Interplay of charge and vorticity quantization in superconducting Coulomb blockaded island

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The angular momentum or vorticity of Cooper pairs is shown to affect strongly the charge transfer through a small superconducting (S) island of a single electron transistor. This interplay of charge and rotational degrees of freedom in a mesoscopic superconductor occurs through the effect of vorticity on the quantum mechanical spectrum of electron- hole excitations. The subgap quasiparticle levels in vortices can host an additional electron, thus, suppressing the so-called parity effect in the S island. We propose to measure this interaction between the quantized vorticity and electric charge via the charge pumping effect caused by alternating vortex entry and exit controlled by a periodic magnetic field.

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There are two fundamental quantization phenomena which are manifested in different aspects of physics of superconducting (S) metals: (i) quantization of charge of superconducting carriers or Cooper pairs in units of double electron charge $2e$ and (ii) quantization of trapped magnetic flux in units of the flux quantum $\Phi_0 = \hbar c / 2e$ which originates from the quantization of the angular momentum of Cooper pairs or vorticity. Basic superconducting theory shows that these two quantization rules cannot be considered independently.1 Certainly it would be exciting to clarify if this interplay of charge and vorticity quantization can reveal itself in possible magneto-electric phenomena inherent, e.g., to a superconducting state containing topological defects with quantized vorticity, namely, Abrikosov vortices. Unfortunately for the bulk samples it is extremely difficult to observe these effects experimentally. The reason is that for typical metals the large value of the Fermi energy $E_F$ almost completely suppresses all the electrostatic charge phenomena caused by vortices: the vortex core charge is small due to the very small ratio $\Delta / E_F \sim 10^{-5} - 10^{-2}$.2,3 Here $\Delta$ is the superconducting order parameter.

In the present Letter we dare to make a suggestion overcoming the above difficulties based on the powerful methods provided by superconducting single electron devices (see, e.g., Refs. 4 and 5 and references therein). The key point of this idea is that by creating or removing a single vortex in a small superconducting island, either the odd or even electron number will be favored. Thus the vorticity and the charge of the island will be coupled. To address electrons one by one we propose to use a single electron transistor (SET), i.e., a small metallic Coulomb-blockaded island with total electric capacitance $C = C_L + C_R + C_g$ (see Fig. 1) coupled to the leads by tunnel contacts. Large Coulomb energy $E_C = e^2 / (2C)$ of the island compared to the temperature $T$ prevents an additional electron to tunnel in and allows one to manipulate the charge state $n$ of the island in a controllable way by varying the electrical potential of the gate electrode $V_g = e n_g / C_g$ (see Fig. 1), where $C_g$ is the gate electrode capacitance. For an S island this physical picture becomes more complicated due to the electron number parity effect.2,3 This effect consists in the $2e$ - periodic dependence of the observables on an applied gate voltage and provides, thus, a direct confirmation of the Cooper pair charge quantization. At finite temperatures the parity phenomena are controlled by the free energy difference $\delta F = \Delta - T \ln N_{eff}$ between the states with odd and even number of excess electrons in the granule. Here $N_{eff}$ equals the effective number of available states for an additional particle. It is clear that the parity effect is observable only for a positive value of this free energy barrier, i.e., at low enough tempera-

Figure 1. (color online) Setup of the NISIN SET with a bias voltage $\pm V/2$ applied to the normal metal electrodes tunnel coupled to the central S disc with capacitances $C_L$ and $C_R$. Magnetic field is applied perpendicular to the disc plane.
tures: $T < T^* = \Delta / \ln N_{eff}$. Applying an external magnetic field one can suppress partially the S gap and, thus, suppress the parity phenomenon. One can observe this suppression either by measuring the change of periodicity of the SET characteristics vs the gate voltage in an applied magnetic field $H_{app}$ or by using a varying magnetic field for controlled electron transfer through the SET at a fixed gate potential. At low temperatures the guaranteed suppression of the parity effect can be achieved by introducing a vortex line in the granule which provides a natural trap for an entering electron.

