Algebraic Realization of Quark-Diquark Supersymmetry

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Abstract

Algebraic realizations of supersymmetry through $SU(m/n)$ type superalgebras are developed. We show their applications to a bilocal quark-antiquark or a quark-diquark systems. A new scheme based on $SU(6/1)$ is fully exploited and the bilocal approximation is shown to get carried unchanged into it. Color algebra based on octonions allows the introduction of a new larger algebra that puts quarks, diquarks and exotics in the same supermultiplet as hadrons and naturally suppresses quark configurations that are symmetrical in color space and antisymmetrical in remaining flavor, spin and position variables. A preliminary work on the first order relativistic formulation through the spin realization of Wess-Zumino super-Poincaré algebra is presented.

1 Introduction

To kinds of dynamical supersymmetries are observed in nature. The first observation came in nuclear physics developed by Iachello [1], and also by Balantekin, Bars and Iachello [2]. Second observation came in hadronic physics through an extension of Miyazawa’s supersymmetric generalization of the approximate $SU(6)$ symmetry of hadrons, namely $SU(6/21)$ that was shown to arise within the frame of the standard theory of quarks interacting through gluons, built by Gürsey and Catto [3, 4]. Here, the supersymmetry was found to hold in the same approximation for which $SU(6)$ is a good symmetry, provided a diquark structure well separated from the remaining quark emerges inside baryons through an effective string interaction. At this separation the scalar, spin independent confining part of the effective QCD potential is dominant. Since QCD forces are also flavor independent, the force between the quark ($q$) and the diquark ($D = qq$) inside an excited baryon is essentially the same as
the one between $q$ and the antiquark $\bar{q}$ inside an excited meson. Thus, the approximate spin-flavor independence of hadronic physics gets extended to $SU(6/21)$ supersymmetry between the $q$ and $D$, resulting into the parallelism of mesonic and baryonic Regge trajectories. The $SU(6/21)$ superalgebra for hadrons shown below is then applied to a bilocal quark-antiquark $(q - \bar{q})$ or quark- diquark $(q - D)$ system resulting in multiplets that include the $(\ell = 1)$ mesons in the $(35 + 1)$ representation of $SU(6)$ and the $(\ell = 1)$ baryons $(70^{-})$, in addition to the $\ell = 0$ algebra that brings together the $(35^{-} + 1^{-})$ mesons and $(56^{+})$ baryons. Color is incorporated through the algebra of split octonions.

A relativistic effective Hamiltonian of this quark model was built earlier [5], and new mass formulas were derived [3] [6]. This Hamiltonian exhibited an approximate $SU(6/21)$ symmetry which was broken by the mass differences of the constituents and the spin dependent forces caused by gluon exchange. Linear and quadratic mass relationships were derived relating the baryon and meson mass differences true to within one percent of the experiment.

In this paper we show the algebraic realization of supersymmetry through $SU(m/n)$ type superalgebras. A new kind of supersymmetry scheme based on octonions is described. These algebras are shown to be embedded in a larger octonionic algebra which puts mesons and baryons, exotics, quarks and diquarks in the same multiplet. Existence of minimal schemes and properties of the algebraic treatment of the symmetries of three quark system is being published elsewhere.

The last section will deal with some preliminary work on the possible first order relativistic formulation through the spin realization of the Wess-Zumino algebra.

## 2 Algebraic Realization of Supersymmetry

The Miyazawa supersymmetry [4] based on the supergroup $U(6/21)$ acts on a quark and antidiquark situated at the same point $x_1$. At the point $x_2$ we can consider the action of a Miyazawa supergroup with the same parameters, or one with different parameters. In the first case we have a global symmetry. In the second case, if we only deal with bilocal fields the symmetry will be represented by $U(6/21) \times U(6/21)$, doubling the Miyazawa supergroup. On the other hand, if any number of points are considered, with different parameters attached to each point, we are led to introduce a local supersymmetry $U(6/21)$ to which we should add the local color group $SU(3)^c$. Since it is not a fundamental symmetry, we shall not deal with the local Miyazawa group here. However, the double Miyazawa supergroup is useful for bilocal fields since the decomposition of the adjoint representation of the 728-dimensional Miyazawa group with respect to $SU(6) \times SU(21)$ gives

$$728 = (35, 1) + (1, 440) + (6, 21) + (\bar{6}, 21) + (1, 1)$$ (1)

A further decomposition of the double Miyazawa supergroup into its field with respect to its c.m. coordinates, as will be seen below (see equation [8]), leads to the decomposition of the 126-dimensional cosets $(6, 21)$ and $(21, 6)$ into $56^{+} + 70^{-}$ of the diagonal $SU(6)$.

We would have a much tighter and more elegant scheme if we could perform such a decomposition from the start and be able to identify $(1, 21)$ part of the fundamental representation of $U(6/21)$ with the 21-dimensional representation of the $SU(6)$ subgroup, which means going beyond the Miyazawa supersymmetry to a smaller supergroup having $SU(6)$
as a subgroup. Following two sections will be devoted to the development of a bilocal treatment and a tighter new scheme based on $SU(3)^c \times SU(6/1)$ where the bilocal treatment gets carried over unchanged into it.

## 3 Bilocal approximation to hadronic structure and inclusion of color.

Low-lying baryons occur in the symmetric 56 representation $[3]$ of $SU(6)$, whereas the Pauli principle would have led to the antisymmetrical 20 representation. This was a crucial fact for the introduction of color degree of freedom based on $SU(3)^c$ $[10]$. Since the quark field transforms like a color triplet and the diquark like a color antitriplet under $SU(3)^c$, the color degrees of freedom of the constituents must be included correctly in order to obtain a correct representation of the q-D system. Hadronic states must be color singlets. These are represented by bilocal operators $O(r_1, r_2)$ in the bilocal approximation $[11]$ that gives $\bar{q}(1)q(2)$ for mesons and $D(1)\bar{q}(2)$ for baryons. Here $q(1)$ represents the antiquark situated at $r_1$, $q(2)$ the quark situated at $r_2$, and $D(1) = q(1)\bar{q}(1)$ the diquark situated at $r_1$. If we denote the c.m. and the relative coordinates of the constituents by $R$ and $r$, where $r = r_2 - r_1$ and

$$R = \frac{(m_1r_1 + m_2r_2)}{(m_1 + m_2)} \tag{2}$$

with $m_1$ and $m_2$ being their masses, we can then write $O(R, r)$ for the operator that creates hadrons out of the vacuum. The matrix element of this operator between the vacuum and the hadronic state $h$ will be of the form

$$<h|O(R, r)|0>= \chi(R)\psi(r) \tag{3}$$

where $\chi(R)$ is the free wave function of the hadron as a function of the c.m. coordinate and $\psi(r)$ is the bound-state solution of the $U(6/21)$ invariant Hamiltonian describing the $q - \bar{q}$ mesons, $q - D$ baryons, $\bar{q} - \bar{D}$ antibaryons and $D - \bar{D}$ exotic mesons, given by

$$i\partial_t \psi_{\alpha\beta} = \left[\sqrt{(m_\alpha + \frac{1}{2}V_s)^2 + p^2} + \sqrt{(m_\beta + \frac{1}{2}V_s)^2 + p^2} - \frac{4\alpha_s}{3r} + k\frac{s_\alpha \cdot s_\beta}{m_\alpha m_\beta}\right]\psi_{\alpha\beta} \tag{4}$$

Here $p = -i\nabla$ in the c.m. system and $m$ and $s$ denote the masses and spins of the constituents, $\alpha_s$ the strong-coupling constant, $V_s = br$ is the scalar potential with $r$ being the distance between the constituents in the bilocal object, and $k = |\psi(0)|^2$.

