DISCRIMINANTAL BUNDLES, ARRANGEMENT GROUPS, AND SUBDIRECT PRODUCTS OF FREE GROUPS

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ABSTRACT. The Lawrence-Krammer-Bigelow representation of the braid group arises from the monodromy representation on the twisted homology of the fiber of a certain fiber bundle in which the base and total space are complements of braid arrangements, and the fiber is the complement of a discriminantal arrangement. We present a more general version of this construction and use it to construct nontrivial bundles on the complement of an arbitrary arrangement $A$ whose fibers are products of discriminantal arrangements.

This leads us to consider the natural homomorphism $\rho_X$ from the arrangement group $G(A) = \pi_1(C^\ell - \bigcup A)$ to the product of groups $G(A_X), X \in X$, corresponding to a set $X$ of rank-two flats. Generalizing an argument of T. Stanford, we describe the kernel in terms of iterated commutators, when generators of $G(A_X), X \in X$, can be chosen compatibly. We use this to derive a test for injectivity of $\rho_X$. We show $\rho_X$ is injective for several well-studied decomposable arrangements.

If $A$ is central, the homomorphism $\rho_X$ induces a natural homomorphism $\overline{\rho}_X$ from the projectivized group $\overline{G}(A)$ into the product $\prod_{X \in X} \overline{G}(A_X)$, whose factors are free groups. We show $\overline{\rho}_X$ is injective if and only if $\rho_X$ is. In this case $\overline{G}$ is isomorphic to a specific finitely-presented, combinatorially-determined subdirect product of free groups. In particular $\overline{G}$ is residually free, residually torsionfree nilpotent, a-T-menable, and linear. We show the image of $\overline{\rho}_X$ is a normal subgroup with free abelian quotient, and compute the rank of the quotient in terms of the incidence graph of $X$ with $A$. When $\rho_X$ is injective, we conclude $\overline{G}$ is of type $F_{m-1}$ and not of type $F_m, m = |X|$.

1. Introduction

Suppose that $A = \{H_1, \ldots, H_n\}$ is an arrangement of affine hyperplanes in $\mathbb{C}^\ell$. For each $i$, let $\alpha_i: \mathbb{C}^\ell \to \mathbb{C}$ be a linear polynomial with zero locus $H_i$. Let $Q = \prod_{i=1}^n \alpha_i$. Let $M$ denote the complement $\mathbb{C}^\ell - \bigcup A$, where $\bigcup A = \bigcup_{i=1}^n H_i$. In this paper we present a general construction of nontrivial fiber bundles over $M$. The fibers are the complements of affine discriminantal arrangements, in the sense of Schechtman and Varchenko [SV91]. One may then construct representations of the arrangement group $G(A) = \pi_1(M)$ via the monodromy action on the homology of the fiber with coefficients in certain local systems, generalizing the Lawrence-Krammer-Bigelow representation of the pure braid group as described in [PP02].

These bundles are pullbacks of the Fadell-Neuwirth projection fiber bundles of ordered configuration spaces [FN62, Bir75], determined by an integer $k$ and a collection of generating functions $\{f_1, \ldots, f_\mu\}$, continuous functions on $M$ having the property that the zero locus of $f_i - f_j$ is contained in $\bigcup A$, for each $i \neq j$. The fiber

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is the ordered configuration space of \( k \) points in a plane with \( \mu \) punctures, realized as the complement of the affine discriminantal arrangement \( A_{\mu,k} \) in \( \mathbb{C}^k \). We call the pullback a discriminantal bundle over \( M \).

When \( k = 1 \) and the \( f_i \) are linear, one obtains a strictly linear fibration over \( M \) (with punctured plane fiber) as in the definition of fiber-type arrangements. In this case our construction coincides with the ”root map” construction of \( \text{[CS97, Coh01]} \), used to produce the braid-monodromy presentation of the fundamental group of the total space. The fact that bundles involving strictly linearly fibered arrangements may be realized as pullbacks of configuration space bundles was established in \( \text{[Coh01]} \). This was used to show that fundamental groups of complements of fiber-type arrangements are linear in \( \text{[CCP07]} \).

When \( k = 2 \) we obtain analogues of the bundles which arise in the Lawrence and Lawrence-Krammer-Bigelow (LKB) representations of braid groups. Restricted to the pure braid group, the LKB representations are rationally equivalent to the monodromy representations on the second homology of the fiber of the bundle \( PB_{t+2} \to PB_t \) with coefficients in a certain rank-one \( \mathbb{Q}[x^{\pm 1}, y^{\pm 1}] \)-local system \( \text{[PP02]} \). These LKB representations are generally faithful and are related to several polynomial invariants of knots and links \( \text{[Big02, Big07]} \). Our construction could be used in principle to define analogous polynomials for elements of other arrangement groups.

If \( f_i - f_j \) extends continuously and is not identically zero on \( H \in A \), the resulting bundle over \( M \) will have trivial monodromy around \( H \). The support of a generating set is the set of hyperplanes \( H \) in \( A \) such that the associated discriminantal bundle has nontrivial monodromy around \( H \). Any rank-two arrangement supports a generating set. If \( A \) is a rank-three arrangement supporting a multinet structure \( \text{[FY07]} \) then \( A \) supports a generating set. For arbitrary \( A \), we have no method to construct a generating set supported by \( A \). But by taking Whitney sums of discriminantal bundles over \( M \), we obtain bundles with nontrivial monodromy around every hyperplane of \( A \).

To understand when the resulting representation of \( \pi_1(M) \) is faithful, we are led to study kernels of cartesian products of inclusion-induced homomorphisms of \( G = \pi_1(M) \). Let \( X \) be a set of rank-two flats whose union is \( A \), and \( \rho_X : G \to \prod_{S \in X} G_S \) be the product of inclusion-induced homomorphisms, where \( G_S \) is the group of the subarrangement \( S \). Then \( G \) is generated by elements dual to the hyperplanes of \( A \), uniquely defined up to conjugacy, and \( G_S \) is the quotient of \( G \) obtained by killing the generators corresponding to hyperplanes outside of \( S \). When \( G \) has a generating set \( Y = \{ a_H \mid H \in A \} \) with the property that the subgroup of \( \{ a_H \mid H \in S \} \) of \( G \) maps isomorphically to \( G_S \), for all \( S \in X \), we say \( Y \) is adapted to \( X \). In this case we generalize an argument of T. Stanford \( \text{[Sta99]} \) to show the kernel of \( \rho_X \) is generated by iterated commutators of generators and their inverses, whose supports are not contained in any \( S \in X \). As a corollary we obtain a criterion for \( \rho_X \) to be injective. If \( A \) is central, we show that \( \rho_X \) is injective if and only if its restriction to the group of the decone of \( A \) is injective. Using this we show injectivity of \( \rho_X \) for the rank-three wheel arrangement, labelled \( X_3 \) in \( \text{[FR80]} \) and considered in unpublished work by Arvola \( \text{[Arv92]} \), for the group of the Kohno arrangement of seven lines, labelled \( X_2 \) in \( \text{[FR86]} \), and for a pair of seven-line arrangements that appear in \( \text{[Fal97]} \).
When $\mathcal{A}$ is central, the homomorphism $\rho_X : \overline{G} \to \prod_{S \in X} \overline{G}_S$, where $\overline{G} = G/\mathbb{Z}$ and $\overline{G}_S = G_S/\mathbb{Z}$ are the projectivized fundamental groups. Since $S \in X$ is a rank-two flat (of multiplicity greater than two), $\overline{G}_S$ is a (nonabelian) free group, and so the image $N$ is a combinatorially-determined subdirect product of free groups, in the terminology of [BM09]. Then $N$ is residually free, residually torsionfree nilpotent, has a linear representation, and satisfies the Haagerup property (i.e., $N$ is $a$-T-menable in the sense of Gromov). Hence $\overline{G}(\mathcal{A})$ has these properties when $\overline{p}_X$ is injective. (Injectivity of $\overline{p}_X$ is not a priori combinatorially determined, however.) We show $N$ is a normal subgroup, with free abelian quotient. We compute the rank of the quotient in terms of the incidence graph of $X$ with $\mathcal{A}$. Using [MMW98] we obtain precise information about the finiteness type of $N$. We deduce that, for the rank-three wheel arrangement, $\overline{G}(\mathcal{A})$ is isomorphic to the Stallings’ group [Sta63], as originally observed by Matei and Suciu [MS04] (this observation provided motivation for the current project); see also Question 2.10 in Bestvina’s problem list [Bes04]. We also conclude that the group of the Kohno arrangement is of type $F_4$ and not of type $F_5$, and the groups of the two seven-line arrangements from [Fal97] are $F_3$ but not $F_4$. The $X_3$ example has recently been generalized by Artal-Bartolo, Cogolludo-Augustin, and Matei to a large family of arrangements whose groups are Bestvina-Brady groups, as reported in [Mat07]. We reproduce their result using our approach. If $\mathcal{A}$ is a decomposable arrangement [PS06], $X$ is the set of all rank-two flats, and $G(\mathcal{A})$ has a generating set adapted to $X$, our result implies that the kernel of $\rho_X$ is precisely the nilpotent residue of $\overline{G}$. In all our examples of decomposable arrangements, $\overline{p}$ is injective; we conjecture that this is always the case, that is, that all decomposable arrangement groups embed in products of free groups.

2. Constructing Discriminantal Bundles over Arrangement Complements

In this section we will construct a number of bundles, starting with any arrangement complement. Our construction will mimic how one might construct the pure braid space for $\ell + k$ strings, given the space for $\ell$ strings.

The pure braid space $PB_\ell$ is the complement in $\mathbb{C}^\ell$ of the arrangement defined by

$$\prod_{1 \leq i < j \leq \ell} (z_i - z_j) = 0.$$ 

Its fundamental group is the pure braid group $P_\ell$. Then $PB_{\ell+k}$ is the set

$$\{(z_1, \ldots, z_\ell, z_{\ell+1}, \ldots, z_{\ell+k}) \mid z_i \neq z_j\}.$$ 

To see how this can be built from $PB_\ell$ we write the last $k$ variables as $w_1, \ldots, w_k$. Then

$$PB_{\ell+k} = \{(z_1, \ldots, z_\ell, w_1, \ldots, w_k) \mid z_i \neq z_j, z_i \neq w_j, w_i \neq w_j\}.$$ 

The point here is that we can define the total space by introducing new variables, and prohibiting the variables from taking the values given by our “generating set” $\{z_i\}$ or each other. With the pure braid space as the base the construction is clear; for a general hyperplane arrangement the problem is in finding an analogue of the set $\{z_i\}$. 
2.1. Generating Sets. Fix a hyperplane arrangement \( A = \{H_1, \ldots, H_n\} \) in \( \mathbb{C}^\ell \) with \( H_i \) the zero locus of the linear polynomial \( \alpha_i: \mathbb{C}^\ell \rightarrow \mathbb{C} \) for \( 1 \leq i \leq n \). Let \( Q = \prod_{i=1}^n \alpha_i \) and \( M(A) = \mathbb{C}^\ell - \bigcup_{i=1}^n H_i = \{(z_1, \ldots, z_\ell) \in \mathbb{C}^\ell \mid Q(z_1, \ldots, z_\ell) \neq 0\} \).

Definition 2.1. A set \( \mathcal{F} = \{f_1, f_2, \ldots, f_\mu\} \) is called a generating set for the arrangement \( A \) provided that each \( f_i \) is a continuous function on \( M \) and each difference \( f_i - f_j, \ 1 \leq i < j \leq \ell \) is nowhere zero on \( M \).

So \( f_i(z) = z_i, \ 1 \leq i \leq \ell \), defines a generating set for the braid arrangement in \( \mathbb{C}^\ell \). Functions in a generating set need not be linear in general. In most examples the \( f_i \) are rational functions on \( \mathbb{C}^\ell \), regular on \( M \). In this case \( \mathcal{F} \) forms a generating set for \( A \) if and only if the irreducible components of the (possibly non-reduced) quasi-affine hypersurfaces defined by \( f_i(z) = f_j(z) \) are (unions of) hyperplanes of \( A \), for each \( i \neq j \).

Let \( \mathcal{F} = \{f_1, \ldots, f_\mu\} \) be a (labelled) generating set for \( A \) and let \( k \geq 1 \). Introduce new variables \( w_1, \ldots, w_k \) and consider the topological space

\[
V \subset M \times \mathbb{C}^k \subseteq \mathbb{C}^{\ell+k}
\]

defined by

\[
Q(z_1, \ldots, z_\ell) \cdot \prod_{1 \leq i < k} (w_i - f_i(z)) \cdot \prod_{1 \leq i < j \leq k} (w_i - w_j) = 0.
\]

The space \( E_k(A, \mathcal{F}) \) is then defined to be the complement \( \mathbb{C}^{\ell+k} - V \).

If the \( f_i \) are holomorphic on \( M \) then \( V \) is a quasi-affine analytic subset of \( \mathbb{C}^{\ell+k} \).

By the discussion above, \( \mathbb{P}B_{\ell+k} = E_k(A_\ell, \mathcal{F}) \) where \( A_\ell \) is the braid arrangement in \( \mathbb{C}^\ell \) and \( f_i(z) = z_i \) for \( 1 \leq i \leq \ell \). The projection \( E_k(A_\ell, \mathcal{F}) \rightarrow M(A_\ell) \) is the Fadell-Neuwirth bundle \( \mathbb{P}B_{\ell+k} \rightarrow \mathbb{P}B_\ell \). In general we have the following.

Theorem 2.2. Let \( A \) be an arrangement with generating set \( \mathcal{F} = \{f_1, \ldots, f_\mu\} \) and let \( k \geq 1 \). Then projection \( p: E_k(A, \mathcal{F}) \rightarrow M(A) \) is the projection map of a fiber bundle. This bundle is the pullback of the Fadell-Neuwirth bundle \( \mathbb{P}B_{\mu+k} \rightarrow \mathbb{P}B_\mu \) via the function \( f = (f_1, \ldots, f_\mu): M \rightarrow \mathbb{P}B_\mu \).

