Scalar field cosmology in phase space

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Abstract

Using dynamical systems methods, we describe the evolution of a minimally coupled scalar field and a Friedmann-Lemaître-Robertson-Walker universe in the context of general relativity, which is relevant for inflation and late-time quintessence eras. Focussing on the spatially flat case, we examine the geometrical structure of the phase space, locate the equilibrium points of the system (de Sitter spaces with a constant scalar field), study their stability through both a third-order perturbation analysis and Lyapunov functions, and discuss the late-time asymptotics. As we do not specify the scalar field’s origin or its potential, the results are independent of the high-energy model.

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1 Introduction

The standard cosmological model based on the spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric is very successful at describing many observations at different scales. It has become normal to include an inflationary epoch in this model during the early universe. Although there is no direct proof that inflation actually occurred, and other scenarios should still be considered, the 1992 discovery of temperature fluctuations in the cosmic microwave background by the COBE satellite [2] provided evidence of a nearly scale-invariant spectrum of primordial density perturbations of the kind predicted by inflationary scenarios. In addition, the study of these temperature fluctuations initiated by COBE ushered in an era of “precision cosmology” continued with later cosmic microwave background experiments, most notably WMAP and PLANCK [3, 4]. Most models of early universe inflation are based on scalar fields, and those based on

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quadratic quantum corrections to the Einstein-Hilbert action (“Starobinsky inflation” \cite{5}) can be reduced to the study of scalar field degrees of freedom \cite{6}.

A second revolution in cosmology occurred in 1998 with the discovery, obtained by studying type Ia supernovae, that the current expansion of the universe is accelerated \cite{7}. In the context of general relativity, on which the standard \( \Lambda \)-cold dark matter model is based, this acceleration can only be explained with a cosmological constant \( \Lambda \) of extremely fine-tuned, but non-vanishing, magnitude, or with a very exotic fluid having pressure \( P \) and density \( \rho \) related by the equation of state \( P \approx -\rho \), and dubbed “dark energy”. Most models of dark energy are based on a scalar field \( \phi \) (also known as “quintessence”) rolling in a flat section of its potential \( V(\phi) \). Alternative scenarios, seeking to replace the Einstein-Hilbert action (“\( f(R) \)” or “modified” gravity \cite{6}), can again be reduced to the dynamics of a scalar field degree of freedom.

Both inflation and quintessence models mandate a general understanding of scalar field dynamics in general-relativistic cosmology. Furthermore, a scalar field provides the simplest field theory of matter, and although no fundamental classical scalar field has been discovered in nature so far (except possibly for quintessence), they do provide a toy model useful for understanding many basic theoretical features of more realistic field theories, without the extra details and complications. As such, scalar field theory also constitutes an excellent pedagogical tool used in most relativity textbooks.

In this paper we approach the spatially homogeneous and isotropic cosmology of scalar fields minimally coupled to gravity from the phase space point of view. Although dynamical system methods have been widely used in cosmology since the 1960s \cite{8} and this type of analysis has been performed for non-minimally coupled scalar fields \cite{9} and general scalar-tensor or \( f(R) \) gravity \cite{10,11}, we could not find in the literature a complete and self-contained analysis for the simpler case of relativity with a minimally coupled scalar field, apart from specific scenarios corresponding to particular choices of the scalar field potential \( V(\phi) \), which abound in the literature (e.g., \cite{12}-\cite{15}, see also \cite{16}—the literature on specific scenarios is very large and here we limit ourselves to quote the papers whose approach is closest to the general one that we adopt).\(^1\) By contrast, here we do not commit to any particular scenario, and at most, we make general assumptions on properties of the potential (such as boundedness or monotonicity), refraining from choosing specific forms of the function \( V(\phi) \). Given that there is no preferred scenario of inflation or quintessence, general considerations are valuable.

\(^1\)Ref. \cite{17} studies the minimally coupled case of interest here and presents some of the features of scalar field cosmology derived in the following, but its main interest is in initial singularities and particular classes of potentials.
Since the relevant equations, which reduce to ordinary differential equations (ODEs) in this case, are still non-linear and not amenable to exact solution, the phase space view becomes important in gaining a qualitative understanding of the solutions without actually solving the field equations. It is generally believed that in order to say anything about the phase space and the qualitative behaviour of the solutions of the equations, one must first fully specify the scenario of inflation or quintessence being studied. While this is certainly true if one wants a complete qualitative picture of the dynamics, many aspects of the phase space portrait are common to most, if not all, scenarios and the study of these aspects, without committing to any particular scenario or potential $V(\phi)$, is a necessary preliminary for more detailed analyses of specific models. The purpose of this paper is to discuss these general features, specifically the geometry of the phase space, the existence, nature, and stability of the fixed points, and the late-time behaviour of the solutions, without specifying the form of the scalar field potential energy density, and instead making some generic assumptions on its behaviour (boundedness, presence of asymptotes, etc.).

2 Background

We consider a scalar field minimally coupled to the spacetime curvature as the only source of gravity in the Einstein field equations. This assumption is fully justified in inflationary scenarios of the early universe \[14\], and only approximately justified in quintessence models of the late universe \[18\]. In the latter case, scalar field dark energy is present along with a dust fluid, which combine to determine the dynamics of the universe. However, observations suggest that dark energy comes to dominate the dynamics very quickly, starting from redshifts $z \sim 0.5$, thus we can once again neglect the dust fluid and other forms of energy in the late regimes. In short, there is plenty of motivation to study scalar field cosmology.

The Lagrangian density of a scalar field $\phi$ minimally coupled to the spacetime curvature is:

$$L^{(\phi)} = -\frac{1}{2} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi), \quad (2.1)$$

where $V(\phi)$ is the scalar field potential. The action for gravity and the scalar field is

$$S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} + L^{(\phi)}(\phi, g_{\mu\nu}) \right] \equiv S(g) + S(\phi), \quad (2.2)$$

where $g_{\mu\nu}$ is the spacetime metric, $g$ is its determinant, and $R$ is its Ricci scalar. The action (2.2) is also the action for general scalar-tensor gravity.

