CONNECTED COMPONENTS OF MODULI STACKS
OF TORSORS VIA TAMAGAWA NUMBERS

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ABSTRACT. Let $X$ be a smooth projective geometrically connected curve over a finite field with function field $K$. Let $G$ be a connected semisimple group scheme over $X$. Under certain hypotheses we prove the equality of two numbers associated with $G$. The first is an arithmetic invariant, its Tamagawa number. The second is a geometric invariant, the number of connected components of the moduli stack of $G$-torsors on $X$. Our results are most useful for studying connected components as much is known about Tamagawa numbers.

1. Introduction

We work over a finite ground field $k$. Let $X$ be a smooth geometrically connected projective curve over $k$ with function field $K$. Let $G$ be a semisimple group scheme over $X$. This means that $G$ is a smooth group scheme over $X$, all of whose geometric fibres are (connected) semisimple algebraic groups. We denote the generic fibre of $G$ by $G$.

Recall a little bit of terminology: $G$ is split, if it admits a split maximal torus over $K$. By the semisimplicity assumption, this implies that $G$ is a Chevalley group, i.e., a group scheme which comes from a split semisimple group defined over $k$ by base extension. The fundamental group scheme of $G$ has as fibres the fundamental groups of the fibres of $G$. It is a finite abelian group scheme over $X$. Thus $G$, or equivalently $G$, is simply connected if and only if this fundamental groups scheme is trivial.

The roots of this article are a circle of ideas that began to take shape in [Har70], [HN75] and [Har]. In particular, it was observed that there is a connection between Tamagawa numbers and the trace of the Frobenius endomorphism on the cohomology of certain moduli spaces. This led to the following:

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Conjecture 1.1 (G. Harder). If $G$ is split, the Tamagawa number of $G$ is equal to the number of connected components of the moduli space of $G$-torsors on $X$.

We will study a variation in this paper, that does not assume that $G$ is split. We will work with stacks rather than spaces. This has two advantages, firstly there is a precise relationship between the Lefschetz trace formula on the moduli stack with the Tamagawa number without the need for approximations, see §4. The second advantage is that these stacks always exist without any restriction on the characteristic of the ground field. See for example [Bal04].

In order to forge a relationship between the stack and the Tamagawa number we will need to assume that $G$ satisfies the Hasse principle, see §4. The Hasse principle is known to hold for all split groups, see [Har74]. If $\tilde{G}$ is the universal cover of $G$ then Weil’s conjecture asserts that the Tamagawa number of $\tilde{G}$ is one. This conjecture is known in the split case by [Har74]. In the number field case the full conjecture is known by [Kot88]. One expects that a variation of the proof would work in the geometric case but this work has not been completed. The main result of this work is that under certain assumptions on the ground field the Tamagawa number of $G$ is in fact the number of connected components of the moduli stack. Furthermore one can deduce that these components are in fact geometrically connected.

In section 3 we begin by recalling the definition of the Tamagawa number. The section ends with a precise statement of the main results of this paper.

The purpose of section 4 is to give the proof of the main results modulo the proof of the trace formula and Ono’s formula. Section 4.1 recalls basic facts about the Hasse principle. Section 4.2 use the Seigel formula and the trace formula to give a geometric interpretation of the Tamagawa number. Section 4.3 gives a formula for the canonical open compact subgroup in terms of special values of Artin L-functions. Finally the proof is given in 4.4.

Section 5 is devoted to an outline of the proof of the Lefschetz trace formula for the moduli stack of $G$-torsors. The hardest part of the proof is to show that the trace of the Frobenius converges absolutely on the cohomology of the stack. We only sketch many of the proofs as they are standard (although long) and the details can be found in [Beh90]. Section 5.1, describes the main results on semistability for torsors. Section 5.2 introduces the Shatz stratification on the moduli
stack of $G$-torsors. The proof of the trace formula along with some semipurity results for the weight spaces of the cohomology of the moduli stack of $G$-torsors are proved in section 5.3.

The final technical tool needed in section 4.4 is Ono’s formula and some of its consequences. This formula is proved in section 6.

Let us remark, that Ono’s formula implies that the Tamagawa number is equal to the number of elements of $\pi_1(G)$ when $G$ is split. Thus, we prove that for a semisimple Chevalley group the stack of $G$-bundles has $|\pi_1(G)|$ components.

In view of §4 below, it is tempting to try to use the results and methods of [AB82] to prove the main assertions of this paper. However one does not have the necessary base change theorems required to transport the results of the cited paper to positive characteristic. The moduli stacks of $G$-torsors are not proper, indeed they are not even separated. In the case $G$ is split and the ground field is $\mathbb{C}$ it is known by [AB82] and [Tel98] that the moduli stack has $|\pi_1(G,e)|$ components.

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2. Notations and Conventions

$\mathbb{Q}_\ell$ denotes the $\ell$-adic rationals. We fix once and for all an inclusion $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$.

$k, k_n$ ground field. Assumed to be perfect. From §5 onwards we will assume that $k = \mathbb{F}_q$ is a finite field. In this case $k_n$ denotes the extension of $k$ of degree $n$.

$X$ will be a smooth projective geometrically connected curve over $k$ with structure map $\pi : X \to \text{Spec} k$.

$G$ is a smooth connected reductive group scheme over $X$. From §6 onwards it will be assumed to semisimple.

$G$ is the generic fiber of $G$. From §6 onwards it will be assumed to semisimple. From (4.3) onwards we assume that $G$ satisfies the Hasse principle.

$\text{Dyn}(G)$ the scheme of Dynkin diagrams of $G$. See [DG70, exp. XXIV].

$\mathfrak{T}$ the free abelian group on the connected components of $\text{Dyn}(G)$.

$\mathfrak{T}(P)$ the free abelian subgroup of $\mathfrak{T}$ generated by the components of the type of $P$. 

Bun$_{\mathcal{H},X}$ or Bun$_{\mathcal{H}}$ the moduli stack of $\mathcal{H}$-torsors on $X$ where $\mathcal{H}$ is an affine group scheme over $X$.

Bun$_{\mathcal{H}}^\alpha$ denotes the moduli stack of $\mathcal{H}$-torsors of degree $\alpha$.

Bun$_{\mathcal{H}}^{\alpha,\leq m}$ denotes the moduli stack of $\mathcal{H}$-torsors of degree of instability at most $m$ and degree $\alpha$.

Bun$_{\mathcal{H}}^{\alpha,m}$ denotes the moduli stack of $\mathcal{H}$-torsors of degree of instability equal to $m$ and degree $\alpha$.

Bun$_{\mathcal{H}}^{\alpha,\sigma}$ denotes the moduli stack of $\mathcal{H}$-torsors of type of instability $\sigma$ and degree $\alpha$.

$\tau(G), \tau_n(G)$ is the Tamagawa number of $G$ or of $G \otimes_k k_n$.

3. The Main Results

Let us begin by recalling the definition of the Tamagawa number of a semisimple algebraic group. To do this, we begin by constructing the Tamagawa measure on $G(\mathbb{A})$.

For any point $x$ of $X$ (place of $K$), we denote by $K_x$ the completion of $K$ at $x$. The ring of integers inside $K_x$ is $\widehat{\mathcal{O}}_{X,x}$. The ring of adeles of $K$, notation $\mathbb{A}$, is the restricted product of all $K_x$ with respect to the $\widehat{\mathcal{O}}_{X,x}$. Throughout we fix an additive Haar measure $\mu_x$ on each $K_x$, normalized so that $\widehat{\mathcal{O}}_{X,x}$ has volume one.

We fix a section $\omega$ of the line bundle $\wedge^\dim G \text{Lie}(\mathcal{G})$. It induces, in a natural way, a Haar measure $\omega_x$ on each of the analytic varieties $\mathcal{G}(K_x)$, see [Oes84, Section 2]. The subset $\mathcal{G}(\widehat{\mathcal{O}}_{X,x})$ of $\mathcal{G}(K_x)$ is open and its volume is computed by the formula below, which also characterizes this measure.

**Proposition 3.1.** Let $n$ be the order of vanishing of $\omega$ at $x$. Then we have

$$\text{vol}(\mathcal{G}(\widehat{\mathcal{O}}_{X,x})) = |k(x)|^{-d-n}|\mathcal{G}(k(x))|,$$

where $d$ is the dimension of $G$ and $k(x)$ is the residue field of $x$.

**Proof.** See [Oes84, 2.5] $\square$

For a semisimple group scheme $\mathcal{G}$ on $X$ the vector bundle $\text{Lie}(\mathcal{G})$ is of degree 0 on $X$.

The *Tamagawa measure* is a measure on $G(\mathbb{A})$ defined by

$$q^{(1-g)\dim G} \prod_{x \in X} \omega_x.$$
finite by [Har69]. (Also, the Tamagawa number depends only on the
generic fibre $G$ of $\mathcal{G}$, even though we used $\mathcal{G}$ in the definition.)

**Conjecture 3.2.** (Weil) If $G$ is simply connected then $\tau(G) = 1$.

In the number field case this a theorem proved by R. Kottwitz, see
[Kot88].

Let $\text{Bun}_G$ be the moduli stack of $\mathcal{G}$-torsors on $X$. We will show that
under various hypothesis, the Tamagawa number computes the number
of open and closed substacks $\text{Bun}_G$. Precisely:

**Theorem 3.3.** Assume that there is a splitting field $L$ for $G$ whose con-
stant field is $k$ and that $k$ contains all roots of unity dividing the order
of the fundamental group of $G$. Further assume that Weil’s conjecture
holds for the universal cover of $G$ and $G$ satisfies the Hasse princi-
ple. Then $\text{Bun}_G$ has $\tau(G)$ components and each of these components is
geo metrically connected.

Note that given any $G$ we can always find $k_n/k$ such that by base
extending to $k_n$ the first two hypothesis of the theorem are satisfied.

