NEW INEQUALITIES FOR \( n \)-TIME DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, we obtain several inequalities of Ostrowski type that the absolute values of \( n \)-time differentiable functions are convex.

1. INTRODUCTION

In 1938 Ostrowski \cite{14} obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is as follows.

**Theorem 1.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \([a, b]\) and let \( |f'(t)| \leq M \) for all \( t \in (a, b) \), then the following bound is valid

\[
|f(x) - \frac{1}{b-a} \int_a^b f(t)dt| \leq (b-a)M \left[ \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2}_{(1.1)}\right]
\]

for all \( x \in [a, b] \). The constant \( \frac{1}{4} \) is sharp in the sense that it cannot be replaced by a smaller one.

For applications of Ostrowski’s inequality to some special means and some numerical quadrature rules, we refer the reader to the recent paper \cite{9} by S.S. Dragomir and S. Wang who used integration by parts from

\[
\int_a^b p(x, t)f'(t)dt
\]

where \( p(x, t) \) is a Peano kernel given by

\[
p(x, t) = \begin{cases} 
    t-a, & t \in [a, x] \\
    t-b, & t \in (x, b] 
\end{cases}
\]

In \cite{18}, also A. Sofo and S.S. Dragomir extended the result (1.1) in the \( L^p \) norm.

Dragomir \cite{3-8} further extended the result \cite{11} to incorporate mappings of bounded variation, Lipschitzian and monotonic mappings.

Cerone *et al.* \cite{2} as well as Dedić *et al.* \cite{8} and Pearce *et al.* \cite{15} further extended the result (1.1) by considering \( n \)-times differentiable mappings on an interior point \( x \in [a, b] \). Furthermore, for recent results and generalizations concerning Ostrowski’s inequality see \cite{1}, \cite{10-13}, \cite{15} and \cite{17}.

In \cite{2}, Cerone, Dragomir and Roumeliotis proved the following results:

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Lemma 1. Let \( f : [a, b] \to \mathbb{R} \) be a mapping such that \( f^{(n-1)} \) is absolutely continuous on \([a, b]\). Then for all \( x \in [a, b] \) we have the identity:

\[
\int_a^b f(t) dt = \sum_{k=0}^{n-1} \left( \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} \right) f^{(k)}(x)
+ (-1)^n \int_a^b K_n(x, t) f^{(n)}(t) dt
\]

where the kernel \( K_n : [a, b]^2 \to \mathbb{R} \) is given by

\[
K_n(x, t) = \begin{cases} 
\frac{(t-a)^n}{n!} & \text{if } t \in [a, x] \\
\frac{(t-b)^n}{n!} & \text{if } t \in (x, b],
\end{cases}
\]

\( x \in [a, b] \) and \( n \) natural number, \( n \geq 1 \).

Corollary 1. With the above assumptions, we have the representation:

\[
\int_a^b f(t) dt = \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{(k+1)!} \left( \frac{b-a)^{k+1}}{2^{k+1}} f^{(k)} \left( \frac{a+b}{2} \right) \right)
+ (-1)^n \int_a^b M_n(t) f^{(n)}(t) dt
\]

where

\[
M_n(t) = \begin{cases} 
\frac{(t-a)^n}{n!} & \text{if } t \in [a, \frac{a+b}{2}] \\
\frac{(t-b)^n}{n!} & \text{if } t \in (\frac{a+b}{2}, b].
\end{cases}
\]

Corollary 2. With the above assumptions, we have the representation:

\[
\int_a^b f(t) dt = \sum_{k=0}^{n-1} \left( \frac{(b-a)^{k+1}}{(k+1)!} \left[ f^{(k)}(a) + (-1)^k f^{(k)}(b) \right] \right)
+ \int_a^b T_n(t) f^{(n)}(t) dt
\]

where

\[
T_n(t) = \frac{1}{n!} \left[ \frac{(b-t)^n + (-1)^n(t-a)^n}{2} \right],
\]

\( t \in [a, b] \).

