Parameter space for families of parabolic-like mappings

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Abstract

In this paper we study families of degree 2 parabolic-like mappings \((f_\lambda)_{\lambda \in \Lambda}\) (as we defined in \([L]\)). We prove that the hybrid conjugacies between a nice analytic family of degree 2 parabolic-like mappings and members of the family \(\text{Per}_1(1)\) induce a continuous map \(\chi : \Lambda \to \mathbb{C}\), which under suitable conditions restricts to a ramified covering from the connectedness locus of \((f_\lambda)_{\lambda \in \Lambda}\) to the connectedness locus \(M_1 \setminus \{1\}\) of \(\text{Per}_1(1)\). As an application, we prove that the connectedness locus of the family \(C_\alpha(z) = z + az^2 + z^3\), \(\alpha \in \mathbb{C}\) presents baby \(M_1\).

1 Introduction

A degree \(d\) polynomial-like mapping is a degree \(d\) proper holomorphic map \(f : U' \to U\), where \(U'\) and \(U\) are topological disks and \(U'\) is compactly contained in \(U\). This definition captures the behaviour of a polynomial in a neighbourhood of its filled Julia set. The filled Julia set is defined in the polynomial-like case as the set of points which do not escape the domain. The external class of a polynomial-like map is the (conjugacy classes of) the map which encodes the dynamics of the polynomial-like map outside the filled Julia set. The external class of a degree \(d\) polynomial-like map is a degree \(d\) real-analytic orientation preserving and strictly expanding self-covering of the unit circle: the expansivity of such a circle map implies that all the periodic points are repelling, and in particular not parabolic.

In order to avoid this restriction, in \([L]\) we introduce an object, which we call \textit{parabolic-like mapping}, to describe the parabolic case. A parabolic-like mapping is thus similar to a polynomial-like mapping, but with a parabolic external class; that is to say, the external map has a parabolic fixed point. A parabolic-like map can be seen as the union of two different dynamical
parts: a polynomial-like part and a parabolic one, which are connected by a dividing arc.

**Definition 1.1. (Parabolic-like maps)** A parabolic-like map of degree $d$ is a 4-tuple $(f, U', U, \gamma)$ where

- $U'$ and $U$ are open subsets of $\mathbb{C}$, with $U'$, $U$ and $U \cup U'$ isomorphic to a disc, and $U'$ not contained in $U$,
- $f : U' \to U$ is a proper holomorphic map of degree $d$ with a parabolic fixed point at $z = z_0$ of multiplier 1,
- $\gamma : [-1, 1] \to U$ is an arc with $\gamma(0) = z_0$, forward invariant under $f$, $C^1$ on $[-1, 0]$ and on $[0, 1]$, and such that
  \[
  f(\gamma(t)) = \gamma(dt), \quad \forall \ -\frac{1}{d} \leq t \leq \frac{1}{d}.
  \]
  \[
  \gamma([\frac{1}{d}, 1) \cup (-1, -\frac{1}{d}]) \subseteq U \setminus U', \quad \gamma(\pm 1) \in \partial U.
  \]

It resides in repelling petal(s) of $z_0$ and it divides $U'$ and $U$ into $\Omega'$, $\Delta'$ and $\Omega, \Delta$ respectively, such that $\Omega' \subset U$ (and $\Omega' \subset \Omega$), $f : \Delta' \to \Delta$ is an isomorphism and $\Delta'$ contains at least one attracting fixed petal of $z_0$. We call the arc $\gamma$ a dividing arc.

In [L] we extend the theory of polynomial-like maps to parabolic-like maps, and we straighten degree 2 parabolic-like maps to members of the family of quadratic rational maps with a parabolic fixed point of multiplier 1 at infinity and critical points at 1 and $-1$, which is

\[
\text{Per}_1(1) = \{[PA] \mid PA(z) = z + \frac{1}{z} + A\}.
\]

More precisely, we prove the following:

**Straightening Theorem.** Every degree 2 parabolic-like mapping $(f, U', U, \gamma)$ is hybrid equivalent to a member of the family $\text{Per}_1(1)$. Moreover, if $K_f$ is connected, this member is unique.

Note that $[PA] = \{PA, P_{-A}\}$, since the involution $z \to -z$ conjugates $PA$ and $P_{-A}$, interchanging the roles of the critical points. We refer to a member of the family $\text{Per}_1(1)$ as one of the representatives of its class. The family $\text{Per}_1(1)$ is typically parametrized by $B = 1 - A^2$, which is the multiplier of the 'free' fixed point $z = -1/A$ of $PA$. The connectedness locus of $\text{Per}_1(1)$ is called $M_1$. If $f = (f_{\lambda} : U'_{\lambda} \to U_{\lambda})_{\lambda \in \Lambda}$ is a family of degree 2 parabolic-like
maps with parameter space $\Lambda \subset \mathbb{C}$, calling $M_f$ the connectedness locus of $f$, by the uniqueness of the Straightening we can define a map

$$\chi : M_f \to M_1$$

$$\lambda \to B,$$

which associates to each $\lambda$ the multiplier of the fixed point $z = -1/A$ of the member $[P_a]$ hybrid equivalent to $f_\lambda$.

In this paper we will prove that if the family $f$ is analytic and nice (see Def. 2.1 and 2.0.1), the map $\chi$ extends to a map defined on the whole of $\Lambda$ (see 4.1), whose restriction to $M_f$, under suitable conditions (see Def. 5.3) is a ramified covering of $M_1 \setminus \{1\}$ (see Thm. 2.2). The reason why the map $\chi$ covers $M_1 \setminus \{1\}$, instead of the whole of $M_1$, resides in the definition of analytic family of parabolic-like mappings, and it will be explained in section 2.4. As an application, we will show that the connectedness locus of the family $C_\alpha(z) = z + az^2 + z^3, a \in \mathbb{C}$ (see Fig. 1) presents 2 baby $M_1$ (see Fig. 2).

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2 Definitions and statement of the result

In this Section we define an analytic family of parabolic-like maps and its connectedness locus, nice families of parabolic-like maps, and we give an example of nice analytic family of parabolic-like maps. Then we give a review of the Straightening Theorem, an overview of this paper and we state the main result.

**Definition 2.1.** Let $\Lambda \subset \mathbb{C}$, $\Lambda \approx \mathbb{D}$ and let $f = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda}$ be a family of degree $d$ parabolic-like mappings. Set $U' = \{(\lambda, z)| z \in U'_\lambda\}$, $U = \{(\lambda, z)| z \in U_\lambda\}$, $\Omega' = \{(\lambda, z)| z \in \Omega'_\lambda\}$, and $\Omega = \{(\lambda, z)| z \in \Omega_\lambda\}$. Then $f$ is a degree $d$ analytic family of parabolic-like maps if the following conditions are satisfied:

1. $U'$, $U$, $\Omega'$ and $\Omega$ are domains in $\mathbb{C}^2$;
2. the map $f : U' \to U$ is holomorphic in $(\lambda, z)$.
3. all the parabolic-like maps in the family have the same number of attracting petals in the filled Julia set.

For all $\lambda \in \Lambda$ let us call $z_\lambda$ the parabolic-fixed point of $f_\lambda$, and let us set $K_\lambda = K_{f_\lambda}$, and $J_\lambda = J_{f_\lambda}$. Define $M_f = \{\lambda | K_\lambda \text{ is connected}\}$.

### 2.0.1 Nice families

An analytic family of parabolic-like mappings is *nice* if there exists a holomorphic motion of the dividing arcs

$$\Phi : \Lambda \times \gamma_0 \to \mathbb{C},$$

and there exists a holomorphic motion of the ranges

$$B : \Lambda \times \partial U_0 \to \mathbb{C}$$

which is a piecewise $C^1$-diffeomorphism with no cusps in $z$ (for every fixed $\lambda$), and $B_\lambda(\gamma_0(\pm 1)) = \gamma_\lambda(\pm 1)$. 

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2.0.2 Remarks about the definition and motivations

A nice family is basically endowed by definition with a holomorphic motion of a fundamental annulus (see Section 3.0.3). We did not require analytic families to have these properties, because the concept of parabolic-like map is local. On the other hand, since all the maps in an analytic family of parabolic-like maps have the same number of attracting petals in its filled Julia set, it follows from the holomorphic parameter dependence of Fatou coordinates (see Appendix in [Sh]), that in many cases there is a holomorphic motion of the dividing arcs (however, in individual cases further detail might be required according to circumstances). Moreover, since the concept of parabolic-like map is local, in many cases it is not difficult to construct a holomorphic motion of the ranges for an analytic family of parabolic-like mappings.

2.0.3 Degree 2 analytic families of parabolic-like maps

The definition of analytic family of parabolic-like maps is valid for any degree. However, since in this paper we are interested in proving that, under suitable conditions, the map $\chi$ defined in the introduction is a ramified covering between $M_f$ and $M_1 \setminus \{1\}$, in the remainder we will restrict our attention to degree 2 nice analytic families of parabolic-like maps. All the maps of an analytic family of parabolic-like maps have the same number of attracting petals in their filled Julia set, and each (maximal) attracting petal requires a critical point in its boundary. Hence, if $f$ is a degree 2 analytic family of parabolic-like maps, either for each $\lambda \in \Lambda$ the map $f_\lambda$ has no attracting petals in $K_\lambda$, or for each $\lambda \in \Lambda$ the map $f_\lambda$ has exactly one attracting petal in $K_\lambda$.

Consider now the family $\text{Per}_1(1)$. The $\Delta$-part of a parabolic-like mapping requires (at least) one attracting petal, and for all the members of the family $\text{Per}_1(1)$ with $A \neq 0$ the parabolic fixed point has parabolic multiplicity 1. So a parabolic-like restriction of $P_A$, with $A \neq 0$ has no attracting petals in the filled Julia set. On the other hand, $P_0 = z + 1/z$ has a parabolic fixed point of parabolic multiplicity 2 and the Julia set of $P_0$ is the common boundary of the immediate parabolic basins, so a parabolic-like restriction of $P_0$ has exactly one attracting petal in its filled Julia set.