Applying an oscillating magnetic field, i.e., changing periodically the island vorticity, we can induce the even-odd transitions in the number $n$ of electrons trapped on the island. Choosing the gate voltage as shown in Fig. 2 by a black dot, one can switch between the states $n = 0$ and $n = 1$. Without the vortex, the even electron number, $n = 0$, shown by the large white diamond is favored. With the vortex, the odd electron number becomes preferable as shown by the dashed red diamonds. Applying a constant bias voltage to the SET one can convert this modulation of the charge state into unidirectional charge pumping. The above picture of the vortex controlled parity effect can change at low temperatures less than the minigap $\omega_0 < \Delta$ in the spectrum of quasiparticles trapped in the vortex core. This new energy scale $\omega_0$ arises from the quantization of the spectrum of single particle excitations confined within the core by the Cooper pair potential. Though this minigap in most superconductors is small compared to the bulk gap it can still restore the parity effect. The free energy value paid for the even-odd transition in the electron number in the vortex state can be estimated as $\delta F_v = \omega_0 - T \ln N_v$, where $N_v \sim k_F L$, $L$ is the length of the vortex line and $k_F$ is the Fermi momentum. The condition $\delta F_v = 0$ gives us the temperature $T^*_v = \omega_0 / \ln N_v$, separating the regimes with $e$ and $2e$ charge periods in the vortex state and, thus, at temperatures below $T^*_v$ the parity effect can be restored. It is quite useful to emphasize here a simple analogy with the parity effect in the Josephson junction where $\omega_0$ should be replaced by the minigap that depends on the phase difference between the SC leads.

We now proceed with the study of an exemplary SET setup (see Fig. 1) which allows us to illustrate the above charge-vortex interplay. Hereafter we focus on the single electron transport between the normal metal leads and do not consider possible magnetic pumping based on the use of Cooper pair slusses in Josephson systems with S electrodes. The size of the Coulomb blockaded S island is assumed to be of the order of several coherence lengths $\xi$ so that applying an external magnetic field we can introduce at least one vortex in this island. The electronic transport through this device can be described by a standard rate equation accounting for parity effects. For the sake of simplicity we restrict to a two-level approximation assuming low temperature regime $T \ll E_C$ and taking the gate voltage interval $0 < n_g < 1$. The equation for the $n = 1$ charge state probability $p_1$ reads:

$$dp_1 = \Gamma_{0\rightarrow1}p_0 - \Gamma_{1\rightarrow0}p_1, \quad p_0 = 1 - p_1,$$

where $\Gamma_{0\rightarrow1}$ and $\Gamma_{1\rightarrow0}$ are the rates for the electron tunneling into and out of the island, respectively. These rates are, of course, determined by the sum of contributions coming from the transport through the contacts with left and right electrodes: $\Gamma_{0\rightarrow1} = \Gamma_{L}^{0}[U_L] + \Gamma_{R}^{0}[U_R]$ and $\Gamma_{1\rightarrow0} = \Gamma_{L}^{0}[-U_L] + \Gamma_{R}^{0}[-U_R]$, where

$$\Gamma_{j}^{k}[U] = \frac{1}{e^2 R_{Tj}} \int_{-\infty}^{\infty} \nu_j(\varepsilon)f_N(\varepsilon - U)[1 - f_{S}^{k}(\varepsilon)]d\varepsilon$$

is an increasing function of $U$. Here $R_{Tj}$ is the resistance of the $j$th tunneling junction, $\nu_j(\varepsilon)$ is the local density of states (LDOS) of the island near the $j$th junction normalized to its normal state value $\nu_N(0)$, index $j = L, R$ stands for the left and the right junctions, $f_N(\varepsilon) = (e^{\varepsilon/T} + 1)^{-1}$ is the Fermi distribution function in the normal leads, and $f_{S}^{e}(\varepsilon)$ is the distribution function in the S island describing the states with an even (odd) total number of electrons. The Coulomb blockade effect and the bias voltage $V_{L,R} = \pm V/2$ determine the energy cost $U_{L,R} = E_C(2n_g - 1) - eV_{L,R}$ for tunneling.

