Abstract

We study the Felderhof free-fermion six-vertex model, whose wavefunction recently turned out to possess rich combinatorial structure of the Schur polynomials. We investigate the dual version of the wavefunction in this paper, which seems to be a harder object to analyze. We evaluate the dual wavefunction in two ways. First, we give the exact correspondence between the dual wavefunction and the Schur polynomials, for which two proofs are given. Next, we make a microscopic analysis and express the dual wavefunction in terms of strict Gelfand-Tsetlin pattern. As a consequence of these two ways of evaluation of the dual wavefunction, we obtain a dual version of the Tokuyama combinatorial formula for the Schur polynomials. We also give a generalization of the correspondence between the dual wavefunction of the Felderhof model and the factorial Schur polynomials.

Mathematics Subject Classification. 05E05, 05E10, 16T25, 16T30, 17B37.

Keywords. Integrable lattice models, Yang-Baxter equation, Symmetric polynomials, Combinatorial representation theory.

1 Introduction

Integrable lattice models [1, 2, 3, 4] in mathematical physics have played important roles in the developments of algebras, combinatorics and representation theory. One of the most fundamental models in integrable lattice models is the six-vertex models [5, 6]. The most famous six-vertex model is the one whose \( L \)-operator has the quantum group \( U_q(sl_2) \) symmetry. The corresponding one-dimensional integrable quantum spin chain for this two-dimensional six-vertex model is the Heisenberg XXZ chain.

A less well-known six-vertex model is the Felderhof free-fermion model [9], which can be regarded as a free fermion model in an external field. It was found some time before...
that the Felderhof model has also quantum group symmetry \[10, 11\]. The corresponding representation has a property that the quantum group parameter \( q \) must be roots of unity for the representation to be finite-dimensional. A special class of partition functions called the domain wall boundary partition function was also evaluated for the case of the Felderhof model \[12\] in the past.

However, it was only found in recent years that the Felderhof model has rich mathematical structures related with the combinatorial representation theory of Schur polynomials. One of the striking facts found \[13\] was that the Tokuyama formula \[14, 15\], which is a one-parameter deformation of the Weyl character formula, is naturally realized as wavefunction of the Felderhof model. The wavefunction is a special class of partition function, which can be called as an off-shell Bethe vector since it becomes the Bethe eigenvectors of the corresponding one-dimensional spin chain when the Bethe ansatz equation is imposed on the spectral parameters. In this case, the wavefunction is sometimes called as the on-shell Bethe vector. However, we do not impose the Bethe ansatz equation on the spectral parameters in this paper, i.e., the parameters are free variables.

Besides the spectral parameter, one can introduce at least one free parameter in the \( L \) operator of the Felderhof model, which turns out to play the role of the deformation parameter in the Tokuyama formula for the Schur polynomials. The parameter for the deformation can be interpreted as a free parameter which can be introduced when constructing a finite-dimensional representation space of a quantum group when \( q \) is fixed at roots of unity. Since the \( L \)-operator is constructed as an intertwiner of tensor product of two representation spaces, one can in fact introduce at least two free parameters, one in the auxiliary space, and another in the quantum space. The parameters can in principle be different for different auxiliary and quantum spaces. For the Tokuyama formula to be realized, all the parameters are set to be equal in the auxiliary spaces, and all are zero in the quantum spaces \[13\]. Keeping all the parameters in the quantum spaces non-zero and independent, it was found that the wavefunction gives the factorial Schur polynomials \[16\]. The Tokuyama formula for the Schur polynomials can be understood as a consequence of the evaluation of the wavefunction in two ways. One by expressing it as a product of a one-parameter deformation of the Vandermonde determinant and the Schur polynomials, and another one by making a microscopic analysis and derive an expression using the strict Gelfand-Tsetlin pattern. The Tokuyama formula is a consequence of the two evaluations for the same object. This understanding \[13\] opened a new doorway to the combinatorial representation theory of symmetric polynomials via the Felderhof free-fermion model.

In this paper, we study the dual wavefunction of the Felderhof model, and study the combinatorics of the Schur polynomials by analyzing the dual wavefunction, a similar object but seems harder to analyze than the original wavefunction. The dual wavefunction was evaluated for the special case \( t = 1 \) of the deformation parameter \[13, 16\], which was obtained by transforming the original wavefunction to the dual wavefunction by symmetry arguments. We want the exact evaluation when the deformation parameter is generic, since this free parameter plays the role of refining the combinatorics of the Schur polynomials. We evaluate the dual wavefunction in two ways and obtain a combinatorial formula for the Schur polynomials. First, we analyze the dual wavefunction directly, and show the correspondence between the Schur polynomials. We give two proofs for this correspondence, one by using the arguments which is slightly more complicated than, but the same with the one given in \[13\]. Another proof is a modern statistical mechanical approach, which combines the matrix
product method \cite{17,18} and the Izergin-Korepin method of analysis on the domain wall boundary partition function \cite{19,20}. We next give a microscopic analysis of the dual wavefunction. By calculating the matrix elements of a single $B$-operator, we derive an expression of the dual wavefunction in terms of the strict Gelfand-Tsetlin pattern. By comparing the two evaluations of the dual wavefunction, we derive a dual version of the Tokuyama-type formula for the Schur polynomials.

This paper is organized as follows. We introduce the Felderhof model in section 2 and review the relation between the wavefunction and the Schur polynomials in section 3. In sections 4 and 5, we introduce the dual wavefunction, and show the relation with the Schur polynomials by giving two different proofs. In section 5, we evaluate the dual wavefunction based on the calculation of the matrix elements of a single $B$-operator, and express in terms of the strict Gelfand-Tsetlin pattern. Combining the obtained expression with the one proved in sections 4 and 5, we give a combinatorial formula which can be regarded as a dual version of the Tokuyama formula. We give a generalization of the correspondence between the dual wavefunction of a generalization of the Felderhof model and the factorial Schur polynomials in section 6. Section 7 is devoted to the conclusion.

2 Felderhof model

We introduce the Felderhof model in this section, and review the results on the relation between the wavefunction and the Schur polynomials in the next section. We use the $L$-operator in \cite{13} which is best suited for the study of the combinatorics of the Schur polynomials, since the Tokuyama formula is exactly realized as the wavefunction constructed from this $L$-operator. More generic or gauge-transformed ones can be found in \cite{10,11,12} for example. We also use the terminology of the quantum inverse scattering method or the algebraic Bethe ansatz, which is one of the most fundamental methods for the analysis of quantum integrable models.

The most fundamental objects in integrable lattice models are the $R$-matrix and the $L$-operator. For the case of the Felderhof model, the $R$-matrix is given by

\[ R_{ab}(z,t) = \begin{pmatrix} 1 + tz & 0 & 0 & 0 \\ 0 & t(1-z) & t + 1 & 0 \\ 0 & (t+1)z & z - 1 & 0 \\ 0 & 0 & 0 & z + t \end{pmatrix}, \tag{2.1} \]

acting on the tensor product $W_a \otimes W_b$ of the complex two-dimensional space $W_a$. Let us denote the orthonormal basis of $W_a$ and its dual as \{0,1\} and \{\alpha,\beta\}, and the matrix elements of the $R$-matrix as $a\beta|0\beta\langle R_{ab}(z,t)|0\alpha\rangle^\alpha_b = [R(z,t)]^\alpha_\beta_{ab}$. The matrix elements of the $R$-matrix are explicitly given as

\[ a\langle 0|b\langle 0|R_{ab}(z,t)|0\rangle^\alpha_a|0\rangle_b = 1 + tz, \tag{2.2} \]
\[ a\langle 0|b\langle 1|R_{ab}(z,t)|0\rangle^\alpha_a|1\rangle_b = t(1-z), \tag{2.3} \]
\[ a\langle 0|b\langle 1|R_{ab}(z,t)|1\rangle^\alpha_a|0\rangle_b = t + 1, \tag{2.4} \]
\[ a\langle 1|b\langle 0|R_{ab}(z,t)|0\rangle^\alpha_a|1\rangle_b = (t + 1)z, \tag{2.5} \]
\[ a\langle 1|b\langle 0|R_{ab}(z,t)|1\rangle^\alpha_a|0\rangle_b = z - 1, \tag{2.6} \]
\[ a\langle 1|b\langle 1|R_{ab}(z,t)|1\rangle^\alpha_a|1\rangle_b = z + t. \tag{2.7} \]
The $L$-operator of the Felderhof model is given by

$$L_{aj}(z, t) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & t & 1 & 0 \\
0 & (t+1)z & z & 0 \\
0 & 0 & 0 & z \\
\end{pmatrix},$$

acting on the tensor product $W_a \otimes F_j$ of the space $W_a$ and the two-dimensional Fock space at the $j$th site $F_j$. We also denote the orthonormal basis of $F_j$ and its dual as $\{|0\rangle_j, |1\rangle_j\}$ and $\{|j\rangle_0, |j\rangle_1\}$, and the matrix elements of the $L$-operator are explicitly written as (see Figure 1 for a pictorial description)

$$a\langle 0 | j \langle 0 | L_{aj}(z, t) | 0 \rangle_a | 0 \rangle_j = 1,$$
$$a\langle 0 | j \langle 1 | L_{aj}(z, t) | 0 \rangle_a | 1 \rangle_j = t,$$
$$a\langle 0 | j \langle 1 | L_{aj}(z, t) | 1 \rangle_a | 0 \rangle_j = 1,$$
$$a\langle 1 | j \langle 0 | L_{aj}(z, t) | 0 \rangle_a | 1 \rangle_j = (t+1)z,$$
$$a\langle 1 | j \langle 0 | L_{aj}(z, t) | 1 \rangle_a | 0 \rangle_j = z,$$
$$a\langle 1 | j \langle 1 | L_{aj}(z, t) | 1 \rangle_a | 1 \rangle_j = z.$$

The $R$-matrices and the $L$-operators have origins in statistical physics, and $|0\rangle$ or its dual $|0\rangle$ can be regarded as a hole state, while $|1\rangle$ or its dual $\langle 1 |$ can be interpreted as a particle state from the point of view of statistical physics. We use the terms hole states and particle states to describe states constructed from $|0\rangle$, $\langle 0 |$, $|1\rangle$ and $\langle 1 |$ from now on since they are convenient for the description of the states. We also remark that in the language of the quantum inverse scattering method, the Fock spaces $W_a$ and $F_j$ are usually called the auxiliary and quantum spaces, respectively.

