A NOTE ON SHARP MULTIVARIATE BERNSTEIN– AND MARKOV–TYPE INEQUALITIES

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Abstract. Let \( V \) be a symmetric convex body in \( \mathbb{R}^m \). We prove sharp Bernstein–type inequalities for entire functions of exponential type with the spectrum in \( V \) and discuss certain properties of the extremal functions. Markov–type inequalities with sharp constants for algebraic polynomials on \( V \) and certain non-symmetric convex bodies are proved as well.

1. Introduction

In this paper we discuss several new Bernstein– and Markov–type inequalities in the uniform norm with sharp constants for multivariate entire functions of exponential type and algebraic polynomials. In particular, we extend the Bernstein inequality to entire functions \( f \) of exponential type with the spectrum in a symmetric convex body \( V \subset \mathbb{R}^m \) in the form

\[
\sup_{y \in K^*} \left| \sum_{j=1}^m \frac{\partial f(x)}{\partial x_j} y_j \right| \leq M(K, V) \| f \|_{C(\mathbb{R}^m)}, \quad x \in \mathbb{R}^m,
\]

where \( K \) is a symmetric convex body in \( \mathbb{R}^m \) and \( K^* \) is the polar of \( K \). Formulae for the sharp constant \( M(K, V) \) in the above inequality and properties of its extremal functions are discussed as well. We also prove the similar Markov-type inequalities for polynomials on symmetric and certain non-symmetric convex bodies.

Notation. Let \( \mathbb{R}^m \) be the Euclidean \( m \)-dimensional space with elements \( x = (x_1, \ldots, x_m) \), \( y = (y_1, \ldots, y_m) \), \( t = (t_1, \ldots, t_m) \), \( u = (u_1, \ldots, u_m) \), the inner product \((t, y) := \sum_{j=1}^m t_j y_j\), and the norm \(|t| := \sqrt{(t, t)}\). Next, \( \mathbb{C}^m := \mathbb{R}^m + i \mathbb{R}^m \) is the \( m \)-dimensional complex space with elements \( z = (z_1, \ldots, z_m) = x + iy \) and the norm \(|z| := \sqrt{|x|^2 + |y|^2}\); \( \mathbb{Z}^m \) denotes the set of all integral lattice points in \( \mathbb{R}^m \); and \( \mathbb{Z}_+^m \) is a subset of \( \mathbb{Z}^m \) of all points with nonnegative coordinates. We also use a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}_+^m \) with \(|\alpha| := \sum_{j=1}^m \alpha_j\) and \( x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m} \).

Let \( C(E) \) be the space of all continuous complex-valued functions \( F \) on a measurable set \( E \subset \mathbb{R}^m \) with the finite uniform norm \( \| F \|_{C(E)} := \sup_{x \in E} |F(x)| \), and let \( \mathcal{P}_{n,m} \) be the set of all polynomials.
be the gradient vector. 

In addition, \( T_n(\cdot) := \cos(n \arccos \cdot) \in \mathcal{P}_{n, 1} \) is the Chebyshev polynomial of the first kind. For a differentiable function \( f : \mathbb{R}^m \to \mathbb{C}^1 \), let

\[
\nabla f(x) := \left( \frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_m} \right), \quad x \in \mathbb{R}^m,
\]

be the gradient vector.

Throughout the paper \( R, R_1, R_2, C, C_0, C_1, \ldots \) denote constants independent of essential parameters. Occasionally we indicate dependence on certain parameters. The same symbol does not necessarily denote the same constant in different occurrences.

**Convex Bodies.** In this paper we need certain definitions and properties of convex bodies in \( \mathbb{R}^m \), i.e., compact convex sets with non-empty interiors. We first define the width \( w(C) \) of a convex body \( C \) in \( \mathbb{R}^m \) as the minimum distance between two parallel supporting hyperplanes of \( C \) and define the diameter \( d(C) \) of \( C \) as the maximum distance between two points of \( C \). Let \( \mathcal{B}^m := \{ t \in \mathbb{R}^m : |t| \leq 1 \} \), \( Q^m := \{ t \in \mathbb{R}^m : |t| \leq 1, 1 \leq j \leq m \} \), and \( O^m := \{ t \in \mathbb{R}^m : \sum_{j=1}^m |t_j| \leq 1 \} \) be the \( m \)-dimensional ball, cube, and octahedron, respectively.

Next, let \( V \) be a centrally symmetric (with respect to the origin) closed convex body in \( \mathbb{R}^m \) with the boundary \( \partial(V) \), the width \( w(V) \), and the diameter \( d(V) \). In addition, let \( V^* := \{ y \in \mathbb{R}^m : \forall t \in V, |(t, y)| \leq 1 \} \) be the polar of \( V \). It is well known that \( V^* \) is a centrally symmetric (with respect to the origin) closed convex body in \( \mathbb{R}^m \) and \( V^{**} = V \) (see, e.g., [25, Lemma 3.4.7]). The sets \( V \) and \( V^* \) generate the following norms on \( \mathbb{R}^m \) and \( \mathbb{C}^m \) by

\[
\|x\|_V := \max_{t \in V^*} |(t, x)|, \quad \|y\|_{V^*} := \max_{t \in V} |(t, y)|, \quad x, y \in \mathbb{R}^m; \quad \|z\|_{V^*} := \max_{t \in V} \sum_{j=1}^m t_j z_j, \quad z \in \mathbb{C}^m.
\]

Note also that \( V \) and \( V^* \) are the unit balls in the norms \( \| \cdot \|_V \) and \( \| \cdot \|_{V^*} \) on \( \mathbb{R}^m \), respectively. For example, the following convex bodies and their polars:

\[
V_\mu := \left\{ x \in \mathbb{R}^m : \|x\|_{V_\mu} = \left( \sum_{j=1}^m |x_j|^\mu \right)^{1/\mu} \leq 1 \right\},
\]

\[
(V_\mu)^* = \left\{ y \in \mathbb{R}^m : \|y\|_{(V_\mu)^*} = \left( \sum_{j=1}^m |y_j|^\rho \right)^{1/\rho} \leq 1 \right\},
\]

where \( \mu \in [1, \infty], \rho \in [1, \infty], \) and \( 1/\mu + 1/\rho = 1 \), have various applications in analysis. In particular, \( V_1 = (V_\infty)^* = O^m, V_2 = (V_2)^* = \mathcal{B}^m, \) and \( V_\infty = (V_1)^* = Q^m. \)
Throughout the paper $K$ and $V$ are centrally symmetric (with respect to the origin) closed convex bodies in $\mathbb{R}^m$. Next, we define the constant $M(K, V)$ and sets $A$ and $B$.

