Approximation of Chaotic Operators

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Abstract. As well-known, the concept "hypercyclic" in operator theory is the same as the concept "transitive" in dynamical system. Now the class of hypercyclic operators is well studied. Following the idea of research in hypercyclic operators, we consider classes of operators with some kinds of chaotic properties in this article.

First of all, the closures of the sets of all Li-Yorke chaotic operators or distributionally chaotic operators are discussed. We give a spectral description of them and prove that the two closures coincide with each other. Moreover, both the set of all Li-Yorke chaotic operators and the set of all distributionally chaotic operators have nonempty interiors which coincide with each other as well. The article also includes the containing relation between the closure of the set of all hypercyclic operators and the closure of the set of all distributionally chaotic operators. Finally, we get connectedness of the sets considered above.

1. Introduction

We are interested in the dynamical systems induced by continuous linear operators on Banach spaces. From Rolewicz's article [21], hypercyclicity is widely studied. In fact, it coincides with a dynamical property "transitivity". Now there has been got so many improvements at this aspect, Grosse-Erdmann's and Shapiro's articles [8, 23] are good surveys.

In his celebrated work [9, 10, 11], D. A. Herrero studied the chaotic properties (hypercyclic and Devaney's chaotic) of linear operators. It is important since it shows that we can study the chaotic properties of operators in a really operator theory way. As well-known, it is hard to check whether a topological system be chaotic or not for a general object. But following Herrero's idea, we can use the technique of approximation to study the properties of chaotic operators on Hilbert space under compact or small perturbation. An interesting result, obtained by D. A. Herrero and Z. Y. Wang [9] or K. Chan and J. Shapiro [3], shows that the identity operator $I$ can be perturbed by a small compact operator to be hypercyclic. This stronger result implies that a small perturbation of a simple operator can be an operator with complex dynamic properties.

These papers suggest us to consider the following question:

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**Question**: Which kinds of operators can be approximated by chaotic operators?

From the point of approximation, we should consider closure of the set of all operators satisfying some chaotic property. In this paper, Li-York chaotic operators and distributionally chaotic operators will be studied by classical approximation tools developed in [12].

In order to explain the main results, we must introduce some definitions and properties of chaos and Hilbert space operators.

In 1975, Li and Yorke [16] observed complicated dynamical behavior for the class of interval maps with period 3. This phenomena is currently known under the name of Li-Yorke chaos. Recall that a discrete dynamical system is simply a continuous mapping $f : X \to X$ where $X$ is a complete separable metric space. For $x \in X$, the orbit of $x$ under $f$ is $\text{Orb}(f, x) = \{ x, f(x), f^2(x), \ldots \}$ where $f^n = f \circ f \circ \cdots \circ f$ is the $n$th iterate of $f$ obtained by composing $f$ with $n$ times.

**Definition 1.1.** $\{x, y\} \subset X$ is said to be a Li-Yorke chaotic pair, if

$$
\limsup_{n \to \infty} d(f^n(x), f^n(y)) > 0, \quad \liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0.
$$

Furthermore, $f$ is called Li-Yorke chaotic, if there exists an uncountable subset $\Gamma \subseteq X$ such that each pair of two distinct points in $\Gamma$ is a Li-Yorke chaotic pair.

In 1994, Schweizer and Smítal [22] gave the definition of distributional chaos (where it was called strong chaos), which requires more complicated statistical dependence between orbits than the existence of points which are proximal but not asymptotic.

For any pair $\{x, y\} \subset X$ and any $n \in \mathbb{N}$, define distributional function $F_{xy}^n : \mathbb{R} \to [0, 1]$

$$
F_{xy}^n(\tau) = \frac{1}{n} \# \{ 0 \leq i \leq n-1 : d(f^i(x), f^i(y)) < \tau \},
$$

where $\# \{ A \}$ is the cardinality of the set $A$. Furthermore, define

$$
F_{xy}(\tau) = \liminf_{n \to \infty} F_{xy}^n(\tau),
$$

$$
F_{xy}^*(\tau) = \limsup_{n \to \infty} F_{xy}^n(\tau)
$$

Both $F_{xy}$ and $F_{xy}^*$ are nondecreasing functions and may be viewed as cumulative probability distributional functions satisfying $F_{xy}(\tau) = F_{xy}^*(\tau) = 0$ for $\tau < 0$.

**Definition 1.2.** $\{x, y\} \subset X$ is said to be a distributionally chaotic pair, if

$$
F_{xy}^*(\tau) \equiv 1, \quad \forall \ \tau > 0 \quad \text{and} \quad F_{xy}(\epsilon) = 0, \quad \exists \ \epsilon > 0.
$$

Furthermore, $f$ is called distributionally chaotic, if there exists an uncountable subset $\Lambda \subseteq X$ such that each pair of two distinct points in $\Lambda$ is a distributionally chaotic pair. Moreover, $\Lambda$ is called a distributionally $\epsilon$-scrambled set.
From the definitions, we know distributional chaos implies Li-Yorke chaos. But the converse implication is not true in general. In practice, even in the simple case of Li-Yorke chaos, it might be quite difficult to prove chaotic behavior from the very definition. Such attempts have been made in the context of linear operators (see [6, 7]). Further results of [6] were extended in [20] to distributional chaos for the annihilation operator of a quantum harmonic oscillator. Additionally, distributional chaos for shift operators were discussed by F. Martínez-Giménez, et. al. in [19]. More about Li-Yorke chaos and distributional chaos, one can see [1, 17, 18, 24, 25]. In a recent article [13], B. Hou et. al introduced a new dynamical property for linear operators called norm-unimodality which implies distributional chaos, and obtained a sufficient condition for Cowen-Douglas operator being distributional chaotic and Devaney’s chaotic. We introduce the definition of norm-unimodality here.

**Definition 1.3.** Let $X$ be a Banach space and let $T \in B(X)$. $T$ is called norm-unimodal, if we have a constant $\gamma > 1$ such that for any $m \in \mathbb{N}$, there exists $x_m \in X$ satisfying

$$
\lim_{k \to \infty} \|T^k x_m\| = 0, \text{ and } \|T^i x_m\| \geq \gamma^i \|x_m\|, \quad i = 1, 2, \ldots, m.
$$

Furthermore, such $\gamma$ is said to be a norm-unimodal constant for the norm-unimodal operator $T$.

Next, we introduce the notations and properties of Hilbert space operators. Let $H$ be complex separable Hilbert space and denote by $B(H)$ the set of bounded linear operators mapping $H$ into $H$. For $T \in B(H)$, denote the kernel of $T$ and the range of $T$ by Ker$T$ and Ran$T$ respectively. Denote by $\sigma(T), \sigma_c(T), \sigma_{irr}(T)$ and $\sigma_w(T)$ the spectrum, the essential spectrum, the Wolf spectrum and the Weyl spectrum of $T$ respectively. For $\lambda \in \rho_{s-f}(T) := \mathbb{C} \setminus \sigma_{irr}(T)$, $\text{ind}(\lambda - T) = \dim \text{Ker}(\lambda - T) - \dim \text{Ker}(\lambda - T)^*$, $\min \text{ind}(\lambda - T) = \min \{\dim \text{Ker}(\lambda - T), \dim \text{Ker}(\lambda - T)^*\}$. Denote $\rho_{s-f}^n(T) = \{\lambda \in \rho_{s-f}(T); \text{ind}(\lambda - T) = n\}$, where $-\infty \leq n \leq \infty$, $\rho_{s-f}^{+}(T) = \{\lambda \in \rho_{s-f}(T); \text{ind}(\lambda - T) > 0\}$ and $\rho_{s-f}^{-}(T) = \{\lambda \in \rho_{s-f}(T); \text{ind}(\lambda - T) < 0\}$. According to [12] corollary 1.14 we know that the function $\lambda \mapsto \min \text{ind}(\lambda - T)$ is constant on every component of $\rho_{s-f}(T)$ except for an at most denumerable subset $\rho_{s-f}^*(T)$ without limit points in $\rho_{s-f}(T)$. Furthermore, if $\mu \in \rho_{s-f}^*(T)$ and $\lambda$ is a point of $\rho_{s-f}(T)$ in the same component as $\mu$ but $\lambda$ is not in $\rho_{s-f}^*(T)$, then $\min \text{ind}(\mu - T) > \min \text{ind}(\lambda - T)$. $\rho_{s-f}^*(T)$ is the set of singular points of the semi-Fredholm domain $\rho_{s-f}(T)$ of $T$; $\rho_{s-f}^*(T) = \rho_{s-f}(T) \setminus \rho_{s-f}^*(T)$ is the set of regular points. Denote by $\sigma_0(T)$ the set of isolated points of $\sigma(T) \setminus \sigma_c(T)$. Denote by $\overline{E}$ and $E^0$, the closure and the interior of set $E$ respectively. In addition, denote by $LY(H)$, $DC(H)$, $UN(H)$ the set of all Li-Yorke chaotic operators, the set of all distributionally chaotic operators and the set of all norm-unimodal operators on $H$ respectively.
Now we are in a position to state the main results of this article. In section 2, the closures and interiors of the sets of all distributionally chaotic operators or Li-Yorke chaotic operators are considered. Though distributionally chaotic operators require more complicated statistical dependence between orbits than Li-Yorke chaotic operators, we have

I. \[
\overline{DC(H)} = \overline{LY(H)} = \{ T \in B(H); \partial \mathbb{N} \cap \sigma_{trc}(T) \neq \emptyset \} \cup \{ T \in B(H); \partial \mathbb{N} \subseteq \rho_{s-F}(T) \text{ and } \dim \text{Ker}(\lambda - T) > 0, \forall \lambda \in \partial \mathbb{N} \} \quad \text{(Theorem 2.8)}.
\]

II. \[
 learning that \begin{align*}
\overline{DC(H)}^0 &= \overline{LY(H)}^0 = \{ T \in B(H), \exists \lambda \in \partial \mathbb{N} \text{ s.t. } \text{ind}(\lambda - T) > 0 \} \quad \text{(Theorem 2.14)}.
\end{align*}
\]

From the above two results, one can see distributionally chaotic operators and Li-Yorke chaotic operators are very similar. The closure of \( DC(H)^0 \) (i.e. the closure of \( LY(H)^0 \)) is also considered.