The $R$-matrix (2.1) and $L$-operator (2.8) satisfy the Yang-Baxter relation

$$R_{ab}(z_1/z_2, t)L_{aj}(z_1, t)L_{bj}(z_2, t) = L_{bj}(z_2, t)L_{aj}(z_1, t)R_{ab}(z_1/z_2, t),$$

acting on $W_a \otimes W_b \otimes V_j$. We remark that this $RLL$ relation (2.15) can be regarded as a special case of the generalized Yang-Baxter relation for a more general $R$-matrix $[10, 11, 12]$. The $R$-matrix (2.1) and the $L$-operator (2.8) in this section can be regarded as different specializations of the general $R$-matrix from this viewpoint. One advantages of the point of view from the quantum group used was that one can systematically generalize the Felderhof model to higher-dimensional representations $[11]$. From the $L$-operator, we construct the monodromy matrix

$$T_a(z) = L_{aM}(z, t) \cdots L_{a1}(z, t) = \begin{pmatrix}
A(z) & B(z) \\
C(z) & D(z) \\
\end{pmatrix}_a,$$

which acts on $W_a \otimes (F_1 \otimes \cdots \otimes F_M)$. The intertwining relation between the monodromy matrices

$$R_{ab}(z_1/z_2, t)T_a(z_1)T_b(z_2) = T_b(z_2)T_a(z_1)R_{ab}(z_1/z_2, t),$$

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follow from the $RLL$ relation (2.15). One of the elements of (2.17) is the commutation relations between the $B$ operators

$$ (1 + t z_1/z_2) B(z_1) B(z_2) = B(z_2) B(z_1) (z_1/z_2 + t). \quad (2.18) $$

Note that unlike the one constructed from the usual $U_q(sl_2)$ $R$-matrix, the $B$-operators created from the Felderhof model (2.8) do not simply commute, i.e., they produce extra factors. See Figure 2 for a graphical description of the $B$-operator.

3 Wavefunction and Schur polynomials

We introduce the wavefunction which is a special class of partition functions, and review how it is related with the Schur polynomials defined below.

**Definition 3.1.** The Schur polynomials is defined to be the following determinant:

$$ s_\lambda(z) = \frac{\det_N(z_j^{\lambda_k+N-k})}{\prod_{1 \leq j < k \leq N}(z_j - z_k)}, \quad (3.1) $$

where $\{z\}_N = \{z_1, \ldots, z_N\}$ is a set of variables and $\lambda$ denotes a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with weakly decreasing non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$.

Before introducing the wavefunction, we first define the arbitrary $N$-particle state $|\Psi(z_1, \ldots, z_N)\rangle$ with $N$ spectral parameters $\{z\}_N = \{z_1, \ldots, z_N\}$ as a state constructed by a multiple action of $B$ operator on the vacuum state $|\Omega\rangle := |0^M\rangle := |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$

$$ |\Psi(z_1, \ldots, z_N)\rangle = B(z_1) \cdots B(z_N)|\Omega\rangle. \quad (3.2) $$
Figure 2: The $B$-operator $B(z)$, which is a matrix element of the monodromy matrix $T_a(z)$. The $B$-operator is $2^M \times 2^M$ matrix-valued. The leftmost state on the horizontal line (auxiliary space) is fixed as $\oplus (a|0), \text{ whereas the rightmost state is fixed as } \ominus (|1).$

(3.2) is usually called as the off-shell Bethe vector (off-shell means that we do not assume the spectral parameters satisfy the Bethe ansatz equations). Note that due to the commutation relation between the $B$-operators (2.18), the ordering of the $B$-operators in the off-shell Bethe vector (3.2) is important for the Felderhof model.

Next, we introduce the wavefunction $\langle x_1 \cdots x_N | \Psi(z_1,\ldots ,z_N) \rangle$ as the overlap between an arbitrary off-shell $N$-particle state $|\Psi(z_1,\ldots ,z_N)\rangle$ and the (normalized) state with an arbitrary particle configuration $|x_1 \cdots x_N\rangle$ ($1 \leq x_1 < \cdots < x_N \leq M$), where $x_j$ denotes the positions of the particles. The particle configurations are explicitly defined as

$$\langle x_1 \cdots x_N | = \langle \Omega | \prod_{j=1}^N \sigma^+_j,$$

where $\langle \Omega | = \langle 0^M | = \langle 0 | \otimes \cdots \otimes M \langle 0 |$. Here, we define $\sigma^+$ and $\sigma^-$ as operators acting on the basis elements as

$$\sigma^+|1\rangle = |0\rangle, \quad \sigma^+|0\rangle = 0, \quad \langle 0|\sigma^+ = \langle 1|, \quad \langle 1|\sigma^+ = 0, \quad \sigma^-|0\rangle = |1\rangle, \quad \sigma^-|1\rangle = 0, \quad \langle 1|\sigma^- = 0, \quad \langle 0|\sigma^- = 0.$$

The subscript $j$ of $\sigma^+_j$ or $\sigma^-_j$ indicates that the operator acts on the space $F_j$ as $\sigma^+$ or $\sigma^-$, and as an identity on the other spaces.

Bump, Brubaker and Friedberg proved the following relation between the wavefunction of the Felderhof model and the Schur polynomials.
Theorem 3.2. [13] The wavefunction \( \langle x_1 \cdots x_N | \Psi(z_1, \ldots, z_N) \rangle \) is expressed by the Schur polynomials as

\[
\langle x_1 \cdots x_N | \Psi(z_1, \ldots, z_N) \rangle = \prod_{1 \leq j < k \leq N} (z_j + tz_k) s_\lambda(\{z\}_N). \tag{3.6}
\]

Here the Young diagram for the Schur polynomials correspond to the particle configuration under the relation \( \lambda_j = x_{N-j+1} - N + j - 1, j = 1, \ldots, N \).

The authors in [13] moreover found that the investigating the microscopic description of the wavefunction of the Felderhof model naturally leads to the Tokuyama formula for the Schur polynomials, which is a deformation of the Weyl character formula. The idea is as follows. First, we introduce a strict Gelfand-Tsetlin pattern, which is a triangular array of integers

\[
\mathcal{T} = \begin{pmatrix}
a_{0,0} & a_{0,1} & \cdots & a_{0,N-2} & a_{0,N-1} \\
a_{1,1} & \cdots & \cdots & \cdots \\
\ddots & \ddots & \ddots & \ddots \\
a_{N-1,N-1} & \end{pmatrix}, \tag{3.7}
\]

in which the rows interlace \( a_{i-1,j-1} \geq a_{i,j} \geq a_{i-1,j} \), and the entries in horizontal rows are strictly decreasing.

For each strict Gelfand-Tsetlin pattern, we assign the following weight

\[
G(\mathcal{T}, \{z\}_N) = \prod_{i=1}^{N-1} \prod_{j=i}^{N-1} \gamma(a_{i,j}) z_1^{d_1(\mathcal{T})-d_1(\mathcal{T})} z_2^{d_2(\mathcal{T})-d_2(\mathcal{T})} \cdots z_{N-1}^{d_{N-2}(\mathcal{T})-d_{N-2}(\mathcal{T})} z_N^{d_{N-1}(\mathcal{T})}, \tag{3.8}
\]

where \( d_j(\mathcal{T}) = \sum_{k=j}^{N-1} a_{j,k}, j = 0, \ldots, N-1 \) is the sum of the entries of the strict Gelfand-Tsetlin pattern in the \( j \)-th row, and \( \gamma(a_{i,j}) \) is defined as

\[
\gamma(a_{i,j}) = \begin{cases} 
  t & a_{i,j} = a_{i-1,j-1}, \\
  t+1 & a_{i-1,j} \neq a_{i,j} \neq a_{i-1,j-1}, \\
  1 & \text{otherwise}
\end{cases} \tag{3.9}
\]

for pairs of integers \( (i,j) \) satisfying \( 1 \leq i \leq N-1, \ i \leq j \leq N-1 \).

Investigating the inner states making nonzero contributions to the wavefunction, one finds that the corresponding weight for a fixed inner state, which is the product of the matrix elements of the \( L \)-operators of the inner states, can be characterized by a strict Gelfand-Tsetlin pattern with the top row fixed by the Young diagram as \( a_{0,j} = \lambda_{j+1} + N - j - 1 \).

The weight for each inner state is found to be given by (3.8), and the wavefunction can be expressed as a sum of (3.8) for all strict Gelfand-Tsetlin patterns with the top row fixed as \( a_{0,j} = \lambda_{j+1} + N - j - 1 \). Combining this microscopic analysis with Theorem 3.2 one gets the following combinatorial formula for the Schur polynomials.

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Theorem 3.3. [13] We have the following combinatorial formula for the Schur polynomials
\[
\prod_{1 \leq j < k \leq N} (z_j + tz_k) s_{\lambda} \left( \left\{ \frac{z}{t} \right\}_N \right) = \sum_{\mathcal{T}} G(\mathcal{T}, \{z\}_N) = \sum_{\mathcal{T}} \prod_{i=1}^{N-1} \prod_{j=1}^{N-N_i} \gamma(a_{ij}) z_{\lambda_j}^{d_0(\mathcal{T})-d_i(\mathcal{T})} z_{\lambda_2}^{d_2(\mathcal{T})-d_{N-1}(\mathcal{T})} \cdots z_{\lambda_N}^{d_{N-1}(\mathcal{T})-d_{N-2}(\mathcal{T})},
\]
where the sum is over all strict Gelfand-Tsetlin patterns with the top row of the strict Gelfand-Tsetlin pattern is fixed by the Young diagram as \(a_{0,j} = \lambda_j + 1 + N - j - 1\).