We will deduce from this theorem:

**Corollary 3.4.** If the generic fiber of $\mathcal{G}$ is a Chevalley group then
$\text{Bun}_G$ has exactly $\tau(G)$ components each of which are connected.

Using similar techniques we can also prove:

**Theorem 3.5.** If Weil’s Tamagawa number conjecture is true then
$\text{Bun}_G$ is geometrically connected for every simply connected
$G$.

### 4. The Proof

4.1. **The Hasse principle.** We begin by recalling some theorems of
G. Harder on the Hasse principle. Recall that an algebraic group $G$
over $K$ satisfies the Hasse principle, if the map of Galois cohomology
sets

$$H^1(K, G) \longrightarrow \prod_{x \in X} H^1(K_x, G)$$

is injective.

**Theorem 4.1** (G. Harder). The Galois cohomology group $H^1(K, G)$
is trivial, if $G$ is simply connected. In particular, the Hasse principle
holds for such $G$.

**Proof.** See [Har75].

**Corollary 4.2.** The Hasse principle holds when the generic fiber $G$ is
a Chevalley group.
Proof. Let $G'$ be the universal cover of $G$ and $M$ the fundamental group scheme of $G$. As the first Galois cohomology vanishes for $G'$ and all its inner forms, we have an injection

$$H^1(K, G) \hookrightarrow H^2(K, M).$$

As $G'$ is also a Chevalley group, there is an exact sequence

$$0 \to M \to G'_m \to G'_m \to 0,$$

where $G'_m$ is a maximal torus of $G'$ containing $M$. The result follows from Hilbert’s theorem 90 and the fact that there is an injection of Brauer groups

$$\text{Br}(K) \hookrightarrow \prod_{x \in X} \text{Br}(K_x).$$

\[\square\]

Convention 4.3. For the remainder of this section we assume that the generic fiber $G$ satisfies the Hasse principle.

4.2. The Siegel formula and the trace formula. We now begin to give a geometric interpretation of the Tamagawa number of $G$.

Lemma 4.4. Let $x \in X$ be a closed point. The étale cohomology set $\text{H}^1(\text{Spec}(\hat{O}_{X,x}), G_x)$ is trivial.

Proof. We need to show that every $G_x$-torsor over $\text{Spec}(\hat{O}_{X,x})$ has a $\hat{O}_{X,x}$-point. By Lang’s theorem such a torsor has a point over the residue field of $\hat{O}_{X,x}$ which can be lifted to a $\hat{O}_{X,x}$-point by formal smoothness. \[\square\]

Proposition 4.5. Every $G$-torsor is trivial over the generic point of $X$.

Proof. We have a diagram of étale cohomology sets

$$\begin{array}{ccc}
H^1(X, G) & \longrightarrow & H^1(K, G) \\
\downarrow & & \downarrow \\
\prod_{x \in X} H^1(\text{Spec}(\hat{O}_{X,x}), G_x) & \longrightarrow & \prod_{x \in X} H^1(K_x, G).
\end{array}$$

The bottom left corner vanishes and the right vertical map is injective by the Hasse principle. So the top map is trivial. \[\square\]

Recall that the integral model $G$ of $G$ defines an open compact subgroup $\mathfrak{R}$ of $G(\mathbb{A})$ with

$$\mathfrak{R} = \prod_{x \in X} G(\hat{O}_{X,x}).$$
Lemma 4.6. There is a bijection between elements of \( G(\mathbb{A}) \) and (isomorphism classes of) triples \((P, \phi, (\rho_x)_{x \in X})\), where \( P \) is a \( G \)-torsor, \( \phi \) is a trivialization of \( P \) over the generic point of \( P \) and \( \rho_x \) is a trivialization of \( P \) over the formal disc \( \text{Spec}(\hat{O}_{X,x}) \).

Proof. There is an obvious map from such triples to the elements of \( G(\mathbb{A}) \). We construct its inverse as follows. Let \( a = (a_x) \) be an adelic point of \( G \). There is an open Zariski subset \( U \) of \( X \) such that \( a \) is integral over \( U \), that is, \( a_x \in G(\hat{O}_{X,x}) \) for \( x \in U \). Consider the flat cover

\[ U \cup \bigcup_{x \notin U} \text{Spec}(\hat{O}_{X,x}) \]

of \( X \). To construct \( P \) we need only specify descent data with respect to this cover, and apply faithfully flat descent. On the intersection \( U \cap \text{Spec}(\hat{O}_{X,x}) \) they are given by \( a_x \). This gives \( P \), together with a generic trivialization and a trivialization at each \( x \notin U \). For \( x \in U \) the trivialization \( \rho_x \) is given by the trivialization over \( U \) multiplied by \( a_x \).

\[ \square \]

Proposition 4.7. There is a bijection between points of the double coset space

\[ G(K) \backslash G(\mathbb{A})/\mathfrak{K} \]

and the set of isomorphism classes of \( G \)-torsors over \( X \).

Proof. Use the above Lemma together Proposition (4.5). \( \square \)

Theorem 4.8 (Siegel’s Formula). We have

\[ \tau(G) = \text{vol}(\mathfrak{K}) \sum_{P \in \text{Bun}_G(k)} \frac{1}{|\text{Aut}(P)|}. \]

The sum is over isomorphism classes of \( G \)-torsors on \( X \) and \( |\text{Aut}(P)| \) is the order of the automorphism group of \( P \), which is finite.

Proof. We have

\[ \tau(G) = \text{vol}(G(\mathbb{A})/G(K)) \]

\[ = \sum_x \text{vol}(\mathfrak{K}xG(K)/G(K)) \]
(the sum is over a collection of double coset representatives)

\[
= \sum_x \text{vol}(\mathfrak{g}) \frac{1}{|x.Kx^{-1} \cap G(K)|}
= \text{vol}(\mathfrak{g}) \sum_{P \in \text{Bun}_G(k)} \frac{1}{|\text{Aut}(P)|}.
\]

The first equality is by the preceding proposition. One checks in the bijection above that the automorphism group of \(P\) is identified with \(x.Kx^{-1} \cap G(K)\). Note that one can show that \(\text{vol}(\mathfrak{g})\) is finite, see [Kne67]. Furthermore the sum converges by [Har69].

We will prove below a Lefschetz trace formula for the algebraic stack \(\text{Bun}_G\). This formula forges a link between the Siegel formula and the cohomology of \(\text{Bun}_G\) and we will describe it now.

If \(\mathcal{X}\) is an algebraic stack over the finite field \(k\) we define its number of \(k\)-rational points by

\[
\# \mathcal{X}(k) = \sum \frac{1}{|\text{Aut}(x)|}
\]

where the sum is over isomorphism classes of objects in \(\mathcal{X}(k)\). We denote by \(\Phi\) acting on the cohomology of \(\mathcal{X}\). We will prove in §5 the following version of the trace formula for \(\text{Bun}_G\) over \(k\):

**Theorem 4.9.** We have

\[
\sum_{x \in \text{Bun}_G(k)} \frac{1}{|\text{Aut}(x)|} = q^{(g-1)\dim G} \sum (-1)^i \text{tr} \Phi|_{H^i(\text{Bun}_G, \mathbb{Q}_\ell)}
\]

and both sides converge absolutely.

**Remark 4.10.** The trace formula for stacks of finite type is proved in [Beh03]. The stack \(\text{Bun}_G\) is not of finite type but it is naturally filtered by stacks of finite type. Our main task in proving the above theorem is to prove the convergence.

**Corollary 4.11.** In the current setting we have

\[
\tau(G) = \text{vol}(\mathfrak{g}) q^{(g-1)\dim G} \sum_{i=0}^{\infty} (-1)^i \text{tr} \Phi|_{H^i(\text{Bun}_G, \mathbb{Q}_\ell)}.
\]

**Proof.** Combine the above with (4.9).
4.3. Artin L-functions and the volume of $\mathfrak{R}$. We begin by recalling Steinberg’s formula for the number of points of a semisimple group over a finite field. Let $H/k$ be a semisimple connected linear algebraic group. By Lang’s theorem it is necessarily quasi-split so let $B$ be a Borel subgroup and $T \subseteq B$ a maximal torus. Let $k_n/k$ be a splitting field for $H$. We form the Weyl group $W = (N_H(T)/T)(k_n)$. Note that $W$ acts on $X(T \otimes_k k_n) \otimes \mathbb{Q}$ and hence on its symmetric algebra $S$. By a theorem of Chevalley the invariants of this action is the symmetric algebra on finite dimensional graded vector space $V = \oplus_{n \geq 2} V_n$.

**Theorem 4.12.** Let $F$ be the Frobenius of $k_n/k$. We have

$$\frac{|H(k)|}{q^d} = \prod_{n \geq 2} \det((1 - q^{-n}F)|V_n),$$

where $d$ is dimension of $H$ and $q$ is the number of elements of $k$.

**Proof.** See [Ste68, 11.16].

We return to our global situation. Firstly observe that:

**Proposition 4.13.** Let $\mathfrak{R} = \prod_{x \in X} G(\overline{\mathcal{O}_{X,x}})$ be the canonical open compact. Then

$$\text{vol}(\mathfrak{R}) = q^{(1-g)\dim G} \prod_{x} |k(x)|^{-\dim G} |G(k(x))|. $$

**Proof.** This is by 3.1 combined with the fact that the vector bundle $\text{Lie}(\mathcal{G})$ has degree 0.

As there is an integral model $\mathcal{G}$ for $G$ we can by [5.1] find an unramified extension $L/K$ that splits $G$. We may assume that the extension is in fact Galois. Correspondingly we have a Galois cover $Y \to X$.

