Generalized Faraday law derived from classical forces in a rotating frame

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Abstract
We show the additional spin dependent classical force due to the rotation of an electron spin's rest frame is essential to derive a spin-Faraday law by using an analogy with the usual Faraday law. The contribution of the additional spin dependent force to the spin-Faraday law is the same as that of the spin geometric phase. With this observations, Faraday law is generalized to include both the usual Faraday and the spin-Faraday laws in a unified manner.

Key words: geometric phases, dynamical or topological phases, spin-orbit and Zeeman coupling, semiclassical theories for spin
PACS: 03.65.Vf, 71.70.Ej, 03.65.Sq

1. Introduction
Recently generalizations of Faraday law which includes a spin geometric phase or Berry phase [1] have been studied and these motive forces are measured also [2, 3]. The spin motive force is induced when the spin vector potential defined by the spin geometric phase is time varying. The generalization of the motive force to include the spin Berry phase was obtained by using the mathematical equivalence of the Berry phases of electromagnetic and spin origin [4]. This mathematical equivalence, however, has still a lack of a physical origin. In this paper we have studied the physical origin of the generalized motive force using a classical theory of a spin magnetic moment.

The Aharonov-Bohm (AB) phase due to a magnetic vector potential is a manifestation of the Berry phase [5]. In the original work Aharonov and Bohm have introduced a scalar counterpart (SAB) caused by a scalar electric potential in the Schrödinger equation. In 1984 Aharonov-Casher discovered a phase acquired by a neutral particle with a magnetic moment encircling a line of charge has been as the "dual" of the vector AB phase [6] and has been experimentally verified for thermal neutron [7] and for an atomic system [8]. There is also
a scalar AC (SAC) phase accumulated by a neutral particle in magnetic fields during cyclic motion though experiencing no force (9). (In some literature, this is referred to as SAB phase (10).) In a fundamental generalization of Berry’s idea, Aharonov and Anandan (AA) has lifted the adiabatic restriction and could define a nonadiabatic geometric phase, an AA phase (11).

The AC effect has been extended to include electronic systems (12) and the extension for SAC effect is parallel. In AC and SAC effects spin-orbit and Zeeman interactions can be interpreted as non-Abelian vector and scalar potentials coupled to the electronic spin respectively (13; 9). We call these non-Abelian gauge fields as $SU(2)_{\text{spin}}$ gauge fields since the non-Abelian gauge structure stems from the interaction between electromagnetic fields and electronic spins.

When the AB phase is time varying, an electromotive force is induced by Faraday’s law of magnetic induction,

$$E = -\frac{1}{c} \frac{d}{dt} \Phi_B = -\frac{1}{2\pi} \frac{d}{dt} (\Phi_{\text{AB}} \Phi_0) \quad (14),$$

where $\Phi_B$ is a magnetic flux surrounded by a closed path and $\Phi_{\text{AB}}$ and $\Phi_0 = hc/e$ are the AB phase and one flux quantum respectively. The SAB phase does not contribute to the motive force since the electric potential $\phi$ that could generate the SAB phase is associated with a conservative electric field $-\nabla \phi$.

The SAC phase, however, contributes to a motive force when it becomes time-dependent. Stern has noticed that the time-dependent Berry phase accumulated by the electronic spin encircling a ring under the Zeeman interaction induces a motive force motivated by the similarity between the Berry phase and the AB phase (15). Aronov and Lyanda-Geller pointed out that the time-dependent AC flux due to spin-orbit interactions induces a motive force (16). Balatsky and Altshuler argued that a spin motive force can be induced via the Faraday law similarly to Stern’s motive force (17).

According to the dual nature between AB and AC effects the similar Faraday law such as $E_s = -\frac{1}{2\pi} \frac{d}{dt} (\Phi_{\text{AC}} \Phi_0)$ is expected to be satisfied for the magnetic moment, when the AC phase $\Phi_{\text{AC}}$ is time-varying. This could be referred to as spin-Faraday law in analogy with usual Faraday law. Ryu (18) has given a unified view for various spin-motive forces and spin-Faraday laws using a gauge theoretic approach. He has concluded, however, that the exact parallelism may not exist between the spin-motive force and the electromotive force, that is, the spin-Faraday law associated with the AC flux is sometimes not valid. He argued this by obtaining a covariant force on a spin from the classical Lagrangian based on the $SU(2)_{\text{spin}}$ gauge theory.

