Universality of Zipf’s law for time-dependent rank-based systems

Ricardo T. Fernholz\textsuperscript{1} \hspace{1cm} Robert Fernholz\textsuperscript{2}
Claremont McKenna College \hspace{1cm} Intech

May 2, 2018

Abstract

A set of empirical data with positive values follows a Pareto distribution, or power law, if a log-log plot of the data versus rank forms a straight line. A Pareto distribution satisfies Zipf’s law if the slope of the log-log plot is $-1$. Zipf’s law is a form of universality, since many classes of data approximately follow this distribution. We explain why many time-dependent rank-based systems follow Zipf’s law or a weaker form of Zipf’s law in which the log-log plot of the distribution is concave with a tangent line of slope $-1$ at some point on the curve. Under certain regularity conditions, we show that two conditions—conservation and completeness—ensure that at least this weak form of Zipf’s law will hold. We also show that many time-dependent rank-based systems that occur in practice satisfy these conditions.

1 Introduction

A set of empirical data with positive values follows a Pareto distribution, or power law, if the log-log plot of the values versus rank is approximately a straight line. Power laws are ubiquitous in the social and natural sciences (Bak, 1996; Newman, 2005). A Pareto distribution satisfies Zipf’s law if the log-log plot has a slope of $-1$, following Zipf (1935), who noticed that the frequency of written words in English follows such a distribution. We shall refer to these distributions as Zipfian, and such distributions often result from time-dependent rank-based systems of empirical data. However, many time-dependent rank-based systems have log-log plots that are not straight lines but instead are concave with a tangent of slope $-1$ somewhere along the curve, and we shall call these more general distributions quasi-Zipfian.

Zipf’s law and “quasi-Zipf’s law” represent a form of universality, since systems across both the natural and social sciences frequently follow these distributions. This universality has led researchers to try to explain it, especially in the case of Zipf’s law (Simon, 1955; Steindl, 1965; Gabaix, 1999). Examples of Zipfian and quasi-Zipfian distributions include the market capitalization of companies (Simon and Bonini, 1958; Fernholz, 2002), the population of cities (Gabaix, 1999), the employees of firms (Axtell, 2001), the income and wealth of households (Atkinson et al., 2011; Blanchet et al., 2017), and the assets of banks (Fernholz and Koch, 2017). From the comprehensive survey of Newman (2005) we have also the frequency of web site visits, the frequency of telephone calls, and the frequency of family names. Newman (2005) also includes an assortment of non-Zipfian Pareto distributions: citations of scientific papers, copies of books sold, magnitude of earthquakes, diameter of moon craters, intensity of solar flares, and intensity of wars.

The dichotomy between Zipfian and non-Zipfian Pareto distributions is of interest to us here. We show that Zipfian or quasi-Zipfian distributions result from time-dependent rank-based systems, while the non-Zipfian Pareto distributions appear to result from other phenomena, usually of a cumulative nature. For example, the magnitude of earthquakes is a cumulative system: each new earthquake adds a new observation.

\textsuperscript{1}Claremont McKenna College, 500 E. Ninth St., Claremont, CA 91711, rfernholz@cmc.edu.
\textsuperscript{2}Intech Investments, One Palmer Square, Princeton, NJ 08542, bob@bobfernholz.com.
to the data, but once recorded, these observations do not change over time. Such cumulative systems may generate Pareto distributions, but there is no reason to believe that they will be Zipfian.

To model time-dependent rank-based systems, we use Atlas models and first-order models \cite{Fernholz2002}. Atlas models are systems of continuous semimartingales in which all processes have the same growth rates and variance rates at all ranks. First-order models are systems of continuous semimartingales in which the processes have growth rates and variance rates that vary by rank. Atlas models are first-order models that follow the strong form of Gibrat’s law \cite{Gibrat1931}, which requires that the growth rates and variance rates are constant across ranks. The stable distribution for an Atlas model will be Pareto, while first-order models can be constructed to have any stable distribution.

We show that two conditions—conservation and completeness—ensure that Atlas models will generate a Zipfian stable distribution. For the class of first-order models with growth rates that are constant across rank and variance rates that increase with rank, these two conditions will result in a quasi-Zipfian stable distribution. Conservation and completeness are “natural” conditions that are likely to hold for any empirical time-dependent rank-based system that includes a sufficient number of ranked observations, and this results in the universality of Zipfian and quasi-Zipfian distributions.

In the next sections we first develop a general methodology to understand time-dependent rank-based systems, and then consider Zipfian and quasi-Zipfian systems following this methodology. We apply our results to the capitalization of U.S. companies, with an analysis of the corresponding quasi-Zipfian distribution curve. Finally, we consider a number of other time-dependent rank-based systems as well as other approaches that have been used to analyze these systems. Proofs of all propositions are in the appendix, along with some examples.

2 Asymptotically stable systems of continuous semimartingales

We shall use systems of non-negative continuous semimartingales \(\{X_1, \ldots, X_n\}\), with \(n > 1\), to approximate systems of time-dependent empirical data. For such a system we define the rank function to be the random permutation \(r_t \in \Sigma_n\) such that \(r_t(i) < r_t(j)\) if \(X_i(t) > X_j(t)\) or if \(X_i(t) = X_j(t)\) and \(i < j\). Here \(\Sigma_n\) is the symmetric group on \(n\) elements. The rank processes \(X_1(t) \geq \cdots \geq X_n(t)\) are defined by \(X(r_t(i))(t) = X_i(t)\).

Let us suppose that \(X_i(t) > 0\) for \(t \in [0, \infty)\) and \(i = 1, \ldots, n\), a.s., so we can consider the logarithms of these processes. The processes \((\log X_{(k)} - \log X_{(k+1)})\), for \(k = 1, \ldots, n - 1\), are called gap processes, and we define \(\Lambda_{k,k+1}^X\) to be the local time at the origin for \((\log X_{(k)} - \log X_{(k+1)})\), with \(\Lambda_{0,1}^X = \Lambda_{n,n+1}^X = 0\) \cite{Karatzas1991}. If the log \(X_i\) spend no local time at triple points, then the rank processes satisfy

\[
\begin{aligned}
\log X_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{r_t(i) = k\}} \log X_i(t) + \frac{1}{2} d\Lambda_{k,k+1}^X(t) - \frac{1}{2} d\Lambda_{k-1,k}^X(t), \quad \text{a.s.}
\end{aligned}
\tag{2.1}
\]

for \(k = 1, \ldots, n\) \cite{Fernholz2001, Banner and Ghomrasni2008}.

We are interested in systems that show some kind of rank-based stability, at least asymptotically. Since we must apply our definition of stability to systems of empirical data as well as to continuous semimartingales, we use asymptotic time averages rather than expectations for our definitions. For the systems of continuous semimartingales we consider, the law of large numbers implies that the asymptotic time averages are equal to the expectations \cite{Banner et al.2005, Ichiba et al.2011}.

