The effect of the fifth-order nonlinearity on the existence of bright solitons below the modulation instability threshold

I.S. Gandzha\textsuperscript{1} and Yu. V. Sedletsky\textsuperscript{1}

\textsuperscript{1}Institute of Physics, Nat. Acad. of Sci. of Ukraine, Prosp. Nauky 46, Kyiv 03028, Ukraine

(Dated: November 12, 2015)

We analyze three different high-order nonlinear Schrödinger equation (HONLSE) models that have been used in the literature to describe the evolution of slowly modulated gravity waves on the surface of ideal finite-depth fluid. We demonstrate that the inclusion of the fifth-order nonlinear term to the HONLSE model introduces only a small correction to the amplitude of the bright HONLSE soliton solutions obtained without this term. Such soliton solutions behave as quasi-solitons in this more general case.

PACS numbers: 05.45.Yv, 47.35.Bb, 92.40.Qk

There are three different high-order nonlinear Schrödinger equation (HONLSE) models that have been used in the literature to describe the evolution of slowly modulated gravity waves on the surface of finite-depth, irrotational, inviscid, and incompressible fluid with flat coordinate, time, and wave amplitude introduced in Ref. \cite{1}:

\begin{equation}
\begin{aligned}
\hat{u}_t &= -a_1 u_x - i a_2 u_{xx} + i a_0,0,0|u|^2 u \\
&\quad + \big(a_3 u_{xxx} - a_1,0,0 u_x|u|^2 - a_{0,0,1} u^2 u_x^*\big),
\end{aligned}
\tag{1}
\end{equation}

\begin{equation}
\begin{aligned}
\hat{u}_t &= -a_1 u_x - i a_2 u_{xx} + i a_0,0,0|u|^2 u \\
&\quad + \big(a_3 u_{xxx} - a_1,0,0 u_x|u|^2 - a_{0,0,1} u^2 u_x^*\big) \\
&\quad + i a_{0,0,0,0}|u|^4 u + i \big(a_4 u_{xxxxx} - a_2,0,0 u_{xx}|u|^2 \\
&\quad - a_{1,0,0,1} u_x|u|^2 - a_{0,0,2} u^2 u_{xx}^*\big),
\end{aligned}
\tag{2}
\end{equation}

\begin{equation}
\begin{aligned}
\hat{u}_t &= -a_1 u_x - i a_2 u_{xx} + i a_0,0,0|u|^2 u \\
&\quad + \big(-a_{1,0,0} u_x|u|^2 - a_{0,0,1} u^2 u_x^*\big) \\
&\quad + i a_{0,0,0,0,0}|u|^4 u,
\end{aligned}
\tag{3}
\end{equation}

Equation \textsuperscript{(2)} was derived by Slunyaev \textsuperscript{2} (2005). It takes into account nonlinear and nonlinear-dispersive terms in the next order of smallness as compared to Eq. \textsuperscript{(1)}. Note that here we use different notation for the variables and coefficients as compared to the original notation used by Slunyaev in Ref. \textsuperscript{2}. In deriving his more general HONLSE model, Slunyaev also introduced a correction to the coefficients \(a_{1,0,0}\) and \(a_{0,0,1}\) derived earlier in Ref. \textsuperscript{1}:

\begin{equation}
\begin{aligned}
a_{1,0,0} &= a_{1,0,0} + \Delta, \\
a_{0,0,1} &= a_{0,0,1} - \Delta,
\end{aligned}
\tag{4}
\end{equation}

where the correction \(\Delta\) is expressed as follows \textsuperscript{4}

\begin{equation}
\begin{aligned}
\Delta &= - \frac{1}{32 \sigma \nu} \left( (\sigma^2 - 1)^2 (3 \sigma^2 + 1) k^3 h^3 \\
&\quad - \sigma (\sigma^2 - 1)^2 (5 \sigma^4 - 18 \sigma^2 - 3) k^2 h^2 \\
&\quad + 2 \sigma (\sigma^2 - 1)^2 (\sigma^2 - 9) k h + 3 \sigma^3 (\sigma^2 - 1) (\sigma^2 - 5) \right),
\end{aligned}
\tag{5}
\end{equation}

Here \(\sigma = \tanh(kh)\) and

\begin{equation}
\nu = (\sigma^2 - 1)^2 k^2 h^2 - 2 \sigma (\sigma^2 + 1) k h + \sigma^2.
\tag{6}
\end{equation}

