Bounded Pseudo-Amenability and Contractibility of Certain Banach Algebras

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Abstract. The notion of bounded pseudo-amenability was introduced by Y. Choi and et al. [CGZ]. In this paper, similarly, we define bounded pseudo-contractibility and then investigate bounded pseudo-amenability and contractibility of various classes of Banach algebras including ones related to locally compact groups and discrete semigroups. We also introduce a multiplier bounded version of approximate biprojectivity for Banach algebras and determine its relation to bounded pseudo-amenability and contractibility.

1. Introduction

Let $A$ be a Banach algebra and $X$ a Banach $A$-bimodule. A bounded linear map $D : A \to X$ is called a derivation if
\[ D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A), \]
and it is termed inner if there is $x \in X$ such that
\[ D(a) = a \cdot x - x \cdot a \quad (a \in A). \]

The notion of amenability of Banach algebras was established by B. E. Johnson in 1972 ([Joh2]). If every bounded derivation from $A$ into the dual Banach $A$-bimodule $X^*$ is inner for all Banach $A$-bimodules $X$, then $A$ is said to be amenable. A Banach algebra $A$ is called contractible, if every bounded derivation from $A$ into any Banach $A$-bimodule is inner. In 2004, Ghahramani and Loy developed these concepts and introduced new notions of amenability and contractibility ([GhL]). The basic definition of their notions is referred to as approximately inner derivation. For an $A$-bimodule $X$, a derivation $D : A \to X$ is called approximately inner if there is a net of inner derivations $\{D_\alpha : A \to X\}_\alpha$ such that $D(a) = \lim_\alpha D_\alpha(a)$ for any $a \in A$. The Banach algebra $A$ is said to be (boundedly) approximately amenable if for any $A$-bimodule $X$, every derivation $D : A \to X^*$ is the pointwise limit of a (boundedly) net of inner derivations from $A$ into $X^*$. In a similar manner (boundedly) approximate contractibility was defined. All notions of amenability are characterized in terms of approximate diagonals. We recall definitions needed in this article.

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**Definition 1.1.** Let $A$ be a Banach algebra. A net $\{m_i\} \subset A \hat{\otimes} A$ satisfying
\[am_i - m_ia \to 0, \quad am_i \to a,\]
is called an approximate diagonal, where $\pi : A \hat{\otimes} A \to A$ is the diagonal map determined by $\pi(a \otimes b) = ab$. According to [CGZ], we say that the diagonal $\{m_i\}$ is multiplier-bounded if there exists a constant $K > 0$ such that for all $a \in A$ and all $i$,
\[\|am_i - m_ia\| \leq K\|a\|, \quad \|am_i - a\| \leq K\|a\|, \quad \|\pi(m_i)a\| \leq K\|a\|.
\]
Johnson proved in [Joh1] that a Banach algebra $A$ is amenable if and only if there exists a bounded approximate diagonal, i.e. an approximate diagonal $\{m_i\}$ satisfying $\sup_{a \in A} \|m_i\| < \infty$.

According to [GhZh] a Banach algebra $A$ is called pseudo-amenable if it has an approximate diagonal, and it is pseudo-contractible if it possesses a central approximate diagonal $\{m_i\}$, i.e. $am_i = m_ia$ for all $a \in A$ and all $i$.

**Definition 1.2.** A Banach algebra $A$ is called boundedly pseudo-amenable if it has a multiplier-bounded approximate diagonal. The term “$K$-pseudo-amenable” refers to bounded pseudo-amenability with multiplier bound $K > 0$.

Like Definition 1.2 we introduce the concept of bounded pseudo-contractibility.

**Definition 1.3.** A Banach algebra $A$ is called boundedly pseudo-contractible if it has a central multiplier-bounded approximate diagonal, that is to say there are a central approximate diagonal $\{m_i\}$ and a constant $K > 0$ such that
\[\|am_i - a\| \leq K\|a\| \quad (a \in A).
\]
Similarly, the term “$K$-pseudo-contractible” refers to bounded pseudo-contractibility with multiplier bound $K > 0$.

It is needless to say that every boundedly pseudo-contractible Banach algebra is boundedly pseudo-amenable.

Motivated by the earlier investigations, in this paper, we verify bounded pseudo-amenability and contractibility of some important Banach algebras in harmonic analysis such as group and measure algebras of a locally compact group, Fourier algebra of a discrete group and some algebras constructed on discrete semigroups. We also introduce a multiplier-bounded approximate biprojectivity for Banach algebras and verify its relation with bounded pseudo-amenability and contractibility.

2. Bounded pseudo-amenability and contractibility

In this section we give some general properties of bounded pseudo-amenable and contractible Banach algebras including hereditary properties.

Let $A$ be a Banach algebra. We say that a net $(\varepsilon_a)$ is an approximate identity for $A$, if $\|\varepsilon_a - a\| \to 0$ and $\|\varepsilon_a - a\| \to 0$ for all $a \in A$. It is called central if $\varepsilon_a = e_a$ for each $a \in A$. We call $(\varepsilon_a)$ a bounded approximate identity for $A$, if it is also bounded. The net $(\varepsilon_a)$ is termed a multiplier-bounded approximate identity for $A$ if there exists a constant $k > 0$ such that $\|\varepsilon_a\| \leq k\|a\|$ and $\|\varepsilon_a a\| \leq k\|a\|$ for all $a \in A$ and all $\alpha$. It is clear that boundedly pseudo-amenable Banach algebras possess a multiplier-bounded approximate identity and pseudo-contractive Banach algebras have a multiplier-bounded central approximate identity.

