Steady Prandtl Layer Expansions with External Forcing

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Abstract

In this article we apply the machinery developed in [GI18] together with a new compactness estimate and an object called the degree in order to prove validity of steady Prandtl layer expansions with external forcing.

1 Introduction and Notation

We consider the steady, incompressible Navier-Stokes equations on the two-dimensional domain, \((x,Y) \in \Omega = (0,L) \times (0,\infty)\). Denoting the velocity \(U_{NS} := (U_{NS}, V_{NS})\), the equations read:

\[
\begin{align*}
U_{NS} \cdot \nabla U_{NS} + \nabla P_{NS} &= \varepsilon \Delta U_{NS} + g_{ext} \\
\nabla \cdot U_{NS} &= 0
\end{align*}
\]

in \(\Omega\) \hspace{1cm} (1)

The system above is taken with the no-slip boundary condition on \(\{Y = 0\}\):

\[
[U_{NS}, V_{NS}]|_{Y=0} = [0,0]. \hspace{1cm} (2)
\]

Here, the function \(g_{ext} = (g_{ext}^{(u)}, g_{ext}^{(v)})\) is an external force which vanishes in the inviscid limit. The form of the forcing we treat is given in (7).

In this article, we fix an outer Euler shear flow of the form \([u_0^0(Y), 0,0]\), (satisfying certain assumptions given in (14a) - (14c)). A fundamental question is to describe the asymptotic behavior of solutions to (1) as the viscosity vanishes, that is as \(\varepsilon \to 0\). Generically, there is a mismatch of the tangential velocity at the boundary \(\{Y = 0\}\) of the viscous flows, \(\{2\}\), and inviscid flows. Thus, one cannot expect \([U_{NS}, V_{NS}] \to [u_0^0, 0]\) in a sufficiently strong norm (for instance, \(L^\infty\)).

To rectify this mismatch, it was proposed in 1904 by Ludwig Prandtl that there exists a thin fluid layer of size \(\sqrt{\varepsilon}\) near the boundary \(Y = 0\) that bridges the velocity of \(U_{NS}|_{Y=0} = 0\) with the nonzero Eulerian velocity. This layer is known as the Prandtl boundary layer.

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We work with the scaled boundary layer variable:

\[ y = \frac{Y}{\sqrt{\varepsilon}} \]  

(3)

Consider the scaled Navier-Stokes velocities:

\[ U^\varepsilon(x, y) = U^{NS}(x, Y), \quad V^\varepsilon = \frac{V^{NS}(x, Y)}{\sqrt{\varepsilon}}, \quad P^\varepsilon(x, y) = P^{NS}(x, Y). \]  

(4)

Equation (1) now becomes:

\[
\begin{align*}
U^\varepsilon U^\varepsilon_x + V^\varepsilon U^\varepsilon_y + P^\varepsilon_x &= \Delta^\varepsilon U^\varepsilon + \varepsilon^{(u)} g_{\text{ext}}^u, \\
U^\varepsilon V^\varepsilon_x + V^\varepsilon V^\varepsilon_y + P^\varepsilon_y &= \Delta^\varepsilon V^\varepsilon + \varepsilon^{(v)} g_{\text{ext}}^v/\sqrt{\varepsilon}, \\
U^\varepsilon_x + V^\varepsilon_y &= 0,
\end{align*}
\]  

(5)

We expand the solution in \( \varepsilon \) as:

\[ U^\varepsilon = u_0^e + \sum_{i=1}^{n} \sqrt{\varepsilon} (u_i^e + u_i^p) + \varepsilon^p u^e(\varepsilon) := u_s + \varepsilon^{N0} u^e(\varepsilon), \]

(6)

\[ V^\varepsilon = v_0^p + v_1^e + \sum_{i=1}^{n-1} \sqrt{\varepsilon} (v_i^p + v_i^{e+1}) + \sqrt{\varepsilon} v^n_p + \varepsilon^{N0} v^e(\varepsilon) := v_s + \varepsilon^{N0} v^e(\varepsilon), \]

(6)

\[ P^\varepsilon = P_0^p + \sum_{i=1}^{n} \sqrt{\varepsilon} (P_i^e + P_i^p) + \varepsilon^p P^e(\varepsilon) := P_s + \varepsilon^{N0} P^e(\varepsilon), \]

where the coefficients are independent of \( \varepsilon \). Here \([u^e_i, v^e_i]\) are Euler correctors, and \([u^p_i, v^p_i]\) are Prandtl correctors. These are constructed in the Appendix, culminating in Theorem 35.

Correspondingly, we expand the forcing into \( g_{\text{ext}} := \begin{pmatrix} g_{\text{ext}}^{(u)} \\ g_{\text{ext}}^{(v)} \end{pmatrix} \), which is given by:

\[
\begin{pmatrix}
\sum_{i=1}^{n} \sqrt{\varepsilon} (g^{u,i}_{\text{ext},e} + g^{u,i}_{\text{ext},p}) + \varepsilon^{N0} u^e(\varepsilon) \\
\sqrt{\varepsilon} g^{v,1}_{\text{ext},e} + \sum_{i=1}^{n-1} \sqrt{\varepsilon} (g^{v,i}_{\text{ext},e} + g^{v,i+1}_{\text{ext},p}) + \varepsilon^{N0} v^{e}(\varepsilon)
\end{pmatrix}
\]  

(7)

The main assumption on the forcing applies to \( g^{u,1}_{\text{ext},p} \), and is given in (14c).

For our analysis, we will take \( n = 4 \) and \( p = \frac{3}{2} + \). Let us also introduce the following notation:

\[ \tilde{u}_p^i := u_p^i - u_p^i|_{y=0}, \quad \tilde{v}_p^i := v_p^i - v_p^i|_{y=0}, \quad \tilde{v}_c^i := v_c^i - v_c^i|_{y=0}. \]  

(8)

\footnote{Our result also applies if we add a leading order term \( g^{v,0}_{\text{ext},e} = -1 \), which accounts for gravity.}
The profile \( u_0, v_0 \) from (8) is classically known as the “boundary layer”; one sees from (6) that it is the leading order approximation to the Navier-Stokes velocity, \( U^\varepsilon \). The final layer, 
\[
[u^{(\varepsilon)}, v^{(\varepsilon)}, P^{(\varepsilon)}] = [u^s, P^s].
\]
are called the “remainders” and importantly, they depend on \( \varepsilon \). Controlling the remainders uniformly in \( \varepsilon \) is the fundamental challenge in order to establish the validity of (6), and the centerpiece of our article.

We begin by briefly discussing the approximations, \( [u_s, v_s] \). The particular equations satisfied by each term in \( [u_s, v_s] \) is recorded in Appendix A. We record Theorem 35, which is proven in companion paper [GI18]. We are prescribed the shear Euler flow, \( u_0^e \). The profiles \( [u_0^p, v_0^p] \) are Prandtl boundary layers. Importantly, these layers are rapidly decaying functions of the boundary layer variable, \( y \). At the leading order, \( [u_0^p, v_0^p] \) solve the nonlinear Prandtl equation:
\[
\begin{align*}
\bar{u}_0^p & u_0^p + \bar{v}_0^p u_0^p - u_0^p p_y + P_0^p x = 0, \\
v_0^p & = 0, \quad P_0^p = 0, \quad u_0^p |_{x=0} = U_0^p, \quad u_0^p |_{y=0} = -u_0^e |_{Y=0}. 
\end{align*}
\]

Soon after Prandtl’s seminal 1904 paper, Blasius discovered the celebrated self-similar solution to (9) (with zero pressure). This solution reads
\[
[u_0^p, v_0^p] = \left[ f'(\eta), \frac{1}{\sqrt{x + x_0}} \{ \eta f'(\eta) - f(\eta) \} \right], \quad \text{where} \quad \eta = \frac{y}{\sqrt{x + x_0}}.
\]
where \( f \) satisfies
\[
ff'' + f''' = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \quad \frac{f(\eta)}{\eta} \xrightarrow{\eta \to \infty} 1.
\]
Here, \( x_0 \geq 0 \) is a free parameter. It is well known that \( f''(\eta) \) has a Gaussian tail, and that the following hold:
\[
0 \leq f' \leq 1, \quad f''(\eta) \geq 0, \quad f''(0) > 0, \quad f'''(\eta) < 0.
\]
Moreover, the normal velocity satisfies the asymptotics:
\[
\lim_{\eta \to \infty} \frac{1}{\sqrt{x + x_0}} \{ \eta f'(\eta) - f(\eta) \} = v_1^e |_{Y=0} > 0
\]

Such a Blasius profile has been confirmed by experiments with remarkable accuracy as the main validation of the Prandtl theory (see [Sch00] for instance). These profiles are also canonical from a mathematical standpoint in the following sense: the work, [Ser66], has proven that when \( x \) gets large (downstream), solutions to the Prandtl equation, (9), converge to an appropriately renormalized Blasius profile. Therefore, validating the expansions (9) for the Blasius profile is the main objective and motivation in our study.

Our main assumptions are imposed on the prescribed \( \{ x = 0 \} \) data for the first order Euler approximation for the normal velocity and on the leading order shear flow. Denote by \( u_0 | = \bar{u}_0^p |_{x=0} \), and \( v_0 | = \bar{v}_0^p |_{x=0} \). To state our assumptions we need to define the notion of “degree”:
Definition 1 The degree, \( d \), of a function \( f(y) \) is

\[
d(f) := \int_0^\infty K(y)I_y[f] \, dy, \quad K(y) := u_0 e^{-\int_0^y v_1}, \quad I_y[f] := -\int_y^\infty f(z) \, dz.
\]  

(13)

We now delineate our main assumptions:

\( v_1^1 |_{x=0} > 0 \) for \( Y \geq 0 \),

\[
\|u_0^0\|_{Y^2} \|_{\infty} = o(1),
\]

\[
| \int_0^\infty K(y)g_{ext,p}^{u,1}(y) \, dy | \gtrsim 1.
\]

(14a, 14b, 14c)

We will refer to (14c) as the “non-degeneracy condition”. As we will point out below, these assumptions are certainly consistent with the Blasius layer, (10), and include a large class of shear flows for \( u_0^0 \) (see Subsection 2.4).

We will also require the following finiteness assumptions on the Euler layers:

\( v_1^1 |_{x=0} \) decays either algebraically or exponentially as \( Y \uparrow \infty \),

\[
\sup_{Y} |v_1^1 |_{x=0} | < \infty.
\]

(15)

The system satisfied by the remainders, \([u(\varepsilon), v(\varepsilon), P(\varepsilon)]\), in vorticity formulation gives:

\[
-R[\eta(\varepsilon)] - u_{yy}(\varepsilon) + 2\varepsilon u_{xy}(\varepsilon) + \varepsilon^2 v_x(\varepsilon) + v_s \Delta_x u(\varepsilon) - u(\varepsilon) \Delta v_s = \varepsilon N_0 \{ u(\varepsilon) \Delta v(\varepsilon) - v(\varepsilon) \Delta u(\varepsilon) \} + g,
\]

(16)

Here, \( \Delta_x := \partial_{yy} + \varepsilon \partial_{xx} \), \( g \) is a forcing term that we will not discuss further in the introduction, and where we have defined the Rayleigh operator:

\[
R[\eta(\varepsilon)] = \partial_y \{ u_s^2 \partial_y \eta(\varepsilon) \} + \varepsilon \partial_x \{ u_s^2 \partial_x \eta(\varepsilon) \}, \quad \eta(\varepsilon) := \frac{v(\varepsilon)}{u_s}.
\]

(17)

The boundary condition we take are the following:

\[
v(\varepsilon) |_{x=0} = a_0^x(y), \quad v(\varepsilon) |_{x=L} = a_3^x(y), \quad v(\varepsilon) |_{x=0} = a_2^x(y), \quad v(\varepsilon) |_{x=L} = a_3^x(y)
\]

\[
v(\varepsilon) |_{y=0} = v(\varepsilon) |_{y=\infty} = 0.
\]

(18)

Here, the \( a_i^x(y) \) are prescribed boundary data which we assume satisfy:

\[
\| \partial_j^j \partial_s^j \left\{ \frac{1}{\sqrt{\varepsilon}} \langle y \rangle \right\} + \frac{\langle y \rangle}{v_1^1 |_{x=0}} \right\| \leq o(1) \text{ for } j = 0, \ldots, 4,
\]

(19)

which is a quantitative statement that the expansion (10) is valid at \( \{ x = 0 \} \) and \( \{ x = L \} \).

We are now able to state our main result, so long as we remain vague regarding the norm \( \| \cdot \|_X \) that appears below. A discussion of this norm will be in Subsection 1.2.
Theorem 2 (Main Theorem) Assume boundary data and forcing are prescribed as in Theorem 35 and satisfying the assumptions (14a) - (14c), (15), and (19). Let \(0 < \varepsilon \ll L \ll 1\). Then all terms in the expansion (6) exist and are regular, \(\|u_s, v_s\|_\infty \lesssim 1\). The remainders, \([u(\varepsilon), v(\varepsilon)]\) exists uniquely in the space \(X\) and satisfy:

\[
\|u^r\|_X \lesssim 1. \quad (20)
\]

The Navier-Stokes solutions satisfy:

\[
\|U^{NS} - u_0^0 - u_0p\|_\infty \lesssim \sqrt{\varepsilon} \quad \text{and} \quad \|V^{NS} - \sqrt{\varepsilon}v_0^0 - \sqrt{\varepsilon}v_1\|_\infty \lesssim \varepsilon. \quad (21)
\]

Upon establishing the uniform bound (20), the result (21) follows from the following inequalities: \(\|v\|_\infty \lesssim \varepsilon^{-\frac{1}{2}}|u|_X\), and \(\|u\|_\infty \lesssim \varepsilon^{-\frac{1}{2}}|u|_X\). These are established in Lemmas 23 and 31 together with the definitions in (23).

We also note that thanks to (12), the assumption (14a) is consistent with the Blasius profile.

1.1 Notation

Before we state the main ideas of the proof, we will discuss our notation. Since we use the \(L^2\) norm extensively in the analysis, we use \(\|\cdot\|\) to denote the \(L^2\) norm. It will be clear from context whether we mean \(L^2(\mathbb{R}_+)\) or \(L^2(\Omega)\). When there is a potential confusion (for example, when changing coordinates), we will take care to specify with respect to which variable the \(L^2\) norm is being taken (for instance, \(L^2_0\) means with respect to \(dy\), whereas \(L^2_Y\) will mean with respect to \(dY\)). Similarly, when there is potential confusion, we will distinguish \(L^2\) norms along a one-dimensional surface (say \(\{x = 0\}\)) by \(\|\cdot\|_{x=0}\). Analogously, we will often use inner products \((\cdot, \cdot)\) to denote the \(L^2\) inner product. When unspecified, it will be clear from context if we mean \(L^2(\mathbb{R}_+)\) or \(L^2(\Omega)\). When there is potential confusion, we will distinguish inner products on a one-dimensional surface (say \(\{x = 0\}\)) by writing \((\cdot, \cdot)_{x=0}\).

We will often use scaled differential operators: \(\nabla_\varepsilon := (\partial_x, \sqrt{\varepsilon}\partial_y)\) and \(\Delta_\varepsilon := \partial_{yy} + \varepsilon \partial_{xx}\). For functions \(u : \mathbb{R}_+ \to \mathbb{R}\), we distinguish between \(u^\prime\) which means differentiation with respect to its argument versus \(u_y\) which refers to differentiation with respect to \(y\).

Regarding unknowns, the central object of study in our paper are the remainders, \([u(\varepsilon), v(\varepsilon)]\). By a standard homogenization argument (see subsection A.4), we may move the inhomogeneous boundary terms \(a_i^\varepsilon\) to the forcing and consider the homogeneous problem. We call the new unknowns \([u, v]\), and these are actually the objects we will analyze throughout the paper.

When we write \(a \lesssim b\), we mean there exists a number \(C < \infty\) such that \(a \leq Cb\), where \(C\) is independent of small \(L, \varepsilon\) but could depend on \([u_s, v_s]\). We write \(o_L(1)\) to refer to a constant that is bounded by some unspecified, perhaps small, power of \(L\): that is, \(a = o_L(1)\) if \(|a| \leq CL^\delta\) for some \(\delta > 0\).
We will, at various times, require localizations. All such localizations will be defined in terms of the following fixed $C^\infty$ cutoff function:

$$
\chi(y) := \begin{cases} 
1 & \text{on } y \in [0, 1) \\
0 & \text{on } y \in (2, \infty) 
\end{cases} \quad \chi'(y) \leq 0 \text{ for all } y > 0. 
$$  

(22)

We will use $\| \cdot \|_{loc}$ to mean localized $L^2$ norms. More specifically we take for concreteness $\| \cdot \|_{loc} := \| \cdot \chi(\frac{y}{r_1}) \|$.

