Some results on $\eta$-Yamabe Solitons in 3-dimensional trans-Sasakian manifold

By

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Abstract

The object of the present paper is to study some properties of 3-dimensional trans-Sasakian manifold whose metric is $\eta$-Yamabe soliton. We have studied here some certain curvature conditions of 3-dimensional trans-Sasakian manifold admitting $\eta$-Yamabe soliton. Lastly we construct a 3-dimensional trans-Sasakian manifold satisfying $\eta$-Yamabe soliton.

Key words : Yamabe soliton, $\eta$-Yamabe soliton, $\eta$-Einstein manifold, trans-Sasakian manifold.

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1. Introduction

The concept of Yamabe flow was first introduced by Hamilton [6] to construct Yamabe metrics on compact Riemannian manifolds. On a Riemannian or pseudo-Riemannian manifold $M$, a time-dependent metric $g(\cdot, t)$ is said to evolve by the Yamabe flow if the metric $g$ satisfies the given equation,

$$
\frac{\partial}{\partial t} g(t) = -rg(t), \quad g(0) = g_0,
$$

where $r$ is the scalar curvature of the manifold $M$.

In 2-dimension the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t} g(t) = -2S(g(t))$, where $S$ denotes the Ricci tensor). But in dimension $> 2$ the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not in general.

A Yamabe soliton [1] correspond to self-similar solution of the Yamabe flow, is defined on a Riemannian or pseudo-Riemannian manifold $(M, g)$ by a vector field $\xi$ satisfying the equation,

$$
\frac{1}{2} \mathcal{L}_\xi g = (r - \lambda)g,
$$

where $\mathcal{L}_\xi g$ denotes the Lie derivative of the metric $g$ along the vector field $\xi$, $r$ is the scalar curvature and $\lambda$ is a constant. Moreover a Yamabe soliton is said to
be expanding if $\lambda < 0$, steady if $\lambda = 0$ and shrinking if $\lambda > 0$.

Yamabe solitons on a three-dimensional Sasakian manifold were studied by R. Sharma [14].

Now we define the notion of $\eta$-Yamabe soliton as:

$$\frac{1}{2} \mathcal{L}_\xi g = (r - \lambda)g - \mu \eta \otimes \eta, \quad (1.3)$$

where $\lambda$ and $\mu$ are constants and $\eta$ is a 1-form.

Moreover if $\mu = 0$, the above equation reduces to (1.2) and so the $\eta$-Yamabe soliton becomes Yamabe soliton.

Again,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad (1.4)$$

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{(n-2)}[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y], \quad (1.5)$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(QY,Z)X - g(QX,Z)Y], \quad (1.6)$$

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \quad (1.7)$$

$$C^*(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{n} \left[ \frac{a}{n} + 2b \right] [g(Y,Z)X - g(X,Z)Y], \quad (1.8)$$

where $a$, $b$ are constants,

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Z)QY - g(Y,Z)QX], \quad (1.9)$$

are the Riemannian-Christoffel curvature tensor $R$ [9], the conharmonic curvature tensor $H$ [7], the projective curvature tensor $P$ [15], the concircular curvature tensor $\tilde{C}$ [10], the quasi-conformal curvature tensor $C^*$ [16] and the $W_2$-curvature tensor [10] respectively in a Riemannian manifold $(M^n, g)$, where $Q$ is the Ricci operator, defined by $S(X,Y) = g(QX, Y)$, $S$ is the Ricci tensor, $r = tr(S)$ is the scalar curvature, where $tr(S)$ is the trace of $S$ and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of $M$.

Now in (1.8) if $a = 1$ and $b = -\frac{1}{n-2}$, then we get,

$$C^*(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] - \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] = C(X,Y)Z, \quad (1.10)$$

where $C$ is the conformal curvature tensor [4]. Thus the conformal curvature tensor $C$ is a particular case of the tensor $C^*$. 


In the present paper we study $\eta$-Yamabe soliton on 3-dimensional trans-Sasakian manifolds. The paper is organized as follows: After introduction, section 2 is devoted for preliminaries on 3-dimensional trans-Sasakian manifolds. In section 3, we have studied $\eta$-Yamabe soliton on 3-dimensional trans-Sasakian manifolds. Here we examine if a 3-dimensional trans-Sasakian manifold admits $\eta$-Yamabe soliton, then the scalar curvature is constant and the manifold becomes $\eta$-Einstein. We also characterized the nature of the manifold if the manifold is Ricci symmetric and the Ricci tensor is $\eta$-recurrent. Section 4 deals with the curvature properties of 3-dimensional trans-Sasakian manifold. In this section we have shown the nature of the $\eta$-Yamabe soliton when the manifold is $\xi$- projectively flat, $\xi$-concuircularly flat, $\xi$-conharmonically flat, $\xi$-quasi-conformally flat. Here we have obtained some results on $\eta$-Yamabe soliton satisfying the conditions $R(\xi, X) \cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$.

