General Black Hole Solutions in (2+1)-dimensions with a Scalar Field Non-Minimally Coupled to Gravity

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We discuss black hole solutions in (2+1)-dimensions with a scalar field non-minimally coupled to Einstein’s gravity in the presence of a cosmological constant and a self-interacting scalar potential. Without specifying the form of the potential, we find a general solution of the field equations, which includes all the known asymptotically anti-de Sitter (AdS) black hole solutions in (2+1)-dimensions as special cases once values of the coupling constants are chosen appropriately. In addition, we obtain numerically new black hole solutions and for some specific choices of the coupling constants we derive new exact AdS black hole solutions. We also discuss the possibility of obtaining asymptotically de Sitter black hole solutions with or without an electromagnetic field.

I. INTRODUCTION

Despite the wide application of the AdS/CFT correspondence (and more generally the gauge/gravity duality) in a variety of settings, such as quark gluon plasma and condensed matter systems, the properties of asymptotically AdS spacetimes are still far from being completely understood, even at the classical level. A good example of our lack of understanding is the stability issue of asymptotically AdS spacetimes. It came as a surprise that – unlike Minkowski spacetime – AdS spacetime is generically unstable under small perturbations. For Minkowski spacetime, small
perturbations eventually disperse to infinity and thus the spacetime is stable [1]. In the case of AdS spacetime, however, due to the presence of timelike boundary at infinity, perturbations can bounce off and return to the bulk (recall that null rays can bounce back in a finite time), which in turn can focus enough energy to cause black hole formation. The stability of $D$-dimensional AdS spacetime, here denoted AdS$_D$, for $D \geq 4$ was studied in [2, 3]. It was found that for arbitrarily small perturbations of a massless scalar field minimally coupled to AdS gravity, the AdS$_D$ spacetime is unstable and finally results in black hole formation. To be more specific, this instability was understood to be the result of resonant transfer of energy from low to high frequencies [4]. In (2+1)-dimensions, however, AdS gravity has a different behavior. It was found [5] that for a large class of perturbations, a turbulent cascade of energy to high frequencies still occur, which entails instability of AdS$_3$, but cannot be terminated by black hole formation because small perturbations have energy below the black hole threshold (due to mass gap of black hole).

As is well known, in (2+1)-dimensions the local geometry without the presence of any matter field remains trivial even if a cosmological term is introduced, since the Einstein space is a space of constant curvature in (2+1)-dimensions. Surprisingly, the presence of the cosmological constant introduces a scale which allows one to find a black hole solution in the absence of any scalar field. This is known as the BTZ black hole [6]. It is obtained by identifying certain points of the anti-de Sitter spacetime. The BTZ black hole is characterized by the mass, angular momentum and a negative cosmological constant, and has almost all the features of the Kerr-AdS black hole in four-dimensional Einstein gravity.

In the context of gauge/gravity duality, one could construct black hole solutions in the gravity sector that acquires hair below a critical temperature. Even a static and spherically symmetric black hole with scalar hair requires a deep understanding of the matter field near the black hole horizon, as it should satisfy some physical requirements, e.g. being regular on the horizon, and decays sufficiently fast towards infinity, so that the black hole shows its presence as (primary or secondary) hair in the boundary field theory with finite temperature.

The early attempts to couple a scalar field to gravity was first carried out in asymptotically flat spacetimes. Black hole solutions were found [7] but the scalar field was divergent on the horizon and they were unstable under scalar perturbations [8]. Introducing a cosmological constant we expect that the scalar field to have regular behavior on the resulting black hole horizon, while all possible divergences to be hidden behind the horizon. Hairy black hole solutions were found in asymptotically de Sitter spacetime with either a minimally or a non-minimally coupled scalar field but they were unstable [9–13]. In the case of AdS spacetime with a negative cosmological constant,
stable solutions were found numerically for spherical geometries \([14–16]\) and an exact solution in asymptotically AdS spacetime with hyperbolic geometry was presented in \([17]\) and generalized later to include charge \([18]\). In all the above solutions the scalar field is conformally coupled to gravity. A generalization to non-conformal solutions was discussed in \([19]\).

A scale can also be introduced in the scalar sector of the theory. This can be done if in the Einstein-Hilbert action there is a coupling of a scalar field to the Einstein tensor. The derivative coupling has the dimension of length squared and it was shown to act as an effective cosmological constant \([20, 21]\). Spherically symmetric black hole solutions which are asymptotically anti-de Sitter were found \([22–27]\), thus evading the no-hair theorem for Horndeski theory \([28]\).

In \((2+1)\)-dimensions black holes with scalar field non-minimally coupled to gravity are known to exist \([29, 30]\). Furthermore, both static \([31]\) and rotating \([32–34]\) hairy \((2+1)\)-dimensional black holes were constructed by specifying some suitable forms of the potential. An exact dynamical and inhomogeneous solution that represents gravitational collapse was presented in \([35]\). If an electromagnetic field is introduced then it is possible to find hairy charged black hole solutions in \((2+1)\)-dimensions \([36]\). More recently exact dynamical and inhomogeneous solutions in \((2+1)\)-dimensional AdS gravity with a conformally coupled scalar field was discussed in \([37]\).

Gravity in \((2+1)\)-dimensions is interesting in many ways. One of the principal advantages of working in \((2+1)\)-dimensions is that for simple enough topologies, this space can be characterized completely and explicitly, which can help to gain important insights into black hole physics and the structure of quantum gravity. Hairy black hole solutions in the presence of a cosmological constant and conformally coupled matter are good theoretical laboratories to examine further the gauge/gravity correspondence \([38]\) by connecting the three dimensional gravity bulk with the two dimensional boundary field theory and holographically relating to, e.g., condensed matter systems of two dimensional materials and superfluids.

In this work we study general relativity in \((2+1)\)-dimensions with a scalar field non-minimally coupled to the gravity sector, with a general potential in the presence of a cosmological constant. Our aim is – without specifying the form of the potential a priori – to find a general solution of the field equations in the coupled Einstein-scalar field system, which can give a general hairy black hole solution. To this end, we solve the Einstein-scalar field equations with a static and spherically symmetric metric ansatz. The general solution we found can be reduced to the known black hole solutions \([6, 29, 35, 36, 39]\) depending on the value of the coupling constant. Going beyond the known solutions in the literature, we can also derive other new solutions. We also investigate the possibility of finding asymptotically dS black hole solutions in \((2+1)\)-dimensions and we show that
no such black hole exist, consistent with previous no-go results. However, surprisingly, we note that in the presence of an electromagnetic field the solution found in [36] can give an asymptotically de Sitter black hole solution for a specific choice of parameters.

This work is organized as follows: in Section II we discuss the general solution of a scalar field non-minimally coupled to gravity, in Section III we show some explicit – exact solutions – of asymptotically AdS black holes, in Section IV we discuss the possibility of finding dS black hole solutions and finally in the last Section we present our conclusions.

