^2} I_p \varphi = T_m \varphi, \quad (67)
\]

\[
\lim_{\mu_c \to \infty} W_{\text{tot}}(\varphi) = \frac{1}{2} \left[ \mu_{\text{macro}} + 3a_1 \mu \frac{L_c^2}{R^2} \right] I_p \varphi^2 = \frac{1}{2} T_w \varphi^2.
\]

It is highlighted that there is not a one-to-one correspondence between the torque obtained as a limit from the Cosserat model (67) and that obtained using the indeterminate couple stress model from the beginning (78), but of course the energy (or the sum of the two torques) coincides and thus the total torque stiffness \( T_w \) coincides as well.

6.3. Cosserat limit case \( \mu_c \to 0 \).

We have (Figure 17)

\[
\lim_{\mu_c \to 0} M_c(\varphi) = \mu_{\text{macro}} I_p \varphi = T_c \varphi, \quad \lim_{\mu_c \to 0} M_m(\varphi) = \frac{24 \mu a_1 a_3 L_c^2}{a_1 + 8a_3 R^2} I_p \varphi = T_m \varphi, \quad (68)
\]

\[
\lim_{\mu_c \to 0} W_{\text{tot}}(\varphi) = \frac{1}{2} \left[ \mu_{\text{macro}} + \frac{24 \mu a_1 a_3 L_c^2}{a_1 + 8a_3 R^2} \right] I_p \varphi^2 = \frac{1}{2} T_w \varphi^2.
\]

It is highlighted that the Cosserat model does not collapse into a classical linear elastic model for \( \mu_c \to 0 \), but it remains proportional to \((L_c/R)^2\) (see (68)). In this case, the Cosserat model behaves similarly to the indeterminate couple stress model (67) or (78), and it collapses into this model (both the energy and the torques) by formally letting \( a_3 \to \infty \) as can be seen from (67) and (68).

6.4. Sensitivity of the Cosserat model with respect to the curvature parameters \( a_1 \) and \( a_3 \).

Here, we study the sensitivity for the Cosserat model while varying \( a_1 \) and \( a_3 \) independently.

The parametric study represented in Figure 18 has not been carried out for the limit \( a_1 \to 0 \) and \( a_3 \to 0 \) because we would have had an indeterminate form for \( L_c \to \infty \), and that is why we used the symbol \( \sim \). The solution of the problem while having \( a_3 = 0 \) \emph{a priori} is analysed carefully in Section 6.1, and the solution of the problem while having \( a_1 = 0 \) \emph{a priori} makes the relaxed micromorphic model collapse into a classical
Figure 17. Cosserat model. (a) Torsional stiffness for the classical torque \( T_c \), the higher-order torque \( T_m \) and the torque energy \( T_w \) while varying \( L_c \) for the limit \( \mu_c \rightarrow 0 \). The model does not collapse into a classical linear elastic model. The values of the material parameter used are \( \mu = 1, \mu_e = 1/10, a_1 = 1/5, a_3 = 1/7 \) and \( R = 1 \). (b) Sensitivity study on how the Cosserat model behaves while varying \( \mu_c = \{0, 1/3, 1, \infty\} \): for \( \mu_c \rightarrow \infty \) we recover the indeterminate couple stress model, whereas for \( \mu_c \rightarrow 0 \) we still have a nonlinear relation between \( T_w \) and \( R/L_c \) because a classical linear elastic model is not attained (see (68)). The values of the material parameter used are \( \mu = 1, \mu_e = 1/10, a_1 = 12, a_3 = 1/20 \) and \( R = 1 \).

Figure 18. Cosserat model. Response of the Cosserat model while varying (a) the curvature parameter \( a_1 \) having \( a_3 = 20 \) and (b) the curvature parameter \( a_3 \) having \( a_1 = 20 \). The values of the other parameters are \( \mu = 1, \mu_c = 1/5, \mu_{macro} = 1/10 \) and \( R = 1 \).

linear elastic model with torsional stiffness \( T_{macro} \). It is also highlighted that the Cosserat model collapses into the indeterminate couple stress model for \( a_3 \rightarrow \infty \) also in this more general case for which \( \mu_c \) is arbitrary.

7. Torsional problem for the isotropic micro-void model in dislocation tensor format

The strain energy for the isotropic micro-void continuum in dislocation tensor format can be written as \([1, 53]\)

\[
W(Du, \omega, \text{Curl} (\omega \mathbb{1})) = \mu_{macro} \| \text{dev sym} Du \|^2 + \frac{\kappa_c}{2} \text{tr}^2 (Du - \omega \mathbb{1}) + \frac{\kappa_{micro}}{2} \text{tr}^2 (\omega \mathbb{1}) + \frac{\mu L_c^2}{2} a_2 \| \text{Curl} (\omega \mathbb{1}) \|^2.
\]
Here, $\omega : \mathbb{R}^3 \to \mathbb{R}$ is the additional scalar micro-void degree of freedom [53]. The equilibrium equations, in the absence of body forces, are\(^5\)

$$\bar{\sigma} := \text{Div} \left[ 2\mu_{\text{macro}} \text{dev sym} \, D\mathbf{u} + \kappa \text{tr} (D\mathbf{u} - \omega \mathbb{I}) \right] = 0, \quad (70)$$

$$\frac{1}{3} \text{tr} \left[ \bar{\sigma} - \kappa_{\text{micro}} \text{tr} (\omega \mathbb{I}) \mathbb{I} - \mu L_c^2 \text{Curl} (\omega \mathbb{I}) \right] = 0.$$  

The boundary conditions at the free surface are

$$\bar{\mathbf{t}}(r = R) = \bar{\sigma}(r) \cdot \mathbf{e}_r = 0_{\mathbb{R}^3},$$  

$$\eta(r = R) = \frac{1}{3} \text{tr} (\mathbf{m}(r) \cdot \mathbf{e} \cdot \mathbf{e}_r) = \frac{1}{3} \text{tr} (\mathbf{m}(r) \times \mathbf{e}_r) = 0.$$  

According with the reference system shown in Figure 1, the ansatz for the displacement field and the function $\omega$ have to be

$$\mathbf{u}(x_1, x_2) = \left( \begin{array}{c} -x_2 x_3 \\ x_1 x_3 \\ 0 \end{array} \right), \quad \omega (x_2) = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad (72)$$

which clearly reduce the model to a classical linear elastic model. No further calculation will be carried on and the reader is referred to Section 3.

### 8. Torsional problem for the isotropic couple stress continuum

The indeterminate couple stress model [54–60] appears by letting formally the Cosserat couple modulus $\mu_c \to \infty$. This implies the constraint $\mathcal{A} = \text{skew} \, D\mathbf{u} \in \mathfrak{so}(3)$. It is highlighted that for the torsional problem, we do not have any unknown fields in this model since the displacement $\mathbf{u}$ is known \textit{a priori}.\(^4\)

As $\text{tr} (\text{Curl skew} \, D\mathbf{u}) = \|\text{skew} \, \text{Curl skew} \, D\mathbf{u} \|^2 = 0$ in terms of the ansatz (77), the indeterminate couple stress elastic energy for the torsion can be written as

$$W (D\mathbf{u}, \text{Curl skew} \, D\mathbf{u}) = \mu_{\text{macro}} \|\text{sym} \, D\mathbf{u}\|^2 + \frac{\lambda_{\text{macro}}}{2} \text{tr}^2 (D\mathbf{u}) + \frac{\mu L_c^2}{2} a_1 \|\text{sym} \, \text{Curl skew} \, D\mathbf{u}\|^2. \quad (73)$$

The equilibrium equations, in the absence of body forces, are\(^5\)

$$\text{Div} \left[ 2\mu_{\text{macro}} \text{sym} \, D\mathbf{u} + \lambda_{\text{macro}} \text{tr} (D\mathbf{u}) \mathbb{I} + \mu L_c^2 \text{skew} \, \text{Curl} (a_1 \text{dev sym} \, \text{Curl skew} \, D\mathbf{u}) \right] = 0, \quad (74)$$

whereas the (highly nontrivial) boundary traction conditions on the free surface are (for more details see [54, 55])

$$\bar{\mathbf{t}}(r = R) = \pm \left\{ \bar{\sigma} - \frac{1}{2} \text{Anti} (\text{Div} \, \mathbf{m}) \right\} \cdot \mathbf{e}_r \mp \frac{1}{2} \mathbf{e}_r \times \mathcal{D} \left[ \langle \mathbf{e}_r, \text{sym} \, \mathbf{m} \cdot \mathbf{e}_r \rangle \right],$$

$$\frac{1}{2} \mathcal{D} \left[ \text{Anti} \left( (\mathbb{I} - \mathbf{e}_r \otimes \mathbf{e}_r) \cdot \mathbf{m} \cdot \mathbf{e}_r \right) \cdot (\mathbb{I} - \mathbf{e}_r \otimes \mathbf{e}_r) \right] : (\mathbb{I} - \mathbf{e}_r \otimes \mathbf{e}_r) = 0,$$  

$$\pi(r = R) = \pm \left\{ \text{Anti} \left[ (\mathbb{I} - \mathbf{e}_r \otimes \mathbf{e}_r) \cdot \mathbf{m} \cdot \mathbf{e}_r \right] - \text{Anti} \left[ (\mathbb{I} - \mathbf{e}_r \otimes \mathbf{e}_r) \cdot \mathbf{m} \cdot \mathbf{e}_r \right]^\top \right\} \cdot \mathbf{e}_\phi = 0,$$

where $\bar{\sigma} = 2 \mu_c \text{sym} \, D\mathbf{u} + \lambda \text{tr} (D\mathbf{u}) \mathbb{I}$ is the symmetric force stress tensor, $\mathbf{e}_r$ is the radial unit vector and the nonsymmetric second-order moment stress is

$$\mathbf{m} = \mu L_c^2 (a_1 \text{dev sym} \, \text{Curl skew} \, D\mathbf{u} \mp a_2 \text{skew} \, \text{Curl skew} \, D\mathbf{u}). \quad (76)$$

The term $\text{Anti} \left[ (\mathbb{I} - \mathbf{e}_r \otimes \mathbf{e}_r) \cdot \mathbf{m} \cdot \mathbf{e}_r \right]$ is the measure of the discontinuity of $\text{Anti} \left[ (\mathbb{I} - \mathbf{e}_r \otimes \mathbf{e}_r) \cdot \mathbf{m} \cdot \mathbf{e}_r \right]^\top$ across the boundary.
According to the reference system shown in Figure 1, the ansatz for the displacement field and consequently the skew-symmetric part of the gradient of the displacement are

\[
\mathbf{u}(x_1, x_2) = \vartheta \begin{pmatrix} -x_2 x_3 \\ x_1 x_3 \\ 0 \end{pmatrix} \Rightarrow \text{skew } \mathbf{Du} = \frac{\vartheta}{2} \begin{pmatrix} 0 & -2x_3 & -x_2 \\ 2x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}.
\]

(77)

As the ansatz is completely known, it is possible to check that both the equilibrium equations (74) and the boundary conditions (75) are identically satisfied, and it is possible then to evaluate directly the classical torque, the higher-order torque and the energy.

