LOCAL ZETA FUNCTIONS AND NEWTON POLYHEDRA

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ABSTRACT. To a polynomial $f$ over a non-archimedean local field $K$ and a character $\chi$ of the group of units of the valuation ring of $K$ one associates Igusa’s local zeta function $Z(s, f, \chi)$. In this paper, we study the local zeta function $Z(s, f, \chi)$ associated to a non-degenerate polynomial $f$, by using an approach based on the p-adic stationary phase formula and Néron p-desingularization. We give a small set of candidates for the poles of $Z(s, f, \chi)$ in terms of the Newton polyhedron $\Gamma(f)$ of $f$. We also show that for almost all $\chi$, the local zeta function $Z(s, f, \chi)$ is a polynomial in $q^{-s}$ whose degree is bounded by a constant independent of $\chi$. Our second result is a description of the largest pole of $Z(s, f, \chi_{\text{triv}})$ in terms of $\Gamma(f)$ when the distance between $\Gamma(f)$ and the origin is at most one.

1. Introduction

Let $K$ be a non-archimedean local field, and let $\mathcal{O}_K$ be the ring of integers of $K$ and $\mathcal{P}_K$ its maximal ideal. Let $\pi$ be a fixed uniformizing parameter of $K$, and let the residue field of $K$ be $\mathbb{F}_q$ the field with $q = p^r$ elements. For $x \in K$ we denote by $v$ the valuation of $K$ such that $v(\pi) = 1$, $|x|_K = q^{-v(x)}$ its absolute value and $ac(x) = x\pi^{-v(x)}$ its angular component. Let $f(x) \in \mathcal{O}_K[x]$, $x = (x_1, \ldots, x_n)$ be a non-constant polynomial, and $\chi : \mathcal{O}_K^\times \longrightarrow \mathbb{C}^\times$ a character of $\mathcal{O}_K^\times$, the group of units of $\mathcal{O}_K$. We formally put $\chi(0) = 0$. To these data one associates Igusa’s local zeta function,

$$Z(s, f, \chi) = \int_{\mathcal{O}_K^n} \chi(acf(x))|f(x)|_{\mathcal{O}_K}^s dx,$$

for $\text{Re}(s) > 0$, where $|dx|$ denotes the Haar measure on $K^n$, normalized such that $\mathcal{O}_K^n$ has measure 1. In the case of $K$ having characteristic zero, Igusa [I2] and Denef [D1] proved that $Z(s, f, \chi)$ is a rational function of $q^{-s}$.

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A basic problem is to determine the poles of the meromorphic continuation of \(Z(s, f, \chi)\) into \(\text{Re}(s) < 0\). The general strategy is to take a resolution \(h : X \rightarrow K^n\) of \(f\) and study the resolution data \(\{(N_i, n_i)\}\) in which \(N_i\) is the multiplicity of \(f \circ h\) along a exceptional divisor \(D_i\), and \(n_i\) is the multiplicity of \(h^*(dx)\) along \(D_i\). The set of ratios \(\{-\frac{n_i}{N_i}\} \cup \{-1\}\) contains the real parts of the poles of \(Z(s, f, \chi)\) as observed in [I2]. However, many examples show that most of these ratios do not correspond to poles. The problem of the determination of the actual poles of \(Z(s, f, \chi)\) for arbitrary \(n\) is still an open problem. The case \(n = 2\) was solved for irreducible \(f\) and \(\chi = \chi_{\text{triv}}\) for all primes \(p\) by Meuser [Me]. The generalization to reducible \(f\) and \(\chi \neq \chi_{\text{triv}}\) but for almost all primes \(p\) was solved by Veys in [Ve].

In case of non-degenerate polynomials with respect to its Newton polyhedron and \(K = \mathbb{R}\), Varchenko [Va] gave a procedure to compute a set of candidates for the poles of the complex power of \(f\), by using toroidal resolution of singularities (see also [D-S-1], [D-S-2] for further generalizations).

The \(p\)-adic case is entirely similar to the real case. In this case, Lichtin and Meuser [L-M] proved in the case \(n = 2\) that not all candidates provided by the numerical data of a toric resolution of \(f\) are actually poles of \(Z(s, f, \chi)\). In [D3], Denef gave a procedure based on monomial changes of variables to determine a small set of candidates for the poles of \(Z(s, f, \chi_{\text{triv}})\) in terms of the Newton polyhedron of \(f\).

In this paper, we study the local zeta function \(Z(s, f, \chi)\) associated to a globally non-degenerate polynomial \(f\) (see definition 1.1), by using an approach based on the \(p\)-adic stationary phase formula and Néron \(p\)-desingularization. We show the stationary phase formula gives a small set of candidates for the poles of \(Z(s, f, \chi)\) in terms of the Newton polyhedron \(\Gamma(f)\) of \(f\) (cf. theorem A). When \(\chi = \chi_{\text{triv}}\) and \(\text{char}(K) = 0\) this set of poles agree with that obtained in [D3]. We also show that for almost all \(\chi\), the zeta function \(Z(s, f, \chi)\) is a polynomial in \(q^{-s}\) whose degree is bounded by a constant independent of \(\chi\). Our second result shows that the stationary phase formula can be used to describe the largest pole of \(Z(s, f, \chi_{\text{triv}})\) in terms of \(\Gamma(f)\), when the distance between \(\Gamma(f)\) and the origin is at most one (cf. theorem B). This result was previously known for \(\text{char}(K) = 0\). This result allows one to generalize estimates for exponential sums that were obtained in [D-Sp] to the case \(\text{char}(K) \neq 0\) (cf. corollary 6.1).

We set \(\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}\). Let \(f(x) = \sum_l a_l x^l \in K[x]\), \(x = (x_1, x_2, ..., x_n)\) be a polynomial in \(n\) variables satisfying \(f(0) = 0\). The set \(\text{supp}(f) = \{l \in \mathbb{N}^n \mid a_l \neq 0\}\) is called the support of \(f\). The Newton polyhedron \(\Gamma(f)\) of \(f\) is defined as the convex hull in \(\mathbb{R}^+_n\) of the set
\[ \bigcup_{l \in \text{supp}(f)} (l + \mathbb{R}^n_+) \, . \]

We denote by $<,>$ the usual inner product of $\mathbb{R}^n$, and identify $\mathbb{R}^n$ with its dual by means of it. We set

\[ < a_\gamma, x > = m(a_\gamma), \]

for the equation of the supporting hyperplane of a facet $\gamma$ (i.e. a face of codimension 1 of $\Gamma(f)$) with perpendicular vector $a_\gamma = (a_1, a_2, ..., a_n) \in \mathbb{N}^n \setminus \{0\}$, and $|a_\gamma| := \sum_i a_i$.

**Definition 1.1.** A polynomial $f(x) = \sum_i a_i x^i \in K[x]$ is called *globally non-degenerate with respect to its Newton polyhedron* $\Gamma(f)$, if it satisfies the following two properties:

1. (GND1) The origin of $K^n$ is a singular point of $f(x)$;
2. (GND2) For every face $\gamma \subset \Gamma(f)$ (including $\Gamma(f)$ itself), the polynomial $f_\gamma(x) := \sum_{i \in \gamma} a_i x^i$ has the property that there is no $x \in (K \setminus \{0\})^n$ such that $f_\gamma(x) = \frac{\partial f_\gamma}{\partial x_1}(x) = \ldots = \frac{\partial f_\gamma}{\partial x_n}(x) = 0$.

Our first result is the following.

**Theorem A.** Let $K$ be a non-archimedean local field, and let $f(x) \in \mathcal{O}_K[x]$ be a polynomial globally non-degenerate with respect to its Newton polyhedron $\Gamma(f)$. Then the Igusa local zeta function $Z(s, f, \chi)$ is a rational function of $q^{-s}$ satisfying:

1. If $s$ is a pole of $Z(s, f, \chi)$, then
   \[ s = -\frac{|a_\gamma|}{m(a_\gamma)} + \frac{2\pi i}{\log q} \frac{k}{m(a_\gamma)} , \quad k \in \mathbb{Z} \]
   for some facet $\gamma$ of $\Gamma(f)$ with perpendicular $a_\gamma$, and $m(a_\gamma) \neq 0$, or

2. If $\chi \neq \chi_{\text{triv}}$ and the order of $\chi$ does not divide any $m(a_\gamma) \neq 0$, where $\gamma$ is a facet of $\Gamma(f)$, then $Z(s, f, \chi)$ is a polynomial in $q^{-s}$, and its degree is bounded by a constant independent of $\chi$. 
For a polynomial $f(x) \in K[x]$ globally non-degenerate with respect to its Newton polyhedron $\Gamma(f)$, we set
\[
\beta(f) := \max_{\tau_j} \left\{-\frac{|a_j|}{m(a_j)}\right\},
\]
where $\tau_j$ runs through all facets of $\Gamma(f)$ satisfying $m(a_j) \neq 0$. The point
\[
T_0 = (-\beta(f)^{-1}, ... , -\beta(f)^{-1}) \in \mathbb{Q}^n
\]
is the intersection point of the boundary of the Newton polyhedron $\Gamma(f)$ with the diagonal $\Delta = \{(t, .., t) \mid t \in \mathbb{R}\} \in \mathbb{R}^n$. Let $\tau_0$ be the face of smallest dimension of $\Gamma(f)$ containing $T_0$, and $\rho$ its codimension.

If $g(x) \in \mathcal{O}_K[x]$, $x = (x_1, .., x_n)$, we denote by $\overline{g(x)}$ its reduction modulo $\mathcal{P}_K$.

The second result of this paper describes the largest pole of $Z(s, f, \chi_{\text{triv}})$, when $\beta(f) \geq -1$.

**Theorem B.** Let $K$ be a non-archimedean local field, and let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$. If $\beta(f) > -1$, then $\beta(f)$ is a pole of $Z(s, f, \chi_{\text{triv}})$ of multiplicity $\rho$. If $\beta(f) = -1$, then $\beta(f)$ is a pole of $Z(s, f, \chi_{\text{triv}})$ of multiplicity less than or equal to $\rho + 1$. Moreover, if every face $\gamma \supseteq \tau_0$ satisfies $\operatorname{Card}(\{z \in F_q^\times | \overline{\bar{f}}(z) = 0\}) > 0$, then the multiplicity of $\beta(f)$ is exactly $\rho + 1$.

