The generic fixed point model for pseudo-spin-$\frac{1}{2}$ quantum dots in nonequilibrium: Spin-valve systems with compensating spin polarizations

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We study a pseudo-spin-$\frac{1}{2}$ quantum dot in the cotunneling regime close to the particle-hole symmetric point. For a generic tunneling matrix we find a generic fixed point with interesting nonequilibrium properties, characterized by effective reservoirs with compensating spin orientation vectors weighted by the polarizations and the tunneling rates. At large bias voltage we study the magnetic field dependence of the dot magnetization and the current. The fixed point can be clearly identified by analyzing the magnetization of the dot. We characterize in detail the universal properties for the case of two reservoirs.

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Nonequilibrium properties of strongly interacting quantum dots have gained an enormous interest in the last decades. Quantum dots are experimentally controllable systems useful for a variety of applications in nanoelectronics, spintronics and quantum information processing.1 They are of fundamental interest in the field of open quantum systems in nonequilibrium with interesting quantum many-body properties and coherent phenomena at low temperatures.2 Of particular interest are spin-dependent phenomena where the quantum dot is tuned to the Coulomb blockade regime. In the case of a singly-occupied dot the spin can fluctuate between two values leading to a realization of the isotropic spin-$\frac{1}{2}$ antiferromagnetic Kondo model. A hallmark was the prediction and observation of universal conductance for this model.3 The equilibrium properties of the Kondo model have been studied extensively and, most recently, by using renormalization group (RG) methods in nonequilibrium, also the properties at finite bias voltage and the time dynamics have been analyzed in weak and strong coupling and compared to experiments.4

The isotropic Kondo model with unpolarized leads is only a special case out of the whole class of quantum dot models where a single particle on the dot can fluctuate between two different quantum numbers (which we call a pseudo-spin-$\frac{1}{2}$ quantum dot in the following). Besides the case of ferromagnetic leads with arbitrary spin orientations the two quantum numbers can also label two different orbitals or can arise from a mixture of spin and orbital degrees of freedom in the presence of spin-orbit interaction in the leads or on the dot, leading to non-spin-conserving tunneling matrices. In equilibrium (or the linear response regime), it has been found for several cases that exchange fields are generated but if those are canceled by external ones the universality properties of the Kondo model are re-established. This has been confirmed by numerical renormalization group (NRG) calculations for ferromagnetic leads with parallel or antiparallel orientations and for quantum dots with orbital degrees of freedom or Aharonov-Bohm geometries. In Ref. 18 a mapping between these different models and an analytical understanding in terms of the anisotropic Kondo model has been established. Concerning nonequilibrium transport previous studies have focused on exchange fields generated by ferromagnetic leads, spin-orbit interaction or orbital fluctuations. A systematic nonequilibrium RG study of a pseudo-spin-$\frac{1}{2}$ quantum dot with spin-orbit interaction in the cotunneling regime has been performed in Ref. 21, where a Dzyaloshinskii-Moriya (DM) interaction together with exchange fields proportional to the bias voltage have been identified. For special orientations of the DM-vectors interesting asymmetries in resonant transport where reported when a magnetic field of the order of the bias voltage is applied.

All previous references treated special cases of pseudo-spin-$\frac{1}{2}$ quantum dots without aiming at finding generic features common to all these systems, irrespective of the complexity of the geometry, the special interactions and the polarizations of the reservoirs. The purpose of this letter is to establish such features especially in the nonequilibrium regime. Thereby, we will first use a mapping to a pseudo-spin-$\frac{1}{2}$ quantum dot coupled to effective ferromagnetic leads as depicted in Fig. 1, similar to Refs. 18, 19. Based on this model, we will show that in the Coulomb blockade regime close to the particle-hole symmetric point a fixed point model can be identified where the average of the unit vectors of the spin orientations weighted by the polarizations and the tunneling rates...
\[ \bar{d} = \sum_{\alpha} d_\alpha = 0 \quad \text{and} \quad d_\alpha = x_\alpha p_\alpha \tilde{d}_\alpha, \quad x_\alpha = \frac{\Gamma_\alpha}{\Gamma}, \quad (1) \]

with \( \Gamma = \sum_\alpha \Gamma_\alpha \). This explains why the Kondo effect appears generically in the equilibrium case where all reservoirs can be taken together and (1) leads to a vanishing spin polarization, in agreement withRefs. 16–18. However, what has been overlooked so far is that the fixed point model is generically not the one of the Kondo model with one unpolarized lead but rather a spin-1/2 coupled to several leads with different spin vectors \( \tilde{d}_\alpha \). This is particularly important for the nonequilibrium case where the reservoirs cannot be taken together. Thus, an interesting fixed point emerges which, in the equilibrium case, leads to the usual Kondo physics, whereas, in the nonequilibrium regime, shows essentially different universal behavior compared to the Kondo model. We will characterize the universal features by calculating the magnetic field dependence of the dot magnetization and the charge current at zero temperature and large chemical potentials \( \mu_\alpha \) compared to the Kondo temperature \( T_K \) at and away from the fixed point. As a smoking gun to find the fixed point we find that the dot magnetization \( \bar{M} = \langle \hat{S} \rangle \) is minimal for all magnetic fields lying on a sphere defined by

\[ |\bar{\vec{h}} - \bar{\mu}| = |\bar{\mu}|, \quad \bar{\mu} = \sum_\alpha (\mu_\alpha - \bar{\mu}) \tilde{d}_\alpha, \quad (2) \]

where \( \bar{\mu} = \sum_\alpha x_\alpha \mu_\alpha \). We note that \( \bar{\vec{h}} \) denotes the total magnetic field including exchange fields. We choose units \( \hbar = e = 1 \).

Effective model. We start from a generalized Anderson impurity model, where the dot Hamiltonian is given by

\[ H = \sum_\sigma \epsilon_\sigma n_\sigma + U n_\uparrow n_\downarrow, \quad \text{where} \quad \epsilon_\sigma = \epsilon + \sigma \hbar/2 \]