The increasing magnetic field and vortex entry affect both the LDOS $\nu_{L,R}(\varepsilon)$ and distribution function $f_{S}^{e}(\varepsilon)$ in the above expressions. To find the distribution function $f_{S}^{e}(\varepsilon)$ we assume that the zero (single) charge state corresponds to an even (odd) total number of electrons and use the so-called parity projection technique:

$$f_{S}^{e}(\varepsilon) = \frac{f_F(\varepsilon) \mp \exp(-2N_{gp})f_B(\varepsilon)}{1 \mp \exp(-2N_{gp})},$$

where $f_{F,B}(\varepsilon) = (e^{\varepsilon/T} \pm 1)^{-1}$ are the Fermi and Bose distribution functions. The number of quasiparticles can
be expressed as

\[ N_{qp} = 2\nu N(0) \int dV \int_0^\infty \nu(\varepsilon, \mathbf{r}) f_F(\varepsilon) d\varepsilon . \quad (4) \]

In the limit \( J_{FB} \ll 1 \) one can neglect between these distribution functions and reduce \([3]\) to the form

\[ \frac{\partial}{\partial t} f_S^e(\varepsilon) = A_{\varepsilon, o} f_F(\varepsilon) \]

with the factor \( A_{\varepsilon} = A_{\varepsilon}^{-1} = \tanh(N_{qp}) \). In the low temperature limit \( T \ll \varepsilon_{\text{min}} \) with \( \varepsilon_{\text{min}} \) being the minigap in the quasiparticle spectrum of the island, we obtain \( N_{qp} \approx N_{eff} \varepsilon_{\text{min}} / T \), where \( N_{eff} \) is a slow function of temperature \( T \) (see Appendix \( \Delta \) for details).

Within the region of the essential parity effect (when \( |A_k - 1| \sim 1 \) we can rewrite the tunneling rate as follows:

\[ \Gamma^k_j[U > 0] = \frac{I_j(U)}{e} \left[ \frac{1 + \frac{A_k}{eU/T}}{1 - e^{-U/T}} \right], \quad (5a) \]

\[ \Gamma^k_j[U < 0] = \frac{I_j(U)}{e} \left[ \frac{1 - \frac{A_k}{eU/T}}{1 - e^{-U/T}} \right], \quad (5b) \]

where the “seed” IV-characteristic of the tunnel junction in the absence of the Coulomb effects is

\[ I_j(U) = \int_{\varepsilon_{\text{edge}}}^{\infty} \frac{\nu_j(\varepsilon)}{eR_T} \left[ f_N(\varepsilon - U) - f_N(\varepsilon) \right] d\varepsilon . \quad (6) \]

Note that \( I_j(-U) = -I_j(U) \). Further calculations should assume a certain model describing the dependence of the IV curves \( I_j(U) \) and the number of quasiparticles \( N_{qp} \) on the applied magnetic field. For the sake of simplicity we consider the S island to be symmetric (see Fig. 1) assuming LDOS and tunnel resistances at both junctions to be equal, i.e., \( \nu_j = \nu(\varepsilon) \) and \( R_{Tj} = R_T \). In this case key parameters governing the behavior of the IV curve, i.e., the minigaps \( \varepsilon_j \) in the quasiparticle spectrum at the \( j \)th junction are also equal \( \varepsilon_j = \varepsilon_{\text{edge}} \). The most important part of \( I_j(U) \) controlling the charge transfer corresponds to small voltages \( (U \lesssim \varepsilon_{\text{edge}}) \) when the IV curve reveals the temperature activated behavior,

\[ I_j(U) \approx \frac{T}{eR_T} e^{-(\varepsilon_{\text{edge}} - U)/T} \int_0^\infty \nu(\varepsilon_{\text{edge}} + Tx)e^{-x} dx . \quad (7) \]

In the large voltage limit \( (U \gg \varepsilon_{\text{edge}}) \) we assume a linear dependence \( I_j(U) = U/eR_T \). Note that we neglect here a low voltage contribution to the current arising from the exponential tail of the residual density of states localized inside the vortex core.

