Determination of quantum instrument parameters for a Stern-Gerlach non-ideal device

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The paper identifies and determines some parameters with experimental relevance, which could describe the influence of the non-ideality for the measurement of the intrinsic spin of an atom, using a real Stern-Gerlach device.

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I. INTRODUCTION

Stern-Gerlach experiment is usually considered as a prototype for a filter-measurement, i.e. a measurement with two possible answers, which practically consist in the disparition of the measured object or in its preservation. For a filter-measurement it corresponds a projector $\hat{E}(B)$ from the spectral family of the selfadjoint operator associated to the measured physical quantity:

$$\hat{A} = \int \lambda \hat{E}(d\lambda)$$

Actually, these filter-measurement are only idealizations for those realized using real devices. A major epistemological problem for the Quantum Theory is the influence of the non-ideality of the device on the mathematical formalism. This aim is mathematically-coherent followed in Operational Quantum Physics (OQP) \cite{1,2}. Here one will use OQP formalism in order to identify and determine some parameters with experimental relevance, which would describe the influence of the non-ideality for the measurement of the intrinsic spin of an atom, using a real Stern-Gerlach device. A similar method was used for imperfect polarizers in the paper \cite{3}.

If $\hat{\rho}_s$ and $\hat{\rho}_b$ are state-operators for the atom and, respectively for the device at the initial moment $t_0$ when the measurement did not start yet, the state-operator for the compound system can be written in a factorized form:

$$\dot{\hat{\rho}}(t_0) = \hat{\rho}_s(t_0) \otimes \hat{\rho}_b(t_0)$$

Consider that the measurement starts at the subsequent moment $t_i$ and and ends at the moment $t_f$, so the compound system Hamiltonian is:

$$\hat{H}(t) = \begin{cases} \hat{H}_0, & t < t_i, \quad t > t_f \\ \hat{H}_0 + \hat{H}_{\text{int}}(t), & t \in [t_i, t_f] \end{cases}$$

The compound system state for $t > t_f$ is not anymore factorizable, so it is considered that the state of the atom is given by the operator obtained after the partial trace operation on the device degrees of freedom:

$$\dot{\hat{\rho}}_s(t) = \text{tr}_b(\dot{\hat{\rho}}(t))$$

where:

$$\dot{\hat{\rho}}(t) = \hat{U}(t, t_0) \hat{\rho}_s(t_0) \otimes \hat{\rho}_b(t_0) \hat{U}^\dagger(t, t_0)$$

$$\hat{U}(t, t_0) = \mathbb{T} \exp \left( -\frac{i}{\hbar} \int_{t_0}^{t} dt' \hat{H}(t') \right)$$

The result of this operation is that the state $\dot{\hat{\rho}}_s(t)$ is not anymore pure.

Generally, any mixed state can be decompose in a convex combination of pure statistical operators. An orthogonal decomposition is given by the spectral theorem for the selfadjoint operator which is $\dot{\hat{\rho}}_s(t)$. Apart from some degeneracies of the spectral values, it is also unique \cite{3}. The condition which an interaction Hamiltonian at the measurement process for a physical magnitude has to obey, is that $\dot{\hat{\rho}}_s(t)$ has to be convexly orthogonally decomposable after a complete set of spectral operators (projectors). In the Stern-Gerlach case, if one works with the usually bidimensional $\frac{1}{2}$ spin representation (with the base given by vectors $|\uparrow\rangle$ and $|\downarrow\rangle$), the alluded condition is:

$$\dot{\hat{\rho}}_s(t) = w_\uparrow(t) \hat{\rho}_\uparrow(t) + (1 - w_\uparrow(t)) \hat{\rho}_\downarrow(t)$$

II. DESCRIPTION OF THE NON-IDEAL MEASUREMENT

Transformation:

$$\dot{\hat{\rho}}_s(t_0) \rightarrow \dot{\hat{\rho}}_s(t), \quad t > t_f$$

is a completely positive map, which in the bidimensional case we can write, can be written in the form \cite{3}:

$$\dot{\hat{\rho}}_s(t) = \sum_{m=\uparrow, \downarrow} \hat{A}^\dagger_m(\hat{\rho}_s(t_0)) \hat{A}_m$$

(1)

where the operators \{ $\hat{A}_m$ \} \_\_{m=\uparrow, \downarrow} fulfill the condition:

$$\sum_{m=\uparrow, \downarrow} \hat{A}_m^\dagger \hat{A}_m = \hat{1}$$

(2)

In OQP one postulates the existence of another type of transformation:

$$\dot{\hat{\rho}}_s(t) \rightarrow \left( \hat{A}_m^\dagger \hat{\rho}_s(t_0) \hat{A}_m \right)_{\uparrow, \downarrow}$$

(3)

which correspond to the situation when the measurement was done, and the result is already known as one of the values $\uparrow$.
or \( \downarrow \). (1) is named non-selective measurement, while (3) are selective measurements. The two terms of (3):

\[
\hat{F}_m = \hat{A}_m \hat{A}_m
\]

are positive operators, which generalize the projectors on the pure states \( \uparrow \) and \( \downarrow \). In OPQ they are named effects, while the applications (1) and (3) are the duals of some applications on the observables, which are named quantum instruments.