are the single-particle energies and \( U \) denotes a strong Coulomb repulsion. The dot is coupled to the reservoirs by a generic tunneling matrix \( (t_{\nu,\sigma})_\alpha = t_{\nu,\sigma}^{\alpha} \), where \( \nu \) is a channel index labelling the reservoir bands with possibly different density of states (d.o.s.) \( \rho_\nu \) (in dimensionless units). The key observation is that the reservoirs enter only via the retarded self-energy, which is fully characterized by the hybridization matrix \( \Gamma_\alpha = 2\pi \delta_{\nu,\nu'} \rho_\nu \bar{t}_\nu \), with \( (\rho_\nu)_{\nu'\nu'} = \rho_\nu \delta_{\nu'\nu'} \). This means that all models with the same matrix \( \Gamma_\alpha \) give the same result for the dot density matrix and the charge current. Once \( \Gamma_\alpha \) is known, we can write it in various forms to obtain effective models. \( \Gamma_\alpha \) is a positive semidefinite Hermitian 2 × 2-matrix, i.e., it can be diagonalized by a unitary 2 × 2-matrix \( U_\alpha \), such that \( \Gamma_\alpha = U_\alpha \bar{\Gamma}_\alpha U_\alpha^\dagger \) with the diagonal matrix \( (\bar{\Gamma}_\alpha)_{\sigma\sigma'} = \delta_{\sigma\sigma'} \Gamma_{\alpha,\sigma} \). \( \Gamma_{\alpha,\sigma} \geq 0 \) are the positive eigenvalues which can be written as \( \Gamma_{\alpha,\sigma} = \Gamma_{\alpha,\sigma}^{1/2}(1 + \sigma p_\alpha) \), with \( \Gamma_{\alpha,\sigma} \geq 0 \) and \( 0 \leq p_\alpha \leq 1 \). Defining \( \Gamma_{\alpha,\sigma} = 2\pi \delta_{\nu,\nu'} \rho_\nu \bar{t}_\nu^2 \) and \( \Gamma_\alpha = 4\pi \delta_{\nu,\nu'} \rho_\nu \bar{t}_\nu^2 \), with \( \alpha,\nu,\nu' \geq 0 \), we can write \( \Gamma_\alpha \) in the two equivalent forms

\[ \Gamma_\alpha = 2\pi \delta_{\nu,\nu'} \rho_\nu \bar{t}_\nu^2 \quad \text{and} \quad \rho_\nu = U_\alpha (2\pi \delta_{\nu,\nu'} / \Gamma_\alpha) U_\alpha^\dagger, \quad (3) \]

\[ \Gamma_\alpha = 2\pi \delta_{\nu,\nu'} \rho_\nu \bar{t}_\nu^2 \quad \text{and} \quad (\rho_\nu)_{\sigma\sigma'} = t_{\sigma\sigma'} (U_\alpha^\dagger)_{\sigma\nu} (U_\alpha)_{\nu\sigma'}). \quad (4) \]

The first form is the one where the information is fully shifted to an effective d.o.s. \( \rho_\nu \) of the reservoirs with spin-conserving tunneling rates \( \Gamma_\alpha \). Using \( 2\pi \delta_{\nu,\nu'} / \Gamma_\alpha = 1 + p_\alpha \bar{t}_\nu^2 \) and \( U_\alpha = e^{i\pi \bar{t}_\nu^2 / 2} \) we find \( \rho_\nu = 1 + p_\alpha \bar{t}_\nu^2 \), where \( \bar{t}_\nu \) are the Pauli matrices and \( \bar{d}_\alpha = R(\varphi_\alpha) \varepsilon_\alpha \) is a unit vector obtained by rotating the \( z \)-axis with rotation axis \( \varphi_\alpha \).

As a result we find an effective model with ferromagnetic leads with pseudo-spin channels \( \alpha \equiv \uparrow, \downarrow \), spin orientation \( \tilde{d}_\alpha \) and spin polarization \( \rho_\alpha \), see Fig. 1. Alternatively, one can also shift the whole information into an effective tunneling matrix \( t_{\alpha,\sigma} \), as written in Eq. (4), which describes a model with an effective tunneling matrix and reservoirs without spin polarization. This will be the form we will use in the following.

Coulomb blockade regime. We now present a weak coupling RG analysis close to the particle-hole symmetric point in the Coulomb blockade regime, defined by \( D = \sqrt{\epsilon + U} = -e > \Lambda_c = \max \{|\mu_\alpha|\}; h \). Charge fluctuations are suppressed in this regime and, using a Schrieffer-Wolff transformation\(^2\), spin fluctuations are described by the effective interaction \( V_{\text{eff}} = \sum_{kk'} \bar{U}_{kk'} \bar{J}_{kk'} \bar{S} \) where \( \bar{S} \) denotes the dot spin and \( \bar{J} = 2t \bar{t}^\dagger / D \) is an effective exchange matrix. \( (\bar{u}_\alpha)_{\sigma\sigma'} = a_{\sigma\sigma'} \alpha \) is a vector containing all reservoir field operators and \( (\bar{U})_{\alpha,\sigma\sigma'} = (\bar{u}_\alpha)_{\sigma\sigma'} \) is a matrix containing all tunneling matrices. Via a standard poor man scaling RG analysis we integrate out all energy scales between \( D \) and \( \Lambda_c \). In this regime the chemical potentials \( \mu_\alpha \) do not enter and it is convenient to rotate all reservoirs such that only one reservoir couples effectively to the dot. This is achieved by the singular value decomposition \( \bar{t} = \sqrt{\bar{\gamma}} (\bar{W}^\dagger) \), where \( \bar{V} \) and \( \bar{W} \) are unitary transformations in reservoir and dot space, respectively, and \( (\bar{U})_{\alpha,\sigma\sigma'} = (\bar{u}_\alpha)_{\sigma\sigma'} \) is a matrix containing all tunneling matrices. From a standard poor man scaling RG analysis we integrate out all energy scales between \( D \) and \( \Lambda_c \). In this regime the chemical potentials \( \mu_\alpha \) do not enter and it is convenient to rotate all reservoirs such that only one reservoir couples effectively to the dot. This is achieved by the singular value decomposition \( \bar{t} = \sqrt{\bar{\gamma}} (\bar{V}^\dagger) \), where \( \bar{V} \) and \( \bar{W} \) are unitary transformations in reservoir and dot space, respectively, and \( (\bar{U})_{\alpha,\sigma\sigma'} = (\bar{u}_\alpha)_{\sigma\sigma'} \) is a matrix containing all tunneling matrices. For the RG we omit the unitary transformation \( \bar{V} \) such that only one effective reservoir couples to the dot via the tunneling matrix elements \( \lambda_\alpha \). This model has also been studied in Ref. 18 and leads to an effective 2 × 2 exchange coupling matrix \( \bar{J} = 2t \bar{t}^\dagger a / D \) which can be parametrized by two exchange couplings \( J_z = (\lambda_\alpha^2 + 2\lambda_\alpha^2) / D \) and \( J_{\perp} = 2\lambda_\alpha^2 \lambda_\alpha / D \).

