LOCAL ALGEBRAIC APPROXIMATION OF SEMIANALYTIC SETS

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Abstract. Two subanalytic subsets of $\mathbb{R}^n$ are called $s$-equivalent at a common point $P$ if the Hausdorff distance between their intersections with the sphere centered at $P$ of radius $r$ vanishes to order $> s$ when $r$ tends to $0$. In this paper we prove that every $s$-equivalence class of a closed semianalytic set contains a semialgebraic representative of the same dimension. In other words any semianalytic set can be locally approximated to any order $s$ by means of a semialgebraic set and hence, by previous results, also by means of an algebraic one.

1. Introduction

In [FFW1] we introduced a notion of local metric proximity between two sets that we called $s$-equivalence: for a real $s \geq 1$, two subanalytic subsets of $\mathbb{R}^n$ are $s$-equivalent at a common point $P$ if the Hausdorff distance between their intersections with the sphere centered at $P$ of radius $r$ vanishes to order $> s$ when $r$ tends to $0$.

Given a subanalytic set $A \subset \mathbb{R}^n$ and a point $P \in A$, a natural question concerns the existence of an algebraic representative $X$ in the class of $s$-equivalence of $A$ at $P$; in that case we also say that $X$ approximates $A$ of order $s$ at $P$.

The answer to the previous question is in general negative for subanalytic sets which are not semianalytic, even for $s = 1$ (see [FFW3]). Furthermore, in [FFW2] we defined $s$-equivalence of two subanalytic sets along a common submanifold, and studied 1-equivalence of a pair of strata to the normal cone of the pair. By example we showed that a semianalytic normal cone to a linear $X$ may be not 1-equivalent to any semialgebraic set along $X$. It is still an open problem whether a semialgebraic normal cone along a linear $X$ is $s$-equivalent to an algebraic variety along $X$, for all $s$.

On the other hand some partial positive answers were given in [FFW1] and [FFW3]; in particular we proved that a subanalytic set $A \subset \mathbb{R}^n$ can be approximated of any order by an algebraic one in each of the following cases:
- $A$ is a closed semialgebraic set of positive codimension,
- $A$ is the zero-set $V(f)$ of a real analytic map $f$ whose regular points are dense in $V(f)$,
- $A$ is the image of a real analytic map $f$ having a finite fiber at $P$.

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Using the previous results we also obtained that one-dimensional subanalytic sets, analytic surfaces in \( \mathbb{R}^3 \) and real analytic sets having a Puiseux-type parametrization admit an algebraic approximation of any order.

In the present paper we prove that any closed semianalytic set can be locally approximated of any order by a semialgebraic one having the same dimension. Using the main result of [FFW1], it follows that any closed semianalytic set of positive codimension admits an algebraic approximation of any order. Thus we obtain a complete positive answer to our question for the class of semianalytic sets.

The algebraic approximation, elaborating the methods introduced in [FFW3], is obtained by taking sufficiently high order truncations of the analytic functions appearing in a presentation of the semianalytic set.

Finally, let us mention some possible future developments of these notions and ideas. Since we can prove that two subanalytic sets \( A, B \) are 1-equivalent if and only if their tangent cones coincide (see also [FFW1]), it would be interesting to extend the notion of tangent cone associating to \( A \) a sort of “tangent cone of order \( s \)”, say \( C_s(A) \), in such a way that \( A \) and \( B \) are \( s \)-equivalent if and only if \( C_s(A) = C_s(B) \).

There is currently much interest in bi-Lipschitz equivalence of varieties. Most of the work has been in the complex case. One recent such example is [BFGO]. The theory is closely tied up with the notion of the tangent cone, exceptional subcones, and limits of tangent spaces. The real case has been little studied. A good place to start is in the case of surfaces in \( \mathbb{R}^3 \), which is the only real case in which the tangent cone, exceptional lines, and limits of tangent planes have been deeply analyzed (see [OW]). The \( s \)-equivalence classes are Lipschitz invariants, so they should be a useful tool in this analysis.

2. Basic notions and preliminary results

If \( A \) and \( B \) are non-empty compact subsets of \( \mathbb{R}^n \), we denote by \( D(A,B) \) the classical Hausdorff distance, i.e.

\[
D(A,B) = \inf \{ \epsilon \mid A \subseteq N_\epsilon(B), \ B \subseteq N_\epsilon(A) \},
\]

where \( N_\epsilon(A) = \{ x \in \mathbb{R}^n \mid d(x,A) < \epsilon \} \) and \( d(x,A) = \inf_{y \in A} \| x - y \| \).

If we let \( \delta(A,B) = \sup_{x \in B} d(x,A) \), then \( D(A,B) = \max\{\delta(A,B), \delta(B,A)\} \).

We will denote by \( O \) the origin of \( \mathbb{R}^n \) for any \( n \).

We are going to introduce the notion of \( s \)-equivalence at a point; without loss of generality we can assume that this point is \( O \).

**Definition 2.1.** Let \( A \) and \( B \) be closed subanalytic subsets of \( \mathbb{R}^n \) with \( O \in A \cap B \). Let \( s \) be a real number \( \geq 1 \). Denote by \( S_r \) the sphere of radius \( r \) centered at the origin.

