Synthetic Topological Qubits in Conventional Bilayer Quantum Hall Systems

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The idea of topological quantum computation is to build powerful and robust quantum computers with certain macroscopic quantum states of matter called topologically ordered states. These systems have degenerate ground states that can be used as robust “topological qubits” to store and process quantum information. However, a topological qubit has not been realized since the proposed systems either require sophisticated topologically ordered states that are not available yet, or require complicated geometries that are too difficult to realize. In this paper, we propose a new experimental setup which can realize topological qubits in a simple bilayer fractional quantum Hall (FQH) system with proper electric gate configurations. Compared to previous works, our proposal is accessible with current experimental techniques and only involves well-established topological states. Our system can realize a large class of topological qubits, generalizing the Majorana zero modes studied in the recent literature to more computationally powerful possibilities. We propose three tunneling and interferometry experiments to detect the existence and non-local topological properties of the topological qubits.

One of the greatest challenges of modern physics is to harness the power of quantum mechanics in order to realize quantum computation. The fundamental obstacle is the difficulty of maintaining quantum coherence for long times. Among the various approaches, topological quantum computation (TQC) is a proposal which relates the protection of quantum coherence with fundamental properties of some macroscopic quantum states of matter, called topologically ordered states. These are gapped states of matter with a certain ground state degeneracy determined solely by the spatial topology of the system. All degenerate ground states are indistinguishable in all local properties, so that the quantum coherence between them is robust to environmental noise, for a time that scales exponentially in system size. In two-dimensions, all low energy excitations of topological states are point-like quasi-particles; for a subset of them, called non-Abelian states, a set of quasi-particles at fixed positions can also carry a robust, topological degeneracy of states.

The topological degeneracy in both ground states and non-Abelian quasi-particle states can be used as a “topological qubit,” to carry quantum information. Most proposals use the latter since it is considered easier to realize quantum operations by manipulating the quasiparticle positions. While the non-Abelian states are not yet experimentally established, currently there are two types of candidate systems for non-Abelian states: The non-Abelian quantum Hall states, the most promising of which is proposed to be realized at filling $\nu = 5/2$, and topological superconductors with Majorana zero modes. The topological superconductor proposals have the advantage of possibly larger energy gaps and well-controlled location of non-Abelian defects. However, the non-abelian defects that can be realized in these proposals are restricted to Ising anyons, which have limited computational power, and the required proximity between superconductor and semiconductor or topological insulator adds complications to probing the non-Abelian anyons.

In this paper, we propose a new experimental platform that realizes a wide class of non-abelian defects, vastly generalizing Majorana zero modes, using conventional Abelian bilayer quantum Hall states. Our proposal is inspired by recent progress in vertical field effect transistors, recent theoretical developments regarding extrinsic defects in FQH states, As is illustrated in Fig. 1, our setup only requires the simplest bilayer FQH states that have been observed in the laboratory, with certain configurations of top and bottom gates. The key idea is to use the gate configuration to induce inter-layer tunneling and thus build “staircases” that coherently connect the two layers. The end points of such “staircase” lines are non-Abelian defects, which we refer to as *genons*, although the bilayer FQH state itself is Abelian. The non-Abelian statistics of genons are determined by the parent state of the bilayer, which include the Ising anyons realized in topological superconductors as a special case. With a larger gap than known non-Abelian FQH states and well-controlled position of non-Abelian defects, but without requiring superconductivity, our proposal combines the advantage of both types of proposed systems in the literature.

In what follows, we explain the proposed experimental setup and discuss two novel experiments that can be used to detect the topological qubit.

**Experimental Setup**

Our experimental setup (Fig. 1) consists of a double-layer quantum well system, with line junctions in each layer that are offset in the lateral direction. One way to create the line junctions is with top and bottom gates; the gates are used to deplete the layer closest to them, and to increase the interlayer tunneling over a small region in their vicinity. Another way to create the line junctions, which has been employed successfully in single-layer systems, is through a physical obstruction...
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H₀ and δHᵣ are the Hamiltonian densities of the kinetic/interaction terms and the backscattering terms,
α, β = L, R and I, J = 1, 2 denote the chirality and the layer index of the edge states, respectively. The densities nᵦᵢ = 1/2π φᵦᵢ, with
φᵦᵢ the chiral boson fields satisfying the commutation relations [φᵦᵢ, φᵦᵢ] = ±i/4 δᵦᵢ,sgn(x - y). Tᵣ is backscattering amplitudes between
counter-propagating edge states LI and RJ of the two
layers (see Fig. 1). The velocity of the edge modes, v₀, is
assumed for simplicity to be the same for all of the edge
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while the off-diagonal elements of the symmetric matrix
λ parameterize the density-density interaction between
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To realize the TLJ, the following conditions are re-

FIG. 1. (a) Cross-section of proposed device. Top and bot-
tom gates partially deplete the electron fluids, and the fringe
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indicate the gapless chiral edge states, moving out of or into
the page. (b) 3D view. Interlayer backscattering between
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within each layer, such as an undoped Al₀.₉Ga₁−xAs/AsAs
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With larger interlayer interactions, more exotic inter-
layer correlated states are also possible. Here we consider
the simplest case where each layer individually forms a
ν = 1/m Laughlin FQH state. Double layer 1/3-Laughlin
states have routinely been realized experimentally, with
gaps on the order of several K [22–26].
At the boundary of the line junctions, there are gap-
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Therefore, the ground states form an irreducible representation of the low energy Hamiltonian in the ground state subspace. The scaling dimension of the electron back-scattering term is 
\[
\Delta_t = m \sqrt{1 + \lambda_{12}^R/\lambda_{12}^L} \ \ \text{[30].}
\]
This scaling dimension can be computed on the surface of the torus. With \(n\) TLJ’s, the system is topologically equivalent to a genus \(g = n - 1\) surface, with \(2n + 2\) holes cut out on the surface, which will have a topological degeneracy of \(m^g\). The holes cut on the surface contribute chiral edge states with a finite size gap inversely proportional to the perimeter of the hole, but they do not affect the topological degeneracies at all. We note that the gapless edge states surrounding the TLJ’s can also be removed through more complicated interdigitated gate configurations (see Supplemental Materials).

Since each TLJ increases the degeneracy by a factor of \(m\), each end point of the TLJ, viewed as a point defect, can be associated with \(\sqrt{m}\) degrees of freedom. In the case of the Majorana zero mode carrying \(\sqrt{2}\) degrees of freedom, each end point of a TLJ is a non-Abelian topological defect which we refer to as a genon. \[12\] The resulting topological degeneracies can be used to realize a topological qubit: this setup enables manipulations of such topological qubits by braiding the genons. The braiding process can be shown to correspond to topologically non-trivial surgery operations of the genus \(g\) surface. \[12\] These mathematical operations correspond to cutting the space along a non-contractible loop, rotating by \(2\pi\), and gluing back together.

**Proposed Experiments**

1. **Quantized zero-bias quasiparticle conductance**

The \(\sqrt{m}\) topological degeneracy associated with a genon, i.e. an end point of the TLJ, corresponds to a localized parafermion zero mode \([11, 12, 14-17, 32]\), which generalizes the usual Majorana zero modes in topological superconductors. Zero mode operators \(\alpha_i\) can be defined at each genon, which satisfy the parafermion algebra \(\alpha_i \alpha_j = \alpha_j \alpha_i e^{2\pi i/m}, \ i < j\) with a proper ordering of all genons in the system. Physically, the operator \(\alpha_i\) is a quasi-particle tunneling operator between the two layers at the genon point, which commutes with the Hamiltonian. We refer to Ref. \[12\] for a more rigorous definition of the zero mode operators.

