UNIVERSALITY AT THE EDGE OF THE SPECTRUM FOR
UNITARY, ORTHOGONAL AND SYMPLECTIC ENSEMBLES OF
RANDOM MATRICES

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Abstract. We prove universality at the edge of the spectrum for unitary ($\beta = 2$), orthogonal ($\beta = 1$) and symplectic ($\beta = 4$) ensembles of random matrices in the scaling limit for a class of weights $w(x) = e^{-V(x)}$ where $V$ is a polynomial, $V(x) = \kappa_{2m} x^{2m} + \cdots$, $\kappa_{2m} > 0$. The precise statement of our results is given in Theorem 1.1 and Corollaries 1.2, 1.3 below. For a proof of universality in the bulk of the spectrum, for the same class of weights, for unitary ensembles see [DKMVZ2], and for orthogonal and symplectic ensembles see [DG].

Our starting point in the unitary case is [DKMVZ2], and for the orthogonal and symplectic cases we rely on our recent work [DG], which in turn depends on the earlier work of Widom [W] and Tracy and Widom [TW2]. As in [DG], the uniform Plancherel–Rotach type asymptotics for the orthogonal polynomials found in [DKMVZ2] plays a central role.

The formulae in [W] express the correlation kernels for $\beta = 1$ and 4 as a sum of a Christoffel–Darboux (CD) term, as in the case $\beta = 2$, together with a correction term. In the bulk scaling limit [DG], the correction term is of lower order and does not contribute to the limiting form of the correlation kernel.

By contrast, in the edge scaling limit considered here, the CD term and the correction term contribute to the same order: this leads to additional technical difficulties over and above [DG].

1. Introduction

This paper is a continuation of [DG]. In [DG], the authors proved universality in the bulk for orthogonal and symplectic ensembles: here we prove universality at the edge for orthogonal and symplectic ensembles, and also for unitary ensembles.

For the convenience of the reader, and to fix notation, we now summarize some of the basic theory of invariant ensembles ($\beta = 1, 2$ or 4), borrowing freely and extensively from the introduction in [DG]. We are concerned with ensembles of matrices $\{M\}$ with probability distributions

$$P_{N,\beta}(M) dM = \frac{1}{Z_{N,\beta}} e^{-\text{tr} V_\beta(M)} dM,$$

for $\beta = 1, 2$ and 4, the so-called Orthogonal, Unitary and Symplectic ensembles, respectively (see [Me]). For $\beta = 1, 2, 4$, the ensemble consists of $N \times N$ real symmetric matrices, $N \times N$ Hermitian matrices, and $2N \times 2N$ Hermitian self-dual matrices, respectively. In general the potential $V_\beta(x)$ is a real-valued function growing sufficiently rapidly as $|x| \to \infty$, but we will restrict our attention henceforth to $V_\beta$'s which are polynomials,

$$V_\beta(x) = \kappa_{2m,\beta} x^{2m} + \cdots, \quad \kappa_{2m,\beta} > 0.$$
In \( \mathbb{E} \), \( dM \) denotes Lebesgue measure on the algebraically independent entries of \( M \), and \( Z_{N,\beta} \) is a normalization constant. The above terminology for \( \beta = 1, 2 \) and \( 4 \) reflects the fact that \( \mathbb{E} \) is invariant under conjugation of \( M \), \( M \mapsto UMU^{-1} \), by orthogonal, unitary and unitary-symplectic matrices \( U \). It follows from \( \mathbb{E} \) that the distribution of the eigenvalues \( x_1, \cdots, x_N \) of \( M \) is given (see \( \mathbb{M} \)) by

\[
P_{N,\beta}(x_1, \cdots, x_N) = \frac{1}{Z_{N,\beta}} \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta \prod_{j=1}^N w_\beta(x_j)
\]

where again \( Z_{N,\beta} \) is a normalization constant (partition function). Here

\[
w_\beta(x) = \begin{cases} e^{-V_\beta(x)}, & \beta = 1, 2 \\
 e^{-2V_\beta(x)}, & \beta = 4.
\end{cases}
\]

(The factor 2 in \( w_\beta=4 \) reflects the fact that the eigenvalues of self-dual Hermitian matrices come in pairs.) Let \( \{p_j\}_{j \geq 0} \) be the normalized orthogonal polynomials (OP’s) on \( \mathbb{R} \) with respect to the weight \( w \equiv w_{\beta=2} \), and define \( \phi_j \equiv p_j w^{1/2} \). Note that \( \langle \phi_j, \phi_k \rangle = \delta_{jk} \) where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( L^2(\mathbb{R}) \).

For the unitary matrix ensembles an important role is played by the Christoffel–Darboux (CD) kernel

\[
K_N(x, y) \equiv K_{N,2}(x, y) = \sum_{k=0}^{N-1} \phi_k(x) \phi_k(y).
\]

In particular the probability density \( \mathbb{E} \), the \( l \)-point correlation function \( R_{N,l,2} \) and also the gap probability \( E_2(0; J) \) that a set \( J \) contains no eigenvalues, can all be expressed in terms of \( K_N \), see e.g. \( \mathbb{M} \). For example

\[
R_{N,l,2}(x_1, \cdots, x_l) = \det(K_N(x_j, x_k))_{1 \leq j, k \leq l}.
\]

The Universality Conjecture, in our situation, states that the limiting statistical behavior of the eigenvalues \( x_1, \cdots, x_N \) distributed according to the law \( \mathbb{E} \), in the appropriate scale as \( N \to \infty \), should be independent of the weight \( w_\beta \), and should depend only on the invariance properties of \( \mathbb{P}_{N,\beta} \), \( \beta = 1, 2 \) or \( 4 \), mentioned above. Universality has been considered extensively in the physics literature, see e.g. \( \mathbb{D} \) \( \mathbb{E} \) \( \mathbb{H} \) \( \mathbb{W} \) \( \mathbb{S} \) \( \mathbb{V} \).

The kernel \( K_N(x, y) \) can also be expressed via the Christoffel–Darboux formula

\[
K_N(x, y) = b_{N-1} \frac{\phi_N(x)\phi_{N-1}(y) - \phi_N(y)\phi_{N-1}(x)}{x-y},
\]

where \( b_{N-1} \) is a coefficient in the three-term recurrence relation for OP’s, see \( \mathbb{S} \). In view of the preceding remarks it follows that in the case \( \beta = 2 \), the study of the large \( N \) behavior of \( P_{N,2} \), and in particular the proof of universality, reduces to the asymptotic analysis of \( b_{N-1} \) and the OP’s \( p_{N+j} \) with \( j = 0 \) or \( -1 \). By a fundamental observation of Fokas, Its and Kitaev [FoKi] the OP’s solve a Riemann–Hilbert problem (RHP) of a type that is amenable to the steepest descent method introduced by Deift and Zhou in \( \mathbb{D} \) \( \mathbb{Z} \) and further developed in \( \mathbb{D} \) \( \mathbb{V} \) \( \mathbb{Z} \). In \( \mathbb{D} \) \( \mathbb{K} \) \( \mathbb{M} \) \( \mathbb{V} \) \( \mathbb{Z} \) the authors analyzed the asymptotics of OP’s for very general classes of weights. In particular they proved the Universality Conjecture in the bulk in the case \( \beta = 2 \) for weights \( w(x) = e^{-V(x)} \) where \( V(x) \) is a polynomial as above, and also for \( w(x) = e^{-NV(x)} \) where \( V(x) \) is real analytic and \( V(x)/\log |x| \to +\infty \), as \( |x| \to \infty \).
The bulk scaling limit as $N \to \infty$ is described in terms of the so-called *sine kernel* $K_\infty(x-y)$ where

\begin{equation}
K_\infty(t) \equiv \frac{\sin \pi t}{\pi t}.
\end{equation}

For example \cite[Theorem 1.4]{DKMVZ}, for $w(x) = e^{-V(x)}$, $V(x)$ polynomial, and for any $l = 2, 3, \cdots$ and $r, y_1, \cdots, y_l$ in a compact set, one has as $N \to \infty$

\begin{equation}
\frac{1}{(K_N(0,0))^l} R_{N,l,2}(r + \frac{y_1}{K_N(0,0)}, \cdots, r + \frac{y_l}{K_N(0,0)}) \to \det(K_\infty(y_j - y_k))_{1 \leq j,k \leq l}.
\end{equation}

The scale $x = y/K_N(0,0)$ is chosen so that the expected number of eigenvalues per unit $y$-interval is one. This scaling in the bulk is standard in Random Matrix Theory. Indeed for any Borel set $B \subset \mathbb{R}$,

\begin{equation}
\int_B R_{N,l=1,2}(x) \, dx = \mathbb{E}\{\text{number of eigenvalues in } B\}.
\end{equation}

Thus by (1.6) $K_N(0,0) = R_{N,1,2}(0)$ gives the density of the expected number of eigenvalues near zero. From (1.9), we see that, in the appropriate scale, the large $N$ behavior of the eigenvalues is *universal* (i.e. independent of $V$). Pioneering mathematical work on the Universality Conjecture in the bulk was done in \cite{PS} and for the case of quartic two-interval potential $V(x) = N(x^4 - tx^2)$, $t > 0$ (sufficiently) large, in \cite{BI}. We note again that all these results apply only in the case $\beta = 2$.

In the case $\beta = 1$ and $4$ the situation is more complicated. In place of (1.6) one must use $2 \times 2$ matrix kernels (see e.g. \cite{BI, TW2})

\begin{equation}
K_{N,1}(x,y) = \begin{pmatrix}
S_{N,1}(x,y) & (S_{N,1}D)(x,y) \\
(\epsilon S_{N,1})(x,y) & S_{N,1}(y,x)
\end{pmatrix}, \quad N \text{ even},
\end{equation}

and

\begin{equation}
K_{N,4}(x,y) = \frac{1}{2} \begin{pmatrix}
S_{N,4}(x,y) & (S_{N,4}D)(x,y) \\
(\epsilon S_{N,4})(x,y) & S_{N,4}(y,x)
\end{pmatrix}.
\end{equation}

Here $S_{N,\beta}(x,y)$, $\beta = 1, 4$, are certain scalar kernels (see (1.17), (1.18) below), $D$ denotes the differentiation operator, and $\epsilon$ is the operator with kernel $\epsilon(x,y) = \frac{1}{2} \text{sgn}(x-y)^1$. Such matrix kernels were first introduced by Dyson \cite{Dy} in the context of circular ensembles with a view to computing correlation functions. Dyson’s approach was extended to Hermitian ensembles, first by Mehta \cite{M2} for $V(x) = x^2$, and then for more general weights by Mahoux and Mehta in \cite{MaM}. A more direct and unifying approach to the results of Dyson–Mahoux–Mehta was given by Tracy and Widom in \cite{TW}, where formulae (1.17), (1.18) below were derived. We see that once the kernels $S_{N,\beta}(x,y)$ are known, then so are the other kernels in $K_{N,\beta}$.

As in the case $\beta = 2$, the kernels $K_{N,\beta}$ give rise to explicit formulae for $R_{N,l,\beta}$ and $E_\beta(0; J)$. For example for $\beta = 1, 4$

\begin{equation}
R_{N,1,\beta}(x) \equiv R_{1,\beta}(x) = \frac{1}{2} \text{tr} K_{N,\beta}(x,x)
\end{equation}

\footnote{We use the standard notation $\text{sgn} x = 1$, $0$, $-1$ for $x > 0$, $x = 0$, $x < 0$, respectively.}
and
\[ R_{N,2,\beta}(x, y) = \frac{1}{4} \left( \operatorname{tr} K_{N,\beta}(x, x) \right) \left( \operatorname{tr} K_{N,\beta}(y, y) \right) - \frac{1}{2} \operatorname{tr} \left( K_{N,\beta}(x, y) K_{N,\beta}(y, x) \right), \]
and so on, see [TW2]. We will discuss some of the literature on edge scaling after the statement of our results, Theorem 1.1, Corollary 1.2 and 1.3 below. As indicated above, formula (1.11) only applies to the case when \( N \) is even. When \( N \) is odd, there is a similar, but slightly more complicated, formula (see [AFNvM]). As in [DG], throughout this paper, for \( \beta = 1 \), we will restrict our attention to the case when \( N \) is even. We expect that the methods in this paper also extend to the case \( \beta = 1 \), \( N \) odd, and we plan to consider this situation in a later publication.

Let \( \{q_j(x)\}_{j \geq 0} \) be any sequence of polynomials of exact degree \( j \), \( q_j(x) = q_{j,1}x^j + \cdots, q_{j,1} \neq 0 \). For \( j = 0, 1, 2, \cdots \), set
\[
\psi_{j,\beta}(x) = \begin{cases} 
q_j(x)w_1(x), & \beta = 1 \\
q_j(x)(w_4(x))^{1/2}, & \beta = 4.
\end{cases}
\]

Let \( M_{N,1} \) denote the \( N \times N \) matrix with entries
\[
(M_{N,1})_{jk} = (\psi_{j,1}, \psi_{k,1}), \quad 0 \leq j, k \leq N - 1,
\]
and let \( M_{N,4} \) denote the \( 2N \times 2N \) matrix with entries
\[
(M_{N,4})_{jk} = (\psi_{j,4}, D\psi_{k,4}), \quad 0 \leq j, k \leq 2N - 1,
\]
where again \((\cdot, \cdot)\) denotes the standard real inner product on \( \mathbb{R} \). The matrices \( M_{N,1} \) and \( M_{N,4} \) are invertible (see e.g. [AVM] (4.17), (4.20)). Let \( \mu_{N,1}, \mu_{N,4} \) denote the inverses of \( M_{N,1}, M_{N,4} \) respectively. With these notations we have [TW2] the following formulae for \( S_{N,\beta} \) in (1.11), (1.12)
\[
S_{N,1}(x, y) = -\sum_{j,k=0}^{N-1} \psi_{j,1}(x) (\mu_{N,1})_{jk} (\epsilon \psi_{k,1})(y)
\]
\[
S_{N,4}(x, y) = \sum_{j,k=0}^{2N-1} \psi_{j,4}(x) (\mu_{N,4})_{jk} \psi_{k,4}(y).
\]

An essential feature of the above formulae is that the polynomials \( \{q_j\} \) are arbitrary and we are free to choose them conveniently to facilitate the asymptotic analysis of (1.11), (1.12) as \( N \to \infty \) (see discussion in [DG] and [TW2] below).

In order to state our main result we need more notation. For any \( m \in \mathbb{N} \) let \( V(x) \) be a polynomial of degree \( 2m \)
\[
V(x) = \kappa_{2m}x^{2m} + \cdots, \quad \kappa_{2m} > 0
\]
and let \( w(x) \equiv w_{\beta=2}(x) = e^{-V(x)} \) as before. Let \( p_j(x), j \geq 0 \), denote the OP’s with respect to \( w \), and set \( \phi_j(x) \equiv p_j(x)(w(x))^{1/2}, j \geq 0 \), as above. For \( \beta = 1, 4 \) set
\[
V_{\beta}(x) \equiv \frac{1}{2} V(x)
\]
and let \( N \) be even. Then by (1.14), \( w_4 = e^{-2V_4} = e^{-V} \) and \( w_1 = e^{-V_1} = e^{-V/2} \). This ensures that for the choice \( q_j = p_j \) in (1.14)
\[
\psi_{j, \beta=1}(x) = \psi_{j, \beta=4}(x) = \phi_j(x),
\]
which enables us in turn to handle \( S_{N,1} \) and \( S_{N/2,4} \) in (1.17), (1.18) simultaneously (see [DG] Remark 1.3)). Henceforth and throughout the paper, \( K_N \) denotes the Christoffel–Darboux (CD) kernel (1.5), (1.7) constructed out of these functions \( \phi_j \).

