EQUIDISTRIBUTION AND $\beta$ ENSEMBLES

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Abstract. We find the precise rate at which the empirical measure associated to a $\beta$-ensemble converges to its limiting measure. In our setting the $\beta$-ensemble is a random point process on a compact complex manifolds distributed according to the $\beta$ power of a determinant of sections in a positive line bundle. A particular case is the spherical ensemble of generalized random eigenvalues of pairs of matrices with independently distributed Gaussian entries.

1. Background and setting

Let $(X, \omega)$ be a $n$-dimensional compact complex manifold endowed with a smooth Hermitian metric $\omega$. Let $(L, \phi)$ be a holomorphic line bundle with a positive Hermitian metric $\phi$. This has to be understood as a collection of smooth functions $\phi_i$ defined in trivializing neighborhoods $U_i$ of the line bundle. If $e_i(x)$ is a frame in $U_i$, then $\|e_i(x)\|^2 = e^{-\phi_i(x)}$. Thus $\phi_i$ must satisfy the compatibility condition $\phi_i - \phi_j = \log |g_{ij}|$, where $g_{ij}$ are the transition functions.

As usual we denote by $H^0(L, X)$ the global holomorphic sections. If $L$ is a line bundle over $X$ and $M$ is a line bundle over $Y$, we denote by $L \boxtimes M$ the line bundle over the product manifold $X \times Y$ defined as $L \boxtimes M = \pi_X^*(L) \otimes \pi_Y^*(M)$, where $\pi_X : X \times Y \to X$ is the projection onto the first factor and $\pi_Y : X \times Y \to Y$ is the projection onto the second. The line bundle $L \boxtimes M$ carries a metric induced by that of $L$ and $M$.

Given a basis $s_1, \ldots, s_N$ of $H^0(L, X)$ we define $\det(s_i(x_j))$ as a section of $L^\boxtimes N$ over $M^N$ by the identities $\det(s_i(x_j)) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \otimes_{i=1}^N s_i(x_{\sigma_i})$.

We fix a probability measure on $X$, given by the normalized volume form $\omega^n$, that we denote by $\sigma$.

Definition 1. Let $\beta > 0$. A $\beta$-ensemble is an $N$ point random process on $X$ which has joint distribution given by

$$
\frac{1}{Z_N} \|\det s_i(x_j)\|^\beta d\sigma(x_1) \otimes \cdots \otimes d\sigma(x_N),
$$

where $Z_N = Z_N(\beta)$ is chosen so that this is a probability distribution in $X^N$.

Observe that the random point process is independent of the choice of basis $s_j$.

A particularly interesting case is when $\beta = 2$, since then the process is determinantal. Let $K$ denote the Bergman kernel of the Hilbert space $H^0(L, X)$ endowed with the norm $\|s\| = \int_X \|s(x)\|^2 d\sigma(x)$. Here $\| \cdot \|^2_\phi$
denotes the norm induced by the metric $\phi$ (see Section 1.2 for more details). Then
\[ \| \det(s_i(x_j)) \|^2 = \| \det(K(x_i, x_j)) \|. \]

Another interesting situation occurs when $\beta \to \infty$. In this case the probability charges the maxima of the function $\| \det(s_i(x_j)) \|$. A set of points $\{x_j\}_j$ with cardinality $\dim H^0(L, X)$ and maximizing this determinant is known as a Fekete sequence. The distribution of these sequences has been studied in [LOC10], [MOC10], and [BBWN11] and we will draw some ideas from there to study general $\beta$-ensembles.

We consider now the situation where we replace $L$ by a power $L^k$, $k \in \mathbb{N}$, and let $k$ tend to infinity. We denote by $N_k$ the dimension of $H^0(L^k, X)$. It is well-known, by the Riemann-Roch theorem and the Kodaira vanishing theorem, that
\[ \dim H^0(L^k, X) = \frac{c_1(L)^n}{n!} k^n + O(k^{n-1}) \sim k^n, \]
where $c_1(L)$ denotes the first Chern class of $L$.

