Multiple Attribute Decision-Making Based on Three-Parameter Generalized Weighted Heronian Mean

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Abstract: For the aggregation problem of attributes with a correlation relationship, it is often necessary to take the correlation factor into account in order to make the decision results more objective and reasonable. The Heronian mean is an aggregation operator which reflects the interaction between attributes. It is of great theoretical and practical significance to study and popularize the multiple attribute decision-making methods based on the Heronian mean operator. In this paper, we first give a new three-parameter generalized weighted Heronian mean (TPGWHM), which has a series of excellent properties such as idempotency, monotonicity and boundedness. At the same time, the relationship between the TPGWHM and the existing aggregation operators is given. Then, we propose the intuitionistic fuzzy three-parameter generalized weighted Heronian mean (IFTPGWHM) and give its idempotency, monotonicity, boundedness and limit properties. On this basis, a multiple attribute decision-making method based on the TPGWHM and a multiple attribute decision-making method based on the IFTPGWHM are given, and corresponding examples are given and analyzed.

Keywords: three-parameter generalized weighted Heronian mean; intuitionistic fuzzy three-parameter generalized weighted Heronian mean; multiple attribute decision-making

1. Introduction

As an important part of modern decision science, multiple attribute decision-making is widely used in many fields, such as supply chain management, investment decision-making, project evaluation, logistics location and bidding. How to integrate decision information effectively is one of the core problems in the research of multiple attribute decision-making [1]. An effective and appropriate aggregation operator can make the information of the attribute values not be missing during the aggregation so as to reflect the decision-making effect correctly. Many aggregation operators have been given and extended to different application environments [2–13]. Dyckhoff and Pedrycz [14] proposed the generalized mean as connective operators for the fuzzy set theory which easily allow for modeling the degree of compensation in a natural manner, including the arithmetic and geometric means as well as the maximum and minimum operators as special cases. Due to the generalized mean, it can reflect the preference of decision makers and consider decision information from the overall perspective, and research on generalized means has attracted the attention of scholars. Intuitionistic fuzzy information aggregation is an interesting research direction of the intuitionistic fuzzy set theory. Many scholars have already focused on this area and achieved much. Xu [15] developed the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator and intuitionistic fuzzy hybrid aggregation operator. Liao and Xu [16] proposed a family of intuitionistic fuzzy hybrid weighted aggregation operators, such as the intuitionistic fuzzy hybrid weighted averaging operator, the intuitionistic fuzzy hybrid weighted geometric operator, the generalized intuitionistic fuzzy hybrid weighted averaging operator and the...
generalized intuitionistic fuzzy hybrid weighted geometric operator. Zeng [17] developed a new method for intuitionistic fuzzy decision-making problems with induced aggregation operators and distance measures. However, most of the aggregation operators we use assume that the attributes are independent of each other.

In practical multiple attribute decision-making problems, sometimes there is correlation between the attributes. Therefore, in the selection or construction of an aggregation operator, for the aggregation problem of attributes with a correlation relationship, it is often necessary to take this correlation factor into account in order to make the decision results more objective and reasonable. In fact, the Bonferroni mean, Heronian mean and their generalization in a fuzzy environment, intuitionistic fuzzy environment, hesitant fuzzy environment or linguistic environment are the main aggregation operators that can reflect the interaction between attributes. In recent years, there are many examples of research on the Bonferroni mean, Heronian mean and multiple attribute decision-making based on them. Motivated by the ideal of a generalized weighted Bonferroni mean and a generalized weighted geometric Bonferroni mean, Wei and Lin [18] developed the two-tuple linguistic generalized Bonferroni mean operator for aggregating the two-tuple linguistic information and two-tuple linguistic generalized geometric Bonferroni mean operator. Liu [19] defined the three-parameter Heronian mean operator and the three-parameter weighted Heronian mean operator and extended them to a linguistic environment. Jiang and He [20] developed a series of interval-valued dual hesitant fuzzy power Heronian aggregation operators. Yang and Li [21] extended the traditional generalized Heronian mean operators to a multiple-valued picture fuzzy linguistic environment and proposed the multiple-valued picture fuzzy linguistic generalized weighted geometric Heronian mean aggregating operator. Zhang, Zhang, Huang and Wang [22] used Heronian mean information aggregation technology to fuse picture fuzzy numbers and proposed new picture fuzzy aggregation operators. For more examples, please see [23–27].

In summation, the vast majority of operators aggregate information from a certain perspective. In the face of increasingly complex practical decision-making problems, decision makers sometimes need to examine decision objects from multiple perspectives, considering not only the interaction between attributes but also the overall information of decision objects, as well as the risk preference of decision makers. Moreover, most models only consider the correlation between two input parameters. However, in many practical decision scenarios, there may be multiple associations between the input variables. Therefore, in order to adapt to the new situation and aggregate decision-making information from multiple perspectives, an effective aggregation operator is needed which can meet the different needs of decision-makers at the same time. On this basis, a new three-parameter generalized weighted Heronian mean (TPGWHM) is given which has the advantages of both a generalized mean and a Heronian mean operator. It can extract the multi-association information between attributes and transform the parameters to meet the risk preference needs of decision makers. In practical application, intuitionistic fuzzy parameters can effectively represent the uncertainty of decision attributes. In this paper, the new aggregation operator is extended to the intuitionistic fuzzy environment, and the intuitionistic fuzzy three-parameter generalized weighted Heronian mean (IFTPGWHM) is given. On this basis, multiple attribute decision-making methods based on these operators are presented, and examples are given to illustrate the rationality and effectiveness of the methods. Therefore, the proposed methods are useful in real-life situations.

The rest of this paper is organized as follows. In Section 2, a new three-parameter generalized weighted Heronian mean is given, and its idempotency, monotonicity, boundedness and limit properties are studied. In Section 3, the intuitionistic fuzzy three-parameter generalized weighted Heronian mean is given, and its idempotency, monotonicity, permutation, boundedness and limit properties are studied. In Section 4, a multiple attribute decision-making method based on the new three-parameter generalized weighted Heronian mean (TPGWHM) is given, and an example is analyzed and compared with the existing multiple attribute decision-making methods. In Section 5, a multiple attribute
decision-making method based on the intuitionistic fuzzy three-parameter generalized weighted Heronian mean (IFTPGWHM) is given, and an example is analyzed and compared with the existing multiple attribute decision-making methods. The conclusions are given in Section 6.

2. The New Three-Parameter Generalized Weighted Heronian Mean

**Definition 1.** Let \( x_1, x_2, \cdots, x_n \geq 0 \). Then, the Heronian mean aggregation operator is

\[
H(x_1, x_2, \cdots, x_n) = \frac{2}{n(n+1)} \sum_{i=1}^{n} \sum_{j=i}^{n} \sqrt{x_i x_j}
\]

**Definition 2.** Let \( x_1, x_2, \cdots, x_n \geq 0 \). Then, the geometric Heronian mean aggregation operator is

\[
\tilde{H}(x_1, x_2, \cdots, x_n) = \frac{1}{2} \prod_{i=1, j=i}^{n} (x_i + x_j)^{\frac{2}{n(n+1)}}
\]

**Definition 3.** \([28]\) Let \( x_1, x_2, \cdots, x_n \geq 0 \) and \( p, q > 0 \). Then, the weighted geometric Heronian mean aggregation operator is

\[
W\tilde{H}(x_1, x_2, \cdots, x_n) = \frac{1}{p + q} \prod_{i=1}^{n} (px_i + qx_j)^{\frac{2}{n(n+1)}}
\]

Next, we give a new three-parameter generalized weighted Heronian mean aggregation operator.

**Definition 4.** Let \( x_1, x_2, \cdots, x_n \) be a set of nonnegative real numbers and \( s, t \in \mathbb{R} \) and \( s, t \neq 0 \), \( w_i (i = 1, 2, \cdots, n) \) be the weight of \( x_i \), where \( w_i \geq 0 \), \( \sum_{i=1}^{n} w_i = 1 \). Then, the three-parameter generalized weighted Heronian mean (TPGWHM) operator is

\[
TPGWHM(x_1, x_2, \cdots, x_n) = \left[ \frac{1}{\lambda} \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=j}^{n} \left( w_i x_i^{s/t} + w_j x_j^{s/t} + w_k x_k^{s/t} \right) \right]^{1/t}
\]

where \( \lambda = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=j}^{n} \left( w_i + w_j + w_k \right)^{1/2} \).

In order to give some properties of the new three-parameter generalized weighted Heronian mean, we first give the following two lemmas.

**Lemma 1.** \([29]\) Let \( x_1, x_2, \cdots, x_n \geq 0 \). Then, the generalized mean aggregation operator is expressed as

\[
WP(x_1, x_2, \cdots, x_n) = \left( \sum_{i=1}^{n} w_i x_i^t \right)^{1/t}
\]

This operator monotonically increases with respect to the parameter \( t \) and increases with respect to each independent variable.
**Lemma 2.** [29] Let \( x_1, x_2, \ldots, x_n \geq 0 \). Then, the generalized mean aggregation operator is

\[
WP(x_1, x_2, \ldots, x_n) = \left( \frac{1}{n} \sum_{i=1}^{n} w_i x_i^t \right)^{1/t}
\]  

(6)

This operator has the following limit properties:

(a) \( \lim_{t \to 0} \left( \frac{1}{n} \sum_{i=1}^{n} w_i x_i^t \right)^{1/t} = \prod_{i=1}^{n} x_i^{w_i} \),

(b) \( \lim_{t \to +\infty} \left( \frac{1}{n} \sum_{i=1}^{n} w_i x_i^t \right)^{1/t} = \max\{x_1, x_2, \ldots, x_n\} \),

(c) \( \lim_{t \to -\infty} \left( \frac{1}{n} \sum_{i=1}^{n} w_i x_i^t \right)^{1/t} = \min\{x_1, x_2, \ldots, x_n\} \).

