THE HYBRID LANDAU–GINZBURG MODELS OF CALABI–YAU
COMPLETE INTERSECTIONS

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Abstract. We observe that the state space of Landau–Ginzburg isolated singularities is simply
a special case of Chen–Ruan orbifold cohomology relative to the generic fibre of the potential.
This leads to the definition of the cohomology of hybrid Landau–Ginzburg models and its identi-
fication via an explicit isomorphism to the cohomology of Calabi–Yau complete intersections
inside weighted projective spaces. The combinatorial method used in the case of hypersurfaces
proven by the first named author in collaboration with Ruan is streamlined and generalised after
an orbifold version of the Thom isomorphism and of the Tate twist.

1. Introduction

Landau–Ginzburg models play a central role in mirror symmetry. The simplest examples
are the following isolated singularities: weighted-homogeneous polynomials
\[ W: \mathbb{C}^n \to \mathbb{C} \]
with smooth fibres outside the origin and a single isolated singularity in the special fibre over the
origin. They played a key role in the early development of quantum cohomology, giving some
of the first examples of Frobenius manifolds and allowing to verify mirror symmetry statements.
They have recently acquired a new status with the definition of their quantum invariants by Fan,
Jarvis and Ruan [16] (FJRW theory) and later by Polishchuk and Vaintrob [33] (cohomological
field theory of matrix factorisations). These advances make it possible to obtain equivalences
between Landau–Ginzburg models and Calabi–Yau models without necessarily passing through
mirror symmetry [9]. It is now also possible to state mirror symmetry between Landau–Ginzburg
models both at the level of cohomological invariants and at the level of quantum cohomology
invariants (see [16, 27, 2]). There are many consequences: for instance the absence of any
Calabi–Yau conditions in mirror statements can be effectively used (see for instance [16, 27]);
furthermore, the computational power appears higher: We refer to approaches beyond concavity
and in higher genus, see [16, 18, 19, 10]).

This paper starts from the definition by Fan, Jarvis and Ruan [16] of the state space \( \mathcal{H}(W) \)
of the Landau–Ginzburg model \( W: [\mathbb{C}^n/\mu_d] \to \mathbb{C} \), where \( W \) is a potential with isolated singularity
and \( \mu_d \) is the corresponding monodromy action
\[ \mathcal{H}(W) := \bigoplus_{g \in \mu_d} (\mathcal{Q}_{W_g})^{\mu_d}. \]
Here \( \mathcal{Q}_f \) stands for the Jacobian ring, or chiral algebra, attached to a potential \( f \)
\begin{equation}
\mathcal{Q}_f = dx_1 \wedge \cdots \wedge dx_n \otimes \text{Jac}(f) = dx_1 \wedge \cdots \wedge dx_n \otimes \mathbb{C}[x_1, \ldots, x_n]/(\partial_1 f, \ldots, \partial_n f),
\end{equation}
and \( W_g \) is the restriction of \( W \) to the \( g \)-fixed space \( \mathbb{C}^n_g \) in \( \mathbb{C}^n \). As pointed out by the authors of
[16], the crucial ingredient \( \mathcal{Q} \) and the orbifolding procedure appear already in the definition of the
states of Landau–Ginzburg models in Intriligator–Vafa [23] and in Kaufmann’s work [24, 25, 26]
in the definition of B-model invariants. This state space is bi-graded by applying to each term
\( (\mathcal{Q}_{W_g})^{\mu_d} \) an explicit formula: the somewhat ad hoc bigrading
\begin{equation}
(n_g - \deg(f), \deg(f)) + (\text{age}(g), \text{age}(g)) - \left( \sum_i w_i d, \sum_i w_i d \right),
\end{equation}
where age is the additive morphism \( R\mu_r \to \mathbb{Q} \) mapping \( k \in (\mu_r)^\vee = \mathbb{Z}/r \) with \( 0 \leq k < r \) to \( k/r \).
The cohomological LG/CY correspondence for CICY. In this paper we recognise that $\mathcal{H}(W)$ is nothing but an occurrence of relative orbifold Chen–Ruan cohomology: for a Calabi–Yau hypersurface $X_W = V(W)$ we consider the cohomological Landau–Ginzburg model

$$\mathcal{H}^{p,q}(W) = H^{p+1,q+1}_{CR}([\mathbb{C}^n/\mu_d], W^{-1}(t_0); \mathbb{C})$$

for any $t_0 \in \mathbb{C}^\times$. Note that $[\mathbb{C}^n/\mu_d]$ is $\mathcal{O}(-w) := \mathcal{O}(-w_1) \oplus \cdots \oplus \mathcal{O}(-w_r)$ on the zero-dimensional weighted projective space $\mathbb{P}(d)$. Then, when $W_1, \ldots, W_r$ define a smooth complete intersection inside $\mathbb{P}(w)$, we set

$$\mathcal{H}^{p,q}(W_1, \ldots, W_n) := H^{p+r,q+r}_{CR}(\mathcal{O}_{\mathbb{P}(d)}(-w); \mathbb{P}^{-1}(t_0); \mathbb{C}),$$

where $\mathbb{W}$ may be regarded as a $\mathbb{C}$-valued function, via the Cayley trick $\mathbb{W} = \sum_{i=1}^r p_i W_i(x)$ (see for instance [37, 5.4]). In physics literature, it is common to refer to the coordinates of $\mathbb{P}^r$ as massive or ground coordinates and to the coordinates along the fibres of $\mathcal{O}(-w)$ as massless coordinates, and to refer to $\mathbb{C}$-valued potentials such as $\mathbb{W}$ effectively involving both types of variables as hybrid (see, for instance, [37, 5.4] and [20]).

Finally, by a variation of stability condition one gets the counterpart of this hybrid model: the morphism $\mathcal{W}: \mathcal{O}_{\mathbb{P}(w)}(-\mathcal{W}) \to \mathbb{C}$ which, after the same relative cohomology computation as above, turns out to be isomorphic to the cohomology of the complete Calabi–Yau intersection $V(W_1, \ldots, W_r)$ inside $\mathbb{P}(w)$ (see Prop. [3.4] with $G = 1$). The two LG and CY models are related by the following theorem. We refer to Theorem 5.1 and Corollary 5.2 for the complete statement involving group quotients of complete intersections.