**Definition 2.1.** \cite{Fernholz2002} A system of positive continuous semimartingales \(\{X_1, \ldots, X_n\}\) is asymptotically stable if

1. \(\lim_{t \to \infty} t^{-1} (\log X_{(1)}(t) - \log X_{(n)}(t)) = 0\), a.s. (coherence);
2. \(\lim_{t \to \infty} t^{-1} \Lambda_{k,k+1}^X(t) = \lambda_{k,k+1} > 0\), a.s.;
3. \(\lim_{t \to \infty} t^{-1} (\log X_{(k)} - \log X_{(k+1)}) = \sigma_{k,k+1}^2 > 0\), a.s.;
for \( k = 1, \ldots, n - 1 \), where the local time and quadratic-variation parameters \( \lambda_{k,k+1} \) and \( \sigma^2_{k,k+1} \) are positive constants.

For \( k = 1, \ldots, n \), let us define the processes
\[
X_{[k]} \triangleq X_{(1)} + \cdots + X_{(k)},
\]
in which case we can express \( X_{[k]} \) in terms of the \( X_i \) and \( \Lambda^X_{k,k+1} \).

**Lemma 2.2.** Let \( X_1, \ldots, X_n \) be positive continuous semimartingales that satisfy (2.1). Then
\[
dX_{[k]}(t) = \sum_{i=1}^{n} 1_{\{r_{(i)} \leq k\}} dX_i(t) + \frac{1}{2} X_{(k)}(t) d\Lambda^X_{k,k+1}(t), \quad \text{a.s.} \tag{2.3}
\]
for \( k = 1, \ldots, n \).

Lemma 2.2 describes the dynamic relationship between the combined value, or size, \( X_{[k]} \), of the \( k \) top ranks and the local time process \( \Lambda^X_{k,k+1} \). This local time process compensates for turnover into and out of the top \( k \) ranks. Over time, some of the higher-value, larger processes will decrease and exit from the top ranks, while some of the lower-value, smaller processes will increase and enter into those top ranks. The process of entry and exit into and out of the top \( k \) ranks is quantified by the last term in (2.3), which measures the rate of replacement of the top ranks of the system. We should note that (2.3) makes it possible to evaluate the local time process \( \Lambda^X_{k,k+1} \) implicitly by means of the other terms in the equation, all of which are observable, and this will allow us to extend the definition of local time to time-dependent systems of empirical data.

### 3 Atlas models and first-order models

The simplest system we shall consider is an Atlas model \([\text{Fernholz} 2002]\), a system of positive continuous semimartingales \( \{X_1, \ldots, X_n\} \) defined by
\[
d\log X_i(t) = -g \, dt + ng 1_{\{r_{(i)} = n\}} dt + \sigma \, dW_i(t), \tag{3.1}
\]
where \( g \) and \( \sigma \) are positive constants and \( (W_1, \ldots, W_n) \) is a Brownian motion. Atlas models are asymptotically stable with parameters
\[
\lambda_{k,k+1} = 2kg, \quad \text{and} \quad \sigma^2_{k,k+1} = 2\sigma^2, \quad \text{a.s.,} \tag{3.2}
\]
for \( k = 1, \ldots, n - 1 \) \([\text{Banner et al.} 2005]\).

The processes \( X_i \) in an Atlas model are exchangeable, so each \( X_i \) asymptotically spends equal time in each rank and hence has zero asymptotic log-drift. \([\text{Banner et al.} 2005]\) show that the gap processes \( (\log X_{(k)} - \log X_{(k+1)}) \) for Atlas models have stable distributions that are exponentially distributed and satisfy
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T (\log X_{(k)}(t) - \log X_{(k+1)}(t)) \, dt = \frac{\sigma^2_{k,k+1}}{2\lambda_{k,k+1}}, \tag{3.3}
\]
for \( k = 1, \ldots, n - 1 \).

We define the (asymptotic) slope parameters \( s_k \) by
\[
s_k \triangleq k \lim_{T \to \infty} \frac{1}{T} \int_0^T (\log X_{(k)}(t) - \log X_{(k+1)}(t)) \, dt, \tag{3.4}
\]
for \( k = 1, \ldots, n - 1 \). The slope of the tangent to the log-log plot of size versus rank will be
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\log X_{(k)}(t) - \log X_{(k+1)}(t)}{\log(k) - \log(k+1)} \, dt, \quad \text{a.s.,}
\]
at rank $k$, and
\[ -s_k \left( 1 + \frac{1}{2k} \right) < \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{\log X_{(k)}(t) - \log X_{(k+1)}(t)}{\log(k) - \log(k+1)} \, dt < -s_k, \quad (3.5) \]
for $k = 1, \ldots, n-1$, so for large enough $k$ the slope parameters will be approximately equal to minus the slope. For expositional simplicity, we shall treat the slope parameters $s_k$ as if they measured the true log-log slopes between adjacent ranks, but it is important to remember that this equivalence is only as accurate as the range in inequality (3.5).

For an Atlas model, it follows from (3.2) and (3.3) that
\[ s_k = \frac{\sigma^2}{2g}, \quad \text{a.s.,} \quad (3.6) \]
for $k = 1, \ldots, n-1$, so the stable distribution of an Atlas model follows a Pareto distribution, at least approximately, and when
\[ \sigma^2 = 2g, \quad (3.7) \]
it follows Zipf's law.

A modest generalization of the Atlas model is the first-order model [Fernholz, 2002], a system of positive continuous semimartingales $\{X_1, \ldots, X_n\}$ with
\[ d \log X_i(t) = g_{r_i(i)} \, dt + G_n 1_{\{r_i(i) = n\}} \, dt + \sigma_{r_i(i)} \, dW_i(t), \quad (3.8) \]
where $\sigma_1^2, \ldots, \sigma_n^2$ are positive constants, $g_1, \ldots, g_n$ are constants satisfying
\[ g_1 + \cdots + g_k < 0, \quad \text{for } k \leq n, \quad (3.9) \]
$G_n = -(g_1 + \cdots + g_n)$, and $(W_1, \ldots, W_n)$ is a Brownian motion. An Atlas model is a first-order model with $g_k = -g$ and $\sigma_k^2 = \sigma^2$, for $k = 1, \ldots, n$. First-order models are asymptotically stable with parameters
\[ \lambda_{k,k+1} = -2(g_1 + \cdots + g_k), \quad \text{a.s.,} \quad (3.10) \]
and
\[ \sigma_{k,k+1}^2 = \sigma_k^2 + \sigma_{k+1}^2, \quad \text{a.s.,} \quad (3.11) \]
for $k = 1, \ldots, n-1$ [Banner et al., 2005].

The processes $X_i$ in a first-order model are exchangeable, as they are for Atlas models, so again each $X_i$ asymptotically spends equal time in each rank and hence has zero asymptotic log-drift. Moreover, first-order models have asymptotically exponential gaps, and (3.3) continues to hold in this more general case [Banner et al., 2005].

Reasoning as above, for a first-order model the slope parameters are
\[ s_k = \frac{k(\sigma_k^2 + \sigma_{k+1}^2)}{2\lambda_{k,k+1}} = -\frac{k(\sigma_k^2 + \sigma_{k+1}^2)}{4(g_1 + \cdots + g_k)}, \quad \text{a.s.,} \quad (3.12) \]
for $k = 1, \ldots, n-1$, so the stable distribution of a first-order model is not confined to the class of Pareto distributions.

