Naturally Self-Tuned Low Mass Composite Scalars

Christopher T. Hill

1Theoretical Physics Department, Fermi National Accelerator Laboratory, P. O. Box 500, Batavia, IL 60510, USA

Scalar bosons composed of a pair of chiral fermions in a non-confining potential have an effective Yukawa coupling, $g$, to free external chiral fermions. At large distance a Feynman loop of external fermions generates a scale invariant potential, $V_{\text{loop}} \propto -g^2/r^2$, which acts on valence fermions for separation $\rho \approx 2r$. This generally forces the $s$-wave ground state to deform to a static, zero mass, configuration, and for slowly running, perturbative $g$, a large external “shroud” wave-function forms. This is related to old results of Landau and Lifshitz in quantum mechanics. The massless composite scalar boson ground state is then an extended object. Infra-red effects can generate a small mass for the system. This points to a perturbative BEH-boson composed of top and anti-top quarks and a novel dynamical mechanism for spontaneous electroweak symmetry breaking.

I. INTRODUCTION

For approximately fifty years particle physics has dealt with a conundrum: The electroweak hierarchy problem, the apparent unnaturaness of low mass scalar particles, or, why is the Brout-Englert-Higgs (BEH-boson) mass, or weak scale, small compared to e.g., the Planck scale? This has driven much of the thematic research for half a century, from supersymmetry [1], technicolor [2] and extended technicolor [3], top condensation [4-7], “composite models” (where the BEH-boson is a pseudo-Nambu-Goldstone mode [8]), etc. The discovery of the BEH-boson in 2012 at the LHC, and the apparent lack of any nearby new physics to act as a custodian, has exacerbated the conundrum. The BEH-boson appears to be, for all practical purposes, an approximately massless (e.g., on the Planck scale) scalar field. This is seemingly anathema to fifty years of post-modern theoretical physics.

In the present paper we look more closely at the internal dynamics of bound states consisting of chiral fermions in non-confining potentials. We will show that approximate scale invariance, in conjunction with chiral symmetry, manifests itself in an unusual way in bound states and leads to unexpected consequences for composite solutions.

Here we will see that a scalar boson can form as a compact massive object, a “core” wave-function, consisting of a pair of chiral fermions, bound by some short-distance interaction potential. This can happen at an arbitrarily high mass scale, $M$, potentially as high as $M \sim M_{\text{Planck}}$, and one usually assumes this state cannot then have a naturally small mass, $m << M$. However, if the potential is not confining, then chiral and scale symmetries conspire through a Feynman loop, external to the core, to create a large-distance, attractive, scale invariant, $-cg^2/r^2$ potential between the constituents, where $r$ is the radius of the two-body system and $c$ is a loop factor. This is the $O(h)$ vacuum reaction to the presence of the core, and it is an effect usually phrased in momentum space that is central to the Nambu–Jona-Lasinio model [9].

The constituent fermions are virtually emitted and re-absorbed into the core, experiencing the vacuum effects, as in Fig.(1). The induced vacuum potential leads to an enveloping “shroud” wave-function around the core. The shroud is necessarily a massless solution owing to the scale invariance of the vacuum potential. However, for a consistent solution its null time dependence must match, via boundary conditions, onto the core wave-function.

This happens by a deformation of the short-distance core, locking it to a static, zero mass configuration. Indeed, if one allows arbitrary boundary conditions there are generally massless solutions for any core potential, but these don’t become eigenfunctions because they are matched to exterior solutions in a normal vacuum, typically radiation. This then yields the large mass, approximate eigenvalue. With the vacuum loop potential the core wave-function can deform and match onto the exterior massless shroud solution. The full solution becomes an eigenfunction with a zero mass eigenvalue. We exhibit this explicitly in a simple model, but it is a general phenomenon. Due to scale symmetry, the shroud wave-function is an extended object, and the low energy physics becomes insensitive to the core.

At first this seemed surprising to us, but after arriving at this conclusion and the relevant wave-function of the shroud, we found there is a prior (somewhat obscure) discussion of related effects in the immortal “Nonrelativistic Quantum Mechanics” textbook of Landau and Lifshitz [10] (LL). They explored the Schroedinger equation in $-\beta/r^2$ potentials and found the extended wave-functions that apply to our present situation, though the context and some details differ. Moreover, they argued that the existence of a zero energy ground state is guaranteed in any core potential (modulo any negative energy modes) if the $-\beta/r^2$ is present at large distances.

Quoting from Landau and Lifshitz, page 116, of the edition, [10] (we insert our comments in italics and for us “energy” becomes $M^2$):

*Electronic address: hill@fnal.gov
“Next, let us investigate the properties of the solutions of Schrödinger’s equation in a field which diminishes as \( U = -\beta / r^2 \), and has any form at short distances. (For \textit{weak coupling}) it is easy to see that in this case only a finite number of negative energy levels can exist. For with energy \( E = 0 \) Schrödinger’s equation at large distances has the form (35.1) with the general solution (35.4) \( \text{(our eqns. (37, 38). ...} \)

Finally, let the field be \( U = -\beta / r^2 \) in all space. Then for \( \text{(weak coupling)} \) there are no negative energy levels. For the wave-function of the state \( E = 0 \) is of the form (35.7) in all space; \... i.e., it corresponds to the lowest energy level.”

This is essentially the statement that the shroud solution controls the entire solution, i.e. “the tail wags the dog,” and in a not-so “weak coupling” limit, \( g^2 < 8\pi^2 / N_c \), (see eq. (49)) this can generally be the ground state of the system with \( M^2 = 0 \).

The LL solutions provide a potential new mechanism for achieving light composite scalar bosons, based upon internal dynamics and symmetries. We believe this may provide a candidate solution to the electroweak hierarchy problem and the structure of the BEH-boson, though our present discussion is confined to a single complex scalar field with global chiral covariance. The LHC may be seeing the “shroud” of the ground state solution, the extended structure of the BEH-boson.

The spatial extent of the shroud is cut-off when the chiral symmetry of the constituents is broken, which may be triggered by other forces. In this picture, if the BEH-boson is composed of top and anti-top quarks, it would have an extent of order \( r \sim 1 / m_{\text{top}} \). The renormalization group (RG) running of the top Yukawa BEH-coupling may act perturbatively as the trigger for electroweak symmetry breaking. Essentially, we view this in reverse: the top quark gets a mass, which cuts off the shroud solution. Owing to the minus sign of the vacuum loop potential, this leaves a tachyonic mass term (“Mexican hat” potential) for the composite BEH-boson. This in turn causes its vacuum expectation value (VEV) to form, which in turn generates the top quark mass. The self consistency determines the critical value of \( g = g_{\text{top}} \) at which this occurs, and we indeed find \( g \approx 1 \).

There are requisite stability constraints, e.g., no negative \( M^2 \) solution at the short distance core scale is allowed, and the BEH-Yukawa coupling must not run too quickly, e.g., near a quasi-fixed point of the RG to obtain the shroud solution over a large range of scales. The top quark BEH-Yukawa coupling obliges the latter and the exclusion of negative \( M^2 \) follows from weak dynamics, such as barrier potentials, new non-confining gauge interactions, and possibly gravitation. We emphasize that \textit{this is not a strong dynamical theory, and works perturbatively with} \( g \sim O(1) \).

If the BEH-boson is an extended object it would behave coherently as a pointlike particle at LHC processes probed thus far, but perhaps its compositeness can be seen in higher energy or sensitive flavor processes, or perhaps in deep s-channel production in a muon collider [11]. These issues will not be discussed in the present paper.

We believe, however, that this may be pointing to an intimate relationship between three quantities: the BEH-boson mass of 125 GeV, the top quark mass of 175 GeV, and the VEV of the BEH-boson 246 GeV (or 175 GeV when divided by \( \sqrt{2} \)). We sketch a trigger mechanism for the spontaneous breaking of the \( SU(2) \times U(1) \) symmetry, coming from the QCD contribution to the RG running of the top-quark BEH-Yukawa coupling.

After a discussion of formalism and a “warm-up” example in Section II, we derive the relevant Landau-Lifshitz solutions and construct low mass scalar bound states in Section III. In Section IV we discuss infrared mass and normalization, and sketch a theory of the origin of the electroweak scale, the top quark mass and BEH-boson mass. We conclude in Section V, and present detailed loop calculations, particularly of the vacuum loop potential, in Appendix I.

II. COMPOSITE SYSTEMS

A. Hints from the NJL Model

Many years ago Ken Wilson demonstrated how to solve the Nambu-Jona–Lasinio model, (NJL) [9], in a conceptually powerful way by the renormalization group [12]. The NJL model is the simplest field theory of a composite scalar boson, consisting of a pair of chiral fermions. The chiral fermions induce loop effects that lead to the interesting dynamical phenomena at low energies [5, 12].

Consider a pair of chiral fermions,

\[
\begin{pmatrix}
\psi^a_R \psi^b_L \\
\end{pmatrix}
\]  

with \( N_c \) color indices \((a, b)\), and a global chiral symmetry \( U(1)_L \times U(1)_R \). The NJL model with its non-confining, local, chirally invariant interaction takes the form:

\[
L = \frac{g^2}{M^2} \bar{\psi}^a_R \psi^b_R \bar{\psi}^b_L + L_{\text{kinetic}}
\]  

(we’ll henceforth suppress summed color indices).

We factorize this by introducing an auxiliary field \( \Phi \):

\[
L_M = g \bar{\psi}_R \psi_L \phi + h.c - M^2 \phi^4 
\]

Integrating out \( \phi \) in eq. (3) we recover eq. (2).

Wilson viewed this as the effective action at a scale \( M \). He then computed fermion loop corrections that arise because the chiral fermions are unconfined and wander into the vacuum. This yields the theory at a lower mass...
scale $\mu$.

$$L_M \rightarrow L_\mu = g[V_R \psi_L] \Phi + h.c - V_M \Phi^\dagger \Phi + \ldots$$

where, $V_M = \left( M^2 - \frac{N_c g^2}{8\pi^2} (M^2 - \mu^2) \right)$ \hspace{1cm} (4)

Here $\ldots$ includes an induced kinetic term and quartic interaction which we computed in a large $N_c$ fermion loop approximation [5, 13]:

$$Z_H D H^\dagger D - \frac{\lambda}{2} (H^\dagger H)^2; \hspace{0.5cm} Z_H = c_1 + \frac{N_c g^2}{16\pi^2} \ln \left( \frac{M^2}{\mu^2} \right)$$

$$\lambda = c_2 + \frac{2N_c g^4}{16\pi^2} \ln \left( \frac{M^2}{\mu^2} \right). \hspace{1cm} (5)$$

The log terms give the leading large $N_c$ fermion loop corrections to the kinetic and quartic terms, and yield a running of the couplings, e.g., $g \sim 1/\sqrt{Z_H}$, which can be matched onto the full RG equations in the IR [4, 5, 13]. Indeed, the arguments of the logs inform us that the RG is operant on all scales, $\mu$ to $M$. We recover these results in the pointlike limit of our composite field discussion in Appendix I, and they are largely retained when one looks at RG running in $r$. For further pedagogical discussions see [13].