We obtain the antiferromagnetic anisotropic Kondo model together with a potential scattering term from the anisotropy constant \( c \). The weak-coupling RG flow as function of the effective band width \( \Lambda \) leads to an increase of the exchange couplings towards the isotropic fixed point \( J_z = J_{\perp} \) with \( c \) and \( T_K = \Lambda [(J_z - c) / (J_z + c)]^{1/(4c)} \) being the invariants.
At each stage of the RG flow we can replace $D \to \Lambda$ and get the effective hybridization matrix $\sum_\alpha = 2\pi \lambda V_\alpha \Lambda$, where $\Lambda$ contains the renormalized exchange couplings $\sum_\alpha V_\alpha = \Lambda(\lambda \pm c)/2$. The matrices $V_\alpha$ do not flow under the RG and fulfill $\sum_\alpha V_\alpha V_\alpha^\dagger = 1$ since $V$ is unitary. This leads to $\sum_\alpha V_\alpha \Gamma_\alpha = 2\pi \lambda^2 \sum_\alpha$. Comparing this to the form $\sum_\alpha \Gamma_\alpha = \frac{\Gamma}{2}(1 + \vec{d}_s \vec{d}_s)$ from (3) we find $J_z = (2\pi \Lambda) \sum_\alpha$ and $d = |\vec{d}| = c/J_z$. We conclude that the system shows a tendency to minimize the vector $\vec{d}$ during the RG flow and, for $c < J_z$, we can set this vector to zero and obtain the central result (1). This is reached in the scaling limit, formally defined in terms of the initial parameters by $J_z^{(0)} \to 0$ and $D \to \infty$ such that the Kondo temperature $T_K$ and the ratio $J_z^{(0)} / J_z^{(0)}$ are kept fixed. At this isotropic fixed point, we get $\lambda_1 = \lambda_2 = \lambda$ and $\sum_\alpha = 2\pi \lambda^2 V_\alpha V_\alpha^\dagger$. Using the form (3) we find $V_\alpha V_\alpha^\dagger = x_\alpha \frac{1}{2} + \vec{d}_s \vec{d}_s$ providing a recipe to find the parameters $x_\alpha$ and $d_\alpha$ at the fixed point.

As already explained in the introduction, for reservoirs with different chemical potentials $\mu_\alpha$, the fixed point model gives rise to new interesting universal behavior compared to the Kondo model with unpolarized leads $\vec{d}_\alpha = 0$. The latter case is only the fixed point model when the initial spin vectors are all equal $d_\alpha^{(0)} = d^{(0)}$. Whereas a small deviation between the initial polarizations $p_\alpha$ will still end up in a fixed point with $p_\alpha \ll 1$, a small angle between the spin orientations leads to a rotation of the spin orientations but the polarizations remain finite. A special case are reservoirs with full spin polarization $p_\alpha^{(0)} = 1$ which remain fully spinpolarized during the whole RG flow. In conclusion we find that the Kondo model with unpolarized leads will almost never describe the correct universal behavior in nonequilibrium.

The characteristic features at and away from the fixed point can best be visualized by analyzing the stationary dot magnetization $\vec{M}$ and the charge current $I$ in the strong nonequilibrium regime $\Lambda_\alpha = \max(|\mu_\alpha|) \gg T_K$ as function of the magnetic field $h < \Lambda_\alpha$. For $h \gg \gamma \sim J_z^{(0)} \Lambda_\alpha$ (where $\gamma$ sets the scale of the rates) a standard golden rule theory is sufficient to calculate $\vec{M}$ and $I$ up to $O(1)$. In this regime $\vec{M}$ is either parallel or antiparallel to $\vec{h}$ (depending on the nonequilibrium occupations) and the magnetization perpendicular to the field is negligible of $O(J_z^{(0)} \Lambda_\alpha)$. For $h \lesssim \gamma$ quantum interference phenomena are very important and golden rule theory breaks down. A strong component of the magnetization perpendicular to the magnetic field of $O(1)$ is obtained and the nondiagonal matrix elements of the dot density matrix (accounting for a spin component perpendicular to the magnetic field) have to be taken into account. In the supplemental material we present the analytical results for all regimes which can be obtained from a systematic analysis of the effective dot Liouville operator up to $O(J_z^{(0)} \Lambda_\alpha)$. The full formulas are very involved but can be simplified in certain regimes. Here we summarize the most important nonequilibrium features.

**Dot magnetization in golden rule, arbitrary number of reservoirs at or away from the fixed point.** We first start with the regime $h \gg \gamma$ for an arbitrary number of reservoirs. The magnetization $\vec{M}$ in golden rule is zero if the rates between the two spin states are equal. This occurs for magnetic fields lying on the surface of an ellipsoid which can be fully characterized by the two vectors $\vec{d}$ and $\vec{\mu}$ defined in Eqs. (1) and (2), together with the factor $s = J_z/J_z^{(0)} = 1/(1 - d^2) \geq 1$ characterizing the distance to the isotropic fixed point $s = 1$. We find an ellipsoid which is rotationally invariant around $\vec{d}$ and stretched along $\vec{d}$ by the factor $s$

$$
(\vec{h}_\perp - \vec{\mu}_\perp)^2 + \left(\frac{h_\parallel - s^2 \mu_\parallel}{s}\right)^2 = \mu_\perp^2 + s^2 \mu_\parallel^2,
$$

where we have decomposed the two vectors $\vec{h}$ and $\vec{\mu}$ in two components parallel and perpendicular to $\vec{d}$. This result provides an experimental tool to measure the distance to the fixed point model via the stretching factor $s$ and sets a smoking gun for a characteristic universal feature of the fixed point $s = 1$, where the ellipsoid turns into the sphere. These features are essentially different from the Kondo model with unpolarized leads where $\vec{d} = \vec{\mu} = 0$ such that minimal magnetization in golden rule occurs only for $\vec{h} = 0$. We note that at the fixed point the center of the sphere is given by the vector $\vec{\mu}$, which is a characteristic vector determining the exchange field generated by the reservoirs given by $\vec{h}_{\text{exc}} = J(2\vec{\mu} - \vec{h}_{\text{ext}})$, where $\vec{h}_{\text{ext}}$ is the externally applied field (this can be obtained by a perturbative calculation similar to the one of Ref. 19). Outside (inside) the ellipsoid the magnetization is antiparallel (parallel) to $\vec{h}$ but the rotational symmetry around the vector $\vec{d}$ is no longer valid since all scalar products $\vec{d}, \vec{h}$ enter. Only in the special case of two reservoirs $\alpha = L, R$ at the fixed point $d_L = -d_R$ we obtain antiparallel spin orientations of the two reservoirs with rotational symmetry around the reservoir spin axis. The universal properties of this case are shown in Fig. 2 for the dot magnetization and in Fig. 3 for the charge current and will be discussed in more detail in the following including the quantum interference regime $h \ll \gamma$.

