New biharmonic bases 
in commutative algebras of the second rank and 
monogenic functions 
related to the biharmonic equation

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Abstract

Among all two-dimensional commutative algebras of the second rank a totally 
of all their biharmonic bases \(\{e_1, e_2\}\), satisfying conditions \((e_1^2 + e_2^2)^2 = 0, e_1^2 + e_2^2 \neq 0\), is found in an explicit form. A set of ”analytic” (monogenic) functions 
satisfying the biharmonic equation and defined in the real planes generated by the 
biharmonic bases is built. A characterization of biharmonic functions in bounded 
simply connected domains by real components of some monogenic functions is found.

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1 Statement of the Problem

We say that an associative algebra of the second rank with identity commutative over the field of complex (or real) numbers $\mathbb{C}$ ($\mathbb{R}$) is biharmonic if it contains at least one biharmonic basis, i.e., a basis $\{e_1, e_2\}$ satisfying the conditions

$$e_1^4 + 2e_1^2e_2^2 + e_2^4 = 0, \quad e_1^2 + e_2^2 \neq 0.$$  \hspace{1cm} (1)

A problem of describing all biharmonic algebras and all their biharmonic bases has been posed by I. P. Mel’nichenko in [1].

In [1], I. P. Mel’nichenko proved the uniqueness of the biharmonic algebra

$$\mathbb{B} := \{z_1 e + z_2 \rho : z_k \in \mathbb{C}, k = 1, 2\}, \quad \rho^2 = 0, e\rho = \rho, e^2 = e,$$ \hspace{1cm} (2)

here $e$ is the identity of algebra. Note that the algebra (2) is isomorphic to the four-dimensional algebras over the field of real numbers considered by L. Sobrero (cf., e.g., [3]) and A. Douglis [4].

V. F. Kovalev and I. P. Mel’nichenko [2] found a multiplication table for a biharmonic basis $\{e_1, e_2\}$:

$$e_1 = e, \quad e_2^2 = e_1 + 2ie_2,$$ \hspace{1cm} (3)

where $i$ is the imaginary complex unit.

I. P. Mel’nichenko claimed that he found a complete totally of biharmonic bases and described this set of biharmonic bases in [1] also.

There were considered in [5, 6] some $\mathbb{B}$-valued “analytic” functions $\Phi$, defined in domains which lies in the span of the vectors (3) over the field of real numbers, such that real components of function $\Phi$ (over the system of vectors: $e_k, ie_k, k = 1, 2$) satisfy the biharmonic equation

$$(\Delta_2)^2 u(x, y) := \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right) u(x, y) = 0,$$ \hspace{1cm} (4)

where $\Delta_2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, i.e., these components are biharmonic functions (in some domain of the Cartesian plane $xOy$ which depends on the domain where a function $\Phi$ is defined).

It is proved in [5] that for any biharmonic function $u$, defined in the bounded and simply connected domain, exist functions $\Phi$ such that their fixed component (before $e_1$) coincides with $u$, and, described the whole totally of these functions $\Phi$.

Consider a short summary of new results which are described in the proposed paper. A totally of biharmonic bases described in [1] is not complete, and the whole totaly is found also here. A constructive description of monogenic functions, defined in domains belong to the planes generated by any biharmonic bases, is found in an explicit form. It is proved that for any biharmonic function $u$, defined in the bounded and simply connected domain, exist some monogenic functions $\Phi$ such that their fixed component (before $e_1$) coincides with $u$, and, described the whole totally of these functions $\Phi$. 

2 Biharmonic bases

Here and what follows we assume to choose an upper or lower sign in formulas containing the symbol ±. The next theorem gives a constructive description of all biharmonic bases.

**Theorem 1.** All biharmonic bases (of the algebra $\mathbb{B}$) are expressed by the formulas:

$$
e_1 = \alpha_1 e + \beta_1 \rho, \quad e_2 = \pm i (\alpha_1 e + \beta_2 \rho), \quad (5)$$

where complex numbers $\alpha_k, \beta_k, k = 1, 2,$ satisfy conditions: $\alpha_1 \neq 0, \beta_1 \neq \beta_2$.

