ININSTANTONS AND CHIRAL ANOMALY IN FUZZY PHYSICS

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Abstract

In continuum physics, there are important topological aspects like instantons, $\theta$-terms and the axial anomaly. Conventional lattice discretizations often have difficulties in treating one or the other of these aspects. In this paper, we develop discrete quantum field theories on fuzzy manifolds using noncommutative geometry. Basing ourselves on previous treatments of instantons and chiral fermions (without fermion doubling) on fuzzy spaces and especially fuzzy spheres, we present discrete representations of $\theta$-terms and topological susceptibility for gauge theories and derive axial anomaly on the fuzzy sphere. Our gauge field action for four dimensions is bounded by a constant times the modulus of the instanton number as in the continuum.

1 Introduction

Conventional discretizations of quantum fields on a manifold $\mathcal{M}$ replace the latter by a lattice of points. An alternative discretization which leads to fuzzy physics treats $\mathcal{M}$ as a phase space and quantizes it. $\mathcal{M}$ is thereby altered to a 'fuzzy' manifold $\mathcal{M}_F$. Earliest investigations of quantization like those of Planck and Bose show that quantization introduces a short distance cut-off: the number of states in a phase space volume $V$ is reduced from infinity to $V/\hbar^3$ if $\hbar$ plays the role of Planck’s constant. If $\mathcal{M}$ is compact, the total number of states is also finite and we end up with a matrix model for $\mathcal{M}$. Continuum physics in this approach has to do with the ‘classical’

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limit $\hbar \to 0$. Functions on $\mathcal{M}$ commute, but they become noncommutative on quantization. For that reason, the fuzzy path lands us in noncommutative manifolds and their geometries. As there are also reasonably orderly methods to formulate quantum field theories (QFT’s) on $\mathcal{M}_F$, fuzzification promises to be a truly original development in discrete physics.

The earliest contributions to topological features of fuzzy physics came from Grosse, Klimčík and Prešnajder. They dealt with monopoles and chiral anomaly for the fuzzy two-sphere $S^2_F$ and took particular advantage of supersymmetry. Later we further elaborated on their monopole work and also developed descriptions of fuzzy $\sigma$-models and their solitons using cyclic cohomology in an important manner. An attractive feature of this cohomological approach is its ability to write analogues of continuum winding number formulae and derive a fuzzy Belavin-Polyakov bound.

The work of is extended in this paper to gauge theories and chiral anomalies. Instantons, the $\theta$-term and “topological susceptibility” are in particular formulated on $S^2_F$. They are not like the existing proposals in discrete physics (cf. the work of Lüscher reported in ref.), have sound and rugged interpretations and promise to resolve problems of much age. As for chiral fermions and anomaly on $S^2_F$, it is known that fuzzy physics requires no “fermion doubling” for Watamura’s Dirac operator. Unfortunately it has zero modes, also there is another Dirac operator which gives a much better approximation to the continuum spectrum. But it does not have a chirality operator because of its highest frequency mode. This mode recedes to infinity and is totally unimportant in the continuum limit. In it is projected out and thereafter a chirality operator anticommuting with this Dirac operator is found without fermion doubling. This paper adapts this operator to gauge theories and instanton physics and also derives the axial anomaly. An alternative analysis of the latter can be found in Grosse et al. We will briefly review their work and compare it with ours in the final section. An interesting new treatment of chiral fermions and anomaly is also discussed in.

Not all manifolds $\mathcal{M}$ can be phase spaces. $\mathcal{M}$ has to be symplectic and hence even dimensional. It must be quantizable to turn into $\mathcal{M}_F$, and ideally $\mathcal{M}$ must be compact and $\mathcal{M}_F$ admit a Laplacian and Dirac operator with decent symmetry properties. Manifolds with these nice features are quantizable coadjoint orbits of compact Lie groups. For simple and semi-simple Lie groups, they are also adjoint orbits. Examples are $\mathbb{C}P^N$ and $S^2 \times S^2$ ($S^2$ being $\mathbb{C}P^1$). $T^2$ is not in this category. We focus on $S^2$ and
\(S^2 \times S^2\) in this paper as the basics of \(S^2\) are under reasonable control. \(\mathbb{CP}^2\) has been treated by [18] while our own approach to \(\mathbb{CP}^N\) and other orbits is yet to appear in print. Using \(S^2\) and \(S^2 \times S^2\) as examples, we present our considerations in such a manner that they can be easily be adapted to general adjoint orbits once their fuzzy basics are assumed.

2 The Fuzzy Sphere

A sphere \(S^2\) is a submanifold of \(\mathbb{R}^3\):

\[
S^2 = \{ \vec{n} \in \mathbb{R}^3 : \sum_{i=1}^{3} n_i^2 = 1 \}.
\]  

(2.1)

If \(\hat{n}_i\) are the coordinate functions on \(S^2\), \(\hat{n}_i(\vec{n}) = n_i\), then \(\hat{n}_i\) commute and the algebra \(A\) of smooth functions they generate is commutative.

In contrast, the operators \(x_i\) describing \(S^2_F\) are noncommutative:

\[
[x_i, x_j] = \frac{i \epsilon_{ijk} x_k}{l(l+1)^{1/2}}, \quad \sum_{i=1}^{3} x_i^2 = 1, \quad l \in \left\{ \frac{1}{2}, 1, \frac{3}{2}, \ldots \right\}.
\]  

(2.2)

The \(x_i\) approach \(\hat{n}_i\) as \(l \to \infty\). If \(L_i = [l(l+1)]^{1/2} x_i\), then \([L_i, L_j] = i \epsilon_{ijk} L_k\) and \(\sum L_i^2 = l(l+1)\) so that \(L_i\) give the irreducible representation (IRR) of \(SU(2)\) Lie algebra for angular momentum \(l\). \(L_i\) or \(x_i\) generate the algebra \(A = M_{2l+1}\) of \((2l+1) \times (2l+1)\) matrices.

Scalar wave functions on \(S^2\) come from elements of \(A\). In a similar way, elements of \(A\) assume the role of scalar wave functions on \(S^2_F\). A scalar product on \(A\) is \(\langle \xi, \eta \rangle = Tr \xi \eta^\dagger\). \(A\) acts on this Hilbert space by left- and right-multiplications giving rise to the left and right-regular representations \(A^{L,R}\) of \(A\). For each \(a \in A\), we thus have operators \(a^{L,R} \in A^{L,R}\) acting on \(\xi \in A\) according to \(a^L \xi = a \xi, a^R \xi = \xi a\). [Note that \(a^L b^L = (ab)^L, a^R b^R = (ba)^R\]. We assume by convention that elements of \(A^L\) are to be identified with functions on \(S^2\).

Of particular interest are the angular momentum operators. There are two kinds of angular momenta \(L_i^{L,R}\) for \(S^2_F\), while the orbital angular momentum operator, which should annihilate \(1\) is \(\mathcal{L}_i = L_i^L - L_i^R\). \(\mathcal{L}\) plays the role of the continuum \(-i(\vec{r} \times \vec{\nabla})\). The “position” operators are not proportional to \(\mathcal{L}_i\), but are instead \(L_i^L / [l(l+1)]^{1/2}\).
The elements of $A$ have a dual role, one as members of a Hilbert space and the second as operators on this space. We shall hereafter most often denote these Hilbert space vectors and their duals in a ket-bra notation to minimize confusion. Thus $|\eta\rangle$ is the Hilbert space vector for $\eta \in A$, $\langle \xi | \eta \rangle = Tr \xi^\dagger \eta$, $a^L |\xi\rangle = |a\xi\rangle$ and $a^R |\xi\rangle = |\xi a\rangle$.

There are two Dirac operators $D_\alpha$ on $S^2$ that are of particular importance to us:

$$D_1 = \bar{\sigma} [-i(\vec{r} \times \vec{\nabla})] + 1, \quad (2.3)$$

$$D_2 = -\epsilon_{ijk} \sigma_i \hat{n}_j J_k, \quad (2.4)$$

where

$$J_k = [-i(\vec{r} \times \vec{\nabla})]_k + \sigma_k/2 = \text{Total angular momentum operators.} \quad (2.5)$$

There is a common chirality operator $\Gamma$ anticommuting with both:

$$\Gamma = \bar{\sigma} \hat{n} = \Gamma^\dagger, \quad \Gamma^2 = 1, \quad (2.6)$$

$$\Gamma D_\alpha + D_\alpha \Gamma = 0. \quad (2.7)$$

$D_\alpha$ and $\Gamma$ act on spinors $\psi = (\psi_1, \psi_2)$ where $\psi_i \in A$. Also these Dirac operators in the continuum are unitarily equivalent,

$$D_2 = \exp (i\pi \Gamma/4) D_1 \exp (-i\pi \Gamma/4) \quad (2.8)$$

and have the spectrum

$$\text{Spectrum of } D_\alpha = \{|\pm (j + 1/2) : j \in \{1/2, 3/2, \ldots \}\}, \quad (2.9)$$

where $j$ is total angular momentum (spectrum of $\vec{J}^2 = \{j(j + 1)\}$). There is a circle of possibilities $\{e^{(i\theta \Gamma/2)} D_1 e^{(-i\theta \Gamma/2)}\}$, in which the operators $D_\alpha$ are just two points.

