Polynomial Hamiltonian form of General Relativity

M. O. Katanaev *

Steklov Mathematical Institute,
Gubkin St. 8, Moscow, 119991

27 October 2005

Abstract

The phase space of general relativity is extended to a Poisson manifold by inclusion of the determinant of the metric and conjugate momentum as additional independent variables. As a result, the action and the constraints take a polynomial form. We propose a new expression for the generating functional for the Green’s functions. We show that the Dirac bracket defines a degenerate Poisson structure on a manifold and the second class constraints are the Casimir functions with respect to this structure. As an application of the new variables, we consider the Friedmann universe.

1 Introduction

Canonically formulating any model of mathematical physics is the most important step when analyzing equations of motion, in particular, when setting and analyzing the Cauchy problem. It also provides the basis for canonically quantizing models. Canonically formulating general relativity is technically involved, and many papers are devoted to this problem. We mention only a few of them. Dirac first formulated general relativity self-consistently in the second-order formulism [1]. He took the metric components $g_{\alpha\beta}$ as the independent variables and showed that the Hamiltonian of the gravitational field is equal to a linear combination of constraints. Afterwards, Arnowitt, Deser, and Misner in the series of papers resulting in the review [2] essentially simplified the calculations and clarified the geometrical meaning of the canonical momenta, expressing them in terms of the extrinsic curvature of a spacelike hypersurface imbedded in a four-dimensional space-time. The expression for the Hamiltonian was found in the first order formalism when the metric $g_{\alpha\beta}$ and the symmetric affine connection $\Gamma_{\{\alpha\beta\}}^\gamma$ considered independent variables. In essence, this approach simplified the calculations. In the pioneering papers [1, 2], the constraint algebra was not calculated explicitly, but the constraints were shown to be consistent with the equations of motion (the first class constraints). In [2], the role of boundary terms was also analyzed, and the total energy of the gravitational field for an asymptotically flat space-time was defined in terms of the surface integral. The role of boundary terms was discussed in a more general form in [3].

*E-mail: katanaev@mi.ras.ru
In [4], general relativity was canonically formulated and a generalization of the Schrödinger equation for the wave function of the universe, which later was called the Wheeler–DeWitt equation, was considered in detail. The constraint algebra in general relativity was first calculated explicitly there.

The constraint algebra for any model invariant with respect to general coordinate transformations was obtained in [5] assuming that the model is self-contained and that the constraints generate the general coordinate transformations for the canonical variables (see also [?]). We note that the self-consistency of a model (the closedness of the constraint algebra) is a very strong assumption. Conversely, for a given model, the constraint algebra, because it is not known beforehand, must be calculated explicitly to prove the self-consistency of the model.

Dirac [7] and Schwinger [8] started the investigation of the vielbein Hamiltonian formulation of general relativity using the time gauge for simplicity. The Hamiltonian formulation in a general case without gauge fixing was given much later because of serious technical difficulties [9, 10].

The Hamiltonian formulation of general relativity contains constraints that are non-polynomial on space-section metric components, and this is an essential obstacle to analyzing and quantizing the theory of gravity. The polynomial Hamiltonian formulation given by Ashtekar [11] attracted much interest in recent years. He proposed using complex variables in the extended phase space, which are tensor densities and lead to polynomial constraints. Here, we consider a different extension of the phase space [12] with the metric determinant and its conjugate momentum considered additional variables. We show that the Poisson structure on the extended phase space is degenerate and that the initial phase space is mapped on a subspace of the extended phase space by the canonical transformation. All new canonical variables are real tensor densities, and the constraints take a polynomial form.

We propose a functional integral over a Poisson manifold as a new expression for the generating functional for the Green’s functions. This form of the integral reduces to the standard expression for the generating functional over the phase space [13] after integration over the additional variables, which is removed by two supplementary $\delta$-functions. We prove that the corresponding Jacobian of coordinate transformation is equal to unity.

The plan of the paper is as follows. In Secs. 2–6, we describe the transition from the Hilbert–Einstein Lagrangian to the Hamiltonian in detail to avoid sending the reader to the original papers, where a greater part of the calculations is usually omitted. Moreover, we consider a general case of affine space-time geometry in Sec. 3 when describing the geometry of hypersurfaces. In this case, the antisymmetric part of external curvature of a hypersurface is defined by the torsion tensor. This is important for the canonical formulation of general relativity in the vielbein formulation and for the Hamiltonian formulation of models with absolute parallelism. The canonical transformation between the phase space of general relativity and the submanifold of the extended Poisson manifold is described in Sec. 7. We show that all constraints and the action of the model take a polynomial form in the extended space, and we compute the constraint algebra. We propose the expression for the generating functional for the Green’s functions on the Poisson manifold in Sec. 8. In Sec. 9 as an application of the new variables, we consider the case of a homogeneous and isotropic universe, a case where all the calculations can be easily checked.
2 ADM parameterization of a metric

To analyze the Hamiltonian structure of the general relativity equations Arnowitt, Deser, and Misner used the special parameterization of the metric (ADM-parameterization), which essentially simplifies calculations [2]. We consider a manifold \( M, \dim M = n \) equipped with a metric of Lorentzian signature \((+−...−)\). We deliberately do not restrict ourselves to the most important case of four-dimensional space-time because gravity models in higher and lower number of dimensions have attracted much interest recently. Let \( \{x^\alpha\}, \alpha = 0, 1, \ldots, n−1 \) denote the local coordinates. We choose the time coordinate \( t = x^0 \), and then \( \{x^\alpha\} = \{x^0, x^\mu\}, \mu = 1, \ldots, n−1 \). In what follows, the letters from the beginning of the Greek alphabet \((\alpha, \beta, \ldots)\) range all index values, while the letters from the middle \((\mu, \nu, \ldots)\) range only the space-related values. This rule is easily remembered from the inclusions \( \{\mu, \nu, \ldots\} \subset \{\alpha, \beta, \ldots\} \) and \( \{1, 2, \ldots\} \subset \{0, 1, 2, \ldots\} \). The ADM parameterization of the metric has the form

\[
g_{\alpha\beta} = \left( N^2 + N^\rho N_\rho N_\nu g_{\mu\nu} \right), \tag{1}\]

where \( g_{\mu\nu} \) is the metric on \((n−1)-dimensional manifold sections \( x^0 = \text{const} \). In the chosen parameterization, we introduced the same number of functions \( N \) and \( N_\mu \) instead of the \( n \) metric components containing at least one time index \( g_{00} \) and \( g_{0\mu} \). Here, \( N^\rho = \hat{g}^{\rho\mu} N_\mu \), where \( \hat{g}^{\rho\mu} \) is the \((n−1) \times (n−1)\)-matrix inverse to \( g_{\mu\nu} \):

\[
\hat{g}^{\rho\mu} g_{\mu\nu} = \delta^\rho_\nu,
\]

which we call the inverse metric on the sections \( x^0 = \text{const} \). In what follows, we always raise the space indices using the inverse metric \( \hat{g}^{\rho\mu} \), which is marked with the hat and does not coincide with the space part of the metric \( g^{\alpha\beta} \) inverse to \( g_{\alpha\beta} \). The function \( N = N(x) \) is called the lapse function, and the functions \( N_\mu = N_\mu(x) \) are shift functions. Without loss of generality, we assume that the lapse function is positive \((N > 0)\). In this case, the ADM parameterization of the metric \( g_{\alpha\beta} \) is in one-to-one correspondence. The interval corresponding to the parameterization \( g_{\alpha\beta} \) has the form

\[
ds^2 = N^2 dt^2 + g_{\mu\nu}(dx^\mu + N^\mu dt)(dx^\nu + N^\nu dt).
\]

We assume that the coordinate \( x^0 = t \) is the time, i.e., that the vector \( \partial_0 \) tangent to the coordinate \( x^0 \) is timelike. Formally, this condition is written as

\[
(\partial_0, \partial_0) = g_{00} = N^2 + N^\rho N_\rho > 0. \tag{2}
\]

In this case, the metric \( g_{\alpha\beta} \) has the Lorentzian signature if and only if the matrix

\[
g_{\mu\nu} = \frac{N_\mu N_\nu}{N^2 + N^\rho N_\rho} \tag{3}
\]

is negative definite. We note that the metric \( g_{\mu\nu} \) induced on sections \( x^0 = \text{const} \) may not be negative definite. This means that sections \( x^0 = \text{const} \) are not spacelike in a general case. In what follows, we additionally assume that the coordinates are chosen such that all sections \( x^0 = \text{const} \) are spacelike, i.e., the metric \( g_{\mu\nu} \) is also negative definite. This is convenient for posing the Cauchy problem when the initial data are given on a spacelike surface and we consider their evolution in time.
Similarly, a metric on a Riemannian manifold can be parameterized with a positive-definite metric. For this, it suffices to explicitly choose any coordinate instead of time. The metric inverse to (1) is

\[ g^{\alpha\beta} = \left( \begin{array}{cc} \frac{1}{N^2} & -\frac{N^\mu}{N^2} \\ \frac{N^\mu}{N^2} & \hat{g}^{\mu\nu} + \frac{N^\mu N^\nu}{N^2} \end{array} \right). \] (4)

The space matrix in the lower right block

\[ g^{\mu\nu} = \hat{g}^{\mu\nu} + \frac{N^\mu N^\nu}{N^2}, \] (5)

is inverse to metric (3), as can be easily verified. This means that the negative definiteness of metric (3) is equivalent to the negative definiteness of the matrix \( g^{\mu\nu} \).

We note that if the metric on a manifold \( M \) has the Lorentzian signature, then the condition that all sections \( x^0 = \text{const} \) are spacelike is equivalent to the condition \( N^2 > 0 \). Indeed, the negative definiteness of the inverse matrix \( \hat{g}^{\mu\nu} \) follows from that of \( g_{\mu\nu} \). Then the negative definiteness of the matrix

\[ g^{\mu\nu} = \frac{N^\mu N^\nu}{N^2}, \]

follows from Eq. (5). In turn, this is equivalent to the condition \( g^{00} > 0 \) or \( N^2 > 0 \).

We consider a simple example to show details that may arise for ADM parameterization of a metric.

**Example.** We consider the two-dimensional Minkowskian space-time \( \mathbb{R}^{1,1} \) with the Cartesian coordinates \( t, x \). We introduce the new coordinate system \( \xi, \eta \) depending on two real parameters \( a \) and \( b \) (see Figure)

\[ \xi = t + ax, \quad \eta = t - bx, \quad |a| \neq 1, \quad |b| \neq 1, \quad a + b \neq 0. \]

We can easily obtain the formulas for the inverse transformation

\[ t = \frac{b\xi + a\eta}{a + b}, \quad x = \frac{\xi - \eta}{a + b}. \]

The metric has the form

\[ ds^2 = dt^2 - dx^2 = \frac{1}{(a + b)^2} \left[ (b^2 - 1) d\xi^2 + 2(ab + 1) d\xi d\eta + (a^2 - 1) d\eta^2 \right]. \]

in the new coordinates.

We now analyze the ADM parameterization of the metric in the coordinates \( x^0 = \xi, \ x^1 = \eta \):

\[ g_{00} = \frac{b^2 - 1}{(a + b)^2}, \quad g_{01} = \frac{ab + 1}{(a + b)^2}, \quad g_{11} = \frac{a^2 - 1}{(a + b)^2}. \]

The lapse and shift functions are

\[ N^2 = -\frac{1}{a^2 - 1}, \quad N_1 = \frac{ab + 1}{(a + b)^2}. \]
The inequalities \(|b| > 1\) and \(|a| < 1\) follow from the respective conditions \(g_{00} > 0\) and \(g_{11} < 0\). We see that these conditions are necessary and sufficient for the coordinate lines \(\xi\) and \(\eta\) to be respectively timelike and spacelike. It is easy to verify the equivalence of the conditions

\[
\begin{align*}
g_{00} > 0 & \iff g_{11} - \frac{N_1 N_1}{N^2 + N_1 N_1} = -\frac{1}{b^2 - 1} < 0, \\
g^{00} > 0 & \iff \hat{g}^{11} = g^{11} - \frac{N^1 N^1}{N^2} = \frac{(a + b)^2}{a^2 - 1} < 0.
\end{align*}
\]

Using the formula for the determinant of block matrices, we obtain the expression for the determinant of metric \(\Pi\):

\[
\det g_{\alpha\beta} = N^2 \det g_{\mu\nu}.
\]

We hence have the expression for the volume element

\[
e = N \hat{e}, \quad \sqrt{|\det g_{\alpha\beta}|} = \sqrt{|\det g_{\mu\nu}|}.
\]

This formula is a generalization of the well-known school rule: the volume of a prism is equal to the product of the base area and the height. In this case, the base area is \(\hat{e}\) and the height is the lapse function \(N\).

The following formulas, which can be verified straightforwardly, are useful for calculations:

\[
\begin{align*}
g^{00} g^{\mu\nu} - g^{0\mu} g^{0\nu} &= \frac{\hat{g}^{\mu\nu}}{N^2}, \\
g^{\sigma\mu} g^{0\nu} - g^{\sigma\nu} g^{0\mu} &= \frac{N^\mu \hat{g}^{\sigma\nu} - N^\nu \hat{g}^{\sigma\mu}}{N^2}, \\
g^{\mu\nu} g_{\nu\sigma} &= \delta^\mu_\sigma + \frac{N^\mu N_\sigma}{N^2}, \\
g^{\mu\nu} g_{\mu\nu} &= n - 1 + \frac{N^\mu N_\mu}{N^2}.
\end{align*}
\]

3 Geometry of hypersurfaces

In the Hamiltonian formulation of gravity models, we take the space-time as a family of spacelike hypersurfaces \(x^0 = \text{const}\) parameterized by time. In other words, in each instant, the space is a hypersurface embedded in the space-time. It is useful to know what geometry arises on spacelike hypersurfaces because equations of gravity models define geometry of the whole space-time. In the present section, we approach this problem from a general standpoint, assuming that an arbitrary affine geometry is given on the embedding manifold without assuming that the metric signature is Lorentzian.