4 Dual wavefunction

We now introduce the dual wavefunction, and study the exact relation between it and the Schur polynomials. In this section, we use the argument which is more slightly complicated than, but basically the same with the one given in [13]. We analyze by another method based on a modern statistical physical and quantum integrable techniques, which will be given in the next section.

Before defining the dual wavefunction, we introduce another type of arbitrary dual \(N\)-hole state \(\langle \Phi(z_1, \ldots, z_N) | \rangle\) by a multiple action of \(B\) operator on the dual particle occupied state \(\langle 1 \cdots M | := 1 | 1 \otimes \cdots \otimes M | \rangle\)
\[
\langle \Phi(z_1, \ldots, z_N) | = \langle 1 \cdots M | B(z_1) \cdots B(z_N). \tag{4.1}
\]
It is convenient to introduce a notation for the state with an arbitrary hole configuration \(\langle x_1 \cdots x_N \rangle\) \((1 \leq x_1 < \cdots < x_N \leq M)\), where \(x_j\) denotes the positions of holes. Explicitly,
\[
\langle x_1 \cdots x_N \rangle = \prod_{j=1}^{N} \sigma_{x_j}^+ (|1\rangle \otimes \cdots \otimes |1\rangle_M). \tag{4.2}
\]

The dual wavefunction \(\langle \Phi(z_1, \ldots, z_N) | x_1 \cdots x_N \rangle\) is defined as the overlap between the arbitrary dual \(N\)-hole state \(\langle \Phi(z_1, \ldots, z_N) | \rangle\) and hole configurations \(\langle x_1 \cdots x_N \rangle\) (see Figure 3 for an example of a graphical description of the dual wavefunction).

We show the following relation between the dual wavefunction and the Schur polynomials.

Theorem 4.1. The dual wavefunction \(\langle \Phi(z_1, \ldots, z_N) | x_1 \cdots x_N \rangle\) can be expressed by the Schur polynomials as
\[
\langle \Phi(z_1, \ldots, z_N) | x_1 \cdots x_N \rangle = t^{N(M-N)} \prod_{1 \leq j < k \leq N} (z_j + tz_k) s_{\lambda} \left( \left\{ \frac{z}{t} \right\}_N \right). \tag{4.3}
\]
Here the Young diagram for the Schur polynomials correspond to the particle configuration under the relation \(\lambda_j = x_{j-1} - N + j - 1, j = 1, \ldots, N\), and the symmetric variables are
\[
\left\{ \frac{z}{t} \right\}_N = \left\{ \frac{z_1}{t}, \ldots, \frac{z_N}{t} \right\}. \tag{4.4}
\]
Before proving Theorem 4.1 let us make some comments. There is a factor $t^{N(M-N)}$ which depends on the number of sites $M$ and the number of particles $N$ in the right hand side of (4.3). What makes things more complicated is that the symmetric variables of the Schur polynomials are $\{\frac{z}{t}\}_N$, which are the reasons why the relation for the dual wavefunction (4.3) seems hard to find. We actually first found this Theorem by using the statistical physical method given in the next section. One advantages of the proof given in this section following [13] is that the proof naturally lifts to the correspondence between a generalization of the Felderhof model and the factorial Schur polynomials.

Proof. We rewrite (4.3) by rescaling each $z_j$ to $t \cdot z_j$ in the following form

$$t^N (1 \cdots M) \left(\frac{B(tz_j)}{t^M}\right) \cdots \left(\frac{B(tz_N)}{t^M}\right) |x_1 \cdots x_N\rangle = \prod_{1 \leq j < k \leq N} \left(\frac{1}{t} z_j + z_k\right) s_{\lambda}(\{z\}_N).$$

For giving a proof, it is convenient to introduce the rescaled $L$-operator

$$\tilde{L}(z, t) = \frac{1}{t} L(tz, t) = \begin{pmatrix} 1/t & 0 & 0 & 0 \\ 0 & 1 & 1/t & 0 \\ 0 & (t+1)z & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix},$$

and the rescaled monodromy matrix

$$\tilde{T}_a(z) = \tilde{L}_{aM}(z, t) \cdots \tilde{L}_{a1}(z, t) = \begin{pmatrix} \tilde{A}(z) \\ \tilde{B}(z) \\ \tilde{C}(z) \\ \tilde{D}(z) \end{pmatrix}_a.$$
Using these rescaled objects, (4.4) can be expressed as
\[
t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}) = \prod_{1 \leq j < k \leq N} \left( \frac{1}{t} z_j + z_k \right) s_N(\{ z \}_N). \tag{4.7}
\]

Instead of proving (4.3), we show (4.7) since this is equivalent to (4.3) and is the expression which one can use the argument given in [13].

We first show the following lemma.

**Lemma 4.2.**
\[
\prod_{1 \leq j < k \leq N} \left( \frac{1}{t} z_j + z_k \right)^{-1} t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}), \tag{4.8}
\]
does not depend on \( t \).

**Proof.** We prove this lemma by showing the following properties for \( t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}) \):

1. \( t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}) \) is a polynomial of \( t' := t^{-1} \) with highest degree \( N(N - 1)/2 \).

2. \( t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}) \) has \( t' z_j + z_k \), \( 1 \leq j < k \leq N \) as factors.

We first show \( \text{deg}_{t'}(t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N})) \leq N(N - 1)/2 \) by induction on \( N \). The case \( N = 1 \) follows as an special case of the general fact.

We can see easily from the definition of the rescaled \( L \)-operator \( L(z, t) \). Next, let us assume \( \text{deg}_{t'}(t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N})) \leq N(N - 1)/2 \).

\[
t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}) \leq N(N - 1)/2,
\]
by combining the assumption \( \text{deg}_{t'}(t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N})) \leq N(N - 1)/2 \), the fact \( 0 \leq \text{deg}_{t'}(t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N})) \leq N \) and the decomposition
\[
t^{N+1} (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_{N+1}) | \underline{y_1} \cdots \underline{y_{N+1}}) = \sum_{\{ \underline{x} \}} (t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}))(t(\underline{x_1} \cdots \underline{x_N}) \tilde{B}(z_{N+1}) | \underline{y_1} \cdots \underline{y_{N+1}})). \tag{4.9}
\]

Next we show Property 2. The commutation relation (2.18) can be rewritten as the following commutation relation between the rescaled \( B \)-operators
\[
(z_1 + t' z_2) \tilde{B}(z_1) \tilde{B}(z_2) = \tilde{B}(z_2) \tilde{B}(z_1)(t' z_1 + z_2). \tag{4.10}
\]

Applying the commutation relation (4.10) repeatedly, one gets the following equality
\[
t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}) \prod_{1 \leq j < k \leq N} (z_j + t' z_k) \\
= t^N (1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \underline{x_1} \cdots \underline{x_N}) \prod_{1 \leq j < k \leq N} (t' z_j + z_k). \tag{4.11}
\]
Note that in the equality (4.11), the factors $t^N(1 \cdots M|\tilde{B}(z_1) \cdots \tilde{B}(z_N)|\vec{x}_1 \cdots \vec{x}_N)$, $\prod_{1 \leq j < k \leq N}(z_j + t' z_k)$, $t^N(1 \cdots M|\tilde{B}(z_1) \cdots \tilde{B}(z_N)|\vec{x}_1 \cdots \vec{x}_N)$ and $\prod_{1 \leq j < k \leq N}(t' z_j + z_k)$ are polynomials of $t'$. From this fact and that $\prod_{1 \leq j < k \leq N}(z_j + t' z_k)$ is not divided by $\prod_{1 \leq j < k \leq N}(t' z_j + z_k)$, one can see $t^N(1 \cdots M|\tilde{B}(z_1) \cdots \tilde{B}(z_N)|\vec{x}_1 \cdots \vec{x}_N)$ is divided by $\prod_{1 \leq j < k \leq N}(t' z_j + z_k)$.

From Property 2, we have $\deg_{t'}(t^N(1 \cdots M|\tilde{B}(z_1) \cdots \tilde{B}(z_N)|\vec{x}_1 \cdots \vec{x}_N)) \geq N(N-1)/2$. Together with $\deg_{t'}(t^N(1 \cdots M|\tilde{B}(z_1) \cdots \tilde{B}(z_N)|\vec{x}_1 \cdots \vec{x}_N)) \leq N(N-1)/2$ which is proved before, we have Property 1.

□

From Lemma 4.2 one sees that to study the wavefunction, it is enough to examine a particular value of $t$. The case when $t = -1$ in which the six-vertex model reduces to a five-vertex model

$$\tilde{L}(z, -1) = -L(-z, -1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & z \end{pmatrix}, \quad (4.12)$$

is easy to examine, and we have the following relation.

**Lemma 4.3.** We have

$$\prod_{1 \leq j < k \leq N} \left( \frac{1}{t} z_j + z_k \right)^{-1} t^N(1 \cdots M|\tilde{B}(z_1) \cdots \tilde{B}(z_N)|\vec{x}_1 \cdots \vec{x}_N) \bigg|_{t = -1} = s_\lambda(\{z\} N). \quad (4.13)$$

**Proof.** To prove the Lemma is equivalent to show

$$\langle 1 \cdots M|\tilde{B}(z_1) \cdots \tilde{B}(z_N)|\vec{x}_1 \cdots \vec{x}_N \rangle |_{t = -1} = (-1)^N \prod_{1 \leq j < k \leq N} (-z_j + z_k) s_\lambda(\{z\} N). \quad (4.14)$$

To show this, we first note that the matrix elements of a single $B$-operator is given by

$$\langle \vec{x}_1 \cdots \vec{x}_k-1|\tilde{B}(z)|\vec{y}_1 \cdots \vec{y}_k \rangle = (-1)^k(-1)^{j-1}z^{j-1}, \quad (4.15)$$

when the hole configurations $\{\vec{x}\}$ and $\{\vec{y}\}$ satisfy $\vec{x}_1 = \vec{y}_1, \cdots, \vec{x}_{j-1} = \vec{y}_{j-1}, \vec{x}_j = \vec{y}_{j+1}, \cdots, \vec{x}_{k-1} = \vec{y}_k$ for some $j$, and 0 otherwise.