The operator product expansion $[12]$ will give a singular part depending only on $r$ and proportional to the propagator of the field binding the two constituents. There will be a finite number of singular coefficients $c_n(r)$ depending on the dimensionality of the constituent fields. For example, for a meson, the singular term is proportional to the propagator of the gluon field binding the two constituents. Once we subtract the singular part, the remaining part $\tilde{O}(R, r)$ is analytic in $r$ and thus we can write

$$\tilde{O}(R, r) = O_0(R) + r \cdot O_1(R) + o(r^2). \tag{5}$$
Now $O_0(R)$ creates a hadron at its c.m. point $R$ equivalent to a $\ell = 0$, s-state of the two constituents. For a baryon this is a state associated with $q$ and $D \sim qq$ at the same point $R$, hence it is essentially a 3-quark state when the three quarks are at a common location. The $O_1(R)$ can create three $\ell = 1$ states with opposite parity to the state created by $O_0(R)$. Hence, if we denote the nonsingular parts of $\bar{q}(1)q(2)$ and $D(1)q(2)$ by $[\bar{q}(1)q(2)]$ and $[D(1)q(2)]$, respectively, we have

\[ [\bar{q}(1)q(2)]|0 > = |M(R) > + r \cdot |M'(R) > + o(r^2), \]
\[ [D(1)q(2)]|0 > = |B(R) > + r \cdot |B'(R) > + o(r^2), \]

and similarly for the exotic meson states $D(1)\bar{D}(2)$.

Here $M$ belongs to the $(35+1)$-dimensional representation of $SU(6)$ corresponding to $\ell = 0$ bound state of the quark and the antiquark. The $M'(R)$ is an orbital excitation ($\ell = 1$) of opposite parity, which are in the $(35+1,3)$ representation of the group $SU(6) \times O(3)$, $O(3)$ being associated with the relative angular momentum of the constituents. The $M'$ states contain mesons like $B, A_1, A_2$ and scalar particles. On the whole, the $\ell = 0$ and $\ell = 1$ part $\bar{q}(1)q(2)$ contain $4 \times (35 + 1) = 144$ meson states.

Switching to the baryon states, the requirement of antisymmetry in color, and symmetry in spin-flavor indices gives the $(56)^+$ representation for $B(R)$. The $\ell = 1$ multiplets have negative parity and have mixed spin-flavor symmetry. They belong to the representation $(70^-,3)$ of $SU(6) \times O(3)$ and are represented by the states $|B'(R) >$ which are 210 in number. On the whole, these 266 states account for all the observed low-lying baryon states obtained form $56 + 3 \times 70 = 266$. A similar analysis can be carried out for the exotic meson states $D(1)\bar{D}(2)$, where the diquark and the antidiquark can be bound in a $\ell = 0$ or $\ell = 1$ state with opposite parities.

4 Color algebra and Octonions.

The behavior of various states under the color group are best seen if we use split octonion units defined by $[13]$:

\[ u_0 = \frac{1}{2}(1 + ie_7), \quad u_0^* = \frac{1}{2}(1 - ie_7), \]
\[ u_j = \frac{1}{2}(e_j + ie_{j+3}), \quad u_j^* = \frac{1}{2}(e_j - ie_{j+3}), \quad j = 1,2,3. \]

The automorphism group of the octonion algebra is the 14-parameter exceptional group $G_2$. The imaginary octonion units $a_\alpha (\alpha = 1, \ldots, 7)$ fall into its 7-dimensional representation.

Under the $SU(3)^c$ subgroup of $G_2$ that leaves $e_j$ invariant, $u_0$ and $u_0^*$ are singlets, while $u_j$ and $u_j^*$ correspond, respectively, to the representations $\mathbf{3}$ and $\bar{\mathbf{3}}$. The multiplication table can now be written in a manifestly $SU(3)^c$ invariant manner (together with the complex conjugate equations):

\[ u_0^2 = u_0, \quad u_0u_0^* = 0 \]
\[ u_0u_j = u_j^*u_0 = u_j, \quad u_0^*u_j = u_ju_0 = 0, \]
\[ u_ju_j = -u_ju_i = e_{ijk}u_k^*, \]

(10), (11), (12)
\[ u_i u_j^* = -\delta_{ij} u_0 \]  

where \( \epsilon_{ijk} \) is completely antisymmetric with \( \epsilon_{ijk} = 1 \) for \( ijk = 123, 246, 435, 651, 572, 714, 367 \). Here, one sees the virtue of octonion multiplication. If we consider the direct products

\[
\begin{align*}
C : & \quad 3 \otimes 3 = 1 + 8, \\
G : & \quad 3 \otimes 3 = 3 + 6
\end{align*}
\]

for \( SU(3)^c \), then these equations show that octonion multiplication gets rid of \( 8 \) in \( 3 \otimes \bar{3} \), while it gets rid of \( 6 \) in \( 3 \otimes 3 \). Combining (12) and (13) we find

\[
(u_i u_j) u_k = -\epsilon_{ijk} u_0^*.
\]

Thus the octonion product leaves only the color part in \( 3 \otimes \bar{3} \) and \( 3 \otimes 3 \otimes 3 \), so that it is a natural algebra for colored quarks.

The quarks, being in the triplet representation of the color group \( SU(3)^c \), they are represented by the local fields \( q^i_\alpha(x) \), where \( i = 1, 2, 3 \) is the color index and \( \alpha \) the combined spin-flavor index. Antiquarks at point \( y \) are color antitriplets \( \bar{q}^j_\beta(y) \). Consider the two-body systems

\[
\begin{align*}
C_{\alpha\beta}^{ij} &= q^i_\alpha(x_1) \bar{q}^j_\beta(x_2), \\
G_{\alpha\beta}^{ij} &= \bar{q}^i_\alpha(x_1) q^j_\beta(x_2),
\end{align*}
\]

so that \( C \) is either a color singlet or color octet, while \( G \) is a color antitriplet or a color sextet. Now \( C \) contains meson states that are color singlets and hence observable. The octet \( q - \bar{q} \) state is confined and not observed as a scattering state. In the case of two-body \( G \) states, the antitriplets are diquarks which, inside a hadron can be combined with another triplet quark to give observable, color singlet, three-quark baryon states. The color sextet part of \( G \) can only combine with a third quark to give unobservable color octet and color decuplet three-quark states. Hence the hadron dynamics is such that the \( 8 \) part of \( C \) and the \( 6 \) part of \( G \) are suppressed. This can best be achieved by the use of above octonion algebra \[\text{[14]}\]. The dynamical suppression of the octet and sextet states in (17) and (18) is, therefore, automatically achieved. The split octonion units can be contracted with color indices of triplet or antitriplet fields. For quarks and antiquarks we can define the "transverse" octonions (calling \( u_0 \) and \( u_0^* \) longitudinal units)

\[
\begin{align*}
q_\alpha &= u_i q^i_\alpha = \mathbf{u} \cdot \mathbf{q}_\alpha, \\
\bar{q}_\beta &= u_i^* \bar{q}^i_\beta = -\mathbf{u}^* \cdot \mathbf{q}_\beta.
\end{align*}
\]

