A COMBINATION OF SMALL-GAIN AND DENSITY
PROPAGATION INEQUALITIES FOR STABILITY ANALYSIS OF
NETWORKED SYSTEMS

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Abstract. In this paper, the problem of stability analysis of a large-scale interconnection of nonlinear systems for which the small-gain condition does not hold globally is considered. A combination of the small-gain and density propagation inequalities is employed to prove almost input-to-state stability of the network.

Key words. nonlinear systems, input-to-state stability, interconnected systems, large-scale systems, small-gain condition, dual to Lyapunov’s techniques

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1. Introduction. Nonlinear dynamical systems are known for having essentially more complex behavior than linear systems. This property makes the investigation of some properties such as stability and robustness challenging. Consequently, nonlinear systems also require mathematical tools that are different from the linear systems (e.g., [3, 14, 17, 19]). In many applications nonlinear systems appear in form of interconnections [13]. These are often of large scale, especially in modern applications like power networks, interacting agents, logistics networks, etc.

In this paper, a network composed by an arbitrary number of interconnected nonlinear systems is considered. For each subsystem, a gain relating the system input to its state can be defined. This class of systems is known as input-to-state stable (ISS) systems [20] (see below for a precise definition, and see also [23] for the concept of input-output gains). For interconnected ISS systems, tools for the stability analysis include the so called small-gain theorems [6, 11] and constructions of a suitable function whose derivative is analyzed along the solutions of the network [7, 12, 15].

Another approach, called dual to Lyapunov methods [16], analyzes the integration of a suitable function along the solutions of differential equations. This method ensures certain stability properties, provided that mild conditions on the derivatives of functions governing its dynamics are met. These conditions are called density propagation inequalities [1] (see below for a precise definition).

Although both methods provide conditions for global stability properties of solutions, it is not always possible to employ a corresponding condition globally [5]. In this paper, the problem under consideration is to decide if solutions of a network composed by an arbitrary number of ISS systems converge to a neighborhood of the origin whose size is proportional to the magnitude of the input, when the small-gain condition is not satisfied everywhere. To motivate the reader, consider the following example. Let the function $p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be defined by (see also Figure 5.1)

$$p(\xi) = c \left( 25\xi - \frac{205}{8} \xi^2 + \frac{47}{4} \xi^3 - \frac{5}{2} \xi^4 + \frac{1}{5} \xi^5 \right),$$

(1.1)

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where \( c = 0.256 \), and consider the system that evolves on the positive orthant given by

\[
\begin{align*}
\dot{V}_1 &= -p(V_1) + \sin \left( \frac{V_2}{1.6} \right)^2 + \sin \left( \frac{V_3}{1.6} \right)^2 + \frac{u_1}{4}, \\
\dot{V}_2 &= -p(V_2) + \sin \left( \frac{V_1}{1.6} \right)^2 + \sin \left( \frac{V_3}{1.6} \right)^2 + \frac{u_2}{4}, \\
\dot{V}_3 &= -p(V_3) + \sin \left( \frac{V_1}{1.6} \right)^2 + \sin \left( \frac{V_2}{1.6} \right)^2 + \frac{u_3}{4}.
\end{align*}
\]

The reader will see below that each subsystem of \((\Sigma(u))\) is ISS. Note that, when \( u_1 = u_2 = u_3 = 0 \), this system has two equilibrium points: the origin and \((2.5, 2.5, 2.5)\). This implies that the origin is not globally asymptotically stable and small-gain based methods cannot be applied to directly analyze the behavior of the solutions to this system.

In fact the reader will see below that, for the class of systems illustrated by this example, the conditions to employ the small-gain theorem are satisfied in a neighborhood of the origin. In addition to this, they are also satisfied for essentially large values of the system states. The region where the small-gain theorem does not hold is located between these two. In this region, the dual to Lyapunov’s method \([1, 16]\) is employed.

In general the situation can be more complex since several gaps where the latter approach needs to be applied may exist. As shown below, such a case is handled by covering the state space with several (or infinitely many) domains in a way that the small-gain or the dual to Lyapunov’s approach can be employed to each domain. In addition, tools to establish stability properties by combining both approaches are provided. Thus, the main contribution of this paper lies in a suitable combination of the small-gain and density-propagation conditions for this decomposition of the state space. A previous step in this direction was developed in \([22]\), where interconnections of two systems without external inputs were considered. The state space was split into three parts there, where different conditions were applied.

**Paper outline.** The remaining parts of the introduction presents the system under consideration, and recalls known concepts regarding graph theory. The required background on stability notions and known results are recalled in Section 2. Section 3 formulates the problem under consideration. The results are presented in Section 4. In particular, the main result is presented in Section 4.1. An illustration of the proposed approach is provided in Section 5. The proofs are provided in Section 6. A short conclusion and steps for further development of the research are presented in Section 7.

**Notation.** Let \( k \in \mathbb{N} \). For vectors \( x, y \in \mathbb{R}^k \), the notation

- \( x \leq y \) stands for \( x_i \leq y_i \), for every \( i = 1, \ldots, k \);
- \( x < y \) if and only if, for every \( i = 1, \ldots, k \), \( x_i < y_i \);
- \( x \not< y \) if and only if there exists \( i = 1, \ldots, k \) such that \( x_i > y_i \);
- \( x \in \{a, b\} \setminus \{a\} \subset \mathbb{R}^k \) if and only if \( a \leq x < b \) and \( x \neq a \);
- \( \max\{x, y\} \) stands for the vector \( z \) whose components are \( z_i = \max\{x_i, y_i\} \), for every \( i = 1, \ldots, k \).

The notation \( |x| \) stands for the *Euclidean norm* of \( x \in \mathbb{R}^k \). For a function \( u \in \mathcal{L}_\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^k) \), the notation \( |u|_\infty \) stands for \( \text{ess sup}\{|u_i(s)| : s \geq 0\} \). Let \( A \subset \mathbb{R}^k \) be a compact set, \(|x|_A\) denotes the point-set distance \( |x|_A := \inf_{y \in A} |y - x| \) from \( x \) to \( A \). An open (resp. closed) ball centered at the set \( A \in \mathbb{R}^k \) with radius \( r > 0 \) is defined...
as $B_{\leq r}(A) = \{ x \in \mathbb{R}^k : |x|_A < r \}$ (resp. $B_{< r}(A) = \{ x \in \mathbb{R}^k : |x|_A \leq r \}$). Let $S$ be a subset of $\mathbb{R}^k$ containing the origin, the notation $S_{\geq 0}$ stands for $S \cap \{0\}$. The closure of $S$ is denoted by $c1\{S\}$. The convex closure of a set $S$ is denoted by $co\{S\}$.

A continuous function $f : S \rightarrow \mathbb{R}$ defined in a subset $S$ of $\mathbb{R}^k$ that contains 0 is positive definite if, $\forall x \in S_{\geq 0}$, $f(x) > 0$ and $f(0) = 0$. The class of positive definite functions is denoted as $P$. It is proper if it is radially unbounded. By $C^\infty$ we denote the class of $s$-times continuously differentiable functions, by $\mathcal{L}_{P_{loc}}$ the class of locally Lipschitz continuous functions. The set $\mathcal{N}$ denotes the class of strictly increasing functions, the set $\mathcal{K} := C \cap P \cap \mathcal{N}$ denotes the class of continuous, positive definite and strictly increasing functions; it is denoted by $K_\infty$ if, in addition, the functions are bounded; by $\mathcal{L}$ we denote the class of functions which are continuous, decreasing, and converging to 0 as their argument tends to $+\infty$, by $\mathcal{K}\mathcal{L}$ it is denoted the class of functions $S \times S \rightarrow \mathbb{R}$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument. Let $c \in \mathbb{R}_{>0}$, the notation $\mathcal{L}_{oc}(f)$ stands for the subset of $S$ defined as $\{ x \in S : f(x) \circ c \}$, where $\circ$ is a comparison operator (i.e., $=, <, \geq$ etc). The support of the function $f$ is the closure of the set of points where $f$ is nonzero, i.e., supp $f := c1\{x \in S : f(x) \neq 0 \}$. By $\mathcal{L}_{loc}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^k)$ we denote the class of functions $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^k$ that are locally essentially bounded. Let $x, \bar{x} \in \mathbb{R}_{\geq 0}$, the notation $x \nearrow \bar{x}$ (resp. $x \searrow \bar{x}$) stands for $x \rightarrow \bar{x}$ with $x < \bar{x}$ (resp. $x > \bar{x}$).

Consider a set of $n$ interconnected systems. Each system is indexed by an integer $i = 1, \ldots, n$ and is described by the ordinary differential equation

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u_i(t)), \quad (1.2)$$

where, for every positive value of the time $t$, the vector $x_i(t) \in \mathbb{R}^{N_i}$ denotes the system state, the vector $x_j(t) \in \mathbb{R}^{N_j}$ indexed by $j = 1, \ldots, n$ with $j \neq i$ denotes the interconnecting input variable and the vector $u_i(t) \in \mathbb{R}^{M_i}$ denotes the external input variable for some integers, $N_i > 0$, $N_j > 0$ and $M_i > 0$. Moreover, each corresponding function $f_i$ is assumed to have continuous derivatives, i.e., $f_i \in C^1(\mathbb{R}^{N_i+M_i}, \mathbb{R}^{N_i})$ and to satisfy $f_i(0,0) = 0$, where $N = \sum_{i=1}^n N_i$. From now on, the dependence on $t$ will be omitted.

Inputs to system (1.2) are functions $x_j \in \mathcal{L}_{loc}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^j)$ with $j \neq i$ and $u_i \in \mathcal{L}_{loc}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{M_i})$. A solution to (1.2) with initial condition $x_i$, fixed inputs $x_j$ and $u_i$ to system (1.2), and computed at time $t$ is denoted by $X_i(t,x,u_i)$, where $x := (x_1, \ldots, x_n) \in \mathbb{R}^N$.

