THE UNIFICATION OF NON-ASSOCIATIVE STRUCTURES

Florin F. Nichita
Simion Stoilow Institute of Mathematics of the Romanian Academy
21 Calea Grivitei Street, 010702 Bucharest, Romania

Keywords: non-associative structures, associative algebras, Jordan algebras, Lie algebras, Yang-Baxter equation

1. Introduction

The main non-associative structures are Lie algebras and Jordan algebras. Arguably less studied, Jordan algebras have applications in physics, differential geometry, ring geometries, quantum groups, analysis, biology, etc (see [1]).

There are several ways to unify Lie algebras, Jordan algebras and associative algebras. The next section presents structures which unify (non-)associative structures. The last section refers to cases when the unification of (non-)associative structures could be realised just in the conclusions of theorems.

This paper is related to a communication made at the “Workshop on Non-associative Algebras and Applications”, Lancaster University, UK, July 2018.

All tensor products will be defined over the field $k$.

2. Unification Structures

2.1. UJLA structures. The UJLA structures could be seen as structures which comprise the information encapsulated in associative algebras, Lie algebras and Jordan algebras.

Definition 2.1. For a $k$-space $V$, let $\eta : V \otimes V \rightarrow V$, $a \otimes b \mapsto ab$, be a linear map such that:

(1) $(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab)$,

(2) $(a^2b)a = a^2(ba)$, \hspace{0.5cm} $(ab)a^2 = a(ba^2)$, \hspace{0.5cm} $(ba^2)a = (ba)a^2$, \hspace{0.5cm} $a^2(ab) = a(a^2b)$,

$\forall a, b, c \in V$. Then, $(V, \eta)$ is called a UJLA structure.

Remark 2.2. The UJLA structures unify Jordan, Lie and (non-unital) associative algebras.

Remark 2.3. If $(A, \theta)$, where $\theta : A \otimes A \rightarrow A$, $\theta(a \otimes b) = ab$, is a (non-unital) associative algebra, then we define a UJLA structure $(A, \theta')$, where $\theta'(a \otimes b) = \alpha ab + \beta ba$, for some $\alpha, \beta \in k$. For $\alpha = \beta = \frac{1}{2}$, $(A, \theta')$ is a Jordan algebra, and for $\alpha = 1 = -\beta$, $(A, \theta')$ is a Lie algebra.

Theorem 2.4. (Nichita [2]) Let $(V, \eta)$ be a UJLA structure. Then, $(V, \eta')$, $\eta'(a \otimes b) = [a, b] = ab - ba$ is a Lie algebra.
Theorem 2.5. (Nichita [2]) Let \((V, \eta)\) be a UJLA structure. Then, \((V, \eta')\), \(\eta'(a \otimes b) = a \circ b = \frac{1}{2}(ab + ba)\) is a Jordan algebra.

Remark 2.6. The structures from the two above theorems are related by the relation:
\[[a, b \circ c] + [b, c \circ a] + [c, a \circ b] = 0.\]

Remark 2.7. The classification of UJLA structures is an open problem.

2.2. Yang–Baxter equations. The authors of [3] argued that the Yang–Baxter equation leads to another unification of (non-)associative structures.

For \(V\) a \(k\)-space, we denote by \(\tau: V \otimes V \to V \otimes V\) the twist map defined by \(\tau(v \otimes w) = w \otimes v\), and by \(I: V \to V\) the identity map of the space \(V\); for \(R: V \otimes V \to V \otimes V\) a \(k\)-linear map, let
\[R^{12} = R \otimes I, \quad R^{23} = I \otimes R, \quad R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau).\]

Definition 2.8. A Yang-Baxter operator is an invertible \(k\)-linear map, \(R: V \otimes V \to V \otimes V\), which satisfies the braid condition (sometimes called the Yang-Baxter equation):
\[R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}.\]

If \(R\) satisfies (3) then both \(R \circ \tau\) and \(\tau \circ R\) satisfy the quantum Yang-Baxter equation (QYBE):
\[R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}.\]

Therefore, the equations (3) and (4) are equivalent.