Thus, the basic characteristics of our rate equation are determined by the magnetic field dependence of two energy scales: (i) the spectral gap \( \varepsilon_{\text{edge}} \) at the junctions and (ii) the minimal spectral gap \( \varepsilon_{\text{min}} \) over the island. Considering an exemplary geometry shown in Fig. 11 one can see that the energy scale \( \varepsilon_{\text{min}} \) is determined by the maximum of the local superfluid velocity \( v_S \) reached either at the edge of the S disc or in the vortex core. The gap \( \varepsilon_{\text{edge}} \) at the junctions is determined by the geometry of the S leads attached to the disc. Adding these S leads one can control the magnetic field effect on the tunneling DOS and parity phenomenon independently. Taking, e.g., the diffusion limit with the coherence length well exceeding the mean free path \( \ell \) we find (see Refs. 18–20):

\[ \varepsilon_{\text{edge}} = \Delta(H)(1 - \gamma_H^{2/3})^{3/2} , \quad (8) \]

where \( \Delta(H) = \Delta(0)e^{-\pi \gamma_H/4} \), \( \gamma_H = \hbar v_S^2 /[2D\Delta(H)] \), \( D \) is the diffusion coefficient, \( v_S^2 = (\pi DH w/\Phi_0)^2/3 \), and \( w \ll \xi \) is the width of the S lead. Estimating now the energy scale \( \varepsilon_{\text{min}} \) we can use the same expression \([3]\) substituting \( \gamma_H = \hbar v_S^2 /[2D\Delta(H)] \) with the maximum local superfluid velocity \( v_S \). Assuming the screening effects to be small, i.e., when the disc radius is smaller than the effective London penetration depth, we get: max \( v_S = \pi DH R_\Phi /\Phi_0 \). Considering clean limit \( \xi \ll \ell \) we should put \( \varepsilon_{\text{min}} = \Delta(H) - \hbar k_F v_S \), where \( \Delta(H) \approx \Delta(0)[1 - \alpha H^2/H_c^2] \). Here, \( H_c \sim \Phi_0 / R_\xi^2 \) is the field of the first vortex entry and \( \alpha \) is a numerical factor of order unity. Just before the entry of the first vortex \( \varepsilon_{\text{min}} \) tends to zero in the clean limit and remains at finite value in the dirty limit. After the vortex enters \( \varepsilon_{\text{min}} \) equals the minigap \( \omega_0 \) in the clean limit and turns to zero in the dirty regime.

In order to model vortex induced pumping we assume the following time dependence of the spectral gaps with the period \( \tau = t_0 + t_v \) (see Fig. 3) dictated by the piecewise constant magnetic field applied: \( \varepsilon_{\text{edge}} = \varepsilon_{\text{min}} = \Delta(0) \) for \( 0 < t < t_0 \) and \( \varepsilon_{\text{edge}}(H_m) = \Delta_p, \varepsilon_{\text{min}}(H_m) = \varepsilon_0 \) in the interval \( t_0 < t < \tau \). The characteristic times of the vortex entry/exit are assumed to be negligible comparing to \( t_v \) and \( t_0 \). Changing the value \( \varepsilon_0 \) from zero to \( \omega_0 \) we have the crossover from the dirty to the clean limit (at \( H_m > H_c \)). The average current flowing through the \( j \)-th
charging transmitted through both junctions should be equal to the electron charge. Due to the large ratio \( \Gamma^+_{L}/\Gamma^+_{R} = e^{eV/T} \gg 1 \) of the tunneling rates the most of this charge transfer \( e(1-\exp(-|eV|/T)) \) occurs through the left junction, while only the exponentially small part of it \( e \exp(-|eV|/T) \) is transmitted through the right junction. The vortex exit should be accompanied by the discharge of the island which occurs with equal rates through the both junctions. As a result, half of electron charge exits the island through each junction.