This theorem can be proved by similar arguments which were done at the proof of Theorem 1 in [7].

**Remark 1.** Biharmonic bases of the form (5) with $\beta_2 = \beta_1 - \frac{1}{2\alpha_1}$ (this and only this subset of all biharmonic bases (5)) are considered in [5].

3 Monogenic functions of the plane generated by elements of the biharmonic bases

Consider a plane $\mu_{e_1, e_2} := \{ xe_1 + ye_2 : x, y \in \mathbb{R} \}$ where $\{ e_1, e_2 \}$ is a fixed bases in (5). We use the Euclidean norm $\| a \| := \sqrt{|z_1|^2 + |z_2|^2}$, where $a = z_1 e_1 + z_2 e_2 \in \mathbb{B}, z_k \in \mathbb{C}, k = 1, 2.$ With a domain $D$ of the Cartesian plane $xOy$ we associate the congruent domain $D_\zeta := \{ \zeta = xe_1 + ye_2 \in \mu_{e_1, e_2} : (x, y) \in D \} \subset \mu_{e_1, e_2}$. In what follows,

$$(x, y) \in D, \quad \zeta = xe_1 + ye_2 \in D_\zeta, \quad z = x + iy \in \mathbb{C} \equiv \text{Re} z + i\text{Im} z; \quad x, y \in \mathbb{R}.\)$$

Note that every element $\zeta \in \mu_{e_1, e_2}, \zeta \neq 0$, is invertible (exists an element $\zeta^{-1} \in \mathbb{B}: \zeta \zeta^{-1} = e$).

Inasmuch as divisors of zero don’t belong to the plane $\mu_{e_1, e_2}$, one can define the derivative $\Phi'(\zeta)$ of function $\Phi: D_\zeta \longrightarrow \mathbb{B}$ in the same way as in the complex plane:

$$\Phi'(\zeta) := \lim_{h \rightarrow 0, h \not\in \mu_{e_1, e_2}} (\Phi(\zeta + h) - \Phi(\zeta)) h^{-1}.\)$$

We say that a function $\Phi: D_\zeta \longrightarrow \mathbb{B}$ is monogenic in a domain $D_\zeta$ iff the derivative $\Phi'(\zeta)$ exists at every point $\zeta \in D_\zeta$.

Every function $\Phi: D_\zeta \longrightarrow \mathbb{B}$ has a form

$$\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) ie_1 + U_3(x, y) e_2 + U_4(x, y) ie_2, \quad (6)$$

where $\zeta = xe_1 + ye_2, \quad U_k: D \longrightarrow \mathbb{R}, \quad k = 1, 4.$

Every real component $U_k, k = 1, 4$, in expansion (6) we denote by $U_k[\Phi]$, i.e., for $k \in \{1, \ldots, 4\}$: $U_k[\Phi(\zeta)] := U_k(x, y)$ for all $\zeta = xe_1 + ye_2 \in D_\zeta$.  


It follows from (5) formulas:

\[(e_k)^2 = (-1)^{k+1} \alpha_1 (\alpha_1 e + 2\beta_k \rho), \quad k = 1, 2, \quad e_1 e_2 = \pm io_1 (\alpha_1 e + (\beta_1 + \beta_2)) \rho. \quad (7)\]

In (7) and till the end of the paper by the double sign ± we shall mean the same single sign as in (5) for corresponding basis \{e_1, e_2\}.

Analogously to the case of the biharmonic basis (3) (see [2, 5]) we obtain the next theorem.

**Theorem 2.** A function \(\Phi: D_\zeta \longrightarrow \mathbb{B}\) is monogenic in a domain \(D_\zeta\) if and only if components \(U_k, k = 1, 4\), of the expression (6) are differentiable in the domain \(D\) and the following analog of the Cauchy – Riemann conditions is fulfilled:

\[\frac{\partial \Phi(\zeta)}{\partial y} e_1 = \frac{\partial \Phi(\zeta)}{\partial x} e_2. \quad (8)\]

