Role of shallow water waves generated by modified Camassa-Holm equation: A comparative analysis for traveling wave solutions

1 Introduction

Nowadays, nonlinear partial differential equations have been applied in various fields due to their burning significances in fluid dynamics, optical fibers, biological sciences, quantum mechanics and plasma physics. Various researchers have suggested different methods to solve nonlinear partial differential equations such as, Homotopy perturbation method [1], Sine-Gordon expansion method [2, 3], $(1/G')$-expansion method [4–6], variational iteration method [7], improved Bernoulli sub-equation function method [8, 9], $(1/G')$-expansion method [10–12], $(G'/G, 1/G)$-expansion method [13], Exp-function method [14], Auxiliary equation method [15], Laplace perturbation method [16], Adomian’s decomposition method [17, 18], sub-equation method [19], Haar wavelet collocation method [20] and few others. In this context, Camassa-Holm equation is one of the type of nonlinear partial differential equation for which several studies have been presented in open literature; for instance; Gorka and Reyes [21] studied weak solutions and proved their existence and uniqueness for Camassa-Holm equation. Qu et al. [22] investigated dynamical stability of the single peaked soliton and periodic peaked soliton for an integrable Camassa-Holm equation with cubic nonlinearity. The soliton wave solutions using homotopy analysis method for Camassa-Holm equation have been explored by Abbasbandy [23]. Bekir and Guner [24] suggested new study based on topological (dark) soliton solutions subject to solitary wave Ansatz method for Camassa-Holm equation [24]. In brevity, few recent studies can be viewed in [25–27]. Additionally, the recent work on exact [28–32, 34, 35] and analytical solutions can also be seen therein [36–42]. Motivated by above discussion, we have been traced out analytic solutions of the Camassa-Holm equation by using $(G'/G – 1/G)$ and $(1/G')$-expansion methods. We also presented the comparison of $(G'/G – 1/G)$ and $(1/G')$-expansion methods on Camassa-Holm equation. The Camassa-Holm equation can be written in the form of [43]

$$u_t - u_{txt} + 3u^2 u_x - 2u_x u_{xx} - uu_{xxx} = 0. \tag{1}$$
Eq. (1) possesses shallow water waves and such mathematical model is known to be integrable, possessing multisoliton solutions with peaks [44–46]. Additionally, several fractional analytical techniques can be persuaded with classical and non-classical [47–58], local and nonlocal [59–67] and singular kernels [68–74]. In this study, general information about the equations and analytical methods discussed in the introduction is given. In the second section, the general operation of analytical methods is explained. In the third section, the applications of the methods are carried out. In the fourth section, the advantages and disadvantages of the obtained data and methods are discussed. In the last section, the data in the study is compiled.

2 Comparative methods for Camassa-Holm equation

2.1 \((1/G')\)-Expansion method

We consider two-variable general form of nonlinear partial differential equations

\[ P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots \right) = 0, \]  

(2)

in the general form. Here, let \( u = u(x, t) = U(\xi), \ \xi = x + vt, \ v \neq 0 \), where \( v \) is a constant and the velocity of the wave. After this, it can be converted into following nonlinear ordinary differential equation for \( U(\xi) \):

\[ y(U, U', U'', U''', \ldots) = 0. \]  

(3)

The solution of Eq. (3) is assumed to have the form

\[ U(\xi) = a_0 + \sum_{i=1}^{m} a_i \left( \frac{1}{G'} \right)^i, \]  

(4)

where \( a_i, \ (i = 1, 2, 3, \ldots, m) \) are constants, \( m \) is a positive integer, which is balancing term in Eq. (2), and \( G = G(\xi) \) provides the following second order ordinary differential equation as:

\[ G'' + \lambda G' + \mu = 0, \]  

(5)

where \( \lambda \) and \( \mu \) are constants to be determined after,

\[ \frac{1}{G'(\xi)} = \frac{1}{-\lambda^2 + A \cosh(\xi \lambda) - A \sinh(\xi \lambda)}, \]  

(5a)

where \( A \) is constant. The Eq. (5a) is a solution of the Eq. (5). If the desired derivatives of the Eq. (4) are calculated and substituting in the Eq. (3), a polynomial with the argument \((1/G')\) is attained. An algebraic equation system is created by equalizing the coefficients of this polynomial to zero. This equation system is solved with the help of ready package program and put into place in the default Eq. (3) for solution function. Consequently, the solutions of the Eq. (1) are found.

