N = 2 Supersymmetric Sigma Models and D-branes

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Abstract

We study D-branes of N = 2 supersymmetric sigma models. Supersymmetric nonlinear sigma models with 2-dimensional target space have D0,D1,D2-branes, which are realized as A-,B-type supersymmetric boundary conditions on the worldsheet. When we embed the models in the string theory, the Kähler potential is restricted and leads to a 2-dim black hole metric with a dilaton background. The D-branes in this model are susy cycles and consistent with the analysis of conjugacy classes. The generalized metrics with U(n) isometry is proposed and dynamics on them are realized by linear sigma models. We investigate D-branes of the linear sigma models and compare the results with those in the nonlinear sigma models.

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1 Introduction

Superstring theories have D-branes in open string sectors. However in general it is difficult to analyze properties of the branes on curved backgrounds. The CFT is a powerful tool to describe D-branes exactly, but not convenient for understanding D-branes geometrically. We want to know geometrical properties of the branes directly. For this purpose, we study $N = 2$ supersymmetric sigma models in this work.

We investigate nonlinear sigma models with 2-dimensional target space as an example of curved space in the first part of this paper. By considering A-,B-type supersymmetric boundary conditions on the worldsheet, we get D1-branes for the A-type boundary and D0,D2-branes for the B-type boundary. We can construct boundary interaction terms added to the action. When the theory is required to be conformal invariant the Kähler potential leads to a 2-dim black hole metric with a dilaton background. Then the model is reduced to one of the Wess-Zumino-Witten models and the target space has the form of a cigar or a trumpet, on which we give the geometric description of the D-branes. The result is consistent with that in the analysis of conjugacy classes [1, 2, 3, 4, 5].

The 2-dim black hole metric can be generalized to the Kähler metric with $U(n)$ isometry [6], which yields $2n$-dimensional spacetime. D-branes are given by its submanifold. Moreover an effect of a dilaton on a point-like object is discussed.

The discussion of the nonlinear sigma model is a classical level analysis. Generally the sigma model action is modified by quantum corrections. In Ref.[7] the linear sigma model equivalent to 2-dim black hole at quantum level is proposed. Its generalization to a sigma model with the $2n$-dim target space is thought to realize $N = 2$ superconformal models with $c/3 > 1$ in the IR limit. Therefore we consider such a linear sigma model and study its supersymmetric boundary conditions. As a result we get D-branes consistent with those in the 2-dim case.

The organization of this paper is as follows: In section 2, we study supersymmetric nonlinear sigma models with 2-dim target space and their boundary conditions. Especially the model with the 2-dim black hole metric is investigated in detail. In section 3, we analyze the generalized Kähler potential, which leads to $2n$-dim metric and consider the relation between its submanifold and D-branes. In section 4 we consider the linear sigma model realizing dynamics on the above $2n$-dim target space and examine its D-branes. In section 5, we give summaries and conclusions. In an appendix, we discuss a nonlinear sigma model with F-term.
2 Nonlinear Sigma Model and 2-dim Black Hole

In this section we study \( N = (2, 2) \) supersymmetric nonlinear sigma models on curved backgrounds. The action for a world sheet \( \Sigma \) without boundary is described

\[
S = \frac{k}{4\pi} \int_{\Sigma} d^2z \left[ \partial \bar{\partial} K \cdot (\partial_+ \phi \partial_- \bar{\phi} + \partial_- \phi \partial_+ \bar{\phi}) \\
- i \chi^+ \chi^- \left( \partial_+ \phi \cdot \frac{\partial^2 \bar{\phi} K}{\partial \bar{\partial} K} - \partial_+ \bar{\phi} \cdot \frac{\partial^2 \phi K}{\partial \partial K} \right) - i \lambda^+ \lambda^- \left( \partial_- \phi \cdot \frac{\partial^2 \bar{\phi} K}{\partial \bar{\partial} K} - \partial_- \bar{\phi} \cdot \frac{\partial^2 \phi K}{\partial \partial K} \right) \\
+ 2 \lambda^+ \lambda^- \chi^+ \chi^- \left( \frac{\partial^2 \bar{\phi} K}{(\partial \bar{\partial} K)^2} \right) + i(\lambda^+ \partial_- \lambda^- + \lambda^- \partial_- \lambda^+ + \chi^+ \partial_+ \chi^- + \chi^- \partial_+ \chi^+) \\
+ \frac{1}{2} \partial \bar{\partial} K \cdot F F - F \frac{\partial^2 \bar{\phi} K}{\partial \bar{\partial} K} \cdot \lambda^+ \chi^+ + F \frac{\partial^2 \phi K}{\partial \partial K} \cdot \lambda^- \chi^- \right],
\]

where \( k - 2 \) is a level and \( K(\phi, \bar{\phi}) \) is a Kähler potential. We use the following conventions:

\[
z = \xi_1 + i\xi_2, \quad \bar{z} = \xi_1 - i\xi_2, \quad d\xi_1 d\xi_2 = \frac{i}{2} dz d\bar{z} = d^2z,
\]

\[
\partial_+ = \frac{\partial}{\partial z}, \quad \partial_- = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2), \quad \partial = \frac{\partial}{\partial \phi}, \quad \bar{\partial} = \frac{\partial}{\partial \bar{\phi}}.
\]

In the case of \( K(\phi, \bar{\phi}) = \phi \bar{\phi} \), this model equals that on the flat background. Setting

\[
\psi_+ = \sqrt{2}(\partial \bar{\partial} K)^{-1/2} \lambda^+,
\quad \tilde{\psi}_+ = \sqrt{2}(\partial \bar{\partial} K)^{-1/2} \lambda^-,
\quad \psi_- = \sqrt{2}(\partial \bar{\partial} K)^{-1/2} \chi^+,
\quad \tilde{\psi}_- = \sqrt{2}(\partial \bar{\partial} K)^{-1/2} \chi^-,
\]

(2)

reduces the action to a canonical \( N = (2, 2) \) nonlinear supersymmetric sigma model. Note that this redefinition of the fermions breaks down when \( \partial \bar{\partial} K \) blows up. Such a configuration of \( \phi, \bar{\phi} \) corresponds to a coordinate or curvature singularity in the target space.
The action (11) is invariant under the following $N = 2$ supersymmetry transformations:

\[ \delta \phi = \sqrt{2}(\partial \bar{\partial} K)^{-1/2}(\epsilon_+ \chi^+ - \epsilon_- \chi^-), \quad \delta \bar{\phi} = \sqrt{2}(\partial \bar{\partial} K)^{-1/2}(-\bar{\epsilon}_+ \chi^- + \bar{\epsilon}_- \chi^+), \]
\[ \delta F = -2\sqrt{2}i \epsilon_+ \partial_- ((\partial \bar{\partial} K)^{-1/2} \chi^+) - 2\sqrt{2}i \bar{\epsilon}_- \partial_+ ((\partial \bar{\partial} K)^{-1/2} \chi^-), \]
\[ \delta \bar{F} = -2\sqrt{2}i \epsilon_+ \partial_- ((\partial \bar{\partial} K)^{-1/2} \chi^-) - 2\sqrt{2}i \bar{\epsilon}_- \partial_+ ((\partial \bar{\partial} K)^{-1/2} \chi^+), \]
\[ \delta \lambda^+ = \sqrt{2}(\partial \bar{\partial} K)^{1/2} \epsilon_- \left( i \partial_+ \phi - \frac{\partial^2 \bar{\partial} K}{2(\partial \bar{\partial} K)^2} \cdot \lambda^+ \right) \]
\[ + \frac{1}{\sqrt{2}}(\partial \bar{\partial} K)^{1/2} \left[ \epsilon_+ \left( F + \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \chi^+ \right) - \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \epsilon_+ \chi^+ \lambda^+ \right], \]
\[ \delta \chi^+ = \sqrt{2}(\partial \bar{\partial} K)^{1/2} \epsilon_+ \left( -i \partial_- \phi + \frac{\partial^2 \bar{\partial} K}{2(\partial \bar{\partial} K)^2} \cdot \chi^+ \right) \]
\[ + \frac{1}{\sqrt{2}}(\partial \bar{\partial} K)^{1/2} \left[ \epsilon_- \left( F + \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \chi^+ \right) - \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \epsilon_- \chi^+ \lambda^+ \right], \]
\[ \delta \lambda^- = \sqrt{2}(\partial \bar{\partial} K)^{1/2} \epsilon_- \left( -i \partial_+ \bar{\phi} - \frac{\partial^2 \bar{\partial} K}{2(\partial \bar{\partial} K)^2} \cdot \lambda^- \right) \]
\[ + \frac{1}{\sqrt{2}}(\partial \bar{\partial} K)^{1/2} \left[ \bar{\epsilon}_- \left( \bar{F} - \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \chi^- \right) + \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \epsilon_- \chi^- \lambda^- \right], \]
\[ \delta \chi^- = \sqrt{2}(\partial \bar{\partial} K)^{1/2} \epsilon_+ \left( i \partial_- \bar{\phi} + \frac{\partial^2 \bar{\partial} K}{2(\partial \bar{\partial} K)^2} \cdot \chi^- \right) \]
\[ + \frac{1}{\sqrt{2}}(\partial \bar{\partial} K)^{1/2} \left[ \bar{\epsilon}_- \left( \bar{F} - \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \chi^- \right) + \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \cdot \epsilon_- \chi^- \lambda^- \right]. \tag{3} \]

The singular point is invariant under (3) and thus forms a fixed point of the supersymmetry. This model also has $U(1)$ R-symmetry, whose charge is assigned $+1$ for $\chi^+$ and $\lambda^-$, and $-1$ for $\chi^-$ and $\lambda^+$.

