On a reverse extended Hardy–Hilbert’s inequality

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Abstract
By the use of the weight coefficients, the idea of introducing parameters and the Euler–Maclaurin summation formula, a reverse extended Hardy–Hilbert inequality and the equivalent forms are given. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are also considered.

MSC: 26D15

Keywords: Weight coefficient; Hardy–Hilbert’s inequality; Reverse; Equivalent statement; Parameter

1 Introduction
Assuming that \( p > 1, \frac{1}{q} + \frac{1}{p} = 1, a_m, b_n \geq 0, 0 < \sum_{m=1}^{\infty} a_m^p < \infty \) and \( 0 < \sum_{n=1}^{\infty} b_n^q < \infty \), we have the following Hardy–Hilbert inequality with the best possible constant factor \( \frac{\pi \sin(\pi/p)}{\sin(\pi/p)} \) (cf. [1], Theorem 315):

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \left( \frac{m+n}{m+n} \right)^{1/p} \left( \frac{m+n}{m+n} \right)^{1/q} < \frac{\pi \sin(\pi/p)}{\sin(\pi/p)} \left( \sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left( \sum_{n=1}^{\infty} b_n^q \right)^{1/q},
\] (1)

In 2006, by introducing the parameters \( \lambda_i \in (0, 2] \) \((i = 1, 2), \lambda_1 + \lambda_2 = \lambda \in (0, 4]\), an extension of (1) was provided by [2] as follows:

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_m b_n \left( \frac{m+n}{m+n} \right)^{1/p} \left( \frac{m+n}{m+n} \right)^{1/q} < B(\lambda_1, \lambda_2) \left[ \sum_{m=1}^{\infty} m^{(1-\lambda_1)-1} a_m^p \right]^\frac{1}{p} \left[ \sum_{n=1}^{\infty} n^{(1-\lambda_2)-1} b_n^q \right]^\frac{1}{q},
\] (2)

where the constant factor \( B(\lambda_1, \lambda_2) \) is the best possible \((B(u, v) = \int_0^\infty \frac{t^{u-1}}{1+t^v} dt \ (u, v > 0) \) is the beta function). For \( \lambda = 1, \lambda_1 = \lambda_2 = \frac{1}{q}, \lambda_2 = \frac{1}{p} \), inequality (2) reduces to (1); for \( p = q = 2, \lambda_1 = \lambda_2 = \frac{1}{2} \), (2) reduces to Yang’s work in [3]. Recently, applying (2), [4] gave a new inequality with the kernel \( \frac{1}{(m+n)^p} \) involving partial sums.

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If \( f(x), g(y) \geq 0 \), \( 0 < \int_{0}^{\infty} f^p(x) \, dx < \infty \), and \( 0 < \int_{0}^{\infty} g^q(y) \, dy < \infty \), then we still have the following Hardy–Hilbert integral inequality (cf. [1], Theorem 3.16):

\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{x+y} \, dx \, dy < \frac{\pi}{\sin(\pi/p)} \left( \int_{0}^{\infty} f^p(x) \, dx \right)^{1/p} \left( \int_{0}^{\infty} g^q(y) \, dy \right)^{1/q},
\]

where the constant factor \( \pi / \sin(\pi/p) \) is the best possible. Inequalities (1) and (3) with their extensions and reverses are important in analysis and its applications (cf. [5–15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 3.16): If \( K(t) (t > 0) \) is decreasing, \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), \( 0 < \phi(s) = \int_{0}^{\infty} K(t)t^{s-1} \, dt < \infty, a_n \geq 0 \), \( 0 < \sum_{n=1}^{\infty} a_n^p < \infty \), then we have

\[
\int_{0}^{\infty} x^{p-2} \left( \sum_{n=1}^{\infty} K(nx)a_n \right)^p \, dx < \phi^p \left( \frac{1}{q} \right) \sum_{n=1}^{\infty} a_n^q.
\]

In the last ten years, some extensions of (4) with their applications and the reverses were provided by [16–20].

In 2016, by means of the techniques of real analysis, Hong et al. [21] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. Similar work about Hilbert-type integral inequalities is in [22–26].

In this paper, following the way of [2, 21], by the use of the weight coefficients, the idea of introduced parameters and Euler–Maclaurin summation formula, a reverse extended Hardy–Hilbert inequality as well as the equivalent forms are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 2 and Remark 1–2.

### 2 Some lemmas

In what follows, we assume that \( 0 < p < 1 \) (\( q < 0 \)), \( \frac{1}{p} + \frac{1}{q} = 1 \), \( N = \{1, 2, \ldots\} \), \( \lambda \in (0, 6] \), \( \lambda_i \in (0, 2] \cap (0, \lambda) \) \( (i = 1, 2) \),

\[
O\left( \frac{1}{m^{\lambda_2}} \right) := \frac{(1 + \theta_m)^{-\lambda}}{\lambda_2 B(\lambda_2, \lambda - \lambda_2)} m^{\lambda_2} \in (0, 1), \quad \left( \theta_m \in \left( 0, \frac{1}{m} \right), m \in \mathbb{N} \right).
\]

We also assume that \( a_m, b_n \geq 0 \), such that

\[
0 < \sum_{m=1}^{\infty} m^{p(1-\frac{\lambda_2}{p} + \frac{1}{q})-1} a_m^p < \infty, \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda_2}{q} + \frac{1}{p})-1} b_n^q < \infty.
\]

**Lemma 1** Define the following weight coefficient:

\[
\varpi(\lambda_2, m) := m^{\lambda_2 - \lambda_2} \sum_{n=1}^{\infty} \frac{n^{\lambda_2 - 1}}{(m + n)^{\lambda}} \quad (m \in \mathbb{N}).
\]

