Symmetries in subatomic multi-quark systems

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We discuss the role of QCD (Quantum Chromodynamics) to low energy phenomena involving the color-spin symmetry of the quark model. We then combine it with orbital and isospin symmetry to obtain wave functions with the proper permutation symmetry, focusing on multi quark systems.

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1. Motivation for contributing to this volume

It is with humble feelings that JDV is contributing to this review article in the memory of E. M. Henley.

His first countenance with Ernie was through his book with H. Frauenfelder.\textsuperscript{1} This book had a profound influence on JDV. First, as its title indicates, because of its unifying view of particle and nuclear physics, and second due to its fine interplay of theoretical and experimental physics and, above all, its excellent exposition of the symmetries present in subatomic physics, especially the discrete symmetries. No wonder the 3nd edition of this book is still one of the most popular books in the field of particle and nuclear physics.

JDV met him for the first time in the office of H. Primakoff, when he was an Assistant Professor of Physics at the University of Pennsylvania. He also met him again when he visited the University of Washington for a year in 1983, on sabbatical from the University of Ioannina in Greece at the invitation of Gerry Miller. During that time Ernie, Takamitsu Oka and JDV collaborated on a number of physics projects, which will be discussed later. This has perhaps been the most enjoyable collaboration in his life, combining Ernie’s insight on symmetries, Takamitsu’s talent in arithmetic manipulation, the result of the well known Japanese tradition, and JDV’s background on Group Theory. Beyond the specific research done together, JDV was impressed by Ernie’s deep breadth of physics, his kind personality and his wide view of science and education, combined with excellent managerial skills.

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which he obtained as he was moving up the academic ladder. The latter proved extremely useful, when JDV became a Rector of the University of Ioannina 8 years later.

Motivated by the above, some aspects of the subject he enjoyed the most, namely symmetries, will be explored in this article, focusing on the role of Quantum Chromodynamics (QCD) at low energies and, in particular, on understanding the behavior of multiquark systems.

2. Introduction

Multiquark systems, like the pentaquark and tetraquark, have already been found to exist. Why not multiquark systems in the nucleus? So we will examine the possibility of the presence of six quark clusters in the nucleus, a much more complex problem. Such clusters, if present with a reasonable probability in the nucleus, may contribute to various processes, like neutrinoless double beta decay mediated by heavy neutrinos or other exotic particles. In conventional nuclear physics the relevant nuclear matrix elements are suppressed due to the presence of the nuclear hard core. In this presence of such clusters, however, the interacting quarks are in the same hadron. So one can have a contribution even in the case of a $\delta$-function interaction. Symmetries, of course, play a crucial role in reliably estimating the probability of finding such six quark clusters in the nucleus.

3. Quantum chromodynamics (QCD)

It is well known (see, e.g.,\cite{3,45}) that the strong interaction is governed by the group $SU_c(3)$ of the standard model with a set of traceless generator generators, the Gell-Mann matrices $\lambda^a$, such that $tr(\lambda^a)^2 = 2$, $a = 1 \cdots 8$. It is also known that associated with this symmetry we have a set of 8 gauge bosons $^aG^\lambda$, with $\lambda$ a Lorentz index and $a = 1 \cdots 8$, the gluons. The interactions mediated by the gluons is given in Fig. 1.

A comparison of the gluon exchange and the $W$ exchange in weak interactions is given in Fig. 2.

The gluons remain massless even after the spontaneous symmetry breaking. Since, however, it is not Abelian, it does not lead to any long range force. The emerging theory is called Quantum Chromodynamics. This theory is described by the Lagrangian:

$$\mathcal{L}_c = -\frac{1}{4} F^{\mu\nu}_{\lambda} F^{\lambda\mu\nu} + \sum_r \bar{q}_r \gamma^\mu (D_\mu)_{\lambda}^\alpha q_r^\beta,$$

(1)

$^a$Sometimes the Lorentz index is understood and omitted and the gluons are written in the form of a $3 \times 3$ matrix $^aG_\lambda^\mu$ with $\mu$ and $\lambda$ color degrees of freedom taking values, e.g. $r$, $g$, $b$. The two diagonal matrices have the form $\text{diag}(1,-1,0)$ and $(1/\sqrt{3})\text{diag}(1,1,-2)$. 
where \( r \) is a quark flavor index, \( \alpha \) and \( \beta \) are color indices and \( D^\mu \) is the covariant derivative

\[
(D_\mu)^\alpha_\beta = \partial_\mu \delta^\alpha_\beta - ig_s G^j_\mu \frac{1}{2} (\lambda^j)^\alpha_\beta
\]  \hspace{1cm} (2)

and

\[
F^i_{\mu\nu} = \partial_\mu G^i_\nu - \partial_\nu G^i_\mu - g s f_{ijk} G^k_\mu G^j_\nu.
\]  \hspace{1cm} (3)
where $g_s$ is the group gauge coupling constant\(^{b}\) and $f_{ijk}$ are the structure constants of $SU(3)$ given by:

$$[\lambda^i, \lambda^k] = 2if_{ijk}\lambda^j.$$  

Note that, in contrast to the other unbroken Abelian $U_{EM}(1)$ symmetry, the presence of the $SU(3) F^2$ term leads to three and four point gluon self couplings, which is due to the couplings $f_{ijk}$. This effect results in some technical difficulties in QCD as compared to the electrodynamics.

- The gluons tend to polarize the medium. In other words it becomes energetically cheaper to create quark anti-quark pairs out of the vacuum (see Fig. 3).
- The polarization of the vacuum is not like the dipole creation familiar from dielectrics, which tends to decrease the interaction between two charges. Instead it behaves more like quadrupoles in dielectrics, hard to make in macroscopic scale, which tend to increase the interaction between the colored quarks (see Fig. 4). In other words at high energies, or small distances, the quarks behave almost like being free, i.e. we have asymptotic freedom. On the other hand at low energies (large distances of the order of fm) the interaction becomes very strong. Thus we encounter confinement, perpetual quark slavery.
- Since the interaction becomes strong at low energies, one cannot invoke perturbation theory. So multi-gluon exchange or pair creation out of the vacuum diagrams become important (see Fig. 3).

\(^{b}\) We use here $g_s$ instead of $g_3$ commonly used in the standard model.
The strength of the interaction depends on the energy and the number \( n_f \) of active quark pairs:

\[
\alpha_s(E) = \frac{4\pi}{\left(-33 + 2n_f \ln \frac{E^2}{\mu^2}\right)},
\]

where \( \{n_f = \text{the number of quarks active pair production (up to 6)} \)

\( \mu = \text{experimentally determined scale } \approx 0.2\text{GeV} \).

Another common parametrization is given by

\[
\alpha_s(q^2) = \frac{\alpha_s(\mu^2)}{1 + \beta \alpha_s(\mu^2) \left( \ln \frac{q^2}{\mu^2} \right)}, \quad \beta = \frac{33 - 2n_f}{12\pi}.
\]

These formulas yield \( \alpha_s = 0.12 \) at \( q^2 = (100\text{MeV})^2 \).

---

**Fig. 4:**

The strong coupling strength decreases as a function of the energy scale. At large scale it becomes very weak (asymptotic freedom), while at small scales it becomes very strong (imposed slavery). Shown is the prediction of the simple formula (Eq. 5) (a) and the experimental data of the ALPHA collaboration (b).

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4. The color structure of the one gluon exchange potential involving quarks

Let us suppose\(^c\) that the two quarks are \( q_\alpha(1)q_\beta(2) \), where \( \alpha \) and \( \beta \) are color indices, taking values \( r, g, b \), and \( (1) \ (2) \) label the particles. For simplicity of notation we will

\(^c\)For more details and explanations on this and the next two sections see a recent textbook.\(^4\)
drop the particle index, with the understanding that particle one will be first and particle second, by writing $q_\alpha(1)q_\beta(2) \leftrightarrow \alpha\beta = rr, rg, \text{ etc.}$ The Feynman diagrams leading to the interaction between two quarks are exhibited in Fig. 5.

Fig. 5:
The strong interaction mediated by gluons, with the Lorentz index suppressed. Shown are the color changing gluon interaction ($t^a_{\beta,\alpha}$, $\alpha \neq \beta$, which destroys a quark $\beta$ and converts to a quark $\alpha$) (a), the color conserving interaction involving quarks other than $b$ ($t^{1,8}_{\beta,\alpha}$, $\alpha, \beta = r, g$) (b) and the color conserving interaction in which at least one of the quarks is blue ($t^{2,8}_{\beta,\alpha}$, $\alpha = r, g, b$) (c). Note that all operators are normalized to unity $\text{tr}(t^a t^b) = \delta_{a,b}$

Ignoring the overall strength (color charge, $g_r = g_s = 4\pi\alpha_s$) and omitting the gluon propagator, the interaction between similar colors is

$$\langle rr | V | rr \rangle = \langle gg | V | gg \rangle = \left( \frac{1}{6} + \frac{1}{2} \right) = \frac{2}{3}$$

The first contribution comes from the exchange of the gluon $G^8$ and the second one comes from $G^3$ (see Fig. 5b). Similarly from the exchange of the gluon $G^8$ (see Fig. 5c) we get

$$\langle bb | V | bb \rangle = \frac{1}{6} \cdot 2^2 = \frac{2}{3}$$

In a similar fashion we get

$$\langle gr | V | gr \rangle = \langle rg | V | rg \rangle = \left( \frac{1}{6} - \frac{1}{2} \right) = -\frac{1}{3} \text{ (see Fig. 5b)}$$
\[ \langle rb|V|rb \rangle = \langle br|V|br \rangle = \frac{1}{6} (2) = -\frac{1}{3} \] (see Fig. 5c)

Furthermore

\[ \langle \alpha \beta |V|\beta \alpha \rangle = 1, \quad \alpha \neq \beta \] (see Fig. 5a)

In this last case the off diagonal gluon is exchanged, \( G^{1,2,4,5,6,7} \). All other matrix elements are zero.

The same results hold in the case of two antiquarks.

The above product of two quarks constitutes a basis in color space, \( 3 \otimes 3 \), which yields a reducible nine dimensional representation of \( SU(3) \). Diagonalizing this interaction in the nine dimensional space one finds three states with eigenvalue \(-\frac{4}{3}\) and six states with eigenvalue \( \frac{2}{3} \) as follows:

\[ -\frac{4}{3} \Leftrightarrow \tilde{b} = \frac{1}{2} (rg - gr), \tilde{g} = \frac{1}{2} (rb - br), \tilde{r} = \frac{1}{2} (gb - bg) \]

\[ \frac{2}{3} \Leftrightarrow \frac{1}{2} (rg + gr), \frac{1}{2} (rb + br), \frac{1}{2} (gb + bg), \text{rr, gg, bb.} \]

The first is an irreducible three-dimensional representation (anti-triplet) and the second is a six dimensional representation (sextet), which is symmetric in the color indices.

One uses the anti-triplet and the sextet as a basis and thus the color interaction takes the form

\[ \lambda_1 \lambda_2 = \frac{1}{2} (C(\lambda, \mu) - 2C(1, 0)) \]

where \( \lambda \) and \( \mu \) specify the irreducible representation of \( SU(3) \), which are non-negative integers. \( C(\lambda, \mu) \) is the value of the Casimir operator of \( SU(3) \), given by:

\[ C(\lambda, \mu) = \frac{2}{3} \left( \lambda^2 + \mu^2 + \lambda \mu + 3(\lambda + \mu) \right) \]

Thus:

triplet: \( (\lambda, \mu) = (1, 0) \rightarrow C(\lambda, \mu) = \frac{8}{3} \)

anti-triplet: \( (\lambda, \mu) = (0, 1) \rightarrow C(\lambda, \mu) = \frac{8}{3} \rightarrow \lambda_1 \lambda_2 = -\frac{4}{3} \)

sextet: \( (\lambda, \mu) = (2, 0) \rightarrow C(\lambda, \mu) = \frac{20}{3} \rightarrow \lambda_1 \lambda_2 = \frac{2}{3} \)

Let us now examine the interaction between quarks and antiquarks. The anti-quark viewed as the antisymmetric combination of two quarks given above. Than we can evaluate the matrix element of the gluon between the relevant three quark states, noting that one of the quarks of the antisymmetric combination is a spectator (not interacting). Thus we find the non vanishing independent ME involving different colors are:

\[ \langle r\bar{b}|V|r\bar{b} \rangle = \frac{1}{3}, \quad \langle r\bar{g}|V|r\bar{g} \rangle = \frac{1}{3}, \quad \langle g\bar{b}|V|g\bar{b} \rangle = \frac{1}{3} \]
For the similar color combination we get:

\[ \langle r \bar{r} | V | r \bar{r} \rangle = \langle g \bar{g} | V | g \bar{g} \rangle = \langle b \bar{b} | V | b \bar{b} \rangle = -\frac{2}{3}, \]

\[ \langle r \bar{r} | V | g \bar{g} \rangle = \langle r \bar{r} | V | b \bar{b} \rangle = \langle g \bar{g} | V | b \bar{b} \rangle - 1 \]

Note the change in sign compared to the \( q - q \) interaction. This usually is attributed to the "opposite color charge of anti-quarks", but note the notational difference in the last equation. We have seen, however, that it comes as a transformation from the picture of anti-quarks as an antisymmetric combination of quarks.

The product of a quark and anti-quark transforms under \( SU(3) \) like \( \mathbf{3} \otimes \mathbf{3}^* \), which is reducible. One can easily see that the above above six combinations \( q_\alpha \bar{q}_\beta, \alpha \neq \beta \) transform as the six members of the octet. Diagonalizing the above matrix in the case of similar color indices we find the eigensolutions:

\[ \epsilon_0 = -\frac{8}{3} \leftrightarrow r_0 = \frac{1}{\sqrt{3}}(\langle r \bar{r} \rangle + \langle g \bar{g} \rangle + b \bar{b}), \]

\[ \epsilon_1 = \frac{1}{3} \leftrightarrow r_1 = \frac{1}{\sqrt{3}}(\langle r \bar{r} \rangle - \langle g \bar{g} \rangle), \]

\[ \epsilon_2 = \frac{1}{3} \leftrightarrow r_2 = \frac{1}{\sqrt{6}}(\langle r \bar{r} \rangle + \langle g \bar{g} \rangle - 2b \bar{b}). \]

The first transforms like the singlet, while the other two are the additional members of the octet. The similarity with the expressions of the \( SU(3) \) should not come as a surprise.

In summary we have seen that:

\[ \mathbf{3} \otimes \mathbf{3}^* = 1 + \mathbf{8} \]

As a result we can write the quark antiquark interaction as:

\[ \lambda_1, \lambda_2 = \frac{1}{2} (C(\lambda, \mu) - C(1, 0) - C(0, 1)) \]

singlet: \( (\lambda, \mu) = (0, 0) \rightarrow C(\lambda, \mu) = 1 \rightarrow \lambda_1, \lambda_2 = -\frac{8}{3} \)

octet: \( (\lambda, \mu) = (1, 1) \rightarrow C(\lambda, \mu) = \frac{18}{3} \rightarrow \lambda_1, \lambda_2 = \frac{1}{3} \)

Before concluding this section we should mention that in the context of the one

gluon exchange potential between the above allowed color combinations is proportional to \( \lambda_1, \lambda_2 \alpha_s \).
5. Approximations at low energies-Interaction potentials between quarks

We know that the quarks do not appear free and all the observed hadrons are colorless. So all experimental information regarding quarks is necessarily indirect and complicated manifestations of chromodynamics. It looks as though our only access to electrodynamics came from Van der Waals forces between molecules.

We have already seen that due to anti-screening the interaction between quarks become very strong at low energies so perturbative techniques are not going to be effective. So some approximations have to be made. It common to assume that the quarks have a mass, which is cannot be directly determined since they are never free. They are obtained from fits of the appropriate spectra of hadrons viewed as bound states of quarks. This is achieved by assuming a confining potential, which is attractive, with a strength that increases with the distance between the interacting quarks. The most popular are the linear, \( V(r) \propto r \), logarithmic \( V(r) \propto \ln(r/a) \) and quadratic \( V(r) \propto r^2 \). The latter is going to be discussed below (see next section).

