BROWN REPRESENTABILITY FOR DIRECTED GRAPHS

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ABSTRACT. We prove that any contravariant functor from the homotopy category of finite directed graphs to abelian groups satisfying the additivity axiom and the Mayer-Vietoris axiom is representable.

1. INTRODUCTION

The homotopy theory of directed graphs is a discrete analogue of homotopy theory in algebraic topology. In topology, a homotopy between two continuous maps is defined by an interpolating family of continuous maps parametrized by a closed interval $[0,1]$. Its discrete analogue we study uses a directed line graph keeping track of a discrete change of directed graph maps. See for example Grigor’yan, Lin, Muranov, and Yau [8]. Efforts towards a homotopy theory for graphs extend back to the 1970’s and 1980’s with some results by Gianella [6] and Malle [16]. However, the most recent variant of graph homotopy theory appears to have taken off with a paper by Chen, Yau, and Yeh from 2001 [2], before culminating in the 2014 work of Grigor’yan, Lin, Muranov, and Yau [8]. Recently, there is increased interest in this new notion of homotopy for directed graphs because it was shown by Grigor’yan, Jimenez, Muranov, and Yau in [7] that the path-space homology theory of directed graphs is invariant under this version of graph homotopy.

The path-space homology and cohomology theories for directed graphs were studied by Grigor’yan, Lin, Muranov, and Yau [9] and by Grigor’yan, Muranov, and Yau in [10] and [11]. The homology theory developed in the references above is quite natural, computable, and can be non-trivial for degrees greater than one, depending on the lengths of admissible, $\partial$-invariant paths in a directed graph. Thus, the fact that this homology is invariant under this notion of homotopy for directed graphs suggests that both are the appropriate notions for the category of directed graphs. Furthermore, this notion of homotopy is of interest because counting the essential types of cycles in a directed graph is of interest anywhere one finds directed graphs that model a particular phenomenon, for example quivers, neural nets, electrical circuits, etc.

In this paper, we prove the Brown representability theorem for the category of directed graphs by modifying some constructions of Adams [1] for connected CW complexes with base point. More specifically, we prove that any contravariant functor from the homotopy category of finite directed graphs to the category of abelian groups satisfying the additivity axiom and the Mayer-Vietoris axiom is representable. It should be noted that simply attempting to verify Brown’s axioms [4] for the category of directed graphs is insufficient as we shall show later in Section A.2.
The Brown representability theorem is a classical theorem in algebraic topology first proved by Edgar H. Brown [3]. It states that, given a $\text{Set}$-valued functor on the homotopy category of based CW complexes satisfying the wedge axiom and the Mayer-Vietoris axiom, then that functor is representable. Brown went further in [4] by replacing the homotopy category of based CW complexes with an arbitrary category satisfying a list of proposed axioms, and this result has been further generalized in triangulated categories by Neeman [17], closed model categories in Jardine [13, Theorem 19], and homotopy categories of $\infty$-categories in Lurie [15, Section 1.4.1]. The gist of these generalizations is that Brown representability is more of a category-theoretic feature than topological. However, the idea behind the classical theorem of J. H. C. Whitehead that every CW complex is formed by attaching spheres is essential, whereas an analogue of such in the homotopy category of directed graphs is not well-understood yet. A more fundamental issue in here is that the homotopy extension property fails in the category of directed graphs (see Section A.2). This is why the approach and proof of Adams [1] is more relevant for our purpose in that the investigation of representability of functors defined on finite complexes therein leads us to guess that contravariant functors on the homotopy category of finite directed graphs are representable.

The main technical innovation in this paper is in our constructions 2.21 and 2.25 which mimic the topological setting by inserting a middle slice that preserves the directed graph and its homotopy type. One cannot say that the proof of Adams we are using gives the result mutatis mutandis in that the proof will fail with a use of the cofibering therein. One has to notice that the homotopy category of directed graphs are in some respects drastically different from the homotopy category of CW complexes and at times category-theoretically correct constructions are no longer relevant for directed graph homotopy theory. Therefore, a Brown representability theorem for directed graphs is worth pursuing not only because of the importance of Brown’s representability theorem, it being one of the pillars in stable homotopy theory, but also because it is a good starting point for the study of homotopy theory for directed graphs in depth. Furthermore, it is interesting to observe that the category of directed graphs, a discrete and coarsely structured category compared to the category of CW complexes, still admits a Brown representability.

Since we prove the representability for Brown functors on finite directed graphs, an interesting future direction would be investigating the same statement for both finite and infinite directed graphs. As mentioned earlier, this question is basically asking what the discrete analogue of attaching cells in a CW complex is, and is tightly connected to understanding higher homotopy groups of a directed graphs. (Compare Section 4.6 of Grigor’yan, Lin, Muranov, and Yau [8].)

The Brown representability theorem in topology requires the domain category of the functors to be the category of connected CW complexes with a base point. When the domain category is either the category of not-necessarily-connected CW complexes or the category of unbased CW complexes then there exist well-known counterexamples, see for example Heller [12] and Freyd and Heller [5]. The category of directed graphs is not as rich as the category of topological spaces and the Freyd-Heller counterexample is not likely to occur. However, it is an extra layer of difficulties to formally address this point, e.g. the non-existence of the functor $\pi$ of Freyd and Heller [5].

Since the topic of homotopy for directed graphs is a blend of algebraic topology and graph theory, we tried to write this paper as a self-contained account that audiences from both areas would find readable. We also tried to minimize deviating from Adams’ exposition [1] so that readers would find it easy to compare where and how the proof deviates from the one in topology. The paper is organized as follows. Section 2 consists of backgrounds on digraph homotopy theory and constructions therein. It serves the two-fold purpose of establishing notations and conventions, and providing constructions which underpin our arguments and the main result in later sections.
Section 3 defines a Brown functor and discusses its properties. Section 4 gives a construction of the classifying digraph of a Brown functor. In the Appendix A, we shall elucidate why certain constructions in homotopy theory do not serve purpose for directed graphs. Appendix B discusses the generalized inverse limit of Adams [1] for the sake of self-containedness.

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2. Homotopy Theory for Directed Graphs

In this section, we shall give a brief review of directed graph homotopy theory as well as relevant constructions. A good reference on digraph homotopy theory is Grigor’yan, Lin, Muranov, and Yau [8] which has a broader account. Several constructions we give in this section are the technical core of this paper. See 2.21 and 2.25.

2.1. The category of directed graphs.

Definition 2.1. A directed graph (or digraph for short) $\vec{G}$ is a pair $(V, E)$ consisting of a set $V$ specifying labeled points called vertices and another set $E$ of ordered pairs of distinct vertices in $V$ called edges.

Having an edge $(x, y) \in E$ means that there is a directed arrow from $x$ to $y$ and graphically one draws $x \rightarrow y$. Note that from the definition above, loop-edges are excluded from consideration and since $E$ is a set, $(x, y)$ occurs at most once.

Definition 2.2. A point is a digraph consisting of only one vertex and no edges.

Definition 2.3. An $n$-step line digraph, $n > 0$, is a sequence of vertices, $0, 1, 2, \ldots, n$, such that either $(i - 1, i)$ or $(i, i - 1)$, for $1 \leq i \leq n$, is an edge (but not both) and there are no other edges.

Note that an $n$-step line digraph is also called path digraph or a linear digraph. Such a directed graph forms a line with $n$ arbitrarily oriented edges between each of the $n + 1$ vertices. When $n = 1$, there are two possible line digraphs, $I^+ := 0 \rightarrow 1$ and $I^- := 0 \leftarrow 1$.

Notation 2.4. We will denote an arbitrary $n$-step line digraph as $I_n$ for short and let $I_n$ represent the set of all possible $n$-step line digraphs. The set of all line digraphs of any length will be denoted $\mathcal{I} = \bigcup_n I_n$ and we will refer to an arbitrary element of $\mathcal{I}$ as a line digraph $I$, dropping the reference to the number of steps.

Definition 2.5. A digraph map, $f : \vec{G} \rightarrow \vec{H}$, is a function from the vertex set of $\vec{G}$ to the vertex set of $\vec{H}$ such that whenever $(x, y) \in E$ either $f(x) = f(y)$ in $\vec{H}$ or $f(x) \rightarrow f(y)$ is an edge in $\vec{H}$.

If for some edge $(x, y)$, $f(x) = f(y)$ in $\vec{H}$, then we will say that this edge has been collapsed and if $(f(x), f(y)) \in E_{\vec{H}}$, then we say that the edge has been preserved. Since a digraph map must be a function on the discrete set of vertices, the image of a digraph map has at most as many vertices as the domain.

Definition 2.6. The category of directed graphs $\mathcal{D}$ is a category in which the objects are directed graphs, $\vec{G}$, and the morphisms are digraph maps, $f : \vec{G} \rightarrow \vec{H}$.
**Definition 2.7.** A graph $\vec{G} = (V, E)$ is **finite** if the vertex set $V$ is finite.

**Notation 2.8.** (1) We will use the notation $\mathcal{D}_0$ to denote the category whose objects are finite digraphs and morphisms are digraph maps. The category $\mathcal{D}_0$ is a subcategory of $\mathcal{D}$.

(2) Let $\mathcal{C}$ be a category. Throughout this paper, the expression $X \in \mathcal{C}$ means $X$ is an object of the category $\mathcal{C}$. We will write $f \in \mathcal{C}(X, Y)$ to say $f$ is a morphism from $X$ to $Y$ in $\mathcal{C}$.

2.2. **Operations for directed graphs.**

**Definition 2.9.** A sub-digraph $\vec{X}$ of a digraph $\vec{G}$ denoted $\vec{X} \subseteq \vec{G}$ is a digraph for which $V_{\vec{X}} \subseteq V_{\vec{G}}$ and $E_{\vec{X}} \subseteq E_{\vec{G}}$.

Note that even if $u, v \in V_{\vec{X}}$ and $(u, v) \in E_{\vec{G}}$, it is not necessarily the case that $(u, v) \in E_{\vec{X}}$.

