Compact Complex Surfaces with No Nonconstant Meromorphic Functions *

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Abstract

In 1949 Siegel gave an example of a complex two-torus with no nonconstant meromorphic functions. In 1964 Kodaira showed that compact complex surfaces with no nonconstant meromorphic functions must be of the following three types: tori, Hopf type surfaces with first Betti number equal to one, and K3 surfaces. In this paper we show that surfaces of these three types have a dense set of surfaces in their natural moduli spaces with no nonconstant meromorphic functions.

1 Introduction

Let $X$ be a connected compact complex manifold, and let $\mathcal{M}(X)$ be the field of meromorphic functions on $X$. We let $\deg X$ denote the transcendence degree of $\mathcal{M}(X)$ over the field $\mathbb{C}$ of complex numbers, which we here identify with the field of constant complex-valued functions on $X$.

In 1955 Siegel [20] showed that

$$0 \leq \deg X \leq n,$$

where $n = \dim X$. (1)

This result has a long history going back to the first efforts by Riemann and Weierstrass to show that there are not $n + 1$ algebraically independent

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Abelian function on $\mathbb{C}^n$. In his paper Siegel has a very informative history of the results and proofs leading up to (1).

We note that when Riemann first introduced Riemann surfaces in 1857 [17], one of the primary questions he addressed was the question of the existence of meromorphic functions. In this initial paper he did show the existence of nonconstant meromorphic functions on a Riemann surface, this then leading up to the Riemann-Roch theorem. This was the context of a Riemann surface being a branched covering of the compactified complex plane. When Weyl introduced for the first time an abstract Riemann surface [23], he also showed, following the original outline of Riemann, that any compact Riemann surface has nonconstant meromorphic functions, and that the Riemann-Roch theorem was valid as well. Thus, any connected compact complex manifold $X$ of dimension one has nonconstant meromorphic functions. We shall see that for higher dimensions, this is no longer the case.

In this paper we want to concentrate on connected compact complex surfaces, i.e., complex manifolds of dimension two, and we will call them simply surfaces (assumed compact and connected). Thus, if $X$ is a surface, we have

$$0 \leq \deg(X) \leq 2.$$ 

In one of Kodaira’s seminal papers “On the structure of compact complex analytic surfaces. I” [9], he shows that:

1. If $\deg X = 2$, then $X$ must be a projective algebraic manifold\(^1\).

2. If $\deg X = 1$, then $X$ is a type of elliptic surface, a fiber space of elliptic curves (with a finite number of singular curves), parametrized by an algebraic curve.

What can one say about surfaces of degree zero? Siegel gave an example in his 1949 Lecture Notes [19] of a complex two-torus of degree zero, which we will see again later in this paper. In 1964 Kodaira [9] showed that there are topological restrictions on their structure. We let $b_i = b_i(X)$ be the Betti numbers of a surface $X$ and let $K_X = \wedge^2 T^*(X)$ be the canonical bundle of

\(^1\)In the general case, if $\deg X = n$, where $X$ is $n$-dimensional, then $X$ is not necessarily a projective algebraic manifold; instead it is referred to as a Moishezon manifold, which is always bimerorphically equivalent to a projective algebraic manifold. See Shafarevich [18] for a discussion of this topic.
X, then Kodaira showed that if a surface \( X \) has no exceptional curves\(^2\) then there are three possibilities:

1. \( b_1 = 4 \), and \( X \) is a complex torus.

2. \( b_1 = 1 \), and \( X \) is a Hopf surface or other similar non-Kähler surfaces (Inoue, Enriques-Hirzebruch, etc.).

3. \( b_1 = 0 \), the canonical bundle \( K_X \) of \( X \) is trivial, and \( X \) is a K3 surface.

In Shafarevich’s book [18] there is a simple criterion for a Hopf surface to be of degree zero which immediately yield examples of degree zero Hopf surfaces. We did not find in the literature any examples of K3 surfaces of degree zero. In particular, in Kodaira’s paper [9] he only shows that degree zero surfaces must be of the three types above, not that such manifolds exist with this property.

However, these examples suffice to show that the theory of several complex variables again distinguishes itself from function theory of one variable, as has been the case since Hartogs proved his fundamental theorem in 1906 [7] concerning simultaneous analytic continuation across compact subsets of \( \mathbb{C}^2 \).

In this paper we show that for each of these three cases: complex two-tori, Hopf surfaces, and K3 surfaces, the set of degree zero surfaces is dense in the natural moduli spaces for each of these classes of surfaces. We don’t consider the other non-Kähler surfaces here, but their treatment would, in principle, be similar to our study of Hopf surfaces, as they are defined in a similar manner as a quotient of a specific type of domain in \( \mathbb{C}^2 \) by a discrete group.

To be more specific in the first case, we consider complex two-tori defined by period matrices of type

\[
\Omega = (I, Z),
\]

where \( I \) is the \( 2 \times 2 \) identity matrix and \( Z \) is a \( 2 \times 2 \) complex-valued matrix with \( \text{Im} \, Z > 0 \) (positive definite). Let

\[
M = \{ Z : Z \text{ is a } 2 \times 2 \text{ complex-valued matrix with } \text{Im} \, Z > 0 \}.
\]

\(^2\)An exceptional curve in a surface \( X \) is a curve with self-intersection \( c \cdot c = -1 \); Grauert [5] showed that any surface \( X \) with exceptional curves \( c_1, \ldots, c_k \) is biholomorphic to a surface \( \tilde{X} \) with with a finite number of quadratic transforms (blowups) at points \( p_1, \ldots, p_k \), where the curves \( c_i \) are the quadratic transforms of the points \( p_i \) in \( X \).
This is a moduli space of dimension four for the complex tori defined by the period matrices (3) (it is a complete and effectively parametrized moduli space as shown by Kodaira and Spencer [11] (see Theorem 14.3 in [11]).

Our result in this case is that there is a dense set of points \( M_0 \subset M \) such that\(^3\) for each \( Z \in M_0 \), the torus \( T_Z \) defined by the period matrix \( \Omega = (I, Z) \) has degree zero. The other two cases are formulated similarly, and we will consider the detailed results for all three cases in the following three sections.

## 2 Two-dimensional tori

Let \( M_2 \) denote the vector space of \( 2 \times 2 \) complex-valued matrices, which we will denote generically by

\[
Z = \begin{pmatrix}
z_{11} & z_{12} \\
z_{21} & z_{22}
\end{pmatrix}.
\]

We let

\[
M := \{ Z \in M_2 : \text{Im} \ Z > 0 \},
\]

where \( \text{Im} \ Z \) denotes the imaginary parts of the coefficients of \( Z \) and where \( \text{Im} \ Z > 0 \) means that the matrix \( \text{Im} \ M \) is positive definite.

