ON THE RELATIONSHIP BETWEEN EHRHART UNIMODALITY AND EHRHART POSITIVITY

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ABSTRACT. For a given lattice polytope, two fundamental problems within the field of Ehrhart theory are to (1) determine if its (Ehrhart) $h^*$-polynomial is unimodal and (2) to determine if its Ehrhart polynomial has only positive coefficients. The former property of a lattice polytope is known as Ehrhart unimodality and the latter property is known as Ehrhart positivity. These two properties are often simultaneously conjectured to hold for interesting families of lattice polytopes, yet they are typically studied in parallel. As to answer a question posed at the 2017 Introductory Workshop to the MSRI Semester on Geometric and Topological Combinatorics, the purpose of this note is to show that there is no general implication between these two properties in any dimension greater than two. To do so, we investigate these two properties for families of well-studied lattice polytopes, assessing one property where previously only the other had been considered. Consequently, new examples of each phenomena are developed, some of which provide an answer to an open problem in the literature. The well-studied families of lattice polytopes considered include zonotopes, matroid polytopes, simplices of weighted projective spaces, empty lattice simplices, smooth polytopes, and s-lecture hall simplices.

1. Introduction

A subset $P \subset \mathbb{R}^n$ of $n$-dimensional real Euclidean space is called a (lattice) polytope if it is the convex hull of finitely many lattice points (i.e. points in $\mathbb{Z}^n$) that together span a $d$-dimensional affine subspace of $\mathbb{R}^n$. Lattice polytopes play a central role in geometric and algebraic combinatorics and algebraic geometry. In the former context, lattice polytopes are often associated to combinatorial and algebraic objects such that their geometry reflects known facts, and/or reveals new and interesting facts, about these objects. In the latter context, each lattice polytope serves as a “polyhedral dictionary” from which we can read the algebro-geometric properties of an associated toric variety. Consequently, lattice polytopes amount to a large and diverse family of examples within algebraic geometry. In both fields of research, the number of lattice points within the $t$th dilate of a lattice polytope $P$, $tP := \{tp \in \mathbb{R}^n : p \in P\}$, provides information about the associated algebraic and geometric objects. The Ehrhart function of a $d$-dimensional lattice polytope $P$ is the function $i(P; t) := |tP \cap \mathbb{Z}^n|$ for $t \in \mathbb{Z}_{\geq 0}$. It is well-known [14] that $i(P; t)$ is a polynomial in $t$ of degree $d$ (called the Ehrhart polynomial of $P$), and the Ehrhart series of $P$ is the rational function

$$
\text{Ehr}_P(z) := \sum_{t \geq 0} i(P; t)z^t = \frac{h_0^* + h_1^*z + \cdots + h_d^*z^d}{(1 - z)^{d+1}}.
$$

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where the coefficients $h^*_0, h^*_1, \ldots, h^*_d$ are all nonnegative integers \[29\]. The polynomial $h^*(P; z) := h^*_0 + h^*_1 z + \cdots + h^*_d z^d$ is called the $h^*$-polynomial of $P$. In the field of Ehrhart theory, the lattice point combinatorics of a polytope $P$ are studied using both its Ehrhart polynomial and its $h^*$-polynomial. Both $i(P; t)$ and $h^*(P; z)$ often reflect interesting properties of the underlying combinatorial structure of $P$, but the questions and techniques common to the study of each polynomial are drastically different. Consequently, although the Ehrhart polynomial and the $h^*$-polynomial may be analyzed simultaneously, they are often studied in parallel. In particular, two popular, and parallel, endeavors in Ehrhart theory are to analyze when a lattice polytope $P$ is Ehrhart positive and when it is Ehrhart unimodal.

A lattice polytope $P$ is called Ehrhart positive if all the coefficients of the Ehrhart polynomial $i(P; t)$ are positive rational numbers, and it is called Ehrhart unimodal if its $h^*$-polynomial is unimodal; i.e. for some $t \in [d] := \{1, 2, \ldots, d\}$ we have that

\[ h^*_0 \leq h^*_1 \leq \cdots \leq h^*_t \geq \cdots \geq h^*_{d-t} \geq h^*_d. \]

Both Ehrhart positivity and Ehrhart unimodality are popularly investigated properties with deep algebraic and geometric underpinnings. In fact, each property was recently the focus of its own survey article \[4, 21\], and it is common to see both properties conjectured to hold for nice lattice polytopes (see for example \[13\], Conjecture 2). At the 2017 MSRI Introductory Workshop to the Semester on Geometric and Topological Combinatorics, the first author gave a talk on problems and progress in Ehrhart positivity and the second author gave an analogous talk on Ehrhart unimodality. Following these talks, A. Postnikov posed the question as to whether or not there exists any implication between the two problems. The purpose of the note is to answer this question in each dimension. In doing so, we also address this question for some major families of well-studied lattice polytopes.

In the remainder of this note, we show by way of examples in each dimension greater than two that there is no general implication between Ehrhart positivity and Ehrhart unimodality. In doing so, we study the relationship between Ehrhart positivity and Ehrhart unimodality by way of the fundamental examples associated to each property. We determine whether or not classic examples of one property satisfy the other. In section 2 we summarize the lattice polytopes that are known (or conjectured) to be both Ehrhart positive and Ehrhart unimodal. In section 3 we present lattice polytopes that are Ehrhart positive but not Ehrhart unimodal. In doing so, we provide an answer to an open problem posed in \[3\]. In section 4 we present examples that are not Ehrhart positive but are Ehrhart unimodal. Finally, in section 5 we describe families of lattice polytopes that are neither Ehrhart positive nor Ehrhart unimodal. The examples considered here are all classic families of polytopes, including zonotopes, matroid polytopes, simplices of weighted projective spaces, empty lattice simplices, smooth polytopes, and $s$-lecture hall simplices.

