Abstract

We discuss the procedure of Rieffel induction of representations in the framework of formal deformation quantization of Poisson manifolds. We focus on the central role played by algebraic notions of complete positivity.

1 Introduction

In this note we describe how various concepts and constructions in the theory of $C^*$-algebras carry over to the purely algebraic setting of formal de-
formation quantization of Poisson manifolds. Our discussion centers around
the construction of induced representations, due to Rieffel in the framework
of $C^*$-algebras [15], and its interplay with notions of complete positivity.
Although this note is mostly expository, we highlight some aspects of the
theory that we have not made explicit before.

Deformation quantization [1] is a procedure to construct algebras of
quantum observables associated with classical systems. More precisely, a
classical phase space is a Poisson manifold $(M, \{\cdot, \cdot\})$ and its quantization is
a formal associative deformation $\star$, also called a star product, of the classi-
cal observable algebra $C^\infty(M)$ in the direction of the Poisson bracket. Here
$C^\infty(M)$ denotes the algebra of complex-valued smooth functions on $M$ and
$\star$ is a $\mathbb{C}[[\lambda]]$-bilinear associative multiplication on $C^\infty(M)[[\lambda]]$ given by

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f, g), \quad (1)$$

where $C_0(f, g) = fg, C_1(f, g) = C_1(g, f) = i\{f, g\}, 1 \star f = f = f \star 1$ and all $C_r$
are bidifferential operators. The formal parameter $\lambda$ satisfies $\lambda = \lambda$ and plays
the role of Planck’s constant $\hbar$. We require $\star$ to be a Hermitian star product,
in the sense that $\overline{f \star g} = \overline{g} \star \overline{f}$, so that the $\mathbb{C}[[\lambda]]$-algebra $(C^\infty(M)[[\lambda]], \star)$
acquires a $\star$-involution given by pointwise complex conjugation.

Other quantum mechanical concepts can be defined in deformation quan-
tization analogously to the usual $C^*$-algebraic approach to quantum theory.
The starting point is to regard $\mathbb{R}[[\lambda]]$ as an ordered ring by considering
$\sum_{r=0}^{\infty} \lambda^r a_r$ to be positive if $a_{r_0} > 0$, where $a_{r_0}$ is the first nonzero coeffi-
cient. Then a $\mathbb{C}[[\lambda]]$-linear functional

$$\omega : C^\infty(M)[[\lambda]] \rightarrow \mathbb{C}[[\lambda]] \quad (2)$$

is called positive if $\omega(\overline{\mathcal{F}} \star f) \geq 0$ for all $f \in C^\infty(M)[[\lambda]]$; a state is a positive linear functional such that $\omega(1) = 1$, and the value $\omega(f)$ is interpreted as the expectation value
of the observable $f$ in the state $\omega$.

To implement the idea of superposition of states, one needs a notion of
representation in deformation quantization. Given a Hermitian star prod-
cuct, a representation consists of a pre-Hilbert space $\mathcal{H}$ over $\mathbb{C}[[\lambda]]$ (here
one uses the order structure of $\mathbb{R}[[\lambda]]$ for the definition of positive definite
$\mathbb{C}[[\lambda]]$-valued inner products) on which $(C^\infty(M)[[\lambda]], \star)$ acts by adjointable
operators. Many physically interesting examples can be found in [4], see [18]
for a recent review.

As a next step, following the theory of $C^*$-algebras, one is led to the con-
stuction of induced representations. Recall that if $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras,
the procedure of Rieffel induction consists of constructing representations of \( \mathcal{B} \) from representations of \( \mathcal{A} \) with the aid of a suitable \( (\mathcal{B}, \mathcal{A}) \)-bimodule \( \mathcal{B}\mathcal{E}_\mathcal{A} \) possessing an \( \mathcal{A} \)-valued inner product \( \langle \cdot, \cdot \rangle_\mathcal{A} \). For each \( \ast \)-representation of \( \mathcal{A} \) on a Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \), one considers the tensor product \( \mathcal{E} \otimes_\mathcal{A} \mathcal{H} \) over \( \mathcal{A} \) and the natural left action of \( \mathcal{B} \) on it. In order to turn this tensor product into a Hilbert space carrying a representation of \( \mathcal{B} \), the key point is that one can combine \( \langle \cdot, \cdot \rangle_\mathcal{A} \) and \( \langle \cdot, \cdot \rangle \) to produce an inner product on \( \mathcal{E} \otimes_\mathcal{A} \mathcal{H} \) uniquely defined by