In this paper, by using the some classical integral inequalities, Hölder and Power-Mean integral inequality, we establish some new inequalities for functions whose \( n-th \) derivatives in absolute value are convex functions. Our established results generalize some of those results proved in recent papers for functions whose derivatives in absolute value are convex functions.
2. MAIN RESULTS

Theorem 2. For \( n \geq 1 \), let \( f : [a, b] \to \mathbb{R} \) be \( n \)-time differentiable mapping and \( a < b \). If \( f^{(n)} \in L[a, b] \) and \( |f^{(n)}| \) is convex on \([a, b] \), then

\[
\frac{1}{n!(b-a)} \left\{ f^{(n)}(a) \left[ \frac{(x-a)^{n+1}[(n+2)(b-x) + (x-a)]}{(n+1)(n+2)} + \frac{(b-x)^{n+2}}{(n+2)} \right] + f^{(n)}(b) \left[ \frac{(b-x)^{n+1}[(n+2)(x-a) + (b-x)]}{(n+1)(n+2)} + \frac{(x-a)^{n+2}}{(n+2)} \right] \right\}. 
\]

Proof. From Lemma \( \text{II} \) and using the properties of modulus, we write

\[
\int_a^b f(t) \, dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \leq \int_a^b K_n(x,t) f^{(n)}(t) \, dt
\]

\[
= \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(t) \right| \, dt + \int_x^b \frac{(b-t)^n}{n!} \left| f^{(n)}(t) \right| \, dt
\]

\[
= \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)} \right| \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} \right) dt
\]

\[
+ \int_x^b \frac{(b-t)^n}{n!} \left| f^{(n)} \right| \left( \frac{b-t}{b-a} + \frac{t-a}{b-a} \right) dt.
\]

Since \( |f^{(n)}| \) is convex on \([a, b] \), we have

\[
\int_a^b f(t) \, dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \leq \frac{1}{n!} \left\{ \int_a^x \frac{(t-a)^n}{n!} \left| f^{(n)}(a) \right| + \frac{t-a}{b-a} \left| f^{(n)}(b) \right| \, dt \right. 
\]

\[
+ \int_x^b \frac{(b-t)^n}{n!} \left| f^{(n)}(a) \right| + \frac{t-a}{b-a} \left| f^{(n)}(b) \right| \, dt \right\}.
\]

On the other hand, we have

\[
\int_a^x (t-a)^n (b-t) \, dt = \frac{(x-a)^{n+1}[(n+2)(b-x) + (x-a)]}{(n+1)(n+2)},
\]

\[
\int_a^x (t-a)^{n+1} \, dt = \frac{(x-a)^{n+2}}{(n+2)},
\]

\[
\int_x^b (b-t)^{n+1} \, dt = \frac{(b-x)^{n+2}}{(n+2)},
\]

and

\[
\int_x^b (b-t)^n(t-a) \, dt = \frac{(b-x)^{n+1}[(n+2)(x-a) + (b-x)]}{(n+1)(n+2)}.
\]

This completes the proof. \( \square \)
Corollary 3. With the above assumptions, if we choose \( x = \frac{a+b}{2} \), then we get

\[
\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \right| 
\leq \frac{(b-a)^{n+1}}{2^n(n+1)!} \left[ \frac{|f^{(n)}(a)| + |f^{(n)}(b)|}{2} \right].
\]

Corollary 4. In Theorem 2 if we choose \( x = a \) and \( x = b \), respectively, we have

\[
(2.3) \quad \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \right| 
\leq \frac{(b-a)^{n+1}}{(n+2)!} \left[ (n+1) |f^{(n)}(a)| + |f^{(n)}(b)| \right].
\]

\[
(2.4) \quad \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(-1)^k(b-a)^{k+1}}{(k+1)!} f^{(k)}(b) \right| 
\leq \frac{(b-a)^{n+1}}{(n+2)!} \left[ |f^{(n)}(a)| + (n+1) |f^{(n)}(b)| \right].
\]

Corollary 5. Let the conditions of Theorem 2 hold. Then the following result is valid. Namely,

\[
(2.5) \quad \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| 
\leq \frac{(b-a)^{n+1}}{(n+1)!} \left[ \frac{|f^{(n)}(a)| + |f^{(n)}(b)|}{2} \right].
\]

Proof. Summing the inequalities (2.3) and (2.4) and by using the triangle inequality, we have the inequality (2.5). \( \square \)