We will define now the key norms that appear throughout our analysis:

**Definition 3** Given a weight function $w = w(y)$, define:

$$
||v||_{X_w} := \varepsilon^{-\frac{1}{2}||v||_{w}} + ||q||_{w}, \\
||v||_{Y_w} := ||v||_{w} + \sqrt{\varepsilon} ||q||_{w}, \\
||v^{00}||_R := ||u^{00}_{ypp}(y)|| + ||u_{yy}(y)|| + L ||u^{00}_w|| + ||u^0||_{loc} \\
||v^{00}||_B := ||u^{00}_1||_R + \varepsilon \frac{3}{2} \varepsilon |\kappa| + \varepsilon \frac{3}{2} \sqrt{v^{00}_{c}}|_{x=0}, \\
||v^0, v||_X := ||u^0_B + \varepsilon \frac{3}{2} \varepsilon |v||X_1 + \varepsilon \frac{3}{2} \varepsilon |v||Y_1 + \varepsilon \frac{3}{2} \varepsilon |v||Y_{loc}, \\
||q||_{w} := ||\nabla \varepsilon q_{x} \cdot u_{w}|| + ||\sqrt{v_{s}}(q_{ypp} + \sqrt{\varepsilon q_{xyy}, \varepsilon q_{xx}v_{xx}x}w)|| + |q|_{loc},w \\
||v||_{w} := ||\{\varepsilon v_{xyyy}, \varepsilon v_{xxxy}, \varepsilon \frac{3}{2} v_{yxy}, \varepsilon \frac{3}{2} v_{xxy}, \varepsilon v_{xx}v_{xx}x\}w|| + |v|_{loc},w \\
|q|_{loc},w := ||u_{s}q_{x}w||_{x=0} + \varepsilon q_{xy}w||_{y=0} + ||\sqrt{v_{s}}u_{s}w||_{x=L} + ||q_{yy}w||_{y=0} \\
|v|_{loc},w := \varepsilon \frac{3}{2} \sqrt{v_{s}v_{xxy}w||}_{x=0} + ||\varepsilon v_{xyyy}w||_{y=0} + ||\varepsilon u_{s}v_{xyy}w||_{x=0}. 
$$  

(23)

1.2 Overview of Proof

Let us first recap the ideas introduced in [GN14], which treated the case when the boundary $\{ y = 0 \}$ was moving with velocity $u_b > 0$. First, let us extract:

$$
\text{Leading order operators in } (21) = -R[q] - u_{yy}, 
$$  

(24)

Due to the nonzero velocity at the $\{ y = 0 \}$ boundary, the quantity $\bar{u}|_{y=0} > 0$. A central idea introduced by [GN14] is the coercivity of $R[q]$ over $||\nabla q||$. This coercivity relied on the fact that $q = \frac{\bar{u}}{3} = 1 \notin \text{Ker}(R)$, thanks to the nonzero boundary velocity of $\bar{u}|_{y=0}$. Extensive efforts without success have been made to extracting coercivity from $R[q]$ in the present, motionless boundary, case. However, it appears that this procedure interacts poorly with the operator $\partial_{yy}u$, producing singularities too severe to handle.

**Part 0: The Central Objects**

Our main idea is based on the observation that the $x$ derivative of (23) produces, at leading order:

$$
-\partial_{x}R[q] + v_{yyy}. 
$$  

(25)

6
Unlike (24), these two operators enjoy better interaction properties.

To this end, we split the equation (16) into two pieces that are linked together. First, we take $\frac{\partial}{\partial x}$ of (16) (call this “DNS” for Derivative Navier-Stokes) to obtain:

$$\text{DNS}(v) := -\partial_x R[q] + \Delta^2 v + J(v) = -F_{u^0} + \varepsilon N_0 + g(q),$$

$$v|_{x=0} = v_x|_{x=L} = v_{xx}|_{x=0} = v_{xxx} = 0. \quad (26)$$

Here, $F_{u^0}$ contains the $u^0$ dependencies, which arise through $u = u^0 - \int_0^x v_y$, and is defined as

$$F_{u^0} := v_{sx} u_{yy}^0 - u^0 \Delta_x v_{sx}. \quad (27)$$

$N$ contain nonlinear terms and $g(q)$ contains forcing terms, all of which are defined in (111). Note also the change in notation in (26) as we have dropped the superscript $\varepsilon$, and homogenized the boundary conditions on the sides $\{x = 0\}, \{x = L\}$.

The second piece is to study the boundary trace, $u^0 = u|_{x=0}$. By evaluating the vorticity equation (16) at $\{x = 0\}$, we obtain the following system for $u^0$:

$$\mathcal{L} u^0 := -u^0_{yyy} + v_s u^0_{yy} - u^0 \Delta_x v_x = F_{(v)} + g(u),$$

$$F_{(v)} := -2\varepsilon v_s u_{xx} q_x|_{x=0} - 2\varepsilon v_{xyy}|_{x=0} - \varepsilon^2 v_{xxx}|_{x=0} + \varepsilon v_s v_{xy}|_{x=0}.$$ \hspace{1cm} (28)

As is evident, the right-hand side of (28) depends on (derivatives of) $v|_{x=0}$. The term $g(u)$ is a forcing term which is specified in (110).

Thus, the approach we take is to analyze (26) to control $v$ in terms of the boundary trace, $u^0$, and then to analyze (28) in order to control the boundary trace $u^0$ in terms of $v$. We may schematize this procedure via:

$$u^0 \xrightarrow{\text{DNS}} v \xrightarrow{\mathcal{L}^{-1}} u^0. \quad (29)$$

We then recover a solution to the original Navier-Stokes equation (NS) via a fixed point of (29). This structure of analysis gives rise to a linked set of inequalities (see below, (38), for the scheme of estimates).

**Part 1: $\mathcal{L}^{-1}$ and Boundary Estimate of $u^0$**

Let us turn now to the system, (28). We first decompose the coefficient $v_s$ (refer back to (6) for the definition) in two different ways:

$$v_s = \begin{cases} v_{||} + \bar{v}_1^e + \sqrt{\varepsilon} \bar{v}_1^p + \text{higher order terms}, \\ v_0^p + \bar{v}_1^e + \sqrt{\varepsilon} v_0^p + \text{higher order terms} \end{cases}, \quad (30)$$

where $v_{||}$ has been defined above in (32), and $\bar{v}_1^e, \bar{v}_1^p$ are both defined in (8). The key point of these definitions can be gleaned by examining the leading order of
\( v \), which is \( v_p^0 + v_e^1 \). Both of these quantities decay at \( y = \infty \). In the first case of (30), we rewrite this sum as \( \{ v_p^0 - v_p^0 \}_{y=0} + \{ v_e^1 - v_e^1 \}_{y=0} = \hat{v}_p^1 + \hat{v}_e^1 \) where both of these quantities do not decay at \( y = \infty \). Correspondingly, we have a decomposition of the operator \( L \) into:

\[
L u^0 := L \parallel u^0 + \sqrt{\varepsilon} A u^0 + \hat{v}_e^1 u_{yy}^0 + \text{ higher order terms,}
\]

\[
L \parallel u^0 := -u_{yy}^0 + v^0 u_{yy}^0 - v^0 v_{yy},
\]

\[
A := \hat{v}_p^1 u_{yy}^0 - u_{yy}^0 \hat{v}_p^1 v_{yy}.
\]

(31)

It is first important to study the spectrum of \( L \parallel \). Our first key observation is that \( \hat{u}_p^0 |_{x=0} \) is an element in \( \operatorname{Ker}(L \parallel) \) thanks to the Prandtl equation, (9). Correspondingly, we decompose \( u_0 \) in the following manner:

**Definition 4** Define “parallel” profiles:

\[
u_\parallel = \hat{u}_p^0 |_{x=0} \quad \text{and} \quad v_\parallel = \hat{v}_e^1 |_{x=0},
\]

(32)

and the corresponding decomposition:

\[
u^0 = u_\perp + \kappa u_\parallel, \quad \kappa := \frac{\omega[u^0]}{\omega[u_\parallel]},
\]

(33)

Here \( \omega \) is a linear map satisfying \( 0 < \omega[u_\parallel] < \infty \). We refrain at this time from discussing the particular choice for \( \omega \); this level of detail can be found in Section 2. The reason for the use of “parallel” and, correspondingly, “perpendicular” is because \( u_\parallel \) is in the kernel of the crucial operator, \( L \parallel \) (see below, (31)) whereas \( u_\perp \) is orthogonal to the kernel.

Our first estimate, (see Lemma 20), leads to the following lower bound:

\[ \| L \parallel u_\perp \| \geq \| u_\perp \| \quad \text{where} \quad u_\perp \perp u_\parallel. \]

(34)

The outcome of Lemma 20 is then:

\[ \| u_\perp \| \lesssim o(1)\varepsilon \frac{\kappa}{|\kappa|} + \| F(u) \| + \| u_{yy} \| \] contributions.

(35)

Our second ingredient is to control the coefficient \( \kappa \), the “parallel” component of \( u^0 \), in the decomposition, (33). For this, we use the equation (28). In particular, the operator \( S u_\parallel = A u_\parallel + \hat{v}_e^1 u_{yy} - \sqrt{\varepsilon} \Delta v_e^1 u_\parallel \) is utilized to control this projection. To do this, we require the non-degeneracy and smallness condition in (14a) - (14c). Specifically, Lemma 21 gives

\[ \varepsilon \frac{\kappa}{|\kappa|} \lesssim \| u_\perp \| \quad \text{and} \quad \varepsilon \frac{\kappa}{|\kappa|} \| F(u) \| + \| u_{yy} \| \] contributions.

(36)

Lastly, we need to show that \( \hat{v}_e^1 u_{yy} \) is a small perturbation. However, it is evident that \( \hat{v}_e^1 u_{yy} \) is an order 1 term. It is therefore difficult to imagine that these terms can be treated perturbatively. The key feature that we capitalize on is that \( \hat{v}_e^1 \) exhibits Euler scaling, and it thus suffices to localize to the far-field.
region: $Y = \frac{1}{\sqrt{\epsilon}} \gtrsim 1$. To capitalize on this localization, we employ the second decomposition, (30.2), under which most coefficients have decayed rapidly and become negligible in the region $y \gtrsim \frac{1}{\sqrt{\epsilon}}$. It is in this estimate that we demand $v^1_e|_{x=0} > 0$ in order to extract a lower bound from $v^1_e u^0_{yy}$.

Combining the estimates (34), (36) with Lemma 22, we are able to prove the following estimate

$$\|u^0\|_B \lesssim \|F(u)\{w_0 + \epsilon^{\frac{1}{4}} \langle y \rangle^\frac{1}{2}\}\|$$

(37)

Part 2: Solving DNS for $v$

We now turn our attention to (26). The goal is to establish control over the norms $\|\cdot\|_{y_{w_0}}, \|\cdot\|_{Y_1}, \|\cdot\|_{X_1}$. Since the DNS equation is the same as in [GI18], we simply state the following:

**Proposition 5** There exists a unique solution, $v$, to the system (26) that satisfies (at the linear level):

$$\begin{align*}
\|u^0\|_B^2 &\lesssim \epsilon^{\frac{1}{4}} - \|v\|_{Y_1}^2 + \epsilon^{\frac{1}{4}} \|v\|_{y_{w_0}}^2 + \epsilon^{\frac{1}{4}} \|v\|_{X_1}^2 + \text{Data} \\
\|v\|_{Y_1}^2 &\lesssim \epsilon^{-\frac{1}{4}} \|u^0\|_B^2 + \text{Data} \\
\|v\|_{y_{w_0}}^2 &\lesssim \epsilon^{\frac{1}{4}} \|v\|_{X_1}^2 + \epsilon^{\frac{1}{4}} \|u^0\|_B^2 + \text{Data} \\
\|v\|_{X_1}^2 &\lesssim a_L(1) \|v\|_{X_1}^2 + L \epsilon^{\frac{1}{4}} \|u^0\|_B^2 + \text{Data}.
\end{align*}$$

(38)

Above $w_0$ is a specific weight, which for the purposes of the present discussion we will set to be $\frac{\langle w \rangle}{\langle y \rangle^3 \langle x \rangle^3}$. It is clear that the above scheme of estimates closes to yield control over $\|u^0, v\|_X$.

1.3 Other Works

Let us now place this result in the context of the existing literature. To organize the discussion, we will focus on the setting of stationary flows in dimension 2. This setting in particular occupies a fundamental role in the theory, as it was the setting in which Prandtl first formulated and introduced the idea of boundary layers for Navier-Stokes flows in his seminal 1904 paper, [Pr1905].

In this context, one fundamental problem is to establish the validity of the expansions (6). This was first achieved under the assumption of a moving boundary in [GN14] for $x \in [0, L]$, for $L$ sufficiently small. The method of [GN14] is to establish a positivity estimate to control $\|\nabla_x v\|_{L^2}$, which crucially used the assumed motion of the boundary. Several generalizations were obtained in [Ly15], [Ly16], [Ly17], all under the assumption of a moving boundary. First, [Ly16] considered flows over a rotating disk, in which geometric effects were seen, [Ly16] considered flows globally in the tangential variable, and [Ly17] considered outer Euler flows that are non-shear.

The classical setup of a nonmoving boundary, considered by Prandtl, has remained open until recently. We would like to highlight the exciting paper of
as well as \[GI18\] which both treat the no-slip boundary condition. These works are mutually exclusive. The main concern of \[GI18\] treats the classical self-similar Blasius solution which appears to not be covered by \[GVM18\]. On the other hand, \[GI18\] result does not cover a pure shear boundary layer of the form \((U_0(y), 0)\) since such shears are not a solution to the homogeneous Prandtl equation.

For unsteady flows, expansions of the form \((\ref{expansion})\) have been verified in the following works: \[SC98\], \[SC98\], \[GVMM16\], \[Mae14\]. The reader should also see \[As91\], \[MT08\], \[TW02\], \[Ke08\], \[LMN08\], \[BT13\] for related results. There have also been several works \([GGN15a\), \[GGN15b\), \[GGN15c\], \[GrNg17a\], \[GrNg17b\], \[GrNg18\]) establishing generic invalidity of expansions of the type \((\ref{expansion})\) in Sobolev spaces in the unsteady setting.

A related question is that of wellposedness of the Prandtl equation (the equation for \[\bar{u}\], as defined in \((\ref{bar_u})\)). Since this is not the concern of the present article, we very briefly list some works. In the stationary setting, we point the reader to \[OS99\], \[Ol67\], \[DM15\]. In the unsteady setting, the reader should consult \[AWXY15\], \[MW15\], \[KMVW14\], \[SC98\] - \[SC98\], \[KV13\], \[LCS03\], \[IV16\], and \[GVM13\], \[GVD10\], \[GVN12\], \[EE97\], \[KVW15\], \[HH03\] for wellposedness/ill-posedness results and references therein.

The above discussion is not comprehensive; we refer the reader to the review articles, \[E00\], \[GJT16\] and references therein for a more thorough review of the wellposedness theory.

2 \(L^{-1}\) and Boundary Estimates for \(u^0\)

In this section, we study the quantity \(u^0 = u|_{x=0}\).

Remark 6 For this section, we will work exclusively on the boundary \(\{x = 0\}\). Therefore, all functions (even those with natural extensions to all of \(\Omega\)) will be thought of as functions of \(y\) only. Similarly, inner-products and norms will refer to functions defined on \(\mathbb{R}_+\). We will therefore omit the notation \(|_{x=0}\).

We define the norm,

\[
\|h\|_\Upsilon := \|h_{yy}(y)\| + \|h_y(y)\| + \|h\|_{loc} + \|h\|_{loc}.
\] (39)

We define the corresponding Sobolev space \(\Upsilon\) via:

\[\Upsilon := \{h \in L^2 : \|h\|_\Upsilon < \infty\}\.
\]

A standard embedding shows that:

\[
\|h_y\| + \|h(y)^{-1}\| \lesssim \|h\|_\Upsilon.
\]
In this section, the equation we will analyze is:

\[
\begin{align*}
L^\delta u^0 &:= -u^0_{yyy} + (v_s + \delta)u^0_y - u^0 \Delta v_s = F, \\
F &:= -2\varepsilon u_y g_x |_{x=0} - 2\varepsilon v_{xxy} |_{x=0} - \varepsilon v_{xxx} |_{x=0} + g(w), \\
u^0(0) &= 0, \partial_y u^0(\infty) = 0, \partial_{yy} u^0(\infty) = 0.
\end{align*}
\]

We are ultimately interested in the \( \delta = 0 \) case of (40), which is (28). We will first use a decomposition of \( v_s \) in order to decompose the operator \( L^\delta \). We first recall the definitions in (8) and (32), and correspondingly decompose \( v_s \) into:

\[
v_s = v_\parallel + \sqrt{\varepsilon} \bar{v}^1_p + \sum_{i=2}^{n} \sqrt{\varepsilon} \bar{v}^i_p + \sum_{i=1}^{n} \sqrt{\varepsilon}^{-1} v^i_e.
\]

We thus have a decomposition of:

\[
L^\delta = L_\parallel + \sqrt{\varepsilon} A + J,
\]

where

\[
\begin{align*}
L_\parallel u^0 &:= -u^0_{yyy} + v_\parallel u^0_y - u^0 v_{yy}, \\
A u^0 &:= \bar{v}^1_p u^0_y - u^0 v_{py}^1, \\
J u^0 &:= \delta u^0_{yy} + u^0_{yy} \sum_{i=1}^{n} \sqrt{\varepsilon}^{-1} \bar{v}^i_p + u^0_{yy} \sum_{i=2}^{n} \sqrt{\varepsilon} v^i_p - \varepsilon u^0 \sum_{i=0}^{n} \sqrt{\varepsilon} v^i_{pxx} - u^0 \sum_{i=1}^{n} \sqrt{\varepsilon} v^i_{pyy} - \varepsilon u^0 \sum_{i=1}^{n} \sqrt{\varepsilon}^{-1} \Delta v^i_e.
\end{align*}
\]

Define the following bounded linear functional on \( \Upsilon \):

\[
\omega[g] := (g_{yy}, u_{yy}), \quad \omega : \Upsilon \rightarrow \mathbb{R}.
\]

It is clear that \( \omega \) is bounded on \( \Upsilon \). As a result, we define:

\[
\Upsilon_\perp := \{ g \in \Upsilon : \omega[g] = 0 \},
\]

which is a closed subspace of \( \Upsilon \). We will now project via:

\[
u^0 = u_\perp + \kappa u_\parallel, \quad \kappa := \frac{\omega[u^0]}{\omega[u_\parallel]}.
\]

We note that \( 0 < \omega[u_\parallel] < \infty \). We now denote our \( B \) norm by:

\[
\|u^0\|_B := \|u_\parallel\|_\Upsilon + \varepsilon^{\frac{1}{4}} |\kappa| + \|\varepsilon^{\frac{3}{4}} u_{yy} \frac{(y)}{\sqrt{v_c}}\|.
\]

\textbf{Theorem 7} Assume the boundary data satisfy (14a) - (14c) and (15). Then there exists a unique solution to (40) satisfying:

\[
\|u^0\|_B \lesssim \|F\{w_0 + \varepsilon^{\frac{1}{4}} (y) \frac{1}{2} + \}\|.
\]

This section will be devoted to establishing this theorem.
2.1 Existence

We begin with an existence result. To state our existence result, we will assume the a-priori estimate that we will establish in the forthcoming sections:

$$\|u^0\|_B \lesssim \|F w_1\|$$ for \(u^0 \in B\) solutions to (43),

(47)

where \(w_1 := w_0 + \varepsilon^{-\frac{1}{2}}(y)^{\frac{1}{2}}\).