1.1. Preliminaries. Before we begin, we briefly catalogue some basic facts about Ehrhart theory that will be used in the remainder of this note. Let $P \subset \mathbb{R}^n$ be a $d$-dimensional lattice polytope. The first important fact we need is that the Ehrhart polynomial of $P$ can be recovered from its $h^*$-polynomial $h^*(P; z) = \sum_{j=0}^{d} h^*_j z^j$ by way of the formula

\[ i(P; t) = \sum_{j=0}^{d} h^*_j \binom{t + d - j}{d}. \]
Next, recall from the introduction that the coefficients of $h^*(P; z)$ are known to be nonnegative integers [29]. Going a step beyond this, some of the coefficients of $h^*(P; z)$ have very simply-stated formulae. In particular, we know that

$$h^*_0 = 1, \quad \text{and} \quad h^*_1 = |P \cap \mathbb{Z}^n| - (d + 1).$$

Finally, in the following we will utilize some nice implications that hold between properties of the roots of a univariate polynomial and the distribution of its coefficients. A univariate polynomial is called real-rooted if all of its roots are real numbers. It turns out that if this polynomial further has nonnegative coefficients then it is unimodal [8, Theorem 1.2.1]. In the following, we will often use the fact that a given $h^*$-polynomial has only real-roots to recover Ehrhart unimodality.

2. Polytopes that are Ehrhart Positive and Ehrhart Unimodal

The major conjectures on Ehrhart positivity and Ehrhart unimodality are naturally aimed at positive results; i.e., they purport that a given family of lattice polytopes satisfies the desired property. Consequently, to identify families of lattice polytopes in each dimension that are both Ehrhart positive and Ehrhart unimodal, it suffices to compare the positive results in both fields and identify where they overlap. Moreover, substantially difficult conjectures on Ehrhart positivity are often stated in parallel to equally challenging conjectures on Ehrhart unimodality. In subsection 2.2 we catalogue the families of lattice polytopes that are known to be both Ehrhart positive and Ehrhart unimodal. Then, in subsection 2.3 we review which families of lattice polytopes are further conjectured to satisfy both properties. However, we first begin by assessing our question in dimension two; i.e. the case of all lattice polygons.

2.1. Dimension two: the lattice polygons. Since the goal of this note is to assess the relationship (or lack thereof) between Ehrhart unimodality and Ehrhart positivity in each dimension, then it is natural to first consider our question in dimension two. In fact, dimension two turns out to be the only dimension in which there is a definitive relationship between these two properties! A two-dimensional lattice polytope is often called a lattice polygon. It follows from Pick’s Theorem [3, Theorem 2.8] that if $P \subset \mathbb{R}^2$ is a lattice polygon then

$$i(P; t) = At^2 + \frac{1}{2}Bt + 1 \quad \text{and} \quad h^*(P; z) = \left( A - \frac{B}{2} + 1 \right) z^2 + \left( A + \frac{B}{2} - 2 \right) z + 1,$$

where $A$ denotes the area of $P$ and $B$ denotes the number of lattice points on the boundary of $P$. It can be seen directly from these formulae that $P$ is both Ehrhart positive and Ehrhart unimodal. In particular, Ehrhart unimodality follows from the observation that $A \geq 1/2$ and $B \geq 3$.

Remark 2.1. Since lattice polygons are both Ehrhart positive and Ehrhart unimodal, in what remains we only consider examples in dimensions greater than two.

2.2. Known families. For dimensions greater than two, perhaps the simplest example of a lattice polytope that is both Ehrhart positive and Ehrhart unimodal is the standard $d$-simplex, which is the convex hull

$$\Delta_d := \text{conv}(e_1, \ldots, e_d, 0) \subset \mathbb{R}^d,$$
where $e_1, \ldots, e_d$ denote the standard basis vectors and $0$ denotes the origin in $\mathbb{R}^d$. It is well-known [3, Chapter 2.3] that

$$i(\Delta_d; t) = \binom{t + d}{d} \quad \text{and} \quad h^*(\Delta_d; z) = 1,$$

from which it is straightforward to see that $\Delta_d$ is both Ehrhart positive and Ehrhart unimodal. A second famous example is the $d$-dimensional cross-polytope, which is defined as the convex hull

$$\Diamond_d := \text{conv}(e_1, \ldots, e_d, -e_1, \ldots, -e_d) \subset \mathbb{R}^d.$$

The Ehrhart polynomial and the $h^*$-polynomial of $\Diamond_d$ are, respectively [3, Chapter 2.5],

$$i(\Diamond_d; t) = \sum_{k=0}^{d} \binom{d}{k} \binom{t - k + d}{d} \quad \text{and} \quad h^*(\Diamond_d; z) = (z + 1)^d.$$

Since $h^*(P; z)$ is seen to be real-rooted, the unimodality of its coefficients follows from the discussion in subsection 1.1. At the same time, [30, Exercise 4.61(b)] demonstrates that every zero of $i(\Diamond_d; t)$ has real part $-\frac{1}{2}$. Therefore, $i(\Diamond_d; t)$ is a product of polynomials of the form

$$t + \frac{1}{2} \quad \text{and} \quad t^2 + t + \frac{1}{4} + a^2,$$

for some $a \in \mathbb{R}$, and so $\Diamond_d$ is also Ehrhart positive.