So, if all the members an analytic family of degree 2 parabolic-like mappings $f$ have exactly one attracting petal in its filled Julia set, they are all hybrid conjugate to the map $P_0 = z + 1/z$, and $\chi(\lambda) \equiv 1$, (but this case is not really interesting). On the other hand, if all the members of $f$ have no petals in the filled Julia set, there is no $\lambda \in \Lambda$ such that $f_\lambda$ is hybrid...
conjugate to the map \( P_0 = z + 1/z \), and finally the image of \( M_f \) under the map \( \chi \) is not the whole of \( M_1 \), but it belongs to \( M_1 \setminus \{1\} \). This is the case we are interested in.

### 2.1 Example

Consider the family of cubic polynomials

\[ C_a(z) = z + az^2 + z^3, \quad a \in \mathbb{C}. \]

The maps belonging to this family have a parabolic fixed point at \( z = 0 \) of multiplier 1, and critical points at \( c_+(a) = \frac{-a + \sqrt{a^2 - 3}}{3} \) and \( c_-(a) = \frac{-a - \sqrt{a^2 - 3}}{3} \). Call \( C \) the connectedness locus for this family. Let \( \phi_a \) denote the Böttcher coordinates for \( C_a \) tangent to the identity at infinity, call \( \tilde{c}_-(a) \) the co-critical point of \( c_-(a) \) and let \( \Phi : C \setminus C \rightarrow C \setminus \mathbb{D} \) be the conformal representation of \( C \setminus C \) given by

\[ \Phi(a) = \varphi(\tilde{c}_-(a)). \]

Define \( \Lambda \subset C \) as the open set bounded by the external rays of angle \( 1/6 \) and \( 2/6 \) (see [N]). In this section we are going to prove that the family \( (C_a(z) = z + az^2 + z^3)_{a \in \Lambda} \) yields to a nice family of parabolic-like mappings.

### 2.2 For every \( a \in \Lambda \), \( C_a \) presents a parabolic-like restriction

Let us construct a parabolic-like restriction for every member of the family \((C_a)_{a \in \Lambda}\). Call \( \Xi_a \) the immediate basin of attraction of the parabolic fixed point \( z = 0 \). Then \( c_+(a) \) belongs to \( \Xi_a \), while \( c_-(a) \) does not belong to \( \Xi_a \). Let \( \phi_a : \Xi_a \rightarrow \mathbb{D} \) be the Riemann map normalized by setting \( \phi_a(c_+(a)) = 0 \) and \( \phi_a(z) \xrightarrow{z \to 0} 1 \), and let \( \psi_a : \mathbb{D} \rightarrow \Xi_a \) be its inverse. By the Carathéodory theorem the map \( \psi_a \) extends continuously to \( S^1 \). Note that \( \phi_a \circ C_a \circ \psi_a = h_2 \).

Let \( w_a \) be a \( h_2 \) periodic point in the first quadrant, such that the hyperbolic geodesic \( \tilde{\gamma}_a \in \mathbb{D} \) connecting \( w_a \) and \( \overline{w_a} \) separates the critical value \( z = 1/3 \) from the parabolic fixed point \( z = 1 \). Let \( U_a \) be the Jordan domain bounded by \( \tilde{\gamma}_a = \psi_a(\tilde{\gamma}_a) \), union the arcs up to potential level 1 of the external rays landing at \( v_a = \psi_a(w_a) \) and \( \overline{v}_a = \psi_a(\overline{w}_a) \), together with the arc of the level 1 equipotential connecting this two rays around \( c_-(a) \) (see Fig. 3). Let \( U'_a \) be the connected component of \( C_a^{-1}(U_a) \) containing 0 and the dividing arcs \( \gamma_{a \pm} \) be the fixed external rays landing at the parabolic fixed point 0 and parametrized by potential. Then \((C_a, U'_a, U_a, \gamma_a)\) is a parabolic-like map of degree 2 (see Fig. 3).
2.3 The family \((C_a(z) = z + az^2 + z^3)_{a \in \Lambda}\) yields to a nice analytic family of parabolic-like mappings

For every \(a \in \Lambda\) the parabolic fixed point \(0\) of \(C_a\) has parabolic multiplicity 1, and \((C_a, U_a', U_a, \gamma_a)\) is a parabolic-like map with no attracting petals in its filled Julia set. By the construction we gave, it follows easily that \((C_a)_{a \in \Lambda}\) restricts to an analytic family of parabolic-like mappings. Since external rays move holomorphically, to prove that this analytic family of parabolic-like maps is nice it suffices to show that the boundaries of \(U_a\) move holomorphically with the parameter (by construction the motion defines a piecewise \(C^1\)-diffeomorphisms with no cusps in \(z\)). Let us start by proving that the basin of attraction \(\Xi_a\) of 0 depends holomorphically on the parameter.

2.3.1 The basin of attraction of the parabolic fixed point depends holomorphically on \(a\)

Call \(\mathcal{P}_a\) the maximal attracting petal in \(\Xi_a\), and let \(F_a : \mathcal{P}_a \rightarrow \mathbb{H}_l\) be Fatou coordinates for \(C_a\) normalized by sending the critical point \(c_\pm(a)\) to 1. Since the family \((C_a)_{a \in \mathbb{C}}\) depends holomorphically on \(a\), \(F_a\) depends holo-
morphically on \( a \) and the extended Fatou coodinates to the whole parabolic basin \( \mathcal{F}_a : \Xi_a \to \mathbb{C} \) depend holomorphically on \( a \). On the other hand, let \( \Phi_h : \mathbb{D} \to \mathbb{C} \) be extended Fatou coordinates for the map \( h_2 \), normalized by sending the critical point to 1. Since the Riemann map \( \phi_a \) is a holomorphic conjugacy between \( C_a \) and \( h_2 \), \( \Phi_h \circ \phi_a \) are Fatou coordinates for \( C_a \). Since \( \Phi_a(c_+(a)) = 1 = \Phi_h \circ \phi_a(c_+(a)) = \Phi_h(0) = 1 \), we have that \( \Phi_a = \Phi_h \circ \phi_a \). Hence the Riemann map \( \phi_a \) depends holomorphically on \( a \). So (fixing a base point \( a_0 \in \Lambda \)) the dynamical holomorphic motion \( \Phi_{-1}^a \circ \Phi_a^{a_0} : \Lambda \times \Xi_{a_0} \to \mathbb{C} \) (holomorphic in \( z \)) induces a dynamical holomorphic motion of \( \Xi_{a_0} \).

2.3.2 The family \( (C_a(z) = z + az^2 + az^3)_{a \in \Lambda} \) restricts to a nice analytic family of parabolic-like mappings

Since \( \Xi_a \) moves holomorphically, the points \( v_a \) and \( \bar{v}_a \) and the arc \( \hat{\gamma}_a \) defined in [2.2] depend holomorphically on \( a \). Since equipotentials and external rays move holomorphically, for every \( a \in \Lambda \) the set \( \partial U_a \) moves holomorphically. Hence the family \( (C_a(z) = z + az^2 + az^3)_{a \in \Lambda} \) restricts to a degree 2 nice analytic family of parabolic-like maps.

2.4 Review and overview

In [L] we proved that a degree 2 parabolic-like map is hybrid conjugate to a member of the family \( \text{Per}_1(1) \) by changing its external class into the class of \( h_2(z) = \frac{z^2 + 1}{z^2 + 1} \) (see Theorem 6.3 in [L]) and showing that a parabolic-like map is holomorphically conjugate to a member of the family \( \text{Per}_1(1) \) if and only if its external class is given by the class of \( h_2 \) (see Proposition 6.2 in [L]). We defined a (quasiconformal) conjugacy between two parabolic-like maps \((f, U_f, U_f', \gamma_f)\) and \((g, U'_g, U_g, \gamma_g)\) to be a (quasiconformal) homeomorphism between (appropriate) restrictions of \( U_f \) and \( U_g \) which conjugates dynamics on \( \Omega_f \cup \gamma_f \) (see Def. 3.3 in [L]). Let us review how we changed the external class of a degree 2 parabolic-like map \( f \) into the class of \( h_2 \). As first step, we constructed a homeomorphism \( \tilde{\psi} \), quasiconformal everywhere but at the parabolic fixed point, between a fundamental annulus \( A_f = \overline{U_f \setminus \Omega_f} \) of \( f \) and a fundamental annulus \( A = \overline{B \setminus \Omega_B} \) of \( h_2 \). Then we defined on \( A_f \) an almost complex structure \( \sigma_1 \) by pulling back the standard structure by \( \tilde{\psi} \). In order to obtain on \( U_f \) a bounded and invariant (under a map coinciding with \( f \) on \( \Omega_f \)) almost complex structure \( \sigma \) we replaced \( f \) with \( h_2 \) on \( \Delta \), and spread \( \sigma_1 \) by the dynamics of this new map \( \tilde{f} \) (and kept the standard structure on \( K_f \)).
Finally, by integrating $\sigma$ we obtained a parabolic-like map hybrid conjugate to $f$ and with external map $h_2$.