We shall need to consider families of Atlas and first-order models that share the same parameters, and for this purpose we define an Atlas family as a class of Atlas models $\{X_1, \ldots, X_n\}$, for $n \in \mathbb{N}$, with the common parameters $g > 0$ and $\sigma^2 > 0$ each defined as in (3.1). Similarly, we define a first-order family as the class of first-order models $\{X_1, \ldots, X_n\}$, for $n \in \mathbb{N}$, defined as in (3.8) with the common parameters $g_k$ and $\sigma_k^2$ such that, for $k \in \mathbb{N}$,
\[ g_1 + \cdots + g_k < 0, \quad \sigma_k^2 > 0, \quad (3.13) \]
and with $G_n = -(g_1 + \cdots + g_n)$. For these families, the parameters $\sigma^2_{k,k+1}$, $\lambda_{k,k+1}$ and $s_k$ are defined uniquely by (3.2), (3.6), (3.10), (3.11), and (3.12), as the case may be. Let us note that the slope parameters $s_k$ given by (3.12) do not depend on the number of processes in the model as long as $n > k$, so a first order family defines a unique asymptotic distribution curve. Accordingly, these families allow us to derive results about asymptotic distribution curves without repeatedly reciting the characteristics of individual Atlas or first-order models.

The first-order families we shall consider here will usually be of a more restricted form than (3.8) and (3.9). A first-order family is simple if there is a positive constant $g$ such that $g_k = -g$, for $k \in \mathbb{N}$, and the sequence of positive constants $\sigma^2_k$ is nondecreasing. In this case for $\{X_1, \ldots, X_n\}$ we have $G_n = ng$ and (3.8) becomes

$$d \log X_i(t) = -g \, dt + ng \mathbb{1}_{\{r_i(t) = n\}} \, dt + \sigma^2_{r_i(i)} \, dW_i(t),$$

where $(W_1, \ldots, W_n)$ is a Brownian motion. For a simple first-order family,

$$\lambda_{k,k+1} = 2kg,$$  \hspace{1cm} a.s.,  \hspace{1cm} (3.14)

and

$$s_k = \frac{\sigma^2_k + \sigma^2_{k+1}}{4g},$$   \hspace{1cm} a.s.,  \hspace{1cm} (3.15)

and since the $\sigma^2_k$ are nondecreasing, the log-log plot of the stable distribution will be concave.

It appears that real time-dependent rank-based systems frequently behave like simple first-order families, and we include one such example below, the capitalizations of U.S. companies (see Figures 1 and 2, below). That the variation increases at the lower ranks seems natural—even in the original observation of Brown (1827) it would seem likely that the water molecules would have buffeted the smaller particles more vigorously than the larger ones.

A further generalization to hybrid Atlas models, systems of processes with growth rates and variance rates that depend both on rank and on name (denoted by index $i$), was introduced by Ichiba et al. (2011), who showed that these more general systems are also asymptotically stable. In a hybrid Atlas model the processes are not necessarily exchangeable, so processes occupying a given rank need not have the same growth rates and variance rates, and the asymptotic distribution of the gap processes can be mixtures of exponential distributions rather than pure exponentials (Ichiba et al., 2011). Nevertheless, although we can expect (3.3) to hold precisely only for systems in which the growth rates and variance rates are determined by rank alone, in many cases this relation can still provide a reasonably accurate characterization of the invariant distribution of the system. For example, we shall see below in Section 6 that (3.3) often provides an accurate approximation of the asymptotic distribution for actual empirical data, and in particular, for company capitalizations.

4 First-order approximation of rank-based systems

Zipf’s law originally referred to the frequency of words in a written language (Zipf, 1935). To measure the relative frequency of written words it is not possible to observe all the written words in that language. Instead, the data must be sampled, where a random sample is selected (without replacement), and the frequency versus rank of this random sample is studied. For example, in Wikipedia (2018) 10 million words in each of 30 languages were sampled, and the resulting distribution curves created. If the sample is large enough, the distribution of the sampled data should not differ materially from the distribution of the entire data set.

An additional advantage of using sampled data is that the number of data in the sample remains constant over time. The total number of written words that appear in a language is likely to increase over time, and this increase could bias estimates of some parameters. Sampling the data will remove such a trend from the data, since a constant number of words can be sampled at each time. Accordingly, in all cases we shall
assume that global trends have been removed from the data, either by sampling or some other form of detrending, so we need only consider relative changes among the ranks.

Suppose that we represent the word count among the 10 million sampled Wikipedia words at time $t \geq 0$ by $\{Z_1(t), Z_2(t), \ldots \}$, where $Z_i(t)$ represents the number of appearances of the $i$th word. Rather than resort to the tedious time-series notation $t_\nu$ for these data series, we shall use the same notation as for continuous-time stochastic processes, while remembering that for data series $dt$ actually stands for $\Delta t = t_{\nu+1} - t_{\nu}$, and that a limit as $t \to \infty$ stands for the use of all the data we have at our disposal.

As we proceed to general time-dependent rank-based systems, we shall assume that we have sampled or otherwise corrected the data for global trends. As a result of this sampling or detrending, the total value of our sampled data universe will remain constant, so

$$Z_1(t) + Z_2(t) + \cdots = \text{constant},$$

where this constant would be 10 million for the words sampled from Wikipedia. We can define the rank function $\rho_i$ for the data series as the function $\rho_i : \mathbb{N} \to \mathbb{N}$ such that $\rho_i(i) < \rho_i(j)$ if $Z_i(t) > Z_j(t)$ or if $Z_i(t) = Z_j(t)$ and $i < j$, with $\rho_i(i) = i$ for $i > \#\{Z_1(t), Z_2(t), \ldots \}$. As we did above for semimartingales, we can define the ranked values $\{Z(1)(t) \geq Z(2)(t) \geq \cdots \}$ along with

$$Z[k](t) = Z(1)(t) + \cdots + Z(k)(t),$$

for $k \in \mathbb{N}$, and from this we can define local times $\Lambda^Z_{k,k+1}(t)$ implicitly from the equation

$$dZ[k](t) = \sum_{i=1}^{\infty} \mathbb{1}_{\{\rho_i(i) \leq k\}} dZ_i(t) + \frac{1}{2} Z(k)(t) d\Lambda^Z_{k,k+1}(t),$$

for $k \in \mathbb{N}$, as in (2.3). To help clarify the exposition, we shall use boldface fonts for parameters generated from data and normal fonts for parameters related to semimartingales. The local time parameters $\lambda_{k,k+1}$ for the data series will be the time average of the $\Lambda^Z_{k,k+1}(t)$ from Definition (2.1) and the quadratic variation parameters $\sigma^2_{k,k+1}$ for the data series can be estimated from the sample variances

$$\sigma^2_{k,k+1} \triangleq \text{Var}(\log Z(k)(t) - \log Z(k+1)(t)).$$

**Definition 4.1.** The first-order approximation of an asymptotically stable system $\{Z_1(t), Z_2(t), \ldots \}$ is the first-order family with $\{X_1, \ldots, X_n\}$ defined as in (3.8) by

$$d \log X_i(t) = g_{r(i)}(t) dt + G_n \mathbb{1}_{\{r(i) = n\}} dt + \sigma r(i) dW_i(t),$$

where

$$g_k = \frac{1}{2} \lambda_{k-1,k} - \frac{1}{2} \lambda_{k,k+1},$$

$$\sigma^2_k = \frac{1}{4}(\sigma^2_{k-1,k} + \sigma^2_{k,k+1}),$$

for $k \in \mathbb{N}$, $G_n = -(g_1 + \cdots + g_n)$, $(W_1, \ldots, W_n)$ is a Brownian motion, and $\sigma^2_{0,1} = \sigma^2_{1,2}$ by convention.

