Abstract

We construct a hierarchy of regular languages such that the current language in the hierarchy can be accepted by 1-way quantum finite automata with a probability smaller than the corresponding probability for the preceding language in the hierarchy. These probabilities converge to $\frac{1}{2}$.

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1 Introduction

Quantum computation is a most challenging project involving research both by physicists and computer scientists. The principles of quantum computation differ from the principles of classical computation very much. The classical computation is based on classical mechanics while quantum computation attempts to exploit phenomena specific to quantum physics.

One of features of quantum mechanics is that a quantum process can be in a combination (called superposition) of several states and these several states can interact one with another. A computer scientist would call this a massive parallelism. This possibility of massive parallelism is very important for Computer Science. In 1982, Nobel prize winner physicist Richard Feynman (1918-1988) asked what effects the principles of quantum mechanics can have on computation[Fe 82]. An exact simulation of quantum processes demands exponential running time. Therefore, there may be other computations which are performed nowadays by classical computers but might be simulated by quantum processes in much less time.

R.Feynman’s influence was (and is) so high that rather soon this possibility was explored both theoretically and practically. David Deutsch[De 89] introduced quantum Turing machines, quantum physical counterparts of probabilistic Turing machines. He conjectured that they may be more efficient that classical Turing machines. He also showed the existence of a universal quantum Turing machine. This construction was subsequently improved by Bernstein and Vazirani[BV 97] and Yao[Ya 93].

Quantum Turing machines might have remained relatively unknown but two events caused a drastic change. First, Peter Shor[Sh 97] invented surprising polynomial-time quantum algorithms for computation of discrete logarithms and for factorization of integers. Second, unusual quantum circuits having no classical counterparts (such as quantum bit teleportation) have been physically implemented. Hence, there is a chance that universal quantum computers may be built. Moreover, since the modern public-key cryptography is based on intractability of discrete logarithms and factorization of integers, building a quantum computer implies building a code-breaking machine.

In this paper, we consider quantum finite automata (QFAs), a different model of quantum computation. This is a simpler model than quantum Turing machines and it may be simpler to implement.

Quantum finite automata have been studied in[AF 98, BP 99, KW 97, MC 97]. Surprisingly, QFAs do not generalize deterministic finite automata.
Their capabilities are incomparable. QFAs can be exponentially more space-efficient [AF 98]. However, there are regular languages that cannot be recognized by quantum finite automata [KW 97].

This weakness is caused by reversibility. Any quantum computation is performed by means of unitary operators. One of the simplest properties of these operators shows that such a computation is reversible. The result always determines the input uniquely. It may seem to be a very strong limitation. Luckily, for unrestricted quantum algorithms (for instance, for quantum Turing machines) this is not so. It is possible to embed any irreversible computation in an appropriate environment which makes it reversible [Be 89]. For instance, the computing agent could keep the inputs of previous calculations in successive order. Quantum finite automata are more sensitive to the reversibility requirement.

If the probability with which a QFA is required to be correct decreases, the set of languages that can be recognized increases. In particular [AF 98], there are languages that can be recognized with probability 0.68 but not with probability 7/9. In this paper, we extend this result by constructing a hierarchy of languages in which each next language can be recognized with a smaller probability than the previous one.

2 Preliminaries
2.1 Basics of quantum computation

To explain the difference between classical and quantum mechanical world, we first consider one-bit systems. A classical bit is in one of two classical states true and false. A probabilistic counterpart of the classical bit can be true with a probability $\alpha$ and false with probability $\beta$, where $\alpha + \beta = 1$. A quantum bit (qubit) is very much like to it with the following distinction. For a qubit $\alpha$ and $\beta$ can be arbitrary complex numbers with the property $||\alpha||^2 + ||\beta||^2 = 1$. If we observe a qubit, we get true with probability $||\alpha||^2$ and false with probability $||\beta||^2$, just like in probabilistic case. However, if we modify a quantum system without observing it (we will explain what this means), the set of transformations that one can perform is larger than in the probabilistic case. This is where the power of quantum computation comes from.