**Theorem 1.** The TPGWHM operator has the following properties:

1. **Idempotency.** If \( x_i = x \) for \( i = 1, 2, \ldots, n \), then TPGWHM\(^{s,t}(x, x, \ldots, x) = x \) such that

\[
TPGWHM^{s,t}(x, x, \ldots, x) = \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (w_i x_i^s + w_j x_j^s + w_k x_k^s)^{t/s} \right]^{1/t}
\]  

(7)

That is to say, the TPGWHM operator is idempotent.

2. **Monotonicity.** When \( s, t > 0 \), if \( x_i \leq y_i \), then the following is true:

\[
TPGWHM^{s,t}(x_1, x_2, \ldots, x_n) \leq TPGWHM^{s,t}(y_1, y_2, \ldots, y_n)
\]  

(8)

**Proof.** Let \( f(x_1, x_2, \ldots, x_n) = TPGWHM(x_1, x_2, \ldots, x_n) \), and thus

\[
\frac{\partial f}{\partial x_i} = \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (w_i x_i^s + w_j x_j^s + w_k x_k^s)^{t/s} \right]^{1/(t-1)}
\]

\[
\times \left[ \frac{t}{s} \sum_{i=1}^{n} \sum_{j=1}^{n} (w_i x_i^s + w_j x_j^s + w_k x_k^s)^{(t-1)/s} \right] \left( s w_i x_i^{s-1} \right)
\]

\[
= t w_i x_i^{s-1} \left[ \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (w_i x_i^s + w_j x_j^s + w_k x_k^s)^{t/s} \right]^{1/(t-1)}
\]

\[
\times \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} (w_i x_i^s + w_j x_j^s + w_k x_k^s)^{(t-1)/s} \right] \geq 0
\]

Therefore, we know that the function \( f(x_1, x_2, \ldots, x_n) \) is monotonically increasing with respect to each independent variable. As such, for any \( x_i \leq y_i \), the following is true:

\[
TPGWHM^{s,t}(x_1, x_2, \ldots, x_n) \leq TPGWHM^{s,t}(y_1, y_2, \ldots, y_n)
\]  

(10)

**3. Boundedness:**

\[
\min\{x_1, x_2, \ldots, x_n\} \leq TPGWHM^{s,t}(x_1, x_2, \ldots, x_n) \leq \max\{x_1, x_2, \ldots, x_n\}
\]  

(11)
Proof. Without losing generality, let \( \min\{x_1, x_2, \cdots, x_n\} = x_1 \) and \( \max\{x_1, x_2, \cdots, x_n\} = x_n \). Then, we can know from the proof of monotonicity of the TPGWHM operator that the TPGWHM operator is increasing with respect to each independent variable \( x_i \) \((i = 1, 2, \cdots, n)\) such that

\[
\text{TPGWHM}^{s,t}(x_1, x_1, \cdots, x_1) \leq \text{TPGWHM}^{s,t}(x_1, x_2, \cdots, x_n) \leq \text{TPGWHM}^{s,t}(x_n, x_n, \cdots, x_n)
\]  

(12)

From the idempotency of the TPGWHM operator, we can see that

\[
\text{TPGWHM}^{s,t}(x_1, x_1, \cdots, x_1) = x_1, \quad \text{TPGWHM}^{s,t}(x_n, x_n, \cdots, x_n) = x_n.
\]  

(13)

Therefore

\[
x_1 \leq \text{TPGWHM}^{s,t}(x_1, x_2, \cdots, x_n) \leq x_n.
\]  

(14)

That is to say

\[
\min\{x_1, x_2, \cdots, x_n\} \leq \text{TPGWHM}^{s,t}(x_1, x_2, \cdots, x_n) \leq \max\{x_1, x_2, \cdots, x_n\}.
\]

\[\square\]

The relationship between the TPGWHM\(^{s,t}(x_1, x_2, \cdots, x_n)\) operator and the geometric Heronian mean is given as follows:

**Theorem 2.** If we let \( s, t \in \mathbb{R} \) and \( s, t \neq 0 \), then

\[
\lim_{t \to 0} \text{TPGWHM}^{s,t}(x_1, x_2, \cdots, x_n) = \text{TPGWHM}^{s,0}(x_1, x_2, \cdots, x_n)
\]

\[
= \left[ \prod_{i,j,k=1, k=j=i}^{n} \left( \frac{w_i}{w_i + w_j + w_k} x_i^s + \frac{w_j}{w_i + w_j + w_k} x_j^s + \frac{w_k}{w_i + w_j + w_k} x_k^s \right)^{1/s} \right]^{6 \big/(n(n+1)(n+2))}
\]

(15)

In particular, if we have \( s = 1 \), then

\[
\text{TPGWHM}^{1,0}(x_1, x_2, \cdots, x_n)
\]

\[
= \left( \prod_{i,j,k=1, k=j=i}^{n} \frac{w_i}{w_i + w_j + w_k} x_i + \frac{w_j}{w_i + w_j + w_k} x_j + \frac{w_k}{w_i + w_j + w_k} x_k \right)^{6/(n(n+1)(n+2))}
\]

(16)

Proof. According to Lemma 2, we have

\[
\lim_{t \to 0} \text{TPGWHM}^{s,t}(x_1, x_2, \cdots, x_n) = \lim_{t \to 0} \exp \left[ \ln \prod_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=j}^{n} \left( \frac{w_i x_i^s + w_j x_j^s + w_k x_k^s}{w_i + w_j + w_k} \right)^{1/s} \right] - \ln \lambda
\]

(17)

where \( \lambda = \frac{n}{i=1} \sum_{j=1}^{n} \sum_{k=j}^{n} (w_i + w_j + w_k)^{1/2} \).

Moreover, the following is true:
\[
\lim_{t \to 0} \left[ \ln \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( w_i x_i^s + w_j x_j^s + w_k x_k^s \right)^{t/s} - \ln \lambda \right]
\]

\[
= \lim_{t \to 0} \left( \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left( w_i x_i^s + w_j x_j^s + w_k x_k^s \right)^{t/s}}{\lambda} \right) - \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (w_i + w_j + w_k)^{t/s}}{\lambda}
\]

\[
= \frac{n(n+1)(n+2)}{6} \ \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \ln \left( \frac{w_i x_i^s + w_j x_j^s + w_k x_k^s}{w_i + w_j + w_k} \right)^{1/s} \right] - \frac{n(n+1)(n+2)}{6} \ \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \ln \left( \frac{w_i + w_j + w_k}{w_i + w_j + w_k} \right)^{1/s} \right]
\]

Then, we have

\[
\lim_{t \to 0} TPGWHM^{s,t}(x_1, x_2, \ldots, x_n)
\]

\[
= \left[ \prod_{i,j,k=1}^{n} \left( \frac{w_i}{w_i + w_j + w_k} x_i^s + \frac{w_j}{w_i + w_j + w_k} x_j^s + \frac{w_k}{w_i + w_j + w_k} x_k^s \right)^{1/s} \right]^{1/n(1-s)/n}
\]

(19)

The proof is complete. \(\square\)

**Remark 1.** TPGWHM^{1,0}(x_1, x_2, \ldots, x_n) is the weighted geometric Heronian mean \([28]\).

**Theorem 3.** If we let \(s, t \neq 0\), \(w_i > 0, i = 1, 2, \ldots, n\), \(\lambda = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} (w_i + w_j + w_k)^{1/s}, \) then

\[
\lim_{\substack{t \to +\infty \\text{or} \\infty \to +\infty}} \text{TPGWHM}^{s,t}(x_1, x_2, \ldots, x_n)
\]

\[
= \max \{x_1, x_2, \ldots, x_n\}
\]

(20)

\[
\lim_{\substack{t \to -\infty \\text{or} \\infty \to -\infty}} \text{TPGWHM}^{s,t}(x_1, x_2, \ldots, x_n)
\]

\[
= \min \{x_1, x_2, \ldots, x_n\}
\]

(21)

**Proof.** Without losing generality, if we set \(x_1 \leq x_2 \leq \cdots \leq x_n\), then
TPGWHM^d(x_1, x_2, \cdots, x_n)

\begin{equation}
\begin{bmatrix}
x_1^d(w_1 + w_2 + w_3) + x_2^d(w_1 + w_3 + w_4) + w_5 \\
+ x_3^d(w_1 + w_2 + w_4) + x_4^d(w_1 + w_2 + w_3) + w_6 \\
+ \cdots + x_n^d(w_1 + w_2 + w_3) + w_7 \\
\end{bmatrix} \frac{1}{\sqrt[n]{x}}
\end{equation}

\begin{equation}
\begin{bmatrix}
x_1^d(w_1 + w_2 + w_3) + x_2^d(w_1 + w_3 + w_4) + w_5 \\
+ x_3^d(w_1 + w_2 + w_4) + x_4^d(w_1 + w_2 + w_3) + w_6 \\
+ \cdots + x_n^d(w_1 + w_2 + w_3) + w_7 \\
\end{bmatrix} \frac{1}{\sqrt[n]{x}}
\end{equation}