$$M(K, V) := \max_{y \in V} \|y\|_K = \max_{y \in \partial(V)} \|y\|_K = \max_{y \in \mathbb{R}^m \setminus \{0\}} \|y\|_K / \|y\|_V. \quad (1.1)$$

$$A = A(K, V) := \{a \in \partial(V) : \|a\|_K = M(K, V)\}, \quad (1.2)$$

$$B = B(V^*, K^*) := \{b \in \partial(K^*) : \|b\|_{V^*} = M(V^*, K^*)\}. \quad (1.3)$$

Then the following property holds:

**Proposition 1.1.** For any $a = a(K, V) \in A$ there exists $b = b(V^*, K^*) \in B$ such that

$$M(K, V) = M(V^*, K^*) = |(b, a)|. \quad (1.4)$$

**Proof.** Since

$$\max_{y \in V} \|y\|_K = \max_{y \in V} \max_{x \in K^*} |(x, y)| = \max_{x \in K^*} \max_{y \in V} |(x, y)| = \max_{x \in K^*} \|x\|_{V^*}, \quad (1.5)$$

we arrive at the first equality in (1.4). Next, given $a \in A$, there exists $b \in \partial(K^*)$ such that

$$M(K, V) = \max_{y \in V} \|y\|_K = \|a\|_K = |(b, a)|. \quad (1.6)$$

Since by (1.5),

$$\|b\|_{V^*} = \max_{y \in V} |(b, y)| \leq \max_{x \in K^*} \max_{y \in V} |(x, y)| = M(K, V) = |(b, a)| \leq \|b\|_{V^*},$$

we see that $b \in B$. Thus the second equality in (1.4) is established. \hfill \Box

Similarly one can prove that for any $b \in B$ there exists $a \in A$ such that the second equality in (1.4) is valid. More general forms of the first equality in (1.4) are discussed in [16, Theorem 3.13]. The following geometric characterization of $b$ in Proposition 1.1 follows from (1.2), (1.3), and (1.4): given $a \in A$, one can choose $b \in B$ as a point, satisfying the condition $(1/M(K, V))b \in \partial(V^*) \cap H_{\pm}(a)$, where $H_{\pm}(a) := \{x \in \mathbb{R}^m : (a, x) = \pm 1\}$ are parallel supporting hyperplanes of $V^*$. Note that in certain cases $b = \pm (M(K, V)|a|^{-2})a$ (see Example 2.8).

In particular, it follows from (1.1) and (1.4) that if $K$ is the ball $\mathcal{B}^m$ with the norm $\| \cdot \|_K = | \cdot |$, then

$$d(V^*)/2 = M(\mathcal{B}^m, V^*) = M(V, \mathcal{B}^m) = 2/w(V). \quad (1.6)$$

**Entire Functions of Exponential Type.** The set of all trigonometric polynomials $T(x) = \sum_{k \in V \cap \mathbb{Z}^m} c_k e^{i(k,x)}$ with complex coefficients is denoted by $T_V$. A more general set of entire functions is defined below.

**Definition 1.2.** We say that an entire function $f : \mathbb{C}^m \rightarrow \mathbb{C}$ has exponential type $V$ if for any $\varepsilon > 0$ there exists a constant $C_0(\varepsilon, f) > 0$ such that for all $z \in \mathbb{C}$, $|f(z)| \leq C_0(\varepsilon, f) \exp[(1 + \varepsilon)\|z\|_{V^*}].$
The set of all entire function of exponential type $V$ is denoted by $B_V$. In case of $m = 1$, we use the notation $B_\sigma := B_{[-\sigma,\sigma]}$, $\sigma > 0$. Throughout the note, if no confusion may occur, the same notation is applied to $f \in B_V$ and its restriction to $\mathbb{R}^m$ (e.g., in the form $f \in B_V \cap C(\mathbb{R}^m)$). The set $B_V$ was defined by Stein and Weiss [25, Sect. 3.4]. For $V = \sigma Q^m$ and $V = \sigma \mathfrak{B}^m$, $\sigma > 0$, similar sets were defined by Bernstein [4] and Nikolskii [21, Sects. 3.1, 3.2.6], see also [10, Definition 5.1].

The rest of the paper is organized as follows: In Section 2 we obtain Bernstein–type inequalities for functions from $B_V$, and in Section 4 we discuss Markov–type inequalities for polynomials from $\mathcal{P}_{n,m}$ on symmetric and non-symmetric convex bodies. Certain properties of extremal functions in Bernstein–type inequalities are discussed in Section 3.

## 2. Bernstein–type Inequalities

Throughout the section a point $a \in \partial(V)$ satisfies the equality $\|a\|_K = M(K, V)$, i.e., $a \in A(K, V)$ (see (1.1) and (1.2)), and a point $b \in B(V^*, K^*)$ is defined by (1.3) and (1.4).

### Bernstein Inequalities on $\mathbb{R}^1$

The Bernstein–Szegö type inequality for complex-valued functions $\varphi \in B_\sigma \cap C(\mathbb{R}^1), \sigma > 0$, can be presented in the following form:

$$|\sin \alpha \varphi'(\tau) - \sigma \cos \alpha \varphi(\tau)| \leq \sigma \|\varphi\|_{C(\mathbb{R}^1)}, \quad \tau \in \mathbb{R}^1, \quad \alpha \in [0, 2\pi).$$  \hspace{1cm} (2.1)

In particular, the classic Bernstein inequality for complex-valued functions $\varphi \in B_\sigma \cap C(\mathbb{R}^1), \sigma > 0$, immediately follows from (2.1) for $\alpha = \pi/2$,

$$|\varphi'(\tau)| \leq \sigma \|\varphi\|_{C(\mathbb{R}^1)}, \quad \tau \in \mathbb{R}^1. \hspace{1cm} (2.2)$$

The functions $\varphi_0(\tau) := C_1 e^{i\tau} + C_2 e^{-i\tau}, C_1 \in \mathbb{C}^1, C_2 \in \mathbb{C}^1$, with $\|\varphi_0\|_{C(\mathbb{R}^1)} = \left(|C_1|^2 + |C_2|^2\right)^{1/2}$ are the only extremal functions in (2.1) or (2.2), i.e., equality holds in (2.1) or (2.2) for a certain $\tau = \tau_0(\sigma, C_1, C_2) \in \mathbb{R}^1$ and any $\alpha \in [0, 2\pi)$ if and only if $\varphi = \varphi_0$.