In this paper we want to perform this surgery for nice analytic families of degree 2 parabolic-like maps, and prove that the map $\chi : M_f \to M_1$ induced by the family of hybrid conjugacies extends to a continuous map $\chi : \Lambda \to \mathbb{C}$ which under suitable conditions restricts to a branched covering of $M_1 \setminus \{1\}$. We will start by defining a family of quasiconformal maps, depending holomorphically on the parameter, between a fundamental annulus of $h_2$ and fundamental annuli $A_\lambda = \overline{U_\lambda} \setminus \Omega^\prime_\lambda$ of $f_\lambda$, $\lambda \in \Lambda$. In analogy with the polynomial-like setting we will call this family a holomorphic Tubing. In order to construct a holomorphic Tubing, fixed a $\lambda_0 \in \Lambda$, we will start by constructing a quasiconformal homeomorphism $\tilde{\psi}$ between $A$ and $A_{\lambda_0}$ (see Section 3.0.2) and a dynamical holomorphic motion $\hat{\tau} : \Lambda \times A_{\lambda_0} \to A_\lambda$ (see Section 3.0.3). Hence we will obtain a holomorphic Tubing by composing the inverse of $\tilde{\psi}$ with the holomorphic motion (see Section 3.0.4). By Tubing, we will extend the map $\chi$ to the whole of $\Lambda$ (see Section 4.1). We will prove that the map $\chi$ is continuous (see Section 4.3), holomorphic on the interior of $M_f$ (see Section 4.4) and with discrete fibers (see Section 4.5). Finally, we will prove that, on compact subsets of $\Lambda$, the map $\chi$ is a degree $D > 0$ branched covering (see Section 5). By defining proper families of parabolic-like maps we wil give the condition under which, for each neighborhood $U$ of 1, $\chi^{-1}(M_1 \setminus U)$ is a compact subset of $\Lambda$ (see Section 5.1). This implies the following result:

**Theorem 2.2.** Given a proper family of parabolic-like maps $(f_\lambda)_{\lambda \in \Lambda = \mathbb{D}}$, the map $\chi : M_f \to M_1 \setminus \{1\}$ is a degree $D > 0$ branched covering. More precisely, for every neighborhood $U$ of 1 in $\mathbb{C}$ (with $0 \notin U$) there exists a neighborhood $\hat{V}$ of $M_1 \setminus U$ in $\chi(M_f)$ such that the map $\chi : \chi^{-1}(%}
3.0.1 A fundamental annulus \( A \) for \( h_2 \)

The map \( h_2(z) = \frac{z^{2+1/3}}{1+z^{2/3}} \) is an external map of every member of the family \( \text{Per}_1(1) \) (see Prop. 4.2 in [L]). Let \( h_2 : W' \to W \) (where \( W = \{ z : \exp(-\epsilon) < |z| < \exp(\epsilon) \} \) for an \( \epsilon > 0 \), and \( W' = h_2^{-1}(W) \)) be a degree 2 covering extension (this is, an extension such that \( h_2 : W' \to W \) is a degree 2 covering and there exists a dividing arc which devides \( W' \setminus D \) and \( W' \setminus D' \) into \( \Omega'_W, \Delta'_W \) and \( \Gamma'_W, \Delta'_W \) respectively, such that \( \Omega'_W \setminus \Delta'_W \) is a topological quadrilateral; see Def. 5.2 in [L]). Choose \( \lambda_0 \in \Lambda \). Let \( h_{\lambda_0} \) be an external map of \( f_{\lambda_0}, z_0 \) be its parabolic fixed point and define \( \gamma_{h_{\lambda_0}+} = \alpha_{\lambda_0}(\gamma_{\lambda_0+}), \gamma_{h_{\lambda_0}-} = \alpha_{\lambda_0}(\gamma_{\lambda_0-}) \) (where \( \alpha \) is a parabolic equivalence between \( f_{\lambda_0} \) and \( h_{\lambda_0} \)).

Let \( \Xi_{h_{\lambda_0}+} \) and \( \Xi_{h_{\lambda_0}-} \) be repelling petals for the parabolic fixed point \( z_0 \) which intersect the unit circle and \( \phi_{\pm} : \Xi_{h_{\lambda_0} \pm} \to \mathbb{H}_t \) be Fatou coordinates for \( h_{\lambda_0} \) with axis tangent to the unit circle at the parabolic fixed point \( z_0 \). Let \( \Xi_{h_{\lambda_0}+} \) and \( \Xi_{h_{\lambda_0}-} \) be repelling petals which intersect the unit circle for the parabolic fixed point \( z = 1 \) of \( h_2 \), and let \( \phi_{\pm} : \Xi_{h_{\lambda_0} \pm} \to \mathbb{H}_t \) be Fatou coordinates for \( h_2 \) with axis tangent to the unit circle at 1. Define \( \tilde{\gamma}_{+} = \phi_+^{-1}(\phi_{h_{\lambda_0}+}(\gamma_{h_{\lambda_0}+})) \) and \( \tilde{\gamma}_{-} = \phi_{-1}(\phi_{h_{\lambda_0}-}(\gamma_{h_{\lambda_0}-})) \).

Define \( \tilde{\Delta}_W = h_2(\Delta_W \cap \Delta_{W'}), \tilde{W} = \Omega_W \cup \tilde{\gamma} \cup \tilde{\Delta}_W, \tilde{W}' = h_2^{-1}(\tilde{W}), \tilde{\Omega}_W = \Omega_W \cap \tilde{W}', \tilde{\Delta}_W = \Delta_W \cap \tilde{W}' \) and \( Q_W = \Omega_W \setminus \tilde{\Omega}_W \). We call fundamental annulus for \( h_2 \) the topological annulus \( A = \tilde{W} \setminus (\tilde{\Omega}_W \cup D) \).

3.0.2 A fundamental annulus \( A_{\lambda_0} \) for \( f_{\lambda_0} \) and the map \( \tilde{\psi} : A_{\lambda_0} \to A \)

Let \( \Phi_{\Delta_{\lambda_0}} : \Delta_{\lambda_0} \to \Delta_W \) be a homeomorphism coinciding with \( \varphi_{\lambda_0}^{-1} \circ \phi_{h_{\lambda_0} \pm} \circ \alpha_{\lambda_0} \) on \( \gamma_{\lambda_0}, \) quasiconformal on \( \Delta_{\lambda_0} \setminus \{ z_{\lambda_0} \} \) (where \( z_{\lambda_0} \) is the parabolic fixed point of \( f_{\lambda_0} \)) and real-analytic diffeomorphism on \( \Delta_{\lambda_0} \) (see Claim 6.1 in the proof of Thm. 6.3 in [L]). Define \( \tilde{\Delta}_{\lambda_0} = \Phi_{\Delta_{\lambda_0}}^{-1}(\tilde{\Delta}_W), \tilde{\Delta}'_{\lambda_0} = \Phi_{\Delta_{\lambda_0}}^{-1}(\tilde{\Delta}'_W), \) and \( \tilde{U}_{\lambda_0} = (\Omega_{\lambda_0} \cup \gamma_{\lambda_0} \cup \tilde{\Delta}_{\lambda_0}) \subset U_{\lambda_0} \). Consider

\[
\tilde{f}_{\lambda_0}(z) = \begin{cases} 
\Phi_{\Delta_{\lambda_0}}^{-1} \circ h_2 \circ \Phi_{\Delta_{\lambda_0}} & \text{on } \tilde{\Delta}'_{\lambda_0} \\
 f_{\lambda_0} & \text{on } \Omega_{\lambda_0} \setminus \tilde{\Delta}_{\lambda_0}
\end{cases}
\]

Define \( \tilde{U}'_{\lambda_0} = \tilde{f}_{\lambda_0}^{-1}(\tilde{U}_{\lambda_0}), Q_{\lambda_0} = \Omega_{\lambda_0} \setminus \tilde{\Omega}'_{\lambda_0} \), and the fundamental annulus \( A_{\lambda_0} = \tilde{U}'_{\lambda_0} \setminus \Omega_{\lambda_0}' \).

Let \( \Phi_{Q_{\lambda_0}} : \tilde{Q}_{\lambda_0} \to \tilde{Q}_W \) be a quasiconformal map which coincides with \( \varphi_{\lambda_0}^{-1} \circ \phi_{h_{\lambda_0} \pm} \circ \alpha_{\lambda_0} \) on \( \gamma_{\lambda_0} \) (see the proof Claim 6.2 in Thm. 6.3 in [L]). Define
a map \( \tilde{\psi} : A_{\lambda_0} \rightarrow A \) as follows:

\[
\tilde{\psi}(z) = \begin{cases}
\Phi_{\Delta_{\lambda_0}} & \text{on } \Delta_{\lambda_0} \\
\Phi_{Q_{\lambda_0}} & \text{on } Q_{\lambda_0}
\end{cases}
\]

The map \( \tilde{\psi} \) is a homeomorphism, quasiconformal on \( A_{\lambda_0} \setminus \{ z_{\lambda_0} \} \), so the map \( \tilde{\Psi} := \tilde{\psi}^{-1} : A \rightarrow A_{\lambda_0} \) is a homeomorphism, quasiconformal on \( A \setminus \{ 1 \} \).

### 3.0.3 Holomorphic motion of the fundamental annulus \( A_{\lambda_0} \)

Define for all \( \lambda \in \Lambda \) the set \( a_{\lambda} = U_{\lambda} \setminus \Omega_{\lambda}' \). Then the set \( a_{\lambda} \) is a topological annulus. Define the map \( \tilde{\tau} : \Lambda \times \partial a_{\lambda_0} \rightarrow \partial a_{\lambda} \) as follows:

\[
\tilde{\tau}(z) = \begin{cases}
\Phi_{\gamma_{\lambda_0}} & \text{on } \gamma_{\lambda_0} \\
B_{\lambda} & \text{on } \partial U_{\lambda_0} \\
f_{\lambda}^{-1} \circ f_{\lambda_0} & \text{on } \partial U_{\lambda_0}' \cap \partial \Omega_{\lambda_0}'
\end{cases}
\]

Since \( \Phi_{\lambda} \) and \( B_{\lambda} \) are holomorphic motions with disjoint images on \( \partial a_{\lambda_0} \setminus \{ \gamma_{\lambda_0}(1), \gamma_{\lambda_0}(-1) \} \), and \( f_{\lambda} : \partial U_{\lambda}' \rightarrow \partial U_{\lambda} \) is a degree \( d \) covering, \( \tilde{\tau} \) is a holomorphic motion with basepoint \( \lambda_0 \). Since \( \Lambda \approx \mathbb{D} \), by the Slodkowski’s Theorem we can extend \( \tilde{\tau} \) to a holomorphic motion \( \hat{\tau} : \Lambda \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \). In particular we obtain a holomorphic motion of \( \tilde{U}_{\lambda_0} \). For every \( \lambda \in \Lambda \), define \( \tilde{U}_{\lambda} = \tilde{\tau}(\tilde{U}_{\lambda_0}) \), and \( \Delta'_{\lambda} = \tilde{\tau}(\Delta'_{\lambda_0}) \). Define for every \( \lambda \in \Lambda \) the map \( f_{\lambda} \) as follows:

\[
\tilde{f}_{\lambda}(z) = \begin{cases}
\tilde{\tau} \circ \tilde{\Psi} \circ h_{\lambda}^{-1} & \text{on } \tilde{\Delta}_{\lambda} \\
f_{\lambda} & \text{on } \tilde{\Omega}_{\lambda}' \cup \gamma_f_{\lambda}
\end{cases}
\]

and the set \( \tilde{U}'_{\lambda} = \tilde{f}_{\lambda}^{-1}(\tilde{U}_{\lambda}) \). Finally, define for all \( \lambda \in \Lambda \) the set \( A_{\lambda} = U_{\lambda} \setminus \Omega_{\lambda}' \). Then the set \( A_{\lambda} \) is a topological annulus, and we call it fundamental annulus of \( f_{\lambda} \). The holomorphic motion \( \tilde{\tau} : \Lambda \times \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) restricts to a holomorphic motion

\[
\tilde{\tau} : \Lambda \times A_{\lambda_0} \rightarrow A_{\lambda}
\]

which respects the dynamics.

