MODULI SPACES OF MEROMORPHIC CONNECTIONS AND QUIVER VARIETIES

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ABSTRACT. We describe the moduli spaces of meromorphic connections on trivial holomorphic vector bundles over the Riemann sphere with at most one (unramified) irregular singularity and arbitrary number of simple poles as Nakajima’s quiver varieties. This result enables us to solve partially the additive irregular Deligne-Simpson problem.

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1. INTRODUCTION

This paper is devoted to study the relationship between two families of complex symplectic manifolds: one is certain moduli spaces of systems of linear...
ordinary differential equations with rational coefficients (i.e., meromorphic connections on trivial holomorphic vector bundles over the complex projective line $\mathbb{P}^1$), and the other is Nakajima’s quiver varieties [16] (with the real parameter taken to be zero and no framing).

Take positive integers $k_1, k_2, \ldots, k_m$, and for each $i = 1, 2, \ldots, m$ let $O_i$ be a coadjoint orbit of the complex Lie group $G_{k_i} := \text{GL}_n(\mathbb{C}[z_i]/(z_i^{k_i}))$, where $z_i$ is an indeterminate. Define

$$M_s^* = \left\{ (A_i) \in \prod_{i=1}^m O_i \mid (A_i) \text{ is “stable”}, \sum_{i=1}^m \pi_{\text{res}}(A_i) = 0 \right\} / \text{GL}_n(\mathbb{C}),$$

where $\pi_{\text{res}}: (\text{Lie } G_{k_i})^* \to \mathfrak{gl}_n(\mathbb{C})^*$ is the projection. We do not give here the definition of “stability” (see Definition 3.12), which is some open condition to make the quotient a geometrically nice space. Since the map $(A_i) \mapsto \sum_i \pi_{\text{res}}(A_i)$ is a moment map for the diagonal action of $\text{GL}_n(\mathbb{C})$ on $\prod_i O_i$, one can show that $M_s^*$ is a (smooth) complex symplectic manifold.

To view it as a certain moduli space of meromorphic connections on the trivial holomorphic vector bundle $O_{\mathbb{P}^1}^\oplus n$, take distinct points $t_1, t_2, \ldots, t_m \in \mathbb{P}^1$ and (for simplicity) a standard coordinate on $\mathbb{P}^1$ so that $z(t_i) \neq \infty$. For each $i = 1, 2, \ldots, m$, identify each $z_i$ with a coordinate $z - z(t_i)$ and embeds $(\text{Lie } G_{k_i})^*$ into $\mathfrak{gl}_n(\mathbb{C}[z_i^{-1}])z_i^{-1}dz_i$ using the residue and trace operations. Then each $(A_i) \in \prod_i O_i$ gives a meromorphic connection $d - \sum_i A_i$ on $O_{\mathbb{P}^1}^\oplus n$ with poles at $t_i$'s; it is holomorphic at $\infty$ if and only if $\sum_i \pi_{\text{res}}(A_i) = 0$.

The space $M_s^*$ cannot be thought of as a “moduli space of meromorphic connections”; we are not taking into account meromorphic connections on non-trivial holomorphic vector bundles. However, it inherits many interesting structures from the moduli space of meromorphic connections. A problem asking when $M_s^*$ is non-empty is called the additive (irregular) Deligne-Simpson problem. It was solved by Crawley-Boevey [9] in the case where $k_i = 1$ for all $i$ (namely, in the case of logarithmic connections) and by Boalch [6] in the case where one of $k_i$, say $k_1$, is less than 4 and the others are equal to 1, and furthermore $O_1$ contains an unramified “normal form” (see e.g. [25, Definition 10]).

Their approach is to describe $M_s^*$ as a quiver variety; then the problem is immediately solved since we have a criterion [7] for the non-emptiness of quiver varieties. We briefly review their results below.

Let $Q$ be a finite quiver (directed graph) with the set of vertices $Q^v$. To $(v, \zeta) \in \mathbb{Z}_{\geq 0}^Q \times \mathbb{C}^{Q^v}$, one can associate a complex symplectic manifold $\mathfrak{M}_Q^v(v, \zeta)$, called the quiver variety. This is some smooth open subset (the “stable” part) of the holomorphic symplectic quotient $T^* \text{Rep}_Q(V)/\zeta G_V$ at the level specified by
ζ, where $\text{Rep}_Q(V)$ is the space of representations of $Q$ over a collection of vector spaces $V = (V_i)_{i \in Q^v}$ with dimension $v$, and $G_V := \prod_i \text{GL}(V_i)$ (see Section 3.1 for the precise definition). They have rich geometric structures related to the gauge theory, singularity theory and representation theory of (symmetric) Kac-Moody algebras/quantum enveloping algebras of Drinfel’d-Jimbo (see [16–19, 21] and references therein).

Crawley-Boevey [9] showed that if $k_i = 1$ for all $i$, each $\mathcal{M}_s^*$ is isomorphic to a quiver variety. One can easily check that his isomorphism intertwines the symplectic structures (see e.g. [23]). Furthermore, in this case the so-called reflection functors [7, 20] $S_i : \mathcal{M}_Q^*(v, \zeta) \xrightarrow{\sim} \mathcal{M}_Q^*(s_i(v), s_i^T(\zeta)), i \in Q^v$ of the quiver varieties (they satisfy the defining relation for the simple reflections $s_i$ generating the Weyl group associated to the quiver) are expressed in terms of an “additive” analogue [10, 25] of Katz’s middle convolutions [1, 13] and the tensor operation by rank one connections (see [5, 24]).

Also, Boalch [4, 6] showed that if $k_1 \leq 3$ and $k_i = 1$ for $i \neq 1$, and if $O_1$ contains a normal form, then $\mathcal{M}_i^*$ is symplectomorphic to a quiver variety. In the proof he first described $\mathcal{M}_i^*$ as (the stable part of) a holomorphic symplectic quotient of some larger symplectic manifold $\tilde{\mathcal{M}}^*$, called the extended moduli space. This space is, roughly speaking, a certain moduli space of meromorphic connections equipped with a sort of framing at each pole, and was originally introduced by himself [3] to construct intrinsically the “space of singularity data” of Jimbo et al. [11], the phase space for the (lifted) isomonodromic deformation of meromorphic connections. He showed that in the above case there is an equivariant symplectomorphism $\tilde{\mathcal{M}}^* \simeq (T^* \text{GL}_n(C))^{m-1} \times T^* \text{Rep}_Q(V)$ for some $Q$ and $V$, and that it induces a symplectomorphism between $\mathcal{M}_s^*$ and a quiver variety. He also constructed Weyl group symmetries of families of spaces $\mathcal{M}_s^*$ whose parameter behavior coincides with that of $(v, \zeta)$ under the reflection functors through his isomorphism.

In [4, Appendix C] he further “claimed” that there is still an equivariant symplectomorphism $\tilde{\mathcal{M}}^* \simeq (T^* \text{GL}_n(C))^{m-1} \times T^* \text{Rep}_Q(V)$ for some $Q$ and $V$ when one drops the assumption $k_1 \leq 3$, and explained how to construct $Q$ and $V$. However it seems that only the existence of an equivariant isomorphism (not symplectomorphism) was shown there.

In this paper we justify his claim and show that $\mathcal{M}_s^*$ is symplectomorphic to a quiver variety if $k_i = 1$ for $i \neq 1$ and $O_1$ contains a normal form. Thanks to [7], our result immediately gives, under the same assumption, an answer to additive irregular Deligne-Simpson problem, in terms of the root system attached to the quiver.
In the more general case where each $k_i$ is arbitrary and $O_i$ contains a normal form, our method can be applied to some weaker version of the additive irregular Deligne-Simpson problem. This will be written by the first author elsewhere.

The organization of this paper is as follows. In Section 2, we introduce Boalch’s extended moduli spaces and review their basic properties. Section 3 is devoted to prove our main result stated in Section 3.4. Also, we put an appendix (Section 4) consisting of some basic facts on the classical formal reduction theory of meromorphic connections and on coadjoint orbits of general linear groups, which will be respectively used in Sections 2 and 3.

2. Preliminary notions and facts

In this section we define the extended moduli spaces and show their basic properties following Boalch [3]. Strictly speaking, our definition is a slight generalization of that in [3], where the irregular types (see below) are assumed to have regular semisimple top coefficient. Although the generalization is straightforward and already known for specialists, we decided to add the contents of this section to our paper for the reader’s convenience.

Throughout this section we fix a non-empty finite subset $D$ of $\mathbb{P}^1$. Let $\mathcal{O} \equiv \mathcal{O}_{\mathbb{P}^1}$ (resp. $\Omega^1 \equiv \Omega^1_{\mathbb{P}^1}$) be the sheaf of holomorphic functions (resp. one-forms) on $\mathbb{P}^1$ and $\Omega^1(D)$ (resp. $\Omega^1(\log D)$) the sheaf of meromorphic (resp. logarithmic) one-forms on $\mathbb{P}^1$ with poles on $D$ (and no pole elsewhere). For $t \in \mathbb{P}^1$, let $(\mathcal{O}_t, \mathfrak{m}_t)$ be the formal completion of the local ring $(\mathcal{O}_t, \mathfrak{m}_t)$ and $K_t$ (resp. $\hat{K}_t$) the field of fractions of $\mathcal{O}_t$ (resp. $\hat{O}_t$). For $\mathcal{O}_t$-module $M$, we set $\hat{M} = \hat{O}_t \otimes_{\mathcal{O}_t} M$.

2.1. Extended moduli spaces. Let $G$ be a (connected) complex reductive group and $\mathfrak{g} = \text{Lie } G$ its Lie algebra. Fix a maximal torus $t$ of $\mathfrak{g}$. An (unramified) irregular type at $t \in \mathbb{P}^1$ is an element of

$$ t(\hat{K}_t)/t(\hat{O}_t) \simeq (t \otimes_{\mathbb{C}} \hat{K}_t)/(t \otimes_{\mathbb{C}} \hat{O}_t). $$

In terms of a local coordinate $z$ vanishing at $t$, an irregular type may be regarded as an element of

$$ t([z])/(t[1]) \simeq z^{-1}t[z^{-1}]. $$

Let $E \to \mathbb{P}^1$ be a holomorphic principal $G$-bundle on $\mathbb{P}^1$. By a meromorphic connection on $E$ with poles on $D$ we mean a $\mathfrak{g}$-valued meromorphic one-form $A$ on $E$ with poles on $E|_D$ satisfying

$$ \text{Ad}_a(R^*_a A) = A \quad (a \in G), \quad A(X^*) = -X \quad (X \in \mathfrak{g}), $$

where $R^*_a$ is the pullback of $a$.
where $R_a$ is the canonical right action on $E$ by $a$ and $X^*$ is the fundamental vector field associated to $X$ (i.e., $X^*_p = \partial_t(p \cdot e^{tX})|_{t=0}$ for $p \in E$). Beware that our sign convention for the second condition is different to the standard one.

**Definition 2.1.** Let $E \to \mathbb{P}^1$ be a holomorphic principal $G$-bundle on $\mathbb{P}^1$, $A$ a meromorphic connection on $E$ with poles on $D$ and $a = (a_t)_{t \in D} \in \prod_{t \in D} \mathcal{E}_t$ a collection of points of the fibers $\mathcal{E}_t$, $t \in D$ of $E$. The triple $(E, A, a)$ is called a **compatibly framed meromorphic connection** on $(\mathbb{P}^1, D)$ if for each $t \in D$, there exist a germ $\tilde{a}_t$ at $t$ of local sections of $E$ through $a_t$ and an irregular type $T_t \in t(\hat{\mathcal{K}}_t)/t(\hat{\mathcal{O}}_t)$ such that

\[
(1) \quad \tilde{a}_t^*(A) - dT_t \in g \otimes \Omega^1(\log t)_t.
\]

It is known that the above $T_t$ is unique (see Remark 2.10); so it is called the irregular type of $(E, A, a)$ at $t$.

**Remark 2.2.** (i) Any local coordinate $z$ vanishing at $t$ induces an identification

\[
g \otimes \Omega^1(\log t)_t \simeq z^{-1}g(z)dz.
\]

Therefore in terms of the coordinate $z$, condition (1) means that $z(\tilde{a}_t^*(A) - dT_t)$ has no terms of negative degree.