The largest pole of $Z(s, f, \chi_{\text{triv}})$ when $f$ is non-degenerate with respect to its Newton polyhedron $\Gamma(f)$ and $\beta(f) > -1$ follows from observations made by Varchenko in [Va] and was originally noted in the $p$–adic case in [L-M] (although it is misstated there as $\beta(f) \neq -1$). The case $\beta(f) = -1$ is treated in [D-H]. The case of $\beta(f) < -1$ is more difficult and is established in [D-H] with some additional conditions on $\tau_0$ by using a difficult result on exponential sums. Thus our Theorem B gives a different proof of the cases where $\beta(f) \geq -1$.

The organization of this paper is as follows. In section 2, we review Igusa’s stationary phase formula. The results of this section generalize our previous results in [Z-G]. Section 3 contains some basic results about Newton polyhedra. In section 4, we prove theorem A. In section 5, we prove theorem B. Section 6 contains some consequences of the main theorems. More precisely, we give estimates for exponential sums involving globally non-degenerate polynomials (cf. corollary 6.1). In section 7, we compute explicitly the local zeta functions of some polynomials in two variables and discuss the relation between the largest pole of $Z(s, f, \chi_{\text{triv}})$ and $\beta(f)$.

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2. Igusa’s stationary phase formula

In [I3] Igusa introduced the stationary phase formula for $\pi$–adic integrals and suggested that a closer examination of this formula might lead to a new proof of the rationality of $Z(s, f, \chi)$ in any characteristic. Following this suggestion the author proved the rationality of the local zeta function $Z(s, f, \chi_{\text{triv}})$ attached to a semiquasihomogeneous polynomial $f$ over an arbitrary non-archimedean local field $[Z\text{-G}].$ 

Let $L$ be a ring and $f(x) \in L[x]$, we denote by $V_f(L)$ the corresponding $L$–hypersurface and by $Sing_f(L)$ the $L$-singular locus.

We denote by $\pi$ the image of an element of $\mathcal{O}_K$ under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi \mathcal{O}_K \cong \mathbb{F}_q$, i.e. the reduction modulo $\pi$. Given $f(x) \in \mathcal{O}_K[x]$ such that not all its coefficients are in $\pi \mathcal{O}_K$, we denote by $\overline{f(x)}$ the polynomial obtained by reducing modulo $\pi$ the coefficients of $f(x)$.

We fix a lifting $R$ of $\mathbb{F}_q$ in $\mathcal{O}_K$. By definition, the set $R$ is mapped bijectively onto $\mathbb{F}_q$ by the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi \mathcal{O}_K$.

Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial in $n$ variables, $P_1 = (y_1, ..., y_n) \in \mathcal{O}_K^n$, and $m_{P_1} = (m_1, ..., m_n) \in \mathbb{N}^n$. We call a $K^n$–isomorphism $\Phi_{m_{P_1}}(x)$ a dilatation, if it has the form $\Phi_{m_{P_1}}(x) = (z_1, ..., z_n)$, $z_i = y_i + \pi^m_{x_i}$, for each $i = 1, 2, .., n$. The dilatation of $f(x)$ at $P_1$ induced by $\Phi_{m_{P_1}}(x)$ is defined as

$$f_{P_1}(x) := \pi^{-e_{P_1}} f(\Phi_{m_{P_1}}(x)), \quad (2.1)$$

where $e_{P_1}$ is the minimum order of $\pi$ in the coefficients of $f(\Phi_{m_{P_1}}(x))$. We call the $K$–hypersurface $V_{f_{P_1}}(K)$ the dilatation of $V_f(K)$ at $P_1$ induced by $\Phi_{m_{P_1}}(x)$; the number $e_{P_1}$ the arithmetic multiplicity of $f(x)$ at $P_1$ by $\Phi_{m_{P_1}}(x)$, and the set $S(f_{P_1})$, the lifting of $Sing_{\mathcal{F}_{P_1}}(\mathbb{F}_q)$, the first generation of descendants of $P_1$.

Given a sequence of dilatations $(\Phi_{m_{P_k}}(x))_{k \in \mathbb{N}}$, we define inductively $e_{P_1}, ..., P_k$ and $f_{P_1}, ..., P_k(x)$, $S(f_{P_1}, ..., P_k)$ as follows:

$$f_{P_1}, ..., P_k(x) := \begin{cases} f(x), & \text{if } k = 0, \\ \pi^{-e_{P_1}, ..., P_k} f_{P_1}, ..., P_k (\Phi_{m_{P_k}}(x)), & \text{if } k \geq 1, \end{cases} \quad (2.2)$$

where $P_k \in S(f_{P_1}, ..., P_{k-1})$, and $e_{P_1}, ..., P_k$ is the minimum order of $\pi$ in the coefficients of $f_{P_1}, ..., P_{k-1}(\Phi_{m_{P_k}}(x))$. For $k \geq 1$, the set $S(f_{P_1}, ..., P_k) := \bigcup_{P_k} S(f_{P_1}, ..., P_{k-1}, P_k)$ is called the $k^{th}$–generation of descendants of $P_1$. By definition the $0^{th}$–generation of descendants of $P_1$ is $\{P_1\}$. 


Now, we review Igusa’s stationary phase formula, from the point of view of the dilatations. For that, we fix the $m_P$’s equal to $(1, \ldots, 1) \in \mathbb{N}^n$ in (2.1).

Let $\overline{D}$ be a subset of $\mathbb{F}_q^n$ and $D$ its preimage under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K / \pi \mathcal{O}_K \cong \mathbb{F}_q$. Let $S(f, D)$ denote the subset of $\mathbb{F}_q^n$ (the set of representatives of $\mathbb{F}_q^n$ in $\mathcal{O}_K^n$) mapped bijectively to the set $\text{Sing}_f(\mathbb{F}_q) \cap \overline{D}$. We use the simplified notation $S(f)$ in the case of $D = \mathcal{O}_K^n$. Also we define:

$$\nu(\bar{f}, D, \chi) := \begin{cases} q^{-n} \text{Card}\{ \overline{P} \in \overline{D} | \overline{P} \notin V_{\bar{f}}(\mathbb{F}_q) \}, & \text{if } \chi = \chi_{\text{triv}}, \\ q^{-nc_\chi} \sum_{\{P \in D | P \notin V_{f}(\mathbb{F}_q)\mod P_K^c \}} \chi(ac(f(P))), & \text{if } \chi \neq \chi_{\text{triv}}, \end{cases}$$

where $c_\chi$ is the conductor of $\chi$, and

$$\sigma(\bar{f}, D, \chi) := \begin{cases} q^{-n} \text{Card}\{ P \in D | P \text{ is a smooth point of } V_{\bar{f}}(\mathbb{F}_q) \}, & \text{if } \chi = \chi_{\text{triv}}, \\ 0, & \text{if } \chi \neq \chi_{\text{triv}}. \end{cases}$$

If $D = \mathcal{O}_K^n$, we use the simplified notation $\nu(\bar{f}, \chi), \sigma(\bar{f}, \chi)$. We denote by $Z(D, s, f, \chi)$ the integral $\int_D \chi(ac(f(x)))|f(x)|_K^s \ |dx|$. With all this, we are able to establish Igusa’s stationary phase formula for $\pi$–adic integrals ([I3], p. 177):

**Igusa’s Stationary Phase Formula.**

$$Z(D, s, f, \chi) = \nu(\bar{f}, D, \chi) + \sigma(\bar{f}, D, \chi)(1 - q^{-1})q^{-s} \sum_{P \in S(f, D)} q^{-n-eps} \int_{O_K^n} \chi(ac(f_P(x))|f_P(x)|_K^s \ |dx|, \quad (2.3)$$

where $Re(s) > 0$. The proof given by Igusa in [I3], for the case $\chi = \chi_{\text{triv}}$, generalizes literally to arbitrary characters.

In [Z-G] the author introduced the following index of singularity at a point $P \in \mathcal{O}_K^n$, satisfying $P \notin \text{Sing}_f(\mathcal{O}_K)$.

**Definition 2.1.** Let $f(x) \in \mathcal{O}_K[x]$ be a polynomial and $P = (a_1, \ldots, a_n) \in \mathcal{O}_K^n$, such that $P \notin \text{Sing}_f(\mathcal{O}_K)$. We define

$$L(f, P) := \text{Inf} \left( v(f(P)), v\left( \frac{\partial f}{\partial x_1}(P) \right), \ldots, v\left( \frac{\partial f}{\partial x_n}(P) \right) \right).$$
It follows from the definition that \( L(f, P) = 0 \) if and only if the polynomial

\[
\overline{f(x)} = \alpha_0 + \sum_j \alpha_j(x_j - \overline{a_j}) + (\text{degree } \geq 2) \in \mathbb{F}_q[x],
\]

satisfies \( \alpha_j \in \mathbb{F}_q^* \) for some \( j = 0, 1, 2, \ldots, n \).

The index \( L(f, P) \) appears naturally associated to Igusa’s stationary phase, as it was already noted in [Z-G]. In addition, this index plays an important role in the construction of the Néron \( \pi \)-adic desingularization of the special fiber of smooth schemes over Spec\( (\mathcal{O}_K) \) (see [A], [N]).

If \( A \subseteq \mathcal{O}_K^n \), we denote by \( A^c \) the complement of \( A \) with respect to \( \mathcal{O}_K^n \).