Proof. Let \( z = (z_1, \ldots, z_\ell) \) be the coordinates on \( \mathbb{C}^\ell \), and let \( (x_1, \ldots, x_\mu, w_1, \ldots, w_k) \) be coordinates on \( \mathbb{C}^{\mu+k} \). Then the total space of the pullback of \( \mathbb{P}B_{\mu+k} \rightarrow \mathbb{P}B_\mu \) via \( f \) is the set of all points \( (z_1, \ldots, z_\ell, x_1, \ldots, x_\mu, w_1, \ldots, w_k) \in M(A) \times \mathbb{P}B_{\mu+k} \) which satisfy \( f_i(z) = x_i, i = 1, \ldots, \mu \). It is readily checked that the map

\[
(z_1, z_2, \ldots, z_\ell, w_{\ell+1}, \ldots, w_{k+\ell}) \mapsto (z_1, \ldots, z_\ell, f_1(z), \ldots, f_\mu(z), w_1, \ldots, w_k)
\]

from \( E_k(A, \mathcal{F}) \) to the total space of the pullback is a bundle equivalence. \( \square \)

The function \( f = (f_1, \ldots, f_\mu): M(A) \rightarrow \mathbb{P}B_\mu \) is called a generating function for the bundle. (Different generating functions for the same generating set differ by permutation of coordinates in \( M(A) \).)

Definition 2.3. A discriminantal arrangement of type \( (\mu, k) \) is the arrangement \( A_{\mu,k} \) defined by the polynomial

\[
\prod_{i,j} (w_i - m_j) \cdot \prod_{i<j} (w_i - w_j)
\]

where \( m_1, \ldots, m_\mu \) are fixed distinct complex numbers and \( w_1, \ldots, w_k \) are complex variables.
Different choices of \( m_1, \ldots, m_\mu \) lead to lattice-isotopic arrangements. Thus the complement of \( A_{\mu,k} \) is determined up to homeomorphism by \( \mu \) and \( k \) [Ran89]. We denote the complement of \( A_{\mu,k} \) by \( F_{\mu,k} \). The arrangement \( A_{\mu,k} \) is an affine supersolvable arrangement, hence is itself a fiber-type arrangement. In particular \( F_{\mu,k} \) is aspherical (see [Ter86], [FR85]).

The complement \( F_{\mu,k} \) of \( A_{\mu,k} \) may be realized as the configuration space of \( k \) ordered points in \( \mathbb{C} - \{m_1, \ldots, m_\mu\} \). Since \( F_{\mu,k} \) is the fiber of the Fadell-Neuwirth bundle \( PB_{\mu+k} \to PB_\mu \), the fundamental group \( G_{\mu,k} = \pi_1(F_{\mu,k}) \) is a subgroup of the pure braid group \( P_{\mu+k} \). Note that \( G_{\mu,1} \) is a free group of rank \( \mu \).

**Proposition 2.4.** The fiber of \( p: E_k(\mathcal{A}, \mathcal{F}) \to M(\mathcal{A}) \) is homeomorphic to \( F_{\mu,k} \).

In light of the preceding observation, we call such a bundle a discriminantal bundle.

**Corollary 2.5.** The fiber of \( p: E_k(\mathcal{A}, \mathcal{F}) \to M(\mathcal{A}) \) is aspherical with fundamental group isomorphic to the pure braid subgroup of type \( G_{\mu,k} \subset \pi_1(PB_{\mu+k}) \).

Write \( E = E_k(\mathcal{A}, \mathcal{F}) \) and \( M = M(\mathcal{A}) \). The bundle map \( p: E \to M \) is the restriction of a linear projection. If the \( f_i \) are linear and \( k \geq 1 \), then the total space is the complement of an arrangement \( \mathcal{E} \). When \( \mathcal{A} \) is central, the map \( p \) is the bundle projection associated with the modular flat \( \mathcal{A} \) of \( \mathcal{E} \), see [Par00] [FP02]. If \( k = 1 \) then \( p \) is a strictly linear fibration [FR85] [Ter86], and \( f \) is the associated root map as defined in [CS97] [Coh01].

Much of the topology of fiber-type arrangements carries over. The results below all follow from the characterization of these discriminantal bundles as pullbacks of the Fadell-Neuwirth bundle, together with standard results for fiber bundles - see [FR85].

**Theorem 2.6.** The bundle \( p: E \to M \) has a section, and the action of the fundamental group of the base on the fiber is trivial on the first homology.

**Corollary 2.7.** The homology of \( E \) is the tensor product of the homology of the base \( M \) with that of the fiber \( F_{\mu,k} \).

**Corollary 2.8.** If the base arrangement complement \( M \) is aspherical, then so is the total space \( E \).

**Corollary 2.9.** The fundamental group of \( E \) is the semidirect product of the fundamental group of \( M \) with \( G_{\mu,k} \).

In particular, if the fundamental group of \( M \) is an iterated semidirect product of free groups, or more stringently an almost-direct product of free groups, see [FR85] [CS98], then so is the fundamental group of \( E \). In the latter instance, the cohomology ring of the group \( \pi_1(E) \) may be calculated from the almost-direct product structure, see [Coh10]. Additionally, we have the following, as noted in [CCF07] Lem. 6.2.

**Corollary 2.10.** If the fundamental group of the base \( M \) is linear, then so is the fundamental group of the total space \( E \).

**Proof.** The group \( \pi_1(E) \) is a subgroup of the product \( \pi_1(M) \times \pi_1(PB_{\mu+k}) \), which is linear since both factors are. \( \square \)
The fiber $F_{\mu,1}$ of the Fadell-Neuwirth bundle $PB_{\mu+1} \to PB_\mu$ is a copy of $\mathbb{C}$ with $\mu$ punctures. The monodromy of the bundle is the (faithful) Artin representation

$$PB_\mu \to \text{Aut}(F_\mu)$$

of the pure braid group in the group of automorphisms of the free group. With this identification, the pure braid group acts diagonally on $F_{\mu,k}$ for any $k \geq 1$, since the diagonal hyperplanes $w_i = w_j$ are preserved. The bundle $PB_{\mu+k} \to PB_\mu$ associated to $PB_{\mu+1} \to PB_\mu$ via this action.

**Corollary 2.11.** The structure group of the bundle $E_k(A, F) \to M(A)$ reduces to the pure braid group on $\mu = |F|$ strings, and is associated with the bundle $E_1(A, F) \to M(A)$ via the diagonal action of $P_\mu$ on $F_{\mu,k}$.

**Definition 2.12.** Let $F$ be a generating set for $A$ and $H \in A$. We say $F$ is trivial on $H$ if $f_i - f_j$ extends continuously and is not identically zero on $H$, for all $1 \leq i < j \leq \mu$. The support of $F$ is the set of hyperplanes $H \in A$ on which $F$ is not trivial.

The monodromy of the bundle $E(A, F) \to M(A)$ is nontrivial around $H \in A$ if and only if $H$ is in the support of $F$. If $S \subseteq A$ denotes the support of $F$, then $F$ is a generating set for $S$, and $E(A, F) \to M(A)$ is a subbundle of $E(S, F) \to M(S)$, the pullback by the inclusion map $M(A) \hookrightarrow M(S)$.

**Example 2.13.** Let $D_\ell$ be the Coxeter arrangement of type $D_\ell$, with defining equations $z_i = \pm z_j, 1 \leq i < j \leq \ell$. Let $F = \{f_1, \ldots, f_\ell\}$ where $f_i(z) = z_i^2$. Then $F$ is a generating set on $D_\ell$, with support $D_\ell$, and $E_k(D_\ell, F)$ is the complement of the union of $\ell(\ell - 1) + \binom{\ell}{2}$ hyperplanes and $k$ affine quadrics in $\mathbb{C}^{\ell+k}$.

**Example 2.14.** Let $B_\ell$ be the Coxeter arrangement of type $B$, with defining equations $z_i = 0, 1 \leq i \leq \ell$, and $z_i = \pm z_j, 1 \leq i < j \leq \ell$. The set $F = \{-z_\ell, \ldots, -z_1, 0, z_1, \ldots, z_\ell\}$ is a generating set for $B_\ell$, with support $B_\ell$. Here $0$ denotes the zero function. This linear generating set arises from the structure of the projection $M(B_{\ell+1}) \to M(B_\ell)$ as a strictly linear fibration; this is an instance of the root map construction. The corresponding generating function $f: M(B_\ell) \to PB_{2\ell+1}$ realizes the bundle $M(B_{\ell+1}) \to M(B_\ell)$ as the pullback of the Fadell-Neuwirth bundle $PB_{2\ell+2} \to PB_{2\ell+1}$. This is used to determine the structure of the type $B$ pure braid group $\pi_1(M(B_\ell))$ as an almost-direct product of free groups in [Coh11] Thm. 1.4.3.

Another generating set for the arrangement $B_\ell$ is given by $F' = \{f_1, \ldots, f_\ell\}$, where $f_i(z) = \frac{1}{z_i}$ for $1 \leq i \leq \ell$. The support of $F'$ is the entire arrangement $B_\ell$. Note that the hyperplanes $z_i = 0$ and $z_j = 0$ are poles of $f_i - f_j$, of multiplicity two.

**Example 2.15.** Let $A$ be the arrangement consisting of the origin in $\mathbb{C}$. Let $F = \{0, z, 2z\}$. Then $F$ is a generating set for $A$. The total space $E_1(A, F)$ is the complement in $\mathbb{C}^2 = \mathbb{R}^4$ of the real linear subspaces $z = 0, w = 0, w = z$, and $w = 2z$. This complement of four 2-planes in $\mathbb{R}^4$ does not have the homotopy type of the complement of a complex hyperplane arrangement, by [Zie93].

While nonlinear generating functions yield total spaces which are not arrangement complements (as sets), we have not found an example of a generating set consisting of (nonlinear) holomorphic functions on $M$ for which the total space $E_k(A, F)$ does not have the homotopy type of an arrangement complement.
2.2. Existence of Generating Sets. To construct faithful representations of the group \( G(\mathcal{A}) = \pi_1(M) \), we need to know which subarrangements of \( \mathcal{A} \) support generating sets. It turns out the conditions are somewhat restrictive. But one can always construct generating sets supported on rank-two subarrangements.

Let \( \mathcal{A} \) be an arbitrary arrangement, and let \( X \) be a rank-two flat of \( \mathcal{A} \), an intersection of hyperplanes in \( \mathcal{A} \) of codimension two in \( \mathbb{C}' \). Let \( \mathcal{A}_X \) denote the set of hyperplanes of \( \mathcal{A} \) containing \( X \). We explicitly construct a generating set for \( \mathcal{A} \) supported by \( \mathcal{A}_X \). The reader may notice a similarity with the description of a configuration space of distinct points in \( \mathbb{C} \) as a hyperplane complement. This construction gives an indication of our original ideas for producing fibered families of hyperplanes.

We may label the hyperplanes of \( \mathcal{A} \) so that \( \mathcal{A}_X = \{H_1, \ldots, H_\mu\}, \mu = |\mathcal{A}_X| \). We wish to consider the one parameter family of hyperplanes containing \( X \).

Since \( H_1 \) and \( H_2 \) are distinct, \( X \) is the transverse intersection of \( H_1 \) and \( H_2 \). We consider the family \( \{H(m) \mid m \in \mathbb{C}\} \) of hyperplanes, where \( H(m) \) has defining equation \( ma_1(z) = a_2(z) \). This family includes all the hyperplanes of \( \mathcal{A}_X \) except \( H_1 \). Note that \( m = 0 \) gives \( H_2 \), and \( H_1 \) would correspond to \( m = \infty \). There are \( \mu - 2 \) distinct nonzero constants \( m_i, i = 3, \ldots, \mu \) so that \( H(m_i) = H_1 \).

We then define the generating set \( \mathcal{F}_X \) of size \( \mu \) by

\[ \mathcal{F}_X = \{0, a_2, m_3 a_1, \ldots, m_\mu a_1\}. \]

Then \( \mathcal{F}_X \) is a generating set for \( \mathcal{A} \), with support equal to \( \mathcal{A}_X \).

Next we show that, under a mild hypothesis, a polynomial generating set of size three supported on a rank-three arrangement corresponds to a pencil of Čeva type, as studied in [FY07], after a linear change of coordinates in \( PB_3 \). We identify (possibly non-reduced) projective plane curves with their defining polynomials, and say a curve is \textit{completely reducible} if its defining polynomial splits into linear factors (possibly with multiplicities).

\textbf{Definition 2.16.} A \textit{pencil of Čeva type} (or Čeva pencil) is a 1-dimensional linear system of projective plane curves (a rational map \( \mathbb{C}P^2 \rightarrow \mathbb{C}P^1 \)) with no fixed components, connected generic fiber, and three or more completely reducible fibers.

We denote the projectivization of a central arrangement \( \mathcal{A} \) by \( \overline{\mathcal{A}} \). The set of irreducible components of completely reducible fibers in a Čeva pencil forms a projective line arrangement \( \overline{\mathcal{A}} \), which inherits a natural partition and multiplicity function \( m: \mathcal{A} \rightarrow \mathbb{Z}_{>0} \) from the pencil. It is shown in [FY07] that a projective line arrangement \( \overline{\mathcal{A}} \) arises in this way from a Čeva pencil if and only if the associated partition forms a \textit{multinet} for the multiplicity function \( m: \mathcal{A} \rightarrow \mathbb{Z}_{>0} \). (See also [MB09].) Say \( m: \mathcal{A} \rightarrow \mathbb{Z}_{>0} \) is \textit{primitive} if the values of \( m \) are mutually relatively prime.

\textbf{Definition 2.17.} Suppose \( m: \mathcal{A} \rightarrow \mathbb{Z}_{>0} \) is primitive. A \((k, d)\)-\textit{multinet} on the multiarrangement \( (\mathcal{A}, m) \) consists of a partition \( \mathcal{P} = \{\mathcal{A}_1, \ldots, \mathcal{A}_k\} \) of \( \mathcal{A} \) into \( k \geq 3 \) blocks, with the associated “base locus” \( \mathcal{X} \) being the set of intersection points of lines from different blocks, satisfying

(i) each block of \( \mathcal{P} \) has \( d \) lines, counting multiplicity;
(ii) each point of \( \mathcal{X} \) contains the same number of lines from each block, counting multiplicity;
(iii) \( (\bigcup_{L \in \mathcal{A}_i} L) - \mathcal{X} \) is connected for each \( 1 \leq i \leq k \).
Theorem 2.18. Suppose $m: A \to \mathbb{Z}_{>0}$ is primitive and $(A, m)$ supports a $(3, d)$-multinet. Then there is a generating set $\{f_1, f_2, f_3\}$ with support equal to $A$.