For real-valued functions $\varphi \in B_\sigma \cap C(\mathbb{R}^1)$, (2.1) can be reduced to the inequality

$$(\varphi'(\tau))^2 + \sigma^2 (\varphi(\tau))^2 \leq \sigma^2 \|\varphi\|^2_{C(\mathbb{R}^1)}, \quad \tau \in \mathbb{R}^1. \hspace{1cm} (2.3)$$

The functions $\varphi_0(\tau) := R_1 \cos \sigma \tau + R_2 \sin \sigma \tau, R_1 \in \mathbb{R}^1, R_2 \in \mathbb{R}^1$, with $\|\varphi_0\|_{C(\mathbb{R}^1)} = \left(R_1^2 + R_2^2\right)^{1/2}$ are the only extremal functions in (2.3), i.e., equality holds in (2.3) for a certain $\tau = \tau_0(\sigma, R_1, R_2) \in \mathbb{R}^1$ if and only if $\varphi = \varphi_0$.

The proof of (2.1) with the description of all extremal functions in (2.1) and (2.3) can be found in [11, Sect. 84] (see also [9, Ch. 11]).
Bernstein–type Inequalities on $\mathbb{R}^m$. The following theorem extends inequality (2.1) and its extremal functions to multivariate entire functions from $B_V \cap C(\mathbb{R}^m)$.

**Theorem 2.1.** For every $f \in B_V \cap C(\mathbb{R}^m)$, $x \in \mathbb{R}^m$, $y \in \partial(K^*)$, and $\alpha \in [0,2\pi)$, the following inequality holds:

$$|\sin \alpha (\nabla f(x), y) - M(K, V) \cos \alpha f(x)| \leq M(K, V)\|f\|_{C(\mathbb{R}^m)}. \quad (2.4)$$

There exist $x = x_0 \in \mathbb{R}^m$ and $y = y_0 \in \partial(K^*)$ such that for any $\alpha \in [0,2\pi)$ and the functions

$$f(x) = f_0(x) := C_1e^{i(a,x)} + C_2e^{-i(a,x)}, \quad C_1 \in \mathbb{C}^1, \quad C_2 \in \mathbb{C}^1, \quad (2.5)$$

with $\|f_0\|_{C(\mathbb{R}^m)} = (|C_1|^2 + |C_1|^2)^{1/2}$, equality holds in (2.4). Hence inequality (2.4) is sharp.

**Proof. Step 1.** We first need the following elementary lemma:

**Lemma 2.2.** Given two complex numbers $a + ib$ and $c + id$, the function

$$h(\lambda) := \sup_{\alpha \in [0,2\pi)} |\lambda \sin \alpha (a + ib) - \cos \alpha (c + id)|$$

is not decreasing on $(0, \infty)$.

**Proof.** If $(a + ib)(c + id) = 0$, then the statement is trivial. If $(a + ib)(c + id) \neq 0$, then by a straightforward calculation for $\lambda \in (0, \infty)$,

$$h^2(\lambda) = (1/2) \left( \lambda^2 (a^2 + b^2) + c^2 + d^2 + \sqrt{[\lambda^2 (a^2 + b^2) - c^2 - d^2]^2 + 4\lambda^2(ac + bd)^2} \right),$$

$$h'(\lambda) = \frac{\lambda(a^2 + b^2)}{2h(\lambda)} \left( 1 + \frac{\lambda^2 (a^2 + b^2) - c^2 - d^2 + 4(ac + bd)^2}{\sqrt{[\lambda^2 (a^2 + b^2) - c^2 - d^2]^2 + 4\lambda^2(ac + bd)^2}} \right) \geq 0.$$ 

Note that the expression under the square roots above is positive for $\lambda > 0$. Thus the lemma is established.

**Step 2.** Next, let $f \in B_V \cap C(\mathbb{R}^m)$, $x \in \mathbb{R}^m$, and $y \in \mathbb{R}^m \setminus 0$, and let

$$\varphi(w) = \varphi(\tau + i\gamma) := f(x + (\tau + i\gamma)y), \quad \tau \in \mathbb{R}^1, \quad \gamma \in \mathbb{R}^1,$$

be the restriction of $f(z)$ to the one-dimensional complex plane $z = x + wy$ in $\mathbb{C}^m$. Then the following property holds:

**Lemma 2.3.** $\varphi \in B_{\|y\|_V} \cap C(\mathbb{R}^1)$. 

Proof. \( \varphi \) is an entire function of \( w \in \mathbb{C}^1 \), and for any \( \varepsilon > 0 \),
\[
|\varphi(w)| \leq C_0(\varepsilon, f) \exp [(1 + \varepsilon)||x + wy||_{V^*}] \leq C_0(\varepsilon, f) \exp [(1 + \varepsilon)||x||_{V^*}] \exp [(1 + \varepsilon)||w||_{V^*}].
\]

Therefore, \( \varphi \) is a univariate entire function of exponential type \( ||y||_{V^*} \) and \( \varphi \in C(\mathbb{R}^1) \). \( \Box \)

**Step 3.** Using now Step 2 and inequality (2.1) for \( \tau = 0 \), we obtain
\[
|\sin \alpha \langle \nabla f(x), y \rangle - ||y||_{V^*} \cos \alpha f(x) \rangle| = |\sin \alpha \varphi'(0) - ||y||_{V^*} \cos \alpha \varphi(0)|
\leq ||y||_{V^*} \| \varphi \|_{C(\mathbb{R}^1)} \leq ||y||_{V^*} \| f \|_{C(\mathbb{R}^m)}.
\]

Hence for any \( x \in \mathbb{R}^m \) and \( y \in \partial(K^*) \),
\[
\sup_{\alpha \in [0,2\pi]} |\sin \alpha \langle \nabla f(x), y \rangle ||y||_{K^*} / ||y||_{V^*} - \cos \alpha f(x) \rangle| \leq ||f||_{C(\mathbb{R}^m)}.
\] (2.6)

Next, note that for any \( y \in \mathbb{R}^m \setminus \{0\} \),
\[
||y||_{K^*} / ||y||_{V^*} \geq 1/M(V^*, K^*) = 1/M(K, V),
\] (2.7)

by (1.2) and (1.4). Finally applying Lemma 2.2 for \( a + ib = \langle \nabla f(x), y \rangle \) and \( c + id = f(x) \), we obtain (2.4) from (2.6) and (2.7).

