CONTROL AND STABILIZATION OF 2 × 2 HYPERBOLIC SYSTEMS ON GRAPHS

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Abstract. We consider 2 × 2 (first order) hyperbolic systems on networks subject to general transmission conditions and to some dissipative boundary conditions on some external vertices. We find sufficient but natural conditions on these transmission conditions that guarantee the exponential decay of the full system on graphs with dissipative conditions at all except one external vertices. This result is obtained with the help of a perturbation argument and an observability estimate for an associated wave type equation. An exact controllability result is also deduced.

1. Introduction. The 2 × 2 hyperbolic system

\[
\begin{align*}
  p_t + Lq_x + Gp &= 0 & \text{in} \ (0, \ell) \times (0, +\infty), \\
  q_t + Mp_x + Kq &= 0 & \text{in} \ (0, \ell) \times (0, +\infty),
\end{align*}
\]

(1)

is a model for dissipative wave equation on the real interval (0, ℓ). In electrical engineering [30, 42, 22], \( p \) (resp. \( q \)) represents the voltage \( V \) (resp. the electrical current \( I \)) at \( (\ell - x, t) \), where \( L = \frac{1}{C} \), \( M = \frac{1}{L} \), \( G = \frac{G'}{C} \), \( K = \frac{K}{L} \), when \( C > 0 \) is the capacitance, \( L > 0 \) the inductance, \( G' \geq 0 \) the conductance and \( R \geq 0 \) the resistance. This system also models arterial blood flow [11, 12, 13, 15] where \( p \) is the pressure and \( q \) the flow rate at \( (x, t) \), \( L = \frac{1}{C} \), \( M = A \), \( K = -\frac{2\alpha}{\alpha - 1} \nu A \), \( G = 0 \), when \( A > 0 \) is the vessel cross-sectional area, \( C > 0 \) is the vessel compliance, \( \nu \geq 0 \) is the kinematic viscosity coefficient (\( \nu \approx 3.210^{-6}m^2/s \) for blood) and \( \alpha > 1 \) is the Coriolis coefficient or correction coefficient (\( \alpha = 4/3 \) for Newtonian fluids, while \( \alpha = 1.1 \) for non-Newtonian fluids like blood).

The exponential stability of (1) with the boundary conditions

\[
q(0, t) = 0, \quad p(\ell, t) = \gamma q(\ell, t), \quad \forall t > 0,
\]

with \( \gamma > 0 \), can be proved by using Theorem 1 of [17] (see also [5]). Using the method of characteristic we can even show that it can be stabilized in a finite time if \( G = K = 0 \) and \( L = M = \gamma = 1 \), see [37].

But recent applications, like electrical circuits, arterial networks, networks of open channels, traffic flows on networks, involve such a system set on networks. In that case some transmission conditions, that translate some physical preservations,
have to be imposed at the junctions. Hence, our goal is to consider $2 \times 2$ hyperbolic systems on a network and introduce a general class of transmission conditions, including reasonable ones (like the Kirchhoff law or the one introduced in [43]) so that with dissipative boundary conditions at the exterior vertices, we obtain an exponentially stable system. As a consequence we also deduce some exact controllability results. For former results in that direction, we can mention the paper of Zong and Xu [43] where some exact controllability results are obtained for a star-shaped network made of three edges. For non-linear problems, some exact controllability or stabilization results are available in some particular situations, we refer to [27, 18, 20, 11, 57] for a non-exhaustive list.

Note that our proposed system (2) with some particular transmission conditions can be transformed into a $2N \times 2N$ hyperbolic system in $(0,1)$ (in the form (5) of [17], for instance), where $N$ is the number of edges of the network, but the algebraic sufficient conditions from Theorem 1 of [17] become rapidly difficult to check if $N$ increases. Hence, we here take advantage of the network structure and show how our system can be reduced to a wave type equation on the same network. Then, using the multiplier method, we establish an observability estimate and deduce the exponential decay of this wave type system. Coming back to our $2 \times 2$ hyperbolic system, we get the same exponential decay.

Note that some observability estimates for first order hyperbolic systems based on characteristics are available in [26]. Such observability estimates are used in [20] to obtain some exact controllability (or stability) results for second order hyperbolic systems with general boundary conditions. We here use the converse way since for wave equations, observability estimates can be obtained with the help of the multiplier method (see Lemma 6.2). Nevertheless, the observability estimates from [26] could be used to obtain our exponential stability result (see Remark 6.5).

The paper is organized as follows: In section 2 we introduce some notations and the general problem studied later on. The well-posedness of this problem is proved in section 3 by using semi-group theory. In section 4 by using a perturbation argument, we show that our system can be reduced to the homogenous case, i.e., when $K = G = 0$. The link with a wave type equation is extricated in section 5 where natural sufficient conditions are proposed to get strong stability. Section 6 starts with the proof of an observability estimate for our wave type equation (based on a standard identity with multiplier) that is available owing to these sufficient conditions and the exponential decay of the energies of both systems are deduced. Finally, in section 7 we conclude with an exact controllability result.

2. General problems on graphs. We first recall the notion of $C^2$-networks, which is simply those of [7], we refer to [29, 18, 20, 11, 57] for more details.

All graphs considered here are non empty, finite and simple. Let $G$ be a connected topological graph imbedded in $\mathbb{R}^m$, $m \in \mathbb{N}^*$, with $n$ vertices $V = \{v_i : 1 \leq i \leq n\}$ and $N$ edges $E = \{e_j : 1 \leq j \leq N\}$. Each edge $e_j$ is a Jordan curve in $\mathbb{R}^m$ and is assumed to be parametrized by its arc length parameter $x_j$, such that the parametrization

$$\pi_j : [0, l_j] \to e_j : x_j \mapsto \pi_j(x_j)$$

is twice differentiable, i.e., $\pi_j \in C^2([0, l_j], \mathbb{R}^m)$ for all $1 \leq j \leq N$.

We now define the $C^2$-network $\Gamma$ associated with $G$ as the union

$$\Gamma = E \cup V.$$
The valency of each vertex \( v \) is denoted by \( \gamma(v) \). For shortness, we later on denote by \( V_{\text{ext}} = \{ v \in V : \gamma(v) = 1 \} \) the set of boundary (or exterior) vertices and \( V_{\text{int}} = V \setminus V_{\text{ext}} \), corresponding to the set of interior vertices. For each vertex \( v \), we also denote by \( J_v = \{ j \in \{ 1, \ldots, N \} : v \in e_j \} \) the set of edges adjacent to \( v \) and let \( N_v \) be the cardinal of \( J_v \). Note that if \( v \in I_{\text{ext}} \) then \( N_v \) is a singleton that we write \( \{ j_v \} \). For each vertex \( v \) and \( j \in N_v \), we further denote by

\[
\nu_j(v) = \begin{cases} 1 & \text{if } \pi_j(l_j) = v, \\ -1 & \text{if } \pi_j(0) = v, 
\end{cases}
\]

the normal vector in \( e_j \) at \( v \). For any \( v \in V_{\text{int}} \), we also introduce the mapping

\[
T_v : C^{N_v} \rightarrow C^{N_v} : y = (y_j)_{j \in J_v} \rightarrow T_v y = (y_j \nu_j(v))_{j \in J_v}.
\]

Note that \( T_v^2 \) is equal to the identity mapping.

For a function \( u : \Gamma \rightarrow \mathbb{C} \), we set \( u_j = u \circ \pi_j : [0, l_j] \rightarrow \mathbb{C} \), its “restriction” to the edge \( e_j \) and use the abbreviations:

\[
u_j(v) = u_j(\pi_j^{-1}(v)), \quad u'_j(v) = du_j(\pi_j^{-1}(v)), \quad \partial_v u_j(v) = \nu_j(v) u'_j(v),
\]

for a vertex \( v \in e_j \). Finally, differentiations are carried out on each edge \( e_j \) with respect to the arc length parameter \( x_j \).

