ON A CLASS OF C*-PREDUALS OF \( l_1 \)

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Abstract. As it is well known, the Banach space \( l_1 \) of absolutely summable (complex) sequences endowed with the \( \| \cdot \|_1 \) norm is not unique predual. This means that there are many different (i.e. non isometrically isomorphic) Banach spaces \( X \) such that \( X^* \cong l_1 \).

The present note is aimed to point out a simple class of C*-preduals of \( l_1 \): namely the spaces \( C_\tau(\mathbb{N}) \) of continuous functions \( f : \mathbb{N} \to \mathbb{C} \), where the set of natural numbers \( \mathbb{N} \) is equipped with a compact Hausdorff topology \( \tau \).

To be more concrete, we shall explicitly describe a countable collection \( \{ T_n \} \) of such topologies.

Finally, we also provide an abstract characterization of the previous preduals as closed subspaces \( M \subset l_\infty \) rich of positive elements.

As commonly used in the literature, we shall denote by \( l_1 \) the (complex) Banach space of absolutely summable sequences, given of the norm \( \| \cdot \|_1 \) defined by \( \|a\|_1 = \sum_{i=1}^{\infty} |a_i| \) for each \( a \in l_1 \).

It is a very well known fact that \( l_1 \) is a conjugate Banach space, that is there exists at least a Banach space \( X \), such that \( X^* \cong l_1 \) (isometric isomorphism).

Such a space is usually named a predual. The most famous predual of \( l_1 \) is probably represented by the space \( c_0 \) of those (complex) sequences converging to 0, endowed of the \( \sup \)-norm. In this case, the isometric isomorphism \( c_0^* \cong l_1 \) is the map \( \Psi : l_1 \to c_0^* \) given by \( \langle \Psi(y), x \rangle = \sum_{i=1}^{\infty} y_i x_i \) for every \( x \in c_0 \) and \( y \in l_1 \).

In spite of its simple definition, \( l_1 \) is a rather pathological Banach space: for instance the predual is not unique; there is in fact a plenty of (non isomorphic) preduals of \( l_1 \). Some of these are quite "irregular": Y. Benyamini and J. Lindenstrauss \cite{4} proved in 1972 that there is a predual of \( l_1 \) that is not (topologically) complemented in any \( C(K) \)-space, \( K \) being any compact Hausdorff topological space.

On the other hand, the present paper is aimed to discuss a very nice class of C*-preduals of \( l_1 \). In this spirit, the first thing that should be noticed is the following:

**Proposition 1.** If \( \tau \) is a compact Hausdorff topology on the set of natural numbers \( \mathbb{N} \), one has \( C_\tau(\mathbb{N})^* \cong l_1 \).

**Proof.** It is possible to prove the statement by using the Riesz-Markov theorem. Here we perform a proof based on the characterization of separable conjugate

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1 The weak topology of \( l_1 \) is not well behaved: every weakly convergent sequence is indeed norm-convergent, although the weak topology is strictly weaker than the norm topology.
spaces given in [5]. To this aim, we only have to check that \( C_\tau(\mathbb{N}) \subset l^\infty \) is a closed, norm-attaining and 1-norming subspace.

\( C_\tau(\mathbb{N}) \) is closed in \( l^\infty \) as a complete subspace. It is norm-attaining (when it is thought as subspace of bounded linear functionals on \( l_1 \)) thanks to Weierstrass’ theorem, since \((\mathbb{N}, \tau)\) is a compact space by assumption.

