HORIZON OPERATOR APPROACH TO BLACK HOLE

QUANTIZATION.†

G. ’t Hooft

Institute for Theoretical Physics
University of Utrecht, P.O.Box 80 006
3508 TA Utrecht, the Netherlands

Abstract

The $S$-matrix Ansatz for the construction of a quantum theory of black holes is further exploited. We first note that treating the metric tensor $g_{\mu\nu}$ as an operator rather than a background allows us to use a setting where information is not lost. But then we also observe that the ’trans-Planckian’ particles (particles with kinetic energies beyond the Planck energy) need to be addressed. It is now postulated that they can be transformed into ’cis-Planckian particles’ (having energies less that the Planck energy). This requires the existence of a delicate algebra of operators defined at a black hole horizon. Operators describing ingoing particles are mapped onto operators describing outgoing ones, preserving their commutator algebra. At short distance, the transverse gravitational back reaction dictates a discrete lattice of data points, and at large distance the algebra must reproduce known interactions of the Standard Model of elementary particles. It is suggested that further elaboration of these ideas requiring complete agreement with general relativity and unitarity should lead to severe restrictions concerning the inter-particle interactions.

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1. INTRODUCTION

The fact that black holes should emit elementary particles at a well-defined temperature, as can be derived by more-or-less standard techniques in quantum field theory\textsuperscript{[1]} puts them in the same category as the heavy unstable particle like solutions that also exist in many other field theories, and therefore one expects that they exhibit some well-defined spectrum of (excited) states, and one would expect furthermore that these states should be calculable from the local laws of physics. Indeed, from the known mass dependence of the temperature one can readily derive that the density of levels should be given by\textsuperscript{[2]}

\[ \rho(M) = C e^{4\pi M^2}, \]  

in natural units \((G = h = c = 1)\). Here \(C\) is a multiplicative constant. However, in contrast to the situation in conventional field theories, which are sufficiently well understood conceptually, this constant \(C\) seems to be fundamentally uncomputable - indeed, a standard calculation following a background field technique, always gives \(C = \infty\), implying a strictly continuous spectrum.

If this naive result would be correct and the black hole spectrum would indeed be continuous this would require a radical departure from standard quantum theory. It would imply that even the tiniest black hole could absorb infinite amounts of information, and it would never be possible to represent such an object by a finite component wave function, in contrast to all other known physical objects. It has been argued that a quantum mechanical wave function can still be formulated, but only if all possible states in all other "universes" connected to the black holes by analytic extension of its metric, would be taken into account. If we were forced to limit ourselves to our own universe there simply would not exist a Schrödinger equation for black holes. Physicists are now divided mainly in three camps as to what our attitude towards this problem should probably have to be.

The first proposal, particularly defended by Hawking\textsuperscript{[3]}, is that conventional quantum mechanical behavior may be limited to distance scales much larger than the Planck scale, but there are fundamental deviations from that at Planck scale distances. We will have a density matrix \(\rho(t)\), but in stead of its usual propagation law

\[ \frac{d}{dt} \rho(t) = -i[H, \rho(t)] , \]  

he proposes a more general linear evolution law:

\[ \frac{d}{dt} \rho(t) = -i\$\rho(t) , \]  

where \$ is an arbitrary linear operator acting on all components of the matrix \(\rho\). This proposal has been criticised by Banks et al\textsuperscript{[4]}, who argued that conventional conservation
laws such as energy conservation will be violated. We could add to theirs the following consideration. The density matrix may be seen as describing two universes - which under normal circumstances do not interact with each other - since it spans a Hilbert space that is the product of ket- and bra states:
\[ \mathcal{H} = \{ \rho_{ij} \} = \{ |\psi_i\rangle \} \otimes \{ \langle \psi_j | \} , \] (1.4)
which in conventional quantum mechanics evolves according to
\[ \frac{d}{dt} \rho_{ij}(t) = -iH_{ik}\rho_{kj} + iH_{jk}\rho_{ik} \equiv -i(H^{(1)} - H^{(2)})\rho_{ij} . \] (1.5)

We see that in these two universes energy is defined with opposite signs. Imagine now that, if a rule such as eq. (1.3) would hold, some sort of interaction takes place between the two universes:
\[ \frac{d}{dt} \rho(t) = -i(H^{(1)} - H^{(2)} + H^{int})\rho(t) , \] (1.6)
then this would imply that the vacuum state, defined as the zero eigenstate of \( H^{(1)} \) and \( H^{(2)} \), would be able to make transitions to any state of the form \( |E\rangle\langle E| \), since this conserves total energy for both spaces combined. But since phase space for the final states is infinitely larger than that of the vacuum state the transition would never go backwards to the vacuum. Or, in terms of kets alone, transitions would be made into higher energy mixed states, and this would be experienced as an instability of the vacuum state. Only the absence of an interaction Hamiltonian could protect the vacuum against such an instability, but this would precisely correspond to a pure Schrödinger equation rather than a $ matrix evolution law for the density matrix.

The second proposal often defended is that black holes do absorb all information thrown into them, such that their spectrum becomes infinitely degenerate, which then implies that at some lowest energy there must exist an infinitely degenerate state, called a 'black hole remnant'. These remnants then cannot decay any further and hence should be absolutely stable. That this would be the way information is preserved was concluded from calculations in a two dimensional model by Callan et al[5]. The problem with this proposal is that these remnants would not form a Bose-Einstein or Fermi-Dirac gas but rather a 'Boltzman gas', since they are infinitely degenerate. Whatever process would create tiny black holes at the Planck length would destabilize the universe thermally, since phase space of a remnant gas is infinitely larger than that of any other state at the same energy.

