Driven single-band tight-binding dynamics, mesoscopic quantum circuits and realization of generalized para-fermionic polynomial algebras

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Abstract

The quantum dynamics of a driven single-band tight-binding model is considered. The relation between this Hamiltonian and Hamiltonian of the discrete-charge mesoscopic quantum circuits is investigated. It is shown that the former Hamiltonian, with closed boundary conditions, can be considered as a realization of the generalized para-fermionic polynomial algebras.

1 Introduction

The dynamics of a quantum particle in a periodic potential under the effect of an external field is one of the most fascinating phenomena of quantum physics [1,2]. Under this condition, the electronic wave function displays so called Bloch Oscillations (BO) [3,4], the amplitude of which is proportional to the band width. A suitable orthogonal basis for investigating structures with periodic potential is using the localized states. These states are also called the Wannier-Stark states and the nature of this states has a significant influence on the electronic transport properties of solids. The electronic BO was observed for the first time in semiconductor superlattices [5]. In 1992 the first experimental observation of BO was reported [5,6]. Recently, an increasing interest in the dynamics of BO can be observed [6-16].

An explicit time dependence of physical properties can appear under several conditions. The most obvious case is when the external field depends on time. In principle, one has to define an initial state |Ψ(t = 0)> and solve the time-dependent Schrödinger equation. This is a partial differential equation with at

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least two variables which is separable in some simple cases [17,18], but rarely
solved analytically. In particular for the tight-binding model, some analytical
expressions have been derived both for time-dependent and -independent fields
[19-22]. Recently in context of the tight-binding model, a treatment based on the
dynamical Lie algebra is proposed in [23-26]. The time-dependent Hamiltonian
which we will consider is defined by [10,24],
\[ \hat{H} = G \sum_j (|j\rangle\langle j+1| + |j+1\rangle\langle j|) + F(t) \sum_j j|j\rangle\langle j|, \] (1)

where the ket \(|j\rangle\) represents a Wannier state located on site \(j\).

In the present work we show that the Hamiltonian (1) with different boundary
conditions not only appears in discrete-charge mesoscopic quantum circuits
but also it can be a realization of the generalized para-fermionic polynomial
algebras.

Para-fermions of order \(p\) (with \(p\) being a positive integer) have been intro-
duced in [27-29]. The nature of this particles is such that at most \(p\) identical
particles can be found in the same state. The usual spin half fermions corre-
spond to \(p = 1\). The notion of para-fermionic algebra has been extended by
Quesne [30]. The relation between para-fermionic algebras and other algebras
has been given in [31-33]. The simplest para-fermionic polynomial algebra of
order \(p\) is defined by [27]
\[
\begin{align*}
[\hat{M}, \hat{B}] &= -\hat{B}, \\
[\hat{M}, \hat{B}^\dagger] &= \hat{B}^\dagger, \\
\hat{B}^{p+1} &= \hat{B}^{\dagger p+1} = 0, \\
\hat{B}^\dagger \hat{B} &= \hat{M}(p+1-\hat{M}) = [\hat{M}], \\
\hat{B} \hat{B}^\dagger &= (\hat{M} + 1)(p - \hat{M}) = [\hat{M} + 1], \\
\hat{M} &= \frac{1}{2}(\hat{B}^\dagger, \hat{B} + p).
\end{align*}
\] (2)

In the present work a realization of the generalized form of the para-fermionic
polynomial algebra (43) is given and in the special case where there are only
three sites \((N = 3)\) we show that the polynomial algebra (47) is recovered for
\(p = 2\).

The layout of this paper is as follows: In section 2, we introduce the model
and discuss in summary the discrete charge mesoscopic quantum circuits. In
section 3, we discuss the persistent current on a quantum ring as an example
in this approach. In section 4, periodic boundary condition is introduced. In
section 5, a realization of the generalized para-fermionic polynomial algebra is
investigated and the special cases \(N = 2, 3\) are discussed. Section 6, is for
conclusion remarks.
2 The model

The system under study is the derived quantum motion of a charged particle in a one-dimensional array of single-state quantum wells in tight-binding approximation and under the action of an arbitrary time-dependent external field. A simple one-electron tight-binding model is depicted in Fig.1.

