Foreword

These are the English translation of the papers published only on Russian:
V.P. Kotlyarov, Periodic problem for the Schrödinger nonlinear equation,
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"Doklady Akad.Nauk Ukrainian SSR", ser.A, №10, 1976, pp.965-968.

During the last time I have received some requests on these papers. I hope this reprint will be useful for interested readers (V.Kotlyarov).

PERIODIC PROBLEM FOR THE NONLINEAR SCHRÖDINGER EQUATION

V.P. KOTLYAROV

Abstract. The paper offers the method of discovering of some class of solutions for the nonlinear Schrödinger equation

$$i u_t + u_{xx} + 2|u|^2u = 0.$$  

An algorithm of constructive solving of the Cauchy periodic problem with a finite-gap initial condition was also obtained.

1. Introduction

In [1] was discovered a method of the construction of decreasing solutions of the Korteweg de Vries (KdV) equation. The method bases on the inverse scattering theory for the Sturm-Liouville operator. Recently this method was developed [2]-[5] for obtaining periodic solutions to the KdV equation.

In the present paper we investigate the periodic finite-gap problems for the nonlinear Schrödinger equation

$$i v_t + v_{xx} + 2|v|^2v = 0,$$

by using a method from [2].

The periodic finite-gap problem for another type of nonlinear equation

$$i v_t + v_{xx} - 2|v|^2v = 0,$$

is studied in [6], [11]. In this case (equation (6.1)), the spectral problem is connected with the Dirac self-adjoint operator while for equation (1.1) the spectral problem is not self-adjoint.

For the first time these nonlinear equations were studied in [7], where the Cauchy problem was solved for the decreasing (at infinity) initial functions. The authors have developed the inverse scattering method for the both nonlinear equations. For this purpose they gave the formulation and solution of the scattering problems for the self-adjoint and not self-adjoint Dirac operators.
2. Periodic finite-gap potentials

Taking into account ideas of [2], let us introduce a set of complex-valued $l$-periodic potentials

\begin{equation}
(2.1) \quad v(x, x_0, t) := v(x + x_0, t), \quad v(x + l, x_0, t) = v(x, x_0, t)
\end{equation}

and related with this set, the Dirac operators

\begin{equation}
(2.2) \quad L := J \frac{d}{dx} + Q, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\end{equation}

with the not self-adjoint potential matrix

\[ Q = \begin{pmatrix} 0 & iv(x, x_0, t) \\ iv(x, x_0, t) & 0 \end{pmatrix}. \]

The function $v(x, t)$ is supposed to be twice continuously differentiable in $x$ and has the first continuous derivative in $t$.

Let vector $y(x, z)$ is a solution of the equation

\begin{equation}
(2.3) \quad Ly = zy,
\end{equation}

where $z$ is an arbitrary complex number. We introduce additionally two sets of operators:

\[ M_k := D_k + A_k(x, z, x_0, t), \quad D_1 := \frac{\partial}{\partial t}, \quad D_2 := \frac{\partial}{\partial x_0}, \]

where matrices $A_k$ ($k = 1, 2$) are choosing in such a way that

\[ (L - zI)M_k y = \{ JA'_k + JA_k J(Q - zI) + (Q - zI)A_k - D_k Q \} y \equiv B_k(x, x_0, t)y. \]

It is very important that $B_k$ have to be independent on $z$. Such matrices $A_k$ and $B_k$ do exist and have the form:

\begin{align*}
A_1 &= i \begin{pmatrix} 2z^2 - |v|^2 & v_0' - 2izv \\ -2i zv & -2z^2 + |v|^2 \end{pmatrix}, \quad B_1 = - \begin{pmatrix} 0 & v''x + 2|v|^2 v + iv\bar{v} \\ -v''x - 2|v|^2 \bar{v} + iv\bar{v} & 0 \end{pmatrix}, \\
A_2 &= \begin{pmatrix} iz & v \\ -\bar{v} & -iz \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}

Now let $v = v(x, t)$ satisfies equation (2.3). Then $B_1 \equiv 0$ and, hence, operator $M_k$ transforms a solution of equation (2.3) into another solution of the same equation (2.3). Let $\Phi(x, z) \equiv \Phi(x, z, x_0, t)$ be the fundamental matrix of equation (2.3) and let $\Phi(z, x_0, t) \equiv \Phi(l, z, x_0, t)$ be the monodromy matrix on interval $(0, l)$. Then, if $y(x, z)$ is a solution of equation (2.3), then $y(x, z) = \Phi(x, z)y(0, z)$. Since operator $M_k$ transforms a solution of equation (2.3) into another solution of the same equation then

\[ \tilde{y}_k(x, z) := M_k(x)y(x, z) = M_k(x)\Phi(x, z)y(0, z) \]

and

\[ \tilde{y}_k(x, z) := \Phi(x, z)\tilde{y}_k(0, z) = \Phi(x, z)M_k(0)y(x, z). \]

Hence

\[ \{ M_k(x)\Phi(x, z) - \Phi(x, z)M_k(0) \} y(0, z) = 0. \]

Let us put $x = l$ in the last equation. In view of periodicity of the function $v$ we have $A_k(0, z, x_0, t) \equiv A_k(l, z, x_0, t)$ and therefore the last equation gives the following differential equation for the monodromy matrix:

\begin{equation}
(2.4) \quad D_k\Phi(z, x_0, t) = \Phi(z, x_0, t)A_k(l, z, x_0, t) - A_k(l, z, x_0, t)\Phi(z, x_0, t), \quad k = 1, 2
\end{equation}

with initial conditions:

\[ \Phi(z, x_0, t)|_{t=0} = \Phi_1(z, x_0), \quad \Phi(z, x_0, t)|_{x_0=0} = \Phi_2(z, t), \]
where $\Phi_1(z, x_0)$ is the monodromy matrix of operators (2.2) with $v = v(x, x_0, 0)$, and $\Phi_2(z, t)$ is the monodromy matrix of operators (2.2) with $v = v(x, 0, t)$.

