In single-particle or intraparticle entanglement, two degrees of freedom of a single particle, e.g., momentum and polarization of a single photon, are entangled. Single-particle entanglement (SPE) provides a source of non-classical correlations which can be exploited both in quantum communication protocols and in experimental tests of noncontextuality based on the Kochen-Specker theorem. Furthermore, SPE is robust under decoherence phenomena. Here, we show that single-particle entangled states of single photons can be produced from attenuated sources of light, even classical ones. To experimentally certify the entanglement, we perform a Bell test, observing a violation of the Clauser, Horne, Shimony and Holt (CHSH) inequality. On the one hand, we show that this entanglement can be achieved even in a classical light beam, provided that first-order coherence is maintained between the degrees of freedom involved in the entanglement. On the other hand, we prove that filtered and attenuated light sources provide a flux of independent SPE photons that, from a statistical point of view, are indistinguishable from those generated by a single photon source. This has important consequences, since it demonstrates that cheap, compact, and low power entangled photon sources can be used for a range of quantum technology applications.
Entanglement is usually generated by quantum correlation of the degrees of freedom (DoF) of distinct particles, e.g. photons.\textsuperscript{[13]} The generation of entangled photons pairs is typically based on a high power laser that pumps a non-linear crystal, where by either Spontaneous Parametric Down Conversion\textsuperscript{[14]} or by Four Wave Mixing\textsuperscript{[15]} two entangled photons are produced. Another possibility relies on the entanglement of the DoF of a single particle (single-particle entanglement (SPE) or intraparticle entanglement)\textsuperscript{[1, 2]} Since correlated photons pairs are no longer necessary, SPE might use cheaper and simpler light sources. However, single-particle entangled states have been only generated starting from heralded single photons\textsuperscript{[6, 8]} In this letter, we aim to address the following questions: Are pure single photon states needed for SPE? Can single-particle entangled states be produced by any sources of light, even classical ones?

Let us take an attenuated laser light so that the probability that in a given time interval more than one photon reaches a detector is minimized. Laser light can be either described as a superposition of pure states of finite number of photons in a mode\textsuperscript{[13]} or as a mixed superposition of pure states, when random phase fluctuations are considered\textsuperscript{[13]} These two descriptions read, respectively,

\begin{equation}
|\Psi\rangle := \sum_{n=0}^{+\infty} C_n |n_\psi\rangle, \quad \text{with} \quad \sum_{n=0}^{+\infty} |C_n|^2 = 1,
\end{equation}

where $|n_\psi\rangle$ is a Fock state with $n$ photons in the state $\psi$ and the coefficients $|C_n|^2$ are Poisson distributed;

\begin{equation}
\rho := \sum_{n=0}^{+\infty} P_n |n_\psi\rangle\langle n_\psi|, \quad \text{with} \quad \sum_{n=0}^{+\infty} P_n = 1,
\end{equation}

where $\rho$ is the density matrix and the (classical) probabilities $P_n = |C_n|^2$ follow the same Poisson distribution. In both cases, by attenuating the laser intensity we yield a Weak Coherent State (WCS) which is often used as a probabilistic source of single photons\textsuperscript{[23]} On the other hand, a classical light source (e.g. a lamp or a light emitting diode, LED) can be modeled as a collection of independent spontaneous emitters, and its state is described by (2), where the probabilities $P_n$ follow a super-poissonian statistics (ideally a Bose-Einstein statistics\textsuperscript{[13]})

Now, let us inject a pure single photon state in the experimental setup of fig.\textsuperscript{[1]} to generate the SPE states. In the generation stage, we entangle the polarization and the momentum DoF of the photon. For the momentum, we fix two possible different wavevectors $k_0, k_1$ with a common frequency $\nu = \frac{c}{2\pi} |k_0| = \frac{c}{2\pi} |k_1|$ and the associated states $|0\rangle, |1\rangle$. For polarization, we take the vertical and horizontal directions (with respect to the propagation plane) and the associated states $|V\rangle, |H\rangle$. The input polarization and momentum are defined by the input Glan Thompson Polarizer (GTP) and collimator, so that the input state of the photon is $|\psi\rangle = |0V\rangle$. By linear optical elements that act separately on the two DoF, we generate at the output of the generation stage a SPE state\textsuperscript{[1]}:

\begin{equation}
|\Psi_+\rangle = \frac{1}{\sqrt{2}} (|1V\rangle + |0H\rangle).
\end{equation}
Figure 1: **Schematic of the setup to generate and test single particle entanglement.** A light source is coupled to an optical fiber and attenuated to the few photons level. The source is either a gas laser or another light source, here represented by a green LED. The laser is fiber coupled and attenuated by a variable optical attenuator (VOA). If broadband, the light source is filtered by an interference filter (IF) and coupled to an optical fiber. A collimator feeds the input state to the generation stage. Here, a Glan-Thompson Polarizer (GTP) sets the light polarization to vertical. Then, a beam splitter (BS1) splits the signal in two different directions (momenta). Half Wave Plates (HWP) are used to rotate the polarization by the indicated degrees. A piezoelectric transducer (PZT) actuated mirror controls a relative phase shift $\xi$ to compensate for any phase difference in the two arms. This first Mach-Zehnder Interferometer (MZI) performs a rotation of the input state in the momentum and polarization degree of freedom and generates the entangled state. Then, the light enters into a second MZI that transforms the entangled state to a specific state which is determined by the momentum phase $\phi$ and the polarization rotation angle $\theta$ of the two HWP. In the detection stage, two polarizing beam splitters (PBS) on each output arm of the second MZI separate the polarization and momentum states. The light is then coupled to optical fibers and detected by four Si-SPADs. The SPADs are here used since we do a photon counting measurements. The counts from each SPAD are processed by a FPGA interfaced to a computer.
The following stage (preparation stage) transforms the entangled state $|\Psi_+\rangle$ to a prepared state $|\psi'_{a,b}\rangle$ where $a, b$ are unit vectors in the Bloch sphere. Here, the second MZI rotates $|\Psi_+\rangle$ by an angle $\phi = \frac{2\pi \Delta L}{\lambda}$ (\(\lambda\) is the wavelength of the light and $\Delta L$ is the path difference between the two arms). $\phi$ determines $a$ which projects the momentum state on a specific value. Two half wavelength plates (HWPs), one in each output port of the MZI, perform a rotation of the polarization by an angle $\theta$ with respect to the vertical direction. $\theta$ defines $b$, i.e. the orientation of the basis where the polarization states are projected.

Now, let us inject an attenuated light beam in the set-up. The initial multiparticle state has the form $|1\rangle$ or $|2\rangle$ with $|\psi\rangle = |0V\rangle$. Since the generation and preparation stages act linearly, the final multiparticle state entering the following detection stage has still the form respectively $|1\rangle$ or $|2\rangle$ with $|\psi'_{a,b}\rangle$ and the same coefficients $C_n$ and $P_n$.

In the detection stage, the momentum of the state is measured by looking at the two output arms of the second MZI; polarization is measured by using Polarizing Beam Splitters (PBS) to spatially separate the polarization components. Photons are counted by four Silicon single photon avalanche detectors (SPADs). If $N_{xy}^{(a,b)}$ is the number of counts corresponding to a state $xy$ (with $x = 0, 1$ and $y = V, H$), we define the correlation coefficient as

$$E(a, b) = \frac{N_{11}^{(a,b)} + N_{00}^{(a,b)} - N_{01}^{(a,b)} - N_{10}^{(a,b)}}{N_{00}^{(a,b)} + N_{11}^{(a,b)} + N_{10}^{(a,b)} + N_{01}^{(a,b)}}. \tag{4}$$

From $E(a, b)$, the $S$-parameter can be extracted

$$S(a, a', b, b') = E(a, b) - E(a, b') + E(a', b) + E(a', b'), \tag{5}$$

for every choice of $a, a', b$ and $b'$. The $S$-parameter allows writing the Bell inequality in the CHSH model as

$$|S(a, a', b, b')| \leq 2. \tag{6}$$

On the other hand, quantum mechanics produces a theoretical value $E(a, b) = |\langle \psi_{a,b}'|1V\rangle|^2 + |\langle \psi_{a,b}'|0H\rangle|^2 - |\langle \psi_{a,b}'|0V\rangle|^2 - |\langle \psi_{a,b}'|1H\rangle|^2 = \cos(\phi - 2\theta)$, so that the maximal values of $S(\phi, \phi', \theta, \theta')$ are attained for $\phi - 2\theta = -\phi' - 2\theta = \phi' - 2\theta' = \alpha$, where $\alpha$ is a free parameter. The maximal violation of (6) is attained at $\alpha = \pm\pi/4$ where $S = 2\sqrt{2}$ and at $\alpha = \pm3\pi/4$ where $S = -2\sqrt{2}$.