1. We say that \( A \leq_s B \) if either \( O \) is isolated in \( A \) or if \( O \) is non-isolated both in \( A \) and in \( B \) and

\[
\lim_{r \to 0} \frac{\delta(B \cap S_r, A \cap S_r)}{r^s} = 0.
\]

2. We say that \( A \) and \( B \) are \( s \)-equivalent (and we will write \( A \sim_s B \)) if \( A \leq_s B \) and \( B \leq_s A \).

Observe that if \( O \) is non-isolated both in \( A \) and in \( B \), then

\[
A \sim_s B \quad \text{if and only if} \quad \lim_{r \to 0} \frac{D(A \cap S_r, B \cap S_r)}{r^s} = 0.
\]
Moreover, if $A \subseteq B$, then $A \leq_s B$ for any $s \geq 1$. It is easy to check that $\leq_s$ is transitive and that $\sim_s$ is an equivalence relation. The following result shows that $s$-equivalence behaves well with respect to the union of sets:

**Proposition 2.2 ([FFW3]).** Let $A, A', B$ and $B'$ be closed subanalytic subsets of $\mathbb{R}^n$.

1. If $A \leq_s B$ and $A' \leq_s B'$, then $A \cup A' \leq_s B \cup B'$.
2. If $A \sim_s B$ and $A' \sim_s B'$, then $A \cup A' \sim_s B \cup B'$.

Given a closed subanalytic set $A$ and $s \geq 1$, the problem we are interested in is whether there exists an algebraic subset $Y$ which is $s$-equivalent to $A$; in this case we also say that $Y$ approximates $A$ to order $s$. Evidently the question is trivially true when $O$ is an isolated point in $A$.

Among the partial answers to the previous question that have been already achieved, we recall only the following one which will be used later on:

**Theorem 2.3 ([FFW1]).** For any real number $s \geq 1$ and for any closed semialgebraic set $A \subseteq \mathbb{R}^n$ of codimension $\geq 1$, there exists an algebraic subset $Y$ of $\mathbb{R}^n$ such that $A \sim_s Y$.

The following definition introduces a geometric tool which is very useful to test the $s$-equivalence of two subanalytic sets:

**Definition 2.4.** Let $A$ be a closed subanalytic subset of $\mathbb{R}^n$, $O \in A$. For any real $\sigma > 1$, we will refer to the set

$$\mathcal{H}(A, \sigma) = \{ x \in \mathbb{R}^n \mid d(x, A) < \|x\|^\sigma \}$$

as the horn-neighbourhood with center $A$ and exponent $\sigma$.

Note that, if $O$ is isolated in $A$, then $\mathcal{H}(A, \sigma) = \emptyset$ near $O$.

**Proposition 2.5 ([FFW3]).** Let $A, B$ be closed subanalytic subsets of $\mathbb{R}^n$ with $O \in A \cap B$ and let $s \geq 1$. Then $A \leq_s B$ if and only if there exist $\sigma > s$ and an open neighbourhood $\Omega$ of $O$ such that $(A \setminus \{O\}) \cap \Omega \subseteq \mathcal{H}(B, \sigma) \cap \Omega$.

The following technical result suggests that horn-neighbourhoods can be used to modify a subanalytic set producing subanalytic sets $s$-equivalent to the original one:

**Lemma 2.6.** Let $X \subset Y \subset \mathbb{R}^n$ be closed subanalytic sets such that $O \in X$ and let $s \geq 1$. Then:

1. for any $\sigma > s$ we have $Y \sim_s Y \cup \mathcal{H}(X, \sigma)$;
2. if $\overline{Y \setminus X} = Y$, there exists $\sigma > s$ such that $Y \setminus \mathcal{H}(X, \sigma) \sim_s Y$.

**Proof.** (a) Since $Y \cup \mathcal{H}(X, \sigma) \subseteq \mathcal{H}(Y, \sigma)$, by Proposition 2.5 for any $\sigma > s$ we have that $Y \cup \mathcal{H}(X, \sigma) \leq_s Y$ and hence $Y \cup \mathcal{H}(X, \sigma) \sim_s Y$.

(b) Let $\mathcal{U}(X, q) = \{ x \in \mathbb{R}^n \mid \exists y \in X, \|x\| = \|y\|, \|x - y\| < \|y\|^{\eta}\}$.

Arguing as in [FFW1] Corollary 2.6, there exists $q$ such that $Y \setminus \mathcal{U}(X, q) \sim_s Y$. Since $X$ and $Y \setminus \mathcal{U}(X, q)$ are subanalytic sets and meet only in $O$, they are regularly situated, i.e. there exists $\beta$ such that $d(x, X) + d(x, Y \setminus \mathcal{U}(X, q)) > \|x\|^\beta$ for all $x$ near $O$. Then $\mathcal{H}(X, \beta) \subseteq \mathcal{U}(X, q)$ and hence taking $\sigma > \max\{\beta, s\}$ we have that $Y \setminus \mathcal{H}(X, \sigma) \sim_s Y$. \hfill \Box

Another essential tool will be Lojasiewicz’ inequality, which we will use in the following slightly modified version.
Proposition 2.7. Let $A$ be a compact subanalytic subset of $\mathbb{R}^n$. Assume $f$ and $g$ are subanalytic functions defined on $A$ such that $f$ is continuous, $V(f) \subseteq V(g)$, $g$ is continuous at the points of $V(g)$ and such that $|g| < 1$ on $A$. Then there exists a positive constant $\alpha$ such that $|g|^\alpha \leq |f|$ on $A$ and $|g|^\alpha < |f|$ on $A \setminus V(f)$.