For the bulk scaling limit in [DKMVZ1] \( (\beta = 2) \) and [DG] \( (\beta = 1, 4) \), the authors used the standard scale of one (expected) eigenvalue per unit interval. At the edge it is standard (see e.g. [IW3]) to use a slightly different scaling which ensures that the kernel \( K_{\text{Airy}}(\xi, \eta) \) (see (1.25) below) appears in the limiting forms (1.26), (1.27), (1.28) below, without any additional factors. Note that formula (1.10) also holds for \( \beta = 1, 4 \) and so \( R_{N,l=1,\beta}(x) \) gives the density of the expected number of (simple) eigenvalues near \( x \) for \( \beta = 1, 2, 4 \). In view of (1.10), and also in view of (1.11) and (1.12),
\[
R_{N,1,2}(x) = K_N(x, x), \quad R_{N,1,1}(x) = S_{N,1}(x, x), \quad R_{N/2,1,4}(x) = \frac{1}{2} S_{N/2,4}(x, x).
\]

To leading order, the right edge of the spectrum is located at the point \( c_N + d_N \) where \( c_N, d_N \) are the Mhaskar–Rakhmanov–Saff numbers in (3.1), (3.2) below. For all three cases, in the neighborhood of \( c_N + d_N \), we use the scale
\[
\xi \mapsto \xi^{(N)} \equiv c_N \left( 1 + \frac{\xi}{\alpha_N N^{2/3}} + d_N \right)
\]
where \( \alpha_N \) is given in (3.10) (2) below. As we will see (cf. Remark 1.3 below) this scaling differs slightly from a scale of one (expected) eigenvalue per unit interval.

It turns out that the off-diagonal elements in \( K_{N,\beta} \) scale differently as \( N \to \infty \). On the other hand, the statistics of the ensembles are clearly invariant (cf. discussion following (2.8) below) under the conjugation
\[
K_{N,\beta} \mapsto K_{N,\beta}^{(\lambda)} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \cdot K_{N,\beta} \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \begin{pmatrix} (K_{N,\beta})_{11} & \lambda^{-2} (K_{N,\beta})_{12} \\ \lambda^2 (K_{N,\beta})_{21} & (K_{N,\beta})_{22} \end{pmatrix}
\]
for any scalar \( \lambda \). For example, this is obviously true for the cluster functions \( T_{N,1,\beta} \), \( \beta = 1 \) or 4, which have the form
\[
T_{N,1,\beta}(y_1, \ldots, y_l) = \frac{1}{2l} \sum_{\sigma} \text{tr} \left( K_{N,\beta}(y_{\sigma_1}, y_{\sigma_2}) K_{N,\beta}(y_{\sigma_2}, y_{\sigma_3}) \cdots K_{N,\beta}(y_{\sigma_1}, y_{\sigma_1}) \right)
\]
where the sum is taken over all permutations of \( \{1, \ldots, l\} \) (see [TW2] p. 816), etc.

Denote
\[
K_{\text{Airy}}(\xi, \eta) = \frac{\text{Ai}'(\xi) \text{Ai}'(\eta) - \text{Ai}'(\xi) \text{Ai}(\eta)}{\xi - \eta} = \int_0^\infty \text{Ai}(z + \xi) \text{Ai}(z + \eta) \, dz.
\]

Set
\[
\lambda_{(N)} = \left( \frac{c_N}{\alpha_N N^{2/3}} \right)^{-1/2}.
\]

Theorem 1.1 and Corollary 1.2 and 1.3 below are the main results in this paper.
Theorem 1.1. Let $\beta = 2, 1$ or $4$. For any $V(x)$ of degree $2m$ as in \cite{DGKV} define $V_2(x)$ and $w_2(x)$ as in \cite{DG,DGKV}. Fix a number $L_0$. Then there exists $c = c(L_0) > 0$ such that as $N \to \infty^2$ the following holds uniformly for $\xi, \eta \in [L_0, +\infty)$.

In the case $\beta = 2$:

$$E_{N,2} = \frac{1}{\lambda^2_{(N)}} K_N(\xi^{(N)}, \eta^{(N)}) - K_{Airy}(\xi, \eta) \to 0. \tag{1.26}$$

In the case $\beta = 1$:

$$E_{N,1} = \frac{1}{\lambda^2_{(N)}} K_N^{(1)}(\xi^{(N)}, \eta^{(N)}) - K^{(1)}(\xi, \eta) \to 0 \tag{1.27}$$

where

$$(K^{(1)})_{11}(\xi, \eta) = (K^{(1)})_{22}(\eta, \xi) \equiv K_{Airy}(\xi, \eta) + \frac{1}{2} \Ai(\xi) \cdot \int_{-\infty}^{\eta} \Ai(t) \, dt$$

$$(K^{(1)})_{12}(\xi, \eta) \equiv -\partial_{\eta} K_{Airy}(\xi, \eta) - \frac{1}{2} \Ai(\xi) \Ai(\eta)$$

$$(K^{(1)})_{21}(\xi, \eta) \equiv -\int_{\xi}^{\eta} K_{Airy}(t, \eta) \, dt - \frac{1}{2} \int_{\xi}^{\eta} \Ai(t) \, dt + \frac{1}{2} \int_{\xi}^{\eta} \Ai(t) \, dt \cdot \int_{\eta}^{\infty} \Ai(t) \, dt - \frac{1}{2} \sgn(\xi - \eta).$$

In the case $\beta = 4$:

$$E_{N,4} = \frac{1}{\lambda^2_{(N)}} K_N^{(4)}(\xi^{(N)}, \eta^{(N)}) - K^{(4)}(\xi, \eta) \to 0 \tag{1.28}$$

where

$$2(K^{(4)})_{11}(\xi, \eta) = 2(K^{(4)})_{22}(\eta, \xi) \equiv K_{Airy}(\xi, \eta) - \frac{1}{2} \Ai(\xi) \cdot \int_{\eta}^{\infty} \Ai(t) \, dt$$

$$2(K^{(4)})_{12}(\xi, \eta) \equiv -\partial_{\eta} K_{Airy}(\xi, \eta) - \frac{1}{2} \Ai(\xi) \Ai(\eta)$$

$$2(K^{(4)})_{21}(\xi, \eta) \equiv -\int_{\xi}^{\infty} K_{Airy}(t, \eta) \, dt + \frac{1}{2} \int_{\xi}^{\infty} \Ai(t) \, dt \cdot \int_{\eta}^{\infty} \Ai(t) \, dt.$$ 

For the error term we have as $N \to \infty$,

$$E_{N,2} = O(N^{-2/3})e^{-c\xi}e^{-c\eta}$$

$$E_{N,1} = o(1)\begin{pmatrix} e^{-c\xi} & e^{-c\xi}e^{-c\eta} \\ e^{-c\eta} & \end{pmatrix}$$

$$E_{N,4} = o(1)e^{-c\xi}e^{-c\eta}$$

uniformly for $\xi, \eta \in [L_0, +\infty)$. 

Remark 1.1. For $\beta = 4$, but not for $\beta = 1$, our methods actually prove that $E_{N,4} = O(N^{-1/(2m)})e^{-c\xi}e^{-c\eta}$. In order to obtain power law decay for $E_{N,1}$, it would be sufficient to obtain power law decay in the error term in \cite{DG, DGKV} Theorem 2.2]: such power law decay can be obtained using more sophisticated estimates as in \cite{DGKV}.

Footnote: For $\beta = 1, 4$, $N$ is even.
We immediately have the following result. Recall formula \([1.24]\) for the cluster functions for \(\beta = 1, 4\); for \(\beta = 2\), the cluster functions have the form \([1.25]\) p. 815]

\[
T_{N,l,2}(y_1, \cdots, y_l) = \frac{1}{l} \sum_{\sigma} K_N(y_{\sigma_1}, y_{\sigma_2})K_N(y_{\sigma_2}, y_{\sigma_3}) \cdots K_N(y_{\sigma_l}, y_{\sigma_1}).
\]

**Corollary 1.2.** Let \(\beta = 2, 1\) or 4. Let \(V\) be a polynomial of degree \(2m\) and let \(K^{(\beta)}\), \(\beta = 1, 4\) be as in Theorem \([1.1]\). Fix a number \(L_0\). Then for \(\beta = 1\) and \(l = 2, 3, \cdots\) we have uniformly for \(\xi_1, \cdots, \xi_l \geq L_0\)

\[
\lim_{N \to \infty} \frac{1}{(\lambda_2(N))^l} T_{N,l,1}(\xi_1^{(N)}, \cdots, \xi_l^{(N)}) = \frac{1}{2l^2} \sum_{\sigma} \text{tr} \left( K^{(1)}(\xi_{\sigma_1}, \xi_{\sigma_2})K^{(1)}(\xi_{\sigma_2}, \xi_{\sigma_3}) \cdots K^{(1)}(\xi_{\sigma_l}, \xi_{\sigma_1}) \right).
\]

For \(\beta = 4\), the same result is true provided we replace \(T_{N,l,1} \to T_{N/2,l,4}\) and \(K^{(1)} \to K^{(4)}\). For \(\beta = 2\), the same result is true provided we replace \(T_{N,l,1} \to T_{N,l,2}\), \(K^{(1)} \to K_{\text{Airy}}\), \(\frac{1}{2l^2} \to \frac{1}{l}\), and remove the trace.

Together with some additional estimates (see Section \([2]\), Theorem \([1.1]\) also yields the following universality result for the gap probabilities. Recall that for a \(2 \times 2\) block operator \(A = (A_{ij})_{i,j=1,2}\) with \(A_{11}, A_{22}\) in trace class and \(A_{12}, A_{21}\) Hilbert–Schmidt, the regularized 2-determinant (see e.g. \([33]\)) is defined by \(\det_2(I + A) \equiv \det((I + A)e^{-A})e^{\tr(A_{11} + A_{22})}\).

Let \(\lambda_1\) denote the largest eigenvalue of a random matrix \(M\).

**Corollary 1.3.** Let \(\beta = 2, 1\) or 4. Let \(V\) be a polynomial of degree \(2m\) and let \(K^{(\beta)}\), \(\beta = 1, 4\) be as in Theorem \([1.1]\). Fix a number \(L_0\). Then the following holds.

In the case \(\beta = 2\):

\[
\lim_{N \to \infty} \text{Prob} \{ \lambda_1 \leq (L_0)^{(N)} \} = \det(I - K_{\text{Airy}}|_{L^2((L_0, +\infty))}) \equiv F^{(2)}(L_0).
\]

In the case \(\beta = 4\):

\[
\lim_{N \to \infty} \text{Prob} \{ \lambda_1 \leq (L_0)^{(N)} \} = \sqrt{\det(I - K^{(4)}|_{L^2((L_0, +\infty))})} \equiv F^{(4)}(L_0).
\]

In the case \(\beta = 1\), let \(g(\xi) \equiv \sqrt{1 + \xi^2}\), \(G = \text{diag}(g, g^{-1})\). Then

\[
\lim_{N \to \infty} \text{Prob} \{ \lambda_1 \leq (L_0)^{(N)} \} = \sqrt{\det_2(I - GK^{(1)}G^{-1}|_{L^2((L_0, +\infty))})} \equiv F^{(1)}(L_0).
\]

**Remark 1.2.** The regularized 2-determinant is needed for \(\beta = 1\) because the operator with kernel \(\frac{1}{2} \text{sgn}(\xi - \eta)\) is Hilbert–Schmidt but not trace class in \(L^2((L_0, +\infty))\). The auxiliary function \(g\) is needed to ensure that \(GK^{(1)}G^{-1}\) indeed has a 2-determinant; there is considerable freedom in the choice of the function \(g\), see Remark \([2.2]\) below.

**Remark 1.3.** From Theorem \([1.1]\) and \([1.22]\) we have as \(N \to \infty\),

\[
\frac{c_N}{\alpha_N N^{2/3}} R_{N,1,2}(t^{(N)}) = K_{\text{Airy}}(t, t) + o(1)
\]

\(\frac{c_N}{\alpha_N N^{2/3}} R_{N,1,1}(t^{(N)}) = K_{\text{Airy}}(t, t) + \frac{1}{2} \text{Ai}(t) \int_t^\infty \text{Ai}(u) \, du + o(1)\)

\(\frac{c_N}{\alpha_N N^{2/3}} R_{N,2,1,4}(t^{(N)}) = \frac{1}{4} K_{\text{Airy}}(t, t) - \frac{1}{8} \text{Ai}(t) \int_t^\infty \text{Ai}(u) \, du + o(1)\)
uniformly for \( t \) in any fixed half-line \([L_0, +\infty)\). In particular the density of the expected number of eigenvalues at the edge of the spectrum \( c_N + d_N \) is given by

\[
\begin{align*}
\gamma_2 &= (Ai'(0))^2 = 0.066987484 \\
\gamma_1 &= (Ai'(0))^2 + \frac{1}{3} Ai(0) = 0.185330168 \\
\gamma_4 &= \frac{1}{4} (Ai'(0))^2 - \frac{1}{24} Ai(0) = 0.001954035
\end{align*}
\]

for the indicated values of \( \beta = 2, 1, 4 \), where we have used the formula \( K_{\text{Airy}}(t, t) = (Ai'(t))^2 - t(Ai(t))^2 \) and \( \int_{-\infty}^{0} Ai(u) \, du = \frac{2}{3}, \int_{0}^{\infty} Ai(u) \, du = \sqrt{\frac{1}{3}} \) (see [ABSt]). Thus setting \( t \to \hat{t}/\gamma_\beta \), \( \beta = 2, 1, 4 \), rescales the axis so that the density of the expected number of eigenvalues per unit \( \hat{t} \)-interval is one.

The distributions \( F^{(\beta)}(L_0) \), \( \beta = 1, 2, 4 \), are the celebrated Tracy–Widom distributions which turn out to have applications in an extraordinary variety of different areas of pure and applied mathematics (see for example the recent review [TW6]). The distributions \( F^{(\beta)}(L_0) \) can all be expressed in terms of a certain solution of the Painlevé II equation ([TW4, TW5]).

The literature on edge scaling, in particular in the physics community, is vast, and we make no attempt to present an exhaustive survey. Rather we will focus on aspects of the literature which are particularly relevant to this paper. In the physics literature, early work on edge scaling for \( \beta = 2 \) is due to Moore [Mo] and Bowick and Brézin [BoBr]. In the mathematical literature for \( \beta = 2 \) with Gaussian weight \( V(x) = x^2 \), early work can be found in Forrester [F] and in the seminal work of Tracy and Widom [TW4], where the authors derived the Painlevé II representation mentioned above for \( F^{(2)} \). For \( \beta = 1 \) and 4 in the Gaussian case \( V(x) = x^2 \), the Painlevé expressions for \( F^{(\beta)} \) were obtained by Tracy and Widom in [TW3], but without computing directly the edge scaling limit of the Fredholm determinants. The edge scaling limits of matrix kernels \( K_{N,\beta} \), \( \beta = 1, 4 \), in the Gaussian case were obtained by Forrester, Nagao and Honner in [FNH]. The convergence of the Fredholm determinants in the Gaussian case for \( \beta = 1, 4 \) (and also for \( \beta = 2 \)) was first proved only recently by Tracy and Widom in [TW3].

Universality at the edge for \( \beta = 2 \) was considered by many authors in the physics literature (see e.g. [KaFr1]), and for the cases \( \beta = 1, 4 \) see e.g. [SeVe1]. The proof of universality at the edge for \( \beta = 2 \) in Theorem 1.1 above is based on the estimates in [DKMVZ2] and does not use any results from [W, TW2, DG]. Many researchers have noted that universality at the edge for \( \beta = 2 \) is true (see e.g. [CK]), but we believe that the details of the proof (Theorem 1.1, \( \beta = 2 \)) have not been written down previously. In [ST1, ST2, ST3], for \( \beta = 2, 1, 4 \), Stojanovic proves universality at the edge (and also in the bulk) in the special case of an even quartic (two-interval) potential considered previously by Bleher and Its [BI] for \( \beta = 2 \). Stojanovic uses a variant of the formulae in [W] together with the asymptotics for OP’s obtained in [BI]. Universality for the distribution of the largest eigenvalue for a wide class of real and complex Wigner ensembles (see [ABSt]) was proven by Soshnikov in [So]: the methods in [So] are completely different from those in the present paper and are based on the method of moments. Laguerre ensembles have been considered by many authors, see e.g. [F, FNH]. Various universality issues at the soft edge, and also at the hard edge and in the bulk, for generalized Laguerre ensembles for \( \beta = 2 \) were analyzed recently in [V]. The authors are currently completing an analysis of
universality questions for such ensembles in the cases \( \beta = 1 \) and 4, together with Kriecherbauer and Vanlessen, see [DGKV].

