TOTAL EMBEDDING DISTRIBUTIONS OF RINGEL LADDERS

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1

ABSTRACT

The total embedding distributions of a graph is consisted of the orientable embeddings and non-orientable embeddings and have been known for few classes of graphs. The genus distribution of Ringel ladders is determined in [Discrete Mathematics 216 (2000) 235-252] by E.H. Tesar. In this paper, the explicit formula for non-orientable embeddings of Ringel ladders is obtained.

Key words: Graph embedding; Ringel ladders; Overlap matrix; Chebyshev polynomials;
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1. Background

One enumerative aspect of topological graph theory is to count genus distributions of a graph. The history of genus distribution began with J. Gross in 1980s. Since then, it has been attracted a lot of attentions, for the details, we may refer to [1, 8, 9, 11, 13, 16, 17, 19, 22, 26, 28, 31, 32, 33, 34, 35, 36, 37, 38] etc (We only list a few). However, for the total embedding distributions, only few classes are known. For example, Chen, Gross and Rieper [2] computed the total embedding distribution for necklaces of type \((r,0)\), close-end ladders and cobblestone paths, Kwak and Shim [21] computed for bouquets of circles and dipoles. In [3], Chen, Liu and Wang calculated the total embedding distributions of all graphs with maximum genus 1. Furthermore, in [4], Chen, Mansour and Zou obtained explicit formula for total embedding distributions for the necklaces of type \((r,s)\), closed-end ladders and cobblestone path.

It is assumed that the reader is somewhat familiar with the basics of topological graph theory as found in Gross and Tucker [12]. A graph \(G = (V(G), E(G))\) is permitted to have both loops and multiple edges. A surface is a compact closed 2-dimensional manifold without boundary. In topology, surfaces are classified into \(O_m\), the orientable surface with \(m(m \geq 0)\) handles and \(N_n\), the nonorientable surface with \(n(n > 0)\) crosscaps. A graph embedding into a surface means a cellular embedding.

A spanning tree of a graph \(G\) is a tree on its edges has the same order as \(G\). The number co-tree edges of a spanning tree of \(G\) is called the Betti number, \(\beta(G)\), of \(G\). A rotation at a vertex \(v\) of a graph \(G\) is a cyclic order of all edges incident with \(v\). A pure rotation system \(P\) of a graph \(G\) is the collection of rotations at all vertices of \(G\). A general rotation system is a pair \((P, \lambda)\), where \(P\) is a pure rotation system and \(\lambda\) is a mapping \(E(G) \rightarrow \{0,1\}\). The edge \(e\) is said to be twisted (respectively, untwisted) if \(\lambda(e) = 1\) (respectively, \(\lambda(e) = 0\)). It is well known that every orientable embedding of a graph \(G\) can be described by a general rotation

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system \((P, \lambda)\) with \(\lambda(e) = 0\) for all \(e \in E(G)\). By allowing \(\lambda\) to take the non-zero value, we can describe nonorientable embeddings of \(G\), see \([2, 30]\) for more details. A \(T\)-rotation system 
\((P, \lambda)\) of \(G\) is a general rotation system 
\((P, \lambda)\) such that \(\lambda(e) = 0\), for all \(e \in E(T)\).

**Theorem 1.1.** (see \([2, 30]\)) Let \(T\) be a spanning tree of \(G\) and \((P, \lambda)\) a general rotation system. Then there exists a general rotation system \((P', \lambda')\) such that

1. \((P', \lambda')\) yields the same embedding of \(G\) as \((P, \lambda)\), and
2. \(\lambda'(e) = 0\), for all \(e \in E(T)\).

Two embeddings are considered to be the same if their \(T\)-rotation systems are combinatorially equivalent. Fix a spanning tree \(T\) of a graph \(G\). Let \(\Phi^T_G\) be the set of all \(T\)-rotation systems of \(G\). It is known that

\[
|\Phi^T_G| = 2^{\beta(G)} \prod_{v \in V(G)} (d_v - 1)!. 
\]

Suppose that in these \(\Phi^T_G\) embeddings of \(G\), there are \(a_i\), \(i = 0, 1, \ldots\), embeddings into orientable surface \(O_i\) and \(b_j\), \(j = 1, 2, \ldots\), embeddings into nonorientable surface \(N_j\). We call the polynomial

\[
I^T_G(x, y) = \sum_{i=0}^{\infty} a_i x^i + \sum_{j=1}^{\infty} b_j y^j 
\]

the \(T\)-distribution polynomial of \(G\). By the total genus polynomial of \(G\), we shall mean the polynomial

\[
I_G(x, y) = \sum_{i=0}^{\infty} g_i x^i + \sum_{i=1}^{\infty} f_i y^i, 
\]

where \(g_i\) is the number of embeddings (up to equivalence) of \(G\) into the orientable surface \(O_i\) and \(f_i\) is the number of embeddings (up to equivalence) of \(G\) into the nonorientable surface \(N_i\). We call the first (respectively, second) part of \(I_G(x, y)\) the genus polynomial (respectively, crosscap number polynomial) of \(G\) and denoted by \(g_G(x) = \sum_{i=0}^{\infty} g_i x^i\) (respectively, \(f_G(y) = \sum_{i=1}^{\infty} f_i y^i\)). Clearly, \(I_G(x, y) = g_G(x) + f_G(y)\). This means the number of orientable embeddings of \(G\) is \(\prod_{v \in G}(d_v - 1)!\), while the number of non-orientable embeddings of \(G\) is \((2^{\beta(G)} - 1) \prod_{v \in G}(d_v - 1)!\).