![Figure 1: A simple single-band tight-binding model.](image)

The Hamiltonian of such a system can be written as:

$$ \hat{H} = G \sum_j (|j\rangle\langle j+1| + |j+1\rangle\langle j|) + F(t) \sum_j j|j\rangle\langle j|, $$

where the ket $|j\rangle$ represents a Wannier state located on site j. These states fulfill the orthogonality condition $\langle j|j'\rangle = \delta_{jj'}$. The force $F(t)$ is an arbitrary time-dependent force and the real parameter $G$ is the nearest-neighbors coupling strength. In the following we will investigate the Hamiltonian (3) and its applications in derived discrete-charge quantum circuits and specially we will show that this Hamiltonian can be considered as a realization of generalized para-fermionic polynomial algebras.

2.1 Infinite chain

As the first boundary condition let us assume that the Hamiltonian (1) describes an infinite chain of single-state quantum wells i.e., $-\infty < j < +\infty$. In this case the Hamiltonian (1) can be written as [24]

$$ \hat{H} = G(\hat{K} + \hat{K}^\dagger) + F(t)\hat{N}, $$

where

$$ \hat{K} = \sum_{j=-\infty}^{\infty} |j\rangle\langle j+1|, $$

$$ \hat{K}^\dagger = \sum_{j=-\infty}^{\infty} |j+1\rangle\langle j|, $$

$$ \hat{N} = \sum_{j=-\infty}^{\infty} j|j\rangle\langle j|. $$
Operators $K$ and $K^\dagger$ act as ladder operators on Wannier states

$$\hat{K}|j⟩ = |j - 1⟩, \quad \hat{K}^\dagger|j⟩ = |j + 1⟩,$$

and operator $\hat{N}$ acts on Wannier states as a position operator

$$\hat{N}|j⟩ = j|j⟩.$$  

These operators fulfill the following algebra

$$[\hat{N}, \hat{K}] = -\hat{K}, \quad [\hat{N}, \hat{K}^\dagger] = \hat{K}^\dagger, \quad [\hat{K}^\dagger, \hat{K}] = 0.$$  

A quantum theory for mesoscopic circuits was proposed by Li and Chen [34-36] where charge discreteness was considered explicitly. In this case the charge operator $\hat{q}$ takes on discrete eigenvalues when acting on the charge eigenvectors

$$\hat{q}|n⟩ = nq_e|n⟩,$$

where $n \in \mathbb{Z}$ (set of integers) and $q_e$ is the electron charge. Charge operator can be represented in terms of the localized states as

$$\hat{q} = q_e \sum_{j=-\infty}^{\infty} j|j⟩⟨j| = q_e \hat{N}.$$  

Discrete-charge Mesoscopic quantum circuits are described by the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2Lq_e^2}(\hat{Q} + \hat{Q}^\dagger - 2) + V(\hat{q}), \quad \hat{Q} = e^{\frac{i q_e}{\hbar} \hat{p}}, \quad [\hat{q}, \hat{p}] = i\hbar,$$

where $L$ is the inductance of the circuit. The operators $\hat{Q}$ and $\hat{Q}^\dagger$ are charge ladder operators which together with the charge operator $\hat{q}$ fulfill the following commutation relations

$$[\hat{q}, \hat{Q}] = -q_e\hat{Q}, \quad [\hat{q}, \hat{Q}^\dagger] = q_e\hat{Q}^\dagger, \quad [\hat{Q}^\dagger, \hat{Q}] = 0.$$
Therefore the operators \( \hat{N} = q^{-1} \hat{q}, \hat{Q} \) and \( \hat{Q}^\dagger \) satisfy the same algebra (8). When \( V(\hat{q}) = \frac{q^2}{2C} \), that is there is a capacitor in the circuit with capacity \( C \), we have a \( LC \)-design and for \( V(\hat{q}) = 0 \), we have a pure \( L \)-design. In the next section we will consider a \( L \)-design circuit and as an example we will find the persistent current on a quantum ring as a pure \( L \)-design in this approach.