We consider an \((n - 1)\)-dimensional hypersurface \(\mathbb{U}\) embedded in an \(n\)-dimensional manifold \(\mathbb{M}\):

\[
f : \quad \mathbb{U} \to \mathbb{M}.
\]

We let \(x^\alpha, \alpha = 0, 1, \ldots, n - 1\), and \(u^i, i = 1, \ldots, n - 1\), denote the respective coordinates on \(\mathbb{M}\) and \(\mathbb{U}\). Then the embedding of \(\mathbb{U}\) in \(\mathbb{M}\) is locally given by \(n\) functions \(x^\alpha(u)\), which we assume to be sufficiently smooth. An arbitrary vector field \(\{X^i\} \in T(\mathbb{U})\) on the hypersurface is mapped on the vector field \(\{X^\alpha\} \in T(\mathbb{M})\) on \(\mathbb{M}\) by the map differential

\[
f_* : \quad X = X^i \partial_i \in T(\mathbb{U}) \to Y = Y^\alpha \partial_\alpha \in T(\mathbb{M}),
\]
where
\[ Y^\alpha = e^\alpha_i X^i, \quad e^\alpha_i = \partial_i x^\alpha. \]

The Jacobi matrix \( e^\alpha_i \) of the transformation \( f \) is rectangular of the size \( n \times (n - 1) \), has the rank \( n - 1 \), and is obviously irreversible. It is defined not on the whole manifold \( M \) but only on the hypersurface \( U \). In addition, we note that the Jacobi matrix is a vector and covector with respect to the respective coordinates transformations on \( M \) and \( U \). The pullback of the map \( f \) maps each covector field on the image \( f(M) \subset M \) in the covector field on \( U \):

\[ f^* : A = dx^\alpha A_\alpha \in T^*(M) \rightarrow B = du^i B_i \in T^*(U), \]

where
\[ B_i = A_\alpha e^\alpha_i. \]

We identify the hypersurface \( U \) with its image \( U = f(U) \subset M \) in what follows.

The 1-form \( n = dx^\alpha n_\alpha \) defined on the hypersurface \( U \) by the system of algebraic equations
\[ n_\alpha e^\alpha_i = 0, \quad i = 1, \ldots, n - 1, \] (10)

specifies the field of \((n - 1)\)-dimensional subspaces tangent to \( U \) in the tangent bundle \( T(U) \). These equations have a unique solution up to multiplication on an arbitrary nonzero function because the rank of the Jacobi matrix is equal to \( n - 1 \) as a consequence of the definition of embedding.

The Jacobi matrix \( e^\alpha_i \) defines a set of \( n - 1 \) vectors \( e_i = e^\alpha_i \partial_\alpha \) in the tangent spaces \( T_x(M) \), \( x \in U \); these vectors form the basis of the space tangent to the hypersurface.

This is all we can say about a hypersurface \( U \) if there is only embedding (9). The theory becomes much richer in content if there are additional structures on \( M \). We discuss this question in detail.

Let the affine geometry be given on \( M \), i.e., a metric \( g_{\alpha\beta} \) and an affine connection \( \Gamma^\gamma_{\alpha\beta} \). We consider what geometry arises on the hypersurface \( U \subset M \). The pullback of the map \( f^* \) induces a unique metric on the hypersurface:

\[ f^* : g_{\alpha\beta} \rightarrow g_{ij} = g_{\alpha\beta} e^\alpha_i e^\beta_j. \] (11)

The existence of the respective metrics \( g_{\alpha\beta} \) and \( g_{ij} \) on \( M \) and \( U \) allows lowering and raising the indices of the Jacobi matrix:

\[ e^i = g_{ij} e^\beta_j g^{ij}, \]

where \( g^{ij} \) is the metric inverse to \( g_{ij} \). This matrix projects an arbitrary vector from \( T_x(M) \), \( x \in U \), into the space tangent to the hypersurface \( T(U) \)

\[ X^\alpha \rightarrow X^i = X^\alpha e^\alpha_i. \]

We now define the connection on the hypersurface \( U \subset M \) by the relation

\[ \hat{\nabla}_i X^k = (\nabla_\alpha X^\beta) e^\alpha_i e^\beta_k. \] (12)

Opening this relation leads to the expression for the induced connection on the hypersurface \( U \) in the coordinate form:

\[ \hat{\Gamma}^k_{ij} = (\partial^2 \gamma^\alpha + \Gamma^\gamma_{\alpha\beta} e^\alpha_i e^\beta_j) e^\gamma_k. \] (13)
This connection is unique. We note that if the original connection $\Gamma_{\alpha\beta\gamma}$ is symmetric, then the induced connection $\tilde{\Gamma}^{ij}_k$ is also symmetric. As a consequence of Eq. (13), the torsion tensor $T_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma}$ on $\mathbb{M}$ induces the torsion on the hypersurface

$$T_{ij}^k = T_{\alpha\beta\gamma}^i e^\alpha_i e^\beta_j e^\gamma_k.$$  

Furthermore, the connection on $\mathbb{U}$ is uniquely defined only in the case where the metric is given on $\mathbb{M}$ in addition to the connection. All geometric objects related to the hypersurface and constructed using only the induced metric $g_{ij}$ and connection $\tilde{\Gamma}^{ij}_k$ are marked by the hat in what follows.

The metric $g_{\alpha\beta}$ and connection $\Gamma_{\alpha\beta\gamma}$ on $\mathbb{M}$ define the unique metric $g_{ij}$ and connection $\tilde{\Gamma}^{ij}_k$ on the hypersurface $\mathbb{U} \subset \mathbb{M}$. The converse statement is not true. If a metric and a connection are given on the hypersurface $\mathbb{U}$, then they do not induce the geometry on $\mathbb{M}$ uniquely. This is clear because the dimension of the hypersurface is less than that of the manifold itself.

Straightforward calculations yield the expression for the covariant derivative of the induced metric on the hypersurface:

$$\tilde{\nabla}_i g_{jk} = \partial_i g_{ij} - \tilde{\Gamma}^{ij}_l g_{lk} - \tilde{\Gamma}^{ik}_l g_{jl} = (\nabla_{\alpha} g_{\beta\gamma}) e^\alpha_i e^\beta_j e^\gamma_k.$$  

This relation gives the expression for the nonmetricity tensor on the hypersurface

$$Q_{ijk} = Q_{\alpha\beta\gamma} e^\alpha_i e^\beta_j e^\gamma_k.$$  

In particular, if the connection $\Gamma_{\alpha\beta\gamma}$ on $\mathbb{M}$ is metrical ($Q_{\alpha\beta\gamma} = 0$), then the induced connection $\tilde{\Gamma}^{ij}_k$ on $\mathbb{U}$ is also metrical ($Q_{ijk} = 0$).

The existence of metric $g_{\alpha\beta}$ allows forming the unit vector field $n = n^\alpha \partial_\alpha$ orthogonal to the hypersurface. It was already noted that the system of equations $n_\alpha e^\alpha_i = 0$ defines the 1-form $dx^n n_\alpha$ up to multiplication on an arbitrary scalar function. We use this arbitrariness for the vector $n^\alpha = g^{\alpha\beta} n_\beta$ to have the unit length in every point: $(n, n) = n^\alpha n^\beta g_{\alpha\beta} = 1$. This vector is orthogonal to all vectors tangent to the hypersurface by construction:

$$(n, e_i) = n^\alpha e^\beta_i g_{\alpha\beta} = n_\alpha e^\alpha_i = 0.$$  

(15)

If the hypersurface is given on a manifold, then there is a natural basis $\{n, e_i\}$ in the tangent space $T(\mathbb{M})$ defined by this hypersurface. This basis is defined only in points of the hypersurface but not on the whole manifold. The dual basis $\{n = dx^\alpha n_\alpha, e^i = dx^\alpha e^i_\alpha\}$ in the cotangent space $T^*(\mathbb{M})$ corresponds to it. Then an arbitrary vector $X$ and a 1-form $A$ can be decomposed with respect to this basis:

$$X^\alpha = X^\perp n^\alpha + X^i e^\alpha_i, \quad X^\perp = X^\alpha n_\alpha, \quad X^i = X^\alpha e^\alpha_i,$$

$$A_\alpha = A_\perp n_\alpha + A_i e^\alpha_i, \quad A_\perp = A_\alpha n^\alpha, \quad A_i = A_\alpha e^\alpha_i.$$  

A tensor of an arbitrary rank can be decomposed similarly. In particular, a covariant second rank tensor has the decomposition

$$A_{\alpha\beta} = A_{\perp\perp} n_\alpha n_\beta + A_{\perp i} n_\alpha e^i_\beta + A_{i\perp} e^i_\alpha n_\beta + A_{i j} e^i_\alpha e^j_\beta,$$

where

$$A_{\perp\perp} = A_{\alpha\beta} n^\alpha n^\beta, \quad A_{\perp i} = A_{\alpha\beta} n^\alpha e^i_\beta, \quad A_{i\perp} = A_{\alpha\beta} e^i_\alpha n^\beta, \quad A_{i j} = A_{\alpha\beta} e^i_\alpha e^j_\beta.$$  

7
We can easily verify that the decomposition of the metric is essentially simpler:

\[ g_{\alpha\beta} = n_{\alpha} n_{\beta} + e_{\alpha}^{i} e_{\beta}^{j} g_{ij}. \]  

(16)

A similar decomposition holds for the inverse metric:

\[ g^{\alpha\beta} = n^{\alpha} n^{\beta} + e^{\alpha}^{i} e^{\beta}^{j} g^{ij}. \]  

(17)

The summation over Latin indices for the Jacobi matrix follows from the definition of the inverse metric \( g^{\alpha\beta} g_{\beta\gamma} = \delta_{\alpha}^{\gamma} \):

\[ e^{\alpha}_{i} e^{\beta}_{i} = \delta_{\alpha}^{\beta} - n^{\alpha} n_{\beta}. \]  

(18)

As a consequence of Eq. (15) and the definition of the inverse induced metric \( g^{ij} g^{jk} = \delta_{i}^{k}, \) we have the equality

\[ e^{\alpha}_{i} e^{\alpha}_{j} = \delta_{i}^{j}, \]  

(19)

where the summation is over the Greek indices. Using this rule we obtain the representation for the inverse induced metric

\[ g^{ij} = g^{\alpha\beta} e^{i}_{\alpha} e^{j}_{\beta} \]

which follows from (17). Metric (16) and its inverse (17) in the basis \( n, e_{i} \) have the block diagonal form:

\[
\begin{pmatrix}
1 & 0 \\
0 & g_{ij}
\end{pmatrix},
\begin{pmatrix}
1 & 0 \\
0 & g^{ij}
\end{pmatrix}.
\]

This allows raising and lowering the corresponding indices. For example, if \( X_{\alpha} = X^{\beta} g_{\beta\alpha}, \) then \( X_{\perp} = X^{\perp} \) and \( X_{i} = X^{j} g_{ji}. \)

Induced metric (11) and connection (13) define the internal geometry of the hypersurface \( U \subset \mathbb{M}. \) In particular, the induced connection yields the internal curvature tensor of the hypersurface:

\[ \hat{R}_{ijk}^{l}(\hat{\Gamma}) = \partial_{i} \hat{\Gamma}_{jk}^{l} - \hat{\Gamma}_{ik}^{m} \hat{\Gamma}_{jm}^{l} - (i \leftrightarrow j). \]

The embedding \( f \) of the hypersurface allows defining an additional important object which is called the external curvature of the hypersurface,

\[ K_{ij} = -\nabla_{\alpha} n_{\beta} e_{i}^{\alpha} e_{j}^{\beta}, \]  

(20)

which is equal to the covariant derivative of the normal projected on the space tangent to the hypersurface up to a sign. In contrast to the internal curvature tensor, the external curvature is a second-rank tensor, which has no symmetry in the indices in the general case. This tensor characterizes the variation of the normal when it is translated parallel along a curve on the hypersurface. Expanding this definition and using (10) we obtain

\[ K_{ij} = n_{\alpha}(\partial_{i}^{2} e_{j}^{\alpha} + \Gamma_{\beta\gamma}^{\alpha} e_{i}^{\beta} e_{j}^{\gamma}). \]

The antisymmetric part of the external curvature tensor is given by the torsion tensor, \( K_{ij} - K_{ji} = 2K_{[ij]} = n_{\alpha} T_{\beta\gamma}^{\alpha} e_{i}^{\beta} e_{j}^{\gamma} = T_{ij}^{\perp}. \)

As a consequence, the external curvature is symmetrical if and only if the connection \( \Gamma_{\beta\gamma}^{\alpha} \) has no torsion.

