The sensitivity of a quantum PageRank

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Abstract
In this study, we discuss the sensitivity of a quantum PageRank. By utilizing the finite-dimensional perturbation theory, we estimate the change of the quantum PageRank under a small analytical perturbation on the Google matrix. In addition, we will show the way to estimate the lower bound of the convergence radius and the error bound of the finite sum in the expansion of the perturbed PageRank.

Keywords Quantum walk · PageRank · Perturbation theory

Mathematics Subject Classification 81Q15 · 30E15 · 46B28

1 Introduction
Quantum information has recently attracted numerous research attention. Particularly, quantum walks have been substantially discussed from both the theoretical and practical perspectives [1].

A major breakthrough has been the quadratic speedup in the search algorithms of quantum walks [2]. The study [3] extended the random walk concept on a graph to quantum walks on a directed graph.

Based on the ideas presented in [4] and [5], the author introduced a unitary operator that works on a Hilbert space whose elements comprise pairs of edges and the components of the state transition matrix. Recently, studies on the mixing time of the quantum walk on the graph have also been conducted [6].

In parallel with these studies, the ranking process has been substantially discussed in classic network studies. A well-known ranking process is Google’s PageRank algorithm, which properly sorts web pages according to their importance and impact [7, 8].

A quantum walk study on PageRank was pioneered in [9, 10], who proposed a quantum version of the PageRank algorithm. Based on Szegedy’s ideas [3], the authors quantized the behavior of a random walker modeled in the Google matrix.

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and defined the unitary operator. Additionally, they derived some interesting relationships between the quantum and original versions of PageRank. For instance, in the quantum PageRank, the eigenvalues and eigenvectors of the unitary operator can be represented by those of a matrix composed of the Google matrix elements. Besides, they also numerically computed some characteristics of the quantum PageRank as well as revealed its similarities to and differences from the classical PageRank.

We previously studied the behavior of the classical PageRank subjected to small perturbations. Since the Perron–Frobenius theorem ensures good characteristics of the principal eigenvalue of the Google matrix, the PageRank could be analytically perturbed by perturbing the Google matrix. However, the perturbations of the quantum PageRank are less tractable since all the eigenvalues of the unitary operator have unity magnitude. Therefore, under a perturbation, we consider the behavior of all eigenvalues and their corresponding eigenvectors. To overcome these difficulties, we explore the characteristics of normal operators, the reduction process, and the transformation functions in the analytical perturbation theory.

The rest of this study is organized as follows. The next section introduces the terms and notations used in this study. Section 3 reviews the previously conducted studies concerning classical and quantum PageRanks. Furthermore, our main results are demonstrated and proved in Sects. 4 and 5, respectively. The conclusions are presented in Sect. 6.

2 Terms and notations

In this study, $N$ denotes an arbitrary natural number, and the vectors are labeled as $|\psi\rangle$. The tensor product of two spaces $\mathbb{C}^N$ is denoted by $\mathbb{C}^N \otimes \mathbb{C}^N$, and its elements are denoted by $|\phi\rangle \otimes |\psi\rangle$ or simply $|\phi,\psi\rangle$. The transposed row vector of $|\psi\rangle$ is denoted by $\langle \psi |$, and the inner product of two column vectors $|\phi\rangle$ and $|\psi\rangle$ is denoted by $\langle \phi|\psi \rangle$.

A complex number and its conjugate are denoted by $z$ and $z^*$, respectively ($z, z^* \in \mathbb{C}$). Additionally, Re$(z)$ and Im$(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$, respectively. The imaginary unit is $i = \sqrt{-1}$.

The $p$-norm of a general vector $u = (u_1, u_2, \ldots, u_N)^T$ is defined by

$$
\|u\|_p = \left( \sum_{j=1}^{N} |u_j|^p \right)^{\frac{1}{p}} \quad (p \in [1, \infty)),
$$

and $\|u\|_\infty \equiv \max_{1 \leq j \leq N} |u_j|$. For a squared matrix $M = [m_{ij}] \in \mathbb{M}_{N \times N}$ (where $\mathbb{M}_{N \times N}$ denotes a set of $N$ by $N$ matrices), we define

$$
\|M\|_\infty = \max_{1 \leq i \leq N} \sum_{j=1}^{N} |m_{ij}|, \quad \|M\|_1 = \max_{1 \leq j \leq N} \sum_{i=1}^{N} |m_{ij}|.
$$

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We also define the \textit{operator norm} as follows: \( \|M\| \equiv \sup_{x \neq 0} \frac{\|Mx\|}{\|x\|} \), where the norms on the right-hand side are the usual norms of vector spaces: \( \|x\|_p \). Additionally, we simply denote the vector norm \( \| \cdot \|_2 \) by \( \| \cdot \| \) and apply the corresponding operator norm. For a matrix \( M \in \mathbb{M}_{N \times N} \), we denote the sets of eigenvalues and resolvent, trace, and spectral radius by \( \sigma(M) \), \( P(M) \), \( \text{tr}(M) \), and \( \text{spr}(M) \), respectively. That is,

\[
\text{spr}(M) = \lim_{n \to \infty} \|M^n\|^{\frac{1}{n}} = \max_{1 \leq i \leq N} |\lambda_i|.
\]

(See, for example, [11]). The multiplicity of \( \lambda \in \sigma(M) \) is denoted by \( m(\lambda) \). When \( m(\lambda) = 1 \), \( \lambda \) is said to be simple. Let \( R(M) \) be the range of a map represented by \( M \).

For a point \( x \in \mathbb{R}^N \) and a general set \( S \subset \mathbb{R}^N \), we have \( \text{dist}(x, S) \equiv \min_{y \in S} \|x - y\| \), where \( \|x - y\| \) denotes the usual Euclidean distance between two points \( x \) and \( y \) in \( \mathbb{R}^N \). In a unitary space \( H \) with its inner product \( \langle \cdot, \cdot \rangle \), if an operator \( M^* \) satisfies \( \langle M^*x, y \rangle = \langle x, My \rangle \) \( \forall x, y \in H \), then \( M^* \) is called as the \textit{adjoint operator} of \( M \). An operator \( M \) is called \textit{normal} if it commutes with its conjugate operator: \( MM^* = M^*M \). The term isolation distance of an eigenvalue \( \lambda_h \) of a matrix \( M \) refers to the distance between \( \lambda_h \) and the other eigenvalues of \( M \), that is, \( \text{dist}(\lambda_h, \sigma(M) \setminus \{\lambda_h\}) \). It is often convenient to set \( \|\pi\|_1 = 1 \) for a PageRank vector \( \pi \), but Tikhonov’s theorem [12] states that all norms in a finite-dimensional vector space are equivalent. The \( r \)-th derivative of a general function \( f \) with a single argument is sometimes denoted by \( D^r f \) (\( r \in \mathbb{N} \)).

\section{Background}

\subsection{Google matrix and PageRank}

The PageRank algorithm was first proposed in [7]. Numerous researchers have subsequently contributed computationally and theoretically to this algorithm. Although thorough sensitivity analyses of the PageRank vector exist in the literature [8], we remark that more general situations may appear in practical situations, which warrant further discussion. Under small analytic perturbations, the author’s previously conducted study [13] presented a sensitivity analysis of a PageRank. Now, we briefly define a Google matrix and an overview of its characteristics. The Internet is modeled as a directed graph \( G(V, E) \), where \( V \) and \( E \) denote the sets of nodes and edges, respectively. The notation \( N = \#V \) describes the number of nodes in this graph. For later use, the elements of \( V \) are labeled as 1, 2, \ldots, \( N \).

The row-stochastic matrix \( W \) is defined by \( W = H + \frac{a}{N} e^T \), where \( a = (a_j) \), and \( e \) denotes a column vector whose elements are all 1. We set \( a_j = 1 \) if web page \( j \) is a \textit{dangling node} (that is, a node without any outgoing links), and \( a_j = 0 \) otherwise [8]. The \textit{hyperlink matrix} \( H = [h_{ij}] \) is a weighted adjacency matrix defined by \( h_{ij} = 1/D_i \) if there is a link from web page \( i \) to web page \( j \), and \( h_{ij} = 0 \) otherwise, where \( D_i \) denotes the number of the outgoing links from node \( i \). The Google matrix \( G \) is defined by \( G = aW + \frac{(1-a)}{N} ev^T \), where \( v \) denotes a non-negative vector satisfying \( \|v\|_1 = 1 \) (called the \textit{personalization vector}). Moreover, the constant \( a \in (0, 1) \).
is called the damping factor. The quantity \((1 - \alpha)\) denotes the probability that an Internet surfer navigates from one web page \((i)\) to another \((j)\). By definition, \(G\) is completely dense, primitive, and irreducible [8].

The following lemma is the foundation of the Google matrix theory.

**Lemma 1** The Google matrix \(G\) has a simple principal eigenvalue \(\eta_1 = 1\). All eigenvalues \(\{\eta_n\}_n\) of \(G\) satisfy \(\eta_1 = 1 > |\eta_2| \geq |\eta_3| \geq \ldots\). Additionally, there exists a non-negative left-eigenvector \(\pi\) corresponding to the principal eigenvalue \(\eta_1 = 1\), which is known as the PageRank vector.

### 3.2 A quantum PageRank

The PageRank vector denotes the probability that a random walker on the Internet stays on each node during the stationary state. Recently, this concept has been replaced by the quantum walk [9, 10]. To our knowledge, quantum walk on a directed graph was introduced in [3]. Based on this idea, the authors of [9] studied quantum PageRank. They started their arguments by introducing vectors of the following form:

\[
|\psi_j\rangle \equiv |j\rangle_1 \otimes \sum_{k=1}^{N} \sqrt{g_{jk}} |k\rangle_2 \quad (j = 1, 2, \ldots, N),
\]

where \(g_{jk}\) denotes the \((j, k)\) element of the Google matrix \(G\). Here, the indices \(|\cdot\rangle_l\ (l = 1, 2)\) show the start and end nodes of the edges, respectively. It is easy to see that

\[
\langle \psi_j | \psi_k \rangle = \delta_{jk} \quad \forall j, k,
\]

where \(\delta_{jk}\) denotes the Kronecker’s delta. Therefore, \(\{|\psi_j\rangle\}_j\) forms an orthonormal basis in a subspace of \(\mathcal{H} \equiv \mathbb{C}^{N} \otimes \mathbb{C}^{N}\). The authors of [9, 10] also defined the following operators:

\[
B \equiv \sum_{j=1}^{N} |\psi_j\rangle \langle \psi_j|, \quad U \equiv S_w(2B - 1),
\]

where \(I\) denotes the matrix with all elements of unity that acts on \(\mathcal{H}\), and \(S_w\) is the swap operator:

\[
S_w \equiv \sum_{j,k=1}^{N} |k, j\rangle \langle j, k|.
\]

Clearly, \(U\) is a unitary operator on \(\mathcal{H}\) [9, 10]. Now, let us denote its eigenvalues and the corresponding normalized eigenvectors as \(\{|\mu\rangle\}\) and \(\{|\mu\rangle\}\), respectively. Since \(U\) is unitary, it is noteworthy that \(\{|\mu\rangle\}\) forms an orthonormal basis in \(\mathcal{H}\).
Now, let us consider the subspace $H_d$ spanned by $\{|\psi_j\rangle\}_j$ and $\{S_w|\psi_j\rangle\}_j$ [9]. We also consider its orthogonal subspace $H_d^\perp$, in which $U$ acts as $-S_w$ whose eigenvalues are just $\pm1$.

The eigenvalues of $U$ (other than those of $S_w$) can be expressed in the following form: $\mu = \lambda \pm i\sqrt{1 - \lambda^2}$, where the eigenvalues $\lambda$ of the positive symmetric matrix $T = [t_{ij}]$ lie in $[0, 1]$ [3], and $t_{ij} = \sqrt{g_{ij}g_{ji}}$. Hereafter, the number of eigenvalues of $T$ (excluding multiple values) is denoted by $s$. Thus, $\sigma(T) = \{\lambda_h\}_{h=1}^s$. Now, the subspace of $\mathcal{H}$ which is spanned by the eigenvectors of $U$ corresponding to $\mu = \lambda \pm i\sqrt{1 - \lambda^2}$ is denoted by $H_e$ hereafter.

The quantum PageRank for node $i \in V$ at time $m \in \mathbb{N}$ defined in [9, 10] is the following:

$$I_q(P_i, m) = \langle \psi(0)|U^{2m}|i\rangle_2\langle i|U^{2m}|\psi(0)\rangle_2,$$

(3.1)

where $U^\dagger$ denotes the Hermitian transpose of $U$.

