VISCO-ENERGETIC SOLUTIONS TO SOME RATE-INDEPENDENT SYSTEMS IN
DAMAGE, DELAMINATION, AND PLASTICITY

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Abstract. This paper revolves around a newly introduced weak solvability concept for rate-independent systems, alternative to the notions of Energetic (E) and Balanced Viscosity (BV) solutions. Visco-Energetic (VE) solutions have been recently obtained by passing to the time-continuous limit in a time-incremental scheme, akin to that for Energetic solutions, but perturbed by a ‘viscous’ correction term, as in the case of Balanced Viscosity solutions. However, for Visco-Energetic solutions this viscous correction is tuned by a fixed parameter. The resulting solution notion turns out to describe a kind of evolution in between Energetic and Balanced Viscosity evolution.

In this paper we aim to investigate the application of VE solutions to the paradigmatic example of perfect plasticity, and to nonsmooth rate-independent processes in solid mechanics such as damage and plasticity at finite strains. With the limit passage from adhesive contact to brittle delamination, we also provide a first result of Evolutionary Gamma-convergence for VE solutions. The analysis of these applications reveals the wide applicability of this solution concept and confirms its intermediate character.

Keywords: Rate-independent systems, Visco-Energetic solutions, damage, delamination, perfect plasticity, finite-strain plasticity.

1. Introduction

In this paper we explore the application of the newly introduced concept of Visco-Energetic solution to a rate-independent process. We address rate-independent systems in solid mechanics that can be described in terms of two variables \((u, z) \in U \times Z\). Typically, \(u\) is the displacement, or the deformation of the body, whereas \(z\) is an internal variable specific of the phenomenon under investigation, in accordance with the theory of generalized standard materials by HALPHEN & NGUYEN [HN75], cf. also the modeling approach by M. Frémond [Fre02]. In the class of systems we consider here, \(u\) is governed by a static balance law (usually the Euler-Lagrange equation for the minimization of the elastic energy), whereas \(z\) evolves rate-independently. Indeed, when the ambient spaces \(U\) and \(Z\) have a Banach structure, the equations of interest

\[
\begin{align*}
D_u \mathcal{E}(t, u(t), z(t)) &= 0 \quad \text{in } U^*, t \in (0, T), \\
\partial \mathcal{R}(\dot{z}(t)) + D_z \mathcal{E}(t, u(t), z(t)) &\ni 0 \quad \text{in } Z^*, t \in (0, T),
\end{align*}
\]

(1.1a) (1.1b)

feature the derivatives w.r.t. \(u\) and \(z\) of the driving energy functional \(\mathcal{E} : [0, T] \times U \times Z \to (-\infty, \infty]\), and the (convex analysis) subdifferential \(\partial \mathcal{R} : Z \rightrightarrows Z^*\) of a convex, 1-positively homogeneous dissipation potential \(\mathcal{R} : Z \to [0, \infty]\). System (1.1a) reflects the ansatz that energy is dissipated through changes of the internal variable \(z\) only: in particular, the doubly nonlinear evolution inclusion (1.1b) balances the dissipative frictional forces from \(\partial \mathcal{R}(\dot{z})\) with the restoring force \(D_z \mathcal{E}(t, u, z)\).

System (1.1a) is most often only formally written: the very first issue attached to its analysis is the quest of a proper weak solvability notion. In fact, the energy \(\mathcal{E}(t, \cdot, \cdot)\) can be nonsmooth, e.g. incorporating indicator terms to ensure suitable physical constraints on the variables \(u\) and \(z\). However, it is rate-independence that poses the most significant challenges. Since the dissipation potential \(\mathcal{R}\) has linear growth at infinity, one can in general expect only BV-time regularity for \(z\). Thus \(z\) may have jumps as a function of time and the pointwise derivative \(\dot{z}\) in the subdifferential inclusion (1.1b) need not be defined. This has motivated the development of

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various weak solution concepts for system (1.1), suited to the poor time regularity of $z$ and, at the same time, also able to properly capture the behavior of the system at jumps. The latest of these notions, Visco-Energetic solutions, is the focus of this paper. Before illustrating it, let us briefly review the two other notions of Energetic and Balanced Viscosity solutions, with which we shall often compare Visco-Energetic solutions. We refer to [Mic11, MR15] for a thorough survey of all the other weak solvability concepts advanced for rate-independent systems.

From now on, we will leave the Banach setting and simply assume that
- The state spaces $U$ and $Z$ are endowed with two topologies $\sigma_U$ and $\sigma_Z$;
- Dissipative mechanisms are mathematically modeled in terms of a dissipation distance $d_Z$ on $Z$ (in fact, throughout the paper extended, asymmetric quasi-distances will be considered, cf. the general setup introduced in Sec. 2);
- The driving energy $\mathcal{E}(t, \cdot)$ is a $(\sigma_U \times \sigma_Z)$-lower semicontinuous functional.

Henceforth, we will often write $X$ in place of $U \times Z$ and refer to the triple $(X, \mathcal{E}, d_Z)$ as a rate-independent system. On the one hand, this generalized setup comprises the Banach one of (1.1), where $d_Z(z, z') = \mathcal{R}(z' - z)$. On the other hand, working in a metric-topological setting is natural in view of the application to, e.g., fracture, where the state space for the crack variable only has a topological structure, or finite-strain plasticity, where dissipation is described in terms of a Finsler-type distance reflecting the geometric nonlinearities of the model.

1.1. Energetic, Balanced Viscosity, and Visco-Energetic solutions at a glance. Energetic (often abbreviated as $E$) solutions were advanced in [MT99, MT04], cf. also the parallel notion of ‘quasistatic evolution’ in the realm of crack propagation, dating back to [DMT02]. In the context of the rate-independent system $(X, \mathcal{E}, d_Z)$, they can be constructed by recursively solving the time-incremental minimization scheme

$$ (u^n_\tau, z^n_\tau) \in \text{Argmin}_{(u,z) \in X} \left( d_Z(z^{n-1}_\tau, z) + \mathcal{E}(t^n_\tau, u, z) \right), \quad n = 1, \ldots, N_\tau, $$

where $\{t^n_\tau\}_{n=0}^{N_\tau}$ is a partition of $[0, T]$ with fineness $\tau = \max_{n=1,\ldots,N_\tau}(t^n_\tau - t^{n-1}_\tau)$. Under suitable conditions on $\mathcal{E}$, the piecewise constant interpolants $(\tilde{Z}_\tau)_\tau$ of the discrete solutions $(z^n_\tau)_{n=1}^{N_\tau}$ converge as $\tau \downarrow 0$ to an $E$ solution of the rate-independent system $(X, \mathcal{E}, d_Z)$, namely a curve $z \in BV_{d_Z}([0, T]; Z)$, together with

$$ u : [0, T] \to U, \text{ an (everywhere defined) measurable selection } u(t) \in \text{Argmin}_{u \in U} \mathcal{E}(t, u, z(t)), $$

fulfilling
- the global stability condition
  $$ \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u', z') + d_Z(z(t), z') \quad \text{for all } (u', z') \in U \times Z \text{ and all } t \in [0, T] ; $$
- the ‘$E$ energy-dissipation’ balance for all $t \in [0, T]$
  $$ \mathcal{E}(t, u(t), z(t)) + \text{Var}_d(z, [0, t]) = \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_z \mathcal{E}(s, u(s), z(s)) \text{ds}. $$

Due to its flexibility, the Energetic concept has been successfully applied to a wide scope of problems, see e.g. [MR15] for a survey. However, it has been observed that, because of compliance with the global stability condition (S), $E$ solutions driven by nonconvex energy functionals may have to jump ‘too early’ and ‘too long’, c.f., e.g., their characterization for 1-dimensional rate-independent systems obtained in [RS13]. This fact has motivated the introduction of an alternative weak solvability concept, pioneered in [EM06]. The global character of (S) in fact stems from the global minimization problem (IM$_E$), whereas a scheme based on local minimization would be preferable. This localization can be achieved by perturbing (IM$_E$) by a term that penalizes the squared distance from the previous step $z^{n-1}_\tau$, namely

$$ (u^n_\tau, z^n_\tau) \in \text{Argmin}_{(u,z) \in X} \left( d_Z(z^{n-1}_\tau, z) + \frac{\varepsilon}{2\tau} d^2_Z(z^{n-1}_\tau, z) + \mathcal{E}(t^n_\tau, u, z) \right), \quad \text{for } n = 1, \ldots, N_\tau. $$

Here, the viscous correction $\frac{\varepsilon}{2\tau} d^2_Z(z^{n-1}_\tau, z)$, with $d_Z$ a second, possibly different distance on $Z$, is modulated by a parameter $\varepsilon$, vanishing to zero with $\tau$ in such a way that $\frac{\varepsilon}{\tau} \uparrow \infty$. Under appropriate conditions on $\mathcal{E}$ (cf. [MR11, MR16]), the approximate solutions $(\tilde{Z}_\tau)_\tau$ originating from (IM$_{BV}$) converge as $\tau \downarrow 0$ to a Balanced
Viscosity BV solution of the rate-independent system \((X, \mathcal{E}, d_Z)\), namely a curve \(z \in BV_{d_Z}([0, T]; Z)\), with \(u : [0, T] \to U\) as in \((\mathbb{L}_2)\), fulfilling

- the local stability condition

\[ |D_z \mathcal{E}(t, u(t), z(t))| \leq 1 \quad \text{for every } t \in [0, T] \setminus J_z, \quad (SBV) \]

where \(|D_z \mathcal{E}|\) is the metric slope of \(\mathcal{E}\) w.r.t. \(z\), i.e. \(\overline{|D_z \mathcal{E}|(t, u, z)} := \limsup_{w \to z} \frac{(\mathcal{E}(t, u, z) - \mathcal{E}(t, u, w))^+}{d_Z(z, w)}\), and \(J_z\) the set of jump points of \(z\);

- the ‘BV energy-dissipation’ balance for all \(t \in [0, T]\)

\[
\mathcal{E}(t, u(t), z(t)) + \text{Var}_{d_Z, \nu}(z, [0, t]) = \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, ds. \quad (EBV) 
\]

In \((EBV)\) \(\text{Var}_{d_Z, \nu}\) is an augmented notion of total variation, fulfilling \(\text{Var}_{d_Z, \nu} \geq \text{Var}_{d_Z}\) and measuring the energy dissipated at a jump point \(t \in J_u\) in terms of a Finsler-type cost \(\nu(t, \cdot, \cdot)\). Without entering into details, we mention here that \(\nu(t, \cdot, \cdot)\) records the possible onset of viscosity, hence of rate-dependence, into the description of the system behavior at the jump point \(t\), cf. also \([MRS16]\) for more details. Because of the local character of the stability condition \((\mathbb{L}_2)\), BV solutions driven by nonconvex energies have mechanically feasible jumps, as shown by their characterization in \([RS13]\). Nonetheless, a crucial requirement underlying the Balanced Viscosity concept is that the energy \(\mathcal{E}\) complies with a chain-rule type condition. This is ultimately related to convexity/regularity properties of \(\mathcal{E}\) and unavoidably restricts the range of applicability of BV solutions.

That is why, Visco-Energetic (VE) solutions have recently been advanced in \([MJS18]\) as yet another solvability concept for the rate-independent system \((X, \mathcal{E}, d_Z)\). The key idea at the core of this novel notion is to broaden the class of admissible viscous corrections of the original time-incremental scheme \((\mathbb{IM}_E)\). The quadratic perturbation \(\frac{\mu}{2} d_Z^2(z_{\sigma}^{-1}, z)\) in scheme \((\mathbb{IM}_{BV})\) is in fact replaced by the term \(\delta_Z(z_{\sigma}^{-1}, z)\), with \(\delta_Z : Z \times Z \to [0, \infty]\) a general lower semicontinuous functional. This turns \((\mathbb{IM}_E)\) into

\[
(u_n, z_n^+) \in \text{Argmin}_{(u, z) \in X} \left( \mathcal{E}(t_n^+, u, z) + d_Z(z_n^{-1}, z) + \delta_Z(z_n^{-1}, z) \right), \quad n = 1, \ldots, N. \quad (IM_{VE})
\]

For simplicity, we shall confine the exposition in this Introduction to the simpler, but still significant, case in which \(\delta_Z(z, z') = \frac{\mu}{2} d_Z^2(z, z')\) with \(\mu > 0\) a fixed parameter and \(d_Z\) a (possibly different) distance on \(Z\), postponing the discussion of the general case to Sec. 2. This choice gives rise to the time-incremental minimization scheme

\[
(u_n, z_n^+) \in \text{Argmin}_{(u, z) \in X} \left( \mathcal{E}(t_n^+, u, z) + d_Z(z_n^{-1}, z) + \frac{\mu}{2} d_Z^2(z_n^{-1}, z) \right), \quad n = 1, \ldots, N, \quad \mu > 0 \text{ fixed}. \quad (1.3)
\]

In \([MJS16]\) Thm. 3.9) it has been shown that, under suitable conditions (cf. Sec. 2 ahead), the discrete solutions \((Z^\tau)\) of \((IM_{VE})\) converge, as \(\tau \downarrow 0\), to a VE solution of \((X, \mathcal{E}, d_Z)\), i.e. a curve \(z \in BV_{d_Z}([0, T]; Z)\), together with \(u : [0, T] \to U\) as in \((\mathbb{L}_2)\), fulfilling

- the viscously perturbed stability condition

\[
\mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u', z') + d_Z(z, z') + \frac{\mu}{2} d_Z^2(z(t), z') \quad \text{for all } (u', z') \in U \times Z \text{ and all } t \in [0, T] \setminus J_z; \quad (SV_E) \]

- the ‘VE-energy-dissipation’ balance for all \(t \in [0, T]\)

\[
\mathcal{E}(t, u(t)) + \text{Var}_{d_Z, \varphi}(u, [0, t]) = \mathcal{E}(0, u(0)) + \int_0^t \partial_t \mathcal{E}(s, u(s)) \, ds. \quad (VE_E) 
\]

In \((VE_E)\), dissipation of energy is described by the total variation functional \(\text{Var}_{d_Z, \varphi}\), which differs from the ‘BV total variation’ \(\text{Var}_{d_Z, \nu}\) in the contributions at jump points. In the VE-concept, the energy dissipated at jumps is in fact ‘measured’ in terms of a new cost function \(\varphi\), obtained by minimizing a suitable transition cost along curves connecting the two end-point \(z(t^-)\) and \(z(t^+)\) of the curve \(z\) at \(t \in J_z\), namely

\[
c(t, z(t^-), z(t^+)) := \inf \left\{ \text{Trc}_{VE}(t; \vartheta, E) : E \in \mathbb{R}, \vartheta \in C_{\sigma, d_Z}(E; Z), \vartheta(\inf E) = z(t^-), \vartheta(\sup E) = z(t^+) \right\}. \quad (1.4)
\]
The transition cost
\[ \text{Trc}_{VE}(t; \vartheta, E) := \text{Var}_{d_Z}(\vartheta, E) + \text{GapVar}_{\delta_Z}(\vartheta, E) + \sum_{s \in E \setminus \{\sup E\}} \mathcal{R}(t, \vartheta(s)) \]
features (i) the $d_Z$-total variation of the curve $\vartheta$; (ii) a quantity related to the ‘gaps’, or ‘holes’, of the set $E$ (which is just an arbitrary compact subset of $\mathbb{R}$ and may have a more complicated structure than an interval); (iii) the residual function $\mathcal{R} : [0, T] \times Z \to [0, \infty)$ (defined in (2.18) ahead), which records the violation of the VE-stability condition, as it fulfills
\[ \mathcal{R}(t, z) > 0 \text{ if and only if } S_{\text{VE}} \text{ does not hold}. \]

Visco-Energetic solutions are in between Energetic and Balanced Viscosity solutions in several respects:

1. **The structure of the solution concept:** On the one hand, the stability condition $S_{\text{VE}}$, though perturbed by a viscous correction, still retains a global character, like for E solutions. This globality plays a key role in the argument for proving convergence of the discrete solutions of (1.3) to a VE-solution. Indeed, as shown in [MRT18], once $S_{\text{VE}}$ is established for the time-continuous limit, it is sufficient to check the upper estimate $\leq$ to conclude $E_{\text{VE}}$ with an equality sign. In particular, no chain rule for $E$ is needed for the energy balance. On the other hand, VE solutions provide a description of the system behavior at jumps comparable to that of BV solutions. Indeed, optimal jump transitions (i.e., transitions between the two end-points of a jump attaining the inf in (1.4)), exist at every jump point. Moreover, they turn out to solve a minimum problem akin to the time-incremental minimization scheme $L_{\text{VE}}$, cf. (2.15) ahead. Similarly, optimal jump transitions for BV solutions solve a (possibly rate-dependent) evolutionary problem related to the scheme $L_{\text{BV}}$ they originate from.

2. **Their characterization for 1-dimensional rate-independent systems:** In the 1-dimensional setting it was shown in [MRT17] that VE solutions originating from scheme (1.3) where, in addition, $d_Z = d_{Z}$, have a behavior strongly dependent on the parameter $\mu > 0$. If $\mu$ is above a certain threshold, VE solutions exhibit a behavior at jumps akin to that of BV solutions, cf. [MRT17]. With a ‘small’ $\mu$, the behavior is intermediate between $E$ and $BV$ solutions.

3. **The singular limits $\mu \downarrow 0$ and $\mu \uparrow \infty$:** in [RS17], in a general metric-topological setting but, again, with the special viscous correction $\delta_Z = \frac{\mu}{2}d_Z^2$, VE solutions have been shown to converge to $E$ and BV solutions as $\mu \downarrow 0$ and $\mu \uparrow \infty$, respectively.

4. **The assumptions for the existence theory:** Loosely speaking, they turn out to be weaker than for BV solutions, and stronger than for E solutions. Therefore, the range of applicability of VE solutions to rate-independent processes in solid mechanics is intermediate between the $E$ and the BV concepts.

1.2. **Our results.** In this paper we are going to demonstrate the in-between character VE solutions by addressing their application to a rate-independent system for damage, and to a model for finite-strain plasticity; the highly nonlinear and nonsmooth character of these examples also shows the flexibility of the VE concept.

In the case of the damage system, the existence theory for E solutions [MRT06, TMT10, TH13] and for BV solutions [KRZ13, KRZ18, Neg17] seems to be well established. With Theorem 4.1 ahead we will prove the existence of VE solutions by applying the existence result [MRT18, Thm. 3.9] to a quite general damage system. Our assumptions on the constitutive functions of the model and on the problem data will (i) coincide with the conditions for E solutions in the case of the viscous correction $\delta_Z = \frac{\mu}{2}d_Z^2$; (ii) turn out to be slightly stronger than those for E solutions (in particular forcing a stronger gradient regularization for the damage variable), in the case of a ‘nontrivial’ viscous correction $\delta_Z$ involving a distance different from the dissipation distance $d_Z$; (iii) be definitely weaker than those for BV solutions, cf. also Remark 4.3 ahead.

The system for rate-independent elastoplasticity at finite strains we are going to address has been analyzed from the viewpoint of Energetic solutions in [MM09], whereas no result on the existence of BV solutions seems to be available up to now. In fact, the corresponding, viscously regularized system has been only recently tackled in [MRT18], where an existence result has been obtained after considerable regularization of the driving energy
functional to ensure the validity of the chain rule. In contrast, as we will see the existence of VE solutions to the rate-independent finite-strain plasticity system can be checked again under the same conditions as for E solutions in the case of a ‘trivial’ viscous correction. In turn, the ‘nontrivial’ case requires stronger assumptions, cf. Theorem 5.1 and Remark 5.3 ahead.

We are going to examine VE solutions from yet another viewpoint, by testing them on the benchmark example of the Prandtl-Reuss system for associative elastoplasticity. In Theorem 3.5 we are going to show that Visco-Energetic solutions for that system are indeed Energetic. The key point for our argument, cf. Prop. 4.1, will be to deduce that VE solutions comply with the ‘Energetic’ global stability condition (S). Exploiting the ‘global’ character of the VE-stability condition, in fact, we will be able to prove that VE solutions fulfill a characterization of (S) obtained in [DMDM06], and ultimately relying on the convex character of the perfectly plastic system.

Finally, we will tackle the application of VE solutions to a rate-independent system for brittle delamination, which can be thought of as a model for fracture on a prescribed surface. Due to the highly nonconvex and nonsmooth character of the underlying energy functional, the existence results from [MS18] do not directly apply. In fact, in Theorem 6.1 the existence of VE solutions will be proved by passing to the limit in an approximating system that models adhesive contact. In this way we will thus provide a first result on the convergence of VE solutions for systems driven by Γ-converging energies; our proof will rely on a careful asymptotic analysis of optimal jump transitions in the adhesive-to-brittle limit passage. In a future paper we plan to address the issue of Evolutionary Γ-convergence (in the sense of [Mie16]) for VE solutions in a more systematic and comprehensive way.

Plan of the paper. In Section 2 we shall revisit the theory of VE solutions from [MS18] and slightly adapt it to processes described in terms of two variables (u, z) (while [MS18] mostly focused on rate-independent systems in the single variable z). Sections 3, 4, 5 will be centered on the applications to perfect plasticity, damage, and finite-strain plasticity, respectively. Finally, the limit passage in the VE formulation from adhesive contact to brittle delamination will be addressed in Section 6.

Notation 1.1. Throughout the paper, we shall use the symbols c, c′, C, C′, etc., whose meaning may vary even within the same line, to denote various positive constants depending only on known quantities.

Given a topological space X, we will (i) denote by B([0, T]; X) the space of everywhere defined and measurable functions v : [0, T] → X; (ii) if (X, d) is a metric space, denote by BVd([0, T]; X) the space of everywhere defined functions v : [0, T] → X with bounded variation.

Finally, if X is also a normed space, the symbol B_{X}^{r} will denote the closed ball of X of radius r > 0, centered at 0. We will frequently omit the symbol X to avoid overburdening notation. For the same reason, we will often write ∥ · ∥_{X} in place of ∥ · ∥_{X,d} and, in place of X∗⟨·, ·⟩X, we shall write ⟨·, ·⟩X (or even ⟨·, ·⟩ when the duality pairing is clear from the context or has to be specified later).

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2. Setup, definition, and existence result for Visco-Energetic solutions

In this section we recapitulate the basic assumptions and definitions underlying the notion of Visco-Energetic solutions. We draw all concepts from [MS18]. There, however, the focus was on energies depending on the sole dissipative variable z (which was in fact denoted as u in [MS18]), and the case of functionals also depending on the variable at equilibrium u was recovered through a marginal procedure, cf. [MS18, Sec. 4]. Here we will partially revisit the presentation in [MS18] by directly working with energy functionals depending on the two variables (u, z).
2.1. The abstract setup for Visco-Energetic solutions. In what follows we collect the assumptions on the metric-topological setup, on the energy functional, on the dissipation (quasi-)distance, and on the viscous correction, at the core of the existence theory for VE solutions.

2.1.1. The metric-topological setting. Throughout the paper we will denote by σ the product topology on \( X = U \times Z \) induced by the two topologies \( \sigma_U \) and \( \sigma_Z \), and by \( \sigma_R \) the topology induced by \( \sigma \) on \([0,T] \times X\). We will often write \( (u_n, z_n) \xrightarrow{n} (u, z)\) as \( n \to \infty \) to signify convergence w.r.t. \( \sigma \)-topology, and we will use an analogous notation for \( \sigma_R \)-, \( \sigma_Z \)-, and \( \sigma_U \)-convergence.

The mechanism of energy dissipation will be described in terms of an extended, possibly asymmetric quasi-distance

\[
d_Z : Z \times Z \to [0, \infty], \quad \text{l.s.c. on } Z \times Z, \quad \text{s.t.} \quad \begin{cases} \ d_Z(z, z) = 0, \\ \ d_Z(z_0, z) < \infty \text{ for some reference point } z_0 \in Z, \\ \ d_Z(z, w) \leq d_Z(z, \zeta) + d_Z(\zeta, w) \quad \text{for all } z, \zeta, w \in Z. \end{cases} \tag{2.1} \]

We say that \( W \subset Z \) is \( d_Z \)-bounded if \( \sup_{w \in X} d_Z(z_0, w) < \infty \), and that \( d_Z \) separates the points of \( W \) if \( w, w' \in W, \quad d_Z(w, w') = 0 \Rightarrow w = w' \).

Our first condition concerns this metric-topological setting:

\(< T \rangle: \quad \text{We require that} \]

the topological spaces \( (U, \sigma_U) \) and \( (Z, \sigma_Z) \) are Hausdorff and satisfy the first axiom of countability, \( \tag{2.2a} \)

\((U, \sigma_U) \) is a Souslin space, \( \tag{2.2b} \)

namely the image of a Polish (i.e. a separable completely metrizable) space under a continuous mapping. Furthermore, we impose that \( d_Z \) separates the points of \( Z \). \( \tag{2.2c} \)

Let us now recall from [MS18] the definition of \( (\sigma_Z, d_Z) \)-regulated function, encompassing a crucial property that the Visco-Energetic solution component \( z \) shall enjoy at jumps.