**Step 4.** Finally, by Proposition 1.1 for \( \mathbf{a} \in \mathcal{A} \) there exists \( \mathbf{b} = y_0 \in \mathcal{B} \) such that \( \langle \mathbf{a}, y_0 \rangle = ||\mathbf{a}||_K = M(K, V) \). Then the function \( f_0 \) defined by (2.5) belongs to \( B_V \cap C(\mathbb{R}^m) \), and there exists \( x_0 \in \mathbb{R}^m \) such that
\[
|\sin \alpha \langle \nabla f_0(x_0), y_0 \rangle - M(K, V) \cos \alpha f_0(x_0) \rangle|
= M(K, V) \left| e^{-i(\alpha + \pi)} C_1 e^{i(\mathbf{a}, x_0)} + e^{i(\alpha + \pi)} C_2 e^{-i(\mathbf{a}, x_0)} \right| = M(K, V) ||f_0||_{C(\mathbb{R}^m)}.
\]

This completes the proof of Theorem 2.1 \( \Box \)

Multivariate versions of (2.2) and (2.3) are presented below.

**Corollary 2.4.** (a) For every \( f \in B_V \cap C(\mathbb{R}^m) \) and \( x \in \mathbb{R}^m \), the following inequality holds:
\[
||\nabla f(x)||_K \leq M(K, V) ||f||_{C(\mathbb{R}^m)}.
\] (2.8)

There exists \( x = x_0 \in \mathbb{R}^m \) such that for functions (2.5) equality holds in (2.8). Hence inequality (2.8) is sharp.

(b) For a real-valued function \( f \in B_V \cap C(\mathbb{R}^m) \) and every \( x \in \mathbb{R}^m \), the following inequality holds:
\[
||\nabla f(x)||^2_K + M^2(K, V) f^2(x) \leq M^2(K, V) ||f||^2_{C(\mathbb{R}^m)}.
\] (2.9)
There exists \( x = x_0 \in \mathbb{R}^m \) such that for functions

\[
f(x) = f_0(x) := R_1 \cos(a, x) + R_2 \sin(a, x), \quad R_1 \in \mathbb{R}^1, \quad R_2 \in \mathbb{R}^1,
\]

with \( \|f_0\|_{C(\mathbb{R}^m)} = (R_1^2 + R_2^2)^{1/2} \), equality holds in (2.9). Hence inequality (2.9) is sharp.

**Proof.** (a) Choosing in (2.4) \( \alpha = \pi/2 \) and \( y \in \partial(K^*) \) such that \( |(\nabla f(x), y)| = \|\nabla f(x)\|_K \), we arrive at (2.8). In addition, there exists \( x_0 \in \mathbb{R}^m \) such that the function \( f_0 \) defined by (2.5) satisfies the equalities

\[
\|\nabla f_0(x_0)\|_K = \|a\|_K \left| C_1 e^{i(a, x_0)} - C_2 e^{-i(a, x_0)} \right| = M(K, V) \|f_0\|_{C(\mathbb{R}^m)},
\]

(2.11)

(b) Inequalities (2.4) and

\[
(\nabla f(x), y)^2 + M^2(K, V)f^2(x) \leq M^2(K, V)\|f\|^2_{C(\mathbb{R}^m)}, \quad x \in \mathbb{R}^m, \quad y \in \partial(K^*),
\]

(2.12)

are equivalent for real-valued functions \( f \in B_V \cap C(\mathbb{R}^m) \). Then choosing \( y \in \partial(K^*) \) such that \( |(\nabla f(x), y)| = \|\nabla f(x)\|_K \), we obtain (2.9) from (2.12). The case of equality in (2.9) for functions \( f_0 \) can be proved similarly to (2.11). \( \square \)

**Remark 2.5.** Inequalities (2.1), (2.2), and (2.3) are special cases of (2.4), (2.8), and (2.9), respectively, for \( K = V = [-1, 1] \). The author [12, Theorem 1] proved inequality (2.8) for real-valued functions by a different method. Other special cases of inequality (2.8) established earlier include \( K = Q^m, V = \sigma Q^m, M(K, V) = \sigma \) (see Bernstein [4]); \( K = B^m, V = \sigma B^m, M(K, V) = \sigma \) (see Nikolskii [21 Sect. 3.2.6]); and \( K = B^m, M(K, V) = d(V)/2 \) by (1.6) (see the author [11 Theorem 4]).

**Remark 2.6.** Since \( T_\sigma V \subset B_\sigma V, \sigma > 0 \), inequalities (2.4), (2.8), and (2.9) hold for trigonometric polynomials from \( T_\sigma V \). Moreover, these inequalities are asymptotically sharp in \( T_\sigma V \) as \( \sigma \to \infty \). For example, the function \( f_0(x) := C_1 e^{i(k_0, x)} + C_2 e^{-i(k_0, x)} \), where \( k_0 \in \sigma V \cap \mathbb{Z}^m \) is a closest point to \( \sigma a \in \sigma V \), belongs to \( T_\sigma V \), and

\[
\sup_{x \in \mathbb{R}^m} \|\nabla f_0(x)\|_K \geq (\sigma M(K, V) - C) \|f_0\|_{C(\mathbb{R}^m)},
\]

where \( C > 0 \) is a constant independent of \( \sigma \).