For a first-order model of the form (4.4) with parameters (4.5), equations (3.10) and (3.11) imply that

$$\lambda_{k,k+1} = -2(g_1 + \cdots + g_k) = \lambda_{k,k+1}, \quad \text{a.s.,}$$

for $k \in \mathbb{N}$, and

$$\sigma^2_{k,k+1} = \sigma^2_k + \sigma^2_{k+1} = \frac{1}{4}(\sigma^2_{k-1,k} + 2 \sigma^2_{k,k+1} + \sigma^2_{k+1,k+2}), \quad \text{a.s.,}$$

for $k \in \mathbb{N}$. Hence, (3.3) becomes

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T (\log X(k)(t) - \log X(k+1)(t)) dt = \frac{\sigma^2_{k,k+1}}{2 \lambda_{k,k+1}} = \frac{\sigma^2_{k-1,k} + 2 \sigma^2_{k,k+1} + \sigma^2_{k+1,k+2}}{8 \lambda_{k,k+1}}, \quad \text{a.s.,}$$

as $T \to \infty$. The limit stands for the use of all the data we have at our disposal.
for $k \in \mathbb{N}$. If the data satisfy

$$
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} \left( \log Z_{(k)}(t) - \log Z_{(k+1)}(t) \right) dt \equiv \frac{\sigma_{k,k+1}^2}{2\lambda_{k,k+1}},
$$

(4.8)

for $k \in \mathbb{N}$, then the stable distribution (4.7) for the first-order approximation will be a smoothed version of the distribution (4.8) for the data $\{Z_{(1)}(t), Z_{(2)}(t), \ldots\}$. The approximation (4.8) will be accurate for gap series $(\log Z_{(k)}(t) - \log Z_{(k+1)}(t))$ that behave like reflected Brownian motion, which has an exponential stable distribution. We shall say that a system of time-dependent data that satisfies (4.8) is rank-based, and we can expect this approximation to hold when the behavior of the data is determined mostly by rank. The accuracy of this approximation is likely to deteriorate when more idiosyncratic, or name-based, characteristics are present.

5 Zipfian systems

Suppose we have a time-dependent rank-based system with data $\{Z_1(t), Z_2(t), \ldots\}$ and we observe the top $n > 1$ ranks along with

$$Z_{[n]}(t) = Z_{(1)}(t) + \cdots + Z_{(n)}(t).$$

Since the total value of the sampled data in (4.1) is constant, for large enough $n$ it is reasonable to expect the relative change of the top $n$ ranks to satisfy

$$\frac{dZ_{[n]}(t)}{Z_{[n]}(t)} \to 0,$$

as $n$ tends to infinity. This condition is essentially a “conservation of mass” criterion for the system $\{Z_1(t), Z_2(t), \ldots\}$, and we shall extend this criterion to the first-order approximation.

**Definition 5.1.** The first-order family $\{X_1, \ldots, X_n\}$, for $n \in \mathbb{N}$, is conservative if

$$\lim_{n \to \infty} \mathbb{E}\left[ \frac{dX_{[n]}(t)}{X_{[n]}(t)} \right] = 0,$$

where the expectation is for the stable distribution.

The local time process $\Lambda_{n,n+1}^Z(t)$ from (4.2) measures the effect of entry and exit into and out of the top $n$ ranks of a rank-based system. In order that the system $\{Z_1(t), Z_2(t), \ldots\}$ not depend on continual replacement through its lowest ranks, the contribution of the local time term in (4.2) should vanish for sufficiently large $n$, at least on average, so we have

$$\frac{Z_{(n)}(t)}{Z_{[n]}(t)} \lambda_{n,n+1} dt \to 0,$$

as $n$ tends to infinity. This condition is essentially a “completeness” criterion for the system $\{Z_1(t), Z_2(t), \ldots\}$, and we shall extend it to the first-order approximation.

**Definition 5.2.** The first-order family $\{X_1, \ldots, X_n\}$, for $n \in \mathbb{N}$, is complete if

$$\lim_{n \to \infty} \mathbb{E}\left[ \frac{G_n X_{[n]}(t)}{X_{[n]}(t)} \right] = 0,$$

where the expectation is for the stable distribution.

The use of $G_n$ in this definition is appropriate since $\lambda_{n,n+1} = -2(g_1 + \cdots + g_n) = 2G_n$ by (4.6). For an Atlas family or simple first-order family (5.2) is equivalent to

$$\lim_{n \to \infty} \mathbb{E}\left[ \frac{g_n X_{[n]}(t)}{X_{[n]}(t)} \right] = 0.$$

(5.3)
**Definition 5.3.** A first-order family is **Zipfian** if its slope parameters $s_k = 1$, for $k \in \mathbb{N}$. A time-dependent rank-based system is **Zipfian** if its first-order approximation is Zipfian.

**Proposition 5.4.** An Atlas family is Zipfian if and only if it is conservative and complete.

Proposition 5.4 is a rather ideal version of the rank distributions that occur in actual examples. Many time-dependent rank-based systems have empirical distributions that only approximately follow Zipf’s law, with a log-log plot that is concave rather than straight. This motivates us to define a broader class of Zipf-like systems.

**Definition 5.5.** A first-order family is **quasi-Zipfian** if its slope parameters $s_k$ are nondecreasing with $s_1 \leq 1$ and

$$
\lim_{k \to \infty} s_k \geq 1,
$$

(5.4)

where this limit includes divergence to infinity. A time-dependent rank-based system is **quasi-Zipfian** if its first-order approximation is quasi-Zipfian.

Because the slope parameters $s_k$ are approximately equal to minus the slope of a log-log plot of size versus rank, Definition 5.5 implies that a rank-based system will be quasi-Zipfian if this log-log plot is concave with slope not steeper than $-1$ at the highest ranks and not flatter than $-1$ at the lowest ranks. By these definitions, a Zipfian system is also quasi-Zipfian.

**Proposition 5.6.** If a simple first-order family is conservative, complete, and satisfies

$$
\lim_{n \to \infty} E\left[\frac{X^{(1)}(t)}{X^{(n)}(t)}\right] \leq \frac{1}{2},
$$

(5.5)

then it is quasi-Zipfian.

