Logarithmic potential with super-super-exponential kink profiles and tails

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Abstract

We consider a novel one dimensional model of a logarithmic potential which has super-super-exponential kink profiles as well as kink tails. We provide analytic kink solutions of the model—it has 3 kinks, 3 mirror kinks and the corresponding antikinks. While some of the kink tails are super-super-exponential, some others are super-exponential whereas the remaining ones are exponential. The linear stability analysis reveals that there is a gap between the zero mode and the onset of continuum. Finally, we compare this potential and its kink solutions with those of very high order field theories harboring seven degenerate minima and their attendant kink solutions, specifically $\phi^{14}$, $\phi^{16}$ and $\phi^{18}$.

Keywords: Kink tails, Kink stability, Higher order field theories, soliton interaction, logarithmic potentials

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently there has been a growing interest in higher order field theories which admit kink solutions with a power law tail at either both the ends or a power law tail at one end and an exponential tail at the other end [1, 2]. An example of the latter is the $\phi^4$ potential studied in the context of mesons [3]. This is different from almost all the kink solutions that have been discussed during the last four decades where the kink solutions have exponential tail at both the ends [4, 5]. The discovery of these power law kinks [6–9] has raised many interesting questions related to the strength and the range of the kink-kink (KK) and kink-antikink (K-AK) force [10–12], the possibility of resonances [13] and scattering [14], stability analysis of such kinks [6], etc.

Very recently we have introduced a whole family of potentials which exhibits kinks with a power-law tail [15], a super-exponential tail [16] as well as a power-tower tail [17]. The potential with super-exponential tail, $V(\phi) = (1/2)(\phi \ln \phi)^2$ [16] arises in the context of infinite order phase transitions, whereas higher order field theories that we compare with below can model successive (and multiple) first and second order phase transitions [1, 2]. Similarly, logarithmic potentials in two-dimensional condensed matter and related systems play a crucial role, in particular in the context of Kosterlitz-Thouless transition [18]. The latter are widely regarded as infinite order transitions. Thus, the logarithmic potential proposed here is likely to have similar physical significance, i.e. it provides an additional potential that can support infinite order phase transitions.

In addition, in quantum field theories with massless scalar fields it is well known that one loop quantum correction gives rise to logarithmic potentials [19]. Finally, numerical studies of kink-antikink collisions for potentials harboring triplets of solutions with exponential [20, 21] and power-law tails [14] are providing new insights into their collision dynamics, e.g. ‘multi-bounce windows’ and escape. Thus, a similar study of kink collisions with super-exponential and super-super-exponential tails will not only provide additional insights into kink collision dynamics but may also lead to novel phenomena not known in the context of exponential and power-law tails. Therefore, the study of the logarithmic potential discussed here is physically quite important.

Now that one has discovered a variety of power law, super-exponential and power-tower tails (in addition to the well-known exponential tail), it is natural to enquire if there are still new types of possible kink tails. This is our main motivation in this paper and we take the first step in that
direction to discover new kink solutions with a super-super-exponential profile as well as a super-super-exponential tail.

The paper is structured as follows. In section 2 we introduce a logarithmic potential with super-super-exponential kink tail and explicitly obtain the three kinks, three mirror kinks and the corresponding six antikink solutions. We carry out the stability analysis for the different types of kink solutions in section 3 and show that there is a gap between the zero mode and the continuum. In section 4 we briefly describe the interaction between the different kinks and the antikinks. In section 5 we calculate the kink mass for the three kinks. In section 6 we compare these results with the corresponding ones in specific higher order field theories (\(\phi^{14}, \phi^{16}\) and \(\phi^{18}\)). Finally in section 7 we summarize the results obtained in this paper and point out some of the open problems.