Summing up the total charge transmitted through the system per cycle we find \( Q = e((1/2 - e^{-|eV|/T}) \). The above symmetry of the discharging processes results in a rather strong shot noise in the system: the fluctuating transmitted charge equals to \( e/2 \) and the resulting current noise is given by the expression \( \sqrt{(\delta I^2)} = ef/2 \). The last two terms in (12) appear if the time intervals of two stages \( \tau_0 \) and \( \tau_0 \) (with and without vortex, respectively) become comparable or shorter than the characteristic charging times \( 1/\Gamma^+_{L} \) and \( 1/\Gamma^+_{R} \). Therefore the maximum operation frequency \( f = 1/\tau \) is limited by a single quasiparticle tunnel rate \( \Gamma_0/\gamma_{eff} \). Besides the effect of the frequency the average current \( \langle J \rangle \) also deviates from \( ef/2 \) at small bias voltages and/or small magnetic field amplitudes, due to the dependence of total rates \( \Gamma^+_{L,R} \) on these parameters (see Fig. (1)). The terms proportional to \( \gamma_0 \) and \( \gamma_v \) originate from the leakages and lead to currents exceeding \( ef/2 \) shown at Fig. (3) for larger \( V \) and/or \( H_m \) values. Nevertheless satisfying the conditions

\[
N_{eff}/\Gamma_0 \ll \tau_0 \ll \gamma_0^{-1},
\]

\[
e^{(\Delta_v-U_L)/T}\Gamma_0 \ll \tau_v \ll \gamma_v^{-1},
\]

one can obtain the plateau of the average current at \( ef/2 \) (see Appendix C for details). These conditions can be met provided we set \( |eV| > T, \min(T, |eV| - T \ln N_{eff}) \ll U_L \ll \Delta - T \ln N_{eff} \). The breakdown of the above condition at the lower bound of \( U_L \) can signal the presence of the minigap in the vortex core in the clean limit \( \varepsilon_0 = \omega_0 \). Note that for non-symmetric case, the resulting average current will deviate from \( ef/2 \). However, a finite averaged current ranging between 0 and \( ef \) can be obtained in this case as well.

To sum up, we have studied the interplay between vorticity and electric charge which can manifest in conditions of Coulomb blockade through the vortex induced suppression of the parity effect in mesoscopic samples. Vortex entry and exit from the sample is shown to be accompanied by synchronized entry and exit of a single electron charge. Applying the bias voltage and oscillating magnetic field one can observe a vortex governed turnstile phenomenon: the switching between the Meissner and vortex states periodically opens the device for single charge transfer. Thus, we have demonstrated that the SET devices provide a unique tool for manipulating the collective dynamics of charge and vorticity in mesoscopic superconducting samples.
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Appendix A: Derivation of expression for $N_{qp}$ and of Eqs. (5)

In the low temperature limit $T \ll \varepsilon_{\text{min}}$ with $\varepsilon_{\text{min}}$ being the minigap in the quasiparticle spectrum of the island Eq. (7) from main text can be written as follows

$$N_{qp} \approx N_{\text{eff}} e^{-\varepsilon_{\text{min}}/T}, \quad (A1)$$

where

$$N_{\text{eff}} = 2\nu N(0) T \int dV \int_0^\infty \nu(x_{\text{min}} + T \cdot x, r) e^{-x} dx \quad (A2)$$

is a slow function of temperature $T$. Indeed,

$$\frac{\partial N_{\text{eff}}}{\partial T} = 2\nu N(0) T \int dV \int_0^\infty x_{\text{min}} [T \cdot x, r] e^{-x} dx \approx \frac{N_{\text{eff}}}{T} \approx \frac{N_{\text{eff}} \varepsilon_{\text{min}}}{T}. \quad (A3)$$

To derive Eqs. (5) from the main text it is convenient to rewrite Eq. (2) for the tunneling rates in the form:

$$\Gamma_{jj}[U] = \frac{1}{e^{2R_j T_j} \int_{-\infty}^{\infty} \nu_j(x) f_N(x - U)(1 - f_N(x)) dx}$$

$$+ \frac{\Gamma_{jj}[U]}{e^{2R_j T_j} \int_{-\infty}^{\infty} \nu_j(x) f_N(x)(1 - f_N(U - x)) dx}, \quad (A4)$$

where we explicitly separate the parity effect contribution.