Proof. The result will be obtained by adapting the proof given by Łojasiewicz under the stronger hypothesis that $g$ is continuous on $A$ (see [L] Théorème 1); in that paper he used the following lemma ([L] Lemma 4):

If $E \subset [0, \infty) \times \mathbb{R}$ is a compact semianalytic subset of $\mathbb{R}^2$ such that $E \cap \{(0) \times \mathbb{R}\} \subseteq \{(0, 0)\}$, then there exist positive constants $c, \alpha$ such that $E \subseteq \{(x, y) \in \mathbb{R}^2 \mid |y|^\alpha \leq c|x|\}$.

The map $\Phi = (|f|, g) : A \to \mathbb{R}^2$ is subanalytic and bounded; hence $\Phi(A)$ is a subanalytic subset of $\mathbb{R}^2$ and therefore semianalytic ([L] Proposition 2)). Then $E = \Phi(A)$ is a compact semianalytic subset of $[0, \infty) \times \mathbb{R}$.

We have that $E \cap \{(0) \times \mathbb{R}\} \subseteq \{(0, 0)\}$: namely, if $(0, y_0) \in E$, then there exists a sequence $\{a_i\} \subset A$ such that $\lim_{i \to \infty} \Phi(a_i) = (0, y_0)$ with $a_i$ converging to $a_0 \in A$. By continuity $f(a_0) = 0$ and hence $g(a_0) = 0$. By the continuity of $g$ at $a_0$, we have that $y_0 = g(a_0) = 0$.

So $E$ fulfills the hypotheses of the lemma recalled above and therefore there exist positive constants $c, \alpha$ such that $|g|^\alpha \leq c|f|$ on $A$.

Since $|g| < 1$, increasing $\alpha$ if necessary we can obtain the thesis. \qed

3. Main results

This section is devoted to the proof of the local approximation theorem for semianalytic sets.

Since $s$-equivalence depends only on the set-germs at $O$, all the sets we will work with will be considered as subsets of a suitable open ball $\Omega$ in $\mathbb{R}^n$ centered at $O$; we will shrink such a ball whenever necessary without mention.

Definition 3.1. Let $A$ be a closed semianalytic subset of $\Omega$. We will say that $A$ admits a good presentation if the minimal analytic variety $V_A$ containing $A$ is irreducible and there exist analytic functions $f_1, \ldots, f_p$ which generate the ideal $I(V_A)$ and $g_1, \ldots, g_l$ analytic functions on $\Omega$ such that

$$A = \{ x \in \Omega \mid f_i(x) = 0, g_j(x) \geq 0, i = 1, \ldots, p, j = 1, \ldots, l \}.$$ 

We start with a preliminary result concerning a way to decompose and present semianalytic sets:

Lemma 3.2. Let $A$ be a closed semianalytic subset of $\Omega$ with $\dim_{\Omega} A = d > 0$. Then there exist closed semianalytic sets $\Gamma_1, \ldots, \Gamma_r, \Gamma'$ such that

1. $A = (\bigcup_{i=1}^{r} \Gamma_i) \cup \Gamma'$,
2. for each $i$, $\dim_{\Omega} \Gamma_i = d$ and $\Gamma_i$ admits a good presentation,
3. $\dim \Gamma' < d$.

Proof. Let $V_A$ be the minimal analytic variety containing $A$ (in particular $\dim_{\Omega} V_A = d$). Let $V_1 \cup \ldots \cup V_m$ be the decomposition of $V_A$ into irreducible components. Then $A = W_1 \cup \ldots \cup W_m$ where $W_i = A \cap V_i$. Then $V_i$ is the minimal analytic variety containing $W_i$ and $\dim_{\Omega} V_i = \dim_{\Omega} W_i$.

Each $W_i$ is a finite union of sets of the form $\Gamma = \{ h_1 = 0, \ldots, h_q = 0, g_1 \geq 0, \ldots, g_l \geq 0 \}$. 

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Let $\Gamma'$ be the union, over $i = 1, \ldots, m$, of the $\Gamma$'s having dimension less than $d$.

For any $\Gamma \subseteq V_i$ of dimension $d$, $V_i$ is the minimal analytic variety containing $\Gamma$. It follows that $\Gamma = \{ f_1 = 0, \ldots, f_p = 0, g_1 \geq 0, \ldots, g_l \geq 0 \}$ where $f_1, \ldots, f_p$ are generators of the ideal $I(V_i)$. Thus we can take as $\Gamma_1, \ldots, \Gamma_r$ these latter $\Gamma$'s (over $i = 1, \ldots, m$) suitably indexed. □

**Notation 3.3.** Let $g_1, \ldots, g_l$ be analytic functions on $\Omega$ and let $f = (f_1, \ldots, f_p) : \Omega \to \mathbb{R}^p$ be an analytic map. If $A = \{ x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \ldots, l \}$, we will use the following notation:

1. $A_i = \{ x \in \Omega \mid f(x) = O, g_i(x) \geq 0 \}$ for $i = 1, \ldots, l$ (so that $A = \bigcap A_i$),
2. $b(A) = \bigcup_{i=1}^l (V(g_i) \cap A)$.