The third proposal, preferred by the present author[6, 7], is that the information is projected into the Hawking radiation. Mathematically, this assertion means that if two possible initial states for the black hole were mutually orthogonal, the final states will be
orthogonal also, provided that all Hawking particles are included in the considerations. Clearly it will be impossible to check such a statement for macroscopic black holes, so that the latter may in practice be seen to behave exactly as Hawking found. But at a microscopic level it implies that there ought to exist an $S$-matrix with poles corresponding to each metastable black hole configuration. This would imply that the constant $C$ in eq. (1.1) should be strictly finite. Note though that the possibility for it being very large or very small, e.g. $10^{\pm 40}$, is still kept wide open. We have to keep in mind that large numbers may naturally emerge from quantum gravity!

The problem with this third proposal is that it is in conflict with all calculations in the linearized field approximation. These are calculations where quantum operator fields are superimposed onto each other. There are several equivalent ways to see how this happens[6]. First one may consider quantum field theory on the background of the metric in the usual Schwarzschild coordinates $r$ and $t$, where

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2 , \quad (1.7)$$

and $\Omega = (\theta, \varphi)$. The Lagrangian for a scalar field $\phi(x)$ with mass $m$ is

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\left[-\left(1 - \frac{2M}{r}\right)\partial_r\phi^2 + \left(1 - \frac{2M}{r}\right)^{-1}\partial_t\phi^2 - r^{-2}l(l+1)\phi^2 - m^2\phi^2\right]. \quad (1.8)$$

Close to the horizon it becomes manageable in the coordinates

$$\sigma = \log(r - 2M) \quad \text{and} \quad t , \quad (1.9)$$

since then

$$\mathcal{L}d\sigma dt = \frac{1}{2}d\sigma dt \left(r^3\partial_t\phi^2 - r\partial_\sigma\phi^2 - (r - 2M)\left[m^2 r^2 + l(l+1)\right] \phi^2 \right) , \quad (1.10)$$

At $r \approx 2M$ the Euler-Lagrange equations are readily seen to produce plane wave solutions, and since $\sigma$ is bounded at the left only by $-\infty$ we immediately deduce that the spectrum should be continuous. Actually the absence of anything like a boundary condition for linearized fields close to the horizon can also easily be deduced from arguments in Kruskal or Penrose coordinate frames.

Thus, if we suspect that the information bounces back into the Hawking radiation we must assume this to be due to interactions. Clearly, as $e^\sigma$ plummets below the Planck length the gravitational interactions become very strong and here the linearized approximation is invalid. We will therefore always assume that the recovery of information is due to strong gravitational (back) reactions very near the horizon. A ‘boundary condition’
very near the horizon has indeed been proposed by several authors[6, 8]. We do generally assume that Hawking emission occurs as calculated by Hawking when the black hole is large. In that case however the hole is apparently only allowed to be a probabilistic mixture of many states. One must suspect then that general coordinate invariance only holds for average states, not so much for individual states.

It is instructive to compare apparent pure-to-mixed transitions with transitions in a theory with uncertain Hamiltonian. Imagine a conventional quantum mechanical system with a Hamiltonian depending on some parameter $\alpha$:

$$\frac{d}{dt} |\Psi(t)\rangle = -iH(\alpha)|\Psi(t)\rangle,$$  \hspace{1cm} (1.11)

where the value for the parameter $\alpha$ is known to have a probability distribution $P(\alpha)d\alpha$. Let the state at $t = 0$ be given as a pure state $\Psi(0)$. The expectation value of some operator $O$ at time $t$ is then

$$\langle O \rangle = \int d\alpha P(\alpha) \langle \Psi(0) | e^{iH(\alpha)t} O e^{-iH(\alpha)t} | \Psi(0) \rangle = \text{Tr} (\rho(t) O),$$ \hspace{1cm} (1.12)

where $\rho(t)$ is easily seen to be the density matrix for a mixed state.

Apparently the linearized field approximation is tantamount to admitting some uncertainty in the Hamiltonian. The fact that such an interpretation did not follow from calculations in some special models as produced by Ref.[5] may well mean that such special models are not suitable for an accurate representation of quantum gravity. Quite generally there seem to be reasons to complain that these models are not sufficiently explicit when it comes to describing non-perturbative phenomena such as black holes, a complaint one could also utter against (super)string theory and even the existing models of quantum gravity in 2+1 dimensions[9] In any case we will observe that a description of black holes that is completely in accordance to quantum mechanics will require entirely new physics. Admittedly, adhering to this last option (i.e. in an accurate theory black holes remain pure) does not make life easy. Yet we will try to convince the reader that progress from this starting point looks quite promising.

2. THE S-MATRIX ANSATZ

From now on we will make the following assumption (S-matrix Ansatz)[10]:

All physical interaction processes, therefore also all those that involve the creation and subsequent evaporation of a black hole, can be described by one scattering matrix $S$ relating the asymptotic outgoing states $|\text{out}\rangle$ to the ingoing states $|\text{in}\rangle$.

