Orthogonal one-factorizations of complete multipartite graphs

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Abstract The paper provides a complete solution to the existence problem of two orthogonal one-factorizations of a complete balanced multipartite graph $K_{p \times q}$. In particular, new classes of Howell designs are constructed.

Keywords One-factorization · Orthogonality · Latin square · Room square · Howell design

Mathematics Subject Classification 05C70 · 05B15

1 Introduction

We use standard notation $K_{p \times q}$ for a complete balanced $p$-partite graph with each part of cardinality $q$. Let $V(K_{p \times q}) = V_1 \cup V_2 \cup \ldots \cup V_p$, where $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Moreover, we also use the standard symbol $K_{q, q}$ to denote $K_{2 \times q}$, a complete balanced bipartite graph on $2q$ vertices.

A one-factor in a graph $G$ is a regular spanning subgraph of degree one. A one-factorization of $G$ is a set $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ of edge-disjoint one-factors such that $E(G) = \bigcup_{i=1}^{r} E(F_i)$. Two one-factorizations $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$ and $\mathcal{F}' = \{F'_1, F'_2, \ldots, F'_r\}$ are orthogonal if $|F_i \cap F'_j| \leq 1$ for all $1 \leq i, j \leq r$.

Orthogonal one-factorizations of complete graphs are well-studied, mostly in terms of Rooms squares, cf. [7, 12]. Let $m$ be an odd integer and let $S$ be a set of $m + 1$ elements (symbols). A Room square $R$ of side $m$ is an $m \times m$ array which satisfies the following properties:

Communicated by L. Teirlinck.
(1) every cell of $R$ is either empty or contains an unordered pair of symbols from $S$,
(2) every symbol of $S$ occurs exactly once in each row and exactly once in each column of $R$,
(3) every unordered pair of symbols occurs in precisely one cell in $R$.
Thus each row and each column of $R$ contain $\frac{m-1}{2}$ empty cells.

The existence of two orthogonal one-factorizations, $\mathcal{F}$ and $\mathcal{F'}$, of a complete graph $K_{2n}$ is equivalent to the existence of a Room square of side $2n - 1$: each row corresponds to a one-factor in $\mathcal{F}$ whilst each column represents a one-factor in $\mathcal{F'}$.

The existence problem for Room squares is completely settled.

**Theorem 1** [14] A Room square of side $m$ exists if and only if $m$ is odd and $m \neq 3$ and $m \neq 5$.

Two orthogonal one-factorizations of a complete bipartite graph $K_{n,n}$ are equivalent to two orthogonal latin squares of side $n$. A latin square of side $n$ is an $n \times n$ array in which each cell contains a single symbol from an $n$-element set $S$, such that each symbol occurs exactly once in each row and exactly once in each column. Two latin squares, $L$ and $L'$, of side $n$ are orthogonal if the $n^2$ ordered pairs $(L(i, j), L'(i, j))$ are all distinct. Bose, Shrikhande and Parker [3] completely solved the famous Euler’s conjecture.

**Theorem 2** [3] A pair of orthogonal latin squares of side $n$ exists whenever $n \neq 2$ and $n \neq 6$.

The above equivalences can be extended to other classes of regular graphs. Namely, a pair of orthogonal one-factorizations of an $s$-regular graph $G$ on $2n$ vertices corresponds to the existence of a Howell design of type $(s, 2n)$, for which a graph $G$ is called an underlying graph, cf. [15]. Let $S$ be a set of $2n$ symbols. A Howell design $H(s, 2n)$ on the symbol set $S$ is an $s \times s$ array that satisfies the following conditions:
(1) every cell is either empty or contains an unordered pair of symbols from $S$,
(2) every symbol of $S$ occurs exactly once in each row and exactly once in each column of $H$,
(3) every unordered pair of symbols occurs in at most one cell of $H$.