Of course, in the bidimensional case, the state and the instrument operators can be decomposed after the Pauli matrices basis:

\[
\{ \hat{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z \}
\]

where we shall use the usual vectorial notation \( \vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z) \):

\[
\hat{\rho} = \frac{1}{2} (\hat{1} + \vec{\rho} \cdot \vec{\sigma}) , \quad \vec{\rho} = \vec{\rho}^* , \quad |\vec{\rho}| \leq 1
\]

\[
\hat{A} = \alpha \hat{1} + \vec{\beta} \cdot \vec{\sigma} , \quad \alpha = \alpha_1 + i \alpha_2 \in \mathbb{C} , \quad \vec{\beta} = \beta_1 + i \beta_2
\]

where the spin index was omitted.

The condition that a filter Stern-Gerlach experiment has to fulfill is to yield a spatial separation for the two spinorial components, and to place in the zone where one of the components is localized an absorbing screen. In this case, the atoms which are passed after the device can be associated with the other spinorial component.

A major problem is to determine the coefficients which define the instrument. From (2) one gets:

\[
\left( (\alpha \hat{1} + \vec{\beta} \cdot \vec{\sigma}) \cdot (\alpha^* \hat{1} + \vec{\beta}^* \cdot \vec{\sigma}) \right)_\uparrow + (...)_\downarrow = \frac{\alpha \beta^* + \alpha^* \beta + i (\vec{\beta} \times \vec{\beta}^*)}{|\alpha|^2 + |\beta|^2} |\gamma\rangle_{\uparrow} (10)
\]

where:

\[
\vec{\xi}_{\uparrow(\downarrow)} = \frac{\alpha \beta^* + \alpha^* \beta + i (\vec{\beta} \times \vec{\beta}^*)}{|\alpha|^2 + |\beta|^2} |\gamma\rangle_{\uparrow(\downarrow)}
\]

is a real vector. From (3) it comes:

\[
\begin{align*}
& \left( |\alpha|^2 + |\beta|^2 \right)_\uparrow + (...)_\downarrow = 1 \\
& \left( \alpha \beta^* + \alpha^* \beta + i (\vec{\beta} \times \vec{\beta}^*) \right)_\uparrow + (...)_\downarrow = 0 \quad (7)
\end{align*}
\]

The state transformation (1) for the initial state \( \frac{1}{2} (\hat{1} + \vec{k} \cdot \vec{\sigma}) \) gives:

\[
\hat{\rho}_s(\vec{k}) = \frac{1}{2} \left( \hat{1} |\alpha|^2 + |\beta|^2 + (\alpha \beta^* + \alpha^* \beta) \right) \vec{k} + \quad + i \left( \beta \times \beta^* \right) \vec{k} + \vec{F} \left( \alpha \beta^* + \alpha^* \beta + \left( |\alpha|^2 - |\beta|^2 \right) \vec{k} + \right.+ \left. + i \left( \beta \beta^* - \alpha^* \beta \right) \vec{k} + i \left( \vec{F} \times \beta \right) \right) \vec{k} + (......)_\downarrow 
\]

while the state transformations (3) gives only one of the upwards terms. For the last measurements, the probability of obtaining the answer “yes” is given by:

\[
\begin{align*}
& f(\vec{k})_{\gamma(\downarrow)} = \text{tr} \left( \hat{\rho}_s(\vec{k})_{\gamma(\downarrow)} \right) = |\alpha|^2 + |\beta|^2 + \quad + (\alpha \beta^* + \alpha^* \beta) \vec{k} + i \left( \vec{F} \times \beta \right) \vec{k} + (......)_\downarrow \\
& \quad + i \left( \beta \beta^* - \alpha^* \beta \right) \vec{k} + 2i \left( \beta \times \beta^* \right) \vec{k} + (......)_\downarrow \\
& \text{In (3), using (7) one gets:}
\end{align*}
\]

\[
\hat{\rho}_s(\vec{k}) = \frac{1}{2} \left( \hat{1} + \vec{\beta} \cdot \vec{\sigma} \right) \left( |\alpha|^2 - \beta_2 \right) \vec{k} + \left( \alpha \beta^* - \alpha^* \beta \right) \vec{k} + 2i \left( \beta \times \beta^* \right) \vec{k} + (......)_\downarrow 
\]