Finally, perhaps the most substantial family of lattice polytopes that are known to be both Ehrhart positive and Ehrhart unimodal, there exist other large families of lattice polytopes that are conjectured to satisfy both conditions. One substantial family of such polytopes are the **matroid polytopes**. Recall that a matroid $M$ is a finite collection $I$ of subsets of $[d] := \{1, \ldots, d\}$ that satisfies the following three properties:

1. $\emptyset \in I$,
2. if $A \in I$ and $B \subseteq A$ then $B \in I$, and
3. if $A, B \in I$ and $|A| = |B| + 1$ then there exists $i \in A \setminus B$ so that $B \cup \{i\} \in I$.

2.3. **Conjectured families.** While zonotopes constitute the major family of lattice polytopes known to be both Ehrhart positive and Ehrhart unimodal, there do exist other large families of lattice polytopes that are conjectured to satisfy both conditions. One substantial family of such polytopes are the **matroid polytopes**.
The elements of \( \mathcal{I} \) are called the \textit{independent sets} of \( M \) and the inclusion-maximal independent sets are called its \textit{bases}. If \( \mathcal{B} \) denotes the collection of bases of a matroid \( M \), then we defined the matroid polytope for \( M \) to be the convex hull

\[
P(M) := \text{conv} \left( \sum_{i \in B} e_i : B \in \mathcal{B} \right) \subset \mathbb{R}^d.
\]

Conjecture 2.1. [13, Conjecture 2] For any matroid \( M \), the matroid polytope \( P(M) \) is both Ehrhart positive and Ehrhart unimodal.

So far, both aspects of this conjecture have remained elusive despite various attempts and partial results [10, 11, 18]. In general, families for which it is easy to prove Ehrhart unimodality may not be amenable to proofs of Ehrhart positivity (or vice versa). For instance, a family of lattice simplices known as the \textit{simplices for base-}\( r \text{-numeral systems} \), whose combinatorics are tied to representations of integers in the base-\( r \text{-numeral system}, \) were recently shown to have real-rooted (and therefore unimodal) \( h^* \)-polynomials [28, Theorem 4.5]. Given an integer \( r \geq 2 \), the \textit{base-}\( r \text{-d-simplex} \) is defined to be

\[
\mathcal{B}_{(r,d)} := \text{conv} \left( e_1, \ldots, e_d, - \sum_{k=1}^{d} (r-1)r^{k-1} e_k \right) \subset \mathbb{R}^d.
\]

Based on observed data, the author of [28], further conjectured that such simplices also satisfy Ehrhart positivity.

Conjecture 2.2. [28, Section 5] For \( r \geq 2 \) and \( d \geq 1 \), the base-\( r \text{-d-simplex is Ehrhart positive.} \)

3. Polytopes that are Ehrhart Positive but not Ehrhart Unimodal

In this section, we present lattice polytopes in each dimension greater than two that are Ehrhart positive but not Ehrhart unimodal. To start, we define for every weakly increasing vector of positive integers \( q = (q_1, \ldots, q_d) \in \mathbb{R}^d \) a lattice \( d \)-simplex

\[
\Delta_{(1,q)} := \text{conv}(e_1, \ldots, e_d, -q) \subset \mathbb{R}^d.
\]

These lattice simplices have been studied extensively from the perspective of Ehrhart unimodality [5, 6, 23, 28]. For instance, when \( q = ((r−1), (r−1)r, \ldots, (r−1)r^{d−1}) \) for some \( r \geq 2 \), then \( \Delta_{(1,q)} \) is the base-\( r \text{-d-simplex } \mathcal{B}_{(r,d)} \) described in section 2. Moreover, in [23] it is shown that for special choices of \( q \), the \( h^* \)-polynomial of \( \Delta_{(1,q)} \) is non-unimodal. These examples refute (in all dimensions greater than 5) the conjecture of Hibi [16] that every Gorenstein lattice polytope has a unimodal \( h^* \)-polynomial.

Theorem 3.1. [23] Let \( r \geq 0 \), \( s \geq 3 \), and \( k \geq r + 2 \) be integers. If

\[ q = (q_1, \ldots, q_d) = \left( \frac{1,1,\ldots,1,s,s,\ldots,s}{sk-1 \text{ times } r+1 \text{ times}} \right), \]

then

\[
h^*(\Delta_{(1,q)};z) = (1 + z^k + z^{2k} + \cdots + z^{(s-1)k})(1 + z + z^2 + \cdots + z^{k+r}).
\]

Therefore, \( \Delta_{(1,q)} \) is not Ehrhart unimodal.
On the other hand, we can show that for every $q$ as in Theorem 3.1 the simplex $\Delta_{(1,q)}$ is Ehrhart positive. This constitutes a new class of Ehrhart positive lattice polytopes, and this is the first such class of polytopes that are known to be Ehrhart positive but not Ehrhart unimodal.

**Theorem 3.2.** For integers $r \geq 0$, $s \geq 3$, and $k \geq r + 2$, let $\Delta_{(1,q)}$ be defined as in Theorem 3.1. Then $\Delta_{(1,q)}$ is Ehrhart positive.