**Lemma 3.4.** Consider the closed semianalytic set

$$A = \{ x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \ldots, l \},$$

where $f : \Omega \to \mathbb{R}^p$ is an analytic map and $g_1, \ldots, g_l$ are analytic functions on $\Omega$. Assume that $O \in A$. Let $\sigma$ be a real number $> 1$ and let $H \subseteq \mathbb{R}^n$ be an open subanalytic set such that $H \supseteq H(b(A), \sigma)$. Then there exists $\eta$ such that, for each $x \in V(f) \setminus (A \cup H)$, there exists $i$ so that $x \notin H(A_i, \eta)$.

**Proof.** Since the functions $\sum_i d(x, A_i)$ and $d(x, A)$ are subanalytic and vanish exactly on $A$, by Proposition 2.7 there exists $\alpha > 0$ such that, for any $x$,

$$\sum_i d(x, A_i) \geq d(x, A)^\alpha.$$

Let $d_g$ denote the geodesic distance on $V(f)$.

If $x \in V(f) \setminus A$, we have $d_g(x, A) = d_g(x, b(A))$. In a suitable closed ball centered at $O$ we can assume that $V(f)$ is connected; hence, by a result of Kurdyka and Orro ([KO]) for any $\epsilon > 0$ there exists a subanalytic distance $\Delta(x, y)$ on $V(f)$ such that

$$\forall x, y \in V(f) \quad 0 \leq \Delta(x, y) \leq d_g(x, y) \leq (1 + \epsilon)\Delta(x, y).$$

Then, if we take for instance $\epsilon = 1$,

$$\forall x \in V(f) \quad 0 \leq \Delta(x, A) \leq d_g(x, A) \leq 2\Delta(x, A)$$

and so the subanalytic function $\Delta(x, A)$ is continuous at each point of $A$. Hence by Proposition 2.7 there exists $\mu > 0$ such that, for any $x$ in $V(f)$,

$$d(x, A) \geq \Delta(x, A)^\mu$$

and so

$$\sum_i d(x, A_i) \geq \Delta(x, A)^\mu \geq \left( \frac{d_g(x, A)}{2} \right)^{\mu\alpha}.$$ 

Moreover for any $x \in V(f) \setminus (A \cup H)$ we have that

$$d_g(x, A) = d_g(x, b(A)) \geq d(x, b(A)) \geq \|x\|^\sigma.$$

Let us show that the thesis holds choosing $\eta > \sigma \mu \alpha$.

If, for a contradiction, any neighbourhood of $O$ contains a point $x \in \bigcap_i \mathcal{H}(A_i, \eta) \cap (V(f) \setminus (A \cup H))$, then we have that

$$\frac{1}{2\mu \gamma} \|x\|^\sigma \mu \alpha \leq \sum_{i=1}^l d(x, A_i) \leq l\|x\|^\eta,$$

which is impossible when $x$ tends to $O$. □
For any analytic map $\psi$ defined in a neighbourhood of $O$, we will denote by $T_k^k \psi(x)$ the polynomial map whose components are the Taylor polynomials of order $k$ at $O$ of the components of $\psi$.

**Lemma 3.5.** Let $\varphi$ be an analytic function on $\Omega$ such that $\varphi(O) = 0$. Let $X$ be a closed semianalytic subset of $\Omega$, $O \in X$. Then for any real positive $\theta$ there exists $\alpha > 0$ such that, for all integers $k > \alpha$, the function $T_k^k \varphi$ has the same sign as $\varphi$ on $X \setminus \{ \mathcal{H}(X \cap V(\varphi), \theta) \cup \{O\} \}$.

**Proof.** Denote $Z = X \setminus \mathcal{H}(X \cap V(\varphi), \theta)$. Since $V(\varphi) \cap Z = \{O\}$, by Proposition 2.7 there exists $\alpha > 0$ such that $\|x\|^\alpha < |\varphi(x)|$ for all $x \in Z \setminus \{O\}$.

For all integers $k > \alpha$

$$\lim_{x \to O} \frac{\varphi(x) - T_k^k \varphi(x)}{\|x\|^\alpha} = 0.$$ 

If $O$ is isolated in $Z$, there is nothing to prove. Otherwise assume, for a contradiction, that any neighbourhood of $O$ contains a point $x \in Z$ such that $\varphi(x)$ and $T_k^k \varphi(x)$ have different signs (for instance $\varphi(x) > 0$ and $T_k^k \varphi(x) \leq 0$). Then

$$|\varphi(x) - T_k^k \varphi(x)| \geq |\varphi(x)| > \|x\|^\alpha$$

and hence

$$\frac{|\varphi(x) - T_k^k \varphi(x)|}{\|x\|^\alpha} > 1$$

arbitrarily near to $O$, which is impossible. \hfill \Box

**Notation 3.6.** Let $g_1, \ldots, g_l$ be analytic functions on $\Omega$ and let $f: \Omega \to \mathbb{R}^p$ be an analytic map. If $A = \{ x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \ldots, l \}$, for any $h, k \in \mathbb{N}$ let

1. $T^h(A) = \{ x \in \Omega \mid T^h f(x) = O, g_i(x) \geq 0 \text{ for } i = 1, \ldots, l \}$,
2. $T_k^h(A) = \{ x \in \Omega \mid f(x) = O, T^h g_i(x) \geq 0 \text{ for } i = 1, \ldots, l \}$,
3. $T^h_k(A) = \mathcal{H}(T^h_k(A)) = \{ x \in \Omega \mid T^h f(x) = O, T^h g_i(x) \geq 0 \text{ for } i = 1, \ldots, l \}$.

