Domain walls in supersymmetric QCD

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In this talk we construct BPS-saturated domain walls in supersymmetric QCD, for any values of the masses of the chiral matter superfields. We compare our results to those already obtained in the literature and we also discuss their range of applicability, as well as future directions that would be desirable to explore in order to achieve a complete understanding of supersymmetric gluodynamics as a step in improving our knowledge of how QCD works.

1. Introduction

Supersymmetric (SUSY) gluodynamics, the theory of gluons and gluinos with gauge group $SU(N_c)$, shares a lot of common features with ordinary QCD, with the advantage of being supersymmetric. Therefore it seems a good idea to exploit the supersymmetric properties of SUSY gluodynamics in order to try to understand better how QCD works. Although this is an idea which has been around for many years, the recent and positive results of Dvali and Shifman \cite{1}, which we shall briefly explain in detail, triggered a great deal of activity in these past three years.

Our starting point will be the Lagrangian for SUSY gluodynamics, which is given by

$$L = \frac{1}{g_0^2} \left[ \frac{1}{4} G_{\mu\nu}^a G^{\mu\nu}_a + i \lambda^{\alpha\dot{\alpha}} D_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}\dot{\beta}} \right],$$

where $g_0$ is the gauge coupling constant, $G_{\mu\nu}^a$ the usual gluon field strength tensor and $\lambda$ is the gaugino field. This theory has an axial U(1) symmetry which is broken down to $Z_{2N_c}$, the chiral symmetry, by the anomaly. Moreover, the theory is strongly coupled: at a scale $\Lambda$ gaugino condensates (which, from now on, we shall denote as $\langle \lambda \lambda \rangle$) will form. This will break the chiral symmetry from $Z_{2N_c}$ to $Z_2$ and, clearly, the gaugino condensate is the order parameter associated with the symmetry breakdown.

The value of the gaugino condensate was calculated by various methods a few years ago \cite{2}. The $N_c$ degenerate vacua of our theory are given by

$$\langle \lambda \lambda \rangle \equiv \langle \mathrm{Tr} \lambda^a \lambda_a \rangle = \Lambda^3 \frac{e^{i2\pi k}}{\Lambda^2}, \quad k = 0, \ldots, N_c - 1.$$ (2)

The presence of degenerate vacua leads immediately to the formation of domain walls interpolating between them. The key result of Dvali and Shifman \cite{1} was to realize that if these walls are of a certain kind, i.e. BPS-saturated, then they preserve one half of the supersymmetry and their energy density is exactly calculable and given by

$$\epsilon = \frac{N_c}{8\pi^2} |\langle \mathrm{Tr} \lambda^a \lambda_a \rangle_{\infty} - \langle \mathrm{Tr} \lambda^a \lambda_a \rangle_{-\infty}|,$$

i.e. the difference between the values of the condensate at the two vacua -situated at spatial infinities- between which the wall is interpolating.

This is very interesting because it has been suggested \cite{3} that BPS-saturated domain walls would
play an important role in the D-brane description of $N=1$ supersymmetric QCD (SQCD). Also, recent work claims that, in the large $N_c$ limit, these walls are BPS-saturated states, but so far it has not been fully proved for finite $N_c$ configurations. In any case the question to be answered, before making any further connections to other interesting topics, is what the nature of these domain walls is.

In order to do that we are going to discuss first of all different effective approaches to SUSY gluodynamics and the problems associated with them. Then we shall study in detail the Taylor, Veneziano, Yankielowicz (TVY) approach, which is the one used in our calculation. Finally we present results and draw a few conclusions.