(ii) If $E = \mathbb{P}^1 \times G$, then pulling back via the trivial section $\mathbb{P}^1 \to E, x \mapsto (x, 1)$ enables us to regard $A$ as a $g$-valued one-form on $\mathbb{P}^1$ with poles on $D$ and each $a_t$ as an element of $G$. In this case, $(E, A, a)$ is compatibly framed if and only if each $a_t$ admits a lift $\tilde{a}_t \in G(\mathcal{O}_t)$ such that

\[
\tilde{a}_t^{-1}[A] - dT_t \in g \otimes \Omega^1(\log t)_t,
\]

where $\tilde{a}_t^{-1}[A]$ is the gauge transform of $A$ (expressed as $\tilde{a}_t^{-1}A\tilde{a}_t - \tilde{a}_t^{-1}d\tilde{a}_t$ in any representation).

**Definition 2.3.** Two compatibly framed meromorphic connections $(E, A, a)$ and $(E', A', a')$ on $(\mathbb{P}^1, D)$ are said to be **isomorphic** if there exists an isomorphism $\varphi: E \simeq E'$ such that $\varphi^*A' = A$ and $\varphi(a_t) = a'_t$ for each $t \in D$.

If $(E, A, a)$ and $(E', A', a')$ are isomorphic, then they have common irregular type at each $t \in D$.

Let $T = (T_t)_{t \in D}$ be a collection of irregular types $T_t \in t(\hat{\mathcal{K}}_t)/t(\hat{\mathcal{O}}_t)$.

**Definition 2.4.** The **extended moduli space** $\tilde{\mathcal{M}}^*(T)$ is the set of isomorphism classes of compatibly framed meromorphic connections $(E, A, a)$ with irregular type $T_t$ at each $t \in D$ such that $E \simeq \mathbb{P}^1 \times G$. 
2.2. **Extended orbits.** The extended moduli space $\widetilde{\mathcal{M}}^*(T)$ is known to be expressed as a (holomorphic) symplectic quotient of the product of some complex symplectic manifolds, called "extended orbits".

Fix an irregular type $T \in t(\hat{K}_t)/t(\hat{O}_t)$ at some $t \in \mathbb{P}^1$. Let $k > 1$ be the pole order of $dT$ when $T \neq 0$, and $k := 1$ when $T = 0$. Set

$$G_k = G(\hat{O}_t/\hat{m}_t^k) = G(\mathcal{O}_t/m_t^k), \quad g_k = \text{Lie } G_k = g(\hat{O}_t/\hat{m}_t^k),$$

and let $G(\hat{O}_t)_k$ be the kernel of the group homomorphism $G(\hat{O}_t) \to G_k$ induced from the truncation $\hat{O}_t \to \hat{O}_t/\hat{m}_t^k$; so $G_k = G(\hat{O}_t)/G(\hat{O}_t)_k$. Also set

$$B_k = G(\hat{O}_t)_{1}/G(\hat{O}_t)_k, \quad b_k = \text{Lie } B_k.$$

The natural semidirect product decomposition $G_k = B_k \rtimes G$ induces direct sum decompositions $g_k = b_k \oplus g$ and $g_k^* = b_k^* \oplus g^*$. We denote by $\pi_{\text{irr}} : g_k^* \to b_k^*$ and $\pi_{\text{res}} : g_k^* \to g^*$ the projections. The pairing

\begin{equation}
(2) \quad g(\hat{K}_t) \otimes_C (g \otimes C \Omega^1(*t)_{\hat{t}}^\wedge) \to C; \quad X \otimes A \mapsto (X, A) := \text{res } \text{tr } XA
\end{equation}

induces embeddings

$$g_k^* \hookrightarrow (g \otimes C \Omega^1(*t)_{\hat{t}}^\wedge) / (g \otimes C \Omega^1_{\hat{t}}^\wedge) = g \otimes C (\Omega^1(*t)_{\hat{t}}^\wedge / \Omega^1_{\hat{t}}^\wedge),$$

$$b_k^* \hookrightarrow (g \otimes C \Omega^1(*t)_{\hat{t}}^\wedge) / (g \otimes C \Omega^1(\log t)_{\hat{t}}^\wedge) = g \otimes C (\Omega^1(*t)_{\hat{t}}^\wedge / \Omega^1(\log t)_{\hat{t}}^\wedge),$$

by which we may regard $dT$ as an element of $b_k^*$. Note that in terms of a local coordinate $z$ vanishing at $t$, the above embeddings are respectively expressed as $g_k^* \hookrightarrow g[z^{-1}]dz/z$, $b_k^* \hookrightarrow g[z^{-1}]dz/z^2$.

**Definition 2.5.** Let $O_B$ be the $B_k$-coadjoint orbit through $dT$. Define

$$\tilde{O} = \{ (a, A) \in G \times g_k^* | \pi_{\text{irr}}(\text{Ad}_a A) \in O_B \},$$

which we call the *extended orbit* associated to $T$.

The following proposition gives a structure of complex symplectic manifold on the extended orbit $\tilde{O}$.

**Proposition 2.6** (cf. [3, Lemmas 2.2 and 2.4]). There are canonical bijections:

$$\tilde{O} \simeq T^*G_k \times O_B/B_k \simeq T^*G \times O_B,$$

where in the middle term $B_k$ acts by the coadjoint action on $O_B$ and by the standard action on $T^*G_k$ coming from the left multiplication. Furthermore, the last bijection is a symplectomorphism.
Proof. A moment map for the $B_k$-action on $T^*G_k \times O_B$ is given by
\[
\mu: T^*G_k \times O_B \to \mathfrak{b}_k^*: \quad (g, A, B) \mapsto -\pi_{\text{int}}(\text{Ad}_g^* A) + B,
\]
where $T^*G_k \simeq G_k \times \mathfrak{g}_k^*$ via the left trivialization. Define
\[
(3) \quad \chi: \mu^{-1}(0) \to G \times \mathfrak{g}_k^*: \quad (g, A, B) \mapsto (g(t), A).
\]
It is straightforward to check that $\chi$ induces a bijection $T^*G_k \times O_B//B_k \cong \tilde{O}$.

A bijection $\tilde{O} \cong T^*G \times O_B$ and its inverse are respectively given by
\[
(a, A) \mapsto (a, \pi_{\text{res}}(A), \pi_{\text{int}}(\text{Ad}_a A)), \quad (a, R, B) \mapsto (a, \text{Ad}_a^{-1}(B) + R).
\]
Through this bijection, the following map gives a section of $\pi$:
\[
T^*G \times O_B \to T^*G_k \times O_B; \quad (a, R, B) \mapsto (a, \text{Ad}_a^{-1}(B) + R, B),
\]
which is easily seen to be symplectic. \qed

It is easy to see that the $G$-action on $\tilde{O}$ defined by $f \cdot (a, A) = (af^{-1}, \text{Ad}_f A)$ is Hamiltonian with moment map
\[
\mu_G: \tilde{O} \to \mathfrak{g}^*; \quad \mu_G(a, A) = \pi_{\text{res}}(A) \in \mathfrak{g}^*.
\]

**Proposition 2.7** (cf. [3, Proposition 2.1]). Let $T = (T_t)_{t \in D}$ be a collection of irregular types $T_t \in \mathfrak{t}(\tilde{K}_t)/\mathfrak{t}(\tilde{O}_t)$. For $t \in D$ let $\tilde{O}_t$ be the extended orbit associated to $T_t$ and set $\tilde{O} = \prod_{t \in D} \tilde{O}_t$. Then there exists a canonical bijection between the extended moduli space $\tilde{\mathcal{M}}^*(T)$ and the symplectic quotient $\tilde{O}//G$ of $\tilde{O}$ by the action of $G$.

Proof. Suppose that an element of $\tilde{\mathcal{M}}^*(T)$ is given. Take a representative $(\mathcal{E}, A, a)$ of it so that $\mathcal{E} = \mathbb{P}^1 \times G$. Then pulling back via the trivial section $\mathbb{P}^1 \to \mathcal{E}, x \mapsto (x, 1)$, we may regard $A$ as a $\mathfrak{g}$-valued one-form on $\mathbb{P}^1$ with poles on $D$. By the definition, for each $t \in D$, the compatible framing $a_t$ is an element of $G$ and admits a lift $\tilde{a}_t \in G(\tilde{O}_t)$ such that
\[
\tilde{a}_t^{-1}[A] - dT_t \in \mathfrak{g} \otimes \mathcal{O}^1(\log t)_t.
\]
Let $A_t$ be the element of
\[
(\mathfrak{g} \otimes \mathcal{O}^1(*t)_t)/ (\mathfrak{g} \otimes \mathcal{O}^1_1) = \mathfrak{g} \otimes \mathcal{O}^1_1 = \mathfrak{g} \otimes (\mathcal{O}^1(*t)_t/\mathcal{O}^1_1)
\]
represented by the germ of $A$ at $t$. The above condition for $\tilde{a}_t$ now implies that the pair $(a_t^{-1}, A_t)$ is contained in $\tilde{O}_t$; thus we obtain an element of $\tilde{O} = \prod_{t \in D} \tilde{O}_t$. The condition that $A$ is holomorphic away from $D$ is expressed as
\[
\sum_{t \in D} \text{res}_t A_t = 0,
\]
which is exactly the moment map condition $\sum_{t \in D} \mu_G(a_t^{-1}, A_t) = 0$. Changing the choice of representative $(E, A, a)$ corresponds to the simultaneous action of $G$ on $\tilde{O}$; so we may define a map $\tilde{\mathcal{M}}^s(T) \to \tilde{O}/G$. The bijectivity of it immediately follows from the standard fact that taking principal part at each pole gives a bijection

$$\Gamma(\mathbb{P}^1, \Omega^1(*D)) \simeq \left\{ (\alpha_t)_{t \in D} \in \prod_{t \in D} (\Omega^1(*t)_t/\Omega^1_t) \left| \sum_{t \in D} \text{res} \alpha_t = 0 \right. \right\}. \quad \square$$

In what follows, we identify the extended moduli space $\tilde{\mathcal{M}}^s(T)$ with the symplectic quotient $\tilde{O}/G$ via the bijection given above.

### 2.3. Extended orbits and $G_k$-coadjoint orbits.

In this subsection we will give some relationship between extended orbits and $G_k$-coadjoint orbits.

Fix an irregular type $T \in \mathfrak{t}(\tilde{K}_t)/\mathfrak{t}(\tilde{O}_t)$ at some $t \in \mathbb{P}^1$ and let $k$ be as in Section 2.2. Set $H = \{ h \in G \mid \text{Ad}_h T = T \}$ and let $\mathfrak{h}$ be its Lie algebra. The pairing (2) induces a non-degenerate pairing

$$\mathfrak{h} \otimes \mathbb{C} \left( \mathfrak{h} \otimes \mathbb{C} (\Omega^1(\log t)_t^\wedge/\tilde{\Omega}_t^1) \right) \to \mathbb{C}.$$  

In what follows, we identify $\mathfrak{h}^*$ with $\mathfrak{h} \otimes \mathbb{C} (\Omega^1(\log t)_t^\wedge/\tilde{\Omega}_t^1)$ using this pairing.

Let $H$ act on the extended orbit $\tilde{O}$ associated to $T$ by $h \cdot (a, A) = (ha, A)$. This is well-defined because if $\text{Ad}_h^* \pi_{\text{irr}}(aAa^{-1}) = dT$ for some $b \in B_k$, then $hbb^{-1} \in B_k$ and

$$\text{Ad}_h^* \pi_{\text{irr}}(\text{Ad}_h A) = \text{Ad}_h(dT) = dT.$$ 

For $g \in G(\tilde{K}_t)$ and $A \in \mathfrak{g} \otimes \mathbb{C} \Omega^1(*t)_t^\wedge$, let $g[A] = gA g^{-1} + dg \cdot g^{-1}$.

**Proposition 2.8.** Let $A \in \mathfrak{g} \otimes \mathbb{C} \Omega^1(*t)_t^\wedge$ and suppose $A - dT \in \mathfrak{g} \otimes \mathbb{C} \Omega^1(\log t)_t^\wedge$. Then there exists $\hat{b} \in G(\tilde{O}_t)_1$ such that

$$\hat{b}[A] - dT \in \mathfrak{h} \otimes \mathbb{C} \Omega^1(\log t)_t^\wedge.$$  

Furthermore, $\text{res}_t(\hat{b}[A]) \in \mathfrak{h}$ does not depend on $\hat{b}$.