Define the vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^M$, where $M = \sum_{i=1}^n M_i$. The resulting interconnected system, called network, is given by

$$\dot{x} = f(x,u), \quad f \in C^1(\mathbb{R}^{N+M}, \mathbb{R}^N). \quad (1.3)$$

2. Stability Notions and Known Results. The following definition provides a unified framework to deal with systems with multiple inputs.

Definition 2.1 ([7]). A function $\eta \in (C \cap P)(\mathbb{R}_{\geq 0}^p, \mathbb{R}_{\geq 0})$ is said to be a monotone aggregation function (MAF) if, for every vectors $x, y \in \mathbb{R}^p$, the function $\eta$ is $M_1$. Strict increasing: $x < y \Rightarrow \eta(x) < \eta(y)$;

$M_2$. Radially unbounded: $|x| \rightarrow \infty \Rightarrow \eta(x) \rightarrow \infty$.

The space of monotone aggregation functions is denoted by MAF$_p$ and $\eta \in$ MAF$_p^q$ denotes a vector of monotone aggregation functions, i.e., for every $j = 1, \ldots, q$, $\eta_j \in$ MAF$_p$.

The concept of stability that is used throughout this work is recalled in the next definition.

Definition 2.2 ([4, 21]). System (1.2) is said to be input-to-state stable if there exist a monotone aggregation function $\eta_i \in$ MAF$_{n+1}$, ISS gains $\bar{\gamma}_{ij}, \bar{\gamma}_{iu} \in K_{\infty}$, and a
function $\beta_i \in \mathcal{KL}$ such that, for every initial condition $x_i \in \mathbb{R}^N$, and for every inputs $x_j$ and $u_i$ to system (1.2), the solution $X_i(\cdot,x,u_i)$ of (1.2) satisfies the inequality

$$|X_i(t,x,u_i)| \leq \beta_i(|x_i|,t) + \eta_i \left( \bar{\gamma}_{i1}(|x_1|_{0,t}), \ldots, \bar{\gamma}_{in}(|x_n|_{0,t}), \tilde{\gamma}_{iu}(|u_i|_{\infty}) \right),$$

for every $t \geq 0$.

In other words, if the system (1.2) is ISS, then its solutions converge to a closed ball centred at the origin with radius given by a combination of the norms of $x_j$, where $j \neq i$, and $u_i$. This concept is also characterized by the existence of the particular function recalled below.

**Definition 2.3.** A function $V_i \in \mathcal{L}_{w,loc}^p(\mathbb{R}^N, \mathbb{R}_{\geq 0})$ is said to be a candidate-Lyapunov function for (1.2) if there exist functions $\underline{\omega}_i, \overline{\omega}_i \in \mathcal{K}_\infty$ such that the following inequalities

$$\underline{\omega}_i(|x_i|) \leq V_i(x_i) \leq \overline{\omega}_i(|x_i|)$$

hold, for every $x_i \in \mathbb{R}^N$.

For such functions the Clarke’s generalized gradient is given by [4, Theorem 8.1]

$$\partial V_i(x_i) = \text{co} \left\{ \xi_i \in \mathbb{R}^N : \exists x_{ik} \rightarrow x_i, \frac{\partial V_i}{\partial x_i}(x_{ik}) \text{ exists and } \frac{\partial V_i}{\partial x_i}(x_{ik}) \rightarrow \xi_i \right\}.$$

In other words, the Clarke’s generalized gradient is the convex closure of the set of vectors $\xi_i \in \mathbb{R}^N$, for which the derivative of $V_i$ computed at the elements of the convergent sequences exists and converges towards $\xi_i$.

**Definition 2.4 (Based on [7] and [21]).** For each index $i = 1, \ldots, n$ let $V_i$ be a candidate-Lyapunov function for (1.2). The adjective “candidate” is replaced by “ISS” if there exist a monotone aggregation function $\eta_i \in \mathcal{MAF}_{n+1}$, internal ISS-Lyapunov gains $\gamma_{ij} \in \mathcal{K}_\infty$, an external ISS-Lyapunov gain $\gamma_{iu} \in \mathcal{K}_\infty$ and a function $\alpha_i \in \mathcal{K}_\infty$ such that the condition

$$V_i(x_i) \geq \eta_i \left( \gamma_{i1}(V_1(x_1)), \ldots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(u_i) \right)$$

implies that the inequality

$$\xi_i \cdot f_i(x, u_i) \leq -\alpha_i(|x_i|)$$

holds, for every $x \in \mathbb{R}^N$, for every $u_i \in \mathbb{R}^M$, and for every $\xi_i \in \partial V_i(x_i)$.

**Assumption 2.5.** For every $i = 1, \ldots, n$, there exists an ISS-Lyapunov function for the corresponding system (1.2).

Assumption 2.5 implies that each subsystem (1.2) is ISS [7, 21]. This allows the construction of the matrix of internal gains (2.2)

$$\Gamma_{ij} = \gamma_{ij}, \quad \gamma_{ii} \equiv 0, \quad i, j = 1, \ldots, n$$

that collects the ISS-Lyapunov gains and describes the network structure (the network graph).

Consider the map

$$\Gamma_n : \mathbb{R}^n_{\geq 0} \rightarrow \mathbb{R}^n_{\geq 0}

$$

$$\begin{bmatrix}
\eta_1(\gamma_{i1}(s_1), \ldots, \gamma_{in}(s_n), \gamma_{iu}(r)) \\
\vdots \\
\eta_n(\gamma_{n1}(s_1), \ldots, \gamma_{nn}(s_n), \gamma_{nu}(r))
\end{bmatrix}$$

(2.3)
and define
\[ \Gamma_\eta(s) = \Gamma_\eta(s, 0). \] (2.4)

Note that, when \( \Gamma \) is irreducible and due to the fact that \( \Gamma_\eta \in K_n^\infty \), for every vectors \( s_1, s_2 \in \mathbb{R}_0^n \) satisfying \( s_1 \prec s_2 \), the inequality \( \Gamma_\eta(s_1) \prec \Gamma_\eta(s_2) \) holds.

Following [7], the monotone operator \( \Gamma_\eta \) is said to satisfy the global small-gain condition if
\[ \Gamma_\eta(s) \not\geq s \quad \forall s \geq 0, s \neq 0. \] (GSGC)

In other words, for every vector \( s \in \mathbb{R}_0^n \), there exists an index \( j \) such that the inequality \( \eta_j(\gamma_1(s_1), \ldots, \gamma_n(s_n)) < s_j \) holds.

Condition (GSGC) implies the existence of a curve or path such that the monotone operator \( \Gamma_\eta \) is strictly smaller than the identity map, when it is computed along this path. Another important characteristic of this path is that a Lyapunov function can be constructed from such a path called \( \Omega \)-path.

Sufficient conditions for the existence of \( \Omega \)-paths are given in [7, Theorem 5.2]. Also, the problem of finding \( \Omega \)-paths locally has been considered in [8, 5]. The existence of these paths is the main ingredient used in [7, Theorem 5.2] to prove the stability of a network employing the small-gain theorem.

3. Problem Formulation and Main Assumption. Consider the network (1.3) composed of the ISS subsystems described in Equation (1.2). When the small-gain condition (GSGC) holds only in regions of \( \mathbb{R}_0^n \) that correspond to regions in the state space described by the ISS-Lyapunov functions \( V_1, \ldots, V_N \) (see Figure 3.1 below for details), the result ([7, Theorem 5.2]) allowing one to deduce the stability of (1.3) cannot be employed.

The main purpose of this paper is to deal with such a situation. To this end we divide the state space of the network (1.3) into a sequence of closed sets \( A_1 \subset \cdots \subset A_L \) with \( L \in \mathbb{N} \cup \{ \infty \} \) (see Figure 3.1) such that the small-gain theorem can be employed to show that solutions to system (1.3) remain close and eventually converge to a neighborhood of each of the sets \( A_j \), index by \( j = 1, \ldots, L \). Inside the sets \( A_j \), where \( j = 1, \ldots, L \), the dual to Lyapunov’s techniques [1, 16] are employed to show convergence of solutions.

The proposed approach extends the results of [7, 5] and [22]. The first assumption concerns the existence of an appropriate parametrization of the positive orthant along which the monotone operator \( \Gamma_\eta \) is smaller than the identity function.

**Assumption 3.1.** For each index \( i = 1, \ldots, n \), there exists a constant value \( \overline{M}_i \) and an element \( \underline{M}_i \in \mathbb{R}_0^\infty \cup \{ \infty \} \) satisfying the inequalities
\[ 0 \leq \max_{i=1,\ldots,n} M_i = m < \underline{m} = \min_{i=1,\ldots,n} \overline{M}_i \leq \infty. \]

Let the vectors \( \underline{M} = (\underline{M}_1, \ldots, \underline{M}_n) \) and \( \overline{M} = (\overline{M}_1, \ldots, \overline{M}_n) \) of \( \mathbb{R}_0^n \). There exist values \( a, b \in \mathbb{R}_0^\infty \) such that \( a < b \) and a function \( \sigma \in (\mathcal{N} \cap \mathcal{C})([a, b], [\underline{M}, \overline{M}]) \), called as a \( \Gamma \)-path with respect to the monotone operator \( \Gamma_\eta \), satisfying
i. \( \sigma(a) = \underline{M} \) and \( \sigma(b) = \overline{M} \);
ii. The inequality
\[ \Gamma_\eta(\sigma(r)) \prec \sigma(r) \quad \forall r \in (a, b); \] (3.1)
iii. For every index \( i = 1, \ldots, n \), \( \sigma_i^{-1} \in \mathcal{L}_{\text{loc}}((\underline{M}_i, \overline{M}_i), (a, b)); \)
iv. For every compact set $K \subset (m, M)$ there exist constant values $0 < c < C$ such that, for every index $i = 1, \ldots, n$, the inequality

$$m < c \leq (\sigma_i^{-1})'(r) \leq C < m$$

holds at every point $r \in K$ where the function $\sigma_i^{-1}$ is differentiable.

In other words, Assumption 3.1 states that, for every value $r \in (a, b)$, and for every index $i = 1, \ldots, n$, the inequality $\Gamma_{\eta,i}(\sigma(r)) < \sigma_i(r)$ holds. Figure 3.2 illustrates the region $[M, \overline{M}]$ for $n = 3$.