We find

\[
\begin{align*}
q_\alpha(1)\bar{q}_\beta(2) &= u_0 q_\alpha(1) \cdot q_\beta(2), \\
\bar{q}_\alpha(1)q_\beta(2) &= u_0^* \bar{q}_\alpha(1) \cdot q_\beta(2), \\
G_{\alpha\beta}(12) &= q_\alpha(1)q_\beta(2) = \mathbf{u}^* \cdot \mathbf{q}_\alpha(1) \times \mathbf{q}_\beta(2), \\
G_{\beta\alpha}(21) &= q_\beta(2)q_\alpha(1) = \mathbf{u}^* \cdot \mathbf{q}_\beta(2) \times \mathbf{q}_\alpha(1).
\end{align*}
\]

Because of the anticommutativity of the quark fields, we have

\[
G_{\alpha\beta}(12) = G_{\beta\alpha}(21) = \frac{1}{2} \{ q_\alpha(1), q_\beta(2) \}.
\]
If the diquark forms a bound state represented by a field $D_{\alpha\beta}(x)$ at the center-of-mass location $x$

$$x = \frac{1}{2}(x_1 + x_2),$$

(25)

when $x_2$ tends to $x_1$ we can replace the argument by $x$, and we obtain

$$D_{\alpha\beta}(x) = D_{\beta\alpha}(x),$$

(26)

so that the local diquark field must be in a symmetric representation of the spin-flavor group. If the latter is taken to be $SU(6)$, then $D_{\alpha\beta}(x)$ is in the 21-dimensional symmetric representation, given by

$$(6 \otimes 6)_s = 21.$$  

(27)

If we denote the antisymmetric 15 representation by $\Delta_{\alpha\beta}$, we see that the octonionic fields single out the 21 diquark representation at the expense of $\Delta_{\alpha\beta}$. We note that without this color algebra supersymmetry would give antisymmetric configurations as noted by Salam and Strathdee [15] in their possible supersymmetric generalization of hadronic supersymmetry. Using the nonsingular part of the operator product expansion we can write

$$\tilde{G}_{\alpha\beta}(x_1, x_2) = D_{\alpha\beta}(x) + r \cdot \Delta_{\alpha\beta}(x).$$

(28)

The fields $\Delta_{\alpha\beta}$ have opposite parity to $D_{\alpha\beta}$; $r$ is the relative coordinate at time $t$ if we take $t = t_1 = t_2$. They play no role in the excited baryon which becomes a bilocal system with the 21-dimensional diquark as one of its constituents.

Now consider a three-quark system at time $t$. The c.m. and relative coordinates are

$$\mathbf{R} = \frac{1}{\sqrt{3}}(r_1 + r_2 + r_3),$$

(29)

$$\rho = \frac{1}{\sqrt{6}}(2r_3 - r_1 - r_2),$$

(30)

$$r = \frac{1}{\sqrt{2}}(r_1 - r_2),$$

(31)

giving

$$r_1 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\rho + \frac{1}{\sqrt{2}}r$$

(32)

$$r_2 = \frac{1}{\sqrt{3}}\mathbf{R} - \frac{1}{\sqrt{6}}\rho - \frac{1}{\sqrt{2}}r$$

(33)

$$r_3 = \frac{1}{\sqrt{3}}\mathbf{R} + \frac{2}{\sqrt{6}}\rho$$

(34)

The baryon state must be a color singlet, symmetric in the three pairs $(\alpha, x_1)$, $(\beta, x_2)$, $(\gamma, x_3)$. We find

$$(q_\alpha(1)q_\beta(2))q_\gamma(3) = -u_0^* F_{\alpha,\beta,\gamma}(123),$$

(35)

$$q_\gamma(3)(q_\alpha(1)q_\beta(2)) = -u_0 F_{\alpha,\beta,\gamma}(123),$$

(36)
so that
\[ -\frac{1}{2}\{\{q_\alpha(1), q_\beta(2)\}, q_\gamma(3)\} = F_{\alpha\beta\gamma}(123). \] (37)

The operator \(F_{\alpha\beta\gamma}(123)\) is a color singlet and is symmetrical in the three pairs of coordinates. We have
\[ F_{\alpha\beta\gamma}(123) = B_{\alpha\beta\gamma}(R) + \rho \cdot B'(R) + r \cdot B''(R) + C \] (38)
where \(C\) is of order two and higher in \(\rho\) and \(r\). Because \(R\) is symmetric in \(r_1, r_2\) and \(r_3\), the operator \(B_{\alpha\beta\gamma}\) that creates a baryon at \(R\) is totally symmetrical in its flavor-spin indices. In the \(SU(6)\) scheme it belongs to the \((56)\) representation. In the bilocal \(q - D\) approximation we have \(r = 0\) so that \(F_{\alpha\beta\gamma}\) is a function only of \(R\) and \(\rho\) which are both symmetrical in \(r_1\) and \(r_2\). As before, \(B'\) belongs to the orbitally excited \(70^-\) representation of \(SU(6)\). The totally antisymmetrical \((20)\) is absent in the bilocal approximation. It would only appear in the trilocal treatment that would involve the 15-dimensional diquarks. Hence, if we use local fields, any product of two octonionic quark fields gives a \((21)\) diquark
\[ q_\alpha(R)q_\beta(R) = D_{\alpha\beta}(R), \] (39)
and any nonassociative combination of three quarks, or a diquark and a quark at the same point give a baryon in the \((56^+)\) representation:
\[ (q_\alpha(R)q_\beta(R))q_\gamma(R) = -u_0^*B_{\alpha\beta\gamma}(R), \] (40)
\[ q_\alpha(R)(q_\beta(R)q_\gamma(R)) = -u_0B_{\alpha\beta\gamma}(R), \] (41)
\[ q_\gamma(R)(q_\alpha(R)q_\beta(R)) = -u_0B_{\alpha\beta\gamma}(R), \] (42)
\[ (q_\gamma(R)q_\alpha(R))q_\beta(R) = -u_0^*B_{\alpha\beta\gamma}(R). \] (43)

The bilocal approximation gives the \((35 + 1)\) mesons and the \(70^-\) baryons with \(\ell = 1\) orbital excitation.

5 A colored supersymmetry scheme based on \(SU(3)^C \times SU(6/1)\)

We could go to a smaller supergroup having \(SU(6)\) as a subgroup. With the addition of color, such a supergroup is \(SU(3) \times SU(6/1)\). The fundamental representation of \(SU(6/1)\) is 7-dimensional which decomposes into a sextet and singlet under the spin-flavor group. There is also a 28-dimensional representation of \(SU(7)\). Under the \(SU(6)\) subgroup it has the decomposition
\[ 28 = 21 + 6 + 1. \] (44)

Hence, this supermultiplet can accommodate the bosonic antidiquark and fermionic quark in it, provided we are willing to add another scalar. Together with the color symmetry, we are led to consider the \((3, 28)\) representation of \(SU(3) \times SU(6/1)\) which consists of an antidiquark, a quark and a color triplet scalar that we shall call a scalar quark. This boson is in some way analogous to the \(s\) quarks. The whole multiplet can be represented by an octonionic \(7 \times 7\) matrix \(Z\) at point \(x\).
Here $\mathbf{D}^*$ is a $6 \times 6$ symmetric matrix representing the antidiquark, $\mathbf{q}$ is a $6 \times 1$ column matrix, $\mathbf{q}^T$ is its transpose and $\sigma_2$ is the Pauli matrix that acts on the spin indices of the quark so that, if $\mathbf{q}$ transforms with the $2 \times 2$ Lorentz matrix $L$, $\mathbf{q}^T i \sigma_2$ transforms with $L^{-1}$ acting from the right.