Let us put
\[
\begin{align*}
2g &= \Phi_{11} + \Phi_{22}, \\
2if &= \Phi_{11} - \Phi_{22}, \\
\psi &= \Phi_{12}, \\
\varphi &= \Phi_{21},
\end{align*}
\]
where $\Phi_{ik}, i, k = 1, 2$ are entries of the monodromy matrix $\Phi(z, x_0, t)$. Then matrix equation (2.4) gives $D_k g(z, x_0, t) = 0$, i.e. $g(z, x_0, t)$ is independent on $x_0$ and $t$. Other scalar equations for functions $f = f(z, x_0, t)$, $\psi = \psi(z, x_0, t)$, $\varphi = \varphi(z, x_0, t)$ are as follows:
\[
\begin{align*}
\dot{f} &= (v' + 2iz\bar{v})\psi - (v' - 2izv)\varphi, \\
\dot{\psi} &= -2(v' - 2izv)f - 2i(2z^2 - |v|^2)\psi, \\
\dot{\varphi} &= 2(v'' + 2iz\bar{v})f + 2i(2z^2 - |v|^2)\varphi,
\end{align*}
\]
and
\[
\begin{align*}
f' &= iv\psi + iv\varphi, \\
\psi' &= 2ivf - 2iz\psi, \\
\varphi' &= 2ivf + 2iz\varphi,
\end{align*}
\]
where the dot and prime mean differentiation in $t$ and $x_0$ respectively, and $v = v(x_0, t)$, $\bar{v} = \bar{v}(x_0, t)$.

The following symmetry property takes place: if vector $y(x, z) = (y_1(x, z), y_2(x, z))$ is a solution of (2.3) then vector $\tilde{y}(x, z) = (-y_2(x, \bar{z}), y_1(x, \bar{z}))$ is also a solution to (2.3). Therefore entries of the monodromy matrix, being entire in $z$ analytic functions of the exponential type, satisfy the following conditions:
\[
\Phi_{22}(z, x_0, t) = \Phi_{11}(\bar{z}, x_0, t), \quad \Phi_{21}(z, x_0, t) = -\Phi_{12}(\bar{z}, x_0, t).
\]

Hence, entire analytic functions $f, \psi, \varphi$ possess properties:
\[
f(z, x_0, t) = \bar{f}(\bar{z}, x_0, t), \quad \varphi(z, x_0, t) = -\bar{\psi}(\bar{z}, x_0, t).
\]

Let $\Phi_0(z)$ be the monodromy matrix of equation (2.4) with independent on $x_0$ and $t$ potential $v = v(x)$ and let corresponding functions $f_0(z)$, $\psi_0(z)$, $\varphi_0(z)$ have the following factorization:
\[
f_0(z) = \tilde{f}_0(z)h(z), \quad \psi_0(z) = \tilde{\psi}_0(z)h(z), \quad \varphi_0(z) = \tilde{\varphi}_0(z)h(z),
\]
where $h(z)$ is an entire analytic function, and $\tilde{f}_0(z)$, $\tilde{\psi}_0(z)$, $\tilde{\varphi}_0(z)$ are some polynomials. One can put the leading coefficient of the polynomial $f_0(z)$ to be equal to one. Such a requirement defines function $h(z)$ uniquely.

**Definition.** The periodic potential is called finite-gap, if the corresponding monodromy matrix $\Phi_0(z)$ possesses properties (2.8).

Consider a translation $v(x, x_0) = v(x + x_0)$ of the finite-gap potential $v(x)$. Then functions $f(z, x_0)$, $\psi(z, x_0)$ and $\varphi(z, x_0)$ defined by the monodromy matrix $\Phi(z, x_0)$ (such that $\Phi(z, 0) = \Phi_0(z)$) are the solution of the system (2.9) obeying to initial conditions:
\[
f(z, 0) = \tilde{f}_0(z)h(z), \quad \psi(z, 0) = \tilde{\psi}_0(z)h(z), \quad \varphi(z, 0) = \tilde{\varphi}_0(z)h(z)
\]
Lemma 2.1. Under initial conditions (2.7) for equations (2.7) their solution \( f(z, x_0), \varphi(z, x_0) \) \( \psi(z, x_0) \) is represented in the form:

\[
(2.10) \quad f(z, x_0) = \tilde{f}(z, x_0)h(z), \quad \psi(z, x_0) = \tilde{\psi}(z, x_0)h(z), \quad \varphi(z, x_0) = \tilde{\varphi}(z, x_0)h(z)
\]

with the same function \( h(z) \), and \( \tilde{f}, \tilde{\psi}, \tilde{\varphi} \) are polynomials precisely the same degrees that the polynomials \( f_0(z), \psi_0(z), \varphi_0(z) \). In addition, periodic \( n \)-gap potential \( v(x_0) \) is expressed by the leading coefficient of polynomial \( \psi(z, x_0) = \psi_n(x_0)z^n + \psi_{n-1}(x_0)z^{n-1} + \ldots + \psi_0(x_0) \) by the formula:

\[
(2.11) \quad v(x_0) = \tilde{\psi}_n(x_0)
\]

Proof. Let \( C(z, x_0) \) be the fundamental solution of the matrix equation (2.3) on the interval \((0, x_0)\). Due to the periodicity of \( v(x) \), \( C(z, x_0) \) is the fundamental solution of the matrix equation (2.3) on the interval \((l, l + x_0)\). Then the monodromy matrix \( \Phi(z, x_0) \), defined on the interval \((l, l + x_0)\), and the matrix \( \Phi_0(z) \), defined on the interval \((0, l)\), are related by equality:

\[
(2.12) \quad \Phi(z, x_0) = C(z, x_0)\Phi_0(z)C^{-1}(z, x_0)
\]

In view of (2.7) entries of the matrix \( \Phi_0(z) \) have the form:

\[
\Phi_0[11] = g + i\tilde{f}_0 h, \quad \Phi_0[12] = \tilde{\psi}_0 h, \quad \Phi_0[21] = \tilde{\varphi}_0 h, \quad \Phi_0[22] = g - i\tilde{f}_0 h.
\]