**Remark 2.7.** Versions of inequalities (2.4) and (2.8) for continuous \( n \)-homogeneous polynomials on a real or complex Hilbert space were obtained by Anagnostopoulos, Sarantopoulos, and Tonge [2 Theorem 2.2].
Example 2.8. If $K = V_\mu$ and $V = V_\lambda$, $1 \leq \mu, \lambda \leq \infty$, then

\[
M(V_\mu, V_\lambda) = \begin{cases} m^{1/\mu - 1/\lambda}, & 1 \leq \mu < \lambda \leq \infty, \\ 1, & 1 \leq \lambda \leq \mu \leq \infty. \end{cases}
\] (2.13)

Therefore, by Theorem 2.1 and Corollary 2.4, sharp inequalities (2.4), (2.8), and (2.9) hold with $M(V_\mu, V_\lambda)$ defined by (2.13). In addition,

\[
a = a(V_\mu, V_\lambda) = \begin{cases} (\pm m^{-1/\lambda}, \ldots, \pm m^{-1/\lambda}), & 1 \leq \mu < \lambda \leq \infty, \\ (0, \ldots, \pm 1, \ldots, 0), & 1 \leq \lambda \leq \mu \leq \infty, \end{cases}
\]

\[
b = b((V_\lambda)^*, (V_\mu)^*) = \begin{cases} (\pm m^{1/\mu - 1}, \ldots, \pm m^{1/\mu - 1}), & 1 \leq \mu < \lambda \leq \infty, \\ (0, \ldots, \pm 1, \ldots, 0), & 1 \leq \lambda \leq \mu \leq \infty, \end{cases}
\]

\[= \pm (M(V_\mu, V_\lambda) |a|^{-2}) a.
\]

3. Extremal Functions

Throughout the section points $a \in A(K, V)$ and $b \in B(V^*, K^*)$ are defined by (1.2), (1.3), and (1.4).

Theorem 2.1 and Corollary 2.4 show that equalities hold in (2.4) (or (2.8)) and (2.9) for functions (2.5) and (2.10), respectively. In particular, functions (2.10) are extremal in inequalities (2.8) and (2.9), i.e., there exists $x = x_0 \in \mathbb{R}^m$ such that equality holds in (2.8) and (2.9) for functions (2.10).

However, unlike the univariate case, a real-valued extremal function in (2.8) and (2.9) for $m \geq 2$ does not necessarily coincide with (2.10). For example, the function $f_0(x) := R \cos \sigma |x|$, $R \in \mathbb{R}$, $\sigma > 0$, belongs to $B_{\sigma \mathbb{B}^m}$, and it is easy to verify that $f_0$ is an extremal function in (2.8) and (2.9) for $V = \sigma \mathbb{B}^m$, while $f_0$ cannot be represented in the form of (2.10).

Nevertheless, the following theorem shows that there are certain relations between functions (2.10) and real-valued extremal functions in (2.8) and (2.9).

**Theorem 3.1.** Let $f_0 \in B_V \cap C(\mathbb{R}^m)$ be a real-valued extremal function in (2.8) or (2.9). Then there exists a function $g(x) := R_1 \cos (a, x) + R_2 \sin (a, x)$, where $a \in A$, $R_1 \in \mathbb{R}^1$, and $R_2 \in \mathbb{R}^1$, and there exists a straight line $x = x_0 + \tau y_0$ in $\mathbb{R}^m$, where $x_0 \in \mathbb{R}^m$, $y_0 \in \mathcal{B}$ and $\tau \in \mathbb{R}^1$, such that for all $\tau \in \mathbb{R}^1$ the following equalities hold:

\[
f_0(x_0 + \tau y_0) = g(x_0 + \tau y_0),
\]

\[\nabla f_0(x_0 + \tau y_0) = \nabla g(x_0 + \tau y_0).
\] (3.1) (3.2)

In addition, $f_0(x_0) = 0.$
Proof. **Step 1.** We first note that if $f_0(\cdot) \in B_V \cap C(\mathbb{R}^m)$ is a real-valued extremal function in (2.8), then $f_0(\cdot + u)$ is an extremal function in (2.9) for any fixed $u \in \mathbb{R}^m$. So without loss of generality we can assume that $\|f_0\|_{C(\mathbb{R}^m)} = 1$ and

$$\|\nabla f_0(0)\|_K = M(K,V)\sqrt{1-f_0^2(0)},$$

(3.3)

i.e., it suffices to prove (3.1) and (3.2) for $x_0 = 0$.

**Step 2.** Next, we prove (3.1). Setting

$$a := \left(\frac{1}{\sqrt{1-f_0^2(0)}}\right) \nabla f_0(0),$$

(3.4)

we see from (2.9) for $K = V$ that $a \in V$. In addition, $\|a\|_K = M(K,V)$ by (3.3). Therefore, $a \in A$, and by Proposition 1.1, there exists $b = y_0 \in B$ such that $M(K,V) = M(V^*, K^*) = |(a,b)|$.

Then setting $\varphi(\tau) := f_0(\tau y_0), \tau \in \mathbb{R}^1$, we see from Lemma 2.3 that $\varphi$ is a univariate entire function of exponential type $\|y_0\|_{V^*} = M(K,V)$ and $\varphi \in C(\mathbb{R}^1)$. Thus

$$\frac{\|\nabla f_0(0)\|_K}{\sqrt{1-f_0^2(0)}} = \frac{|(\nabla f_0(0), y_0)|}{\sqrt{1-f_0^2(0)}} = \frac{|\varphi'(0)|}{\sqrt{1-\varphi^2(0)}} = M(K,V) = M(K,V)\|\varphi\|_{C(\mathbb{R}^1)},$$

(3.5)

where the last equality in (3.5) follows from (2.3). Then there exist $R_1 \in \mathbb{R}^1$ and $R_2 \in \mathbb{R}^1$ such that $\varphi(\tau) = R_1 \cos [M(K,V) \tau] + R_2 \sin [M(K,V) \tau]$. Taking into account the relations $\|\varphi\|_{C(\mathbb{R}^1)} = |\varphi'(0)|$ and $\|\varphi\|_{C(\mathbb{R}^1)} = 1$, we arrive at the equality $\varphi(\tau) = R_2 \sin [M(K,V) \tau]$, where $R_2 := \text{sgn} (\nabla f_0(0), y_0)$. Therefore, $f_0(0) = \varphi(0) = 0$ and $g(x) := R_2 \sin (a,x)$, where $a := \nabla f_0(0)$ by (3.4). This proves (3.1).