Note, in particular, the behavior of the composite scalar boson mass in $V_M$ of eq.(4). The $-N_c g^2 M^2/8\pi^2$ term arises from the negative quadratic divergence in the loop involving the pair $(\psi_R, \psi_L)$ of Fig.(1), with pointlike vertices and a loop cut-off scale at $M^2$. This is the physical response of the vacuum to the classical interaction $g[V_R \psi_L] \Phi$ in the presence of the bound state $\Phi$. The Dirac sea generates a feedback to reduce $M^2$, and the loop integral is then capturing this physical effect, much like a Casimir effect.

The NJL model allows us in principle to fine-tune the coupling $g^2$ to a critical value, $g_c^2 = 8\pi^2/N_c$, at which point the mass of the bound state becomes zero. In the earliest models of a composite BEH-boson, known as “top condensation,” [4, 5], we tuned the theory to have a massless, or slightly supercritical, bound state, by “human intervention.” Note there is a hint of something special about the critical value, since this corresponds to a cancellation of the large $M^4$ terms in the theory, and an approximate scale symmetry emerges, broken only by log terms and the infrared cut-off, $\mu^2$.

Fine-tuning done by human intervention cannot be viewed as a complete or satisfactory theory. To generate a hierarchy where $M/\mu \sim M_{\text{Planck}}/v_{\text{weak}} \sim 10^{17}$ requires tuning $g^2$ to $g_c^2$ with a precision of $1 : 10^{-34}$. This graphically illustrates the electroweak hierarchy problem. Nevertheless, the NJL model informs us that composite scalar bosons, consisting of a pair of chiral fermions with a non-confining potential, can indeed exist and will have an induced or fundamental Yukawa coupling $g$.

### B. Self-Tuning

In a realistic model with more detailed binding dynamics, however, the possibility of an emergent scale symmetry suggests that the NJL fine-tuning cancellation may actually be a “self-tuning” effect. The internal wave-function of the bound state might adjust itself to find a new ground state which possesses the maximal scale invariance. After all, the NJL model is an effective field theory and only captures physics on IR scales $\mu << M$, but is blind to the detailed internal dynamics, requiring we probe deeper.

The main observation in the present paper is that, viewed in configuration space, external chiral fermions induce an extended, scale invariant, attractive loop potential for the bound state wave-function of the form $-cg^2/r^2$. This particular potential has the nontrivial zero mass solutions of Landau and Lifshitz (LL) [10]. This then leads to the “self-tuning” where the short distance part of the solution becomes locked to the LL exterior solution.

We will also see below that there is an intimate connection between the NJL model and the LL solutions as they share an identical “critical coupling,” even though the former case is controlled by a quantum loop while the latter is a classical result. To us this bolsters the relevance of the LL solution for non-confined bound states of chiral fermions.

We interpret the “custodial symmetry” of the massless system to be the approximate scale invariance of the $-1/r^2$ potential modulo soft RG running of couplings. Various IR effects can subsequently generate a natural small mass for the composite system. This is a self-consistent phenomenon since the valence constituents of the bound state experience a potential due to virtual effects of the same particles in the vacuum via a Feynman loop. It is similar in this sense to a Coleman-Weinberg potential [14] in which quantum fluctuations of a field $\phi$ induce a potential for the VEV of $\phi$.

In this picture, the composite scalar boson becomes an extended object. This is evidently the price one pays for naturalness. In top condensation models we had assumed a pointlike BEH-boson bound state [5], but we were forced to fine-tuning. Presently, we allow the theory, via the vacuum loop potential, to relax the bound state and we obtain the shroud as an extended object. Now we do not require fine tuning, and we can have perturbative coupling.

We can examine the log-terms of eq.(5) in configuration space and see that the usual renormalization group behavior of the BEH-Yukawa coupling is evidently retained as logarithmic functions of the scale $r$, and the LL solution will be maintained if $g$ is approximately constant, i.e. an approximate RG fixed point. The induced potential creates a quasi-conformal window with the wave-function extending from the UV scale of the short-distance binding, to the large IR scale of the mass generation. Hence a hierarchy is dynamically generated.
Any mass for the shroud requires explicit IR modification of the $-1/r^2$ potential. The infrared scalar boson mass can be treated explicitly and it is technically natural when inserted by hand where the potential becomes $→ m^2 - 1/r^2$. However, we expect this will be generated dynamically, e.g., through the IR behavior of the Yukawa coupling, or a Coleman-Weinberg mechanism [14] generates mass through the running of the quartic coupling [15]. This mechanism is general and offers various model building possibilities.

As one possibility, we outline a simple, self-consistent origin of the top quark and BEH-boson masses. The main result here is that the LL solution provides a natural, massless scalar field, due to the inner conformal window.

C. Formalism for Composite Fields

Consider a hypothetical new fundamental interaction associated with a high energy scale, $M$:

$$L' = g_0^2 [\bar{\psi}_L(x)\gamma_\mu T^A \psi_L(x)] D(x-y) [\bar{\psi}_R(y)\gamma_\mu T^A \psi_R(y)]$$

(6)

where $T^A$ are generators of an SU($N_c$) interaction and the $\psi$ fields are in the fundamental representation, e.g. color triplets for SU($N_c=3$) of color. This is a broken gauge theory with massive gluons, analogous to “topcolor” [6], however we will not require that this be a strongly interacting theory, i.e., $g_0$ need not be large.

A Fierz rearrangement of the interaction leads to:

$$L' → -g_0^2 [\bar{\psi}_L(x)\psi_R(y)] D(x-y) [\bar{\psi}_R(y)\psi_L(x)] + O(1/N_c)$$

(7)

where combinations of fields in the [...] are color contracted. We can now factorize this into an effective interaction with a bilocal auxiliary field:

$$L' → g_0^2 [\bar{\psi}_L(x)\psi_R(y)] [\Phi(x,y) + h.c.$$  

$$- \Phi^\dagger(x,y)D^{-1}(x-y)\Phi(x,y).$$

(8)

Note that, apart from normalization, this is a bilocal generalization of eq.(3).

We go to a space-like hyper-surface and impose that the constituent fermions share a single common time coordinate in this frame. The bilocal composite field takes the form, $\Phi(\vec{X},\vec{r},t)$ where $\vec{X}$ is the center-of-mass coordinate, $\vec{r} = 2\vec{r}$ the interparticle separation.\footnote{The single time constraint can be made manifestly Lorentz invariant, e.g. the invariant condition $P_\mu r^\mu = 0$ with the 4-momentum $P_\mu$ and $r^\mu = (x^\mu - y^\mu)/2$. A bilocal invariant action is then “gauged fixed” on a time slice as $S = \int d^4 \rho d^4 X \delta(\sqrt{T^\mu_\rho r^\mu}) = P^\mu_\rho \int d^4 \rho d^4 X$. Fully covariantized expressions rapidly become awkward, as they would be for any conventional composite system, such as proton, atom or molecule. We will work in the rest frame knowing the results can always be boosted with care.}

\[FIG. 1: \text{Fermion loop with wave-function } \phi(r) \text{ vertices which generates the vacuum loop potential term in the action, } -\eta \phi^2(r)/r^2.\]

In the rest frame we have the single time variable $t = X^0$. $\Phi(\vec{X},\vec{r},t)$ may be viewed as a “bosonization” of the s-wave component of the fermion operator product on time slice $t$,

$$\bar{\psi}_R(\vec{X} - \vec{r},t) \psi_L(\vec{X} + \vec{r},t) → \Phi(\vec{X},\vec{r},t) + ..$$

$$X^0 = t, \quad \vec{X} = \frac{\vec{x} + \vec{y}}{2}, \quad \vec{r} = \frac{\vec{x} - \vec{y}}{2}. \tag{9}$$

Presently we will ignore gauge interactions and focus on a singe complex bound state field. We factorize $\Phi$ as:

$$\Phi(x,y) = \chi(X^\mu)\phi(r) \quad (t,X) = X^\mu \tag{10}$$

where the time dependence is carried by the pointlike factor, $\chi(X^\mu)$, and $\phi(r)$ is then a static “internal wave-function.” Typically $\chi(X) \sim \exp(iP_\mu X^\mu)$ describes the motion of the center of mass of the system, such as a plane wave, and $\phi$ describes the bound state structure and dynamics. The action is a function of $\Phi$, and such terms as, e.g., $|\chi^2|$, $|\phi^2|$, etc., are disallowed.

The single time dependence can be carried by either field, though typically we choose $\chi(t)$ hence it is a quantum field with canonical dimension of mass. $\phi(r)$, on the other hand, is typically static, and we presently treat it classically. As a static field $\phi$ has no canonical momentum and satisfies a static differential equation. Since we are working with a classical $\phi$ we will assume it is dimensionless. It forms a static “configuration,” something like an instanton. The full composite field $\Phi = \chi \phi$ is canonical and in the pointlike limit, $\phi → \delta^3(r)$, $\Phi$ becomes a local quantum field, essentially pure $\chi(X^\mu)$.

We will exclusively consider a ground state composed of a pair of fermions in an s-wave, so $\phi(r)$ is spherically symmetric under rotations of the radius $\vec{r} = \vec{r}/2$ with $\vec{X}$ held fixed. Hence the bilocal interaction term of eq.(8) yields the action, including an assumed central potential,
V\hat{0}(\hat{r})$, and kinetic terms, one for the center-of-mass and the other for the radius:

\[ S = \int \frac{d^3r}{V} d^4X (Z_X |\phi|^2 |\partial_X \chi|^2 - Z_0 |\chi|^2 |\partial_generic|_0 |\phi|^2 - V_0(\hat{r}) |\chi|^2) \]

\[ -g \int \frac{d^3r}{V} d^4X [\overline{\psi}_L(\vec{X} + \hat{r}) \psi_R(\vec{X} - \hat{r})] |\chi(\vec{X})\phi(\hat{r}) + h.c. \] (11)

Here $\vec{x} = \vec{X} + \hat{r}$ and $\vec{y} = \vec{X} - \hat{r}$ and we have,

\[ \partial_x^2 + \partial_y^2 = \frac{1}{2} \partial_X^2 + \frac{1}{2} \partial_r^2 \] (12)

Hence we have a “bare” relationship $Z_\chi = Z_\phi = 1/2$.

