Delta shock wave for a $3 \times 3$ hyperbolic system of conservation laws

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Abstract

We study the one-dimensional Riemann problem for a hyperbolic system of three conservation laws of Temple class. This system is a simplification of a recently propose system of five conservation laws by Bouchut and Boyaval that model viscoelastic fluids. An important issue is that the considered $3 \times 3$ system is such that every characteristic field is linearly degenerate. We show the Riemann problem for this system. Under suitable generalized Rankine-Hugoniot relation and entropy condition, both existence and uniqueness of particular delta-shock type solutions are established.

1 Introduction

In 1977, Korchinski [13] in his PhD thesis considered the Riemann problem for system
\begin{align}
\begin{cases}
    u_t + \left( \frac{1}{2}u^2 \right)_x = 0, \\
v_t + \left( \frac{1}{2}uv \right)_x = 0.
\end{cases}
\end{align}

Motivated by his numerical results, he constructed the unique Riemann solution using generalized delta functions to prove singular shocks satisfying (1) in the sense of distributions. In 1994, Tan, Zhang and Zheng in [17] established the existence, uniqueness and stability of delta shock waves to some viscous perturbations in the reduced one-dimensional system
\begin{align}
\begin{cases}
    u_t + \left( u^2 \right)_x = 0, \\
v_t + (uv)_x = 0.
\end{cases}
\end{align}

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and referred to such solutions as delta shock waves. A delta-shock wave is a generalization of an ordinary shock wave. From the physical point of view, a delta shock waves are interpreted as the process of formation of the galaxies in the universe, or the process of concentration of particles (mass) [18]. Other works in this sense are due to Ercole [10] who obtained a delta shock solution as a limit of smooth solutions by the vanishing viscosity method. Sheng and Zhang [15] discussed the Riemann problem for the pressureless gases system. In 2005, Brenier [2] considered the Riemann problem for the Chaplygin gas system. Other works for the Chaplygin gas system can be found in [11, 5, 12].

Finally, recent work on delta shocks for general hyperbolic conservation laws are due to Danilov and Mitrovic [6, 7] where they described delta shock wave generation from continuous initial data by using smooth approximations in the weak sense. An interesting work on new developments of delta shock waves, its applications and historical notes is [18].

In this work, we show existence and uniqueness of delta shock wave for the Suliciu relaxation system [16, 11]

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + s^2 v)_x &= 0, \\
(\rho v)_t + (\rho uv + u)_x &= 0,
\end{align*}
\]

with \( s = \text{const.} > 0 \). The eigenvalues associated to the system (3) are given by,

\[
\lambda_1 = u - \frac{s}{\rho}, \quad \lambda_2 = u, \quad \lambda_3 = u + \frac{s}{\rho},
\]

where the corresponding Riemann invariants are

\[
R_1 = s^2 v - su, \quad R_2 = v + \frac{1}{\rho}, \quad R_3 = s^2 v + su.
\]

From the expressions for the eigenvalues and the Riemann invariants we obtain

\[
\lambda_1 = \frac{R_3}{s} - s R_2, \quad \lambda_2 = \frac{1}{2s} (R_3 - R_1), \quad \lambda_3 = s R_2 - \frac{R_1}{s},
\]

and we can see that system (3) is linearly degenerate.

Recently, Lu et al. [14] showed existence of solutions for the Cauchy problem associated to the Suliciu relaxation system (3) with bounded initial data

\[
(\rho(0, x), u(0, x), v(0, x)) = (\rho_0(x), u_0(x), v_0(x)), \quad x \in \mathbb{R},
\]

\[
\rho_0(x) \geq 0,
\]

subject to the following conditions:

(H1) The functions \( \rho_0, u_0 \) and \( v_0 \) satisfy

\[
c_1 \leq u_0(x) - sv_0(x) \leq c_2, \quad c_3 \leq u_0(x) + sv_0(x) \leq c_4,
\]

\[
v_0(x) + \frac{1}{\varepsilon + \rho_0(x)} > c_5,
\]

where \( \varepsilon \) is a small positive constant and \( c_i, i = 1, \ldots, 5 \), are suitable constants satisfying

\[
c_5 - \frac{c_4 - c_1}{2s} > 0.
\]
The total variations of $u_0(x) - sv_0(x)$ and $u_0(x) + sv_0(x)$ are bounded.