\footnote{We follow the notations of Ref. [19].}
in vacuo, after performing a conformal transformation to the Einstein frame (e.g., \[20\]). The variation of the scalar field action \(S(\phi) = \int d^4x \sqrt{-g} \mathcal{L}^{(\phi)}\) gives the stress-energy tensor

\[
T^{(\phi)}_{\mu \nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}^{(\phi)}}{\delta g^{\mu \nu}} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu \nu} \nabla^\alpha \phi \nabla_\alpha \phi - V(\phi) g_{\mu \nu} .
\] (2.3)

A spatially homogeneous and isotropic universe is described by the FLRW line element

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right) \right] \] (2.4)
in comoving coordinates \((t, r, \theta, \varphi)\), where \(a(t)\) is the scale factor and \(k\) is the curvature index. The Einstein equations

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 8\pi G T_{\mu \nu}
\] (2.5)
(where \(R_{\mu \nu}\) is the Ricci tensor and \(R \equiv R^{\mu}_{\mu}\) reduce to ODEs for the scale factor and matter degrees of freedom. It is customary to approximate the matter content of the universe with a single perfect fluid with four-velocity \(u^\mu = \delta^0_\mu\) in comoving coordinates, energy density \(\rho\), pressure \(P\), and energy-momentum tensor

\[
T_{\mu \nu} = (P + \rho) \, u_\mu u_\nu + P g_{\mu \nu} .
\] (2.6)
The pressure and energy density are usually related by a barotropic equation of state \(P = P(\rho)\), often of the form \(P = w\rho\) where the constant \(w\) is called the “equation of state parameter”. The Einstein equations (2.5) in the presence of a single perfect fluid reduce to

\[
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P) ,
\] (2.7)
\[
\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{k}{a^2} ,
\] (2.8)
\[
\dot{\rho} + 3 \frac{\dot{a}}{a} (P + \rho) = 0 ,
\] (2.9)
where an overdot denotes differentiation with respect to the comoving time \(t\). Eqs. (2.7) and (2.8) are called the acceleration equation and the Hamiltonian constraint, respectively, and the Klein-Gordon equation (2.4) is nothing but the covariant conservation equation \(\nabla^\nu T_{\mu \nu} = 0\) (when \(\phi \neq \text{const.}\)). The Klein-Gordon equation is not independent of eqs. (2.7) and (2.8) and can be derived from them. Excellent pedagogical analyses of the phase space of a FLRW universe coupled to a perfect fluid are available in the literature [21].
In a FLRW universe, a gravitating scalar field must necessarily depend only on the comoving time, \( \phi = \phi(t) \), in order to respect the spacetime symmetries. Therefore, its gradient \( \nabla \phi \) is timelike (or null but trivial if \( \phi = \text{const.} \)). In regions where \( \nabla^\alpha \phi \nabla_\alpha \phi < 0 \), we can introduce the four-vector

\[
u_\mu = \frac{\nabla_\mu \phi}{\sqrt{|\nabla^\alpha \phi \nabla_\alpha \phi|}},
\]

with \( u_\mu u^\mu = -1 \), and the scalar field is equivalent to a perfect fluid with stress-energy tensor of the form (2.6) and energy density and pressure [22]

\[
\rho = \frac{\dot{\phi}^2}{2} + V(\phi),
\]

(2.11)

\[
P = \frac{\dot{\phi}^2}{2} - V(\phi).
\]

(2.12)

One can define the effective equation of state parameter

\[
w(\phi, \dot{\phi}) \equiv \frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}.
\]

(2.13)

The Einstein-Friedmann equations (2.7)–(2.9) become

\[
\ddot{a} = -\frac{8\pi G}{3} \left( \dot{\phi}^2 - V(\phi) \right),
\]

(2.14)

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right) - \frac{k}{a^2},
\]

(2.15)

\[
\dddot{\phi} + 3H \ddot{\phi} + \frac{dV}{d\phi} = 0.
\]

(2.16)

In the following it will be useful to rewrite these equations in terms of the Hubble parameter \( H \equiv \dot{a}/a \) as

\[
\dot{H} = -H^2 - \frac{8\pi G}{3} \left( \dot{\phi}^2 - V(\phi) \right) + \frac{k}{a^2} = -4\pi G \dot{\phi}^2 + \frac{k}{a^2},
\]

(2.17)

\[
H^2 = \frac{8\pi G}{3} \left( \frac{\dot{\phi}^2}{2} + V(\phi) \right) - \frac{k}{a^2},
\]

(2.18)

\[
\dddot{\phi} + 3H \dot{\phi} + V' = 0,
\]

(2.19)

where a prime denotes differentiation with respect to \( \phi \). These equations can also be derived from an effective Lagrangian or Hamiltonian [23].

The equations of scalar field cosmology are non-linear and few exact solutions are known for particular choices of the potential \( V(\phi) \). We would like to discuss the dynamics of the variables \( a(t) \) and \( \phi(t) \) in as much depth as possible without choosing a specific form of \( V(\phi) \). Before we begin, let us note that
• for $V(\phi) = 0$ the scalar field is equivalent to a fluid with stiff equation of state $P = \rho$, which does not seem to be very relevant for inflation and late-time acceleration (although it is relevant for matter at nuclear densities in the core of neutron stars, and possibly near the Big Bang singularity [17]).

• For $V(\phi) = V_0 = \text{const.}$ the potential reduces to a pure cosmological constant $\Lambda$. The scalar field stress-energy tensor reduces to

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi,$$

(2.20)

with $\Lambda = 8\pi GV_0$. Further, for $\phi = \phi_0 = \text{const.}$, one recovers the stress-energy tensor of a pure cosmological constant.

### 3 Phase space

Eqs. (2.14) and (2.16) describe the evolution of $a(t)$ and $\phi(t)$ (remember that there are only two independent equations in the set (2.14)–(2.16) if $\phi$ is not constant). Eq. (2.15) is a first order constraint (contrary to eqs. (2.14) and (2.16) which are of second order). The phase space is, therefore, a 4-dimensional space $(a, \dot{a}, \phi, \dot{\phi})$, but the Hamiltonian constraint (2.15) forces the orbits of the solutions to live on a 3-dimensional hypersurface, introducing a relation between the four variables. For example, one can use the constraint to express $\dot{\phi}$ in terms of the other three variables, $\dot{\phi} = \dot{\phi}(a, \dot{a}, \phi)$.

For particular choices of the scalar field potential, and especially for $k \neq 0$, one can change variables to functions of $a, \dot{a}, \phi, \dot{\phi}$ which can lead to exact solutions or to simpler calculations. In general, however, these new variables do not have an immediate or clear physical meaning and are to be regarded as a mere mathematical trick to perform calculations. Often the results of these calculations cannot be translated explicitly or easily in terms of the variables $(a, \dot{a}, \phi, \dot{\phi})$. However, current observations seem to indicate that we live in a spatially flat ($k = 0$) universe, which is much simpler to analyze than the $k \neq 0$ case. This is the situation that we consider in the following.