2. Model and corresponding kink solutions

Let us consider the following logarithmic potential

\[
V(\phi) = (1/2) \phi^2 [(1/2) \ln(\phi^2)]^2 \times ([((1/2) \ln(1/2) \ln(\phi^2)]^2)^2 ,
\]

which is depicted in figure 1. We then find that

\[
\frac{dV}{d\phi} = \phi [(1/2) \ln(\phi^2)] (1/2) \ln(1/2) \ln(\phi^2)^2
\times ([((1/2) \ln(1/2) \ln(\phi^2)]^2 [(1/2) \ln(\phi^2 e^2) + 1]).
\]

This potential has 7 degenerate minima with \(V_{\text{min}} = 0\) at \(\phi = 0, \pm 1/e, \pm 1, \pm e\) and six maxima (see figure 1) which are solutions of the equation:

\[
((1/2) \ln(1/2) \ln(\phi^2)]^2 [(1/2) \ln(\phi^2 e^2) + 1]) = 0.
\]

It is worth noting that the values of the potential curvature at the seven degenerate minima are

\[
V''(0) = V''(\pm 1) = -\infty, \quad V''(\pm 1/e) = V''(\pm e) = 1.
\]

Note that the potential is smooth at \(\phi = 0\) (and at other minima) and there is no cusp there. Thus this model will have 3 kink solutions, 3 mirror kink solutions and the corresponding 6 antikinks. All these kinks and antikinks are solutions of the self-dual equation

\[
\frac{d\phi}{dx} = \pm \phi [(1/2) \ln(\phi^2)] (1/2) \ln(1/2) \ln(\phi^2)^2 .
\]

2.1. Three kinks, 3 mirror kinks and 6 anti-kink solutions

Solution I

The kink solution from 0 to 1/e is given by

\[
\phi_k^I(x) = e^{-e^{-x}},
\]

and it is easy to check that it is the solution of the self-dual equation (5) with +ve sign. In particular,

\[
\lim_{x \to -\infty} \phi(x) = e^{-e^{-x}}, \quad \lim_{x \to \infty} \phi(x) = \frac{1}{e} = e^{-(x+1)}.
\]

Notice that around \(\phi = 0\), the kink tail is super-super-exponential while around \(\phi = 1/e\) the tail is exponential. To our knowledge, this is the first example of a kink with a super-super-exponential profile and tail. The kink profile is depicted in figure 2 (and its magnified version is shown in figure 3).

On the other hand,

\[
\phi_{\text{mak}}^I(x) = -e^{-e^{x}},
\]

is the solution of the self-dual equation (5) with -ve sign. Note that as \(x \to -\infty\) to \(+\infty\), \(\phi\) goes from 0 to \(-1/e\), i.e. it corresponds to the mirror antikink (maK) associated with the kink solution I, as given by equation (6).

Solution II

The corresponding mirror kink solution from \(-1/e\) to 0 is given by

\[
\phi_{\text{mak}}^I(x) = -e^{-e^{x}}.
\]
It is easy to check that it is the solution of the self-dual equation (5) with +ve sign. Note that as \( x \to -\infty \) to \( +\infty \), \( \phi \) goes from \(-1/e\) to 0.

On the other hand,

\[
\phi^{I}_{mK}(x) = e^{-e^{x}},
\]

is the solution of the self-dual equation (5) with -ve sign. Note that as \( x \to -\infty \) to \( +\infty \), \( \phi \) goes from \( 1/e \) to 0, i.e. it corresponds to the antikink associated with the kink solution (6).

It is useful to mention the relationship

\[
\phi^{I}_{mK}(x) = -\phi^{I}_{K}(x), \quad \phi^{I}_{ak}(x) = -\phi^{I}_{mk}(x).
\]

We will see below that such a relationship also exists for \( \phi^{II}_{K}(x) \) and \( \phi^{II}_{mK}(x) \).

**Solution III**

The kink solution from \( 1/e \) to 1 is given by

\[
\phi^{III}_{K}(x) = e^{-e^{x}},
\]

and it corresponds to the solution of the self-dual equation (5) with +ve sign. In particular,

\[
\lim_{x \to -\infty} \phi(x) = \frac{1}{e} + e^{(x-1)}, \quad \lim_{x \to \infty} \phi(x) = 1 - e^{-e^{x}}.
\]

Notice that around \( \phi = 1 \), the kink tail is super-exponential while it is exponential around \( \phi = 1/e \). The kink profile is depicted in figure 2 (and its magnified version is shown in figure 4).

On the other hand,

\[
\phi^{III}_{mK}(x) = -e^{-e^{x}},
\]

is the solution of the self-dual equation (5) with -ve sign. Note that as \( x \to -\infty \) to \( +\infty \), \( \phi \) goes from \(-1 \) to \(-1/e \), i.e. it corresponds to the mirror antikink associated with the kink solution II, i.e. equation (12).