For each $k$ we consider a collection of $N_k$ points chosen randomly according to the law (1). For each $k$ the collection is picked independently of the previous ones.

Given points $x_1^{(k)}, \ldots, x_{N_k}^{(k)}$ chosen according to (1), consider its associated empirical measure $\mu_k = \frac{1}{N_k} \sum \delta_{x_i^{(k)}}$. For convenience we will drop the superindex $(k)$ hereafter. We are interested in understanding the limiting distribution of the measures $\mu_k$.

The following result is well known; see [BBWN11].

**Theorem** (Berman, Boucksom, Witt). Let $\mu_k$ be the empirical measure associated to a Fekete sequence for the bundle $H^0(L^k, X)$. Then, as $k \to \infty$,
\[ \mu_k \rightharpoonup \nu := \frac{(i\partial \bar{\partial} \phi)^n}{\int_X (i\partial \bar{\partial} \phi)^n} \]
in the weak-$*$ topology.

The measure $\nu$ is called the equilibrium measure.

There is a counterpart of this result for empirical measures of general $\beta$-ensembles (see [Ber14], which gives an estimate for the large deviations of the empirical measure from the equilibrium measure). Our aim is to obtain a different quantitative version of the weak convergence of the empirical measure to the equilibrium measure, measured in terms of the Kantorovich-Wasserstein distance between measures.

This sort of quantification has also been studied, with different tools, in the context of random matrix models, (see for instance [MM13], [Mec13] and [MM14], where similar determinantal point processes arise).

In fact some of the $\beta$-ensembles we are considering admit random matrix models, at least when $\dim_C(M) = 1$. For instance, Krishnapur studied in [Kri09] the following point process: let $A, B$ be $k \times k$ random matrices with i.i.d. complex Gaussian entries. He proved that the generalized eigenvalues associated with the pair $(A, B)$, i.e. the eigenvalues of $A^{-1}B$, have joint
probability density:

\[
\frac{1}{Z_k} \prod_{l=1}^{k} \frac{1}{(1 + |x_l|^2)^{k+1}} \prod_{i<j} |x_i - x_j|^2,
\]

with respect to the Lebesgue measure in the plane.

It was also observed in [Kri09] that, using the stereographic projection

\[
\pi : \mathbb{S}^2 \rightarrow \mathbb{C}
\]

\[
P_j \mapsto x_j,
\]

the joint density (2) (with respect to the product area measure in the product of spheres) is

\[
\frac{1}{Z_k} \prod_{i<j} \|P_i - P_j\|_2^2.
\]

Since this is invariant under rotations of the sphere, the point process is called the spherical ensemble.

A point process with this law had been considered earlier – without a random matrix model – by Caillol [Cai81] as the model of one-component plasma.

One typical instance of the process is as in the picture.

The spherical ensemble has received much attention. We mention a couple of properties related to our results. In [Bor11], Bordenave proves the universality of the spectral distribution of the \(k \times k\)-matrix \(A^{-1}B\) with respect to other i.i.d. random distribution of entries. As an outcome, he proves that the weak* limit of the spectral measures \(\mu_k = \frac{1}{k} \sum_i \delta_{x_i}\), where \(x_i\) are the generalized eigenvalues is the normalized area measure in the sphere. This convergence is rather uniform: in [AZ15] Alishahi and Sadegh Zamani estimate the discrepancy of the empirical measure with respect to its limit and give precise estimates of the Newtonian and the logarithmic energies.