II. GENERAL SOLUTION WITH A SCALAR FIELD NON-MINIMALLY COUPLED TO EINSTEIN’S GRAVITY

In this section we consider a scalar field non-minimally coupled to (2+1)-dimensional Einstein’s gravity with a self-interacting scalar potential. The action is

\[ I = \int d^3x \sqrt{-g} \left[ \frac{R}{2} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} \xi R \Phi^2 - U(\Phi) \right], \]

where \( \xi \) is a coupling constant and \( U(\Phi) \) is the self-interacting potential of the scalar field in which the cosmological constant is included via \( U(\Phi = 0) = \Lambda \) (see details below). The Einstein equations and the Klein-Gordon equation read, respectively,

\[ (1 - \xi \Phi^2) G_{\mu\nu} = \nabla_\mu \Phi \nabla_\nu \Phi - g_{\mu\nu} \left( \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \Phi \nabla_\beta \Phi + U(\Phi) \right) + \xi (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) \Phi^2, \]
\[ \Box \Phi = \xi R \Phi + \frac{dU(\Phi)}{d\Phi}, \]

where \( G_{\mu\nu} \) denotes the Einstein’s tensor. We consider a spherically symmetric ansatz for the metric

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\theta^2. \]

Then, the \( t-t \) and \( r-r \) components of the Einstein equations give

\[ \Phi(r) = \left( \frac{A}{r + B} \right)^{\frac{\delta}{2}}, \]

where \( \delta \equiv 4\xi / (1 - 4\xi) \), with \( A \) and \( B \) being some constants. We first consider \( A > 0, B > 0 \) and \( \delta > 0 \), corresponding to \( 0 < \xi < 1/4 \). Note that \( \delta \neq -1 \), since \( \delta = -1 \) would correspond to unphysical value of \( \xi = \infty \).

The \( r-r \) component of the Einstein equations then gives the metric function

\[ f(r) = \frac{(r + B)^{1+\delta} \left[ C_0 - 8(1 + \delta) \int U(r)rdr \right]}{A^\delta \delta \left[ \delta r - (r + B) \right] + 4(1 + \delta)(r + B)^{1+\delta}}, \]
where $C_0$ is an integration constant. (In our notation, the integration constant is written explicitly, i.e., the integral $\int U(r)rdr$ itself is understood to be without a constant term.)

If we substitute the scalar field (5) and the metric function (6) into the $\theta-\theta$ component of the Einstein equations, we would obtain an integral equation of the potential $U(r)$:

$$C_0 - 8(1 + \delta) \int U(r)rdr = a(r)U(r) + b(r)U'(r),$$

where

$$a(r) = c(r) \left[ A^\delta \delta(3B - \delta r + r) - 4(\delta + 1)(B + r)^\delta(3B - 2\delta r + r) \right],$$

$$b(r) = c(r)(B + r)A^{-\delta}\delta^{-2} \left[ A^2 \delta^2(B - \delta r + r) + 4A^\delta (\delta + 1)(\delta r - 2(B + r))(B + r)^\delta + 16(\delta + 1)^2(B + r)^{2\delta + 1} \right],$$

$$c(r) = -4(\delta + 1)r(B + r) \left[ 4(\delta + 1)(B + r)^{\delta+1} - A^\delta \delta(B - \delta r + r) \right] / \left\{ A^2 \delta (\delta - 1)\delta^2(B + r) - 2A^\delta \delta(\delta + 1) \left[ -4B^2 + B(-3\delta^2 + \delta - 4)r + (\delta - 1)^2\delta r^2 \right] (B + r)^\delta + 8(\delta + 1)^3(\delta r - 2B)(B + r)^{2\delta + 1} \right\}.$$  

By equation (7), the metric function (6) becomes

$$f(r) = \frac{(r + B)^{1+\delta}[a(r)U(r) + b(r)U'(r)]}{A^\delta \delta[r + B] + 4(1 + \delta)(r + B)^{1+\delta}},$$

which can be reduced to the solutions in [29] and [39] when $U(r) = \Lambda = -\ell^{-2}$ and $\delta = 1$. Moreover, when $\delta = 0$, we get the BTZ black hole solution [6].

Taking the derivative of Eq.(7) we obtain, with prime denoting derivative with respect to $r$, the second order differential equation

$$U''(r) + P(r)U'(r) + Q(r)U(r) = 0,$$

where

$$P(r) = \frac{a(r) + b'(r)}{b(r)}, \quad Q(r) = \frac{a'(r) + 8(\delta + 1)r}{b(r)}.$$  

A particular solution of Eq.(12) is

$$U_1(r) = \frac{8(\delta + 1)(r + B)^{2+\delta} - \delta(\delta - 1)(\delta - 2)A^\delta r^2 + 4\delta(\delta - 1)A^\delta B r - 2\delta A^\delta B^2}{8(\delta + 1)(r + B)^{2+\delta}}.$$  

Using Eq.(5) for $\Phi$ and introducing the ratio $\gamma \equiv B/A$, we have

$$U_1(\Phi) = \frac{\delta [\Phi^2(-\delta(\gamma \Phi^{2/\delta} - 1)(\gamma(\delta + 1)\Phi^{2/\delta} - \delta + 3) - 2) + 8]}{8(\delta + 1)}.$$  

In the following we will discuss various solutions for the potential $U(r)$.
A. AdS spacetimes with a stealth structure

If we choose $U(r) = C_1 U_1(r)$ as the potential, then the cosmological constant is simply

$$\Lambda = U(\Phi = 0) = C_1 ,$$

and the metric function becomes

$$f(r) = (r + B)^{1+\delta} \left[ a(r)C_1 U_1(r) + b(r)C_1 U_1'(r) \right] \Bigg/ \left( A^\delta [\delta r - (r + B)] + 4(1+\delta)(r + B)^{1+\delta} \right) = -r^2 C_1 = -r^2 \Lambda ,$$

with constant curvature $R(r) \equiv 6C_1 = 6\Lambda$. It describes AdS spacetimes when $\Lambda = -\ell^{-2} < 0$, where $\ell$ is the length of the AdS space. In fact, this solution has what is known in the literature as a “stealth structure” – this kind of matter configurations have no influence on the geometry. Black holes with stealth scalar field in (2+1)-dimensions have been studied in [40], their potential, which is obtained from the vanishing of energy-momentum, is the same as ours.

B. General AdS black hole solution

Using Liouville’s formula, we can obtain the other linearly independent particular solution to Eq.(12):

$$U_2(r) = U_1(r) \int U_1(r)^{-2} e^{\int P(r) dr} dr .$$

We define

$$P(r) := e^{\int P(r) dr} = \frac{B(1-\delta)}{2r(B+r)^{\delta+3}} + \frac{A^\delta \delta^3 (\delta + 1) [(\delta - 3)r - 4B]}{(B+r)^4 \left[ A^\delta - 4(\delta + 1)(B + r)^\delta \right]^2} \frac{\delta(\delta + 1)(-4B + \delta r + r)}{r(B+r)^3 \left[ 4(\delta + 1)(B + r)^\delta - A^\delta \right]} ,$$

so that

$$U_2(r) = U_1(r) \int U_1(r)^{-2} P(r) dr .$$

The general solution of Eq.(12) is

$$U(r) = C_1 U_1(r) + C_2 U_2(r) ,$$

where $C_1$ and $C_2$ are constants. The integral constant coming from $P(r)$ can be absorbed into $C_2$.