Let \(T\) be a spanning tree of \(G\) and \((P', \lambda')\) be a \(T\)-rotation system. Let \(e_1, e_2, \ldots, \epsilon_{\beta(G)}\) be the cotree edges of \(T\). The overlap matrix of \((P', \lambda')\) is the \(\beta \times \beta\) matrix \(M = [m_{ij}]\) over \(GF(2)\) such that \(m_{ij} = 1\) if and only if either \(i \neq j\) and the restriction of the underlying pure rotation system to \(T + e_i + e_j\) is nonplanar, or \(i = j\) and \(e_i\) is twisted. The following theorem due to Mohar.

**Theorem 1.2.** (see \([24]\)) Let \((P, \lambda)\) be a general rotation system for a graph, and let \(M\) be the overlap matrix. Then the rank of \(M\) equals twice the genus, if the corresponding embedding surface is orientable, and it equals the crosscap number otherwise. It is independent of the choice of a spanning tree.

An \(n\)-rung closed-end ladder \(L_n\) can be obtained by taking the graphical cartesian product of an \(n\)-vertex path with the complete graph \(K_2\), and then doubling both its end edges. Figure 1 presents a 4-rung closed-end ladder.

Ringel ladders, \(R_n\), are the graphs used by Ringel and Youngs in their proof of the Heawood Map Coloring Theorem. In fact, A Ringel ladder, \(R_n\), can be formed by subdividing the end-rungs of the closed-end ladder, \(L_n\), and adding an edge between these two new vertices. Figure 2 shows the Ringel ladder \(R_4\).
2. Homogeneous recurrence relation and Chebyshev polynomials

To begin with the discussion, we give some concepts of the \( n \)-th Chebyshev polynomials of the second kind which is related to the solution of the recurrence relation. Let the recurrence function \( U_n(x) \) be

\[
U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)
\]

with the initial conditions \( U_0(x) = 1, U_1(x) = 2x \), then we derived the \( n \)-th Chebyshev polynomials with the second kind \( U_n(x) \) (see [27]). For instance, \( U_2(x) = 4x^2 - 1, U_3(x) = 8x^3 - 4x, U_4(x) = 16x^4 - 12x^2 + 1 \). Moreover, we have the identity that

\[
U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}(-1)^k(2x)^{n-2k}.
\]

Now, we will build the relation between the recurrence relation and the Chebyshev polynomials with the second kind. Let \( P_n(z) = \sum_{m=0}^{n} C_n(m)z^m \), satisfy the following

\[
P_n(z) = a_1(z)P_{n-1}(z) + a_2(z)P_{n-2}(z),
\]

where \( a_i(z) = \sum_{k=0}^{q} a_{ki} z^k \) for \( i = 1, 2 \), and the initial conditions \( P_0(z) = c_0 \), and \( P_1(z), P_2(z) \) can be derived by the initial values of \( C_n(m) \).

Let \( Q_n(z) = \frac{P_n(z)}{(\sqrt{a_2(z)})^n} \), then it is easy to verify that

\[
Q_n(z) = \frac{a_1(z)}{\sqrt{a_2(z)}}Q_{n-1}(z) - Q_{n-2}(z)
\]
with the initial conditions \( Q_0(z) = P_0(z) = c_0, \) \( Q_1(z) = \frac{P(z)}{a_2(z)}, \) and \( Q_2(z) = \frac{P_2(z)}{a_2(z)}. \) Using the fact that \( U_0(x) = 1, \) \( U_1(x) = 2x, \) \( U_2(x) = 4x^2 - 1, \) by induction on \( n = 0, 1, 2, \) we obtain that

\[
Q_n(z) = A U_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + B U_n-1(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + C U_n-2(\frac{a_1(z)}{2\sqrt{a_2(z)}i}),
\]

where \( A, B, \) and \( C \) are determined by the initial conditions. Thus we have

\[
P_n(z) = (\sqrt{a_2(z)i})^n A U_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + B U_n-1(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) + C U_n-2(\frac{a_1(z)}{2\sqrt{a_2(z)}i}).
\]

Using the fact that

\[
U_n(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} (-1)^k (2x)^{n-2k}.
\]

We can derive that

\[
(i\sqrt{a_2(z)})^n U_n(\frac{a_1(z)}{2\sqrt{a_2(z)}i}) = \sum_{j\geq 0} \binom{n-j}{j} (a_1(z))^{n-2j}.
\]

Since \( a_1(z) \) is a polynomial of degrees less than \( q, \) then \( (a_1(z))^{n-2j} \) can be expressed as the type of power series. Plug the above formula into (3) and comparing the coefficient \( z^n \) in both sides and we can obtain the explicit formulae \( C_n(m) \) for \( 0 \leq m \leq n. \)

3. Total embedding distributions of Ringel ladders

3.1. The rank-distribution polynomial of Closed-end ladders. We adopt the notations of \([4]\), the overlap of matrix of Closed-end ladders \( L_{n-1} \) has the following form \( M_{n}^{X,Y} \) (see \([4]\) for more details).