**Step 1: Highest Order Equation**

We may start with the modified problem:

$$\Theta u := -u'' + \delta u'' = F \in C_0^\infty, \quad u(0) = 0, \quad \partial^k_y u(\infty) = 0 \text{ for } k \geq 1. \quad (48)$$

**Lemma 8** Let \(\delta > 0\). There exists a unique solution, \(u\), to (48) whose derivatives vanishes outside of an interval \([0, I_0]\) for \(I_0 < \infty\). Moreover, \(u\) can be expressed explicitly in terms of \(F\) via the formula \(u = C + u_p\), where \(u_p\) is the particular solution associated to \(F\), as defined below.

**Proof.** By assuming the solution \(u = e^{\gamma y}\), we obtain the system \(-r^3 + \delta r^2 = 0\), which gives the following solutions to the homogeneous problem: \(u_1 = 1, u_2 = y, u_3 = e^{\delta y}\). The task is now to produce a particular solution to the forcing, \(F\).

We may form the Wronskian matrix and its Forcing counterparts:

\[
W(y) = \begin{bmatrix}
1 & y & e^{\delta y} \\
0 & 1 & \delta e^{\delta y} \\
0 & 0 & \delta^2 e^{\delta y}
\end{bmatrix}
\quad W_1(y) = \begin{bmatrix}
0 & y & e^{\delta y} \\
0 & 1 & \delta e^{\delta y} \\
F & 0 & \delta^2 e^{\delta y}
\end{bmatrix}
\quad W_2(y) = \begin{bmatrix}
1 & 0 & e^{\delta y} \\
0 & 0 & \delta e^{\delta y} \\
0 & F & \delta^2 e^{\delta y}
\end{bmatrix}
\quad W_3(y) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & F
\end{bmatrix}
\]

The corresponding determinants are:

\[
W(y) = \det W = \delta^2 e^{\delta y}, \quad W_1(y) = \det W_1 = F e^{\delta y} (\delta y - 1)
\]

\[
W_2(y) = \det W_2 = -\delta F e^{\delta y}, \quad W_3(y) = \det W_3 = F.
\]

We now have the formula for the particular solution: \(u_p = c_1 u_1 + c_2 u_2 + c_3 u_3, \quad c_i = \frac{W_{ij}}{W}\). Solving gives: \(c_1 = -\int_y^\infty F \delta^{-3}(\delta y' - 1) \, dy', \quad c_2 = \int_y^\infty \delta^{-1} F \, dy', \quad c_3 = -\int_y^\infty F \delta^{-2} e^{\delta y'} \, dy'\). Recalling that \(F\) is compactly supported, we thus have the boundary conditions: \(\partial^j_y u_p(\infty) = 0 \text{ for all } j \geq 0, u_p(0) = c_1(0) + c_3(0)\).

We will thus write the full solution as: \(u = C_1 + C_2 y + C_3 e^{\delta y} + u_p\). We now evaluate at \(y = \infty\) to see that \(C_2 = C_3 = 0\). We evaluate at \(y = 0\) to see that \(C_1 = -u_p(0)\). This gives the full solution to (48), \(u = C_1 + u_p\). Given this solution \(u = C_1 + u_p = -u_p(0) + c_1 u_1\), it is clear that the derivatives of \(u\) are compactly supported since the coefficients \(c_i\) are.

Fix \(\delta > 0\). Define the normed space:

\[
\|u\|_{T_{m,n}} := \|u_{yy}(y)^m e^{n y}\| + \delta \|u_{yy}(y)^m e^{n y}\|
\]
Lemma 9 Fix $\delta > 0$. Let $F \in C_0^{\infty}(\mathbb{R})$. The solution, $u$ is in $\mathcal{T}_{m,n}$ and satisfies the following estimate: $\|u\|_{\mathcal{T}_{m,n}} \lesssim \|F(y)^m e^{ny}\|$ for any $m, n$.

**Proof.** We now square both sides of the equation (48) against the weight $\langle y \rangle^{2m} e^{2ny}$ to obtain

$$|u''(y)^m e^{ny}|^2 + \delta^2 |u''(y)^m e^{ny}|^2 - 2\delta(u'', u''(y)^m e^{2ny}) = \|F(y)^m e^{ny}\|^2.$$  

We integrate by parts the cross term, which is possible due to the compact support of $u$, and which generates the positive contribution

$$2\delta m \|u''(y)^m e^{ny}\|^2 + 2\delta n \|u''(y)^m e^{ny}\|^2.$$ 

\[ \Box \]

**Step 2: Arbitrary $F(y)^m e^{ny} \in L^2$**

We now remove the compact support hypothesis on $F$.

Lemma 10 Fix $\delta > 0$. Let $F(y)^m e^{ny} \in L^2$ for any $n, m$. Then there exists a unique solution $u \in \mathcal{T}_{m,n}$ to the system (48) that satisfies $\|u\|_{\mathcal{T}_{m,n}} \lesssim \|F(y)^m e^{ny}\|$.  

**Proof.** Given an arbitrary $F$ satisfying $\|F(y)^m e^{ny}\| < \infty$, there exists $\tilde{F}_j \in C_0^{\infty}$ such that $\tilde{F}_j \to F(y)^m e^{ny}$ in $L^2$. Define $F_j \equiv \frac{1}{y^m e^{ny}} \tilde{F}_j$ so that $F_j(y)^m e^{ny} \to F(y)^m e^{ny}$ in $L^2$ and clearly $F_j \in C_0^{\infty}$ as well. We define $u_j$ as solutions to $\Theta u_j = F_j$. In this case, we have $\|u_j\|_{\mathcal{T}_{m,n}} \lesssim \|F_j(y)^m e^{ny}\| \leq \|F(y)^m e^{ny}\|$ by the previous lemma. Thus, there exists a $\mathcal{T}_{m,n}$-weak* limit, called $u$. We now multiply by a compactly supported test function, $\phi$, and integrate: $(w_j', \phi') + (v_s u_j', \phi) - (\Delta_s v_s u_j, \phi) = (F_j, \phi)$. It is clear we can pass to the limit to obtain a strong solution $u$, which moreover enjoys the estimate $\|u\|_{\mathcal{T}_{m,n}} \lesssim \|F(y)^m e^{ny}\|$. 

\[ \Box \]

**Step 3: Compact Perturbations**

Lemma 11 Fix $\delta > 0$. Let $Fw_0 \in L^2$. Then there exists a unique solution, $u$, satisfying $L_3 u = F$, $u(0) = \partial y u(\infty) = 0$ for $l \geq 1$ which satisfies the bound $\|u\|_B \lesssim \|Fw_1\|$.  

**Proof.** First, express $L_3 u = \Theta u + Ku = F$, where $K = v_s \partial_y^2 - \Delta_s v_s$. Next, it is straightforward to see $\Theta^{-1} K : H^2 \to H^2$ is a compact operator. Thus, by the Fredholm alternative we must exhibit uniqueness of the homogeneous problem, that is with $F = 0$. Assume $u \in H^2$ is a solution to $\Theta u = -Ku$. Then by applying the previous lemma, $u \in \mathcal{T}_{m,n}$ for any $m, n$. This in particular implies that $u \in B_{\gamma}$. Our assumed $a$-priori estimate, (47), may then be used to yield uniqueness of the homogeneous solution and also the bound $\|u\|_B \lesssim \|Fw_1\|$. 

\[ 13 \]
Step 4: $\delta \downarrow 0$

Our estimates are uniform in $\delta$, and thus we can subsequently take the limit as $\delta \downarrow 0$ in the standard manner. We have thus established by combining Steps 1 - 4:

**Proposition 12 (Existence)** Let $Fw_1 \in L^2$, and let $\delta \geq 0$. Assume the a-priori estimate, (47) holds. Then there exists a unique solution to $L_\delta u^0 = F$, $u^0(0) = \partial_y^l u^0(\infty) = 0$ for $l \geq 1$ which satisfies the bound $\|u^0\|_B \lesssim \|Fw_1\|$.

2.2 Coercivity of $L_\parallel$

The main proposition here is:

**Proposition 13** Let $u_\perp$ be defined as in (44). Then:

$$\|L_\parallel u_\perp \cdot \langle y \rangle\| \gtrsim \|u_\perp\|_\mathcal{T}. \quad (49)$$

**Lemma 14** Elements of the three dimensional kernel of $L_\parallel$ can be written as the following linear combination: $c_1 u_\parallel + c_2 \tilde{u}_s + c u_p$, where $c_1, c_2, c \in \mathbb{R}$. Here:

$$\tilde{u}_s := u_\parallel \int_1^y \frac{u_\parallel(1)^2}{u_\parallel} \exp \left[ \int_1^z v_\parallel \, dw \right] \, dz,$$

$$u_p := \tilde{u}_s \int_0^y u_\parallel \exp \left[ - \int_1^z v_\parallel \right] - u_\parallel \int_0^y \tilde{u}_s \exp \left[ - \int_1^z v_\parallel \right].$$

**Proof.** Integrating up once, we define the operator $L_\parallel^1 := -u_{yy} + v_\parallel u_y - w v_\parallel y$. One solution to the homogeneous equation, $L_\parallel^1 u = 0$, is $u_\parallel$. By supposing the second spanning solution is of the form $\tilde{u}_s := u_\parallel a(y)$, we may derive the equation: $a''(y) = \left[ v_\parallel - 2 \frac{u_\parallel(1)^2}{u_\parallel} \right] a'(y)$. Solving this equation gives one solution:

$$a'(y) = \frac{u_\parallel(1)^2}{u_\parallel} \exp \left[ \int_1^y v_\parallel \right], \quad a(y) = \int_1^y \frac{u_\parallel(1)^2}{u_\parallel} \exp \left[ \int_1^z v_\parallel \right] \, dz. \quad (50)$$

We shall need asymptotic information about $\tilde{u}_s$:

**Lemma 15** As defined in Lemma 14, $\tilde{u}_s$ satisfies the following asymptotics:

$$\tilde{u}_s|_{y=0} \sim -1 \text{ and } \tilde{u}_s y \sim 1 \text{ as } y \downarrow 0,$$

$$u_{sy}, \tilde{u}_{sy}, \tilde{u}_s \sim \exp[v_\parallel(\infty) y] \text{ as } y \uparrow \infty. \quad (51)$$

**Proof.** For convenience, denote

$$g(y) = \exp \left[ \int_1^y v_\parallel \right].$$
By rewriting \( v_\| = v_\|(\infty) + [v_\| - v_\|(\infty)] \), and using that the latter difference decays rapidly, we obtain the basic asymptotics \( g \sim \exp[|v_\|(\infty)|] \) as \( y \uparrow \infty \). An expansion of \( a \), given in (30), near \( y = 0 \) gives \( a(y) \approx \int_1^y \frac{1}{x^2} \, dz \sim -\frac{1}{2} \| v_\| = 1 - \frac{1}{y} \).

Thus: \( \tilde{u}_s|_{y=0} \sim u_\|[1 - \frac{1}{y}] \sim -1 \). At \( y = \infty \), we have the asymptotics:

\[
\tilde{u}_s = u_\| \int_1^y \frac{u_\|(1)^2}{u_\|} g(z) \sim \int_1^y \exp[|v_\| y] \, dz \sim \exp[|v_\| y].
\]

We now differentiate to obtain

\[
\tilde{u}_{sy} = u_\| a + u_\| a'(y) = u_\| y a(y) + \frac{u_\|(1)^2}{u_\|} g(y) \sim \exp[|v_\| y].
\]

To evaluate \( \tilde{u}_{sy} \) at \( y = 0 \), we need more precision. Expansions give:

\[
u\| a'(y) = \frac{u_\|(1)^2}{u_\|} g(y) \sim \frac{u_\|(1)^2}{u_\| y_0} g(y) \text{ for } y \sim 0, \text{ and}
\]

\[
u\| y a(y) \sim u_\| y_0 u_\| (1)^2 \int_1^y \frac{1}{u_\|} g(z) \sim \frac{u_\|(1)^2}{u_\| y_0} \int_1^y \frac{g(z)}{z^2} \, dz.
\]

We have used the fact that \( \frac{1}{u_\|} \) does not contribute a factor of \( \frac{1}{z} \) following the singularity of \( \frac{1}{z} \). Indeed, Taylor expanding, using that \( u_\| y_0(0) = u_\| y_0 y_0(0) = 0 \) (see the first identity in (120)), and the elementary identity for any \( a, b \in \mathbb{R} \), \( \frac{1}{a-b} = \frac{1}{b} \), one obtains:

\[
\frac{1}{u_\|(1)^2} = \frac{1}{u_\| y_0(0) z^2} + O(z).
\]

It remains to show \( \int_1^y \frac{g(z)}{z^2} \, dz \sim -\frac{g(y)}{y} \). We decompose the integral into region \([1, y_*]\) and \([y_*, y]\) for \( 0 < y_* \leq z \leq y_* \). The \([1, y_*]\) integral contributes an \( O(1) \) constant. In the \([y_*, y]\) region, the Taylor expansion is valid:

\[
\int_{y_*}^y \frac{g(z)}{z^2} \, dz \sim \int_{y_*}^y \frac{g(y)}{z^2} \, dz + g'(y) \int_{y_*}^y \frac{z - y}{z^2} \, dz \sim g(y)\left[1 - \frac{1}{y_*}\right] + g'(y) \phi(y),
\]

where \( |\phi(y)| \lesssim |\log y| \). We now use that \( v_\| (0) = v_\|'(0) = 0 \) and \( g'(y) = v_\| (y) g(y) \) to show that \( g'(y) \sim y^2 \). Thus, \( g'(y) \phi(y) \) vanishes as \( y \to 0 \). We thus have verified that \( I(y) \sim -\frac{g(y)}{y} \).

We now compute two derivatives:

\[
\tilde{u}_{syy} = u_\| y y a' + 2 u_\| y a'(y) + u_\| a''(y) \\
\sim a''(y) \sim \partial_y \{\frac{1}{u_\|} \exp[|v_\| y]\} \sim \exp[|v_\| y] \text{ as } y \uparrow \infty.
\]

We now establish a general compactness lemma:
Lemma 16 Let $H_D, H$ be Hilbert spaces, with inner-product $(\cdot, \cdot)_D$ and $(\cdot, \cdot)$ and associated norms, $\| \cdot \|_D$, $\| \cdot \|$. Let $T : H_D \to H$ be a bounded operator that is bounded below: $\|Tx\| \geq C_1 \|x\|_D$. Let $T_c : H_D \to H$ be a compact operator. Assume $\text{Ker}(T + T_c) = \{0\}$. Then there exists a $C_0 > 0$ such that

$$\|\{T + T_c\}x\| \geq C_0(\|Tx\| + \|T_c x\|).$$

Proof. We will assume that $\text{Ker}(T + T_c) = \{0\}$, and assume that the lower bound of the lemma does not hold. In this case, for each $k > 0$, we may find a sequence $\{x_n\} \in H$ which satisfies:

$$\|Tx_n\|^2 + \|T_c x_n\|^2 = 1 \text{ and } \|\{T + T_c\}x_n\|^2 = \frac{1}{n}. \quad (52)$$

As $T$ is bounded below, $\{x_n\}$ itself is a bounded sequence in $H_D$, and thus there exists a limit point, $x_s \in H_D$ such that, upon passing to a subsequence and immediately reindexing, $x_n \rightharpoonup x_s$ weakly in $H_D$. This also implies that $Tx_n \to Tx_s$ and $\|T_c x_n - T_c x_s\| \to 0$.