We will often use this Ansatz the following way: we assume some value for one particular transition amplitude $\langle \text{out}_0 | \text{in}_0 \rangle$, after which we make some long series of small changes both
in the in- and in the out-state. The effects of the changes are often directly computable. Hence knowing one matrix element gives access to calculating others, by making use of known laws of physics. It will turn out that using well-established physical laws at large distance scales never leads to direct conflicts; rather, large segments of the $S$-matrix can be computed along these lines

\[ r=2M \text{ horizon} \]
\[ r=0 \text{ singularity} \]
\[ \text{Hawking shift} \]
\[ \text{Penrose diagram for black hole after formation.} \]

One often hears the objection that the $S$-matrix Ansatz appears to violate causality, since matter once fallen through the horizon will be spacelike separated from the region where Hawking radiation is seen, so that this Hawking radiation cannot carry the information. If it did, this would be an example of ”quantum mechanical duplication” – states including information about quantum phase factors seem to occur at two places that are spacelike separated. This objection however is based upon the assumption that one could, at least in principle, independently observe matter that passed through the horizon and Hawking radiation travelling very close to the horizon. We will refer to an observer able to do this as a ‘super observer’. We now claim that such super observers do not exist. To be precise:

If we restrict ourselves to Hilbert space spanned either by all possible asymptotic in-states or by all possible asymptotic out-states, then operators describing fields within the horizon do not commute with operators describing Hawking particles – in fact, all relevant commutators tend to infinity.
Clearly this would be sufficient to exclude their independent observation by some super observer. But how should the above statement be compatible with standard wisdom concerning the commutation of spacelike separated observables? The answer to that is in the smallprint: we limit ourselves to the Hilbert space spanned by the asymptotic states. Consider the Penrose diagram of a black hole just formed, see Fig. 1. In this coordinate frame it is hard to talk about Hawking radiation at all because it is the standard frame for describing the single vacuum state at the onset of the horizon. The Hawking particles live in region $I$, very close to infinity in Fig.1, and very strongly Lorentz transformed. This we have to keep in mind. Now consider just any observable operator $O_H$ acting on the visible Hawking particles. The crucial point is that, as seen by observers in the frame of Fig. 1, these operators create ‘trans-Planckian’ particles.

By themselves, the trans-Planckian particles cannot be excluded from the physical Hilbert space of ingoing particles. But then consider particles passing from region $I$ into region $II$ or $III$, described by operators $O_I$, $O_{II}$ and $O_{III}$. The trans-Planckian Hawking particles created by $O_H$ affect them by a gravitational drag as indicated in the Figure. This shift depends on the position in the transverse coordinates $(\theta, \varphi)$. It causes a mismatch between the operators $O_I$ on the one hand and $O_{II}$ and $O_{III}$ on the other hand. This mismatch is normally forgotten when ingoing particles in regions $II$ and $III$ are described. Since furthermore the shift depends on the transverse coordinates its effect cannot be removed by coordinate redefinitions. Thus we have

$$O_{II}O_H = O_H O_{II}, \text{ shifted}. \quad (2.1)$$

At first sight the need to perform the shift in eq. (2.1) may perhaps not be obvious. The point is however that the Hilbert space we chose to work in is $\mathcal{H}_I$, the one spanned by the states in $I$, not $\mathcal{H}_{II}$ which is spanned by the states in $II$ and $III$. If the latter were the case we could have used the local coordinate frame of $II$ and $III$ and there would be complete commutation. However, the Hilbert space space $\mathcal{H}_{II}$ does not allow the action of $O_H$ without generating a white hole at the onset of the horizon, $S$ in Fig. 1. The absence of a white hole is one of the most essential elements of the $S$-matrix Ansatz. We observe that the non-commutation as described by eq. (2.1) is linked to the need to describe the space-time metric as a quantum operator rather than a c-number background. This was also emphasized in Ref. [11]. It is now not difficult to see that the commutator diverges as $t \to \infty$, where $t$ is time parameter in $O_H$, being the moment Hawking radiation is studied by the external observer.
3. THE NON-SINGULAR METRIC

How should one take the quantum nature of the metric into account? One proposal was made in Ref. [12]. If the in-state is well-specified the metric can be treated as a c-number† until the horizon is reached. Now if we work with the S-matrix Ansatz we can also specify the out-state such that also the metric during evaporation is well-determined as soon as the white hole’s past-horizon is left behind. The philosophy of the S-matrix Ansatz is to consider only small changes both in the in-states and the out-states. To do this we only need the metric \( g_{\mu\nu}(x, t) \) as defined by

\[
g_{\mu\nu} = \frac{\langle \text{out} | \hat{g}_{\mu\nu} | \text{in} \rangle}{\langle \text{out} | \text{in} \rangle},
\]

which is well-defined both during the initial and the final phases. It is therefore proposed to glue the metrics for the in-state and the out-state together, as is depicted in Fig. 2. The relevance of in/out matrix elements of observables as in eq. (3.1) was discussed by Aharonov and Vaidman [13]. They call \( g_{\mu\nu} \) as defined in this equation the ‘weak value’ of the operator \( \hat{g}_{\mu\nu} \).