Let \( I \) be the identity matrix in \( M \), and let

\[
\Omega = (I, Z), \ Z \in M,
\]

which we will call a normalized period matrix. Let \( \{ \omega_1, \omega_2, \omega_3, \omega_4 \} \) be the columns of \( \Omega \), and we see that these four vectors in \( \mathbb{C}^2 \) are linearly independent over the real numbers. Let \( \Lambda \) be the lattice generated by these vectors, i.e., linear combinations of the form

\[
m_1 \omega_1 + m_2 \omega_2 + m_3 \omega_3 + m_4 \omega_4 \quad m_i \in \mathbb{Z},
\]

and let

\[
T_Z := \mathbb{C}^2 / \Lambda,
\]

which is a complex torus of two dimensions defined by the period matrix \( \Omega \).

Let \( Z \in M \), then an Abelian function \( f \) on \( \mathbb{C}^2 \) with respect to the period matrix

\[
\Omega = (I, Z) = (\omega_1, \omega_2, \omega_3, \omega_4),
\]

\(^3\)In fact, \( M_0 \) is a set of second category in the four-dimensional metric space \( M \).
is a meromorphic function on $\mathbb{C}^2$ which satisfies

$$f(z + \omega_j) = f(z), \ j = 1, \ldots, 4.$$  

A *degenerate Abelian function* is an Abelian function $f(z) = f(z_1, z_2)$ which, after a linear change of variables in $\mathbb{C}^2$

$$\left( \begin{array}{c} \zeta_1 \\ \zeta_2 \end{array} \right) = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right), \ \alpha \delta - \gamma \beta \neq 0,$$

becomes a a nonconstant elliptic function of one complex variable, i.e.,

$$\tilde{f}(\zeta_1, \zeta_2) = f(z_1(\zeta_1, \zeta_2), z_2(\zeta_1, \zeta_2)) = \tilde{f}(\zeta_1),$$

where $\tilde{f}(\zeta_1)$ is an elliptic function.

If $\tilde{f}$ is not a constant, then it is an elliptic function in the $\zeta_1$ complex plane which is also periodic with respect to the four complex numbers which are the first row of the transformed period matrix:

$$\left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \Omega,$$

which have the form:

$$(\alpha, \beta, \alpha z_{11} + \beta z_{21}, \alpha z_{12} + \beta z_{22}).$$

Since $\tilde{f}(\zeta_1)$ is an elliptic function, it has two independent periods $\tau_1$ and $\tau_2$ which generate all the periods of $\tilde{f}$ in the complex $\zeta_1$-plane. This means that there are integers $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2$ which satisfy

\begin{equation}
\alpha = p_1 \tau_1 + p_2 \tau_2,
\beta = q_1 \tau_1 + q_2 \tau_2,
\alpha z_{11} + \beta z_{12} = r_1 \tau_1 + r_2 \tau_2,
\alpha z_{21} + \beta z_{22} = s_1 \tau_1 + s_2 \tau_2.
\end{equation}

We will return to this in our analysis below.

An Abelian function $f$ on $\mathbb{C}^2$ is *nondegenerate* if it is not a degenerate Abelian function. See Siegel [19] or Conforto [4] for an overview of nondegenerate and degenerate Abelian functions on $\mathbb{C}^n$. We will use some of the results from these books in our discussion below.
Let
\[ \Omega = (\omega_1, \ldots, \omega_{2n}), \] (5)
be a general period matrix in \( \mathbb{C}^2 \), not necessarily the more specialized normal-
ized period matrices we considered above in (3). Here \( \{\omega_1, \ldots, \omega_{2n}\} \) are
2\( n \) vectors in \( \mathbb{C}^n \) which are linearly-independent over the real numbers, and
we let \( T_\Omega \) denote the complex torus corresponding to \( \Omega \). A period matrix \( \Omega \) of
the form (5) is said to be a Riemann matrix if there exists a skew-symmetric
2\( n \times 2n \) matrix \( P \) with coefficients which are rational numbers such that
\[
\begin{align*}
\Omega P \Omega^t &= 0, \\
-\imath \Omega P \overline{\Omega}^t &> 0.
\end{align*}
\] (6)
Here \( A^t \) denote the transpose of a matrix \( A \), and \( \overline{A} \) denotes the complex
conjugate of the entries of a matrix \( A \).

An important theorem concerning complex tori is that a complex torus
of \( n \)-dimensions is a projective algebraic manifold if and only if the period
matrix \( \Omega \) defining the torus is a Riemann matrix (this is a consequence of the
Kodaira embedding theorem; see, e.g., Wells [22]). A different and related
result is that for a given period matrix \( \Omega \), the matrix \( \Omega \) is a Riemann matrix if
and only if there exists a nondegenerate Abelian function on \( \mathbb{C}^n \) with respect
to \( \Omega \). See Siegel [19] or Conforto [4] for a proof of this. It follows from this
that
\[ \deg T_\Omega = n \] if and only if \( \Omega \) is a Riemann matrix.

In our two-dimensional case of normalized period matrices, we see now
that there are three cases to consider. First, \( \deg T_Z = 2 \) if and only if \((I, Z)\) is
a Riemann matrix. Secondly, if \( \deg T_Z = 1 \), then there must be a nonconstant
degenerate Abelian function \( f \) on \( T_Z \). The third case is that \( \deg T_Z = 0 \), i.e.,
there are no nonconstant meromorphic functions on \( T_Z \). We will give explicit
criteria for the first two cases in terms of the matrices \( Z \in M \), which will
then determine the nature of the set of matrices \( Z \in M \) for which \( \deg T_Z = 0 \).

We now consider period matrices for \( n \)-dimensional complex tori, and we
will restrict ourselves again to the two-dimensional case somewhat later. Two
period matrices \( \Omega_1 \) and \( \Omega_2 \) are said to be equivalent if there is a nonsingular
matrix \( C \in M_k(C) \) and a nonsingular matrix \( N \in M_{2n}(Z) \) such that
\[ \Omega_1 = C\Omega_2 N. \]
Two equivalent period matrices generate tori \( T_{\Omega_1} \) and \( T_{\Omega_2} \) which are biholo-
morphic to each other. Using this equivalence relation, a variety of canonical
forms for Riemann matrices were formulated in the late nineteenth and early twentieth century. These are summarized quite completely in Conforto’s book [4] (see, in particular, the table on p. 90).

Coming back to the two-dimensional case, we see from Conforto’s book that the set of all Riemann matrices are equivalent to the canonical Riemann matrices of the form

$$\Omega_n := (I_n, Z), I_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}, \text{ for } Z \in M, Z = Z^t,$$

for all positive integers $n$. We note that, except for $n = 1$, these are not normalized period matrices. However, by left multiplication by $I_n^{-1}$, we obtain the equivalent family of all normalized Riemann matrices of the form

$$\Omega = (I, I_n^{-1}Z),$$

for $Z \in M$ and where $Z$ is symmetric. Therefore we set

$$S_n := \{ Z \in M : z_{21} = nz_{12}, \text{ } n = 1, 2, \ldots \}$$

and

$$\{ \Omega = (I, Z), Z \in S_n \}, \text{ } n = 1, 2, \ldots,$$

consists of all Riemann matrices in normalized form.