Let \( X = (x_1, x_2, \ldots, x_n) \in (GF(2))^n \) and \( Y = (y_1, y_2, \ldots, y_{n-1}) \in (GF(2))^{n-1}. \) We define the tridiagonal matrix \( M_{n}^{X,Y} \) as

\[
M_{n}^{X,Y} = \begin{pmatrix}
x_1 & y_1 & & 0 \\
y_1 & x_2 & y_2 & \\
& y_2 & x_3 & y_3 \\
& & \vdots & \ddots & \vdots \\
0 & & & y_{n-2} & x_{n-1} \\
& & \vdots & \ddots & \vdots \\
& & & & y_{n-1} & x_n
\end{pmatrix}.
\]

Furthermore, we define \( \mathcal{L}_n = \{M_{n}^{X,Y} \mid X \in (GF(2))^n \text{ and } Y \in (GF(2))^{n-1}\} \), which is the set of all matrices over \( GF(2) \) that are of the type \( M_{n}^{X,Y} \). We define the rank-distribution polynomial to be the polynomial \( \mathcal{L}_n(z) = \sum_{j=0}^{n} D_n(j)z^j \), where \( D_n(j), \ j = 0, 1, \ldots, n, \) is the number of different assignment of the variables \( x_j, y_k, \) where \( j = 1, 2, \ldots, n \) and \( k = 1, 2, \ldots, n-1, \) for which the matrix \( M_{n}^{X,Y} \) in \( \mathcal{L}_n \) has rank \( j \). Similarly, Let \( \mathcal{O}_n = \{M_{n}^{X,Y} \mid Y \in (GF(2))^{n-1}\} \), and \( \mathcal{O}_n(z) = \sum_{j=0}^{n} O_n(j)z^j \) be the rank-distribution polynomial of \( \mathcal{O}_n, \) where \( O_n(j), \ j = 0, 1, \ldots, n, \) is the number of different assignment of the variables \( y_k, \) where \( k = 1, 2, \ldots, n-1, \) for which the matrix \( M_{n}^{X,Y} \) in \( \mathcal{O}_n \) has rank \( j \).

Lemma 3.1. (see \([4]\)) The polynomial \( \Theta_n(z) \) satisfies the recurrence relation

\[
\Theta_n(z) = \Theta_{n-1}(z) + 2z^2 \Theta_{n-2}(z)
\]

with the initial conditions \( \Theta_1(z) = 1 \) and \( \Theta_2(z) = z^2 + 1. \)
Theorem 3.2. (see [4]) For all \( n \geq 1 \),
\[
\Theta_n(z) = \sum_{j \geq 0} \binom{n-j}{j} 2^j z^{2j} - \sum_{j \geq 0} \binom{n-2-j}{j} 2^j z^{2j+2}.
\]

Corollary 3.3. For all \( 1 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \),
\[
\Theta_n(2m+1) = 0,
\]
\[
\Theta_n(2m) = \binom{n-m}{m} \cdot 2^m - \binom{n-m-1}{m-1} \cdot 2^{m-1}.
\]

Lemma 3.4. (see [4]) The polynomial \( L_n(z) \) satisfies the recurrence relation
\[
L_n(z) = (1 + 2iz) L_{n-1}(z) + 4z^2 L_{n-2}(z)
\]
with the initial conditions \( L_1(z) = 1 + z \) and \( L_2(z) = 4z^2 + 3z + 1 \).

Theorem 3.5. (see [4]) For all \( n \geq 1 \),
\[
L_n(z) = (2iz)^n \left[U_n \left( \frac{1+2z}{4iz} \right) + \frac{i}{2} U_{n-1} \left( \frac{1+2z}{4iz} \right) - \frac{1}{2} U_{n-2} \left( \frac{1+2z}{4iz} \right) \right],
\]
where \( U_s(t) \) is the \( s \)-th Chebyshev polynomial of the second kind and \( i^2 = -1 \).

Corollary 3.6. (see [4]) For all \( n \geq 1 \) and \( 0 \leq m \leq n \),
\[
D_n(m) = 2^m \sum_{j=0}^{\left\lfloor m/2 \right\rfloor} \binom{n-j}{j} \binom{n-2j}{n-m} - 2^{m-1} \sum_{j=0}^{\left\lfloor (m-1)/2 \right\rfloor} \binom{n-1-j}{j} \binom{n-1-2j}{n-m}
\]
\[
+ 2^{m-1} \sum_{j=0}^{\left\lfloor (m-2)/2 \right\rfloor} \binom{n-2-j}{j} \binom{n-2-2j}{n-m}.
\]

3.2. The overlap matrix of Ringel ladders. We adopt the same notation used by Ringel [27, p.17]. A cubic graph at each vertex has two cyclic orderings of its neighbors. One of these two cyclic orderings is denoted as clockwise and the other counterclockwise. We color the vertex black, if that vertex has the clockwise ordering of its neighbors, otherwise, we will color the counterclockwise vertices white. This will bring convenient to embed a cubic graph into surfaces, as we can draw an imbedding on the plane and only need to color the vertices black and white.