**Definition 2.10.** An induced sub-digraph $\vec{X}$ of a digraph $\vec{G}$ denoted $\vec{X} \subset \vec{G}$ is a sub-digraph in which whenever $u, v \in V_{\vec{X}}$ and $(u, v) \in E_{\vec{G}}$, then $(u, v) \in E_{\vec{X}}$ too.

**Definition 2.11.** The intersection of digraphs $\vec{G}$ and $\vec{H}$, denoted by $\vec{G} \cap \vec{H}$, is the digraph consisting of $V_{\vec{G} \cap \vec{H}} = V_{\vec{G}} \cap V_{\vec{H}}$ and $E_{\vec{G} \cap \vec{H}} = E_{\vec{G}} \cap E_{\vec{H}}$.

Note that $\vec{G} \cap \vec{H}$ is not necessarily an induced sub-digraph of either $\vec{G}$ or $\vec{H}$.

**Definition 2.12.** The union of digraphs $\vec{G}$ and $\vec{H}$, denoted by $\vec{G} \cup \vec{H}$, is the digraph consisting of $V_{\vec{G} \cup \vec{H}} = V_{\vec{G}} \cup V_{\vec{H}}$ and $E_{\vec{G} \cup \vec{H}} = E_{\vec{G}} \cup E_{\vec{H}}$.

Note that $\vec{G}$ and $\vec{H}$ are necessarily induced subgraphs of $\vec{G} \cup \vec{H}$.

**Definition 2.13.** The disjoint union of two digraphs $\vec{G}$ and $\vec{H}$, denoted $\vec{G} \bigsqcup \vec{H}$ is given by the disjoint union of their respective vertex sets and edge sets, as sets.

The disjoint union is the coproduct in the category $\mathcal{D}$.

**Definition 2.14.** The graph **Cartesian product** $\square$ of two directed graphs $\vec{G}$ and $\vec{H}$ is a directed graph $\vec{G} \square \vec{H}$, where the vertices are all ordered pairs $(u, v)$ such that $u \in V_{\vec{G}}$ and $v \in V_{\vec{H}}$, and $(u_1, v_1) \rightarrow (u_2, v_2)$ is an edge in $\vec{G} \square \vec{H}$ if either $u_1 = u_2$ and $v_1 \rightarrow v_2$ in $\vec{H}$, or $u_1 \rightarrow u_2$ in $\vec{G}$ and $v_1 = v_2$.

Note that the graph Cartesian product is not a product in the category $\mathcal{D}$. Given a fixed vertex $v_0 \in V_{\vec{H}}$, we will denote by $\vec{G} \square \{v_0\}$ the $v_0$-slice of $\vec{G} \square \vec{H}$. It is the induced sub-digraph where the vertices are all ordered pairs $(u, v_0)$ such that $u \in V_{\vec{G}}$ and the edges are those resulting from the edges of $\vec{G}$.

**Definition 2.15.** Let $\vec{G}$ and $\vec{H}$ be digraphs and $\sim$ an equivalence relation on vertex sets of $\vec{G}$ and $\vec{H}$ such that whenever $(g_1, g_2) \in E_{\vec{G}}, g_1 \sim h_1$, and $g_2 \sim h_2$, then either $h_1 = h_2$ or $(h_1, h_2) \in E_{\vec{H}}$.

The **identification digraph** resulting from the relation $\sim$ is a digraph $\vec{G} \bigsqcup \vec{H}/\sim$ whose vertices are equivalence classes and whose edges are the edges between the representatives of the classes.

**Definition 2.16.** A quotient digraph $\vec{G}/\vec{X}$, for $\vec{X} \subseteq \vec{G}$ and $\vec{X}$ not necessarily connected, is an identification digraph $\vec{G} \bigsqcup */\sim$ where $x \sim *$ for all $x \in V_{\vec{X}}$. 

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Definition 2.17. The mapping cylinder of a digraph map $f: \vec{G} \to \vec{H}$ is given by

$$\vec{M}_f := [\vec{G} \sqcup I^- \coprod \vec{H}] / \sim,$$

where $(g, 0) \sim f(g)$ for all $g \in V_\vec{G}$.

Remark 2.18. A digraph map $f: \vec{G} \to \vec{H}$ factors through $\vec{M}_f$. We shall come back to the equivalence $\vec{M}_f \simeq \vec{H}$ in Example 2.28.

\[
\begin{array}{c}
G \\
\downarrow f \\
\vec{M}_f \\
\ldots \\
\uparrow \cong \\
\vec{H}
\end{array}
\]

Definition 2.19. An extension of a mapping cylinder $\vec{M}_f$ for a digraph map $f: \vec{G} \to \vec{H}$ is the digraph $\vec{E}_f = (V_{\vec{E}_f}, E_{\vec{E}_f})$ where

$$V_{\vec{E}_f} = V_\vec{G} \coprod V_\vec{H},$$

and $(u, v)$ is an edge in $E_{\vec{E}_f}$ if $(u, f(v)) \in E_\vec{H}$ for some $u \in V_\vec{H}$ and $v \in V_\vec{G}$ or in the opposite direction $(v, u)$ is an edge if $(f(v), u) \in E_\vec{H}$ for some $u \in V_\vec{H}$ and $v \in V_\vec{G}$.

Definition 2.20. The cone over a digraph $\vec{G}$, denoted by $C\vec{G}$, is the digraph $[\vec{G} \sqcup I^-] / \sim$, where $(g, 0) \sim *$ for all $g \in V_\vec{G}$.

Definition 2.21. The digraph cofiber $\vec{C}(f)$ for a map $f: \vec{G} \to \vec{H}$ is an identification digraph $C(\vec{G}) \coprod (\vec{G} \sqcup I^-) \coprod \vec{H}$ modulo equivalence relation defined by identifying

i) the cone base with $\vec{G} \sqcup \{1\}$ and

ii) $\vec{G} \sqcup \{0\}$ with $\vec{H}$ under $(g, 0) \sim f(g)$ for all $\vec{G}$.

Definition 2.22. The reduced digraph cofiber $\vec{C}(f)_{\text{red}}$ for a map $f: \vec{G} \to \vec{H}$ is an identification digraph $C(\vec{G}) \coprod (\vec{G} \sqcup I^-) \coprod \text{Im}(f)$ modulo equivalence relation defined by identifying

i) the cone base with $\vec{G} \sqcup \{1\}$ and

ii) $\vec{G} \sqcup \{0\}$ with $\text{Im}(f)$ under $(g, 0) \sim f(g)$ for all $\vec{G}$.

Remark 2.23. The resulting structure in the above definition is a “cone on top of the mapping cylinder” with a middle slice that preserves a copy of $\vec{G}$. At least for our purpose in this paper, the above $\vec{C}(f)$ is a cofiber construction that serves the purpose in the category of directed graphs. It should be noted that $\vec{C}(f)$ is not a category-theoretic cofiber. (Compare Definition A.2 and Remark A.3.)

Definition 2.24. A reduced mapping cylinder for the digraph map $f: \vec{G} \to \vec{H}$ is defined to be $\vec{T}_f := (\vec{G} \sqcup I^+) / \sim$ where $(g, 1) \sim f(g)$.

Definition 2.25. A mapping tube between the images of two digraph maps $f: \vec{G} \to \vec{H}$ and $g: \vec{G} \to \vec{H}$ is given by $\vec{M}\vec{T}_{f,g} = \vec{T}_f \cup \vec{T}_g \cup \vec{E}_f \cup \vec{E}_g$. Note that $\vec{T}_f \cap \vec{T}_g = \vec{G}$. See Figure 1.

2.3. Homotopy for digraphs.

Definition 2.26. Two digraph maps $f, g: \vec{G} \to \vec{H}$ are homotopic, denoted $f \simeq g$, if there exists an $n \geq 1$ and a digraph map $F: \vec{G} \sqcup I_n \to \vec{H}$, for some line digraph $I_n \in \mathcal{I}_n$ (Recall Notation 2.4), such that $F|_{\vec{G} \times \{0\}} = f$ and $F|_{\vec{G} \times \{n\}} = g$.ovementary
For every vertex \( i \in V_{I_n} \), \( F \mid_{\vec{G} \times \{i\}} \) must be a digraph map from \( \vec{G} \) to \( \vec{H} \). Thus, if two digraph maps, \( f \) and \( g \), are homotopic, then there must be a sequence of digraph maps, \( \{f_j\}_{j=0}^n \), where \( f_0 = f, f_n = g \), and \( f_j = F \mid_{\vec{G} \times \{j\}} \) for \( 0 < j < n \). We denote by \([f]\) the set of all digraph maps which are homotopic to \( f \).

**Definition 2.27.** Two digraphs are said to be **homotopically equivalent** (or to be of the same homotopy type) if there exist two digraph maps, \( g: \vec{G} \to \vec{H} \) and \( h: \vec{H} \to \vec{G} \), such that \( h \circ g \simeq \text{id}_{\vec{G}} \) and \( g \circ h \simeq \text{id}_{\vec{H}} \).

We will denote the class of all digraphs which are homotopically equivalent to \( \vec{G} \) by \([\vec{G}]\) and every element of this set is said to be of the homotopy type of \( \vec{G} \).

**Example 2.28.** Let \( f: \vec{G} \to \vec{H} \) be a digraph map. The mapping cylinder \( \vec{M}_f \) and the digraph \( \vec{H} \) are homotopically equivalent. This can be shown by taking a homotopy \( F: \vec{M}_f \Box I^- \to \vec{M}_f \) defined by \( F(\cdot, 0) = \text{id}_{\vec{M}_f}, F((g, 0), 1) = (f(g), 1) \) for all \( g \in V_{\vec{G}} \) and \( F((h, 1), 1) = (h, 1) \) for all \( h \in V_{\vec{H}} \).

**Definition 2.29.** A digraph \( \vec{G} \) is said to be **contractible** if there exists a homotopy between \( \text{id}_{\vec{G}} \) and a constant digraph map.

The following example gives some simple examples of contractible digraphs.