**Proposition 2.2.** Let \( D \subseteq \mathcal{O}_K^n \) be an open and compact subset, and let \( f(x) \in \mathcal{O}_K[x] \) be a polynomial such that \( \text{Sing}_f(K) \cap D = \emptyset \). Then there exists a constant \( C(f, D) \in \mathbb{N} \), depending only on \( f \) and \( D \), such that

\[
L(f, P) \leq C(f, D), \quad \text{for all } P \in D.
\] (2.4)

**Proof.** By contradiction, we suppose that \( L(f, P) \) is not bounded on \( D \). Thus there exists a sequence \( (Q_i)_{i \in \mathbb{N}} \) of points of \( D \) satisfying \( \lim L(f, Q_i) \rightarrow \infty \), when \( i \rightarrow \infty \). This sequence has a limit point \( Q_* \in D \). Since \( \text{Sing}_f(K) \) is a closed set, we have that \( Q_* \in \text{Sing}_f(K) \cap D = \emptyset \), contradiction. \( \square \)

From now on, we shall suppose that \( C(f, D) \) is minimal for condition (2.4).

We recall that a subset \( A \) of \( K^n \) is open and compact if and only if there is \( m \geq 0 \) such that \( A \) is the finite union of classes modulo \( \pi^m \). In particular the preimage of any subset of \( \mathbb{F}_q^n \) under the canonical homomorphism \( \mathcal{O}_K \rightarrow \mathcal{O}_K/\pi \mathcal{O}_K \) is an open and compact subset.

The following lemma is a generalization of proposition 2.3 of [Z-G].

**Lemma 2.3.** Let \( D \subseteq \mathcal{O}_K^n \) be the preimage under the canonical homomorphism \( \mathcal{O}_K \rightarrow \mathcal{O}_K/\pi \mathcal{O}_K \) of a subset \( \overline{D} \subseteq \mathbb{F}_q^n \), and let \( f(x) \in \mathcal{O}_K[x] \) be a polynomial such that \( \text{Sing}_f(\mathcal{O}_K) \cap D = \emptyset \), then

(i) \( L(f_{P_1, \ldots, P_k}, 0) \leq L(f, P_1 + \pi P_2 + \ldots + \pi^{k-1}P_k) - k \), for every \( P_k, k \geq 1 \), satisfying:

- (H1) \( P_k \) is in the \( (k - 1)^{\text{th}} \)-generation of descendants of \( P_1 \);
- (H2) \( P_k \) has at least one descendant in the \( k^{\text{th}} \)-generation of descendants of \( P_1 \).

(ii) For any \( P = P_1 \in S(f, D) \), if \( k \geq C(f, D) + 1 \) then \( S(f_{P_1, P_2, \ldots, P_k}) = \emptyset \).
Proof. First, we observe that

\[ f(P_1 + \pi P_2 + \ldots + \pi^{k-1} P_k + \pi^k x) = \pi^{E(P_1,\ldots,P_k)} f_{P_1,\ldots,P_k}(x), \]  

(2.5)

where \( E(P_1,\ldots,P_k) = e_{P_1} + e_{P_1,P_2} + \ldots + e_{P_1,\ldots,P_k} \). The result follows from (2.5), if

\[ e_{P_1,\ldots,P_l} \geq 2, \quad \text{for } l = 1, 2, \ldots, k. \]

This last fact follows from the following reasoning.

By applying the Taylor formula to \( f_{P_1,\ldots,P_l-1}(P_l + \pi x) \), we obtain

\[ f_{P_1,\ldots,P_l-1}(P_l + \pi x) = f_{P_1,\ldots,P_l-1}(P_l) + \pi \sum_j \frac{\partial f_{P_1,\ldots,P_l-1}}{\partial x_j}(P_l)x_j + \pi^2(\text{degree} \geq 2). \]  

(2.6)

From hypothesis (H1) follows that \( v(f_{P_1,\ldots,P_l-1}(P_l)) \geq 1 \) and \( v(\frac{\partial f_{P_1,\ldots,P_l-1}}{\partial x_j}(P_l)) \geq 1 \), and from hypothesis (H1) and (H2) that

\[ v(f_{P_1,\ldots,P_l-1}(P_l)) \geq 2; \]

therefore (2.6) implies that \( e_{P_1,\ldots,P_l} \geq 2, \quad l = 1, 2, \ldots, k. \)

(ii) The second part of the lemma follows immediately from (i). □

We observe that if \( P_l \in S(f_{P_1,\ldots,P_l-1}) \) does not have descendants in the \( l^{th} \)-generation (i.e. \( S(f_{P_1,\ldots,P_l-1,P_l}) = \emptyset \)), then the polynomial

\[ f_{P_1,\ldots,P_l-1,P_l}(P_{l+1} + \pi x) = f_{P_1,\ldots,P_l}(P_{l+1}) + \pi \sum_j \frac{\partial f_{P_1,\ldots,P_l}}{\partial x_j}(P_{l+1})x_j + \pi^2(\text{degree} \geq 2) \]

satisfies \( f_{P_1,\ldots,P_l}(P_{l+1}) \neq 0 \), or \( \frac{\partial f_{P_1,\ldots,P_l}}{\partial x_{j_0}}(P_{l+1}) \neq 0 \), for some \( j_0 \). Thus for any \( P_{l+1} \) satisfying \( f_{P_1,\ldots,P_l}(P_{l+1}) = 0 \), it holds that \( f_{P_1,\ldots,P_{l+1}}(x) \) is a polynomial of degree at most one.

Lemma 2.4. Let \( D \subseteq \mathcal{O}_K^n \) be the preimage under the canonical homomorphism \( \mathcal{O}_K \rightarrow \mathcal{O}_K/\pi \mathcal{O}_K \) of a subset \( \mathcal{D} \subseteq \mathbb{F}_q^n \). Let \( f(x) \in \mathcal{O}_K[x] \) be a polynomial such that \( \text{Sing}_f(K) \cap D = \emptyset \), then

\[ \int_D \chi(\mathfrak{a} f(x))|f(x)|_K^s|dx| = \begin{cases} \frac{T(q^{-s})}{1-q^{-s}}, & \chi = \chi_{\text{triv}}, \\ L(q^{-s}), & \chi \neq \chi_{\text{triv}}, \end{cases} \]
where $T$ and $L$ are polynomials in $q^{-s}$ with rational coefficients. Furthermore, in the case $\chi \neq \chi_{\text{triv}}$, the degree of the polynomial $L(q^{-s})$ is bounded by a constant depending only on $f$ and $D$.

**Proof.** We define inductively $I_k$ as follows:

$$I_1 := S(f, D),$$

$$I_k := \{(P_1, P_2, ..., P_k) \mid (P_1, P_2, ..., P_{k-1}) \in I_{k-1}, \text{ and } P_k \in S(f_{P_1, P_2, ..., P_{k-1}})\}, \quad k \geq 2.$$  

We set $E(P_1, ..., P_k) := e_{P_1} + e_{P_1, P_2} + ... + e_{P_1, P_2, ..., P_k}$.

If $m = C(f, D) + 1$, then $I_{m+1} = \emptyset$, because lemma 2.3 (ii) implies that $S(f_{P_1, P_2, ..., P_m}) = \emptyset$, for every $(P_1, P_2, ..., P_m) \in I_m$. By applying the stationary phase formula $m+1$–times, we obtain

$$Z(D, s, f, \chi) = \nu(\bar{f}, D, \chi) + \sigma(\bar{f}, D, \chi) \frac{(1 - q^{-1})q^{-s}}{(1 - q^{-1-s})} +$$

$$\sum_{k=1}^{m} q^{-kn} \left( \sum_{(P_1, ..., P_k) \in I_k} \nu(\bar{f}_{P_1, ..., P_k}, \chi) q^{-E(P_1, ..., P_k)s} \right) +$$

$$\frac{(1 - q^{-1})q^{-s}}{(1 - q^{-1-s})} \sum_{k=1}^{m} q^{-kn} \left( \sum_{(P_1, ..., P_k) \in I_k} \sigma(\bar{f}_{P_1, ..., P_k}, \chi) q^{-E(P_1, ..., P_k)s} \right).$$  

(2.7)

In the case $\chi \neq \chi_{\text{triv}}$, all $\sigma(\bar{f}_{P_1, ..., P_k}, \chi) = 0$, thus $Z(D, s, f, \chi)$ is a polynomial in $q^{-s}$ and its degree is bounded by the maximum of the $E(P_1, ..., P_m)$, where $P_m$ runs through the descendants of the $C(f, D) + 1$–generation of $S(f, D)$. □

**Corollary 2.5.** Let $D \subseteq \mathcal{O}_K^n$ be the preimage under the canonical homomorphism $\mathcal{O}_K \to \mathcal{O}_K/\pi\mathcal{O}_K$ of a subset $\mathcal{D} \subseteq \mathbb{F}_q^n$. Let $F(x) = f(x) + \pi^\beta g(x) \in \mathcal{O}_K[x]$ be a polynomial such that $\beta \geq C(f, D) + 1$, and

$$\text{Sing}_F(K) \cap D = \text{Sing}_f(K) \cap D = \emptyset.$$  

Then

$$Z(D, s, F, \chi) = Z(D, s, f, \chi).$$  

(2.8)

**Proof.** The result follows immediately from expansion 2.7 and the fact that $C(f, D) = C(F, D)$. □
3. Newton polyhedra

In this section we review some well-known results about Newton polyhedra that we shall use in this paper (see e.g. [K-M-S], [D-3]).

We set \( \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \} \). Let \( f(x) = \sum_l a_l x^l \in K[x], x = (x_1, x_2, ..., x_n) \) be a polynomial in \( n \) variables satisfying \( f(0) = 0 \). The set \( \text{supp}(f) = \{ l \in \mathbb{N}^n \mid a_l \neq 0 \} \) is called the support of \( f \). The Newton polyhedron \( \Gamma(f) \) of \( f \) is defined as the convex hull in \( \mathbb{R}^n_+ \) of the set

\[
\bigcup_{l \in \text{supp}(f)} (l + \mathbb{R}^n_+).
\]

By a proper face \( \gamma \) of \( \Gamma(f) \), we mean the non-empty convex set \( \gamma \) obtained by intersecting \( \Gamma(f) \) with an affine hyperplane \( H \), such that \( \Gamma(f) \) is contained in one of two half-spaces determined by \( H \). The hyperplane \( H \) is named the supporting hyperplane of \( \gamma \). A face of codimension one is named a facet.