**Remark 2.** Using (7), we obtain that in an extended form the condition (8) for the monogenic function (6) is equivalent to the system of four equations with respect to components \(U_k = U_k[\Phi], k = 1, 4\), in (6):

\[
\begin{align*}
\frac{\partial U_1(x,y)}{\partial y} & - (\pm) \frac{\partial U_4(x,y)}{\partial y} \pm \frac{\partial U_2(x,y)}{\partial x} + \frac{\partial U_3(x,y)}{\partial x} = 0, \\
\frac{\partial U_2(x,y)}{\partial y} & \pm \frac{\partial U_3(x,y)}{\partial x} - (\pm) \frac{\partial U_1(x,y)}{\partial x} + \frac{\partial U_4(x,y)}{\partial x} = 0, \\
2\text{Re}\beta_1 \frac{\partial U_1(x,y)}{\partial y} - 2\text{Im}\beta_1 \frac{\partial U_3(x,y)}{\partial y} & - (\pm) \frac{\partial U_4(x,y)}{\partial y} \pm \text{Im} (\beta_1 + \beta_2) \frac{\partial U_1(x,y)}{\partial x} + \\
2\text{Im}\beta_1 \frac{\partial U_1(x,y)}{\partial y} + 2\text{Re}\beta_1 \frac{\partial U_3(x,y)}{\partial y} & \pm \text{Re} (\beta_1 + \beta_2) \frac{\partial U_1(x,y)}{\partial x} - \\
2\text{Im}\beta_2 \frac{\partial U_4(x,y)}{\partial x} - 2\text{Re}\beta_2 \frac{\partial U_3(x,y)}{\partial x} & = 0, \\
\text{Re} (\beta_1 + \beta_2) \frac{\partial U_4(x,y)}{\partial y} + 2\text{Im}\beta_1 \frac{\partial U_3(x,y)}{\partial x} + 2\text{Re}\beta_2 \frac{\partial U_4(x,y)}{\partial x} & = 0.
\end{align*}
\]

(9) (10) (11) (12)

Take into consideration a variable and a domain of the complex plane of the form:

\[Z = \alpha_1 (x \pm iy), \quad D_Z := \{Z = \alpha_1 (x \pm iy) : (x, y) \in D\}. \quad (13)\]

Similar to the proof of theorems 1 and 2 in [5] with use of Theorem 2 we obtain an expression of monogenic functions via holomorphic functions of complex variable \(Z\) in the domain \(D_Z\).
Theorem 3. A function $\Phi: D_\zeta \to \mathbb{B}$ is monogenic in $D_\zeta$ if and only if the following equality if fulfilled

$$\Phi(\zeta) = F(Z) e + \left( \frac{\beta_1}{\alpha_1} Z \pm i (\beta_2 - \beta_1) y \right) F'(Z) + F_0(Z) \rho \equiv \Phi[F,F_0](\zeta) \forall \zeta \in D_\zeta,$$

where $F$, $F_0$ are some holomorphic functions of the complex variable $Z$ in the domain $D_Z$, $F'$ if the derivative of $F$.

In what follows, we mean by $\Phi[F,F_0]$ an expression of the form (14).

Remark 3. An expression (14) for monogenic functions with biharmonic basis (3) is found in [5], and before, for some particular class of domains $D_\zeta$ — in the paper [9].

Remark 4. A relation between monogenic functions (3) (related to the biharmonic basis (3)) with analytic by Douglis functions (considered, for example, in [10,11]) is found in [7].

Remark 5. Consider the biharmonic base $\{e_1, e_2\}$ such that $e_2 = e$. Then in (5) we have

$$e_1 = e, \quad e_2 = (\pm i)e + (\pm i\beta_2)\rho, \quad \beta_2 \in \mathbb{C}, \beta_2 \neq 0.$$

Thus, (3) turns onto

$$\Phi(\zeta) = F(x \pm iy) e + (\pm i\beta_2 y F'(x \pm iy) + F_0(x \pm iy)) \rho \forall \zeta \in D_\zeta.$$

The expression (16) is obtained for “monogenic functions” $\Phi: D_\zeta \to \mathbb{B}$, which are understood as continuous functions and differentiable by Gateaux (along the positive rays) in the paper [12] (a fact that a “convexity” of the domain $D_\zeta$ is omitted can be proved analogous to the similar fact in [7] after Theorem 3). Taking into account that monogenic function (16) is continuous also, obtain that these two kinds of monogeneity are equivalent in the case of the biharmonic bases (15).