Similarly we have

$$Z^c = \mathbf{u}^* \cdot \left( \begin{array}{c} \mathbf{D} \\ \mathbf{q}^T i \sigma_2 \\ \mathbf{S}^* \end{array} \right)$$

(46)

to represent the supermultiplet with a diquark and antidiquark. The mesons, exotic mesons and baryons are all in the bilocal field $Z(1) \otimes Z^c(2)$ which we expend with respect to the center of mass coordinates in order to represent color singlet hadrons by local fields. The color singlets $56^+$ and $70^-$ will then arise as in the earlier section.

Now the $(\hat{D}q)$ system belonged to the fundamental representation of the $SU(6/21)$ supergroup. But $Z$ belongs to the $(28)$ representation of $SU(6/1)$ which is not its fundamental representation. Are there any fields that belong to the 7-dimensional representation of $SU(6/1)$? It is possible to introduce such fictitious fields a 6-dimensional spinor $\xi$ and a scalar $a$ without necessarily assuming their existence as particles. We put

$$\xi = \mathbf{u}^* \cdot \xi, \quad a = \mathbf{u}^* \cdot a,$$

(47)

so that both $\xi$ and $a$ are color antitriplets. Let

$$\lambda = \begin{pmatrix} \xi \\ a \end{pmatrix}, \quad \lambda^c = \begin{pmatrix} \hat{\xi} \\ a^* \end{pmatrix}$$

(48)

where

$$\hat{\xi} = \mathbf{u} \cdot (i \sigma_2 \xi^*), \quad a^* = \mathbf{u} \cdot a^*.$$  

(49)

Consider the $7 \times 7$ matrix

$$W = \lambda \lambda^c = \begin{pmatrix} \xi \hat{\xi}^\dagger & \xi a \\ \xi a^\dagger & 0 \end{pmatrix}.$$  

(50)

$W$ belongs to the 28-dimensional representation of $SU(6/1)$ and transforms like $Z$, provided the components of $\xi$ are Grassmann numbers and $a$ are even (bosonic) coordinates. The identification of $Z$ and $W$ would give

$$\mathbf{s} = a \times a = 0, \quad \mathbf{q}_\alpha = \xi_\alpha \times a, \quad D^*_\alpha \beta = \xi_\alpha \times \xi_\beta.$$  

(51)

A scalar part in $W$ can be generated by multiplying two different $(7)$ representations. The $56^+$ baryons form the color singlet part of the 84-dimensional representation of $SU(6/1)$ while its colored part consists of quarks and diquarks.

Now consider the octonionic valued quark field $q^\prime_A$, where $i = 1, 2, 3$ is the color index and $A$ stands for the pair $(\alpha, \mu)$ with $\alpha = 1, 2$ being the spin index and $\mu = 1, 2, 3$ the flavor index. (If we have $N$ flavors, $A = 1, \ldots, N$). As before
\[ q_A = u_i^q_A = u \cdot q_A. \] (52)

Similarly the diquark \( D_{AB} \) which transforms like a color antitriplet is

\[ D_{AB} = q_A q_B = \epsilon_{ijk} u_k^q_A q_j q_B = u^* \cdot D_{AB}. \] (53)

We note, once again, that because \( q_i^A \) are anticommuting fermion operators, \( D_{AB} \) is symmetric in its two indices. The antiquark and antidiquark are represented by

\[ \bar{q}_A = u^* \cdot \bar{q}_A, \] (54)

and

\[ \bar{D}_{AB} = u \cdot \bar{D}_{AB}, \] (55)

respectively. If we have 3 flavors \( q_A \) has 6 components for each color while \( D_{AB} \) has 21 components.

At this point let us study the system \((q_A, \bar{D}_{BC})\) consisting of a quark and an antidiquark, both color triplets. There are two possibilities: we can regard the system as a multiplet belonging to the fundamental representation of a supergroup \( U(6/21) \) for each color, or as a higher representation of a smaller supergroup. The latter possibility is more economical.

To see what kind of supergroup we can have, we imagine that both quarks and diquarks are components of more elementary quantities: a triplet fermion \( f_A \) and a boson \( C \) which is a triplet with respect to the color group and a singlet with respect to \( SU(2N) \) (\( SU(6) \) for three flavors). The \( f_A \) is taken to have baryon number \( 1/3 \) while \( C \) has baryon number \(-2/3\). The system

\[ q_A^I = \epsilon_{ijk} f_A^J \bar{C}^k \] (56)

will be a color triplet with baryon number \( 1/3 \). It can therefore represent a quark. We can write

\[ q_A = u \cdot q_A = (u^* \cdot f_A)(u^* \cdot \bar{C}). \] (57)

With two anti-f fields we can form bosons that have same quantum numbers as antidiquarks:

\[ \bar{D}_{AB} = u \cdot \bar{D}_{AB} = (u^* \cdot f_A)(u^* \cdot f_B). \] (58)

In this case the basic multiplet is \((f_A, C)\) which belongs to the fundamental representation of \( SU(6/1) \) for each color component. The complete algebra to consider is \( SU(3) \times SU(6/1) \) and the basic multiplet corresponds to the representation \((3, 7)\) of this algebra. Let

\[
F = \begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_6 \\
  C
\end{pmatrix}, \quad f_A = u \cdot f_A, \quad C = u \cdot C.
\] (59)

Also let

\[
f^1 = \begin{pmatrix}
  f_1^1 \\
  f_2^1 \\
  \vdots \\
  f_6^1
\end{pmatrix}, \quad f^2 = \begin{pmatrix}
  f_1^2 \\
  f_2^2 \\
  \vdots \\
  f_6^2
\end{pmatrix}.
\] (60)
Combining two such representations and writing \( \bar{X} = F \times F^T \) we have

\[
\bar{X} = u^* \cdot \bar{X} = \left( \begin{array}{c} f_1^1 \\ C_1 \\
 \end{array} \right) \left( \begin{array}{cc} f_2^2 & C_2 \end{array} \right) - \left( \begin{array}{c} f_1^2 \\ C_2 \end{array} \right) \left( \begin{array}{cc} f_2^1 & C_1 \end{array} \right).
\]

Further identifying

\[
u^* \cdot D_{11} = 2f_1^1 f_1^2, \quad u^* \cdot D_{12} = f_1^1 f_2^1 - f_1^2 f_2^1, \quad \text{etc.,}
\]

and

\[
u^* \cdot \bar{q}_1 = f_1^1 C_2^2 - f_1^2 C_1^1, \quad u^* \cdot \bar{q}_2 = f_2^1 C_2^1 - f_2^2 C_1^2, \quad \text{etc.,}
\]

we see that \( \bar{X} \) has the structure

\[
\bar{X} = u^* \cdot \bar{X} = u^* \cdot \left( \begin{array}{cccc} D_{11} & \ldots & D_{16} & \bar{q}_1 \\
D_{12} & \ldots & D_{26} & \bar{q}_2 \\
D_{13} & \ldots & D_{36} & \bar{q}_3 \\
D_{14} & \ldots & D_{46} & \bar{q}_4 \\
D_{15} & \ldots & D_{56} & \bar{q}_5 \\
D_{16} & \ldots & D_{66} & \bar{q}_6 \\
-q_1 & \ldots & -q_6 & 0
\end{array} \right)
\]

or

\[
(3, 7) \times (3, 7) = (3 \times 27)
\]

The 27 dimensional representation decomposes into 21 + 6 with respect to its SU(6) subgroup.