To prove this we need some basic fact on the classical formal reduction theory of meromorphic connections, which will be shown in Section 4.1 (Corollary 4.3), and the following lemma:

**Lemma 2.9.** Assume $k > 1$ and let $T' \in \mathfrak{t}(\tilde{K}_t)/\mathfrak{t}(\tilde{O}_t)$ be an irregular type at $t$ of pole order at most $k - 1$. Suppose that $g \in G_k$ and $L, L' \in \mathfrak{h}^*$ satisfy

$$\text{Ad}_g^*(dT + L) = dT' + L' \in \mathfrak{g}_k^*.$$  

Then the value $g(t) \in G$ of $g$ at $t$ satisfies $\text{Ad}_{g(t)} T = T'$, $\text{Ad}_{g(t)} L = L'$.
Proof. Replacing \( g, T', L' \) with \( g(t)^{-1}g, \text{Ad}_{g(t)}^{-1} T', \text{Ad}_{g(t)}^{-1} L' \), respectively, we may assume \( g(t) = 1 \). Also, taking a faithful representation of \( G \) if necessary, we may assume \( G = \text{GL}(n, \mathbb{C}) \). Fix a local coordinate \( z \) vanishing at \( t \) and write

\[
g = \sum_{i=0}^{k-1} g_i z^i, \quad T = \sum_{i=1}^{k-1} T_i z^{-i}, \quad T' = \sum_{i=1}^{k-1} T'_i z^{-i}.
\]

Set

\[
h_i = \bigcap_{j=i+1}^{k-1} \ker \text{ad}_{T_j}, \quad (i = 0, 1, \ldots, k - 2), \quad h_{k-1} = g,
\]

\[
h'_i = \text{Im} \left( \text{ad}_{T_{i+1}} |_{h_{i+1}} \right) \quad (i = 0, 1, \ldots, k - 2).
\]

The assumption is expressed as

\[
(4) \quad \sum_{j=0}^{k-1-i} (j+i)(g_j T_j + i - T'_j + i g_j) = 0 \quad (i = 1, \ldots, k - 1),
\]

\[
(5) \quad g_0 \text{res}(L) - \text{res}(L') g_0 = \sum_{j=1}^{k-1} j (g_j T_j - T'_j g_j).
\]

Note that \( g_0 = g(t) = 1 \). Therefore equality \((4)\) for \( i = k - 1 \) implies \( T_{k-1} = T'_{k-1} \).

If \( k > 2 \), equality \((4)\) for \( i = k - 2 \) reads

\[
(k - 2)(T_{k-2} - T'_{k-2}) + (k - 1)[g_1, T_{k-1}] = 0.
\]

Observe that the first term on the left hand side lies in \( \ker \text{ad}_{T_{k-1}} = h_{k-2} \), while the second term is contained in \( \text{Im} \text{ad}_{T_{k-1}} = h'_{k-2} \). Since we have a decomposition

\[
g = h_{k-2} \oplus h'_{k-2},
\]

we see that the both terms are zero; hence \( T_{k-2} = T'_{k-2} \) and \( g_1 \in h_{k-2} \). If \( k > 3 \), look at \((4)\) for \( i = k - 3 \) and use the decomposition \( g = h_{k-3} \oplus h'_{k-3} \oplus h''_{k-2} \); then we obtain \( T_{k-3} = T'_{k-3} \), \( g_1 \in h_{k-3} \) and \( g_2 \in h_{k-2} \). Iterating this argument, we finally obtain

\[
T = T', \quad g_i \in h_i \quad (i = 1, 2, \ldots, k - 2).
\]

Now look at \((5)\); the left hand side is contained in \( h = h_0 \) and each \([g_j, T_j]\) on the right hand side is contained in \( h'_{j-1} \). Using the decomposition

\[
g = h_0 \oplus \bigoplus_{i=0}^{k-2} h'_{i},
\]

we obtain \( g_i \in h_{i-1} \) \((i = 1, \ldots, k - 2)\) and \( L = L' \).

\[\square\]

**Remark 2.10.** The above in particular shows that the irregular type of any compatibly framed meromorphic connection is unique at each pole.
Proof of Proposition 2.8. Take a local coordinate \( z \) vanishing at \( t \). Then Corollary 4.3 provides a desired \( \hat{b} \). The rest assertion follows from Lemma 2.9. \( \square \)

Corollary 2.11. For any \((a, A) \in \tilde{O}\), there exists a unique \( L \in \mathfrak{h}^* \) such that

\[
\text{Ad}_{\text{ba}}^*(A) = dT + L \in \mathfrak{g}_k^*
\]

for some \( b \in B_k \).

We define a map \( \mu_H : \tilde{O} \to \mathfrak{h}^* \) by \( \mu_H(a, A) = -L \), where \( L \in \mathfrak{h}^* \) is given above.

Proposition 2.12 (cf. [3, Lemma 2.3]). (i) The map \( \mu_H \) is a moment map for the \( H \)-action on \( \tilde{O} \).

(ii) For \( L \in \mathfrak{h}^* \), the symplectic quotient \( \tilde{O}/\!/_{O(-L)}H \) of \( \tilde{O} \) along the coadjoint orbit \( O(-L) \) through \(-L\) by the \( H \)-action is naturally isomorphic to the \( G_k \)-coadjoint orbit \( O \) through \( dT + L \).

Proof. (i) Define an injective map \( \iota : G_k \times \mathfrak{h}^* \to \mu^{-1}(0) \subset T^*G_k \times O_B \) by

\[
\iota(g, R) = (g, \text{Ad}_{g^{-1}}^*(dT + R), dT).
\]

Then Corollary 2.11 implies that the composite \( \chi \circ \iota : G_k \times \mathfrak{h}^* \to \tilde{O} \), where \( \chi \) is defined in (3), is surjective. Let \( \text{pr} : \mu^{-1}(0) \to T^*G_k \) be the first projection. Since the \( O_B \)-component of \( \iota \) is constant, the pull-back of the symplectic structure on \( T^*G_k \) along \( \text{pr} \circ \iota \) coincides with the pull-back of the symplectic structure on \( \tilde{O} \) along \( \chi \circ \iota \). Let \( H \) act on \( T^*G_k \) by the standard action coming from the left multiplication, on \( O_B \) by conjugation and on \( G_k \times \mathfrak{h}^* \) by \( h \cdot (g, R) = (hg, R) \). Then \( \mu^{-1}(0) \) is \( H \)-invariant and \( \chi, \iota, \text{pr} \) are all \( H \)-equivariant. A moment map on \( T^*G_k \) is given by

\[
\nu : T^*G_k \to \mathfrak{h}^*; \quad (g, A) \mapsto -\delta(\pi_{\text{res}}(\text{Ad}^*_g A)),
\]

where \( \delta : \mathfrak{g}^* \to \mathfrak{h}^* \) is the projection. The statement now follows from the fact that the pull-back of \( \nu \) along \( \text{pr} \circ \iota \) is the pull-back of \( \mu_H \) along \( \chi \circ \iota \).

(ii) We have a \( G_k \)-action on \( \tilde{O} \) given by \( g \cdot (a, A) = (ag(t)^{-1}, \text{Ad}_g^*(A)) \). This action clearly commutes with the \( H \)-action and the second projection \( (a, A) \mapsto A \in \mathfrak{g}_k^* \) gives a moment map. Therefore it is sufficient to check that each fiber of the projection \( \mu_H^{-1}(-L) \to O; \quad (a, A) \mapsto A \) is exactly a single orbit of the stabilizer \( \text{Stab}_H(L) \subset H \) of \( L \). So assume \( (a, A), (a', A) \in \mu_H^{-1}(-L) \). Then there exist \( b, b' \in B_k \) such that

\[
\text{Ad}_{ba}^*(A) = \text{Ad}_{b'a}^*(A) = dT + L \in \mathfrak{g}_k^*.
\]
Letting \( g = b'a'a^{-1}b^{-1} \in G_k \), we then obtain
\[
\text{Ad}_g^*(dT + L) = dT + L \in \mathfrak{g}_k^*.
\]

Lemma 2.9 shows \( a'a^{-1} = g(t) \in \text{Stab}_H(L) \). \( \square \)

**Remark 2.13.** The above \( dT + L \) for each \( L \in \mathfrak{h} \) is what we call a normal form in Introduction.

**Corollary 2.14.** Let \( T, \tilde{O} \) be as in Proposition 2.7. Set
\[
H_t = \{ h \in G \mid \text{Ad}_h T_t = T_t \}, \quad \mathfrak{h}_t = \text{Lie} H_t \quad (t \in D)
\]
and \( \mathbf{H} = \prod_{t \in D} H_t \). Take arbitrary \( \mathbf{L} = (L_t)_{t \in D} \in \bigoplus_{t \in D} \mathfrak{h}_t^* \) and for \( t \in D \), let \( O_t \) be the \( G_{k_t} \)-coadjoint orbit through \( dT_t + L_t \in \mathfrak{g}_{k_t}^* \) (where \( k_t \) is the pole order of \( dT_t \) when \( T_t \neq 0 \) and \( k_t = 1 \) otherwise). Then there exists a canonical bijection
\[
\tilde{\mathcal{M}}^*(T)\!\!/\!O(-\mathbf{L})\!\!/\!\mathbf{H} \simeq \mathbf{O}\!\!/\!G
\]
between the symplectic quotient of \( \tilde{\mathcal{M}}^*(T) \) along the \( \mathbf{H} \)-coadjoint orbit \( O(-\mathbf{L}) \) through \( -\mathbf{L} \) by the \( \mathbf{H} \)-action and that of the product \( \mathbf{O} := \prod_{t \in D} O_t \) by the \( G \)-action.

Note that the above \( \mathbf{O}\!\!/\!G \) may be singular. The following immediately follows from the arguments in the proof of Proposition 2.7 and the above corollary:

**Corollary 2.15** (cf. [3, Proposition 2.1]). Let \( T, \mathbf{L}, \mathbf{O} \) be as above. Let \( \mathcal{M}^*(T, \mathbf{L}) \) be the set of isomorphism classes of meromorphic connections \( A \in \mathfrak{g} \otimes \mathbb{C} \Omega^1(D) \) on the trivial principal \( G \)-bundle \( \mathbb{P}^1 \times G \) such that for each \( t \in D \) there exists \( g \in G(\mathcal{O}_t) \) satisfying
\[
g[A] - dT_t - L_t \in \mathfrak{g} \otimes \mathbb{C} \Omega^1_t.
\]
Then there exists a canonical bijection between \( \mathcal{M}^*(T, \mathbf{L}) \) and the symplectic quotient \( \mathbf{O}\!\!/\!G \).

### 3. Structure of (extended) moduli spaces

Throughout this section we assume \( G = \text{GL}(n, \mathbb{C}) \).

#### 3.1. Quivers and quiver varieties
Recall that a (finite) quiver is a quadruple \( Q = (Q^v, Q^a, s, t) \) consisting of two finite sets \( Q^v, Q^a \) (the vertices and arrows) and two maps \( s, t : Q^a \rightarrow Q^v \) (the source and target maps). A representation (over \( \mathbb{C} \)) of \( Q \) is a pair \( (V, \rho) \) consisting of a collection \( V = (V_i)_{i \in Q^v} \) of \( \mathbb{C} \)-vector spaces and a collection \( \rho = (\rho_\alpha)_{\alpha \in Q^a} \) of linear maps \( \rho_\alpha : V_{s(\alpha)} \rightarrow V_{t(\alpha)} \). A morphism \( \varphi : (V, \rho) \rightarrow (V', \rho') \) between two representations is a collection \( \varphi = (\varphi_i)_{i \in Q^v} \) of linear maps \( \varphi_i : V_i \rightarrow V'_i \) such that \( \varphi_{t(\alpha)} \circ \rho_\alpha = \rho_\alpha \circ \varphi_{s(\alpha)} \) for all \( \alpha \in Q^a \). Representations of \( Q \) form an abelian category.
Let $V = (V_i)_{i \in Q^a}$ be a non-zero collection of finite-dimensional $\mathbb{C}$-vector spaces. We define

$$\text{Rep}_Q(V) = \bigoplus_{\alpha \in Q^a} \text{Hom}(V_{s(\alpha)}, V_{t(\alpha)}),$$

which is the space of representations $(V, \rho)$ with fixed $V$. The group $G_V := \prod_{i \in Q^a} \text{GL}(V_i)$ acts on it as

$$g = (g_i)_{i \in Q^a} : \rho \mapsto \rho', \quad \rho'_\alpha = g_{t(\alpha)} \circ \rho_\alpha \circ g_{s(\alpha)}^{-1}.$$  

