UNIVERSAL MULTIFRACTAL PROPERTIES OF THE SMALL SCALE INTERMITTENCY IN ANISOTROPIC AND INHOMOGENEOUS TURBULENCE

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The notion of self-similar energy cascades and multifractality has long since been connected with fully developed, homogeneous and isotropic turbulence. We introduce a number of amendments to the standard methods for analysing the multifractal properties of the energy dissipation field of a turbulent flow. We conjecture that the scaling assumption for the moments of the energy dissipation rate is valid within the transition range to dissipation introduced by Castaing et al. (Physica D 46, 177 (1990)). The multifractal spectral functions appear to be universal well within the error margins and exhibit some as yet undiscussed features. Furthermore, this universality is also present in the neither homogeneous nor isotropic flows in the wake very close to a cylinder or the off-centre region of a free jet.

1 Introduction

The concept of an iterated distributive process behind turbulence, commonly called the “energy cascade”, is much older than fractal geometry. It might be said that multifractal theory has its roots, partially, in turbulence research. Of course, by being finite the statistics of the energy cascade do not have to be multifractal, and even if it may be conceived to be so, the expedience of a multifractal description is far from obvious. For several decades one incentive for turbulence research has been the idea of a universal behaviour in small scale intermittency, but it has not been possible to bring this universality to light in the regime of length scales which is accessible to a multifractal model. These questions were addressed in pioneering works by Meneveau and Sreenivasan, and it became clear then that multifractal geometry is an issue in the long process of understanding turbulence.

This paper is devoted to an extensive examination of experimental data in a multifractal framework. We shall not venture to relate our experimental findings to multifractal models of turbulence here, however, we vouch for the multifractality of turbulence.

The central quantity of our investigation, as introduced by Obukhov and Kolmogorov in 1962, is the energy dissipation averaged over a volume of size $r$. In
the case of locally homogeneous and isotropic turbulence this is approximated by

$$\epsilon_r(x) \propto \frac{1}{r} \int_{x-r/2}^{x+r/2} \left( \frac{\partial u}{\partial x'} \right)^2 dx', \quad (1)$$

where $u(x)$ denotes the longitudinal velocity component at point $x$ along a 1-D cut through the energy dissipation field (Figure 1).

We define the multifractal measure $\mu_i$ on a grid of size $r$ to be $\mu_i = r \epsilon_r(ir)$. The $q$-th order moment of $\mu$ is defined as:

$$S_q(r) = \sum_i \mu_i^q \propto r^{\tau(q)}, \quad (2)$$

where the sum is taken over all boxes of some $r$-grid and $\tau(q)$ is the multifractal scaling function.

To derive these quantities from the data, we employ an improved multifractal box counting algorithm recently suggested. We sketch this algorithm in section two. Section three concerns the details of the experimental data and the particular aspects of the selection of the proper scaling range. Section four presents the results obtained for locally isotropic flow conditions indicating a universal multifractal behaviour in the range defined by the transition from the inertial range to the dissipative limit. These universal features are also recovered in inhomogeneous and not fully developed turbulence, as shown in the concluding section.

![Figure 1: The energy dissipation time series (equivalent to a 1-D cut along the mean flow) for experiment J1. Units on the x-axis are in terms of the Kolmogorov microscale $\eta$.](image)

2 An Improved Multifractal Box-Counting Algorithm

The necessity of taking great care when utilising fixed-size box counting algorithms has frequently been expressed. However, here we deem fixed-size algorithms to
be the method of choice for two reasons: their shortcomings are well understood and in some cases be balanced, and they are closer to the theoretical concepts of turbulence than their fixed-mass counterparts in that they do not mix length scales. We also refrain from the direct evaluation of the \( f(\alpha) \) curve, because we find artefacts are easier to trace in the standard approach. The information content of both methods is equivalent since they are based on essentially the same counting procedure.

We want to outline in brief some of the improvements introduced to the method of moments. The prime source of errors was identified as the finite size artefacts due to ill-fitting grids. To reduce the influence of these “clipping errors” we replace in eq. (2) the local measure \( \mu_i \) by a local average \( \langle \mu \rangle_i \) and define

\[
< \mu >_i = \mu_i \prod_{j=1}^{n} \Theta(\mu(x_i + \delta_j)) \quad \text{for} \quad q \geq 0 \tag{3}
\]

\[
< \mu >_i = \left( \prod_{j=1}^{n} \mu(x_i + \delta_j) \right) \frac{1}{n} \quad \text{for} \quad q < 0 , \tag{4}
\]

where \( \{ \delta_j \} \) is a set of displacements smaller than \( r \).

