On the Coherence Properties of Random Euclidean Distance Matrices

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Abstract—In the present paper we focus on the coherence properties of general random Euclidean distance matrices, which are very closely related to the respective matrix completion problem. This problem is of great interest in several applications such as node localization in sensor networks with limited connectivity. Our results can directly provide the sufficient conditions under which an EDM can be successfully recovered with high probability from a limited number of measurements.

Index Terms—Random Euclidean Distance Matrices, Matrix Completion, Limited Connectivity, Subspace Coherence

I. INTRODUCTION

Considering \( N \) points (or nodes) lying in \( \mathbb{R}^d \) with respective positions \( \mathbf{p}_i \in \mathbb{R}^d, i \in \{1, 2, \ldots, N\} \), the \((i, j)\)-th entry of a Euclidean distance matrix (EDM) is defined as

\[
\Delta(i, j) \triangleq \|\mathbf{p}_i - \mathbf{p}_j\|^2_2 \in \mathbb{R}_+, \quad \forall (i, j) \in \mathbb{N}_N \times \mathbb{N}_N.
\]

EDMs appear in a large variety of engineering applications, such as sensor network positioning and localization [1], [2], [3], distributed beamforming problems that rely on second order statistics of the internode channels [4], or molecular conformation [5], where, using NMR spectroscopy techniques, the distances of the atoms forming the protein molecule are estimated, leading to the determination of its structure. Recently, significant attention has been paid to the problem of EDM completion from partial distance measurements in sensor networks, which corresponds to scenarios with limited connectivity among the network nodes (see, e.g., [1]). It can be shown [1] that the rank of an EDM is less or equal than \( d + 2 \) and hence, for a large number of nodes, such a matrix is always of low-rank. Therefore, under certain conditions, matrix completion can be used to recover the missing entries of the matrix. In this paper, we focus on those properties of general random EDMs, which can provide sufficient conditions, under which the EDM completion problem can be successfully solved with high probability.

Relation to the literature - The special case where the node coordinates are drawn from \( U([-1, 1]) \) has been studied in [1], [3], [6]. In our paper, in addition to studying the more general case, we point out some issues in the respective proofs in [3], [6], which result in incorrect results. The correct results can be obtained as a special case of the general results that we propose. More details on the above issues are provided in Section IV.

The paper is organized as follows. In Section II, we provide a brief introduction to the problem of matrix completion, as presented and analyzed in [7]. In Section III, we present our main results. Finally, in Section IV, we discuss the connection of our results with the respective ones presented in [1], [2] and [3].

II. LOW RANK MATRIX COMPLETION

Consider a generic matrix \( \mathbf{M} \in \mathbb{R}^{K \times L} \) of rank \( r \), whose compact Singular Value Decomposition (SVD) is given by

\[
\mathbf{M} = \mathbf{U} \Sigma \mathbf{V}^T \equiv \sum_{i \in \mathbb{N}_r^+} \sigma_i(M) \mathbf{u}_i \mathbf{v}_i^T
\]

and with column and row subspaces denoted as \( U \) and \( V \) respectively, spanned by the sets \( \{ \mathbf{u}_i \in \mathbb{R}^{K \times 1} \}_{i \in \mathbb{N}_r^+} \) and \( \{ \mathbf{v}_i \in \mathbb{R}^{L \times 1}\}_{i \in \mathbb{N}_r^+} \), respectively.

Let \( \mathcal{P}(\mathbf{M}) \in \mathbb{R}^{K \times L} \) denote an entrywise sampling of \( \mathbf{M} \). In all the analysis that follows, we will adopt the theoretical framework presented in [7], according to which one hopes to reconstruct \( \mathbf{M} \) from \( \mathcal{P}(\mathbf{M}) \) by solving the convex program

\[
\begin{align*}
\text{minimize} & \quad \|\mathbf{X}\|_* \\
\text{subject to} & \quad \mathbf{X}(i, j) = \mathbf{M}(i, j), \quad \forall (i, j) \in \Omega,
\end{align*}
\]

where the set \( \Omega \) contains all matrix coordinates corresponding to the observed entries of \( \mathbf{M} \) (contained in \( \mathcal{P}(\mathbf{M}) \)) and where \( \|\mathbf{X}\|_* \) represents the nuclear norm of \( \mathbf{X} \).