Let $\overline{Q}^a = \{ \overline{\alpha} | \alpha \in Q^a \}$ be a copy of $Q^a$ and define $s, t : \overline{Q}^a \rightarrow Q^a$ by $s(\overline{\alpha}) = t(\alpha)$, $t(\overline{\alpha}) = s(\alpha)$. Set $\overline{Q} = (Q^a, \overline{Q}^a, s, t)$, which is the quiver obtained from $Q$ by reversing the orientation of each arrow. Then the representation space

$$\text{Rep}_Q(V) = \bigoplus_{\pi \in \overline{Q}^a} \text{Hom}(V_{s(\pi)}, V_{t(\pi)}) = \bigoplus_{\alpha \in Q^a} \text{Hom}(V_{t(\alpha)}, V_{s(\alpha)})$$

is dual to the vector space $\text{Rep}_Q(V)$ via the trace pairing, and the representation space of the double

$$\hat{Q} = (Q^a, \overline{Q}^a, s, t), \quad \hat{Q}^a := Q^a \sqcup \overline{Q}^a$$

of $Q$ may be identified with the cotangent bundle of $\text{Rep}_Q(V)$:

$$\text{Rep}_{\hat{Q}}(V) = \text{Rep}_Q(V) \oplus \text{Rep}_{\overline{Q}^a}(V) \simeq \text{Rep}_Q(V) \oplus \text{Rep}_Q(V)^* \simeq T^* \text{Rep}_Q(V).$$

The group $G_V$ naturally acts on it and preserves the canonical symplectic form

$$\omega = \sum_{\alpha \in Q^a} \text{tr} d\Xi_{\alpha} \wedge d\Xi_{\overline{\alpha}},$$

where $\Xi_{\alpha} : \text{Rep}_Q(V) \rightarrow \text{Hom}(V_{s(\alpha)}, V_{t(\alpha)}), \Xi \mapsto \Xi_{\alpha}$ denotes the projection. Extend the map $Q^a \rightarrow \overline{Q}^a, \alpha \mapsto \overline{\alpha}$ to an involution of $\hat{Q}^a$ and define $\epsilon : \hat{Q}^a \rightarrow \{ \pm 1 \}$ by $\epsilon(Q^a) = 1$ and $\epsilon(\overline{Q}^a) = -1$. Then $\omega$ is also expressed as

$$\omega = \frac{1}{2} \sum_{\alpha \in Q^a} \epsilon(\alpha) \text{tr} d\Xi_{\alpha} \wedge d\Xi_{\overline{\alpha}}.$$

Let $g_V = \bigoplus_{i \in Q^a} \mathfrak{gl}(V_i)$ be the Lie algebra of $G_V$. A moment map on $\text{Rep}_{\hat{Q}}(V)$ for the $G_V$-action is given by

$$\mu = (\mu_i)_{i \in Q^a} : \text{Rep}_{\hat{Q}}(V) \rightarrow g_V, \quad \mu_i(\Xi) := \sum_{\alpha \in \hat{Q}^a, t(\alpha) = i} \epsilon(\alpha) \Xi_{\alpha} \Xi_{\overline{\alpha}},$$

where we identify $g_V^*$ with $g_V$ using the trace.

**Definition 3.1.** For $\Xi \in \text{Rep}_{\hat{Q}}(V)$, a collection $W = (W_i)_{i \in Q^a}$ of subspaces $W_i \subset V_i$ is called a $\Xi$-invariant subspace if $\Xi_{\alpha}(W_{s(\alpha)}) \subset W_{t(\alpha)}$ for all $\alpha \in \hat{Q}^a$.

A point $\Xi \in \text{Rep}_{\hat{Q}}(V)$ is stable if the representation $(V, \Xi)$ of $\hat{Q}$ is irreducible, i.e., there exist no $\Xi$-invariant subspaces except the trivial ones $0, V$. 


See [14] for the fact that the above stability is one of Mumford’s stability conditions [15]; therefore the stable points form a $G_V$-invariant Zariski open subset of $\text{Rep}_Q(V)$ on which the quotient group $G_V/C^\times$ acts freely and properly (note that the subgroup $\mathbb{C}^\times \subset G_V$ acts trivially on $\text{Rep}_Q(V)$).

**Definition 3.2.** For $\zeta = (\zeta_i)_{i \in Q^e} \in \mathbb{C}^{Q^e}$, set $\zeta_V = (\zeta_i 1_{V_i})_{i \in Q^e} \in \mathfrak{g}_V^{G_V}$ and define

$$\mathcal{M}_Q^s(V, \zeta) = \{ \Xi \in \mu^{-1}(\zeta_V) \mid \Xi \text{ is stable} \}/G_V,$$

which we call the *quiver variety*.

By the definition $\mathcal{M}_Q^s(V, \zeta)$ is a complex symplectic manifold (if non-empty). We also denote it by $\mathcal{M}_Q^s(\mathfrak{v}, \zeta)$, where $\mathfrak{v} = (v_i)_{i \in Q^e} \in \mathbb{Z}_{\geq 0}^{Q^e}$, $v_i := \dim V_i$ is the dimension vector of $V$.

**Remark 3.3.** (i) If $\mu^{-1}(\zeta_V)$ is non-empty, then $\zeta$ must satisfy

$$\zeta \cdot \mathfrak{v} := \sum_{i \in Q^e} \zeta_i v_i = 0.$$

Indeed, for $\Xi \in \mu^{-1}(\zeta_V)$, we have

$$\zeta \cdot \mathfrak{v} = \sum_{i \in Q^e} \text{tr} \mu_i(\Xi) = \sum_{\alpha \in Q^a} \epsilon(\alpha) \text{tr} \Xi_\alpha \Xi_{\alpha^{-1}} = - \sum_{\alpha \in Q^a} \epsilon(\alpha) \text{tr} \Xi_\alpha \Xi_{\alpha^{-1}}$$

and hence $\zeta \cdot \mathfrak{v} = 0$.

(ii) If $\mathcal{M}_Q^s(\mathfrak{v}, \zeta)$ is non-empty, then the dimension formula for symplectic quotients shows

$$\dim \mathcal{M}_Q^s(\mathfrak{v}, \zeta) = 2\Delta(\mathfrak{v}), \quad \Delta(\mathfrak{v}) := \sum_{\alpha \in Q^a} v_{s(\alpha)} v_{t(\alpha)} - \sum_{i \in Q^e} v_i^2 + 1.$$

**3.2. Triangular decomposition of $B_k$-coadjoint orbits.** Fix a non-zero irregular type $T \in \mathfrak{t}(\bar{K}_i)/\mathfrak{t}(\bar{O}_i)$ at some $t \in \mathbb{P}^1$ and let $k > 1$ be the pole order of $dT$. In this subsection we give a “triangular decomposition” of the $B_k$-coadjoint orbit $O_B$ through $dT$, which enables us to describe $O_B$ as the representation space of the double of some quiver.

Take a local coordinate $z$ vanishing at $t$ and write $T(z) = \sum_{i=1}^{k-1} T_i z^{-i}$, $T_i \in \mathfrak{t}$. For $i = 0, 1, \ldots, k - 2$, let $J_i$ be the set of simultaneous eigenspaces of $(T_{i+1}, T_{i+2}, \ldots, T_{k-1})$ and $\mathbb{C}^n = \bigoplus_{p \in J_i} V^{(i)}_p$ the associated decomposition. For each $i$ and $j \leq i$, we have a natural surjection $\pi_i: J_j \to J_i$ such that $V^{(i)}_p \subset V^{(i)}_{\pi_i(p)}$ for $p \in J_j$.

Fix a total ordering on $J_i$ for each $i$ such that

$$i < k - 2, \ p, q \in J_i, \ \pi_{i+1}(p) < \pi_{i+1}(q) \implies p < q.$$
Each decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_1} V_p^{(i)} \) induces a block decomposition of \( \text{Mat}_n(\mathbb{C}) \).
Let \( p_i^+ \) (resp. \( p_i^- \)) \( \subset \text{Mat}_n(\mathbb{C}) \) be the subset of block upper (resp. lower) triangular matrices; explicitly,

\[
p_i^+ = \bigoplus_{p \in J_1, p > q} \text{Hom}(V_p^{(i)}, V_q^{(i)}), \quad p_i^- = \bigoplus_{p \in J_1, p < q} \text{Hom}(V_p^{(i)}, V_q^{(i)}).
\]

We have a decomposition \( p_{i+1}^\pm = \mathfrak{h}_i \oplus \mathfrak{u}_i^\pm \), where

\[
\mathfrak{h}_i = \bigoplus_{p \in J_i} \ker \text{ad}_{T_j}, \quad \mathfrak{u}_i^\pm = \bigoplus_{p \in J_i, p \leq q} \text{Hom}(V_p^{(i)}, V_q^{(i)}).
\]

Note that the property of our total orderings implies

\[
p_{i+1}^\pm = p_i^\pm \oplus (\mathfrak{h}_{i+1} \cap \mathfrak{u}_i^\mp), \quad \mathfrak{u}_i^\pm = \mathfrak{u}_{i+1}^\pm \oplus (\mathfrak{h}_{i+1} \cap \mathfrak{u}_i^\mp), \quad \mathfrak{u}_i^\pm \cap \ker \text{ad}_{T_{i+1}} \subset \mathfrak{u}_{i+1}^\pm.
\]

For convenience we use the convention \( p_{k-1}^\pm = \mathfrak{h}_{k-1} = \mathfrak{g}, \mathfrak{u}_{k-1}^\pm = 0 \). The following lemma immediately follows from the arguments in the proof of Lemma 2.9.

**Lemma 3.4.** An element \( b(z) = \sum_{i=0}^{k-1} b_i z^i \) of \( B_k \) stabilizes \( dT \) if and only if \( b_i \in \mathfrak{h}_i \) for all \( i = 1, 2, \ldots, k-1 \).

Set

\[
\mathcal{P}_\pm = \left\{ b(z) = \sum_{i=0}^{k-1} b_i z^i \in B_k \mid b_i \in p_i^\pm (i = 1, 2, \ldots, k-1) \right\}.
\]

These are Lie subgroups of \( B_k \) since \( p_i \subset p_{i+1} \). Let \( \mathfrak{P}_\pm \) be their Lie algebras. The pairing on \( \mathfrak{b}_k \otimes \mathfrak{b}_k^* \) enables us to identify the duals of \( \mathfrak{P}_\pm \) with

\[
\mathfrak{P}_\pm^* := \left\{ X(z) = \sum_{i=1}^{k-1} X_i z^{-i-1} dz \in \mathfrak{b}_k^* \mid X_i \in p_i^\mp (i = 1, 2, \ldots, k-1) \right\}.
\]

Similarly, set

\[
\mathcal{U}_\pm = \left\{ b(z) = \sum_{i=0}^{k-1} b_i z^i \in B_k \mid b_i \in \mathfrak{u}_i^\pm (i = 1, 2, \ldots, k-1) \right\}.
\]

Note that these are not Lie subgroups of \( B_k \). Let \( \mathfrak{U}_\pm = T_1 \mathfrak{U}_\pm \) be the tangent spaces at the identity, whose duals are identified with

\[
\mathfrak{U}_\pm^* := \left\{ X(z) = \sum_{i=1}^{k-1} X_i z^{-i-1} dz \in \mathfrak{b}_k^* \mid X_i \in \mathfrak{u}_i^\pm (i = 1, 2, \ldots, k-1) \right\}.
\]

**Lemma 3.5.** For any \( b \in B_k \), there uniquely exist \( b_- \in \mathcal{U}_- \) and \( b_+ \in \mathcal{P}_+ \) such that \( b = b_- b_+ \).
Proof. Write

\[ b = \sum_{i=0}^{k-1} b_i z^i, \quad b_\pm = \sum_{i=0}^{k-1} b_i^\pm z^i, \quad b_0 = b_0^\pm = 1_{\mathbb{C^n}}. \]

In terms of the coefficients \( b_i, b_i^\pm \), the relation \( b = b_- b_+ \) is expressed as

\[ b_i^- + b_i^+ = b_i - \sum_{j=1}^{i-1} b_j^- b_{i-j}^+ \quad (i = 1, 2, \ldots, k - 1). \]

The above inductively defines \( b_i^- \in u_i^-, b_i^+ \in p_i^+ \) since \( g = u_i^- \oplus p_i^+ \). \( \square \)

For \( B \in O_B \), take \( b \in B_k \) such that \( B = \text{Ad}_B^*(dT) \) and decompose \( b = b_- b_+ \) as above. Note that \( b_\pm \) does not depend on the choice of \( b \) as the stabilizer of \( dT \) is contained in \( \mathcal{P}_+ \). Set \( B' = b_-^1 B \in b_k^* \) and

\[ Q = b_- - 1_{\mathbb{C^n}} \in \mathfrak{U}_-, \quad P = B'|_{\mathfrak{U}_-} \in \mathfrak{U}_-. \]

**Theorem 3.6.** The map

\[ O_B \to \mathfrak{U}_- \times \mathfrak{U}_+^*; \quad B \mapsto (Q, P) \]

is a symplectomorphism from \( O_B \) onto the space \( \mathfrak{U}_- \times \mathfrak{U}_+^* \) equipped with the symplectic form \( \text{res}_{z=0} \text{tr} \, dQ \wedge dP \).

