ORBITAL STABILITY OF PEAKONS FOR A MODIFIED CAMASSA-HOLM EQUATION WITH HIGHER-ORDER NONLINEARITY

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Abstract. In this paper, we consider the orbital stability of peakons for a modified Camassa-Holm equation with higher-order nonlinearity, which admits the single peakons and multi-peakons. We firstly show the existence of the single peakon and prove two useful conservation laws. Then by constructing certain Lyapunov functionals, we give the proof of stability result of peakons in the energy space $H^1(\mathbb{R})$-norm.

1. Introduction. Recently, the great interest in the well-known Camassa-Holm (CH) equation [2] has inspired the search for various CH-type equations with cubic or higher-order nonlinearity. One of the most concerned is the following modified CH equation:

$$y_t + \left((u^2 - u_x^2)y\right)_x = 0, \quad y = u - u_{xx},$$

which was derived by Fuchssteiner [9] and Olver and Rosenau [16] using the method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified KdV equation. Subsequently, Eq. (1) was shown to arise from the asymptotic theory of surface water waves by Fokas [7]. Moreover, Qiao [17] deduced it from the two-dimensional Euler equations, where $u(t,x)$ denotes the fluid velocity and hence $y(t,x)$ represents its potential density. Hence the modified CH equation is also called FORQ equation in some literature.

The modified CH equation is completely integrable [16]. It has a bi-Hamiltonian structure and also admits a Lax pair [19], and hence can be solved by the inverse scattering transform method. Compared with the classical CH equation, Eq. (1) admits not only peaked solitary waves (peakons), but also possesses cusp solitons (cuspons) and weak kink solutions ($u, u_x, u_t$ are continuous, but $u_{xx}$ has a jump at its peak point) [19, 24]. It has also significant differences from the CH equation about the dynamics of the two-peakons and peakon-kink solutions [18, 24]. The so called white solitons and dark ones of Eq. (1) were presented in [22] and [12], respectively. In [1], the authors applied the geometric and analytic approaches to give a geometric interpretation of the variable $y(t,x)$ and construct an infinite-dimensional Lie algebra of symmetries. The Cauchy problem of Eq. (1) in Besov spaces and the blow-up scenario were studied in [8]. The nonuniform dependence

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on the initial data for Eq. (1) was established in [11]. In [10], the authors considered the formulation of the singularities of solutions and showed that some solutions with certain initial data would blow up in finite time. Then the blow-up phenomena were systematically investigated in [3, 15]. There exist single peakon of the form [10]
\[ u(t, x) = \varphi_c(t, x) = \sqrt{\frac{3c}{2}} e^{-|x-ct|}, \quad c > 0, \]
and the multi-peakons
\[ u(t, x) = \sum_{i=1}^{N} p_i(t) e^{-|x-q_i(t)|}, \]
where \( p_i(t) \) and \( q_i(t) \) satisfy the following system:
\[
\begin{align*}
  p_i' &= 0, \\
  q_i' &= \frac{2}{3} p_i^2 + 2 \sum_{j=1}^{N} p_i p_j e^{-|q_j-q_i|} + 4 \sum_{1 \leq k < i, i < j \leq N} p_k p_j e^{-|q_k-q_j|}.
\end{align*}
\]
It is noticed that, the peakons replicate a feature that is characteristic for the Stokes wave of greatest height which is an exact solution of the governing equations for water waves [4, 5, 23]. On the other hand, the multi-peakon solutions of Eq. (1) in comparison with the CH equation have only constant amplitudes. Later, the orbital stability of single peakon and the train of peakons were proven in [20] and [14], respectively.

In this paper, we consider the following nonlinear equation with higher-order nonlinearity:
\[
y_t + \left((u^2 - u_x^2)^3 y\right)_x = 0, \quad y = u - u_{xx}.
\]
(2)
Indeed, Eq. (2) is the case \( n = 3 \) of the following generalized modified CH equation
\[
y_t + \left((u^2 - u_x^2)^n u\right)_x = 0, \quad 1 \leq n \in \mathbb{N}^+,
\]
which is proposed by Recio and Anco [21] in the study of the family of nonlinear dispersive wave equations involving two arbitrary functions. It is known that one of the main remarkable features of the CH equation (with quadratic nonlinearity) and the modified CH equation (1) (with cubic nonlinearity) is the existence of stable peakons. Interestingly, Eq. (2) (with higher-order nonlinearity) studied here can also represent single peakon and multi-peakons [21]. As far as we know, no attempt has been made here to prove the stability of peakons for Eq. (2) up to now. Thus it is of great interest to identify whether the single peakon of Eq. (2) has the similar stable result as the CH and modified CH equations.

To pursue the above goal, we briefly recall the impressive proof of stability of peakons for the CH equation provided by Constantin and Strauss [6]. The CH equation has two useful invariants
\[
\hat{E}(u) \triangleq \int_{\mathbb{R}} (u^2 + u_x^2) dx \quad \text{and} \quad \hat{F}(u) \triangleq \int_{\mathbb{R}} (u^3 + uu_x^2) dx.
\]
The idea is mainly based on expanding the conserved energy \( \hat{E}(u) \) around the peakon \( \varphi_c \). Then, one can find that the minimum of the \( H^1 \)-norm between the solution \( u(t, x) \) and the peakon \( \varphi_c(-\xi(t)) \) is precisely controlled by an error term \( |M - \max_{x \in \mathbb{R}} \varphi_c| (M \triangleq \max_{x \in \mathbb{R}} u(x)) \). As we know, a crucial ingredient to estimate this error term is the following polynomial inequality between \( \hat{E}(u) \) and \( \hat{F}(u) \)
\[
\hat{F}(u) \leq M \hat{E}(u) - \frac{2}{3} M^3.
\]
The above inequality can be obtained from the observation of the following two integral formulas,
\[
\int_{\mathbb{R}} g^2(x) dx = \tilde{E}(u) - 2M^2 \quad \text{and} \quad \int_{\mathbb{R}} \tilde{h}(x)g^2(x) dx = \tilde{F}(u) - \frac{4}{3}M^3,
\]
where \( \tilde{g} \) is an auxiliary function (see Lemma 2 in [6]) and \( \tilde{h} \triangleq u \). After analyzing the root structure of the above polynomial inequality, one can easily get the desired result.

In order to apply the above approach to Eq. (2), we need to overcome several difficulties. Since the conservation law \( E(u) \) of Eq. (2) is the same as \( \tilde{E}(u) \), we also expect orbital stability of peakons for Eq. (2) in the sense of the energy space \( H^1 \)-norm. While the other conservation law \( F(u) \) is much more complicated than the cases of the CH and modified CH equations. Hence we here require more subtle computation on the introduction of the higher-order conservation \( F(u) \) and new auxiliary function \( h(x) \) (see Lemma 3.3), which is quite different from the case of simply taking \( \tilde{h}(x) = u(x) \) for the CH equation. On the other hand, to get the key inequality related to the conservation laws \( E(u) \) and \( F(u) \) as the CH and modified CH equations, we have to show \( h \leq \frac{64}{35} (\max_{x \in \mathbb{R}} u)^6 \). However, since \( h \) is of the form \( u^6 + \frac{36}{35}u^5u_x - \frac{11}{35}u^4u_x^2 \pm \frac{36}{35}u^3u_x^3 + \frac{13}{35}u^2u_x^4 \pm \frac{2}{7}uu_x^5 \mp \frac{1}{7}u_x^6 \) due to its higher-order nonlinear structure, we here treat it firstly by applying the Cauchy-Schwarz inequality to the term \( \mp u^5u_x \pm u^3u_x^3 \), and then the term \( \mp u^5u_x - u^4u_x^2 \).