The classical torque, the higher-order torque and energy (per unit length \(dz\)) expressions are

\[
M_c(\vartheta) := \int_0^{2\pi} \int_0^R \left[ (\widetilde{\sigma} \mathbf{e}_z, \mathbf{e}_\varphi) \right] r \, dr \, d\varphi = \mu_c I_p \vartheta = T_c \vartheta,
\]

\[
M_m(\vartheta) := \int_0^{2\pi} \int_0^R \left[ (\mathbf{m} \times \mathbf{e}_z, \mathbf{e}_\varphi) - (\mathbf{m} \times \mathbf{e}_z, \mathbf{e}_r) + (\mathbf{m} \times \mathbf{e}_r, \mathbf{e}_z) - (\mathbf{m} \times \mathbf{e}_r, \mathbf{e}_\varphi) \right] r \, dr \, d\varphi
= 3a_1 \mu \frac{L_c^2}{R^2} I_p \vartheta = T_m \vartheta,
\]

(78)

\[
W_{\text{tot}}(\vartheta) := \int_0^{2\pi} \int_0^R W(\mathbf{Du}, \text{Curl skew } \mathbf{Du}) \, r \, dr \, d\varphi = \frac{1}{2} \left[ \mu_{\text{macro}} + 3a_1 \mu \frac{L_c^2}{R^2} \right] I_p \vartheta^2 = \frac{1}{2} T_{\text{w}} \vartheta^2.
\]

The plot of the torsional stiffness for the classical torque, the higher-order torque and the torque energy while varying \(L_c\) is shown in Figure 19.

8.1. Torsional problem for the modified and the ‘pseudo’-consistent isotropic couple stress model

The modified couple stress model [54,59,61–63] consists of choosing \(a_1 > 0, a_2 = 0\) and leads to a symmetric couple stress tensor, whereas the (‘pseudo’) consistent couple stress model [64] appears for \(a_1 = 0, a_2 > 0\) and leads to a skew-symmetric stress tensor \(\mathbf{m}\).

As \(\text{tr} (\text{Curl skew } \mathbf{Du}) = \| \text{skew Curl skew } \mathbf{Du} \|^2 = 0\), the term \(\| \text{dev sym Curl skew } \mathbf{Du} \|^2\) is the only nonzero component in the curvature energy, the form of the energy remains the same. This implies that, for the torsion problem, the modified couple stress model coincides with the indeterminate couple stress model, and that the (‘pseudo’)-consistent couple stress model reduces to a classical linear elastic model without size effects.

According to the notation [64], the constitutive law can be written as

\[
\widetilde{\sigma} = 2\mu_{\text{macro}} \text{ sym } \mathbf{Du} + \lambda_{\text{macro}} \text{ tr } (\mathbf{Du}) \mathbf{1}, \quad \mathbf{m} = \eta (\text{D curl } \mathbf{u})^\top + \eta' \text{D curl } \mathbf{u},
\]

(79)
where, according to the classical Cosserat notation (see Appendix C),

\[
\eta = \beta = \mu_{\text{macro}} \frac{L_c^2}{2} a_1 - a_2, \quad \eta' = \gamma = \mu_{\text{macro}} \frac{L_c^2}{2} a_1 + a_2. \tag{80}
\]

In this notation, the modified couple stress model appears for \( \eta = \eta' \) and the ‘pseudo’-consistent couple stress model appears for \( \eta = -\eta' \).

### 9. Torsional problem for the classical isotropic micromorphic continuum without mixed terms

The strain energy for the isotropic micromorphic continuum without mixed terms ((sym \( P, \) sym (\( Du - P \)), etc.) and simplified isotropic curvature can be written as

\[
W(Du, P, DP) = \mu_e \|\text{dev sym}(Du - P)\|^2 + \frac{\kappa_e}{2} \text{tr}^2(Du - P) + \mu_c \|\text{skew}(Du - P)\|^2 \\
+ \mu_{\text{micro}} \|\text{dev sym}P\|^2 + \frac{\kappa_{\text{micro}}}{2} \text{tr}^2(P) \\
+ \frac{\mu L_c^2}{2} \left( a_1 \|D(\text{dev sym}P)\|^2 + a_2 \|D(\text{skew}P)\|^2 + \frac{2}{9} a_3 \|\text{tr}(P)\|^2 \right). \tag{81}
\]

The equilibrium equations, in the absence of body forces, are as follows

\[
\text{Div}\left[2\mu_e \text{dev sym}(Du - P) + \kappa_e \text{tr}(Du - P) I + 2\mu_c \text{skew}(Du - P)\right] = 0, \\
\tilde{\sigma} - 2\mu_{\text{micro}} \text{dev sym}P - \kappa_{\text{micro}} \text{tr}(P) I \\
+ \mu L_c^2 \left[ a_1 \text{dev sym}\Delta P + a_2 \text{skew}\Delta P + \frac{2}{9} a_3 \text{tr}(\Delta P) I \right] = 0, \tag{82}
\]

where \( \Delta P \in \mathbb{R}^{3 \times 3} \) is taken component-wise. The boundary conditions at the external surfaces are

\[
\tilde{\sigma}(r = R) = \tilde{\sigma}(r) \cdot e_r = 0_{\mathbb{R}^3}, \quad \eta(r = R) = m(r) \cdot e_r = 0_{\mathbb{R}^{3 \times 3}}, \tag{83}
\]

where

\[
m = \mu L_c^2 \left[ a_1 D(\text{dev sym}P) + a_2 D(\text{skew}P) + \frac{2}{9} a_3 D(\text{tr}P) I \right] \tag{84}
\]

is the third-order moment stress tensor, the expression of \( \tilde{\sigma} \) is given in (82), and \( e_r \) is the radial unit vector. According with the reference system shown in Figure 1, the ansatz for the displacement field and the micro-distortion is

\[
u(x_1, x_2, x_3) = u(r, \varphi, z) = \vartheta \begin{pmatrix}
-x_2(r, \varphi) x_3(z) \\
x_1(r, \varphi) x_3(z) \\
0
\end{pmatrix},
\]

\[
P(x_1, x_2, x_3) = P(r, \varphi, z) = \vartheta \begin{pmatrix}
0 & -x_3(z) & -g_2(r)x_2(r, \varphi) \\
x_3(z) & 0 & g_2(r)x_2(r, \varphi) \\
g_1(r)x_2(r, \varphi) & -g_1(r)x_1(r, \varphi) & 0
\end{pmatrix}. \tag{85}
\]
Substituting the ansatz (85) in (82) the 12 equilibrium equations are equivalent to

\[
\begin{align*}
\frac{1}{2} \vartheta \sin \varphi \left( \mu L_c^2 \left( (a_1 - a_2)g_1''(\rho) - (a_1 + a_2)g_2''(\rho) \right) + 2 \mu_c(g_1(\rho) + g_2(\rho) - 1) \right) \\
-2 \left( \mu_c + \mu_{\text{micro}} \right) (g_1(\rho) - g_2(\rho)) - 2 \mu_c + 3 \mu L_c^2 \left( (a_1 - a_2)g_1'(\rho) - (a_1 + a_2)g_2'(\rho) \right) = 0,
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} \vartheta \cos \varphi \left( \mu L_c^2 \left( (a_1 - a_1)g_1''(\rho) + (a_1 + a_2)g_2''(\rho) \right) - 2 \mu_c(g_1(\rho) + g_2(\rho) - 1) \right) \\
+2 \left( \mu_c + \mu_{\text{micro}} \right) (g_1(\rho) - g_2(\rho)) + 2 \mu_c + 3 \mu L_c^2 \left( (a_1 - a_2)g_1'(\rho) + (a_1 + a_2)g_2'(\rho) \right) = 0,
\end{align*}
\]

\[
\begin{align*}
\frac{1}{2} \vartheta \sin \varphi \left( \mu L_c^2 \left( (a_1 - a_2)g_1''(\rho) - (a_1 + a_1)g_2''(\rho) \right) + 2 \mu_c(g_1(\rho) + g_2(\rho) - 1) \right) \\
+ \left( \mu_c + \mu_{\text{micro}} \right) (g_1(\rho) - g_2(\rho)) + 2 \mu_c + 3 \mu L_c^2 \left( (a_1 - a_2)g_1'(\rho) - (a_1 + a_2)g_2'(\rho) \right) = 0.
\end{align*}
\]

(86)

It is important to underline that (82)\(_1\) is identically satisfied, and that between the four equilibrium equations (86) there are only two that are linearly independent because (86)\(_1\) = \tan \varphi (86)\(_2\) and (86)\(_3\) = \tan \varphi (86)\(_4\).

It is also pointed out that the two remaining linearly independent equations (86)\(_{1,3}\) can be uncoupled\(^8\) and have the form of the Bessel ODE if we take their sum and difference, while being careful of substituting \(g_p(\rho) := g_1(\rho) + g_2(\rho)\) and \(g_m(\rho) := g_1(\rho) - g_2(\rho)\) along with their derivatives:

\[
\begin{align*}
\vartheta \sin \varphi \left( a_1 \mu L_c^2 \left( 3g_m''(\rho) + \rho g_m''(\rho) \right) - 2 \rho \mu_c(g_m(\rho) + 1) - 2 \rho g_m(\rho)\mu_{\text{micro}} \right) = 0,
\end{align*}
\]

\[
\begin{align*}
\vartheta \sin \varphi \left( 2 \rho \mu_c(g_p(\rho) - 1) - a_2 \mu L_c^2 \left( 3g_p''(\rho) + \rho g_p''(\rho) \right) \right) = 0.
\end{align*}
\]

As \(g_1(\rho) := \frac{g_p(\rho) + g_m(\rho)}{2}\) and \(g_2(\rho) := \frac{g_p(\rho) - g_m(\rho)}{2}\), the solution in terms of \(g_1(\rho)\) and \(g_2(\rho)\) of (87) is

\[
\begin{align*}
g_1(\rho) &= \frac{1}{2} \left( 1 - \frac{iA_1 I_1 \left( \frac{\mu L_c}{\mu} \right) - A_2 Y_1 \left( -i \frac{\mu L_c}{\mu} \right) + iB_1 I_1 \left( \frac{\mu L_c}{\mu} \right) - B_2 Y_1 \left( -i \frac{\mu L_c}{\mu} \right)}{r} \right) - \frac{\mu_c}{\mu_c + \mu_{\text{micro}}}, \\
g_2(\rho) &= \frac{1}{2} \left( 1 + \frac{iA_1 I_1 \left( \frac{\mu L_c}{\mu} \right) - A_2 Y_1 \left( -i \frac{\mu L_c}{\mu} \right) - iB_1 I_1 \left( \frac{\mu L_c}{\mu} \right) + B_2 Y_1 \left( -i \frac{\mu L_c}{\mu} \right)}{r} \right) + \frac{\mu_c}{\mu_c + \mu_{\text{micro}}}, \\
f_1 := \sqrt{\frac{2\mu_c}{a_2 \mu}}, \quad f_2 := \sqrt{\frac{2(\mu_c + \mu_{\text{micro}})}{a_1 \mu}},
\end{align*}
\]

where \(I_n(\cdot)\) is the modified Bessel function of the first kind, \(Y_n(\cdot)\) is the Bessel function of the second kind (see Appendix B for the formal definitions), and \(A_1, B_1, A_2\) and \(B_2\) are integration constants.

