On A Pair Of Universal Weak Inverse Property Loops *†

Tèmítópé Gbóláhàn Jàíyéólà‡
Department of Mathematics,
Obafemi Awolowo University, Ile Ife, Nigeria.
jaiyeolatemitope@yahoo.com, tjayeola@oauife.edu.ng

Abstract

A new condition called $T$ condition is introduced for the first time and used to study a pair of isotopic loops. Under this condition, a loop in the pair is a WIPL if and only if the other loop is a WIPL. Furthermore, such WIPLs are isomorphic. The translation elements $f$ and $g$ of a CIPL with the $T$ condition (such that its $f$, $g$-isotope is an automorphic inverse property loop) are found to be alternative, flexible, centrum and equal elements. A necessary and sufficient condition for a pair WIPLs with a weak $T$ condition to be isomorphic is shown. A CIPL and an isomorph are observed to have this weak $T$ condition.

1 Introduction

Michael K. Kinyon [15] gave a talk on Osborn Loops and proposed the open problem: ”Is every Osborn Loop universal?” which is obviously true for universal WIP loops and universal CIP loops. Our aim in this work is to introduced for the first time a new condition called the $T$ condition and use it to study a pair of isotopic loops. The work is a special case of that of Osborn [18] for a WIPLs and we want to see if the result of Artzy [3] that isotopic CIP loops are isomorphic is true for WIP loops or some specially related WIPLs (i.e special isotopes). The special relation here is the $T$ condition. But before these, we shall take few basic definitions and concepts in loop theory which are needed here. For more definitions, readers may check [19], [6], [7], [8], [20] and [9].

Let $L$ be a non-empty set. Define a binary operation $\cdot$ on $L$: If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations:

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

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‡All correspondence to be addressed to this author.
have unique solutions for $x$ and $y$ respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. For each $x \in L$, the elements $x^\rho, x^\lambda \in L$ such that $xx^\rho = e = x^\lambda x$ are called the right, left inverses of $x$ respectively. $L$ is called a weak inverse property loop (WIPL) if and only if it obeys the weak inverse property (WIP);

$$xy \cdot z = e \implies x \cdot yz = e \quad \forall \, x, y, z \in L$$

while $L$ is called a cross inverse property loop (CIPL) if and only if it obeys the cross inverse property (CIP);

$$xy \cdot x^\rho = y.$$ 

According to [4], the WIP is a generalization of the CIP. The latter was introduced and studied by R. Artzy [2] and [3] while the former was introduced by J. M. Osborn [18] who also investigated the isotopy invariance of the WIP. Huthnance Jr. [10] did so as well and proved that the holomorph of a WIPL is a WIPL. A loop property is called universal (or at times a loop is said to be universal relative to a particular property) if the loop has the property and every loop isotope of such a loop possesses such a property. A universal WIPL is called an Osborn loop in Huthnance Jr. [10] but this is different from the Osborn loop of Kinyon [15] and Basarab. The Osborn loops of Kinyon and Basarab were named generalised Moufang loops or M-loops by Huthnance Jr. [10] where he investigated the structure of their holomorphs while Basarab [5] studied Osborn loops that are G-loops. Also, generalised Moufang loops or M-loops of Huthnance Jr. are different from those of Basarab. After Osborn’s study of universal WIP loops, Huthnance Jr. still considered them in his thesis and did an elaborate study by comparing the similarities between properties of Osborn loops (universal WIPL) and generalised Moufang loops. He was able to draw conclusions that the latter class of loops is large than the former class while in a WIPL the two are the same.

But in this present work, a new condition called $T$ condition is introduced for the first time and used to study a pair of isotopic loops. Under this condition, a loop in the pair is a WIPL if and only if the other loop is a WIPL. Furthermore, such WIPLs are isomorphic. The translation elements $f$ and $g$ of a CIPL with the $T$ condition (such that its $f, g$-isotope is an automorphic inverse property loop) are found to be alternative, flexible, centrum and equal elements. A necessary and sufficient condition for a pair WIPLs with a weak $T$ condition to be isomorphic is shown. A CIPL and an isomorph are observed to have this weak $T$ condition.

### 2 Preliminaries

**Definition 2.1** Let $(L, \cdot)$ and $(G, \circ)$ be two distinct loops. The triple $\alpha = (U, V, W) : (L, \cdot) \to (G, \circ)$ such that $U, V, W : L \to G$ are bijections is called a loop isotopism $\iff$ $xU \circ yV = (x \cdot y)W \quad \forall \, x, y \in L$. Hence, $L$ and $G$ are said to be isotopic whence, $G$ is an isotope of $L$. 


If \( W = I \), then \( \alpha \) is called a principal isotopism and in addition, if \( U = R_g \) and \( V = L_f \) then \( \alpha \) is called an \( f, g \)-principal isotopism with the ordered pair \((g, f)\) called the pair of translation elements of the principal isotope.

**Definition 2.2** Let \( L \) be a loop. A mapping \( \alpha \in S(L) \) (where \( S(L) \) is the group of all bijections on \( L \)) which obeys the identity \( x^\rho = [(x\alpha)^\rho]\alpha \) is called a weak right inverse permutation. Their set is represented by \( S^\rho(L) \).

Similarly, if \( \alpha \) obeys the identity \( x^\lambda = [(x\alpha)^\lambda]\alpha \) it is called a weak left inverse permutation. Their set is represented by \( S^\lambda(L) \).