Again form the Weyl group $W = (N_G(T)/T)(L)$ which acts on the symmetric algebra of $X(G \otimes_k L) \otimes \mathbb{Q}$. Using Chevalley’s result again we obtain a finite dimensional graded vector space $V = \oplus_{n \geq 2} V_n$. Each of the $V_i$’s are $\text{Gal}(L/K)$-modules so we can form the associated Artin $L$-functions

$$L_n(X, s) = L(X, V_n, s) = \prod_{x \in X} \det((1 - q^{-s} \deg x f_x)|V_n).$$

In the above $f_x$ is the Frobenius for the extension of residue fields $k(y)/k(x)$ and $y$ is an point lying over $x$. Using the above theorem and
we have
\[
\text{vol}(\mathcal{R}) = q^{(1-g) \dim G} \prod_{n \geq 2} L_n(X, n)^{-1}.
\]

In summary:

**Theorem 4.14.** The volume of the open compact is given by
\[
\text{vol}(\mathcal{R}) = q^{(1-g) \dim G} \prod_{n \geq 2} L_n(X, n)^{-1},
\]
where each \( L_n \) is an Artin \( L \)-function described above. Furthermore we have
\[
L_n(X, s) = Z(X, s)^{\gamma_n} \prod_{i=1}^{m_n} p_{n_i}(X, s),
\]
where \( Z(X, s) \) is the Zeta function and
\[
p_{n_i}(X, s) = \prod (1 - \alpha_{n_i} q^{-s})
\]
and \( |\alpha_{n_i}| = q^{1/2} \) and \( \gamma_n \) is some integer.

**Proof.** The only part that needs justification is the last statement. This is by [Mil80] page 126. □

### 4.4. The Proof of the Main Theorems

We are in a position to give the proof of the main theorems modulo some technical results. The proofs of these will be given later.

We denote the Tamagawa number of the base extension by
\[
\tau_n(G) = \tau(G \otimes_k k_n).
\]
Here \( k_n \) is the unique extension of the finite field \( k \) of degree \( n \). A key ingredient in the proofs of the main theorems is the fact that the sequence
\[
\tau_1(G), \tau_2(G), \ldots,
\]
is constant under suitable hypothesis. Using Ono’s formula we will show:

**Proposition 4.15.** Suppose that \( G \) with field of constants \( k \) and all roots of unity dividing the order of \( |\pi_1(G)| \) are in \( k \). Further assume Weil’s conjecture for the universal cover of \( G \) we have
\[
\tau_n(G) = \tau(G).
\]
for every \( n \).

**Proof.** See (6.14). □
We need the following technical result to make the proofs of the main results go more smoothly:

**Definition 4.16.** Let $\sum_{m=1}^{\infty} s_{nm} = t_n$ be a sequence of series of complex numbers. We say that the series *converge uniformly* if for every $\epsilon > 0$ there is an $M_0$ such that for every $M \geq M_0$ we have
\[ \left| \sum_{m=1}^{M} s_{nm} - t_n \right| < \epsilon \]
independently of $n$.

Our theorem will follow from the following lemma

**Lemma 4.17.** Let $\sum_{m=1}^{\infty} s_{nm}$ be a sequence of series that all sum to $t$ independently of $n$. Furthermore assume that the convergence is uniform and that the series $\sum_{m=1}^{\infty} s_{nm}$ converge absolutely for each $m$. Then $t = 0$.

**Proof.** Let $\epsilon > 0$ and $M$ be as in the definition of uniform convergence. We have
\[ |t - \sum_{m=1}^{M} s_{nm}| < \epsilon. \]
However, $\sum_{n=1}^{\infty} \sum_{m=1}^{M} s_{nm}$ converges and hence
\[ \lim_{n \to \infty} \sum_{m=1}^{M} s_{nm} = 0, \]
and we are done. \(\square\)

Recall that the zeta function of $X$ can be written in the form
\[ Z(X, s) = \frac{\prod_{i=1}^{2g} (1 - \alpha_i q^{-s})}{(1 - q^{-s}) (1 - q^{-s+1})}, \]
where $\alpha_i$ are the eigenvalues of the Frobenius on $H^1(X, \mathbb{Q}_\ell)$. It will be important below to note that the $\alpha_i$ have absolute value $q^{1/2}$.

Finally, before giving the proof, we will need two facts about the cohomology of $\text{Bun}_G$. Firstly we will need to know that the vector spaces $H^i(\text{Bun}_G, \mathbb{Q}_\ell)$ are finite dimensional. Secondly we will need that the eigenvalues of $\Phi$ have absolute value at most $q^{-i/2}$ on $H^i(\text{Bun}_G, \mathbb{Q}_\ell)$. Both these facts are proved in §5.
Proof. (proof of 3.3) We have
\[
\tau(G) = \text{vol}(\mathfrak{g}) q^{(g-1)\dim G} \left( \sum_{i=0}^{\infty} \text{tr} \Phi|_{H^i(Bun_G, \mathbb{Q}_\ell)} \right) \quad (4.8)
\]
\[
= \prod_{n \geq 2} L_n(X, s)^{-1} (\text{tr} \Phi|_{H^0(Bun_G, \mathbb{Q}_\ell)}) \quad (4.14).
\]
Let \( \{\beta_j\} \) be the eigenvalues of \( \Phi \) on \( \bigoplus_{i>0} H^i(Bun_G, \mathbb{Q}_\ell) \) and \( \epsilon_j \) their signs in the above formula. So we have
\[
\tau(G) - \text{tr} \Phi|_{H^0(Bun_G, \mathbb{Q}_\ell)} = (\prod_{n \geq 2} L_n(X, n)^{-1} - 1) (\text{tr} \Phi|_{H^0(Bun_G, \mathbb{Q}_\ell)})
\]
\[
+ \prod_{n \geq 2} L_n(X, n)^{-1} (\sum_j \beta_j \epsilon_j).
\]
We remind the reader that \( \ell \)-adic cohomology of a stack \( \mathfrak{X} \) over a finite field is defined by first passing to the algebraic closure, i.e it is really defined on \( \mathfrak{X} \otimes_k \bar{k} \). With this in mind the action of the Frobenius on the cohomology of the base extension \( Bun_{\mathbb{G}_m} \) is just given by \( \Phi^m \).
\[
\tau_m(G) - \text{tr} \Phi^m|_{H^0(Bun_G, \mathbb{Q}_\ell)} = (\prod_{n \geq 2} L_n(X_m, n)^{-1} - 1) (\text{tr} \Phi^m|_{H^0(Bun_G, \mathbb{Q}_\ell)})
\]
\[
+ \prod_{n \geq 2} L_n(X_m, n)^{-1} (\sum_j \beta_j^m \epsilon_j).
\]
For future use, we denote the series on the right hand side of the above equation by \( A_m \).

Note that
\[
L_n(X_m, s) = L_n(X, s) = Z(X_m, s)^{\gamma_n} \prod_{i=1}^{m} p_{n_i}(X_m, s)
\]
where the \( \gamma_n \) is the same as that in (4.14). We have
\[
Z(X_m, s) = \frac{\prod_{i=1}^{2g} (1 - \alpha_i^m q^{-ms})}{(1-q^{-ms})(1-q^m(1-s))},
\]
and
\[
p_{n_i}(X_m, s) = \prod (1 - \alpha_{ni}^m q^{-ms}).
\]
In the above formulas the \( \alpha_i \) and \( \alpha_{ni} \) are the same as those in the formulas for \( X \). Now \( H^0(Bun_G, \mathbb{Q}_\ell) \) is finite dimensional by the results of §5. So there is a \( k_m/k \) such that the connected components of \( Bun_{\mathbb{G}_m} \) are geometrically connected and have a rational point. It follows that
\( \Phi^{lm} \) is the identity on \( H^0(Bun_G, \mathbb{Q}_l) \) for all \( l > 1 \). So using (4.15) the series \( A_{lm} \) satisfy the first of the hypothesis of (4.17). Now using the fact that \( |\beta_j| \leq q^{-1/2} \) by (4.22) the remaining hypothesis are easily checked. It follows that \( A_{lm} = 0 \). It follows that there are exactly \( \tau(G) \) components.

Now consider \( 1 < r < m \). A similar analysis shows that each of the series \( A_{lm+r} \) is zero. So \( tr \Phi^r |_{H^0(Bun_G, \mathbb{Q}_l)} = \tau(G) \) also. It follow that that \( \Phi \) must be the identity and we are done. \( \square \)

**Proof.** (proof of 3.4) Note that Weil's conjecture is true for Chevalley groups by [Har74]. The result is now obtained by combining the above with (6.8). \( \square \)

**Proof.** (proof of 3.5) Argue as in the proof of (3.3). \( \square \)

## 5. The Lefschetz trace formula for \( Bun_G \)

### 5.1. Semistability for \( G \)-torsors

The purpose of this subsection is to recall the main results and constructions of [Beh95]. The main point of that paper is to extend notions such as (semi)stability and Harder-Narasimhan filtration to torsors over a reductive group scheme.

For concepts such as root systems with complementary convex solids, special facets and semistability of root systems the reader is referred to the first three sections of [Beh95]. The relationship of these concepts with what is to follow can be found in section 6 of that paper.

The following construction will be used throughout this work.

**Lemma 5.1.** There is a finite étale cover \( f : Y \to X \) such that \( f^* \mathcal{G} \) is an inner form.

**Proof.** We make use of the notations of [DG70]. Let \( \mathcal{G}_0 \) be the constant reductive group scheme over \( X \) having the same type as \( \mathcal{G} \). Being an inner form means that the scheme \( \text{Isomext}(\mathcal{G}, \mathcal{G}_0) \) has a section over \( X \). By [DG70] XXIV, theorem 1.3 and by [DG70] XXII, corollary 2.3] \( G \) is quasi-isotrivial and hence so is \( \text{Isomext}(\mathcal{G}, \mathcal{G}_0) \). This implies by [DG70] X, corollay 5.4] that \( \text{Isomext}(\mathcal{G}, \mathcal{G}_0) \) is etale and finite over \( X \). So we take \( Y \) to be one of these components and the section is the tautological section. \( \square \)

Note that such an inner form is generically split by the Hasse principle. See (4.2) and (4.5) below.