Take into account that \( \det C(z, x_0) = 1 \) we find from (2.12)

\[
\Phi_0[11] = g + i\left(\tilde{f}_0[C[11]C[22] + C[12]C[21]] + i\tilde{\psi}_0 C[11]C[21] - i\tilde{\varphi}_0 C[12]C[22]\right) h,
\]

\[
\Phi_0[12] = \left(\tilde{\psi}_0 C[11]C[12] - \tilde{\varphi}_0 C[12]C[12] - 2i\tilde{f}_0 C[11]C[12]\right) h,
\]

\[
\Phi_0[21] = \left(\tilde{\varphi}_0 C[22]C[22] - \tilde{\psi}_0 C[22]C[21] + 2i\tilde{f}_0 C[21]C[22]\right) h,
\]

\[
\Phi_0[22] = g + i\left(\tilde{f}_0[C[11]C[22] + C[12]C[21]] + i\tilde{\psi}_0 C[11]C[21] - i\tilde{\varphi}_0 C[12]C[22]\right) h
\]

These equalities give the representations (2.10) with some functions \( \tilde{f}(z, x_0), \tilde{\psi}(z, x_0), \tilde{\varphi}(z, x_0) \). They are, in general, are entire analytic. To prove that these functions are polynomials we use rough asymptotic formulas for the entries of the monodromy matrices \( \Phi(z, x_0) \) and \( \Phi_0(z) \). Thus we find that \( \tilde{f}(z, x_0), \tilde{\psi}(z, x_0), \tilde{\varphi}(z, x_0) \) are polynomials of the same degrees that initial ones. So, if \( v(x) \) is a periodic \( n \)-gap potential then the system of equations (2.6) has the polynomial solution. For \( n \)-gap potential the function \( \tilde{f}(z, x_0) \) is a polynomial of \( n + 1 \)-th degree. If so, then equations (2.6) give that \( \tilde{\psi}(z, x_0) \) and \( \tilde{\varphi}(z, x_0) \) are polynomials of \( n \)-th degrees. Further, since \( \tilde{f}_{n+1}(x_0) = \tilde{f}_{n+1}(0) = 1 \), then the second equation gives equality (2.11) \( \square \)

3. Autonomous system of equations and solutions of the nonlinear Schrödinger equation

In this section we show that the systems of equations (2.5) and (2.6) generate some nonlinear autonomous equations, which lead to a solution of the nonlinear Schrödinger equation (1.1). Let coefficients of equations (2.5) and (2.6) are completely arbitrary, i.e. we will consider the next system of equations:

\[
(3.1) \quad f = (d + 2izb)\psi - (c - 2iza)\varphi,
\]

\[
\dot{\psi} = -2(c - 2iza)f - 2i(2z^2 - ab)\psi,
\]

\[
\dot{\varphi} = 2(d + 2izb)f + 2i(2z^2 - ab)\varphi,
\]
Here we put these differential equations we have algebraic relations:

\[ \begin{align*}
\psi' &= 2ia f - 2iz \psi, \\
\varphi' &= 2ib f + 2iz \varphi,
\end{align*} \]

(3.2)

where \( a = a(x, t), b = b(x, t), c = c(x, t), d = d(x, t) \) are some arbitrary functions, and the dot and prime mean differentiation in \( t \) and \( x \) respectively (from here and below we use \( x \) instead of \( x_0 \)).

First of all we emphasize that each of the system has the conservation law:

\[ f^2(x, t, z) - \psi(x, t, z) \varphi(x, t, z) \equiv P(z), \]

(3.3)

where \( P(z) \) is defined by initial conditions. If \( f, \psi, \varphi \) satisfy (3.1) and (3.2) simultaneously, then \( P(z) \) is independent on \( t \) and \( x \). Below we consider the case when \( a, b, c, d \) are unknown functions. In this case the systems of equations are underdetermined (not closed). However, we can make them to be closed. To this end, let us seek polynomial in \( z \) solutions of these systems. For arbitrary \( n \in \mathbb{N} \), choosing \( f \) as a polynomial of \( n + 1 \)-th degree

\[ f(x, t, z) = \sum_{j=0}^{n+1} f_j(x, t) z^j, \]

it is easy to see that \( \psi \) and \( \varphi \) have to be polynomials of \( n \)-th degree:

\[ \psi(x, t, z) = \sum_{j=0}^{n} \psi_j(x, t) z^j, \quad \varphi(x, t, z) = \sum_{j=0}^{n} \varphi_j(x, t) z^j \]

System (3.1) transforms into the system of equations on polynomial coefficients:

\[ \begin{align*}
\hat{f}_j &= d \psi_j - c \varphi_j + 2ib \psi_{j-1} + 2ia \varphi_{j-1}, & 0 \leq j \leq n + 1, \\
\hat{\psi}_l &= -2cf_l + 4ia f_{l-1} + 2ib \psi_l - 4i \psi_{l-2}, & 0 \leq l \leq n, \\
\hat{\varphi}_m &= 2df_m + 4ib f_{m-1} - 2ia \varphi_m + 4i \varphi_{m-2}, & 0 \leq m \leq n.
\end{align*} \]

(3.4)

Here we put \( f_j, \psi_l, \varphi_m \) are equal zero when \( j, l, m < 0 \) and \( j > n + 1, l, m > n \). Besides these differential equations we have algebraic relations:

\[ \begin{align*}
4ia f_{n+1} - 4i \psi_n &= 0, \\
2cf_{n+1} - 4ia f_n + 4i \psi_{n-1} &= 0, \\
4ib f_{n+1} + 4i \varphi_n &= 0, \\
2df_{n+1} + 4ib f_n + 4i \psi_{n-1} &= 0.
\end{align*} \]

The last relations give unknown functions:

\[ \begin{align*}
a &= \frac{\psi_n}{f_{n+1}}, & b &= -\frac{\varphi_n}{f_{n+1}}, \\
c &= 2i \left( \frac{\psi_n f_n}{f_{n+1}} - \frac{\psi_{n-1}}{f_{n+1}} \right), & d &= 2i \left( \frac{\varphi_n f_n}{f_{n+1}} - \frac{\varphi_{n-1}}{f_{n+1}} \right).
\end{align*} \]