We have introduced a “normalization volume,” $V$, to maintain canonical dimensionality of the overall spacetime action integral. No physical quantities depend upon $V$. Note that with the normalization condition,

\[ \int \frac{d^3r}{V} \phi^2(r) = 1. \] (13)

and with a rescaling $\chi \rightarrow \chi/\sqrt{Z_\chi}$ we can make the $\chi$ kinetic term canonical. A renormalized action is then:

\[ S = \int \frac{d^3r}{V} d^4X (|\phi|^2 |\partial_X \chi|^2 - |\chi|^2 |\partial_r \phi|^2 - V_0(\hat{r}) |\chi|^2) \]

\[ -g \int \frac{d^3r}{V} d^4X [\overline{\psi}_L(\vec{X} + \hat{r}) \psi_R(\vec{X} - \hat{r})] |\chi(\vec{X})\phi(\hat{r}) + h.c. \] (14)

where,

\[ g = Z_\chi^{-1/2} g_0 \quad V_r \rightarrow Z_\chi^{-1} V_0 \] (15)

In general, we have the parameter $z = Z_\phi/Z_\chi$ which is potentially subject to radiative corrections. To simplify the present discussion we will adopt the value $z = 1$.

A global $U_L(1) \times U_R(1)$ chiral symmetry is now the $U(1)$ transformation $\Phi \rightarrow e^{i\theta} \Phi$. We presently ignore a potentially thorny issue of local gauge covariance, which requires internal Wilson lines.

Working in the rest-frame, $\chi \propto e^{iMt}$, the mass $M$ of the bound state is then determined by the eigenvalue of the static equation for the ground state in $\phi$ (see the next subsection, II.D):

\[ \left( \frac{d^2}{dV^2} + 2 \frac{d}{r \, dr} - V_r(r) \right) \phi(r) = -M^2 \phi(r) \] (16)

Here $V_r(r)$ is the core potential that binds the fermions into quasi-stable, approximate eigenstates, but is not confining. Substituting this into the action, eq.(11) where we integrate over $r$ and apply eq.(13) we obtain an effective point-like action for $\Phi(X,r) \rightarrow \Phi(X)$,

\[ S = \int d^4X \left( |\partial_X \phi|^2 - M^2 |\phi|^2 \right) \]

\[ -g \int d^4X [\overline{\psi}_L(X)\psi_R(X)] \Phi(X) + h.c. + ... \] (17)

The ellipsis, ..., includes corrections to $g$ from an expansion in $r$, involving higher derivatives, $\sim r^n \partial^n \psi(X)$, which may provide probes of compositeness.

Note we will require that the relevant solutions to the equation of motion for physical bound states at short distances, $r \sim M^{-1}$ must have a real mass eigenvalue $M$, hence $M^2 \geq 0$. A negative $M^2$ represents a vacuum instability at short distances. For the barrier potential below we can enforce this by positivity of $V_r(r)$.

D. Warm-up: A Simple Composite Model With Barrier Potential

We now consider a simple barrier potential model, which leads to a straightforward textbook quantum mechanics problem. This illustrates the bosonized formalism and the derivation of the BEH-Yukawa coupling in a general potential model, where it may not be present ab initio, as in eq.(11). The present warm-up model ignores the induced vacuum loop potential and the Landau-Lifshitz solutions.

We will presently assume the renormalized action and we’ll neglect the $g$, Yukawa term. Varying $\chi$, from eq.(14) with $z = 1$, it follows that:

\[ \partial^2 \chi \int \frac{d^3r}{V} |\phi|^2 = \chi \int \frac{d^3r}{V} \left| -|\partial_r \phi|^2 - V_r(r) |\phi|^2 \right. \] (18)

We assume an eigenvalue, $M^2$, and varying with respect to $\phi$, we then have the separate equations of motion:

\[ \partial^2 \chi(X) = -M^2 \chi(X), \] (19)

\[ \left( \frac{d^2}{dr^2} + 2 \frac{d}{r \, dr} - V_r(r) \right) \phi(r) = -M^2 \phi(r). \] (20)

Note the $\chi$ equation is a free Klein-Gordon form, while the $\phi$ equation is static.

To simplify we can work in the rest frame, and impose the normalization conditions:

\[ \int \frac{d^3r}{V} |\phi|^2 = 1 \quad \chi = \frac{1}{\sqrt{2MV}} \exp(iMt) \] (21)

where $\chi$ has a conventional plane wave “box normalization” where $V$ is an imaginary volume associated with the internal 3-space, and $V$ is the volume of an imaginary exterior 3-space box (these volume factors cancel in physical quantities). Note that the $\chi$ terms become canonical in eq.(11) with the above $\phi$ normalization.

Consider a “thin wall” barrier potential:

Region I: \quad $V_r(r < R) = 0$

Region II: \quad $V_r(R < r < R + a) = W^2$

Region III: \quad $V_r(r > R + a) = 0$ (22)

In contrast to nonrelativistic quantum mechanics where the barrier has dimensions of energy, here the barrier, $W^2$, has dimensions of (mass)$^2$. 
The general solution for eq.(19) is:

Region I: \[ \phi(r) = N e^{-kr}, \quad k = M \]

Region II: \[ \phi(r) = N e^{-kr}, \quad \kappa = \sqrt{W^2 - k^2} \]

Region III: \[ \phi(r) = a e^{iMr} + b e^{-iMr} \] \hspace{2cm} (23)

Region III is pair radiation, since for \( \Phi = \chi \phi \), we have:

Region III: \[ \Phi = a \chi_0 e^{iM(t+r)} + b \chi_0 e^{iM(t-r)} \] \hspace{2cm} (24)

a sum of incoming (left-moving) and outgoing (right-moving) spherical waves. If we set \( a = 0 \) we have the outgoing \( s \)-wave of a fermion pair from the decay of the bound state.

The matching of Region I to Region II requires:

\[ \tan(kR) = -\frac{k}{\kappa}, \quad N' = N e^{\kappa R} \sin(kR) \] \hspace{2cm} (25)

Note that, as usual, the boundary matching conditions determine \( k \) and the eigenvalue, \( M = k \). For large \( \kappa \) we have \( kR \rightarrow \pi \) and \( \sin(kR) \rightarrow -k/\kappa, \cos(kR) \rightarrow 1 \).

The matching of Region II to Region III requires:

\[ a = \frac{1}{2} N \left( 1 + \frac{\kappa}{k} \right) \sin(kR)e^{-ik(R+a) - \kappa a} \]

\[ b = \frac{1}{2} N \left( 1 - \frac{\kappa}{k} \right) \sin(kR)e^{ik(R+a) - \kappa a} \] \hspace{2cm} (26)

The normalization integral of eq.(21) is dominated by the cavity Region I and yields approximately, with \( kR = MR = \pi \):

\[ 1 = \int_{V} d^3r \frac{\phi^2}{V} = 4\pi \int_0^R N^2 \sin^2(kr) \frac{dr}{V} = 2\pi^2 N^2 \frac{M}{MV} \]

\[ N^2 = \frac{MV}{2\pi^2}. \] \hspace{2cm} (27)

(note, the Region II contribution to the mass, in the large \( \kappa \) and thin wall \( a \ll R \) limit, is negligible \( \sim M^2 a + O(aM/\kappa) \).)

The solution represents a steady state, a balance of an incoming and outgoing radiative part. It cannot be matched to a pure outgoing wave unless the core solution explicitly decays in time, which then requires integrals over Green’s functions. However, if we are interested in an initial state, consisting of one pair of fermions localized in the Regions I+II, then we can switch off the incoming radiation, \( a \rightarrow 0 \), and the state will decay, where the decay amplitude is \( b \). The decay width is obtained semi-classically by the rate of energy loss (power) into the outgoing spherical wave, divided by the mass.

Here’s a cursory semi-classical estimate of the decay width. In the rest-frame with no explicit dependence upon \( \vec{X} \) we see that \( \Phi \) can be viewed as a localized field in \( \vec{r} \) and \( X^0 = t \), hence \( \Phi(r,t) = \phi(r)e^{i\Omega t}/\sqrt{2MV} \). The outgoing power is given by the stress tensor, \( T_{00} \), from the right-mover solution. Note that \( \kappa \sin(kR)/k \rightarrow -1 \) in the large \( \kappa \) limit and

\[ |\phi(r)|^2 \sim |b|^2/r^2 \sim N^2 e^{-2\kappa a}/4r^2, \] \hspace{2cm} (28)

Hence the power, using eq.(27):

\[ P = \int d^3X \frac{4\pi r^2}{V} (\partial_0 \Phi^* \partial_0 \Phi) \bigg|_{r \rightarrow \infty} \]

\[ \approx \pi M^2 V N^2 \frac{e^{-2\kappa a}}{(2MV)} = \frac{1}{4\pi} M^2 e^{-2\kappa a} \] \hspace{2cm} (29)

and the decay width is obtained as the ratio,

\[ \Gamma = \frac{P}{M} = \frac{M}{4\pi} e^{-2\kappa a} \] \hspace{2cm} (30)

We can compare the decay width from a complex point-like field consisting on a single color \( N_c = 1 \), of mass \( M \) with Yukawa coupling \( g \) to the fermions:

\[ \Gamma = \frac{g^2}{16\pi} M \] \hspace{2cm} (31)

Matching this to the composite model calculation gives

\[ g = 2e^{-\kappa a} \] \hspace{2cm} (32)

We therefore have a heavy bound state with mass \( M = k \approx \pi/R \) and an induced Yukawa coupling \( g \propto e^{-\kappa a} \) which is perturbative in the large \( \kappa a \sim \text{Wu} \) limit.

We’ve done this for a single color. In this simple model if we extend to \( N_c \) colors, then \( \Phi \) will receive a color normalization factor of \( 1/\sqrt{N_c} \) and the mass will then become \( M \times N_c/N_c \) unchanged. The decay width we have computed semiclassically is also unchanged as color sums cancel against this normalization factor. When we compare to the field theory decay width with \( N_c \) colors, we see that our model predicts \( g_f = g/\sqrt{N_c} \), and yields a color suppressed decay. The above calculation assumed \( g_0 = 0 \) (zero bare Yukawa coupling) and obtained the induced effective coupling \( g \sim 1/\sqrt{N_c} \). However, the coupling \( g \) need not be induced, and can come directly from a fundamental \( g \) and is then \( \mathcal{O}(1) \) rather than \( 1/\sqrt{N_c} \).