Now we denote by \( PC(\Gamma) \) the set of piecewise continuous functions on \( \Gamma \), which means that \( u : \Gamma \rightarrow \mathbb{C} \) belongs to \( PC(\Gamma) \) if and only if \( u_j \in C([0, l_j]) \), for all \( j = 1, \ldots, N \). Further \( C(\Gamma) \) is the set of continuous functions on \( \Gamma \), which means that \( u \in C(\Gamma) \) if and only if \( u \in PC(\Gamma) \) and

\[
u_j(v) = u_k(v), \forall j, k \in J_v, v \in V_{\text{int}}.
\]

For any \( v \in V \), we finally introduce the “trace” mapping

\[
\gamma_v : PC(\Gamma) \rightarrow C^{N_v} : u \rightarrow \gamma_v u = (u_j(v))_{j \in J_v}.
\]

Similarly, we denote by \( PH^1(\Gamma) \) the set of piecewise \( H^1 \) functions on \( \Gamma \), in other words \( u \in PH^1(\Gamma) \) if and only if \( u_j \in H^1([0, l_j]) \), for all \( j = 1, \ldots, N \). This space is clearly a Hilbert space with its natural inner product.

Let us now fix a \( C^2 \)-network \( \Gamma \). For each edge \( e_j \), we also fix positive real constants \( L_j, M_j \) and non-negative real constants \( G_j, K_j \). For each \( v \in V_{\text{int}} \), we fix a subspace \( Z_v \subset C^{N_v} \) and denote by \( Z_v^\perp \) its orthogonal complement in \( C^{N_v} \) with respect to the euclidean inner product. We finally fix a decomposition of \( V_{\text{ext}} = V_{\text{ext}}^{\text{Dir}} \cup V_{\text{ext}}^{\text{Diss}} \) with two disjoint subsets \( V_{\text{ext}}^{\text{Dir}} \) and \( V_{\text{ext}}^{\text{Diss}} \). With these assumptions, we consider the problem

\[
\begin{cases}
\partial_t p_j + L_j \partial_x q_j + G_j p_j = 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \\
\partial_t q_j + M_j \partial_x p_j + K_j q_j = 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \\
\gamma_v p(\cdot, t) \in Z_v, \forall v \in V_{\text{int}}, t > 0, \\
T_v \gamma_v q(\cdot, t) \in Z_v^\perp, \forall v \in V_{\text{int}}, t > 0, \\
q_j(\cdot, t) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}, t > 0, \\
p_j(\cdot, t) = \alpha \nu_j(v) q_j(\cdot, 0), \forall v \in V_{\text{ext}}^{\text{Diss}}, t > 0, \\
p(\cdot, 0) = p_0, q(\cdot, 0) = q_0 \text{ in } \Gamma,
\end{cases}
\]

where \( Q_j := (0, l_j) \times (0, \infty) \) and \( \alpha \geq 0 \).
In the above system, the transmission conditions at interior nodes, namely
\[ \gamma_v p(v,t) \in Z_v, T_v \gamma_v q(v,t) \in Z_v \perp, \forall v \in V_{\text{int}}, t > 0, \]  
(3) have to be compared with those from [8, 31, 24, 14, 9] for the heat or wave equations on graphs (see also section 5 below). We introduce them because they generalize standard junction conditions. The first one is the so-called Kirchoff condition [40, 11, 12, 13, 15], where we impose at \( v \in V_{\text{int}} \)
\[ \sum_{j \in J_v} \alpha_j \nu_j(v) p_j(v,t) = 0, \]  
(4)\[ p_j(v,t) = p_k(v,t), \forall j, k \in J_v. \]  
(5) The condition (4) describes the mass conservation (resp. charge conservation) for fluid flow models (resp. electrical circuits); on the contrary the condition (5) means that the pressures (resp. voltages) agree at the junction. The Kirchoff condition is a particular case of (3) by taking \[ Z_v = \text{Span } (1, \ldots, 1)^\top. \]  
(6) The second junction conditions were introduced in [43] for three edges with a common vertex \( v \). A generalization to any number of edges is given as follows. Fix one edge \( j_v \in J_v \) with \( v \in V_{\text{int}} \) and take
\[ \sum_{j \in J_v} \alpha_j \nu_j(v) p_j(v,t) = 0, \]  
(6a)\[ q_j(v,t) = \alpha_j q_j(v,t), \forall j \in J_v, \]  
(7) with \( \alpha_j = 1 \) and for some mass partition coefficients \( \alpha_j \in (0, 1) \) such that \( \sum_{j \in J_v \setminus \{j_v\}} \alpha_j = 1 \). Hence, the second condition is a flow rates partition and the first one means that no energy is lost at the bifurcation. Such conditions enter in the framework (3) by taking
\[ Z_v^\perp = \text{Span } (\alpha_j)_{j \in J_v}. \]  
(8) We recognize in the boundary condition
\[ p_j(v,t) = \alpha_j \nu_j(v) q_j(v,t), \forall v \in V_{\text{ext}}, t > 0, \]  
a standard dissipative boundary condition used for \( 2 \times 2 \) hyperbolic systems (like the condition (3) in [17] or the condition (52) in [19]) once the system is transformed into a diagonal one.

As already mentioned, our main problem is to find sufficient conditions that guarantee the exponential decay of the solution \((p, q)\) of this problem. But before going on let us show its well-posedness.

3. An existence result. In order to show that system (3) is well-posed, we introduce the following operator \( A_{\lambda} \) on the Hilbert space \( H = L^2(\Gamma)^2 = \prod_{j=1}^N L^2(0, l_j)^2 \), endowed with the inner product
\[ ((p, q), (r, s))_H = \sum_{j=1}^N \int_0^{l_j} \left( L_j^{-1} p_j(x_j) \bar{r}_j(x_j) + M_j^{-1} q_j(x_j) \bar{s}_j(x_j) \right) dx_j, \]  
as follows:
\[ D(A_{\lambda}) = \{(p, q) \in PH^1(\Gamma)^2 \text{ satisfying (8 to (11) hereafter )}, \]  
\[ A_{\lambda} (p, q) = -(Lq + Gp, Mp' + Kq), \forall(p, q) \in D(A_{\lambda}), \]
with the notations:
\[ Lq' := (L_jq'_j)_{j=1}^N, \quad Mp' := (M_jp'_j)_{j=1}^N, \quad Gp := (G_jp_j)_{j=1}^N, \quad Kq := (K_jq_j)_{j=1}^N. \]

\[ \gamma_v p \in Z_v, \forall v \in V_{int}, \quad (8) \]
\[ T_v \gamma_v q \in Z_v^\perp, \forall v \in V_{int}, \quad (9) \]
\[ q_j(v) = 0, \forall v \in V_{ext}^\text{Dir}, \quad (10) \]
\[ p_{j_v}(v) = \alpha \nu_{j_v}(v)q_{j_v}(v,t), \forall v \in V_{ext}^\text{Diss}. \quad (11) \]

Remark 3.1. Note that we can replace the boundary condition (10) by
\[ p_{j_v}(v) = 0, \forall v \in V_{ext}^\text{Dir}, \]
without changing the results below.

For further uses, introduce
\[ V = \{ q \in PH^1(\Gamma) \text{ satisfying (9) and (10)} \}, \]
that is clearly a Hilbert space.

With these notations, problem (2) enter in the following framework:
\[ \begin{cases} 
  U_t = A_\alpha U, \\
  U(0) = U_0, 
\end{cases} \quad (12) \]

where \( U \) is the vector \((p,q)\). As usual the existence of a solution to (12) is obtained using semigroup theory.

**Theorem 3.2.** Under the above assumptions, the operator \( A_\alpha \) generates a \( C_0 \)-semigroup of contractions on \( H \).

**Proof.** It suffices to prove that \( A_\alpha \) is a maximal dissipative operator, hence, by Lumer-Phillips’ theorem it generates a \( C_0 \)-semigroup of contractions on \( H \).