Each measurement on a single photon in the state $|\psi'\rangle$ performed by the four detectors is represented by tests, i.e. by an orthogonal projection operators $Q_j$ (acting on the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^2$ associated to the single photon) admitting only two outcomes: 0 (FALSE) or 1 (TRUE). The measurement performed on a single photon is therefore described by 4 tests $Q_1, Q_2, Q_3, Q_4$, which are mutually exclusive ($Q_i Q_j = 0$ if $i \neq j$) and exhaustive ($\sum_{j=1}^4 Q_j = I$). When

\[\text{in this paragraph we omit to label } |\psi'\rangle \text{ with } a, b\]
dealing with states of \textit{arbitrarily many} photons, these tests have to be composed into \textit{multiple tests} \( Q_{n_1,n_2,n_3,n_4} \), which are still mutually exclusive and exhaustive. Each \( Q_{n_1,n_2,n_3,n_4} \) commutes with the number of particles operator and acts independently over each subspace of the Fock space with finite number \( n \) of particles. These properties imply that measurements cannot distinguish between an input state in the form (1) or (2). In fact, given \( n \) photons are detected, the conditional probability that for any \( j = 1, 2, 3, 4 \) exactly \( n_j \) photons are collected by the \( j \)-th detector (with \( \sum_j n_j = n \)) is given by a multinomial distribution:

\[
\langle n | \psi' \rangle | Q_{n_1,n_2,n_3,n_4} | n \psi' \rangle = \frac{n!}{n_1!n_2!n_3!n_4!} \langle \psi' | Q_1 | \psi' \rangle^{n_1} \langle \psi' | Q_2 | \psi' \rangle^{n_2} \langle \psi' | Q_3 | \psi' \rangle^{n_3} \langle \psi' | Q_4 | \psi' \rangle^{n_4},
\]

in both cases. The right-hand side coincides with the \textit{classical joint probability distribution} of \( n \) independent random variables with 4 attainable outcomes and elementary probabilities

\[ P(j) = \langle \psi' | Q_j | \psi' \rangle, \]

where \( P(1) = |\langle \psi'_{a,b} | 1H \rangle|^2 \), \( P(2) = |\langle \psi'_{a,b} | 1V \rangle|^2 \), \( P(3) = |\langle \psi'_{a,b} | OV \rangle|^2 \) and \( P(4) = |\langle \psi'_{a,b} | OH \rangle|^2 \). In other words, (7) does not distinguish between experiments where a sequence of single (heralded or not) photons or a multi photon state of the form (1) or (2) are used. Quantum effects are embodied in the probabilities \( \langle \psi' | Q_j | \psi' \rangle \), irrespective of the type of the multiparticle state (coherent or incoherent) entering and handled by the apparatus. The theoretical probabilities (8) are the building blocks for the verification of the Bell inequality (6). The key point is the measurement of single-particle observables. The use of SPADs in the experiment allows us recording a time-ordered sequence of outcomes of measurements of single particle observables.

For a single (heralded) photon, (8) is estimated in terms of empirical frequencies by repeating the experiment a large number of times. For an attenuated laser beam or attenuated filtered light, the estimation of the probabilities (8) relies on the measurement of the empirical frequencies. In fact, if the light intensity is sufficiently low, the probability that more than a photon is simultaneously collected from the same detector (hence the number of photons cannot be resolved) is negligible. There are no fundamental differences between an attenuated light and single (heralded) photons. In both cases, we observe a flux of photons. In the case of the heralded photons, the arrival times can (to some extent) be determined by the experimental procedure, while, in the case of the attenuated light, the arrival times are stochastically distributed. In particular, under the assumption that the weights \( C_n \) in the statistical mixture (2) are Poisson distributed, i.e. \( |C_n|^2 = \frac{\mu^n}{n!} \), the lapses of time between two subsequent detections are exponentially distributed with rate \( \mu \).

This theoretical analysis is clearly reflected by our experimental results: by injecting light from an attenuated HeNe laser in the setup, we witness a violation of the Bell Inequality, as can be seen in fig. 2. If we fix \( \phi = 0 \), we obtain
Figure 2: **Violation of the CHSH Inequality with a WCS produced by an attenuated laser beam.** Data (red point) are taken by varying the polarization angle $\theta$ of the HWPs in the preparation stage of the set-up of fig. 1. For each data point, the average of several measurements is reported. Error bars are the errors propagated from the standard deviations of the measurements. The solid curve represents the $S$-parameter of eq. (9). The dashed curve is the $S^{\text{eff}}$-parameter given by (14), which takes into account the noise ($\eta = 0.95 \pm 0.01$) and the coherence of the source ($\epsilon = 0$). The blue regions indicate the violation of the CHSH inequality (6).

$-2\theta = \alpha$ and (18) becomes:

$$S(\theta) = 3 \cos(2\theta) - \cos(6\theta)$$

Data are well reproduced by (9).

$E(a, b)$ does not depend on whether the input state is a coherent superposition or a statistical mixture of pure states. Furthermore, it is independent of the photon statistics. Therefore, SPE states can –in principle – be generated also by attenuated incoherent sources, such as a thermal lamp or a LED. In this case, the spontaneous nature of the photon...
emission results in a short coherence time $\tau_c$ and in a short coherence length $l_c$. As discussed in the *Supplementary Material*, it is possible to phenomenologically take into account the broad spectrum of any classical light source by replacing the entangled state (16) with the mixed state

$$\rho_e = (1 - \epsilon)|\psi_{\text{entangled}}\rangle\langle \psi_{\text{entangled}}| + \epsilon \rho_{\text{Mixed}},$$

where

$$\rho_{\text{Mixed}} = \frac{1}{2}|0H\rangle\langle 0H| + \frac{1}{2}|1V\rangle\langle 1V|,$$

and $\epsilon$ is a phenomenological parameter which takes coherence properties into account. The pure state (12) is a SPE that differs from (16) by a relative phase which is a function of the time delay $T$ due to the different paths in the generation stage (fig. 1). In the case of a source with a gaussian spectrum centered at $\omega_0$ and spectral half-width $\sigma_\omega$, quantum mechanics predicts that $\epsilon$ depends on $T$ as:

$$\epsilon(T) = 1 - e^{-T^2\sigma_\omega^2/2}.$$  

(13)

Since $\sqrt{\frac{1}{\sigma_\omega}} = \tau_c$, (13) shows that coherence is completely lost for $|T| \gg \tau_c$, where the state can be considered mixed and described by (11), while for $|T| \ll \tau_c$ the state (10) is entangled. From a practical perspective, the value of $\epsilon$ decreases by reducing $\sigma_\omega$ with spectral filtering and $T$ with optical alignment.

In order to measure the entanglement generated by any classical light sources in the coherent regime, we performed the Bell inequality test by using two different filtered broadband sources: a halogen lamp and a LED. By using a 1 nm interference filter, we increased $l_c$ and we witnessed a violation of the inequality (fig. 3a and 4). As predicted by the theory, the amplitude of $N_{0H}^{(a,b)}$ shows constant interference fringes for different values of $\theta$ (fig. 3b). In this case, (9) can be generalized to include both incoherence by means of $\epsilon$ and noise, that reduces the visibility of the detection channels, by means of a parameter $\eta$ (see Supplementary):

$$S^{\text{eff}}(\theta) = \eta(1 - \epsilon)(3 \cos(2\theta) - \cos(6\theta)) + \eta \epsilon (2 \cos^3(2\theta) - 2 \sin^2(2\theta) \cos(6\theta)) .$$

(14)

In figures 3a and 4 we show that (14) reproduces the data.

To further confirm this theory, we repeated the measurement with the LED in the incoherent regime (fig. 3c,d). As expected, the Bell inequality is never violated and the experimental points agree with the theoretical prediction for two independent states entering the measuring setup. Note that in this case, the visibility in $N_{0H}^{(a,b)}$ starts to decrease as we increase the polarization angle $\theta$, reaching the value $\approx 0$ for $\theta \approx \frac{\pi}{4}$. 


Figure 3: CHSH Inequality Measurement for the LED source. (a) Bell inequality measurement for a filtered (1nm bandwidth) LED and for the setup aligned within the coherence regime. In this measurement, the coherence length is $l_c = 46 \pm 0.3\mu m$. Red points are the experimental data. The solid line is the ideal $S$-parameter [9], while the dashed line is the $S^{\text{eff}}$-parameter with full coherence ($\epsilon = 0$) and noise ($\eta = 0.87 \pm 0.02$) [14]. (b) Signal acquired by one single SPAD ($N^{(\phi, \theta)}_{\text{OH}}$) as a function of $\phi$ for different polarization angles $\theta$ in the coherent case. Counts are normalized to the maximum count in the set. Error bars are measurement errors. The lines are guide to the eyes. The visibility ($V = \frac{N_{\text{OH}}(\max) - N_{\text{OH}}(\min)}{N_{\text{OH}}(\max) + N_{\text{OH}}(\min)}$) does not change as a function of $\theta$. (c) $S$-parameters (experimental data and theoretical curves) in the incoherent regime. This is achieved by displacing the mirror in the first MZI of the set-up in fig. 1 by a distance longer than $l_c$. $\eta = 0.89 \pm 0.01$ and $\epsilon = 1$ were used in the calculation of the $S^{\text{eff}}$-parameter by [14]. (d) $N^{(\phi, \theta)}_{\text{OH}}$ as a function of $\phi$ for different $\theta$ for the incoherent case. Here, $V$ changes as a function of $\theta$, approaching 0 for $\theta \approx \frac{\pi}{4}$. 
In conclusion, we have theoretically justified and experimentally demonstrated that SPE can be generated from coherent and classical light sources attenuated to the few photons level and that SPE from such sources does indeed violate the Bell inequality. A crucial condition for SPE to be observed is that self-coherence (first-order coherence) between the involved DoF is preserved. In addition, our theoretical analysis provides a quantum mechanical interpretation of recent results in classical optics, where violation of Bell-type inequalities is interpreted as a quantitative measure of the coherence properties of classical light\textsuperscript{22}. In fact, as soon as single photons are detected, no classical realistic non-contextual description is possible\textsuperscript{11}. The crucial difference between our experimental context and that of Ref. \textsuperscript{22} is the single photon counting technique which allows recording a time-ordered sequence of outcomes of measurements of single particle observables. On the contrary, experiments with classical light deal with measurement of light intensities,
i.e. total number of photons. In other words, they collect the cumulative statistics of the outcomes (i.e. the total number of counts for each channel) losing single detection information. In this case, the cumulative statistics of these measurements can be completely explained in terms of classical realistic theory even if Bell inequalities are violated. On the other hand, if we try to provide a probabilistic hidden variable theory that describes our (single-particle) experimental results in term of sequences of discrete random variables (each with two possible outcomes), then Bell inequalities cannot be violated (see Sect. 1.6.2 of the Supplementary Material). This means that such a realistic non-contextual description of our experimental data cannot exist. In other words, the duality wave-particle is manifest in our experimental context, while it is completely absent in the case of “classical entanglement” described in terms of intensities.

