AN IMPROVED UPPER BOUND FOR THE BONDAGE NUMBER OF GRAPHS ON SURFACES

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Abstract. The bondage number $b(G)$ of a graph $G$ is the smallest number of edges whose removal from $G$ results in a graph with larger domination number. Recently Gagrin and Zverovich showed that, for a graph $G$ with maximum degree $\Delta(G)$ and embeddable on an orientable surface of genus $h$ and a non-orientable surface of genus $k$, $b(G) \leq \min\{\Delta(G) + h + 2, \Delta + k + 1\}$. They also gave examples showing that adjustments of their proofs implicitly provide better results for larger values of $h$ and $k$. In this paper we establish an improved explicit upper bound for $b(G)$, using the Euler characteristic $\chi$ instead of the genera $h$ and $k$, with the relations $\chi = 2 - 2h$ and $\chi = 2 - k$. We show that $b(G) \leq \Delta(G) + \lfloor r \rfloor$ for the case $\chi \leq 0$ (i.e. $h \geq 1$ or $k \geq 2$), where $r$ is the largest real root of the cubic equation $z^3 + 2z^2 + (6\chi - 7)z + 18\chi - 24 = 0$. Our proof is based on the technique developed by Carlson-Develin and Gagarin-Zverovich, and includes some elementary calculus as a new ingredient. We also find an asymptotically equivalent result $b(G) \leq \Delta(G) + \lceil \sqrt{12 - 6\chi - 1}/2 \rceil$ for $\chi \leq 0$, and a further improvement for graphs with large girth.

1. Introduction

All graphs in this paper are finite, undirected, and without loops or multiple edges. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Given a vertex $v$ in $G$, let $N(v)$ be the set of all neighbors of $v$ and let $d(v) = |N(v)|$ be the degree of $v$. The maximum and minimum vertex degrees of $G$ are denoted by $\Delta(G)$ and $\delta(G)$.

A dominating set for a graph $G$ is a subset $D \subseteq V$ of vertices such that every vertex not in $D$ is adjacent to at least one vertex in $D$. The minimum cardinality of a dominating set is called the domination number of $G$. The concept of domination in graphs has many applications in a wide range of areas within the natural and social sciences.

The bondage number $b(G)$ of a graph $G$ is defined as the smallest number of edges whose removal from $G$ results in a graph with larger domination number. The bondage number of $G$ was introduced in [2, 5], measuring to some extent the reliability of the domination number of $G$ with respect to edge removal from $G$ (which may corresponds to link failure in communication networks).

In general it is NP-hard to determine the bondage number $b(G)$ (see Hu and Xu [8]), and thus useful to find bounds for it.

**Lemma 1** (Hartnell and Rall [7]). For any edge $uv$ in a graph $G$, we have

$$b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|.$$  

In particular, $b(G) \leq \Delta(G) + \delta(G) - 1$.

The following two conjectures are still open.

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**Key words and phrases.** Bondage number, graph embedding, genus, Euler characteristic, Euler’s formula, girth.
Conjecture 2 (Teschner [12]). For any graph $G$, $b(G) \leq \frac{3}{2}\Delta(G)$.

Conjecture 3 (Dunbar-Haynes-Teschner-Volkmann [4]). For any planar graph $G$, $b(G) \leq \Delta(G) + 1$.

On the way of attacking Conjecture 3, Kang and Yuan [9] had the following result.

Theorem 4 (Kang and Yuan [9]). For any planar graph $G$, $b(G) \leq \min\{\Delta(G) + 2, 8\}$.

A simpler proof for the above theorem was later given by Carlson and Develin [3], whose ideas were further extended by Gagarin and Zverovich [6] to establish a nice upper bound for arbitrary graphs, a step forward towards Conjecture 2. To state this result we first recall some basic facts about graphs on surfaces below; the readers are referred to Mohar and Thomassen [10] for more details.