**Definition 2.1.** [MS18, Def. 2.3] We call a curve \( z : [0, T] \to Z \) \( (\sigma_Z, d_Z) \)-regulated if for every \( t \in [0, T] \) there exist the left- and right-limits of \( z \) w.r.t. \( \sigma_Z \)-topology, i.e.

\[
z(t-) = \lim_{s \uparrow t} z(s) \quad \text{in } (Z, \sigma_Z), \quad z(t+) = \lim_{s \downarrow t} z(s) \quad \text{in } (Z, \sigma_Z) \tag{2.3a} \]

(with the convention \( z(0-) := z(0) \) and \( z(T+) := z(T) \)), also satisfying

\[
\lim_{s \uparrow t} d_Z(z(s), z(t-)) = 0, \quad \lim_{s \downarrow t} d_Z(z(t+), z(s)) = 0,
\]

\[
d_Z(z(t-), z(t)) = 0 \Rightarrow z(t-) = z(t), \quad d_Z(z(t), z(t+)) = 0 \Rightarrow z(t) = z(t+). \tag{2.3b} \]

We denote by \( \text{BV}_{\sigma_Z, d_Z}([0, T]; Z) \) the space of \( (\sigma_Z, d_Z) \)-regulated functions \( z \) with finite \( d_Z \)-total variation \( \text{Var}_{d_Z}(z, [0, T]) \), where we define, for a subset \( E \subset [0, T] \),

\[
\text{Var}_{d_Z}(z, E) := \sup \left\{ \sum_{j=1}^{M} d_Z(\partial^+ z(t_{j-1}), \partial^+ z(t_j)) : t_0 < t_1 < \ldots < t_M, \{t_j\}_{j=0}^M \in \mathcal{F}(E) \right\} \tag{2.4} \]

with \( \mathcal{F}(E) \) the collection of all finite subsets of \( E \).

If \( (Z, d_Z) \) is a complete metric space, every function \( z \in \text{BV}_{\sigma_Z, d_Z}([0, T]; Z) \) is \( (d_Z) \)-regulated, namely at every \( t \in [0, T] \) there exist the left- and right-limits of \( u \) w.r.t. the metric \( d_Z \). However, since in the present context we are not assuming completeness of \( (Z, d_Z) \), the concept of \( (\sigma_Z, d_Z) \)-regulated function turns out to be significant. Observe that, for every \( z \in \text{BV}_{\sigma_Z, d_Z}([0, T]; Z) \) the jump set

\[
J_z := J^+_z \cup J^-_z, \quad \text{with } J^+_z := \{t \in [0, T] : z(t-) \neq z(t)\}, \quad J^-_z := \{t \in [0, T] : z(t) \neq z(t+)\}, \tag{2.5} \]

\[
\text{if } (Z, d_Z) \text{ is complete, then } J_z \text{ is}\]
eralization was mainly motivated by the need to encompass in the theory and in fact surrogating the partial time derivative only depending on the dissipative variable $z$.

$$\sigma_{U\text{-regulated function}}, \text{i.e. } \forall t \in [0, T] \exists u(t-) = \lim_{s \uparrow t} u(s) \text{ in } (U, \sigma_U), \quad u(t+) = \lim_{s \downarrow t} u(s) \text{ in } (U, \sigma_U). \quad (2.6)$$

### 2.1.2. The energy functional.

We now recall the basic assumptions on the energy functional $E$ enunciated in [MSIS]. In view of Proposition 3.1 ahead, differently from [MSIS] we choose not to encompass lower semicontinuity and compactness requirements into a unique condition.

**Assumption $< A >$**. The RIS $(X, E, d_Z)$ fulfills

$< A.1 >$: **Lower semicontinuity**: The proper domain $D(E(t, \cdot))$ does not depend on $t$, namely there exists $D \subset X$ such that $D(E(t, \cdot)) = D$ for all $t \in [0, T]$. In what follows, we will use the notation

$$D_u := \pi_1(D), \quad D_z := \pi_2(D) \quad (2.7)$$

with $\pi_1 : X \to U$ and $\pi_2 : X \to Z$ the projection operators. We require that

there exists $F_0 \geq 0$ such that the perturbed functional

$$\mathcal{T} : [0, T] \times X \to (-\infty, \infty] \quad \mathcal{T}(t, (u, z)) := E(t, (u, z)) + d_Z(z_o, z) + F_0 \quad (2.8)$$

fulfills $\mathcal{T}(t, (u, z)) \geq 0$ for all $(t, (u, z)) \in [0, T] \times X$,

with $z_o$ the reference point satisfying (2.1). In what follows, with slight abuse of notation we will write

$$E(t, u, z) \text{ in place of } E(t, (u, z)), \quad \text{and analogously for } \mathcal{T}.$$

We impose that $E$ is $\sigma$-l.s.c. on the sublevels of $\mathcal{T}$.

$< A.2 >$: **Compactness**: The sublevels of $\mathcal{T}$ are $\sigma_2$-sequentially compact in $[0, T] \times X$.

$< A.3 >$: **Power control**: The functional $t \mapsto E(t, u, z)$ is differentiable for all $(u, z)$, $\partial_t E : (0, T) \times X \to \mathbb{R}$ is sequentially upper semicontinuous on the sublevels of $\mathcal{T}$, and

$$\exists \Lambda_P, C_P > 0 \quad \forall (t, u, z) \in (0, T) \times X : \quad |\partial_t E(t, u, z)| \leq \Lambda_P \mathcal{T}(t, u, z) + C_P. \quad (2.9)$$

**Remark 2.2.** A natural choice for the reference point $z_o$ in (2.1) and (2.8) is the initial datum $z_0 \in D_z$ for the rate-independent process. In fact, along the evolution there holds $\text{Var}_{d_Z}(z, [0, T]) < \infty$, cf. Remark 2.9 ahead, and therefore $\sup_{t \in [0, T]} d_Z(z_o, z(t)) \leq C < \infty$. That is why, we may suppose without loss of generality that, for every $z \in D_z$ there holds $d_Z(z_0, z) < \infty$.

In [MSIS] a more general version of the power-control condition was assumed, involving a generalized ‘power functional’ $\mathcal{P} : [0, T] \times X \to \mathbb{R}$ satisfying

$$\limsup_{s \uparrow t} \frac{E(s, u, z) - E(t, u, z)}{s - t} \leq \mathcal{P}(t, u, z) \leq \liminf_{s \downarrow t} \frac{E(t, u, z) - E(s, u, z)}{t - s} \quad \text{for all } (t, u, z) \in [0, T] \times X,$$

and in fact surrogating the partial time derivative $\partial_t E$ whenever $E$ is not differentiable w.r.t. $t$. This generalization was mainly motivated by the need to encompass in the theory marginal energies, i.e. functionals only depending on the dissipative variable $z$ and obtained from energies depending on both variables $(u, z)$ via minimization w.r.t. $u$. For simplicity, in this paper we shall not work with this power functional.

Finally, we point out that (2.9) could be weakened by allowing for a (positive) function $\Lambda_P \in L^1(0, T)$, in place of a (positive) constant $\Lambda_P$.

A straightforward consequence of $< A.1 > \& < A.2 >$ is that

$$\inf_{u \in U} E(t, u, z) \neq \emptyset \quad \text{for all } (t, z) \in [0, T] \times D_z. \quad (2.10)$$
In what follows, we will often work with the reduced energy functional

\[ J : [0, T] \times Z \to (-\infty, \infty) \quad J(t, z) := \begin{cases} \inf_{u \in U} \mathcal{E}(t, u, z) = \min_{u \in U} \mathcal{E}(t, u, z) & \text{if } (t, z) \in [0, T] \times D_z, \\ \infty & \text{otherwise}. \end{cases} \quad (2.11) \]

Combining the power-control estimate in (2.9) with the Gronwall Lemma, we conclude that

\[ J(t, u, z) \leq J(s, u, z) \exp(C_P|t - s|) \quad \text{for all } s, t \in [0, T] \text{ and all } (u, z) \in X. \]

In particular,

\[ \sup_{t \in [0, T]} J(t, u, z) \leq \exp(C_P T) J(0, u, z) \quad \text{for all } (u, z) \in X. \quad (2.12) \]

That is why, in what follows we will directly work with the functional

\[ J_0(u, z) := J(0, u, z) \quad \text{for every } (u, z) \in X. \]

Finally, we highlight that the upper semicontinuity of \( \partial_t \mathcal{E} \) required in \(<A.3>\) can be relaxed if \( d_Z \) enjoys an additional continuity property, stated in \(<A.3'>\) below. Indeed, \(<A.3'>\) can replace assumption \(<A.3>\).

\(<A.3'>\): \( d_Z \) is left-continuous on the sublevels of \( J_0 \), i.e. for all sequences \((u_n, z_n)_n \subset U \times Z\) s.t.

\[ J_0(u_n, z_n) \leq C, \quad z_n \overset{\sigma}{\to} z \quad \text{there holds } d_Z(z_n, \zeta) \to d_Z(z, \zeta) \quad \text{for all } \zeta \in Z, \quad (2.13a) \]

and the map \( \partial_t \mathcal{E} : [0, T] \times X \to \mathbb{R} \) satisfies \([FM06] \) and the conditional upper semicontinuity

\[ (t_n, u_n, z_n) \overset{\sigma}{\to} (t, u, z), \quad \mathcal{E}(t_n, u_n, z_n) \to \mathcal{E}(t, u, z) \quad \Rightarrow \quad \limsup_{n \to \infty} \partial_t \mathcal{E}(t_n, u_n, z_n) \leq \partial_t \mathcal{E}(t, u, z). \quad (2.13b) \]

The condition that convergence of the energies implies convergence of the powers is often required for the analysis of rate-independent systems, cf. \[MR15\]. For later use, we recall here a result from where this implication was proved in the case in which \( \partial_t \mathcal{E} \) is uniformly continuous on sublevels of \( \mathcal{E} \), namely

\[ \forall C > 0 \text{ there exists a modulus of continuity } \omega_C : [0, T] \to [0, \infty) \text{ such that } \quad \forall (u, z) \in U \times Z : \quad J_0(u, z) \leq C \Rightarrow |\partial_t \mathcal{E}(t_1, u, z) - \partial_t \mathcal{E}(t_2, u, z)| \leq \omega_C(|t_1 - t_2|) \text{ for all } t_1, t_2 \in [0, T]. \quad (2.14) \]

**Proposition 2.3.** \([FM06] \) Prop. 3.3 Assume \((2.14)\). Then, for every \( t \in [0, T] \) the following implication holds

\[ (u_n, z_n) \overset{\sigma}{\to} (u, z) \text{ in } X, \quad \mathcal{E}(t_n, u_n, z_n) \to \mathcal{E}(t, u, z) \quad \Rightarrow \quad \partial_t \mathcal{E}(t_n, u_n, z_n) \to \partial_t \mathcal{E}(t, u, z). \quad (2.15) \]

**2.1.3. The viscous correction of the time-incremental scheme.** We consider a lower semicontinuous map \( \delta_Z : Z \times Z \to [0, \infty) \) with \( \delta_Z(z, z) = 0 \) for all \( z \in Z \).

We introduce the ‘corrected’ dissipation

\[ D_Z(z, z') := d_Z(z, z') + \delta_Z(z, z'). \]

**Definition 2.4.** Let \( Q \geq 0 \). We say that \( (t, u, z) \in [0, T] \times X \) is \((D_Z, Q)\)-stable if it satisfies

\[ \mathcal{E}(t, u, z) \leq \mathcal{E}(t, u', z') + D_Z(z, z') + Q \quad \text{for all } (u', z') \in X. \quad (2.16) \]

If \( Q = 0 \), we will simply say that \( (t, u, z) \) is \( D_Z \)-stable. We denote by \( J_{D_Z} \) the collection of all the \( D_Z \)-stable points, and by \( J_{D_Z}(t) \) its section at the process time \( t \in [0, T] \).

In view of \(<A.1> \) & \(<A.2>\) (which guarantee \((2.10)\)), the quasi-stability condition \((2.16)\) is equivalent to

\[ J(t, z) \leq J(t, z') + D_Z(z, z') + Q \quad \text{for all } z' \in Z \quad (2.17) \]

involving the reduced energy \( J \) from \((2.11)\). That is why,

- in what follows we will often allow for the abuse of notation \((t, z) \in J_{D_Z} \) (and \( z \in J_{D_Z}(t) \)), in place of \((t, u, z) \in J_{D_Z} \).
we now introduce the residual stability function \( \mathcal{R} : [0,T] \times Z \to \mathbb{R} \) directly in terms of the reduced energy \( \mathcal{I} \), namely we define

\[
\mathcal{R}(t,z) := \sup_{z' \in Z} \{ \mathcal{I}(t,z) - \mathcal{I}(t,z') - D_Z(z,z') \} = \mathcal{I}(t,z) - \mathcal{V}(t,z)
\]  

\[ \mathcal{V}(t,z) = \inf_{z' \in Z} (\mathcal{I}(t,z') + D_Z(z,z')). \tag{2.18} \]

Note that, as soon as the energy functional \( \mathcal{E} \) complies with \(<A.1>\) and \(<A.2>\) (and we will suppose this hereafter), the inf in the definition of \( \mathcal{V} \) is attained, i.e.

\[ M(t,z) := \text{Argmin}_{z' \in Z} (\mathcal{I}(t,z') + D_Z(z,z')) \neq \emptyset. \tag{2.19} \]

Observe that \( \mathcal{R} \) in fact records the failure of the stability condition at a given point \((t,z) \in [0,T] \times Z\), since

\[ \mathcal{R}(t,z) \geq 0 \quad \text{for all } (t,z) \in [0,T] \times Z, \quad \text{with} \]

\[ \mathcal{R}(t,z) = 0 \quad \text{if and only if } (t,z) \in \mathcal{A}_Z. \tag{2.20} \]

Let us now specify the compatibility properties that admissible viscous corrections have to enjoy with respect to the driving distance \( d_Z \).

\(<B.1>\): \( d_Z \)-compatibility: For every \( z, z', z'' \in Z \)

\[ d_Z(z,z') = 0 \Rightarrow \delta_Z(z'',z') \leq \delta_Z(z'',z) \quad \text{and} \quad \delta_Z(z,z'') \leq \delta_Z(z',z''). \tag{2.21} \]

\(<B.2>\): \( d_Z \)-continuity: For every sequence \((u_n,z_n)_n\) and every \((u,z) \in X\) we have

\[ \sup_n \mathcal{F}_0(u_n,z_n) < \infty, \quad z_n \xrightarrow{\mathcal{F}_0} z, \quad d_Z(z_n,z) \to 0 \Rightarrow \lim_{n \to \infty} \delta_Z(z_n,z) = 0. \tag{2.22} \]

\(<B.3>\): \( D_Z \)-stability yields local \( d_Z \)-stability: for all \((t,u,z) \in \mathcal{A}_Z\) and all \( M > 1 \) there exist \( \eta > 0 \) and a neighborhood \( I_U \times I_Z \) of \((u,z)\) such that

\[ \mathcal{E}(s,u',z') \leq \mathcal{E}(s,u,z) + Md_Z(z',z) \quad \text{for all } (s,u',z') \in \mathcal{A}_Z \]

with \( s \in [t-\eta,t] \), for all \((u,z) \in I_U \times I_Z\) with \( d_Z(z',z) \leq \eta \). \[ \tag{2.23} \]

**Remark 2.5.** As already observed in [MS18], \(2.23\) is in fact equivalent to the condition

\[ \limsup_{(s,z') \to (t,z)} \frac{\mathcal{I}(s,z') - \mathcal{I}(t,z)}{d_Z(z',z)} \leq 1, \tag{2.24} \]

involving the reduced energy \( \mathcal{I} \) from \(2.17\), where we have written \((s,z') \xrightarrow{\mathcal{F}_0} (t,z)\) as a place-holder for \((s \to t, z' \xrightarrow{\mathcal{F}_0} z, d_Z(z,z') \to 0, (s,z') \in \mathcal{A}_Z, s \leq t)\). In turn, a sufficient condition for \(2.21\) is

\[ \limsup_{(s,z') \to (t,z)} \frac{\delta_Z(z',z)}{d_Z(z',z)} = 0 \quad \text{for every } z \in \mathcal{A}_Z(t) \text{ and all } t \in [0,T]. \tag{2.25} \]

In particular, any viscous correction of the form

\[ \delta_Z(z,z') = h(d_Z(z,z')) \quad \text{with } h \in C([0,\infty)) \text{ nondecreasing and fulfilling } \lim_{r \to 0} \frac{h(r)}{r} = 0 \tag{2.26} \]

satisfies \(2.25\) and, in fact, the whole Assumption \(<B>\).

**Closedness of the (quasi-)stable set.** Finally, we require

\(<C>\): For every \( Q \geq 0 \) the \((D_Z,Q)\)-quasi-stable sets have \( \sigma \)-closed intersections with the sublevels of \( \mathcal{F}_0 \).
It was proved in [MS18, Lemma 3.11] that \(< C >\) holds if and only if a property akin to the \textit{mutual recovery sequence} condition from [MRS08] holds, namely

for every sequence \((t_n, z_n) \subset [0, T] \times Z\) with \(t_n \to t\), \(z_n \overset{\mathcal{W}}{\to} z\), \(\sup_n d_Z(z_n, z) < \infty\)

and \(\lim_{n \to \infty} J(t_n, z_n) = J(t, z) + \eta, \eta \geq 0,\)

there exists \(z' \in M(t, z)\) and a sequence \((z'_n)_n\) such that

\[
\liminf_{n \to \infty} (J(t_n, z'_n) + D_Z(z_n, z'_n)) \leq J(t, z') + D_Z(z, z') + \eta,
\]

(recall that \(M(t, z)\) denotes the set of minimizers associated with the functional \(\mathcal{Y}\) in (2.18)).

2.2. Definition of Visco-Energetic solution. As already mentioned in the Introduction, the concept of Visco-Energetic solution of the rate-independent system \((X, \mathcal{E}, d_Z)\) (cf. Definition 2.8 ahead) consists of the \(D_{Z}\)-stability condition \([S_{VE}]\) combined with the energy-dissipation balance \([E_{VE}]\). In \([E_{VE}]\) the energy dissipated at jumps is measured in terms of a jump dissipation cost \(c\) that keeps track of the viscous correction \(\delta_Z\). This jump dissipation is obtained by minimizing a suitable transition cost over a class of continuous curves connecting the two end-points of a jump. In what follows,

(1) Firstly, we will specify what we mean by ‘end-points of a jump’ of a curve \((u, z)\) enjoying the properties of a Visco-Energetic solution, viz.

\(z \in BV_{\sigma, d_Z}([0, T]; Z)\) and \(t \mapsto u(t)\) is a measurable selection in \(\text{Argmin}_{u \in U} \mathcal{E}(t, u, z(t))\).

Namely, for a curve \((u, z)\) as in (2.28), we will introduce \textit{surrogate} left- and right-limits for \(u\) at a jump point \(t \in J_z\).

(2) Secondly, we will rigorously introduce the cost \(c\).

1. Surrogate left- and right limits of \(u\): given a curve \((u, z)\) as in (2.28), we extend \(u\) in this way:

at every \(t \in J_z\) we denote by \(\begin{cases} u(t-) & \text{an element Argmin}_{u \in U} \mathcal{E}(t, u, z(t-)), \\ u(t+) & \text{an element in Argmin}_{u \in U} \mathcal{E}(t, u, z(t+)) \end{cases}\)

(2.29)

with the convention that \(u(t-) = u(t+) = u(t)\) if \(t \notin J_z\), such that the extended mapping, still denoted by \(u\), is still measurable.

Observe that this definition is meaningful in view of (2.10). The notation \(u(t-)\) and \(u(t+)\) is used here in an extended sense, as the true left- and right-limits of \(u\) at \(t\) w.r.t. \(\sigma_U\)-topology need not exist. Nonetheless, in Lemma 2.10 ahead, we will provide some sufficient conditions, which can be verified for a reasonable class of examples, ensuring that, if \((u, z)\) is a Visco-Energetic solution, then \(u\) is \(\sigma_U\)-regulated and, in that case, \(u(t-)\) and \(u(t+)\) defined by (2.29) are its left- and right-limits.

2. The Visco-Energetic cost \(c\). It involves minimization of a suitable cost functional over a class of continuous curves, connecting the left- and right-limits \((u(t-), z(t-))\) and \((u(t+), z(t+))\) at a jump point \(t \in J_z\) (with \(u(t-)\) and \(u(t+)\) as in (2.29)). Such curves are in general defined on a compact subset \(E \subset \mathbb{R}\) with a possibly more complicated structure than that of an interval. To describe it, we fix some notation:

\[
E^- := \inf E, \quad E^+ := \sup E.
\]

(2.30a)

We also introduce

the collection \(\mathcal{H}(E)\) of the connected components of the set \([E^-, E^+] \setminus E\).

(2.30b)

Since \([E^-, E^+] \setminus E\) is an open set, \(\mathcal{H}(E)\) consists of at most countably many open intervals, which we will often refer to as the ‘holes’ of \(E\). We are now in a position to introduce the transition cost at the basis of the concept of Visco-Energetic solution, evaluated along curves \(\vartheta = (\vartheta_u, \vartheta_z) \in \mathcal{B}(E; X)\) such that, in addition

\[
\vartheta_z \in C_{\sigma, d_Z}(E; Z) := C_{\sigma, d_Z}(E; Z) \cap C_{d_Z}(E; Z).
\]

(2.31)
Here, \( \text{C}_{\sigma_Z}(E; Z) \) is the space of functions from \( E \) to \( Z \) that are continuous with respect to the \( \sigma_Z \)-topology, while \( \text{C}_{d_Z}(E; Z) \) is the space of functions \( \vartheta : E \to Z \) satisfying the following continuity condition w.r.t. \( d_Z \):
\[
\forall \varepsilon > 0 \quad \exists \eta > 0 \quad \forall s_0, s_1 \in E \text{ with } s_0 \leq s_1 \leq s_0 + \eta : \quad d_Z(\vartheta(s_0), \vartheta(s_1)) \leq \varepsilon .
\]

**Definition 2.6.** Let \( E \) be a compact subset of \( \mathbb{R} \) and \( \vartheta = (\vartheta_u, \vartheta_z) \in \text{B}(E; U) \times \text{C}_{\sigma_Z,d_Z}(E; Z) \). For every \( t \in [0, T] \) we define the transition cost function
\[
\text{Trc}_{\text{VE}}(t, \vartheta, E) := \text{Var}_{d_Z}(\vartheta_z, E) + \text{GapVar}_{d_Z}(\vartheta_z, E) + \sum_{s \in E \setminus \{E^+\}} \mathcal{R}(t, \vartheta_z(s)) , \tag{2.32}
\]

with

1. \( \text{Var}_{d_Z}(\vartheta, E) \) the \( d_Z \)-total variation of the curve \( \vartheta \), cf. [MS18];
2. \( \text{GapVar}_{d_Z}(\vartheta, E) := \sum_{t \in \partial(Z)} \delta_Z(\vartheta_z(I^-), \vartheta_z(I^+)) \); 
3. the (possibly infinite) sum
\[
\sum_{s \in E \setminus \{E^+\}} \mathcal{R}(t, \vartheta_z(s)) := \begin{cases} \sup \{ \sum_{s \in P} \mathcal{R}(t, \vartheta_z(s)) : P \in \mathcal{P}_f(E \setminus \{E^+\}) \} & \text{if } E \setminus \{E^+\} \neq \emptyset , \\ 0 & \text{otherwise} . \end{cases}
\]

Along with [MS18], we observe that, for every fixed \( t \in [0, T] \) and admissible \( \vartheta \), the transition cost fulfills the additivity property
\[
\text{Trc}_{\text{VE}}(t, \vartheta, E \cap [a, c]) = \text{Trc}_{\text{VE}}(t, \vartheta, E \cap [a, b]) + \text{Trc}_{\text{VE}}(t, \vartheta, E \cap [b, c]) \quad \text{for all } a < b < c .
\]

We are now in a position to define the *Visco-Energetic jump dissipation cost* \( c : [0, T] \times X \times X \to [0, \infty) \) between the two end-points of a jump of a curve \((u, z)\) as in (2.32). Namely, we set
\[
c(t, (u_-, z_-), (u_+, z_+)) := \inf \{ \text{Trc}_{\text{VE}}(t, \vartheta, E) : E \in \mathbb{R}, \vartheta = (\vartheta_u, \vartheta_z) \in \text{B}(E; U) \times \text{C}_{\sigma_Z,d_Z}(E; Z), \\
\vartheta(E^-) = (u_-, z_-), \vartheta(E^+) = (u_+, z_+) \} . \tag{2.33}
\]

**Remark 2.7.** In fact, for every admissible transition curve \( \vartheta = (\vartheta_u, \vartheta_z) \) between two pairs \((u_-, z_-)\) and \((u_+, z_+)\), all of the three contributions to the transition cost from (2.32) only depend on the \( \vartheta_z \)-component. That is why, from now on with slight abuse of notation we will simply write
\[
c(t, (u_-, z_-), (u_+, z_+)) \text{ in place of } c(t, (u_-, z_-), (u_+, z_+) \).
\]

Accordingly, we will introduce the concept of *Optimal Jump Transition*, cf. [2.44] ahead, only in terms of the \( \vartheta_z \)-component of an admissible transition curve \( \vartheta = (\vartheta_u, \vartheta_z) \).