Moreover, for any analytic map $\varphi: \Omega \to \mathbb{R}^p$, denote $\Sigma_\varphi = \{ x \in \Omega \mid \text{rk } d_x \varphi < r \}$, and $\Sigma(\varphi) = \Sigma_{\varphi}(\varphi)$.

**Lemma 3.7.** Let $A$ be a closed semianalytic subset of $\Omega$, with $\dim O A = d > 0$. Assume that $A = \{ f(x) = O, g_i(x) \geq 0, i = 1, \ldots, l \}$, with $g_1, \ldots, g_l$ analytic functions on $\Omega$ and $f: \Omega \to \mathbb{R}^{n-d}$ an analytic map. Assume also that

$$\dim O(A \setminus (\Sigma(f) \cup b(A))) > 0.$$ 

Then for any $s \geq 1$ there exist $h_0 > 0, k_0 > 0$ such that, for all integers $h, k$ with $h \geq h_0$ and $k \geq k_0$, we have

1. $T^h_k(A) \leq_s A$,
2. $A \setminus (\Sigma(f) \cup b(A)) \leq_s T^h_k(A)$,
3. $\dim O T^h_k(A) = d$.

**Proof.** Let $s \geq 1$ and let $\sigma > s$. Denote $X = (\Sigma(f) \cap A) \cup b(A)$.

(1) Let $H = \mathcal{H}(X, \sigma)$. By Lemma 3.4 there exists $\eta$ such that, for each $x \in V(f \setminus (A \cup H))$, there exists $\eta_0$ so that $x \notin \mathcal{H}(A_{i_0}, \eta)$.

For all $j$, applying Lemma 3.5 to $V(f)$, $g_j$ and $\eta$, we find $\alpha_j > 0$ such that, for all integers $k > \alpha_j$, the functions $g_j$ and $T^k g_j$ have the same sign on $V(f \setminus (\mathcal{H}(V(f) \cap V(g_j), \eta) \cup \{O\})$. 

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Let $x \in V(f) \setminus (A \cup H)$. Then $x \not\in \mathcal{H}(A_{i_0}, \eta)$ for some $i_0$ and hence $g_{i_0}(x) < 0$; moreover, since $V(f) \cap V(g_{i_0}) \subseteq A_{i_0}$, we have that $x \in V(f) \setminus (\mathcal{H}(V(f) \cap V(g_{i_0}), \eta) \cup \{O\})$ and hence, for all integers $k > \alpha_1$, $T^k g_{i_0}(x) < 0$. This implies that $T_k(A) \subseteq A \cup H$.

Applying Lemma 2.6 (1) to the sets $X$ and $A$, we have $A \sim_s A \cup H$, and so $T_k(A) \subseteq A$.

Let $B_k = \{x \in \Omega \mid T^k g_i \geq 0, \ i = 1, \ldots, l\}$.

Since $T_k(A) = B_k \cap V(f)$, by Proposition 2.7 there exists $\rho > 0$ such that $\|f(x)\| \geq d(x, T_k(A))^{\rho}$ for all $x \in B_k$; then for $x \in B_k \setminus \mathcal{H}(T_k(A), \sigma)$ we have that $\|f(x)\| \geq \|x\|^\rho \sigma$.

Let $h$ be an integer such that $h \geq \rho \sigma$. Then

$$\lim_{x \to O} \frac{\|f(x) - T^h f(x)\|}{\|x\|^\rho \sigma} = 0.$$

We have that $T^h (T_k(A)) \setminus \{O\} \subseteq \mathcal{H}(T_k(A), \sigma)$; otherwise there would exist a sequence of points $y_i \not\in O$ converging to $O$ such that $y_i \in T^h (T_k(A)) \setminus \mathcal{H}(T_k(A), \sigma)$ and hence

$$\lim_{i \to \infty} \frac{\|f(y_i) - T^h f(y_i)\|}{\|y_i\|^\rho \sigma} = \lim_{i \to \infty} \frac{\|f(y_i)\|}{\|y_i\|^\rho \sigma} \geq 1,$$

which is a contradiction.

Thus by Proposition 2.5 we get that $T^h_k(A) \sim_s T_k(A) \subseteq A$.

(2) Let $Y = A \setminus X$. By our hypotheses $O$ is not isolated in $Y$.

