On the hierarchy of natural theories

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No axiom system suffices for the development of all of mathematics; how should we navigate the vast array of axiomatic theories?
The so-called **consistency strength hierarchy** maps out the reasonable axiomatic theories and their relations.
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**Definition**

For a base theory $B$, we say that $T \leq_{\text{Con}}^B U$ if $B$ proves that the consistency of $U$ implies the consistency of $T$. 

**Definition**

$t < B \text{Con } u$ if $T \leq_{\text{Con}}^B U$ and $U \nleq_{\text{Con}}^B T$.

**Definition**

$t$ and $u$ are equiconsistent over $B$ if $T \leq_{\text{Con}}^B U$ and $U \leq_{\text{Con}}^B T$. 

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Theorem (Folklore)

\[<_{\text{Con}} \text{ is not pre-linear, i.e., there are non-equiconsistent } T \text{ and } U \text{ such that } T \not<_{\text{Con}} U \text{ and } U \not<_{\text{Con}} T.\]
Theorem (Folklore)

$\langle \text{Con} \rangle$ is not pre-linear, i.e., there are non-equiconsistent $T$ and $U$ such that $T \nless \text{Con} U$ and $U \nless \text{Con} T$.

Theorem (Folklore)

The ordering $\langle \text{Con} \rangle$ is ill-founded, i.e., there is a sequence $T_0 > \text{Con} T_1 > \text{Con} T_2 > \text{Con} \ldots$ where each $T_i$ is consistent.
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**Empirical Observation:** The restriction of $\prec_{\text{Con}}$ to the theories that arise in practice is a *well-ordering*.

$$\text{EA, } \text{EA}^+, \text{PRA, } I \Sigma_n, \text{PA, } \text{ATR}_0, \Pi^1_n \text{CA}_0, \text{PA}_n, \text{ZF, AD}^L(\mathbb{R})$$
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Explaining this contrast is widely regarded as a major outstanding conceptual problem in mathematical logic.
The fact that “natural” theories, i.e. theories which have something like an “idea” to them, are almost always linearly ordered with regard to logical strength has been called one of the great mysteries of the foundations of mathematics.

S. Friedman, Rathjen, Weiermann
1 Introduction

2 Set theory as a case study

3 The consistency operator

4 Second-order arithmetic
Three reasons for discussing set theory.
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2. ZFC is highly general.
3. ZFC is insufficient for answering many of the problems that motivated the early development of set theory:
   - The Continuum Hypothesis
   - Projective Measure
   - Suslin’s Hypothesis
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Steel has promoted the following maxim:

**MAXIMIZE STRENGTH.**
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The $<_{\text{Con}}$ tells us what mathematics can be developed on the basis of one theory rather than another; (more or less) if $\text{Con}(T)$ implies $\text{Con}(U)$ then $T$ can interpret $U$ and not vice-versa.
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The $\Con$ tells us what mathematics can be developed on the basis of one theory rather than another; (more or less) if Con($T$) implies Con($U$) then $T$ can interpret $U$ and not vice-versa.

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- Dedekind interpreted analysis in set theory.
- Gödel interpreted proof theory in arithmetic.
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When we restrict our attention to natural theories, only the first three possibilities are realized.

This is just to say that natural theories are linearly ordered by consistency strength.
Consider again the axiom systems extending ZFC:

- large cardinal axioms
- axioms of definable determinacy
- forcing axioms

These systems have different motivations, but they are well-ordered by consistency strength.
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*At the level of sentences about $\mathbb{R}$, we know of only one road upward. We are led to it many different ways.*

Steel
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Are there any proper extensions of $T$ that are strictly weaker than $T + \text{Con}_T$?
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$$T \vdash \left( R_T \leftrightarrow \forall x (\text{Pf}_T(x, \neg R_T) \rightarrow \exists y < x \text{Pf}_T(y, \neg \neg R_T)) \right)$$

$R_T$ “says”: If there are any proofs of $R_T$, then they are preceded by proofs of $\neg \neg R_T$. 

We can use Rosser’s trick to produce independent sentences strictly weaker than $\text{Con}_T$. 

$$\text{Con}_T \lor R_T \land \neg \text{Con}_T$$

Yet these sentences are highly unnatural. 

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Instead of focusing on specific theories, we focus on algorithms for **uniformly** extending theories.
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We are particularly interested in $g$ that are monotone:

If $T$ proves $\varphi \rightarrow \psi$, then $T$ proves $g(\varphi) \rightarrow g(\psi)$. 
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Rosser’s trick engenders an algorithm for extending theories, but it is not monotone.

Indeed, the Rosser algorithm is not monotone in virtue of the pathological properties flagged earlier.
The goal is to prove that the consistency operator is the unique weakest monotone algorithm for uniformly extending theories.
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Let $\varphi$ be a true sentence. Then the set of sentences that implies $\varphi$ is a cone.

$$\{\psi : T + \psi \text{ proves } \varphi\}$$
Let’s call a function $g$ bounded if there exists a $k \in \mathbb{N}$ such that, for every $\varphi$, $g(\varphi) \in \Pi^0_k$.

For technical reasons, we restrict our attention to bounded functions.
Introduction
Set theory as a case study
The consistency operator
Second-order arithmetic

Theorem (W.)

Let $g$ be a bounded, computable, and monotone. Then one of the following holds:

1. There is a cone $C$ such that for all $\varphi \in C$, $T + \varphi \vdash g(\varphi)$.
2. There is a cone $C$ such that for all $\varphi \in C$, $T + \varphi + g(\varphi) \vdash \text{Con}_T(\varphi)$.

That is, either $g$ is as weak as the identity on a cone or as strong as the consistency operator on a cone.

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The consistency operator is the unique weakest method for uniformly extending theories.
This contributes to a partial explanation of the well-ordering phenomenon.
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It suggests that the iterates of the consistency operator form a spine of axiomatic theories that is, in some sense, canonical.
We now shift our attention to second-order arithmetic, the joint theory of the natural numbers and the real numbers.
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$\text{ACA}_0$ is our base theory; it is a second-order pendant of PA.
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Definition

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**Fact:** A theory is consistent just in case it is $\Pi_1^0$-sound.
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Definition

$T \models^\Gamma \varphi$ if there is a true $\psi \in \Gamma$ such that $T + \psi \models \varphi$. 

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**Definition**

$T \vdash^\Gamma \varphi$ if there is a true $\psi \in \Gamma$ such that $T + \psi \vdash \varphi$.

**Fact:** For any $T$ and $\varphi$, $T \vdash \varphi$ if and only if $T \vdash \Sigma^0_1 \varphi$. 

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Definition

\[ T \leq_{\text{Con}} U := \text{ACA}_0 \vdash \text{Con}(U) \rightarrow \text{Con}(T). \]
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**Theorem (W.)**

The relation \( \leq_{\Sigma^1_1 \Pi^1_1} \) pre-well-orders the \( \Pi^1_1 \)-sound extensions of \( \text{ACA}_0 \).
Thanks!

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