The values of \(A_1, B_1\) are determined thanks to the boundary conditions (83), whereas, owing to the divergent behaviour of the Bessel function of the second kind at \(r = 0\), we have to set \(A_2 = 0\) and \(B_2 = 0\) in order to have a continuous solution. The fulfilment of the boundary conditions (83) allows us to find the expressions of the integration constants

\[
\begin{align*}
A_1 &= \frac{2iL_c \mu_c}{f_2 (\mu_c + \mu_{\text{micro}}) \left( I_0 \left( \frac{\mu L_c}{\mu} \right) + I_2 \left( \frac{\mu L_c}{\mu} \right) \right)}, \quad B_1 = -\frac{2iL_c}{f_1 \left( I_0 \left( \frac{\mu L_c}{\mu} \right) + I_2 \left( \frac{\mu L_c}{\mu} \right) \right)}.
\end{align*}
\]

(89)
The classical torque, the higher-order torque and energy (per unit length $dz$) expressions are

\[ M_c(\theta) := \int_0^{2\pi} \int_0^R \left[ (\bar{\sigma} e_z, e_r) \right] r \, dr \, d\varphi \]

\[ = \left[ \left( \frac{8\mu_e I_2 \left( \frac{B_1}{I_c} \right)}{f_1^2 I_0 \left( \frac{B_1}{I_c} \right) + f_2^2 I_2 \left( \frac{B_2}{I_c} \right)} \right) + \frac{8\mu_e^2 I_2 \left( \frac{B_2}{I_c} \right)}{(\mu_e + \mu_{\text{micro}}) \left( f_1^2 I_0 \left( \frac{B_1}{I_c} \right) + f_2^2 I_2 \left( \frac{B_2}{I_c} \right) \right)} \right] \frac{L^2}{R^2} + \frac{\mu_e \mu_{\text{micro}}}{\mu_e + \mu_{\text{micro}}} \right] I_p \, \theta, \]

\[ M_m(\theta) := \int_0^{2\pi} \int_0^R \left[ ((\bar{m} e_z) e_r, e_r) - ((\bar{m} e_z) e_r, e_r) \right] r \, dr \, d\varphi = 4a_2 \mu \frac{L^2}{R^2} I_p \, \theta = T_m \, \theta, \] \label{eq:90}

\[ W_{\text{tot}}(\theta) := \int_0^{2\pi} \int_0^R W(Du, P, DP) \ r \, dr \, d\varphi \]

\[ = \frac{1}{2} \left[ \left( \frac{8\mu_e I_2 \left( \frac{B_1}{I_c} \right)}{f_1^2 I_0 \left( \frac{B_1}{I_c} \right) + f_2^2 I_2 \left( \frac{B_2}{I_c} \right)} \right) + \frac{8\mu_e^2 I_2 \left( \frac{B_2}{I_c} \right)}{(\mu_e + \mu_{\text{micro}}) \left( f_1^2 I_0 \left( \frac{B_1}{I_c} \right) + f_2^2 I_2 \left( \frac{B_2}{I_c} \right) \right)} \right] \frac{L^2}{R^2} + \frac{\mu_e \mu_{\text{micro}}}{\mu_e + \mu_{\text{micro}}} \right] I_p \, \theta^2 \]

\[ + 4a_2 \mu \frac{L^2}{R^2} \right] I_p \, \theta^2 = \frac{1}{2} T_w \, \theta^2, \]

and again it holds that

\[ \frac{d}{d\theta} W_{\text{tot}}(\theta) = M_c(\theta) + M_m(\theta), \quad \frac{d^2}{d\theta^2} W_{\text{tot}}(\theta) = T_c + T_m = T_w. \] \label{eq:91}

It is underlined that the boundary conditions for the micromorphic model are consistent with those of the relaxed micromorphic model, being careful of changing $\bar{m} e_z$, with $m \times e_z$. The plot of the torsional stiffness for the classical torque (light blue), the higher-order torque (red) and the torque energy (green) while varying $L_c$ is shown in Figure 20.

**Figure 20. Micromorphic model, classical case.** Torsional stiffness for the classical torque $T_c$, the higher-order torque $T_m$ and the torque energy $T_w$ while varying $L_c$. The torsional stiffness is unbounded as $L_c \to \infty$ ($R \to 0$). The values of the parameters used are $\mu = 1, \mu_e = 1/3, \mu_{\text{micro}} = 1/4, \mu_c = 1/5, a_1 = 1/5, a_2 = 1/6$ and $R = 1$. 

}\]
9.1. Limits

9.1.1. The classical micromorphic model with symmetric forces stresses ($\mu_c \to 0$): nothing special. The classical micromorphic model with symmetric forces stresses ($\mu_c \to 0$) is shown in Figure 21.

![Figure 21. Micromorphic model. Torsional stiffness for the torque energy while varying $L_c$, for different values of $\mu_c = \{0, 1/30, 1/10, 1/5, 1, \infty\}$. The torsional stiffness remains bounded as $L_c \to \infty$ ($R \to 0$) and the model does not collapse in a linear elastic one. The values of the other parameters used are $\mu = 1$, $\mu_c = 1/3$, $\mu_{micro} = 1/4$, $a_1 = 2$, $a_3 = 1/20$ and $R = 1$.](image)

9.1.2. The classical micromorphic model with reduced curvature energy ($a_2 = 0$). The classical micromorphic model with reduced curvature energy ($a_2 = 0$; see Figure 22) collapses into the micro-strain model (Section 10 with symP) thus becoming independent with respect to the Cosserat couple modulus $\mu_c$ (see (47) for the different stiffnesses expressions).

![Figure 22. Micromorphic model. Torsional stiffness for the torque energy while varying $L_c$, for different values of $\mu_c = \{0, 1/30, 1/10, 1/5, 1, \infty\}$. The torsional stiffness remains bounded as $L_c \to \infty$ ($R \to 0$) and the model does not collapse in a linear elastic one. The values of the other parameters used are $\mu = 1$, $\mu_c = 1/3$, $\mu_{micro} = 1/4$, $a_1 = 2$, $a_3 = 1/20$ and $R = 1$. In this case, the stiffness for arbitrary small sample size is governed by $T_e$ and not $T_{micro}$. The reason for this is explained in Appendix D.](image)

10. Torsional problem for the micro-strain model without mixed terms

The micro-strain model [26, 65] is a particular case of the classical Mindlin–Eringen model, in which it is assumed a priori that the micro-distortion remains symmetric, $P = S \in \text{Sym}(3)$.

A torsion solution for a more general case with mixed terms has been derived in [28], but here we employ a reduced isotropic curvature expression to make the calculations more manageable.
It is underlined that the micro-strain model cannot be obtained as a limit case of the relaxed micromorphic model and vice versa, although there are some similarities. The strain energy which we consider is

\[
W(Du, S, DS) = \mu_e \| \text{dev} (\text{sym} Du - S) \|^2 + \frac{\kappa_e}{2} \| Du - S \|^2 + \mu_{\text{micro}} \| \text{dev} S \|^2 + \frac{\kappa_{\text{micro}}}{2} \| \text{tr}^2(S) \|
\]  

(92)

The chosen two-parameter curvature expression represents a simplified isotropic curvature (the full isotropic curvature for the micro-strain model would still count eight parameters [66]). In this form, the micro-strain model can be obtained from the classical micromorphic model (Section 9), in general, by setting \( \mu_e = 0 \) and \( a_2 = 0 \). For the torsion problem, the condition \( a_2 = 0 \) alone is sufficient.

It is underlined that for the ansatz (96), which will be presented later in this section, it holds that \( \text{tr}(S) = 0 \). The equilibrium equations, in the absence of body forces, are therefore the following

\[
\bar{\sigma}(r) := \text{Div} \left[ 2 \mu_e \text{dev} (\text{sym} Du - S) + \kappa_e \text{tr} (Du - S) \| S \| \text{tr}(S) \| \right] = 0,
\]

(93)

\[
2 \mu_e \text{dev} (\text{sym} Du - S) + \kappa_e \text{tr} (Du - S) \| S \| - 2 \mu_{\text{micro}} \text{dev} S - \kappa_{\text{micro}} \text{tr}(S) \| S \| + \mu L_c^2 \text{sym Div} \left[ a_1 D(\text{dev} S) + \frac{2}{9} a_3 D(\text{tr}(S) \| S \|) \right] = 0.
\]

The boundary conditions at the external free surfaces are

\[
\bar{t}(r = R) = \bar{\sigma}(r) \cdot e_r = 0_{\mathbb{R}^3}, \quad \eta(r = R) = \text{sym} (m(r) \cdot e_r) = 0_{\mathbb{R}^{3 \times 3}},
\]

(94)

where

\[
m = \mu L_c^2 \left[ a_1 D(\text{dev} S) + \frac{2}{9} a_3 D(\text{tr}(S) \| S \|) \right]
\]

(95)

is the third-order moment stress tensor, the expression of \( \bar{\sigma} \) is in (93), \( e_r \) is the radial unit vector. According with the reference system shown in Figure 1, the ansatz for the displacement field and the micro-distortion is

\[
u(r, \varphi, z) = \vartheta \left( \begin{array}{c} -x_2(r, \varphi) x_3(z) \\ x_1(r, \varphi) \| x_3(z) \\ 0 \end{array} \right),
\]

(96)

\[
S(r, \varphi, z) = \frac{\vartheta}{2} \left( \begin{array}{cccc} 0 & 0 & g_m(r) x_2(r, \varphi) & -g_m(r) x_2(r, \varphi) \\ 0 & 0 & -g_m(r) x_1(r, \varphi) & 0 \end{array} \right),
\]

where, in relation to the ansatz (38), \( g_m(r) := g_1(r) - g_2(r) \). Substituting the ansatz (96) in (93) the nine equilibrium equations are equivalent to

\[
\frac{1}{2} \vartheta \sin \varphi \left( a_1 \mu L_c^2 \left( 3 g_m''(r) + r g_m''(r) \right) - 2r \mu_e (g_m(r) + 1) - 2r g_m(r) \mu_{\text{micro}} \right) = 0,
\]

(97)

\[
-\frac{1}{2} \vartheta \cos \varphi \left( a_1 \mu L_c^2 \left( 3 g_m''(r) + r g_m''(r) \right) - 2r \mu_e (g_m(r) + 1) - 2r g_m(r) \mu_{\text{micro}} \right) = 0.
\]

Between the two equilibrium equations (97) there is only one independent equation because (97)_1 = - tan \varphi (97)_2. The solution of (97) is

\[
g_m(r) = \frac{A_2 Y_1 \left( \frac{\varphi L_c}{r} \right) - i A_1 I_1 \left( \frac{\varphi L_c}{r} \right)}{r} - \frac{\mu_e}{\mu_e + \mu_{\text{micro}}}, \quad f_1 := \sqrt{\frac{2(\mu_e + \mu_{\text{micro}})}{a_1 \mu}},
\]