Let us start with the dissipativity. For \( U = (p,q) \in D(A_\alpha) \), we have
\[ \Re(A_\alpha U, U)_H = -\Re \sum_{j=1}^N \int_0^{l_j} (q'_j \bar{p}_j + p'_j \bar{q}_j) \, dx_j \]
\[ - \sum_{j=1}^N \int_0^{l_j} (L_j^{-1} G_j |p_j|^2 + M_j^{-1} K_j |q_j|^2) \, dx_j. \]

Therefore, applying Green’s formula in the first term of this right-hand side, we find
\[ \Re(A_\alpha U, U)_H = -\Re \sum_{v \in V} (T_v \gamma_v q) \cdot \gamma_v \bar{p} \]
\[ - \sum_{j=1}^N \int_0^{l_j} (L_j^{-1} G_j |p_j|^2 + M_j^{-1} K_j |q_j|^2) \, dx_j. \]

Now using \( 8 \), \( 9 \) and \( 10 \), we have
\[ (T_v \gamma_v q) \cdot \gamma_v \bar{p} = 0, \forall v \in V_{int} \cup V_{ext}^\text{Dir}, \]
while using \( 11 \) it holds
\[ (T_v \gamma_v q) \cdot \gamma_v \bar{p} = \alpha |\gamma_v q|^2, \forall v \in V_{ext}^\text{Diss}. \]
Hence, we obtain
\[ \Re(A_\alpha U, U)_H = -\alpha \sum_{v \in V_{\text{ext}}} |\gamma_v q|^2 \]
(13)
and therefore, \( A_\alpha \) is dissipative.

Let us go on with the maximality. Let \( \lambda > 0 \) be fixed. Given \( F = (f, g) \in H \), we look for \( U = (p, q) \in D(A_\alpha) \) such that \( (\lambda - A_\alpha)U = F \), or equivalently
\[
\begin{align*}
\begin{cases}
(\lambda + G_j)p_j + L_j q_j' = f_j & \text{in } (0, l_j), \\
(\lambda + K_j)q_j + M_j p_j' = g_j & \text{in } (0, l_j), \forall j = 1, \ldots, N.
\end{cases}
\end{align*}
\]
(14)
Assume for the moment that such a \( U \) exists. Then, the first identity is equivalent to
\[ p_j = \frac{1}{\lambda + G_j} (f_j - L_j q_j'). \]
(15)
Multiplying the second equation by \( \frac{\bar{r}_j}{M_j} \) with a test function \( r \in V \), integrating on \( (0, l_j) \) and summing on \( j \) yields
\[
\sum_{j=1}^N \int_0^{l_j} (\frac{\lambda + K_j}{M_j} q_j + p_j' \bar{r}_j) \, dx_j = \sum_{j=1}^N \frac{1}{M_j} \int_0^{l_j} g_j \bar{r}_j \, dx_j.
\]
Integrating by parts in the second term of this left-hand side, we get
\[
\sum_{j=1}^N \int_0^{l_j} (\frac{\lambda + K_j}{M_j} q_j \bar{r}_j - p_j \bar{r}_j') \, dx_j + \sum_{v \in V} (T_v \gamma_v p) \cdot v = \sum_{j=1}^N \frac{1}{M_j} \int_0^{l_j} g_j \bar{r}_j \, dx_j.
\]
By taking into account the boundary conditions satisfied by \( p \) and \( r \), we find
\[
\sum_{j=1}^N \int_0^{l_j} (\frac{\lambda + K_j}{M_j} q_j \bar{r}_j - p_j \bar{r}_j') \, dx_j + \alpha \sum_{v \in V_{\text{ext}}} \gamma_v q \cdot v = \sum_{j=1}^N \frac{1}{M_j} \int_0^{l_j} g_j \bar{r}_j \, dx_j, \forall r \in V.
\]
(16)
Replacing \( p_j \) by (15), we arrive at
\[
\sum_{j=1}^N \int_0^{l_j} (\frac{\lambda + K_j}{M_j} q_j \bar{r}_j + \frac{L_j}{\lambda + G_j} q_j' \bar{r}_j') \, dx_j + \alpha \sum_{v \in V_{\text{ext}}} \gamma_v q \cdot v = \sum_{j=1}^N \frac{1}{M_j} \int_0^{l_j} g_j \bar{r}_j \, dx_j, \forall r \in V.
\]
(17)
Now this problem has a unique solution \( q \in V \), by Lax-Milgram Lemma because the left-hand side is a continuous and coercive sesquilinear form on \( V \), since
\[ \int_0^{l_j} (\frac{\lambda + K_j}{M_j} |q_j|^2 + \frac{L_j}{\lambda + G_j} |q_j'|^2) \, dx_j \geq C_j(\lambda) \|q\|^2_{L^2(0, l_j)} \geq \Re(A_\alpha U, U)_H \]
with \( C_j(\lambda) = \min \{ \frac{\lambda + K_j}{M_j}, \frac{L_j}{\lambda + G_j} \} > 0 \) and because the right-hand side is a continuous form on \( V \).
Now we define \( p \) by (15), that for the moment is an element in \( L^2(\Gamma) \). But then we remark that (17) is equivalent to (16). In (16) taking as test function \( r \) such that \( r_j \in D(0,t_j) \), for some \( j = 1, \ldots, N \) and \( r_k = 0 \), for all \( k \neq j \), we find

\[
\lambda + K_j \frac{M_j}{M_j} q_j + p_j' = \frac{1}{M_j} g_j \text{ in } D'(0,t_j).
\]

As \( \frac{1}{M_j} g_j - \lambda + K_j \frac{M_j}{M_j} q_j \) belongs to \( L^2(0,t_j) \), we deduce that \( p_j \) belongs to \( H^1(0,t_j) \) and that (14) holds.

Hence, it remains to check the boundary conditions (8) and (11). For that purpose, we take different test functions \( r \in V \) in (16). First by Green’s formula and taking into account (14), we have

\[
-\sum_{v \in V} T_v \gamma_v p \cdot \gamma_v \bar{r} + \alpha \sum_{v \in V_{\text{ext}}} \gamma_v q \cdot \gamma_v \bar{r} = 0, \forall r \in V.
\]

First we fix \( v \in V_{\text{int}} \) and can take \( r \in V \) such that \( T_v \gamma_v (r) = z_v \) is arbitrary in \( Z_v \) and \( r_j(w) = 0 \), for all \( j \in I_w, w \in V \setminus \{v\} \). With this choice, we get

\[
\gamma_v p \cdot z_v = 0, \forall z_v \in Z_v^+,
\]

which means that (8) holds. Second we fix \( v \in V_{\text{ext}} \) and can take \( r \in V \) such that \( \gamma_v (r) = z \) is arbitrary in \( C \) and \( r_j(w) = 0 \), for all \( j \in I_w, w \in V \setminus \{v\} \). Hence, we obtain

\[
(-T_v \gamma_v p + \alpha \gamma_v q) z = 0, \forall z \in \mathbb{C},
\]

which is nothing else than (11).

As usual the energy associated with (12) is defined by

\[
E(t) = \frac{1}{2} \sum_{j=1}^{N} \int_0^{l_j} (L_j^{-1} |p_j(x,t)|^2 + M_j^{-1} |q_j(x,t)|^2) \, dx_j,
\]

that is equal to 1/2 of the norm of \((p,q)\) in \( H \).

**Proposition 3.3.** The solution \((p,q)\) of (12) with initial datum in \( D(A_{\alpha}) \) satisfies

\[
E'(t) = -\alpha \sum_{v \in V_{\text{ext}}} |\gamma_v q(t)|^2 - \sum_{j=1}^{N} \int_0^{l_j} (L_j^{-1} G_j |p_j(x,t)|^2 + M_j^{-1} K_j |q_j(x,t)|^2) \, dx_j \leq 0,
\]

therefore, the energy is non increasing.

