Controller Reduction via Weighted Interpolation

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Abstract—The important analytical control designs which are based on the state-space model of the linear time-invariant system yield a controller whose order is almost the same as that of the plant model. If a plant is described by a high-order model, the resulting controller cannot be implemented without reducing its order to a practically acceptable value. This is achieved using weighted model order reduction wherein the weights represent a specific closed-loop performance criterion. In this paper, we present a weighted model order reduction algorithm, which is computationally efficient and ensures less weighted error. The algorithm tends to achieve the weighted-$\mathcal{H}_2$ error optimality and guarantee the stability of the reduced-order model, unlike the existing weighted interpolation algorithms. The proposed algorithm is an effective design tool to obtain a lower order controller for large-scale plants in a computationally efficient way. The application of the proposed technique in achieving this objective is also demonstrated on benchmark problems.

Index Terms—Controller reduction, model reduction, optimal, weighted interpolation.

I. INTRODUCTION

The optimal controller design procedures like linear-quadratic-Gaussian (LQG), $\mathcal{H}_2$, and $\mathcal{H}_\infty$ controllers are theoretically well-grounded in guaranteeing the desired closed-loop performance. However, their existence and implementation for large-scale plants are the real challenges. The control law in these analytical procedures is generally computed by solving linear matrix equations (Riccati equations) which encounter several numerical difficulties like ill-conditioning. Hamiltonian too close to the imaginary axis, and excessive memory requirements as the size of the plants becomes high. Moreover, the controller obtained is of approximately the same order as that of the plant, which restricts its practical implementation. These issues are tackled using model order reduction (MOR) algorithms [1]-[5]. In MOR, a reduced-order model (ROM) is sought which accurately approximates the original high-order model while preserving its important characteristics. The MOR reduction algorithms that are used to obtain a reduced-order controller for the original high-order plant tend to preserve the closed-loop characteristics. This necessitates the inclusion of both the plant and controller in the approximation criterion of the MOR algorithms, which mostly makes this problem a weighted MOR criterion. There are two indirect methodologies to obtain a lower-order controller for high-order plants, i.e., plant reduction and controller reduction. The former involves the order reduction of the plant, and the latter involves the order reduction of the controller [1]-[3]. Several algorithms have been developed since the late 1980s for this purpose, and several closed-loop approximation criteria have been considered so far like closed-loop stability, closeness of the closed-loop transfer function, and controller input spectrum; see [1] for a detailed survey.

We now briefly review the plant and controller reduction problems and show how these boil down to the weighted MOR problems. Consider a $n_p^{th}$-order plant $P(s)$ and let $\hat{P}_r(s)$ be its $n_r^{th}$-order ($n_r^{th} << n_p^{th}$) ROM. Let $\hat{K}_r(s)$ be a stabilizing controller for $\hat{P}_r(s)$ with the following closed-loop transfer function

$$\hat{C}_r(s) = \hat{P}_r(s)\hat{K}_r(s)\left[1 + \hat{P}_r(s)\hat{K}_r(s)\right]^{-1}.$$ 

Then, according to the robust stability theorem [6], $\hat{K}_r(s)$ is also a stabilizing controller for $P(s)$ with the following closed-loop transfer function

$$\hat{C}(s) = P(s)\hat{K}_r(s)\left[1 + P(s)\hat{K}_r(s)\right]^{-1}$$

if

$$\left\|\left(P(s) - \hat{P}_r(s)\right)\hat{K}_r(s)\left[1 + \hat{P}_r(s)\hat{K}_r(s)\right]^{-1}\right\|_{\mathcal{H}_\infty} < 1, \tag{1}$$

and $\hat{P}_r(s)$ has the same number of poles in the right half of the s-plane as that of $P(s)$. One may notice that [1] is a weighted MOR criterion with the weight

$$W(s) = K_r(s)\left[1 + P(s)\hat{K}_r(s)\right]^{-1}, \text{ i.e.,}$$

$$\left\|\left(P(s) - \hat{P}_r(s)\right)W(s)\right\|_{\mathcal{H}_\infty} < 1.$$ 

