\(\kappa\)-deformation of phase space; generalized Poincaré algebras and \(R\)-matrix

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We deform a phase space (Heisenberg algebra and corresponding coalgebra) by twist. We present undeformed and deformed tensor identities that are crucial in our construction. Coalgebras for the generalized Poincaré algebras have been constructed. The exact universal \(R\)-matrix for the deformed Heisenberg (co)algebra is found. We show, up to the third order in the deformation parameter, that in the case of \(\kappa\)-Poincaré Hopf algebra this \(R\)-matrix can be expressed in terms of Poincaré generators only. This implies that the states of any number of identical particles can be defined in a \(\kappa\)-covariant way.

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A.S. dedicates this work to the memory of his late brother Saša Saftić.
1. INTRODUCTION

Quantum field theory (QFT) appeared as a result of the attempt to construct a theory that describes the many-particle systems in accord with the quantum mechanical principles and the principles of special relativity. The Poincaré symmetry is thus incorporated in the theory from the very beginning. It is believed that for particles with very high energies the gravity effects become significant \([1]\), so that these particles no more see the spacetime as smooth and continuous, but instead they see it as quantized and fuzzy. The QFT constructed on such deformed manifolds requires a new framework. Such framework is provided by the noncommutative geometry, where the search for the diffeomorphisms leaving spacetime invariant leads to deformation of Poincaré symmetry, with \(\kappa\)-deformed Poincaré symmetry being among the most extensively studied \([2,3,4]\). Much interest has been generated in the study of the physical consequences emerging from the \(\kappa\)-deformation of Poincaré symmetry, e.g. construction of field theories \([5,6,7,8]\), electrodynamics \([9,10]\) and geodesic equation \([11]\) on \(\kappa\)-Minkowski spacetime and a modification of the particle statistics \([12,13,14]\). \(\kappa\)-deformed Poincaré symmetry is algebraically described by the \(\kappa\)-Poincaré Hopf algebra and is an example of deformed relativistic symmetry that can possibly describe physical reality at the Planck scale. The deformation parameter \(\kappa\) is usually interpreted as the Planck mass or the quantum gravity scale.

The deformation of the symmetry group can be realized through the application of the Drinfeld twist on that group \([15]\). It is argued that for the twisted (deformed) Poincaré symmetry to be retained in a quantum theory, it is necessary to implement twisted statistics \([16,13,17,18]\), with the form of the interaction also being dictated by the quantum symmetry. The virtue of the twist formulation is that the deformed (twisted) symmetry algebra is the same as the original undeformed one and only the coalgebra structure changes, leading to the same free field structure as the corresponding commutative field theory.

In QFT one deals with the asymptotic (in and out) scattering states that are quantum states of the free theory. Under the action of the Poincaré symmetry algebra, one particle asymptotic states transform according to the irreducible representation of the Poincaré algebra \(\mathfrak{iso}(1,3)\). To extend the action of the symmetry algebra from one particle to two and many particle states, one needs a notion of the coproduct. Hence, if \(D\) is a representation of the symmetry algebra \(\mathfrak{a}\) (which is here \(\kappa\)-Poincaré algebra, \(\mathfrak{a} \equiv \mathfrak{iso}(1,3)\)), which acts on the space \(V\) of physical states (asymptotic in and out states) as \(|\phi\rangle \rightarrow D(\Lambda)|\phi\rangle\), with \(\Lambda \in \mathfrak{a}\) and \(|\phi\rangle \in V\), then this action of the symmetry algebra can be extended to two particle states according to

\[
|\phi_1\rangle \otimes |\phi_2\rangle \rightarrow (D \otimes D)\Delta(\Lambda)|\phi_1\rangle \otimes |\phi_2\rangle,
\]

where \(\Delta: \mathfrak{a} \rightarrow \mathfrak{a} \otimes \mathfrak{a}\) is the coproduct. Since in noncommutative versions of QFT, in much the same way as in the standard QFT, one is concerned with the states of many identical particles, it is of importance to identify the exchange statistics that particles obey upon deformation. It is known that the (anti)symmetrization procedure in NCQFT, required to describe deformed bosons and fermions, is carried out with the help of the statistics flip operator \(\tau\) (intertwiner), which needs to be compatible with the deformation, i.e. with the twisted symmetry algebra. This means that \(\{\Delta(\Lambda), \tau\} = 0\). The symmetry algebra that admits the existence of such flip operator belongs to a class of bialgebras that possess a quasitriangular structure, which means that a universal quantum \(R\)-matrix for such symmetry algebra can be found. We recall that \(R\)-matrix for the symmetry algebra \(\mathfrak{a}\) is an invertible element \(R \in \mathfrak{a} \otimes \mathfrak{a}\) having the property

\[
R\Delta(\Lambda)R^{-1} = \tilde{\Delta}(\Lambda),
\]

for all \(\Lambda \in \mathfrak{a}\). Here \(\tilde{\Delta}(\Lambda)\) is the opposite coproduct of \(\Delta(\Lambda)\), where the factors in the tensor product are interchanged in every term. In this case the statistics flip operator \(\tau\) can be expressed as \(\tau = \tau_0R\), where \(\tau_0\) is the undeformed flip operator, \(\tau_0(|\phi_1\rangle \otimes |\phi_2\rangle) = |\phi_2\rangle \otimes |\phi_1\rangle\), for any \(|\phi_1\rangle, |\phi_2\rangle \in V\).

We believe that the further detailed study of the algebraic structure of \(\kappa\)-Poincaré, in particular the study of its triangular quasibialgebra structure is of high importance. The existence of the triangular quasibialgebra structure ensures that there is a fully \(\kappa\)-covariant way to define states of many identical particles (in any representation) in such a way that there is a one to one correspondence between the states of \(\kappa\)-deformed theory and the states of undeformed theory.