Since the integral in $U_2(r)$ cannot be integrated analytically, special attention should be given to the continuity of the potential and the metric function for further analysis.
For $-1 < \delta < 0$, $\mathbb{P}(r)$ and $U_1(r)^{-2}$ are both continuous. The potential $U(r)$ is also continuous because $U_1(r)$ is continuous. For other cases, namely $\delta > 0$ and $\delta < -1$, the continuity conditions for $\mathbb{P}(r)$ and $U_1(r)^{-2}$ are identical:

$$\delta A^\delta \leq 4B^\delta (1+\delta) \quad \text{i.e.} \quad 4(1+\delta)\gamma^\delta \geq \delta. \quad (22)$$

From equation (6), we can see that if the potential is continuous then so is the metric function. Therefore the continuity condition (22) can ensure the continuity of the potential and the metric.

The potential at infinity plays the role of a cosmological constant

$$\Lambda = U(\Phi = 0) = U(r \to \infty) = C_1 + C_2u_2, \quad (23)$$

where $u_2 = U_2(r \to \infty)$ and $U_1(r \to \infty) = 1$. We can analyze the asymptotic behaviors of the metric function at leading order

$$f(r \to \infty) = -\left[ C_1 + \frac{2C_2u_2(2\delta - 1)}{\delta(1+\delta)} \right]r^2, \quad (24)$$

$$f(r \to 0) = \begin{cases} 
\frac{4C_2(1+\delta)\gamma^\delta B^2[-\delta+4(1+\delta)\gamma^\delta]}{\delta^2[(\delta-1)\delta+4(1+\delta)^2\gamma]} & , 4(1+\delta)\gamma^\delta > \delta \\
C_2B^{1-\delta} & , 4(1+\delta)\gamma^\delta = \delta 
\end{cases} \quad (25)$$

Under the condition $4(1+\delta)\gamma^\delta \geq \delta$, the sign of $f(r \to 0)$ is easy to obtain: $\text{sgn} (f(r \to 0)) = \text{sgn}(C_2)$.

For $\delta = 1$, we have $f(r \to \infty) \propto -\Lambda r^2$, and then we recover the BTZ black hole [6]. However for $\delta \neq 1$, the metric function is not proportional to $\Lambda$. Nevertheless, any constant that arises from the indefinite integral in $U_2(r)$ can be absorbed into $C_1$ and $C_2$. Considering that $C_1$ and $C_2$ are free parameters, $u_2$ can be set to zero and then $\Lambda = C_1$ and $f(r \to \infty) \propto -\Lambda r^2$.

Since the integral in $U_2(r)$ cannot be solved while all parameters being free, we need to fix some parameters and confirm our solution by consistency check. We set $\delta = 1$, $A = 1/q$, $B = 1/(8q)$, $C_1 = -\frac{1}{q^2}$ and $C_2 = 6144\alpha q$ where the additional free parameter $q$ characterizes the strength of the scalar field. We find that the potential and metric function become

$$U(r) = -\frac{813qr + 24q^2r^2 + 64q^3r^3 + 32\alpha \ell^2}{\ell^2 (1+8qr)^3}, \quad (26)$$

$$U(\Phi) = -\frac{1}{\ell^2} + \left( \frac{1}{512\ell^2} - \frac{\alpha}{2} \right) \Phi^6, \quad (27)$$

$$f(r) = \frac{r^2}{\ell^2} - \frac{12\alpha}{q^2} - \frac{\alpha}{q^3}r, \quad (28)$$

which can reduce to the static limit of the black hole solution in [35]. This potential form has been widely studied in static exact black hole solutions dressed with non-minimally coupled scalar field [29, 35, 41].
Let us now discuss the general solution. When $C_1 < 0$ and $C_2 < 0$, from relations (24) and (25), $f(0) < 0$ and $f(\infty) > 0$, so there must be a zero in the metric function, which corresponds to the event horizon of a black hole. Note that what really has physical meaning is the coupling constant $\xi$ rather than $\delta$, so we first plot the relation $\xi(\delta) = \frac{\delta}{4(1+\delta)}$ in Fig. 1.

![Graph of $\xi(\delta)$](image1)

**FIG. 1:** The coupling constant $\xi$ always rises with $\delta$. Note that $\delta \neq -1$ and $\xi \neq 1/4$, the former corresponds to $\xi \neq \pm \infty$.

From the figure we can see that positive $\delta$ corresponds to $0 < \xi < 1/8$, $\delta < -1$ corresponds to $\xi > 1/8$ and $-1 < \delta < 0$ corresponds to negative $\xi$. We plot the potential and metric functions with different ranges of $\delta$ respectively in Fig. 2, Fig. 3, Fig. 4 and Fig. 5, in all of which we have fixed the other parameters as: $A = 1$, $B = 2$, $\Lambda = C_1 = -1$ and $C_2 = -10$. Table I shows the influence of the coupling constant $\xi$ and $\delta$.

![Graphs of potential and metric functions](image2)

**FIG. 2:** The black curve, black dashed curve, pink curve and pink dashed curve correspond to $\delta = 1, 2, 3, 4$, i.e. $\xi = 1/8, 1/6, 3/16, 1/5$, respectively. Large coupling constant $\xi$ corresponds to black holes with small event horizon when $1/8 \leq \xi < 1/4$. 
FIG. 3: The black curve, black dashed curve, pink curve and pink dashed curve correspond to $\delta = 1, 1/2, 1/3, 1/4$, i.e. $\xi = 1/8, 1/12, 1/16, 1/20$, respectively. Large coupling constant $\xi$ corresponds to black holes with small event horizon when $0 < \xi \leq 1/8$.

FIG. 4: The black dashed curve, pink curve and pink dashed curve correspond to $\delta = -2, -3, -4$, i.e., $\xi = 1/2, 3/8, 1/3$ respectively. Large coupling constant $\xi$ corresponds to black holes with large event horizon when $\xi > 1/4$.

From the figures we can see that for $0 < \xi < 1/4$, large coupling constants $\xi$ correspond to black holes with small radius of event horizons while for $\xi < 0$ and $\xi > 1/4$, large coupling constants $\xi$ correspond to large radius of event horizons. (For $\xi < 0$, to be larger means its absolute value is

| TABLE I: The influence of the coupling constant $\xi$ and $\delta$ |
|---------------------------------------------------------------|
| $0 < \delta \leq 1$ | $\delta \geq 1$ | $-1 < \delta < 1$ | $\delta < -1$ |
| $0 < \xi \leq 1/8$ | $1/8 \leq \xi < 1/4$ | $\xi < 0$ | $\xi > 1/4$ |

Large $\xi$ leads to small event horizon. Large $\xi$ leads to large event horizon. Large absolute values of $\delta$ correspond to small radius of event horizon.
FIG. 5: The black dashed curve, pink curve and pink dashed curve correspond to \(\delta = -1/2, -1/3, -1/4\), i.e. \(\xi = -1/4, -1/8, -1/12\) respectively. Large coupling constant \(\xi\) corresponds to black holes with large event horizon when \(\xi < 0\).