Necessary condition for the existence of Howell designs $H(s, 2n)$ is $n \leq s \leq 2n - 1$. The existence of an $H(n, 2n)$ comes from two orthogonal one-factorizations of a complete bipartite graph $K_{n,n}$ if $n \neq 2, 6$ and some 6-regular graph if $n = 6$ [13]. There in no $H(2, 4)$. In the other extreme case, an $H(2n - 1, 2n)$ is a Room square of side $2n - 1$. The existence of Howell designs has been completely determined for all remaining values of $s$.

**Theorem 3** [17] If $s$ is odd and $n < s < 2n - 1$ then there exists an $H(s, 2n)$, except that $H(5, 8)$ does not exist.

**Theorem 4** [2] If $s$ is even and $n < s < 2n - 1$ then there exists an $H(s, 2n)$.

An important question related to Howell designs concerns properties of graphs which are underlying graphs of Howell designs. While for $s = 2n - 1$ and $s = 2n - 2$ these graphs are unique (the complete graph $K_{2n}$ and the cocktail party graph $K_{2n} \setminus F$, respectively, where $F$ is a one-factor), determining these graphs in general seems to be hopeless [15,16]. We have to notice that some known constructions may provide Howell designs for certain classes of underlying graphs; in particular, in the case of a powerful recursive “PBD-construction” (cf.
[2,17]), the structure of an underlying graph strongly depends on the choice of parameters, parallel classes in a PBD as well as Howell subdesigns used in the recursion.

It is known that a necessary and sufficient condition for the existence of a one-factorization of a complete balanced multipartite graph $K_{p\times q}$ is that $pq$ is even [11]. The goal of this paper is to show that balanced complete multipartite graphs are underlying graphs of Howell designs; the main result provides a complete solution to the existence problem of two orthogonal one-factorizations of $K_{p\times q}$.

2 Constructions

We first discuss a general recursive construction which in fact is an application of a standard “expansion by latin squares” method.

**Lemma 5** Let $p$, $q$ and $m$ be integers such that $p \geq 2$, $q \geq 1$, $m \geq 3$ and $m \neq 6$. Suppose there exist two orthogonal one-factorizations of the complete multipartite graph $K_{p\times q}$ and moreover two orthogonal one-factorizations of the complete bipartite graph $K_{m,m}$. Then there exists a pair of orthogonal one-factorizations of the complete multipartite graph $K_{p\times qm}$.

**Proof** Let $X$ be the vertex set of $K_{p\times q}$ and let $(Y,Y)$ be the vertex set of $K_{m,m}$. Let $F^1$, $F^2$ be two orthogonal one-factorizations of $K_{p\times q}$ on the set $X$ such that $F^z = \{F_1^z, F_2^z, \ldots, F_{q(p-1)}^z\}$, $z = 1, 2$. Moreover, let $E_1^z$, $E_2^z$ be a pair of orthogonal one-factorizations of $K_{m,m}$ on $(Y,Y)$ and $E^z = \{E_1^z, E_2^z, \ldots, E_m^z\}$, $z = 1, 2$.

For each $z = 1, 2$ we construct a one-factorization $D^z = \{D_{s,t}^z : s = 1, 2, \ldots, q(p-1), t = 1, 2, \ldots, m\}$ of $K_{p\times qm}$ on vertex set $X \times Y$. We replace each edge of $K_{p\times q}$ with one-factorization $E^z$ as follows: the edge $\{(i,j), (k,l)\}$ belongs to one-factor $D_{s,t}^z$ if $\{i,k\}$ is an edge of $F_s^z$ and $\{j,l\}$ is an edge of $E_t^z$.

To prove orthogonality of $D^1$ and $D^2$ we suppose to the contrary that there are two distinct edges, $\{(i,j), (k,l)\}$ and $\{(i',j'), (k',l')\}$ of $K_{p\times qm}$ that belong together to the same two one-factors, $D_{s,t}^1$ and $D_{s',t'}^2$. We consider two cases:

1. $i = i'$ and $k = k'$. Then $j \neq j'$ and $l \neq l'$. Moreover, $\{j,l\}$ and $\{j',l'\}$ are both in the same two one-factors $E_1^1$ and $E_2^1$, a contradiction to the orthogonality of $E^1$ and $E^2$.
2. $i \neq i'$ or $k \neq k'$. Then $\{i,k\}$ and $\{i',k'\}$ are two distinct edges of both $F_s^1$ and $F_{s'}^2$, a contradiction to the orthogonality of $F^1$ and $F^2$.