2.2 \((G'/G - 1/G)\)-Expansion method

The form of nonlinear partial differential equation containing two or more independent variables for which the solution can be explored by using \((G'/G - 1/G)\)-expansion method is written as follows:

\[ K(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xx}, \ldots) = 0. \]  

(6)

If \( u = u(x, t) = U(\xi), \ \xi = x + vt \) transformations are used in Eq. (6) then \( v \) is a constant, Eq. (6) is converted into a nonlinear ordinary differential equation and this equation can be generally written as:

\[ f(U, U', U'', U''', \ldots) = 0. \]  

(7)

Here, Eq. (7) can be integrated to decrease the operational complexity. By the nature of \((G'/G - 1/G)\)-expansion method, \( G(\xi) \) function is solution function of the second order ordinary differential equation as

\[ G''(\xi) + \lambda G(\xi) = \mu, \]  

(8)

where \( \lambda \) and \( \mu \) are real constants. As, \( \phi = \phi(\xi) = G'/G \) and \( \psi = \psi(\xi) = \frac{1}{G(\xi)} \) provides operational esthetic. We can write the derivatives of the functions defined herein as:

\[ \phi' = -\phi^2 + \mu \psi - \lambda, \ \psi' = -\phi \psi. \]  

(9)

We can present the behaviors of the solution functions of Eq. (8) with respect to the condition of \( \lambda \) by considering the equations given by Eq. (9).

Case I: If \( \lambda < 0 \)

\[ G(\xi) = c_1 \sinh \left( \sqrt{-\lambda} \xi \right) + c_2 \cosh \left( \sqrt{-\lambda} \xi \right) + \frac{\mu}{\lambda}, \]  

(10)

whereas \( c_1 \) and \( c_2 \) are arbitrary constants. By considering Eq. (10);

\[ \psi^2 = \frac{-\lambda}{\lambda^2 \sigma + \mu^2} \left( \phi^2 - 2\mu \psi + \lambda \right), \ \sigma = c_1^2 - c_2^2. \]  

(11)

Eq. (11) is easily written.

Case II: If \( \lambda > 0 \)

\[ G(\xi) = c_1 \sin \left( \sqrt{\lambda} \xi \right) + c_2 \cos \left( \sqrt{\lambda} \xi \right) + \frac{\mu}{\lambda}, \]  

(12)
here $c_1$ and $c_2$ are arbitrary constants. By considering Eq. (12), there is following equation;

$$
\psi^2 = \frac{\lambda}{\lambda^2 a - \mu^2} \left( \phi^2 - 2\mu \psi + \lambda \right), \quad \sigma = c_1^2 + c_2^2, \tag{13}
$$

**Case III:** If $\lambda = 0$

$$
G(\xi) = \frac{\mu}{2} \xi^2 + c_1 \xi + c_2, \tag{14}
$$

here $c_1$ and $c_2$ are arbitrary constants. By considering Eq. (14), there is following equation;

$$
\psi^2 = \frac{1}{c_1^2 - 2\mu c_2} \left( \phi^2 - 2\mu \psi \right). \tag{15}
$$

Finally, the solution of Eq. (7) in terms of $\phi$ and $\psi$ polynomials is expressed as;

$$
U(\xi) = \sum_{i=0}^{m} a_i \phi^i + \sum_{i=1}^{m} b_i \phi^{i-1} \psi. \tag{16}
$$

Here, $a_i$ ($i = 0, 1, \ldots, m$) and $b_i$ ($i = 1, \ldots, m$) numbers are the constants to be determined later. $m$ is a positive equilibrium term which can be attained by comparing maximum order derivative with the maximum order nonlinear term in Eq. (7). If Eq. (16) is written in Eq. (7) along with Eqs. (9, 11, 13) or (15), a polynomial function related to $\phi$ and $\psi$ is written. Each coefficient of $\phi^i \psi^j$ ($i = 0, 1, \ldots, m$) ($j = 1, \ldots, m$) terms of the attained polynomial functions are equated to zero and an algebraic equation system is attained for $a_i$, $b_i$, $\nu$, $\mu$, $c_1$, $c_2$ and $\lambda$ ($i = 0, 1, \ldots, m$). The required coefficients are found by solving this algebraic equation by means of ready package program. These coefficients found are put into Eq. (16) and $U(\xi)$ solution function of the ordinary differential equation given as Eq. (7) is attained and if $\xi = x + vt$ transformation is operated in reverse order, we will obtain the desired $u(x, t)$ traveling wave solution of Eq. (6).