2. Towards an effective theory of SUSY gluodynamics

A first and very complete description of SUSY gluodynamics was provided by Veneziano and Yankielowicz (VY), in terms of a composite chiral superfield $S$ whose lowest component is the gaugino condensate

$$S \equiv \frac{3}{32\pi^2} (\text{Tr}(\omega_\alpha \omega^\alpha)) = \frac{3}{32\pi^2} (\text{Tr}(\lambda_\alpha \lambda^\alpha)) + \ldots \quad (4)$$

The Lagrangian for this model is given by the usual supersymmetric structure,

$$\mathcal{L} = \frac{1}{4} \int d^4\theta \mathcal{K} + \frac{1}{2} \left[ \int d^2\theta W + \text{h.c.} \right], \quad (5)$$

with $\mathcal{K}$ the Kähler potential and $W$ the superpotential. Throughout this talk we shall be working with canonically normalized fields, therefore $\mathcal{K} = (\bar{S}S)^{1/3}$. The superpotential for the VY model is given by

$$W = \frac{2}{3} S \ln \left( \frac{S^{N_c}}{(e\Lambda^3)^{N_c}} \right). \quad (6)$$

The structure of $W$ is uniquely fixed by the anomaly and the symmetries of the theory. Again $\Lambda$ is the scale parameter. It has been widely shown that the VY superpotential describes very well the vacuum structure of SUSY gluodynamics, however this is not the case when we try to describe the dynamics of the theory and, in particular, when we try to construct domain walls.

It was pointed out not so long ago that this Lagrangian is not explicitly $Z_{2N_c}$ invariant. Moreover, the scalar potential derived in global supersymmetry, i.e. $V(S) = \mathcal{K}^{-1}_{SS} |\partial W/\partial S|^2$ is multivalued due to the presence of the logarithm in $W$. That is, a physical state and its equivalent, just rotated by $2\pi$, would give rise to completely different values of $V$, which is totally unacceptable.

In order to cure this problem, several solutions have been proposed. In Refs the idea of glued potential was introduced and developed. Essentially it consists of adding a Lagrange multiplier to the Lagrangian in such a way that the scalar potential is divided into $N_c$ sectors which, glued together, result in a single-valued $Z_{2N_c}$-invariant theory. The problem with such an amendment is that cusps will inevitably form at the joining point of each sector with its nearest neighbours. Any configuration interpolating between two vacua will necessarily cross a cusp and it is doubtful whether it would be possible to correctly interpret the energy density associated with it. Therefore we conclude that the VY model, even when properly modified, is unable to provide us with a good description of domain walls in SUSY gluodynamics.

Other solutions have been proposed to deal with the problems associated to the presence of the logarithm in the VY superpotential. For example, in Ref. it was proposed that the origin of such problems was related to leaving behind some relevant degrees of freedom when deriving the VY effective Lagrangian. It is then suggested that introducing new degrees of freedom in the model, in particular a glueball order parameter, would be enough to enable us to construct well behaved domain walls. This is certainly a possibility worth considering, however we will not be following this approach but that of Taylor, Veneziano and Yankielowicz (TVY), which we describe in detail in the next section.

3. Our model: the TVY approach

Given all the problems of the VY model when trying to construct domain wall solutions of the equations of motion, it is convenient to work with
a slightly more complicated model which respects all the symmetries of the theory. This consists of adding \( N_f \) pairs of chiral superfields, \( Q^i, \bar{Q}_i \) to the VY model. Then, below the condensation scale we shall have matter condensates, \( M_i^j = Q^i \bar{Q}_j \), as well as gaugino condensates. The Lagrangian, given again by Eq. (6), has the following superpotential

\[
\mathcal{W} = \frac{2}{3} S \ln \left( \frac{S_{N_c}^{N_f} \det M}{e^{N_c - N_f} \Lambda^{3N_c - N_f}} \right) - \frac{1}{2} \text{Tr}(m M). \tag{7}
\]

From now on we shall work in a flavour-diagonal basis, i.e. \( m^j_i = \delta^j_i m_j \). Our goal will be to construct domain walls between the vacua of this model, and then try to obtain the limit of large masses where the theory should tend to SUSY gluodynamics. In other words, we start off by working in the weak (Higgs) phase of the model, and we will try to extrapolate the results obtained to strong coupling.