$W(s)$ depends both on $\hat{P}_r(s)$ and $\hat{K}_r(s)$ which are unknown and thus, the plant reduction poses a cart before the horse like situation. In other words, the plant reduction problem requires incorporating both the plant and controller in the approximation criterion to ensure the preservation of the closed-loop characteristics, but the controller is yet unknown. There are some approaches presented in the literature like [2] to remove the dependency on the knowledge of $\hat{P}_r(s)$ and $\hat{K}_r(s)$ in [1] by using the controller design methods wherein the anticipated closed-loop system is approximately known in advance; for instance, in loop shaping design procedures. This approach is criticized for its lack of closed-form solution, inaccuracy, lack of generality, numerical difficulties, and excessive computational cost [1], [3]. Note that the main motivation of the plant reduction (as described in [2]) is to make the control design possible for large-scale plants for which otherwise the solution of large-scale linear matrix equations is required that can be beyond the computational capability of the controller design package. Recently, several low-rank approximation algorithms are developed for solving linear matrix equations which have expanded the solvability of the optimal controllers for linear time-invariant (LTI) systems.
and have further belittled the necessity of the plant reduction. Another approach that is widely used is to obtain a moderate order ROM of $P(s)$ first, which is enough to avoid the numerical difficulties associated with the controller design but still maintains high accuracy. The accuracy in the desired frequency range is further enhanced using the frequency weights which emphasize the frequency region of interest. A compact controller can later be obtained in the controller reduction stage; see for instance [11], [13].

Let $K(s)$ be a stabilizing controller for $P(s)$ with the following closed-loop transfer function

$$C(s) = P(s)K(s)[I + P(s)K(s)]^{-1}.$$ 

Let $K_r(s)$ be the $r^{th}$-order ROM of $K(s)$. Then, according to the robust stability theorem [6], $K_r(s)$ is also a stabilizing controller for $P(s)$ with the following closed-loop transfer function

$$C(s) = P(s)K_r(s)[I + P(s)K_r(s)]^{-1}$$

if

$$\| [K(s) - K_r(s)]P(s)[I + K(s)P(s)]^{-1} \|_{\mathcal{H}_\infty} < 1,$$ 

and $K_r(s)$ has the same number of poles in the right half of the $s$-plane as that of $K(s)$. Again, it may be noticed that (2) is a weighted MOR problem with the weight $W(s) = P(s)[I + K(s)P(s)]^{-1}$, i.e.,

$$\| [K(s) - K_r(s)]W(s) \|_{\mathcal{H}_\infty} < 1.$$ 

Unlike plant reduction, $W(s)$ is known in controller reduction, and hence, it can easily be performed using weighted MOR techniques. Controller reduction is a well-researched and an important problem which has become an important last step of the optimal controller designs like LQG, $\mathcal{H}_2$, and $\mathcal{H}_\infty$ controllers [14]. This is due to the compactness of the ROM, which weighted MOR algorithms offer at a good accuracy.

Balanced truncation (BT) is an important MOR technique for which several algorithms and extensions have been presented over the last three decades [15]. BT is known for its good accuracy, stability preservation, and error bound [16]. It generally produces a suboptimal ROM in terms of $\mathcal{H}_\infty$-error. However, BT can only be applied to models of moderate size due to its excessive computational cost. There exist some generalizations to extend its applicability to large-scale systems by replacing the large-scale Lyapunov equations with their low-rank approximations [17]. Enns generalized BT to weighted BT (WBT) for the plant and controller reduction problems [16]. Several other extensions of WBT are also reported in the literature like [18]-[22] which are surveyed in [23]. The optimal solution for the weighted MOR problem in $\mathcal{H}_\infty$-error sense is hard to find, and the available algorithms based on the solution of large-scale linear matrix inequalities (LMIs) can only be applied to small-scale systems [24].

Moment matching is another important class of MOR techniques which interpolate the transfer function of the original system at some selected frequency points. Unlike BT and its generalizations, moment matching techniques are computationally efficient due to their Krylov subspace-based implementation [25]. Moment matching has been successfully applied to the controller reduction problems [26], [27]. The solution for $\mathcal{H}_2$-optimal MOR problem can be found in a computationally efficient way using Krylov subspace-based moment matching algorithm, i.e., iterative rational Krylov algorithm (IRKA) [28]. Although, most of the closed-loop approximation criteria are specified in terms of $\mathcal{H}_\infty$-error, ensuring $\mathcal{H}_2$-optimality generally leads to high-fidelity ROM in terms of $\mathcal{H}_\infty$-error as well [29]. IRKA is generalized heuristically to the weighted MOR scenario in [30] for single-input single-output (SISO) systems. In [31], a near-optimal weighted interpolation (NOWI) algorithm is presented which generates a ROM which is nearly optimal for the optimal weighted-$\mathcal{H}_2$ MOR problem $\|K(s) - K_r(s)\|_{\mathcal{H}_{W_2}}$ where the weighted-$\mathcal{H}_2$ norm is defined as $\|K(s) - K_r(s)\|_{\mathcal{H}_{W_2}} = \| [K(s) - K_r(s)]W(s) \|_{\mathcal{H}_2}$. NOWI avoids the computation of two large-scale Lyapunov equations required in WBT [16], and thus, it is computationally less expensive. However, it can become computationally expensive if it does not converge in a few iterations when $W(s)$ is of a high order. The order of $W(s)$ in controller reduction is mostly the sum of the orders of $P(s)$ and $K(s)$.