(3.5)

Moreover, the first equation of system (3.1) gives that \( f_n \) and \( f_{n-1} \) is independent on \( t \). Substituting \( a, b, c, d \) into (3.4) we obtain the autonomous system of ODEs:

\[ \begin{align*}
\dot{y}_j &= F_j(y_0, y_1, \ldots, y_N), & j = 0, 1, \ldots, N, & N = 3n + 1,
\end{align*} \]

(3.7)

where \( y_j = f_j (0 \leq j \leq n + 1), y_{n+1+j} = \psi_j (0 \leq j \leq n), y_{2n+2+j} = \varphi_j (0 \leq j \leq n) \), and the right hand sides \( F_j \) are at most the third degree polynomials on \( y_0, y_1, \ldots, y_N \).
In polynomial coefficients the system (3.2) is written in the form:

\[
\begin{align*}
    f' &= ib\psi_j + ia\varphi_j, \\
    \psi' &= 2ia f_l - 2i\psi_{l-1}, \\
    \varphi'_m &= 2ib f_m + 2i\varphi_{m-1},
\end{align*}
\]  

(3.8)

For the functions \(a\) and \(b\) formulas (3.5) are valid as before. The first equation of the system gives that \(f_{n+1}\) and \(f_n\) are independent on \(x\). After the elimination of \(a\) and \(b\) these equations can be also written in the form of autonomous system of ODEs:

\[
y_j' = \Phi_j(y_0, y_1, \ldots, y_N), \quad j = 0, 1, \ldots, N, \quad N = 3n + 1,
\]

(3.9)

where \(\Phi_j\) are at most the second degree polynomials on \(y_0, y_1, \ldots, y_N\).

Thus, the requirement of the existence of the polynomial in \(z\) solution to systems (3.1) and (3.2) defines uniquely the functions \(a, b, c, d\) through solutions of autonomous systems (3.7) and (3.9). On the other hand, if \(f_j, \psi_j\) and \(\varphi_m\) are a solution of autonomous systems (3.7), (3.9), and \(a, b, c, d\) are the function constructed by (3.5), (3.6) then functions (3.10)

\[
f(x, t, z) = \sum_{j=0}^{n+1} f_j(x, t)z^j, \quad \psi(x, t, z) = \sum_{j=0}^{n} \psi_j(x, t)z^j, \quad \varphi(x, t, z) = \sum_{j=0}^{n} \varphi_j(x, t)z^j
\]

are a polynomial solution of systems (3.7), (3.9). In what follows the system (3.4), (3.8) together with (3.5), (3.6) and the system (3.7), (3.9) are identified.

It is well known that necessary and sufficient conditions for the systems (3.7), (3.9) to be compatible are the following ones:

\[
\sum_{j=0}^{N} \left( \frac{\partial F_j}{\partial y_m}\Phi_m - \frac{\partial F_j}{\partial y_m}\Phi_m \right) = 0, \quad j = 0, 1, \ldots, N.
\]

(3.11)

The direct calculation shows that compatibility conditions (3.11) are fulfilled.

**Theorem 3.1.** Let \(f_j(x, t), \psi_j(x, t), \varphi_m(x, t)\) be a compatible local solution of autonomous system (3.7), (3.9). Then defined by (3.7), (3.8) functions \(a(x, t)\) and \(b(x, t)\) are a local infinitely differentiable in \(x\) and \(t\) solution of the following nonlinear equations:

\[
\begin{align*}
    i\dot{a} + a'' + 2a^2b &= 0, \\
    i\dot{b} - b'' - 2b^2a &= 0.
\end{align*}
\]

(3.12)

**Proof.** Systems (3.7), (3.8) contain two pair of equations:

\[
\begin{align*}
    \dot{\psi}_n &= -2cf_n + 4iaf_{n-1} + 2ab\psi_n - 4i\psi_{n-2}, \\
    \dot{\psi}' &= 2ia f_n - 2i\psi_{n-1}
\end{align*}
\]  

(3.13)

and

\[
\begin{align*}
    \dot{\varphi}_n &= 2df_n + 4ibf_{n-1} - 2iab\varphi_n + 4i\varphi_{n-2}, \\
    \dot{\varphi}' &= 2ib f_n + 2i\varphi_{n-1}
\end{align*}
\]

(3.15)

Since right hand sides \(F_j(y_0, y_1, \ldots, y_N)\) and \(\Phi_j(y_0, y_1, \ldots, y_N)\) of systems (3.7), (3.9) are infinitely differentiable in \(y_0, y_1, \ldots, y_N\) then their solutions are also infinitely differentiable in \(t\) and \(x\). Differentiating (3.14) and (3.16) in \(x\), taking into account the independence of \(f_n\) on \(x\) and also equations

\[
\begin{align*}
    \psi''_{n-1} &= 2ia f_{n-1} - 2i\psi_{n-2}, \\
    \varphi''_{n-1} &= 2ib\varphi_{n-1} + 2i\varphi_{n-2}
\end{align*}
\]

we find

\[
\begin{align*}
    \psi'' &= 2ia f_n + 4af_{n-1} - 4\psi_{n-2}, \\
    \varphi'' &= -2ib f_n + 4bf_{n-1} + 4\varphi_{n-2}
\end{align*}
\]
Let us multiply the last equations on $i$ and subtract them from (3.13) and (3.15). Then we have

$$
\dot{\psi}_n - i \psi''_n - 2iab\psi_n = 2(a' - c)f_{n-1}, \quad \dot{\varphi}_n + i \varphi''_n + 2iab\varphi_n = 2(d - b')f_{n-1}.
$$

As $f_{n+1}$ is independent on $t$ and $x$ then formulas (3.5), (3.6) together with equations (3.14), (3.16) give that $a' = c$ and $d = b'$. Finally, since $\psi_n = af_{n+1}$ and $\varphi_n = -bf_{n+1}$, then the last two equations coincide with system (3.12).