The covariant derivative of the Jacobi matrix is

\[ \nabla_{i} e_{j}^{\alpha} = e_{i}^{\beta}(\partial_{\beta} e_{j}^{\alpha} + \Gamma_{\beta\gamma}^{\alpha} e_{j}^{\gamma}) - \hat{\Gamma}_{ij}^{k} e_{k}^{\alpha}. \]

(22)
where we use the connection $\Gamma_{\alpha\beta\gamma}$ on the whole manifold along with the connection $\hat{\Gamma}_{ij}^k$ on the hypersurface. Simple calculations show that this covariant derivative has only the normal component and is proportional to the external curvature:

$$\nabla_i e_{\alpha}^j = n^\alpha K_{ij}. \quad (23)$$

This relation is known as the Gauss–Weingarten formula. As a consequence, we have one more representation for the external curvature tensor:

$$K_{ij} = n^\alpha \nabla_i e_{\alpha}^j. \quad (24)$$

The full curvature tensor $R_{\alpha\beta\gamma\delta}$ of a manifold $M$ projected on the hypersurface can be expressed in terms of the internal curvature tensor $\hat{R}_{ijkl}$ constructed for induced metric (11) and connection (13) and the external curvature tensor. For this, we consider the commutator of covariant derivatives of a vector field, which is given by the curvature and torsion tensors,

$$[\nabla_\alpha, \nabla_\beta] X_\gamma = -R_{\alpha\beta\gamma\delta} X_\delta - T_{\alpha\beta\gamma} X_\delta =$$

$$= -R_{\alpha\beta\gamma} X^l - R_{\alpha\beta\gamma\perp} X_{\perp} - T_{\alpha\beta} X_\gamma - T_{\alpha\beta\perp} \nabla_\perp X_\gamma, \quad (25)$$

where we first compute the covariant derivatives in the right-hand side and then project them on the hypersurface: $\nabla_i X_\gamma = e_i^\alpha \nabla_\alpha X_\gamma$, $\nabla_\perp X_\gamma = n^\alpha \nabla_\alpha X_\gamma$. To project this relation on the hypersurface, we first project the covariant derivative:

$$\nabla_i X_\gamma = e_i^\alpha (\nabla_\alpha X_\beta) e_j^\beta = \hat{\nabla}_i X_\gamma - X_\perp K_{ij},$$

where

$$\hat{\nabla}_i X_\gamma = \partial_i X_\gamma - \hat{\Gamma}_{ij}^k X_k$$

is $(n - 1)$-dimensional covariant derivative on the hypersurface. The second covariant derivative is projected similarly:

$$\nabla_i \nabla_j X_k = \hat{\nabla}_i \nabla_j X_k - \nabla_\perp X_k K_{ij} - \nabla_j X_\perp K_{ik} =$$

$$= \hat{\nabla}_i \hat{\nabla}_j X_k - \hat{\nabla}_i X_\perp K_{jk} - \hat{\nabla}_j X_\perp K_{ik} - X_\perp \hat{\nabla}_i K_{jk} - \nabla_\perp X_k K_{ij},$$

where

$$\nabla_i K_\perp = e_i^\alpha (\nabla_\alpha X_\beta) n^\beta = \hat{\nabla}_i X_\beta + X^j K_{ji}.$$

The antisymmetrization of the obtained expression in the indices $i, j$ yields the projection of commutator (25) on the hypersurface:

$$[\nabla_i, \nabla_j] X_k = -R_{ijkl} X^l - R_{ijk\perp} X_\perp - T_{ij}^l \nabla_l X_k - T_{ij\perp} \nabla_\perp X_k.$$

Taking the independence of $X^l$ and $X_\perp$ and expressions (14) and (21) for the torsion and curvature tensors into account, we obtain the expressions for the projections of the full curvature tensor on the hypersurface:

$$R_{ijkl} = \hat{R}_{ijkl} + K_{ik} K_{jl} - K_{jk} K_{il}, \quad (26)$$

$$R_{ijk\perp} = \hat{\nabla}_i K_{jk} - \hat{\nabla}_j K_{ik} + T_{ij}^l K_{lk}. \quad (27)$$

The obtained relations are generalizations of the Gauss–Peterson–Codazzi equations to the case where an arbitrary affine geometry with nonzero torsion and nonmetricity is given on the embedding manifold $M$ instead of a Riemannian geometry.
To conclude this section, we compute the normal components $G_{\perp\perp}$ and $G_{\perp i}$ of the Einstein tensor

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R,$$

in the Riemannian geometry where the torsion and nonmetricity equal zero. First, we compute the scalar curvature

$$R = g^{\alpha\gamma} g^{\beta\delta} R_{\alpha\beta\gamma\delta} = 2 R_{\perp\perp} + g^{ik} g^{jl} R_{ijkl},$$

where $R_{\perp\perp} = g^{ij} R_{i\perp j\perp}$ is the normal component of the Ricci tensor and the expression for the inverse metric is used. As a consequence of the Gauss–Peterson–Codazzi, we obtain

$$g^{ik} g^{jl} R_{ijkl} = \hat{R} + K^2 - K^{ij} K_{ij},$$

where $\hat{R}$ is the scalar internal curvature of the hypersurface and $K = g^{ij} K_{ij}$ is the scalar external curvature of the hypersurface. As a consequence, we obtain the expressions for the normal components of the Einstein tensor

$$G_{\perp\perp} = -\frac{1}{2} (\hat{R} + K^2 - K^{ij} K_{ij}),$$

$$G_{\perp i} = \hat{\nabla}_j K_i^j - \nabla_i K,$$

It is important that these components of the Einstein tensor do not contain derivatives normal to the hypersurface $\nabla_{\perp}$ of the induced metric and external curvature tensor at all. In the Hamiltonian language, this means that time derivatives are absent and that Einstein’s equations $G_{\perp\perp} = 0$ and $G_{\perp i} = 0$ represent the constraints because the components of the external curvature $K^{ij}$ are shown to be proportional to the momenta canonically conjugate to the induced metric $g_{ij}$ in the next section.

### 4 Curvature in the ADM-parameterization of the metric

The ADM parameterization of metric is convenient for the canonical formulation of general relativity in which the metric components $g_{\alpha\beta}$ and canonically conjugate momenta $p^{\alpha\beta}$ are the independent variables. The passage from the Lagrangian to the Hamiltonian needs a relatively tedious calculations, which are given here.

To essentially simplify calculations, we use the results of the preceding section. Namely, the sections $x^0 = \text{const}$ of the space-time $\mathcal{M}$ yield the family of the hypersurfaces $\mathcal{U} \subset \mathcal{M}$ which are spacelike by the assumption. We take the space coordinates as the coordinates on the hypersurfaces

$$\{u^i\} \rightarrow \{x^\mu\}.$$

As a consequence, we lose the freedom of independent coordinate transformations on the space-time $\mathcal{M}$ and on the spacelike hypersurface $\mathcal{U}$, but many formulas became simpler.

The Jacobi matrix of the hypersurface embedding in the case under consideration is

$$\{e^\alpha_i\} \rightarrow \{0_\nu, \delta^\mu_\nu\}, \quad \{e^i_\alpha\} \rightarrow \{N^\mu, \delta^\mu_\nu\},$$

where $0_\nu$ denotes the row consisting of $n - 1$ zeroes. The embedding induces the metric $g_{\mu\nu}$ on the hypersurfaces according to formula.
We now construct the vector field \( n = n^\alpha \partial_\alpha \) orthogonal to the family of hypersurfaces \( x^0 = \text{const} \). As a consequence of orthogonality condition \( (n, X) = 0 \), where \( X = X^\mu \partial_\mu \) is a vector tangent to the section \( x^0 = 0 \), we obtain
\[
n = n^0 (\partial_0 - N^\mu \partial_\mu).
\]

Moreover, if we set \( n^0 = 1/N \), then the length of the normal vector is equal to unity \( (n^2 = 1) \). The unit vector orthogonal to a section \( x^0 = \text{const} \) has the form
\[
n = \frac{1}{N} (\partial_0 - N^\mu \partial_\mu)
\] (29)
and is always timelike. The corresponding orthonormal 1-form is
\[
n = dx^0 N.
\] (30)

An arbitrary vector on \( \mathbb{M} \) can be decomposed with respect to the basis \( \{ n, e_\mu \} \). In particular, we have the decompositions for vectors and 1-forms
\[
X^\alpha = X^\perp n^\alpha + \tilde{X}^\mu e_\mu^\alpha, \quad X_\alpha = X^0 n_\alpha + \tilde{X}_\mu e_\mu^\alpha,
\]
where
\[
X^\perp = X^0 N, \quad \tilde{X}^\mu = X^0 N^\mu + X^\mu, \\
X_\perp = \frac{1}{N} (X_0 - N^\mu X_\mu), \quad \tilde{X}_\mu = X_\mu.
\]

The representations for the metric (16) of the whole space-time and its inverse (17) are
\[
g_{\alpha\beta} = n_\alpha n_\beta + g_{\mu\nu} e_\alpha^\mu e_\beta^\nu, \\
g^{\alpha\beta} = n^\alpha n^\beta + \hat{g}^{\mu\nu} e_\alpha^\mu e_\beta^\nu.
\] (31)

Connection (13) induced on the hypersurfaces is the Christoffel symbols \( \hat{\Gamma}^\rho_{\mu\nu} \) computed for the space metric \( \hat{g}_{\mu\nu} \).

In the ADM-parameterization of the metric, external curvature tensor (20) on a hypersurface \( x^0 = \text{const} \) has the form
\[
K_{\mu\nu} = \Gamma^0_{\mu\nu} N = \frac{1}{2N} (\hat{\nabla}_\mu N_\nu + \hat{\nabla}_\nu N_\mu - \hat{g}_{\mu\nu}),
\] (32)
where the dot denotes the differentiation with respect to time, \( \hat{g}_{\mu\nu} = \partial_0 g_{\mu\nu} \) and \( \hat{\nabla}_\mu N_\nu = \partial_\mu N_\nu - \hat{\Gamma}^\rho_{\mu\nu} N_\rho \). The external curvature tensor is symmetrical \( (K_{\mu\nu} = K_{\nu\mu}) \) because the torsion vanishes in the metric formulation of general relativity. In what follows, we need the trace of the external curvature tensor
\[
K = K^\mu_\mu = \hat{g}^{\mu\nu} K_{\mu\nu}.
\]

All time derivatives of the space part of the metric \( \hat{g}_{\mu\nu} \) are conveniently expressed in terms of \( K_{\mu\nu} \) when computing the curvature tensor \( R_{\alpha\beta\gamma\delta} \) of the space-time \( \mathbb{M} \). Moreover, to exclude the second time derivatives \( \ddot{g}_{\mu\nu} \), we need the time derivative of the external curvature tensor
\[
\dot{K}_{\mu\nu} = \frac{1}{2N} \left[ \hat{\nabla}_\mu \hat{N}_\nu + \hat{\nabla}_\nu \hat{N}_\mu - \hat{N}^\rho (\hat{\nabla}_\mu \hat{g}_{\nu\rho} + \hat{\nabla}_\nu \hat{g}_{\mu\rho} - \hat{\nabla}_\rho \hat{g}_{\mu\nu}) \right] - \frac{\dot{N}}{N} K_{\mu\nu},
\]
where
\[ \hat{\nabla}_\mu \hat{N}_\nu = \partial_\mu \hat{N}_\nu - \hat{\Gamma}_\mu^\rho N_\rho \]
and where we must substitute the expression for \( \dot{g}_{\mu\nu} \) in terms of \( K_{\mu\nu} \).

We now compute the curvature tensor \( R_{\alpha\beta\gamma\delta} \) for the metric of form (1). Straightforward calculations yield the linearly independent Christoffel symbols

\[
\Gamma^{0}_{00} = \frac{1}{N} \left( \dot{N} + N^\rho \partial_\rho N + N^\sigma N^\rho K_{\rho\sigma} \right),
\]
\[
\Gamma^{0}_{0\nu} = \frac{1}{N} \left( \partial_\mu N + N^\nu K_{\mu\nu} \right),
\]
\[
\Gamma^{0}_{\mu\nu} = \nabla_\mu N^\nu - NK_{\mu\nu} - \frac{N^\nu}{N} (\partial_\mu N + N^\rho K_{\rho\nu}),
\]
\[
\Gamma_{\mu\nu}^{\rho} = \frac{1}{N} K_{\mu\nu}^\rho.
\]

In what follows, we need the following combinations of the Christoffel symbols
\[ \Gamma_\alpha = \Gamma_{\alpha\beta}^\beta, \quad g^{\beta\gamma} \Gamma_{\beta\gamma}^\alpha. \]

Simple calculations give
\[
\Gamma_0 = \frac{\dot{N}}{N} + \hat{\nabla}_\mu N^\mu - NK,
\]
\[
\Gamma_\mu = \dot{\Gamma}_\mu + \frac{\partial_\mu N}{N},
\]
\[
g^{\beta\gamma} \Gamma_{\beta\gamma}^0 = \frac{1}{N} K + \frac{1}{N} (\dot{N} - N^\mu \partial_\mu N),
\]
\[
g^{\beta\gamma} \Gamma_{\beta\gamma}^{\mu} = \left( \hat{g}^{\rho\sigma} + \frac{N^\rho N^\sigma}{N^2} \right) \hat{\Gamma}_{\rho\sigma}^\mu - \frac{N^\mu}{N} K - \frac{N^\mu}{N} (\dot{N} - N^\rho \partial_\rho N) + \frac{1}{N^2} \hat{g}^{\rho\sigma} (\dot{N} - N^\rho \partial_\rho N - N^\sigma \hat{\nabla}_\rho N_\sigma - 2N^\sigma \hat{\nabla}_\sigma N_\rho + 2N N^\sigma K_{\rho\sigma}).
\]

We also write the formulas for the time derivatives of the Christoffel symbols
\[
\partial_0 \hat{\Gamma}_{\mu\nu} = \frac{1}{2} \left( \hat{\nabla}_\mu \dot{g}_{\nu\rho} + \hat{\nabla}_\nu \dot{g}_{\mu\rho} - \hat{\nabla}_\rho \dot{g}_{\mu\nu} \right) + \hat{\Gamma}_{\mu\nu} \dot{g}_{\rho\sigma},
\]
\[
\partial_0 \hat{\Gamma}_{\mu\nu} = \frac{1}{2} \hat{g}^{\rho\sigma} \left( \hat{\nabla}_\mu \dot{g}_{\nu\rho} + \hat{\nabla}_\nu \dot{g}_{\mu\rho} - \hat{\nabla}_\rho \dot{g}_{\mu\nu} \right),
\]
\[
\partial_0 \hat{\Gamma}_\mu = \frac{1}{2} \hat{g}^{\mu\nu} \hat{\nabla}_\nu \dot{g}_{\mu\nu}.
\]

The time derivatives \( \dot{g}_{\mu\nu} \) are also eliminated from these expressions using relation (32).