**Proof.** Notice that the zeros of $h^*(\Delta_{(1,q)}; z)$ are all on the unit circle \( \{ z \in \mathbb{C} : |C| = 1 \} \) of the complex plane. It then follows from the main result in [24] or Theorem 3.2 of [32] that each zero of $i(\Delta_{(1,q)}; t)$ has real part $-\frac{1}{2}$. Therefore, the conclusion follows from the same discussion we give for cross-polytopes in subsection 2.2. □

The proof of Theorem 3.2 uses the same technique used to prove Ehrhart positivity of the $d$-dimensional cross-polytope $\diamondsuit_d$, discussed in Section 2. This special technique for proving Ehrhart positivity is the focus of an open problem posed in [3]. In particular, Theorem 3.2 provides one answer to [3, Open Problem 2.43].

### 3.1. Low dimensions.

As stated in section 1, the goal of this note is to assess the relationship between Ehrhart unimodality and Ehrhart positivity in each dimension greater than two. Since Theorems 3.1 and 3.2 only covers dimensions greater than five, it remains to identify examples of lattice polytopes that are Ehrhart positive but not Ehrhart unimodal in dimensions 3, 4, and 5. For these three dimensions, we then consider the Reeve’s tetrahedron, a well-known 3-dimensional lattice simplex defined as follows: Given a positive integer $h \geq 1$, define the Reeve’s tetrahedron $R_h := \text{conv}((0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, h)) \subset \mathbb{R}^3$.

It is well-known [3, Example 3.22] that the Reeve’s tetrahedron $R_h$ has Ehrhart polynomial $i(R_h; t) = \frac{h}{6} t^3 + t^2 + \left( \frac{12 - h}{6} \right) t + 1,$ and $h^*$-polynomial $h^*(R_h; z) = 1 + (h - 1)z^2.$

In particular, for $2 \leq h \leq 11$ the Reeve’s tetrahedron $R_h$ is Ehrhart positive but not Ehrhart unimodal. These examples settle the question in dimension 3. We can then lift this example into dimensions 4 and 5 by way of lattice pyramids over $R_h$.

If $P \subset \mathbb{R}^n$ is a lattice polytope, then the lattice pyramid over $P$ is the polytope

\[ \text{Pyr}(P) := \text{conv}(P \times \{0\}, e_{n+1}) \subset \mathbb{R}^{n+1}. \]

A well-known fact in Ehrhart theory is that $h^*(P; z) = h^*(\text{Pyr}(P); z)$ [1]. We let $\text{Pyr}^k(P)$ denote the $k$-fold pyramid over the lattice polytope $P$. Using the software Polymake [15] one can quickly compute that the four and five dimensional lattice pyramids $\text{Pyr}(R_h)$ and $\text{Pyr}^2(R_h)$ are both Ehrhart positive. This provides the remaining two examples needed to complete our objective in this section.

**Remark 3.1.** The lattice polytopes $R_h$, $\text{Pyr}(R_h)$, $\text{Pyr}^2(R_h)$, and those identified in Theorems 3.1 and 3.2 collectively demonstrate that in each dimension greater than two there exist lattice polytopes that are Ehrhart positive but not Ehrhart unimodal.
While our low-dimensional examples may suggest that the lattice pyramid operation preserves Ehrhart positivity (in addition to Ehrhart unimodality), we will see in the coming two sections that in fact quite the opposite is true.

4. Polytopes that are not Ehrhart Positive but are Ehrhart Unimodal

In this section, we demonstrate that there exist families of lattice polytopes that are Ehrhart positive but not Ehrhart unimodal in each dimension greater than two. We begin by showing, in subsection 4.1, that there exists a smooth polytope such that for all $i \in [n]$ the lattice point $v + bu_i$ is in $P$ but is not a vertex of $P$. The chiseling off of the vertex $v$ at distance $b$ from $P$ is the polytope

$$P' = \text{conv}\left((\text{Vert}(P) \setminus \{v\}) \cup \{v + bu_1, \ldots, v + bu_d\}\right).$$

For each edge $e$ of $P$ define the lattice edge length of $e$ to be $\ell(e) := |e \cap \mathbb{Z}^d| + 1$, and let $\ell_P := \min(\ell(e) : e$ is an edge of $P)$. For any integer $b < \frac{\ell_P}{2}$, we can define the full chiseling of $P$ to be the smooth polytope $\text{ch}(P,b)$ produced by chiseling every vertex of $P$ at distance $b$. Using the theory of half-open decompositions of lattice polytopes and the results of [12], we can find a smooth lattice polytope in each dimension greater than two that is Ehrhart unimodal but not Ehrhart positive.

4.1. Smooth polytopes. A $d$-dimensional lattice polytope $P$ is smooth if every vertex of $P$ is contained in precisely $d$ edges and the primitive edge directions form a lattice basis for $\mathbb{Z}^d$. In [12] the authors used the concept of chiseling smooth lattice polytopes to obtain smooth polytopes with negative Ehrhart coefficients. Let $P$ be a smooth lattice polytope of dimension $d$ with vertex set $\text{Vert}(P)$. Suppose that $v$ is a vertex of $P$ with primitive edge directions $u_1, \ldots, u_d$, and suppose also that there is an integer $b \in \mathbb{Z}_{>0}$ such that for all $i \in [n]$ the lattice point $v + bu_i$ is in $P$ but is not a vertex of $P$. The chiseling off of the vertex $v$ at distance $b$ from $P$ is the polytope

$$P' = \text{conv}\left((\text{Vert}(P) \setminus \{v\}) \cup \{v + bu_1, \ldots, v + bu_d\}\right).$$

For each edge $e$ of $P$ define the lattice edge length of $e$ to be $\ell(e) := |e \cap \mathbb{Z}^d| + 1$, and let $\ell_P := \min(\ell(e) : e$ is an edge of $P)$. For any integer $b < \frac{\ell_P}{2}$, we can define the full chiseling of $P$ to be the smooth polytope $\text{ch}(P,b)$ produced by chiseling every vertex of $P$ at distance $b$. Using the theory of half-open decompositions of lattice polytopes and the results of [12], we can find a smooth lattice polytope in each dimension greater than two that is Ehrhart unimodal but not Ehrhart positive.