Also note that this change of behavior corresponds to the sign reversal of \(\delta\), so there is a consistent influence from \(\delta\): large absolute values of \(\delta\) correspond to black holes with small radius of event horizon, and the gradient of self-potential of the scalar field becomes steeper at the origin. Finally, the conformal case when \(\delta \to \pm \infty\), corresponds to \(\xi = 1/8\). Note that in all of our plots, we have fixed \(B = 2\) and \(A = 1\), i.e., \(\gamma = 2\). With this value of \(\gamma\), arbitrary value of \(\delta\) can satisfy the continuity condition \(4(1 + \delta)\gamma^\delta \geq \delta\).

Lastly we check for the curvature singularity of the solutions. The Ricci scalar and the Kretschmann scalar are given by

\[
R(r) = -f''(r) - \frac{2f'(r)}{r}, \quad \text{and} \quad K(r) = f''(r)^2 + \frac{2f'(r)^2}{r^d},
\]

respectively. We note that

\[
f'(r \to 0) = \begin{cases} 
-\frac{24(C_2(2\delta+1)^2 B^{\delta-1} \ln r)}{(\delta A^\delta - 4(\delta+1) B^\delta)^2}, & 4(1 + \delta)\gamma^\delta > \delta \\
-C_2 B^{1-\delta}, & 4(1 + \delta)\gamma^\delta = \delta 
\end{cases}
\]

\[
f''(r \to 0) = \begin{cases} 
-\frac{24C_2(2\delta+1)^2 B^{\delta-1}}{r(\delta A^\delta - 4(\delta+1) B^\delta)^2}, & 4(1 + \delta)\gamma^\delta > \delta \\
2C_2 B^{1-\delta}, & 4(1 + \delta)\gamma^\delta = \delta 
\end{cases}
\]

Since \(f'(r \to 0) \to \infty\) and \(f''(r \to 0) \to \infty\), there must be a curvature singularity at \(r = 0\).

III. SPECIAL EXACT SOLUTIONS

Since the integral in \(U_2(r)\) cannot be solved with \(\delta\) unspecified, let us fix \(\delta\) to find some special solutions. In fact, for \(\delta = n, -n, 1/n, -1/n\) with \(n = 1, 2, 3, \ldots\), we can always – in principle – obtain...
the corresponding exact black hole solutions, however this becomes more complicated with increasing \( n \). So we only present the solutions with \( \delta = 1, \delta = 2, \delta = -2, \delta = 1/2 \) and \( \delta = -1/2 \) to illustrate the different properties of spacetimes for \( \delta \geq 1, \delta < -1, -1 < \delta < 0 \) and \( 0 < \delta < 1 \). Besides, the solutions can be greatly simplified when we saturate the inequality and set \( 4(1 + \delta)\gamma^\delta = \delta \) in the continuity condition (22). Note that in all these solutions we impose the same requirement that

\[
\lim_{r \to \infty} \int U_1(r)^{-2} \mathcal{P}(r) dr = 0 ,
\]

which has the same effect as setting \( u_2 = 0 \) when \( \delta \) is positive. We use this choice because Eq.(23) for \( \Lambda \) is no longer suitable for negative \( \delta \). According to the definition of \( u_2 \) we have

\[
u_2 = U_2(r \to \infty) = U_1(r \to \infty) \lim_{r \to \infty} \int U_1(r)^{-2} \mathcal{P}(r) dr ,
\]

where

\[
U_1(r \to \infty) = \begin{cases} 1, & \delta > 0 \\ -\frac{\delta(\delta^2 - 3\delta + 2)A^\delta r^{-\delta}}{8(\delta + 1)}, & \delta < -1 \text{ or } -1 < \delta < 0 . \end{cases}
\]

Therefore for positive \( \delta \), the choice Eq.(32) can lead to \( u_2 = 0 \) while for negative \( \delta \), \( u_2 \) approaches infinity, not part of the cosmological constant as Eq.(23) shows.

### A. \( \delta = 1 \)

When \( \delta = 1 \) the scalar field, the metric and the potential functions are

\[
\Phi(r) = A(r + B)^{-1/2} ,
\]

\[
f(r) = \frac{8A^2C_2(B + 2r) - A^2C_1r^2(A^4 - 16A^2B + 64B^2) - 64B^2C_2}{A^2(A^2 - 8B)^2}
\]

\[
+ \frac{128A^2C_2r^2}{A^2(A^2 - 8B)^3} \ln \frac{r}{8(B + r) - A^2} ,
\]

\[
U(r) = C_1U_1(r) + C_2U_2(r) ,
\]

\[
U_1(r) = \frac{(8(r + B)^3 - A^2B^2)}{8(r + B)^3} ,
\]

\[
U_2(r) = \frac{384\ln 2}{(A^2 - 8B)^3} + \frac{8(B + r)^3 - A^2B^2}{(A^2 - 8B)^3(B + r)^3} \left\{ \frac{16\ln r}{8(B + r) - A^2} + \frac{8A^2(A^2 - 8B)}{(A^2 + 8B)(A^2 - 8(B + r))}
\right.
\]

\[
+ \frac{(A^2 - 8B)(A^2B + 8A^2(2B + r)^2 + 64B(A^2 + 5Br + 2r^2))}{(A^2 + 8B)(A^2B^2 - 8(B + r)^3)} \right\} ,
\]

which agree with the black hole solution presented in [42] except for a clerical error in that paper.

The continuity condition becomes \( A \leq 8B \). In fact this solution only describes the case when \( A < 8B \). If we choose \( A = 8B \), then the solution reduces to the solution [35] as mentioned before.
B. $\delta = 2$

We can obtain a new exact black hole solution with $\delta = 2$, with

\[
\Phi(r) = \frac{A}{r + B}, \quad (40)
\]

\[
f(r) = \frac{1}{A^2 (A^2 - 6B^2)^3} \left\{ - (A^2 - 6B^2) \left[ A^6 C_1 r^2 - 12 A^4 B^2 C_1 r^2 + 9A^2 B (4B^3 C_1 r^2 + BC_2 + 4C_2 r) \right] \\
- 54 B^4 C_2 \right \} + 9A^2 C_2 r^2 (A^2 + 18B^2) \ln \frac{6r^2}{6(B+r)^2 - A^2} \\
+ 27 \sqrt{6} A B C_2 r^2 (A^2 + 2B^2) \ln \frac{\sqrt{6}(B+r)+A}{\sqrt{6}(B+r)-A} , \quad (41)
\]

\[
U(r) = C_1 U_1(r) + C_2 U_2(r) , \quad (42)
\]

\[
U_1(r) = 1 - \frac{A^2 B (B - 2r)}{6(B+r)^4} , \quad (43)
\]

\[
U_2(r) = \frac{18(B+r)^4 - 3A^2 B (B - 2r)}{2(A^2 - 6B^2)^3 (B+r)^4} \left\{ - 2 \left( A^2 - 6B^2 \right) \left[ A^4 - 6A^2 B (B+r) - 108B^3 (B+r) \right] \right \} \\
\frac{A^2 - 6B^2}{(A^2 - 54B^2) (A^2 B (B-2r) - 6(B+r)^4)} [A^4 (6B^2 + 2Br - r^2) \\
+ 12 A^2 B (-24B^3 + 8B^2 r + 11Br^2 + 3r^3) - 108B^3 (34B^3 + 54B^2 r + 39Br^2 + 10r^3)] \\
- (A^2 + 18B^2) \ln \frac{r^2}{(B+r)^2 - A^2/6} - \frac{3\sqrt{6}B}{A} \left( A^2 + 2B^2 \right) \ln \frac{\sqrt{6}(B+r)+A}{\sqrt{6}(B+r)-A} \right \} , \quad (44)
\]

