Testing the origin of cosmological magnetic fields through the large-scale structure consistency relations

P. Berger, A. Kehagias and A. Riotto

Department of Theoretical Physics and Center for Astroparticle Physics (CAP), 24 quai E. Ansermet, Geneva 4, CH-1211 Switzerland

Physics Division, National Technical University of Athens, Zografou Campus, Athens, 15780 Greece

E-mail: nicolas.berger@cern.ch, kehagias@central.ntua.gr, antonio.riotto@pd.infn.it

Received February 7, 2014
Revised April 14, 2014
Accepted April 27, 2014
Published May 19, 2014

Abstract. We study the symmetries of the post-recombination cosmological magnetohydrodynamical equations which describe the evolution of dark matter, baryons and magnetic fields in a self-consistent way. This is done both at the level of fluid equations and of Vlasov-Poisson-Maxwell equations in phase space. We discuss some consistency relations for the soft limit of the \((n + 1)\)-correlator functions involving magnetic fields and matter overdensities. In particular, we stress that any violation of such consistency relations at equal-time would point towards an inflationary origin of the magnetic field.

Keywords: primordial magnetic fields, Magnetohydrodynamics, dark matter theory, inflation

ArXiv ePrint: 1402.1044
1 Introduction

Magnetic fields appear to be ubiquitous in astrophysical and cosmological environments [1]. The largest observable magnetic fields are found inside galaxies with a strength of a (few-totens) of $\mu$G and in clusters of galaxies with a strength of the order of the $\mu$G. Magnetic fields are found at even larger scales; they are not associated to collapsing or gravitationally bound structures, but are coherent on scales greater than the largest known structures (about $10^2$ Mpc or even the Hubble radius) and they permeate the whole universe. The magnetic fields at such different scales are possibly produced by amplification of preexisting magnetic fields via the dynamo mechanism [2] associated to a subsequent compression and turbulent flows generated during the formation of the large-scale structure. This mechanism needs some primordial seeds which may be either produced during some era much earlier than structure formation or during the formation of the first objects. In the first category falls the so-called inflationary magnetogenesis [3, 4] where magnetic fields coherent on very large (super-Hubble) cosmological scales are generated thanks to a coupling which breaks the conformal invariance of electromagnetism [5].

In order to describe Dark Matter (DM), baryons, and magnetic fields in a self-consistent way and within a cosmological set-up, one needs to numerically solve the full set of equations of cosmological MagnetohydroDynamics (MHD). Given the highly non-linear nature of these equations, one relies more and more on numerical simulations (see for instance ref. [6] for a recent work) and it seems clear by now that seed magnetic fields are needed to explain the presence of the observed magnetic fields in galaxy clusters [7].

In this paper we wish to take a different, albeit modest, path to provide some information about the interplay between the large-scale structure formation, the magnetic fields, and their origin. Our method is based on symmetries. There is no doubt that symmetries play a crucial role in high energy physics allowing, for instance, to derive non-perturbative identities among correlation functions which remain valid even after renormalization [8, 9].

Symmetries are also relevant in the cosmological setting. For instance, during inflation [10] the de Sitter isometry group acts as conformal group on $\mathbb{R}^3$ when the fluctuations...
are on super-Hubble scales. In such a regime, the $\text{SO}(1,4)$ isometry of the de Sitter background is realized as conformal symmetry of the flat $\mathbb{R}^3$ sections and correlators are constrained by conformal invariance \cite{11–15}. This applies in the case in which the cosmological perturbations are generated by light scalar fields other than the inflaton (the field that drives inflation). In the opposite case in which the inflationary perturbations originate from only one degree of freedom, one can construct consistency relations relating an $n$-point function to an $(n + 1)$-point function where the additional leg is a soft curvature perturbation, the Goldstone boson associated with a non-linearly realized symmetry. This is possible because the symmetries shift the comoving curvature perturbation $\zeta$ in a non-linear way, precisely like a Goldstone boson \cite{16–19}.

The same arguments can be applied to the case of the large-scale structure \cite{20, 21}. Indeed, there are large-scale structure observables, such as the DM peculiar velocity, which are shifted in a non-linear way under some symmetry transformation, while the same transformation shifts the density contrasts only linearly. This gives rise to new kinds of consistency relations where the Goldstone boson is the peculiar velocity. The power of the consistency relations is that they are non-perturbative and this is very useful when studying the large-scale structure, where one has to deal with small and non-linear scales. The consistency relations in the large-scale structure have been recently the subject of an intense activity \cite{22–30} as they have the virtue of being true also for the galaxy overdensities, independently of the bias between galaxies and DM. As such, they may serve as a guidance in building up the bias theory, that is the theory connecting the observed galaxy correlators to the underlying DM ones.

In this paper we wish to show that the symmetry arguments can be adopted to learn something about the correlators among the magnetic fields, the DM and the baryons in the limit in which one of the momenta is soft. Apart from this restriction, all the other momenta can correspond to non-perturbative non-linear scales. We will see that there is a nice by-product of this sort of soft-pion theorem applied to magnetic fields: the consistency relations are violated at equal-time in the case in which the seeds of the primordial magnetic fields are of inflationary origin. Testing the violation of the consistency relation therefore provides a way to test the ultimate origin of the cosmological magnetic fields we observe in the universe.

The paper is organized as follows. In section 2 we discuss the symmetries of cosmological MHD in the fluid approximation, while the full set of symmetries in phase space is described in the appendix A. In section 3 we derive the consistency relations involving the magnetic fields, and in section 4 we discuss the violation of the consistency relations and its implications. Finally, we conclude in section 5.

2 Symmetries of cosmological magnetohydrodynamics

Our starting point is a system containing the same physical content of the N-body simulations: DM particles treated as a fluid, a baryon component that behaves like an ideal conducting plasma, and the magnetic field in the post-recombination era. Moreover, our considerations are valid even in the presence of a cosmological constant, that is in a $\Lambda$CDM universe. Of course, the fluid approximation breaks down at sufficiently small scales where, for instance, the single stream approximation is no longer valid. However, the symmetries we are going to employ extend also to the Vlasov-Poisson-Maxwell equations in phase space as we show in the appendix A. Our considerations are therefore valid beyond the fluid approximation which we assume in the bulk of the paper for simplicity.
To treat all the components in a self-consistent way, the equations of ideal MHD have to be solved together with the DM fluid equations in a fully cosmological setting. The baryonic plasma is taken to be collisional, with small resistivity so that the equations MHD can be applied. We indicate by

$$\delta_c(\vec{x}, \tau) = \frac{\rho(\vec{x}, \tau)}{\rho_c} - 1, \quad \delta_p(\vec{x}, \tau) = \frac{\rho_p(\vec{x}, \tau)}{\rho_p} - 1$$

(2.1)

the DM and the plasma overdensities defined over the mean DM density $\rho_c$ and the plasma density $\rho_p$, respectively. In addition, we define by $\vec{v}_c(\vec{x}, \tau)$ and $\vec{v}_p(\vec{x}, \tau)$ the peculiar velocities of the DM fluid and plasma, respectively, and by $\Phi(\vec{x}, \tau)$ the gravitational potential due to density fluctuations.

The equations which specify the dynamics of DM, plasma, gravity and the magnetic field $\vec{B}(\vec{x}, \tau)$ are

- the DM mass conservation

$$\frac{\partial \delta_c(\vec{x}, \tau)}{\partial \tau} + \vec{\nabla}(1 + \delta_c(\vec{x}, \tau))\vec{v}_c(\vec{x}, \tau) = 0,$$

(2.2)

- the DM momentum conservation

$$\frac{\partial \vec{v}_c(\vec{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau)\vec{v}_c(\vec{x}, \tau) + \left[\vec{v}_c(\vec{x}, \tau) \cdot \vec{\nabla}\right] \vec{v}_c(\vec{x}, \tau) = -\vec{\nabla}\Phi(\vec{x}, \tau),$$

(2.3)

- the Poisson equation

$$\nabla^2 \Phi(\vec{x}, \tau) = \frac{3}{2} \mathcal{H}^2(\tau) \left( \Omega_c \delta_c(\vec{x}, \tau) + \Omega_p \delta_p(\vec{x}, \tau) \right),$$

(2.4)

- the plasma mass conservation

$$\frac{\partial \delta_p(\vec{x}, \tau)}{\partial \tau} + \vec{\nabla}(1 + \delta_p(\vec{x}, \tau))\vec{v}_p(\vec{x}, \tau) = 0,$$

(2.5)

- the plasma momentum conservation

$$\rho_p(\vec{x}, \tau) \left[\frac{\partial \vec{v}_p(\vec{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau)\vec{v}_p(\vec{x}, \tau) + \left(\vec{v}_p(\vec{x}, \tau) \cdot \vec{\nabla}\right) \vec{v}_p(\vec{x}, \tau) + \vec{\nabla}\Phi(\vec{x}, \tau)\right] =$$

$$\frac{1}{\mu} \left(\vec{B}(\vec{x}, \tau) \cdot \vec{\nabla}\right) \vec{B}(\vec{x}, \tau) - \frac{1}{2\mu} \vec{\nabla}B^2(\vec{x}, \tau) - \vec{\nabla}P_p(\vec{x}, \tau),$$

(2.6)

- the induction equation

$$\frac{\partial \vec{B}(\vec{x}, \tau)}{\partial \tau} + 2\mathcal{H} \vec{B} - \vec{\nabla} \times \left(\vec{v}_p(\vec{x}, \tau) \times \vec{B}(\vec{x}, \tau)\right) = 0,$$

(2.7)

- Gauss’ law for magnetism

$$\vec{\nabla} \cdot \vec{B}(\vec{x}, \tau) = 0.$$  

(2.8)

\footnote{Notice that the plasma velocity should be intended as the mean velocity of ions and electrons, see the appendix A for more details.}
We have denoted by $\bar{x}$ the comoving spatial coordinates, $d\tau = dt/a$ the conformal time, $a$ the scale factor in the FRW metric, $\mathcal{H} = d\ln a/d\tau$ the conformal expansion rate, $\vec{B}(\bar{x}, \tau)$ the magnetic field strength, and by $P_p(\bar{x}, \tau)$ the plasma pressure. Finally, $\Omega_c = 8\pi G \rho a^2 / 3H^2$ and $\Omega_p = 8\pi G \bar{p} a^2 / 3H^2$ are the density parameters, and $\mu$ is the permeability of the plasma.