Throughout this paper a **surface** means a connected compact Hausdorff topological space which is locally homeomorphic to an open disc in $\mathbb{R}^2$. The classification theorem for surfaces [10, Theorem 3.1.3] states that, any surface $S$ is homeomorphic to either $S_h$ ($h \geq 0$) which is obtained from a sphere by adding $h$ handles, or $N_k$ ($k \geq 1$) which is obtained from a sphere by adding $k$ crosscaps. In the former case $S$ is an orientable surface of genus $h$, and in the latter case $S$ is a non-orientable surface of genus $k$. For example, the torus, the projective plane, and the Klein bottle are homeomorphic to $S_1$, $N_1$, and $N_2$, respectively. The Euler characteristic $\chi$ of $S$ is defined as

\[
\chi(S) = \begin{cases} 
2 - 2h, & S \cong S_h, \\
2 - k, & S \cong N_k.
\end{cases}
\]

Any graph $G$ can be embedded on some surface $S$, i.e. it can be drawn on $S$ with no crossing edges; in addition, the surface $S$ can be taken to be either orientable or non-orientable. Denote by $\chi(G)$ the largest integer $\chi$ for which $G$ admits an embedding on a surface $S$ with $\chi(S) = \chi$. For example, $G$ is planar if and only if $\chi(G) = 2$.

Theorem 5 (Gagarin and Zverovich [6]). Let $G$ be a graph embeddable on an orientable surface of genus $h$ and a non-orientable surface of genus $k$. Then $b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}$.

According to Theorem 5, if $G$ is planar ($h = 0, \chi = 2$) or can be embedded on the real projective plane ($k = 1, \chi = 1$), then $b(G) \leq \Delta(G) + 2$. For larger values of $h$ and $k$, it was mentioned in [6] that improvements of Theorem 5 can be achieved by adjusting its proof - for example, with the same assumptions as above,

\[
b(G) \leq \Delta(G) + \begin{cases} 
h + 1, & \text{if } h \geq 8, \\
h, & \text{if } h \geq 11, \\
k, & \text{if } k \geq 3, \\
k - 1, & \text{if } k \geq 6.
\end{cases}
\]

The goal of this paper is to establish the following explicit improvement of Theorem 5.

**Theorem 6.** Let $G$ be a graph embedded on a surface whose Euler characteristic $\chi$ is as large as possible. If $\chi \leq 0$ then $b(G) \leq \Delta(G) + \lfloor r \rfloor$, where $r$ is the largest real root of the following cubic equation in $z$:

\[
z^3 + 2z^2 + (6\chi - 7)z + 18\chi - 24 = 0.
\]

In addition, if $\chi$ decreases then $r$ increases.
Our proof for Theorem 6 is based on the technique developed by Carlson-Develin and Gagarin-Zverovich, and includes some elementary calculus (mainly the intermediate value theorem and the mean value theorem) as a new ingredient.

We will show that $r$ is the unique positive root of the above cubic equation when $\chi \leq 0$. The explicit formula for $r$ is complicated and will be given in Section 3. However, we have a simpler result which turns out to be asymptotically equivalent to Theorem 6.

**Theorem 7.** Let $G$ be a graph embedded on a surface whose Euler characteristic $\chi$ is as large as possible. If $\chi \leq 0$ then $b(G) < \Delta(G) + \sqrt{12 - 6\chi} + 1/2$, or equivalently, $b(G) \leq \Delta(G) + \lceil \sqrt{12 - 6\chi} - 1/2 \rceil$.

We will prove Theorem 6 and Theorem 7 in Section 2. Then some remarks will be given in Section 3, including the explicit formula for $r$, a comparison of Theorem 5 (for $\chi \leq 0$), Theorem 7 and Theorem 6 and a further improvement of Theorem 6 for graphs with large girth.

## 2. Proofs for the main results

Let $G$ be a connected graph which admits an embedding on a surface $S$ whose Euler characteristic $\chi$ is as large as possible. By Mohar and Thomassen [10, §3.4], this embedding of $G$ on $S$ can be taken to be a $2$-cell embedding, namely an embedding with all faces homeomorphic to an open disk.