The simplest situation to be discussed first is the case where the in-state is chosen to contain a single radially symmetric shell of matter moving inwards, and the out-state to contain also a single shell of matter, now moving outwards. Although this out-state does not represent Hawking radiation we expect this amplitude to be of the same order of magnitude as the amplitudes for Hawking radiation, the only reason for this particular out-state to be less likely being its much smaller phase space factor. Within the shells space-time is flat, and this allows us to glue the metrics of in-space and out-space together also at the inside.

The price one pays for obtaining this singularity-free overall metric (Fig. 2b) is that at the point where the in- and out-shells meet, labeled \( S \) in Fig. 2a and b, the curvature is very large. A physical description of this point is to say that a very violent explosion takes place there, sending ingoing matter back out; the curvature results from the huge stress tensor that is needed for this. In the limit of infinitely thin shells we do have a singularity at \( S \) but it is a mild one: a cusp singularity. It is surrounded by nearly-flat space-time such that a space-time journey looping around \( S \) produces a Lorentz transformation.

† We treat the metric operator \( \hat{g}_{\mu\nu}(x, t) \) as if the in-state is an eigenvector for it at all \( (x, t) \) during the collapse, but this of course is not true. \( g_{\mu\nu} \) and \( \dot{g}_{\mu\nu} \) obey canonical commutation relations. However, these can be regarded as the fields describing gravitons, which will be considered as small fluctuations. In the present argument it is the large scale fluctuations of \( g_{\mu\nu} \) that we are describing and trying to keep under control.
a) The metric generated by $g_{\mu\nu}(x, t)$ as it acts on the in state (below) and the out state (above). b) The glued metric as matrix element between out and in state. The shaded regions in (a) are removed since they would become ambiguous. The regions with the slanted lines in (a) are also glued together, so that in this metric information is restored. c) The resulting metric in less deformed coordinates.

One can now proceed to do quantum field theory in this topologically trivial space-time. Certainly no "information gets lost". The $S$-matrix Ansatz does require that we consider weak and low-energy fields only, and this will be an important restriction as we will see. At first sight such a restriction does not seem to be worrisome. We now should be able to consider small perturbations $\delta\langle\text{out}\mid$ and $\delta\langle\text{in}\rangle$ and compute the amplitudes for the perturbed states$^\dagger$. Field theory shows interesting effects near $S$. The cusp is a particle producer: if we have a vacuum preceding $S$ the state behind $S$ will contain particles (in a quantum mechanically coherent state). Details of the mechanism are not difficult to compute; we refer to Ref. [12]. The result is that the intensity of the particle production

$^\dagger$ Note that in terms of distances in Hilbert space the perturbations need not be small; the perturbed states are generally orthogonal to the original ones; the perturbations are small in the sense that they have little gravitational effects.
by a cusp in the $xy$ direction is given by

$$\frac{dN}{dp x d^2 x} = \frac{1}{p_0} \left( \frac{\alpha}{\theta \frac{\alpha}{2}} - 2 \right),$$

(3.2)

where $\alpha$ is the generator of the Lorentz boost surrounding the cusp. We see that this expression vanishes as $\alpha \to \infty$ and it grows linearly in $|\alpha|$ as $|\alpha| \to \infty$.

![Diagram of cusp singularity and residual mass](image)

**Fig. 3**

a) Process with 3 shells in the in-state and 3 shells in the out-state. Here we have $0 < M_1 < M_2 < M_3 < M_4 < M_5$. b) One shell in and a near continuum of shells out. On top the residual mass as a function of the $y$ coordinate. Shaded: region with strong curvature. Dotted lines $\alpha$ and $\beta$: see text.

As a next step one might consider having several shells of matter both in the in-state and in the out-state. The result is sketched in Fig. 3. In the regions $i$ in between the black hole has a mass $M_i$. The most natural prescription seems to be to extend the mass shells as far to the origin as possible without creating negative mass values anywhere. Again the $S$-matrix Ansatz implies unconventional physics at the intersection points, but this was in the postulate that the corresponding amplitude was unequal to zero. Again we can then consider doing quantum field theory in this space-time.

Finally one might hope to consider the ‘most interesting’ case, which is the choice of physically realistic Hawking radiation in the out-state (Fig. 3b). The in-state can be kept at its simplest, a single shell. Doing quantum field theory here might help us to unearth
more precise values for all sorts of amplitudes. Now the $y$ coordinate near the upper right corner of Fig. 3b can be mapped onto the time coordinate $t$ for the distant observer. It is easy to see that during the radiation in a good approximation the mass will vary linearly as a function of $y$. Since time is proportional to $M^3$ we find a cubic relation between $y$ and $t$.

Again, our philosophy is to use this metric as a background for a quantum field theory in order to compute ”neighboring” amplitudes. In practice however there turn out to be important difficulties. Figures 2a,b and 3a,b are Penrose diagrams. This means that the local speed of light is easy to read off; light propagates at a maximum of $45^\circ$. However, distances cannot be derived directly from the figures; they are very distorted. Write the metric as

$$ds^2 = Adx dy + r^2 d\Omega^2,$$  \hspace{1cm} (3.3)

where $A$ and $r$ depend on the coordinates. The function $r$ is quite regular, having a natural zero at the origin. But $A$ behaves very wildly, and this turns out to be a real problem. Demanding the line $r = 0$ to be vertical puts a constraint on the lightcone coordinates $x$ and $y$ which removes some of the arbitrariness in $A$ that is due to redefinitions of $x$ and/or $y$. In the Appendix the metric of Fig. 3b is calculated. The result is given in eqs. (A.14) and (A.15). From these one can read off the red- and blueshift factors associated to trajectories through this space-time. The blueshift along the trajectory labeled $\alpha$ in Fig. 3b is found to be:

$$\frac{\partial r}{\partial x} \to e^{M_0^2/4\lambda}.$$  \hspace{1cm} (3.4)