In last section we gave an example of a 3-dimensional trans-Sasakian manifold satisfying $\eta$-Yamabe soliton.

2. Preliminaries

Let $M$ be a connected almost contact metric manifold with an almost contact metric structure $(\phi, \xi, \eta, g)$ where $\phi$ is a $(1, 1)$ tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \eta \circ \phi = 0, \phi \xi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(X, \phi Y) = -g(\phi X, Y),$$

$$g(X, \xi) = \eta(X),$$

for all vector fields $X, Y \in \chi(M)$.

An almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$ is called a trans-Sasakian structure [8], if $(M \times R, J, G)$ belongs to the class $W_4$ [5], where $J$ is the almost complex structure on $M \times R$ defined by $J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt})$ for all vector fields $X$ on $M$ and smooth functions $f$ on $M \times R$. It can be expressed by the condition [2],

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

for some smooth functions $\alpha, \beta$ on $M$ and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. From the above expression we can write

$$\nabla_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi),$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

For a 3-dimensional trans-Sasakian manifold the following relations hold [3], [11]:

$$2\alpha \beta + \xi \alpha = 0,$$

$$S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi \beta)\eta(X) - X \beta - (\phi X)\alpha.$$
\[ S(X, Y) = \left( \frac{r}{2} + \xi \beta - (\alpha^2 - \beta^2) \right) g(X, Y) - \left( \frac{r}{2} + \xi \beta - 3(\alpha^2 - \beta^2) \right) \eta(X) \eta(Y) \]
\[ - (Y \beta + (\phi Y) \alpha) \eta(X) - (X \beta + (\phi X) \alpha) \eta(Y), \quad (2.10) \]

where \( S \) denotes the Ricci tensor of type \((0, 2)\), \( r \) is the scalar curvature of the manifold \( M \) and \( \alpha, \beta \) are defined as earlier.

For \( \alpha, \beta = \text{constant} \) the following relations hold \([3], [11]\):
\[ S(X, Y) = \left( \frac{r}{2} - (\alpha^2 - \beta^2) \right) g(X, Y) - \left( \frac{r}{2} - 3(\alpha^2 - \beta^2) \right) \eta(X) \eta(Y), \quad (2.11) \]
\[ S(X, \xi) = 2(\alpha^2 - \beta^2) \eta(X), \quad (2.12) \]
\[ R(X, Y) \xi = (\alpha^2 - \beta^2)[\eta(Y) X - \eta(X) Y], \quad (2.13) \]
\[ R(\xi, X) Y = (\alpha^2 - \beta^2)[\eta(X) \xi - \eta(Y) X], \quad (2.14) \]
\[ R(\xi, X) \xi = (\alpha^2 - \beta^2)[\eta(X) \xi - X], \quad (2.15) \]
\[ \eta(R(X, Y) Z) = (\alpha^2 - \beta^2)[\eta(Y) Z \eta(X) - \eta(X) Z \eta(Y)], \quad (2.16) \]

where \( R \) is the Riemannian curvature tensor.
\[ QX = \left( \frac{r}{2} - (\alpha^2 - \beta^2) \right) X - \left( \frac{r}{2} - 3(\alpha^2 - \beta^2) \right) \eta(X) \xi, \quad (2.17) \]

where \( Q \) is the Ricci operator defined earlier.

Again,
\[ (\mathcal{L}_\xi g)(X, Y) = (\nabla_\xi g)(X, Y) - \alpha g(\phi X, Y) + 2\beta g(X, Y) - 2\beta \eta(X) \eta(Y) \]
\[ - \alpha g(X, \phi Y). \]

Then using \((2.3)\), the above equation becomes,
\[ (\mathcal{L}_\xi g)(X, Y) = 2\beta g(X, Y) - 2\beta \eta(X) \eta(Y). \quad (2.18) \]

where \( \nabla \) is the Levi-Civita connection associated with \( g \) and \( \mathcal{L}_\xi \) denotes the Lie derivative along the vector field \( \xi \).