Also in [7], the authors introduce the notion of subspace coherence, in order to derive specific conditions under which the solution of (2) coincides with \( \mathbf{M} \). The formal definition of subspace coherence follows, in a slightly more expanded form compared to the original definition stated in [7].

Definition 1. [7] Let \( \mathbf{U} \equiv \mathbb{R}^r \subseteq \mathbb{R}^N \) be a subspace spanned by the set of orthonormal vectors \( \{ \mathbf{u}_i \in \mathbb{R}^{N \times 1}\}_{i \in \mathbb{N}_r^+} \). Also, define the matrix \( \mathbf{U} \equiv [\mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_r] \in \mathbb{R}^{N \times r} \) and let \( \mathbf{P}_U \equiv \mathbf{UU}^T \in \mathbb{R}^{N \times N} \) be the orthogonal projection onto \( U \). Then the coherence of \( \mathbf{U} \) with respect to the standard basis \( \{ \mathbf{e}_i \}_{i \in \mathbb{N}_N^+} \) is defined as

\[
\mu(\mathbf{U}) \triangleq \frac{N}{r} \sup_{i \in \mathbb{N}_N^+} \|\mathbf{P}_U \mathbf{e}_i\|_2^2 = \frac{N}{r} \sup_{i \in \mathbb{N}_N^+} \sum_{k \in \mathbb{N}_N^+} (\mathbf{U}(i, k))^2.
\]
Additionally, the following crucial assumptions regarding the subspaces $U$ and $V$ are of particular importance [7].

\( A_0 \quad \max \{ \mu(U), \mu(V) \} \leq \mu_0 \in \mathbb{R}^{++}. \)

\( A_1 \quad \left\| \sum_{i \in \mathbb{N}^+} u_i v_i^\top \right\|_{\infty} \leq \mu_1 \sqrt{\frac{r}{KL}}, \quad \mu_1 \in \mathbb{R}^{++}. \)

Indeed, if the constants $\mu_0$ and $\mu_1$ associated with the singular vectors of a matrix $M$ are known, the following theorem holds.

**Theorem 1.** [7] Let $M \in \mathbb{R}^{K \times L}$ be a matrix of rank $r$ obeying $A_0$ and $A_1$ and set $N \triangleq \max \{ K, L \}$. Suppose we observe $m$ entries of $M$ with matrix coordinates sampled uniformly at random. Then there exist constants $C, c$ such that if

\[
m \geq C \max \left\{ \mu_0^2, \mu_1^2, \mu_0 \mu_1^{1/4} \right\} N r \beta \log N \tag{4}
\]

for some $\beta > 2$, the minimizer to the program (2) is unique and equal to $M$ with probability at least $1 - c N^{-\beta}$. For $r \leq \mu_0^{-1} N^{1/5}$ this estimate can be improved to

\[
m \geq C \mu_0 N^{6/5} r \beta \log N, \tag{5}
\]

with the same probability of success.

Of course, the lower the rank of $M$, the less the required number of observations for achieving exact reconstruction. Regarding the rank the EDM $\Delta$ at hand, one can easily prove the following lemma [1].

**Lemma 1.** Let $\Delta \in \mathbb{R}^{N \times N}$ be an EDM corresponding to the distances $N$ of points (nodes) in $\mathbb{R}^d$. Then, $\text{rank}(\Delta) \leq d + 2$.

Thus, in the most common cases of interest, that is, when $d$ equals 2 or 3 and for sufficiently large number of nodes $N$, $\text{rank}(\Delta) \ll N$ and consequently the problem of recovering $\Delta$ from a restricted number of observations is of great interest.

**III. THE COHERENCE OF RANDOM EDMs**

According to Theorem 1, the bound of the minimum number of observations required for the exact recovery of $\Delta$ from $\mathcal{P}(\Delta)$, involves the parameters $\mu_0$ and $\mu_1$. Next, we derive a general result, which provides estimates of these parameters in a probabilistic fashion.