**Theorem (cohomological LG/CY correspondence for CICY).** For any smooth complete intersection $X_{\mathbb{W}} = V(W_1, W_2, \ldots, W_r)$ of Calabi–Yau type inside the weighted projective space $\mathbb{P}(w)$ we have

$$\mathcal{H}^{p,q}(W_1, \ldots, W_r) \cong H^{p,q}_{CR}(X_W; \mathbb{Q}).$$

**Remark 1.1.** A related result by Libgober [29] proves the invariance of the elliptic genus in a more general setup which includes complete intersections and several other GIT quotients (we recall that, after specialisation, the elliptic genus gives a combination of Hodge numbers, see e.g. [3]). It is also worth-while pointing out how, in Libgober’s treatment, Calabi–Yau models, Landau–Ginzburg models and their hybrid versions are precisely characterised in geometric terms, see [29, Defn. 2.3].

Here, our main focus is how the Landau–Ginzburg/Calabi–Yau correspondence follows from several properties of Chen–Ruan cohomology of independent interest. We illustrate it in the rest of the introduction.

**Thom isomorphism in Chen–Ruan cohomology and the Tate twist.** The LG/CY statement, and the more general Thm. 5.1 including group actions, relies on a version of the Thom isomorphism in ordinary cohomology

$$H^*(X; \mathbb{C}) = H^*(\mathbb{P}(w), \mathbb{P}(w) \setminus X; \mathbb{C}) (r),$$

where, by the Tate twist “(r)”, the isomorphism preserves the bi-degree. Notice that, due to its age-shifted degree, Chen–Ruan orbifold cohomology does not satisfy (4) in the above form. However, by replacing $\mathbb{P}(w)$ with the total space of the vector bundle $\mathcal{O}(-d_1) \oplus \cdots \oplus \mathcal{O}(-d_r)$ we get a Thom isomorphism statement (see Prop. [3.4] for the general statement).

**Proposition (Thom isomorphism in orbifold cohomology).** We have a canonical isomorphism

$$H^{p-r,q+r}_{CR}(X_W) \cong H^{p,q}_{CR}(\mathcal{O}_w(-d), \mathbb{W}^{-1}(t_0)).$$
This is interesting in its own right and explains, the *ad hoc* bi-degree shift \(2\) in a geometric way: in Calabi–Yau cases it is merely the Tate twist in the Thom isomorphism.\(^1\)

In the same spirit, let us point out how the present description of the state space clarifies in classical geometric terms the dichotomy between narrow and broad sectors, which, following [16], depends on the condition that the space of \(g\)-fixed points is or is not reduced to the origin. This dichotomy does not generalise as such in the complete intersection case. However, in our relative cohomology picture broad states are simply those that lie in the kernel of the natural morphism \(H^*_{CR}(\mathcal{O}(-w), W^{-1}(t_0)) \to H^*_{CR}(\mathcal{O}(-w))\). In this way “broad” is nothing but a generalisation of the classical notion of primitive.

We also feel that the generalisation of Gysin sequences and Thom isomorphisms in Chen–Ruan orbifold cohomology is worth pursuing further in view of a well-behaved setup for an orbifold cohomology theory.

An algebraic model for the quantum theory of hybrid models. We finally notice how, using the Thom isomorphism, the theorem stated above follows from a bi-degree preserving isomorphism

\[
H^*_{CR}(\mathcal{O}_{\mathbb{P}(w)}(-d); \mathbb{C}) \cong H^*_{CR}(\mathcal{O}_{\mathbb{P}(d)}(-w); \mathbb{C}).
\]

This isomorphism holds by \(K\)-equivalence but can be expressed directly via a combinatorial argument which, with respect to [7], is greatly simplified.

By relying on this identification and on the classical descriptions of the primitive cohomology of complete intersections [13, 30], we derive an algebraic model of the cohomology of Calabi–Yau complete intersection in terms of the Jacobian ring; see Theorem 4.3. This type of algebraic presentation of cohomology classes has been proven crucial on many occasions in the formulation and proof of mirror symmetry statements.

The cohomology groups \(\mathcal{H}(W_1, \ldots, W_n)\) developed in this paper, have already been used to provide a quantum correspondence between Calabi–Yau complete intersections inside projective spaces and hybrid Landau–Ginzburg models, see Clader [11]. Recently, Fan, Jarvis and Ruan provided a generalisation of their theory based on the state space \(\mathcal{H}(W_1, \ldots, W_n)\), see [17]. Clader establishes a correspondence between the quantum D-modules attached to complete intersection of CY type within \(\mathbb{P}(\mathbb{C}^n)\) and the corresponding hybrid LG models. Her correspondence depends on the choice of an analytic continuation which should mirror the parallel transport along the Gauß–Manin connection on the B-side. In this way, all isomorphisms arising at the cohomological level from these quantum correspondences should depend on the path chosen for the parallel transport. In this perspective it is interesting how our proof of Theorem 5.1 provides a combinatorial method for writing an isomorphism explicitly. Therefore, our isomorphism could provide a useful tool to study the monodromy operators in the future.

Structure of the paper. We recall the setup in §2, develop the Calabi–Yau side in §3, the Landau–Ginzburg side in §4, and their correspondence in §5.

Aknowledgements. The first named author is grateful to the IMB, Dijon, for the hospitality during the finalisation of this work. We are extremely grateful to Anatoly Libgober for his comments and suggestions.