There have been attempts in the literature to obtain \(\kappa\)-Poincaré Hopf algebra from the Drinfeld twist, but neither of them so far succeeded to accomplish this completely. Particularly, a full description of deformation of Poincaré algebra in terms of both, the twist and the \(R\)-matrix is missing. The Abelian twists \([15,17,18]\) and Jordanian twists \([19,20]\) compatible with \(\kappa\)-Minkowski spacetime have been constructed. However, the problems with these twists were first that they are not expressed in terms of the Poincaré generators only and the second, that they give rise to a coalgebra that does not close within the \(\kappa\)-Poincaré algebra, but instead runs out, generally into \(\mathcal{U}(\mathfrak{sl}(4)) \otimes \mathcal{U}(\mathfrak{sl}(4))\). On the other side, in Ref. \([21]\) an attempt was made to find the \(R\)-matrix that lies within the Poincaré algebra. While the authors succeeded to find the perturbative expansion of the \(R\)-matrix up to the
fifth order in deformation parameter, they failed to construct its exact form. Also, they missed to give the form of the twist and the corresponding relation between the twist and the $R$-matrix.

In this paper we elucidate and solve the above problems in a different approach, using deformation of the Heisenberg algebra and coalgebra by twist. In Section 2 we introduce the undeformed Heisenberg algebra and construct the corresponding coalgebra. We also present a type of tensor exchange identities and show that the introduced coalgebra is compatible with them. These identities appear to be crucial in our construction. In Section 3 we deform the Heisenberg algebra and coalgebra by the twist deformation and get the deformed tensor exchange identities that are also compatible with the deformed coalgebra structure. This deformed Heisenberg algebra includes the $\kappa$-Minkowski spacetime. By using the homomorphism of the coproduct and the introduced tensor exchange identities, in Section 4, continuing analysis from Ref. [22], we give the coproducts for the Poincaré generators, proposing two new methods of calculation, which are explicitly illustrated in few examples. In Section 5 we give the exact form of the universal $R$-matrix, generally for the deformed Heisenberg coalgebra and especially for the $\kappa$-Poincaré Hopf algebra, and present the method by which it can be cast into a form including Poincaré generators only. Finally, in Section 6, we give a short conclusion.

2. HEISENBERG ALGEBRA AND COALGEBRA

Before going into deformed relativistic symmetries let us start with the undeformed Heisenberg algebra and the deformation of a phase space (deformed Heisenberg algebra) including $\kappa$-Minkowski spacetime. In the undeformed case Heisenberg algebra $H$ can be defined as an algebra generated by 4 coordinates $x_\mu$ and 4 momenta $p_\mu$, satisfying the following relations:

$$\left[ x_\mu, x_\nu \right] = 0, \quad \left[ p_\mu, x_\nu \right] = -i\eta_{\mu\nu} \cdot 1, \quad \left[ p_\mu, p_\nu \right] = 0,$$

where $\mu, \nu = 0, 1, 2, 3$ and $\eta_{\mu\nu} = (-, +, +, +)$ is diagonal metric tensor with Lorentzian signature. Similarly, the algebra in $\{x_\mu\}$ is denoted by $\mathcal{A}$, and the symmetric algebra in $\{p_\mu\}$ is denoted by $T$. Using relation $p_\mu x_\nu - x_\nu p_\mu = -i\eta_{\mu\nu} \cdot 1$, we can write $H = \mathcal{A}T$.

The action $h \triangleright f(x)$ for any $h \in H$, $f(x) \in \mathcal{A}$, is defined by $x_\mu \triangleright f(x) = x_\mu f(x)$, $p_\mu \triangleright f(x) = -i\partial f(x)/\partial x^\mu$ and by the property:

$$h_1 h_2 \triangleright f(x) = h_1 \triangleright (h_2 \triangleright f(x)), \quad (4)$$

where $h_1, h_2 \in H$. Hence, $H \triangleright \mathcal{A} = \mathcal{A}$ is a $H$-module. The Leibniz rule $h \triangleright (f(x)g(x))$, for any $h \in H$, is obtained by using Eq. (4) and from $x_\mu \triangleright (f(x)g(x)) = \alpha(x_\mu f(x))(g(x)) + (1 - \alpha)f(x)(x_\mu g(x))$ and $p_\mu \triangleright (f(x)g(x)) = (p_\mu \triangleright f(x))(g(x)) + (f(x))(p_\mu \triangleright g(x))$, where $\alpha$ is an arbitrary real number.

The guiding principle in this paper is the idea that the Heisenberg algebra $H$ can be endowed with a coalgebra structure. Namely, the Leibniz rule and the coproduct $\Delta_0$ are related by:

$$h \triangleright (f(x)g(x)) = m_0(\Delta_0 h) \triangleright (f(x) \otimes g(x)), \quad (5)$$

for every $h \in H$, and $m_0$ is the multiplication map. From the Leibniz rule for $x_\mu$ it follows:

$$\Delta_0 x_\mu = \alpha x_\mu \otimes 1 + (1 - \alpha) 1 \otimes x_\mu = [x_\mu \otimes 1] = [1 \otimes x_\mu] \in \mathcal{A} \otimes \mathcal{A}/\mathcal{R}_0, \quad (6)$$

where $[x_\mu \otimes 1] = [1 \otimes x_\mu]$ is the equivalence class generated by the relations $\mathcal{R}_0 \equiv x_\mu \otimes 1 - 1 \otimes x_\mu = 0$. The relations $\mathcal{R}_0$ generate the equivalence classes on $\mathcal{A} \otimes \mathcal{A}$. It can be shown that $\Delta_0(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}/\mathcal{R}_0$ is an algebra isomorphic to $\mathcal{A}$. Similarly, from the Leibniz rule for $p_\mu$ it follows:

$$\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu. \quad (7)$$

Note that $\Delta_0(T)$ is an algebra isomorphic to $T$.