where $A^2 < 6B^2$ agrees with our continuity condition (22). When $A^2 = 6B^2$, the solution can be simplified as

\[
\Phi(r) = \frac{\sqrt{6}B}{B+r} , \quad (45)
\]

\[
f(r) = \frac{C_2 r^2}{32B^4} \ln \frac{r}{2B+r} + \frac{C_2 r}{16B^3} + \frac{3C_2}{16B^2} + \frac{C_2}{12Br} - C_1 r^2 , \quad (46)
\]

\[
U(r) = C_1 U_1(r) + C_2 U_2(r) , \quad (47)
\]

\[
U_1(r) = \frac{r (6B^3 + 6B^2 r + 4Br^2 + r^3)}{(B+r)^4} , \quad (48)
\]

\[
U_2(r) = -\frac{-10B^4 + 12B^3 r + 28B^2 r^2 + 15Br^3 + 3r^4}{48B^3 (B+r)^4 (2B+r)} \left\{ - \frac{r (6B^3 + 6B^2 r + 4Br^2 + r^3)}{32B^4 (B+r)^4} \ln \frac{r}{2B+r} . \quad (49)
\]
C. $\delta = -2$

For negative $\delta$, we can obtain an exact solution when $\delta = -2$, with

$$\Phi(r) = \frac{r + B}{A},$$

$$f(r) = \frac{1}{2(2A^2 - B^2)^3} \left\{-2 \left(2A^2 - B^2\right) \left(2A^6 C_2 + A^4 \left(-B^2 C_2 + 4BC_2 r + 4C_1 r^2\right) - 4A^2 B^2 C_1 r^2 + B^4 C_1 r^2\right) + 2A^4 C_2 r^2 \left(2A^2 + 3B^2\right) \ln \frac{r^2}{(B + r)^2 - 2A^2} + \sqrt{2} A^3 B C_2 r^2 \left(6A^2 + B^2\right) \ln \frac{\sqrt{2}(B + r) - 2A}{\sqrt{2}(B + r) - 2A}\right\},$$

$$U(r) = C_1 U_1(r) + C_2 U_2(r),$$

$$U_1(r) = -\frac{2A^2 + B^2 + 6Br + 6r^2}{2A^2},$$

$$U_2(r) = \frac{-2A^2 + B^2 + 6Br + 6r^2}{2A^2 (2A^2 - B^2)^3} \left\{\frac{2 \left(2A^2 - B^2\right) \left(20A^8 + 2A^6 B(18B + 11r) + A^4 B^3 (B + r)\right)}{(50A^2 - B^2) (2A^2 - (B + r)^2)} - \frac{18 \left(2A^2 - B^2\right) \left(20A^8 + 4A^6 B(10B + 13r) - A^4 B^3 (B + 2r)\right)}{(50A^2 - B^2) (-2A^2 + B^2 + 6Br + 6r^2)} + A^4 \left(2A^2 + 3B^2\right) \ln \frac{r^2}{(B + r)^2 - 2A^2} + \frac{A^3 B \left(6A^2 + B^2\right)}{\sqrt{2}} \ln \frac{2A + \sqrt{2}(B + r)}{\sqrt{2}(B + r) - 2A}\right\},$$

which is the solution for $2A^2 < B^2$.

When $A^2 = B^2/2$ we have

$$\Phi(r) = \frac{\sqrt{2}(B + r)}{B},$$

$$f(r) = \frac{8B^3 C_2 - 6B^2 C_2 r + 3C_2 r^3 \ln \frac{r}{2B + r} + 6BC_2 r^2 - 96C_1 r^3}{96r},$$

$$U(r) = C_1 U_1(r) + C_2 U_2(r),$$

$$U_1(r) = -\frac{6r(B + r)}{B^2},$$

$$U_2(r) = \frac{2B \left(B^2 + 6Br + 3r^2\right) + 3r \left(2B^2 + 3Br + r^2\right) \ln \frac{r}{2B + r}}{16B^2 (2B + r)}.$$
We can also obtain an exact black hole solution with $\delta = 1/2$, with
\[
\Phi(r) = \left(\frac{A}{r + B}\right)^{1/4},
\]
\[
f(r) = \frac{1}{\sqrt{AB^{3/2}(A - 2^{4/3}B)^3}} \left\{ \sqrt{B}(2^{4/3}2B - A) \left[ 72A^{3/2} \left( -4B^2C_1r^2 + C_2r\sqrt{B + r} + 2BC_2\sqrt{B + r} \right) \right. \\
+ A^{5/2}BC_1r^2 + 2^{7/4}\sqrt{AB}\left( 2B^2C_1r^2 + 3C_2r\sqrt{B + r} - 2BC_2\sqrt{B + r} \right) + 2^63^3ABC_2(2B + 2r) \right. \\
- 2^103^5B^3C_2 \right\} + 36 \left[ \sqrt{A}C_2r^2 \left( A^2 - 2^53^3AB - 2^83^5B^2 \right) \ln \frac{2\sqrt{B}\sqrt{B + r} + 2B + r}{r} \\
+ 2^93^3AB^{3/2}C_2r^2 \ln \frac{r}{2^14^2(B + r) - A} + 2^93^3AB^{3/2}C_2r^2 \ln \frac{2^14^2\left( \sqrt{A} + 12\sqrt{B + r} \right)}{12\sqrt{B + r} - \sqrt{A}} \right\},
\]
\[
U(r) = C_1U_1(r) + C_2U_2(r),
\]
\[
U_1(r) = 1 - \frac{\sqrt{A}(8B^2 + 8Br + 3r^2)}{96(B + r)^{5/2}},
\]
\[
U_2(r) = \left\{ 18\sqrt{B} \left[ 144A^{5/2}(2B + r)(7B + r)\sqrt{B + r} - A^{7/2}(2B + r)\sqrt{B + r} + 8A^3 \left( 9B^2 + 10Br + 4r^2 \right) \right. \\
+ 2^{15}3^6B^2(B + r) \left( 19B^2 + 28Br + 12r^2 \right) - 2^83^3A^{3/2}B\sqrt{B + r} \left[ 66B^2 + 77Br + 64(B + r)^2 \ln 12 \\
+ 26r^2 \right] + 2^{12}3^5\sqrt{AB}(B + r)^{3/2} \left[ B(30B + 23r) + 64(B + r)^2 \ln 12 \right] \\
+ 2^73^2A^2 \left[ B^3 + 4B \left( 8B^2 + 8Br + 3r^2 \right) \ln 12 - 5B^2r - 10Br^2 - 4r^3 \right] \\
- 2^{11}3^4AB \left( B^3(29 + 12\ln 12) + 4B^2r(13 + 16\ln 12) + 2Br^2(17 + 22\ln 12) + 4r^3(2 + 3\ln 12) \right) \right] \\
+ 3(A - 144(B + r)) \left( \sqrt{A} \left( 8B^2 + 8Br + 3r^2 \right) - 96(B + r)^{5/2} \right) \left[ 13824\sqrt{AB}^{3/2} \\
\ln \frac{r}{144(B + r) - A} + \ln \frac{\sqrt{A} + 12\sqrt{B + r}}{12\sqrt{B + r} - \sqrt{A}} + (A^2 - 864AB - 62208B^2) \right. \\
\ln \frac{2\sqrt{B}\sqrt{B + r} + 2B + r}{r} \right\} \left[ 8B^{3/2}(144B - A)^3(B + r)^{5/2}(144(B + r) - A) \right],
\]
where $\sqrt{A} < 12\sqrt{B}$ agrees with the continuity condition (22).
For $\sqrt{A} = 12\sqrt{B}$ the solution becomes much simpler