\[
(x \otimes \phi, y \otimes \psi) \mapsto \langle \phi, \langle x, y \rangle_\mathcal{A} \cdot \psi \rangle,
\]

where \( x, y \in \mathcal{B}\mathcal{E}_\mathcal{A}, \phi, \psi \in \mathcal{H} \). An important point of this construction where specific properties of \( C^* \)-algebras must come into play is showing that the inner product defined by (3) is positive, see e.g. [14] for a detailed discussion.

Understanding the positivity of inner products of the form (3) in purely algebraic versions of Rieffel induction is the heart of this note, see also [8] [10]. We will discuss Rieffel induction in the framework of \( \ast \)-algebras over ordered rings in Section 3. Applications to deformation quantization are presented in Section 4. The last section contains a brief discussion on strong Morita equivalence, a notion closely related to algebraic Rieffel induction.

2 The general framework of \( \ast \)-algebras over ordered rings

In order to give a unified treatment of \( C^* \)-algebras and the \( \ast \)-algebras over \( \mathbb{C}[[\lambda]] \) defined by Hermitian star products, we work in the following general algebraic setting, see [10] for details: we consider \( \ast \)-algebras \( \mathcal{A} \) over a ring of the form \( \mathbb{C} = \mathbb{R}(i) \); here \( \mathbb{R} \) is an ordered ring, like e.g. \( \mathbb{R} \) or \( \mathbb{R}[[\lambda]] \), so \( \mathbb{C} \) is a ring extension of \( \mathbb{R} \) by a square root of \(-1\).

Along the same lines of the discussion in the introduction, we define a \( \mathbb{C} \)-linear functional \( \omega : \mathcal{A} \to \mathbb{C} \) to be positive if \( \omega(a^*a) \geq 0 \) for all \( a \in \mathcal{A} \), which makes sense since \( \mathbb{R} \subseteq \mathbb{C} \) is ordered. An algebra element \( a \in \mathcal{A} \) is called positive if its expectation values are all non-negative, i.e. \( \omega(a) \geq 0 \) for all positive linear functionals \( \omega \). These notions agree with the usual ones, e.g., for \( C^* \)-algebras, and also make sense for Hermitian star products. We denote the set of positive elements by \( \mathcal{A}^+ \). See [16] for more general concepts of positivity in \( O^* \)-algebras and [17] for a comparison between them.

We now pass to representations. A pre-Hilbert space \( \mathcal{H} \) over \( \mathbb{C} \) is a \( \mathbb{C} \)-module with a positive definite inner product \( \langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \), i.e.
\[ \langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}, \langle \phi, \phi \rangle > 0 \text{ for } \phi \neq 0 \text{ and } \langle \cdot, \cdot \rangle \text{ is linear in the second argument.} \]

The adjointable operators from \( \mathcal{H}_1 \) to \( \mathcal{H}_2 \) are defined as \( \mathbb{C} \)-linear maps for which adjoints exist in the usual sense. It is easy to check that when an adjoint exists, it is is unique. Note that, in the case of complex Hilbert spaces, the Hellinger-Toeplitz theorem ensures that the adjointable operators coincide with bounded operators. The adjointable operators \( \mathfrak{B}(\mathcal{H}) \) on a pre-Hilbert space \( \mathcal{H} \) form a \( * \)-algebra in the natural way, and a \( * \)-representation of a \( * \)-algebra \( \mathcal{A} \) over \( \mathbb{C} \) on \( \mathcal{H} \) is a \( * \)-homomorphism \( \pi : \mathcal{A} \longrightarrow \mathfrak{B}(\mathcal{H}) \). When \( \mathcal{A} \) is unital, we assume that \( \pi(1_{\mathcal{A}}) = \text{id}_{\mathcal{H}}. \)