**Example 2.30.** Consider \( I^+ \). To show that \( I^+ \) is contractible, one is seeking a digraph map \( F \) such that there is a consistent edge orientation for the red lines in the Figure 2 below, while \( F\mid_{\{0\}} = \text{id} \) and \( F\mid_{\{1\}} = * \). And indeed, taking \( F(0, 1) = F(1, 1) = 0 \) and taking \( I = I^- \) will yield the desired digraph map. Note, in the target space, the edge is from vertex 0 to vertex 1. Since \( F(1, 1) = 0 \) and \( F(1, 0) = 1 \) the edge linking vertices in the domain should match the edge from 0 to 1 in the codomain. Hence, \( I = I^- \).

In fact, there is a homotopy between \( I^+ \Box I^- \) and \( I^+ \). See Figure 3. By taking \( F(i, 0) = i \) for \( 0 \leq i \leq 3 \), \( F(0, 1) = F(3, 1) = 0 \), and \( F(1, 1) = F(2, 1) = 1 \), one can see that the necessary edge orientation for \( I \) is \( I^- \) and this completes the homotopy. Thus, by induction, one can show that \( I_1 \Box I_1 \Box \cdots \Box I_1 \) is contractible.

**Definition 2.31.** Digraph maps \( f, g: \vec{G} \to \vec{H} \) are said to be **weakly homotopic**, denoted \( f \simeq_w g \), if for every \( \vec{K} \in \mathcal{D}_0 \) and every digraph map \( h: \vec{K} \to \vec{G} \), compositions \( f \circ h \) and \( g \circ h \) are homotopic. Let \([\vec{G}, \vec{H}]_w\) denote the set of weak homotopy classes of digraph maps from \( \vec{G} \) to \( \vec{H} \).
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Note that, when $\vec{G}$ is a finite digraph, $[\vec{G}, \vec{H}]_w = [\vec{G}, \vec{H}]$.

**Definition 2.32.** The **homotopy category of directed graphs**, denoted Ho$\mathcal{D}$, is a category in which the objects are directed graphs and the morphisms are equivalence classes of digraph maps where $f \simeq g$ whenever $f$ and $g$ are homotopic.

The homotopy category of finite directed graphs Ho$\mathcal{D}_0$ and the weak homotopy category for directed graphs wHo$\mathcal{D}$ are defined in the same manner.

**Remark 2.33.** The category Ho$\mathcal{D}$ is a homotopy category in a category-theoretic sense. It is isomorphic to a category $\mathcal{W}^{-1}\mathcal{D}$ via a localization of the category $\mathcal{D}$ with respect to a collection of morphisms $\mathcal{W}$ consisting of digraph maps admitting a homotopy inverse.

**Lemma 2.34.** Let $f: \vec{G} \to \vec{H}$ and let $i: \vec{H} \to \vec{M}_f$ be the natural inclusion. Then $i \circ f \simeq \text{id}_{\vec{G}}$.

**Proof.** Note, the domain of $i \circ f$ is $\vec{G}$. Hence, letting $I$ be $I^+$, taking $F|_{\vec{G} \times \{0\}} = \text{id}_{\vec{G}}$ and $F|_{\vec{G} \times \{1\}} = f$ will yield the required homotopy.

**Lemma 2.35.** Let $f, g: \vec{G} \to \vec{H}$ be digraph maps. Let $\overrightarrow{MT}_{f,g}$ be the mapping tube between the images of $f$ and $g$ and let $i: \vec{H} \to \vec{H} \cup \overrightarrow{MT}_{f,g}$. Then $i \circ f \simeq i \circ g$.

**Proof.** By the previous lemma, $i \circ f \simeq \text{id}_{\vec{G}}$ and $i \circ g \simeq \text{id}_{\vec{G}}$. Hence the result follows.
3. BROWN FUNCTORS AND THEIR PROPERTIES

In this section, we define a Brown functor and verify its properties. We are closely following Adams [1, Section 3]. With regards to Proposition 3.9 and Lemmas 3.6 and 3.10, we are not claiming any originality but we have “filled a much needed gap in the literature” by adding omitted details for the sake of completeness and readability. Our constructions in 2.21 and 2.22 were created to make Lemmas 3.7 and 3.8 work in the category of directed graphs.

**Notation 3.1.** Consider a diagram $B \xrightarrow{f} A \xrightarrow{g} C$ in the category of abelian groups $\text{Ab}$. We will use the notation $B \times_{A} C$ to denote the subset of $B \times C$ defined by $\{(b, c) : f(b) = g(c)\}$.

**Definition 3.2.** A Brown functor on finite digraphs is a functor $H : \text{Ho}D_{0}^{\text{op}} \rightarrow \text{Ab}$ satisfying the following axioms

1. **Additivity Axiom.** The functor $H$ sends coproduct to product. i.e. $H(\coprod_{\alpha \in \Lambda} \vec{G}_{\alpha}) = \prod_{\alpha \in \Lambda} H(\vec{G}_{\alpha})$ for any family of digraphs $\{\vec{G}_{\alpha}\}_{\alpha \in \Lambda}$.
2. **Mayer-Vietoris Axiom.** For any $\vec{G}_{1}, \vec{G}_{2} \in \text{Ho}D_{0}$, there exists a surjection $H(\vec{G}_{1} \cup \vec{G}_{2}) \twoheadrightarrow H(\vec{G}_{1}) \times_{H(\vec{G}_{1} \cap \vec{G}_{2})} H(\vec{G}_{2})$.

Recall that a set endowed with a relation is partially ordered if the relation is reflexive, antisymmetric, and transitive. A partially ordered set $(\Lambda, \leq)$ is a directed set if for every $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A directed system is a family $\{(\vec{G}_{\alpha}, f_{\alpha \beta})\}_{\alpha, \beta \in \Lambda}$ over a directed set $\Lambda$ if there exists a map $f_{\alpha \beta} : \vec{G}_{\alpha} \rightarrow \vec{G}_{\beta}$ whenever $\alpha \leq \beta$, $f_{\alpha \alpha} = 1_{\vec{G}_{\alpha}}$ and $f_{\alpha \gamma} = f_{\beta \gamma} \circ f_{\alpha \beta}$. An inverse system over $\Lambda$ is a directed system over $\Lambda$ with all arrows reversed.

**Definition 3.3.** Let $\vec{G} \in \mathcal{D}$, $\{\vec{G}_{\alpha}\}_{\alpha \in \Lambda}$ be a directed system of finite subdigraphs over a directed set $\Lambda$, and $\vec{G} = \bigcup_{\alpha} \vec{G}_{\alpha}$. Also let $H : \text{Ho}D_{0}^{\text{op}} \rightarrow \text{Ab}$ be a Brown functor. The inverse limit of the inverse system $\{H(\vec{G}_{\alpha})\}_{\alpha \in \Lambda}$ over $\Lambda$ is

$$\hat{H}(\vec{G}) := \lim_{\alpha} H(\vec{G}_{\alpha}).$$

**Lemma 3.4.** The assignment $\hat{H} : \text{Ho}D_{0}^{\text{op}} \rightarrow \text{Ab}$, $\vec{G} \mapsto \lim_{\alpha} H(\vec{G}_{\alpha})$ is a functor which restricts to the functor $H : \text{Ho}D_{0}^{\text{op}} \rightarrow \text{Ab}$. Also an assignment $\hat{H} : \text{wHo}D_{0}^{\text{op}} \rightarrow \text{Ab}$ defined similarly is a functor that restricts to the functor $H : \text{Ho}D_{0}^{\text{op}} \rightarrow \text{Ab}$.

**Lemma 3.5.** The functor $\hat{H}$ satisfies the additivity axiom.

**Proof.** This is readily seen, since $H$ satisfies the additivity axiom. 

**Lemma 3.6** (Compare Adams [1], Lemma 3.4). Let $\vec{G} \in \mathcal{D}$ and $\{\vec{G}_{\alpha}\}_{\alpha \in \Lambda}$ be any directed system of subdigraphs of $\vec{G}$ over a directed set $\Lambda$ whose union is $\vec{G}$. Then there is a canonical bijection

$$\Theta : \hat{H}(\vec{G}) \rightarrow \lim_{\alpha} \hat{H}(\vec{G}_{\alpha})$$

$$x \mapsto (\iota_{\alpha}^{*} x)_{\alpha \in \Lambda}$$

where $\iota_{\alpha} : \vec{G}_{\alpha} \hookrightarrow \vec{G}$ and $\iota_{\alpha}^{*} = f_{\alpha \beta}^{*} \circ \iota_{\beta}^{*}$ are satisfied for all $\alpha, \beta \in \Lambda$.

**Proof.** We sketch the proof for surjectivity. First we write down $y \in \lim_{\alpha} \hat{H}(\vec{G}_{\alpha})$ into components $(y_{\alpha})_{\alpha \in \Lambda}$ and express each $y_{\alpha}$ as an element of $\lim_{\xi} H((\vec{G}_{\alpha})_{\xi})$, where $\{\vec{G}_{\alpha_{\xi}}\}_{\xi \in M}$ is a directed
system of subdigraphs of $\vec{G}_\alpha$ over the directed set $M$ whose union is $\vec{G}_\alpha$. Then use the Cantor embedding map giving a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$. By looking at which set $H((\vec{G}_\alpha)_\zeta)$ that each element $(y_\alpha)_\zeta$ belongs to, we make a choice of a directed system of finite subdigraphs of $\vec{G}$ whose union is $\vec{G}$, and then form the inverse limit representing $H(\vec{G})$. Then the preimage of $y$ under the map $\Theta$ is simply a sequence of terms $(y_\alpha)_\zeta$ along the Cantor embedding map. Injectivity of $\Theta$ is easy to see.

**Lemma 3.7** (Compare Adams [1], Lemma 3.1). Suppose $f : \vec{G} \to \vec{H}$ is a map of finite digraphs and $H$ as above. Consider the sequence $\vec{G} \xleftarrow{i} \vec{H} \xrightarrow{\epsilon} \vec{C}(f)$ where the map $i$ is the inclusion. The induced sequence $H(\vec{G}) \xleftarrow{f^*} H(\vec{H}) \xrightarrow{i^*} H(\vec{C}(f))$ is exact.