In this paper, we present an iteration-free moment matching algorithm for the weighted-$\mathcal{H}_2$ MOR problem. The algorithm satisfies a subset of the optimality conditions for the problem $\|K(s) - K_r(s)\|_{\mathcal{H}_{W_2}}^2$ and also gives the freedom to the user to place the poles of $K_r(s)$ at the desired location which may include the poles of $K(s)$ and $W(s)$. The preservation of the stability of the ROM is thus natural. We name this algorithm as “pseudo-optimal weighted interpolation” (POWI). Like NOWI, the main computational effort still depends on the order of $W(s)$; however, POWI does not have an iterative framework which saves a lot of computational time. POWI is based on the recent approaches of parameterizing the ROM to ensure the desired properties [32], [33]. The free parameter in these approaches is used to enforce a subset of the optimality conditions on $K_r(s)$ for the problem $\|K(s) - K_r(s)\|_{\mathcal{H}_{W_2}}^2$. The application and usefulness of POWI is demonstrated on two benchmark systems for which reduced controllers are obtained. The numerical results confirm the theory developed in the paper.

II. EXISTING TECHNIQUES

The important mathematical notations used throughout the text are tabulated in Table I.

| Notation | Meaning |
|----------|---------|
| $\| \cdot \|_{\cdot}$ | Hermitian of the matrix |
| $\Lambda_0(\cdot)$ | Eigenvalues of the matrix |
| $\text{ran}(\cdot)$ | Range of the matrix |
| $\text{orth}(\{ \cdot \})$ | Orthogonal basis for the range of the matrix |
| $\text{span}(\{ \cdot \})$ | Span of the set of $r$ vectors |

A. WBT [16]

A state-space realization is called a balanced realization if its states are equally controllable and observable. The
NOWI is the weighted-generalization of IRKA. Let us define $\mathcal{F}[H(s)]$ and $\mathcal{F}[H_r(s)]$ as

$$\mathcal{F}[K(s)] = C_h(sI - A_h)^{-1}B\mathcal{F},$$

$$\mathcal{F}[K_r(s)] = C_h(sI - A_h)^{-1}B\mathcal{F},$$

where

$$H_r(s) = K_r(s)W(s) = C_h(sI - A_h)^{-1}Bh_r + Dh_r,$$

$$B\mathcal{F} = \begin{bmatrix} Z_k C_w^T + B_k D_w D_w^T \\ P_w C_w^T + B_w D_w^T \end{bmatrix},$$

$$A_{hr} = \begin{bmatrix} \hat{A}_r & \hat{B}_r C_w \\ 0 & A_w \end{bmatrix},$$

$$B_{hr} = \begin{bmatrix} \hat{B}_r D_w \\ B_w \end{bmatrix},$$

$$C_{hr} = \begin{bmatrix} \hat{C}_r & \hat{D}_r C_w \\ D_h = \hat{D}_r D_w \end{bmatrix},$$

$$B_{\mathcal{F}} = \begin{bmatrix} \tilde{Z}_r C_w^T + \hat{B}_r D_w D_w^T \\ P_w C_w^T + B_w D_w^T \end{bmatrix},$$

and $\tilde{Z}_r$ solves the following Sylvester equation

$$\hat{A}_r \tilde{Z}_r + \tilde{Z}_r \hat{A}_r^T + \hat{B}_r (C_w P_w + D_w B_w^T) = 0. \quad (7)$$

Let $\tilde{X}, \tilde{P}_r, \tilde{Q}_r,$ and $\tilde{Y}$ solve following Lyapunov equations

$$A_h \tilde{X} + \tilde{X} A_h^T + B_{\mathcal{F}} \tilde{B}_r^T = 0, \quad (8)$$

$$\hat{A}_r \tilde{P}_r + \tilde{P}_r \hat{A}_r^T + \tilde{B}_r (0 \ 0 \ C_w) \tilde{X} + \tilde{X} \tilde{X}^T \begin{bmatrix} 0 & 0 \\ C_w^T & -\hat{B}_r D_w D_w^T \hat{B}_r^T \end{bmatrix} = 0 \quad (9)$$

$$\hat{A}_r^T \tilde{Q}_r + \tilde{Q}_r \hat{A}_r^T + \tilde{C}_r^T \tilde{C}_r = 0 \quad (10)$$

$$A_h^T \tilde{Y} + \tilde{Y} A_h - \begin{bmatrix} C_w^T \\ (D_h - \hat{D}_r C_w) \end{bmatrix} \hat{C}_r^T + \begin{bmatrix} 0 \\ C_w^T \end{bmatrix} \tilde{B}_r^T \tilde{Q}_r = 0. \quad (11)$$