III. \[
\overline{DC(H)}^0 = \overline{LY(H)}^0 = \{ T \in B(H); \partial \mathbb{N} \not\subseteq \rho_{s-F}^0(T) \cup \rho_{lre}^0(F) \}. \quad \text{(i.e. the closure}
\]

\]

The relation between \( UN(H) \) and \( UN(H) \setminus \overline{DC(H)}^0 \) is also obtained.

IV. \[
\overline{UN(H)} = \overline{DC(H)} = \overline{LY(H)}, \overline{DC(H)}^0 = \overline{LY(H)}^0 \subseteq \overline{UN(H)}, \text{ and}
\]

\[
\overline{UN(H)} \setminus \overline{DC(H)}^0 = \overline{DC(H)} \setminus \overline{DC(H)}^0 = \overline{LY(H)} \setminus \overline{LY(H)}^0. \quad \text{(Theorem 3.3)}
\]

It follows from this result that, the norm-unimodal operators are very large in the class of distributionally chaotic operators. Moreover, it is useful for people to prove that an operator is distributionally chaotic as the criterion of hypercyclic operators given by Kitai [15] and refined by Grosse-Erdmann and Shapiro, et al. [8].

In section 4, we consider the connectedness of the sets considered above.

V. \[
\overline{DC(H)}^0, \overline{DC(H)}^0, \overline{DC(H)} \text{ and } \overline{DC(H)} \setminus \overline{DC(H)}^0 \quad \text{(i.e. } \overline{LY(H)}^0, \overline{LY(H)}^0, \overline{LY(H)} \text{ and } \overline{LY(H)} \setminus \overline{LY(H)}^0 \quad ) \text{ are all arcwise connected. (Theorem 4.1)}
\]

2. Closures and interiors of the sets of all distributionally chaotic operators or Li-Yorke chaotic operators

Firstly, we need some lemmas which will be used in the proof theorem 2.8. The definition given by Cowen and Douglas [4] is well known as follows.

\textbf{Definition 2.1.} For \( \Omega \) a connected open subset of \( \mathbb{C} \) and \( n \) a positive integer, let \( B_n(\Omega) \) denotes the operators \( T \) in \( B(H) \) which satisfy:

\begin{align*}
(1) & \quad \Omega \subseteq \sigma(T); \\
(2) & \quad \text{ran}(T - \omega) = H \text{ for } \omega \text{ in } \Omega; \\
(3) & \quad \bigvee_{\omega \in \Omega} \ker(T - \omega) = H; \text{ and}
\end{align*}

(4) dimker(T − ω) = n for ω in Ω.

One often calls the operator T in $B_n(\Omega)$ Cowen-Douglas operator. Denote by $\mathbb{D}$ and $\partial\mathbb{D}$ the unit open disk and its boundary. Then we have the following theorem.

**Theorem 2.2.** [13] Let $T \in B_n(\Omega)$. If $\Omega \cap \partial\mathbb{D} \neq \emptyset$, then T is norm-unimodal. Consequently, T is distributionally chaotic.

**Remark 2.3.** In fact, this result can be extended to $n = \infty$.

**Lemma 2.4.** Let $T \in B(H)$. Then the following statements are equivalent.

1. T is not Li-Yorke chaotic.
2. $\lim_{n \to \infty} \|T^n(x)\| = 0$ implies $\lim_{n \to \infty} \|T^n(x)\| = 0$.

The proof is easy and left to the reader.

**Lemma 2.5.** Let $T \in B(H)$, $\sigma(T) \cap \partial\mathbb{D} = \emptyset$. Then $\lim_{n \to \infty} \|T^n(x)\| = 0$ implies $\lim_{n \to \infty} \|T^n(x)\| = 0$. Moreover, T is neither Li-Yorke chaotic nor distributionally chaotic.

**Proof.** Since $\sigma(T) \cap \partial\mathbb{D} = \emptyset$, then according to Riesz’s decomposition theorem

$$T = \begin{bmatrix} T_1 & H_1 \\ T_2 & H_2 \end{bmatrix},$$

where $\sigma(T_1) = \sigma(T) \cap \mathbb{D}$ and $\sigma(T_2) = \sigma(T) - \sigma(T_1)$. Furthermore,

$$\tilde{T} = \begin{bmatrix} T_1 & H_1 \\ \tilde{T}_2 & H_1^{1/2} \end{bmatrix},$$

where $\tilde{T}_2 \sim T_2$ and then $\sigma(\tilde{T}_2) = \sigma(T_2) = \sigma(T) - \sigma(T_1)$.

By spectral mapping theorem and spectral radius formula,

$$r_1(\tilde{T}_2)^{-1} = r(\tilde{T}_2)^{-1} = \lim_{n \to \infty} \|\tilde{T}_2^{-n}\|^\frac{1}{n}, \quad \text{(where } r_1(\cdot) = \inf\{|\lambda|; \lambda \in \sigma(\cdot)|\}).$$

Notice $r_1(\tilde{T}_2) \geq \delta > 1$, then there is $\epsilon > 0$ such that $r_1(\tilde{T}_2)^{-1} + \epsilon < 1$. Hence we have $M \in \mathbb{N}$ such that for any $n \geq M$,

$$\frac{1}{\|\tilde{T}_2^{-n}\|} \geq (\frac{1}{r_1(\tilde{T}_2)^{-1} + \epsilon})^n.$$ 

Furthermore, for any $y \in H_1^{1/2}$,

$$\|\tilde{T}_2^{-n}(y)\| \geq \frac{1}{\|\tilde{T}_2^{-n}\|} \|y\| \geq (\frac{1}{r_1(\tilde{T}_2)^{-1} + \epsilon})^n \|y\| \geq \|y\|, \text{ when } n \geq M.$$ 

Let $\liminf_{n \to \infty} \|T^n(x)\| = 0$ and $x = x_1 \oplus x_2$, $x_1 \in H_1$, $x_2 \in H_1^{1/2}$, then one can easily obtain $x_2 = 0$ and then $T^n(x) = T_1^n(x_1)$. On the other hand $r(T_1) < 1$, so there exist $0 \leq \rho < 1$ and $N \in \mathbb{N}$ such that for any $n \geq N$, $\|T_1^n(x_1)\| \leq \rho^n \|x_1\|$. Therefore, $\liminf_{n \to \infty} \|T^n(x)\| = \liminf_{n \to \infty} \|T_1^n(x_1)\| = 0$.

Following from Kitaï’s result in [15], no finite dimensional Hilbert space supports a hypercyclic operator. In fact, we have
Lemma 2.6. Let $0 < n < \infty$ be an integer and $T \in B(C^n)$. Then \( \liminf_{m \to \infty} ||T^m(x)|| = 0 \) implies \( \lim_{m \to \infty} ||T^m(x)|| = 0 \). Moreover, $T$ is neither Li-Yorke chaotic nor distributionally chaotic.

From [14], one can see

Lemma 2.7. For any $\epsilon > 0$, there is a compact operator $K_\epsilon \in B(H)$ such that $||K_\epsilon|| < \epsilon$ and $I + K_\epsilon$ is distributionally chaotic.

Now we will give a description of the closures for the sets of distributionally chaotic operators or Li-Yorke chaotic operators.

Theorem 2.8. Let $E_1 = \{T \in B(H); \partial D \cap \sigma_{irr}(T) \neq \emptyset\}$ and $E_2 = \{T \in B(H); \partial D \subseteq \rho_{s-f}(T) \text{ and } \dim \ker(\lambda - T) > 0, \forall \lambda \in \partial D\}$. Then $\overline{DC(H)} = \overline{LY(H)} = E_1 \cup E_2$.

Proof. Clearly, $\overline{DC(H)} \subseteq \overline{LY(H)}$. So it suffices to show $E_1 \cup E_2 \subseteq \overline{DC(H)}$ and $\overline{LY(H)} \subseteq E_1 \cup E_2$.