**Dot magnetization, 2 reservoirs at the fixed point.** For two reservoirs at the fixed point, we choose $d_L = -d_R$ in $z$-direction and characterize the coupling $J$ by the Korringa rate $\gamma = 4x_L x_R \pi J^2 V$, where $V = \mu_L - \mu_R$ is the bias voltage. From $\vec{\mu} = V \vec{d}$ and $|\vec{\mu}| = V x_L p_L$ the minimum of the magnetization in the golden rule regime $h \gg \gamma$ lies on a sphere centered around $h_L = x_L p_L V$, $h_\perp = 0$ with radius $x_L p_L V$. Since $2x_L p_L = 2x_L x_R (p_L + p_R) \leq (p_L + p_R)/2 \leq 1$, the sphere will always lie inside the region $h < V$. At $h = V$ we get $\vec{M} = -\vec{h}/(2V)$. These features follow from energy conservation and the fact that the majority spins in the left/right lead are $\uparrow / \downarrow$. For small $h_\perp$ the upper level of the dot consists mainly of the spin-$\uparrow$ state which will be occupied from the left lead...
FIG. 2. (Color online) The dot magnetization $M$ as function of $h_z$ and $h_\perp$ for $h < V$ with $x_L = x_R = \frac{1}{2}$, $p_L = p_R = \frac{3}{4}$, $J = \frac{1}{100\sqrt{\pi}}$ and $\gamma = 10^{-4} V$. The white line indicates $h_\perp^{\text{min}}(h_z)$ where $M$ is minimal. Inset: The same plot on logarithmic scale for $h_z > 0$.

but has a small probability to escape to the right one. Therefore the magnetization is parallel to the external field and quite large (but not maximal). Increasing $h_\perp$ will lead to transition rates between the upper and lower dot level until they are equal, which defines the minimum of the magnetization. For large $h_\perp \sim O(V)$ the energy phase space for the transition from the lower to the upper level becomes smaller leading to an increase of the population of the lower level. Thus, the magnetization becomes antiparallel to the magnetic field and the magnitude increases until $h = V$, where only the lower level is occupied and the magnetization becomes maximal. For $h_z < 0$ this mechanism does not occur since in this case the lower level will always have a higher occupation. For small magnetic fields $h \lesssim \gamma$ quantum interference processes become important and the minimum position of the magnetization saturates at $h_\perp^{\text{min}}(h_z) \sim O(J V)$, see the inset of Fig. 2. For $h_z \lesssim \gamma$ and $h_\perp \ll V$, the precise line shape follows from $M \approx \sqrt{\pi^2 J^2 x^2 + M_z^2(1 + x^2)}$ with

$$M_z \approx \frac{1}{2} \frac{p_L + p_R - 2\pi J^2 x^2 h_z/\gamma}{1 + p_L p_R + x^2}, \quad x = \frac{h_\perp}{\sqrt{h_z^2 + \gamma^2}}. \quad (6)$$

At $h = 0$ we obtain $M_0 = M_{h=0} = (1/2)(p_L + p_R)/(1 + p_L p_R)$ which, together with $x_L + x_R = 1$, $x L p_L = x R p_R$ and the value $x L p_L$ from the minimum magnetization, determines the four parameters $x L, p L, x R, p R$ of the fixed point model. The coupling $J$ is related to the Korringa rate which follows from the curvature of the magnetization as function of $h_z$ at the origin: $(\partial^2 M/\partial h_z^2)_{h=0} = -\gamma^{-2} M_0 (1 + p_L p_R)/(1 - p_L p_R)$. Furthermore, for vanishing $h_\perp$, the point $h_z = 0$ can be characterized by a jump of the derivative $(\partial M/\partial h_z)_{h=0}$ with a ratio given by the parameters $x L, p L, x R, p R$, see supplementary material.

**Charge current, 2 reservoirs at the fixed point.** The charge current $I$ in units of the Korringa rate is shown in Fig. 3. The current is related to the magnetization in a universal way by the formula

$$(I - I_0)/\gamma = \tilde{M} \hat{h}_\perp/V + (1 + p_L p_R)(M_z - M_0)(h_z/V - 2 M_0), \quad (7)$$

with $I_0/\gamma = I_{h=0}/\gamma = 1/2 + (1 + p_L p_R)(1 - 8 M_0^2)/4$. At fixed $h_z$ the current shows a maximum as function of $h_\perp$ at a value roughly of the same order where the magnetization is minimal. This is caused by enhanced inelastic processes increasing the current in this regime. However, since the current varies only slowly in a wide region around the maximum this is not useful to determine the model parameters. An exception is the axis $h_\perp = 0$, where the maximum current follows from the formula $I_{h=0}^{\text{max}}/\gamma = (3 + p_L p_R)/4$. Another point of interest is $\tilde{h} = V$ where the magnetization is maximal $M = -\hat{h}/(2V)$ (see above). At this point the upper dot level has no occupation and transport happens via elastic cotunneling processes through the lower one. From Eq. (7) we get $I_{\perp h=0}^{\text{max}}/\gamma = (1 - p_L p_R(2h_\perp^2/V^2 - 1))/4$. For $h_z = 0$, $h_\perp = V$ or $h_z = V$, $h_\perp = 0$ this gives $I/\gamma = (1 \pm p_L p_R)/4$. These two values are related to $I_{h=0}^{\text{max}}/\gamma$ in a universal way. Together with $I_0$ the parameters $p_L, p_R$ and $\gamma$ can be determined and $x L, x R$ follow from $x L + x R = 1$ and $x L p_L = x R p_R$. In the quantum interference regime of small magnetic fields the current is shown in the inset of Fig. 3. Analytically the features follow for $h_z \lesssim \gamma$ and $h_\perp \ll V$ from $(I - I_0)/\gamma \approx (p_L + p_R) M_0 x^2/(1 + p_L p_R + x^2)$.

**Conclusions.** We have shown that the Kondo model with unpolarized leads is generically not the appropriate model to describe the nonequilibrium properties of pseudo-spin-$\frac{1}{2}$ quantum dots in the Coulomb blockade regime. Noncollinear spin orientations in effective reservoirs give rise to characteristic features as function of an applied magnetic field in the strong nonequilibrium regime independent of the microscopic details of the model, even away from the fixed point. These features are experimentally accessible. For future research it is of high interest to characterize the universal properties of the model also in the strong coupling regime $V \sim T_K$ where more refined techniques have to be used$^{12-14}$.

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Supplemental material for:
The generic fixed point model for pseudo-spin-$1/2$ quantum dots in nonequilibrium:
Spin-valve systems with compensating spin polarizations

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Here we present the technical details of how to calculate the dot magnetization and the charge current in the stationary limit by a perturbative treatment in the renormalized exchange couplings. We use the Liouville operator method described in Ref. 1. We will show the general scheme for the model away from the scaling limit for an arbitrary number of reservoirs and analyze the magnetization in detail. Furthermore, we subsequently will evaluate the formulas for the special case of two reservoirs in the scaling limit.

