On generators with infinite entropy

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Abstract

Many years ago B.S. Pitskel observed that the metric entropy of the shift transformation in the sample space of a stationary random process $X = \{X_n, n \in \mathbb{Z}\}$ with a countable number of states is equal to the conditional entropy $H(X_0|X_{-1}, X_{-2}, \ldots)$ if $X$ is a stationary Markov chain (in which case the above conditional entropy is $H(X_0|X_{-1})$), whether the entropy $H(X_0)$ is finite or not, while in general the statement is not true. In this note we present a class of processes for which Pitskel’s observation holds, despite the fact that no of these processes is a Markov chain of some order.

1 Introduction

We use standard terminology, notation and basic facts from entropy theory of dynamical systems (see, e.g., [2]). It is well known that every ergodic automorphism of a Lebesgue space has a countable generator (generating partition) and every automorphism with finite entropy has a finite generator. A generator is very useful for studying the automorphism. In particular, if an automorphism has finite entropy, the entropy can be immediately expressed in terms of every generator with finite entropy. However sometimes such a generator is unknown, while another generator, good in all senses but with infinite entropy, is at hand. The following example of this situation was considered in [4].

Let $X = \{X_n, n \in \mathbb{Z}\}$ be a discrete time stationary process whose states form an infinite countable set $A$, and $\Omega = A^\mathbb{Z}$ be its sample space equipped with the corresponding measure $\nu$. Define the shift transformation $T : \Omega \to \Omega$ by

$$(T\omega)_i = (\omega)_{i-1}, \quad i \in \mathbb{Z}.\,$$

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Let $\alpha$ be the partition of $\Omega$ into one-dimensional cylinders

$$C_a := \{ \omega \in \Omega : \omega_0 = a \}, \ a \in A.$$  

Since $\nu$ is a $T$-invariant probability measure and $\alpha$ is a generator for $T$, we have

$$h_\nu(T) = h_\nu(T, \alpha) = H_\nu(\alpha | \cup_{n=1}^{\infty} T^{-n}\alpha),$$  
(1)

whenever $H_\nu(\alpha) < \infty$, while if $H_\nu(\alpha) = \infty$, this is in general not the case. However, as was observed in [4], (1) holds, provided that $\nu$ is a Markov measure, even if $H_\nu(\alpha) = \infty$. We remark that in this case

$$H_\nu(\alpha | \cup_{n=1}^{\infty} T^{-n}\alpha) = H_\nu(\alpha | T^{-1}\alpha).$$  
(2)

The aim of this note is to present a class of non-Markov measures on $\Omega$ for which the situation is exactly as in the Markov case. These measures are induced by stationary random processes $\tilde{X}$ with the same state set $A$. The construction is the following. Let $X = \{ X_n, n \in \mathbb{Z} \}$ be a Markov chain with states $a \in A$, transition probabilities $p_{a,b}$, $a, b \in A$, and stationary probabilities $\pi_a$, $a \in A$. Denote the corresponding Markov measure on $\Omega$ by $\mu$ and assume that

$$-\sum_{a \in A} \pi_a \log \pi_a = \infty, \ -\sum_{a,b \in A} \pi_a p_{a,b} \log p_{a,b} < \infty$$  
(3)

(see Section 3 below for an explicit example of such a Markov chain). Consider a function $f_0 : A \rightarrow \mathbb{N}$ with

$$\sum_{a \in A} \pi_a f_0(a) < \infty$$  
(4)

and define $f(\omega) := f_0(\omega_0)$. One can rewrite (3) and (4) as

$$H_\mu(\alpha) = \infty, \ H_\mu(\alpha | T^{-1}\alpha) < \infty, \ \int_\Omega f d\mu < \infty.$$  

We now define what is called a suspension automorphism $\tilde{T} = (T, f)$ constructed by $T$ and the ‘roof’ function $f$. This $\tilde{T}$ acts in the space

$$\tilde{\Omega} := \{ \tilde{\omega} = (\omega, u) : \omega \in \Omega, \ 0 \leq u \leq f(\omega) \}$$

and is defined by

$$\tilde{T}(\omega, u) = \begin{cases} 
(\omega, u + 1), & u < f(\omega), \\
(T\omega, 0), & u = f(\omega).
\end{cases}$$
It is easy to check that \( \tilde{T} \) preserves the probability measure \( \tilde{\mu} := \gamma(\mu \times \kappa) |_{\tilde{\Omega}} \), where \( \kappa \) is the counting measure on \( \mathbb{Z}_+ \) and \( \gamma = 1/ \int_{\Omega} f \, d\mu \).