If \( y \in l_1 \) and \( \varepsilon > 0 \), there is \( n \in \mathbb{N} \) such that \( \|y\|_1 \leq \sum_{i=1}^{n} |y_i| + \varepsilon \). Let \( \theta_i \in \mathbb{R} \) such that \( y_i = |y_i| e^{i\theta_i} \) for each \( i = 1, 2, \ldots, n \). The subset \( C_n = \{1, 2, \ldots, n\} \subset \mathbb{N} \) is closed (and discrete), hence the function \( f: C_n \to \mathbb{C} \) given by \( f(i) = e^{-i\theta_i} \) for each \( i \in C_n \) is continuous and \( \|f\|_\infty = 1 \). Since \((\mathbb{N}, \tau)\) is a compact Hausdorff space, it is a normal topological space, so Tietze extension theorem applies to get a function \( g \in C_\tau(\mathbb{N}) \) such that \( \|g\|_\infty = 1 \) and \( g(i) = e^{-i\theta_i} \) for each \( i \in 1, 2, \ldots, n \).

We have \( |\langle g, y \rangle| = |\sum_{i=1}^{\infty} g(i)y_i| \geq |\sum_{i=1}^{n} |y_i| - \varepsilon| \geq \|y\|_1 - 2\varepsilon \). The last inequality easily implies that

\[
\sup_{g \in C_\tau(\mathbb{N})} |\langle g, y \rangle| = \|y\|_1
\]

that is \( C_\tau(\mathbb{N}) \subset l^\infty \) is a 1-norming subspace. This ends the proof.

The previous proposition immediately leads to the following corollary in point-set topology:

**Corollary 2.** Every compact Hausdorff topology on the set of natural numbers \( \mathbb{N} \) is metrizable.

**Proof.** Let \( \mathcal{T} \) be such a topology. We have \( C_\tau(\mathbb{N})^* \cong l_1 \), hence \( C_\tau(\mathbb{N}) \) is a separable Banach space, as a predual of the separable Banach space \( l_1 \), so that \((\mathbb{N}, \mathcal{T})\) is metrizable.

**Note 3.** As far as I know, a simple proof of the corollary quoted above does not seem available in the general setting of point-set topology, since it is not apparent that a compact Hausdorff topology on \( \mathbb{N} \) is automatically second countable.

On the other hand, non first countable topologies on \( \mathbb{N} \) are known: Appert topology, for instance, provides an elegant example of such a space. For the reader’s convenience, we recall here that Appert’s topology on \( \mathbb{N} \) is defined as follows: a subset \( A \subset \mathbb{N} \) is open if \( 1 \notin A \) or (when \( 1 \in A \)) if \( \lim_{n \to \infty} N(n, A) \) exists, where \( N(n, A) = |\{k \in A : k \leq n\}| \). Appert space is Lindelöf, separable but it is not first countable, since 1 does not have a countable basis of neighborhoods. For more details, we refer the interested reader to [6] or directly to the original paper by Appert [1].

Here below we shall describe explicitly a countable collection of compact Hausdorff topologies on \( \mathbb{N} \). Before introducing the announced topologies, one should mention that every set \( X \) can be endowed with a compact Hausdorff
topology, by virtue of a straightforward application of the Axiom of Choice\(^4\). Now let \(n \in \mathbb{N}\) be a fixed natural number. Given any \(k \in \{1, 2, \ldots, n\}\), we define the sets \(A_{k,l} = \{k, mn + k : m \geq l\}\). The sets \(A_{k,l}\) allow us to define a topology \(\mathcal{T}_n\), whose basis \(\mathcal{B}_n\) is given by the subset \(B \subset \mathbb{N}\) of the form \(A_{k,l}\) if \(k \in B\) for some \(k \in \{1, 2, \ldots, n\}\), otherwise we do not put any restriction, namely if \(\{1, 2, \ldots, n\} \cap B = \emptyset\) then \(B\) is allowed to be any subset of the natural numbers. Since \(A_{k,l} \cap A_{k',l}' = A_{k,l'}\) and \(A_{k,l} \cap A_{k',l'} = \emptyset\) when \(k, k' \in \{1, 2, \ldots, n\}\) are different, \(\mathcal{B}_n\) is really a basis. It is a straightforward verification to check that \(\mathcal{T}_n\) is a compact Hausdorff topology; the notion of convergence inherited by this topology is clearly the following:

A sequence \(\{n_m : m \in \mathbb{N}\}\) of integers converges to \(k \in \{1, 2, \ldots, n\}\) iff \(n_m\) is eventually in a set \(A_{k,l}\), while converges to \(k > n\) iff it is eventually equal to \(k\).