On top of this one supposes an interaction between the quarks In this section we are going to derive the effective potential between quarks in the one gluon approximation in the non relativistic limit up to including terms of second order in the quark momenta, i.e of order of \( p^2/m_q^2 \). To this end we express the 4-spinor forms into matrix elements involving two component wave functions. Clearly this approximation will be applicable to heavy quarks or assuming a constituent masses, about a third of the nucleon mass, for light quarks.

5.1. One gluon exchange potential in a process involving only baryons

The non relativistic reduction of the one gluon exchange amplitude\(^6\) (see Fig. 2) leads to the effective 2-body operator:

\[
\tilde{V} = -\frac{1}{(2\pi)^3} \lambda_1 \lambda_2 \delta(\tilde{q}_1 + \tilde{q}_2) \frac{4\pi \alpha_s}{q_1^2} \left[ 1 - \frac{1}{(2m_q^2)} \left( \tilde{Q}_1 \cdot \tilde{Q}_1 + i\sigma_1 \cdot (\tilde{q}_1 \times \tilde{Q}_2) + i\sigma_2 \cdot (\tilde{q}_2 \times \tilde{Q}_1) \right) \right],
\]

where \( \lambda_1, \lambda_2 \) the SU(3) invariant, \( \tilde{q}_i = p_i - p_i \) and \( \tilde{Q}_i = p_i' + p_i, i = 1, 2 \). It is traditional to perform a Fourier transform and go to the coordinate space. Thus we get

\[
V = -\alpha_s \lambda_1 \lambda_2 \left[ \frac{1}{r} + \frac{1}{(2m_q^2)^2} \left( \frac{2}{3} V_S + \frac{1}{3} V_T + \frac{1}{r} V_{QQ} + \frac{1}{r^3} V_{qQ} \right) \right],
\]

with \( \alpha_s \) treated as a parameter to be fitted to the spectra. It is assumed to be of order 1, i.e. about five times larger than a typical value used in high energy physics
Furthermore
\[ V_S = 4\pi \delta(r)\sigma_1.\sigma_2, \]
\[ V_T = 3\sigma_1.\dot{r}.\sigma_2.\dot{r} - \sigma_1.\sigma_2, \]
\[ V_{QQ} = \nabla r_1(\leftarrow) \cdot \nabla r_2(\leftarrow) + \nabla r_1(\rightarrow) \cdot \nabla r_2(\rightarrow) - \nabla r_1(\rightarrow) \cdot \nabla r_2(\leftarrow). \]
\[ V_{qQ} = (\sigma_1 + \sigma_2). ((\ell(\rightarrow) - \ell(\leftarrow)) \]
\[ - \frac{1}{2} ((\sigma_2 - \sigma_1). (i \, r \times \nabla_R(\leftarrow) - (\sigma_2 - \sigma_1). (i \, r \times \nabla_R(\rightarrow))). \]

In the above expressions the arrows in parenthesis indicate the direction (bra or ket) on which the non local operator acts. As usual $\ell = \ell_2 - \ell_1$ is the relative orbital angular momentum, $r = r_2 - r_1$ and $R = \frac{1}{2}(r_2 + r_1)$. The relative orbital angular momentum gives no contribution since the relevant matrix element is diagonal (this part of the operator does not depend on the CM coordinates).

5.2. **One gluon exchange potential in processes involving the creation of a $q\bar{q}$ pair.**

In this case in the place of the diagrams of Fig. 5 one obtains a new diagram by replacing a $q - q$ line by a $q - \bar{q}$ pair as in Fig. 6. The non relativistic reduction of the one gluon exchange amplitude, resulting from this diagram, leads to the effective 2-body operator:

\[ \tilde{V} = - \frac{1}{(2\pi)^3} \lambda_1.\lambda_2.\delta(q_1 - Q_2) \frac{4\pi\alpha_s}{q_1^2} \left[ \frac{\sigma_2.Q_1}{2m_q} + i \frac{1}{2m_q} q_1.(\sigma_2 \times \sigma_1) \right]. \quad (9) \]
Where the subscripts 1 and 2 refer to the left and right legs of the diagram. Here \( q_1 = p_1 - p_1' \), \( Q_1 = p_1 + p_1' \), \( Q_2 = p_2 + p_2' \) or \( q_1 = Q_2 \).

Furthermore this diagram, combined with a number of "spectator" (non interacting) quarks, can lead to processes involving only color singlet states (see Fig. 7), as, e.g., meson decay into two mesons. Another approach is to consider the \( ^3P_0 \), which is derived from the quark string model, yielding a \( q-\bar{q} \) pair, "created out of the vacuum".

Fig. 7: The 1 gluon exchange potential acting between the quarks labeled \( q(p_1), q(p_1'), q(p_2) \), while the other quarks do not participate in the interaction (they are spectators) (a). The same effect can be accomplished by creating a \( q\bar{q} \) pair out of the vacuum, marked by \( \times \), (b), in which case the pair is normally in an \( S = 1, \ell = 1, J = 0 \) state (\( ^3P_0 \) model). The process exhibited describes the baryon meson coupling. If the initial baryon is heavier than the final, it provides the decay amplitude \( B_1 \rightarrow B_2 M \) (the lens like curves indicate a baryon or meson bound state). The same diagram is applicable in any process in which the interaction causes an \( 1q \rightarrow 2q - \bar{q} \) transition. For example, if the middle horizontal line is missing and the top arrow is reversed, it describes the decay of one meson into two mesons.

### 5.3. One gluon exchange potential in processes involving meson spectra

The non relativistic reduction of the one gluon exchange amplitude (see Fig. 8) in this case leads to the effective potential:

\[
\tilde{V} = -\frac{1}{(2\pi)^3} \lambda_1 \lambda_2 \delta(\tilde{Q}_1 - \tilde{Q}_2) \frac{4\pi \alpha_s}{Q_1^2} \\
\left\{ \sigma_1 \cdot \sigma_2 + \frac{1}{2} \left[ \sigma_2 \cdot (\tilde{q}_1 \times \tilde{Q}_1) + \sigma_1 \cdot (\tilde{q}_2 \times \tilde{Q}_1) \right] \right\} \\
+ \frac{1}{4} \left[ \tilde{Q}_1^2 - \tilde{q}_1^2 - \tilde{q}_2^2 + (\tilde{q}_1 - \tilde{q}_2) \cdot (\tilde{Q}_1 \cdot (\sigma_1 \times \sigma_2)) \right].
\]

### 6. Low energy formalism

One would like to make suitable approximations to microscopically study the usual baryons, e.g., the proton, neutron, \( \Delta \) resonances etc, and mesons, pions,
The one gluon effective potential relevant for meson spectra.

6.1. The orbital part at the quark level

Orbital wave functions in momentum space are expressed in terms of Jacobi coordinates:

$$\psi_{P_M} = \sqrt{2E_P} \left(2\sqrt{2}\right)^{1/2} \left(2\pi\right)^{3/2} \delta \left(\sqrt{2Q_M} - P_M\right) \phi_M(\rho),$$

(11)

$$\psi_{P_B} = \left(3\sqrt{3}\right)^{1/2} \left(2\pi\right)^{3/2} \delta \left(\sqrt{3}Q_B - P_B\right) \phi_B(\xi, \eta),$$

(12)

where $P_M$, and $P_B$ are the momenta of the meson and the baryon respectively and

$$\xi = \frac{1}{\sqrt{2}}(p_1 - p_2), \quad \eta = \frac{1}{\sqrt{6}}(p_1 + p_2 - 2p_3), \quad Q = \frac{1}{\sqrt{3}}(p_1 + p_2 + p_3),$$

(13)

$$\rho = \frac{1}{\sqrt{2}}(p_1 - p_2), \quad Q_M = \frac{1}{\sqrt{2}}(p_1 + p_2),$$

(14)

where $p_i$, $i = 1 \cdots 3$ are the momenta of the three quarks of the baryon, or $p_i$, $i = 1, 2$ are the momenta of the quark and anti-quark in the meson.

The above wave functions were normalized in the usual way:

$$\langle \psi_{P_B} | \psi_{P'_B} \rangle = (2\pi)^3 \delta(P_B - P'_B), \quad \langle \psi_{P_M} | \psi_{P'_M} \rangle = 2E_M (2\pi)^3 \delta(P_M - P'_M).$$

(15)
The harmonic oscillator wave functions describing the relative coordinates are well known. Thus, e.g. for $0\Sigma$ states take the form:

$$\phi_B(\xi, \eta) = \phi(\xi)\phi(\eta),$$  \hspace{1cm} (16)

$$\phi(\xi) = \phi(0)e^{-(b_B^2\xi^2)/2}, \phi(0) = \sqrt{\frac{b_B^3}{\pi\sqrt{\pi}}} \text{ etc},$$

$$\phi_M(\rho) = \phi_M(0)e^{-(b_M^2\rho^2)/2}, \phi_M(0) = \sqrt{\frac{b_M^3}{\pi\sqrt{\pi}}}.$$  \hspace{1cm} (17)

6.2. The kinetic energy part

The mass of the quarks is assumed to be the constituent quark mass, about a third of the mass of the nucleon for the light quarks $u$, $d$ and $s$. For the heavy quarks one extracts their masses from the mass of the corresponding mesons $J/\Psi$ and $Y$.

6.3. The confining potential

We will assume a confining potential of the form:

$$V_c = -\lambda_1\lambda_2 k(r_1 - r_2)^2,$$  \hspace{1cm} (18)

where $\lambda_1, \lambda_2$ is the two-body part of the $SU(3)$ Casimir operator, quadratic $SU(3)$ invariant $g(\lambda, \mu)$, given by

$$\lambda_1\lambda_2 = C(\lambda, \mu),$$

$$C(\lambda, \mu) = \frac{1}{2}(g(\lambda, \mu) - 2g(1, 0)), \quad g(\lambda, \mu) = \frac{2}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)).$$

The integers $\lambda$ and $\mu$ characterize the $SU(3)$ representation, e.g.,

$$(\lambda, \mu) = (1, 0), (0, 1), (2, 0), (1, 1)$$

for the fundamental $3$ (triplet), $3^*$ (anti-triplet), $6$ (sextet) and the $8$ (octet or adjoined) representations respectively. $k = \omega^2 m_q$ is a constant to be fitted by yielding the correct value of the mass of the nucleon for a given size parameter $a$ (see below).

6.4. Fitting the strength of the confining potential

The strength $k$ of the confining potential can be determined by considering the nucleon as a three quark system. One must first compute the mass of the baryons. To this end we consider:

- The kinetic energy:

$$\epsilon_q = \frac{1}{2m_q} (p_1^2 + p_2^2 + p_3^2), \quad m_q = \frac{m_p}{3}. \quad (19)$$
For harmonic oscillator wave functions in the internal variables we find in the rest frame of the baryon:

\[
\langle \psi_N | \epsilon_q | \psi_N \rangle = \langle \psi_N | \frac{1}{2m_q} \left( \frac{1}{2} (p_1 - p_2)^2 + \frac{1}{6} (p_1 + p_2 - 2p_3)^2 \right) | \psi_N \rangle = \frac{1}{2m_q} \left( \frac{3}{2} \frac{1}{b^2} + \frac{3}{2} \frac{1}{b^2} \right) = \frac{9}{2m_p} \frac{1}{b^2}.
\]

(20)

- The average confining potential.
  
  In this case we find \( ME = \frac{4}{3} \langle V_c \rangle \), with \( \langle V_c \rangle \) the expectation of the radial part of the potential. The negative value is due to the fact that only the color isotriplet pairs contribute (\( C_{(0,1)} = -\frac{4}{3} \)). It is straightforward to show that

\[
\langle V_c \rangle = \frac{3}{2} kb^2, \quad b = \text{the nucleon size} \Rightarrow ME = 2kb^2.
\]

(21)

We thus arrive at the equation:

\[
m_p = \frac{9}{2m_pb^2} + 2kb^2
\]

or

\[
ka^2 = \frac{m_p}{2} \left( -\frac{9}{2m_p b^2} + 1 \right).
\]

(23)

For \( a = 1 \text{fm} \) we find:

\[
kb^2 = 0.05m_p = 50 \text{ MeV} \rightarrow k = 50 \text{MeV/fm}^2.
\]

7. Matrix elements involving two quarks

We need not worry about the 1-body terms arising from the kinetic terms, since it is trivial to compute them for any hadron.

7.1. The confining potential

The confining potential is spin independent and does not change color. Furthermore the two quark matrix elements are independent of isospin. Thus:

\[
\langle (n_1l_1, n_2l_2) | [f_{cs}] (\lambda, \mu) S; JI | V_c | (n_1' l_1', n_2' l_2') | [f'_{cs}] (\lambda', \mu') S'; JI \rangle =
\]

\[
\delta_{L,L'} \delta_{S,S'} \delta_{\lambda,\lambda'} \delta_{\mu,\mu'} \delta_{[f_{cs}],[f'_{cs}]} (-) C_{\lambda,\mu'} \langle (n_1 l_1, n_2 l_2) | V_c | (n_1' l_1', n_2' l_2') \rangle,
\]

where \([f_{cs}] (\lambda, \mu) S\) is the two particle spin-color wave function with permutation symmetry \([f_{cs}]\), color \((\lambda, \mu)\) and spin \(S\). Similarly for \([f'_{cs}] (\lambda', \mu') S'\). \( J \) is the total angular momentum, \(\langle (n_1 l_1, n_2 l_2) | V_c | (n_1' l_1', n_2' l_2') \rangle\) is the radial integral involving the potential and

\[
C_{(\lambda, \mu)} = -\frac{4}{3} \text{ for } (\lambda, \mu) = (0, 1) \text{ (anti-triplet)}
\]
\[ C(\lambda, \mu) = \frac{2}{3} \text{ for } (\lambda, \mu) = (2, 0) \text{ (sextet)}. \]

We will see below that we have the following possibilities:

\[ [f_{cs}] = [2] \text{ (symmetric)} \rightarrow (\lambda, \mu) S = (2, 0)1, (0, 1)0 \]

\[ [f_{cs}] = [1^2] \text{ (antisymmetric)} \rightarrow (\lambda, \mu) S = (2, 0)0, (0, 1)1 \]

### 7.2. The one gluon exchange potential

Even though the above operator in the coordinate space is non local, one can still compute the needed matrix elements by taking gradients of the state vectors and employing the standard Racah algebra techniques. We find it more convenient, however, to work in momentum space. Furthermore most of the elementary interactions are derived in momentum space. So in many other applications one would like to have developed a formalism permitting exploitation of the advantages of momentum space.

We first introduce the dimensionless variables \( Q_i = \frac{1}{\sqrt{2}} \tilde{Q}_i b, q_i = \frac{1}{\sqrt{2}} \tilde{q}_i b, i = 1, 2 \), with \( b \) being the harmonic oscillator size parameters. Then the above operator associated with baryons only (see Eq. (7)) can be brought into a more convenient form:

\[
\tilde{V} = -\frac{1}{(2\pi)^3} \lambda_1 \lambda_2 \frac{1}{2\sqrt{2}} \delta(q_1 + q_2) \frac{4\pi\alpha_s}{2b} \frac{1}{q_1^2} \frac{1}{q_2^2} \left[ 1 - 2\kappa \left( Q_1 \cdot Q_1 + i\sigma_1 \cdot (q_1 \times Q_2) + i\sigma_2 \cdot (q_2 \times Q_1) - (\sigma_1 \times q_1) \cdot (\sigma_2 \times q_2) \right) \right],
\]

with \( \alpha_s = \frac{g^2}{4\pi} \) and \( \kappa = \frac{1}{2m_q g^2} \). In terms of tensor operators it is conveniently rewritten as follows:

\[
\tilde{V} = -\frac{1}{(2\pi)^3} \lambda_1 \lambda_2 \frac{1}{2\sqrt{2}} \delta(q_1 + q_2) \frac{4\pi\alpha_s}{2b} \frac{1}{q_1^2} \frac{1}{q_2^2} \left\{ 1 - 2\kappa \left( Q_1 \cdot Q_1 - \sqrt{2}\sigma_1 \cdot [(Q_2 \times q_1)]^1 - \sqrt{2}\sigma_2 \cdot [(Q_1 \times q_2)]^1 - \frac{2}{3} \sigma_1 \cdot \sigma_2 + \frac{1}{3} T(\sigma_1, \sigma_2, \hat{q}_1) q_2^2 \right) \right\},
\]

with \( T \) the tensor operator defined by

\[
T(\sigma_1, \sigma_2, \hat{q}) = 3\sigma_1 \cdot \hat{q} \sigma_2 \cdot \hat{q} - \sigma_1 \cdot \sigma_2 = \sqrt{\frac{6}{5}} \sqrt{4\pi Y^2(\hat{q})} |\sigma_1 \times \sigma_2|^2.
\]

In other words the operator in momentum space indeed takes a very simple form. The same procedure can be applied in the case of the operators of Eqs (9) and (10).