□

When $q = 1$ we immediately get the following.

**Corollary 6** Let $p$ and $q$ be integers such that $p$ is even, $p \geq 8$, $m \geq 3$ and $m \neq 6$. There exists a pair of orthogonal one-factorizations of a complete multipartite graph $K_{p\times m}$.

The second construction is based on Room frames. Let $\{S_1, S_2, \ldots, S_k\}$ be a partition of the set $S$. An $\{S_1, S_2, \ldots, S_k\}$-Room frame is an $|S| \times |S|$ array, $F$, indexed by $S$, which satisfies the following properties:

1. every cell of $F$ is either empty or contains an unordered pair of symbols from $S$,
2. the subarrays $S_i \times S_i$ are empty, for $1 \leq i \leq k$ (these subarrays are called holes),
3. every symbol $x \notin S_i$ occurs exactly once in each row $s$ and exactly once in each column $t$, for any $s, t \in S_i$,
4. pairs occurring in $F$ are those $\{s, t\}$, where $(s,t) \in (S \times S) \setminus \bigcup_{i=1}^k (S_i \times S_i)$.

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The type of a Room frame $F$ is a multiset $\{|S_i| : 1 \leq i \leq k\}$. An “exponential” notation is used to describe types; a Room frame has type $t_1^{a_1}t_2^{a_2}\cdots t_l^{a_l}$ if there are $u_i$ subsets of cardinality $t_i$, $1 \leq i \leq l$. A Room frame of type $t^u$ (one hole size) is called uniform. In particular, a Room square of side $m$ is equivalent to a Room frame of type $1^m$.

The existence problem for uniform Room frames is completely solved.

**Theorem 7** [5,6,8–10] Suppose $t$ and $u$ are positive integers, $u \geq 4$ and $(t,u) \neq (1,5)$ and $(2,4)$. Then there exists a uniform Room frame of type $t^u$ if and only if $t(u-1)$ is even.

Room frames are key structures in the “filling in holes” construction for Howell designs, cf. [4]. In particular, applying this construction for uniform Room frames yields Howell designs with complete balanced multipartite graphs as underlying graphs.

**Lemma 8** Let $t$ and $u$ be integers such that $t \geq 3$, $t \not= 6$, $u \geq 4$ and $t(u-1)$ is even. Then there exists a Howell design $H(ut, ut+t)$ whose underlying graph is $K_{(u+1)\times t}$.

**Proof** By Theorem 7, there exists a Room frame $F$ of type $t^u$ on a set $S$ of cardinality $tu$. Let $S_1, S_2, \ldots, S_u$ be sets corresponding to holes of $F$, $S_i \subset S$ and $|S_i| = t$ for each $i = 1, 2, \ldots, u$. Let $S_{u+1}$ be a set containing $t$ elements, none of them in the set $S$.

For each pair of sets $(S_i, S_{u+1})$, $i = 1, 2, \ldots, u$, by Theorem 2, there exists a pair of orthogonal Latin squares of side $t$ which correspond to two orthogonal one factorizations of complete bipartite graphs $K_{t,t}$, with bipartition $(S_i, S_{u+1})$, and moreover which are equivalent to a Howell design $H_i$ of type $(t, 2t)$ on the set $S_i \cup S_{u+1}$. It is easy to see that each hole $S_i \times S_j$ of $F$ can be filled with $H_i$. In this way we obtain a Howell design $H$ on the set $S \cup S_{u+1}$. Notice that none of unordered pairs with both elements in the same $S_i$, $i = 1, 2, \ldots, u+1$, occurs in $H$. Thus $K_{(u+1)\times t}$ is an underlying graph of $H$. \hfill $\square$

The well-known starter-adder construction, as a basic method to obtain Room squares, can be generalized for Howell designs, cf. [1]. Let $G$ be an abelian group of order $s$. A Howell starter in $G$, where $s+1 \leq 2n \leq 2s$, is a set $S_{s,n} = \{|x_i, y_i| : 1 \leq i \leq s-n \} \cup \{|x_i| : s-n+1 \leq i \leq n\}$ that satisfies:

1. $\{|x_i| : 1 \leq i \leq n\} \cup \{|y_i| : 1 \leq i \leq s-n\} = G$,
2. $(x_i-y_j) \neq \pm(x_j-y_j)$ if $i \neq j$.

