Reconfiguring 10-colourings of planar graphs

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Abstract

Let $k \geq 1$ be an integer. The reconfiguration graph $R_k(G)$ of the $k$-colourings of a graph $G$ has as vertex set the set of all possible $k$-colourings of $G$ and two colourings are adjacent if they differ on exactly one vertex.

A conjecture of Cereceda from 2007 asserts that for every integer $\ell \geq k + 2$ and $k$-degenerate graph $G$ on $n$ vertices, $R_\ell(G)$ has diameter $O(n^2)$. The conjecture has been verified only when $\ell \geq 2k + 1$. We give a simple proof that if $G$ is a planar graph on $n$ vertices, then $R_{10}(G)$ has diameter at most $n^2$. Since planar graphs are 5-degenerate, this affirms Cereceda’s conjecture for planar graphs in the case $\ell = 2k$.

Let $k \geq 1$ be an integer. The reconfiguration graph $R_k(G)$ of the $k$-colourings of a graph $G$ has as vertex set the set of all possible $k$-colourings of $G$ and two colourings are adjacent if they differ on the colour of exactly one vertex of $G$. A list assignment of a graph is a function $L$ that assigns to each vertex $v$ a list $L(v)$ of colours. The graph $G$ is $L$-colourable if it has a proper colouring $f$ such that $f(v) \in L(v)$ for each vertex $v$ of $G$.

For a positive integer $d$, a graph $G$ is $d$-degenerate if every subgraph of $G$ contains a vertex of degree at most $d$. Expressed in another way, $G$ is $d$-degenerate if there there exists an ordering $v_1, \ldots, v_n$ of the vertices in $G$ such that each $v_i$ has at most $d$ neighbours $v_j$ with $j < i$.

Reconfiguration problems have received much attention in the past decade; we refer the reader to the surveys by van den Heuvel [12] and Nishimura [9].
In this note, we are concerned with a conjecture of Cereceda [3] from 2007 which asserts that for every integer \( \ell \geq k + 2 \) and \( k \)-degenerate graph \( G \) on \( n \) vertices, \( R_{\ell}(G) \) has diameter \( O(n^2) \). Cereceda [3] verified the conjecture whenever \( \ell \geq 2k + 1 \) but the conjecture remains open for every other value \( 2k \geq \ell \geq k + 2 \). It is also known to hold for graphs of bounded tree-width [1] (the claimed shorter proof in [6] yields instead a bound of \( O(tn^2) \) on the diameter, where \( t \) is the tree-width of the graph under consideration) as well as \((\Delta - 1)\)-degenerate graphs [8], where \( \Delta \) is the maximum degree of the graph under consideration. Our aim in this note is to address the conjecture for planar graphs in the following theorem.

**Theorem 1.** For every planar graph \( G \) on \( n \) vertices, \( R_{10}(G) \) has diameter at most \( n^2 \).

Since planar graphs are 5-degenerate, Theorem 1 affirms Cereceda’s conjecture for planar graphs in the case \( \ell = 2k \). In all other cases, some partial results are known. Given a planar graph \( G \) on \( n \) vertices, it is shown in [2] that \( R_\ell(G) \) has diameter \( O(n^c) \) for each \( \ell \geq 8 \) and some (large) positive constant \( c \) (see [7] for a short proof of this result with a weaker bound on \( c \)) while in [5] it is shown that \( R_7(G) \) has diameter \( 2^{O(\sqrt{n})} \).

Let us note that the novelty of our approach lies, in some sense, on a new trick that essentially reformulates the reconfiguration problem as a list colouring problem. In particular, Theorem 1 will follow as a corollary from the following special case of a famous theorem due to Thomassen [10].

**Theorem 2.** Let \( G \) be a planar graph, and let \( v \) be a vertex of \( G \). Suppose that \( L(u) \) is a list of one colour if \( u = v \) and a list of at least five colours if \( u \in V(G) - \{ v \} \). Then \( G \) is \( L \)-colourable.

**Proof of Theorem 2.** Since \( G \) is 5-degenerate, we can order the vertices of \( G \) as \( v_1, \ldots, v_n \) such that each \( v_i \) has at most five neighbours \( v_j \) with \( j < i \).

Let \( \alpha \) and \( \beta \) be 10-colourings of \( G \), and let \( h \) be the lowest index such that \( \alpha(v_h) \neq \beta(v_h) \). Starting from \( \alpha \), we shall describe a sequence of recolourings such that

- for \( i < h \), \( v_i \) is not recoloured,
- for \( i > h \), \( v_i \) is recoloured at most once, and
- \( v_h \) is recoloured with colour \( \beta(v_h) \).
By repeatedly using such sequences, we can recolour $\alpha$ to $\beta$ by at most $n$ recolourings per vertex and the theorem follows.

To describe the sequence, let $H$ be the graph induced by $S = \{v_h, \ldots, v_n\}$. We start by finding a list assignment $L$ of $H$ as follows:

- $L(v_h) = \{\beta(v_h)\}$, and
- for $i > h$, $L(v_i) = \{1, \ldots, 10\} \setminus \{\alpha(v_j) : (v_i, v_j) \in E(G), j < i\}$.

Applying Theorem 2, we obtain an $L$-colouring $f$ of $H$. We then simply recolour $v_k$ with $f(v_k)$ starting with $v_n$ and working backwards through $S$. (It is possible that $f(v_k) = \alpha(v_k)$ in which case the colour of $v_k$ is unchanged.) Each colouring obtained is proper since $v_n$ has no neighbours coloured $f(v_n)$ and when a vertex $v_k$, $k < n$, is recoloured, its neighbours $v_j$ with $j < k$ do not have colour $f(v_k)$ by definition of $L(v_k)$ nor do its other neighbours $v_j$ with $j > k$ since $f$ is a proper colouring. Given that, at the end of the sequence, $v_h$ is recoloured to colour $\beta(v_h)$, this completes the proof.

It is not difficult to prove the following theorem using the same approach as in the proof of Theorem 1.

**Theorem 3.** Let $k$ and $\ell$ be positive integers, let $G$ be a $k$-degenerate graph on $n$ vertices, and let $v$ be a vertex of $G$. Suppose that $L(u)$ is a list of one colour if $u = v$ and a list of at least $\ell$ colours if $u \in V(G) - \{v\}$. If $G$ is $L$-colourable, then $R_{k+\ell}(G)$ has diameter at most $n^2$.

We state two of possibly other consequences of Theorem 3.

**Corollary 1.** Let $G$ be a planar graph on $n$ vertices and of girth 5. Then $R_6(G)$ has diameter at most $n^2$.

**Proof.** Since planar graphs of girth 5 are 3-degenerate, the result is immediate from Theorem 3 combined with Theorem 2.1 in [11].

**Corollary 2.** Let $k$ be a positive integer, and let $G$ be a $k$-degenerate graph on $n$ vertices. If $k + 1$ is prime, then $R_{2k+1}(G)$ has diameter at most $n^2$.

**Proof.** Combine Theorem 3 with Theorem 6 in [4].
Acknowledgements

The author is grateful to Louis Esperet for pointing out Corollary 2. This work is supported by the Research Council of Norway via the project CLAS-SIS. It was conducted while the author was visiting the Department of Applied Mathematics of the Faculty of Mathematics and Physics at Charles University.

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