First step, $E_1 \cup E_2 \subseteq \overline{DC(H)}$. We will show for any $T \in E_1 \cup E_2$ and $\epsilon > 0$, there exists an operator $C$ such that $||C|| < \epsilon$ and $T + C \in DC(H)$. In fact, one can obtain a compact operator $K$ such that $||K|| < \epsilon$ and $T + K \in DC(H)$.

If $T \in E_1$, then choose a $\lambda_0 \in \partial D \cap \sigma_{irr}(T)$. By AFV theorem there exists a compact operator $K_1$ such that $||K_1|| < \epsilon/2$ and

$$T + K_1 = \begin{bmatrix} \lambda_0 I & \ast \\ \ast & H_0 \end{bmatrix},$$

where $\dim H_0 = \infty$.

Following lemma 2.7 there exists a compact operator $K_2$ such that $||K_2|| < \epsilon/2$ and $\lambda_0 I + K_2$ is distributionally chaotic. Let

$$\tilde{K}_2 = \begin{bmatrix} K_2 \\ 0 \end{bmatrix} \begin{bmatrix} H_0 \\ H_0^\perp \end{bmatrix}.$$

Then $T + \tilde{K}_2 = DC(H)$, where $K_1 + \tilde{K}_2$ is a compact operator and $||K_1 + \tilde{K}_2|| < \epsilon$.

If $T \in E_2$, define

$$H_r = \bigvee_{\lambda \in \rho_{s-f}^c(T) \cap \Delta} \ker(\lambda - T),$$

where $\Delta$ is the component of semi-Fredholm domain $\rho_{s-f}(T)$ which contains $\partial D$.

Then $\dim H_r = \infty$ and

$$T = \begin{bmatrix} T_r & \ast \\ \ast & H_r \end{bmatrix} \begin{bmatrix} H_r \\ H_r^\perp \end{bmatrix}.$$

Since

1. $\rho_{s-f}^c(T) \cap \Delta \subseteq \sigma(T_r)$,
2. $\dim \ker(\mu - T_r) = \dim \ker(\mu - T) = n, \forall \mu \in \rho_{s-f}^c(T) \cap \Delta$,
where \( n \in \mathbb{N}^+ \cup \{ \infty \} \),

\[
(3) \quad \bigvee_{\mu \in \rho^*_s(T) \cap \Delta} \text{Ker}(\mu - T_r) = \bigvee_{\mu \in \rho^*_s(T) \cap \Delta} \text{Ker}(\mu - T) = H_r,
\]

\[
(4) \quad \text{Ran}(\mu - T_r) = H_r, \quad \mu \in \rho^*_s(T) \cap \Delta,
\]

we know \( T_r \in B_n(\rho^*_s(T) \cap \Delta) \). Notice \( \rho^*_s(T) \cap \Delta \cap \partial \mathbb{D} \neq \emptyset \), by theorem 2.2 \( T_r \) is norm-unimodal and hence distributionally chaotic. So it is \( T \). The first step is complete.

Second step, \( \overline{LY(H)} \subseteq E_1 \cup E_2 \). Notice

\[ \{E_1 \cup E_2\}^c = \{T \in B(H); \partial \mathbb{D} \subseteq \rho_{s-F}(T) \text{ and } \exists \lambda \in \partial \mathbb{D} \text{ s.t. dimKer}(\lambda - T) = 0\}, \]

by Fredholm theory \( \{E_1 \cup E_2\}^c \) is open. Since \( \{\overline{LY(H)}\}^c = \{LY(H)^c\}^0 \), it suffices to prove \( \{E_1 \cup E_2\}^c \subseteq \overline{LY(H)}^c \).

Let \( T \in \{E_1 \cup E_2\}^c \), define

\[ H_l = \bigvee_{\lambda \in \rho^*_s(T) \cap \Phi} \text{Ker}(\lambda - T)^*, \]

where \( \Phi \) is the component of semi-Fredholm domain \( \rho_{s-F}(T) \) which contains \( \partial \mathbb{D} \). Then

\[ T = \begin{bmatrix} T_0 & * \\ T_l \end{bmatrix} H_l^{\perp}, \quad (H_l \text{ maybe } \{0\}). \]

Claim 1: \( \rho^*_s(T) \cap \Phi \subseteq \rho(T_0) \).

Let \( \mu \in \rho^*_s(T) \cap \Phi \). Since \( \lambda \rightarrow \min \text{ind}(\lambda - T) \) is constant on the semi-Fredholm domain \( \rho^*_s(T) \) and \( \exists \lambda_0 \in \partial \mathbb{D} \text{ s.t. dimKer}(\lambda_0 - T) = 0 \), we have \( \text{Ker}(\mu - T) = \{0\} \).

Hence \( \text{Ker}(\mu - T_0) = \{0\} \). Notice \( \text{Ker}(\mu - T)^* = \text{Ker}(\mu - T_0)^* \) and \( (\mu - T)^*(H_l) = H_l \), then \( \text{Ker}(\mu - T_0)^* = \{0\} \). Therefore, \( \mu - T_0 \) is invertible.

Claim 2: \( \sigma_0(T_0) \cap \Phi = \sigma(T_0) \cap \Phi = \rho^*_s(T) \cap \Phi \).

From claim 1, \( \sigma_0(T_0) \cap \Phi \subseteq \sigma(T_0) \cap \Phi \subseteq \rho^*_s(T) \cap \Phi \). Let \( \lambda \in \rho^*_s(T) \cap \Phi \). If \( \lambda - T_0 \) is invertible, then \( \lambda - T_0 \) is a semi-Fredholm operator and \( \min \text{ind}(\lambda - T_0) = \min \text{ind}(\lambda - T) \). Since \( (\lambda - T) = (\lambda - T_0) = H_l \), \( \dim \text{Ker}(\lambda - T_0) = 0 \) and hence \( \min \text{ind}(\lambda - T) = \min \text{ind}(\lambda - T_0) = 0 \). It is contradict to \( \lambda \in \rho^*_s(T) \). Therefore, \( \lambda - T_0 \) is not invertible and \( \rho^*_s(T) \cap \Phi \subseteq \sigma(T_0) \cap \Phi \). Because \( \lambda - T \) is a left semi-Fredholm operator, \( (\lambda - T)(H_l^{\perp}) \) is closed. Therefore \( \lambda - T_0 \) is a semi-Fredholm operator.

Again notice claim 1 we obtain \( \rho^*_s(T) \cap \Phi \subseteq \sigma_0(T_0) \cap \Phi \).

Since the only limit points of \( \rho^*_s(T) \) belong to \( \partial[\rho^*_s(T)] \), let \( \sigma_0(T_0) \cap \Phi \cap \partial \mathbb{D} = \{\mu_1, ..., \mu_m\}, m < \infty \). By Riesz's decomposition theorem and Rosenblum-Davis-Rosenthal corollary [12],

\[
T_0 = \begin{bmatrix} T_{00} & T_{01} \\ T_{01} \end{bmatrix} H_{01} = \begin{bmatrix} T_{00} & * \\ T_{01} \end{bmatrix} H_l^{\perp} \subset H_{00} \sim \begin{bmatrix} T_{00} & T_{01} \\ T_{01} \end{bmatrix} H_{01} \subset H_{00},
\]
where \(\sigma(T_{00}) = \{\mu_i\}_{i=1}^m, \ m < \infty, \ \sigma(T_{01}) \cap \partial \mathbb{D} = \emptyset, \ T_{01} \sim \tilde{T}_{01}\) and \(\dim H_{00} < \infty\). Hence
\[
T \sim S := \begin{bmatrix}
T_{00} & 0 \\
T_{01} & * \\
T_{1} & *
\end{bmatrix}
\]

Moreover \(H_i = \bigvee_{\lambda \in \rho_{s,F}^{-}(T) \cap \Phi} \ker(\lambda - T)^* = \bigvee_{\lambda \in \partial \mathbb{D} \cup \rho_{s,F}^{-}(T) \cap \Phi} \ker(\lambda - T)^*\), then
\[
T_i = \begin{bmatrix}
\lambda_1 & * & \lambda_2 \\
* & \lambda_3 & * \\
\vdots & \vdots & \cdot.
\end{bmatrix}
\]

where \(\{\lambda_i\}_{i=1}^\infty \subseteq \partial \mathbb{D} \cap \rho_{s,F}^{-}(T) \cap \Phi\) and \(\{e_i\}_{i=1}^\infty\) is an ONB of \(H_i\).