We show in Example A.1 below that a conservative and complete first-order family with $g_k = -g < 0$, for $k \in \mathbb{N}$, can have a Pareto distribution with log-log slope steeper than $-1$ if the $\sigma_k$ are not nondecreasing. We show in Example A.2 below that without a condition of the form (5.5), a simple, conservative, and complete first-order family can fail to be quasi-Zipfian, but it may be possible to improve on the bound of $1/2$.

### 6 Examples and discussion

Here we apply the methods we developed above to an actual time-dependent rank-based system. We also discuss a number of other such systems, as well as other approaches to time-dependent rank-based systems.

**Example 6.1.** Market capitalization of companies.

The market capitalization of U.S. companies was studied as early as Simon and Bonini (1958), and here we follow the methodology of Fernholz (2002). The capitalization of a company is defined as the price of the company’s stock multiplied by the number of shares outstanding. Ample data are available for stock prices, and this allows us to estimate the first-order parameters we introduced in the previous sections.

Figure 1 shows the smoothed first-order parameters $\sigma_k^2$ and $-g_k$ for the U.S. capital distribution for the 10 year period from January 1990 to December 1999. The capitalization data we used were from the monthly stock database of the Center for Research in Securities Prices at the University of Chicago. The market we consider consists of the stocks traded on the New York Stock Exchange, the American Stock Exchange, and the NASDAQ Stock Market, after the removal of all Real Estate Investment Trusts, all closed-end funds, and those American Depositary Receipts not included in the S&P 500 Index. The parameters in Figure 1 correspond to the 5120 stocks with the highest capitalizations each month. The estimates for the $g_k$ were calculated from (4.5) with the $\lambda_{k,k+1}$ derived from the local times $A_{k,k+1}(t)$, as in Fernholz (2017). The $\sigma_k^2$ and the $\sigma_k^2$ were generated as in (4.3) and (4.5). Both plots in Figure 1 are smoothed by convolution.
with a Gaussian kernel with ±3.16 standard deviations spanning 100 months on the horizontal axis, with reflection at the ends of the data.

According to Figure 1, the values of the parameters $-g_k$ are relatively constant compared to the parameters $\sigma^2_k$, which increase almost linearly with rank. The near-constant $-g_k$ suggest that the first-order approximation will generate a simple first-order family. In Figure 2, the average distribution curve for the capitalizations is represented by the black curve. The red curve is the first-order approximation of the distribution following (4.7). The two curves are quite close, and this indicates that the time-dependent system is an Itô process of the form

$$dX(t) = \mu(X(t)) dt + \sigma(X(t)) dW(t),$$

where $W$ is Brownian motion and $\mu$ and $\sigma$ are well-behaved real-valued functions. We can convert this into logarithmic form by Itô’s rule, in which case

$$d\log X(t) = \left(\mu(X(t)) - \frac{\sigma^2(X(t))}{2}\right) dt + \sigma(X(t)) dW(t), \quad \text{a.s.}$$

We shall assume that this equation has at least a weak solution with $X(t) > 0$, a.s., and that the solution has a stable distribution.

Let us construct $n$ i.i.d. copies $X_1, \ldots, X_n$ of $X$, all defined by (6.1) or, equivalently, by (6.2), and assume that the $X_i$ are all in their common stable distribution. Let us assume that the $X_i$ spend no local time at triple points, so we can define the rank processes, and (4.1) and (4.3) will be valid. If the system is asymptotically stable we can calculate the corresponding rank-based growth rates $g_k$, but if we know the stable distribution of the original process (6.1), then there is a simpler way to proceed.

If we know the common stable distribution of the $X_i$, then we can calculate expectations under this stable distribution and let

$$g_k = E\left[\mu(X_{(k)}(t)) - \frac{\sigma^2(X_{(k)}(t))}{2}\right] \quad \text{and} \quad \sigma^2_k = E[\sigma^2(X_{(k)}(t))],$$

for $k = 1, \ldots, n$. These values are (reverse) order statistics, and under weak regularity conditions on the $\mu$ and $\sigma$, the expectations here will be equal to the asymptotic time averages of the functions. Since the $X_i$ are stable, the geometric mean $(X_1 X_2 \cdots X_n)^{1/n} = (X_{(1)} X_{(2)} \cdots X_{(n)})^{1/n}$ will also be stable, so

$$(g_1 + \cdots + g_n) t = E\left[\log (X_{(1)}(t) \cdots X_{(n)}(t)) - \log (X_{(1)}(0) \cdots X_{(n)}(0))\right] = 0.$$ 

Hence,

$$g_1 + \cdots + g_n = 0, \quad \text{with} \quad g_1 + \cdots + g_k < 0, \quad \text{for} \ k < n,$$

so the $g_k$ and $\sigma^2_k$ define the first-order model

$$d\log Y_i(t) = g_{r_i(i)} dt + \sigma_{r_i(i)} dW_i(t),$$

where $W_1, \ldots, W_n$ is $n$-dimensional Brownian motion. (Note that this model is equivalent to (3.8) with $G_n$ and $g_n$ combined into $g_n$.)
If the functions $\mu$ and $\sigma$ in (6.1) are smooth enough, then the system is likely to be rank-based and the stable distributions of the gap processes $(\log X_{(k)} - \log X_{(k+1)})$ will be (close to) exponential. In this case the stable distribution of the first-order model (6.5) will be close to that of the original system (6.1). More conditions are required to ensure that this stable distribution be quasi-Zipfian, and to achieve a true Zipfian distribution, a lower reflecting barrier or other device must be included in the model (Gabaix, 2009).

Example 6.3. Frequency of written words.

Word frequency is the origin of Zipf’s law (Zipf, 1935), but testing our methodology with word-frequency could be difficult. Ideally, we would like to construct a first-order approximation for the data and compare the first-order distribution to that of the original data. However, the parameters $\lambda_{k,k+1}$ and $\sigma^2_{k,k+1}$ for the top-ranked words in a language are likely to be difficult to estimate over any reasonable time frame, since the top-ranked words seldom change ranks. Nevertheless, while the top ranks may require centuries of data for accurate estimates, the lower ranks would probably be amenable to analysis similar to that which we carried out for company capitalizations. Moreover, it might be possible to combine, for example, all the Indo-European languages and generate accurate estimates of the $\lambda_{k,k+1}$ and $\sigma^2_{k,k+1}$ even for the top ranks of the combined data.

We can see from the remarkable chart in Wikipedia (2018) that the log-log plots for 30 different languages are (almost) straight. Actually, these plots are slightly concave, or quasi-Zipfian in nature. It is possible that this slight curvature is due to sampling error at the lower ranks, which would raise the variances and steepen the slope, but this would have to be determined by studying the actual data.

Example 6.4. Population of cities.

The distribution of city populations is a prominent example of Zipf’s law in social science. However, as the comprehensive cross-country investigation of Soo (2005) shows, city size distributions in most countries are not Zipfian but rather quasi-Zipfian. Gabaix (1999) hypothesized that the quasi-Zipfian distribution of U.S. city size was caused by higher population variances at the lower ranks, consistent with Proposition 5.6. Which of the deviations from Zipf’s law uncovered by Soo (2005) are due to population variances that decrease with increasing city size remains an open question.