At this point we should mention that we find it convenient to evaluate the matrix elements of the two body strong interaction using the color-spin symmetry.
7.3. A simple interaction

For illustration purposes we will consider the simple spin color interaction

\[ V_{12} = \sigma_1 \cdot \sigma_2 \lambda_1 \cdot \lambda_2. \]  

(26)

In the case of two quarks the color spin state is of the form

\[ [2] = (2,0)s = 1, (0,1)s = 0, [2] = (2,0)s = 0, (0,1)s = 1, \]  

(27)

We know that

\[ \langle S| \sigma_1 \cdot \sigma_2 |S \rangle = 2S(S+1) - 3, \langle (\lambda, \mu)| \lambda_1 \cdot \lambda_2 |(\lambda, \mu) \rangle = \begin{cases} \frac{2}{3}, & (\lambda, \mu) = (2,0) \\ -\frac{1}{3}, & (\lambda, \mu) = (0,1) \end{cases}, \]  

(see section 4). Thus

\[ \langle [2]|V_{12}|[2] \rangle = \begin{cases} \frac{2}{3}, & (\lambda, \mu) = (2,0), s = 1 \\ -\frac{1}{3}, & (\lambda, \mu) = (0,1), s = 0 \end{cases}, \langle (1^2)|V_{12}|(1^2) \rangle = \begin{cases} -2, & (\lambda, \mu) = (2,0), s = 1 \\ -4, & (\lambda, \mu) = (0,1), s = 0 \end{cases} \]

In the case of a quark-antiquark the color spin state is of the form

\[ [2^1] = (1,1)s = 1, (1,1)s = 0, [16] = (0,0)s = 0, \]  

(28)

\[ \langle S|\sigma_1 \cdot \sigma_2 |S \rangle = 2S(S+1) - 3, \langle (\lambda, \mu)|\lambda_1 \cdot \lambda_2 |(\lambda, \mu) \rangle = \begin{cases} \frac{1}{3}, & (\lambda, \mu) = (1,1) \\ -\frac{2}{3}, & (\lambda, \mu) = (0,0) \end{cases}, \]  

Thus

\[ \langle [2^1]|V_{12}|[2^1] \rangle = \begin{cases} -1, & (\lambda, \mu) = (1,1), s = 0 \\ -8/3, & (\lambda, \mu) = (0,0), s = 1 \end{cases}, \langle (1^6)|V_{12}|(1^6) \rangle = 8 \]  

(29)

In the the spin color symmetry is redundant. The quantum numbers \((\lambda, \mu), S\) are adequate.

The above expressions must be multiplied with the matrix element of the radial part of the operator \(V(|r_1 - r_2|) = \alpha_s |r_1 - r_2|\). The radial functions can be obtained by solving Scröinger’s equation with a confining potential as discussed in section 6. In most applications only s-states are considered. In the present calculation we will employ harmonic oscillator and the MIT bag model \(^{1112}\) wave functions, in order to get analytic expressions for the Coulomb type interaction.

In the case of 0s harmonic oscillator wave function with size parameter \(b\) one finds that

\[ I_{ho} = \langle V(|r_1 - r_2|) \rangle = (\alpha_s \sqrt{\frac{2}{\pi}} \frac{1}{b}) \]  

(30)

In the case of the MIT bag model \(^{1112}\) the nucleon s- wave function is of the form:

\[ R(x) = \sqrt{\frac{2}{a - \sin a \cos a}} \sin \frac{x}{a} \sin a \cos \frac{x}{a}, x = kr, 0 \leq x \leq a \]  

(31)
where \( k \) is the nucleon momentum.

In this case the radial integral of the above potential is

\[
I = \frac{1}{(4\pi)^2} \int \int R(x_1) R(x_2) \frac{\alpha_s k}{|x_2 - x_1|} d^3x_1 d^3x_2
\]

\[
= \frac{2\alpha_s k}{a - \sin a \cos a} \int_0^a dx_1 \int_0^a dx_2 x_1 x_2 \sin x_1 \sin x_2 \left( \frac{1}{x_1} H(x_1 - x_2) + \frac{1}{x_2} H(x_2 - x_1) \right)
\]

where \( H(x) \) is the Heaviside step function. Thus:

\[
I_{bag} = k \alpha_s \left( a (\cos(2a) + 2) - 3 \sin(a) \cos(a) \right) \frac{a - \sin(a) \cos(a)}{a - \sin(a) \cos(a)}
\]

where \( R_c \) is the bag radius.

8. Symmetries in multi-quark systems

Multi-quark systems are Fermions containing more than three quarks and mesons containing more pairs than one quark one anti-quark. Such systems have been examined from the point of view of symmetries a long time ago, see, e.g.,\(^{13-15}\) See also\(^{16-18}\). The symmetries that were found useful were the combined spin color symmetry \( SU_{cs}(6) \) the orbital symmetry and the flavor symmetry. Such multi-quarks were later discovered experimentally, the penta-quark\(^{19,20}\) and the tetra-quark\(^{21,22}\). On the latter see also the recent reviews\(^{23,24,25,26}\).

In the LHCb experiment\(^{19}\) what is observed is the decay of a particle with the quantum numbers of \( \Lambda_b \) as follows:

\[
\Lambda_b \rightarrow J/\psi + p + K^-
\]

The data seem to be consistent with a pentaquark of the type \((udb)\bar{c}\bar{c}\), with the decay process at the quark level being \( b \rightarrow du\bar{u} \) or \( ub \rightarrow du \) weakly.

Motivated with this experimental evidence we will examine the general case of a pentaquark system, from the point of view of symmetries, in the expectation that structures may be found in the future.

In the case of multiquark systems one finds useful to employ the following symmetries:

i) The orbital symmetry. This is described in terms of a Young tableaux \([f]\).
ii) the isospin symmetry. In this case the corresponding Young tableaux \([f]_I\) has at most two rows,\([f]_I = [f_1, f_2]\), and the total isospin is \( I = \frac{1}{2}(f_1 - f_2) \). This symmetry can be handled by the usual angular momentum techniques
iii) the spin group \( SU_{cs}(6) \supset SU_s(2) \otimes SU_c(3) \)

9. Symmetries involved in the case of pentaquarks

In the case of pentaquarks the space part of the wave function is trivial, since it assumed to be made of 0s single particle states and, thus, it is completely symmetric.
For the group theoretical description of the pentaquarks we refer the reader to previous work.\cite{Henley}
Now since the orbital symmetry is trivial, the wave function becomes antisymmetrized if the Young tableaux \([f]\) is the conjugate of that of the isospin \([f]'\), i.e. obtained from that by exchanging rows and columns.

Regarding isospin we use
i) \(SU_f(2)\) for \(u\) and \(d\) quarks,
ii) \(U^f_n\) for \(n = s, c, b\) quarks.

### 9.1. Configurations involved in the case of pentaquarks

We will distinguish the following possibilities:

Configurations of the type:

- \(q^*q, q = u, d\). Then we will consider wave functions of good isospin \([I_1 \otimes I_2]'\), where \(I_2 = 1/2\) is the isospin of the antiquark and \(I_1\) the isospin of the 4-quark state. We have:
  
  \[I_1 = 0 \Leftrightarrow [2, 2] \text{symmetry}, I_1 = 1 \Leftrightarrow [3, 1] \text{symmetry}, I_1 = 2 \Leftrightarrow [4] \text{symmetry}\]

For completely antisymmetric wave functions the color-spin symmetries are obtained by interchanging the rows and columns of the Young diagrams, i.e.

\([2, 2], [2, 1^2], [1^4]\),

respectively. The associated color quantum number must be \((\lambda, \mu) = (1, 0)\) or \(\mathbf{3}\), so that combined with the color of the antiquark \(\mathbf{3}\) to yield a color singlet state \((0, 0)\). Thus, since the color-spin symmetry of the antiquark is redundant, the wave functions can be cast in the form

\[\begin{align*}
\{[2, 2](s_1 = 1(10)), I_1 = 0 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2\}^{I=1/2, s=1/2, 3/2}_{(0, 0)} & \quad (34) \\
\{[2, 1^2](s_1 = 0, (10))I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2\}^{I=1/2, s=1/2}_{(0, 0)} & \quad (35) \\
\{[2, 1^2](s_1 = 0, (10))I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2\}^{I=3/2, s=1/2}_{(0, 0)} & \quad (36) \\
\{[2, 1^2](s_1 = 1(10)), I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2\}^{I=1/2, s=1/2, 3/2}_{(0, 0)} & \quad (37) \\
\{[2, 1^2](s_1 = 1, (1, 0)), I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2\}^{I=3/2, s=1/2, 3/2}_{(0, 0)} & \quad (38) \\
\{[2, 1^2](s_1 = 2(1, 0)), I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2\}^{I=1/2, s=3/2, 5/2}_{(0, 0)} & \quad (39) \\
\{[2, 1^2](s_1 = 2, (1, 0)), I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2\}^{I=3/2, s=3/2, 5/2}_{(0, 0)} & \quad (40)
\end{align*}\]
\[ [1^4](s_1 = 1(1, 0)), I_1 = 2 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2 \]
\[ I = 3/2, s = 1/2, 3/2 \]  

\[ [1^4](s_1 = 1, (0, 0)), I_2 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 1/2 \]
\[ I = 5/2, s = 1/2, 3/2 \]  

We have 7 states of isospin 1/2, 7 states of isospin 3/2 and two states of isospin 5/2. Since the strong interactions conserve isospin, states with different isospin do not get admixed.

ii) \( q^4 \bar{q}_a \), \( q = u, d, \) \( q_a = s, c, b, t \). Then we proceed as above noting that the antiquark is an isospin singlet, \( I = I_1 \) with \( I_1 \) the isospin of the 4-quark state. We have:

\[ [2, 2](s_1 = 1(0)), I_1 = 0 \times (s_2 = 1/2, (0, 1)), I_2 = 0 \]
\[ I = 0, s = 1/2, 3/2 \]  

\[ [2, 1^2](s_1 = 0, (10)), I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 0 \]
\[ I = 1, s = 1/2 \]  

\[ [2, 1^2](s_1 = 1(1, 0)), I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 0 \]
\[ I = 1, s = 1/2, 3/2 \]  

\[ [2, 1^2](s_1 = 2(1, 0)), I_1 = 1 \times (s_2 = 1/2, (0, 1)), I_2 = 0 \]
\[ I = 1, s = 3/2, 5/2 \]  

\[ [1^4](s_1 = 1(1, 0)), I_1 = 2 \times (s_2 = 1/2, (0, 1)), I_2 = 0 \]
\[ I = 3/2, s = 1/2, 3/2 \]  

Since the strong interactions conserve isospin, these states do not admix with those of case i). There exist \( 2 \times n \) isospin 0 and 2 states and \( 5 \times n \) isospin 1 states, where \( n \) is the number of the different antiquarks considered. Again states of different isospin do not get admixed.

iii) \( q^4 \bar{q}_a, q = u, d, \) \( q_a = s, c, b, t \). Then we proceed as above noting that in the case of \( a \) and \( b \) flavors the isospin of each is zero and their combined isospin is zero. Thus the possible isospins are those contained in the \( q^4 \) part, i.e. \( 3/2 \leftrightarrow [3] \) and \( 1/2 \leftrightarrow [21] \), i.e. \( f_{cs}[1^3] \) and \( f_{cs}[21] \) respectively. The spin-color possibilities of the \( q_b \bar{q}_a \) combination are: \( (\lambda, \mu)s = (1, 1)1, (1, 1)0, (0, 0)1 \) and \( (0, 0)0 \). So among the components of [2, 1] relevant for yielding a color singlet state are \((11)1/2, (11)3/2, (0, 0)1/2, (0, 0)3/2, (0, 0)1/2, (0, 0)3/2\). The relevant components of the \([1^3]\) are \((0, 0)1/2, (0, 0)3/2\). The spin-color symmetries of the \( q_b \bar{q}_a \) are \([2, 1^4]\) and \([1^6]\), but they are redundant. In any case one can easily find that the \([2, 1^4]\) contains \((\lambda, \mu)s = (1, 1)1, (1, 1)0, (0, 0)1\). while the \([1^6]\) contains only the \((0, 0)0\). Thus we can have:

\[ [2, 1](s_1 = 1/2, (1, 1)), I_1 = 1/2 \times [2, 1^4](s_{a,b} = 0, (1, 1)), I_{ab} = 0 \]
\[ I = 1/2, s = 1/2 \]  

\[ [2, 1](s_1 = 1/2, (1, 1)), I_1 = 1/2 \times [2, 1^4](s_{a,b} = 1, (1, 1)), I_{ab} = 0 \]
\[ I = 1/2, s = 1/2, 3/2 \]  

\[ [2, 1](s_1 = 3/2, (1, 1)), I_1 = 1/2 \times [2, 1^4](s_{a,b} = 0, (1, 1)), I_{ab} = 0 \]
\[ I = 1/2, s = 3/2 \]
\([2, 1](s_1 = 3/2, (1, 1)), I_1 = 1/2 \times [2, 1^4](s_{a,b} = 1, (1, 1)), I_{ab} = 0^{I=1/2,s=1/2,3/2,5/2}
\]

\([2, 1](s_1 = 1/2, (0, 0)), I_1 = 1/2 \times [2, 1^4](s_{a,b} = 1, (0, 0)), I_{ab} = 0^{I=1/2,s=1/2,3/2}
\]

\([2, 1](s_1 = 1/2, (0, 0)), I_1 = 1/2 \times [1^6](s_{a,b} = 0, (0, 0)), I_{ab} = 0^{I=1/2,s=1/2}
\]

\([1^3](s_1 = 1/2, (1, 1)), I_1 = 3/2 \times [2, 1^4](s_{a,b} = 1, (1, 1)), I_{ab} = 0^{I=3/2,s=1/2,3/2}
\]

\([1^3](s_1 = 3/2, (0, 0)), I_1 = 3/2 \times [1^6](s_{a,b} = 0, (1, 1)), I_{ab} = 0^{I=3/2,s=3/2}
\]

These states can, in principle, admix with the states of class i) with the same isospin.

iv) \(q^2_{c}: q_a q_b, q_c, q_a, q_b = s, c, b, t\). The classification of the states \(q_a q_b\) is exactly as in the previous case. The total isospin is 0 or 1. The allowed \(q^2_{c}\) combinations are

\[q^2_{c}(\lambda_1, \mu_1) I_1 \otimes (1, 0)^{1/2}
\]

where \((\lambda, \mu) = (1, 1), (0, 0), s' \otimes \frac{1}{2} \rightarrow s_1\) and

\[I = 0 \Rightarrow (\lambda_1, \mu_1, s'_1) = (2, 0)1, (0, 1)0, I = 1 \Rightarrow (\lambda_1, \mu_1, s'_1) = (2, 0)0, (0, 1)1
\]

The two sets of states are combined as in the previous case iii).

The remaining flavor combinations are not of interest from the point of symmetries and they are not going to be considered.

At this point we should mention that one can evaluate the matrix elements of the two body strong interaction knowing the one and two particle coefficients of fractional parentage (CFP, see below). For the isospin part these are trivial, i.e. the usual Clebsch-Gordan coefficients \(I_1 \times I_2 \rightarrow I\), indicated symbolically as \(C^I_{I_1, I_2}\). For the color spin the 1-particle CFP’s can be found in the work of So and Strottman.27 Those of interest to the pentaquark systems are given in tables 1-2.