Therefore the first step consists of defining the vacua of the model. These are given by

\[
S_{N_c}^{N_f} = \left( \frac{3}{4} \right)^{N_f-N_c} \det m \tag{8}
\]

\[
(M_i^j) = \delta^j_i \frac{1}{m_i} \frac{4}{3} S^\ast, \tag{9}
\]

where, and from now on, we are taking \( \Lambda = 1 \). That is, the vacuum values for the gaugino and matter condensates are aligned (* denotes values at the vacuum). Also, one can easily evaluate the superpotential to find out that it is proportional to the vacuum values of the fields, i.e. \( W_* = -(2/3)N_c S^\ast \).

A very important point which we are going to discuss in detail is the paths the fields take when going from one vacuum to another. As mentioned before, we are using the matter fields to restore the lost \( Z_{2N_c} \) invariance of the TV Lagrangian and ensure that, when building domain walls, we do not cross the logarithmic branch. In order to formulate this in a more precise way, let us suppose that we are building a wall between vacuum \( a \) and vacuum \( b \). Then we define the field trajectories as follows

\[
S_{b} = e^{i\delta} S_{a} \tag{10}
\]

where \( \omega_i \), the windings of the matter fields, have to be integers, in order to fulfill the alignment condition at the vacua. To ensure that the logarithmic branch is not crossed one must impose the following condition, coming from the fact that the phases in the logarithm of Eq. (7) cancel out

\[
(N_c - N_f)\delta + \sum_{i=1}^{N_f} (\delta - 2\pi\omega_i) = 0. \tag{11}
\]

As an immediate result it follows that \( \delta = 2\pi k / N_c \) where we define \( k = \sum_{i=1}^{N_f} \omega_i \). Note that, if all the windings are equal to one, then \( k = N_f \). This is an important fact to which we will return.

Let us now briefly discuss the method used to calculate the domain wall profiles. We assume that the walls spread along the \( xy \) plane, therefore the profiles are calculated along \( z \). This is done by minimizing the energy functional which, for a generic set of fields, \( X^k \), with Kähler metric \( g^i_j = K_{X^k \bar{X}^l} \), looks like

\[
\epsilon_{ab} = \text{Re}(e^{i\gamma} (\mathcal{W}_b - \mathcal{W}_a) + \frac{1}{2} \int_{-\infty}^{+\infty} dz (g^{i} \bar{g}^j X^k \bar{X}^l + (g^{i} \bar{g}^j)^{-1} \mathcal{W}_i \mathcal{W}_j), \tag{12}
\]

where \( \mathcal{W}_i = \partial \mathcal{W} / \partial X^i \). This equation can be rewritten as

\[
\epsilon_{ab} = \text{Re}(e^{i\gamma} (\mathcal{W}_b - \mathcal{W}_a) + \frac{1}{2} \int_{-\infty}^{+\infty} dz (g^{i} \bar{g}^j)^{-1} (g^{k} \partial_\bar{z} X^k - e^{-i\gamma} \mathcal{W}_i) \tag{13}
\]

where \( e^{i\gamma} \) is an arbitrary phase. In fact, if we choose it in a clever enough way, i.e. \( e^{-i\gamma_{ab}} = (\mathcal{W}_b - \mathcal{W}_a)/|\mathcal{W}_b - \mathcal{W}_a| \), then it is easy to see that we can put a lower bound on the energy density of the wall that interpolates between vacua \( a \) and \( b \), which is

\[
\epsilon_{ab} \geq |\mathcal{W}_b - \mathcal{W}_a|. \tag{14}
\]

This is the so-called BPS bound; if the bound is saturated the wall to which it corresponds is a BPS-saturated domain wall. An immediate consequence of the BPS condition being fulfilled is that the first term in Eq. (12) becomes zero, i.e. in other words, the equations of motion become first order.
order, which represents a significant simplification from the numerical point of view. Further details of this can be found in Ref. [1].