In the topology \(\mathcal{T}_n\) the set \(\{k : k \leq n\}\) is composed by non isolated points, while all the integers \(k > n\) are isolated. In some sense, topologies \(\mathcal{T}_n\) are as best as possible among compact Hausdorff ones, since it is a straightforward application of Baire category theorem that a compact Hausdorff topology on \(\mathbb{N}\) cannot have an infinite set of accumulation points\(^5\). However, what is more important here is that a simple argument can be performed to prove that the topologies \(\mathcal{T}_n\) are not homeomorphic:

**Proposition 4.** With the notations above, if \(n \neq m\) the topological spaces \((\mathbb{N}, \mathcal{T}_n)\) and \((\mathbb{N}, \mathcal{T}_m)\) are not homeomorphic.

**Proof.** Let us suppose that \(m > n\) and let \(\Phi : (\mathbb{N}, \mathcal{T}_m) \to (\mathbb{N}, \mathcal{T}_n)\) be a continuous injective map. If \(k \in \{1, 2, \ldots, m\}\), we can consider a sequence \(\{n_l\}\) converging to \(k\). The sequence \(\{\Phi(n_l)\}\) converges to \(\Phi(k)\) thanks to the continuity of \(\Phi\). Since \(\{n_l\}\) is not constant and \(\Phi\) is an injection, \(\Phi(k)\) is forced to be a natural number belonging to the subset \(\{1, 2, \ldots, n\}\), against the injectivity of \(\Phi\).

Let us denote by \(X_n\) the Banach space \(C_{\mathcal{T}_n}(\mathbb{N})\). Clearly we have \(X_n^* \cong l_1\) and

**Proposition 5.** If \(n \neq m\) the Banach space \(X_n\) and \(X_m\) are \(l_1\)-preduals, which are not isometrically isomorphic.

**Proof.** If they were isometrically isomorphic, the topological space \((\mathbb{N}, \mathcal{T}_n)\) and \((\mathbb{N}, \mathcal{T}_m)\) should be homeomorphic according to the classical Banach-Stein theorem.

The remaining part of the present paper is devoted to provide an intrinsic characterization of the spaces \(C_p(\mathbb{N})\) as suitable subspaces of \(l^\infty\). To this aim, one probably has to remind that any predual \(M\) of a conjugate spaces \(X\) should

\(^4\)The discrete topology \(\mathcal{P}(X)\) on \(X\) is locally compact and Hausdorff. The Alexandroff compactification \(\hat{X}\) of \(X\) is compact and Hausdorff; moreover, if \(X\) is an infinite set, there is a bijection \(\Phi : X \to \hat{X}\). We can use \(\Phi\) to define a compact Hausdorff topology \(\mathcal{T}\) on \(X\), by requiring a set \(U \subset X\) to be open if \(\Phi(U)\) is an open subset of \(X\).

\(^5\)Here \(l^1\) stands for \(\max\{l, h\}\).

\(^6\)Whenever \(\mathcal{T}\) is a compact Hausdorff topology on \(\mathbb{N}\), \((\mathbb{N}, \mathcal{T})\) is a Baire space as a complete metric space, hence it cannot be written as a countable union of rare sets, but every non isolated point \(n \in \mathbb{N}\) gives a rare singleton \(\{n\}\). In particular, the set of natural numbers \(\mathbb{N}\) cannot be given of a connected compact Hausdorff topology, anyway a connected Hausdorff topology on \(\mathbb{N}\) is available: for instance Golomb topology, see \([3]\).
be sought as a closed subspace of the dual space $X^*$, which is 1-norming and norm-attaining, namely each linear functionals belonging to the subspace is required to attain its norm on the unit ball of $X$.