In [9], they redefined Eq. (3.1) as follows:

$$I_q(P_i, m) = \left| \sum_{\mu} \mu^{2m} \langle i|\mu\rangle \langle \mu|\psi(0)\rangle \right|^2.$$  

(3.2)

Based on what we have stated before, the notation $\langle 2|\cdot \rangle$ means the transpose of the basis state of the destination node $|i\rangle_2$. Now, we interpret this definition and rewrite it in a form more suitable for our sensitivity analysis.

First, the content in the norm in the norm of Eq. (3.2) can be obtained by considering the spectral decomposition of $U^{2m}$:

$$U^{2m} = \sum_{\mu} \mu^{2m} |\mu\rangle \langle \mu|.$$ 

Therefore, Eq. (3.2) is equivalent to the following equality:

$$I_q(P_i, m) = \left| \langle 2|U^{2m}|\psi(0)\rangle \right|^2.$$  

(3.3)

Although the notation $\langle 2|\cdot \rangle$ seems a bit unusual, Eq. (3.3) seems to be equal to the following equality in [9, 10]:

$$I_q(P_i, m) = \sum_{j \in V} \left| \langle j, i|U^{2m}|\psi(0)\rangle \right|^2 = \sum_{j \in V} \left| \sum_{\mu} \mu^{2m} \langle j, i|\mu\rangle \langle \mu|\psi(0)\rangle \right|^2.$$ 

(3.4)

This is because Paparo et al. stated the following in [10]: “The instantaneous PageRank of the node $i$ is given by the probability of finding the walker on the node $i$ of the network”, and the same concept is defined in the paper [6] which Paparo also cited.
(See Definition 3.2 in [6]). Here, $|\alpha_j\rangle = U^t|\psi_0\rangle$. By replacing $a \in A$ in Eq. (3.5) by $j \in V$ and $t$ by $2m$, we obtain Eq. (3.4). Paparo et al. [10] also mentioned that “we need to project onto $|i\rangle_2$, and finally to take formally the squared norm of the resulting quantum state,” which seems to be a state $\sum_{\mu} \mu^{2m} |j, i\rangle \langle \mu|\psi(0)\rangle$.

Based on these considerations, we hereafter denote the quantum PageRank introduced by Paparo et al. [9, 10] as follows:

$$P_t(v|\alpha_0) = \sum_{a \in A} |\langle a, v|\alpha_t\rangle|^2.$$  \hspace{1cm} (3.5)

where $|\psi(0)\rangle$ denotes the initial state of the walk, which lies in the subspace $H_d$ spanned by $\{|\psi_j\rangle\}$ and $\{S_w|\psi_j\rangle\}$ [9, 10]. (Although the preceding papers [9, 10] used the notation $|\psi_0\rangle$, we use the notation $|\psi(0)\rangle$ herein to demonstrate that it is the initial state since we use numerous notations accompanying the subindex 0.)

In addition, the notation $\mu \in H_c$ under the summation symbol implies that we take the sum over eigenvalues whose corresponding eigenvectors belong to $H_c$. Definition (3.1) implies that we consider the spectral decomposition of $U$ on the space $H_c$, that is, we project the space $H_d$ onto $H_c$.

### 3.3 Formulation and settings

In this study, we impose a small perturbation on the Google matrix $G \in M_{N \times N}$. The perturbed Google matrix is denoted by $G(\chi) = [g_{ij}(\chi)]$, where $\chi \in \mathbb{C}$. Moreover, the perturbed $W$ is denoted by $W(\chi)$. The perturbation is presumed not to remove any node or edge, as well as preserve the characteristics of $G$ and $W$, both of which are row-stochastic, irreducible, and primitive. In this study, we impose the following analytic perturbation on the Google matrix, as discussed in [13]:

$$G(\chi) = G + \chi G^{(1)} + \chi^2 G^{(2)} + \cdots.$$  \hspace{1cm} (3.7)

We assume that

$$\|G^{(l)}\| \leq A_0 B_0^{-l+1} \quad l = 1, 2, \ldots.$$  \hspace{1cm} (3.8)

where $A_0$ and $B_0$ denote positive constants independent of $l$. Assuming that Eq. (3.7) converges, such constants certainly exist. We also define the matrix $T(\chi) = [t_{ij}(\chi)]$ with $t_{ij}(\chi) = \sqrt{g_{ij}(\chi)g_{ji}(\chi)}$, and $|\psi_j(\chi)\rangle = |j\rangle_1 \otimes \sum_{k=1}^{N} \sqrt{g_{jk}(\chi)} |k\rangle_2$.

Then, $T(\chi)$ is holomorphic with respect to $\chi$ when $|\chi|$ is sufficiently small and can be expanded as an the following infinite sum:
The sensitivity of a quantum PageRank

The corresponding eigenvalues of the perturbed $T(\chi)$ are denoted by $\{\lambda(\chi)\}$. Obviously, $\lambda(0) = \lambda$ for each eigenvalue. To introduce the quantum walk, we further define the following equation:

$$B(\chi) = \sum_{j=1}^{N} |\psi_j(\chi)\rangle\langle \psi_j(\chi)|, \quad U(\chi) = S_w \left( 2B(\chi) - 1 \right).$$

The corresponding eigenvalues and the normalized eigenvectors of the perturbed $U(\chi)$ are denoted by $\{\mu(\chi)\}$ and $\{|\mu(\chi)\rangle\}$, respectively. As defined above, $U$ is a unitary operator on $\mathcal{H}$, and all of its eigenvalues have unity magnitude. Then, the quantum PageRank at time $m$ under perturbation is defined by

$$I_q^{(\chi)}(i, m|\psi(0)) = \sum_{j \in V} \sum_{\mu \in \mathcal{H}_e} (\mu(\chi))^{2m}(j, i|\mu(\chi))\langle \mu(\chi)|\psi(0)\rangle^2,$$

where the summation is taken over the set of the corresponding unperturbed $\mu$’s in $\mathcal{H}_e$. The perturbation is assumed to be very small, such that

$$|\mu(\chi)| < 1 \quad \forall \mu = \mu(0) \in \mathcal{H}_e.$$ 

### 4 Main results

We now give the main statement in this study.

**Theorem 1** Let $m \in \mathbb{N}$ be arbitrary. Assume that there exists at least one dangling node in the network. Then, when $\chi \in \mathbb{C}$ is sufficiently small, the temporal quantum PageRank is holomorphic with respect to $\chi$ on the real axis. Thus, within a neighborhood of the real axis, it can be represented in the following form:

$$I_q^{(\chi)}(i, m|\psi(0)) = I_q(i, m|\psi(0)) + \sum_{n=1}^{\infty} \chi^n I_q^{(n)}(i, m|\psi(0))$$

$$(4.1)$$

where $I_q^{(n)}(i, m|\psi(0))$ denotes the coefficient of $\chi^n$ in the expansion of $I_q^{(\chi)}(i, m|\psi(0))$ with respect to $\chi$.

Based on assumption (3.8), the convergence radius of this expansion is determined by the following: $N, m, A_0, B_0$, the elements of $G$, and the isolation distance of $T$. In addition, each term in the right-hand side of Eq. (4.1) is estimated from above in the region stated above.
where the constants $A_1, B_1 > 0$ depend on the same quantities as the convergence radius.

**Remark 1** The term ‘holomorphic on the real axis’ means that the quantum PageRank is holomorphic in a neighborhood of the real axis.

$I_q(i, m||\psi(0))$ is oscillatory, and it is not temporally convergent in general. However, the following quantity is convergent.

**Definition 1** The average temporal quantum PageRank is defined as follows:

$$\bar{I}_q(t, i||\psi(0)) \equiv \frac{1}{t} \sum_{m=0}^{t-1} I_q(i, m||\psi(0)) \quad \forall t \in \mathbb{N}.$$  

From this definition, we can state the following facts.

**Lemma 2** The average of the temporal quantum PageRank converges as time tends to infinity. The limit is given by

$$I_\infty(i||\psi(0)) \equiv \lim_{t \to \infty} \bar{I}_q(t, i||\psi(0))$$

$$= \sum_{p,q} \sum_{j \in V} \langle \mu_p|\psi(0) \rangle \langle \mu_q|\psi(0) \rangle^* \langle j, i|\mu_p \rangle \langle \mu_q|j, i \rangle,$$

where the first summation is taken over the pairs of eigenvalues $(\mu_p, \mu_q)$ satisfying $\mu_p = \mu_q$.

**Corollary 1** If all eigenvalues of $T$ are simple, then

$$I_\infty(i||\psi(0)) = \sum_{h=1}^{N} \sum_{j \in V} |\langle \mu_h|\psi(0) \rangle|^2 |\langle j, i|\mu_h \rangle|^2$$

This statement directly follows from the results of [6]. They also discussed the upper bound of the mixing time:

**Lemma 3** The following inequality holds:

$$|\bar{I}_q(t, i||\psi(0)) - I_\infty(i||\psi(0))| \leq \sum_{\mu_p, \mu_q \in H_e} \frac{2|\langle \mu_p|\psi(0) \rangle|^2}{t|\mu_p - \mu_q|}.$$  

Under an analytic perturbation on $G$, it is observed that $\bar{I}_q(t, i||\psi(0))$ and $I_\infty(i||\psi(0))$ are not usually analytic. Moreover, we see that some eigenvalues of $U(\chi)$ may actually split into several eigenvalues as $\chi$ grows.
In this case, the terms in the summation on the right-hand side of inequality (4.2) may increase, meaning that they may substantially change under even a small perturbation.

5 Proof of Theorem 1

In this section, we prove Theorem 1. The proof is divided into several steps.

5.1 Perturbation on G

By assuming an analytic perturbation on $G$, we define

$$g_{ij}(\chi) = g_{ij} + \chi g_{ij}^{(1)} + \chi^2 g_{ij}^{(2)} + \cdots \equiv g_{ij} + \tilde{g}_{ij}(\chi).$$

Then, the elements of $T(\chi) = [t_{ij}(\chi)]$ are given as follows:

$$t_{ij}(\chi) = \sqrt{(g_{ij} + \tilde{g}_{ij}(\chi))(g_{ji} + \tilde{g}_{ji}(\chi))} = t_{ij} + \sum_{n=1}^{\infty} \chi^n t_{ij}^{(n)}, \quad (5.1)$$

for small $\chi$ that satisfies $|\tilde{g}_{ij}(\chi)| < 1$. Since we assume $\alpha \in (0, 1)$ in the construction of the Google matrix, note that each element of the Google matrix differs from zero [8], and $t_{ij}^{(n)}$ is defined by

$$t_{ij}^{(n)} = \frac{1}{n!} \left( \frac{d}{d\chi} \right)^n t_{ij}(\chi) \bigg|_{\chi=0},$$

(see [14] for instance). By applying Leibnitz’s rule, we have

$$\left( \frac{d}{d\chi} \right)^n t_{ij}(\chi) = t_{ij} \sum_{k=0}^{n} nC_k \left( \frac{d}{d\chi} \right)^k \left[ \sqrt{1 + \tilde{g}_{ij}(\chi)} \right] \left( \frac{d}{d\chi} \right)^{n-k} \left[ \sqrt{1 + \tilde{g}_{ji}(\chi)} \right],$$

where

$$\tilde{g}_{ij}(\chi) \equiv \frac{\tilde{g}_{ij}(\chi)}{g_{ij}} \equiv \frac{g_{ij}(\chi) - g_{ij}}{g_{ij}}.$$ (Note that $\tilde{g}_{ij}(0) = \tilde{g}_{ij}(0) = 0$). By the derivative formula of composite functions [15] and setting $f(z) = \sqrt{1 + z}$, we see that
\[
\left( \frac{d}{d\chi} \right)^n \sqrt{1 + \tilde{g}_{ij}(\chi)} = \left( \frac{d}{d\chi} \right)^n f(\tilde{g}_{ij}(\chi)) \\
= \sum_{r=1}^{n} \frac{D^r f(\tilde{g}_{ij}(\chi))}{r!} \\
\times \sum_{p_1 + \cdots + p_r = n} \frac{n!}{p_1!p_2!\cdots p_r!} (D^{p_1} \tilde{g}_{ij}(\chi)) (D^{p_2} \tilde{g}_{ij}(\chi)) \cdots (D^{p_r} \tilde{g}_{ij}(\chi)).
\]

