Abstract

A formal symplectic structure on $R \times M$ is constructed for the unsteady flow of an incompressible viscous fluid on a three dimensional domain $M$. The evolution equation for the helicity density is expressed via the divergence of the associated Liouville vector field that generates symplectic dilation. For an inviscid fluid this equation reduces to a conservation law. As an application the symplectic dilation is used to generate Hamiltonian automorphisms of the symplectic structure which are then related to the symmetries of the velocity field.
The helicity which is first discovered in [1] has been recognized to be an important ingredient of the problem of relationship between invariants of fluid motion and the topological structure of the vorticity field [2]–[4]. For three-dimensional flows its ergodic and topological interpretations were introduced and investigated in [5]–[10]. It has also been studied in the context of Noether theorems [11]–[17]. Kinematical aspects of helicity invariants in connection with the particle relabelling symmetries were discussed in [18].

In this work, we shall show that there is also a dynamical content of the helicity density in the sense that the information contained in the Eulerian dynamical equations can be represented in the framework of symplectic geometry by a current vector field governing the dynamics of helicity. More precisely, starting from the Navier-Stokes equations of incompressible fluids we shall construct helicity four-vector whose divergence will define the time-evolution of helicity density. The dynamical properties of the fluid, such as viscosity, are implicit in this vector field. The evolution equation for the helicity density reduces to a conservation law for inviscid Euler flows. For fluid dynamical content of this work we shall refer to [2] and the necessary mathematical background can be found in [19]–[21].

The Navier-Stokes equation for a viscous incompressible fluid in a bounded domain $M \subset \mathbb{R}^3$ is
\begin{equation}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} \tag{1}
\end{equation}
where $\mathbf{v}$ is the divergence-free velocity field tangent to the boundary of $M$, $p$ is the pressure per unit density and $\nu$ is the kinematic viscosity. The identity $\mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{2} \nabla |\mathbf{v}|^2 - \mathbf{v} \times (\nabla \times \mathbf{v})$ can be used to bring the equation (1) into the form
\begin{equation}
\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = \nu \nabla^2 \mathbf{v} - \nabla(p + \frac{1}{2} \nu^2) \tag{2}
\end{equation}
and in terms of the vorticity field $\mathbf{w} \equiv \nabla \times \mathbf{v}$ this gives
\begin{equation}
\frac{\partial \mathbf{w}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{w}) = \nu \nabla^2 \mathbf{w} . \tag{3}
\end{equation}
For a fluid with a potential $\varphi$ and velocity field $\mathbf{v}$ the densities
\begin{equation}
\mathcal{H} = \frac{1}{2} \mathbf{v} \cdot \nabla \times \mathbf{v} , \quad \mathcal{H}_w = \frac{1}{2} \mathbf{w} \cdot \nabla \times \mathbf{w} , \quad q = \mathbf{w} \cdot \nabla \varphi = w(\varphi) \tag{4}
\end{equation}
will be called helicity, vortical helicity and potential vorticity, respectively.
Proposition 1  For a velocity field satisfying Eqs. (4) and for \( q \neq 2\nu H_w \) the two-form
\[
\Omega_\nu = - (\nabla \varphi + v \times w - \nu \nabla \times w) \cdot dx \wedge dt + w \cdot (dx \wedge dx)
\]
is symplectic on \( I \times M \) where \( I \) is an open interval in \( \mathbb{R} \). Moreover, it is exact, \( \Omega_\nu = d\theta \) with the Liouville (or canonical) one-form
\[
\theta = -(\varphi + p + \frac{1}{2}v^2) \, dt + v \cdot dx
\]
which is independent of the viscosity \( \nu \).

Proof: \( \Omega_\nu \) is closed by Eq. (3) and the divergence-free property of the vorticity field. The non-degeneracy follows from the recognition that the density in the symplectic volume \( \Omega_\nu \wedge \Omega_\nu / 2 \) is the function \( q - 2\nu H_w \) which is assumed to be non-zero. The exactness can be verified using Eq. (2).