We complete this introduction with a description of Widom’s result [W] which is basic for our approach in this paper. Widom’s method applies to general weights \( w_\beta \) with the property that \( w'_\beta/w_\beta \) is a rational function. This property certainly holds for our weights as in (1.3), (1.2), and also for general Laguerre type weights which we consider in the forthcoming paper [DGKV]. Introduce the matrices

\[
D_N \equiv ((D\phi_j, \phi_k))_{0 \leq j,k \leq N-1}, \quad \epsilon_N \equiv ((\epsilon\phi_j, \phi_k))_{0 \leq j,k \leq N-1}.
\]

It follows from [W, Section 6] that the matrix \( D_N \) is banded with bandwidth \( 2n+1 \) where

\[
n \equiv 2m - 1.
\]

Thus \( (D_N)_{jk} = 0 \) if \( |j - k| > n \). Next, let \( N \) be greater than \( n \), and introduce the following \( N \)-dependent \( n \)-column vectors

\[
\Phi_1(x) \equiv (\phi_{N-n}(x), \ldots, \phi_{N-1}(x))^T
\]

\[
\Phi_2(x) \equiv (\phi_N(x), \ldots, \phi_{N+n-1}(x))^T
\]

\[
\epsilon\Phi_1(x) \equiv (\epsilon\phi_{N-n}(x), \ldots, \epsilon\phi_{N-1}(x))^T
\]

\[
\epsilon\Phi_2(x) \equiv (\epsilon\phi_N(x), \ldots, \epsilon\phi_{N+n-1}(x))^T
\]

and the following \( 2n \times 2n \) matrices consisting of four \( n \times n \) blocks

\[
B \equiv \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = ((\epsilon\phi_j, \phi_k))_{N-n \leq j,k \leq N+n-1}.
\]

and

\[
A \equiv \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & D_{12} \\ -D_{21} & 0 \end{pmatrix}
\]

where

\[
\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \equiv ((D\phi_j, \phi_k))_{N-n \leq j,k \leq N+n-1}.
\]

Finally, set

\[
C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \equiv \begin{pmatrix} I_n + (BA)_{11} & (BA)_{12} \\ (BA)_{21} & (BA)_{22} \end{pmatrix}.
\]

Note that

\[
C_{11} = I_n + B_{12}A_{21} = I_n - B_{12}D_{21}.
\]

The main result in [W] is the following pair of formulae for \( S_{N,1} \) and \( S_{N,2,4} \)

\[
S_{N,1}(x,y) = K_N(x,y) - (\Phi_1(x)^T, 0^T) \cdot (AC(I_{2n} - BA)^{-1})^T \cdot (\epsilon\Phi_1(y)^T, \epsilon\Phi_2(y)^T)^T
\]

and

\[
S_{N,2,4}(x,y) = K_N(x,y) + \Phi_2(x)^T \cdot D_{21} \cdot \epsilon\Phi_1(y) + \Phi_2(x)^T \cdot D_{21}C_{11}^{-1}B_{11}D_{12} \cdot \epsilon\Phi_2(y).\]
Observe that $S_{N,1}$ and $S_{N/2,4}$ are sums of the $\beta = 2$ kernel $K_N(x,y)$ together with correction terms that depend only on $\phi_{N+j}$ for $j \in \{-n, \ldots, n-1\}$. The $\beta = 4$ case is different from the case $\beta = 1$ since, by (1.18), for any $x \in \mathbb{R}$,

$$S_{N/2,4}(x,+\infty) = 0, \quad K_N(x,+\infty) = 0.$$  

Therefore in (1.43) for any (even) $N$ and for all $x \in \mathbb{R}$

$$\Phi_2(x)^T \cdot D_{21} \cdot c \Phi_1(+\infty) + \Phi_2(x)^T \cdot D_{21} C_{11}^{-1} B_{11} D_{12} \cdot c \Phi_2(+\infty) = 0. \tag{1.45}$$

As the entries of $\Phi_2(x)$ are functionally independent, and as $D_{12}$ is invertible for large $N$ (see [DG (2.13)]), it follows that

$$c \Phi_1(+\infty) + C_{11}^{-1} B_{11} D_{12} \cdot c \Phi_2(+\infty) = 0 \tag{1.46}$$

for large $N$. From the definition of $c$ for any integrable $\psi$

$$\epsilon\psi(y) = \frac{1}{2} \int_{-\infty}^{\infty} \psi(t) \, dt - \int_{y}^{\infty} \psi(t) \, dt = \epsilon\psi(+\infty) - \int_{y}^{\infty} \psi(t) \, dt. \tag{1.47}$$

Hence (1.47), (1.46) imply

$$S_{N/2,4}(x,y) = K_N(x,y) + \Phi_2(x)^T \cdot D_{21} \cdot \left( - \int_{y}^{\infty} \Phi_1(t) \, dt \right)$$

$$+ \Phi_2(x)^T \cdot D_{21} C_{11}^{-1} B_{11} D_{12} \cdot \left( - \int_{y}^{\infty} \Phi_2(t) \, dt \right). \tag{1.48}$$

Formula (1.48) makes clear the decay properties of $S_{N/2,4}(x,y)$ as $x,y \to +\infty$. Note that $S_{N,1}$ does not satisfy (1.44): this is the reason why we introduce auxiliary functions (cf. $G = \text{diag}(g, g^{-1})$) when proving convergence of the determinant in Corollary 1.3. As noted earlier, the question of convergence of the determinants for $\beta = 1, 4$ in the Gaussian case was first treated in [AV3].

The following observations apply to the 21 entries in the matrix kernels in the $\beta = 1$ and 4 cases. Note that by (1.13), $(\epsilon S_{N,1})(x,y)$ is skew symmetric. Thus

$$\epsilon S_{N,1}(x,y) = \epsilon S_{N,1}(x,y) - \epsilon S_{N,1}(y,x) = - \int_{x}^{y} S_{N,1}(t,y) \, dt. \tag{1.49}$$

Also, from (1.18), we see that $(\epsilon S_{N/2,4})(+\infty, y) = 0$ for all $y \in \mathbb{R}$. Together with (1.47), this implies that

$$\epsilon S_{N/2,4}(x,y) = - \int_{x}^{\infty} S_{N/2,4}(t,y) \, dt. \tag{1.50}$$

These observations simplify evaluation of integrals of the CD kernel, and also integrals of the functions $\phi_{N+j}$ in Sections 3 and 4 below.

**Remark 1.4.** We note that (1.49) is also true for $S_{N/2,4}$, but (1.50) is more relevant for the calculations that follow.

In Section 2 we prove Theorem 1.1 and Corollary 1.3 using results on the edge scaling limits of the CD terms and the correction terms in $K_{N,1}$ and $K_{N,4}$. These scaling limits are proved in turn in Section 3 for the CD terms, and in Section 4 for the correction terms. Note that Corollary 1.2 is an immediate consequence of Theorem 1.1.

**Notational remark:** Throughout this paper $c, c', C, C(m), c_1, c_2, \cdots$ refer to constants independent of $N, \xi, \eta$. The symbols $c, c', C, \cdots$ refer to generic constants,
whose precise value may change from one inequality to another. The symbol $c_N$ however always refers to the $N$-dependent constant below.

Acknowledgments. The work of the first author was supported in part by NSF grants DMS–0296084 and DMS–0500923. The second author would like to thank the Courant Institute, New York University, where he has spent a part of the academic year 2004–05, for hospitality and financial support. The second author also would like to thank Caltech for hospitality and financial support. Finally, the second author would like to thank the Swedish foundation STINT for providing basic support to visit Caltech.

2. Proofs of Theorem 1.1 and Corollary 1.3

The key estimates for the proofs of Theorem 1.1 and Corollary 1.3 are obtained below in Section 3 for the CD terms and in Section 4 for the correction terms.

2.1. Proof of Theorem 1.1

Inequality (3.8) proves the result for the $\beta = 2$ case. In the case $\beta = 4$, we use (1.48) and consider the CD part and the correction term separately. The properly scaled 11, 22 and 12 entries of $K^{(\lambda_N)}(\lambda_N N, \eta(\lambda_N))$ converge to the corresponding entries in (1.28) et seq. with the error estimate $o(1)e^{-c_N e^{-c\eta}}$, uniformly for $\xi, \eta \in [L_0, +\infty)$: this follows from (3.8) for the CD kernel part, and from (4.22) and (4.17), respectively, for the correction term. By (1.50), (3.56) and (4.26), the (unscaled) 21 entry $(\epsilon_S N, 1)(\xi(\lambda_N), \eta(\lambda_N))$ of $K^{(\lambda_N)}(\lambda_N N, 1)$ satisfies

\[
\left| 2(\epsilon_S N, 1)(\xi(\lambda_N), \eta(\lambda_N)) - \left[ \left( -\int_{\xi}^{\infty} K_{\text{Airy}}(t, \eta) \, dt \right) + \frac{1}{2} \left( \int_{\xi}^{\eta} \text{Ai}(t) \, dt \right) \left( \int_{\eta}^{\infty} \text{Ai}(t) \, dt \right) \right] \right| \leq o(1)e^{-c_N e^{-c\eta}}
\]

uniformly for $\xi, \eta \in [L_0, +\infty)$. This completes the proof of Theorem 1.1 for $\beta = 4$.

In the case $\beta = 1$, we use (1.48) and again consider the CD part and the correction term separately. The properly scaled 11 and 22 entries of $K^{(\lambda_N)}(\xi(\lambda_N), \eta(\lambda_N))$ converge to the corresponding entries in (1.27) et seq. with the error estimates $o(1)e^{-c_N e^{-c\eta}}$ and $o(1)e^{-c_N e^{-c\xi}}$, respectively, uniformly for $\xi, \eta \in [L_0, +\infty)$: this follows from (3.58) for the CD kernel part (giving rise to a smaller error $o(1)e^{-c_N e^{-c\eta}}$ and from (4.16) for the correction term. The properly scaled 12 entry converges to the corresponding entry in (1.27) et seq. with error $o(1)e^{-c_N e^{-c\eta}}$, uniformly for $\xi, \eta \in [L_0, +\infty)$: this follows from (4.49) for the CD kernel part and from (4.39) for the correction term. Finally, in view of (1.49), (4.50) and (4.50), the (unscaled) 21 entry of $K^{(\lambda_N)}(\xi(\lambda_N), \eta(\lambda_N))$ satisfies

\[
\left| (\epsilon_S N, 1)(\xi(\lambda_N), \eta(\lambda_N)) - \left[ -\int_{\xi}^{\eta} K_{\text{Airy}}(t\eta) \, dt - \frac{1}{2} \int_{\xi}^{\eta} \text{Ai}(s) \, ds \right] \right| \leq o(1)e^{-c_{\min(\xi, \eta)}} = o(1)
\]
with the uniform estimate $o(1)$ for $\xi, \eta \geq L_0$. In order to obtain the same form for the limit as claimed in Theorem 1.1 we note that for all $\xi, \eta \in \mathbb{R}$

\[
- \int_{\xi}^{\eta} K_{\text{Airy}}(t, \eta) \, dt + \frac{1}{2} \left( \int_{\xi}^{\eta} \text{Ai}(t) \, dt \right) \left( \int_{\eta}^{\infty} \text{Ai}(t) \, dt \right)
\]

\[
= - \int_{\xi}^{\eta} K_{\text{Airy}}(t, \eta) \, dt + \frac{1}{2} \left( \int_{\xi}^{\infty} \text{Ai}(t) \, dt \right) \left( \int_{\eta}^{\infty} \text{Ai}(t) \, dt \right).
\]

(2.3)

Indeed, a direct calculation using the representation (1.25) for $K$, since

\[
K \text{ is Hilbert– Schmidt in } L^2(\mathbb{R}).
\]

As the trace class convergence, we only have to prove that

\[
\rho(\xi, \eta) = \chi_{\xi, \eta}
\]

proves (2.5).

Let

\[
\rho \text{ denote differentiation and let } \frac{d}{dx} \text{ for both sides are equal}
\]

and hence the identity follows. This finishes the proof of Theorem 1.1.

2.2. Proof of Corollary 1.3

The following basic fact is well-known (see e.g. [ReSi]). Let $D = d/dx$ denote differentiation and let $\rho(x)$ be any positive function such that $\rho^{-1} \in L^2(\mathbb{R})$. Then the operator

\[
A = \frac{1}{\rho} \frac{1}{D + I}
\]

(2.4)

is Hilbert–Schmidt in $L^2(\mathbb{R})$. Indeed, by the Fourier transform, $A$ is unitarily equivalent to an operator with square integrable kernel $(\rho^{-1})(k-k') \frac{1}{i \pi k}$, $k, k' \in \mathbb{R}$.

2.2.1. The case $\beta = 2$. Let $\lambda_1$ denote the largest eigenvalue of the matrix $M$ in the unitary ensemble. It is well-known (see e.g. [TW2]) that for finite $N$

\[
\text{Prob} \left\{ \lambda_1 \leq c_N \left( \frac{L_0}{\alpha_N N^{2/3}} \right) + d_N \right\}
\]

\[
= \det \left( 1 - \frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) \right|_{L^2([L_0, +\infty))}).
\]

Since $K_N$ is finite rank, it is indeed trace class. As the trace class determinant is continuous under the trace class convergence, we only have to prove that

\[
(\Delta_N(\xi, \eta) \equiv \frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) - K_{\text{Airy}}(\xi, \eta) \to 0, \quad \text{as } N \to \infty,
\]

(2.5)

in the trace norm in $L^2([L_0, +\infty))$, in order to prove Corollary 1.3 for $\beta = 2$. Let $\chi_{L_0}(\xi)$ be a $C^\infty$ function such that $\chi_{L_0}^\#(\xi) = 1$ for $\xi \geq L_0$ and $\chi_{L_0}^\#(\xi) = 0$ for $\xi \leq L_0 - 1$. We will show that

\[
\chi_{L_0}^\# \Delta_N \chi_{L_0} \to 0, \quad N \to \infty,
\]

(2.6)

in the trace norm in $L^2(\mathbb{R})$. But then $\chi_{L_0} \Delta_N \chi_{L_0} = \chi_{L_0}(\chi_{L_0}^\# \Delta_N \chi_{L_0}^\#) \chi_{L_0}$ also converges to zero in trace norm in $L^2(\mathbb{R})$, where $\chi_{L_0}$ is the characteristic function of $[L_0, +\infty)$, and this clearly proves (2.6).

Let $\rho(\xi) = (1 + \xi^2)^{1/2}$ and write

\[
\chi_{L_0}^\# \Delta_N \chi_{L_0}^\# = \frac{1}{\rho(D + I)} \left( (D + I) \rho \chi_{L_0}^\# \Delta_N \chi_{L_0}^\# \right).
\]

The first operator is Hilbert–Schmidt (see (2.4)) and the second operator is of order $O(N^{-2/3})$ in Hilbert–Schmidt norm by (2.5), with $L_0$ replaced with $L_0 - 1$. This proves (2.6).
2.2.2. The case $\beta = 4$. Let $\lambda_1$ denote the largest eigenvalue of the matrix $M$ in the symplectic ensemble. Then in [TW2] the authors prove

\[
\text{Prob}\left\{ \lambda_1 \leq c_N \left( 1 + \frac{L_0}{\alpha_N N^{2/3}} \right) + d_N \right\} = \sqrt{\det \left( 1 - \frac{c_N}{\alpha_N N^{2/3}} K^{(\lambda(N))}_{N/2,4}(\xi(N), \eta(N)) \right)_{L^2([L_0, +\infty])}}.
\]

The proof will therefore be complete if we could prove that all the four entries of $K^{(\lambda(N))}_{N/2,4}(\xi(N), \eta(N))$ converge to the corresponding entries of $K^{(4)}(\xi, \eta)$ in trace class norm in $L^2([L_0, \infty))$. Again we use (4.18) and prove the trace class convergence of the CD part and of the correction term separately. The trace class convergence of the CD parts of all the four entries of $K^{(\lambda(N))}_{N/2,4}$ follows by using (3.8) and (3.56) together with the trace class convergence method in Subsection 2.2.1.