1.1. The Kantorovich-Wasserstein distance. To measure the uniformity and speed of convergence of the empirical measures \(\mu_k\) to the limiting measure \(\nu\) we use the Kantorovich-Wasserstein distance \(W\). Given probability measures \(\mu\) and \(\nu\), it is defined as

\[
W(\mu, \nu) = \inf_{\rho} \int \int_{X \times X} d(x, y) \, d\rho(x, y),
\]

where \(d(x, y)\) is the distance associated to the metric \(\omega\) and the infimum is taken over all admissible transport plans \(\rho\), i.e., all probability measures in \(X \times X\) with marginal measures \(\mu\) and \(\nu\) respectively.
In general, the Kantorovich-Wasserstein distance is defined on probability measures over a compact metric space $X$, and it metrizes the weak-$*$ convergence of measures.

It was observed in [LOC10] that in the definition of $W$ it is possible to enlarge the class of admissible transport plans to complex measures $\rho$ that have marginals $\mu$ and $\nu$ respectively. We include the argument for the sake of completeness.

Let

$$\tilde{W}(\mu, \nu) = \inf_{\rho \in S} \int \int_{X \times X} d(x, y) |d\rho(x, y)|,$$

where the infimum is now taken over the set $S$ of all complex measures $\rho$ on $X \times X$ with marginals $\rho(\cdot, X) = \mu$ and $\rho(X, \cdot) = \nu$.

In order to see that $W(\mu, \nu) = \tilde{W}(\mu, \nu)$, we recall the dual formulation of $W$ (see [Vil09, Formula (6.3)]):

$$W(\mu, \nu) = \sup \left\{ \int_X f (\mu - \nu) : f \in \text{Lip}_{1,1}(X) \right\},$$

where $\text{Lip}_{1,1}(X)$ is the collection of all functions $f$ on $X$ satisfying $|f(x) - f(y)| \leq d(x, y)$.

For any complex measure $\rho$ with marginals $\mu$ and $\nu$ and any $f \in \text{Lip}_{1,1}(X)$ we have

$$\left| \int_X f (\mu - \nu) \right| = \left| \int \int_{X \times X} (f(x) - f(y)) d\rho(x, y) \right| \leq \int \int_{X \times X} d(x, y) |d\rho(x, y)|.$$

Hence

$$W(\mu, \nu) \leq \inf_{\rho \in S} \int \int_{X \times X} d(x, y) |d\rho(x, y)| = \tilde{W}(\mu, \nu).$$

The remaining inequality ($\tilde{W}(\mu, \nu) \leq W(\mu, \nu)$) is trivial.

A standard reference for basic facts on Kantorovich-Wasserstein distances is the book [Vil09].

1.2. Lagrange sections. We fix now a basis of sections $s_1, \ldots, s_{N_k}$ of $H^0(L^k, X)$. Given any collection of points $(x_1, \ldots, x_{N_k})$ we define the Lagrange sections informally as:

$$\ell_j(x) = \begin{bmatrix} s_1(x_1) & \cdots & s_1(x) & \cdots & s_1(x_{N_k}) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ s_{N_k}(x_1) & \cdots & s_{N_k}(x) & \cdots & s_{N_k}(x_{N_k}) \\ s_1(x_1) & \cdots & s_1(x_j) & \cdots & s_1(x_{N_k}) \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ s_{N_k}(x_1) & \cdots & s_{N_k}(x_j) & \cdots & s_{N_k}(x_{N_k}) \end{bmatrix}$$

Clearly $\ell_j \in H^0(L^k, X)$ and $\ell_j(x_i) = 0$ if $i \neq j$ and $\|\ell_j(x_j)\| = 1$.

More formally, we proceed as in [LOC10]: if $e_j(x)$ is a frame in a neighborhood $U_j$ of the point $x_j$, then the sections $s_i(x)$ are represented on each $U_j$ by scalar functions $f_{ij}$ such that $s_i(x) = f_{ij}(x)e_j(x)$. Similarly, the metric $k\phi$ is represented on $U_j$ by a smooth real-valued function $k\phi_j$ such that $\|s_i(x)\|^2 = |f_{ij}(x)|^2 e^{-k\phi_j(x)}$.