Denote by \( \tilde{\alpha} \) the partition of \( \tilde{\Omega} \) into atoms of the form

\[
\tilde{C}_a := \{ (\omega, u) : \omega \in C_a, \ u \leq f_0(a) \}, \quad a \in A,
\]

and let

\[
\tilde{X}_n(\tilde{\omega}) = a \text{ iff } \tilde{T}^n \tilde{\omega} \in \tilde{C}_a, \quad a \in A, \quad n \in \mathbb{Z}.
\]

The stationary process \( \tilde{X} := \{ \tilde{X}_n, \ n \in \mathbb{Z} \} \) is what we wanted to construct. Its properties are studied in Section 2; in Section 3 we present an explicit example of a Markov chain that satisfies conditions (3).

Remark 1. It is easy to check that if \( f \) is unbounded, then \( X_n \) is not a Markov chain (we mean Markov chains of any order).

The above-mentioned result from [4] (see (1)) was obtained due to Pitskel’s observation that, for a countable alphabet Markov chain \( \{X_n\} \) with \( H(X_0) < \infty \) or \( H(X_0) = \infty \) and \( h(T) < \infty \), where \( T \) is the shift in the sample space of \( \{X_n\} \), the Shannon–McMillan–Breiman (SMB) theorem holds true in the following form (we use the above notation):

\[
\lim_{n \to \infty} \left| \frac{1}{n} \log \mu(C^n(\omega)) \right| = H_\mu(\alpha | T^{-1} \alpha),
\]

where \( (C^n(\omega)) \) is the atom of the partition \( \bigvee_{i=0}^{n-1} T^i \alpha \) that contains \( \omega \) and where the \( \mu \)-a.s. convergence is meant. By (1), (2) the right hand side of (5) can be replaced by \( h_\mu(T) \). We proceed in the opposite direction: first we use Pitscel’s result for \( \alpha \) and \( \mu \) to prove, for \( \tilde{\alpha} \) and \( \tilde{\mu} \), a similar fact (Theorem 1), after which we prove for them the SMB theorem.

2 Properties of \( \tilde{X} \) and \( \tilde{T} \)

All properties of \( \tilde{X} \) can be expressed in terms of those of \( \tilde{T} \) and vice versa. We will study \( \tilde{T} \), because this is a little more convenient. We will refer to the set \( \{ (\omega, u) \in \tilde{\Omega} : u = k \} \) as to the \( k \)th level. The \( k \)th level with the maximal possible \( k \) and with the minimal possible \( k \) (\( k = 0 \)), will be called the top level and bottom level, respectively. The former will be denoted by \( L \).

Proposition 1. If \( T \) is ergodic, then the partition \( \tilde{\alpha} \) is a generator for \( \tilde{T} \).

Proof. For every point \( \omega \in \Omega \), we call the sequence of symbols

\[
n(\omega, \alpha, T) = (a_i(\omega), \ i \in \mathbb{Z})
\]
the \((\alpha, T)\)-name of \(\omega\) if \(T^i\omega \in C_{a_i(\omega)}, \; i \in \mathbb{Z}\), and call its finite subsequence corresponding to \(i\) from \(i_1\) to \(i_2\) the \((\alpha, T)\)-subname of \(\omega\) from \(i_1\) to \(i_2\). In a similar way we define \(n(\tilde{\omega}, \tilde{\alpha}, \tilde{T})\), the \((\tilde{\alpha}, \tilde{T})\)-name of \(\tilde{\omega} \in \bar{\Omega}\) and its subnames. To prove the proposition it suffices to find a set \(\bar{\Omega} \subset \Omega\) with \(\tilde{\mu}(\bar{\Omega}) = 1\) such that no two different points in \(\bar{\Omega}\) have identical \((\tilde{\alpha}, \tilde{T})\)-names.