In the case of the Majorana zero modes localized at the ends of a 1D topological superconductor, one of its key signatures is a quantized zero bias conductance peak with conductance \(G = 4e^2/3h\). When a normal metal lead is coupled to the Majorana zero mode \([5, 32, 34]\). This is due to the perfect Andreev reflection in the pres-
ence of the Majorana mode. In our system, the analog of an electron in topological superconductor is a quasiparticle-quasi-hole pair, i.e., a “topological exciton” with charge $(1/m, -1/m)$. Here the two numbers represent the charge of the exciton in the two layers. When such an exciton moves across the TLJ, the two layers are exchanged, such that its charge becomes $(-1/m, 1/m)$; a topological exciton will become its anti-particle when going across a TLJ, just like an electron being transformed into a hole in a superconductor through an Andreev process. The topological exciton has a perfect “Andreev” reflection at presence of the parafermion zero mode, which can be probed experimentally (Fig. 3 (a)). By separately contacting the two layers, the edge states can carry different currents, and the relative current $I_r = I_1 - I_2$ is the exciton current. If we switch off the TLJ (by turning off the gate voltages), the two layers are decoupled and the voltage in each layer is proportional to its current. Therefore $dI_r/dV_e = \frac{1}{m} e^{-\frac{x}{\xi}}$. With the TLJ the two layers are exchanged when the edge states meet the TLJ, so that the two voltages are also reversed, and $dI_r/dV_e = -\frac{1}{m} e^{-\frac{x}{\xi}}$. The cases with and without TLJ correspond to perfect Andreev and perfect normal scattering of the exciton.

To verify that the zero bias peak conductance observed from the perfect “Andreev” reflection experiment is really from a local parafermion zero mode, we can further consider the experimental setup in Fig. 3 (b), which is an analog of using a scanning tunneling microscope (STM) to probe the Majorana zero mode. By contacting source and drain to the two different layers and bringing the edge states close to the TLJ, the edge states carry the exciton current $I_e$, and play the role of an STM tip. It is well-known that the electron tunneling from an STM tip leads to a differential conductance proportional to the local electron density of states. By analogy, the tunneling conductance in our system measures the local density of states of the exciton, related to $G(1, -1)(x, \omega) = \langle V_{1, -1}^\dagger(x, \omega)V_{1, -1}(x, -\omega)\rangle$. $V_{1, -1}(x, t) = e^{i\phi_L(x, t) - i\phi_{RL}(x, t)}$ annihilates an exciton in the upper edge of the TLJ, or equivalently, tunnels a $1/m$ quasiparticle from layer 2 to layer 1. We find $G(1, -1)$ has the following exponential behavior: $G(1, -1)(x, \omega) = f(x)e^{-x/\xi}\delta(\omega)$, where $\xi \sim \hbar v/g$ is the correlation length of the gapped region (with $g$ the gap opened at the TLJ), and $f(x)$ is a power-law function of $x$. The exponential localization is remarkable, since part of the edge states are still gapless. To understand this, consider the edge theory description of the TLJ in Fig. 3 (b). In Eq. 1, 2, we used separate chiral boson fields $\phi_{L(1)}$, $\phi_{L(2)}$ to describe the two edges from each layer. Here, for convenience we describe the whole edge of the first (second) layer by a single chiral boson $\phi_1(x)$. Therefore $\phi_{1, 2}(x)$ each lives on a circle with perimeter $2L$ (with $L$ the length of the TLJ). The edge of each layer is a chiral Luttinger liquid, and it is convenient to group the two chiral Luttinger liquids into a non-chiral Luttinger liquid. For this purpose we define the spatial coordinate $\tilde{x} \in [-L, L]$ of the two edges with opposite chiralities (Fig. 3 (c)). $\tilde{x}$ increases along the counter-clockwise direction for layer 1 and clockwise direction for layer 2. With such a choice, $\phi_1$ and $\phi_2$ has opposite chirality (defined with respect to $\tilde{x}$), and the vertical back-scattering induced at the TLJ occurs locally between $\phi_1(\tilde{x})$ and $\phi_2(\tilde{x})$. The Luttinger liquid Lagrangian can be written as

$$L = \int_{-L}^{L} d\tilde{x} \left[ (\partial_\tilde{x} \varphi)^2 + v(\partial^2 \varphi)^2 + g \theta(-\tilde{x}) \cos(m \varphi) \right].$$

where $\varphi = \phi_1 + \phi_2$ and where $\theta(x)$ is a step function. The tunneling induces a gap in the region $\tilde{x} \in [-L, 0]$, and the region $\tilde{x} \in [0, L]$ remains gapless. In this representation, the $m$ topological degenerate ground states are correctly given by the $m$ minima of the $\cos(m \varphi)$ term, with $\varphi$ pinned to the values $2\pi n/m$, $n = 0, 1, ..., m - 1$. Importantly, in this coordinate the exciton annihilation operator becomes nonlocal: $V_{-1, 1}(x) = e^{i\phi_1(x) - i\phi_2(-x)}$. Due to the non-local commutation relation of chiral boson fields, applying this operator to the ground state causes a domain wall in $\varphi$ in the region $-L < \tilde{x} < 0$. Away from the defects at $x = 0$ and $x = -L \sim L$, such a domain wall costs an energy of order the gap $g$. Therefore, the correlations of this operator must decay exponentially, away from the positions of the defects. At the positions of the defects, this operator keeps the systems in the ground
suring the currents left and right-moving edges in both layers, and measures the interlayer current noise cross-correlation of the two quasiparticle paths in these two loops are induced due to the uncertainty principle. Therefore non-local correlation between interference patterns in these two loops L, L’ measure the electric charge in the loops, which measure the topological qubit state of the genons 1, 2 and 2, 3 respectively. As a key consequence of the non-Abelian nature of genons, the topological qubit degrees of freedom are non-local, and the topological charge carried by genons 1, 2 does not commute with that of 2, 3. Therefore non-local correlation between interference patterns in these two loops are induced due to the uncertainty principle. Alternatively, the non-commutativity of the two quasiparticle paths L and L’ can also be understood from the geometrical picture shown in Fig. 2 with the measured topological charge corresponding to \( W(a), W(b) \).

To be more precise, the interference pattern can be observed by applying a voltage difference \( V \) between the left and right-moving edges in both layers, and measuring the currents \( I_1, I_2 \) (Fig. 4). We find that the interlayer current noise cross-correlation \( S_{12}(t) \) is a dramatic feature:

\[
S_{12}(t) \equiv \frac{1}{2} \langle (I_1(t), I_2(0)) \rangle - \langle I_1(t) \rangle \langle I_2(0) \rangle
\]

Such a long time noise correlation, at finite temperatures, is a striking direct consequence of the non-Abelian nature of the genons, which has never been proposed or observed before in an Abelian state. Intuitively, this is understood as follows. The current \( I_1(t) \) depends on the quasiparticle interference between the two QPCs \( \Gamma_{1,2} \). When a quasiparticle leaves the top layer through the QPC \( \Gamma_1 \), its fractional statistics cause a phase shift in the interference of loop \( L \) and therefore changes the value of \( I_1(t) \). Such a phase shift is permanent until the next quasi-particle tunneling occurs at \( \Gamma_1 \). The converse is also true, that the quasiparticle tunneling at \( \Gamma_2 \) causes a permanent phase shift in \( I_2(t) \). Therefore, the fluctuation of the currents \( I_1(t) \) and \( I_2(0) \) have a long-time correlation for both \( t > 0 \) and \( t < 0 \). It should be noticed that only having three QPC’s \( \Gamma_1, \tilde{\Gamma}_1, \Gamma_2 \) can already induce a long-time correlation of \( S_{12} \), but only for \( t > 0 \).