### 3.0.4 Holomorphic Tubings

Define \( T := \tilde{\tau} \circ \tilde{\Psi} : \Lambda \times A \rightarrow A_{\lambda} \). We call the map \( T \) a holomorphic tubing. A holomorphic tubing is not a holomorphic motion, since \( T_{\lambda_0} = \tilde{\Psi} \neq Id \), but nevertheless it is quasiconformal in \( z \) for every fixed \( \lambda \in \Lambda \) and holomorphic in \( \lambda \) for every fixed \( z \in A \).
3.0.5 Straightening of the members of the family \((f_\lambda)_{\lambda \in \Lambda}\)

Let us now straighten the members of the family \((f_\lambda)_{\lambda \in \Lambda}\) to members of the family \(Per_1(1)\). For every \(\lambda \in \Lambda\) define on \(U_\lambda\) the Beltrami form \(\mu_\lambda\) as follows:

\[
\mu_\lambda(z) = \begin{cases} 
\mu_{\lambda,0} = (T_\lambda)_*(\sigma_0) & \text{on } A_\lambda \\
\mu_{\lambda,n} = (\tilde{f}_\lambda^n)^*\mu_{\lambda,0} & \text{on } (\tilde{f}_\lambda)^{-n}(A_\lambda) \\
0 & \text{on } K_\lambda
\end{cases}
\]

For every \(\lambda\) the map \(T_\lambda\) is quasiconformal, hence \(||\mu_{\lambda,0}||_\infty \leq k < 1\) on every compact subset of \(\Lambda\). On \(\tilde{\Omega}_\lambda\) the Beltrami form \(\mu_{\lambda,n}\) is obtained by spreading \(\mu_{\lambda,0}\) by the dynamics of \(f_\lambda\), which is holomorphic, while on \(\Delta_\lambda\) the Beltrami form \(\mu_{\lambda,n}\) is constant for all \(n\) (by construction of the map \(\tilde{f}_\lambda\)). Thus \(||\mu_\lambda||_\infty = ||\mu_{\lambda,0}||_\infty\) which is bounded. By the measurable Riemann mapping theorem (see \([Ah]\)) for every \(\lambda \in \Lambda\) there exists a quasiconformal map \(\phi_\lambda : U_\lambda \to \mathbb{D}\) such that \((\phi_\lambda)^*\mu_0 = \mu_\lambda\). Finally, for every \(\lambda \in \Lambda\) the map \(g_\lambda = \phi_\lambda \circ \tilde{f}_\lambda \circ \phi^{-1}_\lambda\) is a parabolic-like map hybrid conjugate to \(f_\lambda\) and holomorphically conjugate to a member of the family \(Per_1(1)\) (see Prop. 6.2 in \([L]\)).

**Remark 3.1.** Note that for every \(\lambda \in \Lambda\), the dilatation of the integrating map \(\phi_\lambda\) is equal to the dilatation of the holomorphic Tubing \(T_\lambda\). So the family of integrating maps \((\phi_\lambda)_{\lambda \in \Lambda}\) has locally bounded dilatation.

3.0.6 Lifting Tubings

Let us lift the Tubing \(T_{\lambda_{\hat{\lambda}}}\). Define \(A_{\lambda,0} = A_\lambda\), \(B_{\lambda,1} = \tilde{f}_\lambda^{-1}(A_{\lambda,0})\), \(A_0 = A\) and \(B_1 = h_\lambda^{-1}(A_0)\). Hence \(\tilde{f}_\lambda : B_{\lambda,1} \to A_{\lambda,0}\) and \(h_\lambda : B_1 \to A_0\) are degree 2 covering maps, and we can lift the Tubing \(T_{\lambda_{\hat{\lambda}}}\) to \(T_{\lambda_{\hat{\lambda}}}: = \tilde{f}_\lambda^{-1} \circ T_\lambda \circ h_\lambda : B_1 \to B_{\lambda,1}\) (such that \(T_{\lambda,1} = T_\lambda\) on \(B_1 \cap A_0\)).

Define recursively \(A_{\lambda,n} = B_{\lambda,n} \cap \tilde{U}, B_{\lambda,n+1} = \tilde{f}_\lambda^{-1}(A_{\lambda,n}), A_n = B_n \cap \tilde{W}\) and \(B_{\lambda,n+1} = h_\lambda^{-1}(A_n)\). Hence \(\tilde{f}_\lambda : B_{\lambda,n+1} \to A_{\lambda,n}\) and \(h_\lambda : B_{\lambda,n+1} \to A_n\) are degree 2 covering maps, and we can lift the Tubing to \(T_{\lambda,n+1} : = \tilde{f}_\lambda^{-1} \circ T_{\lambda,n} \circ h_\lambda : B_{\lambda,n+1} \to B_{\lambda,n+1}\) (such that \(T_{\lambda,n+1} = T_{\lambda,\hat{n}}\) on \(B_{\lambda,n+1} \cap B_n\)).

In the case \(K_\lambda\) is connected, we can lift the Tubing \(T_{\lambda_{\hat{\lambda}}}\) to the whole of \((W \cup W') \setminus \overline{\mathbb{D}}\). If \(K_\lambda\) is not connected, the maximum domain we can lift the Tubing \(T_{\lambda_{\hat{\lambda}}}\) to is \(B_{n_0}\), such that \(B_{\lambda,n_0}\) contains the critical value of \(f_\lambda\). Note that the extension is still quasiconformal in \(z\).

4 Properties of the map \(\chi\)

Consider the map \(\chi : M_f \to M_1 \setminus \{1\}\) (defined in Section \([\mathbb{L}]\) which associates to each \(\lambda \in M_f\) the multiplier of the fixed point of the map \(P_A\) hybrid
equivalent to $f_\lambda$. In this section, we will first extend the map $\chi$ to the whole of $\Lambda$ (see Section 4.1), then prove that the map $\chi : \Lambda \to \mathbb{C}$ is continuous at the boundary of $M_f$ (see Prop. 4.3) and that it depends analytically on $\lambda$ for $\lambda \in \partial M_f$ (see Prop. 4.5). Finally, we will prove that the map $\chi$ has discrete fibers (see Prop. 4.6).

4.1 Extending the map $\chi$ to all of $\Lambda$

Let $T$ be a holomorphic tubing for the nice analytic family of parabolic-like maps $f$. Call $c_\lambda$ the critical point of $f_\lambda$ and let $n$ be such that $f_n^{\lambda}(c_\lambda) \in A_\lambda$, $f_n^{\lambda-1}(c_\lambda) \notin A_\lambda$. Lift the holomorphic tubing $T_\lambda$ to $T_{\lambda,n-1}$ (see Section 3.0.6).

We can therefore extend the map $\chi$ by setting:

$$\chi : \Lambda \setminus M_f \to \mathbb{C} \setminus M_1$$

$$\lambda \to \Phi^{-1} \circ T_{\lambda,n-1}^{-1}(c_\lambda)$$

where $\Phi : \mathbb{C} \setminus M_1 \to \mathbb{C} \setminus \overline{D}$ is the canonical isomorphism between the complement of $M_1$ and the complement of the unit disk (see [M2]). Since the tubing $T_\lambda$ has locally bounded dilatation, the map $\chi : \Lambda \setminus M_f \to \mathbb{C} \setminus M_1$ is quasiregular on $\Lambda' \setminus M_f$ for any open $\Lambda' \subset \subset \Lambda$.

4.2 Indifferent periodic points

An indifferent periodic point $z'$ for $f_{\lambda_0}$, $\lambda_0 \in \Lambda$, is called persistent if for each neighborhood $V(z')$ of $z'$ there exists a neighborhood $W(\lambda_0)$ of $\lambda_0$ such that, for every $\lambda \in W(\lambda_0)$ the map $f_\lambda$ has in $V(z')$ an indifferent periodic point $z'_\lambda$ of the same period and multiplier (see [MSS]). Let $(f_\lambda)_{\lambda \in \Lambda}$ be an family of parabolic-like mappings. For all $\lambda \in \Lambda$, the parabolic fixed point is persistent. Since all the other indifferent periodic points are non persistent, in the remainder we will call them indifferent periodic points without further notation.

**Proposition 4.1.** The indifferent parameter values for a family of parabolic-like mappings belong to $\partial M_f$.

**Proof.** The proof follows the proof of Prop. 11 in [DH]. Since for all $\lambda \in \Lambda \setminus M_f$ the critical point $c_\lambda$ of $f_\lambda$ belongs to $(U_\lambda \setminus K_\lambda)$, the map $f_\lambda$ is hyperbolic.