4. Biharmonic functions as components of monogenic functions. The equality (14) and Theorem 2 yield that any monogenic function $\Phi: D_\zeta \to \mathbb{B}$ has a derivative $\Phi^{(n)}$ for every natural $n$, and components $U_k = U_k[\Phi], k = 1,4$, are infinitely differentiable in $D$. Therefore, using the equality

$$\left(\Delta^2_2\right) \Phi(\zeta) = \mathcal{L}(e_1,e_2)\Phi^{(4)}(\zeta) \equiv 0 \forall \zeta \in D_\zeta,$$  

we conclude that components $U_k, k \in \{1, \ldots, 4\}$, from the expansion (6) are biharmonic functions in the domain $D$.

Now and till the end of the paper we mean by the biharmonic basis [5] the following basis:

$$e_1 = e + \rho, \quad e_2 = i(e + 2\rho).$$
Note, that the basis (18) was not found in [1]. Then in (13) we take \( \alpha = 1 \) and the sigh “+”. Thus, the formula (14) turns onto the form

\[
\Phi(\zeta) = F(z) e + ((z + iy) F'(z) + F_0(z)) \rho \quad \forall \zeta \in D_\zeta.
\]  

(19)

With respect to the basis (18) the expression (14) turns onto the form

\[
\Phi(\zeta) = \left( F(z) + \tilde{F}(z) \right) e_1 + \tilde{F}(z) ie_2 \quad \forall \zeta \in D_\zeta,
\]  

(20)

here \( \tilde{F}(z) := F(z) - (z + iy) F'(z) - F_0(z) \) for every \( z \in D_z \).

**Lemma 1.** Let the domain \( D \) be bounded and simply connected. Any monogenic function \( \Phi_0 : D_\zeta \rightarrow \mathbb{B} \) such that

\[
U_1[\Phi_0(\zeta)] = 0 \quad \forall \zeta \in D_\zeta,
\]  

(21)

is expressed in the form:

\[
\Phi_0(\zeta) = a (\zeta + z \rho) + bie_1 + ce_2 + die_2,
\]  

(22)

where \( a, b, c, d \) are any real constants.

**Remark 6.** An analogous result for monogenic functions generated by the biharmonic basis (3) is considered in [5, Lemma 3]. Note, that in this case real components of an analog of the formula (22) are polynomials of degree no more then the second power with respect to real variables \( x \) and \( y \), unlike the formula (22) where components are polynomials of degree no more then the first power with respect to real variables \( x \) i \( y \).

**Lemma 2.** Let \( f_k : D_z \rightarrow \mathbb{C}, \; k = 1,2 \), are holomorphic functions of the complex variable. Then a monogenic function \( \Phi : D_\zeta \rightarrow \mathbb{B} \) defined by the formula

\[
\Phi(\zeta) := \Phi[F,F_0](\zeta) \quad \forall \zeta \in D_\zeta,
\]  

(23)

where

\[
F(z) := 2f_2(z), \; F_0(z) := 4f_2(z) - f_1(z) - 3zf'(z) \quad \forall z \in D_z,
\]  

(24)

satisfies the relation:

\[
U_1[\Phi(\zeta)] = \text{Re} \left( f_1(z) + \overline{z} f_2'(z) \right) (\overline{z} := x - iy) \quad \forall \zeta \in D_\zeta.
\]  

(25)

A proof of Lemma is a simple corollary of expressions (14) and (20).

The well-known fact is that any biharmonic function \( U_1 \) in simply connected domain \( D \) is expressed by the Goursat formula (cf., e.g., [13])

\[
U_1(x,y) = \text{Re} \left( \psi(z) + \overline{z} \varphi(z) \right),
\]  

(26)

where \( \psi, \varphi \) are holomorphic in \( D_z \) functions, \( \overline{z} := x - iy \).