To prove it we need two lemmas.

**Lemma 3.7.** For any \((Q, P) \in \mathfrak{U}_- \times \mathfrak{U}_+^*\), there exists a unique \( B' \in b_k^* \) such that \( B'|_{\mathfrak{U}_-} = P \) and \( B'(1_{\mathbb{C^n}} + Q) - dT \in \mathfrak{U}_+^* \).

Proof. Write \( dT = \sum_{i=1}^{k-1} (dT)_i z^{-i-1} dz \) and

\[ Q = \sum_{i=1}^{k-1} Q_i z^i, \quad P = \sum_{i=1}^{k-1} P_i z^{-i-1} dz, \quad B' = \sum_{i=1}^{k-1} B'_i z^{-i-1} dz. \]

Conditions \( B'|_{\mathfrak{U}_-} = P \) and \( B'(1_{\mathbb{C^n}} + Q) - dT \in \mathfrak{U}_+^* \) are then expressed as

\[ B'_{k-1}|_{\mathfrak{U}_-} = P_{k-1}, \quad B'_{k-1}|_{\mathfrak{U}_-} = (dT)_{k-1} - \sum_{j=1}^{i-1} B'_{k-j} Q_{i-j} |_{\mathfrak{U}_-} \quad (i = 1, 2, \ldots, k-1). \]

Since \( g = u_{k-i}^- \oplus p_{k-i}^- \), the above inductively defines \( B'_{k-i} \). \( \square \)

**Lemma 3.8.** The \( \mathcal{P}_+ \)-orbit through \( dT \) coincides with the subset \( dT + \mathfrak{U}_+^* \subset b_k^* \).

Proof. Write \( dT = \sum_{i=1}^{k-1} (dT)_i z^{-i-1} dz \). For \( B = \sum_{i=1}^{k-1} B_i z^{-i-1} dz \in b_k^* \) and \( b = \sum_{i=0}^{k-1} b_i z^i \in \mathcal{P}_+ \), the relation \( B = \text{Ad}_B^*(dT) \) is equivalent to

\[ B_{k-i} - (dT)_{k-i} + \sum_{j=1}^{i-1} (B_{k-j} b_{i-j} - b_{i-j} (dT)_{k-j}) = 0 \quad (i = 1, 2, \ldots, k - 1). \]
In terms of $X_i := B_i - (dT)_i$ ($i = 1, 2, \ldots, k - 1$), the above is expressed as

$$(6) \quad X_{k-i} = \sum_{j=1}^{i-1} ([b_{i-j}, (dT)_{k-j}] - X_{k-j}b_{i-j}) \quad (i = 1, 2, \ldots, k - 1).$$

Now assume (6). Then $X_{k-1} = 0 \in u^+_{k-1}$, and furthermore we have

$$(7) \quad [p^+_{i-j}, (dT)_{k-j}] \subset [p^+_{k-1-j}, (dT)_{k-j}] \subset u^+_{k-j-1} \subset u^+_{k-i},$$

$$u^+_{k-j} \cdot p^+_{i-j} \subset u^+_{k-j} \cdot p^+_{k-j} \subset u^+_{k-j} \subset u^+_{k-i}$$

for $j < i < k$. Therefore (6) inductively shows $X_{k-i} \in u^+_{k-i}$ for all $i$. Hence $B - dT \in \mathcal{U}^*$. Conversely, assume $B - dT \in \mathcal{U}^*$. To find $b \in \mathcal{P}_+$ satisfying (6), we first project (6) to $u^+_{k-i} \oplus u^+_{k-i+1} = \mathcal{H}_{k-i+1} \cap u^+_{k-i}$ for $i \geq 2$. By (7), we then obtain

$$X_{k-1}|_{u^+_{k-i} \oplus u^+_{k-i+1}} = \sum_{j=1}^{i-1} ([b_{i-j}, (dT)_{k-j}] - X_{k-j}b_{i-j})|_{u^+_{k-i} \oplus u^+_{k-i+1}}$$

$$= [b_1, (dT)_{k-i+1}]|_{u^+_{k-i} \oplus u^+_{k-i+1}} \quad (i = 2, \ldots, k - 1).$$

Note that $[b_1, (dT)_{k-i+1}]|_{u^+_{k-i} \oplus u^+_{k-i+1}}$ depends only on the $u^+_{k-i} \oplus u^+_{k-i+1}$-component of $b_1$ since

$$p^+_1 \oplus u^+_{k-i} \subset \ker \text{ad}_{T_{k-i+1}}, \quad [u^+_{k-i+1}, (dT)_{k-i+1}] \subset u^+_{k-i+1}.$$ By $u^+_{k-i} \cap \ker \text{ad}_{T_{k-i+1}} \subset u^+_{k-i+1}$, we have $u^+_{k-i} \oplus u^+_{k-i+1} \subset \im \text{ad}_{T_{k-i+1}}$. Hence there exists a unique $b_1 \in u_1$ such that $b_1|_{u^+_{k-i} \oplus u^+_{k-i+1}}$ satisfies the above equality for $i = 2, \ldots, k - 1$.

Next, we project (6) to $u^+_{k-i+1} \oplus u^+_{k-i+2}$ for $i \geq 3$. By (7), we then obtain

$$X_{k-1}|_{u^+_{k-i+1} \oplus u^+_{k-i+2}} = \sum_{j=1}^{i-1} ([b_{i-j}, (dT)_{k-j}] - X_{k-j}b_{i-j})|_{u^+_{k-i+1} \oplus u^+_{k-i+2}}$$

$$= ([b_1, (dT)_{k-i+1}] - X_{k-i+1}b_1 + [b_2, (dT)_{k-i+2}])|_{u^+_{k-i+1} \oplus u^+_{k-i+2}}$$

for $i = 3, \ldots, k - 1$. Note that $[b_2, (dT)_{k-i+2}]|_{u^+_{k-i+1} \oplus u^+_{k-i+2}}$ depends only on the $u^+_{k-i+1} \oplus u^+_{k-i+2}$-component of $b_2$ since

$$p^+_2 \oplus u^+_{k-i+1} \subset \ker \text{ad}_{T_{k-i+2}}, \quad [u^+_{k-i+2}, (dT)_{k-i+2}] \subset u^+_{k-i+2}.$$ We have $u^+_{k-i+1} \oplus u^+_{k-i+2} \subset \im \text{ad}_{T_{k-i+2}}$. Hence there exists a unique $b_2 \in u_2$ such that $b_2|_{u^+_{k-i+1} \oplus u^+_{k-i+2}}$ satisfies the above equality for $i = 3, \ldots, k - 1$.

Repeating this argument, we see that (6) uniquely determines $b_i \in u_i^*$, $i = 1, 2, \ldots, k - 2$, which shows the existence (and uniqueness) of $b \in \mathcal{U}_+ \subset \mathcal{P}_+$. □
Proof of Theorem 3.6. Lemmas 3.7 and 3.8 imply that the map \( B \mapsto (Q, P) \) is bijective (note that \( B \) is uniquely determined from \( Q \) and \( B' \)). We show that it is also symplectic. Let \( B = Ad_b^*(dT) \in O_B \). Decompose \( b = b_- b_+ \) as in Lemma 3.5 and let \( B_+ = Ad_{b_+}^*(dT) \). For \( i = 1, 2 \), let \( v_i \in T_{p_i}O_B \) and \((\xi_i, \eta_i)\) the corresponding tangent vector at \((Q, P) \in \mathfrak{M}_- \times \mathfrak{M}_-^*\). Take \( X_i \in b_k \) such that \([X_i, B] = v_i\). By differentiating \( b = b_- b_+ \), we find \( \xi_i' \in \mathfrak{M}_+^* \) such that

\[
X_i = (\xi_i + b_- \xi_i')b_+^{-1}.
\]

Now we calculate the symplectic form evaluated at \( v_1, v_2 \):

\[
\omega_B(v_1, v_2) = \text{res} \text{ tr}(B[X_1, X_2])
\]

\[
= \text{res} \text{ tr}(B[(\xi_1 + b_- \xi_1')b_+^{-1}, (\xi_2 + b_- \xi_2')b_+^{-1}])
\]

\[
= \text{res} \text{ tr}(B[\xi_1 b_+^{-1}, \xi_2 b_+^{-1}]) + \text{res} \text{ tr}(B[b_- \xi_1 b_+^{-1}, b_- \xi_2 b_+^{-1}])
\]

\[
+ \text{res} \text{ tr}(B[\xi_1 b_+^{-1}, b_- \xi_2 b_+^{-1}]) + \text{res} \text{ tr}(B[b_- \xi_1 b_+^{-1}, \xi_2 b_+^{-1}]).
\]

Lemma 3.8 implies \( B_+ - dT \in \mathfrak{M}_-^* \); thus the second term in the most right hand side of (8) is simplified as follows:

\[
\text{res} \text{ tr}(B[b_- \xi_1 b_+^{-1}, b_- \xi_2 b_+^{-1}]) = \text{res} \text{ tr}(B_+ [\xi_1', \xi_2']) = \text{res} \text{ tr}(dT[\xi_1', \xi_2']).
\]

Since the coefficient of \([dT, \xi_i']\) in \( z^{-i-1}dz \ (i \geq 1) \) is contained in

\[
\sum_{j=1}^{k-i-1} [(dT)_{i+j}, p^+_j] \subset \sum_{j=1}^{k-i-1} [(dT)_{i+j}, p^+_{i+j-1}]
\]

\[
\subset \sum_{j=1}^{k-i-1} u^+_{i+j-1} \subset u^+_i,
\]

we find \([dT, \xi_i'] \in \mathfrak{M}_-^* \) and thus

\[
\text{res} \text{ tr}(dT[\xi_1', \xi_2']) = \text{res} \text{ tr}([dT, \xi_1'][\xi_2']) = 0.
\]

Next, look at the third term in the most right hand side of (8):

\[
\text{res} \text{ tr}(B[\xi_1 b_+^{-1}, \xi_2 b_+^{-1}]) = \text{res} \text{ tr}(B_+ [b_+^{-1} \xi_1, \xi_2'])
\]

\[
= \text{res} \text{ tr}([\xi_2', B_+] b_+^{-1} \xi_1).
\]

Differentiating the relation \( P = b_+^{-1}B|_{\mathfrak{M}_-^*} \) yields

\[
[\xi_1', B_+] b_+^{-1}|_{\mathfrak{M}_-^*} = \eta_1 + b_+^{-1}B(\xi_1 b_+^{-1})|_{\mathfrak{M}_-^*}.
\]

Hence the third term is expressed as

\[
\text{res} \text{ tr}([\xi_2', B_+] b_+^{-1} \xi_1) = \text{res} \text{ tr}(\eta_1 \xi_1) + \text{res} \text{ tr}(B(\xi_1 b_+^{-1})|_{\mathfrak{M}_-^*}).
\]
Similarly, the fourth term is
\[ \text{res } \text{tr}(B[b_\xi b^{-1}_2, \xi_2 b^{-1}_2]) = - \text{res } \text{tr}(\eta_1 \xi_2) - \text{res } \text{tr}(B \xi_1 b^{-1}_2 \xi_2 b^{-1}_2). \]
Thus we obtain
\[ \omega_B(v_1, v_2) = \text{res } \text{tr}(B[\xi_1 b^{-1}_2, \xi_2 b^{-1}_2]) + \text{res } \text{tr}(\eta_2 \xi_1) + \text{res } \text{tr}(B \xi_2 b^{-1}_2 \xi_2 b^{-1}_2) \]
\[ - \text{res } \text{tr}(\eta_1 \xi_2) - \text{res } \text{tr}(B \xi_1 b^{-1}_2 \xi_2 b^{-1}_2) \]
\[ = \text{res } \text{tr}(\xi_1 \eta_2 - \xi_2 \eta_1), \]
which shows \( \omega = \text{res } z = 0 \text{ tr } dQ \wedge dP. \)