If \( \alpha \) satisfies both, it is called a weak inverse permutation. Their set is represented by \( S'(L) \).

It can be shown that \( \alpha \in S(L) \) is a weak right inverse if and only if it is a weak left inverse permutation. So, \( S'(L) = S^\rho(L) = S^\lambda(L) \).

**Remark 2.1** Every permutation of order 2 that preserves the right(left) inverse of each element in a loop is a weak right (left) inverse permutation.

**Example 2.1** If \( L \) is an extra loop, the left and right inner mappings \( L(x, y) \) and \( R(x, y) \forall x, y \in L \) are automorphisms of orders 2 \([16]\). Hence, they are weak inverse permutations by Remark 2.1.

Throughout, we shall employ the use of the bijections; \( J^\rho : x \mapsto x^\rho \), \( J^\lambda : x \mapsto x^\lambda \), \( L_x : y \mapsto xy \) and \( R_x : y \mapsto yx \) for a loop and the bijections; \( J'_\rho : x \mapsto x'^\rho \), \( J'_\lambda : x \mapsto x'^\lambda \), \( L'_x : y \mapsto xy \) and \( R'_x : y \mapsto yx \) for its loop isotope. If the identity element of a loop is \( e \) then that of the isotope shall be denoted by \( e' \).

**Lemma 2.1** In a loop, the set of weak inverse permutations that commute form an abelian group.

**Remark 2.2** Applying Lemma 2.1 to extra loops and considering Example 2.1, it will be observed that in an extra loop \( L \), the Boolean groups \( \text{Inn}_\lambda(L), \text{Inn}_\rho \leq S'(L) \) . \( \text{Inn}_\lambda(L) \) and \( \text{Inn}_\rho(L) \) are the left and right inner mapping groups respectively. They have been investigated in \([17]\), and \([16]\). This deductions can’t be drawn for CC-loops despite the fact that the left (right) inner mappings commute and are automorphisms. And this is as a result of the fact that the left(right) inner mappings are not of exponent 2.

**Definition 2.3** \((T\text{-conditions})\)

Let \((G, \cdot)\) and \((H, \circ)\) be two distinct loops that are isotopic under the triple \((A, B, C)\). \((G, \cdot)\) obeys the \( T_1 \) condition if and only if \( A = B \). \((G, \cdot)\) obeys the \( T_2 \) condition if and only if
\[
J'_\rho = C^{-1}J_\rho B = A^{-1}J_\rho C.
\]
\((G, \cdot)\) obeys the \( T_3 \) condition if and only if
\[
J'_\lambda = C^{-1}J_\lambda A = B^{-1}J_\lambda C.
\]
So, \((G, \cdot)\) obeys the \(T\) condition if and only if it obey \(T_1\) and \(T_2\) conditions or \(T_1\) and \(T_3\) conditions since \(T_2 \equiv T_3\). Furthermore, \((G, \cdot)\) obeys the \(T_{31}\) condition if and only if \(J'_\rho = C^{-1}J_\rho B\), \((G, \cdot)\) obeys the \(T_{22}\) condition if and only if \(J'_\rho = A^{-1}J_\rho C\), \((G, \cdot)\) obeys the \(T_{31}\) condition if and only if \(J'_\lambda = C^{-1}J_\lambda A\) and \((G, \cdot)\) obeys the \(T_{32}\) condition if and only if \(J'_\lambda = B^{-1}J_\lambda C\).

So when \((H, \circ) = (G, \circ)\) is an \(f, g\)-principal isotope of \((G, \cdot)\) under the triple \((R_g, L_f, I)\): \(T_1\) condition \(\equiv R_g = L_f\), \(T_2\) condition \(\equiv J'_\rho = J_\rho L_f = R_g^{-1}J_\rho\), \(T_3\) condition \(\equiv J'_\lambda = J_\lambda R_g = L_f^{-1}J_\lambda\), \(T_{21}\) condition \(\equiv J'_\rho = J_\rho L_f\), \(T_{22}\) condition \(\equiv J'_\rho = R_g^{-1}J_\rho\), \(T_{31}\) condition \(\equiv J'_\lambda = J_\lambda R_g\) and \(T_{32}\) condition \(\equiv J'_\lambda = L_f^{-1}J_\lambda\).

In case \((G, \cdot)\) and \((H, \circ)\) are two distinct non-isotopic loops, then they are said to obey the weak \(T_{21}\) condition if and only if \(J'_\rho = A^{-1}J_\rho A\) or \(J'_\lambda = A^{-1}J_\lambda A\) for some \(A : G \rightarrow H\) where \(J'_\rho\), \(J_\rho\) and \(J'_\lambda\), \(J_\lambda\) still retain their earlier definitions as right and left inverse mappings on \(G\) and \(H\) respectively.

It must here by be noted that the \(T\)-conditions refer to a pair of isotopic loops at a time. This statement might be omitted at times. That is whenever we say a loop \((G, \cdot)\) has the \(T\)-condition, then this is relative to some isotope \((H, \circ)\) of \((G, \cdot)\).

A loop \(L\) is called a left inverse property loop(LIPL) if it obeys the left inverse property (LIP):
\[x^\lambda(xy) = y \forall x, y \in L\]
and a right inverse property loop(RIPL) if it obeys the right inverse property (RIP):
\[(xy)^\rho = x \forall x, y \in L\]

If it has both properties, then it is said to have the inverse property (IP) hence called an inverse property loop (IPL).