**Proof.** If \( U_0 \in D(A_{\alpha}) \), we can derive the energy (18) because \( U \in C^1([0,\infty), H) \) and obtain

\[
E'(t) = \Re(U, U_t)_H.
\]

Using problem (12), we get

\[
E'(t) = \Re(A_{\alpha} U, U)_H.
\]

We conclude thanks to (13). \( \square \)
4. Reduced problems. Our main goal is to find sufficient conditions on the spaces $Z_v$ so that system (1) is exponentially stable by a boundary feedback, i.e., when $\alpha > 0$. But in a first step we show that we can reduce our system to the case $G_j = K_j = 0$. We start by an abstract perturbation result that uses the following result (see [38] or [21]):

**Lemma 4.1.** Let $(e^{tA})_{t\geq 0}$ be a $C_0$ semigroup on a Hilbert space $H$ satisfying

$$\|e^{tA}\| \leq M, \forall t \geq 0,$$

for some $M > 0$. Then, it is exponentially stable, i.e., satisfies

$$\|e^{tA}U_0\| \leq C e^{-\omega t}\|U_0\|_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants $C$ and $\omega$ if and only if

$$\rho(A) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R}, \quad (19)$$

and

$$\sup_{\beta \in \mathbb{R}} \|(i\beta - A)^{-1}\|_{\mathcal{L}(H)} < \infty, \quad (20)$$

where $\rho(A)$ denotes the resolvent set of the operator $A$.

**Theorem 4.2.** Let $A$ be the generator of an exponentially stable $C_0$ semigroup of contraction on a Hilbert space $H$ with a compact resolvent and let $B$ be a bounded operator on $H$ such that $-B$ is a non negative selfadjoint operator. Suppose further that the $C_0$ semigroup $(e^{t(A+B)})_{t\geq 0}$ generated by $A + B$ on $H$ satisfies

$$\|e^{t(A+B)}\| \leq M, \forall t \geq 0, \quad (21)$$

for some $M > 0$. Then, $(e^{t(A+B)})_{t\geq 0}$ is exponentially stable.

**Proof.** First recall that a standard perturbation result (see Theorem 3.1.1 of [36] for instance) yields that $A + B$ generates a $C_0$ semigroup (that does not necessarily satisfy (21)). Due to the assumption (21), according to the previous Lemma, it suffices to show that $A + B$ satisfies (19) and (20). For the first property, as $A + B$ has also a compact resolvent, it suffices to check that $A + B$ has no eigenvalue on the imaginary axis. Therefore, let $\xi \in \mathbb{R}$ and $u \in D(A)$ be such that

$$(i\xi - A - B)u = 0.$$ 

Then, taking the inner product in $H$ with $u$ and taking the real part, we find that

$$\Re\langle Au, u\rangle_H + \langle Bu, u\rangle_H = 0.$$

As $A$ is dissipative, we deduce that $(Bu, u)_H = 0$, or equivalently

$$\|(B)^{\frac{1}{2}}u\|^2_H = 0.$$

This implies that $Bu = 0$ and therefore, $u$ satisfies

$$(i\xi - A)u = 0,$$

and by (19) applied to $A$, we deduce that $u = 0$. Hence, $A + B$ satisfies (19).

For the second property, by (19) and the compactness of the resolvent of $A + B$, for all $\xi \in \mathbb{R}$ and $f \in H$, there exists a unique solution $u \in D(A)$ of

$$(i\xi - A - B)u = f.$$ 

(22)

Again taking the inner product in $H$ with $u$, taking the real part and using the dissipativeness of $A$, we find that

$$-\langle Bu, u\rangle \leq \Re\langle f, u\rangle_H.$$
By Cauchy-Schwarz’s and Young’s inequalities, we deduce that
\[ \|(-B)^{\frac{1}{2}}u\|_H^2 \leq \frac{1}{2\varepsilon} \|f\|_H^2 + \frac{\varepsilon}{2} \|u\|_H^2, \]
for all \( \varepsilon > 0 \). Since \((-B)^{\frac{1}{2}}\) is bounded we deduce that
\[ \|Bu\|_H \leq \frac{C}{\varepsilon} \|f\|_H + \varepsilon \|u\|_H, \quad (23) \]
for all \( \varepsilon > 0 \) and some \( C > 0 \) independent of \( \varepsilon \).

Now we come back to (22) and notice that it can be equivalently written as
\[(i\xi - A)u = f + Bu.\]
Using the property (20) satisfied by \( A \), we deduce that
\[ \|u\|_H \leq K(\|f\|_H + \|Bu\|_H), \]
for some \( K > 0 \) independent of \( \xi \). With the help of (23),
\[ \|u\|_H \leq K(1 + C\varepsilon)\|f\|_H + K\varepsilon \|u\|_H, \]
for all \( \varepsilon > 0 \). By choosing \( \varepsilon = \frac{1}{2K} \), we deduce that
\[ \|u\|_H \leq 2K(1 + 2KC)\|f\|_H, \]
which means that \( A + B \) satisfies (20).

**Corollary 4.3.** Let \( A_{\alpha}^{(0)} \) be the operator \( A_{\alpha} \) defined before with \( G_j = K_j = 0 \) for all \( j = 1, \ldots, N \). If \( A_{\alpha}^{(0)} \) generates an exponentially stable \( C_0 \) semigroup on \( H \), then, \( A_{\alpha} \) generates an exponentially stable \( C_0 \) semigroup on \( H \).

**Proof.** It suffice to notice that
\[ A_{\alpha} = A_{\alpha}^{(0)} + B, \]
where \( B \) is defined by
\[ B(p, q) = -((G_j p_j)_{j=1}^N, (K_j q_j)_{j=1}^N). \]
As \( G_j \) and \( K_j \) are non negative, we directly deduce that \(-B\) is a non negative and bounded selfadjoint operator. The conclusion then follows from Theorems 3.2 and 4.2.

From now on we then assume that \( G_j = K_j = 0 \) for all \( j = 1, \ldots, N \).

5. **Link with a wave equation.** We now make the relation between our system (2) and a wave type equation.

**Theorem 5.1.** Let \( (p, q) \in C([0, \infty); D(A_{\alpha}^2)) \cap C^1([0, \infty); D(A_{\alpha})) \cap C^2([0, \infty); H) \) be a solution of (12) with \( K_j = G_j = 0 \). Then, for all \( t > 0 \), \( q(\cdot, t) \in V \) satifies the wave equation
\[ \partial_t^2 q_j - M_j L_j q_j'' = 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \quad (24) \]
the boundary conditions
\[ T_v \gamma_v q(\cdot, t) \in Z_v^\perp, \forall v \in V_{\text{int}}, t > 0, \quad (25) \]
\[ q_j (v, t) = 0, \forall v \in V_{\text{ext}}^\text{Dir}, t > 0, \quad (26) \]
\[ \gamma_v (L_j q)(v, t) \in Z_v, \forall v \in V_{\text{int}}, t > 0, \quad (27) \]
\[ \partial_v L_j q_j (v, t) = -\alpha \partial_v q_j (v, t), \forall v \in V_{\text{ext}}^\text{Diss}, t > 0, \quad (28) \]
and the initial conditions
\[ q(\cdot, 0) = q_0, \partial_t q(\cdot, 0) = -L_p' \tag{29} \]

**Proof.** The regularity on \((p, q)\) allows to deduce that \((p, q)\) satisfies (2) strongly, implying that
\[
\begin{align*}
\partial_t p_j + L_j \partial_x q_j &= 0 \text{ in } Q_j, \forall j = 1, \ldots, N, \\
\partial_t q_j + M_j \partial_x p_j &= 0 \text{ in } Q_j, \forall j = 1, \ldots, N. 
\end{align*}
\tag{30}
\]
Furthermore we can derive the first identity in space and the second one in time and obtain
\[
\begin{align*}
\partial_t p'_j + L_j q''_j &= 0 \text{ in } Q_j, \\
\partial^2_t q_j + M_j \partial_t p'_j &= 0 \text{ in } Q_j, 
\end{align*}
\]
eliminating \(\partial_t p'_j\) we arrive at (24).

Now as \((\partial_t p(t), \partial_t q(t))\) belongs to \(D(A_\alpha)\), we have
\[ \gamma_v \partial_t p(\cdot, t) \in Z_v, \forall v \in V_{\text{int}}, t > 0, \]
and
\[ \partial_t p_j(v, t) = \alpha v_j(v) \partial_t q_j(v, t), \forall v \in V_{\text{ext}}, t > 0, \]
and since \(\partial_t p = -Lq'\), we obtain (27) and (28).