**Proof.** Consider the following sequence of digraph maps $\vec{G} \xleftarrow{f} \text{Im}(f) \xleftarrow{\epsilon} \vec{C}(f)$, where digraph maps $e$ and $\epsilon$ are inclusions (cf. Definitions 2.21 and 2.22). Since $\epsilon \circ e \circ f = i \circ f$ and $\vec{C}(f)$ is contractible, the induced map $f^* \circ e^* \circ i^*$ factors through zero, and hence $f^* \circ i^* = 0$. Thus, $\text{Im}(i^*) \subseteq \ker(f^*)$.

Consider the following commutative diagrams:

\[
\begin{array}{ccc}
\vec{G} & \xrightarrow{f} & \vec{H} \\
\downarrow \epsilon & & \downarrow i \\
\vec{C}\vec{G} & \xrightarrow{\epsilon} & \vec{C}(f) \\
\end{array}
\quad \quad \quad
\begin{array}{ccc}
H(\vec{G}) & \xleftarrow{f^*} & H(\vec{H}) \\
\downarrow \epsilon^* & & \downarrow i^* \\
H(\vec{C}\vec{G}) & \xrightarrow{\epsilon^*} & H(\vec{C}(f)) \\
\end{array}
\]

where $\epsilon$ is the inclusion map. Suppose $x \in H(\vec{H})$ and $f^*(x) = 0$. Since $C\vec{G}$ is contractible, the group $H(C\vec{G})$ is trivial. It follows that $(0, x) \in H(C\vec{G}) \times_{H(\vec{G})} H(\vec{H})$. Since the map $f$ factors through $\vec{M}_f$ via $\vec{G} \xrightarrow{f} \vec{M}_f \xrightarrow{\sim} \vec{H}$, we may identify $H(\vec{H})$ and $H(\vec{M}_f)$ up to isomorphism. By the Mayer-Vietoris axiom, there is a surjection $H(\vec{C}(f)) \to H(C\vec{G}) \times_{H(\vec{G})} H(\vec{H})$. Post-composing with the projection map onto the second factor, this surjection is the same as the map $i^*$. Hence $x \in \text{Im}(i^*)$.

**Lemma 3.8** (Compare Adams [1], Lemma 3.2). Let $\vec{G}$ and $\vec{H}$ be finite digraphs. There is a long exact sequence

\[
H(\vec{G}) \times H(\vec{H}) \cong H(\vec{G} \coprod \vec{H}) \xleftarrow{\epsilon} H(\vec{G} \cup \vec{H}) \xleftarrow{\epsilon} H(\vec{C}(f)) \xleftarrow{\epsilon} H(\vec{C}(g)),
\]

which is natural in $\vec{G}$ and $\vec{H}$.

**Proof.** Apply Lemma 3.7 to the following digraph cofiber sequence

\[
\vec{G} \coprod \vec{H} \xrightarrow{h} \vec{C}(f) \xrightarrow{b} \vec{C}(g),
\]

we get a long exact sequence

\[
H(\vec{G}) \times H(\vec{H}) \cong H(\vec{G} \coprod \vec{H}) \xleftarrow{\epsilon} H(\vec{G} \cup \vec{H}) \xleftarrow{\epsilon} H(\vec{C}(f)) \xleftarrow{\epsilon} H(\vec{C}(g)),
\]

where the equivalence on the far-left is by the additivity axiom. The naturality is clear.

**Proposition 3.9** (Compare Adams [1], Proposition 3.5). Let $\vec{G}, \vec{H} \in \mathcal{D}$ and $\vec{G} \cap \vec{H} \in \mathcal{D}_0$. Then the map $H(\vec{G} \cup \vec{H}) \to H(\vec{G}) \times_{\alpha(\vec{G} \cap \vec{H})} H(\vec{H})$ induced by inclusion maps $\vec{G} \hookrightarrow \vec{G} \cup \vec{H}$ and $\vec{H} \twoheadrightarrow \vec{G} \cup \vec{H}$ is onto.
We claim that \( g \) action is given by adding an element in the image of the map \( y \) that it indeed defines an action on \( (1) \) from Lemma 3.8 and take any \( w \) of \( D \) in the category \( \vec{K}_D \). In the possible image of the induced map \( \text{Lemma } 3.10 (2) \), it follows that the objects of \( \textbf{lim}_C \) are objects that restricts to \( x \), \( y \) \( \in \vec{H}(\vec{G}_\alpha) \) and \( x_\beta \in \vec{H}(\vec{H}_\beta) \) so that \( (x_\alpha,y_\beta) \in H(\vec{G}_\alpha) \times H(\vec{G}_\beta) \) \( H(\vec{H}_\alpha) \) \( H(\vec{H}_\beta) \). Notice here that \( \vec{G}_\alpha \cap \vec{H}_\beta \subseteq \vec{G} \cap \vec{H} \). We consider the following subcategory \( \mathcal{C} \) of Set whose objects are the set \( H(\vec{G}_\alpha \cup \vec{H}_\beta) \) that restricts to \( x_\alpha \in H(\vec{G}_\alpha) \) and \( x_\beta \in H(\vec{H}_\beta) \) so that \( (x_\alpha,y_\beta) \in H(\vec{G}_\alpha) \times H(\vec{G}_\beta) \) \( H(\vec{H}_\alpha) \) \( H(\vec{H}_\beta) \). For any \( H(\vec{G}_\alpha,H(\vec{H}_\beta)) \) \( \mathcal{C}(\vec{H}_\gamma,H(\vec{G}_\alpha,H(\vec{H}_\gamma))) \) is a singleton \( \iota^* \), if there is an obvious inclusion \( \iota: \vec{G}_\alpha \cup \vec{H}_\beta \rightarrow \vec{G}_\gamma \cup \vec{H}_\delta \).

We claim that \( \iota^*: \vec{G}_\gamma \rightarrow H(\vec{G}_\alpha,H(\vec{H}_\beta)) \) is onto. Note that the group \( H(\vec{C}(f)) \) is acting on \( H(\vec{G}_\alpha,H(\vec{H}_\beta)) \). The action is given by adding an element in the image of the map \( g^*: H(\vec{C}(f)) \rightarrow H(\vec{G}_\gamma \cup \vec{H}_\beta) \). Note that it indeed defines an action on \( H(\vec{G}_\alpha,H(\vec{H}_\beta)) \) and moreover, it is transitive. To see the latter, consider \( (1) \) from Lemma 3.8 and take any \( w_1,w_2 \) \( H(\vec{G}_\alpha,H(\vec{H}_\beta)) \) since \( w_1 - w_2 \) \( H(\vec{G}_\alpha,H(\vec{H}_\beta)) \) \( H(\vec{H}_\alpha) \) \( H(\vec{H}_\beta) \). By naturality of the sequence \( (1) \), the transitive action of \( H(\vec{C}(f)) \) commutes with \( \iota^* \). Take an element \( x \in H(\vec{G}_\gamma,H(\vec{H}_\beta)) \) \( H(\vec{C}(f)) \). Now any \( y \in H(\vec{G}_\alpha,H(\vec{H}_\beta)) \) can be written as \( \iota^*(x) + g^*(h) \) for some \( h \in H(\vec{C}(f)) \). Since the action commutes with \( \iota^* \), a preimage of \( y \) is \( x + g^*(h) \).

In \( \mathcal{D}_0 \) we can consider a countably many set of objects such that each set contains at least one representative \( \vec{K} \) of every existing homotopy type in \( \mathcal{D}_0 \). Now we consider the map \( h: \vec{C}(f) \rightarrow \vec{K} \) in the category \( \mathcal{D}_0 \). For each \( \vec{K} \) there are only countably many homotopy classes of \( h \), and hence the possible image of the induced map \( h^*: H(\vec{K}) \rightarrow H(\vec{C}(f)) \) is also countably many. Now by Lemma 3.10 \( (2) \), it follows that the objects of \( \vec{C} \) fall into countably many equivalence classes. By Corollary B.12, \( \text{lim}_C \mathcal{C} \) is nonempty.

Recall maps \( g \) and \( h \) in Lemma 3.8.

**Lemma 3.10** (Compare Adams [1], Lemma 3.6). (1) The hom-set \( \vec{C}(H_{\theta,\varphi},H_{\alpha,\beta}) \) is nonempty if and only if the image of \( h^*_{\theta,\varphi}: H(\vec{C}(g_{\theta,\varphi})) \rightarrow H(\vec{C}(f)) \) is contained in the image of \( h^*_{\alpha,\beta}: H(\vec{C}(g_{\alpha,\beta})) \rightarrow H(\vec{C}(f)) \).

(2) \( H_{\theta,\varphi} \) and \( H_{\alpha,\beta} \) are equivalent in \( \vec{C} \) if and only if \( h^*_{\theta,\varphi} = h^*_{\alpha,\beta} \).

**Proof.** Note that (2) is a consequence of (1). We prove (1).

Consider \( \vec{G}_\zeta \cup \vec{H}_\zeta \) that contains both \( \vec{G}_\theta \cup \vec{H}_\varphi \) and \( \vec{G}_\alpha \cup \vec{H}_\beta \). We have the following maps:
Recall from the proof of Proposition 3.9 that $i_{\alpha,\beta}^*$ and $i_{\alpha,\beta}$ are onto. We will use the same notation $g$ with subindices. Assume $\text{Im}(h_{\theta,\varphi}^*) \subseteq \text{Im}(h_{\alpha,\beta}^*)$. Take any $x \in H(\overline{C}(f))$ that belongs to $\text{Im}(h_{\alpha,\beta}^*)$ and not $\text{Im}(h_{\theta,\varphi}^*)$. Then an arbitrary element of $H_{\theta,\varphi}$ can be written as $i_{\theta,\varphi}^*g_{\xi,\zeta}^*(x) + g_{\theta,\varphi}(\delta)$ for all $\delta \in H(\overline{C}(f))$. We set up a map $j: H_{\theta,\varphi} \to H_{\alpha,\beta}$ defined by $i_{\theta,\varphi}^*g_{\xi,\zeta}^*(x) + g_{\theta,\varphi}(\delta) \to i_{\alpha,\beta}^*g_{\xi,\zeta}^*(x) + g_{\alpha,\beta}(\delta)$ which satisfies $j \circ i_{\theta,\varphi}^* = i_{\alpha,\beta}^*$. Hence $j \in \overline{C}(H_{\theta,\varphi}, H_{\alpha,\beta})$.