Proof. By [FY07], there are completely reducible polynomials $Q_1, Q_2,$ and $Q_3$, pairwise relatively prime, whose zero loci are the blocks $A_1, A_2,$ and $A_3$ of the multinet structure, and $Q_3 = aQ_1 + bQ_2,$ for some $a, b \in \mathbb{C}$. Without loss of generality, $a = b = 1.$ Then one easily checks that $\{0, Q_1, -Q_2\}$ is a generating set with support equal to $A$. $\square$

Given a generating set $\{f_1, \ldots, f_\mu\}$ consisting of homogeneous polynomials of the same degree $d$, we may set $Q_i = f_i - f_{i+1}$ for $1 \leq i \leq \mu - 1.$ Then the $(\mu - 2)$-dimensional linear system corresponding to the rational mapping

$$[Q_1 : \cdots : Q_{\mu-1}]: \mathbb{CP}^{\mu-1} \to \mathbb{CP}^{\mu-2}$$

has $\binom{\mu}{2}$ completely reducible fibers $f_i - f_j$. This linear system may have fixed components and/or disconnected general fiber.

Theorem 2.19. Let $A$ be an arrangement of rank three with generating set $F = \{f_1, f_2, f_3\}$, consisting of homogeneous polynomial functions of degree $d$ on $\mathbb{C}^d$. Suppose the support of $F$ is $A$, and the polynomials $f_1 - f_2$ and $f_2 - f_3$ are relatively prime. Then $A$ supports a $(k, d)$-multinet structure for some $k \geq 3.$

Proof. With notation as above, the pencil determined by $[Q_1 : Q_2]: \mathbb{CP}^2 \to \mathbb{CP}^1$ has three completely reducible fibers $Q_1 = f_1 - f_2, Q_2 = f_2 - f_3,$ and $Q_1 + Q_2 = f_1 - f_3,$ whose irreducible components comprise $A.$ Since $Q_1$ and $Q_2$ are relatively prime, the pencil has no fixed components. The associated partition of $A$ satisfies (i) and (ii) of the definition of multinet. We can then refine this partition to a multinet, by [FY07], Remark 2.6. $\square$

2.3. Linear generating sets. Any set of linear forms is a generating set. Indeed, if $F = \{f_1, \ldots, f_\mu\}$ is a set of $\mu$ distinct linear forms, then $F$ is a generating set whose support $A$ has defining polynomial $\prod_{1 \leq i < j \leq \mu}(f_i - f_j)$. In this case, as observed earlier, $E_k(A,F)$ is the complement of an arrangement $E$, which contains $A$ as a modular flat in the case where $A$ is central, and $E_k(A,F) \to M(A)$ is the associated bundle projection. Rescaling the $f_i$ may result in a different supporting arrangement $A$, so we cannot replace the generating set of linear forms $F$ with its associated arrangement (or matroid), and retain a well-defined operation $F \mapsto A$.

If $F$ consists of the coordinate functions $\{z_1, \ldots, z_\ell\}$ then its support $A$ is the braid arrangement. If $F$ consists of the natural defining forms $z_i - z_j$ for the braid arrangement, then $A$ is the $p = 2$ center-of-mass arrangement defined in [CK07], whose complement parametrizes the labelled configurations of $\ell$ distinct points in $\mathbb{R}^2$ with pairwise distinct midpoints. In fact, if $F$ consists of the natural defining forms

$$\sum_{k=1}^{p} z_{i_k} - \sum_{k=1}^{p} z_{j_k},$$

for the $p$-fold center-of-mass arrangement on $\ell$ points, then the associated arrangement $A$ is the $2p$-fold center-of-mass arrangement on $\ell$ points.

The correspondence $F \mapsto A$ has a nice interpretation at the level of the Grassmannian. Suppose $\mu > \ell$ and $F$ contains $\ell$ linearly independent forms. Then the image of $(f_1, \ldots, f_\mu): \mathbb{C}^\ell \to \mathbb{C}^\mu$ is an $\ell$-dimensional linear subspace $L$ of $\mathbb{C}^\mu$. The arrangement of hyperplanes $\hat{F}$ determined by $F$, with defining polynomial
$\prod_{1 \leq i \leq \mu} f_i$, is linearly isomorphic to the arrangement $L^{\text{bool}}$ cut out on $L$ by the coordinate hyperplanes $w_i = 0$ in $\mathbb{C}^\mu$. The support arrangement $A$ with generating set $F$ is isomorphic to the arrangement $L^{\text{braid}}$ cut out on the same subspace $L$ by the hyperplanes $w_i = w_j$ of the braid arrangement in $\mathbb{C}^\mu$. (The fact that $A$ is not determined by the arrangement $\hat{F}$ means that this operation is not invariant under the torus action on subspaces of $\mathbb{C}^\mu$.)

If $f_1, \ldots, f_\mu$ are distinct linear forms, then the set $F = \{ \frac{1}{f_1}, \ldots, \frac{1}{f_\mu} \}$ of reciprocals is also a generating set, whose support is the arrangement defined by

$$\prod_{1 \leq i \leq \mu} f_i \cdot \prod_{1 \leq i < j \leq \mu} (f_i - f_j).$$

3. Products of localization homomorphisms

Given an arbitrary arrangement $A$, we would like to build a bundle with base $M(A)$ which is sufficiently twisted to yield a faithful representation of $\pi_1(M)$. For $A$ itself to support a discriminantal bundle requires fairly special circumstances, as we have seen, but $A$ may have several proper subarrangements supporting such bundles. Indeed, any rank-two subarrangement, and any rank-three subarrangement supporting a multinet, will have that property. We propose to pull back the product of all such discriminantal bundles supported on subarrangements, to obtain a bundle over $M(A)$.

More precisely, let $D$ denote the set of subarrangements of $A$ supporting generating sets, and let

$$\varphi_D : M(A) \to \prod_{S \in D} M(S)$$

be the product of inclusion maps. Choosing a generating set $F_S$ of size $\mu_S$ and a positive integer $k_S$ for each $S \in D$, we have discriminantal bundles $E_{k_S}(S, F_S) \to M(S)$, and hence a product bundle

$$\prod_{S \in D} \rho_S : \prod_{S \in D} E_{k_S}(S, F_S) \to \prod_{S \in D} M(S).$$

(Note: the codomain is also an arrangement complement.) The pullback $\varphi_D^{-1}(\prod \rho_S)$ gives a bundle over $M(A)$ whose fiber $F$ is $\prod_{S \in D} F_{k_S, \mu_S}$. This bundle will have nontrivial monodromy around every hyperplane of $A$, since $\bigcup D = A$.

To use the product bundle $\varphi_D^{-1}(\prod \rho_S)$ constructed above to produce faithful representations of $\pi_1(M(A))$, one would first build faithful representations of $\pi_1(M_S)$ for $S \in D$, using the monodromy of discriminantal bundles, and then show that $\varphi_D$ induces an injection on fundamental groups. We can carry out the first step at least in case $S$ comes from a rank-two lattice element.

3.1. Monodromy representations associated to rank-two lattice elements.

Fix now a single lattice element $X$ of rank two. Let $A_X$ denote the arrangement consisting of just those hyperplanes which contain $X$, and let $M_X$ denote the complement of $A_X$. We have an inclusion-induced homomorphism

$$i_X : \pi_1(M) \to \pi_1(M_X).$$

Let $F_X$ be the generating set with support $A_X$ constructed in the preceding section. Let $\mu = |A_X|$ and $f : M_X \to PB_\mu$ be the associated generating function. Let $k = k(X) \geq 1$, and let $p : E_{k,X} \to M_X$ be the associated discriminantal bundle.
Proposition 3.1. The induced homomorphism $f_* : \pi_1(M_X) \to P_\mu$ is injective.

Proof. Recalling the construction of $f$ from [2,2] we may assume that $A_X = \{H_1, \ldots, H_{\mu}\}$, and we can choose coordinates so that $H_1$ is defined by $x_1 = 1$, $H_2$ by $x_2 = 0$, and $H_\ell$ is defined by $x_\ell = m_\ell x_1$ for $3 \leq \ell \leq \mu$. Then $f : M_X \to PB_\mu$ is given by $f(x_1, \ldots, x_\mu) = (0, x_2, m_3 x_1, \ldots, m_\mu x_1)$. Let $p : PB_\mu \to PB_{\mu-1}$ be the projection $(y_1, \ldots, y_\mu) \to (y_1, y_3, \ldots, y_\mu)$. Then $p$ is a discriminant bundle projection; in particular the kernel of $p_*$ is the fundamental group of the fiber of $p$, a free group on $\mu - 1$ generators.

For $i = 1, \ldots, \mu$, let $a_i$ be a loop in $M_X$ dual to $H_i$. Choosing the base point in the hyperplane $x_1 = 1$, we may assume that the loops $a_2, \ldots, a_\mu$ lie in the subspace $x_1 = 1, x_3 = m_3, \ldots, x_\mu = m_\mu$. Then $p_\mu f_*$ sends $a_i$ to $1$ for $2 \leq i \leq \mu$. Then any element of the kernel of $f_*$ lies in $(a_2, \ldots, a_\mu)$, a free group of rank $\mu - 1$. Moreover, $f_*$ sends this subgroup to the fundamental group of the fiber of $p$. Then $f_*$ is injective by the Hopfian property of free groups. □

Corollary 3.2. The LKB-type representation arising from the bundle $p : E_{2,X} \to M_X$ is faithful.

Proof. Since $p$ is the pullback of the Fadell-Neuwirth bundle $p_k : PB_{\mu+2} \to PB_\mu$ via $f$, and $f_*$ is injective, it suffices that the LKB-type representation of Fadell-Neuwirth bundles is faithful. This is true by [Big01, Kra02, PP02]. □

3.2. The kernel of $(\varphi_{\mathcal{S}})_*$. Next we consider a general product mapping $\varphi_{\mathcal{S}} = \prod_{S \in \mathcal{S}} i_S : M \to \prod_{S \in \mathcal{S}} M_S$, where $\mathcal{S}$ is an arbitrary set of subarrangements of $A$, $M_S = M(S)$, and $i_S : M \to M_S$ is the inclusion. We pick a base point in $M$ and obtain an induced homomorphism,

$$(\varphi_{\mathcal{S}})_* : \pi_1(M) \to \prod_{S \in \mathcal{S}} \pi_1(M_S).$$

For simplicity we denote $(\varphi_{\mathcal{S}})_*$ by $\rho_{\mathcal{S}}$. Let us also denote $(i_S)_*$ by $\rho_S$, so that $\rho_{\mathcal{S}} = \prod_{S \in \mathcal{S}} \rho_S$. For consistency with [MKS04] and [Fal93], for the remainder of this section we adopt the conventions $x^y = y^{-1}xy$ and $[x,y] = x^{-1}y^{-1}xy$ for group elements $x$ and $y$.

Recall that $\pi_1(M)$ is generated by small loops around the hyperplanes of $A$. For each $S \in \mathcal{S}$, $\rho_S$ kills the generators corresponding to hyperplanes in $A - S$. For the pure braid group, and a certain sets of flats $\mathcal{S}$, this is the effect of deleting strands. So elements in the kernel of the product mapping $\rho_{\mathcal{S}}$ are analogous to Brunnian braids, braids that become trivial upon deletion of any strand.

Example 3.3. Let $A$ be the braid arrangement in $\mathbb{C}^4$, so that $\pi_1(M) = P_4$, the 4-string pure braid group. Denote the pure braid generators by $A_{ij}$, for $1 \leq i < j \leq 4$, corresponding to the hyperplanes $H_{ij}$ given by $x_i = x_j$. By considering the projection to $\mathbb{C}^3$ along the $x_4$ axis, we see that the subgroup $U$ generated by $A_{14}, A_{24}$ and $A_{34}$ is a free subgroup on three generators.

Let $\mathcal{S} = \{S_{123}, S_{124}, S_{134}, S_{234}\}$ be the set of rank-two flats of $A$ of multiplicity greater than two: $S_{ijk} = \{H_{ij}, H_{ik}, H_{jk}\}$. Consider the commutator $g = [A_{14}, [A_{24}, A_{34}]]$. Then, for every $i, j, k$, $\rho_{S_{ijk}}(g) = 1$. Indeed, one of $H_{ij}, H_{24}$ or $H_{34}$ lies outside of $S_{ijk}$, hence at least one factor of the commutator is sent to $1$ by $\rho_{S_{ijk}}$. Thus $\rho_{\mathcal{S}}(g) = 1$. Clearly $g \neq 1$, since $g$ is a nontrivial reduced word lying in the free subgroup $U$. Consequently, $\rho_{\mathcal{S}}$ is not injective in general. Interpreted
as a map on pure braids, the homomorphism $\rho_\mathcal{S}$ has the effect of deleting, in turn, each of the four strands, Thus $g$ corresponds to a nontrivial Brunnian pure braid on four strands. (The closure of this braid is the Borromean rings link.)

The same argument used in this example shows that $\rho_\mathcal{S}$ is not injective for complement of any strictly linearly-fibered arrangement which is not a product, and any set of flats $\mathcal{S}$.

Stanford showed that any braid (necessarily pure) which becomes trivial upon deletion of the strands outside a set $S \subseteq \{1, \ldots, \ell\}$ is generated by the pure braid generators $A_{ij}$ for $\{i, j\} \not\subseteq S$ along with iterated commutators of pure braid generators and their inverses which include at least one factor of this type. His argument can be cast in a more general setting so as to apply to other groups, including some arrangement groups.

Let $G$ be a group with finite generating set $Y$, and let $\mathcal{S}$ be a family of subsets of $Y$.

**Definition 3.4.** The support of an element $q \in G$ is relative to $\mathcal{S}$ is $\bigcap\{S \in \mathcal{S} \mid q \in \langle S \rangle\}$.

We will write $\text{supp}(q)$ for the support of $q$, $\mathcal{S}$ being understood. The support of $1$ is $\bigcap_{S \in \mathcal{S}} S$, which may be empty. Nonidentity elements of $G$ may also have empty support. In particular, $q \in \langle \text{supp}(q) \rangle$ need not hold.