Secondly, we introduce the concept of wandering intercepts which describe the deviations from linear scaling of the partition function. For a given \( q \) we define

\[
C(q, r) = \log S_q(r) - \hat{\tau}(q) \log r , \tag{5}
\]

where \( \hat{\tau}(q) \) is an estimate of \( \tau(q) \), e.g. the result of a least squares line fit to \( \log S_q(\log r) \). The \( C(q, r) \) can be thought of as a “lacunarity function”. Note that the concept of Extended Self-Similarity introduced by Benzi is a special case of eq. (5). Our development relies on the assumption that the \( C(q, r) \) does not behave too erratically in \( q \) and can ideally be factorised into functions of \( r \) and \( q \) only. Although we shall not make detailed assumptions about the \( C(q, r) \) in general, a lack of coherence with respect to \( q \) at a point \( q_0 \) betrays the presence of a phase transitional behaviour of the multifractal. For these purposes we employ plots of \( C(q_1, r_1) \) against \( C(q_2, r_1) \) for \( q_1 < q_0 < q_2 \) and expect that a lack of correlation between the fluctuations in \( r \) of the wandering intercepts appears as a scattered plot.

The greatest obstacle in the multifractal analysis of turbulence data (as with any other physical data) is the determination of the proper scaling range. For our findings it is characteristic that the multifractal scaling assumption applies only to the transitional range to dissipation, which can loosely be defined by the lengthscale \( \eta \) where the local Reynolds number becomes of the order of unity: \( \frac{\bar{u} \eta}{\nu} = 1 \). In most cases, the lower bound is dictated by the detector resolution, which usually exceeds \( \eta \) quite considerably for turbulent flows of a reasonable Reynolds number. The upper bound on the scaling range is much more indistinct, since typically one can only discern a sweeping transition to a power law behaviour of \( S_q(r) \), which in turn sometimes extends over less than one order of magnitude. Although it is tempting to expand the scaling range both to acquire a better statistical basis and to extend
the validity of the analysis, we handle the problem quite restrictively. Since the essence of numerical multifractal analysis is extracting to a high precision minor differences in interrelated scaling laws, we deem our approach justified.

3 The Statistical Properties of Experimental Data

The turbulence velocity data were obtained in three series of experiments of flows in air. Series J was a free jet through a nozzle of $d = 8$ mm diameter and an output velocity of 40 m/s. The estimated $R_\lambda$ in the centre was 210. The jet was scanned axially and radially at an axial distance of $60d = 48$ cm. The time series had a size of $1.25 \times 10^7$ data points. Series CF was a wind tunnel experiment with a cylinder of diameter $D = 5$ cm. The turbulent wake was examined at a distance of $32d = 160$ cm for a number of velocities, yielding $R_\lambda$ from 200 to 520. The size of the time series was $1.25 \times 10^6$. Series CV consisted of a variation of the location of the probe in the wake field of the cylinder at $21 \text{m/s}$ velocity, ranging from $4d = 20\text{cm}$ to $40d = 200\text{cm}$. Again, $1.25 \times 10^7$ points were sampled. The equipment was a DANTEC anemometer StreamLine 90CN10 with single wire probes 55P01 (sensitive length 1.25mm) for the jet measurements and x-wire probes 55P61 (sensitive length 1mm) for the cylinder measurements. The use of the latter probe facilitated a more accurate measurement of the longitudinal velocity component.

An extensive analysis was performed for all time series. This included the calculation of characteristic mean-field quantities (see table 1) and the generation of probability density functions (PDFs) of velocity increments. The changing form of the PDFs as a function of the spatial resolution $r$ was quantified by a parameter $\Lambda^2(r)$ according to Castaing [18]. The power law behaviour of the resulting function $\Lambda^2(r)$ locates the transitional range between the classical inertial range and the dissipative limit [19] (see figure 2). In the following we restrict our investigations of the scaling behaviour of $<e^q_0>$, respectively $S_q(r)$, to this transitional range. Critical evidence for the coincidence of these scaling ranges was deduced from our experimental data. The chosen ranges for each example are given in tables 1 and 2.