3. \( \eta \)-Yamabe soliton on 3-dimensional trans-Sasakian manifold

Let \( M \) be a 3-dimensional trans-Sasakian manifold. Consider the \( \eta \)-Yamabe soliton on \( M \) as:
\[ \frac{1}{2}(\mathcal{L}_\xi g)(X, Y) = (r - \lambda) g(X, Y) - \mu \eta(X) \eta(Y), \quad (3.1) \]

for all vector fields \( X, Y \) on \( M \).

Then from \((2.18)\) and \((3.1)\), we get,
\[ (r - \lambda - \beta) g(X, Y) = (\mu - \beta) \eta(X) \eta(Y). \quad (3.2) \]

Taking \( Y = \xi \) in the above equation and using \((2.1)\), we have,
\[ (r - \lambda - \mu) \eta(X) = 0. \quad (3.3) \]

Since \( \eta(X) \neq 0 \), so we get,
\[ r = \lambda + \mu. \quad (3.4) \]

Now as both \( \lambda \) and \( \mu \) are constants, \( r \) is also constant.

So we can state the following theorem:
**Theorem 3.1.** If a 3-dimensional trans-Sasakian manifold $M$ admits an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of $M$, then the scalar curvature is constant.

In (3.4) if $\mu = 0$, we get $r = \lambda$ and so (3.1) becomes,

$$\mathcal{L}_\xi g = 0.$$  

Thus $\xi$ is a Killing vector field and consequently $M$ is a 3-dimensional $K$-trans-Sasakian manifold.

Then we have,

**Corollary 3.2.** If a 3-dimensional trans-Sasakian manifold $M$ admits a Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of $M$, then the manifold is a 3-dimensional $K$-trans-Sasakian manifold.

Now from (2.11) and (3.4), we have,

$$S(X, Y) = \left(\frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2)\right)g(X, Y) - \left(\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right)\eta(X)\eta(Y),$$  

(3.5)

for all vector fields $X, Y$ on $M$.

This leads to the following:

**Corollary 3.3.** If a 3-dimensional trans-Sasakian manifold $M$ admits an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of $M$, then the manifold becomes $\eta$-Einstein manifold.

We know,

$$(\nabla_X S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z),$$  

(3.6)

for all vector fields $X, Y, Z$ on $M$ and $\nabla$ is the Levi-Civita connection associated with $g$.

Now replacing the expression of $S$ from (3.5), we obtain,

$$(\nabla_X S)(Y, Z) = -\left[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right][\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z].$$  

(3.7)

for all vector fields $X, Y, Z$ on $M$.

Now let the manifold be Ricci symmetric i.e $\nabla S = 0$.

Then from (3.7), we have,

$$\left[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right][\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z] = 0,$$  

(3.8)

for all vector fields $X, Y, Z$ on $M$.

Taking $Z = \xi$ in the above equation and using (2.7), (2.1), we get,

$$\left[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right][\beta g(\phi X, \phi Y) - \alpha g(\phi X, Y)] = 0$$  

(3.9)
for all vector fields $X, Y$ on $M$.

Hence we get,

$$\lambda + \mu = 6(\alpha^2 - \beta^2).$$

This leads to the following:

**Proposition 3.4.** Let a 3-dimensional trans-Sasakian manifold $M$ admit an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of $M$. If the manifold is Ricci symmetric then $\lambda + \mu = 6(\alpha^2 - \beta^2)$, where $\lambda, \mu, \alpha, \beta$ are constants.

Now if the Ricci tensor $S$ is $\eta$-recurrent, then we have,

$$\nabla S = \eta \otimes S,$$

which implies that,

$$\nabla_X S(Y, Z) = \eta(X) S(Y, Z),$$

for all vector fields $X, Y, Z$ on $M$.

Then using (3.7), we get,

$$-\left[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right][\eta(Z)(\nabla_X \eta)Y + \eta(Y)(\nabla_X \eta)Z] = \eta(X) S(Y, Z),$$

for all vector fields $X, Y, Z$ on $M$.