We wish now to show that the above equations are invariant under the following set of transformations

$$
\tau' = \tau, \quad \bar{x}' = \bar{x} + \vec{n}(\tau),
$$

(2.9)

$$
\delta_c'(\bar{x}, \tau) = \delta_c(\bar{x}', \tau'),
$$

(2.10)

$$
\vec{v}_c'(\bar{x}, \tau) = \vec{v}_c(\bar{x}', \tau') - \vec{n}(\tau),
$$

(2.11)

$$
\delta_p'(\bar{x}, \tau) = \delta_p(\bar{x}', \tau'),
$$

(2.12)

$$
\vec{v}_p'(\bar{x}, \tau) = \vec{v}_p(\bar{x}', \tau') - \vec{n}(\tau),
$$

(2.13)

$$
\vec{B}'(\bar{x}, t) = \vec{B}(\bar{x}', t'),
$$

(2.14)

$$
P_p(\bar{x}, t) = P_p(\bar{x}', t'),
$$

(2.15)

$$
\Phi'(\bar{x}, \tau) = \Phi(\bar{x}', \tau') + \left(\vec{n}(\tau) + \mathcal{H}(\tau) \vec{n}(\tau)\right) \cdot \bar{x},
$$

(2.16)

where $\vec{n}(\tau)$ is an arbitrary time-dependent vector. In other words, if $\delta_c(\bar{x}, \tau), \vec{v}_c(\bar{x}, \tau), \cdots$ are solutions, then also $\delta_c'(\bar{x}, \tau), \vec{v}_c'(\bar{x}, \tau), \cdots$ are solutions of the cosmological MHD equations. The invariance of the eqs. (2.2)–(2.8) under the transformations (2.9)–(2.16) can be easily checked by noticing that temporal and spatial derivatives transform as

$$
\frac{\partial}{\partial \tau} \bigg|_{\bar{x}} = \frac{\partial}{\partial \tau'} \bigg|_{\bar{x}'} + \vec{n} \cdot \vec{\nabla}', \quad \vec{\nabla} = \vec{\nabla}'.
$$

(2.17)

This implies that the operators

$$
D^c'' = \frac{\partial}{\partial \tau} + \vec{v}_c(\bar{x}, \tau) \cdot \vec{\nabla}, \quad D^p'' = \frac{\partial}{\partial \tau} + \vec{v}_p(\bar{x}, \tau) \cdot \vec{\nabla}
$$

(2.18)

are left invariant. Indeed, the invariance of eqs. (2.2)–(2.4) has been proven in ref. [20], but we repeat here for the benefit of the reader. We start with the DM momentum conservation (2.3)

$$
0 = \frac{\partial \vec{v}_c'(\bar{x}, \tau)}{\partial \tau} + \mathcal{H}(\tau) \vec{v}_c'(\bar{x}, \tau) + \left[\vec{v}_c'(\bar{x}, \tau) \cdot \vec{\nabla}\right] \vec{v}_c'(\bar{x}, \tau) + \vec{\nabla} \Phi'(\bar{x}, \tau)
$$

$$
= \frac{\partial \vec{v}_c(\bar{x}', \tau')}{\partial \tau'} - \vec{n} + \mathcal{H}(\tau') \vec{v}_c(\bar{x}', \tau') - \mathcal{H}(\tau') \vec{n} + \left[\vec{v}_c(\bar{x}', \tau') \cdot \vec{\nabla}'\right] \vec{v}_c(\bar{x}', \tau') + \vec{\nabla} \Phi(\bar{x}', \tau)
$$

$$
= -\vec{n} - \mathcal{H}(\tau') \vec{n} - \vec{\nabla}' \Phi(\bar{x}', \tau') + \vec{\nabla} \Phi'(\bar{x}, \tau),
$$

(2.19)

from which we deduce the transformation

$$
\vec{\nabla} \Phi'(\bar{x}, \tau) = \vec{\nabla} \Phi(\bar{x}', \tau') + \vec{n} + \mathcal{H}(\tau') \vec{n}
$$

(2.20)

or

$$
\Phi'(\bar{x}, \tau) = \Phi(\bar{x}', \tau') + \left(\vec{n} + \mathcal{H}(\tau') \vec{n}\right) \cdot \bar{x}.
$$

(2.21)

The invariance of the Poisson equation (2.4) automatically follows. The DM and plasma mass conservations (2.2) and (2.5) respectively also follow from rewriting them as

$$
D^c'' \delta(\bar{x}, \tau) + (1 + \delta(\bar{x}, \tau)) \vec{\nabla} \cdot \vec{v}_c(\bar{x}, \tau) = 0, \quad D^p'' \delta_p(\bar{x}, \tau) + (1 + \delta_p(\bar{x}, \tau)) \vec{\nabla} \cdot \vec{v}_p(\bar{x}, \tau) = 0.
$$

(2.22)
The invariance of the plasma momentum conservation (2.6) is proved similarly to what is done for the DM momentum conservation equation (2.3) with the addition that the plasma density contrast \(\delta_p(x, \tau)\), the magnetic field \(B(x, \tau)\), and the pressure \(P_p(x, \tau)\) are scalars under the transformations. The same reasoning applies to the Gauss’ law (2.8).

Finally, let us consider the induction equation (2.7). Using the fact that

\[
\left[ \nabla \times (\vec{v}_p \times \vec{B}) \right] = - \left( \nabla \cdot \vec{v}_p \right) \vec{B} - \left( \vec{v}_p \cdot \nabla \right) \vec{B} + \vec{v}_p \left( \nabla \cdot \vec{B} \right) + \left( \vec{B} \cdot \nabla \right) \vec{v}_p \tag{2.23}
\]

and exploiting \(\nabla \cdot \vec{B} = 0\), we can recast the induction equation in the form

\[
0 = \frac{\partial \vec{B}}{\partial \tau} + 2\mathcal{H} \vec{B} + \left( \nabla \cdot \vec{v}_p \right) \vec{B} + \left( \vec{v}_p \cdot \nabla \right) \vec{B} - \left( \vec{B} \cdot \nabla \right) \vec{v}_p
\]

\[
= D_{\nu \nu} \vec{B} + 2\mathcal{H} \vec{B} + \left( \nabla \cdot \vec{v}_p \right) \vec{B} - \left( \vec{B} \cdot \nabla \right) \vec{v}_p,
\]

which is invariant since it involves the invariant derivative \(D_{\nu \nu}\) and gradients of \(\vec{v}_p\).

We close this section with two comments. First, the symmetry of the cosmological MHD equations are correct even though they are highly non-linear. This crucial point will allow us to find consistency relations involving the magnetic field at any scales. Secondly, if one introduces back in the plasma momentum conservation equation (2.6) and the induction equation (2.7) the terms depending on the shear viscosity and conductivity respectively, our symmetry arguments do not change. This is because the shear viscosity terms are proportional to gradients of the peculiar velocities and the conductivity terms are proportional to the Laplacian of the magnetic field.

3 Cosmological magnetohydrodynamics consistency relations

In this section we wish to analyze the implications of the symmetry discussed previously. Consider the \(n\)-point correlation function of short modes of the magnetic field. The symmetries of the cosmological MHD equations equations imply, for instance, that

\[
\langle \vec{B}(t_1, \vec{x}_1) \cdots \vec{B}(t_n, \vec{x}_n) \rangle = \langle \vec{B}(\tau_1, \vec{x}_1) \cdots \vec{B}(\tau_n, \vec{x}_n) \rangle = \langle \vec{B}(\tau'_1, \vec{x}'_1) \cdots \vec{B}(\tau'_n, \vec{x}'_n) \rangle. \tag{3.1}
\]

The meaning of the eq. (3.1) is that the correlators in the two coordinate systems have to be the same as both \(\vec{B}(\tau_i, \vec{x}_i)\) and \(\vec{B}(\tau'_i, \vec{x}'_i) = \vec{B}(\tau'_i, \vec{x}'_1)\) \((i = 1, \ldots, n)\) satisfy the equations on motion.