**Lemma 2.2** ([19]) Let \(L\) be a loop. The following are equivalent.

1. \(L\) is a WIPL
2. \(y(xy)^\rho = x^\rho \forall x, y \in L\).
3. \((xy)^\lambda x = y^\lambda \forall x, y \in L\).

**Lemma 2.3** Let \(L\) be a loop. The following are equivalent.

1. \(L\) is a WIPL
2. \(R_yJ_\rho L_y = J_\rho \forall y \in L\).
3. \(L_xJ_\lambda R_x = J_\lambda \forall x \in L\).
3 Main Result

3.1 Isotopes of Weak Inverse Property Loops

Theorem 3.1 Let \((G, \cdot)\) and \((H, \circ)\) be two distinct loops that are isotopic under the triple \((A, B, C)\).

1. If the pair of \((G, \cdot)\) and \((H, \circ)\) obey the \(T\) condition, then \((G, \cdot)\) is a WIPL if and only if \((H, \circ)\) is a WIPL.

2. If \((G, \cdot)\) and \((H, \circ)\) are WIPLs, then

\[
J_{\lambda}R_{\rho}J_{\lambda}B = C J_{\lambda}' R'_{xA} J'_{\rho} \quad \text{and} \quad J_{\rho}L_{x} J_{\lambda}A = C J_{\rho}' L'_{xB} J'_{\lambda} \quad \forall \ x \in G.
\]

Proof

1. \((A, B, C) : G \rightarrow H\) is an isotopism \(\iff\) \(x A \circ y B = (x \cdot y) C \iff y B L'_{xA} = y L_{x} C \iff \)

\[
L_{x} = B L'_{xA} C^{-1}
\]

Also, \((A, B, C) : G \rightarrow H\) is an isotopism \(\iff\) \(x A R'_{yB} = x R_{y} C \iff A R'_{yB} = R_{y} C \iff R'_{yB} = A^{-1} R_{y} C \iff \)

\[
R_{y} = A R'_{yB} C^{-1}
\]

Applying (1) and (2) to Lemma 2.3 separately, we have:

\[
R_{y} J_{\rho} L_{y} = J_{\rho}, \quad L_{x} J_{\lambda} R_{x} = J_{\lambda} \Rightarrow (A R'_{xB} C^{-1}) J_{\rho} (B L'_{xA} C^{-1}) = J_{\rho}, \quad (B L'_{xA} C^{-1}) J_{\lambda} (A R'_{xB} C^{-1}) = J_{\lambda} \Rightarrow \]

\[
A R'_{xB} (C^{-1} J_{\rho} B) L'_{xA} C^{-1} = J_{\rho}, \quad B L'_{xA} (C^{-1} J_{\lambda} A) R'_{xB} C^{-1} = J_{\lambda} \Rightarrow \]

\[
R'_{xB} (C^{-1} J_{\rho} B) L'_{xA} = A^{-1} J_{\rho} C, \quad L'_{xA} (C^{-1} J_{\lambda} A) R'_{xB} = B^{-1} J_{\lambda} C.
\]

Let \(J'_{\rho} = C^{-1} J_{\rho} B = A^{-1} J_{\rho} C, \ J'_{\lambda} = C^{-1} J_{\lambda} A = B^{-1} J_{\lambda} C\). Then, from (3) and by Lemma 2.3, \(H\) is a WIPL if \(xB = xA\) and \(J'_{\rho} = C^{-1} J_{\rho} B = A^{-1} J_{\rho} C\) or \(xA = xB\) and \(J'_{\lambda} = C^{-1} J_{\lambda} A = B^{-1} J_{\lambda} C\) \(\iff\) \(B = A\) and \(J'_{\rho} = C^{-1} J_{\rho} B = A^{-1} J_{\rho} C\) or \(A = B\) and \(J'_{\lambda} = C^{-1} J_{\lambda} A = B^{-1} J_{\lambda} C\) \(\iff\) \(A = B\) and \(J'_{\rho} = C^{-1} J_{\rho} B = A^{-1} J_{\rho} C\) or \(J'_{\lambda} = C^{-1} J_{\lambda} A = B^{-1} J_{\lambda} C\). This completes the proof of the forward part. To prove the converse, carry out the same procedure, assuming the \(T\) condition and the fact that \((H, \circ)\) is a WIPL.

2. If \((H, \circ)\) is a WIPL, then

\[
R_{y} J'_{\rho} L'_{y} = J'_{\rho}, \quad \forall \ y \in H
\]

while since \(G\) is a WIPL,

\[
R_{x} J_{\rho} L_{x} = J_{\rho} \forall \ x \in G.
\]

The fact that \(G\) and \(H\) are isotopic implies that

\[
L_{x} = B L'_{xA} C^{-1} \quad \forall \ x \in G \quad \text{and}
\]

\[5\]
\[ R_x = AR_{xB}^{-1}C^{-1} \forall x \in G. \quad (7) \]

From (4),
\[ R_y' = J_\rho L_y'^{-1}J_\lambda' \forall y \in H \quad (8) \]
\[ L_y' = J_\lambda R_y'^{-1}J_\rho' \forall y \in H \quad (9) \]

while from (5),
\[ R_x = J_\rho L_x^{-1}J_\lambda \forall x \in G \quad (10) \]
\[ L_x = J_\lambda R_x^{-1}J_\rho \forall x \in G. \quad (11) \]