**Theorem 2.** Consider $N$ points (nodes) lying almost surely in a $d$-dimensional convex polytope defined as $\mathcal{H}_d \triangleq [a, b]^d$ with $a < 0 < b$ and let $p_i \triangleq [x_{i1}, x_{i2}, \ldots, x_{id}]^\top \in \mathcal{H}_d$, $i \in \mathbb{N}^+$ denote the position of node $i$, where $x_{ij} \in \mathcal{H}_{1,j} \in \mathbb{N}^+_d$ represents its respective $j$-th position coordinate. Assume that each $x_{ij}, (i, j) \in \mathbb{N}^+_N \times \mathbb{N}^+_d$ is drawn independently and identically according to an arbitrary but atomless probability measure $P$ with finite moments up to order 4. Define

\[
m_k \triangleq \mathbb{E}\left\{ x_{ij}^k \right\} = \int x^k \, dP < \infty,
\]

$k \in \mathbb{N}^+_d, \forall (i, j) \in \mathbb{N}^+_N \times \mathbb{N}^+_d$ and let, without any loss of generality, $m_1 \equiv 0$. Also, define $c \triangleq \max \{ |a|, |b| \}$ and pick $a \in [0, 1]$ and a probability of failure $\gamma \in [0, 1]$. Then, as long as

\[
N \geq \frac{2 \theta (\log (d + 2) - \log(\gamma))}{(1 - t)^2}, \tag{6}
\]

the associated Euclidean distance matrix $\Delta$ with worst case rank $d + 2$ obeys the assumptions $A_0$ and $A_1$ with

\[
\mu_0 = \frac{\theta}{t (d + 2)} \quad \text{and} \quad \mu_1 = \frac{\theta}{t \sqrt{d + 2}}, \tag{7}
\]

with probability of success at least $1 - \gamma$, where the constant $\theta$ (that is, independent of $N$) is defined as

\[
\theta \triangleq \frac{1 + 3c^2 + d^2 c^4}{\lambda^*}, \tag{8}
\]

with $\lambda^* \triangleq \min \{ \lambda_1, \lambda_2, \lambda_3, m_2 \}$, and where $\lambda_1, \lambda_2, \lambda_3$ are the real and positive solutions to the cubic equation

\[
\sum_{i \in \mathbb{N}^+} \alpha_i \lambda^i = 0. \tag{9}
\]

with

\[
\alpha_0 \triangleq \left( m_4 m_2 - m_2^3 - m_3^2 \right) d, \tag{10}
\]

\[
\alpha_1 \triangleq -m_2^3 d^2 + \left( m_2^3 + m_3^2 + m_2^2 - m_4 - m_4 m_2 \right) d - m_2, \tag{11}
\]

\[
\alpha_2 \triangleq m_2^2 d^2 + \left( m_4 - m_2^2 \right) d + m_2 + 1 \quad \text{and} \tag{12}
\]

\[
\alpha_3 \triangleq -1. \tag{13}
\]

Before we proceed with the proof of the theorem, let us state a well known result from random matrix theory, the matrix Chernoff bound (exponential form - see [8], Remark 5.3), which will come in handy in the last part of the proof.

**Lemma 2.** [8] Consider a finite sequence of $N$ Hermitian and statistically independent random matrices $\{ F_i \in \mathbb{C}^{K \times K} \}_{i \in \mathbb{N}^+_N}$ satisfying

\[
F_i \geq 0 \quad \text{and} \quad \lambda_{\text{max}}(F_i) \leq R, \quad \forall i \in \mathbb{N}^+_N \tag{14}
\]

and define the constants

\[
\xi_{\text{min(max)}}(F_i) \triangleq \lambda_{\text{min(max)}} \left( \sum_{i \in \mathbb{N}^+_N} \mathbb{E}\{ F_i \} \right). \tag{15}
\]

Define $F_* \triangleq \sum_{i \in \mathbb{N}^+_N} F_i$. Then, it is true that

\[
\mathbb{P}\{ \lambda_{\text{min}}(F_*) \leq t \xi_{\text{min}} \} \leq K \exp \left( -\frac{\xi_{\text{min}}}{2} (1 - t)^2 \right), \tag{16}
\]

\[
\forall t \in [0, 1], \quad \text{and}
\]

\[
\mathbb{P}\{ \lambda_{\text{max}}(F_*) \geq t \xi_{\text{max}} \} \leq K \left( \frac{e^{\xi_{\text{max}}}}{c} \right)^{c \xi_{\text{max}}}, \quad \forall t \geq e. \tag{17}
\]