Since $Y \setminus X = Y$, applying Lemma 2.6 (2) to the sets $X \cap Y$ and $Y$ and increasing $\sigma$ if needed, we have that $Y \setminus \mathcal{H}(X \cap Y, \sigma) \sim_s Y$. Denote

$$Y' = Y \setminus \mathcal{H}(X \cap Y, \sigma) \quad \text{and} \quad H_i = \mathcal{H}(V(g_i) \cap Y, \sigma).$$

If for each $i$ we apply Lemma 3.3 to $Y$, $g_i$ and $\sigma$, we can find $\alpha_2 > 0$ such that, for all integers $k > \alpha_2$, the functions $g_i$ and $T^k g_i$ have the same sign on $Y \setminus (H_i \cup \{O\})$.

Since $V(g_i) \cap Y \subseteq X \cap Y$ for each $i$, then $\bigcup H_i \subseteq \mathcal{H}(X \cap Y, \sigma)$, and therefore $Y' \setminus \{O\} \subseteq \bigcap_i (Y \setminus (H_i \cup \{O\}))$. In particular

$$Y' \setminus \{O\} \subseteq \{T^k g_1 > 0, \ldots, T^k g_l > 0\}.$$

From now on, assume that $k > \alpha_2$. We will get the result by replacing $f$ with a suitable truncation of it in the presentation of $T_k(A)$. We will denote by $B(x, r)$ the open ball centered at $x$ of radius $r$.

By the last inclusion, the distance $d(x, b(B_k))$ is subanalytic and positive on $Y' \setminus \{O\}$ so, by Proposition 2.7, there exists $\nu > 0$ (and we can assume $\nu > s$) such that $d(x, b(B_k)) > \|x\|^{\nu}$ for all $x$ in $Y' \setminus \{O\}$. As a consequence

$$B(x, \|x\|^{\nu}) \subseteq \{T^k g_1 > 0, \ldots, T^k g_l > 0\}.$$

Following FFW3 consider the real-valued function

$$\Lambda f(x) = \begin{cases} 0 & \text{if } \text{rk } d_x f < n - d \\ \inf_{v \in \ker d_x f, \|v\| = 1} \|d_x f(v)\| & \text{if } \text{rk } d_x f = n - d \end{cases}.$$

Observe that $\Lambda f(x)$ is subanalytic, continuous and positive where $f$ is submersive, in particular on $Y' \setminus \{O\}$. Hence, again by Proposition 2.7, there exists $\beta > 0$ such that $\Lambda f(x) > \|x\|^\beta$ for all $x$ in $Y' \setminus \{O\}$.
Consider the subanalytic set \( W = \{(x, y) \in Y' \times \Omega \mid \Lambda f(y) \geq \|x\|^\beta \} \) and let \( W_0 = \{(x, y) \in Y' \times \Omega \mid \Lambda f(y) = \|x\|^\beta \} \); then the set \( \{(x, x) \mid x \in Y' \setminus \{O\} \} \) is contained in the open subanalytic set \( W \setminus W_0 \).

The function \( \varphi : Y' \setminus \{O\} \to \mathbb{R} \) defined by \( \varphi(x) = d((x, x), W_0) \) is subanalytic and positive. Then again by Proposition 2.7 there exists \( \tau > 0 \) (and we can assume \( \tau > \nu \)) such that \( \varphi(x) > \|x\|^\tau \) on \( Y' \setminus \{O\} \). Then for all \( x \in Y' \setminus \{O\} \) and for all \( y \in B(x, \|x\|^\tau) \) we have

\[
\|(x, y) - (x, x)\| = \|y - x\| < \|x\|^\tau < \varphi(x).
\]

Hence \( (x, y) \in W \setminus W_0 \), i.e. for all \( x \in Y' \setminus \{O\} \) and for all \( y \in B(x, \|x\|^\tau) \) we have \( \Lambda f(y) > \|x\|^\beta \). In particular \( \Lambda f(y) > 0 \) and hence \( d_yf \) is surjective for all \( y \in B(x, \|x\|^\tau) \).

Let \( h \) be an integer such that \( h > \beta + 1 \) and let \( \bar{f}(x) = T^hf(x) \).

Then \( T^{h-1}d_yf = d_y\bar{f} \); thus we have that \( \|d_yf - d_y\bar{f}\| \leq \|y\|^{h-1} \) for all \( y \) near to \( O \), where we consider \( \text{Hom}(\mathbb{R}^n, \mathbb{R}^{n-d}) \) endowed with the standard norm

\[
\|L\| = \max_{u \neq 0} \frac{\|L(u)\|}{\|u\|}.
\]

Thus by [FFW3, Proposition 3.3] we have

\[
|\Lambda f(y) - \Lambda \bar{f}(y)| \leq \|y\|^{h-1}.
\]

Claim. For \( x \in Y' \setminus \{O\} \) and for \( y \in B(x, \|x\|^\tau) \), we have

\[
\Lambda \bar{f}(y) \geq \|x\|^\beta + 1.
\]

To see this, assume for a contradiction that there exist a sequence \( x_i \in Y' \setminus \{O\} \) converging to \( O \) and a sequence \( y_i \in B(x_i, \|x_i\|^\tau) \) such that \( \Lambda \bar{f}(y_i) < \|x_i\|^\beta + 1 \). Thus we have

\[
\frac{\Lambda f(y_i) - \Lambda \bar{f}(y_i)}{\|x_i\|^\beta} > \frac{\|x_i\|^\beta - \|x_i\|^\beta + 1}{\|x_i\|^\beta} = 1 - \|x_i\|.
\]