There is another phenomenon that occurs with city size distributions. Suppose that rather than studying a large country like the U.S. we consider instead the populations of the cities in New York State. According to the 2010 U.S. census, the largest city, New York City, had a population of 8,175,133, while the second largest, Buffalo, had only 261,310, so this distribution is non-Zipfian. The corresponding population of New York State was 19,378,102, so hypothesis (5.5) of Proposition 5.6 is satisfied, but nevertheless the proposition fails. This calls for an explanation, and we conjecture that while the population of the cities of New York State comprise a time-dependent system, this system is not rank-based. The population of New York City is not determined merely by its rank among New York State cities, but is highly city-specific in nature. Hence, we cannot expect the stable distribution for the gap process between New York City and second-ranked Buffalo to be exponential, and we cannot expect the distribution of the system to be quasi-Zipfian.

Example 6.5. Assets of banks.

Fernholz and Koch (2016) show that the distribution of assets held by U.S. bank holding companies, commercial banks, and savings and loan associations are all quasi-Zipfian. This is true despite the fact that these distributions have undergone significant changes over the past few decades. However, as Fernholz and Koch (2017) show, the first-order approximations of these time-dependent rank-based systems generally do not satisfy the hypotheses of Proposition 5.6, since the parameters $\sigma^2_{k,k+1}$ are, in most cases, lower for higher values of $k$. Nonetheless, the parameters $\lambda_{k,k+1}$ vary with $k$ in such a way as to generate quasi-Zipfian distributions.
Example 6.6. Employees of firms.

Axtell (2001) shows that the distribution of employees of U.S. firms is close to Zipfian, with only slight concavity. A number of empirical analyses have shown that for all but the tiniest firms, employment growth rates of U.S. firms do not vary with firm size (Neumark et al., 2011). This result suggests that the first-order approximation to the system has the same parameter $g_k$ at all ranks. Based on this observation together with the slight concavity demonstrated by Axtell (2001), it is possible that the first-order approximation of U.S. firm employees is simple, which would explain its quasi-Zipfian nature.

7 Conclusion

We have introduced a methodology for understanding the structure of time-dependent rank-based systems with stable distributions that are Zipfian or quasi-Zipfian. A time-dependent rank-based system with the same growth rate and variance rate at each rank will have a stable distribution that satisfies Zipf’s law if and only if the system is conservative and complete. A time-dependent rank-based system with the same growth rate at each rank and variance rates that increase with rank will have a quasi-Zipfian stable distribution if the system is conservative and complete, provided that the largest weight is not greater than one half. Because these conditions of conservation and completeness should be satisfied by most real-world time-dependent systems that include a large number of ranked observations, our results offer an explanation for the universality of Zipfian and quasi-Zipfian distributions.

A Proofs and examples

Proof of Lemma 2.2. Suppose that the rank processes $X_{(k)}$ satisfy (2.1), so we have

$$d \log X_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{r_i(t) = k\}} d \log X_i(t) + \frac{1}{2} d \Lambda^X_{k,k+1}(t) - \frac{1}{2} d \Lambda^X_{k-1,k}(t), \quad \text{a.s.,}$$

for $k = 1, \ldots, n$. By Itô’s rule this is equivalent to

$$\frac{d X_{(k)}(t)}{X_{(k)}(t)} = \sum_{i=1}^{n} \mathbb{1}_{\{r_i(t) = k\}} \frac{d X_i(t)}{X_i(t)} + \frac{1}{2} d \Lambda^X_{k,k+1}(t) - \frac{1}{2} d \Lambda^X_{k-1,k}(t)$$

$$= \sum_{i=1}^{n} \mathbb{1}_{\{r_i(t) = k\}} \frac{d X_i(t)}{X_{(k)}(t)} + \frac{1}{2} d \Lambda^X_{k,k+1}(t) - \frac{1}{2} d \Lambda^X_{k-1,k}(t), \quad \text{a.s.,}$$

for $k = 1, \ldots, n$. From this we have

$$d X_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{r_i(t) = k\}} d X_i(t) + \frac{1}{2} X_{(k)}(t) d \Lambda^X_{k,k+1}(t) - \frac{1}{2} X_{(k)}(t) d \Lambda^X_{k-1,k}(t)$$

$$= \sum_{i=1}^{n} \mathbb{1}_{\{r_i(t) = k\}} d X_i(t) + \frac{1}{2} X_{(k)}(t) d \Lambda^X_{k,k+1}(t) - \frac{1}{2} X_{(k-1)}(t) d \Lambda^X_{k-1,k}(t), \quad \text{a.s.,}$$

for $k = 1, \ldots, n$, since the support of $d \Lambda^X_{k-1,k}$ is contained in the set $\{t : \log X_{(k-1)}(t) = \log X_{(k)}(t)\}$. Now we can add up $d X_{(1)}(t) + \cdots + d X_{(k)}(t) = d X_{[k]}(t)$ and we have

$$d X_{[k]}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{r_i(t) \leq k\}} d X_i(t) + \frac{1}{2} X_{(k)}(t) d \Lambda^X_{k,k+1}(t), \quad \text{a.s.,}$$

for $k = 1, \ldots, n$. \qed
Proof of Proposition 5.4. For an Atlas model \( \{X_1, \ldots, X_n\} \) with parameters \( g > 0 \) and \( \sigma > 0 \), Itô’s rule implies that
\[
dX_i(t) = \left( \frac{\sigma^2}{2} - g + ng \mathbb{1}_{\{r_i = n\}} \right) X_i(t) \, dt + \sigma X_i(t) \, dW_i(t), \quad \text{a.s.,}
\]
for all \( i = 1, \ldots, n \). Hence,
\[
dX_n(t) = \left( \frac{\sigma^2}{2} - g \right) X_n(t) \, dt + \sum_{i=1}^{N} \mathbb{1}_{\{r_i \leq n\}} X_i(t) \, dW_i(t) + ng X_n(t) \, dt,
\]
\[
= \left( \frac{\sigma^2}{2} - g \right) X_n(t) \, dt + X_n(t) \, dM(t) + ng X_n(t) \, dt, \quad \text{a.s.,}
\]
where \( M \) is a martingale incorporating all of the terms \( \sigma W_i \). From this we have
\[
\frac{dX_n(t)}{X_n(t)} = \left( \frac{\sigma^2}{2} - g \right) \, dt + dM(t) + \frac{ng X_n(t)}{X_n(t)} \, dt, \quad \text{a.s.,}
\]
so
\[
\mathbb{E} \left[ \frac{dX_n(t)}{X_n(t)} \right] = \left( \frac{\sigma^2}{2} - g \right) \, dt + \mathbb{E} \left[ \frac{ng X_n(t)}{X_n(t)} \right] \, dt, \quad \text{(A.1)}
\]
where the expectation is for the stable distribution.