We will establish $\|\{T + T_c\}x_s\| = 0$, but $\|Tx_s\| + \|T_c x_s\| = 1$, thereby contradicting the triviality of $\text{Ker}(T + T_c)$. For the first step, we combine weak lower-semicontinuity with a polarization argument.

$$0 \leq \|Tx_s + T_c x_s\|^2 = \|Tx_s\|^2 + \|T_c x_s\|^2 + 2(Tx_s, T_c x_s)$$

$$\leq \|Tx_n\|^2 + \|T_c x_n\|^2 + 2(Tx_n, T_c x_n) + P_n$$

$$= \|Tx_n + T_c x_n\|^2 + P_n$$

$$\leq \frac{1}{n} + |P_n|.$$

Here, $P_n := (Tx_n, T_c x_n - T_c x_s) + (T_c x_s - Tx_n, T_c x_s)$ is a result of polarizing, and vanishes as $n \to \infty$:

$$(T_x, T_c x) = (Tx_n, T_c x_n) + (T_c x_s - Tx_n, T_c x_n)$$

$$= (Tx_n, T_c x_n) + (Tx_n, T_c x_n - T_c x_n) + (T_c x_s - Tx_n, T_c x_s)$$

$$= (Tx_n, T_c x_n) + P_n.$$

For the first term in $P_k$, we use strong convergence of $\{T_c x_n\}$:

$$\|(T_c x_n - T_c x_s)\| \leq \|Tx_n\|\|T_c x_s - T_c x_n\| \to 0.$$

For the second term in $P_k$, we use weak convergence of $\{Tx_n\}$:

$$(T_c x_s - Tx_n, T_c x_s) \to 0.$$

Thus $\|\{T + T_c\}x_s\| = 0$.

By squaring, we see that to establish $\|Tx_s\|^2 + \|T_c x_s\|^2 = 1$, it suffices to establish $-2(Tx_s, T_c x_s) = 1$. The two equations in (52) imply that $-2(Tx_n, T_c x_n) = 1 + o(\frac{1}{n})$. Again a polarization argument establishes that we may pass to the limit in the cross term to obtain: $-2(Tx_n, T_c x_n) \to -2(Tx_s, T_c x_s)$. Applying the implication of the lemma to $x_s$ gives the desired contradiction.

We will now apply Lemma [16] twice, starting with:
Thus, (54) contributes exponentially at this elementary bound with Lemma 15 yields:

$$\|[-h_{yy} + v]\|_{L^2(y)}^2 \gtrsim \|h_{yy}(y)\|^2 + \|h_{yy}(y)\|^2.$$  

**Proof.** Define $H = L^2(\langle y \rangle)$ and $H_D$ via the norm: $\|h\|_D := \|h_{yy}(y)\| + \|h_{yy}(y)\|$. We take $T_h = -h_{yy} + v\|h_{yy}(y)\|$ and $T_c = (v - v_\|h_{yy}(y)\|$). A standard argument shows that $T_c(H_D) \hookrightarrow L^2(\langle y \rangle)$ and thus $T_c$ is a compact operator. Thus to establish the inequality, we must rule out elements of $\text{Ker}(T + T_c)$, which are zero solutions to $h'' = v_\|h_{yy}(y)\|$. An explicit integration yields all solutions are of the form $h'' = C_0 \exp(\int_0^y v_\|h_{yy}(y)\|)$. Since $v_\| \geq 0$, this is in contradiction to $h''(\infty) = 0$. To conclude, we establish positivity of $T$, which follows by a straightforward integration by parts:

$$\|T_h(y)\|^2 = \|h_{yy}(y)\|^2 + |v_\|\|^2(\|h_{yy}(y)\|^2 - 2(h_{yy}(y), v_\|h_{yy}(y)\|^2))$$

$$= \|h_{yy}(y)\|^2 + |v_\||^2(\|h_{yy}(y)\|^2 + 2v_\|\|h_{yy}(y)\|^2) + h_{yy}v_\|\|y = 0.$$  

Lemma 18. Let $u_\| \in \mathbb{Y}_\|$. The following coercivity estimate holds:

$$\|T_{\|}[u_\| \cdot \langle y \rangle\|^2 \gtrsim \|[-u_\|yy + v_\|u_\|yy}\|\|^2 + \|u_\| v_\|yy\|^2.$$  

**Proof.** Here, we define $T = -h_{yy} + v\|h_{yy}(y)$, and $T_c = h\|v_{yy}$. We take $H_D = \mathbb{Y}_\|$. That $T$ is bounded below follows from the previous lemma. We proceed to rule out elements of $\text{Ker}(T + T_c) \subset \mathbb{Y}_\|$. An appeal to Proposition 14 shows that a general solution for the operator $L_v(h) = 0$ can be expressed as: $h = c_1 u_\| + c_2 \tilde{u}_s + c u_0$. As $u_\|0 = \exp(0) = 0$, the boundary condition $h(0) = 0$ eliminates $c_2$. We next use $h_{yy}(\infty) = 0$ to eliminate $c$. In particular, $h_{yy} = c_1 u_\|'' + c u_0''$ is computed via:

$$\lim_{y \to \infty} u_\|'' = \lim_{y \to \infty} \left\{ \tilde{u}_{yy} \int_0^y u_\| \exp\left[-\int_1^z v_\|\right] + \tilde{u}_{sy} u_\| \exp\left[-\int_1^y v_\|\right] \right\}.$$  

The first term is dominant, thereby preventing cancellation between these two factors. To see this, for the first term, as $v_\| \geq 0$ and $\lim_{y \to \infty} v_\| \geq 0$, we have, for large enough $y$, that the integral $\int_0^y u_\| \exp[-\int_1^z v_\|] \, dz \geq 1$. Coupling this elementary bound with Lemma 15 yields:

$$\tilde{u}_{sy} \int_0^y u_\| \exp[-\int_1^z v_\|] \, dz \sim \tilde{u}_{yy} \sim \exp[v_\|y] as y \uparrow \infty.$$  

On the other hand, Lemma 15 also gives:

$$\tilde{u}_{sy} \int_1^y u_\| \exp[-\int_1^z v_\|] \, dz \sim \exp[v_\|y] \exp[-v_\|y] \lesssim 1.$$  

Thus, (53) contributes exponentially at $y = \infty$, and we must conclude $c = 0$. Finally, we recall (19), and we use the condition $\omega[h] = 0$ for elements $h \in \mathbb{Y}_\|$ to conclude that $c_1 = 0$.  

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2.3 \textit{B} Estimate

We begin with the following lemma:

**Lemma 19** If \( h \in \mathcal{L}_y(\mathcal{T}) \) then \( d(h) = 0 \).

**Proof.** Assume that \( h \in \text{Range}(\mathcal{L}_y) \). Then there exists an element, \( \tilde{u} \in \mathcal{T} \), such that \( \mathcal{L}_y(\tilde{u}) = h \). By decomposing \( \tilde{u} = \tilde{u}_l + m_2u_y \), we see that \( \mathbb{L}_y[\tilde{u}_l] = h \). Generically, such a solution, \( \tilde{u}_l \), is of the form:

\[
\tilde{u}_l = \alpha u_l + \alpha_2 \tilde{u}_s + C_0u^p - u_{(h)}^p,
\]

where

\[
u_{(h)}^p := \tilde{u}_s \int_0^y u_y \exp \left[ - \int_1^y v_y \right] I_y[h] - u \int_0^y \tilde{u}_s \exp \left[ - \int_1^y v_y \right] I_y[h].
\]

Above, we note that \( I_y[h] \) is well defined as \( h \in L^2((y)) \hookrightarrow L^1 \).

First, we use that \( u_{(h)}(0) = 0 \) to eliminate the coefficient in front of \( \tilde{u}_s \). Next, we take the indefinite integral of \( \mathbb{L}_y[\tilde{u}_l] \) and \( h \) (thereby creating the constant \( C_0 \) below) to obtain:

\[
-\tilde{u}_{\perp yy} + v_y\tilde{u}_{\perp y} - v_{\perp y}\tilde{u}_l = C_0 + I_y[h]. \tag{55}
\]

Evaluating this equation as \( y \uparrow \infty \) shows that \( C_0 = 0 \). We have thus determined that \( \tilde{u}_l = \alpha u_l - u_{(r)}^p \). We now compute \( \tilde{u}_l \)' directly using the expression for \( u_{(h)}^p \), which yields:

\[
\lim_{y \uparrow \infty} \tilde{u}_l'(y) = \lim_{y \uparrow \infty} \tilde{u}_s\begin{bmatrix} - \int_0^y I_y[h]K \end{bmatrix} = 0,
\]

which implies that \( d(h) = 0 \). \( \blacksquare \)

**Lemma 20** Assume (13a) - (13c). Let \( u^0 = u_{\perp} + \kappa u_y \) be a solution to (40). Then the following estimate holds:

\[
\|u_{\perp}\|_\mathcal{T} \lesssim \delta_\varepsilon \varepsilon^{\frac{1}{2}}|\kappa| + \|u_{\perp yy}\langle y \rangle\{1 - \chi\left(\frac{Y}{\gamma}\right)\} + \|F\langle y \rangle\|. \tag{56}
\]

**Proof.** We will decompose the operator \( \mathcal{L}_y \) in the following manner:

\[
\mathcal{L}_y^0 = \mathcal{L}_y u^0 + \sqrt{\varepsilon}Au^0 + \tilde{v}_s u^0 + \varepsilon\Delta v^0 u^0 + \tilde{j} u^0
\]

\[
= \mathcal{L}_y u_{\perp} + \sqrt{\varepsilon}Au_{\perp} + \kappa \tilde{v}_s u_{\perp yy} - \kappa \varepsilon \Delta v^0 u_{\perp}
\]

\[
+ \sqrt{\varepsilon}Au_{\perp} + \tilde{v}_s u_{\perp yy} - \varepsilon \Delta v^0 u_{\perp} + \tilde{j} u_{\perp}^0
\]

\[
= \mathcal{L}_y u_{\perp} + \sqrt{\varepsilon}Au_{\perp} + \sqrt{\varepsilon}S(u_{\perp}) + \sqrt{\varepsilon}S(u_{\perp}) + \tilde{j} u_{\perp}^0,
\]

where \( \tilde{j} u_{\perp}^0 = J u^0 - \{\tilde{v}_s u_{\perp yy} - \varepsilon \Delta v^0 u_{\perp}\} \) is explicitly given by

\[
\tilde{j} := \delta_0 u_{\perp yy} + u_{\perp yy} \sum_{i=2}^n \sqrt{\varepsilon} \tilde{v}_i + u_{\perp yy} \sum_{i=2}^n \sqrt{\varepsilon} \tilde{v}_i - \varepsilon u^0 \sum_{i=0}^n \sqrt{\varepsilon} v^0_{i,xx}
\]

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\[ -u^0 \sum_{i=2}^n \sqrt{\varepsilon} v^i_{yy} - \varepsilon u^0 \sum_{i=2}^n \sqrt{\varepsilon}^{-1} \Delta v^i_e, \]

and we have defined \( S \) via
\[
S_g := \sqrt{\varepsilon} A g + \bar{v}_g g - \varepsilon \Delta v^1_g. \]

We use (57) to write the equation as
\[
L_\|u\| = F - \sqrt{\varepsilon} S(\|u\|) - \varepsilon S(u_\perp) - \bar{J} u^0. \tag{58}
\]

We place both sides of the equation in \( L^2(\langle y \rangle) \). On the left-hand side of (58), we produce coercivity over \( \|u_\perp\|_\Upsilon \). Next, we have
\[
\|\sqrt{\varepsilon} S(\|u\|)\| \leq \|\sqrt{\varepsilon} A u(\|u\|)\| + \|\kappa \bar{v}^1_{yy} u(\|u\|)\| + \|\kappa \varepsilon \Delta v^1_e u(\|u\|)\|
\leq \sqrt{\varepsilon} \|\kappa\| \|A u(\|u\|)\| + \sqrt{\varepsilon} \|\bar{v}^1_e(0)\| \|u_{yy}\|_2
\]
\[
+ \varepsilon^{\frac{1}{2}} \delta_\varepsilon |\kappa| \|\frac{u^0_{YY}}{u^0_e} Y\|_{L^2_Y}
\]
\[
\lesssim (\delta_\varepsilon \varepsilon^{\frac{1}{4}} + \sqrt{\varepsilon}) |\kappa|.
\]

We treat the terms in \( \sqrt{\varepsilon} S(u_\perp) \) perturbatively. First,
\[
\|\sqrt{\varepsilon} Au_\perp(\langle y \rangle)\| \lesssim \sqrt{\varepsilon} \|u_\perp\|_\Upsilon.
\]

Next, we treat the \( \bar{v}^1_{Y_e} = v^1_{Y_e} - v^1_{Y_e}|_{y=0} \) term by localizing. Recalling the definition of \( \chi \) in (22), we will fix \( \gamma > 0 \) to be a small parameter, and then split:
\[
\|\bar{v}^1_{u_{yy}}(\langle y \rangle)\| \leq \|\bar{v}^1_{u_{yy}}(\langle y \rangle)\chi(\frac{Y}{\gamma})\| + \|\bar{v}^1_{u_{yy}}(\langle y \rangle)\{1 - \chi(\frac{Y}{\gamma})\}\|
\]

The localized contribution is estimated upon using \( |\bar{v}^1_{Y_e}| \lesssim Y \) via:
\[
\|\bar{v}^1_{u_{yy}}(\langle y \rangle)\chi(\frac{Y}{\gamma})\| \leq \|\bar{v}^1_{u_{yy}}(\langle y \rangle)\chi(\frac{Y}{\gamma})\|_{\infty} |\gamma| \|u_{yy}\| \lesssim |\gamma| \|u_{yy}\|.
\]

As \( \gamma \ll 1 \), we absorb this into the \( \|u_\perp\|_\Upsilon \) term appearing on the left-hand side. The nonlocal component contributes to the right-hand side of estimate (56).

Next, for the shear contribution we use that \( \varepsilon^{(y)^2} \lesssim \langle Y \rangle^2 \) to estimate:
\[
\|\varepsilon \Delta v^1_{u_{\perp}}(\langle y \rangle)\| = \|\varepsilon \Delta v^1_{u_{\perp}}(\langle y \rangle)^2\|
\leq \|\Delta v^1_{u_{\perp}}(\langle Y \rangle)^2\|_{\infty} \|u_{\perp}\| = o(1) \|u_{\perp}\|_\Upsilon,
\]

which is absorbed to the left-hand side thanks to (14m) - (14e) and since the Euler equation for \( v^1_e \) reads:
\[
\Delta v^1_e = u^0_{YY} \frac{u^0_e}{u^0_e} v^1_e.
\]
We now move to the terms in $\tilde{J}$, which are all treated perturbatively. First, using $\delta \leq \sqrt{\epsilon}$:

$$\|\delta u^0_{yy}(y)\| \leq \delta \|u_{\perp yy}(y)\| + \frac{\delta}{\sqrt{\epsilon}} |\sqrt{\epsilon}\kappa| \cdot \|u_{yy}(y)\|.$$  

The $\tilde{v}_i^0 u^0_{\perp yy}$ terms are treated the same as in the $\tilde{v}_1^0 u^0_{\perp yy}$ term. For the parallel contribution, we estimate for $i \geq 2$:

$$\|\sqrt{\epsilon}^{-1} \tilde{v}_i^0 u_{||yy} \kappa(y)\| \leq \sqrt{\epsilon}^{-1} \left[ \|\tilde{v}_i^0 u_{||yy} \kappa(y) \chi(Y)\| + \|\tilde{v}_i^0 u_{||yy} \kappa(y) \{1 - \chi(Y)\}\| \right] \lesssim \epsilon |\kappa|.$$  

The final $u^0_{yy}$ term may be estimated directly as:

$$\|u^0_{yy} \sum_{i=2}^n \sqrt{\epsilon} \tilde{v}_i^0(y)\| \lesssim \epsilon \|u_{\perp yy}(y)\| + \sqrt{\epsilon} \|u_{yy}(y)\|.$$  

We now move to the $u^0$ terms from $\tilde{J}$, (43). The rapid decay of $v^0_p$ gives:

$$\|\tilde{v}^0_{pxx} u^0_{\perp}(y)\| \leq \|\tilde{v}^0_{pxx} \kappa \{1 + \kappa u\}(y)\| \leq \epsilon \|\tilde{v}^0_{pxx}(y)^2 \|_{\infty} \|u_{\perp}(y)^{-1}\| + \sqrt{\epsilon} |\sqrt{\epsilon}\kappa| \|v^0_{pxx}(y)^2 \|_{\infty} \|u\|_{(y)}\|.$$  

The same computation is performed for all intermediate Prandtl layers, $v^i_p$, $i = 1, \ldots, n - 1$. Next, we perform the same estimate for $v^0_{pxx}$, this time using that the support is on $y \leq \frac{1}{\sqrt{\epsilon}}$:

$$\|\epsilon \tilde{v}^0_{pxx} u^0_{\perp}(y)\| \leq \|\epsilon \tilde{v}^0_{pxx} \kappa \{1 + \kappa u\}(y)\| \leq \epsilon \|\tilde{v}^0_{pxx}(y)^2 \|_{\infty} \|u_{\perp}(y)^{-1}\| + \sqrt{\epsilon} |\sqrt{\epsilon}\kappa| \|v^0_{pxx}(y)^2 \|_{\infty} \|u\|_{(y)}\|.$$  

Next, we treat contributions from $v^i_{pyy}$ for $i = 2, \ldots, n$, again using the rapid decay of this quantity:

$$\|u^0 \sum_{i=2}^n \sqrt{\epsilon} v^i_{pyy}(y)\| \lesssim \epsilon \|v^i_{pyy}(y)^2 \|_{\infty} \|u^1_{\perp}(y)^{-1}\| + \|u\|_{(y)}\|.$$  