Consider now an antiquark-diquark system at point \( x_1 = x - \frac{1}{2} \xi \) and another quark-antidiquark system at point \( x_2 = x + \frac{1}{2} \xi \); Hence we take the direct product of \( X(x_1) \) and \( X(x_2) \). In other words

\[
(D_{AB}(x_1), \bar{q}_D(x_1)) \otimes (\bar{D}_{EF}(x_2), q_C(x_2))
\]

consisting of the pieces

\[
H(x_1, x_2) = \left( \begin{array}{cc}
\bar{q}_D(x_1)q_C(x_2) & D_{AB}(x_1)q_C(x_2) \\
\bar{q}_D(x_1)\bar{D}_{EF}(x_2) & D_{AB}(x_1)\bar{D}_{EF}(x_2)
\end{array} \right)
\]

The diagonal pieces are bilocal fields representing color singlet 1 + 35 mesons and 1 + 35 + 405 exotic mesons respectively with respect to the subgroup SU(3)\(^c\) × SU(6) of the algebra. The off diagonal pieces are color singlets that are completely symmetrical with respect to the indices (ABC) and (DEF). They correspond to baryons and antibaryons in the representations 56 and 56 respectively of SU(6).

We can write \( F_{ABC} = -\frac{1}{2} \{D_{AB}, q_C\} \) so that

\[
D_{AB}q_C = (u_1^1 q_A^2 q_B^3 + u_2^2 q_A^2 q_B^3 + u_3^3 q_A^2 q_B^3) + (u_1^1 q_A^2 q_B^3 + u_2^2 q_A^2 q_B^3 + u_3^3 q_A^2 q_B^3) \times
\]

\[
(u_1^1 q_C^2 + u_2^2 q_C^2 + u_3^3 q_C^2),
\]

becomes

\[
D_{AB}q_C = -u_0^*((q_A^2 q_B^3 + q_B^2 q_A^3)^1 + (q_A^2 q_B^3 + q_B^2 q_A^3)^2 + (q_A^2 q_B^3 + q_B^2 q_A^3)^3).
\]
and similarly
\[ q_C D_{AB} = -u_0((q_A^2 q_B^3 + q_B^2 q_A^3)q_C^1 + (q_A^3 q_B^1 + q_B^3 q_A^1)q_C^2 + (q_A^1 q_B^2 + q_B^1 q_A^2)q_C^3). \] (70)

Since \( u_0 + u_0^* = 1 \), we have
\[ F_{ABC} = -\frac{1}{2} \{ D_{AB}, q_C \} = (q_A^1 q_B^2 + q_B^1 q_A^2)q_C^3 + (q_A^2 q_B^3 + q_B^2 q_A^3)q_C^1 + (q_A^3 q_B^1 + q_B^3 q_A^1)q_C^2 \] (71)

which is completely symmetric with respect to indices \((ABC)\), corresponding to baryons.

In the limit \( x_2 - x_1 = \xi \rightarrow 0 \), \( H \) can be represented by a local supermultiplet with dimension \( 2 \times 56 + 2(1 + 35) + 405 = 589 \) of the original algebra. This representation includes 56 baryons, antibaryons, mesons and \( q^2 \bar{q}^2 \) exotic mesons.

### 6 Transformation properties

Since \( F = u \cdot F \) consists of three 7-dimensional representations of \( SU(6/1) \) we have
\[ \delta F = Z \cdot F \] (72)
where \( Z \in SU(6/1) \),
\[ F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_6 \\ C \end{pmatrix}, \]
(73)

Hence \( Z \) is super antihermitian color singlet:
\[ Z = \begin{pmatrix} iH & \eta \\ \eta^\dagger & i\omega \end{pmatrix}, \] (74)

with \( H = H^\dagger, \omega = \omega^*, \text{Str} Z = 0 \) (\( \text{Str} = \text{supertrace} \)), \( (\omega = tr H) \), and
\[ \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_6 \end{pmatrix}, \quad \{\eta_\alpha, \eta_\beta\} = 0. \] (75)

\( H \) is an antihermitian \( 6 \times 6 \) matrix.

Then by taking supertransposed quantities \( (sT = \text{supertransposed}) \)
\[ \delta F^{sT} = \delta F^T = F^T \cdot Z^{sT} \] (76)
we have
\[ \begin{pmatrix} \delta f \\ \delta C \end{pmatrix} = \begin{pmatrix} iH & \eta \\ \eta^\dagger & i\omega \end{pmatrix} \begin{pmatrix} f \\ C \end{pmatrix}, \] (77)
\begin{pmatrix} \delta f^T & \delta C \end{pmatrix} = \begin{pmatrix} iH^* & -i\eta^* \\ \eta^T & i\omega \end{pmatrix} \tag{78}.

We have
\[ \bar{X} = F \quad F^T = F \quad F^{sT} \tag{79} \]
so that
\[ \delta \bar{X} = Z \quad \bar{X} + \bar{X} \quad Z^{sT}. \tag{80} \]

Writing
\[ \bar{X} = \begin{pmatrix} \bar{D} & q \\ -q^T & 0 \end{pmatrix}, \quad (D = D^T). \tag{81} \]
under \( U(6/21) \), \( \delta X \) gives:
\[ \delta \bar{D} = i(H\bar{D} + \bar{D}H^T) - (\eta q^T - q\eta^T), \tag{82} \]
\[ \delta q = i(H + \omega)q - \bar{D}\eta^*. \tag{83} \]

For supertransformation \( SU(6/1) \mid U(6) \), the change in \( D \) and \( q \) are
\[ \delta \bar{D} = q\eta^T - \eta q^T, \quad \delta q = -\bar{D}\eta^*. \tag{84} \]

Since
\[ \delta f = Zf, \tag{85} \]
or in component notation
\[ \delta f_A = Z_{AB}f_B \tag{86} \]
where \( (A, B = 0, 1, \ldots, 6) \) with \( f_0 = C \), we have
\[ \delta f^{sT} = \delta f^\dagger = f^\dagger Z^{sT}. \tag{87} \]