By utilizing the fact that
\[
\left( \frac{d}{dz} \right)^r \left[ \sqrt{1 + z} \right] = r! \frac{1}{2} C_r (1 + z)^{\frac{(2r-1)}{2}},
\]
and \(D^{p_i} \tilde{g}_{ij}(\chi)\) \(\big|_{\chi=0} = \frac{p_i! g^{(p_i)}}{g_i} (i = 1, 2, \ldots, r)\), we have
\[
\left( \frac{d}{d\chi} \right)^n \left[ \sqrt{1 + \tilde{g}_{ij}(\chi)} \right] \big|_{\chi=0} = n! \sum_{r=1}^{n} \frac{1}{2} C_r \left( \sum_{p_1 + \cdots + p_r = n} \prod_{r=1}^{r} g^{(p_r)}_{ij} \right).
\]

By considering this expression, \(t_{ij}^{(n)}\) becomes
\[
t_{ij}^{(n)} = t_{ij} \left[ \sum_{r=1}^{n} \frac{1}{2} C_r \left( \sum_{p_1 + \cdots + p_r = n} \prod_{r=1}^{r} g^{(p_r)}_{ij} \right) \right] \\
+ t_{ij} \sum_{k=1}^{n-1} \left[ \sum_{r=1}^{k} \frac{1}{2} C_r \left( \sum_{p_1 + \cdots + p_r = n - k} \prod_{r=1}^{r} g^{(p_r)}_{ij} \right) \right] \\
\times \left[ \sum_{r'=1}^{n-k} \frac{1}{2} C_{r'} \left( \sum_{p_1 + \cdots + p_{r'} = n - k} \prod_{r'=1}^{r'} g^{(p_{r'})}_{ij} \right) \right] \quad (5.2) \\
+ t_{ij} \sum_{r=1}^{n} \left[ \sum_{p_1 + \cdots + p_r = n} \prod_{r=1}^{r} g^{(p_r)}_{ij} \right] = I_{1(n)}^{(ij)} + I_{2(n)}^{(ij)} + I_{3(n)}^{(ij)}.
\]
For instance, we have the following equality:

\[ t_{ij}^{(1)} = \frac{1}{2t_{ij}} \left( s_{ij}^{(1)} g_{ji} + g_{ji}^{(1)} g_{ij} \right). \]

### 5.2 Perturbation on \( T \)

Based on the discussion in the previous subsection, we rewrite equality (5.1) in the following vector form:

\[ T(\chi) = T + \sum_{n=1}^{\infty} \chi^n T^{(n)} \quad (|\tilde{g}_{ij}(\chi)| < 1 \ \forall i,j \in V). \]  

(5.3)

Note that \( T \) and \( T(\chi) \) for \( \chi \in \mathbb{R} \) are symmetric, and consequently semisimple. The resolvent of \( T \) is defined by

\[ R(\zeta) \equiv (T - \zeta I)^{-1}, \]

and the eigenprojection corresponding to eigenvalue \( \lambda_h \) is given by

\[ P_h \equiv -\frac{1}{2\pi i} \int_{\Gamma_h} R(\zeta) \, d\zeta. \]

Here, \( \Gamma_h \) denotes an arbitrary convex loop enclosing \( \lambda_h \) in the complex plain, excluding all other eigenvalues of \( T \). Now, we represent the resolvent and eigenprojection under a small perturbation. We denote the resolvent of \( T(\chi) \) by \( R(\zeta, \chi) \equiv (T(\chi) - \zeta I)^{-1} \), where \( \zeta \in \mathbb{C} \cap \sigma(T(\chi)) \). \( R(\zeta, \chi) \) is defined for all \( \zeta \notin \sigma(T(\chi)) \). It is known that \( R(\zeta, \chi) \) is holomorphic with respect to \( \zeta \) and \( \chi \) for such a \( \zeta \) (see Theorem II.1.5 in [11]), and it can be expanded as follows (see also, II. (1.13) in [11] ) :

\[ R(\zeta, \chi) = R(\zeta) \left[ I + A(\chi) R(\zeta) \right]^{-1} = R(\zeta) + \sum_{l=1}^{\infty} \chi^l R^{(l)}(\zeta), \]  

(5.4)

where \( A(\chi) = T(\chi) - T \), and

\[ R^{(l)}(\zeta) \equiv \sum_{\nu_1 + \cdots + \nu_p = l \atop \nu_j \geq 1} (-1)^p R(\zeta) T^{(\nu_1)} R(\zeta) T^{(\nu_2)} \cdots R(\zeta) T^{(\nu_p)} R(\zeta). \]

Here, the summation on the right-hand side is taken over all possible values of \( p \in \mathbb{N} \) and \((\nu_1, \ldots, \nu_p)\) satisfying the conditions below the summation symbol. By integrating the infinite series (5.4) over \( \Gamma_h \), the perturbation of the projection operator \( P_h(\chi) \) is defined by

\[ P_h(\chi) = -\frac{1}{2\pi i} \int_{\Gamma_h} R(\zeta, \chi) \, d\zeta = P_h + \sum_{l=1}^{\infty} \chi^l P_h^{(l)}. \]  

(5.5)
\[ P_h(0) = P_h = -\frac{1}{2\pi i} \int_{I_h} R(\zeta) \, d\zeta. \] In Eq. (5.5), \( P_h^{(l)} \) is given by

\[ P_h^{(l)} = -\frac{1}{2\pi i} \sum_{v_1 + \cdots + v_p = l, \ v_k \geq 1} (-1)^p \int_{I_h} R(\zeta) T^{(v_1)} R(\zeta) T^{(v_2)} \cdots R(\zeta) T^{(v_p)} R(\zeta) \, d\zeta. \]

We also define

\[ S_h = -\frac{1}{2\pi i} \int_{I_h} (\zeta - \lambda_h)^{-1} R(\zeta) \, d\zeta, \]

which satisfies \( P_h S_h = S_h P_h = O \), where \( O \) denotes the zero matrix. Additionally, we have \( TP_h = \lambda_h P_h + D_h \), where \( D_h \) is called the eigennilpotent operator of \( T \). For later use, we define for \( q \in \mathbb{Z} \):

\[ S_h^{(0)} = -P_h, \quad S_h^{(q)} = S_h^q (q > 0), \quad S_h^{(q)} = O (q < 0). \quad (5.6) \]

Here, we used the fact that \( U \) is unitary and therefore normal, implying that its corresponding eigennilpotent operator vanishes. Using the same fact, we later discuss the form of the eigenvalues of \( U \) under a perturbation. For simplicity, we omit the subindex \( h \) provided there is no ambiguity.

### 5.3 Perturbation on eigenvalue of \( T \)

We now expand \( \lambda_h(\chi) \), which is the eigenvalues of \( T(\chi) \). By utilizing the resultant representation, we later observe the form of the perturbed eigenvalues of \( U \). We first discuss whether \( \lambda_h(\chi) \) is holomorphic on a certain region in the complex plane. To answer this question, we need the following facts from [11] (see Theorems II.1.10 and II.6.1).

**Theorem 2** Let \( X \) be a unitary space and \( \chi_0 \in D_0 \) for a certain simply connected region \( D_0 \subset \mathbb{C} \). In addition, suppose a sequence \( \{\chi_n\} \) converges to \( \chi_0 \), and let \( T(\chi_n) \) \((n = 1, 2, \ldots)\) be normal.

All eigenvalues and eigenprojections of \( T(\chi) \), which denoted by \( \lambda_h(\chi) \) and \( P_h(\chi) \), respectively, are then holomorphic at \( \chi = \chi_0 \).

**Corollary 2** Let \( \{T(\chi)\} \) be a general family of normal matrices, that are holomorphic with respect to \( \chi \in \mathbb{R} \). Then, their eigenvalues \( \{\lambda_h(\chi)\} \) are also holomorphic with respect to \( \chi \) on \( \mathbb{R} \).

In our case, \( T \) and \( U \) are symmetric and unitary operators, respectively, so they both satisfy Theorem 2 and Corollary 2, respectively. Therefore, \( \lambda_h(\chi) \) and \( \mu_h(\chi) \) are holomorphic on the real axis. Within a neighborhood of the real axis, they can be expressed in the following form:
\[ \lambda_h(\chi) = \lambda_h + \sum_{n=1}^{\infty} \chi^n \lambda_h^{(n)}, \quad \mu_h(\chi) = \mu_h + \sum_{n=1}^{\infty} \chi^n \mu_h^{(n)}. \]  

(5.7)

Hereafter, the first and second equalities in (5.7) are denoted by (5.7)\(_1\) and (5.7)\(_2\), respectively.

Our next question is the following: how do we find and estimate the coefficients of the above expansion? The estimates of the coefficients are also required for finding the error bounds, as discussed later. If \( \lambda_h \) is a simple eigenvalue of \( T \), we can write the following equation (see II.(2.31) in \cite{11})

\[
\lambda_h^{(n)} = \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \sum_{v_1 + \cdots + v_p = n} \text{tr} \ T^{(v_1)} S_h^{(k_1)} \cdots T^{(v_p)} S_h^{(k_p)}. \]

(5.8)

In addition, we also have an estimate of the following form:

\[
|\lambda_h^{(n)}| \leq \rho_h r_h^{-n},
\]

(5.9)

where \( r_h \) denotes the convergence radius of (5.7)\(_1\), and \( \rho_h = \max_{\zeta \in \Gamma_h} |\zeta - \lambda_h| \) with \( \Gamma_h \) as defined in Sect. 5.2. To find the convergence radius of (5.7)\(_1\), we first define \( r_h(\zeta) > 0 \) where \( \zeta \in \Gamma_h \) is a number satisfying the following equality:

\[
\sum_{n=1}^{\infty} (r_h(\zeta))^n \| T^{(n)} R(\zeta) \| = 1.
\]

Then, for \( |\chi| < r_h(\zeta) \), \( \sum_{n=1}^{\infty} |\chi|^n \| T^{(n)} R(\zeta) \| < 1 \) holds. For such \( \chi \), the infinite sum

\[
\| A(\chi) R(\zeta) \| = \left\| \left( \sum_{n=1}^{\infty} \chi^n T^{(n)} \right) R(\zeta) \right\|
\]

converges to a value less than 1, implying that Eqs. (5.4) and (5.5) also converge there. Thus, \( r_h = \min_{\zeta \in \Gamma_h} r_h(\zeta) \). Since \( T \) is normal, the lower bound of \( r_h \) is actually given by \( 2(\tilde{A}/d_h + \tilde{B})^{-1} \) for some constants \( \tilde{A}, \tilde{B} > 0 \) and the isolation distance \( d_h \) of \( \lambda_h \). Then, the convergence radius is given by

\[
r_1 = \min \left\{ \min_{i,j \in V} r_{ij}, \min_{h=1,2,\ldots,s} \{ r_h \} \right\},
\]

(5.10)

where \( r_{ij} \) denotes the value of \( \chi \) satisfying \( g_h(\chi) = 1 \). For lower \( n \)'s, Eq. (5.8) can be simplified to

\[
\lambda^{(1)} = (T^{(1)} \phi, \psi),
\]

\[
\lambda^{(2)} = (T^{(2)} \phi, \psi) - \sum_j (\lambda_j - \lambda_h)^{-1} (T^{(1)} \phi, \psi_j) (T^{(1)} \phi_j, \psi).
\]
Here, \( \phi \) denotes an eigenvector corresponding to \( \lambda \); \( \psi \) denotes the eigenvector of \( T^* \) corresponding to \( \lambda^* \) (where \( T^* \) denotes the adjoint operator of \( T \)); \{\( \lambda_j \)\} denote the eigenvalues of \( T \) with corresponding eigenvectors \{\( \phi_j \)\}; \{\( \psi, \psi_j \)\} denotes the basis of \( X^* \) adjoint to the basis \{\( \phi, \phi_j \)\} of \( X \).

In general, some eigenvalues may not be simple when \( \chi = 0 \); they may split as \( \chi \) grows. In this case, Eq. (5.7) only sums the perturbed eigenvalues, and the expansion of each eigenvalue remains unknown. To calculate the explicit expansion of each eigenvalue, we apply the reduction process (for details, see [11]).

Algorithm 1 shows the general process for finding the explicit expansions of all eigenvalues along with the descriptions of each process. Our idea is to construct a “tree of eigenvalues” (see Fig. 1).