The discrete versions of $D_\alpha$ are

$$D_1 = \bar{\sigma} \vec{L} + 1, \quad (2.10)$$

$$D_2 = -\epsilon_{ijk} \sigma_i a^L_j J_k = \epsilon_{ijk} \sigma_i x^L_j L^R_k, \quad (2.11)$$

where

$$J_k = \vec{L}_k + \sigma_k/2 = \text{Total angular momentum operators.} \quad (2.12)$$
The $D_{\alpha}$ are no longer unitarily equivalent, their spectra being

\[
\text{Spectrum of } D_1 = \left\{ \pm \left( j + \frac{1}{2} \right) : j \in \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, 2l - \frac{1}{2} \right\} \right\} \\
\cup \left\{ (j + \frac{1}{2}) : j = 2l + \frac{1}{2} \right\},
\]

(2.13)

\[
\text{Spectrum of } D_2 = \left\{ \pm \left( j + \frac{1}{2} \right) \left[ 1 + \frac{1 - (j + 1/2)^2}{4l(l + 1)} \right]^{1/2} : \\
j \in \left\{ \frac{1}{2}, \frac{3}{2}, \ldots, 2l - \frac{1}{2} \right\} \cup \left\{ 0 : j = 2l + \frac{1}{2} \right\} \right\}.
\]

(2.14)

$j$ once more is total angular momentum, the spectrum of $\vec{J}^2$ being \{j(j + 1)/2\}.

The first operator has been used extensively by Grosse et al \[3–5\] while the second was first introduced by the Watamuras \[6, 7\].

The Hilbert space $A$ has naturally to be enhanced to $A^2 = A \otimes \mathbb{C}^2$ once the Dirac operator comes into the picture. It is spanned by vectors \{|$\xi$\} : $\xi = (\xi_1, \xi_2), \xi_i \in A$\} with the scalar product \langle $\xi$|$\eta$\rangle = $\text{Tr}$$\xi_i^\dagger \eta_i$. $A^{L,R}$ act in an obvious manner on this space, $a^L|$ $\xi$\rangle = $|a\xi$\rangle, $a^R|$ $\xi$\rangle = $|\xi a$\rangle, while $\sigma$ $|$ $\xi$\rangle is just $|\sigma_i\xi$\rangle. Here $(a\xi)_r = a\xi_r, (\xi a)_r = \xi_r a$ and $(\sigma_i\xi)_r = (\sigma_i)_r s\xi_s$.

It is easy to derive (2.13) by writing

\[
D_1 = \left( \vec{L} + \frac{\vec{\sigma}}{2} \right)^2 - \vec{L}^2 - \left( \frac{\vec{\sigma}}{2} \right)^2 + 1,
\]

(2.15)

\[
\left( \frac{\vec{\sigma}}{2} \right)^2 = \frac{3}{4} 1.
\]

(2.16)

Then for $\vec{L}^2 = k(k + 1), k \in \{0, 1, \ldots, 2l\}$, if $j = k + 1/2$ we get $+(j + 1/2)$ as eigenvalue, while if $j = k - 1/2$ we get $-(j + 1/2)$. The absence of $-(2l + 1/2)$ in (2.13) is just because $k$ cuts off at $2l$.

It is remarkable that (2.13) coincides exactly with those of $D_{\alpha}$ up to $j = (2l - 1/2)$. So $D_1$ is an excellent approximation to $D_{\alpha}$.

But $D_1$ as it stands admits no chirality operator unless its eigenspace with top eigenvalue is treated as a chiral singlet, or better still is projected out. Then it does admit a chirality operator as shown in detail in [17]. We
can summarize the results of \[17\] as follows. Define the operators
\[
\epsilon_\alpha = \frac{D_\alpha}{|D_\alpha|}, \quad |D_\alpha| = \text{Positive square root of } D^2_\alpha,
\]
on the subspace \(V\) with \(j \leq 2l - 1/2\),
\[
= 0 \quad \text{on the subspace } W \text{ with } j = 2l + 1/2.
\]
(2.17)

Then \(\epsilon_1, \epsilon_2\) and \(i\epsilon_1\epsilon_2\) anticommute, are hermitian and square to the projection operator \(P\) where
\[
P\xi = \xi, \quad \xi \in V,
\]
\[
= 0, \quad \xi \in W.
\]
(2.18)

Furthermore they all commute with \(|D_\alpha|\) so that \(i\epsilon_1\epsilon_2\) is a chirality operator for \(D_\alpha\).

As for \(D_2\), its spectrum (2.14) has been calculated in \[6,7\]. It has the following chirality too, anticommuting with \(D_2\), commuting with \(\vec{J}\) and squaring to 1 in the entire Hilbert space \(V \oplus W = A^2\):
\[
\gamma_2 = \gamma^\dagger_2 = -\frac{\vec{\sigma} \vec{L}^R - 1/2}{l + 1/2},
\]
(2.19)
\[
\gamma^2_2 = 1.
\]
(2.20)

As shown in \[17\], on \(V\), it is a linear combination of \(\epsilon_1\) and \(i\epsilon_1\epsilon_2\) for each fixed \(\vec{J}^2 \equiv J^2\):
\[
P\gamma_2 P = e^{(i\theta(J^2)\epsilon_2)/2}(i\epsilon_1\epsilon_2)e^{(-i\theta(J^2)\epsilon_2)/2},
\]
\[
= \cos \theta(J^2)(i\epsilon_1\epsilon_2) + \sin \theta(J^2)\epsilon_1.
\]
(2.21)

The angle \(\theta\) is rotationally invariant and vanishes as \(l \to \infty\). In that limit, \(\gamma_2\) and \(P\gamma_2 P\) approach the continuum \(\Gamma\) as they should.

The operator \(\gamma_2\) has the important property that it commutes with \(x_i^L\). That is not the case with \(i\epsilon_1\epsilon_2\). It is thus useful to replace \(D_1\) by
\[
e^{(i\theta(J^2)\epsilon_2)/2}D_1e^{(-i\theta(J^2)\epsilon_2)/2}.
\]

Noticing also that \(P\) is a function of only \(\vec{J}^2\) and that \(\vec{J}^2\) commutes with \(\gamma_2\) and \(D_\alpha\), we construct our basic operators
\[
D = e^{(i\theta(J^2)\epsilon_2)/2}(PD_1 P)e^{(-i\theta(J^2)\epsilon_2)/2},
\]
(2.22)
\[
F = e^{(i\theta(J^2)\epsilon_2)/2}\left(P\frac{D_1}{|D_1|} P\right)e^{(-i\theta(J^2)\epsilon_2)/2},
\]
(2.23)
and
\[ \gamma = P\gamma_2 P. \]  
(2.24)

They are all zero on \( W \) and leave its orthogonal complement \( V \) invariant. They have the following additional fundamental properties on \( V \):

(i) They are self-adjoint:
\[ D^\dagger = D, \quad F^\dagger = F, \quad \gamma^\dagger = \gamma. \]  
(2.25)

(ii) \( F \) and \( \gamma \) square to 1 on \( V \):
\[ F^2 = \gamma^2 = P. \]  
(2.26)

(iii) \( \gamma \) anticommutes with \( F \) and \( D \):
\[ \{\gamma, F\} = \{\gamma, D\} = 0. \]  
(2.27)

So \( \gamma \) is a chirality operator on \( V \). There is no fermion doubling, and \( D \) is an excellent approximation to \( D_\alpha \).

### 3 The Projector \( P \) and the Operator Algebra

Our presentation above must have betrayed our intention to project all to the subspace \( V \). But we can consistently do so without disturbing the mathematical formalism (even by small amounts) only if no operator we deal with has matrix elements between \( V \) and \( W \). We must therefore work only with operators commuting with \( P \).

This criterion is satisfied by \( D, F \) and \( \gamma \), but not by \( a^L \) and \( a^R \). As we cannot avoid their use, we correct them to commute with \( P \).