**Definition 5.2.** Let \( \mathcal{H} \) be a smooth affine group scheme over \( X \) with connected fibers. We define the the degree of \( \mathcal{H} \) to be

\[
\deg \mathcal{H} = \deg \text{Lie}(\mathcal{H}),
\]
where \( \text{Lie}(H) \) is the Lie algebra of \( H \) viewed as a vector bundle on \( X \).

By (5.1) a reductive group scheme has degree 0.

**Definition 5.3.** (i) We say that \( G \) is **semistable** if for every parabolic subgroup \( P \) of \( G \) we have \( \deg P \leq 0 \).

(ii) We say that \( G \) is **stable** if for every parabolic subgroup \( P \) of \( G \) we have \( \deg P < 0 \).

(iii) The largest integer \( d \), such that there exists a parabolic subgroup \( P \) of \( G \) of degree \( d \) is called the **degree of instability** of \( G \) and is denoted \( \text{deg}_i(G) \).

By [Beh95, Lemma 4.3] the integer \( \text{deg}_i(G) \) is finite.

Let \( \text{Dyn}(G) \) be the scheme of Dynkin diagrams of \( G \), see [DG70, XXIV]. The power scheme of \( \text{Dyn}(G) \), denoted \( P(\text{Dyn}(G)) \) is the scheme that represents the functor

\[
\text{schemes}/X \to \text{sets} \quad T \mapsto P(\text{Dyn}(G_T)),
\]

here \( P \) means set of open and closed subschemes. For a parabolic subgroup \( P \) of \( G \) recall the definition of the type of \( P \), denoted \( t(P) \), from [Beh95, pg. 294]. The type \( t(P) \) is a section of \( P(\text{Dyn}(G)) \to X \).

In a nutshell, the type of \( P \) can be thought of in the following way: think of \( G \) as a family of reductive groups over \( X \) and then \( \text{Dyn}(G) \) is their Dynkin diagrams glued together in the appropriate way. Over a point \( x \in X \) choose a Borel subgroup contained inside \( P_x \). This Borel gives a choice of simple roots which correspond to the vertices of the Dynkin diagram over \( x \). We consider \( \text{Lie}(P)_x \subseteq \text{Lie}(G)_x \), and let \( R \) be the subset of the simple roots that consists of those roots \( \alpha \) such that the weight space for \( -\alpha \) is in \( \text{Lie}(P) \) with respect to the above inclusion. The value of \( t(P) \) over \( x \) is the complement of \( R \).

Let \( \Sigma \) be the free abelian group on \( \pi_0(\text{Dyn}(G)) \). By definition of power scheme, the section \( t(P) \) chooses some connected components of \( \text{Dyn}(G) \) and let \( \Sigma(P) \) be the free abelian group on these components.

Let \( \sigma \) be a positive element of \( \Sigma(P) \), that is an element of the form \( \sum n_i \sigma_i \) with the \( n_i \) positive. Given such an \( \sigma \) one can construct a vector bundle \( W(P, \sigma) \), we refer the reader to [Beh95, pg. 293] for the construction and basic properties.

**Definition 5.4.** Let \( P \) be a parabolic subgroup of \( G \) and let \( \sigma \) be a component of its type. We define the **numerical invariant** of \( P \) with respect to \( \sigma \) to be \( \deg W(P, \sigma) \). The collection of such numbers as \( \sigma \) varies over the components of the type of \( P \) are called the **numerical invariants** of \( P \).
Definition 5.5. A parabolic subgroup $\mathcal{P} \subseteq \mathcal{G}$ is called canonical if
\begin{enumerate}[(i)]  \item The numerical invariants of $\mathcal{P}$ are all positive.  \item The Levi component $\mathcal{P}/R_u(\mathcal{P})$ of $\mathcal{P}$ is semistable. \end{enumerate}

The main results of [Beh95] can be summarised in the following.

Theorem 5.6. There is a unique canonical parabolic subgroup of $\mathcal{G}$. It is maximal among parabolic subgroups of maximal degree. It commutes with pullback under separable covers.

The above constructions and definitions apply to a $\mathcal{G}$-torsor $E$ as follows. One forms the inner form
\[ E\mathcal{G} = E \times_{\mathcal{G},\text{Ad}} \mathcal{G}. \]
Then $E\mathcal{G}$ is a reductive group scheme over $X$ and we define the degree of $E$, etc to be that of $E\mathcal{G}$.

5.2. The Shatz stratification on $\text{Bun}_\mathcal{G}$. We will describe in this subsection the Shatz stratification on $\text{Bun}_\mathcal{G}$ and state its elementary properties. The proofs are often just generalizations of facts about the usual Shatz stratification for vector bundles. When new ideas are involved we sketch these. The interested reader is referred to [Beh90] for complete proofs.

Let $\text{Bun}_\mathcal{G}$ be the moduli stack of $\mathcal{G}$-torsors. Let $X(\mathcal{G})$ be the group of characters of $\mathcal{G}$. Each $\mathcal{G}$-torsor $E$ defines a map
\[ \text{deg } E : X(\mathcal{G}) \to \mathbb{Z}, \]
\[ \phi \mapsto \text{deg}(E \times_\phi \mathbb{G}_m). \]

For $\alpha \in X(\mathcal{G})^\vee$ we denote by $\text{Bun}_\mathcal{G}^\alpha$ the open and closed substack of $\text{Bun}_\mathcal{G}$ of torsors of degree $\alpha$. For $m$ an integer we denote by $\text{Bun}_\mathcal{G}^{\alpha \leq m}$ the substack of torsors of degree of instability at most $m$. It is an open substack of $\text{Bun}_\mathcal{G}^\alpha$ that is in fact of finite type. To show this last fact we proceed in several steps.

By a vector group over $X$ we mean the underlying additive group of a vector bundle over $X$.

Proposition 5.7. Let $V$ be a vector group on $X$. Then the natural map
\[ \text{Bun}_V \to H^1(X, V) \]
makes $\text{Bun}_V$ into an affine gerbe over the vector space $H^1(X, V)$. This gerbe is trivial, i.e., isomorphic to $BH^0(X, V) \times H^1(X, V)$. It follows that $\text{Bun}_V$ is a smooth stack of finite type of dimension $r(g - 1) - d$ where $r$ is the rank of $V$ and $d$ its degree.
Proof. A torsor for $V$ defines, via cocycles, a cohomology class and this defines the canonical map. It is easy to see it is a gerbe. Let $t : T \to H^1(X, V)$ be an affine morphism. The $T$-points of $H^1(X, V)$ are in bijection with $H^1(X_T, V_T)$, thinking of $T$ as a $k$-scheme by composition of structure maps. Hence $t$ defines a cohomology class $\xi \in H^1(X_T, V_T)$ which corresponds to a $V_T$-torsor $E$. This gives a map $\tilde{t} : T \to \text{Bun}_V$ that lifts $t$. Hence the triviality result. \hfill \Box

**Proposition 5.8.** Let $\mathcal{P}$ be a parabolic subgroup of $G$ and let $\mathcal{H} = \mathcal{P}/R_u(\mathcal{P})$. The natural map

$$\text{Bun}_\mathcal{P} \to \text{Bun}_\mathcal{H}$$

is a smooth epimorphism of stacks that is of finite type and relative dimension

$$\dim_X R_u(\mathcal{P})(g - 1) - \deg(E),$$

where $E$ is the universal $G$-torsor. It induces an isomorphism on cohomology.

**Proof.** The unipotent radical is filtered by subgroups all of whose quotients are vector bundles, see [DG70, XXVI, 2.1]. One then proceeds by induction. \hfill \Box

**Proposition 5.9.** Let $B$ be a Borel subgroup of $G$ and assume that $G$ is split over the generic point of $X$. Then for each $\beta \in X(B)^{\vee}$ the stack $\text{Bun}_B^\beta$ is of finite type.

**Proof.** The quotient $B/R_u(B)$ is a split torus. It is well known that the components of $\text{Bun}_{G_m}$ are of finite type and the result follows from the above proposition. \hfill \Box

**Proposition 5.10.** Let $Z$ be a projective scheme over $k$ and let $f : Z' \to Z$ be a projective flat cover. Let $\mathcal{H}$ be a smooth affine group scheme over $Z$. Then the natural pullback map

$$\text{Bun}_\mathcal{H} \to \text{Bun}_{f^*\mathcal{H}}$$

is affine and of finite presentation.

**Proof.** Straightforward. See [Beh90, 4.4.3]. \hfill \Box

Before we get to the proof of the fact that $\text{Bun}^{\alpha \leq m}_G$ is of finite type we need some constructions.

Let $\mathcal{P}$ be a parabolic subgroup of $G$. The type of $\mathcal{P}$ is an open and closed subscheme of $\text{Dyn}(G)$. Its connected components $\alpha_1, \alpha_2, \ldots, \alpha_s$ generate a subgroup $\mathcal{I}(\mathcal{P})$ of $\mathcal{I}$. There is an action of $\mathcal{P}$ on $W(P_0, \alpha_i)$. Taking the determinant of the action produces a character $\chi_i$ of $\mathcal{P}$. 
Definition 5.11. We say that an element $\alpha$ of $X(P)^\vee$ is positive if $\alpha(\chi_i) > 0$ for $i = 1, \ldots, s$. We denote by $X(P)^\vee_+$ the collection of all such positive elements.