### 3 Solutions of modified Camassa-Holm equation

#### 3.1 ($G'/G - 1/G$)-Expansion method

We consider Camassa-Holm Eq. (1). Using transmutation $u = u(x, t) = U(\xi), \quad \xi = x + vt$ and taking once the integral of Eq. (1), we get

$$
\nu \left( U - U'' \right) + U^3 - \frac{1}{2} \left( U' \right)^2 - UU'' = 0. \tag{17}
$$

Where, $\nu$ is the wave velocity. Thus, by finding the equilibrium term $m = 2$ in Eq. (17), and in Eq. (16) we obtain to following form of the solution

$$
U(\xi) = a_0 + a_1 \phi[\xi] + b_1 \psi[\xi] + a_2 \phi[\xi]^2 + b_2 \phi[\xi] \psi[\xi]. \tag{18}
$$

If we substitute the Eq. (18) in the Eq. (17) and the coefficients of the algebraic equation are equal to zero, we can establish the following algebraic equation systems

$$
\begin{align*}
\text{Const} & : \quad \nu a_0 + a_0^3 - \frac{1}{2} \lambda^2 a_0^2 - \frac{\lambda^2 \mu^2 a_0^2}{2 - \mu^2 + \lambda^2} - 2\nu \lambda^2 a_2 - 2\nu \lambda^2 \mu^2 a_2 - 4\lambda^2 a_0 a_2 + \frac{4\lambda^2 \mu^2 b_1 a_0}{\mu^2 + \lambda^2} + \frac{4\lambda^2 \mu^2 b_1 a_2}{\mu^2 + \lambda^2} - \frac{3\lambda^2 a_0 b_1^2}{\mu^2 + \lambda^2} + \frac{2\lambda^2 a_1 b_2}{\mu^2 + \lambda^2} - \frac{\lambda^2 b_2^2}{2 (\mu^2 + \lambda^2)} = 0, \\
\phi[\xi] & : \quad \nu a_1 - 2\nu \lambda a_1 - 2\nu a_0 a_1 + 3a_0^2 a_1 - 4\lambda^2 a_1 a_2 + \frac{4\lambda^2 \mu^2 a_1 a_2}{\mu^2 + \lambda^2} + \frac{5\lambda^2 \mu a_1 b_1}{\mu^2 + \lambda^2} + \frac{3\lambda^2 a_1 b_2^2}{\mu^2 + \lambda^2} + \frac{6\nu \lambda^2 a_0 b_2}{\mu^2 + \lambda^2} + \frac{6\lambda^2 a_0 b_2}{\mu^2 + \lambda^2} = 0,
\end{align*}
$$
(\phi [\xi])^2 : \quad -3\lambda a_1^2 - \lambda^2 a_1^2 \frac{2}{\lambda + \lambda^2} + 3a_0 a_1^2 + \nu a_1 a_2 - 2\nu a_1 a_2 - \frac{2\nu a_0 a_2}{\mu^2 + \lambda^2} - 8\lambda a_0 a_2

- \frac{2\lambda^2 a_0 a_2}{\mu^2 + \lambda^2} + a_0 a_2 - 4\lambda a_2^2 - \frac{4\lambda^2 a_2^2}{-\mu^2 + \lambda^2} + \nu a_2 b_1 + \frac{\lambda a_0 b_1}{\mu^2 + \lambda^2} + \frac{\lambda}{\lambda^2} \frac{b_1}{b_1}

+ \frac{\lambda}{\lambda^2} \frac{b_1}{b_1} - 7\lambda^2 b_1^2 + 3\lambda a_0 b_1