For a $d$-dimensional lattice polytope $P$, let $P = P_1 \cup \cdots \cup P_m$ be a decomposition of $P$ into lattice polytopes $P_1, \ldots, P_m$. i.e., every $P_i$ is a $d$-dimensional lattice polytope such that $P_i$ does not intersect the relative interior of $P_j$ for any $j \neq i$. We say that a point $\omega \in \mathbb{R}^d$ is in general position with respect to the decomposition $P_1 \cup \cdots \cup P_m$ if $\omega$ does not lie in any facet-defining hyperplane for any $P_i$, $i \in [m]$. We further say that $\omega$ is beyond a facet $F$ of $P_i$ if the facet-defining hyperplane for $F$ separates $\omega$ from the relative interior of $P_i$. Let $\text{Bey}(P_i, \omega)$ denote the collection of facets $F$ of $P_i$ for which $\omega$ is beyond $F$. Then the half-open polytope associated to $P_i$ and $\omega$ is

$$\mathbb{H}_\omega P_i := P_i \setminus \bigcup_{F \in \text{Bey}(P_i, \omega)} F.$$

**Lemma 4.1.** [19, Theorem 3] Let $P$ be a lattice polytope and let $P = P_1 \cup \cdots \cup P_m$ be a decomposition of $P$ into lattice polytopes. If $\omega \in P$ is in general position with respect to $P_1 \cup \cdots \cup P_m$ then $P$ is the disjoint union

$$P = \mathbb{H}_\omega P_1 \cup \cdots \cup \mathbb{H}_\omega P_m,$$
The same chiseling applied to \([-1,1]^d\) for \(d \geq 7\) will fail to be Ehrhart positive but will still be Ehrhart unimodal.

\[
 h^*(P; z) = h^*(\mathbb{H}_0 P_1; z) + \cdots + h^*(\mathbb{H}_m P_m; z).
\]

The results of [12] and Lemma 4.1 allow us to identify the desired smooth lattice polytopes. In fact, the resulting examples have a very classical combinatorial flavor.

**Theorem 4.2.** For \(d \geq 7\), the chiseling \(\text{ch}([-1,1]^d,1)\) of the \(d\)-dimensional cube \([-1,1]^d\) is Ehrhart unimodal but not Ehrhart positive.

**Proof.** It follows from the proof of [12, Proposition 1.3] that the linear term of 
\[
 i(\text{ch}([-1,1]^d,1); t) = 2d - \frac{2^d}{d},
\]
which is seen to be negative for \(d \geq 7\). Therefore, we need only observe that the \(h^*\)-polynomial of \(\text{ch}([-1,1]^d,1)\) is unimodal. To see this, we first recall that the \(h^*\)-polynomial of the cube \([-1,1]^d\) is the Type B Eulerian polynomial, which is well-known to be real-rooted and unimodal [9]. Recall from subsection 1.1 that the linear coefficient of the \(h^*\)-polynomial of a \(d\)-dimensional lattice polytope \(P\) is always \(|P \cap \mathbb{Z}| - (d + 1)\). Thus, we know that
\[
 [z^k] h^*([-1,1]^d; z) = 3^d - (d + 1),
\]
(Here, \([z^k]. f(z)\) denotes the coefficient of \(z^k\) in the polynomial \(f(z)\).) Now, for each vertex \(v\) of \([-1,1]^d\) with primitive edge directions \(u_1, \ldots, u_d\), define the unimodular \(d\)-simplex \(S_v := \text{conv}(v, v + u_1, \ldots, v + u_d)\). It follows that \([-1,1]^d\) admits the decomposition into lattice polytopes
\[
 [-1,1]^d = \text{ch}([-1,1]^d,1) \cup \bigcup_{v \in \text{Vert}([-1,1]^d)} S_v,
\]
and the origin \(0 \in \mathbb{R}^d\) is in general position with respect to this decomposition for all \(d \geq 2\). Thus, when \(d \geq 2\), \(0\) is beyond no facet of \(\text{ch}([-1,1]^d,1)\), and for all \(v \in \text{Vert}([-1,1]^d)\), \(0\) is only beyond the facet \(F_v\) of \(S_v\) that does not contain \(v\). In particular, 
\[
 h^*(\mathbb{H}_0 \text{ch}([-1,1]^d,1); z) = h^*(\text{ch}([-1,1]^d,1); z),
\]
and for all \(v\) in
Theorem 4.3. For every $h^*(\mathbb{R}_0 S^n; z) = z$; i.e., the $h^*$-polynomial of the standard $d$-simplex with precisely one facet removed. It then follows from Lemma 1.1 that

$$h^*(\text{ch}([-1,1]^d,1); z) = h^*([-1,1]; z) - 2^d z.$$ 

Since $3^d - 2^d - (d + 1) > 0$ for all $d \geq 7$ and $h^*([-1,1]; z)$ is unimodal, it follows that $h^*(\text{ch}([-1,1]^d,1); z)$ is also unimodal. Thus, for every dimension $d \geq 7$, $\text{ch}([-1,1]^d,1)$ is Ehrhart unimodal but not Ehrhart positive.