From these one can obtain the 2-particle CFP’s in the usual way.28 Those of interest for the pentaquarks are listed in table 3. The 2-particle CFP’s for three quark states coincide with the corresponding one particle CFP’s.

The CFPs are very useful in constructing the many body Hamiltonian matrix. For this one needs in addition the matrix elements of the interaction between two particle states, e.g. those discussed in section 5.

### 9.2. Evaluation of the many body matrix elements of the 2-body interaction

These can be obtained by using the CFPs calculated above for the pentaquark states. The results are given as follows:
Table 1: The one particle CFP's involving the spin color symmetry, needed in the case of a 4-quark state, with color symmetry (10) and spin 1.

| $|f_1\rangle|\lambda_1,\mu_1\rangle S_1$ | $|f\rangle|\lambda,\mu\rangle S$ | CFP |
|--------------------------------------|---------------------------------|-----|
| $[2, 1](1, 1)^\frac{1}{2}$           | $[2, 1^2](1, 0)\frac{1}{2}$     | $\frac{\sqrt{2}}{3}$ |
| $[2, 1](1, 1)^\frac{1}{2}$           | $[2, 1^2](0, 0)\frac{1}{2}$     | $\frac{1}{\sqrt{6}}$ |
| $[2, 1](0, 0)\frac{1}{2}$            | $[2, 1^2](1, 0)\frac{1}{2}$     | $\frac{1}{\sqrt{6}}$ |

Table 2: The one particle CFP's involving the spin color symmetry, needed in the case of a 3-quark state, with color symmetry (11) and (0,0) with spin 1/2.

| $|f_1\rangle|\lambda_1,\mu_1\rangle S_1$ | $|f\rangle|\lambda,\mu\rangle S$ | CFP |
|--------------------------------------|---------------------------------|-----|
| $[1^3](1, 1)^\frac{1}{2}$            | $[1^3](10)\frac{1}{2}$         | $\frac{\sqrt{2}}{3}$ |
| $[1^3](0, 0)^\frac{1}{2}$            | $[1^3](10)\frac{1}{2}$         | $\frac{1}{\sqrt{3}}$ |

Case i):

$$
\langle q^4\bar{q}\phi^{f_{sc}}\rangle_{1}(0, 1)S_1, I_1, S, I|V|q^4\bar{q}\phi^{f_{sc}}\rangle_{1}(0, 1)S_1', I_1', S, I =
\delta_{I_1, I_1'}\delta_{S_1, S_1'}\langle q^4\phi^{f_{sc}}\rangle_{1}(0, 1)S_1, I_1|V|q^4\phi^{f_{sc}}\rangle_{1}(0, 1)S_1', I_1'
$$

$$+4\sum_{x, y}\frac{n_{|f_1\rangle}}{\sqrt{n_{|f\rangle}n_{|f_{sc}\rangle}}}(|f_{x}(\lambda_x, \mu_x)S_x|f_{sc}\rangle_{1}(0, 1)S_1)(|f_{x}(\lambda_x, \mu_x)S_x|f_{sc}\rangle_{1}(0, 1)S_1')$$

$$U(I_x, \frac{1}{2}, \frac{1}{2}, I_y, I_y)U(I_x, \frac{1}{2}, \frac{1}{2}, I_y, I_y)U(S_x, \frac{1}{2}, S_y, S_y)U(S_x, \frac{1}{2}, S_y, S_y)$$

$$C_{I_x, I_y, S_x}^{I_{x}', I_{y}', S_y}^{I, I_1, I_1'}(q\bar{q}I_y(\mu_x, \lambda_x)S_y|V|q\bar{q}I_y(\mu_x, \lambda_x)S_y)$$

Where $n_{|f_1\rangle}, n_{|f\rangle}, n_{|f_{sc}\rangle}$ are the dimensions of the corresponding representations of the symmetric group $S_4$, $(\mu_x, \lambda_x) = (1, 1), (0, 0)$, $U=$the usual unitary Racah
Table 3: The two particle CFP’s involving the spin color symmetry for 4 identical quarks, needed in the study of pentaquarks, corresponding to color state (0,1) and spin one.

| $|f_1\rangle (\lambda_1, \mu_1) S_1$ | $|f_2\rangle (\lambda_2, \mu_2) S_2$ | $|f\rangle (\lambda, \mu) S$ | CFP | $|f_1\rangle (\lambda_1, \mu_1) S_1$ | $|f_2\rangle (\lambda_2, \mu_2) S_2$ | $|f\rangle (\lambda, \mu) S$ | CFP |
|---|---|---|---|---|---|---|---|
| $[2](0,1)0$ | $[2](2,0)1$ | $[2^2](10)1$ | $\frac{1}{\sqrt{2}}$ | $[2](0,1)0$ | $[2](2,0)1$ | $[2, 1^2](10)1$ | 0 |
| $[2](2,0)1$ | $[2](0,1)0$ | $[2^2](10)1$ | $\frac{1}{\sqrt{2}}$ | $[2](2,0)1$ | $[2](0,1)0$ | $[2, 1^2](10)1$ | 0 |

functions, $\langle [f_x](\lambda_x, \mu_x) S_x | [f^{sc}]_1(0,1) S_1 \rangle$ etc are the one particle CFPs for 4 particles and the last factor is the elementary two particle interaction. Furthermore

$$\langle q^4 [f^{sc}]_1(0,1) S_1, I_1 | V | q^2 [f^{sc}]_1(0,1) S_1, I_1 \rangle = \frac{1}{2} 4 \times 3$$

and

$$\sum_{x,y} \frac{n_{[f_x]}}{\sqrt{n_{[f^{sc}]}} \sqrt{n_{[f^{sc}]}}} \langle [f_x]_2(\lambda_x, \mu_x) S_x | f_y_2(\lambda_y, \mu_y) S_y | [f^{sc}]_1(0,1) S_1 \rangle$$

$$C_{I_1, I_y}^T C_{I_x, I_y}^{T'} \langle [f_x]_2(\lambda_x, \mu_x) S_x | f_y_2(\lambda_y, \mu_y) S_y | [f^{sc}]_1(0,1) S_1 \rangle$$

where $[f_x] = [2]$ or $[1^2]$ and $[f_y] = [2]$ or $[1^2]$ as selected by the two particle CFPs.

Case ii):

$$\langle q^4 \bar{q}_\alpha [f^{sc}]_1(0,1) S_1, S, I | V | q^4 \bar{q}_\beta [f^{sc}]_1(0,1) S'_1, S, I \rangle =$$

$$\delta_{\bar{q}_\alpha, \bar{q}_\beta} \delta_{S_1, S'_1}$$

and

$$\langle q^4 [f^{sc}]_1(0,1) S_1, I | V | q^2 [f^{sc}]_1(0,1) S_1, I \rangle$$

$$+ 4 \sum_{x,y} \frac{n_{[f_x]}}{\sqrt{n_{[f^{sc}]}} \sqrt{n_{[f^{sc}]}}} \langle [f_x](\lambda_x, \mu_x) S_x | [f^{sc}]_1(0,1) S_1 \rangle \langle [f_x](\lambda_x, \mu_x) S_x | [f^{sc}]_1(0,1) S'_1 \rangle$$

$$C_{I, \frac{1}{2}}^T C_{I, \frac{1}{2}}^{T'} U(S_x, \frac{1}{2} S_{\frac{1}{2}}, S_1, S_y) U(S_x, \frac{1}{2} S_{\frac{1}{2}}, S'_1, S_y)$$

$$\langle q \bar{q}_\alpha (\mu_x \lambda_x) S_y | V | q \bar{q}_\beta (\mu_x \lambda_x) S_y \rangle$$
\[ \langle q^4[f^*]_1(0,1)S_1, I|V|q^4[f^*]_1(0,1)S'_1, I \rangle \] is given as in the previous case with the obvious substitutions.

Next we get case iii):

\[ \langle q^4[q]\langle f^*\rangle_1(0,1)S_1, I, S, I|V|q^4[q\beta][f^*]_1(0,1)S'_1, S, I \rangle = 0 \]

Case iv):

\[ \langle q^4[q]\langle f^*\rangle_1(0,1)S_1, I, S, I|V|q^4[q\beta][f^*]_1(0,1)S'_1, I, q_\alpha q_\beta(\lambda, \lambda)S'_2, S, I \rangle = \]

\[ +4 \sum_{x,y} \frac{n_{[f^*]}}{\sqrt{n_{[f^*]}n_{[f^*]}}} \langle [f]_x(\lambda, \lambda)S'_1|[f^*]_1(0,1)S_1 \rangle U(S'_1, \frac{1}{2}S_1, \frac{1}{2}, S_1, S_2) \]

\[ C_{I, I'}^{f_1} C_{I, I'}^{f_2}(qq\lambda, \lambda)S'_2|V|q_\alpha q_\beta(\lambda, \lambda)S'_2 \]

Case v):

\[ q^3[f^*]_1(\lambda, \lambda)S_1 I, q_\alpha q_\beta(\lambda, \lambda)S_2, S, I|V|q^3[f^*]_1(\lambda', \lambda')S'_1 I, q_\alpha' q_\beta(\lambda, \lambda)S'_2, S, I \rangle = V_1 \delta_{\beta, \beta'} + V_2 \delta_{\alpha, \alpha'} \]

\[ \delta_{S_1, S'_1} \delta_{S_2, S'_2} \delta_{\lambda, \lambda'} \left( \langle [f^*]_1, [f^*]_1 \rangle V_1(\lambda, \lambda)S_2 \rangle + \langle q^3[f^*]_1(\lambda, \lambda)S_1|V|q^3[f^*]_1(\lambda, \lambda)S_1 \rangle \right) \]

with

\[ V_1 = \sum_x \langle q^3[f^*]_1(\lambda, \lambda)S_1 I, q_\alpha(1,0)\frac{1}{2}S_2; (10)\rangle \langle V|q^3[f^*]_1(\lambda, \lambda)S'_1 I, q_\alpha(1,0)\frac{1}{2}S'_2; (10) \rangle \]

\[ U(S_1, \frac{1}{2}S_1, S_2)U(S'_1, \frac{1}{2}S'_1, S'_2) = \]

\[ 3 \sum_{x,y} \frac{1}{\sqrt{n_{[f^*]}n_{[f^*]}}} U(S_1, \frac{1}{2}S_1, S_2)U(S'_1, \frac{1}{2}S'_1, S'_2)U(S_1, \frac{1}{2}S_1, S_1, S_y) \]

\[ U(S_2, \frac{1}{2}S_2, S'_1, S_y)U((\lambda, \mu)(1,0)(1,0)(1,0)(1,0)(\lambda, \lambda, \mu)) U(I_z, \frac{1}{2}, I, 0, I, \frac{1}{2})^2 \]

\[ \langle [f]_2(\lambda, \mu)S_2|[f^*]_1(\lambda, \lambda)S_1 \rangle \langle [f]_2(\lambda, \mu)S_2|[f^*]_1(\lambda, \lambda)S'_1 \rangle \]

\[ C_{I, I'}^{f_1} C_{I, I'}^{f_2}(qq_\lambda(\lambda, \lambda)S_y)|V|qq_\lambda(\lambda, \lambda)S_y \]

The isospin \(U\)-function is unity, but it was introduced to indicate the angular momentum constraint on \(I_z\), which in turn constrains the two particle spin isospin symmetry in the CFP. Clearly \((\lambda_y, \mu_y) = (2, 0), (0, 1)\)

\[ V_2 = \sum_x \langle q^3[f^*]_1(\lambda, \lambda)S_1 I, \tilde{q}_\beta(0,1)\frac{1}{2}S_2; (10)\rangle \langle V|q^3[f^*]_1(\lambda, \lambda)S'_1 I, \tilde{q}_\beta(0,1)\frac{1}{2}S'_2; (10) \rangle \]
\[ U(S_1, \frac{1}{2}S_1, S_x, S_2)U(S_1', \frac{1}{2}S_1, S_x', S_2') = \]

\[ 3 \sum_{x,z,y} \frac{1}{\sqrt{|f_{x}^{'}| H |f_{x}^{'}|}} U(S_1, \frac{1}{2}S_1, S_x, S_2)U(S_1', \frac{1}{2}S_1, S_x', S_2') U(S_z, \frac{1}{2}S_z, S_1, S_y) \]

\[ U(S_z, \frac{1}{2}S_z, S_1, S_y)U((\lambda_z, \mu_z)(1,0)(0,1)(0,1)(\lambda, \lambda)(\lambda_y, \mu_y))^{2} U(I_z, \frac{1}{2}I, 0, I, \frac{1}{2})^2 \]

\[ \langle f_{1}^{'}|\langle \lambda_z, \mu_z \rangle S_z|f^{sc}\rangle_1(\lambda, \lambda)S_1 \langle f_{1}^{'}|\langle \lambda_z, \mu_z \rangle S_z|f^{sc}\rangle_1(\lambda, \lambda)S_1' \rangle \]

\[ C_{I_1}^{I_1} \frac{q \bar{q}_{\beta}(\lambda_y, \mu_y)S_y}{V|q \bar{q}_{\beta}(\lambda_y, \mu_y)S_y\rangle} \]

Now \((\lambda_y, \mu_y) = (1, 1), (0, 0)\)

Finally in the simple case of three identical spin 1/2 quarks we have the following simple cases:

i) \(I = 1/2\)

\[ \langle [2, 1](1, 1) \frac{1}{2}|V|[2, 1](1, 1) \frac{1}{2} \rangle = 3 \times \frac{1}{2} \times \frac{1}{2} \]

\[ \left( \langle (2, 0)|V|(2, 0)1 \rangle + \langle (2, 0)|0V|(2, 0)0 \rangle + \frac{1}{3} \langle (0, 1)|V|(0, 1)1 \rangle + \langle (0, 1)|0V|(0, 1)0 \rangle \right) \]

(3 is the number of pairs one can make out of 3 particles, the first 1/2 is the ratio of the dimensions of the symmetry group involved, the last 1/2 is the square of the CFP, while 1/3 is the relevant isospin CFP). Similarly

\[ \langle [2, 1](1, 1) \frac{3}{2}|V|[2, 1](1, 1) \frac{3}{2} \rangle = 3 \times \frac{1}{2} \times 1 \left( \langle (2, 0)|V|(2, 0)1 \rangle + \frac{1}{3} \langle (0, 1)|V|(0, 1)1 \rangle \right) \]

\[ \langle [2, 1](0, 0) \frac{1}{2}|V|[2, 1](0, 0) \frac{1}{2} \rangle = 3 \times \frac{1}{2} \times 1 \left( \frac{1}{3} \langle (0, 1)|V|(0, 1)1 \rangle + (0, 1)|0V|(0, 1)0 \rangle \right) \]

i) \(I = 3/2\)

\[ \langle [3](1, 1) \frac{1}{2}|V|[3](1, 1) \frac{1}{2} \rangle = 3 \times 1 \times \frac{1}{2} \left( \langle (2, 0)|0V|(2, 0)0 \rangle + \frac{1}{3} \langle (0, 1)|V|(0, 1)1 \rangle \right) \]

\[ \langle [3](0, 0) \frac{3}{2}|V|[3](0, 0) \frac{3}{2} \rangle = 3 \times 1 \times \frac{1}{3} \langle (0, 1)|V|(0, 1)1 \rangle \]
### 9.3. A pentaquark with the quantum numbers of the proton

We will consider a pentaquark with spin $1/2$ and isospin $1/2$. We will demand that there will be at least three quarks of the type $q$ with isospin $I=1/2$. Then we have the following possibilities:

Two states of the form

$$q^4[2^2](1,0)S_1 = 1, I_1 = 0 \times \bar{q} \quad \text{and} \quad q^4[2,1^2](1,0)S_1 = 1, I_1 = 1 \times \bar{q}$$

and five states of the form:

$$q^3[2,1](\lambda, \lambda)S_1 = 1, I_1 = \frac{1}{2} \times (q_\alpha \bar{q}_\alpha(\lambda, \lambda), S_2), \alpha = s, c, b, t$$

where $(\lambda, \lambda)$ are the self conjugate color functions $(1,1)$, octet, and $(0,0)$, singlet.