If $\Sigma$ has a boundary on $z = \bar{z}$ ($\xi_2 = 0$), one should consider the model on the upper half plane $\Sigma = \{ (\xi_1, \xi_2) \mid \xi_2 > 0 \} = \{ z \mid \text{Im} z > 0 \}$. Under the above susy transformation, we can calculate the variation of $S$ as

\[ \delta S = \frac{k}{4\pi} \frac{1}{2} \int_{\partial \Sigma} d\xi_1 \left[ i\sqrt{2}(\partial \bar{\partial} K)^{1/2} \left( \bar{\epsilon}_- \chi^- \partial_+ \phi + \bar{\epsilon}_- \lambda^- \partial_- \phi - \bar{\epsilon}_+ \chi^+ \partial_+ \bar{\phi} - \bar{\epsilon}_- \chi^+ \partial_- \bar{\phi} \right) \right] \]
\[ + \frac{1}{\sqrt{2}}(\partial \bar{\partial} K)^{1/2} \left[ \epsilon_+ \left( F + 2 \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \chi^+ \phi \right) \right] \lambda^- \]
\[ + \frac{1}{\sqrt{2}}(\partial \bar{\partial} K)^{1/2} \left[ \bar{\epsilon}_- \left( \bar{F} - 2 \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \chi^- \phi \right) \right] \lambda^+ \]
\[ + \epsilon_+ \left( F - 2 \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \chi^- \phi \right) \chi^+ - \bar{\epsilon}_- \left( \bar{F} - 2 \frac{\partial^2 \bar{\partial} K}{(\partial \bar{\partial} K)^2} \chi^- \phi \right) \chi^- \right]. \tag{4} \]

Here we note that all the terms except those in the first line vanish due to the equations of motion of $F , \bar{F}$. If the $\partial \bar{\partial} K$ does not become zero nor blow up, this leads us to the discussion of the supersymmetric sigma models. There exist two types of boundary conditions depending on how to mix left moving and right moving parts of the supersymmetry $\delta \xi_1$ $\delta \xi_2$ $\delta \xi_3$ $\delta \xi_4$.
• A-type boundary $\epsilon_+ = \bar{\epsilon}_- = \epsilon, \quad \epsilon_- = \bar{\epsilon}_+ = \bar{\epsilon};$

an example of the boundary condition is

$$\lambda^- \pm \chi^+ = \lambda^+ \pm \chi^- = 0, \quad \partial_- \phi \pm \partial_+ \bar{\phi} = 0, \quad \partial_- \bar{\phi} \pm \partial_+ \phi = 0,$$

• B-type boundary $\epsilon_+ = -\epsilon_- = \epsilon, \quad \bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon};$

$$\lambda^- \pm \chi^- = \lambda^+ \pm \chi^+ = 0, \quad \partial_- \phi \pm \partial_+ \bar{\phi} = 0, \quad \partial_- \bar{\phi} \pm \partial_+ \phi = 0. \quad (5)$$

For the case of B-type $\epsilon_+ = -\epsilon_- \quad (\bar{\epsilon}_+ = -\bar{\epsilon}_-)$, we can introduce the following interaction terms on the boundary

$$S_{bdy} = -\frac{k}{4\pi} \frac{1}{2} \int \xi_1 (\chi^+ \lambda^- + \chi^- \lambda^+) \, ,$$

so that the supersymmetry variation vanishes $\delta (S + S_{bdy}) = 0$. The ordinary variation of the fields on the boundary is proportional to

$$\delta \phi \left( - \partial \bar{\partial} K \cdot \partial_2 \bar{\phi} + \frac{\partial^2 \bar{\partial} K}{\partial \partial K} (\chi^+ \chi^- - \lambda^+ \lambda^-) \right)$$

$$+ \delta \bar{\phi} \left( - \partial \bar{\partial} K \cdot \partial_2 \phi - \frac{\partial \bar{\partial}^2 K}{\partial \partial K} (\chi^+ \chi^- - \lambda^+ \lambda^-) \right)$$

$$+ (\delta \chi^+ - \delta \lambda^+) (\chi^- + \lambda^-) + (\delta \chi^- - \delta \lambda^-) (\chi^+ + \lambda^+),$$

which yields boundary conditions on $z = \bar{z}$;

$$\chi^+ \pm \lambda^+ = \chi^- \pm \lambda^- = 0, \quad \delta \phi = \delta \bar{\phi} = 0 \text{ (Dirichlet)} \quad \text{or} \quad \partial_2 \phi = \partial_2 \bar{\phi} = 0 \text{ (Neumann)}.$$

These boundary conditions are consistent with Eqs.(5). The former condition for the bosonic part means the existence of D0-brane and the latter means that of D2-brane.

We can choose Kähler metrics arbitrarily as a sigma model, but the geometry is restricted when we embed the model in the superstring theories. Here we require the theory to be conformal invariant, namely $\beta^G_{\mu \nu} = 0 \quad [6, 12]$. The metric, Christoffel connections and Ricci tensors are given as:

$$ds^2 = \partial \bar{\partial} K \cdot d\phi d\bar{\phi}, \quad K = K(\phi, \bar{\phi}),$$

$$\Gamma^\phi_{\phi \phi} = \frac{\partial^2 \bar{\partial} K}{\partial \partial K}, \quad \Gamma^{\bar{\phi}}_{\phi \bar{\phi}} = \frac{\partial \bar{\partial}^2 K}{\partial \partial K}, \quad \text{ (other connections)} = 0,$$

$$R_{\phi \phi} = - \partial \bar{\partial} \log(\partial \bar{\partial} K), \quad R_{\phi \bar{\phi}} = R_{\bar{\phi} \phi} = 0.$$ 

The condition for the $\beta$-function to vanish is

$$R_{\mu \nu} = - 2 \nabla_\mu \nabla_\nu \Phi,$$
Table 1: Classification of the metric. The $\phi$ and $\bar{\phi}$ are rescaled appropriately. The line element $ds^2$ is calculated by using the parameterization \[1\] or \[12\]. Each model is equivalent to a WZW model.

| $A, D$ | metric | $ds^2$ | WZW model |
|--------|--------|--------|------------|
| (I) $A > 0$, $D > 0$ | $\partial\bar{\partial}K = \frac{1}{\phi+1}$ | $dr^2 + \tanh^2 r d\theta^2$ | $\text{SL}(2;\mathbb{R})/U(1)_A$ |
| (II) $A > 0$, $D < 0$ | $\partial\bar{\partial}K = \frac{1}{\phi-1}$ | $dr^2 + \coth^2 r d\bar{\theta}^2$ | $\text{SL}(2;\mathbb{R})/U(1)_V$ |
| (III) $A < 0$, $D > 0$ | $\partial\bar{\partial}K = \frac{1}{-\phi+1}$ | $dr^2 + \tan^2 r d\theta^2$ | $\text{SU}(2)/U(1)$ |

where $\Phi$ is a dilaton field, or

$$2\partial\bar{\partial}\Phi = \partial\bar{\partial}\log(\partial\bar{\partial}K),$$

(7)

$$\partial^2\Phi - \Gamma_{\phi\bar{\phi}}^\phi \Phi = 0,$$

(8)

$$\bar{\partial}^2\Phi - \Gamma_{\phi\bar{\phi}}^{\bar{\phi}} \Phi = 0.$$  

(9)

From \(\square\) we can choose $2\Phi = \log(\partial\bar{\partial}K)$. With this relation Eqs.\(\Box\) and \(\Box\) are rewritten into $\partial^2 e^{-2\Phi} = 0$, $\bar{\partial}^2 e^{-2\Phi} = 0$. Thus the following relation is obtained

$$e^{-2\Phi} = (\partial\bar{\partial}K)^{-1} = A\phi\bar{\phi} + B\phi + C\bar{\phi} + D, \quad A, B, C, D : \text{constant}.$$

If $A \neq 0$, we can set $B = C = 0$ after a shift of $\phi, \bar{\phi}$ and then get

$$\partial\bar{\partial}K = \frac{1}{A\phi\bar{\phi} + D}.$$

In the case of $D = 0$ the model is reduced to trivial flat space geometry after an appropriate transformation of $\phi, \bar{\phi}$. So we look into the case of $AD \neq 0$. We classify the metrics depending on signs of $A$ and $D$ in Table \ref{table1}. When $K(\phi, \bar{\phi})$ is represented by the dilogarithm function

$$K(\phi, \bar{\phi}) = -\text{Li}_2(\pm\phi\bar{\phi}) = -\sum_{k=1}^{\infty} \frac{(\pm\phi\bar{\phi})^k}{k^2},$$

this model has target space geometry with a 2-dim black hole metric \[13\] \[14\]

$$ds^2 = \frac{d\phi d\bar{\phi}}{\phi \bar{\phi} \pm 1}.$$  

The upper sign corresponds to the case (I) in Table \ref{table1}. For this case if we set $\phi = u$, $\bar{\phi} = -v$ the action \[11\] leads to

$$S = \frac{k}{4\pi} \int d^2z \left[ -\frac{\partial u\bar{\partial}v + \partial v\bar{\partial}u}{1 - uv} + i\chi^+ \chi^- \frac{u\partial v - v\partial u}{1 - uv} + i\lambda^+ \lambda^- \frac{u\partial v - v\partial u}{1 - uv} \right.$$  

$$-\frac{2}{1 - uv} \chi^+ \chi^- \lambda^+ \lambda^- + i \left( \lambda^+ \partial_+ \lambda^- + \lambda^- \partial_- \lambda^+ + \chi^+ \partial_+ \chi^- + \chi^- \partial_+ \chi^+ \right),$$

(10)
after eliminating $F$, $\hat{F}$ by the equations of motion. This action is equal to one of the SL(2,R)/U(1) gauged Wess-Zumino-Witten models after gauge-fixing and known as axial model. On the other hand, the lower sign corresponds to the case (II). For this case the same action as (10) in which $λ^\pm$ are replaced with $λ^\mp$ and $(u, v)$ with $(a, b)$ is obtained and known as vector model. This model is achieved by using a twisted chiral superfield instead of a chiral superfield in constructing a nonlinear sigma model and setting $φ = a, \tilde{φ} = b$.