We have the following inequality:

\[
B(\lambda_2, \lambda - \lambda_2) \left( 1 - O\left( \frac{1}{m^{\lambda_2}} \right) \right) < \varpi(\lambda_2, m) < B(\lambda_2, \lambda - \lambda_2) \quad (m \in \mathbb{N}).
\]
Proof For fixed \( m \in \mathbb{N} \), we set function \( g(m, t) := \frac{t^{\lambda_2-1}}{(m + t)^\lambda} \) \((t > 0)\). Using the Euler–Maclaurin summation formula (cf. [2, 3]), for \( \rho(t) := t - \lfloor t \rfloor - \frac{1}{2} \), we have

\[
\sum_{n=1}^{\infty} g(m, n) = \int_1^{\infty} g(m, t) \, dt + \frac{1}{2} g(m, 1) + \int_1^{\infty} \rho(t) g'(m, t) \, dt
\]

\[
= \int_0^{\infty} g(m, t) \, dt - h(m),
\]

\( h(m) := \int_0^1 g(m, t) \, dt - \frac{1}{2} g(m, 1) - \int_1^{\infty} \rho(t) g'(m, t) \, dt. \)

We obtain \(-\frac{1}{2} g(m, 1) = -\frac{1}{2(m+1)^\lambda} \),

\[
\int_0^1 g(m, t) \, dt
\]

\[
= \frac{1}{\lambda_2} \int_0^1 \frac{t^{\lambda_2-2}}{(m + t)^\lambda} \, dt = \int_0^1 \frac{t^{\lambda_2}}{(m + t)^{\lambda+1}} = \frac{1}{\lambda_2} \int_0^1 \frac{t^{\lambda_2}}{(m + t)^{\lambda+1}} + \frac{\lambda_2}{\lambda_2} \int_0^1 \frac{t^{\lambda_2+1}}{(m + t)^{\lambda+1}}
\]

\[
> \frac{1}{\lambda_2} \int_0^1 \frac{1}{(m + t)^{\lambda}} + \frac{\lambda_2}{\lambda_2} \int_0^1 \frac{t^{\lambda_2+1}}{(m + t)^{\lambda+1}}
\]

\[
= \frac{1}{\lambda_2} \frac{1}{(m + t)^\lambda} + \frac{\lambda_2}{\lambda_2} \frac{1}{(m + t)^{\lambda+1}} + \frac{\lambda_2}{\lambda_2} \frac{1}{(m + t)^{\lambda+1}}
\]

We find

\[
g'(m, t) = -\frac{(\lambda_2 - 1)t^{\lambda_2-2}}{(m + t)^\lambda} + \frac{\lambda_2 t^{\lambda_2-1}}{(m + t)^{\lambda+1}} = \frac{(1 - \lambda_2)t^{\lambda_2-2}}{(m + t)^\lambda} + \frac{\lambda_2 t^{\lambda_2-2}}{(m + t)^{\lambda+1}} = \frac{\lambda_2 t^{\lambda_2-2}}{(m + t)^{\lambda+1}}
\]

\[
and for 0 < \lambda_2 < 2, \lambda_2 < \lambda \leq 6, it follows that
\]

\[
(-1)^i \frac{d^i}{dt^i} \left[ \frac{t^{\lambda_2-2}}{(m + t)^\lambda} \right] > 0, \quad (-1)^i \frac{d^i}{dt^i} \left[ \frac{t^{\lambda_2-2}}{(m + t)^{\lambda+1}} \right] > 0 \quad (i = 0, 1, 2, 3).
\]

Still by the Euler–Maclaurin summation formula (cf. [2, 3]), we obtain

\[
(\lambda_2 - 1) \int_1^{\infty} \rho(t) t^{\lambda_2-2} \, dt > -\frac{\lambda_2 - 1}{12(m + 1)^\lambda},
\]

and

\[
-\frac{\lambda_2}{12(m + 1)^{\lambda+1}} = \frac{\lambda_2}{720} \left[ \frac{t^{\lambda_2-2}}{(m + t)^{\lambda+1}} \right]^{t=1}
\]

\[
> \frac{(m + 1)\lambda - \lambda}{12(m + 1)^{\lambda+1}} = \frac{(m + 1)\lambda}{720} \left[ \frac{(\lambda_2 + 1)(\lambda + 2)}{(m + 1)^{\lambda+1}} + \frac{2(\lambda_2 + 1)(\lambda - 2)}{(m + 1)^{\lambda+3}} + \frac{(2 - \lambda_2)(3 - \lambda_2)}{(m + 1)^{\lambda+1}} \right]
\]
\[
\frac{\lambda}{12(m + 1)^{\lambda}} - \frac{\lambda}{12(m + 1)^{\lambda+1}} - \frac{\lambda}{720} \left[ (\lambda + 1)(\lambda + 2) + 2(\lambda + 1)(2 - \lambda_2) + (2 - \lambda_2)(3 - \lambda_2) \right]
\]

Hence, we have \( h(m) > h_1(m) + h_2(m) + h_3(m) \), where

\[
h_1 := \frac{1}{\lambda_2} \left( 2 - \frac{1}{12} - \frac{\lambda_2(2 - \lambda_2)(3 - \lambda_2)}{720} \right),
\]

\[
h_2 := \frac{1}{\lambda_2(\lambda_2 + 1)} - \frac{1}{12} \left( \frac{\lambda_2 + 1)(2 - \lambda_2)}{360} \right),
\]

and

\[
h_3 := \frac{1}{\lambda_2(\lambda_2 + 1)(\lambda_2 + 2)} - \frac{\lambda_2^3}{720}.
\]

For \( \lambda \in (0, 6) \), \( \frac{\lambda_2^3}{720} < \frac{1}{24} \), \( \lambda_2 \in (0, 2] \), we find

\[
h_1 > \frac{1}{\lambda_2} - \frac{1}{2} - \frac{1 - \lambda_2}{12} - \frac{(2 - \lambda_2)(3 - \lambda_2)}{24} = \frac{24 - 20\lambda_2 + 7\lambda_2^2 - \lambda_2^3}{24} > 0.
\]