For \(\bar{\Omega}\) we take the set of points \(\tilde{\omega} = (\omega, u)\) such that the \((\alpha, T)\)-name of \(\omega\) contains no infinite tails of identical symbols (neither in \(-\infty\) or in \(+\infty\)). From ergodicity of \(T\) it follows that \(\tilde{\mu}(\bar{\Omega}) = 1\).

We prove that for every \(\tilde{\omega} = (\omega, u) \in \bar{\Omega}\), the \((\tilde{\alpha}, \tilde{T})\)-name of \(\tilde{\omega}\) determines the \((\alpha, T)\)-name of \(\omega\), from which it follows that \(\tilde{\alpha}\) is a generator for \(\tilde{T}\), because \(\alpha\) is a generator for \(T\).

Given \(\tilde{\omega} \in \bar{\Omega}\), we introduce the jump set

\[
J(\tilde{\omega}) := \{ i \in \mathbb{Z} : a_i(\tilde{\omega}) \neq a_{i-1}(\tilde{\omega}) \}.
\]

It is clear that if \(i \in J(\tilde{\omega})\), then \(\tilde{T}^i\tilde{\omega}\) lies at the bottom level and \(a_i(\tilde{\omega}) = a_l(\omega)\) for some \(l\) belongs to \(n(\omega, \alpha, T)\). Moreover, if \(i' > i\), \(i' \in J(\tilde{\omega})\), and no \(j\) between \(i\) and \(i'\) belongs to \(J(\tilde{\omega})\), then \(i' = i + m'(f_0(a_i(\tilde{\omega}) + 1)\) for some \(m' \in \mathbb{N}\). Hence all \(a_j(\tilde{\omega})\) with \(j\) of the form \(i + m(f_0(a_i(\tilde{\omega}) + 1)\), \(m \in \mathbb{Z}_+, \; 0 \leq m \leq m'\), belong to \(n(\omega, \alpha, T)\) (with other indices), while the remaining \(a_j(\tilde{\omega})\) with \(j\) between \(i\) and \(i'\) do not belong to \(n(\omega, \alpha, T)\). Thus we have described \(n(\omega, \alpha, T)\) in terms of \(n(\tilde{\omega}, \tilde{\alpha}, \tilde{T})\).

**Proposition 2.** If conditions (2), (4) are satisfied, then \(H_{\tilde{\mu}}(\tilde{\alpha}) = \infty\).

**Proof.** By definition

\[
H_{\tilde{\mu}}(\tilde{\alpha}) = -\sum_{a \in A} \tilde{\mu}(\tilde{C}_a) \log \tilde{\mu}(\tilde{C}_a) = -\sum_{a \in A} \gamma_{\pi_a f_0(a)} \log[\gamma_{\pi_a f_0(a)}] \\
= -\gamma \sum_{a \in A} \pi_a f_0(a) \log \gamma - \gamma \sum_{a \in A} \pi_a f_0(a) \log \pi_a \\
= -\gamma \sum_{a \in A} \pi_a f_0(a) \log f_0(a). \tag{6}
\]

The first sum in the right hand side of (6) is \(-\log \gamma\), the second sum equals \(+\infty\) (see (3)). At last, due to the concavity of the function \(v \mapsto -v \log v\),

\[
-\gamma \sum_{a \in A} \pi_a f_0(a) \log f_0(a) = \gamma \sum_{a \in A} \pi_a(-f_0(a) \log f_0(a)) \\
\leq \gamma \sum_{a \in A} \pi_a f_0(a) \log \sum_{a \in A} \pi_a f_0(a) = -\log \gamma.
\]

\[\square\]
The main result of this section is the following

**Theorem 1.** If $T$ is ergodic, then

$$H_{\tilde{\mu}}(\tilde{\alpha} | \bigvee_{i=1}^{\infty} \tilde{T}^{-i} \tilde{\alpha}) = h_{\tilde{\mu}}(\tilde{T}).$$