The main takeaway is that the eigenvalue \( M^2 \) is generated by the matching of Regions I, II and the radiative Region III. It is the matching that determines the eigenvalue and dictates the relevant solutions to the differential equations in each region.
III. FERMION INDUCED VACUUM LOOP POTENTIAL

A. Discussion

The full action for $\Phi$, including only $V_0(r)$, is incomplete. Since there is a Yukawa coupling to the exterior fermions, either fundamental or induced, we must include the feedback effect arising from the last term in eq. (11) of integrating out fermion fields, as in eq. (4). The chiral fermions roam through the surrounding space and affect the vacuum. The Feynman loop of Fig. (1) takes the form of an attractive, approximately scale invariant “vacuum loop potential” which we denote as $V_{\text{loop}}(r)$. This can be seen by direct calculation in Appendix I:

$$V_{\text{Loop}}(r) = -\frac{\eta}{r^2}, \quad \eta = \frac{N_c g^2}{32 \pi^2}$$

(33)

Note that $r$ is radial and not a Compton wavelength, hence its associated momentum is $1/r$.

We can intuit the form of eq. (33) by comparing to the momentum space form of the loop, the $O(h)$ term in $V_M$ of eq. (4),

$$V_M = M^2 - \hbar \frac{N_c g^2}{8 \pi^2} (M^2 - \mu^2)$$

(34)

At large distances, $r \sim L \gg R$, the bound state will acquire mass, which provides an IR cut-off on the potential in the Lagrangian,

$$V_{\text{Loop}}(r) \sim -\left(\frac{\eta}{r^2} - \frac{\eta}{L^2}\right)$$

(35)

This matches the sign of the $\mu^2 \sim 1/4L^2$ term in $V_M$. Likewise, the short-distance behavior $\sim -\frac{\eta}{r^2}$ with $r^2 \sim 1/4M^2$ matches the $-g_0^2 N_c M^2/8\pi^2$ term in $V_M$.

The key feature for us is that $V_{\text{Loop}}(r)$ contains no mass scales if $g^2$ is constant, i.e., if $g^2$ is an approximate fixed point of the RG evolution into the IR. This means that there is a region outside of $V(r)$, such as Region III in our previous example, in which the potential is the scale invariant $V_{\text{Loop}}$, and in this region the solution is scale invariant, with $M = 0$.

These are the solutions studied by Landau and Lifshitz in [10].

B. Scale Invariant Landau-Lifshitz Solutions

We assume that $V_{\text{Loop}}$ grows more negative until the scale of the core radius of the bound state $r = R$ is reached, then becoming a constant negative vacuum energy in the core, $-\eta/R^2$ for $r < R$. We will presently assume an additional constant core potential, $-U^2$, so that $V_r(r < R) = -U^2 - \eta/R^2$.

Remarkably we can omit $U^2$ altogether and we still have binding from the vacuum loop potential alone. That is, if we simply pinch a pair of chiral fermions together they will generate a self-binding potential and a nontrivial self-consistent solution, hence we expect that even a comparatively weak force, such a gravity, can trigger the formation of these states.

We therefore have:

Region I: $V_r(r < R) = -U^2 - \frac{\eta}{R^2}$

Region II: $V_{\text{loop}}(r > R) = -\frac{\eta}{r^2}$

(36)

Assuming $\eta$ is approximately constant we find for the form of the Region II solutions, following LL. The scale invariant spatial differential equation eq. (19) becomes,

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{\eta}{r^2}\right) \phi(r) = -M^2 \phi(r)$$

(37)

Following Landau and Lifshitz, this is solved with the anzatz $\phi(r) = r^p$, [10], to obtain:

$$p^2 + p + \eta = 0, \quad M^2 = 0$$

(38)

hence, we find:

$$p_1 = -\frac{1}{2} + \frac{1}{2} \sqrt{(1 - 4\eta)}$$

$$p_2 = -\frac{1}{2} - \frac{1}{2} \sqrt{(1 - 4\eta)}$$

(39)

The solution has $M^2 = 0$, hence these are static massless solutions.

We see that there are two distinct cases: $\eta < 1/4$ and $\eta > 1/4$. Our main interest will be in the weak coupling case, $\eta < 1/4$. Remarkably, $4\eta > 1$ and $4\eta < 1$, which emerge from the solutions to the classical differential equation, anticipate the critical behavior of the NJL model, which arises from the loop integral. That is, $4\eta = g_0^2 N_c / 8\pi^2 = 1$ corresponds to the critical value, and $\eta < 1/4 (\eta > 1/4)$ is an unbroken phase (broken phase) of the NJL model, as seen in eq. (4). The classical LL solutions thus anticipate the critical coupling and distinct phases of the NJL model!

In Region I, for any core potential $V_r(r)$, we can generally find a static solution as well. Presently we take,

$$\phi(r) = N \frac{\sin(kr)}{r}$$

(40)

and find that with the choice,

$$k^2 = U^2 + \frac{\eta}{R^2}$$

(41)

we have a zero mass $M^2 = 0$. Note that $U^2$ can have any sign and magnitude, and if $k^2 < 0$ the $\sin(kr) \rightarrow \sinh(|k| r)$.

It is of key importance to note that $k$ is now determined by eq. (41) alone, and not by the matching boundary condition of I to II to radiation III, as in our previous example. Since this is possible for any potential (we can slice any potential into multiple segments with different
values of $U^2$, and match at each segment boundary) and
the scale invariant solution will always exist [10]. There
may be negative $M^2$ solutions which are model depen-
dent but must be disallowed. These are disallowed here
for negative $U^2$ where the potential in I+II resembles a
“castle with moat.”

The full solution is then:

Region I: \( r < R \) \( \phi(r) = \mathcal{N} \frac{\sin(kr)}{r} \)

\[ k^2 = U^2 + \frac{\eta}{R^2} \]

Region II: \( r > R \) \( \phi(r) = \frac{A}{R} \left( \frac{r}{R} \right)^{p_1} + \frac{B}{R} \left( \frac{r}{R} \right)^{p_2} \) \( (42) \)

where matching of Region I to Region II requires,

\[ \mathcal{N} \sin(\sqrt{\beta}) = A + B \]

\[ \mathcal{N} \sqrt{\beta} \cos(\sqrt{\beta}) = A (1 + p_1) + B (1 + p_2) \] \( (43) \)

where:

\[ kR = \sqrt{\beta} \] \( (44) \)

hence,

\[ A = \frac{\mathcal{N}}{(p_2 - p_1)^{-1}} \left( (1 + p_2) \sin(\sqrt{\beta}) - \sqrt{\beta} \cos(\sqrt{\beta}) \right) \]

\[ B = -\frac{\mathcal{N}}{(p_2 - p_1)^{-1}} \left( (1 + p_1) \sin(\sqrt{\beta}) - \sqrt{\beta} \cos(\sqrt{\beta}) \right) \] \( (45) \)

In the case of strong coupling we have a complex expres-
sion, since \( \eta > \frac{1}{4} \),

\[ p_1 = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\eta} = -\frac{1}{2} + \frac{1}{2} i\xi \]

\[ p_2 = -\frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\eta} = -\frac{1}{2} - \frac{1}{2} i\xi \] \( (46) \)

where \( \xi = \sqrt{(1 - 4\eta)} \). Hence,

\[ A = i\xi^{-1} \mathcal{N} \left( \frac{1}{2} (1 - i\xi) \sin(\sqrt{\beta}) - \sqrt{\beta} \cos(\sqrt{\beta}) \right) \]

\[ B = -i\xi^{-1} \mathcal{N} \left( \frac{1}{2} (1 + i\xi) \sin(\sqrt{\beta}) - \sqrt{\beta} \cos(\sqrt{\beta}) \right) \] \( (47) \)

and the general solution is:

\[ \phi(r) = \frac{|A| e^{i\sigma}}{R} \left[ \left( \frac{r}{R} \right)^{(-1+i\xi)/2} + \left( \frac{r}{R} \right)^{(-1-i\xi)/2} \right] \]

\[ e^{2i\sigma} = -\frac{(1 - i\xi) \sin(\sqrt{\beta}) - 2\sqrt{\beta} \cos(\sqrt{\beta})}{(1 + i\xi) \sin(\sqrt{\beta}) - 2\sqrt{\beta} \cos(\sqrt{\beta})} \] \( (48) \)

The case where \( U = 0 \) we have \( \beta = \eta \). The solution in
the weak coupling limit, \( \eta < \frac{1}{4} \), is a simpler real expres-
sion:

\[ p_1 = -\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\eta} \approx -\eta + O(\eta^2) \]

\[ p_2 = -\frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\eta} \approx -1 + \eta + O(\eta^2) \] \( (49) \)

and,

\[ A = -(1 - 2\eta)^{-1} N ((\eta) \sin(\sqrt{\eta}) - \sqrt{\eta} \cos(\sqrt{\eta})) \]

\[ \approx N \sqrt{\eta} + \frac{1}{2} N\eta^2 + O(\eta^2) \]

\[ B = (1 - 2\eta)^{-1} N ((1 - \eta) \sin(\sqrt{\eta}) - \sqrt{\eta} \cos(\sqrt{\eta})) \]

\[ \approx -\frac{2}{3} N\eta^2 + O(\eta^2) \] \( (50) \)

hence,

\[ \phi(r) = \frac{N \sqrt{\eta}}{R} \left( \frac{r}{R} \right)^{-\eta} - \frac{2N\eta^{3/2}}{3R} \left( \frac{r}{R} \right)^{-1+\eta} \] \( (51) \)

or for small \( \eta \).

\[ \phi(r) = \frac{\mathcal{N} \sqrt{\eta}}{R} \left( \frac{r}{R} \right)^{-\eta} - \frac{2N\eta^{3/2}}{3R} \left( \frac{r}{R} \right)^{-1+\eta} \] \( (52) \)

Note in the real case the \( B \) term falls off faster in the IR
while the magnitude of both \( A \) and \( B \) is the same in
the complex case at large distance.