In particular, for the TVY model we are working with, the BPS equations are

$$
\mathcal{K}_{SS} \partial_z \bar{S} = e^{i\gamma} \frac{\partial \mathcal{W}}{\partial S},
$$

$$
\mathcal{K}_{MM} \partial_z \bar{M}_i = e^{i\gamma} \frac{\partial \mathcal{W}}{\partial M}_i.
$$

In order to simplify the analysis, from now on we will work with a Kähler potential which is canonical for the dimension one matter fields.

For the case of degenerate windings, which we will analyze next, this is equivalent to taking $\text{Tr}(MM)^{1/2}$ modulo factors of $\sqrt{N_f}$. The case of non-degenerate windings is slightly more involved and will be fully discussed in the future. Also we choose $\gamma = -\frac{1}{2}(\delta + \pi) = -\frac{4\pi}{N_f} - \frac{\pi}{2}$, where we stress again, $k$ is the sum of all the windings associated with the matter fields. From now on, in order to find solutions to these equations, we adopt the following parametrization

$$
S(z) = |S_i| R(z) e^{i\beta(z)}
$$

$$
M_i(z) = |M_i| \rho_i(z) e^{i\alpha_i(z)},
$$

where $R$, $\beta$, $\rho_i$, and $\alpha_i$ are real functions of the coordinate $z$. Note that the moduli of the fields, $R$ and $\rho_i$, are normalized to one at the vacua. The boundary conditions, when going from vacuum $j$ to vacuum $j + 1$, are

$$
S \rightarrow S e^{i2\pi \frac{1}{N_f}} , \quad M_i \rightarrow M_i e^{i2\pi (\frac{1}{N_f} - \omega_i)}. \quad (16)
$$

We also have a constraint, which is a consequence of the equations of motion, Eqs (14), namely

$$
\text{Im}(e^{i\gamma} \mathcal{W}(S, M_i)) = \text{constant}, \quad (17)
$$

which, evaluated at the centre of the domain wall (i.e. $z = 0$) becomes

$$
- R_0 \left[ \left( 1 - \frac{N_f}{N_c} \right) (\ln R_0 - 1) + \frac{1}{N_c} \sum_i \ln \rho_i \right] - \frac{1}{N_c} \sum_i (-1)^{\omega_i} \rho_i = \cos \left( \pi \frac{k}{N_c} \right). \quad (18)
$$

This constraint will turn out to be extremely relevant to the understanding of the results we are presenting in the next section.

4. Results

4.1. Equal windings

To start the discussion of the results, we are going to present those corresponding to models where the windings $\omega_i$ are all equal to one (remember that, in such case, $k = N_f$). This is the standard choice that other authors in the literature have made, and it is therefore the one we should consider in order to compare with previous results. In this case the constraint given by Eq. (18) becomes

$$
- R_0 \left[ \left( 1 - \frac{N_f}{N_c} \right) (\ln R_0 - 1) + \frac{N_f}{N_c} \ln \rho_0 \right] - \frac{N_f}{N_c} \rho_0 = \cos \left( \pi \frac{N_f}{N_c} \right). \quad (19)
$$

Let us summarize the results. The case $N_f = N_c - 1$ was analyzed by Smilga and Veselov in a series of very interesting papers [1], and recently reanalyzed by Binosi and ter Veldhuis [11]. They found that the TVY model had BPS-saturated domain walls up to a certain value of the mass of the chiral matter condensate, $m_*$. Between $m_*$ and another, higher, value of the mass, $m_{**}$, they found domain wall solutions which were not BPS-saturated. Finally, above $m_*$ there were no solutions at all. The values of $m_*$, $m_{**}$ seem to depend on the value of $N_c$, and they decrease when the latter increases. Moreover there seem to be two branches of BPS-saturated solutions which merge at $m_*$. These results are very well illustrated by Fig. 1 of the third paper in Ref. [11].