On the other hand

\[
\frac{\Lambda f(y_i) - \Lambda \bar{f}(y_i)}{\|x_i\|^\beta} \leq \frac{\|y_i\|^{h-1}}{\|x_i\|^\beta} \leq \frac{(\|y_i - x_i\| + \|x_i\|)^{h-1}}{\|x_i\|^\beta} = \left( \frac{\|y_i - x_i\|}{\|x_i\|^\beta} + \|x_i\|^{1-q} \right)^{h-1} \leq \left( \|x_i\|^{1-q} + \|x_i\|^{1-q} \right)^{h-1}
\]

where \( q = \frac{\beta}{h-1} \). Since \( \tau > 1 \) and \( q < 1 \), we have that

\[
\frac{\Lambda f(y_i) - \Lambda \bar{f}(y_i)}{\|x_i\|^\beta} \text{ converges to 0, which is a contradiction. So the Claim is proved.}
\]

Consequently, for all \( x \in Y' \setminus \{O\} \) the map \( \bar{f} \) is a submersion on \( B(x, \|x\|^\tau) \). Hence, using [FFW3, Lemma 3.5], we get \( \bar{f}(B(x, \|x\|^\tau)) \supseteq B(\bar{f}(x), \|x\|^\lambda) \) with \( \lambda = \beta + 1 + \tau \).

Observe that if \( x \in Y' \setminus \{O\} \), we have that

\[
\lim_{x \to O} \frac{\|\bar{f}(x)\|}{\|x\|^h} = \lim_{x \to O} \frac{\|\bar{f}(x) - f(x)\|}{\|x\|^h} = 0.
\]
So, for any \( h \geq \lambda \) and \( x \in Y' \), the point \( O \) belongs to \( B(f(x), \|x\|^\tau) \) and hence there exists \( y \in B(x, \|x\|^\tau) \) such that \( f(y) = O \).

Since \( \tau > \nu > s \), then \( y \in B(x, \|x\|^\nu) \) so that \( T^k g_i(y) > 0 \) for all \( i \), i.e. \( y \in T^h_k(A) \); hence \( Y' \setminus \{O\} \subseteq \mathcal{H}(T^h_k(A), \lambda) \). Then by Proposition \( \ref{theo:approximation} \) we have that \( Y' \leq T^h_k(A) \) and hence, since \( Y' \sim Y \), we have that

\[
A \setminus (\Sigma(f) \cup b(A)) = Y \leq T^h_k(A).
\]

Therefore, taking \( h_0 = \max\{\rho \sigma, \lambda\} \) and \( k_0 = \max\{\alpha_1, \alpha_2\} \), we have the thesis.

(3) The previous argument shows that, for all \( h \geq h_0 \) and \( k \geq k_0 \), there exist points \( y \in V(T^h f) \) arbitrarily near to \( O \) where \( T^h f \) is submersive and such that \( T^k g_i(y) > 0 \) for all \( i \). Hence \( \dim O T^h_k(A) = d \). \( \square \)

**Theorem 3.8.** Let \( A \) be a closed semianalytic subset of \( \Omega \) with \( O \in A \). Then for any \( s \geq 1 \) there exists a closed semialgebraic set \( S \subseteq \Omega \) such that \( A \sim_s S \) and \( \dim O S = \dim O A \).

**Proof.** We will prove the thesis by induction on \( d = \dim O A \).

If \( d = 0 \) the result holds trivially. So let \( d \geq 1 \) and assume that the result holds for all semianalytic germs of dimension less than \( d \).

By Lemma \( \ref{thm:approximation} \) by Proposition \( \ref{prop:approximation} \) and by the inductive hypothesis, we can assume that

\[
A = \{x \in \Omega \mid f(x) = O, g_i(x) \geq 0, i = 1, \ldots, l\}
\]

with \( f = (f_1, \ldots, f_p) \) such that \( V(f) \) is irreducible, \( V(f) \) is the minimal analytic variety containing \( A \) and \( f_1, \ldots, f_p \) generate the ideal \( I(V(f)) \). In particular \( \dim O(\Sigma_{n-d}(f) \cap A) < d \). Moreover, removing from the previous presentation of \( A \) the inequalities \( g_i(x) \geq 0 \) where \( g_i \) vanishes identically on \( A \) (if any), we can assume that \( \dim O b(A) < d \).

If \( p = n - d \), the thesis follows easily by using Lemma \( \ref{lem:approximation} \). In general \( p \) can be larger than \( n - d \); in this case we introduce a semianalytic set \( \tilde{A} \) of dimension \( d \) which is \( s \)-equivalent to \( A \) and which satisfies the hypotheses of Lemma \( \ref{lem:approximation} \). In order to prove the thesis it will be sufficient to approximate \( \tilde{A} \) by means of a semialgebraic set having the same dimension.

Denote by \( \Pi \) the set of surjective linear maps from \( \mathbb{R}^p \) to \( \mathbb{R}^{n-d} \) and consider the smooth map \( \Phi : (\mathbb{R}^n - V(f)) \times \Pi \to \mathbb{R}^{n-d} \) defined by \( \Phi(x, \pi) = (\pi \circ f)(x) \) for all \( x \in \mathbb{R}^n - V(f) \) and \( \pi \in \Pi \).