Definition 3.7. An edge is called matched if it has the same color at both ends, otherwise it is called unmatched.

We fix a spanning tree \( T \) of \( R_{n-1} \) shown as the thicker lines in Figure 3, that is to say, the cotree edges are \( e, a_1, a_2, \ldots, a_n \).

Property 3.8. Two cotree edges \( e \) and \( a_i \), for \( i = 1, 2, \ldots, n \), overlap if and only if the edge \( c_i \) is unmatched.

Property 3.9. Two cotree edges \( a_i \) and \( a_{i+1} \), for \( i = 1, 2, \ldots, n-1 \), overlap if and only if the edge \( b_i \) is unmatched.
It is easy to see that the overlap matrix of $R_{n-1}$ has the following form.

$$M_{n+1}^{X,Y,Z} = \begin{pmatrix} x_0 & z_1 & z_2 & z_3 & \cdots & z_{n-1} & z_n \\ z_1 & x_2 & y_1 & y_2 & & & \\ z_2 & y_1 & x_2 & y_2 & & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ z_{n-1} & 0 & y_{n-2} & x_{n-1} & y_{n-1} & x_n \\ z_n & & & & & & \end{pmatrix},$$

where $X = (x_0, x_1, \ldots, x_n) \in (GF(2))^n$, $Y = (y_1, y_2, \ldots, y_{n-1}) \in (GF(2))^{n-1}$ and $Y = (z_1, z_2, \ldots, z_n) \in (GF(2))^{n-1}$. Note that $x_0 = 1$ if and only if the edge $e$ is twisted, $x_i = 1$ if and only if the edge $a_i$ is twisted, for all $i = 1, 2, \ldots, n$. If $z_i = 1$ if and only if $b_j$ is unmatched, for all $j = 1, 2, \ldots, n - 1$, and $z_k = 1$ if and only if $c_k$ is unmatched, for all $k = 1, 2, \ldots, n$.

**Property 3.10.** For a fixed matrix of the form $M_{n+1}^{X,Y,Z}$, there are exactly 2 different $T$-rotation systems corresponding to that matrix.

**Proof.** Given a matrix $M_{n+1}^{X,Y,Z}$, the values of $z_1, z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ are determined.

- $z_0 = 0$. If we color the vertex $v_1$ black, by Property 3.8, the color of $v_2$ is black. Since the values of $z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ are given, by Property 3.8 and Property 3.9, all the colors of $v_2, v_3, \cdots, v_n, u_n, v_{n+1}$ are determined. That is to say, all the rotations of vertices of $R_n$ is determined. Otherwise the vertex $v_1$ is colored white, by Property 3.8, the color of $v_2$ is also white, by the values of $z_2, \cdots, z_n$ and $y_1, y_2, \cdots, y_{n-1}$ and by Property 3.8 and Property 3.9, the color all vertices of $R_n$ is determined.
- $z_1 = 1$. Similar discuss like the case $z_1 = 0$, the details are omitted. \hfill $\Box$

Now, we denote $\mathcal{R}_{n+1}$ be the set of all matrices over $GF(2)$ that are of the form $M_{n+1}^{X,Y,Z}$. The we calculate the rank distribution of the set $\mathcal{R}_{n+1}$.

Let $\mathcal{R}_{n+1}(z) = \sum_{j=0}^{n+1} C_{n+1}(j) z^j$ be the rank-distribution polynomial of the set $\mathcal{R}_{n+1}$. In other words, for $j = 0, 1, \ldots, n + 1$, $C_{n+1}(j)$ is the number of different assignment of the variables $x_i, i = 0, 1, \cdots, n$, $y_k, k = 1, 2, \cdots, n - 1$, and $z_l, l = 1, 2, \cdots, n$ for which the matrix $M_{n+1}^{X,Y,Z}$ in $\mathcal{R}_{n+1}$ has rank $j$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{ringel_ladders.png}
\caption{}
\end{figure}
Similarly, let \( \mathcal{P}_{n+1} \) be the set of all matrices over \( GF(2) \) that are of the form \( M^O_{n+1} \). Then we calculate the rank distribution of the set \( \mathcal{P}_{n+1} \). Let \( \mathcal{P}_{n+1}(z) = \sum_{j=0}^{n+1} D_n(j) z^j \) be the rank-distribution polynomial of the set \( \mathcal{C}_{n+1} \). In other words, for \( j = 0, 1, \ldots, n, 1, D_n(j) \) is the number of different assignment of the variables \( y_k, k = 1, 2, \ldots, n-1 \), and \( z_l, l = 1, 2, \ldots, n \) for which the matrix \( M^O_{n+1} \) in \( \mathcal{P}_{n+1} \) has rank \( j \).