Generally, $\Delta_0 h$ for any $h \in H$, can be obtained by the homomorphism $\Delta_0(h h_2) = (\Delta_0 h_1)(\Delta_0 h_2)$ that follows from Eq. (4). Note that $[p_\mu \otimes 1, x_\nu \otimes 1 - 1 \otimes x_\nu] \neq 0$, but $[\Delta_0 p_\mu, x_\nu \otimes 1 - 1 \otimes x_\nu] = 0$ and generally the consistency requirement $[\Delta_0 h, x_\nu \otimes 1 - 1 \otimes x_\nu] = [\Delta_0 h, \mathcal{R}_0] = 0$, for all $h \in H$, is satisfied. Taking $h_1 = p_\mu$, $h_2 = x_\nu$, and using Eqs. (6), (7) and $\mathcal{R}_0$, one gets

$$\Delta_0(p_\mu x_\nu) = (\Delta_0 p_\mu)(\Delta_0 x_\nu) = \left\{ \begin{array}{ll} (p_\mu \otimes 1 + 1 \otimes p_\mu)(x_\nu \otimes 1) = p_\mu x_\nu \otimes 1 + x_\nu \otimes p_\mu = p_\mu x_\nu \otimes 1 + 1 \otimes x_\nu p_\mu \\ (p_\mu \otimes 1 + 1 \otimes p_\mu)(1 \otimes x_\nu) = p_\mu \otimes x_\nu + 1 \otimes p_\mu x_\nu = x_\nu p_\mu \otimes 1 + 1 \otimes p_\mu x_\nu \end{array} \right. \quad (8)$$
In complete analogy, one also obtains four expressions for \( \Lambda_0(x, p) \). It now follows, from these expressions, that
\[
\Lambda_0\left(p_{\mu} x_{\nu}\right) = \Lambda_0\left(p_{\mu} x_{\nu}\right) - \Lambda_0\left(x_{\nu} p_{\mu}\right) = [\Lambda_0 p_{\mu}, \Lambda_0 x_{\nu}] = -i\eta_{\mu\nu} \cdot 1 \otimes 1,
\]
showing that the coproduct \( \Delta_0 h, h \in H \), is compatible with the relations \( \mathcal{R}_0 \). Hence, \( \Lambda_0(H) = \Delta_0(\mathcal{A}) \Delta_0(T) \) is the algebra isomorphic to \( H \). The coalgebra on the undeformed Heisenberg algebra \( H \) is also undeformed. The coalgebra of the Heisenberg algebra \( H \) does not lead to a Hopf algebra. It is related to an algebraic construction whose mathematical structure is described in terms of a Hopf algebroid \([23]\). Note that the elements \( L_{\mu\nu} = x_{\mu} p_{\nu} \) generate the \( \mathfrak{gl}(4) \) algebra and satisfy \( \Delta_0 L_{\mu\nu} = L_{\mu\nu} \otimes 1 + 1 \otimes L_{\mu\nu} \).

In the following, we interpret \( \Delta_0 h \) as the operator \( \Delta_0 h : \Lambda_0(H) \rightarrow \Lambda_0(x) \), defined by \( (\Delta_0 h)(\Lambda_0 h_0) = \Delta_0(h h_0) \) for all \( h, h_0 \in H \).

### 3. \( \kappa \)-DEFORMATION OF HEISENBERG ALGEBRA AND COALGEBRA BY TWIST

We shall be interested in a deformation of the Heisenberg algebra \( H \). We consider the family of twists given by
\[
\mathcal{F} = \exp \left( i \left( \lambda A_0 p_k \otimes A - (1 - \lambda) A \otimes x_k p_k \right) \right),
\]
where \( A = -a \cdot p, \lambda \) is a real parameter and \( a \) is a deformation fourvector, whose only non-zero component is the time component \( a_0 \). The deformation parameter \( a_0 \) can be used to model deformation at the quantum gravity scale, in which case it corresponds to the inverse of the Planck mass, \( a_0 \sim \frac{1}{M} \). The given family of twists represents a subfamily of Abelian twists which satisfy all required properties, including a cocycle and counit condition \([13]\). Now the deformed coalgebra structure can be obtained by using this twist operator \( \mathcal{F} \) to get
\[
\Delta h = \mathcal{F} \Delta_0 h \mathcal{F}^{-1},
\]
for every \( h \) in \( H \). By the same twist operation applied to relations \( \mathcal{R}_0 \), namely \( \mathcal{F} \mathcal{R}_0 \mathcal{F}^{-1} = \mathcal{R} \), they transform into
\[
x_{\mu} \otimes 1 = Z^{\mu\nu} \otimes x_{\nu} Z^{-\mu\nu},
\]
\[
x_{\mu} \otimes 1 = 1 \otimes x_{\mu} - a_0 ((1 - \lambda) \cdot 1 \otimes x_k p_k + \lambda x_k p_k \otimes 1),
\]
where \( Z = e^\lambda \). Let us name the relations \([12], [13]\) obtained in this way by \( \mathcal{R} \). Similarly as before, these relations induce the partition of the algebra \( \mathcal{A} \otimes \mathcal{A} \) into equivalence classes, which together with the algebra \( \Delta(T) \) forms the algebra \( \Delta(H) = \Delta(\mathcal{A}) \Delta(T) \). This algebra contains all deformed coproducts \( \Delta h \), for all \( h \in H \). Again \( \Delta h \) can be understood as the operator acting on \( \Delta(H) \) and acquiring values in the same algebra, namely \( \Delta h : \Delta(H) \rightarrow \Delta(H) \). In this case the twist operator is defined accordingly as \( \mathcal{F} : \Lambda_0(H) \rightarrow \Lambda(H) \), and vice versa for the inverse twist \( \mathcal{F}^{-1} \).