2. Setup

We consider a complete intersection of \(r < n\) hypersurfaces in the weighted projective space \(\mathbb{P}(w) := \mathbb{P}(w_1, \ldots, w_n)\). Let \(W_1, \ldots, W_r\) be quasi-homogeneous polynomials of degree \(d_1, \ldots, d_r\) in

\(^1\) This interpretation of the overall shift-by-total-charge \(\sum \frac{w_i}{d_i} \) in \(2\), may lead to the correct interpretation of the age shift too; it may be worth-while to point out here Denef–Loeser’s [12] view on the age grading as a representation-theoretic analogue of the weight in Hodge theory, see Miles Reid [31, §2].
the variables \(x_1, \ldots, x_n\) of coprime weights \(w_1, \ldots, w_n\). We assume that \(X_W\) is a non-degenerate complete intersection, i.e.,

(i) the choice of weights \(w_1, \ldots, w_n\) is unique;

(ii) \(\{W_1 = 0, \ldots, W_r = 0\} \subset \mathbb{C}^n\) is smooth outside the origin.

Assumption (ii) holds if and only if the Jacobian matrix \(J = (\partial_j W_i(p))_{i,j}\) has rank \(r\) for any \(p \in \{x \in \mathbb{C}^n \mid W_1 = \cdots = W_r = 0\} \setminus \{0\}\). Then, inside the Deligne–Mumford stack \(\mathbb{P}(w)\), we get a smooth complete intersection

\[X_W = \{W_1 = \ldots = W_r = 0\} \subset \mathbb{P}(w)\].

We further assume that \(X_W\) is of Calabi–Yau type in the following sense: we impose that the canonical sheaf is trivial, in other words

\[(\operatorname{dim} \Gamma \times \rho) / \Gamma \subset \mathbb{C}^n\] is the quotient stack \([X/\Gamma] = \mathbb{P}(w)\]. By (i), \(G\) is finite (see for instance [16, Lem. 2.18]). The quotient stack \([U/\Gamma]\), with \(U = \{W_1 = W_2 = \cdots = W_r\} \subset \mathbb{C}^n \setminus \{0\}\), is a smooth Deligne–Mumford stack and, following Romagny’s treatment [55 Rem. 2.4] of actions on stacks, it is canonically equivalent to the 2-stack

\[\mathbb{C} \times \mathbb{C}/\rho = \mathbb{P}(w)\].

We introduce for all \(\gamma \in \Gamma\) the following notation

\[C^n_{\gamma} := \{x \in \mathbb{C}^n \mid \gamma \cdot x = x\}, \quad n_{\gamma} := \operatorname{dim} C^n_{\gamma}, \quad W_{i,\gamma} := W_i |_{C^n_{\gamma}}\].

**Lemma 2.1.** Consider \(\gamma = \alpha \overline{x} := \alpha(x^{w_1}, \ldots, x^{w_n})\) in \(\Gamma\). We have \(\gamma^* W_i = W_i(\gamma \cdot x) = \lambda^{d_i} W_i\).

Therefore \(W_{i,\gamma} = 0\) if \(\lambda\) is not a \(d_i\)-th root of unity. Otherwise, for \(\lambda \in \mu_{d_i}\), \(W_i\) is fixed by \(\gamma\), and the following conditions are satisfied: \(W_{i,\gamma} \neq 0\) and its partial derivatives coincide with those of \(W_i\) on \(C^n_{\gamma}\).

**Proof.** Indeed, if \(\lambda \in \mu_{d_i}\), then \(W_i\) is of the form \(W_{i,\gamma} + f\), where \(f\) belongs to the maximal ideal \(m_{\gamma}\) spanned by the variables \(x_j\) not fixed by \(\gamma\) (i.e. \(\gamma_j = \alpha_j x^{w_j} \neq 1\)). In fact, since the action is diagonal, we have \(f \in (m_{\gamma})^2\), which implies that, after restriction to \(C^n_{\gamma}\), the partial derivatives \(\partial_j W_i\) and \(\partial_j W_{i,\gamma}\) coincide for each \(j = 1, \ldots, n\). In particular, by (ii), we have \(W_{i,\gamma} \neq 0\). \(\square\)

3. **Calabi–Yau side.**

On the Calabi–Yau side of the correspondence, we consider the Chen–Ruan orbifold cohomology of \([X_W/\Gamma] = [U/\Gamma]\). Since the groups acting are Abelian, we can express this explicitly as a direct sum of \(\Gamma\)-invariant parts of cohomology groups

\[H^\alpha_{\operatorname{CR}}([X_W/\Gamma]; \mathbb{Q}) = \bigoplus_{\gamma \in \Gamma} H^{p-a_{\gamma}, q-a_{\gamma}}([X_W,\gamma]; \mathbb{Q})\],

where \(a_\gamma\) is the age of the action of \(\gamma\) on the tangent bundle of \(X_W\) at a point of \(X_W,\gamma\) and \(X_W,\gamma\) is the quotient stack

\[\{W_1,\gamma = \cdots = W_r,\gamma = 0\} \subset \mathbb{C}^n / \Gamma\] \(\subset \mathbb{P}(w),\gamma\),

where \(\mathbb{P}(w),\gamma\) is the weighted projective space spanned by the \(\gamma\)-fixed coordinates \(x_j\).

**Remark 3.1.** We remark that the \(r_\gamma := \# \{W_{i,\gamma} \mid W_{i,\gamma} \neq 0\} \in [0, r]\)

restricted polynomials that do not vanish on \(C^n_{\gamma}\) define a smooth complete intersection of \(r_\gamma\) hypersurfaces; indeed, the Jacobian matrix of the complete intersection in \(C^n_{\gamma}\) has \(r_\gamma\) rows which,
as pointed out above, coincide with the restrictions of the corresponding rows in the Jacobian matrix of $X_W$ and, by (ii), form a matrix of rank $r_\gamma$ at the points of $\mathbb{C}_\gamma^n$ where $W_{i,\gamma} = 0$ for all $i$.