Equation \textsuperscript{(4)} was derived by Slunyaev \textsuperscript{2} (2005). It takes into account nonlinear and nonlinear-dispersive terms in the next order of smallness as compared to Eq. \textsuperscript{(1)}. Note that here we use different notation for the variables and coefficients as compared to the original notation used by Slunyaev in Ref. \textsuperscript{2}. In deriving his more general HONLSE model, Slunyaev also introduced a correction to the coefficients \(a_{1,0,0}\) and \(a_{0,0,1}\) derived earlier in Ref. \textsuperscript{1}:

\begin{equation}
\begin{aligned}
a_{1,0,0} &= a_{1,0,0} + \Delta, \\
a_{0,0,1} &= a_{0,0,1} - \Delta,
\end{aligned}
\tag{4}
\end{equation}

where the correction \(\Delta\) is expressed as follows \textsuperscript{4}

\begin{equation}
\begin{aligned}
\Delta &= - \frac{1}{32 \sigma \nu} \left( (\sigma^2 - 1)^2 (3 \sigma^2 + 1) k^3 h^3 \\
&\quad - \sigma (\sigma^2 - 1)^2 (5 \sigma^4 - 18 \sigma^2 - 3) k^2 h^2 \\
&\quad + 2 \sigma (\sigma^2 - 1)^2 (\sigma^2 - 9) k h + 3 \sigma^3 (\sigma^2 - 1) (\sigma^2 - 5) \right),
\end{aligned}
\tag{5}
\end{equation}

Here \(\sigma = \tanh(kh)\) and

\begin{equation}
\nu = (\sigma^2 - 1)^2 k^2 h^2 - 2 \sigma (\sigma^2 + 1) k h + \sigma^2.
\tag{6}
\end{equation}

It was demonstrated in our earlier work \textsuperscript{4} that the correction \(\Delta\) is small as compared to the values of coefficients \(a_{1,0,0}\) and \(a_{0,0,1}\). It is well seen from Fig. \textsuperscript{1} reproduced here for clarity. Actually, this was the reason why this correction was originally ignored by Sedletsky in Ref. \textsuperscript{1}. Since this correction is small, we deliberately ignore it in Ref. \textsuperscript{5} as well.

Slunyaev’s model \textsuperscript{(2)} has one major drawback: the coefficients \(a_{2,0,0}, \bar{a}_{0,0,2}\) and \(\bar{a}_{1,0,1}\) are asymptotically divergent at \(kh \to \infty\). To avoid this problem in the vicinity of \(kh = 1.363\), where the third-order nonlinear coefficient \(a_{0,0,0}\) vanishes (modulation instability threshold), Slunyaev renormalized high-order terms to get a simplified HONLSE model \textsuperscript{6}, which we will address to as truncated Slunyaev’s equation. It is important to note that this truncated model was formulated by Slunyaev only...
The purpose of this short note is to demonstrate that the new bright soliton solution \( \Psi \) presented in Ref. \( \cite{31} \) is not significantly modified by Slunyaev’s correction \( \Delta \) and that it also exists in the framework of a more general HONLSE model with the fifth-order nonlinearity taken into account.

The effect of Slunyaev’s correction. First, it is important to note that the correction \( \Delta \) does not modify the parameters \( S \) and \( K \) in formula \( \left( \Psi \right)^{2a} \) and the range of solution existence \( \left( \Psi \right) \), since it is canceled out in the sum \( a_{1,0,0} + a_{0,0,1} \), inasmuch as \( \Delta \) makes equal corrections with opposite signs to the both coefficients, as seen from Eq. \( \left( 5 \right) \). Therefore, the correction \( \Delta \) has effect only on the value of parameter \( \kappa \), where it appears in the denominator of expression \( \left( \Psi \right)^{2a} \) through the coefficient \( a_{0,0,1} \).

For \( kh = 1.2 \), the corrected value of parameter \( \kappa \) is 0.216, which makes only a 6% difference to our original estimate \( \kappa \approx 0.230 \). Thus, neglecting Slunyaev’s correction \( \Delta \) is quite legitimate within the limits of accuracy of HONLSE model \( \left( 1 \right) \), but it should be taken into account in the next order of smallness introduced in a more general HONLSE model \( \left( 2 \right) \).

The effect of the fifth-order nonlinearity. The bright soliton solution in form \( \left( 4 \right) \) and \( \left( 5 \right) \) does not exist in the framework of truncated Slunyaev’s equation \( \left( 3 \right) \) because the condition \( a_{3} \neq 0 \) does not hold true in that case. Therefore, we need to consider a more general equation with the \( a_{3} \) term preserved:

\[
\begin{align*}
\tau & = -a_{1}u_{\chi} - a_{2}u_{\chi \chi} + i a_{0,0,0}|u|^{2}u \\
& + \left( a_{3}u_{\chi \chi \chi} - a_{1,0,0}u_{\chi}|u|^{2} - a_{0,0,1}u_{\chi}^{2}u_{\chi}^{*} \right) \\
& + i a_{0,0,0,0}|u|^{4}u . 
\end{align*}
\]

In terms of dimensionless variables used in Ref. \( \cite{31} \), the fifth-order coefficient is expressed as

\[
a_{0,0,0,0} = -\frac{\alpha_{31}}{\omega k^{4} c} .
\]

The expression for the coefficient \( \alpha_{31} \) is given in Slunyaev’s work \( \cite{2} \), and the expression for the dimensionless phase speed \( c \) is given in Ref. \( \cite{7} \). Here, the sign minus at the coefficient \( \alpha_{31} \) is due to the fact that Slunyaev used the conjugate carrier wave basis as compared to our notation. For \( kh = 1.2 \), we have \( \alpha_{31} = 0.30 \).