To prove the convergence in trace class for the 11 and 22 correction terms, we must show that

\[
\Delta_N(\xi, \eta) = \frac{c_N}{\alpha_N N^{2/3}} \left[ \Phi_2(\xi^{(N)})^T \cdot D_{21} \cdot \left( -\int_{\eta(N)}^\infty \Phi_1(t) \, dt \right) + \Phi_2(\xi^{(N)})^T \cdot D_{21} \cdot C^{-1}_{41} B_{11} D_{12} \cdot \left( -\int_{\eta(N)}^\infty \Phi_2(t) \, dt \right) - \left( -\frac{1}{2} \text{Ai}(\xi) \int_\eta^\infty \text{Ai}(t) \, dt \right) \right]
\]

(cf. (1.13), (4.23)) converges to zero in trace class in $L^2([L_0, \infty))$. But $\Delta_N$ is an operator with finite rank at most $n + 1 = 2m = \text{deg} \, V$, independent of $N$. For such operators we have the following inequality

\[\|\Delta_N\|_1 \leq \sqrt{2m}\|\Delta_N\|_{HS}\]

where $\| \cdot \|_1$, $\| \cdot \|_{HS}$ denote the trace norm, Hilbert–Schmidt norm in $L^2([L_0, \infty))$, respectively. Indeed, $\|\Delta_N\| = \sqrt{\Delta_N^* \Delta_N}$ is also an operator of rank at most $2m$, and hence it has at most $2m$ nonzero eigenvalues, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_j > 0$, $0 \leq j \leq 2m$. Thus

\[
\|\Delta_N\|_1 = \text{tr} |\Delta_N| = \sum_{i=1}^j \sigma_i \leq \sqrt{j} \left( \sum_{i=1}^j \sigma_i^2 \right)^{1/2} \leq \sqrt{2m}\|\Delta_N\|_{HS}.
\]

But from (4.22), $\|\Delta_N\|_{HS} = o(1) \left( \int_{L_0}^\infty \int_{L_0}^\infty e^{-c\xi} e^{-c\eta} \, d\xi \, d\eta \right)^{1/2} = o(1)$, $N \to \infty$, and we conclude that $\|\Delta_N\|_1 \to 0$, $N \to \infty$, as desired. A similar argument using (4.17) for the 12 entry and (4.20) for the 21 entry, completes the proof of Corollary 1.3 for $\beta = 4$.

2.2.3. The case $\beta = 1$. Let $\lambda_1$ denote the largest eigenvalue of the matrix $M$ in the orthogonal ensemble. Let $g(\xi) = \sqrt{1 + \xi^2}$ and set $G(\xi) = \begin{pmatrix} g(\xi) & 0 \\ 0 & g^{-1}(\xi) \end{pmatrix}$. Note that $g^{-1}(\xi) \in L^2(\mathbb{R})$. Let $g_{(N)}(t) = \sqrt{1 + \frac{\alpha_N N^{2/3}}{c_N} (t - c_N - d_N)^2}$ and $G_{(N)}(\xi) = \begin{pmatrix} g_{(N)}(\xi) & 0 \\ 0 & g_{(N)}^{-1}(\xi) \end{pmatrix}$. Note that $g_{(N)}(\xi^{(N)}) = g(\xi)$. Recall the definition of $\det_2$
in the Introduction. A slight modification of the calculations in [TW2] Section 9 shows that

\[(2.8) \quad \text{Prob}\left\{ \lambda_1 \leq c_N \left( 1 + \frac{L_0}{\alpha_N N^{2/3}} \right) + d_N \right\} = \sqrt{\frac{c_N}{\alpha_N N^{2/3}} \left| G^{(N)} K_N^{(\lambda_{N/2}, N)} G_N^{-1} (\xi^{(N)}, \eta^{(N)}) \right|_{L^2([0, +\infty])}} \cdot \det(1 - g) = \sqrt{\frac{c_N}{\alpha_N N^{2/3}} \left| G N^{(N/2, N)} G_N^{-1} (\xi^{(N)}, \eta^{(N)}) \right|_{L^2([0, +\infty])}} \cdot \det(1 - g) = \sqrt{\frac{c_N}{\alpha_N N^{2/3}} \left| G N^{(N/2, N)} G_N^{-1} (\xi^{(N)}, \eta^{(N)}) \right|_{L^2([0, +\infty])}} \cdot \det(1 - g)\]

In [TW2] Section 9 the authors use the fact that \( \det(1 + AB) = \det(1 + BA) \)
for appropriate operators \( A \) and \( B \). But one clearly has the freedom to write
\( AB = AG^{(N)}_N G^{(N)}_N B \), and so we also have \( \det(1 + AB) = \det(1 + AG^{-1}_N G^{(N)}_N B) = \det(1 + G^{(N)}_N BAG^{-1}_N) \)
and this leads to \( \left(2.8\right) \). We have chosen \( G^{(N)}_N \) as above in \n
such a way as to ensure that \( 1 + G^{(N)}_N BAG^{-1}_N \) has a 2-determinant, but there is clearly
great freedom in the choice of \( g^{(N)} \), and hence of \( G^{(N)}_N \). From \( \left(2.8\right) \)
we see that in order to prove \( \left(1.33\right) \) it is enough to show \( \xi \) that the diagonal (respectively
the off-diagonal) entries of \( G^{(N)}_N K_N^{(\lambda_{N/2}, N)} G_N^{-1} (\xi^{(N)}, \eta^{(N)}) \)
converge to the respective entries of \( (G K^{(1)} G^{-1})(\xi, \eta) \) in trace (respectively Hilbert–Schmidt
norm in \( L^2([L_0, \infty]) \)).

We consider first the \( 11 \) entry (again the \( 22 \) entry can be considered similarly). This
entry has the form

\[ \frac{c_N}{\alpha_N N^{2/3}} g^{(N)}(\xi^{(N)}) S_{N,1}(\xi^{(N)}, \eta^{(N)}) g^{-1}(\eta) \]

where \( S_{N,1} \) is given by the CD part and the correction term as in \( \left(1.32\right) \). The proof
that \( g(\xi) \left[ \frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) - K_{\text{Airy}}(\xi, \eta) \right] g^{-1}(\eta) \to 0, N \to \infty, \)
in trace norm in \( L^2([L_0, \infty]) \) is completely analogous to the \( \beta = 2 \) case in Subsection \( \left(2.1\right) \)
(note that \( g \) and its derivative are polynomially bounded) and the details are left to the
reader.

As in the \( \beta = 4 \) case above, the fact that the correction term in the \( 11 \) entry has
a fixed maximal rank independent of \( N \) implies that the trace norm convergence
follows from the Hilbert–Schmidt convergence. But by \( \left(1.40\right), \left(1.39\right) \)

\[
\left| \int_{-\infty}^{\xi} \Phi_1(\xi^{(N)}) T \cdot G_{11} \cdot \left( - \int_{0}^{\infty} \Phi_1(t) dt \right) \right| \leq o(1)g(\xi) e^{-\xi} g^{-1}(\eta)
\]

which is \( o(1) \) in Hilbert–Schmidt norm in \( L^2([L_0, \infty]) \). This proves the trace class
convergence of the \( 11 \) (and similarly of the \( 22 \) entry).

Finally, we note from the uniform pointwise bounds in \( \left(1.29\right) \) that the error terms
in the \( 12 \) and \( 21 \) entries are bounded by \( o(1)g(\xi)e^{-\xi} e^{-\xi} g(\eta) \) and \( o(1)g^{-1}(\xi)g^{-1}(\eta) \),
respectively, uniformly for \( \xi, \eta \geq L_0 \). This immediately implies the Hilbert–Schmidt
convergence of the off-diagonal entries to their appropriate limits. This completes
the proof of Corollary \( \left(1.23\right) \).

\textbf{Remark 2.1.} With a little more work one can show that in the \( \beta = 1 \) case
the off-diagonal entries (apart from the term \( g^{-1}(\xi) \operatorname{sgn}(\xi - \eta) g^{-1}(\eta) \)) in fact converge
in trace class norm, and not just in Hilbert–Schmidt norm.
Remark 2.2. As noted earlier, there is considerable freedom in the choice of the auxiliary function \( g \). We see that all we need is that \( g, g' \) are polynomially bounded and \( g^{-1} \in L^2(\mathbb{R}) \).

3. The edge scaling limits of the Christoffel–Darboux (\( \beta = 2 \)) kernel, and of its derivatives and integrals

3.1. Auxiliary facts from \([\text{DKMVZ}2]\). We now recall some notation from \([\text{ibid.}]\).

Let \( d\mu_N^{(eq)}(x) \) denote the equilibrium measure (see e.g. \([\text{SaTo}]\)) for OP’s corresponding to the rescaled weight \( e^{-N V_N(x)} \), \( V_N = \frac{1}{N} V(c_N x + d_N) \), where \( c_N, d_N \) are the so-called Mhaskar–Rakhmanov–Saff (MRS) numbers (see \([\text{MhSa}], [\text{Ra}]\)). For \( V(x) = \kappa_{2m} x^{2m} + \kappa_{2m-1} x^{2m-1} + \cdots \) as in \([1.19] \), we have \([\text{ibid.}, \text{Thm. } 2.1]\) to any order \( q \) as \( N \to \infty \)

\[
(3.1) \quad c_N = \left( \frac{1}{\kappa_{2m}} \frac{(2m)!}{m!(2m-1)!} \right)^{1/(2m)} N^{1/(2m)} + \sum_{j=0}^{q} \frac{c_{(j)}}{N^{j/(2m)}} + O(N^{-(q+1)/(2m)})
\]

and

\[
(3.2) \quad d_N = -\frac{\kappa_{2m-1}}{2m \kappa_{2m}} + \sum_{j=1}^{q} \frac{d(j)}{N^{-j/(2m)}} + O(N^{-(q+1)/(2m)}).
\]

As \( N \to \infty \), the equilibrium measure is absolutely continuous with respect to Lebesgue measure, \( d\mu_N^{(eq)}(x) = \psi_N^{(eq)}(x) \, dx \), and is supported on the (single) interval \([-1, 1] \),

\[
(3.3) \quad \psi_N^{(eq)}(x) \equiv \psi_N(x) = \frac{1}{2\pi} [1 - x^2]^{1/2} \chi_{[-1, 1]}(x) \, h_N(x)
\]

(see \([\text{ibid., (2.4)]}\) where \( h_N(x) \) is a real polynomial of degree \( 2m - 2 \) satisfying \([\text{ibid., Prop. } 5.3]\)

\[
(3.4) \quad h_N(x) \geq h_{\min} > 0, \quad x \in \mathbb{R}, \quad N \geq N_1(V).
\]

Set \([\text{ibid., (5.33]}\]

\[
(3.5) \quad \Xi_N(z) \equiv g_+(z) - g_-(z) = i \int_{-1}^{1} |1 - x^2|^{1/2} h_N(x) \, dx.
\]

We also use the same symbol for the analytic continuation of \( \Xi_N \) to \( \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty)) \).

Notational remark: Here we denote by \( \Xi_N \) what was denoted by \( \xi_N \) in \([\text{ibid.}]\).

For a fixed \( \delta > 0 \) sufficiently small (cf. \([\text{DG}, \text{Rem. } 4.3]\)), let \( R \) denote the matrix function defined in \([\text{DKMVZ}2], (7.47)\). The function \( R \) is analytic in the complement of the contour \( \Sigma_R \) as in \([\text{ibid., Fig. } 7.6]\) and is continuous up to the boundary. Furthermore by \([\text{ibid., Thm. } 7.10]\), it has an asymptotic expansion

\[
(3.6) \quad R(z) \sim I + N^{-1} \sum_{k=0}^{\infty} r_k(z) N^{-k/(2m)}
\]
where \( \{ r_k(z) \} \) are bounded functions that are analytic in the complement of \( \{|z - 1| = \delta\} \cup \{|z + 1| = \delta\} \). The expansion \( (3.0) \) is uniform for \( z \in \mathbb{C} \setminus \hat{\Sigma}_R \). Moreover, by the proof of [ibid., Thm 7.10] and Cauchy’s theorem, it follows that \((3.0)\) can be differentiated term by term,

\[
\frac{d^j}{dz^j} R(z) \sim N^{-1} \sum_{k=0}^\infty \frac{d^j}{dz^j} r_k(z) N^{-k/(2m)}, \quad j = 1, 2, \ldots,
\]

where again the expansion is uniform for \( z \in \mathbb{C} \setminus \hat{\Sigma}_R \). Also, each \( \frac{d^j}{dz^j} r_k(z) \) is bounded (and analytic) in the complement of \( \{|z - 1| = \delta\} \cup \{|z + 1| = \delta\} \).

3.2. Estimates on the CD kernel and its derivatives. We will only consider the end point 1 (the end point -1 can be considered similarly). Let \( L_0 \in \mathbb{R} \) be fixed. Recall the notation in \( (1.25), (1.23) \) and \( (1.5) \). Our goal in this Subsection is to prove that for \( j, k = 0, 1, \) and some \( C = C(L_0) > 0 \), \( c = c(L_0) > 0 \), one has uniformly for \( \xi, \eta \in [L_0, +\infty) \)

\[
(3.8) \quad \left| \frac{d^j}{dz^j} \right| \frac{C_N}{\alpha_N N^{2/3}} \left| K_N(\xi(N), \eta(N)) - K_{\text{Airy}}(\xi, \eta) \right| \leq C N^{-2/3} e^{-c \xi} e^{-cn}.
\]

Note that \( |\hat{\alpha}(\xi)| \leq C(1 + |\xi|)^{-1/4}, \quad |\hat{\alpha}'(\xi)| \leq C(1 + |\xi|)^{1/4}, \quad \xi \in \mathbb{R}, \)

\[
|d^q \hat{\alpha}(\xi)|/d\xi^q \leq C_1 e^{-\xi} \leq C_2, \quad \xi \in [L_0, +\infty), \quad q = 0, 1, 2, \ldots.
\]

3.2.1. Auxiliary notation. Set (see [DKMVZ2] (2.15) and also [DG] (4.10)))

\[
(3.10) \quad f_N(x) = \alpha_N N^{2/3} (x - 1) \hat{f}_N(x)
\]

which satisfies the following (see (the proof of) [DKMVZ2] Proposition 7.3))

1. \( \hat{f}_N(x) \) is real analytic on \((1 - 2\delta, 1 + 2\delta)\), and to any order \( q = 0, 1, 2, \ldots \)

\[
\hat{f}_N(x) = \sum_{j=0}^q N^{-j/(2m)} \hat{f}_{(j)}(x) + O(N^{-(q+1)/(2m)})
\]

uniformly for \( x \) in the interval. Moreover, the functions \( \hat{f}_{(j)}(x) \) are also real analytic on \((1 - 2\delta < x < 1 + 2\delta)\)

2. to any order \( q = 1, 2, \ldots \)

\[
\alpha_N \equiv (h_N^2(1)/2)^{1/3} = 2m^{2/3} + \sum_{j=1}^q N^{-j/(2m)} \alpha_{(j)} + O(N^{-(q+1)/(2m)})
\]

3. \( f_N'(x) = -\alpha_N N^{2/3} W_N(x) \), where \( W_N(x) = \hat{f}_N(x) + (x - 1) \hat{f}_N'(x) \) also has an expansion uniform in \( x \) to any order \( q = 0, 1, 2, \ldots \) as above

\[
W_N(x) = \sum_{j=0}^q N^{-j/(2m)} W_{(j)}(x) + O(N^{-(q+1)/(2m)}).
\]

The terms \( W_{(j)}(x) \) are real analytic on \((1 - 2\delta < x < 1 + 2\delta)\)

4. \( \max_{0 < m < 1/2} \max_{1 - 2\delta \leq x \leq 1 + 2\delta} |d^k \hat{f}_N(x)|/dx^k | \leq M < \infty \) for \( N \geq N_2(V) \)

5. \( \hat{f}_N(1) = 1 = W_N(1) \) and \( \min_{1 - 2\delta \leq x \leq 1 + 2\delta} \hat{f}_N(x) \geq 1 \) for \( N \geq N_2(V) \). Also \( \hat{f}_{0}(1) = 1 = W_{0}(1) \).
Denote
\begin{equation}
\begin{aligned}
\xi_N &\equiv \xi/(\alpha_N N^{2/3}), \\
\eta_N &\equiv \eta/(\alpha_N N^{2/3}), \\
I_N &\equiv [L_0, \alpha_N N^{2/3} \delta], \\
II_N &\equiv [\alpha_N N^{2/3} \delta, +\infty).
\end{aligned}
\end{equation}

Thus, recalling (3.10)(1)(5) and the formula
\begin{equation}
c
\end{equation}
As above, let \( \delta > 0 \) be fixed and sufficiently small. Consider first \( \xi_N, \eta_N \) in a neighborhood of 0. Set
\begin{equation}
\begin{aligned}
g_N(\xi) &\equiv \xi \hat{f}_N(1 + \xi_N) \\
\hat{F}_N(1 + \xi_N) &\equiv (2 + \xi_N)^{1/4} \cdot (\hat{f}_N(1 + \xi_N))^{1/4} \\
F_N(1 + \xi) &\equiv N^{1/6} \alpha_N^{1/4} \hat{F}_N(1 + \xi_N)
\end{aligned}
\end{equation}
and also
\begin{equation}
\begin{aligned}
A_0(\xi) &\equiv N^{1/6} \alpha_N^{1/4} \hat{F}_N(1 + \xi_N) \cdot \text{Ai}(g_N(\xi)) \\
A_1(\xi) &\equiv N^{-1/6} \alpha_N^{-1/4} \left( \hat{F}_N(1 + \xi_N) \right)^{-1} \cdot \text{Ai}'(g_N(\xi)).
\end{aligned}
\end{equation}

Note that in view of (3.11)(1)(5) and the formula
\begin{equation}
g_N'(\xi) = \hat{f}_N(1 + \xi_N) + \xi_N \frac{d}{d\xi} \hat{F}_N(1 + \xi_N)
\end{equation}
there exist \( c_2 > c_1 > 0 \) such that
\begin{equation}
c_1 \leq \frac{g_N(\xi)}{\xi} \leq c_2, \quad \xi \in I_N
\end{equation}
and
\begin{equation}
c_1 \leq g_N'(\xi) \leq c_2, \quad |g_N''(\xi)| \leq c_2 N^{-2/3}, \quad \xi \in I_N.
\end{equation}

Similarly one has uniformly for \( \xi \in I_N \)
\begin{equation}
c_1 \leq \hat{F}_N(1 + \xi_N) \leq c_2, \quad \left| \frac{d^k}{dz^k} \hat{F}_N(z) \right|_{z = 1 + \xi_N} \leq C(k)
\end{equation}
for some \( C(k) \), \( k = 1, 2, \ldots \).