Indeed, an experimental evaluation of the theoretical probabilities can be performed by using four different detectors able to count the photons. In this case, it is no longer necessary to use attenuated light, and the intensities measured by the four detectors are proportional to the (quantum mechanical) probabilities. We believe that our results allow to give a unified view of quantum SPE and “classical entanglement”: both can be ascribed to a particular form of the state of the single photons in the light beam (see (10)). Even if \( E(a, b) \) and the \( S \)-parameter are evaluated in terms of classical quantities (e.g. as intensities of light in \( \mathbb{R} \)), they still admit a quantum mechanical interpretation. In this picture, the coherence properties of the light can still be taken into account by means of (10) and (13), so that a violation of the Bell inequality is related to first-order coherence and can be interpreted as single-particle entanglement of the one-photon states.

Single-particle or two-particles entanglement violations of the Bell inequalities are interpreted as a signature of a non-classical nature of the measured system: contextuality or non-locality, respectively\[^{11}\]. However, from an experimental point of view, the use of SPE is advantageous with respect to two-particles entanglement. In fact, to observe a violation of Bell inequalities with two-particles entanglement the knowledge of the joint distribution of the outcomes \( N_{xy}^{(a,b)} \) is needed. These data are available only if we are able to recognize when a pair of measurements is referred to a single entangled pair. In other words, a strict control of the pair arrival times is fundamental. This experimentally demanding requirement is not necessary in the case of SPE because, in this case, the numbers \( N_{xy}^{(a,b)} \) can be simply obtained by looking at the number of counts in the four different channels of the experimental apparatus. That is why the times of arrival do not play a fundamental role and Bell inequalities are violated even with intense light beams (see Sect 1.6 of Supplementary Material).
Our results confine the need of expensive sources of single photons, e.g. heralded photons, to those Quantum Information protocols that require a deterministic time of arrivals of the single photons. Even if the statistics of the input light does not affect the SPE, it must be taken into account in practical applications to Quantum Information tasks. For example, SPE has been suggested for increasing the security of BB84-type QKD protocols. In fact, security check based on violation of Bell-type inequalities increases the robustness against side-channel attacks. On the other hand, one should also consider that thermal sources feature a super-poissonian emission statistics, and the presence of multiphoton components opens the risk of photon number splitting attacks. Other applications of SPE can be in the implementation and certification of quantum random number generators (QRNG). Indeed, the violation of Bell inequalities allows to prove a lower bound for the entropy of the random sequence produced. In this case, the photon statistics plays no role. The presence of multiphoton states does not affect the randomness of the results as the system acts independently on each photon. The only consequence is the possibility of coincidence counts in the detectors, which cannot contribute to the random string and, therefore, decreases the generation rate. However, by using classical attenuated sources, the number of coincidence counts over the total signal can be easily controlled by the source intensity. Though, the signal-to-coincidence ratio is expected to be worse for classical light sources than for an attenuated laser. Therefore, using incoherent light compromises the performances with the feasibility (cost, size, power consumption, weight) of the QRNG compared to a laser, but it may still be a good trade-off given the advantage of using an incoherent light source.

**Methods**

**Setup Description**

The three sources used for the measurement were:

- An attenuated single mode green HeNe laser source from Melles Griot, emitting at 541nm with nominal power of 5mW.

- A commercial through-hole 5mm LED with peak wavelength of 517 nm and spectral width of 30nm, which was filtered by an interference filter centered at 531 nm with a bandwidth of 1nm.

- A Halogen lamp, model: HL-2000-FHSA-LL from Ocean Optics with a broad spectrum (360 – 2400nm) , which was filtered at 531nm to a 1nm wide peak with the same interference filter used for the LED.
The sources were coupled to a single mode optical fiber. For the LED and the lamp, this ensures to collect photons from different spatial modes. At the output of the set-up, the light was coupled to long (> 1 m) optical fibers and interfaced to four silicon SPADs (Excelitas). This long fibers are used to prevent cross-correlation, i.e. false counts, between the SPADs. The best detector efficiency was measured at 52%. Since the different detectors had different efficiencies we equalized their efficiencies by controlling the fiber-coupling of the signal to the detector. The entire setup (sources and detector excluded) is enclosed in a dark box to isolate it from external environment and reduce the noise. The choice of the working wavelength is mainly motivated by the availability of both sources and detectors, but also by the possibility of having best performances (e.g. constant retardance of the polarizers and equal transmittance and reflectance of the beam splitters) for the optics. Since the signals are symmetric with respect to the chosen direction of measurement (|0⟩ and |1⟩ respectively), as described by the theory [6], the $S$-parameter was estimated by the projections over |0H⟩ and |0V⟩ in the case of the LED and of the Halogen lamp. In the case of the laser, the $S$-parameter is estimated by acquiring all the four signals.

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**Acknowledgements**

We acknowledge helpful discussions with P. Bettotti in the initial phase of the experiment and with S. Azzini on single particle entanglement. G. Fontana developed the acquisition system based on FPGA. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under grant agreement No 820405 project QRange, and by the India-Trento Programme of Advanced Research ITPAR phase IV project. The work of N.L. was supported by a Q@TN grant and the one of D.P. by Fondazione Caritro.

**Author contribution**

S.M., V.M., D.P. and L.P. conceived the experiment. S.M., V.M. and D.P. elaborated the theory with the help of M.P. who designed the experiment and mounted the setup. M.P. performed the experiment and analyzed the data with the help of N.L. All the authors contributed to the manuscript. L.P. supervised the work.

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1 Supplementary Material

1.1 Single photon entanglement and experimental setup

This work deals with single-particle entangled states of the momentum and polarization degrees of freedom (DoF) of a single photon. First, consider a two-state basis for each of them. For momentum, we fix two possible different wave vectors \( k_0, k_1 \), i.e., two directions of propagation in the experimental setup, together with a common frequency \( \nu = \frac{c}{2\pi} \left| k_0 \right| = \frac{c}{2\pi} \left| k_1 \right| \). Hence, in this setting the Hilbert space describing the momentum DoF reduces to a 2-dimensional space \( \mathcal{H}_M \) spanned by the associated qubit basis states \( |0\rangle, |1\rangle \).

For polarization, we take the vertical and horizontal directions (with respect to the propagation plane), defining the basis \( |V\rangle, |H\rangle \) of the polarization Hilbert space \( \mathcal{H}_P \).

The space of our two-qubit composite system is the four-dimensional \( \mathcal{H}_S = \mathcal{H}_M \otimes \mathcal{H}_P \) spanned by

\[
\{|0V\rangle, |0H\rangle, |1V\rangle, |1H\rangle\},
\]

where henceforth \( |XY\rangle = |X\rangle \otimes |Y\rangle \). Generation of single photon entangled states only needs linear optical elements that act separately on the two DoF (see Fig. 5a). We focus on states belonging to a Bell Basis of \( \mathcal{H}_S \)

\[
|\Psi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|0H\rangle \pm |1V\rangle).
\]

1.2 Detection of Entanglement using CHSH Inequality

Detection of entanglement has become an increasingly fundamental task in Quantum Information for the validation of experiments and the development of protocols. As in [2,3], we exploit here the version of Bell Inequality [4] due to Clauser, Horne, Shimony and Holt (CHSH) [5], which is generally accepted as an entanglement test, and it is not too experimentally demanding.

The used experimental setup to test CHSH inequality (Fig. 1 in the main text) can be functionally divided in three stages: (I) generation, (II) preparation, and (III) detection.