Lastly we reach to an obviously correct inequality (34) by the delicate combination of the terms of \( h \). Moreover, it is noted to point out that the reason why we do not study the case \( n = 2 \) of the generalized modified CH equation, even though the existence of peakons and the conservation laws \( E(u), F(u) \) hold for this case. As mentioned above, the current approach requires principally one ingredient: the precise polynomial inequality between the conservation laws. However, when \( n = 2 \) in the generalized modified CH equation, the auxiliary function \( h(x) \) is the form of \( u^4 \mp \frac{5}{7}u^3u_x - \frac{2}{7}u^2u_x^2 \pm \frac{2}{7}uu_x^3 \mp \frac{1}{7}u_x^4 \), which cannot generally be bounded by \( \frac{2}{7}u^4 \) due to the positive term \( \frac{2}{7}u_x^4 \). Thus the choice of the generalized modified CH equation when \( n \) is a positive odd integer seems to be the best adapted to our approach used here.

The remainder of this paper is organized as follows. In Section 2, we mainly demonstrate that the existence of peakons which can be understood as weak solutions for Eq. (2), and then prove two conserved quantities which are crucial in the proof of the stability theorem. In Section 3, the orbital stability of peakons for Eq. (2) is proved based on several useful lemmas. In Section 4, we give the proof of last inequality of (36) in Lemma 3.3.

Notation. In the following, the symbols \( \lesssim \) is used to express the corresponding inequality that includes a universal constant. For example, \( f \lesssim g \) denotes that there exists a constant \( C > 0 \) such that \( f \leq Cg \).

2. Preliminaries. As shown in [25], one can easily obtain the following local well-posedness result by solving a transport equation satisfied by the momentum \( y \) instead of \( u \).

Lemma 2.1. Let \( y_0(x) = (1 - \partial_x^2)u(0, x) \in H^s(\mathbb{R}) \) with \( s > \frac{1}{2} \). Then there exists a time \( T > 0 \) such that the Cauchy problem (2) has a unique strong solution \( y(t, x) = (1 - \partial_x^2)u(t, x) \in C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{s-1}(\mathbb{R})) \) and the map
Proof. For any $y_0$ is continuous from a neighborhood of $y_0$ in $H^s(\mathbb{R})$ into $C([0, T); H^s(\mathbb{R})) \cap C^1([0, T); H^{-1}(\mathbb{R}))$.

Next we restate Eq. (2) in a more convenient form. Note that Eq. (2) is equivalent to the following one:

$$u_t - u_{txx} + 7u^6u_x - 15u^4u_x^3 - 12u^5u_x u_{xx} - u^6u_{xxx} + 9u^2u_x^5 + 24u^3u_x^2u_{xx} + 6u_x^4u_{xx} + 3u_x^2u_{xxx} - u_x^7 - 12u^5u_{xx} - 12u^2u_x^2u_{xx} - 3u_x^4u_{xxx} + 6u_x^5u_{xx} + u_x^6u_{xxx} = 0. \quad (3)$$

Applying the operator $(1 - \partial_x^2)^{-1}$ to the both sides of (3), we can obtain

$$u_t + [u^6 - (u^4u_x^2 - \frac{3}{5}u^2u_x^4 + \frac{1}{7}u_x^2)]u_x + (1 - \partial_x^2)^{-1}\partial_x(\frac{6}{7}u^7 + 3u^5u_x^2 - u_x^4u_x) + \frac{1}{5}u_xu_x^6) + (1 - \partial_x^2)^{-1}(\frac{1}{7}u_x^2) = 0. \quad (4)$$

Recall that if $p(x) \triangleq \frac{1}{2}e^{-|x|}, x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1}f = p * f$ for all $f \in L^2$. In order to understand the meaning of a peakon solution to Eq. (2), we can apply Eq. (4) to define the notion of weak solutions for Eq. (2).

**Definition 2.2.** Let $u_0 \in W^{1,7}(\mathbb{R})$ be given. If $u(t,x) \in L^\infty([0, T); W^{1,7}(\mathbb{R}))$ and satisfies

$$\int_0^T \int_\mathbb{R} [u\phi_t + \frac{1}{7}u^7\phi_x + (u^4u_x^3 - \frac{3}{5}u^2u_x^4 + \frac{1}{7}u_x^2)\phi + p * (\frac{6}{7}u^7 + 3u^5u_x^2 - u_x^4u_x) + \frac{1}{5}u_xu_x^6)\phi_x - p * (u^4u_x^3 - \frac{3}{5}u^2u_x^4 + \frac{1}{7}u_x^2)\phi dxdt + \int_\mathbb{R} u_0(x)\phi(0,x)dx = 0,$$

for all functions $\phi \in C^\infty_0([0, T) \times \mathbb{R})$, then $u(t,x)$ is called a weak solution to Eq. (2). If $u$ is a weak solution on $[0, T)$ for every $T > 0$, then it is called a global weak solution.

In the following theorem, we discuss the existence result of the single peakons for Eq. (2).

**Theorem 2.3.** For any $a \neq 0$, the peaked functions of the form

$$\varphi_c(t,x) = ae^{-|x-ct|}, \quad \text{where} \quad c = \frac{16}{35}a^6, \quad (5)$$

is a global weak solution to Eq. (2) in the sense of Definition 2.2.

**Proof.** For any test function $\phi(\cdot) \in C^\infty_c(\mathbb{R})$, integrating by parts, we infer

$$\int_\mathbb{R} e^{-|x|}\phi'(x)dx = \int_\mathbb{R} e^x\phi(x)dx + \int_\mathbb{R} e^{-x}\phi(x)dx = e^x\phi(x)|^{0}_{\infty} - \int_\mathbb{R} e^x\phi(x)dx + e^{-x}\phi(x)|^{\infty}_{0} + \int_\mathbb{R} e^{-x}\phi(x)dx = -\int_\mathbb{R} e^x\phi(x)dx + \int_\mathbb{R} e^{-x}\phi(x)dx = \int_\mathbb{R} \text{sign}(x)e^{-|x|}\phi(x)dx.$$

Thus, for all $t \geq 0$, we have

$$\partial_x \varphi_c(t,x) = -\text{sign}(x-ct)\varphi_c(t,x), \quad (6)$$

in the sense of distribution $\mathcal{S}'(\mathbb{R})$. Letting $\varphi_{0,c}(x) \triangleq \varphi_c(0,x)$, then we get

$$\lim_{t \to 0^+} \|\varphi_c(t, \cdot) - \varphi_{0,c}(x)\|_{W^{1,\infty}} = 0. \quad (7)$$
The same computation as in (6), for all $t \geq 0$, yields,

$$\partial_t \varphi_c(t, x) = c \text{ sign}(x - ct) \varphi_c(t, x) \in L^\infty.$$  

(8)

Combining (6)-(8) and using integration by parts, for any test function $\phi(t, x) \in C_c^\infty([0, \infty) \times \mathbb{R})$, we obtain

\[
\int_0^{+\infty} \int_{\mathbb{R}} \left[ \varphi_c \partial_t \phi + \frac{1}{7} \varphi_c^7 \partial_x \phi + (\varphi_c^4 \partial_x \varphi_c)^3 - \frac{3}{5} \varphi_c^2 (\partial_x \varphi_c)^5 + \frac{1}{7} (\partial_x \varphi_c)^7 \right] dx dt \\
+ \int_{\mathbb{R}} \varphi_{0, c}(x) \phi(0, x) dx \\
= - \int_0^{+\infty} \int_{\mathbb{R}} \left[ \partial_t \varphi_c + \varphi_c^6 \partial_x \varphi_c - (\varphi_c^4 \partial_x \varphi_c)^3 - \frac{3}{5} \varphi_c^2 (\partial_x \varphi_c)^5 + \frac{1}{7} (\partial_x \varphi_c)^7 \right] \phi dx dt
\]