**Step 3.** Furthermore, we prove (3.2). Without loss of generality we can assume that $(\nabla f_0(0), y_0) > 0$, i.e., $g(x) = \sin (a,x)$. Next, let us define the function $h := f_0 - g$ and the straight line

$$L := \{x \in \mathbb{R}^m : x = \tau y_0, \tau \in \mathbb{R}^1\}.$$  

(3.6)

It is clear that $h \in B_V \cap C(\mathbb{R}^m)$. Then

$$h(x) = 0, \quad x \in L,$$

by (3.1) and

$$\nabla h(0) = 0,$$  

(3.7)

since by the definition of $a$ and $g$, $\nabla f_0(0) = a = \nabla g(0)$. Next, let

$$\tau_k := \frac{(2k + 1)\pi}{2M(K,V)}, \quad U_k := \tau_k y_0, \quad k = 0, \pm 1, \ldots,$$

be points on $\mathbb{R}^1$ and $L$, respectively. In addition, let $H_k$ be the hyperplane passing through $U_k, k = 0, \pm 1, \ldots$, that is orthogonal to $a$. Note that $U_k$ is the only element of $H_k \cap L, k = 0, \pm 1, \ldots$, since $(a, y_0) > 0$. 


Then $g$ is constant on $H_k$ and

$$g(x) = g(U_k) = \sin \tau_k = (-1)^k, \quad x \in H_k, \quad k = 0, \pm 1, \ldots.$$ 

Therefore,

$$(-1)^k f_0(x) \leq \|f_0\|_{C(\mathbb{R}^m)} = (-1)^k g(x), \quad x \in H_k, \quad k = 0, \pm 1, \ldots,$$

which implies

$$(-1)^k h(x) \leq 0, \quad x \in H_k, \quad k = 0, \pm 1, \ldots \quad (3.8)$$

Since by (3.6), $h(U_k) = 0$, $k = 0, \pm 1, \ldots$, we see from (3.8) that $h$ has a relative maximum (minimum) on $H_k$ at $U_k$ for even (odd) numbers $k = 0, \pm 1, \ldots$. Choosing now $m - 1$ linearly independent vectors $\{y^{(d,k)}\}_{d=1}^{m-1}$ in $H_k$, we obtain by a necessary condition for a relative extremum that

$$\left(\nabla h(U_k), y^{(d,k)}\right) = 0, \quad 1 \leq d \leq m - 1, \quad k = 0, \pm 1, \ldots \quad (3.9)$$

In addition, using (3.6), we have that

$$\left(\nabla h(U_k), y_0\right) = 0, \quad k = 0, \pm 1, \ldots \quad (3.10)$$

Then the system of vectors $\{y_0, y^{(1,k)}, \ldots, y^{(m-1,k)}\}$ is linearly independent (we recall that $H_k \cap \mathbb{L} = \{U_k\}, k = 0, \pm 1, \ldots$), and combining (3.9) with (3.10), we arrive at the equality

$$\nabla h(U_k) = 0, \quad k = 0, \pm 1, \ldots \quad (3.11)$$

Next, $h_j(x) := \partial h(x)/\partial x_j \in C(\mathbb{R}^m)$ by (2.8) and, in addition, $h_j \in B_{V_j}, 1 \leq j \leq m$ (see [21, Sect. 3.1] and [13, Lemma 2.1 (d)]). Then Step 2 of the proof of Theorem 2.1 shows that the restriction $\varphi_j$ of $h_j$ to $\mathbb{L}$ belongs to $B_{M(K,V)} \cap C(\mathbb{R}^1), 1 \leq j \leq m$. We also take account of the equations $\varphi_j(\tau_k) = 0, k = 0, \pm 1, \ldots, 1 \leq j \leq m$, that follow from (3.11).

Therefore, by a well-known result (see, e.g., [27, Sect. 4.3.1]), $\varphi_j(\tau) = C_j \cos [M(K,V) \tau]$, and taking account of $\varphi_j(0) = 0$ that follows from (3.7), we obtain that for all $\tau \in \mathbb{R}^1, \varphi_j(\tau) = 0, 1 \leq j \leq m$. Thus (3.2) is established. \hfill \Box

**Remark 3.2.** Theorem 3.1 is an analogue of properties of extremal polynomials in Markov–type inequalities proved by Kroó [18, Corollary 1] and Révész [22, Theorem 1], and certain ideas from [18, 22] are used in the proof of Theorem 3.1. We do not know as to whether the corresponding analogues of the criteria for the uniqueness of extremal polynomials discussed in [18, Theorem 2] and [22, Theorem 2] are valid for extremal functions in Bernstein–type inequalities.
4. Markov-type Inequalities

Throughout the section a point $a^* = (a^*_1, \ldots, a^*_n) \in \partial (V^*)$ satisfies the equality $\|a^*\|_K = M(K, V^*)$, i.e., $a^* \in A(K, V^*)$ (see (1.1) and (1.2)), and $I^m := \{ u \in \mathbb{R}^m : u_j \geq 0, 1 \leq j \leq m \}$ is the first orthant of $\mathbb{R}^m$.

Markov-type inequalities on convex bodies with sharp constants have been a hot topic in approximation theory since the early 1990s (see surveys [19] and [8]).

**Symmetric Bodies.** The following multivariate version of the Markov inequality for polynomials with real coefficients was proved by Sarantopoulos [23 Theorem 2] who elegantly applied the inscribed ellipse method (IEM) to this problem.

**Theorem 4.1.** For $P \in \mathcal{P}_{n,m}$,

$$|(\nabla P(x), y)| \leq \|y\|_V n^2 \|P\|_{C(V)}, \quad x \in V, \quad y \in \mathbb{R}^m \setminus \{0\}. \quad (4.1)$$

Independently and by the pluripotential theory approach (PTA), Baran [3 Theorem 2 (c)] established (4.1) and extended this inequality to any $P \in \mathcal{P}_{n,m}$ with complex coefficients. It turned out later that IEM and PTA are equivalent (see Burns, Levenberg, Ma’u, and Révész [7 Corollary 4.3]). In case of $V = \mathfrak{B}^m$, (4.1) was earlier obtained by Kellogg [17 Theorem VI].