If we define \( g \) by
\[ g = \begin{pmatrix} I & 0 \\ 0 & i \end{pmatrix}, \tag{88} \]
then
\[ \delta(f^\dagger g f) = f^\dagger(gZ + Z^{sT}g)f. \tag{89} \]
It is easy to show \( gZ + Z^{sT}g = 0 \) so that \( \delta(f^\dagger g f) = 0. \) If we look at \( U(2/1) \) parts
\[ \delta F = \begin{pmatrix} iH & 0 \\ 0 & i\omega \end{pmatrix} \tag{90} \]
giving
\[ \delta f = iHf, \quad \delta C = i\omega C \tag{91} \]
and
\[ \delta F^\dagger = F^\dagger \begin{pmatrix} -iH & 0 \\ 0 & -i\omega \end{pmatrix} \tag{92} \]
giving
\[ \delta f^\dagger = -if^\dagger H, \quad \delta C^* = -i\omega C^*, \tag{93} \]
we obtain
\[ \delta(f^\dagger f) = 0, \quad \text{and} \quad \delta(C^*C) = 0. \] (94)

Similarly
\[ \delta F = \begin{pmatrix} 0 & \eta \\ i\eta^\dagger & 0 \end{pmatrix} F \]
(95)
gives
\[ \delta f = \eta C, \quad \delta C = i\eta^\dagger f, \]
and using
\[ \delta f^\dagger = C^*\eta^\dagger, \quad \text{and} \quad \delta c^* = -i\eta^T f^* = if^\dagger \eta, \]
(97)
we arrive at
\[ \delta(f^\dagger f + iC^*C) = 0 \] (98)
or
\[ \delta(if^\dagger f - C^\dagger C) = 0 \] (99)
Defining \( \bar{f} = if^\dagger \), we have
\[ \delta(\bar{f} f - C^\dagger C) = 0. \] (100)

If we now define
\[ h_{AB\dot{A}\dot{B}} = (\bar{X})_{AB}(x)(X)_{\dot{A}\dot{B}}(x) \]
(101)
where \( A, B = 0, 1, \ldots, 6 \) as before, then the subset \( h_{\dot{a}\dot{b}a\dot{b}} \) antisymmetric in the first and last pairs of indices would describe \( q\bar{q} \) mesons, \( h_{ab\dot{a}b} \) symmetric in first and last pairs of indices would describe \( q^2\bar{q} \) exotic mesons, \( h_{\dot{a}\dot{b}a\dot{b}} \) antisymmetric in the first and symmetric in the last pair of indices would describe \( q^2\bar{q} \) baryons, and \( h_{ab\dot{a}b} \) symmetric in the first and antisymmetric in the last pair of indices would describe \( \bar{q}q \) antibaryons.

Aside from \( h_{AB\dot{A}\dot{B}} \) describing baryons, antibaryons, mesons and exotics, this algebra can be extended to include preons \( F_A \), antipreons \( \dot{F}_A \), \( X_{AB} \) describing \( \bar{q}^2q \), and \( \dot{X}_{\dot{A}\dot{B}} \) describing \( \bar{q}^2\bar{q} \). Gauge bosons and gauginos can be in the adjoint representation \( V_{AB} \). We shall explore these aspects as well as building meson-baryon Lagrangians in another publication.

We note that, since
\[ (Z_{AB})^* = (Z^*)_{\dot{A}\dot{B}} \]
(102)
we have
\[ \delta F_A = Z_{AB} F_B \quad \text{and} \quad \delta F_\dot{A} = Z^*_{\dot{A}\dot{B}} F_\dot{B} \]
(103)

7 Particle multiplets including a giant supermultiplet

Multiplet \( X \) that sits in the adjoint representation of \( SU(6/21) \) given by
\[ X = \begin{pmatrix} M & B \\ \bar{B} & N \end{pmatrix}. \]
(104)
Here \( M \) and \( N \) are mesons and exotics, and \( B \) and \( \bar{B} \) are fermions. The \( M \) and \( N \) are square matrices, and \( B \) is a rectangular matrix. Specifically \( M = 6 \times 6 \), \( B = 6 \times 21 \), \( \bar{B} = 21 \times 6 \), and \( N = 21 \times 21 \). \( M \) and \( N \) are taken to be Hermitian. If we have three flavors the \( SU(6) \)
content of these matrices are \( M = 1 + 35 \) (negative parity), \( N = 1 + 35 + 405 \) (positive parity), and \( B = 56 + 70 \) (positive parity). The fundamental representation \( \mathbf{F} \) is the color triplet

\[
\mathbf{F} = \begin{pmatrix} \mathbf{Q} \\ \mathbf{D} \end{pmatrix} = \begin{pmatrix} \mathbf{q} \\ \mathbf{q} \times \overline{\mathbf{q}} \end{pmatrix},
\]

(105)

with \( \mathbf{q} = 6 \times 1 \) and \( \mathbf{q} \times \overline{\mathbf{q}} = 21 \times 1 \). Now let \( \Xi \) be the superalgebra element of \( SU(6/21) \) (\( SU(3) \) singlet). \( Xi \) is a color singlet given by

\[
\Xi = \begin{pmatrix} m \\ b \\ n \end{pmatrix}
\]

(106)

and the transformation law for the fundamental representation \([3, (6 + 21)]\) is

\[
\delta \mathbf{F} = \Xi \mathbf{F} = \begin{pmatrix} m \\ b \\ n \end{pmatrix} \begin{pmatrix} M & B \\ B & N \end{pmatrix} = \begin{pmatrix} mM + bB & mb + bN \\ bM + nB & bB + nN \end{pmatrix},
\]

(107)

\[
\delta \overline{\mathbf{F}} = \mathbf{F} \Xi = (Qm + DB, Qb + DDn) = (\delta Q, \delta D)
\]

(108)

or in the index notation

\[
\delta q^i_\alpha = m^\beta q^i_\beta + b_\alpha \delta (D^i)^{\beta \gamma},
\]

(109)

\[
(\delta \overline{D}^i)^{\beta \gamma} = \overline{b}^{\alpha \beta \gamma} q^i_\alpha + n^{\beta \gamma} (\overline{D}^i)^{\rho \sigma}.
\]

(110)

On the other hand, the transformation law for the adjoint representation is

\[
\delta \Xi = i[\Xi, X],
\]

(111)

where

\[
\Xi X = \begin{pmatrix} m \\ b \\ n \end{pmatrix} \begin{pmatrix} M & B \\ B & N \end{pmatrix} = \begin{pmatrix} Mm + Bb & MB + bN \\ bM + nB & bB + nN \end{pmatrix},
\]

(112)

\[
X \Xi = \begin{pmatrix} Mm + Bb \\ Bm + nb \end{pmatrix} \begin{pmatrix} Mb + bN \\ Bb + nN \end{pmatrix},
\]

(113)

so that

\[
\begin{pmatrix} \delta M & \delta B \\ \delta B & \delta N \end{pmatrix} = i \begin{pmatrix} [M, m] + bB - Bb & Mb - Bn + bN - MB \\ -Bm + nB - nb + bM & [n, N] + bB - Bb \end{pmatrix}.
\]

(114)

Next we build a giant supermultiplet containing \( M, N, L, B, \overline{B}, Q, \overline{Q}, D, \) and \( \overline{D} \). The fundamental representation of \( U(6/21) \times [SU(3)^c]_{\text{triplet}} \) and the adjoint representation of \( U(6/21) \times [SU(3)^c]_{\text{singlet}} \) fits in the adjoint representation of an octonionic version of \( U(6/22) \) denoted by \( Z \), given by

\[
Z = u_0 \begin{pmatrix} M & B & 0 \\ B^\dagger & N & 0 \\ 0 & 0 & 0 \end{pmatrix} + u_0^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L \end{pmatrix} + u \cdot \begin{pmatrix} 0 & 0 & \mathbf{Q} \\ 0 & 0 & \mathbf{D}^* \end{pmatrix} +
\]

\[
u^* \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon \mathbf{Q}^\dagger & \epsilon \mathbf{D}^T & 0 \end{pmatrix} = \begin{pmatrix} u_0 M & u_0 B \\ u_0 B^\dagger & u_0 N \\ \epsilon u^* \cdot \mathbf{Q}^\dagger & \epsilon u^* \cdot \mathbf{D}^T & u_0^* L \end{pmatrix}
\]