Note that the wave type equation (24)-(29) in \(q\) is well-posed. Indeed, assume for simplicity that \(V_{\text{Dir}}^{\text{ext}}\) is non empty (hence, \(V_{\text{ext}}\) is non empty as well), then it can be written in the following abstract form
\[
\begin{align*}
q''(t) + A q(t) + B B^* q'(t) &= 0 \text{ in } V', \\
q(0) &= q_0, \partial_t q(0) = q_1. 
\end{align*}
\tag{31}
\]
Indeed, we first equip the Hilbert spaces \(L^2(\Gamma)\) and \(V\) with the inner product (reminding that \(V_{\text{Dir}}^{\text{ext}}\) is non empty)
\[
\begin{align*}
(u, w)_V &= \sum_{j=1}^N \int_0^{l_j} L_j u_j \bar{w}_j \, dx_j, \\
(u, w)_{L^2(\Gamma)} &= \sum_{j=1}^N \int_0^{l_j} M_j^{-1} u_j \bar{w}_j \, dx_j.
\end{align*}
\]
Then, we introduce the selfadjoint operator \(A\) in \(L^2(\Gamma)\) associated with the triple \((H, V, a)\), where \(a\) is the sesquilinear, symmetric and coercive form defined by
\[ a(v, w) = (u, w)_V, \forall u, w \in V. \]
Finally, if we denote by \(N_{\text{Diss}}\) the cardinal of \(V_{\text{ext}}^{\text{Diss}}\), we define
\[ B : C^{N_{\text{Diss}}} \rightarrow V' : (z_v)_{v \in V_{\text{ext}}^{\text{Diss}}} \rightarrow \sqrt{\alpha} \sum_{v \in V_{\text{ext}}^{\text{Diss}}} z_v \delta_v, \]
where for all \(v \in V_{\text{ext}}^{\text{Diss}}\)
\[ \langle \delta_v, w \rangle = w_{j_v}(v), \forall w \in V. \]
Hence, an easy exercise yields that
\[ B^* : V \rightarrow C^{N_{\text{Diss}}} : w \rightarrow \sqrt{\alpha}(w_{j_v}(v))_{v \in V_{\text{ext}}^{\text{Diss}}} \]
is bounded from \(V\) into \(C^{N_{\text{Diss}}}\).
In conclusion all assumptions of Theorem 1.2.1 of [3] are satisfied and we conclude the next result.

**Theorem 5.2.** Assume that $V^\text{Dir}_{\text{ext}}$ is non empty. If $(q_0, q_1) \in V \times L^2(\Gamma)$, then problem (31) has a unique solution $q \in C([0, \infty), V) \cap C^1([0, \infty), L^2(\Gamma))$. Furthermore if $(q_0, q_1) \in D(\mathcal{A}_o) \times V$, then $(q, \partial_t q) \in C([0, \infty), D(\mathcal{A}_o) \times V) \cap C^1([0, \infty), V \times L^2(\Gamma))$.

where

$$D(\mathcal{A}_o) = \{(q, r) \in (PH^2(\Gamma) \cap V) \times V : (q, r) \text{ satisfies (32) and (33) below}\},$$

$$\gamma_v(Lq')(v) \in Z_v, \forall v \in V_{\text{ext}},$$

$$\partial_v L_j, r_j(v) = -\alpha r_j(v), \forall v \in V^\text{Diss}_{\text{ext}},$$

and $q$ satisfies (24)-(29) strongly. The energy

$$\mathcal{E}(t) = \frac{1}{2} \sum_{j=1}^N \int_0^T (L_j q_j'(x, t))^2 + M_j^{-1} |\partial_t q_j(x, t)|^2 \, dx_j,$$

is non increasing and

$$\mathcal{E}(0) - \mathcal{E}(T) = \alpha \int_0^T \sum_{v \in V^\text{Diss}_{\text{ext}}} |\partial_t q_{j_v}(v, t)|^2 \, dt, \forall T > 0. \quad (34)$$

**Remark 5.3.** Let us remark that the transmission conditions (25) and (27) are particular self-adjoint nodal conditions from [23]. But it has been shown in [24, Thm 6] (see also [32]) that the nodal conditions

$$T_v \gamma_v p \in Z_v^\perp,$$

$$\gamma_v(Lq') + A_v \gamma_v q \in Z_v,$$

with $A_v$ a symmetric matrix on $T_v Z_v^\perp$ is the most general ones leading to self-adjoint operators. Our transmission condition (27) corresponds to the choice $A_v = 0$ and is guided by the examples mentioned in section 2. Hence, a more general problem would be to replace in (2) the transmission condition $\gamma_v p(\cdot, t) \in Z_v$ by $\gamma_v p(\cdot, t) - A_v \gamma_v q(\cdot, t) \in Z_v$. Our results can surely be extended to this case.

Note that the transmission condition (9) can be equivalently written as

$$\gamma_v q \in T_v Z_v^\perp, \forall v \in V_{\text{int}}. \quad (35)$$

Hence, if for all $v \in V_{\text{int}}$, we chose $T_v Z_v^\perp$ as the linear span of $(1, 1, \ldots, 1)^T$, system (24)-(29) is nothing else than the wave equation on $\Gamma$ with the so-called Kirchoff condition at the interior node and a standard damping condition at the node of $V^\text{Diss}_{\text{ext}}$, see for instance [25, 16]. Accordingly owing to [39, 25, 16], if $\Gamma$ has some cycles or if two exterior nodes are uncontrolled, we cannot expect to have an exponential decay from a boundary dissipation. Therefore, in the remainder of the paper, we assume that $\Gamma$ is a tree, that $V^\text{Dir}_{\text{ext}}$ is reduced to one external vertex, called the root of the tree and that $V^\text{Diss}_{\text{ext}}$ is the set of all other external vertices. To simplify the arguments below, we now parametrize each edge $e_j$ in such a way that the vertex $\pi_j(0)$ is more close to the root than $\pi_j(l_j)$. For all $v \in V_{\text{int}}$, we denote by $Y_v := T_v Z_v^\perp \subset \mathbb{C}^N_v$ (hence, $Y_v^\perp = T_v Z_v$) and assume without loss of generality that the first component of $\mathbb{C}^N_v$ corresponds to the edge $j$ such that $\pi_j(l_j) = v$ (or equivalently the edge $j \in J_v$ closer to the root); let us denote this edge $j_v$. We
further denote by $\Pi_v$ the orthogonal projection on $(1,0,\ldots,0)^T$ in $\mathbb{C}^{N_v}$. Now we make the following assumption:

$$\forall v \in V_{\text{int}} : \Pi_v Y_v \neq \{0\} \text{ and } \Pi_v Y_v^\perp \neq \{0\}. \quad (36)$$

**Remark 5.4.** The two examples mentioned in section 2 satisfy condition (36), indeed, as $\Pi_v Y_v = \Pi_v T_v Z_v = \nu_{j_v}(v) \Pi_v Z_v$ (resp. $\Pi_v Y_v^\perp = \nu_{j_v}(v) \Pi_v Z_v^\perp$), it suffices to check this condition for $Z_v$. For the Kirchhoff condition, $Z_v$ being given by (6), we clearly have $\Pi_v Z_v \neq \{0\}$, since $\Pi_v (1,1,\ldots,1)^T = 1$. Similarly, as $(1,-1,0,\ldots,0)^T \in Z_v^\perp$, $\Pi_v Z_v^\perp \neq \{0\}$.

For the flow rates partition model, $Z_v$ is determined by (7) and therefore, $\Pi_v Z_v \neq \{0\}$, since $\Pi_v (\alpha_j)_{j \in J_v} = 1$, and $\Pi_v Z_v^\perp \neq \{0\}$ because $(1,-1,-1,\ldots,-1)^T$ belongs to $Z_v^\perp$.

This remark and the next lemma show that this assumption is quite natural.