Under some conditions the systems (3.4), (3.8) have a global solution.

**Lemma 3.1.** Let initial conditions satisfy the symmetry properties:

$$
(3.17) \quad f_j = \bar{f}_j, \quad \varphi_j = -\bar{\varphi}_j.
$$

Then system (3.4), as well as the system (3.8), has a solution defined for all $t \in \mathbb{R}$ or $x \in \mathbb{R}$ respectively. The solution possesses the same property (3.17) for all $t \in \mathbb{R}$.

**Proof.** The proof will be done for system (3.4). We first show that the symmetry property is conserved for all $t$. Indeed, let $f_j(t), \psi_l(t), \varphi_m(t)$ be a solution of (3.4). Consider a new functions: $\tilde{f}_j(t) = \bar{f}_j(t), \tilde{\psi}_l(t) = -\bar{\psi}_l(t), \tilde{\varphi}_m(t) = -\bar{\varphi}_m(t)$. After complex conjugation of system (3.4) it is easy to see that $\tilde{f}_j(t), \tilde{\psi}_l(t), \tilde{\varphi}_m(t)$ satisfy (3.4). Under condition $t = t_0$, in view of (3.17), these two sets of functions are coincide. Then the uniqueness theorem for ODEs gives that they are coincide for all $t$. Further, the existence theorem for ODEs gives the local solution of (3.4), which inherits the symmetry properties (3.17). In turn these properties lead to the uniform boundedness of the solution and, hence, the solution has an extension for all $t \in \mathbb{R}$.

The uniform boundedness is a consequence of the conservation law (3.3):

$$
(3.18) \quad P(z) \equiv f^2(z, t) - \psi(z, t)\varphi(z, t),
$$

where polynomials $f, \psi, \varphi$ are constructed by $f_j(t), \psi_l(t), \varphi_m(t)$. In view of (3.17), $f(z, t) = \bar{f}(\bar{z}, t)$ and $\varphi(z, t) = -\bar{\varphi}(\bar{z}, t)$. Hence $P(z) = \bar{P}(\bar{z})$. Therefore the polynomial $P(z)$ can be factorized:

$$
P(z) = P^+(z)P^-(z),
$$

where $P^+(z)$ is a polynomial of $n+1$-th degree. Its zeroes are located in the closed upper half plane. The second polynomial is the complex conjugated to the first one: $P^-(z) = \bar{P}^+(\bar{z})$.

For real $z$ the conservation law (3.18) takes the form

$$
f^2(z, t) + |\psi(z, t)|^2 \equiv |P^+(z)|^2.
$$

Hence

$$
|f(z, t)| \leq |P^+(z)|, \quad |\psi(z, t)| = |\varphi(z, t)| \leq |P^+(z)|, \quad z \in \mathbb{R}.
$$

By using the Bernstien inequalities we find

$$
\left| \frac{\partial^m f(z, t)}{\partial z^m} \right| \leq \left| \frac{d^m P^+(z)}{dz^m} \right|, \quad m = 1, 2, \ldots, \quad z \in \mathbb{R}.
$$

Choosing $m = 0, 1, 2, \ldots, n$ and $z = 0$ we obtain

$$
|f_m(t)| \leq |P^+_m|, \quad |\psi_m(t)| = |\varphi_m(t)| \leq |P^+_m|, \quad m = 1, 2, \ldots, n + 1,
$$

where $P^+_m$ are the coefficients of polynomial $P^+(z)$, which are uniquely defined by initial conditions of the autonomous system. The analogous proof goes for the system (3.8).
Corollary 3.1. If initial conditions for the systems (3.12) and (3.8) possess properties (3.17) then there exists the compatible global infinitely differentiable and bounded solution $\tilde{f}_j(t) = f_j(t)$, $\tilde{\psi}(t) = \tilde{\phi}(t)$, $\tilde{\varphi}_m(t) = -\tilde{\psi}_m(t)$. Moreover, the function $a(x, t) = \psi(x, t)/f_{n+1}(0, 0)$ is a solution of the nonlinear Schrödinger equation (1.1).

Indeed, due to the symmetry properties,

$$b(x, t) = -\frac{\varphi_n(x, t)}{f_{n+1}(0, 0)} = \frac{-\tilde{\psi}_n(x, t)}{f_{n+1}(0, 0)} = \tilde{a}(x, t).$$

Therefore the first equation in (3.12) is the NLS equation (1.1), and the second is complex-conjugated with the first (1.1).

4. Periodic Cauchy problem for the nonlinear Schrödinger equations with finite-gap initial data

On the whole line $-\infty < x < \infty$ let us consider the Cauchy periodic problem for equation (1.1), whose an initial datum $u(x)$ is $l$-periodic $n$-gap potential. Consider also the set of shifts $v(x, x_0) = u(x + x_0)$. Then, due to the Lemma 1, the monodromy matrix $\Phi(z, x_0)$ and corresponding functions $f(z, x_0), \psi(z, x_0), \varphi(z, x_0)$ have the form (2.8) with a function $h(z)$ that independent on $x_0$. Therefore the coefficients of the polynomials $\tilde{f}(z, x_0), \tilde{\psi}(z, x_0), \tilde{\varphi}(z, x_0)$ are periodic solution of autonomous system (3.8). Further we solve (for every fixed $x_0$) the system (3.12) with initial data that is defined by the coefficients of polynomials $\tilde{f}(z, x_0), \tilde{\psi}(z, x_0), \tilde{\varphi}(z, x_0)$.

Theorem 4.1. For any periodic $n$-gap initial datum, the Cauchy periodic problem to the Schrödinger nonlinear equation (1.1) is uniquely solvable. Moreover, the solution is also periodic $n$-gap potential for every fixed $t$.