where one can introduce the notation:

\[
\hat{A}(\vec{k}) = \left( |\alpha|^2 - 2 \beta_2 \right) \vec{k} + i \left( \alpha \beta^* - \alpha^* \beta \right) \vec{k} + \quad + 2i \left( \vec{F} \times \beta \right) \vec{k} + (......)_\downarrow 
\]

so the state is given by:

\[
\hat{\rho}_s(\vec{k}) = \frac{1}{2} \left( \hat{1} + \vec{\beta} \cdot \vec{\sigma} \right) \hat{A}(\vec{k}) \quad (11)
\]

If the device is rotated with the angle \( \phi \) around the \( \vec{n} \) axis, the quantum instrument has to fulfill a covariance condition (11): (2) (3):

\[
\hat{A}(\theta)_m = \hat{U}(\theta) \hat{A}(\vec{k}) \hat{U}(\theta), \quad \hat{U}(\theta) = \exp \left( -i \frac{\vec{n} \cdot \vec{S}}{\hbar} \right) \quad (12)
\]

where \( \hat{U}(\theta) \) is the symmetry group of the rotations, whose generator is the total kinetic moment operator, which can be identified here with the intrinsic spin kinetic moment.

\[
\hat{U}(\phi) = \cos \left( \frac{\phi}{2} \right) \hat{1} + i \sin \left( \frac{\phi}{2} \right) \vec{n} \cdot \vec{\sigma} \quad (13)
\]

The operators (11) are cf. (2) (3):

\[
\hat{A}(\theta (\phi)) = \left[ \alpha \hat{1} + \cos \phi \left( \vec{\beta} \cdot \vec{\sigma} \right) \right] + \\
\left. + \sin \phi \left( \left( \vec{n} \times \vec{\sigma} \right) \cdot \vec{\sigma} \right) + 2 \sin^2 \frac{\phi}{2} \left( \vec{n} \times \vec{\sigma} \right) \left( \vec{n} \cdot \vec{\sigma} \right) \right]_\gamma(\downarrow) \quad (14)
\]
Introducing the notation:

\[
\tilde{B}(\phi) = \cos \phi \tilde{\beta} + \sin \phi (\vec{n} \times \vec{\beta}) + 2\sin^2 \frac{\phi}{2} (\vec{n} \cdot \vec{\beta}) \vec{n}
\]

one has:

\[
\hat{A}_{\gamma_\parallel}(\phi) = \left( \alpha \hat{1} + \tilde{B}(\phi) \cdot \vec{\sigma} \right)_{\gamma_\parallel}
\]

(15)

If one makes first the nonselective measurement (1) followed by a selective one using a rotated instrument \(\vec{n}\), the final states are given by:

\[
\rho_s(\vec{k}, \phi) = \frac{1}{2} \left\{ |\alpha|^2 + |\tilde{B}(\phi)|^2 + (\alpha \tilde{B}(\phi)^* + \alpha^* \tilde{B}(\phi)) \right\} \hat{A}(\vec{k}) + \frac{i}{2} \left[ \tilde{B}(\phi)^* \right. + \tilde{B}(\phi)^{\ast} ] \hat{A}(\vec{k}) + \\left. \right\}
\]

(16)

and the probabilities by:

\[
f(\vec{k}, \phi)_{\gamma_\parallel} = \text{tr} \left( \rho_s(\vec{k}, \phi)_{\gamma_\parallel} \right) = |\alpha|^2 + |\tilde{B}(\phi)|^2 + |\tilde{B}(\phi)^* + \alpha^* \tilde{B}(\phi)\hat{A}(\vec{k}) + \frac{i}{2} \left[ \tilde{B}(\phi) \times \tilde{B}(\phi)^* \right. \hat{A}(\vec{k}) \left. \right]\}
\]

(17)

An interesting case is that of the angle \(\phi_m = m \cdot \frac{2\pi}{3}\) (\(m = 0, 1, 2\)) rotation around \(\vec{n} = \frac{1}{\sqrt{3}}(1, 1, 1)\), which yield even circular permutations for the 3 axis:

\[
\tilde{B}_0 = \tilde{\beta}
\]

\[
\tilde{B}_1 = \beta_x \vec{e}_x + \beta_y \vec{e}_y + \beta_z \vec{e}_z
\]

\[
\tilde{B}_2 = \beta_x \vec{e}_x + \beta_y \vec{e}_y + \beta_z \vec{e}_z
\]