We now compute the linearly independent components of the curvature tensor:
\[
R_{0\mu0\nu} = - NK_{\mu\nu} + \hat{R}_{\mu\rho\sigma\nu} N^\rho N^\sigma + N N^\mu (\hat{\nabla}_\mu K_{\nu\rho} + \hat{\nabla}_\nu K_{\mu\rho} - \hat{\nabla}_\rho K_{\mu\nu}) +
\]
\[
+ N \hat{\nabla}_\mu \hat{\nabla}_\nu N + K_{\mu\nu} N^\rho N^\sigma K_{\rho\sigma} + N (K_{\mu\nu} \hat{\nabla}_\rho N_\rho + K_{\nu\rho} \hat{\nabla}_\mu N_\rho) -
\]
\[
- N^2 K_{\mu\nu}^\rho K_{\nu\rho} - N^\rho N^\sigma K_{\mu\rho} K_{\nu\sigma},
\]
\[
R_{\mu\nu\rho\delta} = \hat{R}_{\mu\rho\sigma\nu} N^\sigma + N (\hat{\nabla}_\mu K_{\nu\rho} - \hat{\nabla}_\nu K_{\mu\rho}) + (K_{\mu\rho} K_{\nu\sigma} - K_{\nu\rho} K_{\mu\sigma}) N^\sigma,
\]
\[
R_{\mu\nu\rho\sigma} = \hat{R}_{\mu\rho\sigma\nu} + K_{\mu\rho} K_{\nu\sigma} - K_{\mu\sigma} K_{\nu\rho},
\]

12
where we use the formula for the commutator of covariant derivatives

\[ (\hat{\nabla}_\mu \hat{\nabla}_\nu - \hat{\nabla}_\nu \hat{\nabla}_\mu) N_\rho = -\hat{\nabla}_\rho N_\sigma. \]

The components of the curvature tensor having at least one time index seem simpler when expressed in the basis \( n, e_\mu \):

\[
R_{\mu \rho \pm} = \hat{\nabla}_\mu K_{\rho \pm} - \hat{\nabla}_\nu K_{\nu \rho}.
\]

The components of the curvature tensor \( R_{\mu \nu \rho \sigma} \) and \( R_{\mu \nu \rho \pm} \) were actually obtained in the preceding section (26), (27) without straightforward calculations.

The Ricci tensor has the linearly independent components

\[
R_{00} = -N \hat{g}^{\mu \nu} \hat{K}_{\mu \nu} + \hat{R}_{\mu \nu} N^\mu N^\nu + N N^\mu (2 \hat{\nabla}_\nu K_{\mu \nu} - \partial_\mu K) + N \hat{\nabla}^\mu \hat{\nabla}_\mu N +
\]

\[ + \frac{N^\mu N^\nu}{N} \left( -\hat{K}_{\mu \nu} + N^\rho \hat{\nabla}_\mu K_{\nu \rho} + \hat{\nabla}_\mu \hat{\nabla}_\nu N + K_{\mu \rho} \hat{\nabla}_\nu N_\rho + K_{\nu \rho} \hat{\nabla}_\mu N_\rho \right) \]

\[
R_{0\mu} = \frac{N^\nu}{N} \left( -\hat{K}_{\mu \nu} + N^\rho \hat{\nabla}_\nu K_{\mu \rho} + \hat{\nabla}_\nu \hat{\nabla}_\mu N + K_{\nu \rho} \hat{\nabla}_\rho N_\mu + K_{\mu \rho} \hat{\nabla}_\rho N_\nu \right)
\]

\[ + \hat{R}_{\mu \nu} N^\nu + N \left( \hat{\nabla}_\nu K_{\mu \nu} - \partial_\mu K \right) + + K_{\mu \rho} N^\nu K - 2 K_{\mu \rho} K_{\nu \rho} N^\nu,
\]

\[
R_{\mu \nu} = \hat{R}_{\mu \nu} + \frac{1}{N} \left( -\hat{K}_{\mu \nu} + \hat{\nabla}_\mu \hat{\nabla}_\nu N + K_{\mu \rho} \hat{\nabla}_\nu N_\rho + K_{\nu \rho} \hat{\nabla}_\mu N_\rho \right)
\]

\[ + \frac{N^\rho}{N} \hat{\nabla}_\rho K_{\mu \nu} + K_{\mu \rho} K - 2 K_{\mu \rho} K_{\nu \rho}.
\]

For reference, we also write the Ricci tensor components with respect to the basis \( n, e_\mu \):

\[
R_{\perp \perp} = \frac{1}{N} \hat{g}^{\mu \nu} \hat{K}_{\mu \nu} + \frac{1}{N} \hat{\nabla}^\mu \hat{\nabla}_\mu N + \frac{2}{N} K^{\mu \nu} \hat{\nabla}_\mu N_\nu - K^{\mu \nu} K_{\mu \nu} + \frac{N^\mu}{N} \partial_\mu K,
\]

\[
R_{\perp \mu} = \hat{\nabla}_\nu K_{\mu \nu} - \partial_\mu K.
\]

Finally, we compute the scalar curvature:

\[
R = \hat{R} + \frac{2}{N} \left( -\hat{g}^{\mu \nu} \hat{K}_{\mu \nu} + \hat{\nabla}^\mu \hat{\nabla}_\mu N + 2 K^{\mu \nu} \hat{\nabla}_\mu N_\nu + N^\mu \partial_\mu K \right) - 3 K^{\mu \nu} K_{\mu \nu} + K^2.
\]

\section{The Hamiltonian}

The scalar curvature contains second derivatives of the metric components with respect to time as well as the space coordinates and is therefore not suitable for canonically formulating general relativity. It suffices to eliminate only the second time derivatives from the Lagrangian to obtain a canonical formulation. The Lagrangian density takes the simplest form after adding the boundary term:

\[
\mathcal{L}_{\text{ADM}} = N \dot{\epsilon} R + 2 \partial_0 (\dot{\epsilon} K) - 2 \partial_\mu (\dot{\epsilon} g^{\mu \nu} \partial_\nu N).
\]

Straightforward calculations yield the expression

\[
\mathcal{L}_{\text{ADM}} = N \dot{\epsilon} \left( K^{\mu \nu} K_{\mu \nu} - K^2 + \hat{R} \right).
\]
The transition to the Hamiltonian formalism is now easy. First, the ADM Lagrangian does not contain time derivatives of the lapse and shift functions $N$ and $N_\mu$. This means that the theory contains $n$ primary constraints

$$
p^\perp = \frac{\partial L_{\text{ADM}}}{\partial N} = 0, \quad p^\mu = \frac{\partial L_{\text{ADM}}}{\partial N_\mu} = 0, \quad (43)
$$

their number coinciding with the number of independent functions that parametrize diffeomorphisms.

The momenta canonically conjugate to the space metric $g_{\mu\nu}$ are proportional to the external curvature tensor,

$$
p^{\mu\nu} = \frac{\partial L_{\text{ADM}}}{\partial \dot{g}^{\mu\nu}} = -\frac{1}{2N} \frac{\partial L_{\text{ADM}}}{\partial \dot{K}^{\mu\nu}} = -\dot{\hat{e}} (K^{\mu\nu} - \hat{g}^{\mu\nu} K). \quad (44)
$$

We note that the momenta are not tensors with respect to the coordinate transformations $x^\mu$ but tensor densities of the degree $-1$, as is the determinant of the vielbein, which degree is

$$\text{deg} \hat{e} = -1$$

by definition.

To eliminate the velocities $\dot{g}_{\mu\nu}$ from the ADM Lagrangian, we decompose the momenta on the irreducible components, extracting the trace from $p^{\mu\nu}$

$$p^{\mu\nu} = \tilde{p}^{\mu\nu} + \frac{1}{n-1} p \hat{g}^{\mu\nu}, \quad (45)$$

where we introduce the trace of the momenta

$$p = p^{\mu\nu} g_{\mu\nu} = \hat{e} (n-2) K \quad (46)$$

and the symmetric traceless part

$$\tilde{p}^{\mu\nu} = \tilde{p}^{\nu\mu} = -\hat{e} \left( K^{\mu\nu} - \frac{1}{n-1} \hat{g}^{\mu\nu} K \right), \quad \tilde{p}^{\mu\nu} g_{\mu\nu} = 0.$$ We can now solve Eq. (44) for the velocities using relation (32),

$$\dot{g}_{\mu\nu} = \frac{2N}{\hat{e}} \left( p_{\mu\nu} - \frac{1}{(n-2)} p g_{\mu\nu} \right) + \hat{\nabla}_\mu N_\nu + \hat{\nabla}_\nu N_\mu.$$ Simple calculations yield the Hamiltonian density

$$H = p^{\mu\nu} \dot{g}_{\mu\nu} - L_{\text{ADM}} = NH_\perp + N^\mu H_\mu + 2 \partial_\mu (p^{\mu\nu} N_\nu), \quad (47)$$

where

$$H_\perp = \frac{1}{\hat{e}} \left( p^{\mu\nu} p_{\mu\nu} - \frac{1}{(n-2)} p^2 \right) - \hat{e} \hat{R},$$

$$H_\mu = -2 \hat{\nabla}_\nu p^{\nu\mu} = -2 \partial_\nu p^{\nu\mu} + \partial_\mu g_{\nu\rho} p^{\nu\rho}, \quad (48)$$

and $p_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} p^{\rho\sigma}$. We note that the covariant derivative of the momenta contains only one term with Christoffel symbols because the momenta are tensor densities. The
expressions for $H_\perp$ and $H_\mu$ are proportional to the $G_{\perp\perp}$ and $G_{\perp\mu}$ components of Einstein tensor (28), which justifies the chosen notations.

Dropping the divergence in the expression for the Hamiltonian density (47), we obtain the final expression for the Hamiltonian:

$$H_{\text{ADM}} = \int d^3 x H_{\text{ADM}} = \int d^3 x (NH_\perp + N^\mu H_\mu).$$  \hfill (49)

We now rewrite the expression for $H_\perp$ in terms of the irreducible components of momenta:

$$H_\perp = \frac{1}{e} \left[ p^{\mu\nu} p_{\mu\nu} - \frac{1}{(n-1)(n-2)} p^2 \right] - e \bar{R}.$$  \hfill (50)

In particular, the quadratic part of the momenta in $H_\perp$ for $n \geq 3$ is consequently not positive definite.

6 Secondary constraints

To finish constructing the Hamiltonian formalism, we must analyze the consistency of primary constraints (43) with the equations of motion. The phase space of general relativity is described by $n(n+1)$ conjugate coordinates and momenta: $(N, p^\perp)$, $(N_\mu, p^\mu)$, $(g_{\mu\nu}, p_{\mu\nu})$ on which the canonical equal-time Poisson bracket

$$\{N, p^{\prime \perp}\} = \delta, \quad \{N_\mu, p^{\prime \nu}\} = \delta^\nu_\mu \delta, \quad \{g_{\mu\nu}, p^{\prime \rho\sigma}\} = \delta_{\rho\sigma}^\mu \delta_{\nu}^\nu,$$

is given, where primed field variables are considered at a point $x' = (x'^1, \ldots, x'^n-1)$. All fields are considered at the same instant $t = x^0$. For brevity, we use the notation

$$\delta = \delta^{(n-1)}(x - x') = \delta(x^1 - x'^1) \ldots \delta(x^{n-1} - x'^{n-1}),$$

$$\delta^\rho_\mu = \frac{1}{2} (\delta_\mu^\rho \delta^\sigma_\nu + \delta^\rho_\mu \delta_\nu^\sigma),$$

for the $(n-1)$-dimensional $\delta$-function and the symmetric combination of the Kronecker symbols in the right-hand sides of the Poisson brackets. We write all $\delta$-functions on the right to distinguish them from a field variation.

We now consider the Hamiltonian equations of motion for primary constraints (43):

$$\dot{p}^\perp = \{p^\perp, H_{\text{ADM}}\} = -H_\perp, \quad \dot{p}^\mu = \{p^\mu, H_{\text{ADM}}\} = -H^\mu.$$  \hfill (51)

The consistency of the primary constraints with the equations of motion $\dot{p}^\perp = 0$, $\dot{p}^\mu = 0$ leads to the secondary constraints

$$H_\perp = 0, \quad H_\mu = 0,$$  \hfill (52)

where $H_\perp = H^\perp$ and $H_\mu = g_{\mu\nu} H^\nu$. We note that the secondary constraints are not tensors but tensor densities of degree $-1$. Moreover, it is more convenient to consider the equivalent set of constraints with lowered index $H_\mu$ instead of the constraints $H^\mu$. Below, we show that these constraints define the generators of coordinate transformations on the sections $x^0 = \text{const}$ and satisfy a simpler algebra.

The constraints $H_\mu$ are linear in the momenta and metric. The constraint $H_\perp$ is quadratic in the momenta and nonpolynomial in the metric $g_{\mu\nu}$ because it depends on
the square root of the metric \( \hat{e} \) and the inverse metric \( \hat{g}^{\mu\nu} \). The last circumstance raises essential technical difficulties in the perturbation theory.

The secondary constraints are independent on the canonical variables \((N, p^\perp)\) and \((N^\mu, p^\mu)\), and they can be eliminated by considering the \(n(n-1)\)-dimensional phase space of the variables \(g_{\mu\nu}\) and \(p^{\mu\nu}\) on which constraints \((52)\) are imposed. In this case, the lapse and shift functions \(N\) and \(N^\mu\) are regarded as Lagrange multipliers in the problem for the conditional extremum for the action

\[
S = \int d^n x (p^{\mu\nu} \dot{g}_{\mu\nu} - \mathcal{H}_{\text{ADM}}).
\]

Because Hamiltonian \([19]\) in general relativity is equal to a linear combination of the secondary constraints, we must compute the Poisson brackets of the constraints between themselves to analyze the consistency of secondary constraints \((52)\) with the equations of motion. The constraint algebra in general relativity is well known,

\[
\begin{align*}
[H^\perp, H^\perp] &= -(H^\mu \hat{g}^{\mu\nu} + H'^\mu \hat{g}'^{\mu\nu}) \delta_\nu, \\
[H^\perp, H'^\mu] &= -H^\perp \delta_\mu, \\
[H^\mu, H'^\nu] &= -H^\nu \delta_\mu - H'^\mu \delta_\nu,
\end{align*}
\]

where we use the shorthand notation for the derivative of the \(\delta\)-function,

\[
\delta_\mu = \frac{\partial}{\partial x^\mu} \delta(x' - x).
\]

Straightforward calculations of constraint algebra \((53)-(55)\) are very tedious. This algebra was first written by Dirac \([14]\) using symmetry considerations. The assumptions on the form of the constraints made in the course of derivation are not satisfied in general relativity, and the existence of the corresponding canonical transformation is now questionable. Therefore, Dirac’s derivation of the constraints algebra cannot be considered satisfactory.