In fact, setting \( g(\sigma) := 24 - 20\sigma + 7\sigma^2 - \sigma^3 (\sigma \in (0, 2]) \), we obtain

\[
g'(\sigma) = -20 + 14\sigma - 3\sigma^2 = -3\left( \sigma - \frac{7}{3} \right)^2 - \frac{11}{3} < 0,
\]

and then

\[
h_1 > \frac{g(\lambda_2)}{24\lambda_2} \geq \frac{g(2)}{24\lambda_2} = \frac{4}{24\lambda_2} > 0 \quad (\lambda_2 \in (0, 2]).
\]

We obtain \( h_2 > \frac{1}{6} - \frac{13}{12} - \frac{18}{360} = \frac{1}{3} > 0 \), and \( h_3 \geq \frac{1}{24} - \frac{10}{720} = \frac{1}{36} \geq 0 \). Hence, we have \( h(m) > 0 \), and then setting \( t = \mu \), it follows that

\[
\sigma(\lambda_2, m) = m^{\lambda - \lambda_2} \sum_{n=1}^{\infty} g(m, n) < m^{\lambda - \lambda_2} \int_0^\infty g(m, t) \, dt
\]

\[
= m^{\lambda - \lambda_2} \int_0^\infty \frac{t^{\lambda_2-1}}{(m + t)^{\lambda}} \, dt = \int_0^\infty \frac{\mu^{\lambda_2 mass} - \lambda}{(1 + \mu)} \, d\mu = B(\lambda_2, \lambda - \lambda_2).
\]

On the other hand, we also have

\[
\sum_{n=1}^{\infty} g(m, n) = \int_0^\infty g(m, t) \, dt + \frac{1}{2} g(m, 1) + \int_1^\infty \rho(t) g'(m, t) \, dt
\]

\[
= \int_0^\infty g(m, t) \, dt + H(m),
\]

\[
H(m) := \frac{1}{2} g(m, 1) + \int_1^\infty \rho(t) g'(m, t) \, dt.
\]

We have obtained \( \frac{1}{2} g(m, 1) = \frac{1}{2(m + 1)^{\lambda}} \) and

\[
g'(m, t) = \frac{-(\lambda + 1 - \lambda_2)t^{\lambda_2-2}}{(m + t)^{\lambda}} + \frac{\lambda m t^{\lambda_2-2}}{(m + t)^{\lambda_2+1}}.
\]
For $\lambda_2 \in (0, 2] \cap (0, \lambda), 0 < \lambda \leq 6$, by the Euler–Maclaurin summation formula, we obtain

\[-(\lambda + 1 - \lambda_2) \int_1^\infty \rho(t) \frac{t^{\lambda_2 - 2}}{(m + t)^\lambda} dt > \frac{\lambda + 1 - \lambda_2}{12(m + 1)^\lambda} - \frac{\lambda + 1 - \lambda_2}{720} \left[ \sum_{k=1}^{m} \frac{t^{\lambda_2 - 2}}{(m + t)^\lambda} \right] \]

\[= \frac{\lambda + 1 - \lambda_2}{12(m + 1)^\lambda} - \frac{\lambda + 1 - \lambda_2}{720} \left[ \frac{(2 - \lambda_2)(3 - \lambda_2)}{(m + t)^\lambda} + \frac{2\lambda(2 - \lambda_2)}{(m + t)^{\lambda + 1}} \right] \]

\[= \frac{\lambda + 1 - \lambda_2}{12(m + 1)^\lambda} - \frac{\lambda + 1 - \lambda_2}{720} \left[ \frac{(2 - \lambda_2)(3 - \lambda_2)}{(m + t)^\lambda} + \frac{2\lambda(2 - \lambda_2)}{(m + t)^{\lambda + 1}} + \frac{\lambda(\lambda + 1)}{(m + t)^{\lambda + 2}} \right] \]

\[m\lambda \int_1^\infty \rho(t) \frac{t^{\lambda_2 - 2}}{(m + t)^{\lambda + 1}} dt > -\frac{m\lambda}{12(m + 1)^{\lambda + 1}} = -\frac{(m + 1)\lambda - \lambda}{12(m + 1)^{\lambda + 1}} = -\frac{\lambda}{12(m + 1)^{\lambda + 1}} + \frac{\lambda}{12(m + 1)^{\lambda + 1}}. \]

Hence, we have $H(m) > \frac{H_1}{(m + 1)^\lambda} + \frac{H_2(m)}{(m + 1)^{\lambda + 1}}$, where

\[H_1 := \frac{7 - \lambda_2}{12} - \frac{(\lambda + 1 - \lambda_2)(2 - \lambda_2)(3 - \lambda_2)}{720}, \]

\[H_2(m) := \frac{1}{12} - \frac{(\lambda + 1 - \lambda_2)(2 - \lambda_2)}{720} - \frac{(\lambda + 1 - \lambda_2)(\lambda + 1)}{720(m + 1)}. \]

For $\lambda_2 \in (0, 2] \cap (0, \lambda), 0 < \lambda \leq 6$, we find $H_1 > \frac{5}{12} - \frac{42}{720} > 0$, and

\[H_2(m) > \frac{1}{12} - \frac{14}{360} - \frac{49}{1440} = \frac{15}{1440} > 0. \]

It follows that $H(m) > 0$, and then

\[m^{\lambda - \lambda_2} \sum_{n=1}^\infty g(m, n) > m^{\lambda - \lambda_2} \int_1^\infty g(m, t) dt = m^{\lambda - \lambda_2} \int_0^1 g(m, t) dt - m^{\lambda - \lambda_2} \int_1^\infty g(m, t) dt = B(\lambda_2, \lambda - \lambda_2) \left[ \frac{1}{B(\lambda_2, \lambda - \lambda_2)} \int_0^1 \frac{u^{\lambda_2 - 1}}{(1 + u)^\lambda} du \right] > 0. \]