With the jump dissipation cost \( c \) we associate the *incremental cost* \( \Delta c : [0, T] \times X \times X \to [0, \infty] \) defined at all \( t \in [0, T] \) and \((u_-, z_-), (u_+, z_+) \in X \) by
\[
\Delta c(t, (u_-, z_-), (u_+, z_+)) = \Delta c(t, z_-, z_+) := c(t, z_-, z_+) - d_Z(z_-, z_+) \tag{2.35}
\]
(in fact, observe that \( c(t, z_-, z_+) \geq d_Z(z_-, z_+) \), so that \( \Delta c(t, z_-, z_+) \geq 0 \), for all \( t \in [0, T] \) and \( z_\pm \in Z \)). We will also use the notation
\[
\Delta c(t, z_-, z_+) := \Delta c(t, z_-, z_+) + \Delta c(t, z_+, z_-) .
\]

The *augmented total variation* functional induced by \( c \) is defined, along a curve \((u, z) \in \text{BV}([0, T]; X)\), by
\[
\text{Var}_{d_Z,c}((u, z), [t_0, t_1]) := \text{Var}_{d_Z}(z, [t_0, t_1]) + \text{Jmp}_{\Delta c}((u, z); [t_0, t_1]) \quad \text{for any sub-interval } [t_0, t_1] \subset [0, T] , \tag{2.36}
\]
where the *incremental jump variation* of \((u, z)\) on \([t_0, t_1]\) is given by
\[
\text{Jmp}_{\Delta c}((u, z); [t_0, t_1]) := \Delta c(t_0, z(t_0), z(t_0+)) + \Delta c(t_1, z(t_1-), z(t_1)) + \sum_{t \in J_+(t_0, t_1)} \Delta c(t, z(t-), z(t)) \cdot \tag{2.37}
\]
Ultimately, also this jump contribution only depends on the \( z \)-component, namely
\[
\text{Jmp}_{\Delta c}((u, z); [t_0, t_1]) = \text{Jmp}_{\Delta c}(z; [t_0, t_1]) .
\]
Therefore, hereafter we shall write
\[ \text{Var}_{d_Z,c}(z, [t_0, t_1]) \text{ in place of } \text{Var}_{d_Z,c}((u, z), [t_0, t_1]). \]

As observed in [MS18], although it is not canonically induced by a distance, the total variation functional \( \text{Var}_{d_Z,c} \) still enjoys the additivity property
\[ \text{Var}_{d_Z,c}(z, [a, c]) = \text{Var}_{d_Z,c}(z, [a, b]) + \text{Var}_{d_Z,c}(z, [b, c]) \text{ for all } 0 \leq a \leq b \leq c \leq T. \]

We are now in a position to define the concept of Visco-Energetic solution \((u, z)\) of the rate-independent system \((X, \mathcal{E}, d_Z)\), featuring the \(d_Z\)-stability condition, and the energy-dissipation balance with the total variation functional \(\text{Var}_{d_Z,c}\). Let us stress in advance that, since \(\text{Var}_{d_Z,c} \geq \text{Var}_{d_Z} \) only controls the \(z\)-component of the curve \((u, z)\), it will be for \(z\) only that we shall claim \(z \in \text{BV}_{d_Z}([0, T]; Z)\) (in fact, \(z \in \text{BV}_{d_Z,c}([0, T]; Z)\)), while for the \(u\) component only measurability will be a priori asked for.

**Definition 2.8 (Visco-Energetic solution).** A curve \((u, z) : [0, T] \to X, \) with \(u \in B([0, T]; U)\) and \(z \in \text{BV}_{d_Z,c}([0, T]; Z)\), is a Visco-Energetic (VE) solution of the rate-independent system \((X, \mathcal{E}, d_Z)\) with the viscous correction \(\delta_Z\), if it satisfies

- the minimality condition
  \[ u(t) \in \text{Argmin}_{u \in U} \mathcal{E}(t, u, z(t)) \text{ for all } t \in [0, T]; \]  
  \[ \tag{2.38} \]

- the \(d_Z\)-stability condition
  \[ \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u', z') + d_Z(z(t), z') \]
  \[ = \mathcal{E}(t, u', z') + d_Z(z(t), z') + \delta_Z(z(t), z') \text{ for all } (u', z') \in X \text{ and all } t \in [0, T] \setminus J_z, \]  
  \[ \tag{SVE} \]

- the \((d_Z, c)\)-energy-dissipation balance
  \[ \mathcal{E}(t, u(t), z(t)) + \text{Var}_{d_Z,c}(z, [0, t]) = \mathcal{E}(0, u(0), z(0)) + \int_0^t \partial_s \mathcal{E}(s, u(s), z(s)) \, ds \text{ for all } t \in [0, T]. \]  
  \[ \tag{EVE} \]

**Remark 2.9.** From the energy-dissipation balance, exploiting the power-control condition \(E_{a,b}\) to estimate the power term on the right-hand side of \(E_{a,b}\), we easily deduce that
\[ \sup_{t \in [0, T]} |\mathcal{E}(t, u(t), z(t))| \leq \sup_{t \in [0, T]} \mathcal{T}(t, u(t), z(t)) \leq C_0, \]
\[ \text{Var}_{d_Z}(z, [0, T]) \leq \text{Var}_{d_Z,c}(z, [0, T]) \leq C_0 \]
\[ \tag{2.39} \]
for a constant \(C_0 > 0\) only depending on \((u(0), z(0)).\)

Observe that the \(d_Z\)-stability condition, tested with \((u', z') = (u', z(t))\) and \(u'\) arbitrary in \(U\), in particular ensures that \(u(t) \in \text{Argmin}_{u \in U} \mathcal{E}(t, u, z(t))\) for all \(t \in [0, T] \setminus J_z\). We want to claim this property at all \(t \in [0, T]\), though. That is why, \(2.38\) is required, as a separate property, at all \(t \in [0, T]\).

2.3. Characterization, properties, and main existence result for Visco-Energetic solutions. In all of the following statements we will implicitly assume that the rate-independent system \((X, \mathcal{E}, d_Z)\) satisfies conditions \(< T >, < A >, < B >, \) and \(< C >\) enunciated in Sec. 2.1, we will impose them explicitly only in the statement of Theorem 2.12.

**Lemma 2.10.** Suppose that
\[ \text{Argmin}_{u \in U} \mathcal{E}(t, u, z) \text{ is a singleton for every } (t, z) \in [0, T] \times D_z. \]
\[ \tag{2.40} \]
Let \((u, z)\) be a Visco-Energetic solution to \((X, \mathcal{E}, d_Z)\). Then, \(u\) is \(\sigma_U\)-regulated, with left- and right-limits given by \(E_{a,b}\).

**Proof.** Let us fix \(t \in [0, T]\). In order to show that the only element \(u(t+)\) in \(\text{Argmin}_{u \in U} \mathcal{E}(t, u, z(t+))\) is the right-limit of \(u\) w.r.t. the \(\sigma_U\)-topology, it is sufficient to show that, for all \((s_n)_n \subset (0, T)\) with \(s_n \downarrow t\), there holds \(u(s_n) \to u(t+)\) in \((U, \sigma_U)\). Since \(J_z\) is at most countable, we may suppose that \((s_n)_n \subset (0, T) \setminus J_z\). It follows from \(2.39\) and \(< A.2 >\) that there exists some \(u^* \in U\) such that, up to a (not relabeled) subsequence,
We recall that (cf. [MS18, Def. 3.13]), given a compact set \( E \) satisfying (recall the definition (2.19) of the set \( M \) in the statement below we also encompass the convergence result (cf. [MS18, Thm. 7.2]) for the \( \delta_z \) stability condition if and only if \( z \) satisfies, in addition, at every jump point \( t \) of the elements \((\bar{\vartheta}_n^z)_{n\in\mathbb{N}}\) of the set \( M(t,z) \)

\[
\vartheta_z(z^-) = z_-, \quad \vartheta_z(z^+) = z_+, \quad \text{Trc}_{\text{VE}}(t, \vartheta_z, E) = c(t, z_-, z_+).
\]

Furthermore, we say that \( \vartheta_z \) is a

1. sliding transition, if \( \mathcal{R}(t, \vartheta_z(s)) = 0 \) for all \( s \in E \);

2. viscous transition, if \( \mathcal{R}(t, \vartheta_z(s)) > 0 \) for all \( s \in E \setminus \{E^-, E^+\} \).

It has been shown in [MS18, Rmk. 3.15, Cor. 3.17] that, for a viscous transition \( \vartheta_z \) between \( z_- \) and \( z_+ \) the compact set \( E \setminus \{E^-, E^+\} \) is discrete, i.e. all of its points are isolated: namely, \( \vartheta_z \) is a pure jump transition. In fact, \( \vartheta_z \) may be represented as a finite, or countable, sequence \((\vartheta_n^z)_{n\in\mathbb{N}}\) with \( O \) a compact interval of \( Z \), satisfying (recall the definition (2.19) of the set \( M(t,z) \))

\[
\vartheta_n^z \in M(t, \vartheta_{n-1}^z) = \text{Argmin}_{z\in\mathbb{Z}} \left( \partial_z + D_z(\vartheta_{n-1}^z, z') \right) \quad \text{for all } n \in \mathbb{N} \setminus \{O^-\}.
\]

Furthermore, it has been proved in [MS18, Prop. 3.18] that any optimal jump transition can be canonically decomposed into (at most) countable collections of sliding and viscous, pure jump transitions. Finally, it has been shown in [MS18, Thm. 3.14] that, at every jump point \( t \) of a VE solution \( z \) there exists an optimal jump transition \( \vartheta_z \) between \( z(t-) \) and \( z(t+) \) such that \( \vartheta_z(s) = z(t) \) for some \( s \in E \).

We conclude this section by giving an existence result for VE solutions, proved in [MS18, Thm. 4.7]. For completeness, in the statement below we also encompass the convergence result (cf. [MS18, Thm. 7.2]) for the (left-continuous) piecewise constant interpolants

\[
\mathbf{Z}_n : [0, T] \to U, \quad \mathbf{Z}_n(0) := z_0, \quad \mathbf{Z}_n(t) := z^n \quad \text{for } t \in (t_n^{u-1}, t_n^u], \quad n = 1, \ldots, N_T
\]

associated with the discrete solutions \((z_n^u)_{n=1}^{N_T}\) of the time-incremental minimization problem [LMVE]. We shall discuss the convergence of the interpolants \((\mathbf{U}_T)_T\) of the elements \((u_n^*)_{n=1}^{N_T}\) with \( u_n^* \) minimizers for time-incremental minimization problem [LMVE], right after the statement of Thm. 2.12.
Theorem 2.12. [MIST] Thm. 4.7) Under Assumptions $< T >, < A >, < B >$, and $< C >$, let $z_0 \in D_z$. Then, for every sequence $(\tau_k)_k$ of time steps with $\tau_k \downarrow 0$ as $k \to \infty$ there exist a (not relabeled) sequence $(Z_{\tau_k})_k$ and $z \in \text{BV}_{\sigma_2,d_2}([0,T];Z)$ such that

1. $z(0) = z_0$, and
2. there exists $u \in B([0,T];U)$ such that $(u,z)$ is a VE solution to the rate-independent system $(X, \mathcal{E}, d_Z)$, with the viscous correction $\delta$.

In fact, the curve $u$ in the above statement is obtained as a measurable selection in $\text{Argmin}_{u \in U} \mathcal{E}(t, u, z(t))$. It is not, in general, related to the limit of the piecewise constant interpolants $U_{\tau_k}$. However, if, in addition, property (2.40) holds, and the functional $\mathcal{E}$ fulfills the following $\Gamma$-lim sup estimate, i.e.

for all $(t_k)_k \subset [0,T]$ and $(z_k)_k \subset Z$ with $t_k \to t$, $z_k \rightharpoonup z$ in $Z$ then

for all $v \in U$ there exists $(v_k)_k \subset U$ such that $\limsup_{k \to \infty} \mathcal{E}(t_k, v_k, z_k) \leq \mathcal{E}(t, v, z), \quad (2.48)$
then it is possible to prove convergence to the curve $u$. Namely, that

To check this, we may observe that from (2.49) it follows that

(2.50)
(with $\bar{U}_{\tau_k}$ the left-continuous piecewise constant interpolant associated with the partition of $[0,T]$). From the energy bound $\mathcal{F}_0(U_{\tau_k}(t), Z_{\tau_k}(t)) \leq C$ for a constant independent of $k \in \mathbb{N}$ and $t \in [0,T]$, cf. [MIST] Thm. 7.1, combined with Assumption $< A.2 >$, we infer that there exists a compact subset $U \subset U$ such that $U_{\tau_k}(t) \in U$ for all $t \in [0,T]$ and $k \in \mathbb{N}$. Then, for all $t \in [0,T]$ there exists $u_*(t) \in U$ such that, along a (not relabeled) subsequence possibly depending on $t$, there holds

(2.51)

Combining (2.50) and (2.51) with (2.48) and taking into account the lower semicontinuity $< A.1 >$ we find that $\mathcal{E}(t, u_*(t), z(t)) \leq \liminf_{k \to \infty} \mathcal{E}(U_{\tau_k}(t), Z_{\tau_k}(t)) \leq \liminf_{k \to \infty} \mathcal{E}(U_{\tau_k}(t), v, Z_{\tau_k}(t))$ for all $v \in U$ and all $t \in [0,T]$. Exploiting (2.48), we conclude that $u_*(t) \in \text{Argmin}_{u \in U} \mathcal{E}(t, u(t), z(t))$. Since the latter set is a singleton by (2.40), convergence (2.51) holds for the whole sequence $(\tau_k)_k$, and we conclude (2.49).

3. When Visco-Energetic solutions are Energetic: the case of perfect plasticity

The following result characterizes the situation in which VE solutions turn out to be E solutions as well. Note that it holds under the sole conditions $< A.1 >$ and $< A.3 >$.

Proposition 3.1. Assume $< T >, < A.1 >$, and $< A.3 >$. Then, a Visco-Energetic solution $(u,z)$ of the rate-independent system $(X, \mathcal{E}, d_Z)$ is an Energetic solution if and only if it satisfies the global stability condition $\mathcal{S}$ at every $t \in [0,T]$. In that case, at every jump point $t \in \text{Jump}_z$ the curves $(u, z)$ fulfill the jump conditions

$\mathcal{E}(t, u(t-), z(t-)) - \mathcal{E}(t, u(t), z(t)) = d_Z(z(t-), z(t)), \quad \mathcal{E}(t, u(t), z(t)) - \mathcal{E}(t, u(t+), z(t+)) = d_Z(z(t), z(t+)). \quad (3.1)$

Proof. Clearly, if $(u, z)$ is an E solution, then $\mathcal{S}$ holds.

Conversely, let $(u, z)$ be a VE solution complying with $\mathcal{S}$. Since $\text{Var}_{d_Z}(z, [0,t]) \geq \text{Var}_{d_Z}(z, [0,t])$ for all $t \in [0,T]$, from the energy-dissipation balance $\mathcal{E}_{\text{EVE}}$ we deduce that $(u, z)$ fulfills the Energetic energy-dissipation upper estimate (2.42) on $[0,t]$. Then, taking into account $\mathcal{S}$, we may apply either [MIST] Prop. 2.1.23 or [MIST] Lemma 6.2, mimicking the argument of the proof of Thm. 6.5 therein. In this way we conclude that $(u, z)$ is an Energetic solution. Hence, comparing $\mathcal{E}_{\text{EVE}}$ and $\mathcal{E}$ we ultimately find

$\text{Var}_{d_Z}(z, [0,t]) = \text{Var}_{d_Z,c}(z, [0,t]) \overset{2.19}{=} \text{Var}_{d_Z}(z, [0,t]) + \text{Jump}_{\Delta}(z; [0,t])$ for all $t \in [0,T]$. 
Therefore, at every \( t \in \text{Jump}_z \) there holds \( \Delta_z(t, z(t), z(t)) = \Delta_z(t, z(t), z(t+)) = 0 \), i.e.
\[
\bar{c}(t, z(t+), z(t)) = d_z(z(t), z(t)) \quad \text{and} \quad \bar{c}(t, z(t), z(t+)) = d_z(z(t), z(t+)).
\] Combining (3.2) with the Visco-Energetic jump conditions (2.43) we immediately deduce (3.1).

Notation 3.2. We will use the symbol \( \mathbb{M}^{d \times d} \) for the space of \( d \times d \) matrices, endowed with the Frobenius inner product \( \eta : \xi := \sum_{ij} \eta_{ij} \xi_{ij} \) for two matrices \( \eta = (\eta_{ij}) \) and \( \xi = (\xi_{ij}) \). We will denote by \( | \cdot | \) the induced matrix norm and, in accordance with Notation 1.1, by \( \mathbb{B}_r \) the closed ball with radius \( r \) centered at 0 in \( \mathbb{M}^{d \times d}_{\text{sym}} \). The latter symbol denotes the subspace of symmetric matrices, while \( \mathbb{M}^{d \times d}_{\text{dev}} \) stands for the subspace of symmetric matrices with null trace. In fact, every \( \eta \in \mathbb{M}^{d \times d}_{\text{sym}} \) can be written as \( \eta = \eta_{\text{dev}} + \frac{\text{tr}(\eta)}{d} I \) with \( \eta_{\text{dev}} \) the orthogonal projection of \( \eta \) into \( \mathbb{M}^{d \times d}_{\text{dev}} \). We will refer to \( \eta_{\text{dev}} \) as the deviatoric part of \( \eta \). With the symbol \( \odot \) we will denote the symmetrized tensor product of two vectors \( a, b \in \mathbb{R}^d \), defined as the symmetric matrix with entries \( \frac{1}{2}(a_i b_j + a_j b_i) \). Finally,
\[
\mathbb{B}(\Omega; \mathbb{R}^d) := \{ \bar{u} \in L^1(\Omega; \mathbb{R}^d) : \varepsilon(\bar{u}) \in \mathbb{M}(\Omega; M^{d \times d}_{\text{sym}}) \}
\]
is the space of functions with bounded deformation, such that the (distributional) strain tensor \( \varepsilon(\bar{u}) \) is a Radon measure on \( \Omega \), valued in \( \mathbb{M}^{d \times d}_{\text{sym}} \), and
\[
\mathbb{M}(\Omega \cup \Gamma_D; \mathbb{M}^{d \times d}_{\text{dev}}) \quad \text{is the space of (} \mathbb{M}^{d \times d}_{\text{dev}} \text{-valued) Radon measures on } \Omega \cup \Gamma_D.
\]

The PDE system governing perfect plasticity, formulated in a (bounded, Lipschitz) domain \( \Omega \subset \mathbb{R}^d \) (the reference configuration) consists of
- the equilibrium equation
\[
- \text{div}(\mathbb{C}\varepsilon) = f \quad \text{in } \Omega \times (0, T), \tag{3.3a}
\]
where \( f \) is a time-dependent body force, \( \mathbb{C} \) is the (symmetric, positive definite) elasticity tensor, \( \varepsilon \) the strain, which enters into the additive decomposition of the (symmetric) linearized strain tensor \( \varepsilon(\bar{u}) = \frac{1}{2}(\nabla \bar{u} + \nabla^T \bar{u}) \) (with \( \bar{u} : \Omega \to \mathbb{R}^d \) the displacement and \( A^T \) the transpose of a matrix \( A \)), into an elastic and a plastic part, i.e.
\[
\varepsilon(\bar{u}) = \varepsilon + p \quad \text{in } \Omega \times (0, T); \tag{3.3b}
\]
- the flow rule for the plastic tensor \( p \)
\[
\partial \mathbb{R}(\bar{p}) \ni \sigma_{\text{dev}} \quad \text{in } \Omega \times (0, T), \tag{3.3c}
\]
where \( \sigma_{\text{dev}} \) is the deviatoric part of the stress \( \sigma := \mathbb{C}\varepsilon \), the 1-homogeneous dissipation potential \( \mathbb{R} \) is the support function of the closed convex subset \( K \subset \mathbb{M}^{d \times d}_{\text{dev}} \) to which the (deviatoric part of the) stress is constrained to belong, and \( \partial \mathbb{R} : \mathbb{M}^{d \times d}_{\text{dev}} \rightharpoonup \mathbb{M}^{d \times d}_{\text{dev}} \) is the convex analysis subdifferential of \( \mathbb{R} \). Along the footsteps of [DMDM06], we will suppose hereafter that
\[
\mathbb{B}_{r_k} \subset K \subset \mathbb{B}_{R_K} \quad \text{for some } 0 < r_K \leq R_K. \tag{3.4a}
\]
Furthermore, we will have \( \partial \Omega = \Gamma_D \cup \Gamma_N \cup \partial \Gamma \), with \( \Gamma_D \) and \( \Gamma_N \) disjoint open sets and \( \partial \Gamma \) their common boundary, and we will denote by \( \nu \) the external unit normal to \( \partial \Omega \). We will assume that
\[
\mathcal{H}^{d-1}(\Gamma_D) > 0 \quad \text{and} \quad \partial \Omega, \partial \Gamma \text{ are of class } C^2 \quad \text{(3.4b)}
\]
(with \( \mathcal{H}^{d-1} \) the \((d-1)\)-dimensional Hausdorff measure). On the Dirichlet part of the boundary \( \Gamma_D \) we will prescribe a Dirichlet condition through an assigned function
\[
w_D \in C^1([0, T]; H^1(\Omega; \mathbb{R}^d)), \quad \text{(3.4c)}
\]
with trace on \( \Gamma_0 \) still denoted by \( u_b \). On the Neumann part \( \Gamma_N \) we will apply a non-zero traction \( g \). A standard condition in perfect plasticity is that the body and surface forces

\[
f \in C^1([0, T]; L^d(\Omega; \mathbb{R}^d)), \quad g \in C^1([0, T]; L^\infty(\Gamma_N; \mathbb{R}^d))
\]

satisfy the safe-load condition:

\[
\exists \varepsilon \in AC([0, T]; L^2(\Omega; M_{\text{dev}}^{d \times d})) \text{ with } \varepsilon_{\text{dev}} \in AC([0, T]; L^\infty(\Omega; M_{\text{sym}}^{d \times d})) \text{ s.t. } \begin{cases} -\text{div}(\varepsilon) = f & \text{in } \Omega \\ \varepsilon \nu = g & \text{in } \Gamma_N \end{cases} \text{ on } (0, T)
\]

and fulfilling \( \exists \tau > 0 \forall t \in [0, T] \) for a.a. \( x \in \Omega : \varepsilon_{\text{dev}}(t, x) + \mathbf{K} \tau \subset K \).

\((3.4d)\)

With \( f \) and \( g \) we associate the total load function

\[
\ell : [0, T] \to BD(\Omega; \mathbb{R}^d)^*, \quad (\ell(t), v)_{BD(\Omega; \mathbb{R}^d)} := \int_\Omega f(t) v \, dx + \int_{\Gamma_N} g(t) v \, d\mathcal{H}^{d-1}(x).
\]

\((3.5)\)

Indeed, the above integrals are well defined for any \( v \in BD(\Omega; \mathbb{R}^d) \) due to the embedding and trace properties of \( BD(\Omega; \mathbb{R}^d) \). Clearly, \( \ell(t) \) is also an element of \( H^1(\Omega; \mathbb{R}^d)^* \) for every \( t \in [0, T] \); in what follows, to avoid overburdening notation, we will often omit to specify the spaces when writing the duality pairing \( (\ell(t), v) \).

With the boundary datum \( u_b \) we associate the set \( \mathcal{A}(u_b) \) of the kinematically admissible states \((\tilde{u}, p)\), viz.

\[
(\tilde{u}, p) \in \mathcal{A}(u_b) \text{ if and only if } \begin{cases} (i) \quad \tilde{u} \in BD(\Omega; \mathbb{R}^d), \quad p \in M(\Omega \cup \Gamma_0; M_{\text{dev}}^{d \times d}), \\
(ii) \quad e = \varepsilon(\tilde{u}) - p \in L^2(\Omega; M_{\text{sym}}^{d \times d}), \\
(iii) \quad p = (u_b - \tilde{u}) \circ \nu \mathcal{H}^{d-1} \text{ on } \Gamma_D. 
\end{cases}
\]

\((3.6)\)

We set \( \mathcal{A} := \mathcal{A}(0) \).

Indeed, an admissible \( \tilde{u} \) may have jumps (i.e., the measure \( \varepsilon(\tilde{u}) \) can concentrate on) \( \partial \Omega \). Hence, the boundary condition \( \tilde{u} = u_b \) on \( \Gamma_0 \) has to be relaxed in terms of \( 3.6(iii) \) (to be understood as an equality between measures on \( \Gamma_0 \)), which expresses the fact that any jump of \( \tilde{u} \) violating the Dirichlet condition \( \tilde{u} = u_b \) is due to a localized plastic deformation. From now on, we will use the splitting

\[
\tilde{u} = u + u_b
\]

\((3.7)\)

and work with the state variables \((u, p)\).

The Energetic formulation (cf. [MDM06]) of the perfectly plastic system \((3.3)\) is given in this setup: **Ambient space:**

\[
X = U \times Z \text{ with } U = BD(\Omega; \mathbb{R}^d), \quad Z = M(\Omega \cup \Gamma_0; M_{\text{dev}}^{d \times d})
\]

\((3.8a)\)

and (1) \( \sigma_Z \) is the weak*-topology on \( M(\Omega \cup \Gamma_0; M_{\text{dev}}^{d \times d}) \), identified with the dual of the space of \( M_{\text{dev}}^{d \times d} \)-valued continuous functions with compact support on \( \Omega \cup \Gamma_0 \); (2) \( \sigma_U \) is the weak* topology on \( BD(\Omega; \mathbb{R}^d) \) (which has in fact a predual, cf. e.g. [TS80]), inducing the following notion of weak*-convergence: \( u_k \rightharpoonup^* u \) in \( BD(\Omega; \mathbb{R}^d) \) if and only if \( u_k \to u \) in \( L^1(\Omega; \mathbb{R}^d) \) and \( \varepsilon(u_k) \rightharpoonup^* \varepsilon(u) \) in \( M(\Omega; M_{\text{sym}}^{d \times d}) \).