Let
\[ \Gamma(P) = 2P - 1, \quad \Gamma(P)^2 = 1. \]  
(3.1)

Then to any operator \( \alpha \), we can associate another:
\[ P(\alpha) = \alpha + \frac{1}{2}\Gamma(P)[\alpha, \Gamma(P)] = \frac{1}{2}[\alpha + \Gamma(P)\alpha\Gamma(P)]. \]  
(3.2)

It commutes with \( \Gamma(P) \),
\[ P(\alpha)\Gamma(P) = \Gamma(P)P(\alpha) \]  
(3.3)
and its projections to $V$ and $W$ coincide with those of $\alpha$:

$$PP(\alpha)P = P\alpha P, \quad (1 - P)P(\alpha)(1 - P) = (1 - P)\alpha(1 - P).$$

(3.4)

We can think of $\frac{1}{4}\Gamma(P)[\alpha, \Gamma(P)]$ as the connection canonically extending the action of $\alpha$ from $V \oplus W$ to $V$ and $W$.

It is interesting that

$$P(\alpha)P(\beta) - P(\alpha\beta) = -\frac{1}{4}\Gamma(P)[\alpha, \Gamma(P)]\Gamma(P)[\beta, \Gamma(P)]$$

$$= \frac{1}{4}[\alpha, \Gamma(P)][\beta, \Gamma(P)]$$

(3.5)

where the right-hand side commutes with $\Gamma(P)$. It vanishes certainly as $l \to \infty$.

In future we will change operators like $a^{L,R}$ not commuting with $P$ to their $P$-images. Let $P(A^{L,R})$ be the algebra generated by $P(a^{L,R})$. From (3.5), we see that they will contain small operators with no pre-image in $A^{L,R}$ and going to zero as $l \to \infty$.

## 4 Connections and Curvatures on $S^2_F$

Connes’ approach to gauge theories is based on spectral triples and $K$-cycles [9–14]. Fredholm modules and cyclic cohomology [3] seem better suited for fuzzy physics, especially for maintaining continuum topological features like instanton bounds and $\theta$-states. We have already seen this in work on fuzzy monopoles [15], and give further supporting evidence in this paper. Later on, we will comment on the continuum limit of this approach, its full details being reserved to future work [21].

Alternative approaches to gauge theories on the fuzzy sphere have also been developed by Grosse and Prešnajder [4] and Klimčík [22].

In mathematics, Fredholm modules and cyclic cohomology have played a central role in the $K$-theory of algebras for many years [4], while Rajeev and coworkers [23, 24] have long appreciated their importance in the physics of large-$N$ gauge theories. Here, we suggest their importance for fuzzy physics too.

We first introduce the concept of forms and then go on to formulate gauge theories. All that we now do can be restricted to $V$. They are also suitable
for adaptation to general fuzzy spaces. However, the treatment of instantons and monopoles is being postponed till Section 6.

4.1 Forms

These are constructed using $F, \gamma$ and $P(A^L)$. For this work, in addition to (2.27, 2.26, 2.27), it is important that

$$[\gamma, P(a^L)] = [\gamma, a^L + \frac{1}{2} \Gamma(P)[a^L, \Gamma(P)]] = 0. \quad (4.1)$$

This result is implied by the fact that $\gamma$ commutes with both $a^L$ and $P$.

Forms are linear spans of elements

$$\omega_{\lambda} = P(a^L_0) [F, P(a^L_1)] \cdots [F, P(a^L_{\lambda})] \quad (4.2)$$

for all $\lambda$. Their product can also be written as linear combinations of terms like (1.2) using properties of derivations. (For instance, $[F, P(a^L_0)] P(b^L) = [F, P(a^L_0) P(b^L)] - P(a^L_0) [F, P(b^L)]$).

Forms are $\mathbb{Z}_2$-graded by $\gamma$: those $\omega_{\lambda}$ with $\lambda$ even commute with $\gamma$ and give us the space of even forms $\Omega^0[P(A^L)]$ while those $\omega_{\lambda}$ with $\lambda$ odd anticommute with $\gamma$ and give us the space of odd forms $\Omega^1[P(A^L)]$.

We can assign degrees $\partial \omega_{\lambda}$ to $\omega_{\lambda}$ ($=0$ or $1$ if $\lambda = 0$ or $1$) which is additive (mod 2). The degree of $\omega_{\lambda} \omega_{\mu}$ is $\lambda + \mu$ (mod 2).

There is a derivation $d$ between $\Omega^0[P(A^L)]$ and $\Omega^1[P(A^L)]$ which squares to zero. It is given by the graded commutator with $F$:

$$d \omega_{\lambda} = F \omega_{\lambda} - (-1)^{\partial \omega_{\lambda}} \omega_{\lambda} F \equiv \{F, \omega_{\lambda}\}, \quad (4.3)$$

$$d^2 = 0. \quad (4.4)$$

The operator $d$ resembles the differential in manifold theory and leads to a homology theory for $P(A^L)$.

4.2 Connections and Curvatures

Connections $\nabla(\omega)$ and curvatures $\mathcal{F}(\omega)$ depend on “Lie-algebra valued one-forms” $\omega$. For $U(M)$ gauge theories, if $-iT(\alpha)$ ($\alpha = 0, 1, \ldots M^2 - 1$) are $M \times M$ antihermitian matrices spanning the Lie algebra $\underline{U(M)}$ of $U(M)$,
then

\[ \omega = T(\alpha)P(\omega_1^\alpha), \]

\[ P(\omega_1^\alpha) = \sum_j P(a_{\alpha}^{L,j})[F, P(b_{\alpha}^{L,j})], \] (4.6)

\[ P(\omega_1^\alpha) = P(\omega_1^\alpha)^*, \quad P(a_{\alpha}^{L,j}), \quad P(b_{\alpha}^{L,j}) \in P(A^L). \] (4.7)

\( \omega \) is a linear operator on

\[ A^M \otimes \mathbb{C}^2 = \{ |\xi = (\xi_{\lambda j}), \lambda = 1, \ldots M; j = 1, 2, \xi_{\lambda j} \in A \}. \] (4.8)

Here, \( \lambda \) is the “internal symmetry” index on which \( T(\alpha) \) act and \( j \) is the spin index. The scalar product for \( A^M \otimes \mathbb{C}^2 \) is the evident one:

\[ \langle \eta | \xi \rangle = Tr \sum_{\lambda, j} \eta_{\lambda j}^* \xi_{\lambda j}. \] (4.9)

\( \nabla(\omega) \) and \( \mathcal{F}(\omega) \) are the following operators:

\[ \nabla(\omega) = d + \omega, \] (4.10)

\[ \mathcal{F}(\omega) = d\omega + \omega^2 \equiv \{ F, \omega \} + \omega^2. \] (4.11)

[Both act on \( P(A^L)^M \otimes \mathbb{C}^2 \equiv P(A^L)^{M+2} \), with \( d \) being the derivation. \( \mathcal{F}(\omega) \)

is also a linear operator on \( A^M \otimes \mathbb{C}^2 \).]

The gauged version of \( F \) on \( A^M \otimes \mathbb{C}^2 \) is this:

\[ \text{Gauged version of } F = F + \omega. \] (4.12)

The \( U(M) \) gauge group consists of \( M \times M \) unitary matrices \( u \) with \( u_{ij} \in P(A^L) \). They act on \( \nabla(\omega) \) and \( \mathcal{F}(\omega) \) in the usual way:

\[ \nabla(\omega) \rightarrow u\nabla(\omega)u^\dagger, \] (4.13)

\[ \mathcal{F}(\omega) \rightarrow u\mathcal{F}(\omega)u^\dagger. \] (4.14)

We can describe gauge theories for a subgroup \( G \subset U(M) \) by restricting \( \omega \). That is, if \( \tilde{T}(\alpha) \) span the Lie algebra \( \mathfrak{g} \) of \( G \), we can decide to consider only \( \omega = \tilde{T}(\alpha)P(\omega_1^\alpha) \). (But that would not guarantee that \( \mathcal{F}(\omega) \) has an expansion containing only \( \tilde{T}(\alpha) \) since \( P(\omega_1^\alpha) \) and \( P(\omega_1^\beta) \) may not anticommute in the term \( \omega^2 = \tilde{T}(\alpha)\tilde{T}(\beta)P(\omega_1^\alpha)P(\omega_1^\beta) \).) In the same vein, when considering covariant derivative in a representation \( \Gamma \) of \( U(M) \), we change \( T(\alpha) \) to its representative in the Lie algebra \( \mathfrak{g} \) of \( \Gamma \).
5 The Actions and Quantization