(115)
where mesons $M (6 \times \bar{6})$ and exotics $N (\bar{21} \times 21)$ are Hermitian; $B (6 \times 21)$, $\bar{B} (\bar{21} \times \bar{6})$, $Q (6 \times 1)$, $\bar{Q} (1 \times \bar{6})$, $D (1 \times 21)$, $\bar{D} (\bar{21} \times 1)$, and $L (1 \times 1)$; $\epsilon$ can be taken as $1$ if $u^\dagger = \bar{u}^* = -u^*$, $-1$ if $u^\dagger = u^*$, and zero. Closure properties of $Z$ matrices are such that

$$[Z, Z'] = iZ'', \quad \{Z, Z'\} = Z'''$$  \hspace{1cm} (116)

and in general they are nonassociative (Jacobian $J = f(Q, D) \neq 0$), except in the case when $\epsilon = 0$ we have

$$[[Z, Z'], Z''] + [[Z', Z''], Z] + [[Z'', Z], Z'] = 0.$$  \hspace{1cm} (117)

Then we have a true superalgebra (non-semisimple) which is a contraction of a simple algebra that closes but does not satisfy the Jacobi identity. In both cases we get an extension of $U(6/21)$ considered by Miyazawa [7].

We now consider the element of the algebra

$$\Omega = \begin{pmatrix} u_0 m & u_0 b & u \cdot \xi \\ u_0 b^\dagger & u_0 n & u \cdot d^* \\ \epsilon u^* \cdot \xi^\dagger & \epsilon u^* \cdot d^T & u_0^\dagger \ell \end{pmatrix}$$  \hspace{1cm} (118)

where

$$\begin{pmatrix} u_0 m & u_0 b \\ u_0 b^\dagger & u_0 n \end{pmatrix}$$  \hspace{1cm} (119)

are the color singlet parameters $[U(6/21)]$,

$$\begin{pmatrix} u \cdot \xi \\ u \cdot d^* \end{pmatrix} \quad \text{and} \quad (\epsilon u^* \cdot \xi^\dagger, \epsilon u^* \cdot d^T)$$  \hspace{1cm} (120)

are the colored parameters,

$$\begin{pmatrix} u_0 b & u \cdot \xi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_0 b^\dagger \\ \epsilon u^* \cdot \xi^\dagger \end{pmatrix}$$  \hspace{1cm} (121)

are the fermionic parameters, and $m \in SU(6)$.

The change in $Z$ is given by

$$\delta Z = [\Omega, Z]$$  \hspace{1cm} (122)

which leads to

$$\delta Q = mQ - M\xi + bD^* - Bd^* + \xi L - Q\ell,$$  \hspace{1cm} (123)

and

$$\delta D^* = b^\dagger Q - B^\dagger \xi + nD^* - Nd^* + d^* L - D^* \ell.$$  \hspace{1cm} (124)

The $U(6/21)$ subgroup is obtained by taking

$$\xi = 0, \quad d = 0, \quad \ell = 0$$  \hspace{1cm} (125)

so that

$$\delta Q = mQ + bD^*$$  \hspace{1cm} (126)

$$\delta D^* = b^\dagger Q + nD^*$$  \hspace{1cm} (127)
which, in index form, is equivalent to equations (109) and (110). This subgroup is valid for a Hamiltonian describing $q(x_1)$ and $\bar{q}(x_2)$, $q(x_1)$ and $D(x_2)$, $\bar{q}(x_1)$ and $D(x_2)$ interacting through a scalar potential $V = br$ as we have seen earlier \[3\].

In general for $\frac{m}{2}$ flavors and $n = \frac{1}{2}m(m+1)$, we have

$$Z = \begin{pmatrix}
    u_0 M & u_0 B & u \cdot Q \\
    u_0 B^\dagger & u_0 N & u \cdot D^* \\
    \epsilon u^* \cdot Q^\dagger & \epsilon u^* \cdot D^T & u_0^* L
\end{pmatrix} = \begin{pmatrix}
    m \times m & m \times n & m \times 1 \\
    n \times m & n \times n & n \times 1 \\
    1 \times m & 1 \times n & 1 \times 1
\end{pmatrix}.$$  (128)

As examples, for 2 flavors, $M = 4 \times 4$, $N = 10 \times 10$; for 6 flavors (including the top quark), $M = 12 \times 12$, $N = 78 \times 78$.

The automorphism group of this algebra includes $SU(m) \times SU(n) \times SU(3)^c$. If $m = 6$, it includes $SU(6) \times SU(3)^c$. If $Q$ is Majorana and $D$ real, then the group becomes $Osp(n/m) \times SU(3)^c$, with subgroup $Sp(2n/R) \times O(m) \times SU(3)^c$. We shall explore quark models built by use of such groups as well as use of auxiliary octonions of quadratic norm $\frac{1}{2}$ which are related to the split octonion units we used in this paper in a subsequent publication [16].

8 Relativistic formulation through the spin realization of the Wess-Zumino algebra

It is possible to use a spin representation of the Wess-Zumino algebra to write first order relativistic equations for quarks and diquarks that are invariant under supersymmetry transformations. In this section we briefly deal with such Dirac-like supersymmetric equations. A discussion of experimental possibilities for the observation of the diquark structure and exotic $\bar{D} - D = (\bar{q} \bar{q})(qq)$ mesons will be given elsewhere [3] [14]. For a very nice discussion of the experimental situation we refer the reader to a recent preprint by Anselmino, et.al. [17]. Also, for a historical review of dynamical supersymmetries we refer the reader to a recent article by Iachell [1].

There is a spin realization of the Wess-Zumino super-Poincaré algebra

$$[p_\mu, p_\nu] = 0, \quad [D_\alpha, p_\mu] = 0,$$  (129)

$$[\bar{D}_\beta, p_\mu] = 0, \quad [D^\alpha, \bar{D}^\beta] = \sigma^{\alpha\beta}_\mu p^\mu$$  (130)

with $p_\mu$ transforming like a 4-vector and $D^\alpha, \bar{D}^\beta$ like the left and right handed spinors under the Lorentz group with generators $J_{\mu\nu}$.