Since the matrix elements of a single $B$-operator are essentially the same with the ones for the original wavefunction at $t = -1$ in [13] except the sign $(-1)^k$ (we also have to translate the hole configurations to particle configurations), the same argument can be applied. One observes that the number of the inner states whose weights gives non-zero contributions to the dual wavefunction is $N!$. The weight of each nonvanishing inner state corresponds to one term $(-1)^{N} \prod_{j=1}^{N} \frac{z_j^{\lambda(j)+N-\sigma(j)}}{z_j^{\lambda(j)}}$ of the determinant expansion of the numerator (3.1) of the Schur polynomials multiplied by the extra factor $(-1)^{N(N+1)/2}$. The factor $(-1)^{N(N+1)/2}$ appears since the dual wavefunction is constructed from $N$ layers of $B$-operators, and the $k$-th layer of the $B$-operator has the extra sign $(-1)^k$ in the right hand side of (4.13), hence the total contribution of $N$ layers of the $B$-operators gives the extra sign $\prod_{k=1}^{N}(-1)^k = (-1)^{N(N+1)/2}$. We further rewrite the extra factor $(-1)^{N(N+1)/2}$ as

$$(-1)^{N(N+1)/2} = (-1)^{N(-1)^{N(N-1)/2} = (-1)^{N} \prod_{1 \leq j < k \leq N}(-z_j + z_k) \prod_{1 \leq j < k \leq N}(z_j - z_k), \quad (4.16)$$
to get
\[
\langle 1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle |_{t=-1} = (-1)^{N(N+1)/2} \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{j=1}^{N} \bar{z}_j^{\sigma(j)+N-\sigma(j)}
\]
\[
=(-1)^{N} \prod_{1 \leq j < k \leq N} (-z_j + z_k) \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{j=1}^{N} \bar{z}_j^{\sigma(j)+N-\sigma(j)}
\]
\[
=(-1)^{N} \prod_{1 \leq j < k \leq N} (-z_j + z_k) s_{\lambda}(\{z\}_N). \quad (4.17)
\]

From Lemma 4.2 and (4.13), we have
\[
\prod_{1 \leq j < k \leq N} \left( \frac{1}{t} z_j + z_k \right)^{-1} t^N \langle 1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle |_{t=-1} = \prod_{1 \leq j < k \leq N} \left( \frac{1}{t} z_j + z_k \right)^{-1} t^N \langle 1 \cdots M | \tilde{B}(z_1) \cdots \tilde{B}(z_N) | \overline{x_1} \cdots \overline{x_N} \rangle |_{t=-1} \]
\[
\quad = s_{\lambda}(\{z\}_N), \quad (4.18)
\]
which is exactly (4.17), hence Theorem 4.1 is proved.

**Example** Let us check Theorem 4.1 by an example $M = 4, N = 2$, $(x_1, x_2) = (2, 4)$. The corresponding Young diagram is $\lambda = (\lambda_1, \lambda_2) = (x_2 - 2, x_1 - 1) = (4 - 2, 2 - 1) = (2, 1)$. The left hand side of (4.3) can be calculated graphically by noting that there are three inner states whose weights make nonzero contributions to the dual wavefunction $\langle \Phi(z_1, z_2) | \overline{x_1} = 2, \overline{x_2} = 4 \rangle$, which are given in Figures 4, 5 and 6. From its graphical description and using the data of the matrix elements of the $L$-operator (2.8) and multiplying them, one sees that each of the configurations have weights $t^2 z_1^3, t(t+1)z_1^2z_2$ and $t^2z_1z_2^3$. Summing up the weights and noting that one can extract $(z_1 + t z_2)$ as an overall factor, we have
\[
t^2 z_1^3 + t(t+1)z_1^2z_2 + t^2z_1z_2^3 = t^4(z_1 + t z_2) \left( \frac{z_1}{t} \right)^2 \left( \frac{z_2}{t} \right) + \left( \frac{z_1}{t} \right) \left( \frac{z_2}{t} \right)^2
\]
\[
= t^4(z_1 + t z_2) s_{(2,1)} \left( \frac{z_1}{t}, \frac{z_2}{t} \right), \quad (4.19)
\]
which is nothing but the right hand side of (4.3).

5 Another proof

We give another proof of Theorem 4.1 by using a modern statistical mechanical method and an analysis on a fundamental object in quantum integrable models, i.e., we use the
Figure 4: An inner state of the dual wavefunction $\langle \Phi(z_1, z_2) | x_1 = 2, x_2 = 4 \rangle$ giving the weight $t^2 z_1 z_2^3$.

matrix product method and the domain wall boundary partition function, as was done in the case of the Grothendieck polynomials in [21] (see also [22] in which we demonstrate a proof of Theorem 3.2 by using the same arguments given in this section). We prove Theorem 4.1 as follows. We first rewrite the dual wavefunction into a matrix product form, following [17][18]. The matrix product form can be expressed as a determinant with some overall factor which remains to be calculated. The information of the hole configuration $\{x_1, x_2, \ldots, x_N\}$ is encoded in the determinant. On the other hand, the overall factor is independent of the hole positions, and this factor can be determined by considering the specific configuration: we explicitly evaluate the overlap of the consecutive hole configuration (i.e. $x_j = j$) whose evaluation essentially reduces to that of the domain wall boundary partition function.

Proof. Let us begin to compute the wavefunction $\langle \Phi(\{z\}_N) | x_1 \cdots x_N \rangle$. We first rewrite it into the matrix product representation. With the help of graphical description, one finds that the wavefunction can be written as

$$\langle \Phi(\{z\}_N) | x_1 \cdots x_N \rangle = \text{Tr}_{W \otimes \cdots \otimes W} \left[ (1 \cdots M) \prod_{a=1}^N T_a(z_a) | x_1 \cdots x_N \rangle \langle 0 \cdots 0 \right], \quad (5.1)$$

where $P = |1^N \rangle \langle 0^N|$ is an operator acting on the tensor product of auxiliary spaces $W_1 \otimes \cdots \otimes W_N$. The trace here is also over the auxiliary spaces.

Changing the viewpoint of the products of the monodromy matrices, we have

$$\prod_{a=1}^N T_a(z_a) = \prod_{j=1}^M T_j(\{z\}_N), \quad (5.2)$$

where $T_j(\{z\}_N) := \prod_{a=1}^N L_a(z_a) \in \text{End}(W^\otimes N \otimes V_j)$ can be regarded as a monodromy matrix consisting of $L$-operators acting on the same quantum space $V_j$ (but acting on different
auxiliary spaces). The monodromy matrix is decomposed as
\[ T_j(\{z\}_N) := \begin{pmatrix} A_N(\{z\}_N) & B_N(\{z\}_N) \\ C_N(\{z\}_N) & D_N(\{z\}_N) \end{pmatrix}, \] (5.3)
where the elements \((A_N, \text{etc.})\) act on \(W_1 \otimes \cdots \otimes W_N\).

The wavefunction (5.1) can then be rewritten by \(T_j(\{z\}_N)\) as
\[ \langle \Phi(\{z\}_N) | \mathcal{X}_1 \cdots \mathcal{X}_N \rangle = \text{Tr}_{W \otimes N} \left[ \right. \left. (1 + t)^{M-1} D_{M-1}^{t_1} C_{N-1}^{t_2} \cdots C_{N-N}^{t_N} \right]. \] (5.4)

For these operators, one finds the following recursive relations:
\[ D_{n+1}(\{z\}_{n+1}) = D_n(\{z\}_n) \otimes \begin{pmatrix} t & 0 \\ 0 & z_{n+1} \end{pmatrix} + C_n(\{z\}_n) \otimes \begin{pmatrix} 0 & 0 \\ (1 + t)z_{n+1} & 0 \end{pmatrix}, \] (5.5)
\[ C_{n+1}(\{z\}_{n+1}) = D_n(\{z\}_n) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + C_n(\{z\}_n) \otimes \begin{pmatrix} 1 & 0 \\ 0 & z_{n+1} \end{pmatrix}, \] (5.6)
with the initial condition
\[ D_1 = \begin{pmatrix} t & 0 \\ 0 & z_1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \] (5.7)

By using the recursive relations (5.5) and (5.6), one sees that these operators satisfy the following simple algebra.
Lemma 5.1. There exists a decomposition of $C_n : C_n = \sum_{j=1}^{n} c_n^{(j)}$ such that the following algebraic relations hold for $D_n$ and $C_n^{(j)}$:

$$C_n^{(j)} D_n = \frac{z_j}{t} D_n C_n^{(j)},$$  \hspace{1cm} (5.8)

$$(C_n^{(j)})^2 = 0,$$  \hspace{1cm} (5.9)

$$C_n^{(j)} C_n^{(k)} = -\frac{z_j}{z_k} C_n^{(k)} C_n^{(j)}, \quad (j \neq k).$$  \hspace{1cm} (5.10)

Proof. We show by induction on $n$. For $n = 1$, from (5.7) $D_1$ is diagonal and one can directly see that the relations are satisfied. For $n$, we assume that $D_n$ is diagonalizable and write the corresponding diagonal matrix as $D_n = G_n^{-1} D_n G_n$. Also writing $C_n = G_n^{-1} C_n G_n$ and $\varphi_n = \sum_{j=1}^{n} \varphi_n^{(j)}$, and noting that the algebraic relations above do not depend on the choice of basis, we suppose by the induction hypothesis that the same relations are satisfied by $D_n$ and $\varphi_n^{(j)}$.