It is one of the traits of box-counting, that it delivers a result under almost all circumstances. However, it is sometimes open to arbitrariness to find out which results comprise significant information about the multifractal. As it turned out, statistical resolution is a severe issue for moments as small as 3 for about 1 million data points, and 10 million points improve the situation only marginally. It has been shown [4], that $f(\alpha)$ will assume negative values due to the 1-D nature of the data set as opposed to the three dimensional energy field, which infers that the main contribution to a moment that corresponds to a negative $f(\alpha)$ comes from very few singularities. This being the case, it comes as no surprise to find that the correlation plots for $q_1, q_2 > 5$ are extremely well correlated. In general, we attribute little credibility to any moment larger than 4. For negative $q$, noise and digitalisation artefacts put an end to the reliability of the results at sometimes quite moderate values of $q \approx -1$. A special case are the inhomogeneous examples, where due to the background of the intertwined laminar flow no multifractal fine structure can be resolved with negative moments.
4 Multifractal Spectra in the Locally Isotropic Case

In the following, four examples of classical fully developed turbulence are investigated in the centre of the flows, where local isotropy is known to hold for small scales. The experiments were chosen to cover a range of $R_{\lambda}$ from 190 to 520. Table 1 gives the relevant details. An example of an energy time series is given in figure 1.

In the context of this paper, we want to establish experiment J1 as the standard for a simple reason: the detector resolution and $\eta$ are quite close, so that one can assume that the fine structure of the original velocity signal is well resolved. In fact, a direct comparison with experiment J2 shows that this point is crucial for the calculation of extreme moments. Figures 3.1 and 3.2 show the $\tau(q)$ and $f(\alpha)$ curve for the transitional range.

Figure 4 shows a superposition of the respective $f(\alpha)$ spectra for all experiments of table 1. The errorbars are significantly larger than the deviations between the curves. Given these results, we have to assume universality of the spectra of singularities.

Although it is notoriously difficult to establish negative values of $f$ reliably, we have confidence in the results we have given. A self consistent analysis of the $\tau(q)$ curve showed that statistics are still sound for $q \approx 3$, whereas the intersection with the $\alpha$-axis occurs for $q = 3.2$ c.f. figure 3.

There is a couple of interesting, but less obvious results. Firstly, from correlation plots of $C(q, r)$ one is led to deduce the presence of a phase transition at $q_0 = 1$, which does not express itself in a kink in $f(\alpha)$ since both tangents necessarily have slope 1. We interpret this as the presence of two phases: the highly intermittent or turbulent phase which dominates the moments for $q > 1$, and the "laminar" phase which does so by the less pronounced singularities for moments with $q < 1$.

Secondly, if one sets a threshold for the analysis of the measure, one finds that for all (very small) thresholds the laminar branch of the multifractal spectrum
3.1: The Experiment J1 serves as a standard against which the others can be compared since the detector resolution is close to \( \eta \). As it turns out, resolution has an impact both on positive and negative moments. This figure shows the \( \tau(q) \) curve of this experiment.

3.2: The \( f(\alpha) \) curve corresponding to experiment J1, see also figure 7. This curve was obtained by a Legendre transform of \( \tau(q) \). Errors are amplified due to the great leverage of \( q \).

disappears. The conclusion is, that all energy dissipation must be concentrated on the set which corresponds to the Hölder exponent \( \alpha(q = 1) \). Whilst this is necessarily so for mathematical multifractals, it is remarkable for a physical, hence finite one. In other words, the steepest singularities feed on most of the energy and dissipation attracts further energy to dissipate.
Inherent to most theories pertinent to small scale turbulence is the assumption of isotropy and homogeneity of the turbulent velocity field. A question of practical importance is if similar properties can be found where these conditions are violated. To approach this we investigated a free jet with a radial offset of 9 cm in 48 cm axial distance from the nozzle, i.e. in the outer layers where laminar phases are mixed in. As a second inhomogeneous situation the wake of a cylinder was examined in 4 diameters distance, i.e. under conditions of not fully developed turbulence. Figure 5 shows the energy dissipation field for these cases. In comparison to figure 1 we clearly see intermittent burst of turbulent activity in an otherwise smooth flow.