General scheme.— After renormalization we have obtained the following effective hybridization matrix

$$\Gamma^\alpha = 2\pi \Lambda c V \sigma^\alpha \sigma^\alpha , \quad \Lambda^2 = \frac{1}{2} \Lambda_c (J^z \frac{1}{2} + c \sigma^z) ,$$

(1)

where $\Lambda_c$ is the low-energy cutoff, $c = \sqrt{J^2 - J_1^2}$ one invariant of the flow equations, and $J_{z,\perp}$ are the renormalized exchange couplings from the anisotropic Kondo model. The matrices $V^\alpha$ follow from the first two columns of the matrix $V$, which was defined by the singular value decomposition of the full tunneling matrix of the original model. Since $V$ is unitary we have the important property $\sum_{\alpha} V^\dagger_{\alpha} V_{\alpha} = \frac{1}{2}$. The hybridization matrix can be decomposed into the unity and Pauli matrices as

$$\Gamma^\alpha = \frac{1}{2} \Gamma_{\alpha} x^\alpha , \quad x^\alpha = x_\alpha \frac{1}{2} + \vec{d}_\alpha \vec{\sigma} , \quad \sum_{\alpha} x_\alpha = 1 , \quad |\vec{d}_\alpha| = x_\alpha p_\alpha , \quad 0 \leq x_\alpha, p_\alpha \leq 1 ,$$

(2)

where $x_\alpha$, $p_\alpha$, and $\vec{d}_\alpha$ describe the parameters of an effective model with ferromagnetic leads. Using $\sum_{\alpha} V^\dagger_{\alpha} V_{\alpha} = \frac{1}{2}$ we obtain the properties

$$\Gamma = 2\pi \Lambda_c J^z , \quad d = |\vec{d}| = \frac{c}{J^z} , \quad \vec{d} = \sum_{\alpha} \vec{d}_\alpha = \vec{d} \vec{e}_z .$$

(3)

This result shows that the $z$-axis has been chosen such that the vector $\vec{d}$ points in $z$-direction. This choice has been taken when omitting the unitary dot matrix $W$ in the singular value decomposition of the full tunneling matrix. Independent of the scale $\Lambda_c$ the matrix $x^\alpha$ containing the effective model parameters can be defined from the equation

$$x^\alpha = x_\alpha \frac{1}{2} + \vec{d}_\alpha \vec{\sigma} = (\frac{1}{2} + \frac{c}{J^z} \sigma^z)^{1/2} V^\dagger_{\alpha} \frac{1}{2} + \frac{c}{J^z} \sigma^z)^{1/2} V^\dagger_{\alpha} \frac{1}{2} + \frac{c}{J^z} \sigma^z)^{1/2} .$$

(4)

This provides an algorithm how to determine the parameters of the effective model from the knowledge of $V^\alpha$ and $J_{z,\perp}$. In the scaling limit, where $c/J^z \to 0$, the r.h.s. of this equation contains only $V^\dagger V^\alpha$.

From (2) we can easily define an effective tunneling matrix via

$$\Gamma^\alpha = 2\pi \frac{t^\dagger_{\alpha} t_{\alpha}}{x^\alpha} , \quad t^\alpha = \sqrt{\Gamma/(4\pi)} x^{1/2} = \sqrt{\Lambda c J^z/2} x^{1/2} .$$

(5)

From this tunneling matrix we can calculate the effective exchange coupling matrix describing the model in the Coulomb blockade regime

$$\vec{J}^\alpha_{\alpha'} = \frac{2}{\Lambda_c} t^\alpha \vec{\sigma} t^\dagger_{\alpha'} = J^z x^{1/2} \vec{\sigma} x^{1/2} .$$

(6)

This coupling matrix defines the input for the following perturbation theory to obtain the dot magnetization and the charge current in the stationary state.

Using the formalism of Ref. 1, the stationary dot density matrix $\rho$ and the stationary charge current $I^\gamma$ flowing from lead $\gamma$ into the quantum dot follow from

$$L \rho = 0 , \quad I^\gamma = -i \text{Tr} \Sigma^\gamma \rho .$$

(7)
where \( \text{Tr} \) denotes the trace over the two dot states, \( L \) is the effective Liouvillian and \( \Sigma^\gamma \) is the current kernel. \( L \) and \( \Sigma^\gamma \) are superoperators acting on usual operators, i.e. they are \( 4 \times 4 \)-matrices in the Liouville basis \( \uparrow \uparrow, \downarrow \downarrow, \uparrow \downarrow, \downarrow \uparrow \), whereas \( \rho \) is an ordinary operator or a 4-component vector in the Liouville basis. From the matrix elements of the density matrix the stationary dot magnetization \( \rho = \langle \vec{S} \rangle \) follows as

\[
M_x = \text{Re} \rho_{\uparrow \downarrow}, \quad M_y = \text{Im} \rho_{\uparrow \downarrow}, \quad M_z = \frac{1}{2} (\rho_{\uparrow \uparrow} - \rho_{\downarrow \downarrow}),
\]

with \( \rho_{\uparrow \uparrow} + \rho_{\downarrow \downarrow} = 1 \) and \( \rho_{\uparrow \downarrow} = \rho_{\downarrow \uparrow}^* \).

In second order perturbation theory in the exchange coupling matrix, the Liouvillian and the current kernel follow from

\[
L = \hat{G}_{\alpha \sigma, \alpha' \sigma'} \left\{ -i \frac{\pi}{2} (\mu_\alpha - \mu_{\alpha'} - L_0) \hat{G}_{\alpha' \sigma, \alpha \sigma} + H(\mu_\alpha - \mu_{\alpha'} - L_0) \hat{G}_{\alpha' \sigma, \alpha \sigma} \right\},
\]

\[
\Sigma^\gamma = \hat{I}^\gamma_{\alpha \sigma, \alpha' \sigma'} \left\{ -i \frac{\pi}{2} (\mu_\alpha - \mu_{\alpha'} - L_0) \hat{G}_{\alpha' \sigma, \alpha \sigma} + H(\mu_\alpha - \mu_{\alpha'} - L_0) \hat{G}_{\alpha' \sigma, \alpha \sigma} \right\},
\]

where we sum implicitly over all reservoir indizes \( \alpha, \alpha' \) and all spin indizes \( \sigma, \sigma' \), and we have defined the superoperator vertices

\[
\hat{G}_{\alpha \sigma, \alpha' \sigma'} = \hat{f}_{\alpha \sigma, \alpha' \sigma'} (\hat{L}_+ + \hat{L}_-) \quad \hat{G}_{\alpha \sigma, \alpha' \sigma'} = \hat{f}_{\alpha \sigma, \alpha' \sigma'} (\hat{L}_+ - \hat{L}_-) \quad \hat{I}^\gamma_{\alpha \sigma, \alpha' \sigma'} = \hat{c}_{\alpha \sigma, \alpha' \sigma'} \quad \hat{c}_{\alpha \sigma, \alpha' \sigma'} = -\frac{1}{2} (\delta_{\alpha \gamma} - \delta_{\alpha' \gamma})
\]

\( \hat{L}_\pm \) are two dot superoperators defined by the following action on ordinary dot operators \( A \)

\[
\hat{L}_+ A = \vec{S} A \quad \hat{L}_- A = -A \vec{S}
\]

and \( \vec{S} \) denotes the spin operator of the dot. \( L_0 \) and \( H(x) \) in Eqs. (9) and (10) are defined by

\[
L_0 = \vec{h} (\hat{L}_+ + \hat{L}_-), \quad H(x) = x \ln \frac{-ix}{\Lambda_c} = -i \frac{\pi}{2} |x| + x \ln \frac{x}{\Lambda_c} \approx -i \frac{\pi}{2} |x|
\]