### 4 Spatially flat FLRW scalar field cosmologies

The description of the phase space greatly simplifies for $k = 0$ as, in this case, the scale factor $a(t)$ appears in the dynamical equations only through the combination $\dot{a}/a \equiv H$, the Hubble parameter, which is a physical observable obtained by fitting theoretical models to cosmological data. Since $\phi$ is the only matter field in the theory, it is natural from the field theory point of
view to choose it as another dynamical variable. By choosing $H$ and $\phi$ as dynamical variables, the phase space reduces to the 3-dimensional space $(H, \phi, \dot{\phi})$, but the orbits of the solutions of eqs. (2.17)–(2.19) with $k = 0$ are forced to move on a 2-dimensional subset of the phase space by the Hamiltonian constraint (2.18).

Let us examine the structure of the “energy surface” on which the orbits are forced to move. We choose to eliminate $\dot{\phi}$ by expressing it in terms of the other variables $(H, \phi)$ in eq. (2.18) with $k = 0$, which can then be viewed formally as a quadratic algebraic equation for $\dot{\phi}$ and solved, obtaining

$$\dot{\phi} = \pm \sqrt{\frac{3H^2}{4\pi G} - 2V(\phi)}. \quad (4.1)$$

For certain choices of the potential $V(\phi)$, an arbitrary choice of values of the pair $(H, \phi)$ could make the argument of the square root on the right hand side negative. Therefore, in general, there can be a region of the phase space forbidden to the orbits of the dynamical solutions,

$$\mathcal{F} \equiv \left\{ (H, \phi, \dot{\phi}) : 3H^2 < 8\pi G V(\phi) \right\} \quad (4.2)$$

(“forbidden region”). This region may or may not exist depending on the form of $V(\phi)$.

There are two portions of the phase space region accessible to the dynamics (the “energy surface” corresponding to vanishing effective Hamiltonian [23]), corresponding to the two signs of the right hand side of eq. (4.1). These sets are symmetric with respect to the $\dot{\phi} = 0$ plane of the $(H, \phi, \dot{\phi})$ space. We call these two subsets of the energy surface “upper sheet” and “lower sheet”, corresponding to the positive and negative sign, respectively. In the upper sheet $\phi$ is always increasing ($\dot{\phi} > 0$), while on the lower sheet $\phi$ is always decreasing ($\dot{\phi} < 0$). The two sheets are either disconnected, or always join on the plane $\dot{\phi} = 0$, on the boundary of the forbidden region

$$B \equiv \partial \mathcal{F} = \left\{ (H, \phi, \dot{\phi}) : \dot{\phi} = 0 \Leftrightarrow 3H^2 = 8\pi G V(\phi) \right\}. \quad (4.3)$$

Figs. 1 and 2 show the upper and lower sheet for the example potential $V(\phi) = m^2 \phi^2 / 2$. The dynamics of the spatially curved ($k \neq 0$) scalar field universe are confined to either side of the “energy surface” corresponding to $k = 0$ in the phase space—this fact can be deduced by reducing the Hamiltonian constraint to

$$\dot{\phi} = \pm \sqrt{\frac{3H^2}{4\pi G} - 2V(\phi) + \frac{3k}{4\pi Ga^2}}. \quad (4.4)$$

An astronomer would instead choose the density of the matter field $\Omega_\phi$ (in units of the critical density) as another variable.
Figure 1: The upper sheet corresponding to the positive sign in eq. (4.1), for the quadratic potential $V(\phi) = \frac{m^2 \dot{\phi}^2}{2}$ (in arbitrary units).

Trajectories corresponding to $k > 0$ would exist above the $k = 0$ upper sheet (i.e., for larger values of $\dot{\phi}$ than those corresponding to the $k = 0$ upper sheet) and below the $k = 0$ lower sheet (i.e., for lower values of $\dot{\phi}$). Trajectories corresponding to $k < 0$ would exist between each $k = 0$ sheet (i.e., for values of $\dot{\phi}$ comprised between those given by eq. (4.1)). This property was realized in Ref. [13] for the specific inflationary potential $V = \frac{m^2 \phi^2}{2}$, but the conclusion is general. Trajectories in a region corresponding to $k > 0$ cannot cross the $k = 0$ sheet and move to regions corresponding to $k < 0$, and vice-versa. Such dynamical transitions between different topologies of the universe are forbidden (the topology of spacetime is not ruled by the dynamics).

Finally, the lower dimension of the “energy surface” leads one to believe that chaos is impossible in the dynamical system under study. This statement is not trivial given that the standard results on the absence of chaos in a two-dimensional phase space are proven for a plane, not for a curved surface or for a subset of a higher-dimensional phase space obtained by gluing two 2-dimensional sheets [24]. However, it is not difficult to reduce this situation to the standard case, as has been shown for scalar-tensor gravity in [25]. The theory of a minimally coupled scalar field in Einstein gravity is contained in this reference as a special case.
Figure 2: The surface describing the Hamiltonian constraint \((4.1)\) for the quadratic potential \(V(\phi) = m^2\phi^2/2\) (in arbitrary units). The upper and lower sheets join at the \(\dot{\phi} = 0\) plane to form a cone.

5 Equilibrium points

Having chosen \(H\) and \(\phi\) as dynamical variables, the equilibrium points of the dynamical system (when they exist) are, by definition, of the form \((\dot{H}, \dot{\phi}) = (0, 0)\) and \((\ddot{H}, \ddot{\phi}) = (0, 0)\), or \((H, \phi) = (H_0, \phi_0) = (\text{const.}, \text{const.})\), and they must all lie in the \(\dot{\phi} = 0\) plane, and therefore, on the boundary \(B\) of the forbidden region (if this region exists). These equilibrium points are de Sitter spaces with a constant scalar field. When they exist, they are the only de Sitter spaces possible in this theory. In fact, eq. \((2.17)\) with \(k = 0\) reduces to \(\dot{H} = -4\pi G \dot{\phi}^2\), and a de Sitter space with \(H = \text{const.}\) necessarily has \(\phi = \text{const.}\) as well.\(^4\) A degenerate case is \(H_0 = 0\), which corresponds to Minkowski space. de Sitter spaces are important in cosmology because they are usually attractors in inflation and quintessence models \([15, 18]\).

For \(\phi = \text{const.}\), \(\mathcal{L}^{(\phi)}\) and \(T^{(\phi)}_{\mu\nu}\) reduce to \(\mathcal{L}^{(\phi)} = -V_0 \equiv -V(\phi_0)\) and \(T^{(\phi)}_{\mu\nu} = -V_0 g_{\mu\nu}\), i.e., to a pure cosmological constant term with \(\Lambda = 8\pi GV_0\).