5.1 Critical Dimensions

In Connes’ approach, the Euclidean actions for free massless scalars, spinors and gauge fields on a manifold of dimension \( n \) and Dirac operator \( \mathcal{D} \) are respectively,

\[
S(\Phi) = \text{constant } Tr^+ \left[ \frac{1}{[\mathcal{D}]^n} ([\mathcal{D}, \Phi]^\dagger [\mathcal{D}, \Phi]) \right],
\]

\[
S(\Psi) = \text{constant } Tr^+ \left[ \frac{1}{[\mathcal{D}]^n} \Psi^\dagger \mathcal{D} \Psi \right],
\]

\[
S(\hat{\omega}) = \text{constant } Tr^+ \left[ \frac{1}{[\mathcal{D}]^n} ([\mathcal{D}, \hat{\omega}] + \hat{\omega}^2)^\dagger ([\mathcal{D}, \hat{\omega}] + \hat{\omega}^2) \right].
\]

Here \( Tr^+ \) is Dixmier trace, \( n \) is spacetime dimension, and \( \Phi, \Psi \) and \( \hat{\omega} \) are scalar, spinor and gauge fields respectively. [Issues involving “junk forms” are being ignored.]

Under the scaling transformation \( \mathcal{D} \rightarrow \lambda \mathcal{D} \), the response of \( \omega \) is \( \omega \rightarrow \lambda \omega \). Hence under \( \mathcal{D} \rightarrow \lambda \mathcal{D} \),

\[
S(\Phi) \rightarrow \lambda^{2-n} S(\Phi),
\]

\[
S(\Psi) \rightarrow \lambda^{1-n} S(\Psi),
\]

\[
S(\hat{\omega}) \rightarrow \lambda^{4-n} S(\hat{\omega}).
\]

Gauging \( \mathcal{D} \) does not affect (5.4) and (5.5).

The critical dimensions where actions are scale-invariant are thus
\( n = 2 \) for \( \Phi \), \( n = 1 \) for \( \Psi \) and \( n = 4 \) for \( \hat{\omega} \).

We first propose actions for fuzzy scalars and gauge fields in their critical dimensions, \( n = 1 \) being outside our modeling scope.

5.2 \( n = 2 \) Fuzzy Massless Scalar Fields

A fuzzy scalar field \( \phi \) for \( n = 2 \) is a polynomial in \( P(a^L) \). If it has internal degrees of freedom, it is a vector with each component \( \phi_p \) being such a polynomial.

Our (Euclidean) action for a zero mass “non-interacting” fuzzy scalar field \( \phi \) is

\[
S(\phi) = f^2 Tr P[dP(\phi)]^\dagger [dP(\phi)]
\]

(5.7)
where the internal index $\rho$, if any, is summed within the trace. It can be gauged in an evident manner by replacing $d$ by $\nabla(\omega)$. It is scale invariant just like (5.1).

In ref [15], the analogue of this action for fuzzy $\sigma$-models was proposed and its Belavin-Polyakov bound [25] was discussed. As suggested there, we conjecture that as $l \to \infty$, $f^2$ can be scaled in such a way that $S(\phi)$ approaches $S(\Phi)$. Work on this matter is in progress [21].

5.3 $n = 4$ Fuzzy Gauge Field

The formalism of cyclic cohomology and gauge theories depends only on the knowledge of suitable $d$ and chirality operators. For $n = 4$, there are two fuzzy spaces that are susceptible to our analysis, namely $S^2_F \times S^2_F$ and $(\mathbb{C}P^2)_F$. The former is enough for illustration. The algebra for that space is $A \otimes \mathbb{C}A$ while its $d$ and chirality operators are $\frac{1}{\sqrt{2}}(F \otimes 1 + \gamma \otimes C F)$, respectively ($F$ and $\gamma$ being those of $S^2_F$.)

Our proposed action is the evident one:

$$S(\omega) = \frac{1}{2\epsilon^2} Tr P \{ F(\omega) F(\omega) \}.$$  \hspace{1cm} (5.8)

It is accompanied by a conjecture like that for $S[\phi]$ about its $l \to \infty$ limit. It is scale invariant like $S(\hat{\omega})$ of $n = 4$.

For $(\mathbb{C}P^2)_F$ as well, our action looks the same as (5.8). But we have yet to tell what $d$ and the chirality operator are, a task postponed to later work.

5.4 Away from Criticality

Our guide for the choice of actions for any $n$ continues to be scaling properties and gauge invariance. Also in our formulas for general $n$, the precise definitions of $P, D, F$ and $D_1$ require future elucidation.

(a) Scalars
The action for $\phi$ for any $n$ is suggested by (5.4) to be

$$S(\phi) = f^2 Tr P |D|^{2-n}[\nabla(\omega)P(\phi)]^\dagger[\nabla(\omega)P(\phi)]$$ \hspace{1cm} (5.9)

[For $n = 2$, $|D|$ is invertible on $V$.]

(b) The field $g$

The formulation of gauge invariant actions for fuzzy spinor and gauge fields
requires the introduction of a matrix ‘field’ $g$. Its components $g_{ab}$ are polynomials in $P(a^L)$. It is unitary and commutes with $P$:

\[
g^\dagger g = 1, \\
gP = Pg.
\]  
(5.10)  
(5.11)

The index $a$ carries the action of the gauge transformations $u$:

\[
u : g \rightarrow ug
\]  
(5.12)

(c) Spinors

A fuzzy spinor $\psi$ is an element of the Hilbert space on which operators like $\gamma, D$ and elements of $A^{L,R}$ can act. In the presence of internal symmetry, it has components $\psi_{\lambda,j}$ (c.f. 4.8). The gauged action is suggested by (5.5):

\[
S(\psi, g) = \kappa \langle \psi | P g \left( \frac{1}{|D|^{(n-1)/2}} \right) g^\dagger (F + \omega) g \left( \frac{1}{|D|^{(n-1)/2}} \right) g^\dagger P |\psi\rangle.
\]  
(5.13)

It depends on both $\psi$ and $g$.

We can now define a new Dirac field

\[
|\chi\rangle = g \left( \frac{1}{|D|^{(n-1)/2}} \right) g^\dagger |\psi\rangle
\]  
(5.14)

which transforms in the same way as $|\psi\rangle$ under the gauge group,

\[
u : |\chi\rangle \rightarrow u|\chi\rangle,
\]  
(5.15)

but which scales differently:

\[
|\chi\rangle \rightarrow \lambda^{(1-n)/2} |\chi\rangle \quad \text{under} \quad D \rightarrow \lambda D.
\]  
(5.16)

The spinor action is thus

\[
S(\chi) = \kappa \langle \chi | P [F + \omega] P |\chi\rangle
\]  
(5.17)

where $\chi$ scales as in (5.16). (We assume as for $n = 2$ that $[|D|, P] = 0$.)

(d) Gauge Fields

The $n \neq 4$ gauge field action, like (5.13), depends on both $\omega$ and $g$ and reads

\[
S(\omega, g) = \frac{1}{2e^2} TrP \left[ g|D|^{4-n} g^\dagger \right] F(\omega)^2.
\]  
(5.18)
compatibly with \((5.6)\).

There is a certain freedom in the choice of gauge field action. Any one of the following actions are a priori equally acceptable:

\[
S_a(\omega, g) = \frac{1}{2e^2} TrP|D|^a g^+ \mathcal{F}(\omega) |D|^b g^+ \mathcal{F}(\omega), \quad a + b = 4 - n.
\]

\[(5.19)\]

(e) A Remark
In the continuum limit, if \(Tr\) goes over to \(Tr^+\) as we conjecture, we can cancel \(g\) with \(g^+\) in \((5.13), (5.18)\) and \((5.19)\) \([9,12,14]\) and the dependence on \(g\) disappears.

(f) Mass and Interaction Terms
Mass and interaction terms can also be introduced with guidance from continuum scaling properties and from gauge invariance. We omit the simple details.

(g) Quantization
Quantization can be done using functional integration. We can for example try to expand the fields in normal modes and integrate \(\exp(-\text{action})\) over the coefficients in the mode expansions to define the partition function. This method is especially useful for the Dirac field for which the normal modes are given by an orthonormal basis of the Hilbert space and the coefficients of the expansion are Grassmann numbers.