More generally, we consider quantum systems with $m$ basis states. We denote the basis states $|q_1\rangle$, $|q_2\rangle$, ..., $|q_m\rangle$. Let $\psi$ be a linear combination of
them with complex coefficients

\[ \psi = \alpha_1 |q_1\rangle + \alpha_2 |q_2\rangle + \ldots + \alpha_m |q_m\rangle. \]

The \(l_2\) norm of \(\psi\) is

\[ \|\psi\| = \sqrt{|\alpha_1|^2 + |\alpha_2|^2 + \ldots + |\alpha_m|^2}. \]

The state of a quantum system can be any \(\psi\) with \(\|\psi\| = 1\). \(\psi\) is called a superposition of \(|q_1\rangle, \ldots, |q_m\rangle\). \(\alpha_1, \ldots, \alpha_m\) are called amplitudes of \(|q_1\rangle, \ldots, |q_m\rangle\). We use \(l_2(Q)\) to denote the vector space consisting of all linear combinations of \(|q_1\rangle, \ldots, |q_m\rangle\).

Allowing arbitrary complex amplitudes is essential for physics. However, it is not important for quantum computation. Anything that can be computed with complex amplitudes can be done with only real amplitudes as well. This was shown for quantum Turing machines in [BV 93] and the same proof works for QFAs. However, it is important that negative amplitudes are allowed. For this reason, we assume that all amplitudes are (possibly negative) reals.

There are two types of transformations that can be performed on a quantum system. The first type are unitary transformations. A unitary transformation is a linear transformation \(U\) on \(l_2(Q)\) that preserves \(l_2\) norm. (This means that any \(\psi\) with \(\|\psi\| = 1\) is mapped to \(\psi'\) with \(\|\psi'\| = 1\).)

Second, there are measurements. The simplest measurement is observing \(\psi = \alpha_1 |q_1\rangle + \alpha_2 |q_2\rangle + \ldots + \alpha_m |q_m\rangle\) in the basis \(|q_1\rangle, \ldots, |q_m\rangle\). It gives \(|q_i\rangle\) with probability \(\alpha_i^2\). (\(\|\psi\| = 1\) guarantees that probabilities of different outcomes sum to 1.) After the measurement, the state of the system changes to \(|q_i\rangle\) and repeating the measurement gives the same state \(|q_i\rangle\).

In this paper, we also use partial measurements. Let \(Q_1, \ldots, Q_k\) be pairwise disjoint subsets of \(Q\) such that \(Q_1 \cup Q_2 \cup \ldots \cup Q_k = Q\). Let \(E_j\), for \(j \in \{1, \ldots, k\}\), denote the subspace of \(l_2(Q)\) spanned by \(|q_j\rangle\), \(j \in Q_i\). Then, a partial measurement w.r.t. \(E_1, \ldots, E_k\) gives the answer \(\psi \in E_j\) with probability \(\sum_{i \in Q_j} |\alpha_i|^2\). After that, the state of the system collapses to the projection of \(\psi\) to \(E_j\). This projection is \(\psi_j = \sum_{i \in Q_j} \alpha_i |q_i\rangle\).

2.2 Quantum finite automata

Quantum finite automata were introduced twice. First this was done by C. Moore and J.P. Crutchfield [MC 97]. Later in a different and non-equivalent way these automata were introduced by A. Kondacs and J. Watrous [KW 97].

\(^1\)For unknown reason, this proof does not appear in [BV 97].
The first definition just mimics the definition of 1-way probabilistic finite automata only substituting stochastic matrices by unitary ones. We use a more elaborated definition [KW 97].