Assume that for $\lambda_0 \in M_f$ the map $f_{\lambda_0}$ has an indifferent periodic point $\alpha_0$ of period $k$, and assume first $(f^k)'(\alpha_0) \neq 1$. By the Implicit Function Theorem there exist $W, V$ neighborhoods of $\lambda_0$ and $\alpha_0$ respectively, with $W \subset M_f$, where the indifferent cycle and the critical point move holomorphically with the parameter, and where the multiplier map $\rho(\lambda) = (f^k_\lambda)'(\alpha_\lambda)$
is a holomorphic non constant map. Set $\alpha(\lambda) = \alpha_{\lambda}$. By taking a restrictions if necessary, we can assume $\lambda_0$ is the only parameter in $W$ for which $f_\lambda$ has in $V$ an indifferent periodic point. Let $(\lambda_n)$ be a sequence in $W$ converging to $\lambda_0$, such that for all $n$, $|\rho(\lambda_n)| < 1$. Hence for all $n$, there exists a $\alpha^i(\lambda_n) \in \{\alpha^0(\lambda_n), \ldots, \alpha^{k-1}(\lambda_n)\}$ such that

$$f_{\lambda_n}^{i+k_p}(c_{\lambda_n}) \to \alpha^i(\lambda_n) \text{ as } p \to \infty$$

We can assume $i$ independent of $\lambda$ by choosing a subsequence. Let us define for all $\lambda \in W$ the sequence $F_p(\lambda) = f_{\lambda}^{i+k_p}(c_{\lambda})$.

Since $(F_p)$ is a family of analytic maps bounded on any compact subset of $W$, it is a normal family. Let $F_{p_n}$ be a subsequence converging to some function $h$. Then $h(\lambda_n) = \alpha(\lambda_n)$ for all $n$, hence $h = \alpha$ and for all $\lambda \in W$, $F_p(\lambda) \to \alpha(\lambda)$. But in $W$ there are parameters $\lambda^*$ for which $\alpha(\lambda^*)$ is a repelling periodic point, and thus it cannot attract the sequence $F_p(\lambda^*)$.

In the case $(f^k)'(a_0) = 1$, let $\Lambda_0$ be a neighborhood of $\lambda_0 \in \hat{M}_f$, let $\lambda : W(0) \to \Lambda_0$, $t \to t^2 + \lambda_0$, be a branched covering of $\Lambda_0$ branched at 0 for some neighborhood $W(0)$ of 0, and repeat the previous argument.

4.3 Continuity of $\chi$ on $\partial M_f$

In this Section we prove that the map $\chi : \Lambda \to \mathbb{C}$ is continuous on the boundary of $M_f$.

**Proposition 4.2.** Suppose $A_1, A_2 \in \mathbb{C}$, with $B_1 = 1 - (A_1)^2 \in \partial M_1$. If the maps $P_{A_1}$ and $P_{A_2}$ are quasiconformally conjugate, then $B_1 = B_2$.

**Proof.** Let $(P_1, U', U, \gamma_1)$ and $(P_2, V', V, \gamma_2)$ be parabolic-like restrictions of $P_{A_1}$ and $P_{A_2}$ respectively (for the construction of a parabolic-like restriction of members of the family $\text{Per}_1(1)$ see the proof of Prop. 4.2 in [L]), and let $\varphi : U \to V$ be a quasiconformal conjugacy between them. If $K_{P_1}$ is of measure zero (where $K_{P_1} = \hat{\mathbb{C}} \setminus B_\infty$, and $B_\infty$ is the parabolic basin of attraction of infinity, see Section 1 in [L]), then $\varphi$ is a hybrid conjugacy and the result follows from Prop. 6.5 in [L].

Let $K_{P_1}$ be not of measure zero. Define on $\hat{\mathbb{C}}$ the following Beltrami form:

$$\tilde{\mu}(z) := \begin{cases} (\phi)^* \mu_0 & \text{on } K_{P_1} \\ 0 & \text{on } \hat{\mathbb{C}} \setminus K_{P_1} \end{cases}$$

Since $\phi$ is quasiconformal, $||\tilde{\mu}||_\infty = k < 1$. Therefore for $|t| < 1/k$ we can define on $\hat{\mathbb{C}}$ the family of Beltrami form $\mu_t = \tilde{\mu}t$, and $||\mu_t||_\infty < 1$. The family
\(\mu_t\) depends holomorphically on \(t\). Let \(\Phi_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\) be the family of integrating maps fixing \(\infty, 1\) and \(0\). Hence the family \(\Phi_t\) depends holomorphically on \(t\), \(\Phi_1 = \phi\) and \(\Phi_0 = \text{Id}\). The family of holomorphic maps \(F_t = \Phi_t \circ P_{A_t} \circ \Phi_t^{-1}\) has the form \(F_t(z) = z + 1/z + A(t)\) (since it is a family of quadratic rational maps with a parabolic fixed point at \(z = \infty\) with preimage at \(z = 0\) and a critical point at \(z = -1\)) and it depends holomorphically on \(t\). Therefore the map \(\alpha : t \to B(t) = 1 - A^2(t)\) is holomorphic, hence it is either an open or constant, and \(\alpha(0) = B_1 \in \partial M_1\). If \(\alpha(t)\) is open, there exists a neighborhood \(W\) of \(0\) such that \(\alpha(W) \subset M_1\). Hence the map \(\alpha(t)\) is constant, so for all \(t\), \(\alpha(t) = B_1\). In particular \(\alpha(1) = B_1\), and \(F_1 = P_{A_1}\).

Finally the map \(\phi \circ \Phi_t^{-1}\) is a quasiconformal conjugacy between \(P_{A_1}\) and \(P_{A_2}\) with \((\phi \circ \Phi_t^{-1})^*\mu_0 = \mu_0\) on \(K_P\). So \(P_{A_1}\) and \(P_{A_2}\) are hybrid equivalent, and the result follows by Prop. 6.5 in [L].

**Proposition 4.3.** The map \(\chi : \Lambda \to \mathbb{C}\) is continuous at any point \(\lambda \in \partial M_f\), and moreover \(\chi(\lambda) \in \partial M_1\).

**Proof.** In order to prove continuity at any point \(\lambda \in \partial M_f\), we have to show that for every sequence \(\lambda_n \in \Lambda\) converging to \(\lambda_0 \in \partial M_f\) there exists a subsequence \(\lambda_{n^*}\) such that \(B_{n^*} = \chi(\lambda_{n^*})\) converges to \(B_0 = \chi(\lambda_0) \in \partial M_1\). Let us start by proving that \(B_0 \in \partial M_1\).

Let \(\lambda_m\) be a sequence of indifferent parameters converging to \(\lambda_0\). Hence there exists a sequence \(B_m = \chi(\lambda_m)\) such that, for each \(m\), \(f_{\lambda_m}\) is hybrid conjugate to \(P_{A_m}\) by some quasiconformal map \(\phi_m\). The sequence \(\phi_m\) is a sequence of quasiconformal maps with locally bounded dilatation (see Remark [L1]), hence it is precompact in the topology of uniform convergence on compact subsets of \(U_{\lambda_0}\) (see [Ly]). Therefore there exists a subsequence \(\phi_{\lambda_{m^*}}\) which converges to some quasiconformal limit map \(\hat{\phi}\), which conjugates \(f_{\lambda_0}\) to some \(P_{\hat{A}}\), so \(B_{m^*} \to \hat{B}\). For all \(m\) the map \(f_{\lambda_m}\) has an indifferent periodic point, hence \(P_{A_m}\) has an indifferent periodic point, thus \(B_m \in \partial M_1\) and finally \(\hat{B} \in \partial M_1\). Since the map \(f_{\lambda_0}\) is hybrid conjugate to \(P_{A_0}\) and quasiconformally conjugate to \(P_{\hat{A}}\), and \(\hat{B} \in \partial M_1\), by Prop. [L2] \(B_{0} = \hat{B} \in \partial M_1\).

Let now \(\lambda_n \in \Lambda\) be a sequence converging to \(\lambda_0 \in \partial M_f\). Since the sequence \(\phi_n\) is precompact, there exists a subsequence \((\lambda^*_n)\) such that \(\phi_{\lambda^*_n}\) converges to some limit map \(\hat{\phi}\) which conjugates \(f_{\lambda_0}\) to \(P_{\hat{A}} = \hat{\phi} \circ f_{\lambda_0} \circ \hat{\phi}^{-1}\), so \(B_{n^*} \to \hat{B}\). Finally, since \(B_0 \in \partial M_1\) and \(f_{\lambda_0}\) is quasiconformally conjugate to both \(P_{\hat{A}}\) and \(P_{A_0}\), by Prop. [L2] \(\hat{B} = B_0\).
4.4 Analicity of $\chi$ on the interior of $M_f$

The proof of the analycity of the map $\chi$ on the interior of $\hat{M}_f$ (see Proposition 4.5) follows the proof Lyubich gave in the setting of polynomial-like mappings (see [LV]). We will prove that the map $\chi$ is holomorphic on hyperbolic components first, and then on queer components. To prove that $\chi$ is holomorphic on queer components, we first need the following Proposition.

**Proposition 4.4.** Let $Q$ be a queer component of $\hat{M}_1$. Then for every $A \in Q$, $J_{P_A}$ admits an invariant Beltrami form with positive support. In particular, area $J(P_A) > 0$.

**Proof.** Choose $B_0 \in Q$ and set $P_0 = P_{A_0}$. Let us start by proving that there exists a dynamical holomorphic motion $\eta_A : Q \times \hat{C} \to \hat{C}$ with base point $A_0$, holomorphic in $z$.