**Definition 3.5.** A monic commutator in $Y$ is an element of $G$ defined recursively as follows:

- if $x \in Y$ then $x$ and $x^{-1}$ are monic commutators;
- if $x$ and $y$ are monic commutators, then $[x, y]$ is a monic commutator.

In other words, a monic commutator is an iterated commutator of generators and their inverses.

Let $\{T_1, \ldots, T_{2n}\}$ be the family of all subsets of $Y$, linearly ordered so that $T_i \subseteq T_j$ implies $i \leq j$. In particular, $T_1 = \emptyset$.

**Lemma 3.6.** Every element $q$ of $G$ can be written in the form $q = q_1 \cdots q_{2n}$, where each $q_i$ is a product of monic commutators with support equal to $T_i$, or $q_i = 1$.

**Proof.** The proof follows Stanford [Sta09] mutatis mutandis. We start with $q_1 = 1$, with support $T_1$. Assume inductively that $q = q_1 \cdots q_{i-1} s$ where $q_i$ is a product of monic commutators whose support is $T_i$, or $q_i = 1$, for $1 \leq i \leq r$, and $s$ is 1 or a product of monic commutators whose support is greater than or equal to $T_r$ in the linear order. Assume $s \neq 1$ and fix a factorization of $s$ into monic commutators. Suppose some monic commutator factor occurring in $s$ has support equal to $T_r$. We can then reduce by one the number of monic commutator factors in $s$ with support equal to $T_r$, as follows. Write $s$ as a product of monic commutators $x_1 \cdots x_3yz$ where each $x_i$ is a monic commutator with support greater than $T_r$, $y$ is a monic commutator with support $T_r$, and $z$ is a product of monic commutators with support greater than or equal to $T_r$. Then

$$s = yx_1[x_1, y]x_2[x_2, y]x_3 \cdots x_r[x_r, y]z.$$  

We replace $q_r$ by $q_r' = q_r y$ and $s$ by $s' = x_1[x_1, y]x_2[x_2, y]x_3 \cdots x_r[x_r, y]z$. Then $q = q_1 \cdots q_{i-1} q_r' s'$, with $q_r'$ a product of monic commutators with support $T_r$, and with $s'$ having one fewer monic commutator factors with support $T_r$ than $s$. 


Iterating the process, we finally may write \( q = q_1 \cdots q_n \) as above, but with \( s \) a product of monic commutators having support strictly greater than \( T_r \) in the linear order. Then, setting \( q_{r+1} = 1 \), we have \( q = q_1 \cdots q_{r+1}s \) with \( s \) a product of monic commutators with support greater than or equal to \( T_{r+1} \). This completes the inductive step. Setting \( r = 2^n \) yields the theorem. \( \square \)

Note, in the theorem above, if \( T_i \) is not an intersection of elements of \( \mathcal{S} \), then \( q_i = 1 \).

For \( S \) a subset of \( Y \), let \( G_S \) be the quotient of \( G \) by \( \langle Y - S \rangle \), the normal closure of \( Y - S \). Let \( \rho_T : G \to G_S \) be the canonical projection. Thus \( \rho_S \) kills generators not in \( S \).

**Definition 3.7.** A subset \( S \) of \( Y \) is retractive if \( \rho_S : G \to G_S \) restricts to an injection \( \langle S \rangle \to G_S \). A retractive family is a family of retractive subsets of \( Y \), all of whose intersections are also retractive.

When \( S \) is retractive, we may tacitly identify \( \langle S \rangle \) with \( G_S \). Then \( \rho_S : G \to G_S \) is a retraction in the usual sense: \( \rho_S(q) = q \) for \( q \in G_S \). By convention, \( \emptyset \) is retractive.

**Definition 3.8.** A subset \( T \) of \( Y \) is transverse to a family \( \mathcal{S} \) of subsets of \( Y \) if \( T \not\subseteq S \), or equivalently, \( T \cap (Y - S) \neq \emptyset \), for every \( S \in \mathcal{S} \).

Let
\[
\rho_{\mathcal{S}} = \prod_{S \in \mathcal{S}} \rho_S : G \to \prod_{S \in \mathcal{S}} G_S.
\]

**Theorem 3.9.** Suppose \( \mathcal{S} \) is a retractive family. Then the kernel of \( \rho_{\mathcal{S}} \) is generated by monic commutators whose support is transverse to \( \mathcal{S} \).

**Proof.** Again we adapt Stanford’s argument. It is easy to show by induction that \( \rho_S(q) = 1 \) if \( q \) is a monic commutator whose support meets \( Y - S \), since \([\rho_S(x), \rho_S(y)] = 1\) if \( \rho_S(x) = 1 \) or \( \rho_S(y) = 1 \).

Conversely, suppose \( q \in \ker(\rho_{\mathcal{S}}) \). Write \( q = q_1 \cdots q_{2^n} \) as in the preceding lemma. Fix \( S \in \mathcal{S} \). It suffices to show \( q_i = 1 \) if \( T_i \subseteq S \). We prove this by induction on \( |T_i| \).

We may assume \( T_i \) is an intersection of elements of \( \mathcal{S} \), by our earlier observation. If \( T_i \subseteq S \), then \( \rho_T = \rho_T^S \circ \rho_S \), where \( \rho_T^S : G_S \to G_T \) kills the images of the elements of \( S - T_i \). Then we have \( \rho_T(q) = 1 \), since \( \rho_S(q) = 1 \).

By the inductive hypothesis, \( q_j = 1 \) if \( T_j \) is a proper subset of \( T_i \). On the other hand, \( \rho_T(q_j) = 1 \) if \( T_j \not\subseteq T_i \), by the preceding paragraph. Then \( \rho_T(q_j) = 1 \) for \( j \neq i \), and thus, since \( \rho_T(q) = 1 \), we have \( \rho_T(q_i) = 1 \). Then \( q_i = 1 \) since \( T_i \) is an intersection of elements of \( \mathcal{S} \). \( \square \)

The hypothesis that \( \mathcal{S} \) is retractive is necessary. For example, consider the full braid group on three strands, \( G = \langle a, b \mid aba = bab \rangle \), with \( S = \{a\} \). Then \( G_S \) is the trivial group. Then \( a \neq 1 \) is in the kernel of \( \rho_S \), but is not a product of monic commutators whose supports contain \( b \), as can be seen by considering induced permutations.

The following is an immediate consequence of Theorem 3.9, which results in a condition for injectivity of \( \rho_{\mathcal{S}} \) that one can check by hand.

**Proposition 3.10.** Let \( Y \) be a generating set for \( G \) and \( S \subseteq Y \). If \( G_S \) is a free group of rank \( |S| \), then every subset of \( S \) is retractive relative to \( Y \).

Finally, we establish a corollary of Theorem 3.9, which results in a condition for injectivity of \( \rho_{\mathcal{S}} \) that one can check by hand.
Suppose $\mathcal{F}$ is a retractive family relative to the generating set $Y$ of $G$. Let $F$ be the free group on $Y$, and observe that $\mathcal{F}$ is also a retractive family in $F$, relative to $Y$. For $w$ in $F$, denote the image of $w$ in $G$ by $\overline{w}$. We will consider $Y$ to be a subset of $F$ and of $G$ - this should not cause undue confusion, and simplifies the statement of the theorem.

The support of a nontrivial monic commutator in $F$ is precisely the set of generators that appear in $w$. If $w$ is a monic commutator in $F$ with support transverse to $\mathcal{F}$, then $\overline{w} \in \ker(\rho_\mathcal{F})$. For any group $\Gamma$, denote the lower central series of $\Gamma$ by $\Gamma = \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_r \supseteq \cdots$, where $\Gamma^{k+1} = [\Gamma, \Gamma_k]$ for $k \geq 1$. The commutator length $\ell(x)$ of $x \in \Gamma$ is the smallest $r$ such that $x \in \Gamma_r$.

**Theorem 3.11.** Suppose $\mathcal{F}$ is a retractive family relative to a generating set $Y$ of $G$. Suppose $\mathcal{F}$ covers $Y$, and the elements of $\mathcal{F}$ are pairwise incomparable. Suppose

(i) $[a, b] = 1$ if $\{a, b\} \subseteq Y$ is transverse to $\mathcal{F}$, and

(ii) $[a, G_2^S] = 1$ for every $S \in \mathcal{F}$ and $a \in Y - S$.

Then $\rho_\mathcal{F}$ is injective.

**Proof.** Denote $\rho_\mathcal{F}$ by $\rho$. Suppose (i) and (ii) hold. Let $w$ be a monic commutator in $F$ relative to $Y$ with support transverse to $\mathcal{F}$. We show $\overline{w} = 1$ in $G$ by induction on the length of $w$. Since $\mathcal{F}$ covers $\mathcal{A}$, $\ell(w) \geq 2$. Thus $w = [x, y]$ for monic commutators $x$ and $y$. If $\ell(w) = 2$ then $\overline{w} = 1$ by (i). Suppose $\ell(w) > 2$. If $y$ is transverse to $\mathcal{F}$ then $\overline{y} = 1$ by the inductive hypothesis. Thus we may assume $y \in F_2^S$ for some $S \in \mathcal{F}$ and $r \geq 2$. If $\ell(x) = 1$ then $x = a$ for some $a \in Y$, and $\overline{w} = [\overline{x}, \overline{y}] \in [a, G_2^S] = 1$ by (ii).

Then we can assume $\ell(x) > 1$. Write $x = [u, v]$ for monic commutators $u$ and $v$. Fix $S \in \mathcal{F}$ with $\text{supp}(y) \subseteq S$. Since $\text{supp}(w)$ is transverse to $\mathcal{F}$, if $\text{supp}(u) \subseteq S$ or $\text{supp}(v) \subseteq S$, then $x = [u, v]$ is transverse to $\mathcal{F}$, hence $\overline{x} = 1$ by induction, and $\overline{w} = 1$. Then $\text{supp}(u) \not\subseteq S$ and $v \not\subseteq S$. Then, by the incomparability assumption, $\text{supp}([u, y])$ and $\text{supp}([v, y])$ are transverse to $\mathcal{F}$.

Then $[\overline{u}, \overline{y}] = [\overline{v}, \overline{y}] = 1$ by induction, hence $\overline{uv} = \overline{y}$. Applying the commutator identity $[u, v] = [v, u][[u, v], y] = 1$, we conclude that $\overline{w} = [\overline{x}, \overline{y}] = [[\overline{u}, \overline{v}], \overline{y}] = 1$. □

### 3.3 Retractive families for arrangement groups.

Let $\mathcal{A}$ be an arrangement of affine hyperplanes in $\mathbb{C}^k$. Let $M = M(\mathcal{A}) = \mathbb{C}^k - \bigcup_{H \in \mathcal{A}} H$ and $G = G(\mathcal{A}) = \pi_1(M, x_0)$ where $x_0 \in M$.

Let $L = L(\mathcal{A})$ be the set of nonempty intersections of elements of $\mathcal{A}$. Define a partial order on $L$ by

$$X \leq Y \text{ if } Y \subseteq X$$

The meet of a pair of elements $X, Y \in L(\mathcal{A})$ is $X \wedge Y = \bigcap \{H \in \mathcal{A} \mid H \supseteq X + Y\}$. $L(\mathcal{A})$ is closed under this operation, and forms a meet-semilattice, with minimal element $\mathbb{C}^k$; it is a lattice if and only if $\bigcap_{H \in \mathcal{A}} H \neq \emptyset$. The latter case we say $\mathcal{A}$ is central. In any case we will call $L(\mathcal{A})$ the intersection lattice of $\mathcal{A}$. $L(\mathcal{A})$ is a ranked poset, with rank function given by $\text{rank}(X) = \text{codim}(X)$. All maximal elements of $\mathcal{A}$ have the same rank [OT92], which we call the rank of $\mathcal{A}$. If $\mathcal{A}$ has rank $\ell$ we say $\mathcal{A}$ is essential. A subset $C$ of $L(\mathcal{A})$ is meet-closed if $X \wedge Y \in C$ for all $x, y \in C$, and the meet-closure of a subset of $L(\mathcal{A})$ is the intersection of all meet-closed sets containing it.
If $A$ is central, $L = L(A)$ is isomorphic to the lattice of flats of the underlying matroid of $A$. A flat of $A$ is a subset of $A$ of the form \{ $H \in A \mid p \in H$ \} for $p \in C^\ell$. The flat corresponding to a subspace $X \in L$ is $A_X = \{ H \in A \mid X \subseteq H \}$, obtained by choosing $p$ to be a generic point on $X$. The arrangement $A_X$ is called a localization of $A$. We have $X \subseteq Y$ in $L$ if and only if $A_X \subseteq A_Y$. Also $A_{X \wedge Y} = A_X \cap A_Y$. We denote $M(A_X)$ by $M_X$.

For $X \in L(A)$, choose a generic point $p_X$ of $X$, so that $p_X \in H$ if and only if $X \subseteq H$, for $H \in A$. Let $B_X$ be a small ball centered at $p_X$ satisfying $B_X \cap M_X \subseteq M$. Note that the inclusion $B_X \cap M_X \hookrightarrow M_X$ factors through the inclusion $M \hookrightarrow M_X$.

Let $H \in A_X$. Let $\beta_{X,H}$ be the oriented boundary of a planar topological disk in $B_X$ that intersects $H$ transversely in a single interior point and misses the other hyperplanes in $A_X$. Let $\gamma_{X,H}$ be a path in $M$ from the base point $x_0$ to a point of $\beta_{X,H}$. Then let $a_{X,H} \in G$ be represented by $\gamma_{X,H} \beta_{X,H}(\gamma_{X,H})^{-1}$. The element $a_{X,H}$ depends on the choice of $p_X, \beta_{X,H}$, and $\gamma_{X,H}$, but is well-defined up to conjugacy in $G$. Let $S_X = \{ a_{H,X} \mid H \in A_X \}$.

**Definition 3.12.** A subset $Y$ of $\{ a_{X,H} \mid X \in L, H \in A_X \}$ that contains exactly one element of the form $a_{X,H}$ for each $H \in A$ is a standard generating set of $G$.