**Proof of Theorem 2:** In order to make it more tractable, we will divide the proof into the following subsections.
A. Characterization of the SVD of $\Delta$

To begin with, observe that $\Delta$ admits the rank-1 decomposition

$$\Delta = 1_{N} \otimes p^{T} - 2 \sum_{i \in N_{d}^{+}} x_{i} x_{i}^{T} + p 1_{N \times 1},$$

(17)

where

$$p = [\|p_{1}\|_{2}^{2}, \|p_{2}\|_{2}^{2}, \ldots, \|p_{N}\|_{2}^{2}] \in \mathbb{R}_{N}^{+},$$

(18)

$$p_{i} \triangleq [x_{i1}, x_{i2}, \ldots, x_{id}]^{T} \in \mathcal{H}_{d}, \quad i \in N_{d}^{+},$$

(19)

$$x_{i} \triangleq [x_{i1}, x_{i2}, \ldots, x_{in}]^{T} \in \mathcal{H}_{N}, \quad i \in N_{d}^{+}.$$ (20)

Then, we can equivalently express $\Delta$ as

$$\Delta = XDX^{T},$$

(21)

where

$$X \triangleq \text{col} (X_{1}, X_{2}, \ldots, X_{N}) \in \mathbb{R}_{N}^{N \times (d+2)},$$

(22)

$$X_{i} \triangleq [1 \otimes p_{i}^{T}] \in \mathbb{R}_{N}^{1 \times (d+2)}, \quad i \in N_{d}^{+},$$

(23)

$$D \triangleq \text{diag} (1, -21_{1 \times d, 1}) \in \mathbb{R}_{N}^{(d+2) \times (d+2)}.$$ (24)

Since $\Delta$ is a symmetric matrix, it can be easily shown that its SVD possesses the special form

$$\Delta = U |A| (U \cdot \text{sign} (A))^{T} \equiv U \Sigma U^{T},$$

(25)

$$\equiv \sum_{i \in N_{d}^{+}} |\lambda_{i} (\Delta)| u_{i} \left(\text{sign} (\lambda_{i} (\Delta)) u_{i}^{T}\right),$$

(26)

where $|A|$ and $\text{sign} (A)$ denote the entrywise absolute value and sign operators on the matrix $A \in \mathbb{R}^{R \times L}$, respectively. In the expressions above, $\Sigma \equiv |A|, \Delta \in \mathbb{R}_{d \times (d+2)}^{(d+2)}$ is the diagonal matrix containing the (at most $d + 2$) non zero eigenvalues of $\Delta$ in decreasing order of magnitude, denoted as $\lambda_{i} (\Delta), i \in N_{d+2}^{+}$, whose absolute values coincide with its singular values, that is, $\sigma_{i} (\Delta) \equiv |\lambda_{i} (\Delta)|$, and $U \in \mathbb{R}_{N \times (d+2)}^{N \times (d+2)}$ contains as columns the eigenvectors of $\Delta$ corresponding to its non zero eigenvalues, denoted as $u_{i}, i \in N_{d+2}^{+}$, which essentially coincide with its left singular vectors.

Due to this special form of the SVD of $\Delta$, if we denote its column and row subspaces with $U$ and $U_{\pm}$, respectively, it is true that

$$\mu (U_{\pm}) = \frac{N}{d + 2} \sup_{i \in N_{d+2}^{+}} \sum_{k \in N_{d+2}^{+}} (\text{sign} (\lambda_{i} (\Delta)) U (i, k))^{2}$$

$$= \frac{N}{d + 2} \sup_{i \in N_{d+2}^{+}} \sum_{k \in N_{d+2}^{+}} (U (i, k))^{2} \equiv \mu (U).$$

(27)

As a result, at least regarding the Assumption A0, it suffices to study the coherence of only one subspace, say $U$. It is then natural to consider how the SVD of $\Delta$, given by (25), is related to its alternative representation given by (21).