Therefore

\[ \lim_{s \to \infty} TPGWHM^d(x_1, x_2, \cdots, x_n) = \frac{6}{n(n+1)(n+2)} \begin{bmatrix}
x_1^d + x_2^d + \cdots + x_n^d \\
+ x_2^d + x_3^d + \cdots + x_n^d \\
+ \cdots + x_n^d
\end{bmatrix} \frac{1}{\sqrt[n]{x}} \begin{bmatrix}
x_1^d + x_2^d + \cdots + x_n^d \\
+ x_2^d + x_3^d + \cdots + x_n^d \\
+ \cdots + x_n^d
\end{bmatrix} \frac{1}{\sqrt[n]{x}}
\]

\[ = x_n \left\{ \frac{6}{n(n+1)(n+2)} \begin{bmatrix}
x_1^d + x_2^d + \cdots + x_n^d \\
+ x_2^d + x_3^d + \cdots + x_n^d \\
+ \cdots + x_n^d
\end{bmatrix} \frac{1}{\sqrt[n]{x}} \right\}
\]

and then

\[ \lim_{l \to \infty} \lim_{s \to \infty} TPGWHM^d(x_1, x_2, \cdots, x_n) = x_n = \max\{x_1, x_2, \cdots, x_n\}
\]

Similarly, it can be proven that

\[ \lim_{l \to \infty} \lim_{s \to \infty} TPGWHM^d(x_1, x_2, \cdots, x_n) = x_1 = \min\{x_1, x_2, \cdots, x_n\}
\]

The proof is complete. □

**Theorem 4.** If we let \( s, t \in \mathbb{R} \) and \( s, t \neq 0 \), then

\[ \lim_{s \to 0} \lim_{t \to 0} TPGWHM^d(x_1, x_2, \cdots, x_n) = \left( \prod_{i=1}^{n} X_i \right)^{1/n}
\]

**Proof.** According to Theorem 2, we have
\[
\lim_{t \to 0} TPGWHM^x(x_1, x_2, \cdots, x_n)
= \left[ \prod_{i, j, k = 1}^{n} \left( \frac{w_i}{w_i + w_j + w_k} x_i^i + \frac{w_j}{w_i + w_j + w_k} x_j^j + \frac{w_k}{w_i + w_j + w_k} x_k^k \right)^{1/s} \right]^{6 \pi(n + 1)(n + 2)}
\]

(27)

Then

\[
\lim_{s \to 0} \lim_{t \to 0} TPGWHM^x(x_1, x_2, \cdots, x_n)
= \left[ \prod_{i, j, k = 1}^{n} \left( \frac{w_i}{w_i + w_j + w_k} x_i^i + \frac{w_j}{w_i + w_j + w_k} x_j^j + \frac{w_k}{w_i + w_j + w_k} x_k^k \right)^{1/s} \right]^{6 \pi(n + 1)(n + 2)}
\]

(28)

If \( w_i = 1/n \) and \( i = 1, 2, \cdots, n \), then by a simple computation, we can obtain that Equation (26) holds. \( \square \)

3. The New Intuitionistic Fuzzy Three-Parameter Generalized Weighted Heronian Mean

In this section, the intuitionistic fuzzy three-parameter generalized weighted Heronian mean (IFTPGWHM) operator is given, and the properties of this operator and the relationship between it and other aggregation operators are discussed.

3.1. The Related Concepts of the Intuitionistic Fuzzy Set

First, the concept of the Atanassov intuitionistic fuzzy set is given.

Definition 5. [30,31] If we let \( U \) be a given set, then

\[
E = \{(x, \mu_E(x), \nu_E(x)) | x \in U\}
\]

(29)

is called an intuitionistic fuzzy set, where \( \mu_E(x) : U \to [0,1] \) and \( \nu_E(x) : U \to [0,1] \) represent the membership degree and non-membership degree of element \( x \in U \) to the set \( E \), respectively, and satisfy \( 0 \leq \mu_E(x) + \nu_E(x) \leq 1 \).

Definition 6. [30,31] For any intuitionistic fuzzy set \( E \), if \( \pi_E(x) = 1 - \mu_E(x) - \nu_E(x) \), then \( \pi_E(x) \) is called the degree of indeterminacy of element \( x \in U \) belonging to \( E \).
For convenience, Xu and Yager [32] called the combination \((μ_E(x), ν_E(x))\) the intuitionistic fuzzy number, abbreviated as \(α = (μ_α, ν_α)\), where \(μ_α, ν_α ≥ 0, μ_α + ν_α ≤ 1\), \(π_α(x) = 1 - μ_α(x) - ν_α(x)\).

Obviously, \(α^∗ = (1,0)\) and \(α_∗ = (0,1)\) are the maximum and minimum intuitionistic fuzzy numbers, respectively. Every intuitionistic fuzzy number \(α\) has its practical significance [33]. For example, if \(α = (0.6, 0.3)\), then \(μ_α = 0.6, ν_α = 0.3\) and \(π_α = 0.1\). This can be explained as follows: 10 people vote on the resolution, and the result is 6 in favor, 3 against and 1 abstention. In order to compare and rank intuitionistic fuzzy numbers, Chen and Tan [34] proposed the concept of the score function:

\[
s(α) = μ_α - ν_α
\]

They called \(s(α)\) the score value of \(α\), and obviously \(s(α) ∈ [-1, 1]\). In order to further distinguish the case of equal scores, Hong and Choi [35] proposed the concept of the precise function:

\[
h(α) = μ_α + ν_α
\]

They called \(h(α)\) the accuracy degree of intuitionistic fuzzy number \(α\). From these, we can get the comparison and ranking method of intuitionistic fuzzy number \(α\).

**Definition 7. [15,32]** When letting \(α = (μ_α, ν_α), α_1 = (μ_{α_1}, ν_{α_1})\) and \(α_2 = (μ_{α_2}, ν_{α_2})\) be intuitionistic fuzzy numbers, if the scores of \(α_1\) and \(α_2\) are \(s(α_1) = μ_{α_1} - ν_{α_1}\) and \(s(α_2) = μ_{α_2} - ν_{α_2}\), respectively, and the accuracy degree of \(α_1\) and \(α_2\) are \(h(α_1) = μ_{α_1} + ν_{α_1}\) and \(h(α_2) = μ_{α_2} + ν_{α_2}\), respectively, then the following applies:

1. If \(s(α_1) < s(α_2)\), then \(α_1 < α_2\);
2. If \(s(α_1) = s(α_2)\), then the following applies:
   1. If \(h(α_1) < h(α_2)\), then \(α_1 = α_2\), which also indicates that \(μ_{α_1} = μ_{α_2}\) and \(ν_{α_1} = ν_{α_2}\);
   2. If \(h(α_1) > h(α_2)\), then \(α_1 > α_2\);
3. If \(h(α_1) < h(α_2)\), then \(α_1 < α_2\).

**Definition 8. [33]** Let \(α = (μ_α, ν_α), α_1 = (μ_{α_1}, ν_{α_1})\) and \(α_2 = (μ_{α_2}, ν_{α_2})\) be intuitionistic fuzzy numbers. Then, the following algorithms apply:

1. \(α_1 ⊕ α_2 = (μ_{α_1} + μ_{α_2} - μ_{α_1} · μ_{α_2}, ν_{α_1} · ν_{α_2})\);
2. \(α_1 ⊙ α_2 = (μ_{α_1} · μ_{α_2}, ν_{α_1} + ν_{α_2} - ν_{α_1} · ν_{α_2})\);
3. \(λα = \left( 1 - (1 - μ_α)^λ, μ_α^λ \right), λ > 0\);
4. \(α^λ = \left( μ_α^λ, 1 - (1 - ν_α)^λ \right), λ > 0\).

The algorithms listed above also have the following operation laws:

1. Commutative law. \(α_1 ⊕ α_2 = α_2 ⊕ α_1, α_1 ⊙ α_2 = α_2 ⊙ α_1\);
2. Distributive law. \(λ(α_1 ⊕ α_2) = λα_1 + λα_2, (α_1 ⊙ α_2)^λ = α_1^λ ⊙ α_2^λ\);
3. Associative law. \(λ_1α ⊕ λ_2α = (λ_1 + λ_2)α, α^{λ_1} ⊙ α^{λ_2} = α^{λ_1 + λ_2}, λ_1, λ_2 > 0\).

### 3.2. The Intuitionistic Fuzzy Three-Parameter Generalized Weighted Heronian Mean Operator

**Definition 9.** Let \(α_i = (μ_{α_i}, ν_{α_i}), i = 1, 2, \ldots, n\) be a set of intuitionistic fuzzy numbers such that the weight \(w_i ≥ 0\) satisfies \(\sum_{i=1}^{n} w_i = 1\), and let \(λ = \sum_{i=1}^{n} \sum_{j=1}^{n} (w_i + w_j)^{1/s}\). If \(s, t > 0\), then

\[
IFTPGWHM^{st}(α_1, α_2, \ldots, α_n) = \left( \frac{1}{λ} \oplus_{i,j,k=1}^{n} (w_i α_i^s ⊕ w_j α_j^s ⊕ w_k α_k^s)^{1/s} \right)^{1/t}
\]
is called the intuitionistic fuzzy three-parameter generalized weighted Heronian mean, denoted as IFTPGWHM.