Following the trajectory $\beta$ one finds a redshift of only

$$\frac{\partial y}{\partial r} \to \frac{\lambda}{M_0 M^2}.$$  \hspace{1cm} (3.5)

Both of these results present us with a problem. The redshift (3.5) seems to imply that the information from ingoing material will be spread over extremely long wavelengths in the outgoing radiation. But (3.4) is even more disastrous. Information entering via trajectory $\alpha$ will be turned into extreme trans-Planckian particles. Indeed the blueshift is so extreme that the energies of outgoing particles would tend to surpass quickly the energy of the entire universe, which of course would be nonsense. The $S$-matrix Ansatz forces us to omit such final states; yet we seem to obtain that precisely here the information of ingoing material goes.

A – possibly related – problem is the observation that when we apply field theory in these non-singular space-times we will of course respect all symmetries of the initial quantum field model, including possible baryon number conservation. Now surely the background metric does violate baryon number conservation (in most interesting cases),
but now we see that the changes we can consider in the in- and out-states will not bring us from one channel with baryon violation $B_{\text{out}} - B_{\text{in}} = \Delta B$ to any other channel with different $\Delta B$. This will make it difficult for us to study the details of any baryon number violating phenomena in black holes, even though our theory will permit these violations.

4. TRANS-PLANCKIAN TO CIS-PLANCKIAN MAPPING: OPERATOR ALGEBRA

From the previous section we conclude that a unitary field theory in a non-singular space-time may lead to a unitary $S$-matrix only if we allow for states that should actually be considered inadmissible: particles with energies far beyond the Planck mass, even far beyond the black hole mass. Our Ansatz forces us to limit ourselves to particles with energies up to the Planck mass but not much beyond. It is here where we are really forced to modify the laws of physics as we know them. At the points $A$ and $B$ in Fig. 2b one would normally expect only near-vacuum states whereas our field theories seem to generate strongly blueshifted particles there. It seems to be inevitable that a transformation law has to exist allowing us to transform states with particles much beyond the Planck energy into states where all particles have less than a Planck unit of energy. Again the $S$-matrix Ansatz can be helpful here to obtain this transformation rule.

This procedure has been described in Ref. [10]. Making use of the fact that ingoing material always affects outgoing waves via gravitational interactions (if not other interactions as well) we can consider a small change $\delta_{\text{in}}$ in the ingoing state $|\text{in}\rangle$, having a momentum distribution $\delta p^{\text{in}}(\theta, \varphi)$. The geodesics of outgoing particles will be shifted, as one can easily derive [10], and the shift $\delta y$ in the $y$ coordinate is given by

$$\delta y(\Omega) = \int d\Omega' G(\Omega, \Omega') \delta p^{\text{in}}(\Omega') ,$$

where $\Omega$ stands for $(\theta, \varphi)$ and $G(\Omega, \Omega')$ is a Green function on the transverse coordinates, determined by the equation

$$(1 - \partial_{\Omega}^2)G(\Omega, \Omega') = C \delta^2(\Omega, \Omega') .$$

Here $C$ is a known constant depending on the units used and the gravitational constant. The solution to this equation is

$$G(\Omega, \Omega') = \kappa \int_{2\pi - \theta}^{2\pi} dz (\cos \theta - \cos z)^{-\frac{1}{2}} e^{-\frac{1}{2} \sqrt{3} z} ,$$

where $\kappa$ is related to $C$, and $\theta$ is the angle between $\Omega$ and $\Omega'$. It is important however to stress that eq. (4.1) is an approximation: it was assumed that the ingoing particles were massless and had negligible transverse momenta. Also all non-gravitational interactions were ignored.
Since the shift operator is
\[ \exp \left( i \int d^2 \Omega p^\text{out} (\Omega) \delta y(\Omega) \right), \] (4.4)
we notice that all information concerning the in-states being used is the momentum distribution \( p^\text{in}(\Omega) \). Also the outgoing states are only distinguished by their momentum distribution \( p^\text{out}(\Omega) \). We now make the following essential step: we assume that all information concerning these states is in these momentum distributions. From a physical point of view this seems to be reasonable. If we were able to determine \( p(\Omega) \) with a Planckian resolution we really would have a lot of information which seems to be more than sufficient in practical situations. In physics at long distance scales our assumption therefore seems to imply little restrictions; at small distances however this restriction is crucial.