We have a homomorphism
\[ T(P) \to X(P) \]
and taking duals and identifying the dual of $T(P)$ with itself via the basis $o_1, o_2, \ldots, o_s$ we obtain
\[ \sigma : X(P)^\vee \to T(P). \]

As $P$ acts on $R_u(P)$ and this group is filtered by subgroups with vector bundle quotients we may take determinants to obtain a character $\chi_0$. Evaluation at $\chi_0$ gives a map
\[ m : X(P)^\vee \to \mathbb{Z}. \]
Finally the inclusion $P \subseteq G$ gives a map
\[ \delta : X(P)^\vee \to X(G)^\vee. \]

Proposition 5.12. The map
\[ \delta \times \sigma : X(P)^\vee \to X(G)^\vee \times T(P) \]
is injective with finite cokernel.

Proof. The details can be found in [Beh90, 7.3.11] but the idea is as follows. Using (5.1) one can assume that $G$ is generically split. To see the reduction observe the cover in (5.1) may be taken to be Galois with group $\Gamma$. The character groups of the original groups are just the groups of $\Gamma$ invariants.

In the split case one uses the correspondences set up in [Beh95, §6] to reduce the question to questions about root systems with complementary solids.

Finally:

Theorem 5.13. The stack $Bun_{G}^{\alpha, \leq m}$ is of finite type.

Proof. See [Beh90, 8.2.6] for full details. Again, by passing to Galois covers, we may assume that $\mathcal{G}$ is generically split as the natural map
\[ Bun_{G}^{\alpha, \leq m} \to Bun_{\chi}^{tr(\alpha), \leq m} \]
is of finite type by (5.10). Choose a Borel $B \subseteq \mathcal{G}$ and let $\chi_1, \chi_2, \ldots, \chi_s$ be the associated characters.
\[ \coprod_{\beta} Bun_{B}^{\beta} \to Bun_{G}^{\alpha, \leq m} \]
where the disjoint union is over all characters such that

1. $d(\beta) = \alpha$
2. $m(\delta) \leq m$
3. $\beta(\chi_i) \geq -2g$.

The above Proposition shows this disjoint union is finite. A calculation shows that the morphism exists, i.e., the degree of instability of the torsors in the image is at most $m$. Furthermore, the morphism is surjective. By (5.9) we are done. □

Every parabolic subgroup $P$ of $G$ determines an element

$$\sum_{i=1}^{s} n(P, o_i) o_i \in \mathcal{T},$$

where $o_i$ are the connected components of the type of $P$. For a reductive group scheme $G$ on a family of curves $X \to S$ we define a function

$$n : S \to \mathcal{T}$$

as follows. For a point $s \in S$ choose an algebraic closure $\overline{k(s)}$ of the residue field at $s$. Define $n(s)$ to be $n(P_s)$ where $P_s$ is the canonical parabolic subgroup of $G_s$.

**Proposition 5.14.** Let $S_d$ be the locally closed subscheme of $S$ where the degree of instability of $G$ is $d$. Then $n$ is a continuous function on $S_d$.

**Proof.** See [Beh90] (7.2.9) □

Denote by $\text{Bun}_G^{\alpha,m}$ the locally closed substack of torsors of degree $\alpha$ and degree of instability $m$. Denote by $\text{Bun}_G^{\alpha,o}$ the locally closed substack of torsors of degree $\alpha$ and type of instability $o$.

If $E$ is a $P$ torsor of degree $\alpha$ then the torsor $E \times_P G$ has degree $\delta(\alpha)$. If $\sigma(\alpha) = \sum_{i=1}^{s} n_i o_i$ then $n_i = n(E \times_{P,Ad} P)$ where we think of $E \times_{P,Ad} P$ as a parabolic subgroup of $E \times_{P,Ad} G$. Furthermore, $\deg_i(E \times_{P,Ad} P) = m(\alpha)$.

**Theorem 5.15.** Denote by $\tilde{\mathcal{G}}$ the reductive group scheme $G \times_k \overline{k}$ over the curve $X \times_k \overline{k}$. Let $P$ be a parabolic subgroup of $\tilde{\mathcal{G}}$ and let $\alpha \in X(\mathcal{P})^{\vee}$. The natural map

$$\text{Bun}_{\mathcal{P}}^{\alpha,0} \to \text{Bun}_{\mathcal{G}}^{\delta(\alpha),m(\alpha)}$$

is finite radical and surjective.

**Proof.** Recall that a morphism is radical if induces a bijection on $L$-points for every field $L$. Representability of this morphism is easy to show. The fact that it is radical and surjective amounts to the existence and uniqueness of the canonical parabolic. □
5.3. The Lefschetz trace formula for \( \text{Bun}_G \).

**Proposition 5.16.** Let \( \mathcal{P} \) be a parabolic subgroup of \( \mathcal{G} \). Let \( \alpha \in X(\mathcal{P})_+ \). Let \( \mathcal{H} = \mathcal{P}/R_u(\mathcal{P}) \). Then there is a natural isomorphism

\[
H^i(\text{Bun}_{\mathcal{G}}^{d(\alpha),\sigma(\alpha)}, \mathbb{Q}_\ell) \to H^i(\text{Bun}_{\mathcal{H}}^{\alpha,0}).
\]

**Proof.** Use (5.15) and (5.8). Note that a finite radical and surjective morphism induces an isomorphism on cohomology. For this last fact see [AGV72, Expose VII].

**Lemma 5.17.** There is a function \( r: \mathfrak{T} \to \mathbb{Z} \) such that if \( E \) is a \( \mathcal{G} \)-torsor of type of instability \( \mathfrak{o} \) then \( r(\mathfrak{o}) = \dim_X R_u(\mathcal{P}) \) where \( \mathcal{P} \) is the canonical parabolic of \( E \).

**Proof.** This is just because two parabolic subgroups having the same type are twisted forms of each other.

**Proposition 5.18.** The closed immersion

\[
\text{Bun}_G^{\alpha,\mathfrak{o}} \to \text{Bun}_G^{\alpha,\leq m(\mathfrak{o})}
\]

is of codimension \( c(\mathfrak{o}) = r(\mathfrak{o})(g - 1) + m(\mathfrak{o}) \).

**Proof.** This is a standard dimension calculation.

Define \( \gamma(i) \) to be the smallest integer such that

\[
\gamma(i) \geq \begin{cases} 
1 + i/2 & \text{if } g > 0 \\
1 + i/2 + |\Phi| & \text{if } g = 0
\end{cases}
\]

where \( |\Phi| \) is the number of roots of \( \mathcal{G} \).

**Proposition 5.19.** Let \( i \geq 0 \) be such that \( m \geq \gamma(i) \). Then the canonical map

\[
H^i(\text{Bun}_G^{\alpha,\leq m}, \mathbb{Q}_\ell) \to H^i(\text{Bun}_G^\alpha, \mathbb{Q}_\ell)
\]

is an isomorphism.

**Proof.** Let \( c \) be the codimension of \( \text{Bun}_G^{\alpha,\mathfrak{o}} \) in \( \text{Bun}_G^{\alpha,\leq m} \) where \( m(\mathfrak{o}) = m \). Using the above one shows that for \( m \geq \gamma(i) \) we have \( i \leq 2c - 2 \). A Gysin sequence yields the result.

For a divisor \( D \) on \( X \) we denote by \( \text{Bun}_G(D) \) the moduli stack of \( \mathcal{G} \)-torsors with level structure over \( D \). By level structure over \( D \) for a torsor \( E \) we mean a section of \( i^*_DE \) where

\[
i_D: D \hookrightarrow X
\]

is the natural inclusion.
Proposition 5.20. Let $D$ be a divisor on $X$. Let $\text{Bun}^{\alpha \leq m}_G(D)$ be the moduli stack of $G$-torsors on $X$ with level structure at $D$. Let $E$ be the universal torsor on $\text{Bun}^{\alpha \leq m}_G$. If
$$p_* E \text{Lie}(G)(-D)|_{\text{Bun}^{\alpha \leq m}_G} = 0$$
then $\text{Bun}^{\alpha \leq m}_G(D)$ is a Deligne-Mumford stack.

Proof. We need to show that the diagonal morphism of $\text{Bun}^{\alpha \leq m}_G(D)$ is unramified. Let $K$ be a field and consider a $K$-point $\text{Spec} K \to \text{Bun}^{\alpha \leq m}_G$. It corresponds to a pair $(E, s)$ where $E$ is a torsor on $X_K$ and $s$ is a section. Denote by $\text{Aut}(E, s)$ the automorphism group of $E$ compatible with $s$. If $\pi : X \to \text{Spec} k$ is the structure morphism then we need to show that $\pi_K(\text{Aut}(E, s))$ is unramified over $\text{Spec} K$. A calculation shows that one may identify the Zariski tangent spaces of this group with $H^0(X_K, \text{Lie}(G)(-D))$ which vanishes. \[\square\]

Theorem 5.21. The eigenvalues of the arithmetic Frobenius acting on $H^i(\text{Bun}^{\alpha \leq m}_G, \mathbb{Q}_\ell)$ have absolute value at most $q^{-i/2}$.

Proof. First, for a smooth Deligne-Mumford stack the analogous statement is true by comparison to its coarse moduli space. See [Beh93]. Now for a quotient stack $[X/\Gamma]$ where $X$ is a smooth Deligne-Mumford stack and $\Gamma$ any algebraic group, we proceed as follows. First we may assume that $\Gamma = \text{GL}_n$ by choosing a faithful representation $\Gamma \hookrightarrow \text{GL}_n$ and replace $X$ by $X \times_\Gamma \text{GL}_n$. The Leray spectral sequence for $[X/\text{GL}_n] \to \text{BGL}_n$ has $E_2$ term $H^i(\text{BGL}_n, \mathbb{Q}_\ell) \otimes H^j(X, \mathbb{Q}_\ell)$. The result follows for the quotient as it is known for $X$ and $\text{BGL}_n$. \[\square\]

Corollary 5.22. The eigenvalues of the arithmetic Frobenius acting on $H^i(\text{Bun}^{\alpha}_G)$ have absolute value at most $q^{-i/2}$.