Corollary 6. In Theorem 2 if we have \( n = 1 \), then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| 
\leq \frac{1}{(b-a)^2} \left\{ \left[ \frac{(x-a)^2 [3(b-x) + (x-a)]}{6} + \frac{(b-x)^3}{3} \right] |f'(a)| 
+ \left[ \frac{(b-x)^2 [3(x-a) + (b-x)]}{6} + \frac{(x-a)^3}{3} \right] |f'(b)| \right\}.
\]
Theorem 3. Let \( f : [a, b] \to \mathbb{R} \) be \( n \)-time differentiable mapping and \( a < b \). If \( f^{(n)} \in L[a, b] \) and \( |f^{(n)}|^q \) is convex on \([a, b] \), then we have the following inequalities:

\[
\int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \leq \frac{1}{n!(b-a)^\frac{1}{q}} \left\{ \left( \int_a^b (t-a)^n dt \right)^\frac{1}{n} \left( \int_a^b |f^{(n)}(t)|^q dt \right)^\frac{1}{q} \right. \\
+ \left. \left( \int_x^b (b-t)^n dt \right)^\frac{1}{n} \left( \int_x^b |f^{(n)}(t)|^q dt \right)^\frac{1}{q} \right\}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. From Lemma 11 we have

\[
\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right|
\leq \left| \int_a^b K_n(x, t)f^{(n)}(t)\,dt \right|
= \int_a^x \frac{(t-a)^n}{n!} |f^{(n)}(t)|\,dt + \int_x^b \frac{(b-t)^n}{n!} |f^{(n)}(t)|\,dt.
\]

By Hölder inequality, we obtain

\[
\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right|
\leq \frac{1}{n!} \left\{ \left( \int_a^x (t-a)^{np} dt \right)^\frac{1}{p} \left( \int_a^x |f^{(n)}(t)|^q dt \right)^\frac{1}{q} \right. \\
+ \left. \left( \int_x^b (b-t)^{np} dt \right)^\frac{1}{p} \left( \int_x^b |f^{(n)}(t)|^q dt \right)^\frac{1}{q} \right\}
\]

Since \( |f^{(n)}|^q \) is convex on \([a, b] \) and \( t = \frac{b-t}{b-a} + \frac{a-t}{b-a} \), we have

\[
\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right|
\leq \frac{1}{n!} \left\{ \left( \int_a^x (t-a)^{np} dt \right)^\frac{1}{p} \left( \int_a^x \frac{b-t}{b-a} |f^{(n)}(a)|^q + \frac{t-a}{b-a} |f^{(n)}(b)|^q dt \right)^\frac{1}{q} \right. \\
+ \left. \left( \int_x^b (b-t)^{np} dt \right)^\frac{1}{p} \left( \int_x^b \frac{b-t}{b-a} |f^{(n)}(a)|^q + \frac{t-a}{b-a} |f^{(n)}(b)|^q dt \right)^\frac{1}{q} \right\}
\]
valid. Namely, Corollary 9. Let the conditions of Theorem 3 hold. Then the following result is

Proof. Summing the inequalities (2.7) and (2.8) and by using the triangle inequality, we have the inequality (2.9). □

Corollary 7. Assume that \( f \) is as in Theorem 3. If we choose \( x = \frac{a+b}{2} \), then we have

\[
\begin{align*}
\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{1 + (-1)^k}{(k+1)!} \left( \frac{b-a}{2} \right)^{k+1} f^{(k)} \left( \frac{a+b}{2} \right) \right| \\
\leq \frac{(b-a)^{np+1+\frac{1}{q}}}{(np+1)n!} \left[ \frac{3 |f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{4} \right]^{\frac{1}{q}} + \left[ \frac{|f^{(n)}(a)|^q + 3 |f^{(n)}(b)|^q}{4} \right]^{\frac{1}{q}} .
\end{align*}
\]

Corollary 8. With the above assumptions, if we choose \( x = a \) and \( x = b \), respectively, we have

\[
\begin{align*}
(2.7) & \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{b-a}{(k+1)!} f^{(k)}(a) \right| \\
& \leq \frac{(b-a)^{np+1+\frac{1}{q}}}{(np+1)n!} \left[ \frac{|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{2} \right]^{\frac{1}{q}} ,
\end{align*}
\]

\[
\begin{align*}
(2.8) & \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(-1)^k(b-a)^{k+1}}{(k+1)!} f^{(k)}(b) \right| \\
& \leq \frac{(b-a)^{np+1+\frac{1}{q}}}{(np+1)n!} \left[ \frac{|f^{(n)}(a)|^q + |f^{(n)}(b)|^q}{2} \right]^{\frac{1}{q}} .
\end{align*}
\]