**Example 2.1** If \( \mathcal{A} \) is a \( * \)-algebra over \( \mathbb{C} \), an important class of examples of representations is given by an algebraic version of the GNS construction for \( C^* \)-algebras. Following [4], for each positive linear functional \( \omega : \mathcal{A} \rightarrow \mathbb{C} \), one forms the space \( \mathcal{H}_\omega := \mathcal{A}/J_\omega \), where \( J_\omega \) consists of elements \( a \in \mathcal{A} \) with \( \omega(a^*a) = 0 \). The space \( \mathcal{H}_\omega \) is a pre-Hilbert space with inner product \( \langle \psi_a, \psi_b \rangle := \omega(a^*b) \), where \( \psi_a \) denotes the class of \( a \in \mathcal{A} \) in \( \mathcal{H}_\omega \); the GNS \( * \)-representation of \( \mathcal{A} \) on \( \mathcal{H}_\omega \) is defined by \( \pi(a)\psi_b := \psi_{ab} \).

This construction in deformation quantization gives rise to important formal representations of Hermitian star products, such as the Bargmann-Fock representation of Wick star products, or the Schrödinger representation of Weyl star products on cotangent bundles, see [4] [3].

For a \( * \)-algebra \( \mathcal{A} \) over \( \mathbb{C} \), we define \( * \)-rep\((\mathcal{A}) \) to be the category whose objects are \( * \)-representations of \( \mathcal{A} \) on pre-Hilbert spaces over \( \mathbb{C} \) and with adjointable intertwiners as morphisms. We refer to this category as the representation category (or representation theory) of \( \mathcal{A} \). In these terms, the procedure of Rieffel induction, to be discussed in the next section, can be seen as an explicit construction of functors between representation categories. Functors which establish equivalence of categories of representations will be briefly discussed in the last section.

### 3 Complete positivity and algebraic Rieffel induction

In order to describe Rieffel induction in the algebraic framework of Section 2 we need to consider algebraic analogs of Hilbert \( C^* \)-modules, see e.g. [13]. The reader may consult [10] for details.

Let \( \mathcal{A} \) be a \( * \)-algebra over \( \mathbb{C} \), and let \( \mathcal{E} \) be a (right) \( \mathcal{A} \)-module (we may write \( \mathcal{E}_{\mathcal{A}} \) to stress the \( \mathcal{A} \)-action). An \( \mathcal{A} \)-valued inner product on \( \mathcal{E} \) is a
C-sesquilinear map (linear in the second argument)
\[ \langle \cdot, \cdot \rangle_A : E \times E \rightarrow A, \]  
\[ \text{4) such that } \langle x, y \rangle_A = \langle y, x \rangle_A^* \text{ and } \langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a \text{ for all } x, y \in E \text{ and } a \in A. \]
We call \( \langle \cdot, \cdot \rangle_A \) non-degenerate if \( \langle x, y \rangle_A = 0 \) for all \( x \) implies \( y = 0 \), in which case the pair \((E, \langle \cdot, \cdot \rangle_A)\) is called an inner-product \( A \)-module. The inner product \( \langle \cdot, \cdot \rangle_A \) is called positive if \( \langle x, x \rangle_A \in A^+ \). Finally, \( \langle \cdot, \cdot \rangle_A \) is called strongly non-degenerate if the map \( E \ni x \mapsto \langle x, \cdot \rangle_A \in \text{Hom}_A(E, A) \) is a bijection. Similar definitions hold for left modules (the only difference is that we have \( C \) and \( A \)-linearity in the first argument).

If \( B \) is another \( \ast \)-algebra over \( C \), then a \((B, A)\)-inner-product bimodule is an inner-product \( A \)-module \((E, \langle \cdot, \cdot \rangle_A)\) together with a \( \ast \)-homomorphism \( B \rightarrow \mathcal{B}(E) \), where \( \mathcal{B}(E) \) is the \( \ast \)-algebra of adjointable operators with respect to \( \langle \cdot, \cdot \rangle_A \). Consider an object in \( \ast\text{-}\text{rep}(A) \), i.e., a pre-Hilbert space \((H, \langle \cdot, \cdot \rangle)\) carrying a \( \ast \)-representation of \( A \). In order to obtain an object in \( \ast\text{-}\text{rep}(B) \) from \((E, \langle \cdot, \cdot \rangle_A)\) and \( H \), we follow [15] and consider the algebraic tensor product \( E \otimes_A H \), which carries a left \( B \)-action, equipped with the inner product determined by
\[ (x \otimes \phi, y \otimes \psi) \mapsto \langle \phi, \langle x, y \rangle_A \cdot \psi \rangle, \]  
\[ \text{5) for } x, y \in E \text{ and } \phi, \psi \in H. \] In the framework of \( C^* \)-algebras, one can prove that if \( \langle \cdot, \cdot \rangle_A \) is positive, then so is the induced inner product (5) (see e.g. [13,14]). The following proposition indicates what is algebraically needed in general.