The logarithmic real part of the function \( H(x) \) contributes only to a weak renormalization of the dot levels and changes the final results only in higher orders. Therefore it is omitted in the following. Inserting the various definitions into (9) and (10) we obtain

\[
L = -i \frac{\pi}{2} \left( \text{Tr}_\sigma \hat{f}_{\alpha \sigma, \alpha' \sigma'} \hat{f}_{\alpha' \sigma', \alpha \sigma} \right) \left( (\mu_\alpha - \mu_{\alpha'} - L_0) (\hat{L}_+^2 + \hat{L}_-^2) + |\mu_\alpha - \mu_{\alpha'} - L_0| (\hat{L}_+^2 + \hat{L}_-^2) \right),
\]

\[
\Sigma^\gamma = -i \frac{\pi}{2} \hat{c}_{\alpha \sigma, \alpha' \sigma'} \left( \text{Tr}_\sigma \hat{f}_{\alpha \sigma, \alpha' \sigma'} \hat{f}_{\alpha' \sigma', \alpha \sigma} \right) \left( (\mu_\alpha - \mu_{\alpha'} - L_0) (\hat{L}_+^2 - \hat{L}_-^2) + |\mu_\alpha - \mu_{\alpha'} - L_0| (\hat{L}_+^2 + \hat{L}_-^2) \right)
\]

where \( \text{Tr}_\sigma \) denotes the trace over the reservoir spins in the space of the matrices \( \hat{f}_{\alpha \sigma} \). This trace can be calculated explicitly by using the form (6) of the exchange coupling matrix with \( x_\alpha \) given by (4). After some algebra one obtains for fixed indizes \( \alpha, \alpha', i, j \)

\[
\text{Tr}_\sigma \hat{f}_{\alpha \sigma, \alpha' \sigma'} \hat{f}_{\alpha' \sigma', \alpha \sigma} = 2 J_z^2 \left\{ \delta_{ij} (x_\alpha x_{\alpha'} - \vec{a}_\alpha \cdot \vec{a}_{\alpha'}) + i (\vec{e}_i \wedge \vec{e}_j) (\vec{d}_\alpha \cdot \vec{d}_{\alpha'}) + (\vec{d}_{\alpha} \vec{d}_{\alpha'} + \vec{d}_{\alpha'} \vec{d}_{\alpha}) \right\}
\]

where \( \vec{e}_i \wedge \vec{e}_j \) denotes the vector product of the unit vectors \( \vec{e}_i \) in \( i \)-direction. This expression can be simplified when summing over \( \alpha \) or \( \alpha' \) by using the properties (3) but this is not always possible when inserting (17) in (15) and (16).

The formulas (15), (16) and (17) together with (7) conclude the general formalism how to obtain the dot magnetization and the charge current for the generic case. The only task which remains is to insert the \( 4 \times 4 \)-matrices \( \hat{L}_\pm \) and work out the algebra. This can be done either numerically or analytically. Here we will first present the analytical result for the general case and then discuss the special case of two reservoirs in the scaling limit in more detail.

**Arbitrary number of reservoirs away from the scaling limit.**— Starting from the rotational invariant equations (15), (16) and (17) we choose the basis such that the magnetic field is parallel or antiparallel to the \( z \)-direction \( h_z = \sigma h, h_x = h_y = 0 \) with \( \sigma = \pm \). In matrix form we represent the Liouvillian as

\[
L = \begin{pmatrix}
\uparrow \uparrow & 0 \\
0 & \downarrow \downarrow \\
\downarrow \uparrow & 0 \\
0 & \uparrow \downarrow
\end{pmatrix}
\]
After a lengthy algebra, we obtain

\[ L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \]

\[ A = -i\Gamma_1 (1 - \sigma_x) + i\Gamma_2 (\sigma_z + i\sigma_y), \]
\[ B = -\gamma_1 (1 - \sigma_x) + i\gamma_2 \sigma_z + i\gamma_3 (1 + \sigma_x), \]
\[ C = \gamma_1 (1 + \sigma_x) + i\gamma_2 (\sigma_z - i\sigma_y) + i\gamma_3 (1 + \sigma_x), \]
\[ D = \sigma h\sigma_z - i\gamma_5 (1 + i\gamma_6 \sigma_x + i\gamma_7 \sigma_y), \]

with the parameters

\[ \Gamma_1/ (\pi J_z^2) = \frac{1}{2} (x_a x_{\alpha'} - d_{\alpha}^2 d_{\alpha'}) (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h|) - \sigma d_{\alpha} x_{\alpha'} (|\mu_{\alpha\alpha'} + h| - |\mu_{\alpha\alpha'} - h|), \]
\[ \Gamma_2/ (\pi J_z^2) = 2d_{\alpha} x_{\alpha'} \mu_{\alpha\alpha'} - \sigma h (1 - (d^2)^2), \]
\[ \gamma_1/ (\pi J_z^2) = \frac{\sigma}{2} d_{\alpha} x_{\alpha'} (|\mu_{\alpha\alpha'} + h| - |\mu_{\alpha\alpha'} - h|) + d_{\alpha}^2 d_{\alpha'}^2 (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h|), \]
\[ \gamma_2/ (\pi J_z^2) = \frac{\sigma}{2} d_{\alpha} x_{\alpha'} (|\mu_{\alpha\alpha'} + h| - |\mu_{\alpha\alpha'} - h|) + d_{\alpha}^2 d_{\alpha'}^2 (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h|), \]
\[ \gamma_3/ (\pi J_z^2) = 2d_{\alpha} x_{\alpha'} \mu_{\alpha\alpha'} + \sigma d_{\alpha} d_{\alpha'} h, \]
\[ \gamma_4/ (\pi J_z^2) = 2d_{\alpha} x_{\alpha'} \mu_{\alpha\alpha'} + \sigma d_{\alpha} d_{\alpha'} h, \]
\[ \gamma_5/ (\pi J_z^2) = (x_a x_{\alpha'} - d_{\alpha}^2 d_{\alpha'}) |\mu_{\alpha\alpha'}| + \frac{1}{2} [x_a x_{\alpha'} - (d_{\alpha}^2 d_{\alpha'} + d_{\alpha}^2 d_{\alpha'}) (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h|)], \]
\[ \gamma_6/ (\pi J_z^2) = (d_{\alpha}^2 d_{\alpha'}^2 - d_{\alpha}^2 d_{\alpha'}^2) |\mu_{\alpha\alpha'}|, \]
\[ \gamma_7/ (\pi J_z^2) = 2d_{\alpha} d_{\alpha'} |\mu_{\alpha\alpha'}|. \]