Two Poisson brackets \((54)\) and \((55)\) can in fact be found without straightforward calculations. For this, we consider the functional

\[
T_u = -\int d\bm{x} \ u^\mu H_\mu,
\]

where \(u^\mu(x)\) is an infinitesimal vector field. Calculating the Poisson brackets of the phase-space coordinates \(g_{\mu\nu}\) and \(p^{\mu\nu}\) with \(T_u\) yields

\[
\begin{align*}
\delta_u g_{\mu\nu} &= [g_{\mu\nu}, T_u] = -\partial_\mu u^\rho g_{\rho\nu} - \partial_\nu u^\rho g_{\mu\rho} - u^\rho \partial_\nu g_{\mu\rho}, \\
\delta_u p^{\mu\nu} &= [p^{\mu\nu}, T_u] = \partial_\rho u^\mu p^{\rho\nu} + \partial_\nu u^\mu p^{\rho\rho} - \partial_\rho (u^\mu p^{\rho\nu}).
\end{align*}
\]

This means that the functional \(T_u\), which is defined by the constraints \(H_\mu\), is the generator of general coordinate transformations on the hypersurfaces \(x^0 = \text{const}\). We recall that the momenta \(p^{\mu\nu}\) are tensor densities of degree \(-1\). The algebra of general coordinate transformations is well known and defined by the Poisson bracket \((55)\). We also can avoid computing the Poisson bracket \((54)\) explicitly. Its form follows from the fact that the constraint \(H^\perp\) is the scalar density of weight \(-1\). Therefore, only the Poisson bracket \((53)\) must be computed. These calculations, being very cumbersome, were apparently first performed much later by DeWitt \([4]\).
bracket after the canonical transformation, which casts the constraints into a polynomial form, essentially simplifying the calculations.

We say that the constraints $H_\mu$ are kinematical because they define only space diffeomorphisms. They are also independent of the coupling constants in the action if there are any. The constraint $H_\perp$ is said to be dynamical because it governs the evolution of the initial data in time and depends essentially on the original action, in particular, on the coupling constants.

For comparison, we write the Poisson brackets of the constraints $H^\mu = 0$ with a contravariant index which are equivalent to the constraints $H_\mu = 0$:

$$[H^\mu, H^\nu] = (\dot{g}^{\mu\nu} H^\rho + \dot{g}^{\mu\nu} H^{\rho'}) \delta_\rho + (\dot{g}^{\rho\rho} \partial_\rho g_{\sigma\lambda} \dot{g}^{\sigma\lambda} - \dot{g}^{\rho\rho} \partial_\rho g_{\sigma\lambda} \dot{g}^{\mu\sigma}) H^\lambda \delta.$$  

We see that this seems more complicated than bracket (55).

7 The canonical transformation

The idea of the canonical transformation is as follows. The momenta $p^{\mu\nu}$ are reducible and decompose into the traceless part and the trace (15). Usually, working with irreducible components is more convenient for calculations because many terms automatically cancel. We pose the question: “Is it possible to perform a canonical transformation such that the irreducible components $\tilde{p}^{\mu\nu}$ and $p$ become new canonical momenta?” This question is nontrivial because the decomposition of the momenta involves the metric, whose components themselves are coordinates of the phase space. The answer to this question is negative because the Poisson brackets between the momenta are nonzero. For example, $[\tilde{p}^{\mu\nu}, p'] \neq 0$. Nevertheless, there is a canonical transformation such that the new momenta are proportional to the irreducible components $\tilde{p}^{\mu\nu}$ and $p$. Constructing this canonical transformation is our subject in this section.

We consider the canonical transformation

$$(g_{\mu\nu}, p^{\mu\nu}) \rightarrow (k_{\mu\nu}, P^{\mu\nu}), (\rho, P),$$ \hspace{1cm} (56)$$

to the new pairs of canonically conjugate coordinates and momenta with the additional constraints on the coordinates $k_{\mu\nu} = k_{\nu\mu}$ and conjugate momenta $P^{\mu\nu} = P^{\nu\mu}$

$$|\text{det} k_{\mu\nu}| = 1, \quad P^{\mu\nu} k_{\mu\nu} = 0,$$ \hspace{1cm} (57)

and $\rho > 0$. We choose the space integral as the generator of the canonical transformation

$$F = -\int d\mathbf{x} \rho^m k_{\mu\nu} p^{\mu\nu}, \quad m \in \mathbb{R}, \quad m \neq 0,$$ \hspace{1cm} (58)$$

depending on the new coordinates $\rho, k_{\mu\nu}$, old momenta $p^{\mu\nu}$, and the real parameter $m$. The old coordinates and new momenta are then given by the variational derivatives (see, e.g., [15])

$$g_{\mu\nu} = -\frac{\delta F}{\delta p^{\mu\nu}} = \rho^m k_{\mu\nu},$$ \hspace{1cm} (59)$$

$$P^{\mu\nu} = -\frac{\delta F}{\delta k_{\mu\nu}} = \rho^m \tilde{p}^{\mu\nu},$$ \hspace{1cm} (60)$$

$$P = -\frac{\delta F}{\delta \rho} = \frac{m}{\rho} p.$$ \hspace{1cm} (61)
Computing the variational derivative with respect to $k_{\mu\nu}$, we use the condition $|\det k_{\mu\nu}| = 1$, which restricts the variations $k_{\mu\nu} \delta k_{\mu\nu} = 0$, where $k_{\mu\nu}$ is the tensor density inverse to $k_{\mu\nu}$. Hence, the vanishing of momentum traces (57) follows automatically from the unit determinant condition for the density $k_{\mu\nu}$ for generating functional (58). In Eq. (61), we use relation (59).

In essence, the determinant of the metric raised to some power is singled out from the metric,

$$
\rho = |\det g_{\mu\nu}|^{\frac{1}{m(n-1)}},
$$

as a consequence of (59). For brevity in what follows, we also call the symmetrical tensor density with the unit determinant $k_{\mu\nu}$ the metric.

Variables (56) for $n = 4$ and $m = 1/2$ were considered in [12], although the canonical transformation was not noted there.

Straightforward calculations yield the expression for the scalar curvature of the section $x^0 = \text{const}$ in the new coordinates:

$$
\hat{R} = \rho^{-m-2} \left[ \rho^2 R^{(k)} + m(n-2)\rho \partial_\mu(k_{\mu\nu} \partial_\nu \rho) + m(n-2) \left( m \frac{n-3}{4} - 1 \right) k_{\mu\nu} \partial_\mu \rho \partial_\nu \rho \right].
$$

The “scalar curvature” for the metric $k_{\mu\nu}$ takes a particular simple form,

$$
R^{(k)} = \partial_{\mu\nu}^2 k_{\mu\nu} + \frac{1}{2} k_{\mu\nu} \partial_\rho k_{\mu\sigma} \partial_\nu k^{\rho\sigma} - \frac{1}{4} k_{\mu\nu} \partial_\mu k_{\rho\sigma} \partial_\nu k^{\rho\sigma}.
$$

This expression is not a scalar with respect to coordinate transformations of $x^\mu$, because $k_{\mu\nu}$ is not a tensor but tensor density. But we note that the group of diffeomorphisms of sections $x^0 = \text{const}$ has a subgroup consisting of coordinates transformations of $x^\mu$ with a unit determinant. The density $k_{\mu\nu}$ is a tensor and $R^{(k)}$ is a scalar with respect to this subgroup.

As a consequence of the unit determinant of the metric, we obtain the components of the inverse metric $k^{\mu\nu}$ as polynomials of degree $n-2$ in the components $k_{\mu\nu}$,

$$
k^{\mu\nu} = \frac{1}{(n-2)!} \varepsilon^{\mu_1\cdots\mu_{n-2}} \varepsilon_{\nu_1\cdots\nu_{n-2}} k_{\mu_1\nu_1} \cdots k_{\mu_{n-2}\nu_{n-2}},
$$

where $|\varepsilon^{\mu_1\cdots\mu_{n-1}}| = 1$ is the totally antisymmetric tensor density of rank $n-1$. Therefore, the scalar curvature $R^{(k)}$ is polynomial in the metric $k_{\mu\nu}$ as well as in its inverse $k^{\mu\nu}$.

The dynamical constraint in the new variables becomes

$$
H_{\perp} = \rho^{-m(n-1)/2} \left[ P^{\mu\nu} P_{\mu\nu} - \frac{\rho^2}{m^2(n-1)(n-2)} P^2 \right]
- \rho^{-m(n-1)/2 - m-2} \left[ \rho^2 R^{(k)} + m(n-2)\rho \partial_\mu(k_{\mu\nu} \partial_\nu \rho) + m(n-2) \left( m \frac{n-3}{4} - 1 \right) k_{\mu\nu} \partial_\mu \rho \partial_\nu \rho \right],
$$

where $P^\mu_{\nu} = k_{\mu\rho} k^{\nu\sigma} P_{\rho\sigma}$.

We now analyze the possibility of choosing the constant $m$ such that the dynamical constraint becomes polynomial. Both expression in square brackets are polynomial in all dynamical variables. Because $n \geq 3$, we need the inequality $m < 0$ to ensure a positive power of the density $\rho$ before the first square bracket. In this case, the power of $\rho$ before the second square bracket is negative. We therefore cannot ensure that the constraint $H_{\perp}$ itself is polynomial by choosing the constant $m$. But a constraint can be multiplied by
an arbitrary nonzero factor without changing the surface defined by the constraint in the
phase space. The power of $\rho$ by which $H_\perp$ be multiplied is minimum when the powers of
$\rho$ before the square brackets are equal. We consequently obtain the equality

$$m = \frac{2}{n - 2}.$$  

Then multiplying the dynamical constraint

$$K_\perp = \rho^{\frac{n-1}{n-2}} H_\perp = \hat{c} H_\perp,$$  

we obtain the equivalent polynomial constraint

$$K_\perp = P^{\mu \nu} P_{\mu \nu} - \frac{n-2}{4(n-1)} \rho^2 P^2 - \rho^2 \tilde{R} = 0,$$  

where the scalar curvature density

$$\tilde{R}(\rho, k) = R^{(k)} + 2 \frac{\partial_p (k^{\mu \nu} \partial_\nu \rho)}{\rho} - \frac{n-1}{n-2} \frac{k^{\mu \nu} \partial_\mu \rho \partial_\nu \rho}{\rho^2}, \quad \text{deg} \tilde{R} = -\frac{2}{n-1},$$

is introduced. We note that the “scalar” curvature $R^{(k)}$ constructed for the metric density
$k_{\mu \nu}$ is not a scalar density. Therefore, using $\tilde{R}$ instead of $R^{(k)}$ simplifies many formulas
and calculations.

Multiplying the dynamical constraint $H_\perp$ by the nonzero factor leads to modifying the
Lagrange multiplier (the lapse function) by the inverse factor,

$$N \to \tilde{N} = N \rho^{\frac{n-1}{n-2}}.$$  

In turn, the modification of the Lagrange multipliers in general relativity is equivalent to
coordinate changes that do not change the physical content of the theory.

The kinematical constraints in the new dynamical variables remain polynomial:

$$H_\mu = -2 \nabla_\nu (P^{\mu \nu}) - \frac{n-2}{n-1} \nabla_\mu (P \rho) = 0,$$  

where $\nabla_\mu$ is the covariant derivative constructed for the metric

$$g_{\mu \nu} = \rho^{\frac{2}{n-2}} k_{\mu \nu},$$

and indices are lowered using the tensor density $P^{\mu \nu} = P^{\nu \mu} k_{\mu \nu}$. It can be easily verified
that

$$\nabla_\mu \rho = 0, \quad \nabla_\mu k_{\nu \rho} = 0.$$  

Lowering and raising indices with the metric density $k_{\mu \nu}$ therefore commutes with the
covariant differentiation operation.

We note that the covariant derivative of a tensor density $\phi$ of degree $\text{deg} \phi = r$ in our
notation is given by the expression:

$$\nabla_\mu \phi = \partial_\mu \phi + r \Gamma_\mu \phi, \quad \Gamma_\mu = \Gamma_{\nu \mu}^{\nu} = \frac{n-1}{n-2} \frac{\partial_\mu \rho}{\rho}.$$  

All new canonical variables are tensor densities of the degrees

$$\text{deg} \, k_{\mu \nu} = \frac{2}{n-1}, \quad \text{deg} \rho = \frac{n-2}{n-1},$$

$$\text{deg} \, P^{\mu \nu} = -\frac{n+1}{n-1}, \quad \text{deg} \, P = \frac{1}{n-1}.$$
and this should be taken into account for covariant differentiation.

We now compute the basic Poisson brackets for the new canonical variables, which follow from explicit expressions (59)–(61). Only three brackets are nonzero:

\[
[\rho, P_\nu] = \delta, \quad \tag{68}
\]

\[
[k_{\mu\nu}, P^{\rho\sigma}] = \left( \delta^{\rho\sigma}_{\mu\nu} - \frac{1}{n-1} k_{\mu\nu} k^{\rho\sigma} \right) \delta, \quad \tag{69}
\]

\[
[P^{\mu\nu}, P^{\rho\sigma}] = \frac{1}{n-1} (P^{\mu\nu} k^{\rho\sigma} - P^{\rho\sigma} k^{\mu\nu}) \delta. \quad \tag{70}
\]

Poisson brackets (69) and (70) do not have the canonical form for the phase variables. This occurs because the fields \( k_{\mu\nu} \) and \( P^{\mu\nu} \) are subjected to additional constraints (57).