If $S_{s,n}$ is a Howell starter, then an ordered set $A_{s,n} = \{|a_i| : 1 \leq i \leq n\}$ is an adder for $S_{s,n}$ if elements in $A_{s,n}$ are distinct and $\{|x_i+a_i| : 1 \leq i \leq s-n\} \cup \{|y_i+a_i| : 1 \leq i \leq s-n\} = G$.

In what follows, we use notation $S A_{s,n} = \{|x_i, y_i|^{a_i} : 1 \leq i \leq s-n\} \cup \{|x_i|^{a_i} : s-n+1 \leq i \leq n\}$ for a Howell starter $S_{s,n}$ together with an adder $A_{s,n}$. Moreover, we take the cyclic group $Z_s$ as $G$.

**Lemma 9** Suppose that there exist a Howell starter $S_{s,n}$ together with an adder $A_{s,n}$ in $Z_s$ such that $q = 2n-s$ is a divisor of $s$ and moreover none of the pairs in $S_{s,n}$ has the difference of its elements divisible by $p = s/q$. Then a Howell design of type $(s, 2n)$, generated by $S_{s,n}$ and $A_{s,n}$, has an underlying graph $K_{(p+1)\times q}$.

**Proof** An $H(s, 2n)$ is constructed on the symbol set $V$ of cardinality $2n$. Let $(V_0, V_1, \ldots, V_p)$ be a partition of $V$ such that $V_j = \{j, j+p, j+2p, \ldots, j+(q-1)p\}$, where $j = 0, 1, \ldots, p-1$, and $V_p = \{\infty_1, \infty_2, \ldots, \infty_q\}$.

Let us label rows and columns of $H(s, 2n)$ by elements of $Z_s$. The first row consists of pairs $\{x_i, y_i\}$, for $i = 1, 2, \ldots, s-n$, and pairs $\{x_i, \infty_{i-s+n}\}$, for $i = s-n+1, s-n+2, \ldots, n$, each of them in column $-a_i$. It is easy to see that all these pairs form a 1-factor of $K_{(p+1)\times q}$.
on \( V \). The remaining cells of the Howell design are filled out by developing the square via
the group \( \mathbb{Z}_6 \); that is, the pair \( \{x_i + k, y_i + k\} \) is placed in row \( k \) and column \( -a_i + k \), and also the pair \( \{x_i + k, \infty_{i-s+n}\} \) in a cell in row \( k \) and column \( -a_i + k \), where all arithmetic is modulo \( s \). Thus, in particular, the first column consists of pairs \( \{x_i + a_i, y_i + a_i\} \), for \( i = 1, 2, \ldots, s - n \), and pairs \( \{x_i + a_i, \infty_{i-s+n}\} \), for \( i = s - n + 1, s - n + 2, \ldots, n \), which obviously constitute a 1-factor of \( K_{(p+1)\times q} \). Due to cyclic rotation of rows and columns we obtain two orthogonal one factorizations of \( K_{(p+1)\times q} \).

An elementary verification shows that the following sets \( SA_{s,n} \) are Howell starters and adders for Howell designs of type \((s, 2n)\) whose underlying graphs are \( K_{4s,n} \), where \( q = n/2 \).

Notice that none of the pairs in starters has the difference of elements divisible by 3.