Lemma 5.23. For the group $\mathcal{G}$ there is a function $m : \mathcal{X} \to \mathbb{Z}$ such that if $E$ is a $\mathcal{G}$-torsor with type of instability $\mathfrak{o} \in \mathcal{X}$ then $m(\mathfrak{o})$ is the degree of instability of $E\mathcal{G}$. If $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$ then...
the diagram

\[
\begin{array}{ccc}
\mathcal{X}(\mathcal{P})^\vee & \xrightarrow{\sigma} & \mathcal{X} \\
\downarrow & & \downarrow m \\
\mathbb{Z} & & \\
\end{array}
\]

commutes.

**Proof.** If \(E\) and \(F\) are torsors with the same type of instability \(\mathfrak{o}\). There is a parabolic \(\mathcal{P}\) of type \(\eta\) where \(\eta\) is the support of \(\mathfrak{o}\) and \(E\) and \(F\) have reductions \(E'\) and \(F'\) to \(\mathcal{P}\). Then \(\mathfrak{o}(\deg E') = \mathfrak{o} = \mathfrak{o}(\deg F')\) and our lemma follows from the fact that \(m\) factors through \(\mathcal{X}(\mathcal{P})\). \(\square\)

Let \(\eta\) be a closed and open subscheme of \(\text{Dyn}(\mathcal{G})\). Let

\[C(\eta, \mu) = |\{\mathfrak{o} \in \mathcal{X}(\eta)^+ | m(\mathfrak{o}) = \mu\}|.\]

Here \(\mathcal{X}(\eta)^+\) denotes the set of linear combinations in the support of \(\eta\) all of whose coefficients are positive.

**Lemma 5.24.** We have \(C(\eta, \mu) = O(\mu^s)\) where \(s\) is the number of components of the type of \(\mathcal{P}\).

**Proof.** Recall the definition of the characters \(\chi_i\) and \(\chi_0\) from the discussion after Proposition (5.10). The result follows from the fact that there are positive rational numbers such that

\[\chi_0 = \sum_{i=1}^{s} y_i \chi_i.\]

\(\square\)

**Lemma 5.25.** Let \(R\) be the radical of \(\mathcal{G}\). There exist finitely many \(d_1, \ldots, d_n \in X(\mathcal{G})^\vee\) such that for every \(d \in X(\mathcal{G})^\vee\) there is an \(R\)-torsor of degree \(d_i - d\).

**Proof.** Let \(M = \{\delta \in X(R)^\vee | \text{Bun}_R^\delta \neq \emptyset\}\). By looking at dual objects one constructs an exact sequence of tori over \(X\)

\[1 \rightarrow S \rightarrow R \rightarrow G'_m \rightarrow 1\]

where the last map has a quasi-section. We can identify \(X(R)^\vee\) with \(X(G'_m)^\vee\) and using this identification and the quasi-section we observe that \(M\) has finite index in \(X(R)^\vee\). It follows that \(M\) has finite index in \(X(\mathcal{G})^\vee\) and we take \(d_i\) to be a set of coset representatives for \(X(\mathcal{G})^\vee / M\). \(\square\)
Let \( A \) be the set of open and closed subschemes \( \eta \subseteq \text{Dyn} \mathcal{G} \) such that there is a torsor \( E \) whose canonical parabolic has type \( \eta \). For each \( \eta \in A \) fix a torsor \( E \). Let \( \mathcal{P}_\eta \subseteq \mathcal{E}_n \mathcal{G} \) be the canonical parabolic and let \( H_\eta = \mathcal{P}_\eta / R_u(\mathcal{P}_\eta) \) be its Levi factor. We choose for each \( \eta \) degrees \( d(\eta, 1), d(\eta, 2), \ldots, d(\eta, n) \in X(H_\eta) \) according to the above lemma. We get a finite family of stacks \( \text{Bun}_{H_\eta}^{d(\eta,j),0} \) parameterized by \( A \times \{1, \ldots, n\} \). Set

\[
b_i(\eta, j) = \dim_{\mathbb{Q}_\ell} H^i(\text{Bun}_{H_\eta}^{d(\eta,j),0}, \mathbb{Q}_\ell).
\]

Choose \( B(i) = \sup b_i(\eta, j) \).

**Lemma 5.26.** There is an integer \( N \) so that \( B(i) = O(i^N) \).

**Proof.** The stacks in question are quotients of smooth Deligne-Mumford stacks by \( (5.20) \). The result follows from the usual spectral sequence. \( \square \)

Recall the definitions of \( \gamma(i) \) after \( (5.18) \) and \( r(\sigma) \) in \( (5.17) \). We set

\[
D(i) = \sum_{\eta \in A} \sum_{\mu = 0}^{\gamma(i)} C(\eta, \mu) B(i - 2\mu - 2r(\eta)(g - 1)).
\]

**Lemma 5.27.** The following sum converges:

\[
\sum_{\eta \in A} \sum_m q^{-m+(1-g)R_u P} \sum_i \dim H^i(\text{Bun}_{H_\eta}^{d(\eta,j),0}, \mathbb{Q}_\ell) q^{-i/2}.
\]

**Proof.** One use the above estimate. \( \square \)

The convergence now follows from some general observations about cohomology of stacks that we outline below. Denote by \( W^i H^j(X, \mathbb{Q}_\ell) \) the \( i \) th weight space of the \( j \)th cohomology group.

Recall that a morphism \( Z \to \tilde{Z} \) is called a universal homeomorphism if it is finite radical and surjective. Such a morphism induces an equivalence of etale sites by \([GR71, Expose IX 4.10]\).

**Theorem 5.28.** Let \( \mathfrak{X} \) be a smooth stack with a countable stratification by locally closed stacks \( \mathfrak{Z}_i \). We assume that the union \( \mathfrak{X}_n = \bigcup_{i \geq 0} \mathfrak{Z}_i \) is open. Suppose that there are universal homeomorphisms \( \mathfrak{Z}_i \to \mathfrak{Z}_i \) with each of the \( \mathfrak{Z}_i \)'s smooth and the following sum converges:

\[
\sum_{n=0}^{\infty} q^{-\text{codim}(\mathfrak{Z}_i, \mathfrak{X})} \sum_{i,j} \dim G^1_i H^j(\mathfrak{Z}_n, \mathbb{Q}_\ell) q^{-i/2} < \infty.
\]

Then the trace of the Frobenius converges absolutely on \( \mathfrak{X} \).
Proof. Let $Z \to X$ be a morphism of finite type smooth schemes which factors as $Z \to \tilde{Z} \to X$, where $\pi : Z \to \tilde{Z}$ is a universal homeomorphism and $i : \tilde{Z} \to X$ a closed immersion with complement $U$. We have a long exact sequence

$$\ldots \to H^*(\tilde{Z}, i^! \mathbb{Q}_\ell) \to H^*(X, \mathbb{Q}_\ell) \to H^*(U, \mathbb{Q}_\ell) \to \ldots$$

Let $c = \dim X - \dim Z$. We have

$$H^{*-2c}(Z, \mathbb{Q}_\ell(-c)) = H^*(Z, \pi^! i^! \mathbb{Q}_\ell)$$

because $Z$ and $X$ are smooth. Now pulling back via $\pi$ induces an isomorphism of étale sites (see [GR71, Expose IX,4.10]). As $\pi_*$ is the right adjoint of $\pi^*$, it is the inverse of $\pi^*$ and hence also a left adjoint of $\pi^*$. Since $\pi$ is proper, we conclude that $\pi^! = \pi^*$. Thus, we have

$$H^*(Z, \pi^! i^! \mathbb{Q}_\ell) = H^*(Z, \pi^* i^! \mathbb{Q}_\ell) = H^*(\tilde{Z}, i^! \mathbb{Q}_\ell),$$

Thus we have a natural long exact sequence

$$\ldots \to H^{*-2c}(Z, \mathbb{Q}_\ell(-c)) \to H^*(X, \mathbb{Q}_\ell) \to H^*(U, \mathbb{Q}_\ell) \to \ldots$$

This result extends to stacks and filtrations of schemes and stacks consisting of more than two pieces. Assembling these long exact sequences we get the required result and some simple analysis gives the required result.

Theorem 5.29. We have

$$\sum_{x \in \text{Bun}_G^m(k)} \frac{1}{\text{Aut}(x)} = q^{(g-1)(\dim G)} \sum (-1)^i \text{tr} \Phi |_{H^i(\text{Bun}_G, \mathbb{Q}_\ell)}$$

and both sides converge absolutely.

Proof. The convergence of the trace is by the aboove proposition and lemma, noting that the natural maps

$$\text{Bun}_P^{m, 0} \to \text{Bun}_G$$

are finite radical and surjective onto their image by the uniqueness of the canonical parabolic. The stratification being used is Shatz stratification induced by reduction to the canonical parabolic. The open substacks of bounded degree of instability are of finite type by 5.13. As the trace formula holds for these the result follows.
6. Ono’s Formula and Applications

Let $M$ be the fundamental group scheme of $G$. So $M$ is an Abelian group scheme over $K$ and there is an exact sequence

$$1 \to M \to \tilde{G} \to G \to 1,$$

where $\tilde{G}$ is the universal cover of $G$. For a continuous $\text{Gal}(\overline{K}/K)$-module $N$ let

$$\text{III}^1(K, N) = \ker \left( H^1(K, N) \to \prod_{x \in X} H^1(K_x, N) \right)$$

Here $K_x$ is the completion of the global field $K$ at $x$.

**Theorem 6.1.** (Ono’s Formula) Assume that Weil’s conjecture holds true for the universal cover $\tilde{G}$ of $G$, that is

$$\tau(\tilde{G}) = 1.$$

Then we have

$$\tau(G) = \frac{|H^0(K, \hat{M})|}{|\text{III}^1(\hat{M})|},$$

where $\hat{M} = \text{Hom}(M, \mathbb{G}_m)$ is the dual Galois module.