So, using (9) and (11) in (6) we get
\[ J_\lambda R_x J_\rho B = CJ_\lambda' R_x' A J_\rho' \forall x \in G \quad (12) \]

while using (8) and (10) in (7) we get
\[ J_\rho L_x J_\lambda A = CJ_\rho' L_x' B J_\lambda' \forall x \in G. \quad (13) \]

**Corollary 3.1** Let \((G, \cdot)\) and \((H, \circ)\) be two distinct loops that are isotopic under the triple \((A, B, C)\). If \(G\) is a WIPL with the \(T\) condition, then \(H\) is a WIPL:

1. there exists \(\alpha, \beta \in S'(G)\) i.e \(\alpha\) and \(\beta\) are weak inverse permutations and
2. \(J_\rho' = J_\lambda' \Leftrightarrow J_\rho = J_\lambda\).

**Proof**
By Theorem 3.1 \(A = B\) and \(J_\rho' = C^{-1}J_\rho B = A^{-1}J_\rho C\) or \(J_\lambda' = C^{-1}J_\lambda A = B^{-1}J_\lambda C\).

1. \(C^{-1}J_\rho B = A^{-1}J_\rho C \Leftrightarrow J_\rho B = CA^{-1}J_\rho C \Leftrightarrow J_\rho = CA^{-1}J_\rho CB^{-1} = CA^{-1}J_\rho CA^{-1} = \alpha J_\rho \alpha\) where \(\alpha = CA^{-1} \in S(G, \cdot)\).
2. \(C^{-1}J_\lambda A = B^{-1}J_\lambda C \Leftrightarrow J_\lambda A = CB^{-1}J_\lambda C \Leftrightarrow J_\lambda = CB^{-1}J_\lambda CA^{-1} = CB^{-1}J_\lambda CB^{-1} = \beta J_\lambda \beta\) where \(\beta = CB^{-1} \in S(G, \cdot)\).
3. \(J_\rho' = C^{-1}J_\rho B, J_\lambda' = C^{-1}J_\lambda A\). \(J_\rho' = J_\lambda' \Leftrightarrow C^{-1}J_\rho B = C^{-1}J_\lambda A = C^{-1}J_\lambda B \Leftrightarrow J_\lambda = J_\rho\).

**Lemma 3.1** Let \((G, \cdot)\) be a WIPL with the \(T\) condition and isotopic to another loop \((H, \circ)\), \((H, \circ)\) is a WIPL and \(G\) has a weak inverse permutation.

**Proof**
From the proof of Corollary 3.1 \(\alpha = \beta\), hence the conclusion.

**Theorem 3.2** With the \(T\) condition, isotopic WIP loops are isomorphic.

**Proof**
From Lemma 3.1 \(\alpha = I\) is a weak inverse permutation. In the proof of Corollary 3.1 \(\alpha = CA^{-1} = I \Rightarrow A = C\). Already, \(A = B\), hence \((G, \cdot) \cong (H, \circ)\).
3.2 \( f, g\)-Principal Isotopes of Weak Inverse Property Loops

**Lemma 3.2** Let \((G, \cdot)\) be a WIPL with a WIP \(f, g\)-principal loop isotope \((G, \circ)\) under the triple \(\alpha = (R_g, L_f, I)\).

1. \(J_f R_f J_\rho L_f = L_f^{-1}, J_\rho L_g J_\lambda = R_g^{-1}, J_\lambda R_g J_\rho = L_f, J_\rho L_f J_\lambda = R_g\).

2. \(J_\rho L_f = R_f^{-1} J_\rho, J'_f L_f = R'_g J'_f, J_\lambda R_g = L_g^{-1} J_\lambda, J'_\lambda R_g = L_f J'_\lambda \).

3. \(J_\rho R_f^{-1} J_\rho = J_\lambda R_g J_\rho, J_\rho L_g^{-1} J_\lambda = J'_\rho L_f J'_\lambda \).

4. \(\alpha = (J_\rho L_g^{-1} J_\lambda, J_\lambda R_f^{-1} J_\rho, I)\) and \(\alpha = (J'_\rho L_f J'_\lambda, J'_\lambda R_g J'_\rho, I)\).

**Proof**

Using the second part of Theorem 3.1

\[ J_\lambda R_f J_\rho L_f = J'_\lambda R'_g J'_\rho \quad \text{and} \quad J_\rho L_x J_\lambda R_g = J'_\rho L_f J'_\lambda \forall x \in G. \]  \(14\)

\[ J_\rho L_x J_\lambda R_g = J'_\rho L_f J'_\lambda \forall x \in G. \]  \(15\)

(1) to (4) are achieved by taken \(x = f, x = e\) in (14) and \(x = g, x = e\) in (15).

**Remark 3.1** In [I.4.1 Theorem, [19]], it is shown that in an IP quasigroup ; \(J_f R_f J_\rho = L_x\lambda, J_\rho L_x J_\lambda = R_x\rho\). These equations are true relative to the translational elements \(f, g\) in a universal WIPL as shown in Lemma 3.2 (1).