Let us for simplicity now turn to Rindler space in stead of the original black hole. This is not much else than replacing the angular variables \( \Omega = (\theta, \varphi) \) by \( \tilde{x} = (x^1, x^2) \). Let us describe the in-state as \( |p^\text{in}(\tilde{x})\rangle \) and introduce its functional Fourier transform \( |u^\text{in}(\tilde{x})\rangle \) by demanding
\[ \langle u^\text{in}|p^\text{in} \rangle = \exp \left[ i \int d^2 \tilde{x} p^\text{in}(\tilde{x}) u^\text{in}(\tilde{x}) \right]. \] (4.5)
As was shown in ref.[10] the \( S \)-matrix is generated by giving the following inner product between the out-states and the in-states:
\[ \langle p^\text{out}|p^\text{in} \rangle = \mathcal{N} \exp \left[ -i \int d^2 \tilde{x} d^2 \tilde{x}' p^\text{out}(\tilde{x}) f(\tilde{x} - \tilde{x}') p^\text{in}(\tilde{x}') \right], \] (4.6)
where \( \mathcal{N} \) is a normalization factor and \( f \) is now the Green function defined by
\[ \tilde{\partial}^2 f(\tilde{x}) = -\delta^2(\tilde{x}) , \] (4.7)
in units were \( 4\pi G = 1 \) (notice that this normalization is now different from what was used in the earlier sections; in black hole descriptions one usually avoids the \( 4\pi \)).

For black holes this definition of the inner products just generates the \( S \)-matrix. However, if we look at these identities from the point of view of local observers at the horizon, they provide for the required mapping between trans-Planckian and cis-Planckian particles.

It is now very convenient to introduce operators describing in- or outgoing particles. The operators \( p^\text{in}(\tilde{x}) \) and \( p^\text{out}(\tilde{x}) \) just measure these quantities. The operators \( u^\text{in} \) and \( u^\text{out} \) are the canonically conjugated operators. They satisfy
\[ [p^\text{in}(\tilde{x}), p^\text{in}(\tilde{x}')] = 0 ; \]
\[ [p^\text{in}(\tilde{x}), u^\text{in}(\tilde{x}')] = -i\delta^2(\tilde{x} - \tilde{x}') ; \]
\[ [p^\text{out}(\tilde{x}), p^\text{out}(\tilde{x}')] = 0 ; \]
\[ [p^\text{out}(\tilde{x}), u^\text{out}(\tilde{x}')] = -i\delta^2(\tilde{x} - \tilde{x}') . \] (4.8)
Furthermore we have

\[ u^{\text{out}}(\tilde{x}) = \int d^2\tilde{x}' f(\tilde{x} - \tilde{x}')p^{\text{in}}(\tilde{x}') , \]  

(4.9)

from which

\[ [p^{\text{out}}(\tilde{x}), p^{\text{in}}(\tilde{x}')] = i\tilde{\partial}^2 \delta^2(\tilde{x} - \tilde{x}'); \]

\[ [u^{\text{out}}(\tilde{x}), u^{\text{in}}(\tilde{x}')] = -if(\tilde{x} - \tilde{x}'), \]

(4.10)

so that

\[ \tilde{\partial}^2 u^{\text{out}} = -p^{\text{in}}, \quad \tilde{\partial}^2 u^{\text{in}} = p^{\text{out}}. \]

(4.11)

Apparently our equations are very symmetric under time reversal. So-far this algebra has been used in Ref. [10], where it is also explained how remarkably the resulting functional integral expressions for the S-matrix resemble the ones in string theory.

There are however two serious shortcomings still present in these expressions:

i) The spectrum of states is still continuous, since the \( p \) and \( u \) are continuous operators, and also since they depend on the continuous transverse coordinates \( \tilde{x} \).

ii) The relations between representations of our operator algebra on the horizon on the one hand and ordinary Fock space of elementary particles in the surrounding four-dimensional space-time on the other, seem to be difficult to recover. Fock space after all is not only described by the momentum distribution but also by the essential particle counting operators. It seems that we replaced the usual Fock space by a space where particles approaching each other extremely closely in the transverse coordinates become indistinguishable, even if they stay far apart in the longitudinal (light cone) coordinates.

In order to cure these shortcomings we observe that the above algebra is not yet accurate. Not only did we ignore all non-gravitational interactions, we also have not yet taken into account the transverse parts of the gravitational interactions. These transverse parts become important as soon as particles approach each other at distance scales shorter than the Planck length in the transverse directions. The inclusion of non-gravitational forces is in principle straightforward. In particular the electro-magnetic force can be added in an elegant way[14]. With all these new forces we do also obtain new degrees of freedom at small distances, such as the charge density \( \rho(\tilde{x}) \) with its canonically associated variable: a periodic fifth Kaluza-Klein coordinate[14]. It is the transverse gravitational force that will provide the answer to the questions raised above. We have to note that there will also be a momentum distribution in the sideways direction, \( \tilde{p}^{\text{in}}(\tilde{x}) \) and \( \tilde{p}^{\text{out}}(\tilde{x}) \). These operators generate sideways displacements, and as such can be easily defined:

\[ \tilde{p}^{\text{in}}(\tilde{x}) = p^{\text{in}}(\tilde{x})\tilde{\partial}u^{\text{in}}(\tilde{x}) , \quad \tilde{p}^{\text{out}}(\tilde{x}) = p^{\text{out}}(\tilde{x})\tilde{\partial}u^{\text{out}}(\tilde{x}) . \]

(4.12)
The commutation rules with the other operators, taking either all operators referring to
the in-space, or all referring to out-space, are:

\[
\begin{align*}
[\tilde{p}(\tilde{x}), u(\tilde{x}')] &= -i\delta^2(\tilde{x} - \tilde{x}')\tilde{\partial}u(\tilde{x}) ; \\
[\tilde{p}(\tilde{x}), p(\tilde{x}')] &= ip(\tilde{x})\tilde{\partial}\delta^2(\tilde{x} - \tilde{x}') ; \\
[\tilde{p}_i(\tilde{x}), \tilde{p}_j(\tilde{x}')] &= -\tilde{p}_j(\tilde{x})\delta_{ij}\delta^2(\tilde{x} - \tilde{x}') - i\tilde{p}_i(\tilde{x}')\partial_j\delta^2(\tilde{x} - \tilde{x}') .
\end{align*}
\]