A QFA is a tuple \( M = (Q; \Sigma; V; q_0; Q_{acc}; Q_{rej}) \) where \( Q \) is a finite set of states, \( \Sigma \) is an input alphabet, \( V \) is a transition function, \( q_0 \in Q \) is a starting state, and \( Q_{acc} \subset Q \) and \( Q_{rej} \subset Q \) are sets of accepting and rejecting states. The states in \( Q_{acc} \) and \( Q_{rej} \) are called halting states and the states in \( Q_{non} = Q - (Q_{acc} \cup Q_{rej}) \) are called non halting states. \( \kappa \) and \( $ \) are symbols that do not belong to \( \Sigma \). We use \( \kappa \) and \( $ \) as the left and the right endmarker, respectively. The working alphabet of \( M \) is \( \Gamma = \Sigma \cup \{\kappa; $\} \).

The transition function \( V \) is a mapping from \( \Gamma \times l_2(Q) \) to \( l_2(Q) \) such that, for every \( a \in \Gamma \), the function \( V_a : l_2(Q) \to l_2(Q) \) defined by \( V_a(\psi) = V(a, \psi) \) is a unitary transformation.

The computation of a QFA starts in the superposition \( |q_0\rangle \). Then transformations corresponding to the left endmarker \( \kappa \), the letters of the input word \( x \) and the right endmarker \( $ \) are applied. The transformation corresponding to \( a \in \Gamma \) consists of two steps.

1. First, \( V_a \) is applied. The new superposition \( \psi' \) is \( V_a(\psi) \) where \( \psi \) is the superposition before this step.
2. Then, \( \psi' \) is observed with respect to \( E_{acc}, E_{rej}, E_{non} \) where \( E_{acc} = \text{span}\{|q\rangle : q \in Q_{acc}\} \), \( E_{rej} = \text{span}\{|q\rangle : q \in Q_{rej}\} \), \( E_{non} = \text{span}\{|q\rangle : q \in Q_{non}\} \) (see section 2.1).

If we get \( \psi' \in E_{acc} \), the input is accepted. If we get \( \psi' \in E_{rej} \), the input is rejected. If we get \( \psi' \in E_{non} \), the next transformation is applied.

We regard these two transformations as reading a letter \( a \). We use \( V'_a \) to denote the transformation consisting of \( V_a \) followed by projection to \( E_{non} \). This is the transformation mapping \( \psi \) to the non-halting part of \( V_a(\psi) \). We use \( \psi_y \) to denote the non-halting part of QFA’s state after reading the left endmarker \( \kappa \) and the word \( y \in \Sigma^* \).

We compare QFAs with different probabilities of correct answer. This problem was first considered by A. Ambainis and R. Freivalds [AF 98]. The following theorems were proved there:

**Theorem 2.1** Let \( L \) be a language and \( M \) be its minimal automaton. Assume that there is a word \( x \) such that \( M \) contains states \( q_1, q_2 \) satisfying:

1. \( q_1 \neq q_2 \),
2. If \( M \) starts in the state \( q_1 \) and reads \( x \), it passes to \( q_2 \).
3. If $M$ starts in the state $q_2$ and reads $x$, it passes to $q_2$, and

4. $q_2$ is neither "all-accepting" state, nor "all-rejecting" state.

Then $L$ cannot be recognized by a 1-way quantum finite automaton with probability $7/9 + \epsilon$ for any fixed $\epsilon > 0$.

**Theorem 2.2** Let $L$ be a language and $M$ be its minimal automaton. If there is no $q_1, q_2, x$ satisfying conditions of Theorem 2.1 then $L$ can be recognized by a 1-way reversible finite automaton (i.e. $L$ can be recognized by a 1-way quantum finite automaton with probability 1).

**Theorem 2.3** The language $a^*b^*$ can be recognized by a 1-way QFA with the probability of correct answer $p = 0.68...$ where $p$ is the root of $p^3 + p = 1$.

**Corollary 2.1** There is a language that can be recognized by a 1-QFA with probability $0.68...$ but not with probability $7/9 + \epsilon$.

For probabilistic automata, the probability of correct answer can be increased arbitrarily and this property of probabilistic computation is considered as evident. Theorems above show that its counterpart is not true in the quantum world! The reason for that is that the model of QFAs mixes reversible (quantum computation) components with nonreversible (measurements after every step).