6 On Twisted Bundles and Fuzzy Physics

In the continuum, instantons are particular connection fields \(\omega\) on certain twisted bundles over the base manifold \(\mathcal{M}\). On \(S^2\), they are monopole bundles, on \(S^4\) or \(\mathbb{C}P^2\), they can be \(SU(2)\) bundles. For such reasons, we may henceforth refer to monopoles also as instantons.

In algebraic \(K\)-theory \([14,27]\), it is well-known that these bundles are associated with projectors \(P\). \(P\) is a matrix of some dimension \(M\) with \(P_{ij} \in \mathcal{A} \equiv \mathcal{C}\infty(\mathcal{M})\), \(P^2 = P = P^\dagger\). The physical meaning of \(P\) is the following. Let \(\mathcal{A}^M = \mathcal{A} \otimes \mathcal{C}^M = \{\xi = (\xi_1, \xi_2, \ldots, \xi_M) : \xi_i \in \mathcal{A}\}\). Then \(P\mathcal{A}^M = \{P\xi : \xi \in \mathcal{A}^M\}\) consists of smooth sections (or wave functions) \(P\xi\) of a vector bundle over \(\mathcal{M}\). For suitable choices of \(P\), we get monopole or instanton vector bundles. These projectors are known \([13,27]\) and those for monopoles have been reproduced in \([15]\).
The projectors \( p^{(\pm N)} \) for fuzzy monopoles of charge \( \pm N \) have also been found in \[15\]. They act on \( A^{2N} = \{ \xi \text{ with components } \xi_{b_1...b_N} \in A, b_i \in \{1,2\} \} \). Let \( \tilde{\tau}^{(i)} (i = 1, 2, \ldots N) \) be commuting Pauli matrices. \( \tilde{\tau}^{(i)} \) has the normal action on the index \( b_i \) and does not affect \( b_j \) (\( j \neq i \)). Then \( \tilde{K} = \tilde{L}^L + \sum_i \tilde{\tau}^{(i)} / 2 \) generates the \( SU(2) \) Lie algebra and \( p^{(N)} \) (\( p^{(-N)} \)) is the projector to the maximum (minimum) angular momentum \( k_{\text{max}} = l + N/2 \) \( (k_{\text{min}} = l - N/2) \). \[ \tilde{K}^2 p^{(N)} = k_{\text{max}}(k_{\text{max}} + 1)p^{(N)}, \quad \tilde{K}^2 p^{(-N)} = k_{\text{min}}(k_{\text{min}} + 1)p^{(-N)}. \] Fuzzy analogues of monopole wave functions are \( p^{(\pm N)} A^{2N} \). Explicit expressions for \( p^{(\pm N)} \) may be found in \[15\].

When spin is included, we must enhance \( p^{(\pm N)} A^{2N} \) to \( p^{(\pm N)} A^{2N} \otimes \mathbb{C}^2 = p^{(\pm N)} A^{2N+1} = \{ \xi \text{ with components } \xi_{b_1...b_N,j} \in A : b_i, j \in \{1,2\} \} \).

As for four-dimensional instantons, we do not know their projectors even for \( S^2_F \times S^2_F \), so we shall just assume their existence in what follows. That is enough for the presentation because of its generality.

The discussion below focuses on \( S^2_F \), but one can readily see how to go beyond this space, once the basic ingredients become available.

### 6.1 Cyclic Cohomology of Twisted Sectors

All the complications resolved here are caused by the need to project out a subspace of \( A^{2N+1} \). It is the analogue of the subspace projected out by \( P \) for \( N = 0 \). In its absence, for example in the continuum, there is a canonical way to extend cyclic cohomology to twisted bundles. It is also due to Connes \[9\].

The material being explained now has been partially reported in \[17\]. It is not essential reading in all its details for what follows once it is accepted that a certain subspace of \( A^{2N+1} \) can be consistently projected out.

In the \( N = 0 \) sector, the projector \( P \) cuts out the subspace \( W \) of \( A^2 \). The function of the map \( \alpha \rightarrow P(\alpha) \) of operators is to make them compatible with the splitting \( A^2 = V \oplus W \).

When we pass to \( p^{(\pm N)} A^{2N} \) and thence to \( p^{(\pm N)} A^{2N+1} \) by including spin, the subspace to be projected out is not determined by \( P \) if \( N \neq 0 \), as we shall see below. Rather, we can explain it as follows: Let \( \tilde{J} = \tilde{K} - \tilde{L} \tilde{R} + \tilde{\sigma} / 2 \) be the “total angular momentum”. Calling \( \tilde{J} \) by this name is appropriate as it becomes \( 21 \) for \( N = 0 \) and displays the known “spin-isospin mixing” \[19, 24\] for \( N \neq 0 \). The maximum of \( \tilde{J}^2 \) on \( p^{(\pm N)} A^{2N+1} \) is \( J_{\text{max}}(J_{\text{max}} + 1) \), \( J_{\text{max}} = (l \pm N/2) + l + 1/2 = 2l \pm N/2 + 1/2 \). [We assume that \( 2l \geq (N - 1)/2 \).] The vectors to be projected out are those with total angular momentum \( J_{\text{max}} \). If
\(\mathcal{J}^{(\pm N)}\) are the corresponding projectors [with \(\mathcal{J}^{(0)} = P\)], the twisted space we work with is thus \(\mathcal{J}^{(\pm N)} p^{(\pm N)} A^{2N+1}\). Since \(p^{(\pm N)}\) commute with \(\mathcal{J}\) and hence with \(\mathcal{J}^{(\pm N)}\), \(Q^{(\pm)} = \mathcal{J}^{(\pm N)} p^{(\pm N)}\) are also projectors.

There is no degeneracy for angular momentum \(J_{\text{max}}\) in \(p^{(\pm N)} A^{2N+1}\). That is because there is only one way to couple \(l \pm N/2, l\) and \(1/2\) to \(J_{\text{max}}\). The space \((1 - \mathcal{J}^{(\pm N)}) p^{(\pm N)} A^{2N+1}\) is thus of dimension \(2J_{\max} + 1\). We want to get rid of this subspace.

The operators \(T = D, F\) or \(\gamma\) of Section 2 are zero on \((1 - P) A^2\) where \(P\) cuts out states of angular momentum \(2l + 1/2\). There is no degeneracy for this angular momentum in \(A^2\). \(T\) and \(P\) extend canonically to \(A^2 \otimes \mathbb{C}^{2^N} (\equiv A^{2N+1})\) as \(T \otimes 1\) and \(P \otimes 1\). Let us call them once more as \(T\) and \(P\). \(T\) and \(P\) commute with \(\mathcal{J}\) and hence with \(\mathcal{J}^{(\pm N)}\). There is only one way to couple \(N\) “isospin” 1/2’s to \((2l + 1/2)\) to get \(J_{\text{max}}\) so that \((1 - \mathcal{J}^{(\pm N)})(1 - P) A^{2N+1}\) is also of dimension \(2J_{\text{max}} + 1\). And \(T\) is zero on this subspace.

The projectors \((1 - \mathcal{J}^{(\pm N)}) p^{(\pm N)}\) and \((1 - \mathcal{J}^{(\pm N)})(1 - P)\) being of the same rank, there exists a unitary operator \(U\) on \(A^{2N+1}\) transforming one to the other:

\[
(1 - \mathcal{J}^{(\pm N)}) p^{(\pm N)} = U (1 - \mathcal{J}^{(\pm N)})(1 - P) U^{-1}.
\]

(6.1)

If we transport \(T\) by \(U\),

\[
T' = UTU^{-1},
\]

(6.2)

then \(T' = D', F'\) or \(\gamma'\) vanishes on \((1 - \mathcal{J}^{(\pm N)}) p^{(\pm N)} A^{2N+1}\). On its orthogonal complement \([1 - (1 - \mathcal{J}^{(\pm N)}) p^{(\pm N)}] A^{2N+1}\), invariant under \(T'\), \(F'\) and \(\gamma'\) square to unity and \(\gamma'\) anticommutes with \(D'\) and \(F'\), just as we want. What replaces \(P\) now is not \(\mathcal{J}^{(\pm N)}\), but rather

\[
P^{(\pm N)} = [1 - (1 - \mathcal{J}^{(\pm N)}) p^{(\pm N)}],
\]

(6.3)

\[
P^{(0)} = P.
\]

(6.4)

As \(l \to \infty\), \(J_{\text{max}}\) becomes dominated by \(2l\) and so we have the freedom to let \(U\) approach 1. That is, no \(U\) is needed in the continuum limit.

Since \(p^{(\pm N)}\) define different topological sectors, we also have the freedom to choose different \(U\)'s for these sectors. But as both these sectors come from \(A^{2N+1}\), it is convenient to find a single \(U\) valid for both.