To investigate geometrical shapes of the D-branes in the target space for the A-type model, we parameterize

$$
\begin{pmatrix}
\ a \\
\ u \\
\ -v \\
\ b
\end{pmatrix} = \begin{pmatrix}
\ e^{i\tilde{θ}} \cosh r & e^{iθ} \sinh r \\
\ e^{-iθ} \sinh r & e^{-i\tilde{θ}} \cosh r
\end{pmatrix},
$$

where $ab + uv = 1$, $0 \leq θ < 2\pi$, $0 ≤ \tilde{θ} < 2\pi$ and $0 < r < ∞$. For the axial model the integrand in Eq.(11) is rewritten into the formula proportional to

$$
e [λ^- \partial_-(e^{iθ} \sinh r) - λ^+ \partial_+(e^{-iθ} \sinh r)] + \tilde{e} [\chi^- \partial_+(e^{iθ} \sinh r) - λ^+ \partial_-(e^{-iθ} \sinh r)]
= \frac{1}{2} \tilde{e} [(λ^+ e^{-iθ} - λ^+ e^{-iθ})(\cosh r \partial_1 r - \sinh r \partial_2 θ) + i(λ^+ e^{iθ} + λ^+ e^{-iθ})(\sinh r \partial_1 θ + \cosh r \partial_2 r)]
+ \frac{1}{2} \tilde{e} [i(λ^+ e^{iθ} + λ^+ e^{-iθ})(\sinh r \partial_1 r - \cosh r \partial_2 θ) + (λ^+ e^{iθ} - λ^+ e^{-iθ})(\cosh r \partial_1 r + \sinh r \partial_2 θ)].
$$

We can set the boundary conditions in various ways;

(i) $\chi^+ e^{-iθ} + λ^- e^{iθ} = χ^- e^{iθ} + λ^+ e^{-iθ} = 0$, $\partial_1 r = \partial_2 θ = 0$,
(ii) $\chi^+ e^{-iθ} = χ^- e^{iθ} = λ^+ e^{-iθ} = λ^- e^{iθ} = 0$, $\partial_1 θ = \partial_2 r = 0$,
(iii) $\chi^+ + λ^- = χ^- + λ^+ = 0$, $\partial_1 (\cos θ \sinh r) = \partial_2 (\sin θ \sinh r) = 0$,
(iv) $\chi^+ - λ^- = χ^- - λ^+ = 0$, $\partial_1 (\sin θ \sinh r) = \partial_2 (\cos θ \sinh r) = 0$,

and illustrate the D1-branes expressed by these in Fig.1. Similarly for the vector model we can set the boundary conditions;

(i) $\chi^+ e^{-iθ} - λ^- e^{iθ} = χ^- e^{iθ} - λ^+ e^{-iθ} = 0$, $\partial_1 r = \partial_2 \tilde{θ} = 0$,
(ii) $\chi^+ e^{-iθ} + λ^- e^{iθ} = χ^- e^{iθ} + λ^+ e^{-iθ} = 0$, $\partial_1 θ = \partial_2 r = 0$,
(iii) $\chi^+ + λ^- = χ^- + λ^+ = 0$, $\partial_1 (\cos \tilde{θ} \cosh r) = \partial_2 (\sin \tilde{θ} \cosh r) = 0$,
(iv) $\chi^+ + λ^+ = χ^- + λ^- = 0$, $\partial_1 (\sin \tilde{θ} \cosh r) = \partial_2 (\cos \tilde{θ} \cosh r) = 0$,

and illustrate them in Fig.2. These results are consistent with the semiclassical descrip-

Figure 1: The D-branes in the axial model. The target space metric of this model has the form of a cigar and the horizon at $r = 0$ (regular).
The target space metric of this model has the form of a trumpet and a singularity at \( r = 0 \). We set \( C = \cosh r \cos \tilde{\theta} \) for (iii) or \( C = \cosh r \sin \tilde{\theta} \) for (iv).

To investigate D-branes inside the horizon, \( r \) needs to be analytically continued. The model corresponds to the case (III) in Table I and leads to an SU(2)/U(1) WZW model. In this case we parameterize

\[
\begin{pmatrix}
a & u \\
-v & b
\end{pmatrix} = \begin{pmatrix}
e^{i\tilde{\theta}} \cos r & ie^{i\tilde{\theta}} \sin r \\
e^{-i\tilde{\theta}} \sin r & e^{-i\tilde{\theta}} \cos r
\end{pmatrix},
\]

where \( 0 \leq \theta, \tilde{\theta} < 2\pi \) and \( 0 < r < \pi/2 \). By setting \((\phi, \bar{\phi}) = (u, v)\) for the axial model or \((a, b)\) for the vector model, several boundary conditions are obtained by the same procedure as above (see Fig. 3).

In the above discussion, we neglect a dilaton. Its effect makes some of D-branes unstable and several D-branes in these figures disappear.

## 3 Generalized Metrics

In the previous section, we studied brane configurations in the 2-dimensional black hole. In this section, we study 2n-dimensional metrics with U(n) isometries, whose spacetimes are the generalization of the 2-dimensional black hole.
The set of holomorphic coordinates is given by \((u_1, u_2, \cdots, u_n)\) and Kähler potential of the space is a function of \(x = \sum_{i=1}^{n} |u_i|^2\) on the appropriate local coordinate patch. Before writing down the metric, we parameterize \(u_i\)'s by \(x, \theta\) and \(w_i\)'s

\[
u_i = x^{1/2} F^{-1/2} e^{i\theta/n} \cdot w_i \quad (i = 1, 2, \cdots, n; w_n = 1),
\]
\[
F = 1 + \sum_{i=1}^{n-1} |w_i|^2.
\]

This expression is valid for \(x \neq 0\) and \(w_i\)'s are coordinates of \(\mathbb{CP}^{n-1}\). (The case of \(x = 0\) corresponds to the point \(u_i = 0\) \((i = 1, 2, \cdots, n)\).) Let us introduce a new coordinate \(Y\) through an equation

\[
B x^n = h_n \left( \frac{2nY}{k} \right),
\]
\[
h_n(y) = \frac{(-1)^{n-1}}{(n-1)!} \int_{0}^{y} dt \ t^{n-1} e^t = -1 + e^y \sum_{m=0}^{n-1} \frac{(-y)^m}{m!} \quad (B : \text{constant}).
\]

We plot the function \(h_n(y)\) for \(n = 1, 2, 3\) in Fig. 4. The \(h_n(y)\) becomes zero only at \(y = 0\), while \(h_n(y)\) approaches \(-1\) in the \(y \to -\infty\) limit. If \(n\) is even, \(h_n(y) \leq 0\) and \(B\) must be a negative constant. If \(n\) is odd, \(h_n(y)\) increases monotonously and \(B\) can be either positive or negative. Since the associated Kähler potential \(K = K(x)\) is defined by a solution of \(x \frac{dK}{dx} = Y(x)\), the metric is expressed as

\[
ds^2 = \frac{g_n(Y)}{2} dY^2 + \frac{2}{n^2 g_n(Y)} \left( d\theta - nA \right)^2 + 2Y ds^2_{FS},
\]
\[
ds^2_{FS} = F^{-1} \sum_{i,j=1}^{n-1} \left( \delta_{ij} - F^{-1} \bar{w}_i w_j \right) dw_i d\bar{w}_j,
\]
\[
\Phi = -\frac{nY}{k} + C \quad (C : \text{constant}), \quad e^{2C} = B \cdot n! (-1)^{n-1} \left( \frac{2n}{k} \right)^{-n},
\]

where we define

\[
g_n(Y) = \frac{2n}{k} \left\{ f_n \left( \frac{2nY}{k} \right) \right\}^{-1}, \quad f_n(Y) = nY^{1-n} e^{-Y} \int_{0}^{Y} dt \ t^{n-1} e^t,
\]
\[
A = -\frac{i}{2} F^{-1} \sum_{i=1}^{n-1} \left( w_i d\bar{w}_i - \bar{w}_i dw_i \right).
\]

Figure 4: The function \(h_n(y)\) for \(n = 1, 2, 3\)
The line element $ds^2_{\mathcal{F}}$ represents the Fubini-Study metric of $\mathbb{C}P^{n-1}$ and $A$ is a connection 1-form on $\mathbb{C}P^{n-1}$. The Kähler geometry is a fibered space over the $\mathbb{C}P^{n-1}$. The dilaton $\Phi$ is introduced so that the one-loop beta function becomes zero. The case of $n = 1$ corresponds to the 2-dim black hole treated in Section 2.

At $Y = 0$ the function $f_n(Y)$ is zero and $g_n(Y)$ blows up. However the scalar curvature

$$R = \frac{4n}{k} \left[ n - f_n \left( \frac{2nY}{k} \right) \right]$$

is equal to $4n^2/k$ and the geometry is regular there. Thus $Y = 0$ is just a coordinate singularity of this system. In fact $Y = 0$ corresponds to the tip of the cigar of the 2-dim black hole in the case of $n = 1$. In the $Y \to \infty$ limit, $R$ approaches zero, while $Y \to -\infty$ limit $R$ blows up.