We present in the Supplemental Materials an explicit calculation of \( S_{12} \) based on the edge theory; the result, for \( |t| \gg 1/T \) and to lowest order in the tunneling strength at the QPC’s, takes the form:

\[
S_{12}(t) = T^\frac{m}{2} |\tilde{\Gamma}_1 \tilde{\Gamma}_2| \theta(-t) S_{12}^{(1)} + |\tilde{\Gamma}_1 \Gamma_2| \theta(t) S_{12}^{(2)} + |\Gamma_1 \Gamma_2 \tilde{\Gamma}_1 \tilde{\Gamma}_2| (S_{12}^{(3)} \theta(t) + S_{12}^{(4)} \theta(-t)) \]  

(5)

The dimensionless functions \( S_{12}^{(i)} \), whose explicit form is given in the Supplemental Materials, depend on the quasiparticle statistics (they are non-zero only when \( m \neq 1 \) and decay with the dimensionless ratios \( \hbar eV/mk_B T \), and \( k_B T x/hv \), which are optimally of order one, where \( x \) is the distance scale between the QPCs.

On much longer time scales, thermally excited mobile quasiparticles can decohere the state of the qubit, which will wash out the noise signature considered here. However this time scale is expected to be exponentially large in \( 1/T \), (ie t \( \propto e^{S/T} \)), because the quasiparticle density is exponentially small in \( 1/T \).

**Discussion**

A longer term goal is to implement unitary gates on the topological qubit by braiding genons. This can be realized with further sets of gates to tune interactions between various zero modes localized at the genons. The computational power of genons in generic Abelian states is stronger than that provided by Majorana zero modes. The setup proposed here can
be generalized to realize universal TQC in non-abelian states that by themselves would not be universal for TQC [12]. Examples include two layers of the non-abelian phase of Kitaev’s honeycomb model [12, 38], or the odd-denominator Bonderson-Slingerland FQH states [39, 40].

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FIG. 5. The defects (genons) can be viewed as domain walls between two different ways of generating an energy gap in the counterpropagating edge states (double arrows indicate interedge electron tunneling). The zero mode operator $\alpha_{2i-1}$ corresponds to the quasiparticle hopping process shown in the figure, projected into the ground state subspace of the edge theory.

**Supplemental Materials**

**I. Parafermion Zero Modes**

As mentioned in the main text, an important property of the TLJs is that their ends localize exotic forms of zero modes, called parafermion zero modes, which generalize the usual Majorana zero mode algebra. Here we will briefly recall the definition and physical meaning of these zero mode operators, reviewing results from $[12]$.

Let us imagine aligning all TLJs, and cutting the whole system along the line, yielding the counterpropagating edge states. Then, the normal regions of the fluid can be obtained by gluing the edges back together with the electron tunneling terms $\sum_I (\Psi^\dagger_{eL_I} \Psi_{eR_I} + H.c.) \propto \sum_I \cos(m\phi_I)$, while the twisted line junctions can be described by gluing the edges back together with the twisted tunneling $\Psi^\dagger_{eL_I} \Psi_{eR_1} + \Psi^\dagger_{eL_2} \Psi_{eR_1} + H.c. \propto \sum_I \cos(m\tilde{\phi}_I)$, where we have defined $\phi_I = \phi_{LI} + \phi_{RI}$, and $\tilde{\phi}_I = \phi_{LI} + \phi_{R_1} + \phi_{R_2}/2$. Here, we have included also the tunneling term $\Psi^\dagger_{eL_2} \Psi_{eR_1} + H.c.$, which is not present in the experimental setup of Fig. 1.

The ends of the TLJ can therefore be considered as domain walls between these two different ways of gapping the counterpropagating edge states:

$$\delta H_{tun} = g \left\{ \begin{array}{ll} \sum_I \cos(m\phi_I) & \text{if } x \in A_i \\ \sum_I \cos(m\tilde{\phi}_I) & \text{if } x \in B_i \end{array} \right. \quad (6)$$

where $A_i$ indicates the normal regions, while $B_i$ are the regions associated with the TLJs.

As shown in $[12]$, we can define quasiparticle tunneling operators (see Fig. [5]) $\alpha_{2i-1} = e^{i\phi_1(x_{A_i})} e^{-i\phi_1(x_{B_i})} \Psi^\dagger_{eL_I} \Psi_{eR_1} + \Psi^\dagger_{eL_2} \Psi_{eR_1} + H.c. \propto \sum_I \cos(m\tilde{\phi}_I)$, which are zero modes of the edge Hamiltonian, $[\alpha_i, H_{edge}] = 0$, and which satisfy a $\mathbb{Z}_m$ 'parafermion' algebra: $\alpha_i \alpha_j = \alpha_j \alpha_i e^{2\pi i/m}$, and $\alpha_m = 1$.

Other ways of creating domain walls between two different ways of generating an energy gap in a single pair of counterpropagating chiral Luttinger liquids has been explored in $[14][16]$, where similar parafermion zero modes are obtained.

**II. Interdigitated gating**

In the experimental setup described in the main text, there are gapless edge states surrounding the TLJs. These edge states do not affect the topological degeneracies, the exponential localization of the zero modes, or the possibility of braiding the genons by tuning the interactions between the parafermion zero modes. Nevertheless, it might be desirable to remove these gapless edge states, while still preserving the topological degeneracy. This can be done simply by alternating the offset in space (see Fig. [6]). The tunneling term in the edge states can be modelled as follows:

$$\delta H_{tunn} = \alpha(x) \cos(m\tilde{\phi}_1) + \beta(x) \cos(m\tilde{\phi}_2). \quad (7)$$

While $\alpha(x)$ and $\beta(x)$ are never both non-zero simultaneously, at long wavelengths only the average value of $\alpha$ and $\beta$ is important. Therefore, both edge states can be gapped in the twisted way on average in this setup.
FIG. 6. Alternating the offset of the top and bottom gate will allow us to generate an energy gap for all edge states. At long wavelengths, both twisted tunneling terms, $\sum \cos(m \phi_I)$ are simultaneously present.

III. Interlayer Current Noise Cross-Correlation

A. Summary and Results

Here, will present the derivation of the current noise cross-correlation for the interferometer summarized in the main text. We consider a tunneling Hamiltonian with four QPCs [41, 42]:

$$H_{\text{tun}}(t) = \sum_{i=1}^{2} [H_i(t) + \tilde{H}_i(t)],$$

$$H_i(t) = v_i \hat{O}^+(x_i) + v_i^* \hat{O}^-(x_i),$$

$$\tilde{H}_i(t) = \tilde{v}_i \hat{\tilde{O}}^+(\tilde{x}_i) + \tilde{v}_i^* \hat{\tilde{O}}^-(\tilde{x}_i),$$

where

$$v_i = \Gamma_i e^{-i \omega_i t}, \quad \tilde{v}_i = \tilde{\Gamma}_i e^{-i \omega_i t}. \quad (9)$$

$\Gamma_i, \tilde{\Gamma}_i$ are the tunneling matrix elements for the different QPCs, and $\omega_i = e^* V_i$, where $V_i$ are the potential differences across the QPCs and $e^* = e/m$ is the elementary fractional charge of the quasiparticles. We note that in the setup of Fig. 4 to lowest order in the tunneling matrix elements, the voltage across each QPC is $V_i = V$. For what follows, it will be useful to define:

$$x_{12} \equiv x_1 - x_2, \quad \tilde{x}_{12} = \tilde{x}_1 - \tilde{x}_2. \quad (10)$$