The unitization of a Banach algebra $A$ is denoted by $A^\#$ which is $\mathcal{A} \hat{\otimes} \mathbb{C}$ with the following product:
\[(a, \lambda) \cdot (b, \mu) = (ab + \mu a + \lambda b, \lambda \mu) \quad (a, b \in A, \lambda, \mu \in \mathbb{C}).
\]
It is obvious that with $l^1$-norm $A^\#$ is a Banach algebra as well.

**Proposition 2.1.** ([CGZ, Proposition 2.2]) A Banach algebra $A$ is boundedly approximately contractible if and only if its unitization $A^\#$ is boundedly pseudo-amenable.
The next proposition provides an example of a pseudo-amenable Banach algebra which is not boundedly pseudo-amenable.

**Proposition 2.2.** There is a unital Banach algebra which is pseudo-amenable but not boundedly pseudo-amenable.

**Proof.** Consider the Banach algebra $A$ constructed in [GhR] which is boundedly approximately amenable but not boundedly approximately contractible. Then it follows from [CGZ, Proposition 2.4] that $A^*$ is boundedly approximately amenable and so $A^*$ is pseudo-amenable by [Pou1, Corollary 3.7]. Using Proposition 2.1 and the fact that $A$ is not boundedly approximately contractible we conclude that $A^*$ is not boundedly pseudo-amenable.

**Theorem 2.3.** Let $A$ be a $K$-pseudo-amenable (contractible) Banach algebra, $B$ a Banach algebra and $\theta : A \to B$ a continuous epimorphism. Then $B$ is boundedly pseudo-amenable (contractible) with bound $K' = \max\{K\|\theta\|^2, K\|\theta\|\}$.

**Proof.** By the assumption there is a net $(m_i)$ in $A \hat{\otimes} A$ such that

$$am_i - m_ia \to 0, \quad a\pi(m_i) \to a,$$

$$\|am_i - m_ia\| \leq K\|a\|, \quad \|a\pi(m_i) - a\| \leq K\|a\|, \quad \|\pi(m_ia) - a\| \leq K\|a\|.$$

For each $i \in \mathbb{N}$ let $[a^*_n]_{n=1}^\infty, [b^*_n]_{n=1}^\infty \subset A$ be sequences such that $m_i = \sum_{n=1}^\infty a^*_n \otimes b^*_n$ and $\sum_{n=1}^\infty \|a^*_n\| \|b^*_n\| < \infty$. Set $C = \|\theta\|$ and define

$$M_i = (\theta \otimes \theta)(m_i) = \sum_{n=1}^\infty \theta(a^*_n) \otimes \theta(b^*_n).$$

Then $\|M_i\| \leq C^2\|m_i\|$ and for each $a \in A$,

$$\|\theta(a)M_i - M_i\theta(a)\| = \|((\theta \otimes \theta)(am_i - m_ia))\| \leq C^2\|am_i - m_ia\| \leq C^2K\|a\|,$$

$$\|\theta(a)\pi(M_i) - \theta(a)\| = \|\theta(a)\pi(\theta \otimes \theta(m_i)) - \theta(a)\| = \|\theta(a)\pi(\pi(m_i)) - \theta(a)\| = \|\theta(a)\pi(m_i) - \theta(a)\| \leq C\|\pi(m_i) - a\| \leq CK\|a\|,$$

and similarly

$$\|\pi(M_i)\theta(a) - \theta(a)\| \leq CK\|a\|.$$

Therefore, $(M_i)$ is a multiplier-bounded approximate diagonal for $B$, with bound $K' = \max\{KC^2, KC\}$.

**Corollary 2.4.** Let $A$ be a $K$-pseudo-amenable (contractible) Banach algebra and $I$ be a closed two-sided ideal of $A$. Then $A/I$ is $K$-pseudo-amenable (contractible).

**Corollary 2.5.** Let $A$ and $B$ be two Banach algebras such that $A \hat{\otimes} B$ is boundedly pseudo-amenable (contractible) and $B$ has a non-zero character. Then $A$ is boundedly pseudo-amenable (contractible).

**Proof.** Suppose that $A \hat{\otimes} B$ is $K$-pseudo amenable, $\phi$ is a non-zero character of $B$ and consider the epimorphism $\theta(A \hat{\otimes} B) \to A$ by $\theta(a \otimes b) = \phi(b)a$. Now Theorem 2.3 implies that $A$ is $K$-pseudo-amenable.

**Theorem 2.6.** Suppose that $A$ is a boundedly pseudo-amenable Banach algebra and $J$ is a two-sided closed ideal of $A$. Suppose also $[e_n] \subset A$ is a central approximate identity for $J$ that is multiplier-bounded in $A$. Then $J$ is also boundedly pseudo-amenable.