**Solution IV**

The corresponding mirror kink solution from \(-1 \) to \(-1/e \) is given by

\[
\phi^{IV}_{mK}(x) = -e^{-e^{x}}.
\]

It is easy to check that it is the solution of the self-dual equation (5) with +ve sign. Note that as \( x \to -\infty \) to \( +\infty \), \( \phi \) goes from \(-1 \) to \(-1/e \).

On the other hand,

\[
\phi^{IV}_{mK}(x) = e^{-e^{x}},
\]

is the solution of the self-dual equation (5) with -ve sign. Note that as \( x \to -\infty \) to \( +\infty \), \( \phi \) goes from \( 1/e \) to 0, i.e. it corresponds to the mirror antikink associated with the kink solution (12).

It is worth pointing out that the second kink solution and the corresponding mirror kink and antikinks as given by equation (12) and equations (14) to (16) also satisfy the relationships in equation (11).

**Solution V**

The kink solution from \( 1 \) to \( e \) is given by

\[
\phi^{V}_{K}(x) = e^{e^{x}},
\]

and it corresponds to the solution of the self-dual equation (5) with +ve sign. In particular,

\[
\lim_{x \to -\infty} \phi(x) = 1 + e^{-(x-1)}, \quad \lim_{x \to \infty} \phi(x) = e - e^{-(x-1)}.
\]

Notice that around \( \phi = 1 \), the kink tail is super-exponential while it is exponential around \( \phi = e \). The kink profile is depicted in figure 2 (and its magnified version is shown in figure 5).

On the other hand,

\[
\phi^{V}_{mK}(x) = -e^{e^{x}},
\]

is the solution of the self-dual equation (5) with -ve sign. Note that as \( x \to -\infty \) to \( +\infty \), \( \phi \) goes from \(-1 \) to \(-e \), i.e. it corresponds to the mirror antikink associated with the kink solution III, i.e. equation (17).

**Solution VI**

The corresponding mirror kink solution from \(-e \) to \(-1 \) is given by

\[
\phi^{VI}_{mK}(x) = -e^{e^{x}}.
\]
It is easy to check that it is the solution of the self-dual equation (5) with $-ve$ sign. Note that as $x \to -\infty$ to $+\infty$, $\phi$ goes from $e$ to $-1$.

On the other hand the mirror kink,

$$\phi_{ak} = e^{-x},$$

(21)
is the solution of the self-dual equation (5) with $+ve$ sign. Note that as $x \to -\infty$ to $+\infty$, $\phi$ goes from $e$ to 1, i.e. it corresponds to the antikink associated with the kink solution (17).

It is worth pointing out that the third kink solution and the corresponding mirror kink and antikinks as given by equation (17) and equations (19) to (21) also satisfy the relationships in equation (11).

3. Stability of kink solutions

We now perform the kink stability analysis for all three kink solutions as given by equations (6), (12) and (17) and show that for all of them there is a gap between the zero mode and the onset of the continuum.

The kink potential $V_k(x)$ which appears in the stability equation

$$-\frac{d^2\psi(x)}{dx^2} + V_k(x)\psi(x) = \omega^2\psi(x),$$

(22)
can be calculated using the relationship $V_k(x) = \frac{dV(\phi)}{d\phi}$ evaluated at $\phi = \phi_k(x)$. Using the three distinct kink solutions given by equations (6), (12) and (17) we now carry out the stability analysis in each of the three cases.

3.1. Stability of kink solution $\phi_K^I(x)$

On using the kink potential as given by equation (1) and the first kink solution as given by equation (6) we find that

$$V_k(x) = e^{-2x}e^{-2x} - 3e^{-x}e^{-x}[e^{-x} + 1] + e^{-2x} + 3e^{-x} + 1,$$

(23)

which is depicted in the inset of figure 3. It may be noted that $V(\infty) = 1$ while $V(-\infty) \to \infty$ so that the continuum begins at $\omega^2 = 1$. The corresponding kink zero mode is given by

$$\psi_0(x) = \frac{d\phi_K^I(x)}{dx} = e^{-e^{-x}}e^{-e^{-x}}e^{-x}.$$

(24)

The above zero mode is clearly nodeless and vanishes both as $x \to \pm \infty$. Further, it is easy to check that the zero mode eigenfunction (24) satisfies the stability equation (22) with the potential $V_k(x)$ given by equation (23) and with $\omega^2 = 0$.