The analysis is to some extent different from the homogeneous case. Firstly, most quantities like $R_\lambda$ and $\eta$ lose their physical meaning. Secondly, the $\Lambda^2$ plots become distorted by the laminar phases, so that the analysis is open to some greater degree of arbitrariness than in the homogeneous case. Table 2 gives the approximate values.

From the spectra in Figure 6 it could be deduced that the small scale properties of the energy dissipation field are to some degree independet of the prerequisites of global uniformity and isotropy. We cannot state yet to which extent these conditions do not hold within a turbulent burst, but we accept the possibility that the multifractal aspects of the energy dissipation field are in fact universal.

Notice that the ‘laminar’ branch of the spectrum is not accessible due to the presence of a laminar background. Thus the right hand part of the spectrum collapses to a point at the maximum. The peak at unity indicates that the energy

| flow          | free jet | free jet | wake of a cylinder | wake of a cylinder |
|---------------|----------|----------|--------------------|--------------------|
| reference     | J1       | J2       | CF1                | CF2                |
| position of probe | 125 diam. | 60 diam. | 32 diam.           | 32 diam.           |
| number of points | $1.25 \times 10^7$ | $1.25 \times 10^7$ | $1.25 \times 10^6$ | $1.25 \times 10^6$ |
| mean velocity  | 2.26 m/s | 4.2 m/s  | 22.2 m/s           | 4.6 m/s           |
| sampling rate  | 8 kHz    | 50 kHz   | 120 kHz            | 20 kHz            |
| Kolmogorov microscale $\eta$ | 0.25 mm  | 0.1 mm   | 0.07 mm            | 0.22 mm            |
| Taylor microscale $\lambda$ | $27 \eta$ | $28 \eta$ | $44 \eta$         | $29 \eta$         |
| Taylor-Reynolds number $R_\lambda$ | $190$ | $210$ | $520$ | $220$ |
| transitional range | 4 .. 113 $\eta$ | 8 .. 83 $\eta$ | 26 .. 79 $\eta$ | 8 .. 85 $\eta$ |
| detector resolution | 4 $\eta$ | 10 $\eta$ | 14 $\eta$ | 5 $\eta$ |
| permissible moments | -2 .. 5 | -0.2 .. 4 | -0.6 .. 4 | -4 .. 4 |

Table 1 Summary of experimental conditions and evaluation parameters for the locally isotropic flows.

5 Multifractal Spectra in the Inhomogeneous Case
Figure 4: The superposition of the $f(\alpha)$ curves of all locally isotropic experiments ($J1, J2, CF1, CF2$) are in reasonable agreement. The solid line is the standard $J1$, the errorbars given for $CF1$ are representative for all experiments with 'poor' statistics.

5.1: The energy time series of experiment $J3$ in the off-centre region of a free jet. The width was chosen to match figure 4 in terms of $\eta$. Laminar phases are mixed with turbulent bursts which were transported out of the central region of the jet.

5.2: The energy time series of experiment $CV1$ in the wake close behind the cylinder.
flow | free jet | wake
---|---|---
reference | J3 | CV1
position of probe | 60 diam. | 4 diam.
11.25 diam. off-centre
number of points | $1.25 \times 10^7$ | $1.25 \times 10^7$
mean velocity | 0.43 m/s | 21.2 m/s
sampling rate | 6 kHz | 200 kHz
Kolmogorov microscale $\eta$ | 0.17 mm | 0.035 mm
Taylor microscale $\lambda$ | 11 $\eta$ | 53 $\eta$
Taylor-Reynolds number $R_\lambda$ | 34 | 720
transitional range | 4 .. 25 $\eta$ | 18 .. 129 $\eta$
detector resolution | 6 $\eta$ | 28 $\eta$
permissible moments | 0 .. 4 | 0 .. 4

Table 2: Summary of experimental conditions and evaluation parameters for the inhomogeneous flows.

Figure 6: In the inhomogeneous case, the right branch of $f(\alpha)$ is not accessible. The solid line again is the standard J1, errorbars are given for J3. Statistics are quite poor for the jet measurement due to the great dilatation of the turbulent bursts.
dissipation is still space filling locally, despite its sparse appearance.