3.3. **Quiver description of \( B_k \)-coadjoint orbits.** Using Theorem 3.6, we will describe \( O_B \) as the representation space of the double of the following quiver \( Q \). Let \( Q^v \) be the set of simultaneous eigenspaces for coefficients \((T_1, T_2, \ldots, T_{k-1})\) of \( T \) and \( V = (V_p)_{p \in Q^v} \) the collection of corresponding simultaneous eigenspaces:
\[ Q^v = J_0, \quad V_p = V_p^{(0)} \quad (p \in Q^v). \]
We have a natural surjection \( \pi_i : Q^v \to J_i \) for \( i = 0, 1, \ldots, k-2 \). For each \( p, q \in Q^v \) with \( p < q \), draw \( m_{p,q} \) arrows from \( p \) to \( q \), where
\[ m_{p,q} := \max \{ i \mid \pi_i(p) < \pi_i(q) \} \in \{0, 1, \ldots, k-2\}. \]
Let \( Q \) be the resulting quiver and label the arrows from \( p \) to \( q \) as \( \alpha_{pq,i}, i = 1, 2, \ldots, m_{p,q} \). Then \( G_V = H \) and
\[
\begin{align*}
\mathcal{U}_- &= \bigoplus_{i=1}^{k-1} U_i^- z^i = \bigoplus_{i=1}^{k-1} \bigoplus_{p,q \in J_i, p < q} \text{Hom}(V_p^{(i)}, V_q^{(i)}) z^i \\
&= \bigoplus_{i=1}^{k-1} \bigoplus_{p,q \in J_i, \pi_i(p) < \pi_i(q)} \text{Hom}(V_p, V_q) z^i \simeq \text{Rep}_Q(V),
\end{align*}
\]
where the last isomorphism sends the factor \( \text{Hom}(V_p, V_q) z^i \) to the set of linear maps associated to the arrow \( \alpha_{pq,i} \). Therefore Theorem 3.6 implies the following:

**Corollary 3.9.** The map given in (9) induces a \( G_V \)-equivariant symplectic isomorphism between the \( B_k \)-coadjoint orbit \( O_B \) and \( \text{Rep}_Q(V) \).

The following lemma will be used in the next subsection.

**Lemma 3.10.** Let \( B = \sum_{i=1}^{k-1} B_i z^{-i-1} dz \in O_B \) and \( \Xi \in \text{Rep}_Q(V) \) the corresponding element under the above isomorphism.

(i) For any \( \Xi \)-invariant subspace \( W = (W_p)_{p \in J_0} \) of \( V \), the direct sum \( S := \bigoplus_p W_p \subset \mathbb{C}^n \) is invariant under all \( B_i \).
(ii) Any subspace \( S \subset \mathbb{C}^n \) invariant under all \( B_i \) is homogeneous with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_0} V_p \) and the collection \( W = (W_p)_{p \in J_0}, W_p := S \cap V_p \) is a \( \Xi \)-invariant subspace of \( V \).

**Proof.** We first show that any subspace \( S \subset \mathbb{C}^n \) invariant under all \( B_i \) is homogeneous with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_0} V_p \). Write \( dT = \sum_{i=1}^{k-1}(dT)_{k-1} z^{-i-1} dz \) and take \( b(z) = \sum_{i=0}^{k-1} b_i z^i \in B_k \) so that \( B = \text{Ad}_b(dT) \). Then \( B_{k-1}(S) \subset S \) and \( B_{k-1} = (dT)_{k-1} \) imply that \( S \) is homogeneous with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_{k-1}} V_p^{(k-1)} \). Now assume that \( S \) is homogeneous with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_{k-l}} V_p^{(k-l)} \), where \( 2 \leq l < k \). We have

\[
(dT)_{k-l} = B_{k-l} + \sum_{i=1}^{l-1} [(dT)_{k-l+i}, b_i],
\]

in which each \([ (dT)_{k-l+i}, b_i ] \) is block off-diagonal with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_{k-l}} V_p^{(k-l)} \). Therefore the assumption \( B_{k-l}(S) \subset S \) implies that each \( S \cap V_p^{(k-l)} \) is invariant under \((dT)_{k-l} \), and hence that \( S \) is homogeneous with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_{k-l+1}} V_p^{(k-l-1)} \). This inductive argument shows that \( S \) is homogeneous with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_0} V_p \).

We have a one-to-one correspondence between the collections \( W = (W_p)_{p \in J_0} \) of subspaces \( W_p \subset V_p \) and the subspaces \( S \subset \mathbb{C}^n \) homogeneous with respect to the decomposition \( \mathbb{C}^n = \bigoplus_{p \in J_0} V_p \):

\[
W \mapsto S := \bigoplus_{p \in J_0} W_p, \quad S \mapsto W = (W_p)_{p \in J_0}, \quad W_p := S \cap V_p.
\]

We next show that under this correspondence, \( W \) is \( \Xi \)-invariant if and only if \( S \) is invariant under all \( B_i \). For given \( W \), take a subspace \( W'_p \subset V_p \) complimentary to \( W_p \) for each \( p \in J_0 \) and set \( S' = \bigoplus_p W'_p \). Define a one-parameter subgroup \( \tau : \mathbb{C}^x \rightarrow H \) by

\[
\tau(u) = \begin{bmatrix} u & 0 \\ 0 & 1_{s'} \end{bmatrix} : \mathbb{C}^n = S \oplus S' \rightarrow S \oplus S'.
\]

If we express each \( \Xi_{\alpha} (\alpha \in \hat{Q}^a) \) as

\[
\Xi_{\alpha} = \begin{bmatrix} \Xi_{11}^\alpha & \Xi_{12}^\alpha \\ \Xi_{21}^\alpha & \Xi_{22}^\alpha \end{bmatrix} : V_{s(\alpha)} = W_{s(\alpha)} \oplus V_{s(\alpha)}' \rightarrow W_{t(\alpha)} \oplus V_{t(\alpha)}' = V_{t(\alpha)},
\]

then the action by \( \tau(u) \) transforms it to

\[
\begin{bmatrix} \Xi_{11}^\alpha & u \Xi_{12}^\alpha \\ u^{-1} \Xi_{21}^\alpha & \Xi_{22}^\alpha \end{bmatrix}.
\]
Hence the limit $\lim_{u \to 0} \tau(u) \cdot \Xi$ exists if and only if $\Xi_{\alpha} = 0$ for all $\alpha$, i.e., $W$ is $\Xi$-invariant. Similarly, if we express each $B_i$ as

$$B_i = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}: \mathbb{C}^n = S \oplus S' \to S \oplus S' = \mathbb{C}^n,$$

then the limit $\lim_{u \to 0} \tau(u) B \tau(u)^{-1}$ exists in $O_B$ (note that $O_B \subset \mathfrak{g}_k^*$ is closed as $B_k$ is a unipotent algebraic group; see [22, Theorem 2]) if and only if $B_{21}^i = 0$ for all $i$, i.e., $S$ is invariant under all $B_i$.

Since the map $O_B \simeq \text{Rep}_{\hat{Q}}(V)$, $B \mapsto \Xi$ is an $H$-equivariant homeomorphism, the existence of $\lim_{u \to 0} \tau(u) \cdot \Xi$ is equivalent to that of $\lim_{u \to 0} \tau(u) B \tau(u)^{-1}$; we are done. 

3.4. Moduli spaces and quivers. As in Corollary 2.14, let $T = (T_t)_{t \in D}$ be a collection of irregular types and set

$$H_t = \{ h \in G \mid hT_t h^{-1} = T_t \}, \quad \mathfrak{h}_t = \text{Lie} H_t \quad (t \in D), \quad H = \prod_{t \in D} H_t.$$

Assume that $D_{\text{irr}} := \{ t \in D \mid T_t \neq 0 \}$ is non-empty as the empty case is not interesting. For each $t \in D_{\text{irr}}$, let $Q_t, V_t$ be the quiver and collection of vector spaces associated to $T_t$ (so $G_{V_t} = H_t$). Take a base point $\infty \in D_{\text{irr}}$ and set $D_0 = D \setminus \{ \infty \}$.

**Corollary 3.11.** The extended moduli space $\tilde{\mathcal{M}}^*(T)$ is $H$-equivariantly symplectomorphic to the product

$$\prod_{t \in D_0} T^*G \times \prod_{t \in D_{\text{irr}}} \text{Rep}_{\hat{Q}_t}(V_t).$$

Here $H_\infty$ acts on each copy of $T^*G$ as the standard action coming from the right multiplication and on $\text{Rep}_{\hat{Q}_\infty}(V_\infty)$ in the obvious way, while for $t \in D_0$, the group $H_t$ acts on the $t$-th copy of $T^*G$ as the action coming from the left multiplication and on $\text{Rep}_{\hat{Q}_t}(V_t)$ (if $t \in D_{\text{irr}}$) in the obvious way.

**Proof.** Recall that if $M$ is a complex symplectic manifold on which $G$ acts in a Hamiltonian fashion with a moment map $\mu: M \to \mathfrak{g}$, then the symplectic quotient of the product $T^*G \times M$ by the diagonal action of $G$ (where the $G$-action on $T^*G$ comes from the right multiplication) is canonically symplectomorphic to $M$:

$$(T^*G \times M)/G \simeq M; \quad [a, R, x] \mapsto a \cdot x.$$ 

Under this map, the $G$-action on $M$ corresponds to the one on $(T^*G \times M)/G$ defined by $f \cdot [a, R, x] = [fa, R, x].$
Let $k$ be the pole order of $dT_\infty$ and $O_B$ the $B_k$-coadjoint orbit through $dT_\infty$. For $t \in D$, let $\tilde{O}_t$ be the extended orbit associated to $T_t$. Propositions 2.6 and 2.7 together with the above fact imply

$$\tilde{\mathcal{M}}^*(T) \simeq \prod_{t \in D} \tilde{O}_t // G \simeq O_B \times \prod_{t \in D_0} \tilde{O}_t.$$  

Corollary 3.9 now shows the assertion. □

In the rest of this section we assume $D_{\text{irr}} = \{\infty\}$. In this case, the space $\tilde{\mathcal{M}}^*(T)$ is $H$-equivariantly symplectomorphic to the product

$$\prod_{t \in D_0} T^*G \times \text{Rep}_{\hat{Q}_\infty}(V_\infty).$$

Hence the symplectic quotient $\mathcal{M}^*(T, L)$ of $\tilde{\mathcal{M}}^*(T)$ (see Corollary 2.15) by the $H$-action along the coadjoint orbit through each $-L_t = (-L_t)_{t \in D} \in \bigoplus_{t \in D} \mathfrak{h}_t^*$ is isomorphic to

$$\prod_{t \in D_0} O(L_t) \times \text{Rep}_{\hat{Q}_\infty}(V_\infty) // O(-L_\infty)H_\infty,$$

where $O(\pm L_t)$ is the $H_t$-coadjoint orbit through $\pm L_t$ for $t \in D$ (note that $H_t = G$ for $t \in D_0$). By the shifting trick, we thus obtain

$$\mathcal{M}^*(T, L) \simeq O(L_\infty) \times \prod_{t \in D_0} O(L_t) \times \text{Rep}_{\hat{Q}_\infty}(V_\infty) // H_\infty.$$  

Denoting by $V_p$, $p \in Q^v_\infty$ the simultaneous eigenspaces for the coefficients of $T_\infty(z)$, we can express $O(L_\infty)$ as the product $\prod_p O_p(L_\infty)$, where $O_p(L_\infty)$ is the $\text{GL}(V_p)$-coadjoint orbit through $L_\infty|_{V_p}$. Now recall that coadjoint orbits of general linear groups admit a sort of quiver description; see Section 4.2. This fact leads to the definition of the following quiver $Q$.