We will show in this paper that a spin dependent force newly added due to the rotation of a rest frame of the electron spin is essential to derive the spin-Faraday law by using an analogy with the usual Faraday law. In the rotating frame the spin direction is set equal to the $z$-axis of the rest frame so that the additional spin dependent force depends on the electron spin precession. As a result, the line integral of this force for a cyclic path is proportional to the solid angle subtended by the closed trace of the electron spin. Therefore the contribution of this spin dependent force to the spin-Faraday law is the same as that of the spin geometric phase which is half the solid angle subtended by the spin precession. In section 2 we will re derive the AC phase for self-containdedness using invariant operator method, which is useful to solve the Schrödinger-type
equations. In section 3 we will discuss the classical theory for an electron with spin interactions in the rotating rest frame of the electron spin and show the spin-Faraday law is valid in the AC case. The extension of Stern’s result to nonadiabatic case and its observability are discussed. With this observations, a Faraday law is generalized to include both the usual Faraday and the spin-Faraday laws in a unified manner. In section 4 we summarize and discuss our results.

2. Model and AC phase revisited

The nonrelativistic Hamiltonian for an electron in external electromagnetic potentials reduced from the Dirac Hamiltonian in the low-energy field limit is written by

\[ H = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} - \frac{\mu}{c} \sigma \times \mathbf{E} \right)^2 + eA_0 - \mu \sigma \cdot \mathbf{B}, \] (1)

where \( \sigma_i \) with \( i = 1, 2, 3 \) are the Pauli matrices and \( \mu = e\hbar/(2mc) \) is a magnetic moment of the electron. This Hamiltonian has \( U(1) \times SU(2)_{\text{spin}} \) gauge symmetry with \( U(1) \) gauge potential \( A_\nu = (\phi, \mathbf{A}) \) and \( SU(2)_{\text{spin}} \) gauge potential \( b_\nu = (b_0, \mathbf{b}) = (-\sigma \cdot \mathbf{B}, \sigma \times \mathbf{E}) \). The electric field \( \mathbf{E} \) and the magnetic field \( \mathbf{B} \) are measured in the inertial laboratory coordinate system in which Hamiltonian (1) is written.

This Hamiltonian includes all gauge interactions related to our general case, however, we first focus on the AC effect to concentrate on studying additional spin dependent forces due to the rotation of the rest frame of the spin and so it is supposed \( A_\nu = 0 \) and \( b_0 = 0 \). Without loss of generality, the electron is supposed to move in a ring of radius \( a \) for simplicity. Then in cylindrical coordinates \((r, \phi, z)\), the Hamiltonian of the electron becomes \( H = \frac{1}{2ma^2} \left( -i\hbar \frac{d}{d\phi} - \frac{\mu a}{c} b_\phi \right)^2 \), where \( b_\phi = \mathbf{b} \cdot \hat{\phi} \).

The gauge potential \( \mathbf{b} \) helps us to compare two wavefunctions at different space. Then the evolution of the wavefunction \( \Psi(\phi) \) along the ring is described by the equation of parallel transport

\[ i\hbar \frac{d}{d\phi} \Psi(\phi) = -\frac{\mu a}{c} b_\phi \Psi(\phi). \] (2)

This equation is a Schrödinger-type equation as a function of the azimuthal angle \( \phi \), instead of time \( t \). Let \( \hat{H}(\phi) = -\frac{\mu a}{c} b_\phi \), then an invariant operator \( \hat{I}(\phi) \) satisfies the quantum Liouville-type equation

\[ i\hbar \frac{\partial}{\partial \phi} \hat{I}(\phi) + \left[ \hat{I}(\phi), \hat{H}(\phi) \right] = 0. \] (3)