**Proof.** By the assumption there is a constant $M \geq 1$ such that for all $a$ and $n \in A$,

$$\|ae_n\| \leq M\|a\|, \quad \|e_na\| \leq M\|a\|.$$
So for each \( \alpha \) and \( m \in A \hat{\otimes} A \) we infer that
\[
||m e_\alpha|| \leq M ||m||, \quad ||e_\alpha m|| \leq M ||m||.
\]
Let \( \{m_i\} \subset A \hat{\otimes} A \) be a net satisfying conditions of Definition 1.2 with bound \( K > 0 \). For any \( \varepsilon > 0 \) and finite set \( F \subset J \), there are \( i \) and \( \alpha \) such that
\[
||am_i - m_i a||M^2 \leq \varepsilon / 2, \quad ||\pi(m_i)a - a||M \leq \varepsilon / 2 \quad (a \in F),
\]
and
\[
||e_\alpha a - a|| \leq \varepsilon / 4, \quad ||\pi(m_i)(e_\alpha a - a)||M \leq \varepsilon / 4 \quad (a \in F).
\]
Similar to the proof of [GhZh, Proposition 2.6], we obtain
\[
||ae_\alpha m e_\alpha - e_\alpha m e_\alpha a|| \leq \varepsilon, \quad ||\pi(e_\alpha m e_\alpha a - a)|| < \varepsilon \quad (a \in F).
\]
Passing to a subnet we may suppose that \( \{e_\alpha m e_\alpha\} \subset J \otimes J \) constitutes an approximate diagonal for \( J \). Since \( \{e_\alpha\} \) is central, for each \( i \) and \( a \in J \) we have
\[
||ae_\alpha m e_\alpha - e_\alpha m e_\alpha a|| = ||e_\alpha am e_\alpha - e_\alpha m e_\alpha a|| = ||e_\alpha (am_i - m_i a)e_\alpha|| \\
\leq M^2 ||am_i - m_i a|| \leq M^2 K ||a||,
\]
and
\[
||\pi(e_\alpha m e_\alpha a - a)|| = ||e_\alpha \pi(m_i)e_\alpha a - a|| \\
= ||e_\alpha \pi(m_i)e_\alpha a - e_\alpha e_\alpha a + e_\alpha e_\alpha a - a|| \\
\leq ||e_\alpha (\pi(m_i)e_\alpha a - e_\alpha a)|| + ||e_\alpha e_\alpha a - a|| \\
\leq M ||\pi(m_i)e_\alpha a - e_\alpha a|| + ||e_\alpha e_\alpha a - a|| \\
\leq MK ||e_\alpha a|| + M ||e_\alpha a|| + ||a|| \\
\leq M^2 K ||a|| + M^2 ||a|| + ||a|| \\
= (M^2 K + M^2 + 1)||a||.
\]
Likewise, \( ||\pi(e_\alpha m e_\alpha a) - a|| \leq (M^2 K + M^2 + 1)||a|| \). These imply that \( J \) is \((M^2 K + M^2 + 1)\)-pseudo-amenable. \( \square \)

**Corollary 2.7.** Suppose that \( A \) is a boundedly pseudo-amenable Banach algebra, \( J \) a closed two-sided ideal of \( A \) with a bounded central approximate identity. Then \( J \) is boundedly pseudo-amenable.

The proof of the next proposition is the same as that of [GhZh, Proposition 3.3] and is omitted.

**Proposition 2.8.** Let \( A \) be a \( M \)-boundedly approximately contractible Banach algebra. If \( A \) has a bounded central approximate identity \( \{e_\alpha\} \) with bound \( K \), then \( A \) is \((2K^2 + M)\)-pseudo-amenable.

**Corollary 2.9.** Let \( A \) be a boundedly approximately contractible commutative Banach algebra. Then \( A \) is boundedly pseudo-amenable.

**Proof.** Every boundedly approximately contractible Banach algebra has a bounded approximate identity. \( \square \)

**Theorem 2.10.** Suppose that \( A \) is a boundedly pseudo-amenable Banach algebra and \( X \) is a Banach \( A \)-bimodule for which each multiplier bounded left (right) approximate identity of \( A \) is a multiplier bounded left (right) approximate identity for \( X \). Then

1. Every derivation \( D : A \to X \) is boundedly approximately inner.
2. Every derivation $D : A \to X'$ is boundedly weak$^\ast$ approximately inner.

Proof. (1): Let $\Phi : A \otimes A \to X$ be defined by $\Phi(a \otimes b) = D(a) \cdot b$ and let $\{m_i\}$ be a net satisfying conditions of Definition 1.2 with corresponding bound $K > 0$. If we set $\psi_i = -\Phi(m_i)$, then as in [GhZh, Proposition 3.5] for each $a \in A$ we obtain

$$D(a) = \lim_i (a\psi_i - \psi_i a),$$

and also we get

$$\|a \cdot \psi_i - \psi_i a\| - \|D(a)\pi(m_i)\| \leq \|a \cdot \psi_i - \psi_i a - D(a)\pi(m_i)\| = \|\Phi(a \cdot m_i - m_i \cdot a)\|$$

$$\leq \|\Phi\| \|a \cdot m_i - m_i \cdot a\| \leq K\|\Phi\| \|\|a\| \leq K\|D\| \|\|a\|,$$

and so

$$\|a \cdot \psi_i - \psi_i a\| \leq K\|D\| \|\|a\| + \|D(a)\pi(m_i)\| \leq K\|D\| \|\|a\| + (K' + 1)\|D(a)\| \leq K''\|D(a)\|.$$ 

Whence $D$ is boundedly approximately inner.

(2) can be proven similarly. \(\square\)

Obviously, every contractible Banach algebra is boundedly pseudo-contractible. We end this section by presenting an example of a boundedly pseudo-contractible Banach algebra which is not amenable and consequently not contractible.

Example 2.11. For $1 \leq p < \infty$ let $\ell^p$ be the usual Banach sequence algebra with pointwise multiplication. Since $\ell^p$ does not have a bounded approximate identity, it is not amenable. Now for each $i \in \mathbb{N}$ let $\delta_i$ be the characteristic function of the singleton $\{i\}$. Then every $f \in \ell^p$ is of the form $\sum_{i=1}^{\infty} f(i)\delta_i$. For each $n \in \mathbb{N}$ put $u_n := \sum_{i=1}^{n} \delta_i \otimes \delta_i$. It is seen that

$$f \cdot u_n = \sum_{i=1}^{n} f(i)\delta_i \otimes \delta_i = \sum_{i=1}^{n} \delta_i \otimes \delta_i f(i) = u_n \cdot f,$$

and

$$\|f\pi(u_n) - f\|_p = \left\|\sum_{i=1}^{n} f(i)\delta_i - \sum_{i=n}^{\infty} f(i)\delta_i\right\|_p \to 0, \quad \|f\pi(u_n)\| \leq \|f\|.$$ 

Hence, $\ell^p$ is 1-pseudo-contractible. We also remark that $\ell^p$ is not approximately amenable[DLZh]. Therefore $(\ell^p)^\#$ is not approximately amenable and thus $(\ell^p)^\#$ is not pseudo-amenable by [GhZh, Proposition 3.2]. Therefore, bounded pseudo-contractibility of a Banach algebra $A$ does not imply not only bounded pseudo-contractibility but also bounded pseudo-amenability of $A^\#$.