(I) In the first stage, the single-photon entangled state is generated. The input polarization and momentum are defined, for the photon entering the apparatus, using a Glan Thompson Polarizer and a collimator so that the state of the photon is \( |\psi\rangle = |0V\rangle \). Next, we generate an entangled state from the initial state \( |\psi\rangle \), as schematically shown in 5(a). In the actual setup (Fig. 1 in the main text) the relative phase \( \xi \) of the superposition is fixed by changing the optical path difference between the two beams before recombinining them, using a piezoelectric transducer (PZT) actuated mirror.

With the said setup, the state entering the second beam splitter is \( |\Psi_{+}\rangle \) in [16].
Figure 5: Single Photon Entanglement Generation and Manipulation. (a) Setup for the generation of single photon entangled states. A photon is prepared in a definite momentum-polarization state, then a beam splitter (BS) puts it in a superposition of momentum states and a half wave plate (HWP) rotates the polarization of photons with momentum $k_1$. The phase shift accumulated in the reflection or other dispersion is compensated by moving a mirror in the actual setup in Fig 1 (main text). Here we indicate this action by the generic phase shift (PS) element. (b) Momentum and polarization bases in the Bloch sphere and the optical elements constituting the preparation stage of Fig. 1 in the main text. A rotation in the $x$-$z$ plane of the momentum sphere can be performed using a Mach Zehnder Interferometer, while the polarization state can be rotated in the same axis using a half wave birefringent plate.

The following stage performs the rotations in the qubit spaces transforming the entangled state $|\Psi_+\rangle$ to a prepared state $|\psi_a'b\rangle$. It can be done in different ways depending on a couple $(a, b)$ of unit vectors in the Bloch Sphere (Fig. 5(b)). All transformations are performed by suitable unitary maps in the space of the states (see (42)-(43) in Section 1.5). The Mach Zehnder interferometer (MZI) in the second part of the setup acts as a momentum-qubit gate, rotating the state by an angle $\phi = \frac{2\pi}{\lambda} \Delta L$ that corresponds to the path difference $\Delta L$ in the two arms. This angle determines the vector $a$ in the Bloch Sphere (see Fig. 5(b)) associated to the orientation of the basis of $C^2$ where the momentum states are represented or, equivalently, it determines the 1-particle observable $O^a = a \cdot \sigma$ related to the momentum DoF, where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the associated Pauli matrices.
Two HWPs, one in each output port of the MZI, with the fast axis rotated by the same amount $\vartheta$, perform a rotation in the polarization space by an angle $\theta = 2\vartheta$ with respect to the vertical direction. Analogously, this angle defines the vector $b$ in the Bloch Sphere, i.e. the 1-particle observable $O^b = b \cdot \sigma$ related to the polarization DoF.

(III) In the third stage, the measurement of the observable $O^a \otimes O^b$ is performed on the state $|\Psi_+\rangle$. This aim is achieved by measuring momentum and polarization of the prepared states $|\psi_{a,b}'\rangle$, i.e. by computing their components with respect to the elements of standard base (15). From the experimental point of view, momentum is measured by looking at the two output arms of the setup, while polarization is measured by using Polarizing Beam Splitters (PBS) to spatially separate the polarization components. Four collimators, one for each state, couple the photons into optical fibers that lead them to four Silicon SPADs.

If $N_{xy}^{(a,b)}$ is the number of counts in the detector corresponding to the outcome $xy$ (with $x = 0, 1$ and $y = V, H$), the correlation coefficient is defined as:

$$E(a, b) = \frac{N_{1V}^{(a,b)} + N_{0H}^{(a,b)} - N_{0V}^{(a,b)} - N_{1H}^{(a,b)}}{N_{0V}^{(a,b)} + N_{1H}^{(a,b)} + N_{1V}^{(a,b)} + N_{0H}^{(a,b)}}. \quad (17)$$

The CHSH inequality reads $|S(a, a', b, b')| \leq 2$ for every choice of $a, a', b$ and $b'$ associated to the corresponding angles $\phi, \phi', \theta, \theta'$, where

$$S(a, a', b, b') = E(a, b) - E(a, b') + E(a', b) + E(a', b'). \quad (18)$$

Quantum mechanics produces a theoretical value

$$E(a, b) = |\langle \psi_{a,b}' | 1V \rangle |^2 + |\langle \psi_{a,b}' | 0H \rangle |^2 - |\langle \psi_{a,b}' | 0V \rangle |^2 - |\langle \psi_{a,b}' | 1H \rangle |^2 = \cos(\phi - 2\theta).$$

At first glance, this seems a complex function of four parameters, but in fact only three of the arguments are mutually independent. Indeed, the following equality holds:

$$\phi - 2\theta = -\phi' - 2\theta = \phi' - 2\theta' = \alpha, \quad (19)$$

where $\alpha$ is a free parameter we can vary, giving

$$S(\alpha) = 3 \cos \alpha - \cos(3\alpha). \quad (20)$$

The maximal violation of (18) foreseen by Quantum Theory is attained at $\alpha = \pm \pi/4$ where $S = 2\sqrt{2}$ and at $\alpha = \pm 3\pi/4$ where $S = -2\sqrt{2}$. 

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1.3 Types of in-going states

Referring to Fig. 1 in the main text, this work considers two possibilities for the type of quantum state entering the first beam splitter. The former is typically obtained by a short time laser pulse, the latter is typically obtained after frequency filtration of an incoherent source (LED/halogen lump) or a laser beam.

(1) Coherent superposition of pure states of finite number of particles in the same mode

$$| \Psi \rangle := \sum_{n=0}^{+\infty} C_n |n_\psi\rangle, \quad \text{where} \quad \sum_{n=0}^{+\infty} |C_n|^2 = 1 \quad (21)$$

(2) Incoherent superposition of pure states of finite number of particles in the same mode

$$\rho := \sum_{n=0}^{+\infty} P_n |n_\psi\rangle \langle n_\psi|, \quad \text{where} \quad \sum_{n=0}^{+\infty} P_n = 1 \quad (22)$$

Above, $|\psi\rangle$ denotes the one-particle in-going state selected by mode filtration. That is a state with defined momentum $\hbar k$ and polarization, $|0V\rangle$ in our case. The used notation is the standard of 2nd quantization in the Fock space:

$$|n_\psi\rangle = \frac{1}{\sqrt{n!}} (a_\psi^\dagger)^n |\text{vac}\rangle \quad (23)$$

where $|\text{vac}\rangle$ is the vacuum state of photons.

In case of an ideal laser, (1) is valid, far above threshold and at atomic physics time-scale, with

$$C_n = e^{-\mu/2} \frac{\mu^{n/2}}{\sqrt{n!}},$$

where $\mu = \langle n \rangle$ and standard deviation $\sqrt{\sigma_n} = \sqrt{\langle n \rangle}$. However, the phase between states with different number of particles quickly become undefined, through a process of phase diffusion and the emitted state settles in the form (2) with a Poissonian distribution

$$P_n = e^{-\mu} \frac{\mu^n}{n!},$$

so that $\langle n \rangle = \mu$ and $\sqrt{\sigma_n} = \sqrt{\langle n \rangle}$ is still valid. In the case of a mode filtered thermal light at temperature $T$, but this model is also valid for a LED source, the form (2) of the entering state is valid with

$$P_n = \frac{1}{1 + \langle n \rangle} \left( \frac{\langle n \rangle}{\langle n \rangle + 1} \right)^n,$$

where

$$\langle n \rangle = \frac{1}{\exp\{\hbar \omega/k_B T\} - 1}$$

and $\omega = c|k|$. 
1.4 Analysis of the coherence length/time

Before the initial generally multi-particle state (either $|\Psi\rangle$ or $\rho$ as in Sect. 1.3) encounters the final stage of detectors, it is transformed by the intermediate stage of the circuit. As all elements of the circuit are linear, the net action on the multi-particle state directly arises form the action on a single-particle state as will be discussed in Sect. 1.6. For the moment we stick to the analysis of the action of a one-particle state and to some issues concerning temporal and spatial coherence which can be tackled at one-particle level. Since the source is not perfectly monochromatic and has a finite coherence length/time, the finite-dimensional description of the one-particle photon states as presented in Sect. 1.1 by means of the space of the states $\mathbb{C}^4 \simeq \mathbb{C}^{2_{\text{momentum}}} \otimes \mathbb{C}^{2_{\text{polarization}}}$ is correct as long as the difference between the length of the arms of the MZI is less than the coherence length of the light source or, equivalently, the accumulated delay between the two photons belongs to the interval of coherence time. In a preliminary test we measured the autocorrelation of the LED source to analyze its coherence properties. To do so, we used the apparatus represented in Fig. 6. The spectrum of the LED can be approximated by a gaussian, and we fitted it with the function $f(\omega) = Ae^{-\left(\frac{(\omega - \omega_0)^2}{2\sigma_\omega^2}\right)}$. From this fit we obtain $A = 0.932 \pm 0.002$, $\omega_0 = (3611.4 \pm 0.4)\text{THz}$ and $\sqrt{\sigma_\omega} = (134 \pm 9)\text{THz}$ (see Fig. 7(a)). This gives $\tau_c = \frac{1}{\sqrt{\sigma_\omega}} = (7.43 \pm 0.02)\text{fs}$, or a coherence length of $l_c = \tau_c c = (2.227 \pm 0.006)\mu\text{m}$. In order to increase the coherence length of the LED, we filtered it by a 1nm interference filter centered at 531nm. In this way we obtain
\( \tau_c = (154 \pm 1) \text{ fs} \) and \( l_c = (46.0 \pm 0.3) \mu \text{m} \). The difference coming from the increase in coherence time can be seen by comparing the filtered and unfiltered autocorrelations (see Fig. 7(b) and (d)). By manually displacing the arms of the MZI we can reach the incoherent regime, as shown in Fig. 3(c) in the main text, where the the S-parameter [18] is calculated in the incoherent regime for the LED.