**Remark 4.2.** For the reader’s convenience, we present here a shorter and more straightforward proof of (4.1), compared with [3], for complex-valued polynomials (cf. [15 Theorem 1.1]). Let $y \in \mathbb{R}^m \setminus \{0\}$ and let $P = P_1 + iP_2 \in \mathcal{P}_{n,m}$ be a nonconstant polynomial, where $P_j \in \mathcal{P}_{n,m}, j = 1, 2$, are polynomials with real coefficients. Then there exists $\xi \in V$ such that $\max_{x \in V} |(\nabla P(x), y)| = |(\nabla P(\xi), y)|$. Let us define $\gamma \in [0, 2\pi)$ by the equality $e^{i\gamma} = (\nabla P(\xi), y) / |(\nabla P(\xi), y)|$. Then the polynomial $D := \cos \gamma P_1 + \sin \gamma P_2 \in \mathcal{P}_{n,m}$ has real coefficients and satisfies the relations: $|D(x)| \leq |P(x)|$ for $x \in V$ and $|\nabla D(\xi, y)| = |(\nabla P(\xi), y)|$. Therefore, for any $x \in V$ and $y \in \mathbb{R}^m \setminus \{0\}$,

$$\frac{|(\nabla P(x), y)|}{\|P\|_{C(V)}} \leq \frac{|(\nabla P(\xi), y)|}{\|P\|_{C(V)}} \leq \frac{|(\nabla D(\xi, y)|}{\|D\|_{C(V)}} \leq \|y\|_V n^2,$$

by [23 Theorem 2]. Thus (4.1) is established.

Inequality (4.1) along with (4.1) and (1.6) immediately imply the following relations:

$$|(\nabla P(x)| = \|\nabla P(x)\|_{2m} \leq M(\mathfrak{B}^m, V) n^2 \|P\|_{C(V)} = \frac{2n^2}{w(V)} \|P\|_{C(V)}, \quad x \in V, \quad (4.2)$$

(see, e.g., [20 Eq. (7)]). The polynomial $P_0(x) := CT_n((a^*, x)), C \in \mathbb{C}^1$, is an extremal polynomial in (4.2), i.e., equality holds in (4.2) for $P = P_0$ and for $x = x_0 \in V$, satisfying the equality $(a^*, x_0) = \pm 1$. 
However, unlike the univariate case, a real-valued extremal polynomial in (4.2) for $m \geq 2$ does not necessarily coincide with $P_0$. For example, the polynomial $P(x) := RT_n(|x|)$, $R \in \mathbb{R}^1$, for an even $n$ is an extremal polynomial in (4.2) for $V = \mathfrak{B}^m$ (not $T_{n/2}(|x|^2)$ as stated in (22) p. 466), while $P$ cannot be represented in the form of $P_0$.

Kroó [18] and Révész [22] studied properties of the extremal polynomials in (4.2). In particular, they found certain relations (like (3.1) and (3.2)) between polynomials $P_0$ for $V = B_m$ (not $T_n/2(|x|^2)$ as stated in [22, p. 466]), while $P$ cannot be represented in the form of $P_0$.

The following result is a polynomial version of (2.8) and a more general version of (4.2).

**Corollary 4.3.** For every $P \in \mathcal{P}_{n,m}$ and $x \in V$, the following inequality holds:

$$\|\nabla P(x)\|_K \leq M(K, V^*) n^2 \|P\|_{C(V)}.$$  \hspace{1cm} (4.3)

Equality holds in (4.3) for $P(x) = P_0(x) := CT_n((a^*, x))$, $C \in \mathbb{C}^1$, and $x = x_0 \in V \cap H_\pm(a^*)$, where $H_\pm(a^*) := \{x \in \mathbb{R}^m : (a^*, x) = \pm 1\}$ are parallel supporting hyperplanes of $V$. Hence inequality (4.3) is sharp.

**Proof.** It follows from Theorem 4.1 that for every $y \in K^*$,

$$|(\nabla P(x), y)| \leq \|y\|_V/\|y\|_K \cdot n^2 \|P\|_{C(V)} \leq M(V, K^*) n^2 \|P\|_{C(V)} = M(K, V^*) n^2 \|P\|_{C(V)},$$  \hspace{1cm} (4.4)

by (1.4). Then (4.3) follows from (4.4). Finally, $\|\nabla P_0(x_0)\|_K = \|a^*\|_K n^2 \|P_0\|_{C(V)}$. \hspace{1cm} \Box

**Remark 4.4.** Note that for $K = \mathfrak{B}^m$ (4.3) is reduced to (4.2). In addition, we conjecture that using the techniques from [18] and [22], it is possible to extend the properties of the extremal polynomials in (4.2) to the extremal polynomials in (4.3).

**Non-symmetric Bodies.** Bia/ls-Cie˙ z and Goetgheluck [5, Sect. 3] found out that (4.2) fails for certain non-symmetric convex bodies, in particular, for the simplex

$$\Delta_0 := \left\{u \in I^m : 0 \leq \sum_{j=1}^m u_j \leq 1\right\}.$$

Certain estimates of the sharp constant in the Markov-type inequality on triangles in $\mathbb{R}^2$ were obtained by Kroó and Révész [20, Theorem 4]. Other Markov-type inequalities on convex bodies were obtained by Wilhelmsen [28] and Ditzian [9]. Skalyga [24, Theorems 1 and 2] proved that for every convex body $\mathcal{C} \subset \mathbb{R}^m$ and $Q \in \mathcal{P}_{n,m}$,

$$\|\nabla Q\|_{C(\mathcal{C})} \leq \frac{2}{w(\mathcal{C})} n \cot \left(\frac{\pi}{4n}\right) \|Q\|_{C(\mathcal{C})},$$

and this inequality cannot be improved on the class of all convex bodies in $\mathbb{R}^m$. The same result was independently announced by Subbotin and Vasiliev [26] as well.