**Theorem 5.** Let \( a_i = (\mu_{a_i}, \nu_{a_i}), i = 1, 2, \cdots, n \) be a set of intuitionistic fuzzy numbers where the weight \( w_i \geq 0 \) satisfies \( \sum_{i=1}^{n} w_i = 1 \). If \( s, t > 0 \), then the result aggregated by the IFTPGWHM operator is still an intuitionistic fuzzy number, and

\[
\text{IFTPGWHM}_s^t(a_1, a_2, \cdots, a_n) = \left( \frac{1}{n} \sum_{i,j,k=1, k=j=i}^{n} \left( w_i a_i^s + w_j a_j^s + w_k a_k^s \right)^{1/s} \right)^{1/t}
\]

\[
= \left\{ \begin{array}{c}
1 - \left( \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - (1 - (1 - \mu_{a_i})^s)^w_i \left( 1 - (1 - \nu_{a_i})^s \right)^{w_i} \right)^{1/s} \right)^{1/t}
\end{array} \right\}
\]

**Proof.** According to the operation laws of intuitionistic fuzzy numbers, we have

\[
\alpha_i^s = \left( \mu_{a_i}, 1 - \left( 1 - \nu_{a_i} \right)^s \right), \quad \alpha_j^s = \left( \mu_{a_j}, 1 - \left( 1 - \nu_{a_j} \right)^s \right), \quad \alpha_k^s = \left( \mu_{a_k}, 1 - \left( 1 - \nu_{a_k} \right)^s \right)
\]

\[
\sum_{i,j,k=1, k=j=i}^{n} \left( w_i \alpha_i^s + w_j \alpha_j^s + w_k \alpha_k^s \right)^{1/s}
\]

\[
= \left( 1 - \left( 1 - \mu_{a_i}^s \right)^{w_i} \left( 1 - \mu_{a_j}^s \right)^{w_j} \left( 1 - \mu_{a_k}^s \right)^{w_k} \left( 1 - (1 - \nu_{a_i})^s \right)^{w_i} \left( 1 - (1 - \nu_{a_j})^s \right)^{w_j} \left( 1 - (1 - \nu_{a_k})^s \right)^{w_k} \right)^{1/s}
\]

Then

\[
= \left\{ \begin{array}{c}
1 - \left( \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - (1 - \mu_{a_i})^s \left( 1 - \mu_{a_j}^s \left( 1 - \mu_{a_k}^s \right)^{w_i} \left( 1 - \nu_{a_i}^s \right)^{w_j} \left( 1 - \nu_{a_j}^s \right)^{w_k} \right)^{1/s} \right)^{1/t}
\end{array} \right\}
\]

In addition...
\[
\left( \frac{1}{\pi} \right)^{\frac{n}{2}} \prod_{i,j,k=1, k=j=i}^{n} \left( w_i w_j w_k \right)^{1/s} \left( \prod_{i,j,k=1, k=j=i}^{\lambda} \left( 1 - \left( 1 - \mu_{a_i}^{w_i} \right)^{w_i} \left( 1 - \mu_{a_j}^{w_j} \right)^{w_j} \left( 1 - \mu_{a_k}^{w_k} \right)^{w_k} \right)^{1/s} \right)^{1/t}
\]

Since

\[
0 \leq \left( \prod_{i,j,k=1, k=j=i}^{\lambda} \left( 1 - \left( 1 - \mu_{a_i}^{w_i} \right)^{w_i} \left( 1 - \mu_{a_j}^{w_j} \right)^{w_j} \left( 1 - \mu_{a_k}^{w_k} \right)^{w_k} \right)^{1/s} \right)^{1/t} \leq 1
\]

and for any \( i, 1, 2, \cdots, n, \mu_{a_i} + \nu_{a_i} \leq 1 \), we have

\[
0 \leq \left( \prod_{i,j,k=1, k=j=i}^{\lambda} \left( 1 - \left( 1 - \mu_{a_i}^{w_i} \right)^{w_i} \left( 1 - \mu_{a_j}^{w_j} \right)^{w_j} \left( 1 - \mu_{a_k}^{w_k} \right)^{w_k} \right)^{1/s} \right)^{1/t} \leq 1
\]
The proof is complete. □

Next, some properties of the IFTPGWHM operator are given.

**Property 1.** (Idempotency) If all \( a_i \) \((i = 1, 2, \cdots, n)\) are equal, that is, \( a_i = a = (\mu_a, \nu_a) \), \( i = 1, 2, \cdots, n \), then

\[
\text{IFTPGWHM}^{s,d}(a_1, a_2, \cdots, a_n) = \text{IFTPGWHM}^{s,d}(a, a, \cdots, a)
\]

\[
= \left( \frac{1}{\lambda} \sum_{i,j,k=1, \atop k=j=i}^{n} \left( w_i a^s \oplus w_j a^s \oplus w_k a^s \right)^{1/s} \right)^{1/\lambda}
= \left( \frac{1}{\lambda} \sum_{i,j,k=1, \atop k=j=i}^{n} a^i (w_i \oplus w_j \oplus w_k)^{1/s} \right)^{1/\lambda}
(41)
\]

\[
= \left( \frac{\alpha^i}{\lambda} \sum_{i,j,k=1, \atop k=j=i}^{n} (w_i \oplus w_j \oplus w_k)^{1/s} \right)^{1/\lambda}
= \alpha
\]

In particular, if all \( a_i(i = 1, 2, \cdots, n) \) are the minimum intuitionistic fuzzy numbers, (i.e., \( a_i = a_s = (0, 1) \)), then

\[
\text{IFTPGWHM}^{s,d}(a_s, a_s, \cdots, a_s) = (0, 1).
\]

This is to say that after aggregation, they are also the minimum intuitionistic fuzzy numbers. If all \( a_i \) \((i = 1, 2, \cdots, n)\) are the maximum intuitionistic fuzzy numbers, that is, \( a_i = a^s = (1, 0) \), then

\[
\text{IFTPGWHM}^{s,d}(a^s, a^s, \cdots, a^s) = (1, 0).
\]

This is to say that they are also the maximum intuitionistic fuzzy numbers after aggregation.

**Property 2.** (Monotonicity) If we let \( a_i = (\mu_{a_i}, \nu_{a_i}) \) and \( \beta_i = (\mu_{\beta_i}, \nu_{\beta_i}) \) \((i = 1, 2, \cdots, n)\) be two sets of intuitionistic fuzzy numbers where \( \mu_{a_i} \leq \mu_{\beta_i}, \nu_{a_i} \geq \nu_{\beta_i}, \) \( i = 1, 2, \cdots, n \), then

\[
\text{IFTPGWHM}^{s,d}(a_1, a_2, \cdots, a_n) \leq \text{IFTPGWHM}^{s,d}(\beta_1, \beta_2, \cdots, \beta_n)
(42)
\]

**Proof.** On the one hand, since \( 0 \leq \mu_{a_i} \leq 1, 0 \leq \mu_{\beta_i} \leq 1, 0 \leq w_i \leq 1, i = 1, 2, \cdots, n \) and \( s, t > 0 \), we have

\[
\prod_{i,j,k=1, \atop k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \mu_{a_i}^s \right) w_i \right) \left( 1 - \mu_{a_k}^s \right) ^{w_k} \right)^{1/s}
\geq \prod_{i,j,k=1, \atop k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \mu_{\beta_i}^s \right) w_i \right) \left( 1 - \mu_{\beta_k}^s \right) ^{w_k} \right)^{1/s}
(43)
\]

Then

\[
1 - \left( \prod_{i,j,k=1, \atop k=j=i}^{n} \left( 1 - \left( 1 - \mu_{a_i}^s \right) w_i \right) \left( 1 - \mu_{a_k}^s \right) ^{w_k} \right)^{1/s} \leq 1 - \left( \prod_{i,j,k=1, \atop k=j=i}^{n} \left( 1 - \left( 1 - \mu_{\beta_i}^s \right) w_i \right) \left( 1 - \mu_{\beta_k}^s \right) ^{w_k} \right)^{1/s}
(44)
\]
Therefore, we have

\[
1 - \left\{ 1 - \left( \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \mu_{\alpha_i} \right)^{w_i} \left( 1 - \mu_{\beta_j} \right)^{w_j} \left( 1 - \mu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right) \right\}^{1/t}
\]

\[
\leq \left\{ 1 - \left( \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \mu_{\alpha_i} \right)^{w_i} \left( 1 - \mu_{\beta_j} \right)^{w_j} \left( 1 - \mu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right) \right\}^{1/t}
\]

On the other hand, since \(1 \geq \nu_{a_i} \geq \nu_{\beta_i} \geq 0, \omega_i > 0\) and \(i = 1, 2, \ldots, n\), then

\[
\prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \nu_{a_i} \right)^{w_i} \left( 1 - \nu_{\beta_j} \right)^{w_j} \left( 1 - \nu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right)
\]

\[
\geq \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \nu_{a_i} \right)^{w_i} \left( 1 - \nu_{\beta_j} \right)^{w_j} \left( 1 - \nu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right)
\]