Now we come to end the proof. Since Li-Yorke chaos is invariant under similar and lemma 2.4, it suffices to show \(\liminf_{n \to \infty} \|S^n(x)\| = 0\) implies \(\lim_{n \to \infty} \|S^n(x)\| = 0\). Let \(\liminf_{n \to \infty} \|S^n(x)\| = 0\), then there exist \(\{n_k\}_{k=1}^\infty\) such that \(\lim_{n_k \to \infty} \|S^{n_k}(x)\| = 0\). Notice \(x = x_0 \oplus \tilde{x}_0 \oplus x_l, \ x_0 \in H_{00}, \ \tilde{x}_0 \in H_l^+ \ominus H_{00}, \ x_l \in H_l,\) we have \(\lim_{n_k \to \infty} \|T_l^{n_k}(x_l)\| = 0\). Following the matrix representation of \(T_l, x_l = 0, \) Hence,
\[
S^{n_k}(x) = \begin{bmatrix}
T_{00} & 0 \\
T_{01} & 0
\end{bmatrix}^{n_k} \begin{bmatrix}
x_0 \\
\tilde{x}_0
\end{bmatrix}.
\]

So by lemma 2.6 and lemma 2.5 \(\lim_{n \to \infty} \|T_{00}(x_0)\| = 0\) and \(\lim_{n \to \infty} \|T_{01}(\tilde{x}_0)\| = 0\). That is \(\lim_{n \to \infty} \|S^n(x)\| = 0\). The second step is complete. \(\Box\)

Theorem 2.8 also includes the information of the interior for \(DC(H)^c\). Obviously, the operator \(T\) satisfying, \(\sigma(T) \cap \partial \mathbb{D} = \emptyset\), is in \(\{DC(H)^c\}^0\) (i.e. \(\{DC(H)\}^c\)). There exists an operator \(T\) in \(\{DC(H)^c\}^0\), whose spectrum \(\sigma(T)\) intersects the unit circle, i.e. \(\sigma(T) \cap \partial \mathbb{D} \neq \emptyset\).

**Example 2.9.** Let \(A \in B(H)\) satisfying
\[
\left\{
\begin{array}{ll}
Ae_i = \frac{1}{2}e_{i+1}, & i \leq -2, \\
Ae_i = 2e_{i+1}, & i > -2,
\end{array}
\right.
\]
where \(\{e_i\}_{i=-\infty}^\infty\) is an ONB of \(H\). Then \(A\) is in \(\{DC(H)^c\}^0\).

**Proof.** Through easy compute, one can obtain \(\sigma(A) = \{z \in \mathbb{C}; \ 1/2 \leq |z| \leq 2\}\) and \(\text{ind}(\lambda - A) = -1, \ \dim \ker(\lambda - A) = 0\) for \(\lambda \in \{z \in \mathbb{C}; \ 1/2 < |z| < 2\}\). According to theorem 2.8, \(A \in \overline{\{DC(H)^c\}}\) (i.e. \(\{DC(H)^c\}^0\)).
We consider the dynamical property of $A$. For any $x$ in $H$, $x = \sum_{i=1}^{\infty} x_i e_i$, and

$$A^{2n+1}(x) = (\cdots, \frac{1}{2^{2n+1}}x_{-(2n+2)}, \frac{1}{2^{2n-1}}x_{-(2n+1)}, \frac{1}{2^{2n-3}}x_{-2n}, \cdots)$$

$$A^{2n}(x) = (\cdots, \frac{1}{2^{2n}}x_{-2n-1}, \frac{1}{2^{2n-2}}x_{-2n}, \frac{1}{2^{2n-4}}x_{-2n+1}, \cdots)$$

where the position under $\wedge$ is the 0 position corresponding to the ONB $\{e_i\}_{i=-\infty}^{\infty}$.

One can easily obtain if $x \neq 0$, then $\|A^n(x)\| \to \infty$. Hence $A$ is not distributionally chaotic.

Next, we consider the interiors of the sets of all Li-Yorke chaotic operators or distributionally chaotic operators. Before proving the result theorem 2.14, it is convenient to cite in full length a result of Apostol and Morrel. Let $\Gamma = \partial \Omega$, where $\Omega$ is an analytic Cauchy domain, and let $L^2(\Gamma)$ be the Hilbert space of (equivalent classes of) complex functions on $\Gamma$ which are square integrable with respect to $(1/2\pi)$-times the arc-length measure on $\Gamma$; $M(\Gamma)$ will stand for the operator defined as multiplication by $\lambda$ on $L^2(\Gamma)$. The subspace $H^2(\Gamma)$ spanned by the rational functions with poles outside $\Omega$ is invariant under $M(\Gamma)$. By $M_+(\Gamma)$ and $M_-(\Gamma)$ we shall denote the restriction of $M(\Gamma)$ to $H^2(\Gamma)$ and its compression to $L^2(\Gamma) \ominus H^2(\Gamma)$, respectively, i.e.

$$M(\Gamma) = \begin{bmatrix} M_+(\Gamma) & Z \\ \overline{Z} & M_-(\Gamma) \end{bmatrix} \begin{bmatrix} H^2(\Gamma) \\ H^2(\Gamma) \end{bmatrix}$$

**Definition 2.10.** [12] $S \in B(H)$ is a simple model, if it has the form

$$S = \begin{bmatrix} S_+ & * & * \\ * & A & * \\ * & * & S_- \end{bmatrix},$$

where

1. $\sigma(S_+), \sigma(S_-), \sigma(A)$ are pairwise disjoint;
2. $A$ is similar to a normal operator with finite spectrum;
3. $S_+$ is (either absent or) unitarity equivalent to $\oplus_{i=1}^{m} M_+(\partial \Omega_i)^{k_i}$, $1 \leq k_i \leq \infty$, where $\{\partial \Omega_i\}_{i=1}^{m}$ is a finite family of analytic Cauchy domains with pairwise disjoint closures;
4. $S_-$ is (either absent or) unitarity equivalent to $\oplus_{j=1}^{n} M_-((\partial \Phi_j)^{h_j}$, $1 \leq h_j \leq \infty$, where $\{\partial \Phi_j\}_{j=1}^{n}$ is a finite family of analytic Cauchy domains with pairwise disjoint closures.
Theorem 2.11. [12] The simple models are dense in $B(H)$. More precisely: Given $T \in B(H)$ and $\epsilon > 0$ there exists a simple model $S$ such that

1. $\sigma(S_+) \subseteq \rho_{{s^{-}}_{-}}(T) \subseteq \sigma(S_+) \cap \sigma(S_-)$, $\sigma(S_-) \subseteq \rho_{{s^{+}}_{-}}(T) \subseteq \sigma(S_-)$, and $\sigma(A) \subseteq \sigma(T)$.
2. $\text{ind}(\lambda - S) = \text{ind}(\lambda - T)$, for each $\lambda \in \rho_{{s^{-}}_{-}}(S_+) \cup \rho_{{s^{+}}_{-}}(S_-)$.
3. $\|T - S\| < \epsilon$.

Additionally, we give a lemma which appeared in another article [14]. But for convenience to read this article, we also give the details of the proof.

Lemma 2.12. Let $N \in B(H)$ be a normal operator. Then $\lim \inf_{n \to \infty} \|N^n(x)\| = 0$ implies $\lim_{n \to \infty} \|N^n(x)\| = 0$. Moreover, $N$ is neither Li-Yorke chaotic nor distributionally chaotic.

Proof. Since $N$ is a normal operator, then there exist a locally compact space $X$, a finite positive regular Borel measure $\mu$ and a Borel function $\eta \in L^\infty(X, \mu)$ such that $N$ and $M_\eta$ are unitarily equivalent. $M_\eta$ is multiplication by $\eta$ on $L^2(X, \mu)$. Let $\lim \inf_{n \to \infty} \|M_\eta^n(f)\| = 0$ and

$$\Delta_1 = \{z \in X; |\eta(z)| \geq 1\},$$
$$\Delta_2 = \{z \in X; |\eta(z)| < 1\},$$
$$\Delta_3 = \{z \in X; f(z) = 0 \text{ a.e. } [\mu]\},$$
$$\Delta_4 = \{z \in X; f(z) \neq 0 \text{ a.e. } [\mu]\}.$$

Then there exists a sequence of positive integers $\{n_k\}_{k=1}^\infty$ such that $\lim_{n_k \to \infty} \|M_\eta^{n_k}(f)\| = 0$ and

$$\|M_\eta^{n_k}(f)\|^2 = \int_X |\eta^{n_k}f|^2 d\mu$$
$$\geq \int_{\Delta_1 \cap \Delta_4} |f|^2 d\mu + \int_{\Delta_2 \cap \Delta_4} |\eta^{n_k}f|^2 d\mu$$
$$\geq \int_{\Delta_1 \cap \Delta_4} |f|^2 d\mu + \int_{\Delta_2 \cap \Delta_4} |\eta^{n_k}f|^2 d\mu.$$ 

Consequently $\mu(\Delta_1 \cap \Delta_4) = 0$. For any $n \in \mathbb{N}$, there exists a positive integer $k$ such that $n_k \leq n < n_{k+1}$. Therefore,

$$\|M_\eta^n(f)\|^2 = \int_{\Delta_2 \cap \Delta_4} |\eta^n f|^2 d\mu$$
$$= \int_{\Delta_2 \cap \Delta_4} |\eta^{n_k} f|^{2n-k} |\eta^{-n_k}|^{2} d\mu$$
$$\leq \int_{\Delta_2 \cap \Delta_4} |\eta^{n_k} f|^2 d\mu$$
$$= \|M_\eta^{n_k}(f)\|^2,$$

and hence $\lim_{n \to \infty} \|M_\eta^n(f)\| = 0$. Notice the property which we considered is invariant under unitarily equivalence, we obtain the result. \qed
Corollary 2.13. Let $T \in B(H)$ be a subnormal operator. Then $\liminf_{n \to \infty} ||T^n(x)|| = 0$ implies $\lim_{n \to \infty} ||T^n(x)|| = 0$. Moreover, $T$ is neither Li-Yorke chaotic nor distributionally chaotic.