Let us suppose the invariant operator \( \hat{I}(\phi) \) be known by some technique, then the exact quantum state of the Schrödinger-type equation (2) is given by (19)

\[ \Psi(\phi) = e^{i\int_0^\phi \hat{\Psi}_\lambda(\phi')(\mathbf{A}(\phi')+ib_\phi(\phi')) d\phi'} \hat{\Psi}_\lambda(\phi) \] (4)
where \( \tilde{\Psi}_\lambda(\phi) \) is an eigenfunction of the invariant operator \( \hat{I}(\phi) \), 
\[ \hat{I}(\phi) \tilde{\Psi}_\lambda(\phi) = \lambda \tilde{\Psi}_\lambda(\phi). \]

To study a specific and illustrative model, consider a cylindrically symmetric electric field 
\( E(\phi) = E(\cos \chi \hat{r} - \sin \chi \hat{z}) \), \( \chi \) is the tilt angle with respect to the plane on which the ring lies. Then \( b_\phi \) becomes 
\[ E(\phi, \chi, \beta) = E(\cos \phi \sin \beta \hat{r} + \sin \phi \sin \beta \hat{z}). \]

It is easily shown that the invariant operator 
\[ \hat{I}(\phi) = (\cos \phi \sin \beta \sigma_1 + \sin \phi \sin \beta \sigma_2 + \cos \beta \sigma_3) \]
with 
\[ \mu a E/(2\hbar c) = \tan \beta/ (\cos \chi \tan \beta - \sin \chi) \]
 satisfies the quantum Liouville-type equation (3).

The eigenvalue equation of 
\[ \hat{I}(\phi) \]
 is solved with
\[ \tilde{\Psi}_\pm(\phi) = e^{i \int_0^\phi \tilde{\Psi}_\pm(\phi') \left( \frac{\hbar a \sigma_3}{\mu a E(\phi') \hat{I}(\phi')} \right) d\phi'}, \]
where \( \alpha = \mu E a/(2\hbar c) \). This geometric phase is obtained for non-adiabatic case so that this phase is a kind of AA phase. These \( \Phi_1^\pm \) and \( \Phi_2^\pm \) are the same as the dynamical and nonadiabatic geometric (AA) phases in Refs. (20).

3. Classical theory and generalized Faraday law

One of our main purposes is to understand the spin motive force of an electron in a classical context. In non-relativistic quantum mechanics classical forces associated with spin could be given by the generalized Ehrenfest theorem (21)
\[ m \frac{dv^i}{dt} = \left\langle \frac{\partial}{\partial \hat{O}} \left( \hat{p}^i - \frac{\hbar}{c} b^i \right) \right\rangle + i \hbar \left\langle \left[ \hat{O}, \hat{p}^i - \frac{\hbar}{c} b^i \right] \right\rangle \]
for \( v^i = \frac{d}{dt} \langle x^i \rangle = \frac{1}{m} \langle \hat{p}^i - \frac{\hbar}{c} b^i \rangle \), where \( \langle \hat{O} \rangle \) is the expectation value of an operator \( \hat{O} \) in the state \( |\Psi\rangle \).

In non-relativistic quantum mechanics spin is described by the covering group of the rotation group, \( SU(2) \) (22). The representation of the spin operator
is given by Pauli matrices representing non-Abelian properties of the $SU(2)_{\text{spin}}$ gauge potential in our case. The force in Eq. (9) is determined by average values of observables for an wavefunction of the electron. The expectation values of observables become $c$-numbers even for non-commuting observables so that the force in Eq. (9) is composed of $c$-numbers only. Therefore the force in AC effect depends on the expectation values of the electronic spin operators. Since the average values of the spin operators are $c$-numbers, the classical spin operators in a proper classical theory are also desirable to have an Abelian nature.