Let $\Xi^0$ be an attracting petal of $P_0$ containing the critical value $z = 2$, and let $\Phi_0 : \Xi^0 \to \mathbb{H}_+$ be the incoming Fatou coordinates of $P_0$ normalized by $\Phi_0(2) = 1$. For $A \in Q$, let $\Xi^A$ be an attracting petal of $P_A$ and let $\Phi_A : \Xi^A \to \mathbb{H}_+$ be the incoming Fatou coordinates of $P_A$ with $\Phi_A(2 + A) = 1$. The map $\eta_A = \Phi_A^{-1} \circ \Phi_0 : \Xi^0 \to \Xi^A$ is a conformal conjugacy between $P_0$ and $P_A$. Defining $\Xi^0_{-n}$, $n > 0$ as the connected component of $P_0^{-n}(\Xi^0)$ containing $\Xi^0$, and $\Xi^A_{-n}$, $n > 0$ as the connected component of $P_A^{-n}(\Xi^A)$ containing $\Xi^A$, we can lift the map $\eta$ to $\eta_n : \Xi^0_{-n} \to \Xi^A_{-n}$. Since $K_{P_0}$ and $K_{P_A}$ are connected (where $K_{P_0}$ and $K_{P_A}$ are the complements of the basin of attraction of the parabolic fixed point for $P_0$ and $P_A$ respectively), by iterated lifting we can extend $\eta_n$ to $\eta_A : \hat{C} \setminus K_{P_0} \to \hat{C} \setminus K_{P_A}$. The map $\eta_A$ is a holomorphic conjugacy between $P_0$ and $P_A$, and since the family $\text{Per}_1(1)$ is a family of holomorphic maps depending holomorphically on the parameter and so Fatou coordonates depend holomorphically on the parameter, the family $(\eta_A)_{A \in Q}$ depends holomorphically on the parameter. Hence $\eta_A : Q \times \hat{C} \setminus K_{P_0} \to \hat{C} \setminus K_{P_A}$ is a dynamical holomorphic motion with base point $A_0$, holomorphic in $z$. Since for every $A \in Q$ all the periodic points of $P_A$ (but the parabolic fixed point) are repelling, $K_A$ is nowhere dense. Hence $K_{P_0} = J_{P_0}$ and thus by the $\lambda$-Lemma we can extend $\eta_A$ to $\hat{\eta}_A : Q \times \hat{C} \to \hat{C}$. Note that $\hat{\eta}_A$ still conjugates dynamics, and it is conformal on $\hat{C} \setminus K_{P_0}$.

Define $\mu_A = (\hat{\eta}_A)^* \mu_0$. By construction, $\mu_A = \mu_0$ on $\hat{C} \setminus K_{P_0}$. On the other hand, if $\mu_A = \mu_0$ on $\hat{C}$, by the Weyl’s Lemma for every $A \in Q$ the map $\hat{\eta}_A$ is holomorphic, hence for all $A \in Q$ the maps $P_A$ are conformally equivalent. Therefore $\mu_A \neq \mu_0$ on $K_{P_0}$, and thus area$(\text{supp} \mu_A) > 0$ on $K_{P_0}$. In particular, this implies area$(J_{P_0}) > 0$. \hfill \Box

**Proposition 4.5.** The map $\chi : \Lambda \to \hat{C}$ depends analytically on $\lambda$ for $\lambda \in \hat{M}_f$. 

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Proof. Let us start by proving that, for every hyperbolic component \( P \subset M_f \), there exists a hyperbolic component \( Q \subset M_1 \) such that \( \chi|_P : P \to Q \) is a holomorphic map. By the Implicit Function Theorem and Prop. 4.1, all the parameter values in \( P \) are hyperbolic. Hence for \( \lambda \in P, f_\lambda \) has an attracting cycle, thus \( P_{A_\lambda} = \phi_\lambda \circ f_\lambda \circ (\phi_\lambda)^{-1} \) has an attracting cycle and \( Q \subset M_1 \). Since \( \phi_\lambda \) is conformal on \( K_\lambda \), calling \( \rho_P, \rho_Q \) the multiplier maps on \( P, Q \) respectively, \( \rho_P(\lambda) = \rho_Q(A_\lambda) \). Hence on \( P \) we can write the map \( \chi \) as \( \chi = \rho_Q^{-1} \circ \rho_P \). Since \( f_\lambda \) is holomorphic in both \( \lambda \) and \( z \), and by the Implicit Function Theorem the attracting cycle moves holomorphically on \( P \), the map \( \rho_P \) is holomorphic. For the same reason, \( \rho_Q \) is holomorphic as well. Since \( \rho_Q \) has degree 1 (see [PT]), it is conformal and then \( \chi \) is holomorphic.

Let now \( C \) be a queer component in \( M_f \), and let \( \lambda_0 \in C \). Since \( C \subset M_f \) we can lift the holomorphic motion \( \hat{\tau}_\lambda : A_{\lambda_0} \to A_\lambda \) constructed in 3.0.3 to \( \tau_\lambda : U_{\lambda_0} \setminus K_{\lambda_0} \to U_\lambda \setminus K_\lambda \) (as we did for the holomorphic Tubing, see 3.31.6). Since for all \( \lambda \in C \), \( K_\lambda \) is nowhere dense, by the \( \lambda \)-Lemma we can extend \( \tau_\lambda \) to a dynamical holomorphic motion \( \tau_\lambda : U_{\lambda_0} \to U_\lambda \). Let \( P_{A_{\lambda_0}} \) be the member of the family \( \text{Per}_1(1) \) hybrid conjugate to \( f_{\lambda_0} \), let \( \phi_{\lambda_0} \) be a hybrid conjugacy between them and set \( K_0 = K_{P_{A_{\lambda_0}}} \). Note that, for all \( \lambda \in C \), the map \( \tau_\lambda \circ \phi_{\lambda_0}^{-1} : \phi_{\lambda_0}(U_{\lambda_0}) \to U_\lambda \) is a quasiconformal conjugacy between \( P_{A_{\lambda_0}} \) and \( f_\lambda \). Define on \( \hat{\mathbb{C}} \) the following family of Beltrami forms:

\[
\nu_\lambda(z) := \left\{ \begin{array}{ll} (\tau_\lambda \circ \phi_{\lambda_0}^{-1})^* \mu_0 & \text{on } K_0 \\ \mu_0 & \text{on } \hat{\mathbb{C}} \setminus K_0 \end{array} \right.
\]

The family \( \nu_\lambda \) is a family of \( P_{A_{\lambda_0}} \)-invariant Beltrami forms depending holomorphically on \( \lambda \). By Prop. 4.3, for every \( \lambda \in C \), on \( K_0 \) area(supp\( \nu_\lambda \)) > 0. Let \( \psi_\lambda : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be the family of integrating maps fixing \( -1, 0 \) and \( \infty \), then the family \( P_{A(\lambda)} = \psi_\lambda \circ P_{A_{\lambda_0}} \circ (\psi_\lambda)^{-1} \) has the form \( P_{A(\lambda)}(z) = z + 1/z + A(\lambda) \), where \( A(\lambda) \) depends holomorphically on the parameter. Finally, for every \( \lambda \in C \), the map \( \psi_\lambda \circ \phi_{\lambda_0} \circ \tau_\lambda^{-1} \) is a hybrid conjugacy between \( f_\lambda \) and \( P_{A(\lambda)} \), hence \( P_{A(\lambda)} \) is the straightening of \( f_\lambda \) and the map \( \chi|_C \) is holomorphic.

\[ \square \]

4.5 Discreteness of fibers

Set \( B = \chi(\Lambda) \).

Proposition 4.6. For every \( B \in \mathcal{B}, \chi^{-1}(B) \) is discrete.

Proof. This follows the proof of Lemma 10.13 in [LY]. Let us assume there exists a \( B \in \mathcal{B} \) such that there exists a sequence \( \lambda_n \in \chi^{-1}(B) \) and \( \lambda_n \to \hat{\lambda} \in \chi^{-1}(B) \). The map \( \chi \) is quasiregular on \( \Lambda \setminus M_f \) and holomorphic on \( M_f \),
hence $\hat{\lambda} \in \partial M_f$ (or $\hat{\lambda}$ in a queer component of $M_f$ for which $\chi$ is constant, and then we can replace $\hat{\lambda}$ with a boundary point).

Note that, for every $n$, the maps $f_{\lambda_n}$ are hybrid equivalent to $f_{\hat{\lambda}}$ by some hybrid equivalence $\beta_{\lambda_n}$. Let us assume that for all $\lambda$ in a neighborhood of $\hat{\lambda}$, $f_{\lambda}^{-1}(\Delta_{\lambda}) \subset \Delta_{\lambda}$ (in other case, take a nice analytic family of parabolically restrictions for which the assumption holds). For every $\lambda$, call $z_{\lambda}$ the parabolic fixed point of $f_\lambda$, $c_{\lambda}$ its critical point and $v_{\lambda}$ its critical value. Consider $\hat{\lambda}$ as the base point of a holomorphic motion $\hat{\tau} : \Lambda \times A_{\hat{\lambda}} \to A_{\lambda}$, extend it by the Slodkowski’s Theorem and then restrict it to a holomorphic motion $\tau : \Lambda \times \hat{\mathbb{C}} \setminus \hat{\Omega}_{\hat{\lambda}}’ \to \mathbb{C} \setminus \Omega_{\lambda}’$. Define for every $n$ the map $\alpha_{\lambda_n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows:

$$\alpha_{\lambda_n}(z) := \begin{cases} \tau_{\lambda_n}^{-1} \circ f_{\lambda_n}^{-1} \circ \tau_{\lambda}^{-1}(z) & \text{on } A_{\lambda,n} \\ \beta_{\lambda_n} \circ f_{\lambda}^{-1} \circ \tau_{\lambda}^{-1}(z) & \text{on } K_{\lambda} \end{cases}$$

where the maps $\hat{f}_{\lambda}$ are as in 3.0.3 and the sets $A_{\lambda,n}$ are constructed in 3.0.6. The proof of Prop. 6.4 in [L] shows that, for every $n$, the map $\alpha_{\lambda_n}$ is continuous and hence quasiconformal. Therefore, for every $n$, $\alpha_{\lambda_n}$ restricts to a hybrid equivalence between $f_{\hat{\lambda}}$ and $f_{\lambda_n}$. Consider on $\hat{\mathbb{C}}$ the family of Beltrami forms $\nu_{\lambda_n} = (\alpha_{\lambda_n})^*\mu_0$. Note that trivially $\alpha_{\lambda_n}$ integrates $\nu_{\lambda_n}$, and for some subsets $O_n$ of $\hat{\Omega}_{\lambda_n}$ and $O$ of $\hat{\Omega}_{\hat{\lambda}}$, $(f_{\lambda_n})_{\mid O_n} = \alpha_{\lambda_n} \circ f_{\hat{\lambda}} \circ (\alpha_{\lambda_n})_{\mid O}^{-1}$.