We let

$$S := \bigcup_{n=1}^{\infty} S_n.$$

Each $S_n$ is a linear hyperplane in the vector space $M_2(\mathbb{C})$ intersecting the open set $M \subset M_2(\mathbb{C})$, and, as such, it is a closed subset of $M$ with no interior points, thus $S$ is a set of first category in $M$. Therefore we obtain that $T_Z$ has degree two if and only if $Z \in S$. This is a useful criterion for complex tori of degree two.

We now want to give a similar criterion for complex two-tori of degree one. We suppose now that $Z \in M$ and $\deg T_Z = 1$. Thus there is a nonconstant degenerate Abelian function $f$ on $X_Z$. It follows from our earlier discussion that there is a change of variables of the form

$$C = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

7
and periods $\tau_1, \tau_2 \in \mathbb{C}$ with $\text{Im} \frac{\tau_1}{\tau_2} \neq 0$ and integers $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2$ satisfying (4). Moreover, the matrix

$$
\begin{pmatrix}
  p_1 & p_2 \\
  q_1 & q_2 \\
  r_1 & r_2 \\
  s_1 & s_2 \\
\end{pmatrix}
$$

has maximal rank.

Eliminating $\alpha$ and $\beta$ from the four equations in (4) we find that

$$
(p_1 \tau_1 + p_2 \tau_2) z_{11} + (q_1 \tau_1 + q_2 \tau_2) z_{21} = r_1 \tau_1 + r_2 \tau_2
$$

$$
(p_1 \tau_1 + p_2 \tau_2) z_{12} + (q_1 \tau_1 + q_2 \tau_2) z_{22} = s_1 \tau_1 + s_2 \tau_2,
$$

which gives

$$
(p_1 z_{11} + q_1 z_{21} - r_1) \tau_1 + (p_2 z_{11} + q_2 z_{21} - r_2) \tau_2 = 0,
$$

$$
(p_1 z_{12} + q_1 z_{22} - s_1) \tau_1 + (p_2 z_{12} + q_2 z_{22} - s_2) \tau_2 = 0.
$$

Since $\tau_1$ and $\tau_2$ are linearly independent over the real numbers, it follows that the determinant of this linear system must be zero. Hence

$$
(r_1 s_2 - s_1 r_2) + (s_1 p_2 - p_1 s_2) z_{11} + (p_1 r_2 - r_1 p_2) z_{12} + (s_1 q_2 - q_1 s_2) z_{21} + (q_1 r_2 - r_1 q_2) z_{22} + (p_1 q_2 - q_1 p_2)(z_{11} z_{22} - z_{12} z_{21}) = 0
$$

(7)

Let $m = (m_0, m_1, m_2, m_3, m_4, m_5)$ be a sextuple of integers, and assume that $m_i \neq 0$ for some $i, 1 \leq i \leq 5$, which we call an admissible sextuple. Then, for such an admissible sextuple, we define the algebraic variety $R_m$ in $M$ by the equation

$$
R_m := \{ Z \in M : m_0 + m_1 z_{11} + m_2 z_{12} + m_3 z_{21} + m_4 z_{22} + m_5(z_{11} z_{22} - z_{12} z_{21}) \}.
$$

(8)

We see that $R_m$ is of codimension one.

We note that the coefficients of the polynomial in (7) are precisely the minors of the $4 \times 2$ matrix

$$
\begin{pmatrix}
  p_1 & p_2 \\
  q_1 & q_2 \\
  r_1 & r_2 \\
  s_1 & s_2 \\
\end{pmatrix}
$$

(9)
Now we are assume that $Z \in M$ and that $\deg T_Z = 1$, then it follows that $Z$ is a point on $R_m$. Namely, at least one of the minors must be nonzero. If the minor $r_1s_2 - s_1r_2$ is the only nonzero minor, then equation (7) couldn’t be satisfied, and so at least one other minor must also be nonzero, and thus $Z \in R_m$ for some suitable admissible $m$.

Let now

$$R = \bigcup_m R_m,$$

where we sum over admissible sextets $m$. Then any $Z \in M$ with $X_Z$ of degree one must be a point in $R$, and as we saw earlier, any $Z \in M$ with $X_Z$ of degree two must be a point in $S$, so we have the theorem.

**Theorem 1** Let $X_Z$ be a two-torus with the period matrix $(I, Z)$, then, if $\deg X_Z \geq 1$, then $Z \in S \cup R$.

Since $S \cup R$ is a countable union of subvarieties of $M$, each of codimension one, we have the following theorem. Let

$$M_0 := M - (S \cup R).$$

**Theorem 2** $M_0$ is of second category in $M$, and in particular is dense, and for each $Z \in M_0$,

$$\deg T_Z = 0.$$

We will close this section with a discussion of two examples. As mentioned in the Introduction, Siegel gave in his lecture notes [19] an example of a two-torus of degree zero which we describe now. Let

$$\tilde{\Omega}_1 = \begin{pmatrix} 1 & 0 & \sqrt{-2} & \sqrt{-5} \\ 0 & 1 & \sqrt{-3} & \sqrt{-7} \end{pmatrix},$$

This is Siegel’s example, but we need to choose a sign for the square roots (other choices work as well), and we can reorder the vectors slightly, so we let

$$\Omega_1 = \begin{pmatrix} 1 & 0 & \sqrt{3i} & \sqrt{2i} \\ 0 & 1 & \sqrt{7i} & \sqrt{3i} \end{pmatrix},$$

and we let

$$Z_1 = \begin{pmatrix} \sqrt{3i} & \sqrt{2i} \\ \sqrt{7i} & \sqrt{3i} \end{pmatrix},$$
which we see is a point in the moduli space $M$. We see immediately that $\Omega_1$ is not a Riemann matrix since $\sqrt{7}i$ is not of the form $n\sqrt{2}i$, for some positive integer $n$.

Suppose that $Z \in R$. then there is an admissible sextuple $m$ such that

$$m_0 + m_1(\sqrt{2}i) + m_2(\sqrt{5}i) + m_3(\sqrt{3}i) + m_4(\sqrt{7}i) + m_5(\sqrt{15} - \sqrt{14}) = 0.$$

This means that

$$m_0 + m_5(\sqrt{15} - \sqrt{14}) = 0,$$

$$m_1\sqrt{2} + m_2\sqrt{5} + m_3\sqrt{3} + m_4\sqrt{7} = 0.$$

It is easy to conclude from these two equations that $m_i = 0, i = 0, \ldots, 5$, which contradicts $m$ being an admissible sextuple. Hence $Z_1$ is not in either $S$ or $R$, and must therefore correspond to a torus $T_1$ which is of degree zero.

Let us now look at Shafarevich’s example of a two-torus of degree zero (in his book [18]). We let

$$\Omega_2 = \begin{pmatrix} 1 & 0 & i & \sqrt{2} \\ 0 & 1 & 0 & i \end{pmatrix},$$

and we let

$$Z_2 = \begin{pmatrix} i & \sqrt{2} \\ 0 & i \end{pmatrix},$$

which is also a point in $M$.