**Proposition 3.1** Let us assume, for simplicity, that \( A \) and \( B \) are unital, and let \((E, \langle \cdot, \cdot \rangle_A)\) be a \((B, A)\)-inner-product bimodule. Then the following are equivalent:

1. The inner product (5) is positive for any \( \ast \)-representation of \( A \).
2. For all \( n \) and all \( x_1, \ldots, x_n \in E \), the matrix \( (\langle x_i, x_j \rangle_A) \) is a positive element in \( M_n(A) \) (viewing \( M_n(A) \) as a \( \ast \)-algebra over \( C \) in the natural way).

For the proof, we need the following simple lemma:

**Lemma 3.2** Let \( A \) be unital. If \( \Omega : M_n(A) \rightarrow C \) is a positive linear functional then there exists a \( \ast \)-representation \((\mathcal{H}, \pi)\) of \( A \) and vectors \( \phi_1, \ldots, \phi_n \in \mathcal{H} \) such that
\[ n\Omega(A) = \sum_{i,j} \langle \phi_i, \pi(a_{ij})\phi_j \rangle \]  
\[ \text{6) for } a_{ij} \in A. \]
where $A = (a_{ij}) \in M_n(A)$. Conversely, for any $^*$-representation $(\mathcal{H}, \pi)$ of $A$ and any choice of vectors $\phi_1, \ldots, \phi_n \in \mathcal{H}$, the right hand side of (6) defines a positive linear functional of $M_n(A)$ (and this defines a positive $\Omega$ if $1/n \in \mathbb{C}$).

**Proof:** This is a simple application of the GNS construction and should be well-known. For the reader’s convenience we outline the proof. Let $E_{ij} \in M_n(A)$ be the elementary matrices with 1 at the $(i,j)$-position and 0 elsewhere. Then $nA = \sum_{i,j,k,l} E_{ji}^* a_{il} E_{kl}$. Now let $(\mathcal{H}_\Omega, \Pi_\Omega)$ be the GNS representation of $M_n(A)$ with respect to $\Omega$. Then define $\phi_i = \sum_j \psi_j E_{ji} \in \mathcal{H}_\Omega$. Clearly $\pi(a) := \Pi_\Omega(a E_{11})$ is a $^*$-representation of $A$ on $\mathcal{H}_\Omega$ and we now have

$$n\Omega(A) = n \langle \psi_1, \Pi_\Omega(A) \psi_1 \rangle = \sum_{i,j} \langle \phi_i, \pi(a_{ij}) \phi_j \rangle.$$ 

The converse statement can be easily checked. \qed

**Proof:** We can now complete the proof of the proposition. Let $A = ((x_i, x_j)) \in M_n(A)$. Using the assumption of (1) and the lemma, we have $\Omega(A) \geq 0$ for all positive linear functionals $\Omega : M_n(A) \rightarrow \mathbb{C}$. The converse implication will follow in much more generality in Theorem 3.4. \qed

An $A$-valued inner product on $\mathcal{E}$ satisfying the condition in (2) is called **completely positive** [10]. If $\langle \cdot, \cdot \rangle_A$ is completely positive, we call $(\mathcal{E}, \langle \cdot, \cdot \rangle_A)$ a **pre-Hilbert $A$-module**; a $(\mathcal{B}, A)$-inner product bimodule for which the $A$-valued inner product is completely positive is called a **pre-Hilbert bimodule**.

**Example 3.3**

1. If $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a pre-Hilbert space over $\mathbb{C}$, then $\langle \cdot, \cdot \rangle$ is automatically completely positive;

2. If $A$ is a $C^*$-algebra, then any positive $A$-valued inner product is completely positive;

3. If $A$ is a $^*$-algebra over $\mathbb{C}$ and $\mathcal{E}$ is the right projective $A$-module $PA^n$, where $P \in M_n(A)$ is a projection, then the restriction of the natural $A$-valued inner product on $A^n$ to $\mathcal{E}$ is completely positive.