On the other hand, define on $\hat{\mathbb{C}}$ the family of Beltrami forms $\mu_{\lambda}$ as follows:

$$\mu_{\lambda}(z) := \begin{cases} \mu_{\lambda,0} = (\tau_{\lambda})^*\mu_0 & \text{on } \hat{\mathbb{C}} \setminus \hat{\Omega}_{\hat{\lambda}}’ \\ \tau_{\lambda}^\prime \circ \mu_{\lambda,0} & \text{on } \hat{A}_{\lambda,n} \\ 0 & \text{on } K_{\lambda} \end{cases}$$

where for every $\lambda$ the map $\hat{f}_{\lambda}$ which defines the sets $\hat{A}_{\lambda,n}$ and spreads $\mu_{\lambda}$ is defined as follows:

$$\hat{f}_{\lambda}(z) = \begin{cases} \tau_{\lambda}^{-1} \circ f_{\lambda} \circ \tau_{\lambda} & \text{on } \hat{\Omega}_{\hat{\lambda}}’ \\ \tau_{\lambda}^{-1}(f_{\lambda}^{-1}(\Delta_{\lambda})) & \text{on } \hat{\Omega}_{\hat{\lambda}}’ \end{cases}$$

Note that $\hat{f}_{\hat{\lambda}}$ and $\hat{f}_{\lambda}$ coincide on $\hat{\Omega}_{\hat{\lambda}}’$, hence for every $n$, $(\bigcup_n A_{\lambda,n}) \cap \hat{\Omega}_{\hat{\lambda}}’ = (\bigcup_n \hat{A}_{\lambda,n}) \cap \hat{\Omega}_{\hat{\lambda}}’$. Therefore, for all $n$, $\mu_{\lambda,n} = \nu_{\lambda,n}$.

The family $\mu_{\lambda}$ depends holomorphically on $\lambda$, because $\tau_{\lambda}$ is a holomorphic motion, on $\Delta_{\hat{\lambda}}$ it is constant and on $\hat{\Omega}_{\hat{\lambda}}’$ it is spread by the dynamics of $f_{\hat{\lambda}}$ (which does not depends on $\lambda$).

Let $H_{\lambda} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the holomorphic family of integrating maps mapping $z_{\hat{\lambda}}$ to $z_{\lambda}$, $c_{\hat{\lambda}}$ to $c_{\lambda}$ and $v_{\hat{\lambda}}$ to $v_{\lambda}$. The family $G_{\lambda} = H_{\lambda} \circ f_{\hat{\lambda}} \circ H_{\lambda}^{-1} : H_{\lambda}(U_{\hat{\lambda}}) \to$
$H_\lambda(U_\lambda)$ is a holomorphic family of parabolic-like mappings hybrid equivalent to $f_\lambda$. For all $n$, $\alpha_{\lambda_n} = H_{\lambda_n}$ (since they are solutions of the same family of Beltrami form and they coincide on 3 points), and therefore on the $O_n$, $G_{\lambda_n}(z) = f_{\lambda_n}(z)$.

Choose a neighborhood $\Lambda_*$ of $\hat{\lambda}$ in $\Lambda$ and an open set $V$ in $\cap_n O_n$ such that the maps $G_\lambda$ and $f_\lambda$ are well-defined in $\Lambda_* \times V$. Since $G_\lambda(z) = f_\lambda(z)$ on $\lambda_n \times V$, and $\lambda_n$ converges to $\hat{\lambda}$, $G_\lambda = f_\lambda$ on $\Lambda_* \times V$. This is impossible, because in $\Lambda_*$ there are $\lambda$ for which $f_\lambda$ has disconnected Julia set (since $\hat{\lambda} \in \partial M_f$), while for all $\lambda$, $G_\lambda$ has connected Julia set (since it is equivalent to $f_\lambda$).

5 The map $\chi: \Lambda \to \mathbb{C}$ is a ramified covering from the connectedness locus $M_f$ to $M_1 \setminus \{1\}$

In this chapter, we will first prove that for any closed and connected subset $K$ of $B = \chi(\Lambda)$, if $C = \chi^{-1}(K)$ is compact, then $\chi|_C$ is a proper map of degree $d$ (Prop. 5.1), and then that $\chi: C \to K$ is a $d$-fold branched covering (Prop. 5.2). Finally, we will prove that, under certain conditions (Def. 5.3), for every neighborhood $U$ of the root of $M_1$ (without specifications, we consider a neighborhood open), the set $\chi^{-1}(M_1 \setminus U)$ is compact in $\Lambda$. This implies Theorem 2.2.

Denote by $i_{\chi}(\chi)$ the local degree of $\chi$ at $\lambda$. Note that, since $\chi: \Lambda \to B$ is quasiregular on $\Lambda \setminus M_f$ and holomorphic on $\hat{M}_f$, for all $\lambda \in \Lambda$, $i_{\chi}(\chi) > 0$.

**Proposition 5.1.** Let $K$ be a closed and connected subset of $B$, and $C = \chi^{-1}(K)$. If $C$ is compact, then there exist neighborhoods $\hat{V}$ of $K$ in $B$ and $\hat{U}$ of $C$ in $\Lambda$ such that $\chi: \hat{U} \to \hat{V}$ is a proper map of degree $d$, where, for every $y \in K$, $d = \sum_{x \in \chi^{-1}(y)} i_x(\chi)$.

**Proof.** The proof follows the analogous one in [DH]. Let $N$ be a closed neighborhood of $C$ in $\Lambda$ with $\text{dist}(C, \partial N) > 0$ (it exists since $C$ compact). Hence $C \subset N \subset \Lambda$, $\chi: N \to \chi(N)$ is proper and $\partial K \cap \chi(\partial N) = \emptyset$. Call $\hat{V}$ the connected component of $B \setminus \chi(\partial N)$ which contains $K$, and set $\hat{U} = \chi^{-1}(\hat{V}) \cap N$. Then $\chi^{-1}(\hat{V}) \cap \partial N = \emptyset$, hence the map $\chi|_{\hat{U}}: \hat{U} \to \hat{V}$ is proper.

Since $\chi$ is continuous and $\hat{V}$ is connected, $\hat{U}$ is the union of connected components. Set $\hat{U} = \bigcup_j \hat{U}_j$. The restriction $\chi: \hat{U}_j \to \hat{V}$ is then a proper map between connected sets, thus it has a degree $d_j$, and since $\chi$ has discrete
fibers, for every \( v \in \hat{V} \) (see [III] pg. 136):

\[
d_j = \sum_{u \in \chi^{-1}(v) \cap \hat{U}_j} i_u(\chi)
\]

(note that \( d_j > 0 \)). Therefore \( \chi : \hat{U} \to \hat{V} \) has a degree \( d = \deg \chi|_{\hat{U}} = \sum_j d_j \) and in particular for all \( y \in K \), \( d = \deg \chi|_{\hat{U}} = \deg \chi|_{C} = \sum_{x \in \chi^{-1}(y) \cap C} i_x(\chi) \).

**Proposition 5.2.** In the hypothesis of Prop. 5.1, the map \( \chi|_{\hat{U}} : \hat{U} \to \hat{V} \) is a branched covering of degree \( d \).

**Proof.** The map \( \chi|_{\hat{U}} : \hat{U} \to \hat{V} \) is continuous, and by the previous proposition it is a proper surjective map of degree \( d \). Let \( y \in \hat{V} \), and let \( Y \) be a neighborhood of \( y \) in \( V \) such that for all \( x \in X = \chi^{-1}(Y) \) and \( x \neq \chi^{-1}(y) \), \( x \) is a regular point (such a \( Y \) exists since the fiber of \( \chi \) are finite). Hence \( X = \bigcup_{j \in J} U_j \), with the \( U_j \) disjoint and \( 1 \leq j \leq d \). If \( j = d \), \( y \) is a regular point, and for all \( j, \chi|_{U_j} \) is a homeomorphism.

If \( j < d \), we want to show that for every \( w \in Y \setminus \{y\} \) there exists a neighborhood \( W \subset Y \) of \( w \) such that \( \chi^{-1}(W) = \bigcup_{1 \leq j \leq d} T_j \), with the \( T_j \) disjoint and \( \chi|_{T_j} \) homeomorphism. This is clear because all the points in \( X \) different from the preimage of \( y \) are regular points.

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### 5.1 Proper families of parabolic-like maps

As we saw in 2.4, the range \( B \) of the map \( \chi \) is not the whole of \( \mathbb{C} \) but a proper subset, because there is no \( \lambda \in \Lambda \) for which \( f_\lambda \) is hybrid equivalent to \( P_0 = z + 1/z \). Hence \( M_1 \notin B \). However, we could hope that \( B = 1 \) is the only point of \( M_1 \) missing from \( B \), or in other words, that as \( B \to \partial B \) \( B \notin M_1 \) or \( B \to 1 \). Indeed this is the case under appropriate conditions (e.g. the following one).

**Definition 5.3.** Let \( f = (f_\lambda : U'_\lambda \to U_\lambda)_{\lambda \in \Lambda} \) be a nice analytic family of parabolic-like maps of degree 2, such that, for \( \lambda \to \partial \Lambda \):

1. \( \lambda \notin M_f \) or
2. \( \chi(\lambda) \to 1 \).

Then we call \( f \) a *proper family of parabolic-like mappings.*
Proposition 5.4. Let \( f \) be a proper family of parabolic-like mappings. Then, for every \( U(1) \) neighborhood of 1 in \( \mathbb{C} \), setting \( K = M_1 \setminus U(1) \), the set \( C = \chi^{-1}(K) \) is compact in \( \Lambda \).