Note that, when $M = 0$ and $\overline{M} = \infty$, the D-path is an $\Omega$-path. Condition (i) states that, along the path parametrized by $\sigma$, $\Gamma_{\eta}$ is strictly smaller than its argument. Conditions (ii) and (iii) are employed to show that, if given inequalities are satisfied, an ISS-Lyapunov function can be constructed for system (1.3).

4. Results. The existence of a D-path $\sigma$ with respect to the monotone operator $\Gamma_{\eta}$ is the main ingredient for the construction of an ISS-Lyapunov function for system (1.3). This is formalized in the main result of this paper.

**Proposition 4.1.** Under Assumptions 2.7 and 3.1, consider the D-path $\sigma : [a, b] \to [M, \overline{M}]$. There exists a function $\varphi \in (C \cap N)((a, b), \mathbb{R}_{\geq 0})$ such that the inequality

$$\Gamma_{\eta}(\sigma(r), \varphi(r)) < \sigma(r)$$

holds, for every $r \in (a, b)$, given the function

$$V : \mathbb{R}^N \to \mathbb{R}_{\geq 0}, \quad x \mapsto \max_{i=1, \ldots, n} \sigma_i^{-1}(V_i(x_i)),$$

there exist a constant value $\delta_u > 0$ and a function $\alpha \in K_{\infty}$ such that the condition

$$V(x) \geq \max_{i=1, \ldots, n} \{\varphi^{-1}(\gamma_{\varphi}(|u_i|))\}$$

holds.
Figure 3.2. The region $[M, \overline{M}]$ of Assumption 3.1, in the case $n = 3$. All of the points $p$ in the rectangular set delimited by the solid and dashed lines satisfy $M \preceq p \preceq \overline{M}$.

implies that the inequality

$$\xi \cdot f(x, u) \leq -\alpha(|x|_A)$$

holds, for every $(x, u) \in (B \setminus A) \times B_{\leq \delta_u}$, and for every $\xi \in \partial V(x)$, where the sets $A$ and $B$ are given by

$$A = \bigcap_{i=1}^{n} \{ x \in \mathbb{R}^N : V_i(x_i) \leq \underline{g} \}, \quad (4.3a)$$

where $\underline{g} = \max_{i=1, \ldots, n} \{ m_i, \Gamma_{\eta,i}(m, \ldots, m) \}$ and

$$B = \bigcap_{i=1}^{n} \{ x \in \mathbb{R}^N : V_i(x_i) < m \}. \quad (4.3b)$$

In other words, Proposition 4.1 states that solutions to system (1.3) that start in the set $B \setminus A$ converge to a closed ball centered at the set $A$ with radius proportional to the norm of the input. This proof is based on the proof of [7, Theorem 5.2] and it is provided in Section 6.1.

Two particular cases follow from Proposition 4.1.

Corollary 4.2 (Local stability). Under Assumptions 2.5 and 3.1 with vectors $\underline{M} = 0$ and $\overline{M} = M_0 < \infty$, consider a D-path $\sigma : (0, b_0) \rightarrow [0, M_0]$ with respect to $\Gamma_{\eta_0}$. There exists a function $\varphi_0 \in (C \cap N)((0, b_0), \mathbb{R}_{\geq 0})$ satisfying inequality (4.1), the set of inequalities (4.2) holds for system (1.3) with the function $V$ and sets $A$ and $B$.
given and denoted, respectively, as

\[ V_0(x) = \max_{i=1,\ldots,n} \sigma_{0,i}^{-1}(V_i(x_i)), \quad A_0 = \{0\}, \]

\[ B_0 = \bigcap_{i=1}^n \{x \in \mathbb{R}^N : V_i(x_i) < m_0\}, \quad (4.4) \]

where \( m_0 = \min_{i=1,\ldots,n} M_{0,i} \).

**Corollary 4.3 (Global attractivity).** Under Assumptions 2.5 and 3.1 with \( M = M_\infty > 0 \) and \( M = \infty \), consider a D-path \( \sigma_\infty : (a_\infty, \infty) \to (M_\infty, \infty) \) with respect to \( \Gamma_{\sigma_\infty} \) and the set

\[ A_\infty = \bigcap_{i=1}^n \{x \in \mathbb{R}^N : V_i(x_i) \leq g_\infty\}, \quad (4.5) \]

where \( g_\infty = \max_{i=1,\ldots,n} \{m_\infty, \sigma_{\infty,i}(m_\infty, \ldots, m_\infty)\} \) and \( m_\infty = \max_{i=1,\ldots,n} M_{\infty,i} \). There exists a function \( \varphi_\infty \in (\mathcal{C} \cap N)(b_\infty, 0, \mathbb{R}_\geq 0) \) satisfying inequality (4.1), the set of inequalities (4.2) holds for system (1.3) with the function \( V \) and set \( B \) given and denoted, respectively, as

\[ V_\infty(x) = \max_{i=1,\ldots,n} \sigma_{\infty,i}^{-1}(V_i(x_i)), \quad B_\infty = \{\mathbb{R}^N \setminus A_\infty\}. \]

The proof of these corollaries can be derived from Proposition 4.1 using the techniques from [5, 7].

ISS of system (1.3) follows trivially if the constants \( M_0 \) and \( M_\infty \) satisfy the inequality \( g_\infty < m_0 \). When this is not the case, solutions to (1.3) starting in \( B_\infty \) may converge to a set contained in \( A_\infty \setminus B_0 \) (see [10, Birkhoff’s Theorem]). This set may lie outside a closed ball centered in the origin and with radius proportional to the norm of the input. Motivated by this discussion, the next result presents a criterion to check the behavior of solution inside this set.

Before stating it, a concept of stability based on the one introduced in [11] is given.

**Definition 4.4.** System (1.3) is said to be almost input-to-state practically stable (aISpS) if it is locally uniformly asymptotically stable, and there exists a function \( \tilde{\gamma} \in (\mathcal{C} \cap N)(\mathbb{R}_\geq 0, \mathbb{R}_\geq 0) \) satisfying \( \tilde{\gamma}(0) = 0 \) (possibly discontinuous at zero) such that, for every input \( u \) for system (1.3), the inequality

\[ \limsup_{t \to \infty} |X(t, x, u)| \leq \tilde{\gamma}(|u|_\infty) \quad \text{(AG)} \]

holds, for almost every \( x \in \mathbb{R}^N \). If \( \tilde{\gamma} \) can be chosen as a function of class \( \mathcal{K}_\infty \) then system (1.3) is said to be almost input-to-state stable (aISS). In particular, when \( |u|_\infty = 0 \), system (1.3) is said to be almost globally asymptotically stable (aGAS).

The reader may note that difference between the aISpS and aISS concepts consists of the discontinuity of \( \gamma \) at zero. The next assumption concerns the behaviour of the derivative of the vector field \( f \) in the regions where the D path is not defined.

**Assumption 4.5.** Let \( D \subset \mathbb{R}^N \) be an open subset. There exist

- A function \( \rho \in \mathcal{C}^1(D, \mathbb{R}_\geq 0) \);
- A function \( Q \in \mathcal{C}(D, \mathbb{R}_\geq 0) \) satisfying the inequality \( Q(x) > 0 \), for almost every \( x \in D \);
A function $\gamma \in \mathcal{K}_{\infty}$ (resp. $\gamma \in (\mathbb{C} \cap \mathcal{N})(\mathbb{R}_{>0}, \mathbb{R}_{\geq 0})$ with $\gamma(0) = 0$) such that the condition

$$\max_{i=1,\ldots,n} \{ V_i(x) \} \geq \gamma(|u|)$$

(4.6a)

implies that the inequality

$$\text{div}(\rho f)(x, u) := \sum_{i=1}^{N} \frac{\partial (\rho f)}{\partial x_i}(x, u) \geq Q(x)$$

(4.6b)

holds, for every $x \in D$, and for every $u \in \mathbb{R}^{M}$.

At this point, the second result of this paper can be stated.

**Theorem 4.6.** Under Assumptions 2.5, 3.1 and 4.5 consider the constants $M_0$ and $M_\infty$ of Corollaries 4.2 and 4.3, respectively and define the set $S = \text{cl}\{ A_\infty \setminus B_0 \}$.

If, there exists an index $i = 1, \ldots, n$ such that the inequality

$$\max \{ M_\infty, i, \Gamma_{\eta, i}, (M_\infty, i-1, M_\infty, i+1, \ldots, M_\infty, n) \} > M_{0, i}$$

(4.7)

holds and $S \subsetneq \text{cl}\{ D \}$, then system (1.3) is aISS (resp. aISpS).

In other words, under Assumptions 2.5, 3.1 and 4.5 for almost every initial condition issuing solutions to (1.3) converge to a ball centered at the origin with radius proportional to the norm of the input. Moreover, the trajectories that do not converge to this ball have Lebesgue measure zero. This is the result needed to solve the problem under consideration in this paper. Its proof is provided in Section 6.2.

**Corollary 4.7 (Based on [22]).** Let $u = 0$ and $n = 2$. Under Assumptions 2.5, 3.1 and 4.5 consider the constant values $M_0$ and $M_\infty$ of Corollaries 4.2 and 4.3, respectively. Define the set $S = \text{cl}\{ A_\infty \setminus B_0 \}$ and let the open subset $D$ of $\mathbb{R}^N$ contain the set $S$. If inequality (4.7) holds and there exists a function $\rho \in C^1(D, \mathbb{R}_{>0})$ satisfying $\text{supp} \rho \supset D$ and the inequality

$$\text{div}(\rho f)(x) := \sum_{i=1}^{N} \frac{\partial (\rho f)}{\partial x_i}(x) > 0$$

(4.8)

holds, for almost every $x \in D$, then system (1.3) is aGAS.

The proof of Corollary 4.7 can be sketched as follows. Since the input $u$ is fixed to zero, from Corollary 1.2 (resp. 1.3) the origin (resp. the set $A_\infty$) is locally (resp. globally) asymptotically stable (resp. attractive). From the fact that inequality (4.7) holds and the existence of a function $\rho$ satisfying the inequality (4.8), for almost every initial condition, solutions of (1.3) will remain close and converge to the origin. Thus, system (1.3) is aGAS.