Theorem 2.14. Let $F = \{ T \in B(H), \exists \lambda \in \partial \mathbb{D} \text{ s.t. ind}(\lambda - T) > 0 \}$. Then $DC(H)^0 = LY(H)^0 = F$.

Proof. Obviously, $DC(H)^0 \subseteq LY(H)^0$. So we only need to show $F \subseteq DC(H)^0$ and $LY(H)^0 \subseteq F$.

First step, $F \subseteq DC(H)^0$. By Fredholm theory, $F$ is open. Consequently, it suffices to prove $F \subseteq DC(H)$.

Let $T \in F$. Define

$$H_r = \bigvee_{\mu \in \rho_{s-p}^{(c)}(T) \cap \Delta} \operatorname{Ker}(\mu - T),$$

where $\Delta$ is the component of $\rho_{s-p}^{(c)}(T)$ which contains a point in $\partial \mathbb{D}$. Then $\dim H_r = \infty$ and

$$T = \begin{bmatrix} T_r & * \\ * & H_r^* \end{bmatrix}, \quad T_r = \begin{bmatrix} H_r & \Delta \\ \Delta^* & H_r^* \end{bmatrix}.$$

Similar to theorem 2.8, we have $T_r \in B_n(\rho_{s-p}^{(c)}(T) \cap \Delta)$, moreover $T$ is normal-unimodal and distributionally chaotic. The first step is complete.

Next step, $LY(H)^0 \subseteq F$. Because $(LY(H)^0)^c = \overline{LY(H)^c}$, we only need to show for any $T \in F^c$ and $\epsilon > 0$, there exists $C \in B(H)$ such that $||C|| < \epsilon$ and $T + C$ is not Li-Yorke chaotic.

Let $T \in F^c$ and $\epsilon > 0$. According to Theorem 2.11, there exists a simple model

$$S = \begin{bmatrix} S_+ & * & * \\ * & A & * \\ * & S_- \end{bmatrix}$$

such that $\sigma(S_-) \subseteq \rho_{s-p}^{(c)}(T) \subseteq \sigma(S_-)$, and $||T - S|| < \epsilon$. Since $\rho_{s-p}^{(c)}(T) \cap \partial \mathbb{D} = \emptyset$, we can ensure $\sigma(S_-) \cap \partial \mathbb{D} = \emptyset$ according to the proof of theorem 2.11. So it suffices to prove $S$ is impossible to be Li-Yorke chaotic.

Notice $\sigma(S_+), \sigma(S_-), \sigma(A)$ are pairwise disjoint, according to Rosenblum-Davis-Rosenthal corollary [12],

$$S \sim \begin{bmatrix} S_+ & A \\ A & S_- \end{bmatrix}.$$  

Because Li-Yorke chaotic is invariant under similar and $A$ is similar to a normal operator $N$ with finite spectrum, we directly let $S = S_+ \oplus N \oplus S_-.$

If $\liminf_{n \to \infty} ||S^n(x)|| = 0$, since $x = x_+ \oplus x_0 \oplus x_-$ corresponding to the space decomposition, following lemma 2.5 and corollary 2.13 we have

$$\lim_{n \to \infty} ||S_-^n(x_-)|| = 0 \text{ and } \lim_{n \to \infty} ||S_+^n \begin{bmatrix} x_+ \\ N \\ x_0 \end{bmatrix}|| = 0.$$

Hence $\lim_{n \to \infty} ||S^n(x)|| = 0$. $S$ is not Li-Yorke chaotic. The second step is complete. \qed
We give an example which is distributionally chaotic but not in $DC(H)^0$.

**Example 2.15.** Let $A \in B(H)$ satisfying
\[
\begin{cases}
Ae_i = 2e_{i-1}, & i \neq 0, \\
Ae_0 = 0,
\end{cases}
\]
where $\{e_i\}_{i=-\infty}^{\infty}$ is an ONB of $H$. Then $A$ is distributionally chaotic but not in $DC(H)^0$.

**Proof.** Since $H_0 = \bigvee_{i=0}^{\infty} \{e_i\}$ is an invariant space of $A$ and $A|_{H_0} = 2B$ is distributionally chaotic, where $B$ is backward unilateral shift, then $A$ is distributionally chaotic. One can easily obtain $\text{ind}(\lambda - A) = 0$ for $|\lambda| < 2$, so $A$ is not in $DC(H)^0$.

We prove it directly. For any $\epsilon > 0$, let $K \in B(H)$ satisfying $Ke_0 = e\epsilon_{-1}$, $Ke_i = 0, i \neq 0$. Then $K$ is compact and $|K| = \epsilon$. Since $\sigma(A + C) = \{z \in \mathbb{C}; |z| = 2\}$, $A + K$ is not distributionally chaotic. Hence $A$ is not in $DC(H)^0$. \qed

Unfortunately, the closure of $DC(H)^0$ (i.e. the closure of $LY(H)^0$) is not equal to the closure of $DC(H)$ (i.e. the closure of $LY(H)$). It means there exists a class of distributionally chaotic operators (Li-Yorke chaotic operators) which are more complicated. We give these descriptions.

**Theorem 2.16.** Let $G_0 = \{T \in B(H); \partial \Omega \notin \rho_{\sigma_{\text{tr}}(T)}(T) \cup \rho_{\sigma_{\text{tr}}(N)}(T)\}$, $G_1 = \{T \in B(H); \partial \Omega \subseteq \rho_{\sigma_{\text{tr}}(N)}(T) \cup \rho_{\sigma_{\text{tr}}(N)}(T) \text{ and dim Ker}(\lambda - T) > 0, \forall \lambda \in \partial \Omega\}$ and $G_2 = \{T \in B(H); \partial \Omega \cap \sigma_{\text{tr}}(T) \neq \emptyset \text{ and } \rho_{\sigma_{\text{tr}}(N)}(T) \cap \partial \Omega = \emptyset\}$. Then $\overline{DC(H)^0} = \overline{LY(H)^0} = G_0$ and $\overline{DC(H)^0} \setminus \overline{DC(H)^0} = \overline{LY(H)^0} = G_1 \cup G_2$.

**Proof.** First, we prove $\overline{DC(H)^0} = \overline{LY(H)^0} = G_0$. Clearly, $DC(H)^0 = LY(H)^0 = F \subseteq G_0$, where $F$ is denoted in theorem 2.14. By Fredholm theory, $G_0$ is closed. Hence $\overline{DC(H)^0} = \overline{LY(H)^0} \subseteq G_0$. So we only need to show for any $T \in G_0$ and $\epsilon > 0$, there exists $C \in B(H)$ such that $\|C\| < \epsilon$ and $T + C \in DC(H)^0$ (i.e. $LY(H)^0$).

Let $T \in G_0$ and $\epsilon > 0$. Then
\[
\begin{align*}
(1) & \exists \lambda \in \partial \Omega \text{ s.t. } \text{ind}(\lambda - T) > 0, \\
(2) & \rho_{\sigma_{\text{tr}}(T)}^{(+)}(T) \cap \partial \Omega = \emptyset, \text{ but } \exists \lambda \in \partial \Omega \text{ s.t. } \text{ind}(\lambda - T) = 0, \text{ or} \\
(3) & \rho_{\sigma_{\text{tr}}(N)}^{(+)\prime}(T) \cap \partial \Omega = \emptyset, \text{ but } \sigma_{\text{tr}}(T) \cap \partial \Omega \neq \emptyset.
\end{align*}
\]

Case (1) is obvious.

Case (2). First it implies $\sigma_{\text{tr}}(T) \cap \partial \Omega \neq \emptyset$. Then choose a $\lambda_0 \in \sigma_{\text{tr}}(T) \cap \partial \rho_{\sigma_{\text{tr}}(N)}^{(+)\prime}(T) \cap \partial \Omega$. According to AFV theorem, there exists a compact operator $K_1$ such that $\|K_1\| < \epsilon/2$ and
\[
T + K_1 = \begin{bmatrix} \lambda_0 I & * \\ A & H_0 \end{bmatrix} H_0^{-1},
\]
where \( \sigma(A) = \sigma(T), \sigma_{trc}(A) = \sigma_{trc}(T) \) and \( \text{ind}(\lambda - A) = \text{ind}(\lambda - T) \), for \( \lambda \in \rho_{s-F}(T) \). Let
\[
B_\epsilon = \begin{bmatrix}
0 & \epsilon/2 \\
0 & \epsilon/2 \\
& & \ddots \\
& & & \ddots
\end{bmatrix} e_1 \\
& & & \ddots \\
& & & & \ddots \\
& & & & & \ddots
\]
where \( \{e_i\}^\infty_{i=1} \) is an ONB of \( H_0 \) and
\[
K_2 = \begin{bmatrix} B_\epsilon & H_0 \\
0 & H_0^\perp \end{bmatrix}.
\]
Obviously, \( ||K_1 + K_2|| < \epsilon \) and there exists \( \lambda \in \partial \mathbb{D} \) such that \( \text{ind}(T + K_1 + K_2 - \lambda) = 1 > 0 \). Hence \( T + K_1 + K_2 \in DC(H)^0 \).