Corollary 9. Let the conditions of Theorem 3 hold. Then the following result is valid. Namely,

\[
\begin{align*}
(2.9) & \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{b-a}{(k+1)!} \left[ \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right] \right| \\
& \leq \frac{(b-a)^{n+1}}{(n+1)!} \left[ \frac{|f^{(n)}(a)| + |f^{(n)}(b)|}{2} \right] .
\end{align*}
\]

Proof. Summing the inequalities (2.7) and (2.8) and by using the triangle inequality, we have the inequality (2.9). □
Corollary 10. In the inequalities (2.6), if we choose $n = 1$, then we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{(b-a)^{1+\frac{1}{q}}} \left\{ \frac{(x-a)^{p+1+\frac{1}{q}}}{p+1} \left[ \frac{(2b-a-x)}{2} |f'(a)|^q + \frac{(x-a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right. \\
+ \left. \frac{(b-x)^{p+1+\frac{1}{q}}}{p+1} \left[ \frac{(b-x)}{2} |f'(a)|^q + \frac{(b+x-2a)}{2} |f'(b)|^q \right]^{\frac{1}{q}} \right\}.
\]

Theorem 4. Let $f : [a, b] \to \mathbb{R}$ be $n$-time differentiable mapping and $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is convex on $[a, b]$, then we have

\[
(2.10) \quad \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right| \\
\leq \frac{1}{n!(b-a)^{q\frac{1}{p}+2}} \left( \frac{q-1}{pq+q-p-1} \right)^{1-\frac{1}{q}} \\
\times \left\{ (x-a)^{n+1} \left[ \frac{(p+2)(b-x) + (x-a)}{(p+1)} \right] |f^{(n)}(a)|^q + (x-a)^{p+1} \left[ f^{(n)}(b) \right]^{\frac{1}{q}} \right. \\
+ (b-x)^{n+1} \left[ (b-x)^{p+1} \left[ f^{(n)}(a) \right] + \frac{(p+2)(x-a) + (b-x)}{(p+1)} \right] \left[ f^{(n)}(b) \right]^{\frac{1}{q}} \right\}.
\]

Proof. From Lemma 1 and using the properties of modulus, we have

\[
\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right| \\
\leq \left| \int_a^b K_n(x, t)f^{(n)}(t)dt \right| \\
= \int_a^b \frac{(t-a)^n}{n!} \left| f^{(n)}(t) \right| dt + \int_x^b \frac{(b-t)^n}{n!} \left| f^{(n)}(t) \right| dt \\
= \frac{1}{n!} \left\{ \int_a^x (t-a)^n \left| f^{(n)}(t) \right| dt + \int_x^b (b-t)^n \left| f^{(n)}(t) \right| dt \right\} \\
= \frac{1}{n!} \left\{ \int_a^x (t-a)^n(t-a)^{\frac{q}{p}} \left| f^{(n)}(t) \right| dt + \int_x^b (b-t)^n(b-t)^{\frac{q}{p}} \left| f^{(n)}(t) \right| dt \right\}
\]
By Hölder inequality, we obtain

\[
\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right| \\
\leq \frac{1}{n!} \left\{ \left( \int_a^x \left( \frac{(t-a)^n}{(t-a)^{\frac{q}{n}}} \right)^{\frac{1}{q}} \,dt \right)^{1 - \frac{1}{q}} \left( \int_a^x \left( f^{(n)}(t) \right)^q \,dt \right)^{\frac{1}{q}} \right\}^{\frac{1}{2}} \\
+ \left\{ \left( \int_x^b \left( \frac{(b-t)^n}{(b-t)^{\frac{q}{n}}} \right)^{\frac{1}{q}} \,dt \right)^{1 - \frac{1}{q}} \left( \int_x^b \left( f^{(n)}(t) \right)^q \,dt \right)^{\frac{1}{q}} \right\}^{\frac{1}{2}}.
\]