**Corollary 3.2** Let \((G, \cdot)\) be a WIPL with an \(f, g\)-principal loop isotope \((G, \circ)\). If the \(T\) condition holds in \((G, \cdot)\), then \((G, \circ)\) is a WIPL. But, provided any of the following holds :

1. \(T_1\) and \(T_{21}\) conditions

2. \(T_1\) and \(T_{22}\) conditions,

if \((G, \circ)\) is a WIPL, then the \(T\) condition holds. Hence, \(L_f, R_g \in S'(G, \cdot)\).

**Proof**

The proof of the first part is like the proof of the first part of Theorem 3.1. The second part is achieved using (2) of Lemma 3.2.

**Corollary 3.3** Let \((G, \cdot)\) be a loop with an \(f, g\)-principal isotope \((G, \circ)\). If \((G, \cdot)\) is a WIPL, then \((G, \circ)\) is a WIPL provided the \(T\) condition holds in \((G, \cdot)\). But, if \((G, \circ)\) is a WIPL then \(J_\lambda R_f J_\rho L_f = J'_\lambda R'_g J'_\rho\) and \(J_\rho L_x J_\lambda R_g = J'_\rho L_f J'_\lambda \forall x \in G\).

**Proof**

In Theorem 3.1 let \((A, B, C) = (R_g, L_f, I)\) and \(G = H\), then \(A = R_g, B = L_f\) and \(C = I\). Putting these in the results ; \(A = R_g = B = L_f\) and \(J'_\rho = I^{-1} J_\rho L_f = R_g^{-1} J_\rho I\) or \(J'_\rho = I^{-1} J_\rho R_g = L_f^{-1} J_\rho I \iff R_g = L_f\) and \(J'_\rho = J_\rho L_f = R_g^{-1} J_\rho\) or \(J'_\lambda = J_\lambda R_g = L_f^{-1} J_\lambda\).

This completes the proof of the first part. The second part follows by just using the above replacements.
Corollary 3.4 Let $G$ be a WIPL with either the $\mathcal{T}_2$ or $\mathcal{T}_3$ condition. For any arbitrary $f, g$-principal isotope $G'$ of $G$, the principal isotopism is described by the triple $(J_\rho J'_\lambda, J_\lambda J'_\rho, I)$.

Proof

By the $\mathcal{T}_2$ or $\mathcal{T}_3$ condition, $J_\rho J'_\lambda = J_\lambda J'_\rho = L_f = J_\lambda J'_\rho$. Thus the triple $(R_g, L_f, I) = (J_\rho J'_\lambda, J_\lambda J'_\rho, I)$. 

Lemma 3.3 For every WIPL with the $\mathcal{T}_2$ condition, if all $f, g$-principal isotopes have the same right(left) inverse mappings, then there exists a unique $f, g$-principal loop isotope. Thence, all loop isotopes of such a loop are isomorphic loops. Furthermore, with the $\mathcal{T}$ condition, if all $f, g$-principal isotopes have the same right(left) inverse mappings then, there exists a unique WIPL $f, g$-principal loop isotope. Thence, all loop isotopes of such a loop are isomorphic WIP loops.

Proof

Let $G$ be the WIPL in consideration. Let $G'$ and $G''$ be any two distinct principal isotopes of $G$ under the triples $\alpha = (R_{g'}, L_{f'}, I)$ and $\beta = (R_{g''}, L_{f''}, I)$ respectively. If the $\mathcal{T}_2$ condition holds in $G$, then $L_{f'} = J_\lambda J'_\rho$, $R_{g'} = J_\rho J'_\lambda$ and $L_{f''} = J_\lambda J'_\rho$, $R_{g''} = J_\rho J'_\lambda$. By hypothesis, $J_\rho = J_\rho$ or $J_\lambda = J_\lambda$. So, $R_{g'} = R_{g''}$ and $L_{f'} = L_{f''} \Rightarrow f' = f''$ and $g' = g''$. Thus, $\alpha = \beta \Rightarrow G' = G''$. This proves the uniqueness of $f, g$-principal loop isotope. Recall that if $H'$ and $H''$ are any two distinct loop isotopes of $G$ then there exists $G'$ and $G''$ such that $H' \cong G'$ and $H'' \cong G''$. So, $H' \cong H''$.

Assuming the $\mathcal{T}$ condition in $G$, the further statement follows by the same argument and the isotopes are therefore WIP loops since this property is isomorphic invariant.

Centrum The set of elements that commute with all other elements in a loop $L$ is denoted by $C(L)$.

Lemma 3.4 1. Let $(G, \cdot)$ be a WIPL with the $\mathcal{T}$ condition for an $f, g$-principal isotope $(G, \circ)$ or

2. if an $f, g$-principal isotope of a WIPL $(G, \cdot)$ with the $\mathcal{T}_1$ and $\mathcal{T}_{21}$ conditions or $\mathcal{T}_1$ and $\mathcal{T}_{22}$ conditions is also a WIPL, then

(a) $xg = fx$, $f, g \in C(G)$.
(b) $x_{\circ} = fx_{\circ}$.
(c) $x_{\cdot} = x_{\circ}g$.
(d) $gg = ff = fg = gf$.
(e) $f_{\circ} = g_{\cdot} = e$

Proof

Using Corollary 3.2.
(a) \( R_g = L_f \iff xR_g = xL_f \iff xg = fx \)

(b) \( x' = xJ' = xJ\rho L_f = x^\circ L_f = fx^\circ \)

(c) \( x^\lambda = xJ^\lambda = xJ^\lambda R_g = x^\lambda R_g = x^\lambda g \)

(d) From (a), with \( x = f \), \( fg = ff \). With \( x = g \), \( gg = fg \Rightarrow fg = ff = gg \).