For an arbitrary smooth function \( f \) of \((t, x)\) the unique Hamiltonian vector field \( X_f \) defined by the symplectic two-form (4) via \( i(X_f)(\Omega_\nu) = -df \) is given by
\[
X_f = \frac{1}{q - 2\nu H_w}[-w(f)(\frac{\partial}{\partial t} + v) + \frac{df}{dt}w + ((\nabla \varphi - \nu \nabla \times w) \times \nabla f) \cdot \nabla].
\]

Here, \( d/dt \) denotes the convective derivative \( \partial_t + v \cdot \nabla \) which, viewed as a vector field on \( I \times M \), is not Hamiltonian. In fact, with the notation \( v \equiv v \cdot \nabla \), one can check that the one-form
\[
i(\partial_t + v)(\Omega_\nu) = (\nabla \varphi - \nu \nabla \times w) \cdot (dx - v dt)
\]
is not closed and hence \( \partial_t + v \) is not even locally Hamiltonian.

Next proposition describes invariantly the connection between the symplectic structure (5) and the helicity density.

Proposition 2  The identity
\[
d(\theta \wedge \Omega_\nu) - \Omega_\nu \wedge \Omega_\nu \equiv 0
\]
gives the equation
\[
\frac{\partial \mathcal{H}}{\partial t} + \nabla \cdot (\mathcal{H} v + \frac{1}{2}(p - \frac{1}{2}v^2)w) = \nu \cdot \nabla^2 w - \nu \mathcal{H}_w
\]
for the evolution of helicity density.
Proof: We have $\Omega_\nu \wedge \Omega_\nu = -2(q - 2\nu \mathcal{H}_w)dx \wedge dy \wedge dz \wedge dt$ and we compute

$$\nabla \cdot [\varphi \mathbf{w} + \mathbf{v} \times (\nabla \varphi - \nu \nabla \times \mathbf{w})] = 2q - 2\nu \mathcal{H}_w - \nu \mathbf{v} \cdot \nabla^2 \mathbf{w} \quad (11)$$

for the derivative of certain terms in the expression

$$\theta_\nu \wedge \Omega_\nu = 2\mathcal{H} dx \wedge dy \wedge dz - \left[(\varphi + p - \frac{1}{2}v^2)\mathbf{w} + 2\mathcal{H} \mathbf{v} + \mathbf{v} \times (\nabla \varphi - \nu \nabla \times \mathbf{w})\right] \cdot dx \wedge dx \wedge dt \quad (12)$$

for the three-form. Putting them together in the identity (11) we obtain Eq. (10). Upon integration, the term $\nu \mathbf{v} \cdot \nabla^2 \mathbf{w}$ in Eq. (10) gives the integral of $-\nu \mathcal{H}_w$ and one obtains the usual expression for the time change of total helicity as given in, for example, Ref. [2].

Note that the helicity flux in Eq. (10) is independent of the function $\varphi$ which we have introduced by hand to make the symplectic form non-degenerate.

Using the invariant description (1) of the evolution of helicity density, we shall introduce a current vector $J_\nu$ and show that it is an infinitesimal symplectic dilation of $\Omega_\nu$. $J_\nu$ will be defined as the one-dimensional kernel of the three-form $\theta \wedge \Omega_\nu$. Since the symplectic two-form is nondegenerate, it can be obtained as the unique solution of

$$i(J_\nu)(\Omega_\nu \wedge \Omega_\nu/2) = \theta \wedge \Omega_\nu \, , \quad (13)$$

that is, as the dual of the three-form $\theta \wedge \Omega_\nu$ with respect to the symplectic volume. We find

$$J_\nu = \frac{1}{q - 2\nu \mathcal{H}_w} [2\mathcal{H}(\partial_t + v) + (\varphi + p - \frac{1}{2}v^2)w + \mathbf{v} \times (\nabla \varphi - \nu \nabla \times \mathbf{w}) \cdot \nabla] \quad (14)$$

as the expression for the helicity current.