3 \textit{L-design}

The time-dependent Hamiltonian for a \( L \)-design circuit in charge representation under the external potential \( \epsilon(t) \) is

\[
\hat{H} = -\frac{\hbar^2}{2Lq_c^2} (\hat{Q} + \hat{Q}^\dagger - 2) + \epsilon(t) \hat{q}.
\]

This Hamiltonian is in fact equivalent to the Hamiltonian (4) of a tight-binding system. In terms of the localized states we have

\[
\hat{Q} = \sum_{j=-\infty}^{\infty} |j\rangle\langle j + 1| = \hat{K}, \quad \hat{Q}^\dagger = \sum_{j=-\infty}^{\infty} |j + 1\rangle\langle j| = \hat{K}^\dagger.
\]

The current operator can be determined from the Heisenberg equation \( \hat{I} = -\frac{i}{\hbar}[\hat{H}, \hat{q}] \) as

\[
\hat{I} = \frac{\hbar}{2iLq_c} \sum_{j=-\infty}^{\infty} (|j\rangle\langle j + 1| - |j + 1\rangle\langle j|).
\]

The eigenvalues \( I_\theta \) and eigenvectors \( |I_\theta\rangle \) of the current operator are

\[
I_\theta = \frac{\hbar}{Lq_c} \sin(\theta), \quad |I_\theta\rangle = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{\infty} \exp(i\theta) |j\rangle,
\]

where \( \theta \in \mathbb{R} \) is a real continuous quantum number.

We can find the evolution operator \( \hat{U}(t) \) for the Hamiltonian (13) using the method introduced in [24]. For this purpose let us decompose the Hamiltonian (13) as

\[
\hat{H}_V = -\frac{\hbar^2}{2Lq_c^2} (\hat{Q} + \hat{Q}^\dagger - 2), \quad \hat{H}_W = \epsilon(t) \hat{q},
\]

then the time evolution operator can be written as a product
\[ \dot{U} = U_W \dot{U}_V, \]  

(18)

where

\[ \frac{i \hbar}{dt} \dot{U}_W = \hat{H}_W \hat{U}_W, \quad \frac{i \hbar}{dt} \dot{U}_V = (\hat{U}_W^{-1} \hat{H}_V \hat{U}_W) \hat{U}_V, \]

(19)

From equations (19) and using the Hamiltonian (13), the operator \( \hat{U}_W(t) \) can be found easily as

\[ \hat{U}_W(t) = \exp(-i f(t) \hat{q}), \quad f(t) = \int_0^t \frac{\epsilon(t')}{\hbar} dt'. \]

(20)

Direct calculation also shows that

\[ \hat{U}_W^{-1} \hat{H}_V \hat{U}_W = \frac{\hbar^2}{2Lq_c^2} \left( \exp(-i f(t) \hat{q}) \hat{Q} + \exp(i f(t) \hat{q}) \hat{Q}^\dagger - 2 \right). \]

(21)

The operator (21) commutes at different times such that the operator \( \hat{U}_V(t) \) can be obtained easily as

\[ \hat{U}_V(t) = \exp \left( \frac{i \hbar}{Lq_c^2} t \right) \exp \left( -i \frac{\hbar}{2Lq_c^2} (g \hat{Q} + g^* \hat{Q}^\dagger) \right), \]

(22)

where the function \( g(t) \) is defined by

\[ g(t) = \int_0^t \exp(-i f(t')dt'). \]

(23)

Therefore the time evolution operator \( \hat{U}(t) \) can be written as

\[ \frac{i \hbar}{Lq_c^2} \exp(-i \mu_0) \exp \left( \frac{i \hbar}{2Lq_c^2} (g \hat{Q} + g^* \hat{Q}^\dagger) \right). \]

(24)

Knowing the evolution operator we can find the current through the system easily. Let \( |I_\theta \rangle \) be an eigenvector of the current operator \( \hat{I} \) in \( t = 0 \), at \( t \neq 0 \) this state is given by

\[ |I_\theta, t \rangle = \hat{U}_W \hat{U}_V |I_\theta \rangle. \]

(25)