By the integral mid-value theorem, we find

\[\int_0^1 \frac{u^{\lambda_2 - 1}}{(1 + u)^\lambda} du = \frac{1}{(1 + \theta_m)^\lambda} \int_0^1 \frac{u^{\lambda_2 - 1}}{(1 + u)^\lambda} du = \frac{1}{(1 + \theta_m)^\lambda} \frac{1}{\lambda_2 m^{\lambda_2}} \left( \theta_m \in \left(0, \frac{1}{m}\right) \right), \]

namely, (7) follows. □
Lemma 2 We have the following reverse extended Hardy–Hilbert inequality:

\[ I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} > B^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)B^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \]

\times \left[ \sum_{m=1}^{\infty} \left[ 1 - O \left( \frac{1}{m^{\lambda_2}} \right) \right] m^{\theta_1 - (\frac{1}{2} + \frac{1}{p})} - 1 \right] d_m \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{\theta_2 - (\frac{1}{2} + \frac{1}{q})} - 1 \right] b_n^{\frac{1}{q}}. \quad (8) \]

Proof In the same way as obtaining (7), for \( n \in \mathbb{N} \), we obtain the following inequality of the weight coefficient:

\[ \omega(\lambda_1, n) := n^{\lambda_1 - 1} \sum_{m=1}^{\infty} \frac{n^{\lambda_1 - 1}}{(m+n)^\lambda} < B(\lambda_1, \lambda - \lambda_1). \quad (9) \]

By the reverse Hölder inequality (cf. [27]), we obtain

\[ I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^\lambda} \left[ \frac{n^{(1-\lambda_1)/p}}{m^{(1-\lambda_2)/q}} d_m \right] \left[ \frac{m^{(1-\lambda_2)/p}}{n^{(1-\lambda_1)/q}} b_n \right] \]

\[ \geq \left[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m+n)^\lambda} n^{\lambda_2 - 1} m^{\lambda_1 - 1} \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^\lambda} n^{\lambda_1 - 1} m^{\lambda_2 - 1} \right]^{\frac{1}{q}} \]

\[ = \left[ \sum_{m=1}^{\infty} \sigma(\lambda_2, m) m^{\theta_1 - (\frac{1}{2} + \frac{1}{p})} - 1 \right] d_m \left[ \sum_{n=1}^{\infty} \omega(\lambda_1, n) n^{\theta_2 - (\frac{1}{2} + \frac{1}{q})} - 1 \right] b_n \].

Then, by (7) and (9), in view of \( 0 < p < 1, q < 0 \), we have (8). \( \square \)

Remark 1 By (8), for \( \lambda_1 + \lambda_2 = \lambda \in (0, 4), 0 < \lambda_i \leq 2 \ (i = 1, 2) \), we find

\[ \omega(\lambda_1, n) < B(\lambda_1, \lambda_2), \]

\[ B(\lambda_1, \lambda_2) \left( 1 - O \left( \frac{1}{m^{\lambda_2}} \right) \right) < \sigma(\lambda_2, m) < B(\lambda_1, \lambda_2) \quad (m, n \in \mathbb{N}), \]

\[ O \left( \frac{1}{m^{\lambda_2}} \right) = \frac{(1 + \theta_m)^{-\lambda}}{\lambda_2 B(\lambda_1, \lambda_2) m^{\lambda_2}} \in (0, 1) \quad \left( \theta_m \in \left( 0, \frac{1}{m} \right) \right), \]

\[ 0 < \sum_{m=1}^{\infty} m^{\theta_1 - (\lambda_1 - 1)} d_m < \infty, \quad 0 < \sum_{n=1}^{\infty} n^{\theta_2 - (\lambda_2 - 1)} b_n < \infty. \]

and the following reverse inequality:

\[ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^\lambda} > B(\lambda_1, \lambda_2) \left[ \sum_{m=1}^{\infty} \left[ 1 - O \left( \frac{1}{m^{\lambda_2}} \right) \right] m^{\theta_1 - (\lambda_1 - 1)} d_m \right]^{\frac{1}{p}} \left[ \sum_{n=1}^{\infty} n^{\theta_2 - (\lambda_2 - 1)} b_n \right]^{\frac{1}{q}}. \quad (10) \]
Lemma 3 For any $\varepsilon > 0$, we have

$$L := \sum_{m=1}^{\infty} O\left(\frac{1}{m^{2+\varepsilon+1}}\right) = O(1).$$ (11)

Proof There exists a constant $M > 0$, such that

$$|L| \leq M \sum_{m=1}^{\infty} \frac{1}{m^{2+\varepsilon+1}} = M \left(1 + \sum_{m=2}^{\infty} \frac{1}{m^{2+\varepsilon+1}}\right).$$

By the decreasing property of the series, it follows that

$$|L| \leq M \left(1 + \int_{1}^{\infty} \frac{1}{x^{2+\varepsilon+1}} \, dx\right) < M \left(1 + \frac{1}{\lambda_2}\right) < \infty.$$ 

Hence, Eq. (11) follows. \qed

Lemma 4 For $\lambda_1 + \lambda_2 = \lambda \in (0, 4]$, the constant factor $B(\lambda_1, \lambda_2)$ in (10) is the best possible.