**Proof.** 1. Denote

$$\hat{\alpha}_{m}^{n}(\tilde{T}) := \bigvee_{i=m}^{n} \tilde{T}^{i} \tilde{\alpha}, \quad m,n \in \mathbb{Z}, \quad m < n, \quad \tilde{\alpha}_{\tilde{T}} := \bigvee_{i=1}^{\infty} \tilde{T}^{-i} \tilde{\alpha}.$$  

Clearly, $\tilde{\alpha}_{-n}^{-1}(\tilde{T}) \not\supset \tilde{\alpha}_{\tilde{T}}$, and we know that $H(\tilde{\alpha} | \tilde{\alpha}_{-n}^{-1}(\tilde{T})) \subset H(\tilde{\alpha} | \tilde{\alpha}_{\tilde{T}})$ if $H(\tilde{\alpha} | \tilde{T}^{-1} \tilde{\alpha}) < \infty$ (see [5], 5.11). We first show that this condition is satisfied.

To this end we find, for all $a, b \in A$, the conditional measure

$$\hat{\mu}(\tilde{C}_{b} | \tilde{T}^{-1} \tilde{C}_{a}) = \frac{\hat{\mu}(\tilde{C}_{b} \cap \tilde{T}^{-1} \tilde{C}_{a})}{\hat{\mu}(\tilde{T}^{-1} \tilde{C}_{a})}. \tag{7}$$

By the definition of $T$, $\tilde{T}$ and $\mu$, $\hat{\mu}$, if $b \neq a$, then

$$\hat{\mu}(\tilde{C}_{b} \cap \tilde{T}^{-1} \tilde{C}_{a}) = \hat{\mu}(\{\tilde{\omega} \in \tilde{C}_{b} : \tilde{T} \tilde{\omega} \in \tilde{C}_{a}\})$$

$$= \hat{\mu}(\{\tilde{\omega} = (\omega, u) : \omega \in \tilde{C}_{b}, \tilde{T} \omega \in \tilde{C}_{a}, \ u = f_{0}(b)\}) = \gamma \pi_{a} p_{a,b}, \tag{8}$$

while

$$\hat{\mu}(\tilde{C}_{a} \cap \tilde{T}^{-1} \tilde{C}_{a}) = \hat{\mu}(\{\tilde{\omega} = (\omega, u) : \omega \in \tilde{C}_{a}, \ u < f_{0}(a)\})$$

$$+ \hat{\mu}(\{\tilde{\omega} = (\omega, u) : \omega, \tilde{T} \omega \in \tilde{C}_{a}, \ u = f_{0}(a)\})$$

$$= \gamma (\pi_{a}(f_{0}(a) + p_{a,a}). \tag{9}$$

From (8) – (9) we obtain $\hat{\mu}(\tilde{C}_{b} | \tilde{T}^{-1} \tilde{C}_{a}) = p_{a,b}(1 + f_{0}(a))^{-1}$ if $a \neq b$, and $\hat{\mu}(\tilde{C}_{a} | \tilde{T}^{-1} \tilde{C}_{a}) = (p_{a,a} + f_{0}(a))(1 + f_{0}(a))^{-1}$.

Therefore,

$$H(\tilde{\alpha} | \tilde{T}^{-1} \tilde{\alpha}) = - \sum_{a \in A} \mu(\tilde{T}^{-1} \tilde{C}_{a}) [\hat{\mu}(\tilde{C}_{a} | \tilde{T}^{-1} \tilde{C}_{a}) \log \hat{\mu}(\tilde{C}_{a} | \tilde{T}^{-1} \tilde{C}_{a})$$

$$+ \sum_{b \in A \setminus \{a\}} \hat{\mu}(\tilde{C}_{a} | \tilde{T}^{-1} \tilde{C}_{a}) \log \hat{\mu}(\tilde{C}_{b} | \tilde{T}^{-1} \tilde{C}_{a})]$$

$$= - \gamma \sum_{a \in A} (f_{0}(a) + 1) \pi_{a} [\frac{f_{0}(a) + p_{a,a}}{f_{0}(a) + 1} \log \frac{f_{0}(a) + p_{a,a}}{f_{0}(a) + 1}$$

$$+ \sum_{b \in A \setminus \{a\}} \frac{p_{a,b}}{f_{0}(a) + 1} \log \frac{p_{a,b}}{f_{0}(a) + 1}]. \tag{10}$$
We open the square brackets in (10) and estimate each sum obtained. The first sum is

\[ S_1 := \gamma \sum_{a \in A} \pi_a (f_0(a) + 1) \frac{f_0(a) + p_{a,a}}{f_0(a) + 1} \log \frac{f_0(a) + p_{a,a}}{f_0(a) + 1} \]