The existence result for the Cauchy problem \((3) - (6)\) includes solutions in vacuum regions \([14, \text{Theorem } 1]\). When $\rho_0(x) \geq \rho > 0$, the condition \(H1\) becomes:

\[
(H1) \text{ The functions } \rho_0, u_0 \text{ and } v_0 \text{ satisfy }
\]
\[
c_1 \leq u_0(x) - sv_0(x) \leq c_2, \quad c_3 \leq u_0(x) + sv_0(x) \leq c_4,
\]
\[
v_0(x) + \frac{1}{\rho_0(x)} > c_5,
\]
where $c_i, i = 1, \ldots, 5$, are suitable constants satisfying
\[
c_5 - \frac{c_4 - c_1}{2s} > 0. \tag{8}
\]

In \([8]\) we show uniqueness of solutions for the Cauchy problem associated to the Suliciu relaxation system \((3) - (6)\) with $\rho_0(x) \geq \rho > 0$. Moreover, the Theorem 2 in \([8]\) ensures that when we consider $v_0(x) = -\frac{1}{\rho_0(x)}$, the Suliciu relaxation system is reduced to following Chaplygin gas system

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 - \frac{\alpha^2}{\rho})_x &= 0.
\end{align*} \tag{9}
\]

The classical Riemann problem associated to the Suliciu relaxation system \((3) - (10)\) has been extensively studied, for instance in \([1, 3, 4, 8]\). In \([9]\), the authors show uniqueness for the generalized Riemann problem for the Suliciu relaxation system.

Now, we show the existence and uniqueness of solutions for the Riemann problem associated with the Suliciu relaxation system with initial data

\[
(\rho_0(x), u_0(x), v_0(x)) = \begin{cases} (\rho_l, u_l, v_l), & \text{if } x < 0, \\
(\rho_r, u_r, v_r), & \text{if } x > 0,
\end{cases} \tag{10}
\]

in which the left and right constant states $(\rho_l, u_l, v_l)$ and $(\rho_r, u_r, v_r)$, with $\rho_l, \rho_r > 0$, satisfies the conditions \(H2\) and $\lambda_1(\rho_l, u_l, v_l) \geq \lambda_3(\rho_r, u_r, v_r)$, i.e., they do not satisfy condition \(H1\) globally. In this way, we have the following situations:

1. $(\rho_l, u_l, v_l)$ and $(\rho_r, u_r, v_r)$ satisfy locally the condition \(H1\), i.e.,
   - $(\rho_l, u_l, v_l)$ satisfy
     \[
     \alpha_1 \leq u_l - sv_l \leq \alpha_2, \quad \alpha_3 \leq u_l - sv_l \leq \alpha_4 \quad \text{and} \quad v_l + \frac{1}{\rho_l} > \alpha_5
     \]
     where $\alpha_i, i = 1, \ldots, 5$, are suitable constants satisfying $\alpha_5 - \frac{\alpha_4 - \alpha_1}{2s} > 0$,
   - $(\rho_r, u_r, v_r)$ satisfy
     \[
     \beta_1 \leq u_r - sv_r \leq \beta_2, \quad \beta_3 \leq u_r - sv_r \leq \beta_4 \quad \text{and} \quad v_r + \frac{1}{\rho_r} > \beta_5
     \]
     where $\beta_i, i = 1, \ldots, 5$, are suitable constants satisfying $\beta_5 - \frac{\beta_4 - \beta_1}{2s} > 0$.  

• Let $c_1 = \min\{\alpha_1, \beta_1\}$ and $c_4 = \max\{\alpha_1, \beta_1\}$. Then there are $c_i$, $i = 1, \ldots, 4$ such that

$$c_1 \leq \left\{ \frac{u_l - s v_l}{u_r - s v_r} \right\} \leq c_2 \quad \text{and} \quad c_3 \leq \left\{ \frac{u_l + s v_l}{u_r + s v_r} \right\} \leq c_4,$$

but is not possible to find a constant $c_5$ such that $c_5 - \frac{c_4 - c_1}{2s} > 0$.

2. Only $(\rho_l, u_r, v_r)$ satisfy locally the condition H1, i.e., $(\rho_l, u_l, v_l)$ satisfy

$$\beta_1 \leq u_l - s v_l \leq \beta_2, \quad \beta_3 \leq u_l - s v_l \leq \beta_4 \quad \text{and} \quad v_l + \frac{1}{\rho_l} > \beta_5,$$

where $\beta_i$, $i = 1, \ldots, 5$, are suitable constants satisfying $\beta_5 - \frac{\beta_4 - \beta_1}{2s} \leq 0$.