Proof For any $0 < \varepsilon < p \lambda_1$, we set

$$\tilde{a}_m := m^{\lambda_1 - \frac{\varepsilon}{p} - 1}, \quad \tilde{b}_n := n^{\lambda_2 - \frac{\varepsilon}{q} - 1} \quad (m, n \in \mathbb{N}).$$

If there exists a constant $M \geq B(\lambda_1, \lambda_2)$, such that (10) is valid when replacing $B(\lambda_1, \lambda_2)$ by $M$, then in particular, substitution of $a_m = \tilde{a}_m$ and $b_n = \tilde{b}_n$ in (10), we have

$$\tilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(m+n)^{\lambda}} \tilde{a}_m \tilde{b}_n$$

$$> M \left\{ \sum_{n=1}^{\infty} \left[ 1 - O\left(\frac{1}{m^{p/2}}\right) \right] \sum_{m=1}^{\infty} n^{\left(p(1-\lambda_1) - 1\right) - 1} \tilde{a}_m \right\} - \frac{1}{p} \left\{ \sum_{n=1}^{\infty} n^{\left(p(1-\lambda_2) - 1\right) - 1} \tilde{b}_n \right\}^{\frac{1}{q}}.$$ 

By (11) and the decreasing property of series, we obtain

$$\tilde{I} > M \left\{ \sum_{m=1}^{\infty} \left[ 1 - O\left(\frac{1}{m^{p/2}}\right) \right] \sum_{m=1}^{\infty} n^{\left(p(1-\lambda_1) - 1\right) - 1} \tilde{a}_m \right\} - \frac{1}{p} \left\{ \sum_{n=1}^{\infty} n^{\left(p(1-\lambda_2) - 1\right) - 1} \tilde{b}_n \right\}^{\frac{1}{q}}$$

$$= M \left( \sum_{m=1}^{\infty} m^{-\varepsilon - 1} - \sum_{m=1}^{\infty} O\left(\frac{1}{m^{p/2+\varepsilon+1}}\right) \right)^{\frac{1}{p}} \left(1 + \sum_{n=2}^{\infty} n^{-\varepsilon - 1}\right)^{\frac{1}{q}}$$

$$> M \left( \int_{1}^{\infty} x^{-\varepsilon - 1} \, dx - O(1) \right)^{\frac{1}{p}} \left(1 + \int_{1}^{\infty} y^{-\varepsilon - 1} \, dy\right)^{\frac{1}{q}}$$

$$= \frac{M}{\varepsilon} (1 - \varepsilon O(1))^{\frac{1}{p}} (\varepsilon + 1)^{\frac{1}{q}}.$$
Hence, we have

\[ \tilde{I} = \sum_{n=1}^{\infty} \left[ n^{\frac{1}{p+\frac{1}{\tilde{\lambda}}}} \sum_{m=1}^{\infty} \frac{1}{(m+n)^{\frac{1}{p}}} m^{\frac{1}{\tilde{\lambda}}-1} \right] n^{-1} \]

\[ = \sum_{n=1}^{\infty} \omega(\tilde{\lambda}, n) n^{-\epsilon-1} < B(\tilde{\lambda}_1, \tilde{\lambda}_2) \left( 1 + \sum_{n=2}^{\infty} n^{-\epsilon-1} \right) \]

\[ < B(\tilde{\lambda}_1, \tilde{\lambda}_2) \left( 1 + \int_{1}^{\infty} y^{-\epsilon-1} \, dy \right) = \frac{\epsilon + 1}{\epsilon} B\left( \lambda_1 - \frac{\epsilon}{p}, \lambda_2 + \frac{\epsilon}{p} \right). \]

Then we have

\[ (\epsilon + 1)B\left( \lambda_1 - \frac{\epsilon}{p}, \lambda_2 + \frac{\epsilon}{p} \right) > \epsilon \tilde{I} > M(1 - \epsilon O(1))^\frac{1}{2} (\epsilon + 1)^\frac{1}{2}. \]

For \( \epsilon \to 0^+ \), in view of the continuity of the beta function, we find \( B(\tilde{\lambda}_1, \tilde{\lambda}_2) \geq M \). Hence, \( M = B(\tilde{\lambda}_1, \tilde{\lambda}_2) \) is the best possible constant factor of (10). \( \square \)

Setting \( \tilde{\lambda}_1 := \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}, \tilde{\lambda}_2 := \frac{\lambda_1}{q} + \frac{\lambda_2}{p}, \) we find

\[ \tilde{\lambda}_1 + \tilde{\lambda}_2 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p} + \frac{\lambda}{q} = \lambda, \]

and we can reduce (8) to the following:

\[ I = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n)^{\frac{1}{p}}} \]

\[ > B^{\frac{1}{2}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{2}}(\lambda_1, \lambda - \lambda_1) \]

\[ \times \left[ \sum_{n=1}^{\infty} \left( 1 - O\left( \frac{1}{m^{p/2}} \right) \right) m^{p(1-\tilde{\lambda}_2)-1} a_m \right]^{\frac{1}{2}} \left[ \sum_{n=1}^{\infty} m^{p(1-\tilde{\lambda}_1)-1} b_n \right]^{\frac{1}{2}}. \]  

(12)

**Lemma 5** If \( \lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1)) \), the constant factor \( B^{\frac{1}{2}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{2}}(\lambda_1, \lambda - \lambda_1) \) in (12) is the best possible, then we have \( \lambda - \lambda_1 - \lambda_2 = 0 \), namely, \( \lambda = \lambda_1 + \lambda_2 \).

**Proof** For \( \lambda - \lambda_1 - \lambda_2 \in (-p\lambda_1, p(\lambda - \lambda_1)) \), we obtain

\[ \tilde{\lambda}_1 = \frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q} > \frac{(1-p)\lambda_1}{p} + \frac{\lambda_1}{q} = 0, \quad \tilde{\lambda}_2 < \frac{\lambda_1 + p(\lambda - \lambda_1)}{p} + \frac{\lambda_1}{q} = \lambda, \]

\[ 0 < \tilde{\lambda}_2 = \lambda - \tilde{\lambda}_1 < \lambda. \]

Hence, we have \( B(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \mathbb{R}_+ = (0, \infty) \).