(9)

Using the definition of $\varphi_c$ and $c = \frac{16}{35} a^6$, we have, for $x > ct$,

$$\text{sign}(x - ct) \varphi_c(c - \frac{16}{35} \varphi_c^6) = \frac{16}{35} a^7 \left(e^{ct-x} - e^{7(ct-x)}\right),$$  

(10)

and for $x \leq ct$,

$$\text{sign}(x - ct) \varphi_c(c - \frac{16}{35} \varphi_c^6) = -\frac{16}{35} a^7 \left(e^{-ct-x} - e^{7(x-ct)}\right).$$  

(11)

On the other hand, by Definition 2.2, we derive

\[
\int_0^{+\infty} \int_{\mathbb{R}} \left[ (1 - \partial_x^2)^{-1} \left( \frac{6}{7} \varphi_c^7 + 3 \varphi_c^5 (\partial_x \varphi_c)^2 - \varphi_c^3 (\partial_x \varphi_c)^4 + \frac{1}{5} \varphi_c^2 (\partial_x \varphi_c)^6 \right) \partial_x \phi \\
- (1 - \partial_x^2)^{-1} \left( \varphi_c^4 (\partial_x \varphi_c)^3 - \frac{3}{5} \varphi_c^2 (\partial_x \varphi_c)^5 + \frac{1}{7} (\partial_x \varphi_c)^7 \right) \phi \right] dx dt
\]

(12)

We calculate from (6) that,

$$6 \varphi_c^6 \partial_x \varphi_c + \varphi_c^4 (\partial_x \varphi_c)^3 - \frac{3}{5} \varphi_c^2 (\partial_x \varphi_c)^5 + \frac{1}{7} (\partial_x \varphi_c)^7$$

$$= 6 \varphi_c^6 \left( - \text{sign}(x - ct) \varphi_c \right) + \varphi_c^4 \left( - \text{sign}(x - ct) \varphi_c \right)^3 - \frac{3}{5} \varphi_c^2 \left( - \text{sign}(x - ct) \varphi_c \right)^5$$

$$+ \frac{1}{7} \left( - \text{sign}(x - ct) \varphi_c \right)^7$$

$$= \frac{229}{35} \left( - \text{sign}(x - ct) \varphi_c \right) = \frac{229}{245} \partial_x (\varphi_c^7),$$

which together with (12) leads to

\[
\int_0^{+\infty} \int_{\mathbb{R}} \left[ (1 - \partial_x^2)^{-1} \left( \frac{6}{7} \varphi_c^7 + 3 \varphi_c^5 (\partial_x \varphi_c)^2 - \varphi_c^3 (\partial_x \varphi_c)^4 + \frac{1}{5} \varphi_c^2 (\partial_x \varphi_c)^6 \right) \partial_x \phi \\
+ \frac{6}{7} \varphi_c^7 \partial_x \varphi_c + \varphi_c^4 (\partial_x \varphi_c)^3 - \frac{3}{5} \varphi_c^2 (\partial_x \varphi_c)^5 + \frac{1}{7} (\partial_x \varphi_c)^7 \right] dx dt.
\]
\[-(1 - \partial_x^2)^{-1}(\varphi_c^4(\partial_x \varphi_c)^3 - \frac{3}{5}\varphi_c^2(\partial_x \varphi_c)^5 + \frac{1}{7}(\partial_x \varphi_c)^7)\phi] dx dt = - \int_0^{+\infty} \int_{\mathbb{R}} \left[ \phi \cdot \partial_x p \ast (3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7) \right] dx dt. \] (13)

Inserting the above equalities \(\partial_x p(x) = \frac{1}{2} \text{sign}(x) e^{-|x|} \) for \(x \in \mathbb{R}\), we deduce

\[
\partial_x p \ast (3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7)(t, x) = - \frac{1}{2} \int_{-\infty}^{+\infty} \text{sign}(x - y) e^{-|x-y|} \left( \frac{229}{245} + 3\text{sign}^2(y - ct) - \text{sign}^4(y - ct) \right) + \frac{1}{5}\text{sign}^6(y - ct) a^7 e^{-7|y-ct|} dy. \] (14)

For \(x > ct\), we can split the right hand side of (14) into the following three parts,

\[
\partial_x p \ast (3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7)(t, x) = - \frac{384}{245} a^7 \left( \int_{-\infty}^{ct} + \int_{ct}^{x} + \int_{x}^{+\infty} \right) \text{sign}(x - y) e^{-|x-y|} e^{-7|y-ct|} dy \\
\Delta I_1 + I_2 + I_3. \] (15)

A direct calculation for each one of the terms \(I_i\), \(1 \leq i \leq 3\), yields

\[
I_1 = - \frac{384}{245} a^7 \int_{-\infty}^{ct} e^{-(x-y)} e^{7(y-ct)} dy = - \frac{384}{245} a^7 e^{-(x+7ct)} \int_{-\infty}^{ct} e^{8y} dy = - \frac{48}{245} a^7 e^{ct-x},
\]

\[
I_2 = - \frac{384}{245} a^7 \int_{ct}^{x} e^{-(x-y)} e^{-7(y-ct)} dy = - \frac{384}{245} a^7 e^{-(x-7ct)} \int_{ct}^{x} e^{-6y} dy = \frac{64}{245} a^7 e^{-(x-7ct)}(e^{-6x} - e^{-6ct}) = - \frac{64}{245} a^7 (e^{ct-x} - e^{7(ct-x)}),
\]

and

\[
I_3 = \frac{384}{245} a^7 \int_{x}^{+\infty} e^{x-y} e^{-7(y-ct)} dy = \frac{384}{245} a^7 e^{(x+7ct)} \int_{x}^{+\infty} e^{-8y} dy = \frac{48}{245} a^7 e^{7(ct-x)}.
\]

Inserting the above equalities \(I_1-I_3\) into (15), we find that for \(x > ct\),

\[
\partial_x p \ast (3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7) = - \frac{16}{35} a^7 (e^{ct-x} - e^{7(ct-x)}). \] (16)
While for $x \leq ct$, we also split the right hand side of (14) into three parts,
\[
\partial_x p \left( 3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7 \right)(t, x) \\
= -\frac{384}{245}a^7 \left( \int_{-\infty}^{x} + \int_{x}^{ct} + \int_{ct}^{+\infty} \right) \text{sign}(x-y)e^{-|x-y|e^{-7(y-ct)}}dy \\
\triangleq II_1 + II_2 + II_3.
\] (17)

We now directly compute each one of the terms $II_i, 1 \leq i \leq 3$, as follows,
\[
II_1 = -\frac{384}{245}a^7 \int_{-\infty}^{x} e^{-(x-y)}e^{7(y-ct)}dy \\
= -\frac{384}{245}a^7 e^{-(x+7ct)} \int_{-\infty}^{x} e^{8y}dy = -\frac{48}{245}a^7 e^{(ct-x)},
\]
\[
II_2 = \frac{384}{245}a^7 \int_{x}^{ct} e^{x-y}e^{7(y-ct)}dy = \frac{384}{245}a^7 e^{x-7ct} \int_{x}^{ct} e^{6y}dy \\
= \frac{64}{245}a^7 e^{x-7ct}(e^{6ct} - e^{6x}) = \frac{64}{245}a^7(e^{x-ct} - e^{7(x-ct)}),
\]
and
\[
II_3 = \frac{384}{245}a^7 \int_{ct}^{+\infty} e^{x-y}e^{-7(y-ct)}dy \\
= \frac{384}{245}a^7 e^{(x+7ct)} \int_{ct}^{+\infty} e^{-8y}dy = \frac{48}{245}a^7 e^{x-ct}.
\]