4. If $A = C^\infty(M)$, then any positive strongly nondegenerate $A$-valued inner product on a finitely generated projective (f.g.p.) $A$-module is completely positive.

To see why (4) holds, note that it follows from (3) that any f.g.p. module over $A$ can be equipped with a completely positive $A$-valued inner product.
and in the case where \( A = C^\infty(M) \), any two \( A \)-valued inner products on the same f.g.p. module are equivalent. This is because, by Serre-Swan’s theorem, each f.g.p. module \( E \) is given by the space of sections of a complex vector bundle \( E \to M \), and strongly non-degenerate \( A \)-valued inner products on \( E \) correspond to hermitian fibre metrics on \( E \). But any two such metrics on \( E \) are isometric.

With the assumption of complete positivity on inner products, it turns out that Rieffel induction can be carried out in an even broader setting, as we now recall.

Let \( A, B \) and \( D \) be \( * \)-algebras over \( C \) (not necessarily unital), and let \((\_ E_A, \langle \cdot, \cdot \rangle_A^E)\) and \((\_ H_D, \langle \cdot, \cdot \rangle_D)\) be right inner-product bimodules. Let \( \_ E_A \otimes_A \_ H_D \) be the algebraic tensor product over \( A \), seen as a \((B, D)\)-bimodule in the usual way. It carries a \( D \)-valued inner product, generalizing \( \langle \cdot, \cdot \rangle^E \), determined by

\[
\langle x \otimes \phi, y \otimes \psi \rangle_{\_ E_A \otimes_A \_ H_D}^E := \langle \phi, \langle x, y \rangle_A^E \cdot \psi \rangle_D,
\]

for \( x, y \in \_ E_A \) and \( \phi, \psi \in \_ H_D \). The main observation is [10]:

**Theorem 3.4** If \( \langle \cdot, \cdot \rangle_A^E \) and \( \langle \cdot, \cdot \rangle_D \) are completely positive, then \( \langle \cdot, \cdot \rangle_{\_ E_A \otimes_A \_ H_D}^E \) is completely positive.

To obtain a pre-Hilbert module, we consider the quotient

\[
\_ E_A \otimes_A \_ H_D = \_ E_A \otimes_A \_ H_D / (\_ E_A \otimes_A \_ H_D)^\perp,
\]

which now carries a nondegenerate, completely positive inner product induced by \( \langle \cdot, \cdot \rangle_{\_ E_A \otimes_A \_ H_D}^E \). So \( \_ E_A \otimes_A \_ H_D \) is a \((B, D)\)-pre-Hilbert bimodule. In fact, the tensor product \( \otimes_A \) defines a functor

\[
\otimes_A : \text{*-rep}_A(B) \times \text{*-rep}_D(A) \to \text{*-rep}_D(B),
\]

where \( \text{*-rep}_D(A) \) denotes the category of \( * \)-representations of \( A \) on (right) pre-Hilbert \( D \)-modules (in other words, pre-Hilbert \((A, D)\)-bimodules). Note that, by Example 3.3 part (1), if \( D = C \), then \( \text{*-rep}_D(A) \) agrees with \( \text{*-rep}(A) \), the representation category of \( A \) defined in Section 2.

By fixing the bimodule \( \_ E_A \), we obtain the **Rieffel induction** functor

\[
R_E = \_ E_A \otimes_A : \text{*-rep}_A(A) \to \text{*-rep}_D(B),
\]

which allows to compare the representation theories of \( A \) and \( B \) for any auxiliary \( * \)-algebra \( D \). When \( A, B \) and \( D \) are \( C^* \)-algebras, one recovers the original construction of Rieffel after suitable topological completions.
Remark 3.5 Note that condition (1) in Proposition 3.1 coincides with property $P$ used in [8] for the description of Rieffel induction; hence Proposition 3.1 relates the approaches of [10] and [8].