A couple of years ago, two of us [12] analyzed the case of $N_f = 1$ flavour. We found that there were BPS-saturated solutions for any value of the mass parameter $m$. This can be seen in Fig. 1 where we plot the magnitude of the gaugino condensate, $R$, versus the spatial coordinate, $z$, for several values of $m$. As the mass increases it can be seen that the profiles tend to a unique one, represented by the thick line. We shall return to that point later on. These results were subsequently confirmed by the work of Ref. [11]. In general it
is possible to make the following statement:

- for \( \frac{N_f}{N_c} \geq \frac{1}{2} \) there are BPS-saturated domain walls only up to \( m_* \)
- for \( \frac{N_f}{N_c} < \frac{1}{2} \) there are BPS-saturated domain walls for any value of \( m \)

Also, a more thorough study of the \( \frac{N_f}{N_c} < \frac{1}{2} \) case [13] leads us to claim that, for \( N_f > 1 \), these domain walls will cross the logarithmic branch and therefore we are uncertain of their physical meaning. We illustrate this point in Fig. 2, where we plot the phase of the logarithm that appears in the second equation of Eqs (7) as a function of the spatial coordinate, \( z \), for a TVY model with \( N_c = 5 \) and \( N_f = 1, m = 1000 \) (dashed) and \( N_f = 2, m = 200 \) (solid).

Figure 1.
Plot of the magnitude of the gaugino condensate, \( R \), as a function of the spatial coordinate, \( z \), for \( N_c = 3, N_f = 1 \) and different values of the mass of the matter condensate: \( m = 2 \) (dotted), \( m = 20 \) (dash-dotted), \( m = 100 \) (dashed), \( m = 200 \) (solid).

Figure 2.
Plot of the phase of the logarithm of the TVY superpotential (see Eq. (7)), in units of \( 2\pi \), as a function of the spatial coordinate, \( z \), for the TVY model with \( N_c = 5 \) and \( N_f = 1, m = 1000 \) (dashed) and \( N_f = 2, m = 200 \) (solid).

It immediately follows that the evolution of the phases of the fields from \( z = 0 \) towards the vacuum, when \( m \) is large, follows a very precise pattern: first, the phases of the matter condensates, \( \alpha_i \), change, more or less at the edge of the centre of the domain wall (defined by \( z = \pm 1/m \)); after that, the gaugino condensate phase, \( \beta \), starts to change as well, to compensate the previous changes in such a way that the phase of the logarithm cancels again at the vacuum. This can be formulated by saying that, when \( z = 1/m \), the different phases are given by

\[
\begin{align*}
\beta_0 &= \frac{k}{2N_c}, \\
\alpha_{01} &= \frac{k}{2N_c} - \frac{\omega_1}{2}.
\end{align*}
\]

where, as usual, \( k = \sum_i \omega_i \). Note that, by definition, the phase of the logarithm (given by \((N_c - N_f)\beta + \sum_i \alpha_i \), modulo \( 2\pi \)) cancels here.

This point is actually straightforward to realize once the large mass regime has been explained, see Eq. (24).
This is the point at which the phase of the logarithm will acquire its biggest absolute value, before it starts decreasing again. This is given by the changes produced both at $\pm 1/m$, i.e.

$$\frac{\Delta(\text{phase log})}{2\pi} \leq 2\frac{|k|}{2} = |k|.$$  \hfill (22)

Therefore, in order to have a change of the phase less or equal to $2\pi$, $|k| = 1$ (remember that $k$ must be an integer).

In practice it means that, for equal windings, only models with $N_f = 1$ will give domain walls profiles with no crossing of the branch. This implies a series of conceptual problems which are beyond the scope of this talk, but which should certainly be addressed at some point.

Now that the main result has been presented, let us try to understand why the different situations arise. In order to do that, we consider the cases where this model could be described by just one of the two condensates, i.e. either by the gaugino or by the matter condensate. This corresponds to the limits of very small and very big mass, respectively.

- For small masses (i.e. $m \ll \Lambda$) one can integrate out the gaugino condensate $S$ and describe the theory in terms of the matter condensate $Q\bar{Q}$ only. The resulting model accounts for the results we obtained for $N_f = 1$ \hbox{[12]} and also for the upper branch of the $N_f = N_c - 1$ case, studied by Smilga et al \hbox{[10]}.