We can take advantage of this relation in the following way. Suppose that the \(n\) points are contained in a sphere of radius \(R\) much smaller than a long wavelength mode of size \(\sim 1/q\) and centered at the origin of the coordinates. Since the cosmological MHD equations are invariant under the generic transformation \(\tau \to \tau\) and \(\vec{x} \to \vec{x} + \vec{n}(\tau)\), and the DM peculiar velocity transforms non-linearly under such a symmetry (like a Goldstone boson)

\[
\vec{v}_c'(\vec{x}, \tau) = \vec{v}_c(\vec{x}', \tau') - \vec{n}(\tau), \tag{3.2}
\]

this means that we can go to a new coordinate system to cancel (or generate) a long wavelength mode for the DM velocity perturbation \(\vec{v}_{cL}(\tau, \vec{0})\) just by choosing properly the vector \(\vec{n}(\tau)\)

\[
\vec{n}(\tau) = \int d\eta \vec{v}_{cL}(\eta, \vec{0}) + O(qRv_{cL}^2). \tag{3.3}
\]
This is true if we may consider a sufficiently long wavelength mode 1/q such that the gravitational potential and its gradient can be considered constant in space on a scale of size R. This amounts to removing the time-dependent but homogeneous gravitational force via a change of coordinates.\footnote{To get convinced about this point, let us suppose that the impact of baryons and the magnetic field onto the peculiar DM velocity is negligible. If so, one finds that the transformation (in a matter-dominated universe) becomes τ′ = τ, \vec{x′} = \vec{x} + \int^\tau d\eta \vec{v}_{\ell M}(\eta) = \vec{x} + (1/6)\tau^2 \vec{\Phi}_L. Neglecting the magnetic field contribution to \( \delta_{cL}(\vec{q}, \tau) \), and therefore to the DM peculiar velocity, is a good approximation because the linear evolution of the DM density contrast \( \delta_{cL}(\vec{q}, \tau) \) receives corrections from the magnetic field (through the gravitational coupling to the baryon fluid) which are at least of the order of \((B_{in}/n G)^2(q \text{ Mpc})^2 \tau^2\), where \( B_{in} \) is the initial condition for the comoving magnetic field on large scales [31]. For linear scales \( 1/q \sim 10^{-2} \text{ Mpc} \) and for \( B_{in} \sim n G \), or less, the corresponding contribution is totally negligible. This is also confirmed by N-body simulation results [6].}

The correlative of the short wavelength modes of the magnetic field in the background of the long wavelength mode perturbation should satisfy the relation

\[
\left\langle \vec{B}(\tau_1, \vec{x}_1) \vec{B}(\tau_2, \vec{x}_2) \cdots \vec{B}(\tau_n, \vec{x}_n) \right\rangle_{v_L} \equiv \left\langle \vec{B}(\tau'_1, \vec{x}'_1) \vec{B}(\tau'_2, \vec{x}'_2) \cdots \vec{B}(\tau'_n, \vec{x}'_n) \right\rangle. \tag{3.4}
\]

This equation tells us that effect of a physical long wavelength DM velocity perturbation onto the short modes of the magnetic field should be indistinguishable from the long wavelength mode velocity generated by the transformation with \( \vec{x} \leftrightarrow \vec{x} + \vec{n}(\tau) \). Correlating with the linear long wavelength mode of the DM density contrast in momentum space, we obtain

\[
\left\langle \delta_{cL}(\vec{q}, \tau) \vec{B}(\vec{k}_1, \tau_1) \cdots \vec{B}(\vec{k}_n, \tau_n) \right\rangle_{q \to 0} = \left\langle \delta_{cL}(\vec{q}, \tau) \vec{B}(\vec{k}_1, \tau_1) \cdots \vec{B}(\vec{k}_n, \tau_n) \right\rangle_{v_L}. \tag{3.5}
\]

The variation of the \( n \)-point correlator under the infinitesimal transformation is given at first-order in \( \delta \vec{x}_a = n^i(\tau_a) \) by

\[
\delta_n \left\langle \vec{B}(\tau_1, \vec{x}_1) \cdots \vec{B}(\tau_n, \vec{x}_n) \right\rangle \simeq \int \frac{d^3 \vec{k}_1}{(2\pi)^3} \cdots \frac{d^3 \vec{k}_n}{(2\pi)^3} \left\langle \vec{B}(\vec{k}_1, \tau_1) \cdots \vec{B}(\vec{k}_n, \tau_n) \right\rangle \times \sum_{a=1}^n \delta \vec{x}_a(i_k^a) e^{i(\vec{k}_1 \cdot \vec{x}_1 + \cdots + \vec{k}_n \cdot \vec{x}_n)} \tag{3.6}
\]

Therefore we find that

\[
\left\langle \delta_{cL}(\vec{q}, \tau) \vec{B}(\vec{k}_1, \tau_1) \cdots \vec{B}(\vec{k}_n, \tau_n) \right\rangle_{q \to 0} = \left\langle \delta_{cL}(\vec{q}, \tau) \vec{B}(\vec{k}_1, \tau_1) \cdots \vec{B}(\vec{k}_n, \tau_n) \right\rangle_{v_L} = i \sum_{a=1}^n \left\langle \delta_{cL}(\vec{q}, \tau) n^i(\tau_a) \right\rangle_{v_L} k_a^i \vec{B}(\vec{k}_1, \tau_1) \cdots \vec{B}(\vec{k}_n, \tau_n). \tag{3.7}
\]

In a ΛCDM model we have

\[
\int^\tau d\eta \vec{v}_{\ell M}(\vec{q}, \eta) = i \vec{q} \int^\tau d\eta \frac{\partial}{\partial \eta} \delta_{cL}(\vec{q}, \eta) = i \frac{\vec{q}}{q^2} \delta_{cL}(\vec{q}, \tau). \tag{3.8}
\]
We thus obtain the consistency relation
\[
\left\langle \delta_{cL}(q, \tau) \vec{B} \left( \vec{k}_1, \tau_1 \right) \cdots \vec{B} \left( \vec{k}_n, \tau_n \right) \right\rangle'_{q \to 0} = -\sum_{a=1}^{n} \frac{q \cdot k_a}{q^2} \left\langle \delta_{cL}(q, \tau_0) \delta_{cL}(q, \tau_a) \right\rangle' \left\langle \vec{B} \left( \vec{k}_1, \tau_1 \right) \cdots \vec{B} \left( \vec{k}_n, \tau_n \right) \right\rangle',
\]
(3.9)
where the primes indicate that one should remove the Dirac delta’s coming from the momentum conservation. If we now make the reasonable assumption that the back-reaction of the magnetic fields on the DM evolution is negligible, we may write \( \delta_{cL}(q, \tau_0) = D(\tau)/D(\tau_n) \cdot \delta_{cL}(q, \tau_n) \), where \( \tau_n \) is some initial time and \( D(\tau) \) is the linear growth factor. If so, the consistency relation can be rewritten as
\[
\left\langle \delta_{cL}(q, \tau) \vec{B} \left( \vec{k}_1, \tau_1 \right) \cdots \vec{B} \left( \vec{k}_n, \tau_n \right) \right\rangle'_{q \to 0} = -P_{cL}(q, \tau) \sum_{a=1}^{n} \frac{D(\tau_0)}{D(\tau)} \frac{q \cdot k_a}{q^2} \left\langle \vec{B} \left( \vec{k}_1, \tau_1 \right) \cdots \vec{B} \left( \vec{k}_n, \tau_n \right) \right\rangle',
\]
(3.10)
where \( P_{cL}(q, \tau) \) is the linear DM power spectrum. For instance, for the three-point correlator, we obtain
\[
\left\langle \delta_{cL}(q, \tau) \vec{B} \left( \vec{k}_1, \tau_1 \right) \vec{B} \left( \vec{k}_2, \tau_2 \right) \right\rangle'_{q \to 0} = -P_{cL}(q, \tau) \left( \frac{D(\tau_1)}{D(\tau)} - \frac{D(\tau_2)}{D(\tau)} \right) \frac{q \cdot k_1}{q^2} \left\langle \vec{B} \left( \vec{k}_1, \tau_1 \right) \vec{B} \left( \vec{k}_2, \tau_2 \right) \right\rangle'.
\]
(3.11)
Let us check the validity of the consistency relation at second-order for the three-point correlator. From the induction equation (2.7) we deduce that at second-order we have
\[
\vec{B}(\vec{x}, \tau) = \vec{B}_{in}(\vec{x}) + \frac{1}{a^2(\tau)} \int^{\tau} d\eta \frac{a^2(\eta)}{a^2(\tau)} \vec{v}_{cL}(\vec{x}, \eta) \times \left( \nabla \times \left( \vec{v}_{cL}(\vec{x}, \eta) \times \vec{B}_{in}(\vec{x}) \right) \right),
\]
(3.12)
where \( \vec{B}_{in}(\vec{x}) \) is the initial condition for the magnetic field and we have considered it a first-order perturbation. Also, we have replaced in the induction equation \( \vec{v}_{PL} \) by \( \vec{v}_{cL} \). This amounts to ignoring the magnetic back-reaction and the relative motion of the baryons, effectively considering a single fluid. We get
\[
B^j \left( \vec{k}, \tau \right) = \frac{1}{a^2(\tau)} \left[ B^j_{in} \left( \vec{k} \right) - \epsilon_{\ell m} \epsilon_{mrs} \int \frac{d^3q}{(2\pi)^3} \frac{1}{q^2} \left( q^r q^r + k^r q^r \right) \delta_{cL}(q, \tau) B^j_{in} \left( \vec{k} - \vec{q} \right) \right].
\]
(3.13)
Therefore at second-order we obtain
\[
a^2(\tau_1) a^2(\tau_2) \left\langle \delta_{cL}(q, \tau) B^j \left( \vec{k}_1, \tau_1 \right) B^j \left( \vec{k}_2, \tau_2 \right) \right\rangle'_{q \to 0} = \left\langle \delta_{cL}(q, \tau) B^j_{in} \left( \vec{k}_1 \right) B^j \left( \vec{k}_2, \tau_2 \right) \right\rangle'_{q \to 0} + \left\langle \delta_{cL}(q, \tau) B^j \left( \vec{k}_1, \tau_1 \right) \right\rangle'_{q \to 0} \left\langle \delta_{cL}(q, \tau) B^j \left( \vec{k}_2, \tau_2 \right) \right\rangle'_{q \to 0}
= \epsilon_{\ell m} \epsilon_{mrs} \frac{k_1^r q^r}{q^2} \left\langle \delta_{cL}(q, \tau) \delta_{cL}(q, \tau_2) \right\rangle'_{q \to 0} \left\langle B^j_{in} \left( \vec{k}_1 \right) B^j_{in} \left( \vec{k}_2 \right) \right\rangle'
+ \epsilon_{\ell m} \epsilon_{mrs} \frac{k_2^r q^r}{q^2} \left\langle \delta_{cL}(q, \tau) \delta_{cL}(q, \tau_1) \right\rangle'_{q \to 0} \left\langle B^j_{in} \left( \vec{k}_1 \right) B^j_{in} \left( \vec{k}_2 \right) \right\rangle'.
\]
(3.14)
Using the fact that
\[ \epsilon_{i\ell m} \epsilon_{mrs} = -\epsilon_{m\ell i} \epsilon_{mrs} = - (\delta_{\ell r} \delta_{is} - \delta_{\ell s} \delta_{ir}) \] (3.15)
and the condition \( k_i B_{i}^{\text{in}}(\vec{k}) = 0 \), we finally obtain
\[ a^2(\tau_1) a^2(\tau_2) \langle \delta c_L(q, \tau_1) B^i(\vec{k}_1, \tau_1) B^j(\vec{k}_2, \tau_2) \rangle'_q \approx 0 = \]
\[ -\frac{k_2 \cdot \vec{q}}{q^2} \langle \delta c_L(q, \tau_1) B^i(\vec{k}_1) B^j(\vec{k}_2) \rangle'_q - \frac{k_1 \cdot \vec{q}}{q^2} \langle \delta c_L(q, \tau_2) \delta c_L(q, \tau_1) \rangle'_q \langle B^i(\vec{k}_1) B^j(\vec{k}_2) \rangle'_q, \]
(3.16)
which reproduces the consistency relation in the soft limit for the three-point correlator. Notice that, if the correlators are computed all at equal times, the right-hand side of eq. (3.10) vanishes by momentum conservation and the \( 1/q^2 \) infrared divergence will not appear when calculating invariant quantities. The vanishing of the equal-time correlators in the soft limit is rooted in the fact that one can locally eliminate the zero mode and the first spatial gradient of the long and linear wavelength mode of the gravitational potential and that the response of the system on short scales is a uniform displacement. Once more, we stress that these relations are valid beyond linear order for the short wavelength modes which might well be in the non-perturbative regime. Furthermore, one could derive different consistency relations involving, for example, the correlations between the magnetic field and the plasma density contrast.