**Remark 3.2.** All stacks considered in this paper are Deligne–Mumford stacks which are global quotients $[Y/H]$. In these cases when we consider ordinary cohomology with coefficients in $\mathbb{Q}$ of the stack $[Y/H]$ and of the coarse scheme $Y/H$ we get the same result via a canonical isomorphism. The groups $H$ considered in this paper are at worse finite extensions of tori operating with finite stabilisers. Sometimes we consider non-faithful actions: $H \mapsto \text{Aut}(Y)$ with nontrivial kernel $K$. In these cases it is important to notice that, whereas for ordinary orbifold cohomology it does not matter if we consider $[Y/H]$ or $[Y/(H/K)]$, when it comes to Chen–Ruan orbifold cohomology these two stacks yield different cohomology groups, as the formula recalled above obviously shows.

As a consequence of these considerations we point out that in the above formula for Chen–Ruan orbifold cohomology $H_{\text{CR}}^{p,q}(X_W/G; \mathbb{Q})$, we can replace on the right hand side $H^*(X_W; \mathbb{Q})^G$ by $H^*(X_W; \mathbb{Q})^G$; indeed, the identity component $\Gamma^0 \cong \mathbb{C}^\times$ acts trivially on $X_W$ and $\mathbb{P}(w)$ and on each sector $X_{W,\gamma}$ and $\mathbb{P}(w)_\gamma$.

“Landau–Ginzburg models” is an expression often used for $\mathbb{C}$-valued functions defined on vector spaces or, more generally on vector bundles, as in this paper. In this context, the $\mathbb{C}$-valued functions are often called superpotentials. In order to relate the Calabi–Yau complete intersection $X_W$ to a Landau–Ginzburg model, we rephrase its cohomology in terms of relative cohomology of a vector bundle.

For any $i = 1, \ldots, r$, we can define a character $\chi_i \colon \Gamma \to \mathbb{C}^\times$ by mapping $\alpha \mapsto \lambda^d_i$; indeed, if $\alpha$ and $\beta \in (\mathbb{C}^\times)^n$ preserve all polynomials $W_i$ and satisfy $\alpha \lambda = \gamma = \beta \lambda/\in \Gamma$ for some $\lambda, \mu \in \mathbb{C}^\times$, then we have $\lambda^d_i = (\gamma^* W_i)/W_i = \mu^d_i$ all $i$.

**Remark 3.3.** We notice that $r_\gamma$ equals $\#\{i \mid \chi_i(\gamma) = 0\}$. See Lemma 2.1.

In this way, we can define a $\Gamma$-action on $\mathbb{C}^{n+r}$ by

$$
\gamma \cdot (x_1, \ldots, x_n, p_1, \ldots, p_r) = (\gamma_1 x_1, \ldots, \gamma_n x_n, \chi_1^{-1}(\gamma^{-1} p_1, \ldots, \chi_r^{-1}(\gamma) p_r),
$$

or, more explicitly, by $\alpha \lambda \cdot (x, p) = (\alpha_1 \lambda x_1, \ldots, \alpha_n \lambda x_n, \lambda^{-d_1} p_1, \ldots, \lambda^{-d_r} p_r)$. We consider the $\Gamma$-invariant $\mathbb{C}$-valued function

$$
\overline{W} : \mathbb{C}^{n+r} \to \mathbb{C}
$$

$$
(x, p) \mapsto p_1 W_1(x) + \ldots + p_r W_r(x).
$$

The fibre $M = \{\overline{W} = t_0\}$ over any point $t_0 \neq 0$ in $\mathbb{C}$ is smooth and its cohomology does not depend on the choice of $t_0 \in \mathbb{C}^\times$.

We consider

$$
U_{\text{CY}} = \mathbb{C}^n \setminus \{0\} \times \mathbb{C}^r
$$

and the corresponding vector bundle $[U_{\text{CY}}/\mathbb{C}^\times]$

$$
\mathcal{O}_w(-d) := \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}(w)}(-d_i),
$$

which we identify with the total space over $\mathbb{P}(w)$. If we consider the $\Gamma$-action on $\mathbb{C}^{n+r}$, we study the corresponding (total space of the) vector bundle $[\mathcal{O}_w(-d)/G] \to [\mathbb{P}(w)/G]$.

The function $\overline{W}$ descends to a $\mathbb{C}$-valued function $W_{\text{CY}}$ on $[\mathcal{O}_w(-d)/G]$

$$
\overline{W}_{\text{CY}} : [\mathcal{O}_w(-d)/G] \to \mathbb{C}.
$$

**Proposition 3.4.** We have a canonical isomorphism

$$
H^{p,q-r}_{\text{CR}}(X_W/G) \cong H_{\text{CR}}^{p,q}([\mathcal{O}_w(-d)/G], [F/G]),
$$

where $F$ is the quotient stack $[M/\mathbb{C}^\times]$ inside $\mathcal{O}_w(-d)$. 
Remark 3.5. $F$ is the generic fibre of the morphism $\mathcal{O}_w(-d) \to \mathcal{O} \to \mathcal{C}$.

Proof. On both sides we have a direct sum over $\gamma \in \Gamma$ involving $\Gamma$-invariant classes of $X_{W,\gamma}$ and of the pair $(\mathcal{O}_w(-d), F_\gamma)$. We notice that $\gamma$ acts with age $a_\gamma$ on the tangent bundle of $X_W$ and with age $a_\gamma + (r - r_\gamma)$ on the tangent bundle of $\mathcal{O}_w(-d)$. Indeed, using the chain of inclusions $X_W \subset \mathbb{P}(w) \subset \mathcal{O}_w(-d)$ (the last one being obtained by identifying $\mathbb{P}(w)$ with the zero section) one readily shows that the normal bundle of $X_W$ inside $\mathcal{O}_w(-d)$ is $\mathcal{O}_w(d) \oplus \mathcal{O}_w(-d)$. For any fixed element $\gamma \in \Gamma$, in view of age computations, we may ignore the $r_\gamma$ summands of $\mathcal{O}_w(-d)$ where $\gamma$ acts trivially. Since, for a nontrivial character $\chi$, $\text{age}(\chi) + \text{age}(\chi^{-1}) = 1$ we obtain the desired age shift difference $r - r_\gamma$.