Ryu has derived the spin-motive force in AC effect by obtaining the spin-dependent force on a spin using covariant derivative of the potential term $U = -\frac{2e}{c}v \cdot b + \mu b_0$ in the Lagrangian operator $L = T - U$ (18). This spin-dependent force could be called as a spin-Lorentz force, analogue to the Lorentz force for an electric charge. The spin Lorentz force is written by

$$F_s = e \left( E_s + \frac{\nabla}{c} \times B_s \right), \quad (10)$$

where $E_s = \frac{\mu}{c} \left( -\nabla b_0 - \frac{i}{\hbar c} \{b, b_0\} \right)$ and $B_s = \frac{\mu}{c} \left( \nabla \times b - \frac{i}{\hbar c} b \times b \right)$. The spin-motive force was defined from this spin Lorentz force as

$$\mathcal{E} = \frac{1}{e} \oint F_s \cdot dl = -\frac{\mu}{ec} \frac{\partial}{\partial t} \oint b \cdot dl \quad (11)$$

in the analogy to the electromotive force. Here $\oint dl$ is the line integral around the closed path. We have defined the spin-motive force by the quotient of the line integral of the spin Lorentz force divided by the electric charge $e$ not magnetic charge $\mu$ since it is convenient to give a unified view of a generalized Faraday law including both electro- and spin-motive forces. The average value of this spin-motive force (11) for the electron in the ring geometry considered in section 2 is calculated as $\mathcal{E} = -\frac{\mu}{2}\langle \frac{d}{dt} \oint b_0 d\phi \rangle_\pm = -\frac{1}{2\pi} \frac{\partial}{\partial t} \langle \Phi_1 \Phi_0 \rangle_\pm$, where $\langle \rceil \rangle_\pm$ represents the expectation value in the wavefunction $\Psi_\pm$ in Eq. (6). This spin-motive force contains only the contribution from the dynamical phase, $\Phi_1$. The spin-Faraday law could not be satisfied since there is no contribution from the spin geometric phase. This is because the spin Lorentz force (10) is not a proper classical force since it involves non-commutative operators which must be averaged over the quantum state.

Therefore it is desired to find the classical spin Lorentz force which consists of only commuting variables. This Abelian nature is achieved by requiring the $z$-direction of the instantaneous coordinate system in the rest frame of the spin always coincides with the direction of the spin. Then the spin operator $\hbar \sigma / 2$ and the corresponding magnetic moment operator $\mu$ in this coordinate system has only the $z$-component, so that the $SU(2)_{\text{spin}}$ gauge field $b$ becomes effectively Abelian. Note that $\mu = es/2mc$. In our case where the motion of the spin is represented by the wavefunction (6), the spin precesses along the $z$ axis of the inertial laboratory coordinate system so that the $z$-axis of the instantaneous coordinate frame of the spin’s rest frame also rotates with respect to an inertial
laboratory frame. This means the instantaneous rest frame of the spin is not an inertial reference frame, so the total time rate of change of the spin in the inertial laboratory frame is given by

$$\frac{ds}{dt} = s \times \left( \frac{e}{mc} \left( B - \frac{v}{c} \times E \right) - \omega \right).$$

Here electromagnetic fields $E$ and $B$ are in the inertial laboratory frame. $\omega$ is the angular velocity of the precession of the spin with respect to the laboratory frame. The additional term $s \times \omega$ is due to the rotation of the rest frame of the spin.

Then the Lagrangian for the spin can be constructed by $L = T - U$, where $T$ is a kinetic energy and $U$ is a generalized potential energy.

$$U = -\mu \cdot \left( B - \frac{v}{c} \times E \right) + s \cdot \omega$$

is the corresponding energy of interaction for the mechanical torque in Eq. (12). The corresponding Hamiltonian is written by

$$H = \frac{1}{2m} \left( p - \frac{1}{c} \mu \times E + \frac{1}{a} s \times \hat{r} \right)^2 - \mu \cdot B,$$

where $v = a \hat{r} \times \omega$ since the angular velocity of the precession of the spin is the same as the angular velocity of rotation along the ring in this case as seen by (6).

Note that the only difference between this Hamiltonian and the Hamiltonian (1) for $A_\mu = 0$ is the additional term $\frac{1}{a} s \times \hat{r}$ due to the rotation of the rest frame.