![Figure 5: The scalar curvature for n=2. The function $2(n - f_n(y))$ is plotted. We also obtain similar curves for the other values of n.](image)

Here we consider the $N = 2$ supersymmetric nonlinear sigma model on this curved background. The action is given as

$$S = \int_{\Sigma} d^2z \left[ 2g_{ij}(\partial_+ \phi^i \partial_- \bar{\phi}^j + \partial_- \phi^i \partial_+ \bar{\phi}^j) + \frac{i}{2}g_{ij} \bar{\psi}^i_-(D_0 + D_1)\psi^j_- + \frac{i}{2}g_{ij} \psi^i_+(D_0 - D_1)\bar{\psi}^j_- + \frac{i}{2}g_{ij} \psi^i_+(D_0 - D_1)\bar{\psi}^j_- - R_{ik} \xi^i_+ \psi^j_- \bar{\psi}^k_+ \right],$$

where $g_{ij}$ is the Kähler metric realizing the above geometry. Supersymmetric transformation is defined by

$$\delta \phi^i = \epsilon_+ \psi^i_- - \epsilon_- \psi^i_+,$$

$$\delta \psi^i_+ = 2i\bar{\epsilon}_+ \partial_+ \phi^i + \epsilon_+ \Gamma^j_{ik} \psi^j_- \psi^k_-,$$

$$\delta \psi^i_- = -2i\bar{\epsilon}_- \partial_- \phi^i + \epsilon_- \Gamma^j_{ik} \psi^j_+ \psi^k_-. $$

Under this transformation, the variation of the action is calculated

$$\delta S = \int_{\partial \Sigma} d\xi^1 \left[ -\epsilon_+ g_{ij} \psi^i_+ \partial_+ \bar{\phi}^j - \epsilon_- g_{ij} \psi^i_- \partial_- \bar{\phi}^j + \bar{\epsilon}_+ g_{ij} \bar{\psi}^i_- \partial_+ \phi^j + \bar{\epsilon}_- g_{ij} \bar{\psi}^i_+ \partial_- \phi^j \right].$$

Let us examine the A-type boundary with $\epsilon_+ = \bar{\epsilon}_- = \epsilon$ and $\epsilon_+ = \epsilon_- = \bar{\epsilon}$. By decomposing the fields $\phi_i$’s into angular parts $\varphi_i$ and radial parts $|u_i|$ like $\phi_i = |u_i|e^{i\varphi_i}$, the variation is
appropriate coordinate patch). The set of coordinates is labelled by the Kähler form.

From this formula, we can read possible boundary conditions rewritten into dimensional cycles, which are Lagrangian submanifolds. For the discussed Kähler metric, the A-type boundary conditions correspond to middle space can be considered to be a fibered space over this submanifold. Its fiber is a kind of minimal volume surfaces with middle dimension. The entire submanifold. It is nothing else but the condition for the Lagrangian submanifold specified by

\[ \hat{\psi}_i e^{-i\varphi_i} + \bar{\psi}_i e^{i\varphi_i} \]

\[ (K' \delta_{ij} + \frac{1}{2} K'' |u_i||u_j|)(\partial_1 |u_j| - |u_j|\partial_2 \varphi_j) \]

From this formula, we can read possible boundary conditions

\[ \psi_i e^{-i\varphi_i} = 0, \quad \partial_1 |u_i| = 0, \quad \partial_2 \varphi_i = 0 \quad (i = 1, 2, \cdots, n), \]

\[ \psi_i e^{-i\varphi_i} - \bar{\psi}_i e^{i\varphi_i} = 0, \quad \partial_2 |u_i| = 0, \quad \partial_1 \varphi_i = 0 \quad (i = 1, 2, \cdots, n). \]

The former condition fixes \(|u_i|\)'s on the boundary. The latter condition fixes \(\varphi_i\)'s and this configuration corresponds to the Lagrangian submanifold. We look into Kähler form \(J\):

\[ J = \sqrt{-1} K_{u_i u_j} du_i \wedge du_j = \frac{dK(x)}{dx} \sum_{i=1}^{n} d|u_i|^2 \wedge d\hat{\varphi}_i + \frac{d^2 K(x)}{dx^2} \sum_{i,j=1}^{n} |u_j|^2 d|u_i|^2 \wedge d\hat{\varphi}_j. \]

\[ \hat{\varphi}_i = \arg u_i \quad (i = 1, 2, \cdots, n). \]

From this formula it turns out that the radial parts \(|u_i|\)'s and phase parts \(\hat{\varphi}_i\)'s are complementary one another. So we can consider Lagrangian submanifold specified by \(\{\hat{\varphi}_i = \text{constant}\}\) (more precisely, linear combinations of \(\hat{\varphi}_i\)'s are constants on the appropriate coordinate patch). The set of coordinates is labelled by \(|u_i|\)'s. Thus pullback of the Kähler form \(J\) vanish on this submanifold.

To understand the structure of this fibered space, we use the coordinates \(w_i\)'s and \(x\) instead of \(u_i\)'s. Then

\[ J = d \left[ Y \cdot \left( \frac{1}{n} d\theta - A \right) \right] \]

\[ = dY \wedge \left( \frac{1}{n} d\theta - A \right) + Y F^{-1} dF \wedge A + Y F^{-1} \sum_{i=1}^{n-1} |w_i|^2 d\varphi_i, \]

\[ A = -F^{-1} \sum_{i=1}^{n-1} |w_i|^2 d\varphi_i, \quad \varphi_i = \arg w_i \quad (i = 1, 2, \cdots, n - 1). \]

Note that \(dY\) and \(dF\) do not depend on \(\theta\) nor \(\varphi_i\)'s. When one considers a subspace \(\mathcal{M}\) parameterized by the coordinates \(Y\) and \(|w_i|^2\) \((i = 1, 2, \cdots, n - 1)\), the pullback of this Kähler form vanish on \(\mathcal{M}\). It is nothing else but the condition for the Lagrangian submanifold. It is a kind of minimal volume surfaces with middle dimension. The entire space can be considered to be a fibered space over this submanifold. Its fiber is \(n\)-dimensional torus parameterized by \(\theta\) and \(\varphi_i\) \((i = 1, 2, \cdots, n - 1)\).

Generally 2-dim \(N = (2, 2)\) supersymmetric sigma models have A-, B-type supersymmetric boundary conditions. The A-type boundary conditions correspond to middle dimensional cycles, which are Lagrangian submanifolds. For the discussed Kähler metric,
(subsets of) $\mathcal{M}$ belongs to this kind of submanifolds. It corresponds to noncompact curves described by $\theta(\tilde{\theta}) = \text{constant}$ (see Fig.1(ii)) for the 2-dim black hole case ($n = 1$).

For the B-type, the variation of the action can be written with $\epsilon_+ = -\epsilon_- = \epsilon$ and $\bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon}$

$$\delta S = \frac{1}{2} \int_{\partial \Sigma} \xi^1 \left[ \epsilon \left\{ g_{ij}(-\psi^i_- + \psi^i_+) \partial_1 \bar{\phi}^j + ig_{ij}(\psi^i_- + \psi^i_+) \partial_2 \bar{\phi}^j \right\} 
+ \bar{\epsilon} \left\{ g_{ij}(\bar{\psi}^j_- - \bar{\psi}^j_+) \partial_1 \phi^i - ig_{ij}(\bar{\psi}^j_- + \bar{\psi}^j_+) \partial_2 \phi^i \right\} \right].$$

From this formula, it turns out that the B-type boundary corresponds to holomorphic cycles. Especially there are 0-dim point-like objects and objects spreading over $2n$-dim spacetime.

For this Kähler geometry, there exists a dilaton $\Phi$ and the brane tension is proportional to $e^{nY/k} \cdot e^{-C}$. For odd $n$, $B$ must be positive and tension is minimized at $Y = 0$ or $x = 0$. In this case, the point-like object should be localized at $x = 0$. For even $n$, $B$ must be negative. However, there are two branches of the coordinate transformation from $Y$ to $x$. If we start at $Y > 0$ and increase $x$, $Y$ increases and the tension gets greater. In order to minimize the tension, we should be at $Y = 0$ ($x = 0$) on this branch. So the position of this point-like object should be $Y = 0$ ($x = 0$). Meanwhile, if we start at $Y < 0$ and increase $x$, $Y$ decreases and $Bx^n$ approaches $-1$. Then the tension becomes zero. On the other hand, the geometry blows up because of the curvature singularity. The description of geometry breaks down in this limit. Accordingly, if we consider $Y \geq 0$ and $x \geq 0$ branch, we obtain the result that a point-like object is localized at $Y = 0$ ($x = 0$). For the case of $n = 1$, such an object is the D0-brane localized on the tip of the cigar.
4 Linear Sigma Model

In section 2, we discussed the nonlinear sigma model and particularly investigated the model whose geometry is 2-dim black hole. As a result two types of supersymmetric boundary conditions are obtained, but the analysis is at classical level. In general the sigma model action is modified by quantum corrections. In Ref. [7] the linear sigma model equivalent to 2-dim black hole at quantum level is proposed. Its generalization to a sigma model with the target space discussed in Section 3 is believed to realize \( N = 2 \) superconformal models with \( c/3 > 1 \) in the IR limit. So let us study supersymmetric boundary conditions of such a linear sigma model.

We use conventions for 2-dim worldsheet coordinates \((x^0, x^1)\) and derivatives

\[
\begin{align*}
 ds^2 &= -(dx^0)^2 + (dx^1)^2, \quad \eta_{\alpha\beta} = (-1, +1), \\
 x^\pm &= x^0 \pm x^1, \quad \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1).
\end{align*}
\]

At first, we consider a linear sigma model containing chiral superfields \( \Phi \), \( P \) and a vector superfield \( V \)

\[
\begin{align*}
 \Phi &= \phi(y) + \theta^+ \psi_+(y) + \theta^- \psi_-(y) + \theta^+ \theta^- F(y), \\
 P &= p(y) + \theta^+ \chi_+(y) + \theta^- \chi_-(y) + \theta^+ \theta^- F_P(y), \\
 V &= \theta^- \theta^- (v_0 - v_1) + \theta^+ \theta^+ (v_0 + v_1) - \theta^- \theta^+ \sigma - \theta^+ \theta^- \bar{\sigma} \\
 &\quad + i \theta^- \theta^+ (\theta^- \lambda_- + \theta^+ \lambda_+) + i \theta^+ \theta^- (\theta^- \lambda_- + \theta^+ \lambda_+) + \theta^- \theta^+ \theta^- \bar{\theta}^- D,
\end{align*}
\]

where we take Wess-Zumino gauge for \( V \). We use the following conventions

\[
\begin{align*}
 v_{01} &= \partial_0 v_1 - \partial_1 v_1, \quad y^\pm = x^\pm - i \theta^\pm \bar{\theta}^\pm, \quad \bar{y}^\pm = x^\pm \mp i \theta^\pm \bar{\theta}^\pm, \\
 D_\pm &= + \frac{\partial}{\partial \theta^\pm} - i \bar{\theta}^\pm \partial_\pm, \quad \bar{D}_\pm = - \frac{\partial}{\partial \bar{\theta}^\pm} + i \theta^\pm \partial_\pm,
\end{align*}
\]

then the field strength of \( V \) is given by \( \Sigma \)

\[
\Sigma = \bar{D}_+ D_- V = \sigma(\bar{y}) + i \theta^+ \bar{\lambda}_+(\bar{y}) - i \bar{\theta}^- \lambda_-(\bar{y}) + \theta^+ \bar{\theta}^- (D - i v_{01})(\bar{y}).
\]