For each $t \in D_0$, fix a marking $(\lambda_{t,1}, \lambda_{t,2}, \ldots, \lambda_{t,d_t})$ of $O(L_t)$, i.e., a tuple satisfying

$$\prod_{i=1}^{d_t} (\pi_{\text{res}}(L_t) - \lambda_{t,i} l_\infty) = 0$$

(see Definition 4.4). Also for each $p \in Q^v_\infty$, fix a marking $(\lambda_{p,1}, \lambda_{p,2}, \ldots, \lambda_{p,d_p})$ of $O_p(L_\infty) \subset \text{gl}(V_p)^*$. Set

$$Q^v = Q^v_\infty \sqcup \bigcup_{t \in D_0} \{ [t, l] \mid l = 1, 2, \ldots, d_t - 1 \} \sqcup \bigcup_{p \in Q^v_\infty} \{ [p, l] \mid l = 1, 2, \ldots, d_p - 1 \},$$

and draw one arrow from

- each $[t, l]$ ($l \geq 2$) to $[t, l - 1]$,
- each $[p, l]$ ($l \geq 2$) to $[p, l - 1]$,
- each $[t, 1]$ to each $p \in Q^v_\infty$,
each \([p, 1]\) to \(p\).

Denote the set of these arrows by \(Q^a\), and define
\[
Q^a = Q^a_\infty \cup Q^a_0.
\]

Define a collection \(\zeta = (\zeta_i)_{i \in Q^v}\) of complex numbers by
\[
\zeta_p = -\lambda_{p, 1} - \sum_{t \in D_0} \lambda_{t, 1} \quad (p \in Q^v_\infty),
\]
\[
\zeta_{[t, l]} = \lambda_{t, l} - \lambda_{t, l+1},
\]
\[
\zeta_{[p, l]} = \lambda_{p, l} - \lambda_{p, l+1},
\]
and a collection \(V = (V_i)_{i \in Q^v}\) of vector spaces as follows. For \(p \in Q^v_\infty\), let \(V_p\) be the one used above. For \(t \in D_0\) and \(l = 1, 2, \ldots, d_t - 1\), set
\[
V_{[t, l]} = \text{Im} \left( \prod_{i=1}^l (\pi_{\text{res}}(L_t) - \lambda_{t, i}1_{\mathbb{C}^n}) \right),
\]
and also, for \(p \in Q^v_\infty\) and \(l = 1, 2, \ldots, d_p - 1\), set
\[
V_{[p, l]} = \text{Im} \left( \prod_{i=1}^l (\pi_{\text{res}}(L_\infty)|_{V_p} - \lambda_{p, i}1_{V_p}) \right).
\]

Applying Lemma 4.5 to \(O(L_t), O_p(L_\infty)\), we then obtain an open embedding
\[
\varphi: \mathcal{M}^*(T, L) \hookrightarrow \text{Rep}_Q(V) \backslash \zeta V G_V.
\]

We will show that \(\varphi\) maps the subset \(\mathcal{M}^*_s(T, L)\) of \(\mathcal{M}^*(T, L)\) consisting of all stable points in the following sense onto the quiver variety \(\mathfrak{M}^*_Q(V, \zeta)\).

**Definition 3.12.** A meromorphic connection \(A \in g \otimes_{\mathbb{C}} \Omega^1(*D)\) on the trivial vector bundle \(O^{\oplus n}\) is stable if it has no non-zero proper subspace \(S \subset \mathbb{C}^n\) such that \(A(S \otimes_{\mathbb{C}} O) \subset S \otimes_{\mathbb{C}} \Omega^1(*D)\).

**Theorem 3.13.** There exists a symplectomorphism \(\mathcal{M}^*_s(T, L) \simeq \mathfrak{M}^*_Q(V, \zeta)\).

**Proof.** For \(t \in D_0\) and \(p \in Q^v_\infty\), let \(\alpha_{p, t}\) be the arrow from \([t, 1]\) to \(p\). Then the image of \(\varphi\) is exactly the set of \(G_V\)-orbits \([\Xi]\) in \(\mu^{-1}(\zeta V)\) such that
\[
- \text{Ker} \Xi_\alpha = 0 \quad \text{and} \quad \text{Im} \Xi_\alpha = V_d(\alpha) \quad \text{for any} \quad \alpha \in Q^v_\infty \setminus \{ \alpha_{p, t} \mid p \in Q^v_\infty, t \in D_0 \},
\]
\[
- \bigcap_{p \in Q^v_\infty} \text{Ker} \Xi_{\alpha_{p, t}} = 0 \quad \text{and} \quad \sum_{p \in Q^v_\infty} \text{Im} \Xi_{\alpha_{p, t}} = \mathbb{C}^n \quad \text{for all} \quad t \in D_0.
\]

We will check that the map \(\varphi\) gives a symplectomorphism between the stable parts. Let \(A\) be a meromorphic connection representing a point in \(\mathcal{M}^*(T, L)\) and \([\Xi] = \varphi([A])\). Take a standard coordinate \(z\) on the affine line \(\mathbb{P}^1 \setminus \{\infty\}\). Then \(A\) is expressed as
\[
A = \left( \sum_{i=0}^{k-2} A_i z^i + \sum_{t \in D_0} \frac{R_t}{z - z(t)} \right) \, dz, \quad A_i, R_t \in g.
\]
Set $B = \sum_{i=0}^{k-2} A_i z^i dz \in \mathfrak{g}^*$. By the definition of $\varphi$, we may assume that $B$ is contained in the $B_k$-coadjoint orbit through $dT_\infty$ and each $R_t$ is expressed as

$$R_t = (\Xi_{\alpha,p,t}, \Xi_{\alpha,q,t})_{p,q} + \lambda_{t,1} 1_{\mathbb{C}^n}.$$ 

Now suppose that $A$ is stable and let $W = (W_i)_{i \in Q^v}$ be a $\Xi$-invariant subspace. Then the above expression of $R_t$ and Lemma 3.10 show that the direct sum $S := \bigoplus_{p \in Q^v} W_p$ is invariant under all $A_i$ and $R_t$. Therefore the stability of $A$ implies that $S = 0$ or $S = \mathbb{C}^n$. If $S = 0$, the injectivity conditions for $\Xi_\alpha$, $\alpha \in Q_0^a$ immediately show that $W_i = 0$ for all $i \in Q^v$, and if $S = \mathbb{C}^n$, the surjectivity conditions for $\Xi_\alpha$, $\alpha \in Q_0^a$ show that $W_i = V_i$ for all $i \in Q^v$. Hence $\Xi$ is stable.

Conversely, suppose that $\Xi$ is stable and let $S \subset \mathbb{C}^n$ be a subspace invariant under all $A_i$ and $R_t$. Note that it is then also invariant under $\text{res}_\infty(A)$. Lemma 3.10 shows $S = \sum_{p \in Q^v} (S \cap V_p)$ and that the collection $(W_p)_{p \in Q^v}$ is invariant under $(\Xi_\alpha)_{\alpha \in Q_0^a}$. Applying Lemma 4.5 to all $R_t$ and the block components $\text{res}_\infty(A)|_{\mathfrak{gl}(V_p)}$, we thus obtain a $\Xi$-invariant subspace $W = (W_i)_{i \in Q^v}$ containing $(W_p)_{p \in Q^v}$ as a subcollection. The stability of $\Xi$ then implies $W = 0$ or $W = V$, and hence $S = 0$ or $S = \mathbb{C}^n$. □

The above theorem enables us to apply Crawley-Boevey’s criterion [7] for the non-emptiness of quiver varieties to the additive irregular Deligne-Simpson problem and obtain the following result, generalizing [6, 9] (see [12] for the basic terminology on the root systems attached to quivers):

**Corollary 3.14.** The space $\mathcal{M}_s^* (\mathbf{T}, \mathbf{L}) \simeq \mathcal{M}_Q^* (\mathbf{v}, \zeta)$ is non-empty if and only if the following conditions hold:

(i) $\mathbf{v} \in \mathbb{Z}^Q_{\geq 0}$ is a positive root,

(ii) $\zeta \cdot \mathbf{v} = 0$,

(iii) any non-trivial decomposition $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_l$ of $\mathbf{v}$ by positive roots $\mathbf{w}_j$ with $\zeta \cdot \mathbf{w}_j = 0$ satisfies the inequality

$$\Delta(\mathbf{v}) > \sum_{j=1}^{l} \Delta(\mathbf{w}_j).$$

4. **Appendix**

4.1. **Formal reduction theory.** In this subsection, we recall some basic facts due to Babbitt-Varadarajan [2] on the formal reduction theory of meromorphic connections. Let $G$ be a complex reductive group and $\mathfrak{g}$ its Lie algebra.

**Proposition 4.1 ([2, Proposition 9.3.2]).** Let $A = \sum_{i \geq 0} A_i z^{i-k} dz \in \mathfrak{g}(z)dz$ with $k > 1$, $A_0 \neq 0$. Suppose that $A_0$ is semisimple. Then there exists a unique
\(\hat{g} \in G(\mathbb{C}[z])\) of the form
\[
\hat{g}(z) = \prod_{i=1}^{\infty} \exp(z^i X_i) = \lim_{j \to \infty} \left( \exp(z^j X_j) \exp(z^{j-1} X_{j-1}) \cdots \exp(z X_1) \right)
\]
with \(X_i \in \text{Im ad}_{A_0} (i \in \mathbb{Z}_{\geq 0})\), such that
\[
A' = \sum_{i \geq 0} A'_i z^{i-k} dz := \hat{g}[A]
\]
satisfies
\[
A'_0 = A_0, \quad [A_0, A'] = 0.
\]
If there exists \(p \in \mathbb{Z}_{\geq 0}\) such that \([A_0, A_i] = 0\) for \(i \leq p\), then \(A'_i = A_i\) for \(i \leq p\).

**Proposition 4.2.** Let \(A = \sum_{i \geq 0} A_i z^{i-k} dz \in \mathfrak{g}((z))dz\) with \(k > 1\), \(A_0 \neq 0\).

Suppose that \(A_i, i \leq k-2\) are contained in some torus \(t \subset \mathfrak{g}\). Set
\[
H_i = \bigcap_{j=0}^{k-i-2} \{ h \in G \mid \text{Ad}_h A_j = A_j \} \quad (i = 0, 1, \ldots, k-2), \quad H_{k-1} = G,
\]
and let \(\mathfrak{h}_i\) be the Lie algebra of \(H_i\) for \(i \leq k-1\). Then there exists a unique \((k-1)\)-tuple \((\hat{g}^{(1)}, \hat{g}^{(2)}, \ldots, \hat{g}^{(k-1)})\) with
\[
\hat{g}^{(l)}(z) = \prod_{i=1}^{\infty} \exp(z^i X_i^{(l)}) \in H_{k-l}(\mathbb{C}[z]), \quad X_i^{(l)} \in \text{Im} (\text{ad}_{A_{l-1}} |_{\mathfrak{h}_{k-l}})
\]
such that
\[
A^{(l)} = \sum_{i \geq 0} A_i^{(l)} z^{i-k} dz := \hat{g}^{(l)} \cdots \hat{g}^{(1)}[A]
\]
satisfies
\[
A_i^{(l)} = A_i \quad (i = 0, 1, \ldots, k-2), \quad A^{(l)} \in \mathfrak{h}_{k-l-1}(z))dz
\]
for each \(l \geq 1\).

**Proof.** Proposition 4.1 shows that there uniquely exists
\[
\hat{g}^{(1)}(z) = \prod_{i > 0} \exp(z^i X_i^{(1)}) \in G(\mathbb{C}[z]), \quad X_i^{(1)} \in \text{Im ad}_{A_0}
\]
such that \(A^{(1)} = \sum_{i \geq 0} A_i^{(1)} z^{i-k} dz := \hat{g}^{(1)}[A]\) satisfies
\[
A_i^{(1)} = A_i \quad (i \leq k-2), \quad A^{(1)} \in \mathfrak{h}_{k-2}(z)dz.
\]
If \(k = 2\), we are done. Otherwise, we apply Proposition 4.1 to \(A^{(1)} - A_0 z^{-k} dz \in \mathfrak{h}_{k-2}(z)dz\) with \(G\) replaced by \(H_{k-2}\), and uniquely find
\[
\hat{g}^{(2)}(z) = \prod_{i > 0} \exp(z^i X_i^{(2)}) \in H_{k-2}(\mathbb{C}[z]), \quad X_i^{(2)} \in \text{Im} (\text{ad}_{A_1} |_{\mathfrak{h}_{k-2}})
\]
such that
\[ A^{(2)} = \sum_{i \geq 0} A_i^{(2)} z^{-k} dz := \hat{g}^{(2)}[A^{(1)}] = \hat{g}^{(2)}[A^{(1)} - A_0 z^{-k} dz] + A_0 z^{-k} dz \]
satisfies
\[ A_i^{(2)} = A_i \quad (i \leq k - 2), \quad A^{(2)} \in h_{k-3}(z) dz. \]
Repeating this argument yields the assertion. \(\square\)

**Corollary 4.3.** Under the assumption and notation of Proposition 4.2, there exists \(\hat{g} \in G(\mathbb{C}[z])\) with \(\hat{g}(0) = 1\) such that
\[ A' = \sum_{i \geq 0} A_i' z^{-k} dz := \hat{g}[A] \]
satisfies
\[ A_i' = A_i \quad (i = 0, 1, \ldots, k - 2), \quad A' \in h_0((z)) dz. \]

### 4.2. Coadjoint orbits of general linear groups and quivers of type A.

In this subsection, we assume \(G = \text{GL}(n, \mathbb{C})\) and recall some relation between \(G\)-coadjoint orbits and quivers of type \(A\).