C. Barrier Potential with \( V_{\text{Loop}} \) and LL Solutions

As a simple example of a full solution with core and
shroud, we return to the barrier potential and include
the presence of a nonzero \( g \). Therefore we must match
onto the LL solution in Region III, rather than onto the
radiative solution. Here we assume the potential \( V_r \) as
defined in eqs.(22) for the Regions I and II. This is ac-
tually “unrealistic” in the sense that we are ignoring
the additional \( V_{\text{loop}} \) effect for Region I, \( r < R + a \). However,
this shows the generality of the matching effect with
the tunneling barrier in Region II.

Consider a “castle and moat” barrier potential:

Region I: \( V_r(r < R) = 0 \)

Region II: \( V_r(R < r < R + a) = W^2 \)

Region III: \( V_{\text{loop}}(r > R + a) = -\eta/r^2 \) \( (53) \)

Here we have chosen \( U^2 + \eta/R^2 = 0 \) (negative \( U^2 \)) for
simplicity and comparison to the previous “warm-up” ex-
ample of a barrier potential with \( V_r = 0 \) in its Region I;
Any value of \( U^2 \) will produce an LL solution in Region III.

The solution in the three regions is now, with \( \kappa = |W|:

Region I: \( \phi(r) = \phi_0 \)

Region II: \( \phi(r) = \frac{\phi_0}{2\kappa R} \times \)

\[ (1 + \kappa R)e^{\kappa(r-R)} - (1 - \kappa R)e^{-\kappa(r-R)} \]

Region III: \( \phi(r) = \frac{A}{R} \left( \frac{r}{R} \right)^{p_1} + \frac{B}{R} \left( \frac{r}{R} \right)^{p_2} \) \( (54) \)
where $A$ and $B$ are rather messy expressions which we quote in the limit $a/R << 1$ and $\eta << 1$:

$$
\begin{align*}
A &= \frac{\phi_0 (\kappa R \eta - 1) \cosh(\kappa a) + (\eta - \kappa^2 R^2) \sinh(\kappa a)}{\kappa R(2\eta - 1)}, \\
B &= \frac{\phi_0 \kappa R \eta \cosh(\kappa a) + (\eta + \kappa^2 R^2 - 1) \sinh(\kappa a)}{\kappa R(2\eta - 1)}
\end{align*}
$$

(55)

The solution is shown as (B), with comparison to the “warm-up” example (A), in Fig.(2). First we note that Region I is the solution (B) with free boundary conditions if $\phi = \phi_0 = \text{constant}$. In the solution of Fig.(2) (A), which is the barrier potential of II.C, the boundary matching conditions determined $k$ and the eigenvalue, $M = k$. On the other hand for (B) the Region I solution is a trivial constant, $k = 0$. This reflects that the scale invariance in this case tends to “flatten” the core wavefunction. The matching of Region I to Region II then requires that there are both exponentially increasing and decreasing components in the barrier.

In Region III in Fig.(2) we see that the previous solution (A) with $\eta = 0$ matches onto radiation, while now (B) with $\eta$ nonzero matches onto the LL solution “shroud” in the exterior.

IV. IR MASS AND NORMALIZATION

We emphasize that the LL solutions will always force the overall bound state solutions to be massless. The internal wave-function $\phi(r)$ presently satisfies a linear differential equation and can be freely renormalized (though we will contemplate a quartic interaction below). However, one sees that the massless LL solutions are not compact and are, without an IR cut-off, non-normalizable. We require an IR cut-off to the solution, which we will define to be $L$. This can come from an IR mass term and leads to $L \sim 1/m$.

We also see that these solutions are very insensitive to the core structure, and nearly vanish as $r \to R$. We will presently focus on the dominant LL solution in the IR in the small $\eta$ limit which is the $A$ component. Given the core insensitivity it is inconvenient to maintain explicit dependence upon $R$. Hence we will renormalize the solution and use an IR cutoff $L$ as the unique scale,

$$
\phi(r) \to \phi_r \left( \frac{r}{L} \right)^{-\eta}
$$

(56)

It is now convenient to redefine the normalization:

$$
\int \frac{d^3r}{\bar{V}} \phi^2 \approx \int_0^L \frac{4\pi r^2 dr}{\bar{V}} \phi_r^2 \left( \frac{r}{L} \right)^{-2\eta} \sim \frac{\phi_r^2}{1 - 2\eta/3}
$$

(57)

where we define $\bar{V} = 4\pi L^3/3$ (this differs from eq.(13)). This leads to the normalization integrals:

$$
\mathcal{N}(n) = \int \frac{d^3r}{\bar{V}} \phi^n = \frac{\phi_r^n}{(1 - n\eta/3)}
$$

(58)

In eq.(14) we have the combined action for $\chi$ and $\phi$:

$$
S = \int d^4X \frac{d^3r}{\bar{V}} \left( |\phi \partial_\mu \chi|^2 - |\chi \nabla_r \phi|^2 - V_0(r) |\chi \phi|^2 \right) - g_0 \int d^4X \frac{d^3r}{\bar{V}} \overline{\psi_L} (X + r, t) \psi_R (X - r, t) |\chi (X) \phi (r) + h.c.
$$

(59)

To maintain canonical kinetic terms, the $\chi$ field and other parameters must then be renormalized as:

$$
\chi' = \sqrt{\mathcal{N}(2)} \chi
$$

$$
g' = g \mathcal{N}(1)/\sqrt{\mathcal{N}(2)} = g (1 - 2\eta/3)^{1/2} \approx g
$$

$$
\lambda' = \lambda \mathcal{N}(4)/\sqrt{\mathcal{N}(2)}^4 = \lambda (1 - 2\eta/3)^2 \approx \lambda
$$

(60)

Hence the renormalized action becomes:

$$
S = \int d^4X \left( |\partial_\mu \chi|^2 - M_r^2 |\chi'|^2 \right) - g \int d^4X \overline{\psi_L} (X, t) \psi_R (X, t) |\chi' (X) + h.c. + ...}

(61)
where the ellipsis is series of higher dimension derivative terms by expanding the $g$ term in $r$. Note that in this scheme $\phi$ is dimensionless, while $\chi$ carries dimensions of mass, and $\Phi = \phi \chi$ has canonical dimensions of mass. Recall, we treat $\phi$ as dimensionless since we are only considering it classically at present and it is a static field.

The energy of the massless solution, $V_r = -\eta/r^2$, is finite with an IR cut-off and is given by the spatial integral over the stress tensor, $T_{00}$. Integrating by parts and using the equation of motion for the static massless solution, e.g. (16), we obtain surface terms:

$$
\int_0^L 4\pi r^2 dr \left( |\nabla_r \phi|^2 + V_r(r)|\phi|^2 \right) = \frac{4\pi r^2}{V} \phi^*(r) \nabla_r \phi(r) \right|_0^L
$$

(62)

Clearly we have at the origin $\nabla_r \phi(r \to 0) = 0$. Moreover, with the dominant $A(r)^{-\eta}$ solution we have $\hat{V} \sim L^{3-2\eta}$ and,

$$
m^2 \sim \frac{r^2}{V} \phi^*(r) \nabla_r \phi(r) \sim -\eta L^{-2} \quad \text{as} \quad r \to L
$$

(63)

Hence the mass $\sim 1/L$ and is arbitrarily small in the large $L$ limit and resides on the surface at large $L$.

We remark that this represents the flow of “scale charge” onto the bounding surface in the IR (where $\phi_\eta \phi$ is a Weyl current in the implementation of scale transformations in general relativity with scalar fields and non-minimal couplings [16]). This is somewhat reminiscent of topology in which the Chern current, whose charge represents a topological field configuration in the bulk in odd-$D$ will produce an anomaly on a bounding surface in even-$(D - 1)$.

A. Positive Infrared (Mass)$^2$

We presently discuss the origin of mass in the context of the extended LL solution for a BEH-boson. This will not be a rigorous treatment, but rather a sketch of how we think some mechanisms for mass generation may work. We will return to this in greater detail elsewhere.

To be a physical and normalizable solution, we require an IR cut-off, $L$, which in turn implies a mass, $m \sim 1/L$, for the scalar field. In order to have a mass the $-1/r^2$ potential must deviate from the scale invariant form in the IR. We can add a small IR mass term to the theory by explicitly modifying $V_{\text{loop}}$. If the potential evolves into the form,

$$
V_{\text{loop},m} = -\frac{\eta}{r^2} + m^2
$$

(64)

we see that the $\chi, \phi$ equations of motion for an eigenvalue, $M^2$, separate:

$$
\frac{\partial^2 \chi(X)}{-M^2 \chi(X),
$$

$$
\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \frac{\eta}{r^2} - m^2 \right) \phi(r) = -M^2 \phi(r),
$$

(66)

and now the solution has mass $M^2 = m^2$:

$$
\Phi = \chi(X) \phi(r) = \phi_r \left( \frac{r}{L} \right)^{-\eta} e^{imt}
$$

(67)

However, this is energetically disfavored for $r > L$ and is no longer a pure eigenstate since we can reduce the energy by terminating the LL solution and transitioning into pure radiation. This happens at the scale at which the $m^2$ term dominates the $-\eta/r^2$, approximately where the potential vanishes:

$$
m^2 = \eta/L^2, \quad L = \sqrt{\eta} m^{-1}.
$$

(68)

Hence, at larger distances, $r \gg L$ we expect to have a pure radiative solution, as in Region III of the warm-up barrier potential.

The mass term in principle can be generated by the running of $g$, or equivalently, $\eta$. The potential will evolve by the RG as:

$$
-\frac{\eta}{r^2} \rightarrow \frac{\eta}{r^2} + \frac{\eta^2}{r^2} + m^2
$$

(69)

where $\eta$ develops approximate power law behavior via the RG equation. Eq.(69) is known as a “Mie potential” [17] and dates from early days of molecular physics.

This form would imply a large beta function, $\beta(g)/g \sim n$, passing through integer values. This represents a large trace anomaly, and is the analogue behavior that is seen in the Coleman-Weinberg potential [14] (see [15] for a discussion of the trace-anomaly in this context). This can in principle occur with perturbative $g$, but the full formalism using the improved stress tensor is beyond the scope of this present paper.

This is “hand-waving” at this point and requires a more detailed analysis to understand the solution at the massless LL transition to positive $m^2$ radiation. The negative $m^2$ case discussed next seems to be more well-defined and directly applicable to the BEH-boson.