Since \( f_0(a) \geq 1 \) and \( p_{a,a} \geq 0 \), we have

\[ \frac{1}{2} \leq \frac{f_0(a) + p_{a,a}}{f_0(a) + 1} \leq 1, \quad a \in A, \quad (11) \]

so that, by (4), \( S_1 < \infty \).

The second sum is

\[ S_2 := -\gamma \sum_{a \in A} \sum_{b \in A \setminus \{a\}} \pi_a (f_0(a) + 1) \frac{p_{a,b}}{f_0(a) + 1} \log p_{a,b} \]

\[ = -\gamma \sum_{a \in A} \sum_{b \in A \setminus \{a\}} \pi_a p_{a,b} \log p_{a,b} \]

\[ = -\gamma \sum_{a \in A} \sum_{b \in A \setminus \{a\}} \pi_a p_{a,b} \log p_{a,b} + \gamma \sum_{a \in A} \sum_{b \in A \setminus \{a\}} \pi_a (f_0(a) + 1) \log p_{a,b} \]

\[ \leq \gamma h_\mu(T) + \gamma \sum_{a \in A} \pi_a (f_0(a) + 1) < \infty. \]

Here we used Pitskel’s result from [4] (because \( \mu \) is a Markov measure) and (3).

Thus the inequality \( H(\tilde{\alpha}|\tilde{T}^{-1}\tilde{\alpha}) < \infty \) is proved.

2. We say that an atom

\[ \tilde{C} = \tilde{T}^{-1}\tilde{C}_{a_1} \cap \cdots \cap \tilde{T}^{-n}\tilde{C}_{a_n}, \quad n \geq 2 \]

of the partition \( \tilde{\alpha}^{-1}_n(\tilde{T}) \) is good if there exists \( k < n \) such that \( a_{k+1} \neq a_1 \).

From the definition of \( \tilde{\Omega} \) (see the proof of Proposition 1) it follows that each point of \( \tilde{\Omega} \) belongs to a good atom of \( \tilde{\alpha}^{-1}_n(\tilde{T}) \), beginning with some \( n \). Since \( G_n \), the union of good atoms of \( \tilde{\alpha}^{-1}_n(\tilde{T}) \), does not decrease in \( n \), we see that

\[ \lim_{n \to \infty} \mu(G_n) = 1. \]

We wish to find \( \mu(\tilde{C}_b|\tilde{C}) \), where \( \tilde{C} \subset G_n \) and \( b \in A \). Assume that

\[ a_1 = \cdots = a_k \neq a_{k+1} \]
(see (12)). If \( k \) is not of the form \( k = m(f_0(a_1) + 1) \) for some \( m \in \mathbb{N} \), then no \( \omega \in C \) can be at the top level. Hence \( \mu(C_a | C) = 1 \).

Now let \( k = m(f_0(a_1) + 1) \). We say that such \( C \) is a very good atom and denote the union of very good atoms by \( VG_n \). For such atom we define by induction a sequence of symbols \( a'(1), a'(2), \ldots \), where \( a'(1) = a_{k+1} \) and if \( a'(1), \ldots, a'(m) \) are already defined and \( a'(m) = a_i \), then \( a'(m + 1) := a_i + f_0(a_i) + 1 \).