The remaining, higher order shear terms can be treated as in the $\Delta v^1_{\perp}$ case for $u_{\perp}$, whereas for the parallel component:

$$\|\epsilon \sqrt{\epsilon}^{-1} \Delta v^1_{\perp} \kappa u\|_{(y)} \lesssim \epsilon \frac{\|u\|_{(y)}\|}{\sqrt{\epsilon}}.$$  

On the right-hand side, we majorize using $\|F(y)\|$. This concludes the lemma.
Lemma 21. Let \( u^0 = u_\perp + \kappa u_\parallel \) solve equation (40). Assume the nondegeneracy condition. Then the following estimate is valid:
\[
\sqrt{\varepsilon} |\lambda| \lesssim \varepsilon^{\frac{1}{2}} \| u_\perp \|_\mathcal{H} + \varepsilon^{\frac{3}{4}} \| u^{BY}_{2}(1 - \chi(Y)) \| + \| F(y) \|^{\frac{1}{2} + \epsilon}.
\] (59)

Proof. Our starting point is the equation:
\[
d(Equation \ (57)) = d(F).
\]
We know that \( d(\mathcal{L} u_\perp) = 0 \), and that, by the nondegeneracy condition,
\[
|\sqrt{\varepsilon} d(S(u_\parallel))| \gtrsim \sqrt{\varepsilon} |\lambda|
\]
It remains thus to estimate \( d(\cdot) \) of the latter two terms in (57) and \( d(F) \).
We first estimate:
\[
|d(F)| = |\int_0^{\infty} K(y) I_y[F]| \leq \| K \|_1 \| I_y[F] \|_\infty \lesssim \| F \|_1 \lesssim \| F(y) \|^{\frac{1}{2} + \epsilon}.
\]
The contributions from \( \mathcal{J} \) and \( \sqrt{\varepsilon} Au_\perp \) are majorized easily by:
\[
|d(\mathcal{J})| + |d(\sqrt{\varepsilon} Au_\perp)| \lesssim \| \mathcal{J}(y) \| + \sqrt{\varepsilon} \| Au_\perp(y) \|
\]
We focus on the two main contributions, beginning with:
\[
|d(\bar{v}^1 u_{1yy})| \lesssim \| \bar{v}^1 u_{1yy} \|_1 \leq \| \bar{v}^1 u_{1yy} \chi(Y) \|_1 + \| \bar{v}^1 u_{1yy} \{1 - \chi(Y)\} \|_1.
\]
For the localized term, we use:
\[
\| \bar{v}^1 u_{1yy} \chi(Y) \|_1 \leq \sqrt{\varepsilon} \| u_{1yy}(y) \|_2 \| \chi(Y) \|_2 \leq \varepsilon^{\frac{1}{2}} \| u_\perp \|_\mathcal{H}.
\]
For the far-field term, we estimate
\[
\| \bar{v}^1 u_{1yy} \{1 - \chi(Y)\} \|_1 \leq \| u_{1yy} \{1 - \chi(Y)\} \| \| \langle y \rangle^{-1} \{1 - \chi(Y)\} \|
\]
Above, we have used the inequality:
\[
\| \langle y \rangle^{-1} \{1 - \chi(\frac{y}{\alpha})\} \|_2^2 = \int_0^{\infty} \langle y \rangle^{-2} dy = \alpha^{-1},
\]
with the particular choice of \( \alpha = \varepsilon^{-\frac{1}{2}} \).
We move to the shear \( u_\perp \) term:
\[
\| \varepsilon \Delta v^1_{1u_\perp} \|_1 \leq \| \varepsilon \Delta v^1_{1u_\parallel}(y) \|_1 \leq \sqrt{\varepsilon} \| \Delta v^1_{c}(Y) \| \| u_{1yy} \|
\]
\[
\leq \varepsilon^{\frac{1}{2}} \| \Delta v^1_{c} Y \|_{L^2_{\parallel}} \| u_{1yy} \| \lesssim \varepsilon^{\frac{1}{2}} \| u_\perp \|_\mathcal{H},
\]
where we have used that \( \| \cdot \| = \varepsilon^{-\frac{1}{2}} \| \cdot \|_{L^2_{\parallel}} \), the factor of \( \varepsilon^{\frac{1}{2}} \) arising from the Jacobian. This concludes the proof.
\[\blacksquare\]
Lemma 22 (Multiscale Estimate) Assume (13a) - (13c) and (15). Then:

\[
\|u_{yy}^0 \{ 1 - \chi(Y/\gamma) \} \| \leq \| \epsilon \|^{1/2} u_{yy}^0 \frac{1}{\sqrt{\epsilon v_c^1}} \{ 1 - \chi(Y/\gamma) \} \| (60)
\]

Proof. We compute the inner-product:

\[
(L_{\delta} u_{\delta}^0, u_{yy}^0 \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} )^2 (y) = \left( u_{yy}^0 \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} \right)^2 (y)
\]

On the left-hand side, we begin with:

\[
-(u_{yy}^0, u_{yy}^0 \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} )^2 (y) = \left( \frac{|u_{yy}^0|^2}{2}, \epsilon \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} \right) \frac{1}{w} (y)
\]

First, we note that (61.1), (61.2), and (61.3) are favorable signed terms. First, using the rapid decay of \( v \), it is clear that \( w_y w^{-2} \leq -\frac{1}{2} v_e \). Next, upon using the positivity of \( v_e \), we estimate

\[
\frac{v_e}{w} \{ 1 - \chi(Y/\gamma) \} \geq \frac{v_e}{w} \{ 1 - \chi(Y/\gamma) \} - \sum_{i=2}^{n} \frac{\sqrt{\gamma} \gamma^i v_e^i}{w} \{ 1 - \chi(Y/\gamma) \} \]

\[
- \sum_{i=0}^{n-1} \frac{\sqrt{\gamma} \gamma^i v_e^i}{w} \{ 1 - \chi(Y/\gamma) \} - \frac{\sqrt{\gamma} \gamma^i v_e^i}{w} \{ 1 - \chi(Y/\gamma) \}
\]

Using this lower bound, we see

\[
((v_e + \delta) u_{yy}^0, u_{yy}^0 \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} )^2 (y) \geq \| u_{yy}^0 \{ 1 - \chi(Y/\gamma) \} \|
\]

Next, we move to:

\[
( -\Delta_{\epsilon} v_{\epsilon} u_{\epsilon}^0, u_{yy}^0 \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} )^2 (y) \]

First, using the rapid decay of \( v_{yy}^j \) for \( i = 0, ..., n \) and \( v_{pxx}^j \) for \( j = 0, ..., n-1 \), we immediately estimate:

\[
| \left[ \sum_{i=0}^{n} \sqrt{\gamma} v_{yy}^i + \sum_{j=0}^{n-1} \sqrt{\gamma} v_{pxx}^j \right] u_{yy}^0, u_{yy}^0 \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} | \lesssim \| u_{yy}^0 \frac{1}{w} \{ 1 - \chi(Y/\gamma) \} \| \lesssim \| u_{yy}^0 \|_B^2.
\]
This leaves the terms $\varepsilon^{\frac{n+2}{2}} v_{pxx}^n$ and the Euler contributions. First,

$$\|v_{pxx}^n\| \leq \|u_0 \gamma y \{1 - \chi(\gamma)\}\| \cdot \left[ \varepsilon^{\frac{n+2}{2}} \|u_\perp(y)\| \|Y^2 v_{pxx} w^{-1}\|_\infty + \varepsilon^{\frac{n+2}{2}} \frac{1}{\gamma} \|\sqrt{\varepsilon \kappa}\| \cdot \|u_\parallel(y)\| \|Y^2 v_{pxx} w^{-1}\|_\infty \right].$$

We now estimate the shear terms, with the $i > 1$ case following in a similar manner to the $i = 1$ case:

$$(-\varepsilon \Delta v_{\gamma}^1 u^0, u_{\perp yy} (1 - \chi(\gamma)) \langle y \rangle^{\frac{2}{w}})
= (-\varepsilon \Delta v_{\gamma}^1 u_{\perp}, u_{\perp yy} (1 - \chi(\gamma)) \langle y \rangle^{\frac{2}{w}}) + (-\varepsilon \Delta v_{\gamma}^1 \kappa u_{\parallel}, u_{\perp yy} (1 - \chi(\gamma)) \langle y \rangle^{\frac{2}{w}})
\lesssim \|\Delta v_{\gamma}^1 Y^2 \frac{1}{w} \|_\infty \|u_{\perp yy} (1 - \chi) \langle y \rangle\| + |\varepsilon^{\frac{1}{2}} \kappa| \|\Delta v_{\gamma}^1 Y^2 \frac{1}{w} \|_{L^2} \|u_{\perp yy} (1 - \chi) \langle y \rangle\|
\lesssim \delta_u u_0^0 B \|u_{\perp yy} \langle y \rangle (1 - \chi)\|
\|F \varepsilon^{-\frac{\chi}{\sqrt{v_{\gamma}^1}}} (1 - \chi(\gamma))\| \|u_0^0 (1 - \chi(\gamma)) \langle y \rangle\|$$

The right-hand side is majorized immediately using Cauchy-Schwartz by $\|F \varepsilon^{-\frac{\chi}{\sqrt{v_{\gamma}^1}}} (1 - \chi(\gamma))\| \|u_0^0 (1 - \chi(\gamma)) \langle y \rangle\|$.

We now close the section with a straightforward $L^\infty$ embedding:

**Lemma 23** $|\varepsilon^{\frac{1}{2}} u_{\perp} \|_\infty \lesssim \|u_0^0\|_B$.

**Proof.** First, using $u_{\perp}(0) = 0$, we write: $u_{\perp} = f_0^y u_{\perp y} \lesssim \|u_{\perp y} \langle y \rangle^{\frac{1}{w}}\|$. Next, using $u_{\perp y}(\infty) = 0$, we use Hardy to majorize:

$$\|u_{\perp}\| \lesssim \|u_{\perp y}\|_{loc} + \|u_{\perp yy} \langle y \rangle^{\frac{3}{w}}\|
\lesssim \|u_0^0\|_B + \varepsilon^{\frac{3}{2} - \frac{1}{w}} \varepsilon^{\frac{1}{2}} \|u_{\perp yy} \frac{1}{v_{\gamma}} \|_{L^2} \|1 - \chi(\gamma)\|
+ \varepsilon^{-(1 + \frac{3}{2} - \frac{1}{w})} \|u_{\perp yy} \langle y \rangle^{0+}\|
\lesssim \varepsilon^{\frac{3}{2} - \frac{1}{w}} \|u_0^0\|_B.$$



### 2.4 Computation of Degree

We now compute $d(S(u_{\parallel}))$. We first use the particular form of $f = f^{(1)}$ given in (62) to obtain the condition:

$$\int_0^\infty K(y) r(y) \, dy$$

$$= -u_{\| y=0} |x=0 \rangle u_{\| x=0 \rangle} e^{-\int_0^y v_t} + \int_0^\infty K(y) f(y) \, dy$$

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\[ = -u_{\parallel y}|_{x=0}(0)u_c^1|_{x=0}(0)e^{-\int_0^t v_1} + \int_0^\infty K(y)\{-u_p^0u_{\parallel ex}(0) \\
- u_p^0u_c^1(0) - \bar{v}^1_{cY}(0)yu_p^0 - u_{cY}^0(0)yv_p^0 - u_p^0\bar{v}^1_{cY}(0) \\
+ g^{(1)}_{ext}\}|_{x=0}. \]

Note that we have referred to the definition in (34) and retained the lowest order terms in \( f \). We now record the following identity for future use:

**Lemma 24**

\[ \int K(y)\left[r(y) + v^1_{cY}(0)\{yv_p^0 - u_p^0\}ight] dy \\
= -\int K(y)\left[u_{cY}^0(0)yv_p^0 + v_p^0u_{cY}(0)\right] dy + \int K(y)g^{(1)}_{ext} dy. \] (64)

**Proof.** The \( v^1_{cY}(0) \) terms on the left-hand side cancel out the first and third terms in the integral in (63). This leaves:

\[
\begin{align*}
\text{LHS of (64)} &= - u_{\parallel y}|_{x=0}(0)u_c^1|_{x=0}(0)e^{-\int_0^t v_1} + \int K(y)\{-u_c^1|_{x=0}(0)u_p^0 \\
- u_{cY}^0(0)yv_p^0 - v_p^0u_{cY}(0) + g^{(1)}_{ext}\} \\
&= - u_c^1|_{x=0}(0)\left[u_{\parallel y}|_{x=0}(0)e^{-\int_0^t v_1} + \int K(y)u_p^0\right] \\
&- \int K(y)\left[u_{cY}^0(0)yv_p^0 + u_{cY}^0(0)v_p^0 + g^{(1)}_{ext}\right] \\
&= - \int K(y)\left[-g^{(1)}_{ext} + u_{cY}^0(0)yv_p^0 + u_{cY}^0(0)v_p^0\right].
\end{align*}
\]

Above, we have used the identity:

\[ u_{\parallel y}|_{x=0}(0) + \int u_{\parallel y}u_p^0e^{-\int_0^t f_p^0} = u_{\parallel y}|_{x=0}(0) - \int v_1u_p^0e^{-\int_0^t f_p^0} + \int u_p^0v_p^0e^{-\int_0^t f_p^0} = 0, \]

upon integrating by parts the \( u_p^0u_{\parallel y} \) term. ■

We will next make a reduction of \( d(S(u_{\parallel y})) \) to the quantity \( n \), which we now define:

\[ n := \int_0^\infty K(y)\left[g^{(1)}_{ext} + u_{cY}^0(0)\{yv_p^0 + v_p^0\} + u_c^0(0)\int_0^\infty \Delta v_c^1\right] dy. \] (65)

**Lemma 25** The following inequality holds:

\[ |d(S(u_{\parallel y}))| \geq |n| - \sqrt{\varepsilon}. \]
Proof. We begin with a rewriting of the latter two terms in \( S \):

\[
\bar{v}^1_e u_{\|y\|} - \sqrt{\varepsilon} \Delta v^1_e u^0_p
\]

\[
= \bar{v}^1_e u_{\|y\|} - \sqrt{\varepsilon} \Delta v^1_e u^0_p - \sqrt{\varepsilon} \Delta v^1_e u^0(0)
\]

\[
= \partial_y \left\{ \bar{v}^1_e u_{\|y\|} - v^1_{eY} u^0_p \right\} - \sqrt{\varepsilon} v^1_{eY} u^0_p - \sqrt{\varepsilon} u^0(0) \partial_y [\Delta v^1_e]
\]

Combining now with \( Au_{\|} = r'(y) \), we take \( d(\cdot) \):

\[
d \left( Au_{\|} + \frac{\bar{v}^1_e}{\sqrt{\varepsilon}} u_{\|y\|} - \sqrt{\varepsilon} \Delta v^1_e u^0_p \right)
\]

\[
= \int_0^\infty K(y) \left[ r(y) + \frac{\bar{v}^1_e}{\sqrt{\varepsilon}} u_{\|y\|} - v^1_{eY} u^0_p \right] dy
\]

\[- \int K(y) \sqrt{\varepsilon} v^1_{exx} u^0_p dy - \int K(y) u^0(0) I_Y [\Delta v^1_e] dy.
\]

We estimate the middle term immediately via:

\[
| \int K(y) \sqrt{\varepsilon} v^1_{exx} u^0_p | \lesssim \sqrt{\varepsilon} \| K \| \| u^0_p \| \| v^1_{exx} \|_\infty.
\]

The lowest order terms are now collected from the first integral upon using the identity (64):

\[
\int K(y) \left[ r(y) + \frac{\bar{v}^1_e}{\sqrt{\varepsilon}} u_{\|y\|} - v^1_{eY} (0) u^0_p \right] dy
\]

\[
= - \int K(y) \left[ u^0_{eY} (0) y u^0_p + v^0_{p} u^0_{eY} (0) \right] dy + \int K(y) g_{ext,p} u^1_e dx dy.
\]

We have Taylor expanded:

\[
\bar{v}^1_e = \bar{v}^1_{eY} (0) Y + \phi(Y) Y^2, \quad v^1_{eY} = v^1_{eY} (0) + Y \phi(Y),
\]

which, upon inserting into the first integral produces the following error terms:

\[
| \int K(y) \left[ \sqrt{\varepsilon} Y^2 \phi(Y) u^0_{\|y\|} - \sqrt{\varepsilon} \phi(Y) u^0_p \right] dy | \lesssim \sqrt{\varepsilon}.
\]

The lowest order term from the third integral reads:

\[
- \int K(y) u^0_e (0) I_0 [\Delta v^1_e] dy = - u^0_e (0) \int_0^\infty K(y) dy \int_0^\infty \Delta v^1_e dY.
\]
A similar Taylor expansion shows that the error term can be majorized by \( \sqrt{\varepsilon} |\kappa| \). We thus arrive at the following leading order expression:

\[
\int_{0}^{\infty} K(y) \left[ g_{\text{ext},0}^1 + u_{\varepsilon Y}(0) \{ y u_{p}^{0} + v_{0}^{0} \} + \varepsilon u_{\varepsilon}^{0} \int_{0}^{\infty} \Delta v_{1}^{0} \, dY \right] \, dy,
\]

which proves the desired estimate.

\[\text{Corollary 26}\]

\[|d(S(u_\|))| \gtrsim 1.\]

\[\text{Proof.}\] By the previous lemma, it suffices to provide a lower bound on \( n \). By invoking the assumption that all the shear terms are size \( \delta_s \), we obtain

\[|n| \gtrsim \left| \int_{0}^{\infty} K(y) g_{\text{ext},0}^{1} \, dy \right| - \mathcal{O}(\delta_s) \gtrsim 1,\]

by our non-degeneracy assumption.

