Noncommutative Biorthogonal Polynomials

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The theory of orthogonal polynomials is well established and has many applications. For any sequence \( \{S_i\} \) of elements of a commutative ring \( R \), we can define a biadditive function \(< \cdot, \cdot>: R[x] \times R[x] \to R \) by \( < ax_i, bx_j > = abS_{i+j} \) for \( a, b \in R \) and define a sequence of polynomials \( \{p_n\} \) by

\[
p_n = \begin{vmatrix} S_n & \cdots & S_{2n-1} & x^n \\ \vdots & \ddots & \vdots & \vdots \\ S_0 & \cdots & S_{n-1} & 1 \end{vmatrix}
\]

Then \( < p_n, p_m > \) if and only if \( n \neq m \), i.e. the sequence \( \{p_n\} \) is orthogonal. The \( S_i \) are called the moments of \( \{p_n\} \). For a more detailed introduction, see Chihara’s classic text \[1\] on the subject. The idea of orthogonal polynomials and this method of generating them has been generalized in two ways to achieve new types of polynomials: noncommutative orthogonal polynomials and biorogonal polynomials.

The theory of orthogonal polynomials has been extended to cover rings of noncommutative operators, such as matrices (see \[1\]). In 1994, Gelfand, Krob, ___

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Lascoux, Leclerc, Retakh and Thibon extended the theory to general non-commutative rings by setting $p_n$ equal to the quasideterminant of a similar matrix. The paper also shows that the 3-term recurrence relation, which is well-known for commutative orthogonal polynomials, still holds in this case.

Second, orthogonal polynomials have been generalized in several ways to biorthogonal polynomials. See [2] for more details on these generalizations. One such extension is considered by Bertola, Gekhtman and Szmigielski. A family of biorthogonal polynomials is defined to be two sequences of real polynomials $\{p_n(x)\}$ and $\{q_n(y)\}$ with the property that $\int \int p_n(x)q_m(y)K(x, y)d\alpha(x)d\beta(y) = 0$ when $n \neq m$ for particular $K$, $\alpha$ and $\beta$. In this paper, it is shown that these polynomials can be represented as determinants of matrices whose entries are bimoments and, for a specific $K(x, y)$, a 4-term recurrence relation is obtained.

Here, we define biorthogonal polynomials over a noncommutative ring. We bring together the two different generalizations described above to present a completely algebraic definition of noncommutative biorthogonal polynomials. For our purposes, a biorthogonal family consists of two sequences of polynomials $\{p_n(x)\}$ and $\{q_n(y)\}$, over a division ring $R$, along with a function $\langle \cdot , \cdot \rangle : R[x] \times R[y] \to R$ so that $\langle p_n(x), q_m(y) \rangle = 0$ for all $n \neq m$. Using this definition, we obtain recurrence relations for some types of biorthogonal polynomials and thus generalize the 4-term recurrence relations of [3]. We conclude with a broad extension of Favard’s theorem.
1 Set-Up and Definitions

Let $R$ be a division ring with center $C$. We will view $R[x]$ as an $R$-$C$ bimodule of $R$ and $R[y]$ as a $C$-$R$ bimodule of $R$. That is, elements of $R[x]$ will be of the form $\sum a_i x^i$ and elements of $R[y]$ will be of the form $\sum y^j b_j$ so that $xc = cx$ and $yc = cy$ for all $c \in C$. Let $\langle \cdot, \cdot \rangle: R[x] \times R[y] \to R$ so that

$$\langle \sum a_i x^i, \sum y^j b_j \rangle = \sum a_i \langle x^i, y^j \rangle b_j.$$ 

A system of polynomials $\{p_n\}, \{q_n\}_{n \in \mathbb{N}}$ is biorthogonal with respect to $\langle \cdot, \cdot \rangle$ if $\langle p_n(x), q_m(y) \rangle = 0$ for all $n \neq m$.