Now assume that there is a map $j: H_{\theta,\varphi} \to H_{\alpha,\beta}$ satisfying $j \circ i_{\theta,\varphi}^* = i_{\alpha,\beta}^*$. Suppose there is $x \in \text{Im}(h_{\theta,\varphi}^*)$ but not in $\text{Im}(h_{\alpha,\beta}^*)$. Take any $y \in H_{\theta,\varphi}$ and act by $x$ via $g_{\theta,\varphi}$ and then apply $j$. This yields $j(y + g_{\theta,\varphi}(x)) = j(y) + g_{\alpha,\beta}(x)$. Since $x \in \text{Im}(h_{\theta,\varphi}^*)$ we have $g_{\theta,\varphi}(x) = 0$ whereas $x \notin \text{Im}(h_{\alpha,\beta}^*)$ implies that $g_{\alpha,\beta}(x)$ cannot vanish, which is a contradiction. Hence the result. ■

4. Brown’s Method for Directed Graphs

In this section, we construct a classifying object representing a Brown functor on finite digraphs. Arguments in this section parallels those in Brown [3] and Adams [1]. We have added omitted proofs to Lemmas 4.1 and 4.2. Lemma 4.4 is identical to the one already in [1] but reiterated for the sake of completeness. Notice that our construction in 2.25 is necessary for Lemma 4.3. Our main theorem of this paper is Theorem 4.5. Throughout this section we will use the notation $\text{Nat}(F, G)$ to denote the set of all natural transformations from a functor $F$ to a functor $G$.

Let $H: \text{Ho}D^{\text{op}} \to \text{Set}$ be a functor. Take $\overline{Y} \in D$ and consider $[-, \overline{Y}]$, the set of homotopy classes of digraph maps into $\overline{Y}$. Let $T: [-, \overline{Y}] \to H(-)$ such that for any $y \in H(\overline{Y})$ and any $[f] \in [G, \overline{Y}]$, define $T_y[f] = H(f)(y) = f^*(y)$. Then, by the Yoneda lemma, $\text{Nat}([- \overline{Y}], H(-)) \cong H(\overline{Y})$.

Consider a functor $H: \text{Ho}D^{\text{op}} \to \text{Set}$. To extend $H$ to infinite digraphs such that the extension restricts to $H$ on finite digraphs, consider $\overline{Y} \in D$ and define

$$H(\overline{Y}) := \text{Nat}([- \overline{Y}], H(-))$$

such that for any $f: \overline{Y} \to \overline{Z}$, $H(f)T = Tf_*$, where $T: [-, \overline{Z}] \to H(-)$ and $f_*: [-, \overline{Y}] \to [-, \overline{Z}]$. Given $\{\overline{Y}_\alpha\}$ a sequence of finite subdigraphs of $\overline{Y}$ such that whenever $\overline{Y}_\alpha \subseteq \overline{Y}_\beta$ we have the following diagram of inclusions:

$$\begin{array}{ccc}
\overline{Y}_\alpha & \xrightarrow{i_{\alpha,\beta}} & \overline{Y}_\beta \\
\downarrow{i_\alpha} & & \downarrow{i_\beta} \\
\overline{Y} & & \\
\end{array}$$

Thus, $(H(\overline{Y}_\alpha), H(i_{\alpha,\beta}))$ forms an inverse system and since for finite complexes $H = H$,

$$H(i_{\alpha}): H(\overline{Y}) \to H(\overline{Y}_\alpha).$$

**Lemma 4.1** (Compare Brown [3], Lemma 3.3). There is an isomorphism between sets, $\varphi: H(\overline{Y}) \to \lim H(\overline{Y}_\alpha)$.  

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Proof. Let $T_1, T_2 \in \tilde{H}(\vec{Y})$, let $\vec{G} \in \mathcal{D}_0$, and let $[f] \in [\vec{G}, \vec{Y}]$, then there exists a $\vec{Y}_\alpha$ and a digraph map $f_\alpha : \vec{G} \to \vec{Y}_\alpha$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\vec{G} & \xrightarrow{f_\alpha} & \vec{Y}_\alpha \\
\downarrow f & & \downarrow i_\alpha \\
\vec{Y} & \xrightarrow{i_\alpha} & \vec{Y}_\alpha
\end{array}
$$

Suppose that $\varphi(T_1) = \varphi(T_2)$. Since

$$
\tilde{H}(i_\alpha)T_1[f_\alpha] = T_1i_{\alpha*}[f_\alpha] = T_1[f],
$$
and

$$
\tilde{H}(i_\alpha)T_2[f_\alpha] = T_2i_{\alpha*}[f_\alpha] = T_2[f],
$$
thus, $T_1[f] = T_2[f]$ and so $T_1 = T_2$. Suppose $y \in \lim H(\vec{Y}_\alpha)$ and take $T_y[f] = H(f_\alpha)y_\alpha$ (notice that this is well-defined), where $y_\alpha$ is the projection of $y$ into $H(\vec{Y}_\alpha)$. Then $T_y \in \tilde{H}(\vec{Y})$ and $\tilde{H}(i_\alpha)T_y = y_\alpha$. \qed

Recall the notion of weak homotopy from Definition 2.31. Also $\tilde{H} : \text{wHo}^{\mathcal{D}_0^{op}} \to \text{Set}$ from Definition 3.3 and $H : \text{wHo}^{\mathcal{D}_0^{op}} \to \text{Set}$ in Definition 3.2. Given a digraph $\vec{Y} \in \mathcal{D}$ and an element $y \in \tilde{H}(\vec{Y})$, $y = (y_\alpha)$, let $\hat{T} : [\vec{G}, \vec{Y}]_w \to H(\vec{G})$ be a natural transformation between functors such that for any $f : \vec{G} \to \vec{Y}$ define $\hat{T}$ by

$$
\hat{T} : [\vec{G}, \vec{Y}]_w \to H(\vec{G})
$$

$$
\hat{T}(f) = f^* y.
$$

Lemma 4.2. The following are bijections which are natural in $\vec{Y}$

$$
\tilde{H}(\vec{Y}) \cong \text{Nat}([\vec{G}, \vec{Y}]_w, \tilde{H}(\vec{G})) \cong \text{Nat}([\vec{K}, \vec{Y}], H(\vec{K})),
$$
where the slots filled by $\vec{G}$ indicates that the functor takes an object from $\mathcal{D}$ and the slots with $\vec{K}$ an object from $\mathcal{D}_0$.

Proof. The far-left bijection holds by the Yoneda lemma and the map is given by (2). It follows from Lemma 4.1 and Definition 3.3 that $\text{Nat}([\vec{K}, \vec{Y}], H(\vec{K})) \to \tilde{H}(\vec{Y})$ is a bijection. The naturality in $\vec{Y}$ is clear. \qed

Let $\mathcal{K}$ be the set of homotopy types of finite digraphs in the category of digraphs $\mathcal{D}$. As mentioned at the end of the proof of Proposition 3.9, there are countably many homotopy types for finite directed graphs. Let $\{\vec{K}_\alpha\}_{\alpha \in A}$ be a countable collection of representatives taken from each of the homotopy types in $\mathcal{K}$.

Lemma 4.3. Suppose $\vec{Y}_n$ is a digraph with the element $y_n \in \tilde{H}(\vec{Y}_n)$, then there exists a digraph $\vec{Y}_{n+1}$, an embedding $i : \vec{Y}_n \to \vec{Y}_{n+1}$, and an element $y_{n+1} \in \tilde{H}(\vec{Y}_{n+1})$, such that the following holds.

i) $y_{n+1}$ restricts to $y_n \in \hat{H}(\vec{Y}_n)$, and

ii) for any pair of digraph maps $f, g : \vec{K} \to \vec{Y}_n, \vec{K}$ a finite digraph, if $f^* y_n = g^* y_n$ then $i \circ f \simeq i \circ g$ in $\vec{Y}_{n+1}$.

Proof. Let $\vec{K}_\alpha \in \{\vec{K}_\alpha\}_{\alpha \in A}$ be a representative taken from one of the countable homotopy types and suppose that $[f], [g] : \vec{K}_\alpha \to \vec{Y}_n$ are a pair of homotopy classes of maps such that $f^* y_n = g^* y_n$. \qed

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Take representatives \( f_\alpha \in [f] \) and \( g_\alpha \in [g] \), and consider the mapping tube \( \overrightarrow{M T}_{f_\alpha, g_\alpha} \) between the images of \( f_\alpha \) and \( g_\alpha \) (cf. Definition 2.25). Define

\[
\bar{Y}_{n+1} := \bar{Y}_n \cup \bigcup_{\alpha \in A} \overrightarrow{M T}_{f_\alpha, g_\alpha}.
\]

By Lemma 2.35, \( i \circ f_\alpha \simeq i \circ g_\alpha \) for each \( \alpha \in A \). This proves part ii above.

For the existence of \( y_{n+1} \), let \( B \in \mathcal{P}(A) \), the power set of \( A \), such that there is a \( b \in \hat{H} \left( \bar{Y}_n \cup \bigcup_{\beta \in B} \overrightarrow{M T}_{f_\beta, g_\beta}\right) \) which restricts to \( y_n \in \hat{H}(\bar{Y}_n) \). Consider all such pairs, \( (B, b) \), and define a partial ordering \( \preceq \) by

\[
(B_i, b_i) \preceq (B_j, b_j) \iff B_i \subseteq B_j \text{ and } b_i \text{ restricts to } b_i \in \hat{H} \left( \bar{Y}_n \cup \bigcup_{\beta \in B_i} \overrightarrow{M T}_{f_\beta, g_\beta}\right).
\]

For simplicity let us denote by \( \mathcal{B}_i := \bar{Y}_n \cup \bigcup_{\beta \in B_i} \overrightarrow{M T}_{f_\beta, g_\beta} \), where the subindex \( i \) indicates the corresponding index set \( B_i \) being used. Let \( B_0 = \varnothing \), take \( b_0 = y_n \) and so the pair \( (\varnothing, y_n) \) trivially satisfies the condition above. Hence, \( \{(B, b)\} \) is non-empty. Let \( \{(B_k, b_k)\}_k \) be a subset of \( \{(B, b)\} \), such that for any \( (B_i, b_i), (B_j, b_j) \in \{(B_k, b_k)\}_k \), either \( (B_i, b_i) \preceq (B_j, b_j) \) or \( (B_j, b_j) \preceq (B_i, b_i) \).