4.2. $s$-Lecture hall simplices. A well-studied family of examples of Ehrhart unimodal lattice polytopes are the $s$-lecture hall simplices [26]. Let $s := (s_k)_{k=1}^d$ be a sequence of positive integers. The $d$-dimensional $s$-lecture hall simplex is

$$P_d^s := \left\{ x \in \mathbb{R}^d : 0 \leq \frac{x_1}{s_1} \leq \frac{x_2}{s_2} \leq \cdots \leq \frac{x_d}{s_d} \leq 1 \right\} \subset \mathbb{R}^d.$$ 

The $h^*$-polynomial of $P_d^s$ is called the $s$-Eulerian polynomial, and it enumerates the $s$-inversion sequences

$$J_d^s := \left\{ m \in \mathbb{Z}^d : 0 \leq m_i < s_i \right\}$$

by their number of $s$-ascents; that is, the value

$$\text{asc}_s(m) := \left\{ i \in \{0,1,\ldots,d-1\} : \frac{m_i}{s_i} < \frac{m_{i+1}}{s_{i+1}} \right\},$$

for $m \in J_d^s$, with the convention that $m_0 := 0$ and $s_0 := 1$ [26]. In other words,

$$h^*(P_d^s; z) = \sum_{m \in J_d^s} z^{\text{asc}_s(m)}.$$ 

(2)

In the case that $s = (1,2,\ldots,d)$, the $s$-Eulerian polynomial is the classic Eulerian polynomial, which enumerates the permutations of $[n]$ by the descent statistic. It was shown in [27] that for all choices of $s$ and $d$, the $h^*$-polynomial of $P_d^s$ is real-rooted and therefore unimodal. The $s$-lecture hall simplices are a combinatorially rich family of lattice polytopes (see for example [25]), so it is natural to ask whether or not they are of interest from the perspective of Ehrhart positivity as well. In fact, as we see with the following theorem, there exist infinitely many $s$-lecture hall simplices, even in low dimensions, that are not Ehrhart positive. In the following, for positive integers $a, b, k_2$ and nonnegative integers $k_1, k_3$, we write $(1^{k_1}, a, 1^{k_2}, b, 1^{k_3})$ for the vector $(1, \ldots, 1, a, 1, \ldots, 1, b, 1, \ldots, 1)$.

Theorem 4.3. For every $d \geq 3$, there exist $s$-lecture hall simplices of the form $P_d^{(1^{k_1}, a, 1^{k_2}, b, 1^{k_3})}$ that are not Ehrhart positive.

We will use the following lemma to prove the above theorem, which allows us to write $h^*(P_d^{(1^{k_1}, a, 1^{k_2}, b, 1^{k_3})}; z)$ explicitly in terms of the parameters $a$ and $b$.

Lemma 4.4. For any positive integers $a, b, k_2$, nonnegative integers $k_1, k_3$ and integer $d \geq 3$, we have

$$h^*(P_d^{(1^{k_1}, a, 1^{k_2}, b, 1^{k_3})}; z) = (1 + (a - 1)z)(1 + (b - 1)z).$$ 

(3)

Proof. It follows from equation (2) that

$$h^*(P_d^{(1^{k_1}, a, 1^{k_2}, b, 1^{k_3})}; z) = h^*(P_d^{(a, 1^{k_2}, b)}; z).$$
Thus it suffices to prove the statement for \( s = (a, 1^{d-2}, b) \). Suppose \( m \in J_d^s \). Then \( m_1 \in \{0, 1, \ldots, a-1\}, m_d = \{0, 1, \ldots, b-1\} \), and \( m_i = 0 \) for \( 2 \leq i \leq d-1 \). Notice that only 0 and \( d-1 \) can be \( s \)-ascents. Furthermore, 0 is an \( s \)-ascent in \( m \) if and only if \( m_1 \neq 0 \), and \( d-1 \) is an \( s \)-ascent in \( m \) if and only if \( m_d \neq 0 \). Therefore, \( h^s(P_d^{(a,1^{d-2},b)}; z) = \sum_{m \in J_d^s, \sum \text{asc}_s(m) = a-1} z^{\text{asc}_s(m)} = \sum_{m_1=0}^{a-1} z f(m_1) \sum_{m_d=0}^{b-1} z f(m_d), \) where \( f(x) = 0 \) if \( x = 0 \), and \( f(x) = 1 \) if \( x > 0 \). We then see that the right-hand-side of the above equation becomes \((1 + (a-1)z)(1 + (b-1)z)\). \( \square \)

The proof of Theorem 4.3 is then given as follows.