**Euler’s Formula.** (c.f. [10]) Suppose that a graph $G$ with vertex set $V(G)$ and edge set $E(G)$ admits a $2$-cell embedding on a surface $S$, and let $F(G)$ be the set of faces in this embedding. Then

$$|V(G)| - |E(G)| + |F(G)| = \chi(S).$$

Every edge $uv$ in the $2$-cell embedding of $G$ on $S$ appears on the boundary of either two distinct faces $F \neq F'$ or a unique face $F = F'$; in the former case $uv$ occurs exactly once on the boundary of each of the two faces $F$ and $F'$, while in the latter case $uv$ occurs exactly twice on the boundary of the face $F = F'$. Let $m$ and $m'$ be the number of edges on the boundary of $F$ and $F'$, whether or not $F$ and $F'$ are distinct. For instance, a path $P_n$ with $n$ vertices is embedded on a sphere with only one face, and for any edge in $P_n$ we have $m = m' = 2(n - 1)$. We may assume that $m$ and $m'$ are at least 3, since $m \leq 2$ or $m' \leq 2$ implies $G = P_2$ which is trivial. Following Gagarin and Zverovich [9], we define the curvature of $uv$ to be

$$w(uv) = \frac{1}{d(u)} + \frac{1}{d(v)} - 1 + \frac{1}{m} + \frac{1}{m'} - \frac{\chi}{|E(G)|}.$$

It follows from Euler’s formula that

$$\sum_{uv \in E(G)} w(uv) = |V(G)| - |E(G)| + |F(G)| - \chi = 0.$$  

**Lemma 8.** Let $G$ be a connected graph embedded on a surface whose Euler characteristic $\chi$ is as large as possible. Then $b(G) < \Delta(G) + z$, if $\chi \leq 0$ and $z \geq 0$ satisfy

$$z^2 - z + 4\chi - 6 > 0,$$

$$5z^3 + 6z^2 + (24\chi - 31)z + 48\chi - 70 > 0,$$

$$z^3 + 2z^2 + (6\chi - 7)z + 18\chi - 24 > 0.$$  


Proof. Suppose to the contrary that \(b(G) \geq \Delta(G) + z\). Let \(uv\) be an arbitrary edge in \(G\). Assume \(d(u) \leq d(v)\) and \(m \leq m'\), without loss of generality. By Lemma 1,
\[
\Delta(G) + z \leq b(G) \leq d(u) + d(v) - 1 - |N(u) \cap N(v)|
\]
and thus
\[
0 \leq |N(u) \cap N(v)| \leq d(u) + d(v) - 1 - \Delta(G) - z \leq d(u) - 1 - z.
\]
It follows that \(d(u) \geq z + 1\), and so \(\delta(G) \geq z + 1\) since the edge \(uv\) is arbitrary. We distinguish three cases below for the value of \(d(u)\).

If \(d(u) = [z] + 1\) then \(|E(G)| \geq (z + 1)(z + 2)/2\), and \(|N(u) \cap N(v)| = 0\), which implies \(m' \geq m \geq 4\). Thus
\[
w(uv) \leq \frac{2}{z + 1} + \frac{1}{4} + \frac{1}{4} - 1 - \frac{2\chi}{(z + 1)(z + 2)}
= \frac{z^2 - z + 4\chi - 6}{2(z + 1)(z + 2)} < 0
\]
where the last inequality follows from (3).

If \(d(u) = [z] + 2\) then \(|E(G)| \geq (2(z + 2) + (z + 1)^2)/2\), and \(|N(u) \cap N(v)| \leq 1\), which implies \(m' \geq 4\), \(m \geq 3\). Thus
\[
w(uv) \leq \frac{2}{z + 2} + \frac{1}{4} + \frac{1}{3} - 1 - \frac{2\chi}{2(z + 2) + (z + 1)^2}
= \frac{-5z^3 + 6z^2 + (24\chi - 31)z + 48\chi - 70}{12(z + 2)(z^2 + 4z + 5)} < 0
\]
where the last inequality follows from (4).

If \(d(u) \geq [z] + 3\) then \(|E(G)| \geq (2(z + 3) + (z + 1)(z + 2))/2\), and \(m' \geq m \geq 3\). Thus
\[
w(uv) \leq \frac{2}{z + 3} + \frac{1}{3} + \frac{1}{3} - 1 - \frac{2\chi}{2(z + 3) + (z + 1)(z + 2)}
= \frac{-z^3 + 2z^2 + (6\chi - 7)z + 18\chi - 24}{3(z + 3)(z^2 + 5z + 8)} < 0
\]
where the last inequality follows from (5).