Consider the thin QR decomposition of $X$ given by $X = VA$, where $V \in \mathbb{R}_{N \times (d+2)}^{N \times (d+2)}$ with $V^{T} V = I_{(d+2)}$ and $A \in \mathbb{R}_{(d+2) \times (d+2)}^{(d+2) \times (d+2)}$ constitutes an upper triangular matrix. Then, $\Delta = V A D A^{T} V^{T}$ and since the matrix $A D A^{T} \in \mathbb{R}_{(d+2) \times (d+2)}^{(d+2) \times (d+2)}$ is symmetric by definition, the finite dimensional spectral theorem implies that it is diagonalizable with eigendecomposition given by $A D A^{T} = QA \Lambda Q^{T}$, where $Q \in \mathbb{R}_{(d+2) \times (d+2)}^{(d+2)}$, with $Q Q^{T} = Q^{T} Q = I_{d \times d}$ and $\Lambda \in \mathbb{R}_{(d+2) \times (d+2)}^{(d+2) \times (d+2)}$ is diagonal, containing the eigenvalues of $A D A^{T}$. Thus, we arrive at the expression

$$\Delta = V Q \Lambda Q^{T} V^{T} = V Q |A| \left(V Q \cdot \text{sign} (\Lambda)\right)^{T},$$

(28)

which constitutes a valid SVD of $\Delta$, since $(V Q)^{T} V Q = I_{(d+2)}$ and consequently, by the uniqueness of the singular values of a matrix, $|A| \equiv \Sigma$. Therefore, we can set $U = V Q$ and if $V_{i} \in \mathbb{R}_{(d+2) \times (d+2)}^{1 \times (d+2)}$, $i \in N_{d+2}^{+}$ denotes the $i$-th row of $V$, then

$$\mu (U) = \frac{N}{d + 2} \sup_{i \in N_{d+2}^{+}} \|V_{i} Q_{i}\|_{2}^{2} = \frac{N}{d + 2} \sup_{i \in N_{d+2}^{+}} \|V_{i} \|_{2}^{2},$$

(29)

B. Bounding $\sup_{i \in N_{d+2}^{+}} \|V_{i} \|_{2}^{2}$

Next, consider the hypotheses of the statement of Theorem 2. Since the probability measure $P$ is atomless, the columns of $X$ will be almost surely linearly independent. Then, $\text{rank} (A) = d + 2$ almost surely and consequently

$$\|V_{i}\|_{2}^{2} \leq \frac{\|X_{i}\|_{2}^{2}}{\sigma_{\text{min}}^{2} (A)}, \quad \forall i \in N_{d+2}^{+},$$

(30)

where $X_{i} \in \mathbb{R}_{1 \times (d+2)}^{1 \times (d+2)}$, $i \in N_{d+2}^{+}$ denotes the $i$-th row of $X$. Thus, in order to bound $\mu (U)$ from above, it suffices to bound $\|X_{i}\|_{2}$ from above and $\sigma_{\text{min}} (A)$ from below. Considering that $p_{i}$ is bounded almost surely in $\mathcal{H}_{d}$, we can easily bound $\|X_{i}\|_{2}$ as

$$\|X_{i}\|_{2}^{2} \leq 1 + dc_{2} + d^{2} c_{4}^{2}, \quad \forall i \in N_{d+2}^{+},$$

(31)

where $c \triangleq \max \{|a|, |b| \}$. Regarding $\sigma_{\text{min}}^{2} (A) \equiv \lambda_{\text{min}} \left(A^{T} A\right)$, observe that the Grammian $A^{T} A$ admits the rank-1 decomposition

$$A^{T} A \equiv A T A^{T} V A = X^{T} X = \sum_{i \in N_{d+2}^{+}} X_{T} X_{i},$$

(32)

In general, it seems impossible to find a deterministic lower bound for $\lambda_{\text{min}} \left(A^{T} A\right)$. For this reason, one could resort on probabilistic bounds that are generally a lot easier to derive, providing considerably good estimates. Towards this direction, below we will employ the matrix Chernoff bound (Lemma 2), which fits perfectly to our bounding problem.