Inserting the above equalities $II_1-III$ into (17), we find that for $x \leq ct$,
\[
\partial_x p \left( 3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7 \right) \\
= \frac{16}{35}a^7 (e^{x-ct} - e^{7(x-ct)}).
\] (18)

Hence, by (10)-(11), (16) and (18), we deduce that for all $(t,x) \in (0, +\infty) \times \mathbb{R}$,
\[
\text{sign}(x-ct) \left( \varphi_c(c - \frac{16}{35}\varphi_c^3) \right)(t, x) + \partial_x p \left( 3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7 \right) \\
+ \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 + \frac{229}{245}\varphi_c^7 \right)(t, x) = 0.
\] (19)

Thanks to (9), (13) and (19), we conclude that
\[
\int_{0}^{+\infty} \int_{\mathbb{R}} \left[ \varphi_c \partial_t \phi + \frac{1}{7} \varphi_c^7 \partial_x \phi + (\varphi_c^5(\partial_x \varphi_c)^3 - \frac{3}{5}\varphi_c^2(\partial_x \varphi_c)^5 + \frac{1}{7}(\partial_x \varphi_c)^7) \phi \\
+(1-\partial_x^2)^{-1} \left( \frac{6}{7} \varphi_c^7 + 3\varphi_c^5(\partial_x \varphi_c)^2 - \varphi_c^3(\partial_x \varphi_c)^4 + \frac{1}{5}\varphi_c(\partial_x \varphi_c)^6 \partial_x \phi \right) dxdt \\
-(1-\partial_x^2)^{-1} \left( \varphi_c^4(\partial_x \varphi_c)^3 - \frac{3}{5}\varphi_c^2(\partial_x \varphi_c)^5 + \frac{1}{7}(\partial_x \varphi_c)^7 \right) dxdt \\
+ \int_{\mathbb{R}} \varphi_{0,c}(x) \phi(0,x)dx \right] dxdt = 0,
\]
for every test function $\phi(t,x) \in C^\infty_c([0,\infty) \times \mathbb{R})$. This completes the proof of Theorem 2.3. \qed

In the following lemma, we give two useful conservation laws which are crucial in our development.
Lemma 2.4. If the initial data \( u_0 \in H^s(\mathbb{R}) \) with \( s > \frac{5}{2} \), then the following two functionals

\[
E(u) \triangleq \int_\mathbb{R} (u^2 + u_x^2) \, dx, \quad F(u) \triangleq \int_\mathbb{R} (u^8 + 4u_0^6u_x^2 - 2u_4^4u_x^4 + \frac{4}{5} u^2u_x^6 - \frac{1}{7} u_8^8) \, dx, \tag{20}
\]

are invariants for Eq. (2).

Proof. A direct computation yields

\[
\frac{dE(u)}{dt} = \frac{d}{dt} \int_\mathbb{R} (u^2 + u_x^2) \, dx = 2 \int_\mathbb{R} (uu_t + u_xu_{tx}) \, dx
\]

\[
= 2 \int_\mathbb{R} u(u_t - u_{txx}) \, dx = -2 \int_\mathbb{R} u((u^2 - u_x^2)^3)_x \, dx
\]

\[
= -2 \int_\mathbb{R} u(u^2 - u_x^2)^3 \, dy - 12 \int_\mathbb{R} (u^2 - u_x^2)^2u_xy^2 \, dx
\]

\[
= 12 \int_\mathbb{R} (u^2 - u_x^2)^2u_x y^2 \, dx + 2 \int_\mathbb{R} (u^2 - u_x^2)^3 u_x (u - u_{xx}) \, dx
\]

\[
= \int_\mathbb{R} (u^2 - u_x^2)^3 d(u^2 - u_x^2) = 0.
\]

Set \( v(x,t) = \int_{-\infty}^x u_t(z,t) \, dz \). Consider the equalities

\[
\frac{d}{dt} \int_\mathbb{R} u^8 \, dx = \int_\mathbb{R} 8u^7 u_t \, dx = \int_\mathbb{R} 8u^7 v_x \, dx = -8 \cdot \int_\mathbb{R} 7u^6 u_x v \, dx,
\]

\[
\frac{d}{dt} \int_\mathbb{R} 4u^6u_x^2 \, dx = \int_\mathbb{R} (24u^5u_tu_x^2 + 8u^6u_xu_{tx}) \, dx = \int_\mathbb{R} (24u^5u_x^2 v_x + 8u^6u_x v_{xx}) \, dx
\]

\[
= -24 \int_\mathbb{R} (5u^4u_x^3 + 2u_5^5u_xu_{xx}) \, vdx + 8 \int_\mathbb{R} (30u_4^4u_x^3 + 18u_5^5u_xu_{xx}) \, vdx
\]

\[
+ u^6u_{xxx} \, vdx
\]

\[
= 8 \cdot \int_\mathbb{R} (15u^4u_x^3 + 12u^5u_xu_{xx} + u^6u_{xxx}) \, vdx.
\]

Similarly, one obtains

\[
-2 \frac{d}{dt} \int_\mathbb{R} u^4u_x^4 \, dx = -8 \int_\mathbb{R} (u^3u_x^4 u_t + u^4u_x^3u_{tx}) \, dx = -8 \int_\mathbb{R} (u^3u_x^4 v_x + u^4u_x^3 v_{xx}) \, dx
\]

\[
= -8 \int_\mathbb{R} (-3u^2u_x^5 - 4u^3u_x^3u_{xx}) \, vdx - 8 \int_\mathbb{R} (12u^2u_x^5 + 28u^3u_x^3u_{xx}) \, vdx
\]

\[
+ 6u^4u_x^2u_{xx}^2 + 3u^4u_x^2u_{xxx} \, vdx
\]

\[
= -8 \cdot \int_\mathbb{R} (9u^2u_x^5 + 24u^3u_x^3u_{xx} + 6u^4u_x^2u_{xx} + 3u^4u_x^2u_{xxx}) \, vdx,
\]
\[
\frac{4}{5} \frac{d}{dt} \int_{\mathbb{R}} u^2 u_x^6 dx = \frac{4}{5} \int_{\mathbb{R}} (2u u_x^6 u_t + 6u^2 u_x^5 u_{x2}) dx = \frac{4}{5} \int_{\mathbb{R}} (2u u_x^6 v_x + 6u^2 u_x^5 v_{xx}) dx
\]

\[
= \frac{4}{5} \int_{\mathbb{R}} (-2u^7 - 12u^5 u_x v_x) dx + \frac{4}{5} \int_{\mathbb{R}} (12u^7 + 132u^5 u_x) dx + 120u^2 u_x^3 u_{xx}^2 + 30u^2 u_x^4 u_{xx}^2 dx
\]

\[
= 8 \cdot \int_{\mathbb{R}} (u_x^7 + 12uu_x^5 u_{xx} + 12u^2 u_x^3 u_{xx}^2 + 3u^2 u_x^4 u_{xx}^2) dx,
\]

and

\[
-\frac{1}{7} \frac{d}{dt} \int_{\mathbb{R}} u^8 dx = -\frac{8}{7} \int_{\mathbb{R}} u^3 u_{tx} dx = -\frac{8}{7} \int_{\mathbb{R}} u_x^3 u_{x2} dx
\]

\[
= -8 \int_{\mathbb{R}} (6u^5 u_x^2 + u_x^6 u_{x2}) dx.
\]