The map \( \Phi \) is transverse to \( \{O\} \): namely the partial Jacobian matrix of \( \Phi \) with respect to the variables in \( \Pi \) (considered as an open subset of \( \mathbb{R}^{p(n-d)} \)) is the \( (n-d) \times p(n-d) \) matrix

\[
\begin{bmatrix}
f(x) & 0 & 0 & \ldots & 0 \\
0 & f(x) & 0 & \ldots & 0 \\
\vdots \\
0 & 0 & 0 & \ldots & f(x)
\end{bmatrix};
\]

thus, for all \( x \in \mathbb{R}^n - V(f) \) and for all \( \pi \in \Pi \) the Jacobian matrix of \( \Phi \) has rank \( n - d \).

As a consequence, by a well-known result of singularity theory (see for instance \( \cite{BK} \), Lemma 3.2), we have that the map \( \Phi_\pi : \mathbb{R}^n - V(f) \to \mathbb{R}^{n-d} \) defined by \( \Phi_\pi(x) = \Phi(x, \pi) = (\pi \circ f)(x) \) is transverse to \( \{O\} \) for all \( \pi \) outside a set \( \Gamma \subset \Pi \) of measure zero and hence \( \pi \circ f \) is a submersion on \( V(\pi \circ f) \setminus V(f) \) for all such \( \pi \).
Let \( x \in V(f) \) be a point at which \( f \) has rank \( n - d \). Then there is an open dense set \( U \subset \Pi \) such that for all \( \pi \in U \) the map \( \pi \circ f \) is a submersion at \( x \), and hence off some subvariety of \( V(f) \) of dimension less than \( d \).

Thus, if we choose \( \pi_0 \in (\Pi \setminus \Gamma) \cap U \), the map \( F = \pi_0 \circ f \) has \( n - d \) components, \( \Sigma(F) \cap V(F) \subset V(f) \subset V(F) \), and \( \dim_O V(F) = d \) and \( \dim_O (\Sigma(F) \cap V(F)) < d \). In particular \( V(f) \) is an irreducible component of \( V(F) \).

For each \( m \in \mathbb{N} \) denote \( \bar{A}_m = \{ F(x) = 0, \| x \|^2m - \| f(x) \|^2 \geq 0, g_i(x) \geq 0, i = 1, \ldots, l \} \).

Since \( A \subset \bar{A}_m \subset V(F) \), we have that \( A \leq_s \bar{A}_m \) and \( \dim_O \bar{A}_m = d \).

We claim that there exists \( m \) such that \( \bar{A}_m \sim_s A \); to show that it is sufficient to prove that there exists \( m \) such that \( \bar{A}_m \leq_s A \). Namely, let \( B = \{ g_i(x) \geq 0, i = 1, \ldots, l \} \). Since \( V(\| f \|) \cap B = V(\| d(x, A) \|) \cap B \), by Proposition 2.7 there exists \( q \) such that \( d(x, A)^q \leq \| f(x) \| \) for all \( x \in B \). Let \( m > sq \). Then \( d(x, A) \leq \| f(x) \| \frac{1}{q} \leq \| x \| \frac{m}{q} \) for all \( x \in \bar{A}_m \), i.e. \( \bar{A}_m \subset \mathcal{H}(\mathbb{A}, \frac{m}{q}) \) and hence \( \bar{A}_m \leq_s A \).

Fix \( m \) as above and let \( \bar{A} = \bar{A}_m \). Also let \( \bar{X} = (\Sigma(F) \cap A) \cup b(\bar{A}) \).

Observe that \( b(\bar{A}) \cap A = b(A) \) and so \( \bar{X} \cap A = (\Sigma(F) \cap A) \cup b(A) \).

Denote \( K = \bar{X} \cap (\bar{A} \setminus A) \) so that \( X = (\bar{X} \cap A) \cup K \).

By Lemma 3.7 there exist positive integers \( h, k \) such that

\[
\bar{A} \setminus \bar{X} \leq_s T_k^h(\bar{A}) \leq_s \bar{A} \quad \text{and} \quad \dim_O T_k^h(\bar{A}) = d.
\]

Since \( \dim_O (\bar{X} \cap A) < d \), by induction there exists a semialgebraic set \( S_0 \) such that \( S_0 \sim_s \bar{X} \cap A \) and \( \dim_O S_0 < d \). Moreover, since \( A \subset \bar{A} \setminus K \subset \bar{A} \), we have that \( \bar{A} \setminus K \sim_s \bar{A} \).

Then

\[
\bar{A} \sim_s \bar{A} \setminus K = \bar{A} \setminus \bar{X} \cup (\bar{X} \cap A) \leq_s T_k^h(\bar{A}) \cup S_0 \leq_s \bar{A} \cup (\bar{X} \cap A) = \bar{A}
\]

so we can choose \( S = T_k^h(\bar{A}) \cup S_0 \).

From Theorem 3.8 and from Theorem 2.8 we immediately obtain:

**Theorem 3.9.** Let \( A \) be a closed semianalytic subset of \( \Omega \) of codimension \( \geq 1 \) with \( O \in A \). Then for any \( s \geq 1 \) there exists an algebraic set \( Y \subset \mathbb{R}^n \) such that \( \bar{A} \sim_s Y \).

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