The operator $\hat{O}^\pm$ is the quasiparticle tunneling operator across QPCs enclosing the normal regions of the fluid:

$$\hat{O}^\pm(x_i) = \kappa_i^\pm e^{\pm i(\phi_{1L} - \phi_{1R})(x_i)}, \quad (11)$$

where $\kappa_i$ are Klein factors that ensure that the quasiparticle tunneling operators at different points in space commute with each other, as required. The operators $\hat{\tilde{O}}^\pm$ are quasiparticle tunneling operators across the twisted line junctions (TLJs):

$$\hat{\tilde{O}}^\pm(\tilde{x}_i) = \tilde{\kappa}_i^\pm e^{\pm i(\phi_{1L} - \phi_{2R})(\tilde{x}_i)}, \quad (12)$$

where $\tilde{\kappa}_i^\pm$ are also Klein factors. The commutation relations of $\kappa_i^\pm$ and $\tilde{\kappa}_i^\pm$ can be determined from the boson commutation relations:

$$[\phi_{L1}(x), \phi_{L1}(x')] = i \frac{\pi}{m} \text{sgn}(x - x'),$$

$$[\phi_{R1}(x), \phi_{R1}(x')] = -i \frac{\pi}{m} \text{sgn}(x - x'), \quad (13)$$

from which we determine that $[\kappa_i^+, \kappa_j^+] = 0$, $[\tilde{\kappa}_i^+, \tilde{\kappa}_j^+] = 0$, and

$$\kappa_i^+ \tilde{\kappa}_j^+ = \tilde{\kappa}_j^+ \kappa_i^+ e^{i \theta_{ij}^\pm}, \quad (14)$$

where $\theta_{ij}^\pm = rs \pi \text{sgn}(x_i - \tilde{x}_j)/m.$
In the interferometer discussed in the main text, which probes the non-abelian character of the genons, we require \( x_1 < \tilde{x}_1 < x_2 < \tilde{x}_2 \). However it will also be helpful to compare this with the case \( x_1 < x_2 < \tilde{x}_1 < \tilde{x}_2 \), which does not probe the non-commutativity of the two non-commuting, non-contractible quasiparticle paths.

Now, we let \( I_{b1} \) and \( I_{b2} \) be the current that is backscattered through the normal regions and the TLJs, respectively:

\[
I_{b_j}(t) = \frac{dQ_{Rj}}{dt} = -i[H_{\text{tun}}(t),\langle Q_{Rj}\rangle],
\]

where \( Q_{Rj} = \frac{1}{2e} \int d\bar{x}\partial\bar{x}\phi_{Rj} \) is the total charge on the right edge in the \( I \)th layer. Thus:

\[
I_{b1}(t) = \frac{i}{m} \sum_{i=1}^{2} [\Gamma_i e^{-ie^v t}\Omega^+(x_i) - H.c.] = \frac{i}{m} \sum_{i=1}^{2} [\tilde{v}_i\Omega^+(x_i) - v_i^*\Omega^-(x_i)] = \frac{i}{m} \sum_{i=1}^{2} \epsilon_i\tilde{v}_i^*\tilde{O}^+(x_i)
\]

\[
I_{b2}(t) = \frac{i}{m} \sum_{i=1}^{2} [\tilde{v}_i e^{-ie^v t}\tilde{\Omega}^+(\tilde{x}_i) - H.c.] = \frac{i}{m} \sum_{i=1}^{2} [\tilde{v}_i\tilde{\Omega}^+(\tilde{x}_i) - \tilde{v}_i^*\tilde{\Omega}^-(\tilde{x}_i)] = \frac{i}{m} \sum_{i=1}^{2} \epsilon_i\tilde{v}_i^*\tilde{O}^+(\tilde{x}_i)
\]

In the setup of Fig. 4, we find that:

\[
I_1 = \frac{e^2}{m}\tilde{v} + I_{b1},
\]

\[
I_2 = \frac{e^2}{m}\tilde{v} + I_{b2} + I_{g2},
\]

where \( I_1 \) and \( I_2 \) are the currents flowing through the device in the first and second layer (see Fig. 4). In order to understand \( I_{g2} \), observe that current can be backscattered through the TLJs in two different ways. The first way involves only electron tunneling, will not have long-time cross-correlations between the quasiparticle tunneling current \( I_{b1} \). Therefore:

\[
S_{12}(t) = \frac{1}{2}\langle I_1(t)I_2(0)\rangle - \langle I_1(t)\rangle\langle I_2(0)\rangle
\]

\[
= \frac{1}{2}\langle I_{b1}(t)(I_{b2}(0) + I_{g2}(0))\rangle - \langle I_{b1}(0)\rangle\langle I_{b2}(0) + I_{g2}(0)\rangle
\]

We will be interested in the limit \( |t| \gg 1/T \), in which case it will become clear that \( I_{g2} \), which involves only electron tunneling, will not have long-time cross-correlations between the quasiparticle tunneling current \( I_{b1} \).

Therefore, we will compute the following interlayer noise cross-correlation:

\[
S_{12}(t) = \frac{1}{2}\langle\{I_{b1}(t),I_{b2}(0)\}\rangle - \langle I_{b1}(t)\rangle\langle I_{b2}(0)\rangle.
\]

Note that the expectation values are given by

\[
\langle A(t) \rangle = \langle TCA(t)e^{-i\int_C H_{\text{tun}}(\tau)d\tau} \rangle_0,
\]

where the \( \langle \cdots \rangle_0 \) indicates an expectation value in the unperturbed theory. \( C \) indicates the Keldysh contour, and \( T_C \) is time-ordering in the Keldysh contour. In our calculation, we will drop the subscript and assume all expectation values are in the unperturbed theory.

Let us first present the result of the calculation. We find that to lowest order in the QPC tunneling matrix elements,

\[
S_{12}(|t| \gg 1/T) = \frac{(\pi T)^{4/m-2}}{m^2} [\Gamma^1_\text{eff}\tilde{\Gamma}^2_\text{eff}\theta(-t) S_{12}^{(1)}(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{x}_1,\tilde{x}_2,\tilde{\varphi}_1,\tilde{\varphi}_2) + |\Gamma^1_\text{eff}|^2 \tilde{\Gamma}^2_\text{eff}\theta(t) S_{12}^{(1)}(\tilde{\omega}_1,\tilde{\varphi}_1,\tilde{x}_1,\tilde{x}_2,\tilde{\varphi}_1,\tilde{\varphi}_2)] +
\]

\[
|\Gamma^1_\text{eff}\tilde{\Gamma}^2_\text{eff}|\tilde{\Gamma}^2_\text{eff}\theta(-t) S_{12}^{(2)}(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{x}_1,\tilde{x}_2,\tilde{\varphi}_1,\tilde{\varphi}_2)\theta(t) + S_{12}^{(2)}(\tilde{\omega}_1,\tilde{\omega}_2,\tilde{x}_1,\tilde{x}_2,\tilde{\varphi}_1,\tilde{\varphi}_2,\tilde{\varphi}_2,\tilde{\varphi}_1)\theta(-t))]
\]

where we have defined the dimensionless quantities

\[
\tilde{\omega}_1 \equiv \frac{\hbar}{\pi k_B T} \omega_1, \quad \tilde{\omega}_2 \equiv \frac{\hbar}{\pi k_B T} \omega_2, \quad \tilde{x}_1 \equiv \frac{\hbar}{\pi k_B T} x_{12}, \quad \tilde{x}_2 \equiv \frac{\hbar}{\pi k_B T} x_{12}
\]

(22)
and $\varphi_{12}$ and $\tilde{\varphi}_{12}$ are the phase differences between the tunneling matrix elements of the QPCs: $\Gamma_1\Gamma_2^* = |\Gamma_1\Gamma_2|e^{i\varphi_{12}}$, and $\tilde{\Gamma}_1\tilde{\Gamma}_2^* = |\tilde{\Gamma}_1\tilde{\Gamma}_2|e^{i\tilde{\varphi}_{12}}$. Note that $S_{12}^{(i)}$ depend on only dimensionless quantities.