Example 3.6 Let $A$ be a $^*$-algebra over $C$, and let $\omega : A \to C$ be a positive linear functional. We consider $A$ as an $(A,C)$-bimodule, with $C$-valued inner product $\langle a,b \rangle_\omega := \omega(a^* b)$. Although this is not strictly a pre-Hilbert bimodule according to our definition, since $\langle \cdot,\cdot \rangle_\omega$ may be degenerate, Rieffel induction goes through just as well. The representation of $A$ induced by the canonical representation of $C$ on itself by left multiplication is the GNS representation of Example 2.1.

## 4 Rieffel induction in deformation quantization

In this section we discuss examples of modules over Hermitian star products which carry completely positive inner products, and hence can be used to implement Rieffel induction in the context of deformation quantization. We start by recalling how classical and quantum positive linear functionals are related in this context.

**Theorem 4.1** Let $A := (C^\infty(M)[[\lambda]],\star)$ be a Hermitian deformation quantization, and let $\omega_0$ be a positive linear functional on $C^\infty(M)$. Then one can find $C$-linear functionals $\omega_r : A \to C$, $r = 1, 2, \ldots$, so that $\omega_0 + \sum_{r=1}^\infty \lambda^r \omega_r$ is a positive linear functional of $A$.

In other words, any Hermitian star product is a positive deformation in the sense of [6]. A proof of this theorem can be found in [11].

We now turn our attention to examples of pre-Hilbert modules over Hermitian star products. Let $E$ be a f.g.p. module over a Hermitian deformation quantization $A = (C^\infty(M)[[\lambda]],\star)$, and let

$$h : E \times E \to A, \quad (x,y) \mapsto h(x,y)$$

be an $A$-valued inner product. Then $\tilde{E}_0 := E/(\lambda E)$ is a (f.g.p.) module over $C^\infty(M)$, and $h$ naturally induces an inner product

$$h_0 : \tilde{E}_0 \times \tilde{E}_0 \to C^\infty(M)$$

by $h_0([x],[y]) := h(x,y) \mod \lambda$. We refer to $h_0$ as the classical limit of the inner product $h$. The next result is an analogue in deformation quantization of Example 3.3 part (4).
Theorem 4.2 Let $\mathcal{E}$ be a f.g.p. module over a Hermitian deformation quantization $\mathcal{A} = (C^\infty(M)[[\lambda]], \star)$, let $h$ be a positive, strongly nondegenerate $\mathcal{A}$-valued inner product on $\mathcal{E}$. Then $h$ is completely positive, and its classical limit $h_0$ is a Hermitian fibre metric on the vector bundle $E$ corresponding to $E_0$.

**Proof:** Let $h_0$ be the classical limit of $h$. We first observe that $h_0$ is a positive inner product on $E_0$. Given $x \in \mathcal{E}$, consider $h_0([x], [x]) = h(x, x) \mod \lambda \in C^\infty(M)$, and let $\omega_0$ be a positive linear functional on $C^\infty(M)$. By Theorem 4.1, we can find a positive linear functional on $\mathcal{A}$ of the form $\omega = \omega_0 + \sum_{r \geq 1} \lambda^r \omega_r$. Since $\omega(h(x, x)) \geq 0$ in $C[[\lambda]]$, we have that $\omega_0(h(x, x) \mod \lambda) \geq 0$ in $C$. So $h_0([x], [x]) \geq 0$.

A direct computation shows that $h_0$ is strongly non-degenerate. Thus $(E_0, h_0)$ comes from a vector bundle $E$ over $M$ carrying a Hermitian fibre metric $h_0$, and $(\mathcal{E}, h)$ is an example of a deformation quantization of a Hermitian vector bundle in the sense of [5]. By [10], it follows that $(\mathcal{E}, h)$ is isometric to an $\mathcal{A}$-module as the one in Example 3.3, part (3). Hence $h$ is completely positive.

We note that checking that the classical limit $h_0$ is strongly nondegenerate is sufficient to guarantee the strong nondegeneracy of $h$. In particular, according to [5], any Hermitian vector bundle over $M$ can be deformed into a pre-Hilbert module over $\mathcal{A}$ which can be used for the construction of induced representations. In this context, line bundles over $M$ play a special role. This is because a deformation of a line bundle $L \to M$ with respect to a star product $\star$ defines a pre-Hilbert bimodule for $\star$ and another deformation quantization $\star'$ of $M$. If $M$ is symplectic, the relationship between $\star$ and $\star'$ is that the difference of their characteristic classes (in the sense of e.g. [12]) is $2\pi ic_1(L)$ [9], where $c_1(L)$ denotes the first Chern class of $L$. One can then use Rieffel induction to transfer representations from one quantization to the other.