Case (3). By theorem 2.11 there exists \( C_1 \) such that \( ||C_1|| < \epsilon/2 \) and
\[
T + C_1 = \begin{bmatrix} S_+ & * & * \\
& A & * \\
& & S_- \end{bmatrix},
\]
where \( S_+ \) is either absent or unitarity equivalent to a subnormal operator and \( \partial \mathbb{D} \setminus \sigma(S_+) \) contains a small arc in \( \partial \mathbb{D} \). \( A \) is similar to a normal operator with finite spectrum and \( \sigma_{trc}(A) \cap \partial \mathbb{D} \neq \emptyset \), \( S_- \) is either absent or unitarity equivalent to the adjoint of a subnormal operator and \( \sigma(S_-) \cap \partial \mathbb{D} = \emptyset \); \( \sigma(S_+), \sigma(A), \sigma(S_-) \) are pairwise disjoint. Furthermore,
\[
\sigma_{trc}(T + C_1) = \sigma_{trc}(S_+) \cup \sigma_{trc}(A) \cup \sigma_{trc}(S_-) \text{ and } \rho(T + C_1) = \rho(S_+) \cap \rho(A) \cap \rho(S_-).
\]
Hence
\[
\sigma_{trc}(T + C_1) \cap \partial \mathbb{D} \neq \emptyset \text{ and } \rho(T + C_1) \cap \partial \mathbb{D} \neq \emptyset.
\]
Then we can obtain \( C_2 \) through the technology of case (2) such that \( ||C_2|| < \epsilon/2 \) and \( \text{ind}(T + C_1 + C_2 - \lambda_1) > 0 \) (where \( \lambda_1 \in \partial \mathbb{D} \)). Hence \( T + C_1 + C_2 \in DC(H)^0 \).

The first equation is complete.

Second, we prove \( DC(H) \setminus DC(H)^0 = LY(H) \setminus LY(H)^0 = G_1 \cup G_2 \). Clearly,
\[
DC(H) \setminus DC(H)^0 \subseteq DC(H) \setminus DC(H)^0 \subseteq LY(H) \setminus LY(H)^0 \subseteq LY(H) \setminus LY(H)^0.
\]
Then \( G_1 \subseteq DC(H) \setminus DC(H)^0 \subseteq LY(H) \setminus LY(H)^0 \subseteq G_1 \cup G_2 \). In order to obtain the result, we only need to show \( G_2 \subseteq DC(H) \setminus DC(H)^0 \).

For any \( T \in G_2 \) and \( \epsilon > 0 \), according to AFV theorem there exists a compact operator \( K_1 \) such that \( ||K_1|| < \epsilon/2 \) and
\[
T + K_1 = \begin{bmatrix} \lambda_0 I & \ast \\
& A \end{bmatrix} \begin{bmatrix} H_0 \\
H_0^\perp \end{bmatrix}
\]
where \( \lambda_0 \in \partial \mathbb{D} \setminus \sigma_{trc}(T) \). By lemma 2.7, we know there exists a compact operator \( K_\epsilon \) such that \( ||K_\epsilon|| < \epsilon/2 \) and \( \lambda_0 I + K_\epsilon \) is distributionally chaotic. Let
\[
K_2 = \begin{bmatrix} K_\epsilon \\
0 \end{bmatrix} \begin{bmatrix} H_0 \\
H_0^\perp \end{bmatrix}.
\]
So \( ||K_1 + K_2|| < \epsilon \) and \( T + K_1 + K_2 \) is distributionally chaotic. Notice
\[
\rho_{s-F}(T + K_1 + K_2) \cap \partial \mathbb{D} = \rho_{s-F}(T) \cap \partial \mathbb{D} = \emptyset,
\]
we know $T + K_1 + K_2 \in \{ DC(H)^0 \}$\textsuperscript{c}. The second equation is complete. \hfill \Box

### 3. Some other results

In this section, we consider the relation between hypercyclic operators and distributionally chaotic operators, and the closure of the set of all norm-unimodal operators.

Recall the definition of chaos given by Devaney [5] as follows.

**Definition 3.1.** Suppose that $f : X \to X$ is a continuous function on a complete separable metric space $X$, then $f$ is Devaney’s chaotic if:

- (a) the periodic points for $f$ are dense in $X$,
- (b) $f$ is transitive,
- (c) $f$ has sensitive dependence on initial conditions.

It was shown by Banks et. al. [2] that if $f$ satisfies (a) and (b), then $f$ must have sensitive dependence on initial conditions. Hence only the first two conditions of the definition need to be verified.

Denote by $HC(H)$ and $DE(H)$ the set of all hypercyclic operators and the set of all Devaney’s chaotic operators on $H$ respectively. Obviously, $DE(H) \subseteq HC(H)$.

**Proposition 3.2.** $DE(H) = HC(H) \subseteq DC(H)^0 = LY(H)^0$.

**Proof.** According to [11] proposition 4 and [10], one can obtain $DE(H) = HC(H) = \{ T \in B(H) ; \sigma_w(T) \cup \partial D$ is connected, $\sigma_0(T) = \emptyset$ and $\text{ind}(\lambda - T) \geq 0$, $\lambda \in \rho_s - F(T) \}$. Following theorem 2.16, we obtain the result. \hfill \Box

Next, we can see the set of all norm-unimodal operators is large in the set of all distributionally chaotic operators.

**Theorem 3.3.** $\overline{UN(H)} = DC(H) = LY(H)$, $DC(H)^0 = LY(H)^0 \subseteq UN(H)$ and $\overline{UN(H)} \setminus DC(H)^0 = DC(H) \setminus DC(H)^0 = LY(H) \setminus LY(H)^0$.

**Proof.** First we prove $\overline{UN(H)} = DC(H) = LY(H)$. Obviously, $\overline{UN(H)} \subseteq DC(H) = \overline{LY(H)}$, it suffices to prove $E_1 \cup E_2 \subseteq \overline{UN(H)}$, where $E_1, E_2$ are denoted in theorem 2.8. We will show for any $T \in E_1 \cup E_2$ and $\epsilon > 0$, there exists $C$ such that $||C|| < \epsilon$ and $T + C \in UN(H)$. But different to theorem 2.8, we can not generally find a compact operator satisfying the property.

If $T \in E_1$, then choose any $\lambda_0 \in \partial \mathbb{D} \cap \sigma_{trc}(T)$. According to AFV theorem, there exists a compact operator $K_1$ such that $||K_1|| < \epsilon/2$ and $T + K_1 = \begin{bmatrix} \lambda_0 I & \ast \\ \ast & H_0 \end{bmatrix}_{H_0}$, where $\dim H_0 = \infty$. Let

$$C_1 = \begin{bmatrix} 0 & \epsilon/2 & \vdots \\ 0 & \epsilon/2 & \vdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \epsilon_0 \\ \epsilon_1 \\ \vdots \end{bmatrix}. $$
where \( \{ \epsilon_i \}_{i=0}^{\infty} \) is an ONB of \( H_0 \). Then \( C_1 \) is a Cowen-Douglas operator with \( ||C_1|| = \epsilon/2 \). Let
\[
\tilde{C}_1 = \begin{bmatrix} C_1 & 0 \\ H_0 & H_0 \end{bmatrix}.
\]
Following theorem 2.2, we know
\[
T + K_1 + \tilde{C}_1 \in UN(H), \quad (||K_1 + \tilde{C}_1|| < \epsilon).
\]
Notice \( K_1 + \tilde{C}_1 \) is not compact. In fact, the operator \( T \) in \( E_1 \) satisfying \( \sigma(T) = \sigma_{\text{irr}}(T) \) and \( \sigma(T) \subseteq \mathbb{D}^- \) cannot be perturbed into \( UN(H) \) by compact operator, since \( T + K \) is impossible to be norm-unimodal (one can observe [14] for details).

If \( T \in E_2 \), then according to the first step in the proof of theorem 2.8, we know \( T \) is norm-unimodal. The first equation is complete. The second inclusion is immediate obtained from the first step in the proof of theorem 2.14.

Next we prove \( \tilde{\lambda} \in UN(H) \setminus DC(H)^0 = DC(H) \setminus DC(H)^0 = LY(H) \setminus LY(H)^0 \). Clearly, \( \tilde{\lambda} \in UN(H) \setminus DC(H)^0 \subseteq DC(H) \setminus DC(H)^0 = LY(H) \setminus LY(H)^0 \). We only need to show \( G_1 \cup G_2 \subseteq \tilde{\lambda} \setminus DC(H)^0 \), where \( G_1, G_2 \) are denoted in theorem 2.16. Similar to the first step of theorem 2.8, \( G_1 \subseteq \tilde{\lambda} \setminus DC(H)^0 \).