(4.13)

Now if an incoming particle moves in with a sideways component the gravitational
shift it produces also has a sideways component. Therefore the equations (4.9)–(4.11)
should actually be seen as the third components of vector equations. Hence one could
expect to have, analogously to (4.10), also

\[
[\tilde{p}^\text{out}_i(\tilde{x}), \tilde{p}^\text{in}_j(\tilde{x}')] = i\tilde{\partial}^2\delta^2(\tilde{x} - \tilde{x}')\delta_{ij} .
\]

(4.14)

However, this cannot be right. Eq. (4.14) in combination with the previous commutators
do not obey Jacobi’s identities. So we have to ask how we can alter these equations such
that at long distance scales the effects ingoing particles have on outgoing ones are still
described in accordance with the $S$-matrix Ansatz while at short distances the Jacobi
identities are restored. This now may be regarded as a challenging puzzle. We think
that the present paper gave a rough outline for the rules of the game, but a completely
satisfactory answer has not yet been found. We do have a suggestion as to what direction
one could consider going.

5. DISCRETENESS ON THE HORIZON

Instead of a continuous two dimensional space of transverse points \{\tilde{x}\} suppose that
these points are discretized. Thus we replace the points \tilde{x} by points indicated by single
capital letters \(A, B, \ldots\), and assume that al every point \(A\) we have operators
\(x^\text{in}_A, x^\text{out}_A, p^\text{in}_A\) and \(p^\text{out}_A\), where \(i = 1, 2, 3\). The value \(i = 3\) corresponds to the former \(z\) components
and \(i = 1, 2\) describes the position in transverse space. We can then introduce ordinary
commutators:

\[
[x^\text{in}_A, p^\text{in}_B] = i\delta^i_j\delta_{AB} , \quad \text{etc.}
\]

(5.1)

We now wish to find a relation between the in- and out-operators that would reproduce
Eq. (4.11) in the continuum limit. To do this we introduce a lattice on our horizon. What
this really means is that links are defined between points, and these links then define which
points are to be considered as being each other’s neighbors, whereas the minimal number
of links between any pair of points \(A\) and \(B\) defines a transverse distance between \(A\) and
\(B\). The lattice is two-dimensional. It is reasonable to demand that links do not cross each
other: the lattice is planar. We see no reason yet to restrict ourselves to special kinds
of lattices such as a square or triangular lattice. More satisfactory will be the ‘random’ lattice, see Fig. 4.

It is suggestive to replace eq. (4.11), or $p^{\text{out}} = \bar{\partial}^2 u^{\text{in}}$, by

$$p^{\text{out}}_{i,A} = -x^{\text{in}}_{i,A} + \langle x^{\text{in}} \rangle_{\text{linked to } A},$$

where the average is over the neighbors of $A$ only, with possible weight factors $C_{AB}$:

$$p^{\text{out}}_{i,A} = \sum_B C_{AB} x^{\text{in}}_{i,B},$$

$$C_{AA} = -1, \quad \sum_B C_{AB} = 0.$$  

The coefficients $C_{AB}$ tell us whether points $A$ and $B$ are neighbors. In that case

$$C_{AB} \approx 1/N,$$  

$N$ being the number of neighbors. If $A$ and $B$ are not neighbors $C_{AB}$ vanishes.

Consequently we have

$$[p^{\text{out}}_{i,A}, p^{\text{in}}_{j,B}] = i\delta_{ij} C_{AB},$$

which is beautifully time reversally symmetric provided that in addition to (5.3) we have

$$\sum_A C_{AB} = 0.$$  

This implies that if our lattice is not a regular one (such that the number of neighbors is variable) the coefficients $C_{AB}$ will have to be chosen in a more complicated way than just $1/N$, which is why the ‘approximatively’ sign $\approx$ is used in eq. (5.4).

The radical difference between the algebra of this section with the previous one is not only the discreteness, but also the fact that there is now additional information on the surface: the details of the lattice structure. Coarse graining adds information, there is little to be done about that. Allowing the lattice to have a random structure is mandatory.
If we want to give the horizon of a finite size black hole the necessary $S_2$ topology. But this is not a high price to pay; it implies extra local information of the same type as the information that would be added if we consider new sorts on interaction at extremely high energies. This is because the information of the lattice structure is strictly local.

Finally, we now suspect that our discrete algebra indeed gives the black hole a discrete spectrum. This is not yet completely evident since the spectrum of the $p^\text{in}$ operators alone, or the $p^\text{out}$ operators all by themselves, is still continuous. But if we try to ‘localize’ the black hole by putting some constraints on the in- and the out-operators in combination, for instance if we try to localize the position of the horizon by demanding $x^\text{in2} + x^\text{out2} < R^2$, where $R$ is some limiting size, then clearly only a finite number of states satisfy this constraint since our system in all respects then behaves as a harmonic oscillator\textsuperscript{†}.