In this paper, we consider the best probabilities of acceptance by 1-way quantum finite automata the languages $a^*b^*...z^*$. Since the reason why the language $a^*b^*$ cannot be accepted by 1-way quantum finite automata is the property described in the Theorems 2.1 and 2.2, this new result provides an insight on what the hierarchy of languages with respect to the probabilities of their acceptance by 1-way quantum finite automata may be. We also show a generalization of Theorem 2.3 in a style similar to Theorem 2.2.

## 3 Main results

**Lemma 3.1** For arbitrary real $x_1 > 0$, $x_2 > 0$, ..., $x_n > 0$, there exists a unitary $n \times n$ matrix $M_n(x_1, x_2, ..., x_n)$ with elements $m_{ij}$ such that

\[
m_{11} = \frac{x_1}{\sqrt{x_1^2 + \cdots + x_n^2}}, \quad m_{21} = \frac{x_2}{\sqrt{x_1^2 + \cdots + x_n^2}}, \quad \ldots, \quad m_{n1} = \frac{x_n}{\sqrt{x_1^2 + \cdots + x_n^2}}.
\]
Let \( L_n \) be the language \( a_1^*a_2^*...a_n^* \).

**Theorem 3.1** The language \( L_n \) \((n > 1)\) can be recognized by a 1-way QFA with the probability of correct answer \( p \) where \( p \) is the root of \( p^{\frac{n+1}{n-1}} + p = 1 \) in the interval \([1/2, 1]\).

**Proof:** Let \( m_{ij} \) be the elements of the matrix \( M_k(x_1, x_2, ..., x_k) \) from Lemma \([3]\). We construct a \( k \times (k-1) \) matrix \( T_k(x_1, x_2, ..., x_k) \) with elements \( t_{ij} = m_{i,j+1} \). Let \( R_k(x_1, x_2, ..., x_k) \) be a \( k \times k \) matrix with elements \( r_{ij} = \frac{p_1 x_1 x_j}{x_1 + ... + x_k} \) and \( I_k \) be the \( k \times k \) identity matrix.

For fixed \( n \), let \( p_n \in [1/2, 1] \) satisfy \( p_n^{\frac{n+1}{n-1}} + p_n = 1 \) and \( p_k (1 \leq k < n) = \frac{k+1}{kn} - p_n^{\frac{k-1}{n-1}} \). It is easy to see that \( p_1 + p_2 + ... + p_n = 1 \) and

\[
1 - \frac{p_n(p_k + ... + p_n)^2}{(p_k - 1 + ... + p_n)^2} = 1 - \frac{p_n p_n^{\frac{2(k-1)}{n-1}}}{p_n^{\frac{2(k-2)}{n-1}}} = 1 - p_n^{\frac{n+1}{n-1}} = p_n. \quad (1)
\]

Now we describe a 1-way QFA accepting the language \( L_n \).

The automaton has \( 2n \) states: \( q_1, q_2, ..., q_n \) are non halting states, \( q_{n+1}, q_{n+2}, ..., q_{2n-1} \) are rejecting states and \( q_{2n} \) is an accepting state. The transition function is defined by unitary block matrices

\[
V_k = \begin{pmatrix}
M_n(\sqrt{p_1}, \sqrt{p_2}, ..., \sqrt{p_n}) & 0 \\
0 & I_n
\end{pmatrix},
\]