Total angular momentum \(2l + 1/2 + N/2\) has no multiplicity in \(A^{2N+1}\). As both \((1 - \mathcal{J}^{(N)})(1 - P) A^{2N+1}\) and \((1 - \mathcal{J}^{(N)}) p^{(N)} A^{2N+1}\) have this angular
momentum, we have that
\[
(1 - J^{(N)})(1 - P)A^{2N+1} = (1 - J^{(N)})p^{(N)}A^{2N+1}.
\] (6.5)
So we choose
\[
U = 1 \text{ on } (1 - J^{(N)})(1 - P)A^{2N+1}.
\] (6.6)
Next in accordance with (6.1), we set
\[
U(1 - J^{(-N)})(1 - P)A^{2N+1} = (1 - J^{(-N)})p^{(-N)}A^{2N+1}.
\] (6.7)
We also demand that
\[
[U, J_i] = 0.
\] (6.8)
That fixes \(U\) up to a phase on the subspace \((1 - J^{(-N)})(1 - P)A^{2N+1}\).
We saw in [15] that \((1 \pm \gamma)/2\) are projectors for combining \(-\vec{L}^R\) and \(\vec{\sigma}/2\) to give angular momenta \(l \pm 1/2\). So \(\gamma = +1\) on all the subspaces \((1 - J^{(\pm N)})(1 - P)A^{2N+1}\), \((1 - J^{(\pm N)})p^{(\pm N)}A^{2N+1}\). Also, \((\sum \vec{\tau}^{(i)})^2\) is \(\frac{N^2}{3}(\frac{N}{2} + 1)\) on these subspaces. It follows that (6.6), (6.7) and (6.8) are compatible with a \(U\) commuting with \(J_i\), \(\gamma\), \((-\vec{L}^R + \vec{\sigma}/2)^2\) and \((\sum \vec{\tau}^{(i)})^2\). We now outline an extension of \(U\) to all of \(A^{2N+1}\) consistently with rotational invariance (6.8) and
\[
\left[ U, \left(-\vec{L}^R + \frac{\vec{\sigma}}{2}\right)^2 \right] = \left[ U, \left(\sum \vec{\tau}^{(i)}\right)^2 \right] = 0.
\] (6.9)
An important consequence is that
\[
\gamma' = \gamma
\] (6.10)
so that
\[
[\gamma', p^{(\pm N)}] = [\gamma', J_i] = [\gamma', J^{(\pm N)}] = 0.
\] (6.11)
One way to specify \(U\) more fully is as follows. Let
\[
A^{2N+1} = X \oplus X^\perp = X' \oplus X'^\perp
\] (6.12)
be orthogonal decompositions where
\[ X = (1 - \mathcal{J}^{(N)})(1 - P)A^{2N+1} \oplus (1 - \mathcal{J}^{(-N)})(1 - P)A^{2N+1}, \]  
\[ X' = UX = (1 - \mathcal{J}^{(N)})p^{(N)}A^{2N+1} \oplus (1 - \mathcal{J}^{(-N)})p^{(-N)}A^{2N+1}. \]  
(6.13)

Both \( X \) and \( X' \) are invariant under the self-adjoint operators
\[ J, \left(-\vec{L}R + \vec{a}\right)^2 \quad \text{and} \quad \left(\sum \tau(i)\right)^2. \]

Therefore, the same is the case with \( X^\perp \) and \( X'^\perp \). We can extend \( U \) to a map \( X^\perp \rightarrow X'^\perp \) which commutes with the above operators. There would still be uncertainties about choosing \( U \) requiring further conventions for elimination.

The analogue of the \( N = 0 \) map \( \alpha \rightarrow P(\alpha) \) is just \( \alpha \rightarrow P(\pm N)(\alpha) \).

We can now reproduce Sections 4 and 5 for any \( N \) using \( T' \) and \( P(\pm N)(A^L) \).

Although \( T' \) and forms are operators on \( P(\pm N)A^{2N+1} \), that is not the space of sections for the twisted bundles. The latter is, rather,
\[ Q^{(\pm N)}A^{2N+1} = p^{(\pm N)}\mathcal{J}^{(\pm N)}A^{2N+1} = p^{(\pm N)}P^{(\pm N)}A^{2N+1}. \]
\( (6.15) \)

It is not an invariant subspace for \( D' \) and \( F' \) unless they are projected, or corrected by connections. We shall do that below. However, chirality \( \gamma' \) is well-defined on twisted sections because of (6.11).

For notational simplicity, we now permanently rename \( T', P^{(\pm N)} \) and \( Q^{(\pm N)} \) as follows:
\[ D', F', \gamma' \rightarrow D, F, \gamma, \]  
\[ p^{(\pm N)} \rightarrow p, \]  
\[ P^{(\pm N)} \rightarrow P, \]  
\[ Q^{(\pm N)} \rightarrow Q. \]  
(6.16-6.19)

\( \gamma' \) in any case is \( \gamma \).

7 Fuzzy Instantons, Topological Susceptibility, \( \theta \)-Term

We will now deal with a generic space \( M_F \) and let \( A \) stand for its algebra. We will also assume that their cyclic cohomology and instanton wave functions
are described by operators $D, F, \gamma$ and projectors $p, P$ and $Q = pP$ just as for $S^2_F$.

A generic operator $P(\alpha)$ will not commute with $p$. It must be changed to $pP(\alpha)p$.

It is often more convenient, as with $P(\alpha)$, to work with $p \cdot P(\alpha) := p(P(\alpha))$:

\[
p \cdot P(\alpha) \equiv pP(\alpha)p + (1 - p)P(\alpha)(1 - p) = P(\alpha) + \frac{1}{2}\Gamma(p)[P(\alpha), \Gamma(p)], \tag{7.1}
\]

\[
\Gamma(p) = 2p - 1. \tag{7.2}
\]

The modification of $d$ is accordingly

\[
p(d) = d + \frac{1}{2}\Gamma(p) (d\Gamma(p)). \tag{7.3}
\]

It contains the minimum irreducible gauge term $\Gamma(p)(d\Gamma(p))/2$. Here, there can be further fluctuations of the kind $p\hat{\omega}_1p$ [we can also use $p \cdot \hat{\omega}_1$] so that the general connection or covariant derivative is

\[
\nabla(\omega) = d + \frac{1}{2}\Gamma(p) (d\Gamma(p)) + p\hat{\omega}_1p,
\]

\[
\equiv d + \omega, \quad \hat{\omega}_1 = \text{a matrix of one-forms}. \tag{7.4}
\]

The curvature can be read off now:

\[
\mathcal{F}(\omega) = d\omega + \omega^2 = d \left[ \frac{1}{2}\Gamma(p) (d\Gamma(p)) + p\hat{\omega}_1p \right] + \left[ \frac{1}{2}\Gamma(p) (d\Gamma(p)) + p\hat{\omega}_1p \right]^2. \tag{7.5}
\]

Our action in the twisted sectors is also like (5.18):

\[
S[\omega, g] = \frac{1}{2e^2} TrQ (g|D|^{4-n}g^\dagger) \mathcal{F}(\omega)^2. \tag{7.6}
\]

For $n = 4$, let

\[
S[\omega, g] = \frac{1}{2e^2} \tilde{S}(\omega). \tag{7.7}
\]
\( \hat{S}[\omega] \) has the topological (\( \mathbb{Z} \)-valued) lower bound \( N \) for \( p = p^{(\pm N)} \). It is like the bound on continuum action saturated by instantons. That is enough to identify fuzzy instantons, topological susceptibility and \( \theta \)-term.

The bound follows from the inequality
\[
\left( \frac{1 \pm \gamma}{2} QF(\omega) \right) \left( \frac{1 \pm \gamma}{2} QF(\omega) \right)^\dagger \geq 0
\] (7.8)
by tracing, where \( \geq \) indicates a nonnegative operator:
\[
TrQF(\omega)^2 \geq |Tr\gamma QF(\omega)^2|.
\] (7.9)

In the next section, we show that
\[
Tr\gamma QF(\omega)^2
\] (7.10)
is independent of \( \hat{\omega}_1 \) and is the index of the operator
\[
\frac{1 - \gamma}{2} pFp \frac{1 + \gamma}{2} \equiv F_+ : \frac{1 + \gamma}{2} QA^{K+s} \rightarrow \frac{1 - \gamma}{2} QA^{K+s}.
\] (7.11)
This index is the difference of the dimensions of the subspaces of \( QA^{K+s} \) with \( \gamma = +1 \) and \( \gamma = -1 \).