This result is in agreement with the theoretical analysis summarized below and based on a more detailed description of the photon state.

### 1.4.1 A precise description of the one-particle state and coherence length/time

The main idea of this analysis is that the state of a single photon exiting form the first beam splitter (Fig. (6)) is actually a (superposition) of normalized packets \( \psi_j(k) \otimes |\theta\rangle \), below simply denoted by \( \psi_j(k)|\theta\rangle \), where

\[
|\theta\rangle := \cos \theta |V\rangle + \sin \theta |H\rangle
\]

is the polarization part of the state and the function \( \psi_j \) is sharply concentrated around the value \( k_j \in \mathbb{R}^3 \). As we know, these momenta \( h k_0, h k_1 \) define the sharp states \( |0\rangle, |1\rangle \) which are a rough but effective approximation of the functions \( \psi_0 \) and \( \psi_1 \) we exploited to describe the space of the states as in Sect. 1.1. It holds \( \omega_0 = c|k_0| = c|k_1| \) where \( \frac{\omega_0}{2\pi} \) is the frequency of the filtered light entering the whole apparatus. The functions \( \psi_j \), with \( j = 1, 2 \), vanish outsides two corresponding small balls, respectively, \( B_j \subset \mathbb{R}^3 \), whose centers are the vectors \( k_j \) and such that \( B_1 \) and \( B_2 \) are sharply disjoint. The appearance of a coherence length (or coherence time) is here explained in terms of time evolution of the wave functions

\[
U_t \psi(k)|\theta\rangle = e^{-ic|k|t}\psi(k)|\theta\rangle.
\]

It is clear that, within this more precise view, we have a different state at different times, since the phase \( e^{-ic|k|t} \) depends on \( k \). The Hilbert space can no longer be considered finite dimensional in this description.

The entangled state state entering the second beam splitter in Fig. 6 is no longer the Bell state

\[
\frac{1}{\sqrt{2}} (|1V\rangle + i|0H\rangle),
\]

but:

\[
|\psi^{(\text{precise})}\rangle := \frac{1}{\sqrt{2}} \left( \psi_1(k)|V\rangle + i e^{-icT|k|}\psi_0(k)|\theta\rangle \right).
\]

An overall phase \( e^{-ict|k|} \) irrelevant for our computation will henceforth be omitted. Here \( T \) is the delay time between the two paths exiting from the first beam splitter, i.e. \( cT|k| = \phi \) is therefore the phase difference acquired during
Figure 7: Spectrum and autocorrelation measurement for the LED source before and after introducing the interference filter. (a) Spectrum of the unfiltered LED. The blue curve represents the experimental data, while the red curve shows their gaussian fit $f(\omega) = Ae^{-\frac{(\omega - \omega_0)^2}{2\sigma^2}}$. Fit parameters are $A = (0.932 \pm 0.002)$, $\omega_0 = (3611.4 \pm 0.4)$THz and $\sqrt{\sigma_\omega} = (134 \pm 9)$THz. (b) Autocorrelation of the unfiltered LED obtained by using the experimental setup of Fig. 6. It is acquired by moving the piezoelectric transducer by 20\(\mu\)m and recording the output of one arm of the Mach Zehnder interferometer. Here we report the autocorrelation as a function of the time delay between the two optical paths. (c) Spectrum of the filtered LED. The blue curve represents the experimental data, while the red curve shows their gaussian fit $f(\omega) = Ae^{-\frac{(\omega - \omega_0)^2}{2\sigma^2}}$. Fit parameters are $A = (0.985 \pm 0.006)$, $\omega_0 = (3547.24 \pm 0.04)$THz and $\sqrt{\sigma_\omega} = (6.5 \pm 0.8)$THz. (d) Autocorrelation of the filtered LED obtained by using the experimental setup of Fig. 6. It is acquired by moving the piezoelectric movement by 20\(\mu\)m. Note that in this case, the oscillations of the signal do not decrease significantly over the 20\(\mu\)m range. Here we report the autocorrelation as a function of the time delay between the two optical paths.
the propagation in the two different arms. In order to quantify the robustness of the entanglement, we measure the interference between the two addends of the state through the final stage of the preliminary experiment consisting of a second beam splitter and a pair of detectors as in Fig. (6).

The state entering the detectors exiting the second beam splitter is described as follows, taking the unitary transformation describing the second beam splitter into account.

\[
|\psi_{\text{out}}^{(\text{precise})}\rangle = (U_{BS} \otimes I)|\psi_{\text{entangled}}^{(\text{precise})}\rangle \\
= \frac{1}{2}(i\psi_0(k) + \psi_1(k))|V\rangle + \frac{i}{2}e^{-ict}|k\rangle(\psi_0(k) + i\psi_1(k))|\theta\rangle
\]  

With the sharp packet approximations, the detectors D1 and D2 at the end of the circuit represented in Fig. (6) are mathematically described by orthogonal projectors \(|0\rangle\langle 0|\) and \(|1\rangle\langle 1|\). However, moving on to the more accurate description, we can represent those orthogonal projectors as the multiplicative operators

\[
P_j := \chi_{K_j}(k)
\]

where

\[
\chi_{K_j}(k) = 0 \text{ if } k \notin K_j \quad \text{and} \quad \chi_{K_j}(k) = 1 \text{ if } k \in K_j.
\]

Here \(K_j\) is a set of momenta which necessarily includes the corresponding ball \(B_j\) and such that \(K_0 \cap K_1 = \emptyset\). While the extension of \(B_j\) is decided by the source, the shape of \(K_j\) is fixed by the detector. An expected shape of \(K_j\) is a truncated cone whose axis is parallel to \(k_j\) and whose bases are portions of parallel spherical surfaces whose distance from the origin of the space of momenta is respectively proportional to the minimal and the maximal frequency detectable by the device. The probability to detect the photon in the \(j\)-th detector with a polarization selected by the one-dimensional orthogonal projector \(Q\) acting in \(\mathbb{C}^2\) polarization is

\[
\langle \psi_{\text{out}}^{(\text{precise})} | P_j \otimes Q | \psi_{\text{out}}^{(\text{precise})} \rangle.
\]

We have for \(j = 0, 1\),

\[
\langle \psi_{\text{out}}^{(\text{precise})} | P_j \otimes Q | \psi_{\text{out}}^{(\text{precise})} \rangle = \frac{1}{4}\left[|V\rangle\langle Q|V\rangle + \langle V|Q|\theta\rangle + (-1)^j2Re\left(\langle V|Q|\theta\rangle \int_{\mathbb{R}^3} |\psi_j(k)|^2e^{-icT|k|}d^3k\right)\right].
\]  

In particular, for \(Q = I\):

\[
\langle \psi_{\text{out}}^{(\text{precise})} | P_j \otimes I | \psi_{\text{out}}^{(\text{precise})} \rangle = \frac{1}{2}\left(1 + (-1)^j \cos \theta \int_{\mathbb{R}^3} \cos(cT|k|)|\psi_j(k)|^2d^3k\right).
\]

When dropping the third addend in the parenthesis of (30) and the second in (31) we obtain the same respective results as that of an incoherent superposition

\[
\rho_{\text{Mixed}} = \frac{1}{2}\left(|1\rangle\langle 1| \otimes |V\rangle\langle V| + |0\rangle\langle 0| \otimes |\theta\rangle\langle \theta|\right)
\]  

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entering the second detector in place of the entangled state $|\psi_{\text{entangled}}^{(\text{precise})}\rangle$ defined in (27). The second addend in the right-hand side of (31) and the third in the right-hand side of (30) describe the quantum interference as a function of the delay time $T$. Let us focus on the simplest case of $Q = I$ in (31), since the general case is a trivial extension of this. It holds

$$\int_{R^3} \cos(cT|k|)|\psi_j(k)|^2 d^3 k = \text{Re} \int_{0}^{+\infty} e^{iT\omega} f(\omega) d\omega,$$

where

$$f(\omega) = \frac{\omega^2}{c^2} \int |\psi_j(\omega/c, \vartheta, \varphi)|^2 \sin \vartheta d\vartheta d\varphi. \quad (33)$$

The right-hand side is nothing but $|\psi_j(k)|^2$ integrated only along the two polar angles $\vartheta, \varphi$ of the vector $k$ whose norm is $\omega/c$. We are assuming here that the function $f$ does not depend on $j$, since we are approximating it with a Gaussian sharply concentrated on $\omega_0 = c|k_0| = c|k_1|$ with a standard deviation $\sqrt{\sigma_\omega}$ which is nothing but the width of the filter for the entering light,

$$f(\omega) = \frac{e^{-(\omega-\omega_0)^2/2\sigma_\omega}}{\sqrt{2\pi\sigma_\omega}}. \quad (34)$$

Since $\omega_0 \gg \sqrt{\sigma_\omega}$, we can estimate the integral as

$$\text{Re} \int_{0}^{+\infty} e^{iT\omega} f(\omega) d\omega \simeq \text{Re} \int_{-\infty}^{+\infty} e^{iT\omega} f(\omega) d\omega = \text{Re}(e^{iT\omega_0} g(T)) = \cos(\omega_0 T) g(T)$$

where $g(T) = e^{-\frac{1}{2}\sigma_\omega T^2}$. The shape of this function is qualitatively identical to Fig. 7(b). The function $g$ is, up to normalization terms, the density of a centered Gaussian measure with squared standard deviation $\sigma_T = 1/\sigma_\omega$.