Considering the eigenvector with eigenvalue zero of the Liouvillian and using the equations (8) the magnetization reads

\[ M_z = \frac{\Gamma_2 \left( \frac{g_z + g_\alpha - g_\beta}{2} \right) + 2 [-\gamma_2 \gamma_3 (\gamma_5 - \gamma_6) + \gamma_2 \gamma_4 (\gamma_7 - \gamma h) - \gamma_1 \gamma_4 (\gamma_5 + \gamma_6) + \gamma_1 \gamma_3 (\gamma_7 + \gamma h)]}{2 \Gamma_1 \left( \frac{g_z^2 + g_\alpha^2 - g_\beta^2}{2} \right) + 4 \gamma_2 \gamma_5 (\gamma_5 - \gamma_6) - 2 \gamma_1 \gamma_2 \gamma_7 + \gamma_4^2 (\gamma_5 + \gamma_6)} \]
\[ M_x = \frac{\gamma_3 (\gamma_5 - \gamma_6) - \gamma_4 (\gamma_7 - \gamma h) + 2 \gamma_2 (\gamma_5 - \gamma_6) - \gamma_1 (\gamma_7 - \gamma h)}{h^2 + \gamma_5^2 - \gamma_6^2 - \gamma_7^2} \]
\[ M_y = \frac{-\gamma_4 (\gamma_5 + \gamma_6) + \gamma_3 (\gamma_7 + \gamma h) + 2 \gamma_2 (\gamma_5 - \gamma_6) + \gamma_2 (\gamma_7 + \gamma h)}{h^2 + \gamma_5^2 - \gamma_6^2 - \gamma_7^2} \]

and the current

\[ J^\gamma = -\pi c_{\alpha\alpha'} J_z^2 \left\{ \left( 3 x_{\alpha} x_{\alpha'} - d_{\alpha} d_{\alpha'} \right) \mu_{\alpha\alpha'} - 4 d_{\alpha}^2 x_{\alpha'} \sigma h \\
+ 2 \sigma M_z \left( x_{\alpha} x_{\alpha'} - d_{\alpha} d_{\alpha'} \right) (|\mu_{\alpha\alpha'} + h| - |\mu_{\alpha\alpha'} - h|) - 4 M_z d_{\alpha}^2 x_{\alpha'} (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h|) \\
- 2 \sigma M_x d_{\alpha} d_{\alpha'} (|\mu_{\alpha\alpha'} + h| - |\mu_{\alpha\alpha'} - h|) - 2 M_x d_{\alpha}^2 x_{\alpha'} (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h|) \\
+ 2 \sigma M_y d_{\alpha} d_{\alpha'} (|\mu_{\alpha\alpha'} + h| - |\mu_{\alpha\alpha'} - h|) + 2 M_y d_{\alpha}^2 x_{\alpha'} (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h| + |\mu_{\alpha\alpha'}|) \right\} \]

In the regime \(|h_z| \gg \gamma \sim J_z^2 \Lambda_c\) these results agree with the ones of golden rule. Going back to the rotational invariant case, so no specification of the direction of the magnetic field is assumed, the results are

\[ \dot{\vec{M}} = \frac{-h \left( 1 - (\vec{d} \cdot \vec{\phi})^2 \right) + 2 \vec{\mu} \cdot \vec{\phi}}{\left( x_{\alpha} x_{\alpha'} - d_{\alpha} d_{\alpha'} \right) (|\mu_{\alpha\alpha'} + h| + |\mu_{\alpha\alpha'} - h|) - 2 d_{\alpha} \cdot \vec{\phi} x_{\alpha'} (|\mu_{\alpha\alpha'} + h| - |\mu_{\alpha\alpha'} - h|)} \]

with \(\vec{\phi} = \vec{h}/\hbar\). Choosing the basis such that \(\vec{d} = d \hat{z}\) with \(d = |\vec{d}|\) and using the decomposition \(\vec{\mu} = \sum_{i=x,y,z} \mu_i \vec{e}_i\) one can easily verify that the root of the magnetization is given by an ellipsoid equation

\[ (h_z - \mu_z)^2 + (h_y - \mu_y)^2 + \left( \frac{h_z - s^2 \mu_z}{s} \right)^2 = \mu_x^2 + \mu_y^2 + s^2 \mu_z^2, \]
where \( s = 1/\sqrt{1 - d^2} \) is the stretching factor of the ellipsoid in the direction of \( \vec{d} \), which with the help of Eq. (3) can be written as

\[
s = \frac{1}{\sqrt{1 - d^2}} = \frac{1}{\sqrt{1 - c^2/J_z^2}} = \frac{J_z}{J_{\perp}} \geq 1.
\]

In the scaling limit \( d \to 0 \) (or \( J_z = J_{\perp} \)) we get \( s = 1 \) and the ellipsoid becomes a sphere defined by the equation

\[
|\vec{h} - \vec{\mu}| = |\vec{\mu}|
\]

This result is the “smoking gun” to determine the distance to the scaling limit.

**Two reservoirs in the scaling limit.**— For two reservoirs in the scaling limit we consider \( J_z = J, c = 0, \vec{d}_L = -\vec{d}_R = \vec{e}_z, \) and \( x_{LP}L = x_{RP}R. \) Choosing the magnetic field as \( \vec{h} = (h_x, 0, h_z) \) with \( h = |\vec{h}| = \sqrt{h_x^2 + h_z^2} < V, \) we obtain the following result for the Liouvillian

\[
L = \left( \begin{array}{c}
-i \Gamma_1' (\mathbb{1} - \sigma^x) + i \Gamma_2' (\sigma^z + i \sigma^y) \\
-\frac{1}{2} h_x (\mathbb{1} - \sigma^x) + i \gamma_2' (\sigma^z - i \sigma^y) + i \gamma_3' (\mathbb{1} + \sigma^z)
\end{array} \right)
\]

with

\[
\begin{align*}
\Gamma_1'/\langle \pi J^2 \rangle &= 2x_Lx_R(1 + p_{LP}R)V + \frac{1}{2} \sum_{\alpha} x_{\alpha}^2 (1 - p_{\alpha}^2) (h + h_z^2/h) - 2x_Lx_R(p_L + p_R)h_z, \\
\Gamma_2'/\langle \pi J^2 \rangle &= 2x_Lx_R(p_L + p_R)V - h_z, \\
\gamma_2'/\langle \pi J^2 \rangle &= x_Lx_R(p_L + p_R)h_x - \frac{1}{2} \sum_{\alpha} x_{\alpha}^2 (1 - p_{\alpha}^2) h_x h_z/h, \\
\gamma_3'/\langle \pi J^2 \rangle &= -h_x, \\
\gamma_5'/\langle \pi J^2 \rangle &= 4x_Lx_RV - \frac{1}{2} \sum_{\alpha} x_{\alpha}^2 (1 - 3p_{\alpha}^2) h_z^2/h + \frac{1}{2} \sum_{\alpha} x_{\alpha}^2 (3 - p_{\alpha}^2) h_z, \\
\gamma_6'/\langle \pi J^2 \rangle &= -\frac{1}{2} \sum_{\alpha} x_{\alpha}^2 (1 + p_{\alpha}^2) h_z^2/h.
\end{align*}
\]