The bound is saturated if
\[
\frac{1 \pm \gamma}{2} QF(\omega) = 0.
\] (7.12)
We can thus regard (7.12) as defining fuzzy instantons and anti-instantons.

We can go a step further and propose a topological susceptibility. Let \( tr \) (with a lower case \( t \)) indicate trace over spin and internal indices (the latter labeling components of \( A^K \)). Then fuzzy topological susceptibility is
\[
tr\gamma QF(\omega)^2.
\] (7.13)
Dimensional reasons indicate that it is to be identified with \( d(vol)G^*G \) if \( G \) is the continuum curvature, \( n = 4 \) and \( d(vol) \) the volume form.

The proposal (7.13) for topological susceptibility is valid for any \( p \), so also in the \( N = 0 \) sector (\( p = 1 \)).

The \( \theta \)-term in the fuzzy action “density” is proportional to (7.13), being just
\[
i\theta tr\gamma QF(\omega)^2.
\] (7.14)
Electrodynamics on $S^2$ has the term $\theta \int G$. Its analogue here is

$$i\theta \text{tr}\gamma QF(\omega).$$  \hfill (7.15)

Gauge theories for spacetime dimension $2n$ usually admit the $\theta$-terms

$$i\theta \int \text{tr}(G \wedge G \wedge \ldots G) \quad \text{n factors}$$  \hfill (7.16)

Their analogues too exist for fuzzy spaces, being

$$i\theta \text{tr}\gamma QF(\omega)^n$$  \hfill (7.17)

(7.17) is independent of $\hat{\omega}_1$ for all $n \geq 1$ and is the index of $F_+$. It is thus the same for any $n \in \{1, 2, \ldots \}$.

8 Axial Anomaly

We show three connected results for $S^2_F$ in this section.

(a) On the space $QA^{K+2}$, the Dirac operator $QDQ$ as also $QFQ$ have exactly $N$ zero modes of chirality $\gamma = +1$ if $p$ is $p^{(N)}$ (and chirality $\gamma = -1$ if $p$ is $p^{(-N)}$).

(b) $\pm N$, which are the indices of $F_+$, are given by (7.10) for $\hat{\omega}_1 = 0$.

(c) The expression (7.10) is independent of $\hat{\omega}_1$.

In view of Fujikawa’s argument [26], (a) shows how axial anomaly appears (in its integrated form) in the presence of instantons. (b) and (c) supply the missing arguments for the last section. While the focus is on $S^2_F$, the methodology is not so limited and can generalize to any fuzzy space.

8.1 Instantons and Zero Modes

We prove (a) in this subsection.

The operators $QDQ = pDp$, $QFQ = pFp$ and $\gamma$ commute with $\vec{J} = \vec{K} - \vec{L}^R + \vec{\sigma}/2$. Spaces with fixed $J^2$, $J_3$ are thus invariant under these operators.

For fixed $\vec{J}^2$, $J_3$, let $\Lambda$ be the eigenvalue of either of the operators $T = pDp$ or $pFp$. As $\gamma$ anticommutes with $T$ and commutes with $\vec{J}$, for each eigenstate with $\Lambda \neq 0$, there is another with $-\Lambda$ and same $\vec{J}^2$, $J_3$. Eigenvalues $\pm \Lambda \neq 0$ come in pairs with same $\vec{J}^2$, $J_3$.  

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Consider for specificity $p = p^{(N)}$. Then $k_{\text{max}} = l + N/2$ gives the eigenspace $k_{\text{max}}(k_{\text{max}} + 1)$ for $\tilde{K}^2$. Adding $-\tilde{L}^R$ and then $\tilde{\sigma}/2$ gives the spectrum $J(J+1)$ of $\tilde{J}^2$. We find $J = \{(l+N/2) + l, \cdots (l+N/2) - l\} \pm 1/2$.

Figure 1 shows how the addition of angular momenta is working. This explanation is a bit of a repetition of what we did in 6.1.

All the $J$’s occur with multiplicity 2 except the topmost with $J = J_{\text{max}} = 2l + (N+1)/2$ and the lowermost with $J = J_{\text{min}} = (N-1)/2$. Vectors with $J_{\text{max}}$ and $J_{\text{min}}$ are necessarily zero modes of $T$.

Vectors with $J_{\text{max}}$ are unphysical just like their $N=0$ siblings in Section 2. Now $P$ projects out the unpaired state with $J = J_{\text{max}}$. Hence the remaining zero modes after projection by $pP$ have $J = J_{\text{min}}$. Their multiplicity is $N$ as claimed. Also $-\tilde{L}^R$ and $\sigma/2$ must combine to $l + 1/2$ to create $J_{\text{min}}$ as $l - 1/2$ cannot combine with $k_{\text{max}} = l + N/2$ and give $(N-1)/2$. The projector for getting this $l + 1/2$ is just $(1+\gamma)/2$ [15]. Thus these zero modes have left helicity as claimed.

Next let $p = p^{(-N)}$. Then $k_{\text{max}}$ is changed to $k_{\text{min}} = l - N/2$ and the unpaired $J_{\text{max}} = 2l - (N-1)/2$ is once more eliminated by $P$. All other $J$ are paired except for $(l - 1/2) - (l - N/2) = (N-1)/2$. These give the necessary zero modes. There are $N$ of them. As the projector for $l - 1/2$ is $(1-\gamma)/2$ [15], they are right-chiral. All these are as claimed.
We have now established that $\frac{1}{2}T \frac{1+\gamma}{2}$ has index $\pm N$ if $p = p^{(\pm N)}$.

Let $\xi$ be a smooth perturbation of $T$ (for example due to $\hat{\omega}_1$) changing it from $T(0) = T$ to $T(\xi)$ such that $\gamma T(\xi) + T(\xi) \gamma = 0$. We can say that if $V_{\Lambda(\xi)}$ is the eigenspace of $T(\xi)$ for eigenvalue $\Lambda(\xi) \neq 0 [\Lambda(0) = \Lambda]$, then $\gamma V_{\Lambda(\xi)} = V_{-\Lambda(\xi)}$ is its eigenspace for opposite eigenvalue and with same dimension. Also the multiplicities of states with opposite chiralities in $V_{\Lambda(\xi)} \oplus V_{-\Lambda(\xi)}$ are equal. $V_{\pm\Lambda(\xi)}$ are just the deformations of $V_{\Lambda(0)}$ as $\xi$ is continuously turned on and have dimensions independent of $\xi$. As the zero modes of $T$ have a unique chirality, it follows that they cannot become modes with non-zero eigenvalues as $\xi$ is changed, lacking vectors with opposite chirality to pair with. The index of $\frac{1}{2}T \frac{1+\gamma}{2}$ is hence stable under such perturbations.

### 8.2 The Index and Curvature

Let us prove (b) now.

The curvature $\mathcal{F}(\frac{1}{2}\Gamma(p)d\Gamma(p))$ for $\hat{\omega}_1 = 0$ follows from (7.5):

$$\mathcal{F}(\frac{1}{2}\Gamma(p)d\Gamma(p)) = (dp)(dp).$$

(8.1)

We first show that

$$Tr\gamma Q\mathcal{F}(\frac{1}{2}\Gamma(p)d\Gamma(p))^n = \pm N \quad \text{for} \quad p = p^{(\pm N)} \quad \text{and for any} \quad n.$$  

(8.2)

Here $n = \{1, 2, \cdots \}$ and need not have any spacetime interpretation. The proof is due to Connes [9]. First using $p^2 = p$, we find $p[F, p] + [F, p]p = [F, p]$ or

$$p[F, p] = [F, p](1 - p), \quad (1 - p)[F, p] = [F, p]p.$$  

(8.3)

Also, using (8.3),

$$-p[F, p]^2 = -p[F, p]^2 p = pP - (pFp)^2.$$  

(8.4)

Thus

$$(-1)^nTr\gamma Q\mathcal{F}(\frac{1}{2}\Gamma(p)d\Gamma(p))^n = (-1)^nTr\gamma p [F, p]^{2n} = Tr\gamma p [p - (pFp)^2]^{2n}$$  

(8.5)
\[ TrP \left[ \frac{1+\gamma}{2} p - F_+^\dagger F_+ \right]^n - TrP \left[ \frac{1-\gamma}{2} p - F_+ F_+^\dagger \right]^n. \] (8.7)

The non-zero eigenvalues of \( F_+^\dagger F_+ \) and \( F_+ F_+^\dagger \) are equal and of same multiplicity, as elementary arguments show. So this last expression is the index of \( F_+ \) on \( \left[ (1+\gamma)/2 \right] QA^2N+1 \) (≡ difference in the number of zero modes of \( F_+^\dagger F_+ \) in \( \left[ (1+\gamma)/2 \right] QA^2N+1 \) and \( F_+ F_+^\dagger \) in \( \left[ (1-\gamma)/2 \right] QA^2N+1 \)). That is just \( \pm N \) and is also the difference in dimensions of \( \left[ (1\pm\gamma)/2 \right] QA^2N+1 \):

\[ (-1)^n Tr\gamma Q F \left( \frac{1}{2} \Gamma(p) d\Gamma(p) \right)^n = \pm N. \] (8.8)

It remains to show (c), that is that \( Tr\gamma Q F(\omega)^n \) is independent of \( \hat{\omega}_1 \).