Under current situation, we can define an effective $U(1)$ gauge potential

$$A_{eff} = \frac{1}{e} \mu \times E - \frac{c}{ea} s \times \hat{r}.$$  

This is because the spin operator $s$ and the magnetic moment operator $\mu$ could be considered as ordinary commuting vectors since they have only one component in the rotating rest frame of the electronic spin (23). This effective vector potential $A_{eff}$ gives the spin-dependent electric and magnetic fields by $E_s = -\frac{d}{dt} A_{eff}$ and $B_s = \nabla \times A_{eff}$. Then the second term in the $A_{eff}$ gives additional spin dependent forces, $-\frac{c}{e a} \nabla \times (s \times \hat{r})$ and $\frac{c}{e a m} (s \times \hat{r})$. When the $A_{eff}$ becomes time dependent, the $-\frac{c}{e a} \nabla \times (s \times \hat{r})$ does not contribute to the spin-motive force since this term is the same characteristic as the usual magnetic field. Therefore the spin-motive force is given by

$$E = \frac{1}{e} \oint F_s \cdot d\mathbf{l} = -\frac{1}{c \, dt} \oint A_{eff} \cdot d\mathbf{l},$$

where $F_s = e \left( E_s + \frac{s}{c} \times B_s \right)$ is the spin Lorentz force.

Note that $\oint \frac{1}{e} \oint \mu \times E \cdot d\mathbf{l}$ gives the dynamical phase times $\Phi_0$. And $\oint (-\frac{c}{e a} s \times \hat{r}) = \Phi_0 \cos \beta \pi$. This gives the same contribution to motive force as the AA phase since the difference between this result and the AA phase is just a constant $-\pi$ except another overall constant $\Phi_0$. This implies the effect of the spin geometric phase on the motive force could be explained by the additional spin Lorentz force generated by the rotation of the coordinate.
system. Therefore the spin-motive force is represented as $-\frac{1}{2} \frac{d}{d\phi} (\Phi_{AC} \Phi_0)$, where $\Phi_{AC}$ is the AC phase. This implies the spin Faraday law is satisfied for the AC effect in general.

Next we will consider the spin Faraday law for the scalar AC effect. In the scalar AC effect the spin-motive force is generated by a time-dependent magnetic field. In Stern’s geometry \((15)\), \(B = -B_\phi(t) \sin \phi \hat{x} + B_\phi(t) \cos \phi \hat{y} + B_z \hat{z}\) and \(E = 0\), the Hamiltonian \((1)\) becomes

$$\mathcal{H} = \frac{1}{2ma^2} \left(-i \hbar \frac{d}{d\phi} - \frac{eB_z \pi a^3}{2c}\right)^2 - \mu \mathbf{B} \cdot \mathbf{\sigma}. \quad (16)$$

Let \(H_0 = \frac{1}{2ma^2} \left(-i \hbar \frac{d}{d\phi} - \frac{eB_z \pi a^3}{2c}\right)^2\) and a wavefunction \(\Psi_0\) be the eigenfunction of \(H_0\), i.e., \(H_0 \Psi_0 = E_0 \Psi_0\) with \(E_0 = \frac{1}{2ma^2} \left(n - \frac{eB_z \pi a^3}{2c}\right)^2\), where \(n\) is an integer.

If we write the wavefunction \(\Psi\) which satisfies the time-dependent Schrödinger equation given by the Hamiltonian \((16)\) as the product of \(\Psi_0\) and \(\Psi_1\), then the time-dependent Schrödinger equation for \(\Psi_1\) is acquired as \(i\hbar \frac{d}{dt} \Psi_1 = -\mu \mathbf{B} \cdot \mathbf{\sigma} \Psi_1\). The quantum Liouville equation with \(\mathcal{H}_1 = -\mu \mathbf{B} \cdot \mathbf{\sigma}\) is satisfied by the invariant operator, \(\hat{I} = -\sin \phi \sin \chi \sigma_1 + \cos \phi \sin \chi \sigma_2 + \cos \chi \sigma_3\). The eigenvalue equation \(\hat{I} \Psi_1 = \lambda \Psi_1\) is solved with the eigenfunctions

$$\tilde{\Psi}_1^\pm = \begin{pmatrix} e^{-i\phi} \cos \frac{\chi}{2} \\ i \sin \frac{\chi}{2} \end{pmatrix}, \quad \tilde{\Psi}_1^- = \begin{pmatrix} -e^{-i\phi} \sin \frac{\chi}{2} \\ i \cos \frac{\chi}{2} \end{pmatrix}, \quad (17)$$