3.2.2. Estimates for \( (\xi, \eta) \) \( \in I_N \times I_N \). With the above notation the following holds.

**Proposition 3.1.** For \( (\xi, \eta) \) \( \in I_N \times I_N \)
\begin{equation}
\frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) = K_{\text{Avr}}(\xi, \eta) + \frac{1}{\alpha_N N^{2/3}} \sum_{j=1}^{4} Q_{1,j}(\xi, \eta)
\end{equation}
where
\begin{equation}
Q_{1,1}(\xi, \eta) \equiv - \left( \begin{array}{cc}
A_0(\eta) & A_1(\eta)
\end{array} \right) \cdot \left( \begin{array}{cc}
1 & -i \\
-1 & -i
\end{array} \right)
\end{equation}
\begin{equation}
\cdot \int_0^1 (R^T)'(1 + \xi_N + t(\eta_N - \xi_N)) \, dt
\end{equation}
\begin{equation}
\cdot \left( \begin{array}{cc}
1 & -i \\
-1 & -i
\end{array} \right)^{-1} \left( \begin{array}{cc}
-A_1(\xi) \\
A_0(\xi)
\end{array} \right)
\end{equation}
and
\begin{equation}
Q_{1,2}(\xi, \eta) \equiv A_i(g_N(\xi))A_i'(g_N(\eta)) \cdot T_N(\xi, \eta) - A_i(g_N(\eta))A_i'(g_N(\xi)) \cdot T_N(\eta, \xi)
\end{equation}
\begin{equation}
T_N(\xi, \eta) \equiv \int_0^1 \frac{F_N'(1 + \eta_N + \eta(\xi_N - \eta_N))}{F_N(1 + \eta_N)} \, d\tau
\end{equation}

and
\begin{equation}
Q_{1,3}(\xi, \eta) \equiv E_N(\xi, \eta) \int_0^{\infty} \text{Ai}(z + g_N(\xi)) \text{Ai}(z + g_N(\eta)) \, dz
\end{equation}
\begin{equation}
E_N(\xi, \eta) \equiv \int_0^1 \left[ \pi + \tau(\xi - \eta) \right] \left[ \tilde{f}_N'(1 + \eta_N + \tau(\xi_N - \eta_N)) \right. \\
\left. + \int_0^1 \tilde{f}_N'(1 + \sigma(\eta_N + \tau(\xi_N - \eta_N))) \, d\sigma \right] \, d\tau
\end{equation}

and
\begin{equation}
Q_{1,4}(\xi, \eta) \equiv \xi^2 L_N(\xi) \int_0^{\infty} U_N(\xi, z) \text{Ai}(z + g_N(\eta)) \, dz
\end{equation}
\begin{equation}
+ \eta^2 L_N(\eta) \int_0^{\infty} \text{Ai}(z + \xi) U_N(\eta, z) \, dz
\end{equation}
\begin{equation}
L_N(\xi) \equiv \int_0^1 \tilde{f}_N'(1 + \sigma \xi_N) \, d\sigma
\end{equation}
\begin{equation}
U_N(\xi, z) \equiv \int_0^1 \text{Ai}'(z + \xi + \tau(g_N(\xi) - \xi)) \, d\tau.
\end{equation}

Proof. First, some algebra: let \( Y \) solve the Fokas–Its–Kitaev Riemann–Hilbert problem for the polynomials orthogonal with respect to the weight \( e^{-V(x)} \, dx \) (see [DKMVZ2 Thm. 3.1]). Then as in [DKMVZ1 (6.3)] we find for any \( x, y \in \mathbb{R} \)
\begin{equation}
K_N(x, y) = e^{-(V(x) + V(y))/2} \frac{Y_{11}(y)Y_{21}(x) - Y_{11}(x)Y_{21}(y)}{2\pi i(x - y)}
= -e^{-(V(x) + V(y))/2} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \cdot Y_+^T(y) \cdot Y_+^{-T}(x) \cdot \left( \begin{array}{c} 1 \\ 0 \end{array} \right)^T
\end{equation}
\begin{equation}
= \frac{2\pi i(x - y)}{2\pi i(x - y)}.
\end{equation}

Here and below \(+/-\) refer to the boundary values taken from above/below the real axis. (The choice \( Y_+ \) is made only for definiteness. Formula [3.23] clearly remains true if \( Y_+ \) is replaced with \( Y_- \).) Consider first \( z = 1 + \xi_N \in (1 - \delta, 1] \) for \( \xi \in (-\delta N^{2/3}, 0] \). By [DKMVZ2 (4.2), (4.6), (4.22)] we have for \( S_+ \), the solution of the Riemann–Hilbert problem [ibid., (4.24)–(4.26)], (cf. [ibid., (7.46),(7.47)])
\begin{equation}
S_+(z) = e^{-N^{\sigma_3}} e^{-\frac{N}{2} \sigma_3} Y_+(c_N z + d_N)
\end{equation}
\begin{equation}
\times e^{-N(g_+(z) - \frac{\sigma}{2})} \text{Ai}'(c_N z + d_N)
\end{equation}
\begin{equation}
\times \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\end{equation}
where \( \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and the constant \( l \equiv l_N \) is given by [ibid., (5.35)]. Solving for \( Y_+ \) and substituting in (3.23) we find for \( \xi, \eta \in (-\delta \alpha N N^{2/3}, 0] \)

\[
\frac{e_N}{\alpha N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) = -\frac{e^{-\frac{i\pi}{3}(V_N(1+\xi_N)+V_N(1+\eta_N))}}{2\pi i (\xi - \eta)} \times \left( e^{N(g_+(1+\eta_N)-\frac{1}{2})} e^{N(g_-(1+\xi_N)-\frac{1}{2})} \right) \cdot S^T_+(1 + \eta N) \\
\times S_-^T(1 + \xi N) \cdot \left( -e^{N(g_-(1+\xi_N)-\frac{1}{2})} e^{N(g_+(1+\xi_N)-\frac{1}{2})} \right).
\]

(3.25)

Now note that for \( z \in \{1 - \delta, 1\} \), by [ibid., (7.46), (7.47)], \( S(z) = R(z) P_N(z) \). By [ibid., (7.24), (7.9), (7.23), (7.4)],

\[
P_{N,+}(z) = \sqrt{\pi} e^{-i\pi/6} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \begin{pmatrix} F_N(z) & 0 \\ 0 & 1/F_N(z) \end{pmatrix} \\
\times AI_+(f_N(z)) e^{-i\pi\sigma_3/6} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{N\Xi_N(z)\sigma_3/2}
\]

where

\[
AI(f_N(z)) \equiv \begin{pmatrix} \text{Ai}(f_N(z)) & \text{Ai}(\omega^2 f_N(z)) \\ \text{Ai}'(f_N(z)) & \omega^2 \text{Ai}'(\omega^2 f_N(z)) \end{pmatrix}, \quad \omega = e^{2\pi i/3}.
\]

For \( z \in (-1,1) \), in view of [ibid., (5.38)]

\[-V_N(z) + g_+(z) + g_-(z) - l = 0\]

and we find from (3.26)

\[
\frac{e_N}{\alpha N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) = -\frac{e^{-\frac{i\pi}{3}}}{2\pi i (\xi - \eta)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot AI_+^T(f_N(1 + \eta N)) \\
\times \left( F_N(1 + \eta_N) \quad 0 \\ 0 \quad 1/F_N(1 + \eta_N) \right) \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} R_+^T(1 + \eta N) \\
\times R_-^T(1 + \xi N) \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}^{-1} \begin{pmatrix} 1/F_N(1 + \xi N) & 0 \\ 0 & F_N(1 + \xi N) \end{pmatrix} \\
\times AI_-^T(f_N(1 + \xi N)) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi, \eta \in (-\delta \alpha N N^{2/3}, 0].
\]

Similar calculations for \( z \in [1,1+\delta) \) lead to the same formula for all other cases \( \xi < 0, \eta > 0 \), etc., \( |\xi|, |\eta| \leq \delta \alpha N N^{2/3} \).

Now writing

\[
R^T(1 + \eta N) = R^T(1 + \xi N) + (\eta_N - \xi_N) \int_0^1 (R^T)'(1 + \xi N + t(\eta_N - \xi_N)) \, dt
\]

(3.27)
and taking into account \( \det AI_+(f_N(1+\xi_N)) = -1/(2\pi i^{12}) \) (use [ibid., (8.38)] we obtain from (3.20) that
\[
\frac{c_N}{\alpha_N N^{2/3}} \quad K_N(\xi^{(N)}, \eta^{(N)}) = \frac{1}{\alpha_N N^{2/3}} Q_{1,1}(\xi, \eta) \\
\quad + \frac{1}{\xi - \eta} \left\{ \text{Ai}(g_N(\xi)) \text{Ai}'(g_N(\eta)) \cdot \frac{F_N(1+\xi_N)}{F_N(1+\eta_N)} - (\xi \leftrightarrow \eta) \right\}
\]
where \( Q_{1,1} \) is as in (3.19). Now
\[
\frac{F_N(1+\xi_N)}{F_N(1+\eta_N)} = \frac{\hat{F}_N(1+\xi_N)}{F_N(1+\eta_N)} = 1 + (\xi_N - \eta_N) \int_0^1 \hat{f}_N'(1+\eta_N + \tau(\xi_N - \eta_N)) d\tau
\]
and hence using (1.25) we rewrite (3.28) as
\[
\frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) = \frac{g_N(\xi) - g_N(\eta)}{\xi - \eta} K_{\text{Airy}}(g_N(\xi), g_N(\eta)) \\
\quad + \frac{1}{\alpha_N N^{2/3}} (Q_{1,1}(\xi, \eta) + Q_{1,2}(\xi, \eta))
\]
where \( Q_{1,2} \) is as in (3.20). Next we write \( \frac{g_N(\xi) - g_N(\eta)}{\xi - \eta} = \int_0^1 g_N'(\eta + \tau(\xi - \eta)) d\tau \), and use (3.14) and
\[
\hat{f}_N(1+\eta_N + \tau(\xi_N - \eta_N)) = \hat{f}_N(1+\eta_N + \tau(\xi_N - \eta_N)) \int_0^1 \hat{f}_N'(1+\eta_N + \tau(\xi_N - \eta_N)) d\tau
\]
to conclude that \( \frac{g_N(\xi) - g_N(\eta)}{\xi - \eta} = 1 + E_N(\xi, \eta) \) from (3.31). Hence recalling (1.25) we obtain from (3.30)
\[
\frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) = K_{\text{Airy}}(g_N(\xi), g_N(\eta)) + \frac{1}{\alpha_N N^{2/3}} \sum_{j=1}^3 Q_{1,j}(\xi, \eta)
\]
where \( Q_{1,3} \) is as in (3.21). Finally again using (1.25) we find
\[
K_{\text{Airy}}(g_N(\xi), g_N(\eta)) = \int_0^\infty \text{Ai}(z + g_N(\xi)) \text{Ai}(z + g_N(\eta)) dz
\]
\[
= \int_0^\infty \text{Ai}(z + \xi) \text{Ai}(z + \eta) dz
\]
\[
+ \int_0^\infty \text{Ai}(z + \xi) \left[ \text{Ai}(z + g_N(\eta)) - \text{Ai}(z + \eta) \right] dz
\]
\[
+ \int_0^\infty \left[ \text{Ai}(z + g_N(\xi)) - \text{Ai}(z + \xi) \right] \text{Ai}(z + g_N(\eta)) dz.
\]
The first integral equals \( K_{\text{Airy}}(\xi, \eta) \). To evaluate the third integral we recall (3.12), (3.22) and note that
\[
g_N(\xi) - \xi = \xi \left[ \hat{f}_N(1 + \xi_N) - \hat{f}_N(1) \right]
\]
\[
= \frac{\xi^2}{\alpha_N N^{2/3}} \int_0^1 \hat{f}_N'(1 + \sigma\xi_N) d\sigma
\]
where
\[
\frac{\xi^2}{\alpha_N N^{2/3}} L_N(\xi)
\]
which implies
\[
\text{Ai}(z + g_N(\xi)) - \text{Ai}(z + \xi) = \frac{\xi^2}{\alpha_N N^{2/3}} \cdot \int_0^1 \tilde{f}_N(1 + \sigma \xi_N) d\sigma \\
\times \int_0^1 \text{Ai}'(z + \xi + \tau(g_N(\xi) - \xi)) d\tau \\
= \frac{\xi^2}{\alpha_N N^{2/3}} L_N(\xi) U_N(\xi, z).
\] (3.34)

The second integral in (3.32) is treated analogously. We conclude from (3.31), addition (3.16) we conclude that we can always use the exponential bounds on Ai and its derivatives in (3.2), and hence for any \(m \in \mathbb{N}\) and \(k = 0, 1, 2, 3\), as \(N \to \infty\)
\[
\left| \xi^m \left( \frac{d}{d\xi} \right)^k \text{Ai}(g_N(\xi)) \right| \leq C(m) e^{-c(m) \xi}, \quad \xi \in I_N.
\] (3.35)

Consider \(Q_{1,1}(\xi, \eta)\) first. Recall from (3.7) that, in particular, \(\frac{d}{dz} R(z) = O(N^{-1})\), \(j = 1, 2, 3\), uniformly for \(|z - 1| \leq \delta\). It follows then by (3.14) using (3.16), (3.17), (3.35) that for \(j, k = 0, 1\)
\[
\left| \partial_\xi^j \partial_\eta^k Q_{1,1}(\xi, \eta) \right| \leq \text{const} \cdot N^{-4/3} e^{-c \xi} e^{-c \eta}
\] uniformly for \(\xi, \eta \in I_N\). In the same way we find that for \(j, k = 0, 1\) and \(l = 2, 3, 4\)
\[
\left| \partial_\xi^j \partial_\eta^k \partial_\xi^l Q_{1,1}(\xi, \eta) \right| \leq \text{const} \cdot N^{-2/3} e^{-c \xi} e^{-c \eta}
\] uniformly for \(\xi, \eta \in I_N\). In estimating \(Q_{1,4}\), we use the estimate
\[
|g_N(\xi) - \xi| \leq C \delta |\xi|, \quad |\xi| \leq \delta \alpha_N N^{2/3},
\] which follows from (3.36), together with the uniform boundedness of \(L_N(\xi)\) (see (3.10)) for \(\delta\) sufficiently small this implies that
\[
|U_N(\xi, z)| \leq C e^{-cz} e^{-c \xi}, \quad \xi \in I_N, \quad z \geq 0,
\] with similar estimates for the \(\xi\)- and \(z\)- derivatives. This proves (3.35) for \((\xi, \eta) \in I_N \times I_N\).