Let $I_{a,b} = \langle x^a, y^b \rangle$. The set $I = \{I_{a,b}\}_{a,b \in \mathbb{Z}_{\geq 0}}$ is called the set of bimoments for $\langle \cdot, \cdot \rangle$. The bimoments completely define the function $\langle \cdot, \cdot \rangle$ so we will say that a set of polynomials is biorthogonal with respect to $I$.

In keeping with the notation of [3], we will let $I$ be the matrix of bimoments and write $Id$ for the identity matrix. Note in these cases, and below, all matrices and vectors are infinite, with rows and columns indexed by $\mathbb{Z}_{\geq 0}$.

We extend $\langle \cdot, \cdot \rangle$ to $R[x]^n \times R[y]$ and to $R[x] \times R[y]^n$ in the following way:

If $B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \in R[x]^n$ and $g \in R[y]$, then $\langle B, g \rangle = \begin{bmatrix} \langle b_1, g \rangle \\ \vdots \\ \langle b_n, g \rangle \end{bmatrix}$.

Similarly, if $f \in R[x]$ and $D = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in R[y]^n$, then $\langle f, D \rangle = \begin{bmatrix} \langle f, d_1 \rangle \\ \vdots \\ \langle f, d_n \rangle \end{bmatrix}$.

If $C \in \text{Mat}_{r \times n}(R), B \in R[x]^n$ and $g \in R[y]$, then $\langle CB, g \rangle = C \langle B, g \rangle$. 

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For an \((n+1)\times(n+1)\) matrix \(A\), let \(A_{i,j}\) denote the \(n\times n\) matrix formed by removing the \(i\)th row and \(j\)th column. Then (c.f. [4], def. 2.1) the \(i,j\)-quasideterminant of \(A\) \(|A|_{i,j}\) is

\[
\begin{vmatrix}
 a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n+1} \\
 \vdots & \ddots & \ddots & \ddots & \vdots \\
 a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n+1} \\
 \vdots & \ddots & \ddots & \ddots & \vdots \\
 a_{n+1,1} & \cdots & a_{n+1,j} & \cdots & a_{n+1,n+1}
\end{vmatrix}
\]

\[= a_{i,j} - \begin{bmatrix}
a_{i,1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{i,n+1}
\end{bmatrix} \cdot (A_{i,j})^{-1} \cdot \begin{bmatrix}
a_{1,j} \\
\vdots \\
a_{i-1,j} \\
a_{i+1,j} \\
\vdots \\
a_{n+1,j}
\end{bmatrix}.
\]

Note that, after suitably permuting rows and columns, this the is Schur complement of a block decomposition of \(A\). The quasideterminant \(|A|_{i,j}\) exists if and only if \(A_{i,j}\) is invertible.

2 Constructing Biorthogonal Polynomials Using Bimoments

Throughout, we will assume that the set of bimoments is generic in the sense that all quasideterminants considered exist and are invertible. This is our only restriction on the set of bimoments.
Theorem: Let \( \{I_{a,b} \mid a, b \in \mathbb{Z}_{\geq 0}\} \subseteq \mathbb{R} \). For all \( n \in \mathbb{N} \), define

\[
p_n(x) = |I|_{1,n+1} = \begin{vmatrix}
I_{n,0} & \cdots & I_{n,n-1} & x^n \\
\vdots & \ddots & \vdots & \vdots \\
I_{1,0} & \cdots & I_{1,n-1} & x \\
I_{0,0} & \cdots & I_{0,n-1} & 1 \\
\end{vmatrix}
\]

and

\[
q_n(y) = \begin{vmatrix}
1 & y & \cdots & y^n \\
I_{n-1,0} & I_{n-1,1} & \cdots & I_{n-1,n} \\
\vdots & \vdots & \ddots & \vdots \\
I_{0,0} & I_{0,1} & \cdots & I_{0,n} \\
\end{vmatrix}
\]

Then \( \{p_n\}, \{q_n\} \) is a (monic) biorthogonal system of polynomials with respect to the set of bimoments \( \{I_{a,b}\} \).