The necessary and sufficient conditions for the existence of de Sitter fixed

\(^4\)By contrast, with non-minimally coupled scalar fields, de Sitter spaces with a non-constant scalar field are possible \([9]\).
Figure 3: Trajectories converging to a Minkowski fixed point \( (H, \phi, \dot{\phi}) = (0, 0, 0) \) for the previous example \( V(\phi) = m^2 \phi^2 / 2 \).

Points are easily obtained from eqs. (2.17)–(2.19) with \( k = 0 \):

\[
H_0^2 = \frac{8\pi G}{3} V_0 , \tag{5.1}
\]

\[
V_0' = 0 , \tag{5.2}
\]

which obviously require \( V_0 \geq 0 \) (Minkowski space is obtained for \( V_0 = 0 \)). Eq. (5.2) expresses the condition that \( V(\phi) \) has an extremum or a point with horizontal tangent at \( \phi_0 \).

Fig. 3 shows two trajectories, corresponding to different initial conditions, converging to a Minkowski space attractor point for the example of the \( V(\phi) = m^2 \phi^2 / 2 \) potential.

Attractors (or repellors) could exist as an asymptotic limit in \( a \) or \( \phi \). To check for these we must search for fixed points with infinite values of the variables.

### 5.1 Fixed points at infinity with a Poincaré projection

Fixed points at infinity can be found by adopting polar coordinates \((r, \theta)\) with

\[
H = r \cos \theta , \quad \phi = r \sin \theta , \tag{5.3}
\]

and the standard Poincaré transformation \( r \to \mathbf{r} \) with

\[
r \equiv \frac{\sqrt{\mathbf{r}}}{1 - \mathbf{r}} , \tag{5.4}
\]
which maps infinity onto the circle of radius $\tau = 1$. Since

\[
\dot{H} = \frac{(1 + \tau)}{2\sqrt{\tau}(1 - \tau)^2} \dot{\tau} \cos \theta - \frac{\sqrt{\tau}}{1 - \tau} \dot{\theta} \sin \theta, \quad (5.5)
\]

\[
\dot{\phi} = \frac{(1 + \tau)}{2\sqrt{\tau}(1 - \tau)^2} \dot{\tau} \sin \theta + \frac{\sqrt{\tau}}{1 - \tau} \dot{\theta} \cos \theta, \quad (5.6)
\]

fixed points $(H, \phi) = (\text{const.}, \text{const.})$ correspond to $(\tau, \theta) = (\text{const.}, \text{const.})$ thanks to the linear independence of the sine and cosine functions. The dynamical system \( \text{(2.17)–(2.19)} \) becomes

\[
\frac{\tau}{(1 - \tau)^2} \cos^2 \theta = \frac{8\pi G}{3} \frac{V}{V^0}, \quad (5.7)
\]

\[
\frac{\dot{\tau}}{2\sqrt{\tau}(1 - \tau)^2} \cos \theta - \frac{\sqrt{\tau}}{1 - \tau} \dot{\theta} \sin \theta
\]

\[
= -4\pi G \left[ \frac{(1 + \tau)}{2\sqrt{\tau}(1 - \tau)^2} \sin \theta + \frac{\sqrt{\tau}}{1 - \tau} \dot{\theta} \cos \theta \right]^2, \quad (5.8)
\]

\[
\frac{\tau}{(1 - \tau)^2} \cos^2 \theta + \frac{\dot{\tau}(1 + \tau)}{2\sqrt{\tau}(1 - \tau)^2} \sin \theta + \frac{\sqrt{\tau}}{1 - \tau} \dot{\theta} \cos \theta + \frac{\sqrt{\tau}}{1 - \tau} \left( \dot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta \right)
\]

\[
= -3 \left[ \frac{\tau}{(1 - \tau)^2} \dot{\theta} \cos^2 \theta + \frac{(1 + \tau)}{2(1 - \tau)^3} \sin \theta \right] - V'. \quad (5.9)
\]

Setting \( (\dot{H}, \dot{\phi}) = (0, 0) \) and \( (\ddot{H}, \ddot{\phi}) = (0, 0) \) yields

\[
\frac{\tau \cos^2 \theta}{(1 - \tau)^2} = \frac{8\pi G}{3} V_0, \quad (5.10)
\]

\[
V_0' = 0, \quad (5.11)
\]

where $\phi_0 = \phi(\tau_0, \theta_0)$. In order to satisfy eq. \( (5.10) \) in the limit $\tau \rightarrow 1$ we must have either

1. $\cos \theta = 0$, corresponding to $H \rightarrow 0$, $\phi \rightarrow \pm \infty$, and $V(\phi \rightarrow \pm \infty) = 0$ (this situation includes potentials $V(\phi)$ with compact support).
2. \( \cos \theta = \pm 1 \), corresponding to \( H \rightarrow \pm \infty \), \( \phi \rightarrow 0 \), and \( V (\phi \rightarrow 0) = \infty \) (i.e., \( V \) has a vertical asymptote at \( \phi = 0 \). This is the case of the potentials \( V(\phi) \propto 1/\phi^\alpha, \alpha > 0 \) used in many quintessence models).

3. \( \theta \neq 0, \pi, \pm \pi/2 \), which allows \( H \rightarrow \pm \infty \), \( \phi \rightarrow \pm \infty \), and \( V (\phi \rightarrow \pm \infty) = \infty \). This case includes unbounded monotonic potentials such as \( V(\phi) = V_0 e^{\pm \alpha \phi} \) (scalar field cosmology with exponential potentials is studied in detail in Ref. [14]).

Fixed points corresponding to any of these situations must have a potential that asymptotically satisfies eq. (5.11) as well as the stated conditions. A fixed point satisfying the conditions of situation 1) corresponds to Minkowski space with no potential. Situations 2) and 3) both correspond to extreme cases of de Sitter space, with their potentials diverging. A possible situation corresponding to case 3) is \( V(\phi) = V_0 \ln (\phi/\phi_0) \).

The next question that one can ask regards the stability of these equilibrium points.