\begin{align}
\Phi(r) &= 2\sqrt{3} \left( \frac{B}{B + r} \right)^{1/4}, \\
f(r) &= \frac{C_{2r}^2 \ln \left( \frac{2\sqrt{B} \sqrt{B + r} + 2B + r}{8B^{5/2}} \right)}{8B^{5/2}} + \left( \frac{1}{6B} - \frac{r}{4B^2} + \frac{2}{3r} \right) C_2 \sqrt{B + r} + \frac{C_2}{\sqrt{B}} + \frac{2\sqrt{BC_2}}{3r} - C_1r^2, \\
U(r) &= C_1U_1(r) + C_2U_2(r), \\
U_1(r) &= \frac{-8B^{3/2}r - 8B^{5/2} + 8B^2\sqrt{B + r} - 3\sqrt{B}r^2 + 8r^2\sqrt{B + r} + 16Br\sqrt{B + r}}{8(B + r)^{5/2}}, \\
U_2(r) &= U_1(r) \left[ -\frac{1}{8B^{5/2}} \ln \frac{\sqrt{B + r} + \sqrt{B}}{\sqrt{B + r} - \sqrt{B}} + \left( 124B^{5/2}r^2 + 296B^{7/2}r + 208B^{9/2} + 208B^4\sqrt{B + r} + 936B^3r\sqrt{B + r} + 1314B^2\sqrt{B + r} + 192B^4\sqrt{B + r} + 805Br^3\sqrt{B + r} \right) \right] / \left[ 12B^2r \right].
\end{align}

E. $\delta = -1/2$

To show the effects for negative $\xi$ we present the solution with $\delta = -1/2$ as well,

\begin{align}
\Phi(r) &= \left( \frac{r + B}{A} \right)^{1/4}, \\
f(r) &= \frac{1}{B^{3/2}(B - 16A)^3} \left\{ -\sqrt{B}B - 16A \left( 64A^{3/2}BC_2(B - 2r) - 1024A^{5/2}BC_2 - 8AB \right. \\
&\left. \left( 4BC_1r^2 - 3C_2r\sqrt{B + r} + 2BC_2\sqrt{B + r} \right) + 128A^2 \left( 2B \left( C_2\sqrt{B + r} + C_1r^2 \right) + C_2r\sqrt{B + r} \right) \right) \right. \\
&\left. + B^3C_1r^2 + 128A^{3/2}BC^{3/2}C_2r^2 \ln \frac{r}{8\sqrt{A\sqrt{B + r} + 16A + B + r}} \right. \\
&\left. + 4AC_2r^2 \left( -256A^2 + 96AB + 3B^2 \right) \ln \frac{2\sqrt{B}\sqrt{B + r} + 2B + r}{r} \right\}, \\
U(r) &= C_1U_1(r) + C_2U_2(r), \\
U_1(r) &= \frac{8B^2 + 24Br + 15r^2}{32\sqrt{A}(B + r)^{3/2}} + 1, \\
U_2(r) &= \sqrt{A} \left\{ 2(16A - B)\sqrt{B} \left[ 2048A^{5/2}(5B + 4r) + 256A^2(14B + 15r)\sqrt{B + r} - 3B(B + r)^{3/2}(14B + 15r) \right. \\
&\left. - 128A^{3/2} \left( 22B^2 + 23Br + 4r^2 \right) + 16A \left( 28B^2 + 8Br - 15r^2 \right) \sqrt{B + r} \right. \\
&\left. + 8\sqrt{A}(B + r)(17B + 18r) \right] - (-16A + B + r) \left[ 8B \left( 4\sqrt{A}\sqrt{B + r} + 3r \right) \right. \\
&\left. + r \left( 32\sqrt{A}\sqrt{B + r} + 15r \right) + 8B^2 \right] \left( -256A^2 + 96AB + 3B^2 \right) \ln \frac{2\sqrt{B}\sqrt{B + r} + 2B + r}{r} \\
&\left. + 32\sqrt{A}B^{3/2} \ln \frac{r}{8\sqrt{A\sqrt{B + r} + 16A + B + r}} \right\} / \left( 8B^{3/2}(B - 16A)^3(B + r)^{3/2}(-16A + B + r) \right).
\end{align}
where $16A \neq B$. When $-\delta A^\delta = 4B^\delta(1 + \delta)$, i.e. $16A = B$, the solution is simpler:

$$
\Phi(r) = 2 \left( \frac{B}{B + r} \right)^{-1/4},
$$

$$
f(r) = -\frac{C_2}{12r} \left( 8B^{3/2} + \left( \frac{3r}{B} - 8B - 2r \right) \sqrt{B + r} \right) + \frac{C_2r^2 \ln \left( \frac{2\sqrt{B + r} + 2B + r}{r} \right)}{8B^{3/2}} - C_1 r^2,
$$

$$
U_1(r) = \frac{16B^{3/2}r + 8B^{5/2} + 8B^2 \sqrt{B + r} + 8\sqrt{Br}^2 + 15r^2 \sqrt{B + r} + 24Br \sqrt{B + r}}{8\sqrt{B}(B + r)^2},
$$

$$
U_2(r) = U_1(r) \left[ -\frac{1}{8B^{3/2}} \ln \frac{\sqrt{B + r} + \sqrt{B}}{\sqrt{B + r} - \sqrt{B}} + \left( \sqrt{B + r} \left( 80B^3 + 920B^2r + 1518Br^2 + 675r^3 \right) - 72B^{5/2}r - 80B^{7/2} \right) / (12Br \left( 192B^3 + 624B^2r + 656Br^2 + 225r^3 \right)) \right].
$$