To prove the theorem we need the following lemma:

Lemma: Let \( n \in \mathbb{Z}_{\geq 0} \) and \( p_n, q_n \) be as defined as in the proposition. Then

\[
< x^i, q_n > = < p_n, y^i > = 0 \text{ for all } 0 \leq i \leq n - 1.
\]

Proof of Lemma:

Let \( n \in \mathbb{N} \) and \( 0 \leq i \leq n - 1 \). We see that

\[
< p_n, y^i > = < x^n - [I_{n,0} \cdots I_{n,n-1}] \cdot (I^{1,n+1})^{-1} \cdot \begin{bmatrix} x^{n-1} \\ \vdots \\ 1 \end{bmatrix}, y^i > = 0
\]

\[
I_{n,i} = [I_{n,0} \cdots I_{n,n-1}] \cdot (I^{1,n+1})^{-1} \cdot \begin{bmatrix} I_{n-1,i} \\ \vdots \\ I_{0,i} \end{bmatrix}
\]
Applying the definition of quasideterminant, we see that this is
\[
\begin{vmatrix}
I_{n,0} & \cdots & I_{n,n-1} & I_{n,i} \\
\vdots & \ddots & \vdots & \vdots \\
I_{0,0} & \cdots & I_{0,n-1} & I_{0,i}
\end{vmatrix}
\]
Thus, since \(0 \leq i \leq n-1\), \(< p_n, y^i >\) is the quasideterminant of a matrix whose \(n\)th column is equal to its \((i+1)st\) column and hence is 0 (c.f. [5], prop. 1.4.6).

Similarly,
\[
< x^i, q_n > =
\begin{vmatrix}
I_{i,0} & \cdots & I_{i,n} \\
I_{n-1,0} & \cdots & I_{n-1,n} \\
\vdots & \ddots & \vdots \\
I_{0,0} & \cdots & I_{0,n}
\end{vmatrix}
\]
Thus, since \(0 \leq i \leq n-1\), the top row will be equal to the \((n-i+1)th\) row, again making the quasideterminant 0 (c.f. [5], prop. 1.4.6).

**Proof of Proposition:**

Let \(n, m \in \mathbb{N}\) so that \(n \neq m\). Suppose \(n < m\). Now \(p_n(x) = \sum_{k=0}^{n} a_k x^k\) for some \(a_0, \ldots, a_n \in R\). Thus \(< p_n, q_m > = \sum_{k=0}^{n} a_k < x^k, q_m >\). For all \(0 \leq k \leq n\), \(k < m\) so by the lemma, \(< x^k, q_m > = 0\). Thus \(< p_n, q_m > = 0\). The case for \(n > m\) is similar.

**Remark:** We note here that we can recover the construction of orthogonal polynomials in [4] from the construction above. Let \(R\) be the free associative algebra on generators \(S_0, S_1, \cdots\) with \(S_{a+b} = I_{a,b}\) for all \(a, b \in \mathbb{N}\). Following the notation of [4] let \(^*\) be the anti-automorphism so \((S_k)^* = S_k\) and \((\sum c_i x^i)^* = \sum (c_i)^* x^i\).

A little examination shows that \(q_n = p_n^*\). Thus \(< p_n, q_m > = < p_n, p_m >^*, i.e. the collection \(\{p_n\}\) is orthogonal with respect to the (very similar) inner product \(< \cdot, \cdot >^* \) where \(< \sum c_i x^i, \sum d_j y^j >^* = \sum c_i S_{i+j}(d_j)^*\).
3 Banded Matrices:

For $i, j \in \mathbb{Z}_{\geq 0}$, let $E_{i,j}$ denote the matrix with rows and columns indexed by $\mathbb{Z}_{\geq 0}$ so that the $(i, j)$ entry is 1 and all other entries are 0. Let $a \leq 0$ and $b \geq 0$. $M_{[a,b]}$ is defined to be $\text{span}\{E_{i,j} : a \leq i - j \leq b\}$. We will refer to these matrices as “banded”. For example, the set of diagonal matrices is $M_{[0,0]}$. Let $X \in M_{[a,b]}$ and $Y \in M_{[c,d]}$.