(98)

where \( I_n(\cdot) \) is the modified Bessel function of the first kind, \( Y_n(\cdot) \) is the Bessel function of the second kind (see Appendix B for the formal definitions), and \( A_1, A_2 \) are integration constants.
The value of \( A_1 \) is determined thanks to the boundary conditions (94), whereas, owing to the divergent behaviour of the Bessel function of the second kind at \( r = 0 \), we have to set \( A_2 = 0 \) in order to have a continuous solution. The fulfilment of the boundary conditions (94) allows us to find the expressions of the integration constants

\[
A_1 = \frac{2iL_c}{I_0 \left( \frac{R_f}{L_c} \right) + I_2 \left( \frac{R_f}{L_c} \right) f_1(\mu_c + \mu_{\text{micro}})} \tag{99}
\]

The classical torque, the higher-order torque and the energy (per unit length \( d\zeta \)) expressions are

\[
M_c(\vartheta) := \int_0^{2\pi} \int_0^R \left[ (\tilde{\tau}_z, e_{\varphi}) r \right] dr d\varphi
= \left[ \frac{\mu_c \mu_{\text{micro}}}{\mu_c + \mu_{\text{micro}}} + \frac{\mu_c^2 \mu_a_1}{(\mu_c + \mu_{\text{micro}})^2} \right] I_0 \left( \frac{R_f}{L_c} \right) + I_2 \left( \frac{R_f}{L_c} \right) \frac{L_c^2}{R^2} I_0 \vartheta = T_c \vartheta,
\]

\[
M_m(\vartheta) := \int_0^{2\pi} \int_0^R \left[ \langle \text{sym}(m e_z) e_{\varphi}, e_r \rangle - \langle \text{sym}(m e_z) e_r, e_{\varphi} \rangle \right] r dr d\varphi = 0, \tag{100}
\]

\[
W_{\text{tot}}(\vartheta) := \int_0^{2\pi} \int_0^R W(Du, S, DS) \ r dr d\varphi
= \frac{1}{2} \left[ \frac{\mu_c \mu_{\text{micro}}}{\mu_c + \mu_{\text{micro}}} + \frac{\mu_c^2 \mu_a_1}{(\mu_c + \mu_{\text{micro}})^2} \right] I_0 \left( \frac{R_f}{L_c} \right) + I_2 \left( \frac{R_f}{L_c} \right) \frac{L_c^2}{R^2} I_0 \vartheta^2 = \frac{1}{2} T_w \vartheta^2.
\]

The plot of the torsional stiffness for the classical torque, the higher-order torque and the torque energy while varying \( L_c \) is shown in Figure 14. As the higher-order torque is zero and the following relation holds

\[
\frac{d}{d\vartheta} W_{\text{tot}}(\vartheta) = M_c(\vartheta) + M_m(\vartheta) = M_c(\vartheta), \quad \frac{d^2}{d\vartheta^2} W_{\text{tot}}(\vartheta) = T_c + T_m = T_c = T_w, \tag{101}
\]

only the plot of the energy (per unit length \( d\zeta \)) while changing \( L_c \) is shown in Figure 23. The energy of the model remains bounded, as for the shear and bending problem [1, 2], because for both problems the higher-order moments are zero, and this does not create a conflict with the boundary condition as \( L_c \to \infty \) (see (47) for the different stiffness expressions). Note that there is no simple way to a priori guess that the small size torsional response is given by \( T_c \) because \( S \in \text{Sym}(3) \) is not easily seen to be zero. In Appendix D we show that the variational limit for \( L_c \to \infty \) is indeed realized by \( S(x) = \hat{S} = 0 \) and this shows that the limit stiffness is given by \( T_c \).

## 11. Torsional problem for the second gradient continuum

The expression of the most general isotropic strain energy for the second gradient continuum is [29, 63]

\[
W(Du, D^2u) = \mu_{\text{macro}} \|\text{sym} Du\|^2 + \frac{\lambda_{\text{macro}}}{2} \text{tr}^2(Du) + \tilde{a}_1 \chi_{ijk} \chi_{kj} + \tilde{a}_2 \chi_{ijj} \chi_{ijj} + \tilde{a}_3 \chi_{ijk} \chi_{ijk} + \tilde{a}_4 \chi_{ijk} \chi_{jik} + \tilde{a}_5 \chi_{ijk} \chi_{kji}, \tag{102}
\]
where $\chi = D^2 u$ ($\chi_{ijk} = \frac{\partial^2 u}{\partial x_i \partial x_j}$). The expression we are going to use in the following is a simplified isotropic strain energy with three curvature parameters

$$W(Du, D^2 u) = \mu_{\text{macro}} \| \text{sym } Du \|^2 + \frac{\lambda_{\text{macro}}}{2} \| \text{tr } (Du) \|^2$$

$$+ \frac{\mu L_c^2}{2} \left( a_1 \left\| D\left( \text{dev sym } Du \right) \right\|^2 + a_2 \left\| D\left( \text{skew } Du \right) \right\|^2 + \frac{2}{9} a_3 \left\| D\left( \text{tr } (Du) \right) \right\|^2 \right).$$

The equilibrium equation, in the absence of body forces, is

$$\text{Div} \left[ 2\mu_{\text{macro}} \text{sym } Du + \lambda_{\text{macro}} \text{tr } (Du) \mathbb{1} \right]$$

$$- \mu L_c^2 \left( a_1 \text{dev sym } \Delta (Du) + a_2 \text{skew } \Delta (Du) + \frac{2}{9} a_3 \text{tr } (\Delta (Du)) \mathbb{1} \right) = 0,$$

where $\Delta (Du) \in \mathbb{R}^{3 \times 3}$ is taken component-wise. The non-trivial boundary conditions at the free surface are

$$\tilde{\eta}(r = R) = \vec{e}_r + [(\vec{e}_r \otimes \vec{e}_r) \cdot \nabla \mathbf{m}] \vec{e}_r - 2 [(1 - \vec{e}_r \otimes \vec{e}_r) \cdot \nabla \mathbf{m}] \vec{e}_r$$

$$+ \left( [(1 - \vec{e}_r \otimes \vec{e}_r) \cdot \nabla \mathbf{e}_r] (\vec{e}_r \otimes \vec{e}_r) - [(1 - \vec{e}_r \otimes \vec{e}_r) (\nabla \mathbf{e}_r)^T] : \mathbf{m} = 0_{\mathbb{R}^3},ight.$$  

$$\tilde{\sigma}(r = R) = (\vec{e}_r \otimes \vec{e}_r) : \mathbf{m} = 0_{\mathbb{R}^3},$$

where, because the boundary surface is smooth, one set of boundary condition is identically satisfied (see [29, 62] for all the details). According to the reference system shown in Figure 1, the ansatz for the displacement field and consequently the gradient of the displacement are

$$u(x_1, x_2) = \vartheta \left( \begin{array}{c} -x_2 x_3 \\ x_1 x_3 \\ 0 \end{array} \right) \Rightarrow Du = \frac{\vartheta}{2} \left( \begin{array}{ccc} 0 & -2x_3 & -2x_2 \\ 2x_3 & 0 & 2x_1 \\ 0 & 0 & 0 \end{array} \right).$$

As the ansatz is completely known, it is possible to check that the equilibrium equation (104) and the boundary conditions (106) are identically satisfied and it is possible to evaluate directly the classical torque, the higher-order torque and the energy.
The classical torque, the higher-order torque and energy (per unit length $dz$) expressions are

$$M_c(\vartheta) := \int_0^{2\pi} \int_0^R \left[ (\bar{\sigma} e_z, e \vartheta) r \right] r \, dr \, d\varphi = \mu_{macro} I_p \vartheta = T_c \vartheta,$$

$$M_m(\vartheta) := \int_0^{2\pi} \int_0^R \left[ \langle (m e_z) e_r, e \vartheta \rangle - \langle (m e_z) e_r, e \vartheta \rangle + \langle (m e_r) e_z, e \vartheta \rangle - \langle (m e_r) e_z, e \vartheta \rangle \right] r \, dr \, d\varphi$$

$$= 2\mu (a_1 + 3a_2) \frac{L^2}{R^2} I_p \vartheta = T_m \vartheta,$$

$$W_{tot}(\vartheta) := \int_0^{2\pi} \int_0^R W(Du, D^2 u) r \, dr \, d\varphi = \frac{1}{2} \left[ \mu_{macro} + 2\mu (a_1 + 3a_2) \frac{L^2}{R^2} \right] I_p \vartheta^2 = \frac{1}{2} T_w \vartheta^2. \quad (107)$$

The plot of the torsional stiffness for the classical torque (light blue), the higher-order torque (red) and the torque energy (green) while varying $L_c$ is shown in Figure 24.

11.1. The strain gradient continuum as a limit of the micro-strain model

If we let $\mu_e, \kappa_e \to \infty$ in the micro-strain model, we obtain in the limit a strain gradient model with elastic energy (Figure 25)

$$W(Du, D\text{sym} Du) = \mu_{macro} \| \text{sym} Du \|^2 + \frac{\lambda_{macro}}{2} \text{tr}^2 (Du)$$

$$+ \frac{\mu}{2} \left( a_1 \left\| D \left( \text{dev sym} Du \right) \right\|^2 + a_3 \left\| D \left( \text{tr} (Du) \mathbf{1} \right) \right\|^2 \right). \quad (108)$$

As $\text{tr} (Du) = 0$ for our ansatz (14), the equilibrium equations, in the absence of body forces, are

$$\text{Div} \left[ 2\mu_{macro} \text{sym} Du + \lambda_{macro} \text{tr} (Du) \mathbf{1} - \mu L_c^2 a_1 \text{dev sym} \Delta (Du) \right] = 0, \quad (109)$$

where $\Delta (Du) \in \mathbb{R}^{3 \times 3}$ is taken component-wise.
12. Ad hoc model containing Cosserat and micro-strain effects

Given $S \in \text{Sym}(3)$ and $A \in \mathfrak{so}(3)$, the strain energy which we consider now is

$$W(Du, A, S, \text{Curl} A, DS) = \mu_c \|\text{sym} Du - S\|^2 + \frac{\lambda_c}{2} \text{tr}^2(Du - S) + \mu_c \|\text{skew}(Du - A)\|^2$$

$$+ \mu_{\text{micro}} \|\text{dev}\|^2 + \frac{\kappa_{\text{micro}}}{2} \text{tr}^2(S)$$

$$+ \frac{\mu L_c^2}{2} \left( a_1 \|\text{dev} \text{Curl} A\|^2 + \frac{a_3}{3} \text{tr}^2(\text{Curl} A) + a_4 \|D(\text{dev} S)\|^2 \right),$$

because $\|\text{skew} \text{Curl} A\|^2 = \|D(\text{skew} S)\|^2 = \|D(\text{tr} (S) \mathbb{1})\|^2 = 0$ in terms of the ansatz (114).