Summarizing, we find that indeed there is a gap between the zero mode and the onset of the continuum in the case of the first kink solution.

3.2. Stability of kink solution $\phi_K^II(x)$

On using the kink potential as given by equation (1) and the second kink solution as given by equation (12) we find that

$$V_k(x) = e^{-2x}e^{2x} - 3e^{-x}e^{x}[e^{x} - 1] + e^{2x} - 3e^{x} + 1,$$

(25)

which is depicted in the inset of figure 4. It may be noted that $V(\infty) = 1$ while $V(-\infty) = 1$ so that the continuum begins at $\omega^2 = 1$. The corresponding kink zero mode is given by

$$\psi_0(x) = \frac{d\phi_K^{II}(x)}{dx} = e^{-e^{x}}e^{e^{x}}e^{x}.$$

(26)

The above zero mode is clearly nodeless and vanishes both as $x \to \pm \infty$. Further, it is easy to check that the zero mode eigenfunction (26) satisfies the stability equation (22) with the potential $V_k(x)$ given by equation (25) and with $\omega^2 = 0$.

Summarizing, we find that indeed there is a gap between the zero mode and the onset of the continuum in the case of the second kink solution.

3.3. Stability of the kink solution $\phi_K^{III}(x)$

On using the kink potential as given by equation (1) and the third kink solution as given by equation (17) we find that

$$V_k(x) = e^{-2x}e^{-2x} - 3e^{-x}e^{-x}[e^{-x} - 1] + e^{-2x} - 3e^{-x} + 1,$$

(27)

which is depicted in the inset of figure 5. It may be noted that $V(\infty) = 1$ while $V(-\infty) = \infty$ so that the continuum begins at $\omega^2 = 1$. The corresponding kink zero mode is given by

$$\psi_0(x) = \frac{d\phi_K^{III}(x)}{dx} = e^{-e^{-x}}e^{-e^{-x}}e^{-x}.$$

(28)

The above zero mode is clearly nodeless and vanishes both as $x \to \pm \infty$. Further, it is easy to check that the zero mode
eigenfunction (28) satisfies the stability equation (22) with the potential \( V_g(x) \) given by equation (27) and with \( \omega^2 = 0 \).

We thus have seen that for all three kink solutions, the kink stability equation is such that there is a gap between the zero mode and the beginning of the continuum.

4. Kink-kink and kink-antikink interaction

In this model we have three kinks, three mirror kinks and the corresponding six antikinks. In particular, we have seen that while the kink tail around \( \phi = 0 \) is super-super-exponential, the kink tails around \( \phi = \frac{1}{e} \) and \( \phi = e \) are exponential. Finally the kink tail around \( \phi = 1 \) is super-exponential. Using this information, we can immediately deduce the nature of the KK and K-AK as well as AK-K interactions in various cases.

To begin with, the interaction between the \((-\frac{1}{e}, 0, 0)\) K and \((0, 1/e, 0)\) K will be repulsive and super-super-exponential. On the other hand, the interaction between the \((1/e, 0, 0)\) AK and \((0, 1/e, 0)\) K will be attractive and super-super-exponential while the interaction between the \((0, 1/e)\) K and \((1/e, 0)\) AK will be attractive and exponential. On the other hand, the interaction between the \((0, 1/e)\) K and \((1/e, 1)\) K will be repulsive and exponential while the interaction between the \((1, 1/e)\) AK and \((1/e, 1)\) K, will be attractive and exponential. Similarly, the interaction between the \((1/e, 1)\) K and \((1, 1/e)\) AK will be super-exponential but attractive while the interaction between the \((1/e, 1)\) K and \((1, e)\) K will be repulsive but super-exponential. Likewise, the interaction between the \((1/e, 1)\) AK and \((1, e)\) K will be attractive but super-exponential. Finally, the interaction between the \((1, e)\) K and \((e, 1)\) AK will be attractive and exponential.