All points \( \tilde{\omega} \in \tilde{C}_b \cap \tilde{C} \) have identical \((\tilde{a}, \tilde{T})\)-subnames from 0 to \( n \), while all \( \tilde{\omega} \) such that \( \tilde{\omega} = (\omega, f_0(b)) \in \tilde{C}_b \cap \tilde{C} \) have identical \((a, T)\)-subnames from 0 to \( n' \), where \( n' = \max\{l : f_0(a'(0)) + \cdots + f_0(a'(l)) \leq n\} \) (notice that \( a'(0) = b \)). It follows from the definitions of the measures \( \mu \) and \( \tilde{\mu} \) that

\[
\tilde{\mu}(\tilde{C}_b \cap \tilde{C}) = \gamma \pi_{a'(n')} p_{a'(n'), a'(n'-1) \cdots a'(1), b}.
\]

Since \( \tilde{\mu}(\tilde{C}) = \sum_{b \in A} \tilde{\mu}(\tilde{C}_b \cup \tilde{C}) \), we have

\[
\tilde{\mu}(\tilde{C}_b | \tilde{C}) = p_{a_1, b} = p_{a, b}, \text{ when } \tilde{C} \subset VG_n \cap \tilde{T}^{-1} \tilde{C}_a.
\]

Hence

\[
H_{\tilde{\mu}}(\tilde{\alpha} | \tilde{C}) = - \sum_{b \in A} p_{a, b} \log p_{a, b} \tag{14}
\]

if \( \tilde{C} \) is a very good atom from \( \tilde{T}^{-1} \tilde{C}_a \), and the conditional entropy is zero if \( \tilde{C} \) is a good but not very good atom.

Next consider a bad atom \( \tilde{C} = \cap_{i=1}^n \tilde{T}^{-1} \tilde{C}_a \) for some \( a \in A \). We denote it by \( \tilde{C}(a) \). For every \( b \in A \),

\[
\tilde{\mu}(\tilde{C}_b \cap \tilde{C}(a)) = \tilde{\mu}(\tilde{C}_b \cap \tilde{C}(a) \cap L) + \tilde{\mu}(\tilde{C}_b \cap \tilde{C}(a) \cap (\tilde{\Omega} \setminus L)). \tag{15}
\]

The first term in (15) is

\[
\tilde{\mu}(\tilde{C}_b \cap \tilde{C}(a) \cap L) = \tilde{\mu}(\{\tilde{\omega} \in \tilde{C}_b \cap L : \tilde{T}^i \tilde{\omega} \in \tilde{C}_a, \ i = 1, \ldots, n\}) = \pi_a(p_{a, a}) z p_{a, b}, \tag{16}
\]

where \( z = \frac{n}{f_0(a) + 1} \). The second term vanishes if \( b \neq a \), while

\[
\tilde{\mu}(\tilde{C}_a \cap (\tilde{\Omega} \setminus L)) = \tilde{\mu}(\{\tilde{\omega} \in \tilde{C}_a \cap (\tilde{\Omega} \setminus L) : \tilde{T}^i \tilde{\omega} \in \tilde{C}_a, \ i = 1, \ldots, n\}) = f_0(a) \pi_a(p_{a, a})^z. \tag{17}
\]

Similarly,

\[
\tilde{\mu}(\tilde{C}(a)) = \tilde{\mu}(\tilde{C}(a) \cap L) + \tilde{\mu}(\tilde{C}(a) \cap (\tilde{\Omega} \setminus L)) = \sum_{b \in A} \pi_a(p_{a, a})^z p_{a, b} + f_0(a) \pi_a(p_{a, a})^z = \pi_a(p_{a, a})^z (1 + f_0(a)) \tag{18}
\]
From (15)–(18) we obtain
\[
\hat{\mu}(\tilde{C}_b|\tilde{C}(a)) = \begin{cases} 
\frac{p_{a,b}}{1 + f_0(a)}, & b \neq a, \\
\frac{p_{a,a} + f_0(a)}{1 + f_0(a)}, & b = a.
\end{cases}
\] (19)

Hence
\[
H_{\hat{\mu}}(\alpha|\tilde{C}(a)) = -\frac{p_{a,a} + f_0(a)}{1 + f_0(a)} \log \frac{p_{a,a} + f_0(a)}{1 + f_0(a)} - \sum_{b \in A \setminus \{a\}} \frac{p_{a,b}}{1 + f_0(a)} \log \frac{p_{a,b}}{1 + f_0(a)}
\]
if \(\tilde{C}(a) \subset \tilde{T}^{-1}\tilde{C}_a\) is a bad atom of \(\bar{\alpha}_n^{-1}(\tilde{T})\).