\[
V_{a_1} = \begin{pmatrix}
R_n(\sqrt{p_1}, \sqrt{p_2}, ..., \sqrt{p_n}) & T_n(\sqrt{p_1}, \sqrt{p_2}, ..., \sqrt{p_n}) & 0 \\
T_n^T(\sqrt{p_1}, \sqrt{p_2}, ..., \sqrt{p_n}) & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
V_{a_2} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & R_{n-1}(\sqrt{p_2}, ..., \sqrt{p_n}) & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & T_{n-1}^T(\sqrt{p_2}, ..., \sqrt{p_n}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
V_{a_k} = \begin{pmatrix}
0 & 0 & I_{k-1} & 0 & 0 \\
0 & R_{n+1-k}(\sqrt{p_k}, ..., \sqrt{p_n}) & 0 & T_{n+1-k}(\sqrt{p_k}, ..., \sqrt{p_n}) & 0 \\
I_{k-1} & 0 & 0 & 0 & 0 \\
0 & T_{n+1-k}^T(\sqrt{p_k}, ..., \sqrt{p_n}) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
\ldots
\]
\[
V_{a_n} = \begin{pmatrix}
0 & 0 & I_{n-1} & 0 \\
0 & 1 & 0 & 0 \\
I_{n-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
\[
V_S = \begin{pmatrix}
0 & I_n \\
I_n & 0
\end{pmatrix}.
\]

Case 1. The input is \(\kappa a_1^* a_2^* \ldots a_n^* S\).

The starting superposition is \(|q_1\rangle\). After reading the left endmarker the superposition becomes \(\sqrt{p_1} |q_1\rangle + \sqrt{p_2} |q_2\rangle + \ldots + \sqrt{p_n} |q_n\rangle\) and after reading \(a_1^*\) the superposition remains the same.

If the input contains \(a_k\) then reading the first \(a_k\) changes the non-halting part of the superposition to \(\sqrt{p_k} |q_k\rangle + \ldots + \sqrt{p_n} |q_n\rangle\) and after reading all the rest of \(a_k\) the non-halting part of the superposition remains the same.

Reading the right endmarker maps \(|q_n\rangle\) to \(|q_{2n}\rangle\). Therefore, the superposition after reading it contains \(\sqrt{p_n} |q_{2n}\rangle\). This means that the automaton accepts with probability \(p_n\) because \(q_{2n}\) is an accepting state.

Case 2. The input is \(\kappa a_1^* a_2^* \ldots a_k a_k a_m \ldots (k > m)\).

After reading the last \(a_k\) the non-halting part of the superposition is \(\sqrt{p_k} |q_k\rangle + \ldots + \sqrt{p_n} |q_n\rangle\). Then reading \(a_m\) changes the non-halting part to \(\frac{p_m(p_k + \ldots + p_n)^2}{p_m + \ldots + p_n^2} |q_m\rangle + \ldots + \frac{p_n(p_k + \ldots + p_n)^2}{p_m + \ldots + p_n^2} |q_n\rangle\). This means that the automaton accepts with probability \(\leq \frac{p_n(p_k + \ldots + p_n)^2}{p_m + \ldots + p_n^2}\) and rejects with probability at least

\[
1 - \frac{p_n(p_k + \ldots + p_n)^2}{(p_m + \ldots + p_n)^2} = \frac{p_n}{(p_{k-1} + \ldots + p_n)^2} = p_n
\]

that follows from (1). \(\Box\)

**Corollary 3.1** The language \(L_n\) can be recognized by a 1-way QFA with the probability of correct answer at least \(\frac{1}{2} + \varepsilon n\), for a constant \(c\).

**Proof:** By resolving the equation \(p_n^{a+1} + p = 1\), we get \(p = \frac{1}{2} + \Theta(n)\). \(\Box\)
Theorem 3.2 The language $L_n$ cannot be recognized by a 1-way QFA with probability greater than $p$ where $p$ is the root of

$$(2p - 1) = \frac{2(1 - p)}{n - 1} + 4\sqrt{\frac{2(1 - p)}{n - 1}}$$

in the interval $[1/2, 1]$.

Proof: Assume we are given a 1-way QFA $M$. We show that, for any $\epsilon > 0$, there is a word such that the probability of correct answer is less than $p + \epsilon$.

Lemma 3.2 [AF98] Let $x \in \Sigma^+$. There are subspaces $E_1$, $E_2$ such that $E_{\text{non}} = E_1 \oplus E_2$ and

(i) If $\psi \in E_1$, then $V_x(\psi) \in E_1$,

(ii) If $\psi \in E_2$, then $\|V'_x(\psi)\| \to 0$ when $k \to \infty$.