We see immediately that this is not a Riemann matrix. However, $Z_2$ is a point of $R$ for some admissible sextuplet $m$. Namely, suppose that $m_1 = -m_4 = n$, where $n$ is a nonzero integer, and we set

$$m_0 = m_2 = m_3 = m_5 = 0,$$

then we find that

$$m_0 + m_1(i) + m_2(\sqrt{2}) + m_3(0) + m_4(i) + m_5(-1) = ni - ni,$$

$$= 0.$$

Thus

$$Z_2 \in R_{0,-n,0,0,n},$$

and is not a point in $M_0$, whereas Siegel’s example was.
This shows that the dense subset $M_0$ of Theorem 2 does not contain all the complex two-tori of degree zero. One would need a more precise characterization of two-tori of degree one for this, as we do have for two-tori of degree two. We note that the integers $m_i$ in (8) should be quadratic forms in terms of the integers $p_i, q_j, r_k, s_l$ in (9), which is a type of restriction on the $m_i$, which we have not used in our analysis, but could play a role in finding a better characterization of degree one tori.

In Section 4 we will study degree zero K3 surfaces, but we note here that Kodaira showed in [9] that in the moduli space for K3 surfaces that we will use, projective algebraic K3 surfaces are dense in the moduli space, similar to the density of degree zero tori that we described above. If we let

$$S_0 := \{ Z \in M : z_{12} = 0 \},$$

and set

$$\tilde{S} = S_0 \cup S,$$

then $\tilde{S}$ is a closed subset of $M$, and its complement $\tilde{M} \subset M_0$ is an open subset of $M$. It follows that projective algebraic tori are not dense in $M$, in contrast to Kodaira’s result for K3 surfaces.

### 3 Hopf surfaces

In 1948 Heinz Hopf introduced [8] a class of compact complex surfaces that have become known as Hopf surfaces. These were the first examples of compact complex manifolds which were not Kähler manifolds. Any compact Kähler manifold $X$ has the property that the odd Betti numbers of $X$, $b_1, b_3, \ldots$, are all even integers, and the Hopf surfaces have $b_1 = 1$, as we will see below, and hence are not Kähler (see, e.g., Wells [22] as a reference for the theory of Kähler manifolds).

Let us start with a simple example, as in Hopf’s paper. Let

$$W := \mathbb{C}^2 - \{0\},$$

be the punctured complex 2-plane, which will play an important role throughout this section. This is a simply-connected noncompact complex manifold which will play the role of a universal covering space for many of our compact complex manifolds we will be discussing in the paragraphs below. Let

$$\gamma : W \to W,$$
be a holomorphic mapping defined by
\[ \gamma(z_1, z_2) = (2z_1, 2z_2), \]
and let \( \gamma^n \) denote \( n \) iterations of this mapping. This generates a discrete transformation group \( \Gamma \) which acts on \( W \), and we define the quotient space
\[ X := W/\Gamma. \]

One can easily verify (see, e.g., Wells [22], p. 200) that \( X \) is a compact complex manifold which is diffeomorphic to \( S^1 \times S^3 \), where \( S^1 \) and \( S^3 \) are the one-sphere and three-sphere, respectively, defined by
\[ S^n := \{ x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1 \}. \]
Thus \( X \) has Betti number \( b_1 = 1 \) and is not a Kähler surface, as mentioned above.

In 1958 Kodaira and Spencer introduced their fundamental theory of deformations of compact complex manifolds [11]. They applied this theory to a variety of examples of specific classes of complex manifolds, including tori as we discussed in the previous section, hypersurfaces of complex projective spaces, Hopf surfaces, etc. We will summarize their deformation theory for Hopf surfaces to give us a suitable moduli space with which we can formulate our theorem concerning degree zero Hopf surfaces.

Let \( M \) be the four-dimensional complex manifold of \( 2 \times 2 \) matrices
\[ t = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \]
which are nonsingular linear transformations
\[ t : \mathbb{C}^2 \to \mathbb{C}^2, \]
and which have the property that
\[ |\alpha + \delta| > 3, |(\alpha - \delta)^2 + 4\beta\gamma| < 1. \]

Then the eigenvalues of \( t \in M \) are
\[ \sigma \pm \sqrt{\Delta}, \text{ where } \sigma = \frac{1}{2}(\alpha + \delta), \Delta = \frac{1}{4}(\alpha - \delta)^2 + \beta\gamma. \]
Letting \( W = \mathbb{C}^2 - \{0\} \), as before, we define the holomorphic automorphisms of \( W \times M \) by
\[
\eta : (z, t) \mapsto (tz, t).
\]
Since the eigenvalues of \( t \) satisfy \( |\sigma \pm \sqrt{\Delta}| > 1 \), it follows that
\[
G = \{\eta^m, m \in \mathbb{Z}\}
\]
is an infinite cyclic group which is a properly discontinuous group of biholomorphic mappings with no fixed points, and thus the quotient space
\[
X := (W \times M)/G
\]
is a complex manifold.

Let
\[
p : W \times M \to X
\]
be the canonical projection and thus there is commutative diagram
\[
\begin{array}{ccc}
W \times M & \xrightarrow{p} & X \\
\downarrow & & \downarrow \pi \\
M & \leftarrow & \leftarrow
\end{array}
\]
and the triple \((X, \pi, M)\) is a complex-analytic family of complex manifolds. We let
\[
X_t := \pi^{-1}(t) = W/G_t,
\]
where \( G_t \) is the infinite cyclic group generated by \( t \) acting on \( W \). The manifold \( X_t \), for \( t \in M \), is a Hopf surface generalizing our example (10) above. Again we have that \( X_t \) is diffeomorphic to \( S^1 \times S^3 \) for all \( t \in M \).

Kodaira and Spencer proved the following important theorem in [11].

**Theorem 3** Let \( t, t' \in M \), then \( X_t \) is biholomorphic to \( X_{t'} \) if and only if there is a \( u \in GL(2, \mathbb{C}) \) such that
\[
t' = utu^{-1}.
\]
In other words, the Hopf surfaces are biholomorphic if and only if the transformations \( t \) and \( t' \) correspond to a linear change of variables in \( \mathbb{C}^2 \).
Using this theorem we see that there are naturally three classes of distinct Hopf surfaces. Namely, let

\[
M_0 = \{ t \in M : t = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}, \alpha \neq \delta \},
\]

\[
M_1 = \{ t \in M : t = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \},
\]

\[
M_2 = \{ t \in M : t = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \}.
\]

We note that these are all biholomorphically distinct Hopf surfaces, with the single exception that if \( t \in M_0 \), then changing the order of the two distinct eigenvalues in the matrix \( t \) yields a biholomorphically equivalent Hopf surface. This follows from the identity

\[
\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1}
\]

In a sufficiently small neighborhood \( U \) of any point \( t_0 \in M_0 \), all the Hopf surfaces \( X_t \) will be biholomorphically distinct. In fact Kodaira and Spencer show that \( M_0 \) is a two-dimensional moduli space for this class of Hopf surfaces. Namely, the Kodaira-Spencer mapping

\[
\rho_t : T_t(M_0) \to H^1(X_t, \Theta_t),
\]

is an isomorphism, where here \( \Theta_t \) is the sheaf of holomorphic vector fields on \( X_t \). In particular,

\[
\dim T_t(M_0) = \dim H^1(X_t, \Theta_t) = 2.
\]

The deformation theory for \( M_1 \) and \( M_2 \) is somewhat more complex (see [11]), but this is not necessary for our purposes here, as we will see below.