A standard generating set is indeed a generating set of $G$ - see [CS97].

**Theorem 3.13.** Let $X \in L(A)$ and let $Y$ be a standard generating set of $G$ containing $S_X$, for some choice of point $p_X$, and paths $\gamma_{X,H}$ and $\beta_{X,H}$ for $H \in A_X$. Then $S_X$ is retractive with respect to $Y$.

**Proof.** The composite $B_X \cap M_X \hookrightarrow M \hookrightarrow M_X$ is a homotopy equivalence, by radial retraction centered at $p_X$. The hypothesis on $Y$ implies that $\rho_{S_X}$ is the homomorphism induced by the inclusion $M \hookrightarrow M_X$. Then $\rho_{S_X}$ restricts to an isomorphism on the image of $\pi_1(B_X \cap M_X)$ in $G$. This image coincides with the subgroup $\langle S_X \rangle = \langle a_{X,H} \mid H \in A_X \rangle$. Thus $S$ is retractive. \hfill \Box

**Corollary 3.14.** Let $\mathcal{F} \subseteq L(A)$. Let $Y$ be a standard generating set of $G$ containing $S_X$, for some choice of point $p_X$ and paths $\gamma_{X,H}$ and $\beta_{X,H}$ for $H \in A_X$, for each $X$ in the meet-closure of $\mathcal{F}$. Then $\{ S_X \mid X \in \mathcal{F} \}$ is a retractive family with respect to $Y$.

Let $\mathcal{F}$ be a family of elements of $L(A)$. We say a standard generating set $Y$ of $G$ is adapted to $\mathcal{F}$ if the hypothesis of the preceding corollary is satisfied. In this case let us identify $\mathcal{F}$ with the corresponding family of subsets of $Y$.

If $Y$ is a standard generating set adapted to the family of all rank-two flats of $A$, then we say $G(A)$ has a conjugation-free presentation with generators $Y$. (See also [EGT09].)

**Corollary 3.15.** Suppose $\mathcal{F} \subseteq L(A)$ and $Y$ is a standard generating set of $G(A)$ adapted to $\mathcal{F}$. Then the kernel of $\rho_{\mathcal{F}}$ is generated by monic commutators with support transverse to $\mathcal{F}$.

**Example 3.16.** Let $\mathcal{A}$ be the braid arrangement of rank $\ell$, with hyperplanes $H_{ij}, 1 \leq i < j \leq \ell$. For $I \subseteq \{1, \ldots, \ell\}$, let $X_I = \bigcap_{i,j \in I} H_{ij}$. Let $\mathcal{F}$ be any family of such lattice elements. Note that $X_I \wedge X_J = X_{I \cup J}$.

Let $Y = \{ A_{ij} \mid 1 \leq i < j \leq \ell \}$ be the standard braid generators. Then $Y$ is adapted to $\mathcal{F}$. This is clear from the representation of the $A_{ij}$ as pure braids. On
the other hand, $G(\mathcal{A})$ does not have a conjugation-free geometric presentation - see [LCT09].

With this observation, Stanford’s description of Brunnian braids is a consequence of Corollary 3.15.

For our applications it will be essential to include flats “at infinity” in the preceding construction. We can do this using Proposition 3.10. Let $\mathcal{A}$ be an affine arrangement. Say two hyperplanes $H, H' \in \mathcal{A}$ are parallel if $H \cap H' = \emptyset$. A set $\mathcal{P} \subseteq \mathcal{A}$ of mutually parallel hyperplanes is called a parallel class.

**Proposition 3.17.** Let $Y$ be a standard generating set of $G(\mathcal{A})$. Let $\mathcal{P}$ be a parallel class in $\mathcal{A}$, and $S(\mathcal{P}) = \{a_{X,H} \in Y \mid H \in \mathcal{P}\}$. Then $S(\mathcal{P})$ is retractive relative to $Y$.

**Proof.** $M(\mathcal{P}) = Y - \bigcup_{H \in \mathcal{P}} H$ is homotopy equivalent to the complement of $|\mathcal{P}|$ points in the plane, hence $G_{S(\mathcal{P})}$ is free of rank $|\mathcal{P}|$. Then $S(\mathcal{P})$ is retractive by Proposition 3.10. \qed

If $Y$ is a standard generating set, we will identify a parallel class $\mathcal{P}$ with the corresponding subset $\{a_{X,H} \in Y \mid H \in \mathcal{P}\}$ of $Y$.

**Corollary 3.18.** Let $\mathcal{I}_0 \subseteq L(\mathcal{A})$ and let $\mathcal{I}_\infty$ be a collection of parallel classes of $\mathcal{A}$. Let $Y$ be a standard generating set of $G(\mathcal{A})$ adapted to $\mathcal{I}_0$. Then $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_\infty$ is a retractive family relative to $Y$.

**Proof.** Every intersection of elements of $\mathcal{I}$ corresponds to an element in the meet-closure of $\mathcal{I}_0$, or is a parallel class. \qed

**Corollary 3.19.** Let $\mathcal{I}_0 \subseteq L(\mathcal{A})$, and let $\mathcal{I}_\infty$ be a set of parallel classes of $\mathcal{A}$. Let $Y$ be a standard generating set of $G(\mathcal{A})$ adapted to $\mathcal{I}_0$. Let $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_\infty$. Then the kernel of $\rho_{\mathcal{I}}$ is generated by monic commutators whose support is transverse to $\mathcal{I}$.

Theorem 3.11 then applies to arrangement groups as follows.

**Corollary 3.20.** Suppose $\mathcal{I}_0 \subseteq L(\mathcal{A})$ and let $Y$ be a standard generating set of $G(\mathcal{A})$ adapted to $\mathcal{I}_0$. Let $\mathcal{I}_\infty$ be a set of maximal parallel classes in $\mathcal{A}$, and $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_\infty$. Suppose $\mathcal{I}$ covers $\mathcal{A}$ and the elements of $\mathcal{I}$ are pairwise incomparable. If

(i) $[a_H, a_K] = 1$ if $\{H, K\}$ is transverse to $\mathcal{I}$, and
(ii) $[a_H, G_\delta^S] = 1$ for every $S \in \mathcal{I}$ and $H \notin S$,

then $\rho_{\mathcal{I}}$ is injective.

**Remark 3.21.** Observe that condition (i) above is automatically satisfied if $G(\mathcal{A})$ has a conjugation-free presentation with generators $Y$.

In our applications of Corollary 3.20 $G_\delta$ contains a nonabelian free group for every $S \in \mathcal{I}$. Then $G_\delta$ is an infinitely-generated free group, making condition (ii) above difficult to check. The following result allows us to verify condition (ii) easily for many examples.

**Proposition 3.22.** Let $G$ be a group with generating set $Y$. Let $S = \{x_0, \ldots, x_m\}$ and $T = \{y_0, \ldots, y_n\}$ be subsets of $Y$, with $x_0 = y_0$. Assume that:

(i) $[x_i, y_j] = 1$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and
(ii) $\langle y_0, y_1, \ldots, y_n \rangle = \langle y_0 y_1 \cdots y_n \rangle \times \langle y_1, \ldots, y_n \rangle$. 

Then \([x_i, \langle y_0, \ldots, y_n \rangle^2] = 1\) for \(1 \leq i \leq m\), and \([y_j, \langle x_0, \ldots, x_m \rangle^2] = 1\) for \(1 \leq j \leq n\).

**Proof.** Write \(G_T = \langle y_0, y_1, \ldots, y_n \rangle\), \(G_{T'} = \langle y_1, \ldots, y_n \rangle\), and \(y = y_1 \cdots y_n\). Assumption (ii) may be expressed as \(G_T = (y_0 y) \times G_{T'}\). It follows that \(G_T^2 = G_{T'}^2\). Since \(x_i\) commutes with all generators of \(G_T\) by assumption (i), we have \([x_i, G_T^2] = 1\) for \(1 \leq i \leq m\). The first conclusion follows.

Since \(x_0 = y_0\), assumption (ii) implies that \([y_j, x_0 y] = 1\) for \(1 \leq j \leq n\). Consequently, \(x_0^{-1} y_j x_0 = y y_j y^{-1}\) and \(x_0 y_j x_0^{-1} = y^{-1} y_j y\) for each \(j\). Let \(u, v \in \langle x_0, x_1, \ldots, x_m \rangle\) and \(1 \leq j \leq n\). Write
\[
\begin{align*}
u &= u_1 x_0 \epsilon_1 u_2 x_0^2 \cdots u_r x_0^r u_{r+1}, \\
v &= v_1 x_0^\lambda v_2 x_0^\lambda \cdots v_s x_0^\lambda v_{s+1},
\end{align*}
\]
where \(\epsilon_j, \lambda_j \in \{1, -1\}\) and \(u_k\) and \(v_l\) are words in \(x_1, \ldots, x_m\). Note that \([y_j, u_k] = 1\) and \([y_j, v_l] = 1\) for \(1 \leq k \leq r + 1\) and \(1 \leq l \leq s + 1\). Also the relations \([y_j, x_0 y] = 1\) recorded above imply that \([x_0, y] = 1\).

For the second assertion, it suffices to show that \([y_j, [u, v]] = 1\). We compute
\[
\begin{align*}
u v [y_j, [u, v]] v^{-1} u^{-1} &= u v y_j^{-1} v^{-1} u^{-1} v u y_j u^{-1}. \\
\end{align*}
\]
Write \(\epsilon = \sum_{k=1}^r \epsilon_k\) and \(\lambda = \sum_{l=1}^s \lambda_l\). Using the above relations, we obtain \(u y_j u^{-1} = y^{-\epsilon} y_j y^\epsilon\), \(v y_j v^{-1} = y^{-\lambda} y_j y^\lambda\), and subsequently, \(u v y_j v^{-1} u^{-1} = v y_j u^{-1} v^{-1} = y^{-\lambda - \epsilon} y_j y^{\lambda + \epsilon}\). It follows easily that \(\nu u [y_j, [u, v]] u^{-1} v^{-1} = 1\).

**Example 3.23.** Let \(A\) be the arrangement of lines in \(\mathbb{C}^2\) with defining equations \(x = 0, x = 1, y = 0, y = -1,\) and \(x + 2y = 0\). (See Figure 1(a).)

Using the Randell algorithm [Ran82, Fal93], the fundamental group \(G = G(A)\) of the complement of \(A\) has a presentation with generators \(a_1, a_2, a_3, a_4, a_5\) corresponding to the lines, and relations \([a_1, a_4], [a_2, a_4], [a_2, a_5], [a_2, a_3], [a_1 a_3, a_5]\), and \([a_1, a_3 a_5]\). (This a conjugation-free presentation.) Let \(S_0 = \{\{a_1, a_3, a_5\}\}, S_\infty = \{\{a_1, a_2\}, \{a_4, a_5\}\},\) and \(S = S_0 \cup S_\infty\). Then \(Y\) is adapted to \(S\) and the elements of \(S\) are pairwise incomparable.
We verify the conditions of Corollary 3.20. Condition (i) holds by Remark 3.21. Let \( S = \{a_1, a_2\} \) and \( T = \{a_1, a_3, a_5\} \). The local group \( G_T \cong \langle T \rangle \) is isomorphic to \( \langle a_1a_3a_5 \rangle \times \langle a_3, a_5 \rangle \), and \( [a_2, a_3] = [a_2, a_5] = 1 \). Then \( [a_2, G_T^2], [a_3, G_T^2] \), and \( [a_5, G_T^2] \) all vanish by Proposition 3.22. Similarly, setting \( U = \{a_3, a_4\} \), we see that \( [a_1, G_T^2], [a_1, G_U^2] \), and \( [a_5, G_U^2] \) are trivial. Thus condition (ii) of Corollary 3.20 holds, and \( \rho_\mathcal{F} \) is injective.

**Example 3.24.** Let \( \mathcal{A} \) be the arrangement of lines in \( \mathbb{C}^2 \) with defining equations \( x = 0, y = 0, x + y = 2, y = 0, \) and \( y = 1 \), exhibited in Figure 1(b). The group \( G(\mathcal{A}) \) has generators \( a_1, a_2, a_3, a_4, a_5, a_6 \) corresponding to the lines, with relations

\[
[a_1a_3, a_5], [a_1a_3a_5], [a_2, a_5], [a_2, a_3], [a_6, a_3], [a_4, a_6^2],
[a_1, a_4], [a_2a_4a_6], [a_2, a_4a_6], \text{ and } [a_1, a_6^2].
\]

Let \( \mathcal{F}_0 = \{\{a_1, a_3, a_5\}, \{a_2, a_4, a_6\}\}, \mathcal{F}_\infty = \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}\}, \text{ and } \mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_\infty. \) Then \( Y = \{a_1, a_2, a_3, a_4, a_5, a_6\} \) is adapted to \( \mathcal{F} \), and the elements of \( \mathcal{F} \) are pairwise incomparable.

Since \( [a_4, a_6^2] = [a_1, a_4] = 1 \), we have \( [a_4, a_5] = 1. \) Since \( [a_2a_4, a_6] = 1, a_6^2 = a_6^{-1}. \) Then \( [a_1, a_6^2] = 1 \) and \( [a_1, a_4] = 1 \) together imply \( [a_1, a_6] = 1 \). Along with the given relations, this confirms that condition (i) of Corollary 3.20 holds. Condition (ii) of Corollary 3.20 can be verified using Proposition 3.22 as in the preceding example. It follows that \( \rho_\mathcal{F} \) is injective.

**Example 3.25.** Let \( \mathcal{A} = D_3 \) be the arrangement of type \( D_3 \) as in Example 2.13. The complement of \( \mathcal{A} \) is diffeomorphic to the complement \( PB_4 \) of the rank-three braid arrangement. As shown in Example 3.3, \( \mathcal{A} \) has four rank-two subarrangements of size three, and the kernel of the resulting product homomorphism \( \rho: G(\mathcal{A}) \to \prod_{S \in \mathcal{F}} G_S \) is isomorphic to the group of Brunnian braids on four strands. In particular, \( \rho \) is not injective.