Let \( T \in G_2 \) and \( \epsilon > 0 \). By AFV theorem there exists a compact operator \( K_1 \) such that \( ||K_1|| < \epsilon/3 \) and
\[
T + K_1 = \begin{bmatrix} \lambda_0 I & \ast & \ast \\ A & \ast \\ \lambda_0 I \\ H_0 & H_1, & H_2 \end{bmatrix}.
\]
where \( \lambda_0 \in \partial \mathbb{D} \cap \sigma_{\text{irr}}(T), \rho_{\kappa_{\ast}^{+}}(A) \cap \partial \mathbb{D} = \emptyset \). Let \( N \in B(H_2) \) be an uniform infinite multiplicity normal operator such that \( \sigma(N) = \{ z \in \mathbb{C}; |z| \leq \epsilon/3 \} \) and
\[
B_\epsilon = \begin{bmatrix} 0 & \epsilon/3 \\ \ast & \ast \\ \ast & \ast \end{bmatrix}.
\]
\[
\{ \epsilon_i \}_{i=0}^{\infty} \text{ is an ONB of } H_0.
\]
Then \( ||N|| = \epsilon/3; ||B_\epsilon|| = \epsilon/3 \), \( B_\epsilon \in B_1(\mathbb{D}_{\epsilon/3}) \) and \( \sigma(B_\epsilon) = \mathbb{D}_{\epsilon/3}^- \).

Hence
\[
T + K_1 + B_\epsilon = \begin{bmatrix} B_\epsilon & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} \lambda_0 I + B_\epsilon & \ast & \ast \\ A & \ast \\ \lambda_0 I + N \end{bmatrix}.
\]
Notice
\[
\rho_{\kappa_{\ast}}^{(+)}(\tilde{\lambda}) \cap \partial \mathbb{D} = \emptyset
\]
and \( \lambda_0 I + B_\epsilon \) is norm-unimodal, we obtain the result. \( \square \)
Example 3.4. Let $A \in B(H)$ satisfying
\[
\begin{align*}
Ae_i &= 2e_{i-1}, & i \geq 1, \\
Ae_0 &= e_1, \\
Ae_i &= \frac{|i|}{|i|+1}e_{i-1}, & i \leq -1.
\end{align*}
\]
where $\{e_i\}_{i=-\infty}^\infty$ is ONB of $H$. Then $A$ is norm-unimodal, but not in $DC(H)^0$.

Proof. For any $m \in \mathbb{N}$, $\|A^i(e_m)\| \geq 2^i\|e_m\|$, $1 \leq i \leq m$ and $\lim_{m \to \infty} \|A^n(e_m)\| = 0$.
So $A$ is norm-unimodal. Since $\sigma(A) = \{z \in \mathbb{C}; 1 \leq |z| \leq 2\}$ implies $\partial \mathbb{D} \subseteq \sigma_{\text{trc}}(A)$, then $A$ is not in $DC(H)^0$. \hfill \blacksquare

4. Connectedness

The main purpose in this section is to discuss the connectedness for the sets considered in section 2.

Theorem 4.1. $DC(H)^0$, $\overline{DC(H)^0}$, $DC(H)$ and $DC(H) \setminus DC(H)^0$ (i.e. $LY(H)^0$, $LY(H)^0$, $LY(H)$ and $LY(H) \setminus LY(H)^0$) are all arcwise connected.

Proof. We show $DC(H)^0$ is arcwise connected, others are similar.

First step, for any $T \in DC(H)^0$, it can be connected to $\tilde{T} \in DC(H)^0$, where
$\sigma_{\text{trc}}(\tilde{T}) \cap \partial \mathbb{D} \neq \emptyset$. If $\sigma_{\text{trc}}(T) \cap \partial \mathbb{D} \neq \emptyset$, then obviously.

Let $\sigma_{\text{trc}}(T) \cap \partial \mathbb{D} = \emptyset$. Then $\partial \mathbb{D} \subseteq \rho_{s-F}^{(+)}(T)$. Choose $\lambda_0 \in \sigma_{\text{trc}}(T)$, by AFV theorem there exists a compact operator $K$ such that $\|K\| < \epsilon$ and
\[
T + K = \begin{bmatrix}
\lambda_0 I & * & * \\
* & A & * \\
* & * & \lambda_0 I
\end{bmatrix}
\]
where $\sigma(T) = \sigma(A), \sigma_{\text{trc}}(T) = \sigma_{\text{trc}}(A)$ and $\text{ind}(\lambda - T) = \text{ind}(\lambda - A)$ for all $\lambda \in \rho_{s-F}(A)$. Choose $\mu_0 \in \partial \mathbb{D}$, let
\[
\delta(t) = \begin{bmatrix}
\alpha(t) I & * & * \\
* & A & * \\
* & * & \alpha(t) I
\end{bmatrix}, 
1 < t \leq 2,
\]
where $\alpha(t) = (t - 1)(\mu_0 - \lambda_0) + \lambda_0, 1 < t \leq 2$. Define
\[
\beta(t) = \begin{cases}
T + tK, & 0 \leq t \leq 1, \\
\delta(t), & 1 < t \leq 2.
\end{cases}
\]
Obviously,
\[
\beta(0) = T, \quad \beta(2) = \tilde{T} := \begin{bmatrix}
\mu_0 I & * & * \\
* & A & * \\
* & * & \mu_0 I
\end{bmatrix}, \quad \sigma_{\text{trc}}(\tilde{T}) \cap \partial \mathbb{D} = \{\mu_0\} \neq \emptyset
\]
and $\beta(t)$ is continuous on $[0,2]$.

For any $1 < t \leq 2$, $\rho_{s-F}^{(+)}(\delta(t)) = \rho_{s-F}^{(+)}(A) \setminus \{\alpha(t)\}$, then $\rho_{s-F}^{(+)}(\delta(t)) \cap \partial \mathbb{D} \neq \emptyset$.
Notice $\rho_{s-F}^{(+)}(T + tK) = \rho_{s-F}^{(+)}(T) \supseteq \partial \mathbb{D}$ for all $0 \leq t \leq 1$, we know $\{\beta(t); 0 \leq t \leq 2\} \subseteq DC(H)^0$. The first step is complete.
Second step, we show for any \( T, S \in \text{DC}(H)^0 \), \( \sigma_{\text{tr}}(T) \cap \partial \mathbb{D} \neq \emptyset \), \( \sigma_{\text{tr}}(S) \cap \partial \mathbb{D} \neq \emptyset \), \( T \) and \( S \) can be connected.

Let \( \lambda_0 \in \partial \rho_{s_{-F}}(T) \cap \partial \mathbb{D} \) be the point such that there exists \( \theta_0 > 0 \) s.t. 
\[
\{ \lambda_0 e^{it}; 0 < \theta < \theta_0 \} \subseteq \rho_{s_{-F}}(T).
\]
Similarly, we can obtain \( \lambda_1 \in \partial \rho_{s_{-F}}(S) \cap \partial \mathbb{D} \) and \( \theta_1 > 0 \) such that 
\[
\{ \lambda_1 e^{it}; 0 < \theta < \theta_1 \} \subseteq \rho_{s_{-F}}(S).
\]
Notice there exists \( \theta' \) such that \( \lambda_0 = e^{i\theta} \lambda_1 \), define \( \widetilde{S} = e^{i\theta} S \). Then \( \lambda_0 \in \partial \rho_{s_{-F}}(\widetilde{S}) \cap \partial \mathbb{D} \) and 
\[
\{ \lambda_0 e^{it}; 0 < \theta < \theta_1 \} \subseteq \rho_{s_{-F}}(\widetilde{S}) \text{ (let } \Phi_0 \text{ be the component of } \rho_{s_{-F}}(\widetilde{S}) \text{ which contains } \{ \lambda_0 e^{it}; 0 < \theta < \theta_1 \}).
\]
By AFV theorem and theorem 3.48 [12], for any \( \epsilon > 0 \), there exist compact operators \( K_1, K_2 \) such that \( ||K_1|| < \epsilon, ||K_2|| < \epsilon \) and 
\[
T + K_1 = \begin{bmatrix} \lambda_0 I & C_1 \\ A_1 & H_1 \end{bmatrix}, \quad H_1 \perp, \quad \widetilde{S} + K_2 = \begin{bmatrix} A_2 & C_2 \\ \lambda_0 I & H_2 \end{bmatrix}, \quad H_2 \perp,
\]
where \( \sigma(A_1) = \sigma(T) \), \( \sigma_{\text{tr}}(A_1) = \sigma_{\text{tr}}(T) \) and \( \text{ind}(\lambda - A_1) = \text{ind}(\lambda - T) \) for all \( \lambda \in \rho_{s_{-F}}(T) \); \( \sigma(A_2) = \sigma(\widetilde{S}) \), \( \sigma_{\text{tr}}(A_2) = \sigma_{\text{tr}}(\widetilde{S}) \), \( \text{ind}(\lambda - A_2) = \text{ind}(\lambda - \widetilde{S}) \) for all \( \lambda \in \rho_{s_{-F}}(\widetilde{S}) \) and \( \text{minind}(\lambda - A_2) = 0 \) for all \( \lambda \in \Phi_0 \). Without loss of generality, let \( H_1 = H_2 \perp \). Define
\[
\gamma(t) = \begin{cases} 
\rho_{s_{-F}}(e^{it}S), & 0 \leq t < \theta', \\
\rho_{s_{-F}}(t - \theta)K_2, & \theta' \leq t \leq \theta' + 1, \\
(t - (\theta' + 1))(T + K_1) + (\theta' + 2 - t)(\widetilde{S} + K_2), & \theta' + 1 < t < \theta' + 2, \\
T + [\theta' + 3 - t]K_1, & \theta' + 2 \leq t \leq \theta' + 3.
\end{cases}
\]
Obviously, \( \gamma(0) = S \), \( \gamma(\theta' + 3) = T \) and \( \gamma(t) \) is continuous on \([0, \theta' + 3]\). We prove
\( \{ \gamma(t); 0 \leq t \leq \theta' + 3 \} \subseteq \text{DC}(H)^0 \).