Finally we need to check

$$H^{p-r_\gamma, q-r_\gamma}(X_{W,\gamma}) = H^{p,q}(\mathcal{O}_w(-d), F_\gamma),$$

which requires the following analysis of $F_\gamma$.

Lemma 3.6. For any $\gamma \in \Gamma$, if $r_\gamma < n_\gamma$, then $X_{W,\gamma}$ is nondegenerate and we have

$$H^k(F_\gamma) = \begin{cases} 
H^k(\mathbb{P}(w), \gamma) & 0 \leq k \leq 2r_\gamma - 2 \\
H^{n_\gamma-r_\gamma-1}(X_{W,\gamma}) & k = n_\gamma + r_\gamma - 2 \\
0 & \text{otherwise}
\end{cases}$$

If $r_\gamma \geq n_\gamma$ then $H^k(F_\gamma) \cong H^k(\mathbb{P}(w))$ for all $k$.

Proof. Set $X = X_{W,\gamma}$ and $F = F_\gamma$. By abuse of notation we write $\mathbb{P}(w)$ as $\mathbb{P}(w)$, $n_\gamma$ and $r_\gamma$ as $n$ and $r$. Consider the projection map $\pi: F \to \mathbb{P}(w)$. Since we may regard $F$ as the variety defined by the equation $\sum_i p_i W_i(x) = 1$, the fiber $\pi^{-1}(x)$ is empty if $x \in X$ and an affine hyperplane isomorphic to $\mathbb{A}^{r-1}$ if $x \notin X$. More precisely, $\pi$ is a locally trivial fibration over $\mathbb{P}(w) \setminus X$ with fiber $\mathbb{A}^{r-1}$ (trivialize over the open subsets $U_i = \{x \in \mathbb{P}(w) | W_i(x) \neq 0\}$). Hence $H^*(F) \cong H^*(\mathbb{P}(w) \setminus X)$. As $X$ is a $\mathbb{Q}$-homology manifold we have an isomorphism of Hodge structures

$$H^k(\mathbb{P}(w), \mathbb{P}(w) \setminus X; \mathbb{Q}) \cong H^{k-2r}(X; \mathbb{Q})(-r).$$

Hence the long exact sequence of relative cohomology for the pair $(\mathbb{P}(w), \mathbb{P}(w) \setminus X)$ can be rewritten as

$$H^k(\mathbb{P}(w)) \to H^k(\mathbb{P}(w) \setminus X) \to H^{k-2r+1}(X) \xrightarrow{i_*} H^{k+1}(\mathbb{P}(w))$$

where cohomology is with $\mathbb{Q}$-coefficients and the Gysin map $i_*$ is Poincaré dual to the map

$$i_*: H_{2n-k-3}(X) \to H_{2n-k-3}(\mathbb{P}(w)).$$

The result then follows from the Lefschetz hyperplane theorem.

We now continue the proof of Proposition 3.4. If $r_\gamma < n_\gamma$ and $X_W$ is nondegenerate then

$$H^k(\mathcal{O}_w(-d), F_\gamma) \cong H^{k-2r}(X_\gamma)(-r).$$

because the pullback via $\pi: \mathcal{O}_w(-d) \to \mathbb{P}(w)$ induces an isomorphism

$$H^k(\mathbb{P}(w)_\gamma, \mathbb{P}(w)_\gamma \setminus X_{W,\gamma}) \cong H^k(\mathcal{O}_w(-d), F_\gamma)$$

and $H^k(\mathbb{P}(w)_\gamma, \mathbb{P}(w)_\gamma \setminus X_{W,\gamma}) \cong H^{k-2r}(X_{W,\gamma})(-r)$ as we have seen above. If $r_\gamma \geq n_\gamma$ the same results hold, with $X = \emptyset$. 

\hfill \Box
4. Hybrid Landau–Ginzburg side

On the other side we consider the open set
\[ U_{LG} = \mathbb{C}^n \times (\mathbb{C}^r \setminus \{0\}) \]
and the (total space of the) vector bundle
\[ [U_{LG}/\mathbb{C}^r] = \bigoplus_{j=1}^{n} O_{\mathbb{P}(d)}(-w_i), \]
which we simply denote by \( O_d(-w) \). We notice that the Milnor fiber \( M = \mathbb{W}^{-1}(t_0) \) for \( t_0 \neq 0 \) is contained in both \( U_{CY} \) and \( U_{LG} \). Its cohomology does not depend on \( t_0 \) and, after modding out the \( \mathbb{C}^r \)-action, has been completely described in Lemma 3.6.

We recall that \( \Gamma \) is a group of diagonal symmetries acting on \( \mathbb{C}^n \); as in the previous section we extend its action to \( \mathbb{C}^n \times \mathbb{C}^r \). We now consider the hybrid LG model \( (O_d(-w), \mathbb{W}) \) with superpotential
\[ \mathbb{W}: [O_d(-w)/G] \to \mathbb{C}. \]
As above, the quotient stack \([O_d(-w)/G]\) may be presented as \([U_{LG}/\Gamma]\) and may be regarded as a vector bundle over \([\mathbb{P}(d)/G]\).

**Definition 4.1.** The generalized state space of the hybrid Landau–Ginzburg model \( (O_d(-w), \mathbb{W}) \) is
\[ H_{\Gamma}^{p,q}(W_1, \ldots, W_r) := H_{CR}^{p+r,q+r}([O_d(-w)/G], [F/G]), \]
where \( G \) is the component group \( \Gamma/\Gamma^0 \).

**Remark 4.2.** We point out that the space \( H^{p,q}(W_1, \ldots, W_r) \) defined in \([3]\) coincides with the above definition for \( \Gamma = \mathbb{C}^r \).