Proof of Theorem 4.3. We will show that for any integer \( a > d \), the \( s \)-lecture hall simplex \( P_d^{(1^{k+1},a,1^{k+2},b,1^{k+1})} \) is Ehrhart positive for sufficiently large integer \( b \). It follows from Lemma 4.4 and Formula (1) that for any integers \( a, b > 1 \)
\begin{equation}
\begin{aligned}
i(P_d^{(1^{k+1},a,1^{k+2},b,1^{k+1})}; t) - i(P_d^{(1^{k+1},a,1^{k+2},b,1^{k+1})}; t) & = \left( t + \frac{d-1}{d} \right) + (a-1) \left( t + \frac{d-2}{d} \right),
\end{aligned}
\end{equation}
which is independent of the parameter \( b \). Therefore, it suffices to show that the linear term of the above expression is always negative. However,
\begin{equation}
\begin{aligned}
[t]. \left( t + \frac{d-1}{d} \right) + (a-1) \left( t + \frac{d-2}{d} \right) & = \frac{d-a}{d(d-1)}t,
\end{aligned}
\end{equation}
which is negative if \( a > d \). This completes the proof. \( \square \)

Since all \( s \)-lecture hall simplices in Theorem 4.3 have lattice width one, we can think of them as thin \( s \)-lecture hall simplices. In the special case of Theorem 4.3 when \( d = 3 \), we can further use Lemma 4.4 to explicitly compute the Ehrhart polynomial \( i(P_3^{(a,1,b)}; t) \), from which we can identify a spectrahedral cone containing all lattice points \((a,b) \in \mathbb{Z}_{\geq 0}^2 \) such that \( P_3^{(a,1,b)} \) is not Ehrhart positive.

Corollary 4.5. Suppose that \( s = (a,1,b) \). Then the 3-dimensional \( s \)-lecture hall simplex \( P_3^s \) has Ehrhart polynomial
\begin{equation}
\begin{aligned}
i(P_3^s; t) & = \left( \frac{ab}{6} \right) t^3 + \left( \frac{a+b}{2} \right) t^2 + \left( \frac{6 + 3(a+b) - ab}{6} \right) t + 1.
\end{aligned}
\end{equation}
In particular, all such \( s \)-lecture hall simplices that are not Ehrhart positive are given by the lattice points \((a,b) \in \mathbb{Z}_{\geq 0}^2 \) satisfying \( 6 + 3(a+b) - ab < 0 \).

Figure 2 depicts the spectrahedral cone in \( \mathbb{R}^2 \) defined by the linear term of \( i(P_3^s; t) \) in Corollary 4.5 that captures the collection of lattice points \((a,b) \) yielding \( P_3^s \) that are not Ehrhart positive. From this picture we can see that the vast majority of \( s \)-lecture hall 3-simplices that are thin in the second coordinate are not Ehrhart positive. We also see from this corollary that the Ehrhart polynomials of these \( s \)-lecture 3-simplices are similar to that of the Reeve’s tetrahedra, a family of lattice 3-simplices that were introduced in subsection 3.1. We end this section with a geometric remark that further connects the \( s \)-lecture hall 3-simplices \( P_3^{(a,1,b)} \) to the Reeve’s tetrahedron.
Figure 2. On the left we see the thin lecture hall simplex \( P_{3}^{(7,1,7)} \), and on the right we see the cone of lattice points \((s_1, s_3)\) for which \( P_{3}^{(s_1,1,s_3)} \) is not Ehrhart positive.

Remark 4.1 (Pyramids and \( s \)-lecture hall simplices). Recall from Subsection 3.1 that if \( \text{Pyr}^k(P) \) is the \( k \)-fold lattice pyramid over a lattice polytope \( P \) then \( h^*(P; z) = h^*(\text{Pyr}(P); z) \). Given an \( s \)-lecture hall simplex \( P_{d}^{(s_1,\ldots,s_d)} \), notice that the \( s \)-lecture hall simplices \( P_{d+1}^{(1,s_1,\ldots,s_d)} \) and \( P_{d+1}^{(s_1,\ldots,s_d,1)} \) are lattice pyramids over \( P_{d}^{(s_1,\ldots,s_d)} \). This gives an alternative method by which to observe that
\[
h^*(P_{d}^{(1^{s_1},a,1^{s_2},b,1^{s_3})}; z) = h^*(P_{d}^{(a,1^{s_2},b)}; z),
\]
a fact that we used in the proof of Theorem 4.3. On the other hand, \( P_{d}^{(a,1^{d-2},b)} \) is not a \((d-2)\)-fold lattice pyramid over \( P_{2}^{(a,b)} \), as can already be seen for \( d = 3 \) in the left-hand-side of Figure 2. These examples demonstrate how lattice pyramids can be used to recover non-Ehrhart positive lattice polytopes in high dimensions with a chosen Ehrhart \( h^* \)-polynomial. Analogous to these \( s \)-lecture hall simplices, in the coming section, we will use the pyramid construction in relation to the Reeve’s tetrahedra to derive our final collection of examples.

5. Polytopes that are neither Ehrhart Positive nor Ehrhart Unimodal

In this section we present a family of lattice polytopes containing polytopes in each dimension greater than two that are neither Ehrhart positive nor Ehrhart unimodal. Analogous to the previous sections, these polytopes also have a nice geometric construction that relies on fundamental examples and tools used frequently in polyhedral geometry and Ehrhart theory. Recall that a lattice simplex is called empty if it contains no lattice points apart from its vertices. In the following, we show that there exist infinitely many empty lattice simplices in each dimension greater than two that are neither Ehrhart positive nor Ehrhart unimodal.