**Lemma 5.5.** If the assumption (36) does not hold, then any $(p,q) \in D(\mathcal{A}_\alpha)$ satisfies

$$\exists v \in V_{\text{int}} : p_{j_v}(v) = 0 \text{ or } q_{j_v}(v), \quad (37)$$

and any $(q,r) \in D(\mathcal{A}_\alpha)$ satisfies

$$\exists v \in V_{\text{int}} : q_{j_v}(v) = 0 \text{ or } q_{j_v}(v). \quad (38)$$

**Proof.** Assumption (36) does not hold if and only if

$$\exists v \in V_{\text{int}} : \Pi_v Y_v = \{0\} \text{ or } \Pi_v Y_v^\perp = \{0\}. \quad \text{(36)}$$

Fix such a $v \in V_{\text{int}}$ and let us prove the second property. Indeed, if $\Pi_v Y_v = \{0\}$, then this means equivalently that $(1,0,\ldots,0)^T$ belongs to $Y_v^\perp$. For $(q,r) \in D(\mathcal{A}_\alpha)$, as $q \in Y_v$, we deduce that

$$q_{j_v}(v) = \gamma_v q \cdot (1,0,\ldots,0)^T = 0. \quad \text{(38)}$$

On the contrary if we assume that $\Pi_v Y_v^\perp = \{0\}$. Then, equivalently $(1,0,\ldots,0)^T$ belongs to $Y_v$ and for $(q,r) \in D(\mathcal{A}_\alpha)$, as $T_v \gamma_v (Lq')(v) \in Y_v$, we deduce that

$$L_{j_v} q'_{j_v}(v) = \pm T_v \gamma_v Lq'(v) \cdot (1,0,\ldots,0)^T = 0. \quad \text{(38)}$$

Therefore,

$$q'_{j_v}(v) = 0. \quad \text{(38)}$$

The proof of the first property is similar. \hfill $\Box$

If any $(q,r) \in D(\mathcal{A}_\alpha)$ satisfies (38), this means that there exists at least one interior vertex $v$ such that we have either Dirichlet or Neumann condition on $q$ at the extremity $v$ of $j_v$. Hence, let us denote by $\Gamma'_v$ the subtree made of the edges of $\Gamma$ that are descendant of $v$ and let $\Gamma_v = \Gamma \setminus \Gamma'_v$. Then, we notice that the wave system (24)-(29) can be split into similar but decoupled problems in $\Gamma_v$ and in $\Gamma'_v$. But on $\Gamma'_v$, we have at least two vertices with a non dissipative boundary condition (the root and $v$), since in such a situation, in general we do not have stability of the system with Kirchhoff conditions at interior vertices (see for instance [10,]), the condition (36) seems to be realistic. It turns out that this condition is sufficient to get uniform stability of systems (2) and (24)-(29) (see the next section) if a dissipative boundary condition is applied on $V_{\text{ext}}$, but before let us show that at least it yields their strong stability. We start with a technical Lemma.
Lemma 5.6. The condition (36) holds if and only if for all \( v \in V_{\text{int}} \), there exist coefficients \( a_{v,j}, b_{v,j} \in \mathbb{C} \), for all \( j \in J_v \setminus \{ j_v \} \) such that

\[
y_{j_v} = \sum_{j \in J_v \setminus \{ j_v \}} a_{v,j} y_j, \forall y \in Y_v, \tag{39}
\]

\[
z_{j_v} = \sum_{j \in J_v \setminus \{ j_v \}} b_{v,j} z_j, \forall z \in Y_v^\perp. \tag{40}
\]

Proof. Let \( v \in V_{\text{int}} \). Then, \( \Pi_v Y_v^\perp \neq \{0\} \) if and only if there exists \( z \in Y_v^\perp \) such that \( z_{j_v} \neq 0 \). Since \( y \cdot z = 0 \), for any \( y \in Y_v \), we get

\[
y_{j_v} z_{j_v} = -\sum_{j \in J_v \setminus \{ j_v \}} y_j z_j,
\]

which furnishes (39) with \( a_{v,j} = \frac{z_j}{z_{j_v}} \). Conversely if (39) holds, then \( (1, (-a_{v,j}) \in J_v \setminus \{ j_v \})^T \) belongs to \( Y_v^\perp \) and therefore, \( \Pi_v Y_v^\perp \neq \{0\} \). \(\square\)

Theorem 5.7. Assume that \( \Gamma \) is a tree, that \( V_{\text{ext}}^\perp \) is the root of the tree and that \( V_{\text{ext}}^{\text{Dir}} \) is the set of all other exterior vertices. If \( \alpha > 0 \) and condition (36) holds, then

\[
i \mathbb{R} \subset \rho(A^{(0)}_\alpha),
\]

and

\[
i \mathbb{R} \subset \rho(A_\alpha).
\]

Proof. We only prove the first assertion, since the second one is fully similar. Let \( \xi \in \mathbb{R} \) and \( U = (p, q) \in D(A^{(0)}_\alpha) \) be such that

\[
(i \xi - A^{(0)}_\alpha) U = 0,
\]

or equivalently satisfying

\[
\left\{
\begin{array}{l}
i \xi p_j + L_j q_j' = 0, \\
i \xi q_j + M_j p_j' = 0,
\end{array}\right. \tag{41}
\]

for all \( j = 1, \ldots, N \). Hence, by the dissipativeness of \( A^{(0)}_\alpha \) (inequality (13)), we get

\[
q_j(v) = p_j(v) = 0, \forall v \in V_{\text{ext}}^{\text{Dir}}. \tag{42}
\]

For \( \xi \neq 0 \), by eliminating \( p_j \) from the first identity of (41) we find that

\[
M_j L_j q_j'' = -\xi^2 q_j, \forall j = 1, \ldots, N.
\]

Consequently, for all \( j = 1, \ldots, N \), there exist constants \( c_{j,1}, c_{j,2} \in \mathbb{C} \) such that

\[
q_j(x) = c_{j,1} e^{i \xi x} + c_{j,2} e^{-i \xi x},
\]

\[
p_j(x) = i \frac{c_{j,1} e^{i \xi x} - c_{j,2} e^{-i \xi x}}{L_j}, \forall x \in (0, l_j).
\]

For any \( v \in V_{\text{ext}}^{\text{Dir}} \), the boundary conditions (42) yield the system

\[
\left\{
\begin{array}{l}
c_{j_v,1} e^{i \xi l_{j_v}} + c_{j_v,2} e^{-i \xi l_{j_v}} = 0, \\
c_{j_v,1} e^{i \xi l_{j_v}} - c_{j_v,2} e^{-i \xi l_{j_v}} = 0.
\end{array}\right.
\]

As the determinant of this system is \( -2 \), we deduce that \( c_{j_v,1} = c_{j_v,2} = 0 \). This means that \( p_j = q_j = 0 \) for all \( j = 1, \ldots, N \).

Now fix an arbitrary node \( w \in V_{\text{int}} \) of the last but one generation, in other words, a node having an edge \( j \in J_w \) such that the other extremity of \( e_j \) is in \( V_{\text{ext}}^{\text{Dir}} \) (see Figure 1, where the edge of the first generation is in green, the edges of the last but one in blue and the edges of the last one in red). But for such a vertex, we remark...
that we have just shown that \( p_j = q_j = 0 \) for all \( j \in J_w \setminus \{ j_w \} \), and by the fact that \((p, q) \in D(A_\alpha(0))\) and the assumption (36) (with the help of Lemma 5.6), we deduce that
\[
q_{j_w}(w) = p_{j_w}(w) = 0.
\]
Comparing with (42), we can view this node as a (new) dissipative node and reiterating the previous argument we deduce that
\[
q_{j_w} = p_{j_w} = 0.
\]
From one generation to the previous one we arrive at the root edge and find \( p = q = 0 \).

If \( \xi = 0 \), by (41) we directly have
\[
q_j(x) = c_{j,1}, p_j(x) = c_{j,2}, \forall x \in (0, l_j),
\]
for some constants \( c_{j,1}, c_{j,2} \in \mathbb{C} \) and all \( j = 1, \ldots, N \). Therefore, for all \( v \in V_{\text{ext}}^{\text{Dis}} \), the boundary conditions (42) directly yield
\[
c_{j_v,1} = c_{j_v,2} = 0.
\]
As before using the assumption (36), by iteration from one generation to the previous one, we deduce that \( p = q = 0 \).