3. Only $(\rho_l, u_l, v_l)$ satisfy locally the condition H1, i.e., $(\rho_r, u_r, v_r)$ satisfy

$$\beta_1 \leq u_r - s v_r \leq \beta_2, \quad \beta_3 \leq u_r - s v_r \leq \beta_4 \quad \text{and} \quad v_r + \frac{1}{\rho_r} > \beta_5,$$

where $\beta_i$, $i = 1, \ldots, 5$, are suitable constants satisfying $\beta_5 - \frac{\beta_4 - \beta_1}{2s} \leq 0$.

4. Neither $(\rho_l, u_l, v_l)$ nor $(\rho_r, u_r, v_r)$ satisfy the local condition.

De la cruz et al. [8] studied the first case. The idea used by the authors in [8] can be extended for the other cases.

## 2 Delta shock solutions

Denote by $BM(\mathbb{R})$ the space of bounded Borel measures on $\mathbb{R}$, and then the definition of a measure solution of Suliciu relaxation system in $BM(\mathbb{R})$ can be given as follows.

**Definition 2.1.** A triple $(\rho, u, v)$ constitutes a measure solution to the Suliciu relaxation system, if it holds that

a) $\rho \in L^\infty((0, \infty), BM(\mathbb{R})) \cap C((0, \infty), H^{-s}(\mathbb{R}))$,

b) $u \in L^\infty((0, \infty), L^\infty(\mathbb{R})) \cap C((0, \infty), H^{-s}(\mathbb{R}))$,

c) $v \in L^\infty_{\text{loc}}((0, \infty), L^\infty_{\text{loc}}(\mathbb{R})) \cap C((0, \infty), H^{-s}(\mathbb{R}))$, $s > 0$,

d) $u$ and $v$ are measurable with respect to $\rho$ at almost for all $t \in (0, \infty)$,

and

$$\begin{cases}
I_1 = \int_0^\infty \int_{\mathbb{R}} (\phi_t + u \phi_x) \; dp \; dt = 0, \\
I_2 = \int_0^\infty \int_{\mathbb{R}} u(\phi_t + u \phi_x) \; dp \; dt + \int_0^\infty \int_{\mathbb{R}} s^2 v \phi_x \; dx \; dt = 0, \\
I_3 = \int_0^\infty \int_{\mathbb{R}} v(\phi_t + u \phi_x) \; dp \; dt + \int_0^\infty \int_{\mathbb{R}} u \phi_x \; dx \; dt = 0,
\end{cases} \quad (11)$$

for all test function $\phi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R})$.

**Definition 2.2.** A two-dimensional weighted delta function $w(s)\delta_L$ supported on a smooth curve $L$ parameterized as $t = t(s)$, $x = x(s)$ ($c \leq s \leq d$) is defined by

$$\langle w(s)\delta_L, \phi(t, x) \rangle = \int_c^d w(s)\phi(t(s), x(s)) \; ds$$ \quad (12)

for all $\phi \in C^\infty(\mathbb{R}^2)$. 
Definition 2.3. A triple distribution \((\rho, u, v)\) is called a delta shock wave if it is represented in the form

\[
(p_l, u_l, v_l)(t, x) = \begin{cases} 
(p_l, u_l, v_l)(t, x), & x < x(t), \\
(w(t)\delta(x - x(t)), u_\delta(t), g(t)), & x = x(t), \\
(p_r, u_r, v_r)(t, x), & x > x(t),
\end{cases}
\]

and satisfies Definition 2.1, where \((p_l, u_l, v_l)(t, x)\) and \((p_r, u_r, v_r)(t, x)\) are piecewise smooth bounded solutions of the Suliciu relaxation system (3) if and only if the following relation holds,

\[
\rho(t)^0 = \frac{\int_0^t \rho u \phi dt}{\int_0^t u \phi dt}.
\]

We set \(\frac{dx}{dt} = u_\delta(t)\) since the concentration in \(\rho\) need to travel at the speed of discontinuity. Hence, we say that a delta shock wave \((13)\) is a measure solution to the Suliciu relaxation system (3) if and only if the following relation holds,

\[
\begin{align*}
\frac{dx(t)}{dt} &= u_\delta(t), \\
\frac{dw(t)}{dt} &= -[\rho]u_\delta(t) + [pu], \\
\frac{dw(t)u_\delta(t)}{dt} &= -[pu]u_\delta(t) + [pu^2 + s^2v], \\
\frac{dw(t)g(t)}{dt} &= -[pv]u_\delta(t) + [pvu + u].
\end{align*}
\]