**Algorithm 1: Logical flow of getting expansions of \( \lambda_h(\chi) \)**

**Input:** Google matrix \( G \)

**Output:** Explicit representation of each eigenvalue of \( T(\chi) \)

1. Define \( T \) and \( T(\chi) \) as in Subsections 3.2 and 3.3.
2. Extract the set \( \Phi(T) \) of eigenvalues of \( T \) that are not simple.
3. if \( \Phi(T) = \phi \) then
   4. Find the explicit expansion of all eigenvalues of \( T \) by using the formula (5.7)_1;
5. else
   6. \( k \leftarrow 0 \)
7. if \( k = 0 \) then
   8. \( \Phi(k) = \Phi \);
   9. else
   10. Define \( T_{0,1,...,k} \) and \( \Phi(k) \equiv \{ \lambda \in \sigma(T_{0,1,...,k}) | m(\lambda) > 1 \} \).
11. while \( \Phi(k) \neq \phi \) do
12. For eigenvalues in the \( k \)-th level that are simple, calculate and store their expansions.
13. Create the level \( (k+1) \), \( T_{0,1,...,k} \) and \( T_{0,1,...,k}(\chi) \) to compute the descendant eigenvalues of those in \( \Phi(k) \), and for each element \( \lambda^{(k)}_j \in \Phi(k) \), connect \( \lambda^{(k)}_j \) and its descendant eigenvalues in level \( (k+1) \).
14. \( k \leftarrow k + 1 \)
15. \( \Phi(k) \equiv \{ \lambda \in \sigma(T_{0,1,...,k}) | m(\lambda) > 1 \} \).
16. Trace all paths from \( \lambda_h \in \sigma(T) \) to the ending eigenvalues, to make the expansion of each one in \( \sigma(T) \).

The top-level contains the eigenvalues of \( T \), that may include simple and non-simple types. Besides, the edges of the non-simple eigenvalues are extended to the descendant eigenvalues in the next level observed through the reduction process. On the other hand, simple eigenvalues have no further edges. We now derive the descendant eigenvalues from the non-simple eigenvalues at the former level.

For a certain non-simple eigenvalue \( \lambda_h \) of \( T \), we can expand a perturbed eigenprojection of the form (5.5). Thus, we obtain the following equation (see II.(2.16)–(2.17) in [11]):
Using this equation, we introduce (see II.(2.37) in [11]):

\[
\tilde{T}_0(\chi) = \frac{\left( T(\chi) - \lambda_h I \right) P_h(\chi)}{\chi} = \sum_{n=0}^{\infty} \chi^n \tilde{T}_0^{(n+1)}
\]

Since \( T \) is symmetric, \( \lambda_h \) is semisimple, so the terms are simplified. For example, the spectral decomposition of \( \tilde{T}_0^{(1)} \) in \( R(P_h) \) is given as follows (see II.(2.38)–(2.39) in [11]):
\[ \hat{T}_0^{(1)} = P_h T^{(1)} P_h = \sum_{j^{(1)}} \lambda_{0,j^{(1)}} P_{0,j^{(1)}}. \]

Here, \( \{ \lambda_{0,j^{(1)}} \}_{j^{(1)}} \) denote the eigenvalues of \( \hat{T}_0^{(1)} \) in \( R(P_h) \). Generally, we call these eigenvalues the descendant eigenvalues of \( \lambda_0 \) in level-1. Note that \( \hat{T}_0^{(1)} = \tilde{T}_0(0) = \tilde{T}_0 \).

Moreover, if we set \( \hat{\lambda}_h \) as the top node of the tree, the descendant eigenvalues \( \{ \lambda_{0,j^{(1)}} \}_{j^{(1)}} \) are clearly derived from \( \hat{\lambda}_h \).

At the beginning of the process, we initialize a set of non-simple eigenvalues of \( T \) as \( \Phi(0) \); thus, if \( \Phi(0) = \phi \), no further processing is required, and each eigenvalue of \( T \) is explicitly expanded by Eq. (5.7).

Otherwise, we find the descendant eigenvalues of each element in \( \Phi(0) \), and we initialize a set \( \Phi(1) \) of non-simple eigenvalues in level-1 (line 12-14 in Algorithm 1). If \( \Phi(1) = \phi \), then each eigenvalue in level-1 is explicitly expanded by Eq. (5.7).

Then, the eigenvalues \( \lambda_{0,j^{(1)}}(\chi) \) are explicitly expanded as follows (see Theorem 2.3 in [11]):

\[ \lambda_{0,j^{(1)}}(\chi) = \lambda_{0,j^{(1)}} + \sum_{n=1}^{\infty} \chi^n \lambda_{0,j^{(1)}}^{(n)}, \]

and the corresponding eigenvalues of \( T(\chi) \) are expressed as follows:

\[ \hat{\lambda}_h(\chi) = \hat{\lambda}_h + \chi \lambda_{0,j^{(1)}}(\chi) \]
\[ = \hat{\lambda}_h + \chi \lambda_{0,j^{(1)}} + \sum_{n=1}^{\infty} \chi^{n+1} \lambda_{0,j^{(1)}}^{(n)}. \]

The \( \lambda_{0,j^{(1)}}^{(n)} \) are given by Eq. (5.8) with \( T^{(v)} \) and \( S_h^{(k)} \) replaced by \( \hat{T}_0^{(v+1)} \) and \( S_1^{(k)} \) (to be defined later), respectively (see II.(2.48) in [11]).

If \( \Phi(1) \neq \phi \), then the reduction process is reiterated to find the descendant eigenvalues of those in \( \Phi(1) \).

We now explain our procedure in details; for simplicity, let us represent \( \lambda_{0,j^{(1)}} \) by \( \lambda_{0,1} \) and \( P_h \) by \( P_0 \). Let \( \lambda_{0,1} \in \Phi(1) \) be the eigenvalue of \( \hat{T}_0 \), and the descendant eigenvalue of a certain \( \lambda_0 \). In addition, we also take the corresponding perturbed eigenvalue \( \hat{\lambda}_{0,1}(\chi) \) as the eigenvalue of \( \hat{T}_0(\chi) \). The eigenprojection \( P_{0,1} \) corresponding to this \( \lambda_{0,1} \) in \( R(P_0) \) can be computed as follows:

\[ P_{0,1} = \int_{I_{0,1}} R_{0,1}(\zeta) \, d\zeta, \]

where \( R_{0,1}(\zeta) = (\hat{T}_0 - \zeta I)^{-1} \), and \( I_{0,1} \) denotes a closed curve isolating \( \lambda_{0,1} \) from all other eigenvalues of \( \hat{T}_0 \). Therefore, we define \( \tilde{T}_{0,1} = P_{0,1} \hat{T}_0^{(2)} P_{0,1} \) and
\[ P_{0,1}^{(n)} = -\sum_{p=1}^{n} (-1)^p \sum_{v_1 + \cdots + v_p = n} S_1^{(k_1)} T_0^{(v_1+1)} \cdots S_1^{(k_p)} T_0^{(v_p+1)} S_1^{(k_{p+1})} , \]

\[ \tilde{T}_{0,1}^{(n)} = -\sum_{p=1}^{n} (-1)^p \sum_{v_1 + \cdots + v_p = n} S_1^{(k_1)} T_0^{(v_1+1)} \cdots S_1^{(k_p)} T_0^{(v_p+1)} S_1^{(k_{p+1})} . \]

Here,

\[ S_1^{(k)} = \begin{cases} S' - \frac{1}{\lambda_{0,1}'} (I - P_0) & (k \geq 1), \\ -P_{0,1} & (k = 0) \end{cases} \]

with

\[ S' = -\sum_{k' \neq 0,1} \frac{P_{k',1}}{|\lambda_{0,1} - \lambda_{0,1}'|}. \]

For the equalities above, see II. (2.12) and II. (2.43)–(2.45) in [11]. We assumed above that \( \lambda_{0,1} \neq 0 \) and \( \lambda_{0,1}' \) denotes an eigenprojection corresponding to \( \lambda_{0,1}' \). The summation runs over all eigenvalues of \( \tilde{T}_0 \) except for \( \lambda_{0,1} \). We then define the following equalities:

\[ P_{0,1}(\chi) = P_{0,1} + \chi P_{0,1}^{(1)} + \chi^2 P_{0,1}^{(2)} + \cdots , \]

\[ \tilde{T}_{0,1}(\chi) = \begin{cases} (\tilde{T}_0(\chi) - \lambda_{0,1} I) P_{0,1}(\chi) = \sum_{n=0}^{\infty} \chi^n \tilde{T}_{0,1}^{(n+1)} & \chi \neq \lambda_{0,1} \end{cases} . \]

From Eqs. (5.6) and (5.11), it is observed that \( \tilde{T}_{0,1} \equiv \tilde{T}_{0,1}(0) = \tilde{T}_0^{(1)} = P_{0,1} \tilde{T}_0 P_{0,1} \). Now, the eigenvalues of \( \tilde{T}_{0,1} \) and \( \tilde{T}_{0,1}(\chi) \) (denoted by \( \lambda_{0,1,2} \) and \( \lambda_{0,1,2}(\chi) \), respectively) are the descendant eigenvalues of \( \lambda_{0,1} \) in level-2. This procedure is applied to all eigenvalues in \( \Phi(1) \) in level-1. Now, if the eigenvalue \( \lambda_{0,1,2} \) of \( \tilde{T}_{0,1}(0) = \tilde{T}_{0,1} \) is simple, then its perturbed value can be represented by

\[ \lambda_{0,1,2}(\chi) = \sum_{n=0}^{\infty} \chi^n \lambda_{0,1,2}^{(n)} . \]

The \( \lambda_{0,1,2}^{(n)} \) are given by Eq. (5.8) by replacing \( T^{(v)} \) and \( S_1^{(k)} \) by \( T^{(v+1)} \) and \( S_1^{(k+1)} \) (to be defined later), respectively. The corresponding eigenvalue of \( \tilde{T}(\chi) \) is represented as follows:

\[ \lambda(\chi) = \lambda_0 + \chi \lambda_{0,1} + \chi^2 \sum_{n=0}^{\infty} \chi^n \lambda_{0,1,2}^{(n)} . \]
Moreover, if all of these eigenvalues are simple, then they can be determined by utilizing Eq. (5.7), and the procedure terminates. Otherwise, we should recheck whether all eigenvalues in level-2 are simple. For this purpose, we initialize a set $\Phi_{(2)}$ of non-simple eigenvalues in level-2 and discuss the case in which $\Phi_{(2)} \neq \phi$. We first introduce the following equality:

$$P_{0,1,2} = \int_{\Gamma_{0,1,2}} R_{0,1,2}(\zeta) \, d\zeta,$$

where $R_{0,1,2}(\zeta) = (\tilde{T}_{0,1} - \zeta I)^{-1}$, and $\Gamma_{0,1,2}$ denotes a closed curve that isolates $\lambda_{0,1,2}$ from all other eigenvalues of $\tilde{T}_{0,1}$. We also define the following equality:

$$P_{0,1,2}^{(n)} = -\sum_{p=1}^{n} (-1)^p \sum_{v_1 + \ldots + v_p = n} S_{2}^{(k_1)\gamma(v_1+1)} \ldots S_{2}^{(k_p)\gamma(v_p+1)} S_{2},$$

$$\tilde{T}_{0,1,2}^{(n)} = -\sum_{p=1}^{n} (-1)^p \sum_{v_1 + \ldots + v_p = n} S_{2}^{(k_1)\gamma(v_1+1)} \ldots S_{2}^{(k_p)\gamma(v_p+1)} S_{2},$$

where

$$S_{2}^{(k)} = \begin{cases} [S_{2}' - \frac{1}{\lambda_{0,1,2} - \lambda_{0,1,2}'} (I - P_{0,1,2})]^{k} & (k \geq 1), \\ -P_{0,1,2} & (k = 0) \end{cases}$$

with

$$S_{2}' = -\sum_{\lambda_{0,1,2}' \neq \lambda_{0,1,2}} \frac{P_{\lambda_{0,1,2}'}}{|\lambda_{0,1,2} - \lambda_{0,1,2}'|}.$$  

We assume that $\lambda_{0,1,2} \neq 0$, and let $P_{\lambda_{0,1,2}'}$ be an eigenprojection corresponding to $\lambda_{0,1,2}'$. The summation of the right-hand side is taken over all eigenvalues of $\tilde{T}_{0,1}$ except for $\lambda_{0,1,2}$. Then, we introduce the following equalities:

$$P_{0,1,2}(\chi) = P_{0,1,2} + \sum_{j=1}^{\infty} \chi^j P_{0,1,2}^{(j)},$$

$$\tilde{T}_{0,1,2}(\chi) = \frac{(\tilde{T}_{0,1}(\chi) - \lambda_{0,1,2} I)}{\chi} P_{0,1,2}(\chi) = \sum_{n=0}^{\infty} \chi^n \tilde{T}_{0,1,2}^{(n+1)}.$$  

Again, we note that $\tilde{T}_{0,1,2} \equiv \tilde{T}_{0,1,2}(0) = \tilde{T}_{0,1,2}^{(1)} = P_{0,1,2} \tilde{T}_{0,1} P_{0,1,2}$. Now, the eigenvalues of $\tilde{T}_{0,1,2}$ and $\tilde{T}_{0,1,2}(\chi)$, denoted by $\lambda_{0,1,2,3}$ and $\lambda_{0,1,2,3}(\chi)$, respectively, are the descendant eigenvalues of $\lambda_{0,1,2}$ in level-3. In level-2, this procedure is applied to all the eigenvalues in $\Phi_{(2)}$. By repeating this procedure until none of the eigenvalues splits, we obtain the exact representations of all the eigenvalues. Note that the procedure
The sensitivity of a quantum PageRank stops within a finite number of iterations (at most level-N iterations). Without loss of generality, we have assumed that all eigenvalues (including the descendant ones) do not vanish. If some eigenvalues vanish, then the unit of $\chi$ can be adjusted so that all eigenvalues that appear in the iteration procedure always remain non-vanishing.