The equilibrium equations, in the absence of body forces, are as follows

$$\text{Div} \left[ 2\mu_c (\text{sym} Du - S) + \frac{\lambda}{2} \text{tr} (Du - S) \mathbb{1} + 2\mu_c (\text{skew} Du - A) \right] = 0,$$

$$2\mu_c \text{skew} (Du - A) - \mu L_c^2 \text{skew} \text{Curl} \left( a_1 \|\text{dev} \text{Curl} A\|^2 + \frac{a_3}{3} \text{tr} (\text{Curl} A) \mathbb{1} \right) = 0$$

$$2\mu_c (\text{sym} Du - S) + \frac{\lambda}{2} \text{tr} (Du - S) \mathbb{1} - 2\mu_{\text{micro}} S - \frac{\lambda_{\text{micro}}}{2} \text{tr} (S) \mathbb{1} + \mu L_c^2 a_4 \text{sym} \Delta (\text{dev} S) = 0.$$
and the micro-distortion is

\[
\begin{align*}
u(r, \varphi, z) &= 2 \left( \begin{array}{c}
-x_2(r, \varphi) x_3(z) \\
x_1(r, \varphi) x_3(z) \\
0
\end{array} \right), \\
A(r, \varphi, z) &= 2 \left( \begin{array}{ccc}
0 & -2x_3(z) & -g_p(r) x_2(r, \varphi) \\
g_p(r) x_2(r, \varphi) & 0 & g_p(r) x_1(r, \varphi) \\
g_m(r) x_2(r, \varphi) & -g_m(r) x_1(r, \varphi) & 0
\end{array} \right), \\
S(r, \varphi, z) &= 2 \left( \begin{array}{ccc}
0 & 0 & g_m(r) x_2(r, \varphi) \\
0 & 0 & g_m(r) x_1(r, \varphi) \\
g_m(r) x_2(r, \varphi) & -g_m(r) x_1(r, \varphi) & 0
\end{array} \right),
\end{align*}
\]

where, in relation to the ansatz (38), \(g_m(r) := g_1(r) - g_2(r)\) and \(g_p(r) := g_1(r) + g_2(r)\). Substituting the ansatz (114) into (111) the 15 equilibrium equations are equivalent to

\[
\begin{align*}
\frac{1}{6} \varphi \sin \varphi \left( 6r \mu_c (g_p(r) - 1) - \mu L_c^2 (a_1 + 2a_3) \left( 3g_p'(r) + r g_p''(r) \right) \right) &= 0, \\
\frac{1}{6} \varphi \cos \varphi \left( 6r \mu_c (g_p(r) - 1) - \mu L_c^2 (a_1 + 2a_3) \left( 3g_p'(r) + r g_p''(r) \right) \right) &= 0, \\
\frac{1}{2} \varphi \sin \varphi \left( A_4 \mu L_c^2 \left( 3g_p'(r) + r g_p''(r) \right) - 2r \mu_c (g_m(r) + 1) - 2r g_m(r) \mu_{micro} \right) &= 0, \\
- \frac{1}{2} \varphi \cos \varphi \left( A_4 \mu L_c^2 \left( 3g_p'(r) + r g_p''(r) \right) - 2r \mu_c (g_m(r) + 1) - 2r g_m(r) \mu_{micro} \right) &= 0.
\end{align*}
\]

Between the two equilibrium equations (115) there are only two independent equations because (115)_1 = −\tan \varphi (115)_2 and (115)_3 = \tan \varphi (115)_4. The solution of (115) is

\[
\begin{align*}
g_p(r) &= 1 - \frac{i A_1 I_1 \left( \frac{rf_1}{Le} \right)}{r} + \frac{A_2 Y_1 \left( \frac{irf_1}{Le} \right)}{r}, \\
g_m(r) &= \frac{A_2 Y_1 \left( \frac{-irf_1}{Le} \right) - i A_1 I_1 \left( \frac{rf_1}{Le} \right)}{r} - \frac{\mu_e}{\mu_e + \mu_{micro}}, \\
f_1 &= \frac{6 \mu_e}{(a_1 + 2a_3) \mu}, \\
f_2 &= \frac{2(\mu_e + \mu_{micro})}{a_4 \mu},
\end{align*}
\]

where \(I_n(\cdot)\) is the modified Bessel function of the first kind, \(Y_n(\cdot)\) is the Bessel function of the second kind (see Appendix B for the formal definitions), and \(A_1, A_2, A_3, A_4\) are integration constants.

The values of \(A_1\) and \(A_2\) are determined thanks to the boundary conditions (112), whereas, owing to the divergent behaviour of the Bessel function of the second kind at \(r = 0\), we have to set \(A_2 = A_4 = 0\) in order to have a continuous solution. The fulfilment of the boundary conditions (112) allows us to find the expressions of the integration constants

\[
\begin{align*}
A_1 &= -\frac{i RL_c}{f_1 Rz_1 \left( I_0 \left( \frac{Rf_1}{Le} \right) + I_2 \left( \frac{Rf_1}{Le} \right) \right) + \frac{i Rz_1}{L_c} \left( \frac{Rf_1}{Le} \right)}, \\
A_3 &= \frac{2i L_c}{I_0 \left( \frac{Rf_1}{Le} \right) + I_2 \left( \frac{Rf_1}{Le} \right)} \frac{\mu_e}{f_2(\mu_e + \mu_{micro})}, \\
z_1 &= \frac{a_1 + 2a_3}{3a_1}, \\
z_2 &= \frac{4a_3 - a_1}{3a_1}.
\end{align*}
\]
The classical torque, the higher-order torque and the energy (per unit length $dz$) expressions are

$$M_c(\vartheta) := \int_0^{2\pi} \int_0^R \left[ \langle \tilde{\sigma} e_z, e_\vartheta \rangle r \right] r \, dr \, d\varphi$$

$$= \left[ 4 \mu_c I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} f_1 \left( 2 f_1 z_1 I_0 \left( \frac{R f_1}{L_c} \right) + (z_2 - 2z_1) I_1 \left( \frac{R f_1}{L_c} \right) \frac{L_c}{R} \right) \right.$$  

$$+ \frac{\mu_c \mu_{micro}}{\mu_c + \mu_{micro}} \frac{\mu_c^2 \mu a_4}{(\mu_c + \mu_{micro})^2} 4 I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} I_0 \left( \frac{R f_1}{L_c} \right) + I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} \right] I_0 \vartheta = T_c \vartheta,$$

$$M_m(\vartheta) := \int_0^{2\pi} \int_0^R \left[ \langle \text{sym}(m e_z) e_\vartheta, e_\vartheta \rangle - \langle \text{sym}(m e_z) e_\vartheta, e_\vartheta \rangle \right] r \, dr \, d\varphi$$

$$= \left[ 2 \mu \left( 3 a_1 f_1 z_1 I_0 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} - 2(a_1 - a_3) I_1 \left( \frac{R f_1}{L_c} \right) \frac{L_c}{R} \right) \right.$$  

$$+ \frac{\mu_c \mu_{micro}}{\mu_c + \mu_{micro}} \frac{\mu_c^2 \mu a_4}{(\mu_c + \mu_{micro})^2} 4 I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} I_0 \left( \frac{R f_1}{L_c} \right) + I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} \right] = T_m \vartheta,$$  

$$W_{\text{tot}}(\vartheta) := \int_0^{2\pi} \int_0^R \left( W(Du, A, S, \text{Curl}A, DS) \right) r \, dr \, d\varphi$$

$$= \frac{1}{2} \left[ 4 \mu_c I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} f_1 \left( 2 f_1 z_1 I_0 \left( \frac{R f_1}{L_c} \right) + (z_2 - 2z_1) I_1 \left( \frac{R f_1}{L_c} \right) \frac{L_c}{R} \right) \right.$$  

$$+ \frac{\mu_c \mu_{micro}}{\mu_c + \mu_{micro}} \frac{\mu_c^2 \mu a_4}{(\mu_c + \mu_{micro})^2} 4 I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} I_0 \left( \frac{R f_1}{L_c} \right) + I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} \right]$$  

$$+ \frac{2 \mu \left( 3 a_1 f_1 z_1 I_0 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} - 2(a_1 - a_3) I_1 \left( \frac{R f_1}{L_c} \right) \frac{L_c}{R} \right) \right.$$  

$$+ \frac{\mu_c \mu_{micro}}{\mu_c + \mu_{micro}} \frac{\mu_c^2 \mu a_4}{(\mu_c + \mu_{micro})^2} 4 I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} I_0 \left( \frac{R f_1}{L_c} \right) + I_2 \left( \frac{R f_1}{L_c} \right) \frac{L_c^2}{R^2} \right] I_0 \vartheta^2 = \frac{1}{2} T_w \vartheta^2.$$  

It is highlighted that, similar to the micro-strain model (Section 10), the higher-order torque contribution $\langle (m e_z) e_\vartheta, e_\vartheta \rangle$ is equal to zero. The plot of the torsional stiffness for the classical torque, the higher-order torque and the torque energy while varying $L_c$ is shown in Figure 26. Again, it holds that

$$\frac{d}{d\vartheta} W_{\text{tot}}(\vartheta) = M_c(\vartheta) + M_m(\vartheta), \quad \frac{d^2}{d\vartheta^2} W_{\text{tot}}(\vartheta) = T_c + T_m = T_w.$$  

13. Summary and conclusions

We have derived the analytical expressions of the torsional rigidity for a family of generalized continua capable of modelling size dependence in the sense that more slender specimens are comparatively stiffer. We only consider (simplified) isotropic expressions so as to better compare the different models with each other. For example, a strain gradient continuum, by construction, does not have mixed energy terms. Therefore, we omitted
Figure 26. Ad-hoc model. Torsional stiffness for the classical torque $T_c$, the higher-order torque $T_m$ and the torque energy $T_w$ while varying $L_c$. The torsional stiffness is unbounded as $L_c \to \infty$ ($R \to 0$) owing to the Cosserat effects. The values of the parameters used are $\mu = 1, \mu_c = 1/2, \mu_e = 1/3, \mu_{\text{micro}} = 1/4, a_1 = 1/5, a_3 = 1/6, a_4 = 1/7$ and $R = 1$.

these terms in all models. Excluding the mixed terms such as $\langle \text{sym } D u, \text{sym } D u - P \rangle$ also simplifies considerably the investigation of positive-definiteness. Indeed, all presented models are positive definite if the usual relations

$$\mu_{\text{macro}} > 0, \quad \kappa_{\text{macro}} = \frac{2\mu_{\text{macro}} + 3\lambda_{\text{macro}}}{3} > 0,$$

$$\mu_{\text{micro}} > 0, \quad \kappa_{\text{micro}} = \frac{2\mu_{\text{micro}} + 3\lambda_{\text{micro}}}{3} > 0,$$

$$\mu_{\text{micro}} > \mu_{\text{macro}} \Rightarrow \mu_e > 0, \quad \kappa_e = \frac{2\mu_e + 3\lambda_{\text{micro}}}{3} > 0,$$

are satisfied together with individual positivity of all curvature parameters. In all cases, the displacement follows the classical pure torsion solution. Despite the conceptual simplicity of the models, we observe already a delicate interplay between the used kinematics and the assumed curvature energy expression. For example, let us compare the relaxed micromorphic model with the micro-strain model (Section 10). Both models have a similar looking lower-order energy term (if the Cosserat couple modulus $\mu_c \equiv 0$), but different kinematics and different curvature energy. For arbitrary slender specimens, the torsional stiffness of the micro-strain model is governed by $\mu_e$, whereas the torsional stiffness of the relaxed micromorphic model is determined by $\mu_{\text{micro}}$. Thus, the physical interpretation of the material parameters in both models is completely different. This is surprising at first glance but the reason for this response is finally explained in Appendix D.