3. Banach algebras on locally compact groups

In this section we will verify Bounded pseudo-amenability and contractibility of some important Banach algebras on locally compact groups. We commence with the convolution group and measure algebras $L^1(G)$ and $M(G)$ and their second duals.

Proposition 3.1. For a locally compact group $G$, $L^1(G)$ is boundedly pseudo-amenable if and only if $G$ is amenable.

Proof. If $G$ is amenable then $L^1(G)$ is amenable and so it is boundedly pseudo-amenable. If $L^1(G)$ is boundedly pseudo-amenable, then it is pseudo-amenable. Thus $G$ is amenable by [GhZh, Proposition 4.1]. \(\square\)
The next proposition is a consequence of [GhZh, Proposition 4.2].

**Proposition 3.2.** Let $G$ be a locally compact group. Then

1. the convolution measure algebra $M(G)$ is boundedly pseudo-amenable if and only if $G$ is discrete and amenable.
2. $L^1(G)^{\ast\ast}$ is boundedly pseudo-amenable if and only if $G$ is finite.

The following proposition determines the bounded pseudo-amenability and contractibility of the Fourier algebra $A(G)$ of a discrete group $G$ which provides an example of a non-amenable, boundedly pseudo-contractible Banach algebra.

**Proposition 3.3.** Let $G$ be a discrete group and $A(G)$ be its Fourier algebra. Then the following are equivalent.

1. $A(G)$ has a multiplicity-bounded approximate identity.
2. $A(G)$ is boundedly pseudo-contractive.
3. $A(G)$ is boundedly pseudo-amenable.

**Proof.** (1) $\implies$ (2): Let $\{e_\alpha\}$ be a multiplicity-bounded approximate identity of $A(G)$ with bound $M$. As it is mentioned in Remark 3.4 of [GhS], we may suppose that every $e_\alpha$ has finite support, say $S_\alpha$. Now let

$$m_\alpha = \sum_{x \in S_\alpha} e_\alpha(x) \delta_x \otimes \delta_x,$$

where $\delta_x$ is the evaluational function at $x$. For each $f \in A(G)$ and $x \in G$ we have

$$f \cdot (\delta_x \otimes \delta_x) - (\delta_x \otimes \delta_x) \cdot f = (f \delta_x) \otimes \delta_x - \delta_x \otimes (\delta_x f) = (f(x)\delta_x) \otimes \delta_x - \delta_x \otimes (\delta_x f) = f(x)\delta_x \otimes \delta_x - \delta_x \otimes \delta_x = 0.$$

Therefore, $f \cdot m_\alpha = m_\alpha \cdot f$. Since $\pi(m_\alpha) = e_\alpha$, for all $f \in A(G)$ we have $\pi(m_\alpha) f - f \to 0$. Hence $\{m_\alpha\}$ is central approximate diagonal for $A(G)$. Furthermore, for any $f \in A(G)$ we have

$$\|\pi(m_\alpha) f - f\| = \|f - f\| \leq (M+1)||f||.$$

Hence, $A(G)$ is $(M+1)$-pseudo-contractive.

(2) $\implies$ (3) is clear.

(3) $\implies$ (1): This is immediate inasmuch as every boundedly pseudo-amenable Banach algebras has a multiplicity-bounded approximate identity. \qed

The following example shows that bounded pseudo-contractibility does not imply amenability.

**Example 3.4.** Let $G$ be a free group. It is shown in [Haa, Theorem 2.1] that $A(G)$ has a multiplicity-bounded approximate identity consisting of functions with finite support. Thus the Fourier algebra of a free group is boundedly pseudo-contractible. Nonetheless, free groups with at least 2 generators are not amenable and so, by Leptin’s theorem, their Fourier algebras lack a bounded approximate identity; consequently they are not amenable.

For a locally compact group $G$, let $PF_p(G)$ denote the Banach algebra of $p$-pseudofunctions on $G$ which is the norm closure of the image of $L^1(G)$ in $B(L^p(G))$, the space of bounded operators on $L^p(G)$, under the left regular representation. It is shown in [CGZ, Theorem 7.1] that for a discrete group $G$, amenability and pseudo amenability of $PF_p(G)$ is equivalent to the amenability of $G$. We therefore have the following proposition.

**Proposition 3.5.** Let $G$ be a discrete group and $p \in (1, \infty)$. Then $PF_p(G)$ is boundedly pseudo-amenable if and only if $G$ is amenable.
4. Banach algebras on discrete semigroups

This section is devoted to the Bounded pseudo-amenability and contractibility of many significant Banach algebras constructed on semigroups.

Like Example 3.4, the following is an example of a boundedly pseudo-contractible Banach algebra which is not amenable and consequently is not contractible.

**Example 4.1.** Let $\Lambda$ be non-empty, totally ordered set which is a semigroup if the product of two elements is defined to be their maximum. In fact it is a semilattice and is denoted by $\Lambda$. Proposition 6.2 of [CGZ] shows that the semigroup algebra $\ell^1(\Lambda)$ is boundedly pseudo-amenable.