As regards the kink-antikink sequences (on an infinite chain) in this model are concerned, there will be topological restrictions on the location of kinks and antikinks that are much more elaborate than those considered in the \( \phi^4 \) model for a first order transition [22] and the \((\phi \ln \phi)^2 \) potential for an infinite order transition [16].

5. Kink masses

One can easily calculate the masses of all three kinks. As we show now, the formal expression for the kink mass is the same in all three cases, the only difference in the three cases comes from the different limits. The kink mass is given by

\[
M_k = \int_{\phi_a}^{\phi_b} d\phi \phi [(1/2)\ln(\phi^2)]\ln[(1/2)\ln(\phi^2)],
\]

(29)

where \( \phi_a, \phi_b \) correspond to two contiguous minima (see figure 1) between which there is a kink solution. This integral is straightforward to evaluate, one possible way is by using the substitution \( t = (1/2)\ln(\phi^2) \). We obtain

\[
M_k = \frac{\phi^2}{4} [\ln(\phi^2) - 1]\ln[(1/2)\ln(\phi^2)] - \frac{\phi^2}{4} + (1/4) Ei[\ln(\phi^2)],
\]

(30)

which is to be evaluated between the two limits \( \phi_a \) and \( \phi_b \). Here \( Ei(x) \) is the exponential integral function [23, 24]. Let us now use appropriate limits and estimate the kink mass for all the three cases.

Mass of Kink I

The kink-I goes from \( \phi = 0 \) to \( \phi = 1/e \) as \( x \) goes from \(-\infty \) to \( +\infty \). Hence its kink mass is obtained by evaluating the expression for \( M_k \) as given by equation (30) between the limits \( \phi_a = 0 \) and \( \phi_b = 1/e \). We find that

\[
M_k^I = \frac{1}{4} \left[ Ei(-2) - \frac{1}{e^2} \right].
\]

(31)

Mass of Kink II

The kink-II goes from \( \phi = 1/e \) to \( \phi = 1 \) as \( x \) goes from \(-\infty \) to \(+\infty \). Hence its kink mass is obtained by evaluating the expression (30) between the limits \( \phi_a = 1/e \) to \( \phi_b = 1 \). We find that

\[
M_k^II = -\frac{1}{4} \left[ Ei(2) - \frac{1}{e^2} + 1 \right].
\]

(32)

Mass of Kink III

The kink-III goes from \( \phi = 1/e \) to \( \phi = 1 \) as \( x \) goes from \(-\infty \) to \(+\infty \). Hence its kink mass is obtained by evaluating the expression (30) between the limits \( \phi_a = 1 \) to \( \phi_b = e \). We find that

\[
M_k^III = \frac{1}{4} \left[ Ei(2) - e^2 + 1 \right].
\]

(33)

6. Comparison with higher order field theories

It is instructive to compare the kink tails in the present case with other potentials that have seven minima, e.g. in the \( \phi^{14} \) field theory model as given by equation (14) and even the higher order field theories \( \phi^{16} \) and \( \phi^{18} \), briefly mentioned in [1, 2] where the kinks have either a power law or an exponential tail.

6.1. Seven minima of the \( \phi^{14} \) field theory

Let us consider the following specific \( \phi^{14} \) field theory model

\[
V(\phi) = (1/2)\phi^2 (\phi^2 - 1/e^2)^2(\phi^2 - 1)^2(\phi^2 - e^2^2). \]

(34)

This model has 7 degenerate minima at \( \phi = 0, \pm 1/e, \pm 1 \) and at \( \pm e \) and hence 3 kink solutions and the corresponding three mirror kink solutions as well as the corresponding six antikinks. The potential is depicted in figure 6 and also as a semilog plot in figure 7.

It is worth noting that the values of the potential curvature at the seven degenerate minima are

\[
V''(0) = 1, \quad V''(\pm 1/e) = \frac{4(e^2 - 1)^4(e^2 + 1)^2}{e^{12}}, \quad V''(\pm e) = \frac{4(e^2 - 1)^4}{e^4},
\]

(35)
Let us now determine the three kink solutions, i.e. from 0 to \(1/e\), from \(1/e\) to 1 and from 1 to \(e\). The solutions for the corresponding three mirror kinks and the corresponding six antikinks can then be easily written down.