We now have our basic theorems concerning the transcendence degrees of Hopf surfaces. Let \( m \) and \( n \) be nonzero integers and define

\[
Z_{m,n} := \{ t = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} : \alpha^m = \delta^n \}.
\]

We see that, for each \( m \) and \( n \), \( Z_{m,n} \) is a nonsingular hypersurface in \( M_0 \) of complex dimension one.
Theorem 4 If 

\[ t \in Z_{m,n}, M_1, \text{ or } M_2 \]

then 

\[ \deg X_t = 1. \]

Proof: First we note that for \( t \in M \), \( \deg X_t \leq 1 \). Namely, if \( \deg X_t = 2 \), it follows from a theorem of Kodaira [9] that \( X_t \) would be projective algebraic and hence Kähler. But this would imply that \( b_1(X_t) \) would be even, but that is not the case since \( X_t \) is diffeomorphic to \( S^1 \times S^3 \).

To show that \( \deg X_t = 1 \) in each of these three cases, it suffices to find a nonconstant meromorphic function \( f \) on \( C^2 \) which satisfies 

\[ f(tz) = f(z). \]

Suppose that \( t \in Z_{m,n} \), then let 

\[ f(z) = f(z_1, z_2) = \frac{z_1^m}{z_2^n} \]

then 

\[ f(tz) = f(\alpha z_1, \delta z_2) = \frac{(\alpha z_1)^m}{(\delta z_2)^n} = \frac{\alpha^m z_1^m}{\delta^n z_2^n} = f(z_1, z_2) = f(z). \]

If \( t \in M_1 \), simply choose 

\[ f(z) = f(z_1, z_2) = \frac{z_1}{z_2}, \]

and argue in the same way.

If \( t \in M_2 \), then it is somewhat more complicated but quite straightforward. Choose 

\[ f(z_1, z_2) = \frac{z_1 + 2z_2 + c_1}{z_1 + 2z_2 + c_2}, \quad (11) \]

where 

\[ c_1 = -\frac{2\alpha^2 + 9\alpha}{3}, \]

\[ c_2 = \frac{2\alpha^2 + 15\alpha}{3(\alpha - 1)}. \]
It is easy to verify that

\[ f(tz) = f(\alpha z_1 + z_2, \alpha z_2) = f(z_1, z_2), \]

and thus defines a nonconstant meromorphic function on \( X_t \). We note that in this example (11) the meromorphic function \( f \) on \( \mathbb{C}^2 \) depends on the parameter \( \alpha \), which was not the case in the first two examples here. q.e.d.

We now have a classical result concerning Hopf surfaces \( X_t \), for \( t \in M_0 \) (see, for instance, Shafarevich [18] or Barth et al [1]).

**Theorem 5**  Let \( t \in M_0 \), then

\[ \deg X_t = 1 \]

if and only if there exist nonzero integers \( m \) and \( n \) such that

\[ \alpha^m = \delta^n. \]

**Proof:** By Theorem 4 we have seen that if \( \alpha^m = \beta^n \), for some integers \( m \) and \( n \), then \( \deg X_t = 1 \). We need to show the converse. Suppose that

\[ \alpha^m \neq \beta^n, \]

for any integers \( m \) and \( n \). If \( \deg X_t = 1 \), then there is a meromorphic function \( f \) on \( W \) such that

\[ f(tz) = f(z). \]

By Hartogs’ theorem for meromorphic functions due to E. E. Levi [14], \( f \) extends as a meromorphic function to \( \mathbb{C}^2 \) (see, e.g., the classical monographs by Osgood [15] and Behnke and Thullen [2] for a discussion of this theorem). We can assume that \( f \) has either a zero or a pole at the origin. Namely, if \( f \) has a pole at \( z = 0 \), then this \( f \) suffices. If \( f \) does not have a pole, then simply replace \( f \) by \( f(z) - f(0) \), and this will also satisfy our requirement.

By an even older theorem of Poincaré from 1883 [16], the function \( f \) can be expressed as the quotient of two holomorphic functions \( g(z) \) and \( h(z) \) on \( \mathbb{C}^2 \),

\[ f(z) = \frac{g(z)}{h(z)}. \]
Let
\[ g(z) = \sum_{ij} a_{ij} z_i^1 z_j^2, \]
\[ h(z) = \sum_{kl} b_{kl} z_k^1 z_l^2, \]
be power series representations for \( g \) and \( h \) on \( \mathbb{C}^2 \), then we must have
\[ g(tz) = g(z), \]
\[ h(tz) = h(z), \]
which gives
\[ \left( \sum_{ij} a_{ij} \alpha^i \delta^j z_i^1 z_j^2 \right) \left( \sum_{kl} b_{kl} z_k^1 z_l^2 \right) = \left( \sum_{ij} a_{ij} z_i^1 z_j^2 \right) \left( \sum_{kl} b_{kl} \alpha^k \delta^l z_k^1 z_l^2 \right). \] (12)

Let \( \mu \) be the order of \( g \) and let \( \nu \) be the order of \( h \), and we have that \( \mu + \nu \geq 1 \). Consider the homogenous terms in (12) of lowest order and we have that
\[ \left( \sum_{i+j=\mu} a_{ij} \alpha^i \delta^j z_i^1 z_j^2 \right) \left( \sum_{k+l=\nu} b_{kl} z_k^1 z_l^2 \right) = \left( \sum_{i+j=\mu} a_{ij} z_i^1 z_j^2 \right) \left( \sum_{k+l=\nu} b_{kl} \alpha^k \delta^l z_k^1 z_l^2 \right). \] (13)

In this equation, at least one \( a_{ij} \neq 0 \) and at least one \( b_{kl} \neq 0 \). Let \( a_{i_0j_0} \) be the unique nonzero coefficient with the property that
\[ a_{ij} = 0, \text{ for } 0 \leq i < i_0. \]

Similarly, let \( b_{k_0l_0} \) be the unique nonzero coefficient such that
\[ b_{kl} = 0, \text{ for } 0 \leq k < k_0. \]

Then we can rewrite (13) as
\[ (a_{\mu_0} \alpha^\mu z_1^\mu + \cdots + a_{i_0j_0} \alpha^{i_0} \delta^{j_0} z_1^{i_0} z_2^{j_0}) (b_{k_0} z_1^{k_0} + \cdots + b_{k_0l_0} z_1^{k_0} z_2^{l_0}) = (a_{\mu_0} z_1^\mu + \cdots + a_{i_0j_0} z_1^{i_0} z_2^{j_0}) (b_{k_0} \alpha^\nu z_1^\nu + \cdots + b_{k_0l_0} \alpha^{k_0} \delta^{l_0} z_1^{k_0} z_2^{l_0}). \] (14)

We see by expanding these products that the two terms on the left side of the equation
\[ a_{\mu_0} b_{\nu_0} \alpha^\mu z_1^{\mu+\nu}, \ a_{i_0j_0} b_{k_0l_0} \alpha^{i_0} \delta^{j_0} z_1^{i_0+k_0} z_2^{j_0+l_0}, \]
are both unique terms of their respective bidegrees, and this is similar for the terms on the right hand side of the equation\textsuperscript{4}.