Combining the above five equalities, we have

\[
\frac{dF(u)}{dt} = \frac{d}{dt} \int_{\mathbb{R}} (u^8 + 4u^6 u_x^2 - 2u^4 u_x^4 + \frac{4}{5} u^2 u_x^6 - \frac{1}{7} u_x^8) dx
\]

\[
= -8 \int_{\mathbb{R}} (7u^6 u_x - 15u^4 u_x^3 - 12u^2 u_x^5 u_{xx} - 9u^2 u_x^5 u_{xx} + 24u^3 u_x^3 u_{xx} + 6u^4 u_x^2 u_{xxx} + 3u^4 u_x^2 u_{xxx} - u_x^7 - 12uu_x^5 u_{xx} - 12u^2 u_x^3 u_{xx}^2 - 3u^2 u_x^4 u_{xx} + 6u_x^5 u_{xx}^2 + u_x^6 u_{xx}) dx,
\]

which along with (3) yields,

\[
\frac{dF(u)}{dt} = 8 \int_{\mathbb{R}} (u_t - u_{txx}) dx = \int_{\mathbb{R}} 8(vv_x - vv_{xxx}) dx = 0.
\]

This completes the proof of Lemma 2.4. □

The proof of the following lemma is very similar to Lemmas 2.8-2.9 in [13], thus we omit it here.

**Lemma 2.5.** Assume \( u_0(x) \in H^s(\mathbb{R}), s > \frac{5}{2} \). If \( y_0(x) = (1 - \partial_x^2)u_0(x) \) does not change sign, then \( y(t, x) \) will not change sign for all \( t \in [0, T) \). It follows that if \( y_0 \geq 0 \), then the corresponding solution \( u(t, x) \) of Eq. (2) is positive for \( (t, x) \in [0, T) \times \mathbb{R} \). Furthermore, if \( y_0 \geq 0 \), then the corresponding solution \( u(t, x) \) of Eq. (2) satisfies

\[
(1 \pm \partial_x)u(t, x) \geq 0, \quad \text{for} \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]

### 3. Stability of peakons

In this section, we prove the orbital stability of peakons for Eq. (2). Our main theorem reads:

**Theorem 3.1.** The peaked soliton \( \varphi_c(t, x) \) defined in (5) traveling with the speed \( c > 0 \) is orbitally stable in the following sense. If \( u_0(x) \in H^s(\mathbb{R}) \), for some \( s > \frac{5}{2} \), \( y_0(x) = (1 - \partial_x^2)u_0 \neq 0 \) is a nonnegative Radon measure of finite total mass, and

\[
\|u(0, \cdot) - \varphi_c\|_{H^1(\mathbb{R})} < \varepsilon, \quad \text{for} \quad 0 < \varepsilon < (3 - 2\sqrt{2})a,
\]

where \( a = \sqrt{\frac{35\varepsilon}{16}} \) by (5) in Theorem 2.1. Then the corresponding solution \( u(t, x) \) of Eq. (2) with initial data \( u_0 \) satisfies

\[
\sup_{t \in [0, T]} \left\|u(t, \cdot) - \varphi_c(-\xi(t))\right\|_{H^1(\mathbb{R})} \lesssim \sqrt{3\varepsilon + 4a \sqrt{A(c, \|u_0\|_{H^s})}} \varepsilon,
\]

where \( A(c, \|u_0\|_{H^s}) \) is a constant depending on \( c \) and \( \|u_0\|_{H^s} \).
where $T > 0$ is the maximal existence time, $\xi(t) \in \mathbb{R}$ is the maximum point of function $u(t, \cdot)$ and the constant $A(c, \|u_0\|_{H^s}) > 0$ depends only on wave speed $c > 0$ and the norm $\|u_0\|_{H^s}$.

We break the proof of Theorem 3.1 into several lemmas. Note that the assumptions on the initial profile in Theorem 3.1 guarantee the existence of the unique local positive solution Eq. (2) by Lemma 2.1 and Lemma 2.5. It is obvious that $\varphi_c(x) = a\varphi(x) = ae^{-|x|} \in H^1(\mathbb{R})$ has the peak at $x = 0$, and hence $\max_{\mathbb{R}} \varphi_c = \varphi_c(0) = a$. By a simple computation, we have $E(\varphi_c) = 2a^2$ and $F(\varphi_c) = \frac{32}{35}a^8$. Here $a = \sqrt[3]{\frac{32}{35}}$ given by (5).

**Lemma 3.2.** For every $u \in H^1(\mathbb{R})$ and $\xi \in \mathbb{R}$, we have

$$E(u) - E(\varphi_c(\cdot - \xi)) = \|u - \varphi_c(\cdot - \xi)\|_{H^1}^2 + 4a(u(\xi) - a).$$

**Proof.** Using (5) and integration by parts, as [6], we calculate

$$\|u - \varphi_c(\cdot - \xi)\|_{H^1}^2 = \int_{\mathbb{R}} (u^2 + u_\xi^2)dx + \int_{\mathbb{R}} (\varphi_c^2 + (\partial_x\varphi_c)^2)dx - 2\int_{\mathbb{R}} u_x(x)\partial_x\varphi_c(x - \xi)dx - 2\int_{\mathbb{R}} u(x)\varphi_c(x - \xi)dx$$

$$= E(u) + E(\varphi_c) - 2\int_{-\infty}^{\xi} u_x(x)\varphi_c(x - \xi)dx + 2\int_{\xi}^{\infty} u_x(x)\varphi_c(x - \xi)dx - 2\int_{\mathbb{R}} u(x)\varphi_c(x - \xi)dx$$

$$= E(u) + E(\varphi_c) - 4au(\xi)$$

$$= E(u) - E(\varphi_c) - 4a(u(\xi) - a),$$

where we used the fact that $E(\varphi_c) = 2a^2$. This completes the proof of Lemma 3.2. \hfill \Box

Next, we derive a crucial inequality between the two conserved quantities $E(u)$ and $F(u)$.

**Lemma 3.3.** Assume $u_0 \in H^s(\mathbb{R}), s > \frac{5}{2}$, and $y_0 \geq 0$. Let $u(t, x)$ be the solution of Eq. (2) with initial data $u_0$. Denoting $M = \max_{x \in \mathbb{R}} \{u(x)\}$, then

$$F(u) \leq \frac{64}{35} M^6 E(u) - \frac{96}{35} M^8.$$  

**Proof.** Let $M$ be taken at $x = \xi$, and define the same function as in [6]

$$g(x) \triangleq \begin{cases} u(x) - u_\xi(x), & x < \xi, \\ u(x) + u_\xi(x), & x > \xi. \end{cases}$$