Since an Atlas model generates a Pareto distribution with log-log slope \(-\sigma^2/2g\), we can calculate
\[
\mathbb{E} \left[ \frac{ng X_n(t)}{X_n(t)} \right] = \begin{cases} 
O(1) & \text{for } \sigma^2/2 < g, \\
O(1/\log n) & \text{for } \sigma^2/2 = g, \\
O(n(1-\sigma^2/2g)) & \text{for } \sigma^2/2 > g.
\end{cases} \quad \text{(A.2)}
\]

First, [5.1] and [5.3] combined with [A.1] imply that \( \sigma^2 = 2g \). Now, if \( \sigma^2 = 2g \), then by [A.2] the right-hand side of [A.1] is \( O(1/\log n) \, dt \), so the left-hand side must also be, and hence both [5.1] and [5.3] hold. Therefore, [5.1] combined with [5.3] is equivalent to \( \sigma^2 = 2g \), which is equivalent to Zipf’s law by (3.7). \( \square \)

Remark. In the first two cases in (A.2) the infinite series
\[
\sum_{k=1}^{\infty} \mathbb{E} \left[ X_k(t) \right]
\]
for a Pareto distribution would not converge. However, this does not affect us since we consider only finite portions of these series.

Proof of Proposition 5.6. For a simple first-order model \( \{X_1, \ldots, X_n\} \) with parameters \( g > 0 \) and \( 0 < \sigma_1^2 \leq \cdots \leq \sigma_n^2 \), Itô’s rule implies that
\[
dX_i(t) = \left( \frac{\sigma^2_{r_i}}{2} - g + ng \mathbb{1}_{\{r_i = n\}} \right) X_i(t) \, dt + \sigma_{r_i} X_i(t) \, dW_i(t), \quad \text{a.s.,}
\]
for \( i = 1, \ldots, n \). Hence,
\[
dX_n(t) = \sum_{k=1}^{n} X_k(t) \left( \frac{\sigma^2_k}{2} - g \right) \, dt + dM(t) + ng X_n(t) \, dt, \quad \text{a.s.,}
\]
where \( M \) is a martingale incorporating all of the terms \( \sigma_{r_i} X_i(t) dW_i(t) \), so
\[
\mathbb{E} \left[ \frac{dX_n(t)}{X_n(t)} \right] = \left( \sum_{k=1}^{n} \mathbb{E} \left[ \frac{X_k(t)}{X_n(t)} \right] \frac{\sigma^2_k}{2} - g \right) \, dt + \mathbb{E} \left[ \frac{ng X_n(t)}{X_n(t)} \right], \quad \text{(A.3)}
\]
where the expectation is for the stable distribution. If the system is conservative (5.1) and complete (5.3), then as \( n \) tends to infinity the first and last terms of (A.3) vanish and we have

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{X[n]} \frac{X_{(k)}(t)}{X_{(n)}(t)} \frac{\sigma_{k}^2}{2g} = 1.
\] (A.4)

Let us now show that (5.5) implies that \( s_1 \leq 1 \). The \( \sigma_{k}^2 \) are nondecreasing, so (A.4) implies that

\[
1 \geq \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{X[n]} \frac{X_{(k)}(t)}{X_{(n)}(t)} \frac{\sigma_{k}^2}{2g} + \left( 1 - \lim_{n \to \infty} \frac{1}{X[n]} \frac{X_{(k)}(t)}{X_{(n)}(t)} \frac{\sigma_{k}^2}{2g} \right)
\]

\[
\geq \frac{1}{2} \frac{\sigma_{1}^2}{2g} + \frac{1}{2} \frac{\sigma_{1}^2}{2g} = s_1,
\]

where the second inequality is a consequence of (5.5).

We must now show that either \( \lim_{k \to \infty} s_k \geq 1 \) or the \( s_k \) diverge to infinity. Since the \( \sigma_{k}^2 \) are nondecreasing, as \( k \) tends to infinity they must either converge to a finite value or diverge to infinity. If the \( \sigma_{k}^2 \) diverge to infinity, the same will be true for the \( s_k \). If \( \lim_{k \to \infty} \sigma_{k}^2 = \sigma^2 \) then \( \lim_{k \to \infty} s_k = \sigma^2/2g \), and since the \( \sigma_{k}^2 \) are nondecreasing,

\[
1 = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{X[n]} \frac{X_{(k)}(t)}{X_{(n)}(t)} \frac{\sigma_{k}^2}{2g} \leq \frac{\sigma^2}{2g}.
\]

It follows that \( \lim_{k \to \infty} s_k \geq 1 \).

**Example A.1.** A conservative and complete first-order family with a non-Zipfian Pareto distribution. Consider the first-order family such that for \( k \in \mathbb{N} \),

\[
\begin{align*}
g_k &= g, \\
\sigma_{2k-1}^2 &= \rho^2, \\
\sigma_{2k}^2 &= 2\sigma^2 - \rho^2,
\end{align*}
\] (A.5)

where \( g > 0, \sigma^2 > 2g \), and \( 0 < \rho^2 < 2\sigma^2 \). In this case,

\[
\sigma_k^2 + \sigma_{k+1}^2 = 2\sigma^2,
\]

so, according to (3.15), the slope parameters \( s_k \) for this model will be

\[
s_k = \frac{\sigma_k^2}{2g},
\]

for \( k \in \mathbb{N} \). The log-log of the distribution for this family is a straight line with slope \(-\sigma^2/2g < -1\), i.e., a Pareto distribution.

Because this first-order model has the same stable distribution as an Atlas model with \( \sigma^2/2 > g \), we can use the argument in (A.2) to conclude that \( \mathbb{E}[ngX_{(n)}(t)/X_{(n)}(t)] \to 0 \) as \( n \to \infty \), so the family is complete. In addition, we have

\[
\mathbb{E}\left[ \frac{dX_{(n)}(t)}{X_{(n)}(t)} \right] = \left( \sum_{k=1}^{n} \mathbb{E}\left[ \frac{X_{(k)}(t)}{X_{(n)}(t)} \frac{\sigma_k^2}{2} - g \right] dt + \mathbb{E}\left[ ngX_{(n)}(t)/X_{(n)}(t) \right] dt, \right.
\]

so if we can show that for some choice of \( \rho^2 \),

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \mathbb{E}\left[ \frac{X_{(k)}(t)}{X_{(n)}(t)} \frac{\sigma_k^2}{2} \right] = g,
\] (A.6)
then the conservation condition,
\[
\lim_{n \to \infty} \mathbb{E} \left[ \frac{dX_{[n]}(t)}{X_{[n]}(t)} \right] = 0,
\]
will also hold for the family with that value of \( \rho^2 \).