\[
S_{12}^{(1)}(\tilde{\omega}_2, \tilde{\omega}_1, x, \varphi_{12}) = 4c_r(\tilde{\omega}_2, 2x)[(\cos(2\pi/m) - 1) \cos\varphi_{12} f_-(\tilde{\omega}_1, 0) - \sin\varphi_{12} \sin(2\pi/m) f_+(\tilde{\omega}_1, 0)],
\]
\[
S_{12}^{(2)}(\tilde{\omega}_1, \tilde{\omega}_2, x, y, \tilde{\varphi}_{12}, \varphi_{12}) = 8c_r(\tilde{\omega}_1, 2x) \cos\varphi_{12} [2(\cos(\pi/m) - 1) \cos(\varphi_{12}) c_r(\tilde{\omega}_2, 2y) - \sin(\pi/m) \sin(\varphi_{12}) f_+(\tilde{\omega}_2, y)],
\]

(23)

The functions $f_{\pm}(\tilde{\omega}, \tilde{x})$ are defined as

\[
f_{\pm}(\tilde{\omega}, \tilde{x}) \equiv \int_{-\infty}^{\infty} e^{i\tilde{\omega}t} dt [e^{-\frac{\tilde{\omega}}{m}(\text{sgn}(t+x)+\text{sgn}(t-x))} \pm e^{\frac{\tilde{\omega}}{m}(\text{sgn}(t+x)+\text{sgn}(t-x))}]{\frac{1}{\sinh |t+x| \sinh |t-x|}}{1/m}
\]

(24)

while $c_r(a, b)$ is defined as

\[
c_r(a, b) \equiv -2\sin(\pi/m) \text{ Im} \left\{ e^{iab/2} \int_0^\infty dt e^{iat} \left( \frac{1}{\sinh |t+b|} \right)^{1/m} \right\}
\]

(25)

As a check, observe that when $m = 1$ and $m \to \infty$, the two non-contractible paths for quasiparticle propagation commute, and therefore this noise cross-correlation should vanish at long times, which it indeed does.

In Fig. 7, we plot the functions $f_{\pm}(a, b)$ and $-c_r(a, b)/\sin(\pi/m)$. In Fig. 8, we plot $S_{12}^{(i)}(a, b, a, b, 0, 0)$, for $i = 1, 2$; that is, for the case where the voltage differences between all QPCs is the same, $x_{12} = \tilde{x}_{12}$, and for zero phase differences $\varphi_{12}$ and $\tilde{\varphi}_{12}$. We see that when $a$ and $b$ are of order one, $S_{12}^{(i)}$ reach their maximum value, and decay quickly as $a$ and $b$ increase. From this, we see that the noise cross-correlation will require the dimensionless parameters $\tilde{\omega}_1$, $\tilde{x}_{12}$, and $\tilde{x}_{12}$ to all be of order one. If we assume $T \approx 50 - 100mK$, and $v \approx 10^5m/s$, we see that this requires $x_{12}$ and $\tilde{x}_{12}$ to be on the order of several $\mu m$, which is reasonable for realistic device dimensions.

**B. Klein Factors**

Here, we will determine the Keldysh-contour ordered correlation functions of the Klein factors, which are required for the calculation. To do this, first we observe that the two point function for Klein factors that commute with each other should be one. Next, we observe that the correlation functions of several Klein factors should factor into two-point functions. In order to obtain the two-point function of non-commuting Klein factors, it is helpful to represent the Klein factors in terms of bosonic operators [43]:

\[
\kappa_i^\pm = e^{\pm i\sqrt{1/m} \theta_i}, \quad \tilde{\kappa}_i^\pm = e^{\pm i\sqrt{1/m} \tilde{\theta}_i}
\]

(26)

where $\theta_i, \tilde{\theta}_i$ are bosonic operators and for what follows we will leave the normal ordering $\cdots :$ implicit. Using the fact that $e^A :: e^B := e^B :: e^A : e^{[A,B]}$ for cases where $[A, B]$ is a c-number, we find that the commutation relation of

![FIG. 7. Density plots of (a) $-\frac{c_r(a,b)}{2 \sin(\pi/m)}$, (b) $f_-(a,b)$, and (c) $f_+(a,b)$, for $m = 3$.](https://example.com/figure7.png)
To calculate this, we use the fact that the Klein factors is reproduced when

\[
[\theta_i, \theta_j] = -i\pi \text{sgn}(x_i - x_j), \quad [\theta_i, \theta_j] = [\theta_i, \theta_j] = 0
\] (27)

Since \(\theta_i\) and \(\theta_j\) are conjugates, we can form a harmonic oscillator algebra:

\[
a_{ij} = \frac{1}{\sqrt{2\pi}} (\theta_i + i\theta_j), \quad a_{ij}^\dagger = \frac{1}{\sqrt{2\pi}} (\theta_i - i\theta_j),
\] (28)

such that

\[
[a_{ij}, a_{ij}^\dagger] = -\text{sgn}(x_i - x_j), \quad (a_{ij} a_{ij}^\dagger) = \frac{1 - \text{sgn}(x_i - x_j)}{2}.
\] (29)

We suppose that the Hamiltonian for these bosonic fields is zero, as they have no dynamics. From this we conclude:

\[
\langle \theta_i(t)\theta_i(0) \rangle = \langle \theta_i(t)\theta_i(0) \rangle = \pi/2, \quad \langle \theta_i(t)\theta_j(0) \rangle = -\langle \theta_j(t)\theta_i(0) \rangle = -\frac{i\pi}{2} \text{sgn}(x_i - x_j).
\] (30)

From this, we conclude that the Keldysh contour ordered correlation function is:

\[
g_{ij}^{n_i n_j}(t) \equiv \langle T_C \theta_i(t^n)\theta_j(0^{n_j}) \rangle = -\frac{i\pi}{2} \text{sgn}(x_i - x_j)\chi_{n_i n_j}(t),
\] (31)

where

\[
\chi_{n_i n_j}(t) = \frac{\eta_i + \eta_j}{2} \text{sgn}(t) - \frac{\eta_i - \eta_j}{2}.
\] (32)

Therefore:

\[
\langle T_C \kappa_i^{\dagger}(t^n)\kappa_j^\dagger(0^{n_j}) \rangle = e^{i\frac{\pi}{2} \text{sgn}(x_i - x_j)}\chi_{n_i n_j}(t) \text{sgn}(x_i - x_j)
\] (33)

For later reference, we will need the following correlation function:

\[
\langle T_C \kappa_i^{\dagger}(t^n)\kappa_{\beta_1}^{-\epsilon_1}(t^{n_1})\kappa_{\alpha_2}^{\epsilon_2}(0^{-})\kappa_{\beta_2}^{-\epsilon_2}(0^{n_2}) \rangle = \langle T_C e^{i\sqrt{1/m\epsilon_1\theta_{\alpha_1}(t^n)}} e^{-i\sqrt{1/m\epsilon_1\theta_{\beta_1}(t^{n_1})}} e^{i\sqrt{1/m\epsilon_2\theta_{\alpha_2}(0^{-})}} e^{-i\sqrt{1/m\theta_{\beta_2}(0^{n_2})}} \rangle
\] (34)