where \(\tan \chi = \mu B_\phi/\hbar \omega + \mu B_z\) and \(\omega = d\phi/dt\) is the angular velocity of the particle encircling the ring. The geometric phase accumulated by the particle encircling the ring is obtained as \(\phi^g_\pm = -\int (\hat{I} \tilde{\Psi}_1^\pm) i \frac{d}{dt} (\tilde{\Psi}_1^\pm) = \pi (1 \pm \cos \chi)\), which is half the solid angle subtended by the cone swept by the spin magnetic moment. It is observed when \(\hbar \omega \ll \mu B_z\), the adiabatic condition holds and the angle \(\chi\) between the \(z\) axis and the spin magnetic moment becomes approximately equal to the angle between the \(z\) axis and the magnetic field which is the same as \(\alpha\) in Ref. \((15)\).

The dynamical phase \(\int_0^\tau (\hat{I} \tilde{\Psi}_1^\pm) (-\mu \mathbf{B} \cdot \mathbf{\sigma}) (\tilde{\Psi}_1^\pm) d\tau\) does not contribute to the motive force since the spin-dependent force \(\mu \mathbf{\nabla} (\mathbf{B} \cdot \mathbf{\sigma})\) related to the dynamical phase is always conservative, that is, \(\oint \mu \mathbf{\nabla} (\mathbf{B} \cdot \mathbf{\sigma}) \cdot d\mathbf{l} = 0\). Where \(\tau\) is the time spent for one cyclic motion. When the magnetic field is time dependent the geometric phase becomes time-dependent, and the particle is subject to a motive force \(\mathcal{E}^\pm = -\frac{d(\hat{I} \phi^g_\pm)}{dt}\) in analogy with Faraday law.

In this case the spin is also precessing under the magnetic field \(\mathbf{B}\) so that the rest frame of the spin is rotating with respect to the laboratory frame. Therefore a classical Lagrangian for this spin has the additional term \(-\mathbf{v} \cdot (\mathbf{s} \times \mathbf{r})/a\) originated from the rotation of the electron’s rest frame. Then the motive force derived by the classical spin-dependent force becomes

$$\mathcal{E} = \frac{1}{c} \frac{d}{dt} \int \left(\frac{c}{ea} \mathbf{s} \times \dot{\mathbf{r}}\right) \cdot d\mathbf{l} = -\frac{d}{dt} \left(\Phi_g^+ \Phi_0\right), \quad (18)$$
since the only spin Lorentz force contributing to spin motive force is \( \frac{1}{c} \frac{\partial}{\partial t} (s \times \hat{r}) \). Therefore the motive force derived by the classical force satisfies the spin Faraday law.

The motive force is easily generalized for the case of the Hamiltonian (Ⅲ) in which both \( U(1) \) electromagnetic and \( SU(2)_{\text{spin}} \) interactions coexist. The AB and SAB phases could be understood on equal footing as the phase accumulated to a wavefunction through a cyclic evolution by \( \left( \frac{\hbar}{e c} \int_{C} A_{\nu} dx_{\nu} \right) \), where \( \nu = 0, 1, 2, 3 \) and \( A_{\nu} = (\phi, A) \) is an \( U(1) \) electromagnetic four vector potential. Using the \( SU(2)_{\text{spin}} \) gauge field we could also describe the AC phase and the SAC phase together. The total phase acquired by the electron in cyclic motion is given by an exponent of the phase factor \( \exp \left( \frac{\mu}{\hbar c} \int_{C} b_{\nu} dx_{\nu} \right) \) for the special initial state, which returns to the initial one after a cyclic evolution apart from a phase factor. The total phase, however, is not the simple sum of the AC phase and the SAC phase since the \( SU(2)_{\text{spin}} \) gauge potentials do not commute.