In the end, the more restricted the used kinematics, the less viable a model may become. In this respect, only the full micromorphic kinematics degree of freedom (=12) can be advised. In addition, the curvature energy should not intervene too strongly. For example, penalizing a full gradient $D P$ in the curvature energy of the classical micromorphic model leads to a stiffness singularity for arbitrary slender specimens, whereas penalizing only $\text{ Curl } P$ in the relaxed micromorphic model does not show the same singular response. Moreover, in the relaxed micromorphic model the interpretation of the lower-order material parameters ($\mu_e, \mu_{\text{micro}}, \mu_{\text{micro}}, \text{ etc.}$) does not, in principle, change when different curvature energies are considered. In the end, it is therefore the relaxed micromorphic model that produces sensible and consistent response in all considered cases. It remains to be investigated if, together with the previously developed solution for bending and shear [1, 2], the present analytical solution allows the complete set of micromorphic parameters of a material to be identified from bending, shear and torsion experiments at specimens with different diameters.

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7. Setting

6. In Hadjesfandiari and Dargush [55] the discussion of higher traction boundary conditions seems to be missing some terms in (75),

2. For the torsion problem,

11. The values of the four terms for a generic second-order tensor

4. As we can show that the classical torsion displacement solution satisfies the external balance equation (74) as well as the higher-order traction boundary conditions (75).

5. Using Nye’s formula [59] \( \text{Curl} A = \text{tr} \left( (\text{D axl} A^T)^T - (\text{D axl} A)^T \right) \) for \( A \in se(3) \) we can rewrite \( \text{Curl} \text{ skew} Du = - (\text{D axl} (\text{skew} Du))^T = \frac{1}{2} \text{curl} u \), because \( \text{tr} (\text{Curl} \text{ skew} Du) = 0 \).

6. In Hadjesfandiari and Dargush [55] the discussion of higher traction boundary conditions seems to be missing some terms in (75), letting the authors erroneously conclude that the classical displacement pure torsion solution does not satisfy the higher-order boundary conditions.

7. Setting \( a_1 \mu_{\text{macro}} L_2^2 = 8 \eta \) we obtain the rigidity as \( T_w := \frac{d^2}{d \phi^2} W_{\text{tot}}(\phi) = \mu_{\text{macro}} \left( 1 + 24 \frac{\eta}{\mu_{\text{macro}}} \frac{1}{R^2} \right) I_P \). In (44) of Hadjesfandiari and Dargush [26] we have the relation \( \ell^2 = \frac{a}{\mu_{\text{macro}}} \), while in (55) we have the formula \( T_w = \mu_{\text{macro}} \left( 1 + 24 \left( \frac{\ell}{R} \right)^2 \right) I_P = \mu_{\text{macro}} \left( 1 + 24 \frac{a}{\mu_{\text{macro}}} \frac{1}{R^2} \right) I_P \).

8. That this uncoupling takes place at all seems to be connected to the chosen form of the curvature energy. It remains unclear at present whether this feature holds for the most general isotropic curvature expression as well.

9. Shaat [56] used the micro-strain model with mixed terms and a degenerate curvature expression in DS, omitting \( S_{11,1}, S_{22,2} \) and \( S_{33,3} \).

10. In index notation \( (1 - e_r \otimes e_r) \nabla m = (\delta_{ip} - n_i n_p) m_{ijk,p} \).

11. The values of the four terms for a generic second-order tensor \( m \) are \( \langle \text{skew}(m \times e_z) e_{\psi}, e_r \rangle = \frac{1}{2} (m_{11} + m_{22}), \langle \text{skew}(m \times e_z) e_r, e_{\psi} \rangle = - \frac{1}{2} (m_{11} + m_{22}), \langle (m \times e_z) e_{\psi}, e_r \rangle = m_{22} \sin^2 \varphi + m_{11} \cos^2 \varphi + (m_{12} + m_{21}) \sin \varphi \cos \varphi \) and \( \langle (m \times e_z) e_r, e_{\psi} \rangle = -m_{22} \cos^2 \varphi - m_{11} \sin^2 \varphi + (m_{12} + m_{21}) \sin \varphi \cos \varphi \). Note that \( \langle (m \times e_z) e_{\psi}, e_r \rangle \neq \langle (m \times e_z) e_r, e_{\psi} \rangle \neq \langle (m \times e_z) e_r, e_r \rangle \).

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Appendix A. Cylindrical coordinates

The relation between the derivatives for a vector field \( \mathbf{u}(x) \), \( \tilde{\mathbf{u}}(r) \) and a second-order tensor \( \mathbf{P}(x) \), \( \tilde{\mathbf{P}}(r) \) with respect to an orthogonal set of coordinates \( x = \{x_1, x_2, x_3\} \) and a cylindrical set of coordinates \( r = \{r, \varphi, z\} \) are as follows

\[
\nabla_x \mathbf{u}(x) = (\nabla_x \tilde{\mathbf{u}}(r)) \mathbf{Q}^{(1)}, \\
\mathbf{u}_{ij} = \tilde{\mathbf{u}}_{ij} \mathbf{Q}^{(1)}_{ij}, \\
\nabla_x \mathbf{P}(x) = (\nabla_x \tilde{\mathbf{P}}(r)) \mathbf{Q}^{(1)}, \\
\mathbf{P}_{ij,k} = \tilde{\mathbf{P}}_{ij,k} \mathbf{Q}^{(1)}_{ij,k},
\]

(122) (123)
Moreover, the symmetric micro-distortion tensor and its classical formulation (gradient of the micro-rotation vector \( \phi \))

\[
Q^{(1)} = (\nabla_r x(r))^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

and

\[
\nabla_r^2 u(x) = (\nabla_r^2 \hat{u}(r)) : Q^{(2)} + (\nabla_r \hat{u}(r)) : Q^{(3)}, \quad \nabla_r P(x) = (\nabla_r^2 \hat{P}(r)) : Q^{(2)} + (\nabla_r \hat{P}(r)) : Q^{(3)}.
\]

\[
\begin{pmatrix}
\cos^2 \varphi & \sin \varphi \cos \varphi & 0 \\
\sin \varphi \cos \varphi & \sin^2 \varphi & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
-\sin \varphi \cos \varphi & \cos(2\varphi) & 0 \\
\cos \varphi \sin \varphi & \sin \varphi \cos \varphi & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cos \varphi \\
0 & 0 & \sin \varphi \\
0 & 0 & 0
\end{pmatrix},
\]

\[
Q^{(3)} = \begin{pmatrix}
\frac{\sin^2 \varphi}{2r} & \frac{\sin \varphi \cos \varphi}{2r} & 0 \\
\frac{\sin \varphi \cos \varphi}{2r} & \frac{\cos^2 \varphi}{2r} & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

**Appendix B. Bessel functions**

The Bessel functions are the solutions \( y(x) \) of the Bessel differential equation [65]

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2)y = 0.
\]

For the particular case in which \( \alpha = n \) is an integer, the solution of (128) can be expressed as a linear combination of the Bessel function of the first \( J_n(x) \) and second \( Y_n(x) \) kind

\[
y(x) = A_1 J_n(x) + A_2 Y_n(x),
\]

whose definitions are

\[
J_n(x) = \int_0^\pi \cos(n \tau - x \sin(\tau)) \, d\tau, \quad Y_n(x) = \frac{J_n(x) \cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}.
\]

Moreover, the modified Bessel function of the first kind is defined as \( I_n(x) = i^{-n} J_n(ix) \).

**Appendix C. Classical Cosserat formulation in micro-rotation vector format**

An overview of the different classical notation for the Cosserat model has been given in [68]. In [50] we have presented the correspondence between the Cosserat model expressed in dislocation format (Curl of the skew-symmetric micro-distortion tensor) and in its classical formulation (gradient of the micro-rotation vector \( \phi \)).
The relation between the coefficients in the two sets of notation is

\[
\begin{align*}
\alpha &= \frac{\mu L_c^2}{2} \frac{1}{3} (4a_3 - a_1), \\
\beta &= \frac{\mu L_c^2}{2} \frac{a_1 - a_2}{2}, \\
\gamma &= \frac{\mu L_c^2}{2} a_1 + a_2, \\
a_1 &= \frac{\gamma + \beta}{\mu L_c^2}, \\
a_2 &= \frac{\gamma - \beta}{\mu L_c^2}, \\
a_3 &= \frac{3\alpha + \beta + \gamma}{4\mu L_c^2}.
\end{align*}
\]

(131)

Setting \( \phi := \text{axl}(A) \) and taking into account (131), the expression of the strain energy for the isotropic Cosserat continuum can be equivalently expressed as

\[
W(Du, A, \text{Curl} A) = \mu_{\text{macro}} \| \text{sym} Du \|^2 + \frac{\lambda_{\text{macro}}}{2} \text{tr}^2 (Du) + \mu_c \| \text{skew} Du - A \|^2
\]

\[
+ \frac{\mu L_c^2}{2} \left( a_1 \| \text{dev sym} \text{Curl} A \|^2 + a_2 \| \text{skew} \text{Curl} A \|^2 + \frac{a_3}{3} \text{tr}^2 (\text{Curl} A) \right)
\]

(132)

dislocation tensor format

\[
= W(Du, \phi, D\phi) = \mu_{\text{macro}} \| \text{sym} Du \|^2 + \frac{\lambda_{\text{macro}}}{2} \text{tr}^2 (Du) + \mu_c \| \text{curl} u - 2\phi \|^2
\]

\[
+ \frac{1}{2} \left( \alpha \text{tr}^2 (D\phi) + \beta (\text{D} \phi^T, \text{D} \phi) + \gamma \| \text{D} \phi \|^2 \right),
\]

classical micro-rotation vector format

because

\[
\| \text{skew} Du - A \|^2 = 2 \| \text{axl}(\text{skew} Du - \text{Anti}(\phi)) \|^2 = 2 \| \frac{1}{2} \text{curl} u - \phi \|^2 = \frac{1}{2} \| \text{curl} u - 2\phi \|^2.
\]

(133)

The equilibrium equations, in the absence of body forces, in the classical notation are

\[
\text{Div} \left[ 2\mu_{\text{macro}} \text{sym} Du + \lambda_{\text{macro}} \text{tr} (Du) \mathbb{1} \right] - \mu_c \text{curl} \left( \text{curl} u - 2\phi \right) = 0,
\]

(134)

\[
\text{Div} \left[ \alpha \text{tr} (D\phi) \mathbb{1} + \beta \left( \text{D} \phi^T, \text{D} \phi \right) + \gamma \| \text{D} \phi \|^2 \right] + 2\mu_c \left( \text{curl} u - 2\phi \right) = 0.
\]