We show that the relations hold for $n + 1$. To this end, we first construct $G_{n+1}$. Noting from (5.9) that $D_{n+1}$ is an upper triangular block matrix whose block diagonal elements are written in terms of $D_n$, we assume that $G_{n+1}$ is written as

$$G_{n+1} = \begin{pmatrix} G_n & 0 \\ G_n H_n & G_n \end{pmatrix},$$  \hspace{1cm} (5.11)

where $2n \times 2n$ matrix $H_n$ remains to be determined. Using the induction hypothesis for $n$, one obtains

$$G_{n+1}^{-1} D_{n+1} G_{n+1} = \begin{pmatrix} t \varphi_n & 0 \\ z_{n+1} \varphi_n H_n + (1 + t) z_{n+1} \varphi_n - t H_n \varphi_n & z_{n+1} \varphi_n \end{pmatrix}. $$  \hspace{1cm} (5.12)

The above matrix is guaranteed to be diagonal when

$$z_{n+1} \varphi_n H_n + (1 + t) z_{n+1} \varphi_n - t H_n \varphi_n = 0.$$  \hspace{1cm} (5.13)
Utilizing the above relation and recalling \( D_n \) and \( C^{(j)}_n \) satisfy the relation same as that in (5.8), one finds

\[
H_n = D_n^{-1} \sum_{j=1}^{n} \frac{(1 + t)z_{j+1}C^{(j)}_n}{z_j - z_{n+1}}.
\]

(5.14)

One thus obtains the diagonal matrix \( D_{n+1} \):

\[
D_{n+1} = \begin{pmatrix} tD_n & 0 \\ 0 & z_1D_n \end{pmatrix}.
\]

(5.15)

The remaining task is to derive \( C^{(j)}_{n+1} \) and to prove the relations (5.8)–(5.10) hold for \( n + 1 \). Combining (5.6), (5.11) and (5.14), and also inserting the relations (5.9) and (5.10), one arrives at

\[
C^{(j)}_{n+1} = \sum_{j=1}^{n+1} \left\{ \begin{array}{l}
\frac{z_j + tz_{j+1}}{z_j - z_{n+1}} \left( \frac{C^{(j)}_n}{t} - \frac{z_{n+1}}{t} \right) \\
\left( \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right)
\end{array} \right. 
\]

(5.16)

for \( 1 \leq j \leq n \)

\[
0D_n 0z_{n+1}D_n
\]

for \( j = n + 1 \).

Finally recalling that \( D_n \) and \( C^{(j)}_n \) are supposed to satisfy the relations (5.8)–(5.10) and using the explicit form of \( D_{n+1} \) (5.15) and \( C^{(j)}_{n+1} \) (5.16), one sees they satisfy the same algebraic relations as those in (5.8)–(5.10) for \( n + 1 \).

Due to the algebraic relations (5.8), (5.9) and (5.10) in Lemma 5.1, the matrix product form for the wavefunction (5.4) can be rewritten as

\[
\langle \Phi(\{z\}_N)|x_1 \cdots x_N \rangle = \prod_{j=1}^{N} \left( \frac{t}{z_j} \right)^j \text{Tr}_{W \otimes N} \left[ D_{N-M}^N C^{(N)}_N \ldots C^{(1)}_N \right] \sum_{\sigma \in S_N} (-1)^{\sigma} \prod_{j=1}^{N} \left( \frac{z_{\sigma(j)}}{t} \right)^{\sigma} \prod_{j=1}^{N} \left( \frac{t}{z_j} \right)^{\sigma-1} \det_N \left( \left( \frac{z_j}{t} \right) \overrightarrow{X} \right) \],
\]

(5.17)

where we have used the translation rule \( \overrightarrow{X}_j = x_{N-j+1} - N + j - 1 \) between the hole configuration and the Young diagram. One easily notes that (5.17) can be further rewritten in terms of the Schur polynomials:

\[
\langle \Phi(\{z\}_N)|x_1 \cdots x_N \rangle = \text{KS}_N \left( \left( \frac{z}{t} \right) \right),
\]

(5.18)
where the prefactor $K$ given below remains to be determined:

$$K = \prod_{j=1}^{N} \left( \frac{t}{z_j} \right)^{j-1} \prod_{1 \leq j < k \leq N} \frac{z_k - z_j}{t} \text{Tr}_{W} \left[ D_{N}^{M-N} C_{N}^{(N)} \cdots C_{N}^{(1)} P \right]. \quad (5.19)$$

In (5.18), we notice that the information of the hole configuration $\{x_1, x_2, \ldots, x_N\}$ is encoded in the determinant, while the overall factor $K$ is independent of the configuration. This fact means that one can determine the factor $K$ by evaluating the overlap for a particular hole configuration. In fact, we find the following explicit expression for the case $x_j = j$ ($1 \leq j \leq N$):

**Proposition 5.2.** The wavefunction $\langle \Phi(\{z\}) | x_1 \cdots x_N \rangle$ for the case $x_j = j$ ($1 \leq j \leq N$) has the following form:

$$\langle \Phi(\{z\}) | x_1 = 1, \cdots, x_N = N \rangle = t^{N(M-N)} \prod_{1 \leq j < k \leq N} (z_j + tz_k). \quad (5.20)$$

**Proof.** We can easily show by its graphical description that $\langle \Phi(\{z\}) | x_1 = 1, \cdots, x_N = N \rangle$ can be factorized as

$$\langle \Phi(\{z\}) | x_1 = 1, \cdots, x_N = N \rangle = t^{N(M-N)} Z_{N}(\{z\}), \quad (5.21)$$

where $Z_{N}(\{z\})$ is the domain wall boundary partition function on an $N \times N$ lattice. The domain wall boundary partition function on an $M \times M$ lattice is defined as

$$Z_{M}(\{z\}_M) = \langle 1 \cdots M | B(z_1) \cdots B(z_M) | \Omega \rangle, \quad (5.22)$$

where $M$ $B$-operators are inserted between the vacuum vector $| \Omega \rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_M$ and the dual state occupied by particles $\langle 1 \cdots M | = 1_1 \otimes \cdots \otimes 1_M$.

The domain wall boundary partition function can be analyzed by generalizing it to the one with inhomogeneities introduced in the quantum spaces

$$Z_{M}(\{z\}_M|\{v\}_M) = \langle 1 \cdots M | B(z_1|\{v\}_M) \cdots B(z_M|\{v\}_M) | \Omega \rangle, \quad (5.23)$$

where

$$B(z|\{v\}_M) = a(0|L_{aM}(z/\{v\}_M) \cdots L_{a1}(z/v_1)|1)_{a}. \quad (5.24)$$

One can show the following factorization formula for the inhomogeneous domain wall boundary partition function.

**Lemma 5.3.** cf. [12] The domain wall boundary partition function with inhomogeneities has the following form

$$Z_{M}(\{z\}_M|\{v\}_M) = \prod_{j=1}^{M} \frac{1}{v_M - j} \prod_{1 \leq j < k \leq M} (z_j + tz_k). \quad (5.25)$$
Lemma 5.3 can be proved by using the Izergin-Korepin technique [4] [19] [20], i.e., show that both hand sides of (5.25) satisfy the same recursive relation, initial condition and the degree counting of polynomials.

Taking the homogeneous limit \( v_j \to 1 \) \((j = 1, \cdots, M)\) of (5.25), we have

\[
Z_M(\{z\}_M) = \prod_{1 \leq j < k \leq M} (z_j + tz_k). \tag{5.26}
\]

Replacing \( M \) by \( N \) in (5.26) and inserting into (5.21), we have

\[
\langle \Phi(\{z\}_N)|x_T = 1, \cdots, x_N = N\rangle = t^{N(M-N)} \prod_{1 \leq j < k \leq N} (z_j + tz_k), \tag{5.27}
\]

which is exactly (5.20). \( \square \)

Using Proposition 5.2 into (5.18), one can see that the prefactor \( K \) in (5.18) is determined by the special case of the dual wavefunction \( \langle \Phi(\{z\}_N)|x_T = 1, \cdots, x_N = N\rangle \) as

\[
K = \langle \Phi(\{z\}_N)|x_T = 1, \cdots, x_N = N\rangle = t^{N(M-N)} \prod_{1 \leq j < k \leq N} (z_j + tz_k). \tag{5.28}
\]

From (5.18) and (5.28), we have (6.1), hence Theorem 4.1 is proved. \( \square \)

6 Combinatorial formula

In the previous two sections, we showed the correspondence between the dual wavefunction and the Schur polynomials by giving two different proofs. To derive a dual Tokuyama-type combinatorial formula for the Schur polynomials, one needs to investigate the microscopic structure and find the partition function expression for the dual wavefunction.

The essential thing to find the expression is to view the dual wavefunction as an object constructed from \( N \) layers of \( B \)-operators, and analyze the matrix elements of a single \( B \)-operator. One can show the following formula.