By twisting the undeformed coproducts according to Eq. \((11)\), one finds the following deformed coproducts
\[
\Delta x_{\mu} = x_{\mu} \otimes Z_{\mu\nu} = Z^{\mu\nu} \otimes x_{\nu},
\]
\[
\Delta x_{0} = x_{0} \otimes 1 + a_0 ((1 - \lambda) \cdot 1 \otimes x_k p_k = 1 \otimes x_{0} - a_0 x_k p_k \otimes 1),
\]
\[
\Delta p_{\mu} = p_{\mu} \otimes Z_{\mu\nu} + Z^{\mu\nu} \otimes p_{\nu},
\]
\[
\Delta p_{0} = p_{0} \otimes 1 + 1 \otimes p_{0}.
\]

It is worth mentioning that due to the twist deformation being expressed in terms of the similarity transformation, the homomorphic property of the undeformed coproduct \( \Lambda_0 \) naturally extends onto the coproduct \( \Delta \), so that it is also a homomorphism,
\[
\Delta(h_1 h_2) = (\Delta h_1)(\Delta h_2), \quad h_1, h_2 \in H.
\]

In analogy to the concluding remark of section 2, choosing \( h_1 = p_{\mu}, h_2 = x_{\nu} \) in \((18)\), together with relations \( \mathcal{R} \), gives us four expressions for \( \Delta(p_{\mu} x_{\nu}) \). Repeating the procedure for \( \Delta(x_{\nu} p_{\mu}) \), we find that \( \Delta\left(p_{\mu} x_{\nu}\right) = \Delta(p_{\mu}, x_{\nu}) \), i.e. \( \Delta h, h \in H \) is compatible with relations \( \mathcal{R} \) (\( \mathcal{R} \Delta h = \Delta h \mathcal{R} \)).

The basic idea in this paper is that the coproduct for any element in \( H \) can be calculated by using only two things. The first one is the homomorphism of the coproduct \( \Delta \) and the second one are the relations \( \mathcal{R} \), expressed in terms of the identities, Eqs. \((12)\) and \((13)\). Particularly, the coproducts for the Lorentz generators can be calculated by using this method, which we show in the next section. This is made possible since according to the theory of realizations \([24],[25],[26]\), the Lorentz generators can be expressed in terms of \( x \) and \( p \) generators of the Heisenberg algebra. To make the picture closed, let us mention that in the theory of realizations, there is one realization corresponding to each twist element. More concretely, having a twist element \( \mathcal{F} \), the corresponding realization can be obtained as
\[
\hat{\mathcal{F}} = \mathcal{F}^{-1} \ast (x_{\mu} \otimes id),
\]
where \( \ast \) denotes the convolution product.
where $m_0$ is the multiplication map. The family of twists \([10]\) leads to
\[
\hat{x}_i = x_i Z^{-\frac{1}{2}}, \quad \hat{x}_0 = x_0 - a_0 (1 - \lambda) x_k p_k.
\]

One can check that realizations \([20]\) for $\hat{x}_\mu$ satisfy the commutation relations
\[
[\hat{x}_\mu, \hat{x}_\nu] = i \left( a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu \right),
\]
that are known to describe $\kappa$-Minkowski spacetime algebra. In this way the $\kappa$-deformation of the Heisenberg algebra $H$ is made explicit.

Eq. \((20)\), along with \([13]-[15]\), can be used to derive $\Delta \hat{x}_\mu$
\[
\Delta \hat{x}_i = \Delta (x_i Z^{-\frac{1}{2}}) = (\Delta x_i)(\Delta Z^{-\frac{1}{2}}) = \left\{ \begin{array}{l}
(x_i \otimes Z^I)(Z^{-1} \otimes Z^{-1}) = x_i Z^{-1} \otimes 1 = \hat{x}_i \otimes 1 \\
(Z^{-1} \otimes x_i)(Z^{-I} \otimes Z^{-I}) = Z^{-1} \otimes x_i Z^{-1} = Z^{-1} \otimes \hat{x}_i,
\end{array} \right.
\]
and analogously for $\hat{x}_0$, with the result
\[
\Delta \hat{x}_i = \hat{x}_i \otimes 1 = 1 \otimes \hat{x}_i - a_0 p_i Z^{-\frac{1}{2}} \otimes \hat{x}_i.
\]
These results can be written compactly as
\[
\Delta \hat{x}_\mu = \hat{x}_\mu \otimes 1 = Z^{-\frac{1}{2}} \otimes \hat{x}_\mu - a_\mu p_\mu \otimes \hat{x}_\mu,
\]
where $a_\mu p_\mu = (1 - Z^{-\frac{1}{2}})/a_0$ and $p_\mu = p_\mu Z^{-\frac{1}{2}}$. The calculated coproducts are in accordance with the Leibnitz rule for $\hat{x}_\mu$ (see Eq.\((31)\) in \([20]\)).

4. GENERALIZED POINCARÉ ALGEBRAS AND COALGEBRAS

There are infinitely many ways of implementing (deformed) Poincaré algebras compatible with the $\kappa$-Minkowski spacetime \([22]\). If one writes an ansatz for the Lorentz generators in the form
\[
\hat{M}_{\mu 0} = x_i p_0 F_1 (A, b) - x_0 p_i F_2 (A, b) + a_0 (x_i p_k) p_j F_3 (A, b) + a_0 x_i p^2 F_4 (A, b)
\]
\[
\hat{M}_{ij} = M_{ij} = x_i p_j - x_j p_i,
\]
the (deformed) commutation relations can be found in \([22]\). The form of the Lorentz generators given by \([22]\) is compatible with the deformed Lorentz algebra (with $\kappa$-Poincaré algebra included), where the Minkowski metric is replaced with the more general metric, having generally nonlinear dependence on momenta. This is unlike the form of the metric dependence of momenta that emerged within the idea of relative locality \([27],[28]\), where the metric dependence on momenta is linear.