Determining the eigenvector with eigenvalue zero of the Liouvillian gives the following result for the magnetization

\[
M_x = \frac{1}{h_z^2 + (\gamma_5')^2 - (\gamma_6')^2} \{ h_x h_z M_z + 2(\gamma_5' + \gamma_6') \gamma_2' M_x + (\gamma_5' + \gamma_6') \gamma_3' \},
\]

\[
M_y = \frac{1}{h_z^2 + (\gamma_5')^2 - (\gamma_6')^2} \{ (\gamma_5' - \gamma_6') h_x M_z - 2\gamma_2' h_z M_z - \gamma_3' h_z \},
\]

\[
M_z = \frac{1}{2} \left[ \Gamma_1'' (h_x^2 + (\gamma_5')^2 - (\gamma_6')^2) - 4\gamma_2' h_x h_z + (4\gamma_5' - 2\gamma_6') h_z^2 - 2(\gamma_5' - \gamma_6') h_z^2 - 4(\gamma_5' + \gamma_6')(\gamma_7')^2 \right].
\]

For the current we get the result

\[
I/\gamma = \frac{1}{4} (3 + p_{LP}R) - (p_L + p_R) M_x + M_x h_x/V - \frac{1}{2} (p_L + p_R) h_z/V + (1 + p_{LP}R) M_z h_z/V
= I_0/\gamma + \tilde{M}_z \tilde{h}_z/V + (1 + p_{LP}R) (M_z - M_0) (h_z/V - 2M_0),
\]

where \( \tilde{M}_z \) and \( \tilde{h}_z \) are the components perpendicular to the z-axis, \( \gamma = 4x_Lx_R\pi J^2 V \) is the Korringa rate, and

\[
M_0 = M_{h=0} = \frac{1}{2} \frac{p_L + p_R}{1 + p_{LP}R}, \quad I_0/\gamma = I_{h=0}/\gamma = \frac{1}{4} (3 + p_{LP}R) - (p_L + p_R) M_0.
\]

These complicated formulas can be simplified in certain regimes. For \( h_z \gg \gamma \) or \( h_x/V \sim O(1) \) or \( h_x \ll \gamma \) the magnetization is parallel or antiparallel to the magnetic field and agrees with the golden rule result

\[
\tilde{M} \approx \tilde{h} \frac{1}{2} \frac{2x_Lx_R(p_L + p_R)V h_z - h^2}{2x_Lx_R(V(h^2 + p_{LP}R h_z^2)) + h^3 - 2x_Lx_R(h + p_L h_z)(h + p_R h_z)}.
\]
This shows that the magnetization has a minimum for $2x_L x_R (p_L + p_R) V h_z = 2x_L p_L V h_z = h_z^2 + (h_x^\text{min}(h_z))^2$, which defines a circle of radius $x_L p_L V$ centered at $h_z = x_L p_L V$, $h_z = 0$ for $h_x^\text{min}(h_z)$. For the special case $h_x = 0$ we obtain the result $M_x = M_y = 0$ and

$$M_z = \frac{1}{2} \frac{2x_L x_R (p_L + p_R) V - h \text{sign}(h_z)}{h + 2x_L x_R (1 + p_L p_R) (V - h) - 2x_L x_R (p_L + p_R) h \text{sign}(h_z)}.$$  

(32)

This gives rise to a characteristic jump of the first derivative of the magnetization at $h_z = 0$ with the ratio

$$\frac{\partial M}{\partial h_z} \bigg|_{h_z=0^+, h_x=0^+} = \frac{(1 + p_L)(1 + p_R)}{(1 - p_L)(1 - p_R)} \cdot \frac{1 - 2x_L x_R (p_L + p_R)}{1 + 2x_L x_R (p_L + p_R)}.$$  

(33)

For small magnetic fields $h_z \lesssim \gamma$ and $J^2 V \lesssim h_x \lesssim JV$ golden rule theory breaks down and a finite component $M_y$ appears. We obtain for $h_z \sim \gamma$ and $h_x \ll V$

$$M_x \approx \frac{1}{2} \frac{h_x h_z}{h_z^2 + \gamma^2} \frac{p_L + p_R - \frac{h_z^2}{2x_L x_R h_z V}}{1 + p_L p_R + x^2}, \quad M_y \approx \frac{1}{2} \frac{\gamma h_z}{h_z^2 + \gamma^2} \frac{p_L + p_R}{1 + p_L p_R + x^2}, \quad M_z \approx \frac{1}{2} \frac{p_L + p_R - \frac{h_x^2}{2x_L x_R V}}{1 + p_L p_R + x^2},$$

(34)

where $x = h_x / \sqrt{h_z^2 + \gamma^2}$. The same formulas can also be used for $h_x \ll \gamma$ and $h_x \ll V$, except for the case $h_z \ll \gamma$ and $h_x \lesssim \gamma$, where $M_x$ has to be changed to

$$M_x \approx \frac{1}{4x_L x_R V} \left[ -1 + \frac{x_L x_R (p_L + p_R)^2}{1 + p_L p_R + x^2} \right].$$  

(35)

However, since $M_x \lesssim O(J^2)$ in this regime, it can be neglected. Therefore, we get in all cases for $h_z \lesssim \gamma$ and $h_x \ll V$ for the total magnetization

$$M \approx |\vec{M}| \approx \sqrt{\pi^2 J^4 x^2 + M_z^2 (1 + x^2)}, \quad M_z \approx \frac{1}{2} \frac{p_L + p_R - \frac{h_x^2}{2x_L x_R V}}{1 + p_L p_R + x^2}.$$  

(36)

The current in the golden rule regime can easily be evaluated by inserting (31) in (29). For small magnetic fields $h_z \lesssim \gamma$ and $h_x \ll V$, we can neglect all terms in the current formula (29) which are proportional to the magnetic fields yielding $(I - I_0) / \gamma \approx -2M_0 (1 + p_L p_R) (M_z - M_0) = -(p_L + p_R) (M_z - M_0)$. Inserting $M_z$ from (34) one can see that the term $\sim h_x x^2 / V$ in the numerator can be neglected since it plays only an important role for $x^2 \sim V/h_z \gtrsim 1/J^2$ where $M_z \sim O(J^2)$ can anyhow be neglected. Therefore we obtain for all $h_z \lesssim \gamma$ and $h_x \ll V$ the result

$$(I - I_0) / \gamma \approx -\frac{1}{2} (p_L + p_R)^2 \left( \frac{1}{1 + p_L p_R + x^2} - \frac{1}{1 + p_L p_R} \right) = \frac{(p_L + p_R) M_0}{1 + p_L p_R + x^2} \cdot$$  

(37)

\[1\] H. Schoeller, Eur. Phys. J. Special Topics 168, 179 (2009).