4 Violation of the cosmological magnetohydrodynamics consistency relations

The consistency relations we have found in the previous section are due to “projection effects” as they are a consequence of the change of coordinates induced by the long velocity mode. This implies that one expects a violation of the consistency relation in the case in which the effect of the long DM mode on the initial conditions of the magnetic field is not a coordinate transformation. If cosmic magnetic fields are produced during inflation, this is precisely what happens. This is because they are likely to be correlated with the scalar curvature perturbation \( \zeta \) which is responsible for both the cosmic microwave background anisotropies and the large-scale structure. Such cross-correlation might reveal the primordial nature of cosmic magnetic fields. Indeed, consider a simple model where the interaction Lagrangian between the scalar field \( \phi \) driving inflation and the electromagnetic field is of the form:

\[ \mathcal{L} \supset f(\phi) F_{\mu \nu}^2 \supset - \frac{f'}{H} \zeta_L F_{\mu \nu}^2, \] (4.1)

where we have used the fact that the comoving curvature perturbation is related to the linear inflation fluctuation by \( \delta \phi_L = - (\phi'/H) \zeta_L \) (in the spatially flat gauge) and primes indicate differentiation with respect to the conformal time. Let us write the initial non-Gaussian initial condition for the magnetic field as

\[ \vec{B}_{\text{in}}(\vec{x}) = \vec{B}_g(\vec{x}) + \frac{1}{2} b_{\text{NL}} \zeta_L \vec{B}_g(\vec{x}), \] (4.2)

\[ ^3 \text{The simplest gauge invariant models of inflationary magnetogenesis are known to suffer from the problems of either large backreaction or strong coupling} \ [32, 33]. \text{For a possible solution see ref.} \ [34]. \]
where $\vec{B}_{\text{g}}$ is the Gaussian initial condition for the magnetic field. One therefore gets $b_{\text{NL}} = \mathcal{O}(f'/fH)$ \cite{35,36}. In the squeezed limit, with the parametrization of the coupling of the form $f(\phi(\tau)) \sim \tau^n$, the index $n$ can be related to the spectral index of the magnetic field power spectrum $n_B = (4 - 2n)$, implying $b_{\text{NL}} = (n_B - 4)$. In the most interesting case of a scale-invariant magnetic field spectrum $n_B = 0$, the non-linear parameter is non-vanishing and given by $b_{\text{NL}} = -4$.

In the presence of such cross-correlations between the scalar perturbation $\zeta$ and the initial condition of the magnetic field, we therefore expect modulation effects from the curvature perturbation long mode in the magnetic field power spectrum and a corresponding violation of the consistency relation at equal-time. This effect is not easy to gauge as the short modes of the magnetic field are typically in the non-linear regime. However, when analyzing the effects of mildly non-linear clustering on magnetic fields frozen into a collapsing proto-cloud that is falling into a CDM potential well \cite{37}, one finds that the primordial tiny seeds of the magnetic fields are to be amplified in the gravitational collapse and one may consider a parametrization of the magnetic field of the form

\[
\vec{B} \sim \vec{B}_{\text{in}} \exp \left( \int \frac{d\eta}{\eta^2} \gamma(\eta) \right),
\]

where $\gamma$ is directional growth rate \cite{37}. An anisotropic collapse typically eliminates one of the magnetic components, confining the field in the pancake plane \cite{37}. Indeed, by ignoring the magnetic back-reaction on the baryons as well as the relative velocity between baryons and DM, the authors of ref. \cite{37} found that DM dominates the collapse and the gravitational anisotropy that amplifies the magnetic field. The magnetic field loses one of its components and is confined in the plane of the pancake. At any rate, the important point is that the exponential amplification depends on the initial condition \cite{4.2} linearly, so that

\[
\frac{\delta B}{\delta \zeta_L} \approx \frac{1}{2} b_{\text{NL}} \vec{B}.
\]

Maybe, a more robust argument is the following. Consider the induction equation \cite{2.7}. Its formal solution is

\[
\vec{B}(\vec{x}, \tau) = \frac{\vec{B}_{\text{in}}(\vec{x})}{a^2(\tau)} + \frac{1}{a^2(\tau)} \int_{\tau} d\eta a^2(\eta) \vec{\nabla} \times \left( \vec{v}_p(\vec{x}, \eta) \times \vec{B}(\vec{x}, \eta) \right)
\]

\[
= \frac{\vec{B}_{\text{in}}(\vec{x})}{a^2(\tau)} + \frac{1}{a^2(\tau)} \int_{\tau} d\eta a^2(\eta) \vec{\nabla} \times \left( \vec{v}_p(\vec{x}, \eta) \times \frac{\vec{B}_{\text{in}}(\vec{x})}{a^2(\eta)} \right)
\]

\[
+ \frac{1}{a^2(\tau)} \int_{\tau} d\eta a^2(\eta) \vec{\nabla} \times \left[ \vec{v}_p(\vec{x}, \eta) \times \frac{1}{a^4(\eta)} \int_{\eta} d\eta' a^2(\eta') \vec{\nabla} \times \left( \vec{v}_p(\vec{x}, \eta') \times \frac{\vec{B}_{\text{in}}(\vec{x})}{a^2(\eta')} \right) \right] + \cdots.
\]

Working at linear order in the non-Gaussian parameter $b_{\text{NL}}$, one can pick up the cross-correlation with the curvature perturbation $\zeta_L$ either from the $\vec{B}_{\text{in}}$ which explicitly appears

\footnote{This is also true in dynamo amplification where the fastest growing mode of the linear system is enhanced over the others and one has to deal with the projection of the initial field and not the initial field itself.}

\footnote{Of course, at small scales the non-linear saturation can make the field very different from the initial one and one expects that the dependence on the initial condition is lost \cite{7}. To avoid such a problem one has to consider large enough scales, but cope with the fact that there will not be much amplification, except due to flux freezing during collapse.}
in every term of the previous expression or from the one which appears implicitly in $\vec{v}_p$. In the first case, using eq. (4.2), one immediately recovers (4.4). The second contribution is more difficult to disentangle, but we expect it to be smaller as $\vec{v}_p$ depends on the magnetic field quadratically.