Following the results of the previous section (relations $\mathcal{R}$, $\Delta_0$ and $\Delta$ for $x_\mu$ and $p_\mu$), we present two (equivalent) new methods for calculating the coproduct of $\hat{M}_{\mu 0}$. The first method includes the twist deformation
\[
\Delta \hat{M}_{\mu 0} = \mathcal{F} \Delta_0 \hat{M}_{\mu 0} \mathcal{F}^{-1}
\]
with the help of the twist, Eq.\((19)\), while the second method uses the realizations \([25]\) and \([26]\) and the homomorphism of the coproduct $\Delta$,
\[
\Delta \hat{M}_{\mu 0} = \Delta x_i \Delta p_i \Delta F_1 - \Delta x_0 \Delta p_i \Delta F_2 + a_0 \Delta x_i \Delta p_k \Delta p_j \Delta F_3 + a_0 \Delta x_i \Delta p^2 \Delta F_4.
\]

In Eq.\((27)\) $\Delta_0 \hat{M}_{\mu 0}$ is not simply $\hat{M}_{\mu 0} \otimes 1 + 1 \otimes \hat{M}_{\mu 0}$, but needs to be calculated by using Eqs.\((3)\) and \((7)\) ($\Delta_0 \hat{M}_{\mu 0} = \Delta_0 x_i \Delta_0 p_0 \Delta_0 F_1 - ...$), and $\Delta x$, $\Delta p$ are given by Eqs.\((13)-(17)\). The coproduct of $M_{ij}$, calculated in an analogous way, is primitive.

We consider three examples of the above
(i) $F_1 = \frac{Z^I - Z^{-I}}{2}$, $F_2 = Z^I$, $F_3 = (1 - \lambda) Z^{-I}$, $F_4 = - \frac{Z^I}{2}$.
This is the case of the undeformed Lorentz algebra which, if extended by noncommutative coordinates, forms a Lie algebra \([22]\). For this case we use $\hat{M}$ instead of $\hat{M}$. Using Eq.\((27)\) or \((28)\) and the relations $\mathcal{R}$, gives
\[
\Delta \hat{M}_{\mu 0} = \hat{M}_{\mu 0} \otimes 1 + Z \otimes \hat{M}_{\mu 0} - a_0 Z^2 p_j \otimes M_{ij}.
\]

(ii) $F_1 = \frac{\sinh \lambda}{\lambda}$, $F_2 = 1$, $F_3 = F_4 = 0$, $\lambda = \frac{1}{2}$.
In this example (known as the standard basis in Refs.\([2]\)) the Lorentz algebra is deformed ($[\hat{M}_{\mu 0}, \hat{M}_{\mu 0}] = -i M_{ij} \cosh \lambda$). $\Delta \hat{M}_{\mu 0}$, calculated by \((27)\) or \((28)\) and $\mathcal{R}$ (relations \((12)\) and \((13)\) for $\lambda = \frac{1}{2}$), is
\[
\Delta \hat{M}_{\mu 0} = \hat{M}_{\mu 0} \otimes Z^{-\frac{1}{2}} + Z^\frac{1}{2} \otimes \hat{M}_{\mu 0} + \frac{a_0}{2} \left( M_{ij} Z^\frac{1}{2} \otimes p_j - p_j \otimes M_{ij} Z^{-\frac{1}{2}} \right).
\]
The same result is trivially obtained using the twist because

\[ \Delta \tilde{M}_\theta = x_i p_0 \otimes Z^i + Z^{(-1,i)} \otimes x_i p_0 - x_0 p_i \otimes Z^{-1} - Z^{-1} \otimes x_0 p_i - (1 - \lambda) a_0 p_i \otimes x_k p_k Z^{-1} + \lambda a_0 x_k p_k Z^{-1} \otimes p_i. \]  

(31)

An illustration of the method is given by calculating the coproduct of \( M_{i,j} \) (which is the same in all three cases):

\[ \begin{align*}
\Delta M_{i,j} & = \Delta x_i \Delta p_j - \Delta x_j \Delta p_i \\
& = x_i p_j \otimes 1 + x_i Z^{(-1)} \otimes p_j Z^{-1} - x_j p_i \otimes 1 - x_j Z^{(-1)} \otimes p_i Z^{-1} \\
& = M_{i,j} \otimes 1 + 1 \otimes M_{i,j}.
\end{align*} \]

The same result is trivially obtained using the twist because \( \Delta_0 M_{i,j} \) is primitive and \( M_{i,j} \) commutes with \( A \) and \( x_k p_k \).

Expressions for the coproducts obtained in (i), (ii) and (iii) coincide with the known results in the literature. The cases (i) and (ii) are the examples of \( \kappa \)-Poincaré Hopf algebra, while in the case (iii) the algebra needs to be extended to \( \mathfrak{gl}(4) \) in order to get a \( \kappa \)-deformed \( \mathfrak{gl}(4) \) Hopf algebra, consistent with \( \kappa \)-Minkowski spacetime.