**Lemma 3.11.** The polynomial \( \mathcal{P}_n(z) \) ( \( n \geq 3 \)) satisfies the recurrence relation

\[
\mathcal{P}_{n+1}(z) = \mathcal{P}_n(z) + 8z^2 \mathcal{P}_{n-1}(z) + 2^{n-1}z^2 \Theta_{n-1}(z),
\]

with the initial condition \( \mathcal{P}_2(z) = z^2 + 1, \mathcal{P}_3(z) = 7z^2 + 1 \) and \( \mathcal{P}_4(z) = 12z^4 + 19z^2 + 1 \) where \( \Theta_{n-1}(z) \) is rank-distribution polynomial of closed-end ladders \( L_{n-2} \).

**Proof.** To obtain the relation between \( \mathcal{P}_{n+1}(z) \) and \( \mathcal{P}_n(z) \), we consider the four different ways to assign the variables \( y_{n-1} \) and \( z_n \) in the matrix \( M^Y_{n+1} \).

**Case 1:** \( y_{n-1} = 0 \).

- **Subcase 1:** \( z_n = 0 \). Then the rank of \( M^Y_{n+1} \) is the same as the upper left \( n \times n \) submatrix, which is a matrix of the form \( M^Y_n \). We conclude that this case contributes to the polynomial \( \mathcal{P}_{n+1}(z) \) by a term \( \mathcal{P}_n(z) \).

- **Subcase 2:** \( z_n = 1 \). It is easy to see that, no matter what assignments of the variables \( z_1, z_2, \ldots, z_{n-1} \), we can transform \( M^Y_{n+1} \) to the following form.

\[
M_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & y_1 & 0 & y_2 \\
0 & y_1 & 0 & y_2 \\
\vdots & \ddots & \ddots & \ddots \\
0 & y_{n-2} & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

We firstly delete the first column and the last column then delete the first row and the last row of \( M_1 \), then we obtain a matrix which is a overlap matrix of closed ladders \( L_{n-2} \). Since there are \( 2^{n-1} \) different assignments of the variables \( z_1, z_2, \ldots, z_{n-1} \), it contributes to the polynomial \( \mathcal{P}_{n+1}(z) \) by a term \( 2^{n-1}z^2 \Theta_{n-1}(z) \).

**Case 2:** \( y_{n-1} = 1 \). If \( z_n = 1 \), we first add the last row to the first row, then add the last column to the first column. A similar discussion for \( y_{n-2} \) and \( z_{n-1} \), we transform \( M^Y_{n+1} \) to the following form.

\[
M = \begin{pmatrix}
0 & z_1 & z_2 & \cdots & z_{n-2} & 0 & 0 \\
z_1 & 0 & y_1 & \cdots \\
z_2 & y_1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
z_{n-2} & y_{n-3} & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

Note that the upper left \( (n-1) \times (n-1) \) submatrix of \( M_2 \), which is a matrix of the form \( M^Y_{n-1} \). There are \( 2^4 \) different assignments of the variables \( y_{n-2}, z_{n-1} \) and \( z_n \) in the matrix \( M^Y_{n+1} \). In this case, it contributes to the polynomial \( \mathcal{P}_{n+1}(z) \) by a term \( 8z^2 \Theta_n(z) \).
Lemma 3.12. The polynomial $\mathcal{R}_n(z)$ ($n \geq 3$) satisfies the recurrence relation

$$\mathcal{R}_{n+1}(z) = (4z + 1)\mathcal{R}_n(z) + 16z^2\mathcal{R}_{n-1}(z) + 2^n z^2 \mathcal{L}_{n-1}(z).$$

with the initial condition $\mathcal{R}_2(z) = 4z^2 + 3z + 1$, $\mathcal{R}_3(z) = 28z^3 + 28z^2 + 7z + 1$, where $\mathcal{L}_{n-1}(z)$ is rank-distribution polynomial of closed-end ladders $L_{n-2}$.

Proof. To obtain the relation between $\mathcal{R}_{n+1}(z)$ and $\mathcal{R}_n(z)$, we consider the eight different ways to assign the variables $x_n$, $y_{n-1}$ and $z_n$ in the matrix $M_{X,Y,Z}$.

Case 1: $x_n = 0$.

- **Subcase 1:** $y_{n-1} = z_n = 0$. Then the rank of $M_{X,Y,Z}^{X,Y,Z}$ is the same as the upper left $n \times n$ submatrix, which is a matrix of the form $M_{n-1}^{X,Y,Z}$. We conclude that this case contributes to the polynomial $\mathcal{R}_{n+1}(z)$ by a term $\mathcal{R}_n(z)$.

- **Subcase 2:** $y_{n-1} = z_n = 1$. We first add the last row to the first row, then add the last column to the first column. If $x_{n-1} = 1$, we add the last column to the $n$-th column. A similar discussion for $y_{n-2}$ and $z_{n-1}$, we transform $M_{n+1}^{X,Y,Z}$ to the following form.

$$M = \begin{pmatrix}
  x_0 & z_1 & z_2 & \ldots & z_{n-2} & 0 & 0 \\
  z_1 & x_1 & y_1 \\
  z_2 & y_1 & x_2 & \ddots \\
  \vdots & \ddots & \ddots & \ddots & y_{n-3} \\
  z_{n-2} & y_{n-3} & x_{n-2} & 0 \\
  0 & 0 & 0 & 0 & 1 \\
  0 & 1 & 0
\end{pmatrix},$$

Note that the upper left $(n-1) \times (n-1)$ submatrix, which is a matrix of the form $M_{n-1}^{X,Y,Z}$. There are $2^4$ different assignments of the variables $x_{n-1}$, $y_{n-2}$ and $z_{n-1}$, in these case it contributes to the polynomial $\mathcal{R}_{n+1}(z)$ by a term $8z^2 \mathcal{R}_{n-1}(z)$.