As \( A_\alpha(0), A_\alpha \) are maximal dissipative and have a compact resolvent, by the Theorem of Arendt and Batty [4], we directly deduce the following result.

**Corollary 5.8.** Under the assumptions of Theorem 5.7, the semi-groups generated by \( A_\alpha(0) \) and \( A_\alpha \) are strongly stable.

### 6. Uniform stability results.

#### 6.1. The wave equation.

We start with the so-called identity with multiplier (compare with Lemma 3.1 of [35]) which is the key identity for an observability estimate.

**Lemma 6.1.** Let \( T > 0 \) and let \( m : \Gamma \to \mathbb{R} \) be a multiplier with the regularity \( m_j \in C^1([0, l_j]) \), for all \( j = 1, \cdots, N \). Then, for all \((q_0, q_1) \in D(A_\alpha) \times V\), the solution \((q, \partial_t q) \in C([0, \infty), D(A_\alpha) \times V) \cap C^1([0, \infty), V \times L^2(\Gamma))\) of (31) satisfies
\[
\frac{1}{2} \sum_{v \in V} \sum_{j \in J_v} \int_0^T (L_j \nu_j(v, t)^2 + M_j^{-1}|\partial_t q_j(v, t)|^2) m_j(v) \nu_j(v) \, dt
\]
\[
= \frac{1}{2} \sum_{j=1}^N \int_0^T \int_0^{l_j} m_j^{-1}(M_j^{-1}|\partial_t q_j|^2 + L_j|q_j|^2) \, dx_j \, dt + \sum_{j=1}^N M_j^{-1} \int_0^{l_j} \partial_t q_j m_j q_j' \, dx_j \bigg|_0^T.
\]
Proof. This identity is obtained as follows: first multiply the equation

$$\partial^2_t q_j - M_j L_j q''_j = 0,$$

by $M_j^{-1} m_j q'_j$ and integrate on $(0, l_j) \times (0, T)$, secondly perform some integrations by part in space and time as in Lemma I.3.7 of [28, p. 40] for instance and thirdly take the sum on $j$ from 1 to $N$.\[ \square \]

This identity allows to prove the next observability estimate.

**Lemma 6.2.** Let the assumptions of Theorem 5.7 be satisfied. Then, there exist a positive constant $C'$ and a positive constant $C$ such that

$$C'(T - T_0)E(T) \leq \sum_{v \in V_{\text{ext}}} \int_0^T |\partial_t q_{j_v}(v, t)|^2 dt, \forall T > 0.$$  \hspace{1cm}(44)

Proof. First we prove (44) for smooth solutions, namely for $(q, \partial_t q) \in C([0, \infty); V) \cap C^1([0, \infty); L^2(\Gamma))$ such that for all $(q_0, q_1) \in V \times L^2(\Gamma)$, the solution $q \in C([0, \infty); V) \cap C^1([0, \infty); L^2(\Gamma))$ of (31) satisfies

$$C'(T - T_0)E(T) \leq \sum_{v \in V_{\text{ext}}} \int_0^T |\partial_t q_{j_v}(v, t)|^2 dt, \forall T > 0.$$  \hspace{1cm}(44)

Proof. First we prove (44) for smooth solutions, namely for $(q, \partial_t q) \in C([0, \infty), D(A_v) \times V) \cap C^1([0, \infty), V \times L^2(\Gamma))$. Now in the identity (43) we restrict ourselves to a multiplier $m$ such that in the left-hand side of (43) the contribution of the interior nodes is nonpositive and the contribution of the root is zero. So we look for $m$ in the form

$$m_j(x_j) = x_j + \beta_j, \forall j = 1, \ldots, N,$$  \hspace{1cm}(45)

with $\beta_j \in \mathbb{R}$ and satisfying $m_j(r) = 0$ for the root $r$ and

$$\sum_{j \in N_v} M^{-1}_j m_j(v)\nu_j(v)|q'_j(v, t)|^2 \leq 0,$$  \hspace{1cm}(46)

$$\sum_{j \in N_v} L_j m_j(v)\nu_j(v)|\partial_t q_j(v, t)|^2 \leq 0, \forall t > 0, v \in V_{\text{int}}.$$  \hspace{1cm}(47)

If these properties hold, then (43) implies that

$$\frac{1}{2} \sum_{j=1}^N \int_0^T \int_0^{l_j} (M_j^{-1}|\partial_t q_j|^2 + L_j|q'_j|^2) dx_j dt \leq \frac{1}{2} \sum_{v \in V_{\text{ext}}} \int_0^T (L_j|q'_j|(v, t)|^2 + M_j^{-1}|\partial_t q_j(v, t)|^2)m_j(v) dt$$

$$- \sum_{j=1}^N M_j^{-1} \int_0^{l_j} \partial_t q_j m_j q'_j dx_j |_0^T.$$

Hence, there exists a positive constant $C$ such that

$$\int_0^T E(t) dt \leq C \sum_{v \in V_{\text{ext}}} \int_0^T (q'_j(v, t)|^2 + |\partial_t q_j(v, t)|^2) dt + C(E(0) + E(T)).$$

By using the boundary condition (25), we obtain

$$\int_0^T E(t) dt \leq C \sum_{v \in V_{\text{ext}}} \int_0^T |\partial_t q_j(v, t)|^2 dt + C(E(0) + E(T)).$$
Since the energy is non-increasing, we deduce that
\[ TE(T) \leq C \sum_{v \in V_{\text{int}}} \int_0^T \left| \partial_t q_{j_v}(v, t) \right|^2 dt + C(E(0) + E(T)). \]

Owing to (34), we arrive at
\[ TE(T) \leq (1 + \alpha)C \sum_{v \in V_{\text{ext}}} \int_0^T \left| \partial_t q_{j_v}(v, t) \right|^2 dt + 2C\mathcal{E}(T). \]

This yields (44) with \( T_0 = 2C \) for smooth \( q \). For \( q \in C([0, \infty); V) \cap C^1([0, \infty); L^2(\Gamma)) \), we use a density argument.

The proof is then complete if one can build a multiplier \( m \) satisfying the constraints mentioned above. First we notice that for smooth solutions \( q \), for all \( v \in V_{\text{int}} \), we have
\[ \gamma_v \partial_t q(\cdot, t) \in Y_v \text{ and } T_v \gamma_v q(\cdot, t) \in Y_v^\perp \text{, } \forall t > 0. \]

Hence, (46)-(47) hold if
\begin{align*}
\sum_{j \in N_v} M_j^{-1} m_j(v) \nu_j(v) |z_j|^2 &\leq 0, \forall z \in Y_v^\perp, \quad (48) \\
\sum_{j \in N_v} L_j m_j(v) |y_j|^2 &\leq 0, \forall y \in Y_v, v \in V_{\text{int}}, \quad (49)
\end{align*}

With our chosen parametrization, we find equivalently
\begin{align*}
M_j^{-1} m_j(v) |z_j|^2 &\leq \sum_{j \notin N_v \cup \{j_v\}} M_j^{-1} m_j(v) |z_j|^2, \forall z \in Y_v^\perp, \\
L_j m_j(v) |y_j|^2 &\leq \sum_{j \notin N_v \cup \{j_v\}} L_j m_j(v) |y_j|^2, \forall y \in Y_v, v \in V_{\text{int}}.
\end{align*}

But owing to the assumption (50) (see Lemma 5.6), for all \( v \in V_{\text{int}} \), we have
\begin{align*}
M_j^{-1} m_j(v) |z_j|^2 &\leq M_j^{-1} m_j(v)(N_v - 1) \sum_{j \notin N_v \cup \{j_v\}} |b_{v,j}|^2 |z_j|^2, \forall z \in Y_v^\perp, \\
L_j m_j(v) |y_j|^2 &\leq L_j m_j(v)(N_v - 1) \sum_{j \notin N_v \cup \{j_v\}} |a_{v,j}|^2 |y_j|^2, \forall y \in Y_v.
\end{align*}