+ \frac{\lambda}{\lambda^2} b_1^2 - \frac{2}{\mu^2 + \lambda^2} + \frac{\lambda^2 a_2^2}{-\mu^2 + \lambda^2} + \frac{\lambda}{\lambda^2} \frac{a_0 b_2}{b_2} + 14\lambda^2 a_0 b_2

+ \frac{\lambda}{\lambda^2} \frac{a_0 b_2}{b_2} - \frac{2}{\mu^2 + \lambda^2} + \frac{3\lambda^2 a_2^2}{-\mu^2 + \lambda^2} = 0,

(\phi [\xi])^3 : \quad -2\nu a_1 - 2a_0 a_1 + a_1^3 - 14a_1 a_2 - \frac{4\lambda a_1 a_2}{\mu^2 + \lambda^2} + 6a_0 a_1 a_2 + \frac{5\lambda a_1 b_1}{\mu^2 + \lambda^2}

+ \frac{3a_1 b_1}{b_1} + \frac{6\nu a_0 b_2}{\mu^2 + \lambda^2} + 6\lambda a_0 b_2 + 3a_0 a_2 b_2 - 17\lambda^2 b_2

+ \frac{6a_0 b_1}{b_1} + \frac{6\lambda a_2 b_1}{b_1} + 3\lambda^2 a_2^2

+ \frac{2}{\mu^2 + \lambda^2} + \frac{3\lambda^2 a_2^2}{\mu^2 + \lambda^2} = 0,

(\phi [\xi])^4 : \quad -\frac{5a_1^2}{2} - 6\nu a_2 - 6a_0 a_2 + 3a_1 a_2 - 12\lambda a_2 - \frac{4\lambda a_2^2}{\mu^2 + \lambda^2} + 3a_0 a_2

+ \frac{13\lambda a_1 b_1}{b_1} + \frac{5\lambda^2 b_1^2}{b_1} + \frac{3a_2 b_1}{b_1} - \frac{2}{\mu^2 + \lambda^2} + \frac{12\lambda a_1 b_2}{b_2} + \frac{6a_0 b_1}{b_1} + \frac{3\lambda a_2 b_2}{b_2}

+ \frac{15\lambda^2 b_2^2}{b_2} + \frac{3\lambda^2 a_2 b_2}{b_2} + \frac{2}{\mu^2 + \lambda^2} + \frac{3\lambda^2 a_1 b_1}{b_1} + \frac{3\lambda^2 a_2 b_2}{b_2} = 0,

(\phi [\xi])^5 : \quad -10a_1 a_2 + 3a_1 a_2^2 + \frac{22a_2 b_2}{\mu^2 + \lambda^2} + \frac{10a_1 b_1}{\mu^2 + \lambda^2} + 6a_2 b_2 + \frac{3a_1 b_2}{b_2}

- \frac{\lambda}{\mu^2 + \lambda^2}

- \frac{8\lambda b_2}{\mu^2 + \lambda^2} + \frac{3\lambda^2 a_2 b_2}{b_2} = 0,

(\phi [\xi])^6 : \quad -8a_2^2 + a_1^3 - \frac{8\lambda b_2}{\mu^2 + \lambda^2} + \frac{3\lambda^2 a_2 b_2}{b_2} = 0,

\psi [\xi] : \quad \lambda a_2 + \frac{\lambda^3 a_2}{-\mu^2 + \lambda^2} + \frac{4\nu a_0 a_2}{\mu^2 + \lambda^2} + 4\lambda a_0 a_2 + \frac{4\lambda^3 a_0 a_2}{\mu^2 + \lambda^2}

+ \frac{\lambda}{\mu^2 + \lambda^2} b_1 - \nu a_0 b_1 - \lambda a_0 b_1 - \frac{2\lambda a_2 a_1}{\mu^2 + \lambda^2} + 3a_2 b_1 - 2\lambda a_2 b_1 + \frac{8\lambda^2 a_2 b_1}{-\mu^2 + \lambda^2}

+ \frac{2\lambda^2 b_1}{\mu^2 + \lambda^2} - 6\lambda a_1 b_1 - \frac{4\lambda^2 a_2 b_2}{-\mu^2 + \lambda^2} + \frac{\lambda}{\mu^2 + \lambda^2} b_1 = 0,