To construct the Lagrange sections we denote by $A$ the matrix

$$(e^{-k\phi_j(x)}f_{ij}(x_j))_{i,j},$$
and define
\[ l_j(x) := \frac{1}{\det(A)} \sum_{i=1}^{N_k} (-1)^{i+j} A_{ij} s_i(x), \]
where \( A_{ij} \) is the determinant of the submatrix obtained from \( A \) by removing the \( i \)-th row and \( j \)-th column. Clearly \( l_j \in H^0(L^k, X) \), and it is not difficult to check that \( \| l_j(x_i) \| = \delta_{ij}, \ 1 \leq i, j \leq N_k. \)

Notice that if we denote by \( \rho_k(x_1, \ldots, x_{N_k}) = \| \det(s_i(x_j)) \| \) then
\[ \| \ell_j(x) \|^\beta = \frac{\rho_k(x_1, \ldots, x_{N_k})}{\rho_k(x_1, \ldots, x_{N_k})}. \]
and thus \( \mathbb{E}(\| \ell_j \|^\beta) \leq 1. \) In the case of the Fekete points \( (\beta = \infty) \), \( \sup_X \| \ell_j(x) \| = 1 \) by definition.

2. Main result

**Theorem 1.** Consider the empirical measure \( \mu_k \) associated to the \( \beta \)-ensemble given in Definition 1 and let \( \nu = \frac{\mu_k}{\int_X \mu_k} \) be the equilibrium measure. Then
\[ \text{EW}(\mu_k, \nu) = O(1/\sqrt{k}). \]

**Remark.** The rate of convergence cannot be improved. Let \( \sigma \) be any nowhere vanishing smooth probability distribution on \( X \). Let \( E_k \) be any discrete set on \( X \) with cardinality \( \# E_k \approx k^n \approx N_k \), and let \( \mu_k = \frac{1}{\# E_k} \sum_{x \in E_k} \delta_x \). Then the distance \( W(\mu_k, \sigma) \geq 1/\sqrt{k}. \)

To obtain a lower bound for \( W(\mu_k, \sigma) \) we use the dual formulation of the Kantorovich-Wasserstein distance and the function \( f(x) = d(x, E_k) \), which is in \( \text{Lip}_{1,1}(X) \). Since \( d(x, E_k) = 0 \) on the support of \( \mu_k \) we obtain
\[ W(\mu_k, \sigma) \geq \int_X d(x, E_k) d\sigma. \]

Vitali’s covering lemma ensures that for each \( k \) and for some \( \varepsilon \) small enough, independent of \( k \), there are at least \( 2\# E_k \) pairwise disjoint balls of radius \( \varepsilon/\sqrt{k} \). Since the number of balls is twice the number of points in \( E_k \), at least half the balls contain no point of \( E_k \). We consider one such ball, \( B(x_i, \varepsilon/\sqrt{k}) \). In the smaller ball \( B(x_i, 0.5\varepsilon/\sqrt{k}) \) we have \( d(x, E_k) \geq 0.5\varepsilon/\sqrt{k}. \) Thus
\[ \int_X d(x, E_k) d\sigma \geq \sum_i \int_{B(x_i, \varepsilon/\sqrt{k})} d(x, E_k) d\sigma \geq \sum_i \frac{1}{\sqrt{k}} \sigma(B(x_i, \varepsilon/\sqrt{k})) \]
\[ \geq \# E_k \frac{1}{\sqrt{k}} k^{-n} \approx \frac{1}{\sqrt{k}}. \]

**Proof of Theorem 2.** To prove this we provide a (complex) transport plan between the probability measure \( b_k(x) = \frac{1}{N_k} K_k(x, x) - b_k \) stand for Bergman measure – and the empirical measure \( \mu_k \). We are going to prove that
\[ \text{EW}(\mu_k, b_k) = O\left(\frac{1}{\sqrt{k}}\right). \]
Standard estimates for the Bergman kernel provide:

$$W(b_k, \nu) = O\left(\frac{1}{\sqrt{k}}\right).$$

Actually one can prove that the total variation (which dominates the Kantorovich-Wasserstein distance) satisfies:

$$\|\frac{K_k(x,x)}{N_k} - \nu\| \leq \frac{C}{\sqrt{k}}.$$  \hspace{1cm} (6)

This follows for instance by the expansion in powers of $1/k$ of the Bergman kernel. In this context this is due to Tian, Catlin and Zelditch, \cite{Tia90, Cat99, Zel98}.