One has also to keep in mind that the dependence on the initial condition $\vec{B}_i(x)$, for the magnetic field at recombination, is washed out for comoving momenta larger than $k_d$ because of the dissipation of magnetic energy due to the generation of MHD waves [1, 38, 39]. Alfven waves are the most effective in dissipating magnetic energy, and damping occurs at scales $k^{-1} \lesssim k_d^{-1} = v_A \tau$, where $v_A$ is the Alfven speed. Strictly speaking, Alfven waves are oscillatory perturbations superimposed on a homogeneous magnetic component and the Alfven speed depends on the amplitude of the homogeneous component. In the cosmological context where the magnetic field is purely stochastic, the amplitude of this component can be taken as the one of a low frequency component obtained by smoothing the magnetic field amplitude over the scale $\sim k^{-1}$

$$v_A^2(k) = \frac{1}{24\pi} \frac{\langle B^2 \rangle_{k^{-1}}}{(\vec{p}_p + \vec{p}_b)},$$

(4.6)

where $\vec{p}_p$ and $\vec{P}_p$ indicate the background energy density and pressure density of the plasma. During the radiation epoch the Alfven speed is a constant and the damping scale $k_d^{-1}$ scales like the conformal time. During the matter-dominated epoch, the Alfven speed decays like $a^{-1/2} \sim 1/\tau$ and therefore the damping scale stops growing. Taking a scale-invariant power spectrum for the magnetic field, we estimate that at recombination the comoving wavenumber is of the order of

$$k_d \simeq \frac{1}{v_A(k)\tau_{\text{rec}}} \simeq \frac{3 \cdot 10^2}{\tau_{\text{rec}} \ln k_{\text{rec}}} \left( \frac{nG}{B_0} \right) \approx 70 \left( \frac{nG}{B_0} \right) \text{ Mpc}^{-1},$$

(4.7)

where $B_0$ is the present-day value of the large-scale magnetic field and $\tau_{\text{rec}}$ is the conformal time at recombination. On scales larger than $k_d^{-1}$ the magnetic field is not damped and it retains the information about its initial conditions when the modes of the magnetic field re-enter the Hubble radius till the recombination epoch.

The three-point correlator (3.11) gets then modified to

$$\langle \delta c_L(q, \tau) \vec{B}(k_1, \tau_1) \vec{B}(k_2, \tau_2) \rangle_{q \to 0} = - P_{cL}(q, \tau) \left( \frac{D(\tau_1)}{D(\tau)} - \frac{D(\tau_2)}{D(\tau)} \right) \frac{\delta c_{\text{NL}}(q)}{q^2} \langle \vec{B}(k_1, \tau_1) \vec{B}(k_2, \tau_2) \rangle_{q \to 0}^\prime$$

$$+ \frac{5}{2} b_{\text{NL}} \frac{\mathcal{H}(\tau_0) \Omega(\tau_0)}{q^2 T(q)} \frac{D(\tau_0)}{D(\tau)} P_{cL}(q, \tau) \langle \vec{B}(k_1, \tau_1) \vec{B}(k_2, \tau_2) \rangle_{q \to 0}^\prime,$$

(4.8)

where $T(q)$ is the DM linear transfer function and we have assumed that the effect of baryons are negligible, so that we can take $\zeta_L(q) = - (5/3) \Phi_L(q) = (5/3) \cdot (3/2) q^{-2} \mathcal{H}^2 \Omega_c \delta c_L(q)$. As expected, the equal-time three-point correlator is reduced to

$$\langle \delta c_L(q, \tau) \vec{B}(k_1, \tau) \vec{B}(k_2, \tau) \rangle_{q \to 0}^\prime = \frac{5}{2} b_{\text{NL}} \frac{\mathcal{H}(\tau_0) \Omega(\tau_0)}{q^2 T(q)} \frac{D(\tau_0)}{D(\tau)} P_{cL}(q, \tau) \langle \vec{B}(k_1, \tau) \vec{B}(k_1, \tau) \rangle_{q \to 0}^\prime.$$

(4.9)

A measurement of a non-vanishing equal-time three-point correlator between the magnetic field and the large-scale DM fluctuation in the soft-limit will be an indication of the inflationary origin of the magnetic field. Maybe, to estimate the strength of the expected
signal one could exploit the fact that the three-point correlator in the squeezed limit is in fact the magnetic field two-point correlator modulated by the long wavelength mode. In such a way, one could measure the equal-time three-point correlator in N-body simulations without actually employing three-point function estimators. Specifically, one could adopt a “position-dependent power spectrum”, that is the power spectrum of the magnetic fields measured in smaller subvolumes of the survey (or simulation box), and correlate it with the mean overdensity of the corresponding subvolume. This correlation should directly measures an integral of the bispectrum dominated by the squeezed configurations.

We reiterate that, even though the exact form of the violation might be different from the one given in eq. (4.9), any violation of the consistency relation at equal-time would be a sign of the fact that the primordial seeds for the magnetic fields were generated during inflation. This is because no other origin can correlate with the long-wavelength DM fluctuations. As we mentioned in the introduction, there are two broad classes of models for the origin of the seed fields: either the seed fields are produced in the early universe, during epochs preceding the structure formation, or the seeds are created during the gravitational collapse leading to structure formation. The second possibility would not explain a correlation with the DM fluctuations at very large scales, that is in the initial conditions. The first possibility would also exclude causal mechanisms, such as the one where the magnetic fields are generated during phase transitions: their coherent lengths will be much smaller than the horizon scales at a given epoch. The only option left which creates a correlation between the initial magnetic field and DM seeds is a common inflationary origin. All this is worth further investigation.

5 Conclusions

In this paper we have focused on the symmetries enjoyed by the cosmological MHD equations. While we have certainly not described all the possible symmetries, we have identified a subset of them which allow to write well-defined consistency relations involving the soft-limit of \((n+1)\)-correlators between the magnetic fields and the DM fluctuations. These consistency relations have important properties: they vanish when the equal-time limit is considered, unless a cross-correlation between the magnetic field and the DM fluctuations already exists on very large-scales, that is in the initial conditions. If a violation of the equal-time consistency relation is observed, this would suggest an inflationary origin of the magnetic field seeds.

Acknowledgments

We thank R. Durrer, J. Noreña, and M. Sloth for very enlightening discussions and for reading the draft of the paper. The research of A.K. was implemented under the “Aristeia” Action of the “Operational Programme Education and Lifelong Learning” and is co-funded by the European Social Fund (ESF) and National Resources. A.K. is also partially supported by European Union’s Seventh Framework Programme (FP7/2007-2013) under REA grant agreement n. 329083. A.R. is supported by the Swiss National Science Foundation (SNSF), project “The non-Gaussian Universe” (project number: 200021140236).

A Symmetries of the cosmological MHD in phase space

The Vlasov equation in physical coordinates \((\vec{r}, t)\) is written as

\[
\frac{\partial f}{\partial t} + \frac{\partial \vec{r}}{\partial t} \cdot \vec{\nabla}_{\vec{r}} f + \frac{\partial p_{\vec{r}}}{\partial t} \nabla_{\vec{p}_{\vec{r}}} f = 0,
\]  

(A.1)
where \( \vec{p}_r = m\vec{v} \) and \( f = f(\vec{p}_r, \vec{r}, t) \) is the phase space density. In comoving coordinates \( \vec{x} = \vec{r}/a \), with peculiar velocity \( \vec{v} = a\vec{x} \), and comoving momenta \( p = ma^2\vec{x} \) we have the relations

\[
\frac{\partial f}{\partial t} |_{\vec{p},\vec{x}} = \frac{\partial f}{\partial t} |_{\vec{p}_r,\vec{r}} + \left( \frac{\dot{a}}{a^2} \vec{v} \cdot \vec{\nabla}_p f \right) |_{\vec{x},t} + \frac{\dot{a}}{a} \left( \vec{\nabla}_x f \big|_{\vec{p},t} - m\dot{a} \vec{\nabla}_p f \big|_{\vec{x},t} \right),
\]

(A.2)

\[
\vec{\nabla}_x f |_{\vec{p}_r,t} = a\vec{\nabla}_x f |_{\vec{p},t} + m\dot{a} \vec{\nabla}_p f |_{\vec{x},t},
\]

(A.3)

\[
\vec{\nabla}_p f |_{\vec{x},t} = \vec{\nabla}_p f |_{\vec{x},t}.
\]

(A.4)

It can then be verified that the Vlasov equation in comoving coordinates is written as

\[
\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2} \vec{\nabla}_x f + \frac{\partial \vec{p}}{\partial t} \vec{\nabla}_p f = 0.
\]

(A.5)

### A.1 Vlasov-Poisson equations

In particular, in the case in which there is a gravitational force, the comoving momentum satisfies

\[
\frac{d\vec{p}}{d\tau} = -ma\vec{\nabla}_x \Phi(\vec{x}, \tau),
\]

(A.6)

where we have switched now to conformal time. Recall that \( \Phi(\vec{x}, \tau) \) satisfies the Poisson equation

\[
\nabla^2 \Phi(\vec{x}, \tau) = 4\pi G \rho a \delta(\vec{x}, \tau)
\]

(A.7)

and therefore the Vlasov-Poisson equation is written as

\[
\frac{\partial}{\partial \tau} f(\vec{p}, \vec{x}, \tau) + \frac{1}{am} \vec{p} \cdot \vec{\nabla}_x f(\vec{p}, \vec{x}, \tau) - am\vec{\nabla}_x \Phi \cdot \vec{\nabla}_p f(\vec{p}, \vec{x}, \tau) = 0.
\]

(A.8)

We will try to find symmetries of the Vlasov-Poisson equations (A.7) and (A.8) of the form

\[
\tau \rightarrow \tau' = T(\tau), \quad \vec{x} \rightarrow \vec{x}' = \vec{y}(\vec{x}, \tau), \quad \vec{p} \rightarrow \vec{p}' = \vec{\Pi}(\vec{p}, \vec{x}, \tau).
\]