Define \( \mathcal{B} = \bigcup_k B_k \) and let \( f_{ij} : \mathcal{B}_i \to \mathcal{B}_j \) be the natural inclusion map for whenever \( B_i \subseteq B_j \).

There is a natural family of maps \( f_1 : \mathcal{B} \to \mathcal{B} \), where \( \mathcal{B} \) denotes \( \bar{Y}_n \cup \bigcup_{\beta \in B} \overrightarrow{M T}_{f_\beta, g_\beta} \). Hence, we have commutative triangles:

\[
\begin{array}{ccc}
\mathcal{B}_i & \xrightarrow{f_{ij}} & \mathcal{B}_j \\
\downarrow f_i & & \downarrow f_j \\
\mathcal{B} & \xrightarrow{f_j} & \mathcal{B}
\end{array}
\]

Applying the functor \( \hat{H} \) the following inverse system is obtained:

\[
\hat{H}(\mathcal{B}_i) \xrightarrow{f_{ij}^*} \hat{H}(\mathcal{B}_j) \xleftarrow{f_j^*} \hat{H}(\mathcal{B}).
\]

By Lemma 3.6, \( \hat{H}(\mathcal{B}) \cong \varprojlim \hat{H}(\mathcal{B}_i) \) and so there is a \( b \in \hat{H}(\mathcal{B}) \) such that \( f_1^*(b) = b_i \). Thus, \( (\mathcal{B}, b) \) is an upper bound for the chain \( \{(B_k, b_k)\}_k \). Since the chain was arbitrary, the set of all pairs \( \{(B, b)\} \) will, by Zorn’s lemma, have a maximal element, call it \( (B', b') \). Assume that \( B' \neq A \), recall that \( A \) is the full indexing set for the countable collection of representatives of each homotopy type for finite digraphs, then \( \exists \alpha \in A \) such that \( \alpha \notin B' \). For simplicity, let \( \bar{G} = \bar{Y}_n \cup \bigcup_{\beta \in B'} \overrightarrow{M T}_{f_\beta, g_\beta} \) and let \( \bar{H} = \overrightarrow{M T}_{f_\alpha, g_\alpha} \). Then one has the following diagram:

\[
\begin{array}{ccc}
H(\bar{G}) & \xrightarrow{\hat{H}} & H(\bar{H}) \\
\downarrow i_1 & & \downarrow j_1 \\
\hat{H}(\bar{G}) & \xleftarrow{\hat{H}} & \hat{H}(\bar{G} \cup \bar{H})
\end{array}
\]

Applying the Mayer-Vietoris Axiom for \( \hat{H} \) as in Proposition 3.9, to the setting where \( b \in \hat{H}(\bar{G}) \), along with some \( h \in \hat{H}(\bar{H}) \) such that \( i_1^*(b) = i_2^*(h) \), then there must exist an extension of \( b \) to some \( a \in \hat{H}(\bar{G} \cup \bar{H}) \) such that \( j_1^*(a) = b \) and \( j_2^*(a) = h \). Hence, any maximal element of \( \{(B, b)\} \)
must be of the form \((A, a)\) and we define \(y_{n+1} = a\). Thus, a \(y_{n+1}\) with the required properties exists.

The preceding lemma is the inductive step in the construction of a classifying directed graph. What remains is the base case.

**Lemma 4.4.** Given a digraph \(\hat{Y}_0\) with an element \(y_0 \in \hat{H}(\hat{Y}_0)\), there exists a digraph \(\hat{Y}\), an embedding \(i: \hat{Y}_0 \rightarrow \hat{Y}\), and there is an element \(y \in \hat{H}(\hat{Y})\) such that the following hold:

i) \(y\) restricts to \(y_0 \in \hat{H}(\hat{Y}_0)\), and

ii) the natural transformation \(T: [-, \hat{Y}] \rightarrow H(-)\) is an isomorphism whenever the input is a finite digraph.

**Proof.** Let \(\vec{K}_\alpha \in \{\vec{K}_\alpha\}_{\alpha \in A}\) be a representative taken from one of the countable homotopy types and let \(\alpha_j \in H(\vec{K}_\alpha)\). Define \(\vec{Y}_1 \coprod \prod_{\alpha} \left( \prod_j \vec{K}_\alpha \right)\). Note, for each fixed \(\alpha\) this disjoint union will contain \(n_\alpha = |H(\vec{K}_\alpha)|\) copies of \(\vec{K}_\alpha\), one for each of the possible \(\alpha_j \in H(\vec{K}_\alpha)\). Since the additivity axiom holds for \(\hat{H}\), we have \(\hat{H}(\vec{Y}_1) = \hat{H}(\vec{Y}_0) \times \prod_{\alpha} \left( \prod_j \hat{H}(\vec{K}_\alpha) \right)\). Take \(y_1 \in \hat{H}(\vec{Y}_1)\) such that \(y_1\) restricts to \(y_0 \in \hat{H}(\vec{Y}_0)\) and restricts to the \(\alpha_j\) element on the \((\alpha, j)\)-th place. By Lemma 4.2,

\[\hat{H}(\vec{Y}_1) \cong \text{Nat} \left( \left[\vec{K}_\alpha, \vec{Y}_1\right], H(\vec{K}_\alpha) \right)\]

Therefore, the \(y_1\) is associated to some natural transformation,

\(T_1: \left[\vec{K}_\alpha, \vec{Y}_1\right] \rightarrow H(\vec{K}_\alpha)\),

and the \(T_1\) associated to \(y_1\) is a surjection for all \(\vec{K}_\alpha\).

Using Lemma 4.3, construct an ascending chain of digraphs

\[\vec{Y}_1 \subset \vec{Y}_2 \subset \vec{Y}_3 \subset \cdots \subset \vec{Y}_n \subset \cdots\]

along with elements \(y_2 \in \hat{H}(\vec{Y}_2), y_3 \in \hat{H}(\vec{Y}_3), y_4 \in \hat{H}(\vec{Y}_4), \) etc, such that \(y_{n+1}\) restricts to \(y_n \in \hat{H}(\vec{Y}_n)\) for \(0 \leq n\). Now, take \(\vec{Y} = \bigcup_n \vec{Y}_n\) along with all the natural inclusions, yielding:

\[
\begin{array}{c}
\vec{Y}_n \\
\downarrow_{i_n} \\
\vec{Y} \end{array}
\]

\[
\begin{array}{c}
\vec{Y}_m \\
\downarrow_{i_m} \\
\vec{Y} \end{array}
\]

Applying the functor \(\hat{H}\), the following inverse system is obtained:

\[
\begin{array}{c}
\hat{H}(\vec{Y}_n) \\
\downarrow_{i_{nm}} \\
\hat{H}(\vec{Y}_m) \\
\downarrow_{i_m} \\
\hat{H}(\vec{Y}) \\
\downarrow_{i_n} \\
\hat{H}(\vec{Y}) 
\end{array}
\]

By Lemma 3.6, there is a \(y \in \hat{H}(\vec{Y})\) such that \(i_n^*(y) = y_n\) for each \(n\). Again by Lemma 4.2, there is a natural transformation associated to \(y\),

\(T: \left[\vec{K}_\alpha, \vec{Y}\right] \rightarrow H(\vec{K}_\alpha)\),

and this transformation is still surjective for each \(\vec{K}_\alpha\).

Let \(f_\alpha, g_\alpha: \vec{K}_\alpha \rightarrow \vec{Y}\) be any two maps such that \(f^*y = g^*y\), i.e. \(T(f) = T(g)\). From \(\vec{K}_\alpha\) being a finite digraph, there exists an \(m\) such that \(\text{Im}(f) \subseteq \vec{Y}_m\), \(\text{Im}(g) \subseteq \vec{Y}_m\), and \(f^*y_m = g^*y_m\). Then, by Lemma 4.3, \(f \simeq g\) in \(\vec{Y}_{m+1}\). Hence, if given two maps \(f, g\) such that \(T(f) = T(g)\), then they
had to have been equivalent at some finite step long before \( \vec{Y} \). Thus, \( T \) is also injective and the resulting \( T : [-, \vec{Y}] \to H(-) \) is an isomorphism of sets for finite digraphs.

Therefore, we have the following theorem.

**Theorem 4.5.** Let \( H : \text{Ho}\mathcal{D}_0^{\text{op}} \to \text{Ab} \) be a Brown functor. Then there exist a digraph \( \vec{Y} \) and a natural isomorphism \( T : [-, \vec{Y}] \to H(-) \).

**Proof.** It was already shown in Lemma 4.4 that \( T : [-, \vec{Y}] \to H(-) \) is a natural isomorphism as \( \text{Set} \)-valued functors. The functor \([-, \vec{Y}] : \text{Ho}\mathcal{D}_0^{\text{op}} \to \text{Set} \) lands in the subcategory \( \text{Ab} \) by inheriting the group structure of \( H(\vec{K}) \) for each \( \vec{K} \in \mathcal{D}_0 \).

APPENDIX A. HOMOTOPY-THEORETIC CONSTRUCTIONS AND PROPERTIES IN DIRECTED GRAPHS

In this section we exhibit some constructions and properties available in the category of \( \text{CW} \) complexes failing to serve purposes in the category of digraphs. This article is self-contained without this section, however, we believe that our discussion in this section would better illuminate interesting features of homotopy theory in a discrete setting. In Section A.1 we will explain why homotopy-theoretic cofiber is the correct construction to consider in the category of directed graphs. In Section A.2 we will discuss a failure of homotopy extension property in the category of directed graphs.