5.2 Stability with respect to homogeneous perturbations

It seems intuitive that if \( V(\phi) \) has a local minimum at \( \phi_0 \), and a de Sitter equilibrium point \((H_0, \phi_0)\) satisfying eqs. (5.1) and (5.2) exists, it will be stable, and \textit{vice-versa}, it will be unstable if \( V(\phi) \) has a local maximum. However, this statement would be a bit naive because \( \phi(t) \) couples to the other variable \( H(t) \) and one must consider both variables simultaneously. Here we consider homogeneous perturbations of the fixed point \((H_0, \phi_0)\), i.e., we write

\[
H(t) = H_0 + \epsilon \delta_1 H(t) + \epsilon^2 \delta_2 H(t) + \epsilon^3 \delta_3 H(t) + \ldots ,
\]

\[
\phi(t) = \phi_0 + \epsilon \delta_1 \phi(t) + \epsilon^2 \delta_2 \phi(t) + \epsilon^3 \delta_3 \phi(t) + \ldots ,
\]

where \( \epsilon \) is a smallness parameter and \( \delta_i H \) and \( \delta_i \phi \) depend only on time. In general, one should consider more general perturbations \( \delta H(t, \mathbf{x}) \), \( \delta \phi(t, \mathbf{x}) \) and even anisotropic perturbations. Inhomogeneous perturbations are subject to notorious gauge dependence problems and can only be treated rigorously in the context of a gauge-invariant formalism [26]. This kind of formalism is necessarily very detailed and complicated and the corresponding gauge-invariant variables are susceptible to physical interpretation only after a gauge is fixed (and then, different gauges produce different interpretations). For clarity, and to avoid the high degree of sophistication needed, we will confine our analysis to homogeneous perturbations. Therefore, if a de Sitter fixed point is stable with respect to homogeneous perturbations it may still be unstable with respect to inhomogeneous ones. In the following we assume that an equilibrium point exists.
By inserting eqs. (5.12) and (5.13) into eqs. (2.17)–(2.19) with \( k = 0 \) and using the zero order eqs. (5.1) and (5.2) for the equilibrium point \((H_0, \phi_0)\), one obtains the perturbed Hamiltonian constraint

\[
2\epsilon H_0 \delta_1 H + 2\epsilon^2 H_0 \delta_2 H + \epsilon^2 \delta_1 H^2 + 2\epsilon^3 H_0 \delta_3 H + 2\epsilon^3 \delta_1 H \delta_2 H
\]

\[
= \frac{8\pi G}{3} \left[ \frac{1}{2} \left( \epsilon^2 \delta_1 \dot{\phi}^2 + 2\epsilon^3 \delta_1 \dot{\phi} \delta_2 \phi \right) \right]
\]

\[
+ \frac{1}{2} V_0'' \left( \epsilon^3 \delta_1 \phi^2 + 2\epsilon^3 \delta_1 \phi \delta_2 \phi \right) + \frac{1}{6} V_0''' \left( \epsilon^3 \delta_1 \phi^3 \right),
\]

(5.14)

the acceleration equation

\[
\epsilon \dot{H} + \epsilon^2 \delta_2 \dot{H} + \epsilon^3 \delta_3 \ddot{H} = - \left( 2\epsilon H_0 \delta_1 H + 2\epsilon^2 H_0 \delta_2 H + 2\epsilon^3 H_0 \delta_3 H \right)
\]

\[
+ 2\epsilon^3 \delta_1 \dot{H} \delta_2 H + \epsilon^2 \delta_1 H^2 \right) - \frac{8\pi G}{3} \left[ \epsilon^2 \delta_1 \dot{\phi}^2 + 2\epsilon^3 \delta_1 \dot{\phi} \delta_2 \phi \right]
\]

\[
- \frac{1}{2} V_0'' \left( \epsilon^3 \delta_1 \phi^2 + 2\epsilon^3 \delta_1 \phi \delta_2 \phi \right) - \frac{1}{6} V_0''' \left( \epsilon^3 \delta_1 \phi^3 \right) \right],
\]

(5.15)

and the Klein-Gordon equation

\[
\left( \epsilon \delta_1 \ddot{\phi} + \epsilon^2 \delta_2 \ddot{\phi} + \epsilon^3 \delta_3 \ddot{\phi} \right)
\]

\[
+ 3 \left( \epsilon H_0 \delta_1 \dot{\phi} + \epsilon^2 H_0 \delta_2 \dot{\phi} + \epsilon^2 \delta_1 H \delta_1 \ddot{\phi} + \epsilon^3 H_0 \delta_3 \dot{\phi} + \epsilon^3 \delta_1 H \dot{\phi} \delta_2 \phi + \epsilon^3 \delta_2 H \dot{\phi} \right)
\]

\[
+ V_0'' \left( \epsilon \delta_1 \dot{\phi} + \epsilon^2 \delta_2 \dot{\phi} + \epsilon^3 \delta_3 \dot{\phi} \right) + \frac{V_0'''}{2} \left( \epsilon^2 \delta_1 \phi^2 + 2\epsilon^3 \delta_1 \phi \delta_2 \phi \right)
\]

\[
+ \frac{V_0^{(IV)}}{6} \epsilon^3 \delta_1 \phi^3 = 0,
\]

(5.16)

where \( V_0'' \equiv V''(\phi_0) \), etc. \( V(\phi) \) and \( dV/d\phi \) have been expanded to third order as

\[
V(\phi) = V_0 + \frac{V_0''}{2} \left[ \epsilon \delta_1 \phi(t) + \epsilon^2 \delta_2 \phi(t) + \ldots \right]^2 + \frac{V_0'''}{6} \left[ \epsilon \delta_1 \phi(t) + \ldots \right]^3 + \ldots,
\]

(5.17)

\[
V'(\phi) = V_0'' \left[ \epsilon \delta_1 \phi(t) + \epsilon^2 \delta_2 \phi(t) + \epsilon^3 \delta_3 \phi(t) + \ldots \right]
\]

\[
+ \frac{V_0'''}{2} \left[ \epsilon \delta_1 \phi(t) + \epsilon^2 \delta_2 \phi(t) + \ldots \right]^2 + \frac{V_0^{(IV)}}{6} \left[ \epsilon \delta_1 \phi(t) + \ldots \right]^3 + \ldots,
\]

(5.18)
using the fact that $V'_0 = 0$. To first order in $\epsilon$, eq. (5.14) yields $2H_0 \delta_1 H = 0$ and, if the de Sitter equilibrium point is not a degenerate Minkowski space with $H_0 = 0$, then\footnote{It is an old adage in cosmological perturbation theory that there are no linear perturbations of de Sitter space sourced by a (minimally coupled) scalar field [27].}

$$\delta_1 H = 0$$

(5.19)

and eq. (5.15) then yields $\delta_1 \dot{H} = 0$. The Klein-Gordon equation (5.16) then decouples and reduces to the equation of the damped harmonic oscillator

$$\delta_1 \ddot{\phi} + 3H_0 \delta_1 \dot{\phi} + V''_0 \delta_1 \phi = 0,$$

(5.20)

a second order ODE with constant coefficients. The associated algebraic equation is

$$\lambda^2 + 3H_0 \lambda + V''_0 = 0,$$

(5.21)

which has the roots

$$\lambda_{1,2} = -3H_0 \pm \frac{\sqrt{9H_0^2 - 4V''_0}}{2} = -\frac{3H_0}{2} \left( 1 + \frac{4V''_0}{9H_0^2} \right).$$

(5.22)

If $9H_0^2 - 4V''_0 \neq 0$, the general solution of eq. (5.20) is

$$\delta_1 \phi(t) = e^{-3H_0 t/2} \left[ C_1 \exp \left( \frac{-\sqrt{9H_0^2 - 4V''_0}}{2} t \right) + C_2 \exp \left( \frac{-\sqrt{9H_0^2 - 4V''_0}}{2} t \right) \right] = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t},$$

(5.23)

where $C_{1,2}$ are arbitrary integration constants. It is easy to see that $V''_0 \geq 0$ is required for stability. In fact, if $V''_0 \geq 0$, then $9H_0^2 - 4V''_0 \leq 9H_0^2$.