If the constant factor \( B^{\frac{1}{2}}(\lambda_2, \lambda - \lambda_2) B^{\frac{1}{2}}(\lambda_1, \lambda - \lambda_1) \) in (12) is the best possible, then in view of (10), the unique best possible constant factor must be \( B(\tilde{\lambda}_1, \tilde{\lambda}_2) \) \((\in \mathbb{R}_+)\), namely,

\[ B(\tilde{\lambda}_1, \tilde{\lambda}_2) = B^{\frac{1}{2}}(\lambda - \lambda_2, \lambda_2) B^{\frac{1}{2}}(\lambda_1, \lambda - \lambda_1). \]
By the reverse Hölder inequality, we find

\[
B(\tilde{\lambda}_1, \tilde{\lambda}_2) = B\left(\frac{\lambda - \lambda_2}{p} + \frac{\lambda_1}{q}, \frac{\lambda - \lambda_1}{q} + \frac{\lambda_2}{p}\right)
\]

\[
= \int_0^\infty \frac{1}{(1 + u)^{\frac{1}{p} + \frac{\lambda - \lambda_2}{p}}} du = \int_0^\infty \frac{1}{(1 + u)^{\frac{1}{q} + \frac{\lambda - \lambda_1}{q}}} du
\]

\[
\geq \int_0^\infty \frac{1}{(1 + u)^{\frac{1}{p} + \frac{\lambda - \lambda_2}{p}}} du = \int_0^\infty \frac{1}{(1 + u)^{\frac{1}{q} + \frac{\lambda - \lambda_1}{q}}} du = B_1^{\frac{1}{p}}(\lambda - \lambda_2, \lambda_2)B_1^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1).
\]

We observe that (13) keeps the form of equality if and only if there exist constants \(A\) and \(B\), such that they are not all zero and (cf. [27])

\[A u^{\lambda - \lambda_2 - 1} = B u^{\lambda - \lambda_1 - 1}\] a.e. in \(\mathbb{R}_+\).

Assuming that \(A \neq 0\), it follows that

\[u^{\lambda - \lambda_2 - \lambda_1} = \frac{B}{A}\] a.e. in \(\mathbb{R}_+\),

and then \(\lambda - \lambda_2 - \lambda_1 = 0\), namely, \(\lambda = \lambda_1 + \lambda_2\).

\[\square\]

3 Main results and some particular cases

Theorem 1 Inequality (8) is equivalent to the following inequalities:

\[
J := \left\{ \sum_{n=1}^\infty n^{\frac{1}{p}} \left( \frac{1 - \lambda_1}{q} + \frac{1 - \lambda_2}{p} \right)^{-1} \left[ \sum_{m=1}^\infty \frac{1}{(m + n)^{\frac{1}{q}}} b_m \right]^{\frac{1}{q}} \right\}^p
\]

\[
> B_1^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)B_1^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left\{ \sum_{m=1}^\infty \left[ 1 - O\left( \frac{1}{m^{\frac{1}{q}}} \right) \right] \left[ m^{\frac{1}{p}} \left( \frac{1 - \lambda_1}{q} + \frac{1 - \lambda_2}{p} \right)^{-1} a_m \right]^{\frac{1}{q}} \right\}^p,
\]

\[
J_1 := \left\{ \sum_{n=1}^\infty n^{\frac{1}{p}} \left( \frac{1 - \lambda_1}{q} + \frac{1 - \lambda_2}{p} \right)^{-1} \left[ \sum_{m=1}^\infty \frac{1}{(m + n)^{\frac{1}{q}}} b_m \right]^{\frac{1}{q}} \right\}^p
\]

\[
> B_1^{\frac{1}{p}}(\lambda_2, \lambda - \lambda_2)B_1^{\frac{1}{q}}(\lambda_1, \lambda - \lambda_1) \left\{ \sum_{n=1}^\infty n^{\frac{1}{p}} \left( \frac{1 - \lambda_1}{q} + \frac{1 - \lambda_2}{p} \right)^{-1} b_n \right\}^{\frac{1}{q}}.
\]

If the constant factor in (8) is the best possible, then so is the constant factor in (14) and (15).

Proof Suppose that (14) is valid. By the Hölder inequality, we have

\[
I = \sum_{n=1}^\infty n^{\frac{1}{p}} \left( \frac{1 - \lambda_1}{q} + \frac{1 - \lambda_2}{p} \right) \left[ \sum_{m=1}^\infty \frac{1}{(m + n)^{\frac{1}{q}}} a_m \right] \left[ n^{\frac{1}{p}} \left( \frac{1 - \lambda_1}{q} + \frac{1 - \lambda_2}{p} \right) b_n \right]^{\frac{1}{q}}
\]

\[
\geq \left\{ \sum_{n=1}^\infty n^{\frac{1}{p}} \left( \frac{1 - \lambda_1}{q} + \frac{1 - \lambda_2}{p} \right)^{-1} b_n \right\}^{\frac{1}{q}}.
\]

(16)
Then, by (14), we obtain (8). On the other hand, assuming that (8) is valid, we set

\[ b_n := n^{p \left( \frac{1}{p} - \frac{\lambda_1}{\varphi}, \frac{1}{p} + \frac{\lambda_2}{\varphi} \right) - 1} \left[ \sum_{m=1}^{\infty} \frac{1}{(m+n)^{q}} a_m \right]^{p-1}, \quad n \in \mathbb{N}. \]