III. RESULTS FOR THE SMALL NON-IDEALITY CASE

In the small non-ideality case, the system of equations becomes linear, and the determination and compatibility conditions can be studied, either by direct inspection of the equations, or using a specialized software for algebraic computations. In the ideal case, the measurement of the \(Oz\) component of the intrinsic spin is given by the projectors:

\[
\hat{E}_{\gamma_\parallel} = \frac{1}{2} (I \pm \hat{\sigma}_z)
\]

The parameters:

\[
a_{\gamma_\parallel} = \frac{1}{2}
\]

\[
\tilde{\beta}_{\gamma_\parallel} = \pm \frac{1}{2} \vec{e}_z + \eta \hat{\sigma}_z
\]

(18)

are the most simple solution for the equation:

\[
\hat{E}_{\gamma_\parallel} = \left( |\alpha|^2 + |\tilde{\beta}|^2 \right) I + \left\{ |\alpha \tilde{\beta}^* + \alpha^* \tilde{\beta} + \right\}
\]

(19)

\[
+ \frac{i}{2} \left[ \tilde{\beta} \times \tilde{\beta}^* \right] \hat{\sigma}_x \}
\]

(20)

so, one will take as a small non-ideality measurement that given by the parameters:

\[
a_{\gamma_\parallel} = \frac{1}{2} + \eta a_{\gamma_\parallel}
\]

(21)

\[
\tilde{\beta}_{\gamma_\parallel} = \pm \frac{1}{2} \vec{e}_z + \eta \hat{\sigma}_z
\]

(22)

where \(\eta\) is a small real parameter, while \(a_{\gamma_\parallel}\) and \(\tilde{\beta}_{\gamma_\parallel}\) are complex magnitudes. Introducing (21) and (22) in (1) and (14), and developing in the first order of parameter \(\eta\), one can obtain the predictions \(f\) for the probabilities of obtaining the positive result at various experimental setups. Of course, one will consider only the deviations from ideality, noted with \(\delta f\).

Taking the spherical angle representation for the unit vector \(\vec{k} = \sin \theta \cos \varphi \cdot \vec{e}_x + \sin \theta \sin \varphi \cdot \vec{e}_y + \cos \theta \cdot \vec{e}_z\):

\[
\delta f(\vec{k})_{\gamma_\parallel} = [a_r + b_r x + (a_r + b_r y) \cos \theta] \hat{\sigma}_x + (b_r y - b_r x) \sin \theta \sin \varphi + (b_r y + b_r x) \cos \theta \hat{\sigma}_y
\]

(23)

Fitting the experimental data for the deviations \(\delta f(\vec{k})_{\gamma_\parallel}\) with the function:

\[
\delta f(\vec{k})_{\gamma_\parallel} = [c_0 + c_1 \sin \theta \cos \varphi + \right\}
\]

(24)

\[
+ c_2 \sin \theta \sin \varphi + c_3 \cos \theta]_{\gamma_\parallel}
\]

from (1) and identification between (23) and (24) one gets the following equations for the parameters \(a_{\gamma_\parallel}\) and \(\tilde{\beta}_{\gamma_\parallel}\) (a total of 16 real parameters):
\[
(a_r + b_{rz})_\uparrow + (a_r + b_{rz})_\downarrow = 0
\]
\[
(b_{rx} - b_{iy})_\uparrow + (b_{rx} - b_{iy})_\downarrow = 0
\]
\[
(b_{ry} + b_{ix})_\uparrow + (b_{ry} + b_{ix})_\downarrow = 0
\]

\[
a_r\uparrow + b_{rz}\uparrow = c_0\uparrow
\]
\[
b_{rx}\uparrow - b_{iy}\uparrow = c_2\uparrow
\]
\[
b_{ry}\uparrow + b_{ix}\uparrow = c_3\uparrow
\]

An important test for OQP is the compatibility between predictions and experimental results, which can be expressed by the confidence parameters of the fitting operation. In the case of an acceptable compatibility, the upward system of equations is enough for the identification of the parameters for the effects (4), which are given, in the small non-ideality case, by:

\[
\hat{F}_{\uparrow(\downarrow)} = \frac{1}{2} \left( \hat{1} \pm \hat{\sigma}_z \right) + \eta \left( (a_r + b_{rz}) \hat{1} + \frac{a_r + b_{rz}}{b_{rx} - b_{iy}} \hat{\sigma}_y \right) \left( \frac{b_{ry} + b_{ix}}{b_{rx} - b_{iy}} \hat{\sigma}_x \right)_{\uparrow(\downarrow)}
\]

where the second index from the experimental parameters stay for the main axis of the device after the rotation (17), which, again, is expected to be sufficient for the description of the two-step successive measurements. For more-step successive measurements one can proceed in the same manner.

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