In the particular case of the spherical ensemble, the kernel is explicit and invariant under rotations, and the estimate is even better: the Bergman measure is the equilibrium measure, i.e. $b_k = \nu$.

Consider the transport plan

$$p(x,y) = \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{x_j}(y)(K_k(x,x_j), \ell_j(x)) \, d\sigma(x).$$

It has the correct marginals – $b_k$ and $\mu_k$ respectively – and thus

$$W(b_k, \mu_k) \leq \int_{X \times X} d(x,y) |p(x,y)|$$

$$\leq \frac{1}{N_k} \sum_{j=1}^{N_k} \int_{X} d(x,x_j)|\ell_j(x)||K_k(x,x_j)| \, d\sigma(x).$$

Now, letting $\beta'$ be the conjugate exponent of $\beta$ (so that $1/\beta + 1/\beta' = 1$), we have

$$\left(\mathbb{E} W\right)^\beta \leq$$

$$\leq \int_{X^{N_k}} \frac{1}{N_k} \sum_{j=1}^{N_k} \left( \int_{X} d(x,x_j)|\ell_j(x)||K_k(x,x_j)| \, d\sigma(x) \right)^\beta \rho_k(x_1, \ldots, x_{N_k})$$

$$\int_{X^{N_k}} \frac{1}{N_k} \sum_{j=1}^{N_k} \left( \int_{X} d(x,x_j)|K_k(x,x_j)| \right)^{\beta/\beta'} \left( \int_{X} |\ell_j(x)|^2 |K_k(x,x_j)| \, d(x,x_j) \right)^{\beta/\beta'} \rho_k(x_1, \ldots, x_{N_k}).$$

Assume for the moment that the following off-diagonal decay of the Bergman kernel holds:

$$\operatorname{sup}_{y \in X} \int_{X} d(x,y)|K_k(x,y)| \, d\sigma(x) \leq \frac{C}{\sqrt{k}}.$$  \hspace{1cm} (7)

Then, by \cite{[3]}, we obtain:

$$\left(\mathbb{E} W\right)^\beta \leq \left(\frac{C}{\sqrt{k}}\right)^{\beta/\beta'} \int_{X^{N_k}} \frac{1}{N_k} \sum_{j=1}^{N_k} \int_{X} |\ell_j(x)|^2 |K_k(x,x_j)| \, d(x,x_j) \rho_k(x_1, \ldots, x_j, \ldots, x_{N_k})$$

$$= \left(\frac{C}{\sqrt{k}}\right)^{\beta/\beta'} \int_{X^{N_k}} \frac{1}{N_k} \sum_{j=1}^{N_k} \int_{X} |K_k(x,x_j)| \, d(x,x_j) \rho_k(x_1, \ldots, x, \ldots, x_{N_k}).$$
Finally, integrating first in $x_j$ and applying again (7) we obtain
\[
(EW)^\beta \leq \left( \frac{C}{\sqrt{k}} \right)^{\beta/\beta'} \left( \frac{C}{\sqrt{k}} \right) = O\left( \frac{1}{\sqrt{k}} \right)\beta,
\]
as desired.

Estimate (7) follows from the pointwise estimate for the Bergman kernel
\[
|K_k(x, y)| \leq C N_k e^{-C \sqrt{k} \delta (x, y)},
\]
which holds when the line bundle is positive, see [Ber03].