(e) Putting \( x = f \) in (b), \( f\rho = ff = e \). Putting \( x = g \) in (c), \( g\lambda = g^\lambda g = e \).

**Corollary 3.5** If an \( f, g \)-principal isotope of a CIPL \( G \) with the \( T \) condition is an AIPL, then \( f \) and \( g \) are

1. Alternative elements (i.e \( (xx)y = x(xy) \) and \( y(xx) = (yx)x \) \( \forall \ y \in G \) and \( x \in \{f, g\} \)).

2. Flexible elements (i.e \( x(yx) = (yx)x \) \( \forall y \in G \) and \( x \in \{f, g\} \)).

3. Centrum elements (i.e \( xy = yx \) \( \forall \ y \in G \) and \( x \in \{f, g\} \)).

4. Equal elements (i.e \( f = g \)).

**Proof**
These are achieved using Lemma 3.4 and some results in [3].

**Remark 3.2** The properties of \( f \) and \( g \) proved in Corollary 3.5 above were not gotten in [3]. In fact, in addition to some identities and autotopisms stated in [3] satisfied by \( f \) and \( g \), we have the following:

**Identities;**

\[
\begin{align*}
  xg \cdot gy &= gg \cdot xy = (xy)g = (g \cdot xy)g = ff \cdot xy = f(xy \cdot g) = f(f \cdot xy). \\
  xg \cdot g &= g \cdot gx = gg \cdot x = xg \cdot g = gx \cdot g = ff \cdot x = f \cdot xg = f \cdot fx.
\end{align*}
\]

for all \( x \) and \( y \) in a CIPL with the \( T \) condition.

**Autotopisms;**

\[
(R_g, L_g, L_{gg}), (R_g, L_g, R_g^2), (R_g, L_g, L_g R_g), (L_f, L_g, L_f R_g) \\
(L_f, L_g, L_g L_f), (L_f, L_g, R_g L_f), (L_f, L_g, L_f^2)(L_f, L_g, L_{ff})
\]

for translation elements \( f \) and \( g \).

**Corollary 3.6** If a WIPL \( G \) has the \( T \) condition, then

(a) \( f^{\rho'} = g^\lambda \) in \( G' \).

(b) \( gg = ff = e' \) in \( G' \).

**Proof**
These follows from Lemma 3.4

(a) This follows immediately by (e); \( f^{\rho'} = e = g^\lambda \).

(b) In \( G' \), \( fg \) is the identity element. Thus from (d), \( gg = ff = fg = e' \).
Automorphic Inverse Property Elements In a loop $L$, an element $g \in L$ is said to be a right automorphic inverse property element ($\rho$-AIPE) if and only if $(xg)^{-1} = x^{-1}g^{-1} \forall x \in L$ while $g$ is called a left automorphic inverse property element ($\lambda$-AIPE) if and only if $(gx)^{-1} = g^{-1}x^{-1}$ hence $g$ is called an automorphic inverse property element if and only if these two conditions hold together.

We also use these definitions for anti-automorphic inverse property elements (AAIPE) by simply interchanging the positions of $g^{-1}$ and $x^{-1}$ on the right hand side.

**Lemma 3.5** A LIPL(RIPL) is a WIPL if and only if it is a RIPL(LIPL).

**Theorem 3.3** If an $f, g$-principal isotope of a LIP(RIP), WIPL $G$ is again a WIPL, then

(a) $g$ is a $\rho$-AIPE

(b) $f$ is a $\lambda$-AIPE

(c) $g, f \in C(G)$.

**Proof**

As shown in Lemma 3.5, a LIPL(RIPL) is a WIPL if and only if it is a RIPL(LIPL). Using Corollary 3.3 and Lemma 3.4;

(a) $x^\rho = fx^\rho = (xg)^{-1} \Rightarrow fx^\rho = (xg)^{-1}$.

(b) $x^\lambda = x^\lambda g = (f^{-1}x)^\lambda \Rightarrow x^{-1}g = (f^{-1}x)^{-1}$.

(c) By (a), $x^{-1}g^{-1} = (xg)^{-1} = g^{-1}x^{-1} \Rightarrow x^{-1}g^{-1} = g^{-1}x^{-1} \Rightarrow xg = gx \Rightarrow g \in C(G)$.

Similarly by (b), $f^{-1}x^{-1} = (fx)^{-1} \Rightarrow f^{-1}x^{-1} = x^{-1}f^{-1}$.

**Remark 3.3** In [7], it is shown that if a loop $L$ has the RIP(L. I. P.) then,

(i) $(xy)^{-1} = y^{-1}x^{-1} \forall y \in N_\rho(L)((N_\lambda(L)) \text{ and } x \in L$

(ii) $(yx)^{-1} = x^{-1}y^{-1} \forall y \in N_\rho(L)((N_\lambda(L)) \text{ and } x \in L$

where $N_\rho(L)((N_\lambda(L))$ is the right(left) nucleus of $L$. This implies right(left) nuclear elements are both $\rho$-AAIPE and $\lambda$-AAIPE.
3.3 Isomorphic Weak Inverse Property Loops

Theorem 3.4 Let \((G, \cdot)\) and \((H, \circ)\) be two distinct WIP loops with the weak \(\mathcal{T}_{21}\) condition. \((G, \cdot) \cong (H, \circ)\) if and only if \(AJ_\rho B = C J_\lambda D \Rightarrow A = C\) or \(B = D\) for some \(A, B, C, D \in S(G, \cdot)\) where \(J_\rho\) is the right inverse mapping on \((G, \cdot)\).