We can describe in greater detail the relative Chen–Ruan orbifold cohomology
\[ H_{CR}^p([O_F(d)(-w)/G], [F/G]) = \bigoplus_{g \in G} H_{CR}^{p-a_g,q-a_g}([O_F(d)(-w)/G]_g, [F/G]_g), \]
or, more explicitly,
\[ H_{CR}^p([U_{LG}/\Gamma], [M/\Gamma]) = \bigoplus_{\gamma \in \Gamma} H^{p-a_\gamma,q-a_\gamma}(O_{d_\gamma}(-w_{\gamma}), F_{\gamma}), \]
where \( d_\gamma \) and \( w_\gamma \) are the multi-indices obtained by suppressing the entries \( i \) and \( j \) for which \( \gamma_i \neq 1 \), and \( a_\gamma \) is the age of the action of \( \gamma \) on \( \mathbb{C}^n \times \mathbb{C}^r \). We recall again that \( H^*(O_{d_\gamma}(-w_{\gamma}), F_{\gamma})^\Gamma \) is just the group of \( G \)-invariant classes \( H^*(O_{d_\gamma}(-w_{\gamma}), F_{\gamma})^G \).

This allows us to give an explicit presentation of FJRW cohomology as a direct sum of two types of terms reflecting the hybrid nature of our Landau–Ginzburg models: a chiral algebra term computing the primitive cohomology via the Jacobian ring, and a term computing the fixed cohomology.

The analogue of the chiral algebra of the Jacobian ring of an isolated singularity is defined as follows. For any smooth complete intersection of \( \tilde{r} \) hypersurfaces \( \{W_i = 0\} \) (with \( i = 1, \ldots, \tilde{r} \)) inside a weighted projective space of \( \tilde{n} > \tilde{r} \) coordinates we consider the generalised chiral algebra
\[ \mathcal{Q}_{\tilde{W}} = dx_1 \wedge \ldots \wedge dx_{\tilde{n}} \wedge dp_1 \wedge \ldots \wedge dp_{\tilde{r}} \otimes \mathbb{C}[x_1, \ldots, x_{\tilde{n}}, p_1, \ldots, p_{\tilde{r}}]/(\partial_{x_j} \tilde{W}, \partial_{p_i} \tilde{W}), \]
where \( \tilde{W} = \sum_{i=1}^{\tilde{r}} p_i W_i(x) \). This is the Jacobian ring from \([13]\) or \([30]\) tensored with the top-degree form. Its \( \mathbb{C}^r \)-invariant part is finite dimensional. For \( \tilde{D} = \tilde{n} - \tilde{r} - 1 \), by \([13\) Thm. 7], \([30\) Thm. 2.16], we have
\[ (\mathcal{Q}_{\tilde{W}})^{\mathbb{C}^r} \cong H_{\tilde{D}}^p(X_{W_\gamma})(-r). \]
If we assign bi-degree \((\tilde{D} - k, k)\) to each term of \(Q_{\tilde{W}}\): degree \(k\) in the variables \(p_1, \ldots, p_{\tilde{r}}\) we may regard the above isomorphism as a bi-degree preserving isomorphism. Note that in both papers cited above the result is stated for complete intersections in projective space; the same proof goes through in the case of weighted projective spaces [13, Remark 18 (i)].

Finally for any non-negative integer \(\tilde{n}\) and for any \(\tilde{r} \geq \tilde{n}\) we need to consider the cohomology of \(\mathbb{P}(d)\) in degrees \(0, \ldots, 2r - 2n - 2\). We write the bi-degree preserving identification

\[
dt(-\tilde{n}) \otimes \mathbb{C}[t]/(t^{\tilde{r}-\tilde{n}-1}) \cong \bigoplus_{k=0}^{\tilde{r}-\tilde{n}-1} H^{2k}(\mathbb{P}(d)).
\]

where \(dt(-\tilde{n}) \otimes t^k\) has bidegree \((\tilde{n}, \tilde{n}) + (k, k)\).

**Theorem 4.3.** We have

\[
\mathcal{H}_r^*(W_1, \ldots, W_r) = \oplus_{r=0}^n \mathcal{H}_r(-a_r + 1),
\]

where \(\mathcal{H}_r\) with its double grading is given by

\[
\mathcal{H}_r = \begin{cases} (Q_{W_r})^r & \text{if } r < n_r, \\ dt(-n_r) \otimes \mathbb{C}[t]/(t^{\gamma_r-n_{\gamma_r}}) & \text{if } r \geq n_r. \end{cases}
\]

The proof follows from the analysis of the relative cohomology of \((\mathcal{O}_{\mathbb{P}(d)}(-w), F)\) for each sector. We assume \(G = 1\) for simplicity; as we saw above, nontrivial groups \(G\) can be treated easily by the same argument. By abuse of notation we write \(d, w\) and \(F\) instead of \(d, w, F\) and \(F_r\). The following lemma provides a complete picture.

**Lemma 4.4.** If \(r < n\) and \(X\) is a nondegenerate complete intersection, then

\[
H^k(\mathcal{O}_d(-w), F) = \begin{cases} H^{n-r-1}(\mathbb{P}(d))(n) & \text{if } k = n + r - 2 \\ 0 & \text{if } k \neq n + r - 2. \end{cases}
\]

If \(r \geq n\) then

\[
H^k(\mathcal{O}_d(-w), F) \cong \begin{cases} H^{k-2n}(\mathbb{P}(d))(n) & \text{if } 2n \leq k \leq 2r - 2 \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Let \(Z_1 \subset \mathcal{O}_w(-d)\) and \(Z_2 \subset \mathcal{O}_d(-w)\) be the zero sections. Write

\[
\mathcal{O}_w(-d)^X = \mathcal{O}_w(-d) \setminus Z_1, \quad \mathcal{O}_d(-w)^X = \mathcal{O}_d(-w) \setminus Z_2.
\]

The vector bundles \(\mathcal{O}_d(-w)\) and \(\mathcal{O}_w(-d)\) contain a common open subset

\[
[U_{CY} \cap U_{LG} / \mathbb{C}^\times] = \mathcal{O}_w(-d)^X \cong \mathcal{O}_d(-w)^X.
\]