An interesting physical example is discussed in [9], where it is shown that the formal representations of star products on cotangent bundles with a “magnetic term” studied in [2] can be obtained by Rieffel induction of the formal Schrödinger representation of the standard Weyl star product. Here, the pre-Hilbert bimodule used to implement the induction is a deformation of the line bundle associated with a magnetic charge satisfying Dirac’s quantization condition; see [18] for a detailed physical discussion of this example.
5  A unified view of strong Morita equivalence

We now briefly recall how to obtain an equivalence of categories of representations using the functor (10). This leads to a generalization of the notion of strong Morita equivalence in $\mathbb{C}^*$-algebras to the algebraic framework of Section 2; details can be found in [10].

**Definition 5.1** Let $A$, $B$ be $\mathbb{C}^*$-algebras over $\mathbb{C}$ and $\mathcal{E}_A$ a $(B, A)$-bimodule so that $B \cdot \mathcal{E} = \mathcal{E}$ and $\mathcal{E} \cdot A = \mathcal{E}$. Suppose that $\mathcal{E}$ is equipped with completely positive and non-degenerate inner products $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$ such that

1. $\langle b \cdot x, y \rangle_A = \langle x, b^* \cdot y \rangle_A$,
2. $\langle x \cdot a, y \rangle_B = \langle x, y \cdot a^* \rangle_B$,
3. $\langle x, y \rangle \cdot z = x \cdot \langle y, z \rangle_A$,
4. $\mathcal{C}$-span $\{ \langle x, y \rangle_A | x, y \in \mathcal{E} \} = A$,
5. $\mathcal{C}$-span $\{ \langle x, y \rangle_B | x, y \in \mathcal{E} \} = B$.

Then $\mathcal{E}_A$ is called a strong Morita equivalence bimodule. If there exists such a bimodule then $A$ and $B$ are called strongly Morita equivalent.

As discussed in [10] [7], one recovers Rieffel’s notion of strong Morita equivalence of $\mathbb{C}^*$-algebras from this purely algebraic definition by passing to minimal dense ideals.

The following theorem summarizes some of the properties of strong Morita equivalence that have well-known counterparts in ring theory and $\mathbb{C}^*$-algebra theory.

**Theorem 5.2**

1. Strong Morita equivalence is an equivalence relation among nondegenerate and idempotent $\mathbb{C}^*$-algebras over $\mathbb{C}$.

2. If $\mathcal{E}_A$ is a strong equivalence bimodule then the Rieffel induction functor

$$\mathcal{E}_A \otimes_A : \mathcal{C} \cdot \text{Rep}_D(A) \longrightarrow \mathcal{C} \cdot \text{Rep}_D(B)$$

(11)

establishes an equivalence of categories for any fixed $\mathbb{C}^*$-algebra $D$.

3. If $\mathcal{E}_A$ is a strong Morita equivalence bimodule for unital $\mathbb{C}^*$-algebras $A$ and $B$, then there exist Hermitian dual bases $(\xi_i, \eta_i)$ and $(x_j, y_j)$, respectively, such that

$$x = \sum_{i=1}^n \xi_i \cdot \langle \eta_i, x \rangle_A = \sum_{j=1}^m \langle x, y_j \rangle_B \cdot x_j$$

(12)
for all \( x \in \mathcal{E}_A \). In particular, \( \mathcal{E}_A \) is finitely generated and projective as right \( A \)-module and also as left \( B \)-module.

4. If \( A \) and \( B \) are unital, then strong Morita equivalence implies ring-theoretic Morita equivalence.

Some comments are in order. The crucial point in (1) is to show transitivity, which relies on the fact that completely positive inner products behave well under tensor products, see Theorem 3.4 in (2), \( ^*\text{-Rep}_D(A) \) denotes the subcategory of \( ^*\text{-rep}_D(A) \) consisting of pre-Hilbert bimodules \( \mathcal{H}_D \) satisfying the extra nondegeneracy condition \( A\mathcal{H} = \mathcal{H} \); property (3) essentially implies (4) and is also used to show that the inner products on equivalence bimodules of unital \( ^* \)-algebras are strongly nondegenerate.