**Proposition 6.1.** The matrix elements of a single \( B \)-operator is given by

\[
\langle x_T \cdots x_N | B(z)|y_T \cdots y_{N+1}\rangle = (t+1)^{(|\{y_T\} - j-1, \cdots, N | | y_T \neq y_j, y_T \neq y_{N+1}|)} \times t^{\sum_{j=1}^{N+1} \max(|y_T - y_{j-1}, 0) \sum_{j=1}^{N} (y_{j+1} - y_j)}, \tag{6.1}
\]

for hole configurations \( \{\bar{x}\} \) \((1 \leq \bar{x}_1 < \cdots < \bar{x}_N \leq M)\) and \( \{\bar{y}\} \) \((1 \leq \bar{y}_1 < \cdots < \bar{y}_{N+1} \leq M)\) satisfying the interlacing relation \( \bar{y}_T \leq \bar{x}_T \leq \bar{y}_2 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_N \leq \bar{y}_{N+1}, \) and 0 otherwise. Here we also set \( \bar{x}_{N+1} = M + 1. \)

Translating into the language of Young diagram via \( \bar{\lambda}_j = \bar{y}_{N-j+1} - N + j - 1, j = 1, \ldots, N, \)

\[
\bar{\mu}_j = \bar{x}_{N-j+2} - N + j - 1, j = 1, \ldots, N + 1\) and also setting \( \bar{\lambda}_0 = M - N \) one gets the following formula for the nonzero matrix elements when the interlacing relation \( 0 \leq \bar{\mu}_{N+1} \leq \bar{\lambda}_N \leq \bar{\mu}_{N-1} \leq \cdots \leq \bar{\lambda}_1 \leq \bar{\mu}_1 \leq M - N \) is satisfied

\[
\langle x_T \cdots x_N | B(z)|y_T \cdots y_{N+1}\rangle = (t+1)^{(|\bar{\lambda}_j = 1, \cdots, N | | \bar{\lambda}_j \neq \bar{\mu}_j, \bar{\lambda}_j \neq \bar{\mu}_{j+1}|)} \times t^{\sum_{j=1}^{N+1} \max(\bar{\lambda}_{j-1} - \bar{\mu}_{j-1}, 0) \sum_{j=1}^{N} (\bar{\mu}_{j+1} - \bar{\lambda}_j) + N}. \tag{6.2}
\]
Proof. Let us first count the powers of the spectral parameter \( z \). If the hole configurations \( \{\overline{\tau} \} \) and \( \{\overline{\mu} \} \) are fixed and satisfies the interrelating relation \( \overline{\mu}_1 \leq \overline{\mu}_2 \leq \cdots \leq \overline{\mu}_N \leq \overline{\mu}_{N+1} \), the inner states in the auxiliary space is fixed uniquely, which is a sequence of 0’s and 1’s. We observe that for each sequence 01\cdots 10 of the inner states in the auxiliary space, all the matrix elements of the \( L \)-operators (2.8) in between contribute to the power \( z \), and we also have the factor \( z^{\sum_j (\overline{\tau}_j - \overline{\mu}_j)}/\{ \overline{\tau}_j \neq \overline{\mu}_j, \overline{\tau}_j \neq \overline{\tau}_{j+1} \} \) in total.

Let us turn to count the powers of \( t+1 \) and \( t \). We get a factor \( t+1 \) for each case when both \( \overline{\tau}_j \neq \overline{\mu}_j \) and \( \overline{\mu}_j \neq \overline{\mu}_{j+1} \) are satisfied since the matrix element of the \( L \)-operator is \( [L(z,t)]^{01}_{01} = (t+1)z \) at the \( \overline{\tau}_j \)-th site for this case. One gets \( (t+1) \{ [L(z,t)]^{j,j+1}_{j,j+1} \} \) for some sum over \( j \). Taking all of the 01\cdots 10 sequences into account, we have the factor \( z^{\sum_{j=1}^N (\overline{\tau}_j - \overline{\mu}_j)} \).

Next, we count the powers of \( t \). If \( \overline{\tau}_j < \overline{\mu}_j \) is satisfied, the matrix elements of the \( L \)-operators are all \( [L(z,t)]^{01}_{01} = t \) from the \( (\overline{\tau}_j + 1) \)-th site to the \( (\overline{\mu}_j + 1) \)-th site. On the other hand, \( [L(z,t)]^{01}_{01} \) does not appear if \( \overline{\mu}_j = \overline{\tau}_j \), and there is no contribution to the power of \( t \) for this case. The contributions from \( t \) is given by \( t^{\sum_{j=1}^N \max (\overline{\tau}_j - \overline{\mu}_j, 1)} \).

Having calculated all factors, one finds the matrix elements are given by (6.1) in the coordinate representation. Translating into the language of Young diagram, we get (6.2). □

Example (coordinate representation) Let \( M = 10, N = 2, \tau = (3,6) \) and \( \overline{\mu} = (1,6,8) \). We also set \( \overline{\tau}_3 = 10 + 1 = 11 \). From Max(\( \overline{\tau}_1 - \overline{\mu}_1 - 1,0 \)) = Max(3 − 1 − 0) = 1, Max(\( \overline{\tau}_2 - \overline{\mu}_2 - 1,0 \)) = Max(6−6−1,0) = 0, Max(\( \overline{\tau}_3 - \overline{\mu}_3 - 1,0 \)) = Max(11−8−1,0) = 2, we have the factor \( t^{2+0+1} = t^3 \). The relations \( \overline{\mu}_1 = \overline{\tau}_1 \neq \overline{\mu}_2, \overline{\overline{\tau}}_2 = \overline{\tau}_2 \neq \overline{\mu}_2 \) give the factor \( (t+1)^1 \), and we also have the factor \( z^5 \) from \( (\overline{\mu}_1 - \overline{\tau}_1) + (\overline{\mu}_2 - \overline{\tau}_2) = (6 - 3) + (8 - 6) = 3 + 2 = 5 \). In total, the right hand side of (6.1) is calculated as \( (t + 1)^3 z^5 \). One can check that this matches the left hand side of (6.1), i.e., the matrix elements of the corresponding \( B \)-operator by explicit calculation (see Figure 7 for a graphical description of the corresponding matrix element).

Example (Young diagram representation) Let \( M = 10, N = 2, \tau = (3,6) \) and \( \overline{\mu} = (1,6,8) \). We have \( \overline{\lambda}_1 = (6−2,3−1) = (4,2) \) and \( \overline{\mu}_1 = (8−3,6−2,1−1) = (5,4,0) \). We also set \( \overline{\lambda}_0 = 10 − 2 = 8 \). From Max(\( \overline{\lambda}_0 - \overline{\mu}_1 - 1,0 \)) = Max(8 − 5 − 1,0) = 2, Max(\( \overline{\lambda}_1 - \overline{\mu}_2 - 1,0 \)) = Max(4 − 4 − 1,0) = 0, Max(\( \overline{\lambda}_2 - \overline{\mu}_3 - 1,0 \)) = Max(2 − 0 − 1,0) = 1, we have the factor \( t^{2+0+1} = t^3 \). The relations \( \overline{\mu}_1 + 1 = \lambda_1 = \overline{\mu}_2, \overline{\overline{\lambda}}_2 + 1 = \lambda_2 = \overline{\mu}_3 \) give the factor \( (t+1)^1 \), and we also have the factor \( z^5 \) from \( (\overline{\mu}_1 - \lambda_1) + (\overline{\mu}_2 - \lambda_2) + 2 = (5 − 4) + (4 − 2) + 2 = 5 \). Altogether, the right hand side of (6.2) is calculated as \( (t + 1)^3 z^5 \).

In order to describe the microscopic structure, we introduce the following strict dual Gelfand-Tsetlin patterns

\[
\tau = \begin{pmatrix}
0 & a_{0,1} & a_{0,2} & \ldots & a_{0,N-1} \\
0 & 0 & a_{1,1} & \ldots & a_{1,N-1} \\
0 & 0 & 0 & \ldots & a_{N-1,N-1}
\end{pmatrix},
\]

(6.3)
in which the rows interlace \( a_{i,j} = a_{i+1,j+1} \), \( a_{i,j} \geq a_{i-1,j} \), and the entries in horizontal rows are strictly decreasing. We use this picture so as to be consistent with the dual wavefunction
Figure 7: The matrix element $\langle x_1 \cdots x_N | B(z) | y_1 \cdots y_{N+1} \rangle$ for $M = 10$, $N = 2$, $\tau = (3, 6)$ and $\overline{\tau} = (1, 6, 8)$. One sees that the inner state is uniquely fixed, and the matrix element is calculated by multiplying the matrix elements of the $L$-operators $t \times t \times 1 \times z \times z \times z \times (t+1)z \times t \times 1 = (t+1)t^3 z^5$.

description. The strict Gelfand-Tsetlin patterns essentially label the inner states by recording the positions of the holes in the auxiliary spaces.

For each strict Gelfand-Tsetlin pattern, we assign the following weight:

$$G(\mathcal{T}, \{z\}_N) = \prod_{i=0}^{N-2} \prod_{j=i+1}^{N-1} \gamma(a_{i,j}) z_{N-1}^{d_0(\mathcal{T})-d_1(\mathcal{T})-d_2(\mathcal{T})-\cdots-d_{N-2}(\mathcal{T})-d_{N-1}(\mathcal{T})} z_1^{d_N(\mathcal{T})-d_{N-1}(\mathcal{T})},$$

(6.4)

where $d_j(\mathcal{T}) = \sum_{k=j}^{N-1} a_{k,j}$, $j = 0, \ldots, N-1$ is the sum of the entries of the Gelfand-Tsetlin pattern in the $j$-th row counted from the bottom, and $\gamma(a_{i,j})$ is defined as

$$\gamma(a_{i,j}) = t^{\max(\overline{a}_{i+1,j} - \overline{a}_{i,j}, 0)} \times \begin{cases} t+1 & \overline{a}_{i-1,j} \neq \overline{a}_{i,j} \neq \overline{a}_{i-1,j}, \\ 1 & \text{otherwise} \end{cases},$$

(6.5)

for pairs of integers $(i, j)$ satisfying $0 \leq i \leq N-1$, $i \leq j \leq N-1$. Note that we define $\gamma(a_{0,j})$, $j = 0, \ldots, N-1$ since we need these weights to describe the dual wavefunction and the dual Tokuyama-type formula (whereas one does not need to define $\gamma(a_{0,j})$ to describe the original wavefunction and the Tokuyama formula). We also define $\overline{a}_{j,j-1} = M$ for $j = 1, \cdots, N$.

As again, the inner states making non-zero contributions can be characterized by the strict Gelfand-Tsetlin pattern with the bottom row fixed by the Young diagram as $\overline{a}_{0,j} = \lambda_{j+1} + N - j - 1$. 

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Theorem 6.2. We have the following combinatorial formula for the Schur polynomials

\[ t^{N(M-N)} \prod_{1 \leq j < k \leq N} (z_j + tz_k) s_\left( \binom{z}{l} \right)_N \]

\[ = \sum_T \mathcal{G}(T, \{z\}_N) \]

\[ = \sum_{i=0}^{N-1} \prod_{j=i}^{N-1} \prod_{i=0}^{N-1} \tau(a_{i,j}) z_N^{d_0(T) - d_1(T) - d_2(T) - \ldots - d_{N-2}(T) - d_{N-1}(T)} z_1^{d_{N-1}(T)}, \quad (6.6) \]

where the sum is over all strict Gelfand-Tsetlin patterns with the bottom row of the strict Gelfand-Tsetlin pattern is fixed by the Young diagram as \( a_{0,j} = \lambda_{j+1} + N - j - 1 \).