**Remark 2** While tracing the tree, the convergence radius should be updated as the minimum of $r_1$ computed by employing Eq. (5.10) or the convergence radius of the eigenvalue at the bottom of each path. Since this procedure terminates after finitely many iterations, we can obtain the final convergence radius, and assign it to $r_1$. In addition, the estimated expression (inequality (5.9)) holds for all $\lambda_h$. By resetting the constants, we can replace $\rho_h$ and $r_h$ with constants that are independent of $h$; hereafter denoted by $\rho_1$ and $r_1$, respectively.

This fact is summarized as a proof at the end of this subsection.

**Lemma 4** Assume that inequality (3.8) holds. In the above expressions of $\lambda_h(\chi)$, we have $|\lambda_h^{(n)}| \leq \rho_1 r_1^{-n}$ ($n = 1, 2, \ldots, h = 1, 2, \ldots, s$) with constants $\rho_1$, $r_1 > 0$, which are independent of $h$.

**Proof** For the eigenvalues of $T$, it is sufficient to assume inequality (5.30). If this actually holds, then we have the lower bound of the convergence radius of Eq. (5.3). From this, we have the desired estimate (see, Example II.3.1 in [11], and the foregoing arguments. Additionally, we can directly estimate $|\lambda^{(n)}|$ following Eq. (5.8) (see, [16] and Sect. II.3.6 in [11]). It remains to show that Lemma 4 holds for the descendant eigenvalues. Now, we take a simple eigenvalue $\lambda_{0,1,\ldots,L+1}$ of $T_{0,1,\ldots,L}$ in level-$(L + 1)$ and perturb it as $\lambda_{0,1,\ldots,L+1}^{(n)}(\chi)$. Then, the perturbed eigenvalue is an eigenvalue of $T_{0,1,\ldots,L} = \sum_{n=0}^{\infty} \chi^n T_{0,1,\ldots,L}^{(n+1)}$. The corresponding eigenvalue of $T$ is then defined by

$$\lambda(\chi) = \lambda_h + \chi^l + \lambda_{0,1} + \cdots + \chi^L \lambda_{0,1,\ldots,L} + \chi^{L+1} \sum_{n=0}^{\infty} \chi^n \lambda_{0,1,\ldots,L+1}^{(n)}.$$  

It is sufficient to show that $T_{0,1,\ldots,L}^{(n)}$ can be estimated as follow:

$$\|T_{0,1,\ldots,L}^{(n)}\| \leq \tilde{A}_L \tilde{B}_L^{n-1} n = 1, 2, \ldots$$  

(5.12)

We show this by induction. First, consider the case wherein $L = 0$. Since $T$ is normal, we have $\|S_h\| \leq 1 / \min_{\lambda_k \neq \lambda_h} |\lambda_k - \lambda_h|$. (See, Problem I.6.43 in [11]). Using this, we can write
\[ \| \tilde{T}^{(n)}_0 \| \leq \sum_{p=1}^{n} \left| \frac{1}{\min_{\lambda_k \neq \lambda_h} |\lambda_k - \lambda_h|} \right|^{p-1} \sum_{\nu_1 + \cdots + \nu_p = n} A^p_2 B^n_2 \]

\[ = B^n_2 \sum_{p=1}^{n} \left| \frac{1}{\min_{\lambda_k \neq \lambda_h} |\lambda_k - \lambda_h|} \right|^{p-1} A^p_2 n - 1 C_{n-p} 2p - 1 C_{p-1} \]

\[ \leq 2^{2n-1} B^n_2 C^n_h \left[ C_h A_2 \left( 1 + C_h A_2 \right)^{n-1} \right], \]

where \( C_h \equiv \left| \frac{1}{\min_{\lambda_k \neq \lambda_h} |\lambda_k - \lambda_h|} \right|. \) Thus, inequality (5.12) holds for \( L = 0. \) Next, assume that inequality (5.12) holds for \( L = (l - 1). \) Then, for \( L = l, \) we have

\[ \tilde{T}^{(n)}_{0,1, \ldots, l} = \]

\[ - \sum_{p=1}^{\nu} (-1)^p \sum_{\nu_1 + \cdots + \nu_p = n} S^{(k)}_l \tilde{T}^{(v_1+1)}_{0,1, \ldots, l-1} \cdots S^{(v_p+1)}_{l-1} \tilde{T}^{(v_p+1)}_{0,1, \ldots, l} S^{(k+1)}_{l}. \]

Assuming inequality (5.12), we have \( \| \tilde{T}^{(n)}_{0,1, \ldots, l-1} \| \leq A_{l-1} B^{n-1}_{l-1}. \) Recall also that

\[ S^{(k)}_l = \begin{cases} \left[ S'_l - \frac{1}{\lambda_{0,1, \ldots, l}} (I - P_{0,1, \ldots, l}) \right]^k & (k \geq 1), \\ -P_{0,1, \ldots, l} & (k = 0) \end{cases} \]

with

\[ S'_l = - \sum_{j'_{0,1, \ldots, l} \neq j_{0,1, \ldots, l}} \frac{P_{j'_{0,1, \ldots, l}}}{\min |\lambda_{0,1, \ldots, l} - \lambda_{j'_{0,1, \ldots, l}}|}, \]

\[ P_{0,1, \ldots, l} = \int_{\Gamma_{0,1, \ldots, l}} R_{0,1, \ldots, l} (\zeta) \, d\zeta, \quad R_{0,1, \ldots, l} (\zeta) = \left( \tilde{T}^{(1)}_{0,1, \ldots, l-1} - \zeta I \right)^{-1}, \]

where \( \Gamma_{0,1, \ldots, l} \) encloses the eigenvalue \( \lambda_{0,1, \ldots, l}. \) Since \( \tilde{T}^{(1)}_{0,1, \ldots, l} \) is normal, we have \( \| P_{0,1, \ldots, l} \| \) and \( \| P_{j'_{0,1, \ldots, l}} \| = 1. \) Moreover, we also observe that

\[ \| S^{(k)}_l \| \leq \max \left\{ 1, \left( \frac{1}{\min |\lambda_{0,1, \ldots, l} - \lambda_{0,1, \ldots, l}|} + \frac{1}{\lambda_{0,1, \ldots, l}} \right)^k \right\} \equiv \max \{ 1, C_l^k \}. \]

Thus, we obtain

\[ \Box \]
The sensitivity of a quantum PageRank

\[ \| \widetilde{T}_{0,1,\ldots,d}^{(n)} \| \leq \sum_{p=1}^{n} \sum_{v_1 + \cdots + v_p = n}^{k_1 + \cdots + k_{p+1} = p-1, v_j \geq 1, k_j \geq 0} \max\{1, C_l^{p-1}\} \tilde{A}_{l-1}^{p} \tilde{B}_{l-1}^{p} \]

where \( \tilde{A}_{l-1} \) and \( \tilde{B}_{l-1} \) denote some positive constants. The desired estimate \( \| \widetilde{T}_{0,1,\ldots,d}^{(n)} \| \leq A_{l} B_{l}^{-1} \) is then deduced as described for inequality (5.12).

5.4 Perturbation on \( U \)

This subsection discusses the form of \( U \) under a perturbation. We first consider the expansion of \( |\psi_j(\chi)\rangle = |j\rangle \otimes \sum_{k=1}^{N} \sqrt{g_{jk}(\chi)} |k\rangle \) with respect to \( \chi \). Thus, we find the expansion of \( \sqrt{g_{jk}(\chi)} \). Now, \( \sqrt{g_{jk}(\chi)} = \sqrt{g_{jk}(1 + \tilde{g}_{jk}(\chi))} \), where \( \tilde{g}_{jk}(\chi) \equiv (g_{jk}(\chi) - g_{jk})/g_{jk} \). Hereafter, the coefficient of \( \chi^n \) in the expansion of \( \sqrt{1 + \tilde{g}_{jk}(\chi)} \) is denoted by \( a_{(j,k)}^{(n)} \), that is,

\[ \sqrt{1 + \tilde{g}_{jk}(\chi)} = \sum_{n=0}^{\infty} a_{(j,k)}^{(n)} \chi^n. \]

Again, from the elementary theory of complex analysis [14], we have

\[ a_{(j,k)}^{(n)} = \frac{1}{n!} \left( \frac{d}{d\chi} \right)^n \left[ \sqrt{1 + \tilde{g}_{jk}(\chi)} \right]_{\chi=0}. \]

By employing the argument in Sect. 5.1, we have

\[ a_{(j,k)}^{(n)} = \sum_{r=1}^{n} \frac{C_r}{g_{jk}^r} \sum_{p_1 + \cdots + p_r = n}^{p_j \geq 1} \prod_{i=1}^{r} g_{jk}^{(p_i)}, \]

which implies that

\[ |\psi_j(\chi)\rangle = |\psi_j\rangle + \sum_{n=1}^{\infty} |\psi_j^{(n)}\rangle \chi^n. \quad (5.13) \]

The convergence radius of (5.13) is lower-bounded by \( r_1 \). \( B(\chi) \) is then expanded as follows:
\[ B(\chi) = \sum_{j=1}^{N} |\psi_j(\chi)\rangle\langle \psi_j(\chi)| = B + \sum_{n=1}^{\infty} B^{(n)} \chi^n, \quad (5.14) \]

where

\[ B^{(n)} = \sum_{j=1}^{N} \sum_{l=0}^{n} |\psi_j^{(l)}\rangle\langle \psi_j^{(n-l)}|. \]

For simplicity, we have utilized the notation \(|\psi_j^{(0)}\rangle = |\psi_j\rangle\). By applying Eqs. (5.13) and (5.14), the operator \(U\) under a perturbation is represented by

\[ U(\chi) = S_w \left( 2B(\chi) - I \right) = U + \sum_{n=1}^{\infty} U^{(n)} \chi^n, \quad (5.15) \]

where

\[ U^{(n)} = 2S_w B^{(n)} = 2S_w \sum_{j=1}^{N} \sum_{l=0}^{n} |\psi_j^{(l)}\rangle\langle \psi_j^{(n-l)}|. \quad (5.16) \]

The infinite sum (5.15) converges as far as \(|\psi(\chi)\rangle\), so their convergence radii are also \(r_1\). We now define the eigenprojection of \(U\) for each eigenvalue of \(U\).