The arrangement \( \mathcal{A} \) itself also supports the generating function

\[
\sigma = (z_1^2, z_2^2, z_3^2): M(\mathcal{A}) \to PB_3,
\]

and one can consider the product homomorphism \( \rho \times \sigma_*: G(\mathcal{A}) \to \prod_{S \in \mathcal{F}} G_S \times P_3 \) given by all five generating sets on \( \mathcal{A} \). The target is a product of free groups. One can still find nontrivial Brunnian braids in the kernel of \( \sigma \), hence \( \rho \times \sigma_* \) is not injective.

4. **Arrangement groups and right-angled Artin groups**

We are mostly interested in the case where \( \mathcal{F}_0 \) is a set of rank-two flats, as in Examples 3.23 and 3.24. In this case we will denote \( \mathcal{F}_0 \cup \mathcal{F}_\infty \) by \( \mathcal{X} \), for clarity.

A right-angled Artin group is a group that has a finite presentation in which all relations are commutators of two generators. The group is determined by the undirected graph whose vertices are the generators, with edges connecting pairs of commuting generators. This family includes products of free groups. Suppose \( \mathcal{A} \) is an affine arrangement, and \( S \) is a rank-two flat or parallel class in \( \mathcal{A} \). Then \( G_S \) is isomorphic to \( F_{r-1} \times \mathbb{Z} \), where \( r = |S| \). The infinite cyclic center of \( G_S \) is generated by the product \( \prod_{j \in S} a_j \), in some order. The free factor is generated by any \( r - 1 \) of the \( a_i, i \in S \). Then, if \( \mathcal{X} \subset 2^4 \) is a set of rank-two flats and parallel classes, the target of \( \rho_\mathcal{X}: G \to \prod_{S \in \mathcal{X}} G_S \) is a product of free groups, hence is a right-angled Artin group. But it will be more convenient to replace \( G_S \) in this...
product by the free group $\overline{G}_S = G/\langle \prod_{i \in S} a_i \rangle$. In this section we show that the image of the resulting homomorphism is normal, the cokernel is free abelian, we compute its rank, and show that projecting to $\prod_{S \in X} \overline{G}_S$ does not affect the kernel of $\rho_X$. As a result we are able to realize some arrangement groups as subgroups of right-angled Artin groups, drawing conclusions about their qualitative properties and homological finiteness type.

4.1. The cokernel of $\rho_X$. For our purposes it will be beneficial to change notation. For the remainder of this section, let $A = \{H_1, \ldots, H_n\}$ be a central arrangement in $\mathbb{C}^d$, with complement $M = \mathbb{C}^d - \bigcup_{1 \leq i \leq n} H_i$ and group $G = \pi_1(M)$. Generators $a_1, \ldots, a_n$ of $G$ may be chosen in such a way that $a_1 \cdots a_n$ is central in $G$; let $C$ denote the quotient $G/(a_1 \cdots a_n)$. (We always assume that standard generating sets satisfy this identity.) The group $C$ is isomorphic to the fundamental group of the projective image $\overline{M}$ of $M$ in $\mathbb{C}P^{d-1}$, which is diffeomorphic to the complement $M(dA)$ of the affine arrangement $dA$ obtained by deconing $A$ - see [OT92]. The group $C$ is generated by $a_1, \ldots, a_{n-1}$. A set $\mathcal{X}$ of rank-two flats of $A$ corresponds bijectively (upon deletion of $H_n$) to a set of rank-two flats and maximal parallel classes of $dA$. If $S \in \mathcal{X}$ then the homomorphism $C \to \overline{G}_S$ described in the previous paragraph is induced by the inclusion $\overline{M} \hookrightarrow \overline{M}_S$ of projectivized arrangement complements in $\mathbb{C}P^{d-1}$.

With this setup, we formulate our result more generally. Let $G$ be a group with finite generating set $Y = \{a_1, \ldots, a_n\}$. Assume that $z = a_1 \cdots a_n$ is central in $G$.

Let $\mathcal{X} = \{S_1, \ldots, S_m\}$ be a set of subsets of $Y$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, let $a_{ij}$ denote the image of $a_j$ in $G_{S_i}$. Then $G_{S_i}$ is generated by $\{a_{ij} \mid j \in S_i\}$, and $a_{ij} = 1$ if $j \not\in S_i$. Since $z$ is central in $G$, $z_i = a_1 \cdots a_n$ is central in $G_{S_i}$, for each $i$. Let $C = G/\langle z \rangle$ and $\overline{G}_{S_i} = G_{S_i}/\langle z_i \rangle$. Let $\rho = \rho_X = \prod \rho_{S_i} : G \to \prod_{i=1}^m \overline{G}_{S_i}$. The image of $a_j$ under $\rho$ is $\prod_{i=1}^m a_{rij}$. Since $\rho S_i(z) = z_i$, $\rho$ induces a well-defined homomorphism $\overline{\rho} : G \to \prod_{i=1}^m \overline{G}_{S_i}$.

Assume further that $\overline{G}_{S_i}$ is a free group of rank $|S_i| - 1$, so that the images of any $|S_i| - 1$ of the elements $a_{ij}, j \in S_i$ form a free basis. Then $\prod_{i=1}^m \overline{G}_{S_i}$ is a right-angled Artin group, whose graph is the complete multipartite graph with parts of sizes $|S_1| - 1, \ldots, |S_m| - 1$. Viewing $\overline{G}_S$ as a subgroup of $\prod_{S \in X} \overline{G}_S$, we have $[a_{ij}, a_{ik}] = 1$ for $i \neq r$.

Finally, assume $|S_i \cap S_r| \leq 1$ for all $1 \leq i \neq r \leq m$. Let $A = \prod_{S \in X} G_S$ and $\overline{A} = \prod_{S \in X} \overline{G}_S$. We have the following commutative diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\rho} & A \\
\downarrow & & \downarrow \\
\overline{G} & \xrightarrow{\overline{\rho}} & \overline{A}.
\end{array}
$$

In our application to arrangement groups, $G$ will be $G(A)$ for a central arrangement $A$, $\overline{G}$ is the fundamental group of the projectivized complement $\overline{M}$, and $\overline{\rho} : \overline{G} \to \overline{A}$ is the product of the homomorphisms induced by inclusions of projectivized complements. Rank-two flats are not retractive in this context, so our analysis of the kernel does not apply to $\overline{\rho}$, only to $\rho$.

**Theorem 4.1.** The image of $\rho$ is a normal subgroup of $A$.

**Proof.** Fix integers $i, j,$ and $k$ with $1 \leq i \leq m$ and $1 \leq j, k \leq n$, and consider the conjugate $\rho(a_k)^{a_{ij}}$. If $S_i$ doesn’t contain both $j$ and $k$, then $\rho(a_k)^{a_{ij}} = \rho(a_k)$. 

Suppose $S_i$ contains both $j$ and $k$. If $r \neq i$ then $S_r$ doesn’t contain both $j$ and $k$, so $a_{ij}^r = a_{rk} = a_{rk}^{a_{ij}}$. If $r = i$ then $a_{ij}^r = a_{rk}^{a_{ij}}$. Then we have

$$\rho(a_k)^{a_{ij}} = \prod_{r=1}^m a_{rk}^{a_{ij}} = \prod_{r=1}^m a_{rk}^{a_{ij}} = \prod_{r=1}^m a_{rk}^r = (\prod_r a_{rk}) \rho(a_{ij}) \rho(a_j).$$

So, in either case, $\rho(a_k)^{a_{ij}}$ lies in the image of $\rho$. \qed

**Corollary 4.2.** The image of $\overline{\rho}$ is a normal subgroup of $\overline{\mathcal{A}}$.

**Proof.** The surjection $A \to \overline{\mathcal{A}}$ maps $\rho(G)$ onto $\overline{\rho(G)}$. \qed

**Proposition 4.3.** The cokernel of $\rho: G \to A$ is abelian.

**Proof.** As in the preceding proof, we observe that $[a_{ij}, a_{ik}] = [\rho(a_j), \rho(a_k)]$ if $S_i$ contains both $j$ and $k$, and is trivial otherwise. Since $[a_{ij}, a_{rk}] = 1$ if $r \neq i$ this shows that $A/\rho(G)$ is abelian. \qed

**Corollary 4.4.** The cokernel of $\overline{\rho}: \overline{G} \to \overline{A}$ is abelian.

**Proof.** The group $\overline{A}/\overline{\rho(G)}$ is a quotient of $A/\rho(G)$. \qed

The fact that $\overline{A}/\overline{\rho(G)}$ is abelian can also be deduced directly from the normality of $\overline{\rho(G)}$ and the fact that it surjects onto each factor of $\overline{A}$, by a result of [BM09].

We denote the abelianization of a group or homomorphism by appending the subscript $ab$. So, for example, $A_{ab} = A/[A, A]$.

Assume that $G_{ab}$ and $A_{ab}$ are free abelian, with bases given by the images of $a_1, \ldots, a_n$ and $a_{ij}, j \in S_i$, respectively. This implies in particular that the central elements $\bar{z}_i, 1 \leq i \leq m$ have infinite order. Again, this hypothesis holds if $G = G(A)$ for a central arrangement $A$ and $\{a_1, \ldots, a_n\}$ is a standard generating set of $G$, ordered appropriately.

Denote the images of $a_k$ and $a_{ij}$ in $G_{ab}$ and $A_{ab}$ by $b_k$ and $b_{ij}$, respectively. Then $\overline{G_{ab}}$ is the quotient of $G_{ab}$ by the subgroup generated by $\sum_k b_k$, and $\overline{A_{ab}}$ is the quotient of $A_{ab}$ by the subgroup generated by $\{\sum_i b_{ij} \mid 1 \leq i \leq m\}$. The latter subgroup will be denoted by $J$. Since $\overline{A}/\overline{\rho(G)}$ is abelian, we have the following:

**Corollary 4.5.** The cokernel of $\overline{\rho}_{ab}: G_{ab} \to \overline{A}_{ab}$ is isomorphic to $\overline{A}/\overline{\rho(G)}$.

The homomorphism $\overline{\rho}_{ab}$ can be understood in graph-theoretical terms, as follows. Let $\Lambda_X$ be the bipartite graph with vertex set $X \cup A$ and edges $\{S_i, H_j \}$ for $H_j \in S_i$. The group $A_{ab}$ is naturally identified with the additive group of integer edge-labellings of $\Lambda_X$, with the canonical generator $b_{ij}$ corresponding to the labelling with 1 on the edge $(H_j, S_i)$ and 0 on all other edges. The map $\rho_{ab}$ maps $b_{ij}$ to the labelling that assigns 1 to each edge incident with $H_j$, and 0 to the other edges of $\Lambda_X$. Then the subgroup $\rho_{ab}(G_{ab})$ is generated by the $n$ edge-labellings that assign 1 to each edge incident with $H_j$, and 0 to all other edges, for $1 \leq j \leq n$. This subgroup will be denoted by $I$.

Similarly, the subgroup $J$ of $A$ defined above is generated by the labellings having value 1 on all edges incident with $S_i$, and 0 on all other edges, for each $i$. Let $M$ be the subgroup of $A_{ab}$ consisting of edge-labellings which sum to zero at each vertex $S_i$, $1 \leq i \leq m$. Then $A_{ab} = M \oplus J$, hence $M$ is isomorphic to $\overline{A}_{ab} = A_{ab}/J$.

Let us identify $A_{ab}$ with $\mathbb{Z}^n$, with the standard basis corresponding to the edges of $\Lambda_X$. Let $R$ denote the incidence matrix of $\Lambda_X$, with rows indexed by the vertices $H_j, 1 \leq j \leq n$, and $S_i, 1 \leq i \leq m$, and columns indexed by edges $(H_j, S_i), H_j \in S_i$. 


By the discussion above, \(I + J\) is the subgroup of \(A_{ab}\) generated by the rows of \(R\). The following observation is then clear.

**Lemma 4.6.** The group \(I + J\) consists of edge-labelings of \(\Lambda_X\) that are induced by vertex labelings, with the label on an edge given by the sum of the labels at its endpoints.

The following lemma is a standard result in graph theory (see [GR01]) that will be useful for us.

**Lemma 4.7.** The left null space of \(R\) has a free basis with one generator for each (bipartite) connected component of \(\Lambda_X\).

**Proof.** Vectors in the left null space of \(R\) correspond to vertex-labellings that sum to zero along each edge. Clearly such a vector is uniquely determined once the value at one vertex of each connected component of \(\Lambda_X\) is specified. As long as that component is bipartite, this vertex label can be any integer - otherwise it must be zero. But \(\Lambda_X\) is bipartite, hence each component is bipartite. \(\square\)

**Lemma 4.8.** The cokernel of \(\rho_{ab}: G_{ab} \to A_{ab}\) is free abelian.

**Proof.** Lifting to \(A_{ab}\), we have that \(A_{ab}/\rho_{ab}(G_{ab})\) is isomorphic to \(A_{ab}/(I + J)\), where \(I = \rho_{ab}(G_{ab})\) and \(J = \ker(A_{ab} \to A_{ab})\) are the subgroups defined above. Suppose \(e \in A_{ab}\) is an edge-labeling of \(\Lambda_X\) and \(ke \in I + J\) for some positive integer \(k\). Then \(ke\) is induced by a vertex-labeling \(\nu\). We can replace \(\nu\) by \(\nu + \kappa\) where \(\kappa\) is in the left nullspace of \(R\) - the induced edge-labeling will still equal \(ke\) because \(\kappa\) sums to zero along each edge. Fixing one vertex in each component of \(\Lambda_X\), and using Lemma 4.7, we can choose \(\kappa\) so that \(\nu + \kappa\) has value zero on the specified vertices. Then, since each value in the induced edge-labeling is divisible by \(k\), it follows that each value in the vertex-labeling \(\nu + \kappa\) is divisible by \(k\). Then \(\nu + \kappa = k\mu\) for some integer vertex-labeling \(\mu\), and \(\mu\) induces \(e\). Thus \(e \in I + J\). We conclude that \(A_{ab}/(I + J)\) is torsion-free. \(\square\)

Recall \(n = |A|\) and \(m = |X|\).

**Theorem 4.9.** The rank of \(\overline{A}_{ab}/\overline{\rho}_{ab}(\overline{G})\) is \(\sum_{S \in X} |S| - n - m + c\), where \(c\) is the number of components of \(\Lambda_X\).