Let \(O \subseteq \mathfrak{g}^*\) be a \(G\)-coadjoint orbit. We identify \(\mathfrak{g}^*\) with \(\mathfrak{g}\) via the trace pairing.

**Definition 4.4.** A marking of \(O\) is an ordered tuple \((\lambda_1, \lambda_2, \ldots, \lambda_d)\) of complex numbers such that
\[ \prod_{i=1}^{d} (A - \lambda_i 1_{\mathbb{C}^n}) = 0 \]
for some (and hence all) \(A \in O\).

Fix a marking \((\lambda_1, \lambda_2, \ldots, \lambda_d)\) of \(O\) and define a quiver \(Q\) with vertices \(Q^v = \{0, 1, \ldots, d - 1\}\) by drawing one arrow from each \(l \in Q^v (l \geq 1)\) to \(l - 1\). Fix \(L \in O\) and define a collection of vector spaces \(V = (V_l)_{l \in Q^v}\) by
\[ V_0 = \mathbb{C}^n, \quad V_l = \text{Im} \left( \prod_{i=1}^{l} (L - \lambda_i 1_{\mathbb{C}^n}) \right) \quad (l \geq 1). \]

For \(\Xi \in \text{Rep} \hat{Q}(V)\), we denote its components by \(\Xi_{l-1,l} \in \text{Hom}(V_{l-1}, V_l), \Xi_{l+1,l} \in \text{Hom}(V_l, V_{l+1}), l = 1, 2, \ldots, d - 1\). Note that in this case the moment map \(\mu = (\mu_l) : \text{Rep} \hat{Q}(V) \to \mathfrak{g}_V\) is expressed as
\[
\mu_l(\Xi) = \begin{cases} 
\Xi_{0,1} \Xi_{1,0} & (l = 0), \\
\Xi_{l+1,l} \Xi_{l+1,l} - \Xi_{l-1,l} \Xi_{l-1,l} & (1 \leq l < d - 1), \\
-\Xi_{d-2,d-1} \Xi_{d-2,d-1} & (l = d - 1).
\end{cases}
\]
Lemma 4.5. Let $Z$ be the subvariety of $\text{Rep}_Q(V)$ defined by

$$Z = \left\{ \Xi \in \text{Rep}_Q(V) \mid \begin{array}{l}
\mu_l(\Xi) = (\lambda_l - \lambda_{l+1})1_{C^n} \quad (l \geq 1), \\
\Xi_{l-1,l} \text{ is injective for } l \geq 1, \\
\Xi_{l,l-1} \text{ is surjective for } l \geq 1
\end{array} \right\}. $$

Then it is smooth, the group $\prod_{l=1}^{d-1} \text{GL}(V_l)$ acts freely there, and the shift of the moment map $\mu_0$ by $\lambda_11_{C^n}$ induces a $G$-equivariant symplectomorphism

$$Z/\prod_{l=1}^{d-1} \text{GL}(V_l) \to O; \quad [\Xi] \mapsto \Xi_{0,1}\Xi_{1,0} + \lambda_11_{C^n}. $$

Furthermore, for any $\Xi \in Z$ and any subspace $S$ of $C^n$ invariant under $\Xi_{0,1}\Xi_{1,0} + \lambda_11_{C^n}$, there exists a $\Xi$-invariant subspace $W = (W_l)_{l \in Q^n}$ of $V$ such that $W = 0$ (resp. $W = V$) if and only if $S = 0$ (resp. $S = C^n$).

Proof. It is known that Mumford’s stability condition for a point $\Xi \in \text{Rep}_Q(V)$ with respect to the action of $\prod_{l=1}^{d-1} \text{GL}(V_l)$ associated to the trivial linearization is equivalent to that the following two conditions hold [14, 20]:

- if a collection $(W_l)_{l \geq 1}$ of subspaces $W_l \subset V_l$ satisfies $\Xi_{l,l+1}(W_{l+1}) \subset W_l$, $\Xi_{l+1,l}(W_l) \subset W_{l+1}$ for all $l \geq 1$ and $W_1 \subset \text{Ker} (\Xi_{0,1})$, then $W_l = 0$ for all $l \geq 1$;

- if a collection $(W_l)_{l \geq 1}$ of subspaces $W_l \subset V_l$ satisfies $\Xi_{l,l+1}(W_{l+1}) \subset W_l$, $\Xi_{l+1,l}(W_l) \subset W_{l+1}$ for all $l \geq 1$ and $W_1 \supset \text{Im} (\Xi_{1,0})$, then $W_l = V_l$ for all $l \geq 1$.

We will show that for a point in

$$\overline{Z} := \{ \Xi \in \text{Rep}_Q(V) \mid \mu_l(\Xi) = (\lambda_l - \lambda_{l+1})1_{C^n} \quad (l \geq 1) \}, $$

the above condition is equivalent to that $\Xi_{l-1,l}$ is injective and $\Xi_{l,l-1}$ is surjective for all $l \geq 1$ (which we call the injectivity/surjectivity condition); then the geometric invariant theory shows that the action of $Z$ is free and proper, and a basic property of moment maps shows that $Z$ is smooth.

Observe first that for $\Xi \in \overline{Z}$, the following relations hold:

$$\begin{align*}
(10) \quad (\Xi_{0,1}\Xi_{1,2} \cdots \Xi_{l,l+1})\Xi_{l+1,l} &= (\Xi_{0,1}\Xi_{1,0} + (\lambda_1 - \lambda_{l+1})1_{C^n})\Xi_{0,1}\Xi_{1,2} \cdots \Xi_{l-1,l}, \\
(11) \quad \Xi_{l,l+1}(\Xi_{l+1,l} \cdots \Xi_{2,1}\Xi_{1,0}) &= \Xi_{l,l-1} \cdots \Xi_{2,1}\Xi_{1,0}(\Xi_{0,1}\Xi_{1,0} + (\lambda_1 - \lambda_{l+1})1_{C^n}).
\end{align*}$$

Indeed, we have

$$\begin{align*}
\Xi_{0,1}\Xi_{1,2}\Xi_{2,1} &= \Xi_{0,1}(\Xi_{1,0}\Xi_{0,1} + (\lambda_1 - \lambda_2)1_{C^n}) = (\Xi_{0,1}\Xi_{1,0} + (\lambda_1 - \lambda_2)1_{C^n})\Xi_{0,1}, \\
\Xi_{1,2}\Xi_{2,1}\Xi_{1,0} &= (\Xi_{1,0}\Xi_{0,1} + (\lambda_1 - \lambda_2)1_{C^n})\Xi_{1,0} = \Xi_{2,1}\Xi_{1,0}(\Xi_{0,1}\Xi_{1,0} + (\lambda_1 - \lambda_2)1_{C^n}),
\end{align*}$$

etc.
and the general case follows from an inductive argument. Now assume \( \Xi \in Z \) is stable in the above sense. We define subspaces \( W_i, W_i' \subset V_i \) by

\[
W_i = \text{Ker}(\Xi_{0,1} \Xi_{1,2} \cdots \Xi_{l-1,l}), \quad W_i' = \text{Im}(\Xi_{l,l-1} \cdots \Xi_{2,1} \Xi_{1,0}).
\]

Then (10) shows \( \Xi_{l,l+1}(W_{l+1}) \subset W_l, \ Xi_{l+1,l}(W_l) \subset W_{l+1} \) for all \( l \geq 1 \) and \( W_1 \subset \text{Ker} \Xi_{0,1} \); hence \( W_l = 0 \) for all \( l \geq 1 \), which implies that all \( \Xi_{l-1,l} \) are injective. Similarly, (11) shows \( \Xi_{l,l+1}(W_{l+1}) \subset W'_l, \ Xi_{l+1,l}(W'_l) \subset W'_{l+1} \) for all \( l \geq 1 \) and \( W'_l \supset \text{Im} \Xi_{1,0} \); hence \( W'_l = V_l \) for all \( l \geq 1 \), which implies that all \( \Xi_{l,l-1} \) are surjective.

Conversely, assume the injectivity/surjectivity condition: all \( \Xi_{l-1,l} \) are injective and all \( \Xi_{l,l-1} \) are surjective. Let \( W_i \subset V_i \), \( l \geq 1 \) be subspaces satisfying \( \Xi_{l,l+1}(W_{l+1}) \subset W_l, \ Xi_{l+1,l}(W_l) \subset W_{l+1} \) for all \( l \geq 1 \) and \( W_1 \subset \text{Ker} \Xi_{0,1} \). Then \( W_1 = 0 \), and inductively we find

\[
W_l \subset \Xi_{l-1,l}(W_{l-1}) = \Xi_{l-1,l}^{-1}(0) = 0
\]

for all \( l \geq 2 \). Let \( W_l \subset V_l \) be subspaces satisfying \( \Xi_{l,l+1}(W_{l+1}) \subset W_l, \ Xi_{l+1,l}(W_l) \subset W_{l+1} \) for all \( l \geq 1 \) and \( W_1 \supset \text{Im} \Xi_{1,0} \). Then \( W_1 = V_1 \), and inductively we find

\[
W_l \supset \Xi_{l,l-1}(W_{l-1}) = \Xi_{l,l-1}(V_{l-1}) = V_l
\]

for all \( l \geq 2 \). Hence the stability is equivalent to the injectivity/surjectivity condition.

Let \( \Xi \in Z \) and define \( X = \Xi_{0,1}^{1,0} + \lambda_1 1_{C^n} \in g \). Using (10) or (11) inductively, we then find

\[
\prod_{i=1}^{l}(X - \lambda_i 1_{C^n}) = (\Xi_{0,1}^{1,2} \cdots \Xi_{l-1,l})(\Xi_{l,l-1} \cdots \Xi_{2,1} \Xi_{1,0}) \quad (l = 1, 2, \ldots, d - 1).
\]

Hence \( \text{rank} \prod_{i=1}^{l}(X - \lambda_i 1_{C^n}) = \text{dim} V_l \) \( (l \geq 1) \), which implies \( X \in O \). Furthermore, the above relation also shows that the induced map \( Z/ \prod_{l \geq 1} \text{GL}(V_l) \to O \) is injective.

Conversely let \( X \in O \). Define

\[
V'_l = \prod_{i=1}^{l}(X - \lambda_i 1_{C^n}).
\]

Let \( \Xi_{l,l+1}' : V'_{l+1} \to V'_l \) be the inclusion and set \( \Xi_{l+1,l}' = (X - \lambda_i 1_{C^n})|_{V'_l} : V'_l \to V'_l \). Then the injectivity/surjectivity condition is satisfied and \( \text{dim} V'_l = \text{dim} V_l \) for all \( l \geq 1 \). Hence these data give a point \([\Xi] \in Z/ \prod_{l \geq 1} \text{GL}(V_l) \) satisfying \( \Xi_{0,1}^{1,0} + \lambda_1 1_{C^n} = \Xi_{0,1}^{1,0} + \lambda_1 1_{C^n} = X \).
The proof of the rest assertion is now easy. Let $\Xi \in Z$ and $X = \Xi_{0,1}\Xi_{1,0} + \lambda_1 C_n \in O$. If $S \subset C^n$ is $X$-invariant, define

$$W_0 = S, \quad W_l = (\Xi_{0,1}\Xi_{1,2} \cdots \Xi_{l-1,l-1})^{-1}(S) \quad (l \geq 1).$$

Then the above arguments show that $W = (W_l)_{l \in \mathbb{Q}^v}$ satisfies all the desired conditions. \hfill \Box

**Remark 4.6.** It is also known that $\mu_0 + \lambda_1 C_n$ induces an isomorphism between the categorical quotient of $\overline{Z}$ by $\prod_{l=1}^{d-1} GL(V_l)$ and the closure of $O$; see [8, Appendix].

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