The action \( S \) of the linear sigma model [7, 15] is

\[
S = \frac{1}{2\pi} \int dx^0 dx^1 (L_K + L_P + L_g + L_\Sigma),
\]
and each Lagrangian density is expressed by the component fields;

\[ L_K = \int d^4 \theta \bar{\Phi} e^V \Phi \]

\[ = \frac{i}{2} \bar{\psi}(-D_0 + D_1) \psi + \frac{i}{2} \bar{\psi}(D_0 + D_1) \psi \]

\[ + \frac{i}{2} \bar{\psi}_+(D_0 - D_1) \psi_+ + \frac{i}{2} \bar{\psi}(D_0 - D_1) \psi + |\phi|^2 + |F|^2 - |\sigma|^2 |\phi|^2 \]

\[ - \sigma \bar{\psi}_- \psi_+ - \bar{\sigma} \psi \bar{\psi}_+ - i \bar{\phi}(\lambda_+ \psi_+ - \lambda_+ \psi_-) - i \bar{\phi}(\bar{\lambda}_- \bar{\psi}_- - \bar{\psi}_+ \bar{\lambda}_-) ; \]

\[ L_P = \frac{k}{4} \int d^4 \theta (P + \bar{P} + V)^2 \]

\[ = \frac{k}{2} \left( D_0 \bar{p} D_0 p - D_1 \bar{p} D_1 p + i \chi_+ \partial_- \bar{\chi}_+ + i \bar{\chi}_+ \partial_- \chi_+ \right. \]

\[ + i \bar{\chi}_- \partial_+ \chi_- + i \chi_- \partial_+ \bar{\chi}_- + i \chi_+ \lambda_- - i \bar{\chi}_- \lambda_+ \]

\[ + i \bar{\chi}_+ \lambda_- - i \chi_- \bar{\lambda}_+ + |F|^2 - |\sigma|^2 + D(p + \bar{p}) \right) ; \]

\[ L_g = \frac{1}{2} \left( \partial_0 \sigma \partial_0 \sigma - \partial_1 \sigma \partial_1 \sigma + i \lambda_+ \partial_\lambda_- \bar{\lambda}_+ + i \bar{\lambda}_+ \partial_\lambda_- \lambda_+ \right. \]

\[ + i \bar{\lambda}_- \partial_+ \lambda_- + D^2 + v_0^2 \right) , \]

\[ L_\Sigma = \frac{1}{2} \int d^2 \bar{\theta} (-t) \Sigma + \frac{1}{2} \int d^2 \bar{\theta} (-t) \bar{\Sigma} = -r D + \theta v_0 , \]

where we used the following formulae

\[ D_a \phi = (\partial_a + iv_a) \phi, \quad D_a \bar{\phi} = (\partial_a - iv_a) \bar{\phi} , \]

\[ D_a p = \partial_a p + iv_a, \quad D_a \bar{p} = \partial_a \bar{p} - iv_a \quad (a = 0, 1), \]

\[ \bar{t} = r - i \theta . \]

The system is invariant under the gauge transformation;

\[ V \rightarrow V - i \Lambda + i \bar{\Lambda} , \quad \Phi \rightarrow e^{i \Lambda} \Phi , \quad P \rightarrow P + i \Lambda . \]

The \( \Lambda \) (\( \bar{\Lambda} \)) is a chiral (an anti-chiral) superfield. This action also has \( N = (2, 2) \) supersymmetry on \( \Sigma \)

\[ \delta \phi = \epsilon_+ \psi_- - \epsilon_- \psi_+ , \]

\[ \delta \psi_+ = + i (D_0 + D_1) \phi \cdot \bar{\epsilon}_+ + e_+ F - \phi \sigma \bar{\epsilon}_+ , \quad \delta \psi_- = - i (D_0 - D_1) \phi \cdot \bar{\epsilon}_+ + e_- F + \phi \bar{\sigma} \bar{\epsilon}_+ , \]

\[ \delta F = - i \{ \bar{\epsilon}_+ (D_0 - D_1) \psi_+ + \bar{\epsilon}_- (D_0 + D_1) \psi_- \} + \bar{\epsilon}_+ \bar{\sigma} \bar{\psi}_+ + \bar{\epsilon}_- \sigma \psi_+ + i \bar{\phi}(\bar{\epsilon}_+ \bar{\lambda}_+ - \bar{\epsilon}_+ \bar{\lambda}_-) ; \]

\[ \delta p = \epsilon_+ \chi_- - \epsilon_- \chi_+ , \]

\[ \delta \chi_+ = + i (D_0 + D_1) p \cdot \bar{\epsilon}_+ - e_+ F - \sigma \bar{\epsilon}_+ , \quad \delta \chi_- = - i (D_0 - D_1) p \cdot \bar{\epsilon}_+ + e_- F + \sigma \bar{\epsilon}_- , \]

\[ \delta F_p = - 2 i (\bar{\epsilon}_+ \partial_+ \chi_- + \bar{\epsilon}_- \partial_+ \chi_-) + i (\bar{\epsilon}_+ \bar{\lambda}_+ - \bar{\epsilon}_+ \bar{\lambda}_-) , \]
\[ \delta v_0 = \frac{i}{2}(\bar{\epsilon}_+ \lambda_+ + \bar{\epsilon}_- \lambda_- + \epsilon_+ \bar{\lambda}_+ + \epsilon_- \bar{\lambda}_-), \quad \delta v_1 = \frac{i}{2}(\bar{\epsilon}_+ \lambda_+ - \bar{\epsilon}_- \lambda_- + \epsilon_+ \bar{\lambda}_+ - \epsilon_- \bar{\lambda}_-), \]
\[ \delta \sigma = -i(\bar{\epsilon}_+ \lambda_+ + \epsilon_- \bar{\lambda}_+), \quad \delta D = -\bar{\epsilon}_+ \partial_- \lambda_+ - \bar{\epsilon}_- \partial_+ \lambda_- + \epsilon_+ \partial_+ \bar{\lambda}_+ + \epsilon_- \partial_- \bar{\lambda}_-, \]
\[ \delta \lambda_+ = + i \epsilon_+(D + i v_{01}) + 2 \partial_+ \sigma \cdot \epsilon_-, \quad \delta \lambda_- = + i \epsilon_-(D - i v_{01}) + 2 \partial_- \sigma \cdot \epsilon_+. \]

To compare with the nonlinear sigma model, we examine the potential terms \( U \) of scalar fields \( \phi, p \) and \( \sigma \)
\[ U = D|\phi|^2 + |F|^2 - |\sigma|^2|\phi|^2 - \sigma \bar{\psi}_- \psi_+ - \bar{\sigma} \psi_+ \bar{\psi}_- \]
\[ + \frac{k}{2} D(p + \bar{p}) + \frac{k}{2} |F_p|^2 - k |\sigma|^2 + \frac{1}{2e^2}(v_{01}^2 + D^2) - rD + \theta v_{01}. \]

The equations of motion leads to
\[ F = F_P = 0, \quad v_{01} = -\theta e^2, \]
\[ -\frac{D}{e^2} = |\phi|^2 + \frac{k}{2} (p + \bar{p}) - r. \]

The target space metric of the model is originally flat because this system is a linear sigma model. In the \( e^2 \rightarrow \infty \) limit, we get the equations
\[ \sigma = - \left( |\phi|^2 + \frac{k}{2} \right)^{-1} \bar{\psi}_+ \psi_-, \quad \bar{\sigma} = - \left( |\phi|^2 + \frac{k}{2} \right)^{-1} \bar{\psi}_- \psi_+. \]

After one eliminates the field \( \sigma \), the four-fermion interaction term is induced in \( U \)
\[ U = \left( |\phi|^2 + \frac{k}{2} \right)^{-1} \bar{\psi}_+ \psi_- \bar{\psi}_+ \psi_+ + \cdots, \]
which corresponds to that of the nonlinear sigma model. It encodes information on curvature of the resulting target space geometry.