Therefore, if $|T| \gg 1/\sqrt{\sigma_\omega}$, the interference term is negligible in (31) and the state (29) can be safely replaced by the incoherent superposition (32) as we shall better discuss in the next section. When the delay $T$ is inside the coherence region, it makes sense to come back to the initial rougher approximation (see the next section) of very sharply picked packets, approximating the phase $e^{-icT|k|}$ in (31) with $e^{-iT\omega_0}$, i.e. $\xi = -T\omega_0$, and defining the entangled state entering the second beam splitter as

$$|\psi_{\text{entangled}}\rangle := \frac{1}{\sqrt{2}} \left( |1\rangle V + ie^{i\xi}|0\theta\rangle \right) \quad (35)$$

so that, the state exiting from the second beam splitter is

$$|\psi_{\text{out}}\rangle = \frac{i}{2} |0\rangle \left[ (1 + e^{i\xi} \cos \theta)|V\rangle + e^{i\xi} \sin \theta |H\rangle \right]$$

$$+ \frac{1}{2} |1\rangle \left[ (1 - e^{i\xi} \cos \theta)|V\rangle - e^{i\xi} \sin \theta |H\rangle \right]. \quad (36)$$

finding

$$\langle \psi_{\text{out}} | P_j \otimes I | \psi_{\text{out}} \rangle = \frac{1}{4} \left[ |1 + (-1)^j e^{i\xi} \cos \theta|^2 + \sin^2 \theta \right] = \frac{1}{2} (1 + (-1)^j \cos \theta \cos \xi). \quad (37)$$
1.4.2 Effective analysis of the coherence length/time

To complete the discussion about the length/time coherence, we show how our picture is in agreement (and actually explains it) with the phenomenological description proposed by\(^8\) where the action of the state \(|\psi_{\text{entangled}}\rangle\langle\psi_{\text{entangled}}|\) when measuring relevant observables is effectively represented in terms of a phenomenological density matrix \(\rho_c\) in the 4-dimensional Hilbert space \(\mathbb{C}^2_{\text{momentum}} \otimes \mathbb{C}^2_{\text{polarization}}\) as follows

\[
\rho_c = (1 - \epsilon)|\psi_{\text{entangled}}\rangle\langle\psi_{\text{entangled}}| + \epsilon \rho_{\text{Mixed}} \tag{38}
\]

where \(\rho_{\text{Mixed}}\) is given by \((32)\), \(|\psi_{\text{entangled}}\rangle\) is the pure vector in \(\mathbb{C}^2 \otimes \mathbb{C}^2\) defined in \((35)\) where we make explicit here the relation \(\phi = -T\omega_0\) for future convenience,

\[
|\psi_{\text{entangled}}\rangle = \frac{1}{\sqrt{2}} \left( |1\rangle_{V} + ie^{-iT\omega_0}|0\rangle_{\theta} \right). \tag{39}
\]

Finally, \(0 \leq \epsilon \leq 1\) is a phenomenological parameter which takes coherence properties into account: \(\epsilon = 0\) means that we are inside the coherence length, while \(\epsilon = 1\) means that coherence is lost.

We expect that \(\epsilon\) is a function of the delay time \(T\) and we go to investigate this relation with the help of a more precise analysis of the coherence performed in the previous section.

The effective state of the system after the action of the second beam splitter is given by

\[
\rho_{c,\text{out}} = (U_{BS} \otimes I)\rho_c(U_{BS} \otimes I)^\dagger.
\]

What really matters in our approach are the expectation values of observables \(|j\rangle\langle j| \otimes Q_j\) in the Hilbert space \(\mathbb{C}^2_{\text{momentum}} \otimes \mathbb{C}^2_{\text{polarization}}\) corresponding to observables \(P_j \otimes Q\) in the infinite dimensional Hilbert space where the precise time evolution \((25)\) takes place, where \(Q\) works in the polarization space. Hence, \(\rho_{c,\text{out}}\) is supposed to satisfy

\[
\text{tr}(\rho_{c,\text{out}}|j\rangle\langle j| \otimes Q) = \langle\psi_{\text{out}}^{\text{precise}}|P_j \otimes Q|\psi_{\text{out}}^{\text{precise}}\rangle
\]

and identity \((38)\) has to be interpreted as

\[
\langle\psi_{\text{out}}^{\text{precise}}|P_j \otimes Q|\psi_{\text{out}}^{\text{precise}}\rangle =
\]

\[
= (1 - \epsilon)\langle\psi_{\text{entangled}}|(U_{BS} \otimes I)^\dagger(|j\rangle\langle j| \otimes Q)(U_{BS} \otimes I)|\psi_{\text{entangled}}\rangle + \epsilon \text{tr}[(U_{BS} \otimes I)\rho_{\text{Mixed}}(U_{BS} \otimes I)^\dagger(|j\rangle\langle j| \otimes Q)] \tag{40}
\]

Expanding the left-hand side of the identity above, with \(\langle V|Q|\theta\rangle = |\langle V|Q|\theta\rangle|e^{i\phi}\) taking \((31)\) into account and exploiting \((32)\) and \((37)\) in the right-hand side, taking the made approximations into account and \((33), (34)\) in particular,
we eventually find the relation linking $\epsilon$ to $T$ and $\sigma_\omega$.

$$
\epsilon(T) = 1 - \int_{-\infty}^{+\infty} \cos(\omega T + \phi) \frac{e^{-\frac{(\omega - \omega_0)^2}{2\sigma_\omega^2}}}{\sqrt{2\pi\sigma_\omega}} \, d\omega = 1 - e^{-T^2\sigma_\omega^2/2}.
$$

As expected, coherence is completely lost for $|T| \gg \sqrt{\frac{2}{\sigma_\omega}}$, there the state can be considered mixed and described by (32) in that regime. For $|T| \ll \sqrt{\frac{2}{\sigma_\omega}}$ the approximated state (36) can be safely used in place of the more accurate one (29).

1.5 Theoretical violation of CHSH inequality for partially incoherent states

We want to compute here the value of the CHSH parameter $S$ when the measured state is the partially incoherent one arising from (38) with the value $\epsilon$ compatible with the experimental setup. The discussion of Section 1.2 can be restated simply replacing the pure state $|\Psi_+\rangle$ entering the stage (II) of the apparatus with the mixed state

$$
\rho_\epsilon = (1 - \epsilon)|\Psi_+\rangle \langle \Psi_+| + \epsilon \rho_{\text{Mixed}}
$$

with $\rho_{\text{Mixed}} = \frac{1}{2}(|0H\rangle \langle 0H| + |1V\rangle \langle 1V|)$. The $\phi$ phase shift in the second MZ induces a rotation $U(\phi/2)$ of an angle $\phi/2$ in the momentum Hilbert space. Setting $\tilde{\phi} = \phi/2$ we have:

$$
U(\tilde{\phi})|0\rangle = \cos \tilde{\phi}|0\rangle - \sin \tilde{\phi}|1\rangle, \quad U(\tilde{\phi})|1\rangle = \cos \tilde{\phi}|1\rangle + \sin \tilde{\phi}|0\rangle.
$$

(42)

The polarization rotators in the final stage of the apparatus produce a rotation $U(\theta)$ in the polarization Hilbert space of an angle $\theta$:

$$
U(\theta)|V\rangle = \cos \theta|V\rangle + \sin \theta|H\rangle, \quad U(\theta)|H\rangle = \cos \theta|H\rangle - \sin \theta|V\rangle.
$$

(43)

By setting $\rho' = U(\tilde{\phi}) \otimes U(\theta) \rho_{\text{Mixed}}(U(\tilde{\phi}) \otimes U(\theta))^\dagger$ we obtain the following probabilities

$$
P_{0V} = \text{Tr}[\rho'|0V\rangle \langle 0V|] = \frac{1}{2}(\cos^2 \tilde{\phi} \sin^2 \theta + \sin^2 \tilde{\phi} \cos^2 \theta)
$$

$$
P_{0H} = \text{Tr}[\rho'|0H\rangle \langle 0H|] = \frac{1}{2}(\cos^2 \tilde{\phi} \cos^2 \theta + \sin^2 \tilde{\phi} \sin^2 \theta)
$$

$$
P_{1V} = \text{Tr}[\rho'|1V\rangle \langle 1V|] = \frac{1}{2}(\cos^2 \tilde{\phi} \cos^2 \theta + \sin^2 \tilde{\phi} \sin^2 \theta)
$$

$$
P_{1H} = \text{Tr}[\rho'|1H\rangle \langle 1H|] = \frac{1}{2}(\cos^2 \tilde{\phi} \sin^2 \theta + \sin^2 \tilde{\phi} \cos^2 \theta)
$$

and

$$
P(\tilde{\phi}, \theta) = P_{1V} + P_{0H} - P_{0V} - P_{1H} = \cos(2\tilde{\phi}) \cos(2\theta).
$$