In fact, for any test function \(\phi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R})\), from (11), we obtain

\[
I_1 = \int_0^\infty \int_\mathbb{R} (\phi_t + u\phi_x) \rho \phi dt = \int_0^\infty \left\{-u_\delta(t)[\rho] + [pu] - \frac{dw(t)}{dt} \right\} \phi dt,
\]

\[
I_2 = \int_0^\infty \left\{-u_\delta(t)[\rho u] + [pu^2 + s^2v] - \frac{dw(t)u_\delta(t)}{dt} \right\} dt,
\]

\[
I_3 = \int_0^\infty \left\{-u_\delta(t)[pv] + [pvu + u] - \frac{dw(t)g(t)}{dt} \right\} \phi dt.
\]

Relations (14) are called the generalized Rankine-Hugoniot condition. It reflects the exact relationship among the limit states on two sides of the discontinuity, the weight, propagation speed and the location of the discontinuity. In addition, to guarantee uniqueness, the delta shock wave should satisfy the admissibility (entropy) condition

\[
\lambda_3(\rho_r, u_r, v_r) \leq u_\delta(t) \leq \lambda_1(\rho_l, u_l, v_l).
\]

Now, the generalized Rankine-Hugoniot condition is applied to the Riemann problem (3)–(10) with left and right constant states \(U_\pm = (\rho_\pm, u_\pm, v_\pm)\) and \(U_\pm = (\rho_\pm, u_\pm, v_\pm)\), respectively, satisfying the condition H2, the fact \(\lambda_3(\rho_+, u_+, v_+) \leq \lambda_1(\rho_-, u_-, v_-)\) and

\[
(u_\pm - u_\mp)^2 \geq \frac{s^2}{\rho_+}(v_\pm - v_\mp) - \frac{s^2}{\rho_-}(v_\pm - v_\mp).
\]

Thereby, the Riemann problem is reduced to solving (14) with initial data

\[
t = 0, \quad x(0) = 0, w(0) = 0, g(0) = 0,
\]

under entropy condition

\[
u_\pm + \frac{s}{\rho_\pm} \leq u_\delta(t) \leq u_\pm - \frac{s}{\rho_-}.
\]
From (14) and (17), it follows that
\[ w(t) = -[\rho]x(t) + [\rho u]t, \]
\[ w(t)u_\delta(t) = -[\rho u]x(t) + [\rho u^2 + s^2 v]t, \text{ and} \]
\[ w(t)g(t) = -[\rho u]x(t) + [\rho u^2 + u]t. \]
Multiplying the first equation in (19) by \( u_\delta(t) \) and then subtracting it from the second one, we obtain that
\[ [\rho]x(t)u_\delta(t) - [\rho u]u_\delta(t)x(t) + [\rho u^2 + s^2 v]t = 0, \]
which is equivalent to
\[ \frac{d}{dt} \left( \frac{[\rho]}{2} x^2(t) - [\rho u]x(t)t + \frac{[\rho u^2 + s^2 v]}{2}t^2 \right) = 0, \]
From condition (16), because
\[ u_\delta = \rho_-(u_- - u_+), \] and
\[ \lambda_1(U_-) - \lambda_3(U_+) < \frac{1}{2} \left\{ \lambda_1(U_-) - \lambda_3(U_+) \right\}, \]
From (22), one can find \( u_\delta(t) = u_\delta \) is a constant and \( x(t) = u_\delta t \). Then, (22) can be rewritten
\[ [\rho]u_\delta^2 - 2[\rho u]u_\delta + [\rho u^2 + s^2 v] = 0. \]
When \([\rho] = \rho_- - \rho_+ = 0\), the situation is very simple and one can easily calculate the solution
\[
\begin{aligned}
  u_\delta &= \frac{u_- + u_+}{2} + s^2 \frac{[v]}{2\rho_- |u|}, \\
  x(t) &= u_\delta t, \\
  w(t) &= \rho_- (u_- - u_+)t, \\
  g(t) &= \frac{[\rho u_+ u_- - u_\delta]}{|\rho u|},
\end{aligned}
\]
which obviously satisfies the entropy condition (18). From condition (16),
\[ s^2 \frac{[v]}{\rho_-} \leq \frac{1}{2} (\lambda_1(U_-) - \lambda_3(U_+))^2 < \frac{1}{2} [u](\lambda_1(U_-) - \lambda_3(U_+)) \]
and
\[ u_\delta - \left( \frac{u_- - s}{\rho_-} \right) = \frac{1}{2} \left( \left( \frac{u_+ + s}{\rho_-} \right) - \left( \frac{u_- - s}{\rho_-} \right) + s^2 \frac{[v]}{\rho_- |u|} \right) \leq 0. \]
Similarly, we can deduce that
\[ u_\delta - \left( \frac{u_+ + s}{\rho_-} \right) = \frac{1}{2} \left( \left( \frac{u_- - s}{\rho_-} \right) - \left( \frac{u_+ + s}{\rho_-} \right) + s^2 \frac{[v]}{\rho_- |u|} \right) \geq 0, \]
because
\[ -s^2 \frac{[v]}{\rho_-} \leq \frac{1}{2} (\lambda_1(U_-) - \lambda_3(U_+))^2 < \frac{1}{2} [u](\lambda_1(U_-) - \lambda_3(U_+)). \]
When \([\rho] = \rho_- - \rho_+ \neq 0\), the discriminant of the quadratic equation (23) is
\[ \Delta = 4[\rho u]^2 - 4[\rho u^2 + s^2 v] = \rho_- \rho_+ [u]^2 - s^2 [\rho] [v] > 0 \]
and then we can find