Now we consider susy conditions on the boundary \( \partial \Sigma = (x^0, 0) \). The susy transformation of \( L = L_K + L_P + L_g + L_{\Sigma} \) is
\[ \delta L = \frac{1}{2} \partial_1 Q + \partial_0 (\cdots), \]
\[ Q = \epsilon_+ \left[ -\psi_-(D_0 + D_1) \bar{\phi} + i \sigma \bar{\phi} \psi_+ + i \bar{\psi}_+ F + \frac{i}{e^2} \bar{\lambda}_- \partial_+ \sigma - T \bar{\lambda}_+ \right. \]
\[ - \frac{k}{2} \chi_- (D_0 + D_1) \bar{p} + \frac{i}{2} \chi_- \sigma + \frac{i}{2} \bar{\chi}_+ F_p \]
\[ \left. + \epsilon_- \left[ -\psi_+(D_0 - D_1) \phi + i \sigma \phi \bar{\psi}_- + i \bar{\psi}_- \bar{F} - \frac{i}{e^2} \bar{\lambda}_+ \partial_- \sigma + T \lambda_- \right. \right. \]
\[ - \frac{k}{2} \chi_+ (D_0 - D_1) \bar{p} + \frac{i}{2} \chi_+ \bar{\sigma} - \frac{i}{2} \bar{\chi}_- F_p \]
\[ + \epsilon_+ \left[ \bar{\psi}_-(D_0 + D_1) \phi + i \sigma \phi \bar{\psi}_+ + i \psi_+ \bar{F} + \frac{i}{e^2} \lambda_- \partial_+ \sigma + T \lambda_+ \right. \]
\[ \left. + \frac{k}{2} \chi_-(D_0 + D_1) p + \frac{i}{2} \chi_+ \bar{\sigma} + \frac{i}{2} \bar{\chi}_+ F_p \right] \]
\[ + \epsilon_- \left[ \bar{\psi}_+(D_0 - D_1) \phi + i \sigma \phi \bar{\psi}_- - i \psi_- \bar{F} - \frac{i}{e^2} \lambda_+ \partial_- \sigma - T \lambda_- \right. \]
\[ \left. + \frac{k}{2} \chi_+(D_0 - D_1) p + \frac{i}{2} \chi_- \sigma - \frac{i}{2} \bar{\chi}_- F_p \right]. \]
where we used
\[ T = |\phi|^2 + \frac{k}{2}(p + \bar{p}) + \frac{1}{2\epsilon^2}(D + iv_0) - t. \]

There are two types of boundary conditions for four susy parameters \((\epsilon_+, \epsilon_-, \bar{\epsilon}_+, \bar{\epsilon}_-)\). The one is the A-type boundary, which is realized by setting \(\epsilon_+ = \bar{\epsilon}_- = \epsilon\) and \(\bar{\epsilon}_+ = \epsilon_- = \bar{\epsilon}\) on the boundary \(x^1 = 0\)
\[ \theta^+ = -\bar{\theta} = \theta, \quad \bar{\theta}^+ = -\bar{\theta} = \bar{\theta}, \]
\[ \Phi e^V \Phi = c_1, \quad P + \bar{P} + V = c_2, \quad \Sigma = 0 \quad (c_1, c_2 : \text{constant}). \]

These lead to Dirichlet boundary conditions for \(|\phi|, \text{Re}(p)\) and \(\sigma\)
\[ |\phi|^2 = c_1, \quad \phi_+ \bar{\phi} + \bar{\phi}_- \phi = 0, \quad \phi_+ \bar{\phi} + \bar{\phi}_+ \phi = 0, \]
\[ \bar{\phi}(F + iD_1 \phi) + \phi(\bar{F} - iD_1 \bar{\phi}) + \bar{\phi}_+ \psi + \bar{\psi}_- \phi = 0, \]
\[ p + \bar{p} = c_2, \quad \chi_+ + \chi_- = 0, \quad \bar{\chi}_+ + \bar{\chi}_- = 0, \quad F_P + \bar{F}_P + i(D_1 p - D_1 \bar{p}) = 0, \]
\[ \sigma = 0, \quad \lambda_+ + \lambda_- = 0, \quad \bar{\lambda}_+ + \bar{\lambda}_- = 0. \]

The other is the B-type boundary, which is realized by setting \(\epsilon_+ = -\epsilon_- = \epsilon\) and \(\bar{\epsilon}_+ = -\bar{\epsilon}_- = \bar{\epsilon}\). For \(\Phi\) and \(\Sigma\), the boundary conditions are given by
\[ \theta^+ = \theta^- = \theta, \quad \bar{\theta}^+ = \bar{\theta}^- = \bar{\theta}, \]
\[ D_\pm = e^{-V} D_\pm e^V, \quad D_+ \Phi = D_- \Phi, \quad \Sigma = \bar{\Sigma}, \]
which are gauge invariant formulae and expressed in the component fields as
\[ \sigma = \bar{\sigma}, \quad \lambda_+ + \lambda_- = 0, \quad \bar{\lambda}_+ + \bar{\lambda}_- = 0, \quad -2v_0 = \partial_1(\sigma + \bar{\sigma}), \]
\[ \psi_+ = \psi_-, \quad F = 0, \quad D_1 \phi = 0, \quad D_1(\psi_+ + \psi_-) = 0. \quad (15) \]

The condition for \(\text{Im}(\sigma)\) is a Dirichlet type, and the conditions for \(\text{Re}(\sigma)\) and \(\phi\) have modified forms of Neumann types in the presence of vector field \(v_a\) \((a = 0, 1)\). The B-type boundary condition for the remaining superfield \(P\) is realized by imposing \(D_+(P + V) = D_-(P + V)\). This equation is gauge invariant since \(P, V\) transform as Eqs.13, and \(\bar{\Lambda}\) is the anti-chiral superfield satisfying \(D_\pm \bar{\Lambda} = 0\). This boundary condition can be written down in the component fields
\[ \chi_+ = \chi_-, \quad F_P = 0, \quad D_1 p = 0, \quad \partial_1(\chi_+ + \chi_-) = 0. \quad (16) \]

In the case of B-type, we can construct the following boundary interaction terms for \(\Sigma = \{(x^0, x^1); x^1 \geq 0\}\) to cancel the boundary terms in Eq.14 after susy transformation
\[ S_{bdy} = \frac{1}{4\pi} \int_{\partial \Sigma} \frac{dx^0}{\partial \Sigma} \left[ i(\psi_+ \bar{\psi}_- + \bar{\psi}_+ \psi_-) + \frac{k}{2}(\chi_+ \bar{\chi}_- - \chi_- \bar{\chi}_+) \right] + \frac{1}{2\epsilon^2} \left[ \partial_1(\sigma^2 + 2\text{Im}(\sigma(D + iv_0))) \right] - i(t\sigma - \bar{\sigma} \sigma) \]
\[ + i(\sigma - \bar{\sigma})(|\phi|^2 + \frac{k}{2}(p + \bar{p})) - \frac{1}{2\epsilon^2} \left[ \partial_1|\sigma|^2 + 2\text{Im}(\sigma(D + iv_0)) \right] - i(t\sigma - \bar{\sigma} \sigma) \].

When we plug the boundary conditions 15, 16 into this boundary action, \(S_{bdy}\) vanishes for \(\theta = 0\). However interaction terms on the boundary are induced for non-zero \(\theta\)
\[ S_{bdy} = -\frac{\theta}{4\pi} \int_{\partial \Sigma} dx^0 (\sigma + \bar{\sigma}) = \frac{1}{4\pi} \theta \int_{\partial \Sigma} dx^0 \left( |\phi|^2 + \frac{k}{2} \right)^{-1} (\bar{\psi}_+ \psi_- + \bar{\psi}_- \psi_+), \quad (17) \]
which corresponds to Eq.(6) in the nonlinear sigma model.

We can easily generalize the above model to the one with \( N \) chiral superfields \( \Phi_i \) (\( i = 1, 2, \ldots, N \)), \( M \) vector superfields \( V_\ell \) (\( \ell = 1, 2, \ldots, M \)) and chiral superfields \( P_\ell \) (\( \ell = 1, 2, \ldots, M \)). The Lagrangian is defined by

\[
L = L_K + L_g + L + P + L_\Sigma,
\]

\[
L_K = \sum_\ell \left[ D_0 \bar{\phi}_i D_0 \phi_i - D_1 \bar{\phi}_i D_1 \phi_i + \frac{i}{2} \bar{\psi}_- i(D_0 + D_1) \psi_- i + \frac{i}{2} \bar{\psi}_- i(D_0 + D_1) \bar{\psi}_- i
- \sum_\ell (Q_\ell^I)^2 |\sigma_\ell|^2 |\phi_i|^2 - \sum_\ell Q_\ell^I \sigma_\ell \bar{\psi}_- i \psi_+ i - \sum_\ell Q_\ell^I \bar{\sigma}_\ell \bar{\psi}_+ i \psi_- i
- i \sum_\ell Q_\ell^I \bar{\phi}_i (\lambda_- \psi_- i - \lambda_+ \psi_+ i) - i \sum_\ell Q_\ell^I \phi_i (\bar{\psi}_- i \bar{\lambda}_+ - \bar{\psi}_+ i \bar{\lambda}_-)
\right],
\]

\[
L_g = \sum_\ell \frac{1}{2c_\ell^2} \left( \partial_0 \bar{\sigma}_\ell \partial_0 \sigma_\ell - \partial_1 \bar{\sigma}_\ell \partial_1 \sigma_\ell + i\lambda_+ \partial_+ \bar{\lambda}_+ + i\lambda_- \partial_- \bar{\lambda}_-
+ i\bar{\lambda}_- \partial_+ \lambda_- + D_\ell^2 + v_0^2 \right),
\]

\[
L_P = \sum_\ell \frac{k_\ell}{2} \left( D_0 \bar{p}_\ell D_0 p_\ell - D_1 \bar{p}_\ell D_1 p_\ell + i\chi_- \partial_- \bar{\chi}_- + i\bar{\chi}_- \partial_+ \chi_- + i\chi_+ \partial_+ \bar{\chi}_+
+ i\bar{\chi}_+ \partial_- \chi_+ + i\chi_- \partial_- \bar{\chi}_- + i\bar{\chi}_- \partial_+ \chi_- - i\chi_+ \lambda_-
+ i\bar{\chi}_+ \lambda_- - i\bar{\chi}_- \lambda_+ + |F_{P,\ell}|^2 - |\sigma_\ell|^2 + D_\ell (p_\ell + \bar{p}_\ell) \right),
\]

\[
L_\Sigma = \sum_\ell (-r_\ell D_\ell + \theta_\ell v_{0,\ell}),
\]

where we set

\[
D_a \phi_i = (\partial_a + i Q^a_i v_{a,\ell}) \phi_i, \quad D_a \bar{\phi}_i = (\partial_a - i Q^a_i v_{a,\ell}) \bar{\phi}_i, \\
D_a p_\ell = \partial_a p_\ell + i v_{a,\ell}, \quad D_a \bar{p}_\ell = \partial_a \bar{p}_\ell - i v_{a,\ell}, \quad (a = 0, 1)
\]

\[
t_\ell = r_\ell - i\theta_\ell.
\]