Note that \(U\) and \(U^{(\lambda)}\) are unitary, and consequently semisimple. The resolvent of \(U\) is defined by \(\hat{R}(\zeta) \equiv (U - \zeta I)^{-1}\), and the eigenprojection corresponding to eigenvalue \(\mu_h\) (recall that \(\mu_h\) can be \(\pm 1\)) is given by

\[ \hat{P}_h \equiv -\frac{1}{2\pi i} \int_{\hat{L}_h} \hat{R}(\zeta) \, d\zeta. \]

Here, \(\hat{L}_h\) denotes an arbitrary convex loop in the complex plain that isolates \(\mu_h\) from all other eigenvalues of \(U\). Under a small perturbation, we now represent the resolvent and eigenprojection. We denote the resolvent of \(U(\chi)\) by \(\hat{R}(\zeta, \chi) \equiv (U(\chi) - \zeta I)^{-1}\), where \(\hat{R}(\zeta, \chi)\) is defined for all \(\zeta \notin \sigma(U(\chi))\). For such a \(\zeta\), we know that \(\hat{R}(\zeta, \chi)\) is holomorphic with respect to both \(\zeta\) and \(\chi\), and it can be expanded as follows:

\[ \hat{R}(\zeta, \chi) = \hat{R}(\zeta) \left[ I + \hat{A}(\chi)\hat{R}(\zeta) \right]^{-1} = \hat{R}(\zeta) + \sum_{l=1}^\infty \chi^l \hat{R}^{(l)}(\zeta), \quad (5.17) \]

where \(\hat{A}(\chi) = U(\chi) - U\), and

\[ \hat{R}^{(l)}(\zeta) \equiv \sum_{\nu_1 + \cdots + \nu_p = l \atop \nu_j \geq 1} (-1)^p \hat{R}(\zeta) U^{(\nu_1)} \hat{R}(\zeta) U^{(\nu_2)} \cdots \hat{R}(\zeta) U^{(\nu_p)} R(\zeta). \]

(II.(1.13)–(1.14) in [11]). By integrating the infinite series (5.17) over \(\hat{L}_h\), we have
\[
\hat{P}_h(\chi) = -\frac{1}{2\pi i} \int_{\hat{h}} \hat{R}(\zeta, \chi) \, d\zeta = \hat{P}_h + \sum_{l=1}^{\infty} \chi^l \hat{P}_h^{(l)}. \tag{5.18}
\]

Equation (5.18) satisfies \(\hat{P}_h(0) = \hat{P}_h = -\frac{1}{2\pi i} \int_{\hat{h}} \hat{R}(\zeta) \, d\zeta\). In Eq. (5.18), \(\hat{P}_h^{(l)}\) is given by

\[
\hat{P}_h^{(l)} = -\frac{1}{2\pi i} \sum_{\nu_1 + \cdots + \nu_p = l} (-1)^p \int_{\hat{h}} \hat{R}(\zeta) U^{(\nu_1)} \hat{R}(\zeta) U^{(\nu_2)} \cdots \hat{R}(\zeta) U^{(\nu_p)} \, d\zeta.
\]  

Besides, we define

\[
\hat{S}_h = -\frac{1}{2\pi i} \int_{\hat{h}} (\zeta - \mu_h)^{-1} \hat{R}(\zeta) \, d\zeta,
\]

which satisfies \(\hat{P}_h \hat{S}_h = \hat{S}_h \hat{P}_h = 0\). Additionally, we have \(U \hat{P}_h = \mu_h \hat{P}_h\). For later use, we define the following equation when \(q \in \mathbb{Z}\):

\[
\hat{S}_h^{(0)} = -\hat{P}_h, \quad \hat{S}_h^{(q)} = \hat{S}_h^{(q)} (q > 0), \quad \hat{S}_h^{(q)} = 0 (q < 0).
\]

Here, we have used the fact that \(U\) is unitary, and therefore normal. Later, these expressions are used for deriving the transformation function of \(\{|\mu\}\).

### 5.5 Perturbation on the eigenvalues of \(U\)

We now represent the eigenvalues of \(U(\chi)\). The eigenvalues of \(U\) (other than those of \(-S_w\)) are given by

\[
\mu = \lambda \pm i\sqrt{1 - \lambda^2}.
\]

Accordingly, the eigenvalues of \(U(\chi)\) are \(\mu(\chi) = \lambda(\chi) + i\left(1 - (\lambda(\chi))^2\right)^{\frac{1}{2}}\) for \(\chi\) sufficiently close to the real axis, where \(\lambda(\chi)\)'s denote the eigenvalues of \(T(\chi)\). Note that \(|\chi|\) is sufficiently small that \(|\lambda(\chi)| < 1\). Let \(r_2\) denote the value with which \(|\lambda(\chi)| = 1\) hold for \(|\chi| = r_2\). Then, the convergence radius of Eq. (5.7) is \(r_0 = \min\{r_1, r_2\}\). As far as \(|\lambda(\chi)| < 1\) is satisfied, the branches of \(\left(1 - (\lambda(\chi))^2\right)^{\frac{3}{2}}\) are holomorphic with respect to \(\chi\). Therefore, by expanding \(\mu(\chi)\) as a series of \(\chi\), we get

\[
\mu(\chi) = \mu + \sum_{n=1}^{\infty} \chi^n \mu^{(n)}.
\]

To show this, we first expand \(\sqrt{1 - (\lambda(\chi))^2}\), and recall that
\[
\mu^{(n)} = \frac{1}{n!} \left( \frac{d}{d\lambda} \right)^n \mu(\lambda) \bigg|_{\lambda=0} \quad (n = 1, 2, \ldots).
\]

We now discuss
\[
\frac{1}{n!} \left( \frac{d}{d\lambda} \right)^n \left( \sqrt{1 - (\lambda(\chi))^2} \right) \bigg|_{\lambda=0}.
\]

By setting \( f(z) = \sqrt{1 - z^2} \), we have
\[
\left( \frac{d}{d\lambda} \right)^n f(\lambda(\chi)) \bigg|_{\lambda=0} = \sum_{r=1}^{n} \frac{D^r f(\lambda(\chi))}{r!} \sum_{p_1 + \cdots + p_r = n, \quad \max(p_i) \geq 1} \frac{n!}{p_1! \cdots p_r!} \left( D^{p_1} \lambda(\chi) \right) \cdots \left( D^{p_r} \lambda(\chi) \right) \bigg|_{\lambda=0}.
\]

(5.20)

where we have used the fact that \( D^{p_i} \lambda(\chi) \big|_{\chi=0} = p_i \lambda^{(p_i)} \). To express the term \( D^r f \) in Eq. (5.20), we set \( h(w) = \sqrt{1 - w} \) and \( l(t) = t^r \); thus, we obtain the following equation:
\[
\left( \frac{d}{dt} \right)^r h(l(t)) = \sum_{r'=1}^{r} \frac{D^{r'} h(l(t))}{r'!} \sum_{p_1' + \cdots + p_{r'}' = r, \quad \max(p_i') \geq 1} \frac{r!}{p_1'! \cdots p_{r'}'!} \left( D^{p_1'} l \right) \cdots \left( D^{p_{r'}'} l \right).
\]

We also note that \( p_i' \in \{1, 2\} \) \( (i = 1, 2, \ldots, r') \) and
\[
D^{p_i'} l = \begin{cases} 
2t & (p_i' = 1), \\
2 & (p_i' = 2).
\end{cases}
\]

Furthermore, we can write
\[
D^{r'} \left[ \sqrt{1 - w} \right] = (-1)^{r'} \frac{r'}{2} C_{r'}(1 - w)^{-\frac{r' - 1}{2}}.
\]

Thus, we have
The sensitivity of a quantum PageRank which gives an explicit representation of $\mu(\chi)$.  

### 5.6 Perturbation on the eigenvectors of $U$

This subsection considers the perturbed eigenvectors of $U$. Note that the space $\mathcal{H} = \mathbb{C}^N \otimes \mathbb{C}^N$ has the following orthogonal decomposition:

$$\mathcal{H} = H_d + H_d^\perp,$$

where $H_d$ denotes the space spanned by $\{|\psi_j\rangle\}_j$ and $\{|s_w\rangle\}_j$ (called the dynamical space in [9]), in which $U$ operates as a non-trivial operator. Meanwhile, on $H_d^\perp$ (which is spanned by vectors orthogonal to $\{|\psi_j\rangle\}_j$), $U$ acts as $-S_w$.

Since $U$ is unitary, its eigenvectors (here called eigenstates) that are normalized to unit length form an orthonormal basis in $\mathcal{H}$. Particularly, the eigenvectors in $H_e$ form an orthonormal basis there. Since the imposed perturbation preserves the unitarity of $U(\chi)$, we consider a set of its eigenvectors $\{|\mu(\chi)\rangle\}$ of unit length as an orthogonal basis of $H_e$ under small perturbation. This method is demonstrated later on.

Similarly, a set of perturbed eigenvectors $\{|\mu(\chi)\rangle\}$ corresponding to the eigenvalues of $-S_w$ forms an orthogonal basis of the space orthogonal to $H_e$ (denoted as $H_e^\perp$). Now, applying the transformation function, we can write

$$|\mu(\chi)\rangle = V(\chi)|\mu\rangle,$$

where $V(\chi)$ denotes an operator satisfying

$$\begin{cases} 
\frac{d}{d\chi} V(\chi) = Q(\chi)V(\chi), \\
V(0) = I,
\end{cases}$$

(5.22)

with $Q(\chi) = -\sum_{h=1}^s \hat{P}_h(\chi)\frac{d}{d\chi}\hat{P}_h(\chi)$. Since $Q(\chi)$ is holomorphic within a certain region of $\mathbb{C}$, then Eq. (5.22) is guaranteed to have a unique holomorphic solution $V(\chi)$ with an inverse matrix $V(\chi)^{-1}$. Actually, this solution satisfies

$$V(\chi)\hat{P}_h(0)V(\chi)^{-1} = \hat{P}_h(\chi) \quad (h = 1, 2, \ldots, s).$$
Using the expansion of $\hat{P}_h(\chi)$ in Eq. (5.18), we expand $Q(\chi)$ as a series in terms of $\chi$:

$$Q(\chi) = -\sum_{h=1}^{s} \left\{ \hat{P}_h + \sum_{n'=1}^{\infty} \chi^{n'} \hat{P}_h^{(n')} \right\} \left\{ \hat{P}_h^{(1)} + \sum_{n=1}^{\infty} (n + 1) \chi^n \hat{P}_h^{(n+1)} \right\} \equiv Q + \chi Q^{(1)} + \chi^2 Q^{(2)} \ldots .$$

By comparing the coefficients, we have

$$Q = -\sum_{h=1}^{s} \hat{P}_h \hat{P}_h^{(1)},$$

$$Q^{(r)} = -\sum_{h=1}^{s} \left[ (r + 1) \hat{P}_h \hat{P}_h^{(r+1)} + \sum_{r_1=1}^{r} (r - r_1 + 1) \hat{P}_h^{(r_1)} \hat{P}_h^{(r-r_1+1)} \right] \quad (r = 1, 2, \ldots).$$

(5.23)

For example, $Q^{(1)} = -2 \sum_{h=1}^{s} \hat{P}_h \hat{P}_h^{(2)} - \sum_{h=1}^{s} (\hat{P}_h^{(1)})^2$. We now find the solution of Eq. (5.22) as an infinite series of $\chi$:

$$V(\chi) = V_0 + \sum_{n=1}^{\infty} \chi^n V^{(n)},$$

and formally represent $V^{(n)}$ by comparing the coefficients. Then, we observe that $V^{(n)}$s are estimated from above with certain quantities to estimate the convergence radius.

First, from the initial condition, we have $V_0 = I$. It is easy to see that

$$V^{(1)} = Q, \quad V^{(2)} = \frac{Q^2 + Q^{(1)}}{2}, \quad V^{(3)} = \frac{Q^3}{6} + \frac{Q^{(1)} Q}{3} + \frac{Q Q^{(1)}}{6} + \frac{Q^{(2)}}{3}.$$

Similarly, we observe that

$$V^{(n)} = \sum_{j=0}^{n-1} \frac{Q^{(j)} V^{(n-1-j)}}{n} \quad (n = 4, 5, 6, \ldots).$$

(5.24)

Using this, we expand $|\mu(\chi)\rangle$ as follows:

$$|\mu(\chi)\rangle = |\mu\rangle + \sum_{n=1}^{\infty} \chi^n |\mu^{(n)}\rangle,$$

where

$$|\mu^{(n)}\rangle = V^{(n)} |\mu\rangle.$$ 

(5.25)

When discussing a PageRank, it is sufficient to consider $|\mu\rangle \in H_\ast$. Near the real axis, $N_q(\chi) (i, j, m | \psi(0)\rangle)$ can be expanded as follows (recall (3.6); that is, $I_q^\chi(i, m | \psi(0))$ is the squared sum of the magnitudes of $N_q(\chi) (i, j, m | \psi(0))$):

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\[ N^{(\chi)}_q(i, j, m \mid \psi(0)) = N_q(i, j, m \mid \psi(0)) + \sum_{r=1}^{\infty} \chi^r N^{(\chi)}_q(i, j, m \mid \psi(0)) \quad \forall i, \]

where \n
\[ N^{(\mu)}_q(i, j, m \mid \psi(0)) \]

\[ = \sum_{\mu \in H} \sum_{0 \leq r_1 + r_2 \leq n} \mu^{2m} \frac{2m C_{r'}}{\mu^{r'}} \sum_{p_1 + \cdots + p_{r'} = r_1} \prod_{r=1}^{r'} \mu^{(p_r)} \]

\[ \times \langle \mu^{(r_2)} \mid \psi(0) \rangle \langle j, i \mid \mu^{(n-r_1-r_2)} \rangle. \]

This can be derived by observing the coefficients of \( \chi^r \) in the expansion of \( \mu(\chi)^{2m} \), which can be represented by

\[ a_1(r_1; m) = \mu^{2m} \frac{\min\{2m, r_1\}}{r_1!} \frac{2m C_{r'}}{\mu^{r'}} \sum_{p_1 + \cdots + p_{r'} = r_1} \prod_{r=1}^{r'} \mu^{(p_r)}. \]