B. Negative Infrared (Mass)$^2$: RG Trigger for EW Mass Generation

The LL solution will terminate at a scale at which the $-\eta/r^2$ potential becomes constant $\sim -\eta/L^2$. This happens naturally if the Feynman loop that induces the potential decouples, and generally requires that the chiral fermions in the loop acquire a mass, $m_f$. The $1/r^2$ potential then “freezes” at the scale $L \sim 1/m_f$ and becomes a negative $-m^2\Phi^2$ term where $m^2 \propto \eta m_f^2$. In addition we will have a $\lambda(\Phi^2\Phi^2)/2$ term, induced by fermion loops (see Appendix I). Hence, the composite $\Phi$ field will develop a VEV in the usual way.

In the case of the BEH-boson composed of top and anti-top quarks this would occur when the top quark acquires a mass. However, the top quark mass comes from the electroweak symmetry breaking and the VEV of the BEH-boson. The formation of the VEV and top
quark mass will then occur in a self consistent way. A consistency condition then determines the effective value of the running $g$. While we haven’t fully developed the SU$(2) \times U(1)$ isodoublet BEH boson, we can get an idea of how this might work for electroweak symmetry breaking within our present understanding of the composite system described here. We presently assume that the BEH boson is composed of top and anti-top quarks [4, 5, 13].

The vacuum loop potential, in terms of the physical separation of the constituents, $\rho = 2r$ is given by

$$V_{\text{Loop}} = -\frac{4\eta}{\rho^2} = -\frac{g^2 N_c}{8\pi^2 \rho^2}$$  \hspace{1cm} (70)

g$^2$ evolves by the RG equation as a running in length scale $\rho$. For the top quark this is [18],

$$16\pi^2 \frac{\partial g}{\partial \ln(\rho)} = -g \left( \frac{g^2}{2} - 8g_{QCD}^2 \right)$$ \hspace{1cm} (71)

The solution shows $g$ gradually increasing at large distances, $\rho = \mu^{-1}$, due to the effects of QCD, which cause it to be slightly asymptotically free as seen in Fig.(3).

However, suppose that the running $\eta = g^2 N_c / 32\pi^2$ halts at some scale $\rho_0$ due to the “freezing.” We then have the potential for $\Phi$:

$$V = -\frac{4\eta}{\rho_0^2} |\Phi|^2 + \frac{\lambda}{2} |\Phi|^4$$ \hspace{1cm} (72)

We expect the normalizations of the $g$ and $\lambda$ are not far from the their standard model values, as seen by the normalization discussion above. In the standard model we have the phenomenological values, $\lambda \approx 1/4$ and $g = g_{top} \approx 1$. Hence the fermion loop freezing will lead to a spontaneous breaking and $\Phi$ develops a VEV:

$$\langle |\Phi|^2 \rangle = 4\eta/\rho_0^2 \lambda$$ \hspace{1cm} (73)

In principle this can happen for any composite field given the negative mass term. However, for the BEH-boson this in turn implies that the top quark develops a mass given by

$$m_{top} = g\langle |\Phi| \rangle = 2g\sqrt{\eta/\rho_0}\sqrt{\lambda}$$ \hspace{1cm} (74)

Hence spontaneous symmetry breaking happens if a consistency condition for $g$ is fulfilled:

$$m_{top}^2 \rho_0 \approx 4g^2 \eta/\lambda \approx \frac{g^4 N_c}{2\pi^2}$$ \hspace{1cm} (75)

We might expect a cut-off when $\rho_0$ is of order a half wavelength:

$$\rho_0 \sim 1/2m_{top}$$ \hspace{1cm} (76)

Therefore we find:

$$g = g_c \approx (\pi^2/6)^{1/4} \approx 1.13$$ \hspace{1cm} (77)

slightly larger than the known experimental value, $g = 1$.

Our crude result indicates that $g \sim O(1)$ can trigger a symmetry breaking mechanism that normalizes the LL solution. There will be enhancements by t-channel gluon and $Z$ exchange that tend to reduce the requisite $g^2$. Moreover, the fermion loop diagram with insertion of $m_{top}$ and a single $\phi(r)$ is expected to generate a $\sim gm/r$ Coulomb interaction in addition to the $\eta/r^2$ potential, so the system is expected to be described by a Mie potential [17], and we again expect a further corrections tending to reduce the value of $g$. We will return this problem in greater detail elsewhere.

V. CONCLUSIONS

In summary, we began by formulating the bound state problem for a pair of chiral fermions in a bosonized wavefunction that represents an s-wave, which can be either a bound state or a pair radiation field. This is a convenient way to describe the s-wave sector, with the correlated colors, spins and flavor quantum numbers, then requiring using only complex scalars.

As an example of the method, we first considered a short distance dynamics that produced a localized ground state wave-function in a simple non-confining barrier potential, $V_4(r)$. This leads to an approximate, quasi-stable, eigenstate, since it can decay to free unbound fermions, and we estimated semiclassically the induced Yukawa coupling $g$ to free fermions.

The main point of this paper is that the presence of the Yukawa coupling to free chiral fermions nontrivially affects the vacuum. The coupling, via a Feynman loop, induces a scale invariant potential, $V_{\text{Loop}}(r) = -\frac{\eta}{\rho^2}$. This acts upon the valence wave-function. This is the quantum loop effect normally considered in momentum space for the Nambu–Jona-Lasinio model, where it subtracts...
from the bound state mass. In the present case we obtain the potential coefficient, \( \eta = N_c g^2/32\pi^2 \), computed explicitly in Appendix I.

This potential, which is approximately scale invariant, causes the bound state solution to be of the Landau-Lifshitz form external to the core \([10]\). We discovered that the LL solutions have a natural affinity with the NJL model. We see that the NJL critical value of the coupling, defined by \( N_c g^2/8\pi^2 = 1 \) is identical to the critical value defined by the LL solution, i.e., \( 4\eta = 1 \). It is remarkable that the classical LL solutions anticipate the critical coupling of the NJL model, where the latter is obtained by a loop calculation. This kind of classical-quantum correspondence is reminiscent of topological solutions, Chern-Simons terms and anomalies.

The wave-function solutions in this potential were first studied by Landau and Lifshitz in nonrelativistic quantum mechanics \([10]\). When these LL solutions are present, boundary condition matching to the short distance solution will always force an overall massless scalar bound state solution. This requires no fine-tuning and is inherently perturbative. The pure scale invariant potential therefore implies these massless solutions, which we term the “shroud.” These become the exterior of the ground state for any system containing the chiral fermions in a non-confining potential. Here the RG running of \( g^2 \) is soft and can be treated approximately as a fixed point \([18]\).

This behavior appears to be a general property of non-confining potential solutions with chiral fermions. Landau and Lifshitz commented on this in the context of quantum mechanics in their textbook, and apart from negative energy states (for us, \( -M^2 \) at short distances, that must be excluded), the zero energy ground state is always present.

Far infrared scale breaking can terminate the LL wavefunction, makes the LL solution normalizable and is associated with a naturally small mass for the solution. We think that the \( \text{a priori} \) non-normalizability of these solutions may have previously blocked their application. Presently we have natural IR cut-off, i.e., the BEH mass scale naturally solves the normalization problem, and in this scheme the BEH boson resembles an inflated balloon. Scale breaking can come: (a) explicitly; (b) via a Coleman Weinberg mechanism (which involves the RG evolution of \( \lambda \), or (c) from a new mechanism involving the the RG evolution of the Yukawa coupling \( g \). A negative \( m^2 \) appears to arise naturally in the IR.

This mechanism can therefore yield, with no fine-tuning, a low mass bound state and an arbitrarily large hierarchy dynamically generated between its core and its mass. For the BEH-boson composed of a top quark pair, the RG evolution is a slowly increasing \( g \) in approaching the IR, due to QCD. We argue that this may be the trigger mechanism for the electroweak scale, the BEH-boson mass and the top quark mass as one unified phenomenon. This happens for a perturbative value of the BEH-Yukawa coupling to top of \( O(1) \). A crude calculation gives \( g \approx (\pi^2/6)^{1/4} \approx 1.13 \), compared to 1.0 experimentally. While this is in need of further elaboration, which we will pursue elsewhere, optimistically we may be able to precisely predict the electroweak scale.

Are there any loop-holes in the arguments we have presented? The massless ground state relies on the deformation of the core wave-function, and we believe that is a general phenomenon (as did Landau and Lifshitz). One might posit a pointlike fundamental scalar boson with fixed, nondeformable mass, as in the usual momentum space formulation of the NJL model. In this case the shroud solution would apparently not exist and the exterior is radiation. We believe this is an ambiguity of the NJL model in that it doesn’t specify a short distance dynamics and is treated only as a cut-off theory in momentum space. A better specification of the NJL model is therefore required in configuration space, and it may be that the shroud solution is mandated, since we have seen the equivalence of the critical behavior of the LL solution and the NJL model. Can we construct a non-deformable core solution which would forbid the matching to the LL solution? How does the dynamical symmetry breaking in the NJL model manifest itself through the LL supercritical, \( \eta > 1/4 \), solution? A pair of chiral fermions bound into a black hole \([21]\) poses an intriguing problem problem along these lines. These are questions to be addressed.

We are also relying on the non-existence of negative \( M^2 \) states at short distance \textit{when the vacuum loop potential is included}. If such solutions exist then the chiral symmetry is spontaneously broken at short distance and the chiral fermions acquire mass of order \( M \). This would be a disaster for any composite BEH-boson scenario. We have not formally proven that we can always exclude such solutions, but we know that negative energy bound states in spherical potentials are restricted and often do not exist in weak coupling. Ref.\([10]\) assert that no such negative energy states exist when \( -\beta^2/r^4 \) fills all of space and one has weak coupling. This is realized in the case of barrier potentials, together with \( \eta < 1/4 \). Hence, we believe there is likely a large, non-fine-tuned range of parameters over which the massless ground state exists with no negative \( M^2 \) solutions at short distances.

We have only treated \( \phi \) classically at present and we are able to normalize it as a dimensionless field. It is a static configuration and it is not subject to canonical normalization, though it enters the path integral and would presumably be integrated (perhaps in analogy to instantons). Classically it satisfies a static differential equation that generates the LL solution. We haven’t investigated fully the conceptual issues associated with the composite field factorization or path integration over \( \phi \).

There is much to do to further develop and test and apply this theory. For example, the extension to the many flavors of the standard model requires some kind of novel interactions, suggestive of something akin to extended technicolor interactions \([3]\). The softness of the BEH-boson above the threshold implies a significant and potentially observable, non-pointlike form factor. This
may be probed in sensitive measurements of decay modes and coupling constants. It may be optimally probed in a machine such as a muon collider, a BEH-factory with s-channel production of the BEH-boson [11].