We use $n - 1$ such decompositions: for $x = a_2, x = a_3, \ldots, x = a_n$. The subspaces $E_1, E_2$ corresponding to $x = a_m$ are denoted $E_{m,1}$ and $E_{m,2}$.

Let $m \in \{2, \ldots, n\}$, $y \in a_1^*a_2^* \ldots a_{m-1}^*$. Remember that $\psi_y$ denotes the superposition after reading $y$ (with observations w.r.t. $E_{\text{non}} \oplus E_{\text{acc}} \oplus E_{\text{rej}}$ after every step). We express $\psi_y$ as $\psi^1_y + \psi^2_y$, $\psi^1_y \in E_{m,1}$, $\psi^2_y \in E_{m,2}$.

Case 1. $\|\psi^2_y\|^2 \leq \sqrt{\frac{2(1 - p)}{n - 1}}$ for some $m \in \{2, \ldots, n\}$ and $y \in a_1^* \ldots a_{m-1}^*$.

Let $i > 0$. Then, $ya_{m-1} \in L_n$ but $ya^i_{m-1}a_{m-1} \notin L_n$. Consider the distributions of probabilities on $M$’s answers “accept” and “reject” on $ya_{m-1}$ and $ya^i_{m-1}a_{m-1}$. If $M$ recognizes $L_n$ with probability $p + \epsilon$, it must accept $ya_{m-1}$ with probability at least $p + \epsilon$ and reject it with probability at most $1 - p - \epsilon$. Also, $ya^i_{m-1}a_{m-1}$ must be rejected with probability at least $p + \epsilon$ and accepted with probability at most $1 - p - \epsilon$. Therefore, both the probabilities of accepting and the probabilities of rejecting must differ by at least

$$(p + \epsilon) - (1 - p - \epsilon) = 2p - 1 + 2\epsilon.$$  

This means that the variational distance between two probability distributions (the sum of these two distances) must be at least $2(2p - 1) + 4\epsilon$. We show that it cannot be so large.

First, we select an appropriate $i$. Let $k$ be so large that $\|V'_{a^i_{m-1}}(\psi^2_y)\| \leq \delta$ for $\delta = \epsilon/4$. $\psi^1_y, V'_{a^i_{m-1}}(\psi^1_y), V'_{a^i_{m-1}}(\psi^1_y), \ldots$ is a bounded sequence in a finite-dimensional space. Therefore, it has a limit point and there are $i, j$ such that

$$\|V'_{a^i_{m-1}}(\psi^1_y) - V'_{a^j_{m-1}}(\psi^1_y)\| < \delta.$$
We choose $i, j$ so that $i > k$.

The difference between the two probability distributions comes from two sources. The first source is the difference between $\psi_y$ and $\psi_{ya_i^m}$ (the states of $M$ before reading $a_{i-1}^m$). The second source is the possibility of $M$ accepting while reading $a_i^m$ (the only part that is different in the two words). We bound each of them.

The difference $\psi_y - \psi_{ya_i^m}$ can be partitioned into three parts.

$$\psi_y - \psi_{ya_i^m} = (\psi_y - \psi^1_y) + (\psi^1_y - V_{a_i^m}(\psi^1_y)) + (V_{a_i^m}(\psi^1_y) - \psi_{ya_i^m}). \quad (3)$$

The first part is $\psi_y - \psi^1_y$ and $\|\psi^1_y\| \leq \sqrt{\frac{2(1-p)}{n-1}}$. The second and the third parts are both small. For the second part, notice that $V_{a_i^m}$ is unitary on $E_{m,1}$ (because $V_{a_m}$ is unitary and $V_{a_m}(\psi)$ does not contain halting components for $\psi \in E_{m,1}$). Hence, $V_{a_i^m}$ preserves distances on $E_{m,1}$ and 

$$\|\psi^1_y - V_{a_i^m}(\psi^1_y)\| = \|V_{a_i^m}(\psi^1_y) - V_{a_{i+1}^m}(\psi^1_y)\| < \delta$$