Because the Hamiltonian

\[
H = \int d\mathbf{x} (\tilde{N} K_\perp + N^\mu H_\mu)
\]

is polynomial in the new variables, the equations of motion are also polynomial. Straightforward calculations with Poisson brackets (68)–(70) yield the equations of motion:

\[
\dot{\rho} = -\frac{n-2}{2(n-1)} \tilde{N} \rho^2 P + \frac{n-2}{n-1} \rho \nabla_\mu N^\mu, 
\]

\[
\dot{P} = -\frac{n-2}{2(n-1)} \tilde{N} \rho P^2 + 2 \tilde{N} \rho \dot{\tilde{N}} + 2 \rho \nabla \tilde{N} + \frac{1}{n-1} P \nabla_\mu N^\mu + N^\mu \nabla_\mu P, 
\]

\[
\dot{k}_{\mu\nu} = 2 \tilde{N} P_{\mu\nu} + \nabla_\mu N_\nu + \nabla_\nu N_\mu - \frac{2}{n-1} k_{\mu\nu} \nabla_\rho N^\rho, 
\]

\[
\dot{P}^{\mu\nu} = -2 \tilde{N} P^{\rho\sigma} P_\rho^{\nu} - \rho^2 \left( \tilde{N} k^{\rho\sigma} k^{\mu\nu} \tilde{R}_{\rho\sigma} - \frac{1}{n-1} \tilde{N} \dot{\tilde{N}} k^{\mu\nu} + \nabla_\mu \nabla_\nu \tilde{N} - \frac{1}{n-1} k^{\mu\nu} \nabla \tilde{N} \right) + 
\]

\[
+ \nabla_\rho (N^\rho P^{\mu\nu}) + \frac{2}{n-1} P^{\mu\nu} \nabla_\rho N^\rho - P^{\rho\sigma} \nabla_\rho N^\nu - P^{\nu\rho} \nabla_\rho N^\mu,
\]

where

\[
N_\mu = N^\nu k_{\mu\nu}, \quad \nabla^\mu = k^{\mu\nu} \nabla_\nu, \quad \quad \square = k^{\mu\nu} \nabla_\mu \nabla_\nu,
\]

and

\[
\dot{R}_{\mu\nu} = R^{(k)}_{\mu\nu} + \frac{n-3}{n-2} \frac{\partial^2 \rho}{\rho} - \left( \frac{n-3}{n-2} \frac{\partial \rho \partial_\nu \rho}{\rho^2} - \frac{n-3}{n-2} \Gamma^{(k)}_{\mu\nu} \frac{\partial \rho}{\rho} - \frac{1}{(n-2)^2} k^{\mu\lambda} \partial_\lambda \partial_\rho \partial_\mu \rho \right) + \frac{1}{n-2} k^{\mu\nu} \frac{\partial_\sigma (k^{\sigma\lambda} \partial_\lambda \rho)}{\rho^2}. \]

We now consider the constraints algebra. It is changed because we introduced the new constraint \( K_\perp \) instead of the dynamical constraint \( H_\perp \). Simple calculations yield

\[
[K_\perp, K_\perp'] = -\left( \rho^2 H_\mu k^{\mu\nu} + \rho^2 H_\mu' k^{\mu\nu} \right) \delta_\nu, \quad \tag{71}
\]

\[
[K_\perp, H_\mu'] = -(K_\perp + K_\perp') \delta_\mu, \quad \tag{72}
\]

\[
[H_\mu, H_\mu'] = -H_\nu \delta_\mu - H_\mu' \delta_\nu. \quad \tag{73}
\]

Changes occur in Poisson brackets (71) and (72) as compared with the original algebra (53)–(54). The second Poisson bracket, bracket (72), has a kinematical origin and is defined by the fact that the new constraint is not a scalar function but a tensor density.
of degree \( \deg K_\lambda = -2 \). Poisson bracket (71) from direct calculations. We note that calculating that bracket is much simpler in the new variables than in the original ones.

So far we have considered the metric density \( k_{\mu\nu} \) and its conjugate momenta \( P^{\mu\nu} \) together with additional constraints (57). This is possible in the classic theory, but problems arise in the quantum theory of gravity. In the functional integral, considered in the next section, we integrate over all values of \( k_{\mu\nu} \) and \( P^{\mu\nu} \). In principle, we can solve the constraints explicitly, but doing so is not interesting, because we then no longer have polynomials. Therefore, we describe the manifold \( N \) given by the coordinates \( k_{\mu\nu} \) and \( P^{\mu\nu} \) in detail. For simplicity, we assume that the coordinates take all possible real values, and the manifold \( N \) is consequently topologically trivial and diffeomorphic to the Euclidean space \( \mathbb{R}^{n(n-1)} \). We have the coordinate Poisson brackets (69) and (70) on that manifold, thus defining the Poisson structure. It can be easily verified that this structure is degenerate. This means that the manifold \( N \) is not a symplectic and is only a Poisson manifold (see, i.e., [16]). Because the rank of the Poisson structure is \( n(n-1)-2 \), there are two functionally independent Casimir functions on the Poisson manifold \( N \),

\[
C_1 = \det k_{\mu\nu}, \quad C_2 = \frac{1}{n-1} P^{\mu\nu} k_{\mu\nu} \tag{74}
\]

(introducing the constant factor \( 1/(n-1) \) in \( C_2 \) simplifies several formulas in what follows.) Indeed, the Poisson brackets of the functions \( C_1 \) and \( C_2 \) with all coordinates are zero,

\[
[C_{1,2}, k'_\mu] = [C_{1,2}, P^{\mu\nu}] = [C_{1,2}, \rho'] = [C_{1,2}, P'] = 0
\]

as a consequence of the definition of Poisson structure (69), (70). The Poisson brackets of these functions with an arbitrary differentiable function \( f \in C^1(N) \) then vanish, \( [C_{1,2}, f'] = 0 \), and \( C_1 \) and \( C_2 \) are therefore Casimir functions. The Poisson structure projected on sections \( V \subset N \) defined by the equations \( C_{1,2} = \text{const} \) is nondegenerate. These sections are hence symplectic.

It is always possible to choose local coordinates on the Poisson manifold \( N \) that are connected with the symplectic leaves \( C_{1,2} = \text{const} \). We let \((q_\lambda, p^\lambda), \, \lambda = 1, \ldots, n(n-1)/2 - 1, \) denote the coordinates on these leaves. We choose coordinates \( q_\lambda \) and \( p^\lambda \) such that the representation

\[
k_{\mu\nu} = |C_1|^{\frac{1}{n-1}} k_{\mu\nu}, \quad P^{\mu\nu} = P^{\mu\nu} + C_2 k^{\mu\nu}, \tag{75}
\]

is satisfied, where the respective matrix elements \( k_{\mu\nu}(q) \) and \( P^{\mu\nu}(p) \) depend only on \( q_\lambda \) and \( p^\lambda \). Obviously, such representations always exist. We choose the Casimir functions themselves as the lacking coordinates on \( N \). We thus obtain the local coordinate system

\[
(k_{\mu\nu}, P^{\mu\nu}) \leftrightarrow (q_\lambda, C_1), (p^\lambda, C_2). \tag{76}
\]

Constraints (57) have a simple form in the new coordinates,

\[
C_1 = \pm 1, \quad C_2 = 0. \tag{77}
\]

We choose the plus or minus sign for \( C_1 \) if the space has the respective even or odd dimensionality.

We consider the canonical transformation (59)–(61) from a different standpoint. Strictly speaking, a canonical transformation considered in this section is canonical only between coordinates

\[
(g_{\mu\nu}, p^{\mu\nu}) \leftrightarrow (\rho, P), (q_\lambda, p^\lambda). \tag{78}
\]
The new phase space of general relativity in the considered case is the manifold $\mathbb{R}_+ \times \mathbb{R} \times V$, where $\rho \in \mathbb{R}_+$, $P \in \mathbb{R}$, and the submanifold $V \subset N$ is defined by two values of Casimir functions \( \Omega \). The Poisson brackets on $V$ have the canonical form by construction,

$$[q_\lambda, p^0] = \delta_\lambda^0, \quad [q_\lambda, q_0] = 0, \quad [p^\lambda, p^0] = 0.$$

The constraints are made polynomial by extending the space $V$ to the Poisson manifold $N$ with Poisson brackets \( \Omega \). When additional constraints \( \Omega \) are solved explicitly, the polynomiality is lost. This is not surprising. For example, electrodynamics contains constraints whose explicit solution even leads to a nonlocal action for physical degrees of freedom (see, i.e., [17]).

The Poisson brackets of the Casimir functions between themselves equal zero \( [C_1, C_2] = 0 \). From the standpoint of the Hamiltonian formalism, they could be regarded as the first-class constraints generating gauge transformations. But these transformations are trivial because the Poisson brackets of Casimir functions with all phase-space coordinates vanish. This is possible only on a Poisson manifold with a degenerate Poisson structure. There are no Casimir functions on a symplectic manifold.

The Poisson manifold $N$ can be equipped with the second, now canonical, Poisson bracket. With respect to this new canonical Poisson structure, the submanifold $V$ is defined by two second-class constraints \( \Omega \). Then the original degenerate Poisson structure \( \Omega \), \( \Omega \) is just the Dirac bracket with respect to the canonical Poisson structure on $N$.

We also note the following. In gravity, we assume that a space-time metric $g_{\alpha \beta}$ has the Lorentzian signature and therefore is nondegenerate. In quantum gravity, the functional integral is integrated over all independent metric components, and this property can be taken into account only by restricting the integration domain. When the phase space is extended to a Poisson manifold, the integration domain is extended to the Euclidean space. The nondegeneracy of the metric is automatically provided by the presence of $\delta$-functions in the integrand.

## 8 Generating functional for the Green functions

Here, we summarize the results calculating in the most interesting case of the four-dimensional space-time and write the explicit expression for the generating functional for Green’s functions. The polynomial Hamiltonian formulation of general relativity on the Poisson manifold $\mathbb{R}_+ \times \mathbb{R} \times N$ with coordinates \( (\rho, P) \in \mathbb{R}_+ \times \mathbb{R} \) and \( (k_{\mu \nu}, P^{\mu \nu}) \in N \) is given in the preceding section. The dimensionality of this manifold for $n = 4$ equals 14. The Poisson structure is defined by the nonzero Poisson brackets

\[
\begin{align*}
[q_\rho, P] &= \delta, \\
[k_{\mu \nu}, P^\rho] &= \left(\delta^\rho_\mu - \frac{1}{3} k_{\mu \sigma} k^{\rho \sigma}\right) \delta, \\
[P^\mu, P^\rho] &= \frac{1}{3} \left(P^{\mu \sigma} k^{\rho \sigma} - P^{\rho \sigma} k^{\mu \nu}\right) \delta.
\end{align*}
\]

It is degenerate and has rank 12, which coincides with the dimensionality of the phase space of general relativity. There are two Casimir functions \( \Omega \) on $N$. The section $C_1 = -1$, $C_2 = 0$ is a symplectic submanifold $V \in N$ and defines the phase space of general relativity.
The action of general relativity in the new coordinates has the form

\[ S_{\text{HE}} = \int dx (P \dot{\rho} + P^\mu{}_{\nu} \dot{k}^\mu{}_{\nu} - H - \partial_\mu B^\mu), \]  

(79)

where the Hamiltonian density \( H \) is equal to a linear combinations of constraints,

\[ H = \tilde{N} K_\perp + N^\mu H_\mu, \]

and \( \tilde{N} \) and \( N^\mu \) are Lagrange multipliers. To action (79), we added the boundary term \( \partial_\mu B^\mu \) on a space section \( x^0 = \text{const} \), which is written as the divergence of some function of the canonical variables \( B^\mu(\rho, k^\mu{}_{\nu}, P, P^\mu{}_{\nu}) \). We briefly discuss the necessity of adding this important term in the action below without specifying its form. The constraints

\[ K_\perp = P^{\mu\nu} P^\mu{}_{\nu} - \frac{1}{6} \rho^2 P^2 - \rho^2 R^{(k)} - 2 \rho \partial_\mu (k^{\mu\nu} \partial_\nu \rho) + \frac{3}{2} k^{\mu\nu} \partial_\mu \rho \partial_\nu \rho, \]  

(80)

\[ H_\mu = -2 \partial_\nu (P^{\mu\sigma} k_{\sigma\mu}) + P^{\nu\sigma} \partial_\mu k^{\nu\sigma} - \frac{2}{3} \partial_\mu (P \rho) + P \partial_\mu \rho, \]  

(81)

are polynomial first-class constraints and satisfy the algebra (71)–(73). The scalar curvature \( R^{(k)} \) for the metric density \( k^\mu{}_{\nu} \) with unit determinant has the form (63). The constraint \( K_\perp \) is quadratic in the momenta and the variable \( \rho \). It is a fifth-order polynomial in metric density \( k^\mu{}_{\nu} \) (and its partial derivatives). The constraint \( H_\mu \) is linear in the momenta and also in the coordinates.