The associated boundary conditions are summarized:

- **B-type boundary** \( \epsilon_+ = -\epsilon_-; \)
  \[
  \theta^+ = \theta^-, \quad \bar{\theta}^+ = \bar{\theta}^- ,
  \quad \Sigma_\ell = \bar{\Sigma}_\ell , \quad D_+(P_\ell + V_\ell) = D_-(P_\ell + V_\ell) , \quad D_+ \Phi_i = D_- \Phi_i ,
  \]

- **A-type boundary** \( \epsilon_+ = +\bar{\epsilon}_-; \)
  \[
  \theta^+ = -\bar{\theta}^- , \quad \bar{\theta}^+ = -\theta^- ,
  \quad P_\ell + \bar{P}_\ell + V_\ell = \hat{c}_\ell , \quad \Phi_i e^{Q_i \cdot V} \Phi_i = c_i , \quad \Sigma_\ell = 0 .
  \]
By eliminating the auxiliary fields in the potential terms, several constraints are obtained:

\[-\frac{D_\ell}{e^2} = \sum_i Q^\ell_i |\phi_i|^2 + \frac{k_\ell}{2}(p_\ell + \bar{p}_\ell) - r_\ell, \quad F_i = F_{P,\ell} = 0.\]

In the limit $e^2 \to \infty$, the model is reduced to a nonlinear sigma model and $\sigma_\ell$'s are written down

\[
\sigma_\ell = -\left( \sum_i (Q^\ell_i)^2 |\phi_i|^2 + \frac{k_\ell}{2} \right)^{-1} \sum_i Q^\ell_i \bar{\psi}_i \psi_{-i},
\]

\[
\bar{\sigma}_\ell = -\left( \sum_i (Q^\ell_i)^2 |\phi_i|^2 + \frac{k_\ell}{2} \right)^{-1} \sum_i Q^\ell_i \bar{\psi}_{-i} \psi_i,
\]

which leads to the boundary terms as in Eq. (17).

\[
\sigma_\ell + \bar{\sigma}_\ell = -\left( \sum_i (Q^\ell_i)^2 |\phi_i|^2 + \frac{k_\ell}{2} \right)^{-1} \sum_i Q^\ell_i (\bar{\psi}_i \psi_{-i} + \bar{\psi}_{-i} \psi_i).
\]

Here we investigate the FI-term. For simplicity, let us consider the case of $\ell = 1$. In the $e^2 \to \infty$ limit, the explicit forms of the gauge field $v_0$, $v_1$ is obtained,

\[
v_0 = \frac{1}{2} \left( \sum_i Q^2_i |\phi_i|^2 + \frac{k}{2} \right)^{-1} \left[ i \sum_i Q_i (\bar{\phi}_i \partial_0 \phi_i - \phi_i \partial_0 \bar{\phi}_i) + \frac{i k}{2} \partial_1 (p - \bar{p}) \right]
+ \sum_i Q_i (\bar{\psi}_i \psi_{+i} + \bar{\psi}_{-i} \psi_{-i}),
\]

\[
v_1 = \frac{1}{2} \left( \sum_i Q^2_i |\phi_i|^2 + \frac{k}{2} \right)^{-1} \left[ i \sum_i Q_i (\bar{\phi}_i \partial_1 \phi_i - \phi_i \partial_1 \bar{\phi}_i) + \frac{i k}{2} \partial_1 (p - \bar{p}) \right]
+ \sum_i Q_i (\bar{\psi}_i \psi_{+i} - \bar{\psi}_{-i} \psi_{-i}).
\]

The configuration represented by the above $v_0$, $v_1$ contributes to the FI-terms, in particular to the theta-term

\[
S_\Sigma = \frac{1}{2\pi} \int dx^0 dx^1 \left(-rD + \theta v_{01}\right)
- \frac{\theta}{2\pi} \int dx^0 dx^1 \left( \sum_i Q^2_i |\phi_i|^2 + \frac{k}{2} \right)^{-1} \sum_i \left[ iQ_i \{(D_1 \phi_i) (\bar{D}_0 \phi_i) - (\bar{D}_1 \bar{\phi}_i) (\bar{D}_0 \bar{\phi}_i)\} \right]
+ \frac{\theta}{2\pi} \int dx^0 dx^1 \sum_i \left[ -Q_i \partial_+ \left\{ \left( \sum_j Q^2_j |\phi_j|^2 + \frac{k}{2} \right)^{-1} \psi_{-i} \bar{\psi}_{-i} \right\} \right]
+ Q_i \partial_- \left\{ \left( \sum_j Q^2_j |\phi_j|^2 + \frac{k}{2} \right)^{-1} \psi_{+i} \bar{\psi}_{+i} \right\} + \cdots,
\]

\[
\bar{D}_a \phi_i = \partial_a \phi_i + iQ_i \tilde{v}_a \phi_i,
\]

\[
\tilde{v}_a = \frac{1}{2} \left( \sum_i Q^2_i |\phi_i|^2 + \frac{k}{2} \right)^{-1} \left[ iQ_i (\bar{\phi}_i \partial_a \phi_i - \phi_i \partial_a \bar{\phi}_i) + \frac{i k_\ell}{2} \partial_a (p - \bar{p}) \right].
\]
Note that the elimination of gauge potential induces anti-symmetric fields.

Now let us introduce the linear sigma model realizing the Kähler geometry in 2n-dimensional target space \cite{7} discussed in section 3. The action contains \( n \) chiral superfields \( \Phi \), one chiral superfield \( P \) and one \( U(1) \) gauge field \( V \)

\[
L = L_K + L_P + L_g,
\]

\[
L_K = \sum_{i=1}^{n} \int d^4 \theta \bar{\Phi}_i e^V \Phi_i, \quad L_P = \frac{k}{4} \int d^4 \theta (P + \bar{P} + V)^2, \quad L_g = -\frac{1}{2e^2} \int d^4 \theta \bar{\Sigma} \Sigma.
\]

There are \( n \) complex scalars \( \phi_i \)'s and one complex scalar \( p \), but there is a D-flatness condition

\[
-\frac{D}{e^2} = \sum_{i=1}^{n} |\phi_i|^2 + \frac{k}{2} (p + \bar{p}) = 0,
\]

and the imaginary part of \( p \) can be gauged away. So the remaining degrees of freedom is \( 2n = 2n + 2 - 1 - 1 \) and dimension of the target space geometry is \( 2n \). The target space of the sigma model is characterized by its kinetic terms and one can get the metric of the spacetime:

\[
ds^2 = 2 \sum_{i=1}^{n} d\phi_i d\bar{\phi}_i + kdpd\bar{p}.
\]

We parameterize \( \phi_i \)'s as

\[
\sum_{i=1}^{n} |\phi_i|^2 = r^2, \quad \phi_i = rF^{-1/2}e^{\frac{k}{4}F} \cdot w_i \quad (i = 1, 2, \cdots n; w_n = 1), \quad F = 1 + \sum_{i=1}^{n-1} |w_i|^2.
\]

From the D-flatness condition, one has to impose a relation \( r^2 + k\text{Re}(p) = 0 \) for the real part of \( p \). On the other hand, the imaginary part \( \text{Im}(p) \) is related to \( \varphi \) through gauge transformation; \( \varphi \rightarrow \varphi + n\alpha, \text{Im}(p) \rightarrow \text{Im}(p) + \alpha \). From these relations one can write the spacetime metric \cite{7}

\[
ds^2 = 2 \hat{f}(r)dr^2 + \frac{2r^2}{n^2 \hat{f}(r)}(d\varphi - nA)^2 + 2r^2 ds^2_{FS},
\]

\[
\hat{f}(r) = 1 + \frac{2r^2}{k} \quad A = -\frac{i}{2} F^{-1} \sum_{i=1}^{n-1} (w_i d\bar{w}_j - \bar{w}_i dw_j),
\]

\[
ds^2_{FS} = F^{-1} \sum_{i,j} (\delta_{ij} - F^{-1} w_i \bar{w}_j) d\bar{w}_i dw_j.
\] (18)

It is a Kähler metric and one can consider \( N = (2, 2) \) nonlinear sigma models with this target space. By the same procedure as the analysis of the nonlinear sigma model in section 3 we can get A-, B-type susy boundary conditions.

Here we consider the linear sigma model. For the A-type case, one gets boundary conditions \( |\phi_i|^2 = c_i \geq 0 \) \( (i = 1, 2, \cdots , n) \), which represents an \( n \)-dimensional object in the geometry. The \( r \) is determined uniquely by \( \sum_{i=1}^{n} c_i = r^2 \). Thus the boundary condition
for the \( r \) direction is a Dirichlet type at least classically. This boundary conditions also impose the constraints \( F^{-1}|w_i|^2 = c_i r^{-2} \) for the absolute values of \( w_i \) (\( i = 1, 2, \ldots, n \)) of the base space \( \text{CP}^{n-1} \) on each local coordinate patch. But we can choose phases (angular coordinates) of \( w_i \)'s arbitrarily and the object is extended in the angular directions. For the 2-dim black hole, this configuration corresponds to Fig. (i). This consideration might be true only at classical level (discussion of the sigma model at tree level). The geometry discussed now is classical and changes into another one through RG flow. The resulting metric is thought to the one studied in section 3. Comparing these two metrics, we see that both the metrics have similar forms. The radial part \( f(r) \) is changed into \( g_n(Y) \), while the \( w_i \)'s are coordinates of the base \( \text{CP}^{n-1} \) space in both metrics. Also there exists an effect of a dilaton and the brane tension restricts the shape of the susy objects. The tension of the brane is proportional to \( e^{nY/k} \) and in particular the A-type branes considered above should shrink to \( r = 0 \). The radial parts \( Y, |w|^2 \)'s and angular parts \( \theta, \arg w_i \)'s play a complementary role in the Kähler form because this geometry has the \( U(n) \) isometry. So it is natural to consider non-compact objects extended to the radial parts. In the case of 2-dim black hole there are the D-branes extended in the noncompact direction (see Fig. (ii) etc). Therefore noncompact branes corresponding to them should exist in this model.