The phase accumulated to a wavefunction in the general case is obtained by the parallel transporter

\[
\exp \left( \frac{e}{\hbar c} \int_{C} A_{\nu} dx_{\nu} \right) P \exp \left( \frac{\mu}{\hbar c} \int_{C} b_{\nu} dx_{\nu} \right). \tag{19}
\]

Since the electric potential \( \phi \) is given by a conservative force \( -\nabla \phi \), only the AB phase given by the magnetic vector potential contributes to motive forces. The second path integral is path-ordered since it contains noncommuting operators. This path integral gives AC and scalar AC phase under a cyclic evolution with a special wavefunction which returns its original form apart from the overall phase factor. The total dynamical phase becomes the simple sum of the dynamical phase in the AC effect and that in the scalar AC effect. This is because the path-ordered integral could be performed by invariant operator method. In Eq. (Ⅲ) the dynamical phase is given by the integral of the averaged operators which becomes effectively commutative. The nonadiabatic geometric phase, however, is determined by the geometry of the precession of the spin determined by the combined effects of both \( SU(2)_{\text{spin}} \) gauge fields, \( b \) and \( b_{0} \). That is, the precession angle of the spin is not the simple sum of the precession angles for AC and scalar AC effects. Therefore the spin geometric phase (AA phase) does not become just the simple sum of each AA phases in the AC and SAC effects.

The dynamical phase in the AC effect contributes to the motive force as shown in section Ⅱ. The dynamical phase in the SAC effect, however, does not contribute to motive forces as we have seen in the above Stern’s example. Therefore the phases contributing to motive forces are the AB phase and the dynamical phase of the AC phase and the spin geometric phase which is given by the combined effect of the scalar and vector \( SU(2)_{\text{spin}} \) gauge fields. The existence of the geometric phase implies the precession of the spin so that the instantaneous rest frame of the spin is rotating. This rotation of the rest frame gives an additional effective \( U(1) \) gauge potential \( -\frac{e}{c} s \times \hat{r} \) in the classical Hamiltonian as shown. Hence in the rotating rest frame of the electron spin, the effective \( U(1) \) gauge potential under \( SU(2)_{\text{spin}} \) gauge interaction becomes the same as \( A_{\text{eff}} \) of Eq. (Ⅲ).
The spin dependent forces that produce generalized motive force are \(-\frac{e}{c} \frac{d}{dt} A\) and \(-\frac{e}{c} \frac{d}{dt} A_{\text{eff}}\), where \(A\) is the magnetic vector potential. Therefore when these phases vary in time, they generates the generalized motive force as

\[
\mathcal{E} = \frac{1}{e} \oint \mathbf{F} \cdot d\mathbf{l} = -\frac{d}{dt} \oint (A + A_{\text{eff}}).
\]

(20)

This generalized motive force is easily calculated and could be represented as

\[-\frac{1}{2\pi} \frac{d}{dt} (\Phi_{AB} + \Phi_{\text{dyn},AC} + \Phi_{\text{geo}}) \Phi_0.\]

Here \(\Phi_{AB}\), \(\Phi_{\text{dyn},AC}\), and \(\Phi_{\text{geo}}\) are the AB phase, the dynamical phase in the AC effect, and the spin geometric phase respectively. Let \(\Phi_T = \Phi_{AB} + \Phi_{\text{dyn},AC} + \Phi_{\text{geo}}\), then the same form with the ordinary Faraday law is acquired for the generalized Faraday law including both the electro- and the spin-motive forces as \(\mathcal{E} = -\frac{1}{2\pi} \frac{d}{dt} (\Phi_T \Phi_0)\).

4. Conclusion and Discussion

In summary we have obtained the generalized Faraday law in a unified manner using the classical spin Lorentz forces in the rotating rest frame of the electron spin. The spin is intrinsically quantum object, however we have shown that the spin can be successfully described by a classical magnetic moment when the rotation of the rest frame is correctly considered. In this picture the effect on the spin-motive force of the time varying spin geometric phase is understood as the same as that of the new spin-dependent classical force generated by the spin precession.