Since \( |f^{(n)}|_q \) is convex on \([a, b]\) and \( t = \frac{b-t}{b-a}a + \frac{t-a}{b-a}b \), we have

\[
\left| \int_a^b f(t)\,dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right| \\
\leq \frac{1}{n!} \left\{ \left( \int_a^x \left( \frac{(t-a)^n}{(t-a)^{q/p}} \right)^{\frac{1}{q/p}} \,dt \right)^{1 - \frac{1}{q/p}} \left( \int_a^x \left( f^{(n)}(t) \right)^q \,dt \right)^{\frac{1}{q/p}} \right\}^{\frac{1}{2}} \\
+ \left\{ \left( \int_x^b \left( \frac{(b-t)^n}{(b-t)^{q/p}} \right)^{\frac{1}{q/p}} \,dt \right)^{1 - \frac{1}{q/p}} \left( \int_x^b \left( f^{(n)}(t) \right)^q \,dt \right)^{\frac{1}{q/p}} \right\}^{\frac{1}{2}}
\]

\[
= \frac{1}{n!(b-a)^{\frac{q}{p+1}}(p+2)^{\frac{q}{p+1}}} \left( \frac{q-1}{nq + q - p - 1} \right)^{\frac{1}{q/p}} \times \left\{ (x-a)^{n+1} \left[ (p+2)(b-x) + (x-a) \right] f^{(n)}(a)^q + (x-a)^{p+1} \left[ f^{(n)}(a) \right]^q \right\}^{\frac{1}{q/p}} \\
+ \left( b-x \right)^{n+1} \left[ (b-x)^{p+1} \left[ f^{(n)}(b) \right]^q + \left( p+2 \right)(x-a) + (b-x) \right] f^{(n)}(b)^q \right\}^{\frac{1}{q/p}}.
\]

By using the fact that

\[
\int_a^x (t-a)^{\frac{nq+p-1}{p+1}} \,dt = \frac{q-1}{nq + q - p - 1} (x-a)^{\frac{nq+p-1}{p+1}},
\]
\[
\int_x^b (b-t)^{\frac{nq+p-1}{p+1}} \,dt = \frac{q-1}{nq + q - p - 1} (b-x)^{\frac{nq+p-1}{p+1}}
\]

we get the inequality (2.10), which completes the proof of the theorem. □
Corollary 11. Assume that $f$ is as in Theorem 4. If we choose $x = \frac{a + b}{2}$, then we have
\[
\left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} (\frac{b-a}{2})^k f^{(k)} \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^{n+1}}{n!(p+2)^{p+1} nq + q - p - 1} \left( \frac{q-1}{nq + q - p - 1} \right)^{1-\frac{1}{q}} \times \left\{ \frac{p+3}{p+1} |f^{(n)}(a)|^q + \frac{b-a}{2} |f^{(n)}(b)|^q \right\}^{\frac{1}{q}}.
\]

Corollary 12. With the above assumptions, if we choose $x = a$ and $x = b$, respectively, we have
\[
\text{(2.11)} \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)} (a) \right| \leq \frac{(b-a)^{n+1}}{n!(p+2)^{p+1} nq + q - p - 1} \left( \frac{q-1}{nq + q - p - 1} \right)^{1-\frac{1}{q}} \times \left\{ \frac{1}{p+1} |f^{(n)}(a)|^q + (b-a)^p |f^{(n)}(b)|^q \right\}^{\frac{1}{q}}.
\]

Corollary 13. Let the conditions of Theorem 4 hold. Then the following result is valid. Namely,
\[
\text{(2.12)} \quad \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(-1)^k (a-b)^{k+1}}{(k+1)!} f^{(k)} (b) \right| \leq \frac{(b-a)^{n+1}}{n!(p+2)^{p+1} nq + q - p - 1} \left( \frac{q-1}{nq + q - p - 1} \right)^{1-\frac{1}{q}} \times \left\{ \frac{1}{p+1} |f^{(n)}(a)|^q + (b-a)^p |f^{(n)}(b)|^q \right\}^{\frac{1}{q}}.
\]
Theorem 5. For the following inequality:

Proof. Summing the inequalities (2.11) and (2.12) and by using the triangle inequality, we have the inequality (2.13). □

Corollary 14. In the inequalities (2.10), if we choose $n = 1$, then we have

\[ \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{(b-a)^\frac{q}{p}(p+2)^\frac{q}{p}} \left( \frac{q-1}{2q-p-1} \right)^{1-\frac{q}{p}} \times \left\{ (x-a)^2 \left[ \frac{(p+2)(b-x) + (x-a)}{(p+1)} \right] |f'(a)|^q + (x-a)^{p+1} |f'(b)|^q \right\}^{\frac{1}{q}} + (b-x)^{p+1} |f'(a)|^q + \frac{(p+2)(x-a) + (b-x)}{(p+1)} |f'(b)|^q \right\}^{\frac{1}{q}}. \]