Therefore, we have

\[
1 - \left\{ 1 - \left[ \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \nu_{a_i} \right)^{w_i} \left( 1 - \nu_{\beta_j} \right)^{w_j} \left( 1 - \nu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right) \right] \right\}^{1/t}
\]

\[
\leq \left\{ 1 - \left[ \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \nu_{a_i} \right)^{w_i} \left( 1 - \nu_{\beta_j} \right)^{w_j} \left( 1 - \nu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right) \right] \right\}^{1/t}
\]

Thus

\[
1 - \left\{ 1 - \left[ \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \nu_{a_i} \right)^{w_i} \left( 1 - \nu_{\beta_j} \right)^{w_j} \left( 1 - \nu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right) \right] \right\}^{1/t}
\]

\[
\geq 1 - \left\{ 1 - \left[ \prod_{i,j,k=1, k=j=i}^{n} \left( 1 - \left( 1 - \left( 1 - \nu_{a_i} \right)^{w_i} \left( 1 - \nu_{\beta_j} \right)^{w_j} \left( 1 - \nu_{\kappa_k} \right)^{w_k} \right)^{1/s} \right) \right] \right\}^{1/t}
\]
According to Equations (45) and (48), we have

\[
\left\{ 1 - \left( \prod_{i,j,k=1, \atop k \neq j}^n \left( 1 - (1 - \mu_{a_i}^{w_i} \nu_i \nu_j \nu_k) \right)^{1/\lambda} \right) \right\}^{1/\lambda} \leq \left\{ 1 - \left( \prod_{i,j,k=1, \atop k \neq j}^n \left( 1 - (1 - \nu_{a_i}) \nu_i \nu_j \nu_k \right)^{1/\lambda} \right) \right\}^{1/\lambda}
\]

\[
\leq \left\{ 1 - \left( \prod_{i,j,k=1, \atop k \neq j}^n \left( 1 - (1 - \mu_{\beta_i}^{w_i} \nu_i \nu_j \nu_k) \right)^{1/\lambda} \right) \right\}^{1/\lambda}
\]

(49)

Let \( \alpha = \text{IFTPGWHM}^{\otimes i}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \beta = \text{IFTPGWHM}^{\otimes i}(\beta_1, \beta_2, \ldots, \beta_n) \), and use \( s_\alpha \) and \( s_\beta \) to represent the scores of \( \alpha \) and \( \beta \), respectively, such that the inequality in Equation (49) is equivalent to \( s_\alpha \leq s_\beta \). The following two cases are considered:

(i) If \( s_\alpha < s_\beta \), then we immediately get

\[
\text{IFTPGWHM}^{\otimes i}(\alpha_1, \alpha_2, \ldots, \alpha_n) < \text{IFTPGWHM}^{\otimes i}(\beta_1, \beta_2, \ldots, \beta_n)
\]

(50)

(ii) If \( s_\alpha = s_\beta \), then

\[
\left\{ 1 - \left( \prod_{i,j,k=1, \atop k \neq j}^n \left( 1 - (1 - \mu_{a_i}^{w_i} \nu_i \nu_j \nu_k) \right)^{1/\lambda} \right) \right\}^{1/\lambda} \leq \left\{ 1 - \left( \prod_{i,j,k=1, \atop k \neq j}^n \left( 1 - (1 - \nu_{a_i}) \nu_i \nu_j \nu_k \right)^{1/\lambda} \right) \right\}^{1/\lambda}
\]

\[
= \left\{ 1 - \left( \prod_{i,j,k=1, \atop k \neq j}^n \left( 1 - (1 - \mu_{\beta_i}^{w_i} \nu_i \nu_j \nu_k) \right)^{1/\lambda} \right) \right\}^{1/\lambda}
\]

\[
\leq \left\{ 1 - \left( \prod_{i,j,k=1, \atop k \neq j}^n \left( 1 - (1 - \nu_{\beta_i}) \nu_i \nu_j \nu_k \right)^{1/\lambda} \right) \right\}^{1/\lambda}
\]

(51)

Since \( \mu_{a_i} \leq \mu_{\beta_i} \), \( \nu_{a_i} \geq \nu_{\beta_i} \) and \( i = 1, 2, \ldots, n \), then
\[
\left\{ 1 - \left( \prod_{i,j,k = 1, \atop k = j = i}^n \left( 1 - \left( 1 - v_{a_i} \right)^s w_{j} \left( 1 - v_{a_j} \right)^s w_{i} \left( 1 - v_{a_k} \right)^s w_{k} \right)^{t/s} \right) \right\}^{1/\lambda} \right\}^{1/t}
\]

Therefore

\[
h_\alpha = \left\{ 1 - \left( \prod_{i,j,k = 1, \atop k = j = i}^n \left( 1 - \left( 1 - \mu_{a_i} \right)^{w_{j}} \left( 1 - \mu_{a_j} \right)^{w_{i}} \left( 1 - \mu_{a_k} \right)^{w_{k}} \right)^{t/s} \right) \right\}^{1/\lambda} \left\}^{1/t}
\]

\[
+1 - \left\{ 1 - \left( \prod_{i,j,k = 1, \atop k = j = i}^n \left( 1 - \left( 1 - \left( 1 - v_{a_i} \right)^s w_{j} \left( 1 - v_{a_j} \right)^s w_{i} \left( 1 - v_{a_k} \right)^s w_{k} \right)^{t/s} \right) \right\}^{1/\lambda} \left\}^{1/t}
\]

\[
= \left\{ 1 - \left( \prod_{i,j,k = 1, \atop k = j = i}^n \left( 1 - \left( 1 - \mu_{b_i} \right)^{w_{j}} \left( 1 - \mu_{b_j} \right)^{w_{i}} \left( 1 - \mu_{b_k} \right)^{w_{k}} \right)^{t/s} \right) \right\}^{1/\lambda} \left\}^{1/t}
\]

\[
+1 - \left\{ 1 - \left( \prod_{i,j,k = 1, \atop k = j = i}^n \left( 1 - \left( 1 - \left( 1 - v_{b_i} \right)^s w_{j} \left( 1 - v_{b_j} \right)^s w_{i} \left( 1 - v_{b_k} \right)^s w_{k} \right)^{t/s} \right) \right\}^{1/\lambda} \left\}^{1/t}
\]

\[
= h_\beta
\]
Then
\[ \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \text{IFTPGWHM}^{s;\lambda}(\beta_1, \beta_2, \ldots, \beta_n) \] (55)

According to (i) and (ii), we have
\[ \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \text{IFTPGWHM}^{s;\lambda}(\beta_1, \beta_2, \ldots, \beta_n) \] (56)

which is to say that the monotonicity holds.

The proof is complete. \( \square \)

**Property 3.** (Permutation) If we let \( \alpha_i = (\mu_{\alpha_i}, v_{\alpha_i}) \) \((i = 1, 2, \ldots, n)\) be a set of intuitionistic fuzzy numbers and \((\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)\) be any permutation of \((\alpha_1, \alpha_2, \ldots, \alpha_n)\), then
\[ \text{IFTPGWHM}^{s;\lambda}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) \] (57)

**Proof.** Since \((\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n)\) is any permutation of \((\alpha_1, \alpha_2, \ldots, \alpha_n)\), then
\[ \left( \frac{1}{n} \mathop{\oplus} \sum_{i,j,k=1, k=j=i}^{n} \left( w_i \tilde{\alpha}_i^s \oplus w_j \tilde{\alpha}_j^s \oplus w_k \tilde{\alpha}_k^s \right)^{1/s} \right)^{1/l} = \left( \frac{1}{n} \mathop{\oplus} \sum_{i,j,k=1, k=j=i}^{n} \left( w_i \tilde{\alpha}_i^s \oplus w_j \tilde{\alpha}_j^s \oplus w_k \tilde{\alpha}_k^s \right)^{1/s} \right)^{1/l} \] (58)

and then \( \text{IFTPGWHM}^{s;\lambda}(\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n) = \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) \).

The proof is complete. \( \square \)

**Property 4.** (Boundedness) Let \( \alpha_i = (\mu_{\alpha_i}, v_{\alpha_i}) \) \((i = 1, 2, \ldots, n)\) be a set of intuitionistic fuzzy numbers, and let
\[ \alpha^- = \left\{ \min_{i} \{ \mu_{\alpha_i} \}, \max_{i} \{ v_{\alpha_i} \} \right\}, \alpha^+ = \left\{ \max_{i} \{ \mu_{\alpha_i} \}, \min_{i} \{ v_{\alpha_i} \} \right\}, \]

Then, \( \alpha^- \leq \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+ \).

**Proof.** Since \( \mu_{\alpha_i} \geq \min_{i} \{ \mu_{\alpha_i} \} \) and \( v_{\alpha_i} \leq \max_{i} \{ v_{\alpha_i} \} \), according to the idempotency and monotonicity of the IFTPWHM operator, we have
\[ \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) \geq \text{IFTPGWHM}^{s;\lambda}(\alpha^-, \alpha^-, \ldots, \alpha^-) = \alpha^- \] (59)

Similarly, we can obtain
\[ \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \text{IFTPGWHM}^{s;\lambda}(\alpha^+, \alpha^+, \ldots, \alpha^+) = \alpha^+ \] (60)

Therefore, the IFTPWHM operator is bounded.