A notational clarification is in order here. The object $\hat{M}$ is to be viewed as functor on field extensions of $K$. We have

$$\hat{M}(L) = X(M) \otimes_K L.$$

We will often write $H^i(L, \hat{M})$ when we really mean $H^i(L, \hat{M}(\bar{L}_{\text{sep}}))$.

The above theorem is the main result of [Ono65]. It was originally only proved in the number field case and some modifications are needed in the function field case. We will detail these below.

To prove this result we need to generalize the theory of Tamagawa measures to reductive groups. We refer the reader to [Oes84, 1.4] for the definition of the Tamagawa measure $d\tau_H$ for a reductive group $H$.

In [Ono65], the theorem is proved by reducing to the case of an isogeny of tori. This was treated in [Ono63]. However this last paper contains a small error in the function field case which was corrected in [Oes84, pg.23 and Chapter IV].

We need some background results before giving the proof of the above theorem. Let $\Lambda_1 \subseteq \Lambda$ be an inclusion of free abelian groups of the same rank $r$. Let

$$x = \{x_1 = 0, x_1, \ldots x_t\}$$

be a set of coset representatives for $\Lambda/\Lambda_1$. A function $f : \Lambda \to \mathbb{R}$ is said to be $x$-compatible if
(i) $f$ has finite support.
(ii) $f(\alpha) = f(\alpha + x_i)$ for all $\alpha \in \Lambda_1$ and all $i$.

Such a function is said to be $\Lambda_1$-compatible if it is $x$-compatible for some choice of coset representatives containing 0.

**Lemma 6.2.** Suppose we have three lattices
\[ \Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \]
of the same rank $r$. Let $f : \Lambda_3 \to \mathbb{R}$ be $\Lambda_1$ compatible then
\[ \sum_{y \in \Lambda_2} f(y) = \left( \sum_{y \in \Lambda_3} f(Y) \right) \frac{1}{[\Lambda_3 : \Lambda_2]} . \]

**Proof.** This is elementary. \(\square\)

We denote by $D_q$ the set $\{q^i | i \in \mathbb{Z}\}$. Choose a basis $\{\chi_1, \ldots \chi_r\} = \chi$ for the group $X(H)$ of rational characters of $H$. We define
\[ \psi^\chi_H = \psi_H : H(\mathbb{A}) \to D_q^r \]
by sending
\[ x \mapsto (||\chi_1(x)||, ||\chi_2(x)||, \ldots, ||\chi_r(x)||) . \]
In the above $\chi_i$ is really the adelization of $\chi_i$. The image of $\psi$ is of finite index in $D_q^r$. Denote by $H(\mathbb{A})^1$ the kernel of $\psi$. We denote by $d\tau^1_H$ the measure on $H(\mathbb{A})^1$ that is the quotient of $d\tau_H$ by $[D_q^r : \text{Im}(\psi)](\log q)^r$. The Tamagawa number of $H$ is
\[ \tau(H) = \int_{H(\mathbb{A})^1/H(K)} d\tau^1_H . \]

**Proposition 6.3.** Let $\Lambda \subseteq \text{Im}(\psi) \subseteq D_q^r$ be a sublattice of maximal rank. If $F$ is $\Lambda$-compatible then
\[ \int_{H(\mathbb{A})/H(K)} F(\psi_H(x)) d\tau_H = \tau(H) \left( \sum_{x \in D_q^r} F(x) \right) (\log q)^r . \]

**Proof.** We have, noting the previous lemma,
\[ \int_{H(\mathbb{A})/H(K)} F(\psi_H(x)) d\tau_H = \sum_{y \in \text{Im}(\psi)} F(y) \int_{H(\mathbb{A})^1/H(K)} d\tau \]
\[ = \sum_{y \in D_q^r} F(y) \int_{H(\mathbb{A})^1/H(K)} \frac{d\tau}{[D_q^r : \text{Im}(\psi)]} \]
\[ = \sum_{y \in D_q^r} F(y) \int_{H(\mathbb{A})^1/H(K)} d\tau^1_H (\log q)^r . \]
\(\square\)
**Proposition 6.4.** Consider an exact sequence

\[ 1 \rightarrow H' \xrightarrow{i} H \rightarrow H'' \rightarrow 1. \]

Suppose \( H' \) is a torus, \( H'' \) is semisimple and the sequence is generically split. Then

\[ \tau(H')\tau(H'') = \tau(H)|\cok \hat{i}|. \]

In the above formula \( \hat{i} \) is the dual map on character groups.

**Proof.** (cf. [Ono65, Proposition(1.2.2)]) The fact that the sequence is generically split implies that the induced map on adelic and \( K \)-points is exact. Furthermore \( \mathfrak{X}(H) \) is a subgroup of \( \mathfrak{X}(H') \) of finite index \( |\cok \hat{i}| \).

By the elementary divisors theorem we may choose a basis \( \chi_1 \ldots \chi_r \) of \( \mathfrak{X}(H') \) such that \( m_1\chi_1 \ldots m_r\chi_r \) is a basis of \( \mathfrak{X}(H) \). We have a diagram

\[
\begin{array}{c}
D^r_q \xrightarrow{m} D^r_q \\
\downarrow \quad \downarrow \\
\text{Im}(\psi_{H'}) \xrightarrow{\psi_{H'}} \text{Im}(\psi_H) \\
\downarrow \quad \downarrow \\
H'(\mathbb{A}) \xrightarrow{\psi_{H}} H(\mathbb{A}).
\end{array}
\]

We have a sequence of inclusions

\[ \text{Im}(\psi_{H'}) \hookrightarrow D^r_q \subset D^r_q. \]

In what follows we write the group operation on \( D^r_q \) additively. There are basis of the two outside lattices of the form

\[ \{e_1, \ldots e_r\} \quad \{d_1e_1, \ldots d_re_r\}, \]

by the elementary divisors theorem. Define a function on \( D^r_q \) by

\[ f(e_1^{\alpha_1} + \ldots + e_r^{\alpha_r}) = \begin{cases} 1 & 0 \leq \alpha_i < d_i \\ 0 & \text{otherwise} \end{cases} \]

This function has the property that

\[
\int_{H'/(\mathbb{A})/H'(\mathbb{K})} f(\psi_H(x') + t)d\tau_{H'} = \int_{H'/(\mathbb{A})/H'(\mathbb{K})} f(\psi_H(x'))d\tau_{H'},
\]

which just follows from the definitions. Using the compatibility properties of \( f \) and (6.2) one shows that

\[
\int_{H'/(\mathbb{A})/H'(\mathbb{K})} f(\psi_{H'}(x'))d\tau_{H'} = \frac{1}{|\cok \hat{i}|} \int_{H'/(\mathbb{A})/H'(\mathbb{K})} f(\psi_{H'}(x')).
\]
This follows from the definition of $f$ and the fact that $d\tau_{H'}$ is a Haar measure. We have

$$
\tau(H)(\sum_{y \in D_q} f(y))(\log q)^r = \int_{H(H)/H(K)} f(\psi_H(x))d\tau_H
$$

$$
= \int_{H''(H)/H''(K)} d\tau_{H''} \int_{H'/(H'/K)} f(\psi_{H'}(x')\psi(x))d\tau_{H'}
$$

$$
= \int \tau(H') \int_{H''/(H''(K)} f(\psi_{H'}(x'))d\tau_{H'}
$$

$$
= \frac{\tau(H')\tau(H'')}{|cok_i|} (\sum_{y \in D_q} f(y))(\log q)^r,
$$

which finishes the proof. \[\square\]

**Lemma 6.5.** Let

$$
1 \to H' \to H \to H'' \to 1
$$

be an exact sequence of linear algebraic groups over $K$. Assume that $H'$ is semisimple simply connected, $H''$ is a torus and $H$ is reductive. Then

(i) $\kappa(H_A) \cap H'' = \kappa(H_A)$

(ii) $H'' = \kappa(H_A)$.

**Proof.** Let $x \in \kappa(H_A) \cap H''$. Then $\kappa^{-1}(x)$ is a torsor for $H'$. By [Har75] this torsor is trivial which yields the result.

(ii) For each $x \in X$ the map $\kappa_x$, obtained by base change to $K_x$ is surjective. This is because the Galois cohomology $H^1(K_x, H')$ vanishes since $H'$ is simply connected. \[\square\]

**Proposition 6.6.** Let

$$
1 \to H' \to H \to H'' \to 1
$$

be an exact sequence of connected reductive groups with $H'$ semisimple simply connected and $H''$ a torus. Then

$$
\tau(H')\tau(H'') = \tau(H)
$$

**Proof.** Compare with [Oto65, Prop 1.2.3]. Let $F$ be compatible with $\text{Im}\psi_{H''}$. Consider the integral

$$
J = \int_{H(H)/H(K)} F(\psi_{H''}(\kappa(x)))d\tau_H
$$

$$
= \tau(H)(\sum_{x \in D_q} F(x))(\log q)^r.
$$
To see this note that \( \hat{\kappa} \) induces an isomorphism on character groups as \( H' \) is semisimple. Then apply the above lemma along with (6.3). Again by the lemma we may apply \cite[Theorem 2.4.4]{Wei82} to this integral and obtain:

\[
J = \tau(H') \int_{H(\hat{k})/H(K)} F(\psi_{H''}(y)) d\tau_{H''} = \tau(H')\tau(H'')(\sum_{x \in D_q} F(x))(\log q)^r.
\]

Again we have made use of the above lemma. \( \square \)

We now recall Ono’s construction of crossed diagrams. Let \( \tilde{G} \) be the universal cover of \( G \), so that we have an exact sequence

\[
1 \to M \to \tilde{G} \to G \to 1.
\]

Note that \( M \) is of multiplicative type, see \cite[IX]{DG70}, over the field \( K \). Recall that

**Proposition 6.7.** The category of groups of multiplicative type over \( K \) is antiequivalent to the category of \( \text{Gal}(\bar{K}^s/K) \)-modules that are finitely generated as abelian groups.