The order of the states is

$$(1,1)S_1 = \frac{1}{2}, S_2 = 0, (1,1)S_1 = \frac{1}{2}, S_2 = 1, (1,1)S_1 = \frac{3}{2}, S_2 = 1,$$

$$(0,0)S_1 = \frac{1}{2}, S_2 = 0, (0,0)S_1 = \frac{1}{2}, S_2 = 1$$

states of the form $q^4[f^{exc}] \times \bar{q}_\alpha$ cannot yield an $I = 1/2$ state and states of the form:

$$q^3[2,1](\lambda, \lambda)S_1 = 1, I_1 = \frac{1}{2} \times (q_\alpha \bar{q}_\beta(\lambda, \lambda), S_2), \alpha, \beta = s, c, b, t$$

essentially do not require any new symmetry input and they will not be considered in this discussion. Also states of the form

$$[q^2[f^{exc}](\lambda_x, \mu_x)S_x, I_x \times q_\beta]^{(\lambda, \lambda), S_1, I_1 \times (q_\alpha \bar{q}_\alpha(\lambda, \lambda), S_2)},$$

like the pentaquark actually discovered, involve only more complicated combinatorics in the evaluation of the relevant matrix elements and do not require any additional CFP’s. So these are not going to be included in our discussion, except to say that it will be interesting to see whether the charmonium component corresponds to one of the eigenstates of the type $B$ matrix discussed below. We should also mention that other types of symmetries maybe involved in evaluation the energies resulting from realistic wave functions, obtained, e.g., with a linear potential, can be computed as perturbations of the Coulomb potential using the non compact SO[2,1] algebra, using standard techniques.

Anyway the above Hamiltonian matrix takes the form:

$$\mathcal{H} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$$

with $A$ $(2 \times 2)$ matrix, $C$ $(2 \times 5)$ matrix, $B$ $(5 \times 5)$ matrix

In evaluating the matrix $A$ we need the following combined spin-color and isospin CFP’s

$$[2^2](1,0)1 : \frac{\sqrt{2}}{\sqrt{2}}, -\frac{\sqrt{2}}{\sqrt{2}}, \frac{\sqrt{6}}{\sqrt{6}}, \frac{\sqrt{6}}{\sqrt{6}}, \frac{\sqrt{6}}{\sqrt{6}}, \frac{\sqrt{3}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{3}}, -\frac{\sqrt{3}}{\sqrt{3}}$$

$$[2,1^2](1,0)1 : 0, 0, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{3}}, \frac{\sqrt{3}}{\sqrt{3}}, -\frac{\sqrt{3}}{\sqrt{3}}$$
with ratios of the dimensions of the symmetric group $\frac{1}{2}$ and $\frac{1}{3}$ respectively and 6 particle combinations allowed. In the order of the pairs with spin color combinations:

$(0, 1), (2, 0), (2, 0), (0, 1), (0, 1), (2, 0), (0, 1), (2, 0), (0, 1), (2, 1), (0, 1), (0, 1)$

We also need the spin-color and isospin 1-particle CFP's:

$[2^2](1, 0)1 : \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{2}}$

$[2, 1^2](1, 0)1 : \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}} \times \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}} \times \frac{1}{\sqrt{2}}$

in the order of $(1, 1), (1, 1), (0, 0), (0, 0)$ states of spin-color symmetry $[2, 1]$. With this information and using the formulas of case i) section 9.2 we can construct the matrix $A$. We can also construct the matrix $C$, using the formulas of case iii).

We come next to the evaluation of matrix $B$, using the formulas of case v) of section 9.2. We now need the following spin-color and isospin 1-particle CFP's:

$[2, 1](1, 1), (1, 0)1 : \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

$[2, 1](1, 1), (1, 1)3 : 1, 0, 0, -1 \times \frac{1}{\sqrt{3}}$

These matrices are expressed in units of the relevant radial integrals, e.g. $I_{HO}$ or $I_{bag}$ (see Eqs 30 and 32).

To proceed further one must also add to the Hamiltonian the 1-body terms arising from the quark masses. So in the case of heavy quarks the mixing is small. So we will not proceed further in this direction. So we will draw some conclusions based on the form of $A$ and $B$. In the first place we see that in the case of $A$ there is very little mixing. The lowest state is almost pure $[2, 1^2](1, 0)1$ with the $[2^2](1, 0)1$ being higher at 5.5 times the radial integral.

Table 4: The one particle CFP’s involving the spin color symmetry, needed in the case of a 4-quark state, with color symmetry (10) and spin 1.

$$A = \begin{pmatrix}
\frac{8}{3} & 2 \\
\frac{2}{3\sqrt{3}} & \frac{2}{\sqrt{3}}
\end{pmatrix},
C = \begin{pmatrix}
\frac{\sqrt{3}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} & \frac{8\sqrt{2}}{3\sqrt{3}} \\
-\frac{2}{\sqrt{3}} & \frac{2}{3\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{8}{3\sqrt{3}}
\end{pmatrix}
$$

$$B = \begin{pmatrix}
-\frac{214}{81} & -\frac{14}{81\sqrt{3}} & \frac{14}{81\sqrt{3}} & 0 & 0 \\
-\frac{14}{81\sqrt{3}} & -\frac{290}{243} & \frac{290}{243} & 0 & 0 \\
-\frac{14}{81\sqrt{3}} & \frac{243}{243} & -\frac{243}{243} & \frac{56}{81\sqrt{3}} & \frac{56}{81\sqrt{3}} \\
0 & 0 & -\frac{56}{81\sqrt{3}} & -\frac{8}{81\sqrt{3}} & -\frac{8}{81\sqrt{3}} \\
0 & 0 & -\frac{56}{81\sqrt{3}} & -\frac{8}{81\sqrt{3}} & -\frac{8}{81\sqrt{3}}
\end{pmatrix}
$$

$I_{bag}$ (see Eqs 30 and 32).
Regarding the spectrum of $B$ we find the eigenvalues
\[-5.31, -3.07, -2.64, -1.19, -0.46\]
in units of the radial integrals. The lowest state is predominantly $|q^3(2,1)(1,1)S_1 = \frac{3}{2} \times (11)S_2 = 1\rangle$

10. symmetries of six-quark clusters

To further illustrate the role of the symmetries involved in multi-quark structures, we will discuss here in some detail the case of the six-quark clusters, which can be present in the nucleus. We will restrict ourselves in the case that the six quark cluster has the same quantum numbers with those of the two nucleon system, i.e. being colorless, $(\lambda, \mu) = (0, 0)$, and with isospin 0 or 1.

In describing the six quark clusters we will use standard group theory. This involves:

1. The isospin structure.
   Here both the initial and the final cluster have isospin $I = 1$, with $I_3 = -1$ (initial state with the quantum numbers of two neutrons) and $I_3 = 1$ (final state with the quantum numbers of two protons). The corresponding $SU(2)$ symmetry is described by a Young Tableaux, which has at most 2 rows $\tilde{f} = [42], [33]$ corresponding to isospin $I = 1, 0$ respectively.

2. The orbital spin color structure.
   In order to get a totally antisymmetric wave function the overall symmetry must be the conjugate of the above, namely $f = [2^2, 1^2]$. This symmetry can be described by the product of an orbital symmetry $f_L$ associated with $SU(6)$ and a color spin symmetry $f_{cs}$ also associated with $SU(6)$.

3. Possible $f_{cs}$ structures.
   Clearly $f_{cs}$ can be further analyzed in terms of the symmetries $SU_s(2)$ for the spin and $SU_c(3)$ for the color group. Only the trivial representations of the latter symmetry, namely the $s = 0, 1$ and the color singlet $(0,0)$ of $SU_c(3)$ are relevant.

4. The possible $f_L$ structures.
   We will consider the quarks moving in a confining potential. We will choose a basis set obtained in a harmonic oscillator potential considering positive parity states with $E \leq 2\hbar \omega$ excitations. We thus encounter:
   - $0s^6$ configurations. The allowed symmetry is $f_L = [6]$ with $L = 0$
   - $0s^51s$ configurations. The allowed symmetries are $f_L = [6]$ and $[5,1]$ with $L = 0$.
   - $0s^2, 0p^2$ configurations. The allowed symmetries are $f_L = [6,0], [5,1]$ and $[4,2]$ with $L = 0, 2$.

Using the standard group theory techniques we find the following color
singlets:

\[ f_L = [6], \quad f_{cs} = [2^2, 1^2] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 0, 2, I = 1 \]

\[ f_L = [6], \quad f_{cs} = [2^3] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1, 3, I = 0 \]

\[ f_L = [51], \quad f_{cs} = [321] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1, 2 \]

It is not relevant since it has more than two columns.

\[ f_L = [51], \quad f_{cs} = [2^3] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1, 3, I = 0 \]

\[ f_L = [51], \quad f_{cs} = [2^21^2] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 0, 2, I = 1 \]

\[ f_L = [51], \quad f_{cs} = [31^3] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1 \]

It is not relevant since it has more than two columns.

\[ f_L = [51], \quad f_{cs} = [21^4] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1 \]

It is not relevant since it is characterized by isospin 1=2.

\[ f_L = [42], \quad f_{cs} = [321] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1, 2 \]

It is not relevant since it has more than two columns.

\[ f_L = [42], \quad f_{cs} = [41^2] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 0 \]

It is not relevant since it has more than two columns.

\[ f_L = [42], \quad f_{cs} = [2^3] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1, 3, I = 0 \]

\[ f_L = [42], \quad f_{cs} = [31^3] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1 \]

It is not relevant since it has more than two columns.

\[ f_L = [42], \quad f_{cs} = [2^21^2] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 0, 2, I = 1 \]

\[ f_L = [42], \quad f_{cs} = [21^4] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 1, I = 2 \]

It is not relevant since it does not have an isospin quantum number corresponding to a two nucleon system.

\[ f_L = [42], \quad f_{cs} = [33] \quad \text{with} \quad (\lambda, \mu) = (0, 0) \quad \text{and} \quad s = 0 \]

It is not relevant since it has more than two columns.
Table 5: The one particle CFP's involving the spin color symmetry, needed in the case of a 6-quark cluster, colorless with spins 0, 1, 2 and 3.

| $[f_1(\lambda_1, \mu_1)S_1]$ | $[f(\lambda, \mu)S]$ | CFP |
|-----------------------------|----------------------|-----|
| $[2^2, 1](0, 1)\frac{1}{2}$ | $[2^2](00)\frac{1}{2}$ | $\frac{\sqrt{2}}{4}$ |
| $[2^2, 1](0, 1)\frac{1}{2}$ | $[2^2](00)\frac{1}{2}$ | $\frac{1}{4}$ |

Table 6: The two particle CFP's involving the spin color symmetry, needed in the case of a 6-quark cluster, colorless with spin zero and 2.

| $[f_1(\lambda_1, \mu_1)S_1]$ | $[f_2(\lambda_2, \mu_2)S_2]$ | $[f(\lambda, \mu)S]$ | CFP |
|-----------------------------|-----------------------------|----------------------|-----|
| $[2, 2](0, 2)0$ | $[1^2](2, 0)0$ | $[2^12^1](00)0$ | $\frac{{\sqrt{3}}}{3}$ |
| $[2, 2](0, 2)0$ | $[1^2](2, 0)0$ | $[2^12^1](00)0$ | $\frac{1}{3}$ |

| $[f_1(\lambda_1, \mu_1)S_1]$ | $[f_2(\lambda_2, \mu_2)S_2]$ | $[f(\lambda, \mu)S]$ | CFP |
|-----------------------------|-----------------------------|----------------------|-----|
| $[2, 1^2](0, 2)1$ | $[2](2, 0)1$ | $[2^12^1](00)0$ | $\frac{\sqrt{3}}{3}$ |
| $[2, 1^2](0, 2)1$ | $[2](2, 0)1$ | $[2^12^1](00)0$ | $1$ |

Table 7: The two particle CFP's involving the spin color symmetry, needed in the case of a 6-quark cluster, colorless with spin two.

| $[f_1(\lambda_1, \mu_1)S_1]$ | $[f_2(\lambda_2, \mu_2)S_2]$ | $[f(\lambda, \mu)S]$ | CFP |
|-----------------------------|-----------------------------|----------------------|-----|
| $[2, 2](0, 2)2$ | $[1^2](2, 0)0$ | $[2^12^1](00)2$ | $\frac{\sqrt{2}}{2}$ |
| $[2, 2](0, 2)2$ | $[1^2](2, 0)0$ | $[2^12^1](00)2$ | $\frac{1}{2}$ |

| $[f_1(\lambda_1, \mu_1)S_1]$ | $[f_2(\lambda_2, \mu_2)S_2]$ | $[f(\lambda, \mu)S]$ | CFP |
|-----------------------------|-----------------------------|----------------------|-----|
| $[2, 1^2](1, 0)1$ | $[1^2](0, 1)1$ | $[2^12^1](00)2$ | $\frac{\sqrt{3}}{3}$ |
| $[2, 1^2](1, 0)1$ | $[1^2](0, 1)1$ | $[2^12^1](00)2$ | $1$ |
Table 8: The two particle CFP’s involving the spin color symmetry, needed in the case of a 6-quark cluster, colorless with spin two.

| $f_1(\lambda_1, \mu_1)S_1$ | $f_2(\lambda_2, \mu_2)S_2$ | $f(\lambda, \mu)S$ | CFP       | $f_1(\lambda_1, \mu_1)S_1$ | $f_2(\lambda_2, \mu_2)S_2$ | $f(\lambda, \mu)S$ | CFP       |
|--------------------------|--------------------------|------------------|-----------|--------------------------|--------------------------|------------------|-----------|
| [2, 2](0, 2)2           | [1]2)(0, 0)             | [2]212)(00)2     | $\sqrt{3}$ | [2, 1]2)(0, 2)1          | [2](2, 0)1              | [2]212)(00)2     | $-\sqrt{3}$ |
| [2, 2](1, 0)1           | [1]2)(0, 1)1            | [2]212)(00)2     | $-\sqrt{3}$| [2, 1]2)(1, 0)2          | [2](0, 1)0             | [2]212)(00)0     | $-\sqrt{3}$ |

| $f_1(\lambda_1, \mu_1)S_1$ | $f_2(\lambda_2, \mu_2)S_2$ | $f(\lambda, \mu)S$ | CFP       |
|--------------------------|--------------------------|------------------|-----------|
| [2, 1]2)(1, 0)1          | [1]2)(0, 1)1            | [2]212)(00)2     | $\sqrt{3}$ |
| [2, 1]2)(1, 0)2          | [1]2)(0, 1)2            | [2]212)(00)2     | $-\sqrt{3}$|

Table 9: The two particle CFP’s involving the spin color symmetry, needed in the case of a 6-quark cluster, colorless with spins 1 and 3.

| $f_1(\lambda_1, \mu_1)S_1$ | $f_2(\lambda_2, \mu_2)S_2$ | $f(\lambda, \mu)S$ | CFP       |
|--------------------------|--------------------------|------------------|-----------|
| [2, 2](0, 2)0           | [2](2, 0)1              | [2]3)(00)1       | $\sqrt{5}$ |
| [2, 2](0, 2)2           | [2](2, 0)1              | [2]3)(00)1       | $\sqrt{5}$ |
| [2, 2](1, 0)0           | [2](0, 1)0              | [2]3)(00)1       | $\sqrt{5}$ |

11. Obtaining the lowest six quark cluster.

In order to achieve this goal, one must diagonalize the QCD Hamiltonian in the above basis, using the standard two particle coefficients of fractional parentage for the orbital part, for the spin color part and for isospin (for notation see sec. 13 below).

Those involved in the symmetry $f_{cs}$ to $SU_c(3) \times SU_f(2)$ can be obtained from the one particle CFP’s calculated by So and Strottman.

From those we obtained the needed 2-particle CFP’s which are given in tables 6 and 8.