Note that in contrast to Theorem 4.6 in Corollary 4.7 there is no need to assume the existence of the continuous function $Q$ satisfying the inequality $Q(x) > 0$ for almost every $x \in D$. If (4.8) holds, then $Q$ can be obtained by letting $Q(\cdot) = \text{div}(\rho f)(\cdot)/2$. However one cannot do the same in Theorem 4.6 since inequality (4.6) should hold uniformly with respect to the input $u$.

**4.1. Main Result.** Theorem 4.6 is generalized for the case of multiple regions that does not satisfy small-gain condition as follows. This is the main result of this paper. Consider an element $L \in \mathbb{N} \cup \{\infty\}$ and the collection of $L$ vectors $\overline{M}^j \in \mathbb{R}_{\geq 0}^N$ and elements $\overline{M}^j \in \mathbb{R}_{\geq 0}^N \cup \{\infty\}$ indexed by $j = 1, \ldots, L$ and satisfying the inequalities

$$0 \leq \overline{M}^j < \overline{M}^j \leq \infty \quad \text{and} \quad \overline{M}^{j+1} \neq \overline{M}^j.$$
For every index $j = 1, \ldots, L$, define the values

$$m_j = \max_{i=1, \ldots, n} M_i^j, \quad \overline{m}_j = \min_{i=1, \ldots, n} M_i^j$$
and $g_j = \max_{i=1, \ldots, n} \{m_j, \Gamma_{\eta,i}(m_j, \ldots, m_j)\}$

and the sets

$$A_j = \bigcap_{i=1}^n \{x \in \mathbb{R}^N : V_i(x_i) \leq g_j\}$$
$$B_j = \bigcap_{i=1}^n \{x \in \mathbb{R}^N : V_i(x_i) < m_j\}$$.

$$S_j = \begin{cases} 
\text{cl}\{A_1\}, & \text{if } g_1 > 0, \\
\emptyset, & \text{if } g_1 = 0,
\end{cases} \quad \text{if } j = 0,$$
$$S_j = \begin{cases} 
\text{cl}\{A_j+1 \setminus B_j\}, & \text{if } j = 1, \ldots, L-1, \\
\text{cl}\{\mathbb{R}^N \setminus B_L\}, & \text{if } m_L < \infty, \\
\emptyset, & \text{if } m_L = \infty,
\end{cases} \quad \text{if } j = L,$$

**Theorem 4.8.** Under Assumption 2.5, for every index $j = 1, \ldots, L$,

- Let Assumption 3.1 holds with
  - Vectors $M_i^j$ and $\overline{M}_i^j$;
  - Intervals $[a_j, b_j] \subset \mathbb{R}$ with $b_j < a_{j+1}$;
  - $D$ paths $\sigma_j : [a_j, b_j] \to [M_j, \overline{M}_j]$;
- Let also Assumption 4.5 holds with
  - Functions $\rho_j$ and sets $D_j$.

If, for every index $j = 0, \ldots, L$, the inclusion $S_j \subset \text{cl}\{D_j\}$ holds, then system (1.3) is aISS (resp. aISpS). Moreover, if for some index $j = 1, \ldots, L$, Assumption 3.1 (resp. 4.5) fails then, for every input $u$ for system (1.3), and for almost every initial condition, solutions to system (1.3) will converge to a neighborhood of the set $B_j$ (resp. $A_{j+1}$).

Theorem 4.8 states that, if there exists a countable union of regions where the Assumption 3.1 holds, and another countable union of regions where Assumption 4.5 holds, and if the overall union covers the state space, then system (1.3) is aISS (resp. aISpS). The proof of Theorem 4.8 is provided at section 6.3.

Note that Theorem 4.6 is a corollary of Theorem 4.8.

**5. Illustration.** Despite an intuitive clarity of the theoretical result a composition of a suitable illustrative example was a very challenging and nontrivial task. The main difficulty is that the domains where the small-gain and the “almost positiveness” of divergence conditions hold should cover the whole phase space of the network (1.3) without any gaps. For this purpose, recall the function

$$p : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad \xi \mapsto c \left(25\xi - \frac{205}{8}\xi^2 + \frac{47}{4}\xi^3 - \frac{5}{2}\xi^4 + \frac{1}{5}\xi^5\right),$$

where $c = 2/p(2.5) = 0.2560$. Figure 5.1 shows (1.1) evaluated in the interval $[0, 5]$. 
Let $n = 3$ and recall the system evolving in the positive orthant described by

$$
\begin{align*}
\dot{V}_1 &= -p(V_1) + \sin\left(\frac{V_2}{1.6}\right)^2 + \sin\left(\frac{V_3}{1.6}\right)^2 + \frac{u_1}{4}, \\
\dot{V}_2 &= -p(V_2) + \sin\left(\frac{V_1}{1.6}\right)^2 + \sin\left(\frac{V_3}{1.6}\right)^2 + \frac{u_2}{4}, \\
\dot{V}_3 &= -p(V_3) + \sin\left(\frac{V_1}{1.6}\right)^2 + \sin\left(\frac{V_2}{1.6}\right)^2 + \frac{u_3}{4}.
\end{align*}
$$

There is no loss of generality in considering this kind of system, since it may also represent the derivative of candidate Lyapunov functions computed along the directions of interconnected higher dimensional systems (see also [2]). Using vector notation, system $(\Sigma(u))$ is denoted by $\dot{V} = f(V,u)$. The notation $(\Sigma(0))$ stands for $\dot{V} = f(V,0)$ with $u \equiv 0$.

Note that system $(\Sigma(0))$ has two equilibrium points: the origin and the point $V = (2.5, 2.5, 2.5)$. Thus, the origin is not globally asymptotically stable for $(\Sigma(0))$. Hence, small-gain based methods cannot be applied in this case. It remains to check whether the system $(\Sigma(u))$ is aISS/aISpS.

Consider the first order approximation of $(\Sigma(0))$ at $V$, i.e., the system $\dot{V} = AV$, where $A = \frac{\partial f}{\partial V}(V,0)$. Note that the matrix $A$ has its three eigenvalues equal to zero. This implies that $V$ is locally marginally stable [9, pp. 73].

Subsystems are ISS. To show that each subsystem of $(\Sigma(u))$ is ISS, the ISS-Lyapunov gains of each subsystem are computed. As the reader will see below, there exist no ISS-Lyapunov gains such that the small-gain condition can be satisfied. To show this claim, an analysis of the polynomial $p$ is provided. Note that $p$ has a root at zero and three critical points: 1, 2.5 (a saddle point), and 4. Thus, it cannot be inverted. Define, for every $r \geq 0$, the discontinuous and strictly increasing unbounded
function
\[ I_p(\tau) = \begin{cases} \min(\text{roots}(p(\xi) - r = 0)), & \text{if } r \in [0,1.75) \\ \max(\text{roots}(p(\xi) - r = 0)), & \text{otherwise,} \end{cases} \tag{5.1} \]

where \( \text{roots}(p(\xi) - r = 0) \) denotes the real roots of the equation \( p(\xi) - r = 0 \). Note that, in the intervals where \( p \) is strictly monotone, the definition of \( I_p \) corresponds to the inverse function of \( p \).

Define, for every \( r_1, r_2, r_3 \in \mathbb{R}_{\geq 0} \), the functions
\[
\kappa(r_1, r_2, r_3) := r_1 + r_2 + r_3
\]
\[
\eta(r_1, r_2, r_3) := \frac{1}{1 - \varepsilon} I_p(\kappa(\gamma_{ij}(r_1), \gamma_{ik}(r_2), \gamma_{iu}(r_3)))
\]
\[
\gamma_{ij}(r_1) := \max_{0 \leq i \leq r_1} \left\{ 0.5(r - 2.5) + \sin \left( \frac{2.5}{1.6} \right)^2, \sin \left( \frac{r}{1.6} \right)^2 \right\}
\]
\[
\gamma_{iu}(r_1) := \frac{r_1}{4},
\]
where \( \varepsilon \in (0,1) \) is a constant value.

From the definition of these functions, for every \( \varepsilon \in (0,1) \) the condition
\[ V_i \geq \eta(\gamma_{ij}(V_j), \gamma_{ik}(V_k), \gamma_{iu}(u_i)) \tag{5.2a} \]
implies that the inequality
\[ \dot{V}_i \leq -\frac{\varepsilon}{2} V_i, \tag{5.2b} \]
holds, for every \( (V_i, V_j, V_k) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}, \) for every \( u_i \geq 0 \), and for every indexes \( i, j, k = 1, 2, 3 \) with \( j \not= i \not= k \) and \( j \not= i \). Note that the set of inequalities (5.2) also holds if \( I_p \) is replaced by any function \( I^*_p \) of class \( \mathcal{K}_\infty \) satisfying \( I^*_p(\cdot) \geq I_p(\cdot) \). Thus, the subsystems of (5.2) are ISS (cf. 2.1). Figure 5.2 shows a plot of the functions \( I_p, \eta \) and \( \gamma_{ij} \) on a particular vector.

Note that the set of inequalities (5.2) do not hold if \( I_p \) is replaced by a function \( I^*_p \in \mathcal{K}_\infty \) satisfying \( I^*_p(\tau^*) < I_p(\tau^*) \) at some point \( \tau^* \geq 0 \). More precisely,

**Claim 5.1.** For any function \( I^*_p \in \mathcal{K}_\infty \) such that the inequality \( I^*_p(\tau^*) < I_p(\tau^*) \) holds for some \( \tau^* \in \mathbb{R}_{\geq 0} \), there exists a quadruple \((V_i, V_j, V_k, v) \in \mathbb{R}_{\geq 0}^4 \) such that the condition \( V_i > I^*_p(\gamma_{ij}(V_j), \gamma_{ik}(V_k), \gamma_{iu}(v)) \) implies that the inequality \( \dot{V}_i > 0 \) holds.