3.2.3. Estimates for \((\xi, \eta) \in I_{II} \times I_{II}\). Recall from [ibid., (4.30), (4.31), (6.16)]
\[
S^{(\infty)}(z) \equiv N(z) = \frac{1}{2} \begin{pmatrix} a(z) + a(z)^{-1} & i(a(z)^{-1} - a(z)) \\ i(a(z) - a(z)^{-1}) & a(z) + a(z)^{-1} \end{pmatrix}
\] (3.39)
where \(a(z) \equiv (\frac{z - 1}{z + 1})^{1/4} \to 1\) as \(z \to \infty\).

Proposition 3.2. For \(j, k = 0, 1\) and some \(C, c > 0\)
\[
\left| \partial_\xi^j \partial_\eta^k \left( c_N \frac{\alpha_N N^{2/3}}{N} K_N(\xi(\eta), \eta(N)) \right) \right| \leq C e^{-cN} e^{-c(N(\xi - \delta))} e^{-c(N(\eta - \delta))}
\] uniformly for \(\xi, \eta \in I_{II}\).
Proof. Note first of all that (3.44) still holds. For \( z = 1 + \xi_N \in [1 + \delta, +\infty) \) we now have in place of (3.41)

\[
S_+ (z) = c_N^{-N \sigma_3} e^{-\frac{N}{2} \sigma_3} Y_+(c_N z + d_N) e^{-N g_+ (z) - \frac{N}{2} \sigma_3}
\]

where again the constant \( l \equiv l_N \) is given by [ibid., (5.38)]. Solving for \( Y_+ \) and substituting in (3.23) we find for \( \xi, \eta \in II_N \)

\[
\frac{e_N}{\alpha_N N^{2/3}} K_N (\xi^{(N)}, \eta^{(N)}) = -e^{-\frac{N}{2} \tau V_N (1 + \xi_N) - 2g_+ (1 + \xi_N) + l} e^{-\frac{N}{2} \tau V_N (1 + \eta_N) - 2g_+ (1 + \eta_N) + l} \times \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot S_+^T (1 + \eta_N) S_+^{-T} (1 + \xi_N) \cdot \begin{pmatrix} 0 & 1 \end{pmatrix}^T}{2\pi i (\xi - \eta)}.
\]

In view of [ibid., (5.38)]

\[-V_N (1 + \xi_N) + 2g_+ (1 + \xi_N) - l = \Xi_{N,+} (1 + \xi_N), \quad \xi \in II_N.
\]

Now by [ibid., (2.14), (5.34)] for some \( C_1(\delta), C_2(\delta) > 0 \) and \( c > 0 \) for \( N \) large enough

\[
\Xi_{N,+} (1 + \xi_N) = -\left( \int_1^{1+\delta} + \int_{1+\delta}^{1+\xi_N} \right) \sqrt{t^2 - 1} h_N (t) dt \leq -\int_1^{1+\delta} \sqrt{t^2 - 1} h_{min} dt - \int_{1+\delta}^{1+\xi_N} \sqrt{t^2 - 1} h_{min} dt \leq -C_1 - C_2 (\xi_N - \delta), \quad \xi \in (\delta \alpha N N^{2/3}, +\infty).
\]

By [ibid., (7.46), (7.47)] for \( z \geq 1 + \delta, S(z) = R(z) S^{(\infty)} (z) \). Using (3.21), which is still valid for \( \xi, \eta \in II_N \), we obtain

\[
S_+^T (1 + \eta_N) S_+^{-T} (1 + \xi_N) = S^{(\infty) T} (1 + \eta_N) S^{(\infty)-T} (1 + \xi_N)
\]

\[
+ (\eta_N - \xi_N) S^{(\infty) T} (1 + \eta_N) \left( \int_0^1 (R_T^*)' (1 + \xi_N + t(\eta_N - \xi_N)) dt \right) S^{(\infty)-T} (1 + \xi_N).
\]

Substituting

\[
S^{(\infty) T} (1 + \eta_N) = S^{(\infty) T} (1 + \xi_N)
\]

\[
+ (\eta_N - \xi_N) \left( \int_0^1 (S^{(\infty) T})' (1 + \xi_N + t(\eta_N - \xi_N)) dt \right)
\]

in the first term in the RHS of (3.41) and noting that \( \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot I \cdot \begin{pmatrix} 0 & 1 \end{pmatrix}^T = 0 \), we obtain an expression for \( \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot S^{(\infty) T} (1 + \eta_N) S^{(\infty)-T} (1 + \xi_N) \cdot \begin{pmatrix} 0 & 1 \end{pmatrix}^T \)

which is proportional to \( (\xi - \eta) \). The exponential bounds (3.40) then follow from (3.43) and the properties of \( S^{(\infty)} \) and \( R \) (see (3.39) and (3.40), respectively). \( \square \)

Now we prove (3.3) for \( \xi, \eta \in II_N \) by showing that both of the two terms on the LHS of (3.3) satisfy the exponential bound. More precisely, let \( \xi \in II_N \). Then either \( \xi_N \geq 2\delta \) or \( \Xi_N \in [\delta, 2\delta] \). In the former case

\[
e^{-N(\xi_N - \delta)} = e^{-cN((\xi_N/2) - \delta)} e^{-(cN/(2\alpha N N^{2/3}) \xi)} \leq e^{-\xi}, \quad N \to \infty,
\]

since \( \alpha_N \to (2m)^{2/3} \neq 0 \) as \( N \to \infty \). In the latter case

\[
e^{-\xi} = e^{-\alpha N N^{2/3} \xi_N} \geq e^{-\alpha N N^{2/3} 2\delta} \geq e^{-(c/2) N}, \quad N \to \infty
\]
and hence
\begin{equation}
\frac{c_N}{\alpha_N N^{2/3}} e^{-c/2} \leq e^{\frac{c}{2}} N \leq e^{-\frac{c}{2}} N \leq e^{-c} \xi, \quad \xi, \eta \in [\delta, 2\delta].
\end{equation}
Combining (3.45) and (3.46) we conclude that Proposition 3.2 implies
\begin{equation}
\left| \partial^k_{\xi} \partial^k_{\eta} \left( \frac{c_N}{\alpha N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) \right) \right| \leq C e^{-c} \xi e^{-\eta}, \quad \xi, \eta \in II_N.
\end{equation}
Now we consider $K_{\text{Airy}}(\xi, \eta)$ for $\xi, \eta \in II_N$. It follows from [AbSt] that
\begin{equation}
\left| \text{Ai}(x) \right|, \left| \text{Ai}'(x) \right| \leq C(L_0)e^{-c(L_0)|x|^{3/2}}, \quad x \geq L_0.
\end{equation}
Using the integral representation we estimate for $\xi, \eta \in II_N$
\begin{equation}
\left| K_{\text{Airy}}(\xi, \eta) \right| \leq C \int_0^\infty e^{-c(z+\xi)^{3/2}} e^{-c(z+\eta)^{3/2}} dz.
\end{equation}
Let $\xi \in II_N$. Then $\xi \geq 1$ for large $N$. It is elementary to verify that
\begin{equation}
(z + \xi)^{3/2} \geq z^{3/2} + \xi^{3/2}, \quad z \geq 0, \quad \xi \geq 1.
\end{equation}
Next, $\xi^{3/2} \geq (\delta \alpha_N)^{3/2} N \geq cN, \quad N \to \infty$ and hence
\begin{equation}
\xi^{3/2} - \xi = \xi^{3/2}(1 - \xi^{-1/2}) \geq e''N, \quad N \to \infty.
\end{equation}
Inserting (3.50), (3.51) and their analogs for $\eta$ in (3.49) we find
\begin{equation}
\left| K_{\text{Airy}}(\xi, \eta) \right| \leq C e^{-cN} e^{-c\xi} e^{-c\eta}, \quad \xi, \eta \in II_N.
\end{equation}
A similar argument using (3.48) also shows that the derivatives of $K_{\text{Airy}}$ satisfy the same bound. Combining (3.47) and (3.52) completes the proof of (3.8) for $\xi, \eta \in II_N$.

3.2.4. The “mixed” neighborhoods of the end point 1: $(\xi, \eta) \in (I_N \times II_N) \cup (II_N \times I_N)$. Let us consider the case $(\xi, \eta) \in I_N \times II_N$ (the other case is treated analogously). For $K_{\text{Airy}}$, we use the bound in (3.39) for $\xi$,
\begin{equation}
\left| \text{Ai}(z + \xi) \right|, \left| \text{Ai}'(z + \xi) \right| \leq C(L_0) e^{-c} e^{-\xi}, \quad z \geq 0, \quad \xi \geq L_0,
\end{equation}
together with the bound (3.53) for $\eta$. Inserting these bounds in (1.20) we obtain for $j, k = 0, 1$
\begin{equation}
\left| \partial^j_{\xi} \partial^k_{\eta} K_{\text{Airy}}(\xi, \eta) \right| \leq C e^{-cN} e^{-c\xi} e^{-c\eta}, \quad (\xi, \eta) \in I_N \times II_N
\end{equation}
as before. For $K_N(\xi^{(N)}, \eta^{(N)})$, there are two cases: $|\xi_N - \eta_N| \leq \delta/2$ and $|\xi_N - \eta_N| > \delta/2$. In the first case we can treat both points as lying in a $I_N \times I_N$ region corresponding to a larger (fixed) value of $\delta$ (more precisely, set $\delta \to 3\delta/2$) and hence (3.38) follows by the arguments in Subsection 3.2.2.

It remains to consider the case $(\xi, \eta) \in I_N \times II_N$, $|\xi_N - \eta_N| \geq \delta/2$. For such $\xi, \eta$, we have
\begin{equation}
|\xi - \eta|^{-1} \leq N^{-2/3} \alpha_N^{-1} \delta^{-1}.
\end{equation}
The computations that led to (3.26) and (3.42) now imply for $\xi \in I_N, \eta \in I_N$

\[
\frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) = -\frac{e^{-\pi i/6} e^{-N\Xi_{\xi}(1+\xi)}/2}{2\pi i(\xi - \eta)}
\]
\[
\times \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} S^{(\infty)}(1 + \eta N) R^T(1 + \eta N) R^T_+(1 + \xi N)
\]
\[
\times \begin{pmatrix} 1 \quad -i \\ -1 \quad -i \end{pmatrix}^{-1} \begin{pmatrix} 1/F_N(1 + \xi N) & 0 \\ 0 & F_N(1 + \xi N) \end{pmatrix}
\]
\[
\times A I^T_+(f_N(1 + \xi N)) \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(3.55)

and using the preceding estimates we find for $j, k = 0, 1$

\[
\left| \frac{\partial_j^k}{\partial \xi^j \partial \eta^k} \frac{c_N}{\alpha_N N^{2/3}} K_N(\xi^{(N)}, \eta^{(N)}) \right| \leq CN^{-2/3} N^{1/6} e^{-cN} e^{-c \xi e^{-\eta}} \leq C e^{-c N} e^{-c \xi e^{-\eta}}
\]

uniformly for $(\xi, \eta) \in I_N \times I_N, |\xi_N - \eta_N| > \delta/2$.

There is a similar estimate for $(\xi, \eta) \in I_N \times I_N$ which, together with (3.58), then proves (3.55) for $(\xi, \eta) \in (I_N \times I_N) \cup (I_N \times I_N)$.

3.3. Estimates on integrals of the CD kernel. For $\xi, \eta \in [L_0, +\infty)$, making a change of variables $s = c_N(1 + t_N) + d_N$, and using (3.58) with $j = k = 0$, we readily find

(3.56)

\[
- \int_{\xi(N)}^{\eta(N)} K_N(s, \eta^{(N)}) ds - \left( - \int_{\xi}^{\eta} K_{\text{Airy}}(t, \eta) dt \right) \leq CN^{-2/3} e^{-c \xi e^{-\eta}}
\]

\[
- \int_{\xi(N)}^{\eta(N)} K_N(s, \eta^{(N)}) ds - \left( - \int_{\xi}^{\eta} K_{\text{Airy}}(t, \eta) dt \right) \leq CN^{-2/3} e^{-c \min(\xi, \eta) e^{-\eta}}
\]

uniformly for $\xi, \eta \in [L_0, +\infty)$.

4. The contribution of the correction term for $\beta = 1$ and 4

4.1. Auxiliary facts concerning integrals of the orthogonal functions $\phi_j$.

It was shown in [DG] (4.14) that for a fixed $j \in \mathbb{N}$ the following holds as $N \to \infty$

(4.1)

\[
\int_{-\infty}^{+\infty} \phi_{N+j}(y) dy = c_N^{1/2} N^{-1/2} (2m)^{-1/2} (1 + (-1)^{N+j} + O(N^{-1/2m}))
\]

where $2m = \deg V$. Introduce the following column vectors of size $2m - 1$

\[
a \equiv (1, 0, 1, 0, \cdots, 1)^T, \quad b \equiv (0, 1, 0, 1, \cdots, 0)^T
\]

(4.2)

\[
e \equiv a + b = (1, 1, 1, \cdots, 1)^T.
\]

By (1.21) and (4.1) as (even) $N \to \infty$

\[
eg \Phi_1(+ \infty) = \frac{1}{2} \int_{-\infty}^{+\infty} \Phi_1(y) dy = c_N^{1/2} N^{-1/2} (2m)^{-1/2} (b + o(1))
\]

(4.3)

\[
eg \Phi_2(+ \infty) = \frac{1}{2} \int_{-\infty}^{+\infty} \Phi_2(y) dy = c_N^{1/2} N^{-1/2} (2m)^{-1/2} (a + o(1)).
\]

We need also the following result. Recall the notation (1.23), (3.11).
Proposition 4.1. For any fixed \( j \in \mathbb{N} \) there exist \( C, c > 0 \) such that the following holds as \( N \to \infty \),

\[
\tag{4.4} \left| \phi_{N+j}(t^{(N)}) - \frac{\alpha_{N}^{1/4} N^{1/6} 2^{1/4}}{c_{N}^{1/2}} \text{Ai}(t) \right| \leq Cc_{N}^{-1/2} N^{-1/6} e^{-ct}, \quad t \in I_{N} \cup II_{N}.
\]

This estimate implies that for a fixed \( j \in \mathbb{Z} \) there exist \( C, c > 0 \) such that

\[
\tag{4.5} \left| \int_{\xi(N)}^{\infty} \phi_{N+j}(s) \, ds - \frac{c_{N}^{1/2}}{N^{1/2}} \frac{2^{1/4}}{\alpha_{N}^{1/4}} \int_{\xi}^{\infty} \text{Ai}(t) \right| \leq Cc_{N}^{1/2} N^{-5/6} e^{-ct}, \quad \xi \in I_{N} \cup II_{N}.
\]

Proof. Assume first that \( j = 0 \). It was shown in \cite[Thm. 2.2]{DKMVZ2} that (in our notation)

\[
\tag{4.6} \phi_{N}(t^{(N)}) = c_{N}^{-1/2} \left[ \alpha_{N}^{1/4} N^{1/6} \hat{F}_{N}(1 + t_{N}) \, \text{Ai}(g_{N}(t)) \, (1 + O(N^{-1})) \right. \\
- \left. \alpha_{N}^{-1/4} N^{-1/6} (\hat{F}_{N}(1 + t_{N}))^{-1} \, \text{Ai}'(g_{N}(t)) \, (1 + O(N^{-1})) \right]
\]

where the error terms are uniform for \( t \in I_{N} \). Using \cite[8.38]{DKMVZ2} we immediately estimate the second term above by \( Cc_{N}^{-1/2} N^{-1/6} e^{-ct} \) uniformly for \( t \in I_{N} \). The part of the first term that corresponds to \( O(N^{-1}) \) is estimated similarly by \( Cc_{N}^{-1/2} N^{-5/6} e^{-ct}, t \in I_{N} \). To estimate the leading part of the first term we write

\[
\hat{F}_{N}(1 + t_{N}) \, \text{Ai}(g_{N}(t)) = \hat{F}_{N}(1) \, \text{Ai}(t) \\
+ \hat{F}_{N}(1) \left[ \text{Ai}(g_{N}(t)) - \text{Ai}(t) \right] + \text{Ai}(g_{N}(t)) \left[ \hat{F}_{N}(1 + t_{N}) - \hat{F}_{N}(1) \right].
\]

By \cite[(3.10) and (3.12)]{DKMVZ2}, \( \hat{F}_{N}(1) = 2^{1/4} \). By formula \cite[8.34]{DKMVZ2} and \cite[8.38]{DKMVZ2}

\[
|\text{Ai}(g_{N}(t)) - \text{Ai}(t)| \leq CN^{-2/3} t^{2} e^{-ct} \leq C' N^{-2/3} e^{-ct}, \quad t \in I_{N}.
\]

Also using \cite[8.35]{DKMVZ2} and \cite[8.14]{DKMVZ2} we obtain

\[
|\text{Ai}(g_{N}(t))| \cdot |\hat{F}_{N}(1 + t_{N}) - \hat{F}_{N}(1)| \leq CN^{-2/3} t^{2} e^{-ct} \leq C' N^{-2/3} e^{-ct}, \quad t \in I_{N}.
\]

Combining the above estimates we find that

\[
|\hat{F}_{N}(1 + t_{N}) \, \text{Ai}(g_{N}(t)) - 2^{1/4} \text{Ai}(t)| \leq CN^{-2/3} t e^{-ct}, \quad t \in I_{N},
\]

which completes the proof of \cite[4.3]{DKMVZ2} for \( j = 0 \) and \( t \in I_{N} \). We now consider \cite[4.3]{DKMVZ2} for \( j \neq 0 \) and \( t \in II_{N} = [\delta_{N} N^{2/3}, \infty) \). For such \( t \), by \cite[8.39]{DKMVZ2},

\[
|\text{Ai}(t)| \leq Ce^{-t} \leq Ce^{-cN^{2/3} e^{-t/2}}
\]

and hence

\[
\tag{4.7} |\phi_{N+j}(t^{(N)})| \leq Cc_{N}^{-1/2} N^{-1/6} e^{-ct}.
\]

Also from \cite[(4.8)]{DKMVZ2}, we find

\[
|\phi_{N}(t^{(N)})| \leq Cc_{N}^{-1/2} e^{-cN} e^{-ct}.
\]

These two estimates for \( t \in II_{N} \), together with the previous estimate for \( t \in I_{N} \), yield \cite[4.3]{DKMVZ2} in the case \( j = 0 \) for all \( t \in [L_{0}, \infty) \).