**Lemma:** $X + Y \in M_{[\min(a,c),\max(b,d)]}$ and $XY \in M_{[a+c,b+d]}$.

**Proof:**

The proof that $X + Y \in M_{[\min(a,c),\max(b,d)]}$ is trivial. Suppose $[XY]_{u,v} \neq 0$. Then $[X]_{u,w} \neq 0$ and $[Y]_{w,v} \neq 0$ for some w. This implies $a \leq w - u \leq b$ and $c \leq v - w \leq d$. Adding these equations shows that $a + c \leq v - u \leq b + d$. Thus $XY \in M_{[a+c,b+d]}$.

4 Recurrence Relations:

In the commutative case, Bertola, Gekhtman and Szmigielski [3] obtain a 4 term recurrence relation when $I_{a+1,b} + I_{a,b+1} = \alpha_{a}\beta_{b}$. This means there is a formula for $p_{n+1}$ in terms of $p_n, p_{n-1}$, and $p_{n-2}$ and a similar formula for $q_{n+1}$.

$K$ is called the kernel of a system of biorthogonal polynomials if $<a(x), b(y)> = \int \int a(x)b(y)K(x, y)dxdy$. The condition above corresponds to what the authors of this paper called the “Cauchy kernel”: $K(x, y) = \frac{1}{x + y}$. Below, we achieve similar, but longer, recurrences that correspond to kernels of the form $\frac{1}{f(x) + g(y)}$ where f and g are polynomials.

For all $n \in \mathbb{N}$, let

\[
p_n = \begin{vmatrix}
I_{n,0} & \cdots & I_{n,n} \\
\vdots & \ddots & \vdots \\
I_{0,0} & \cdots & I_{0,n}
\end{vmatrix}^{-1}
\begin{vmatrix}
I_{n,0} & \cdots & I_{n,n-1} & x^n \\
\vdots & \ddots & \vdots & \vdots \\
I_{0,0} & \cdots & I_{0,n-1} & 1
\end{vmatrix}
\]

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and let

\[ q_n = \begin{bmatrix}
1 & \cdots & y^n \\
I_{n-1,0} & \cdots & I_{n-1,n} \\
\vdots & \ddots & \vdots \\
I_{0,0} & \cdots & I_{0,n}
\end{bmatrix} \]

These are scalar multiples of the polynomials constructed in Section 2. Therefore they are biorthogonal. A quick check will show that we also have that \(< p_n, q_n > = 1\) for all \(n \in \mathbb{N}\). Thus this system of polynomials is biorthonormal.

**Theorem:** Let \(\{p_k\}, \{q_k\}\) be any biorthonormal polynomials with bimoments \(I\). Suppose there exist polynomials over the center of \(R\) \(f(x) = \sum_{i=0}^{n} a_i x^i\) and \(g(y) = \sum_{j=0}^{m} y^j b_j\) so that \(\sum_{i=0}^{n} a_i I_{r+1,s} + \sum_{j=0}^{m} I_{r,s+j} b_j = \alpha_r \beta_s\) for all \(r, s \in \mathbb{N}\). Then there exist \(n+m+2\) term recurrence relations for \(p_i\) and \(q_i\). That is, we can express \(p_{i+1}\) in terms of \(p_i, \ldots, p_{i-n-m-2}\) and \(q_{i+1}\) in terms of \(q_i, \ldots, q_{i-n-m-2}\). The recurrences we achieve for \(p_{i+1}\) and \(q_{i+1}\) have polynomial coefficients for \(p_i, p_{i-1}, q_i, \text{ and } q_{i-1}\) and scalar coefficients for all other terms.