(A.9)

Since

\[
\frac{\partial}{\partial \tau} = \dot{T} \frac{\partial}{\partial \tau} + \dot{\vec{y}}^i \frac{\partial}{\partial y^i} + \dot{\vec{\Pi}}^i \frac{\partial}{\partial \Pi^i},
\]

(A.10)

\[
\frac{\partial}{\partial x^i} = A_{ij} \frac{\partial}{\partial y^j} + B_{ij} \frac{\partial}{\partial \Pi^j},
\]

(A.11)

\[
\frac{\partial}{\partial p^i} = C_{ij} \frac{\partial}{\partial \Pi^j},
\]

(A.12)

where

\[
A_{ij} = A_{ij}(\vec{x}, \tau) = \frac{\partial y^j}{\partial x^i}, \quad B_{ij} = B_{ij}(\vec{x}, \tau) = \frac{\partial \Pi^j}{\partial x^i}, \quad C_{ij} = C_{ij}(\vec{x}, \tau) = \frac{\partial \Pi^j}{\partial p^i},
\]

(A.13)

we get from

\[
\frac{\partial}{\partial \tau} f(\vec{p}, \vec{x}, \tau) + \frac{1}{am} \vec{p} \cdot \frac{\partial}{\partial x^i} f(\vec{p}, \vec{x}, \tau) - a'(-\tau) \frac{\partial}{\partial x^i} \vec{\Phi}(\vec{p}, \vec{x}, \tau) = 0
\]

(A.14)
we get that
\[ \frac{T}{\partial \tau} f(\bar{p'}, \bar{x'}, \tau') + \frac{\partial}{\partial y^i} f(\bar{p'}, \bar{x'}, \tau') + \frac{\partial}{\partial \Pi} f(\bar{p'}, \bar{x'}, \tau') + \frac{1}{a' m} p^i A_{ij} \frac{\partial}{\partial y^j} f(\bar{p'}, \bar{x'}, \tau') + \frac{1}{a' m} p^i B_{ij} \frac{\partial}{\partial \Pi} f(\bar{p'}, \bar{x'}, \tau') = 0. \] (A.15)

The above equation gives the condition (vanishing of the $\partial f/\partial y^i$ term)
\[ \frac{T}{\partial \tau} f(\bar{p'}, \bar{x'}, \tau') + \frac{\partial}{\partial y^i} f(\bar{p'}, \bar{x'}, \tau') = 0, \] (A.16)

whereas, from the vanishing of the $\partial f/\partial \Pi$ term, we get
\[ \frac{\partial}{\partial x^i} \Phi' (\bar{x}, \tau) = \frac{T a' (\tau)}{a' (\tau)} C_{ij}^{-1} \frac{\partial}{\partial y^j} \Phi (\bar{x'}, \tau') + \frac{1}{a' (\tau) m} C_{ij}^{-1} \left( \Pi^j + \frac{1}{a' (\tau) m} p^k B_{kj} \right). \] (A.17)

Now, we have the two following additional conditions to satisfy
\[ \bar{p}' (\tau) a' (\tau)^3 = \bar{p} (\tau) a (\tau)^3, \quad \bar{p}' (\tau) \left( 1 + \delta (\bar{x}, \tau) \right) = \bar{p} (\tau) \left( 1 + \delta (\bar{x'}, \tau') \right). \] (A.18)

Since
\[ \rho (\bar{x}, \tau) = \frac{1}{a^3 (\tau)} \int d^3 p f(\bar{p}, \bar{x}, \tau) \] (A.19)
and
\[ \bar{p} (\tau) = \frac{1}{a^3 (\tau)^3} \int d^3 x d^3 p f(\bar{p}, \bar{x}, \tau), \] (A.20)

we get that
\[ \det C = \frac{a' (\tau)^3}{a^3 (\tau)^3}, \quad \det \mathcal{A} = \frac{a' (\tau)^3}{a (\tau')^3}, \] (A.21)

where $C$ and $\mathcal{A}$ are the 3 matrices $(C_{ij})$ $(A_{ij})$, respectively. Therefore, we find that $\bar{p}$ transforms as
\[ \bar{p} (\tau) \rightarrow \bar{p}' (\tau) = \frac{1}{\det \mathcal{A}} \bar{p} (\bar{x}, \tau). \] (A.22)

Acting on eq. (A.17) with $\partial/\partial x^i$ and using Poisson equation we get
\[ 4 \pi G \bar{p}' (\tau) a' (\tau)^2 \delta (\bar{x}, \tau) = \frac{T a' (\tau)}{a' (\tau)} C_{ij}^{-1} A_{ik} \frac{\partial^2}{\partial y^k \partial y^j} \Phi (\bar{x'}, \tau') + \frac{1}{a' (\tau) m} C_{ij}^{-1} \frac{\partial}{\partial x^j} \left( \Pi^j + \frac{1}{a' (\tau) m} p^k B_{kj} \right). \] (A.23)

The only way this equation determines the transformation property of the overdensity $\delta$ is if
\[ C_{ij}^{-1} A_{ik} = M (\tau) \delta_{jk}, \] (A.24)
which from (A.21) gives that
\[ M (\tau) = (\det \mathcal{A})^{2/3}. \] (A.25)
In this case we get
\[
\delta'(\vec{x}, \tau) = \dot{T}M(\tau)\delta(\vec{x}', \tau') + \frac{C_{ij}^{-1}}{4\pi Gp'(\tau)a'(\tau)^3m} \frac{\partial}{\partial x^i} \left( \dot{\Pi}^j + \frac{1}{a'(\tau)m} p^k B_{kj} \right). \tag{A.26}
\]

Then, from the second condition of eq. (A.18) we get that
\[
1 + \delta'(\vec{x}, \tau) = \det A(1 + \delta(\vec{x}', \tau')) , \tag{A.27}
\]
so that
\[
\dot{T}M(\tau) = \det A , \quad \frac{C_{ij}^{-1}}{4\pi Gp'(\tau)a'(\tau)^3m} \frac{\partial}{\partial x^i} \left( \dot{\Pi}^j + \frac{1}{a'(\tau)m} p^k B_{kj} \right) = \det A - 1. \tag{A.28}
\]

Therefore we find
\[
\dot{T} = (\det A)^{1/3} , \tag{A.29}
\]
so that
\[
\delta'(\vec{x}, \tau) = \det A \delta(\vec{x}', \tau') + \det A - 1. \tag{A.30}
\]

Note that from eq. (A.13) we get that
\[
y^j = A_{ij}(\tau)x^i + n^i(\tau) , \quad P^j = C_{ij}(\tau)p^j + B_{ij}(\tau)x^j + s^i(\tau) , \tag{A.31}
\]
since from the second condition of eq. (A.28), if $B_{ij}$ was a function of $x^i$, then (A.28) will depend on the momenta $p^i$ as well. Using that
\[
\Pi^i = (\det A)^{-2/3} A_{ij}p^j + (\det A)^{-1/3} a(\tau)m\ddot{y}^i
\]
\[
= (\det A)^{-2/3} A_{ij}p^j + (\det A)^{-1/3} a(\tau)m \frac{\partial}{\partial \tau} (A_{ij}x^j + n^i(\tau)) \tag{A.32}
\]
we can identify
\[
B_{ij} = (\det A)^{-2/3} a'(\tau)m \dot{A}_{ij} , \quad s^i(\tau) = (\det A)^{-2/3} a'(\tau)m n^i(\tau) , \tag{A.33}
\]
and find that the second condition of (A.28) is written as
\[
\frac{(\det A)^2}{4\pi Gp'(\tau)a'(\tau)^3} A_{ij}^{-1} \frac{\partial}{\partial \tau} \left( \frac{a'(\tau)}{(\det A)^{2/3}} \frac{\partial}{\partial \tau} A_{ij} \right) = \det A - 1. \tag{A.34}
\]

This can be expanded to obtain
\[
\frac{1}{4\pi Gp'(\tau)a'(\tau)^3} \left[ \ddot{A}_{ij} + \left( \frac{a'(\tau)}{a'(\tau)} \frac{2}{3} \frac{\partial}{\partial \tau} (\det A) \right) \dot{A}_{ij} \right] + (1 - \det A) A_{ij} = 0 \tag{A.35}
\]