\psi [\xi] \psi [\xi] : \quad 3\nu a_1 + 3a_0 a_1 + 8\lambda a_1 a_2 + \frac{8\lambda^2 a_1 a_2}{\mu^2 + \lambda^2} - \frac{4\lambda a_1 b_1}{-\mu^2 + \lambda^2} + \frac{10\lambda a_0 b_1}{b_1} - \frac{6a_0 a_1 b_1}{b_1}

- \frac{6\lambda a_1 b_1}{b_1} + \frac{2\mu b_2}{b_2} - \nu b_2 - 5\nu b_2 - 12\nu b_2 - \frac{12\mu b_2}{\mu^2 + \lambda^2} + 3a_0 b_2 - 12\lambda a_0 b_2

+ \frac{3a_0 b_2}{b_2} - 6\lambda^2 a_2 b_2 - \frac{16\lambda^2 b_2}{b_2} + \frac{14\lambda^2 b_2}{b_2} = 0,

\phi [\xi] \psi [\xi] : \quad 4\mu a_1^2 + 10\nu a_1 + 10a_0 a_2 + 8\lambda a_2 + \frac{8\lambda a_1 a_2}{\mu^2 + \lambda^2} - 2\nu b_1 - 2a_0 b_1 + 3a_1 b_1

- 11\lambda a_1 a_1 - \frac{26\mu a_1 b_1}{b_1} + \frac{6a_0 a_2 b_1}{b_1} + \frac{5\mu b_1}{b_1} - 6\lambda a_0 b_1

+ \frac{24\lambda^2 a_1 b_1}{b_1} + \frac{6a_0 a_2 b_2}{b_2} - \frac{12a_0 a_2 b_2}{b_2} - \frac{14\lambda^2 b_2}{b_2} + 14\lambda^2 b_2 + \frac{6a_0 a_2 b_2}{b_2} = 0,

\phi [\xi] \psi [\xi] : \quad 17\mu a_1 - 5a_1 b_1 + 6a_1 a_1 b_1 - 6\nu b_2 - 6a_0 b_2 - 3a_1 b_2 - 19a_1 b_2

- 44\mu a_2 b_2 + 6a_0 a_2 b_2 + \frac{20\mu b_1 b_1}{b_1} + \frac{12a_0 a_2 b_2}{b_2} - 6\lambda a_1 b_2 = 0,

\phi [\xi] \psi [\xi] : \quad 14\mu a_1^2 - 10a_1 b_1 + 3a_1 b_2 - 10a_1 b_2 - 6a_1 a_2 b_2 - \frac{16\mu b_2}{b_2} - 6\lambda a_1 b_2 = 0,

\phi [\xi] \psi [\xi] : \quad -16a_2 b_2 + 3a_2^2 b_2 = 0,
\( \psi[\xi]^3 : -2\mu^2a_2b_1 + \mu b_2^2 + b_1^2 - \mu^2a_1b_2 + \lambda \mu b_2^2 = 0, \)

\( \phi[\xi] \psi[\xi]^3 : -4\mu^2a_2b_2 + 8\mu b_1b_2 + 3b_1^2b_2 = 0, \)

\( \phi[\xi]^2 \psi[\xi]^3 : 8\mu b_2^2 + 3b_1b_2^2 = 0, \)

\( \phi[\xi] : b_2^2 = 0, \)

\( \psi[\xi]^3 : -2\mu^2a_2b_1 + \mu b_2^2 + b_1^2 - \mu^2a_1b_2 + \lambda \mu b_2^2 = 0, \)

aims with ready package program, reaching the solutions of system (19) then we obtained the following cases:

**Case I:** If \( \lambda > 0, \)

\[
a_0 = 1, a_1 = 0, a_2 = 8, b_1 = 0, b_2 = 0, v = -1, \mu = 0, \lambda = \frac{1}{4},
\]

(20)

replacing the values of Eq. (20) into Eq. (18) then we have the following trigonometric traveling wave solution for Eq. (1):

\[
\xi = x + vt,
\]

\[
u_1(x, t) = 1 + \frac{8}{c_2} \left( \frac{\xi}{2} \right) - \frac{1}{2} c_1 \sin \left( \frac{\xi}{2} \right),
\]

(21)