A.1. Cofibers in category theory. In Definition 2.21 we have defined a digraph cofiber \( \vec{C}(f) \) and discussed in Remark 2.23 that it is not a cofiber in category theory. For comparison purposes, it is worth mentioning what the category-theoretic cofiber of a map \( f : \vec{G} \to \vec{H} \) in the category of directed graphs is.

**Definition A.1.** A pushout of the diagram \( \vec{H}_2 \xrightarrow{f_2} \vec{G} \xrightarrow{f_1} \vec{H}_1 \) in the category of directed graphs is the digraph \( (\vec{H}_1 \coprod \vec{H}_2) / \sim \) such that \( f_1(g) \sim f_2(g) \) for all \( g \in V_G \).

The pushout defined as above satisfies the universal property. It is the triple \((\vec{Z}, i_1, i_2)\) in the diagram below which is universally repelling.

\[
\begin{array}{ccc}
\vec{G} & \xrightarrow{f_1} & \vec{H}_1 \\
\downarrow f_2 & & \downarrow i_1 \\
\vec{H}_2 & \xrightarrow{i_2} & \vec{Z}
\end{array}
\]

**Definition A.2.** A cofiber of a digraph map \( f : \vec{G} \to \vec{H} \) in \( \mathcal{D} \) is the pushout \( \vec{H} \cup_f \vec{C}\vec{G} \) in the following diagram:

\[
\begin{array}{ccc}
\vec{G} & \xrightarrow{j_1} & \vec{M}_f \\
\downarrow j_2 & & \downarrow \\
\bullet & \xrightarrow{\star} & \vec{H} \cup_f \vec{C}\vec{G}
\end{array}
\]

where \( j_1 \) is the inclusion and \( j_2 \) is the constant map.
Remark A.3. The cofiber $\vec{H} \cup_f \vec{C} \vec{G}$ is a category-theoretically correct construction. However, there are several reasons that we cannot use it for our purpose in digraph homotopy theory. Even at a glimpse the $\vec{H} \cup_f \vec{C} \vec{G}$ above is not analogous to the cofiber of a continuous map $f : X \to Y$ in the category of topological spaces in that it lacks a slice of the domain digraph $\vec{G}$. Furthermore, in general any subdigraph $\vec{K} \subseteq \text{Im}(f)$ cannot be homotoped to the cone vertex in that there might be edges in $\vec{H}$ whose source and target are not entirely in $\text{Im}(f)$.

A.2. Homotopy extension property in the category of digraphs. The digraph homotopy is a natural analogue of the homotopy in topology, however, it lacks certain properties available in topology. For example, the homotopy extension property stated as follows does not hold.

Definition A.4. Let $\vec{X} \subseteq \vec{G}$, the pair $(\vec{G}, \vec{X})$ is said to have the homotopy extension property, if given a map $F_0 : \vec{G} \to \vec{H}$ and a homotopy $f_t : \vec{X} \to \vec{H}$ such that $f_0 = F_0|_{\vec{X}}$, there exists an extension of $f_t$ to a homotopy $F_t : \vec{G} \to \vec{H}$ such that $F_t|_{\vec{X}} = f_t$.

It is well-known that if $X$ is a CW complex and $A$ a subcomplex, then the pair $(X, A)$ has the homotopy extension property. As we shall see from the following example, it is not true in general that a pair $(\vec{G}, \vec{X})$ consisting of a digraph $\vec{G}$ and its (induced) subdigraph $\vec{X}$ satisfies the homotopy extension property as defined above.

Example A.5. Let the digraph in Figure 4 be $\vec{G}$. Take $A \leftarrow C$ to be the $\vec{X}$ subdigraph of $\vec{G}$ and let $\vec{H} = \vec{G}$.

![Figure 4. A cycle digraph on three vertices](image)

Take $F_0 : \vec{G} \to \vec{G}$ be the identity map on $\vec{G}$. As we have seen in Example 2.30, there is a homotopy between the identity map on $A \leftarrow C$ and a point. Let $f_t$ be that homotopy. Thus, $f_0$ is the identity map on $\vec{X}$ and $f_1 \equiv C$. In order to extend this homotopy to all of $\vec{G}$, one needs to find a digraph map $F : \vec{G} \sqcup I_n \to \vec{G}$ such that $F_0|_{\vec{X}} = \text{id}_{\vec{X}}$ and $F_1|_{\vec{X}} = C$. The extension depends on where $F_1$ maps the vertex $(B, 1)$. Note, $F_1(A, 1) = C$, $F_1(C, 1) = C$ and the homotopy restricted to $\vec{X}$ forces an edge orientation for the $I_n$ to be $I^-$. Since, $F_0(B, 0) = B$, then either $F_1(B, 1)$ is $A$, $B$, or $C$. If $F_1(B, 1) = C$, then the edge $((B, 1), (B, 0))$ is not preserved. If $F_1(B, 1) = A$, then the edge $((B, 1), (C, 1))$ is not preserved. Lastly, if $F_1(B, 1) = B$, then the edge $((A, 1), (B, 1))$ is not preserved. Hence there is no possible extension of $f_t$ to a homotopy $F$ on $\vec{G}$. Therefore, it is not generally the case that a digraph pair $(\vec{G}, \vec{X})$ has the homotopy extension property.
Appendix B. Generalized inverse limits à la J. F. Adams

This section is identical to Adams [1, Section 2] and there is no originality of ours in this subsection. However, we iterate it by adding omitted proofs in [1] to make this paper more accessible for a broader audience. In this section, a category is always a subcategory of Set, the category of sets.

Definition B.1. The inverse limit of a category \( \mathcal{C} \) is a set \( \prod_{X \in \mathcal{C}} X \) consisting of set maps

\[
e : \mathcal{C} \to \prod_{X \in \mathcal{C}} X
\]

\[
X \mapsto e_X \in X
\]
satisfying that for any \( f \in \mathcal{C}(X, Y) \), we have \( e_Y = f(e_X) \).

We shall impose the following two conditions on a category \( \mathcal{C} \).

Axiom B.2. (1) For any \( X, Y \in \mathcal{C} \), there is at most one element in \( \mathcal{C}(X, Y) \).
(2) For any \( X, Y \in \mathcal{C} \), there exists \( Z \in \mathcal{C} \) and morphisms \( Z \to X \) and \( Z \to Y \).

Remark B.3. The usual definition of an inverse limit uses an inverse system indexed by a poset (compare Lang [14, p.51]). Such an index set (and hence the indexed family) has to satisfy the antisymmetry: If \( i \leq j \) and \( j \leq i \), then \( i = j \). The inverse limit of Definition B.1 is more general than the usual inverse limit in that the category \( \mathcal{C} \) may have morphisms \( X \to Y \) and \( Y \to X \) but \( X \) and \( Y \) are not required to be equal or isomorphic. When the category \( \mathcal{C} \) is an inverse system, then the set \( \prod_{X \in \mathcal{C}} X \) is the usual inverse limit.

Definition B.4. A subcategory \( \mathcal{C} \) of a category \( \mathcal{A} \) is cofinal in \( \mathcal{A} \) if for every \( X \in \mathcal{A} \), there exist \( Y \in \mathcal{C} \) and a morphism \( Y \to X \) in \( \mathcal{A} \).

Lemma B.5. If \( \mathcal{C} \) is cofinal in \( \mathcal{A} \), the restriction map \( \prod_{X \in \mathcal{A}} A \to \prod_{X \in \mathcal{C}} X \) is an isomorphism of sets.

Proof. Let \( E : \mathcal{A} \to \prod_{X \in \mathcal{A}} X \), \( X \mapsto E_X \in X \) be an element in \( \prod_{X \in \mathcal{A}} X \). The map res takes \( E \) to its restriction \( e := E|_C : \mathcal{C} \to \prod_{X \in \mathcal{C}} X \). We verify that the map res is onto. Take any \( e \in \prod_{X \in \mathcal{C}} X \) and any \( X \in \mathcal{A} \). Because \( \mathcal{C} \) is cofinal, there exist \( Y \in \mathcal{C} \) and a morphism \( Y \to X \) in \( \mathcal{A} \). We define \( E : \mathcal{A} \to \prod_{X \in \mathcal{A}} X \) by \( X \mapsto E_X := f_{YX}e_Y \). We now verify the injectivity. Take \( E^1, E^2 \in \prod_{X \in \mathcal{A}} A \) and suppose \( E^1|_C = E^2|_C \). Take any \( X \in \mathcal{A} \). Again by cofinality of \( \mathcal{C} \), there exist \( Y \in \mathcal{C} \) and a morphism \( Y \to X \). It follows that \( E^1_X = fE^1_Y = fE^2_Y = E^2_X \). Thus the map res is one-to-one. \( \square \)

Definition B.6. A sequence \( S \) is a category whose objects are \( \{X_n : n \in \mathbb{Z}^+ \} \) and the hom-set \( S(X_n, X_m) \) contains only one element whenever \( n \geq m \).

Lemma B.7. Let \( S \) be a sequence in which the objects are nonempty sets and the morphisms are surjective. Then \( \lim_{\leftarrow} S \) is nonempty.