Observe that the sequence $\{X_{T} X_{i} \in \mathbb{R}_{N \times N}^{N \times N}\}_{i \in N_{d+2}^{+}}$ consists of $N$ statistically independent random Grammians, which implies that $X_{T} X_{i} \geq 0, \forall i \in N_{d+2}^{+}$. Also, since all these Grammians are rank-1, it can be trivially shown that the one and only non zero eigenvalue of $X_{T} X_{i}$ coincides with $\|X_{i}\|_{2}^{2}$ and consequently

$$\lambda_{\text{max}} \left(X_{T} X_{i}\right) \leq 1 + dc_{2} + d^{2} c_{4}^{2}, \quad \forall i \in N_{d+2}^{+}.$$ (33)

As a result, the hypotheses of Lemma 2 are satisfied and our next task involves specifying the constant $\xi_{\text{min}}$. Since
the coordinates \( x_{ij}, (i, j) \in \mathbb{N}_N^+ \times \mathbb{N}_d^+ \) are independent and identically distributed, it can be easily shown that

\[
\mathbb{E} \left\{ A^T A \right\} = N \mathbb{E} \left\{ X_i^T X_1 \right\} = N \begin{bmatrix} 1 & \cdots & 0_{1 \times d} \\ 0_{d \times 1} & \cdots & m_1 \mathbf{1}_{d} \end{bmatrix} \begin{bmatrix} dm_2 \\ m_3 \mathbf{1}_{1 \times d} \end{bmatrix} d \left( m_4 - m_2^2 + d^2 m_2^2 \right) \triangleq N R_d.
\]

(34)

Thus,

\[
\xi_{min} = \lambda_{min} \left( \mathbb{E} \left\{ A^T A \right\} \right) = N \lambda_{min} (R_d)
\]

and it suffices to characterize the eigenvalues of \( R_d \in \mathbb{R}^{(d+2) \times (d+2)} \).

1) Positive Definiteness of \( R_d \): We first argue that \( R_d \) is a positive definite matrix. We can prove this argument using the strong Law of Large Numbers (LLN). Since \( A \) is almost surely of full rank, the Gramian \( A^T A \) will be almost surely positive definite. Consider the sequence \( \{ A_i^T A_i \}_{i \in \mathbb{N}} \), consisting of independent realizations of \( A^T A \). Of course, \( A_i^T A_i \geq 0, \forall i \in \mathbb{N} \). Then, the strong LLN implies that

\[
\frac{1}{N} \sum_{i \in \mathbb{N}} A_i^T A_i \overset{a.s.}{\to} \mathbb{E} \left\{ A^T A \right\}, \text{ as } N \to \infty.
\]

(35)

As a consequence of the fact that the set of positive definite matrices is closed under linear combinations with nonnegative weights, \( \mathbb{E} \left\{ A^T A \right\} > 0 \) and, therefore, by (34), \( R_d > 0, \forall d \in \mathbb{N}^+ \).

2) Characterization of the eigenvalues of \( R_d \): Next, in order to find the eigenvalues of \( R_d \), we would like to solve the equation

\[
\det (R_d - \lambda I_{d+2}) = 0.
\]

(36)

Using a well known identity, we can rewrite (36) as

\[
\det (R_d - \lambda I_{d+2}) = \det (E - \lambda I_{d+1}) \det \left( (H - \lambda) - G (E - \lambda I_{d+1})^{-1} F \right) = (1 - \lambda) (m_2 - \lambda)^d \cdot \left( d \left( m_4 - m_2^2 \right) + d^2 m_2^2 - \lambda - \frac{d^2 m_2^2}{1 - \lambda} - \frac{dm_2}{m_2 - \lambda} \right) = 0,
\]

where

\[
E \triangleq \begin{bmatrix} 1 & \cdots & 0_{1 \times d} \\ 0_{d \times 1} & \cdots & m_1 \mathbf{1}_{d} \end{bmatrix}, \quad F \triangleq \begin{bmatrix} dm_2 \\ m_3 \mathbf{1}_{1 \times d} \end{bmatrix},
\]

\[
G \triangleq [dm_2 \ m_3 \mathbf{1}_{1 \times d}], \quad H \triangleq d \left( m_4 - m_2^2 \right) + d^2 m_2^2,
\]

and which, after lots of dull algebra, yields the equation

\[
(m_2 - \lambda)^{d-1} \left( \sum_{i \in \mathbb{N}_3} \alpha_i \lambda^i \right) = 0,
\]

(37)

where the coefficients \( \{ \alpha_i \}_{i \in \mathbb{N}_3} \) are given by (10), (11), (12) and (13), respectively.