To analyze the effect of the fifth-order nonlinear term on the evolution of bright HONLSE soliton \( \Psi \), we used this wave form as the initial condition in Eq. \( \left( 10 \right) \) at \( kh = 1.2 \). Figure \( \Psi \) demonstrates that in this case the fifth-order term makes only a small perturbation to the third-order dispersive term and the third-order nonlinear term. It can also be seen that the inputs of the third-order dispersive and nonlinear terms have the same order of magnitude. This fact proves once again the the third-order dispersion cannot be neglected in the correct description of wave evolution at \( kh = 1.2 \). Figure \( \Psi \)
the framework of a more general HONLSE model \( \text{[1]} \), the modulation instability threshold slightly shifts to higher \( kh \), depending on the amplitude \( u_0 \) of a homogeneous solution. In particular, at \( u_0 = 0.17 \), the modulation instability threshold shifts to \( kh \approx 1.365 \) \( \text{[2]} \). In the framework of HONLSE model \( \text{[10]} \) with the fifth-order nonlinear term, the modulation instability criterion should be modified appropriately. A homogenous solution to Eq. \( \text{[10]} \) has the following form:

\[
 u(x, \tau) = u_0 e^{i(a_0, o, o,u_0) + a_0, o, o, o, o|u_0|^4}) \tau. \tag{12}
\]

Its instability criterion can be determined by introducing a small perturbation to the amplitude \( u_0 \):

\[
 u(x, \tau) = (u_0 + \epsilon(x, \tau)) e^{i(a_0, o, o, n, o|u_0|^2 + a_0, o, o, o, o|u_0|^4}) \tau,
\epsilon(x, \tau) = c_0 e^{i\chi x - i\Omega\tau} + c_0 e^{-i\chi x + i\Omega \tau}. \tag{13}
\]

Here, we assume the perturbation frequency \( \Omega \) to be complex-valued and the perturbation wave number \( \kappa \) to be real. Substituting this ansatz in Eq. \( \text{[10]} \) leads to the following dispersion relation between \( \Omega \) and \( \kappa \):

\[
 \Omega = (a_1 + a_1, o, o|u_0|^2) + a_3 \kappa^3 \pm \kappa \sqrt{R}, \tag{14}
\]

\[
 R = 2a_2 (a_0, o, o, o, o|u_0|^2)|u_0|^2 + a_0^2, o, l|u_0|^4 + a^2 \kappa^2.
\]

A homogeneous solution is modulalionly unstable when the perturbation exponentially grows with time. This happens when \( \text{Im} \Omega > 0 \), which effectively requires the radicand \( R \) in Eq. \( \text{[14]} \) to be negative. This condition is satisfied when

\[
 a_0, o, o < -(a_0, o, o, o, o, o, o)|u_0|^2, \tag{15}
\]

where we took into account that \( a_2 = \frac{1}{2} \). Criterion \( \text{[15]} \) was derived earlier in Ref. \( \text{[2]} \). The fifth-order nonlinearity makes a significant input in the modulation instability criterion, so that the instability threshold shifts in the direction of smaller \( kh \), inasmuch as the expression \( a_0^2, o, o + 2a_0, o, o, o, o \) is negative at \( kh \gtrsim 1.2 \). In particular, at \( u_0 = 0.17 \), the modulation instability threshold shifts to \( kh \approx 1.353 \). This point, however, lies far above the region of existence of the bright soliton solution discussed above, and all our predictions regarding the evolution of this new solution remain valid.

Conclusions. In this short note we proved that

1. Slunyaev’s correction to Sedletsky’s coefficients is small and can readily be ignored within the limits of accuracy of HONLSE model \( \text{[1]} \).
2. At \( kh = 1.2 \), the third-order dispersion and the third-order nonlinearity make the comparative contributions to the evolution equation and, therefore, the third-order dispersion cannot be ignored.
3. At $kh = 1.2$, the effect of the fifth-order nonlinearity is small as compared to the third-order dispersion and the third-order nonlinearity. It manifests in the slight oscillations of the soliton amplitude around the undisturbed initial level (the so-called quasi-soliton behavior).

4. The fifth-order nonlinearity makes a significant correction to the modulation instability threshold, so that it shifts in the direction of smaller $kh$. However, this effect is important only in the small vicinity of $kh \approx 1.363$ and is insignificant at smaller depths, in particular, at $kh = 1.2$.

5. The criterion of existence of HONLSE bright solitons and the criterion of modulation instability are two different and independent criteria, and this conclusion holds true in the case of the fifth-order nonlinearity.

* [gandzha@iop.kiev.ua, sedlets@iop.kiev.ua]

[1] Yu. V. Sedletsky, JETP 97, 180 (2003).
[2] A. V. Slunyaev, JETP 101, 926 (2005).
[3] R. H. J. Grimshaw and S. Y. Annenkov, Stud. Appl. Math. 126, 409 (2011).
[4] I. S. Gandzha, Yu. V. Sedletsky, and D. S. Dutykh, Ukr. J. Phys. 59, 1201 (2014) [arXiv:1501.05933].
[5] I. S. Gandzha and Yu. V. Sedletsky, [arXiv:1501.06902].