Proof. The proof follows from evaluating the dual wavefunction \( \langle \Phi(z_1, \ldots, z_N) | x_1 \cdots x_N \rangle = \langle 1 \cdots M | B(z_1) \cdots B(z_N) | x_1 \cdots x_N \rangle \) in two ways. First, we note from Theorem 4.1 that \( \langle 1 \cdots M | B(z_1) \cdots B(z_N) | x_1 \cdots x_N \rangle \) is expressed using Schur polynomials as (4.3).

Another way of evaluation can be accomplished by viewing the dual wavefunction as a partition function constructed from \( N \) layers of \( B \)-operators, inserting the completeness relation and decomposing it as sums of products of matrix elements of the \( B \)-operators. That is, we decompose the dual wavefunction as

\[ \langle 1 \cdots M | B(z_1) \cdots B(z_N) | x_1 \cdots x_N \rangle \]

\[ = \sum_T \langle 1 \cdots M | x_1 \cdots x_N \rangle \]

\[ \times \prod_{j=1}^{N} \frac{\langle a_{N-j+1,N-1} + 1, \cdots, a_{N-j+1,N-j+1} \rangle \langle a_{N-j+1,N-1} + 1, \cdots, a_{N-j+1,N-j+1} + 1 | B(z_j) \rangle}{\langle a_{N-j+1,N-1} + 1, \cdots, a_{0,N-1} + 1 | 1 \rangle \langle 1 | 1 \rangle \cdots \langle 1 | 1 \rangle \prod_{k=j}^{N-1} \sigma_{a_{j,k+1}}}, \quad (6.7) \]

where

\[ \langle a_{j,N-1} + 1 \cdots a_{0,j} + 1 \rangle = \prod_{k=j}^{N-1} \sigma_{a_{j,k+1}} \langle 1 \rangle_{1 \cdots 1 \cdots 1 \cdots 1} M, \quad (6.8) \]

\[ \langle a_{j,N-1} + 1 \cdots a_{0,j} + 1 \rangle = \langle 1 | 1 \rangle \cdots \langle 1 | 1 \rangle \prod_{k=j}^{N-1} \sigma_{a_{j,k+1}}. \quad (6.9) \]

and use the formula for the matrix elements of a single \( B \)-operator (6.1) in Proposition 6.1.

Then one finds the product of the matrix elements of the \( B \)-operators in (6.7) corresponding to each strict Gelfand-Tsetlin pattern \( T \) can be expressed as (6.4)

\[ \mathcal{G}(T, \{z\}_N) = \prod_{i=0}^{N-1} \prod_{j=i}^{N-1} \tau(a_{i,j}) z_N^{d_0(T) - d_1(T) - d_2(T) - \ldots - d_{N-2}(T) - d_{N-1}(T)} z_1^{d_{N-1}(T)}, \quad (6.10) \]
Hence, the identity (6.7) can be rewritten in the following form for the dual wavefunction

\[ \langle 1 \cdots M | B(z_1) \cdots B(z_N) | \mathcal{T} \cdots \mathcal{T}_N \rangle = \sum_{\mathcal{T}} \mathcal{C}(\mathcal{T}, \{ z \}_N) \]

\[ = \sum_{\mathcal{T}} \prod_{i=0}^{N-1} \prod_{j=1}^{N-i-1} \gamma(a_{i,j}) z_N^{d_0(\mathcal{T}) - d_i(\mathcal{T})} z_{N-i-1}^{d_i(\mathcal{T}) - d_{i-1}(\mathcal{T})} \cdots z_1^{d_{N-2}(\mathcal{T}) - d_{N-1}(\mathcal{T})} z_1^{d_{N-1}(\mathcal{T})}. \]  

(6.11)

Comparing the two expressions (6.3) and (6.11) evaluated by two ways, we get (6.6).

\[ \square \]

The combinatorial formula (6.6) in Theorem 6.2 can be rewritten into the following form by scaling every spectral parameter \( z_j \) to \( t z_j \) and cancelling powers of \( t \) of both hand sides and simplifying.

**Theorem 6.3.** We have the following combinatorial formula for the Schur polynomials

\[ \prod_{1 \leq j < k \leq N} (z_j + t z_k) \gamma(\{ z \}_N) = \sum_{\mathcal{T}} \prod_{i=0}^{N-1} \prod_{j=1}^{N-i-1} \gamma(a_{i,j}) z_N^{d_0(\mathcal{T}) - d_i(\mathcal{T})} z_{N-i-1}^{d_i(\mathcal{T}) - d_{i-1}(\mathcal{T})} \cdots z_1^{d_{N-2}(\mathcal{T}) - d_{N-1}(\mathcal{T})} z_1^{d_{N-1}(\mathcal{T})}, \]

(6.12)

where the sum is over all strict Gelfand-Tsetlin patterns with the bottom row of the strict Gelfand-Tsetlin pattern is fixed by the Young diagram as \( a_{0,j} = \lambda_{j+1} + N - j - 1 \).

Let us discuss the differences between Theorem 6.3 and Theorem 6.3. In the original wavefunction, the factor \( \gamma(a_{i,j}) \) (3.9) in (3.10) depends only on three neighbors \( a_{i,j}, a_{i-1,j} \) and \( a_{i-1,j-1} \) in the strict Gelfand-Tsetlin pattern. On the other hand, for the case of the dual wavefunction, the factor \( \gamma(a_{i,j}) \) (6.5) in (6.12) depends on four neighbors \( a_{i,j}, a_{i-1,j}, a_{i-1,j-1} \) and \( a_{i+1,j} \). Note also the order of the symmetric variables (spectral parameters) in (3.10) is \( z_1, \ldots, z_N \), while it is \( z_N, \ldots, z_1 \) in (6.12), i.e., the order is reversed. Moreover, the right hand side of (6.12) has powers of \( t \) as factors, which explicitly depends on the size of the Young diagram, the total number of sites \( M \) and the total number of holes \( N \). This explicit dependence cannot be found in (3.10).

**Example** Let us check (6.12) by an example. Consider the case \( M = 4, N = 2, \lambda = (\lambda_1, \lambda_2) = (2, 1) \). The bottom row of the dual strict Gelfand-Tsetlin pattern is fixed as \( a_{0,0} = \lambda_1 + N - 0 - 1 = 2 + 2 - 0 - 1 = 3, a_{0,1} = \lambda_2 + N - 1 - 1 = 2 + 1 - 1 - 1 = 1 \). From the interlacing relation \( 3 = a_{0,0} \geq a_{1,1} \geq a_{0,1} = 1 \), we have \( a_{1,1} = 1, 2 \) or 3. Therefore, there are three strict Gelfand-Tsetlin patterns in the sum of (6.12)

\[ \mathcal{T} = \left\{ \begin{array}{c} \begin{array}{c} 0 \ \ 1 \\ 1 \end{array} \end{array} \right\}, \quad \left\{ \begin{array}{c} \begin{array}{c} 2 \\ 1 \end{array} \end{array} \right\}, \quad \left\{ \begin{array}{c} \begin{array}{c} 3 \\ 1 \end{array} \end{array} \right\}. \]  

(6.13)
We also set $a_{1,0} = a_{2,1} = 4$ for each strict Gelfand-Tsetlin pattern. Keeping this in mind, let us calculate the weights for each pattern.

1. $\overline{\mathcal{T}} = \left\{ \begin{array}{ll} 1 & 3 \\ 1 & 1 \end{array} \right\}$. We have $\gamma(a_{0,0}) = t^{\text{Max}(4-3-1,0)} = 1$, $\gamma(a_{0,1}) = t^{\text{Max}(1-1-1,0)} = 1$, $\gamma(a_{1,1}) = t^{\text{Max}(4-1-1,0)} = t^2$, $d_0(\overline{\mathcal{T}}) = 3 + 1 = 4$, $d_1(\overline{\mathcal{T}}) = 1$. Thus we have $\gamma(a_{0,0})\gamma(a_{0,1})\gamma(a_{1,1})z_2^{d_0(\overline{\mathcal{T}})-d_1(\overline{\mathcal{T})}}z_1^{d_1(\overline{\mathcal{T})}} = t^2z_2^4z_1^4$.

2. $\overline{\mathcal{T}} = \left\{ \begin{array}{ll} 2 & 3 \\ 1 & 1 \end{array} \right\}$. We have $\gamma(a_{0,0}) = t^{\text{Max}(4-3-1,0)} = 1$, $\gamma(a_{0,1}) = t^{\text{Max}(2-1-1,0)} = 1$, $\gamma(a_{1,1}) = t^{\text{Max}(4-2-1,0)}(t + 1) = t(t + 1)$, $d_0(\overline{\mathcal{T}}) = 3 + 1 = 4$, $d_1(\overline{\mathcal{T}}) = 2$. The corresponding weight is $\gamma(a_{0,0})\gamma(a_{0,1})\gamma(a_{1,1})z_2^{d_0(\overline{\mathcal{T}})-d_1(\overline{\mathcal{T})}}z_1^{d_1(\overline{\mathcal{T})}} = t(t + 1)z_2^2z_1^2$.

3. $\overline{\mathcal{T}} = \left\{ \begin{array}{ll} 3 & 3 \\ 1 & 1 \end{array} \right\}$. We have $\gamma(a_{0,0}) = t^{\text{Max}(4-3-1,0)} = 1$, $\gamma(a_{0,1}) = t^{\text{Max}(3-1-1,0)} = t$, $\gamma(a_{1,1}) = t^{\text{Max}(4-3-1,0)} = 1$, $d_0(\overline{\mathcal{T}}) = 3 + 1 = 4$, $d_1(\overline{\mathcal{T}}) = 3$. We have $\gamma(a_{0,0})\gamma(a_{0,1})\gamma(a_{1,1})z_2^{d_0(\overline{\mathcal{T}})-d_1(\overline{\mathcal{T})}}z_1^{d_1(\overline{\mathcal{T})}} = t_2z_2^3$ in total.