If $V''_0 > 9H_0^2/4 = 6\pi G V_0$, then $\lambda_{1,2} = \left( -3H_0 \pm i \sqrt{|9H_0^2 - 4V''_0|} \right) / 2$ have imaginary parts and the two independent modes are of the form

$$\exp \left( \frac{-3H_0 t}{2} \right) \cdot \exp \left( \frac{\pm i \sqrt{9H_0^2 - 4V''_0}}{2} t \right),$$

(5.24)

which decay because of the first exponential if $H_0 > 0$, and increase without bound if $H_0 < 0$.

If $V''_0 = 9H_0^2/4$ then $\lambda_1 = \lambda_2 = -3H_0/2$ and the solutions of eq. (5.20) are

$$\delta_1 \phi(t) = e^{-3H_0 t/2} (C_1 + C_2 t),$$

(5.25)
again, stable if $H_0 > 0$.

If $0 < V''_0 < 9H_0^2/4$, then $0 < 9H_0^2 - 4V''_0 < 9H_0^2$, or $\sqrt{9H_0^2 - 4V''_0} < 3H_0$ and $\lambda_{1,2} < 0$, so that the independent modes $e^{\lambda_{1,2}t}$ do not grow.

Finally, if $V''_0 = 0$ in eq. (5.20), the solution is

$$\delta_1 \phi(t) = C_1 + C_2 e^{-3H_0t} ;$$

(5.26)

this solution is stable for $H_0 > 0$, but not asymptotically stable, as it does not fully decay. We can therefore conclude that:

- If $H_0 > 0$, then the de Sitter equilibrium point $(H_0, \phi_0)$ is asymptotically stable if $V''_0 > 0$, unstable if $V''_0 < 0$, and stable but not asymptotically stable if $V'_0 = 0$.

- If $H_0 < 0$, the de Sitter equilibrium point $(H_0, \phi_0)$ is always unstable (to first order and all higher orders).

Furthermore, to first order there is no perturbation $\delta_1 H$ and the perturbations $\delta H$ and $\delta \phi$ are decoupled.

The second order equations yield

$$\delta_2 H = \frac{2\pi G}{3H_0} \left( \delta_1 \dot{\phi}^2 + V''_0 \delta_1 \phi^2 \right) ,$$

(5.27)

$$\delta_2 \dot{H} = -4\pi G \delta_1 \dot{\phi}^2 ,$$

(5.28)

$$\delta_2 \ddot{\phi} + 3H_0 \delta_2 \dot{\phi} + V''_0 \delta_2 \phi = -\frac{V''_0}{2} \delta_1 \phi^2 .$$

(5.29)

The second order perturbations $\delta_2 H$ and $\delta_2 \phi$ depend on the first order ones $\delta_1 \phi$ and their derivatives. These act as a source term in the Klein-Gordon equation (5.20) for $\delta_2 \phi$, which is a damped forced harmonic oscillator equation. From the general theory of ODEs, it is clear that if $H_0 > 0$, the friction term $3H_0 \delta_2 \dot{\phi}$ will correspond to positive friction, while if $H_0 < 0$, there is “anti-friction” leading to instability. Therefore,

- For $H_0 > 0$, there is stability if $V''_0 \geq 0$, and instability if $V''_0 < 0$.

- If $H_0 < 0$, the perturbations $\delta_2 \phi$ grow without bound and the de Sitter equilibrium point $(H_0, \phi_0)$ is unstable.
To third order in $\epsilon$, one obtains

$$H_0 \delta_3 H = \frac{4\pi G}{3} \left( \delta_1 \dot{\phi} \delta_2 \dot{\phi} + V_0'' \delta_1 \phi \delta_2 \phi + \frac{1}{6} V_0''' \delta_1 \phi^3 \right),$$  \hspace{1cm} (5.30)

$$\delta_3 \dot{H} = -8\pi G \delta_1 \dot{\phi} \delta_2 \dot{\phi},$$  \hspace{1cm} (5.31)

$$\delta_3 \ddot{\phi} + 3H_0 \delta_3 \dot{\phi} + V_0'' \delta_3 \phi = -\frac{2\pi G}{3H_0} \left( \delta_1 \ddot{\phi} + V_0'' \delta_1 \phi^2 \delta_1 \dot{\phi} \right) - V_0'' \delta_1 \phi \delta_2 \dot{\phi}$$

$$- \frac{V_0' (\text{IV})}{6} \delta_1 \phi^3.$$  \hspace{1cm} (5.32)

By considering only situations in which we have stability at the lower orders, the lower order perturbations and their derivatives, which are acting as sources in the higher order equations, are bounded, and eqs. (5.30) and (5.31) guarantee that $\delta_3 H$ and its derivatives are bounded by (small) initial conditions. Eq. (5.32) again takes the form of a driven, damped harmonic oscillator with instability if $H_0 < 0$, or if $H_0 > 0$ with $V_0'' < 0$, and stability for $H_0 > 0$ and $V_0'' \geq 0$.