The proof is complete. \( \square \)

**Corollary 1.** Let \( \alpha_i = (\mu_{\alpha_i}, v_{\alpha_i}) \) \((i = 1, 2, \ldots, n)\) be a set of intuitionistic fuzzy numbers and the weight be \( w_i \geq 0 \) to satisfy \( \sum_{i=1}^{n} w_i = 1 \). If \( s > 0 \), then the result aggregated by the \( \text{IFTPGWHM}^{s;\lambda}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) operator is still an intuitionistic fuzzy number, and
lim_{t \to \infty} IFTPGWHM^d(a_1, a_2, \cdots, a_n) = IFTPGWHM^O(a_1, a_2, \cdots, a_n)

= \left[ \prod_{i,j,k=1, k=j=i}^n \left( 1 - \left( 1 - \mu_{a_i}^t \right) \frac{\nu_j}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_j}^t \right) \frac{\nu_k}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_k}^t \right) \frac{\nu_i}{\nu_i + \nu_j + \nu_k} \right)^{1/s} \right]^{1/\alpha}

= \left[ \prod_{i,j,k=1, k=j=i}^n \left( 1 - \left( 1 - \mu_{a_i}^t \right) \frac{\nu_j}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_j}^t \right) \frac{\nu_k}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_k}^t \right) \frac{\nu_i}{\nu_i + \nu_j + \nu_k} \right)^{1/s} \right]^{1/\alpha}

In particular, when s = 1, then

lim_{t \to \infty} IFTPGWHM^d(a_1, a_2, \cdots, a_n) = IFTPGWHM^O(a_1, a_2, \cdots, a_n)

= \left[ \prod_{i,j,k=1, k=j=i}^n \left( 1 - \left( 1 - \mu_{a_i}^t \right) \frac{\nu_j}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_j}^t \right) \frac{\nu_k}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_k}^t \right) \frac{\nu_i}{\nu_i + \nu_j + \nu_k} \right)^{1/s} \right]^{1/\alpha}

Proof. According to the operation laws of intuitionistic fuzzy numbers, we have

\[
\prod_{i,j,k=1, k=j=i}^n \left( 1 - \left( 1 - \mu_{a_i}^t \right) \frac{\nu_j}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_j}^t \right) \frac{\nu_k}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_k}^t \right) \frac{\nu_i}{\nu_i + \nu_j + \nu_k} \right)^{1/s} \]

(63)

Then

\[
\prod_{i,j,k=1, k=j=i}^n \left( 1 - \left( 1 - \mu_{a_i}^t \right) \frac{\nu_j}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_j}^t \right) \frac{\nu_k}{\nu_i + \nu_j + \nu_k} \left( 1 - \mu_{a_k}^t \right) \frac{\nu_i}{\nu_i + \nu_j + \nu_k} \right)^{1/s} \]

(64)

Therefore
\[
\left[ \begin{array}{cccc}
\frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\
\mu_j & \mu_j & \mu_j & \mu_j \\
\eta_j & \eta_j & \eta_j & \eta_j \\
\alpha_j & \alpha_j & \alpha_j & \alpha_j
\end{array} \right]^{1/s}
\]

(65)

Since

\[
0 \leq \left( \prod_{i,j,k=1, k=j=1}^n \left( 1 - \left( 1 - \mu_{ij} \right)^{\frac{1}{\eta_j + \eta_j}} \left( 1 - \mu_{ij} \right)^{\frac{1}{\tau_j + \tau_j}} \left( 1 - \mu_{ij} \right)^{\frac{1}{\nu_j + \nu_j}} \right)^{1/s} \right) \leq 1
\]

(66)

\[
0 \leq 1 - \left( \prod_{i,j,k=1, k=j=1}^n \left( 1 - \left( 1 - \nu_{ij} \right)^{\frac{1}{\eta_j + \eta_j}} \left( 1 - \nu_{ij} \right)^{\frac{1}{\tau_j + \tau_j}} \left( 1 - \nu_{ij} \right)^{\frac{1}{\nu_j + \nu_j}} \right)^{1/s} \right) \leq 1
\]

(67)

In addition, for any \( i = 1, 2, \cdots, n \), there is \( \mu_{ij} + \nu_{ij} \leq 1 \), and then

\[
0 \leq \left( \prod_{i,j,k=1, k=j=1}^n \left( 1 - \left( 1 - \mu_{ij} \right)^{\frac{1}{\eta_j + \eta_j}} \left( 1 - \mu_{ij} \right)^{\frac{1}{\tau_j + \tau_j}} \left( 1 - \mu_{ij} \right)^{\frac{1}{\nu_j + \nu_j}} \right)^{1/s} \right) \leq 1
\]

(68)

Similarly, the case of \( s = 1 \) can be obtained. The proof is complete. \( \square \)
Similarly, it can be proven that the $IFTPGWHM^{s,0}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ operator and $IFTPGWHM^{s,1}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ operator also satisfy idempotency, monotonicity, boundedness and permutation.

4. Multiple Attribute Decision-Making Method Based on the New Three-Parameter Generalized Weighted Heronian Mean

In this part, the multiple attribute decision-making method based on the new three-parameter generalized weighted Heronian mean is given. Then, the new three-parameter generalized weighted Heronian mean multiple attribute decision-making method is analyzed by an example, which is compared with the existing multiple attribute decision-making methods. The effectiveness of the proposed method is illustrated, and the influence of parameters $s, t$ on the decision-making results is analyzed.

4.1. Multiple Attribute Decision-Making Methods Based on the New Three-Parameter Generalized Weighted Heronian Mean

Let the scheme set be $X = \{x_1, x_2, x_3, \ldots, x_m\}$, the attribute set be $U = \{u_1, u_2, u_3, \ldots, u_n\}$ and the attribute weight vector be $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)$, $\omega_i \geq 0, \sum_{i=1}^{n} \omega_i = 1$. Several decision makers are organized to decide the alternatives.

Step 1. The attribute value of scheme $x_i$ with respect to attribute $u_j$ is $a_{ij}$, which is given by the decision maker, and the decision attribute matrix is $A = (a_{ij})_{mn}$ where $a_{ij} \geq 0$.

Step 2. The decision matrix $A$ is normalized, and the normalized decision matrix $R = (r_{ij})_{mn}, a_{ij} \geq 0 (i = 1, 2, 3, \ldots, m, j = 1, 2, 3, \ldots, n)$ is obtained. In order to eliminate the influence of different physical dimensions on the decision results, the decision matrix $A$ can be normalized according to the following formula:

$$r_{ij} = \frac{a_{ij}}{\max_i (a_{ij})}, r_{ij} = \frac{\min_i (a_{ij})}{a_{ij}}. \quad (69)$$

Step 3. According to the new three-parameter generalized weighted Heronian mean, the comprehensive evaluation value of each scheme is calculated.

Step 4. The comprehensive evaluation values of each scheme are sorted, and the decision results are given.

4.2. Illustrative Example of MADM Based on the New Three-Parameter Generalized Weighted Heronian Mean

In order to illustrate the effectiveness of the decision-making method given in this paper, the example in [19] is used for illustration, and the decision results are compared with the well-established methods.

Example 1. An investment bank will invest in four enterprises $x_i (i = 1, 2, 3, 4)$ in a city. The evaluation indexes are the output value ($u_1$), investment cost ($u_2$), sales volume ($u_3$), proportion of national income ($u_4$) and environmental pollution degree ($u_5$). The investment bank inspected the above indicators of the four enterprises in the previous year (the pollution level was detected and quantified by relevant environmental protection departments), and the evaluation results are shown in Table 1. The investment cost and environmental pollution degree were of the cost type, and the others were of the benefit type. The attribute weight was $\omega = (0.36, 0.16, 0.16, 0.16, 0.16)$ . Try to determine the best investment enterprise.

|   | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ |
|---|---|---|---|---|---|
| $x_1$ | 8350 | 5300 | 6135 | 0.82 | 0.17 |
| $x_2$ | 7455 | 4952 | 6257 | 0.65 | 0.13 |
| $x_3$ | 11,000 | 8001 | 9008 | 0.59 | 0.15 |
| $x_4$ | 9624 | 5000 | 8892 | 0.74 | 0.28 |
Obviously, the output value, investment cost, sales volume, proportion of national income and environmental pollution degree are not independent but interactive. For example, the level of the output value will affect the investment cost and sales \[19\]. Therefore, the new three-parameter generalized weighted Heronian mean operator given in this paper was considered for information aggregation.

First, the decision attribute matrix is normalized to obtain the normalized matrix \(R\).

\[
R = \begin{bmatrix}
0.7591 & 0.9343 & 0.6811 & 1 & 0.7647 \\
0.6777 & 1 & 0.7246 & 0.7926 & 1 \\
1 & 0.6189 & 1 & 0.7195 & 0.8667 \\
0.8749 & 0.9904 & 0.9871 & 0.9024 & 0.4643
\end{bmatrix}
\]

Then, take the different values of parameters \(s\) and \(t\), use the TPGWHM to calculate the comprehensive evaluation value of each scheme, and rank the schemes according to the comprehensive attribute values. The ranking results and the best scheme are obtained. The results are shown in Table 2.