Then we have

$$\int_{\mathbb{R}} g^2(x)dx = E(u) - 2M^2.$$  

Next we introduce the other function as

$$h(x) \triangleq \begin{cases} (u^6 - \frac{58}{35}u^5u_x - \frac{11}{11}u^4u_x^2 + \frac{36}{35}u^3u_x^3 + \frac{13}{35}u^2u_x^4 - \frac{2}{7}uu_x^5 - \frac{4}{7}u_x^6)(x), & x < \xi, \\ (u^5 + \frac{11}{35}u^4u_x - \frac{11}{11}u^3u_x^2 - \frac{36}{35}u^2u_x^3 + \frac{13}{35}u^2u_x^4 + \frac{2}{7}uu_x^5 - \frac{4}{7}u_x^6)(x), & x > \xi. \end{cases}$$
Integrating by parts, we calculate
\[
\int_{\mathbb{R}} h(x)g^2(x)dx = \int_{-\infty}^{\xi} (u - u_x)^2(u^6 - \frac{58}{35}u^5u_x - \frac{11}{35}u^4u_x^2 + \frac{36}{35}u^3u_x^3 + \frac{13}{35}u^2u_x^4 \\
- \frac{2}{7}uu_x^5 - \frac{1}{7}u_x^6)dx + \int_{\xi}^{\infty} (u + u_x)^2(u^6 + \frac{58}{35}u^5u_x - \frac{11}{35}u^4u_x^2 - \frac{36}{35}u^3u_x^3 \\
+ \frac{13}{35}u^2u_x^4 + \frac{2}{7}uu_x^5 - \frac{1}{7}u_x^6)dx
\]
\[
= \int_{-\infty}^{\xi} (u^8 + 4u^6u_x^2 - 2u^4u_x^4 + \frac{4}{5}u^2u_x^6 - \frac{1}{7}u_x^8)dx - \frac{128}{35} \int_{-\infty}^{\xi} u^7u_xdx + \int_{\xi}^{\infty} (u^8 + 4u^6u_x^2 - 2u^4u_x^4 + \frac{4}{5}u^2u_x^6 - \frac{1}{7}u_x^8)dx + \frac{128}{35} \int_{\xi}^{\infty} u^7u_xdx
\]
\[
= F(u) - \frac{16}{35} u^8(x) \bigg|_{-\infty}^{\xi} + \frac{16}{35} u^8(x) \bigg|_{\xi}^{\infty} = F(u) - \frac{32}{35} u^8(\xi)
\]
\[
= F(u) - \frac{32}{35} M^8. \tag{25}
\]

By Lemma 2.5, the solution \(u\) of Eq. (2) satisfies
\[
u(t, x) \geq 0, \ (u \pm u_x)(t, x) \geq 0 \text{ and } (u^2 - u_x^2)(t, x) \geq 0 \quad \text{for } \forall (t, x) \in [0, T) \times \mathbb{R}. \tag{26}
\]

We now claim that
\[
h(t, x) \leq \frac{64}{35} u^6(t, x) \quad \text{for } \forall (t, x) \in [0, T) \times \mathbb{R}. \tag{27}
\]

To see this, it suffices to show that
\[
\pm \frac{58}{35} u^5u_x - \frac{11}{35} u^4u_x^2 \pm \frac{36}{35} u^3u_x^3 + \frac{13}{35} u^2u_x^4 \pm \frac{2}{7} uu_x^5 \leq \frac{29}{35} u^6 + \frac{1}{7} u_x^6,
\]
that is,
\[
\pm 58u^5u_x - 11u^4u_x^2 \pm 36u^3u_x^3 + 13u^2u_x^4 \pm 10uu_x^5 \leq 29u^6 + 5u_x^6. \tag{28}
\]

By the Cauchy-Schwarz inequality and (26), we deduce
\[
\pm u^5u_x \pm u^4u_x^2 = (\pm uu_x)u^2(u^2 - u_x^2) \leq \frac{u^2 + u_x^2}{2} \cdot u^2(u^2 - u_x^2) = \frac{1}{2} (u^6 - u^2u_x^4). \tag{29}
\]

Thanks to (29), to prove (28), we only need to show that
\[
\pm 22u^5u_x - 11u^4u_x^2 - 5u^2u_x^4 \pm 10uu_x^5 \leq 11u^6 + 5u_x^6. \tag{30}
\]

In a similar manner,
\[
\pm u^5u_x - u^4u_x^2 = (\pm uu_x)u^3(u \pm u_x) \leq \frac{u^2 + u_x^2}{2} \cdot (u^4 \pm u^3u_x) = \frac{1}{2} (u^6 \pm u^5u_x + u^4u_x^2 \pm u^3u_x^3). \tag{31}
\]

Owing to (31), to prove (30), it suffices to show that
\[
\pm \frac{11}{2} u^5u_x + \frac{11}{2} u^4u_x^2 \pm \frac{11}{2} u^3u_x^3 - 5u^2u_x^4 \pm 10uu_x^5 \leq \frac{11}{2} u^6 + 5u_x^6. \tag{32}
\]
Using (29) again, to prove (32), we only need to show that
\[
\frac{11}{2} u^4 u_x^2 - \frac{31}{4} u^2 u_x^4 + 10 u u_x^5 \leq \frac{11}{4} u^6 + 5 u_x^6.
\] (33)
Since \(\mp u u_x^5 \leq u_x^4 \cdot \frac{u^2 + u_x^2}{2}\), to prove (33), it suffices to show that
\[
2 u^4 u_x^2 - u^2 u_x^4 \leq u^6.
\] (34)
Obviously, (34) holds. This proves our claim (27).

Combining (24)-(25) and (27), we obtain
\[
F(u) - \frac{32}{35} M^8 = \int_R h(x) g^2(x) dx \leq \frac{64}{35} M^6 \int_R g^2(x) dx = \frac{64}{35} M^6 (E(u) - 2 M^2),
\]
which gives the desired inequality (23). This completes the proof of Lemma 3.3. \(\square\)

**Lemma 3.4.** For \(u \in H^s(R), s > \frac{5}{4}, \) if \(\|u - \varphi_c\|_{H^1(R)} < \varepsilon, \) with \(0 < \varepsilon < (3 - 2\sqrt{2})a, \) then
\[
|E(u) - E(\varphi_c)| \leq 3\varepsilon \quad \text{and} \quad |F(u) - F(\varphi_c)| \lesssim A(c, \|u\|_{H^s}) \cdot \varepsilon,
\]
where the constant \(A(c, \|u\|_{H^s}) > 0\) depends only on wave speed \(c > 0\) and the norm \(\|u\|_{H^s}.\)