**Energy functional:**

\[
\mathcal{E}(t, u, p) := \frac{1}{2} \int_\Omega C(\varepsilon(u + u_b(t)) - p) : (\varepsilon(u + u_b(t)) - p) \, dx + I_A(u, p) - \langle \ell(t), u + u_b(t) \rangle_{BD(\Omega; \mathbb{R}^d)}.
\]

\((3.8b)\)

Here, the indicator function \( I_A \) forces the constraint \((u, p) \in \mathcal{A}, \) so that \( \tilde{u} = u + u_b \in \mathcal{A}(u_b) \);

**Dissipation distance:** it is defined in terms of the support function

\[
R : M_{\text{dev}}^{d \times d} \to [0, \infty), \quad R(\pi) := \sup_{\omega \in K} \omega : \pi
\]

of the set \( K \) from \([3.4] \), via

\[
d_Z(p, \bar{p}) := R(\bar{p} - p) \quad \text{with } R(\pi) := \int_{\Omega \cup \Gamma_0} \frac{R(|\pi|)}{|\pi|} |\pi| \, dx \text{ for all } \pi \in M(\Omega \cup \Gamma_0; M_{\text{dev}}^{d \times d}),
\]

\((3.8c)\)

where \( |\pi| \) is the variation of \( \pi \) and \( \frac{R}{|\pi|} \) its Radon-Nykodim derivative w.r.t. \(|\pi|\).
It is straightforward to check that in the above metric-topological setting \( < T > \) is fulfilled. For the reader’s convenience, we recapitulate here the arguments from [DMDM06] to show that

**Lemma 3.3.** **Under conditions** [3.4], the energy functional \( \mathcal{E} \) from [3.8] fulfills \( < A_1 >, < A_2 >, < A_3 >. \)

**Proof.** In view of the safe-load condition [3.4d] and [DMDM06] Lemma 3.1, the loading term rewrites as

\[
\langle \ell(t), u + u_b(t) \rangle_{BD(\Omega; \mathbb{R}^d)} = \langle \varrho, (u + u_b(t)) - p \rangle + \langle \varrho_{dev}, p \rangle + \langle \ell(t), u_b(t) \rangle - \langle \varrho, (u_b(t)) \rangle,
\]

where the duality pairing \( \langle \varrho_{dev}, p \rangle \) involving the measure \( p \) has been carefully defined in [DMDM06] Sec. 2, and the other duality pairings are not specified for notational simplicity. Let us now fix any reference point \( p_0 \in \mathbb{Z} \) satisfying the kinematical admissibility condition [3.0] (i.e., such that there exists \( u_0 \in U \) such that \( (u_0, p_0) \in \mathcal{A}(u_b(0)) \)). Therefore, suitably choosing \( F_0 \) (cf. (3.10) below), we find for all \((u, p) \in \mathcal{A} \)

\[
\mathcal{F}(t, u, p) = \mathcal{E}(t, u, p) + d_Z(p_0, p) + F_0
\]

\[
= \frac{1}{2} \int_0^T \mathbb{C}(\varepsilon((u + u_b(t))) - p) : (\varepsilon((u + u_b(t))) - p) \, dx - \langle \varrho(t), \varepsilon(u + u_b(t)) - p \rangle + \mathbb{R}(p - p_0) - \langle \varrho_{dev}(t), p - p_0 \rangle
\]

\[
\geq \frac{\gamma_c}{4} \| \varepsilon(u + u_b(t)) - p \|^2_{L^2(\Omega)} - \frac{1}{\gamma_c} \| \varrho(t) \|^2_{L^2(\Omega)} + \| p - p_0 \|_{M(\Omega; \mathbb{R}^d)}
\]

(observe that the duality pairing \( \langle \varrho_{dev}(t), p_0 \rangle \) is well defined since \( p_0 \) is a kinematically admissible strain, cf. [DMDM06] Sec. 2)). Here, (i) follows from (i) the estimate \( \frac{1}{2} \mathbb{C} \varepsilon : \varepsilon - \varrho : \varepsilon \geq \frac{1}{2} \gamma_c |\varepsilon|^2 - \frac{1}{2} \gamma_c |\varepsilon|^2 - \frac{1}{4} |\varepsilon|^2 \) by the positive-definiteness of \( \mathbb{C} \) and Young’s inequality; (ii) [DMDM06] Lemma 3.2, which ensures the estimate \( \mathbb{R}(p - p_0) - \langle \varrho_{dev}(t), p - p_0 \rangle \geq \| p - p_0 \|_{M(\Omega; \mathbb{R}^d)} \), with \( \gamma > 0 \) from the safe-load condition [3.4d]; (iii) choosing

\[
F_0 \geq \sup_{t \in [0, T]} \left( \langle \varrho_{dev}(t), p_0 \rangle + \langle \ell(t), u_b(t) \rangle + \langle \varrho(t), \varepsilon(u_b(t)) \rangle \right)
\]

thanks to [3.4c] and [3.4d]. From [3.9] and again [3.4e] we thus deduce that

\[
\exists \gamma, C > 0 \quad \forall (t, u, p) \in [0, T] \times U \times Z \text{ with } (u, p) \in \mathcal{A}:
\]

\[
\mathcal{F}(t, u, p) \geq c \left( \| \varepsilon(u) \|_{\mathbb{M}(\Omega)} + \| p \|_{M(\Omega; \mathbb{R}^d)} + \| \varepsilon(u) - p \|_{L^2(\Omega)} \right) - C.
\]

From the bound for \( p \) and the information that \( u \in \mathcal{H}^{d-1} \) we conclude a bound for \( u \) in \( L^1(\Gamma_0; \mathbb{R}^d) \). Therefore, a Poincaré-type estimate for BD-functions (cf. e.g., [Tem83] Prop. 2.4, Rmk. 2.5) yields a bound for \( u \) in \( BD(\Omega; \mathbb{R}^d) \). We thus conclude that the sublevels of \( \mathcal{F} \) are bounded in \( [0, T] \times U \times Z \), and thus sequentially relatively compact w.r.t. the \( \sigma_\mathbb{R} \)-topology, whence \( < A_2 > \).

Relying on estimate (3.11) and on the closedness properties of the set \( \mathcal{A}(u_b) \) (cf. [DMDM06] Lemma 2.1), it is standard to show that \( \mathcal{E} \) is sequentially l.s.c. on sublevels of \( \mathcal{F} \) w.r.t. the \( (\sigma_\mathbb{R} \times \sigma_Z) \)-topology, i.e. \( < A_1 >. \)

It follows from [3.4c] and [3.4d] that \( \partial_u \mathcal{E}(t, u, p) \) exists and

\[
\partial_u \mathcal{E}(t, u, p) = \int_\Omega \mathbb{C}(\varepsilon(u) - p + \varepsilon(u_b(t))) : \varepsilon(\dot{u}_b(t)) \, dx - \langle \dot{\ell}(t), u_b(t) \rangle_{H^1(\Omega; \mathbb{R}^d)} - \langle \ell(t), \dot{u}_b(t) \rangle_{H^1(\Omega; \mathbb{R}^d)}
\]

for a.a. \( t \in (0, T) \) and for all \((u, p) \in D \).

Therefore, in view of (3.11) we find that

\[
|\partial_u \mathcal{E}(t, u, p)| \leq C \| \varepsilon(u) - p \|_{L^2(\Omega)} \| \varepsilon(\dot{u}_b(t)) \|_{L^2(\Omega)} + C \| \varepsilon(u_b(t)) \|_{H^1(\Omega)} \| \varepsilon(u_b(t)) \|_{L^2(\Omega)}
\]

\[
+ \| \dot{\ell}(t) \|_{H^1(\Omega)} \| u_b(t) \|_{H^1(\Omega)} + \| \ell(t) \|_{H^1(\Omega)} \| \dot{u}_b(t) \|_{H^1(\Omega)}
\]

\[
\leq \Lambda_p \left( \mathcal{F}(t, u, p) + C \rho \right)
\]

with \( \Lambda_p (t) = C \sup_{t \in [0, T]} \left( \| \dot{u}_b(t) \|_{H^1(\Omega)} + \| \dot{\ell}(t) \|_{H^1(\Omega)} \right) \) and \( C \rho = \| \ell(t) \|_{L^2(0, T; H^1(\Omega))} + \| u_b(t) \|_{L^2(0, T; H^{-1}(\Omega))} \). It is straightforward to check that, again thanks to [3.4c] [3.4e] and [3.11], \( \partial_u \mathcal{E}(t, u, p) \) is indeed (sequentially) continuous w.r.t. the \( \sigma_\mathbb{R} \)-topology on the sublevels of \( \mathcal{F} \). This concludes the proof of \( < A_3 > \).

\( \square \)
The viscous correction: Let us now consider the family of viscous corrections
\[ \delta_Z(p, \tilde{p}) := h(d_Z(p, \tilde{p})) = h(\mathcal{R}(\tilde{p} - p)) \quad \text{for all } p, \tilde{p} \in Z \text{ and } h \text{ as in (2.24)} \] (cf. Remark 3.6 for a discussion on more general viscous corrections). With our next result we show that, in the frame of the rate-independent system \((X, E, d_Z)\) given by (3.8), and with this choice of \(\delta_Z\), the Visco-Energetic stability condition \([\mathcal{S}, \mathcal{E}, \mathcal{F}]\) in indeed equivalent to the Energetic stability \([\mathcal{S}]\).

**Proposition 3.4.** Assume (3.4) and let \((u, p) \in B([0, T]; BD(\Omega; \mathbb{R}^d)) \times BV([0, T]; M(\Omega; \mathbb{R}^d; M_{d_{\text{sym}}}))\) fulfill \((u(t), p(t)) \in A \text{ for all } t \in [0, T]\). Then, the following conditions are equivalent at a given \(t \in [0, T]\):

1. \((u, p)\) fulfill the stability condition \([\mathcal{S}, \mathcal{E}, \mathcal{F}]\) for the rate-independent system \((X, E, d_Z)\) (3.8), with the viscous correction \(\delta_Z\) from (3.12);
2. there holds \(\sigma(t) = C(\varepsilon(u(t) + u_b(t)) - p(t)) \in \mathcal{X}(\Omega) \cap \mathcal{K}(\Omega) \) with

\[
\begin{align*}
\mathcal{X}(\Omega) &:= \{ \sigma \in L^2(\Omega; M_{d_{\text{sym}}}) : \text{div}(\sigma) \in L^d(\Omega; \mathbb{R}^{d \times d}), \sigma_{\text{dev}} \in L^\infty(\Omega; M_{d_{\text{dev}}}) \}, \\
\mathcal{K}(\Omega) &:= \{ \sigma \in L^2(\Omega; M_{d_{\text{sym}}}) : \sigma_{\text{dev}}(x) \in K \text{ for a.a. } x \in \Omega \},
\end{align*}
\]

and \(-\text{div}(\sigma(t)) = f(t) \ a.e. \text{ in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N\); \quad (3.13b)

3. \((u, p)\) fulfill the stability condition \([\mathcal{S}]\) for the rate-independent system \((X, E, d_Z)\) from (3.8).

**Proof.** First of all, we show that (1) \(\Rightarrow\) (2). Indeed, in the stability condition \([\mathcal{S}, \mathcal{E}, \mathcal{F}]\), i.e.

\[
\frac{1}{2} \int_\Omega \mathcal{C}(\varepsilon(u + u_b(t)) - p) : (\varepsilon(u + u_b(t)) - p) \, dx - \langle \ell(t), u(t) \rangle_{BD(\Omega; \mathbb{R}^d)}
\]

\[
\leq \frac{1}{2} \int_\Omega \mathcal{C}(\varepsilon(u' + u_b(t)) - p') : (\varepsilon(u' + u_b(t)) - p') \, dx - \langle \ell(t), u' \rangle_{BD(\Omega; \mathbb{R}^d)} + \mathcal{R}(p' - p(t)) + h(\mathcal{R}(p' - p(t)))
\]

for all \((u', p') \in A\), we choose \((u', p') := (u(t) + \eta v, p(t) + \eta q)\), with arbitrary \(\eta \in \mathbb{R}\) and \((v, q) \in A\). With straightforward calculations we find

\[
0 \leq \frac{1}{2} \int_\Omega (\eta \varepsilon(v) - \eta q) : (\eta \varepsilon(v) - \eta q) \, dx + \int_\Omega \mathcal{C}(\varepsilon(u + u_b(t)) - p) : (\eta \varepsilon(v) - \eta q) \, dx - \langle \ell(t), \eta v \rangle_{BD(\Omega; \mathbb{R}^d)}
\]

\[
+ \mathcal{R}(\eta q) + h(\mathcal{R}(\eta q))
\]

Hence, by the positive homogeneity of \(\mathcal{R}\) we conclude

\[
0 \leq \eta^2 \frac{1}{2} \int_\Omega \mathcal{C}(\pm \varepsilon(v) \mp q) : (\pm \varepsilon(v) \mp q) \, dx + \eta \int_\Omega \mathcal{C}(\varepsilon(u + u_b(t)) - p) : (\pm \varepsilon(v) \mp \eta q) \, dx - \eta \langle \ell(t), \pm v \rangle_{BD(\Omega; \mathbb{R}^d)}
\]

\[
+ \eta \mathcal{R}(\pm q) + h(\mathcal{R}(\pm q)) \quad \text{for all } \eta > 0.
\]

Dividing by \(\eta\) and letting \(\eta \downarrow 0\), and using that

\[
\lim_{\eta \downarrow 0} \frac{h(\mathcal{R}(\eta(\pm q)))}{\eta} = \lim_{\eta \downarrow 0} \frac{h(\mathcal{R}(\eta(\pm q)))}{\mathcal{R}(\eta(\pm q))} \frac{\mathcal{R}(\eta(\pm q))}{\eta} = 0
\]

thanks to property (2.26) we find that

\[
\begin{align*}
\begin{cases}
-\mathcal{R}(q) \leq \int_\Omega \sigma(t) : (\varepsilon(v) - q) \, dx - \langle \ell(t), v \rangle_{BD(\Omega; \mathbb{R}^d)}, \\
\int_\Omega \sigma(t) : (\varepsilon(v) - q) \, dx - \langle \ell(t), v \rangle_{BD(\Omega; \mathbb{R}^d)} \leq \mathcal{R}(q)
\end{cases}
\end{align*}
\]

for all \((v, q) \in A\). \quad (3.15)

It has been shown in [DMDM06, Prop. 3.5] that (3.15) is equivalent to (3.13). This shows (2).

In turn, (2) \(\Leftrightarrow\) (3) by [DMDM06, Thm. 3.6]. Finally, we clearly have that (3) \(\Rightarrow\) (1). This concludes the proof.

We are now in a position to prove
Theorem 3.5. Assume \(X, \mathcal{E}, d_Z\) and let \((u, p) \in B([0, T); \text{BD}(\Omega; \mathbb{R}^d)) \times BV([0, T]; M(\Omega, \Gamma_D; M_{\text{dev}}^{d \times d}))\) be a VE solution of the rate-independent system \((X, \mathcal{E}, d_Z)\) from (3.3), with the viscous correction \(\delta_Z\) from (3.3). Suppose that \((u, p)\) fulfills \(t = 0\) the stability condition
\[
\mathcal{E}(t, u(0), p(0)) \leq \mathcal{E}(t, u', p') + \mathcal{R}(p' - p(0)) \quad \text{for all } (u', p') \in A.
\]
Then, \((u, p)\) is an Energetic solution of the rate-independent system \((X, \mathcal{E}, d_Z)\).

Proof. We have that \((u, p)\) fulfills the stability condition \(S\) at \(t = 0\) and, in view of Prop. 3.4 whenever it fulfills \(S_\text{VE}\), i.e. at every \(t \in [0, T] \setminus J_z\). Passing to the limit in \(S\) we conclude that it holds also at the end-points of every jump, i.e.
\[
\mathcal{E}(t, u(t\pm), p(t\pm)) \leq \mathcal{E}(t, u', p') + \mathcal{R}(p' - p(t\pm)) \quad \text{for all } (u', p') \in U \times Z \text{ and all } t \in J_z.
\]
With the very same argument as in the proof of [RS17, Thm. 1], using the upper energy-dissipation estimate (3.16) we deduce that
\[
\mathcal{E}(t, u(t), p(t)) + \mathcal{R}(p(t) - p(-)) \leq \mathcal{E}(t, u(t-), p(t-)) \quad \text{for all } t \in J_z,
\]
which, combined with (3.17) and the triangle inequality for \(\mathcal{R}\) delivers the stability \(S\) at all \(t \in J_z\). In view of Lemma 3.3, we may then apply Prop. 3.4 and conclude that \((u, p)\) is an Energetic solution.

Remark 3.6. Indeed, Theorem 3.5 carries over to VE solutions of the perfectly plastic system arising from a more general viscous correction \(\delta_Z : Z \times Z \to [0, \infty]\), provided that it fulfills the compatibility condition
\[
\lim_{\hat{p} \to p \text{ strongly in } Z} \frac{\delta_Z(p, \hat{p})}{\mathcal{R}(\hat{p} - p)} = 0.
\]
Note that (3.18) is a strengthened version of (2.22), in turn implying < B.3 >. As a matter of fact, (3.18) guarantees the analogue of (3.11), and then the proof of Proposition 3.4 still goes through. This is sufficient to extend the proof of Thm. 3.5.

4. Visco-Energetic solutions for a damage system

We consider a rate-independent damage process in a nonlinearly elastic material, located in a bounded Lipschitz domain \(\Omega \subset \mathbb{R}^d\). The body is subject to a time-dependent external force and it is clamped on a portion \(\Gamma_D\) of its boundary \(\partial \Omega\), fulfilling \(\mathcal{H}^{d-1}(\Gamma_D) > 0\). Hence, on \(\Gamma_D\) the displacement field \(\tilde{u} : (0, T) \times \Omega \to \mathbb{R}^d\) is prescribed by the time-dependent Dirichlet condition
\[
\tilde{u}(t) = u_b(t) \quad \text{on } \Gamma_D, \ t \in (0, T).
\]
From now on, in Sec. 3 we will use the splitting \(\tilde{u} = u + u_b\) with \(u = 0\) on \(\Gamma_D\) and, with slight abuse of notation, \(u_b\) the extension of the Dirichlet datum into the domain \(\Omega\). The state variables of the damage process will thus be \(u\) and a scalar damage variable \(z : (0, T) \times \Omega \to \mathbb{R}\), with values in the interval \([0, 1]\), such that \(z(t, x) = 1\) means no damage and \(z(t, x) = 0\) means maximal damage in the neighborhood of the point \(x \in \Omega\), at the process time \(t \in [0, T]\).

We will confine the discussion to a gradient theory for damage, thus accounting for an internal length scale. Namely, we allow for the gradient regularizing contribution \(\int_{\Omega} |\nabla d| \, dx\) to the driving energy, along the footsteps of [MR06, TM10, Tho13] analyzing Energetic solutions. More precisely, the condition \(r > d\) imposed in [MR06] was weakened to \(r > 1\) in [TM10] and, further, to \(r = 1\) (i.e. a BV-gradient) in [Tho13]. Here we will stay with the case \(r > 1\), possibly strengthening this condition to \(r > d\) when considering a viscous correction that involves a norm different from that of the rate-independent dissipation potential, cf. Thm. 3.11 ahead.

All in all, we consider the rate-independent PDE system for damage
\[
-\text{div}(D_z W(x, \varepsilon(u + u_b), z)) = f \quad \text{in } \Omega \times (0, T),
\]
\[
\partial R(x, \dot{z}) - \Delta_r z + \partial I_{[0,1]}(z) \ni -D_z W(x, \varepsilon(u + u_b), z) \quad \text{in } \Omega \times (0, T),
\]
(4.2)
supplemented with the homogeneous Dirichlet condition \( u = 0 \) on \( \Gamma_0 \), with the Neumann boundary conditions
\[
\varepsilon(u + u_b)\nu = g \text{ on } \Gamma_N = \partial\Omega \setminus \Gamma_D \quad (\text{where } \nu \text{ is the exterior unit normal to } \partial\Omega), \quad \partial_z u = 0 \text{ on } \partial\Omega.
\]

The conditions on the elastic energy density \( W = W(x, e, z) \) (whose Gâteaux derivatives w.r.t. \( e \) and \( z \) are denoted by \( D_e \) and \( D_z \), respectively), and on the body and surface forces \( f, g \) will be specified in (4.3) and (4.6) ahead; \( -\Delta_r \) is the \( r \)-Laplacian operator and \( \partial I_{[0,1]} : \mathbb{R} \to \mathbb{R} \) is the subdifferential of the indicator function \( I_{[0,1]} \); enforcing the constraint \( 0 \leq r \leq 1 \) a.e. in \( \Omega \). The dissipation potential \( R : \Omega \times \mathbb{R} \to [0, \infty) \) is given by
\[
R(x, v) := \begin{cases} \kappa(x)v \text{ if } v \leq 0, \\ \infty & \text{otherwise} \end{cases} \quad \kappa \in L^\infty(\Omega), \quad 0 < \kappa_0 < \kappa(x) \text{ for a.a. } x \in \Omega. \quad (4.3)
\]

The Energetic formulation of the damage system (4.2) is given in the following setup:

**Ambient space:** \( X = U \times Z \) with
\[
U = W_{1,0}^1(\Omega; \mathbb{R}^d) := \{ u \in W_{1,0}^{1,p}(\Omega; \mathbb{R}^d) : u = 0 \text{ on } \Gamma_D \} \text{ and } Z = W^{1,r}(\Omega). \quad (4.4a)
\]

Here, \( p \) is as in (4.5c) below, and \( r > 1 \). The topology \( \sigma_U \) on the space of admissible displacements is the weak topology of \( W_{1,0}^{1,p}(\Omega; \mathbb{R}^d) \); analogously, \( \sigma_Z \) is the weak \( W^{1,r}(\Omega) \)-topology.

**Energy functional:** \( \mathcal{E} : [0, T] \times X \to (-\infty, \infty) \) is given by the sum of (1) the stored elastic energy \( W \); (2) a term \( \beta \) encompassing the gradient regularization and the indicator term \( I_{[0,1]}(z) \); (3) the power of the external loadings, with the force term \( \ell \) comprising volume and surface forces \( f \) and \( g \) via
\[
\langle \ell(t), v \rangle := \langle f(t), v \rangle + \langle g(t), v \rangle,
\]

where the duality pairings involving the forces \( f \) and \( g \) are not specified for simplicity, and the duality pairing between \( \ell \) and \( u + u_b(t) \) will be settled below, cf. (4.6). Namely, \( \mathcal{E} \) is defined by
\[
\mathcal{E}(t, u, z) := W(t, u, z) + \beta(z) - \langle \ell(t), u + u_b(t) \rangle \text{ with } \begin{cases} W(t, u, z) := \int_\Omega W(x, \varepsilon(u) + \varepsilon(u_b(t)), z) \, dx, \\ \beta(z) := \int_\Omega \left( \frac{1}{2} |\nabla z|^r + I_{[0,1]}(z) \right) \, dx, \quad r > 1. \end{cases} \quad (4.4b)
\]

Then,
\[
D_z := \{ z \in W^{1,r}(\Omega) : z(x) \in [0, 1] \text{ for a.a. } x \in \Omega \}.
\]

**Dissipation distance:** We consider the asymmetric extended quasi-distance \( d_z : Z \times Z \to [0, \infty] \) defined by
\[
d_Z(z, z') := \mathcal{R}(z' - z) \quad \text{with } \mathcal{R} : L^1(\Omega) \to [0, \infty], \quad \mathcal{R}(\zeta) := \int_\Omega R(x, \zeta(x)) \, dx. \quad (4.4c)
\]

Along the footsteps of [10], for the elastic energy density \( W \) we assume
\[
W(x, \cdot, \cdot) \in C^0(\mathbb{M}_{sym}^{d \times d} \times \mathbb{R}) \quad \text{for a.a. } x \in \Omega, \quad W(\cdot, e, z) \text{ measurable on } \Omega \quad \text{for all } (e, z) \in \mathbb{M}_{sym}^{d \times d} \times \mathbb{R}; \quad (4.5a)
\]
\[
W(x, \cdot, \cdot) \text{ is convex for every } (x, z) \in \Omega \times \mathbb{R}; \quad (4.5b)
\]
\[
\exists c_1, C_1 > 0 \forall p \in (1, \infty) \forall (x, e, z) \in \Omega \times \mathbb{M}_{sym}^{d \times d} \times \mathbb{R} : \quad W(x, e, z) \geq c_1 |e|^p - C_1; \quad (4.5c)
\]
for all \( (x, z) \in \Omega \times [0, 1] \) we have \( W(x, \cdot, \cdot) \in C^1(\mathbb{M}_{sym}^{d \times d}) \) and
\[
\exists c_2, C_2 > 0 \forall (x, e, z) \in \Omega \times \mathbb{M}_{sym}^{d \times d} \times \mathbb{R} : \quad |D_e W(x, e, z)| \leq c_2(W(x, e, z) + C_2); \quad (4.5d)
\]
\[
\exists c_3, C_3 > 0 \forall (x, e, z) \in \Omega \times \mathbb{M}_{sym}^{d \times d} \times \mathbb{R} \quad \text{with } \tilde{z} \leq z \quad \text{there holds} \quad W(x, e, \tilde{z}) \leq W(x, e, z) \leq c_3(W(x, e, z) + C_3). \quad (4.5e)
\]

While referring to [10] Sec. 3 for all details, here we may comment that (4.5c) enters in the proof of the power-control condition \( \langle A.3 \rangle \) for the energy functional \( \mathcal{E} \) (4.4b), whereas the ‘monotonicity’ type requirement (4.5c) is helpful for the closedness condition \( \langle C. \rangle \). As for the data \( \ell \) and \( u_b \), we require
\[
\begin{align*}
u_b & \in C^1([0, T]; W_{1,\infty}(\Omega; \mathbb{R}^d)); \\
\ell & \in C^1([0, T]; W^{-1,p'}(\Omega; \mathbb{R}^d)).
\end{align*}
\]
so that the power of the external loadings features the duality pairing between $W^{-1,p'}_0(\Omega; \mathbb{R}^d)$ and $W^{1,p}_0(\Omega; \mathbb{R}^d)$.