IV. DE SITTER BLACK HOLE SOLUTIONS IN (2+1)-DIMENSIONS

To study the possibility of de Sitter black hole, we study the zeroes of the metric function. The equivalent equation for $f(r_h) = 0$ is $a(r_h)U(r_h) + b(r_h)U'(r_h) = 0$, where the roots $r_h$ correspond to the horizons. After substituting in the concrete expressions we obtain a rather complicated equation

$$
R(r_h) \equiv \int^{r_h} U_1(r)^{-2} \Phi(r) dr - \left\{ 16C_2(\delta + 1)^2 r_h^2 \left( B + r_h \right)^\delta \left( A^\delta(\delta - 1)\delta + 4(\delta + 1)(B + r_h)^\delta \right) + 2B^2 \left( 4(\delta + 1)(B + r_h)^\delta - A^\delta \right) \left[ A^\delta C_1^2 \delta^2 r_h^2 \left( A^\delta - 4(\delta + 1)(B + r_h)^\delta \right) + 8C_2(\delta + 1)^2 (B + r_h)^\delta \right] 
- A^\delta C_1 \delta^2 r_h^2 \left( A^\delta(\delta - 3\delta + 2) - 4A^\delta(\delta^3 - 2\delta^2 + \delta + 4)(B + r_h)^\delta + 32(\delta + 1)^2 (B + r_h)^\delta \right) 
- 4Br_h \left[ A^\delta C_1^2 \delta^2 r_h^2 \left( A^\delta(\delta - 1\delta^2 + 4A^\delta(\delta^2 - \delta - 2)(B + r_h)^\delta + 16(\delta + 1)^2 (B + r_h)^\delta \right) 
- 4C_2(\delta + 1)^2 (B + r_h)^\delta \left( A^\delta(\delta - 2)\delta + 8(\delta + 1)(B + r_h)^\delta \right) \right] \right\} \left\{ A^\delta C_2^2 \delta^2 r_h^2 \left( 4(\delta + 1)(B + r_h)^\delta - A^\delta \right) \right\} 
\left[ B^2 \left( 8(\delta + 1)(B + r_h)^\delta - 2A^\delta \delta + r_h^2 \left( 8(\delta + 1)(B + r_h)^\delta - A^\delta(\delta^2 - 3\delta + 2) \right) 
+ 4Br_h \left( A^\delta(\delta - 1\delta^2 + 4(\delta + 1)(B + r_h)^\delta \right) \right) \right] = 0.
$$

where we have defined the root function $R(r_h)$. It monotonically increases for positive $r_h$:

$$
R'(r_h) = \frac{16(\delta + 1)^2(B + r_h)^\delta}{A^\delta C_2^2 \delta^2 r_h^2 \left( 4(\delta + 1)(B + r_h)^\delta - A^\delta \right)} > 0, \quad R(\infty) = \frac{C_1}{C_2}.
$$

For asymptotically dS spacetimes, $\Lambda = C_1 > 0$. If $C_2 < 0$, then $R(\infty) < 0$ and $R(r_h)$ is always negative and there is no horizon. When $C_2 > 0$, there is only one horizon, representing the cosmological horizon in pure de Sitter spacetime without any black hole.

Negative $\delta$ will not change this discussion; the sign of $R'(r_h)$ is fixed only if the continuity condition is fixed. If we consider negative $A$ and $B$, the scalar field is restricted to the region
\[ 0 \leq r < -B. \] Because of the continuity condition \(4(1 + \delta)\gamma^\delta \geq \delta,\) the function \(R(r_h)\) is divided into two monotonic pieces by the line \(r_h = -B.\) For \(0 \leq r < -B,\) there is at most one horizon, but no asymptotically de Sitter black hole. If the continuity condition is not valid then there are three turning points

\[ r_h = \pm \left[ \frac{\delta}{4(\delta + 1)} \right]^{1/\delta} - B, r_h = -B. \tag{81} \]

The function \(R(r_h)\) is still monotonic in the region without divergence but there is still no dS black hole.

On the other hand, an asymptotically flat spacetime requires \(\Lambda = C_1 = 0, U(r) = C_2 U_2(r)\) and the metric function becomes

\[ f(r) = \frac{C_2 (r + B)^{1+\delta} [a(r)U_2(r) + b(r)U_2'(r)]}{A^\delta \delta [\delta r - (r + B)] + 4(1 + \delta)(r + B)^{1+\delta}}. \tag{82} \]

At infinity we have

\[ U(r \to \infty) = -\frac{C_2}{(\delta + 2)r^{\delta + 2}}, \tag{83} \]

\[ f_\infty \equiv f(r \to \infty) = \frac{2C_2(\delta + 1)}{A^\delta \delta^2}, \tag{84} \]

which imply \(\text{sgn}(f(r \to 0)) = \text{sgn}(C_2) = \text{sgn}(f_\infty).\)

In this case, the function \(R(r_h)\) becomes simpler

\[
R(r_h) = \int_{r_h}^{\infty} U_1(r)^{-2} IP(r) dr - 16(\delta + 1)^2 (B + r_h)^{\delta + 1} \left[-A^\delta B\delta + A^\delta (\delta - 1)\delta r_h \right.
\]

\[ +4B(\delta + 1) (B + r_h)^{\delta} + 4(\delta + 1)r_h (B + r_h)^{\delta} \bigg]\left/ \bigg\{ A^\delta \delta^2 r_h^2 \left(4(\delta + 1)(B + r_h)^{\delta} - A^\delta \delta \right) \right.
\]

\[ \left. \left[B^2 \left(8(\delta + 1)(B + r_h)^{\delta} - 2A^\delta \delta \right) + r_h^2 \left(8(\delta + 1)(B + r_h)^{\delta} - A^\delta \delta \left(\delta^2 - 3\delta + 2 \right) \right) \right] \bigg) \right\}, \tag{85} \]

whose derivative is the same as Eq.(80),

\[ R'(r_h) = \frac{16(\delta + 1)^2(B + r_h)^{\delta}}{A^\delta \delta^2 r_h^3 \left(4(\delta + 1)(B + r_h)^{\delta} - A^\delta \delta \right)} > 0, \quad R(\infty) = 0, \tag{86} \]

We observe that we have neither a cosmological horizon nor a black hole horizon. When \(C_2 > 0\) the spacetime is asymptotically flat without black hole solutions.

Therefore, we conclude that there is only asymptotically AdS \((2+1)-dimensional black hole in non-minimally coupled theory without electromagnetic field, at least under our metric ansatz assumption.
A. Charged de Sitter black hole solution

In [36], a charged scalar black hole solution is obtained, with
\[
\phi(r) = \pm \sqrt{\frac{8B}{r + B}}, \tag{87}
\]
\[
f(r) = \left(3\beta - \frac{Q^2}{4}\right) + \left(2\beta - \frac{Q^2}{9}\right) \frac{B}{r} - Q^2 \left(\frac{1}{2} + \frac{B}{3r}\right) \ln r + \frac{r^2}{\ell^2}, \tag{88}
\]
\[
V(\phi) = -\frac{1}{\ell^2} + \frac{1}{512} \left(1 - \frac{\beta}{B^2}\right) \phi^6 - \frac{Q^2}{18432B^2} (192\phi^2 + 48\phi^4 + 5\phi^6) \\
+ \frac{Q^2}{3B^2} \left[\frac{2\phi^2}{(8 - \phi^2)^2} - \frac{1}{1024} \phi^6 \ln B \left(\frac{8 - \phi^2}{\phi^2}\right)\right]. \tag{89}
\]