We set \(<,>\) for the usual inner product in \( \mathbb{R}^n \), and identify the dual vector space with \( \mathbb{R}^n \). For \( a \in \mathbb{R}^n_+ \), we define

\[
m(a) := \inf_{x \in \Gamma(f)} \{ <a, x> \}.
\]

The first meet locus of \( a \in \mathbb{R}^n_+ \setminus \{0\} \) is defined by

\[
F(a) := \{ x \in \Gamma(f) \mid <a, x> = m(a) \}.
\]

The first meet locus \( F(a) \) of \( a \) is a proper face of \( \Gamma(f) \).

We define an equivalence relation on \( \mathbb{R}^n_+ \setminus \{0\} \) by

\[
a \preceq a' \text{ if and only if } F(a) = F(a').
\]

If \( \gamma \) is a face of \( \Gamma(f) \), we define the cone associated to \( \gamma \) as

\[
\Delta_\gamma := \{ a \in (\mathbb{R}_+)^n \setminus \{0\} \mid F(a) = \gamma \}.
\]

The following two propositions describe the geometry of the equivalences classes of \( \preceq \) (see e.g. [D-3]).
Proposition 3.2. Let $\gamma$ be a proper face of $\Gamma(f)$. Let $w_1, w_2, \ldots, w_e$ be the facets of $\Gamma(f)$ which contain $\gamma$. Let $a_1, a_2, \ldots, a_e$ be vectors which are perpendicular to respectively $w_1, w_2, \ldots, w_e$. Then

$$\Delta_\gamma = \{ \sum_{i=1}^{e} \alpha_i a_i \mid \alpha_i \in \mathbb{R}, \ \alpha_i > 0 \}.$$ 

If $a_1, a_2, \ldots, a_e \in \mathbb{R}^n$, we call $\{ \sum_{i=1}^{e} \alpha_i a_i \mid \alpha_i \in \mathbb{R}, \ \alpha_i > 0 \}$ the cone strictly positive spanned by the vectors $a_1, a_2, \ldots, a_e$. Let $\Delta$ be a cone strictly positive spanned by the vectors $a_1, a_2, \ldots, a_e$. If $a_1, a_2, \ldots, a_e$ are linearly independent over $\mathbb{R}$, the cone $\Delta$ is called a simplicial cone. In this last case, if $a_1, a_2, \ldots, a_e \in \mathbb{Z}^n$, the cone $\Delta$ is called a rational simplicial cone. If $\{a_1, a_2, \ldots, a_e\}$ can be completed to be a basis of $\mathbb{Z}$–module $\mathbb{Z}^n$, the cone $\Delta$ is named a simple cone.

A vector $a \in \mathbb{R}^n$ is called primitive if the components of $a$ are positive integers whose greatest common divisor is one.

For every facet of $\Gamma(f)$ there is a unique primitive vector in $\mathbb{R}^n$ which is perpendicular to this facet. Let $\mathcal{D}$ be the set of all these vectors.

Proposition 3.3. Let $\Delta$ be the cone strictly positively spanned by vectors $a_1, a_2, \ldots, a_e \in \mathbb{R}^n_+ \setminus \{0\}$. Then there is a partition of $\Delta$ into cones $\Delta_i$, such that each $\Delta_i$ is strictly positively spanned by some vectors from $\{a_1, a_2, \ldots, a_e\}$ which are linearly independent over $\mathbb{R}$.

The two previous propositions imply the existence of a partition of $\Delta_\gamma$ into rational simplicial cones.

Proposition 3.4. ([K-M-S], p. 32-33) Let $\Delta$ be a rational simplicial cone. Then there exists a partition of $\Delta$ into simple cones.

Summarizing, given a polynomial $f(x) \in K[x]$, $f(0) = 0$, with Newton polyhedron $\Gamma(f)$, there exists a finite partition of $\mathbb{R}^n_+$ of the form:

$$\mathbb{R}^n_+ = \{(0, \ldots, 0)\} \bigcup \bigcup_i \Delta_i,$$

where each $\Delta_i$ is a simplicial cone contained in an equivalence class of $\simeq$. Furthermore, by proposition 3.4, it is possible to refine this partition in such a way that each $\Delta_i$ is a simple cone contained in an equivalence class of $\preceq$. 
4. Local zeta functions of globally non-degenerate polynomials

In this section we prove theorem A. First, we give some preliminary results.

If \( A \subseteq \mathbb{Z}_+^n \), we set
\[
E_A := \{ (x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid (v(x_1), \ldots, v(x_n)) \in A \},
\]
and
\[
Z_A(s, f, \chi) := \int_{E_A} \chi(acf(x)) |f(x)|_K^s \, dx.
\]
Also, if \( B \subseteq \mathcal{O}_K^n \), we set
\[
Z(B, s, f, \chi) := \int_B \chi(acf(x)) |f(x)|_K^s \, dx.
\]
Thus \( Z_A(s, f, \chi) = Z(E_A, s, f, \chi) \).

**Proposition 4.1.** Let \( f(x) \in \mathcal{O}_K[x] \) be a globally non-degenerate polynomial with respect to its Newton polyhedron \( \Gamma(f) \), \( \gamma \subseteq \Gamma(f) \) a proper face, and \( \Delta_\gamma \) its associated cone. If \( \Delta_\gamma \) is a simple cone spanned by \( a_1, a_2, \ldots, a_e \in \mathcal{D} \), and \( f(x) = f_\gamma(x) + \pi^{g_0} H(x) \), where \( g_0 \geq C(f_\gamma, \mathcal{O}_K^\times) + 1 \) (the constant whose existence was established in proposition 2.2), and all monomials of \( H(x) \) are not in \( \gamma \), then
\[
Z_{\Delta_\gamma}(s, f, \chi) = Z(\mathcal{O}_K^\times, s, f_\gamma, \chi) q^{-\sum_{j=1}^e (|a_j| + m(a_j)s)} \prod_{j=1}^e \frac{1}{1 - q^{-|a_j| - m(a_j)s}}.
\]

**Proof.** The hypothesis \( \Delta_\gamma \) is a simple cone spanned by \( a_{j} = (a_{1,j}, a_{2,j}, \ldots, a_{n,j}), j = 1, 2, \ldots, e \), implies that
\[
\Delta_\gamma \cap \mathbb{N}^n = \bigoplus_{j=1}^e a_j(\mathbb{N} \setminus \{0\}).
\]
From (4.2), we obtain the following expansion for \( Z_{\Delta}(s, f, \chi) \):
\[
Z_{\Delta_\gamma}(s, f, \chi) = \sum_{y_1=1}^\infty \ldots \sum_{y_e=1}^\infty \int_{\omega(y_1, \ldots, y_e)} \chi(acf(x)) |f(x)|_K^s \, dx,
\]
where
\[
\omega(y_1, \ldots, y_e) := \{ (x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid x_i = \pi^{y_i} a_i \mu_i, \mu_i \in \mathcal{O}_K, i = 1, 2, \ldots, n \}.
\]
In order to compute the integral in (4.3), we introduce the dilatation

\[ \Phi_{(y_1, \ldots, y_e)}(x) = (\Phi_1(x), \ldots, \Phi_n(x)) : K^n \rightarrow K^n, \]

where

\[ \Phi_i(x) = \pi^{\sum_j a_{ij} y_j x_i}, \quad i = 1, 2, \ldots, n. \quad (4.4) \]

By using the dilatation (4.4) as a change of variables in (4.3), it holds that

\[ \int_{\omega_{(y_1, \ldots, y_e)}} \chi(acf(x)) \left| f(x) \right|_K^+ \, dx = \]

\[ q^{-\sum_{j=1}^s y_j (|a_j| + m(a_j)s)} \left( \int_{O_K^{\times n}} \chi(ac(f_{(y_1, \ldots, y_e)}(x))) \left| f_{(y_1, \ldots, y_e)}(x) \right|_K^+ \, dx \right), \quad (4.5) \]

where \( f_{(y_1, \ldots, y_e)}(x) = f_\gamma(x) + \pi^{g(y_1, \ldots, y_e) + g_0} H_{(y_1, \ldots, y_e)}(x) \), and \( g(y_1, \ldots, y_e) \geq 1 \). The result follows from (4.5) by using corollary 2.5 and expansion (4.3).

**Proposition 4.2.** Let \( f(x) \in O_K[x] \) be a globally non-degenerate polynomial with respect to its Newton polyhedron \( \Gamma(f) \), \( \gamma \subseteq \Gamma(f) \) a proper face, and \( \Delta_\gamma \) its associated cone. If \( \Delta_\gamma \) is a simple cone spanned by \( a_1, a_2, \ldots, a_e \in D \), then

\[ Z_{\Delta_\gamma}(s, f, \chi) = \sum_{y \text{ finite}} A_y(q^{-s}) Z(O_K^{\times n}, s, f_y, \chi) + \sum_{I \subseteq \{1, 2, \ldots, e\}} A_I(q^{-s}) Z(O_K^{\times n}, s, f_I, \chi) / \prod_{j \in I} (1 - q^{-|a_j| - m(a_j)s}), \]

where \( y \) runs through a finite number of points in \( \mathbb{N}^n \), \( A_y(q^{-s}) \), \( A_I(q^{-s}) \in \mathbb{Q}[q^{-s}] \), \( f_y(x) \) and \( f_I(x) \) are polynomials in \( O_K[x] \) satisfying \( \text{Sing}_{f_y}(K) \cap (K \setminus \{0\})^n = \emptyset \), for every \( y \in \mathbb{N} \), \( \text{Sing}_{f_I}(K) \cap (K \setminus \{0\})^n = \emptyset \), for every \( I \), respectively. Furthermore, if \( \gamma_{a_i} \) denotes the facet with perpendicular \( a_i \), and \( \gamma_I = \bigcap_{i \in I} \gamma_{a_i} \), then \( f_I(x) = f_{\gamma_I}(x) \).