Substituting \( |I_\theta \rangle \) from (16) in (25) we find

\[ |I_\theta, t \rangle = \exp \left( \frac{i \hbar}{Lq_c^2} t \right) \exp \left( -i \frac{\hbar \mu_0}{2Lq_c^2} \right) \sum_j \exp(i j (\theta - f(t)q_c)) |j \rangle. \]

(26)

From (26) we can obtain the current \( I_\theta(t) \) as

\[ I_\theta(t) = \langle I_\theta, t | \hat{I} | I_\theta, t \rangle = \frac{\hbar}{Lq_c} \sin(\theta - f(t))q_c. \]

(27)
In the case of a quantum ring (a natural L-design) pierced by a time-dependent magnetic field, we have from Faraday's law $\epsilon(t) = -\frac{d\phi(t)}{dt}$, where $\phi(t)$ is the magnetic flux. In this case formula (27) gives the persistent current on the quantum ring. In the limiting case $q_e \to 0$ we have

$$I_{\theta}(t) = \frac{\hbar}{L}(\theta - f(t)).$$  (28)

It is remarkable to note that for a constant induced electric field or equivalently for a linearly increasing magnetic flux the current according to (28) linearly increases unlimitedly contrary to relation (27) which shows a periodic behavior with frequency $\omega = \frac{d f(t)}{dt}$.

## 4 Periodic boundary condition

We can also take the periodic boundary condition $|n + N\rangle = |n\rangle$, in this case the Hamiltonian (1) and the corresponding operators are defined as

$$\hat{H} = G \sum_{j=1}^{N} (|j\rangle\langle j + 1| + |j + 1\rangle\langle j|) + F(t) \sum_{j=1}^{N} j|j\rangle\langle j|, $$  \hspace{1cm} (29)

$$\hat{K} = \sum_{j=1}^{N} |j\rangle\langle j + 1|, $$

$$\hat{K}^\dagger = \sum_{j=1}^{N} |j + 1\rangle\langle j|, $$

$$\hat{N} = \sum_{j=1}^{N} j|j\rangle\langle j|. $$  \hspace{1cm} (30)

The commutation relations are as before and we can obtain the evolution operator following the same strategy as

$$\hat{U}(t) = \hat{U}_Q(t)\hat{U}_P(t), $$  \hspace{1cm} (31)

where

$$\hat{U}_Q(t) = \exp(-i\eta(t)\hat{N}), $$

$$\hat{U}_P(t) = \exp(-i(\xi\hat{K} + \xi^*\hat{K}^\dagger)), $$

$$\eta(t) = \int_0^t \frac{F(t')}{\hbar}dt', $$

$$\xi(t) = \int_0^t \exp(-i\eta(t'))dt'. $$  \hspace{1cm} (32)
knowing the evolution operator one can calculate the transition and survival probabilities between the localized states.

5 Realization of generalized para-fermionic polynomial algebras

In this section we take a closed boundary condition and show that the Hamiltonian (3) with this new boundary condition turns out to be a realization of generalized para-fermionic polynomial algebras. By closed boundary condition we mean that the particle can not escape from the ends of the chain i.e., in site \( j = 1 \) the particle tunnels only to the nearest site \( j + 1 \) and in site \( j = N \) the particle can only tunnel to the site \( j = N - 1 \). This situation can be met for example in one-dimensional ion traps or optical lattices. In this case the Hamiltonian and the related operators are

\[
\hat{H} = G \sum_{j=1}^{N-1} (|j\rangle\langle j+1| + |j+1\rangle\langle j|) + F(t) \sum_{j=1}^{N} j|j\rangle\langle j|, \\
\hat{K} = \sum_{j=1}^{N-1} |j\rangle\langle j+1|, \\
\hat{K}^\dagger = \sum_{j=1}^{N-1} |j+1\rangle\langle j|, \\
\hat{N} = \sum_{j=1}^{N} j|j\rangle\langle j|. 
\] (33)

The Hamiltonian in terms of the generators can be rewritten as

\[
\hat{H}(t) = G(\hat{K} + \hat{K}^\dagger) + F(t)\hat{N}, 
\] (34)

and the new commutation relations are now

\[
[\hat{N}, \hat{K}] = -\hat{K}, \\
[\hat{N}, \hat{K}^\dagger] = \hat{K}^\dagger, \\
[\hat{K}^\dagger, \hat{K}] = |N\rangle\langle N| - |1\rangle\langle 1|. 
\] (35)