\[\text{3 Solution to DNS and NS}\]

\[3.1 \text{ Nonlinear } a\text{-priori Estimate}\]

Define the following linear combinations:

\[\text{Definition 27}\]

\[N_{X_1} := (\partial_x N, q_x + q_{xx} + q_{yy} + \varepsilon^{-\frac{3}{4}} \varepsilon^{2} v_{xxxx} + \varepsilon^{-\frac{3}{4}} \varepsilon u_{sx} v_{xyy}),\]

\[N_{Y_0} := (\partial_x N, \{ \varepsilon q_x + \varepsilon q_{xx} + \varepsilon q_{yy} + \varepsilon^{2} v_{xxxx} + \varepsilon u_{sx} v_{xyy} \} w^2\]

\[+ \{ \varepsilon^{2} v_{xxxx} + \varepsilon^{-\frac{3}{4}} \varepsilon u_{sx} v_{xyy} \})\]

\[B_{X_1} := (F u_0, q_x + q_{xx} + q_{yy} + \varepsilon^{-\frac{3}{4}} \varepsilon^{2} v_{xxxx} + \varepsilon^{-\frac{3}{4}} \varepsilon u_{sx} v_{xyy}),\]

\[B_{Y_0} := (F u_0, \{ \varepsilon q_x + \varepsilon q_{xx} + \varepsilon q_{yy} + \varepsilon^{2} v_{xxxx} + \varepsilon u_{sx} v_{xyy} \} w^2\]

\[+ \{ \varepsilon^{2} v_{xxxx} + \varepsilon^{-\frac{3}{4}} \varepsilon u_{sx} v_{xyy} \})\]

\[F_{X_1} := (g(q), q_x + q_{xx} + q_{yy} + \varepsilon^{-\frac{3}{4}} \varepsilon^{2} v_{xxxx} + \varepsilon^{-\frac{3}{4}} \varepsilon u_{sx} v_{xyy}),\]

\[F_{Y_0} := (g(q), \{ \varepsilon q_x + \varepsilon q_{xx} + \varepsilon q_{yy} + \varepsilon^{2} v_{xxxx} + \varepsilon u_{sx} v_{xyy} \} w^2\]

\[+ \{ \varepsilon^{2} v_{xxxx} + \varepsilon^{-\frac{3}{4}} \varepsilon u_{sx} v_{xyy} \})\]

\[\text{Lemma 28 (Boundary Estimates)}\] Let \( j = 0, 1 \). The following estimate holds:

\[
\| \partial_x^j \{ F u_0 \} w \|_{x=0} \leq \begin{cases} 
\varepsilon^{-\frac{1}{4}} \| u_0 \|_B & \text{if } w = 1, \\
\varepsilon^{-\frac{3}{4}} \| u_0 \|_B & \text{if } w = \frac{1}{\nu} (y) \\
\varepsilon^{-\frac{1}{4}} \| u_0 \|_B & \text{if } w = \frac{1}{\nu} 
\end{cases}.
\]

(67)
Proof. First, set $j = 0$. The $j = 1$ case follows in an identical manner. First, let $w = 1$. We estimate immediately upon consulting (45):

$$
\|v_{sx} u_{yy}^0(y)\| \leq \|v_{sx} \| \infty \|u_{\perp yy}^0(y)\| + \varepsilon^{-\frac{1}{4}} |\varepsilon^{\frac{3}{4}} \kappa \|u_{yy}^0(y)\| \lesssim \varepsilon^{-\frac{1}{4}} \|u^0\|_B.
$$

Above we have used that $v_{sx}|_{y = 0} = 0$, and so $v_{sx}(0,0) = 0$. Next, we treat $|u^0 \Delta_x v_{sx} \frac{(y)}{u^0}\|$. For the region $y \leq 1$, we estimate easily using the boundary condition $u^0(0) = 0$:

$$
\|u^0 \Delta_x v_{sx} \frac{(y)}{u^0}\chi\| \leq \|\Delta_x v_{sx} \chi \| \infty \|u^0\|, u^0 \|_{loc} \lesssim \varepsilon^{-\frac{1}{4}} \|u^0\|_B.
$$

For the region $y \geq 1$, we may omit the weight of $u_s$ and simply estimate:

$$
\|u^0 \Delta_x v_{sx} y^2 \frac{1}{v_{s}}\| \lesssim \varepsilon^{-\frac{1}{4}} \|u^0\|_B.
$$

Next, we let $w = \frac{1}{v_{s}}(y)^m$ for $m = 0, 1$.

$$
\|v_{sx} u_{yy}^0(y)\| \lesssim \|Y^m \chi\| \infty \|\frac{1}{v_{s}} u_{yy}^0(y)[1 - \chi(Y)]\| \lesssim \varepsilon^{-\frac{1}{2}} \|u^0\|_B.
$$

Next,

$$
\|v_{sx} u_{\perp yy}^0(y)\| \lesssim \varepsilon^{-\frac{1}{2}} \|Y^m \frac{1}{v_{s}} \chi(Y)\| \infty \|u_{\perp yy}^0(y)\| \lesssim \varepsilon^{-\frac{1}{2}} \|u^0\|_B,
$$

$$
\|v_{sx} \kappa u_{yy}^0(y)\| \lesssim \varepsilon^{-\frac{1}{2}} |\varepsilon^{\frac{3}{4}} \kappa \|u^0\|_B.
$$

We now move to $\|u^0 \Delta_x v_{sx} y^2 \frac{1}{v_{s}}\|$. It is convenient to split:

$$
\Delta_x v_{sx} = \sum_{i=0}^{n} \sqrt{\varepsilon} \Delta_x v_{px}^i + \varepsilon \sum_{j=1}^{n} \sqrt{\varepsilon^{j-1}} \Delta v_{cx}^j.
$$

We treat the $i = 0$ case, with the higher order Prandtl terms following similarly. First, using the rapid decay of $v_{p}^0$, we estimate:

$$
\|u_{\perp} \Delta_x v_{px} \frac{(y)}{v_{c}}\| \lesssim \|u_{\perp} \| \lesssim \|u^0\|_B
$$

$$
\|\kappa u_{\perp} \Delta_x v_{px} \frac{(y)}{v_{s}}\| \lesssim \varepsilon^{-\frac{1}{4}} \|u^0\|_B.
$$

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Next, we move to the Euler contributions. We treat the $j = 1$ case, with the higher order Euler terms following similarly:

\[
\begin{align*}
\|u_\perp \varepsilon \Delta v_{ex}^1 \langle y \rangle^{m+1} v_{e|x=0}^{-1} \| & \lesssim \varepsilon e^{-\frac{1}{2} n(m+2)} \| \frac{u_\perp}{\langle y \rangle} \|_2 \| \Delta v_{ex}^1 Y^{m+1} \| v_{e|x=0}^{-1} \| \infty, \\
& \lesssim e^{-\frac{1}{2}} \| u^0 \|_B. \\
\| \kappa u_\perp \varepsilon \Delta v_{ex}^1 \langle y \rangle^{m+1} v_{e|x=0}^{-1} \| & \lesssim \varepsilon e^{-\frac{1}{2} n(m+1)} \varepsilon e^{-\frac{1}{2}} \| \kappa \varepsilon \varepsilon \Delta v_{ex}^1 Y^{m+1} \| v_{e|x=0}^{-1} \| L_Y^2 \\
& \lesssim e^{-\frac{1}{2}} \| u^0 \|_B.
\end{align*}
\]

This concludes the proof.

Lemma 29

\[
\begin{align*}
|B_{X_1}| & \lesssim \varepsilon^{-\frac{1}{2}} \| u^0 \|_B^2 + o(1) \| u \|_{X_1}^2, \\
|B_{Y_{uw}}| & \lesssim L \varepsilon^{-\frac{1}{2}} \| u^0 \|_B^2 + o(1) \| v \|_{Y_{wo}}^2, \\
|B_{Y_{w1}}| & \lesssim \varepsilon^{-\frac{1}{2}} \| u^0 \|_B^2 + o(1) \| v \|_{Y_{w1}}^2.
\end{align*}
\]

Proof. First, fix $w = 1$. We compute by integrating by parts first in $y$ and second in $x$ for the first term below:

\[
\begin{align*}
(F_{w}, q_{xx}) & = (\partial_y \{ v_{xx} u_y^0 - v_{xxy} u_x^0 \}, q_{xx}) - (\varepsilon v_{xxxx} u_x^0, q_{xx}) \\
& = (v_{xx} u_y^0 - v_{xxy} u_x^0, q_{xy}) - (\varepsilon v_{xxxx} u_x^0, q_{xx}) \\
& = (v_{xx} u_y^0 - v_{xxy} u_x^0, q_{xy}) - (v_{xx} u_y^0 - v_{xxy} u_x^0, q_{xy})_{x=L} \\
& + (v_{xx} u_y^0 - v_{xxy} u_x^0, q_{xy})_{x=0} - (\varepsilon v_{xxxx} u_x^0, q_{xx}). 
\end{align*}
\]

We first estimate (71.3) and (71.4):

\[
\begin{align*}
|71.3| & \lesssim \left[ \| v_{xx} u_x \|_{\infty} \| u_x^0 \| + \| v_{xxy} \langle y \rangle \| \frac{u^0}{u_x \langle y \rangle} \| \right] \| u_x q_{xy} \|_{x=0} \\
& \lesssim \varepsilon^{-\frac{1}{2}} \| u^0 \|_B \| q \|_1, \\
|71.4| & \lesssim \| v_{xxxx} \sqrt{\varepsilon} \langle y \rangle \| \frac{u^0}{\langle y \rangle} \| \sqrt{\varepsilon} q_{xx} \| \lesssim \varepsilon^{-\frac{1}{2}} \| u^0 \|_B \| q \|_1.
\end{align*}
\]

Above, we have used that $v_{xx}|_{y=0} = 0$ and that:

\[
\begin{align*}
\| v_{xxy} \langle y \rangle \|_{\infty} & \leq \sum_{i=0}^{n} \sqrt{\varepsilon} \| u^i_{pxx} \langle y \rangle \|_{\infty} + \sum_{i=1}^{n} \sqrt{\varepsilon}^{-1} \| v_{xxy} \sqrt{\varepsilon} y \|_{\infty} \lesssim 1, \\
\| v_{xxxx} \sqrt{\varepsilon} \langle y \rangle \| & \lesssim \sqrt{\varepsilon} \sum_{i=0}^{n} \sqrt{\varepsilon} \| u^i_{xxx} \langle y \rangle \|_{\infty} + \sum_{i=1}^{n} \sqrt{\varepsilon}^{-1} \| v_{xxy} \sqrt{\varepsilon} y \|_{\infty}.
\end{align*}
\]
For the $w \neq 1$ case, we invoke the bottom estimate in (67):

$$ \| (v_{xx} u_{yy}^0 - u^0 \Delta v_{xx}, q_{xx} w^2) \| \leq \varepsilon^{-\frac{1}{2}} \sqrt{L} \| (v_{xx} u_{yy}^0 - u^0 \Delta v_{xx}) w \| \| \sqrt{\varepsilon} q_{xx} w \| \leq \varepsilon^{-\frac{1}{2}} \sqrt{L} \| u^0 \|_B \| q \|_w. $$

$$ \| (F_{u^0}, q_{yy} w^2) \| \leq \sqrt{L} \| (v_{xx} u_{yy}^0 - u^0 \Delta v_{xx}) w \| \| q_{yy} w \| \leq \sqrt{L} \| q \|_w \varepsilon^{\frac{1}{4}} \| u^0 \|_B. $$

$$ \| (F_{u^0}, q_{x} w^2) \| \leq \sqrt{L} \| (v_{xx} u_{yy}^0 - u^0 \Delta v_{xx}) w \| \| q_{x} w \| \leq \sqrt{L} \| q \|_w \varepsilon^{\frac{1}{4}} \| u^0 \|_B. $$

(72)

$$ \| (F, \varepsilon v_{xxyy} u_w w^2) \| = - \{ (v_{xx} u_{yy}^0 - u^0 \Delta v_{xx}, \varepsilon v_{xxyy} u_w w^2) \} = 0 $$

$$ - \{ (v_{xxx} u_{yy}^0 - u^0 \Delta v_{xxx}, \varepsilon v_{xxyy} u_w w^2) \} = 0 $$

$$ - \{ (v_{xx} u_{yy}^0 - u^0 \Delta v_{xx}, \varepsilon v_{xxyy} u_w w^2) \} = 0 $$

$$ \varepsilon^{\frac{1}{4}} \| u^0 \|_B \| q \|_w \varepsilon^{\frac{1}{4}} \| u^0 \|_B. $$

$$ \| (F, \varepsilon v_{xxyy} u_w w^2) \| = - \{ (v_{xx} u_{yy}^0 + u^0 \Delta v_{xx}) w_{xxx} w^2 \} = 0 $$

$$ - \{ (v_{xx} u_{yy}^0 + u^0 \Delta v_{xx}) w_{xxx} w^2 \} = 0 $$

$$ \varepsilon^{\frac{1}{4}} \| u^0 \|_B \| q \|_w \varepsilon^{\frac{1}{4}} \| u^0 \|_B. $$

**Proposition 30**

$$ \| u^0 \|_B^2 \leq \varepsilon^{\frac{1}{4}} \| v \|_{L^2} + \varepsilon^{\frac{1}{4}} \| v \|_{L^2} + \varepsilon^{\frac{1}{4}} \| g \|_{L^2}^2 + \varepsilon^{\frac{1}{4}} \| g \|_{H^1}^2 + \| g \|_{H^1}^2 + \| g \|_{H^1}^2 $$

$$ \| v \|_{L^2}^2 \leq B_{X_1} + F_{X_1} + N_{X_1} $$

$$ \| v \|_{L^2}^2 \leq B_{Y_0} + F_{Y_0} + N_{Y_0} $$

$$ \| v \|_{L^2}^2 \leq O_{L}(1) \| v \|_{L^2}^2 + B_{Y_0} + F_{Y_0} + N_{Y_0}. $$
Proof. First bring together estimates (56), (59), and (60) to obtain
\[ ||u^0||_y^2 \lesssim ||F(\gamma)||^2 + \varepsilon^{-\frac{1}{2}} ||F(\gamma)^{\frac{1}{2}}||^2.\]

Next, consider \( F = F(v) + g(u) \) as specified in (40). The \( g(u) \) quantities appear in the desired estimate, so we do not treat them further. We interpolate the \( F(v) \) terms in the following manner:
\[ \varepsilon^{-\frac{1}{2}} ||F(v)(\gamma)^{\frac{1}{2}}|| \leq \varepsilon^{-\frac{1}{2}} ||F(v)(\gamma)^{\frac{1}{2}}|| + \varepsilon^{-\frac{1}{2}} ||F(v)(\gamma)^{\frac{1}{2}}||. \]
\[ \leq \varepsilon^{-\frac{1}{4}} ||F(v)(\gamma)^{\frac{1}{2}}|| + \varepsilon^{-\frac{1}{4}} ||F(v)(\gamma)^{\frac{1}{2}}||. \]
\[ (74) \]

We now estimate each term in \( F(v) \) which we write here from (40) for convenience:
\[ F(v) := -2\varepsilon u_u u_{xx} q_x |_{x=0} - 2\varepsilon v_{xy} |_{x=0} - \varepsilon^2 v_{xxx} |_{x=0} + \varepsilon v v_{xy} |_{x=0}. \]
\[ (75) \]
Starting with the higher order terms,
\[ \|\varepsilon^2 v_{xxx} w\|_{x=0} \leq \|\varepsilon^2 v_{xxx} w(1 - \chi)\|_{x=0} + \|\varepsilon^2 v_{xxx} w\chi\|_{x=0} \]
\[ \leq \sqrt{\varepsilon^{1/4} \|u u_{xxx} w\|_{x=0} + \varepsilon^{1/4} \|v_{xxx}\| + \varepsilon^{1/4} \|v_{xxx}\|^{3/2}} \]
\[ \lesssim \|v\|_{Y} + \varepsilon^{1/2 + 3/2} \|v\|_{X_1}. \]
The identical argument is performed for (75)2.