Theorem 5. For $n \geq 1$, let $f : [a,b] \to \mathbb{R}$ be $n$-time differentiable mapping and $a < b$. If $f^{(n)} \in L[a,b]$ and $|f^{(n)}|_q$ is convex on $[a,b]$ and $q \geq 1$, then we have the following inequality:

\[
\left| \frac{1}{(n+1)!(b-a)^\frac{q}{p}(n+2)^\frac{q}{p}} \times \left\{ (x-a)^{n+1} \left[ [(n+2)(b-x) + (x-a)] |f^{(n)}(a)|^q + (n+1)(x-a) |f^{(n)}(b)|^q \right\}^{\frac{1}{q}} + (b-x)^{n+1} \left[ (n+1)(b-x) |f^{(n)}(a)|^q + [(n+2)(x-a) + (b-x)] |f^{(n)}(b)|^q \right\}^{\frac{1}{q}}. \]

Proof. From Lemma 14 and using the properties of modulus, we obtain

\[
\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k(x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \right| \leq \left| \frac{1}{n!} \left\{ \int_a^b K_n(x,t) f^{(n)}(t)dt \right\} \right| + \left| \frac{1}{n!} \left\{ \int_x^b (t-a)^n |f^{(n)}(t)| dt + \int_x^b (b-t)^n |f^{(n)}(t)| dt \right\} \right|. \]
By Power-mean inequality, we obtain
\[
\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( b - x \right)^{k+1} + \frac{(-1)^k (x - a)^{k+1}}{(k+1)!} \right| f^{(k)}(x) \leq \frac{1}{n!} \left\{ \left( \int_a^t (t-a)^n dt \right)^{1-\frac{1}{q}} \left( \int_a^t (t-a)^n f^{(n)}(t)^q dt \right)^{\frac{1}{q}} + \left( \int_x^b (b-t)^n dt \right)^{1-\frac{1}{q}} \left( \int_x^b (b-t)^n f^{(n)}(t)^q dt \right)^{\frac{1}{q}} \right\}.
\]

Since \(|f^{(n)}|^q\) is convex on \([a, b]\) and \(t = \frac{b-a}{x-a} + \frac{b-a}{x-b}\), we have
\[
\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( b - x \right)^{k+1} + \frac{(-1)^k (x - a)^{k+1}}{(k+1)!} \right| f^{(k)}(x) \leq \frac{1}{n!} \left\{ \left( \int_a^t (t-a)^n \frac{b-t}{b-a} f^{(n)}(a)^q + \frac{t-a}{b-a} f^{(n)}(b)^q \right)^{\frac{1}{q}} + \left( \int_x^b (b-t)^n \frac{b-t}{b-a} f^{(n)}(a)^q + \frac{t-a}{b-a} f^{(n)}(b)^q \right)^{\frac{1}{q}} \right\}
\]
\[
= \frac{1}{(n+1)! (b-a)^{\frac{n}{q}} (n+2)^{\frac{1}{q}}}
\times \left\{ (x-a)^{n+1} \left[ (n+2)(b-x) + (x-a) \right] f^{(n)}(a)^q + (n+1)(x-a) f^{(n)}(b)^q \right\}^{\frac{1}{q}}
\times \left\{ (n+1) f^{(n)}(a)^q + (n+3)(x-a) f^{(n)}(b)^q \right\}^{\frac{1}{q}}.
\]
Hence the proof of the theorem is completed. \(\square\)

**Corollary 15.** With the above assumptions, if we choose \(x = \frac{a+b}{2}\), then we have
\[
\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left( b - \frac{a+b}{2} \right)^{k+1} f^{(k)} \left( \frac{a+b}{2} \right) \right| \leq \frac{(b-a)^{n+1}}{(n+1)! (n+2)^{\frac{1}{q}}}
\times \left\{ (n+3) f^{(n)}(a)^q + (n+1) f^{(n)}(b)^q \right\}^{\frac{1}{q}}
\times \left\{ (n+1) f^{(n)}(a)^q + (n+3) f^{(n)}(b)^q \right\}^{\frac{1}{q}}.
\]