For \(A\) be a (unitary) associative \(k\)-algebra, and \(\alpha, \beta, \gamma \in k\), the authors of [4] defined the \(k\)-linear map \(R^{A}_{\alpha, \beta, \gamma}: A \otimes A \to A \otimes A\),
\[a \otimes b \mapsto \alpha ab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b\]
which is a Yang-Baxter operator if and only if one of the following cases holds:
(i) \(\alpha = \gamma \neq 0, \beta \neq 0\); (ii) \(\beta = \gamma \neq 0, \alpha \neq 0\); (iii) \(\alpha = \beta = 0, \gamma \neq 0\).

An interesting property of (5), can be visualized in knot theory, where the link invariant associated to \(R^{A}_{\alpha, \beta, \gamma}\) is the Alexander polynomial.

For \((L, [\cdot, \cdot])\) a Lie algebra over \(k\), \(z \in Z(L) = \{z \in L : [z, x] = 0 \ \forall x \in L\}\), and \(\alpha \in k\), the authors of the papers [5] and [6] defined the following Yang-Baxter operator: \(\phi^{L}_{\alpha}: L \otimes L \to L \otimes L\),
\[x \otimes y \mapsto \alpha [x, y] \otimes z + y \otimes x.\]

Remark 2.9. The formulas (5) and (6) lead to the unification of associative algebras and Lie algebras in the framework of Yang-Baxter structures. At this moment, we do not have a satisfactory answer to the question how Jordan algebras fit in this framework (several partial answers were given).
3. Unification of the Conclusions of Theorems

Sometimes it is not easy to find structures which unify theorems for (non-)associative structures, but we could unify just the conclusions of theorems, as we will see in the next theorems.

**Theorem 3.1.** If $A$ is a Jordan algebra, a Lie algebra or an associative algebra, and if $a, b \in A$, then

$$D : A \to A, \quad D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b$$

is a derivation.

**Proof.** We consider three cases.

If $A$ is a Jordan algebra, then $D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b = a(bx) - (xa)b = a(bx) - b(ax)$. According to [7], $D$ is a derivation.

If $A$ is a Lie algebra, then $D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b = a(bx) - b(ax) = a(bx) + b(xa) = (ab)x$. So, $D$ is a derivation.

If $A$ is an associative algebra, then $D(x) = a(bx) + b(ax) + (ax)b - a(xb) - (xb)a - (xa)b = a(bx - (xa)b) = (ab + ba)x - x(ab + ba)$. So, $D$ is a derivation.

□

**Theorem 3.2.** If $A$ is a Jordan algebra, a Lie algebra or an associative algebra, and if $a, b \in A$, then $D : A \to A, \quad D(x) = a(bx) - (xa)b$ is a derivation.

**Proof.** We consider three cases, and follow similar steps as in the previous proof.

□

**References**

[1] Iordanescu, R. *Jordan structures in geometry and physics with an Appendix on Jordan structures in analysis*, Romanian Academy Press, 2003.

[2] Nichita, F.F. *On Jordan algebras and unification theories*, REVUE ROUMAINE DE MATHEMATIQUES PURES ET APPLIQUES (ROMANIAN JOURNAL OF PURE AND APPLIED MATHEMATICS) Selected papers from the 12th International Workshop on Differential Geometry and its Applications (23-26 September, 2015) TOME LXI NO 4 2016, 305-316.

[3] Iordanescu, R.; Nichita, F.F.; Nichita I.M., *The Yang-Baxter Equation, (Quantum) Computers and Unifying Theories*, Axioms 2014, 3(4), 360-368.

[4] S. Dăscălescu and F. F. Nichita, *Yang-Baxter operators arising from (co)algebra structures*. Comm. Algebra 1999, 27, 5833–5845.

[5] S. Majid, *Solutions of the Yang-Baxter equation from braided-Lie algebras and braided groups*. J. Knot Theory and Its Ramifications 1995, 4, 673-697.

[6] F.F. Nichita and B.P. Popovici, *Yang-Baxter operators from $(G,\theta)$-Lie algebras*. Romanian Reports in Physics 2011, 63(3), 641-650.

[7] Ivan Todorov, Michel Dubois-Violette, *Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra*, arXiv:1806.09450 [hep-th].