If \( J = \infty \), then (14) is naturally valid; if \( J = 0 \), then it is impossible to make (14) valid, namely, \( J > 0 \). Suppose that \( 0 < J < \infty \). By (8), we have

\[ \sum_{n=1}^{\infty} n^{q(1-(\frac{1}{p}, \frac{1}{p} - \frac{\lambda_1}{\varphi})) - 1} b_n \]

\[ = \frac{I}{p} = I \]

\[ > B^\frac{1}{p}(\lambda_2, \lambda - \lambda_2)B^\frac{1}{q}(\lambda_1, \lambda - \lambda_1) \times \left\{ \sum_{m=1}^{\infty} \left[ 1 - O\left( \frac{1}{m^{\lambda_2}} \right) \right] m^{p\left[ 1-(\frac{1}{p}, \frac{1}{p} + \frac{\lambda_2}{\varphi}) \right] - 1} a_m \right\} \]

\[ J = \left[ \sum_{n=1}^{\infty} n^{q(1-(\frac{1}{p}, \frac{1}{p} - \frac{\lambda_1}{\varphi})) - 1} b_n \right]^{\frac{1}{p}} \]

\[ > B^\frac{1}{p}(\lambda_2, \lambda - \lambda_2)B^\frac{1}{q}(\lambda_1, \lambda - \lambda_1) \left\{ \sum_{m=1}^{\infty} \left[ 1 - O\left( \frac{1}{m^{\lambda_2}} \right) \right] m^{p\left[ 1-(\frac{1}{p}, \frac{1}{p} + \frac{\lambda_2}{\varphi}) \right] - 1} a_m \right\} \]

namely, (14) follows. Hence, inequality (8) is equivalent to (14).

Suppose that (15) is valid. By the Hölder inequality, we have

\[ I = \sum_{m=1}^{\infty} \left[ \left( 1 - O\left( \frac{1}{m^{\lambda_2}} \right) \right) \right] \frac{1}{m^{\frac{1}{p} + \frac{\lambda_2}{\varphi}}} a_m \left[ \frac{m^{\frac{\lambda_1}{\varphi}} + \frac{\lambda_2}{\varphi}}{(1 - O(\frac{1}{m^{\lambda_2}}))^{1/p}} \sum_{n=1}^{\infty} \frac{1}{(m+n)^{q}} b_n \right] \]

\[ \geq \left[ \sum_{m=1}^{\infty} \left[ 1 - O\left( \frac{1}{m^{\lambda_2}} \right) \right] m^{p\left[ 1-(\frac{1}{p}, \frac{1}{p} + \frac{\lambda_2}{\varphi}) \right] - 1} a_m \right]^{\frac{1}{p}} \]

(17)

Then, by (15), we obtain (8). On the other hand, assuming that (8) is valid, we set

\[ a_m := m^{p\left( \frac{1}{p} - \frac{\lambda_2}{\varphi}, \frac{1}{p} + \frac{\lambda_1}{\varphi} \right) - 1} \left[ \sum_{n=1}^{\infty} \frac{1}{(m+n)^{q}} b_n \right]^{q-1}, \quad m \in \mathbb{N}. \]

If \( J_1 = \infty \), then (15) is naturally valid; if \( J_1 = 0 \), then it is impossible to make (15) valid, namely, \( J_1 > 0 \). Suppose that \( 0 < J_1 < \infty \). By (8), we have

\[ \sum_{m=1}^{\infty} \left( 1 - O\left( \frac{1}{m^{\lambda_2}} \right) \right) m^{p\left[ 1-(\frac{1}{p}, \frac{1}{p} + \frac{\lambda_2}{\varphi}) \right] - 1} a_m \]

\[ = J_1^p = I \]

\[ > B^\frac{1}{p}(\lambda_2, \lambda - \lambda_2)B^\frac{1}{q}(\lambda_1, \lambda - \lambda_1) \left\{ \sum_{m=1}^{\infty} \left( 1 - O\left( \frac{1}{m^{\lambda_2}} \right) \right) m^{p\left[ 1-(\frac{1}{p}, \frac{1}{p} + \frac{\lambda_2}{\varphi}) \right] - 1} a_m \right\} \]
in (8) is not the best possible. Namely, (15) follows. Hence, inequality (8) is equivalent to (15) and then inequalities (8), (14) and (15) are equivalent.

If the constant factor in (8) is the best possible, then so is the constant factor in (14) and (15). Otherwise, by (16) or (17), we would reach a contradiction that the constant factor in (8) is not the best possible.

\[ \Box \]

\textbf{Theorem 2} The following statements (i), (ii), (iii) and (iv) are equivalent:

(i) \( B^\frac{1}{p} (\lambda_2, \lambda_1, \lambda - \lambda_1) \) is independent of \( p, q \);

(ii) \( B^\frac{1}{p} (\lambda_2, \lambda_1, \lambda - \lambda_1) \) is expressible as a single integral;

(iii) \( B^\frac{1}{p} (\lambda_2, \lambda_1, \lambda - \lambda_1) \) in (8) is the best possible constant factor;

(iv) if \( \lambda - \lambda_1 - \lambda_2 \in (-p \lambda_1, p(\lambda - \lambda_1)) \), then \( \lambda = \lambda_1 + \lambda_2 \).

If the statement (iv) follows, namely, \( \lambda = \lambda_1 + \lambda_2 \), then we have (10) and the following equivalent inequalities with the best possible constant factor \( B(\lambda_1, \lambda_2) \):

\[ \sum_{n=1}^{\infty} n^{p q_2 - 1} \left[ \sum_{m=1}^{\infty} \frac{1}{(m + n)^q} a_m \right]^\frac{q}{p} > B(\lambda_1, \lambda_2) \left[ \sum_{m=1}^{\infty} \frac{1}{(m + n)^q} b_m \right]^\frac{q}{p}, \tag{18} \]

\[ \left\{ \sum_{n=1}^{\infty} \frac{m^{p q_1 - 1}}{[1 - O (\frac{1}{m^q})]} \left[ \sum_{n=1}^{\infty} \frac{1}{(m + n)^q} b_m \right]^q \right\}^\frac{1}{q} > B(\lambda_1, \lambda_2) \left[ \sum_{n=1}^{\infty} n^{p q - 1} b_n \right]^\frac{1}{q}. \tag{19} \]