To calculate this, we use the fact that \(\langle T_C e^{A_1 \cdots A_n} \rangle = e^{\sum_{i<j} A_i A_j} \langle T_C A_i A_j \rangle\), where in the Klein factor case, we only include correlation functions between non-commuting variables. Therefore:

\[
\begin{align*}
\langle T_C \kappa_i^{\dagger}(t^n)\kappa_{\beta_1}^{-\epsilon_1}(t^{n_1})\kappa_{\alpha_2}^{\epsilon_2}(0^{-})\kappa_{\beta_2}^{-\epsilon_2}(0^{n_2}) \rangle &= e^{i\frac{\pi}{2} \text{sgn}(x_i - x_j) s_{\alpha_1 \alpha_2} - s_{\beta_1 \beta_2}} e^{-i\frac{\pi}{2} \text{sgn}(x_i - x_j) s_{\alpha_1 \alpha_2} + s_{\beta_1 \beta_2}} \\
&= \begin{cases} 
- e^{i\frac{\pi}{2} \text{sgn}(x_i - x_j) s_{\alpha_1 \alpha_2} - s_{\beta_1 \beta_2} + \eta_i s_{\beta_1 \beta_2} + \eta_j s_{\alpha_1 \alpha_2} - s_{\beta_1 \beta_2}} & t > 0 \\
- e^{i\frac{\pi}{2} \text{sgn}(x_i - x_j) s_{\alpha_1 \alpha_2} + \eta_i s_{\beta_1 \beta_2} - \eta_j s_{\alpha_1 \alpha_2} - s_{\beta_1 \beta_2}} & t < 0
\end{cases}
\end{align*}
\] (35)
where $s_{\alpha\beta} \equiv sgn(x_{\alpha} - x_{\beta})$. When $x_1 < x_2 < \tilde{x}_1 < \tilde{x}_2$, $s_{\alpha\beta} = -1$ for all $\alpha$, $\beta$, and therefore
\[
\langle TC e^{i\phi(x,t^n)} e^{-i\phi(x',t'^n)} \rangle = e^{\langle TC \phi(x,t^n) \phi(x',t'^n) \rangle}.
\]
(38)

At finite temperatures, this is (setting the velocities of the left- and right-moving modes to unity):
\[
G_{\eta_1\eta_2}(x,t) = \left( \frac{(\pi T)^2}{\sin[\pi T(\delta + i\chi_{\eta_1\eta_2}(t)(t + x))]} \right)^{1/m} \left( \frac{(\pi T)^2}{\sin[\pi T(t + x)]} \right)^{1/m},
\]
(39)

where
\[
\chi_{\eta_1\eta_2}(t) = \frac{(\eta_1 + \eta_2)}{2} \text{sgn}(t) - \frac{(\eta_1 - \eta_2)}{2}.
\]
(40)

Taking the limit $\delta \to 0^+$, we can replace the above by
\[
G_{\eta_1\eta_2}(x,t) = e^{-i\frac{\pi T}{m} \chi_{\eta_1\eta_2}(t)(\text{sgn}(t+)+\text{sgn}(t-))} \left( \frac{(\pi T)^2}{\sin[\pi T(t + x)]} \right)^{1/m} \left( \frac{(\pi T)^2}{\sin[\pi T(t - x)]} \right)^{1/m}.
\]
(41)

The chiral correlator, involving only the left or right-moving modes (and re-inserting the velocities $v_L/R$ for the left/right movers), is:
\[
G^{L/R}_{\eta_1\eta_2}(x,t) = \langle TC e^{i\phi_{L/R}(x,t^n)} e^{-i\phi_{L/R}(0,0^n)} \rangle = e^{-i\frac{\pi T}{m} \chi_{\eta_1\eta_2}(t)(\text{sgn}(t+)+\text{sgn}(t-))} \left( \frac{(\pi T)^2}{\sin[\pi T(t + x)]} \right)^{1/m} \left( \frac{(\pi T)^2}{\sin[\pi T(t - x)]} \right)^{1/m}.
\]
(42)

### C. Boson correlation functions

We will need the following correlation functions:
\[
G_{\eta \eta'}(x-x',t-t') \equiv \langle TC e^{i\phi(x,t^n)} e^{-i\phi(x',t'^n)} \rangle = e^{\langle TC \phi(x,t^n) \phi(x',t'^n) \rangle}.
\]
(43)

### D. Details of calculation

Here we describe in detail the calculation of $S_{12}(t)$, to lowest order in the tunneling matrix elements. We have (recall eq. (39, 20)):
\[
\frac{1}{2} \langle \{ I_{b_1}(t), I_{b_2}(0) \} \rangle = \frac{1}{4} \sum_{\eta = \pm} (-i)^2 \int d\tau_1 d\tau_2 \langle TC I_{b_1}(t^n) H_{\text{Tun}}(\tau_1) I_{b_2}(0^{-\eta}) H_{\text{Tun}}(\tau_2) \rangle
\]
\[
= -\frac{1}{2} \sum_{\eta_1, \eta_2 = \pm} \int_{-\infty}^{\infty} \eta_1 \eta_2 \tau_1 \tau_2 \langle TC I_{b_1}(t^n) H(\tau_1) I_{b_2}(0^{-\eta}) \tilde{H}(\tau_2) \rangle.
\]
(44)

Consider the correlation function:
\[
\langle TC I_{b_1}(t^n) H(\tau_1) I_{b_2}(0^{-\eta}) \tilde{H}(\tau_2) \rangle = \frac{i^2}{m^2} \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2} \sum_{\epsilon_{\alpha_1}, \epsilon_{\beta_1}, \epsilon_{\alpha_2}, \epsilon_{\beta_2}} v_{\alpha_1}^c v_{\beta_1}^c v_{\alpha_2}^c v_{\beta_2}^c \epsilon_{\alpha_1} \epsilon_{\alpha_2} \langle TC O^{\epsilon_{\alpha_1}}(x_{\alpha_1}, t^n) O^{\epsilon_{\beta_1}}(x_{\beta_1}, \tau_1^n) \tilde{O}^{\epsilon_{\alpha_2}}(x_{\alpha_2}, 0^{-\eta}) \tilde{O}^{\epsilon_{\beta_2}}(x_{\beta_2}, \tau_2^n) \rangle
\]
(45)
Recall that in our notation, the superscript $\bar{}$ implies complex conjugation, not inverse.

\begin{align}
\langle T C I_b (t^\nu) H (\tau_1^{\nu}) I_{b2} (0^{-}) \bar{H} (\tau_2^{\nu}) \rangle &= - \frac{1}{m^2} \sum_{\epsilon_1, \epsilon_2 = \pm} \sum_{\alpha_1 , \beta_1 , \alpha_2 , \beta_2} \epsilon_1 \epsilon_2 \Gamma_{\alpha_1}^{\epsilon_1} \Gamma_{\beta_1}^{-\epsilon_1} \Gamma_{\alpha_2}^{\epsilon_2} \Gamma_{\beta_2}^{-\epsilon_2} e^{i \epsilon_1 \omega_1 (t - \tau_1) + i \epsilon_2 \omega_2 (t - \tau_2)} \\
&\quad \langle T C K_{\alpha_1}^{\epsilon_1} (t^\nu) K_{\beta_1}^{-\epsilon_1} (\tau_1^{\nu}) \bar{K}_{\alpha_2}^{\epsilon_2} (0^{-}) \bar{K}_{\beta_2}^{-\epsilon_2} (\tau_2^{\nu}) \rangle \\
&\quad G_{\eta \eta}^{R,R} (\vec{x}_{\alpha_2} - \vec{x}_{\beta_2}, - \tau_2) G_{\eta \eta}^{R,R} (x_{\alpha_1} - x_{\beta_1}, t - \tau_1) \\
&\quad \langle T C e^{i \epsilon_1 \phi_1 (x_{\beta_1}, t^\nu) e^{-i \epsilon_2 \phi_1 (\vec{x}_{\beta_2}, \tau_2^{\nu})} e^{i \epsilon_2 \phi_1 (\vec{x}_{\beta_2}, \tau_2^{\nu})} \rangle
\end{align}

(46)