5.1. Empty simplices. In dimension three there exists a well-known family of empty lattice simplices that are neither Ehrhart positive nor Ehrhart unimodal. This family is collectively known as the Reeve’s tetrahedra, which we introduced
in subsection 3.1. Recall from subsection 3.1 that the Reeve’s tetrahedron $R_h$ has $h^*$-polynomial
\[ h^*(R_h; z) = 1 + (h - 1)z^2. \]
Thus, the Reeve’s tetrahedron $R_h$ exhibits the special shape of the $h^*$-polynomial of empty simplices; namely, $h^*_1 = 0$ for any empty lattice simplex. Moreover, any nonunimodular empty lattice $d$-simplex will not be Ehrhart unimodal. In addition, recalling that
\[ i(R_h; t) = \frac{h}{6}t^3 + t^2 + \left(\frac{12 - h}{6}\right)t + 1, \]
we see that for any $h \geq 13$ the Reeve’s tetrahedron $R_h$ will also not be Ehrhart positive. The following theorem shows that both phenomena can be lifted into higher dimensions using the techniques we applied to $s$-lecture hall simplices in subsection 4.2, and the pyramid construction defined in subsection 3.1.

**Theorem 5.1.** For $d \geq 3$ let
\[ H := \left\lceil \frac{1}{(d-2)!} \left\lfloor \frac{d+1}{2} \right\rfloor \right\rceil + 1, \]
where $\left\lfloor \frac{n}{k} \right\rfloor$ denotes the unsigned Stirling number of the first kind. For all $h \geq H$ the empty lattice $d$-simplex
\[ \text{Pyr}_{d-3}(R_h) \]
is neither Ehrhart positive nor Ehrhart unimodal.

**Proof.** The proof of this theorem is another application of the techniques applied in subsection 4.2. First, recall that for all $d - 3 \geq 0$ the $(d - 3)$-fold pyramid over $R_h$ satisfies
\[ h^* \left( \text{Pyr}_{d-3}(R_h); z \right) = h^* \left( R_h; z \right) = 1 + (h - 1)z^2, \]
and therefore $\text{Pyr}_{d-3}(R_h)$ is not Ehrhart unimodal. Again by (1), we have
\[ i \left( \text{Pyr}_{d-3}(R_h); t \right) = \left( t + \frac{d}{d} \right) + (h - 1) \left( t + \frac{d - 2}{d} \right). \]
Hence,
\[ i \left( \text{Pyr}_{d-3}(R_{h+1}); t \right) - i \left( \text{Pyr}_{d-3}(R_h); t \right) = \left( t + \frac{d - 2}{d} \right). \]
Similar to the proof of Theorem 4.3 we can show that the linear term of the above expression is always negative. In particular, we notice that
\[ [t] \left( t + \frac{d - 2}{d} \right) = -\frac{1}{d(d - 1)} < 0. \]
It then follows that for some sufficiently large $h \in \mathbb{Z}_{>0}$, the lattice pyramid $\text{Pyr}_{d-3}(R_h)$ is not Ehrhart positive if and only if $h \geq H$. Thus, it remains to prove that the value $h$ is indeed $H$, as defined above. To check this, notice that
\[ [t]i \left( \text{Pyr}_{d-3}(R_d); t \right) = [t] \left( t + \frac{d}{d} \right) = \sum_{k=1}^{d} \frac{1}{k}. \]
So by the above argument we have that for all $h > 1$,
\[ [t]i \left( \text{Pyr}_{d-3}(R_h); t \right) = \sum_{k=1}^{d} \frac{1}{k} - \frac{h - 1}{d(d - 1)}. \]
Thus, the linear term of \( i \left( \text{Pyr}^{d-3}(R_h); t \right) \) is negative whenever
\[
\sum_{k=1}^{d} \frac{1}{k} - \frac{h - 1}{d(d - 1)} < 0.
\]

Since,
\[
\sum_{k=1}^{d} \frac{1}{k} = \frac{1}{d!} \left[ \frac{d + 1}{2} \right],
\]
this completes the proof. \(\square\)

6. Final Remarks

In this note, we examined the relationship between the properties of Ehrhart unimodality and Ehrhart positivity of lattice polytopes within each dimension greater than (or equal to) two. We focused on well-studied families of polytopes that were previously investigated with respect to one property but not the other. These families of polytopes included simplices of weighted projective spaces, smooth polytopes, \(s\)-lecture hall simplices, and empty lattice simplices arising as \(k\)-fold pyramids over the well-known Reeve’s tetrahedron. Through this analysis, we showed that in each dimension greater than two there is no relationship between the properties of Ehrhart unimodality and Ehrhart positivity. That is, in each such dimension there exists a lattice polytope that is (1) both Ehrhart positive and Ehrhart unimodal, (2) Ehrhart positive but not Ehrhart unimodal, (3) Ehrhart unimodal but not Ehrhart positive, and (4) neither Ehrhart positive nor Ehrhart unimodal. These results provide new examples in regards to both Ehrhart unimodality and Ehrhart positivity for well-studied families of lattice polytopes, and at the same time make explicit the relationship (or lack thereof) between these two properties with respect to dimension.

On the other hand, the results in this note do not exclude the possibility that there exist special families of polytopes for which there is some implication between Ehrhart unimodality and Ehrhart positivity. Such examples would be of general interest, since they would constitute a setting in which techniques for proving one property are utilizable in the analysis of the other. In this direction, one useful tool pointed out by the various examples in this paper that could pertain to such case analyses is the lattice pyramid operation. Suppose we are interested in analyzing the Ehrhart polynomials of a collection \(\Omega\) of lattice polytopes. The examples presented here suggest that if \(\Omega\) (or a subset thereof) is closed under lattice pyramids, then this operation can be used to identify members of \(\Omega\) exhibiting both Ehrhart positivity and non-Ehrhart positivity. This purports the lattice pyramid operation not only as a useful tool in analyzing the shape of \(h^*\)-polynomials, but also in assessing the likely validity of conjectures on Ehrhart positivity for special families of polytopes.

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