Combining the previous results we get the following theorem.
Theorem 4. Let a function $U_1$ is biharmonic in the bounded and simply connected domain $D$ of the Cartesian plane $xOy$. Then all monogenic functions $\Phi: \mathbb{D}_\zeta \rightarrow \mathbb{B}$ such that $U_1[\Phi] \equiv U_1$ are expressed by the form

$$\Phi(\zeta) = \Phi[2\mathcal{F}, 4\mathcal{F} - \psi - 3z\varphi](\zeta) + \Phi_0(\zeta) \quad \forall \zeta \in \mathbb{D}_\zeta, \quad (27)$$

where $\Phi_0$ is defined by the formula (22), $\mathcal{F}$ is a primitive function for $\varphi$.

Remark 7. An analogous result for monogenic functions generated by the biharmonic basis (3) was found in [5, Theorem 5].

Remark 8. An analogous results for monogenic functions related to some bases generated by the stress equations for certain orthotropic deformations were found in [8,14].

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References

[1] Mel’nichenko I. P. *Biharmonic bases in algebras of the second rank*, Ukr. Math. J., **38** (1986), No. 2, 252–254.

[2] Kovalev V. F. Mel’nichenko I. P. *Biharmonic functions on the biharmonic plane*, Reports Acad. Sci. USSR, ser. A., No. 8 (1981), 25–27 [in Russian].

[3] Sobrero L. *Nuovo metodo per lo studio dei problemi di elasticità, con applicazione al problema della piastra forata*. Ricerche di Ingegneria. **13** (1934), No. 2, 255–264 [in Italian].

[4] Douglis A., *A function-theoretic approach to elliptic systems of equations in two variables*, Communications on Pure and Applied Mathematics, **6** (1953), No. 2, 259–289.

[5] Grishchuk S. V. (the same as: Gryshchuk S. V.), Plaksa S. A. *Monogenic functions in a biharmonic algebra*, Ukr. Mat. Zh., **61** (2009), No. 12, 1587–1596 [in Russian]; *English translation*: Ukr. Math. J., **61** (2009), No. 12, 1865–1876.

[6] Gryshchuk S. V., Plaksa S. A. *Basic Properties of Monogenic Functions in a Biharmonic Plane*, in: “Complex Analysis and Dynamical Systems V”, Contemporary Mathematics, **591** (2013), Amer. Math. Soc., Providence, RI, 127–134.

[7] Gryshchuk S. V. *Commutative omplex algebras of the second rank with unity and some cases of plane orthotropy I*, Ukr. Mat. Zh., **70** (2018), No. 8, 1058–1071 [in Ukrainian]; *English translation*: Ukr. Math. J., **70** (2019), No. 8, 1221–1236.

[8] Gryshchuk S. V. *Commutative omplex algebras of the second rank with unity and some cases of plane orthotropy II*, Ukr. Mat. Zh., **70** (2018), No. 10, 1382–1389 [in Ukrainian]; *English translation*: Ukr. Math. J., **70** (2019), No. 10, 1594–1603.

[9] V. F. Kovalev, *Biharmonic Schwarz problem*, Preprint No. 86.16. Institute of Mathematics, Acad. Sci. USSR, Inst. of Math. Publ. House, Kiev, 1986 [in Russian].

[10] Soldatov A. P. *Hyperanalytic functions and their applications*, J. of Math. Sci. **132** (2006), No. 6, 827–882.

[11] Soldatov A. P. *On the theory of anisotropic plane elasticity*, Contemporary Mathematics. Fundamental Trends, RUDN, Moscow, **60** (2016), pp. 114–163 [in Russian].

[12] Shpakivskyi V. S. *Monogenic functions in finite-dimensional commutative associative algebras*, Zb. Pr. Inst. Mat. NAN Ukr., **12** (2015), No. 3, 251–268.

[13] Muskhelishvili N.I. *Some basic problems of the mathematical theory of elasticity. Fundamental equations, plane theory of elasticity, torsion and bending*. English transl. from the 4th Russian edition by R.M. Radok, Noordhoff International Publishing: Leiden, 1977.
[14] Gryshchuk S. V. *Monogenic functions in two dimensional commutative algebras to equations of plane orthotropy*, Pratsi Inst. Prikl. Mat. Mikh. NANU (Proc. of In-te of Appl. Math. and Mech. of NAS of Ukraine), **32** (2018), 18–29 [in Ukrainian].