**Proof.** By induction on \( l \), the number of generators of the simple cone \( \Delta_\gamma \).

**Case 1=1**

Let \( m_0 = C(f_\gamma, O_K^\times) + 1 \), and

\[ S := \Delta_\gamma \cap \mathbb{N}^n = \{ a_1 y \mid y \in \mathbb{N}, y \geq 1 \}. \]

The set \( S \) can be partitioned into the subsets \( S_0, S_1 \), defined as follows:

\[ S_0 := \{ a_1 y \mid y = 1, 2, \ldots, m_0 - 1 \}, \quad S_1 := \{ a_1 y \mid y \in \mathbb{N}, y \geq m_0 \}. \]
Also we define
\[ E_0 := \{(x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid (v(x_1), \ldots, v(x_n)) \in S_0 \}, \]
\[ E_1 := \{(x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid (v(x_1), \ldots, v(x_n)) \in S_1 \}. \]

Thus \( Z_{\Delta, \gamma}(s, f, \chi) = Z(E_0, s, f, \chi) + Z(E_1, s, f, \chi), \) and by making a change of variables of type (4.4), we obtain
\[
Z_{\Delta, \gamma}(s, f, \chi) = \sum_{y=1}^{m_0-1} q^{-y(|a_1| + m(a_1)s)} Z(\mathcal{O}_K^x, s, f_y, \chi) + q^{-m_0(|a_1| + m(a_1)s)} Z_{\Delta, \gamma}(s, f_{a_1}(x) + \pi^{m_0}H(x), \chi),
\]
(4.6)

where \( f_y(x) \) are obtained from \( f(x) \) by a change of variables of type (4.4) followed by a division by a power of \( \pi \), \( f_{a_1}(x) \) is the restriction of \( f(x) \) to the facet \( \gamma_{a_1} \) with perpendicular \( a_1 \), and all monomials of \( H(x) \) are not in \( \gamma_{a_1} \). The result follows from (4.6), by means of the following equality (cf. proposition 4.1)
\[
q^{-m_0(|a_1| + m(a_1)s)} Z_{\Delta, \gamma}(s, f_{a_1}(x) + \pi^{m_0}H(x), \chi) = \frac{q^{-(m_0+1)(|a_1| + m(a_1)s)}}{1-q^{-(|a_1| + m(a_1)s)}} Z(\mathcal{O}_K^x, s, f_{a_1}, \chi).
\]

**Induction hypothesis** Suppose that the lemma is valid for every polynomial \( f(x) \) globally non-degenerate with respect its Newton polyhedron, and for every simple cone spanned by at most \( e-1 \) vectors of \( \mathcal{D} \).

**Case \( l > 1 \)**

Let \( f(x) \) be globally non-degenerate polynomial and \( \Delta, \gamma \) a simple cone spanned by \( a_1, a_2, \ldots, a_e \), satisfying the conditions of proposition 4.2.

We set \( m_0 = C(f_{\gamma}, \mathcal{O}_K^x) + 1, \) and
\[
S := \Delta, \gamma \cap \mathbb{N}^e = \bigoplus_{j=1}^e a_j(\mathbb{N} \setminus \{0\}),
\]
(4.7)

\( a_j = (a_{1,j}, \ldots, a_{n,j}), j = 1, 2, \ldots, e. \) For each subset \( I \subseteq \{1, 2, \ldots, e\}, \) we put \( r_I \in \mathbb{N}^{e-\text{Card}(I)}, \)
\( r_I = (r_{i_1}, r_{i_2}, \ldots, r_{i_{e-\text{Card}(I)}}), \) with \( 0 < r_{i_l} \leq m_0 - 1, l = 1, 2, \ldots, e - \text{Card}(I). \) The set \( S \) admits the following partition:
\[
S = \bigcup_{I, r_I} S_{I, r_I},
\]
(4.8)
with
\[ S_{I,r} = \{ \sum_{j \in I} a_jy_j + \sum_{j \notin I} a_j\gamma_j \mid y_j \geq m_0, \text{ if } j \in I, \text{ and } y_j = r_{ij}, \text{ if } j \notin I \}, \]
where for each \( I \subseteq \{1, 2, \ldots, e\} \), the corresponding \( r_I \)'s run through all possible different integer vectors satisfying the above mentioned conditions. We set
\[ E_{I,r} := \{(x_1, \ldots, x_n) \in O_K^n \mid (v(x_1), \ldots, v(x_n)) \in S_{I,r}\}. \]

It follows from partition (4.8) that
\[ Z_{\Delta,s}(s, f, \chi) = \sum_{I,r} Z(E_{I,r}, s, f, \chi). \tag{4.9} \]

By a change of variables of type
\[ \Phi_i(x) = \pi((\sum_{j \in I} a_i,jy_j+\sum_{j \notin I} a_i,j\gamma_j)x_i, \ i = 1, \ldots, n; \]
the integral \( Z(E_{I,r}, s, f, \chi) \) equals
\[ q^{-m_0\sum_{j \in I}(|a_j|+m(a_j)s)-\sum_{j \notin I}r_j(|a_j|+m(a_j)s)} Z_{\Delta,I}(s, f, \chi), \tag{4.10} \]
where \( \Delta_I \) is a simple cone generated by \( a_i, \ i \in I \), and \( f_I(x) \) is obtained from \( f(\Phi_i(x)) \) by division by a power of \( \pi \). From these observations and (4.9), we obtain
\[ Z_{\Delta,s}(s, f, \chi) = \sum_{I \subseteq \{1, 2, \ldots, e\}} A_I(q^{-s})Z_{\Delta,I}(s, f, \chi) + \]
\[ q^{-m_0\sum_{j=1}^e(|a_j|+m(a_j)s)} Z_{\Delta,s}(s, f_\gamma + \pi^{g_0}H(x), \chi), \tag{4.11} \]
where \( I \) runs through all proper subsets of \( \{1, 2, \ldots, e\} \), \( A_I(q^{-s}) = \sum_k q^{-a_k(I)-b_k(I)s}, \ a_k(I), b_k(I) \in \mathbb{N}, g_0 \geq m_0 \), and all monomials of \( H(x) \) are not in \( \gamma \). From (4.11) and proposition 4.1, we obtain
\[ Z_{\Delta,s}(s, f, \chi) = \sum_{I \subseteq \{1, 2, \ldots, e\}} A_I(q^{-s})Z_{\Delta,I}(s, f, \chi) + \]
\[ q^{-(1+m_0)\sum_{i=1}^e(|a_i|+m(a_i)s)} Z(O_K^\times, s, f, \gamma, \chi) \frac{1}{\prod_{j=1}^e(1-q^{-|a_j|+m(a_j)s})}. \tag{4.12} \]

The result follows from the induction hypothesis and (4.12). □

We observe that each \( A_I(q^{-s}) \) in proposition 4.1 is a finite sum of monomials of type \( q^{-a_I-b_I s} \), with \( a_I, b_I > 0 \). We also note that a facet with supporting hyperplane \( x_{i_0} = 0 \) contributes to the denominator of \( Z_{\Delta,s}(s, f, \chi) \) with a constant factor \( \frac{1}{1-q^{-1}} \).

The proof of proposition 4.2 can be easily adapted to state the following more general result.
Corollary 4.3. Let \( f(x) \in \mathcal{O}_K[x] \) be a globally non-degenerate polynomial with respect to its Newton polyhedron \( \Gamma(f) \), \( \gamma \subseteq \Gamma(f) \) a proper face, and \( \Delta_\gamma \) its associated cone. Let \( \{a_1, a_2, \ldots, a_f\} \subseteq \mathcal{D} \) be a set of generators of \( \Delta_\gamma \), \( \{a_1, a_2, \ldots, a_e\} \subseteq \{a_1, a_2, \ldots, a_f\} \) of \( \mathbb{R} \)–linearly independent vectors, and \( b \in \Delta_\gamma \cap (\mathbb{N} \setminus \{0\})^n \). We set \( \Delta := b + \bigoplus_{j=1}^e a_j\mathbb{N} \). Then

\[
Z_\Delta(s, f, \chi) = \sum_y A_y(q^{-s})Z(\mathcal{O}_K^\times, s, f_y, \chi) + \sum_{I \subseteq \{1, 2, \ldots, e\}} \frac{A_I(q^{-s})Z(\mathcal{O}_K^\times, s, f_I, \chi)}{\prod_{j \in I}(1 - q^{-|a_j| - m(a_j)s})},
\]

where \( y \) runs through a finite number of points in \( \mathbb{N}^n \), \( A_y(q^{-s}) \), \( A_I(q^{-s}) \) \( \in \mathbb{Q}[q^{-s}] \), with \( A_I(q^{-s}) = \sum_k q^{-a_k(I) - b_k(I)s} \), \( a_k(I), b_k(I) \in \mathbb{N} \), \( f_y(x) \) and \( f_I(x) \) are polynomials in \( \mathcal{O}_K[x] \) satisfying \( \text{Sing}_{f_y}(K) \cap (K \setminus \{0\})^n = \emptyset \), for every \( y \), \( \text{Sing}_{f_I}(K) \cap (K \setminus \{0\})^n = \emptyset \), for every \( I \), respectively. Furthermore, if \( \gamma_{a_i} \) denotes the facet with perpendicular \( a_i \), and \( \gamma_I = \bigcap_{i \in I} \gamma_{a_i} \), then \( f_I(x) = f_{\gamma_I}(x) \).

