Controllability to the origin implies state-feedback stabilizability for discrete-time nonlinear systems *

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Abstract

The problem of state-feedback stabilizability of discrete-time nonlinear systems has been considered in this note. Two assertions have been proved. First, if the system is N-step controllable to the origin, then there is a state feedback control law for which the trajectory of the closed-loop system converges to the origin in N steps. Second, if the system is asymptotically controllable to the origin and satisfies the controllability rank condition at the origin, then there is a state feedback control law for which the trajectory of the closed-loop system converges to the origin in finite steps.

keywords Nonlinear systems, discrete-time systems, controllability, stabilizability, state feedback, discontinuous

1 Introduction

From control theoretic point of view, one of the most important properties of a system is stability. Controllability assures the existence of an open-loop control law, but in many cases, a state-feedback control law is preferable. For continuous-time systems, the relation between asymptotic controllability and state-feedback stabilizability has been established in [1]. In [1], it has been shown that there is a discontinuous state feedback stabilizing control law for a system which is asymptotically controllable. Because the discontinuity of the

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control law arises naturally in stabilization and optimization problems, discontinuous control laws have been studied in many papers, e.g. [2, 3, 4, 5]. For discrete-time systems, problems related to controllability have been extensively studied in [6, 7, 8]. However, these works do not deal with state-feedback stabilization problem. State feedback stabilization problem of discrete-time nonlinear systems has been studied for past decades (e.g. [9, 10, 11, 12, 13]), but researches dealing with nonsmooth or discontinuous control laws are relatively rare [14, 15]. Especially, to the best of the author’s knowledge, discrete-time analogue of the results obtained in [1] have not been established yet. The objective of this note is to fill the gap.

In the following, we prove two facts. First, if a discrete-time nonlinear system is $N$-step controllable to the origin (precise definition of this notion is given below), then there is a (possibly discontinuous) state feedback control law for which the trajectory of the closed-loop system converges to the origin in $N$ steps. Second, if the system is asymptotically controllable to the origin and satisfies the controllability rank condition at the origin (again, precise definitions of these notions are given below), then there is a (possibly discontinuous) state feedback control law for which the trajectory of the closed-loop system converges to the origin in finite steps (the required steps may differ for different initial conditions.) Our construction explicitly uses the axiom of choice.

2 Main Results

Consider a discrete-time nonlinear system of the form

$$x(t + 1) = f(x(t), u(t)),$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $t \in \mathbb{N}$ is the time. It is assumed that $(x, u) = (0, 0)$ is an equilibrium of (1), that is, $f(0, 0) = 0$. Henceforth, we use the following notations: $u[t_0, t_1]$ denotes the finite sequence of inputs $(u(t_0), \ldots, u(t_1))$, and $u[t_0, \infty)$ denotes the infinite sequence of inputs $(u(t_0), u(t_0 + 1), \ldots)$; $\phi(t, t_0, x_0, u)$ denotes the trajectory of (1) initialized at $t = t_0$ by $x_0$ and driven by the input. The notation $\phi(t, t_0, x_0, u)$ implicitly assumes that the input is at least defined over the time interval $[t_0, t - 1]$. Because we are dealing with a time-invariant system, the value of the initial time $t_0$ is immaterial. Hence, without loss of generality, we assume that $t_0 = 0$.

As for controllability, we first assume the following.

Assumption 1 (N-step controllability to the origin) $\exists N > 0, \forall x_0 \in \mathbb{R}^n, \exists u[0, N - 1], \phi(N, 0, x_0; u) = 0$.

Our first objective is to show the following proposition.

Proposition 1 If (1) is $N$-step controllable to the origin, then there is a (possibly discontinuous) state feedback control law $v(x)$ for which the solution of

$$x(t + 1) = f(x(t), v(x(t)))$$

converges to the origin in $N$-steps for any initial condition.
The sequence \( \{ x \in \mathbb{R}^n : \exists u, f(x, u) \in A_{k-1} \} \) and let \( N \) be nonempty. Pick an element \( x_0 \in U_x \) (here, we use the axiom of choice), and let \( v(x) = u_x \). Because \( i_x \) is uniquely determined for each \( x \), \( v(x) \) is well defined. Let \( x(t) \in \mathbb{R}^n \) be the trajectory of \( x(t) \) for which the solution of (2) converges to the origin in finite steps for any initial condition. Next, instead of Assumption 1, we assume that Assumption 2 (Asymptotic controllability to the origin) and Assumption 3 (Rank controllability at the origin) are satisfied. Then, there is a (possibly discontinuous) state feedback control law \( u \) of the form (2) initialized with \( x_0 \) and driven by the control law \( v(x) \). We prove that \( i_{x_0} = 0 \) for some \( t \leq N \) by contradiction. Suppose that \( \forall t, i_{x_0} > 0 \). Then, \( x_0 \in A_{i_{x_0} - 1} \), \( i_{x_0} \leq N \) and hence \( x_1 = f(x_0, v(x_0)) \in A_{i_{x_0} - 1} \), and by (3), \( i_{x_1} \leq i_{x_0} - 1 \). Inductively, assume that \( x_j \in A_{i_{x_j}} \) with \( i_{x_j} \leq i_{x_0} - j \). Then, \( x_{j+1} = f(x_j, v(x_j)) \in A_{i_{x_j} - 1} \), hence \( i_{x_{j+1}} \leq i_{x_0} - (j + 1) \). Therefore, for any \( j \), \( i_{x_j} \leq i_{x_0} - j \). But this is impossible, because \( 0 \leq i_{x_0} \leq N \) and \( 0 \leq i_{x_j} \leq N \). □

Thus far, we have not assumed differentiability of \( f(x, u) \) and constructed a discontinuous state feedback control law merely under the assumption of \( N \)-step controllability to the origin. Next, instead of Assumption [1], we assume that \( f(x, u) \) is at least \( C^1 \), and additionally, the following two assumptions are fulfilled.