$K_r(s)$ is a local optimum for the problem $\|K(s) - K_r(s)\|_{\mathcal{H}_2}^2$ if the following first-order optimality conditions (derived by Halevi [35]) are met

$$\tilde{Y}^T \tilde{X} + \tilde{Q}_r \tilde{P}_r = 0 \quad (12)$$

$$C_h \hat{X} - \hat{C}_r \hat{P}_r - \hat{D}_r (0 \ 0 \ C_w) \hat{X} = 0 \quad (13)$$

$$\tilde{Y}^T B_{\mathcal{F}} + \tilde{Q}_r (\tilde{B}_r D_w D_w^T + \tilde{X} (0 \ 0 \ C_w^T)) = 0 \quad (14)$$

$$\tilde{C}_r \hat{X}^T \begin{bmatrix} 0 \\ C_w^T \end{bmatrix} M - C_k Z_k C_k^T M = 0 \quad (15)$$

where $M$ is a basis for the null space of $D_w^T$.

Let $K_r(s)$ has only simple poles and it can be represented in its pole-residue form as

$$K_r(s) = \sum_{i=1}^{r} \frac{c_i b_i^T}{s - \lambda_i} + \hat{D}_r. \quad (16)$$

**B. NOWI** [37]

IRKA [28] generates a (local) $\mathcal{H}_2$-optimal ROM which interpolates the original system at the mirror images of its poles in the direction of their associated right and left residuals.
Then, $K_r(s)$ satisfies the first-order optimality condition \((12)-(15)\) if
\[
\mathcal{F}[K(-\lambda_i)]b_i = \mathcal{F}[K_r(-\lambda_i)]b_i
\]
\[
c_t^T \mathcal{F}[K(-\lambda_i)] = c_t^T \mathcal{F}[K_r(-\lambda_i)]
\]
\[
c_t^T \mathcal{F}'[K(-\lambda_i)]b_i = c_t^T \mathcal{F}'[K_r(-\lambda_i)]b_i
\]
\[
\tilde{D}_r = C_k(Z_k - \tilde{V}_r \tilde{Z}_r)C_k^T M = (M^T C_w P_w C_w^T M)^{-1} M^T.
\]
The reduction subspaces $\tilde{V}_r$ and $\tilde{W}_r$ which satisfy the bitangential Hermite interpolation conditions \((17)-(19)\) for $\mathcal{F}[K(s)]$ are computed as
\[
\text{Ran} \left[ \begin{matrix} \tilde{V}_r^{(a)} \\ \tilde{V}_r^{(b)} \end{matrix} \right] = \text{span} \left\{ (-\lambda_i I - A_h)^{-1} B \right\}_{i=1}^{r}
\]
\[
\text{Ran} \left[ \begin{matrix} \tilde{W}_r^{(a)} \\ \tilde{W}_r^{(b)} \end{matrix} \right] = \text{span} \left\{ (-\lambda_i I - A_h^T)^{-1} C^T \right\}_{i=1}^{r}
\]
where $\text{Ran}(\tilde{V}_r) \supset \text{Ran}(\tilde{V}_r^{(a)})$, $\text{Ran}(\tilde{W}_r) \supset \text{Ran}(\tilde{W}_r^{(a)})$, and $\tilde{W}_r^T \tilde{V}_r = I$. The poles $\lambda_i$, right residues $b_i$, and left residues $c_i$ of $K_r(s)$ are not known a priori. Therefore, an iterative algorithm similar to IRKA is proposed in \cite{11} which starts with a random set of interpolation points and tangential directions, and after each iteration, the interpolation points are updated as the mirror images of the poles of $K_r(s)$, and the tangential directions are updated as the respective right and left residues associated with these poles. At convergence, $K_r(s)$ which satisfies \((17)-(19)\) is obtained. The choice of $\tilde{D}_r$ according to \((20)\) can affect the interpolation conditions \((17)-(19)\) because $C_h$ depends on $\tilde{D}_r$. If $Z_k = \tilde{V}_r \tilde{Z}_r$, the optimality conditions including that on $\tilde{D}_r$ are satisfied exactly, however, it is generally not the case. Thus, NOWI generates a nearly (local) optimal ROM.