Lemma 4.4. Let \( f(x) \in \mathcal{O}_K[x] \) be a globally non-degenerate polynomial with respect to its Newton polyhedron \( \Gamma(f) \), \( \gamma \subseteq \Gamma(f) \) a proper face, and \( \Delta_\gamma \) its associated cone. Let \( \{a_1, a_2, \ldots, a_e\} \subseteq \mathcal{D} \) be a set of generators of \( \Delta_\gamma \). Then

\[
Z_{\Delta_\gamma}(s, f, \chi) = \sum_y A_y(q^{-s})Z(\mathcal{O}_K^\times, s, f_y, \chi) + \sum_{I \subseteq \{1, 2, \ldots, e\}} \frac{A_I(q^{-s})Z(\mathcal{O}_K^\times, s, f_I, \chi)}{\prod_{j \in I}(1 - q^{-|a_j| - m(a_j)s})},
\]

where \( y \) runs through a finite number of points in \( \mathbb{N}^n \), \( A_y(q^{-s}) \), \( A_I(q^{-s}) \) \( \in \mathbb{Q}[q^{-s}] \), with \( A_I(q^{-s}) = \sum_k q^{-a_k(I) - b_k(I)s} \), \( a_k(I), b_k(I) \in \mathbb{N} \), \( f_y(x) \) and \( f_I(x) \) are polynomials in \( \mathcal{O}_K[x] \) satisfying \( \text{Sing}_{f_y}(K) \cap (K \setminus \{0\})^n = \emptyset \), for every \( y \), and \( \text{Sing}_{f_I}(K) \cap (K \setminus \{0\})^n = \emptyset \), for every \( I \), respectively. Furthermore, if \( \gamma_{a_i} \) denotes the facet with perpendicular \( a_i \), and \( \gamma_I = \bigcap_{i \in I} \gamma_{a_i} \), then \( \Gamma(f_I) = \gamma_I \).

Proof. By proposition 3.3 there exists a finite partition of \( \Delta_\gamma \) into cones \( \Delta_j \), such that each \( \Delta_j \) is strictly positively spanned by some vectors from \( \{a_1, a_2, \ldots, a_e\} \) which are linearly independent over \( \mathbb{R} \). Now, each cone \( \Delta_j \) can be partitioned into a finite number of cones satisfying the conditions of corollary 4.3. In order to verify this last assertion, we observe that the set \( \Delta_j \cap \mathbb{N}^n \) admits the following partition:
\[
\Delta_j \cap \mathbb{N}^n = \left( \bigoplus_{i=1}^e a_i(\mathbb{N} \setminus \{0\}) \right) \bigcup \bigcup_b \left( b + \bigoplus_{i=1}^e a_j \mathbb{N} \right),
\]

(4.14)

where \( b \) runs through a finite number of vectors in

\[
\mathbb{N}^n \cap \{ \sum_{i=1}^e a_i \lambda_i \mid \lambda_i \in \mathbb{R}, 0 \leq \lambda_i < 1, i = 1, \ldots, e \}.
\]

Now the result follows from corollary 4.3. \( \square \)

In the proof of the above result, we did not use a partition of the cone \( \Delta \) into simple cones, because this approach produces a bigger list of candidates for the poles of \( Z_{\Delta,\gamma}(s, f, \chi) \).

**Proof of Theorem A.**

(i) Given a polynomial \( f(x) \in \mathcal{O}_K[x], f(0) = 0 \), there exists a partition of \( \mathbb{R}_+^n \) of the form:

\[
\mathbb{R}_+^n = \{(0, \ldots, 0)\} \bigcup \bigcup_{\gamma} \Delta_\gamma,
\]

(4.15)

where \( \gamma \) runs through all proper faces of \( \Gamma(f) \), and \( \Delta_\gamma \) is a cone strictly positive spanned by some vectors \( a_1, \ldots, a_e \in \mathcal{D} \). In addition, \( \Delta_\gamma \) is contained in an equivalence class of \( \succsim \). From the above partition we obtain the following expression for \( Z(s, f, \chi) \):

\[
Z(s, f, \chi) = \int_{\mathcal{O}_K^n} \chi(acf(x)) \mid f(x) \mid_K^s dx + \sum_{\gamma} Z_{\Delta,\gamma}(s, f, \chi).
\]

(4.16)

In (4.16) there are two different types of integrals: \( Z(\mathcal{O}_K^n, s, f, \chi) \), and \( Z_{\Delta,\gamma}(s, f, \chi) \). The integrals of the first type are rational functions of \( q^{-s} \) with poles satisfying \( \text{Re}(s) = -1 \) (cf. lemma 2.4). The second type of integrals are rational functions of \( q^{-s} \) with poles satisfying condition (i) in the statement of theorem A (cf. lemma 4.4).

(ii) If \( \chi \neq \chi_{\text{triv}} \), from (4.16) and lemma 2.4 follow that \( Z(s, f, \chi) \) is equal to a polynomial, with degree bounded by a constant independent of \( \chi \), plus a finite sum of functions of the form

\[
\frac{A_I(q^{-s})Z(\mathcal{O}_K^n, s, f_I, \chi)}{\prod_{j \in I}(1 - q^{-|a_j| - m(a_j)s})},
\]

(4.17)

where \( f_I(x) \) denotes the restriction of \( f(x) \) to the face \( \gamma_I = \bigcap_{i \in I} \gamma_{a_i} \), and \( \gamma_{a_i} \) denotes the facet with perpendicular \( a_i \). The second part of the theorem follows from (4.17) by
the following fact:

$$Z(\mathcal{O}_K^n, s, f_I, \chi) = 0,$$

if the order of $\chi$ does not divide some $m(a_j) \neq 0$, $j \in I$. \hspace{1cm} (4.18)

If the order of $\chi$ does not divide $m(a_j)$, with $a_j = (a_{1,j}, a_{2,j}, \ldots, a_{n,j})$, then there exists an $u \in \mathcal{O}_K^n$ such that

$$\chi^{m(a_j)}(u) \neq 1.$$ \hspace{1cm} (4.19)

We set

$$\phi_u : \mathcal{O}_K^n \xrightarrow[]{} \mathcal{O}_K^n,$$

$$(x_1, x_2, \ldots, x_n) \xrightarrow[]{} (x_1u^{a_{1,j}}, x_2u^{a_{2,j}}, \ldots, x_nu^{a_{n,j}}).$$ \hspace{1cm} (4.20)

The map $\phi_u$ establishes a bijection of $\mathcal{O}_K^n$ to itself that preserves the Haar measure.

By using (4.20) as change of variables in the integral $Z(\mathcal{O}_K^n, s, f_I, \chi)$, it verifies that

$$(1 - \chi^{m(a_j)}(u))Z(\mathcal{O}_K^n, s, f_I, \chi) = 0.$$  

Therefore, (4.19) implies $Z(\mathcal{O}_K^n, s, f_I, \chi) = 0$. \hspace{1cm} $\square$

5. The largest pole of $Z(s, f, \chi_{\text{triv}})$

In this section we prove theorem B. Its proof will be accomplished by means of three preliminary results.

For a polynomial $f(x) \in \mathcal{O}_K[x]$ globally non-degenerate with respect to its Newton polyhedron $\Gamma(f)$, we set

$$\beta(f) := \max_{\tau_j} \left\{-\frac{|a_j|}{m(a_j)}\right\},$$

where $\tau_j$ runs through all facets of $\Gamma(f)$ satisfying $m(a_j) \neq 0$. The point

$$T_0 = (-\beta(f)^{-1}, \ldots, -\beta(f)^{-1}) \in \mathbb{Q}^n$$

is the intersection point of the boundary of the Newton polyhedron $\Gamma(f)$ with the diagonal $\Delta = \{(t, \ldots, t) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^n$. Let $\tau_0$ be the face of smallest dimension of $\Gamma(f)$ containing $T_0$, and $\rho$ its codimension, i.e. $\rho = \dim \Delta_{\tau_0}$.

**Proposition 5.1.** Let $f(x) \in \mathcal{O}_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$. If $\beta(f) > -1$, then $\beta(f)$ is a pole of $Z(s, f, \chi_{\text{triv}})$ and its multiplicity is equal to $\rho$. 
Proof. First, we note that the multiplicity of the possible pole $\beta(f)$ is less than or equal to $\dim \Delta_{\tau_0} = \rho$ (cf. formulas (4.16), (4.13), (2.7)). In order to prove that $\beta(f)$ is a pole of $Z(s,f,\chi_{\text{triv}})$, it is sufficient to show that

$$\lim_{s \to \beta(f)} \left( 1 - q^{\beta(f) - s} \right)^\rho Z(s,f,\chi_{\text{triv}}) > 0. \quad (5.1)$$

This last assertion is a consequence of the following result (cf. (4.16), (4.13)):

**Claim A**

(i) $$\lim_{s \to \beta(f)} \left( 1 - q^{\beta(f) - s} \right)^\rho \left( \frac{A_I(q^{-s})Z(O_K^{x^s},s,I,\chi_{\text{triv}})}{\prod_{j \in I}(1 - q^{-|a_j| - m(a_j)s})} \right) \geq 0, \quad (5.2)$$

for every cone $\Delta_{\gamma} = \{ \sum_{i=1}^e a_i y_i \mid y_i \geq 0, \text{ for all } i \}$, and every $I \subseteq \{1,2,..,e\}$.

(ii) There is a cone $\Delta_0$ and a subset $I_0$ of generators of this cone such that inequality (5.2) is strict.