Now fix any \( j \in \mathbb{Z} \) and write

\[
\phi_{N+j} \left( c_{N} \left( 1 + \frac{t}{\alpha_{N} N^{2/3}} \right) + d_{N} \right) = \phi_{N+j} \left( c_{N+j} \left( 1 + \frac{t_{N+j}}{\alpha_{N+j} (N+j)^{2/3}} \right) + d_{N+j} \right)
\]
where
\[ t_{N,j} = t \cdot \frac{c_N \alpha_{N+j} (N + j)^{2/3}}{c_{N+j}} + \left( \frac{c_N}{c_{N+j}} - 1 \right) \cdot \alpha_{N+j}(N + j)^{2/3} + \frac{d_{N+j}}{c_{N+j}} \cdot \left( \frac{d_N}{d_{N+j}} - 1 \right) \cdot \alpha_{N+j}(N + j)^{2/3} = (1 + O(N^{-1/3})) t \]
(4.8)

by (3.11), (3.12), (3.10) (2). In particular, as \( N \to \infty, t_{N,j} \geq (1 - \frac{1}{2} \text{sgn} L_0) L_0. \) Now the RHS of (4.7) can be written as \( \phi'((t_{N,j})^{(N)'} \) where \( N' = N + j. \) Applying the estimate (4.4) just derived for \( j = 0, \) with \( L_0 \) replaced by \( (1 - \frac{1}{2} \text{sgn} L_0) L_0, \) we obtain

\[ |\phi'((t_{N,j})^{(N)'}) - \frac{1/4}{c_{N+j}}(N + j)^{1/6} 2^{1/4} \alpha_{N+j} |(t_{N,j})| \leq Cc_{N+j}(N + j)^{-1/6} e^{-c t N_{j}} \]
(4.9)
for all \( t \geq L_0. \) Using (4.8), and also (3.1), (3.10) (2), together with the elementary estimate

\[ |\text{Ai}(t_{N,j}) - \text{Ai}(t)| \leq C' N^{-1/3} e^{-c't} \]
(4.10)

(see (4.9)), we obtain (4.4) from (4.10) for any fixed \( j \in \mathbb{Z}. \)

Finally (4.10) follows readily by integrating (4.10). \( \square \)

Recall the notation (1.12). Proposition 4.1 implies that for \( j = 1, 2 \) one has uniformly for \( t, \xi, \eta \geq L_0 \)

\[ \left| \Phi_j(t^{(N)}) - \frac{1/4}{c_N^{1/2}} N^{1/6} 2^{1/4} \alpha_N^{1/2} \text{Ai}(t^{(N)}) \right| \leq Cc_{N}^{-1/2} N^{-1/6} e^{-c t} \]
(4.11)

\[ \left| \int_{\xi^{(N)}}^{\eta^{(N)}} \Phi_j(s) ds - \left( \frac{c_N^{1/2}}{N^{1/2}} 2^{1/4} \alpha_N^{1/2} \int_{\xi}^{\eta} \text{Ai}(t) dt \right) \right| \leq Cc_{N}^{1/2} N^{-5/6} e^{-c \xi} \]
(4.12)

4.2. The case \( \beta = 4. \)

4.2.1. The contribution of the correction term to the 12 entry of \( K_{N,4}. \) Since \((SD)(x,y) = -\partial_y S(x,y), \) the correction term in (1.13) has the form

\[ -(\Phi_2(x)^T \cdot D_{21} \cdot \Phi_1(y) - \Phi_2(x)^T \cdot D_{21} C_{11}^{-1} B_{11} D_{12} \cdot \Phi_2(y). \]

Set \( x = \xi^{(N)}, y = \eta^{(N)}. \) Recall \( n = 2m - 1, 2m = \text{deg} V. \) Note that by [DG] (2.13)

\[ D_{21} = \begin{pmatrix} \varepsilon_{N} & \varepsilon_{N} & \cdots \varepsilon_{N}^{n-1/2} \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} + o(1) \]
(4.12)

and \( D_{21} = O(c_{N}^{2m-1}) = O(N^{1/2(2m)}) \) as \( N \to \infty \) (see [5.11]). Also since \( C_{11} = I - B_{12} D_{21}, \) we see from [ibid., (2.19)] that \( C_{11}^{-1} B_{11} \) is skew symmetric, being the lower right corner of the skew symmetric matrix \( D_{N}^{-1}. \) (Note that \( B_{11} \) is the lower right \( n \times n \) corner of \( \varepsilon_{N}.) \) Hence \( D_{21} C_{11}^{-1} B_{11} D_{12} \) in (4.11) is also skew, and using
[ibid., (2.13)] and the fact that \( C_{11}^{-1} \) is bounded as \( N \to \infty \), we see that \( D_{21} C_{11}^{-1} B_{11} D_{12} = O(N^{1-1/(2m)}) \) as \( N \to \infty \). Recall that the 12 entry in \( K_{N,4}(\xi^{(N)},\eta^{(N)}) \) has an overall scaling factor \( \frac{c_{N}^{2}}{\alpha_{N}^{2} N^{4/3}} \). Substituting the leading term in the representation of \( \Phi_{j} \) in (4.10) into the first term in (4.11), and using (4.12), we obtain

\[
\frac{c_{N}^{2}}{\alpha_{N}^{2} N^{4/3}} \frac{m_{2m}}{2^{m-1}} \frac{c_{N}^{2m-1}}{N} \frac{1/2 N^{1/3} 2^{1/2}}{c_{N}} (\Sigma_{n} + o(1)) \text{Ai}(\xi) \text{Ai}(\eta)
\]

where \( o(1) \) is independent of \( \xi, \eta \) and \( \Sigma_{n} \) denotes the sum of all elements of the first (binomial) matrix on the RHS in (4.12). Using the formula preceding [ibid., (6.7)] one finds

\[
\Sigma_{n} = \frac{1}{2} \frac{m(m!)^{2}}{(m!)^{2}}.
\]

Recall that by (3.10)(2), [ibid., (2.14)] and (3.9),

\[
\alpha_{N} = 2m^{2/3} + o(1), \quad \frac{c_{N}^{2m} m_{2m}}{N} \frac{1/2 N^{1/3} 2^{1/2}}{c_{N}}(\Sigma_{n} + o(1)) \text{Ai}(\xi) \text{Ai}(\eta) \leq C e^{-\xi}, \quad \xi \geq L_{0}.
\]

Inserting these estimates, (4.13) becomes

\[
-\frac{1}{2} \text{Ai}(\xi) \text{Ai}(\eta) + o(1) e^{-\xi} e^{-\eta}
\]

uniformly for \( \xi, \eta \geq L_{0} \) and \( o(1) \) is independent of \( \xi, \eta \). The error that was made by substituting only the leading term in (4.10) in the first term in (4.11), is estimated as follows:

\[
O(c_{N}^{2} N^{-4/3}) O(c_{N}^{2m-1}) \left\{ O(N^{1/6} c_{N}^{-1/2}) c_{N}^{-1/2} N^{-1/6} (|\text{Ai}(\xi)| e^{-c_{N} \xi} + |\text{Ai}(\eta)| e^{-c_{N} \eta})
\]

\[
+ O(N^{-1/3} c_{N}^{-1}) e^{-c_{N} \xi} e^{-c_{N} \eta} \right\}
\]

\[
\leq O(N^{-1/3}) e^{-c_{N} \xi} e^{-c_{N} \eta}
\]

uniformly for \( \xi, \eta \geq L_{0} \), and independent of the degree \( 2m \) of \( V \).

Next we substitute the leading terms in the representation of \( \Phi_{2} \) in (4.11) in the second term in (4.11). By the skew symmetry of \( D_{21} C_{11}^{-1} B_{11} D_{12} \) noted above, the result is precisely zero. The error that is made by such a substitution is estimated in exactly the same way as in (4.15) and is also of order

\[
O(N^{-1/3}) e^{-c_{N} \xi} e^{-c_{N} \eta}
\]

uniformly for \( \xi, \eta \geq L_{0} \). We conclude that the contribution of the correction term to the 12 entry is given by

\[
-\frac{1}{2} \text{Ai}(\xi) \text{Ai}(\eta) + o(1) e^{-c_{N} \xi} e^{-c_{N} \eta}
\]

uniformly for \( \xi, \eta \geq L_{0} \).
4.2.2. The contribution of the correction term to the 11 and 22 entries of $K_{N,4}$. We consider the 11 entry of $K_{N,4}$ (the 22 entry is analyzed in the same way). The correction term in (4.18) has the form

$$\Phi_2(x)^T \cdot D_{21} \cdot \left( - \int_y^\infty \Phi_1(t) \, dt \right)$$

(4.18)

$$+ \Phi_2(x)^T \cdot D_{21} c_{11}^{-1} B_{11} D_{12} \cdot \left( - \int_y^\infty \Phi_2(t) \, dt \right).$$

We set $x = \xi^{(N)}$, $y = \eta^{(N)}$ in (4.18). The 11 (and 22) entry in $K_{N,4}(\xi^{(N)}, \eta^{(N)})$ has an overall scaling factor $\alpha_N N^{-1/3}$. Hence, substituting the leading terms in the representation of $\Phi_2, \int \Phi_1$ in (4.10) into the first term in (4.18) and using (4.12), we obtain

$$\frac{c_N}{\alpha_N N^{2/3}} \frac{m k_2 m}{2^{m-1} c_N^{2m-1}} \frac{c_N^{1/4} N^{1/2} 21/4}{\alpha_N^{3/4} N^{1/2}} (\Sigma_n + o(1)) \text{Ai}(\xi) \left( - \int_\eta^\infty \text{Ai}(t) \, dt \right)$$

(4.19)

where $o(1)$ is independent of $\xi, \eta$. Computing the factor and using (4.14) as above we see that (4.19) becomes

$$- \frac{1}{2} \text{Ai}(\xi) \int_\eta^\infty \text{Ai}(t) \, dt + o(1)e^{-\xi}e^{-\eta}$$

uniformly for $\xi, \eta \geq L_0$ and $o(1)$ is independent of $\xi, \eta$. The error that was made by substituting only the leading terms for $\Phi_2, \int \Phi_1$ in (4.14) into the first term in (4.18), is estimated as follows:

$$O(c_N N^{-2/3}) O(c_N^{2m-1}) \left( \frac{N^{1/6} c_N^{1/2}}{N^{5/6}} + \frac{1}{N^{1/6} c_N^{1/2}} \frac{N^{1/2}}{N^{1/6} c_N^{1/2}} \frac{1}{N^{5/6}} \frac{N^{1/6}}{N^{1/6} c_N^{1/2}} \right) e^{-c_\xi}e^{-c_\eta}$$

(4.20)

$$= O(N^{1/3})(N^{-2/3} + N^{-2/3} + N^{-1})e^{-c_\xi}e^{-c_\eta}$$

$$= O(N^{-1/3})e^{-c_\xi}e^{-c_\eta}$$

uniformly for $\xi, \eta \geq L_0$, and again independent of the degree $2m$ of $V$.

Next we substitute the leading terms in the representation of $\Phi_2, \int \Phi_2$ into (4.10) in the second term in (4.18). Again by skew symmetry, the result is precisely zero. The error that is made by such a substitution is estimated in exactly the same way as in (4.20) and also has order

$$O(N^{-1/3})e^{-c_\xi}e^{-c_\eta}$$

(4.21)

uniformly for $\xi, \eta \geq L_0$. We conclude that the contribution of the correction term to the 11 entry is given by

$$- \frac{1}{2} \text{Ai}(\xi) \int_\eta^\infty \text{Ai}(t) \, dt + o(1)e^{-c_\xi}e^{-c_\eta}$$

(4.22)

uniformly for $\xi, \eta \geq L_0$ (for the 22 entry $\xi$ and $\eta$ should be interchanged).
4.2.3. The contribution of the correction term to the 21 entry of $K_{N,4}$. By (4.10), the correction term in the 21 entry of $K_{N,4}$ is given by
\[
\int_x \Phi_2^T(s) \, ds \cdot D_{21} \cdot \int_y \Phi_1(t) \, dt
\]
(4.23)
\[
+ \int_x \Phi_2^T(s) \, ds \cdot D_{21} C_{11}^{-1} B_{11} D_{12} \cdot \int_y \Phi_2(t) \, dt.
\]
Again we replace $x = \xi^{(N)}$, $y = \eta^{(N)}$. Recall that the 21 entry in $K_{N,4}(\xi^{(N)}, \eta^{(N)})$ has no overall scaling factor. Substituting the leading terms in the representation of $\int \Phi_j, j = 1, 2$, in (4.10) into the first term in (4.23), the result is again uniformly for $\xi, \eta$.

Finally, we substitute the leading terms in the representation of $\int \Phi_j, j = 1, 2$, in (4.10) into the second term in (4.23). By the skew symmetry the result is again precisely zero. The error that is made by such a substitution is estimated in exactly the same way as in (4.24) and is also of order
\[
O(c_N^{2m-1}) \left( 2 c_N^{1/2} \frac{c_N^{1/2}}{N^{1/2} N^{5/6}} + c_N N^{5/3} \right) e^{-c_N^2} e^{-c_N \eta}
\]
(4.25)
\[
= O(N^{1/3}) (N^{-4/3} + N^{-5/3}) e^{-c_N^2} e^{-c_N \eta}
\]
\[
= O(N^{-1/3}) e^{-c_N^2} e^{-c_N \eta}
\]
uniformly for $\xi, \eta \geq L_0$, here all order factors are independent of $\xi, \eta$. (Here we have used $|\int_0^\infty \Phi(s) \, ds| \leq C e^{-c_N^2}$ uniformly for $\xi \geq L_0$.)

Finally, we substitute the leading terms in the representation of $\int \Phi_j, j = 1, 2$, in (4.10) into the second term in (4.23). By the skew symmetry the result is again precisely zero. The error that is made by such a substitution is estimated in exactly the same way as in (4.24) and is also of order
\[
O(c_N^{2m-1}) \left( 2 c_N^{1/2} \frac{c_N^{1/2}}{N^{1/2} N^{5/6}} + c_N N^{5/3} \right) e^{-c_N^2} e^{-c_N \eta}
\]
(4.26)
\[
\frac{1}{2} \int_x \Phi_i(s) \, ds \int_y \Phi(t) \, dt + o(1) e^{-c_N^2} e^{-c_N \eta}
\]
uniformly for $\xi, \eta \geq L_0$.