\textbf{Proof} (i) \( \Rightarrow \) (ii). By (i), we have

\[ B^\frac{1}{p} (\lambda_2, \lambda - \lambda_2) B^\frac{1}{q} (\lambda_1, \lambda - \lambda_1) = \lim_{p \to 1^-, q \to -\infty} B^\frac{1}{p} (\lambda_2, \lambda - \lambda_2) B^\frac{1}{q} (\lambda_1, \lambda - \lambda_1) = B(\lambda_2, \lambda - \lambda_2). \]

Namely, \( B^\frac{1}{p} (\lambda_2, \lambda - \lambda_2) B^\frac{1}{q} (\lambda_1, \lambda - \lambda_1) \) is expressible as a single integral

\[ B(\lambda_2, \lambda - \lambda_2) = \int_0^\infty \frac{1}{(1 + u)^p} u^{\lambda_2 - 1} \, du. \]

(ii) \( \Rightarrow \) (iv). If \( B^\frac{1}{p} (\lambda_2, \lambda - \lambda_2) B^\frac{1}{q} (\lambda_1, \lambda - \lambda_1) \) is expressible as a convergent single integral

\[ B\left( \frac{\lambda - \lambda_2}{p}, \frac{\lambda_1}{q}, \frac{\lambda - \lambda_1}{q}, \frac{\lambda_2}{p} \right), \]
then (13) keeps the form of equality. In view of the proof of Lemma 5, it follows that \( \lambda = \lambda_1 + \lambda_2 \).

(iv) \( \Rightarrow \) (i). If \( \lambda = \lambda_1 + \lambda_2 \), then

\[
B^\frac{1}{\lambda} (\lambda_2, \lambda - \lambda_2) B^\frac{1}{\lambda_1} (\lambda_1, \lambda - \lambda_1) = B(\lambda_1, \lambda_2),
\]

which is independent of \( p, q \). Hence, it follows that (i) \( \Leftrightarrow \) (ii) \( \Leftrightarrow \) (iv).

(iii) \( \Rightarrow \) (iv). By Lemma 5, we have \( \lambda = \lambda_1 + \lambda_2 \).

(iv) \( \Rightarrow \) (iii). By Lemma 4, for \( \lambda = \lambda_1 + \lambda_2 \),

\[
B^\frac{1}{\lambda} (\lambda_2, \lambda - \lambda_2) B^\frac{1}{\lambda_1} (\lambda_1, \lambda - \lambda_1) \quad (= B(\lambda_1, \lambda_2))
\]

is the best possible constant factor of (8). Therefore, we have (iii) \( \Leftrightarrow \) (iv).

Hence, the statements (i), (ii), (iii) and (iv) are equivalent.

Remark 2 For \( \lambda_1 = \lambda_2 = \frac{\lambda}{2} \in (0, 2] \) \((0 < \lambda \leq 4)\) in (10), (18) and (19), we have the following equivalent inequalities with the best possible constant factor \( B(\frac{\lambda}{2}, \frac{\lambda}{2}) \):

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m + n)^\lambda} > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left[ \frac{1}{1 - O\left(\frac{1}{m}\right)} \right] \left[ \sum_{n=1}^{\infty} a_n^p \right]^{\frac{1}{\lambda}} \left[ \sum_{n=1}^{\infty} b_n^q \right]^{\frac{1}{\lambda}},
\]

\[
\left\{ \sum_{n=1}^{\infty} \left[ \frac{1}{1 - O\left(\frac{1}{m}\right)} \right] a_n^p \right\}^{\frac{1}{\lambda}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{\lambda}} > B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left[ \sum_{n=1}^{\infty} b_n^q \right]^{\frac{1}{\lambda}},
\]

In particular, (i) for \( \lambda = 2 \), we have the following equivalent inequalities:

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m + n)^2} > \left\{ \sum_{n=1}^{\infty} \left[ 1 - O\left(\frac{1}{m}\right) \right] a_n^p \right\}^{\frac{1}{\lambda}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{\lambda}},
\]

\[
\left\{ \sum_{n=1}^{\infty} \left[ 1 - O\left(\frac{1}{m}\right) \right] a_n^p \right\}^{\frac{1}{\lambda}} > \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{\lambda}},
\]

(ii) for \( \lambda = 4 \), we have the following equivalent inequalities:

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m + n)^4} > \frac{1}{6} \left\{ \sum_{m=1}^{\infty} \left[ 1 - O\left(\frac{1}{m^2}\right) \right] a_m^p \right\}^{\frac{1}{\lambda}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{\lambda}},
\]
\[
\left\{ \sum_{n=1}^{\infty} n^{2p-1} \left[ \sum_{m=1}^{\infty} \frac{a_m}{(m+n)^{3}} \right]^{\frac{1}{p}} \right\} > \frac{1}{6} \left\{ \sum_{m=1}^{\infty} \left[ 1 - O \left( \frac{1}{m^2} \right) \right] \right\}^{\frac{1}{4}} \left( \sum_{n=1}^{\infty} \frac{b_n^{\frac{1}{2}}}{(m+n)^{4}} \right)^{\frac{1}{4}},
\]

(27)

\[
\left\{ \sum_{m=1}^{\infty} m^{2q-1} \left[ \sum_{n=1}^{\infty} \frac{1}{(m+n)^{4}} \right]^{\frac{1}{q}} \right\} > \frac{1}{6} \left( \sum_{n=1}^{\infty} \frac{b_n^{\frac{1}{2}}}{n^{4}} \right)^{\frac{1}{4}}.
\]

(28)

4 Conclusions
In this paper, by the use of the weight coefficients, the idea of introducing parameters and the Euler–Maclaurin summation formula, a reverse extended Hardy–Hilbert inequality as well as the equivalent forms are given in Lemma 2 and Theorem 1. The equivalent statements of the best possible constant factor related to a few parameters and some particular cases are considered in Theorem 2 and Remark 1, 2. The lemmas and theorems provide an extensive account of this type of inequalities.

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Authors’ contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. ZH and YS participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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