In [10], we describe a class of unital \( ^* \)-algebras, including both unital \( C^* \)-algebras and Hermitian deformation quantizations, for which the converse of part (3) holds, i.e., strong and ring-theoretic Morita equivalences define the same equivalence relation. The comparison between these two types of Morita equivalence becomes more interesting at the level of Picard group(oid)s, see [10] for a discussion.

The fact that, for star products, strong Morita equivalence coincides with Morita equivalence in the classical sense of ring theory is used in [9] to classify strong Morita equivalent Hermitian deformation quantizations on symplectic manifolds. An interesting problem is to investigate the precise connection between Morita equivalence for star products and their counterparts in \( C^* \)-algebraic versions of deformation quantization.

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**References**

[1] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: *Deformation Theory and Quantization*. Ann. Phys. **111** (1978), 61–151.
[2] Bordemann, M., Neumaier, N., Pflaum, M. J., Waldmann, S.: On representations of star product algebras over cotangent spaces on Hermitian line bundles. J. Funct. Anal. 199 (2003), 1–47.

[3] Bordemann, M., Neumaier, N., Waldmann, S.: Homogeneous Fedosov star products on cotangent bundles II: GNS representations, the WKB expansion, traces, and applications. J. Geom. Phys. 29 (1999), 199–234.

[4] Bordemann, M., Waldmann, S.: Formal GNS Construction and States in Deformation Quantization. Commun. Math. Phys. 195 (1998), 549–583.

[5] Bursztyn, H., Waldmann, S.: Deformation Quantization of Hermitian Vector Bundles. Lett. Math. Phys. 53 (2000), 349–365.

[6] Bursztyn, H., Waldmann, S.: On Positive Deformations of *-Algebras. In: Dito, G., Sternheimer, D. (eds.): Conférence Moshé Flato 1999. Quantization, Deformations, and Symmetries, Mathematical Physics Studies no. 22, 69–80. Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[7] Bursztyn, H., Waldmann, S.: *-Ideals and Formal Morita Equivalence of *-Algebras. Int. J. Math. 12.5 (2001), 555–577.

[8] Bursztyn, H., Waldmann, S.: Algebraic Rieffel Induction, Formal Morita Equivalence and Applications to Deformation Quantization. J. Geom. Phys. 37 (2001), 307–364.

[9] Bursztyn, H., Waldmann, S.: The characteristic classes of Morita equivalent star products on symplectic manifolds. Commun. Math. Phys. 228 (2002), 103–121.

[10] Bursztyn, H., Waldmann, S.: Completely positive inner products and strong Morita equivalence. Preprint (FR-THEP 2003/12) math.QA/0309402 (September 2003), 36 pages. To appear in Pacific J. Math.

[11] Bursztyn, H., Waldmann, S.: Hermitian star products are completely positive deformations. Preprint (FR-THEP 2004/18) math.QA/0410350 (October 2004), 8 pages. To appear in Letters in Mathematical Physics.
[12] Gutt, S., Rawnsley, J.: *Equivalence of star products on a symplectic manifold; an introduction to Deligne’s Čech cohomology classes.* J. Geom. Phys. 29 (1999), 347–392.

[13] Lance, E. C.: *Hilbert C*-modules. A toolkit for operator algebraists,* vol. 210 in *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1995.

[14] Raeburn, I., Williams, D. P.: *Morita equivalence and continuous-trace C*-algebras,* vol. 60 in *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1998.

[15] Rieffel, M. A.: *Induced representations of C*-algebras.* Adv. Math. 13 (1974), 176–257.

[16] Schmüdgen, K.: *Unbounded Operator Algebras and Representation Theory,* vol. 37 in *Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, Boston, Berlin, 1990.

[17] Waldmann, S.: *The Picard Groupoid in Deformation Quantization.* Lett. Math. Phys. 69 (2004), 223–235.

[18] Waldmann, S.: *States and Representation Theory in Deformation Quantization.* Rev. Math. Phys. 17 (2005), 15–75.