The expression for the generating functional for Green’s functions as a functional integral over the phase space \([13]\) is easily generalized on a Poisson manifold. For brevity, we introduce a new notation for the secondary constraints and Lagrange multipliers:

\( \{ H_a \} = \{ K_\perp, H_\mu \}, \quad \{ N^a \} = \{ \tilde{N}, N^\mu \}, \quad a = 0, 1, 2, 3. \)

We now fix the invariance under general coordinate transformations using four gauge conditions \( F^a = 0 \). The gauge is assumed to be canonical,

\[ \det [H_a, F^b] \neq 0. \]

The canonical form of the generating functional for the Green’s functions for the metric is given by the functional integral up to a normalization factor \([13]\):

\[ Z(J) = \int D(g^\mu{}_{\nu}) D(p^{\mu\nu}) D(N^a) \exp \left\{ \frac{i}{\hbar} \int dx (p^{\mu\nu} \dot{g}^\mu{}_{\nu} - N^a H_a - \partial_\mu B^\mu + g^\mu{}_{\nu} J^{\mu\nu}) \right\} \times \]  

\[ \times \det [H_a, F^b] \prod_a \delta(H_a) \prod_a \delta(F^a), \]  

(82)

where

\[ D(g^\mu{}_{\nu}) = \prod_x d g^\mu{}_{\nu}, \quad D(p^{\mu\nu}) = \prod_x d p^{\mu\nu}, \quad D(N^a) = \prod_x d N^a \]

and the \( J^{\mu\nu} \) are the sources for the metric. The functional integral \( Z(J) \) is the generating functional only for “coordinate” Green’s functions because the sources are written only for the space metric \([17]\). Because the Jacobian of any canonical transformation and,
in particular, of transformation (78) is equal to unity, the expression for the functional
integral can be rewritten in the equivalent form

\[ Z(J) = \int D(\rho)D(P)D(q_\lambda)D(p^\lambda)D(N^a) \times \]
\[ \times \exp \left\{ \frac{i}{\hbar} \int dx \left( P \dot{\rho} + p^\lambda \dot{q}_\lambda - N^a H_a - \partial_\mu B^\mu + \rho J + q_\lambda J^\lambda \right) \right\} \times \]
\[ \times \det [H_a, F^b] \prod_a \delta(H_a) \prod_a \delta(F^a). \]

The constraints \( H_a \) are nonpolynomial in coordinates \( q_\lambda \) and momenta \( p^\lambda \) in this
form. The integration must be extended over the whole Poisson manifold \( N \) to make the
constraints polynomial. We provide this by introducing two additional \( \delta \)-functions:

\[ Z(J) = \int D(\rho)D(P)D(k_{\mu\nu})D(P^{\mu\nu})D(N^a) \times \]
\[ \times \exp \left\{ \frac{i}{\hbar} \int dx \left( P \dot{\rho} + P^{\mu\nu} k_{\mu\nu} - N^a H_a - \partial_\mu B^\mu + \rho J + k_{\mu\nu} \ddot{J}^{\mu\nu} \right) \right\} \times \]
\[ \times \det [H_a, F^b] \delta(C_1 + 1) \delta(C_2) \prod_a \delta(H_a) \prod_a \delta(F^a). \]

Two new \( \delta \)-functions remove the integration over the additional variables and restrict the
integration over the Poisson manifold \( N \) to integration over symplectic section (77). To
perform the corresponding integration, we must perform coordinates transformation (76).

We must prove that the Jacobian of the coordinates transformation (76)

\[ D(q_\lambda)D(C_1)D(p^\lambda)D(C_2) = D(k_{\mu\nu})D(P^{\mu\nu}) \left| \frac{\partial (q_\lambda, C_1, p^\lambda, C_2)}{\partial (k_{\mu\nu}, P^{\mu\nu})} \right| \]

equals unity on constraint surface (77) to prove that the proposed expression for generating
functional (83) over the Poisson manifold is equivalent to the original functional integral
(82) over the phase space. This can be easily done. Let the coordinates \( q_\lambda \) parametrize the
matrix \( k_{\mu\nu} \) with a unit determinant arbitrarily. Then

\[ \left| \frac{\partial (q_\lambda, C_1, p^\lambda, C_2)}{\partial (k_{\mu\nu}, P^{\mu\nu})} \right| = \left| \frac{\partial (q_\lambda, C_1)}{\partial (k_{\mu\nu})} \right| \left| \frac{\partial (p^\lambda, C_2)}{\partial (P^{\mu\nu})} \right| , \]

because the matrix

\[ \frac{\partial (q_\lambda, C_1)}{\partial (P^{\mu\nu})} = 0. \]

As a consequence of the definition of canonical momenta (60), we obtain

\[ p^\lambda = -\frac{\delta F}{\delta q_\lambda} = -\frac{\delta F}{\delta k_{\mu\nu}} \frac{\partial k_{\mu\nu}}{\partial q_\lambda} = P^{\mu\nu} \frac{\partial k_{\mu\nu}}{\partial q_\lambda}. \]

(84)

Representation (75) for the metric \( k_{\mu\nu} \) yields

\[ k_{\mu\nu} = (n - 1) C_1 \frac{\partial k_{\mu\nu}}{\partial C_1} . \]

Therefore,

\[ \frac{\partial C_2}{\partial P^{\mu\nu}} = \frac{1}{n - 1} k_{\mu\nu} = C_1 \frac{\partial k_{\mu\nu}}{\partial C_1} . \]
Consequently,
\[
\frac{\partial (p^A, C_2)}{\partial (P_{\mu\nu})} = \frac{\partial k_{\mu\nu}}{\partial (q_\lambda, C_1)} C_1.
\]
This implies that for the constraint surface, the modulus of the Jacobian of the coordinates transformation is equal to unity:
\[
\left| \frac{\partial (q_\lambda, C_1, p^A, C_2)}{\partial (k_{\mu\nu}, P_{\mu\nu})} \right|_{C_1=-1} = 1.
\]
This motivates introducing the numerical factor in the Casimir function \( C_2 \) in Eq. (74). Without it, the Jacobian of the coordinate transformation would equal some nonzero constant, which could be included in the definition of the normalization factor of the generating functional.

We now say a few words about including the boundary term \( \partial_\mu B^\mu \) in the action of general relativity. The original expression for the generating functional is justified by assuming that the functional integral over just physical degrees of freedom with the unit measure appears after solution of all the constraints and gauge conditions. The Hamiltonian on the constraint surface becomes zero in this case, \( N^a H_a = 0 \). On the other hand, we know that the dynamics of the physical degrees of freedom are nontrivial. A possible way out from this contradiction was proposed in [2]. If the gauge condition depends explicitly on time, then the nontrivial Hamiltonian for the physical degrees of freedom on the constraint surface arises from the kinetic term \( p^{\mu\nu} \dot{g}_{\mu\nu} \). But this is not a unique possibility. To obtain a nontrivial Hamiltonian for the physical degrees of freedom in the canonical gauge, which does not depend on time explicitly, the boundary term was added to the action [17]. The importance of the boundary term in general relativity is now universally recognized, but its role still remains obscure in many cases because of large technical difficulties. In two-dimensional gravity where the constraints are solved explicitly, it was proved that the Hamiltonian for the physical degrees of freedom appears from the boundary term for gauge conditions, which do not depend explicitly [18]. Namely, the term \( \partial_\mu B^\mu \) on the constraint surface is equal to the Hamiltonian density for the physical degrees of freedom and is not equal to the divergence of some function. This statement is local and is independent of whether the universe is closed.

We note one more important point. For an asymptotically flat space-time, the volume integral of \( \partial_\mu B^\mu \) is equal to the surface integral of \( B^\mu \) and coincides with the mass of the Schwarzschild solution. That is why this integral was proposed as the definition of the total energy of a gravitational field [2]. It is usually assumed that the total energy of a closed universe vanishes because it has no boundary, and no surface integral arises. This is true if the constraints admit smooth solutions on compact manifolds for the nonphysical degrees of freedom in the general case. The example of two-dimensional gravity (which includes spherically symmetric solutions of general relativity) shows that constraints do not admit smooth solutions on a circle in general. In this case, we must make a cut on a compact space and add a boundary term there in order to pose the variational problem. This yields a nontrivial expression for the total energy of a closed universes. We can say this in other words. The divergence \( \partial_\mu B^\mu \) on the constraint surface expressed in terms of the physical degrees of freedom is no longer the divergence of some function. This Hamiltonian density for the physical degrees of freedom has the same form independently of whether the universe is closed or not. Therefore, this expression can be taken as the definition of the energy density of a gravitational field. Of course, this definition is not covariant and depends on the choice of a coordinate system.
9 Homogeneous and isotropic universe

We consider the Friedman universe \[19, 20\] to demonstrate the new variables introduced in Sec. 7. First, we recall the derivation of the equations in the Lagrangian formalism. For comparison, we then reformulate the model in the Hamiltonian language in the old and new variables in which the Hamiltonian becomes polynomial.

In the Friedman model of the universe, which is the basis for most contemporary cosmological models, we assume that the space is a Riemannian manifold of constant curvature at each instant (all sections \(x^0 = t = \text{const}\)). This assumption corresponds to a homogeneous isotropic universe. The metric satisfying this requirement is

\[
ds^2 = dt^2 + a^2 \tilde{g}_{\mu\nu} dx^\mu dx^\nu , \tag{85}\]

where \(a = a(t)\) is the scale factor depending only on time, and \(\tilde{g}_{\mu\nu}\) is a (negative-definite) constant-curvature space metric, which is independent of time by assumption. The specific form of the constant-curvature space metric \(\tilde{g}_{\mu\nu}\) depends on the coordinate system and is not important for the following consideration. For simplicity, we restrict ourselves to the most important case of four-dimensional space-time \((n = 4)\).

The metric of the space-time satisfies Einstein’s equations

\[
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \frac{1}{2} \Lambda g_{\alpha\beta} = -\frac{1}{2} T_{\alpha\beta} , \tag{86}\]

where we introduce the cosmological constant \(\Lambda\) and the matter energy-momentum tensor \(T_{\alpha\beta}\). The energy-momentum tensor in the comoving frame has the form \[21\]

\[
T^\alpha_\beta = \begin{pmatrix}
\mathcal{E} & 0 & 0 & 0 \\
0 & -\mathcal{P} & 0 & 0 \\
0 & 0 & -\mathcal{P} & 0 \\
0 & 0 & 0 & -\mathcal{P}
\end{pmatrix} , \tag{87}\]

where \(\mathcal{E}\) and \(\mathcal{P}\) are the respective energy density and pressure of matter. For a homogeneous isotropic universe in the chosen coordinate system, these densities depend only on time \(\mathcal{E} = \mathcal{E}(t)\) and \(\mathcal{P} = \mathcal{P}(t)\).

We also assume that the matter equation of state is given by some dependence of energy density on pressure,

\[
\mathcal{E} = \mathcal{E}(\mathcal{P}) . \tag{88}\]

Because energy-momentum tensor \[87\] was not obtained by varying some invariant action for matter fields with respect to a metric, the energy density \(\mathcal{E}\) cannot be an arbitrary function. Indeed, the covariant divergence of the left hand side of Einstein’s equations is identically zero as a consequence of the Bianchi identities. Therefore, Einstein’s equations yield the equation for the energy-momentum tensor:

\[
\nabla_\alpha T^\alpha_\beta = 0 .
\]

For metric \[85\] and energy-momentum tensor \[87\], these four relations reduce to one nontrivial equation

\[
\dot{\mathcal{E}} + \frac{3 \dot{a}}{a} (\mathcal{E} + \mathcal{P}) = 0 . \tag{89}\]

For a given equation of state \[88\], we have one first-order differential equation, which we rewrite in the form

\[
\frac{d\mathcal{E}}{\mathcal{E} + \mathcal{P}(\mathcal{E})} = -3 \frac{da}{a} . \tag{90}\]
Solving this equation yields the energy density $E$ as a function of the scale factor $a$.

We obtain the equation for the scale factor because metric (85) must satisfy Einstein’s equations (86). Simple calculations yield the expressions for the Einstein tensor

$$R_{0}^{0} - \frac{1}{2}R = 3 \frac{K_{0} - \dot{a}^{2}}{a^{2}},$$

$$R_{\mu}^{\mu} = 0,$$

$$R_{\mu}^{\nu} - \frac{1}{2}R\delta_{\mu}^{\nu} = -\frac{1}{a^{2}} (2a\ddot{a} + \dot{a}^{2} - K_{0}) \delta_{\mu}^{\nu},$$

where $K_{0}$ in our notation is the scalar curvature of a three-dimensional sphere ($K_{0} = 1$), Euclidean space ($K_{0} = 0$), or one-sheet hyperboloid ($K_{0} = -1$). We can now easily see that Einstein’s equations lead to only two nontrivial equations on the scale factor:

$$3 \frac{K_{0} - \dot{a}^{2}}{a^{2}} + \Lambda + \frac{1}{2} E = 0,$$

$$-\frac{1}{a^{2}} (2a\ddot{a} + \dot{a}^{2} - K_{0}) + \Lambda - \frac{1}{2} \mathcal{P} = 0. \quad (91)$$

It is easy to verify that Eq. (92) is a consequence of Eqs. (91) and (89) because Einstein’s equations are linearly dependent as soon as Eq. (89) is satisfied. Therefore, Eq. (92) can be dropped, but we do not do this, because it is needed for the canonical treatment of the Friedman universe.

We thus find the dependence $E = E(a)$ of the energy density on the scale factor by solving Eq. (90) for a given equation of state (88). Substituting this function in Eq. (91), we obtain the first-order ordinary differential equation for the scale factor. This is precisely the main equation in the standard cosmological models for a homogeneous isotropic universe.