Let $\{A_i\}_{i\in I}$ be a family of Banach algebras and $1 \leq q < \infty$. Then their $\ell^q$-direct sum

$$A = \ell^q - \bigoplus_{i\in I} A_i = \left\{ a = (a_i)_{i\in I} \middle| a_i \in A_i, \|a\|_A = \left( \sum_{i\in I} \|a_i\|_{A_i}^q \right)^{1/q} < \infty \right\},$$

is a Banach algebra under componentwise product.

**Theorem 4.2.** Let $\{A_i\}_{i\in I}$ be a family of $K$-pseudo-amenable (contractible) Banach algebras, $1 \leq q < \infty$ and $A = \ell^q - \oplus_{i\in I} A_i$. Then $A$ is $(K + 1)$-pseudo-amenable (contractible).

**Proof.** We follow the proof of Proposition 2.1 of [GhZh]. For arbitrary $\varepsilon > 0$ and a finite set $F \subset A$, there is a finite set $J \subset I$ such that $\|P_i(a) - a\|_A < \frac{\varepsilon}{2}$ for $a \in A$, where $P_i : A \to \ell^q - \oplus_{i\in I} A_i$ is the natural projection and $P_i$ is defined to be $P_{i|_I}$. Since $A_i$ is K-pseudo-amenable, there are $i \in J$ and $u_i \in A_i \otimes A_i$ such that

$$\|P_i(a)u_i - u_iP_i(a)\| < \frac{\varepsilon}{2|I|^1} \quad \|\pi_i(u_i)P_i(a) - P_i(a)\| < \frac{\varepsilon}{2|I|^1} \quad (a \in F),$$

and for all $b \in A_i$

$$\|P_i(b)u_i - u_iP_i(b)\| < K\|P_i(b)\|, \quad \|\pi_i(u_i)P_i(b) - P_i(b)\| < K\|P_i(b)\|, \quad \|P_i(b)\pi_i(u_i) - P_i(b)\| < K\|P_i(b)\|,$$

where $\pi_i : A_i \otimes A_i \to A_i$ is also the diagonal map. Setting $u = \{x_i\}_{i\in I}$ where $x_i = u_i$ for $i \in J$ and $x_i = 0$ for $i \in I \setminus J$ implies that $ua = uP_i(a)$ and $au = P_i(a)u$. Hence for each $a \in F$,

$$\|ua - ua\|_A = \|P_i(a)u - uP_i(a)\|_A = \left( \sum_{i\in J} \|P_i(a)u_i - u_iP_i(a)\|_i^q \right)^{1/2} < \varepsilon;$$

and

$$\|an(a) - a\|_A = \|P_i(a)\pi_i(u) - P_i(a)\|_A \leq \|P_i(a)\pi_i(u) - P_i(a)\|_A + \|P_i(a) - a\|_A$$

$$= \sum_{i\in J} \|P_i(a)\pi_i(u) - P_i(a)\|_i^q + \|P_i(a) - a\|_A \leq \varepsilon / 2 + \varepsilon / 2 = \varepsilon.$$

Also for each $b \in A$ we have

$$\|bu - ub\|_A = \|P_i(b)u - uP_i(b)\|_A = \left( \sum_{i\in J} \|P_i(b)u_i - u_iP_i(b)\|_i^q \right)^{1/2} \leq \left( \sum_{i\in J} K\|P_i(b)\|_i^q \right)^{1/2} = K\|P_i(b)\|_A \leq K\|b\|_A,$$

(1)
and

\[
||\pi(u) - b||_A \leq ||P_j(b)\pi(u) - P_j(b)||_A + ||P_j(b) - b||_A \\
\leq (\sum_{i\in I} ||P_j(b)\pi_i(u) - P_j(b)||_A^p)^{\frac{1}{p}} + ||b||_A \\
\leq (\sum_{i\in I} K^p||P_j(b)||_A^p)^{\frac{1}{p}} + ||b||_A = K||P_j(b)||_A + ||b||_A \\
\leq (k + 1)||b||_A, \tag{2}
\]

and similarly

\[
||\pi(u) - b||_A \leq (K + 1)||b||_A, \quad (b \in A). \tag{3}
\]

\[\square\]

So Theorem 4.2 shows that there are a large class of bounded pseudo-amenable(contractible) Banach algebras that are not amenable. We remark that \(A = \ell^p - \bigoplus_i A_i\) is amenable if and only if \(|l| < \infty\) and each \(A_i\) is amenable.

**Example 4.3.** Since \(\ell^p = \ell^p - \bigoplus_{i=1}^\infty C_i\), it is 2-pseudo-amenable invoking Theorem 4.2. Notice that, it is in fact \(\ell^p\) is 1-pseudo-contractible by Example 2.11.

**Proposition 4.4.** Let \(A\) be a Banach algebra and \(M_n(A)\) be its \(\ell^1\)-Munn algebra \((n \in \mathbb{N})\). Then \(M_n(A)\) is K-pseudo-amenable if and only if \(A\) is K-pseudo-amenable.