Indeed, consider the function $h(s) = se^{-C \sqrt{k} s}$ strictly decreasing in $\left[ \frac{1}{C \sqrt{k}}, +\infty \right)$.

For any $y \in X$ we bound the integral in (7) as
\[
\int_X d(x, y)|K_k(x, y)| \, d\sigma(x) \lesssim N_k \int_0^{+\infty} |h'(s)| \sigma(\{x \in X : h(d(x, y)) > s\}) \, ds \lesssim \frac{1}{\sqrt{k}},
\]
where the last estimate follows from $\sigma(B(y, s)) \lesssim s^{2n}$ and $N_k \sim k^n$.

In the particular case of the spherical ensemble, the kernel is explicit and the decay is even faster:
\[
|K_k(z, w)|^2 = k^2 \left( 1 - \frac{|z - w|^2}{(1 + |z|^2)(1 + |w|^2)} \right)^{k-1} \leq K k^2 \exp \left( -C_k \frac{|z - w|^2}{(1 + |z|^2)(1 + |w|^2)} \right) = K k^2 \exp \left( -C_k d(z, w)^2 \right),
\]
where here $d(z, w)$ coincides with the chordal metric. \(\square\)

3. The determinantal setting

Now we turn our attention to the almost sure convergence of the empirical measure. Using the fact that Lipschitz functionals of determinantal process concentrate the measure around the mean we prove the following result.

**Corollary 2.** If $\mu_k$ is the empirical measure associated with the determinantal point process given by (1) with $\beta = 2$, and $\nu$ denotes the equilibrium measure, then

- If $\dim C(M) > 1$ then $W(\mu_k, \nu) = O(1/\sqrt{k})$ almost surely.
- If $\dim C(M) = 1$ then $W(\mu_k, \nu) = O(\log k/\sqrt{k})$ almost surely.

In particular, any realization of the spherical ensemble satisfies $W(\mu_k, \nu) = O(\log k/\sqrt{k})$ almost surely.

Let $\nu$ be, as before, the normalized equilibrium measure. Let us define the functional $f$ on the set of measures of the form $\sigma = \sum_{i=1}^n \delta_{x_i}$ by
\[
f(\sigma) = nW\left( \frac{\sigma}{n}, \nu \right).
\]
As the Kantorovich-Wasserstein distance is controlled by the total variation, $f$ is a Lipschitz functional with Lipschitz norm one with respect to the total
variation distance. Here we use the following result of Pemantle and Peres [PP14, Theorem 3.5].

**Theorem** (Pemantle-Peres). Let $Z$ be a determinantal point process of $N$ points. Let $f$ be a Lipschitz-1 functional defined in the set of finite counting measures (with respect to the total variation distance). Then

$$
P(f - Ef \geq a) \leq 3 \exp \left( -\frac{a^2}{16(a + 2N)} \right)
$$

Take now $a = 10\alpha_k N_k/\sqrt{k}$, where $\alpha_k = \sqrt{\log k}$ for $n = 1$ and $\alpha_k = 1$ for $n > 1$. Then

$$
P \left( W(\mu_k, \nu) > \frac{11\alpha_k}{\sqrt{k}} \right) \leq P \left( N_k W(\mu_k, \nu) > N_k EfW(\mu_k, \mu) + 10\alpha_k \frac{N_k}{\sqrt{k}} \right)
$$

$$
\leq 3 \exp \left( -\frac{100\alpha_k^2 N_k^2/k}{16(10\alpha_k N_k/\sqrt{k} + 2N_k)} \right)
\lesssim \exp(-\alpha_k^2 N_k/k) \leq \frac{1}{k^2}.
$$

Finally, a standard application of the Borel-Cantelli lemma shows that, with probability one,

$$W(\mu_k, \nu) \leq \frac{11\sqrt{\alpha_k}}{\sqrt{k}}.$$ 

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