Proof. Take an arbitrary element \( e_{X_1} \in X_1 \) and choose an arbitrary preimage of \( e_{X_1} \) under the map \( X_2 \to X_1 \) and call it \( e_{X_2} \). Repeating the process of taking preimage ad infinitum, we get an assignment \( S \to \prod_{i \in \mathbb{Z}^+} X_i \), \( X_i \mapsto e_{X_i} \) satisfying \( e_{X_j} = f_{ij}e_{X_i} \) for the map \( f_{ij} : X_i \to X_j \) for any \( i \geq j \). \( \square \)

Lemma B.8. Let \( \mathcal{C} \) be a category. Suppose the objects of \( \mathcal{C} \) fall into countably many equivalence classes. Then \( \mathcal{C} \) contains a cofinal sequence.
Proof. We denote by \( \sigma = \{ \sigma_i \}_{i \in \mathbb{Z}^+} \) the family of countably many equivalence classes of objects in \( \mathcal{C} \); i.e. \( \sigma \) is a partition of the totality of all objects of \( \mathcal{C} \). We shall find a sequence \( S \) in \( \mathcal{C} \). Take representatives \( s_1 \in \sigma_1 \) and \( s_2 \in \sigma_2 \). We set \( X_1 := s_1 \). By Axiom B.2 (2), there exist \( X_2 \) and morphisms \( X_2 \to X_1 \) and \( X_2 \xrightarrow{\gamma} s_2 \). Now take a representative \( s_3 \in \sigma_3 \) and apply the same axiom to get an object \( X_3 \) and morphisms \( X_3 \to X_2 \) and \( X_3 \xrightarrow{\gamma} s_3 \). Repeating this process of choosing a representative \( s_i \in \sigma_i \) for \( i \geq 3 \) to get an object \( X_i \) and morphisms \( X_i \to X_{i-1} \) and \( X_i \xrightarrow{\gamma} s_i \) determines a sequence \( S \) whose objects are \( \{ X_i \}_{i \in \mathbb{Z}^+} \). The sequence \( S \) is clearly cofinal because any object \( Y \) of \( \mathcal{C} \) belongs to some \( \sigma_i \) and there always exists \( X_i \xrightarrow{\gamma} s_i \in \sigma_i \).  

Corollary B.9. Let \( \mathcal{C} \) be a category in which the objects are nonempty sets and the morphisms are surjective. Suppose the objects of \( \mathcal{C} \) fall into countably many equivalence classes. Then \( \lim_\leftarrow \mathcal{C} \) is nonempty.

Proof. By Lemma B.8, there is a cofinal sequence \( S \) in \( \mathcal{C} \). By Lemma B.7, \( \lim_\leftarrow S \) is nonempty. It follows from Lemma B.5 that the restriction map \( \lim_\leftarrow \mathcal{C} \to \lim_\leftarrow S \) is bijective. Hence \( \lim_\leftarrow \mathcal{C} \) is nonempty.

In practice we encounter a category \( \mathcal{C} \) whose elements fall into uncountably many equivalence classes. The strategy for such a case is throwing in more morphisms to form a category \( \overline{\mathcal{C}} \) consisting of the same objects so that two objects which are not equivalent in \( \mathcal{C} \) may be equivalent in the category \( \overline{\mathcal{C}} \).

Definition B.10. Let \( \mathcal{C} \) be a category such that every morphism in \( \mathcal{C} \) is onto. The category \( \overline{\mathcal{C}} \) is defined by the following data. Its objects are the same as the objects of \( \mathcal{C} \). The hom-set \( \overline{\mathcal{C}}(X,Y) \) consists of \( f \in \text{Set}(X,Y) \) satisfying that \( f \circ a = b \) for some \( Z \in \mathcal{C} \), \( a \in \mathcal{C}(Z,X) \), and \( b \in \mathcal{C}(Z,Y) \).

Lemma B.11. The above \( \overline{\mathcal{C}} \) is a category satisfying Axiom B.2 and containing \( \mathcal{C} \) as a cofinal subcategory. The objects of \( \overline{\mathcal{C}} \) are nonempty sets and every morphism of \( \overline{\mathcal{C}} \) is a surjection.

Proof. In verifying the axioms that \( \overline{\mathcal{C}} \) is indeed a category, we look at how the composition is defined. Let \( f \in \overline{\mathcal{C}}(X,Y) \) with \( a \in \mathcal{C}(W,X) \) and \( b \in \mathcal{C}(W,Y) \) such that \( b = f \circ a \) and \( g \in \overline{\mathcal{C}}(Y,Z) \) with \( c \in \mathcal{C}(U,Y) \) and \( d \in \mathcal{C}(U,Z) \) such that \( d = g \circ c \). The composition \( g \circ f \), a morphism in \( \text{Set} \), is in \( \overline{\mathcal{C}}(X,Z) \), because there are \( \alpha \in \mathcal{C}(T,W) \), \( \beta \in \mathcal{C}(T,U) \), and the fact that \( b \circ \alpha = c \circ \beta \), since \( \mathcal{C}(T,W) \) has at most one element in it.  

It is readily seen that every \( f \in \overline{\mathcal{C}}(X,Y) \) is onto. It is also easy to see that \( \mathcal{C} \) is a subcategory of \( \overline{\mathcal{C}} \).

Note that any \( f : X \to Y \in \mathcal{C} \) is of the form \( f = \overline{f} \circ \alpha \) where \( \alpha \) is in \( \mathcal{C}(X,Y) \) and \( \overline{f} \) is an element in \( \overline{\mathcal{C}}(X,Y) \). That the inclusion \( \mathcal{C} \to \overline{\mathcal{C}} \) is functorial is evident.

We verify Axiom B.2 (1) for the category \( \overline{\mathcal{C}} \) as follows. Suppose \( f,g \in \overline{\mathcal{C}}(X,Y) \) such that \( a \in \mathcal{C}(Z,X) \), \( b \in \mathcal{C}(Z,Y) \), \( b = f \circ a \), \( c \in \mathcal{C}(W,X) \), \( d \in \mathcal{C}(W,Y) \), and \( d = g \circ c \). By Axiom B.2 (2) for \( \mathcal{C} \), there exist \( U \in \mathcal{C} \), \( \alpha \in \mathcal{C}(U,Z) \), and \( \beta \in \mathcal{C}(U,W) \). Note that \( f \circ a \circ \alpha = b \circ \alpha = d \circ \beta = g \circ c \circ \beta \). Here the middle equality holds because \( \mathcal{C}(U,Y) \) has at most one element. Since all morphisms in \( \mathcal{C} \) and \( \overline{\mathcal{C}} \) are onto, we have \( f = g \). Axiom B.2 (2) follows from the argument in the preceding paragraph. Finally, the cofinality of \( \mathcal{C} \) follows from Axiom B.2 for \( \overline{\mathcal{C}} \).

Corollary B.12. Suppose that the objects of \( \overline{\mathcal{C}} \) fall into countably many equivalence classes. Then \( \lim_\leftarrow \mathcal{C} \) is nonempty.
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Proof. By Lemma B.11 and B.8, \( \mathcal{C} \) has a cofinal sequence \( \mathcal{S} \). By Lemma B.7, \( \lim \leftarrow \mathcal{S} \) is nonempty. By cofinality of \( \mathcal{S} \) in \( \mathcal{C} \), \( \lim \leftarrow \mathcal{C} \) is nonempty by Lemma B.5 and the cofinality of \( \mathcal{C} \) in \( \mathcal{C} \) implies that \( \lim \leftarrow \mathcal{C} \) is nonempty.

\[ \square \]

References

[1] J. F. Adams, A variant of E. H. Brown’s representability theorem, Topology 10 (1971), 185–198, DOI 10.1016/0040-9383(71)90003-6.
[2] Beifang Chen, Shing-Tung Yau, and Yeong-Nan Yeh, Graph homotopy and Graham homotopy, Discrete Math. 241 (2001), no. 1-3, 153–170, DOI 10.1016/S0012-365X(01)00115-7.
[3] Edgar H. Brown Jr., Cohomology theories, Ann. of Math. (2) 75 (1962), 467–484, DOI 10.2307/1970209.
[4] ______., Abstract homotopy theory, Trans. Amer. Math. Soc. 119 (1965), 79–85, DOI 10.2307/1994231.
[5] Peter Freyd and Alex Heller, Splitting homotopy idempotents. II, J. Pure Appl. Algebra 89 (1993), no. 1-2, 93–106, DOI 10.1016/0022-4049(93)90088-B.
[6] Gian Mario Gianella, Su una omotopia regolare dei grafi, Rend. Sem. Mat. Univ. Politec. Torino 35 (1976/77), 349–360 (1978).
[7] Alexander Grigor’yan, Rolando Jimenez, Yuri Muranov, and Shing-Tung Yau, On the path homology theory of digraphs and Eilenberg-Steenrod axioms, Homology Homotopy Appl. 20 (2018), no. 2, 179–205, DOI 10.4310/HHA.2018.v20.n2.a9.
[8] Alexander Grigor’yan, Yong Lin, Yuri Muranov, and Shing-Tung Yau, Homotopy theory for digraphs, Pure Appl. Math. Q. 10 (2014), no. 4, 619–674, DOI 10.4310/PAMQ.2014.v10.n4.a2.
[9] ______., Cohomology of digraphs and (undirected) graphs, Asian J. Math. 19 (2015), no. 5, 887–931, DOI 10.4310/AJM.2015.v19.n5.a5.
[10] Alexander Grigor’yan, Yuri Muranov, and Shing-Tung Yau, On a cohomology of digraphs and Hochschild cohomology, J. Homotopy Relat. Struct. 11 (2016), no. 2, 209–230, DOI 10.1007/s40062-015-0103-1.
[11] ______., Homologies of digraphs and K"unneth formulas, Comm. Anal. Geom. 25 (2017), no. 5, 969–1018, DOI 10.4310/CAG.2017.v25.n5.a4.
[12] Alex Heller, On the representability of homotopy functors, J. London Math. Soc. (2) 23 (1981), no. 3, 551–562, DOI 10.1112/jlms/s2-23.3.551.
[13] J. F. Jardine, Representability theorems for presheaves of spectra, J. Pure Appl. Algebra 215 (2011), no. 1, 77–88, DOI 10.1016/j.jpaa.2010.04.001.
[14] Serge Lang, Algebra, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.
[15] Jacob Lurie, Higher algebra, available on author’s webpage. September 18, 2017 version (2017).
[16] G. Malle, A homotopy theory for graphs, Glas. Mat. Ser. III 18 (1983), no. 1, 3–25.
[17] Amnon Neeman, The Grothendieck duality theorem via Bousfield’s techniques and Brown representability, J. Amer. Math. Soc. 9 (1996), no. 1, 205–236, DOI 10.1090/S0894-0347-96-00174-9.

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