Thus, \( R_d \) has an eigenvalue \( \lambda_0 \triangleq m_2 \) with multiplicity \( d-1 \) for sure, and three additional eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \), which of course are the roots to the cubic polynomial appearing in (37) and can be computed easily in closed form. Since \( R_d > 0, \forall d \in \mathbb{N}^+ \), all its eigenvalues must be strictly positive and, therefore,

\[
\xi_{min} = N \lambda_{min} (R_d) = N \lambda^*, \quad (38)
\]

where \( \lambda^* \triangleq \min \{ \lambda_1, \lambda_2, \lambda_3, m_2 \} \in \mathbb{R}_{++} \).

C. Putting it altogether

We can now directly apply the matrix Chernoff bound (Lemma 2) to upper bound the probability of the event \( \mathcal{Z} \triangleq \{ \sigma_{min}^2 (A) \leq t N \lambda^* \}, \forall t \in [0, 1] \) as

\[
\mathbb{P} (\mathcal{Z}) \leq (d + 2) \exp \left( - \frac{N (1 - t)^2}{2 \theta} \right) \triangleq \epsilon (t),
\]

(39)

where

\[
\theta \triangleq 1 + dc^2 + d^3 c^4 \lambda^*.
\]

(40)

The inequality (39) is equivalent to the statement that \( \sigma_{min}^2 (A) \geq t N \lambda^* \) will hold with probability at least \( 1 - \epsilon (t) \), \( \forall t \in [0, 1] \). It is natural to select \( N \) such that \( 1 - \epsilon (t) \in [0, 1] \). Since the involved exponential function is strictly decreasing in \( N \), we can choose a \( \gamma \in [0, 1] \) such that \( \epsilon (t) \leq \gamma \), yielding the condition

\[
N \geq \frac{2 \theta (\log (d + 2) - \log (\gamma))}{(1 - t)^2}.
\]

(41)

Consequently, as long as (41) holds, the inequality \( \sigma_{min}^2 (A) \geq t N \lambda^* \) will hold with probability at least \( 1 - \gamma, \forall \gamma \in [0, 1] \).

Hence, since \( ||X_i||^2 \leq 1 + dc^2 + d^3 c^4 \), \( \forall i \in \mathbb{N}_N^+ \) with probability 1, we can write

\[
\mu (U) \leq \frac{N}{d + 2} \sup_{i \in \mathbb{N}_N^+} \frac{||X_i||^2}{\sigma_{min}^2 (A)} = \frac{1 + dc^2 + d^3 c^4}{t \lambda^* (d + 2)} \frac{\mu_0}{t (d + 2)} = \frac{\theta}{t (d + 2)} \mu_0, \quad \forall t \in [0, 1],
\]

(42)

holding true with probability at least \( 1 - \gamma, \forall \gamma \in [0, 1] \) and with \( N \) satisfying (41).

Finally, regarding the Assumption A1, by a simple argument involving the Cauchy-Schwarz inequality, it can be shown that (42)

\[
\mu_1 \triangleq \mu_0 \sqrt{d + 2} = \frac{\theta}{t \sqrt{d + 2}}, \quad \forall t \in [0, 1],
\]

(43)

under the same of course circumstances as \( \mu_0 \), therefore completing the proof.

If the coordinates of the nodes are drawn from a symmetric probability distribution, the following more compact theorem holds.

Theorem 3. Consider \( N \) points (nodes) lying almost surely in a \( d \)-dimensional hypercube defined as \( \mathcal{H}_d \triangleq [-a, a]^d \) with \( a > 0 \) and let the rest of the hypotheses of Theorem 2 hold,
with the additional assumption $m_3 \equiv 0$. Pick a $t \in [0, 1]$ and a probability of failure $\gamma \in [0, 1]$. Then, as long as the condition (6) holds, the associated Euclidean distance matrix $\Delta$ with worst case rank $d + 2$ obeys the assumptions $A0$ and $A1$ with constants $\mu_0$ and $\mu_1$ defined as in (7), respectively, with probability of success at least $1 - \gamma$ and
\[
\theta \triangleq \frac{2(1 + da^2 + d^2a^4)}{\lambda},
\]
where
\[
\lambda^* \triangleq \min \left\{ \zeta - \sqrt{\zeta^2 - 4d(m_4 - m_2^2)}, 2m_2 \right\}
\]
and $\zeta \triangleq d(m_4 - m_2^2) + d^2m_2^2 + 1$.