Using (2.7), the above equation becomes,

$$-\left[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right][\eta(Z)(-\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y))$$

$$+ \eta(Y)(-\alpha g(\phi X, Z) + \beta g(\phi X, \phi Z))] = \eta(X) S(Y, Z).$$

(3.13)

Now taking $Y = \xi, Z = \xi$ and using (2.1), (3.5), the above equation becomes,

$$2(\alpha^2 - \beta^2)\eta(X) = 0.$$

Since $\eta(X) \neq 0$, for all $X$ on $M$, we have,

$$\alpha = \pm \beta.$$ 

(3.14)

This leads to the following:

**Proposition 3.5.** Let a 3-dimensional trans-Sasakian manifold $M$ admit an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of $M$. If the Ricci tensor $S$ is $\eta$-recurrent then $\alpha = \pm \beta$.

Now if the manifold is Ricci symmetric and the Ricci tensor $S$ is $\eta$-recurrent, then using (3.14) in $\lambda + \mu = 6(\alpha^2 - \beta^2)$ and from (3.4), we have the following:

**Proposition 3.6.** Let a 3-dimensional trans-Sasakian manifold $M$ admit an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field of $M$. If the manifold is Ricci symmetric and the Ricci tensor $S$ is $\eta$-recurrent then the manifold becomes flat.

Let an $\eta$-Yamabe soliton be defined on a 3-dimensional trans-Sasakian manifold $M$ as,

$$\frac{1}{2} \mathcal{L}_V g = (r - \lambda) g - \mu \eta \otimes \eta,$$

(3.15)
where $\mathcal{L}_V g$ denotes the Lie derivative of the metric $g$ along a vector field $V$, $r$ is defined as (1.2) and $\lambda, \mu$ are defined as (1.3).

Let $V$ be pointwise co-linear with $\xi$ i.e. $V = b\xi$ where $b$ is a function on $M$. Then the equation (3.15) becomes,

$$ (\mathcal{L}_b g)(X, Y) = 2(r - \lambda)g(X, Y) - 2\mu \eta(X)\eta(Y), $$

(3.16)

for any vector fields $X, Y$ on $M$.

Applying the property of Lie derivative and Levi-Civita connection we have,

$$ b(g(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) = 2(r - \lambda)g(X, Y) - 2\mu \eta(X)\eta(Y). $$

(3.17)

Using (2.6) and (2.3), the above equation reduces to,

$$ 2b \beta[g(X, Y) - \eta(X)\eta(Y)] + (Xb)\eta(Y) + (Yb)\eta(X) = 2(r - \lambda)g(X, Y) - 2\mu \eta(X)\eta(Y). $$

(3.18)

Now taking $Y = \xi$ in the above equation and using (2.1), (2.4), we obtain,

$$ Xb + (\xi b)\eta(X) = 2(r - \lambda)\eta(X) - 2\mu \eta(X). $$

(3.19)

Again taking $X = \xi$, we get,

$$ \xi b = r - \lambda - \mu. $$

(3.20)

Then using (3.20), the equation (3.19) becomes,

$$ Xb = (r - \lambda - \mu)\eta(X). $$

(3.21)

Applying exterior differentiation in (3.21), we have,

$$ (r - \lambda - \mu)d\eta = 0. $$

(3.22)

Since $d\eta \neq 0$ [3], the above equation gives,

$$ r = \lambda + \mu. $$

(3.23)

Using (3.23), the equation (3.21) becomes,

$$ Xb = 0, $$

(3.24)

which implies that $b$ is constant.

So we can state the following theorem:

**Theorem 3.7.** Let $M$ be a 3-dimensional trans-Sasakian manifold admitting an $\eta$-Yamabe soliton $(g, V)$, $V$ being a vector field on $M$. If $V$ is pointwise co-linear with $\xi$, then $V$ is a constant multiple of $\xi$, where $\xi$ being the Reeb vector field of $M$.

Using (3.23), the equation (3.15) becomes,

$$ (\mathcal{L}_V g)(X, Y) = 2\mu[g(X, Y) - \eta(X)\eta(Y)], $$

(3.25)

for all vector fields $X, Y, Z$ on $M$.

Then we have,

**Corollary 3.8.** Let $M$ be a 3-dimensional trans-Sasakian manifold admitting an $\eta$-Yamabe soliton $(g, V)$, $V$ being a vector field on $M$ which is pointwise co-linear with $\xi$, where $\xi$ being the Reeb vector field of $M$. $V$ is a Killing vector field iff the
soliton reduces to a Yamabe soliton.

From the equation (3.5), we get,

$$QX = \left(\frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2)\right)X - \left(\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right)\eta(X)\xi, \quad (3.26)$$

for any vector field $X$ on $M$ and $Q$ is defined as earlier.