**Table 2.** Scheme ranking the results for different \(s\) and \(t\) values.

| \(s = 0.001, t = 0.001\) | \(c(x_1)\) | \(c(x_2)\) | \(c(x_3)\) | \(c(x_4)\) | Scheme Ranking Results |
|------------------------|-------------|-------------|-------------|-------------|------------------------|
| \(0.807\) | 0.7955 | 0.8587 | 0.8262 | \(x_3 > x_4 > x_1 > x_2\) |
| \(0.807\) | 0.7956 | 0.8589 | 0.8265 | \(x_3 > x_4 > x_1 > x_2\) |
| \(0.8181\) | 0.8187 | 0.849 | 0.8365 | \(x_3 > x_4 > x_2 > x_1\) |
| \(0.818\) | 0.8186 | 0.8492 | 0.8367 | \(x_3 > x_4 > x_2 > x_1\) |
| \(0.8141\) | 0.8067 | 0.8728 | 0.85 | \(x_3 > x_4 > x_1 > x_2\) |
| \(0.7894\) | 0.7497 | 0.9528 | 0.8725 | \(x_3 > x_4 > x_1 > x_2\) |
| \(0.7703\) | 0.7063 | 0.9864 | 0.8754 | \(x_3 > x_4 > x_1 > x_2\) |

For example, let \(s = 1, t = 5\), and then use the TPGWHM to calculate the comprehensive evaluation value. We can get

\[
c(x_1) = 0.7894, \ c(x_2) = 0.7497, \ c(x_3) = 0.9528, \ c(x_4) = 0.8725
\]

By ranking the comprehensive values of each scheme, we can obtain \(x_3 > x_4 > x_1 > x_2\). Therefore, enterprise three is the best investment enterprise.

Next, we further analyze the influence of parameters \(s\) and \(t\) on the decision results.

It can be seen from Table 2 that when \(s\) and \(t\) took different values, the scheme ranking results were relatively stable, and the best scheme remained unchanged.

Fix the value of parameter \(s(s = 1)\), and set the value of \(t\) to be from 0.001 to 1. The change of the comprehensive attribute value of each scheme is shown in Figure 1.

As can be seen from Figure 1, when \(s\) was fixed and \(t\) increased, scheme three \((x_3)\) was always the best scheme. With the increase of parameter \(t\), the difference between the comprehensive attribute values became larger.

Fix the value of parameter \(t(t = 1)\) and set parameter \(s\) to be from 0.001 to 1. The change of the comprehensive attribute value of each scheme is shown in Figure 2.
As can be seen from Figure 2, when \( t \) was fixed and \( s \) increased, scheme three \((x_3)\) was always the best scheme.

When \( s \) and \( t \) changed, the comprehensive attribute values of the four schemes changed as shown in Figures 3–6.

Figure 1. \( s = 1, \ t \in [0.001, 1] \).

Figure 2. \( t = 1, \ s \in [0.001, 1] \).

Figure 3. Comprehensive attribute values of \( x_1 \) obtained by the TPGWHM \((t, \ s \in [0.001,1])\).
In summation, we can see that with the change of $s$ and $t$, the ranking scheme also changed, which indicates that the ranking scheme was affected by the subjectivity of the decision-maker’s parameter selection. Therefore, in the process of decision-making, experts can select the appropriate parameters according to their own risk preference. The results illustrate the flexibility and robustness of the proposed method. Therefore, the proposed method is effective and feasible, and it is sufficient to deal with practical MADM problems.
4.3. Comparative Analysis

In this part, the proposed method is compared with the existing multiple attribute decision-making methods. The method based on the OWA operator, the method introduced by Liu [19] based on the three-parameter weighted Heronian mean (TPWHM) operator, and the proposed method based on the TPGWHM operator were used to solve the above example, and the comparison of the results is shown in Table 3.

Table 3. Comparison of the results.

| Method | Comprehensive Attribute Value | Ranking Result |
|--------|--------------------------------|----------------|
| The method based on the OWA operator | $c(x_1) = 0.8623, c(x_2) = 0.8712, c(x_3) = 0.8728, c(x_4) = 0.8731$ | $x_4 > x_3 > x_2 > x_1$ |
| Liu’s [19] method based on the TPWHM operator ($p = q = r = 1$) | $c(x_1) = 0.8095, c(x_2) = 0.7954, c(x_3) = 0.8921, c(x_4) = 0.8566$ | $x_3 > x_4 > x_1 > x_2$ |
| The proposed method based on the TPGWHM operator ($s = 1, t = 1$) | $c(x_1) = 0.8141, c(x_2) = 0.8067, c(x_3) = 0.8728, c(x_4) = 0.85$ | $x_3 > x_4 > x_1 > x_2$ |

From Table 3, we can see that the ranking result obtained by the method based on the OWA operator was different from that derived by the proposed method ($s = t = 1$). The main reason for this is that the OWA operator assumes that attributes are independent when integrating information, while the TPGWHM operator considers the interaction between attributes. Therefore, the decision results obtained by the proposed method were more realistic. The ranking result obtained by the proposed method ($s = t = 1$) was consistent with that derived by method based on TPWHM ($p = q = r = 1$) in [19], because both methods are based on the Heronian operator, which can reflect the interaction between attributes. The proposed method can not only reflect the interaction between attributes but also reflect the advantages of the generalized mean operator. According to the risk preference of different decision makers, different parameters can be selected. Therefore, the method presented in this paper has more extensive application.

5. Multiple Attribute Decision-Making Method Based on the New Intuitionistic Fuzzy Three-Parameter Generalized Weighted Heronian Mean

In this part, the multiple attribute decision-making method based on the new intuitionistic fuzzy three-parameter generalized weighted Heronian mean is given. Then, the method is analyzed by an example which is compared with the existing multiple attribute decision-making methods. The effectiveness of the proposed method is illustrated, and the influence of parameters $s, t$ on the decision-making results is analyzed.

5.1. Multiple Attribute Decision-Making Method Based on the Intuitionistic Fuzzy Three-Parameter Generalized Weighted Heronian Mean

Let the scheme set be $X = \{x_1, x_2, x_3, \ldots, x_m\}$, the attribute set be $U = \{u_1, u_2, u_3, \ldots, u_n\}$ and the attribute weight vector be $\omega = (\omega_1, \omega_2, \omega_3, \ldots, \omega_n)$, $\omega_i \geq 0, \sum_{i=1}^{n} \omega_i = 1$. The intuitionistic fuzzy set which is used to represent the feature information of the scheme is

$$x_i = \{(u_j, \mu_{x_i}(u_j), \upsilon_{x_i}(u_j)) | u_j \in U \} \ (i = 1, 2, \cdots, n),$$

where $\mu_{x_i}(u_j)$ is the degree to which scheme $x_i$ satisfies attribute $u_j$, $\upsilon_{x_i}(u_j)$ is the degree to which scheme $x_i$ does not satisfy attribute $u_j$ and $\mu_{x_i}(u_j) \in [0, 1], \upsilon_{x_i}(u_j) \in [0, 1] | \mu_{x_i}(u_j) + \upsilon_{x_i}(u_j) \leq 1$.

For convenience, the characteristic of scheme $x_i$ with respect to attribute $u_j$ is represented by intuitionistic fuzzy number $d_{ij} = (\mu_{ij}, \upsilon_{ij})$ (i.e., $\mu_{ij}$ and $\upsilon_{ij}$ represent the degree to which scheme $x_i$ satisfies attribute $u_j$ and does not satisfy attribute $u_j$, respectively). Therefore, we can use an intuitionistic fuzzy decision matrix $D = (d_{ij})_{n \times m}$ to represent the characteristic information of all alternatives $x_i (i = 1, 2, \cdots, n)$ with respect to all attributes $u_j (j = 1, 2, \cdots, m)$, where $d_{ij} = (\mu_{ij}, \upsilon_{ij}), \mu_{ij}, \upsilon_{ij} \in [0, 1]$ satisfies $\mu_{ij} + \upsilon_{ij} \leq 1$. 
Step 1. According to the data types in the scheme set, the decision matrix $D$ is normalized to $R = (r_{ij})_{n \times m}$ by the following formula:

$$
(r_{ij})_{n \times m} = (\mu_{ij}', \nu_{ij}') = \begin{cases} 
  \overline{d}_{ij} & \text{If scheme } u_i \text{ is benefit type} \\
  d_{ij} & \text{If scheme } u_i \text{ is cost type}
\end{cases}
$$

(70)

where $\overline{d}_{ij}$ is the complement of $d_{ij}$.

Step 2. The IFTPGWHM is used to aggregate the characteristic information $r_{ij}$ ($i = 1, 2, \ldots, m$) of $x_i$ with respect to all its attributes $u_j$ ($j = 1, 2, \ldots, m$), and the comprehensive attribute value $\tilde{r}_i$ ($i = 1, 2, \ldots, n$) of scheme $x_i$ is obtained.

Step 3. According to the score function $s(\alpha) = \mu_\alpha - \nu_\alpha$, the score $s(\tilde{r}_i)$ of the comprehensive attribute value $\tilde{r}_i$ of each scheme $x_i$ is calculated, where $i = 1, 2, \ldots, n$.