These expressions read

$$\Gamma_{jj}[U] = \frac{I_j(U) / e}{1 - e^{-U/T}}$$

$$+ \left( \frac{\Gamma_{jj}[U]}{e^{U/T}} - \frac{\Gamma_{jj}[U]}{1 - e^{-U/T}} \right), \quad (A5)$$

where $I_j(U)$ is the “seed” IV-characteristic of the tunnel junction in the absence of the Coulomb effects given by the Eq. (4) of the main text and

$$\Gamma_{qp} = \frac{T}{e^{2R_j T_j} \int_0^\infty \nu_j(x_j + T x) e^{-x} dx}.$$  \quad (A6)

Within the region of the essential parity effect (when $|A_k - 1| \sim 1$) we can obviously neglect the term proportional to $\Gamma_{qp}$ and obtain the Eqs. (5) from the main text:

$$\Gamma_{j}[U > 0] = I_j(U)/e \left[ 1 + \frac{A_k}{e^{U/T} - 1} \right], \quad (A7a)$$

$$\Gamma_{j}[U < 0] = \frac{I_j(U)/e}{1 - e^{-U/T} A_k}. \quad (A7b)$$

Appendix B: Derivation of Eq. (12)

Using the magnetic field protocol considered in the main text $\varepsilon_{\text{edge}} = \varepsilon_{\text{min}} = \Delta(0)$ for $0 < t < t_0$ and $\varepsilon_{\text{edge}}(H_m) = \Delta_1, \varepsilon_{\text{min}}(H_m) = \varepsilon_0$ in the interval $t_0 < t < \tau$, and the periodicity condition

$$p_1(0) = p_1(\tau) = \frac{\int_0^\infty \Gamma_{0-1} e^{-E \gamma dt''} dt''}{1 - e^{-\gamma \Gamma_{0-1} t''}}. \quad (B1)$$

one can rewrite the solution (10) in the main text as follows

$$p_1(t < t_0) = p_{1,cq}^{0} + (p_{1,cq}^{0} - p_{1,cq}^{v}) \frac{e^{-\Gamma_{0-1}' 0 t}}{1 - e^{-\Gamma_{0-1}' 0 t}}, \quad (B2)$$

$$p_1(t > t_0) = p_{1,cq}^{0} + (p_{1,cq}^{0} - p_{1,cq}^{v}) e^{-\Gamma_{0-1}''(t-t_0)} \frac{1 - e^{-\Gamma_{0-1}'' t}}{1 - e^{-\Gamma_{0-1}'' t_0}}. \quad (B3)$$

Here the superscript 0 (v) corresponds to the time interval $0 < t < t_0$ ($t_0 < t < \tau$, $\tau = t_0 + t_v$) and $p_{1,cq}^{k} = \Gamma_{0-1}^{k}/\Gamma_{0-1}^{k}$ is the adiabatic solution in the corresponding time interval.

Substituting this solution to Eqs. (9) (12) from the main text one can obtain

$$\langle J \rangle = J_{eq}^{0} t_0 + J_{eq}^{v} t_v + e \left( \frac{\Gamma_{R}^{k}}{\Gamma_{\Sigma}^{k}} - \frac{\Gamma_{R}^{v}}{\Gamma_{\Sigma}^{v}} \right)$$

$$\times \left( p_{0,eq}^{0} - p_{0,eq}^{v} \right) \left[ 1 - e^{-\Gamma_{0-1}' 0} \right] \left[ 1 - e^{-\Gamma_{0-1}'' t} \right]. \quad (B4)$$

where $\Gamma_{R}^{k} = \Gamma_{0-1}^{R,k} + \Gamma_{1-0}^{R,k}$ and $J_{eq}^{k} = e(\Gamma_{R,k}^{0-1} \Gamma_{0-1}^{L,q} - \Gamma_{0-1}^{L,k} \Gamma_{0-1}^{R,q})/\Gamma_{\Sigma}^{k}$ is the current flowing through the island when the probabilities follow the adiabatic solution $p_{eq}^{k}$.