From (33) it is clear that the operators \( \hat{K} \) and \( \hat{K}^\dagger \) are nilpotent of order \( N \) that is

\[
\hat{K}^N = \hat{K}^\dagger^N = 0. 
\] (36)
The last commutation relation in (35) can be expressed as a polynomial in terms of the operator $\hat{N}$. This polynomial is not necessarily unique and we can choose a polynomial with the smallest degree. For this purpose let the unknown polynomial $f(\hat{N})$ be

$$f(\hat{N}) = |N\rangle\langle N| - |1\rangle\langle 1|.$$  

(37)

The form of $f(\hat{N})$ can be determined from (37) using the following conditions

$$f(\hat{N})|n\rangle = 0, \quad 2 \leq n \leq N - 1,$$

$$f(\hat{N})|N\rangle = |N\rangle,$$

$$f(\hat{N})|1\rangle = -|1\rangle.$$  

(38)

After some simple algebra one finds

$$f(\hat{N}) = \frac{1}{(N-2)!} \prod_{j=2}^{N-1} (\hat{N} - j), \quad \text{for odd } N,$$

$$f(\hat{N}) = \frac{1}{(N-1)!} (2\hat{N} - (N + 1)) \prod_{j=2}^{N-1} (\hat{N} - j), \quad \text{for even } N.$$  

(39)

The Casimir operator of this algebra can be written as,

$$\hat{C} = \hat{K}^\dagger \hat{K} + g(\hat{N}).$$  

(40)

Where $g(\hat{N})$ is a polynomial in $\hat{N}$ defined by

$$f(\hat{N}) = g(\hat{N} + 1) - g(\hat{N}).$$  

(41)

From (41) one can find $g(\hat{N})$ up to a constant as

$$g(\hat{N}) = \left(\frac{2}{N!}\right) \hat{N} - \frac{N + 2}{N!} \prod_{j=2}^{N} (\hat{N} - j), \quad \text{for even } N,$$

$$g(\hat{N}) = \frac{1}{(N-1)!} \prod_{j=2}^{N} (\hat{N} - j), \quad \text{for odd } N.$$  

(42)

Gathering the related commutation relations we find the following polynomial algebra
\[ [\hat{N}, \hat{K}^\dagger] = \hat{K}^\dagger, \]
\[ [\hat{N}, \hat{K}] = -\hat{K}, \]
\[ [\hat{K}^\dagger, \hat{K}] = f(\hat{N}) = g(\hat{N} + 1) - g(\hat{N}), \]
\[ [\hat{K}^\dagger \hat{K}, \hat{K}^\dagger] = 0, \]
\[ \hat{K}^N = \hat{K}^{1N} = 0, \]
\[ \hat{K}^\dagger \hat{K} = 1 - g(\hat{N}) =: \phi(\hat{N}), \]
\[ \hat{K} \hat{K}^\dagger = 1 - g(\hat{N} + 1) =: \phi(\hat{N} + 1), \]

(43)

where

\[ \phi(1) = \phi(N + 1) = 0. \]

(44)

It is remarkable to note the similarity between this algebra and the deformed algebras [37,38]. Recently the algebras of the operators of single spinor with a fixed spin value \( j = \frac{p}{2} \) have been mapped onto polynomial algebras, which constitute a quite recent subject of investigations in physics. The \( j = \frac{p}{2} \) spinor algebra \( S = \{\hat{S}_\pm, \hat{S}_0\} \) is mapped onto the para-fermionic algebra \( L = \{\hat{B}, \hat{B}^\dagger, \hat{M}\} \), which is a polynomial algebra given by the commutation relations [27],

\[ [\hat{M}, \hat{B}] = -\hat{B}, \]
\[ [\hat{M}, \hat{B}^\dagger] = \hat{B}^\dagger, \]
\[ \hat{B}^{p+1} = \hat{B}^{1p+1} = 0, \]
\[ \hat{B}^\dagger \hat{B} = \hat{M}(p + 1 - \hat{M}) = [\hat{M}], \]
\[ \hat{B} \hat{B}^\dagger = (\hat{M} + 1)(p - \hat{M}) = [\hat{M} + 1], \]
\[ \hat{M} = \frac{1}{2}(\hat{B}^\dagger \hat{B} + p) \]