For the third part of (3), remember that $\psi_{ya_i^m} = V_{a_i^m}(\psi_y)$. Therefore, 

$$\psi_{ya_i^m} - V_{a_i^m}(\psi^1_y) = V_{a_i^m}(\psi_y) - V_{a_i^m}(\psi^1_y) = V_{a_i^m}(\psi_y - \psi^1_y) = V_{a_i^m}(\psi^2_y)$$

and $\|\psi^2_{ya_i^m}\| \leq \delta$ because $i > k$. Putting all three parts together, we get

$$\|\psi_y - \psi_{ya_i^m}\| \leq \|\psi_y - \psi^1_y\| + \|\psi^1_y - \psi^1_{ya_i^m}\| + \|\psi^1_{ya_i^m} - \psi_{ya_i^m}\| \leq \sqrt{\frac{2(1-p)}{n-1}} + 2\delta.$$

**Lemma 3.3** [BV 97] Let $\psi$ and $\phi$ be such that $\|\psi\| \leq 1$, $\|\phi\| \leq 1$ and $\|\psi - \phi\| \leq \epsilon$. Then the total variational distance resulting from measurements of $\phi$ and $\psi$ is at most $4\epsilon$.

This means that the difference between any probability distributions generated by $\psi_y$ and $\psi_{ya_i^m}$ is at most

$$4\sqrt{\frac{2(1-p)}{n-1}} + 8\delta.$$

In particular, this is true for the probability distributions obtained by applying $V_{a_{m-1}}$, $V_S$ and the corresponding measurements to $\psi_y$ and $\psi_{ya_i^m}$. 

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The probability of $M$ halting while reading $a_m^i$ is at most $\|\psi_m^2\|^2 = \frac{2(1-p)}{n-1}$. Adding it increases the variational distance by at most $\frac{2(1-p)}{n-1}$. Hence, the total variational distance is at most

$$2\left(1 - p\right) \frac{n-1}{n-1} + 4\sqrt{\frac{2\left(1 - p\right)}{n-1}} + 8\delta = 2\left(1 - p\right) \frac{n-1}{n-1} + \frac{2(1-p)}{n-1} + 2\epsilon.$$ 

By definition of $p$, this is the same as $(2p - 1) + 2\epsilon$. However, if $M$ distinguishes $y$ and $ya_m^i$ correctly, the variational distance must be at least $(2p - 1) + 4\epsilon$. Hence, $M$ does not recognize one of these words correctly.

**Case 2.** $\|\psi_p^2\| > \sqrt{\frac{2\left(1 - p\right)}{n-1}}$ for every $m \in \{2, \ldots, n\}$ and $y \in a_1^* \ldots a_{m-1}^*$. We define a sequence of words $y_1, y_2, \ldots, y_m \in a_1^* \ldots a_n^*$. Let $y_1 = a_1$ and $y_k = y_{k-1}a_k^i$ for $k \in \{2, \ldots, n\}$ where $i_k$ is such that

$$\|V_{a_k^i}^\prime (\psi_{y_{k-1}}^2)\| \leq \sqrt{\frac{\epsilon}{n-1}}.$$ 

The existence of $i_k$ is guaranteed by (ii) of Lemma 3.2.