5 Conclusions

In this paper we have studied the D-branes of the \( N = 2 \) supersymmetric sigma model. We made it clear that the nonlinear sigma model has D0,D1,D2-branes in the 2-dim target space by considering A-,B-type supersymmetric boundary conditions on the worldsheet, and achieved the boundary interaction term added to the action. Especially in the case of black hole metric, we obtained the geometrical descriptions for the D-branes consistent with the result in the analysis of conjugacy classes.

Furthermore we investigated the linear sigma model realizing dynamics on the general-
ized metrics with \( U(n) \) isometry. We considered two types of boundary conditions and obtained the D-branes consistent with those in the 2-dim case.

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A Nonlinear Sigma Model with F-term

In section 2 we investigated nonlinear sigma models without F-term. It is possible to have similar discussion even in the presence of F-term (superpotential \( W(\phi) \)).
The action is

\[
S = \frac{k}{4\pi} \int d^2z \left[ \partial \bar{\partial} K \cdot (\partial_+ \phi \partial_- \bar{\phi} + \partial_- \phi \partial_+ \bar{\phi}) - i \chi^+ \chi^- \left( \partial_+ \bar{\phi} \cdot \frac{\partial^2 \bar{K}}{\partial \bar{\partial} K} - \partial_+ \phi \cdot \frac{\partial^2 K}{\partial \partial K} \right) - i \lambda^+ \lambda^- \left( \partial_- \phi \cdot \frac{\partial^2 \bar{K}}{\partial \bar{\partial} K} - \partial_- \bar{\phi} \cdot \frac{\partial^2 K}{\partial \partial K} \right) + 2 \lambda^+ \lambda^- \chi^+ \chi^- \frac{\partial^2 \bar{K}}{\partial \bar{\partial} K} + i (\lambda^+ \partial_- \lambda^+ - \lambda^- \partial_+ \lambda^- + \lambda^+ \partial_+ \lambda^- + \lambda^- \partial_+ \lambda^+) \right] + \frac{1}{2} \partial \bar{\partial} K - \bar{F} \frac{\partial^2 \bar{K}}{\partial \bar{\partial} K} \cdot \lambda^+ \chi^+ + F \frac{\partial^2 K}{\partial \partial K} \cdot \lambda^- \chi^- - (\partial \bar{\partial} K)^{-1} \partial^2 W \cdot \lambda^+ \chi^+ + (\partial \bar{\partial} K)^{-1} \partial^2 \bar{W} \cdot \lambda^- \chi^- + \frac{1}{2} F \partial W + \frac{1}{2} \bar{F} \partial \bar{W} \right].
\]

If \( \partial \bar{\partial} K \) does not blow up, this is reduced to a canonical form of \( N = (2,2) \) supersymmetric nonlinear sigma model by the redefinition of fermionic fields \( \bar{\epsilon} \). The supersymmetry transformation is expressed as

\[
\begin{align*}
\delta \phi &= \sqrt{2} (\partial \bar{\partial} K)^{-1/2} (\epsilon_+ \chi^+ - \epsilon_- \lambda^+) , \\
\delta \bar{\phi} &= \sqrt{2} (\partial \bar{\partial} K)^{-1/2} (-\epsilon_+ \chi^- + \epsilon_- \lambda^-) , \\
\delta F &= -2 \sqrt{2} i \epsilon_+ \partial_- ((\partial \bar{\partial} K)^{-1/2} \lambda^+) - 2 \sqrt{2} i \epsilon_- \partial_+ ((\partial \bar{\partial} K)^{-1/2} \chi^+) , \\
\delta \bar{F} &= -2 \sqrt{2} i \epsilon_- \partial_- ((\partial \bar{\partial} K)^{-1/2} \lambda^-) - 2 \sqrt{2} i \epsilon_+ \partial_+ ((\partial \bar{\partial} K)^{-1/2} \chi^-) , \\
\delta \lambda^+ &= \sqrt{2} (\partial \bar{\partial} K)^{1/2} \epsilon_- \left( i \partial_+ \phi - \frac{\partial \bar{\partial} K}{2 (\partial \partial K)} \cdot \lambda^+ \chi^- \right) + \frac{1}{\sqrt{2}} (\partial \bar{\partial} K)^{1/2} \left[ \epsilon_+ \left( F + \frac{\partial^2 \bar{K}}{\partial \bar{\partial} K} \cdot \chi^+ \lambda^+ \right) - \frac{\partial \bar{\partial} K}{\partial \bar{\partial} K} \cdot \epsilon_+ \chi^- \lambda^+ \right] , \\
\delta \chi^+ &= \sqrt{2} (\partial \bar{\partial} K)^{1/2} \epsilon_+ \left( -i \partial_- \phi + \frac{\partial \bar{\partial} K}{2 (\partial \partial K)} \cdot \lambda^+ \chi^+ \right) + \frac{1}{\sqrt{2}} (\partial \bar{\partial} K)^{1/2} \left[ \epsilon_- \left( F + \frac{\partial^2 \bar{K}}{\partial \bar{\partial} K} \cdot \chi^+ \lambda^+ \right) - \frac{\partial \bar{\partial} K}{\partial \bar{\partial} K} \cdot \epsilon_- \chi^+ \lambda^- \right] , \\
\delta \lambda^- &= \sqrt{2} (\partial \bar{\partial} K)^{1/2} \epsilon_- \left( -i \partial_+ \bar{\phi} - \frac{\partial^2 \bar{K}}{2 (\partial \partial K)} \cdot \lambda^+ \lambda^- \right) + \frac{1}{\sqrt{2}} (\partial \bar{\partial} K)^{1/2} \left[ \bar{\epsilon}_+ \left( \bar{F} - \frac{\partial \bar{\partial} K}{\partial \bar{\partial} K} \cdot \lambda^- \chi^- \right) + \frac{\partial \bar{\partial} K}{\partial \bar{\partial} K} \cdot \epsilon_+ \chi^- \lambda^+ \right] , \\
\delta \chi^- &= \sqrt{2} (\partial \bar{\partial} K)^{1/2} \epsilon_+ \left( i \partial_- \bar{\phi} + \frac{\partial \bar{\partial} K}{2 (\partial \partial K)} \cdot \chi^+ \chi^- \right) + \frac{1}{\sqrt{2}} (\partial \bar{\partial} K)^{1/2} \left[ \bar{\epsilon}_- \left( \bar{F} - \frac{\partial \bar{\partial} K}{\partial \bar{\partial} K} \cdot \lambda^- \chi^- \right) + \frac{\partial \bar{\partial} K}{\partial \bar{\partial} K} \cdot \epsilon_- \chi^- \lambda^+ \right] ,
\end{align*}
\]

where \( \epsilon_+, \epsilon_- , \bar{\epsilon}_+ , \bar{\epsilon}_- \) are susy parameters. Under the above transformation, we obtain the
variation of the action

\[ \delta S = \frac{k}{4\pi^2} \int_{\partial \Sigma} d\xi \left[ -i\sqrt{2} (\partial \bar{\partial} K)^{1/2} \left( \epsilon_- \lambda^+ \partial_- \bar{\phi} - \bar{\epsilon}_- \lambda^- \partial_- \phi + \epsilon_+ \chi^+ \partial_+ \bar{\phi} - \bar{\epsilon}_+ \chi^- \partial_+ \phi \right) \\
+ \frac{1}{\sqrt{2}} (\partial \bar{\partial} K)^{-1/2} \left( \epsilon_- \chi^- \partial \bar{W} + \bar{\epsilon}_- \chi^+ \partial W - \epsilon_+ \lambda^- \partial \bar{W} - \bar{\epsilon}_+ \lambda^+ \partial W \right) \\
+ \frac{1}{\sqrt{2}} (\partial \bar{\partial} K)^{-1/2} \left( \epsilon_- \chi^- - \epsilon_+ \lambda^- \right) \left( F \cdot \partial \bar{\partial} K - 2 \frac{\partial^2 \bar{\partial} K}{\partial \partial K} \lambda^+ \chi^+ + \partial \bar{W} \right) \\
+ \frac{1}{\sqrt{2}} (\partial \bar{\partial} K)^{-1/2} \left( \epsilon_- \chi^+ - \epsilon_+ \lambda^+ \right) \left( \bar{F} \cdot \partial \bar{\partial} K - 2 \frac{\partial^2 \partial K}{\partial \partial K} \lambda^- \chi^- + \partial W \right) \right]. \]

When one uses equations of motion of \( F \) and \( \bar{F} \), the last two lines vanish. The second line contains contributions of \( W \) and \( \bar{W} \). But these terms vanish at the singular point given by \((\partial \bar{\partial} K)^{-1} = 0\).

We shall write down susy boundary conditions;

1. A-type \( \epsilon_+ = \bar{\epsilon}_-, \bar{\epsilon}_+ = \epsilon_- \);

\[
\lambda^- \pm \chi^+ = \lambda^+ \pm \chi^- = 0, \\
\partial_- \phi \pm \partial_+ \bar{\phi} = 0, \quad \partial_- \bar{\phi} \pm \partial_+ \phi = 0, \quad \partial W \pm \partial \bar{W} = 0,
\]

2. B-type \( \epsilon_+ = -\epsilon_-, \bar{\epsilon}_+ = -\bar{\epsilon}_- \);

\[
\lambda^+ - \chi^+ = \lambda^- - \chi^- = 0, \\
\partial_2 \phi = \partial_2 \bar{\phi} = 0, \quad \partial W = \partial \bar{W} = 0,
\]

or

\[
\lambda^+ + \chi^+ = \lambda^- + \chi^- = 0, \\
\partial_1 \phi = \partial_1 \bar{\phi} = 0.
\]

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