The first part of the previous claim follows from the following two facts. The first fact is

$$\lim_{s \to \beta(f)} (A_I(q^{-s})Z(O_K^{x^s},s,f_I,\chi_{\text{triv}})) > 0. \quad (5.3)$$

Since $A_I(q^{-s}) = \sum_k q^{a_k(I) - b_k(I)s}$, with $a_k(I), b_k(I) \in \mathbb{N}$, inequality (5.3) follows from noticing that

$$\lim_{s \to \beta(f)} \left( \frac{(1 - q^{-1})q^{-s}}{1 - q^{1-s}} \right) > 0, \text{ when } \beta(f) > -1.$$

The second fact is

$$\lim_{s \to \beta(f)} \left( 1 - q^{\beta(f) - s} \right)^\rho \left( \frac{1}{\prod_{j \in I}(1 - q^{-|a_j| - m(a_j)s})} \right) \geq 0. \quad (5.4)$$

The second part of the claim follows from the following reasoning. Let $a_1, a_2, .., a_e$ be the unique primitive vectors perpendicular to the facets which contain $\tau_0$. There exists a cone $\Delta_0$ in the partition into simplicial cones of $\Delta_{\tau_0}$ given by proposition 3.3. and $I_0 \subseteq \{1,2,..,e\}$ such that $\{a_i \mid i \in I_0\}$ is a set of $\rho$ linearly independent generators of $\Delta_0$, because the dimension of $\Delta_{\tau_0}$ is $\rho$. Then inequality (5.2) is strict for the cone $\Delta_0$ and $I_0$. Thus, $\beta(f)$ is a pole of $Z(s,f,\chi_{\text{triv}})$ of multiplicity $\rho$. □
Proposition 5.2. Let \( f(x) \in \mathcal{O}_K[x] \) be a globally non-degenerate polynomial with respect to its Newton polyhedron \( \Gamma(f) \), and \( \gamma \subseteq \Gamma(f) \) a proper face. If \( \sigma(\bar{f}_\gamma, \mathcal{O}_K^x) = \sigma(\bar{f}_\gamma, \mathcal{O}_K^x, \chi_{\text{triv}}) > 0 \) then

\[
\lim_{s \to -1} (1 - q^{-1-s}) Z(\mathcal{O}_K^x, s, f, \chi_{\text{triv}}) \neq 0.
\] (5.5)

Proof. By using expansion (2.7), with \( D = \mathcal{O}_K^x \), and \( m = \mathcal{C}(f, \mathcal{O}_K^x) + 1 \), we have that

\[
\lim_{s \to -1} (1 - q^{-1-s}) Z(\mathcal{O}_K^x, s, f, \chi_{\text{triv}}) = (q - 1)\sigma(\bar{f}_\gamma, \mathcal{O}_K^x, \chi_{\text{triv}}) + (q - 1)\sum_{k=1}^{m} q^{-kn} \left( \sum_{(P_1, \ldots, P_k) \in I_k} \sigma(\bar{f}_\gamma P_1, \ldots, P_k, \chi_{\text{triv}}) q^E(P_1, \ldots, P_k) \right).
\] (5.6)

Since the right side of (5.6) is a sum of positive numbers, the result follows from the hypothesis \( \sigma(\bar{f}_\gamma, \mathcal{O}_K^x, \chi_{\text{triv}}) > 0 \). \( \square \)

Proposition 5.3. Let \( f(x) \in \mathcal{O}_K[x] \) be a globally non-degenerate polynomial with respect to its Newton polyhedron \( \Gamma(f) \). Let \( a_1, a_2, \ldots, a_e \) be the unique primitive vectors perpendicular to the facets which contain \( \tau_0 \). If \( \beta(f) = -1 \), then \( \beta(f) \) is a pole of \( Z(s, f, \chi_{\text{triv}}) \) with multiplicity less than or equal to \( \rho + 1 \). Furthermore, if every face \( \gamma \supseteq \tau_0 \) satisfies \( \sigma(\bar{f}_\gamma, \mathcal{O}_K^x) > 0 \), then the multiplicity of the pole \( \beta(f) \) is \( \rho + 1 \).

Proof. In the case \( \beta(f) = -1 \) the multiplicity of the possible pole \( \beta(f) \) is less than or equal to \( \rho + 1 \) because \( Z(\mathcal{O}_K^x, s, f, \chi_{\text{triv}}) \) may have a pole at \( s = -1 \) (cf. formulas (4.16), (4.13), (2.7)). As in the case \( \beta(f) > -1 \), the result follows from inequality (5.1) by claim A. In the case \( \beta(f) = -1 \), we may suppose that

\[
Z(\mathcal{O}_K^x, s, f, \chi_{\text{triv}}) = \frac{c_I(q^{-s})}{(1 - q^{-1-s})},
\] (5.9)

where \( c_I(q^{-s}) \) is a polynomial with positive coefficients (cf. expansion 2.7). The proof of claim A involves the same ideas as in the case \( \beta(f) > -1 \).

The second part of the proposition is proved as follows. There exists a simplicial cone \( \Delta_0 \subseteq \Delta_{\tau_0} \) with \( \dim \Delta_0 = \rho \) (cf. final part of the proof of proposition 5.1). Let \( I_0 \) be a set of \( \rho \) linearly independent generators of \( \Delta_0 \). By duality this cone corresponds to a face \( \gamma \supseteq \tau_0 \), and \( Z(\mathcal{O}_K^x, s, f_{I_0}, \chi_{\text{triv}}) \) has a pole of multiplicity 1 at \( s = -1 \) (cf. proposition 5.2), thus

\[
\lim_{s \to -1} (1 - q^{-1-s})^{\rho+1} \left( \frac{A_{I_0}(q^{-s})Z(\mathcal{O}_K^x, s, f_{I_0}, \chi_{\text{triv}})}{\prod_{j \in I_0} (1 - q^{-|a_j| - m(a_j)s})} \right) > 0.
\] (5.10)
Proof of Theorem B. The theorem follows from proposition 5.1 and proposition 5.3.

6. Exponential sums

Let $\Psi$ be an additive character trivial on $O_K$ but not on $P_K^{-1}$. A such character is named standard. We put $z = u\pi^{-m}$, $m \in \mathbb{N} \setminus \{0\}$, $u \in O_K^\times$. To these data one associates the following exponential sum:

$$E(z, K, f) = q^{-nm} \sum_{x \mod \mathcal{P}_K^m} \Psi(uf(x)/\pi^m).$$

The following corollary follows theorem A, theorem B above, and proposition 1.4.5 of [D-2], by writing $Z(s, f, \chi)$ in partial fractions.

Corollary 6.1. (i) Let $f(x) \in O_K[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, then for $|z|$ big enough $E(z, K, f)$ is a finite $\mathbb{C}$-linear combination of functions of the form

$$\chi(ac(z)) |z|^\lambda_K (\log_q(|z|_K))^{\beta},$$

with coefficients independent of $z$, and with $\lambda \in \mathbb{C}$ a pole of $(1 - q^{-1-s})Z(s, f, \chi_{\text{triv}})$ or of $Z(s, f, \chi)$, $\chi \neq \chi_{\text{triv}}$, and $\beta \in \mathbb{N}$, $\beta \leq (\text{multiplicity of pole } \lambda) - 1$. Moreover all poles $\lambda$ appear effectively in this linear combination.

(ii) Let $L$ be a global field, and let $f(x) \in L[x]$ be a globally non-degenerate polynomial with respect to its Newton polyhedron $\Gamma(f)$, and suppose that $\beta(f) > -1$. For almost all non-archimedean completions $L_v$ of $L$, there exists a constant $C(L_v) \in \mathbb{R}$ satisfying

$$|E(z, L_v, f)| \leq C(L_v) |z|_{L_v}^{-\beta(f)} \log_q(|z|_{L_v})^{\rho-1}, \text{ for all } z \in L_v.$$

Igusa has conjectured that $C(L_v) = 1$ for almost all $v$ [I2]. This conjecture was proved by Denef and Sperber when $K$ has characteristic zero, $f$ is a non-degenerate polynomial, and the face of the Newton polyhedron which cuts the diagonal does not have vertex in $\{0, 1\}^n$ [D-Sp]. Corollary 6.1 permits us to extent the result of Denef and Sperber to positive characteristic using the methods in [D-Sp].
7. Examples

Example 7.1. In this example, we compute $Z(s, f, \chi_{triv}) = Z(s, f)$, for $f(x, y) = x^2 + xy + y^2$, when the characteristic of $K$ is different from 2, 3, and analyze the behavior of the pole $s = -1$. In this case $Sing_f(K) = \{(0, 0)\}$, and the Newton polygon has only a compact segment with supporting hyperplane $x + y = 2$. The polynomial $f$ is globally non-degenerate with respect to its Newton polygon.

One easily verifies that $\mathbb{R}^2_+ \setminus \{(0, 0)\}$ can be partitioned into equivalence classes modulo $\cong$, as follows.

If

$$
\Delta_1 := \{(0, a) \mid a > 0\},$
$$
$$
\Delta_2 := \{(b, a + b) \mid a, b > 0\},$
$$
$$
\Delta_3 := \{(a, a) \mid a > 0\},$
$$
$$
\Delta_4 := \{(a + b, a) \mid a, b > 0\},$
$$
$$
\Delta_5 := \{(a, 0) \mid a > 0\},$

then

$$
\mathbb{R}^2_+ = \{(0, 0)\} \bigcup \bigcup_{i=1}^{5} \Delta_i,$$

and

$$
Z(s, f) = Z(O_K^{x^2}, s, f) + \sum_{i=1}^{5} Z_{\Delta_i}(s, f).
$$

**Calculation of $Z(O_K^{x^2}, s, f)$, and $Z_{\Delta_1}(s, f)$**

By using the stationary phase formula, we obtain

$$
Z(O_K^{x^2}, s, f) = \nu(\bar{f}, O_K^{x^2}) + \sigma(\bar{f}, O_K^{x^2}) \frac{(1 - q^{-1})q^{-1}}{(1 - q^{-1-s})}. \tag{7.1}
$$

On the other hand, it is simple to verify that $Z_{\Delta_1}(s, f) = q^{-1}(1 - q^{-1})$.