**Proof.** Suppose that \(\{\Psi_a\}\) is an approximate diagonal of \(M_n(A)\) with bound \(K\). Keeping \(M_n(A) \otimes M_n(A) \cong M_{n^2}(A \otimes A)\) in mind, we may assume that

\[
\Psi_a = \begin{bmatrix}
m_{11}^a & m_{12}^a & \cdots & m_{1n^2}^a \\
m_{21}^a & m_{22}^a & \cdots & m_{2n^2}^a \\
\vdots & \vdots & \ddots & \vdots \\
m_{n^2-1}^a & m_{n^2-2}^a & \cdots & m_{n^2n^2}^a
\end{bmatrix},
\]

where \(m_{ij}^a \in A \otimes A\). For each \(a \in A\) we have

\[
\begin{bmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{bmatrix} - \begin{bmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{bmatrix} = \begin{bmatrix}
am_{11}^a & am_{12}^a & \cdots & am_{1n^2}^a \\
am_{21}^a & am_{22}^a & \cdots & am_{2n^2}^a \\
\vdots & \vdots & \ddots & \vdots \\
am_{n^2-1}^a & am_{n^2-2}^a & \cdots & am_{n^2n^2}^a
\end{bmatrix} - \begin{bmatrix}
am_{11}^a & m_{12}^a & \cdots & m_{1n^2}^a \\
m_{21}^a & m_{22}^a & \cdots & m_{2n^2}^a \\
\vdots & \vdots & \ddots & \vdots \\
m_{n^2-1}^a & m_{n^2-2}^a & \cdots & m_{n^2n^2}^a
\end{bmatrix}.
\]

Hence \(am_{ij}^a - m_{ij}^a \to 0\) and \(||am_{ij}^a - m_{ij}^a|| \leq K||a||\). With a similar fashion we can get \(a\pi(m_{ij}^a) \to a, \pi(m_{ij}^a)a \to a, \)

\(||a\pi(m_{ij}^a) - a|| \leq K||a||\) and \(||\pi(m_{ij}^a)a - a|| \leq K||a||\).

Conversely, suppose that \(A\) is K-pseudo-amenable and \(\{m_a\}\) is an approximate diagonal for it, and set

\[
\psi_a = \begin{bmatrix}
m_a & 0 & \cdots & 0 \\
0 & m_a & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & m_a
\end{bmatrix}.
\]
Definition 4.5. A (discrete) semigroup $S$ is called an inverse semigroup if for any $s \in S$, $ss^*s = s$ and $ss^*s = s$. The set of idempotent elements of $S$ is denoted by $E(S)$, that is $E(S) = \{ss^* : s \in S\}$.

Let $S$ be a inverse semigroup. For $e \in E(S)$, $G_e = \{s \in S : ss^* = s's = e\}$ constitutes a group called maximal subgroup of $G$ at $e$.

For all $s, t \in S$ the relation $D$ defined on an inverse semigroup $S$ by $sDt$ if and only if there exists $x \in S$ with

$$Ss \cup \{s\} = Sx \cup \{x\}, \quad ts \cup \{t\} = xs \cup \{x\},$$

is an equivalence relation. There is also a natural partial order on $S$ given by $s \leq t \iff s = ss't$. For $p \in S$ we set $(p) = \{q \in S : q \leq p\}$.

Definition 4.6. An inverse semigroup $S$ is called locally finite whenever $|\{p\}| < \infty$ for all $p \in S$, and it is called uniformly locally finite (ULF) if $\sup_{p \in S}|\{p\}| < \infty$.

We recall that a Banach algebra $A$ is called biflat if there exists a Banach $A$-bimodule morphim $\rho : (A \hat{\otimes} A)^* \to A^*$ such that $\rho \circ \pi^* (y) = y$ for all $y \in A^*$, where $\pi^* : A^* \to (A \hat{\otimes} A)^*$ is adjoint of the diagonal map $\pi$.

Proposition 4.7. Let $S$ be a ULF inverse semigroup and $\{D_\lambda : \lambda \in \Lambda\}$ be the family of its $D$-classes such that for all $\lambda \in \Lambda$, $|E(D_\lambda)| < \infty$. For each $\lambda \in \Lambda$ let $p_\lambda \in E(D_\lambda)$. Then the following statements are equivalent.

1. For each $\lambda \in \Lambda$ the maximal subgroup $G_{p_\lambda}$ is amenable.
2. $\ell^1(S)$ is pseudo-amenable.
3. $\ell^1(S)$ is boundedly pseudo-amenable.

Moreover, in this case $\ell^1(S)$ is biflat.

Proof. From [Ram, Theorem 2.18] we have the following isometric isomorphism

$$\ell^1(S) \cong \ell^1 - \bigoplus_{\lambda \in \Lambda} [M_{E(D_\lambda)}, \ell^1(G_{p_\lambda})] : \lambda \in \Lambda].$$

The proposition now follows from Propositions 3.1, 4.4, Theorem 4.2, and [Ram, Therem 3.7].

Definition 4.8. An inverse semigroup $S$ is called a Clifford semigroup if for all $s \in S$, $ss^* = s's$.}

Theorem 4.9. Let $S$ be a Clifford semigroup and $A(S)$ be its Fourier algebra introduced in [MP]. Then the following statements are equivalent.

1. $A(S)$ has a multiplier-bounded approximate identity.
2. $A(S)$ is boundedly pseudo-contractible.
3. $A(S)$ is boundedly pseudo-amenable.

Proof. $(1) \implies (2)$: Suppose that $A(S)$ has a multiplier-bounded approximate identity with bound $M$. By [MP] we have the following useful decomposition

$$A(S) = \ell^1 - \bigoplus_{e \in E(S)} A(G_e).$$

Thus it can be readily seen that for each $e \in E(S)$, $A(G_e)$ has a multiplier-bounded approximate identity with bound $M$. From Proposition 3.3 we conclude that $A(G_e)$ is $(M + 1)$-pseudo-contractible for all $e \in E(S)$. Now Theorem 4.2 implies that $A(S)$ is $(M + 2)$-pseudo-contractible. The other parts of proof are obvious.
Applying the above decomposition, as it is done in [MP], for a Clifford semigroup $S$ with abelian maximal subgroups $G_e$, we obtain $A(S) \cong \ell^1 - \bigoplus_{e \in E(S)} L^1(G_e)$, where $\hat{G}_e$ is the Pontrjagin dual of $G_e$. Since $\hat{G}_e$ is compact, it is amenable and so $L^1(\hat{G}_e)$ is 1-amenable. Hence $L^1(\hat{G}_e)$ is 1-pseudo-amenable for all $e \in E(S)$. From Theorem 4.2 it can be inferred that $A(S)$ is 2-pseudo-amenable.