One can easily check that the second, lower branch present in this latter case corresponds to $S \sim 0$. If we have a look at the constraint equation (19) evaluated at $R_0 \sim 0$ the result is

$$\rho_0 = -\frac{N_c}{N_f} \cos \left( \frac{\pi N_f}{N_c} \right).$$  \hfill (23)

The immediate conclusion one can draw is that there will only exist domain wall solutions with $\rho_0 > 0$ if and only if $\frac{N_f}{N_c} > 1/2$. It has been argued in Ref. \hbox{[14]} that the existence of this lower branch is directly connected to the fact that the Kähler metric for the $S$ field is, in this case, singular. In that case, this lower branch would be nothing but an artifact due to the choice of metric. It should also be mentioned here that, according to Ref. \hbox{[1]}, the total number of BPS-saturated domain walls for a given value of $N_f/N_c$ can only change by a multiple of two, when $m$ is continuously varied, and that is the reason why the upper and lower branches end up collapsing into a non-BPS domain wall above $m_c$. However this does not explain why they have to collapse at all.

- For large masses (i.e. $m \gg \Lambda$) one can integrate out the matter condensate $(Q\bar{Q})$ and our model will be described by the gaugino condensate only (i.e. we recover SUSY gluodynamics).

In order to do that, it is worth noticing that, in this large mass regime, $M \sim S$, and we can do the following identifications

$$\rho(z) e^{i\alpha(z)} = R(z) e^{i\beta(z)} , \quad z \ll -1/m$$  \hfill (24)

$$\rho(z) e^{i\alpha(z)} = R(z) e^{i(\beta(z)-2\pi)} , \quad z \gg 1/m .$$

That is, to the left of the centre of the domain wall, both condensates behave in the same way whereas, to the right of it, their phase difference is $2\pi$. This is illustrated in Fig. 3, where we plot the phase difference between the two condensates (in units of $2\pi$) as a function of the spatial coordinate, for several values of the mass parameter $m$. One can also analyze the constraint equation (10) in this large mass limit, which will be given in terms of the modulus of the condensate $R_0$ as

$$R_0 (1 - \ln R_0) = \cos \left( \pi \frac{N_f}{N_c} \right).$$  \hfill (25)

A quick glance at this equation tells us that, in order to have $R_0 < 1$, which corresponds to finite-energy, well-defined domain walls, we must be in the case where $\frac{N_f}{N_c} < 1/2$. In other words, it does not seem possible to reach the $m \to \infty$ limit, continuously from small $m$, in a model where $\frac{N_f}{N_c} > 1/2$. The fact that, in that case, there are no BPS-saturated domain walls at large $m$, whereas we had two branches of solutions at small $M$ explains why, at some point, those must have annihilated each other at $m_c$.

On the other hand, Eq. (23) is telling us that there is an analytic limit $m \to \infty$ limit that we can construct when $N_f/N_c < 1/2$; using Eq. (24) we can write down two BPS equations for $S$, one for the right hand side of the domain wall and another one for the left hand side. The solution is given by the thick line in Fig. 1 and, as we
can see, coincides very well with what we would expect for the large \( m \) limit by looking at the finite mass results.

4.2. Different windings

Now that we have presented and explained most of the results in the literature, let us step onto new ground. As mentioned above, up to now the standard procedure was to consider all the matter fields transforming in the same way when going from one vacuum of the theory to another. We shall now break this degeneracy and assign different winding numbers to the different matter fields. At this stage we present results concerning a particular example, that of \( N_c = 3 \) and \( N_f = 2 \), while a more detailed and general analysis will be presented elsewhere [13].