6 Conclusion

We have reported on a correspondence in small scale energy dissipation spectra in both locally isotropic and inhomogeneous turbulence. The results are plausible if one takes into consideration that firstly the analysis is restricted to small scales of less than $100\eta$, and secondly, by the observation that the inhomogeneous turbulence in the boundary layer of the free jet can be thought of as a composition of zones of turbulence transported out of the centre and inserted from the surrounding laminar flow. Less accessible to intuition are our findings for the non-developed turbulent wake.

To set our findings in context with previous work on small scale turbulence, we want to stress once more that our work focuses on the transitional regime, a range of length scales between the inertial range and the dissipative limit. Here we expect that the dissipation plays a role of increasing importance. Traditionally, scaling behaviour is investigated on larger scales in the inertial range, where the dissipation should be of minor influence. Some recent work has shed new light on the turbulent cascade in the inertial range. Our results suggest that in the neighbouring regime the statistics of the energy dissipation field $\epsilon_r$ become universally multifractal. The challenge remains to match the various regimes in which turbulence is described to finally arrive at a truly comprehensive view.

7 References

1. B. B. Mandelbrot. *J. Fluid. Mech.*, 1974.
2. U. Frisch. *Turbulence: The Legacy of AN Kolmogorov*. Cambridge University Press, Cambridge, 1995.
3. C. Meneveau and K. R. Sreenivasan. The multifractal spectrum of the dissipation field in turbulent flows. *Nucl. Phys. B*, 1987.
4. K. R. Sreenivasan. Fractals and multifractals in fluid turbulence. *Annu. Rev. Fluid. Mech.*, 23:539–600, 1991.
5. K. R. Sreenivasan and R. A. Antonia. The phenomenology of small-scale turbulence. *Annu. Rev. Fluid. Mech.*, 29:435–472, 1997.
6. A. M. Obukhov. *J. Fluid. Mech.*, 1962.
7. A. N. Kolmogorov. *J. Fluid. Mech.*, 1962.
8. M. Alber and J. Peinke. Improved multifractal box-counting algorithm, phase transitions and negative dimensions. *Phys. Rev. E*, 57, 1998.
9. W. E. Caswell and J. A. Yorke. Invisible errors in dimension estimates. In G. Mayer-Kress, editor, *Dimensions and Entropies in Chaotic Systems*. Springer, Berlin, 1986.
10. S. Borgani, G. Murante, A. Provenzale, and R. Valdarinini. Multifractal analysis of the galaxy distribution: reliability of results from finite data sets. *Phys. Rev. E*, 47:3879, 1993.
11. R. Pastor-Satorras and R. H. Riedi. Numerical estimates of the generalized dimensions of the hénon attractor for negative $q$. *J. Phys. A*, 29:L391, 1996.
12. R. Badii and G. Broggi. Measurement of the dimension spectrum $f(\alpha)$: fixed-mass approach. *Phys. Lett. A*, 1988.

13. A. Chhabra and R. V. Jensen. Direct determination of the $f(\alpha)$ singularity spectrum. *Phys. Rev. Lett.*, 62:1327, 1989.

14. C. Meneveau and K. R. Sreenivasan. The multifractal nature of turbulent energy dissipation. *J. Fluid Mech.*, 1991.

15. C. D. Cutler. Some results on the behaviour and estimation of the fractal dimensions of distributions on attractors. *J. Stat. Phys.*, 62:651, 1991.

16. C. D. Cutler. A review of the theory and estimation of fractal dimension. In H. Tong, editor, *Nonlinear Time Series and Chaos, Vol. I,Dimension Estimation and Models*. World Scientific, Singapore, 1993.

17. R. Benzi, S. Ciliberto, C. Baudet, and G. R. Chavarria. On the scaling of three-dimensional homogeneous and isotropic turbulence. *Physica D*, 80:385, 1995.

18. B. Castaing, Y. Gagne, and E. J. Hopfinger. Velocity probability density functions of high reynolds number turbulence. *Physica D*, 1990.

19. B. Chabaud, A. Naert, J. Peinke, F. Chill`a, B. Castaing, and B. Hébral. *Phys. Rev. Lett.*, 1994.

20. J. Griesemann, M. Greiner, and P. Lipa. Wavelet cascades. preprint.

21. M. Nelkin and G. Stolovitzky. *Phys. Rev. E*, 1996.

22. A. Naert. Fokker-planck equation for the energy cascade in turbulence. *Phys. Rev. E*, 1997.