**Assumption 2 (Asymptotic controllability to the origin)** \( \forall x_0 \in \mathbb{R}^n, \exists u(0, x_0; t) \) converges to the origin as \( t \to \infty \).

**Assumption 3 (Rank controllability at the origin)** \( \exists N > 0, \) \( \operatorname{rank} \left( \frac{\partial \phi(N, 0, x_0, u)}{\partial u}[0, N-1] \right) = n \) on an open neighborhood of \( (x, u(0), \ldots, u(N-1)) = (0, 0, \ldots, 0) \), where \( \partial \phi(N, 0, x_0, u)/\partial u[0, N-1] \) denotes the partial derivative of \( \phi \) with respect to the variable \( u[0, \ldots, N-1] \) with the re-interpretation that \( u[0, \ldots, N-1] \) is the vector \( (u^T(0), u^T(1), \ldots, u^T(N-1))^T \), where \( .^T \) denotes the transpose of a vector.

Our next objective is to show that under the assumptions that \( f(x, u) \) is \( C^1 \), Assumption 2 and Assumption 3 a similar result to Proposition [1] holds.

**Proposition 2** Assume that \( f(x, u) \) is \( C^1 \) and Assumption 2 and Assumption 3 are satisfied. Then, there is a (possibly discontinuous) state feedback control law \( u = v(x) \) for which the solution of (2) converges to the origin in finite steps for any initial condition.

**Proof.** As in the proof of Proposition [1] let \( A_0 = \{0\} \), and inductively define \( A_k = \{x \in \mathbb{R}^n : \exists u, f(x, u) \in A_{k-1} \} \). We first show that \( A_N \) contains an open neighborhood of \( x = 0 \). This is a direct consequence of the implicit function theorem. For, let \( p \) be the collection of components of \( u[0, N-1] \) for which \( \operatorname{rank} \partial \phi(N, 0, x, u[0, N]/\partial u = n \). Let us rewrite \( u[0, N-1] = (p^T, q^T)^T \) by re-arranging the variables, and rewrite \( \phi(N, 0, x, u[0, N-1]) \) as \( \psi(x, p, q) \). Because
ψ(0, 0, 0) = 0 and rank ∂ψ/∂p = n, by applying the implicit function theorem, 0 = ψ(x, p, q) may be solved for p in the form p = h(x, q) for some smooth function h(x, q) on a neighborhood of (x, q) = (0, 0), and hence the trajectory initialized at this neighborhood of x may be driven to the origin by applying the corresponding input sequence. Hence AN contains an open neighborhood of x = 0 (similar idea to above analysis has been used in Lemma 2 of [16].)

Next, we show that \( \bigcup_{k=0}^{\infty} A_k = \mathbb{R}^n \). For, suppose that \( \mathbb{R}^n \setminus \bigcup_{k=0}^{\infty} A_k \neq \emptyset \). Let \( x_0 \in \mathbb{R}^n \setminus \bigcup_{k=0}^{\infty} A_k \). Then, \( \phi(t, 0, x_0, u) \notin A_k \) for any \( t, k \) and \( u \) because otherwise \( x_0 \) is in some \( A_k \). However, \( A_N \) contains an open neighborhood of the origin. Let the neighborhood be \( G \). Assumption 2 implies that, by choosing an adequate \( u \in [0, \infty) \), \( \phi(t, 0, x_0, u) \in G \) residually, whereas we have seen that \( \phi(t, 0, x_0, u) \notin G \). This is a contradiction. Hence, \( \mathbb{R}^n \setminus \bigcup_{k=0}^{\infty} A_k \) must be empty, whence \( \bigcup_{k=0}^{\infty} A_k = \mathbb{R}^n \).

Because \( \bigcup_{k=0}^{\infty} A_k = \mathbb{R}^n \), the control law may be constructed in the same fashion as that of Proposition 1. Again, for each \( x \in \mathbb{R}^n \), let \( i_x = \min \{ i : x \in A_i \} \). Because \( \bigcup_{k=0}^{\infty} A_k = \mathbb{R}^n \), \( i_x \) is well defined and finite. Then, the set \( U_x = \{ u : f(x, u) \in A_{i_x-1} \} \) is nonempty. Pick an element \( u_x \in U_x \) (here, we use the axiom of choice), and let \( \nu(x) = u_x \). The proof that this state feedback control law makes the trajectory converge to the origin in finite steps is identical to that of Proposition 1. Note that, for each \( x \in \mathbb{R}^n \), the trajectory initialized at \( x \) converges to the origin in at most \( i_x \) steps. The required steps may differ for different initial conditions.

\[ \square \]

3 Conclusion

In this note, we have proved two facts. First, if a discrete-time nonlinear system is N-step controllable to origin then there is a state feedback control law for which the trajectory of the closed-loop system converges to the origin in \( N \) steps. Second, if the system is asymptotically controllable to the origin and satisfies the controllability rank condition at the origin, then there is a state feedback control law for which the trajectory of the closed-loop system converges to the origin in finite steps (but the required steps may differ for different initial conditions.)

The obtained control law is not always continuous. The pursuit of the possibility of constructing a continuous control law is left to further research.

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