**Case II:** If \( \lambda < 0, \)

\[
a_0 = -2, a_1 = 0, a_2 = 8, b_1 = 0, b_2 = 0, v = -2, \mu = 0, \lambda = -\frac{1}{4},
\]

(22)

replacing Eq. (22) into Eq. (18) then we obtain the following hyperbolic traveling wave solution of Eq. (1):

\[
\xi = x + vt,
\]

\[
u_2(x, t) = -2 + \frac{8}{c_2} \left( \frac{\xi}{2} \right) + \frac{1}{2} c_1 \sin \left( \frac{\xi}{2} \right),
\]

(23)

**Case III:** If \( \lambda = 0, \)

\[
a_0 = 1, a_1 = 0, a_2 = 8, b_1 = 0, b_2 = 0, v = -1, \mu = 0,
\]

(24)

replacing values of Eq. (24) into Eq. (18), then we obtain the following rational traveling wave solution for Eq. (1):

\[
\xi = x + vt,
\]

\[
u_3(x, t) = \frac{8c_2^2}{(c_1 + c_2x)^2},
\]

(25)

\[\text{Figure 1: 3-D, 2-D and contour graphs for } c_2 = 0.5, \quad c_1 = -1 \text{ values in Eq. (21).}\]

\[\text{Figure 2: 3-D, 2-D and contour graphs for } c_2 = 0.5, \quad c_1 = -1 \text{ values in Eq. (23).}\]

\[\text{Figure 3: 3-D, 2-D and contour graphs for } c_2 = -0.2, \quad c_1 = 1.5 \text{ values in Eq. (25).}\]

### 3.2 \((1/G')\)-Expansion method

We consider Eq. (1). For which using transmutation \( u = u(x, t) = U(\xi), \quad \xi = x + vt, \quad v \neq 0, \quad \nu \neq 0, \) and taking once the integral of Eq. (1), we obtain

\[
u (U'' - U') + U^3 - \frac{1}{2} (U')^2 - UU'' = 0,
\]

(26)

where, \( \nu \) represents the velocity of the wave. Taking into account the Eq. (26), we find the equilibrium term \( m = 2 \) and in Eq. (4), we attain to following form of the solution

\[
U(\xi) = a_0 + a_1 \left( \frac{1}{G'} \right) + a_2 \left( \frac{1}{G'} \right)^2.
\]

(27)
If we substitute the Eq. (27) in the Eq. (26) and the coefficients of the algebraic equation are equal to zero, we can establish the following algebraic equation systems

\[
\text{Const : } va_0 + a_1^3 = 0,
\]

\[
\left( \frac{1}{G'(\xi)} \right)^1: \;\; va_1 - v\lambda^2a_1 - \lambda^2a_0a_1 + 3a_0^2a_1 = 0,
\]

\[
\left( \frac{1}{G'(\xi)} \right)^2: \;\; -3v\lambda\mu a_1 - 3\mu a_0a_1 - \frac{3}{2}\lambda^2a_1^2 + 3a_0a_1^2 + va_2 - 4v\lambda^2a_2 - 4\lambda^2a_0a_2 + 3a_0^2a_2 = 0,
\]

\[
\left( \frac{1}{G'(\xi)} \right)^3: \;\; -2v\mu^2a_1 - 2\mu^2a_0a_1 - 4\mu a_1^2 + a_1^3 - 10v\lambda a_2 - 10\lambda\mu a_0a_2 - 7\lambda^2a_1a_2 + 6a_0a_1a_2 = 0,
\]

\[
\left( \frac{1}{G'(\xi)} \right)^4: \;\; -\frac{5}{2}v\mu^2a_1 - 6v\mu^2a_2 - 6\mu^2a_0a_2 - 17\lambda\mu a_1a_2 + 3a_1^2a_2 - 6\lambda^2a_2^2 + 3a_0a_2^2 = 0,
\]

\[
\left( \frac{1}{G'(\xi)} \right)^5: \;\; -10v\mu^2a_1a_2 - 14\lambda^2a_2^2 + 3a_1a_2^2 = 0,
\]

\[
\left( \frac{1}{G'(\xi)} \right)^6: \;\; -8\mu^2a_2^2 + a_2^3 = 0.
\]