It follows that we have the equality
\[ a_{i_0j_0} b_{k_0l_0} \alpha^{i_0} \delta^{k_0} z_1^{i_0+k_0} z_2^{j_0+l_0} = a_{i_0j_0} b_{k_0l_0} \alpha^{k_0} \delta^{l_0} z_1^{i_0+k_0} z_2^{j_0+l_0}, \]
and both sides of this equality are nonzero for \( z_1 \) and \( z_2 \) both being nonzero. This implies that
\[ \alpha^{i_0-k_0} = \delta^{j_0-l_0}, \]
and hence we have a contradiction. q.e.d.

As a simple example, take
\[ t = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}, \]
and we see by the fundamental theorem of arithmetic that \( 3^m \neq 5^n \), for any integers \( m \) and \( n \), and hence
\[ \deg X_t = 0, \]
in this case.

The following theorem is now an easy consequence of this result. Let
\[ Z := \bigcup_{m,n} Z_{m,n}. \]
Here, as before, \( m \) and \( n \) range over the nonzero integers. We then see that \( Z \) is a countable union of hypersurfaces of dimension one in \( M_0 \).

\textbf{Theorem 6} The set
\[ M_0 - Z \]
is a set of second category in the metric space \( M_0 \), and, in particular, is dense in \( M_0 \). Moreover, for each \( t \in M_0 - Z \),
\[ \deg X_t = 0. \]

Thus we see that Hopf surfaces also have the property that generically Hopf surfaces have no nonconstant meromorphic functions.

\textsuperscript{4}I would like to thank Georges Dloussky for pointing out and correcting an error in this part of my argument that appeared in an earlier draft of this paper.
4 K3 surfaces

In the previous two sections we studied two-dimensional complex tori and Hopf surfaces. Both of these classes of surfaces are defined as quotients of a fixed Euclidean space ($\mathbb{C}^2$ or $\mathbb{C}^2 - \{0\}$) by discrete transformation groups acting on these spaces. In contrast, K3 surfaces are defined by abstract cohomological conditions, and they were first formulated by Weil in 1958 [21] in honor of Kummer, Kähler and Kodaira (as well as a tribute to the Himalayan Peak K2). They have been studied quite intensely since, and a very readable and informative recent survey is in Buchdahl’s paper [3]. In this paper we want to show that a dense set of K3 surfaces, in a suitably defined moduli space, have transcendence degree zero, i.e., have no nonconstant meromorphic functions.

We will use the standard sheaf-theoretical cohomology theory of several complex variables and complex manifold theory (see, e.g., Wells [22] or Griffiths and Harris [6]). In particular, we will use the standard invariants for compact complex manifolds. Let $X$ be an $n$-dimensional compact complex manifold, then

\[ b_i = \dim H^i(X, \mathbb{C}), \quad \text{Betti numbers,} \]
\[ b^+ = \text{no. of positive eigenvalues of fundamental quadratic form on} \]
\[ H^n(X, \mathbb{C}), \]
\[ h^{p,q} = \dim H^{p,q}(X, \Omega^p), \quad \text{Hodge numbers.} \]

Moreover, we let $c_i(E)$ denote the Chern classes of a holomorphic vector bundle on $X$.

There are various equivalent definitions of K3 surfaces, and we choose the following one. Let $S$ be a surface, and let

\[ q = h^{0,1} = \dim H^1(S, \mathcal{O}), \]

be the irregularity of $S$, and let

\[ p_g = h^{2,0} = \dim H^0(S, \Omega^2), \]

be the geometric genus of $S$. We define a K3 surface to be a surface which has a trivial canonical bundle $K_S = \wedge^2 T^*(S)$, where $T(S)$ is the tangent bundle to $S$, and where the irregularity $q = 0$. It follows that the geometric genus for a K3 surface is simply $p_g = 1$. 

19
Moreover, for any surface $S$ with $q = 0$, we note that for any holomorphic line bundle $F$ on $S$, $c_1(F) = 0$ if and only if $F$ is trivial. The Chern class of a surface is defined as the Chern class of its tangent bundle, and thus we have

\[
  c_1(S) = c_1(T(S)),
  = c_1(\wedge^2 T(S)),
  = -c_1(K_S).
\]

It follows that $c_1(S) = 0$ if and only if the canonical bundle is trivial, which is an alternative definition of a K3 surface.

Kodaira also shows in [9] that for any surface,

\[
  q = \begin{cases} 
  b_1/2, & \text{if } b_1 \text{ is even,} \\
  (b_1 - 1)/2, & \text{if } b_1 \text{ is odd.}
  \end{cases}
\]

Thus, the two defining numerical invariants for K3 surfaces are topological invariants.

In Kodaira’s paper [9] there are various relationships derived between the various cohomological invariants for the case of surfaces, and, in particular for K3 surfaces, using the above definition. The resulting Betti and Hodge numbers for K3 surfaces are as follows:

\[
  b_1 = 0, \\
  b_2 = 22, \\
  b^+ = 3, \\
  h^{2,0} = h^{0,2} = 1, \\
  h^{1,1} = 20.
\]

There are many examples of K3 surfaces which are projective algebraic, in particular, any quartic in $\mathbb{P}_3$, such as\(^5\)

\[
  z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0.
\]

is a K3 surface (see, e.g., [6]).

As an illustration, we give the simple proof here. Let

\[
  Q = \{ z \in \mathbb{P}_3 : f(z) = 0 \},
\]

\(^5\)We denote projective space over the complex numbers $\mathbb{P}_m(\mathbb{C})$ simply as $\mathbb{P}_m$ in this paper.
where \( f \) is a homogeneous polynomial of degree four. By the Lefschetz hyperplane section theorem (e.g., [6]),
\[
H^0(Q, \mathbb{C}) \cong H^0(P_3, \mathbb{C}) \cong \mathbb{C}
\]
\[
H^1(Q, \mathbb{C}) \cong H^1(P_3, \mathbb{C}) = 0,
\]
and thus \( b_1(Q) = 0 \). Since \( Q \) is a projective algebraic manifold,
\[
b_1 = h^{1,0} + h^{0,1},
\]
and hence
\[
q(Q) = h^{0,1} = 0.
\]
On the other hand, if \([Q]\) is the line bundle on \( P_3 \) of the divisor defined by the hypersurface \( Q \), then
\[
[K_Q] = H^4,
\]
where \( H \) is the hyperplane section bundle on \( P_3 \). The adjunction formula for a hypersurface in a complex manifold (e.g., [6]) gives us the canonical bundle of the hypersurface in terms of the divisor defined by the hypersurface and the canonical bundle of the complex manifold. Namely, in this case here we have
\[
K_Q = K_{P_3} \otimes [Q]
\]
\[
= H^{-4} \otimes H^4,
\]
\[
= H^0,
\]
which is the trivial bundle. Thus \( Q \) is a K3 surface.