**The viscous correction:** We will either take a viscous correction of the form

$$\delta_Z(z, z') = h(\mathbb{R}(z' - z)),$$

with $h$ as in (2.20), or consider the viscous correction

$$\delta_Z : Z \times Z \to [0, \infty] \text{ defined by } \delta_Z(z, z') := \begin{cases} \frac{1}{q} \| z - z' \|_{L^q(\Omega)}^q & \text{if } z, z' \in L^q(\Omega), \\ \infty & \text{otherwise,} \end{cases} \quad \text{and } q > 1$$

(cf. also Remark 4.2 ahead).

The main result of this section guarantees the existence of VE solutions of the rate-independent damage system $(X, \mathcal{E}, d_Z)$ given by (4.4).

**Theorem 4.1.** Assume (4.3), (4.5), and (4.6). If the viscous correction $\delta_Z$ is given by (4.7a), suppose in addition that

$$r > d. \quad \text{ (4.8)}$$

Then, for every $z_0 \in D_z$ there exists a VE solution $(u, z)$ of the rate-independent damage system $(X, \mathcal{E}, d_Z)$ (4.4) with the viscous correction $\delta_Z$ from (4.7), such that $z(0) = z_0$ and

$$u \in L^\infty(0, T; W^{1,p}(\Omega; \mathbb{R}^d)), \quad z \in L^\infty(0, T; W^{1,r}(\Omega)) \cap BV([0, T]; L^1(\Omega)). \quad \text{ (4.9)}$$

The proof will be carried out in Sec. 4.1 below.

**Remark 4.2.** The condition $r > d$ can be weakened to the requirement

$$r > \frac{qd}{q + d}, \quad \text{ (4.10)}$$

on $r$, $q$, and the space dimension $d$, provided that we replace the viscous correction (4.7a) by

$$\delta_Z : Z \times Z \to [0, \infty] \text{ given by } \delta_Z(z, z') := \begin{cases} \frac{1}{q} \| z - z' \|_{L^q(\Omega)}^q & \text{if } z, z' \in L^q(\Omega), \\ \infty & \text{otherwise} \end{cases} \quad \text{ (4.11)}$$

and $\gamma > 1$ satisfying a further compatibility condition with $r$ and $q$, cf. (4.20) in Remark 4.2 ahead.

**Remark 4.3** (VE solutions are in between E and BV solutions (I)). The application of the VE concept to damage reveals that this weak solvability notion has an intermediate character between Energetic and Balanced Viscosity solutions. Indeed,

- When the viscous correction is given by (4.7a), then the existence theory for VE-solutions works under the same conditions as for E solutions, cf. [TM10]. In particular, it is possible to consider a gradient regularization with an arbitrary exponent $r > 1$; the restriction $r > d$ (or (4.10)) comes into play only upon choosing the viscous correction (4.7b) (or (4.11)).

- Balanced Viscosity solutions to the rate-independent system (4.2) have been in turn addressed in [KRZ18], with a quadratic viscous regularization (modulated by a vanishing parameter). The vanishing-viscosity analysis developed in [KRZ18] crucially relies on the requirement $r > d$ and, additionally, on the quadratic character of the elastic energy density $W$, as well as on smoothness requirements on the reference domain $\Omega$ (the smoothness of $\Omega$ can be dropped if the nonlinear $r$-Laplacian is replaced by a less standard fractional Laplacian regularization, cf. [KRZ13]). Here, instead, we can allow for an energy density $W$ of arbitrary $p$-growth and we do not need to restrict to smooth domains.

4.1. **Proof of Theorem 4.1.** In what follows, we are going to check that the rate-independent damage system $(X, \mathcal{E}, d_Z)$ given by (4.4) complies with Assumptions $< A >$, $< B >$, and $< C >$ of Theorem 2.12 (it is immediate to see that $< T >$ is satisfied). As it will be clear from the ensuing proof, $< A >$ and $< C >$ can be checked under the sole condition that the exponent $r$ is strictly bigger than 1. It is in the proof of $< B >$, in the case the viscous correction $\delta$ is given by (4.7b), that the restriction $r > d$ comes into play.
\[ \exists c_4, C_4 > 0 \forall (t, u, z) \in [0, T] \times U \times \mathcal{Z} : \mathcal{E}(t, u, z) \geq c_4(\|u\|^p_{W^{1,p}(\Omega; \mathbb{R}^d)} + \|z\|^q_{W^{1,q}(\Omega)}) - C_4. \] (4.12)

Hence, the sublevels of \( \mathcal{E}(t, \cdot, \cdot) \) are bounded in \( W^{1,p}(\Omega; \mathbb{R}^d) \times W^{1,r}(\Omega) \), uniformly w.r.t. \( t \in [0, T] \). In [TM10 Lemma 3.4] it was proved that \( \mathcal{E}(t, \cdot, \cdot) \) is sequentially lower semicontinuous w.r.t. the weak topology on \( W^{1,p}(\Omega; \mathbb{R}^d) \times W^{1,r}(\Omega) \). In view of (4.10), a standard modification of that argument yields the lower semicontinuity of \( \mathcal{E} \), hence \( \mathcal{E} \) satisfies the coercivity estimate (4.13). 

It was shown in [TM10 Thm. 3.7] that there exist constants \( c_5, C_5 > 0 \) such that for all \( (u, z) \in [0, T] \times U \times D_z \) the function \( t \mapsto \mathcal{E}(t, u, z) \) belongs to \( C^1([0, T]) \), with

\[
\partial_t \mathcal{E}(t, u, z) = \int_{\Omega} \mathcal{D}_s \mathcal{W}(x, \varepsilon(u + w_{b_l}(t)), z) : \varepsilon(\delta_{b_l}(t)) dx - \langle \ell(t), u + w_{b_l}(t) \rangle - \langle \ell(t), \delta_{b_l}(t) \rangle 
\]

and

\[
|\partial_t \mathcal{E}(t, u, z)| \leq c_5(\mathcal{E}(t, u, z) + C_5) \quad \text{for all} \quad (t, u, z) \in (0, T) \times D_r
\]

whence (2.29). We now check \( \mathcal{E} \) is left-continuous on the sublevels of \( \mathcal{F}_0 \) since the latter subsets are bounded in \( W^{1,r}(\Omega) \) by (4.12) and \( W^{1,r}(\Omega) \subseteq L^1(\Omega) \). It remains to prove the conditional upper semicontinuity (2.13b) of \( \partial_t \mathcal{E} \). For this, we apply [TM10 Lemma 3.11], ensuring that \( \partial_t \mathcal{E} \) complies with (2.14). Then, we are in a position to apply Proposition 2.23 and conclude the validity of property (2.15).

**Assumption < C >:** We will verify property (2.27) in the case of the viscous correction (4.13) (the case (4.15) can be handled with similar calculations). Let \( (t_n, u_n, z_n) \in (0, T) \times D_r \) fulfill the conditions of (2.27), and let \( (u', z') \) be any element in \( M(t, z) \). Preliminarily, from \( \sup_n \mathcal{E}(t_n, u_n, z_n) < \infty \) we deduce, via (4.12), that the sequence \( (z_n)_{n \in \mathbb{N}} \) is bounded in \( W^{1,r}(\Omega) \) and, thus, that \( z_n \rightarrow z \) in \( W^{1,r}(\Omega) \) as \( n \rightarrow \infty \). Since \( 0 \leq z_n \leq 1 \) a.e. in \( \Omega \), we then infer that \( z_n \rightarrow z \) in \( L^r(\Omega) \) for all \( s \in [1, \infty) \). For the sequence \( (u'_n, z'_n) \), we borrow the construction for the \textit{mutual recovery sequence} devised in the proof of [TM10 Thm. 3.14]. Note that this construction is in fact applicable to any \( (u', z') \in U \times D_z \) such that \( \mathcal{R}(z' - z) < \infty \). In particular, we pick \( u' \in \text{Argmin}_{u \in U} \mathcal{E}(t, u, z') \). Namely, we set for every \( n \in \mathbb{N} \)

\[
u'_n := u' \]
\[z'_n := \min\{(z' - \delta_n)^+, z_n\} = \begin{cases} (z' - \delta_n)^+ & \text{if } (z' - \delta_n)^+ \leq z_n, \\ z_n & \text{if } (z' - \delta_n)^+ > z_n, \end{cases} \quad \text{with } \delta_n := \|z_n - z\|^{1/r}_{L^r(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty.\] (4.13)

Observe that this construction gives \( z'_n \in W^{1,r}(\Omega) \) as well as \( 0 \leq z'_n \leq 1 \) a.e. in \( \Omega \), so that \( \mathcal{R}(z'_n - z_n) < \infty \). In the proof of [TM10 Thm. 3.14] it is shown that

\[
z'_n \rightarrow z' \quad \text{in } W^{1,r}(\Omega) \quad \text{as } n \rightarrow \infty. \] (4.14)

Slightly adapting the argument from [TM10 Thm. 3.14] to allow for a sequence \( (t_n)_{n \in \mathbb{N}} \) of times converging to \( t \), we find that

\[
\limsup_{n \rightarrow \infty} (\mathcal{E}(t_n, u'_n, z'_n) + d_Z(z_n, z'_n)) \leq \mathcal{E}(t, u', z') + d_Z(z, z').
\]

Therefore, for the reduced energy \( \mathcal{I}(t, z) = \min_{u \in U} \mathcal{E}(t, u, z) \) we deduce

\[
\limsup_{n \rightarrow \infty} (\mathcal{I}(t_n, z'_n) + d_Z(z_n, z'_n)) \leq \mathcal{I}(t, z') + d_Z(z, z'), \] (4.15)

where we have used that \( \mathcal{I}(t_n, z'_n) \leq \mathcal{E}(t_n, u'_n, z'_n) \) and that \( \mathcal{I}(t, z') = \mathcal{E}(t, u', z') \) by our choice of \( u' \). On the other hand, again using that \( 0 \leq z'_n \leq 1 \) a.e. in \( \Omega \), from (4.14) we infer that \( z'_n \rightarrow z' \) in \( L^r(\Omega) \) for every \( s \in [1, \infty) \). All in all, we gather that \( z_n \rightarrow z \) and \( z'_n \rightarrow z' \) in \( L^r(\Omega) \). Therefore,

\[
\lim_{n \rightarrow \infty} \delta_Z(z_n, z'_n) = \delta_Z(z, z'). \] (4.16)

which, combined with (4.15), finishes the proof of property (2.27).
\textbf{Assumption} $< B >$: The viscous correction $\delta$ from (4.10) clearly complies with $< B.1 >$ and $< B.2 >$. To check $< B.3 >$, we verify property (2.23). Preliminarily, observe that with the Gagliardo-Nirenberg inequality we have

$$
\delta_Z(z',z) = \frac{1}{q} \|z-z''\|_{L^q(\Omega)} \leq C \frac{\|z-z''\|_{W^{1,r}(\Omega)}^{\theta q}}{\|z-z''\|_{L^1(\Omega)}^{(1-\theta)q}} \quad (4.17)
$$

with

$$
\frac{1}{q} = \theta \left( \frac{1}{r} - \frac{1}{d} \right) + 1 - \theta \quad (4.18)
$$

Since $r > d$, there exists $\theta \in (0,1)$ complying with (4.18).

Let us now consider $(t, z) \in \mathcal{S}_D$ and a sequence $(t_n, z_n)_{n} \subset \mathcal{S}_D$, $t_n \uparrow t$, $z_n \rightharpoonup z$ in $W^{1,r}(\Omega)$, $\mathcal{R}(z_n-z) \to 0$. Then,

$$
\sup_{n \in \mathbb{N}} \|z_n\|_{W^{1,r}(\Omega)} \leq C \quad (4.19)
$$

Therefore,

$$
\limsup_{(t_n, z_n) \rightharpoonup (t, z)} \frac{\delta_Z(z_n, z)}{d_Z(z_n, z)} \overset{(1)}{\leq} C \limsup_{(t_n, z_n) \rightharpoonup (t, z)} \frac{\|z-z_n\|_{W^{1,r}(\Omega)}}{\|z-z_n\|_{L^1(\Omega)}} \overset{(2)}{\leq} C' \limsup_{(t_n, z_n) \rightharpoonup (t, z)} \|z-z_n\|_{L^1(\Omega)}^{(1-\theta)q-1} \overset{(3)}{=} 0,
$$

where (1) follows from (4.17), (2) from (4.19), and (3) from the fact that, since $r > d$, the exponent $\theta$ in (4.18) fulfills $(1-\theta)q > 1$. This finishes the proof of (2.23).

\textbf{Conclusion of the proof:} Theorem 2.12 applies, yielding the existence of a VE solution. The summability properties (4.19) for $u$ and $z$ follow from combining the coercivity property (4.12) with the energy bound $\sup_{t \in [0,T]} |\mathcal{E}(t, u(t), z(t))| \leq C$, cf. (2.39) in Remark 2.9

\begin{remark}

Remark 4.4. Observe that (4.10) is the sharpest condition ensuring that $\theta$ given by (4.15) is in $(0,1)$. The requirement $r > d$ can be weakened to (4.10), provided that we replace the viscous correction $\delta$ from (4.15) by that in (4.11), with $\gamma > 1$ chosen in such a way that $\theta$ from (4.15) fulfills $(1-\theta)\gamma > 1$. This amounts to imposing the following condition on $\gamma$

$$
\gamma \left( \frac{1}{q} - \frac{1}{q-1} \frac{d-r}{dr+d-r} \right) > 1 \quad (4.20)
$$

For instance, if $d = 3$ and $r = 2$ (i.e. we consider the standard Laplacian regularization), then $q = 2$ complies with the compatibility condition (4.10). An admissible viscous correction would then be

$$
\delta_Z(z, z') := \frac{1}{2} \|z-z\|_{L^2(\Omega)}^\gamma \quad \text{with} \quad \gamma > \frac{5}{2}.
$$

\end{remark}

\section{Visco-Energetic solutions for plasticity at finite strains}

We consider a model for elastoplasticity at finite strains in a bounded body $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Finite plasticity is based on the multiplicative decomposition of the gradient of the elastic deformation $\varphi : \Omega \to \mathbb{R}^d$ into an elastic and a plastic part, i.e. $\nabla \varphi = F_{el} P$ with $P \in \mathbb{R}^{d \times d}$ the plastic tensor, usually assumed with determinant $\det(P) = 1$. While the elastic part $F_{el} = \nabla \varphi P^{-1}$ contributes to energy storage and is at elastic equilibrium, energy is dissipated through changes of the plastic tensor, which thus plays the role of a (dissipative) internal variable.

The model for rate-independent finite-strain plasticity we address was first analyzed in [MM09] within the framework of energetic solutions. The PDE system in the unknowns $(\varphi, P)$ can be formally written as

$$
\varphi(t) \in \text{Argmin} \left( \int_{\Omega} W(x, \nabla \varphi P^{-1}(t)) \, dx - \langle \ell(t), \varphi \rangle : \varphi \in \mathcal{F} \right), \quad t \in (0,T),
$$

$$
\partial_t \dot{P} P^{-T} + (\nabla \varphi P^{-1})^T D_{\varphi} W(x, \nabla \varphi P^{-1}) P^{-T}
+ D_P H(x, P, \nabla P) - \text{div}(D_{\nabla P} H(x, P, \nabla P)) = 0, \quad (x, t) \in \Omega \times (0,T). \quad (5.1a)
$$
Here, $W = W(x, F)$ is the elastic energy density, $\ell$ is a time-dependent loading, e.g. associated with an applied body force $f$ and a traction $g$ on the Neumann part $\partial \Omega$, $\mathcal{F}$ is the set of admissible deformations (cf. (5.1b) below), the dissipation potential $R(x, \cdot)$ is 1-homogeneous, and the energy density $H$ encompasses hardening and regularizing effects through the term $\int_{\Omega} |\nabla P|^r \, dx$, for some $r > 1$ specified later. System (5.1) is further supplemented with a time-dependent Dirichlet condition for $\varphi$

$$\varphi(t, x) = \phi_b(t, x) \quad (t, x) \in [0, T] \times \Gamma_D, \quad (5.1b)$$

with $\phi_b : [0, T] \times \Gamma_D \to \mathbb{R}^d$ given on the Dirichlet boundary $\Gamma_D \subset \partial \Omega$ such that $\mathcal{R}^{d-1}(\Gamma_D) > 0$. Following [FM06, MM09], to treat (5.1b) compatibly with the multiplicative decomposition of $\nabla \varphi$, we will seek for $\varphi$ in the form of a composition

$$\varphi(t, x) = \phi_b(t, y(t, x)) \quad \text{with } y(t, \cdot) \text{ fulfilling } y = \text{Id} \text{ on } \Gamma_D, \quad (5.2)$$

where we have denoted by the same symbol the extension of $\phi_b$ to $[0, T] \times \mathbb{R}^d$, cf. (5.6a) below.

Therefore, we consider the pair $(y, P)$ as state variables and, accordingly, the Energetic formulation of system (5.1) is given in the following setup:

**Ambient space:** we take $X = U \times Z$, with

$$U := \{ y \in W^{-q_Y}((\Omega; \mathbb{R}^d) : \text{Id on } \Gamma_D \} \quad \text{for } q_Y > 1 \text{ to be specified later, and}$$

$$Z = \{ P \in W^{1,r}((\Omega; \mathbb{R}^{d \times d}) \cap L^{q_P}((\Omega; \mathbb{R}^{d \times d}) : P(x) \in G \quad \text{for a.a. } x \in \Omega \}, \quad q_P, r > 1 \text{ specified below.} \quad (5.3a)$$

Here, $G$ is a Lie subgroup of $\text{GL}^+(d) := \{ P \in \mathbb{R}^{d \times d} : \det(P) > 0 \}$. From now on, we will focus on the case $G = \text{SL}(d) := \{ P \in \mathbb{R}^{d \times d} : \det(P) = 1 \}$

**Energy functional:** $\mathcal{E} : [0, T] \times X \to (-\infty, \infty]$ is given by

$$\mathcal{E}(t, y, P) := \mathcal{E}_1(P) + \mathcal{E}_2(t, y, P). \quad (5.3b)$$

The functional $\mathcal{E}_1 : [0, T] \times Z \to \mathbb{R}$ includes the hardening and gradient regularizing terms, i.e.

$$\mathcal{E}_1(P) = \int_{\Omega} H(x, P(x), \nabla P(x)) \, dx \quad \text{with } H : \Omega \times \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \to \mathbb{R} \text{ fulfilling (5.4) below.}$$

The stored elastic energy $\mathcal{E}_2$ reflects the multiplicative split for the deformation gradient $\nabla \varphi = \nabla \phi_b(t, y) \nabla y$ due to (5.2), and it is thus of the form

$$\mathcal{E}_2(t, y, P) := \int_{\Omega} W(x, \nabla \phi_b(t, y) \nabla y P^{-1}) \, dx - \langle \ell(t), \phi_b(t, y) \rangle_{W^{1,q_Y}},$$

with the elastic energy density $W$ specified ahead and $\nabla \phi_b$ the gradient of $\phi_b$ w.r.t. the variable $y$.

**Dissipation distance:** Along the footsteps of [MM09] (cf. also [Mie02, HMM03]), we consider on $X$ dissipation distances of the form

$$d_{Z}(P_0, P_1) := \int_{\Omega} R(P_1(x) P_0^{-1}(x)) \, dx, \quad (5.3c)$$

where the functional $R : \text{SL}(d) \to [0, \infty)$ (for simplicity, we omit the possible $x$-dependence of $R_1$) is generated by a norm-like function $R$, cf. (5.5) below, on the Lie-algebra $T_1 \text{SL}(d)$ via the formula

$$\mathcal{R}(\Sigma) := \inf \left\{ \int_0^1 R(\Xi(s) \Xi(s)^{-1}) \, ds : \Xi \in C^1([0, 1]; G), \, \Xi(0) = 1, \, \Xi(1) = \Sigma \right\}.$$
while we require the following conditions on the elastic energy density $W : \Omega \times \mathbb{R}^{d \times d} \to [0, \infty]$: Firstly,
\[
\text{dom}(W) = \Omega \times \text{GL}^+(d), \quad \text{i.e. } W(x, F) = \infty \text{ for } \det F \leq 0 \text{ for all } x \in \Omega,
\]
(5.4b)
\[
\exists c_2 > 0 \exists j \in L^1(\Omega) \exists q_F > d \forall (x, F) \in \text{dom}(W) : \quad W(x, F) \geq j(x) + c_2 |F|^q_F,
\]
(5.4c)
and we impose a further compatibility condition between the integrability powers $q_Y, q_F, q_P$, i.e.
\[
\frac{1}{q_F} + \frac{1}{q_P} = \frac{1}{q_Y} < \frac{1}{d}.
\]
(5.4d)
Secondly, $W(x, \cdot) : \mathbb{R}^{d \times d} \to (-\infty, \infty]$ is polyconvex for all $x \in \Omega$, i.e. it is a convex function of its minors:
\[
\exists W : \Omega \times \mathbb{R}^{d \times d} \to (-\infty, \infty] \text{ such that}
\]
(i) $W$ is a normal integrand,
(ii) $\forall (x, F) \in \Omega \times \mathbb{R}^{d \times d} : W(x, F) = W(x, M(F))$,
(iii) $\forall x \in \Omega : \ W(x, \cdot) : \mathbb{R}^{d \times d} \to (-\infty, \infty]$ is convex,
(5.4e)
where $M : \mathbb{R}^{d \times d} \to R^{d \times d}$ is the function which maps a matrix to all its minors, with $\mu_d := \sum_{s=1}^{d} \binom{d}{s}^2$. Thirdly, $W$ satisfies the multiplicative stress control conditions
\[
\exists \delta > 0 \exists c_3, c_4 > 0 \forall (x, F) \in \text{dom}(W) \forall N \in N_\delta :
\]
(i) $W(x, \cdot) : \text{GL}^+(d) \to \mathbb{R}$ is differentiable,
(ii) $|D_P W(x, F) F^T| \leq c_3(W(x, F) + 1),$
(iii) $|D_P W(x, F) F^T - D_P W(x, NF) (NF)^T| \leq c_4|N - 1|(W(x, F) + 1),$
(5.4f)
with $N_\delta := \{N \in \mathbb{R}^{d \times d} : |N - 1| < \delta \}$. We refer to [MM09] for examples of functionals $H$ and $W$ complying with (5.4). Finally, the functional (whose possible dependence on $x$ is neglected by simplicity)
\[
R : T_1\text{SL}(d) \to [0, \infty) \text{ is } 1\text{-positively homogeneous and fulfills}
\]
\[
\exists c_R, C_R > 0 \forall \Sigma \in T_1\text{SL}(d) : \quad c_R|\Sigma| \leq R(\Sigma) \leq C_R|\Sigma|,
\]
(5.5)
cf. [HMM03] for examples in von-Mises and single-crystal plasticity. For the Dirichlet loading $\phi_0$ we require
\[
\phi_0 \in C^1([0, T] \times \mathbb{R}^d; \mathbb{R}^d), \quad \nabla \phi_0 \in \text{BC}^1([0, T] \times \mathbb{R}^d; \mathbb{R}^{d \times d}),
\]
\[
\exists c_5 > 0 \forall (t, x) \in [0, T] \times \mathbb{R}^d : \quad |\nabla \phi_0(t, x)^{-1}| \leq c_5,
\]
(5.6a)
where BC stands for bounded continuous. Finally, on the external load $\ell$ we impose
\[
\ell \in C^1([0, T]; W^{1,q_Y}(\Omega; \mathbb{R}^d)^*)
\]
(5.6b)
The viscous correction: We will take viscous corrections
\[
(1) \text{ either of the form}
\]
\[
\delta_Z(P_0, P_1) = h(d_Z(P_0, P_1)) \quad \text{with } h \text{ as in [2.2.6]},
\]
(5.7a)
\[
(2) \text{ or we define } \delta_Z : Z \times Z \to [0, \infty] \text{ by}
\]
\[
\delta_Z(P_0, P_1) := \begin{cases} \int_\Omega R_q((P_1(x) - P_0(x))P_1(x)^{-1})dx = \int_\Omega R_q(P_1(x)P_0(x)^{-1} - 1)dx & \text{if } R_q(P_1P_0^{-1} - 1) \in L^1(\Omega), \\
\infty & \text{otherwise}, \end{cases}
\]
\[
\text{for a given convex lower semicontinuous functional } R_q : T_1\text{SL}(d) \to [0, \infty) \text{ fulfilling}
\]
\[
R_q(\Sigma) = R_q(-\Sigma) \text{ for all } \Sigma \in T_1\text{SL}(d) \quad \text{and } \lim_{\Sigma \to 0} \frac{R_q(\Sigma)}{|\Sigma|^q} = C_q \in (0, \infty) \text{ for some } q > 1.
\]
(5.7b)
For our existence result of VE solutions to the rate-independent system $(X, E, d_Z)$ from (5.3), like for the damage system in Sec. 4 we shall strengthen the condition $r > 1$ to $r > d$ when addressing the non-trivial viscous correction (5.7b).