Needless to state that many questions are left unanswered. It should be possible to obtain more information about the rules at a black hole horizon by exploiting more of the physical information we already have from the Standard Model concerning the Hilbert space we are trying to construct. This way we might get a handle on questions concerning the choice of the coefficients $C_{AB}$ and any other possible local degrees of freedom.

**APPENDIX: METRIC FOR IN- AND OUTGOING SHELLS OF LIGHTLIKE MATERIAL.**

The metric of Fig. 2, produced by discrete shells of matter which are themselves delta-distributed can of course be written down in closed form: these space-times consist of various pieces of Schwarzschild geometry glued together on lightlike seams, such that at least the transverse part of the metric, given by $r(x, y)$, is continuous there. Everywhere except at the conical singularities we have that the components $T_{xy}, T_{\theta\theta}, T_{\theta\varphi}$ and $T_{\varphi\varphi}$ of the stress-energy tensor vanish. At the conical singularities the transverse part of $T_{ij}$ has a delta-distribution.

In this appendix we remove the conical singularities and consider the continuum limit of the case of radially symmetric ingoing and outgoing lightlike shells passing through each other. In the longitudinal direction we use the lightcone coordinates $x$ and $y$. The metric is then

$$ ds^2 = A(x, y) dx dy + r^2(x, y) d\Omega^2, \quad (A.1) $$

In the discrete case there was a well-defined Schwarzschild mass parameter everywhere between the shells. In the continuum limit we still have such a parameter $M(x, y)$. It is

\textsuperscript{†} The resemblance to a harmonically vibrating membrane becomes even more evident if we make a restriction in momentum space: give a bound to $\sum_A (p_A^{\text{in2}} + p_A^{\text{out2}})$.  

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defined by

\[ A \equiv \frac{2 \, r \, r_x \, r_y}{r - 2M}, \quad (A.2) \]

where \( r_x \) stands for \( \frac{\partial r}{\partial x} \). The conditions \( T_{xy} = T_{\theta \theta} = 0 \), corresponding to \( R_{xy} = R_{\theta \theta} = 0 \) imply

\[ r_{xy} = \frac{2M \, r_x \, r_y}{r(r - 2M)}; \quad M_{xy} = -\frac{2M_x \, M_y}{r - 2M}, \quad (A.3) \]

and these can be integrated to give

\[ 2M_x \, r_x = g(x) \left( 1 - \frac{2M}{r} \right); \quad 2M_y \, r_y = h(y) \left( 1 - \frac{2M}{r} \right), \quad (A.4) \]

where \( g(x) \) and \( h(y) \) are arbitrary functions. A physical constraint is that \( g \) and \( h \) should be non-negative since

\[ T_{xx} = \frac{2g(x)}{r^2}, \quad T_{yy} = \frac{2h(y)}{r^2}. \quad (A.5) \]

By redefining the coordinates \( x \) and \( y \) one could normalize the functions \( g \) and \( h \) to be equal to one. Physically then the coordinates \( x \) and \( y \) measure the amount of material entering and leaving the hole. The equations (A.4) however cannot be solved in the completely general case.

In Fig. 3 however we have one delta-distributed ingoing shell and a fairly arbitrary outgoing shell. Outside the ingoing shell we then have

\[ g(x) = 0. \quad (A.6) \]

The physically interesting special case is then

\[ M_x = 0; \quad M = M(y). \quad (A.7) \]

Let us now assume that far away from the hole the mass loss per unit of time is given:

\[ \frac{dM}{dt} = -F(M) = -\frac{\lambda}{M^2}, \quad (A.8) \]

the latter being the expected intensity of the Hawking radiation. \( \lambda \) is of order one and will here be taken to be constant although in reality it will depend slightly on \( M \). Since the radiation goes with the speed of light we can take at large \( r \)

\[ \frac{\partial M}{\partial t} \bigg|_r \rightarrow \frac{\partial M}{\partial r} \bigg|_t \rightarrow \frac{\partial M/\partial y}{\partial r/\partial y}, \quad (A.9) \]

leading to

\[ \frac{\lambda}{M^2} \rightarrow \frac{2M_y^2}{h(y)}. \quad (A.10) \]
Let us normalize the $y$ coordinate by (see Fig. 3b)

$$M(y) = M_0 y,$$

(A.11)

so that the range of the $y$ coordinate is $[0,1]$, and $M_0$ is the initial mass. From eqs (A.10) and (A.4) we get

$$h(y) = \frac{2M^2M_y^2}{\lambda}; \quad \frac{\partial r}{\partial M} = \frac{M^2}{\lambda} \left(1 - \frac{2M}{r}\right).$$

(A.12)

Close to the horizon, where $r \approx 2M$, the solution to this equation is

$$r(M) = 2M + \frac{4\lambda}{M} + Ce^{M^2/4\lambda} + \text{higher orders}.$$  

(A.13)

The $x$ coordinate can now be normalized with the integration constant $C$. We do this such that at $y = 1$ the $x$ dependence is regular:

$$r(x,y) = 2M_0y + \frac{4\lambda}{M_0y} + x e^{M_0^2(y^2-1)/4\lambda} + \text{small corrections}.$$  

(A.14)

With eq. (A.1) this leads to

$$A(x,y) \approx \frac{2M_0^3y^2}{\lambda} e^{M_0^2(y^2-1)/4\lambda}.$$  

(A.15)

It is the very strong $y$ dependence of these expressions that is further discussed in section 3.

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