**Proposition 3** $J_\nu$ is a vector field of divergence 2 with respect to the symplectic volume and it is an infinitesimal symplectic dilation for $\Omega_\nu$. The evolution of helicity density $\mathcal{H}$ can be described by the identity

$$\text{div}_{\Omega_\nu}(J_\nu) - 2 \equiv 0 \, . \quad (15)$$
Proof: The exterior derivative of Eq. (13) gives

\[ di(J_\nu)(\Omega_\nu \wedge \Omega_\nu/2) = \mathcal{L}_{J_\nu}(\Omega_\nu \wedge \Omega_\nu/2) \equiv div_{\Omega_\nu}(J_\nu) \Omega_\nu \wedge \Omega_\nu/2 \] (16)

\[ = d(\theta \wedge \Omega_\nu) = \Omega_\nu \wedge \Omega_\nu \] (17)

where we used the identity \( \mathcal{L}_J = i(J) \circ d + d \circ i(J) \) in the first equality and the second equality is the definition of the divergence. We see that \( J_\nu \) is a vector field whose divergence is 2. From the last equality, we conclude that the equation (15) is equivalent to Eq. (14) describing the evolution of helicity density. \( J_\nu \) is the unique vector field satisfying

\[ i(J_\nu)(\Omega_\nu) = \theta \] (18)

and it follows from this that \( J_\nu \) fulfills the condition

\[ \mathcal{L}_{J_\nu}(\Omega_\nu) = di(J_\nu)(\Omega_\nu) = d\theta = \Omega_\nu \] (19)

of being an infinitesimal symplectic dilation for \( \Omega_\nu \) [22]. \( J_\nu \) is also called to be the Liouville vector field of \( \Omega_\nu \) [21].

We observed that the local existence of a Hamiltonian function for \( \partial_t + v \) is being prevented by the viscosity term [23]. Moreover, the viscosity term causes the helicity not to be conserved. We shall now show that, for the case of inviscid incompressible fluids described by the Euler equation, namely Eq. (1) with \( \nu = 0 \), \( \partial_t + v \) is Hamiltonian and that the helicity density \( H \) is conserved. To this end, we assume that the scalar field \( \varphi \) is advected by the fluid motion

\[ \frac{\partial \varphi}{\partial t} + v \cdot \nabla \varphi = 0 \] (20)

and that the potential vorticity \( q \neq 0 \).

**Proposition 4** [23] Let \( v \) and \( \varphi \) satisfy Eq. (1) with \( \nu = 0 \) and Eq. (20), respectively. Then, the suspended velocity field \( \partial_t + v \) on \( I \times M \) and \( q^{-1} w \) are Hamiltonian vector fields for the exact symplectic two-form

\[ \Omega_0 = - (\nabla \varphi + v \times w) \cdot dx \wedge dt + w \cdot (dx \wedge dx) = d\theta \] (21)

with the Hamiltonian functions \( \varphi \) and \( t \), respectively. The evolution equation (14) reduces to the conservation law in divergence form for the helicity density.
Proof: Using Eq. (20) \( \partial_t + v \) can be written in Hamiltonian form \( i(\partial_t + v)(\Omega_\nu) = -d\varphi \). More generally, the Hamiltonian vector field with the symplectic two-form (21) for an arbitrary function \( f \) on \( I \times M \) is given by

\[
X_f = \frac{1}{q} [-w(f)\left(\frac{\partial}{\partial t} + v\right) + \frac{df}{dt}w + (\nabla \varphi \times \nabla f) \cdot \nabla]
\]

which clearly reduces to \( \partial_t + v \) for \( f = \varphi \) and to \( q^{-1}w \) for \( f = t \). The conservation of helicity density is obvious. 

For the inviscid flow of the Euler equation the helicity current takes the form

\[
J_0 = \frac{1}{q} [2\mathcal{H}(\partial_t + v) + (\varphi + p - \frac{1}{2}v^2)w + v \times \nabla \varphi \cdot \nabla]
\]

while the canonical one-form remains to be the same. That means, the difference between the dynamics of fluid motion with \( \nu = 0 \) and \( \nu \neq 0 \) is contained in the helicity current. Thus, the dynamical content of the helicity is encoded in its current and this, in turn, is connected with the symplectic structure on \( I \times M \) which was constructed as a consequence of the Eulerian dynamical equations.

The realization of dynamics of fluid motion in the symplectic framework is useful in the study of the geometry of the motion on \( M \) and of the hypersurfaces in \( I \times M \) defined by the time-dependent Lagrangian invariants, that is, the invariants of the velocity field. The present framework also provides geometric tools for the investigation of scaling properties of the fluid motion because the action by the Lie derivative of helicity current on tensorial objects corresponds to infinitesimal scaling transformations [21]. Leaving the discussions of these issues elsewhere, we shall conclude this work with an application to the symmetry structure of the velocity field which is also related to the results presented in [24].