Proof

\(G\) is a WIPL \iff
\[
R_x J_\rho L_x = J_\rho \forall x \in G
\] (16)

\(H\) is a W. I. P. L. \iff
\[
R'_y J'_\rho L'_y = J'_\rho \forall y \in H.
\]

Let \(y = xA\) such that \(A : (G, \cdot) \to (H, \circ)\) is a bijection. Let \(J'_\rho = A^{-1} J_\rho A\), then \(R'_y (A^{-1} J_\rho A) L'_y = A^{-1} J_\rho A \Leftrightarrow AR'_y (A^{-1} J_\rho A) L'_y A^{-1} = J_\rho \Leftrightarrow\)
\[
(AR'_y A^{-1}) J_\rho (AL'_y A^{-1}) = J_\rho
\] (17)

Using the hypothesis combined with (16) and (17) ; \(R_x = AR'_{x, A} A^{-1}\) or \(L_x = AL'_{x, A} A^{-1} \Leftrightarrow (G, \cdot) \cong (H, \circ)\).

Conversely ; \(G\) is a WIPL \iff
\[
R_x J_\rho L_x = J_\rho \forall x \in G
\]

(18)

(19)

If \(G \cong H\), then \(\exists\) a bijection \(A : G \to H \ni xA \circ yA = (x \cdot y)A \forall x, y \in G \iff\)
\[
L_x = AL'_{x, A} A^{-1}\ \text{or} \ R_y = AR'_y A^{-1} \forall x, y \in G
\] (20)

Putting \(J'_\rho = A^{-1} J_\rho A\) in (19), we have \(R'_y (A^{-1} J_\rho A) L'_y = A^{-1} J_\rho A \Leftrightarrow (AR'_y A^{-1}) J_\rho (AL'_y A^{-1}) = J_\rho\). Let \(B = AR'_y A^{-1}, C = AL'_y A^{-1} \in S(G, \cdot)\) and \(D = R_x, E = L_x \in S(G, \cdot)\), then
\[
BJ_\rho C = J_\rho
\] (21)

\[
DJ_\rho E = J_\rho
\] (22)

Thus, (21) and (22) implies (20) i.e \(BJ_\rho C = J_\rho\) and \(DJ_\rho E = J_\rho \Rightarrow B = D\) or \(C = E\). Hence the proof.

Remark 3.4 By Theorem 3.4, WIP loops with the weak \(\mathcal{T}_{21}\) condition are isomorphic under a necessary and sufficient condition. This condition is therefore an isomorphy condition and not an isotopy-isomophy condition which is characteristic of the Wilson’s Identity ([21]) and the condition given in [Lemma 2, [13]] by Osborn.

Theorem 3.5 CIP is an isomorphic invariant property if and only if \(C J'_\rho = J_\rho C\) or \(DJ'_\lambda = J_\lambda D\) where \(C\) and \(D\) are permutations while \(J_\rho, J_\lambda\) and \(J'_\rho, J'_\lambda\) are the right, left inverse mappings of the loop and its isomorph respectively.
Proof
Let \((G, \cdot)\) be a CIPL isomorphic to a loop \((H, \circ)\) i.e there exists \(A : G \rightarrow H \cong G^A H\).

\[ xA \circ yA = (x \cdot y)A \iff R_y' A = A^{-1} R_y A \quad (23) \]

\[ xA \circ yA = (x \cdot y)A \iff L_x' A = A^{-1} L_x A \quad (24) \]

\(G\) is a CIPL \(\iff xy \cdot x^\rho \iff R_x y = L^{-1}_x y^\rho\) \(\iff \rho\).

Let \(z = x^\rho\), then \(x = z^\lambda\). So, \(R_z = L^{-1}_z\). Putting this in (23);

\[ R_y' A = A^{-1} L^{-1}_y A \quad (25) \]

From (24), \(L_x x = AL_x^\lambda A^{-1}\). Putting this in (25), get \(R_y' A = A^{-1} L^{-1}_y A = A^{-1} (AL_x^\lambda A^{-1})^{-1} A = L_y^\lambda A' R_y A = I \iff zAL_y^\lambda A' R_y A = zA \iff (yJ_y A \circ zA) \circ yA = zA\).

Let \(C = J_y A \iff A = J_{\rho} C\), so

\[ (yC \circ zA) \circ yJ_{\rho} C = zA \quad (26) \]

If CIP is an isomorphic invariant property then \((y' \circ z') \circ y'^\rho = z' \forall y', z' \in H\). Thus, with \(y' = yC, z' = zA\),

\[ (yC \circ zA) \circ yCJ_{\rho} = zA \quad (27) \]

Comparing (26) and (27), \(J_{\rho} C = CJ_{\rho}'\).

The converse is proved by doing the reverse. The proof of the second part is similar.

End of proof.

Corollary 3.7 A CIPL and a loop isomorph have the weak \(T_{21}\) condition.

Proof
Since the CIP is an isomorphic invariant property, the proof of this claim follows from Theorem 3.5.