**Kink from 0 to 1/e**

In this case the self-dual first order equation is

\[
\frac{d\phi}{dx} = \phi \left(1/e^2 - \phi^2\right)(1 - \phi^2)(e^2 - \phi^2). \tag{36}
\]

Thus in this case

\[
x = \int \frac{d\phi}{\phi(1/e^2 - \phi^2)(1 - \phi^2)(e^2 - \phi^2)}. \tag{37}
\]

Using partial fractions the integrand on the right hand side can be written as

\[
\frac{A_1}{\phi} + \frac{B_1\phi}{1/e^2 - \phi^2} + \frac{C_1\phi}{1 - \phi^2} + \frac{D_1\phi}{e^2 - \phi^2}. \tag{38}
\]

where

\[
A_1 = 1, \quad B_1 = \frac{e^6}{(e^2 - 1)^2(e^2 + 1)},
\]

\[
C_1 = -\frac{e^2}{(e^2 - 1)^2}, \quad D_1 = \frac{1}{(e^2 - 1)^2(e^2 + 1)}. \tag{39}
\]

This is easily integrated with the solution

\[
x = \ln(\phi) - (B_1/2)\ln(1/e^2 - \phi^2) - (C_1/2)\ln(1 - \phi^2) - (D_1/2)\ln(e^2 - \phi^2). \tag{40}
\]

Equation (40) is numerically inverted and the kink solution is depicted in figure 8. Thus, asymptotically

\[
\lim_{x \to -\infty} \phi(x) = f_1(e)e^x, \quad \lim_{x \to \infty} \phi(x) = 1/e - g_1(e)e^{-2x/B_1}. \tag{41}
\]

Here \(f_1(e)\) and \(g_1(e)\) are known constants.

**Kink from 1/e to 1**

In this case the self-dual first order equation is

\[
\frac{d\phi}{dx} = \phi \left(\phi^2 - 1/e^2\right)(1 - \phi^2)(e^2 - \phi^2). \tag{42}
\]

Thus in this case

\[
x = \int \frac{d\phi}{\phi(\phi^2 - 1/e^2)(1 - \phi^2)(e^2 - \phi^2)}. \tag{43}
\]

Again, using partial fractions the integrand on the right hand side can be written as

\[
\frac{A_2}{\phi} + \frac{B_2\phi}{\phi^2 - 1/e^2} + \frac{C_2\phi}{1 - \phi^2} + \frac{D_2\phi}{e^2 - \phi^2}. \tag{44}
\]

where

\[
A_2 = -1, \quad B_2 = \frac{e^6}{(e^2 - 1)^2(e^2 + 1)},
\]

\[
C_2 = \frac{e^2}{(e^2 - 1)^2}, \quad D_2 = -\frac{1}{(e^2 - 1)^2(e^2 + 1)}. \tag{45}
\]
This is easily integrated with the solution
\[
x = -\ln(\phi) + (B_2/2)\ln(\phi^2 - 1/e^2) - (C_2/2)\ln(1 - \phi^2) - (D_2/2)\ln(e^2 - \phi^2).
\]
(46)

Equation (46) is numerically inverted and the kink solution is depicted in figure 9. Thus, asymptotically
\[
\lim_{x \to -\infty} \phi(x) = 1/f_2(e) e^{2x/B_2},
\]
\[
\lim_{x \to \infty} \phi(x) = 1 - g_2(e) e^{-2x/C_2}.
\]
(47)

Here \(f_2(e)\) and \(g_2(e)\) are known constants.

**Kink from 1 to \(e\)**

In this case the self-dual first order equation is
\[
\frac{d\phi}{dx} = \phi (\phi^2 - 1/e^2)(\phi^2 - 1)(e^2 - \phi^2).
\]
(48)

Thus in this case
\[
x = \int \frac{d\phi}{\phi(\phi^2 - 1/e^2)(\phi^2 - 1)(e^2 - \phi^2)}.
\]
(49)

Again, using partial fractions the integrand on the right hand side can be written as
\[
\frac{A_3}{\phi} + \frac{B_3 \phi}{\phi^2 - 1/e^2} + \frac{C_3 \phi}{\phi^2 - 1} + \frac{D_3 \phi}{e^2 - \phi^2},
\]
(50)

where
\[
A_3 = 1, \quad B_3 = -\frac{e^6}{(e^2 - 1)^2(e^2 + 1)},
\]
\[
C_3 = \frac{e^2}{(e^2 - 1)^2}, \quad D_3 = \frac{1}{(e^2 - 1)^2(e^2 + 1)}.
\]
(51)