Claim 5.1 gives a lower estimate for the tightest gain function. The proof of Claim 5.1 is provided at the end of this example.

The monotone operator (2.3) is given, for every \((s, r) \in \mathbb{R}_{\geq 0}^3 \times \mathbb{R}_{\geq 0} \) by
\[
\Gamma_\eta(s, r) = \begin{bmatrix} \eta(\gamma_{12}(s_2), \gamma_{13}(s_3), \gamma_{1u}(r)) \\ \eta(\gamma_{21}(s_1), \gamma_{23}(s_3), \gamma_{2u}(r)) \\ \eta(\gamma_{31}(s_1), \gamma_{32}(s_2), \gamma_{3u}(r)) \end{bmatrix}.
\]

Note that, for every \( r \in (1.75, 5.5) \), the inequality \( I_p(r) > r \) holds. This implies that the monotone operator \( \Gamma_\eta \) is such that \( \Gamma_\eta(s) > s \), for every \( s \in (1.75, 5.5)^3 \). Moreover, from Claim 5.1 there exist no other ISS-Lyapunov gains such that \( \Gamma_\eta(s) \not< s \), for every \( s > 0 \). This motivates the method proposed in this paper.
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Figure 5.2. Functions $I_p$, $\eta$, and $\gamma_{ij}$ for $\varepsilon = 0.05$ on the interval $[0, 5]$.

Application of the proposed method. Since function (5.1) is strictly increasing, two functions of class $K_\infty$ can be defined as

$$I_{p_0} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$$

$$r \mapsto \begin{cases} I_p(r), & \text{if } r \in [0, 1.75) \\ h_0(r), & \text{otherwise} \end{cases}$$

and

$$I_{p_\infty} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$$

$$r \mapsto \begin{cases} h_\infty(r), & \text{if } r \in [0, 1.75) \\ I_p(r), & \text{otherwise} \end{cases}$$

where $h_0$ (resp. $h_\infty$) is a function of class $K_\infty$ (resp. $K$) satisfying the inequality $h_0(\cdot) \geq I_p(\cdot)$ (resp. $h_\infty(\cdot) \leq I_p(\cdot)$).

Two monotone operators can be defined with the functions $I_{p_0}$ and $I_{p_\infty}$ as follows. For every $(s, r) \in \mathbb{R}_{\geq 0}^3 \times \mathbb{R}_{\geq 0}$, let

$$\Gamma_{\eta_0}(s, r) = \begin{bmatrix} \eta_0(\gamma_{12}(s_2), \gamma_{13}(s_3), \gamma_{1u}(r)) \\ \eta_0(\gamma_{21}(s_1), \gamma_{23}(s_3), \gamma_{2u}(r)) \\ \eta_0(\gamma_{31}(s_1), \gamma_{32}(s_2), \gamma_{3u}(r)) \end{bmatrix}, \quad \Gamma_{\eta_\infty}(s, r) = \begin{bmatrix} \eta_\infty(\gamma_{12}(s_2), \gamma_{13}(s_3), \gamma_{1u}(r)) \\ \eta_\infty(\gamma_{21}(s_1), \gamma_{23}(s_3), \gamma_{2u}(r)) \\ \eta_\infty(\gamma_{31}(s_1), \gamma_{32}(s_2), \gamma_{3u}(r)) \end{bmatrix}.$$

Figure 5.3 shows a plot of functions $2I_{p_\phi}$ and $2\eta_\phi$, where $\phi \in \{0, \infty\}$, on a particular vector.

Note also that letting the constant value $M_0 = (1.75, 1.75, 1.75)$ (resp. $M_\infty = (12, 12, 12)$) the inequality $2\Gamma_{\eta_0}(s) \prec s$ (resp. $2\Gamma_{\eta_\infty}(s) \prec s$) holds, for every $s \in (0, M_0)$ (resp. $s \in (M_\infty, \infty)$). This implies that Assumption 3.1 holds on the sets $(0, M_0)$ and $(M_\infty, \infty)$ of $\mathbb{R}^3$ with the D paths

$$\sigma_0 : [0, b_0] \rightarrow [0, M_0]$$

$$r \mapsto \frac{1}{b_0} M_0 r \quad (5.3)$$
and

\[ \sigma_\infty : [a_\infty, \infty) \to [M_\infty, \infty) \]
\[ r \mapsto \frac{1}{a_\infty} M_\infty r \]  \hspace{1cm} (5.4) \]

defined for any values \( b_0 > 0 \) and \( a_\infty > 0 \). Consequently, the following inequalities

\[ 2 \Gamma_{\sigma_0}(\sigma_0(0), 0) \prec \sigma_0(r), \ \forall r \in (0, b_0), \]
\[ 2 \Gamma_{\sigma_\infty}(\sigma_\infty(0), 0) \prec \sigma_\infty(r), \ \forall r \in (a_\infty, \infty) \]

hold.

Define the function \( \sigma_{\infty}(\cdot) = \min_{i=1,2,3} \{\sigma_0(\cdot)_i\} \), where \( \in \{0, \infty\} \). Since \( 2 \Gamma_{\sigma_0}(0, r) \prec (r, r, r)^T \), for every \( r > 0 \), from the obtained D-paths,

\[ 2 \Gamma_{\sigma_0}(0, \zeta_0(r)) \prec \zeta_0(r)(1,1,1)^T \ \forall r \in (0, b_0) \]
\[ 2 \Gamma_{\sigma_\infty}(0, \zeta_\infty(s)) \prec \zeta_\infty(r)(1,1,1)^T \ \forall r \in (a_\infty, \infty). \]

This implies that the inequalities

\[ \Gamma_{\sigma_0}(\sigma_0, \zeta_0) \prec \frac{1}{2} (\sigma_0 + \zeta_0)(1,1,1)^T \leq \sigma_0(r), \ \forall r \in (0, b_0) \]
\[ \Gamma_{\sigma_\infty}(\sigma_\infty, \zeta_\infty) \prec \frac{1}{2} (\sigma_\infty + \zeta_\infty)(1,1,1)^T \leq \sigma_\infty(r), \ \forall r \in (a_\infty, \infty) \]

hold.

Note that \( \zeta_\infty = \max_{i=1,2,3} \{12, \Gamma_{\sigma_\infty, i}(12, 12)\} \leq 12 \). From Corollary 4.2 (resp. 4.3) solutions to system (1.3) will remain close and eventually converge towards a ball.
centered at the origin (resp. at the set $A_{\infty} = \{ V \in \mathbb{R}_{\geq 0}^3 : V_i \leq 12, \ i = 1, 2, 3 \}$) with radius proportional to the norm of the input.

Since $1.75 = m_0 < g_{\infty} = 12$, it remains to check whether the solutions of $\Sigma(u)$ starting in the set $S := c I \{ A_{\infty} \setminus B_0 \}$ (i.e. the difference between the global attractor and the estimation of the region of local attraction), where $B_0 = \{ V \in \mathbb{R}_{\geq 0}^3 : V_i < 1.75, \ i = 1, 2, 3 \}$, will converge to a closed ball centered at the origin with radius proportional to the norm of the input.

Direct calculation shows that the inequality $\text{div} f(V, u) \leq 0$ holds, for every $(V, u) \in \mathbb{R}_{\geq 0}^3 \times \mathbb{R}_{\geq 0}^3$. To satisfy the assumptions of Theorem 4.6 the following density function

$$
\rho : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}
$$

is employed.

The divergence of the function $\rho f$ yields, for every $(V, V_2, V_3) \in \mathbb{R}_{\geq 0}^3$, the equation

$$
(\text{div} \rho f)(V, u) = (\text{grad} \rho)(V) \cdot f(V, u) + \rho(V) \cdot (\text{div} f)(V, u)
$$

where

$$
g(V, u) = 10 \left( -p(V_1) - p(V_2) - p(V_3) + 2 \left( \sin \left( \frac{V_1}{1.6} \right)^2 + \sin \left( \frac{V_2}{1.6} \right)^2 + \sin \left( \frac{V_3}{1.6} \right)^2 \right) 
+ u_1 + u_2 + u_3 \right) - (p'(V_1) + p'(V_2) + p'(V_3))
$$

Figure 5.4 shows the function $g$ evaluated along the axis $(1, 1, 1)$ with $u \equiv 0$, and on the interval $(0, 6)$. Note that the inequality $(\text{div} \rho f)(V, 0) > 0$ holds, for almost every $V \in (1, 1.3)^3$.

The function $\text{div} \rho f$ has the same sign as the function $g$. Moreover, denoting $\bar{u} = (\max_{i=1,2,3} u_i) / 10$, the inequality

$$
g(V, u) \geq -10 \left( -p(V_1) - p(V_2) - p(V_3) + 2 \left( \sin \left( \frac{V_1}{1.6} \right)^2 
+ \sin \left( \frac{V_2}{1.6} \right)^2 + \sin \left( \frac{V_3}{1.6} \right)^2 \right) \bar{u} 
- (p'(V_1) + p'(V_2) + p'(V_3)) \right) 
\geq g(V, 0) - \bar{u},
$$

holds, for every $(V, u) \in \mathbb{R}^3 \times \mathbb{R}_{\geq 0}^3$. Note that, for any $\varepsilon \in (0, 1)$, the condition $g(V, 0) \geq \bar{u}/(1 - \varepsilon)$ implies that the inequality $g(V, u) > 0$ holds. From now on, fix $\varepsilon = 0.05$. To see that the density propagation condition (4.6) holds, define the function $I_g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where each component $i = 1, \ldots, 3$ is given by

$$
(I_g)_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}
$$

$$
r \mapsto \begin{cases} 0, & \text{if } r = 0, \\
\max \{ V_i : g(V, 0) - r = 0 \}, & \text{if } r > 0,
\end{cases}
$$
where max\{\(V_i : g(V,0) - r = 0\)\} returns the largest \(V_i \in \mathbb{R}_{\geq 0}\) that corresponds to the level set \(g(V,0) = r\). Note that \((I_g)\), \(i = 1, \ldots, n\) is strictly increasing and continuous everywhere except the origin.\(^1\)

**Claim 5.2.** There exists a continuous function \(Q : \mathbb{R}_{\geq 0}^3 \to \mathbb{R}_{\geq 0}\) such that \(Q(V) > 0\), for almost every \(V\in (1,13)^3\). Moreover, in this set the condition

\[
V \geq I_g \left( \frac{u}{0.95} \right)
\]

implies that the inequality

\[
(\text{div } \rho f)(V,u) \geq Q(V)
\]

holds.