Collecting our results we find that Vlasov-Poisson system is invariant under the transformations
\[
\tau \to \tau' = \int^{\tau} (\det A)^{1/3} d\eta , \quad x^i \to x'^i = A_{ij}(\tau)x^j + n^i(\tau) , \tag{A.36}
\]
\[
a'(\tau) = (\det A)^{1/3} a(\tau') , \tag{A.37}
\]
\[
p'(\tau) = \frac{1}{\det A} p(\tau') , \tag{A.38}
\]
\[
\delta'(\vec{x}, \tau) = \det A \delta(\vec{x}, \tau) + \det A - 1 , \tag{A.39}
\]
\[
p^i \to p'^i = (\det A)^{-2/3} A_{ij}p^j + (\det A)^{-2/3} a'(\tau)m \left( \dot{A}_{ij}(\tau)x^j + n^i(\tau) \right) , \tag{A.40}
\]
\[
\frac{\partial}{\partial x^i} \Phi(\vec{x}, \tau) = (\det A)^{2/3} A_{ij}^{-1} \frac{\partial}{\partial y^j} \Phi(\vec{x}', \tau') + \frac{1}{a'(\tau)m} A_{ij}^{-1} \left( (\det A)^{2/3} \dot{p}^j + \dot{A}_{kj}p^k \right) . \tag{A.41}
\]
The transformations (A.36)–(A.41) are generated by the matrix $A_{ij}(\tau)$ which satisfies eq. (A.34). As $A_{ij}$ has necessarily determinant non-zero, it belongs to the group $GL(3, \mathbb{R})$ of $3 \times 3$ non-singular matrices. Every such matrix can be decomposed as

$$A_{ij} = \frac{1}{3} \text{Tr} A \delta_{ij} + A_{ij}^T + A_{[ij]} \quad (A.42)$$

where

$$A_{ij}^T = \frac{1}{2} (A_{ij} + A_{ji}) - \frac{1}{3} \text{Tr} A \delta_{ij} \quad (A.43)$$

and

$$A_{[ij]} = \frac{1}{2} (A_{ij} - A_{ji}) \quad (A.44)$$

are the traceless symmetric $A_{ij}^T$ and antisymmetric parts $A_{[ij]}$ of any matrix $A_{ij} \in GL(3, \mathbb{R})$. Clearly the trace part $A_{ij}$ is the one that has been employed in the previous section. It is also clear that its antisymmetric part generates vorticity since in this case

$$\left(\vec{\nabla} p'\right)_{ij} = a'((\tau)m(\det A)^{-1/3} \dot{A}_{ij}(\tau) \quad (A.45)$$

is non-zero if the antisymmetric part of $A_{ij}$ is non-vanishing. Notice that when $\det A = 1$ the matrix $A_{ij}$ belongs to the special linear group $SL(3, \mathbb{R})$ and the transformations (A.36)–(A.41) read

$$\tau \rightarrow \tau' = \tau, \quad x^i \rightarrow x'^i = A_{ij}(\tau)x^j + n^i(\tau), \quad (A.46)$$

$$a'((\tau) = a(\tau'), \quad (A.47)$$

$$\vec{p}'(\vec{x}, \tau) = \vec{p}(\vec{x}', \tau), \quad (A.48)$$

$$\delta'(\vec{x}, \tau) = \delta(\vec{x}, \tau), \quad (A.49)$$

$$p^i \rightarrow p'^i = A_{ij}p^j + a(\tau)m\left(\dot{A}_{ij}(\tau)x^j + \dot{n}^i(\tau)\right), \quad (A.50)$$

$$\frac{\partial}{\partial x^i} \Phi(\vec{x}, \tau) = A_{ij}^{-1} \frac{\partial}{\partial y^j} \Phi(\vec{x}', \tau) + \frac{1}{a(\tau)m} A_{ij}^{-1} \left(\dot{p}'' + \dot{A}_{kj} p^k\right). \quad (A.51)$$

For $A_{ij} = \delta_{ij}$ we get the transformation used in this paper. However, we can still generate vorticity with a matrix $A_{ij}$ with non-zero antisymmetric part.

### A.2 Vlasov-Poisson-Maxwell

Let us now consider a collisionless plasma in the presence of gravitational, electric, and magnetic fields, $\vec{E}$ and $\vec{B}$. Let us assume that the plasma is composed of two fluids, of particles of masses $m_i$ and charges $e_i$, $(i = 1, 2)$, electrons and baryons, with corresponding particle distribution functions $f_i(\vec{p}, \vec{x}, t)$, and which obey the Vlasov-Poisson-Maxwell equation

$$\frac{\partial}{\partial \tau} f_i(\vec{p}, \vec{x}, \tau) + \frac{\vec{p}_i}{m_i} \cdot \vec{\nabla}_x f_i(\vec{p}, \vec{x}, \tau) - \left[am_i \vec{\nabla}_x \Phi - ae_i \left(\vec{E} + \frac{\vec{p}_i}{am_i} \times \vec{B}\right)\right] \cdot \vec{\nabla}_p f_i(\vec{p}, \vec{x}, \tau) = 0. \quad (A.52)$$

Note that $\vec{B} = \vec{B}_\Sigma$ and $\vec{E} = \vec{E}_\Sigma$ are the electric and magnetic fields in the comoving frame and they are related to the corresponding fields $\vec{B}_\Sigma$ and $\vec{E}_\Sigma$ in the coordinate frame by

$$\vec{E}_\Sigma = a\vec{E}_{\Sigma} + \dot{a} \vec{r} \times \vec{B}_{\Sigma}, \quad \vec{B}_\Sigma = a\vec{B}_{\Sigma}. \quad (A.53)$$
As usual, we define the number density field of the fluids
\[ n_i(\vec{x}, t) = a^{-3} \int d^3p f_i \] (A.54)
and the mean velocity of the fluids
\[ \vec{v}_i(\vec{x}, t) = \frac{\int d^3p \vec{p}_i m_i f_i}{\int d^3p f_i} \] (A.55)

By taking moments of the Vlasov-Poisson-Maxwell equation, we obtain the evolution equations for the corresponding moments. For example, the zeroth-order moment gives
\[ \partial_t (n_i a^{-3}) + \nabla \cdot (a^{-3} n_i \vec{v}_i) = 0 \] (A.56)

Similarly, the first moment equation gives
\[ \frac{\partial (a^3 \rho_i \vec{v}_i)}{\partial \tau} + a^3 \left( \mathcal{H} \rho_i \vec{v}_i + \vec{v}_i \nabla \rho_i \vec{v}_i + \rho_i \left( \vec{v}_i \cdot \nabla \right) \vec{v}_i + \rho_i \nabla \Phi - e_i n_i \left( \vec{E} + \vec{v}_i \times \vec{B} \right) + \nabla P_i \right) = 0 \] (A.57)

for the case of homogeneous and isotropic collapse, where \( P_i \) is the pressure and \( \rho_i = m_i n_i \).

Summing over the fluids and defining the plasma mass density function \( \rho_p \), the velocity \( \vec{v}_p \) and the pressure as
\[ \rho_p = \sum_i e_i n_i = e (n_1 - n_2), \quad \vec{v}_p = \sum_i e_i n_i \vec{v}_i = e (n_1 \vec{v}_1 - n_2 \vec{v}_2) \] (A.58)
we find the plasma continuity equation [40]
\[ \frac{\partial (\rho_p a^3)}{\partial \tau} + \nabla \cdot (a^3 \rho_p \vec{v}_p) = 0 \] (A.60)
as well as momentum conservation
\[ \rho_p \left( \frac{\partial \vec{v}_p}{\partial \tau} + \mathcal{H} \vec{v}_p + \left( \vec{v}_p \cdot \nabla \right) \vec{v}_p + \nabla \Phi \right) - \rho_e \vec{E} - \vec{j}_p \times \vec{B} + \nabla P_p = 0 \] (A.61)

In the last equation we have assumed \( m_1 \gg m_2 \); this is a good approximation because ultimately the two particles have to be identified with protons (ions) and electrons and therefore one can ignore the contribution of the light electrons to the non-linear advection term. In addition, in MHD we assume quasi-neutrality
\[ \rho_e = 0 \] (A.62)
and ideal Ohm’s law, i.e. zero resistivity, so that \( \vec{E} = \vec{B} \times \vec{v}_p \) in the physical frame. Then Maxwell’s equation
\[ \nabla \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu \vec{j}_p \] (A.63)
turns out to be
\[
\nabla \times \vec{B} + \frac{1}{c^2} \frac{\partial}{\partial t} (\vec{v}_p \times \vec{B}) = \mu \vec{J}_p. \tag{A.64}
\]

Therefore, in the non-relativistic limit \( v \ll c \), the displacement current vanishes and we are left with Ampere’s law which is written as
\[
\mu \vec{j}_p = \frac{1}{a^2} \nabla \times \vec{B}, \tag{A.65}
\]
in the comoving frame. By plugging eqs. (A.62) and (A.65) into eq. (A.61), and returning \( \vec{B} \) to its physical value, we get the plasma momentum conservation (2.6).

It is now straightforward to verify that the Vlasov-Poisson-Maxwell equation (A.52) is invariant under the transformations (A.46)–(A.51). Indeed, we have shown already that this system is invariant for \( \vec{B} = \vec{E} = 0 \). Thus (A.52) is invariant for non-vanishing \( \vec{E} \) and \( \vec{B} \) if the term
\[
I_i = a e_i \left( \vec{E} + \vec{v}_i \times \vec{B} \right) \cdot \nabla f(p, \vec{x}, \tau) \tag{A.66}
\]
is also invariant. In other words, we must have
\[
a(\tau') e_i \left( \vec{E}'(\vec{x}', \tau') + \frac{\vec{p}_i}{ma(\tau')} \times \vec{B}(\vec{x}', \tau') \right) \cdot \nabla f(p', \vec{x}', \tau') = \\
a(\tau) e_i \left( \vec{E}(\vec{x}, \tau) + \frac{\vec{p}_i}{ma(\tau)} \times \vec{B}(\vec{x}, \tau) \right) \cdot \nabla f(p, \vec{x}, \tau) \tag{A.67}
\]
in order for the Vlasov-Poisson-Maxwell equation (A.52) to be invariant. Then, by using (A.36)–(A.41) it is straightforward to verify that (A.66) specifies the transformation properties of the electric and magnetic fields to be
\[
E_j'(\vec{x}, \tau) = (\det A)^{1/3} A^{-1}_{ij} E_j'(\vec{x}', \tau') + \epsilon_{jkf} A_{kn} x^n + \epsilon^{kn} B^f(\vec{x}', \tau') A^{-1}_{ij}, \tag{A.68}
\]
\[
B'_i(\vec{x}, \tau) = \frac{1}{2} \epsilon_{ijk} \epsilon_{qlm} A^{-1}_{jq} A_{kl} B^m(\vec{x}', \tau'). \tag{A.69}
\]

For the consistency relation described in the text we have employed the particular case of the general transformations (A.36)–(A.41), (A.68) and (A.69), namely
\[
T(\tau) = \tau, \quad A_{ij} = \delta_{ij}. \tag{A.70}
\]
In this case, (A.68) and (A.69) turn out to be
\[
\vec{E}'(\vec{x}, \tau) = \vec{E}(\vec{x}', \tau') + \vec{n} \times \vec{B}(\vec{x}', \tau'), \tag{A.71}
\]
\[
\vec{B}'(\vec{x}, \tau) = \vec{B}(\vec{x}', \tau'). \tag{A.72}
\]
which is the transformation we employed in section 2. Of course, since the electric field is multiplied by the charge density \( \rho_e \) and the latter vanishes for a quasi-neutral plasma, it does not enter in the momentum conservation equation.
References

[1] For a review and references therein, see R. Durrer and A. Neronov, Cosmological Magnetic Fields: Their Generation, Evolution and Observation, Astron. Astrophys. Rev. 21 (2013) 62 [arXiv:1303.7121] [SPIRE].