Lemma 3.6 Let \(G\) be a CIPL. Then, \(D = J^2_{\rho} C \) and \(C = J^2_{\lambda} D\). If in addition the loop has the RIP or LIP, then \(D = C\). Hence, \(J_{\rho} = J_{\lambda}\).

Proof
Using the notations in Theorem 3.5 since \(C = J_\lambda A\) and \(D = J_\rho A\) implies that \(A = J_\lambda^{-1} C = J_\rho C\) or \(A = J_\rho^{-1} D = J_\lambda D\) then \(D = J^2_{\rho} C\) and \(C = J^2_{\lambda} D\).

If in addition, \(G\) is a RIPL or LIPL then by earlier result and since \(J^2_{\rho} = I\) or \(J^2_{\lambda} = I\) respectively, \(C = D\) in each case. Consequently, \(J_{\rho} = J_{\lambda}\).

Remark 3.5 In [3], isotopic CIP loops are shown to be isomorphic and this is true for commutative Moufang loops as shown in [19].
4 Conclusion and Future Study

Karkliniūnas and Karkliń [11] introduced m-inverse loops i.e loops that obey any of the equivalent conditions

\[(xy)_\rho, J^m \cdot xJ^{m+1}_\rho = yJ^m_\rho \quad \text{and} \quad xJ^{m+1}_\lambda \cdot (yx)_\lambda = yJ^m_\lambda.\]

They are generalizations of WIPLs and CIPLs, which corresponds to \(m = -1\) and \(m = 0\) respectively. After the study of \(m\)-loops by Keedwell and Shcherbacov [12], they have also generalized them to quasigroups called \((r, s, t)\)-inverse quasigroups in [13] and [14]. It will be interesting to study the universality of \(m\)-inverse loops and \((r, s, t)\)-inverse quasigroups. These will generalize the works of J. M. Osborn and R. Artzy on universal WIPLs and CIPLs respectively. Furthermore, we raise the question of studying \(m\)-inverse loops and \((r, s, t)\)-inverse quasigroups under a generalized type of \(T\) condition say a \(T_m\) condition and \(T_{(r,s,t)}\) condition just like we have done it for WIPLs? Also, we need to know if under the \(T\) condition, a loop in a pair is a CIPL if and only if the other loop is a CIPL as we have shown it to be true for WIPLs in Theorem 3.1.

References

[1] J. O. Adeniran and A. R. T. Solarin (1998), A note on automorphic inverse property loops, Collections of Scientific Papers of the Faculty of Science Krag. 20, 47–52.

[2] R. Artzy (1955), On loops with special property, Proc. Amer. Math. Soc. 6, 448–453.

[3] R. Artzy (1959), Crossed inverse and related loops, Trans. Amer. Math. Soc. 91, 3, 480–492.

[4] R. Artzy (1960), Relations between loops identities, Proc. Amer. Math. Soc. 11,6, 847–851.

[5] A. S. Basarab (1994), Osborn’s G-loop, Quasigroups and Related Systems 1, 51–56.

[6] R. H. Bruck (1966), A survey of binary systems, Springer-Verlag, Berlin-Göttingen-Heidelberg, 185pp.

[7] O. Chein, H. O. Pflugfelder and J. D. H. Smith (1990), Quasigroups and Loops : Theory and Applications, Heldermann Verlag, 568pp.

[8] J. Dene and A. D. Keedwell (1974), Latin squares and their applications, the English University press Lts, 549pp.

[9] E. G. Goodaire, E. Jespers and C. P. Milies (1996), Alternative Loop Rings, NHMS(184), Elsevier, 387pp.

[10] E. D. Huthnance Jr.(1968), A theory of generalised Moufang loops, Ph.D. thesis, Georgia Institute of Technology.
[11] B. B. Karklinüsh and V. B. Karkliñ (1976), *Inverse loops*, In 'Nets and Quasigroups', Mat. Issl. 39, 82-86.

[12] A. D. Keedwell and V. A. Shcherbacov (2002), *On m-inverse loops and quasigroups with a long inverse cycle*, Australas. J. Combin. 26, 99-119.

[13] A. D. Keedwell and V. A. Shcherbacov (2003), *Construction and properties of (r, s, t)-inverse quasigroups I*, Discrete Math. 266, 275-291.

[14] A. D. Keedwell and V. A. Shcherbacov, *Construction and properties of (r, s, t)-inverse quasigroups II*, Discrete Math. 288 (2004), 61-71.

[15] M. K. Kinyon (2005), *A survey of Osborn loops*, Milehigh conference on loops, quasigroups and non-associative systems, University of Denver, Denver, Colorado.

[16] M. K. Kinyon, K. Kunen (2004), *The structure of extra loops*, Quasigroups and Related Systems 12, 39–60.

[17] M. K. Kinyon, K. Kunen, J. D. Phillips (2004), *Diassociativity in conjugacy closed loops*, Comm. Alg. 32, 767–786.

[18] J. M. Osborn (1961), *Loops with the weak inverse property*, Pac. J. Math. 10, 295–304.

[19] H. O. Pflugfelder (1990), *Quasigroups and Loops : Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 147pp.

[20] W. B. Vasantha Kandasamy (2002), *Smarandache loops*, Department of Mathematics, Indian Institute of Technology, Madras, India, 128pp.

[21] E. Wilson (1966), *A class of loops with the isotopy-isomorphy property*, Canad. J. Math. 18, 589–592.