We know,

$$\nabla_\xi QX = \nabla_\xi QX - Q(\nabla_\xi X), \quad (3.27)$$

for any vector field $X$ on $M$.

Then using (3.26), the equation (3.27) becomes,

$$\nabla_\xi QX = -\left[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right](\nabla_\xi \eta)X\xi. \quad (3.28)$$

Using (2.7) in the above equation, we get,

$$\nabla_\xi QX = 0, \quad (3.29)$$

for any vector field $X$ on $M$.

Hence $Q$ is parallel along $\xi$.

Again from (3.7), we obtain,

$$\nabla_\xi S(X, Y) = -\left[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\right][\eta(Y)(\nabla_\xi \eta)X + \eta(X)(\nabla_\xi \eta)Y], \quad (3.30)$$

for any vector fields $X, Y$ on $M$.

Using (2.7) in the above equation, we get,

$$\nabla_\xi S(X, Y) = 0, \quad (3.31)$$

for any vector fields $X, Y$ on $M$.

Hence $S$ is parallel along $\xi$.

So we can state the following theorem:

**Theorem 3.9.** Let $M$ be a 3-dimensional trans-Sasakian manifold admitting an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field on $M$. Then $Q$ and $S$ are parallel along $\xi$, where $Q$ is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$ and $S$ is the Ricci tensor of $M$.

4. **Curvature properties on 3-dimensional trans-Sasakian manifold admitting $\eta$-Yamabe soliton**

   From the definition of projective curvature tensor (1.6), defined on a 3-dimensional trans-Sasakian manifold and using the property $g(QX, Y) = S(X, Y)$, we have,

   $$P(X, Y)Z = R(X, Y)Z - \frac{1}{2}[S(Y, Z)X - S(X, Z)Y], \quad (4.1)$$
for any vector fields $X, Y, Z$ on $M$.

Putting $Z = \xi$ in the above equation and using (2.13) and (3.5), we obtain,

$$P(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - \frac{1}{2}[2(\alpha^2 - \beta^2)\eta(Y)X - 2(\alpha^2 - \beta^2)\eta(X)Y],$$

which implies that,

$$P(X, Y)\xi = 0.$$  \hfill (4.2)

So we can state the following theorem:

**Theorem 4.1.** A 3-dimensional trans-Sasakian manifold $M$ admitting $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field on $M$ is $\xi$- projectively flat.

From the definition of concircular curvature tensor (1.7), defined on a 3-dimensional trans-Sasakian manifold, we have,

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{6}[g(Y, Z)X - g(X, Z)Y],$$

for any vector fields $X, Y, Z$ on $M$.

Putting $Z = \xi$ in the above equation and using (2.4) and (2.13), we obtain,

$$\tilde{C}(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - \frac{r}{6}[\eta(Y)X - \eta(X)Y].$$  \hfill (4.5)

Now using (3.4), we get,

$$\tilde{C}(X, Y)\xi = [(\alpha^2 - \beta^2) - \frac{\lambda + \mu}{6}] [\eta(Y)X - \eta(X)Y].$$  \hfill (4.6)

This implies that $\tilde{C}(X, Y)\xi = 0$ iff $\lambda + \mu = 6(\alpha^2 - \beta^2)$.

So we can state the following theorem:

**Theorem 4.2.** A 3-dimensional trans-Sasakian manifold $M$ admitting $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field on $M$ is $\xi$-concircularly flat iff $\lambda + \mu = 6(\alpha^2 - \beta^2)$.

Now if the Ricci tensor $S$ is $\eta$- recurrent then using (3.14) in (4.5), we have,

**Corollary 4.3.** Let $M$ be a 3-dimensional trans-Sasakian manifold admitting an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field on $M$. If the manifold is $\xi$-concircularly flat and the Ricci tensor is $\eta$- recurrent then the manifold $M$ becomes flat.

From the definition of conharmonic curvature tensor (1.5), defined on a 3-dimensional trans-Sasakian manifold, we have,

$$H(X, Y)Z = R(X, Y)Z - [g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y],$$

for any vector fields $X, Y, Z$ on $M$.

Putting $Z = \xi$ in the above equation and using (2.4), (2.13), (3.5) and (3.26), the
above equation becomes,

\[ H(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - \frac{\lambda + \mu}{2} + (\alpha^2 - \beta^2)\eta(Y)X - \eta(X)Y. \]  

Hence we get,

\[ H(X, Y)\xi = -\frac{\lambda + \mu}{2}[\eta(Y)X - \eta(X)Y]. \]  

This implies that \( H(X, Y)\xi = 0 \) iff \( \lambda + \mu = 0 \).