**Theorem 5.1.** Assume (5.3), (5.6), and (5.7). Furthermore, if the viscous correction $\delta_Z$ is given by (5.7b), suppose in addition that $r > d$. Then, for every $P_0 \in Z$ there exists a VE solution $(y, P)$ of the rate-independent finite-plasticity system $(X, E, d_Z)$, with the viscous correction $\delta_Z$ from (5.7b), such that $P(0) = P_0$ and

$$y \in L^\infty(0, T; W^{1,q}\left(\Omega; \mathbb{R}^d\right)), \quad P \in L^\infty(0, T; W^{1,r}\left(\Omega; \mathbb{R}^{d \times d}\right)) \cap BV\left([0, T]; L^1(\Omega; \mathbb{R}^{d \times d})\right). \tag{5.9}$$

The proof will be carried out in Section 5.1 ahead.

**Remark 5.2** (Extensions). The model for finite plasticity considered in [MM09] is actually more general than that addressed here, as it features a further internal variable $p \in \mathbb{R}^m$, $m \geq 1$, besides the plastic tensor $P$. The vector $p$ possibly encompasses hardening variables/slip strains and, like $P$, it is subject to a gradient regularization. Under the very same conditions as in [MM09, Thm. 3.1], it is possible to show that the energy functional comprising $p$ complies with condition $< A >$ in the metric topological setup where

$$Z = \left(L^{q_p}(\Omega; \mathbb{R}^{d \times d}) \cap W^{1,r}(\Omega; \mathbb{R}^{d \times d})\right) \times \left(L^{q_p}(\Omega; \mathbb{R}^m) \cap W^{1,r}(\Omega; \mathbb{R}^m)\right).$$

A typical example where the additional variable $p$ comes into play is isotropic hardening, cf. [MM09, Example 3.3]. There, the scalar $p \in \mathbb{R}$ measures the amount of hardening and the variables $(P, p)$ are subject to some constraint. The relevant dissipation distance accounts for such constraint and takes $\infty$ as a value.

Actually, our analysis could be extended to dissipation distances with values in $[0, \infty]$ under the very same conditions enunciated in [MM09] formula (3.4). In particular, if we take the ‘trivial’ viscous correction $\delta_Z$ from (5.7a), then the same argument as in [MM09, Sec. 5.3] allows us to check condition (2.27), whence the validity of assumption $< C >$ of the general existence Thm. 2.12. With the viscous correction in (5.7b) we can generalize our existence Thm. 5.1 for VE solutions also in the other directions outlined in [MM09, Sec. 6].

**Remark 5.3** (VE solutions are in between E and BV solutions (II)). The statement of Thm. 5.1 as well as Remark 5.2 highlight the fact that, in the case of the viscous correction (5.7a), the existence theory for VE solutions to the finite-strain plasticity system works under the very same conditions as for E solutions. Nonetheless, when bringing into play a different viscous correction such as that in (5.7b), like for the damage system in Sec. 4 we need to strengthen our conditions on the gradient regularization and in fact impose $r > d$. For E solutions to the finite-plasticity system, this requirement was made only in the cases in which the dissipation distance took values in $[0, \infty]$, cf. [MM09]. Instead, in the case of the viscous correction from (5.7b) we cannot weaken this condition even when $d_Z$ is valued in $[0, \infty)$, cf. also Remark 5.6 ahead.

At any rate, the existence of VE solutions is proved here under weaker conditions than for BV solutions. Although the latter have not yet been addressed in the context of finite plasticity, we may observe that a prerequisite for tackling them is the existence of solutions to the corresponding viscously regularized problem, which has been recently done in [MRS18]. Such viscous solutions have to fulfill an energy-dissipation balance that, in turn, relies on the validity of a suitable chain rule for the driving energy. Actually this chain rule is at the very core of the existence argument. In [MRS18] it has been possible to prove this condition, and to ultimately conclude the existence of solutions to the viscoplastic finite-strain system, only for a considerably regularized version of the energy functional $E$ from (5.3b).

### 5.1. Proof of Theorem 5.1

Preliminarily, we collect the properties of $\mathcal{R}_1$ in the following result.

**Lemma 5.4.** Assume (5.3). Then, the functional $\mathcal{R}_1: \text{SL}(d) \to [0, \infty]$ is continuous, strictly positive for $\Sigma \neq 1$, satisfies the triangle inequality $\mathcal{R}_1(\Sigma_1 \Sigma_0) \leq \mathcal{R}_1(\Sigma_0) + \mathcal{R}_1(\Sigma_1)$ for all $\Sigma_0, \Sigma_1 \in T_1\text{SL}(d)$, as well as the estimate

$$\exists C_1 > 0, \exists q_P \in [1, q_P) \forall \Sigma_0, \Sigma_1 \in \text{SL}(d) : \quad \mathcal{R}_1(\Sigma_1 \Sigma_0^{-1}) \leq C_1 (1 + |\Sigma_0|^{q_P} + |\Sigma_1|^{q_P}). \tag{5.10}$$
Moreover,
\[ \forall M > 0 \quad \exists c_M > 0 \quad \forall \Sigma \in \text{SL}(d) : \quad \mathcal{R}_1(\Sigma) \leq M \quad \Rightarrow \quad \mathcal{R}_1(\Sigma) \geq c_M |\Sigma - 1|. \] (5.11)

**Proof.** In order to check (5.11) (we refer to [MM99] Sec. 3) for the proof of all the other properties of \( \mathcal{R}_1 \), let \( \Sigma \) fulfill \( \mathcal{R}_1(\Sigma) \leq M \): we choose an infimizing sequence \( (\Xi_n)_n \subset C^1([0, 1]; G) \) such that \( \Xi_n(0) = 1 \) and \( \Xi_n(1) = \Sigma \), fulfilling \( \lim_{n \to \infty} \int_0^1 R_1(\dot{\Xi}_n(s)\Xi_n(s)^{-1}) \, ds = \mathcal{R}_1(\Sigma) \). We define
\[ s_n : [0, 1] \to [0, 1] \quad \text{by} \quad s_n(t) := c_n \int_0^t \left( 1 + R_1(\dot{\Xi}_n(s)\Xi_n(s)^{-1}) \right) \, ds, \]
with the normalization constant \( c_n := \left( 1 + \int_0^1 R_1(\dot{\Xi}_n(s)\Xi_n(s)^{-1}) \, ds \right)^{-1} \), and set
\[ t_n := s_n^{-1}, \quad \tilde{\Xi}_n := \Xi_n \circ t_n. \]
Therefore, for \( n \) sufficiently big we have
\[ 2 + M \geq 1 + \int_0^1 R_1(\dot{\Xi}_n(s)\Xi_n(s)^{-1}) \, ds = \frac{1}{c_n} \geq \frac{1}{c_n} \frac{R_1(\dot{\Xi}_n(t_n(s))\Xi_n(t_n(s))^{-1})}{(1 + R_1(\dot{\Xi}_n(t_n(s))\Xi_n(t_n(s))^{-1}))} = R_1(\dot{\Xi}_n(s)\Xi_n(s)^{-1}) \geq c_R|\dot{\Xi}_n(s)\Xi_n(s)^{-1}|, \]
for all \( s \in [0, 1] \), where the latter estimate ensues from (5.5). Hence the function \( s \mapsto \Lambda_n(s) := \dot{\Xi}_n(s)\Xi_n(s)^{-1} \) is uniformly bounded in \( L^\infty(0, 1; \mathbb{R}^{d \times d}) \). Writing
\[ \tilde{\Xi}_n(s) := 1 + \int_0^s \Lambda_n(r)\Xi_n(r) \, dr \]
we conclude, via the Gronwall Lemma, that
\[ \exists \tilde{c}_M > 0 \quad \forall n \in \mathbb{N} : \quad \|\tilde{\Xi}_n\|_{L^\infty(0, 1; \mathbb{R}^{d \times d})} \leq \tilde{c}_M. \]

Therefore,
\[ \mathcal{R}_1(\Sigma) = \lim_{n \to \infty} \int_0^1 R_1(\dot{\Xi}_n(s)\Xi_n(s)^{-1}) \, ds \geq c_R \liminf_{n \to \infty} \int_0^1 |\dot{\Xi}_n(s)\Xi_n(s)^{-1}| \, ds \geq \frac{c_R}{\tilde{c}_M} \liminf_{n \to \infty} \int_0^1 |\dot{\Xi}_n(s)| \, ds \geq \frac{c_R}{\tilde{c}_M} |\Sigma - 1| \]
where we have used the estimate \(|AB^{-1}| \geq \frac{|A|}{|B|}\). \( \square \)

**Corollary 5.5.** Assume \( \Xi \) and \( \Sigma \). Then, \( d_Z \) from (5.3) is a (possibly asymmetric) quasi-distance separating the points of \( Z \), and fulfilling
\[ \forall M > 0 \quad \exists \tilde{c}_M > 0 \quad \forall P_0, P_1 \in Z : \quad \|P_0\|_{L^\infty(\Omega)} + \|P_1\|_{L^\infty(\Omega)} \leq M \quad \Rightarrow \quad d_Z(P_0, P_1) \geq \tilde{c}_M \int_\Omega |P_1(x)P_0(x)^{-1} - 1| \, dx. \] (5.12)

Furthermore, the viscous correction \( \delta_Z \) from (5.7b) is \( \sigma_Z \)-lower semicontinuous on \( Z \times Z \).

**Proof.** To check that \( d_Z \) separates the points of \( Z \), we observe that
\[ d_Z(P_0, P_1) = 0 \quad \Rightarrow \quad \mathcal{R}_1(P_1(x)P_0^{-1}(x)) = 0 \quad \text{for a.a.} \quad x \in \Omega \quad \Rightarrow \quad P_0(x) = P_1(x) \quad \text{for a.a.} \quad x \in \Omega \]
since \( \mathcal{R}_1(\Sigma) > 0 \) if \( \Sigma \neq 1 \).

Let us now show how (5.12) derives from (5.11). From \( \|P_0\|_{L^\infty} + \|P_1\|_{L^\infty} \leq M \) it follows that \( \|P_0^{-1}\|_{L^\infty} + \|P_1\|_{L^\infty} \leq M \). To check this, we use that
\[ P_0^{-1} = \frac{1}{\det(P_0)} \operatorname{cof}(P_0)^\top = \operatorname{cof}(P_0)^\top \] (5.13)
(cof($P_0$) denoting cofactor matrix of $P_0$), as $P_0 \in \mathbb{SL}(d)$. Since $\mathcal{R}_1$ is continuous, we deduce that
\[
\exists \tilde{M}^* > 0 : \sup_{x \in \Omega} \mathcal{R}_1(p_i(x)P_0^{-1}(x)) \leq \tilde{M}^*,
\]
so that (5.11) yields $\tilde{c}_M > 0$ such that
\[
\mathcal{R}_1(p_i(x)P_0^{-1}(x)) \geq \tilde{c}_M |p_i(x)P_0^{-1}(x) - 1| \quad \text{for almost all } x \in \Omega.
\]

Then, (5.12) follows.

Finally, let $(P_n)_n \subset Z$ fulfill $P_n \to P_i$ as $n \to \infty$ in $L^{q_p}(\Omega; \mathbb{R}^{d \times d}) \cap W^{1,r}(\Omega; \mathbb{R}^{d \times d})$, for $i = 0, 1$. Therefore, $P_n \to P_i$ in $L^r(\Omega; \mathbb{R}^{d \times d}) \cap L^{q_n-\epsilon}(\Omega; \mathbb{R}^{d \times d})$ for every $\epsilon \in (0, q_P - 1]$. This implies that
\[
P_n(x) \to P_i(x), \text{ whence } (P_n(x))^{-1} \overset{\text{a.e.}}{\to} (P_i(x))^{-1} \quad \text{for a.a. } x \in \Omega, \ i = \{0, 1\},
\]
as $\det(P_n(x)) = 1$ for a.a. $x \in \Omega$, and hence $\det(P_i) \equiv 1$ a.e. in $\Omega$. All in all, we conclude that
\[
P_i(x)(P_0(x))^{-1} \to P_i(x)(P_0(x))^{-1} \quad \text{for a.a. } x \in \Omega.
\]
Therefore, if $\liminf_{n \to \infty} \delta_Z(P_n, P_n) < \infty$, we easily conclude that $\delta_Z(P_0, P_1) < \infty$ and
\[
\lim_{n \to \infty} \delta_Z(P_n, P_n) = \liminf_{n \to \infty} \int_\Omega R_q(P_n(x)(P_0(x))^{-1} - 1) \, dx \geq \int_\Omega R_q(P_i(x)(P_0(x))^{-1} - 1) \, dx = \delta_Z(P_0, P_1),
\]
i.e. the claimed lower semicontinuity of $\delta_Z$.

We are now in a position to carry out the proof of Theorem 5.1 by verifying the validity of the conditions of Theorem 2.12. As we will see, the requirement $r > d$ enters in the proof of $< B > \& < C >$, only in the case the viscous correction is given by (5.11).

Assumption $< T >$: It follows from Corollary 5.5.

Assumption $< A >$: In the proof of [MM09, Thm. 3.1] it was shown that
\[
\exists C_2, C_3 > 0 \forall (t, y, P) \in [0, T] \times U \times Z : \quad \mathcal{E}(t, y, P) \leq C_2(\|\nabla y\|_{L^{q_V}(\Omega)} + \|P\|_{L^{q_p}(\Omega)} + \|\nabla P\|_{L^r(\Omega)}) - C_3.
\]
In view of Korn’s inequality, this yields that the sublevels of $\mathcal{E}(t, \cdot, \cdot)$ are bounded in the space $V := W^{1,q_V}(\Omega; \mathbb{R}^d) \times W^{1,r}(\Omega; \mathbb{R}^{d \times d})$, uniformly w.r.t. $t \in [0, T]$, i.e.
\[
\forall S > 0 \exists R_S > 0 \forall (t, y, P) \in [0, T] \times U \times Z : \quad |\mathcal{E}(t, y, P)| \leq S \Rightarrow (y, P) \in \overline{B}_R^V
\]
cf. Notation (1.1). We will now show that
\[
\left( t_n \to t \text{ in } [0, T], \ y_n \to y \text{ in } W^{1,q_V}(\Omega; \mathbb{R}^d), \ P_n \to P \text{ in } W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \cap L^{q_p}(\Omega; \mathbb{R}^{d \times d}) \right)
\]
\[
\Rightarrow \liminf_{n \to \infty} \mathcal{E}(t_n, y_n, P_n) \geq \mathcal{E}(t, y, P).
\]

The (sequential) lower semicontinuity of the functional $\mathcal{E}$ w.r.t. $\sigma_Z$ follows from [MM09, Thm. 5.2]. We adapt the arguments from the latter result to show the lower semicontinuity of $\mathcal{E}_1$. First of all, since $q_V > d$ by (5.4e), from $y_n \to y$ in $W^{1,q_V}(\Omega; \mathbb{R}^d)$ we deduce that $y_n \to y$ in $C^0(\overline{\Omega}; \mathbb{R}^d)$. Therefore, by (5.6a) we deduce that
\[
\nabla \phi_b(t_n, y_n) \to \nabla \phi_b(t, y) \quad \text{in } C^0(\overline{\Omega}; \mathbb{R}^d).
\]

All in all, we conclude that $\phi_b(t_n, y_n) \to \phi_b(t, y)$ in $W^{1,q_V}(\Omega; \mathbb{R}^d)$ so that, since $\ell(t_n) \to \ell(t)$ in $W^{1,q_V}(\Omega; \mathbb{R}^d)^*$ by (5.6c), we ultimately find
\[
\langle \ell(t_n), \phi_b(t_n, y_n) \rangle_{W^{1,q_V}(\Omega; \mathbb{R}^d)} \to \langle \ell(t), \phi_b(t, y) \rangle_{W^{1,q_V}(\Omega; \mathbb{R}^d)} \quad \text{as } n \to \infty.
\]

To conclude (5.17), it remains to check that
\[
\liminf_{n \to \infty} \int_\Omega W(x, \nabla \phi_b(t_n, y_n)(x)) \, dx \geq \int_\Omega W(x, \nabla \phi_b(t, y)(x)) \, dx.
\]
For this, we follow the very same arguments as in the proof of [MM09, Thm. 5.2], also exploiting (5.18). Clearly, (5.15) and (5.17) ensure the validity of $< A_1 >$ and $< A_2 >$.
It was shown in [MRS18, Lemma 6.1] that for every \((y, P) \in U \times Z\) the mapping \(t \mapsto \mathcal{E}(t, y, P)\) is differentiable on \([0, T]\), with
\[
\partial_t \mathcal{E}(t, y, P) = \int_\Omega K(x, \nabla \phi_b(t, y(x)) \nabla y(x) P(x)^{-1} : V(t, y(x)) dx - \langle \ell(t), \phi_b(t, y) \rangle_{W^{1,q_y}}, \]
with the short-hand notation \(K(x, F) := D_F W_F(x, F) F^T\) for the (multiplicative) Kirchhoff stress tensor, and \(V(t, y) := \nabla \phi_b(t, y) (\nabla \phi_b(t, y))^{-1}\). The power-control estimate \((2.9)\) holds too, cf. again [MRS18, Lemma 6.1].

Now, for all \(\Xi \in Z\) the functional \(d_Z(\cdot, \Xi)\) is left-continuous on \((Z, \sigma_Z)\) in the sense of \((2.13a)\). Indeed, from \(P_n \to P\) in \(W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \cap L^{q} (\Omega; \mathbb{R}^{d \times d})\) as \(n \to \infty\) we have that \(P_n \to P\) in \(L^{q_p - \epsilon}(\Omega; \mathbb{R}^{d \times d})\) for all \(\epsilon \in (0, q_p - 1]\). Combining the growth condition \((5.10)\) of \(\mathcal{R}_1\) and the dominated convergence theorem we deduce that
\[
d_Z(P_n, \Xi) = \int_\Omega \mathcal{R}_1(\Xi(x) P_n^{-1}) dx \to \int_\Omega \mathcal{R}_1(\Xi(x) P^{-1}) dx = d_Z(P, \Xi). \tag{5.19}\]

Therefore, we can check \(< A.3' >\), namely the conditional upper semicontinuity \((2.13b)\). This has been done in [MM09, Prop. 4.4] by resorting to Prop. \((2.9)\)

\[\Rightarrow \text{Assumption } < C > :\] We will in fact check \((2.27)\). Let \((t_n, y_n, P_n)_n\), converging to \((t, y, P)\), be a sequence as in \((2.27)\); with the very same arguments used for \(< A.3' >\), from \(\sup_{n \in \mathbb{N}} \mathcal{E}(t_n, y_n, P_n) \leq C\) we deduce that \(P_n \to P\) in \(L^{q_p - \epsilon}(\Omega; \mathbb{R}^{d \times d})\) for all \(\epsilon \in (0, q_p - 1]\). Let us now pick any \((y', P') \in U \times Z\) with \(y' \in \text{Argmin}_{u \in U} \mathcal{E}(t, y, P)\) and take the constant recovery sequence \((y'_n, P'_n)_n = (y', P')\) for all \(n \in \mathbb{N}\). Clearly, \(\lim_{n \to \infty} \mathcal{E}(t, y'_n, P'_n) = \mathcal{E}(t, y', P')\), which entails \(\lim \sup_{n \to \infty} J(t, P'_n) \leq J(t, P')\) for the reduced energy. Arguing as in the above lines, we also find \(d_Z(P_n, P'_n) = d_Z(P_n, P') \to d_Z(P, P')\) as \(n \to \infty\), which concludes the proof of \((2.27)\) in the case the viscous correction \(\delta_Z\) is the ‘trivial’ one, as in \((5.7a)\).

When \(\delta_Z\) is instead given by \((5.7b)\), we rely on the compact embedding \(W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \subseteq C^0(\overline{\Omega}; \mathbb{R}^{d \times d})\) due to \(r > d\). This guarantees that the sequence \((P_n)_n\), bounded in \(W^{1,r}(\Omega; \mathbb{R}^{d \times d})\), in fact fulfills \(P_n \to P\) in \(C^0(\overline{\Omega}; \mathbb{R}^{d \times d})\). Therefore, \(\text{cof}(P_n)^T \to \text{cof}(P)^T\) in \(C^0(\overline{\Omega}; \mathbb{R}^{d \times d})\) and thus we find
\[
P'_n P_n^{-1} = P' \text{cof}(P_n)^T \to P' \text{cof}(P)^T = P' P^ {-1} \quad \text{in } C^0(\overline{\Omega}; \mathbb{R}^{d \times d}) \tag{5.20}\]
(here we have again used that \(P_n^{-1} = \text{cof}(P_n)^T\), and analogously for \(P'\), in view of \((5.13)\) and of the fact that \(\det(P') = \det(P_n) = 1\) for every \(n \in \mathbb{N}\). Thus, by the continuity of \(R_q\) we have that \(\sup_{x \in \Omega} R_q(P'(x) P_n(x)^{-1} - 1) \leq C\). With the dominated convergence theorem we then infer that \(\delta_Z(P_n, P'_n) \to \delta_Z(P, P')\), which establishes the validity of \((2.27)\).

\[\Rightarrow \text{Assumption } < B > :\] Since \(R_q(0) = 0\) by \((5.8)\), we easily check that the viscous correction \(\delta_Z\) from \((5.7b)\) complies with \(< B.1 >\). Condition \(< B.2 >\) follows from the very same arguments as in the above lines. We will prove \(< B.3 >\) through \((2.29a)\). Let us now consider \((t, P) \in \mathscr{D}\) and a sequence \((t_n, P_n)_n \subseteq \mathcal{D}, t_n \uparrow t, P_n \to P\) in \(W^{1,r}(\Omega), d_Z(P_n, P) \to 0\). Since \(P_n \to P\) in \(C^0(\overline{\Omega}; \mathbb{R}^{d \times d})\), we may use that
\[
\exists \bar{c} > 0 \forall n \in \mathbb{N} : \quad d_Z(P_n, P) \geq \bar{c} \|PP_n^{-1} - 1\|_{L^1(\Omega; \mathbb{R}^{d \times d})} \tag{5.21}\]
thanks to \((5.11)\). Moreover, observing that, indeed, we even have that
\[
PP_n^{-1} \to 1 \quad \text{in } C^0(\overline{\Omega}; \mathbb{R}^{d \times d}) \tag{5.22}\]
(cf. \((6.20)\), in view of \((5.8)\) we find, for \(n\) sufficiently big,
\[
R_q(P(x) P_n^{-1}(x)) \leq \left( C_q + \frac{1}{2} \right) \|P(x) P_n^{-1}(x) - 1\|_q^q \quad \text{for all } x \in \Omega.
\]
Therefore,
\[
\delta_Z(P_n, P) \leq \left( C_q + \frac{1}{2} \right) \|PP_n^{-1} - 1\|_q^q \tag{5.23}\]
Ultimately, we conclude
\[
\lim_{(t_n, P_n) \rightharpoonup (t, P)} \frac{\delta_Z(P_n, P)}{d_Z(P_n, P)} \leq C \lim_{(t_n, P_n) \rightharpoonup (t, P)} \|PP_n^{q-1} - 1\|_{L^q(\Omega)}^{nq}
\]
\[
\leq C \lim_{(t_n, P_n) \rightharpoonup (t, P)} \frac{\|PP_n^{q-1} - 1\|_{L^q(\Omega)}}{\|PP_n^{q-1} - 1\|_{L^q(\Omega)}}
\]
\[
\leq C \lim_{(t_n, P_n) \rightharpoonup (t, P)} \|PP_n^{q-1} - 1\|_{L^q(\Omega)}^{(1-\theta)q} = 0.
\]

Here we have used the Gagliardo-Nirenberg inequality in the very same way as in the proof of Thm. 4.1 and the previously established convergence (5.22). Hence, we conclude condition (2.25), yielding \( < B.3 > \).

Thus, we are in a position to apply Thm. 2.12 and conclude the existence of VE solutions. The summability properties (5.9) follows from the energy bound \( \sup_{t \in (0, T)} |\mathcal{E}(t, y(t), P(t))| \leq C \), cf. (2.39), combined with the coercivity estimate (5.15). We have thus finished the proof of Thm. 5.1.

Remark 5.6. A close perusal of the proof of the validity of conditions \( < B > \) and \( < C > \), in the case of the non-trivial viscous regularization \( \delta_Z \) from (5.7b), reveals the key role played by the condition \( r > d \) (which has been for instance used in the proof of (5.21)). Unlike for the damage system tackled in Sec. 4, it is not clear how to weaken this requirement.