5. Flip Operator and the Universal R-Matrix

The ordinary flip operator \( \tau_0 : H \otimes H \rightarrow H \otimes H \) is defined by \( \tau_0(h_1 \otimes h_2) = h_2 \otimes h_1 \), for any two elements \( h_1, h_2 \) in \( H \), and has the properties

\[ \tau_0^2 = 1 \otimes 1, \quad \tau_0 \Delta_0 h = \Delta_0 h \tau_0, \]  

(33)

for all elements \( h \) in \( H \). We use it to define the twist \( \widetilde{F} \) and the coproduct \( \tilde{\Delta} \)

\[ \begin{align*}
\widetilde{F} : \Delta_0(H) & \rightarrow \tilde{\Delta}(H), \\
\tilde{\Delta}h : \tilde{\Delta}(H) & \rightarrow \tilde{\Delta}(H),
\end{align*} \]

(34)

\[ \tau_0 = \tau_0 \Delta h \tau_0 = \widetilde{F} \Delta h \widetilde{F}^{-1}, \]  

(35)

for every \( h \) in \( H \). Here \( \tilde{\tau} = \tau_0 R \tau_0 = \widetilde{F} R \widetilde{F}^{-1} \) are the relations coming from the coproduct \( \tilde{\Delta} \)

\[ \begin{align*}
x_i \otimes 1 & = Z^i \otimes x_i Z^{-1}, \\
x_0 \otimes 1 & = 1 \otimes x_0 + a_0 1 \otimes 1 \otimes x_k p_k + a_0 (1 - \lambda) x_k p_k \otimes 1
\end{align*} \]  

(36)

(37)

and \( \tilde{\Delta}(H) = \tilde{\Delta}(\mathcal{A}) \tilde{\Delta}(T) \), where \( \tilde{\Delta}(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A} / \mathcal{R} \). The algebra \( \tilde{\Delta}(H) \) contains all coproducts \( \tilde{\Delta}h \), for all \( h \) in \( H \). Note that which one among the relations \( R_0, R, \tilde{\tau} \) needs to be applied in a given situation depends on the codomain of the operator in question.

The flip operator \( \tau \) is defined with \( \tau \Delta h = \Delta h \tau \). It follows that

\[ \tau = \widetilde{F} \tau_0 \widetilde{F}^{-1} = \tau_0 \widetilde{F} \widetilde{F}^{-1} = \tau_0 R. \]  

(38)

The last equality in (38) is the defining relation of the \( R \)-matrix

\[ R = \widetilde{F} \widetilde{F}^{-1}, \quad R : \Delta(H) \rightarrow \tilde{\Delta}(H). \]  

(39)

From the property \( \tau^2 = 1 \otimes 1 \) and the definitions of \( \tau \) and \( R \), one gets

\[ R \Delta h R^{-1} = \tilde{\Delta}h, \]  

(40)

for every \( h \) in \( H \). For the family of twists (10), the corresponding \( R \)-matrix is given by

\[ R = \exp(i(A \otimes x_k p_k - x_k p_k \otimes A)), \]  

and it satisfies

\[ \tau_0 R \tau_0 = R^{-1}, \quad R^{-1} : \tilde{\Delta}(H) \rightarrow \Delta(H). \]  

(42)
We note that if the $\star$-product is defined by $(f \star g)_R = m_0 F^{-1} \triangleright (f \otimes g)_R$, where $R$ in the subscript denotes that $(f \otimes g)_R \in \Delta(\mathcal{A})$, and we define $(f \star g)_\bar{R} = m_0 \bar{F}^{-1} \triangleright (f \otimes g)_{\bar{R}}$, then it follows that

$$(f \star g)_R = (g \star f)_\bar{R} \Rightarrow \bar{F}^{-1} \triangleright (f \otimes g)_R = \bar{F}^{-1} \triangleright (g \otimes f)_\bar{R},$$

where $(g \otimes f)_\bar{R} \in \bar{\Delta}(\mathcal{A})$. Multiplying the second equation in (43) by $\bar{F}$ from the left, one finds

$$R \triangleright (f \otimes g)_R = (g \otimes f)_\bar{R}.$$  

(44)

By direct calculation (inserting (41) and (14)-(17) into the LHS of (40)), it is easy to check that (41) is the exact $R$-matrix for the generators of the Heisenberg algebra $H$

$$R \Delta x_i R^{-1} = x_i \otimes Z^{i-1} + Z^{-i} \otimes x_i = \Delta x_i$$

$$R \Delta x_0 R^{-1} = x_0 \otimes 1 - a_0 \lambda \cdot 1 \otimes x_k p_k = 1 \otimes x_0 + a_0 (1 - \lambda) x_k p_k \otimes 1 = \Delta x_0$$

$$R \Delta p_i R^{-1} = p_i \otimes Z^{i-1} + Z^{-i} \otimes p_i = \Delta p_i$$

$$R \Delta p_0 R^{-1} = p_0 \otimes 1 + 1 \otimes p_0 = \Delta p_0,$$

(45)\text{--} \text{(48)}

and because of the homomorphism property, also for the entire algebra $H$. The first equalities in Eqs. (45)\text{--} \text{(48)} were calculated by using $e^{\rho} he^{-\rho} = e^{ad_{h}(\rho)}$ and in the second equality of Eqs. (45)\text{--} \text{(46)} relations $\bar{R}$ (46, 47) were used.

In (21) the authors have used a perturbative approach to construct the universal $R$-matrix for the $\kappa$-deformed Poincaré algebra generated by $\mu$ and $\bar{M}_\mu$ from the standard basis of $\mathcal{A}$ (case (ii) in Section 1) up to terms of order $\kappa^{-5}$ ($a_0 = -1/\kappa$). The $R$-matrix they presented is expanded in terms of the deformed Poincaré generators and possesses a wedge-product structure. They give no statement about the existence or appearance of the matrix at higher orders of the deformation parameter. We have shown here that the exact $R$-matrix is precisely given by (41).