The derivation is similar to that of \( t_0(\chi) \) in Sect. 5.1. It follows that

\[ a_1(r_1; m) = \frac{1}{r_1!} \left( \frac{d}{d\chi} \right)^{r_1} \mu(\chi)^{2m} \bigg|_{\chi=0} = \mu^{2m} \frac{\min\{2m, r_1\}}{r_1!} \left( \frac{d}{d\chi} \right)^{r_1} \left( 1 + \tilde{\mu}(\chi) \right)^{2m} \bigg|_{\chi=0}, \]

where \( \tilde{\mu}(\chi) = (\mu(\chi) - \mu) / \mu = \sum_{n=1}^{\infty} \chi^n \mu^{(n)} / \mu. \) By setting \( f(z) = (1 + z)^{2m} \) and applying \( D^0 \tilde{\mu}(\chi) \big|_{\chi=0} = p_1 ! \mu^{(p_1)} / \mu, \) we have

\[ \left( \frac{d}{d\chi} \right)^n f(\tilde{\mu}(\chi)) \bigg|_{\chi=0} = \sum_{r_1=1}^{\min\{2m, r_1\}} \frac{n!}{p_1! \cdots p_{r_1}!} \left( D^{p_1} \tilde{\mu}(\chi) \right) \cdots \left( D^{p_{r_1}} \tilde{\mu}(\chi) \right) \bigg|_{\chi=0} \]

\[ = n! \sum_{r=1}^{\min\{2m, r_1\}} \frac{D^r f(\tilde{\mu}(\chi))}{r! \mu^r} \bigg|_{\chi=0} \sum_{p_1 + \cdots + p_r = n} \prod_{j=1}^{r} \mu^{(p_j)}. \]

Because

\[ D^r \left[ (1 + z)^{2m} \right] = \begin{cases} r! 2m C_r (1 + z)^{2m-r} & (r = 0, 1, \ldots, 2m), \\ 0 & (r \geq 2m + 1), \end{cases} \]

the index \( r \) in (5.27) actually runs from 1 to \( \min\{2m, n\}. \) It follows that
We now estimate the latter part of Theorem 1. To this end, we estimate each term \( I^{(r)}_q(i, m|\psi(0)) \) \((r = 1, 2, \ldots)\). By definition, we have

\[
a_1(r_1;m) = \mu^{2m} \sum_{r' = 1}^{\min\{2m, r_1\}} \frac{2mC_{r'}}{\mu^{r'}} p_1 + \cdots + p_{r'} = r_1 \prod_{i=1}^{r'} \mu^{(r_i)}, \tag{5.28}
\]

which yields Eq. (5.26). We now seek the form of \( I^{(r)}_q(i, m|\psi(0)) \) \((r = 1, 2, \ldots)\). By definition, we have

\[
I^{(r)}_q(i, m|\psi(0)) = \sum_{j \in V} \left( \sum_{\mu(x)} (\mu(x))^{2m} \sum_j \langle j, i | \mu(x) \rangle \langle \mu(x) | \psi(0) \rangle \right) \times \left( \sum_{\mu(x)} (\mu(x))^{2m} \sum_j \langle j, i | \mu(x) \rangle \langle \mu(x) | \psi(0) \rangle \right)^* \\
= \sum_{j \in V} \left( N_q(i, j, m|\psi(0)) + \sum_{n=1}^{\infty} \chi^n N^{(n)}_q(i, j, m|\psi(0)) \right) \times \left( N_q(i, j, m|\psi(0)) + \sum_{n=1}^{\infty} \chi^n N^{(n)}_q(i, j, m|\psi(0)) \right)^* \\
= I_q(i, m|\psi(0)) + \sum_{n=1}^{\infty} \chi^n I^{(n)}_q(i, m|\psi(0)).
\]

By comparing the coefficients, we obtain

\[
I^{(r)}_q(i, m|\psi(0)) = \sum_{j \in V} \left[ N^{(r)}_q(i, j, m|\psi(0)) N^{(n)}_q(i, j, m|\psi(0)) \right] \\
+ \sum_{l=1}^{r-1} N^{(r-l)}_q(i, j, m|\psi(0)) N^{(l)}_q(i, j, m|\psi(0)) \\
+ N_q(i, j, m|\psi(0)) N^{(r)}_q(i, j, m|\psi(0)) \right] \tag{5.29}
\]

\((r = 1, 2, \ldots)\).

### 5.7 Error bounds

We now estimate the latter part of Theorem 1. To this end, we estimate each term \( I^{(l)}_q(i, m|\psi(0)) \) \((l = 1, 2, \ldots)\) in the perturbed quantum PageRank. We first prove some lemmas to be utilized in the sequel.

**Lemma 5** Assume that inequality (3.8) holds. Then, for \( l = 1, 2, \ldots \), we have

\[
\|\psi_j^{(l)}\| \leq A_1 B_1^{l-1} \ (j = 1, 2, \ldots, N, \ l = 1, 2, \ldots),
\]

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where the constants $A_1$, $B_1 > 0$ depend on $N$ and $G$.

**Proof** This can be proved using the representation of $|\psi_j(\chi)\rangle$ (Eq. (5.13) in Sect. 5.4). Since $|\psi_j\rangle$ and $|k\rangle$ are unit vectors, it is sufficient to estimate $a_{(j,k)}^{(n)}$. Under assumption (3.8), we have

$$\left\| |\psi_j^{(n)}\rangle \right\| = \left\| |\psi_j\rangle \otimes \sum_{k=1}^{N} a_{(j,k)}^{(n)} |k\rangle \right\|$$

$$\leq \sum_{k=1}^{N} \left( \sum_{r=1}^{n} g_{jk}^{\frac{1}{2} - r} C_r \sum_{p_1 + \cdots + p_r = n} \prod_{i=1}^{r} A_{ij}^{(p_i)} B_0^{-(n-r)} \right) \right\| \right\|_p \geq 1$$

$$\leq \sum_{k=1}^{N} \sum_{r=1}^{n} g_{jk}^{\frac{1}{2} - r} C_r \left( \sum_{p_1 + \cdots + p_r = n} A_0^{n-r} B_0^{-(n-r)} \right) \right\| \right\|_p \geq 1$$

$$= B_0^n \sum_{k=1}^{N} \sum_{r=1}^{n} g_{jk}^{\frac{1}{2} - r} C_r \left( \sum_{p_1 + \cdots + p_r = n} A_0^{n-r} B_0^{-(n-r)} \right) \right\| \right\|_p \geq 1$$

By taking $\delta \in (0, 1)$, and defining $A_0/B_0 g_{jk} = \delta \times A_0/\delta B_0 g_{jk} \equiv \delta \epsilon_{(k)}$, we have

$$B_0^n \sum_{k=1}^{N} \sum_{r=1}^{n} g_{jk}^{\frac{1}{2} - r} C_r \left( A_0/B_0 g_{jk} \right)^{r-n} C_{r-1} \right\| \right\|_p \geq 1$$

$$\leq B_0^n \sum_{k=1}^{N} \left( \sum_{r=1}^{n} g_{jk}^{\frac{1}{2} - r} C_r \left( \sum_{p_1 + \cdots + p_r = n} A_0^{n-r} B_0^{-(n-r)} \right) \right) \right\| \right\|_p \geq 1$$

$$\leq B_0^n \left\{ 1 - (1 - \delta)^{\frac{1}{2}} \right\} \epsilon_{(k)} \left( 1 + \epsilon_{(k)} \right)^{n-1}.$$  

Here, we have used the fact $g_{jk} \in (0, 1)$. Now, by taking $A_1 = \max_k N \left\{ 1 - (1 - \delta)^{\frac{1}{2}} \right\} \epsilon_{(k)}$ and $B_1 = \max_k (1 + \epsilon_{(k)})$, our proof is complete. $\square$

From the above lemma, we have the following lemma.

**Lemma 6** Assume inequality (3.8) holds. Then, for $n = 1, 2, \ldots$, we have

$$\|U^{(n)}\| \leq A_2 B_2^{n-1} (n = 1, 2, \ldots),$$
where the constants $A_2, B_2 > 0$ depend on $N$ and $G$.

**Proof** By employing Eqs. (5.15)–(5.16), it is sufficient to estimate $\|B^{(n)}\|$. From the definition and Lemma 5, we have

$$\|B^{(n)}\| \leq \sum_{j=1}^{N} \sum_{l=0}^{n} (A_1 B_1^{l-1}) (A_1 B_1^{n-l}) = (n + 1) N A_1^2 B_1^{n-1}.$$  

Since $(n + 1) \leq e^n$ $(n = 1, 2, \ldots)$, we have $(n + 1) B_1^{n-1} \leq e(e B_1)^{n-1}$, yielding the desired estimate. $\square$

**Lemma 7** Assume inequality (3.8) holds. Then, for $n = 1, 2, \ldots$, we have

$$\|T^{(n)}\| \leq A_3^2 B_3^{n-1} (n = 1, 2, \ldots),$$  

(5.30)

where the constants $A_3, B_3 > 0$ depend on $N$ and $G$.

**Proof** By recalling Eq. (5.2) in Sect. 5.1 and following the proof of Lemma 5, we obtain an estimate of the following form:

$$|I^{(i,j)}_{r(n)}| \leq A_4^2 B_4^{n-1} \quad (\tau = 1, 3).$$

Similarly, we have

$$\left| \sum_{r=1}^{k} \frac{1}{C_{r}} \sum_{(r)}^{r} g_{ij}^{(p_{r})} \right| \leq A_5^2 B_5^{k-1}, \quad \left| \sum_{r=1}^{n-k} \frac{1}{C_{r}} \sum_{(r)}^{r} g_{ij}^{(p_{r})} \right| \leq A_5^2 B_5^{n-k-1},$$

which implies that

$$|I^{(i,j)}_{2(n)}| \leq |t_{ij}| \sum_{k=1}^{n-1} A_5^2 B_5^{n-2} = |t_{ij}| (n - 1) A_5^2 B_5^{n-2}.$$  

As described in the proof of Lemma 6, this is estimated by $A_5 B_5^{n-1}$ with sufficiently large $\tilde{B}$. $\square$

**Lemma 8** Assume inequality (3.8) holds. Then, for $n = 1, 2, \ldots$, we have

$$\|\hat{P}^{(n)}_h\| \leq A_6^2 B_6^{n-1} \quad (h = 1, 2, \ldots, s),$$

where the constants $A_6, B_6 > 0$ depend on $N$, elements of $G$, and the isolation distance of $T$.

**Proof** From Eq. (5.19), we have
The sensitivity of a quantum PageRank

where $\hat{\gamma}_h$ is introduced in Eq. (5.18). Additionally, the summation on the right-hand side is taken over all possible values of $p \in \mathbb{N}$ and $(v_1, \ldots, v_p)$ satisfying the condition below the summation symbol. By the normality of $U$, we obtain

$$\|\hat{\mathbf{R}}(\zeta)\| = \frac{1}{\text{dist}(\zeta, \sigma(U))} \quad \forall \zeta \in P(U). \quad (5.31)$$

In this case, note that the convergence radius $r_h$ of the series (5.15) and (5.17) is lower-bounded by

$$r_h = \left( \frac{2A_2}{\hat{d}_h} + B_2 \right)^{-1},$$

where $A_2$ and $B_2$ are given in Lemma 6, and $\hat{d}_h$ is the isolation distance of $\mu_h$. By taking $\hat{I}_h$ as a circle of radius $\hat{d}_h/2$ surrounding $\mu_h$, (5.31) becomes

$$\|\hat{\mathbf{R}}(\zeta)\| = 2/\hat{d}_h \quad \forall \zeta \in \hat{I}_h.$$ 

Then, it follows that

$$\int_{\hat{I}_h} \|\hat{\mathbf{R}}(\zeta)\|^{p+1} \, d\zeta \leq \pi\hat{d}_h \times \left( 2/\hat{d}_h \right)^{(p+1)} = 2^{(p+1)} \pi \hat{d}_h^{-p}.$$ 

Thus, we have

$$\|\hat{\mathbf{P}}_h^{(n)}\| \leq B_2^n \sum_{v_1 + \cdots + v_p = n} \left( \frac{2A_2}{B_2\hat{d}_h} \right)^p \quad \text{for} \quad v_1, \ldots, v_p \geq 1.$$ 

From the relationship $\mu_h = \lambda_h \pm i \sqrt{1 - \lambda_h^2}$, note that the isolation distance $\hat{d}_h$ is estimated from below by that of the corresponding $\lambda_h$. Now, let us consider the summation on the right-hand side. This sum is taken over all possible values of $p$ and $(v_1, \ldots, v_p)$ satisfying the condition below the summation symbol, implying that $p$ runs from 1 to $n$. For fixed $p$ ($1 \leq p \leq n$), the number of sets $(v_1, \ldots, v_p)$ satisfying $v_1 + \cdots + v_p = n$ and $v_1, \ldots, v_p \geq 1$ is $n_{\lambda} C_{n-p}$. Thus, by defining $c_0 \equiv \left( \frac{2A_2}{B_2\hat{d}_h} \right)$, we have
leading to an estimate of the desired form. □

The following lemma easily follows by applying Eqs. (5.23)–(5.24) and Lemma 8.