**Proposition 5** Let \( X_f \) be a Hamiltonian vector field for \( \Omega_\nu \). Then, the vector fields \( (L_{J_\nu})^k(X_f), k = 0, 1, 2, ... \) are infinitesimal Hamiltonian automorphisms of \( \Omega_\nu \).

**Proof:** The symplectic two-form is invariant under the flows of Hamiltonian vector fields because \( L_{X_f}(\Omega_\nu) = di(X_f)(\Omega_\nu) = df \equiv 0 \) where we used the identity \( L_X = i(X) \circ d + d \circ i(X) \) for the Lie derivative, \( d\Omega_\nu = 0 \) and the Hamilton’s equations \( i(X_f)(\Omega_\nu) = -df \). It then follows from the identity

\[
L_{[J_\nu, X_f]} = L_{J_\nu} \circ L_{X_f} - L_{X_f} \circ L_{J_\nu}
\]
evaluated on \( \Omega_{\nu} \) that \([J_{\nu}, X_f]\) also leaves \( \Omega_{\nu} \) invariant. Replacing \( X_f \) with \([J_{\nu}, X_f]\) in Eq. (24) we see that one can generate an infinite hierarchy of invariants of the symplectic two-form \( \Omega_{\nu} \). To see that these are Hamiltonian vector fields we compute

\[
i([J_{\nu}, X_f])(\Omega_{\nu}) = \mathcal{L}_{[J_{\nu}, X_f]}(\Omega_{\nu}) - i(X_f)(\mathcal{L}_{J_{\nu}}(\Omega_{\nu})) \tag{25}
\]

where we used Eq. (19). Thus, \([J_{\nu}, X_f]\) is Hamiltonian with the function \( J_{\nu}(f) - f \).

By induction one can find similarly that \((L_{J_{\nu}})^2(X_f)\) is Hamiltonian with \((J_{\nu})^2(f) - 2J_{\nu}(f) + f\) and so on. Interchanging \( J_{\nu} \) and \( X_f \) in the identity (25) we also obtain

\[
i(X_f)(\theta) = J_{\nu}(f). \tag{26}
\]

In particular, we let \( \nu = 0, f = t \) so that \( X_t = q^{-1}w \) and consider the infinitesimal Hamiltonian automorphisms \((L_{J_0})^k(q^{-1}w)\), \( k = 0, 1, 2, \ldots \) of \( \Omega_0 \).

The identity (24) evaluated on the vector field \( \partial_t + v \) gives

\[
\mathcal{L}_{[J_0, q^{-1}w]}(\partial_t + v) = -\mathcal{L}_{q^{-1}w}([J_0, \partial_t + v]) \tag{27}
\]

where the vector field \([J_0, \partial_t + v]\) is, by proposition (5), Hamiltonian with the function \( J_0(\varphi) - \varphi = p - v^2/2 \). By the Lie algebra isomorphism \([X_f, X_g] = X_{\{f, g\}}\) defined by the symplectic structure \( \Omega_0 \), the right hand side of Eq. (27) is a Hamiltonian vector field with the function

\[
\{t, p - \frac{1}{2}v^2\} = \frac{1}{q}w(p - \frac{1}{2}v^2). \tag{28}
\]

On the level surfaces defined by the constant values of the function (28) we have \([[J_0, q^{-1}w], \partial_t + v] = 0\). In fact, if we restrict to the constant values of the function \( p - v^2/2 \) the hierarchy of Hamiltonian automorphisms of \( \Omega_0 \) can be identified as the infinitesimal symmetries of the velocity field. This can be seen by replacing \( q^{-1}w \) with \([J_0, q^{-1}w]\) in Eq. (27). We thus proved that

**Proposition 6** For the Euler flow, the hierarchy of infinitesimal Hamiltonian automorphisms \((L_{J_0})^k(q^{-1}w)\), \( k = 0, 1, 2, \ldots \) of \( \Omega_0 \) generate infinitesimal time-dependent symmetries of the velocity field on the level surfaces \( p - v^2/2 = \text{constant} \).

As a matter of fact, the function \( p - v^2/2 \) is related, in Ref. [23], to the invariance under particle relabelling symmetries of the Lagrangian density of the variational formulation of the Euler equation.
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