**Proof.** Since \(\overline{A}_{ab}/\overline{\rho}_{ab}(\overline{G}) \cong A_{ab}/(I + J)\), is free abelian, we tensor with \(\mathbb{Q}\) and calculate dimension. The vector space \((I + J) \otimes \mathbb{Q}\) is the row space of \(R\) over \(\mathbb{Q}\), so \(A_{ab}/(I + J) \otimes \mathbb{Q}\) is isomorphic to the kernel of \(R\) over \(\mathbb{Q}\). We compute

\[
\dim(\ker(R)) = \sum_{i=1}^{m} |S_i| - \rank(R) = \sum_{i=1}^{m} |S_i| - \rank(R^T)
\]

\[
= \sum_{i=1}^{m} |S_i| - [n + m - \dim(\ker(R^T))] = \sum_{i=1}^{m} |S_i| - n - m + c
\]

by Lemma 4.7. \(\square\)

4.2. **Injectivity of \(\overline{\rho}\).** The results of the previous section do not apply directly in the projective setting because the projectivized local groups are not retractive. At the same time it is problematical to apply our injectivity criteria to central rank-three arrangements. In this subsection we resolve these issues.
Proposition 4.10. Suppose \( \Lambda_X \) is connected. Then the kernel of \( \rho \) projects isomorphically onto the kernel of \( \overline{\rho} \).

Proof. Denote the projections \( G \to \overline{G} \) and \( A \to \overline{A} \) by \( p \) and \( q \) respectively. The kernel of \( p \) is generated by the central element \( a_1 \cdots a_n \), which intersects \( \ker(\rho) \) trivially. Thus \( \ker(\rho) \) injects into \( \ker(\overline{\rho}) \).

Let \( \delta = \rho(a_1 \cdots a_n) \). Then \( \delta \in \ker(q) \). To prove that \( \ker(\rho) \) maps onto \( \ker(\overline{\rho}) \) it suffices to show that \( \rho(G) \cap \ker(q) = \langle \delta \rangle \). The kernel of \( q \) is the free abelian group with basis \( \{ \prod_j a_{ij} \mid 1 \leq i \leq m \} \). The image of \( \rho \) is generated by \( \{ \prod_i a_{ij} \mid 1 \leq j \leq n \} \). Suppose \( a \in \rho(G) \cap \ker(q) \), and let \( g \in G \) with \( a = \rho(g) \). Since \( a \in \ker(q) \), we may write \( a = \prod_i (\prod_j a_{ij})^{k_i} \) for some integers \( k_i, 1 \leq i \leq m \). Note that \( k_i \) is equal to the exponent sum of \( a_j \) in \( g \), for any \( H_j \in S_i \).

Replacing \( a \) by \( a' = a\delta^{-k_1} \) we may assume \( k_1 = 0 \). This implies the exponent sum on \( a_j \) is zero, for any \( H_j \in S_1 \). Let \( 2 \leq i \leq m \), and choose a path \( (S_1, H_{i_1}, S_{i_2}, H_{i_2}, \ldots, H_{i_k}, S_i) \) in \( \Lambda_X \) from \( S_1 \) to \( S_i \). The exponent sum on \( a_{i_1} \) in \( g \) is zero, which implies \( k_{i_3} = 0 \) by the observation above. Then the exponent sum on \( a_{i_2} \) in \( g \) is zero. Then \( k_{i_3} = 0 \). Continuing in this way we conclude that \( k_i = 0 \). Thus \( a' = 1 \), so \( a = \delta^{k_1} \in \langle \delta \rangle \).

Finally we adapt the preceding result to groups of affine arrangements - this will make it easier to check the conditions of Corollary 3.20 in our examples. Let \( \hat{G} \) be the subgroup of \( G \) generated by \( \{a_1, \ldots, a_{n-1}\} \). Assume that \( G \cong \hat{G} \times \langle z \rangle \). If \( G \) is isomorphic to \( \overline{G} \). Similarly, for \( 1 \leq i \leq m \), let \( \hat{G}_{S_i} \) be the subgroup of \( G_{S_i} \) generated by \( \{a_{ij} \mid 1 \leq j \leq n \} \). Then \( \hat{G}_{S_i} = G_{S_i} \) if \( n \notin S_i \), and \( \hat{G}_{S_i} \) is a free group of rank \( |S_i| - 1 \) isomorphic to \( G_{S_i} \) if \( a_n \in S_i \). Assume \( \hat{G}_{S_i} \) is normal in \( G_{S_i} \). Then, if \( n \in S_i \), \( G_{S_i} \cong \hat{G}_{S_i} \times \langle z_i \rangle \). Write \( \hat{A} = \prod_{i=1}^m \hat{G}_{S_i} \), and observe that \( \rho(\hat{G}) \subseteq \hat{A} \). Let \( \hat{\rho} : \hat{G} \to \hat{A} \) be the restriction of \( \rho \).

If \( G = G(A) \) for a central arrangement \( A \), and \( \{a_1, \ldots, a_n\} \) is a standard generating set of \( G \), then \( \hat{G} = G(dA) \) and all of the assumptions in the preceding paragraph hold.

Corollary 4.11. Assume \( a_i \notin S_i \) for some \( i \). Then \( \overline{\rho} \) is injective if and only if \( \hat{\rho} \) is injective.

Proof. Necessity is immediate since the restriction of \( p \) to \( \hat{G} \) is an isomorphism. Suppose \( \overline{g} \in \ker(\overline{\rho}) \). By Proposition 4.10, there exists \( g \in \ker(\rho) \) with \( p(g) = \overline{g} \). Also there exists \( g_0 \in \hat{G} \) such that \( \rho(g_0) = \overline{g} \). Then \( g_0 = g z^k \) for some \( k \in \mathbb{Z} \). Then \( \rho(g_0) = \rho(z^k) \). The quotient of \( \hat{A} \) by the normal subgroup \( \hat{A} \) is free abelian, generated by \( \{z_i \mid a_n \notin S_i \} \), and is not trivial by hypothesis. The image of \( \rho(z)^k \) in this quotient is trivial, since \( \rho(\hat{G}) \subseteq \hat{A} \), and this implies \( k = 0 \). Thus \( g = g_0 \) so \( \rho(g_0) = \rho(g) = 1 \). Then \( g_0 = 1 \), and \( \rho(g_0) = \overline{g} = 1 \).

4.3. Qualitative and finiteness properties of arrangement groups. Now we combine the descriptions of the kernel and cokernel of \( \rho_X \) to draw some conclusions about arrangement groups.

Example 4.12. Let \( A \) be the arrangement in \( \mathbb{C}^3 \) obtained by "coning" the affine arrangement of Example 3.25. It has defining polynomial

\[ Q(x, y, z) = x(x - z)(x + 2y)(y + z)yz. \]
Let $\mathcal{X} = \{\{1, 3, 5\}, \{1, 2, 6\}, \{3, 4, 6\}\}$, where the hyperplanes are labelled as in Figure 2(a). The graph $\Lambda_X$ is illustrated in Figure 2(b). $\Lambda_X$ is connected.

We have $\overline{\mathcal{A}} \cong F_2 \times F_2 \times F_2$, while $\overline{\mathcal{A}}/\rho(G)$ is free abelian of rank $(3 + 3 + 3) - 6 - 3 + 1 = 1$, i.e., $\overline{\mathcal{A}}/\rho(G) \cong \mathbb{Z}$. The matrix $R$ is

$$
\begin{array}{cccccccc}
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
5 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
126 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
135 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
346 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{array}
$$

One can choose the generators of $F_2 \times F_2 \times F_2$ so that each maps to the same generator of the quotient. (See Proposition 4.23 for a more general result.). Then $\rho(G)$ is isomorphic to Stallings’ group $\text{Sta63}$, the kernel of the map $F_2 \times F_2 \times F_2 \rightarrow \mathbb{Z}$ sending every canonical generator to 1.

The standard generating set for the deconed arrangement given in Example 3.23 can be extended to a standard generating set $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ of $G$, with $a_6$ corresponding to the line at infinity, and $a_1 \cdots a_6$ central. Then the restriction $\hat{\rho} : \hat{G} \rightarrow \hat{A}$ is exactly the homomorphism analyzed in that example, where it was shown to be injective. Then $\hat{\rho}$ is injective by Corollary 4.11 so in fact $G$ is isomorphic to the Stallings group. This was first observed by D. Matei and A. Suciu [MS04]; their discovery motivated our research.

We observe some properties of $\overline{\rho(G)}$, immediate from the definition and properties of free groups. Recall that a discrete group has the Haagerup property, or is a-T-menable, if it acts properly and isometrically on an affine Hilbert space. Groups with this property satisfy the Baum-Connes and Novikov conjectures [CCJ+01]. Free groups are a-T-menable, as are subgroups and finite direct products of a-T-menable groups.
Theorem 4.13. Let $\overline{\mathcal{G}}: \mathcal{G} \rightarrow \prod_{S \in \mathcal{X}} \mathcal{G}_S$ as above. Then

(i) $\overline{\mathcal{P}}(\mathcal{G})$ is determined by the underlying matroid of $\mathcal{A}$.
(ii) $\overline{\mathcal{P}}(\mathcal{G})$ is residually free.
(iii) $\overline{\mathcal{P}}(\mathcal{G})$ is torsionfree.
(iv) $\overline{\mathcal{P}}(\mathcal{G})$ is residually torsionfree nilpotent.
(v) $\overline{\mathcal{P}}(\mathcal{G})$ has solvable word and conjugacy problems.
(vi) $\overline{\mathcal{P}}(\mathcal{G})$ has a linear representation.
(vii) $\overline{\mathcal{P}}(\mathcal{G})$ is residually finite.
(viii) $\overline{\mathcal{P}}(\mathcal{G})$ has the Haagerup property.

If $\overline{\mathcal{P}}$ is injective, then $\overline{\mathcal{G}}$ has properties (ii) - (viii).

Injectivity of $\overline{\mathcal{P}}$ may depend on the geometry of $\mathcal{A}$, so one cannot conclude that $\overline{\mathcal{G}}$ is determined by the underlying matroid.

We can apply the results of [MMW98] to say something about the homological finiteness properties of $\overline{\mathcal{P}}(\mathcal{G})$. Recall a group $G$ is of type $F_k$ if there is a $K(G, 1)$ with finite $k$-skeleton. Let $\Gamma$ be the graph associated with the right-angled Artin group $\mathcal{A}$; $\Gamma$ is the complete multipartite graph with parts of sizes $|S_i| - 1$, $1 \leq i \leq m$. The vertices of $\Gamma$ correspond to the generators $a_{ij}, H_j \in S_i - \max(S_i)$ of $\mathcal{A}$. Following [MMW98], say a vertex $a_{ij}$ is living if $a_{ij}$ maps to a nonzero element of the quotient $\mathcal{A}/\overline{\mathcal{P}}(\mathcal{G})$.

Let Flag($\Gamma$) denote the flag complex of $\Gamma$, the simplicial complex whose $p$-simplices are the cliques of size $p + 1$ in $\Gamma$. Since $\Gamma$ is a complete multipartite graph with $m$ parts, Flag($\Gamma$) is a join of $m$ zero-dimensional complexes, hence is a homeomorphic to a bouquet of $(m - 1)$-spheres. Let $K(\Gamma)$ be the full subcomplex of $\Gamma$ on the set of living vertices - we call $K(\Gamma)$ the living subcomplex of Flag($\Gamma$). We refer the reader to [MMW98] for the definition of an “$n$-acyclic dominating” subcomplex. The complex Flag($\Gamma$) is an $(m - 1)$-acyclic dominating subcomplex of itself.

Corollary 4.14. The group $\overline{\mathcal{P}}(\mathcal{G})$ is of type $F_k$ if and only if $K(\Gamma)$ is $(k - 1)$-connected and is a $(k - 1)$-acyclic dominating subcomplex of Flag($\Gamma$).

Corollary 4.15. If every vertex of $\Gamma$ is living, then $\overline{\mathcal{P}}(\mathcal{G})$ is of type $F_{m-1}$ and not of type $F_m$.

Corollary 4.16. If $\overline{\mathcal{P}}_\mathcal{X}$ is injective, then $\mathcal{M}(\mathcal{A})$ is not aspherical.

In particular, $\overline{\mathcal{P}}_\mathcal{X}$ is not injective if $\mathcal{A}$ is a fiber-type arrangement.

Remark 4.17. Since $\mathcal{A}/\overline{\mathcal{P}}(\mathcal{G})$ is abelian, $a_{ij}$ is living if and only if $b_{ij} \not\in \overline{\mathcal{P}}(\mathcal{G})_{ab} \subseteq \mathcal{A}_{ab}$. Then it is easy to see that $a_{ij}$ is a living vertex of $\Gamma$ if $j$ is an element of more than one $S_i \in \mathcal{X}$, or, equivalently, the vertex $H_j$ has degree greater than one in the graph $\Gamma$. In fact, to guarantee that all vertices of $\Gamma$ are living, it suffices to find a subgraph of $\Gamma$ that has $|S_i| - 1$ edges incident with each $S_i$, and no pendant vertices. These edges correspond to canonical generators of the free factor $\mathcal{G}_{S_i}$.

Example 4.18. Let $\mathcal{A}$ be the arrangement of Example 4.12. Identify $\mathcal{A} \cong F_2 \times F_2 \times F_2$ with the subgroup $\hat{\mathcal{A}}$ of $\mathcal{A}$. Let $S_1 = \{a_1, a_2, a_6\}, S_2 = \{a_1, a_4, a_5\}, S_3 = \{a_3, a_4, a_6\}$, and $\mathcal{X} = \{S_1, S_2, S_3\}$. Recall that $a_{1j}$ is the generator of $\mathcal{A}$ corresponding to the pair $(S_i, H_j)$. Choose the generators of $\mathcal{A}$ to be $a_{11}, a_{16}, a_{21}, a_{23}, a_{33},$ and $a_{36}$. Observe that none of these correspond to edges $(S_i, H_j)$ of $\Gamma$ with $H_j$ of degree one. Thus every vertex of $\Gamma$ is living.
The flag complex \( \text{Flag}(\Gamma) \) is a triangulation of \( S^2 \). Hence \( \mathcal{G} \) is of type \( F_2 \) (i.e., \( \mathcal{G} \) is finitely-presented) but not of type \( F_3 \). This was first established by different methods in unpublished work of Arvola [Arv92], which motivated Matei and Suciu’s identification of this group with Stallings’ group.