The inclusions \(F \subset \mathcal{O}_d(-w)^X \subset \mathcal{O}_d(-w)\) give an inclusion of pairs

\[
(\mathcal{O}_d(-w)^X, F) \subset (\mathcal{O}_d(-w), F) \subset (\mathcal{O}_d(-w), \mathcal{O}_d(-w)^X).
\]

We shall calculate the relative cohomology \(H^k(\mathcal{O}_d(-w), F)\) using the exact sequence of relative cohomology

\[
H^k(\mathcal{O}_d(-w), \mathcal{O}_d(-w)^X) \to H^k(\mathcal{O}_d(-w), F) \to H^k(\mathcal{O}_d(-w)^X, F).
\]

We have

\[
H^k(\mathcal{O}_d(-w), \mathcal{O}_d(-w)^X) = H^k_{Z_2}(\mathcal{O}_d(-w)) \cong H^{k-2n}(Z_2) \cong H^{k-2n}(\mathbb{P}(d)).
\]

To calculate \(H^k(\mathcal{O}_d(-w)^X, F)\), we use the isomorphism \(H^k(\mathcal{O}_d(-w)^X, F) \cong H^k(\mathcal{O}_w(-d)^X, F)\) and the previous results for \(\mathcal{O}_w(-d)^X\). Using the long exact sequence

\[
\ldots \to H^k(\mathcal{O}_w(-d), \mathcal{O}_w(-d)^X) \to H^k(\mathcal{O}_w(-d), F) \to H^k(\mathcal{O}_w(-d)^X, F) \to \ldots
\]

coming from the inclusion of pairs

\[
(\mathcal{O}_w(-d)^X, F) \subset (\mathcal{O}_w(-d), F) \subset (\mathcal{O}_w(-d), \mathcal{O}_w(-d)^X),
\]
Proof of Theorem 4.3. The claim follows immediately from the formula (7) and Lemma 4.4.

By the previous calculations we have
\[ H^k(\mathcal{O}_d(-d), F) \cong H^{k-2r}(\mathbb{P}(d)) \]
and the five lemma, we obtain
\[ H^k(\mathcal{O}_w(-d)^\times, F) \cong H^{k-2r+1}(\mathbb{P}(w), X). \]

We now turn to the calculation of \( H^k(\mathcal{O}_d(-w), F) \) in the case \( r < n \). If \( k \geq 2r \) the exact sequence
\[ H^{k-1}(F) \to H^k(\mathcal{O}_d(-w), F) \to H^k(\mathcal{O}_d(-w)) \]
shows that
\[ H^k(\mathcal{O}_d(-w), F) \cong H^{k-1}(F) = \begin{cases} H^{n-r-1}(X) & \text{if } k = n + r - 1 \\ 0 & \text{otherwise.} \end{cases} \]
If \( k \leq 2r-1 \) the exact sequence (8) shows that \( H^k(\mathcal{O}_d(-w), F) = 0 \) since the terms \( H^{k-2n}(\mathbb{P}(d)) \) and
\[ H^k(\mathcal{O}_d(-w)^\times, F) = H^k(\mathcal{O}_w(-d)^\times, F) \cong H^{k-2r+1}(\mathbb{P}(w), X) \]
vanish (also for \( k = 2r - 1 \), since \( H^0(\mathbb{P}(w)) \cong H^0(X) \)). Hence in the case \( r < n \) we find
\[ H^k(\mathcal{O}_d(-w), F) = \begin{cases} H^{n-r-1}(X) & \text{if } k = n + r - 1 \\ 0 & \text{otherwise.} \end{cases} \]

We now consider the case \( r \geq n \). The calculations go through as before, with \( X = \emptyset \). Using the exact sequence of relative cohomology for the pair \( (\mathcal{O}_d(-w), F) \) we find that \( H^k(\mathcal{O}_d(-w), F) = 0 \) if \( k \geq 2r \).

Next we consider the case \( k \leq 2r-2 \). The exact sequence (8) then shows that \( H^k(\mathcal{O}_d(-w), F) \cong H^{k-2n}(\mathbb{P}(d)) \), hence
\[ H^k(\mathcal{O}_d(-w), F) = \begin{cases} \mathbb{Q} & \text{if } 2n \leq k \leq 2r - 2, k \text{ even} \\ 0 & \text{otherwise.} \end{cases} \]
(Note that if \( r = n \) the formula says that \( H^k(\mathcal{O}_d(-w), F) = 0 \) for all \( k \leq 2r - 2 \).)

For the remaining case \( k = 2r - 1 \) we use an Euler characteristic calculation. The exact sequence (8) shows that
\[ \sum_i (-1)^i(h^i(\mathcal{O}_d(-w), \mathcal{O}_d(-w)^\times) - h^i(\mathcal{O}_d(-w), F) + h^i(\mathcal{O}_d(-w)^\times, F)) = 0. \]

By the previous calculations we have
\[ \sum_i (-1)^i h^i(\mathcal{O}_d(-w), \mathcal{O}_d(-w)^\times) = \sum_i (-1)^i h^{i-2n}(\mathbb{P}(d)) = r \]
\[ \sum_i (-1)^i h^i(\mathcal{O}_d(-w), F) = r - n - h^{2r-1}(\mathcal{O}_d(-w), F) \]
\[ \sum_i h^i(\mathcal{O}_d(-w)^\times, F) = \sum_i (-1)^i h^{i-2r+1}(\mathbb{P}(w)) = -n. \]

Hence we find that \( H^{2r-1}(\mathcal{O}_d(-w), F) = 0 \).