4.3. The case $\beta = 1$. As we will see, this case is more involved than the case $\beta = 4$. Consider the $2n \times 2n$ ($n = 2m - 1$, $2m = \deg V$) matrix $AC(I_{2n} - BAC)^{-1}T$ in the $\beta = 1$ correction term in (4.12) as a two by two block matrix with blocks of size $n \times n$. Denote the upper left and the upper right blocks by $G_{11}$ and $G_{12}$, respectively. With this notation the correction term has the form
\[
-\Phi_1(x)^T \cdot G_{11} \cdot c \Phi_1(y) - \Phi_1(x)^T \cdot G_{12} \cdot c \Phi_2(y).
\]
(4.27)
As in (4.12) let $R = R_n$ denote the $n \times n$ matrix with all entries zero apart from ones on the anti-diagonal (thus $R_{i,j} = 1$ if $j = n - i + 1, 1 \leq i \leq n$, and $R_{i,j} = 0$ for others).
otherwise). Note that \( R^2 = I_n \). Define

\[
\tilde{G}_{11} \equiv -RD_{21}C_{11}^{-1}B_{11}D_{12}R. 
\]

Note from Subsection 4.2.1 that \( D_{21}C_{11}^{-1}B_{11}D_{12} \) is skew and of order \( O(N^{1-1/(2m)}) \) as \( N \to \infty \). Hence \( \tilde{G}_{11} \) is also skew and has the same order as \( N \to \infty \). We need the following result.

**Proposition 4.2.** As (even) \( N \to \infty \) we have \( G_{11}, G_{12} = O(N^{1-1/(2m)}) \), more precisely

\[
G_{11} = \tilde{G}_{11} + o(N^{1-1/(2m)}), \quad N \to \infty, 
\]

and also

\[
G_{12} = D_{12} + o(N^{1-1/(2m)}), \quad N \to \infty. 
\]

**Proof.** It was shown in [DG, Theorem 2.3] that, as \( N \to \infty \),

\[
BA_{22} = -R(BA)_{11}R + o(1) 
\]

\[
BAC = \begin{pmatrix}
0 & 0 \\
(BA)_{21} + o(1) & (BA)_{22} + o(1)
\end{pmatrix}. 
\]

Denote

\[
T \equiv I_n - (BAC)_{22} = I_n - (BA)_{22} + o(1) = I_n + R(BA)_{11}R + o(1) = RC_{11}R + o(1). 
\]

It was shown in [DG, Theorem 2.6] that, as \( N \to \infty \), \( T \) approaches a constant nondegenerate matrix. Thus

\[
(I_{2n} - BAC)^{-1} = \begin{pmatrix}
I_n & 0 \\
T^{-1}((BA)_{21} + o(1)) & T^{-1}
\end{pmatrix}
\]

and simple algebra using [139], [140] now shows that in the product \( AC(I_{2n} - BAC)^{-1} \) we have by (4.31)

\[
(4.33) \quad G_{11}^T = A_{12}[I_n + (BA)_{111} + (BA)_{12}T^{-1}((BA)_{21} + o(1))] \\
G_{12}^T = A_{21}[I_n + (BA)_{111} + (BA)_{12}T^{-1}((BA)_{21} + o(1))].
\]

Using (4.32), this implies

\[
N^{-1+1/(2m)}G_{11}^T = N^{-1+1/(2m)}A_{12}[I_n + (BA)_{22}T^{-1}((BA)_{21} + o(1))] \\
= N^{-1+1/(2m)}A_{12}[T + (BA)_{22}][T^{-1}((BA)_{21} + o(1))] \\
= N^{-1+1/(2m)}A_{12}T^{-1}((BA)_{21} + o(1)).
\]

Now from

\[
(4.35) \quad BA = \begin{pmatrix}
B_{12}A_{21} & B_{11}A_{12} \\
B_{22}A_{21} & B_{21}A_{12}
\end{pmatrix}
\]
and (4.31), (4.32) we obtain

\[ N^{-1+1/(2m)} G_{ii}^T = N^{-1+1/(2m)} A_{12} (R R + R (B A)_{11})^{-1} B_{22} A_{21} + o(1) \]
\[ = N^{-1+1/(2m)} A_{12} R (I_n + (B A)_{11})^{-1} R B_{22} A_{21} + o(1) \]
\[ = N^{-1+1/(2m)} A_{12} R C_{11}^{-1} R B_{22} R A_{21} + o(1). \]

Using the asymptotic relations from [DG, Subsec. 5.2] we see that

\[ N^{-1+1/(2m)} R A_{12} R = N^{-1+1/(2m)} A_{21} + o(1) \]
\[ N^{-1+1/(2m)} R B_{22} R = -N^{1-1/(2m)} B_{11} + o(1) \]

from [DG] Subsec. 5.2] we see that

\[ N^{-1+1/(2m)} R G_{i1}^T R = N^{-1+1/(2m)} (R A_{12} R) C_{11}^{-1} (R B_{22} R) (R A_{21} R) + o(1) \]
\[ = -N^{-1+1/(2m)} A_{21} C_{11}^{-1} B_{11} A_{12} + o(1) \]
\[ = N^{-1+1/(2m)} D_{21} C_{11}^{-1} B_{11} D_{12} + o(1). \]

As noted above, the matrix \( D_{21} C_{11}^{-1} B_{11} D_{12} \) is skew symmetric and hence

\[ N^{-1+1/(2m)} G_{11} = -N^{-1+1/(2m)} R D_{21} C_{11}^{-1} B_{11} D_{12} R + o(1) \]

which proves (4.29).

Now let us prove (4.30). From (4.35) we derive

\[ N^{-1+1/(2m)} G_{12}^T = N^{-1+1/(2m)} A_{21} [I_n + (B A)_{11} + (B A)_{12} T^{-1} (B A)_{21}] + o(1) \]

and hence, because \( A_{21}^T = A_{12} = D_{12} \), we note that we just have to prove

\[ N^{-1+1/(2m)} A_{21} [(B A)_{11} + (B A)_{12} T^{-1} (B A)_{21}] = o(1). \]

Since \( N^{-1+1/(2m)} A_{21} = O(1) \), it is sufficient to prove

\[ (B A)_{11} + (B A)_{12} T^{-1} (B A)_{21} = o(1). \]

By (4.35) the LHS is \( B_{12} A_{21} + B_{11} A_{12} T^{-1} B_{22} A_{21} \) and so we see that it is sufficient to show that

\[ B_{12} + B_{11} A_{12} T^{-1} B_{22} = o(N^{-1+1/(2m)}). \]

Using (4.32) this reduces to showing that

\[ B_{12} + B_{11} A_{12} R C_{11}^{-1} R B_{22} = o(N^{-1+1/(2m)}) \]

or

\[ R B_{12} R + (R B_{11} R) (R A_{12} R) C_{11}^{-1} (R B_{22} R) = o(N^{-1+1/(2m)}). \]

Using \( N^{-1+1/(2m)} R B_{12} R = -N^{-1+1/(2m)} B_{21} + o(1) \) which follows as in (4.34), we are reduced to proving finally

\[ -B_{21} + B_{22} A_{21} C_{11}^{-1} B_{11} = o(N^{-1+1/(2m)}). \]

But

\[ -B_{21} + B_{22} A_{21} C_{11}^{-1} B_{11} = 0 \]

by (taking the transposes of) [DG (5.12)]. The proof of Proposition 4.2 is complete. \( \square \)
Remark 4.1. The second relation in (4.31) was sharpened recently by Kriecherbauer and Vanlessen [KV] who showed that the $o(1)$ terms are in fact identically zero. One might hope that this improved result could be used to strengthen the estimates in (4.29), (4.30). This is indeed the case for (4.30): one can show that $G_{12} = D_{12}$ identically. However we have not been able to use [KV] to improve the estimate in (4.29).

4.3.1. The contribution of the correction term to the 12 entry of $K_{N,1}$. In view of (4.27), since $(SD)(x,y) = -\partial_y S(x,y)$, the correction term has the form

\begin{equation}
\Phi_1(x)^T \cdot G_{11} \cdot \Phi_1(y) + \Phi_1(x)^T \cdot G_{12} \cdot \Phi_2(y).
\end{equation}

Again set $x = \xi^{(N)}$, $y = \eta^{(N)}$. Using Proposition 4.2 and proceeding in the same way as in Subsection 4.2.1, we find that as $N \to \infty$, the term (4.38), multiplied as before by \( \frac{c_N}{\alpha_N N^{2/3}} \), becomes

\begin{equation}
 \frac{1}{2} \text{Ai}(\xi) \text{Ai}(\eta) + o(1)e^{-c\xi}e^{-c\eta}
\end{equation}

uniformly for $\xi, \eta \geq L_0$. Note that the sum of all elements of (the binomial matrix in the limiting form) $D_{12}$ is, up to a sign, the same as for $D_{21}$.

Remark: Note also that the only new element in the above proof as compared with the case $\beta = 4$ in Subsection 4.2.1 is that the matrix $G_{11}$ is only asymptotically (and not identically) skew symmetric. This leads to the estimate $o(1)e^{-c\xi}e^{-c\eta}$ in place of (4.39).

4.3.2. The contribution of the correction term to the 11 and 22 entries of $K_{N,1}$. We consider the 11 entry of $K_{N,1}$ (the 22 entry is considered in the same way). Using (4.27), we rewrite the correction term as

\begin{equation}
-\Phi_1(x)^T \cdot G_{11} \cdot \left( - \int_y^\infty \Phi_1(t) \, dt \right) - \Phi_1(x)^T \cdot G_{12} \cdot \left( - \int_y^\infty \Phi_2(t) \, dt \right)
- \Phi_1(x)^T \cdot G_{11} \cdot c\Phi_1(+\infty) - \Phi_1(x)^T \cdot G_{12} \cdot c\Phi_2(+\infty).
\end{equation}

Again set $x = \xi^{(N)}$, $y = \eta^{(N)}$. The first two terms can be treated in the same way as in Subsections 4.2.2 and 4.3.1. More precisely we find that the first two terms in (4.40), multiplied by \( \frac{c_N}{\alpha_N N^{2/3}} \), become, as $N \to \infty$

\begin{equation}
- \frac{1}{2} \text{Ai}(\xi) \int_\eta^\infty \text{Ai}(t) \, dt + o(1)e^{-c\xi}e^{-c\eta}
\end{equation}

uniformly for $\xi, \eta \geq L_0$. Now consider the (scaled) sum of the last two terms in (4.40)

\begin{equation}
- \frac{c_N}{\alpha_N N^{2/3}} \left( \Phi_1^{(N)}(\xi)^T \cdot G_{11} \cdot c\Phi_1(+\infty) + \Phi_1^{(N)}(\xi)^T \cdot G_{12} \cdot c\Phi_2(+\infty) \right).
\end{equation}

By (4.3), (4.10), Proposition 4.2, this becomes as $N \to \infty$

\begin{equation}
\frac{c_N}{\alpha_N N^{2/3}} \frac{1}{\alpha_N N^{1/6}2^{1/4}} \frac{1}{c_N^{1/2}} \frac{1}{(2m)^{1/2}N^{1/2}} \frac{1}{2^{1/2}} \left\{ e^{T} \cdot G_{11} \cdot (b + o(1)) + e^{T} \cdot G_{12} \cdot (a + o(1)) \right\} \text{Ai}(\xi) + o(1)e^{-c\xi}.
\end{equation}
Setting $a = e - b$, we find that (4.44) reduces to
\[ 1/2 \text{Ai}(\xi) + o(1)e^{-c\xi} + \left( e^T \cdot G_{11} \cdot b - e^T \cdot G_{12} \cdot b \right) \cdot \text{Ai}(\xi) \cdot O(N^{-1+1/(2m)}) . \]

So if we could prove
\[ e^T \cdot G_{11} \cdot b - e^T \cdot G_{12} \cdot b = o(N^{1-1/(2m)}), \quad N \to \infty, \]
then we would find that (4.44) equals
\[ 1/2 \text{Ai}(\xi) + o(1)e^{-c\xi} \]
uniformly for $\xi \geq L_0$.

We prove (4.45). We will, perhaps surprisingly, use a property of the $\beta = 4$ correlation kernel $S_{N/2,4}$: it is not clear how to prove (4.45) directly using the asymptotic properties of $(D\Phi_{N,j}^N, \Phi_{N+k}^N)$ and $(\epsilon\Phi_{N+j}^N, \Phi_{N+k}^N)$ given in [DG]. More precisely, (4.45) follows from (4.29), (4.30) and the relation
\[ b + C_{11}^{-1}B_{11}D_{12} \cdot a = o(1), \quad N \to \infty, \]
which is proved by dividing (4.46) by $(\frac{cN^2}{N})^{1/2}$ and using (4.3) as $N \to \infty$. Multiplying (4.47) from the left by $e^T \cdot RD_{21}$ and noting $b = R \cdot b$, $a = R \cdot a$, we find
\[ e^T \cdot RD_{21} \cdot b + e^T \cdot RD_{21}C_{11}^{-1}B_{11}D_{12} \cdot (e - b) = o(N^{1-1/(2m)}). \]

But the second matrix is skew symmetric (see (4.28) et seq.). By (4.29) the above relation becomes
\[ e^T \cdot RD_{21} \cdot b + e^T \cdot G_{11} \cdot b = o(N^{1-1/(2m)}). \]

But from (4.30)
\[ RD_{21} = -D_{12} + o(N^{1-1/(2m)}) \]
and hence (4.48), (4.30) imply (4.45).

Collecting the estimates (4.41), (4.46) we see that since $\int_{-\infty}^{\infty} \text{Ai}(t) \, dt = 1$, the correction term in the 11 entry has the form
\[ 1/2 \text{Ai}(\xi) \left( 1 - \int_\eta^\infty \text{Ai}(t) \, dt \right) + o(1)e^{-c\xi} \]
\[ = 1/2 \text{Ai}(\xi) \int_{-\infty}^{\eta} \text{Ai}(t) \, dt + o(1)e^{-c\xi} \]
uniformly for $\xi, \eta \geq L_0$. The correction term in the 22 entry has the same asymptotic form with $\xi$ and $\eta$ interchanged.

4.3.3. The contribution of the correction term to the 21 entry of $K_{N,1}$. By (1.29), (4.27) the correction term in this case has the form
\[ \left( \int_x^y \Phi_1(t)^T \, dt \right) \cdot G_{11} \cdot \epsilon \Phi_1(y) + \left( \int_x^y \Phi_1(t)^T \, dt \right) \cdot G_{12} \cdot \epsilon \Phi_2(y) \]
which equals
\[
\left( \int_x^y \Phi_1(t)^T \, dt \right) \cdot G_{11} \cdot \left( - \int_y^\infty \Phi_1(t) \, dt \right) \\
+ \left( \int_x^y \Phi_1(t)^T \, dt \right) \cdot G_{12} \cdot \left( - \int_y^\infty \Phi_2(t) \, dt \right) \\
+ \left( \int_x^y \Phi_1(t)^T \, dt \right) \cdot G_{11} \cdot \mathcal{E}_1(+) + \left( \int_x^y \Phi_1(t)^T \, dt \right) \cdot G_{12} \cdot \mathcal{E}_2(+) 
\]

by (1.47). Again set \( x = \xi^{(N)} \), \( y = \eta^{(N)} \). A calculation very similar to the one in Subsection 4.3.2, using the last estimate in (4.10) in place of the first, leads to the following asymptotic form for the 21 correction as \( N \to \infty \)

\[
(4.50) \quad -\frac{1}{2} \int_\xi^\eta \text{Ai}(s) \, ds + \frac{1}{2} \left( \int_\xi^\eta \text{Ai}(s) \, ds \right) \left( \int_\eta^\infty \text{Ai}(t) \, dt \right) + o(1)e^{-c \min(\xi, \eta)}
\]

uniformly for \( \xi, \eta \geq L_0 \). (Recall that there is no overall scaling factor for the 21 entry.) This completes the analysis of the contribution of the correction term to the 21 entry of \( K_{N,1} \).

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