**Calculation of $Z_{\Delta_2}(s, f)$ and $Z_{\Delta_3}(s, f)$**

$$
Z_{\Delta_2}(s, f) = \sum_{a, b=1}^{\infty} q^{-a - 2b} \int_{O_K^{x^2}} |\pi^{2b} x^2 + \pi^{a+2b} xy + \pi^{2a+2b} y^2|_K |dx dy|
$$
\[
Z_{\Delta_3}(s, f) = \sum_{a \geq 1} q^{-2a} \int_{\mathcal{O}_K^{\times 2}} |\pi^{2a} x^2 + \pi^{2a} xy + \pi^{2a} y^2|_K^s \, dx dy = \frac{q^{-2-2s}}{(1-q^{-1-s})(1+q^{-1-s})} \left( \nu(f, \mathcal{O}_K^{\times 2}) + \sigma(f, \mathcal{O}_K^{\times 2}) \frac{(1-q^{-1})q^{-s}}{(1-q^{-1-s})} \right).
\]

(7.3)

Calculation of \(Z_{\Delta_4}(s, f)\) and \(Z_{\Delta_5}(s, f)\)

\[
Z_{\Delta_4}(s, f) = \sum_{a, b \geq 1} q^{-2a-b} \int_{\mathcal{O}_K^{\times 2}} |\pi^{2a+2b} x^2 + \pi^{2a+b} xy + \pi^{2a} y^2|_K^s \, dx dy = \frac{q^{-3-2s}(1-q^{-1})}{(1-q^{-1-s})(1+q^{-1-s})}.
\]

(7.4)

\[
Z_{\Delta_5}(s, f) = q^{-1}(1-q^{-1}).
\]

(7.5)

From the above calculations, we obtain

\[
\lim_{s \to -1} (1-q^{-1-s})^2 Z(s, f) = \frac{\sigma(f, \mathcal{O}_K^{\times 2})(q-1)}{2}.
\]

(7.6)

Now suppose that \(K = \mathbb{Q}_p\), with \(p \neq 2, 3\). Since

\[
\sigma(f, \mathcal{O}_K^{\times 2}) = p^2 \text{Card} \left( \{(u, v) \in \mathbb{F}_p^{\times 2} \mid f(u, v) = 0 \} \right) = \begin{cases} 0, & \text{if } p \equiv 5, 11 \mod 12, \\ 2p^{-2}(p-1), & \text{if } p \equiv 1, 7 \mod 12, \end{cases}
\]

it follows from (7.6) that

\[
\lim_{s \to -1} (1-p^{-1-s})^2 Z(s, f) = \begin{cases} 0, & \text{if } p \equiv 5, 11 \mod 12, \\ p^{-2}(p-1)^2, & \text{if } p \equiv 1, 7 \mod 12. \end{cases}
\]

(7.7)

Thus \(Z(s, f)\) has a pole at \(s = -1\) of multiplicity \(\rho + 1 = 2\), when

\[
\text{Card} \left( \{(u, v) \in \mathbb{F}_p^{\times 2} \mid \overline{f}(u, v) = 0 \} \right) = \text{Card} \left( \{(u, v) \in \mathbb{F}_p^{\times 2} \mid \overline{f}(u, v) = 0 \} \right) > 0.
\]

Otherwise the multiplicity is \(\rho = 1\).
**Example 7.2.** In this example, by using the method of lemma 4.4, we compute the local zeta function attached to the polynomial \( f(x, y) = x^2 y^2 + x^5 + y^5 \in K[x, y] \), when the characteristic of \( K \) is different from 2, 5. This polynomial is globally non-degenerate with respect to its Newton polyhedron.

One easily verifies that \( \mathbb{R}_+^2 \setminus \{(0,0)\} \) can be partitioned into equivalence classes modulo \( \simeq \), as follows.

If

\[
\begin{align*}
\Delta_1 & := \{(0, a) \mid a > 0\}, \\
\Delta_2 & := \{(2b, a + 3b) \mid a, b > 0\}, \\
\Delta_3 & := \{(2a, 3a) \mid a > 0\}, \\
\Delta_4 & := \{(2a + 3b, 3a + 2b) \mid a, b > 0\}, \\
\Delta_5 & := \{(3a, 2a) \mid a > 0\}, \\
\Delta_6 & := \{(3a + b, 2a) \mid a, b > 0\}, \\
\Delta_7 & := \{(a, 0) \mid a > 0\},
\end{align*}
\]

then

\[
\mathbb{R}_+^2 = \{(0, 0)\} \bigcup \bigcup_{i=1}^{7} \Delta_i,
\]

where each \( \Delta_i \) is exactly an equivalence class modulo \( \simeq \).

**Calculation of** \( Z(\mathcal{O}_K^\times, s, f) \), **and** \( Z_{\Delta_1}(s, f) \)

By using the stationary phase formula, we obtain

\[
Z(\mathcal{O}_K^\times, s, f) = \nu(\mathcal{O}_K^\times, \mathcal{O}_K^\times) + \sigma(\mathcal{O}_K^\times, \mathcal{O}_K^\times) \frac{(1 - q^{-1-s})}{1 - q^{-1-s}}.
\]

(7.8)

On the other hand, it is simple to verify that \( Z_{\Delta_1}(s, f) = q^{-1}(1 - q^{-1}) \).

**Calculation of** \( Z_{\Delta_2}(s, f) \) **and** \( Z_{\Delta_3}(s, f) \)

The cone \( \Delta_2 \) is not a simple. In this case, one verifies that there is only one element in \( \Delta_2 \cap \mathbb{N}^2 \) satisfying \( 0 \leq a < 1, \ 0 \leq b < 1 \). This element is \( (1, 2) = (0, 1)\frac{1}{2} + (2, 3)\frac{1}{2} \). Thus

\[
\Delta_2 \cap \mathbb{N}^2 = \{(0, 1)(\mathbb{N} \setminus \{0\}) + (2, 3)(\mathbb{N} \setminus \{0\})\} \bigcup \{(1, 2) + (0, 1)\mathbb{N} + (2, 3)\mathbb{N}\}.
\]

(7.9)
From the partition \((7.9)\), we obtain that
\[
Z_{\Delta_2}(s, f) = \sum_{a,b=1}^{\infty} q^{-a-5b} \int_{\mathcal{O}_K^2} | \pi^{2a+10b}x^2y^2 + \pi^{10b}x^5 + \pi^{5a+15b}y^5 |^s_K |dxdy| + \sum_{a,b=0}^{\infty} q^{-a-5b-3} \int_{\mathcal{O}_K^2} | \pi^{2a+10b+6}x^2y^2 + \pi^{10b+5}x^5 + \pi^{5a+15b+10}y^5 |^s_K |dxdy| =
\frac{q^{-5-10s}}{1-q^{-5-10s}} q^{-1}(1-q^{-1}) + \frac{q^{-3-5s}}{1-q^{-5-10s}} (1-q^{-1}) = \frac{(1-q^{-1})(q^{-3-5s} + q^{-6-10s})}{1-q^{-5-10s}}.
\]
(7.10)

By applying proposition \(4.1\), and then the stationary phase formula to \(Z_{\Delta_3}(s, f)\), one obtains
\[
Z_{\Delta_3}(s, f) = \sum_{a=1}^{\infty} q^{-5a-10as} \int_{\mathcal{O}_K^2} | y^2 + x^3 |^s_K |dxdy| =
\frac{q^{-5-10s}}{1-q^{-5-10s}} \left( \nu(\mathcal{J}, \mathcal{O}_K^{x2}) + \sigma(\mathcal{J}, \mathcal{O}_K^{x2}) \right) \left( 1-q^{-1} \right) q^{-s}.
\]
(7.11)

**Calculation of \(Z_{\Delta_4}(s, f)\) and \(Z_{\Delta_5}(s, f)\)**

The cone \(\Delta_4\) is not a simple, thus we proceed as in the computation of \(Z_{\Delta_2}(s, f)\), i.e. we find \(0 \leq a < 1, 0 \leq b < 1\), such that
\[
(2,3)a + (3,2)b \in \mathbb{N}^2 \cap \Delta_4.
\]

If \(a = b\), one finds immediately that \((2,3)\frac{a}{5} + (3,2)\frac{b}{5} \in \mathbb{N}^2 \cap \Delta_4, i = 1,2,3,4\). The case \(a \neq b\) cannot occur. Suppose that \((m, n) \in \mathbb{N}^2 \cap \Delta_4\), with \(b > a, a \neq 0, b \neq 0\), \((a = 0 \text{ or } b = 0 \text{ cannot occur})\), i.e.
\[
m = 2a + 3b, \quad n = 3a + 2b, \quad m, n \in \mathbb{N} \setminus \{0\}, \quad 0 < a < b < 1.
\]
(7.12)

From (7.12), we get \(b - a = m - n\), but this is impossible because \(0 < b - a < 1\), and \(m - n \geq 1\). If \(a > b\) then \(a - b = n - m\) and the same argument applies.

Therefore, we have the following partition for \(\mathbb{N}^2 \cap \Delta_4\):
\[
\mathbb{N}^2 \cap \Delta_4 = \{(2,3)(\mathbb{N} \setminus \{0\}) + (3,2)(\mathbb{N} \setminus \{0\})\} \cup \bigcup_{i=1}^{4} \{(i,i) + (2,3)\mathbb{N} + (3,2)\mathbb{N}\}.
\]
(7.13)
From the partition (7.13), we obtain that
\[ Z_{\Delta_4}(s, f) = \left( \frac{1 - q^{-1}}{1 - q^{-5-10s}} \right)^2 + \left( \sum_{i=1}^{4} q^{-2i-4is} \right) \left( \frac{1 - q^{-1}}{1 - q^{-5-10s}} \right)^2. \quad (7.14) \]

For \( Z_{\Delta_5}(s, f) \), we get
\[ Z_{\Delta_5}(s, f) = q^{-5-10s} \left( \nu(f, O_K^\times) + \sigma(f, O_K^\times) \frac{1 - q^{-1}q^{-s}}{1 - q^{-1-s}} \right). \quad (7.15) \]

**Calculation of \( Z_{\Delta_6}(s, f) \)**

In the computation of the integral \( Z_{\Delta_6}(s, f) \), we use the following partition:
\[ \Delta_6 \cap \mathbb{N}^2 = \{(3, 2)(\mathbb{N} \setminus \{0\}) + (1, 0)(\mathbb{N} \setminus \{0\})\} \cup \{(2, 1) + (3, 2)\mathbb{N} + (1, 0)\mathbb{N}\}. \quad (7.16) \]

From the above partition, we get
\[ Z_{\Delta_6}(s, f) = (1 - q^{-1}) \frac{q^{-3-5s} + q^{-6-10s}}{1 - q^{-5-10s}}. \quad (7.17) \]

**Calculation of \( Z_{\Delta_7}(s, f) \)**

\[ Z_{\Delta_7}(s, f) = q^{-1}(1 - q^{-1}). \quad (7.18) \]

Now, with \( \beta(f) = -\frac{1}{2} \), and \( \rho = 2 \), it holds that
\[ \lim_{s \to \beta(f)} \left( 1 - q^{\beta(f)-s} \right)^{\rho} Z(s, f) = \lim_{s \to \beta(f)} \left( 1 - q^{\beta(f)-s} \right)^{\rho} Z_{\Delta_4}(s, f) = \frac{(1 - q^{-1})^2}{50}. \]

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