For the fourth term, we expand \( v_{xy} |_{x=0} = u_u q_{xy} |_{x=0} + u_{xy} q_x |_{x=0} \), perform a Hardy type inequality for the \( q_x \) term to obtain
\[ \|v_{xy} v_{xy} w\|_{x=0} \leq \|v_{xy} u_u q_{xy} w\|_{x=0} + \|v_{xy} u_{xy} q_x w\|_{x=0} \]
\[ \leq \sqrt{\varepsilon^2 \|v\|_{Y} + \varepsilon^2 \|u u_{xy} q_{xy} w\|_{x=0} \]
\[ \leq \sqrt{\varepsilon^2 \|v\|_{Y} + \varepsilon^{1/2} \|v\|_{X_1}} \]
\[ \lesssim \sqrt{\varepsilon^2 \|v\|_{Y} + \varepsilon^{1/2} \|v\|_{X_1}}. \]
To estimate the first term from (75), we split into Euler and Prandtl:
\[ \|\varepsilon u_{xx} q_x w\|_{x=0} \leq \|\varepsilon u_{xx} q_x w\|_{x=0} + \varepsilon^{1/2} \|u_{xx} q_x w\|_{x=0} \]
\[ \leq \|u_{xx} w(\gamma)\|_{\infty} \|\varepsilon q_x \sqrt{\varepsilon w}\| + \|\varepsilon q_{xx} \sqrt{\varepsilon w}\| \]
\[ \lesssim \varepsilon^{1/2} \|w\|_{X_1} + \sqrt{\varepsilon} \|w\|_{Y}. \]
We have thus established:
\[ \|F(v) w\| \lesssim \sqrt{\varepsilon} \|w\|_{Y} + \varepsilon^{1/2 + 3/2} \|w\|_{X_1}. \]
Inserting this inequality into (74) with \( w = 1 \) and \( w = w_0 \) respectively gives the desired bound for \( ||u^0||_y^2 \). This concludes the first estimate of (73).
We now move to the $X_1$ estimate in (73). The following two bounds hold:

\[ \|v\|^2 \lesssim \varepsilon \|q\|^2 + (F_{v^0} + g(q) + \partial_x N, \varepsilon v_{xxxx} + \varepsilon^{-\frac{1}{2}} \varepsilon u_x v_{xyy}) \]

\[ \|q\|^2 \lesssim o_L(1) \|v\|^2 + (F_{v^0} + g(q) + \partial_x N, q_{xx} + q_{yy} + q_x). \]

We multiply the top equation by $\varepsilon^{-\frac{1}{2}}$ and add it to the bottom equation, which establishes the desired estimate for $X_1$. To establish the $Y_1$ estimate, we multiply the bottom equation by $\varepsilon$ and add it to the top equation.

We now turn to the $Y_0$ estimate in (77). The following two bounds hold:

\[ \|v\|^2 \lesssim o_L(1) \|v\|^2 + \|q\|^2 \|\varepsilon\|_{\sqrt{\sigma}} \|q\|_{\wedge} \]

\[ \|q\|^2 \lesssim o_L(1) \|v\|^2 + o_L(1) \|q_{xx}\|^2 + \|L \varepsilon^{-\frac{1}{2}} \|u\|^2 + \varepsilon \|\|q\|\|^2 \]

\[ \|q_x\|^2_{\wedge} \lesssim \|q_{xx}\|_{\sqrt{\sigma}}^2 + \|q_{xx}\|^2. \]

According to our definition of $o_L(1)$, there exists some $\sigma_1$ such that the first two majorizing terms in the $\|q\|_{\wedge}$ estimate can be written as:

\[ L^{\sigma_1} \{ \|q\|^2_{\wedge} + \|q_{xx}\|^2_{\wedge} \} \leq L^{\sigma_1} \|q\|^2_{\wedge} + L^{\sigma_1} \|q_{xx}\|^2_{\sqrt{\sigma}} + L^{\sigma_1} \|q_{xx}\|^2. \]

We now turn to the $\|q\|_{\sqrt{\sigma}}$ estimate above, and split the product:

\[ \|q\|_{\sqrt{\sigma}} \|q\| \|q\|_{\wedge} \lesssim L^{\sigma_2} \|q\|_{\sqrt{\sigma}} \|q\|_{\wedge} \]

\[ \lesssim L^{\sigma_2} \|q\|_{\sqrt{\sigma}} \|q\|_{\wedge} \]

We select $0 < \sigma_2 < \sigma_1$. Then we multiply the $\|q\|^2_{\wedge}$ equation by $\varepsilon L^{-\sigma_2}$ and add it to the $\|q\|^2_{\wedge}$ estimate. This then concludes the proof.

The above scheme closes to yield:

\[ \|u\|^2 \lesssim \varepsilon \frac{1}{2} |\mathcal{N}| + \varepsilon \frac{1}{2} - \mathcal{N} \|v\|^2 + \varepsilon |\mathcal{N}| + \mathcal{F}, \]

where

\[ \mathcal{F} : = \varepsilon \frac{1}{2} \mathcal{F}_1 + \varepsilon \frac{1}{2} - \mathcal{F}_2 + \varepsilon \frac{1}{2} \mathcal{F}_3 + \varepsilon \frac{1}{2} \mathcal{F}_4 + \varepsilon \frac{1}{2} \mathcal{F}_5, \]

\[ + \|g(u)u_0\|^2 + \varepsilon \frac{1}{2} \|g(u)\|_{\sqrt{\sigma}}. \]

**Lemma 31**

\[ \|N_x\| \leq \varepsilon^{p-(1+)} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}}. \]

\[ \|N_{\bar{y}}\| \leq \varepsilon^{p-(1+)} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}} + \varepsilon^{p-\frac{1}{2}} \|v\|_{\sqrt{\sigma}} \|v\|_{\sqrt{\sigma}}. \]
Proof. We begin with the immediate estimates:
\[ |N_{X_1}| \lesssim \varepsilon^{-\frac{1}{2}} \|\partial_x N\| \|v\|_{X_1}, \quad |N_{Y_0}| \lesssim \|\partial_z N w\| \|v\|_{Y_0}. \]

We now establish the following bound:
\[ \varepsilon^{-\frac{1}{2}} \|\partial_x N \cdot w\| \lesssim \{\varepsilon^{-\frac{1}{2}} \|\vec{u}^0\|_B + \varepsilon^{-\frac{1}{2}} \|\vec{v}\|_{X_1}\} \|\vec{q}\|_{w}. \]

To establish this, we go term by term:
\[
\begin{align*}
\|\vec{v} \Delta_x \vec{v} w\| &\leq \|\vec{v}\|_\infty \|\Delta_x \vec{v} w\| \\
\|I_x[\vec{v}]_x \Delta_x \vec{v} w\| &\leq \|\vec{v}\|_\infty \|\Delta_x \vec{v} w\| \\
\|\vec{v} I_x[\vec{v}]_{yy} w\| &\leq \varepsilon^{-\frac{1}{4}} \|\vec{v}\|_\infty \|\vec{v}_{yy} w\| \\
\|\vec{v} \varepsilon \vec{v}_x w\| &\leq \sqrt{\varepsilon} \|\vec{v}\|_\infty \|\vec{v}_x w\| \\
\|\vec{v} \Delta_x \vec{v} y\| &\leq \varepsilon^{-\frac{1}{4}} \|\vec{v}\|_\infty \|\Delta_x \vec{v} y\| \\
\|\vec{u}^0 \Delta_x \vec{v} w\| &\leq \|\vec{u}^0\|_\infty \|\Delta_x \vec{v} w\| \lesssim \varepsilon^{-\frac{1}{4}} \|\vec{u}^0\|_B \|\vec{q}\|_{w}, \\
\|\vec{u}^0_y \vec{v} w\| &\leq \|\vec{u}^0_y (\vec{y})\|_{L^\infty_x L^2_y} \|\vec{v} (\vec{y})^{-1} w\|_{L^2_x L^2_y} \lesssim \|\vec{u}^0\|_B \|\vec{q}\|_{w}. 
\end{align*}
\]

Above, we have used the following interpolation:
\[ \|\vec{v} \vec{y} (\vec{y})^{-\frac{1}{2}} w\|_{L^2_x L^2_y} \lesssim \|\vec{v} \vec{y} (\vec{y})^{-1} w\|^{\frac{1}{2}} \|\vec{v}_x y w\|^{\frac{1}{2}}, \]

and Hardy’s inequality. The result follows upon remarking the following basic fact. For any function \(g(x, y)\) such that \(g_{x=0} = 0\) and \(g_{y=\infty} = 0\):
\[ |g|^{\frac{1}{2}} \leq \|g_x\| ||g\| + \|g\| ||g_{xy}||. \]

This immediately gives:
\[ \|\varepsilon \vec{v} w\|_\infty + \|\vec{v} (\vec{y})^{-1/2} w\|_\infty + \|\nabla \vec{v} w\|_\infty \lesssim \|\vec{q}\|_1. \]

A basic interpolation also gives:
\[ \|\varepsilon \vec{v} x \|_{L^2_y} \leq \|\sqrt{\varepsilon} \vec{v} x \|^{\frac{1}{2}} \|\vec{v} x y \|^{\frac{1}{2}}. \]

We now select \(N_0 = 1^+\) and \(n = 4\) to obtain

**Corollary 32** Let \(u^0\) solve (40) and \(v\) solve (111). Then the following estimate holds:
\[ \|u\|^4_X \lesssim \|\vec{u}\|^4_X + o(1)\|\vec{u}\|^2_X + o(1). \]
Appendix

A Derivation of Equations

We will assume the expansions:

\[ U^\varepsilon = \tilde{u}^n + \varepsilon N_0 u, \quad V^\varepsilon = \hat{v}^n + \varepsilon N_0 v, \quad P^\varepsilon = \hat{P}^n + \varepsilon N_0 P. \]  (80)

We will denote the partial expansions:

\[ u^i_s = \sum_{j=0}^{i} \sqrt{\varepsilon} u^j_e + \sum_{j=0}^{i-1} \sqrt{\varepsilon} u^j_p, \quad \tilde{u}^i_s = u^i_s + \sqrt{\varepsilon} u^i_p, \]  \( (81) \)

\[ v^i_s = \sum_{j=1}^{i} \sqrt{\varepsilon} v^j_e + \sum_{j=0}^{i-1} \sqrt{\varepsilon} v^j_p, \quad \tilde{v}^i_s = v^i_s + \sqrt{\varepsilon} v^i_p, \]  \( (82) \)

\[ P^i_s = \sum_{j=0}^{i} \sqrt{\varepsilon} P^j_e, \quad \hat{P}^i_s = P^i_s + \sqrt{\varepsilon} \{ P^i_p + \sqrt{\varepsilon} P^i,a \}. \]  \( (83) \)

We will also define \( u^{E,i}_s = \sum_{i=0}^{n} \sqrt{\varepsilon} u^i_e \) to be the “Euler” components of the partial sum. Similar notation will be used for \( u^{P,i}_s, v^{E,i}_s, v^{P,i}_s \). The following will also be convenient:

\[ u^E_s := \sum_{i=0}^{n} \sqrt{\varepsilon} u^i_e, \quad v^E_s := \sum_{i=1}^{n} \sqrt{\varepsilon} v^{i-1}_e, \]

\[ u^P_s := \sum_{i=0}^{n} \sqrt{\varepsilon} u^i_p, \quad v^P_s := \sum_{i=0}^{n} \sqrt{\varepsilon} v^i_p, \]

\[ u_s = u^P_s + u^E_s, \quad v_s = v^P_s + v^E_s. \]  \( (84) \)

The \( P^{i,a}_p \) terms are “auxiliary Pressures” in the same sense as those introduced in \cite{GN14} and \cite{Iy15} and are for convenience. We will also introduce the notation:

\[ \tilde{u}_p^i := u^i_p - u^i_p|_{y=0}, \quad \tilde{v}_p^i := v^i_p - v^i_p(x,0), \quad \tilde{v}_e^i = v^i_e - v^i_e|_{Y=0}. \]  (85)

A.1 \( i = 0 \)

We first record the properties of the leading order \((i = 0)\) layers. For the outer Euler flow, we will take a shear flow, \([u^0_e(Y), 0, 0]\). The derivatives of \( u^0_e \) decay rapidly in \( Y \) and that is bounded below, \(|u^0_e| \gtrsim 1\).

For the leading order Prandtl boundary layer, the equations are:

\[ \begin{align*}
\tilde{u}_p^0 u_{px}^0 &+ u_p^0 u_{py}^0 - u_{pyy}^0 + P_{px}^0 = 0, \\
u_{px}^0 + u_{py}^0 = 0, \quad P_{py}^0 = 0, \quad u_{p|x=0} = U_T^0, \quad u_{p|y=0} = -u_e^0|_{Y=0}. \end{align*} \]  (86)
It is convenient to state results in terms of the quantity \( \overline{u}^0_p \), whose initial data is simply \( \overline{U}^0_p := u^0_p(0) + U^0_p \). Our starting point is the following result of Oleinik in [OS99], P. 21, Theorem 2.1.1:

**Theorem 33 (Oleinik)** Assume boundary data is prescribed satisfying \( U^0_p \in C^\infty \) and exponentially decaying \( |\partial_y^j \{ U^0_p - u^0_p(0) \}| \) for \( j \geq 0 \) satisfying:

\[
\overline{U}^0_p > 0 \quad \text{for} \; y > 0, \quad \partial_y \overline{U}^0_p(0) > 0, \quad \partial_y^2 \overline{U}^0_p \sim y^2 \quad \text{near} \; y = 0
\]  

(87)

Then for some \( L > 0 \), there exists a solution, \( [\overline{u}^0_p, \overline{v}^0_p] \) to (86) satisfying, for some \( y_0, m_0 > 0 \),

\[
\sup_{x \in (0, L)} \sup_{y \in (0, y_0)} |\overline{u}_p^0, \partial_y [\overline{u}_p^0, \partial_y \overline{u}_p^0, \partial_y^2 \overline{u}_p^0]| \lesssim 1, \quad \sup_{x \in (0, L)} \sup_{y \in (0, y_0)} \partial_y \overline{u}_p^0 > m_0 > 0.
\]  

(88)

(89)

By evaluating the system (86) and \( \partial_y \) of (86) at \( \{ y = 0 \} \) we conclude:

\[
\overline{v}^0_{p,yy}|_{y=0} = \overline{u}^0_{p,yy}|_{y=0} = 0.
\]

**A.2** \( 1 \leq i \leq n - 1 \)

We now list the equations to be satisfied by the \( i \)th layers, starting with the \( i \)th Euler layer:

\[
\begin{aligned}
&u_e^i \partial_x u_e^i + \partial_y u_e^i v_e^i + \partial_x P_e^i =: f_{E,1}^i, \\
&u_e^i \partial_x v_e^i + \partial_y P_e^i =: f_{E,2}^i, \\
&\partial_x u_e^i + \partial_y v_e^i = 0, \\
&v_e^i|_{y=0} = -v_{p,0}^i|_{y=0}, \quad u_e^i|_{x=0, L} = V_{E,(0, L)}^i, \quad u_e^i|_{x=0} = U_E^i.
\end{aligned}
\]  

(90)

For the \( i \)th Prandtl layer:

\[
\begin{aligned}
&\bar{u} \partial_x u_p^i + u_p^i \partial_x \bar{u} + \partial_y [v_p^i - v_p^i|_{y=0}] + \bar{v} \partial_y u_p^i + \partial_x P_p^i - \partial_y u_p^i := f^{(i)}, \\
&\partial_x u_p^i + \partial_y v_p^i = 0, \quad \partial_y P_p^i = 0, \\
&u_p^i|_{y=0} = -u_p^i|_{y=0}, \quad [u_p^i, v_p^i]|_{y=0} = 0, \quad v_p^i|_{x=0} = \text{prescribed initial data}.
\end{aligned}
\]  

(91)

The relevant definitions of the above forcing terms are given below. Note that as a matter of convention, summations that end with a negative number are empty sums.

**Definition 34 (Forcing Terms)**

\[
-f_{E,1}^i := u_{ex}^{-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{-1} \{ u_e^j + w_p^j(x, \infty) + w_{e}^{i-1} \sum_{j=1}^{i-2} \sqrt{\varepsilon}^{-1} w_{ex}^j \}
\]

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\[ + \sqrt{\varepsilon}^{-2}\left[\{u_e^{i-1} + u_p^{i-1}(x, \infty)\}u_{ex}^{i-1} + u_e^{i-1}u_{ey}^{i-1}\right] \\
+ u_{eY}^{i-1}\sum_{j=1}^{i-2}\sqrt{\varepsilon}^{-1}v_e^j + v_e^{i-1}\sum_{j=1}^{i-2}\sqrt{\varepsilon}^{-1}u_{ey}^j - \sqrt{\varepsilon}\Delta u_e^{i-1} - g_{ext,e} \\
- f_{E,2}^{i-1} := v_{eY}^{i-1}\sum_{j=1}^{i-2}\sqrt{\varepsilon}^{-1}v_e^j + v_e^{i-1}\sum_{j=1}^{i-2}\sqrt{\varepsilon}^{-1}u_{ey}^j + \sqrt{\varepsilon}^{-2}[v_e^{i-1}u_{eY}^{i-1} + u_e^{i-1}v_{ex}^{i-1}] \\
+ \{u_e^{i-1} + u_p^{i-1}(x, \infty)\}\sum_{j=1}^{i-2}\sqrt{\varepsilon}^{-1}v_e^j + v_e^{i-1}\sum_{j=1}^{i-2}\sqrt{\varepsilon}^{-1}\{u_e^j + u_p^j(x, \infty)\} \\
- \sqrt{\varepsilon}\Delta v_e^{i-1} - g_{ext,e}. \]

For \(i = 1\) only, we make the following modifications. The aim is to retain only the required order \(\sqrt{\varepsilon}\) terms into \(f^{(1)}\). \(f^{(2)}\) will then be adjusted by including the superfluous terms. Moreover, \(f^{(1)}\) will contain the important \(g_{ext,p}^{u_1}\) external forcing term. Specifically, define:

\[ f^{(1)} := g_{ext,p}^{u_1} - u_p^{0}u_{ex}^{1}Y = 0 - u_p^{0}u_{ex}^{1}|Y = 0 \\
- u_e^{0}Y(0)u_{py}^{0} - v_p^{0}u_{ey}^{0} - v_e^{0}Y(0)u_{py}^{0}. \]
A.3  $i = n$