Let $\{A_i\}_{i \in I}$ be a family of Banach algebras. Their $c_0$-direct sum

$$A = c_0 - \bigoplus_{i \in I} A_i = \left\{ a = (a_i)_{i \in I} \mid a_i \in A_i, \|a_i\|_{A_i} \to 0, \|a\|_A = \sup_{i \in I} \|a_i\|_{A_i} \right\},$$

is a Banach algebra under componentwise product.

The next theorem gives the $c_0$-analogue of Theorem 4.2. Since the proof is similar, we omit it.

**Theorem 4.10.** Let $\{A_i\}_{i \in I}$ be a family of $K$-pseudo-amenable (contractible) Banach algebras and $A = c_0 - \bigoplus_{i \in I} A_i$. Then $A$ is $(K + 1)$-pseudo-amenable (contractible).

**Corollary 4.11.** Let $S$ be a Clifford semigroup and consider the Banach algebra $PF_p(S)$ of $p$-pseudofunctions on $S$ introduced in [Pou2]. Then $PF_p(S)$ is boundedly pseudo-amenable if and only if every maximal subgroup $G_e$ of $S$ is amenable.

**Proof.** By [Pou2] we have the following decomposition

$$PF_p(S) \cong c_0 - \bigoplus_{e \in E(S)} PF_p(G_e).$$

Combining Theorem 4.10 and Proposition 3.5 the corollary follows.

**5. Multiplier-bounded approximate biprojectivity**

In this section we introduce an approximate version of biprojectivity and then investigate its relation with (bounded) pseudo-amenability.

**Definition 5.1.** ([Pou1]) A Banach algebra $A$ is said to be approximately biprojective if there is a net $\{\rho_\alpha\} \subset B(A \hat{\otimes} A, A)$ such that for each $a, b \in A$:

$$\pi \circ \rho_\alpha(a) \to a, \quad \rho_\alpha(ab) - a \rho_\alpha(b) \to 0, \quad \rho_\alpha(ab) - \rho_\alpha(a)b \to 0.$$  

We say that, $A$ is called boundedly approximately biprojective when $\sup_{\alpha} \|\rho_\alpha\| < \infty$.

**Definition 5.2.** An approximately biprojective Banach algebra $A$ is termed multiplier-boundedly approximately biprojective if there is a $K > 0$ such that for each $a, b \in A$:

$$\|\pi \circ \rho_\alpha(a) - a\| \leq K\|a\|, \quad \|\rho_\alpha(ab) - a \rho_\alpha(b)\| \leq K\|a\||b||, \quad \|\rho_\alpha(ab) - \rho_\alpha(a)b\| \leq K\|a\||b||,$$

where $\{\rho_\alpha\}$ satisfies condition of Definition 5.1.

Obviously, every boundedly approximately biprojective Banach algebra is multiplier-boundedly approximately biprojective.

**Corollary 5.3.** Let $A$ be a boundedly pseudo-amenable Banach algebra. Then $A$ is multiplier-boundedly approximately biprojective.
Hence A is boundedly pseudo-amenable. Let $\{m_i\}$ be an approximate diagonal of A with multiplier bound $K > 0$. Define $\rho_a : A \rightarrow A \hat{\otimes} A$ by $\rho_a(a) = a \cdot m_a$. By [Pou1, Proposition 3.4], we have

$$\pi \circ \rho_a(a) \rightarrow a, \quad \rho_a(ab) - a \cdot \rho_a(b) \rightarrow 0, \quad \rho_a(ab) - \rho_a(a) \cdot b \rightarrow 0, \quad (a, b \in A).$$

Moreover, for each $a \in A$ and for each $\alpha \in A$ we have

$$\|\pi \circ \rho_a(a) - a\| = \|\pi(a \cdot m_a) - a\| = \|\alpha \tau(m_a) - a\| \leq K\|a\|$$

On the other hand, for all $a$ and every $b \in A$, $\rho_a(ab) - a \cdot \rho_a(b) = 0$ and

$$\|\rho_a(ab) - \rho_a(a) \cdot b\| = \|ab \cdot m_a - (a \cdot m_a) \cdot b\| \leq \|a\||b\| \cdot m_a - m_a \cdot b\| \leq K\|a\||\|b\|$$

Therefore A is multiplier-boundedly approximately biprojective. □

**Proposition 5.4.** Let A be a multiplier-boundedly approximately biprojective Banach algebra with a central bounded approximate identity $\{e_\alpha\}$. Then A is boundedly pseudo-amenable.

Proof. Let $\{\rho_a\}$ be a net satisfying Definition 5.2. As in Proposition 3.5 of [Pou1], there are subnets $\{e_\beta\}$ of $\{e_\alpha\}$ and $\{\rho_a\}$ of $\{\rho_a\}$ such that $m_i := \rho_a(e_\beta)$ is an approximate diagonal for A. We show that $\{m_i\}$ is a multiplier-boundedly approximately diagonal. Let $\{e_\alpha\}$ be bounded by $K_0$. Then for each $a \in A$ we have

$$\|a \cdot m_i - m_i \cdot a\| = \|a \cdot \rho_a(e_\beta) - \rho_a(e_\beta) \cdot a\|$$

$$\leq \|a \cdot \rho_a(e_\beta) - \rho_a(ae_\beta) + \rho_a(e_\beta a) - \rho_a(e_\beta) \cdot a\|$$

$$\leq K\|a\||\|e_\beta\|\|a\|$$

and

$$\|\pi(m_i)a - a\| = \|\pi \circ \rho_a(e_\beta)a - a\| \leq \|\pi \circ \rho_a(e_\beta)a - e_\beta a\| + \|e_\beta a - a\|$$

$$\leq K\|a\||\|e_\beta\|\| + \|e_\beta\|\|a\| + \|a\| = (KK_0 + K_0 + 1)|a|.$$ 

Hence A is boundedly pseudo-amenable. □

The following example gives an approximately biprojective Banach algebra that is not multiplier-boundedly approximately biprojective.