6. Passing from adhesive contact to brittle delamination with Visco-Energetic solutions

In this section we construct VE solutions to a rate-independent system modeling brittle delamination between two elastic bodies, by passing to the limit in the Visco-Energetic formulation of an approximating system for adhesive contact. Besides providing the existence of VE solutions for brittle delamination, Theorem 6.1 below is, in fact, a first result on the Evolutionary Gamma-Convergence of Visco-Energetic solutions.

First of all, let us briefly sketch the model. We consider delamination between two bodies \( \Omega_+, \Omega_- \subset \mathbb{R}^d \), \( d \in \{2,3\} \) along their common boundary. More precisely, throughout this section we shall suppose that
\[
\Omega_\pm, \quad \Omega := \Omega_+ \cup \Gamma_C \cup \Omega_-
\]
are Lipschitz domains,
\[
\partial \Omega = \Gamma_b \cup \Gamma_n \text{ with } \begin{cases} \mathbb{R}^{d-1}(\partial \Omega_\pm \cap \Gamma_b) > 0, \\ \Gamma_n \cap \Gamma_b = \emptyset. \end{cases}
\]

The process is modeled with the aid of an internal delamination variable \( z : [0; T] \rightarrow \Gamma_C, 0 \leq z \leq 1 \) on \( \Gamma_C \), which describes the state of the adhesive material located on \( \Gamma_C \) during a time interval \( [0; T] \). In particular, in our notation \( z(x, t) = 1 \), resp. \( z(x, t) = 0 \), shall indicate that the glue is fully intact, resp. broken, at the point \( x \in \Gamma_C \) and at the process time \( t \in [0, T] \). Within the assumption of small strains, we also consider the displacement variable \( u : \Omega \rightarrow \mathbb{R}^d \). Brittle delamination is characterized by the

\[
\text{brittle constraint} \quad z(x, t)[u(x, t)] = 0 \quad \text{on } \Gamma_C \times (0, T),
\]

where \( [u] := u^+|\Gamma_C - u^-|\Gamma_C \) is the difference of the traces on \( \Gamma_C \) of \( u^\pm = u|_{\Omega_{\pm}} \). This condition allows for displacement jumps only at points \( x \in \Gamma_C \) where the bonding is completely broken, i.e. \( z(x, t) = 0 \); at points where \( z(x, t) > 0 \) it ensures \( [u(x, t)] = 0 \), i.e. the continuity of the displacements. Therefore, (6.2) distinguishes between the crack set, where the displacements may jump, and the complementary set with active bonding, where it imposes a transmission condition on the displacements.
The (formally written) rate-independent system for brittle delamination reads

\[- \text{div}(C\varepsilon(\ddot{u})) = f \quad \text{ in } \Omega \times (0, T),
\]

\[\ddot{u} = u_b \quad \text{ on } \Gamma_0 \times (0, T), \quad C\varepsilon(\ddot{u})|\Gamma_N \nu = g \quad \text{ on } \Gamma_N \times (0, T), \]

\[\varepsilon(\ddot{u})|\Gamma_C n + \partial_u I_C(\ddot{u}, z) + \partial I_{U(x)}([\ddot{u}]) \geq 0 \quad \text{ on } \Gamma_C \times (0, T),
\]

\[\partial \mathcal{R}(x, \dot{z}) + \partial_1 I_C([\ddot{u}], z) + \partial I_{[0,1]}(z) \ni a_0 \quad \text{ on } \Gamma_C \times (0, T). \]

The static momentum balance \((6.3a)\), where \(C\) is the (positive definite, symmetric) elasticity tensor and \(f\) a body force, is coupled with a time-dependent Dirichlet condition on the Dirichlet portion \(\Gamma_0\) of the boundary \(\partial \Omega\), with outward unit normal \(\nu\) (cf. \(6.1\) below). On the Neumann part \(\Gamma_N\) a surface force \(g\) is assigned. The evolutions of \(u\) and \(z\) are coupled by the Robin-type boundary condition \((6.3c)\) on the contact surface \(\Gamma_C\), where \(\partial_u I_C : \mathbb{R}^d \rightrightarrows \mathbb{R}^d\) is the (convex analysis) subdifferential w.r.t. \(u\) of the indicator function of the set

\[C := \{(v, z) \in \mathbb{R}^d \times \mathbb{R} : \begin{bmatrix}v \\ z\end{bmatrix} \geq 0\},\]

while \(\partial I_{U(x)} : \mathbb{R}^d \rightrightarrows \mathbb{R}^d\) is the subdifferential of the indicator of

\[U(x) = \{v \in \mathbb{R}^d : v \cdot n(x) \geq 0\}, \quad x \in \Gamma_C,
\]

with \(n\) the unit normal to \(\Gamma_C\), oriented from \(\Omega_+\) to \(\Omega_-\). Hence, besides \((6.2)\), we are also imposing the non-penetration constraint \([u] \cdot n \geq 0\) in \(\Omega\) between \(\Omega_+\) and \(\Omega_-\). Finally, the flow rule \((6.3d)\) for the delamination parameter \(z\) involves the very same dissipation density \(\mathcal{R}\) from \((4.3)\), the subdifferential w.r.t. \(z\) of \(I_C\), and the coefficient \(a_0\), i.e. the phenomenological specific energy per area which is stored by disintegrating the adhesive.

From now on, we will again use the splitting \(\ddot{u} = u + u_b\), with \(u_b\) an extension of the Dirichlet datum to the whole of \(\Omega\). In view of \((6.1)\), without loss of generality we may assume that this extension fulfills

\[u_b|\Gamma_C = 0 \quad \text{on } \Gamma_C, \quad \text{so that } \begin{bmatrix}u \\ z\end{bmatrix} = \begin{bmatrix}u \\ u_b + z\end{bmatrix} = \begin{bmatrix}u\end{bmatrix}. \quad (6.4)\]

The Energetic formulation of the brittle system \((6.3)\) thus involves the following:

**Ambient space:** \(X = U \times Z\) with

\[U = H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d) := \{u \in H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d) : u = 0 \text{ on } \Gamma_0\}, \quad Z := \{z \in L^\infty(\Gamma_C) : 0 \leq z \leq 1 \text{ on } \Gamma_C\}, \quad (6.5a)\]

endowed with the weak topology \(\sigma_U\) of \(H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d)\) and with the weak*-topology \(\sigma_Z\) of \(L^\infty(\Gamma_C)\), respectively.

**Energy functional:** \(\mathcal{E} : [0, T] \times X \to (-\infty, \infty)\) is given by

\[\mathcal{E}(t, u, z) := \frac{1}{2} \int_{\Omega \setminus \Gamma_C} C\varepsilon(u + u_b) : \varepsilon(u + u_b) \, dx
\]

\[+ \int_{\Gamma_C} \left(I_{U(x)}([u]) + I_C([u], z) + I_{[0,1]}(z) - a_0 z\right) \, d\mathcal{H}^{d-1}(x) - \langle \ell(t), u + u_b(t) \rangle_{H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d)}, \quad (6.5b)\]

where the function \(\ell : [0, T] \to H^1(\Omega; \mathbb{R}^d)^*\) subsumes the body and surface forces \(f\) and \(g\). Observe that the domain of \(\mathcal{E}\) does not depend on the time variable, i.e.

\[D(\mathcal{E}(\cdot, \cdot)) = \{(u, z) \in U \times Z : [u(x)] \in U(x), \ z(x)[u(x)] = 0, \ z(x) \in [0, 1] \text{ for a.a. } x \in \Gamma_C\} \quad \text{for all } t \in [0, T].\]

**Dissipation distance:** We consider the extended asymmetric quasi-distance \(d_Z : Z \times Z \to [0, \infty]\) defined by

\[d_Z(z, z') := \mathcal{R}(z' - z) \quad \text{with } \mathcal{R} : L^1(\Gamma_C) \to [0, \infty], \quad \mathcal{R}(\zeta) := \int_{\Gamma_C} R(x, \zeta(x)) \, d\mathcal{H}^{d-1}(x) \quad (6.5c)\]

and the dissipation density \(\mathcal{R}\) from \((4.3)\). Due to the highly nonconvex character of the brittle constraint \((6.2)\), the existence of Energetic solutions to the rate-independent system \((X, \mathcal{E}, d_Z)\) from \((6.5)\) cannot be proved by directly passing to the time-continuous limit in the associated time-incremental minimization scheme. Indeed, an existence result was obtained in \([RSZ09]\) by passing to the limit in the Energetic formulation for a penalized
version of system (6.3). The resulting system is in fact a model for *adhesive contact*. The relevant energy functional, in the very same displacement and delamination variables, is given by

\[ E_k(t, u, z) := \frac{1}{2} \int_{\Omega \setminus \Gamma_c} C \varepsilon(u + u_b) : \varepsilon(u + u_b) \, dx \]

\[ + \int_{\Gamma_c} \left( I_{0}(\varepsilon)[u] + \frac{\kappa}{2} \| \varepsilon[u] \|^2 + I_{[0,1]}(z) - a_0 z \right) \, \mathcal{H}^{d-1}(x) - \langle \ell(t), u + u_b(t) \rangle_{H^1}, \quad k > 0. \]  

(6.6)

Note that the brittle constraint (6.2) is penalized by the term \( \frac{\kappa}{2} \| \varepsilon[u] \|^2 \). Via the Evolutionary Gamma-convergence theory from [MRS08], in [RSZ09] it was shown that \( E \) solutions to the adhesive contact system \((X, E_k, d_Z)\) converge as \( k \to \infty \) to \( E \) solutions to the brittle delamination system \((X, E, d_Z)\).

We aim to extend this approach to the existence of VE solutions of the brittle system. In fact, VE solutions of the adhesive contact system were tackled in [MS18] Example 4.5 with the

**Viscous correction:** \( \delta_Z : Z \times Z \to [0, \infty) \) of the form

\[ \delta_Z(z, z') := h(d_Z(z, z')) \quad \text{with } h \text{ as in (2.26)}, \]

(6.7)

cf. also Remark 6.2 below. Under the condition that

\[ u_b \in C^1([0, T]; H^1(\Omega; \mathbb{R}^d)), \quad \ell \in C^1([0, T]; H^1(\Omega; \mathbb{R}^d)^*), \]

(6.8)

the existence of VE solutions \((u_k, z_k) \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^d)) \times (L^\infty(\Gamma_c \times (0, T)) \cap BV([0, T]; L^1(\Gamma_c)))\) to the adhesive contact system \((X, E_k, d_Z)\) with the viscous correction from (6.7) was derived in [MS18] (again, observe that the summability property \( u \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^d)) \) derives from the energy bound \( \sup_{t \in (0, T]} |E_k(t, u(t), z(t))| \leq C \), cf. (2.33)).

We now address the limit passage in the VE formulation of \((X, E_k, d_Z)\) as \( k \to \infty \). From now on, we will assume for simplicity that \( k \in \mathbb{N} \).

**Theorem 6.1.** Assume (6.1), (6.4), and (6.8).

Let \((u_k, z_k)_{k \in \mathbb{N}} \subset L^\infty(0, T; H^1(\Omega; \mathbb{R}^d)) \times (L^\infty(\Gamma_c \times (0, T)) \cap BV([0, T]; L^1(\Gamma_c)))\) be a sequence of VE solutions to the rate-independent systems \((X, E_k, d_Z)\), with \( \delta_Z \) from (6.7) and initial datum \( z_0 \in D_Z \).

Then, for any sequence \((k_j)_{j \in \mathbb{N}}\) with \( k_j \to \infty \) as \( j \to \infty \) there exist a (not relabeled) subsequence \((u_{k_j}, z_{k_j})_{j \in \mathbb{N}}\) and \((u, z) \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^d)) \times (L^\infty(\Gamma_c \times (0, T)) \cap BV([0, T]; L^1(\Gamma_c)))\) such that

1. \( z(0) = z_0; \)
2. the following convergences hold as \( j \to \infty \)
   \[ u_{k_j}(t) \to u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^d) \quad \text{for all } t \in [0, T], \]
   \[ z_{k_j}(t) \rightharpoonup^* z(t) \quad \text{in } L^\infty(\Gamma_c) \quad \text{for all } t \in [0, T], \]

(6.9a)

(6.9b)

3. \((u, z)\) is a VE solution of the brittle delamination system \((X, E, d_Z)\) (6.5), with the viscous correction from (6.7), such that the minimality property (2.33) holds at all \( t \in [0, T] \setminus \bar{J} \), with \( \bar{J} \) a negligible subset of \( \{0, T\} \).

Furthermore, we have the additional convergences as \( j \to \infty \)

\[ E_{k_j}(t, u_{k_j}(t), z_{k_j}(t)) \to E(t, u(t), z(t)) \quad \text{and } \text{Var}_{d_Z,c}(z_{k_j}, [0, t]) \to \text{Var}_{d_Z,c}(z, [0, t]) \quad \text{for all } t \in [0, T]. \]

(6.10)

Observe that we are able to recover the minimality property (2.33) only almost everywhere on \((0, T)\). The proof of Thm. 6.1 will be carried out throughout Sec. 6.1 also relying on a technical result, Lemma 6.2 ahead, proved in Sec. 6.2.

**Remark 6.2.** The existence of VE solutions to the adhesive contact system \((X, E_k, d_Z)\) could be extended to the case of the ‘non-trivial’ viscous correction

\[ \delta_Z(z, z') := \frac{1}{q} \| z' - z \|^q_{L^q(\Gamma_c)}, \quad q, \gamma > 1, \]

(6.11)
as soon as a gradient regularizing term of the type $|\nabla z|^r$ is added to the energy functional $E_k$ (under the additional, technical condition that $\Gamma_C$ is a ‘flat’ $(d-1)$-dimensional surface, so that Laplace-Beltrami operators can be avoided). The exponents $r, q, \gamma$ should satisfy the compatibility condition \((1.10)\). For instance, in the case $\Omega \subseteq \mathbb{R}^3$, so that $\Gamma_C \subseteq \mathbb{R}^2$, with $r = 2$ and $q = 2$ one would have to take $\gamma > 2$.

We could perform the adhesive-to-brittle limit passage with $\delta_Z$ from \((6.11)\) by straightforwardly adapting the arguments in the proof of Thm. \((6.1)\). Anyhow, we have preferred not to do so in order to focus on the analytical difficulties related to the limit passage in the notion of VE solution.

### 6.1. Proof of Theorem \((6.1)\)

Preliminarily, let us recall the $\Gamma$-convergence properties of the adhesive contact energies $(E_k)_k$. These properties are at the core of the proof of Thm. \((6.1)\).

**Lemma 6.3.** [RSZ09 Corollary 3.2] Assume \((6.1), (6.4), \) and \((6.8)\). Then the functionals $E_k$ from \((6.6)\) $\Gamma$-converge as $k \to \infty$ to $E$ w.r.t. to the $\sigma_R$-topology of $[0,T] \times H^1(\Omega \setminus \Gamma_C; \mathbb{R}^d) \times L^\infty(\Gamma_C) \ (\text{i.e., the weak}^*\text{-topology})$, namely there hold the

- **$\Gamma$-liminf estimate:** $(t_k,u_k,z_k) \xrightarrow{k\to\infty} (t,u,z) \Rightarrow \liminf_{k \to \infty} E_k(t_k,u_k,z_k) \geq E(t,u,z)$

- **$\Gamma$-limsup estimate:** $\forall (t,u,z) \exists (t_k,u_k,z_k)_k : (t_k,u_k,z_k) \xrightarrow{k\to\infty} (t,u,z), \limsup_{k \to \infty} E_k(t_k,u_k,z_k) \leq E(t,u,z). \tag{6.12}$

In order to pass to the limit in the VE-formulation, we also need to investigate the closure, as $k \to \infty$, of the stable (in the Visco-Energetic sense) sets

$$\mathcal{K}_{Dz}^k := \{(t_k,u_k,z_k) \in [0,T] \times U \times Z : E_k(t_k,u_k,z_k) = E_k(t_k,u_k,z_k) + d_Z(z_k,z') + h(d_Z(z_k,z')) \}$$

for all $(t_k,u_k,z_k) \in \mathcal{K}_{Dz}^k$, such that $(t_k,u_k,z_k) \xrightarrow{k \to \infty} (t,u,z)$.

Recall that, by \((2.20)\) $\mathcal{K}_{Dz}^k$ is the zero set of the residual stability function

$$R_k(t,z) := \sup_{z' \in Z} \{J_k(t,z) - J_k(t,z') - d_Z(z,z') - h(d_Z(z,z'))\} \tag{6.15a}$$

with the reduced energy

$$J_k(t,z) := \inf_{u \in H^1_D(\Omega \setminus \Gamma_C; \mathbb{R}^d)} E_k(t,u,z) = \min_{u \in H^1_D(\Omega \setminus \Gamma_C; \mathbb{R}^d)} E_k(t,u,z), \tag{6.15b}$$

(the inf in the definition of $J_k$ is attained since for every $(t,z) \in [0,T] \times Z$ the functional $u \mapsto E_k(t,u,z)$ has sublevels bounded in $H^1_D(\Omega \setminus \Gamma_C; \mathbb{R}^d)$ by Korn’s inequality, cf. \((6.20)\) ahead, and is lower semicontinuous w.r.t. $H^1$-weak convergence). In fact, the study of $\lim_{k \to \infty} \mathcal{K}_{Dz}^k$ is related to the $\Gamma$-liminf (w.r.t. $\sigma_R$-topology) of the functionals $(R_k)_k$. That is why, we will further obtain the limit-inf-inequality \((6.14)\) below. Such estimate will also play a crucial role for the limit passage in the Visco-Energetic energy-dissipation balance as $k \to \infty$.

**Lemma 6.4.** Assume \((6.1), (6.4), (6.8)\). Then,

$$\lim_{k \to \infty} \mathcal{K}_{Dz}^k \subseteq \mathcal{K}_{Dz} \tag{6.13}$$

and, in fact, for every $(t_k,z_k) \in [0,T] \times Z$ there holds

$$(t_k,z_k) \xrightarrow{k \to \infty} (t,u,z) \Rightarrow \liminf_{k \to \infty} R_k(t_k,z_k) \geq R(t,z). \tag{6.14}$$

**Proof.** We start by showing \((6.14)\). We use that

$$R_k(t_k,z_k) = \sup_{z' \in Z} \{J_k(t_k,z_k) - J_k(t_k,z') - d_Z(z_k,z') - h(d_Z(z_k,z'))\}, \tag{6.15a}$$

$$R(t,z) = \sup_{z' \in Z} \{J(t,z) - J(t,z') - d_Z(z,z') - h(d_Z(z,z'))\}. \tag{6.15b}$$
Then, we will have
\[
\lim_{k \to \infty} \sup_{k} (J_k(t_k, z'_k) + d_Z(z_k, z'_k) + h(d_Z(z_k, z'_k))) \leq (\mathcal{J}(t, z') + d_Z(z, z') + h(d_Z(z, z'))) .
\]
(6.16)
Then, we will have
\[
\liminf_{k \to \infty} \mathcal{J}_k(t_k, z_k) \geq \liminf_{k \to \infty} (J_k(t_k, z'_k) - J_k(t_k, z'_k) - d_Z(z_k, z'_k) - h(d_Z(z_k, z'_k)))
\]
\[
\geq \lim_{k \to \infty} (\mathcal{J}(t, z) - \mathcal{J}(t, z') - d_Z(z, z') - h(d_Z(z, z'))).
\]
where we have also exploited the \(\Gamma\)-lim inf-estimate in (6.12). Then, (6.14) shall follow from the arbitrariness of \(z'\). We borrow the definition of the sequence \((z'_k)\) from the proof of [RSZ09, Thm. 3.3], letting
\[
z'_k := \begin{cases} z_k & \text{if } z' > 0, \\ 0 & \text{otherwise.} \end{cases}
\]
(6.17)
Since \(z' \leq z\) a.e. in \(\Gamma_c\), it is immediate to verify that \(0 \leq z'_k \leq z_k \leq 1\) a.e. in \(\Gamma_c\). Furthermore, \(z_k \to z\) in \(L^\infty(\Gamma_c)\) gives \(z'_k \to z'\) in \(L^\infty(\Gamma_c)\). Therefore,
\[
\lim_{k \to \infty} d_Z(z_k, z'_k) = \lim_{k \to \infty} \int_{\Gamma_c} \kappa(z_k(x) - z'_k(x)) d\mathcal{H}^{d-1}(x) = \lim_{k \to \infty} \int_{\Gamma_c} \kappa(z(x) - z'(x)) d\mathcal{H}^{d-1}(x) = d_Z(z, z'),
\]
(6.18)
whence \(\lim_{k \to \infty} h(d_Z(z_k, z'_k)) = h(d_Z(z, z'))\), too. Let us now consider the (unique) minimizer \(u' \in U\) for \(\mathcal{E}(t, z', z')\). We have
\[
\limsup_{k \to \infty} \mathcal{J}_k(t_k, z'_k) \leq \limsup_{k \to \infty} \mathcal{E}_k(t_k, u', z'_k) = \lim_{k \to \infty} \mathcal{E}_k(t_k, u', z'_k) = \mathcal{E}(t, u', z') = \mathcal{J}(t, z').
\]
(6.19)
Indeed, for (1) we have used the fact that \(z'_k[ u'] = 0\) a.e. in \(\Gamma_c\), which follows from \(z'[ u'] = 0\) and from the definition (6.17) of \(z'_k\). From (6.18) and (6.19) we clearly conclude (6.14), whence (6.14).

In order to show that every element \((t, u, z)\) in \(\text{Lip}_{\infty}(\mathcal{H})\) fills the \(d_Z\)-stability condition with the brittle energy functional, for every \((u', z')\) we need to exhibit a recovery sequence \((u'_k, z'_k)\) such that
\[
\limsup_{k \to \infty} (\mathcal{E}_k(t_k, u'_k, z'_k) + d_Z(z_k, z'_k) + h(d_Z(z_k, z'_k))) \leq (\mathcal{E}(t, u', z') + d_Z(z, z') + h(d_Z(z, z'))) .
\]
The sequence \((u'_k, z'_k)_k := (u'_k, z'_k)_k\) with \((z'_k)_k\) from (6.17), does the job. This finishes the proof. \(\square\)

The proof of Thm. (6.1) will be carried out in the following steps:

(1) First of all, we will show that the sequence \((z_k)_k\) of VE solutions in the statement of the theorem does admit a subsequence converging in the sense of (6.9a) to \(z\);
(2) Secondly, we will prove that \(z\) complies with the stability condition \((\text{SVE})\) for the brittle system \((X, \mathcal{E}, d_Z)\) from (6.5) and, as a byproduct, obtain convergence (6.9a) for \((u_k)_j\) from (6.5).
(3) Thirdly, we will show that \((u, z)\) fulfills the upper energy-dissipation estimate (2.41) for the brittle system also relying on Proposition 5.5 ahead;
(4) We shall thus conclude that \((u, z)\) is a VE solution to the brittle system \((X, \mathcal{E}, d_Z)\) from (6.5).

\(\triangleright\) Step 1: Since the constant \(C_0\) in (2.39) only depends on the initial data \((u_0, z_0)\), which in turn do not depend on \(k_j\), for the VE solutions \((u_k, z_k)_j\) to the adhesive contact system the following bounds are valid
\[
\exists C > 0 \forall j \in \mathbb{N} \forall t \in [0, T] : \sup_{\mathcal{E}_k(t_k(t), u_k(t), z_k(t))] + \text{Var}(z_k, [0, T]) \leq C.
\]
In turn, it follows from the positive definiteness of \(C\), Korn’s inequality and from (6.8) that
\[
\exists c_1, c_2 > 0 \forall (t, u, z) \in [0, T] \times U \times Z : \mathcal{E}_k(t, u, z) \geq c_1 \|u\|^2_{H(\Omega, \Gamma_c)} - c_2.
\]
(6.20)
We ultimately conclude that the sequences \((u_{kj})_j\) and \((z_{kj})_j\) are bounded in \(L^\infty(0,T;H^1_0(\Omega;\mathbb{R}^d))\) and in \(L^\infty(\Gamma_0 \times (0,T)) \cap BV([0,T];L^1(\Gamma_0))\), respectively. An infinite-dimensional version of Helly’s compactness theorem (cf., e.g., [MM05 Thm. 3.2]) yields that, up to a not relabeled subsequence, convergence \((6.23)\) for \((z_{kj})_j\) holds. As for \((u_{kj})_j\), for every \(t \in (0,T]\) there exist a subsequence \((k'_j)\), possibly depending on \(t\), and \(\tilde{u}(t) \in H^1_0(\Omega;\mathbb{R}^d)\) such that

\[ u_{kj}(t) \rightharpoonup \tilde{u}(t) \quad \text{in} \quad H^1_0(\Omega;\mathbb{R}^d). \quad (6.21) \]

Furthermore, mimicking the arguments in the proof of [MS13 Thm. 7.2], we also find a finer approximation property at every \(t\) in the jump set \(J_z\) of \(z\), namely

\[ \forall t \in J_z \cap (0,T) \exists (\alpha_{kj})_j, (\beta_{kj})_j \subset [0,T] \quad \text{such that} \quad \begin{cases} \alpha_{kj} \uparrow t \quad \text{and} \quad z_{kj}(\alpha_{kj}) \rightharpoonup z(t-) \quad \text{in} \quad L^\infty(\Gamma_0), \\ \beta_{kj} \downarrow t \quad \text{and} \quad z_{kj}(\beta_{kj}) \rightharpoonup z(t+) \quad \text{in} \quad L^\infty(\Gamma_0), \end{cases} \quad (6.22) \]

with obvious modifications at \(t \in J_z \cap \{0,T\}\).