We also note that

$$[J_{\mu\nu}, p_\lambda] = \delta_{\mu\nu} p_\lambda - \delta_{\nu\lambda} p_\mu,$$  (131)

and

$$[J, J] = J.$$  (132)

The finite non unitary spin realization is in terms of $4 \times 4$ matrices for $J_{\mu\nu}$ and $p_\nu$

$$J_{\mu\nu} = \frac{1}{2} \sigma_{\mu\nu} = \frac{1}{4i} [\gamma_\mu, \gamma_\nu],$$  (133)
\[ J^L_{\mu\nu} = \frac{1 - \gamma_5}{2} \frac{1}{2} \sigma_{\mu\nu} = \Sigma^L_{\mu\nu}, \quad (134) \]
\[ p_{\mu} = \Pi^L_{\mu} = \frac{1 - \gamma_5}{2} \gamma_{\mu}. \quad (135) \]

Introducing two Grassmann numbers \( \theta_\alpha (\alpha = 1, 2) \) that transforms like the components of a left handed spinor and commute with the Dirac matrices \( \gamma_\mu \), we have the representation
\[ D_\alpha = \Delta_\alpha = \frac{\partial}{\partial \theta_\alpha}; \quad (136) \]
\[ \bar{D}_{\dot{\beta}} = \bar{\Delta}_{\dot{\beta}} = \theta_\alpha \sigma^\alpha_{\dot{\beta}} \Pi^L_{\mu}. \quad (137) \]

Such a representation of the super-Poincaré algebra acts on a Majorana chiral superfield
\[ S(x, \theta) = \psi(x) + \theta_\alpha B^\alpha(x) + \frac{1}{2} \theta_\alpha \theta^\alpha \chi(x). \quad (138) \]

Here \( \psi \) and \( \chi \) are Majorana superfields associated with fermions and \( B^\alpha \) has an unwritten Majorana index and a chiral spinor index \( \alpha \), so that it represents a boson.

Note that the sum of the two representations we wrote down is also a realization of the Wess-Zumino algebra.

On the other hand we have the realization of \( p_{\mu} \) in terms of the differential operator \(-i \partial_{\mu} = -i \frac{\partial}{\partial x^\mu}\). In the Majorana representation, the operator \( \gamma_\mu \partial_{\mu} = i \gamma_\mu p_{\mu} \) is real, and \( \psi = \psi^c = \psi^* \). Let us now define \( \psi_L \) and \( \psi_R \) by
\[ \psi_L = \frac{1}{2} (1 + \gamma_5) \psi, \quad (139) \]
and
\[ \psi_R = \frac{1}{2} (1 - \gamma_5) \psi = \psi^*_L. \quad (140) \]

The free particle Dirac equation can now be written as
\[ \Pi^L_{\mu} \partial_{\mu} \psi_L = m \psi^*_L, \quad (141) \]
or
\[ \Pi_{\mu} p^\mu \psi_L = -im \psi^*_L. \quad (142) \]

We can introduce
\[ S_L = \frac{1}{2} (1 + \gamma_5) S, \quad (143) \]
\[ S_R = \frac{1}{2} (1 - \gamma_5) S = S^*_L. \quad (144) \]

Then equation (141) generalizes to the superfield equation
\[ \Pi^L_{\mu} \partial_{\mu} S_L = m S^*_L, \quad (145) \]
or
\[ \Pi_{\mu} p^\mu S_L = -im S^*_L. \quad (146) \]
Now consider the supersymmetry transformation
\[ \delta S_L = (\xi^\alpha \Delta_\alpha + \bar{\xi}^\dot{\alpha} \bar{\Delta}^{\dot{\alpha}})S_L = \Xi S_L. \] (147)

This transformation commutes with the operator \( \Pi^L_\mu \partial_\mu \) so that
\[ \Pi^L_\mu \partial_\mu (S_L + \delta S_L) = m(S_L + \delta S_L)^* \] (148)

If \( \psi_L \) is a left handed quark and \( B^\alpha(x) \) an antidiquark with the same mass as the quark, equation (147) provides a relativistic form of the quark antidiquark symmetry which is in fact broken by the quark-diquark mass difference. The scalar supersymmetric potential is introduced through \( m \rightarrow m + V_\text{s} \) as before and the equation (145) remains supersymmetric. By means of this formalism, it is possible to reformulate the treatments given in the earlier sections in first order relativistic form.

To write equations in the first order form, we consider \( V \) and \( \Phi \) given in terms of the boson fields by
\[ V = i \gamma_\mu V_\mu - \frac{1}{2} \sigma_{\mu\nu} \] (149)
and
\[ \Phi = i \gamma_5 \phi + i \gamma_5 \gamma_\mu \phi_\mu. \] (150)

In the Majorana representation
\[ V^* = -V, \quad \text{and} \quad \Phi = \Phi^*. \] (151)

We now define the left and right handed component fields by
\[ V_L = \frac{1 - \gamma_5}{2} V, \quad \text{and} \quad V_R = \frac{1 + \gamma_5}{2} V \text{ so that} \]
\[ V_R = -V_L^* \] (152)
with
\[ V_L = i \frac{1 - \gamma_5}{2} \gamma_\mu V_\mu - \frac{1 - \gamma_5}{2} \frac{1}{2} \sigma_{\mu\nu} V_{\mu\nu} \] (153)
and using equation (135) we have
\[ V_L = i \Pi^L_\mu V_\mu - \frac{1 - \gamma_5}{2} \frac{1}{2} \sigma_{\mu\nu} V_{\mu\nu}. \] (154)

Noting that for any \( a_\mu \) and \( b_\nu \)
\[ \Pi_\mu \Pi^*_\nu a_\mu b_\nu = \frac{1 - \gamma_5}{2} \gamma_\mu \frac{1 + \gamma_5}{2} \gamma_\nu a_\mu b_\nu = \frac{1 - \gamma_5}{2} \gamma_\mu \gamma_\nu a_\mu b_\nu \] (155)
so that using the definition of \( \sigma_{\mu\nu} \) we have
\[ \frac{1}{2} (\Pi_\mu \Pi^*_\nu - \Pi^*_\nu \Pi_\mu) = \frac{1 - \gamma_5}{2} i \sigma_{\mu\nu}. \] (156)
Incorporating this in equation (155) leads to

\[ V_L = i\Pi^L_\mu V_\mu + \frac{i}{2}(\Pi_\mu \Pi^*_{\nu} - \Pi_\nu \Pi^*_{\mu})V_{\mu\nu}. \]  

(158)

Letting \( \Sigma^L_{\mu\nu} = \Pi_\mu \Pi^*_{\nu} - \Pi_\nu \Pi^*_{\mu} \), equation (158) becomes

\[ V_L = i\Pi^L_\mu V_\mu + \frac{i}{2}\Sigma^L_{\mu\nu}V_{\mu\nu}. \]  

(159)

So, we now have a first order equation

\[ \Pi^L_\mu \partial_\mu V_R = mV_L. \]  

(160)

Also,

\[ \Pi^R_\mu \partial_\mu V_L = mV_R. \]  

(161)

Therefore

\[ \Pi^L_\mu \Pi^R_\mu \partial_\mu \partial_\nu V_L = m\Pi^L_\mu \partial_\mu V_R = m^2V_L \]  

(162)

which after substitution of equation (160) gives

\[ \Box V_L = m^2V_L. \]  

(163)

Similarly, for the \( \Phi \) part we can write

\[ \Pi^L_\mu \partial_\mu \Phi_R = m\Phi_L \]  

(164)

again, as before

\[ \Phi_L = \frac{1 - \gamma_5}{2}\Phi, \quad \text{and} \quad \Phi_R = \frac{1 + \gamma_5}{2}\Phi \]  

(165)

with

\[ \Phi_R = \Phi^*_L. \]  

(166)

We can then repeat the above procedure for the \( \Phi \) fields.

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