According to the results of the previous subsection, when \( N_c = 3 \) and \( N_f = 2 \), then \( N_f/N_c > 1/2 \) and we should expect two branches of BPS-saturated domain walls that annihilate each other at \( m_* \) (see Ref. [1]). And that is indeed what happens when \( \omega_1 = \omega_2 = 1 \). However, if we consider a different choice for the windings, i.e. \( \omega_1 = 1, \, \omega_2 = 0 \), then it turns out that one can construct BPS-saturated domain walls for any value of the mass \( m \). The profiles of these fields can be seen in Fig. 4, where we plot the Argand diagram for \( S, M_1 \) and \( M_2 \) with masses \( m_1 = m_2 = 250 \). There it can be seen that the matter condensate with \( \omega_1 = 1 \) follows a different path than the one with no winding, which simply mimics the gaugino condensate, \( S \); and it is exactly the same path described by the matter field for \( N_c = 3, \, N_f = 1 \). Actually it is possible to increase the value of \( m_2 \) while keeping \( m_1 \) finite and constant, in order to integrate out the second condensate and recover the results we obtained for \( N_c = 3, \, N_f = 1 \). The key point to perform this exercise is to notice that the relevant ratio that classifies the solutions is not \( N_f/N_c \) but \( k/N_c \) where, as defined before, \( k = \sum_i \omega_i \). In the previous case, \( k = N_f \), whereas now we have \( N_f = 2 \) with \( k = 1 \).

Therefore it should be in principle possible to construct BPS-saturated domain walls for any value of \( N_f, \, N_c \), just by choosing the windings.
of the matter fields in such a way that their sum, \( k \), over the number of colours, \( N_c \), is less than \( 1/2 \). However it must be pointed out once again that, only when \( |k| \leq 1 \), the paths described by the different fields between two vacua do not cross the logarithmic branch. Those are the only physically meaningful domain walls that we know, so far, how to construct.

5. Conclusions

Let us summarize first the results obtained, before discussing what would be interesting to pursue in order to improve our understanding of these objects. We have looked in detail at domain walls in SUSY-QCD, as a way of understanding SUSY gluodynamics, the nearest relative to ordinary QCD. Using the TVY effective Lagrangian, we have constructed BPS-saturated domain walls for certain values of \( N_f \), the number of matter condensates, and \( m \), the mass of those condensates. The results so far obtained can be summarized as follows:

- All matter fields transforming in the same way (equal windings)
  - If \( N_f/N_c < 1/2 \) there are BPS-saturated domain walls for any value of \( m \).
  - If \( N_f/N_c \geq 1/2 \) there are BPS-saturated domain walls up to \( m_\ast \).
  - The logarithmic branch of the scalar potential is not crossed only when \( N_f = 1 \).

- Non-degenerate flavours (different windings): any choice of \( \omega_i \) such that \( k = \sum_i \omega_i = 1 \) gives BPS-saturated solutions for any \((N_f,N_c)\) and \( m \) with no crossing of the branch.

- The analysis of the constraint \( \text{Im}(e^{i\gamma}W) = \text{const} \) at the origin \( (z = 0) \) is the key to understand the results.

So, at least, we have now a criterion to construct well-behaved BPS-saturated domain walls but, of course, this project is far from over. The next issue that should be addressed is that of the branches associated with the logarithm in the potential. As mentioned at the end of section 2, several suggestions have been put forward but none has so far provided us with a satisfactory answer.

Another issue which is rather controversial is the dependence of these results on the choice of Kähler metric. In fact it has been widely claimed in the literature that the origin of the lower branch found in Refs. [10] is a Kähler metric which blows up near the centre of those solutions. This is not a problem that one encounters in theories with higher supersymmetries, where the Kähler function is totally determined by the symmetries of the theory. Therefore an obvious way of trying to understand this dependence is through the connection of the \( N = 1 \) SUSY constructions here presented with those attempted with higher supersymmetries (for example, those of Refs. [14,15]).

Finally it would be desirable to connect the results obtained here with constructions of BPS-saturated domain walls in the large \( N_c \) limit in the context of SUSY gluodynamics, which have been performed in Ref. [4]. As mentioned in the introduction, these objects could play a very important role in the D-brane description of SUSY-QCD.

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