The boundary conditions at the free surface are

\[
\vec{t}(r = R) = \vec{\sigma}(r) \cdot \vec{e}_r = 0_{\mathbb{R}^3}, \quad \vec{n}(r = R) = \vec{m}(r) \cdot \vec{e}_r = 0_{\mathbb{R}^{3 \times 3}},
\]

(135)

where

\[
\vec{\sigma} = 2\mu_{\text{macro}} \text{sym} Du + \lambda_{\text{macro}} \text{tr} (Du) \mathbb{1} + 2\mu_c \left( \text{skew} Du - \text{Anti}(\phi) \right),
\]

(136)

\( \vec{e}_r \) is the radial unit vector, and the second-order moment stress tensor

\[
\vec{m} = \alpha \text{tr} (D\phi) \mathbb{1} + \beta \left( \text{D} \phi^T, \text{D} \phi \right) + \gamma \text{D} \phi.
\]

(137)

According to the reference system shown in Figure 1, the ansatz for the displacement field and the micro-rotation vector turns into

\[
\begin{align*}
\vec{u}(x_1, x_2, x_3) &= \vec{u}(r, \varphi, z) = \vec{\varphi} \left( \begin{array}{ccc}
-x_2(r, \varphi) x_3(z) \\
x_1(r, \varphi) x_3(z) \\
0
\end{array} \right), \\
\vec{\phi}(x_1, x_2, x_3) &= \vec{\phi}(r, \varphi, z) = \vec{\varphi} \left( \begin{array}{ccc}
-g_r(r) x_3(r, \varphi) \\
g_r(r) x_1(r, \varphi) \\
2x_3(z)
\end{array} \right),
\end{align*}
\]

(138)

Substituting the ansatz (138) in (134) the equilibrium equations are equivalent to

\[
-\frac{1}{2} \vec{\varphi} \cdot \vec{\sigma} \cdot \vec{\varphi} (4\rho \mu_c (g(\rho) - 1) - (\alpha + \beta + \gamma) (3g'(\rho) + \rho g''(\rho))) = 0,
\]

(139)

\[
-\frac{1}{2} \vec{\varphi} \cdot \vec{\sigma} \cdot \vec{\varphi} (4\rho \mu_c (g(\rho) - 1) - (\alpha + \beta + \gamma) (3g'(\rho) + \rho g''(\rho))) = 0,
\]
which are completely equivalent to (62) in Section 6 once we have used the relations (131). As also the boundary conditions (135) are equivalent to the boundary condition (59) in Section 6, further calculations are avoided. Here, we recall the relations between the two moment stress tensors expressed in the classical format ($\mathbf{m}$) and in the dislocation format ($\mathbf{\bar{m}}$)

$$
\text{dev sym } \mathbf{m} = -\text{dev sym } \mathbf{\bar{m}}, \quad \text{skew } \mathbf{m} = \text{skew } \mathbf{\bar{m}}, \quad \text{tr } \mathbf{m} = \frac{1}{2} \text{tr } \mathbf{\bar{m}},
$$

(140)

where skew $\mathbf{m} = \text{skew } \mathbf{\bar{m}} = 0$ for the torsional problem, and

$$
\mathbf{m} = \mu L_c^2 \left( a_1 \text{dev sym } \text{Curl } \mathbf{A} + \frac{a_3}{3} \text{tr } (\text{Curl } \mathbf{A}) \mathbf{1} \right), \quad \mathbf{\bar{m}} = \alpha \text{tr } (\text{D}(\text{axl}(\mathbf{A}))) \mathbf{1} + \beta (\text{D}(\text{axl}(\mathbf{A})))^T + \gamma \text{D}(\text{axl}(\mathbf{A})).
$$

(141)

It is also interesting to show the relation between the two higher-order torques expressed in terms of $\mathbf{m}$ and $\mathbf{\bar{m}}$, respectively. First, we observe

$$
\langle \text{skew } (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_\psi, e_r \rangle - \langle \text{skew } (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_r, \mathbf{e}_\psi \rangle = \langle (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_z, e_r \rangle - \langle (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_r, \mathbf{e}_\psi \rangle,
$$

(142)

which does not hold component-wise.\textsuperscript{11}

Using that the cross product between two unit vectors gives the third, and

$$
(\mathbf{m} \times \mathbf{v}) = \mathbf{m} (\mathbf{v} \times \mathbf{w}) \quad \forall \mathbf{v}, \mathbf{w} \in \mathbb{R}^3 \quad \text{and} \quad \forall \mathbf{m} \in \mathbb{R}^{3 \times 3},
$$

(143)

it is possible to write

$$
\langle (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_\psi, e_r \rangle - \langle (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_r, \mathbf{e}_\psi \rangle = \langle \mathbf{m} (\mathbf{e}_\psi \times \mathbf{e}_z), e_r \rangle - \langle \mathbf{m} (\mathbf{e}_r \times \mathbf{e}_z), \mathbf{e}_\psi \rangle = -\left[ \langle \mathbf{m} \mathbf{e}_r, \mathbf{e}_r \rangle + \langle \mathbf{m} \mathbf{e}_\psi, \mathbf{e}_\psi \rangle \right].
$$

(144)

As $\langle \mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\psi \otimes \mathbf{e}_\psi + \mathbf{e}_z \otimes \mathbf{e}_z \rangle = \mathbf{1}$ we may convert the double dot-product into a dyadic product as follows

$$
-\left[ \langle \mathbf{m} \mathbf{e}_r, \mathbf{e}_r \rangle + \langle \mathbf{m} \mathbf{e}_\psi, \mathbf{e}_\psi \rangle \right] = -\langle \mathbf{m}, (\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\psi \otimes \mathbf{e}_\psi) \rangle = -\langle \mathbf{m}, (\mathbf{1} - \mathbf{e}_z \otimes \mathbf{e}_z) \rangle.
$$

(145)

Substituting the relation (140) between $\mathbf{m}$ and $\mathbf{\bar{m}}$ we have

$$
-\langle \mathbf{m}, (\mathbf{1} - \mathbf{e}_z \otimes \mathbf{e}_z) \rangle = -\langle \left( -\text{dev } \mathbf{\bar{m}} + \text{skew } \mathbf{\bar{m}} + \frac{1}{3} \frac{\text{tr } (\mathbf{\bar{m}})}{2} \mathbf{1} \right), (\mathbf{1} - \mathbf{e}_z \otimes \mathbf{e}_z) \rangle.
$$

(146)

As $\langle \mathbf{1}, \mathbf{1} \rangle = 3$, $\langle \mathbf{1}, (\mathbf{e}_z \otimes \mathbf{e}_z) \rangle = 1$ and $\mathbf{\bar{m}}$ is decomposed into its three orthogonal components (except for multiplying factors), we can write

$$
-\langle \left( -\text{dev } \mathbf{\bar{m}} + \text{skew } \mathbf{\bar{m}} + \frac{1}{3} \frac{\text{tr } (\mathbf{\bar{m}})}{2} \mathbf{1} \right), (\mathbf{1} - \mathbf{e}_z \otimes \mathbf{e}_z) \rangle = -\langle \left( -\text{dev } \mathbf{\bar{m}} + \frac{1}{3} \frac{\text{tr } (\mathbf{\bar{m}})}{2} \mathbf{1} \right), (\mathbf{1} - \mathbf{e}_z \otimes \mathbf{e}_z) \rangle =
$$

(147)

$$
\langle \text{dev } \mathbf{\bar{m}}, (\mathbf{1} - \mathbf{e}_z \otimes \mathbf{e}_z) \rangle - \frac{1}{3} \frac{\text{tr } (\mathbf{\bar{m}})}{2} [3 - 1] = -\langle \text{dev } \mathbf{\bar{m}}, \mathbf{e}_z \otimes \mathbf{e}_z \rangle - \frac{1}{3} \text{tr } (\mathbf{\bar{m}}) =
$$

(148)

$$
-\langle \text{dev } \mathbf{\bar{m}}, \mathbf{e}_z \otimes \mathbf{e}_z \rangle - \frac{1}{3} \text{tr } (\mathbf{\bar{m}}) \langle \mathbf{1}, \mathbf{e}_z \otimes \mathbf{e}_z \rangle = -\langle \text{dev } \mathbf{\bar{m}} + \frac{1}{3} \text{tr } (\mathbf{\bar{m}}) \mathbf{1}, \mathbf{e}_z \otimes \mathbf{e}_z \rangle = \langle \mathbf{\bar{m}}, \mathbf{e}_z \otimes \mathbf{e}_z \rangle = -\langle \mathbf{\bar{m}} \mathbf{e}_z, \mathbf{e}_z \rangle = -\mathbf{\bar{m}} \mathbf{e}_z.
$$

(149)

The last relation is a pure algebraic relation valid for all $\mathbf{m}, \mathbf{\bar{m}}$ related by (138) and $\mathbf{e}_r, \mathbf{e}_\psi, \mathbf{e}_z$ are given in (11). Thus, we have shown that

$$
\langle \text{skew } (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_\psi, e_r \rangle - \langle \text{skew } (\mathbf{m} \times \mathbf{e}_z) \mathbf{e}_r, \mathbf{e}_\psi \rangle = \mathbf{\bar{m}} \mathbf{e}_z.
$$

(150)
This solution is valid for a generic second-order tensor and for a generic vector triplet. Using (149), we finally see that

$$\int_{\Gamma} F_{zz} r \, dr \, d\varphi = \int_{\Gamma} -\left[\langle \text{skew} \left( m \times e_z \right) e_{\varphi}, e_r \rangle - \langle \text{skew} \left( m \times e_z \right) e_r, e_{\varphi} \rangle \right] r \, dr \, d\varphi. \quad (151)$$

The ratio $\Omega$ between the Cosserat torsional stiffness and the classical value that can be found, e.g., in [9, 69, 70] (see also Figure 27) is

$$\Omega = 1 + 6 \left( \frac{\ell_a}{R} \right)^2 \left[ \frac{1 - 4/3 \Psi \chi}{1 - \Psi \chi} \right], \quad \ell_a^2 = \frac{\beta + \gamma}{2 \mu_{\text{macro}}}, \quad \Psi = \frac{\beta + \gamma}{\alpha + \beta + \gamma}, \quad (152)$$

where $\ell_a$ is the characteristic length for torsion, $\Psi$ is the polar ratio, $N$ is the Cosserat coupling number, $\alpha$, $\beta$, and $\gamma$ are the curvature coefficients in the classical Cosserat formulation, $\mu_{\text{macro}}$ is the classical Cauchy shear modulus, $\mu_c$ is the Cosserat couple modulus and $I_n$ is the modified Bessel function of the first kind of order $n$.