In some cases the corresponding 1-particle cfp’s for the orbital part may be needed. For the cases of interest in the present study they are shown in tables 13, 14, and 15.
Table 10: The 2-particle coefficients of fractional parentage (cfp) involved in the orbital symmetries [6] and [51] for the configurations 0s^5ℓ for ℓ = 1s and 0d. See the main text for notation.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
[f_1] & [f_2] & [6] & [51]_1 & [51]_2 & [51]_3 & [51]_4 & [51]_5 \\
\hline
s^4ℓ[3, 1]_1 & [2] & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
s^3ℓ[3, 1]_2 & [2] & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
s^3ℓ[3, 1]_3 & [2] & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
s^3ℓ[4] & [2] & \sqrt{2} / 3 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & -\sqrt{2} / 15 \\
\hline
4 sl & [2] & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \sqrt{2} / 5 & -\frac{2}{\sqrt{15}} \\
\hline
4 sl & 0 & 0 & 0 & 0 & \sqrt{5} / 5 & -\frac{3}{\sqrt{15}} \\
\hline
\end{array}
\]

Table 11: The 2-particle coefficients of fractional parentage (cfp) involved in the orbital symmetries [6] and [51] for the configurations 0s^4p^2. See the main text for notation.

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
[f_1] & [f_2] & [6] & [51]_1 & [51]_2 & [51]_3 & [51]_4 & [51]_5 \\
\hline
s^4p^2[2, 2]_1 & [2] & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
s^4p^2[2, 2]_2 & [2] & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
s^4p^2[3, 1]_1 & [2] & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
\hline
s^4p^2[3, 1]_2 & [2] & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
\hline
s^4p^2[3, 1]_3 & [2] & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
\hline
s^4p^2[4] & [2] & \sqrt{2} / 3 & 0 & 0 & 0 & -\frac{3}{10} & -\frac{1}{10} \\
\hline
s^3p[3, 1]_1 & sp[2] & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
\hline
s^3p[3, 1]_2 & sp[2] & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
\hline
s^3p[3, 1]_3 & sp[2] & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
\hline
s^3p[4] & sp[2] & \sqrt{6} / 15 & 0 & 0 & 0 & \frac{1}{10} & \frac{1}{15} \\
\hline
s^3p[3, 1]_1 & sp[11] & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
s^3p[3, 1]_2 & sp[11] & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
s^3p[3, 1]_3 & sp[11] & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
s^3p[4] & sp[11] & 0 & 0 & 0 & 0 & -\frac{3}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\
\hline
[p^2][4] & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{5}} & \sqrt{2} / 15 \\
\hline
\end{array}
\]
Table 12: The 2-particle coefficients of fractional parentage (cfp) involved in the orbital symmetry $[42]$ for the configurations $0s^40p^2$. See the main text for notation.

| $[f_1]$ | $[f_2]$ | $[42]_1$ | $[42]_2$ | $[42]_3$ | $[42]_4$ | $[42]_5$ | $[42]_6$ | $[42]_7$ | $[42]_8$ | $[42]_9$ |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $s^2p^2[2, 2]_1$ | $[2]$ | $\frac{1}{\sqrt{2}}$ | $0$ | $0$ | $-\frac{\sqrt{3}}{4}$ | $-\frac{1}{4}$ | $0$ | $\frac{1}{\sqrt{2}}$ | $0$ | $0$ |
| $s^2p^2[2, 2]_2$ | $[2]$ | $\frac{1}{\sqrt{2}}$ | $0$ | $0$ | $\frac{1}{4}$ | $\frac{1}{\sqrt{2}}$ | $0$ | $-\frac{1}{\sqrt{2}}$ | $0$ | $0$ |
| $s^2p^2[3, 1]_1$ | $[2]$ | $0$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{2\sqrt{3}}$ | $-\frac{1}{4\sqrt{3}}$ | $-\frac{1}{12}$ | $\frac{1}{2\sqrt{6}}$ | $-\frac{1}{6\sqrt{2}}$ | $-\frac{\sqrt{3}}{4}$ | $-\frac{1}{4\sqrt{3}}$ |
| $s^2p^2[3, 1]_2$ | $[2]$ | $0$ | $-\frac{1}{2\sqrt{2}}$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{3\sqrt{2}}$ | $\frac{1}{12\sqrt{6}}$ | $-\frac{1}{6\sqrt{2}}$ | $-\frac{1}{6\sqrt{6}}$ | $-\frac{1}{4}$ | $-\frac{1}{2\sqrt{3}}$ |
| $s^2p^2[3, 1]_3$ | $[2]$ | $0$ | $0$ | $0$ | $-\frac{\sqrt{3}}{3}$ | $-\frac{1}{4}$ | $-\frac{2}{\sqrt{3}}$ | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{12}}$ | $\frac{1}{\sqrt{12}}$ |
| $s^2p^2[4]$ | $[2]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{4}$ | $0$ | $\frac{1}{4\sqrt{2}}$ | $-\frac{1}{4\sqrt{2}}$ |
| $s^3p[3, 1]_1$ | sp$[2]$ | $0$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{4\sqrt{2}}$ | $\frac{1}{12}$ | $\frac{1}{2\sqrt{6}}$ | $\frac{1}{6\sqrt{2}}$ | $\frac{1}{6\sqrt{6}}$ | $\frac{1}{4\sqrt{2}}$ |
| $s^3p[3, 1]_2$ | sp$[2]$ | $0$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{2\sqrt{2}}$ | $\frac{1}{4\sqrt{2}}$ | $\frac{1}{12}$ | $\frac{1}{2\sqrt{6}}$ | $\frac{1}{6\sqrt{2}}$ | $\frac{1}{6\sqrt{6}}$ | $\frac{1}{4\sqrt{2}}$ |
| $s^3p[3, 1]_3$ | sp$[2]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{4}$ | $0$ | $\frac{1}{4\sqrt{2}}$ | $-\frac{1}{4\sqrt{2}}$ |
| $s^3p[4]$ | sp$[2]$ | $0$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{4\sqrt{3}}$ | $\frac{1}{12}$ | $\frac{1}{2\sqrt{6}}$ | $\frac{1}{6\sqrt{2}}$ | $\frac{1}{6\sqrt{6}}$ | $\frac{1}{4\sqrt{2}}$ |
| $s^3p[3, 1]_1$ | sp$[11]$ | $0$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{2\sqrt{3}}$ | $\frac{1}{4\sqrt{3}}$ | $\frac{1}{12}$ | $\frac{1}{2\sqrt{6}}$ | $\frac{1}{6\sqrt{2}}$ | $\frac{1}{6\sqrt{6}}$ | $\frac{1}{4\sqrt{2}}$ |
| $s^3p[3, 1]_2$ | sp$[11]$ | $0$ | $0$ | $0$ | $-\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ | $\frac{1}{\sqrt{3}}$ | $0$ | $\frac{1}{\sqrt{6}}$ | $\frac{1}{\sqrt{6}}$ |
| $s^3p[3, 1]_3$ | sp$[11]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-\frac{\sqrt{3}}{3}$ | $\sqrt{3}$ | $\frac{1}{\sqrt{3}}$ | $\frac{1}{\sqrt{6}}$ |
| $s^3p[4]$ | sp$[11]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $[4]$ $p^5[2]$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\frac{\sqrt{2}}{2}$ | $0$ | $\sqrt{3}$ | $\frac{1}{4\sqrt{2}}$ |

Table 13: The 1-particle coefficients of fractional parentage (cfp) involved in the orbital symmetries $[6]$ and $[51]$ for the configurations $0s^5\ell$ with $\ell = 0s$ or 0d. The rows are labeled by the five particle symmetry $[f_1]$ and the angular momentum of the last particle. See the main text for notation.

| $[f_1]$ | $\ell$ | $[6]$ | $[51]_1$ | $[51]_2$ | $[51]_3$ | $[51]_4$ | $[51]_5$ |
|---------|---------|---------|---------|---------|---------|---------|---------|
| $[41]_1$ | $0s$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ |
| $[41]_1$ | $0s$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ |
| $[41]_1$ | $0s$ | $0$ | $0$ | $0$ | $1$ | $0$ | $0$ |
| $[41]_1$ | $0s$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ |
| $[5]$ | $0s$ | $\frac{1}{\sqrt{6}}$ | $0$ | $0$ | $0$ | $0$ | $\sqrt{\frac{3}{2}}$ |
| $[5]$ | $1s$ or 0d | $\sqrt{\frac{3}{6}}$ | $0$ | $0$ | $0$ | $0$ | $-\frac{1}{\sqrt{6}}$ |
Table 14: The 1-particle coefficients of fractional parentage (cfp) involved in the orbital symmetries [6] and [51] for the configurations 0s^40p^2. The rows are labeled by the five particle symmetry [f_1] and the angular momentum of the last particle. See the main text for notation.

| [f_1] | ℓ  | [6] | [51]_1 | [51]_2 | [51]_3 | [51]_4 | [51]_5 |
|-------|-----|-----|--------|--------|--------|--------|--------|
| [41]_1 | 0s  | 0   | 0      | 0      | 0      | 0      | 0      |
| [41]_2 | 0s  | 0   | 0      | 0      | 0      | 0      | 0      |
| [41]_3 | 0s  | 0   | 0      | 0      | 0      | 0      | 0      |
| [41]_4 | 0s  | 0   | 0      | 0      | 0      | 0      | 0      |
| [41]_1 | 0p  | 0   | 0      | 0      | 0      | 0      | 0      |
| [41]_2 | 0p  | 0   | 0      | 0      | 0      | 0      | 0      |
| [41]_3 | 0p  | 0   | 0      | 0      | 0      | 0      | 0      |
| [41]_4 | 0p  | 0   | 0      | 0      | 0      | 0      | 0      |
| [5]   | 0s  | 0   | 0      | 0      | 0      | 0      | 0      |
| [5]   | 0p  | 0   | 0      | 0      | 0      | 0      | 0      |

12. Computing the mixing of the lowest six quark cluster and with the two nucleon wf.

In this case we will concentrate on the orbital symmetry and transform the original wf into the Jacobi coordinates defined as follows:

\[ \xi_1 = \frac{1}{\sqrt{2}}(r_1 - r_2) \]

\[ \eta_1 = \frac{1}{\sqrt{6}}(r_1 + r_2 - 2r_3) \]

\[ \xi_2 = \frac{1}{\sqrt{2}}(r_4 - r_5) \]

\[ \eta_2 = \frac{1}{\sqrt{6}}(r_4 + r_5 - 2r_6) \]

\[ r = \frac{1}{\sqrt{6}}(r_1 + r_2 + r_3 - r_4 - r_5 - r_6) \]

\[ r = \frac{1}{\sqrt{6}}(r_1 + r_2 + r_3 + r_4 + r_5 + r_6) \]
Table 15: The 1-particle coefficients of fractional parentage (cfp) involved in the orbital symmetry [42] for the configurations 0s40p2. The rows are labeled by the five particle symmetry [f1] and the angular momentum of the last particle. See the main text for notation.

| [f1]  | ℓ   | [42]1 | [42]2 | [42]3 | [42]4 | [42]5 | [42]6 | [42]7 | [42]8 | [42]9 |
|-------|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| [32]1 | 0s   | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [32]2 | 0s   | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [32]3 | 0s   | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     |
| [32]4 | 0s   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [32]5 | 0s   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]1 | 0s   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]2 | 0s   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]3 | 0s   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]4 | 0s   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]1 | 0p   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]2 | 0p   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]3 | 0p   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| [41]4 | 0p   | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |

13. Expressions regarding the orbital symmetry

- The representation f = [6] is one-dimensional.
- The representation f = [5, 1] is five-dimensional.

A basis can be found using the Yamanouchi symbols.32 We chose them in the order Y1=[111121], Y2=[111211] Y3=[112111], Y4=[121111], Y5=[211111]. The corresponding states are not orthogonal. We orthogonalize them using the Gram-Schmidt procedure in the above i.e we start with first state [5, 1]1 ⇔ Y1, the second state [5, 1]2 is an orthogonal combination of the first two associated with Y1 and Y2 etc. The orthogonal states will be indicated as [51]i, i=1,2,...,5. Thus:

\[
[51]1 = Y_1, \quad [51]2 = \frac{1}{\sqrt{3}} (-Y_1 + 2Y_2), \quad [51]3 = \frac{(-Y_1 - Y_2 + 3Y_3)}{\sqrt{6}}, \quad (56)
\]

\[
[51]4 = \frac{(-Y_1 - Y_2 - Y_3 + 4Y_4)}{\sqrt{10}}, \quad [51]5 = \frac{(-Y_1 - Y_2 - Y_3 - Y_4 + 5Y_5)}{\sqrt{15}}. \quad (57)
\]

- The representation f = [4, 2] is nine-dimensional.
A basis is found using the Yamanouchi symbols: \( Y_1 = [112211], \) \( Y_2 = [121211], \) \( Y_3 = [211211], \) \( Y_4 = [122111], \) \( Y_5 = [211121], \) \( Y_6 = [112111], \) \( Y_7 = [212111], \) \( Y_8 = [121121], \) and \( Y_9 = [211121]. \) An orthonormal basis is obtained as in the previous case. The obtained states will be denoted as \([4,2]_i\), \( i=1,2,...,9.\) Thus:

\[
[42]_1 = Y_1, \quad [42]_2 = \frac{(-Y_1 + 2Y_2)}{\sqrt{3}}, \quad [42]_3 = \frac{(-Y_1 - Y_2 + 3Y_6)}{\sqrt{6}} \quad (58)
\]

\[
[42]_4 = \frac{(-Y_2 + 2Y_4)}{\sqrt{3}}, \quad [42]_5 = \frac{(Y_2 - 2Y_3 - 2Y_4 + 4Y_5)}{3}, \quad (59)
\]

\[
[42]_6 = \frac{(-Y_3 - Y_5 - 3Y_6)}{\sqrt{6}}, \quad [42]_7 = \frac{(-3Y_1 - Y_2 - Y_3 + 2Y_4 + 2Y_5 - 6Y_7)}{3\sqrt{2}}, \quad (60)
\]

\[
[42]_8 = \frac{(2Y_1 - 2Y_2 - Y_3 - 2Y_4 - Y_5 + 3Y_6 - 4Y_7 + 6Y_8)}{2\sqrt{3}}, \quad (61)
\]

\[
[42]_9 = \frac{(2Y_1 + 2Y_2 - 3Y_3 + 2Y_4 - 3Y_5 - 3Y_6 - 4Y_7 - 6Y_8 + 12Y_9)}{2\sqrt{15}}. \quad (62)
\]

In computing the cfp's we need a description of the two particle and four particle states. If only 0's states are involved we use the standard labels \([4],[31],[22]\) for the four particles and \([2]\) and \([11]\) for the two particles. If configurations other than 0's are involved we attach to them suitable configuration labels. If the representation is more than one dimensional we specify the relevant Yamanouchi symbols.

- The representation \([2,2]\).
  The relevant Yamanouchi symbols are \( Y_1 = [2121] \) and \( Y_2 = [2211]. \) The first state \([2,2]_1\) is associated with \( Y_1 \) and the other is orthogonal to it. Thus:

\[
[22]_1 = Y_1, \quad [22]_2 = \frac{1}{\sqrt{3}} (-Y_1 + 2Y_2). \quad (63)
\]

- The representation \([3,1]\).
  Now we have \( Y_1 = [1121], \) \( Y_1 = [1211], \) \( Y_1 = [2111]. \) We orthogonalize them in this order \(([3,1] \leftrightarrow Y_1 \text{ etc}). \) Thus:

\[
[31]_1 = Y_1, \quad [31]_2 = \frac{1}{\sqrt{3}} (-Y_1 + 2Y_2), \quad [31]_3 = \frac{1}{\sqrt{6}} (-Y_1 - Y_2 + 3Y_3) \quad (64)
\]

In the case of 1-particle cfp's we need the 5-particle symmetries \([5],[41]\) and \([32]\). The first is uniquely specified. In the case of \([4,1]\) we follow a procedure analogous to that for \([5,1]\) discussed above. In the case of \([32]\) the five orthogonal states were obtained as follows:

\[
[32]_1 = Y_1, \quad [32]_2 = (-Y_1 + 2Y_2)/\sqrt{3}, \quad [32]_3 = (-Y_2 + 2Y_3)/\sqrt{3},
\]
$$[32]_4 = (-2Y_1 - Y_2 + 2Y_3 + 4Y_4)/3 , \quad [32]_5 = \frac{\sqrt{2}}{3}(Y_1 - Y_2 - Y_3 - 2Y_4 + 3Y_5)$$

where the Yamanouchi symbols are: $Y_1 = [12121], Y_2 = [21211], Y_3 = [22111], Y_4 = [12211], Y_1 = [21121]$.