5. Dynamics of zeroes of polynomials

Consider the compatible solution $f_j(x, t), \psi_m(x, t), \varphi_m(x, t)$ (0 ≤ $j \leq n$, 0 ≤ $l, m \leq n - 1$) of autonomous systems (3.1) and (3.2), whose initial data have the properties (3.17). Then, due to the previous considerations, formula

$$v(x, t) = a(x, t) = \frac{\psi_n(x, t)}{f_{n+1}(0, 0)}$$
defines some solution of nonlinear equation (1.1). Moreover, constructed by (3.10) polynomials \( f(z, x, t), \psi(z, x, t), \phi(z, x, t) \) satisfy properties (2.7). They are the solution of linear systems (2.5) and (2.6). It allow to give another representation for the solution \( v(x, t) \). Indeed, let \( \mu_j = \mu_j(x, t) \) \((j = 1, 2, \ldots, n)\) be zeroes of the polynomial \( \psi(z, x, t) \). Zeroes of the polynomial \( \phi(z, x, t) \) are complex conjugated to \( \mu_j \). The Vieta formulas give

\[
\sum_{j=1}^{n} \mu_j(x, t) = -\frac{\psi_n-1(x, t)}{\exp(x, t)},
\]

(5.2)

\[
\sum_{j=1}^{n} \nu_j(x, t) = -\frac{f_n(0, 0)}{f_{n+1}(0, 0)},
\]

(5.3)

where \( \nu_j(x, t) \) \((j = 1, 2, \ldots, n)\) are zeroes of polynomial \( f(z, x, t) \). Without loss of generality we put \( f_{n+1}(0, 0) = 1 \). Then, due to conservation law (3.18),

\[
f_n(0, 0) \int f_{n+1}(0, 0) = -\frac{1}{2} \sum_{j=1}^{n+1} (E_j + \bar{E}_j) = K,
\]

where \( E_j \) and \( \bar{E}_j \) are zeroes of polynomial

\[
P(z) = \prod_{j=1}^{2n+2} (z - E_j) \equiv f^2(z, x, t) - \psi(z, x, t) \phi(z, x, t).
\]

The second equation of system (3.2) with \( \mu_j \) together with (5.2), (5.3) give

\[
\frac{\partial}{\partial x} \ln \psi_n(x, t) = \Phi(x, t), \quad \Phi(x, t) = 2i \sum_{k=1}^{n} \mu_k(x, t) + 2iK.
\]

(5.5)

By analogy, the second equation of system (3.1) with \( \mu_j \) together with (5.4), (5.5) give

\[
\frac{\partial}{\partial t} \ln \psi_n(x, t) = F(x, t),
\]

(5.6)

where

\[
F(x, t) = 2i \left[ \sum_{j>k} E_j E_k - \frac{3}{4} \left( \sum_{k=1}^{2n+2} E_k \right)^2 \right] - 4i \left[ K \sum_{k=1}^{n} \mu_k(x, t) + \sum_{j>k} \mu_j(x, t) \mu_k(x, t) \right].
\]

Here \( \mu_j(x, t) \) are zeroes of polynomial \( \psi(z, x, t) \), and \( E_j \) are zeroes of the polynomial \( P(z) \). An integration of (5.5), (5.6) lead to the representation for \( v(x, t) \):

\[
v(x, t) = \psi_n(0, 0) \exp \left\{ \int_0^t f(0, s) ds + \int_0^z \Phi(\xi, t) d\xi \right\}.
\]

The systems of equations (3.1), (3.2) gives differential equations for \( \mu_j(x, t) \). Indeed, dividing the second equation of the system (3.1) on \( \psi \) and using (5.5), (5.6) we find

\[
\frac{\partial}{\partial t} \ln \psi(z, x, t) = \frac{\partial}{\partial t} \ln \left( \psi_n \prod_{l=1}^{n} (z - \mu_l) \right) = -4iv \left( \sum_{k=1}^{n} \mu_k(x, t) + K - z \right) \frac{f(z)}{\psi(z)}.
\]
Here and below we suppose that zeroes \( \mu_j \) are simple. The multiplication of this equation on \( z - \mu_j \) and the passage (as \( z \to \mu_j \)) to the limit give the following equations:

\[
\frac{\partial}{\partial t} \mu_j(x, t) = 4i \left( \sum_{l=1}^{n} \mu_l + K - \mu_j \right) \frac{f(\mu_j)}{\partial \psi/\partial z(\mu_j)}.
\]

Since

\[
\psi(z) = \psi_n \prod_{l=1}^{n} (z - \mu_l)
\]

then

\[
\frac{\partial \psi(z)}{\partial z} \bigg|_{z=\mu_j} = \psi_n \prod_{l \neq j}^{n} (\mu_j - \mu_l).
\]

Finally, since \( f^2(\mu_j) = P(\mu_j) \) and \( \nu = \psi_n \), then

\[
(5.7) \quad \frac{\partial}{\partial t} \mu_j(x, t) = 4i \left( \sum_{l=1}^{n} \mu_l + K - \mu_j \right) \frac{\sqrt{P(\mu_j)}}{\prod_{l \neq j}^{n} (\mu_j - \mu_l)}, \quad j = 1, 2, \ldots, n
\]

By the same way we find

\[
(5.8) \quad \frac{\partial}{\partial x} \mu_j(x, t) = -2i \frac{\sqrt{P(\mu_k)}}{\prod_{l \neq j}^{n} (\mu_j - \mu_l)}, \quad j = 1, 2, \ldots, n.
\]

Identity (5.4) means that if an arbitrary polynomial \( P(z) \) is positive on the real axis then initial values \( \mu_j(0, 0) \) and \( \psi_n(0, 0) \) as parameters have to satisfy the condition:

\[
\sum_{j=1}^{n+1} (z - E_j)(z - \bar{E}_j) - |\psi_n(0, 0)|^2 \prod_{j=1}^{n} (z - \mu_j(0, 0))(z - \bar{\mu}_j(0, 0)) = f^2(z),
\]

where \( f(z) \) is a polynomial of \( n \)-th degree with real coefficients.

Earlier such systems, like (5.7), (5.8), and related to the Korteweg de Vries equation, were considered in [2], [4], [5]. In [4], [5] they were integrated with the help of the Abel map and the Jacobi inversion problem. Our systems (5.7), (5.8) are also integrated by the same procedure. Below we reproduce the corresponding result [12].

