On safe orbits of a space station tethered to a precessing asteroid by two cables

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Abstract. We study the problem of existence of safe motion for a space station tethered to a small planet by two cables. We consider the case of the dynamically symmetric asteroid and of the cables which opposite ends are fixed to the poles of the small planet. We establish that the station equilibria with respect to axes of precession and of dynamical symmetry are the safe orbits if gravitation of the asteroid satisfy some restrictions. We illustrate these restrictions by geometrical diagrams. Also, we deduce general conditions for existence of the safe orbits of other types.

1. Introduction

One of the goals for the modern space missions is to master the Asteroid Belt. Each of the small planets might be interesting as an immediate base on the way into deep space, as a source of mineral raw materials or as a potentially dangerous object. Now we know that landing on the small planet surface is an elaborate procedure. Generally, the reasons are the asteroids complex shapes, and, consequently, theirs complex spin motion and theirs intricate gravitational potentials. Apparently, to place a space station close to an asteroid without direct landing might be more convenient decision. Taking into account that the asteroid gravitation is very small in comparison with the solar gravitation, one can place the space station on the asteroid heliocentric orbit. If the station will be sufficiently far from the small planet then the mission will be restricted to observations only. But if the station will be directly nearby the small planet then one can anchor the station to the asteroid surface, in other words, one might link the station and the asteroid by a cable. (For the first time, such linking have been suggested in [1] as non-gravitational method of orbits correction). Factually, in this case the tether is the simplest transport system between the station and the asteroid. Nevertheless, note that if the distance from the station to the asteroid surface is comparable-sized with the asteroid diameter then gravitation of the asteroid is several orders more than the other forces sum (of course, here we must take into account forces of inertia also). In this case the station becomes satellite of the asteroid and some problems of the station safety take place. Evidently, the tether eliminates running away of the station, but the station might fall down on the asteroid surface. Moreover, the tether might slack. In this case the tether might taut again after some time with impact that can be catastrophic for the station equipment. Moreover, the long-term missions is realizable only if the station motion is stable (may be in orbital sense only). One might suggest that the second anchor might stabilize the station orbit, but it is not guaranteed. Note that the systems like ‘space elevators’ is not realizable for the problem under consideration by the reasons formulated above. Nevertheless, there are small planets that are close to bodies of rotation.
One might assume that such asteroids are dynamically symmetric rigid bodies and theirs spin motions are close to regular precessions. In this case, one might fix the tether(s) end(s) at the asteroid poles, which arise as intersection of the asteroid surface and the axis of dynamical symmetry. (In practice, the ‘anchors’ may be some towers constructed on the poles). Evidently, the station doesn’t fall down if there are two tethers and both of them are taut throughout the motion. Also, in this case, the station moves with respect to the asteroid surface, so the present construction is more universal then the traditional space elevator. Moreover, the station orbit can be stable [2, 3].

2. Formulation of the problem
Consider a dynamically symmetric rigid body modeling an asteroid. Let $C$ be mass center of the body, $Cz$ be the axis of dynamical symmetry, $Cz_1$ be the axis of precession with angular velocity $\omega$ and with angle of acute nutation $\theta$ (see figure 1). In our case $\omega = const, \theta = const$. Let us restrict to the general case of acute $\theta$. Let $S$ be a particle modeling a space station. Suppose that $S$ tethered to poles $P_1$ and $P_2$ via two cables. As coordinate of $P_1$ on $Cz$ is positive, this pole is called the “north pole”, unlike $P_2$ that is called the “south pole”.

The space station trajectory is called the safe orbit if the particle $S$ motion along this trajectory is stable (maybe in orbital sense) and both tethers are taut throughout this motion. Suppose that the tethers are weightless, inextensible and absolutely flexible. Then the particle $S$ motion is restricted by two unilateral constraints that are spheres with centers in the poles $P_1$ and $P_2$ and radii that are equal to lengths of the north and south tethers respectively. The spheres surfaces are called the constraint boundaries. Intersection of constraints boundaries is some circle with center on $Cz$ (the point $H$ in fig.1). This circle is orthogonal to the axis of dynamical symmetry. Therefore, the station $S$ safe orbit is a point or an arc (maybe complete) of a circle with center on $Cz$ in the plane that is orthogonal to $Cz$. Let us remark that we have two frame of references: axes $Cz$ and $Cz_1$ and the asteroid. Evidently, if some safe orbit in the first frame of reference is point or arc then this orbit is the complete circle with respect to the asteroid, as the asteroid rotates about $Cz$. Motion along the safe orbit is called the constrained motion.

Let $Cx$ be orthogonal to $Cz$ and $Cz_1$ and $Cxyz$ and $Cxyz_1$ are right-handed coordinate systems. Let $x, y, z$ be coordinates of $S$ in $Cxyz$ and $x = \rho \cos \varphi, y = \rho \sin \varphi$. Moreover, let $\beta_1 = \angle SP_1C$, $\beta_2 = \angle SP_2C$, $\alpha = \angle HSC$. Note that if the particle $S$ moves along the safe orbit then $\alpha, \beta_1, \beta_2, \rho, z$ don’t change. Factually, they are parameters of the safe orbit. Without loss of generality one can assume that $z \geq 0$ (equivalently, $0 \leq \alpha < \pi/2$) and that $\beta_2$ is acute.