\[
\begin{align*}
    u_\delta &= \frac{|\rho u| - \sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v}}{|\rho|}, \\
    x(t) &= \frac{|\rho u| - \sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v}}{|\rho|} t, \\
    w(t) &= \sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v} t, \\
    g(t) &= \frac{-|\rho u||\rho v| + |\rho|\sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v} + |\rho|\rho u + u}{|\rho|\sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v}} t,
\end{align*}
\]  

(26)

or,

\[
\begin{align*}
    u_\delta &= \frac{|\rho u| + \sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v}}{|\rho|}, \\
    x(t) &= \frac{|\rho u| + \sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v}}{|\rho|} t, \\
    w(t) &= -\sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v} t, \\
    g(t) &= \frac{-|\rho u||\rho v| - |\rho|\sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v} + |\rho|\rho u + u}{|\rho|\sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v}} t.
\end{align*}
\]  

(27)

Next, with the help of the entropy condition (18), we will choose the admissible solution from (26) and (27). Observe that by the entropy condition and since the system is strictly hyperbolic, we have that

\[
u_+ - \frac{s}{\rho_+} < \nu_- \leq \nu_+ + \frac{s}{\rho_+} < \nu_- < \nu_+ + \frac{s}{\rho_-}. \]

Observe that,

\[
\begin{align*}
    -|\rho|\lambda_1(\rho_-, \nu_-, \nu_-) + |\rho u| &= \rho_+ \left( \left( u_- - \frac{s}{\rho_-} \right) - \left( u_+ - \frac{s}{\rho_+} \right) \right) > 0, \\
    -|\rho|\lambda_3(\rho_+, \nu_+, \nu_+) + |\rho u| &= \rho_- \left( \left( u_- + \frac{s}{\rho_-} \right) - \left( u_+ + \frac{s}{\rho_+} \right) \right) > 0,
\end{align*}
\]

\[
\begin{align*}
    |\rho|\lambda_1(\rho_-, \nu_-, \nu_-)^2 - 2|\rho u|\lambda_1(\rho_-, \nu_-, \nu_-) + |\rho u^2| + s^2 |v| &= \\
    -\rho_+ \left( u_- - u_+ - \frac{s}{\rho_-} \right)^2 + s^2 \rho_+ + s^2 |v| \leq 0,
\end{align*}
\]

\[
\begin{align*}
    |\rho|\lambda_3(\rho_+, \nu_+, \nu_+)^2 - 2|\rho u|\lambda_3(\rho_+, \nu_+, \nu_+) + |\rho u^2| + s^2 |v| &= \\
    \rho_- \left( u_- - u_+ - \frac{s}{\rho_+} \right)^2 - s^2 \rho_- + s^2 |v| \geq 0,
\end{align*}
\]

then, for the solution given in (26), we have

\[
u_\delta - \lambda_1(\rho_-, \nu_-, \nu_-) \leq 0 \quad \text{and} \quad \nu_\delta - \lambda_3(\rho_+, \nu_+, \nu_+) \geq 0,
\]

which imply that the entropy condition (18) is valid. When \(\lambda_1(\rho_-, \nu_-, \nu_-) = \lambda_3(\rho_+, \nu_+, \nu_+)\), we have trivially that \(\lambda_1(\rho_-, \nu_-, \nu_-) = \nu_\delta = \lambda_3(\rho_+, \nu_+, \nu_+)\).