### III. MAIN WORK

In this section, we present an iteration-free algorithm which generates a ROM $K_r(s)$ which satisfies a subset of the first-order optimality conditions for the problem $\|K(s) - K_r(s)\|_{H_\infty}^2$ with a computational cost which is a fraction of that of NOWI. Note that the closed-loop stability criterion \((2)\) is defined in terms of $H_\infty$-norm and not $H_2$-norm. Thus, achieving (local) $H_\infty$-optimality is merely a theoretical interest, and the actual aim is to reduce $\|K(s) - K_r(s)\|_{H_\infty}$, however, $H_2$-MOR techniques generally also ensure good $H_\infty$ error characteristics, as shown in \cite{29}. Therefore, we temporarily shift our focus on achieving low $H_2$-norm error in this section as it will indirectly ensure less $H_\infty$-norm error, which is the main goal from a controller reduction perspective.

#### A. POWI

We first consider the case when $D_k = 0$. Let us define $\tilde{V}_r$, $S_h$ and $L_h$ as $\tilde{V}_r = [\tilde{V}_1^T \tilde{V}_2^T]^T$, $S_h = \text{diag}(\sigma_1 \cdots \sigma_r)$, and $L_h = [\tilde{r}_1 \cdots \tilde{r}_r]$, respectively where $\sigma_i$ are the interpolation points in the respective tangential directions $\tilde{r}_i$. Suppose $\tilde{V}$ satisfies the following Sylvester equation
\[
A_h \tilde{V} - S_h \tilde{V} - B \beta L_h = 0
\]
\[
\begin{bmatrix}
A_k & B_k C_w \\
0 & A_w
\end{bmatrix}
\begin{bmatrix}
\tilde{V}_1 \\
\tilde{V}_2
\end{bmatrix}
= \begin{bmatrix}
\tilde{V}_1 \\
\tilde{V}_2
\end{bmatrix}
S_h
\]
\[
- Z_k C_w^T + B_k D_w C_w^T + B_k D_w C_w^T L_h = 0.
\]
Then, $\tilde{V}_2$ and $\tilde{V}_1$ satisfy the following Sylvester equations
\[
A_w \tilde{V}_2 - \tilde{V}_2 S_h - (P_w C_w^T + B_w D_w^T)L_h = 0
\]
\[
A_k \tilde{V}_1 - \tilde{V}_1 S_h - (Z_k C_w^T + B_k D_w^T) L_h + B_k C_w \tilde{V}_2 = 0.
\]
Suppose that the pair $(-S_h, L_h)$ is observable and has a positive-semidefinite weighted observability Gramian $Q_{sw}$ which solves the following Lyapunov equation
\[
-S_h Q_{sw} - Q_{sw} S_h + L_{sw}^T L_{sw} = 0
\]
where
\[
L_{w_1} = [L_h^* \tilde{Z}_B B_w L_h^* D_w^*],
\]
\[
L_{w_2} = [\tilde{Z}_B B_w L_h^* L_h^* D_w^*],
\]
and $\tilde{Z}_s$ solves the following Sylvester equation
\[
-S_h^T \tilde{Z}_s + \tilde{Z}_s A_w + L_h^* (B_w^T Q_s + D_w C_w) = 0.
\]
$Q_{sw}$ is the observability Gramian of the pair $(A_w, C_w)$ which solves the following Lyapunov equation
\[
A_{sw}^T Q_{sw} + Q_{sw} A_w + C_{sw}^T C_w = 0.
\]
The state-space matrices of $K_r(s)$ in POWI are calculated as the following
\[
\tilde{A}_r = -Q_{sw}^T S_h^T Q_{sw}, \quad \tilde{B}_r = -Q_{sw}^T L_h^*,
\]
\[
\tilde{C}_r = C_k \tilde{V}_1, \quad \tilde{D}_r = 0.
\]

**Theorem:** If $\tilde{A}_r$, $\tilde{B}_r$, $\tilde{C}_r$, and $\tilde{D}_r$ are computed as in equation \((32)\), the following statements are true:

(i) The poles of $K_r(s)$ are at the mirror images of the interpolation points.

(ii) $\tilde{r}_1$ is the right residual of the pole $-\sigma_1$ of $K_r(s)$.

(iii) $Q_{sw}^T$ is the weighted controllability Gramian of the pair $(\tilde{A}_r, \tilde{B}_r)$.

(iv) $K_r(s)$ satisfies the optimality condition \((13)\).

(v) $\mathcal{F}[K(-\lambda_i)]b_i = \mathcal{F}[K_r(-\lambda_i)]b_i$.