**Proof:**

Let

\[ \Lambda = \begin{bmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & \ddots \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix} \]

Let \(p(x)\) and \(q(y)\) be column vectors with entries \(p_k\) and \(q_k\) respectively. Note that for each \(k \in \mathbb{Z}_{\geq 0}\), \(p_k\) and \(q_k\) are polynomials of degree \(k\) so for each so the products \(p_k f(x)\) and \(g(y) q_k\) can be written as a linear combination of \(p_{n+k}, \cdots, p_1, p_0\) and \(q_{m+k}, \cdots, q_1, q_0\) respectively.

Let \(X\) and \(Y\) be the infinite scalar matrices so that \(p(x) f(x) = X p(x)\) and \(g(y) q^T(y) = q^T(y) Y^T\). Since \(< p(x), q^T(y) > = Id\), we know that \(< p(x) f(x), q^T(y) > = X\) and \(< p(x), g(y) q^T(y) > = Y^T\).
Suppose $p_k(x) = \sum_{i=0}^{k} c_i x^i$ and $q_l(y) = \sum_{i=0}^{l} y^i d_i$. Let $\pi_k = \sum_{i=0}^{k} c_i \alpha_i$ and $\eta_l = \sum_{i=0}^{l} \beta_i d_i$.

$$
(X + Y^T)_{k,l} = <p_k(x), q_l(y)> + <p_k(x), g(y)q_l(y)> = 
\sum_{i,j} c_i <f(x)x^i, y^j> + \sum_{i,j} c_i <x^i, y^j g(y)> = \sum_{i,j} c_i \alpha_i \beta_j d_j = \pi_k \eta_l.
$$

If $\pi$ and $\eta$ are vectors with entries $\pi_n$ and $\eta_n$, respectively, then $X + Y^T = \pi \eta^T = D_\pi (I \otimes I^T) D_\eta$ where $D_\pi$ and $D_\eta$ are diagonal matrices with (i,i) entries $\pi_i$ and $\eta_i$, respectively.

Let $A = (\Lambda - \text{Id}) D_\pi^{-1} X$ and $B^T = Y^T D_\eta^{-1} (\Lambda^T - \text{Id})$. Since $\mathbf{1}$ is a null vector of $\Lambda - \text{Id}$, $(\Lambda - \text{Id}) D_\pi^{-1} (X + Y^T) = 0$ and $(X + Y^T) D_\eta^{-1} (\Lambda^T - \text{Id}) = 0$. Then $A = -(\Lambda - \text{Id}) D_\pi^{-1} Y^T$ and $B^T = -X D_\eta^{-1} (\Lambda^T - \text{Id})$.

We claim that $A$ and $B$ are banded matrices. Note that $X \in M_{[-\infty,n]}$ since $X_{i,j} = <p_i(x), q_j(y)> = 0$ if $i + n < j$ (because the degree $p_i \ast f(x)$ is less than the degree of $q_j$) and that $Y^T \in M_{[-m,\infty]}$ since $Y^T_{i,j} = <p_i(x), g(y)q_j(y)> = 0$ if $i > m + j$. Note also that $(\Lambda - \text{Id}) \in M_{[0,1]}$.

Applying the results we obtained for banded matrices, we see that $A = (\Lambda - \text{Id}) D_\pi^{-1} X \in M_{[-\infty,n+1]}$ and that $A = -(\Lambda - \text{Id}) D_\pi^{-1} Y^T \in M_{[-m,\infty]}$. Thus $A \in M_{[-m,n+1]}$. Similarly, $B^T \in M_{[-\infty,m+1]}$ and $B^T \in M_{[-n,\infty]}$ so $B^T \in M_{[-n,m+1]}$ and $B \in M_{[-m-1,n]}$. 