If the asteroid is close to body of rotation then one might assume that gravitational potential of such asteroid is invariant with respect to rotations about axis of dynamical symmetry. In this case, the asteroid gravitation force $F$, acting on the station, throughout the constrained motion is directed on the same point on $Cz$ ($K$ in fig.1). Moreover, $|F| = const$ in points of any safe orbit. One might present $F$ as sum $F = F_1 + F_2$, where each of $F_i$ is directed to $P_i$. 

![Figure 1. Designations and coordinate systems](image)
3. Equations of motion and conditions for existence of safe orbits

If the station motion is constrained then reaction of each tether might be presented by product of Lagrange’s multiplier and vector $\overrightarrow{P_iS}$ (here $i = 1$ for the north tether and $i = 2$ for the south one). In the case of constrained motion two of three equations of motion for the particle $S$ can be written as

$$\begin{align*}
\varphi^' + 2\cos\vartheta - \sin \varphi \sin \vartheta \tan \beta_1 \varphi^' + 1 + \sin^2 \vartheta (\tan \beta_1 \tan \alpha - \sin^2 \varphi) - \\
- \sin \vartheta \cos \vartheta \sin \varphi (\tan \beta_1 + \tan \alpha) - f_2 \sin (\beta_1 + \beta_2) / \cos \beta_1 &= -\lambda_2, \\
\varphi^2 + 2\cos\vartheta + \sin \varphi \sin \vartheta \tan \beta_2 \varphi^' + 1 - \sin^2 \vartheta (\tan \beta_2 \tan \alpha + \sin^2 \varphi) - \\
- \sin \vartheta \cos \vartheta \sin \varphi (\tan \beta_2 - \tan \alpha) - f_1 \sin (\beta_1 + \beta_2) / \cos \beta_2 &= -\lambda_1.
\end{align*}$$

(1)

where the prime (’) denotes derivative with respect to dimensionless time $\tau$ ($\tau = \omega t$), $f_i = |\overrightarrow{F_i}|/[mS|\omega|^2]$ are dimensionless components of the gravitation force, $m$ is the station mass, $\lambda_i$ depend on Lagrange’s multipliers. (Deriving (1-2) we took into account that $\dot{\rho} = \rho^' = z^' = z^2 = 0$ for constrained motion). Evidently, each of Lagrange’s multipliers must be negative throughout constrained motion, and, consequently, for any safe orbit. Analyzing (1-2) one might see that Lagrange’s multipliers will be negative for acute $\beta_1$ if $\lambda_1 < 0$, $\lambda_2 < 0$, for obtuse $\beta_1$ if $\lambda_1 < 0$, $\lambda_2 > 0$ and for right $\beta_1$ if

$$\lambda_1 < 0, \quad 2 \sin \varphi \left(\varphi^' + \frac{\cos \vartheta}{2}\right) - \tan \alpha \sin \vartheta < -f_2 \cos \beta_2 / \sin \vartheta.$$  

(3)

If the particle $S$ motion is constrained then the last equation of motion is integrable and could be written as

$$\varphi^2 + \sin \varphi \sin \vartheta (\sin \varphi \sin \vartheta + 2 \tan \alpha \cos \vartheta) = 2h = \text{const}$$

(4)

after onetime integration.

4. Relative equilibria

Analyzing left part of (4) one might see that there are several equilibrium points of the particle $S$ in rotating coordinate system $Cxyz$, i.e. in the frame of reference of axes of precession and dynamical symmetry. Such points set is intersection of planes $Cyz_1$ (coincides with $Cyz$ and contains axes of precession and dynamical symmetry) and $Cxy_1$ (is orthogonal to axis of precession) with the circle of constrained motion. Evidently, this circle always has two points of intersection with $Cyz_1$. These points are the ‘upper equilibrium’ $\varphi = \pi/2$ and the ‘lower equilibrium’ $\varphi = -\pi/2$. The upper equilibrium is always unstable. Hence, this point is not a safe orbit. The lower equilibrium is stable only if $\alpha \geq \vartheta$. Note that if $\alpha > \vartheta$ then the circle of constrained motion doesn’t intersect $Cxy_1$, and if $\alpha = \vartheta$ this circle is tangent to $Cxy_1$ in the point $\varphi = -\pi/2$. Thus, if $\alpha \geq \vartheta$ then there is only one stable equilibrium in $Cyz$. If $\alpha < \vartheta$ then the circle of constrained motions has two intersection points with $Cxy_1$. These points are $\varphi = -\arcsin(tg\alpha \ cot\alpha)$ and $\varphi = \pi + \arcsin(tg\alpha \ cot\alpha)$. They are called the ‘horizontal equilibria’. Both of them are stable.