- **Subcase 3:** $y_{n-1} = 1, z_n = 0$. Similarly discuss like subcase 2, it contributes to the polynomial $\mathcal{R}_{n+1}(z)$ by a term $8z^2 \mathcal{R}_{n-1}(z)$.

- **Subcase 4:** $y_{n-1} = 0, z_n = 1$. It is easy to see that, no matter what assignments of the variables $x_0, z_1, z_2, \ldots, z_{n-1}$, we can transform $M_{n+1}^{X,Y,Z}$ to the following form.

$$M_1 = \begin{pmatrix}
  0 & 0 & 0 & 0 & \ldots & 0 & 1 \\
  0 & x_1 & y_1 \\
  0 & y_1 & x_2 & y_2 \\
  0 & y_2 & x_3 & \ddots \\
  \vdots & \ddots & \ddots & \ddots & y_{n-2} \\
  0 & y_{n-2} & x_{n-1} & 0 \\
  1 & 0 & 0
\end{pmatrix},$$

We firstly delete the first column and the last column then delete the first row and the last row of $M_1$, then we obtain a matrix which is a overlap matrix of closed ladders $L_{n-2}$. Since there are $2^n$ different assignments of the variables $x_0, z_1, z_2, \ldots, z_{n-1}$, it contributes to the polynomial $\mathcal{R}_{n+1}(z)$ by a term $2^n z^2 \mathcal{L}_{n-1}(z)$.

Case 2: $x_n = 1$. If $z_n = 1$, we first add the last column to the first column then add the last row to the first row. Similarly, if $y_{n-1} = 1$, we add the last column to the $n$-th column and add
the last row to the \( n \)-th row. As last we can transfer the matrix \( M_{n+1}^{X,Y,Z} \) to the matrix \( M_2 \) of following form.

\[
M_2 = \begin{pmatrix}
x_0 & z_1 & z_2 & z_3 & \cdots & z_{n-1} & 0 \\
z_1 & x_1 & y_1 & & & & \vdots \\
z_2 & y_1 & x_2 & y_2 & & & \vdots \\
z_3 & y_2 & x_3 & & & & \vdots \\
& & & \ddots & & & \vdots \\
z_{n-1} & \cdots & \cdots & \cdots & y_{n-2} & 0 & 0 \\
0 & & & & & y_{n-2} & x_{n-1} \\
0 & & & & & 0 & 1
\end{pmatrix},
\]

Note that the upper left \( n \times n \) submatrix of \( M_2 \), which is a matrix of the form \( M_n^{X,Y,Z} \). There are \( 2^2 \) different assignments of the variables \( y_{n-1} \) and \( z_n \) in the matrix \( M_n^{X,Y,Z} \). In this case, it contributes to the polynomial \( \mathcal{P}_n(z) \) by a term \( 4z \mathcal{P}_n(z) \).

**Theorem 3.13.** For all \( n \geq 2 \),

\[
\mathcal{P}_n(z) = (2\sqrt{2iz})^n \left\{ U_n\left(\frac{1}{2\sqrt{2iz}}\right) - \frac{z^2 + 1}{4\sqrt{2iz}} U_{n-1}\left(\frac{1}{4\sqrt{2iz}}\right) + \frac{17z^2 - 1}{16z^2} U_{n-2}\left(\frac{1}{4\sqrt{2iz}}\right) \right\} + 2^{n-1}z^2 \mathcal{P}_{n-1}(z),
\]

where \( U_s(t) \) is the \( s \)-th Chebyshev polynomial of the second kind, \( i^2 = -1 \) and \( \mathcal{P}_{n-1}(z) \) is rank-distribution polynomial of closed-end ladders \( L_{n-2} \).

**Proof.** Note that

\[
\mathcal{P}_{n+1}(z) = \mathcal{P}_n(z) + 8z^2 \mathcal{P}_{n-1}(z) + 2^{n-1}z^2 \mathcal{P}_{n-1}(z).
\]

We first consider the homogeneous recurrence relation part of (11).

\[
\mathcal{P}_{n+1}(z) = \mathcal{P}_n(z) + 8z^2 \mathcal{P}_{n-1}(z).
\]

By the method of subsection 2, we have a solution of (12).

\[
\mathcal{P}_n(z) = \left(\frac{1}{2\sqrt{2iz}}\right)^n \left\{ A U_n\left(\frac{1}{2\sqrt{2iz}}\right) + BU_{n-1}\left(\frac{1}{2\sqrt{2iz}}\right) + CU_{n-2}\left(\frac{1}{2\sqrt{2iz}}\right) \right\}
\]