Therefore \(w(uv) < 0\) for all edges \(uv\) in \(G\). This contradicts Equation (2). \(\Box\)

**Lemma 9.** Let \(z \geq 0\) and \(\chi \leq 0\). Then the inequalities (3) hold if and only if \(z > r\), where \(r\) is the largest real root of the following cubic equation in \(z\):
\[
z^3 + 2z^2 + (6\chi - 7)z + 18\chi - 24 = 0.
\]

**Proof.** Fix a \(\chi \leq 0\) and consider the left hand side of (3) as polynomials in \(z\):
\[
A(z) = z^2 - z + 4\chi - 6,
B(z) = 5z^3 + 6z^2 + (24\chi - 31)z + 48\chi - 70,
C(z) = z^3 + 2z^2 + (6\chi - 7)z + 18\chi - 24.
\]

We first show that the largest real root \(r\) of \(C(z)\) is larger than or equal to the real roots of \(A(z)\) and \(B(z)\), by using the intermediate value theorem and the limits
\[
\lim_{z \to \infty} A(z) = \lim_{z \to \infty} B(z) = \lim_{z \to \infty} C(z) = \infty.
\]

The polynomial \(A(z)\) has two roots
\[
z_1 = \frac{1}{2} + \frac{1}{2} \sqrt{25 - 16\chi} > 0, \quad z_2 = \frac{1}{2} - \frac{1}{2} \sqrt{25 - 16\chi} < 0.
\]
Substituting $z_1$ in $C(z)$ gives

$$C(z_1) = (\chi + 1)\sqrt{25 - 16\chi + 7\chi - 5}$$

which is negative if $\chi \leq -1$ and is 0 if $\chi = 0$. By the intermediate value theorem, $C(z)$ has a real root larger than or equal to $z_1$, and so $r \geq z_1 > z_2$.

Next consider $B(z)$. If $\chi = 0$ then $B(z)$ has a unique real root $14/5$ and $C(z)$ has a unique real root $3 > 14/5$. Assume $\chi \leq -1$ below. Then $B(3) = 120\chi + 26 < 0$. Applying the intermediate value theorem to $B(z)$ gives the existence of real root(s) of $B(z)$ in $(3, \infty)$; let $z_3$ be the largest one. Then

$$B(z_3) - 4C(z_3) = z_3^3 - 2z_3^2 - 3z_3 + 26 - 24\chi = z_3(z_3 + 1)(z_3 - 3) + 26 - 24\chi > 0$$

which implies $C(z_3) < 0$. Again by the intermediate value theorem, $C(z)$ has a root larger than $z_3$, and thus $r > z_3$.

Therefore $r$ is larger than or equal to any real root of $A(z)$ and $B(z)$. It follows that $A(z)$, $B(z)$, and $C(z)$ are all positive for all $z > r$; otherwise the intermediate value theorem would imply that $A(z)$, $B(z)$, or $C(z)$ has a root larger than $r$, a contradiction.

Conversely, suppose that $A(z)$, $B(z)$, and $C(z)$ are all positive at some point $z = s \geq 0$. Then $s \neq r$ since $C(r) = 0$. If $s < r$, then there exists a point $t$ in $(s, r)$ such that

$$C'(t) = 3t^2 + 4t + 6\chi - 7 < 0$$

by the mean value theorem. It follows that

$$A(t) = \frac{2}{3}C'(t) - t^2 - \frac{11}{3}t - \frac{4}{3} < 0.$$

We have seen that the upward parabola $A(z)$ has two roots $z_1 > 0$ and $z_2 < 0$. Then $A(s) > 0$ and $s \geq 0$ imply $s > z_1$, and $t > s$ implies $A(t) > 0$, which contradicts what we found above. Hence $s > r$.

Proof of Theorem 2 Let $r(\chi)$ be the largest root of $C(z; \chi) = z^3 + 2z^2 + (6\chi - 7)z + 18\chi - 24$ for $\chi \leq 0$. We first show that $r(\chi)$ increases as $\chi$ decreases. We have seen in the proof of Lemma 7 that $r(\chi) \geq 3$. It follows from

$$C(z; \chi) - C(z; \chi - 1) = 6z + 18$$

that $C(r(\chi); \chi - 1) = -6r(\chi) - 18 < 0$. By the intermediate value theorem, $C(z; \chi - 1)$ has a root larger than $r(\chi)$, and thus its largest root $r(\chi - 1)$ is also larger than $r(\chi)$.