Proof. Assume \( C \) is not compact in \( \Lambda \). Then there exists a sequence \((\lambda_n) \in C\) such that \( \lambda_n \to \partial \Lambda \) as \( n \to \infty \). On the other hand, for all \( n \), \( \chi(\lambda_n) \in K \). Let \( \lambda_n \) be a subsequence converging to some parameter \( B \). Since \( K \) is compact, the limit point \( B \) belongs to \( K \subset M_1 \setminus \{1\} \). This is a contradiction, because \( f \) is a proper family of parabolic-like mappings. Therefore \( C \) is compact in \( \Lambda \).

Hence if \( f \) is a proper family of parabolic-like mappings, \( U(1) \) a neighborhood of \( B = 1 \), \( K = M_1 \setminus U(1) \), and \( \hat{V} \) a neighborhood of \( K \) given by Prop. 5.4, then calling \( c_{\lambda} \) the critical point of \( f_\lambda \) and \( \hat{U} = \chi^{-1}(\hat{V}) \), the degree \( D \) of the branched covering \( \chi : \hat{U} \to \hat{V} \) is equal to the number of times \( f_\lambda(c_{\lambda}) - c_{\lambda} \) turns around \( 0 \) as \( \lambda \) describes \( \partial C \).

Proof. The proof follows the analogous one in [DH]. Let \( c_{\lambda} \) be the critical point of \( f_\lambda \). Choose \( \lambda_0 \) such that \( f_{\lambda_0}(c_{\lambda_0}) = c_{\lambda_0} \). Let \( P_{\lambda_0} \) be the member of the family \( \text{Per}_1(1) \) hybrid equivalent to \( f_{\lambda_0} \). Therefore \( P_{\pm}\lambda_0(-1) = -1 \), hence \( \chi(\lambda_0) = 0 \). This means that the multiplicity of \( \lambda_0 \) as zero of the map \( \lambda \to f_\lambda(c_{\lambda} - c_{\lambda} \) is the multiplicity of \( \lambda_0 \) as zero of the map \( \lambda \to \chi(\lambda) \). This last one is \( \sum_{\lambda \in \chi^{-1}(0)} i_\lambda(\chi) = D \).

5.1.1 Extension to the root

Let \((f_\lambda)_{\lambda \in \Lambda} \) be a proper family of parabolic-like mappings. For every \( \lambda \in \Lambda \), \( f_\lambda \) is the restriction of some map \( F_\lambda \). Consequently, \( \Lambda \) is the restriction of the parameter plane of the maps \( F_\lambda \), call it \( G \). Call \( M_F \) the connectedness locus of \( F_\lambda \), hence \( M_f \subset M_F \).
Theorem 5.6. Let $f$ be a proper family of parabolic-like mappings. If the map $\chi : M_f \to M_1 \setminus \{1\}$ is a homeomorphism, and $\partial \Lambda \cap \partial M_f \subset G$, then $\chi$ extends to a homeomorphism $\chi : M_f \cup \{\lambda_*\} \to M_1$ for a unique $\lambda_* \in \partial \Lambda$. More generally, if the map $\chi : M_f \to M_1 \setminus \{1\}$ is a degree $D$ branched covering, and $\partial \Lambda \cap \partial M_f \subset G$, then map $\chi$ extends continuously to $\chi : M_f \cup \{\lambda_1\} \cup \ldots \cup \{\lambda_D\} \to M_1$ for exactly $D$ points in $\partial \Lambda$.

Proof. Let $f$ be a proper family of parabolic-like mappings for which the map $\chi : M_f \to M_1 \setminus \{1\}$ is a degree $D$ covering and $\partial \Lambda \cap \partial M_f \subset G$. Since $f$ is a proper family, as $\chi(\lambda) \to 1$, $\lambda \to \partial \Lambda \cap \partial M_f$. We will prove that for every $\lambda \in \partial \Lambda \cap \partial M_f$, $\chi(\lambda) = 1$, and that $\partial \Lambda \cap \partial M_f$ is a discrete set. Then by continuity, $\partial \Lambda \cap \partial M_f = \{\lambda_1, \ldots, \lambda_D\}$.

The original family $F_\lambda$ has a persistent parabolic fixed point of multiplier 1 and it depends holomorphically on $\lambda$. Take a succession $\lambda_i \in M_f$ such that $\chi(\lambda_i) \to 1$, and call $\lambda_*$ the limit of the $\lambda_i$ in $G$. Since for every $i$, $\lambda_i$ is a hyperbolic parameter, the limit $\lambda_*$ is a hyperbolic or indifferent parameter. So, if $\chi(\lambda_*) \neq 1$, $F_{\lambda_*}$ presents a degree 2 parabolic-like restriction and $\lambda_* \in \Lambda$. Since $\lambda_* \notin \Lambda$, $\chi(\lambda_*) = 1$.

Let us prove now that the set $\partial \Lambda \cap \partial M_f$ is discrete. Note that this is the set of parameters for which the parabolic fixed point $z_\lambda$ of $F_\lambda$ has parabolic multiplicity $n + 1$, where $n$ is the multiplicity of $z_\lambda$ for $\lambda \in M_f$. Then, in a neighborhood of $z_\lambda$ we can consider $F_\lambda$ as

$$z + a_\lambda z^{n+1} + \text{h.o.t.},$$

with $n \geq 1$ and $a_\lambda$ holomorphic in $\lambda$. Hence the set $\lambda_j^*$ for which $a_{\lambda_j^*} = 0$ is a discrete set.

5.2 The parameter plane of the family $C_a(z) = z + az^2 + z^3$ presents baby-$M_1$

Let us show that the family of parabolic-like mappings $(C_a(z) = z + az^2 + z^3)_{a \in \Lambda}$ is proper. Call $M_a$ the connectedness locus of $(C_a)_{a \in \Lambda}$. The finite boundaries of $\Lambda$ are the external rays of angle $1/6$ and $2/6$, which cannot intersect the connectedness locus $M_a$ in other point than the landing point, if they land. Since these rays land at $a = 0$ (see [N]), for $a \to \partial \Lambda$ either $a \notin M_a$, hence $B \notin M_1$, or $a \to 0$, hence $B \to 1$.

Finally, by the relation $\Phi(a) = \varphi(c_c(a))$ between external rays in dynamical and parameter plane, the degree of $\chi|_{M_a}$ is 1. Therefore $C$ presents a baby $M_1$. By symmetry, we can repeat the construction for the family
\((C_a(z) = z + a z^2 + z^3)_{a \in \Lambda'},\) where \(\Lambda'\) is the open set bounded by the external rays of angle \(\frac{4}{6}\) and \(\frac{5}{6}\). Hence the connectedness locus of the family \((C_a(z) = z + a z^2 + z^3)_{a \in \mathbb{C}}\) presents two baby \(M_1\), namely in the connected component bounded by the external rays of angle \(\frac{1}{6}\) and \(\frac{2}{6}\), and in the connected component bounded by the external rays of angle \(\frac{4}{6}\) and \(\frac{5}{6}\) (see Fig. [1] and [2] in the Introduction).

References

[A] K. Astala, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Univ. Press, (2008).

[Ah] L. Ahlfors, Lectures on quasiconformal mappings, Second edition. AMS University Lecture series, Vol. 38. (2006).

[BH] B. Branner & J. H. Hubbard, The iteration of cubic polynomials II: Patterns and parapatterns, *Acta Math.*,157 (1986), no. 1-2, 23–48.

(DE) A. Douady & C. J. Earle, Conformally natural extension of homeomorphisms of the circle, *Acta Math.*,169 (1992), no. 3-4, 229-325.

[DH] A. Douady & J. H. Hubbard, On the Dynamics of Polynomial-like Mappings, *Ann. Sci. École Norm. Sup.*, (4), Vol.18 (1985), 287-343.

[F] O. Forster, *Lectures on Riemann Surfaces*, Springer, (1981).

[H] A. Hatcher, *Algebraic Topology*, Cambridge Univ. Press, (2002).

[Hö] L. Hörmander, *The analysis of Linear Partial Differential Operators I*, Springer-Verlag, (1983).

[Hu] J. Hubbard, *Teichmüller Theory, Volume 1: Teichmüller Theory*, Matrix Editions, (2006).

[L] L. Lomonaco, Parabolic-like maps, [arXiv:1111.7150](https://arxiv.org/abs/1111.7150).

[L1] L. Lomonaco, *Parabolic-like maps*, IMFUFA tekst, (2013).

[Ly] M. Lyubich, *Conformal Geometry and Dynamics of Quadratic Polynomials*, [www.math.sunysb.edu/~mlyubich/book.pdf](http://www.math.sunysb.edu/~mlyubich/book.pdf).

[M] J. Milnor, *Dynamics in One Complex Variable*, Annals of Mathematics Studies, (2006).
[M2] J. Milnor, On Rational Maps with Two Critical Points, *Experimental Mathematics*, (4), Vol. 9 (2000), 481-522.

[MSS] R. Mañé, P. Sad & D. Sullivan, On the Dynamics of Rational maps, *Ann. Sci. École Norm. Sup.*, (4), Vol.16 (1983), 193-217.

[S] D. Sullivan, Quasiconformal Homeomorphisms and Dynamics III, *Ann. Sci. École Norm. Sup.*, (4), Vol.16 (1983), 193-217.

[N] S. Nakane, Capture components for cubic polynomials with parabolic fixed points, *ACADEMIC REPORTS Fac. Eng. Tokio Polytech. Univ.*, (1), Vol.28 (2005), 33-41.

[PT] C. Petersen & L. Tan, Branner-Hubbard motions and attracting dynamics, *Dynamics on the Riemann sphere*, (45-70), Eur. Math. Soc. (2006).

[Sh] M. Shishikura, Bifurcation of parabolic fixed points, *The Mandelbrot set, Theme and Variations*, (325-363), *London Math. Soc. Lecture Note Ser.*, 274 Cambridge Univ. Press, (2000).