Our method for expanding $R = e^{\rho}$, $\rho = i(\bar{A} \otimes x_k p_k - x_k p_k \otimes A)$ in terms of $p_\mu$ and $\bar{M}_\mu$, is the following. We wish to find $r_1, r_2, \ldots$ such that $R = \exp(r_1 + r_2 + r_3 + \ldots)$, where $r_1$ is a series in $a_0$, starting with $a_0^0$ and is linear in $\bar{M}_\mu$ (we do not consider terms containing the momenta alone), for all $i = 1, 2, 3, \ldots$. It follows that $\rho + \frac{\rho^2}{2} + \frac{\rho^3}{6} + \ldots = r_1 + r_2 + \frac{\rho^2}{3} + \frac{r_1^2}{2} + \frac{r_1 r_2 r_1}{6} + r_1 + \frac{r_1^3}{6} + \ldots$.

In the first order we get $r_1 = \rho$. The next step is to write down an ansatz for $r_1$, linear in $\bar{M}$ and $a_0$, with undetermined coefficients $: -ia_0(c_1 M_{\bar{a}} \otimes p_1 + c_2 M_{\bar{a}} p_1 \otimes 1 + d_1 + 1 \otimes M_{\bar{a}} p_1 + d_2 p_1 \otimes M_{\bar{a}})$). In order to compare this expression with $\rho$, in all the terms from $\rho$ and the ansatz, we move $\bar{x}_k$ to the right side of the tensor product using the relations $\bar{R}$ (relations (46) and (47) for $\lambda = 1/2$). We also use the fact that $\bar{M}_{\bar{a}} = x_i p_0 - x_0 p_i + O(a_0^2)$ ($\bar{M}_{\bar{a}} = x_i p_0 \frac{\sinh A}{A} - x_0 p_i$, see Section 4). After equating the expressions, we get 7 equations for the 4 coefficients $(c_1, c_2, d_1, d_2)$. The system of equations has a unique solution given by $c_1 = -d_2 = -1, c_2 = d_1 = 0$, so that the final result up to the first order in the deformation parameter $a_0$ is

$$r_1 = -ia_0(p_k \otimes M_{\bar{a}} - M_{\bar{a}} p_k),$$

(49)

namely, $r_1 = -ia_0(p_k \otimes M_{\bar{a}} - M_{\bar{a}} p_k) = \rho + O(a_0^2)$. Using the relations to move $x_k$ generates terms of higher order in the deformation parameter, e.g., $\rho = -ia_0(p_k \bar{Z}^{i} \otimes x_k p_i) - p_0 \otimes x_k p_k = -ia_0 \cdot 1 \otimes x_k (p_k \otimes p_0 - p_0 \otimes p_k) + O(a_0^2).$

In the second order, $r_2$ is found to be equal to $\frac{3}{4} \rho^2 - \frac{1}{2} r_1^2$ terms of order $a_0^2$ coming from the use of relations $\bar{R}$ ($\lambda = 1/2$) in $\rho - r_1$. It follows that

$$r_2 = 0.$$  

(50)

The general ansatz linear in $\bar{M}$, of order $a_0^2$, has 10 coefficients. Equating it with 0 gives a system of 16 homogeneous equations. The equations have a unique solution given by all the constant equal to 0.

In the third order, $r_3 = \frac{1}{2} \rho^3 - \frac{3}{4} r_1^2$ + terms of order $a_0^3$ coming from the use of relations $\bar{R}$ ($\lambda = 1/2$) in $(\rho + \frac{3}{4} \rho^2 - r_1 - \frac{1}{2} r_1^2) +$ terms of order $a_0^3$ coming from the expansion of $M_{\bar{a}}$ in $r_1$. Summing up all the contributions, we find

$$r_3 = \frac{-ia_0^3}{24} \cdot 1 \otimes x_1 \left( \frac{3}{4} p_i \otimes p_i - 3 p_i \otimes p_i + p_0 \otimes p_i p_i - 2 p_0 \otimes p_i p_i + 2 p_i \otimes p_0 - 2 p_0 \otimes p_0 \right)$$

$$- 2 p_i p_0 \otimes p_j + 2 p_j \otimes p_i p_0 - 2 p_j \otimes p_j p_0 + 2 p_j \otimes p_j p_0 - 2 p_j p_0 \otimes p_j + 2 p_j p_j \otimes p_0 - 2 p_0 \otimes p_i p_j + 2 p_i p_j \otimes p_0$$

$$- 2 p_0 \otimes p_i p_j + 2 p_i p_j \otimes p_0).$$

(51)

As for the first and second order, the obtained expression is compared with the most general ansatz linear in $\bar{M}$ and of order $a_0^3$, which in this case contains 28 constants. We get a system of 35 equations for the 28 coefficients. Solving the equations leaves 3
coefficients undetermined, giving us a three-parameter family of solutions for $r_3$

$$r_3 = -\frac{ia^2}{24}(-\alpha_1 M_{ij} p_i p_0 \otimes p_j + \beta_1 p_i \otimes M_{ij} p_0 p_j - \alpha_2 M_{ij} p_j \otimes p_i p_0 + \alpha_2 p_i p_0 \otimes M_{ij} p_j - (\alpha_1 + 2) \hat{M}_{00} p_i \otimes p_j + (\beta_1 + 2) p_i \otimes \hat{M}_{00} p_j - \alpha_1 \hat{M}_{00} p_j \otimes p_i - \beta_1 p_i \otimes \hat{M}_{00} p_j - 2 \hat{M}_{00} \otimes p_j \otimes \hat{M}_{00} + 3 \hat{M}_{00} p_0^2 \otimes p_i - 3 p_i \otimes \hat{M}_{00} p_0^2 + 3 p_i p_0 \otimes \hat{M}_{00} + (\alpha_2 + 2) M_{00} p_i \otimes p_j + (\alpha_2 + 2) p_j \otimes \hat{M}_{00} p_i + (\alpha_2 + 2) \hat{M}_{00} p_j \otimes p_i p_0 - (\alpha_2 + 2) p_i p_0 \otimes \hat{M}_{00} p_j).$$  \hspace{1cm} (52)  

If we choose $\alpha_1 = \beta_1$, we get a two-parameter family of solutions for $r_3$, which all have a wedge-product structure. $r_1$, $r_2$ and the special case of $r_3$ with $\alpha_1 = \beta_1 = -6$, $\alpha_2 = -2$ coincide with the results found in [21].