(vi) $\text{Ran}(\tilde{V}_1) \supset \text{Ran}(\tilde{V}_r^{(a)})$.

**Proof:**

(i) Since $\tilde{A}_r = -Q_{sw}^T S_h^T Q_{sw}$, $\lambda_i [\tilde{A}_r] = \lambda_i [-S_h]$.

Thus, $K_r(s)$ has poles at $-\sigma_i$ (note that the complex poles are in conjugate pairs).

(ii) $(-Q_{sw})^{-1}(-S_h)(-Q_{sw})$ is the spectral factorization of $\tilde{A}_r$. Also, $\tilde{B}_r = (-Q_{sw})^{-1} [\tilde{r}_1 \cdots \tilde{r}_r]$. Thus, $\tilde{r}_1$ is the right-residuals of the poles $-\sigma_i$.

(iii) The weighted-controllability Gramian $\tilde{P}_{rw}$ of the pair $(\tilde{A}_r, \tilde{B}_r)$ satisfies the following Lyapunov equation
\[
\tilde{A}_r \tilde{P}_{rw} + \tilde{P}_{rw} \tilde{A}_r^* + \tilde{B}_1 \tilde{B}_2^* = 0
\]
where
\[
\tilde{B}_1 = [\tilde{B}_r \tilde{Z}_r C_w^T \tilde{B}_r D_w] \text{ and } \\
\tilde{B}_2 = [\tilde{Z}_r C_w^T \tilde{B}_r \tilde{B}_r D_w].
\]
By putting the values of \(\hat{A}_r\) and \(\hat{B}_r\), equation (33) becomes
\[
-Q_{sw} h^2 S_h Q_{sw} \hat{P}_r - Q_{sw} \hat{P}_r Q_{sw} S_h - Q_{sw} L_h \tilde{Z}_r C_w = 0.
\]
Multiplying by \(Q_{sw}\) from the left and right yields
\[
-Q_{sw} \tilde{Z}_r C_w^T L_h + L_h D_w D_w^T L_h = 0.
\]
Q_{sw}, \(\tilde{Z}_r\), Q_{sw}, \(\tilde{Z}_r\), and \(P_w\) can be written in integral form as
\[
Q_{sw} = \int_0^\infty e^{-S_h^r T} L_w^w e^{-S_h^r} d\tau,
\]
\[
\hat{Z}_r = \int_0^\infty e^{-S_h^r T} (B^T Q_{sw} + D_w B_w^T) e^{A_w^r} d\tau,
\]
\[
Q_w = \int_0^\infty e^{A_w^r} C_w^T C_w e^{A_w^r} d\tau,
\]
\[
\hat{Z}_r = \int_0^\infty e^{A_w^r} C_r \hat{E} (C_w P_w + D_w B_w^T) e^{A_w^r} d\tau,
\]
\[
P_w = \int_0^\infty e^{A_w^r} B_w B_w^T e^{A_w^r} d\tau.
\]
By putting all these integrals into the product \(-Q_{sw} \tilde{Z}_r C_w^T\) and after some tedious simplifications, one reaches the following equality
\[
-Q_{sw} \tilde{Z}_r C_w^T = \tilde{Z}_s B_w.
\]
By putting (40) in (34), equation (34) becomes
\[
-S_h Q_{sw} \hat{P}_r Q_{sw} - Q_{sw} \hat{P}_r Q_{sw} S_h + L_h B_w^T \tilde{Z}_s + \tilde{Z}_s B_w L_h + L_h D_w D_w^T L_h = 0.
\]
Due to uniqueness, \(Q_{sw} \hat{P}_r Q_{sw} = Q_{sw}, Q_{sw} \hat{P}_r = I,\) and \(\hat{P}_r = Q_{sw}^{-1}\.\)
(iv) Consider the following Lyapunov equation
\[
A_h \hat{V} \hat{P}_r + \hat{V} \hat{P}_r A_r + B_r B_r^T = (\hat{V} S_h + B_r L_h) \hat{P}_r - \hat{V} \hat{P}_r Q_{sw} S_h Q_{sw} - B_r L_h Q_{sw} = 0.
\]
Due to uniqueness, \(\hat{V} \hat{P}_r = \hat{X}\). Let us partition \(\hat{X}\) as \(\hat{X} = [\hat{X}^T_{n_r \times r} \hat{X}^T_{n_w \times r}]^T\). Then, equation (41) can be written as
\[
[A_k B_k C_w [\hat{X}^T_{n_r \times r} + [\hat{X}^T_{n_r \times r} \hat{X}^T_{n_w \times r} A_r^T + Z_h C_w^T + B_k D_w D_w^T]
\]
\[
= 0. \quad \text{[42]}
\]
It can be noted from (42) that \(\hat{X}^T_{n_r \times r}\) is actually \(\tilde{Z}_r^T\), i.e., \(\hat{X}^T_{n_r \times r} = \tilde{Z}_r^T\), and thus, \(\hat{P}_r = \hat{P}_r \hat{P}_r\). Since \(\hat{V} \hat{P}_r = \hat{X}\), \(C_h V P_r = C_h \hat{X}\) and \(C_r \hat{P}_r = C_h \hat{X}\) and thus, \(C_h \hat{X} - C_r \hat{P}_r - \hat{D}_r [0 \ C_w] \hat{X} = 0\.\)
(v) As shown in [31], conditions [13] and [17] are equivalent.