The details of the proof of Claim 5.2 are provided at the end of this section. Note that, applying a parallel reasoning as in Claim 5.1, there exists no class \(K\) function \(I_{g}^{C}\) satisfying \(I_{g}^{C}(r^*) < I_{g}(r^*)\) at some point \(r^* \geq 0\) and such that the set of inequalities (5.5) holds. From Claim 5.2, Assumption 4.5 holds.

From Theorem 4.6, system \(\Sigma(u)\) is alSpS. Figures 5.5 and 5.6 show, respectively, the time evolution of the solution of systems \(\Sigma(0)\) and \(\Sigma(2)\) for different initial conditions.

In conclusion, the stability analysis of this system could not be performed using small-gain condition, since it does not hold globally. Also, for the chosen function \(\rho\), the “positiveness” of the divergence condition does not hold globally as well (it is negative on a set of measure larger than zero containing the origin, and vanishes when \(|V| \to \infty\)).

**Proof.** [of Claim 5.1] Assume that \(\tau^* \in (0,1.75)\). This value corresponds to the existence of a triple \((V_j,V_k,\nu) \in \mathbb{R}_{\geq 0}^3\) satisfying \(\tau^* = \gamma_{ij}(V_j) + \gamma_{ik}(V_k) + \gamma_{iu}(\nu)\). By\(^1\) see that \(I_g\) is discontinuous at the origin, note that \(\limsup_{r \to 0} I_g(r) = (2.5,2.5,2.5)\).
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\[ V_1(t) \]

\[ V_2(t) \]

\[ V_3(t) \]

Figure 5.5. Solutions of system $\Sigma(0)$ for different initial conditions (circles). The red circle is the equilibrium point $\overline{V}$.

Figure 5.6. Solutions of system $\Sigma(2)$ for different initial conditions (circles).

assumption, there exists a function $I_p^* : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

\[(1 - \varepsilon)\eta(\gamma_{ij}(V_j), \gamma_{ik}(V_k), \gamma_{iu}(\nu)) > V_i > I_p^*(\gamma_{ij}(V_j), \gamma_{ik}(V_k), \gamma_{iu}(\nu)).\]

From the definition of the functions $I_p$, $\gamma_{ij}$, $\gamma_{ik}$ and $\gamma_{iu}$, together with the fact that $p$ is strictly increasing on $(0, 1.75)$, the above equation is equivalent to the inequality

\[ \gamma_{ij}(V_j) + \gamma_{ik}(V_i) + \gamma_{iu}(\nu) > p(V_i) > p \circ I_p^*(\gamma_{ij}(V_j), \gamma_{ik}(V_k), \gamma_{iu}(\nu)).\]
This implies, from the definition of the each subsystem of \( \Sigma(u) \), that \( \dot{V}_i > 0 \). The case for \( \tau^* \geq 1.75 \) is parallel. This concludes the proof of Claim 5.1.

Proof. [of Claim 5.2] Let \( u \) be such that \( \bar{u} = (\max_{i=1,2,3} u_i)/10 \neq 0 \). This implies that the inequality \( I_g(\bar{u}/0.95) > (2.5,2.5,2.5) \) holds. Applying the function \( V \to g(V,0) \) on this inequality yields \( g(V,0)0.95 \geq \bar{u} \), because the function \( V \to g(V,0) \) is strictly increasing for \( V \geq (2.5,2.5,2.5) \). Thus, the inequality \( g(V,0) - \bar{u} \geq 0.05g(V,0) =: Q(V) \) holds. Moreover, the function \( Q \) is strictly positive for almost every \( V \in (1.3,12) \). This concludes the proof of Claim 5.2.

6. Proofs. The proofs presented here are based on the techniques from \([1,6,7,16,18,22]\).

6.1. Proof of Proposition 4.1. Under Assumptions 2.5 and 3.1 consider the D path \( \sigma : [a,b] \to [\underline{M},\overline{M}] \). For each \( r \in (a,b) \), define the function
\[
\phi(r) = \sup \left\{ s \in (a,b) : \Gamma_\eta(\sigma(s),s) < \sigma(r) \right\}.
\]
From the continuity of the monotone operator \( \Gamma_\eta \) and the strict inequality, the function \( \phi \) is defined, for every point on its domain. Moreover, \( \phi(\cdot) > 0 \). Thus, there exists a function \( \varphi \in K_\infty \) satisfying the inequality \( \varphi(r) < \phi(r) \), for every \( r \in (a,b) \).

Define the set
\[
\mathbf{R} = \bigcap_{i=1}^n \left\{ x \in \mathbb{R}^N : \underline{M}_i \leq V_i(x_i) \leq \overline{M}_i \right\},
\]
pick an \( x \in \text{int}(\mathbf{R}) \) and let \( I \) be the set of indexes \( i \) for which
\[
V(x) = \sigma_i^{-1}(V_i(x_i)) \geq \max_{j \neq i} \sigma_j^{-1}(V_j(x_j)),
\]
where \( \sigma_i \) is the \( i \)-th component of the D-path \( \sigma \).

For the time being, assume (the proof of the existence is provided below) that there exists a constant value \( \delta_u > 0 \) such that the inequality
\[
V(x) \geq \max_{i=1,\ldots,n} \left\{ \varphi^{-1}(\gamma_i(u(|u|))) \right\}
\] holds, for every \( u \in B_{\leq \delta_u} \). Using the abbreviation \( r := V(x) \), the inequality
\[
\Gamma_\eta(\sigma(r),\varphi(r)) < \sigma(r)
\] holds, for every \( r \in (a,b) \).

Assume, without loss of generality, that \( I = \{1\} \). Denote the first component of \( \Gamma_\eta \) by \( \Gamma_{\eta,1} \). Inequalities 6.2 and 3.1 implies that the inequality
\[
V_1(x_1) = \sigma_1(r) > \Gamma_{\eta,1}(\sigma(r),\varphi(r)),
\]
holds.

Since \( \varphi \) is continuous and strictly increasing on its domain, inequality 4.2a implies that \( \gamma_i(u(|u|)) \leq \varphi(r) \) holds, for every index \( i = 1,\ldots,n \). This implies that the inequality
\[
\Gamma_{\eta,1}(\sigma(r),\varphi(r)) \geq \eta_1 \left( \gamma_{i2}(\sigma_2(r)),\ldots,\gamma_{in}(\sigma_n(r)),\gamma_{i1}(u_1) \right)
\] holds.
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holds.

Since, from (6.2), \( r = \sigma^{-1}_1(V_i(x_1)) \), inequalities (6.3) and (6.4) imply that the condition

\[
V_i(x_1) \geq \eta_i \left( \gamma_{12} \circ V_2(x_2), \ldots, \gamma_{1n} \circ V_n(x_n), \gamma_{11u}(|u_1|) \right),
\]

(6.5)

holds. From Assumption 2.5, this implies that the inequality

\[
\xi_1 \cdot f_1(x, u) \leq -\alpha_1(|x_1|)
\]

(6.6)

holds, for every \( \xi_1 \in \partial V_i(x_1) \). It now remains to show that (6.6) implies that the inequality (4.2b) holds.

From the chain rule for Lipschitz continuous functions [4, Theorem 2.5], for each index \( i = 1, \ldots, n \), the inclusion

\[
\partial[\sigma^{-1}_i \circ V_i](x_i) \subset \{ c_i \xi_i : c_i \in \partial\sigma^{-1}_i(V_i(x_i)), \xi_i \in \partial V_i(x_i) \}
\]

holds. Note that \( c_i \) is bounded away from \( M_i \) and \( \overline{M}_i \), because of (3.2).

Pick \( \delta \) satisfying \( \overline{m} < \delta < m \) and define the positive definite function \( \bar{\alpha}_i(\delta) = c_{\delta,i} \alpha_i(\delta) \), where \( c_{\delta,i} \) is the constant corresponding to the compact set

\[
K = \left\{ x_i \in \mathbb{R}^{N_i} : m + \frac{m - \delta}{2} \leq |x_i| \leq m + \frac{m - \delta}{2} \right\} \subset (\overline{m}, m).
\]

Define, for every scalar \( y \in (0, |x - \overline{M}|) \), the class-\( \mathcal{K}_\infty \) function

\[
\alpha(y) = \min_{i=1, \ldots, n} \{ \bar{\alpha}_i(|x_i|) : |x - \overline{M}| = y, V(x) = \sigma^{-1}_i(V_i(x)) \}.
\]

(6.7)

Note that, for such a choice of \( y \) and the fact that \( |x - \overline{M}| = y \), the norm \( |x_i| \) such that \( V(x) = \sigma^{-1}_i(V_i(x)) \) is strictly larger than \( \overline{M} \).

Inequality (6.6) and function (6.7) imply that the inequality

\[
\xi_1 \cdot f_1(x, u) \leq -\alpha(|x|)
\]

holds, for every \( \xi_1 \in \partial V_i(x_1) \). Note that the right-hand side of this inequality depends on \( x \) and not only on \( x_1 \). The same argument applies for all other indexes \( i \in \mathbb{I} \).