**Case I:** If

\[
a_0 = -1, \quad a_1 = \pm 8i\mu, \quad a_2 = 8\mu^2, \quad v = \pm 1, \quad \lambda = -i, \quad (i = \sqrt{-1}),
\]

replacing values Eq. (29) into Eq. (27) and we have the following new type complex hyperbolic traveling wave solution for Eq. (1):

\[
\xi = x + vt,
\]

\[
u_0(x, t) = -1 + \frac{8\mu^2}{(-i\mu + A \cos [t - x] - iA \sin [t - x])^2} - \frac{8i\mu}{(-i\mu + A \cos [t - x] - iA \sin [t - x])},
\]

**Figure 4:** 3-D, 2-D and contour graphs for \(A = 0.5, \mu = 1\) values in Eq. (30).

**Case II:** If

\[
a_0 = 0, \quad a_1 = -8\mu, \quad a_2 = 8\mu^2, \quad v = -2, \quad \lambda = -1,
\]

replacing values of Eq. (31) into Eq. (27) and we have following hyperbolic traveling wave solution for Eq. (1):

\[
\xi = x + vt,
\]

\[
u_2(x, t) = \frac{8\mu^2}{(\mu + A \cosh [2t - x] - A \sinh [2t - x])^2} - \frac{8\mu}{(\mu + A \cosh [2t - x] - A \sinh [2t - x])}.
\]
4 Results and discussions

Shock waves of nonlinear partial differential equations (NLPDEs) have been discussed on the basis of modeling of physical phenomena. The comparative analysis has been attained by two methods namely \((G'/G - 1/G)\)-expansion and \((1/G')\)-expansion method for Camassa-Holm equation. It has been traced out that the solutions are different than the existing solutions in literature. This assured that the results have disclosed new phenomenon for Shock waves based on two different methods. For the sake of physical aspects, it provides the opportunity to understand the dynamics of solitary waves obtained by two expansion methods their states. The solutions obtained with the \((G'/G - 1/G)\)-expansion method are trigonometric, hyperbolic, and rational traveling wave solutions. From comparison point of view, only hyperbolic and complex hyperbolic traveling wave solutions have been obtained via \((1/G')\)-expansion method. The solutions obtained by both methods were found to be different from each other. In this case, the existence of many methods expresses the richness of the solutions of the differential equation. \((G'/G - 1/G)\)-expansion method more complicated and \((1/G')\)-expansion method is less difficult. In this case, we can determine the degree of difficulty by referring to the system of Eq. (19) and (28). It was also observed that the processing time in \((G'/G - 1/G)\)-expansion method was longer by using a ready package program with the same features. The excess of the number of equations in the equation system (19) is effective on the extension of period. It has been observed that all obtained exact solutions, the \((1/G')\)-expansion method is advantages in terms of process complexity, while \((G'/G - 1/G)\)-expansion method is more advantages in terms of number of solutions.

In this study, the application of two different analytical methods is included, and the solutions obtained at the end of this application are important both mathematically and physically. Mathematically important is the generation of traveling wave solutions. Physically, traveling wave solutions, which play an important role in the transport of energy, will shed light on many problems. If the parameters in the traveling wave solution gain physical meaning by considering the physical properties of the problem under consideration, the obtained traveling wave solutions will be much more valuable. It was observed that the traveling wave solutions obtained by both analytical methods satisfy the modified Camassa-Holm equation. At the end of this observation, it can be said that the methods are reliable, useful and applicable methods for obtaining traveling wave solution. Both methods are recommended for obtaining traveling wave solution of NLPDEs in the future.

5 Conclusion

In this letter, as a result, trigonometric, hyperbolic, complex hyperbolic and rational traveling wave solutions of modified Camassa-Holm equation have successfully constructed using \((G'/G - 1/G)\) and \((1/G')\) expansion methods. 3-D, 2-D and contour graphs are presented for the arbitrary values of the parameters in the solutions obtained. The solutions obtained by both methods have different properties and can shed light on some physical events such as different shallow water waves. Advantages and disadvantages of two methods discussed. In the future, it can be used to find traveling wave solutions of many NLPDEs. Because both methods are powerful methods for obtaining traveling wave solutions of NLPDEs.

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