**Proof.** A direct computation yields
\[
|E(u) - E(\varphi_c)| = \left| \left( \|u\|_{H^1} - \|\varphi_c\|_{H^1} \right) \left( \|u\|_{H^1} + \|\varphi_c\|_{H^1} \right) \right|
\leq \|u - \varphi_c\|_{H^1} \left( \|u\|_{H^1} + 2 \|\varphi_c\|_{H^1} \right)
\leq \varepsilon (\varepsilon + 2\sqrt{2}a) \leq 3\varepsilon,
\] (35)
under the assumption \(0 < \varepsilon < (3 - 2\sqrt{2})a.\) For the estimation \(|F(u) - F(\varphi_c)|,\) we have
\[
|F(u) - F(\varphi_c)| = \left| \int_R \left( u^8 + 4u^6 u_x^2 - 2u^4 u_x^4 + \frac{4}{5} u^2 u_x^6 - \frac{1}{7} u_x^8 \right) dx - \left( \varphi_c^8 + 4\varphi_c^6 (\partial_x \varphi_c)^2 \right. \\
- 2\varphi_c^4 (\partial_x \varphi_c)^4 + \frac{4}{5} \varphi_c^2 (\partial_x \varphi_c)^6 - \frac{1}{7} (\partial_x \varphi_c)^8 \bigg) dx \right|
\leq \left| \int_R \left( u^8 + 4u^6 u_x^2 - \varphi_c^8 - 4\varphi_c^6 (\partial_x \varphi_c)^2 \right) dx \right| + \frac{1}{7} \int_R \left( u_x^8 - (\partial_x \varphi_c)^8 \right) dx
+ 2 \int_R \left( u^4 u_x^4 - \varphi_c^4 (\partial_x \varphi_c)^4 \right) dx + \frac{4}{5} \int_R \left( u^2 u_x^6 - \varphi_c^2 (\partial_x \varphi_c)^6 \right) dx
\leq \left| \int_R \left( u^6 - \varphi_c^6 \right) \left( u^2 + 4u_x^2 \right) dx \right| + \left| \int_R \varphi_c^6 \left( (u^2 - \varphi_c^2) + 4(u_x^2 - (\partial_x \varphi_c)^2) \right) dx \right|
+ \frac{1}{7} \left| \int_R \left( u_x^4 + (\partial_x \varphi_c)^4 \right) \left( u_x^4 - (\partial_x \varphi_c)^4 \right) dx \right| + 2 \left| \int_R u^4 (u_x^2 - (\partial_x \varphi_c)^4) dx \right|
+ 2 \left| \int_R (\partial_x \varphi_c)^4 (u^4 - \varphi_c^4) dx \right| + \frac{4}{5} \left| \int_R u^2 (u_x^6 - (\partial_x \varphi_c)^6) dx \right|
+ \frac{4}{5} \left| \int_R (\partial_x \varphi_c)^6 (u^2 - \varphi_c^2) dx \right|
\triangleq J_1 + J_2 + \ldots + J_7 \lesssim A(c, \|u\|_{H^s}) \cdot \varepsilon,
\] (36)
where we give the proof of the last inequality of (36) in Appendix for convenience. This completes the proof of Lemma 3.4. \(\square\)
Proof of Theorem 3.1. Let \( u \in C([0, T); H^s(\mathbb{R})) \), \( s > \frac{5}{2} \) be the solution of Eq. (2) with the initial data \( u_0 \in H^s(\mathbb{R}), s > \frac{5}{2} \) satisfying \( \| u(0, \cdot) - \varphi_c \|_{H^s} < \varepsilon \), with \( 0 < \varepsilon < (3 - 2\sqrt{2})a \), and \( y_0 = (1 - \partial_x^2)u_0 \geq 0 \). Since \( E(u) \) and \( F(u) \) are both invariants by Eq. (2), we have

\[
E(u(t, \cdot)) = E(u_0) \quad \text{and} \quad F(u(t, \cdot)) = F(u_0), \quad \forall \ t \in [0, T).
\]

In view of (23) in Lemma 3.3, by \( E(\varphi_c) = 2a^2 \) and \( F(\varphi_c) = \frac{32}{35}a^8 \), we have

\[
0 \geq 3M^8 - 2M^6E(u) + \frac{35}{32} F(u)
\]

\[
= 3M^8 - 4a^2M^6 + a^8 - 2M^6(E(u) - E(\varphi_c)) + \frac{35}{32}(F(u) - F(\varphi_c))
\]

\[
= (M - a)^2(3M^8 + 6aM^5 + 5a^2M^4 + 4a^3M^3 + 3a^4M^2 + 2a^5M + a^6)
\]

\[
- 2M^6(E(u) - E(\varphi_c)) + \frac{35}{32}(F(u) - F(\varphi_c))
\]

\[
\geq (M - a)^2a^6 - 2M^6(E(u) - E(\varphi_c)) + \frac{35}{32}(F(u) - F(\varphi_c)),
\]

where \( M = \max_{x \in \mathbb{R}} \{u(t, x)\} \). Then we deduce from (37)-(38) that

\[
(M - a)^2 \leq \frac{2M^6|E(u_0) - E(\varphi_c)| + \frac{35}{32}|F(u_0) - F(\varphi_c)|}{a^6}.
\]

By Lemma 3.4, (39) and \( M \leq \frac{\|u_0\|_{H^1}}{\sqrt{2}} \leq \sqrt{\frac{2a + \varepsilon}{\sqrt{2}}} \), we find that for \( 0 < \varepsilon < (3 - 2\sqrt{2})a \),

\[
|M - a| \leq \sqrt{\left(\frac{32}{4}(\sqrt{2}a + \varepsilon)^6 + A(c, \|u_0\|_{H^1})\right)} \cdot \varepsilon
\]

\[
\leq \sqrt{\left(48a + \frac{A(c, \|u_0\|_{H^1})}{a^6}\right)} \cdot \varepsilon \leq \sqrt{A(c, \|u_0\|_{H^1})} \varepsilon,
\]

where we still used \( A(c, \|u_0\|_{H^1}) \) for simplicity. By Lemma 3.2 and (37), we obtain

\[
\|u - \varphi_c(\cdot - \xi(t))\|_{H^1} = E(u_0) - E(\varphi_c) - 4a(u(t, \xi(t)) - a),
\]

where \( \xi(t) \in \mathbb{R} \) is the maximum point of the fluid velocity \( u(t, \cdot) \). Hence, combining Lemma 3.4 with (40)-(41), we have

\[
\|u - \varphi_c(\cdot - \xi(t))\|_{H^1} \leq \sqrt{|E(u_0) - E(\varphi_c)| + 4a|u(t, \xi(t)) - a|}
\]

\[
\leq \sqrt{3a\varepsilon + 4a\sqrt{A(c, \|u_0\|_{H^1})}\varepsilon}.
\]

This completes the proof of Theorem 3.1.

Appendix. In this section, we complete the proof of the last inequality of (36) in Lemma 3.4.

\[
J_1 + J_2 + \cdots + J_7 \lesssim A(c, \|u\|_{H^1}) \cdot \varepsilon.
\]