When $X$ is a separable conjugate space, the conditions above are also sufficient for a closed subspace $M \subset X^*$ to be canonically a predual of $X$ as it is shown in [3].

Here canonically means that the isometric isomorphism $X \cong M^*$ is nothing but the restriction of the canonical injection $j : X \to X^{**}$ to $M$.

Before stating the result announced, let us fix some notations: $e \in l^\infty$ is the sequence constantly equal to 1, $M_+$ stands for the positive cone of a subspace $M \subset l^\infty$, while $\alpha^+$ is the square root of a positive element $\alpha \in l^\infty$.

According to the next theorem the spaces $C_\tau(N)$ are precisely those $l_1$-predual rich of positive elements:

**Theorem 6.** Let $M \subset l^\infty$ be a predual of $l_1$, such that:

(a) $e \in M$.

(b) $M_+$ is weakly*-dense in $l^\infty$.

(c) If $x \in M_+$, then $x^\frac{1}{2} \in M_+$.

Then $M = C_\tau(N)$ for a suitable compact Hausdorff topology on the set of natural numbers $N$.

**Proof.** Let be $\mathfrak{A} \subset l^\infty$ be the unital C*-algebra generated by $M$. If $\omega$ is a pure (multiplicative) state on $\mathfrak{A}$, we can consider its restriction $\omega|_M$. Since $M^* = l_1$, we have $\omega(x) = \varphi_y(x) = \sum y_i x_i$ for each $x \in M$, where $y$ is a suitable sequence in $l_1$. Now pick a positive element $\alpha \in l^\infty$. Thanks to (b), there is a sequence $\{x_n\}_{n \in N} \subset M_+$ such that $\lim x_n = \alpha^+$ (in the weak* topology of $l^\infty$). Then we have

$$\varphi_y(\alpha) = \lim_n \varphi_y(x_n) = \lim_n \varphi\left(x_n^\frac{1}{2} x_n^\frac{1}{2}\right) =$$

$$\lim_n \omega\left(x_n^\frac{1}{2} x_n^\frac{1}{2}\right) = \lim_n \omega\left(x_n^\frac{1}{2}\right) = \omega(\frac{1}{2}) \alpha = \varphi_y(\alpha)^2$$

where the last equality holds since $x_n^\frac{1}{2} \to \alpha^+$ (the weak* convergence in $l^\infty$ is nothing but the bounded pointwise convergence).

If $e_i \in l^\infty$ is the sequence given by $e_i(k) = \delta_{i,k}$, we get $\varphi_y(e_i) = \varphi_y(e_i)^2$, because $e_i^\frac{1}{2}$ is $e_i$ itself. It follows that, for each $i \in N$, $\varphi_y(e_i)$ is 0 or 1. Since $\sum |y_i| = \|\varphi_y\| = 1$, one has $y = e_k$ for some $k$. It easily follows that $\omega$ is the evaluation map at $k$.

This means that $\sigma(\mathfrak{A}) \cong N$, hence $\mathfrak{A} = C_\tau(N)$, $\mathcal{T}$ being the weak* topology on the spectrum of $\mathfrak{A}$.

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6A subspace $M \subset X^*$ is said to be 1-norming if for each $x \in X$, one has $\|x\| = \sup\{\varphi(x) : \varphi \in M_1\}$

$M_1$ being the unit ball of $M$.

7An element $x \in l^\infty$ is said to be positive if $x_i \geq 0$ for each $i \in N$; in this case one writes $x \geq 0$.

8If $x \geq 0$, then $x^\frac{1}{2}$ is the positive sequence given by $x^\frac{1}{2}(i) = x_i^\frac{1}{2}$ for each $i \in N$.

9For a basic treatment of C*-algebras theory, we refer the reader to [4].
Thanks to proposition \( \text{Proposition}\) we have \( C_r(\mathbb{N}) \cong l_1 \); since no proper inclusion relationships are allowed between preduals, we finally get \( M = \mathfrak{A} \). This concludes the proof.

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