**Corollary 16.** In Theorem 5 if we choose \(x = a\) and \(x = b\), respectively, we have
\[
(2.15) \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} f^{(k)}(a) \right| \leq \frac{(b-a)^{n+1}}{(n+1)! (n+2)^{\frac{1}{q}}}
\times \left\{ (n+1) f^{(n)}(a)^q + (n+3) f^{(n)}(b)^q \right\}^{\frac{1}{q}}.
\]
\[ \left| \int_a^b f(t) \, dt - \sum_{k=0}^{n-1} \frac{(-1)^k (b-a)^{k+1}}{(k+1)!} f^{(k)}(b) \right| \leq \frac{(b-a)^{n+1}}{(n+1)!(n+2)^{\frac{q}{2}}} \left[ \left( n+1 \right) \left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}}. \]

**Corollary 17.** Let the conditions of Theorem 5 hold. Then the following result is valid:

\[ \left| \int_a^b f(t) \, dt - \sum_{k=0}^{n-1} \frac{(b-a)^{k+1}}{(k+1)!} \frac{f^{(k)}(a) + (-1)^k f^{(k)}(b)}{2} \right| \leq \frac{(b-a)^{n+1}}{2(n+1)!(n+2)^{\frac{q}{2}}} \times \left\{ \left( n+1 \right) \left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right\}^{\frac{1}{q}}. \]

**Proof.** Summing the inequalities (2.15) and (2.16) and by using the triangle inequality, we have the inequality (2.17).

**Corollary 18.** In the inequalities (2.17), if we choose \( n = 1 \), then we have

\[ \left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{2(b-a)^{\frac{q}{2}}} \left\{ (x-a)^2 \left[ \frac{(3b-2x-a)}{3} \left| f'(a) \right|^q + \frac{2(x-a)}{3} \left| f'(b) \right|^q \right]^{\frac{1}{q}} + (b-x)^2 \left[ \frac{2(b-x)}{3} \left| f'(a) \right|^q + \frac{(b+2x-3a)}{3} \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right\}. \]

### 3. APPLICATIONS TO SPECIAL MEANS

We now consider the means for arbitrary real numbers \( \alpha, \beta \) (\( \alpha \neq \beta \)). We take

1. **Arithmetic mean**:
   \[ A(\alpha, \beta) = \frac{\alpha + \beta}{2}, \quad \alpha, \beta \in \mathbb{R}^+. \]

2. **Logarithmic mean**:
   \[ L(\alpha, \beta) = \frac{\alpha - \beta}{\ln |\alpha| - \ln |\beta|}, \quad |\alpha| \neq |\beta|, \quad \alpha, \beta \neq 0, \quad \alpha, \beta \in \mathbb{R}^+. \]

Now using the results of Section 2, we give some applications for special means of real numbers.
Proposition 1. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$, then, the following inequality holds:

$$|L_n^n(a, b) - x^n| \leq \frac{|n|}{(b-a)^2} \left\{ \frac{(x-a)^2(3b-a-2x) + 2(b-x)^3}{6} \cdot a^{n-1} \right. \left. + \frac{(b-x)^2(b-3a+2x) + 2(x-a)^3}{6} \cdot b^{n-1} \right\}.$$ 

Proof. The proof is obvious from Corollary applied to the convex mapping $f(x) = x^n$, $x \in [a, b]$, $n \in \mathbb{Z}$. □

Proposition 2. Let $a, b \in \mathbb{R}$, $0 < a < b$ and $n \in \mathbb{Z}$, $|n| \geq 1$, then, for all $q \geq 1$, the following inequality holds:

$$|L_n^n(a, b) - x^n| \leq \frac{|n|}{2(b-a)^{\frac{q}{2}}} \left\{ (x-a)^2 \left\{ \frac{(3b-2x-a) (a^{n-1})^q + 2(x-a) (b^{n-1})^q}{3} \right\}^{\frac{1}{q}} \right. \left. + (b-x)^2 \left\{ \frac{2(b-x) (a^{n-1})^q + (b+2x-3a) (b^{n-1})^q}{3} \right\}^{\frac{1}{q}} \right\}.$$ 

Proof. The proof is obvious from Corollary applied to the convex mapping $f(x) = x^n$, $x \in [a, b]$, $n \in \mathbb{Z}$. □

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