(45)

Now let us consider the special \( (N = 3) \), if we make the following redefinitions

\[ \hat{B} = \sqrt{2}\hat{K}, \]
\[ \hat{B}^\dagger = \sqrt{2}\hat{K}^\dagger, \]
\[ \hat{M} = \hat{N} - 1, \]

(46)

we find
which clearly corresponds to \( j = 1 \) \((p = 2)\), spinor algebra.

For \( N \geq 4 \) we will have a generalized para-fermionic algebra as can be seen from (43).

Therefore the Hamiltonian (33) with the closed boundary conditions can be considered as a realization of the generalized para-fermionic polynomial algebras.

5.1 Special cases \( N=2, N=3 \)

Setting \( N = 2 \) in equations (42) we have

\begin{align*}
[\hat{N}, \hat{K}] &= -\hat{K}, \\
[\hat{N}, \hat{K}^\dagger] &= \hat{K}^\dagger, \\
[\hat{K}^\dagger, \hat{K}] &= 2\hat{N} - 3, \\
\hat{K}^2 &= \hat{K}^{12} = 0, \\
\hat{N} &= 1 + \hat{K}^\dagger \hat{K}.
\end{align*}

(48)

From \( \{\hat{K}, \hat{K}^\dagger\} = I \) and \( \hat{K}^2 = \hat{K}^{12} = 0 \), it is clear that the set \( \{\hat{K}, \hat{K}^\dagger, I\} \) is a realization of the usual spin-half fermionic algebra. By defining

\begin{align*}
\hat{J}_0 &= \hat{N} + \frac{3}{2}, \\
\hat{J}_+ &= \hat{K}^\dagger, \\
\hat{J}_- &= \hat{K},
\end{align*}

(49)

we find
\[ [\hat{J}_0, \hat{J}_\pm] = \pm \hat{J}_\pm, \quad [\hat{J}_+, \hat{J}_-] = 2\hat{J}_0, \quad (50) \]

which is the usual \( su(2) \) algebra. In terms of \( \hat{J}_\pm \) and \( \hat{J}_0 \), the Hamiltonian (33) can be rewritten as

\[ \hat{H}(t) = G(\hat{J}_+ + \hat{J}_-) + F(t)(\hat{J}_0 - \frac{3}{2}). \quad (51) \]

For this Hamiltonian the Casimir operator \( \hat{J}^2 = \hat{J}_0^2 + \frac{1}{2}(\hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+) \) is a constant of motion.

Setting \( N = 3 \) in equations (42) we have

\[
\begin{align*}
[\hat{N}, \hat{K}] &= -\hat{K}, \\
[\hat{N}, \hat{K}^\dagger] &= \hat{K}^\dagger, \\
[\hat{K}^\dagger, \hat{K}] &= \hat{N} - 2, \\
\hat{K}^3 &= \hat{K}^{13} = 0, \\
\hat{N} &= 1 + \hat{K}^\dagger \hat{K} + \hat{K}^{12} \hat{K}^2, 
\end{align*}
\]

Again we can define the \( su(2) \) generators as

\[
\begin{align*}
\hat{J}_+ &= \sqrt{2} \hat{K}^\dagger, \\
\hat{J}_- &= \sqrt{2} \hat{K}, \\
\hat{J}_0 &= \hat{N} - 2 
\end{align*}
\]

The Hamiltonian in terms of this generators becomes

\[ \hat{H}(t) = \frac{G}{\sqrt{2}}(\hat{J}_+ + \hat{J}_-) + F(t)(\hat{J}_0 - 2). \quad (54) \]

6 concluding remarks

The similarity between a driven single-state tight-binding Hamiltonian and the Hamiltonian of discrete-charge mesoscopic quantum circuits is explicitly shown. Using this similarity the persistent current on a quantum ring is obtained and discussed. It is shown that the former Hamiltonian can be considered as a realization of the generalized para-fermionic polynomial algebras.
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