We consider the probability that $M$ halts on $y_n = a_1a_2^i \ldots a_n^i$ before seeing the right endmarker. Let $k \in \{2, \ldots, n\}$. The probability of $M$ halting while reading the $a_k^i$ part of $y_n$ is at least

$$\|\psi_{y_{k-1}}^2\|^2 - \|V_{a_k^i}^\prime (\psi_{y_{k-1}}^2)\|^2 > \frac{2(1-p)}{n-1} - \frac{\epsilon}{n-1}.$$ 

By summing over all $k \in \{2, \ldots, n\}$, the probability that $M$ halts on $y_n$ is at least

$$(n-1) \left(\frac{2(1-p)}{n-1} - \frac{\epsilon}{n-1}\right) = 2(1-p) - \epsilon.$$ 

This is the sum of the probability of accepting and the probability of rejecting. Hence, one of these two probabilities must be at least $(1-p) - \epsilon/2$. Then, the probability of the opposite answer on any extension of $y_n$ is at most $1 - (1-p - \epsilon/2) = p + \epsilon/2$. However, $y_n$ has both extensions that are in $L_n$ and extensions that are not. Hence, one of them is not recognized with probability $p + \epsilon$. □

By solving the equation (2), we get

**Corollary 3.2** $L_n$ cannot be recognized with probability greater than $\frac{1}{2} + \frac{3}{\sqrt{n-1}}$. 

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Proof: The right-hand side of (2) is at most \( \frac{1}{n-1} + 4\sqrt{\frac{1}{n-1}} \) because \( p \geq 1/2 \) and, hence, \( 1 - p \leq 1/2 \). This implies

\[
2p - 1 \leq \frac{1}{n-1} + 4\sqrt{\frac{1}{n-1}},
\]

\[
p \leq \frac{1}{2} + 2\sqrt{\frac{1}{n-1}} + \frac{1}{2(n-1)} \leq \frac{1}{2} + 3\sqrt{\frac{1}{n-1}},
\]

and \( L_n \) cannot be recognized with probability greater than \( p \) by Theorem 3.2.

Let \( n_1 = 2 \) and \( n_k = \frac{9n_k^2 - 1}{c^2} + 1 \) for \( k > 1 \) (where \( c \) is the constant from Theorem 3.1). Also, define \( p_k = \frac{1}{2} + \frac{c}{n_k} \). Then, Corollaries 3.1 and 3.2 imply

**Theorem 3.3** For every \( k > 1 \), \( L_{n_k} \) can be recognized with by a 1-way QFA with the probability of correct answer \( p_k \) but cannot be recognized with the probability of correct answer \( p_{k-1} \).

Proof: By Corollary 3.1, \( L_{n_k} \) can be recognized with probability \( \frac{1}{2} + \frac{c}{n_k} = p_k \).

On the other hand, by Corollary 3.2, \( L_{n_k} \) cannot be recognized with probability \( \frac{1}{2} + \frac{3}{\sqrt{n_k-1}} \). The definition of \( n_k \) implies \( n_k - 1 = \frac{9n_k^2 - 1}{c^2} \), \( \sqrt{n_k-1} = \frac{3n_k-1}{c} \).

\[
\frac{1}{2} + \frac{3}{\sqrt{n_k-1}} = \frac{1}{2} + \frac{c}{n_k-1} = p_{k-1}.
\]

Thus, we have constructed a sequence of languages \( L_{n_1}, L_{n_2}, \ldots \) such that, for each \( L_{n_k} \), the probability with which \( L_{n_k} \) can be recognized by a 1-way QFA is smaller than for \( L_{n_{k-1}} \).

Our final theorem is a counterpart of Theorem 2.2. It generalizes Theorem 2.3.

**Theorem 3.4** Let \( L \) be a language and \( M \) be its minimal automaton. If there is no \( q_1, q_2, q_3, x, y \) such that

1. the states \( q_1, q_2, q_3 \) are pairwise different,
2. If \( M \) starts in the state \( q_1 \) and reads \( x \), it passes to \( q_2 \),
3. If \( M \) starts in the state \( q_2 \) and reads \( x \), it passes to \( q_2 \), and
4. If $M$ starts in the state $q_2$ and reads $y$, it passes to $q_3$.

5. If $M$ starts in the state $q_3$ and reads $y$, it passes to $q_3$.

6. both $q_2$ and $q_3$ are neither "all-accepting" state, nor "all-rejecting" state,

then $L$ can be recognized by a 1-way quantum finite automaton with probability $p = 0.68$....

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