To go from (65) to (152) we have to use the relations (131), while remembering to incorporate the term $\mu L_c^2$ (the terms not reported do not change between the two sets of notation)

$$\Omega = 1 + 6 \left( \frac{\ell_t}{R} \right)^2 \left[ \frac{1 - 4/3 \Psi \chi}{1 - \Psi \chi} \right], \quad \ell_t^2 = \frac{a_1}{2 \mu_{\text{macro}}}, \quad \Psi = \frac{3a_1}{2a_1 + 4a_3}, \quad p^2 = \frac{6\mu_c}{a_1 + 2a_3}. \quad (153)$$

In Figure 28 we report how the torsional stiffness divided by the radius of the cylindrical rod squared ($T_w/R^2$) varies with respect to the radius squared $R^2$ for the Cosserat model and the relaxed micromorphic model where

$$\ell_a = \mu L_c^2 \frac{12\pi a_1 a_3}{a_1 + 8a_3} \frac{\pi(\beta + \gamma)(3\alpha + 2\beta + \gamma)}{2\alpha + \beta + \gamma}, \quad \ell_b = \mu L_c^2 \frac{3}{2\pi a_1} \frac{3}{a_1 + 2a_3}. \quad (154)$$

It is highlighted that the Cosserat model does not tend to a classical linear elastic model for $\mu_c \to 0$ as can be seen from (153) or (68). It is underlined that for the relaxed micromorphic model the stiffness is bounded by that obtained for $L_c \to 0$ (macro) and $L_c \to \infty$ (micro): the macro-stiffness ($L_c \to 0$) is the limit to which all curves with finite $L_c$ tend asymptotically to for $R^2 \to \infty$ (this limit has been cut in order make it possible to distinguish among all the curves), whereas the micro-stiffness ($L_c \to \infty$) is the limit to which all the curves tend asymptotically to for $R^2 \to 0$ (see Figure 29).
Figure 28. (a) Cosserat model and (b) relaxed micromorphic model. The values of the coefficient used are $\mu = 1$, $\mu_c = 1/2$, $\mu_{\text{macro}} = 1/14$, $\mu_{\text{micro}} = 1/4$, $a_1 = 1/5$ and $a_3 = 1/37$.

Figure 29. (a) Cosserat model and (b) relaxed micromorphic model. The values of the coefficient used are $\mu = 1$, $\mu_{\text{macro}} = 1/14$, $\mu_{\text{micro}} = 1/4$, $a_1 = 2$, $a_3 = 1/50$ $L_\text{c} = 3$. In the Cosserat model, the solution for $\mu_c \to \infty$ (the indeterminate couple stress model) shows a jump.

Appendix D. Ad hoc minimization for $L_\text{c} \to \infty$ in the full micromorphic model and in the micro-strain model

Looking at the curvature energy of the full micromorphic model (or the micro-strain model) it is clear that for $L_\text{c} \to \infty$ the micro-distortion tensor field $\mathbf{P}$ must be constant $\mathbf{P} = \overline{\mathbf{P}}$, provided all curvature coefficients are strictly positive. We calculate this constant in the following. Thus, we consider

$$
\min_{u, \mathbf{P}} \left[ \int_\Omega \mu_c \left\| \text{dev sym} (\nabla u - \overline{\mathbf{P}}) \right\|^2 + \frac{\kappa_2}{2} \text{tr}^2 (\nabla u - \overline{\mathbf{P}}) + \mu_c \left\| \text{skew} (\nabla u - \overline{\mathbf{P}}) \right\|^2 + \mu_{\text{micro}} \left\| \text{dev sym} \mathbf{P} \right\|^2 + \frac{\kappa_{\text{micro}}}{2} \text{tr}^2 (\mathbf{P}) \right] \ dV.
$$

The weak form is given by

$$
\int_\Omega 2\mu_c \left\langle \text{dev sym} (\nabla u - \overline{\mathbf{P}}), -\delta \mathbf{P} \right\rangle + \kappa_c \text{tr} (\nabla u - \overline{\mathbf{P}}) \left\langle \mathbf{I}, -\delta \mathbf{P} \right\rangle + 2\mu_c \left\langle \text{skew} (\mathbf{P}), -\delta \mathbf{P} \right\rangle + 2\mu_{\text{micro}} \left\langle \text{dev sym} (\mathbf{P}), \delta \mathbf{P} \right\rangle + \kappa_{\text{micro}} \text{tr} (\mathbf{P}) \left\langle \mathbf{I}, \delta \mathbf{P} \right\rangle \ dV = 0 \quad \forall \delta \mathbf{P},
$$
\[ \int_{\Omega} \left[ 2\mu_e \text{dev sym } (Du - \mathbf{P}) + \kappa_e \text{tr} (Du - \mathbf{P}) \right] + 2\mu_c \text{skew } (Du - \mathbf{P}) - 2\mu_{\text{micro}} \text{dev sym } (\mathbf{P}) - \kappa_{\text{micro}} \text{tr } (\mathbf{P}) \cdot \mathbb{1} \] \quad \forall \delta \mathbf{P}. \tag{157}

For constant \(\delta \mathbf{P}\) this can be rewritten as
\[ \left\{ \int_{\Omega} 2\mu_e \text{dev sym } (Du - \mathbf{P}) + \kappa_e \text{tr} (Du - \mathbf{P}) \mathbb{1} + 2\mu_c \text{skew } (Du - \mathbf{P}) - 2\mu_{\text{micro}} \text{dev sym } (\mathbf{P}) - \kappa_{\text{micro}} \text{tr } (\mathbf{P}) \mathbb{1} \right\} dV = 0 \quad \forall \delta \mathbf{P}. \tag{158}

As \(\delta \mathbf{P}\) is arbitrary, this implies that
\[ \int_{\Omega} 2\mu_e \text{dev sym } (Du - \mathbf{P}) + \kappa_e \text{tr} (Du - \mathbf{P}) \mathbb{1} + 2\mu_c \text{skew } (Du - \mathbf{P}) - 2\mu_{\text{micro}} \text{dev sym } (\mathbf{P}) - \kappa_{\text{micro}} \text{tr } (\mathbf{P}) \mathbb{1} dV = 0 \tag{159} \]

or
\[ \int_{\Omega} 2\mu_e \text{dev sym } Du + \kappa_e \text{tr } (Du) \mathbb{1} + 2\mu_c \text{skew } Du dV = \quad \int_{\Omega} 2\mu_e \text{dev sym } \mathbf{P} + \kappa_e \text{tr } (\mathbf{P}) \mathbb{1} + 2\mu_c \text{skew } \mathbf{P} + 2\mu_{\text{micro}} \text{dev sym } \mathbf{P} + \kappa_{\text{micro}} \text{tr } (\mathbf{P}) \mathbb{1} dV. \tag{160} \]

Using the orthogonality of dev sym-, skew- and tr(·) \(\mathbb{1}\) we obtain
\[ \int_{\Omega} 2\mu_e \text{dev sym } Du dV = \int_{\Omega} 2\mu_e \text{dev sym } \mathbf{P} + 2\mu_{\text{micro}} \text{dev sym } \mathbf{P} dV, \tag{161} \]
\[ \int_{\Omega} \kappa_e \text{tr } (Du) dV = \int_{\Omega} \kappa_e \text{tr } (\mathbf{P}) + \kappa_{\text{micro}} \text{tr } (\mathbf{P}) dV, \quad \int_{\Omega} 2\mu_c \text{skew } Du dV = \int_{\Omega} 2\mu_c \text{skew } \mathbf{P} dV, \]
and because \(\mathbf{P}\) is constant we can write
\[ \text{dev sym } \mathbf{P} = \frac{1}{|\Omega|} \int_{\Omega} \frac{\mu_e}{\mu_e + \mu_{\text{micro}}} \text{dev sym } Du dV, \quad \text{tr } (\mathbf{P}) = \frac{1}{|\Omega|} \int_{\Omega} \frac{\kappa_e}{\kappa_e + \kappa_{\text{micro}}} \text{tr } (Du) dV, \quad \text{skew } \mathbf{P} = \frac{1}{|\Omega|} \int_{\Omega} \text{skew } Du dV. \tag{162} \]

As dev sym, skew and tr are linear operators, we obtain equivalently
\[ \text{dev sym } \mathbf{P} = \frac{\mu_e}{\mu_e + \mu_{\text{micro}}} \text{dev sym } \left( \frac{1}{|\Omega|} \int_{\Omega} Du dV \right), \quad \text{tr } (\mathbf{P}) = \frac{\kappa_e}{\kappa_e + \kappa_{\text{micro}}} \text{tr } \left( \frac{1}{|\Omega|} \int_{\Omega} Du dV \right), \quad \text{skew } \mathbf{P} = \text{skew } \left( \frac{1}{|\Omega|} \int_{\Omega} Du dV \right). \tag{163} \]

Substituting the ansatz (85) into (163) we obtain \(\mathbf{P} = 0\). Analogous calculations can be carried out for the micro-strain model for which skew \(\mathbf{P} = 0\) and \(\mu_e = 0\)
\[ \text{dev } \mathbf{S} = \frac{\mu_e}{\mu_e + \mu_{\text{micro}}} \text{dev sym } \left( \frac{1}{|\Omega|} \int_{\Omega} Du dV \right), \quad \text{tr } (\mathbf{S}) = \frac{\kappa_e}{\kappa_e + \kappa_{\text{micro}}} \text{tr } \left( \frac{1}{|\Omega|} \int_{\Omega} Du dV \right). \tag{164} \]

Substituting the ansatz (96) into (164) we obtain \(\mathbf{S} = 0\).
The integral on the circular cross-section $\Gamma$ of the gradient of the displacement is
\[
\int_\Gamma D\mathbf{u}(x) dV = \int_0^{2\pi} \int_0^R D\mathbf{u}(r, \varphi, z) r \, dr \, d\varphi = \left( \begin{array}{ccc} 0 & -\pi R^2 \varphi \, z & 0 \\ \pi R^2 \varphi \, z & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) = \pi R^2 \Theta(z) \left( \begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).
\]
(165)

From (165) it is possible to see that the symmetric part of the integral of $D\mathbf{u}$ on the circular cross-section is zero, whereas the skew-symmetric part is zero only if the domain is symmetric with respect to $z$.

For the Cosserat model, letting $L_c \to \infty$ still implies that $A(x) = \overline{A} = \text{const.}$ must be constant. The same calculations as before yield
\[
\overline{A} = \text{skew} \left( \frac{1}{|\Omega|} \int_\Omega D\mathbf{u} \, dV \right) = \frac{1}{|\Omega|} \int_\Omega D\mathbf{u} \, dV, \quad \text{because } D\mathbf{u} \in \mathfrak{so}(3).
\]
(166)

For $\Omega = [0, L] \times \Gamma$ we have
\[
\frac{1}{|\Omega|} \int_\Omega D\mathbf{u} \, dV = \frac{1}{L(\pi R^2)} \int_0^L \int_0^{2\pi} \int_0^R D\mathbf{u}(r, \varphi, z) r \, dr \, d\varphi \, dz = \frac{1}{L(\pi R^2)} \left( \varphi \, \pi R^2 z^2 |_0^L \right) = \frac{\varphi}{2} L = \frac{1}{2} \Theta(L).
\]
(167)

We remark that the same limit $L_c \to \infty$ in the relaxed micromorphic model yields a linear elastic response with stiffness $C_{\text{micro}}$ because $\text{Curl} \mathbf{P} = 0$ does not imply that $\mathbf{P} = \text{const.}$, but $\mathbf{P} = \nabla \zeta$ for some $\zeta : \Omega \in \mathbb{R}^3 \to \mathbb{R}^3$, see [24].