Summing the three weights calculated above and noting $\sum_{j=1}^2 \overline{\lambda_j} - N(M-N) = 2+1-4 = -1$, the right hand side of (6.12) is $t^{-1}(t^2z_2^3z_1 + t(t + 1)z_2^2z_1^2 + tz_2z_1^3) = t_2z_2^3z_1 + (t + 1)z_2^2z_1^2 + z_2z_1^3$, which can be factorized as $(z_1 + t_2)(z_2^2 + z_1z_2^2) = (z_1 + t_2)s_{(2,1)}(z_1, z_2)$, which is exactly the left hand side of (6.12).

7 A generalization of the correspondence

We have showed Theorem 4.1 which gives the relation between the dual wavefunction and the Schur polynomials, for which we gave two proofs. The one given in section 4 can be applied to a generalization of the Felderhof model, where inhomogeneous parameters are now introduced in the quantum spaces. Since the original wavefunction was found to give the factorial Schur polynomials [10], one expects the dual wavefunction also gives the factorial Schur polynomials.

The $L$-operator which constructs the wavefunction now has dependence on the quantum space $\mathcal{F}_j$: at the $j$-th site in the quantum space, we introduce the following $L$-operator

$$L_{\alpha_j}(z, t, \alpha_j) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & t & 1 & 0 \\ 0 & (t + 1)z & z + \alpha_j & 0 \\ 0 & 0 & 0 & z - t\alpha_j \end{pmatrix}. \tag{7.1}$$

The $L$-operators $L_{\alpha_j}(z, t, \alpha_j)$ now has inhomogeneous parameters $\alpha_j$, $j = 1, \cdots M$ besides the spectral parameter and the deformation parameters.

We consider the wavefunction $\langle x_1 \cdots x_N | \Psi(z_1, \ldots, z_N, \{\alpha\}) \rangle$ by introducing the $N$-particle state

$$\Psi(z_1, \ldots, z_N, \{\alpha\}) = B(z_1, \{\alpha\}) \cdots B(z_N, \{\alpha\}) |\Omega\rangle, \tag{7.2}$$
where the $B$-operator
\begin{equation}
B(z, \{\alpha\}) = a(0|L_{aM}(z, t, \alpha_M) \cdots L_{a1}(z, t, \alpha_1)|1)_a,
\end{equation}
now has dependence on the inhomogeneous parameters $\{\alpha\} = \{\alpha_1, \ldots, \alpha_M\}$, which turns out to be the factorial parameters of the factorial Schur polynomials defined below.

**Definition 7.1.** The factorial Schur polynomials is defined to be the following determinant:
\begin{equation}
s_\lambda(\{z\}_N|\{\alpha\}) = \frac{F_{\lambda+\delta}(\{z\}_N|\{\alpha\})}{\prod_{1 \leq j < k \leq N}(z_j - z_k)},
\end{equation}
where $\{z\} = \{z_1, \ldots, z_N\}$ is a set of variables and $\lambda$ denotes a Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ with weakly decreasing non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$, and $\delta = (N - 1, N - 2, \ldots, 0)$. $F_\mu(\{z\}_N|\{\alpha\})$ is an $N \times N$ determinant
\begin{equation}
F_\mu(\{z\}_N|\{\alpha\}) = \det_N \left( \prod_{j=1}^\mu (z_k + \alpha_j) \right).
\end{equation}
We remark that one must respect the ordering of the factorial parameters $\{\alpha\} = \{\alpha_1, \ldots, \alpha_M\}$.

Bump, McNamara and Nakasuji showed the following correspondence between the wavefunction of the Felderhof model with inhomogeneties and the factorial Schur polynomials.

**Theorem 7.2.** [16] The wavefunction $\langle x_1 \ldots x_N|\Psi(z_1, \ldots, z_N, \{\alpha\}) \rangle$ is expressed by the factorial Schur polynomials as
\begin{equation}
\langle x_1 \ldots x_N|\Psi(z_1, \ldots, z_N, \{\alpha\}) \rangle = \prod_{1 \leq j < k \leq N} (z_j + tz_k)s_\lambda(\{z\}_N|\{\alpha\}),
\end{equation}
under the relation $\lambda_j = x_{N-j+1} - N + j - 1$, $j = 1, \ldots, N$.

This Theorem was proved by noting that the arguments in [13] naturally lift to this inhomogeneous setting. One first shows that the wavefunction is a polynomial of $t$ with highest weight degree $N(N - 1)/2$. Then one evaluates the wavefunction at $t = -1$, at which the six-vertex model reduces to a five-vertex model, and each configuration making nonzero contribution to the wavefunction essentially corresponds to each term of the determinant expansion of the numerator of the factorial Schur polynomials (7.3).

Let us now state the result for the dual wavefunction $\langle \Phi(z_1, \ldots, z_N, \{\alpha\})|\overline{x_1} \cdots \overline{x_N}\rangle$ which is the overlap between the hole configurations $|\overline{x_1} \cdots \overline{x_N}\rangle$ and the dual $N$-particle state
\begin{equation}
\langle \Phi(z_1, \ldots, z_N, \{\alpha\})| = (1 \cdots M)B(z_1, \{\alpha\}) \cdots B(z_N, \{\alpha\}).
\end{equation}

By applying the argument in section 4, one gets the following relation between the dual wavefunction and the factorial Schur polynomials.

**Theorem 7.3.** The dual wavefunction $\langle \Phi(z_1, \ldots, z_N, \{\alpha\})|\overline{x_1} \cdots \overline{x_N}\rangle$ can be expressed by the factorial Schur polynomials as
\begin{equation}
\langle \Phi(z_1, \ldots, z_N, \{\alpha\})|\overline{x_1} \cdots \overline{x_N}\rangle = i^{N(M-N)} \prod_{1 \leq j < k \leq N} (z_j + tz_k)s_\lambda\left(\left\{\frac{z_j}{t}\right\}_N, \{-\alpha\}\right).
\end{equation}
Here the Young diagram for the factorial Schur polynomials correspond to the particle configuration under the relation $\lambda_j = x_{N-j+1} - N + j - 1$, $j = 1, \ldots, N$, and the symmetric variables are $\left\{ \frac{z}{t} \right\}_N = \left\{ \frac{z_1}{t}, \ldots, \frac{z_N}{t} \right\}$. Moreover, the signs of the parameters of the factorial Schur polynomials in the right hand side of (7.8) are now inverted simultaneously: $\{-\alpha\} = \{-\alpha_1, \ldots, -\alpha_M\}$.

The correspondence (7.8) includes the special case $t = 1$ of the relation between the dual wavefunction and factorial Schur polynomials in [16], which was proved by starting from the result for the relation between the original wavefunction and the factorial Schur polynomials, using arguments on the symmetry of the $L$-operators to transform the original correspondence to the dual correspondence. This argument seems very difficult for the case $t \neq 1$ even for the ordinary Schur polynomials. However, one can naturally lift the arguments given in section 4 to this inhomogeneous setting. The problem reduces to the case of the $t = -1$, where the six-vertex model reduces to the five-vertex model. Since we now have the inhomogeneous parameters, this introduction of additional parameters is reflected in the final expression of the correspondence in (7.8).

8 Conclusion

We investigated the Felderhof free-fermion model, and analyzed the dual wavefunction in two ways. We first showed the precise relation between the dual wavefunction and the Schur polynomials, in which we gave two proofs in sections 4 and 5 respectively. One by using the arguments by [13], and another one by combining the matrix product method and the analysis on the domain wall boundary partition function. Next, by calculating the matrix elements of a single $B$-operator, we give a combinatorial expression of the Schur polynomials in terms of strict Gelfand-Tsetlin patterns. By comparing the two expressions, we obtained a combinatorial formula of the Schur polynomials, which can be regarded as a dual version of the Tokuyama formula, since it was found [13] that the original wavefunction naturally gives a realization of the Tokuyama formula for the Schur polynomials, and we are now dealing with the dual wavefunction.

We also generalized the relation between the dual wavefunction to the Felderhof model with inhomogeneous parameters in the quantum space and the factorial Schur polynomials, which is motivated by the fact that the wavefunction of the Felderhof model with inhomogeneities are given by the factorial Schur polynomials [16]. The expression can be extended furthermore to the Felderhof model with two types of inhomogeneous parameters, and there are correspondences between the original and the dual wavefunctions and a generalization of the factorial Schur polynomials [22, 23].

One of the important problems related to this paper is to study the dual wavefunction for the case of other boundary conditions and find combinatorial formulas for other symmetric polynomials such as the symplectic Schur and Schur $Q$ functions. See [24, 25, 26, 27, 28] for examples for the relation with the wavefunctions and the Felderhof model with other boundary conditions.

The Schur polynomials appears not only as the wavefunction of the Felderhof model, but also as special limits of the wavefunction XXZ-type six-vertex model. The integrable five-vertex model which is the $t = 0$ limit of the $L$-operator [28], which gives the Schur
polynomials, can be regarded as special limits of both the Felderhof model and the XXZ model. See [21][22][29][30][31][32][33] for examples on the recent investigations on the combinatorics of the symmetric polynomials from the viewpoint of partition functions, in which the combinatorial identities of various symmetric polynomials such as the Schur, Grothendieck, Hall-Littlewood and their noncommutative versions are derived.

We finally remark that in recent works, it is revealed by number theorists that the six-vertex model considered in this paper can be regarded as a special case of the “metaplectic ice”, which is a six-vertex model over a non-archimedean local field (see [34] for example). It seems worthwhile to study these models and find novel combinatorial formulas by means of modern statistical physical methods and techniques developed to analyze quantum integrable models.

Acknowledgements

Thus work was partially supported by grant-in-Aid for Research Activity start-up No. 15H06218 and Scientific Research (C) No. 16K05468.

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