Supplementary material for ‘Pure single photons from a trapped atom source’

D. B. Higginbottom1,2,* L. Slodička3, G. Araneda2, L. Lachman3, R. Filip3, M. Hennrich2, and R. Blatt2,4
1Australian National University, Canberra ACT 0200, Australia
2Institut für Experimentalphysik, Universität Innsbruck, Technikerstr. 25, 6020 Innsbruck, Austria
3Department of Optics Palacký University, 17. Listopadu 12, 77146 Olomouc, Czech Republic and
4Institut für Quantenoptik und Quanteninformation, Österreichische Akademie der Wissenschaften, Technikerstraße 21a, 6020 Innsbruck, Austria

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Derivation of a QNG threshold

The quantum non-Gaussian (QNG) witness presented in this paper differs in form from similar thresholds presented before [1, 2]. The witness distinguishes optical QNG states by the probability \( P_c \) of coincident photon detections, conditioned on the probability \( P_s \) of single photon detection in an optical HBT. The derivation of this threshold follows the method of [2], with the threshold formulated in terms of observables directly measured in this experiment.

The Wigner function of the vacuum state is

\[
W_{\text{vac}}(x,p) = \frac{1}{2\pi} e^{-\frac{x^2+p^2}{2}},
\]

where \( x \) and \( p \) are generalized coordinates for position and momentum. The Wigner function of a general, pure squeezed state is reached by squeezing the variance of the \( x \) and \( p \) coordinates by parameter \( V \)

\[
W_{V}(x,p) = W_{\text{vac}} \left( \frac{x}{\sqrt{V}}, \sqrt{V}p \right),
\]

followed by rotation by an angle \( \phi \)

\[
W_{V,\phi}(x,p) = W_{V}(x \cos \phi + p \sin \phi, -x \sin \phi + p \cos \phi)
\]

and displacement in \( x \) coordinate by distance \( r \)

\[
W_{V,\phi,r} = W_{V,\phi}(x-r, p).
\]

The probability \( P_n \) of measuring this general, pure squeezed state with photon number \( n \) is the overlap of the state with projectors \( W_n \) for the \( n \) photon Fock state. The projectors of the zero and one photon Fock states are [3]

\[
W_0(x,p) = 2e^{-\frac{x^2+p^2}{2}},
\]

\[
W_1(x,p) = 2(x^2+p^2-1)e^{-\frac{x^2+p^2}{2}}.
\]

Integration \( P_n = \int_{-\infty}^{\infty} W_n W_{V,\phi,r} \, dx \, dp \) yields the vacuum, single photon and higher order Fock state probabilities [4]

\[
P_0 = \frac{2\sqrt{V}}{1+V} e^{\frac{r(-1-V+\sqrt{1-V^2}-2\phi)}{4(1+V)}}
\]

\[
P_1 = \frac{r(1+V^2)-(V^2-1)\cos(2\phi))}{2(1+V)^2} P_0
\]

\[
P_{2+} = 1 - P_0 - P_1
\]

Taking into account the probability of false single photon measurements, in which two photons arrive simultaneously at the same detector, the measurement probabilities \( P_s \) and \( P_c \) are related to the Fock state probabilities by

\[
P_s = P_1 + (P_{2+}/2)
\]

\[
P_c = P_{2+}/2
\]

To derive the QNG threshold for these measured experimental probabilities we consider the linear functional

\[
F(a) = P_s + a P_c
\]

where \( a \) is a free parameter. Following the method of [2] we see that the squeezed states that maximise this linear, unconstrained function are the border states forming the QNG threshold in the directly measurable \( P_c, P_s \) parameter space. For a given \( P_c \) these states have the maximum \( P_s \) possible on the set of Gaussian states. The angle \( \phi \) between the squeezing and displacement directions appears only in the argument of the cosine, so \( F(a) \) is necessarily maximised by \( \phi = 0 \). The maximum of the functional also satisfies \( \frac{\partial F}{\partial V} = 0 \) and \( \frac{\partial F}{\partial r} = 0 \) which gives

\[
\frac{\partial P_s}{\partial V} = -a \frac{\partial P_c}{\partial V}
\]

\[
\frac{\partial P_c}{\partial r} = -a \frac{\partial P_s}{\partial r}
\]

Together these give the condition

\[
\frac{\partial P_s}{\partial V} \frac{\partial P_c}{\partial r} = \frac{\partial P_s}{\partial r} \frac{\partial P_c}{\partial V}
\]

*Electronic address: daniel.higginbottom@anu.edu.au
For $P_s$ and $P_c$ of our general squeezed state this condition evaluates to

$$V^2 + rV - 1 \over (1 + V)^6} e^{-V^2/2} = 0 \quad (15)$$

which is satisfied only when

$$r = 1 - V^2 \over V \quad (16)$$

with this $r-V$ relation the threshold $P_c, P_s$ values can be reduced to the pair of equations presented in the main body of the paper.

$$P_s = \frac{1}{2} + \frac{(1 - V(2 + V))e^{V^2/2}}{\sqrt{V(1 + V)^2}} \quad (17)$$

$$P_c = \frac{1}{2} - \frac{(1 + V^2)e^{V^2/2}}{\sqrt{V(1 + V)^2}} \quad (18)$$

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