Since \( V \) is obtained through maximization, from [4, Exercise 4.6.c], the inclusion

\[
\partial V(x) \subset \text{co} \left\{ \bigcup_{i \in \mathbb{I}} \partial[\sigma^{-1}_i \circ V_i \circ \pi_i](x) \right\}
\]

(6.8)

holds. This implies that every \( \xi \in \partial V(x) \) can be decomposed into a sum of \( n \) components. More precisely, the equation

\[
\xi = \sum_{i \in \mathbb{I}} \lambda_i c_i \xi_i
\]

holds, for suitable scalars \( \lambda_i \geq 0 \) satisfying \( \lambda_1 + \cdots + \lambda_n = 1 \), \( \xi_i \in \partial[\sigma^{-1}_i \circ V_i \circ \pi_i](x) \) and \( c_i \in \partial\sigma^{-1}_i(V_i(x_i)) \). Consequently,

\[
\xi \cdot f(x, u) = \sum_{i \in \mathbb{I}} \lambda_i (c_i \xi_i \cdot f(x, u)) = \sum_{i \in \mathbb{I}} \lambda_i (c_i \pi_i(\xi) \cdot f_i(x, u)) \leq -\sum_{i \in \mathbb{I}} \lambda_i \alpha(|x|) = -\alpha(|x|).
\]
It now remains to show that the above conclusion also holds for every \( x \) in the set
\[
\tilde{B} = \bigcap_{i=1}^{n} \{ x \in \mathbb{R}^N : V_i(x_i) < M_i \}
\]
with at least one component \( i = 1, \ldots, n \) satisfying \( V_i(x_i) > M_i \) and (possibly) other components \( j = 1, \ldots, n \) with \( j \neq i \) satisfying the inequality \( V_j(x_j) \leq M_j \). To see this claim, first note that \( \tilde{B} \) is nonempty, pick an \( x \in \tilde{B} \) and assume, without loss of generality, that \( I = \{1\} \) and \( V_2(x_2) \leq M_2 \). Inequality (6.2) is rewritten as
\[
V(x) = \sigma_1^{-1}(V_1(x_1)) \geq \max \left\{ \sigma_2^{-1}(M_2), \max_{i=3,\ldots,n} \sigma_i^{-1}(V_i(x_i)) \right\}
\]
and implies that \( V_1(x_1) = \sigma_1(V(x)) \).

Let \( r := V(x) \), inequality (3.1) implies that
\[
V_1(x_1) = \sigma_1(r) > \Gamma_{\eta,1}(\sigma(r), \varphi(r)). \tag{6.3}
\]

By assumption, inequality (4.2a) holds. This implies that the inequality \( \gamma_{iu}(|u_i|) \leq \varphi(r) \) holds for every index \( i = 1, \ldots, n \). Consequently, the inequality
\[
\Gamma_{\eta,1}(\sigma(r), \varphi(r)) \geq \eta_1 \left( \gamma_{12}(\sigma_2(r)), \ldots, \gamma_{1n}(\sigma_n(r)), \gamma_{1u}(|u_i|) \right) \tag{6.4}
\]
holds.

From the fact that \( V_2(x_2) \leq M_2 \) and \( \gamma_{12} \in K_{\infty} \), inequalities (6.3) and (6.4) imply that
\[
V_1(x_1) \geq \eta_1 \left( \gamma_{12}(M_2), \gamma_{13} \circ V_3(x_3), \ldots, \gamma_{1n} \circ V_n(x_n), \gamma_{1u}(|u_i|) \right) \\
\geq \eta_1 \left( \gamma_{12} \circ V_2(x_2), \gamma_{13} \circ V_3(x_3), \ldots, \gamma_{1n} \circ V_n(x_n), \gamma_{1u}(|u_i|) \right).
\]

Thus, inequality (6.6) holds and the previous reasoning can be repeated for different choices of the set \( I \) and indexes \( j \).

For the sake of completeness, assume that \( x \in \tilde{B} \) is such that \( I = \{1\} \), and for every index \( j = 2, \ldots, n \), the inequality \( V_j(x_j) \leq M_j \). This implies that the inequality
\[
V(x) = \sigma_1^{-1}(V_1(x_1)) > \max_{j=2,\ldots,n} \sigma_j^{-1}(M_j)
\]
holds. From (3.1), inequality (6.3) holds. By assumption, inequality (4.2a) holds which implies (6.4). Inequalities (6.3) and (6.4) imply that
\[
V_1(x_1) \geq \eta_1 \left( \gamma_{12}(M_2), \gamma_{13}(M_3), \ldots, \gamma_{1n}(M_n), \gamma_{1u}(|u_i|) \right) \\
\geq \eta_1 \left( \gamma_{12} \circ V_2(x_2), \gamma_{13} \circ V_3(x_3), \ldots, \gamma_{1n} \circ V_n(x_n), \gamma_{1u}(|u_i|) \right).
\]

Thus, inequality (6.6) holds.
Let the set
\[ \tilde{A} = \bigcap_{i=1}^{n} \{ x \in \mathbb{R}^N : V_i(x_i) \leq \max\{M_i, \Gamma_{\eta,i}(M_i, \ldots, M_{i-1}, M_{i+1}, \ldots, M_n)\} \} \]

From the above reasoning, the condition
\[ V(x) \geq \max_{i=1, \ldots, n} \{ \varphi^{-1}(\gamma_i(|u_i|)) \} \quad (4.2a) \]
implies that the inequality
\[ \xi \cdot f(x, u) \leq -\alpha(|x|) \]
holds, for every \((x, u) \in (\tilde{B} \setminus \tilde{A}) \times B_{\leq \delta_u}\), and for every \(\xi_i \in \partial V_i(x_i)\).

Since \(\Gamma_n\) is monotonically increasing and \(m = \max_{i=1, \ldots, n} M_i\), the inequality
\[ \max \{ M_i, \Gamma_{\eta,i}(M_i, \ldots, M_{i-1}, M_{i+1}, \ldots, M_n) \} \leq \max \{ m_i, \Gamma_{\eta,i}(m, \ldots, m, m, \ldots, m) \} \]
holds, for each index \(i = 1, \ldots, n\). This implies that \(\tilde{A} \subset A\). Note also that \(B \subset \tilde{B}\).

Thus, the condition \((4.2a)\) implies that inequality \((4.2b)\) holds. It remains to show that the constant value \(\delta_u > 0\) always exists.

Note that, when \(u \equiv 0\), the above proof implies that solutions to system \((1.3)\) starting in set \(B_0 \setminus A\) will remain close to the set \(A\) and eventually converge to it. Consequently, solutions to system \((1.3)\) starting in set \(A\) will remain in this set for all positive times. This implies \((21)\) the existence of a function \(\tilde{\gamma} \in \mathcal{K}_\infty\) and a constant value \(\delta_u\) such that, for every \((x, u) \in (B \setminus A) \times \mathcal{L}_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^M)\) with \(|u|_\infty < \delta_u\), the inequality \(\lim_{t \to \infty} |X(t, t, x, u)|_A \leq \tilde{\gamma}(|u|_\infty)\) holds. Thus, there exists a value \(\delta_u > 0\) such that the condition \((4.2a)\) implies the inequality \((4.2b)\). This concludes the proof of Proposition \(1.1\). \(\square\)

**6.2. Proof of Theorem 4.6** This proof is divided into three parts. The first one shows that solutions starting in \(B_\infty\) converge to a vicinity of \(S\) and the solutions starting in \(B_0\) converge to the ball centered at the origin with radius proportional to the norm of the input. The second part shows that almost all solutions starting in \(S\) converge to a vicinity of \(B_0\). The third part concludes the aISS/aSpS of the system \((1.3)\).

1ST PART. Under Assumptions \(2.5\) and \(3.1\), Corollaries \(4.3\) and \(4.2\) holds. Consider the sets \(A_\infty\) and \(B_0\) defined in Equations \((4.4)\) and \((4.5)\), respectively. By assumption, inequality \((4.7)\) holds. This implies that
\[ \min_{i=1, \ldots, n} m_i < g_\infty = \max_{i=1, \ldots, n} \{ m_{\infty, i}, \Gamma_{\eta,i}(m_{\infty, i}, \ldots, m_{\infty, n}) \} \]
Consequently, \(B_0 \subset A_\infty\).

Corollary \(4.3\) (resp. Corollary \(4.2\)) implies that the solutions to system \((1.3)\) starting in the set \(B_\infty\) (resp. in \(B_0\)) converge to a ball centered at \(A_\infty\) (resp. to the origin) with radius proportional to the norm of the input.

In other words, there exists a function \(\gamma_\infty \in \mathcal{K}_\infty\) (resp. \(\gamma_0 \in \mathcal{K}_\infty\)) such that, for every initial condition \(x \in B_\infty\) (resp. \(x \in B_0\)), and for every input \(u\) for system \((1.3)\), the limit
\[ \lim_{t \to \infty} V_i(X_i(t, x, u)) \leq g_\infty + \gamma_\infty(|u|_\infty) \quad (6.9) \]
\[ \left( \text{resp. } \lim_{t \to \infty} V_i(X_i(t, x, u)) \leq \gamma_0(|u|_\infty) \right) \quad (6.10) \]
exists, for every component \(i = 1, \ldots, n\) of the solution \(X\) to system (1.3).

2ND PART. Let the constant

\[
d = \inf_{x \in \mathbb{R}^N \setminus (D \cup A_\infty)} \{|x|_{A_\infty}\}.
\]

In other words, the value \(d\) is the distance between the sets \(A_\infty\) and \(\mathbb{R}^N \setminus (D \cup A_\infty)\).

Since the strict inclusion \(c1\{D\} \supset S\) holds, \(d > 0\) Thus, if the values of input \(u\) of system (1.3) are such that \(\gamma_\infty(|u|) \leq d\), for a fixed \(\varepsilon \in (0, 1)\), then from the analysis of the 1st part the components of solutions to (1.3) with initial condition \(\bar{x} \in B_\infty\) satisfy the inequality

\[
\lim_{t \to \infty} V_i(X_i(t, \bar{x}, u)) \leq g_{\infty} + d\varepsilon,
\]

for every index \(i = 1, \ldots, n\). Consequently, there exists a finite time \(T\) such that \(X(T, \bar{x}, u) \in D \cap B_\infty\). Note that if \(\gamma_\infty(|u|) \geq d\), then there might not exist a finite time \(T\) such that the solution \(X\) belongs to the set \(D \cap B_\infty\), because this intersection is an open set. In this case, however, solutions will remain in a ball centered at \(A_\infty\) with radius \(\gamma_\infty(|u|)\).