Hence, (48)-(49) hold if
\begin{align*}
M_j^{-1} m_j(v)(N_v - 1) |b_{v,j}|^2 &\leq M_j^{-1} m_j(v), \\
L_j m_j(v)(N_v - 1) |a_{v,j}|^2 &\leq L_j m_j(v), \forall j \in N_v \setminus \{j_v\}, v \in V_{\text{int}}.
\end{align*}

In other words we need that
\[ m_j(v) \geq m_j(v)(N_v - 1) \max \left\{ \frac{M_j |b_{v,j}|^2}{M_j}, \frac{L_j |a_{v,j}|^2}{L_j} \right\}, \forall j \in N_v \setminus \{j_v\}, v \in V_{\text{int}}. \quad (50) \]

Now we build \( m \) from one generation of edges to the next one. The first generation is the edge \( e_j \), that has the root node as node, the second generation is made of the edges (different from \( e_j \)) which have a node in common with \( e_j \), and by iteration the \( (i + 1)^{th} \) generation is the edges which do not belong to the \( i^{th} \) generation and have a node in common with an edge of the \( i^{th} \) generation.
For the edge of the first generation, we take $\beta_{j_0} = 0$ (in (45) for $j_0$). Now for all edge $j$ of the second generation, we chose $\beta_j$ large enough so that (50) holds at its common node $v$ with $e_{j_0}$. Indeed, this condition reduces to

$$\beta_j \geq l_j, (N_v - 1) \max\left\{ \frac{M_j |b_{v,j}|^2}{M_{j_0}}, \frac{L_j |b_{v,j}|^2}{L_{j_0}} \right\},$$

hence, such a $\beta_j$ exists.

This construction can be iterated from one generation to the next one and allow to obtain a multiplier $m$ fulfilling the requested properties. $\square$

**Theorem 6.3.** Let the assumptions of Theorem 5.7 be satisfied. Then system (24) is exponentially stable, i.e., there exist two positive constants $M$ and $\omega$ such that for all $(q_0, q_1) \in V \times L^2(\Gamma)$, the solution $q \in C([0, \infty); V) \cap C^1([0, \infty); L^2(\Gamma))$ of (24) satisfies

$$\mathcal{E}(t) \leq M e^{-\omega t} \mathcal{E}(0), \forall t \geq 0.$$

**Proof.** The proof is a direct consequence of (44) and the identity (34) that yield for $T > T_0$

$$\mathcal{E}(T) \leq C(T)(\mathcal{E}(0) - \mathcal{E}(T)),$$

with $C(T) = \frac{1}{e(T-T_0)}$. Hence, the result follows from a standard argument, see for instance Lemma 1.3.1 of [3]. $\square$

6.2. The hyperbolic system. We now come back to our system (2) and show its exponential stability.

**Theorem 6.4.** Let the assumptions of Theorem 5.7 be satisfied. Then system (2) is exponentially stable, i.e., there exist two positive constants $C$ and $\omega$ such that for all $(p_0, q_0) \in H$, the solution $(p, q) \in C([0, \infty); H)$ of (2) satisfies

$$\mathcal{E}(t) \leq M e^{-\omega t} \mathcal{E}(0), \forall t \geq 0.$$  \hfill (51)

**Proof.** For $(P_0, Q_0) \in D(A_{\alpha}^2)$, let $(P, Q) \in C([0, \infty); D(A_{\alpha}^2)) \cap C^1([0, \infty); D(A_{\alpha})) \cap C^2([0, \infty); H)$ be the solution of (2) with $K_j = G_j = 0$ and initial data $(P_0, Q_0)$. Then, by Theorem 5.1 we know that $Q$ is a strong solution of (24)-(29). Hence, by Theorem 6.3 we deduce that

$$\mathcal{E}(t) \leq M e^{-\omega t} \mathcal{E}(0), \forall t \geq 0.$$  \hfill (52)

Now we notice that

$$\mathcal{E}(t) = \frac{1}{2} \sum_{j=1}^N \int_0^{t_j} (L_j |q_j'(x_j, t)|^2 + M_j^{-1} |\partial_t q_j(x_j, t)|^2) \, dx_j$$

$$= \frac{1}{2} \left( L_j |q_j'(x_j, t)|^2 + M_j |p_j'(x_j, t)|^2 \right) dx_j$$

$$= \frac{1}{2} \|A_{\alpha}^{(0)}(p, q)\|^2_H,$$

owing to (50) and the definition of $A_{\alpha}^{(0)}$. Hence, (52) is equivalent to

$$\|A_{\alpha}^{(0)}(P(t), Q(t))\|^2_H \leq M e^{-\omega t} \|A_{\alpha}^{(0)}(P_0, Q_0)\|^2_H, \forall t \geq 0.$$  \hfill (53)

For any $(p_0, q_0) \in D(A_{\alpha})$, let $(p, q) \in C([0, \infty); D(A_{\alpha})) \cap C^1([0, \infty); H)$ be the solution of (2) with $K_j = G_j = 0$. As $A_{\alpha}^{(0)}$ is an isomorphism (see Theorem 5.7),
we can consider \((P_0, Q_0) = (A_\alpha^{(0)})^{-1}(p_0, q_0)\) that belongs to \(D(A_\alpha^2)\). As the solution \((P, Q)\) of (2) with \(K_j = G_j = 0\) and initial data \((P_0, Q_0)\) is given by
\[
(P(t), Q(t)) = (A_\alpha^{(0)})^{-1}(p(t), q(t)),
\]
the estimate (53) lead to
\[
\|(p(t), q(t))\|_H^2 \leq M e^{-\omega t} \|(p_0, q_0)\|_H^2, \forall t \geq 0.
\]
This proves (51) for \(A_\alpha^{(0)}\) by a density argument.

Remark 6.5. Note that, under the assumptions of Theorem 5.7, an iterative use of the observability estimate (5.15) of [26] combined with Lemma 5.6 allow to obtain the observability estimate
\[
E(T) \leq C \sum_{v \in V_{\text{ext}}} |\gamma_v p(t)|^2,
\]
for \(T\) large enough. As in Theorem 6.3, this estimate and the identity (consequence of Proposition 5.3)
\[
E(0) - E(T) = \alpha \sum_{v \in V_{\text{ext}}} |\gamma_v q(t)|^2,
\]
allow also to deduce the exponential decay (51).

7. Exact controllability results. Using the exponential decay of system (2) and Russell’s principle (see for instance Theorem 4.1 of [34]) we directly deduce the following exact controllability result.

**Theorem 7.1.** Let the assumptions of Theorem 5.7 be satisfied. Then for \(T > 0\) sufficiently large and for all \((p_0, q_0) \in H\), there exist controls \(u_v \in L^2(0, T)\), for all \(v \in V_{\text{ext}}\) such that the solution \((p, q) \in C([0, \infty); H)\) of
\[
\begin{align*}
\partial_t p_j + L_j \partial_x q_j + G_j p_j &= 0 \text{ in } (0, l_j) \times (0, T), \forall j = 1, \ldots, N, \\
\partial_t q_j + M_j \partial_x p_j + K_j q_j &= 0 \text{ in } (0, l_j) \times (0, T), \forall j = 1, \ldots, N, \\
\gamma_v p(\cdot, t) &\in Z_v, \forall v \in V_{\text{int}}, t \in (0, T), \\
T_v \gamma_v q(\cdot, t) &\in Z_v^\perp, \forall v \in V_{\text{int}}, t \in (0, T), \\
qu_j(v, t) &= 0, \forall v \in V_{\text{ext}}, t \in (0, T), \\
p_j(v, t) = u_v(t), \forall v \in V_{\text{ext}}, t \in (0, T), \\
p(\cdot, 0) &= p_0, q(\cdot, 0) = q_0 \text{ in } \Gamma,
\end{align*}
\]
satisfies
\[
p(T) = q(T) = 0.
\]

Notice that this result extends significantly the results from [43]. Indeed, our Theorem can be applied for a star-shaped network made of three edges with flow rates partition condition at the common node, Dirichlet boundary condition on \(q\) at one vertex and control at the two other vertices (as treated in [43]), and therefore, we obtain the exact controllability result (and the exact observability) of this problem without any supplementary conditions (contrary to Theorem 5.3 of [43]).
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