a) \( \rho_{s_{-F}}(e^{it}S) = e^{it}\rho_{s_{-F}}(S), 0 \leq t \leq \theta' \) and \( \rho_{s_{-F}}(S) \cap \partial \mathbb{D} \neq \emptyset \) implies \( \{ e^{it}S; 0 \leq t < \theta' \} \subseteq \text{DC}(H)^0 \).

b) \( \rho_{s_{-F}}(\widetilde{S} + (t - \theta)K_2) = \rho_{s_{-F}}(\widetilde{S}) = e^{it}\rho_{s_{-F}}(S), \theta' \leq t \leq \theta' + 1 \) and \( \rho_{s_{-F}}(S) \cap \partial \mathbb{D} \neq \emptyset \) implies \( \{ \widetilde{S} + (t - \theta)K_2; \theta' \leq t \leq \theta' + 1 \} \subseteq \text{DC}(H)^0 \).

c) For any given \( \theta' + 1 < t < \theta' + 2 \), since
\[
\rho_{s_{-F}}(t - (\theta' + 1))\lambda_0 + [\theta' + 2 - t]A_2 = [t - (\theta' + 1)]\lambda_0 + [\theta' + 2 - t]\rho_{s_{-F}}(A_2) = \lambda_0 + (\theta' + 2 - t)(\rho_{s_{-F}}(A_2) - \lambda_0),
\]
and
\[
\rho_{s_{-F}}(t - (\theta' + 1))A_1 + [\theta' + 2 - t]A_2 = [t - (\theta' + 1)]\rho_{s_{-F}}(A_1) + [\theta' + 2 - t]\lambda_0 = \lambda_0 + [t - (\theta' + 1)](\rho_{s_{-F}}(A_1) - \lambda_0),
\]
we know there exists \( \theta_t > 0 \) such that
\[
\{ \lambda_0 e^{it}; 0 < \theta < \theta_t \} \subseteq \rho_{s_{-F}}(t - (\theta' + 1))A_1 + [\theta' + 2 - t]A_2 \cap \rho_{s_{-F}}([t - (\theta' + 1)]A_1 + [\theta' + 2 - t]A_2).
For each $\lambda \in \{\lambda_0 e^{i\theta}; 0 < \theta < \theta_1\}$, notice $\minind(\mu - A_2) = 0$ for $\mu \in \Phi_0$, so $\minind((t - (\theta' + 1))\lambda_0 + [\theta' + 2 - t]A_2 - \lambda) = 0$. Moreover, $\ind((t - (\theta' + 1))\lambda_0 + [\theta' + 2 - t]A_2 - \lambda) > 0$, then we have $(t - (\theta' + 1))\lambda_0 + [\theta' + 2 - t]A_2 - \lambda$ is epic.

Consequently

$$\ind((t - (\theta' + 1))(T + K_1) + [\theta' + 2 - t](\tilde{S} + K_2) - \lambda)$$

$$= \ind((t - (\theta' + 1))\lambda_0 + [\theta' + 2 - t]A_2 - \lambda) +$$

$$\ind((t - (\theta' + 1))A_1 + [\theta' + 2 - t]\lambda_0 - \lambda) > 0.$$

Hence $\{(t - (\theta' + 1))(T + K_1) + (\theta' + 2 - t)(\tilde{S} + K_2); \theta' + 1 < t < \theta' + 2\} \subseteq DC(H)^0$.

d) $\rho_{s,F}^{-}(T + [\theta' + 3 - t]K_1) = \rho_{s,F}^{-}(T), \theta' + 2 \leq t \leq \theta' + 3$ and $\rho_{s,F}^{-}(T) \cap \partial \mathbb{D} \neq \emptyset$ implies $\{T + [\theta' + 3 - t]K_1; \theta' + 2 \leq t \leq \theta' + 3\} \subseteq DC(H)^0$.

Therefore, $\gamma(t); 0 \leq t \leq \theta' + 3 \} \subseteq DC(H)^0$. The second step is complete.

Thus, $DC(H)^0$ is arcwise connected. 

**Example 4.2.** Let $B$ be backward unilateral shift. Then there exists an arc $\alpha(t)$ in $DC(H)^0$ which connects $5B$ and $5B^2$.

**Proof.** Through easy compute, $\sigma(5B) = \sigma(5B^2) = 5\mathbb{D}^-$ and $\ind(\lambda - 5B) = 1$, $\ind(\lambda - 5B^2) = 2$ for $|\lambda| < 5$. First there exist compact operators $K_1$, $K_2$ such that

$$5B + K_1 = \begin{bmatrix} -5I & C_1 \\ A_1 & H_1 \end{bmatrix} \quad \text{and} \quad 5B^2 + K_2 = \begin{bmatrix} A_2 & C_2 \\ 5I & H_2 \end{bmatrix} \hat{H}_2,$$

where $\sigma(A_1) = \sigma(5B)$, $\sigma_{irr}(A_1) = \sigma_{irr}(5B)$ and $\ind(\lambda - A_1) = \ind(\lambda - 5B)$ for all $\lambda \in \rho_{s,-F}(5B)$; $\sigma(A_2) = \sigma(5B^2)$, $\sigma_{irr}(A_2) = \sigma_{irr}(5B^2)$ and $\ind(\lambda - A_2) = \ind(\lambda - 5B^2)$ for all $\lambda \in \rho_{s,-F}(5B^2)$. Without loss of generality, let $H_1 = H_2^\perp$. Define

$$\alpha(t) = \begin{cases} 5B + (1 + t)K_1, & -1 \leq t \leq 0, \\ \delta(t), & 0 < t < 1, \\ 5B^2 + (2 - t)K_2, & 1 \leq t \leq 2, \end{cases}$$

where $\delta(t) = \begin{bmatrix} -5(1 - t) + tA_2 & (1 - t)C_1 + tC_2 \\ (1 - t)A_1 + 5t & 5I \end{bmatrix}, 0 < t < 1$.

Obviously, $\alpha(-1) = 5B$, $\alpha(2) = 5B^2$ and $\alpha(t)$ is continuous on $[-1, 2]$. It suffices to show $\alpha(t) \in DC(H)^0$ for any $-1 \leq t \leq 2$.

(1) $\rho_{s,F}^{-}((5B + (1 + t)K_1) \subseteq \rho_{s,F}^{-}(5B), -1 \leq t \leq 0$ and $\partial \mathbb{D} \subseteq \rho_{s,F}^{-}(5B)$ implies $\{5B + (1 + t)K_1; -1 \leq t \leq 0\} \subseteq DC(H)^0$.

(2) $\rho_{s,F}^{-}(\delta(t)) = \rho_{s,F}^{-}((-5(1 - t) + tA_2)) \cup \rho_{s,F}^{-}((1 - t)A_1 + 5t)) = \{-5(1 - t) + 5t\mathbb{D}\} \cup \{5(1 - t)\mathbb{D} + 5t\}, 0 < t < 1$ implies $\{\delta(t); 0 < t < 1\} \subseteq DC(H)^0$.

One can read spectrum properties from the picture as follows. The number 1 or 2 in the picture means the index of the open disk which it lies respectively. We choose five moments.
\( (3) \rho^{(t)}_{s-F}(5B^2 + (2-t)K_2) = \rho^{(t)}_{s-F}(5B^2), \quad 1 \leq t \leq 2 \) and \( \partial \mathbb{D} \subseteq \rho^{(t)}_{s-F}(5B^2) \) implies 
\( \{5B^2 + (2-t)K_2; \quad 1 \leq t \leq 2\} \subseteq DC(H)^0. \)
Hence \( \{\alpha(t), \quad -1 \leq t \leq 2\} \subseteq DC(H)^0. \)

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