\(\triangleright\) \textbf{Step 2}: Let us introduce the lim sup of the jump sets \((J_{z_{kj}})_j\), i.e. \(\bar{J} := \cap_{m \in \mathbb{N}} \cup_{j \geq m} J_{z_{kj}}\). Observe that for every \(t \in (0,T]\) \(\setminus \bar{J}\) there exists \(m_t \in \mathbb{N}\) such that for every \(j \geq m_t\) we have \(t \in [0,T] \setminus J_{z_{kj}}\). Therefore, up to taking a bigger \(m_t\) if necessary, we have \((t, u_{kj}(t), z_{kj}(t)) \in \mathcal{S}^k_{\bar{J}}\) for all \(j \geq m_t\). By virtue of \((6.19)\), we conclude that

\[ (t, \tilde{u}(t), z(t)) \in \mathcal{S}_{\bar{J}} \quad \text{for all} \quad t \in [0,T] \setminus \bar{J}. \quad (6.23) \]

From \((6.23)\) we gather, in particular, that \(\tilde{u}(t) \in \text{Argmin}_{u' \in U} E(t, u', z(t))\). Since the latter set is a singleton by Korn’s inequality, we ultimately find that \(\tilde{u}(t)\) is uniquely determined. Therefore, convergence \((6.21)\) holds at every \(t \in (0,T] \setminus \bar{J}\) along the whole sequence \((k_j)_j\). This shows \((6.19a)\) at all \(t \in (0,T] \setminus \bar{J}\).

Finally, we conclude the validity of \((6.19a)\) at every \(t \in [0,T]\) by observing that, at every \(t\) in the \textit{countable} set \(\bar{J}\) we can extract a subsequence of \((k_j)_j\) such that \((6.21)\). With a diagonal procedure we thus construct a subsequence fitting all \(t \in \bar{J}\) and \((6.19a)\) follows.

We now show that

\[ z(t-), z(t+) \in \mathcal{S}_{\bar{J}}(t) \quad \text{for all} \quad t \in (0,T), \quad z(0+) \in \mathcal{S}_{\bar{J}}(0), \quad z(T-) \in \mathcal{S}_{\bar{J}}(T). \quad (6.24) \]

In order to prove the assert at \(t \in (0,T)\) and, e.g., for \(z(t-)\), we pick a sequence \((t_n)_n \subset [0,T] \setminus \bar{J}\) with \(t_n \uparrow t\) as \(n \to \infty\), so that \(z(t_n) \rightharpoonup z(t-) \) in \(L^\infty(\Gamma_0)\) (cf. Definition \(2.1\)). From \((6.23)\) we have that \(\mathcal{S}(t_n, z(t_n)) = 0\) for all \(n \in \mathbb{N}\). With the very same arguments as in the proof of Lemma \(6.4\) it can be shown that \(\mathcal{S}\) is lower semicontinuous w.r.t. the weak-*topology of \([0,T] \times Z\). Thus, we conclude that \(\mathcal{S}(t, z(t)) = 0\).

From \((6.24)\) we clearly conclude that \((u, z)\) fulfills the stability condition \((2.41)\) at all \(t \in [0,T] \setminus J_z\), which in particular yields the minimality property \((2.38)\) at all \(t \in [0,T] \setminus J_z\). All in all, \((2.38)\) holds at every \(t \in [0,T] \setminus J_z\) with \(J = \bar{J} \cap J_z\).

\(\triangleright\) \textbf{Step 3}: Let us now take the \(\liminf\) as \(k \to \infty\) in the (upper) energy-dissipation estimate \((2.41)\) for the adhesive contact system. We handle the terms on the left-hand side by observing that

\[ \liminf_{j \to \infty} E_{kj}(t, u_{kj}(t), z_{kj}(t)) \geq \mathcal{E}(t, u(t), z(t)) \quad \text{and} \quad \liminf_{j \to \infty} \text{Var}_{dz,c}(z_{kj}, [0,t]) \geq \text{Var}_{dz,c}(z, [0,t]) \quad \text{for all} \quad t \in [0,T], \]

where the first inequality is due to the \(\Gamma\)-lim inf estimate \((6.12)\), and the second one follows from Proposition \(6.5\) below. As for the right-hand side, we observe that

\[ \partial_t E_{kj}(t, u_{kj}(t), z_{kj}(t)) = - \langle \ell(t), u_{kj}(t) + u_b(t) \rangle_{H^1} - \langle \ell(t), \dot{u}_b(t) \rangle_{H^1} \to - \langle \ell(t), u(t) + u_b(t) \rangle_{H^1} - \langle \ell(t), \dot{u}_b(t) \rangle_{H^1} = \partial_t \mathcal{E}(t, u(t), z(t)) \]

for every \(t \in [0,T]\), with \(|\partial_t E_{kj}(t, u_{kj}(t), z_{kj}(t))| \leq C\) by \((6.38)\) and the previously obtained bound for \((u_{kj})_j\) in \(L^\infty(0,T;H^1_{D0}(\Omega;\mathbb{R}^d))\). Then,

\[ \lim_{j \to \infty} \int_0^t \partial_t E_{kj}(s, u_{kj}(s), z_{kj}(s)) \, ds = \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, ds \quad \text{for all} \quad t \in [0,T], \quad (6.25) \]

and we thus conclude the upper energy-dissipation estimate \((2.41)\) for the brittle system.
\[ \textbf{Step 4, conclusion of the proof:} \] Since we have proved the stability condition (\textbf{SVE}) and the upper energy-dissipation estimate (\textbf{VE}) thanks to Proposition \textbf{2.11}, we conclude that \((u, z)\) is a VE solution of the brittle system. The energy convergence (\textbf{6.10}) ensues from the following standard argument:

\[
\begin{align*}
\limsup_{j \to \infty} (\mathcal{E}_{k_j}(t, u_{k_j}(t), z_{k_j}(t)) + \text{Var}_{d_z, c}(z_{k_j}, [0, t])) &= \mathcal{E}(0, u_0, z_0) + \lim_{j \to \infty} \int_0^t \partial_t \mathcal{E}_{k_j}(s, u_{k_j}(s), z_{k_j}(s)) \, ds \\
&\geq \mathcal{E}(0, u_0, z_0) + \int_0^t \partial_t \mathcal{E}(s, u(s), z(s)) \, ds \\
&= \mathcal{E}(t, u(t), z(t)) + \text{Var}_{d_z, c}(z, [0, t]),
\end{align*}
\]

with (1) due to (\textbf{SVE}) for the adhesive system, (2) due to (\textbf{6.25}), and (3) following from the energy balance (\textbf{VE}) for the brittle system. This finishes the proof of Theorem \textbf{6.4}.  

With the following result we obtain the key lower semicontinuity estimate for the total variation functionals exploited in Step 3 of the proof of Theorem \textbf{6.4}.

\begin{proposition}
Assume (\textbf{0.1}), (\textbf{0.4}), and (\textbf{0.8}). Let \((z_k)_k, z \in L^\infty(\Gamma C \times (0, T)) \cap BV([0, T]; L^1(\Omega))\) fulfill
\[
\begin{align*}
z_k(t) \to^* z(t) \text{ in } L^\infty(\Gamma C) \text{ for all } t \in [0, T], & \quad (6.26a) \\
\forall t \in J_z \exists (\alpha_k)_k, (\beta_k)_k \subset [0, T] \text{ such that } & \quad (6.26b)
\end{align*}
\end{proposition}

Then,

\[ \liminf_{k \to \infty} \text{Var}_{d_z, c}(z_k, [a, b]) \geq \text{Var}_{d_z, c}(z, [a, b]) \text{ for all } [a, b] \subset [0, T]. \]  

The proof follows the very same lines as the argument for [RS17, Thm. 4], to which we shall refer for all details. Let us just outline it: up to an extraction we may suppose that \(\sup_{k \in \mathbb{N}} \text{Var}_{d_z, c}(z_k, [0, T]) \leq C\). Therefore, the non-negative and bounded Borel measures \(\eta_k\) on \([0, T]\) defined by \(\eta_k([a, b]) := \text{Var}_{d_z, c}(z_k, [a, b])\) for all \([a, b] \subset [0, T]\) weakly* converge (in the duality with \(C^0([0, T])\)) to a measure \(\eta\). We observe that

\[ \eta([a, b]) \geq \limsup_{k \to \infty} \eta_k([a, b]) \geq \limsup_{k \to \infty} \text{Var}_{d_z}(z_k, [a, b]) \geq \text{Var}_{d_z}(z, [a, b]), \]  

with (1) due to the upper semicontinuity of the weak* convergence of measures on closed sets, (2) due to the fact that \(\text{Var}_{d_z, c} \geq \text{Var}_{d_z}\), and (3) due to (\textbf{6.26a}). It follows from Lemma \textbf{6.6} ahead that, at any \(t \in J_z\) and for all sequences \((\alpha_k)_k, (\beta_k)_k \subset [0, T]\) fulfilling (\textbf{6.26b}) there holds

\[ \eta(\{t\}) \geq \limsup_{k \to \infty} \eta_k(\{\alpha_k, \beta_k\}) \geq c(t, z(t^-), z(t^+)). \]  

Combining (\textbf{6.26}) and (\textbf{6.29}) and arguing in the very same way as in the proof of [RS17, Thm. 4] (cf. also [MRS16, Prop. 7.3]), we establish (\textbf{6.27}).

We conclude this section by stating a crucial lower estimate for the Visco-Energetic total variation of a sequence \((z_k)_k\) of solutions to the adhesive contact system (for notational simplicity, we drop the subsequence \((k_j)_j\) and revert to the original sequence of indexes \((k)_k\)). The total variation of the curves \(z_k\) is considered on a sequence of intervals shrinking as \(k \to \infty\) to a jump point of the limit curve \(z\).

\begin{lemma}
Assume (\textbf{0.1}), (\textbf{0.4}), and (\textbf{0.8}). Let \((z_k)_k, z \in L^\infty(\Gamma C \times (0, T)) \cap BV([0, T]; L^1(\Omega))\) fulfill (\textbf{6.26}). For any \(t \in J_z\) pick two sequences \((\alpha_k)_k\) and \((\beta_k)_k\) converging to \(t\) and fulfilling (\textbf{6.26}). Then,

\[ \liminf_{k \to \infty} \text{Var}_{d_z, c}(z, [\alpha_k, \beta_k]) \geq c(t, z(t^-), z(t^+)). \]  

The proof will be given in Sec. \textbf{6.2}.
6.2. Proof of Lemma 6.6 Let us briefly outline the proof, partially borrowed from that of [RST17 Prop. 3]:

(1) for every \( k \in \mathbb{N} \), the curve \( z_k \) has countably many jump points \( (t^k_m)_{m \in M_k} \) between \( \alpha_k \) and \( \beta_k \). Along the footsteps of [RST17], we will suitably reparameterize both the continuous pieces of the trajectory \( z_k \) and the optimal transitions \( \vartheta^k_{z,m} \) connecting the left and right limits \( z_k(t^k_m) \) and \( z_k(t^k_m+) \) at a jump point \( t^k_m \). We will then glue the (reparameterized) continuous pieces and the (reparameterized) jump transitions together.

(2) In this way, we shall obtain a sequence of curves \( (\zeta_k)_k \), defined on compact sets \( (\mathcal{C}_k)_k \), to which we will apply a refined compactness argument from [MST18], yielding the existence of a limiting Lipschitz curve \( \zeta \), defined on a compact set \( \mathcal{C} \subseteq \mathbb{R} \), connecting the left and the right limits \( z(t-) \) and \( z(t+) \).

(3) We will then show that

\[
\liminf_{k \to \infty} \text{Var}_{d,z,c}(z_k, [\alpha_k, \beta_k]) \geq \text{Trc}_{VE}(t, \zeta, \mathcal{C}).
\]

(4) From (6.31) we shall conclude (6.30).

\[\triangleright\text{ Step 1 (reparameterization):}\]

We set

\[
m_k := \beta_k - \alpha_k + \text{Var}_{d,z,c}(z_k, [\alpha_k, \beta_k]) + \sum_{m \in M_k} 2^{-m}
\]

and define the rescaling function \( g_k : [\alpha_k, \beta_k] \to [0, m_k] \) by

\[
g_k(t) := t - \alpha_k + \text{Var}_{d,z,c}(z_k, [\alpha_k, t]) + \sum_{m \in M_k \cap \{ t^k_m \leq t \}} 2^{-m}.
\]

Observe that \( g_k \) is strictly increasing, with jump set \( J_{g_k} = (t^k_m)_{m \in M_k} \). We set

\[
I^k_{m} := (g_k(t^k_m-) - g_k(t^k_m+), g_k(t^k_m+)) \quad \text{and} \quad I_k := \bigcup_{m \in M_k} I^k_{m}, \quad \Lambda_k := [g_k(\alpha_k), g_k(\beta_k)].
\]

On \( \Lambda_k \setminus I_k \) the inverse \( t_k : \Lambda_k \setminus I_k \to [\alpha_k, \beta_k] \) of \( g_k \) is well defined and Lipschitz continuous. We introduce

\[
\zeta_k(s) := (u_k \circ t_k)(s) \quad \text{for all } s \in \Lambda_k \setminus I_k
\]

and observe that \( \zeta_k \) is Lipschitz as well.

We now reparameterize the ‘jump pieces’ of the trajectory. Recall that at every jump point \( t^k_m \) there exists an optimal jump transition \( \vartheta^k_{z,m}(E^k_m; Z) \), fulfilling

\[
\begin{align*}
z(t^k_m-) &= \vartheta^k_{z,m}(E^k_m), & z(t^k_m+) &= \vartheta^k_{z,m}(E^k_m\setminus t^k_m), & z(t^k_m) \in \vartheta^k_{z,m}(E^k_m), \\
\mathcal{E}_k(t^k_m, u(t^k_m), z(t^k_m-)), \mathcal{E}_k(t^k_m, u(t^k_m+)) \in \mathcal{E}_k(t^k_m, z(t^k_m-), z(t^k_m+)) &= \text{Trc}_{VE}(t^k_m, \vartheta^k_{z,m}, E^k_m).
\end{align*}
\]

We define the rescaling function \( \sigma^k_m \) on \( E^k_m \) by

\[
\sigma^k_m(t) := \frac{1}{2m} \frac{t - (E^k_m)^-}{(E^k_m)^+ - (E^k_m)^-} \text{Var}_{d,z,c}(\vartheta^k_{z,m}, E^k_m \cap [(E^k_m)^-, t]) + \text{Var}_{d,z,c}(\vartheta^k_{z,m}, E^k_m \cap [(E^k_m)^-, t]) + \sum_{r \in [(E^k_m)^-, t] \setminus (E^k_m)^+} \mathcal{D}_k(t^k_m, \vartheta^k_{z,m}(r)) + g_k(t^k_m-) \quad \text{for all } t \in E^k_m.
\]

It can be checked that \( \sigma^k_m \) is continuous and strictly increasing, with image a compact set \( S^k_m \subset I^k_{m} \) such that

\[
(S^k_m)^\pm = \sigma_m((E^k_m)^\pm) = g_k(t^k_m\pm).
\]

Its inverse function \( \sigma^k_m : S^k_m \to E^k_m \) is Lipschitz continuous.

Finally, we introduce the compact set

\[
\mathcal{C}_k := (\Lambda_k \setminus I_k) \cup (\cup_{m \in M_k} S^k_m) \subset \Lambda_k \subset [0, m_k]
\]

and extend the functions \( t_k \) and \( \zeta_k \), so far defined on \( \Lambda_k \setminus I_k \), only, to the set \( \mathcal{C}_k \) by setting

\[
t_k(s) := t^k_m \quad \text{and} \quad \zeta_k(s) := \vartheta^k_m(\tau^k_m(s)) \quad \text{whenever } s \in S^k_m \text{ for some } m \in M_k.
\]
It has been checked in [RS17] that the extended curve $\zeta_k$ is in $C_{\sigma z,dz}(\mathcal{C}_k; X) \cup BV_{dz}(\mathcal{C}_k; X)$, with

$$\text{Var}_{dz}(\zeta_k,[s_0,s_1]) \leq \text{Var}_{dz}(\zeta_k,[t_k(s_1),t_k(s_1)]) + (t_k(s_1) - t_k(s_0)) \quad \text{for all } s_0, s_1 \in \Lambda_k \setminus I_k \text{ with } s_0 < s_1,$$

$$\text{Var}_{dz}(\zeta_k,S^k_m) = \text{Var}_{dz}(\sigma^k_{z,m},E^k_m), \quad \text{GapVar}_{dz}(\zeta_k,S^k_m) = \text{GapVar}_{dz}(\sigma^k_{z,m},E^k_m),$$

$$\sum_{s \in S^k_m \setminus \{s^k_m\}} \mathcal{R}_k(t^k_m,\zeta_k(s)) = \sum_{r \in E^k_m \setminus \{(E^k_m)\}} \mathcal{R}_k(t^k_m,\sigma^k_{z,m}(r)). \quad (6.34)$$

**Step 2 (a priori estimates and compactness):** We refer to the proof of [RS17, Prop. 3] for the calculations leading to these a priori estimates:

$$\exists \overline{c} > 0 \forall k \in \mathbb{N} : \begin{cases} \mathcal{C}_k^+ \leq \overline{c}, \\
\text{Var}_{dz}(\zeta_k, \mathcal{C}_k) \leq \overline{c}, \\
\text{Var}_{dz}(\zeta_k, \mathcal{C}_k \cap [s_0,s_1]) \leq (s_1 - s_0) \quad \text{for all } s_0, s_1 \in \mathcal{C}_k \text{ with } s_0 < s_1, \\
\sup_{s \in \mathcal{C}_k} \mathcal{T}_{0,k}(u_k(s),\zeta_k(s)) \leq \overline{c}, \end{cases} \quad (6.35)$$

where $u_k(s)$ is the unique element in $\text{Argmin}_{u \in L^z} \mathcal{E}_k(s,u,\zeta_k(s))$ and $\mathcal{T}_{0,k}$ the perturbed functional associated with $\mathcal{E}_k$ via (2.38).

Therefore, we are in a position to apply the compactness result from [MS18, Thm. 5.4] and conclude that there exist a (not relabeled) subsequence, a compact set $\mathcal{C} \subset [0,\overline{c}]$ with $\overline{c}$ as in (6.35), and a function $\zeta \in C_{\sigma z,dz}(\mathcal{C};X)$ such that, as $k \to \infty$, there hold

1. $\mathcal{C}_k \to \mathcal{C}$ à la Kuratowski, namely $L_{S_k \to \infty} \mathcal{C}_k = \mathcal{C}$ with $L_{S_k \to \infty} \mathcal{C}_k := \{ t \in [0,\infty) : \exists t_k \in \mathcal{C}_k \text{ s.t. } t_k \to t \}$, $L_{S_k \to \infty} \mathcal{C}_k := \{ t \in [0,\infty) : \exists j \to k_j \text{ increasing and } t_{k_j} \in \mathcal{C}_k \text{ s.t. } t_{k_j} \to t \};$

2. for every $s \in \mathcal{C}$ there exists a sequence $(s_k)_k$, with $s_k \in \mathcal{C}_k$ for all $k \in \mathbb{N}$, such that $s_k \to s$ and $\sigma^k_{z,s_k}(Z) \to Z$ in $Z$ as $k \to \infty$;

3. whenever $s_k \in \mathcal{C}_k$ converge to $s \in \mathcal{C}$, then $\zeta_k(s_k) \xrightarrow{\sigma} \zeta(s)$ in $Z$;

4. $\zeta_k(\mathcal{C}_k^\pm) \xrightarrow{\sigma} \zeta(\mathcal{C}^\pm)$;

5. for every $I \in \mathfrak{b}(\mathcal{C})$ (recall (2.30)) there exists a sequence $(J_k)_k$ with $J_k \in \mathfrak{b}(\mathcal{C}_k)$ for all $k \in \mathbb{N}$ and $J_k^- \to I^+$, $J_k^- \to I^-$.

Therefore, $\zeta(C) = \lim_{n \to 0} \zeta(\mathcal{C}), \zeta(C_+) = \lim_{n \to 0} \zeta(\mathcal{C}^+)$. Finally, for later use we observe that

$$\lim_{k \to \infty} \sup_{s \in \mathcal{C}_k} |t_k(s) - t| = 0, \quad (6.37)$$

since the functions $t_k$ take values in the intervals $[\alpha_k, \beta_k]$ shrinking to the singleton $\{t\}$.

**Step 3 (proof of (6.30)):** Repeating the very same arguments as in the proof of [MS18, Thm. 5.3], from the above convergence properties we conclude

$$\text{Var}_{dz}(\zeta, \mathcal{C}) \leq \liminf_{k \to \infty} \text{Var}_{dz}(\zeta_k, \mathcal{C}_k). \quad (6.38)$$

Let us now address the term in the transition cost involving the residual stability function. To this end, we fix a finite set $\{\sigma^0 < \sigma^1 < \ldots < \sigma^N := \mathcal{C}^+\} \subset \mathcal{C}$ such that $\mathcal{R}(\sigma^n,\zeta(\sigma^n)) > 0$ for all $n = 1,\ldots,N - 1$. We use that for every $n \in \{1,\ldots,N\}$ there exists a sequence $(\sigma^n_k)_k$ with $\sigma^n_k \in \mathcal{C}_k$ for all $k \in \mathbb{N}$, $\sigma^n_k \to \sigma^n$ and $\zeta_k(\sigma^n_k) \xrightarrow{\sigma} \zeta(\sigma^n)$ as $k \to \infty$. Furthermore, in view of (6.37) we have that $t_k(\sigma^n_k) \to t$ as $k \to \infty$ for all $n \in \{0,\ldots,N\}$. By the $\Gamma$-lim inf estimate (6.14), we infer that

$$\liminf_{k \to \infty} \mathcal{R}_k(t_k(\sigma^n_k),\zeta_k(\sigma^n_k)) \geq \mathcal{R}(t,\zeta(\sigma^n)) \quad \text{for all } n \in \{0,\ldots,N\},$$

therefore there exist $c > 0$ and an index $\tilde{k} \in \mathbb{N}$ such that

$$\mathcal{R}_k(t_k(\sigma^n_k),\zeta_k(\sigma^n_k)) \geq c > 0 \quad \text{for all } n \in \{1,\ldots,N-1\}.$$
This entails that for every $n \in \{1, \ldots, N-1\}$ and $k \geq \bar{k}$ there exists $m^n_k \in M_k$ (the countable set of jump points of $z_k$ between $\alpha_k$ and $\beta_k$) such that $t_k^k(\sigma^n_k) = t^n_m$. All in all, we conclude that

$$\sum_{n=1}^{N-1} \mathcal{H}(t, \zeta(\sigma^n)) \leq \liminf_{k \to \infty} \sum_{n=1}^{N-1} \mathcal{H}_k(t^n_m, \zeta_k(\sigma^n_k)) \leq \liminf_{k \to \infty} \sum_{n=1}^{N-1} \mathcal{H}_k(t^n_m, \zeta_k(\sigma^n_k))$$

$$\leq \liminf_{k \to \infty} \sum_{m \in M_k} \sum_{s \in S^n_m \setminus \{s^n_m\}} \mathcal{H}_k(t^n_m, \zeta_k(s))$$

$$= \liminf_{k \to \infty} \sum_{m \in M_k} \sum_{r \in E^n_m \setminus \{E^n_m\}} \mathcal{H}_k(t^n_m, \zeta_k(r)),$$

the latter identity due to (5.34). Taking the supremum of the left-hand side over all finite subsets of $\mathcal{C} \setminus \{\mathcal{C}^+\}$, we then conclude that

$$\sum_{\sigma \in \mathcal{C} \setminus \{\mathcal{C}^+\}} \mathcal{H}(t, \zeta(\sigma)) \leq \liminf_{k \to \infty} \sum_{m \in M_k} \sum_{r \in E^n_m \setminus \{E^n_m\}} \mathcal{H}_k(t^n_m, \zeta_k(r)). \tag{6.39}$$

Finally, (6.36) and, again, the very same arguments as in the proof of [MS18, Thm. 5.3] yield that

$$\text{GapVar}_{d_Z}(\zeta, C) = \sum_{l \in h(\mathcal{C})} \delta_Z(\zeta^l(I^n), \zeta^l(I^n+)) \leq \liminf_{k \to \infty} \sum_{J \in h(\mathcal{C}_k)} \delta_Z(\zeta_k(J^-), \zeta_k(J^+))$$

$$= \liminf_{k \to \infty} \sum_{m \in M_k} \text{GapVar}_{d_Z}(\zeta_k, S^n_m) \tag{6.40}$$

$$(^1) = \liminf_{k \to \infty} \sum_{m \in M_k} \text{GapVar}_{d_Z}(\zeta_k, S^n_m)$$

with (1) due to (6.34). Combining (6.38), (6.39) and (6.40), we deduce (6.31).

\[\rightarrow \text{ Step 4 (conclusion):} \text{ Observe that} \]

$$\zeta(t, z(t-), z(t+)) \leq \text{Trc}_{\mathcal{C}}(t, \zeta, \mathcal{C}).$$

Therefore, (6.30) follows from (6.31). This finishes the proof of Lemma 6.6. \[\blacksquare\]

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