We start with the canonical formulation. For metric (85), we have

$$N = 1, \quad N_{\mu} = 0, \quad g_{\mu\nu} = a^{2} \bar{g}_{\mu\nu},$$

External curvature tensor (32) and the volume element are

$$K_{\mu\nu} = -a \dot{a} \bar{g}_{\mu\nu}, \quad K = -3 \frac{\dot{a}}{a},$$

$$\dot{\varepsilon} = a^{3} \bar{\varepsilon}, \quad \bar{\varepsilon} = \sqrt{\det \bar{g}_{\mu\nu}}.$$  

Canonical momenta (44) conjugate to the space metric become

$$p_{\mu\nu} = -2 a \bar{\varepsilon} \bar{g}^\mu^\nu.$$  

The traceless part of these momenta is identically zero,

$$\bar{p}_{\mu\nu} = 0, \quad p = -6 a^{2} \dot{a} \bar{\varepsilon}. \quad (93)$$

The Hamiltonian density is given by the single dynamical constraint (48) because the shift function for the Friedman universe is equal to zero

$$H_{\perp} = -6 \bar{\varepsilon} \left[ (\dot{a}^{2} - K_{0})a - \frac{1}{3} \Lambda a^{3} - \frac{1}{6} \mathcal{E} a^{3} \right], \quad (94)$$
where we take the contribution from cosmological constant and matter fields into account. We cannot insert the momentum trace $p$ instead of the time derivative $\dot{a}$ in this expression, because it is not the variable conjugate to the scale factor.

To find the momentum conjugate to $a$, we rewrite Lagrangian (12) for metric (85),

$$\mathcal{L}_{\text{ADM}} = -6\varepsilon \left[ (\dot{a}^2 + K_0) a + \frac{1}{3} \Lambda a^3 + \frac{1}{6} \mathcal{E} a^3 \right].$$

This expression can be integrated over the space because all the dependence on the space coordinates is contained in the volume element $\varepsilon$. Dropping the constant factor $-6V$, where $V$ is the volume of space (infinite for the Euclidean space and one sheet hyperboloid), we obtain the Lagrangian for the scale factor

$$L = (\dot{a}^2 + K_0) a + \frac{1}{3} \Lambda a^3 + \frac{1}{6} \mathcal{E} a^3, \quad (95)$$

which is independent on the space coordinates. This is the standard Lagrangian for a point particle moving in a one-dimensional space with the coordinate $a \in \mathbb{R}_+$. Deriving the Lagrangian for the scale factor, we dropped the negative factor to obtain the positive sign of the kinetic term $a\dot{a}^2$. We thus changed the total sign of the action and hence the sign of the energy. We note that the contribution of the kinetic term of the scale factor to the energy is negative.

The expression for the momentum conjugate to the scale factor follows from Lagrangian (95):

$$p_a = \frac{\partial L}{\partial \dot{a}} = 2a\dot{a},$$

which differs from momentum trace (93) by a factor and is independent on the space coordinates. The Hamiltonian for the scale factor corresponding to Lagrangian (95) is

$$H = \frac{1}{4a} p_a^2 - K_0 a - \frac{1}{3} \Lambda a^3 - \frac{\mathcal{E}}{6} a^3. \quad (96)$$

We see that this expression for the Hamiltonian coincides up to the factor $-6V$ with the expression obtained by integrating dynamical constraint (94) over the space. This observation is nontrivial because some equations of motion may be lost when specific expressions for field variables are inserted into the Lagrangian.

The Hamiltonian equations for the scale factor are

$$\dot{a} = \frac{1}{2a} p_a, \quad \dot{p}_a = \frac{1}{4d^2} p_a^2 + K_0 + \Lambda a^2 + \frac{1}{2} \mathcal{P} a^2, \quad (97)$$

where we use Eq. (90) to calculate the Poisson bracket

$$[p_a, \mathcal{E}] = [p_a, a] \frac{d\mathcal{E}}{da} = -3 \frac{\mathcal{E} + \mathcal{P}}{a}. $$

It can be easily verified that Hamiltonian equations (97) are equivalent to the second-order Lagrangian equation (92), and Hamiltonian (96) is proportional to the left-hand side of Eq. (94).
We have thus formulated equations for the Friedman universe in the Hamiltonian language. In contrast to the Hamiltonian particle dynamics, we have an additional constraint along with canonical equations of motion (97),
\[ H(a, p_a) = 0. \] (98)

In other words, we only seek those solutions of the equations of motion for which the energy is equal to zero. This problem is self-consistent because the energy is conserved. We have thus formulated equations (91) and (92) for the scale factor in the Hamiltonian form and proved that one constraint (98) on the scale factor and the corresponding momentum is lost if the expression for the metric (85) is substituted not in Einstein’s equations but in the action.

Hamiltonian (96) and equations of motion (97) are nonpolynomial in the scale factor. We show what happens with the equations under the canonical transformation described in section 7. As a consequence of Eqs. (59)–(61), we obtain expressions for the canonical variables
\[ \rho = a^2 e^{2/3}, \quad k_{\mu\nu} = \eta_{\mu\nu}, \] (99)
\[ P = -2\frac{\dot{\rho}}{\sqrt{\rho}}, \quad P^{\mu\nu} = 0, \] (100)

external curvature
\[ K_{\mu\nu} = -\frac{1}{2} \dot{\rho} \eta_{\mu\nu}, \quad K = -\frac{3\dot{\rho}}{2\rho} \]
and volume element
\[ \hat{e} = \rho^{3/2} \]
after the canonical transformation.

To simplify calculations, it is easiest to separate variables by extracting the factor \( q(t) \) depending only on time from \( \rho \),
\[ \rho(t, x^\mu) = q(t) e^{2/3}. \]

The Lagrangian density can then be integrated over the space as previously, and the Hamiltonian reformulation of the equations reduces to the redefinition of the scale factor
\[ q = a^2. \] (101)

Lagrangian (95) for the new variable is
\[ L = \frac{1}{4\sqrt{q}} q^2 + K_0 \sqrt{q} + \frac{1}{3} \Lambda q^{3/2} + \frac{1}{6} \mathcal{E} q^{3/2}. \]

The momentum conjugate to the new dynamical variable \( q(t) \) is
\[ p_q = \frac{1}{2\sqrt{\rho}} \dot{q}. \]

The corresponding Hamiltonian contains a nonpolynomial factor,
\[ H = \sqrt{q} \left( p_q^2 - K_0 - \frac{1}{3} \Lambda q - \frac{1}{6} \mathcal{E} q \right). \]
During the construction of the polynomial Hamiltonian formulation, the dynamical constraint was multiplied by factor $64$. As a result, we obtain a new constraint, which is now polynomial, and the new Hamiltonian

$$K = q^{3/2}H = q^2 \left( p_q^2 - K_0 - \frac{1}{3} \Lambda q - \frac{1}{6} \mathcal{E} q \right).$$

(102)

To preserve the Hamiltonian form of the equations of motion we must also redefine time as well $t \to \tau$, where the new parameter is defined by the differential equation

$$\frac{d\tau}{dt} = q^{-3/2} = a^{-3}.$$

Redefining time corresponds to redefining Lagrange multiplier $66$. The equations of motion for Hamiltonian (102) are

$$\frac{dq}{d\tau} = 2q^2 p_q,$$

$$\frac{dp_q}{d\tau} = q^2 \left( \frac{1}{3} \Lambda - \frac{1}{12} \mathcal{E} - \frac{1}{4} P \right) - \frac{2}{q} K,$$

where we once again used Eq. (90). The second term in the second equation can be dropped because the model contains the constraint

$$K(q, p_q) = 0.$$

(103)

Thus in the new variables, we have a “particle” described by the coordinate $q(\tau) \in \mathbb{R}_+ \ 	ext{and momentum} \ p_q(\tau) \in \mathbb{R}$ with Hamiltonian (102) and with constraint (103) imposed on the canonical variables. The Hamiltonian and equations of motion are polynomial (if the energy density $\mathcal{E}(q)$ is polynomial in $q$) and equivalent to the original Einstein’s equations for the scale factor (91), (92).

For the homogeneous isotropic universe we can go even further and eliminate the common factor $q^2$ from Hamiltonian (102) by redefining time $t \to \tau'$, where

$$d\tau' = q^{-1/2} dt.$$

We then have a “particle” with the simple Hamiltonian

$$K' = p_q^2 - K_0 - \frac{1}{3} \Lambda q - \frac{1}{6} \mathcal{E} q.$$

and the corresponding equations of motion

$$\frac{dq}{d\tau'} = 2p_q,$$

$$\frac{dp_q}{d\tau'} = \frac{1}{3} \Lambda - \frac{1}{12} \mathcal{E} - \frac{1}{4} P.$$

(104)

The space curvature $K_0$ now does not contribute to the equations of motion at all. The system of equations of motion is solved under the zero constraint on the Hamiltonian, $K' = 0$. Just as for the original system of equations (91) and (92), the Hamiltonian equations of motion in the Lagrangian form follow from the equation $K' = 0$, where instead of the momentum, we must substitute its expression in terms of the time derivative of the scale factor (104). This equation is equivalent to Eq. (91).
For dust matter, we have $\mathcal{P} = 0$, and Eq. (90) can be easily integrated,

$$\mathcal{E} = \frac{M}{a^3}, \quad M = \text{const},$$

which provides a massless particle moving in the potential

$$\frac{1}{3} \Lambda q - \frac{M}{6q^2}.$$

Introducing new canonical variables thus allows reformulating the equations for the Friedman universe in polynomial form. Because Eq. (91) has been well studied in recent decades, the new formulation is unlikely to yield new results for the analysis of classical solutions in the considered case. It may be useful for constructing a quantum model of the Friedman universe, but this question requires a separate investigation and is beyond the scope of this paper. Our consideration here is intended only to illustrate how the general method works in a simple case where all steps can be verified by simple calculations.

## 10 Conclusion

We have demonstrated that considering the determinant of the metric and the conjugate momentum as independent additional variables leads to a Hamiltonian formulation of general relativity in which all the constraints are polynomial. The model is formulated in polynomial form not in the phase space but on a Poisson manifold where the Poisson bracket is degenerate. We stress that the resulting model is equivalent to general relativity.

In the new variables, the canonical momenta are proportional to the irreducible components of the momenta in the standard metric formulation. This property essentially simplifies the calculations, in particular, the calculations of the Poisson bracket of the dynamical constraint.

The proposed canonical formulation of general relativity allows writing the functional integral on the Poisson manifold. We proved that this integral is equivalent to the functional integral over the phase space. The advantage of the new expression for the generating functional for the Green’s functions is that the action and all arguments of $\delta$-functions are polynomial in independent variables. This leads to the presence of only a finite number of vertices in the diagram techniques. This seems to simplify calculations in the quantum theory of gravity.

As an example of using new variables, we considered the Friedman model of the universe. In these variables, the Hamiltonian and the equations of motion for the scale factor are simplified and take a polynomial form.

A similar transformation of variables in the configuration space leading to a polynomial Hilbert–Einstein action was proposed in [22] and recently rediscovered [23].

The author is sincerely grateful to I. V. Volovich for the discussion of the paper. This work is supported by Russian Foundation of Basic Research (Grant No. 05-01-00884) and the Program for Supporting Leading Scientific Schools (Grant No. NSh-6705.2006.1).

## References

[1] P. A. M. Dirac. *Proc. Roy. Soc. London*, A246:333–343, 1958.
[2] R. Arnowitt, S. Deser, and S. W. Misner. The dynamics of general general relativity. In L. Witten, editor, *Gravitation: an introduction to current research*, New York – London, 1962. John Wiley & Sons, Inc.

[3] T. Regge and C. Teitelboim. Role of surface integrals in the Hamiltonian formulation of general relativity. *Ann. Phys.*, 88:286–318, 1974.

[4] B. S. DeWitt. Quantum theory of gravity. I. The canonical theory. *Phys. Rev.*, 160(5):1113–1148, 1967.

[5] C. Teitelboim. How commutators of constraints reflect the spacetime structure. *Ann. Phys.*, 79:542–557, 1973.

[6] K. Kuchar. Kinematics of tensor fields in hyperspace. II. *J. Math. Phys.*, 17(5):792–800, 1976.

[7] P. A. M. Dirac. Interacting gravitational and spinor fields. In *Recent Developments in General Relativity*, Oxford, 1962. Pergamon Press.

[8] J. Schwinger. Quantized gravitational field. *Phys. Rev.*, 130(3):1253–1258, 1963.

[9] S. Deser and C. J. Isham. Canonical vierbein form of general relativity. *Phys. Rev.*, D14(10):2505–2510, 1976.

[10] J. M. Charap, M. Henneaux, and J. E. Nelson. Explicit form of the constraint algebra in tetrad gravity. *Class. Quantum Grav.*, 5(11):1405–1414, 1988.

[11] A. Ashtekar. New Hamiltonian formulation of general relativity. *Phys. Rev.*, D36(6):1587-1602, 1987.

[12] R. S. Tate. Polynomial constraints for general relativity using real geometrodynamical variables. *Class. Quantum Grav.*, 9:101–119, 1992.

[13] L. D. Faddeev. Feinman integral for singular Lagrangians. *TMP*, 1(1):3–18, 1969.

[14] P. A. M. Dirac. The Hamiltonian form of field dynamics. *Can. J. Math.*, 3(1):1–23, 1951.

[15] H. Goldstein. *Classical mechanics*. Addison-Wesley press, Inc., Cambridge, Mass., 1950.

[16] Y. Choquet-Bruhat and C. DeWitt-Morette. *Analysis, Manifolds and Physics*. North–Holland, Amsterdam, 1989.

[17] D. M. Gitman and I. V. Tyutin. *Quantization of Fields with Constraints*. Springer–Verlag, Berlin–Heidelberg, 1990.

[18] M. O. Katanaev. Effective action for scalar fields in two-dimensional gravity. *Ann. Phys.*, 296(1):1–50, 2002.

[19] A. Friedmann. Über die Krümmung des Raumes. *Zs. Phys.*, 10: 377, 1922.

[20] A. Friedmann. Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes. *Zs. Phys.*, 21: 326, 1924.
[21] L. D. Landau, E. M. Lifshitz. The Classical Theory of Fields. Second edition. Pergamon, New York, 1962.

[22] A. Peres. Polynomial expansion of gravitational lagrangian. *Nuovo Cimento*, 28(4):865–867, 1963.

[23] M. O. Katanaev. Polynomial form of the Hilbert–Einstein action. *Gen. Rel. Grav.*, 38:1233–1240, 2006; [gr-qc/0507026](http://arxiv.org/abs/gr-qc/0507026) (2005).