Step 4. The schemes $x_i$ ($i = 1, 2, \ldots, n$) are sorted according to the scores $s(\tilde{r}_i)$ ($i = 1, 2, \ldots, n$). If $s(\tilde{r}_i)$ and $s(\tilde{r}_j)$ are equal, we need to further calculate the accuracy degrees $h(\tilde{r}_i)$ and $h(\tilde{r}_j)$ of the comprehensive attribute values $\tilde{r}_i$ and $\tilde{r}_j$ (the accuracy degree is calculated by the accuracy function $h(\alpha) = \mu_\alpha + \nu_\alpha$) and then use the size of $h(\tilde{r}_i)$ and $h(\tilde{r}_j)$ to sort the schemes $x_i$ and $x_j$ so as to give the decision results.

5.2. Illustrative Example of MADM Based on the New Intuitionistic Fuzzy Three-Parameter Generalized Weighted Heronian Mean

In order to illustrate the effectiveness of the decision-making method given in this paper, the example in [28] is used for illustration, and the decision results are compared with the reference.

Example 2. A management school in a Chinese university wants to introduce a teacher from four alternatives. A set of four factors is considered: morality ($c_1$), research capability ($c_2$), teaching skill ($c_3$) and education background ($c_4$). The attribute weight is $\omega = (0.2, 0.3, 0.3, 0.2)^T$. The experts evaluated four alternatives $x_i$ ($i = 1, 2, 3, 4$) in relation to the factors $C = (c_1, c_2, c_3, c_4)$. The evaluation information for $x_i$ ($i = 1, 2, 3, 4$) under the factors $C = (c_1, c_2, c_3, c_4)$ were represented by the IFNs as shown in Table 4.

Table 4. The intuitionistic fuzzy decision matrix.

|     | $c_1$       | $c_2$       | $c_3$       | $c_4$       |
|-----|-------------|-------------|-------------|-------------|
| $x_1$ | (0.9, 0.0)  | (0.5, 0.4)  | (0.8, 0.1)  | (0.5, 0.4)  |
| $x_2$ | (0.7, 0.2)  | (0.9, 0.0)  | (0.6, 0.3)  | (0.8, 0.1)  |
| $x_3$ | (0.4, 0.5)  | (0.7, 0.2)  | (0.9, 0.0)  | (0.7, 0.2)  |
| $x_4$ | (0.6, 0.3)  | (0.6, 0.3)  | (0.7, 0.2)  | (0.9, 0.0)  |

All the attribute values were of the benefit type; therefore, the decision attribute matrix did not need to be normalized. Then, by taking different values of parameters $s$ and $t$, we could use the IFTPGWHM to calculate the comprehensive evaluation value of each scheme. Then, we calculated the score of the comprehensive attribute value and ranked the schemes according to the scores. The ranking results and the best scheme were obtained. The results are shown in Table 5.

Table 5. Scheme ranking results for different values of $s$ and $t$.

|     | $s(x_1)$ | $s(x_2)$ | $s(x_3)$ | $s(x_4)$ | Scheme Ranking Results |
|-----|----------|----------|----------|----------|-----------------------|
| $s = 0.1, t = 0.1$ | 0.714    | 0.7786   | 0.7428   | 0.7163   | $x_2 > x_3 > x_4 > x_1$ |
| $s = 5, t = 0.1$   | 0.7127   | 0.7555   | 0.7274   | 0.7032   | $x_2 > x_3 > x_1 > x_4$ |
| $s = 1, t = 1$     | 0.7247   | 0.7831   | 0.7521   | 0.7219   | $x_2 > x_3 > x_1 > x_4$ |
| $s = 1, t = 10$    | 0.8332   | 0.8988   | 0.8923   | 0.803    | $x_2 > x_3 > x_1 > x_4$ |
| $s = 10, t = 1$    | 0.7609   | 0.7902   | 0.7685   | 0.7452   | $x_2 > x_3 > x_1 > x_4$ |
For example, let $s = 1$, $t = 1$, and use the IFTPGWHM to calculate the comprehensive evaluation value. Then, by calculating the score of the comprehensive attribute value, we can obtain

$$s(x_1) = 0.7247, s(x_2) = 0.7831, s(x_3) = 0.7521, s(x_4) = 0.7219.$$  

By ranking the score of each scheme, we can obtain $x_2 > x_3 > x_1 > x_4$. Therefore, alternative two is the best.

Next, we further analyzed the influence of parameters $s$ and $t$ on the decision results.

It can be seen from Table 5 that when $s$ and $t$ took on different values, the scheme ranking results were relatively stable, and the best scheme remained unchanged.

Next, fix the value of parameter $s(s = 1)$ and let the value of $t$ be from 0.001 to 1. The change of scores of each scheme is shown in Figure 7.

![Figure 7. $s = 1$, $t \in [0.001, 1]$.](image)

As can be seen from Figure 7, when $s$ was fixed and $t$ increased, alternative two ($x_2$) was always the best scheme. With the increase of parameter $t$, the score values increased.

Next, fix the value of parameter $t(t = 1)$ and let the value of $s$ be from 1 to 11. The change of scores of each scheme is shown in Figure 8.

![Figure 8. $t = 1$, $s \in [1, 11]$.](image)

As can be seen from Figure 8, when $t$ was fixed and $s$ increased, alternative two ($x_2$) was always the best scheme. When $s > 2$, with the increase of parameter $s$, the score values increased.

It can be seen from Figures 7 and 8 that with the increase of parameters $s$ and $t$, the score values became larger, but the best scheme remained unchanged. This shows the
flexibility and robustness of the proposed method. In actual decision-making, the parameters can be selected according to the risk preference of the decision maker. The decision maker who takes a gloomy view could select smaller parameters, and the optimistic decision maker could select larger parameters.

When $s$ and $t$ change, the score values of the four schemes change as shown in Figures 9–12.

![Figure 9. Scores of $x_1$ obtained by the IFTPGWHM ($t, s \in [0.001, 1]$).](image)

![Figure 10. Scores of $x_2$ obtained by the IFTPGWHM ($t, s \in [0.001, 1]$).](image)

![Figure 11. Scores of $x_3$ obtained by the IFTPGWHM ($t, s \in [0.001, 1]$).](image)
In summation, we can see that with the change of \( s \) and \( t \), the ranking scheme also changed, which indicates that the ranking scheme was affected by the subjectivity of the decision maker’s parameter selection. Therefore, in the process of decision-making, experts can select the appropriate parameters according to their own risk preference. The results illustrate the flexibility and robustness of the proposed method. Therefore, the proposed method is effective and feasible, and it is sufficient to deal with practical MADM problems.

5.3. Comparative Analysis

In this part, the proposed method is compared with the existing multiple attribute decision-making methods. The method based on intuitionistic fuzzy weighed geometric IFWG operator, the method introduced by Yu [28] based on the intuitionistic fuzzy geometric weighed Heronian mean (IFGWHM) operator and the proposed method based on the IFTPGWHM operator are used to solve the above example, and the comparison of the results is shown in Table 6.

Table 6. Comparison of the results.

| Methods | Score Value | Ranking Result |
|---------|-------------|----------------|
| The method based on the IFWG operator | \( s(x_1) = 0.3980, s(x_2) = 0.5816, s(x_3) = 0.4535, s(x_4) = 0.4640 \) | \( x_2 > x_4 > x_3 > x_1 \) |
| Yu’s [28] method based on the IFGWHM operator (\( p = q = 1 \)) | \( s(x_1) = 0.8230, s(x_2) = 0.8811, s(x_3) = 0.8418, s(x_4) = 0.8463 \) | \( x_2 > x_4 > x_3 > x_1 \) |
| The proposed method based on the IFTPGWHM operator \((s = 1, t = 1)\) | \( s(x_1) = 0.7247, s(x_2) = 0.7831, s(x_3) = 0.7521, s(x_4) = 0.7219 \) | \( x_2 > x_3 > x_1 > x_4 \) |

From Table 6, we can see that the ranking results obtained by the method based on the IFWG operator were different from those derived by the proposed method \((s = 1, t = 1)\). The main reason for this is that the IFWG operator assumes that attributes are independent when integrating information, while the IFTPGWHM operator considers the interaction between attributes. Therefore, the decision results obtained by the proposed method were more realistic. The ranking results obtained by the proposed method \((s = 1, t = 1)\) were different from those derived by the method based on the IFGWHM \((p = q = 1)\) in [28]. Although both of them are based on the Heronian mean, the method proposed in this paper considers the multiple associations between attributes and has the characteristics of the generalized mean operator. According to the risk preference of different decision makers, different parameters can be selected. Therefore, the method presented in this paper has more extensive application.
6. Conclusions

This paper further extends the Heronian mean. In this paper, a new three-parameter generalized weighted Heronian mean is given. The new three-parameter generalized weighted Heronian mean is extended to the intuitionistic fuzzy environment, and the intuitionistic fuzzy three-parameter generalized weighted Heronian mean is given. On this basis, a multiple attribute decision-making method based on the new three-parameter generalized weighted Heronian mean and a multiple attribute decision-making method based on the intuitionistic fuzzy three-parameter generalized weighted Heronian mean are given, and several examples are analyzed. In the future, we will further study the generalization of the proposed operators and make a profound study of their flexibility and robustness.

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