For the final Prandtl layer, we must enforce the boundary condition $v_p^n|_{y=0} = 0$. Define the quantities $[u_p, v_p, P_p]$ to solve

$$
\begin{align*}
\bar{u}\partial_x u_p + u_p \partial_x \bar{u} + \partial_y \bar{v} u_p + \bar{v} \partial_y u_p + \partial_x P_p - \partial_y u_p &:= f^{(n)}, \\
\partial_x u_p + \partial_y v_p &= 0, \\
\partial_y P_p &= 0
\end{align*}
\quad (93)
$$

\[ [u_p, v_p]|_{y=0} = [-u^n_c, 0]|_{y=0}, \quad u_p|_{y \to \infty} = 0 \quad v_p|_{x=0} = V^n_p. \]

Note the change in boundary condition of $v_p|_{y=0} = 0$ which contrasts the $i = 1, \ldots, n-1$ case. This implies that $v_p = \int_0^y u_{px} \, dy$. For this reason, we must cut-off the Prandtl layers:

$$
u^n_p := \chi(\sqrt{\varepsilon}y) u_p + \sqrt{\varepsilon} \chi'(\sqrt{\varepsilon}y) \int_0^y u_p(x, y') \, dy',
$$

$$
v^n_p := \chi(\sqrt{\varepsilon}y) v_p.
$$

Here $E^n$ is the error contributed by the cut-off:

$$
E^{(n)} := \bar{u}\partial_x u_p^n + u_p^n \partial_x \bar{u} + \bar{v} \partial_y u_p^n + v_p^n \partial_y \bar{u} - u^{n}_{pyy} - f^{(n)}.
$$

Computing explicitly:

$$
E^{(n)} := (1 - \chi) f^{(n)} + \bar{u} \sqrt{\varepsilon} \chi' \sqrt{\varepsilon} y u_p(x, y) + \bar{u} \chi' \sqrt{\varepsilon} \int_0^y u_p
$$

$$
+ \bar{v} \sqrt{\varepsilon} \chi' u_p + \varepsilon \bar{v} \chi'' \int_0^y u_p + \sqrt{\varepsilon} \chi' u_p
$$

$$
+ \varepsilon \frac{\partial}{\partial y} \chi'' \int_0^y u_p + 2 \varepsilon \chi'' u_p + \sqrt{\varepsilon} \chi' u_p.
$$

We will now define the contributions into the next order, which will serve as the forcing for the remainder term:

$$
F^{(n+1)} := \sqrt{\varepsilon} \left[ E^n u^{nx}_{px} + v^n_p \{ \bar{u}_{sy} - u^0_{py} \} + \{ u^0_c - u^0_c(0) \} u^n_p
\right.
$$

$$
+ u^{nx}_{px} \sum_{j=1}^n \sqrt{\varepsilon} (u^j_c + u^j_p) + \{ u^n_{sx} - \bar{v}^0_{sx} \} u^n_p + \{ v^n_s - v^n_s \} u^n_{py}
$$

$$
+ \{ u^n_c - u^n_c(x, 0) \} u^n_{py}
\right] + \sqrt{\varepsilon} E^{(n)} + \sqrt{\varepsilon}^{n+1} \Delta u^n_c
$$

$$
+ \sqrt{\varepsilon}^{n+1} u_{ex}^n \sum_{j=1}^{n-1} \sqrt{\varepsilon} u_{c}^j + \sqrt{\varepsilon}^{n+1} u_{c}^n \sum_{j=1}^{n-1} \sqrt{\varepsilon} u_{ex}^j + \sqrt{\varepsilon}^{2n} [u^n_c u_{ex}^n
$$

$$
+ v^n_p u^n_{cY} + \sqrt{\varepsilon}^{n+1} u_{cY}^n \sum_{j=1}^{n-1} \sqrt{\varepsilon}^{-j} u_{c}^j + \sqrt{\varepsilon}^{n-1} v^n_c \sum_{j=1}^{n-1} \sqrt{\varepsilon}^{j+1} u_{cY}^j].
$$

$$
E^{(n+1)} := \sqrt{\varepsilon} \left[ v^n_s \partial_y v^n_p + \partial_y v^n_s v^n_p + \partial_x v^n_s u^n_p + u^n_s \partial_x v^n_p - \Delta_c v^n_p\right.
$$

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A straightforward linearization yields:
\[
\sqrt{\varepsilon} u_0 \partial_x v_0^\varepsilon + v_0^\varepsilon \partial_y v_0^\varepsilon \bigg) \bigg] + (\sqrt{\varepsilon})^n \partial_y v_0^n \sum_{j=1}^{n-1} (\sqrt{\varepsilon})^{j-1} v_0^j
\]
\[
+ \sqrt{\varepsilon} u_0^n \sum_{j=1}^{n-1} \sqrt{\varepsilon}^j \partial_y v_0^j + \sqrt{\varepsilon}^{2n-1} [v_0^n \partial_y v_0^n + u_0^n \partial_y v_0^n]
\]
\[
+ \sqrt{\varepsilon} u_0^n \sum_{j=1}^{n-1} (\sqrt{\varepsilon})^{j-1} \partial_x v_0^j + \sqrt{\varepsilon}^{n-1} \partial_x v_0^n \sum_{j=0}^{n-1} \sqrt{\varepsilon}^j v_0^j + \sqrt{\varepsilon}^{n+1} \Delta v_0^n.
\]

A.4 Remainder System

A straightforward linearization yields:
\[
- \Delta \varepsilon u^{(\varepsilon)} + S_u + \partial_x P^{(\varepsilon)} = \varepsilon^{-N_0} \Delta f^{(n+1)} + \varepsilon N_0 \left[ u^{(\varepsilon)} \partial_x u^{(\varepsilon)} + v^{(\varepsilon)} \partial_y u^{(\varepsilon)} \right]
\]
\[
- \Delta \varepsilon v^{(\varepsilon)} + S_v + \frac{\partial_b}{\varepsilon} P^{(\varepsilon)} = \varepsilon^{-N_0} g^{(n+1)} + \varepsilon N_0 \left[ u^{(\varepsilon)} \partial_x v^{(\varepsilon)} + v^{(\varepsilon)} \partial_y v^{(\varepsilon)} \right]
\]
\[
\partial_x u^{(\varepsilon)} + \partial_y v^{(\varepsilon)} = 0.
\]

Denote:
\[
u_s := \tilde{u}_s^n, \quad \tilde{v}_s := \tilde{\tilde{v}}_s^n.
\]
Here we have defined:
\[
S_u = u_s \partial_x u^{(\varepsilon)} + u_{sx} u^{(\varepsilon)} + v_s \partial_y u^{(\varepsilon)} + u_{sy} v^{(\varepsilon)},
\]
\[
S_v = u_s \partial_x v^{(\varepsilon)} + v_{sx} u^{(\varepsilon)} + v_s \partial_y v^{(\varepsilon)} + v_{sy} v^{(\varepsilon)}.
\]

We will also define:
\[
\tilde{g}^{(u)} := \varepsilon^{-N_0} \{ \partial_x f^{(n+1)} \} - \varepsilon \partial_x \Delta^{(n+1)}, \quad \text{and} \quad \tilde{g}^{(q)} := \partial_x \tilde{g}^{(u)}.
\]

Going to the vorticity formulation of (97), we obtain the following:
\[
-R[q^{(\varepsilon)}] - u^{(\varepsilon)}_{yyy} + 2 \varepsilon \nu^{(\varepsilon)}_{xyy} + \varepsilon^2 \nu^{(\varepsilon)}_{xxx} + A_1 + A_2
\]
\[
= \tilde{g}^{(u)} + \varepsilon N_0 \{ v \Delta u - \Delta v \},
\]

where we have defined the Rayleigh operator:
\[
R[q^{(\varepsilon)}] = \partial_y \{ u^2 \partial_y q^{(\varepsilon)} \} + \varepsilon \partial_x \{ u^2 q^{(\varepsilon)} \},
\]
and where:
\[
A_1 := v_s u^{(\varepsilon)}_{yy} - v_{sy} u^{(\varepsilon)}, \quad A_2 := \varepsilon v_s u^{(\varepsilon)}_{xx} - \varepsilon v_{sx} u^{(\varepsilon)}.
\]

Note for future reference that we may alternatively write:
\[
A_1 + A_2 = v_s \Delta \varepsilon u^{(\varepsilon)} - u^{(\varepsilon)} \Delta \varepsilon v_s.
\]
In Section 2, our main object of analysis with the vorticity equation evaluated at the \( \{x = 0\} \) boundary, \( \{102\}_x=0 \), which reads:

\[
\mathcal{L}u^{0,\nu} := -u_{yy}^{0,\nu} + v_s u_{yy}^{0,\nu} - u^{0,\nu}\Delta_x v_s = F,
\]

\[
F := -2\varepsilon u_s u_{sx} q^{(\nu)}|_{x=0} - 2\varepsilon v_{syy}^{(\nu)}|_{x=0} - \varepsilon^2 v_{sxx}^{(\nu)}|_{x=0} - \varepsilon v_s u_{xx}^{(\nu)}|_{x=0} + \tilde{g}(u), \tag{105}
\]

\[
u^{0,\nu}(0) = 0, \partial_y u^{0,\nu}(\infty) = 0, \partial_{yy} u^{0,\nu}(\infty) = 0.
\]

The \( x \)- differentiated vorticity equation, which we refer to as DNS, \( \partial_x \{102\} \), reads:

\[
-\partial_x R[q^{(\nu)}] + \Delta_x^2 v^{(\nu)} + \partial_x \{A_1\} = \varepsilon N_0 N + \tilde{g}(q),
\]

\[
v^{(\nu)}|_{x=0} = a_0^{(\nu)}, v^{(\nu)}|_{x=L} = a_3^{(\nu)}, \quad v^{(\nu)}|_{y=0} = 0, \quad v^{(\nu)}|_{y=0} = 0.
\tag{106}
\]

It is useful to consider \( N = N(\tilde{u}, \bar{v}) \) which is more suitable to apply a contraction mapping argument. This therefore has the expression:

\[
\varepsilon N_0 N = \varepsilon N_0 \left( \bar{v}_y \Delta_x \tilde{v} + I_x[\bar{v}_y] \Delta_x \tilde{v} - \bar{v}_x I_x[\bar{v}_{yy}]ight) + \varepsilon \bar{v}_x \bar{v}_{xy} - \varepsilon \Delta_x \bar{v}_y - \bar{u} \Delta_x \bar{v}_x + \bar{v}_x \bar{v}_{yy} \tag{107}
\]

The forcing \( g(q) \) has been defined above in \( \{101\} \). The term \( b(q) \) arises as a result of homogenizing the boundary conditions. Define the boundary corrector

\[
\bar{v} := a_0^{\nu} + x \left( a_1^{\nu} - L a_2^{\nu} - \frac{L^2}{2} a_3^{\nu}\right) + x^3 \frac{a_5^{\nu}}{6}.
\]

One checks immediately that \( \bar{v} \) achieves the boundary conditions in \( \{106\} \). We homogenize at the level of the vorticity equation, \( \{102\} \). Define the homogenized quantities:

\[
v := v^{(\nu)} - \bar{v}, \quad u := u^{(\nu)} + \int_0^x \tilde{v}_y, \quad u_s q := v. \tag{108}
\]

It is clear that the divergence free condition is satisfied by the pair \([u, v]\). Writing \( \{102\} \) gives:

\[
-R[q] - u_{yy} + 2\varepsilon v_{xyy} + \varepsilon^2 v_{xxx} + \{v_s + \varepsilon N_0 \bar{v}\} \Delta_x u
\]

\[
- u \Delta_x \{v_s + \varepsilon N_0 \bar{v}\} - \varepsilon N_0 I_x[\bar{v}_y] \Delta_x v + \varepsilon N_0 v \Delta_x I_x[\bar{v}_y] \tag{109}
\]

\[
g(u) + b(u) + \varepsilon N_0 \{\partial_y N_u(u, v) - \varepsilon \partial_y N_v(u, v)\},
\]

where:

\[
b(u) = -R[\bar{v}] + I_x[\bar{v}_{yyy}] + 2\varepsilon v_{xyy} + \varepsilon^2 \bar{v}_{xxx} - \varepsilon \bar{v}_{xy} + v_s I_x[\bar{v}_{yy}] - \Delta_x v_s I_x[\bar{v}_y],
\]

\[
b(q) = \partial_x b(u).
\]
We then solve for \([u, v]\) using the equations (111) and (110). By definition, the pair \([u := u^0 - \int_0^t v_\tau, v]\) solves the vorticity equation, (109). We then obtain \([u^{(2)}, v^{(2)}]\) using (108). We now summarize the two systems we will be studying.

First the \(L\)-system:

\[
\begin{align*}
\mathcal{L} u^0 &:= -u^0_{yy} + v_s u^0_{yy} - u^0 \Delta_x v_s = F, \\
F &:= -2 \varepsilon u_s u_{ss} q_x|_{x=0} - 2 \varepsilon v_{syy}|_{x=0} - \varepsilon^2 v_{xxx}|_{x=0} - \varepsilon v_s u_{xx}|_{x=0} + g(u),
\end{align*}
\]

where \(u^0, v^0 \in H^0(0, \infty) = 0, \partial_y u^{0, \varepsilon}(\infty) = 0, \partial_y g u^{0, \varepsilon}(\infty) = 0.

and

\[
- \partial_\tau R[q] + \Delta^2_v \varepsilon + \partial_\tau \{A_1\} = \partial_\tau N + g(q),
\]

\[
v|_{x=0} = 0, v_x|_{x=0} = 0, v_x|_{x=L} = 0, v_{xxx}|_{x=L} = 0,
\]

Finally, we define the quantities \(H_{(u)}\) and \(H_{(q)}\) that appear above:

\[
\begin{align*}
H_{(u)} &:= u \Delta_x \varepsilon^{N_0} \Delta_x u + \varepsilon^{N_0} I_x[\bar{v} y] \Delta_x u - \varepsilon^{N_0} \Delta_x I_x[\bar{v} y], \\
H_{(q)} &:= \partial_\tau H_{(u)}.
\end{align*}
\]

Recall the definition of \(\mathcal{F}\) from (77). The following estimates are clear that according to the assumption, (109):

\[
\begin{align*}
\mathcal{F}(b_{(q)}, b_{(u)}) &\leq o(1), \\
\mathcal{F}(H_{(q)}, H_{(u)}) &\leq o(1)\|u\|_L^2.
\end{align*}
\]

The following proposition summarizes the profile constructions from [8]:

**Theorem 35** Assume the shear flow \(u^0_{p}(Y) \in C^\infty\), whose derivatives decay rapidly. Assume (87) regarding \(u^0_{p, x=0}\), and the conditions

\[
\begin{align*}
\bar{v}^i_{p,yy}|_{x=0} &= \partial_x g_1|_{x=0, y=0}, \\
\bar{v}^i_{p, x=0} &= \partial_{xy} g_{1}|_{y=0}(x = 0), \\
\bar{u}^0_{py}|_{x=0} u^i_{c, x=0} &= 0 - \int_{0}^{\infty} u^0_{p, e} - f^i \bar{v}^0 \{f^{(i)}(y) - r^{(i)}(y)\} dy = 0,
\end{align*}
\]

where \(r^{(i)}(y) := \bar{v}^i_{p, p, y} - \bar{u}^0_{p, x} v^i_{p, y}\). We assume also standard higher order versions of the parabolic compatibility conditions (113), (116). Let \(v^i_{c, x=0}, v^i_{c, x=L}, u^i_{c, x=0}\) be prescribed smooth and rapidly decaying Euler data. We assume on the data
standard elliptic compatibility conditions at the corners \((0,0)\) and \((L,0)\) obtained by evaluating the equation at the corners. In addition, assume
\[
v^i_\epsilon|_{x=0} \sim Y^{-m_1} \text{ or } e^{-m_1 Y} \text{ for some } 0 < m_1 < \infty, \tag{118}
\]
\[
\| \partial^k Y \{ v^i_\epsilon|_{x=0} - v^i_\epsilon|_{x=L} \} (Y)^M \|_{\infty} \lesssim L \tag{119}
\]

Then all profiles in \([u_s, v_s]\) exist and are smooth on \(\Omega\). The following estimates hold:
\[
\tilde{u}^0_p > 0, \tilde{u}^0_{py}|_{y=0} > 0, \tilde{u}^0_{pyy}|_{y=0} = \tilde{u}^0_{pyyy}|_{y=0} = 0
\]
\[
\| \nabla^K \{ u^0_p, v^0_p \} e^{M_y} \|_{\infty} \lesssim 1 \text{ for any } K \geq 0,
\]
\[
\| u^i_p \|_{\infty} + \| \nabla^K u^i_p e^{M_y} \|_{\infty} + \| \nabla^J v^i_p e^{M_y} \|_{\infty} \lesssim 1 \text{ for any } K \geq 1, M \geq 0, \tag{120}
\]
\[
\| \nabla^K \{ u^1_\epsilon, v^1_\epsilon \} w_{m_1} \|_{\infty} \lesssim 1 \text{ for some fixed } m_1 > 1
\]
\[
\| \nabla^K \{ u^i_\epsilon, v^i_\epsilon \} w_{m_i} \|_{\infty} \lesssim 1 \text{ for some fixed } m_i > 1,
\]

where \(w_{m_i} \sim e^{m_i Y} \text{ or } (1 + Y)^{m_i} \).

In addition the following estimate on the remainder forcing (recall (77)) holds:
\[
\mathcal{F} \lesssim \sqrt{\varepsilon}^{n-1-2N_0}. \tag{121}
\]

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