**Lemma 9** Assume inequality (3.8) holds. Then, for \( n = 1, 2, \ldots \), we have

\[
\| Q^{(n)} \| \leq A_7 B_7^{n-1},
\]

where the constants \( A_7, B_7 > 0 \) depend on \( N \), elements of \( G \), and the isolation distance of \( T \).

**Proof** Since \( \mathbf{U} \) is normal, \( \| \mathbf{\hat{P}}_h \| = 1 \) for \( h = 1, 2, \ldots, s \). By utilizing Eqs. (5.23)–(5.24), we have \( \| \mathbf{Q} \| \leq s \) and

\[
\| Q^{(n)} \| \leq \sum_{h=1}^{s} \left[ \sum_{r_1}^{r} (n - r_1 + 1) \| \mathbf{\hat{P}}_h^{(r)} \| \| \mathbf{\hat{P}}_h^{(n-r_1+1)} \| + (n + 1) \| \mathbf{\hat{P}}_h^{(n+1)} \| \right]
\]

\((n = 1, 2, \ldots)\).

From Lemma 8, we obtain

\[
\| \mathbf{\hat{P}}_h^{(r+1)} \| \leq A_6 B_6^r,
\]

\[
\| \mathbf{\hat{P}}_h^{(r)} \| \| \mathbf{\hat{P}}_h^{(r+1)} \| \leq A_6 B_6^{r-1} \times A_6 B_6^{r+1} = A_6^2 B_6^{r-1} \quad (r_1 = 1, 2, \ldots, r),
\]

\[
\| \mathbf{\hat{P}}_h^{(r)} \| \| \mathbf{\hat{P}}_h^{(1)} \| \leq A_6 B_6^{r-1} \times A_6 = A_6^2 B_6^{r-1},
\]

and therefore

\[
\| Q^{(n)} \| \leq \sum_{h=1}^{s} \left[ (n + 1) A_6 B_6^n + A_6^2 B_6^{n-1} + \sum_{r_1=1}^{n-1} (n - r_1 + 1) A_6 B_6^{n-1} \right],
\]

leading to the desired estimate. □

Moreover, Lemma 10 below follows from Lemma 9.

**Lemma 10** Assume inequality (3.8) holds. Then, for \( n = 1, 2, \ldots \), we have
The sensitivity of a quantum PageRank

where the constants \( A_0, B_0 > 0 \) depend on \( N \), elements of \( G \), and the isolation distance of \( T \).

**Proof** From Eq. (5.25) and given that \( \| \mu \| = 1 \), it suffices to show that

\[
\| V^{(n)} \| \leq A_9 B_9^{n-1} \quad (n = 1, 2, \ldots).
\]

The desired estimate can be obtained by induction. Assuming \( \| V^{(j)} \| \leq A_9 B_9^{j-1} \) for \( j = 2, 3, 4, \ldots, n-1 \), and noting that \( \| Q \| \leq s \), we have

\[
\| V^{(n)} \| \leq \frac{1}{n} \left\{ s + A_7 A_9 \sum_{j=2}^{n-2} B_7^{j-1} B_9^{n-2-j} \right\} \leq s + A_7 A_9 (B_7 + B_9)^{n-1}.
\]

By setting \( \bar{B} = \max\{1, (B_7 + B_9)\} \), we get that

\[
\| V^{(n)} \| \leq s \bar{B} + A_7 A_9 \bar{B}^{n-1} \leq (s + A_7 A_9) \bar{B}^{n-1},
\]

which is the desired estimate. \( \square \)

### 5.8 An estimate of \( |N_q^{(n)}(i, j, m \mid \Omega(0))| \)

We now estimate \( |N_q^{(n)}(i, j, m \mid \psi(0))| \) for \( n = 1, 2, \ldots \).

**Lemma 11** Assume inequality (3.8) holds. Then, for \( n = 1, 2, \ldots \), we have

\[
|N_q^{(n)}(i, j, m \mid \psi(0))| \leq A_{11} B_{11}^{n-1},
\]

where the constants \( A_{11}, B_{11} > 0 \) depend on \( N \) and \( m \), as well as on the elements and isolation distance of \( G \).

**Proof** Recall the representation (5.26). From Lemma 10, \( \| \psi(0) \| = 1 \), and given that \( |j, i \rangle \) forms an orthonormal basis in \( \mathcal{H} \), we have

\[
\left| \langle \mu^{(r_2)} \mid \psi(0) \rangle \right| \left| \langle j, i \mid \mu^{(n-r_1-r_2)} \rangle \right| \leq A_8^2 B_8^{n-r_1-2}.
\]

To estimate \( |\mu^{(n)}| \), we recall (5.21) and estimate the right-hand side. By applying Eq. (5.8), Remark 3, and \( r_0 \) in Sect. 5.5, we have \( |\lambda^{(j)}| \leq \rho_0^{-p_j} \) \( (j = 1, 2, \ldots, r) \). Additionally, we also note that \( p_1' \cdots p_r' \geq 1 \). Then, we have
\[
\frac{1}{n!} \left( \frac{d}{d\chi} \right)^n \left( 1 - (\lambda(\chi))^2 \right)^{\frac{1}{2}} \bigg|_{\chi=0} \leq r_0^n \sum_{r=1}^n \phi_0^r \sum_{r'=1}^r 2^{r'-1} C_{r'-1}^r \left( 1 - \lambda^2 \right)^{-\frac{\varphi(r'-1)}{2}}.
\] (5.32)

By letting \(|\lambda|^2 < 1 - \varepsilon_1\), and defining \(\varepsilon_1^{-1} = \delta_1 \times (\delta_1 \varepsilon_1)^{-1}\) with some \(\delta_1 \in (0, 1)\), we can write

\[|1 - \lambda^2|^{-r'} < \delta_1^r (\delta_1 \varepsilon_1)^{-r'}.
\]

It follows that

\[
\sum_{r'=1}^r \left( \frac{2}{(\delta_1 \varepsilon_1)^{-1}} \right)^{r'-1} C_{r'-1}^r (1 - \lambda^{2 - r'}) \leq \left( \sum_{r'=1}^r \left( \frac{2}{\delta_1 \varepsilon_1} \right)^{r'-1} C_{r'-1}^r \right) \left( \sum_{r'=1}^r C_{r'-1}^r \delta_1^{r'} \right) \leq \left( \frac{2}{\delta_1 \varepsilon_1} \right) \left( 1 + \frac{2}{\delta_1 \varepsilon_1} \right)^{r-1} \left\{ 1 - \left( 1 - \delta_1 \right)^{\frac{1}{2}} \right\}
\]

\[\equiv A_{12} B_{12}^r.
\]

From these expressions, we can estimate the right-hand side of inequality (5.32). We have

\[
\frac{1}{n!} \left( \frac{d}{d\chi} \right)^n \left( 1 - (\lambda(\chi))^2 \right)^{\frac{1}{2}} \bigg|_{\chi=0} \leq A_{12} r_1^{n-1} \sum_{r=1}^n (\varphi_1 B_{12})^r n_{r-1} C_{r-1}
\]

\[= A_{12} B_{12}^{r_1} \varphi_1 \left( 1 + \varphi_1 B_{12} \right)^{n-1}.
\]

By combining this inequality with the estimate of \(|\lambda^{(n)}|\), we obtain the estimate \(|\mu^{(n)}| \leq A_{13} B_{13}^{n-1}\) with constants \(A_{13}, B_{13} > 0\).

Based on this, we estimate \(|N^{(n)}_q(i, j, m \parallel \psi(0))|\). From Eq. (3.1), the quantum PageRank \(I_q(i, m \parallel \psi(0))\) is the sum of the squared magnitude of a complex number \(N_q(i, j, m \parallel \psi(0))\), and \(N^{(n)}_q(i, j, m \parallel \psi(0))\) are the coefficients in the expansion of \(N_q(i, j, m \parallel \psi(0))\) with respect to \(\chi\). Recall that we have denoted it by

\[
N^{(n)}_q(i, j, m \parallel \psi(0)) = \sum_{\mu \in \mathcal{H}_r} \sum_{0 \leq r_1 + r_2 \leq n} \sum_{r_1 \geq 0} \langle \mu^{(r_2)} | \psi(0) \rangle \langle j, i | \mu^{(n-r_1-r_2)} \rangle a_1(r_1 ; m).
\]

Using Eq. (5.28) and noting that \(|\mu| = 1\), we obtain
The sensitivity of a quantum PageRank

\[ |a_1(r_1; m)| \leq \sum_{r=1}^{\min\{2m, r_1\}} 2mC_r A_{13}^r B_{13}^{r_1-r} r_1 - C_{r_1-r} \]

\[ \leq B_{13}^{r_1} \left( \sum_{r=1}^{2m} 2mC_r \right) \left( \sum_{r=1}^{r_1} r_1 - C_{r_1-r} \left( A_{13}/B_{13} \right)^r \right) \]

\[ = B_{13}^{r_1} 2^{2m} \left( A_{13}/B_{13} \right) \left( 1 + \left( A_{13}/B_{13} \right) \right)^{r_1-1}, \]

which yields an estimate of the following form

\[ |a_1(n; m)| \leq A_{14} B_{14}^{r_1-1} \]

with constants \( A_{14}, B_{14} > 0 \). Thus, we have

\[ |N_q^{(n)}(i, j, m||\psi(0))| \leq \sum_{\mu,j} \sum_{0 \leq r_1 + r_2 \leq n} A_{14} B_{14}^{r_1-1} A_{8}^2 B_8^{n-r_1-2} \]

\[ \leq n A_{14}^2 B_{14}^{n-2} B_8^{-1} \sum_{r_1=0}^{n} (B_{14}/B_8)^{r_1} \]

\[ \leq n A_{14} B_{14}^n, \]

where \( A_{11} = A_{8}^2 A_{14} B_8^{-2} / (B_{14} - B_8) \). Without loss of generality, we assume that \( B_8 < B_{14} \) and also assume a sufficiently large constant \( B_{11} > 0 \) (in case \( B_8 > B_{14} \), we have a similar estimate). The right-most-hand side in the above inequality is then estimated from above by \( A_{11} B_{11}^{n-1} \).

We now estimate \( |t_q^{(n)}(i, m||\psi(0))| \). By employing Lemma 10, we have \( |N_q(i, j, m||\psi(0))| \leq A_{11} \) and \( |N_q^{(j)}(i, j, m||\psi(0))| \leq A_{11} B_{11}^{r_1-1} (l = 1, 2, \ldots). \) The utilization of Eq. (5.32) gives that

\[ |t_q^{(n)}(i, m||\psi(0))| \leq (2B_{11} + (n - 1)) A_{11}^2 B_{11}^{n-2} \quad (n \geq 2). \]

Note that \( (2B_{11} + (n - 1)) \frac{1}{n-1} \leq c_0 \) where \( c_0 \) denotes some positive constant. Without loss of generality, we can assume that \( B_{11} > 1 \) and

\[ \log \{2B_{11} + (n - 1)\} \frac{1}{n-1} = \frac{1}{n-1} \log \{2B_{11} + (n - 1)\}. \]

Moreover, \( \log \{2B_{11} + (n - 1)\} \leq 2B_{11} + n - 2 \), implying that \( \log \{2B_{11} + (n - 1)\} \frac{1}{n-1} \) is bounded. Accordingly, \( \{2B_{11} + (n - 1)\} \frac{1}{n-1} \) is also bounded. This completes the proof of Theorem 1.
6 Conclusion

This study discussed the sensitivity analysis of quantum PageRank proposed in [9, 10]. We analytically showed that perturbing the original Google matrix altered the temporal value of quantum PageRank. In addition, we obtained the lower limit of the convergence radius of the expansion of the perturbed PageRank. Moreover, the temporally averaged value of quantum PageRank and its temporal limitation did not analytically depend on the perturbation in general. In our future study, we will extend the definition of quantum PageRank and the sensitivity analysis to graph limits (called graphons).

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