Now, let \( Y_n(z) = 2^n f(z) \mathcal{P}_n(z) \) be one special solution of \( \mathcal{P}_n(z) \), plug it into (11), using the relation

\[
\mathcal{P}_n(z) = (1 + 2z) \mathcal{P}_{n-1}(z) + 4z^2 \mathcal{P}_{n-2}(z),
\]

it leads to

\[
Y_n(z) = 2^{n-1}z^2 \mathcal{P}_{n-1}(z) = \sum_{m \geq 0} 2^{n-1}O_{n-1}(m)z^{m+2}.
\]

Thus,

\[
\mathcal{P}_n(z) = (2\sqrt{2iz})^n \left\{ U_n\left(\frac{1}{2\sqrt{2iz}}\right) + BU_{n-1}\left(\frac{1}{2\sqrt{2iz}}\right) + CU_{n-2}\left(\frac{1}{2\sqrt{2iz}}\right) \right\} + 2^{n-1}z^2 \mathcal{P}_{n-1}(z).
\]
Plug the initial values \( R_2(z), R_3(z) \) into (14), it follows that
\[
\begin{align*}
-8z^2 \left( 2\frac{1}{\sqrt{2iz}} + B \right) & + \frac{1}{\sqrt{2iz}} + (C - 1) \right) + 2z^2 = z^2 + 1 \\
-16\sqrt{2iz}^3 \left( \frac{1}{\sqrt{2iz}} + B \right) & \left( - \frac{1}{8z^2} - 1 + \frac{1}{\sqrt{2iz}} \right) + 4z^2(z^2 + 1) = 7z^2 + 1.
\end{align*}
\]
By simple computation, we immediately obtain
\[
B = -\frac{z^2 - 1}{4\sqrt{2iz}}, \quad C = \frac{17z^2 - 1}{16z^2}.
\]
Then according to the identity (1), the formula (14) is as follows
\[
R_n(z) = \sum_{j \geq 0} \binom{n - j}{j} (8z^2)^j - \frac{z^2 + 1}{2} \times \sum_{j \geq 0} \binom{n - 1 - j}{j} (8z^2)^j
\]
\[
- \frac{17z^2 - 1}{2} \left\{ \sum_{j \geq 0} \binom{n - 2 - j}{j} (8z^2)^j \right\} + 2^{n-1}z^2 \mathcal{L}_{n-1}(z).
\]
Comparing the coefficient of \( z^m \) in both sides, thus for all \( n \geq 2 \) and \( 0 \leq m \leq n \), we have the following result.

**Theorem 3.14.** For all \( n \geq 2 \),
\[
R_n(z) = (4z^2 + 1)R_{n-1}(z) + 16z^2R_{n-2}(z) + 2^{n-1}z^2 \mathcal{L}_{n-2}(z).
\]
where \( R_n(t) \) is the \( s \)-th Chebyshev polynomial of the second kind, \( i^2 = -1 \) and \( \mathcal{L}_{n-1}(z) \) is rank-distribution polynomial of closed-end ladders \( L_{n-2} \).

**Proof.** Note that
\[
R_n(z) = (4z + 1)R_{n-1}(z) + 16z^2R_{n-2}(z) + 2^{n-1}z^2 \mathcal{L}_{n-2}(z). \tag{11}
\]
We first consider the homogeneous recurrence relation part of (11).
\[
R_n(z) = (4z + 1)R_{n-1}(z) + 16z^2R_{n-2}(z). \tag{12}
\]
By the method of subsection 2, we have a solution of (12).
\[
R_n(z) = \left( \sqrt{2a_2(z)}i \right)^n \left\{ A U_n \left( \frac{a_1(z)}{2\sqrt{a_2(z)i}} \right) + B U_{n-1} \left( \frac{a_1(z)}{2\sqrt{a_2(z)i}} \right) + C U_{n-2} \left( \frac{a_1(z)}{2\sqrt{a_2(z)i}} \right) \right\} \tag{13}
\]
Now, let \( Y_n(z) = 2^n f(z) \mathcal{L}_{n}(z) \) be one special solution of \( R_n(z) \), plug it into (11), using the relation
\[
\mathcal{L}_{n}(z) = (1 + 2z)\mathcal{L}_{n-1}(z) + 4z^2\mathcal{L}_{n-2}(z),
\]
it leads to \( f(z) = \frac{z^2 \mathcal{L}_{n-1}(z)}{\mathcal{L}_{n}(z)} \).
Thus we obtain a special solution of non-homogeneous recurrence (11)
\[
Y_n(z) = 2^n z^2 \mathcal{L}_{n-1}(z) = \sum_{m \geq 0} 2^n C_{n-1}(m)z^{m+2}.
\]
Corollary 3.15. Thus,
\[
\mathcal{R}_n(z) = (4zi)^n \left\{ U_n \left( \frac{1 + 4z}{8iz} \right) + BU_{n-1} \left( \frac{1 + 4z}{8iz} \right) + CU_{n-2} \left( \frac{1 + 4z}{8iz} \right) \right\} + 2^n z^2 \mathcal{L}_{n-1}(z).
\]
Plug the initial values $\mathcal{R}_2(z), \mathcal{R}_3(z)$ into (14), it follows that
\[
\begin{align*}
-16z^2 \left\{ (2\left( \frac{1 + 4z}{8iz} \right) + B) U_1 \left( \frac{1 + 4z}{8iz} \right) + C - 1 \right\} + & 4z^2(1 + z) = 4z^2 + 3z + 1 \\
-64iz^3 \left\{ (2\left( \frac{1 + 4z}{8iz} \right) + B) U_1 \left( \frac{1 + 4z}{8iz} \right) + U_1 \left( \frac{1 + 4z}{8iz} \right) (C - 1) \right\} + & 8z^2(4z^2 + 3z + 1) \\
& = 28z^3 + 28z^2 + 7z + 1.
\end{align*}
\]
By simple computation, we immediately obtain
\[
B = -\frac{2z^2 - 7z - 1}{8iz}, \quad C = \frac{34z^2 - z - 1}{32z^2}.
\]
Then according to the identity (1), the formula (14) is as follows
\[
\mathcal{R}_n(z) = \sum_{j \geq 0} \left( \frac{n - j}{j} \right) (1 + 4z)^{n-2j} (4z)^{2j} \left\{ \sum_{j \geq 0} \left( \frac{n - 1 - j}{j} \right) (1 + 4z)^{n-1-2j} (4z)^{2j} \right\} + \frac{34z^2 - z - 1}{2} \sum_{j \geq 0} \left( \frac{n - 2 - j}{j} \right) (1 + 4z)^{n-2-2j} (4z)^{2j} + 2^n z^2 \mathcal{L}_{n-1}(z).
\]
Comparing the coefficient of $z^m$ in both sides of (15), thus for all $n \geq 2$ and $0 \leq m \leq n$, we have the following result.