This gives:

\begin{align}
\langle T C I_b (t^\nu) H (\tau_1^{\nu}) I_{b2} (0^{-}) \bar{H} (\tau_2^{\nu}) \rangle &= - \frac{1}{m^2} \sum_{\epsilon_1, \epsilon_2 = \pm} \sum_{\alpha_1 , \beta_1 , \alpha_2 , \beta_2} \epsilon_1 \epsilon_2 \Gamma_{\alpha_1}^{\epsilon_1} \Gamma_{\beta_1}^{-\epsilon_1} \Gamma_{\alpha_2}^{\epsilon_2} \Gamma_{\beta_2}^{-\epsilon_2} e^{i \epsilon_1 \omega_1 (t - \tau_1) + i \epsilon_2 \omega_2 (t - \tau_2)} \\
&\quad \langle T C K_{\alpha_1}^{\epsilon_1} (t^\nu) K_{\beta_1}^{-\epsilon_1} (\tau_1^{\nu}) \bar{K}_{\alpha_2}^{\epsilon_2} (0^{-}) \bar{K}_{\beta_2}^{-\epsilon_2} (\tau_2^{\nu}) \rangle \\
&\quad G_{\eta \eta}^{R,R} (\vec{x}_{\alpha_2} - \vec{x}_{\beta_2}, - \tau_2) G_{\eta \eta}^{R,R} (x_{\alpha_1} - x_{\beta_1}, t - \tau_1) \\
&\quad G_{\eta \eta}^{L,R} (x_{\alpha_1} - x_{\beta_1}, t - \tau_1) G_{\eta \eta}^{L,R} (x_{\beta_1} - \vec{x}_{\beta_2}, \tau_1 - \tau_2)) e^{i \epsilon_2} \\
&\quad G_{\bar{L},L}^{L,R} (x_{\alpha_1} - x_{\beta_1}, t) G_{\bar{L},L}^{L,R} (x_{\beta_1} - \vec{x}_{\beta_2}, \tau_1 - \tau_2))
\end{align}

(47)

The Green’s functions $G_{\eta \eta}^{L,R} (x, t)$ decay exponentially in $t$; therefore, the integral will only be dominated by regions where $\tau_1 \approx t$ and $\tau_2 \approx 0$. Furthermore, we are interested in the long time behavior, where $t$ is large. In this limit, the ratio in parantheses above will go to one:

\begin{equation}
\left( \frac{G_{\eta \eta}^{L,R} (x_{\alpha_1} - x_{\beta_1}, t - \tau_2) G_{\eta \eta}^{L,R} (x_{\beta_1} - \vec{x}_{\beta_2}, \tau_1)}{G_{\eta \eta}^{R,R} (x_{\alpha_1} - x_{\beta_1}, t) G_{\eta \eta}^{R,R} (x_{\beta_1} - \vec{x}_{\beta_2}, \tau_1 - \tau_2)} \right) \to 1
\end{equation}

(48)

when $t \approx \tau_1 \to \infty$ and $|t| \gg |\tau_2|$, $t \gg |x_{\alpha_1} - x_{\beta_1}|$. Therefore:

\begin{align}
\langle T C I_b (t^\nu) H (\tau_1^{\nu}) I_{b2} (0^{-}) \bar{H} (\tau_2^{\nu}) \rangle &\approx - \frac{1}{m^2} \sum_{\epsilon_1, \epsilon_2 = \pm} \sum_{\alpha_1 , \beta_1 , \alpha_2 , \beta_2} \epsilon_1 \epsilon_2 \Gamma_{\alpha_1}^{\epsilon_1} \Gamma_{\beta_1}^{-\epsilon_1} \Gamma_{\alpha_2}^{\epsilon_2} \Gamma_{\beta_2}^{-\epsilon_2} e^{i \epsilon_1 \omega_1 (t - \tau_1) + i \epsilon_2 \omega_2 (t - \tau_2)} \\
&\quad \langle T C K_{\alpha_1}^{\epsilon_1} (t^\nu) K_{\beta_1}^{-\epsilon_1} (\tau_1^{\nu}) \bar{K}_{\alpha_2}^{\epsilon_2} (0^{-}) \bar{K}_{\beta_2}^{-\epsilon_2} (\tau_2^{\nu}) \rangle \\
&\quad G_{\eta \eta}^{R,R} (\vec{x}_{\alpha_2} - \vec{x}_{\beta_2}, - \tau_2) G_{\eta \eta}^{R,R} (x_{\alpha_1} - x_{\beta_1}, t - \tau_1)
\end{align}

(49)

Therefore, so far we have, for large $t$,

\begin{align}
S_{12} (t) &= - \frac{1}{m^2} \sum_{\eta_1 , \eta_2 , \alpha_1 = \pm} \sum_{\beta_1 , \alpha_2 = \pm} \sum_{\beta_2} \epsilon_1 \epsilon_2 \Gamma_{\alpha_1}^{\epsilon_1} \Gamma_{\beta_1}^{-\epsilon_1} \Gamma_{\alpha_2}^{\epsilon_2} \Gamma_{\beta_2}^{-\epsilon_2} \\
&\quad \langle T C K_{\alpha_1}^{\epsilon_1} (t^\nu) K_{\beta_1}^{-\epsilon_1} (\tau_1^{\nu}) \bar{K}_{\alpha_2}^{\epsilon_2} (0^{-}) \bar{K}_{\beta_2}^{-\epsilon_2} (\tau_2^{\nu}) \rangle - \langle T C K_{\alpha_1}^{\epsilon_1} (t^\nu) K_{\beta_1}^{-\epsilon_1} (\tau_1^{\nu}) \rangle \langle T C \bar{K}_{\alpha_2}^{\epsilon_2} (0^{-}) \bar{K}_{\beta_2}^{-\epsilon_2} (\tau_2^{\nu}) \rangle \\
&\quad \eta_1 \eta_2 \int_{-\infty}^{\infty} e^{i \epsilon_2 \omega_2 (-\tau_2)} G_{\eta_2 \eta_2}^{R,R} (\vec{x}_{\alpha_2} - \vec{x}_{\beta_2}, - \tau_2) d\tau_2 \int_{-\infty}^{\infty} d\tau_1 e^{i \epsilon_1 \omega_1 (-\tau_1)} G_{\eta_1 \eta_1}^{R,R} (x_{\alpha_1} - x_{\beta_1}, t - \tau_1)
\end{align}

(50)

In the case where $x_1 < x_2 < \bar{x}_1 < \bar{x}_2$, using the formula for the Klein factor expression found earlier, we find that the difference of Klein factor correlation functions vanishes, so that $S_{12} (t) \to 0$ for $|t| \gg 1/T$.