Now we prove the upper bound for $b(G)$. If $G$ has multiple components $G_1, \ldots, G_t$, then $\chi \leq \chi_i = \chi(G_i)$ for all $i$, since an embedding of $G$ on a surface $S$ automatically includes an embedding of $G_i$ on $S$. It follows from the definition that $b(G) = \min\{b(G_1), \ldots, b(G_t)\}$. By Theorem 5 we define $r(1) = r(2) = 2$ which is always less than $r(\chi)$ for $\chi \leq 0$. If we could establish our upper bound for connected graphs, then

$$b(G) \leq b(G_i) \leq \Delta(G_i) + |r(\chi_i)| \leq \Delta(G) + |r(\chi)|$$

and we are done. Therefore we assume $G$ is connected below.

It follows from Lemma 8 and Lemma 9 that $b(G) < \Delta(G) + z$ for all $z > r(\chi)$. Writing $z = r(\chi) + \varepsilon$ and taking the one-sided limit as $\varepsilon \to 0^+$ gives $b(G) \leq \Delta(G) + r(\chi)$. The result then follows immediately from $b(G)$ being an integer. \qed
Proof of Theorem 7. We can assume $G$ is connected for the same reason as discussed in the proof of Theorem 6. Let $z = \sqrt{12 - 6\chi} + 1/2$. Then for $\chi \leq 0$ we have

$$A(z) = \frac{23}{4} - 2\chi > 0,$$

$$B(z) = (\sqrt{12 - 6\chi})^3 + \frac{107}{4}\sqrt{12 - 6\chi} + \frac{629}{8} - 21\chi > 0,$$

$$C(z) = \frac{31}{4}\sqrt{12 - 6\chi} + \frac{121}{8} > 0.$$ 

The result follows immediately from Lemma 8 and $b(G)$ being an integer. □

3. Remarks

Using the cubic formula (c.f. M. Artin [1]) one can show that the largest real root of $C(z)$ is

$$r = \frac{25 - 18\chi}{3 \left(253 - 189\chi + 3\sqrt{5376 - 6876\chi + 1269\chi^2 + 648\chi^3}\right)^{1/3}} + \frac{1}{3} \left(253 - 189\chi + 3\sqrt{5376 - 6876\chi + 1269\chi^2 + 648\chi^3}\right)^{1/3} - \frac{2}{3}.$$ 

Some explanations are needed to make this formula work. Let $f = 5376 - 6876\chi + 1269\chi^2 + 648\chi^3$. If $-4 \leq \chi \leq 0$ then $f \geq 0$ and the formula works within $\mathbb{R}$, giving the unique real root of $C(z)$. If $\chi \leq -5$ then $f < 0$ and we need to allow complex numbers when applying the formula. We may take $\sqrt[3]{f}$ to be either of the two square roots of $f$. Then there are three choices for the cubic roots of $253 - 189\chi + 3\sqrt{f}$, giving three distinct real roots of $C(z)$, and we take $r$ to be the largest one.

One can also see that $r$ is the unique positive root of $C(z)$ when $\chi \leq 0$, since $C(0) = 18\chi - 24 < 0$ for $\chi \leq 0$ and $C''(z) = 6\varepsilon + 4 > 0$ for $z > 0$.

Next we consider Theorem 7. By Lemma 9, $r < \sqrt{12 - 6\chi} + 1/2$. Hence Theorem 7 is implied by Theorem 6. We show that these two results are asymptotically equivalent, i.e.

$$\lim_{\chi \to -\infty} \frac{\sqrt{12 - 6\chi} + 1/2}{r} = 1.$$ 

In fact, for any $\varepsilon \in (0, 1)$, substituting $z = (1 - \varepsilon) (\sqrt{12 - 6\chi} + 1/2)$ in $C(z)$ gives

$$\frac{121}{8} - \frac{79\varepsilon}{8} + \frac{631\varepsilon^2}{8} - \frac{145\varepsilon^3}{8} + 3\varepsilon(3\varepsilon^2 - 13\varepsilon + 16)\chi + \left(\frac{31}{4} - \frac{141\varepsilon}{4} + \frac{161\varepsilon^2}{4} - \frac{51\varepsilon^3}{4} + 6\varepsilon(\varepsilon^2 - 3\varepsilon + 2)\chi\right) \sqrt{12 - 6\chi}.$$ 

Since $3\varepsilon^2 - 13\varepsilon + 16 > 0$ and $\varepsilon^2 - 3\varepsilon + 2 > 0$, the above expression is negative when $\chi$ is small enough. It follows from the intermediate value theorem that $r > (1 - \varepsilon) (\sqrt{12 - 6\chi} + 1/2)$. Therefore (4) holds.