**Example 5.5.** Suppose that A is the algebra introduced in Proposition 2.2. Approximate amenability of $A^\ast$ implies its approximate biprojectivity [Pou1, Proposition 3.4]. On the other hand, $A^\ast$ is not boundedly pseudo-amenable and so by Proposition 5.4 is not multiplier-boundedly approximately biprojective.

Here we give an example of multiplier-boundedly approximately biprojective Banach algebra which is not boundedly approximately biprojective.

**Example 5.6.** Suppose that S is an infinite non-empty set and consider the Banach algebra $L^2(S)$ with pointwise multiplication. Let $\{e_i\}_{i \in S}$ be the canonical basis for $L^2(S)$ and let $\Lambda$ be the set of finite subsets of S, which is an ordered set with respect to inclusion. For any $F \in \Lambda$ define $m_F = \sum_{i \in F} e_i \otimes e_i$. Then $\{m_F\}_{F \in \Lambda}$ is a central approximate diagonal for $L^2(S)$ satisfying conditions of Definition 1.2. Therefore it is boundedly pseudo contractible and consequently, by Proposition 5.3, multiplier-boundedly approximately biprojective. However, it is known that $L^2(S)$ is not boundedly approximately biprojective (see [Pou1, Example 4.1]).

**Corollary 5.7.** If G is an infinite Abelian compact group, then $L^2(G)$ is a multiplier-boundedly approximately biprojective Banach algebra.

Proof. Suppose $\Gamma$ is the dual group of G. From Plancherel Theorem we have $L^2(G) \cong L^2(\Gamma)$ and so Example 5.6 gives the desired result. □
The last example provides a boundedly pseudo-amenable Banach algebra which is not boundedly approximately biprojective.

**Example 5.8.** Consider the inverse semigroup $S = (\mathbb{N}, \ast)$ with $s \ast t = \min\{s, t\}$ for all $s, t \in \mathbb{N}$. By [GLZ, Example 4.6], the convolution semigroup algebra $\ell^1(S)$ is sequentially approximately contractible. So the uniform boundedness principle implies that $\ell^1(S)$ is boundedly approximately contractible. Hence by Proposition 2.1, $(\ell^1(S))^\#$ is boundedly pseudo amenable. Nevertheless, since $S$ is a locally finite, non-uniformly locally finite inverse semigroup, by [Ram, Theorem 3.7], $\ell^1(S)$ is not biflat and consequently its unitization $(\ell^1(S))^\#$ is not biflat. It now follows from [Ari, Theorem 3.6(A)] that $(\ell^1(S))^\#$ is not boundedly approximately biprojective.

**References**

[Ari] O. Yu. Aristov, *On approximation of flat Banach modules by free modules*, Sbornik, Mathematics (2005), 1553–1583.

[CGZ] Y. Choi, F. Ghahramani and Y. Zhang, *Approximate and pseudo-amenability of various classes of Banach algebras*, J. Funct. Anal., 256 (2009), 3158-3191.

[Dal] H. G. Dales, *Banach algebras and Automatic continuity*, Oxford university Press, 2001.

[DL] H. G. Dales and R. J. Loy, *Approximate amenability of semigroup algebras and Segal algebras*, Diss. Math., 474 (2010), 1-58.

[DLZh] H. G. Dales, R. J. Loy and Y. Zhang, *Approximate amenability for Banach sequence algebras*, Studia Math., 177 (2006), 81-96.

[GhL] F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, J. Funct. Anal., 79 (2004), 229-260.

[GLZ] F. Ghahramani, R. J. Loy and Y. Zhang, *Generalized notions of amenability. II*, J. Funct. Anal., 254 (2008), 1776-1810.

[GhR] F. Ghahramani and C. J. Read, *Approximate identities in approximate amenability*, J. Funct. Anal., 262 (2012), 3929-3945.

[GhS] F. Ghahramani and R. Stokke, *Approximate and pseudo-amenability of the Fourier algebra*, Indiana Univ. Math. J., 56 (2007), 909-930.

[GhZh] F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Comb. Phil. Soc., 142 (2007), 111-123.

[GhHS] M. Ghandhari, H. Hatami and N. Spronk, *Amenability constants for semilattice algebras*, Semigroup Forum, 79 (2009), 279-297.

[Haa] U. Haagerup, *An example of a nonnuclear C*-algebra, which has the metric approximation property*, Invent. Math., 50 (1978/79), 279-293.

[Joh1] B. E. Johnson, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math., 94 (1972), 685-698.

[Joh2] B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc., 127 (1972).

[MP] A. R. Medghalchi and H. Pourmahmood-Aghababa, *Figa-Talamanca–Herz algebras for restricted inverse semigroups and Clifford semigroups*, J. Math. Anal. Appl., 395 (2012), 473-485.

[Pou1] H. Pourmahmood-Aghababa, *Approximately biprojective Banach algebras and nilpotent ideals*, Bull. Austral. Math. Soc., 87 (2013), 158-173.

[Pou2] H. Pourmahmood-Aghababa, *Pseudomeasures and pseudofunctions on inverse semigroups*, Semigroup Forum, 90 (2015), 632-647.

[Ram] P. Ramsden, *Biflatness of semigroup algebras*, Semigroup Forum, 79 (2009), 515-530.