So in the case \( r \geq n \) we obtain
\[ H^k(\mathcal{O}_d(-w), F) = \begin{cases} H^{k-2n}(\mathbb{P}(d)) & \text{if } 2n \leq k \leq 2r - 2 \\ 0 & \text{otherwise.} \end{cases} \]

\( \square \)

Proof of Theorem 4.3. The claim follows immediately from the formula (7) and Lemma 4.4. \( \square \)
5. The correspondence

**Theorem 5.1.** Let \( X_W = \{ W_1 = 0, \ldots, W_r = 0 \} \subset \mathbb{P}(w_1, \ldots, w_n) \) be a non-degenerate Calabi-Yau complete intersection. There exists an explicit bidegree preserving isomorphism

\[
H^*_{\text{CR}}([\mathcal{O}_{\mathbb{P}(d)}(-w)/G]) \cong H^*_{\text{CR}}([\mathcal{O}_{\mathbb{P}(d)}(-w)/G]).
\]

As an immediate corollary we get the desired isomorphism.

**Corollary 5.2.** There exists an explicit bidegree preserving isomorphism

\[
H^r_t(W_1, \ldots, W_r) \cong H^*_{\text{CR}}([X_W/G]).
\]

**Proof of Theorem 5.1.** Let us refer to the left hand side as the CY side, and to the right hand side as the LG side. By definition, both sides decompose as the direct sum over \( \Gamma \)

\[
\text{CY} = \bigoplus_{\gamma \in \Gamma} \bigoplus_{0 \leq k < n_{\gamma}} [\text{CY}^k_{\gamma}]_{\mathbb{Q}} \quad \text{and} \quad \text{LG} = \bigoplus_{\gamma \in \Gamma} \bigoplus_{0 \leq k < r_{\gamma}} [\text{LG}^k_{\gamma}]_{\mathbb{Q}},
\]

where all terms \( \text{CY}^k_{\gamma} \) and \( \text{LG}^k_{\gamma} \) are elements of the form \( H^k \cap \mathbf{1}_{\gamma} \), where \( H \) is an hyperplane section and \( \mathbf{1}_{\gamma} \) is the fundamental class of the corresponding sector. The bidegree \( (p, q) \) of such elements is \( p = k - 1 + a_{\gamma} \) by definition.

We shall identify the CY and the LG side for every connected component of \( \Gamma \). Each component is of the form \( g\Lambda \) for a fixed diagonal symmetry of \( W_1, \ldots, W_n \). We express \( g \) as the diagonal matrix whose \( n + r \) entries on the diagonal are \( \exp(2\pi i a_1), \ldots, \exp(2\pi i a_n), 1 \) with \( a_i \in [0, 1[ \). We write \( \lambda = \exp(-2\pi it) \).

Let \( \gamma = g\Lambda \) with \( g \) and \( \lambda \) as above. We denote the fractional part of \( x \) by

\[
\langle x \rangle = x - [x].
\]

By definition we have

\[
a_{g\Lambda} = \sum_{j=1}^{n} \langle -tw_j + a_j \rangle + \sum_{j=1}^{r} \langle td_i \rangle = \sum_{j=1}^{n} \langle -tw_j + a_j \rangle = \sum_{i=1}^{r} \langle -|d_i|t \rangle - \sum_{j=1}^{n} \langle a_j - w_jt \rangle = \sum_{i=1}^{r} \langle td_i \rangle - n + \sum_{j=1}^{n} \left\lfloor w_jt - a_j \right\rfloor - \sum_{j=1}^{n} \left\lfloor w_jt - a_j \right\rfloor \leq Z \quad w_jt - a_j \in \mathbb{Z}.
\]

The generators of CY and LG arising from \( \Gamma \) can be easily pictured in the following diagram representing on concentric circles the angular coordinates of the elements of \( \mu_{w_1} \sqcup \cdots \sqcup \mu_{w_N} \) (black dots) and of \( \mu_{d_1} \sqcup \cdots \sqcup \mu_{d_r} \) (white dots).

Let us illustrate how the white and black dots correspond to generators of LG and CY, respectively. Notice that, by construction, a ray \( \{ \rho \lambda \mid \rho \in \mathbb{R}^+ \} \) carries \( r_{\lambda} \) white dots and \( n_{\lambda} \) black dots. The white dots correspond to the states \( \text{LG}^0_{\gamma}, \ldots, \text{LG}^{r_{\gamma}-1}_{\gamma} \) whereas the black dots correspond to the states \( \text{CY}^0_{\gamma}, \ldots, \text{CY}^{n_{\gamma}-1}_{\gamma} \). The total number of black dots equals the total number of white dots, i.e. \( D = \sum_j w_j = \sum_i d_i \). We can order all the dots in lexicographic order starting from \( \text{CY}^0 \); we get \( x_1, x_2, \ldots, x_{2D} \). Then, by the above formulae, the cohomological Chen–Ruan degree of a state corresponding to \( x_i \) is \( \sum_j a_j \) plus the following function

\[
f(x_i) = \# \{ y \text{ black} \mid x_1 \leq y < x_i \} - \# \{ y \text{ white} \mid x_1 \leq y \leq x_i \}.
\]
We can identify each term $x_i$ with the integer $i$ and extend this degree function to a piecewise linear real function defined on $]\frac{1}{2}, 2D + \frac{1}{2}]$

$$f(x) = \begin{cases} 
  f(x_i) + (x - i) & \text{if } x \text{ black and } x \in ]x_i - \frac{1}{2}, x_i + \frac{1}{2}], \\
  f(x_i) - (x - i) & \text{if } x \text{ white and } x \in ]x_i - \frac{1}{2}, x_i + \frac{1}{2}]. 
\end{cases}$$

This function is continuous and its values at the boundary of the interval coincide, so it can be regarded as a continuous function on a circle. It is strictly increasing at all black dots and strictly decreasing at all white dots; its relative maxima occur for values of the form $\frac{1}{2}(x_i + x_{i+1})$ where $x_i$ is black and $x_{i+1}$ is white. Similarly, with inverted colours, for the relative minima. In this way if $f$ attains a given (integer) value at a given number of black dots it must attain the same value at the same number of white dots. (We illustrate this in Fig. 1 by writing the value of $f$ next to each dot). This yields the desired bidegree preserving LG/CY isomorphism.

We can use the function $f$ to define an explicit isomorphism, simply by pairing each black dot for which $f = k$ to the following white dot for which $f = k$. This yields an explicit isomorphism. □

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