A.R. Its and V.P. Kotlyarov

**Explicit formulas for solutions of the nonlinear Schrödinger equation**

(Presented by the academician V.A. Marchenko 11.XII.1975)

**Summary**

The explicit formulas are obtained for solutions of the Schrödinger nonlinear equation

\[
iu_t + uu_{xx} + 2|u|^2u = 0.
\]

The formulas are constructed by means of \( \theta \)-functions.

The study of periodic and almost periodic solutions of some nonlinear partial differential equations has shown a relationship of such solutions with classical objects of the algebraic geometry [9]. This gave the possibility to construct explicit formulas for solutions of some nonlinear equations by using theta functions. The present report is devoted to the construction of such type of solutions to the equation:

\[
iu_t + uu_{xx} + 2|u|^2u = 0.
\]
Below we will use results and methods of [10] and [11].

Let coefficients $f_k(x, t), \psi_l(x, t), \varphi_m(x, t)$ ($0 \leq k \leq n + 1, 0 \leq l, m \leq n$) of polynomials $f(z, x, t), \psi(z, x, t), \varphi(z, x, t)$ satisfy following conditions: $f_k = f_k$, $\psi_m = -\varphi_m$. It is proved in [10] that, if functions $f_k(x, t), \psi_l(x, t), \varphi_m(x, t)$ are a compatible solution to the special autonomous system of differential equations, then function $u(x, t) = \psi_n(x, t)/f_{n+1}(0, 0)$ is a bounded infinitely differentiable solution to equation (6.1). Moreover, the following formulas hold:

\begin{equation}
\frac{\partial}{\partial x} \ln u(x, t) = 2i \sum_{k=1}^{n} \mu_k(x, t) + 2iK, \quad K = -\frac{1}{2} \sum_{k=1}^{2n+2} E_k,
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} \ln u(x, t) = 2i \left[ \sum_{j>k} E_j E_k - \frac{3}{4} \left( \sum_{k=1}^{2n+2} E_k \right)^2 \right] - 4i \left[ K \sum_{k=1}^{n} \mu_k(x, t) + \sum_{j>k} \mu_j(x, t) \mu_k(x, t) \right],
\end{equation}

where $\mu_j(x, t)$ are zeroes of polynomial $\psi(z, x, t)$, and $E_j$ are zeroes of the polynomial

\begin{equation}
P(z) = \prod_{j=1}^{2n+2} (z - E_j) \equiv f^2(z, x, t) - \psi(z, x, t) \varphi(z, x, t),
\end{equation}

which is positive on the real axis. Without loss of generality we have put $f_n(0, 0) = 1$. We emphasize that $f^2 - \psi \varphi$ is independent on $x$ and $t$. Therefore complex numbers $E_j$ are also independent on $x$ and $t$. The functions $\mu_j(x, t)$ satisfy the following equations:

\begin{equation}
\frac{\partial}{\partial x} \mu_k(x, t) = -2i \frac{\sqrt{P(\mu_k)}}{\prod_{j \neq k} (\mu_k - \mu_j)}, \quad k = 1, 2, \ldots, n;
\end{equation}

\begin{equation}
\frac{\partial}{\partial t} \mu_k(x, t) = 4i \left( \sum_{j=1}^{n} \mu_j + K - \mu_k \right) \frac{\sqrt{P(\mu_k)}}{\prod_{j \neq k} (\mu_k - \mu_j)}.
\end{equation}

Equations (6.5) and (6.6) have to be considered on the Riemann surface $\sigma$ of the function $\sqrt{P(z)}$. The upper or lower sheet of the Riemann surface is defined by the sign of the square root. Point $\mu_k(x, t)$ can be located only on one sheet that is defined uniquely by equality: $f(\mu_k(x, t), x, t) = \sqrt{P(\mu_k(x, t))}$. We will use in the usual way (cf. [11]) a realization of the Riemann surface $\sigma$ and basis $(a_j, b_j)$ of cycles on it. The Abel map, after applying to equations (6.5), (6.6), gives that points $\mu_k(x, t)$ are a solution of the hyper-elliptic Jacobi inversion problem of Abelian integrals:

\begin{equation}
\sum_{k=1}^{n} U_j(\mu_k(x, t)) = -2iC_j^1 x - 4i(C_j^2 - KC_j^1) + \sum_{k=1}^{n} U_j(\mu_k(0, 0)),
\end{equation}

where Abelian integrals

\begin{equation}
U_j(z) = \int_{E_{2n+2}}^{z} \left( C_j^1 \lambda^{n-1} + C_j^2 \lambda^{n-2} + \ldots + C_j^n \right) \frac{d\lambda}{\sqrt{P(\lambda)}}, \quad j = 1, 2, \ldots, n
\end{equation}

are normalized by conditions:

\begin{equation}
\int_{a_k} dU_j(z) = \delta_{kj}, \quad k, j = 1, 2, \ldots, n.
\end{equation}
Equations (6.7) give the following relations:

\[(6.8) \quad \sum_{k=1}^{n} \mu_k(x,t) = K_1 - \frac{i}{2} \frac{\partial}{\partial x} \ln J_1(x,t), \quad K_1 = \sum_{k=1}^{n} \int z dU_k(z),\]

\[(6.9) \quad \sum_{k=1}^{n} \mu_k^2(x,t) = K_2 - \frac{i}{4} \frac{\partial}{\partial t} \ln J_1(x,t) + \frac{1}{4} \frac{\partial^2}{\partial x^2} \ln J_2(x,t), \quad K_2 = \sum_{k=1}^{n} \int z^2 dU_k(z),\]

where

\[J_1(x,t) = \theta(g(x,t) - r) \theta(g(x,t) + r),\]

\[J_2(x,t) = \theta(g(x,t) - r) \theta(g(x,t) + r),\]

\[\theta(p) = \sum_{m \in \mathbb{Z}} \exp[\pi i(Bm, m) + 2\pi i(p, m)], \quad g(x,t), r \in \mathbb{C}^n,\]

\[g_j(x,t) = -2iC_j^1 x - 4i(C_j^2 - KC_j^1) t + \sum_{k=1}^{n} U_j(\mu_k(0,0)) + \frac{1}{2} \left( \sum_{k=1}^{n} B_{kj} - j \right),\]

\[r_j = \int_{E_{2n+2}}^{\infty} dU_j(z), \quad B_{kj} = \int_{b_k} dU_j(z),\]

\[\infty^+ \text{ infinite point on the Riemann surface } \sigma.\]