Now let us focus on the case $x_1 < \bar{x}_1 < x_2 < \bar{x}_2$. In this case, for $t > 0$,

\begin{align}
S_{12} (t > 0) &= - \frac{1}{2} \sum_{\eta_1 , \eta_2 , \alpha_1 = \pm} \sum_{\beta_1 , \alpha_2 = \pm} \sum_{\beta_2} \epsilon_1 \epsilon_2 \Gamma_{\alpha_1}^{\epsilon_1} \Gamma_{\beta_1}^{-\epsilon_1} \Gamma_{\alpha_2}^{\epsilon_2} \Gamma_{\beta_2}^{-\epsilon_2} [e^{-i \frac{\pi}{2} (\alpha_1 - \beta_1) \pi [\eta_1 \eta_2 + \eta_2 \beta_2]} - 1] \\
&\quad \eta_1 \eta_2 \int_{-\infty}^{\infty} e^{i \epsilon_2 \omega_2 (-\tau_2)} G_{\eta_2 \eta_2}^{R,R} (\vec{x}_{\alpha_2} - \vec{x}_{\beta_2}, - \tau_2) d\tau_2 \int_{-\infty}^{\infty} d\tau_1 e^{i \epsilon_1 \omega_1 (-\tau_1)} G_{\eta_1 \eta_1}^{R,R} (x_{\alpha_1} - x_{\beta_1}, t - \tau_1)
\end{align}

(51)

In what follows, let us assume that the edge velocities are the same on the left and right edges, and in the two layers, which we set to unity: $v_{1L} = v_{1R} = v_{2R} = 1$. 
Now let us define:
\[ A^\omega_{\eta_1 \eta_2}(x) = \int_{-\infty}^{\infty} e^{i\omega \tau} d\tau G_{\eta_1 \eta_2}(x, \tau) \]
\[ = \int_{-\infty}^{\infty} dt e^{i \omega t} e^{-i \frac{1}{m^2} \chi_{\eta_1 \eta_2}(t)(sgn(t+x)+sgn(t-x))} \left( \frac{(\pi T)^2}{\sinh[\pi T|t + x|] \sinh[\pi T|t - x|]} \right)^{1/m} \]  

(52)

Note that this satisfies
\[ A^\omega_{\eta_1 \eta_2}(x) = A^\omega_{\eta_1 \eta_2}(-x), \quad A^\omega_{\eta_1 \eta_2}(x) = A^\omega_{\eta_1 \eta_2}(x), \quad A^\omega_{\eta_1 \eta_2}(x) + A^\omega_{\eta_1 \eta_2}(x) = A^\omega_{\eta_1 \eta_2}(x) + A^\omega_{\eta_1 \eta_2}(x). \]  

(53)

Then we can write the noise as a simple expression:
\[ S_{12}(t > 0) = \frac{1}{2} \sum_{\eta_1, \eta_2 \in \mathbb{Z}, \epsilon_2 = \pm \alpha_1, \beta_2} \epsilon_1 \epsilon_2 \eta_1 \eta_2 |\Gamma_1 \Gamma_2| \left( e^{i\epsilon_1 \varphi_1 \alpha_1 + i\epsilon_2 \varphi_2 \alpha_2} [e^{-i\frac{1}{m^2} \pi \eta_{\delta \
olimits{\alpha_2} + \eta_{\delta \
olimits{\beta_2}}} - 1] A^\omega_{\eta_1 \eta_2}(x_{\alpha_1 \beta_1}) A^{+\omega \omega}_{\eta_2}(\tilde{x}_{\alpha_2 \beta_2}) \right) \]

(54)

and
\[ S_{12}(t < 0) = \frac{1}{2} \sum_{\eta_1, \eta_2 \in \mathbb{Z}, \epsilon_2 = \pm \alpha_1, \beta_2} \epsilon_1 \epsilon_2 \eta_1 \eta_2 |\Gamma_1 \Gamma_2| \left( e^{i\epsilon_1 \varphi_1 \alpha_1 + i\epsilon_2 \varphi_2 \alpha_2} [e^{-i\frac{1}{m^2} \pi \eta_{\delta \
olimits{\alpha_2} + \eta_{\delta \
olimits{\beta_2}}} - 1] A^\omega_{\eta_1 \eta_2}(x_{\alpha_1 \beta_1}) A^{+\omega \omega}_{\eta_2}(\tilde{x}_{\alpha_2 \beta_2}) \right) \]

(55)

Now let us try to make more explicit the above expressions. We start with the \( t > 0 \) case. Summing over \( \alpha_1 \) and \( \beta_1 \) gives:
\[ S_{12}(t > 0) = \frac{1}{2} \sum_{\eta_1, \eta_2 \in \mathbb{Z}, \epsilon_2 = \pm \alpha_1, \beta_2} \epsilon_1 \epsilon_2 \eta_1 \eta_2 |\Gamma_1 \Gamma_2| \left( e^{i\epsilon_1 \varphi_1 \alpha_1 + i\epsilon_2 \varphi_2 \alpha_2} [e^{-i\frac{1}{m^2} \pi \eta_{\delta \
olimits{\alpha_2} + \eta_{\delta \
olimits{\beta_2}}} - 1] A^\omega_{\eta_1 \eta_2}(x_{\alpha_1 \beta_1}) A^{+\omega \omega}_{\eta_2}(\tilde{x}_{\alpha_2 \beta_2}) \right) \]

(56)

For \( S_{12}(t < 0) \), we perform the sum over \( \alpha_2, \beta_2 \) to get:
\[ S_{12}(t < 0) = \frac{1}{2} \sum_{\eta_1, \eta_2 \in \mathbb{Z}, \epsilon_2 = \pm \alpha_1, \beta_1} \epsilon_1 \epsilon_2 \eta_1 \eta_2 |\Gamma_1 \Gamma_2| \left( e^{i\epsilon_1 \varphi_1 \alpha_1 + i\epsilon_2 \varphi_2 \alpha_2} [e^{-i\frac{1}{m^2} \pi \eta_{\delta \
olimits{\alpha_2} + \eta_{\delta \
olimits{\beta_2}}} - 1] A^\omega_{\eta_1 \eta_2}(x_{\alpha_1 \beta_1}) A^{+\omega \omega}_{\eta_2}(\tilde{x}_{\alpha_2 \beta_2}) \right) \]

(57)

This can be written as the sum of two terms:
\[ S_{12}(t > 0) = S_{12}^1(t > 0) + S_{12}^2(t > 0), \]

(58)

where \( S_{12}^1(t > 0) \) contains the terms where \( \alpha_2 \neq \beta_2 \) in the above sum. \( S_{12}^2(t > 0) \) contains terms in the sum \( S_{12}(t > 0) \) where \( \alpha_2 = \beta_2 \). Similarly, we write
\[ S_{12}(t < 0) = S_{12}^1(t < 0) + S_{12}^2(t < 0), \]

(59)

where \( S_{12}^1(t < 0) \) contains terms in the sum \( 57 \) where \( \alpha_1 \neq \beta_1 \), while \( S_{12}^2(t < 0) \) contains terms in the sum where \( \alpha_1 = \beta_1 \). After simplifying, we find
\[ S_{12}^1(t > 0) = \frac{(\pi T)^{4/m - 2}}{m^2} |\Gamma_1 \Gamma_2| \tilde{\Gamma}_1 \tilde{\Gamma}_2 |S_{12}^{(2)}| (\omega_1, \omega_2, \pi T \tilde{x}_{12}, \pi T \tilde{x}_{12}, \varphi_{12}, \varphi_{12}), \]
\[ S_{12}^2(t > 0) = \frac{(\pi T)^{4/m - 2}}{m^2} |\tilde{\Gamma}_1|^2 |\Gamma_1 \Gamma_2| \tilde{\Gamma}_1 \tilde{\Gamma}_2 |S_{12}^{(1)}| (\omega_1, \omega_2, \pi T \tilde{x}_{12}, \varphi_{12}), \]
\[ S_{12}^1(t < 0) = \frac{(\pi T)^{4/m - 2}}{m^2} |\Gamma_1 \Gamma_2| \tilde{\Gamma}_1 \tilde{\Gamma}_2 |S_{12}^{(2)}| (\omega_2, \omega_1, \pi T \tilde{x}_{12}, \varphi_{12}), \]
\[ S_{12}^2(t < 0) = \frac{(\pi T)^{4/m - 2}}{m^2} |\tilde{\Gamma}_1|^2 |\Gamma_1 \Gamma_2| \tilde{\Gamma}_1 \tilde{\Gamma}_2 |S_{12}^{(1)}| (\omega_2, \omega_1, \pi T \tilde{x}_{12}, \varphi_{12}), \]

(60)

where \( S_{12}^{(i)} \) are presented in [23].