Using the expressions for all tunneling rates, assuming the conditions of Eqs. (13) from the main text to be valid and keeping only the first order corrections in small parameters $e^{-eV/T}, \gamma_0 t_0, \gamma_v t_v, e^{-\Gamma_{0-1}' 0}$, and $e^{-\Gamma_{0-1}'' t}$ we obtain

$$p_{eq}^{0} \approx \frac{\gamma_0}{2 \Gamma_{\Sigma}^{0}} \gamma_v \approx 1 - \frac{\gamma_v}{\Gamma_{\Sigma}^{v}} \gamma_v \approx \frac{\gamma_0}{2 \Gamma_{\Sigma}^{0}}. \quad (B5)$$

$$\Gamma_{R}^{v} \approx 1 - \frac{\gamma_v}{2 \Gamma_{\Sigma}^{v}} e^{-eV/T}, \quad \Gamma_{R}^{0} \approx \frac{\gamma_0}{2 \Gamma_{\Sigma}^{0}} \gamma_v \approx \frac{1}{2} + \frac{\gamma_0}{2 \Gamma_{\Sigma}^{0}}. \quad (B6)$$

Substituting these expressions into (B4) we get Eq. (12) from the main text.
Appendix C: Range of parameters for current plateau observation

In this section we consider ranges of the parameters where the plateau of the current \( J_s(t) \) averaged over the period \( \tau \) is close to \( ef/2 \), i.e., \( \langle J \rangle = ef(1 \pm \epsilon)/2 \) with a certain small \( \epsilon \ll 1 \). Using the result \(^{12}\) from the main text one can roughly rewrite this condition as follows

\[
e^{-|eV|/T}, \gamma_0 t_0, \gamma_v t_v, e^{-R_d t_0}, e^{-R_v t_v} \lesssim \epsilon. \tag{C1}
\]

Further we focus on IV characteristic for \( \langle J \rangle(V) \) assuming that all other parameters \( (T, \Delta, \Delta_v, \varepsilon_0, \Delta_{ch} = E_C(2n_g - 1), N_{eff}, t_0, \text{and } t_v) \) are chosen to be optimal for rather small bias voltages.

Considering all the corrections to the current plateau one can separate them into two groups: (i) voltage-independent corrections

\[
2\Gamma_0 t_0/N_{eff} > \ln(1/\epsilon), \quad \frac{2\Gamma_0 e^{-\Delta_v/T t_v}}{\tanh(N_{eff} e^{-\varepsilon_0/T})} < \epsilon, \tag{C2}
\]

and the necessary condition at \( V = 0 \)

\[
\Gamma_0 e^{-(\Delta - \Delta_{ch})/T t_0} < \epsilon, \tag{C3}
\]

(ii) voltage-dependent corrections

\[
e^{-|eV|/T} < \epsilon, \quad \Gamma_0 e^{-|\Delta - \Delta_{ch}| - |eV|/T t_0} < \epsilon,
\]

\[
\Gamma_0 e^{-|\Delta_v - \Delta_{ch}| - |eV|/T t_v} > \ln(1/\epsilon), \tag{C4}
\]

\[
\Gamma_0 e^{-(\Delta_v + \Delta_{ch} - |eV|)/T t_v} < \epsilon,
\]

which can be rewritten as the conditions on the bias voltage

\[
|eV| > T \ln(1/\epsilon), \tag{C5a}
\]

\[
|eV| > \Delta_v - \Delta_{ch} - T \ln[\Gamma_0 t_v/\ln(1/\epsilon)], \tag{C5b}
\]

\[
|eV| < \Delta - \Delta_{ch} - T \ln[\Gamma_0 t_0/\epsilon], \tag{C5c}
\]

\[
|eV| < \Delta_v + \Delta_{ch} - T \ln[\Gamma_0 t_v/\ln(1/\epsilon)]. \tag{C5d}
\]

The last term in \( \text{(C4)} \) originates from the condition \( \gamma_v t_v \lesssim \epsilon \) for the case \( U_R < -T \ln[1/\tanh(N_{eff} e^{-\varepsilon_0/T})] < 0 \).

One can see that the increase in the minigap \( \varepsilon_0 \) modeling crossover between vortex minigaps in dirty and clean limits breaks first the last voltage-independent condition in Eq. (C2). As a result, the plateau of the averaged current will be shifted to \( \gamma_v t_v \) as a whole without change in the range of the bias voltage \( \text{(C5)} \).

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