[2] For a review and references therein, see R.M. Kulsrud and E.G. Zweibel, The Origin of Astrophysical Magnetic Fields, Rept. Prog. Phys. 71 (2008) 0046091 [arXiv:0707.2783] [SPIRE].

[3] M.S. Turner and L.M. Widrow, Inflation Produced, Large Scale Magnetic Fields, Phys. Rev. D 37 (1988) 2743 [SPIRE].

[4] B. Ratra, Cosmological ‘seed’ magnetic field from inflation, Astrophys. J. 391 (1992) L1 [SPIRE].

[5] L.M. Widrow, D. Ryu, D.R.G. Schleicher, K. Subramanian, C.G. Tsagas and R.A. Treumann, The First Magnetic Fields, Space Sci. Rev. 166 (2012) 37 [arXiv:1109.4052] [SPIRE].

[6] T. Doumler and A. Knebe, Investigating the influence of magnetic fields upon structure formation with AMIGA - a C code for cosmological magnetohydrodynamics, Mon. Not. Roy. Astron. Soc. 403 (2010) 453 [arXiv:0912.1498] [SPIRE].

[7] K. Dolag, M. Bartelmann and H. Lesch, SPH simulations of magnetic fields in galaxy clusters, Astron. Astrophys. 348 (1999) 351 [astro-ph/9906329] [SPIRE].

[8] Y. Takahashi, On the generalized Ward identity, Nuovo Cimento 6 (1957) 370.

[9] J.C. Ward, An Identity in Quantum Electrodynamics, Phys. Rev. 78 (1950) 182 [SPIRE].

[10] D.H. Lyth and A. Riotto, Particle physics models of inflation and the cosmological density perturbation, Phys. Rept. 314 (1999) 1 [hep-ph/9807278] [SPIRE].

[11] I. Antoniadis, P.O. Mazur and E. Mottola, Conformal Invariance, Dark Energy and CMB Non-Gaussianity, JCAP 09 (2012) 024 [arXiv:1103.4164] [SPIRE].

[12] P. Creminelli, Conformal invariance of scalar perturbations in inflation, Phys. Rev. D 85 (2012) 041302 [arXiv:1108.0874] [SPIRE].

[13] A. Kehagias and A. Riotto, Operator Product Expansion of Inflationary Correlators and Conformal Symmetry of de Sitter, Nucl. Phys. B 864 (2012) 492 [arXiv:1205.1523] [SPIRE].

[14] A. Kehagias and A. Riotto, The Four-point Correlator in Multifield Inflation, the Operator Product Expansion and the Symmetries of de Sitter, Nucl. Phys. B 868 (2013) 577 [arXiv:1210.1918] [SPIRE].

[15] M. Biagetti, A. Kehagias, E. Morgante, H. Perrier and A. Riotto, Symmetries of Vector Perturbations during the de Sitter Epoch, JCAP 07 (2013) 030 [arXiv:1304.7785] [SPIRE].

[16] P. Creminelli, J. Noreña and M. Simonovic, Conformal consistency relations for single-field inflation, JCAP 07 (2012) 052 [arXiv:1203.4596] [SPIRE].

[17] K. Hinterbichler, L. Hui and J. Khoury, Conformal Symmetries of Adiabatic Modes in Cosmology, JCAP 08 (2012) 017 [arXiv:1203.6351] [SPIRE].

[18] V. Assassi, D. Baumann and D. Green, On Soft Limits of Inflationary Correlation Functions, JCAP 11 (2012) 047 [arXiv:1204.4207] [SPIRE].

[19] V. Assassi, D. Baumann and D. Green, Symmetries and Loops in Inflation, JHEP 02 (2013) 151 [arXiv:1210.7792] [SPIRE].

[20] A. Kehagias and A. Riotto, Symmetries and Consistency Relations in the Large Scale Structure of the Universe, Nucl. Phys. B 873 (2013) 514 [arXiv:1302.0130] [SPIRE].
[21] M. Peloso and M. Pietroni, *Galilean invariance and the consistency relation for the nonlinear squeezed bispectrum of large scale structure*, *JCAP* 05 (2013) 031 [arXiv:1302.0223] [inSPIRE].

[22] P. Creminelli, J. Noreña, M. Simonović and F. Vernizzi, *Single-Field Consistency Relations of Large Scale Structure*, *JCAP* 12 (2013) 025 [arXiv:1309.3557] [inSPIRE].

[23] A. Kehagias and A. Riotto, *Conformal Symmetries of FRW Accelerating Cosmologies*, arXiv:1309.3671 [inSPIRE].

[24] A. Kehagias, J. Noreña, H. Perrier and A. Riotto, *Consequences of Symmetries and Consistency Relations in the Large-Scale Structure of the Universe for Non-local bias and Modified Gravity*, *Nucl. Phys. B* 883 (2014) 83 [arXiv:1311.0786] [inSPIRE].

[25] P. Creminelli, J. Gleyzes, M. Simonović and F. Vernizzi, *Single-Field Consistency Relations of Large Scale Structure. Part II: Resummation and Redshift Space*, *JCAP* 02 (2014) 051 [arXiv:1311.0290] [inSPIRE].

[26] M. Peloso and M. Pietroni, *Ward identities and consistency relations for the large scale structure with multiple species*, *JCAP* 04 (2014) 011 [arXiv:1310.7915] [inSPIRE].

[27] P. Valageas, *Consistency relations of large-scale structures*, *Phys. Rev. D* 89 (2014) 083534 [arXiv:1311.1236] [inSPIRE].

[28] P. Creminelli, J. Gleyzes, L. Hui, M. Simonović and F. Vernizzi, *Single-Field Consistency Relations of Large Scale Structure. Part III: Test of the Equivalence Principle*, arXiv:1312.6074 [inSPIRE].

[29] P. Valageas, *Angular averaged consistency relations of large-scale structures*, arXiv:1311.4286 [inSPIRE].

[30] A. Kehagias, H. Perrier and A. Riotto, *Equal-time Consistency Relations in the Large-Scale Structure of the Universe*, arXiv:1311.5524 [inSPIRE].

[31] I. Wasserman, *On the Origin of Galaxies, Galactic Angular Momenta, and Galactic Magnetic Fields*, *Astrophys. J.* 224 (1978) 337.

[32] N. Barnaby, R. Namba and M. Peloso, *Observable non-Gaussianity from gauge field production in slow roll inflation and a challenging connection with magnetogenesis*, *Phys. Rev. D* 85 (2012) 123523 [arXiv:1202.1469] [inSPIRE].

[33] T. Fujita and S. Mukohyama, *Universal upper limit on inflation energy scale from cosmic magnetic field*, *JCAP* 10 (2012) 034 [arXiv:1205.5031] [inSPIRE].

[34] R.J.Z. Ferreira, R.K. Jain and M.S. Sloth, *Inflationary magnetogenesis without the strong coupling problem*, *JCAP* 10 (2013) 004 [arXiv:1305.7151] [inSPIRE].

[35] R.K. Jain and M.S. Sloth, *Consistency relation for cosmic magnetic fields*, *Phys. Rev. D* 86 (2012) 123528 [arXiv:1207.4187] [inSPIRE].

[36] R.K. Jain and M.S. Sloth, *On the non-Gaussian correlation of the primordial curvature perturbation with vector fields*, *JCAP* 02 (2013) 003 [arXiv:1210.3461] [inSPIRE].

[37] M. Bruni, R. Maartens and C.G. Tsagas, *Magnetic field amplification in CDM anisotropic collapse*, *Mon. Not. Roy. Astron. Soc.* 338 (2003) 785 [astro-ph/0208126] [inSPIRE].

[38] K. Jedamzik, V. Katalinic and A.V. Olinto, *Damping of cosmic magnetic fields*, *Phys. Rev. D* 57 (1998) 3264 [astro-ph/9606080] [inSPIRE].

[39] K. Subramanian and J.D. Barrow, *Magnetohydrodynamics in the early universe and the damping of nonlinear Alfvén waves*, *Phys. Rev. D* 58 (1998) 083502 [astro-ph/9712083] [inSPIRE].

[40] E. Marsch, lectures on *Space plasma physics*, International Max Planck Research School for Solar System Science at the University of Göttingen, Germany, 11–15 June 2007.