So we can state the following theorem:

**Theorem 4.4.** A 3-dimensional trans-Sasakian manifold \( M \) admitting \( \eta \)-Yamabe soliton \((g, \xi)\), \( \xi \) being the Reeb vector field on \( M \) is \( \xi \)-conharmonically flat iff \( \lambda + \mu = 0 \).

From the definition of quasi-conformal curvature tensor (1.8), defined on a 3-dimensional trans-Sasakian manifold, we have,

\[
C^*(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] - \frac{r}{3}a^2 + 2b[g(Y, Z)X - g(X, Z)Y],
\]  

for any vector fields \( X, Y, Z \) on \( M \) and \( a, b \) are constants.

Putting \( Z = \xi \) in the above equation and using (2.4), (2.13), (3.4), (3.5) and (3.26), the above equation becomes,

\[
C^*(X, Y)\xi = a(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y]
+ b[\frac{\lambda + \mu}{2} + (\alpha^2 - \beta^2)][\eta(Y)X - \eta(X)Y]
- \frac{\lambda + \mu}{3}\left[\frac{a}{2} + 2b\right][\eta(Y)X - \eta(X)Y].
\]  

Hence we have,

\[
C^*(X, Y)\xi = [a(\alpha^2 - \beta^2) + b\frac{\lambda + \mu}{2} + (\alpha^2 - \beta^2)] - \frac{\lambda + \mu}{3}\left[\frac{a}{2} + 2b\right][\eta(Y)X - \eta(X)Y].
\]  

This implies that \( C^*(X, Y)\xi = 0 \) iff \( a(\alpha^2 - \beta^2) + b\frac{\lambda + \mu}{2} + (\alpha^2 - \beta^2) - \frac{\lambda + \mu}{3}\left[\frac{a}{2} + 2b\right] = 0 \).

Then by simplifying, we obtain, \( C^*(X, Y)\xi = 0 \) if \( (a + b)[(\alpha^2 - \beta^2) - \frac{\lambda + \mu}{6}] = 0 \), i.e either \( a + b = 0 \) or \( \lambda + \mu = 6(\alpha^2 - \beta^2) \).

So we can state the following theorem:

**Theorem 4.5.** A 3-dimensional trans-Sasakian manifold \( M \) admitting \( \eta \)-Yamabe soliton \((g, \xi)\), \( \xi \) being the Reeb vector field on \( M \) is \( \xi \)-quasi-conformally flat iff either \( a + b = 0 \) or \( \lambda + \mu = 6(\alpha^2 - \beta^2) \).
Now if the Ricci tensor $S$ is $\eta$-recurrent then using (3.14) in (4.12), we get,

$$C^*(X, Y)\xi = -\frac{a + b}{6}(\lambda + \mu)[\eta(Y)X - \eta(X)Y].$$

(4.13)

Hence using (3.4) in (4.13), we have,

**Corollary 4.6.** Let a 3-dimensional trans-Sasakian manifold $M$ admit an $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field on $M$. If the manifold is $\xi$-quasi-conformally flat and the Ricci tensor is $\eta$-recurrent then the manifold $M$ becomes flat, provided $a + b \neq 0$.

We know,

$$R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z),$$

(4.14)

for any vector fields $X, Y, Z$ on $M$.

Now let the manifold be $\xi$-semi symmetric, i.e $R(\xi, X) \cdot S = 0$.

Then from (4.14), we have,

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0,$$

(4.15)

for any vector fields $X, Y, Z$ on $M$.

Using (2.14), the above equation becomes,

$$S((\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X), Z) + S(Y, (\alpha^2 - \beta^2)(g(X, Z)\xi - \eta(Z)X)) = 0.$$  

(4.16)

Replacing the expression of $S$ from (3.5) and simplifying we get,

$$(\alpha^2 - \beta^2)[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)][g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0.$$  

(4.17)

Taking $Z = \xi$ in the above equation and using (2.1), (2.4), we obtain,

$$(\alpha^2 - \beta^2)[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)][g(X, Y) - \eta(x)\eta(Y)] = 0,$$

(4.18)

for any vector fields $X, Y$ on $M$.

Using (2.2), the above equation becomes,

$$(\alpha^2 - \beta^2)[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)]g(\phi X, \phi Y) = 0,$$

(4.19)

for any vector fields $X, Y$ on $M$.