The proof proceeds by showing that, for almost every initial condition in \(D\), issuing solutions of (1.3) converge to a ball centered at \(B_0\) with radius proportional to the norm of the input. To do so, the same lines as in [22] are followed.

Together with Assumption 4.5, let \(Z \subset \mathbb{R}^N\) be the set defined as

\[
Z = \bigcap_{t = 1}^\infty \left\{ x \in D : \max_{i=1,\ldots,n} \{V_i(X_i(t, x, u))\} > m_0 + \gamma(|u|_\infty), \forall t > t \right\}.
\]

In other words, \(Z\) is the set of initial conditions in \(D\) such that the corresponding solutions to (1.3) starting at the set \(D\) remaining outside a closed ball centered at \(B_0\) with radius \(\gamma(|u|)\). The objective of this part of the proof is to show that \(Z\) has Lebesgue measure zero.

For every \(t \in \mathbb{R}\), let the set

\[
X(t, Z, u) = \{X(t, z, u) : z \in Z, t \in \text{dom}(z)\},
\]

where \(\text{dom}(z)\) is the maximum time interval where the solution \(X(t, z, u)\) of (1.3) exists.

Since for every \(\gamma \in \mathcal{K}\), the set \(B_{\leq \gamma(|u|)}(A_\infty)\) is positively invariant, the set \(Z\) is also positively invariant. Thus, given a fixed \(\tau > 0\), for every time \(t \geq \tau\), the inclusion \(X(t, Z, u) \subset X(\tau, Z, u)\) holds. Hence, the inequality

\[
\int_{X(t, Z, u)} \rho(x) \, dx - \int_Z \rho(x) \, dx \leq 0,
\]

holds, for all positive times \(t\), where \(\rho \in C^1(D, \mathbb{R}_{>0})\) and \(\text{supp}(\rho) \subset D\) by assumption.

From [11] Lemma A.1, the equation

\[
\int_0^t \int_{X(s, Z, u)} \text{div}(f\rho)(x, u(s)) \, dxds = \int_{X(t, Z, u)} \rho(x) \, dx - \int_Z \rho(x) \, dx,
\]

for every positive time \(t\). From the inclusion \(Z \subset D\), for every \(t \geq 0\), the inequality

\[
t \int_{X(t, Z, u)} \text{div}(f\rho)(x, u(t)) \, dx \leq \int_0^t \int_{X(s, Z, u)} \text{div}(f\rho)(x, u(s)) \, dxds
\]
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holds. Together with inequalities (6.11) and (4.6),
\[
\forall t \in \mathbb{R}_{\geq 0}, \quad \int_{X(t,z,r)} Q(x) \, dx \leq \int_{X(t,z,r)} \text{div}(f(x,u(t))) \, dx \leq 0.
\]

Consequently, inequality (6.12) implies that, for every positive time \( t \), the equation
\[
\int_{X(t,z,r)} Q(x) \, dx = 0
\]
holds. This follows from the fact that, for almost every \( x \in D \), the inequality \( Q(x) > 0 \) holds. Thus, for every \( t \geq 0 \), the set \( X(t,z,u) \) has Lebesgue measure zero. In particular, \( Z \) has also Lebesgue measure zero. Consequently, for almost every \( x \in D \), the inequality
\[
\limsup_{t \to \infty} \max_{i=1,\ldots,n} \{ V_i(X_i(t,x,u)) \} \leq m_0 + \gamma(||u||) \quad (6.13)
\]
holds.

3rd Part. It remains to verify if solutions to (1.3) starting in the set \( B_{\infty} \) and converging to \( Z \) have also measure zero.

Since \( Z \) is positively invariant, for every two time instants satisfying \( t_1 < t_2 \leq 0 \), the inclusion \( X(t_2,z,u) \subset X(t_1,z,u) \) holds. Let the set
\[
X = \bigcup_{t \leq 0} \{ X(t,z,u) \} = \bigcup_{t \in \mathbb{Z} \leq 0} \{ X(t,z,u) \}
\]
be the union of initial conditions whose solutions to (1.3) converge to the set \( Z \).

To see that \( X \) is measurable, note that the set \( X \) is a countable union of images of \( Z \) by the flow. Moreover, \( Z \) is measurable and, for every time \( t \in \text{dom}(x) \), the map \( Z \ni x \mapsto X(t,x,u) \) is a diffeomorphism.\footnote{Because (1.3) is of class \( C^1 \) and solutions are unique.}

For every time \( t \in \text{dom}(Z) \), the inequality
\[
\int_{X(t,z,u)} dz \leq \int_{Z} |\text{grad} X(t,x,u)| \, dx = 0
\]
holds, because \( Z \) has measure zero. This implies that, for all times \( t \in \text{dom}(Z) \), the set \( X(t,z,u) \) has measure zero. Since \( X \) is a countable union of sets of measure zero, it has also measure zero.\footnote{Recall that \( X \) is the set of initial conditions from which issuing solutions of (1.3) converge to \( Z \).} Hence the set of trajectories of solutions starting in \( B_{\infty} \) that converge to \( Z \) have also measure zero.

From inequalities (6.9), (6.10) and (6.13), for almost every initial condition \( x \in \mathbb{R}^N \), and for every input \( u \in C^1_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^M) \) for system (1.3), the inequality
\[
\lim_{t \to \infty} \sup V(X(t+T,x,u)) \leq \gamma(||u||)
\]
holds, for every component \( i = 1,\ldots,n \) of the solution \( X \) to system (1.3), with the continuous (except possibly at the origin) and strictly increasing function \( \gamma(\cdot) = \min\{ \gamma_0(\cdot), \frac{g}{\gamma_\infty}(\cdot), m_0 + \gamma(\cdot) \} \). Thus, system (1.3) is almost (practically) input-to-state stable. This concludes the proof of Theorem 4.6.

Remark 6.1. Note that the convergence of solutions starting in \( B_{\infty} \) towards \( D \) depends on the value of the constant value \( d \). Since it can be arbitrarily small, solutions may not converge towards \( D \) but they will remain in a ball centered at \( A_{\infty} \) with radius \( \gamma_\infty(||u||) \).
6.3. Proof of Theorem 4.8. The proof of Theorem 4.8 is similar to the proof of Theorem 4.6. It consists of successive applications of the proven results on appropriate regions of the state space.

Consider Assumptions 2.5 and 3.1 holding for each index \( j = 0, \ldots, L \) and the existence of the existence of the function \( \varphi_j \). Proposition 4.1 implies that there exists a function \( \eta_j \in \mathcal{K}_\infty \) such that, for every initial condition \( x \in B_j \setminus A_j \), and for every input \( u \in \mathcal{L}_\text{loc}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^M) \) for system (1.3), the inequality

\[
\lim_{t \to \infty} V_i(X_i(t, x, u)) \leq g_j + \eta_j(|u|_\infty)
\]

(6.14)

holds, for every component \( i = 1, \ldots, n \) of the solution \( X \) to system (1.3). In other words, solutions to system (1.3) starting in the set, \( j = 1, \ldots, L \) (resp. in \( B_1 \setminus A_1 \) in the case when \( S_0 = \emptyset \)) converge to a ball centered at \( A_j \) (resp. to the origin) with radius proportional to the norm of the input.

Analogously to the lines of the second part of the proof of Theorem 4.6 with \( S \equiv S_j \), for every index \( j = 0, \ldots, L \), there exists a function \( \zeta_j \in (C \cap \mathcal{N})(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \) satisfying \( \zeta_j(0) = 0 \) (and possibly discontinuous at zero) such that, for almost every initial condition \( x \in S_j \) (resp. in \( S_0 \)), and for every input \( u \in \mathcal{L}_\text{loc}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^M) \) for system (1.3), the inequality

\[
\limsup_{t \to \infty} V_i(X_i(t, x, u)) \leq \overline{m}_j + \zeta_j(|u|_\infty)
\]

(6.15)

holds, for every component \( i = 1, \ldots, n \) of the solution \( X \) to system (1.3). In other words, solutions to (1.3) converge to a ball centered at \( B_j \) (resp. to the origin) with radius proportional to the norm of the input.

Analogously to the third part of that of the proof of Theorem 4.6 and from inequalities (6.14) and (6.15), for every initial condition \( x \in \mathbb{R}^N \), and for every input \( u \in \mathcal{L}_\text{loc}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^M) \) for system (1.3), the inequality

\[
\limsup_{t \to \infty} V_i(X_i(t + T, x, u)) \leq \omega(|u|_\infty)
\]

holds, for every component \( i = 1, \ldots, n \) of the solution \( X \) to system (1.3) with the the continuous (except possibly at the origin) and strictly increasing function \( \omega(\cdot) = \min_{j=0,\ldots,L}\{g_j + \eta_j(\cdot), \overline{m}_j + \zeta_j(\cdot)\} \).

Now if for some index \( j \in \{1, \ldots, L\} \) Assume 2.5 or 3.1 fails then a solution to (1.3) starting in the set \( B_j \setminus A_j \) (resp. in \( B_1 \setminus A_1 \) in the case when \( S_0 = \emptyset \) and \( j = 0 \)) does not necessarily converge to a vicinity of \( A_j \) (resp. of the origin). However, this solution will always converge to a ball centered at \( B_j \) with radius proportional to the norm of the input due Assumption 4.5 (i.e., due to the density propagation condition (4.6)).

A similar situation happens if for some index \( j \in \{0, \ldots, L\} \) Assumption 4.5 does not hold, this implies that the density propagation condition (4.6) fails. Consequently, solutions to system (1.3) starting in the set \( A_{j+1} \setminus B_j \) will remain in a neighborhood of the set \( A_{j+1} \) with radius proportional to the norm of the input. This concludes the proof of Theorem 4.8. \[\square\]

7. Conclusion. The stability analysis of a network composed by ISS systems is usually performed in terms of the small-gain theorem. This work considered the case in which the small-gain theorem cannot be employed globally.
To tackle this problem, a suitable combination of small-gain and density propagation conditions is proposed by decomposing the state space into regions where the former or the later method can be employed.

An example that motivates and illustrates this research is provided. The developed stability result is rather general and applies to a wide class of nonlinear systems.

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