### 13.1. Separating out the two nucleon like component of the orbital symmetry.

We distinguish the following cases:

1. The symmetry $f = [6]$.
   
   We have the following possibilities:
   
   - The configuration $0s^6$.
     
     $$|0s^6 f_L = [6] L = 0 > = \psi(\xi_1, \eta_1)\psi(\xi_2, \eta_2)\phi_{0s}(r)\phi_{0s}(R)$$
     
     where $\psi$ is the nucleon like w.f. (in the indicated relative coordinates) and $\phi$ the one particle harmonic oscillator w.f. in the standard notation. They are characterized by an oscillation parameter $a \approx 1$fm, as opposed to the size parameter $b$ of the shell model harmonic oscillator which depends on the nuclear mass number $A$.

   - The configuration $0s^5 1s$.
     
     $$|0s^5 1s f_L = [6] L = 0 > = \frac{1}{\sqrt{6}} \sum_{i=1}^{4} \chi(x_i)\phi_{0s}(r)\phi_{0s}(R)$$
     
     $$+ \frac{1}{\sqrt{6}} \psi(\xi_1, \eta_1)\psi(\xi_2, \eta_2)(\phi_{1s}(r)\phi_{0s}(R) + \phi_{0s}(r)\phi_{1s}(R))$$

     In the above expression $\chi(x_i)$ is the product of a $1s$ harmonic oscillator wave function in the coordinate $\xi_1, \eta_1, \xi_2, \eta_2$ respectively for $i = 1, 2, 3, 4$ with $0s$ in all other coordinates.

   - The configuration $0s^5 0d$.
     
     $$|0s^5 0d f_L = [6] L > = \frac{1}{\sqrt{6}} \sum_{i=1}^{4} \chi(x_i)\phi_{0s}(r)\phi_{0s}(R)$$
     
     $$+ \frac{1}{\sqrt{6}} \psi(\xi_1, \eta_1)\psi(\xi_2, \eta_2)(\phi_{0d}(r)\phi_{0s}(R) + \phi_{0s}(r)\phi_{0d}(R))$$

     Now $\chi(x_i)$ is $1d$ harmonic oscillator wave function in the coordinate $\xi_1, \eta_1, \xi_2, \eta_2$ respectively with $0s$ in all other coordinates.
The configuration $0s^4 0p^2$. The resulting expression is now more complicated. We find:

$$0s^4 0p^2 [6] L = A_1(\xi, \eta, \xi_2, \eta_2) + B_1(\xi, \eta, 2, \eta_2) \otimes R + C_1(\xi, \eta, \xi_2, \eta_2) \otimes R$$

$$+ \frac{1}{2\sqrt{15}} \psi(\xi, \eta) \psi(\xi_2, \eta_2) (5(\Theta^2_0(R)\xi) - (\Theta^2_0(r)L))$$

(68)

The functions $A_1, B_1, C_1$ are not expected to contribute significantly to the coupling to the nucleon system.

(2) The symmetry $f_L = [5, 1]$.

Omitting those components in which we do not have a nucleon like structure in the internal coordinates we find:

- The configuration $0s^5 1s$.

$$|0s^5 1s f_L = [5, 1] L = 0 >^i \sim 0 , \ i = 1, 2$$

$$|0s^5 1s f_L = [5, 1] L = 0 >^i \sim \psi(\xi, \eta_1)\psi(\xi_2, \eta_2)C_i[\phi_1p(r)\otimes \phi_1p(R)]L = 0$$

(70)

with $C_i = \frac{1}{\sqrt{10}}, \frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}$ for $i = 3, 4, 5$

- The configuration $0s^5 1d$.

$$|0s^5 0d f_L = [5, 1] L = 2 >^i \sim 0 , \ i = 1, 2$$

$$|0s^5 0d f_L = [5, 1] L = 2 >^i \sim \psi(\xi, \eta_1)\psi(\xi_2, \eta_2)C_i[\phi_1p(r)\otimes \phi_1p(R)]L = 2$$

(72)

with $C_i = -\frac{1}{\sqrt{10}}, -\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}$ for $i = 3, 4, 5$

- The configuration $0s^4 0p^2$.

$$|0s^4 0p^2 f_L = [5, 1] L = 0, 2 >^i \sim 0 , \ i = 1, 2$$

$$|0s^4 0p^2 f_L = [5, 1] L >^i \sim \psi(\xi, \eta_1)\psi(\xi_2, \eta_2)C_i[\phi_1p(r)\otimes \phi_1p(R)]L$$

(74)

with $L = 0, 2$ and $C_i = -\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{5}}, \frac{2}{\sqrt{15}}$ for $i = 3, 4, 5$

(3) The symmetry $f = [4, 2]$.

The only allowed configuration is $0s^4 0p^2$. we have:

$$|0s^4 0p^2 f_L = [4, 2] L = 0, 2 >^i \sim 0 , \ i = 1, 2, 3, 8$$

$$|0s^4 0p^2 f_L = [4, 2] L = 0, 2 >^i \sim \psi(\xi, \eta_1)\psi(\xi_2, \eta_2)C_i[\phi_1p(R)]L = 0, 2$$

(76)

with $C_i = \frac{2}{\sqrt{3}}, \frac{5}{\sqrt{5}}, \frac{2\sqrt{2}}{3\sqrt{5}}, -\frac{2}{5\sqrt{15}}$ for $i = 4, 5, 6, 7, 9$

From the above expressions we can see that the the 6 quark states configurations involving $1s$, $0d$ or $0p^2$ contain spurious components, i.e. the CM is not in a $0s$
state. We could, of course, project them out, but we do not want to do that. The reason being that the 6-quark cluster is going to be admixed in the two nucleon component of the shell model basis (see below). In the two nucleon w.f. the center of mass component of the two particles is not necessarily in $0s$ mode.

13.2. Transformation into the two nucleon relative and CM coordinates.

As it was mentioned above we must transform the above cluster orbital w.f. into those encountered in the shell model. Clearly the wave functions associated with different angular momenta are orthogonal. Thus

$$\phi_{n_0,\ell}(a) = \sum_n |n,\ell > C_{n_0,n,\ell}(c), \text{ with } C_{n_0,n,\ell}(c) = \left< n,\ell(b)|\phi_{n_0,\ell}(a) > , c = \frac{b}{a}\right>$$

Limiting ourselves to two nucleon harmonic oscillator wave functions with $\hbar \omega \leq 5$ we get $C_{n_0,n,\ell}(c)$ as follows:

For $n_0,\ell = 0s$ in the order $(0s, 1s, 2s, 3s, 4s, 5s, 6s, 7s, 8s, 9s, 10s)$:

For $c = 3/2$

$$(0.886864, 0.417762, 0.127026, 0.030467, 0.006214, 0.000183, 0.000027, 0.0000109, 0.0000054, 0, 0)$$

For $c = 2$

$$(0.715541, 0.525813, 0.249415, 0.093322, 0.029695, 0.008356, 0.00213, 0.000500, 0.000109, 0.000022)$$

For $c = 5/2$

$$(0.572727, 0.507942, 0.290787, 0.131313, 0.050428, 0.017128, 0.00527, 0.001493, 0.000394, 0.000097, 0.000022)$$

For $c = 7/2$

$$(0.383992, 0.399305, 0.268029, 0.141916, 0.063902, 0.025448, 0.009181, 0.003049, 0.000943, 0.000274, 0.000075)$$

For $n_0,\ell = 1s$ in the above order we get:

For $c = 3/2$

$$(-0.417762, 0.558881, 0.399709, 0.150981, 0.042037, 0.009611, 0.001902, 0.000336, 0.000054, 0, 0)$$

For $c = 2$

$$(-0.525813, 0.071554, 0.251164, 0.175254, 0.081627, 0.03025, 0.009567, 0.002681, 0.000681)$$
For $c = 5/2$
$(-0.507942, -0.178083, 0.053996, 0.094805, 0.063452, 0.030737, 0.012284, 0.00428, 0.001341, 0.000384, 0.000102)$

For $c = 7/2$
$(-0.399305, -0.308055, -0.134839, -0.033304, 0.002154, 0.007688, 0.005238, 0.002558, 0.001044, 0.000377, 0.000124)$

For $n_0, \ell = 1p$ in the order $(0p, 1p, 2p, 3p, 4p, 5p, 6p, 7p, 8p, 9p, 10p)$ we get:

For $c = 3/2$
$(0.818643, 0.497841, 0.17911, 0.048711, 0.010984, 0.002154, 0.000378, 0.00006, 0, 0)$

For $c = 2$
$(0.572433, 0.543058, 0.30479, 0.129311, 0.045489, 0.013916, 0.000381, 0.00006, 0, 0)$

For $c = 5/2$
$(0.394984, 0.452242, 0.306334, 0.156856, 0.066595, 0.024589, 0.008127, 0.002451, 0.000683, 0.000178, 0.000043)$

For $c = 7/2$
$(0.202864, 0.27234, 0.216298, 0.129859, 0.064644, 0.027986, 0.010846, 0.003835, 0.001254, 0.000383, 0.000111)$

For $n_0, \ell = 1d$ in the order $(0d, 1d, 2d, 3d, 4d, 5d, 6d, 7d, 8d, 9d, 10d)$ we get:

For $c = 3/2$
$(0.75567, 0.543742, 0.221817, 0.066693, 0.016349, 0.003444, 0.000643, 0.000109, 0.000016, 0, 0)$

For $c = 2$
$(0.457946, 0.514043, 0.327135, 0.15344, 0.058679, 0.019284, 0.005622, 0.001485, 0.00036, 0.000081, 0.000017)$
For \( c = 5/2 \)

\[
(0.272403, 0.369034, 0.283442, 0.160452, 0.074056, 0.010335, 0.003295, 0.000966, 0.000263, 0.000067)
\]

For \( c = 7/2 \)

\[
(0.107173, 0.170238, 0.15331, 0.101757, 0.055068, 0.025609, 0.010565, 0.003949, 0.001358, 0.000434, 0.00013)
\]

We notice that for 0s and 1s the convergence is reasonably good even for high \( c \). For 0p and 0d the overlap for high \( c \) is not as good.

14. The combined orbit-spin-color symmetry

In the case of \( I = 1 \) states the isospin state has symmetry \([4, 2]_2\], the orbital and spin-color symmetries can be combined to yield the orbital-spin-color symmetry \([f^{osc}] = [2, 2]_1\) of the group \( S_6 \), using the relevant Clebsch-Gordan coefficients. Similarly for \( I = 0 \) one gets \([f^{osc}] = [2^3]_1\). There exist now routines that provided these coefficients. Since the above computed cfp’s do not depend on the selected Yamanouchi symbols, the symmetry \([f^{osc}]\) is merely a label to select the allowed states. So we do not need either the C-G coefficients or the two particle cfp’s associated with \([f^{osc}]\).

In fact it is easy to show the following:

i) The configuration \((0s)^6\) is associated with the 1-dimensional completely symmetric state. So in this case \( f^{osc} = f^{sc} \). and the two-particle CFPs provided above are adequate.

ii) Configurations of the type \((0s)^51s\). In this case, using the 1-particle CFPs given in appropriate the tables, the corresponding symmetries can be expressed in terms of antisymmetric states of a given \( J \) and \( I \) of the form:

\[
\begin{align*}
\{ (0s)^5[f^{sc}] & = [2^21](0, 1)S_1, I_1 = 1/2 \otimes 1s(10)s_2 = \frac{1}{2} J_2, \frac{1}{2} J_2 = \frac{1}{2} \}^{J, I = 0, 1} \\

s_1 &= |J - 1/2|, J + 1/2. \text{ In this case one needs the CFP of the spin color symmetry } [2^21](0, 1)s_1, s_1 = 1/2, 1/2.
\end{align*}
\]

\[
\begin{align*}
\{ (0s)^5[f^{sc}] & = [21^3](0, 1)S_1, I_3 = 3/2 \otimes 1s(10)s_2 = \frac{1}{2} J_2, \frac{1}{2} J_2 = \frac{1}{2} \}^{J, I = 1, 2} \\

s_1 &= |J - 1/2|, J + 1/2. \text{ Now one needs the CFP of the spin color symmetry } [2, 1^3](0, 1)s_1, s_1 = 1/2, 1/2.
\end{align*}
\]

The needed one particle spin-color CFPs are given in table 16. Using the results of table 16 one can compute the corresponding 2-particle CFPs given in table 17.
Table 16: The one particle CFP’s involving the spin color symmetry, needed in the case of a 5-quark state, with color symmetry (0,1) and spin 1/2, 3/2 and 5/2.

| $|f_1| (\lambda_1, \mu_1) S_1$ | $|f| (\lambda, \mu) S$ | CFP |
|-----------------------------|-----------------------------|-----|
| $[2^1](0, 2) 0$             | $[2^2, 1](0, 1) \frac{1}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $[2^1](1, 0) 1$             | $[2^2, 1](0, 1) \frac{1}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| $[2^2](0, 2) 2$             | $[2^2, 1](0, 1) \frac{3}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $[2^1](0, 1) \frac{1}{2}$  | $[2^2, 1](0, 1) \frac{3}{2}$ | $\frac{\sqrt{3}}{3}$ |

Table 17: The two particle CFP’s involving the spin color symmetry for 5 identical quarks, needed in the study of the six quark clusters, corresponding to color state (0,1) and spin 1/2, and 3/2.

| $|f_1| (\lambda_1, \mu_1) S_1$ | $|f_2| (\lambda_2, \mu_2) S_2$ | $|f| (\lambda, \mu) S$ | CFP |
|-----------------------------|-----------------------------|-----------------------------|-----|
| $[1^4](0, 2) 0$             | $[1^5](0, 1) \frac{1}{2}$  | $\sqrt{\frac{2}{5}}$       |
| $[1^4](0, 1) 1$             | $[1^5](0, 1) \frac{1}{2}$  | $-\sqrt{\frac{2}{5}}$      |

| $|f_1| (\lambda_1, \mu_1) S_1$ | $|f_2| (\lambda_2, \mu_2) S_2$ | $|f| (\lambda, \mu) S$ | CFP |
|-----------------------------|-----------------------------|-----------------------------|-----|
| $[2, 1](1, 1) \frac{1}{3}$  | $[2, 1](0, 0) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{5}{3}}$ |
| $[2, 1](1, 1) \frac{1}{3}$  | $[2, 1](0, 0) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{5}{3}}$ |
| $[2, 1](1, 1) \frac{1}{3}$  | $[2, 1](0, 0) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{5}{3}}$ |
| $[2, 1](1, 1) \frac{1}{3}$  | $[2, 1](0, 0) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{5}{3}}$ |
| $[1^3](1, 1) \frac{1}{3}$  | $[1^3](0, 1) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{3}{3}}$ |
| $[1^3](1, 1) \frac{1}{3}$  | $[1^3](0, 1) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{3}{3}}$ |
| $[1^3](0, 0) \frac{1}{3}$  | $[1^3](0, 1) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{3}{3}}$ |
| $[1^3](0, 0) \frac{1}{3}$  | $[1^3](0, 1) \frac{1}{3}$  | $[2^1](0, 1) \frac{1}{3}$  | $\sqrt{\frac{3}{3}}$ |
iii) Configurations of the type \((0s)^4(0p)^2\). Then, using the 2-particle CFPs given in appropriate the tables, the corresponding symmetries can be expressed in terms of the form:

\[
\{ [(0s)^4[f^{sc}]_1(\mu, \lambda)S_1 = J_1, I_1] \times [(0p)^2L[f]_2L[f^{sc}]_2(\lambda, \mu), S_2, J_2, I_2] J, I \}
\]

where

\[
[f^{sc}]_1 = 2^2, [2, 1^2], [1^4], \text{ for } I_1 = 0, 1, 2 \text{ respectively}
\]

and

\[
I_2 = 0 : L = \text{even } \leftrightarrow [f]_2L = [2], [f^{sc}]_2 = [2], L = \text{odd } \leftrightarrow [f]_2L = [1^2], [f^{sc}]_2 = [1^2]
\]

\[
I_2 = 1 : L = \text{even } \leftrightarrow [f]_2L = [2], [f^{sc}]_2 = [1^2], L = \text{odd } \leftrightarrow [f]_2L = [1^2], [f^{sc}]_2 = [2]
\]