Hence we get,

$$(\alpha^2 - \beta^2)[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)] = 0.$$  

(4.20)

Then either $(\alpha^2 - \beta^2) = 0$, or $\lambda + \mu = 6(\alpha^2 - \beta^2)$.

So we can state the following theorem:

**Theorem 4.7.** If a 3-dimensional trans-Sasakian manifold $M$ admitting $\eta$-Yamabe soliton $(g, \xi)$, $\xi$ being the Reeb vector field on $M$ is $\xi$-semi symmetric then either $(\alpha^2 - \beta^2) = 0$, or $\lambda + \mu = 6(\alpha^2 - \beta^2)$. 

From the definition of \( W_2 \)-curvature tensor \((1.9)\), defined on a 3-dimensional trans-Sasakian manifold, we have,
\[
W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2}[g(X, Z)QY - g(Y, Z)QX],
\]
for any vector fields \( X, Y, Z \) on \( M \).

Again we know,
\[
W_2(\xi, X) \cdot S = S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z),
\]
for any vector fields \( X, Y, Z \) on \( M \).

Replacing the expression of \( S \) from \((3.5)\), we get on simplifying,
\[
W_2(\xi, X) \cdot S = \frac{1}{2} \left[ \frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2) \right] [\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)] [g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)] = 0,
\]
for any vector fields \( X, Y, Z \) on \( M \).

Let in this manifold \( M \), \( W_2(\xi, X) \cdot S = 0 \).

Then from \((4.24)\), we get,
\[
\left[ \frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2) \right] \left[ \frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2) \right] [g(X,Y)\eta(Z) - 2\eta(X)\eta(Y)\eta(Z)] = 0,
\]
for any vector fields \( X, Y, Z \) on \( M \).

Taking \( Z = \xi \) in the above equation and using \((2.1), (2.4)\), we obtain,
\[
\left[ \frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2) \right] \left[ \frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2) \right] [g(X,Y) - \eta(X)\eta(Y)] = 0,
\]
for any vector fields \( X, Y \) on \( M \).

Using \((2.2)\), the above equation becomes,
\[
\left[ \frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2) \right] \left[ \frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2) \right] g(\phi X, \phi Y) = 0,
\]
for any vector fields \( X, Y \) on \( M \).

Hence we get,
\[
\left[ \frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2) \right] \left[ \frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2) \right] = 0.
\]

Then either \( \lambda + \mu = 2(\alpha^2 - \beta^2) \), or \( \lambda + \mu = 6(\alpha^2 - \beta^2) \).

So we can state the following theorem:
Theorem 4.8. If a 3-dimensional trans-Sasakian manifold \( M \) admits an \( \eta \)-Yamabe soliton \((g, \xi)\), \( \xi \) being the Reeb vector field on \( M \) and satisfies \( W_2(\xi, X) \cdot S = 0 \), where \( W_2 \) is the \( W_2 \)-curvature tensor and \( S \) is the Ricci tensor then either 
\[ \lambda + \mu = 2(\alpha^2 - \beta^2), \] or 
\[ \lambda + \mu = 6(\alpha^2 - \beta^2). \]

Now if the Ricci tensor \( S \) is \( \eta \)-recurrent then using (3.14) in (4.28) and from (3.4), we have,

Corollary 4.9. If a 3-dimensional trans-Sasakian manifold \( M \) admits an \( \eta \)-Yamabe soliton \((g, \xi)\), \( \xi \) being the Reeb vector field on \( M \) and satisfies 
\[ W_2(\xi, X) \cdot S = 0, \]
where \( W_2 \) is the \( W_2 \)-curvature tensor and \( S \) is the Ricci tensor which is \( \eta \)-recurrent, then the manifold becomes flat.

5. Example of a 3-dimensional trans-Sasakian manifold admitting \( \eta \)-Yamabe soliton:

In this section we give an example of a 3-dimensional trans-Sasakian manifold with \( \alpha, \beta \) = constant.
We consider the 3-dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\} \), where \( (x, y, z) \) are standard coordinates in \( \mathbb{R}^3 \). Let \( e_1, e_2, e_3 \) be a linearly independent system of vector fields on \( M \) given by,
\[ e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}. \]

Let \( g \) be the Riemannian metric defined by,
\[ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0. \]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \), for any \( Z \in \chi(M) \), where \( \chi(M) \) is the set of all differentiable vector fields on \( M \) and \( \phi \) be the \((1, 1)\)-tensor field defined by,
\[ \phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = 0. \]

Then, using the linearity of \( \phi \) and \( g \), we have
\[ \eta(e_3) = 1, \phi^2(Z) = -Z + \eta(Z)e_3 \]
and
\[ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W), \]
for any \( Z, W \in \chi(M) \).