However, no sharp constant in the Markov-type inequality on the given non-symmetric convex body $C$ is known. Below we find sharp constants for certain bodies $C$ if the gradient $\nabla f(u)$ is replaced by the “weighted” gradient vector

$$\nabla_u f(u) := \left( \sqrt{u_1} \frac{\partial f(u)}{\partial u_1}, \ldots, \sqrt{u_m} \frac{\partial f(u)}{\partial u_m} \right), \quad u \in I^m. \quad (4.5)$$

We first need certain conditions on $K$ and $V$.

**Definition 4.5.** We say that a body $V \subset \mathbb{R}^m$ satisfies the $S$-condition if $V$ is symmetric about all coordinate hyperplanes, that is, for every $x \in V$ the vectors $(\pm |x_1|, \ldots, \pm |x_m|)$ belong to $V$.

The $S$-condition is equivalent to the $\Pi$-condition introduced in [14, Definition 1.1].

**Definition 4.6.** We say that a pair $(K, V), K \subset \mathbb{R}^m, V \subset \mathbb{R}^m$, satisfies the $a^*$-condition if there exists a point $a^* \in A(K, V^*)$ that belongs to a coordinate axis.

Next, given $V \subset \mathbb{R}^m$, we define the body $C = C(V) \subset I^m$ (not necessarily convex) by the formula

$$C := \{ u \in I^m : \| (\sqrt{u_1}, \ldots, \sqrt{u_m}) \|_V \leq 1 \}. \quad (4.6)$$

Then the following result is valid.

**Theorem 4.7.** Let $V$ satisfy the $S$-condition and let $\nabla_u$ and $C$ be defined by (4.5) and (4.6), respectively. Then for every $Q \in \mathcal{P}_{n,m}$ and $u \in C$, the following inequality holds:

$$\| \nabla_u Q(u) \|_K \leq 2M(K, V^*) n^2 \| Q \|_{C(V)}. \quad (4.7)$$

If $(K, V)$ satisfies the $a^*$-condition, then there exists $k, 1 \leq k \leq m$, such that equality holds in (4.7) for $Q(u) = Q_0(u) := CT_{2n} \left( a^*_k \sqrt{u_k} \right), \ C \in \mathcal{C}^1$, and $u = u_0 := \left(0, \ldots, 0, \pm (a^*_k)^{-2}, 0, \ldots, 0\right) \in C$.

**Proof.** First note that the following statement is an immediate consequence of Definition 4.5 and the definition of $C$: $x \in V$ if and only if $u = (x_1^2, \ldots, x_m^2) \in C$. Next, by Corollary 4.3 for every $P(x) = \sum_{|\alpha| \leq n} c_\alpha x^{2\alpha} \in \mathcal{P}_{2n,m}$ and $x \in V$, the following inequality holds:

$$\| \nabla P(x) \|_K \leq 4M(K, V^*) n^2 \| P \|_{C(V)}. \quad (4.8)$$

Therefore, it follows from (4.8) and the aforementioned statement that for every polynomial $Q(u) = \sum_{|\alpha| \leq n} c_\alpha u^\alpha \in \mathcal{P}_{n,m}$ and $u \in C$,

$$\| \nabla_u Q(u) \|_K = (1/2) \| \nabla P(x) \|_K \leq 2M(K, V^*) n^2 \| P \|_{C(V)} = 2M(K, V^*) n^2 \| Q \|_{C(V)}.$$
Furthermore, $V$ is valid.

In particular, for the simplex $\Delta$ with $1 \leq k \leq m$, such that $a^* = (0, \ldots, 0, a^*_k, 0, \ldots, 0)$. Then by Corollary 4.3, equality holds in (4.8) for $P(x) = P_0(x) := CT_{2n}(a^*_k x_k) \in P_{2n,m}$, $C \in C^1$, and $x = (0, \ldots, 0, \pm 1/a^*_k, 0, \ldots, 0) \in V \cap H_\pm(a^*)$, where $H_\pm(a^*) := \{ x \in \mathbb{R}^m : x_k = \pm 1/a^*_k \}$ are parallel supporting hyperplanes of $V$. Therefore, equality holds in (4.7) for $Q(u) = Q_0(u) := CT_{2n}(a^*_k \sqrt{u_k}) \in P_{n,m}$ and $u = u_0 := (0, \ldots, 0, \pm (a^*_k)^{-2}, 0, \ldots, 0) \in C$. \hfill $\square$

**Example 4.8.** If $K = V_\mu$ and $V = V_\lambda$, $1 \leq \mu, \lambda \leq \infty$, then

$$M(V_\mu, (V_\lambda)^*) = \begin{cases} m^{1/\mu+1/\lambda-1}, & 1/\mu + 1/\lambda > 1, \\ 1, & 1/\mu + 1/\lambda \leq 1. \end{cases} \quad (4.9)$$

In addition,

$$a^* = a^*(V_\mu, (V_\lambda)^*) = \begin{cases} (\pm m^{1/\lambda-1}, \ldots, \pm m^{1/\lambda-1}), & 1/\mu + 1/\lambda > 1, \\ (0, \ldots, 0), & 1/\mu + 1/\lambda \leq 1. \end{cases}$$

Furthermore, $V_\mu$ satisfies the $S$-condition for $1 \leq \mu \leq \infty$, and $(V_\mu, V_\lambda)$ satisfies the $a^*$-condition for $1/\mu + 1/\lambda \leq 1$. Then by Theorem 4.7, sharp inequality (4.7) holds for $K = V_\mu$, $C = C_\lambda := V_\lambda/2 \cap I^m$, and $1/\mu + 1/\lambda \leq 1$ with $M(V_\mu, (V_\lambda)^*)$ defined by (4.9). Note that $C_\lambda$ is not convex for $1 \leq \lambda < 2$.

In particular, for the simplex $\Delta_0 = C_2$ and $\mu = 2$ the sharp inequality

$$|\nabla u Q(u)| \leq 2n^2 \|Q\|_{C(\Delta_0)}, \quad Q \in P_{n,m}, \quad u \in \Delta_0,$$

is valid.

**Acknowledgement** We are grateful to Leokadia Biaś-Cież for the provision of references.

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