Corollary 3.15. For all $n \geq 2$ and $0 \leq m \leq n$,
\[
C_n(m) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \left( \frac{n - j}{j} \right) \left( \frac{n - 2j}{n - m} \right) 4^m - \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \left( \frac{n - j - 1}{j} \right) \left( \frac{n - 1 - 2j}{n - m + 1} \right) 4^{m-2}
\]
\[
- \frac{7}{2} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \left( \frac{n - j - 1}{j} \right) \left( \frac{n - 1 - 2j}{n - m} \right) 4^{m-1} - \frac{1}{2} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \left( \frac{n - j - 1}{j} \right) \left( \frac{n - 1 - 2j}{n - m - 1} \right) 4^m
\]
\[
- \frac{17}{2} \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \left( \frac{n - j - 2}{j} \right) \left( \frac{n - 2 - 2j}{n - m} \right) 4^{m-2} + \frac{1}{2} \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \left( \frac{n - j - 2}{j} \right) \left( \frac{n - 2 - 2j}{n - m - 1} \right) 4^{m-1}
\]
\[
+ \frac{1}{2} \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \left( \frac{n - j - 2}{j} \right) \left( \frac{n - 2 - 2j}{n - m - 2} \right) 4^m + 2^n D_{n-1}(m - 2).
\]
where
\[ D_n(m) = 2^m \sum_{j=0}^{\lfloor m/2 \rfloor} \binom{n-j}{j} \binom{n-2j}{n-m} - 2^{m-1} \sum_{j=0}^{\lfloor (m-1)/2 \rfloor} \binom{n-1-j}{j} \binom{n-1-2j}{n-m} \]
\[ + 2^{m-1} \sum_{j=0}^{\lfloor (m-2)/2 \rfloor} \binom{n-2-j}{j} \binom{n-2-2j}{n-m}. \]

Theorem 3.16. The total genus polynomial of Ringel ladders \( R_{n-1}(x, y) \) is as follows:
\[ I_{R_{n-1}}(x, y) = 2 \sum_{j=0}^{n+1} C_{n+1}(j)y^j - I_0(R_{n-1}, y) + I_0(R_{n-1}, x) \]
where \( I_0(R_{n-1}, x) \) is the genus polynomial of Ringel ladder \( R_{n-1} \), which has been derived by E.H. Tesar [35].

Proof. By Property 3.10, the theorem follows. \( \square \)

For instance, the above theorem gives
\[ I_{R_1}(x, y) = 2 + 14x + 14y + 42y^2 + 56y^3, \]
\[ I_{R_2}(x, y) = 2 + 38x + 24x^2 + 22y + 122y^2 + 424y^3 + 392y^4, \]
\[ I_{R_3}(x, y) = 2 + 70x + 184x^2 + 30y + 242y^2 + 1448y^3 + 3272y^4 + 2944y^5, \]
\[ I_{R_4}(x, y) = 2 + 118x + 648x^2 + 256x^3 + 38y + 410y^2 + 3496y^3 + 12952y^4 + 26880y^5 + 20736y^6, \]
\[ I_{R_5}(x, y) = 2 + 198x + 1656x^2 + 2240x^3 + 46y + 642y^2 + 7240y^3 + 36808y^4 + 120832y^5 + 207168y^6 + 147456y^7. \]

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