Hence, we obtain

$$
S(\tilde{\phi}, \tilde{\phi}', \theta, \theta') = P(\tilde{\phi}, \theta) + P(\tilde{\phi}', \theta) + P(\tilde{\phi}', \theta') - P(\tilde{\phi}, \theta')
$$

$$
= \cos(2\tilde{\phi}) \cos(2\theta) + \cos(2\tilde{\phi}') \cos(2\theta) + \cos(2\tilde{\phi}') \cos(2\theta') - \cos(2\tilde{\phi}) \cos(2\theta').
$$

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In the case where $\tilde{\phi} = 0, \theta = \alpha, 2\tilde{\phi}' = 2\alpha, 2\theta' = 3\alpha$, the corresponding value of $S$ is

$$S^{Mixed} = 2\cos^3\alpha - 2\sin^2\alpha\cos(3\alpha).$$

Dealing with the entangled state $|\Psi_+\rangle$ only as in Section 1.2 the S-parameter was as in (20)

$$S(\alpha) = 3\cos\alpha - \cos(3\alpha)$$

Hence, if the effective state is given by Eq. (41), we get

$$S^\epsilon(\alpha) = (1 - \epsilon)\left(3\cos\alpha - \cos(3\alpha)\right) + \epsilon\left(2\cos^3\alpha - 2\sin^2\alpha\cos(3\alpha)\right).$$

(44)

To control the time delay $T$ in our experiment, we change the length difference between the optical paths in the interferometers (Fig. 1 of main text). We can express $\epsilon$ as a function of the length difference $\Delta L$:

$$\epsilon(T) = 1 - e^{-T^2\sigma_\omega^2} = 1 - e^{-(\Delta L)^2/l^2}.$$ When $\Delta L = 0$ the system is in the optical contact condition, and we have $\epsilon = 0$. As we move away from optical contact, increasing $\Delta L$, the parameter $\epsilon$ increases and the coherence of the system is gradually lost.

Moreover, in order to take into account the non ideality of the setup and different sources of noise that contribute to reduce the visibility of the detection channels, we replace the state $\rho^\epsilon$ with an effective state $\rho^{eff}$ obtained as a convex combination of $\rho^\epsilon$ and the maximally mixed state:

$$\rho^{eff} = \eta\rho^\epsilon + (1 - \eta)I_4,$$

where $\eta \in [0, 1]$ is a positive parameter related to the visibility. The corresponding value for the function $S$ is

$$S^{eff}(\alpha) = \eta S^\epsilon(\alpha),$$

with $S^\epsilon$ given by (44).

1.6 Multi-particle states and their final measurement

This section is devoted to the theoretical analysis of the experiment in the case where states of the form (21) or (22) are considered.

Let us come back to the overall description of the complete experimental apparatus, as in Fig. 1 in the main text, using the description in terms of ideal states in the finite dimensional Hilbert space $\mathbb{C}^4 \cong \mathbb{C}^2_{\text{momentum}} \otimes \mathbb{C}^2_{\text{polarization}}$ assuming to work in the coherence regime. The net action of the first two stages of the apparatus in the one-particle space is represented by a unitary operator $R$ which encodes the cumulative action of beam-splitters, polarizers, delayers, etc. excluding the devices in the third stage, final detectors in particular. The net effect of the circuit is

$$|\psi\rangle \rightarrow |\psi'\rangle := R|\psi\rangle.$$
Actually \( R \) and, therefore, \(|\psi'\rangle\) - which should be better denoted by \(|\psi'_{a,b}\rangle\) - depend on the choice of the parameters \( a, b \) used to prepare the state in the second stage of the apparatus as discussed in Section 1.2. However this dependence is not essential in this discussion and we simply ignore it. In the formalism of second quantization, the action on a multi-particle state is next transferred from the one-particle formalism by the unitary operator \( U_R \), associated to \( R \), acting in the Fock space and completely defined by the requirements
\[
U_R a^\dagger \psi U_R^\dagger = a^\dagger \psi' , \quad U_R |\text{vac}\rangle = |\text{vac}\rangle , \quad |\psi'\rangle = R |\psi\rangle .
\]
As a consequence, the net action on the initial (generally multi-particle) state is like this, referring to the states described in Sect.1.3

(1) For a coherent superposition of pure states of finite number of particles the state (21) is transformed as follows
\[
|\Psi\rangle \mapsto |\Psi'\rangle := U_R |\Psi\rangle = \sum_{n=0}^{+\infty} C_n |n_{\psi'}\rangle ,
\]

(2) For an incoherent superposition of pure states of finite number of particles the state (22) is transformed as follows
\[
\rho \mapsto \rho' := U_R \rho U_R^\dagger = \sum_{n=0}^{+\infty} P_n |n_{\psi'}\rangle \langle n_{\psi'}| .
\]

### 1.6.1 Generation stage in the set-up: coherent and incoherent superposition are indistinguishable

As we shall discuss in details in the next section, the final stage of the experiment consists of the measurement of a set of (pairwise compatible, mutually exclusive and exhaustive) tests, that is observables \( Q \) (represented by orthogonal projectors) attaining only values 0 or 1 which can be tested simultaneously producing mutually exclusive outcomes and such that one of them must always produce the outcome 1. For each of these tests \( Q \), the probability to be found true (outcome 1) in the transformed state is like this.

(1) For a coherent superposition of pure states of finite number of particles, the probability that \( Q \) is true is
\[
\langle \Psi' | Q |\Psi'\rangle := \sum_{n=m=0}^{+\infty} C_m C_n \langle m_{\psi'} | Q | n_{\psi'}\rangle ,
\]

(2) For an incoherent superposition of pure states of finite number of particles, the probability that \( Q \) is true is
\[
\text{tr}(\rho' | Q) := \sum_{n=0}^{+\infty} P_n \langle n_{\psi'} | Q | n_{\psi'}\rangle .
\]

The crucial observation is that each used test \( Q \) commutes with the observable number of particle as we shall discuss in detail below (see in particular Remark[1.1]). As a consequence, in the case (1),
\[
\langle m_{\psi'} | Q | n_{\psi'}\rangle = 0 \text{ if } m \neq n
\]
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and
\[ \langle \Psi' | Q | \Psi' \rangle := \sum_{n=0}^{+\infty} |C_n|^2 \langle n_{\psi'} | Q | n_{\psi'} \rangle. \]

Hence the statistics of the final measurement cannot distinguish between the case of an initial state given in terms of a coherent superposition of pure states with given number of particles:
\[ |\Psi\rangle := \sum_{n=0}^{+\infty} C_n |n_{\psi}\rangle, \]
or a corresponding incoherent superposition
\[ \rho := \sum_{n=0}^{+\infty} |C_n|^2 |n_{\psi}\rangle \langle n_{\psi}|. \]

1.6.2 Preparation stage in the set-up: analysis of the type of tests

Let us describe the type of tests we actually finally measure on the states prepared by the second stage of the apparatus (Fig. 1 in the main text). Since, according to the above discussion, we can separately consider states with fixed number of particles, let us assume that the Fock state \( |n_{\psi}\rangle \) enters the first stage of the apparatus. What we actually measure on each photon in the prepared state \( |n_{\psi}\rangle \) (actually \( |n_{\psi',a}\rangle \)) are the values of two observables, each with two possible outcomes or, equivalently an observable with four possible of outcomes henceforth denoted by 1, 2, 3, 4 for notational simplicity. In fact, for \( n = 1 \) (i.e. one single photon) the four tests \( Q_1, Q_2, Q_3, Q_4 \) are one-to-one associated with the four channels of the experimental apparatus and correspond to the orthogonal projectors associated to the basis \( \{15\} \), i.e., \( |0V\rangle \langle 0V|, |1V\rangle \langle 1V|, |0H\rangle \langle 0H|, |1H\rangle \langle 1H|. \) These tests are mutually compatible, pairwise exclusive, and exhaustive:
\[ [Q_i, Q_j] = 0, \quad Q_i Q_j = 0 \quad \text{for} \ i \neq j, \quad Q_1 + Q_2 + Q_3 + Q_4 = I. \]

For \( n = 2 \), taking into account the fact that the photons are indistinguishable, so that the combined tests must be invariant under swap of particles, we find a set of 10 tests (instead of \( 4^2 = 16 \)) which are mutually compatible, pairwise exclusive, and exhaustive and account for all possible outcomes of measurements when the analyzed system is made of two particles:
\[ Q_1 \otimes Q_1, \quad Q_2 \otimes Q_2, \quad Q_3 \otimes Q_3, \quad Q_4 \otimes Q_4, \]
\[ Q_1 \otimes Q_2 + Q_2 \otimes Q_1, \quad Q_1 \otimes Q_3 + Q_3 \otimes Q_1, \quad Q_1 \otimes Q_4 + Q_4 \otimes Q_1, \]
\[ Q_2 \otimes Q_3 + Q_3 \otimes Q_2, \quad Q_2 \otimes Q_4 + Q_4 \otimes Q_2, \quad Q_4 \otimes Q_3 + Q_3 \otimes Q_4. \]