Substituting (6.8) and (6.9) into (6.2) and (6.3) one finds

\[(6.10) \quad \frac{\partial}{\partial x} \mu_k(x,t) = \frac{\partial}{\partial x} \ln J_1(x,t) - iE, \quad E = \sum_{k=1}^{2n+2} E_j - 2K_1,\]

\[(6.11) \quad \frac{\partial}{\partial t} \mu_k(x,t) = \frac{\partial}{\partial t} \ln J_2(x,t) + iN(x,t),\]

where

\[N(x,t) = \frac{i}{2} \frac{\partial}{\partial t} \ln J_1(x,t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \ln J_2(x,t) + \frac{1}{2} \left( \frac{\partial}{\partial x} \ln J_1(x,t) - iE \right)^2 + 2K_2 - \sum_{k=1}^{2n+2} E_j^2.\]

Equations (6.10) and (6.11) give that the function \(N(x,t)\) is independent of \(x\). In principle, equations (6.10) and (6.11) give already the desired explicit formula for the solutions equation (6.1). This formula can be essentially improved. To this end let us show that

\[(6.12) \quad N(x,t) \equiv N_0 := 4K_2 - 2 \sum_{k=1}^{2n+2} E_j^2 - 4R_1,\]

and

\[(6.13) \quad |u(x,t)|^2 = \frac{\partial^2}{\partial x^2} \ln \theta(g(x,t) + r) + 2R_1,\]

where \(R_1\) is a constant value. This value depends only on numbers \(E_j\). We stress that (6.13) is not a consequence of equations (6.10) and (6.11). Below we give a sketch of deducing of equalities (6.12) and (6.13).

Let us consider the operator

\[L := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & iu \\ iu & 0 \end{pmatrix},\]
where \( u = u(x,t) \) is defined by equations (6.10) and (6.11). It can be shown that equation \( Ly = zy \) has such a solution \( \psi(x,z) \) which is single valued and analytic (in \( z \)) on the Riemann surface \( \sigma \), and its first component \( \psi_1(x,z) \) satisfies the following conditions:

1. zeroes of \( \psi_1(x,z) \) are simple and lie over points \( \mu_k(x,t) \);
2. poles of \( \psi_1(x,z) \) are simple and lie over points \( \mu_k(0,t) \);
3. \( \psi_1(x,z) \sim e^{\text{iz}\alpha u(x,t)}u(0,t) \), \( z \to \infty^+ \), on the upper sheet of \( \sigma \);
4. \( \psi_1(x,z) \sim e^{-\text{iz}\alpha} \), \( z \to \infty^- \), on the lower sheet of \( \sigma \).

The existence of solution \( \psi(x,z) \) to equation \( Ly = zy \) is already sufficient for deducing formulas (6.12) and (6.13). Indeed, the properties i-iv lead to a representation for the function

\[ \varphi(x,z) := [u(x,t)]^{-1/2}[u(0,0)]^{1/2}\psi_1(x,z) \]

through the theta functions:

\[ \varphi(x,z) = e^{iz\omega(z)} \frac{\theta(U(z) - g(x,t))}{\theta(U(z) - g(0,t))} \left[ \frac{\theta(g(0,t) - r)}{\theta(g(0,t) + r)} \theta(g(x,t) - r) \theta(g(x,t) + r) \right]^{1/2}, \]

where \( \omega(z) \) is the normalized Abelian integral of the second kind with simple poles at \( \infty^\pm \):

\[ \omega(z) = \pm \left( z - \frac{E}{2} + R_1 \frac{1}{z} + R_2 \frac{1}{z^2} + \ldots \right). \]

Numbers \( R_1, R_2 \) as well as \( E \) are defined via complex numbers \( E_j \). Moreover, by the definition, the function \( \varphi(x,z) \) satisfies equation

\[ \frac{\partial^2 \varphi(x,z)}{\partial x^2} + \left( z^2 - i\frac{\partial}{\partial x} \ln u(x,t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} \ln u(x,t) - \frac{1}{4} \left( \frac{\partial}{\partial x} \ln u(x,t) \right)^2 + |u(x,t)|^2 \right) \varphi(x,z) = 0. \]

As follows from (11), the last equation together with (6.14) lead to equality (6.13). Substituting \( 2i|u|^2 + i\frac{\partial}{\partial x} \ln u + i \left( \frac{\partial}{\partial x} \ln u \right)^2 \) into the expression for \( N(x,t) \) from (6.11), using (6.13) for \( |u(x,t)|^2 \) and (6.10) for \( \frac{\partial}{\partial x} \ln u \) we arrive to (6.12).

Thus from (6.10) and (6.11) we have the final formula for solutions \( u(x,t) \) equation (6.1):

\[ u(x,t) = u(0,0) \frac{J_1(x,t)}{J_1(0,0)} \exp \left( -iE_x + iN_0 t \right). \]

Note that formula (6.13) is true for the modulus of \( u(x,t) \). From (6.3) follows that solution (6.15) is completely defined by parameters connected by a polynomial relation:

\[ \sum_{j=1}^{n+1} (z - E_j)(z - \bar{E}_j) - |u(0,0)|^2 \prod_{j=1}^{n} (z - \mu_j(0,0))(z - \bar{\mu}_j(0,0)) = f^2(z), \]

where \( f(z) \) is a polynomial of \( n + 1 \)-th degree with real coefficients.

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Institute for Low Temperature Physics, 47, Lenin Ave, 61103 Kharkiv, Ukraine
E-mail address: kotlyarov@ilt.kharkov.ua