There remains the case of the Minkowski fixed point $(H_0, \phi_0) = (0, \phi_0)$, a degenerate de Sitter space. This situation occurs if $V_0 = 0$ and $V_0' = 0$. Eqs. (2.17)–(2.19) then reduce, to third order, to

$$\epsilon^2 \delta_1 H^2 + 2\epsilon^3 \delta_1 H \delta_2 H$$

$$= \frac{8\pi G}{3} \left\{ \frac{1}{2} \left( \epsilon^2 \delta_1 \phi^2 + 2\epsilon^3 \delta_1 \phi \delta_2 \phi \right)$$

$$+ \frac{1}{2} V_0'' \left( \epsilon^2 \delta_1 \phi^2 + 2\epsilon^3 \delta_1 \phi \delta_2 \phi \right) + \frac{1}{6} V_0''' \left( \epsilon^3 \delta_1 \phi^3 \right) \right\},$$  \hspace{1cm} (5.33)

$$\epsilon \delta_1 \dot{H} + \epsilon^2 \delta_2 \dot{H} + \epsilon^3 \delta_3 \dot{H} = -2\epsilon^3 \delta_1 H \delta_2 H - \epsilon^2 \delta_1 H^2$$

$$- \frac{8\pi G}{3} \left[ \epsilon^2 \delta_1 \phi^2 + 2\epsilon^3 \delta_1 \phi \delta_2 \phi$$

$$- \frac{1}{2} V_0'' \left( \epsilon^2 \delta_1 \phi^2 + 2\epsilon^3 \delta_1 \phi \delta_2 \phi \right) - \frac{1}{6} V_0''' \left( \epsilon^3 \delta_1 \phi^3 \right) \right],$$  \hspace{1cm} (5.34)
\[
(\epsilon \delta_1 \ddot{\phi} + \epsilon^2 \delta_2 \ddot{\phi} + \epsilon^3 \delta_3 \ddot{\phi})
\]

\[
+ 3 \left( \epsilon^2 \delta_1 H \delta_1 \dot{\phi} + \epsilon^3 \delta_1 H \delta_2 \dot{\phi} + \epsilon^3 \delta_2 H \delta_1 \dot{\phi} \right)
\]

\[
+ V_0'' \left( \epsilon \delta_1 \phi + \epsilon^2 \delta_2 \phi + \epsilon^3 \delta_3 \phi \right) + \frac{V_0'''}{2} \left( \epsilon^2 \delta_1 \phi^2 + 2 \epsilon^3 \delta_1 \phi \delta_2 \phi \right)
\]

\[
+ \frac{V_0^{(IV)}}{6} \epsilon^3 \delta_1 \phi^3 = 0.
\] (5.35)

The friction term is now absent in eq. (5.35). To first order one obtains
\[
\delta_1 \dot{H} = 0 \quad \text{and} \quad \delta_1 \ddot{\phi} + V_0'' \delta_1 \phi = 0,
\] (5.36)

therefore there is stability for \( V_0'' \geq 0 \).

To second order we have
\[
\delta_2 \dot{H} + \delta_1 H^2 = \frac{8 \pi G}{3} \left( \frac{\delta_1 \dot{\phi}^2}{2} + \frac{V_0'' \delta_1 \phi}{2} \right) = \text{const.}
\] (5.37)

(which yields no information on second order perturbations),

\[
\delta_2 \dot{\phi} + V_0'' \delta_2 \phi = -3 \delta_1 H \delta_1 \dot{\phi} - \frac{V_0''}{2} \delta_1 \phi^2.
\] (5.38)

Eq. (5.38) guarantees stability in \( \delta_2 H \), while eq. (5.39) guarantees stability to second order if \( V_0'' \geq 0 \).

To third order we have
\[
\delta_2 H = \frac{4 \pi G}{3 \delta_1 H} \left( \delta_1 \dot{\phi} \delta_2 \dot{\phi} + V_0'' \delta_1 \phi \delta_2 \phi + \frac{1}{6} V_0''' \delta_1 \phi^3 \right),
\] (5.40)

assuming \( \delta_1 H \neq 0 \) (again, this equation provides no information on third order perturbations),

\[
\delta_3 \dot{H} = -2 \delta_1 H \delta_2 H - \frac{8 \pi G}{3} \left( 2 \delta_1 \dot{\phi} \delta_2 \dot{\phi} - V_0'' \delta_1 \phi \delta_2 \phi - \frac{1}{6} V_0''' \delta_1 \phi^3 \right),
\] (5.41)

\[
\delta_3 \ddot{\phi} + V_0''' \delta_3 \phi = - \left( 3 \delta_1 H \delta_2 \dot{\phi} + 3 \delta_2 H \delta_1 \dot{\phi} + V_0''' \delta_1 \phi \delta_2 \phi + \frac{V_0^{(IV)}}{6} \delta_1 \phi^3 \right),
\] (5.42)

which again provides stability for \( V_0'' \geq 0 \).
5.3 Inhomogeneous perturbations

A treatment with respect to inhomogeneous perturbations is necessarily more complicated, requiring the use of a gauge-invariant formalism (see, e.g., [26]). To first order (for which the gauge-invariant formalisms apply), the results on the stability of de Sitter spaces already obtained also hold for inhomogeneous perturbations. In fact, the stability of de Sitter spaces in the very general theory of gravity described by the action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{f(R, \phi)}{2} - \frac{\omega(\phi)}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] \] (5.43)

was studied in [28]. This class of theories contains scalar-tensor gravity if \( f(R, \phi) = f(\phi)R \) and higher order gravity if \( f(R, \phi) = f(R) \) and \( \phi = \text{const.} \). A de Sitter space \((H_0, \phi_0)\) is stable with respect to inhomogeneous perturbations if and only if

\[ \frac{f_{\phi\phi}}{2} - V_{\phi\phi} + \frac{6f_{\phi R}H^2}{f_R} \left| \begin{array}{c} \omega \\ 1 + \frac{6f_R}{2f_{\phi R}} \end{array} \right| (H_0, \phi_0) \leq 0. \] (5.44)

General relativity with a minimally coupled scalar field corresponds to the trivial case

\[ f(\phi, R) = \frac{R}{8\pi G}, \quad \omega \equiv 1, \] (5.45)

which yields the first order stability condition \( V''_0 \geq 0 \). This result is gauge-invariant and reproduces, to first order, the one which we have already obtained for homogeneous perturbations (the equivalence between homogeneous and inhomogeneous perturbations in this respect does not extend to FLRW universes other than de Sitter space).

5.4 Lyapunov functions

The Lyapunov method [24] can be easily applied to scalar field cosmology in order to assess stability non-perturbatively, and to estimate the size of the attraction basins of stable fixed points in the phase space.

Let \((H_0, \phi_0)\) be a fixed point of the dynamical system (2.17)–(2.19); then, if \( V(\phi) \) has a local minimum at \( \phi_0 \), and \( H_0 > 0 \), the \( C^1 \) function

\[ L_1 \left( H, \phi, \dot{\phi} \right) \equiv \frac{\dot{\phi}^2}{2} + V(\phi) - V_0 \] (5.46)

is a Lyapunov function. In fact,

- \( L_1 \left( H, \phi, \dot{\phi} \right) > 0 \) in a domain \( \mathcal{D} \) containing \( \phi_0 \), except at \((H_0, \phi_0)\);
\[ L_1 (H_0, \phi_0, 0) = 0; \]
\[ \frac{dL_1}{dt} = \dot{\phi} (\ddot{\phi} + V') = -3H\dot{\phi}^2 \]  is strictly negative in \( D \), except at the fixed point, where \( \frac{dL_1}{dt} \) vanishes.