The expectations in (A.6) are invariant with respect to the choice of \( \rho^2 \), so by (A.5) the sum in (A.6) will be continuous in \( \rho^2 \). If we evaluate this sum at \( \rho^2 = 0 \), only the even ranks will appear, so for large enough \( \sigma^2/2 \gg g \),
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{X_{[n]}(t)}{X_{[n]}(t)} \right] \frac{\sigma_{2k}^2}{2} = O \left( \sigma^2 \lim_{n \to \infty} \sum_{k=1}^{n} (2k)^{-\sigma^2/2g} \right)
\]
\[
= O \left( \sigma^2 2^{-\sigma^2/2g} \lim_{n \to \infty} \sum_{k=1}^{n} k^{-\sigma^2/2g} \right)
\]
\[
= O(\sigma^2 2^{-\sigma^2/2g}),
\]
and this tends to zero as \( \sigma^2 \) tends to infinity. Hence, for \( \rho^2 \simeq 0 \) the sum in (A.6) will be close to zero by continuity. For large enough \( \sigma^2/2 \gg g \), the log-log slope \( \sigma^2/2g \) of the distribution becomes arbitrarily steep, so we have \( \mathbb{E}[X_{[1]}(t)/X_{[n]}(t)] \simeq 1 \). In this case, for \( \rho^2 \simeq 2\sigma^2 \),
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{X_{[n]}(t)}{X_{[n]}(t)} \right] \frac{\sigma_{2k}^2}{2} \simeq \frac{\sigma^4}{2} = \frac{\rho^2}{2} \simeq \sigma^2 > g.
\]
By continuity, for some \( \rho^2 \in (0, 2\sigma^2) \), (A.6) will hold, so the family will be conservative. Hence, for that value of \( \rho^2 \) the family will be conservative and complete, but it is neither Zipfian nor quasi-Zipfian.

**Example A.2.** A simple first-order family that is conservative and complete but not quasi-Zipfian. Consider the simple first-order family with \( g = 1/2, \sigma_1^2 = .64 \), and \( \sigma_k^2 = 4 \) for \( k > 1 \). Using (3.3), we can calculate numerically that for \( n = 100 \),
\[
\frac{X_{[1]}(t)}{X_{[n]}} \simeq 0.8946, \quad \frac{X_{[2]}(t)}{X_{[n]}} \simeq 0.0879, \quad \frac{X_{[3]}(t)}{X_{[n]}} \simeq 0.0119,
\]
and
\[
\sum_{k=1}^{n} \mathbb{E} \left[ \frac{X_{[n]}(t)}{X_{[n]}(t)} \right] \frac{\sigma_k^2}{2g} \simeq 0.9939821.
\]
The series (A.8) converges rapidly, so this sum remains almost constant as \( n \to \infty \). If we replace \( \sigma_1^2 = .64 \) by \( \sigma_1^2 = .65 \) and leave the other parameters unchanged, then
\[
\frac{X_{[1]}(t)}{X_{[n]}} \simeq 0.8951, \quad \frac{X_{[2]}(t)}{X_{[n]}} \simeq 0.0875, \quad \frac{X_{[3]}(t)}{X_{[n]}} \simeq 0.0118,
\]
and
\[
\sum_{k=1}^{n} \mathbb{E} \left[ \frac{X_{[n]}(t)}{X_{[n]}(t)} \right] \frac{\sigma_k^2}{2g} \simeq 1.001353.
\]
By continuity, for some \( \sigma_1^2 \in (.64, .65) \) we shall have
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \mathbb{E} \left[ \frac{X_{[n]}(t)}{X_{[n]}(t)} \right] \frac{\sigma_k^2}{2} = g.
\]

We see from (A.7) and (A.9) that the log-log distribution plot for the model satisfying (A.10) will be steeper than \( -1 \) at the top ranks. Furthermore, using the approximation (3.5) and the fact that \( (\sigma_k^2 + \sigma_{k+1}^2)/4g = 4 \) for \( k > 1 \), we conclude that this log-log plot is steeper than \( -1 \) at all ranks. Reasoning similar to (A.2) implies that \( \mathbb{E}[ngX_{[n]}(t)/X_{[n]}(t)] \to 0 \) as \( n \to \infty \), so the family is complete. Finally, completeness and (A.10) together with (A.3) imply that the family is also conservative.
References

Atkinson, A. B., T. Piketty, and E. Saez (2011, March). Top incomes in the long run of history. *Journal of Economic Literature 49*(1), 3–71.

Axtell, R. (2001, September). Zipf distribution of U.S. firm sizes. *Science 293*(5536), 1818–1820.

Bak, P. (1996). *How Nature Works*. New York: Springer-Verlag.

Banner, A., R. Fernholz, and I. Karatzas (2005). Atlas models of equity markets. *Annals of Applied Probability 15*(4), 2296–2330.

Banner, A. and R. Ghomrasni (2008, July). Local times of ranked continuous semimartingales. *Stochastic Processes and their Applications 118*(7), 1244–1253.

Blanchet, T., J. Fournier, and T. Piketty (2017, March). Generalized Pareto curves: Theory and applications. Technical report, World Wealth & Income Database.

Brown, R. (1827). Brownian motion. Unpublished experiment.

Fernholz, E. R. (2002). *Stochastic Portfolio Theory*. New York: Springer-Verlag.

Fernholz, R. (2001). Equity portfolios generated by functions of ranked market weights. *Finance and Stochastics 5*, 469–486.

Fernholz, R. T. (2017). Nonparametric methods and local-time-based estimation for dynamic power law distributions. *Journal of Applied Econometrics 32*(7), 1244–1260.

Fernholz, R. T. and C. Koch (2016, February). Why are big banks getting bigger? *Federal Reserve Bank of Dallas Working Paper 1604*.

Fernholz, R. T. and C. Koch (2017, May). Big banks, idiosyncratic volatility, and systemic risk. *American Economic Review: Papers and Proceedings 107*(5), 603–607.

Gabaix, X. (1999, August). Zipf’s law for cities: An explanation. *Quarterly Journal of Economics 114*(3), 739–767.

Gabaix, X. (2009, 05). Power laws in economics and finance. *Annual Review of Economics 1*(1), 255–294.

Gibrat, R. (1931). *Les Inégalités Économiques*. Paris: Sirey.

Ichiba, T., V. Papathanakos, A. Banner, I. Karatzas, and R. Fernholz (2011). Hybrid Atlas models. *Annals of Applied Probability 21*, 609–644.

Karatzas, I. and S. E. Shreve (1991). *Brownian Motion and Stochastic Calculus*. New York, NY: Springer-Verlag.

Neumark, D., B. Wall, and J. Zhang (2011, February). Do small businesses create more jobs? new evidence for the united states from the national establishment time series. *Review of Economics and Statistics 93*(1), 16–29.

Newman, M. E. J. (2005, September-October). Power laws, Pareto distributions, and Zipf’s law. *Contemporary Physics 46*(5), 323–351.

Simon, H. and C. Bonini (1958). The size distribution of business firms. *American Economic Review 48*, 607–617.

Simon, H. A. (1955, December). On a class of skew distribution functions. *Biometrika 42*(3/4), 425–440.
Soo, K. T. (2005, May). Zipf’s law for cities: A cross-country investigation. Regional Science and Urban Economics 35(3), 239–263.

Steindl, J. (1965). Random Processes and the Growth of Firms. New York, NY: Hafner.

Wikipedia (2018). Zipf’s Law. https://en.wikipedia.org/wiki/Zipf%27s_law

Zipf, G. (1935). The Psychology of Language: An Introduction to Dynamic Philology. Cambridge, MA: M.I.T. Press.
Figure 1: U.S. capital distribution first-order parameters (smoothed): $\sigma_k^2$ (black), $-g_k$ (red).

Figure 2: U.S. capital distribution, 1990–1999 (black). First-order approximation (red).