**Example 4.19.** Let \( \mathcal{A} \) be the arrangement in \( \mathbb{C}^3 \) obtained by coning the arrangement of Example 3.24. The generating set of that example can be extended to a standard generating set \( \{a_1, \ldots, a_7\} \) of \( G \) with \( a_1 \cdots a_7 \) central in \( G \). Let \( X = \{S_1, \ldots, S_5\} = \{(1, 3, 5), (2, 4, 6), (1, 2, 7), (3, 4, 7), (5, 6, 7)\} \). By Example 3.24 and Corollary 4.11, \( \rho \) is injective. Thus \( G \) is residually free, residually nilpotent, and linear.

The graph \( \Lambda_X \) has no pendant vertices, so all vertices of the graph \( \Gamma \) corresponding to the right-angled Artin group \( \mathcal{A} \cong F_2 \times F_2 \times F_2 \times F_2 \times F_2 \) are living. The flag complex \( \text{Flag}(\Gamma) \) is a triangulation of \( S^4 \). Thus \( \mathcal{G} \) is of type \( F_4 \) but not \( F_5 \).

**Example 4.20.** Similar calculations apply to two (combinatorially distinct) seven-line arrangements which appear in [Fal97], illustrated in Figure 3. In these cases the groups are of type \( F_3 \) but not \( F_4 \).

### 4.4. Bestvina-Brady arrangement groups

Example [4.18] has been generalized in recent work of Artal-Bartolo, Cogolludo-Augustin, and Matei, as reported in [Mat07]. For any integers \( n_1, \ldots, n_r \), they construct a projective line arrangement \( \mathcal{A}(n_1, \ldots, n_r) \) of \( \sum_{i=1}^r n_i \) lines, forming an \( r \)-gon of points of multiplicities \( n_1 + 1, \ldots, n_r + 1 \). More precisely, there are \( r \) distinguished lines in general position, forming the sides of an \( r \)-gon, with vertices labelled \( 1, \ldots, r \) in consecutive order. Passing through the \( i \)-th vertex there is a “bundle” of \( n_i - 2 \) lines, pairwise intersecting in double points elsewhere. In the language of [EGT09], the graph of multiple points forms an \( r \)-cycle.

We have illustrated this “arrangement schema” in Figure 4, with each bundle of lines represented by a single colored line. For later purposes we have placed one
Figure 4. The Artal-Cogolludo-Matei arrangement schema

of the distinguished lines at infinity. The underlying matroids of these arrangements have path-connected realization spaces, so the diffeomorphism type of the complement is uniquely determined by the integers $n_1, \ldots, n_r$.

Artal, Cogolludo, and Matei show that the corresponding arrangement group is isomorphic to the kernel of the map $F_{n_1} \times \cdots \times F_{n_r} \to \mathbb{Z}$ sending each generator to 1, and hence is a Bestvina-Brady group of type $F_{r-1}$ and not $F_r$. We can reproduce their result using our method.

Theorem 4.21. Let $A = A(n_1, \ldots, n_r)$ as defined above, and $G = G(A)$. Let $X$ be the set of rank-two flats of $A$ of multiplicity greater than two, and $\overline{p} = \overline{p}_X : \overline{G} \to \prod_{S \in X} \overline{G}_S$. Then $\overline{p}$ is injective.

Proof. Let $\hat{G}$ denote the group of a decone $dA$, relative to one of the sides of the $r$-gon. We first show that $\hat{G}$ has a conjugation-free presentation. Assume that the $y$-axis is far to the right in Figure 4 and order the non-vertical lines of $dA$ by increasing $y$-intercept. Then order the vertical lines of $dA$ by increasing $x$-intercept. Denote the distinguished lines of $dA$ by $q_1, \ldots, q_{r-1}$, so line $q_1$ is the top horizontal line, and $q_{r-1}$ is the left-most vertical line. Write $n = \sum_{i=1}^r n_i$.

Sweeping a line of large negative slope from right to left in Figure 4 the Randell algorithm [Ran82] yields a presentation for $G(dA)$ with relations

$$a_{q_i} a_{q_{i+1}} \cdots a_{q_{i+1}-1} a_{q_{i+1}} = a_{q_{i+1}} \cdots a_{q_{i+1}-1} a_{q_{i+1}} a_{q_i} = \cdots = a_{q_{i+1}} a_{q_i} a_{q_{i+1}} \cdots a_{q_{i+1}-1}$$

for $1 \leq i \leq r - 2$, and commutator relations $[a_s, a_t^{w_{s,t}}]$ where $1 \leq s < t \leq n - 1$, $s, t \notin \{q_1, \ldots, q_{r-1}\}$ for $1 \leq i \leq r - 1$, and $w_{s,t}$ is a word in the $a_k$. If $t = q_j$ for some $j$ or $t > q_{r-1}$, then $w_{s,t} = 1$. If $q_j < t < q_{j+1}$, then $w_{s,t} = a_{t+1} \cdots a_{q_{j+1}}$ and $s < q_j$. For each $j$, $1 \leq j \leq r - 2$, the family of commutator relations

$$[a_s, a_{q_j}], [a_s, a_{t}^{w_{s,t}}], [a_s, a_{q_{j+1}}], t = q_j + 1, \ldots, q_{j+1} - 1,$$
provided by the Randell algorithm is easily seen to be equivalent to the family
\[ [a_s, a_{q_j}], [a_s, a_t], [a_s, a_{q_{j+1}}], \quad t = q_j + 1, \ldots, q_{j+1} - 1. \]

Thus, \( G(dA) \) admits a conjugation-free presentation. This may also be seen by applying the main result of [EGT09], since the graph of multiple points has a unique cycle.

We show \( \hat{\rho}: \hat{G} \to \prod_{S \in X} G_S \) is injective using Corollary 4.20 and Proposition 3.22. The result will then follow from Corollary 4.11. The first condition of Corollary 3.20 holds by Remark 3.21. For the decone \( dA \), let \( \mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_\infty \), where \( \mathcal{S}_0 = \{\{q_i, \ldots, q_{i+1}\} \mid 1 \leq i \leq r - 2\} \) and \( \mathcal{S}_\infty = \{\{1, \ldots, q_1\}, \{q_{r-1}, \ldots, n - 1\}\} \). For \( T \in \mathcal{S} \) and \( j \notin T \) we must show that \([a_j, G_T^2]\) = 1. There is a unique \( S \neq T \) in \( \mathcal{S} \) with \( j \in S \). If either \( T \in \mathcal{S}_\infty \) or \( S \cap T = \emptyset \), then clearly \([a_j, G_T^2]\) = 1 since \([a_j, a_t]\) = 1 for all \( t \in T \) in either of these instances. It remains to consider the case where \( T \in \mathcal{S}_0 \) and \( S \cap T \neq \emptyset \). In this case, \( S \) and \( T \) satisfy the hypotheses of Proposition 3.22. The result follows.

**Corollary 4.22.** If \( A = A(n_1, \ldots, n_r) \), then \( \overline{G}(A) \) satisfies properties (i)-(viii) of Theorem 4.13.

**Proposition 4.23.** The injection \( \overline{\rho} \) realizes \( \overline{G} \) as a Bestvina-Brady group.

**Proof.** The graph \( \Lambda_X \) is connected, and \( |S_i| = n_i + 1 \) for \( 1 \leq i \leq r \). Then, according to Corollary 1.3, Lemma 1.8 and Theorem 4.9 the cokernel of \( \overline{\rho} \) is free abelian of rank equal to \( \sum_{i=1}^r (n_i + 1) - \sum_{i=1}^r n_i - r + 1 = 1 \), i.e., \( \text{coker}(\overline{\rho}) \cong \mathbb{Z} \).

In the bipartite graph \( \Lambda_X \), the \( r \) distinguished hyperplanes have degree two, and all the other hyperplanes have degree one. The \( r \) elements \( S_1, \ldots, S_r \) of \( X \) have degrees \( n_1, \ldots, n_r \) in \( \Lambda_X \). Following Remark 4.17, we exhibit a free basis of \( \overline{A} = \prod_{i=1}^r \overline{G}_{S_i} \), whose elements all map to the same generator of \( \text{coker}(\overline{\rho}) \). Fix \( i \) and write \( S_i = \{H_1, \ldots, H_{n_i}\} \) with \( H_1 \) and \( H_2 \) being the distinguished hyperplanes, and let \( a_{i1}, \ldots, a_{in_i} \) be the corresponding canonical generators of \( G_{S_i} \). Each of the \( a_{ij} \) for \( j \geq 3 \) lie in \( \overline{\rho}(G) \), hence map to zero in \( \text{coker}(\overline{\rho}) \). Using the identification of \( \text{coker}(\overline{\rho}) \) with \( \text{coker}(\overline{\rho}_{S_i}) \), it is clear that \( a_{i1} \) and \( a_{i2} \) map to \( \pm 1 \). By reversing orientation we may assume they each map to 1. Then \( \{a_{i1}, a_{i2}, a_{i1}a_{i3}, \ldots, a_{i1}a_{i1n_i-1}\} \) is a free basis of \( \overline{G}_{S_i} \), each element of which maps to 1 in \( \mathbb{Z} \). Repeating the process for each \( i \), \( 1 \leq i \leq r \), yields the desired free basis of \( \overline{A} \). Since \( \overline{G} \cong \overline{\rho}(G) = \ker(\overline{A} \to \text{coker}(\overline{\rho})) \cong \mathbb{Z} \), this proves the claim. \( \square \)

This proposition reproduces the result of Artal, Cogolludo, and Matei reported on in [Mat07]. By [DPS08], these are the only quasi-projective groups that are Bestvina-Brady groups, aside from products of free groups.

### 4.5. Decomposable arrangements

In [PS06], the authors study the class of decomposable arrangements. Let \( A \) be a central arrangement of rank at least three, and let \( X \) be the set of all rank-two flats of \( A \). Let \( X_0 \) be the set of rank-two flats of multiplicity at least three. Let \( \text{Lie}(G) = \oplus_{n=1}^\infty G^n/G^{n+1} \) denote the graded abelian group associated to the lower central series of \( G \). For any field \( k \), the commutator bracket makes \( \text{Lie}(G) \otimes k \) into a graded \( k \)-Lie algebra. We have the product of inclusion-induced homomorphisms \( \text{Lie}(\rho_X)_n: \text{Lie}_n(G) \to \prod_{S \in X} \text{Lie}_n(G_S) \) for each \( n \geq 1 \). Note that \( \text{Lie}_n(G_S) = 0 \) for all \( n \geq 2 \) if \( S \in X - X_0 \).

For \( n \geq 2 \), \( \text{Lie}(\rho_X)_n \otimes k \) is surjective, and it is an isomorphism for \( n = 2 \). The arrangement \( A \) is \( k \)-decomposable if and only if \( \text{Lie}(\rho_X)_n \otimes k \) is an isomorphism for
every $n \geq 3$, and $\mathcal{A}$ is decomposable if it is $k$-decomposable for every $k$. In [PS06] it is shown that $\mathcal{A}$ is decomposable if and only if $\text{Lie}(\rho_X)_n$ is an isomorphism for $n \geq 2$, and the following theorem is established.

**Theorem 4.24.** $\mathcal{A}$ is $k$-decomposable if and only if 

$$\text{Lie}(\rho_X)_3 \otimes k: \text{Lie}_3(G) \otimes k \rightarrow \prod_{S \in X_0} \text{Lie}_3(G_S) \otimes k.$$ 

is an isomorphism.

This can be checked for particular arrangements and particular fields using the holonomy Lie algebra, which is determined combinatorially. All the examples in this section are decomposable over $\mathbb{Q}$. There are examples of arrangements for which $\text{Lie}_n(G)$ has torsion, but the examples are not decomposable. It is not known if every $\mathbb{Q}$-decomposable arrangement is decomposable.

The nilpotent residue $G^\omega$ of $G$ is the intersection $\bigcap_{n=1}^{\infty} G^n$. For decomposable arrangements we can identify the kernel of $\rho_{X_0}: G \rightarrow \prod_{S \in X_0} G_S$.

**Theorem 4.25.** Let $\mathcal{A}$ be a decomposable arrangement. Suppose $X_0$ covers $\mathcal{A}$ and $G(\mathcal{A})$ has a standard generating set adapted to $X_0$. Then the kernel of $\rho_{X_0}$ is equal to $G^\omega$.

**Proof.** Since products of free groups are residually nilpotent, $G^\omega \subseteq \ker(\rho_{X_0})$ for any arrangement $\mathcal{A}$. Suppose $g \in \ker(\rho_{X_0}) - G^\omega$. Choose $n$ minimal with $g \not\in G^n$. By Theorem 3.15, $g$ is a product of monic commutators with support transverse to $X_0$. Since $X_0$ covers $\mathcal{A}$, and $g \neq 1$, these commutators have length at least three. Then $g$ represents a nonzero element of $G^{n-1}/G^n$, with $n \geq 4$, and hence lies in the kernel of $\text{Lie}(\rho_X)_{n-1}$. Then we have a contradiction to the assumption that $\mathcal{A}$ is decomposable. \qed

**Corollary 4.26.** Suppose $\mathcal{A}$ is decomposable, $G = G(\mathcal{A})$ has a standard generating set adapted to $X_0$, and $X_0$ covers $\mathcal{A}$. Then

(i) $\rho_{X_0}$ is injective if and only if $G$ is residually nilpotent, and

(ii) $G/G^\omega$ is isomorphic to a combinatorially-determined subgroup of a finite product of free groups.

It is not known if all decomposable arrangement groups are residually nilpotent, or whether they always have conjugation-free presentations. All the examples in this section, and all the Artal-Cogolludo-Matei arrangements, are $\mathbb{Q}$-decomposable. We offer the following (strong) conjecture, with admittedly scant evidence.

**Conjecture 4.27.** If $\mathcal{A}$ is a decomposable arrangement, then $G(\mathcal{A})$ embeds in a product of free groups.

The conjecture would follow from two assertions: that the group of any decomposable arrangement has a generating set adapted to its set of nontrivial rank-two flats, and that decomposable arrangement groups are residually nilpotent.

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