**Figure 9.** Comparison of the \(1/e \to 1\) kink with the corresponding super-super-exponential (SSE) kink.

**Figure 10.** Comparison of the \(1 \to e\) kink with the corresponding super-super-exponential (SSE) kink.

This is easily integrated with the solution
\[
x = \ln(\phi) + (B_3/2)\ln(\phi^2 - 1/e^2) - (C_3/2)\ln(\phi^2 - 1) - (D_3/2)\ln(e^2 - \phi^2).
\]
(52)

Equation (35) is numerically inverted and the kink solution is depicted in figure 10. Thus, asymptotically
\[
\lim_{x \to -\infty} \phi(x) = 1 + f_3(e) e^{2x/C_3},
\]
\[
\lim_{x \to \infty} \phi(x) = e - g_3(e) e^{-2x/D_3}.
\]
(53)

Here \(f_3(e)\) and \(g_3(e)\) are known constants.

We can also obtain the kink stability potential for the three kinks (similar to the ones shown in the insets of figures 3–5) numerically but we do not pursue this here.

**6.2. Seven Minima of the \(\phi^{16}\) Field Theory**

Consider a specific \(\phi^{16}\) field theory model potential that is given by
\[
V(\phi) = (1/2)\phi^4 (\phi^2 - 1/e^2)^2 (\phi^2 - 1)^2 (\phi^2 - e^2)^2.
\]
(54)

This model also has 7 degenerate minima at \(\phi = 0, \pm1/e, \pm1\) and at \(\pm e\) and hence 3 kink solutions and the corresponding three mirror kink solutions as well as the corresponding six antikinks. These kinks will have exponential tails except around \(\phi = 0\) which will be a power law tail. The potential is depicted in figure 11 and also as a semilog plot in figure 12. Using an analysis similar to the previous subsection we can obtain the kink solutions for the \(\phi^{16}\) model as well but we do not depict them here.

**6.3. Seven minima of the \(\phi^{18}\) field theory**

The potential for a specific \(\phi^{18}\) field theory model is given by
\[
V(\phi) = (1/2)\phi^4 (\phi^2 - 1/e^2)^2 (\phi^2 - 1)^2 (\phi^2 - e^2)^2.
\]
(55)

This model also has 7 degenerate minima at \(\phi = 0, \pm1/e, \pm1\) and at \(\pm e\) and hence 3 kink solutions and the corresponding three mirror kink solutions as well as the corresponding six antikinks.
antikinks. These kinks will have exponential tails except around $\phi = 1$ which will be a power law tail. The potential is depicted in figure 13 and also as a semilog plot in figure 14. Using an analysis similar to subsection 6.1 we can also obtain the kink solutions for the $\phi^{18}$ model but we do not depict them here.

7. Conclusions and some open problems

Recently there has been a surge of interest in potentials harboring non-exponential kink tails [1, 2, 6–9], in particular power law [3], super-exponential [16] and power-tower [17]. Here we have introduced a new logarithmic potential which has a kink solution with a super-super-exponential profile as well as a super-super-exponential tail. According to the stability analysis of such kinks, there is a gap between the zero mode and the onset of continuum. Since there are three different types of kinks between the seven minima there will be multiple topological restrictions [16] on the location of kinks/antikinks on an infinite chain that need to be elucidated.

It would be desirable to have numerical studies of kink-kink collisions for kinks with super-super-exponential tails and to compare with collisions of other kinds of kinks: with exponential [20, 21], super-exponential and power law tails [14]. The new logarithmic potential is likely to find applications in the context of some unusual (such as infinite order) phase transitions or other physical contexts, e.g. multiple successive phase transitions [1, 2] as well as Kosterlitz-Thouless like transitions [18]. We might call the super-super-exponential profile as the ‘super-Gumbel’ distribution as opposed to the Gumbel distribution known in the area of extremal event statistics [25]. The physical relevance of super-Gumbel distribution in statistics remains an interesting open problem.
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