Now, for the solution (27), when \(\rho_- < \rho_+\) we have

\[
u_\delta - \lambda_3(\rho_+, \nu_+, \nu_+) = \frac{\rho_-(\lambda_3(U_-) - \lambda_3(U_+)) + \sqrt{|\rho u|^2 - |\rho|\rho u^2 + s^2 v}}{|\rho|} < 0,
\]
and when $\rho_- > \rho_+$, that
\[
\alpha_\delta - \lambda_1(-\rho_-, u_-, v_-) = \frac{\rho_+ (\lambda_1(U_-) - \lambda_1(U_+)) + \sqrt{\rho u^2 - \rho |u|^2 + s^2 v^2}}{|\rho|} > 0.
\]
showing that the solution (27) does not satisfy the entropy condition (18).
Thus we have proved the following result.

**Theorem 2.1.** Given left and right constant states $(\rho_l, u_l, v_l)$ and $(\rho_r, u_r, v_r)$, respectively, such that satisfy the condition $H2$, $\lambda_1(\rho_l, u_l, v_l) \geq \lambda_3(\rho_r, u_r, v_r)$ and (16), that is,
\[
(u_l - u_r)^2 \geq s^2(v_l - v_r)/\rho_r - s^2(v_l - v_r)/\rho_l.
\]
Then, the Riemann problem (3)–(10) admits a unique entropy solution in the sense of measures. This solution is of the form
\[
(\rho, u, v)(t, x) = \begin{cases} 
(\rho_l, u_l, v_l), & \text{if } x < \alpha_\delta t, \\
(w(t)\delta(x - \alpha_\delta t), u_\delta, g(t)), & \text{if } x = \alpha_\delta t, \\
(\rho_r, u_r, v_r), & \text{if } x > \alpha_\delta t,
\end{cases}
\]
where $u_\delta$, $w(t)$ and $g(t)$ are show in (24) for $|\rho| = 0$ or (26) for $|\rho| \neq 0$.

### 3 Numerical test

In this section, we show numerical evidence of delta shock solution for the Suliciu relaxation system using, once again, the Lax-Friedrichs method.

In the numerical test, with $s = 1$, we consider the initial data given by
\[
(\rho_0, u_0, v_0)(x) = \begin{cases} 
(9, 5, 14/5), & \text{if } x < 0, \\
(1, 3, 2), & \text{if } x > 0.
\end{cases}
\]
and the spatial discretization parameter for $N = 1780$ points and a constant $CFL = 0.1969889$.
The numerical results at final time $t = 0.1$ are presented in Figures 1, 2 and 3. The exact solution at time $t$ is
\[
(\rho, u, v)(t, x) = \begin{cases} 
(9, 5, 14/5), & \text{if } x < \alpha_\delta t, \\
(At\delta(x - \alpha_\delta t), u_\delta, Bt), & \text{if } x = \alpha_\delta t, \\
(1, 3, 2), & \text{if } x > \alpha_\delta t,
\end{cases}
\]
with $u_\delta = \frac{21\sqrt{5} - \sqrt{37}}{4\sqrt{5}}$, $A = \frac{2\sqrt{37}}{\sqrt{5}}$ and $B = \frac{\sqrt{5} + 20\sqrt{37}}{10\sqrt{37}}$.

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Figure 1: Numerical solution of $\rho$. (a) The initial data $\rho_0(x) = \rho(0, x)$. (b) Numerical solution of $\rho$ at time $t = 0.1$.

Figure 2: Numerical solution of $u$. (a) The initial data $u_0(x) = u(0, x)$. (b) Numerical solution of $u$ at time $t = 0.1$.

Figure 3: Numerical solution of $v$. (a) The initial data $v_0(x) = v(0, x)$. (b) Numerical solution of $v$ at time $t = 0.1$. 