Moreover, Kodaira discusses in great detail K3 surfaces of transcendence degree one which are all elliptic surfaces, which are fibre spaces over \( P_1 \) with generic fiber being an elliptic curve and with specific classes of singular fibers.

Let now \( S \) be an arbitrary K3 surface, and, as it is an orientable four dimensional compact differentiable manifold, it has a natural quadratic form acting on \( H^2(S, \mathbb{Z}) \) which can be defined by
\[
A(\xi, \eta) = \int_S \xi \wedge \eta,
\]
where \( \xi \) and \( \eta \) are closed two-forms representing the integral cohomology classes in \( H^2(S, \mathbb{Z}) \), noting that torsion plays no role in this quadratic form. The quadratic form \( A \) can be represented by a skew-symmetric integer-valued
matrix with signature (3, 19), i.e., 3 positive eigenvalues and 19 negative eigenvalues.

Let $\gamma_j, j = 1, \ldots, 22$, be two-cycles in $S$ which generate $H_2(S, \mathbb{Z})$, and let $\eta_i$ be closed two-forms on $S$ which are dual to $\{\gamma_j\}$, i.e.,

$$\int_{\gamma_j} \eta_i = \delta_{ij}.$$  

This is all independent of the complex structure on $S$.

Since the canonical bundle $K_S$ is trivial, then there is a nowhere vanishing holomorphic two form $\psi$ on $S$. Moreover, if $\tilde{\psi}$ is any other nowhere vanishing two-form on $S$, then

$$\tilde{\psi} = c\psi,$$

where $c$ is a constant since,

$$\dim H^0(S, \Omega^2) = 1.$$  

We now define

$$\lambda(S, \psi) = (\lambda_1, \lambda_2, \ldots, \lambda_{22}) \in \mathbb{C}^{22},$$

by

$$\lambda_j = \int_{\gamma_j} \psi.$$  

Since any other nowhere vanishing holomorphic two-form differs from $\psi$ by a constant, we see that $\lambda(S, \psi)$ defines a mapping

$$S \mapsto \mathbb{P}_{21},$$

which maps $S$ to a point in the projective space $\mathbb{P}_{21}$ which is independent of which two-form $\psi$ is used. This is called the period mapping for K3 surfaces.

As Kodaira points out in [9], this period mapping which assigns to any K3 surface $S$ with the given underlying differentiable manifold structure to a point in $\mathbb{P}_{21}$ and some of its properties were introduced by Andreotti and Weil. It will be an essential tool in our investigation of K3 surfaces which are of degree zero, as we shall see in the following paragraphs.

Let now $\psi$ be a holomorphic two-form on $S$. It is also a closed two-form. Namely,

$$d\psi = \partial\psi + \bar{\partial}\psi,$$
and $\partial \psi$ is zero by type and $\overline{\partial} \psi = 0$, since $\psi$ is holomorphic. Thus there are coefficients $(\xi_1, \ldots, \xi_{22})$ such that

$$\psi = \sum_{i=1}^{22} \xi_i \eta_i + d\varphi,$$

where $\varphi$ is a one-form on $S$. It follows that

$$\int_{\gamma_j} \psi = \int_{\gamma_j} \sum_i \xi_i \eta_i + 0,$$

$$= \sum_i \xi_i \int_{\gamma_j} \eta_i = \xi_j,$$

and thus $\xi_i = \lambda_i$. Moreover,

$$0 = \int \psi \wedge \psi,$$

$$= \int_S \left( \sum_i \lambda_i \eta_i \right) \wedge \left( \sum_j \lambda_j \eta_j \right),$$

$$= \sum_{i,j} \lambda_i \lambda_j \int_S \eta_i \wedge \eta_j,$$

$$= \sum_{i,j} a_{ij} \lambda_i \lambda_j.$$

Thus $\lambda(S)$ is a point on the quadric

$$Q := \{ [z_1, \ldots, z_{22}] \in P_{21} : \sum_{i,j} a_{ij} z_i z_j = 0 \}.$$

An important question is: how many K3 surfaces are there? This has been investigated substantively over the past decades as is outlined in Buchdahl’s paper [3], but for our purposes we will use the local deformation setting formulated and used by Kodaira in his 1964 paper [9] using the well known deformation theory of Kodaira and Spencer [11]. We formulate this as the following theorem, and we will outline Kodaira’s proof from [9], as it is relatively short and quite informative. Let $\Theta_S$ be the sheaf of holomorphic vector fields on $S$.

**Theorem 7** Let $S$ be a K3 surface, then there is a complex-analytic family $(\mathcal{F}, \pi, M)$, where $\mathcal{F} \xrightarrow{\pi} M$, and $M$ is an open neighborhood of $0 \in \mathbb{C}^{20}$, and where

$$\mathcal{F}_t := \pi^{-1}(t), t \in M,$$
are $K3$ surfaces diffeomorphic to $S$, and

$$S = F_0 = \pi^{-1}(0).$$

Moreover, the Kodaira-Spencer mapping

$$\rho_0 : T(M)_0 \rightarrow H^1(S, \Theta_S),$$

is an isomorphism and the family is complete and effectively parametrized.

**Proof:** Let $\psi$ be a nowhere vanishing holomorphic two-form on $S$, then it is easy to see that

$$\Theta_S \cong \Omega^2_S.$$ \hspace{1cm} (15)

Namely, in any local coordinate system $(z_1, z_2)$ on $S$, we have

$$\psi = \frac{1}{2} \sum_{\alpha, \beta} \psi_{\alpha\beta} dz_\alpha \wedge dz_\beta, \quad \psi_{\alpha\beta} = -\psi_{\beta\alpha}.$$ 

If

$$v = \sum_{\alpha} v_\alpha \frac{\partial}{\partial z_\alpha},$$

is a locally defined holomorphic vector field on $S$, then the mapping

$$\sum_{\alpha} v_{\alpha} \frac{\partial}{\partial z_\alpha} \mapsto \sum_{\beta} v_{\alpha} \psi_{\alpha\beta} dz_\beta,$$

induces the required isomorphism (15).