Let \( \nabla \) be the Levi-Civita connection with respect to the Riemannian metric \( g \).
Then we have,
\[ [e_1, e_2] = 0, [e_2, e_3] = -e_2, [e_1, e_3] = -e_1. \]

The connection \( \nabla \) of the metric \( g \) is given by,
\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\
- g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),
\]
which is known as Koszul’s formula. Using Koszul’s formula, we can easily calculate,
\[ \nabla_{e_1}e_3 = -e_1, \quad \nabla_{e_2}e_3 = -e_2, \quad \nabla_{e_3}e_3 = 0, \]
\[ \nabla_{e_1}e_1 = e_3, \quad \nabla_{e_2}e_1 = 0, \quad \nabla_{e_3}e_1 = 0, \]
\[ \nabla_{e_1}e_2 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_3}e_2 = 0. \]

We see that,
\[
(\nabla_{e_1}\phi)e_1 = \nabla_{e_1}\phi e_1 - \phi \nabla_{e_1}e_1 = -\nabla_{e_1}e_2 - \phi e_3 = 0
= 0(g(e_1, e_1)e_3 - \eta(e_1)e_1) - 1(g(\phi e_1, e_1)e_3 - \eta(e_1)\phi e_1). \tag{5.1}
\]
\[
(\nabla_{e_1}\phi)e_2 = \nabla_{e_1}\phi e_2 - \phi \nabla_{e_1}e_2 = \nabla_{e_1}e_1 - 0 = e_3
= 0(g(e_1, e_2)e_3 - \eta(e_2)e_1) - 1(g(\phi e_1, e_2)e_3 - \eta(e_2)\phi e_1). \tag{5.2}
\]
\[
(\nabla_{e_1}\phi)e_3 = \nabla_{e_1}\phi e_3 - \phi \nabla_{e_1}e_3 = 0 + \phi e_1 = -e_2
= 0(g(e_1, e_3)e_3 - \eta(e_3)e_1) - 1(g(\phi e_1, e_3)e_3 - \eta(e_3)\phi e_1). \tag{5.3}
\]

Hence from (5.1), (5.2) and (5.3) we can see that the manifold \( M \) satisfies (2.5) for \( X = e_1, \alpha = 0, \beta = -1, \) and \( e_3 = \xi. \) Similarly, it can be shown that for \( X = e_2 \) and \( X = e_3 \) the manifold also satisfies (2.5) for \( \alpha = 0, \beta = -1, \) and \( e_3 = \xi. \) Hence the manifold \( M \) is a 3-dimensional trans-Sasakian manifold of type (0,-1).

Also, from the definition of the Riemannian curvature tensor \( R \) (1.4), we get,
\[
R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_3)e_3 = -e_1, \quad R(e_2, e_1)e_1 = -e_2,
\]
\[
R(e_2, e_3)e_3 = -e_2, \quad R(e_3, e_1)e_1 = -e_3, \quad R(e_3, e_2)e_2 = -e_3.
\]

Then, the Ricci tensor \( S \) is given by;
\[
S(e_1, e_1) = -2, \quad S(e_2, e_2) = -2, \quad S(e_3, e_3) = -2. \tag{5.4}
\]
Then the scalar curvature is,
\[
r = -6. \tag{5.5}
\]

From (3.5), we have,
\[
S(e_1, e_1) = \frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2), \quad S(e_2, e_2) = \frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2), \quad S(e_3, e_3) = 2(\alpha^2 - \beta^2). \tag{5.6}
\]

Then from (5.4) and (5.6), we get,
\[
\frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2) = -2, \tag{5.7}
\]
and
\[
\alpha^2 - \beta^2 = -1. \tag{5.8}
\]

Using (5.8) in (5.7), we obtain,
\[
\lambda + \mu = -6. \tag{5.9}
\]
Then the value of \( \lambda + \mu \) in (5.9) is same as the value of \( r \) in (5.5) and so it satisfies (3.4).
Hence $g$ defines an $\eta$-Yamabe soliton on a 3-dimensional trans-Sasakian manifold $M$.

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