Varying the tethers lengths one makes sure that any point of $Cxy_1$ could be the stable relative equilibrium (of course, excepting $C$). Moreover, any point inside acute angles between $Cz$ and $Cyz$ could be the stable relative equilibrium.

5. Diagrams of safety

Any of mentioned in the previous chapter stable equilibrium points could be the safe orbit, but only if both of Lagrange’s multipliers will be negative in such point. These conditions for the lower equilibrium $\varphi = -\pi/2$ are written as

$$f_2 \sin (\beta_1 + \beta_2) < \cos (\beta_1 + \beta_2) \cos (\alpha - \vartheta) / \cos \alpha,$$  

(5)

$$f_1 \sin (\beta_1 + \beta_2) < \cos (\beta_1 - \beta_2) \cos (\alpha - \vartheta) / \cos \alpha.$$  

(6)
((5-6) are result of reducing (1-3)). Note that inequality (6) is possible only if \( \beta_1 < \pi/2 + \theta \).

There is a simple geometrical interpretation of inequalities (5-6). Let \( H_1 \) be projection of \( S \) on \( Cz \). Moreover, let \( SQ_1H_1Q_2 \) be parallelogram with opposite vertexes \( S \) and \( H_1 \) and with sides being parallel to tethers. \( (SQ_1H_1Q_2 \) is painted over in figure 2). Then the lower equilibrium point \( \varphi = -\pi/2 \) is the safe orbit if \( \alpha \geq \varphi \) and if the head of vector \( \Phi = F/(m\omega^2) \) with tail in \( S \) lays inside this parallelogram. Note that the lower equilibrium could be the safe orbit even if \( \beta_1 \) is obtuse (but less than sum of right angle and angle of nutation), i.e. even if the station moves higher the pole.

\[
\begin{align*}
\text{Figure 2. The diagram of safety for equilibrium point } \varphi = -\pi/2 \\
\text{Figure 3. The diagram of safety for horizontal equilibrium points}
\end{align*}
\]

For any equilibrium point in \( Cxyz \) Lagrange’s multipliers is negative if

\[
\begin{align*}
f_1 \sin(\beta_1 + \beta_2) < \cos(\alpha + \beta_2) / \cos \alpha, \quad (7) \\
f_2 \sin(\beta_1 + \beta_2) < \cos(\alpha - \beta_1) / \cos \alpha, \quad (8)
\end{align*}
\]

These inequalities are result of reducing of (1-3) also. They have a simple geometrical interpretation too. Let \( S_1 \) be the lower equilibrium that corresponds to studied horizontal equilibria. Moreover, let \( SQ_3H_1Q_4 \) be parallelogram with opposite vertexes \( S_1 \) and \( H_1 \) and with sides being parallel to tethers. \( (SQ_3H_1Q_4 \) is painted over in figure 3). Then horizontal equilibrium points are the safe orbits if the head of vector \( \Phi = F/(m\omega^2) \) with tail in \( S_1 \) lays inside this parallelogram.

6. Safe orbits of other types

To check out orbits of the station one might analyze phase trajectories of Jacobi’s integral (4). There are only three distinct types of phase portraits for such trajectories on the phase cylinder \( \varphi \mod 2\pi, \varphi \) (see examples in figure 4-6). Note that some areas must be cut out from the mentioned cylinder. Such areas are called ‘the constraint loss areas’ and consist of points in which at least one of Lagrange’s multipliers is nonnegative. The constraint loss areas have a variety of shapes. Some examples of them are shadowed over in figure 4-6. One might define boundaries of such areas substituting 0 for \( \lambda_1 \) in (1-2) or replacing the sign ‘<’ with ‘=' in (3).

Let phase trajectory be regular if it is not be unstable point of equilibrium, if it is not be separatrix and if it has no intersections with constraint loss areas. One might easily check out that anyone regular phase trajectory is stable in the orbital sense. Hence, each regular trajectory of (4) presents some safe orbit of the station. There are three types of such orbits.

The first type is librations near one of the stable equilibrium points (bold ovals in figure 4). In this case the station orbit in \( Cxyz \) is the arc that has not more than one intersection with \( Cxyz_1 \).

The second type is librations around both horizontal equilibria and the lower equilibria. (The last is unstable in this case). The bold curve in figure 5 is an example of such libration. In this case the station orbit in \( Cxyz \) is the arc that has two intersections with \( Cxyz_1 \).
The third type is rotations around the axis of dynamical symmetry (the bold curves in figure 6). In this case the station orbit in $C_{xyz}$ is the full circle of constrained motion. Note that the safe rotations are impossible if at least one of angles $\beta_1$ and $\beta_2$ is not acute.

References

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[2] Rodnikov A V 2011 *Rus. J. Nonlin. Dyn* 7 pp 295-311
[3] Rodnikov A V 2012 *Rus. J. Nonlin. Dyn* 8 pp 309-322