Remark 1: A nonzero \(D_k\) can affect the interpolation condition proved in the theorem because \(C_h\) depends on \(D_k\). Thus, the optimality condition [13] is satisfied nearly when \(D_k\) is nonzero. In this case, we suggest choosing \(\hat{D}_r\) according to equation (20), which satisfies the optimality condition [15]. This is also reasonable from a structure preservation aspect as we have a proper ROM for the proper original model and vice versa.

Remark 2: When \(W(s) = I\), POWI reduces to pseudo-optimal rational Krylov algorithm [33].

Remark 3: Unlike NOWI, we know the poles and their associated right residuals of the ROM generated by POWI a priori. This information can thus be used for a judicious choice of interpolation points and the tangential directions to achieve less weighted-H2 error. For instance, the interpolation points can be chosen as the mirror images of the poles of \(K(s), W(s), \) or \(C(s)\) which have large associated residues to achieve less weighted \(H_2\)-error [27, 30].

B. Algorithmic Aspects

So far POWI is presented for the conceptual clarity and not for the actual implementation. Therefore, the ROM and the reduction subspace are allowed to be complex. In practice, however, it is desirable to obtain a real ROM for the real original transfer function. The real reduction subspace \(\hat{V}_h\) in POWI can be obtained by using any rational Krylov subspace method such that
\[
\text{Ran}(\hat{V}_h) = \text{span} \{ (\sigma I - A_h)^{-1} B_r \hat{f}_i \}.
\]
The next step is to compute the matrices of the Sylvester equation which \(\hat{V}_h\) satisfies. This can be done in a few simple steps. Choose any output reduction subspace \(\hat{W}_h\), for instance, \(\hat{W}_h = \hat{V}_h\). Then, compute the following matrices
\[
\hat{E} = \hat{W}_h^T \hat{V}_h, \quad \hat{A} = \hat{W}_h^T A_h \hat{V}_h, \quad \hat{B} = \hat{W}_h^T B_r.
\]
\[
\hat{L}_h = (B_r^T B_r)^{-1} B_r (A_h \hat{V}_h - \hat{V}_h \hat{E})^{-1} \hat{A}.
\]
\[
\hat{S}_h = \hat{E}^{-1} \left( \hat{A} - \hat{B} \hat{L}_h \right).
\]
Then, \(\hat{V}_h\) solves the following Sylvester equation
\[
A_h \hat{V}_h - \hat{V}_h \hat{S}_h - B_r \hat{L}_h = 0
\]
where \(\sigma_i\) are the eigenvalues of \(\hat{S}_h\). Next, \(\hat{Q}_{sw}\) can be computed from the following Lyapunov equation
\[
-S_h \hat{Q}_{sw} - \hat{Q}_{sw} \hat{S}_h + \hat{L}_w^T \hat{L}_w = 0
\]
where
\[
\hat{L}_w^T = \left[ \hat{L}_{w_1}^T \hat{Z}_h B_w \hat{L}_{w_2}^T D_w^T \right],
\]
\[
\hat{L}_{w_2} = [\hat{Z}_h B_w \hat{L}_{w_2}^T D_w^T],
\]
and \(\hat{Z}_h\) solves the following Sylvester equation
\[
-S_h \hat{Z}_h + \hat{Z}_h A_w + \hat{L}_h (B_w^T Q_{sw} + D_w C_w) = 0.
\]
Now partition $\hat{V}_h$ as $\hat{V}_h = \begin{bmatrix} \hat{V}_{n \times r}^T & \hat{V}_{n \times r}^T \end{bmatrix}^T$. Finally, the ROM is obtained as

\[
\begin{align*}
\hat{A}_r &= -Q_s^{-1}S_h Q_{sw}, \\
\hat{B}_r &= -Q_s^{-1}L_h^T, \\
\hat{C}_r &= CV_{n \times r}. \\
\end{align*}
\]  

(53)

$\hat{D}_r = 0$ when $D_k = 0$, and $\hat{D}_r$ is as in (20) when $D_k \neq 0$.