Set

\[ c_t = \frac{1}{2} \Gamma(d\Gamma) + tp\hat{\omega}_1 p, \] (8.9)
\[ F(c_t) = dc_t + c_t^2. \] (8.10)

Then \( F(c_1) = F(\omega) \). Also \( F(c_t) \) fulfills the Bianchi identity

\[ dF(c_t) + [c_t, F(c_t)] = [F + c_t, F(c_t)] = 0. \] (8.11)

Now

\[ \frac{d}{dt} TrQ\gamma F(c_t)^n = nTrQ\gamma \frac{dF(c_t)}{dt} F(c_t)^{n-1} \] (8.12)
\[ = nTrQ\gamma \{ F + c_t, p\hat{\omega}_1 p \} F(c_t)^{n-1} \] (8.13)
\[ = nTrQ\gamma \{ F + c_t, p\hat{\omega}_1 p F(c_t)^{n-1} \} \] (8.14)
\[ = -nTr[ F + c_t, Q\gamma p\hat{\omega}_1 p F(c_t)^{n-1} ] = 0. \] (8.15)

So

\[ TrQ\gamma F(c_t)^n = TrQ\gamma F(c_t)^n|_{t=0} = TrQ\gamma F, p]^{2n} \] (8.16)

as required.

9 Final Remarks

In this paper we have proposed a formulation of fuzzy physics using cyclic cohomology. It relies especially on the theory of chiral fermions of \( S^2_F \) (with
no fermion doubling) as elaborated in \[17\]. Its distinct characteristic is the
ease with which it reproduces continuum topological features like instantons,
$\theta$-terms and axial anomaly. We remark in this context that we did not
explicitly write the $N \neq 0$ versions of all the actions in Section 5. But that
is easily done by changing the projector $P$ to $Q$ and reinterpreting symbols
like $\nabla$ and $\mathcal{F}(\omega)$.

This paper offers persuasive evidence that a combination of fuzzy man-
ifolds and cyclic cohomology can become a potent approach to discretization
of continuum physics.

There is overlap of this work with previous research \[4\] on fuzzy phy-
sics, especially as regards the treatment of chiral anomalies. We conclude
the paper with a brief comparison of the two approaches. Peter Preˇsnajder had
an essential role in its composition.

In the formalism of \[4\], a central role is played by a spin-1/2 variable
$z = (z_1, z_2), z \neq 0$. The spatial coordinates $n_i$ of $S^2$ are identified with
$(z^\dagger \sigma_i z)/\sqrt{z^\dagger z}$ while the left- and right- chiral components of the Dirac
spinor on $S^2$ in the presence of a monopole field with monopole number $2k = \pm N$
($N \geq 0$) are

$$\psi^{(+)}(z, z^\ast) = \sum_{|m| = |n| = 2k-1} a^{(+)}_{mn} z^m z^n, \quad (9.1)$$

$$\psi^{(-)}(z, z^\ast) = \sum_{|m| = |n| = 2k+1} a^{(-)}_{mn} z^m z^n. \quad (9.2)$$

A multi-index notation is being used with $m = (m_1, m_2), z^m = z_1^{m_1} z_2^{m_2}$,
$|m| = |m_1| + |m_2|$ etc.

The spinor $z$, after the normalization $z^\dagger z = 1$, describes the three-sphere
$S^3$ as a Hopf fibration over $S^2$. Thus $\psi^{(\pm)}$ are being represented here as
functions of the twisted principal bundle $\mathbb{C}^2 - \{0\}$ over $S^2$ (with structure
group $U(1) \times \mathbb{R}_1, \mathbb{R}_1$ being dilatations). This is an important point.

The description of the fuzzy sphere $S^2_F$ is achieved in \[4\] by replacing
$z_\alpha$ and $z^\ast_\alpha$ by annihilation and creation operators $\chi_\alpha$ and $\chi^\ast_\alpha$ with the only
elementary non-vanishing commutator $[\chi_\alpha, \chi^\ast_\beta] = \delta_{\alpha\beta}$. The fuzzy versions of
the chiral components $\psi^{(\pm)}$ of the Dirac field are

\begin{align*}
f &= \sum_{|m| - |n| = 2k - 1,} a_{mn}^{(+)} \chi^m \chi^n, \quad (9.3) \\
g &= \sum_{|m| - |n| = 2k + 1,} a_{mn}^{(-)} \chi^m \chi^n. \quad (9.4)
\end{align*}

where a restriction on $|m| + |n|$ has been introduced. Since $\chi$ and $\chi^*$ transform as spinors, it eliminates all angular momenta $> j_0 - 1/2$. Further, the value of $|n|$ and the power of annihilators in $f$ (or $g$) is $j_0 - k$ (or $j_0 - k - 1$). [It is assumed the $j_0 \geq |k| + 1$.] The domain of $f$ (or $g$) is accordingly restricted in [4] to the vectors $(\chi^*)^{j_0 - k}0$ (or $(\chi^*)^{j_0 - k - 1}0$) of angular momentum $(j_0 - k)/2$ (or $(j_0 - k - 1)/2$). [0] is the vacuum annihilated by $\chi_\alpha$. With this restriction, $f$ and $g$ can be interpreted as finite-dimensional matrices.

Note that the domains of $f$ and $g$ differ in their spinorial character: if one has integral angular momenta, the other has half-odd integral angular momenta. This suggests the introduction of supersymmetry [4].

As angular momentum is being cut-off at the same value $j_0 - 1/2$ in both $f$ and $g$, each angular momentum in the fuzzy spinor occurs with both chiralities. This technique of cut-off thus projects out the analogue of the unwanted top mode in our work.

The relation between $j_0$ here and our $l$ can be found by comparing the top total angular momenta:

\begin{align*}
j_0 &= 2l + k + 1 = 2l \pm N/2 + 1 \quad \text{for } f, \quad (9.5) \\
j_0 &= 2l + k = 2l \pm N/2 \quad \text{for } g. \quad (9.6)
\end{align*}

Here the choice among the $\pm$ signs is determined by the sign of $k$. Note how the value of $l$ increases by half a unit as we go from $f$ to $g$.

Although until this point there is a good correspondence between the two approaches, there is an important aspect besides supersymmetry which differentiates the two. The chiral fields $\psi^{(\pm)}$ are functions on the bundle $\mathbb{C}^2 - \{0\}$ for any instanton number $2k$. They are like the wave functions for Dirac monopoles [23]. In particular there are no separate operators for orbital angular momentum and spin in this formalism.

In contrast, in our approach, there are separate operators $L^L,R_i$ and $L_i$ for characterizing orbital angular momentum. Furthermore, in the sectors
with instanton numbers $\pm N$, we have to introduce $N$ “isospin” operators $\tau^{(i)}/2$ and combine them with orbital angular momentum and spin to find the total angular momentum $\vec{J}$. This construction is the analogue of the similar construction [19, 20] for the ’t Hooft-Polyakov monopoles. Thus our approach has a close correspondence to the description of the latter on the “sphere at infinity”.

Now in the continuum, it is possible to map one description to the other in a well-understood way [30]. A similar possibility for fuzzy physics has not been investigated.

Although the two approaches seem to differ in this manner, both have a Dirac and chirality operator which mutually anti-commute. That is enough to guarantee the presence of zero modes responsible for chiral anomaly. Thus we can see from (9.3, 9.4) that for $2k = N > 0$ the minimum angular momentum $(|m| - |n|)/2 = (N - 1)/2$ occurs only in $f$ (while the rest occur in both $f$ and $g$). The corresponding zero modes of the Dirac operator are therefore of positive chirality and multiplicity $N$. If $2k = -N < 0$, these zero modes are seen to have negative chirality, but the same multiplicity. All this is exactly as we found.

Further studies contrasting the two methods would be useful to expose their relative merits for particular problems.

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