**Construction 1** \( s \equiv 3 \pmod{24} \), \( s \geq 27 \)
Let \( s = 24m + 3 \) and \( n = 16m + 2, m \geq 1 \).

\[
SA_{s,n} = \{(8m - 2i, 8m + 2 + 4i)^{16m+2-i}, (8m - 1 - 2i, 8m + 4 + 4i)^{4m-i}, (16m + 1 - 4j, 16m + 2 + 2j)^{20m+3+i}, (16m - 1 - 4i, 16m + 3 + 2i)^{8m+2-i}, (20m + 3 + 4j)^{13m+3-j}, (20m + 4 + 4i)^{7m+2-i}, (20m + 5 + 4i)^{m+1-i}, (20m + 6 + 4i)^{19m+3-i} : i, j = 0, 1, \ldots, 2m - 1, j = 0, 1, \ldots, 2m \}.
\]

**Construction 2** \( s \equiv 9 \pmod{24} \), \( s \geq 33 \)
Let \( s = 24m + 9 \) and \( n = 16m + 6, m \geq 1 \).

\[
SA_{s,n} = \{(8m + 2 - 2j, 8m + 4 + 4j)^{16m+6-j}, (8m + 1 - 2i, 8m + 6 + 4i)^{4m+1-i}, (16m + 5 - 4j, 16m + 6 + 2j)^{20m+8+j}, (16m + 3 - 4j, 16m + 7 + 2j)^{8m+4+j}, (20m + 8 + 4j)^{7m+4-j}, (20m + 9 + 4j)^{13m+6-j}, (20m + 10 + 4j)^{19m+8-j}, (20m + 11 + 4i)^{m+1-i} : i, j = 0, 1, \ldots, 2m - 1, j = 0, 1, \ldots, 2m \}.
\]

**Construction 3** \( s \equiv 15 \pmod{24} \), \( s \geq 15 \)
Let \( s = 24m + 15 \) and \( n = 16m + 10, m \geq 0 \).

\[
SA_{s,n} = \{(8m + 4 - 2j, 8m + 6 + 4j)^{16m+10-j}, (8m + 3 - 2j, 8m + 8 + 4j)^{4m+2-j}, (16m + 9 - 4k, 16m + 10 + 2k)^{20m+2+k}, (16m + 7 - 4j, 16m + 11 + 2j)^{8m+6+j}, (20m + 13 + 4k)^{7m+5-k}, (20m + 14 + 4j)^{m+1-j}, (20m + 15 + 4j)^{13m+9-j}, (20m + 16 + 4j)^{19m+11-j} : j = 0, 1, \ldots, 2m, k = 0, 1, \ldots, 2m + 1 \}.
\]

**Construction 4** \( s \equiv 21 \pmod{24} \), \( s \geq 21 \)
Let \( s = 24m + 21 \) and \( n = 16m + 14, m \geq 0 \).

\[
SA_{s,n} = \{(8m + 6 - 2k, 8m + 8 + 4k)^{16m+14-k}, (8m + 5 - 2j, 8m + 10 + 4j)^{4m+3-j}, (16m + 13 - 4k, 16m + 14 + 2k)^{20m+18+k}, (16m + 11 - 4k, 16m + 15 + 2k)^{8m+8+k}, (20m + 18 + 4k)^{7m+6-k}, (20m + 19 + 4k)^{13m+11-k}, (20m + 20 + 4k)^{19m+16-k}, (20m + 21 + 4j)^{m-j} : j = 0, 1, \ldots, 2m, k = 0, 1, \ldots, 2m + 1 \}.
\]

Some examples of small order have to be constructed separately.

**Example 1** Two orthogonal one-factorizations of \( K_{3\times 4} \).
The starter-adder for a Howell design \( H(8, 12) \) is \( SA_{8,6} = \{(0, 1)^1, (2, 5)^2, (3)^5, (4)^7, (6)^1, (7)^6\} \).

**Example 2** Two orthogonal one-factorizations of \( K_{4\times 3} \).
The starter-adder for a Howell design \( H(9, 12) \) is \( SA_{9,6} = \{(2, 4)^5, (3, 7)^3, (5, 6)^7, (0)^2, (1)^4, (8)^0\} \).