IV. Numerical Results

In this section, we apply POWI on two controller reduction problems and compare its performance with that of NOWI and WBT. The plants are taken from the benchmark collection of [36], and the controllers are designed using \textit{"lqg"} command of MATLABs Robust Control Toolbox [37]. The accuracy in the moment matching based methods rely heavily on the selection of interpolation points and the tangential direction. Even if the guidelines on the selection of the interpolation points in [27], [30] to obtain less weighted error are followed, there is no guarantee that the actual ROM yielded by NOWI does satisfy these tangential interpolatory conditions due to its iterative nature. Therefore, for a fair comparison, we initialize NOWI randomly, and we use the final interpolation points and the tangential directions generated by NOWI upon convergence for POWI. This also gives a better idea of the effect of only satisfying a subset of the optimality conditions instead of the whole set. The maximum number iterations in NOWI is set to 1000. If NOWI does not converge in 1000 iterations, the ROM is the one yielded in the 1000th iteration. We also study the effect of using $D$-matrix of the ROM generated by NOWI in POWI. Since the stability of ROM is not guaranteed in NOWI, we reject any ROM generated by NOWI which is unstable and restart NOWI with the different interpolation points and the tangential directions. The Lyapunov and Sylvester equations (5)-(7) can be approximated by their low-rank approximations in a large-scale setting to significantly reduce the computational cost. However, we solve all these equations exactly using MATLABs \textit{lyap} command [37] to effectively observe the true computational cost of the algorithms. All the experiments are performed on a laptop with Intel Core M-5Y10c processor, 8GB of RAM, and Windows 8 operating system.

CD Player: This is a 120th order MIMO model from the benchmark collection of [36] with 2-inputs and 2-outputs. A 120th order LQG controller is designed for this system using \textit{lqg} command of MATLABs Robust Control Toolbox by specifying the weighting matrices as identity. This results in a 240th order weight. The controller is reduced using WBT, NOWI, and POWI. Figure 1 shows the weighted $H_\infty$-norm error of the reduced controllers. The strength of the weighted MOR techniques for controller reduction can be appreciated from Figure 1 as a fairly compact controller which satisfies the closed-loop stability condition (2) can be achieved. It can further be noted in Figure 1 that POWI and NOWI outperformed WBT from to 1 - 4th order. Figure 2 shows that WBT maintained superior accuracy than POWI and NOWI for the order greater than 4. It can also be noticed from Figure 2 that POWI compares well with NOWI even though it only satisfies a subset of the optimality conditions which NOWI nearly satisfies. The effect of adding $\hat{D}_r$ is almost indistinguishable in this case. NOWI converged quickly in this experiment, and there is only a slight difference in the computational time consumed by three techniques. Therefore, we do not plot the computational time consumed by WBT, NOWI, and POWI in this experiment.

International Space Station: This is a 270th order MIMO model from the benchmark collection of [36] with 3-inputs and 3-outputs. A 270th order LQG controller is designed for this system using \textit{lqg} command of MATLABs Robust Control Toolbox by specifying the weighting matrices as identity. This results in a 540th order weight. The controller is reduced using WBT, NOWI, and POWI. The weighted $H_\infty$-norm error is plotted in Figure 3. It can be noted that POWI achieves almost the same accuracy as NOWI except for the 17th order. The effect of $\hat{D}_r$ is also clear in this experiment specially for 1 - 4th order wherein POWI outperformed both WBT and NOWI. As the number of inputs and outputs increases, the convergence of IRKA and IRKA-type algorithms slows down. It can be seen from Figure 3 that POWI cannot converge most of the time in this experiment. The peaks in Figure 3 show the instances when NOWI fails to converge within 1000 iterations. The dips in Figure 4 represent the instances...
when NOWI converged quickly. One may notice that NOWI is almost as efficient as POWI for the 2\textsuperscript{nd} and 4\textsuperscript{th} order when it converged quickly. It can be seen in Figure 3 that when NOWI failed to converge quickly, it is even more computational than WBT.

V. CONCLUSION

We present an iteration-free weighted moment matching technique (POWI) which satisfies a subset of the optimality conditions for the weighted-$H_2$ MOR problem. The simulation results reveal that POWI also ensures less weighted-$H_\infty$ error, and it requires less computations than the existing techniques. POWI also allows the user to place the poles of the ROM at a specified location in the $s$-plane. In conclusion, POWI can effectively be used to obtain a lower order controller for the high order plants.

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