Analogously, the $R$-matrix can be expanded in the generators $p_i$ and $M_{ij}$ (from the case (i) in Section 4). However, an easier way is to insert the known relation [22] between $M$ (for $\lambda = 1/2$) and $\hat{M}$ ($\hat{M}_{00} = M_{00} Z^{-2} + \frac{\omega}{8} M_{ij} p_j, \hat{M}_{ij} = M_{ij}$) into the results presented above.

Contrary to $M$, the generators $\tilde{M}$, from the case (iii) in Section 4 cannot be related to $\hat{M}$ (see Ref [22]). Consequently, the universal $R$-matrix cannot be expanded in $\tilde{M}$ and $p$. One arrives at the same result when trying to repeat the presented procedure with $\tilde{M}$ instead of $\hat{M}$ (the system of equations for the coefficients of $r_3$ doesn’t have a solution). The above conclusions can be easily extended for the case of arbitrary $\lambda$.

6. CONCLUDING REMARKS AND DISCUSSION

In the following we clarify what from our perspective is a major physical motivation for studying the universal $R$-matrix and also try to justify a type of deformation carried out in the paper which keeps the track with the Hopf algebra formalism, resulting in a mathematical structure that can be shown to be related to a Hopf algebroid [23].

Basically, the idea for applying a Hopf algebra formalism is grounded in the fact that the general relativity theory together with the uncertainty principle of quantum mechanics leads to a class of models with spacetime noncommutativity. In this setting the smooth spacetime geometry of classical general relativity is replaced with a Hopf algebra at the Planck scale. In our view, a particularly interesting example of Hopf algebra is $k$-Poincaré Hopf algebra, which provides an algebraic setting for describing symmetry underlying the effective NCFT that results from coupling quantum gravity to matter fields after topological degrees of freedom of gravity are integrated out [24, 52]. This particular type of Hopf algebra also appears in the context of Doubly Special Relativity theories [33, 35], which give one possible explanation for few puzzling experimental observations, such as GZK-paradox found in observations of UHECR’s [34, 57] and multi TeV photons [38] coming from certain distant astronomical objects, as well as recently obtained astrophysical data originating from the GRB’s [39].

An important questions arising from the noncommutative nature of spacetime at the Planck scale is how does this noncommutativity affect the very basic notions of physics, such as the particle statistics, particularly the spin-statistics relation and how these changes can be implemented into a quantum field theory formalism to accommodate for these new features. For example, it has been known for some time that quantum gravity can admit exotic statistics [40, 41]. That quantum gravity may lead to such a situation was first discussed in [40] in the context of quantum geons.

In our view, the answer to the above question, or at least some glimpse of it, is given within the framework of quantum groups, i.e. Hopf algebras and can be expressed in terms of its triangular quasibialgebra structure or universal $R$-matrix. To be more precise, given a scalar field $\phi$ and knowing $R$-matrix, the corresponding modified algebra of creation and annihilation operators (which captures the information about statistics) can be inferred via relation

$$\phi(x) \otimes \phi(y) - R\phi(y) \otimes \phi(x) = 0.$$  \hspace{1cm} (53)  

In other words, we are addressing the issue as to how would the spin-statistics relation of a usual boson (or fermion) look like at the Planck scale.

In this paper we have introduced the undeformed Heisenberg algebra and constructed the corresponding coalgebra. The coalgebra so constructed induces relations $R_0$ and it is shown that the coalgebra and $R_0$ identity are compatible. We deform the Heisenberg algebra and coalgebra by the twist belonging to an Abelian family of twists. This deformed algebra includes $k$-Minkowski spacetime. It is shown that the coalgebra structure is compatible with the deformed tensor identities. The coproduct of any element in $H$ is obtained, particularly the coproducts for the Lorentz generators are found. We point out that these coproducts can also be obtained by applying the same twist deformation that was used to deform the Heisenberg algebra. In this way we have shown that there exists a twist deformation that leads to $k$-Poincaré Hopf algebra. This means that from this twist,
the coalgebra and quasitriangular ($R$-matrix) structures can be obtained that are expressed in terms of Poincaré generators only. With $R$-matrix staying entirely within $\kappa$-Poincaré algebra, we have ensured that states of any number of identical particles can be introduced in a $\kappa$-covariant fashion.

We have presented the exact form of the universal $R$-matrix for the Heisenberg (co)algebra $H$ in general and for the $\kappa$-Poincaré Hopf algebra. We have also outlined the method for finding the $R$-matrix in terms of Poincaré generators only, with calculations being carried out up to the third order in the deformation parameter. We have found that in the 3rd order there is a three-parameter family of solutions. For the certain choice of the parameter values, there is a two-parameter family of solutions having a wedge structure. For one special choice of the parameters, these results coincide with the results obtained in [21].

The general form of $\kappa$-Poincaré and $\kappa$-deformed $\mathfrak{sl}(4)$ Hopf algebras might provide an appropriate setting for capturing the signals coming from Planck scale and Quantum gravity effects. Some of those signals could be found in modified dispersion relations [43–46], which can directly be related to the so-called photon time delay. This phenomenon could be a result of $\kappa$-deformation [47] and it could be connected with the similar effects measured for high energy photons coming from the gamma ray bursts. Other footprints of the Planck scale physics might include a change in the algebra of creation and annihilation operators describing particles moving near the horizon of the black hole [14]. The change in the oscillator algebra leads to deformed statistics and it is directly related to the deformation of the $\kappa$-matrix [13].

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