**Example 3** Two orthogonal one-factorizations of \( K_{4\times 4} \).
The starter-adder for a Howell design \( H(12, 16) \) is \( SA_{12,8} = \{(0, 1)^{11}, (4, 11)^2, (5, 9)^5, (6, 8)^9, (2)^7, (3)^4, (7)^1, (10)^6\} \).
3 Main results

Lemma 10  For every even positive integer \( q \) there exist two orthogonal one-factorizations of \( K_{3 \times q} \).

Proof  We consider separately the following cases. If \( q = 2 \) then \( K_{3 \times 2} \) is the cocktail-party graph and the assertion immediately holds by Theorem 4. For \( q = 4 \) we use two orthogonal one-factorizations of \( K_{3 \times 4} \) from Example 1. For \( q \geq 6 \) and \( q \neq 12 \) we apply the general recursive construction given in Lemma 5 taking as initial graphs \( K_{3 \times 2} \) and \( K_{q \times q} \). If \( q = 12 \) we apply the same construction but we use orthogonal one-factorizations of \( K_{3 \times 4} \) and \( K_{3,3} \).

Lemma 11  For every integer \( q \geq 2 \) there exist two orthogonal one-factorizations of \( K_{4 \times q} \).

Proof  If \( q = 2 \) then two orthogonal one-factorizations of the cocktail party graph \( K_{4 \times 2} \) exist by Theorem 4. If \( q = 3 \) or \( q = 4 \), two orthogonal one-factorizations of \( K_{4 \times 3} \) and \( K_{4 \times 4} \) are given in Examples 2 and 3, respectively. For odd \( q \geq 5 \) the existence is satisfied by Constructions 1–4 and Lemma 9. For even \( q \geq 6 \) and \( q \neq 12 \), the general recursive construction given in Lemma 5 can be used taking as initial graphs \( K_{4 \times 2} \) and \( K_{q \times q} \). If \( q = 12 \) we apply the same construction but we use orthogonal one-factorizations of \( K_{4 \times 4} \) and \( K_{3,3} \).

Lemma 12  Let \( p, q \) be integers such that \( p \) is odd and \( p \geq 5 \), \( q \) is even and \( q \geq 2 \). Then there exist two orthogonal one-factorizations of \( K_{p \times q} \).

Proof  If \( q = 2 \) then \( K_{p \times 2} \) is the cocktail-party graph and the assertion holds by Theorem 4. For \( q \geq 4 \) and \( q \neq 6 \), the existence of two orthogonal one-factorizations of \( K_{p \times q} \) follows directly from Lemma 8. If \( q = 6 \), a construction in Lemma 5 can be applied for initial graphs \( K_{p \times 2} \) and \( K_{3,3} \).

Lemma 13  Let \( p, q \) be integers such that \( p \) is even, \( p \geq 6 \) and \( q \geq 2 \). Then there exist two orthogonal one-factorizations of \( K_{p \times q} \).

Proof  If \( q = 2 \) then \( K_{p \times 2} \) is the cocktail-party graph and the assertion holds by Theorem 4. For \( q \geq 3 \) and \( q \neq 6 \), the existence of two orthogonal one-factorizations of \( K_{p \times q} \) follows from Lemma 8. If \( q = 6 \) then we apply the general recursive construction given in Lemma 5 taking \( K_{p \times 2} \) and \( K_{3,3} \) as initial graphs.

Combining Lemmas 10–13 together with Theorems 1 and 2 gives the main result.

Theorem 14  For any integers \( p \) and \( q \) such that \( pq \) is even, \( p \geq 2 \) and \( q \geq 1 \), a complete balanced multipartite graph \( K_{p \times q} \) admits a pair of orthogonal one-factorizations, except for \((p, q) = (2, 2), (2, 6), (4, 1) \) or \((6, 1)\).

Corollary 15  Let \( p \) and \( q \) be integers such that \( pq \) is even, \( p \geq 2 \), \( q \geq 1 \) and \((p, q)\) is none of the pairs \((2, 2), (2, 6), (4, 1) \) and \((6, 1)\). Then there exists a Howell design of type \((pq - q, pq)\) whose underlying graph is \( K_{p \times q} \).

Acknowledgements  The authors would like to thank the referees for helpful comments and suggestions.

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