It is possible that there are many low mass scalar bound states of the chiral standard model fermions, perhaps due to gravitation. Hence a scalar democracy consisting of low mass s-wave combinations of all SM fermion pairs may exist [19]. This possibility is experimentally accessible at LHC upgrades, searching for the \( b\bar{b} \) combination [20].

We recently pointed out that mini-blackholes are expected to form near the Planck mass \( M_{Planck} \) composed of any pair of chiral fermions with the quantum numbers of the BEH-boson. We argued that they may be very light due to unknown dynamics, appealing to the existence BEH-boson as evidence [21]. Here we offer the present mechanism to further substantiate this claim. It may be interesting to study the LL solutions and shroud surrounding a mini-Reissner-Nordstrom black hole.

We remark that the top quark does not necessarily have to be the constituent of the BEH boson even in the context of the present scheme. It is alternatively possible that we have “BEH as a PNGB” theory [8] (e.g., a “Little Higgs theory”), where the parent scalar is an shroud solution to solve the large hierarchy problem, and spontaneously broken chiral symmetry solves the little hierarchy problem. Hence we think the shroud solution offers an arsenal of model building tools, perhaps even a naturalization of traditional grand unified theories such as SU\(_5\) or SO(10).

While this is a candidate mechanism that may provide a solution to the gauge hierarchy problem and a natural low mass BEH-boson, it may also be partially operant within QCD and account for the unexpectedly low mass of the \( \sigma \)-meson. The \( \sigma \)-meson of QCD appears at a surprisingly lower mass scale, \( \sim 500 \) MeV, rather than the expected \( \sim 1 \) GeV (this is the \( f_0(500) \) and for discussion see [22]). Since it is \( m^2 \) that matters here, this seems to be a mini-hierarchy of order \( \sim 1/4 \). This may be a “mini-shroud effect” extending from the expected scale \( \sim 1 \) GeV, to the observed mass, \( \sim 500 \) MeV, and would be expected in the context of a chiral constituent quark model.

Therefore, it is our conclusion that composite low mass scalars composed of chiral fermions can exist naturally. The “custodial symmetry” is scale invariance together with chirality, acting within the internal wave-functions and dynamically realizing the approximate masslessness. This suggests the BEH-boson is composed perturbatively of top and anti-top quarks, and the BEH boson is an extended object, of order \( \sim 1/m_{top} \) in scale, behaving coherently as a pointlike state in current processes at current LHC energies. This may suggest a rich spectroscopy of other flavor combinations in bound states. We hope to devise ways of testing this in foreseeable experiments.

Appendix A: Calculation of the Vacuum Loop Potential

1. Pointlike Limit

We consider the bilocal action of eq.(14):

\[
S=\int \frac{d^4r}{V} d^4X [(|\phi|^2||\partial \chi| \chi|^2 - |\chi|^2||\partial_r \phi|^2 - V_\chi(r)|\chi|^2||\phi|^2)
- g\int \frac{d^4r}{V} d^4X [\bar{\psi_L}(\vec{X} + \vec{r})\psi_R(\vec{X} - \vec{r})|\chi(\vec{X})|\phi(\vec{r})]
\]

where we have set \( z = 1 \) for simplicity.

To test the composite action, we compute the effective potential that is induced for the field \( \chi(X) \) by the fermions, for a point-like bound state, \( \phi \sim \delta^3(r) \). Assume that we have a short-distance solution of the \( \phi \) static spatial equation:

\[
\nabla_{\vec{r}}^2 \phi(\vec{r}) - V(\vec{r})\phi(\vec{r}) = M^2 \phi(\vec{r}) \quad (A2)
\]

where \( M^2 \) is the eigenvalue, as in our discussion of the barrier potential.\(^2\) We then take a limit in which \( \phi \sim \)

\(^2\) Alternatively we could take a simple harmonic oscillator potential bounded by \( R \) as \( V = \kappa \phi^2 \theta(\vec{r} - R) \) which has a Region I Gaussian solution, and a Region II steady state radiation field. This allows a straightforward pointlike limit where the Gaussian becomes \( \sim \delta^3(\vec{r}) \).
\[ \delta^3(\vec{r}) \text{, and define the pointlike dimensionless field:} \]

\[ \phi(\vec{r}) \rightarrow N\phi_0 \delta^3(\vec{r}), \]

hence,

\[ \int d^3r |\phi|^2 = N^2 |\phi_0|^2 \delta^3(0) = 1 \]

and,

\[ \int d^3r \phi = N |\phi_0| = 1 \quad (A3) \]

where \( N^{-1} = |\phi_0| \), and we define \( \delta^3(0) \). Then the action becomes,

\[ S' = \int d^4X (|\partial_\mu |^2 - M^2 |\chi|^2) \]

\[ -g \int d^4X [\bar{\psi}_L(x) \psi_R(x)] |\chi(X)| + h.c. \quad (A4) \]

The loop integral could now be done using the action if eq. (A4) since, in Fig.(4), \( x = y \) and \( w = z \) having integrated out the pointlike internal field \( \phi \). However, it is useful to do the loop integral from the point of view of the composite field \( \phi \) as a warm-up to the non-pointlike case.

First we note that the four vertex variables of Fig.(4) can be written as:

\[ \vec{r} = \frac{1}{2}(\vec{x} - \vec{y}), \quad \vec{r}' = \frac{1}{2}(\vec{w} - \vec{z}), \]

\[ \vec{X} = \frac{1}{2}(\vec{x} + \vec{y}), \quad \vec{X}' = \frac{1}{2}(\vec{w} + \vec{z}). \]

Hence,

\[ \vec{x} - \vec{z} = \vec{r} + \vec{r}' + \vec{X} - \vec{X}', \]

\[ \vec{w} - \vec{y} = \vec{r} + \vec{r}' - \vec{X} + \vec{X}'. \]

Consider the T-ordered product from eq.(A1) (including an \( i^2 \) factor from \( e^{iS} \) and \(-1\) from anti-commutation), and notation \( f \equiv \int dxdydz \):

\[ (i)^2 \frac{g_0^2}{8\pi^2} \int [0|T[\bar{\psi}_L(x)\psi_R(y)][\bar{\psi}_R(w)\psi_L(z)]|0\rangle \Phi(x,y)\Phi^\dagger(z,w) \]

\[ = g_0^2 N_c \int \text{Tr}(S_F(x - z)S_F(y - w)\mathcal{P}_0) \Phi(x,y)\Phi^\dagger(z,w) \]

\[ = g_0^2 N_c \int d^3r d^3r' \frac{d^4X d^4X'}{V} \chi(X)\phi(r)\chi(X')^*\phi(r')^* \]

\[ \times \text{Tr}(S_F(\vec{r} + \vec{r}') + \vec{X} - \vec{X}')S_F(\vec{r} + \vec{r}' - \vec{X} + \vec{X}')\mathcal{P}_3 \quad (A7) \]

where \( \mathcal{P}_0 = (1 - \gamma^5)/2 \), where we included \( \delta(x^0 - y^0) \) and \( \delta(z^0 - y^0) \) factors for the single time gauge fixing, and the volume normalization, \( \int d^4xd^4yd^4z = 1 \) and \( \int d^4X d^3r/V \).

Now define \( \chi = \chi_0 \exp(-i\mathcal{P}_0 X^\mu) \), with the pointlike \( \phi = \hat{V} \delta^3(\vec{r}) \) as in eq.(A3). We then obtain for eq.(A7) with arbitrary in (out) momenta \( P \) (\( P' \)):

\[ = g_0^2 N_c \chi_0^2 \int d^4Xd^4X' \text{Tr}(S_F(X - X')S_F(X' - X)\mathcal{P}_3) \]

\[ \times e^{-i\mathcal{P}_0 X^\mu} e^{i\mathcal{P}'_0 X'^\mu} \]

(A8)

Note cancellation of \( \hat{V} \) factors. We now use the momentum space Feynman propagator,

\[ S_F(x - z) = \int \frac{d^4q}{(2\pi)^4} \frac{ie^{-iq(x - y)}}{q^2 + m^2} \]

Taking the trace, and omitting a factor of \( g_0^2 N_c |\chi_0|^2 \) which we restore at the end, and integrating over \( X, X' \), we have:

\[ = -\int_{X, X'} \frac{d^4\ell}{(2\pi)^4} \frac{d^4\ell'}{(2\pi)^4} \text{Tr}(\mathcal{P}_0 \frac{\ell}{t_2} \frac{\ell'}{t_2'}) e^{-i\ell \cdot (X - X')} e^{-i\ell' \cdot (X' - X)} \]

\[ \times e^{-i\mathcal{P}_0 X^\mu} e^{i\mathcal{P}'_0 X'^\mu} \]

\[ = -2(2\pi)^4 \delta^4(P - P') \int \frac{d^4\ell}{(2\pi)^4} \ell \cdot (\ell + P) \]

\[ = -2 \int d^4X \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} (\ell^2 - x(1 - x)P^2) \]

Here \( \hat{\ell} = \ell - xP \) and we drop terms odd in \( \hat{\ell} \). We have identified the \( (2\pi)^4 \delta^4(P - P') = \int d^4X \) the volume of space-time. in the \( P = P' \) limit. We perform a Wick rotation: \( \ell_0 \rightarrow i\ell_0 \) so \( d^4\ell \rightarrow id^4\hat{\ell} \) and \( \ell^2 \rightarrow -\hat{\ell}^2 = -\ell^2 \) and \( d^4\ell \rightarrow \pi^2\ell^2 d\ell^2 \), and we impose a cut-off, \( \Lambda \), hence:

\[ \approx \frac{i g^2 N_c}{8\pi^2} |\chi_0|^2 \int d^4X \left( (\Lambda^2 - m^2) + \frac{1}{2} P^2 \ln(\Lambda^2/m^2) \right) \]

(A10)

where we restored the \( g^2 N_c \) factor. This then enters the action as a potential and a kinetic term in the NJL model following [5, 13], upon restoring \( g^2 N_c \),

\[ V = \frac{g^2 N_c}{8\pi^2} (\Lambda^2 - m^2) |\chi|^2 \]

\[ K = \frac{g^2 N_c}{16\pi^2} \ln(\Lambda^2/m^2) \partial_\mu \chi^\dagger \partial_\mu \chi \quad (A11) \]

2. Extended Composite Limit

We are now interested in the non-pointlike composite model. We first require the potential energy as a function of an arbitrary internal field configuration \( \phi(r) \) for a particular value of \( r \).