The proof of Theorem 3 is almost identical to that of Theorem 2. Therefore, it is omitted. Further, if the node coordinates are drawn independently from $U[-1, 1]$, the following corollary constitutes a direct consequence of Theorem 2 and is also presented without proof.

**Corollary 1.** Consider $N$ points (nodes) lying almost surely in the $d$-dimensional hypercube defined as $\mathcal{H}_d \triangleq [-1, 1]^d$ and let $p_i \triangleq [x_{i1}, x_{i2}, \ldots, x_{id}]^T \in \mathcal{H}_d, i \in \mathbb{N}_N^+$ denote the position of node $i$, where $x_{ij} \in \mathcal{H}_1, j \in \mathbb{N}_d$ represents its respective $j$-th position coordinate. Assume that each $x_{ij}, (i, j) \in \mathbb{N}_N^+ \times \mathbb{N}_d$ is drawn independently and identically according to the uniform in $[-1, 1]$ probability measure, $U[-1, 1]$. Pick a $t \in [0, 1]$ and a probability of failure $\gamma \in [0, 1]$. Then, as long as the condition (6) holds, the associated Euclidean distance matrix $\Delta$ with worst case rank $d + 2$ obeys the assumptions $A0$ and $A1$ with constants $\mu_0$ and $\mu_1$ defined as in (7), respectively, with probability of success at least $1 - \gamma$ and
\[
\theta \triangleq \frac{90(1 + d + d^2)}{\zeta - \sqrt{\zeta^2 - 720d}},
\]
where $\zeta \triangleq 5d^2 + 4d + 45$.

**IV. DISCUSSION REGARDING RELATED RESULTS**

It is important to note that for the very special case where the node coordinates are drawn from $U[-1, 1]$ (see Corollary 1), to the best of the authors’ knowledge, there exist two previously published closely related results. In particular, see Theorem 1 in [11] and Remark 4.1.1 and the respective proof in [6], as well as Lemma 2 and the respective proof in [3].

However, there are some subtle technical issues in the aforementioned results. More specifically, in both proofs, the respective authors claim that the minimum eigenvalue of the matrix
\[
\begin{bmatrix}
1 & 0_{1 \times d} & d \\
0_{d \times 1} & \frac{1}{3} & 0_{d \times 1} \\
\frac{d}{3} & 0_{1 \times d} & \left(\frac{d}{3}\right)^2 + \frac{4d}{45}
\end{bmatrix},
\]
which constitutes a special case of the matrix $R_d$ defined in (5), equals $1/3$. One can easily confirm that this is not true.

Additionally, there is a minor oversight in both proofs, where it is implicitly stated that since a Euclidean distance matrix is symmetric, its (real) left singular vectors coincide with its right ones. This argument would be correct only if one could prove that all EDMs belong to the cone of symmetric and positive semidefinite matrices. Further, such an argument is also incorrect since one can easily prove by counterexample that there is at least one EDM that is not positive semidefinite, i.e., with spectrum containing at least one negative eigenvalue. In the case of simply symmetric matrices, the SVD takes the special form of (25). However, we should note here that this mistake by itself does not affect the outcome of the proofs, since the coherence of a matrix essentially depends on the norm of the rows of the matrices whose columns constitute the sets of its left and right singular vectors, respectively.

**V. CONCLUSION**

To the best of the authors’ knowledge, Corollary 1 provides a novel result regarding the coherence of EDMs, for the special case where the node coordinates are independently drawn from $U[-1, 1]$. Furthermore, our main result, Theorem 1 presented above, provides a substantial generalization to Corollary 1, essentially covering any case where the node coordinates are independently drawn according to an arbitrary, non-singular probability measure. Therefore, the theoretical results presented in this paper can provide strong evidence as well as sufficient conditions under which the EDM completion problem can be successfully solved with high probability, a fact that also justifies their direct practical applicability in various modern applications which involve EDMs, such as sensor network localization and estimation of the second order statistics of channel state information in wireless communication networks.

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