Recall that $p(x)f(x) = Xp(x)$ and $g(y)q^T(y) = q^T(y)Y^T$. Then $(\Lambda - Id)D^{-1}p(x)f(x) = (\Lambda - Id)D^{-1}Xp(x) = Ap(x)$ and $g(y)q^T(y)D^{-1}(\Lambda^T - Id) = q^T(y)Y^T D^{-1}(\Lambda^T - Id) = q^T(y)B^T$.

Thus examining the k-1th row of these equations gives the following n+m+2 term recurrence relations, as desired:

$$\left(\pi^{-1}_k p_k - \pi^{−1}_{k−1}p_{k−1}\right)f(x) = \sum_{i=0}^{k+n} A_{k-1,i} p_i,$$

$$g(y)(\eta^{-1}_k q_k - \eta^{−1}_{k−1}q_{k−1}) = \sum_{i=0}^{k+m} B_{k-1,i} q_i.$$

5 Biorthogonal Analogue of Favard’s Theorem:

Favard’s theorem states that if $\{p_n(x)\}$ is a sequence of polynomials which obeys the usual 3-term recurrence relation then there exists an inner product for which these polynomials are orthogonal. Here we show that any two sequences of polynomials are biorthogonal with respect to some function, for which we construct the bimoments. It is important to note that no recurrence relation is required here.

**Theorem:** Let $\{p_n\}, \{q_n\}$ be any set of polynomials over any division ring $R$ so that $p_n$ and $q_n$ are of degree $n$ for all $n \in \mathbb{N}$. For any $\{c_k\}_{k \geq 0}$ in $R$, there exists a unique set of bimoments for which $\{p_n\}, \{q_n\}$ is a biorthogonal system of polynomials and $\langle p_k, q_k \rangle = c_k$.

**Proof:**

It is equivalent to show that there is a set of bimoments so that for all $a, b \in \mathbb{N}$,
the following conditions hold:

1) If $a < b$ then $< x^a, q_b(y) > = 0$.
2) If $a > b$ then $< p_a(x), y^b > = 0$.
3) If $a = b$, then $< p_a(x), q_b(y) > = c_a$.

We will define $I_{a,b}$ inductively on $a + b$. It is pivotal to note that the equations $< x^a, q_b(y) > = 0$, $< p_a(x), y^b > = 0$, and $< p_a(x), q_b(y) > = c_a$ do not involve bimoments of the form $I_{i,j}$ where $i + j > a + b$. Recall that $p_0, q_0 \in R$. Let $I_{0,0} = p_0^{-1}c_0q_0^{-1}$. Then $< p_0, q_0 >= p_0I_{0,0}q_0 = 1$ as desired.

Let $n \geq 1$ and suppose for all $a,b$ such that $a + b < n$, we have defined $I_{a,b}$ to satisfy the previous conditions. For each $0 \leq i \leq n$ define $I_{i,n-i}$ as follows:

**Case 1:** If $i < n-i$ then the equation $< x^i, q_{n-i} >= 0$ is a linear equation whose variables (the bimoments) have all been defined except for $I_{i,n-i}$ due to the order in which the $I_{a,b}$’s are defined. Therefore there is a unique solution which we must define $I_{i,n-i}$ to be.

**Case 2:** Similarly, if $i > n-i$, the equation $< p_i, y^{n-i} >= 0$ has only one unknown and thus has a unique solution which we define $I_{i,n-i}$ to be.

**Case 3:** If $i = n-i$ then, again, the equation $< p_i, q_{n-i} >= c_i$ has one unknown and we define $I_{i,n-i}$ to be the unique solution to this linear equation.

At each step we satisfy all the necessary conditions and have no choice so the bimoments constructed are the unique set for which $\{p_n\}, \{q_n\}$ is a biorthogonal system with $< p_k, q_k >= c_k$.

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