This is analogous to the Coleman-Weinberg potential, where we would be interested in the potential energy when the VEV of a field \( \phi \) is constrained to a particular value \( \phi_0 \). In Schroedinger picture this corresponds to a vacuum wave-functional, \( \Psi(\phi) \), where \( \int D\phi \Psi^*(\phi)\phi\Psi(\phi) = \phi_0 \). To obtain the potential we compute the expectation of the Hamiltonian by integrating over the fluctuations in \( \phi \) subject to this constraint and minimizing wrt all other parameters in \( \Psi \). From a path integral point of view we start on a time slice \( t = -\infty \) in which \( \langle \phi \rangle = \phi_0 \), integrate over all space-time fluctuations in \( \phi \) and end on \( t = \infty \) with \( \langle \phi \rangle = \phi_0 \). Typically the field VEV is obtained by addition of a source, \( J\phi \) followed by a
Legendre transformation to the shifted field (the source cancels linear terms in \( \phi_0 \)). Then \( i \times (\text{the log of the path integral}) \) is the effective potential as a function of \( \phi_0 \).

Note that in our present problem we have four space-time vertices, \((x, y, z, w)\) s in Fig.(4). We fix the single time gauge with insertion into the integrand of \( \delta(x^0 - y^0) \delta(w^0 - z^0) \). We implement the fixed \( r \) constraint by inserting a \( \tilde{V} \delta^3(\vec{r} - \vec{r}') \) into our integrand, with the two vertices, \( \phi(\vec{r}), \phi(\vec{r}') \).

The loop integral of Fig.(4), as in eqs.(A6,A7), becomes,

\[
= g_0^2 N_c |\chi_0|^2 \int \frac{d^3r}{V} \frac{d^3r'}{V} d^4X d^4X' \times \text{Tr} (S_F(\vec{r} + \vec{r}') + \vec{X} - \vec{X} \gamma S_F(\vec{r} + \vec{r}' - \vec{X} + \vec{X} \gamma \mathcal{P}_3) \times \phi(\vec{r}) \phi(\vec{r}')^\dagger e^{-i\mu \cdot \vec{X}} e^{i\mu \cdot \vec{X}'} \tilde{V} \delta^3(\vec{r} - \vec{r}') = -F \int \frac{d^3r}{V} \frac{d^3\ell}{(2\pi)^3} \frac{d^4P}{(2\pi)^4} \text{Tr} \mathcal{P}_3 \ell \frac{\ell'}{\ell^2 \ell'^2} \times |\phi(\vec{r})|^2 e^{2i\tilde{r} \cdot \vec{r}} e^{2i\tilde{r}' \cdot \vec{r}} (2\pi)^4 \delta^4(\ell - \ell' - P) \]  

(A12)

Here we performed the \( x^0, y^0, w^0 \) and \( z^0 \) time integrals, and,

\[
F = g_0^2 N_c |\chi_0|^2 (2\pi)^4 \delta^4(P - P') = g_0^2 N_c \int d^4X |\chi_0|^2. \]  

(A13)

We treat \( P, P' \) as pure timelike (ie, \( \vec{P} \cdot \vec{x} = 0 \), etc.), do the \( \ell' \) integral, and take the trace:

\[
= 2F \int \frac{d^3r}{V} \frac{d^3\ell}{(2\pi)^3} \frac{1}{\ell^2 - \mu^2 + i\epsilon} |\phi(\vec{r})|^2 e^{2i\tilde{r} \cdot \vec{r}} \]

\[
= 2F \int_0^1 dx \int \frac{d^3r}{V} \frac{d^3\ell}{(2\pi)^3} \frac{1}{\ell^2 - x(1-x)P^2} |\phi(\vec{r})|^2 e^{2i\tilde{r} \cdot \vec{r}} \]

\[
\approx F \int \frac{d^3\ell}{(2\pi)^3} \frac{d^3r}{V} \frac{2}{\ell^2 - P^2 + \cdots} |\phi(\vec{r})|^2 e^{2i\tilde{r} \cdot \vec{r}} \]

(A14)

where \( \tilde{\ell} = \ell - xP \). Now we don’t Wick rotate, and do the \( \ell_0 \) integral by residues. We have:

\[
\int \frac{d^3\ell}{(2\pi)^3} \frac{1}{\ell^2 - \mu^2 + i\epsilon} = \frac{i}{2} \int \frac{d^3\vec{r}}{(2\pi)^3} \frac{1}{(\vec{r}^2 + \mu^2)^{1/2}} \]

(A15)

We perform the \( \ell_0 \) integrals and then the polar angle integrals:

\[
= \frac{i}{2} \int \frac{d^3r}{V} \frac{d^3\ell}{(2\pi)^3} \left[ \frac{2}{|\ell|^2} + \frac{P^2}{2|\ell|^4} \right] |\phi(\vec{r})|^2 e^{2i\tilde{r} \cdot \vec{r}} \]

\[
= i \int \frac{d^3r}{V} \int_\mu^{2\pi} 2\pi d|\ell| \left[ \frac{2}{|\ell|^2} + \frac{P^2}{2|\ell|^4} \right] |\phi(\vec{r})|^2 \frac{\sin(4|\ell| \mu)}{4|\ell|} \]

(A16)

and upon restoring overall factors we have the result:

\[
= ig^2 N_c \int d^4X |\chi_0|^2 \int \frac{d^3r}{V} \frac{d^4P}{8\pi^2} \times \left[ \frac{1}{4|\vec{r}|^2} (\cos(4\mu |\vec{r}|) - \cos(4\Lambda |\vec{r}|)) \right] \]

\[
+ \frac{P^2}{2} \left( \frac{\sin(4\mu |\vec{r}|)}{2\mu |\vec{r}|} - 2\gamma - \ln(16\mu^2 |\vec{r}|^2) \right) \]

(A17)

using

\[
\int_{\mu}^{\Lambda} \frac{\sin(2\mu |\vec{r}|)}{x^2} dx = \frac{\cos 2\mu R - \cos 2\Lambda R}{2R} \]

\[
\frac{1}{2} \int_{\mu}^{\Lambda} \sin(2\mu |\vec{r}|) dx = \frac{\sin(2\mu R) - \sin(2\Lambda R)}{2\mu R} \]

(A18)

and we drop the rapidly oscillating terms such as \( \cos(2\Lambda r) \).

Now we assume small \( \mu r \), i.e., separation between the valence fermions smaller than the IR cut-off \( \mu^{-1} \). Restoring an overall factor of \( g^2 N_c \), we see that eq.(A17) leads to the vacuum loop potential:

\[
= ig^2 N_c \int d^4X |\chi_0|^2 \int \frac{d^3r}{V} \frac{|\phi(\vec{r})|^2}{8\pi^2} \left[ \frac{(\cos(4\mu |\vec{r}|) - \cos(4\Lambda |\vec{r}|))}{4|\vec{r}|^2} \right] \]

\[
\approx i \int d^4X |\chi(X)|^2 \int \frac{d^3r}{V} g^2 N_c |\phi(\vec{r})|^2 \frac{2}{32\pi^2 \mu^2 |\vec{r}|^2} \]

(A19)

where \( \cos(4r) \) oscillates rapidly and averages to zero for small fluctuations in \( r \), and we drop it.

Eq.(A19) is our main result, corresponding to \( i \times \text{(action)} \) and we see the sign in the action is positive, denoting an attractive potential:

\[
V_{\text{loop}} = -\eta r^2 \quad \eta = \frac{g^2 N_c}{32\pi^2} \]  

(A20)

where we have renormalized the kinetic terms \( Z_{\chi} \rightarrow 1 \).

Note the behavior of the kinetic term in eq.(A17) :

\[
\rightarrow i \int d^4X |\partial \chi|^2 \int \frac{d^3r}{V} g^2 N_c |\phi(\vec{r})|^2 \frac{2}{16\pi^2} \times (2 - 2\gamma - \ln(16\mu^2 |\vec{r}|^2)) \]

(A21)

We see that the coefficient and argument of the log matches the logarithmic running in the Nambu-Jona-Lasinio model as in eq.(5), with \( 4\mu^2 r^2 \sim \mu^2 / M^2 \)

\[
\rightarrow i \frac{g^2 N_c}{16\pi^2} \int d^4X |\partial \chi|^2 (c + \ln(4\Lambda^2 / r^2)) \]

(A22)

using the normalization, eq.(58) and to order \( g^2 \). This indicates that the logarithmic RG running of renormalized couplings in the variable \( \ln(r) \) will be given consistently with full RG equations.
3. Quartic Interaction

As in the NJL model, the fermion loops will induce a quartic interaction. By the scale symmetry of the factorized bilocal field, we will have a term in the action

\[-\frac{\lambda}{2} \int d^4X \frac{d^3\vec{r}}{V}(\chi^* \chi)^2 (\phi^* \phi)^2 = \quad (A23)\]

We can infer from the previous calculations that the loop will have four bilocal vertices and takes the form:

\[\lambda = 2g^4 N_c \int d^4X |\chi(0)|^2 \int \frac{d^4 \hat{\ell} \cdot d^3 \vec{r}}{(2\pi)^4 \frac{1}{\ell^4}} \left| \phi(\vec{r}) \right|^4 e^{i \vec{\ell} \cdot \vec{r}} \]

\[= 2ig^4 N_c \int d^4X |\chi(0)|^2 \int \frac{d^3 r}{V} \left| \phi(\vec{r}) \right|^4 \left| \sin(8\vec{\ell} \cdot \vec{r}) \right| \]

\[= 2ig^4 N_c \int d^4X |\chi(0)|^2 \int \frac{d^3 r}{V} \left| \phi(\vec{r}) \right|^4 \times \left( \frac{\sin(8\mu |\vec{r}|)}{8\mu |\vec{r}|} - \gamma - \ln(8\mu |\vec{r}|) \right) \quad (A24)\]

The log evolution matches the result for the pointlike case with $4\mu^2 r^2 \sim \mu^2 / M^2$.

\[\lambda = c_2 + \frac{2N_c g^4}{16\pi^2} \ln \left( \frac{M^2}{\mu^2} \right). \quad (A25)\]

Acknowledgments

I thank Bogdan Dobrescu for discussions, and the Fermi Research Alliance, LLC under Contract No. DE-AC02-07CH11359 with the U.S. Department of Energy, Office of Science, Office of High Energy Physics.