In this case one needs the CFP of the spin color symmetry

\[
[f^{sc}]\mu, \lambda S_1, [f^{sc}]_1 = 2^2, [2, 1^2], [1^4]
\]

Those with color spin \((1, 0), 1\) were given above in connection with the pentaquarks. The ones with \([(0, 2), 1\) are given in table 18: These were constructed from the one

Table 18: The two particle CFP’s involving the spin color symmetry for 4 identical quarks , needed in the study of the six quark clusters, corresponding to color state \((0,2)\) and spin 0, 1, and 2.

| \(f_1\) | \(f_2\) | \(f\) | CFP |
|-----|-----|-----|-----|
| \((\lambda_1, \mu_1)S_1\) | \((\lambda_2, \mu_2)S_2\) | \((\lambda, \mu)S\) | \((\lambda_1, \mu_1)S_1\) | \((\lambda_2, \mu_2)S_2\) | \((\lambda, \mu)S\) | CFP |
| \(2\) | \(2\) | \(2\) | \(2\) | \(2\) | \(2\) | \(\frac{\sqrt{2}}{2}\) |
| \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(\frac{\sqrt{2}}{2}\) |
| \(2\) | \(2\) | \(2\) | \(2\) | \(2\) | \(2\) | \(\frac{\sqrt{2}}{2}\) |

| \(f_1\) | \(f_2\) | \(f\) | CFP |
|-----|-----|-----|-----|
| \((\lambda_1, \mu_1)S\) | \((\lambda_2, \mu_2)S\) | \((\lambda, \mu)S\) | \((\lambda_1, \mu_1)S\) | \((\lambda_2, \mu_2)S\) | \((\lambda, \mu)S\) | CFP |
| \(2\) | \(2\) | \(2\) | \(2\) | \(2\) | \(2\) | \(\frac{\sqrt{2}}{2}\) |
| \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(0\) | \(\frac{\sqrt{2}}{2}\) |

Even though, however, in the present calculation we do not explicitly make use of the coefficients involving explicitly \(f^{osc}\), we have constructed them and they will appear in the appendix.
The present paper discusses and develops the formalism of extending Quantum Chromodynamics (QCD) at low energy processes. Then using symmetries we applied this in the case of multiquark-antiquarks. The essential symmetries involved for \( n \) identical quarks were the orbital symmetry \( SU^o(n) \), the isospin symmetry \( SU^I(2) \) and the combined spin color-symmetry \( SU^o(6) \). It was shown how these symmetries can be combined to yield a wave function with the proper symmetry, e.g. totally antisymmetric in the case of fermions. The formalism has been applied in the case of pentaquarks, which have recently been found to exist. Also the essen-

Table 19: The one particle CFP’s involving the spin color symmetry, needed in the case of a 4-quark state, with color symmetry \((0, 2)\) and spin 0, 1, and 2.

| \( f_1(\lambda_1, \mu_1)S_1 \) | \( f_1(\lambda, \mu)S \) | CFP |
|-------------------|-----------------|-----|
| \([2, 1]((1, 1)\frac{1}{2})\) | \([2^2](0, 2)\) | 1 |
| \([2, 1]((1, 1)\frac{3}{2})\) | \([2^2](0, 2)\) | 1 |

15. Evaluation of the matrix elements of the two body interaction

We begin by considering the matrix element involving the orbital \((06)\) configurations:

\[
< (0s)^0 [6]_r [2211]_cs(00)_c S = J, I = 1 | \sum_{i,j} V_{ij} | (0s)^0 [6]_r [2211]_cs(00)_c S = J, I = 1 > 
\]

\[
= \frac{6 \cdot 5}{2} \sum_{f''(\lambda'' \mu''), l'' \langle [f_2|S_2], [f_1|S_1] | 2211\rangle} \frac{n_{f''}}{n_{2211}} \left| f_1'f_2' \right| (\lambda'' \mu'')_c S'' (\lambda_2 \mu_2)_c S_2 |(00)\rangle S \right> 
\]

\[
\times \left< \left(0s\right)^2 [2]_r (\lambda' \mu')_c S' (\lambda'_2 \mu'_2)_c S'_2 |(00)\rangle S \right> 
\]

\[
\times \left< 2 \left(0s\right)^2 [2]_r [f_1 f_2]_c (\lambda_2 \mu_2)_c S_2 I_2 \right> 56 \left(0s\right)^2 [2]_r (\lambda_2 \mu_2)_c S_2 I_2 \right> .
\]

In the above expression the first factor represents the number of ways one can select two particles out of 6. \( n_{f''} \) and \( n_{2211} \) are the dimensions of the representations \( f'' \) and \( [2211] \) of the symmetric group \( S_6 \) respectively. The spin-color two particle CFP’s are shown explicitly. The orbital and isospin two particle CFPs are trivial in this example. The last term is the elementary 2-body interaction matrix element, which is diagonal in isospin.

For the other matrix elements the reader is referred to the literature, see Strottman-Vergados and references therein.

16. Discussion

In the present paper we discussed and developed the formalism of extending Quantum Chromodynamics (QCD) at low energy processes. Then using symmetries we applied this in the case of multiquark-antiquarks. The essential symmetries involved for \( n \) identical quarks were the orbital symmetry \( SU^o(n) \), the isospin symmetry \( SU^I(2) \) and the combined spin color-symmetry \( SU^o(6) \). It was shown how these symmetries can be combined to yield a wave function with the proper symmetry, e.g. totally antisymmetric in the case of fermions. The formalism has been applied in the case of pentaquarks, which have recently been found to exist. Also the essen-
tional tools have been given to obtain the structure of six quark clusters up to $2\hbar \omega$ excitations. Such clusters may be admixed with the standard nuclear wave function and thus they may become useful in estimating the matrix elements of short range operators, such as those appearing in the case of neutrinoless double beta decay mediated by the exchange of heavy particles.

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Table 20: The Clebsch-Gordan coefficients entering in the product $[51] \times [42] \rightarrow [2^4 1^2]$. The columns are labeled by the Yamanouchi symbols of the combined symmetry $[2^4 1^2]$. The rows are labeled by two integers $p$ and $q$, i.e. $pq \leftrightarrow (p,q) \leftrightarrow (Y_p,Y_q)$, where $Y_p,Y_q$ are the Yamanouchi symbols of the $[51]$ and $[42]$ respectively. (see text).

\[
\begin{pmatrix}
13 & -1 & -1 & 11 & 11 & -\sqrt{5} & 12 & -\sqrt{5} & 11 & -\sqrt{5} & 12 & -\sqrt{5} & 13 & -\sqrt{5} & 14 & -\sqrt{5} \\
16 & -4 & -2 & 13 & 13 & -2  & 14 & -2  & 14 & -2  & 15 & -2  & 14 & -2  & 15 & -2  &&
24 & 4 & 2 & 26 & 26 & 2  & 26 & 2  & 26 & 2  & 26 & 2  & 26 & 2  \hline
27 & -4 & -2 & 22 & 22 & -2  & 24 & -2  & 24 & -2  & 24 & -2  & 24 & -2  & 25 & -2  \\
35 & 4 & 2 & 24 & 24 & 2  & 24 & 2  & 24 & 2  & 24 & 2  & 24 & 2  & 25 & 2  \hline
38 & -4 & -2 & 27 & 27 & -2  & 29 & -2  & 29 & -2  & 30 & -2  & 30 & -2  & 31 & -2  \\
49 & 4 & 2 & 35 & 35 & 2  & 35 & 2  & 35 & 2  & 35 & 2  & 35 & 2  & 35 & 2  \\
59 & -4 & -2 & 38 & 38 & -2  & 39 & -2  & 39 & -2  & 39 & -2  & 39 & -2  & 39 & -2  \\
0 & 0 & 0 & 39 & 39 & -2  & 39 & -2  & 39 & -2  & 39 & -2  & 39 & -2  & 39 & -2  \\
0 & 0 & 0 & 45 & 45 & -2  & 45 & -2  & 45 & -2  & 45 & -2  & 45 & -2  & 45 & -2  \\
0 & 0 & 0 & 48 & 48 & -2  & 48 & -2  & 48 & -2  & 48 & -2  & 48 & -2  & 48 & -2  \\
0 & 0 & 0 & 58 & 58 & -2  & 58 & -2  & 58 & -2  & 58 & -2  & 58 & -2  & 58 & -2  \\
0 & 0 & 0 & 0 & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0  \\
0 & 0 & 0 & 0 & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0  \\
0 & 0 & 0 & 0 & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0  \\
0 & 0 & 0 & 0 & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0 & 0  & 0  \\
\end{pmatrix}
\]

17. Appendix: The two particle CFP’s in the symmetric group $S_6$

Even though, however, in the present calculation we did not explicitly make use of the coefficients involving explicitly $[f^{\text{exc}}]$, we have constructed them and they will be discussed below.

In the special case of $[f_o] = [6]$ the spin color symmetry is uniquely specified to be $[2^4 1^2]$. In the case of $f_o = [51]$ and $f_{cs} = [42]$, one possibility encountered in present calculation, we present them in table 20. The same coefficients result for the case $f_o = [42]$ and $f_{cs} = [5, 1]$.

In reading this table we mention that the states are order according with the Yamanouchi symbols as follows:

1. for the $[51]$: 211111, 121111, 112111, 111211, 111121
2. for the $[42]$: 221111, 212111, 211211, 211121, 122111, 121211, 121121, 112211, 111211
3. For the $[2^2, 1^2]$: 432211, 432121, 423211, 423121, 421321, 243211, 243121, 241321, 214321

In general the two particle cfp’s for $S_6$ are very complicated. One has to deal with 6! states. For completeness, however, we will briefly include in the appendix the
two particle cfp’s involved, which may enter in other applications. In general the
two particle cfp’s for \( S_0 \) are very complicated. One has to deal with \( 6! \) states.\(^{35} \) We
will classify them in terms of two particle basis states, which are either symmetric
\( S \leftrightarrow [f] = [2,0] \) and antisymmetric \( A \leftrightarrow [f] = [1,1] \). Thus the 24 four particle
orthonormal states are of the form:

\[
(S \times S)[f_4], \ [f_4] = [4],[3,1], [2,2], \ i = 1, 2, 3 ; \ j = 1, 2
\]

\[
(A \times A)[f_4], \ [f_4] = [1^4], [2,1^2], [2,2], \ i = 1, 2, 3 ; \ j = 1, 2
\]

\[
(S \times A)[f_4], \ [f_4] = [2^2],[1^2], \ i = 1, 2, 3 ; \ j = 1, 2, 3
\]

The coefficients of fractional parentage take the form:

\[
\langle (S_1 \times S_2)[f_4]_{k} | [2]_{\ell} | [f_6]_{n} \rangle, \ (S_1 \times S_2)[f_4]_{k} | [11]_{j} | [f_6]_{n} \rangle, \ S_1 = S, A ; \ S_2 = S, A
\]

where the indices \( k, \ell, n \) are merely labels to completely specify the states of the
given symmetry. The index \( k \) takes values \( 1, 2, ..., 15 \) associated with the allowed
ordered pairs of six particles. The associated \( [f_4] \) symmetry is understood to charac-
terize the remaining particles. The orbital-spin-color symmetries \( [f] = [2^2,1^2], [2^3] \)
are needed, when six quark clusters with the quantum numbers of two nucleons are
considered.

All cfp’s have been obtained. We can summarize our results as follows:

\[
(A \times A)[1^4] \times [1^2] \Rightarrow [1^6] + [2,1^4] + [2^2,1^2],
\]

\[
(S \times S)[4] \times [2] \Rightarrow [6] + [5,1] + [4,2],
\]

\[
(A \times A)[1^4] \times [2] \Rightarrow [3,1^3] + [2,1^4],
\]

\[
(S \times S)[4] \times [1^2] \Rightarrow [4,1^2] + [5,1],
\]

\[
(A \times A)[2,1^2] \times [1^2] \Rightarrow [2,1^4] + [2^2,1^2] + [3^2] + [3,1^3] + [3,2,1],
\]

\[
(S \times S)[3,1] \times [2] \Rightarrow [5,1] + [4,2] + [2^3] + [4,1^2] + [3,2,1],
\]

\[
(A \times A)[2,1^2] \times [2] \Rightarrow [5,1] + [4,2] + [2^3] + [4,1^2] + [3,2,1],
\]

\[
(S \times S)[3,1] \times [1^2] \Rightarrow [2,1^4] + [2^2,1^2] + [3^2] + [3,1^3] + [3,2,1],
\]

\[
(A \times S)[3,1] \times [2] \Rightarrow [5,1] + [4,2] + [2^3] + [4,1^2] + [3,2,1],
\]

\[
(S \times A)[3,1] \times [2] \Rightarrow [5,1] + [4,2] + [2^3] + [4,1^2] + [3,2,1],
\]

\[
(S \times A)[3,1] \times [1^2] \Rightarrow [2,1^4] + [2^2,1^2] + [3^2] + [3,1^3] + [3,2,1],
\]
Taking into account that:

\[
\dim([6]) = \dim([1^6]) = 1, \quad \dim([5,1]) = \dim([2,1^4]) = 5, \quad \dim([4,2]) = \dim([2^2,1^2]) = 9
\]

\[
\dim([4,1^2]) = \dim([3,1^3]) = 10, \quad \dim([3,3]) = \dim([2^6]) = 5
\]

\[
\dim([321]) = 16 \text{ (self-conjugate)}
\]

we find that the first four couplings yield $15 \times 15$ matrices of cfp’s, the next 12 yield $45 \times 45$ matrices and in the last 4 the cfp matrices are $30 \times 30$. These matrices were chosen to be orthogonal. Thus the $6! = 720$-dimensional space has been reduced to the above subspaces. No further reduction in dimensions seems possible. There exist, of course, symmetry relations among some of these matrices, but in our formalism we did not bother to relate them this way. For lack of space we present only the $30 \times 30$ matrices of cfp’s in the case of the two simple one dimensional $[f_4] = [4]$ and $[f_4] = [1^4]$ in tables 21-24.
Table 21: The coefficients of fractional parentage with \([f_4] = [4]\) coupled to the symmetric pairs

\[(p, q) = (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6), (4, 5), (4, 6), (5, 6),\]

which label the rows. The resulting six particle symmetries, labeling the columns, are given in the order \([f_6] = [6], [5, 1], [4, 2], [2, 3]\) : \(i = 1, 2, \ldots, 5\), \(j = 1, 2, \ldots, 9\). The orbital-spin-color symmetry \([f] = [2^2, 1^2]\), encountered in the present calculation, does not arise this way.
Table 22: The coefficients of fractional parentage with \([f_4] = [4]\) with the antisymmetric pairs

\[(p, q) = (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6),
(4, 5), (4, 6), (5, 6),\]

which label the rows. The resulting six particle symmetries, labeling the columns,
are given in the order \([f_6] = [5, 1], [4, 1^2], \ldots; i = 1, 2, \ldots, 5, j = 1, 2, \ldots, 10.\) The orbital-spin-color symmetry \([f] = [2^2, 1^2],\) encountered in the present calculation, does not arise this way.
Table 23: The cfp’s with \([f_4] = [1^4]\) involving the antisymmetric pairs 
\[(p, q) = \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \]
\{(4, 5), (4, 6), (5, 6), \}
which label the rows. The resulting six particle symmetries, labeling the columns, are given in the order \([f_0] = [1^6], [2, 1^4]^i, [2^2, 1^2]^j; i = 1, 2, \ldots, 5, j = 1, 2, \ldots, 9.\)
Table 24: The cfp’s with $[f_4] = [1^4]$ involving the symmetric pairs
$(p,q) = (1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6),$
$(4,5), (4,6), (5,6),$
which label the rows. The resulting six particle symmetries, labeling the columns, are given in the order $[f_6] = [2,1^4], [3,1^3]_j : i = 1,2,..,5, j = 1,2,..,10$. The orbital-spin-color symmetry $[f] = [2^2,1^2]$, encountered in the present calculation, does not arise this way.

$\begin{pmatrix}
0 & 0 & 0 & \frac{1}{2} & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0
\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$