The motive force in Ref. (15) has an amplitude of \(10^{-7} V\) in adiabatic condition. In the nonadiabatic case the magnitude of \(\hbar \omega\) in \(\tan \chi\) is approximately \(10^{-23} J\) and \(\mu B_z \approx 9.27 \times 10^{-24} J\) for \(B_z = 1 T\) so that the effect of time-dependent non-adiabatic geometric phase on the generalized motive force could also be observed. The model Hamiltonian of Refs. (2, 3) have similarities to that of Ref. (15), since the interaction between the internal exchange field and the spin is the Zeeman-type. With our picture the physical origin of the generalized Faraday law mentioned in Refs. (2, 21) is satisfied could be clearly understood.

Acknowledgements

This work was supported by the Korea Research Foundation Grant(KRF-2008-331-C00073), a research grant from Seoul Women’s University(2009). We gratefully acknowledge KIAS members and Dr. Sam Young Cho for helpful discussions.

References

[1] M. V. Berry, Proc. R. Soc. A 392 (1984) 45.
[2] S. E. Barnes and S. Maekawa, Phys. Rev. Lett. 98 (2007) 246601; S. E. Barnes, J. Ieda, and S. Maekawa, Appl. Phys. Lett. 89 (2006) 122507; M. Yamanouchi, J. Ieda, F. Matsukura, S. E. Barnes, S. Maekawa, and H. Ohno, Science 317 (2007) 1726; S. E. Barnes Phys. Lett. A 310 (2007) 2035.

[3] S. A. Yang, G. S. D. Beach, C. Knutson, D. Xiao, Q. Niu, M. Tsoi, and J. L. Erskine, Phys. Rev. Lett. 102 (2009) 067201.

[4] For details see, e.g., D.J. Griffiths, Introduction to Quantum Mechanics (Prentice-Hall, New Jersey, 1995).

[5] Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959); M. Peshkin and A. Tonomura, The Aharonov-Bohm Effect (Springer-Verlag, Berlin, 1989).

[6] Y. Aharonov and A. Casher, Phys. Rev. Lett. 53 (1984) 319.

[7] A. Cimmino, G.I. Opat, A.G. Klein, H. Kaiser, S.A. Werner, M. Arif, and R. Clothier, Phys. Rev. Lett. 63 (1989) 380.

[8] K. Sangster, E.A. Hinds, S.M. Barnett, and E. Riis, Phys. Rev. Lett. 71 (1993) 3641.

[9] J. Anandan, Phys. Lett. A 138 (1989) 347; 152 (1991) 504.

[10] G. Badurek, H. Weinfurter, R. Gähler, A. Kollmar, S. Wehinger, and A. Zeilinger, Phys. Rev. Lett. 71 (1993) 307.

[11] Y. Aharonov and J. Anandan, Phys. Rev. Lett. 58 (1987) 1593.

[12] H. Mathur and A. D. Stone, Phys. Rev. Lett. 68 (1992) 2964.

[13] A. S. Goldhaber, Phys. Rev. Lett. 62 (1989) 482.

[14] See, e.g., J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1998).

[15] A. Stern, Phys. Rev. Lett. 68 (1992) 1022.

[16] A. G. Aronov and Y. B. Lyanda-Geller, Phys. Rev. Lett. 70 (1993) 343.

[17] A. V. Balatsky and B. L. Altshuler, Phys. Rev. Lett. 70 (1993) 1678.

[18] C. -M. Ryu, Phys. Rev. Lett. 76 (1996) 968.

[19] H. R. Lewis, Jr., Phys. Rev. Lett. 27 (1967) 510; J. Math. Phys. 9 (1968) 1976; H. R. Lewis, Jr. and W. B. Riesenfeld, J. Math. Phys. 10 (1969) 1458; S. P. Kim, J. Korean Phys. Soc. 43 (2003) 11.

[20] T. -Z. Qian and Z. -B. Su, Phys. Rev. Lett. 72 (1994) 2311.

[21] K.-H. Yang, Ann. Phys. 101 (1976) 62.
[22] J. J. Sakurai, Modern Quantum Mechanics (Addison-Wesley Publishing Company, 1994).

[23] For the spin with eigenvalue $-\hbar$ in Eq. (5) an effective $U(1)$ gauge potential becomes $-\mathbf{A}_{\text{eff}}$ which is just the negative of the effective $U(1)$ gauge potential for the spin with the $+$ eigenvalue. So we focus on the spin with $+$ in the followings.