Using the relation (24), we have for all \( v \in H^1(\mathbb{R}) \),

\[
\sup_{x \in \mathbb{R}} |v(x)| \leq \frac{\sqrt{E(v)}}{\sqrt{2}} \leq \frac{\|v\|_{H^1}}{\sqrt{2}}.
\]
For the term $J_1$, by (35) and (43), we have
\[
J_1 \leq 4 \int_{\mathbb{R}} |u - \varphi_c| \cdot |u^5 + u^4 \varphi_c + u^3 \varphi_c^2 + u^2 \varphi_c^3 + u \varphi_c^4 + \varphi_c^5| \cdot (u^2 + u_x^2) \, dx \\
\leq 4E(u)|u - \varphi_c|_{L^\infty} (|u|_{L^\infty}^5 + |u|_{L^\infty}^4 |\varphi_c|_{L^\infty} + |u|_{L^\infty}^3 |\varphi_c|^2_{L^\infty} \\
+ |u|_{L^\infty}^2 |\varphi_c|^3_{L^\infty} + |u|_{L^\infty} |\varphi_c|^4_{L^\infty} + |\varphi_c|^5_{L^\infty}) \\
\leq \frac{1}{2} (E(\varphi_c) + 3a\varepsilon) \|u - \varphi_c\|_{H^1} \left( |u|_{H^3}^5 + |u|_{H^3}^7 |\varphi_c|_{H^3} + |u|_{H^3}^3 |\varphi_c|^2_{H^3} + |u|_{H^3}^2 |\varphi_c|^3_{H^3} \\
+ |u|_{H^3} |\varphi_c|^4_{H^3} + |u|_{H^3} |\varphi_c|^5_{H^5} + |\varphi_c|^5_{H^3} \right) \\
\leq (E(\varphi_c) + 3a\varepsilon) \left( |u - \varphi_c|_{H^1} + 2 |\varphi_c|_{H^5} \right)^5 \cdot |u - \varphi_c|_{H^1} \\
\leq (2a^2 + 3a\varepsilon) \cdot (2\sqrt{2}a + \varepsilon)^5 \cdot \varepsilon.
\]
Similarly, for the term $J_2$, we have
\[
J_2 \leq \|\varphi_c\|_{L^\infty} \int_{\mathbb{R}} \left( (u - \varphi_c)^2 + 4(u_x - \partial_x \varphi_c)^2 + 2\varphi_c(u - \varphi_c) \right) \\
\quad + 8\partial_x \varphi_c(u_x - \partial_x \varphi_c) \, dx \\
\leq \left( \frac{\|\varphi_c\|_{H^1}}{\sqrt{2}} \right)^6 \left( 4|u - \varphi_c|_{H^1}^2 + 8|\varphi_c|_{H^3} \|u - \varphi_c\|_{H^1} \right) \\
\leq 4a^6 (2\sqrt{2}a + \varepsilon) \cdot \varepsilon.
\]
For the term $J_3$, by the Hölder inequality, we obtain
\[
J_3 = \frac{1}{7} \int_{\mathbb{R}} \left( u^4_x + (\partial_x \varphi_c)^4 \right) (u_x^2 + (\partial_x \varphi_c)^2) (u_x + \partial_x \varphi_c) \cdot (u_x - \partial_x \varphi_c) \, dx \\
\leq \frac{1}{7} \left( \int_{\mathbb{R}} (u^4_x + (\partial_x \varphi_c)^4)^2 \left( (u_x^2 + (\partial_x \varphi_c)^2)^2 (u_x + \partial_x \varphi_c)^2 \right)^{\frac{1}{2}} \right) \\
\quad \times \left( \int_{\mathbb{R}} (u_x - \partial_x \varphi_c)^2 \, dx \right)^{\frac{1}{2}} \\
\equiv \frac{1}{7} \sqrt{K} \|u - \varphi_c\|_{H^1}.
\]
For $K$, a direct use of Young’s inequality yields,
\[
K = \int_{\mathbb{R}} (u^4_x + 2u^{13}_x \partial_x \varphi_c + 3u^{12}_x (\partial_x \varphi_c)^2 + 4u^{11}_x (\partial_x \varphi_c)^3 + 5u^{10}_x (\partial_x \varphi_c)^4 \\
+ 6u^9_x (\partial_x \varphi_c)^5 + 7u^8_x (\partial_x \varphi_c)^6 + 8u^7_x (\partial_x \varphi_c)^7 + 7u^6_x (\partial_x \varphi_c)^8 + 6u^5_x (\partial_x \varphi_c)^9 \\
+ 5u^4_x (\partial_x \varphi_c)^{10} + 4u^3_x (\partial_x \varphi_c)^{11} + 3u^2_x (\partial_x \varphi_c)^{12} + 2u_x (\partial_x \varphi_c)^{13} + (\partial_x \varphi_c)^{14}) \, dx \\
\leq 32 \left( \int_{\mathbb{R}} u^{14}_x \, dx + \int_{\mathbb{R}} (\partial_x \varphi_c)^{14} \, dx \right).
\]
Since $u \in H^s(\mathbb{R}) \subset H^2(\mathbb{R})$, $s > \frac{5}{2}$, by the Gagliardo-Nirenberg inequality, we have
\[
\|u_x\|_{L^4}^{14} \leq C \|u\|_{L^2}^{\frac{5}{2}} \|u_{xx}\|_{L^2}^{10},
\]
with the constant $C > 0$ independent of $u$. Hence it follows from $\|\partial_x \varphi_c\|_{L^4}^{14} = \frac{1}{4} a^{14}$ that
\[
K \lesssim A^2(a, \|u\|_{H^s}).
\]
where the constant $A(a, \|u\|_{H^s}) > 0$ depends only on $a > 0$ and the norm $\|u\|_{H^s}$. Since $a = \frac{\sqrt{25}}{\sqrt{10}}$, then we have

$$J_5 \lesssim A(c, \|u\|_{H^s}) \cdot \varepsilon.$$  

In a similar manner to treat $J_4$ and $J_6$, using the fact that $\|\partial_x \varphi_c\|_{L^6}^6 = \frac{1}{2}a^6$ and $\|\partial_x \varphi_c\|_{L^6}^{10} = \frac{1}{4}a^{10}$, we obtain

$$J_4 \leq 2\|u\|_{L^\infty}^2 \left( \int_{\mathbb{R}} \left( u_x^2 + (\partial_x \varphi_c)^2 \right)^2 \left( u_x + \partial_x \varphi_c \right)^2 \, dx \right) \left( \int_{\mathbb{R}} \left( u_x - \partial_x \varphi_c \right)^2 \, dx \right)^{1/2}\|u - \varphi_c\|_{H^1},$$

$$J_6 \leq \frac{4}{5} \|u\|_{L^\infty}^2 \left( \int_{\mathbb{R}} \left( u_x^2 + (\partial_x \varphi_c)^2 \right)^2 \left( u_x + \partial_x \varphi_c \right)^2 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} \left( u_x - \partial_x \varphi_c \right)^2 \, dx \right)^{1/2}\|u - \varphi_c\|_{H^1},$$

For the terms $J_5$ and $J_7$, a direct use of the Hölder inequality yields,

$$J_5 \leq 2 \left( \|u\|_{L^\infty}^2 + \|\varphi_c\|_{L^\infty}^2 \right) \left( \|u\|_{L^\infty} + \|\varphi_c\|_{L^\infty} \right) \left( \int_{\mathbb{R}} \left( \partial_x \varphi_c \right)^8 \, dx \right)^{1/2} \left( \int_{\mathbb{R}} \left( u - \varphi_c \right)^2 \, dx \right)^{1/2} \leq \frac{1}{\sqrt{2}} \left( \|u - \varphi_c\|_{H^1}^2 + 2 \|u - \varphi_c\|_{H^1} \|\varphi_c\|_{H^1} + 2 \|\varphi_c\|_{H^1}^2 \right) \|u - \varphi_c\|_{H^1} \leq \frac{a}{2\sqrt{2}} \left( \frac{4a^2 + 2\sqrt{2}a + \varepsilon^2}{\sqrt{2}a + \varepsilon} \right) \varepsilon,$$

and

$$J_7 \leq \frac{4}{5} \left( \|u\|_{L^\infty} + \|\varphi_c\|_{L^\infty} \right) \left( \int_{\mathbb{R}} \left( \partial_x \varphi_c \right)^{12} \, dx \right)^{1/2} \left( \int_{\mathbb{R}} \left( u - \varphi_c \right)^2 \, dx \right)^{1/2} \leq \frac{a}{2\sqrt{2}} \left( \frac{4a^2 + 2\sqrt{2}a + \varepsilon^2}{\sqrt{2}a + \varepsilon} \right) \varepsilon,$$
\[
\begin{align*}
\leq & \frac{2\sqrt{2}}{5} \left( \|u - \varphi_c\|_{H^1} + \|\varphi_c\|_{H^1} \right) \|\partial_x \varphi_c\|_{L^{12}}^6 \cdot \|u - \varphi_c\|_{H^1} \\
\leq & \frac{2a}{5} \sqrt{\frac{1}{3}} (\sqrt{2a} + \varepsilon) \cdot \varepsilon,
\end{align*}
\]
where we used the fact that \(\|\partial_x \varphi_c\|_{L^8} = \frac{1}{2} a^8\) and \(\|\partial_x \varphi_c\|_{L^{12}} = \frac{1}{2} a^{12}\). Therefore, gathering the estimations \(J_1-J_7\), we get the desired result (42).

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