Therefore, the fixed point \((H_0, \phi_0)\) is asymptotically stable. If \( V(\phi) \) has only one minimum at \( \phi_0 \), the attraction basin of \((H_0, \phi_0)\) is the entire phase space, while if there are other minima at \( \phi_1, \phi_2, \ldots \), the attraction basin of \((H_0, \phi_0)\) is finite and will be limited by separatrices between the attraction basins of other fixed points.

If \( H_0 > 0 \) and \( V(\phi) \) has a local maximum at \( \phi_0 \) (with \( V'_0 = 0, V''_0 < 0 \)), then the \( \mathcal{C}^1 \) function

\[ L_2 (H, \phi, \dot{\phi}) = -\frac{\dot{\phi}^2}{2} - V(\phi) + V_0 \]  (5.47)

is such that \( L_2 (H, \phi, \dot{\phi}) < 0 \) in a domain containing \((H_0, \phi_0)\) at which \( L_2 \) vanishes, and \( \frac{dL_2}{dt} = 3H\dot{\phi}^2 > 0 \) except at the fixed point itself where \( \frac{dL_2}{dt} \) vanishes. This guarantees that the fixed point is unstable and is a repellor.

6 Late-time behaviour of the solutions

Some conclusions on the asymptotic behaviour of the solutions at late times can be reached under certain assumptions on the scalar field potential, without fully specifying the form of \( V(\phi) \).

First, for spatially flat universes, eq. (2.17) implies that \( \dot{H} \leq 0 \), with the equality being satisfied only for the de Sitter fixed points. Hence, outside of fixed points, \( H(t) \) is always a decreasing function. Assuming that \( H \) starts out positive and that \( V(\phi) \geq 0 \) (which guarantees that the energy density is positive), the Hamiltonian constraint (2.18) shows that \( H \) cannot become negative because, due to continuity, it would have to vanish first and the trajectory of the solution in phase space would then cross a fixed point, which is impossible. Therefore, if \( H \) starts out positive [negative], it remains positive [negative]. Let us consider, for definiteness, the case \( H > 0 \). Since \( H \) is bounded from below by zero and \( H < 0 \), the graph of \( H(t) \) cannot cross the \( H = 0 \) axis and we must have \( \dot{H} \geq 0 \) (assuming that \( H \in \mathcal{C}^2 \)).

Since \( \dot{H}(t) = -4\pi G \dot{\phi}^2 \), as \( \dot{H}(t) \rightarrow 0 \) at late times so also must \( \dot{\phi}(t) \rightarrow 0 \), or \((H(t), \phi(t)) \rightarrow (H_0, \phi_0) \) (with \( H_0 > 0 \)) at late times. The trajectories must asymptotically approach a de Sitter attractor point. One possibility is that \( H_0 = 0 \), corresponding to Minkowski space. This could be a point at infinity, in which case the energy content of the universe gets diluted in the future expansion and the universe more and more resembles empty
Minkowski space. In general, for a strictly monotonic potential which is non-negative (or otherwise bounded from below), we always have \( \frac{dV}{d\phi} \neq 0 \) and there cannot be equilibrium points. In this case \( |\phi| \to +\infty \) at late times with \( \dot{\phi} \to 0 \) due to friction (in an expanding universe), and then also \( \dot{H} = -4\pi G \dot{\phi}^2 \to 0 \). The phase space orbit tends to a de Sitter point \( (H_0, \phi_0) = \left( \frac{8\pi GV_0}{3}, \pm \infty \right) \), where \( V_0 \) is the asymptotic value of \( V(\phi) \).

A further consequence of the acceleration equation \( \dot{H} = -4\pi G \dot{\phi}^2 \) is that there cannot be limit cycles as \( H \) and \( \phi \) would have to repeat themselves for periodic orbits, while here \( H \) is monotonically decreasing and thus cannot be periodic.

The late-time asymptotics become much more complicated if the scalar field couples non-minimally to the Ricci curvature \( R \) or if it is a phantom field with the “wrong” sign of the kinetic energy \([9, 29, 30]\).

### 7 Conclusions

It is possible to partially analyze the dynamics of a minimally coupled scalar field in FLRW cosmology, described by the coupled Friedmann-Klein-Gordon equations (2.18), (2.17), and (2.19) constituting a dynamical system. It has been shown in the previous sections how the study of the geometrical structure of the phase space and of the stationary points and their attraction/repulsion basins can be carried out without specifying the form of the scalar field potential \( V(\phi) \). Limiting the analysis to spatially flat universes reduces the dimension of the phase space from four to three. Further, the trajectories of the system lie on two intersecting energy “sheets” (corresponding to setting an effective Hamiltonian equal to zero) given by the Hamiltonian constraint (4.1). The two sheets correspond to \( \dot{\phi} > 0 \) and \( \dot{\phi} < 0 \). There is no possibility of chaos in this space due to its lower dimensionality. The dynamics of non-spatially flat universes are described by orbits constrained to take place above the upper sheet and below the lower sheet \( (|\dot{\phi}| > |\dot{\phi}_{\text{flat}}|) \) for \( k > 0 \), and between these sheets for \( k < 0 \).

Any stationary points are necessarily de Sitter universes with a constant scalar field, which includes Minkowski space as the special case \( H_0 = 0 \). These points were found to be stable for \( H_0 \geq 0 \) and \( V_0'' \geq 0 \), by both third order homogeneous perturbation analysis and Lyapunov’s second method. The size of their attraction basins is dictated by the form of the potential. The attraction basins are global if \( V_0' = 0 \) corresponds to a single sink. The possibility of limit cycles is excluded.

In an asymptotic analysis, if we assume that \( H \) starts off positive (an initially expanding universe) and \( V(\phi) > 0 \) (enforcing the weak energy condition), then \( H(t) \) will remain positive for all times and all trajectories meeting these conditions will be asymptotically attracted to de Sitter spaces. Physically, this means that for these conditions the universe will always expand,
with the scale factor $a(t)$ becoming exponential at late times. Stationary points could exist as an infinite limit in $H$ and/or $\phi$ for certain potentials. From the previous results one can see that an FLRW universe containing a single scalar field in general relativity has relatively simple dynamics. The rather straightforward discussion presented here needs, of course, to be supplemented by a more detailed study which can only be carried out by fully specifying the potential $V(\phi)$, and this is the limitation of the present paper. At the same time, not specifying the potential makes our discussion completely general. An added bonus of the material presented here is its pedagogical value for a general introduction to inflation and quintessence models.

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