We will discuss the possibility of obtaining asymptotically de Sitter black hole solution from this metric. Here \(\Lambda = -\ell^{-2}\) appears in \(V(\phi)\) as a constant term, which plays the role of the cosmological constant. In [36] only negative \(\Lambda\) is considered, and so \(\Lambda\) is set to be \(-\ell^{-2}\). However, the constant \(\Lambda\) can either be positive, zero or negative. So we rewrite their solution with general \(\Lambda\),
\[
\phi(r) = \pm \sqrt{\frac{8B}{r + B}}, \tag{90}
\]
\[
f(r) = \left(3\beta - \frac{Q^2}{4}\right) + \left(2\beta - \frac{Q^2}{9}\right) \frac{B}{r} - Q^2 \left(\frac{1}{2} + \frac{B}{3r}\right) \ln r + \Lambda r^2, \tag{91}
\]
\[
V(\phi) = \Lambda + \frac{1}{512} \left(-\Lambda + \frac{\beta}{B^2}\right) \phi^6 - \frac{Q^2}{18432B^2} (192\phi^2 + 48\phi^4 + 5\phi^6) \\
+ \frac{Q^2}{3B^2} \left[\frac{2\phi^2}{(8 - \phi^2)^2} - \frac{1}{1024} \phi^6 \ln B \left(\frac{8 - \phi^2}{\phi^2}\right)\right]. \tag{92}
\]

If \(B > 0\), then there is no black hole for non-negative \(\Lambda\). However, once \(B < 0\), it is possible to have a black hole solution with positive \(\Lambda\), which is a de Sitter black hole solution. The scalar field and the metric functions are shown in Fig. 6. Note that the scalar field is divergent at \(r = -B = 5\), but this does not matter because it lies outside the cosmological horizon.

To check the stability of the spacetime, we study the perturbations of a massless scalar field \(\phi_1\) and a massive scalar field \(\phi_2\) respectively,
\[
\Box \phi_1 = 0, \tag{93}
\]
\[
\Box \phi_2 = m^2 \phi_2, \tag{94}
\]
where \(m\) is the mass of the scalar field \(\phi_2\). The transformations \(\phi_1 = r^{-1/2} \varphi_1 e^{-i\omega_1 t}\) and \(\phi_2 = r^{-1/2} \varphi_2 e^{-i\omega_2 t}\) lead to the differential equations
\[
\frac{d^2 \varphi_1}{dr^2} + (\omega_1^2 - V_{m1}) \varphi_1 = 0, \tag{95}
\]
\[
\frac{d^2 \varphi_2}{dr^2} + (\omega_2^2 - V_{m2}) \varphi_2 = 0, \tag{96}
\]
FIG. 6: A (2+1)-dimensional asymptotically de Sitter charged black hole. In this example we take $\Lambda = 1$, $B = -5$, $Q = 2$ and $\beta = -1$.

where $r_* = \int \frac{dr}{f(r)}$ is the tortoise coordinate and

$$V_{\text{ml}} = -\frac{f(r) (f(r) - 2rf'(r))}{4r^2}, \tag{97}$$

$$V_{\text{ms}} = f(r)m^2 - \frac{f(r) (f(r) - 2rf'(r))}{4r^2}, \tag{98}$$

are the effective potentials of the massless scalar field and the massive scalar field, respectively.

We plot these two effective potentials in Fig. 7. From the figures we can see that there is a negative potential well outside the black hole event horizon. Therefore this dS black hole spacetime is unstable under the massless scalar perturbations and massive scalar perturbations within some ranges of parameters. This means that the three-dimensional charged dS black hole we found is unstable. This behavior is similar to that of four-dimensional Reissner-Nordström dS black hole under neutral scalar perturbation (see Fig. 8), where the potential is negative outside the black hole horizon. Possible instabilities of quasinormal and superradiant spectrum usually appear in such regions in four-dimensions with negative potential, see for example [43].

For comparison, we also plot the simplest BTZ and charged BTZ cases.
FIG. 7: The effective potentials for (a) massless scalar field, (b) massive scalar field, with the black curve, black dashed curve, pink curve and pink dashed curve represent scalar mass $m = 1, 2, 3, 4$ respectively. Here we take $\Lambda = 1, B = -5, Q = 2$ and $\beta = -1$.

FIG. 8: For comparison: the metric function of Reissner-Nordström-dS black hole is $f(r) = -\frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3} + 1$. Here we take $\Lambda = 0.1, M = 0.7$ and $Q = 0.5$, which give Cauchy horizon $r = 0.21$, event horizon $r = 1.27$ and cosmological horizon $r = 4.61$. Figure (a) corresponds to the effective potential of a massless scalar field, while figure (b) depicts effective potential for massive scalar fields: the black curve, black dashed curve, pink curve and pink dashed curve represent scalar mass $m = 1, 2, 3, 4$ respectively.

From these plots we see that the spacetime is more stable with larger mass of the scalar field in BTZ and charged BTZ backgrounds, while it becomes less stable with the increase of the mass in the (2+1)-charged dS black hole spacetime.
FIG. 9: The metric function of BTZ black hole is $f(r) = \frac{r^2}{\ell^2} - M$. Here we take $\ell = 1$ and $M = 1$. In figure (a), we have the effective potential of a massless scalar field. In figure (b), we have the effective potential for massive scalar fields. The black curve, black dashed curve, pink curve and pink dashed curve represent scalar mass $m = 1, 2, 3, 4$ respectively.

FIG. 10: The metric function of charged BTZ black hole is $f(r) = \frac{r^2}{\ell^2} - M - 2q^2 \ln \frac{r}{\ell}$. Here we take $\ell = 1$, $M = 1$ and $q = 2$. Figures (a) and (b) show, respectively, the effective potential for massless and massive scalar fields. In figure (b), the black curve, black dashed curve, pink curve and pink dashed curve represent scalar mass $m = 1, 2, 3, 4$ respectively.

V. CONCLUSIONS

We obtained the general AdS$_3$ black hole solutions in the presence of a scalar field non-minimally coupled to gravity. We solved the Einstein-scalar field equations with a static and spherically symmetric metric ansatz, and a general form of a self-interacting scalar potential. Without specifying the form of the potential we found a general solution of the field equations from which all the known AdS black hole solutions in (2+1)-dimensions can be obtained depending on the values of
Choosing various values of the coupling constant we first obtained numerically new black hole solutions. These solutions are characterized by various sizes of the event horizon depending on the values of the coupling constants and the form of the resulting scalar potential. We were also able to find new exact black hole solutions for specific choices of the coupling constant. These solutions have a simple form of the scalar field but the metric function and the scalar potential are complicated functions of the parameters. However, an appropriate choice of the parameters involved can bring these solutions to a simpler form. The properties of these black holes could be interesting to study in future works.

We also attempt in this general framework to find asymptotically de Sitter black hole solutions. We showed that the requirement that the parameters of our theory should satisfy appropriate constraint conditions does not allow dS black hole solutions in (2+1)-dimensions. Likewise, we found that there cannot exist asymptotically flat black holes. As is well known in (2+1)-dimensions there are strong constrains coming from the energy conditions \cite{44} for the existence of black holes, so this is to be expected, though it is still interesting to see how our construction fails explicitly. However, when an electromagnetic field is introduced, surprisingly it is possible to find a charged-dS black hole solution in some ranges of parameters. Nevertheless, this dS black hole solution is unstable under scalar perturbations. It would be interesting to further explore the energy conditions, and see how such a black hole evades the previous no-go results.

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