The first 4 tests are validated when both particle respectively produce the outcomes 1, 2, 3, and 4. In the remaining 6 cases, indistinguishably shows up: e.g., \( Q_1 \otimes Q_2 + Q_2 \otimes Q_1 \) is the observable which attains the value 1 exactly when
one particle produces the outcome 1 and the other produces the outcome 2. The probabilities to validate these tests are respectively given by

\[ \langle 2_{\psi'}|Q_k \otimes Q_k|2_{\psi'} \rangle = \langle \psi'|Q_k|\psi' \rangle^2 \] (45)

\[ \langle 2_{\psi'}|Q_k \otimes Q_h + Q_h \otimes Q_k|2_{\psi'} \rangle = 2\langle \psi'|Q_k|\psi' \rangle\langle \psi'|Q_h|\psi' \rangle \quad k \neq h. \] (46)

The statistics described by (45) and (46) is the cumulative distribution of a couple of independent identically distributed discrete random variables \( \xi_1, \xi_2 \), with four possible outcomes and distribution \( P(\xi_j = h) = \langle \psi'|Q_h|\psi' \rangle, \) \( h = 1, 2, 3, 4 \), while

\[ P(\{\xi_1 = h, \xi_2 = k\} \cup \{\xi_2 = h, \xi_1 = k\}) = 2\langle \psi'|Q_k|\psi' \rangle\langle \psi'|Q_h|\psi' \rangle \quad k \neq h. \]

**Remark 1.1** Notice that the 10 tests of the states with \( n = 2 \) are compatible and exclusive with the 4 tests of the states with \( n = 1 \) since these operators act in orthogonal sub-spaces of the Fock space. This fact is valid for generic \( n \neq m \).

The discussion can go on along the same reasoning dealing with the generic case of the state \( |n_{\psi'}\rangle\langle n_{\psi'}| \). In the \( n \)-particle subspace of the Fock space, the single-particle tests \( Q_1, Q_2, Q_3, Q_4 \) induce a class of tests \( Q_{(n_1,n_2,n_3,n_4)} \), where \( (n_1, n_2, n_3, n_4) \) ranges in the set of strings of 4 natural numbers (including 0) such that \( n_1 + n_2 + n_3 + n_4 = n \).

By definition, \( Q_{(n_1,n_2,n_3,n_4)} \) is validated if and only if \( n_1 \) particles produce the outcome 1, \( n_2 \) particles produce the outcome 2, \( n_3 \) particles produce the outcome 3, and \( n_4 \) particles produce the outcome 4.

These multi-particle tests are **mutually compatible**, \( \left[ Q_{(n_1,n_2,n_3,n_4)}, Q_{(m_1,m_2,m_3,m_4)} \right] = 0 \)

**pairwise exclusive**

\[ Q_{(n_1,n_2,n_3,n_4)}Q_{(m_1,m_2,m_3,m_4)} = 0 \quad \text{for} \quad (n_1, n_2, n_3, n_4) \neq (m_1, m_2, m_3, m_4) \]

and **exhaustive**

\[ \sum_{n_1+n_2+n_3+n_4=n} Q_{(n_1,n_2,n_3,n_4)} = 1. \]

They account for all possible outcomes of the final stage of the apparatus when the entering state includes \( n \) indistinguishable particles. In terms of one-particle tests \( Q_j \), these new tests are defined as

\[ Q_{(n_1,n_2,n_3,n_4)} \equiv \sum_{\pi} Q_1^{\pi(1)} \cdots Q_1^{\pi(n_1)} Q_2^{\pi(n_1+1)} \cdots Q_2^{\pi(n_1+n_2)} Q_3^{\pi(n_1+n_2+1)} \cdots Q_3^{\pi(n_1+n_2+n_3)} Q_4^{\pi(n_1+n_2+n_3+1)} \cdots Q_4^{\pi(n_1+n_2+n_3+n_4)} \] (47)
where the sum is taken over all permutations \( \pi \) of the set of indexes \( \{1, \ldots, n\} \) and the notation \( A_t^{i_1} \cdots A_t^{i_n} \), with \( \{j_1, \ldots, j_n\} = \{1, \ldots, n\} \), stands for the operator acting on the tensor product of \( n \) copies of the Hilbert space \( \mathbb{C}^4 \), and \( A_t^{j_i} \) indicates the operator

\[
A_t^{j_i} = I \otimes \cdots \otimes I \otimes A_t \otimes I \otimes \cdots \otimes I .
\]

For the state \( |n_{\psi'}\rangle \), the probability to find \( n_1 \) particles with outcome 1, \( n_2 \) particles with outcome 2, \( n_3 \) particles with outcome 3, and \( n_4 \) particles with outcome 4 (where \( n_1 + n_2 + n_3 + n_4 = n \)) can be computed from (47) to be

\[
\langle n_{\psi'}|Q_{(n_1,n_2,n_3,n_4)}|n_{\psi'}\rangle = \frac{n!}{n_1!n_2!n_3!n_4!} \langle \psi'|Q_1|\psi'\rangle^{n_1} \langle \psi'|Q_2|\psi'\rangle^{n_2} \langle \psi'|Q_3|\psi'\rangle^{n_3} \langle \psi'|Q_4|\psi'\rangle^{n_4} .
\]

Again, the right-hand side of (48) coincides with the multinomial distribution, i.e. the cumulative distribution of \( n \) independent random variables with 4 attainable outcomes \( h = 1, 2, 3, 4 \) and elementary probabilities

\[
P(h) = \langle \psi'|Q_h|\psi' \rangle .
\]

By considering the numbers of counts \( N_{xy}^{(a,b)} \) on each channel (where \( x = 0, 1 \) and \( y = V, H \)), the analysis above shows that the relative frequencies \( N_{xy}^{(a,b)}/N_{TOT}^{(a,b)} \) (with \( N_{TOT}^{(a,b)} = \sum_{x,y} N_{xy}^{(a,b)} \)) provide an estimate of the elementary quantum-mechanical probabilities [49].

This result provides a purely quantum mechanical explanation of recent results [710] where the violation of Bell inequalities is achieved even for intense light beams. According to our analysis, this so-called classical entanglement can be actually attributed to single-particle entanglement of the state of the single photons the beam is made of. Indeed, as discussed above, by replacing the relative frequencies \( N_{xy}^{(a,b)}/N_{TOT}^{(a,b)} \) with corresponding light intensities \( I_{xy}^{(a,b)}/I_{TOT}^{(a,b)} \), these still provide an estimate of the quantum-mechanical probabilities [49] and the correlation coefficients [17] and [18] will violate Bell inequalities for suitable choices of the parameters \( a, b, a', b' \).

Even if in the case of intense light beams the violation of CHSH inequalities in terms of light intensities can be also explained by means of the classical theory of electromagnetic fields, as soon as single photons are detected no classical realistic non-contextual description of the experimental results is possible. The crucial difference between our experimental context and the one described in [3] is the presence of single photon detectors allowing to collect a time-ordered sequence \( (x_n^{(a)}, y_n^{(b)}) \) (with \( x_n^{(a)}, y_n^{(b)} \in \{+1, -1\} \)) of outcomes of measurements of single particle observables \( O^a = a \cdot \sigma \) and \( O^b = b \cdot \sigma \), instead of just the cumulative distribution \( N_{xy}^{(a,b)} \). The subscript \( n \) stands for the chronological order of observation. In fact, a realistic non-contextual description of these experimental results is not possible, since a classical probabilistic hidden-variable theory does actually predict that Bell inequalities cannot be violated. Indeed, by describing the sequence of measurements outcomes \( (x_n^{(a)}, y_n^{(b)}) \) in terms of the realizations of sequences of independent
identically distributed discrete random variables \((\xi_n^{(a)}, \xi_n^{(b)})\) and by setting \(E(a, b) = E[\xi_n^{(a)} \xi_n^{(b)}]\) (\(E[x]\) denotes the expectation value of the variable \(x\)) a standard argument shows that the values of the \(S\) parameter defined by Eq. (18) satisfies the inequality \(|S(a, a', b, b')| \leq 2\).

**Remark 1.2** As remarked in [21] the possibility of violating Bell inequalities with multiple-particles states is a peculiarity of single-particle entanglement. Indeed, in the case the entanglement is between the same degree of freedom of two different particles, the relative frequencies \(N_{xy}^{(a,b)} / N_{TOT}^{(a,b)}\) of the possible measurement outcomes can be collected only when it is possible to clearly associate particles belonging to the same entangled pair. From an experimental point of view, this requirement can be fulfilled in terms of a strict control of the arrival times. In the case where multiple-particle states are involved, it is only possible to collect the marginal distributions, namely the numbers \(N_x^{(a)} = \sum_y N_{xy}^{(a,b)}\) and \(N_y^{(b)} = \sum_x N_{xy}^{(a,b)}\), which are not sufficient for the construction of the quantum-mechanical correlations [17].

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