Kodaira, Nirenberg, and Spencer [10] showed that if a compact complex manifold $X$ satisfies $H^2(X, \Theta_X) = 0$, then there is a complex-analytic family $(F, \pi, M)$ with

$$\rho_0 : T_0(M) \rightarrow H^1(X, \Theta_X),$$

being an isomorphism and $X = \pi^{-1}(0)$, and thus

$$\dim M = \dim H^1(X, \Theta_X).$$

So in our case, by duality,

$$h^{2,1} = h^{0,1} = 0,$$

24
and hence,

\[ H^2(S, \Theta_S) = H^2(S, \Omega^1) = 0, \]

so the hypothesis of the Kodaira-Nirenberg-Spencer theorem is satisfied, and

\[ \dim H^1(S, \Theta_S) = \dim H^1(S, \Omega^1) = 20. \]

q.e.d.

We note that in the family \( \mathcal{F} \to M \) with \( \mathcal{F}_0 = S \), all of the deformations of \( S \) are also K3 surfaces. This follows since \( q \) and \( c_1(S) \) are topological invariants, as noted earlier.

Since each \( \mathcal{F}_t \) is a K3 surface, then it follows that

\[ \dim H^1(\mathcal{F}_t, \Theta_t) = 20. \]

It follows from the stability theorem of Kodaira and Spencer [13] and a completeness theorem of Kodaira and Spencer [12] that the family is effectively parametrized and complete. q.e.d.

We now have the following theorem of Kodaira [9] which he attributes to Andreotti and Weil. This is often known as a local Torelli theorem in this context of K3 surfaces. We will omit any summary of the proof here. Essentially, this theorem represents the local moduli parameter space \( M \) in Theorem 7 in a quite specific form as an open subset of the quadric \( Q \subset \mathbb{P}_{21} \).

Here, as before, \( S \) is a given K3 surface and \( (\mathcal{F}, \pi, M) \) is the complex analytic family given in Theorem 7.

**Theorem 8** Let

\[ \Lambda : M \to Q \subset \mathbb{P}_{21} \]

be defined by

\[ \Lambda(t) = \lambda(\mathcal{F}_t), \]

and let

\[ p_0 := \Lambda(0) = \lambda(S) \in Q. \]

Then there is an open spherical neighborhood of \( 0 \in M \) and a neighborhood \( W \subset Q \) of \( p_0 \) such that \( \Lambda \) maps \( U \) biholomorphically onto \( W \).

We remark that the fibres of the family \( \mathcal{F} \to M \) as in Theorem 7 are all diffeomorphic, and for \( t \) and \( t' \) in a sufficiently small neighborhood of \( 0 \in M \), \( \mathcal{F}_t \) and \( \mathcal{F}_{t'} \) are biholomorphic if and only if \( t = t' \), i.e., the fibers are locally biholomorphically distinct.
This theorem allows us to use the coordinate geometry of $Q \subset \mathbb{P}_{21}$ to
investigate the deformations of $S$.

Kodaira uses this geometric setting to show that there exist projective
algebraic K3 surfaces arbitrarily close to any given K3 surface $S$ (in the sense
of the deformation theory above). We want to now show that arbitrarily close
to a given K3 surface $S$, there are K3 surfaces of degree zero.

We first have the following Lemma which gives a simple criterion for the
existence of nonconstant meromorphic functions.

**Lemma 9** Let $X$ be a compact complex manifold, then if $f$ is a noncon-
stant meromorphic function on $X$, then there is a nontrivial holomorphic
line bundle on $X$.

**Proof:** Suppose $f$ is a nonconstant meromorphic function on $X$, then if
$(f)$ is the divisor associated to the meromorphic function $f$, then $(f)$ is the
difference of two effective divisors,

$$(f) = D_+ - D_-,$$

where $D_+$ corresponds to the zero set of $f$ and $D_-$ corresponds to the polar
set of $f$, both of which are subvarieties of $X$. Thus, either of the two divisors
$D_+$ or $D_-$ must then give rise to a nontrivial holomorphic line bundle on $X$.
q.e.d.

Now we have a lemma due to Kodaira [9], which helps characterize non-
trivial holomorphic line bundles on a K3 surface in terms of the bilinear f
orm $A$ on $\mathbb{C}^{22}$. The proof is not difficult, and we refer the reader to Kodaira’s
paper. Again $S$ is a fixed K3 surface.

**Lemma 10** A cohomology class $c \in H^2(S, \mathbb{Z})$ is the Chern class of a holo-
morphic line bundle $F$ over $S$, if and only if the point

$$m = (m_1, \ldots, m_{22}), m_j = \int_{\gamma_j} c,$$

satisfies the linear equation

$$A(\lambda, m) = 0,$$

where $\lambda = \lambda(S)$.

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6 Thanks to Nicholas Buchdahl for suggesting this lemma and its proof to me.
We note that the line bundle $F$ is nontrivial if and only if $m \neq 0$, since the irregularity $q(S) = 0$.

We can now formulate our fundamental result concerning K3 surfaces with no nonconstant meromorphic functions. We let $S$ be an arbitrary K3 surface, and let $\mathcal{F} \to M$ be the local deformation space with $\mathcal{F}_0 = S$, where $t$ is the local parameter in $M$, as given in Theorem 7.

**Theorem 11** For a sufficiently small neighborhood $U$ of 0 in $M$, there is a set $\Omega \subset U$ of second category, which, in particular, is dense in $U$, such that if $t \in \Omega$, then $\deg \mathcal{F}_t = 0$.

**Proof:** We use Theorem 8 to biholomorphically map a sufficiently small neighborhood $U$ of 0 onto a neighborhood $W \subset Q$ of $p_0 = \Lambda(0)$. For any $\Lambda(t)$, for $t \in U$, $\mathcal{F}_t$ has a nontrivial holomorphic line bundle $F$ if and only if

$$\int_{\gamma_j} c(F),$$

is a nonzero integer for some $j, j = 1, \ldots, 22$. Let now

$$m = (m_1, \ldots, m_{22}) \in \mathbb{Z}^{22} - \{0\},$$

and we let

$$\tilde{Z}_m := \{z \in \mathbb{C}^{22} : A(z, m) = 0\}.$$

Each such $\tilde{Z}_m$ defines a linear hyperplane in $\mathbb{P}_{21}$, and we let

$$Z_m := \tilde{Z}_m \cap W,$$

which is a complex subvariety of $W$ with $\dim Z_m \leq 19$, and, of course, the intersection could well be an empty set for some $m$. In any event, $Z_m$ is a closed subset of $W$ with no interior points.

We let

$$Z := \bigcup_{m \in \mathbb{Z}^{22} - \{0\}} Z_m,$$

and this is a countable union of closed subsets of $W$ with no interior points, and hence

$$\Omega := W - Z,$$
is a set of second category in $W$ which is dense in $W$. As we saw earlier, any point $p \in W$ with $\deg \mathcal{F}_{\Lambda^{-1}(p)}(p) \geq 1$ must be a point in $Z_m$ for some $m \neq 0$. It follows that if $p \in \Omega$,

$$\deg \mathcal{F}_{\Lambda^{-1}(p)} = 0.$$ 

q.e.d

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