**Proof.** See \cite[X]{DG70}. \( \square \)

The above duality is induced by \( \text{Hom}(-, \mathbb{G}_m) \), i.e by taking character modules. Now observe that we can find an exact sequence

\[
0 \to M \to T' \to T \to 0,
\]

with \( X(T' \otimes_K K') \) a projective \( \text{Gal}(K'/K) \)-module for some splitting field \( K' \) of \( T' \). To see this set

\[
G_M = \{ g \in \text{Gal}(\bar{K}^s/K) | g \text{ fixes } \hat{M} \}.
\]

Let \( \Gamma = \mathbb{Z}[G_M] \) and we can find an exact sequence

\[
0 \to \text{kernel} \to \Gamma + \Gamma + \ldots + \Gamma \to M \to 0.
\]
Set $G^* = (\tilde{G} \times T')/M$ and we have a diagram

\[
\begin{array}{ccccccccc}
0 & \\
& \downarrow & & \\
& & T' & & \\
& & & i & & \\
& & & & & & & & \\
0 & \rightarrow & \tilde{G} & \rightarrow & G^* & \rightarrow & T & \rightarrow & 0 \\
& & & & & \downarrow & & \\
& & & & & & & & G \\
& & & & & & & & 0
\end{array}
\]

Proof. (of (6.1)) We use the above notations. As in [Ono65, Lemma 2.1.1] the vertical column above has a generic section. Assuming Weil’s conjecture that $\tau(\tilde{G}) = 1$ we obtain

\[
\tau(G) = \frac{\tau(T)|\text{cok}(\hat{i})|}{\tau(T')}
\]

using (6.3) and (6.4). The result (6.1), follows now from arithmetic duality theorems and the arguments in [Ono65, pg.99 - 102]. Also note [Oes84, Corollary 3.3].

Corollary 6.8. Suppose that the group $G$ is split. Then the sequence

\[
\tau_1(G), \tau_2(G), \ldots
\]

is constant. Here $\tau_n(G)$ is the Tamagawa number of the base change $G \times_k k_n$.

Proof. See the cited work of Ono, in particular [Ono65 Theorem 2.1.1] and [Ono63 Proposition 4.5.1]. Essentially, under the stated hypothesis, the Tamagawa number is the cardinality of the fundamental group which is stable under base change.

The remainder of this section will be devoted to studying how the Tamagawa number changes under base extensions of the form $k_n/k$ under various hypothesis. We begin by recalling the explicit construction of the localization maps for $\tilde{M}$ in our particular setting.
We view the Galois module \( \hat{M} \) as a functor on field extensions of \( K \) in the usual way. Given a diagram of fields

\[
\begin{array}{ccc}
K'_1 & \longrightarrow & K'_2 \\
\uparrow & & \uparrow \\
K_1 & \longrightarrow & K_2
\end{array}
\]

with the vertical maps being Galois extensions we obtain morphisms

\[
\text{Gal}(K'_2/K_2) \rightarrow \text{Gal}(K'_1/K_1)
\]

and

\[
\hat{M}(K'_1) \rightarrow \hat{M}(K'_2).
\]

This gives maps

\[
H^i(K'_1/K_1, \hat{M}) \rightarrow H^i(K'_2/K_2, \hat{M}).
\]

In particular we have diagrams

\[
\begin{array}{ccc}
\overline{K}_x & \longrightarrow & \overline{K}_x \\
\uparrow & & \uparrow \\
K & \longrightarrow & K
\end{array}
\]

for every \( x \in X \). This yields the map

\[
H^1(K, \hat{M}) \rightarrow \prod_{x \in X} H^1(K_x, \hat{M}),
\]

whose kernel is \( \text{III}^1(K, \hat{M}) \).

We record here:

**Lemma 6.9.** Consider the projection \( \pi : X_n \rightarrow X \).

(i) The map \( \pi \) is etale.

(ii) Let \( x \in X \) then \( \pi^{-1}(x) \) consists of \( \gcd(n, \deg x) \) points.

(iii) Suppose \( \pi(y) = x \). Then \( K_{n,y}/K_x \) is a cyclic Galois extension of degree

\[
\frac{n}{\gcd(n, \deg x)}.
\]

Its Galois group is generated by the Frobenius.

**Proof.** This is well known. See for example, [Ros00]. \( \square \)

We denote by \( K_n \) the function field of \( X_n \). We assume from now on that there is a splitting field \( L \) for \( G \) that has field of constants \( k \) and further \( k \) contains all roots of unity of order dividing the fundamental group of \( G \). The field \( L_n \) has its obvious meaning.
Lemma 6.10. Let $y \in X_n$ and denote by $\pi$ the projection $X_n \to X$. Under the above hypothesis we have that $\hat{M}(K_n)$ (resp. $\hat{M}(K_{n,y})$) is a trivial $\text{Gal}(K_n/K)$-module (resp. $\text{Gal}(K_{n,y}/K_{\pi(y)})$-module).

Proof. Note that

$$\hat{M}(K_n) = \hat{M}(\bar{K}^*/\bar{K}_n).$$

The group $\text{Gal}(K_n/K)$ is cyclic and generated by the Frobenius. Let $F$ be a lift of the Frobenius to $\text{Gal}(\bar{K}^*/K)$. As the field of constants of $L$ and $K$ are the same, we may assume $F$ fixes $L$. Now by the assumption on the roots of unity we have that $F$ acts on $\hat{M}(\bar{K}^*)$ trivially. The result follows. The other case is similar. \hfill $\Box$

Proposition 6.11. Suppose that $G$ has a splitting field with field of constants $k$ and if $k$ contains all roots of unity dividing the order of $M$ then

$$|H^0(K_n, \hat{M})|.$$ does't depend on $n$.

Proof. Follows from the above lemma. \hfill $\Box$

Lemma 6.12. The natural maps

$$H^i(K_n/K, \hat{M}) \to \prod_{y \in X_n} H^i(K_{n,y}/K_{\pi(y)}, \hat{M})$$

is injective.

Proof. By the Riemann hypothesis for function fields we can find a point $x \in X$ with $\deg x$ coprime to $n$. If $y$ lifts this point we have that

$$\text{Gal}(K_n/K) \cong \text{Gal}(K_{n,y}/K_x).$$

The result follows from (6.10). \hfill $\Box$

Theorem 6.13. Suppose that $G$ has a splitting field with field of constants $k$ and $k$ contains all roots of unity dividing the order of $M$. Then there is a natural isomorphism

$$\mathfrak{III}^1(K, \hat{M}) \cong \mathfrak{III}^1(K_n, \hat{M}).$$
Proof. We have an inflation - restriction sequence inducing the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
H^1(K_n/K, \hat{M}) \\
\downarrow \\
H^1(K, \hat{M}) \xrightarrow{\prod_{x \in X}} \prod_{y \in X_n} H^1(K_y, \hat{M}) \\
\downarrow \\
H^1(K_n, \hat{M}) \xrightarrow{\prod_{y \in X_n}} \prod_{y \in X_n} H^1(K_y, \hat{M}) \\
\downarrow \\
H^2(K_n/K, \hat{M})
\end{array}
\]

The previous lemma shows that we have an injection
\[\mathfrak{M}^1(K, \hat{M}) \hookrightarrow \mathfrak{M}^1(K_n, \hat{M}).\]

Let \(\alpha \in \mathfrak{M}^1(K_n, \hat{M}).\) Also by the lemma we can lift \(\alpha\) to \(\tilde{\alpha} \in H^1(K, \hat{M}).\)

We need to show that \(\tilde{\alpha}\) is in the subgroup \(\mathfrak{M}^1(K, \hat{M})\) modulo the image of \(H^1(K_n/K, \hat{M}).\) Let \(l(\tilde{\alpha}) = (\beta_x)_{x \in X}.\) Choose for each \(x \in X\) a lift \(\tilde{x} \in X_n.\) Since \(\alpha \in \mathfrak{M}^1(K_n, \hat{M})\) we have that
\[\beta_x \in H^1(K_{\tilde{x}}/K_x, \hat{M})\]
for every \(x.\) The hypothesis imply that \(\hat{M}(k(\tilde{x}))\) is a trivial as a \(\text{Gal}(k(\tilde{x}))/k(x))\)-module. So we may think of each \(\beta_x\) as a homomorphism
\[\beta_x : \text{Gal}(K_{\tilde{x}}/K_x) \rightarrow \hat{M}(K_{\tilde{x}}).\]

We have an inclusion of Abelian groups
\[\hat{M}(K_n) = \hat{M}(K_{\tilde{x}})^{\text{Gal}(K_{\tilde{x}}/K_n)} \hookrightarrow \hat{M}(K_{\tilde{x}}).\]

As each \(\beta_x\) comes from \(\tilde{\alpha}\) the above homomorphisms factor through \(\hat{M}(K_n).\) Define a homomorphism
\[\alpha_0 : \text{Gal}(K_n/K) \rightarrow \hat{M}(K_n)\]
by
\[F \mapsto \tilde{\alpha}(F),\]
where \(F\) is the Frobenius and we are choosing a representing cocycle for \(\tilde{\alpha}\) in the above expression. Now an easy diagram chase shows that \(\tilde{\alpha} - \alpha_0\) is in \(\mathfrak{M}^1(K, \hat{M}).\) \(\square\)
Corollary 6.14. Under the hypothesis of the theorem and assuming Weil’s conjecture for the universal cover of $G$ we have
\[ \tau_n(G) = \tau(G) \]
for every $n$.

Proof. Combine the theorem with (6.1) and (6.11).

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