As pointed out by Gagarin and Zverovich [6], if $\chi = \chi(G) \leq 0$ then $\delta(G) \leq \left[\frac{5 + \sqrt{49 - 24\chi}}{2}\right]$ (see Sachs [11], for example). It follows immediately from Lemma 10 that

$$b(G) \leq \Delta(G) + \left[\frac{3 + \sqrt{49 - 24\chi}}{2}\right].$$
Our Theorem 7 improves this by 1 or 2, since
\[ \sqrt{12 - 6\chi} + 1/2 = \frac{1 + \sqrt{48 - 24\chi}}{2}. \]

Now consider the results given in [6]. One can prove Theorem 5 for \( \chi \leq 0 \) by showing that
\[ z = h + 3 = 4 - \chi/2 \] (for even \( \chi \leq 0 \) achieved by embeddings on orientable surfaces) and
\[ z = k + 2 = 4 - \chi \] (for all \( \chi \leq 0 \) achieved by embeddings on non-orientable surfaces) satisfy
the inequalities (3,4,5). By Lemma 9, Theorem 6 implies Theorem 5 for \( \chi \leq 0 \). Similarly
Theorem 6 implies (1).

We give a table below to show our upper bound for \( \chi = 0, -1, \ldots, -21 \).

| \( \chi \) | 0   | -1  | -2  | -3  | -4  | -5  | -6  | -7  | -8  | -9  | -10 |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( r \)    | 3   | 3   | 4   | 5   | 5   | 6   | 6   | 7   | 7   | 7   | 8   |
| \( \chi \) | -11 | -12 | -13 | -14 | -15 | -16 | -17 | -18 | -19 | -20 | -21 |
| \( r \)    | 8   | 9   | 9   | 9   | 10  | 10  | 10  | 11  | 11  | 11  | 11  |

Our result can be further improved when the graph \( G \) has large girth \( g(G) \), defined as the
length of the shortest cycle in \( G \). If \( G \) has no cycle then \( g(G) = \infty \), and \( b(G) \leq 2 \) by [2].

**Proposition 10.** Let \( G \) be a graph embedded on a surface whose Euler characteristic \( \chi \) is
as large as possible. If \( \chi \leq 0 \) and \( g = g(G) < \infty \), then \( b(G) \leq \Delta(G) + \lfloor s \rfloor \) where \( s \) is the
larger root of the quadratic polynomial \( A(z) = (g - 2)z^2 + (g - 6)z + 2\chi g - 2g - 4 \), i.e.
\[ s = \frac{\sqrt{8g(2 - g)\chi + (3g - 2)^2} - (g - 6)}{2(g - 2)}. \]

**Proof.** Assume \( G \) is connected for the same reason as in the proof of Theorem 6. It suffices
to show that \( b(G) < \Delta(G) + z \) for all \( z \geq 0 \) with \( A(z) > 0 \); the result follows from writing
\( z = s + \varepsilon \) and taking the one-sided limit as \( \varepsilon \to 0^+ \).

Suppose to the contrary that \( b(G) \geq \Delta(G) + z \) by Lemma 1. Let \( uv \)
be an arbitrary edge in \( G \). It is clear that \( m, m' \geq g \), and thus
\[ w(uv) \leq \frac{2}{z + 1} + \frac{2}{g} - 1 - \frac{2\chi}{(z + 1)(z + 2)} \]
\[ = \frac{-(g - 2)z^2 + (g - 6)z + 2\chi g - 2g - 4}{g(z + 1)(z + 2)} < 0 \]
where the last inequality follows from \( A(z) > 0 \). This contradicts Equation (2). \( \square \)

For example, we have
\[ b(G) \leq \Delta(G) + \begin{cases} 2, & \text{if } \chi = 0, \ g \geq 5, \\ 1, & \text{if } \chi = 0, \ g \geq 7, \\ 2, & \text{if } \chi = -1, \ g \geq 6, \\ 3, & \text{if } \chi = -2, \ g \geq 5, \\ 2, & \text{if } \chi = -2, \ g \geq 7. \end{cases} \]

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