3. We complete the proof by finding the asymptotics of the sum
\[
\sum_{\tilde{C}} \mu(\tilde{C}) H_{\mu}(\bar{\alpha}|\tilde{C}) = \sum_{\tilde{C} \subset G_n} \mu(\tilde{C}) H_{\mu}(\bar{\alpha}|\tilde{C}) + \sum_{\tilde{C} \subset B_n} \mu(\tilde{C}) H_{\mu}(\bar{\alpha}|\tilde{C})
\]
as \(n \to \infty\). Consider each of these sums separately.

Since \(H_{\mu}(\bar{\alpha}|\tilde{C}) = 0\) if \(\tilde{C} \subset G_n \setminus VG_n\), and due to (14), the first sum equals
\[
\sum_{\tilde{C} \subset G_n} \mu(\tilde{C}) H_{\mu}(\bar{\alpha}|\tilde{C}) = -\sum_{a \in A} \sum_{\tilde{C} \subset VG_n \cap \tilde{T}^{-1}\tilde{C}_a} \mu(\tilde{C}) \sum_{b \in A} p_{a,b} \ln p_{a,b}.
\]

Clearly, \(VG_n \cap \tilde{T}^{-1}\tilde{C}_a = L \cap G_n \cap \tilde{T}^{-1}\tilde{C}_a\), and since \(\tilde{\mu}(G_n) \to 1\) as \(n \to \infty\), we have
\[
\lim_{n \to \infty} \tilde{\mu}(VG_n \cap \tilde{T}^{-1}\tilde{C}_a) = \hat{\mu}(L \cap \tilde{T}^{-1}\tilde{C}_a) = \sum_{b \in A} \gamma \pi_a p_{a,b} = \gamma \pi_a,
\]
so that (see (22))
\[
\lim_{n \to \infty} \sum_{\tilde{C} \subset VG_n} \mu(\tilde{C}) H_{\mu}(\bar{\alpha}|\tilde{C}) = \sum_{a \in A} \gamma \pi_a \sum_{b \in A} p_{a,b} \log p_{a,b} = \gamma h_{\mu}(T).
\] (23)

Before dealing with the second sum in (21) we state the following simple auxiliary assertion. Let \(r_i \geq 0, s_{ij} \geq 0, i, j \in \mathbb{N}\) and \(\sum_{i \in \mathbb{N}} r_i < \infty, \sup_{i,j} s_{i,j} < \infty, \lim_{j \to \infty} s_{i,j} = 0\) for all \(i\). Then
\[
\lim_{j \to \infty} \sum_{i \in \mathbb{N}} r_i s_{i,j} = 0.
\] (24)
For $\tilde{C} \subset B_n \cap \tilde{T}^{-1}\tilde{C}_a$ we write $\dot{C}(a)$. By (20), for each $a \in A$,

$$H_{\tilde{\mu}}(\tilde{\alpha}|\dot{C}(a)) = -\frac{p_{a,a} + f_0(a)}{1 + f_0(a)} \log \frac{p_{a,a} + f_0(a)}{1 + f_0(a)} - \sum_{b \in A \setminus \{a\}} \frac{p_{a,b}}{1 + f_0(a)} \log \frac{p_{a,b}}{1 + f_0(a)},$$

so that

$$\sum_{a \in A} \tilde{\mu}(\dot{C}(a)) H_{\tilde{\mu}}(\tilde{\alpha}|\dot{C}(a)) = -\sum_{a \in A} \pi_a (1 + f_0(a))(p_{a,a}) \log \frac{p_{a,a} + f_0(a)}{1 + f_0(a)} - \sum_{a \in A} \sum_{b \in A \setminus \{a\}} \pi_a (1 + f_0(a))(p_{a,a}) \log \frac{p_{a,b}}{1 + f_0(a)}. \quad (25)$$

By (11) the first sum in (25) does not exceed $\sum_{a \in A} \pi_a (1 + f_0(a))(p_{a,a}) z \log 2$. It is clear that $0 \leq p_{a,a} < 1$ (the Markov chain $X$ is assumed to be ergodic) and $z = z(a, n) \to \infty$ as $n \to \infty$. We now can use the above auxiliary assertion by putting $r_a := \pi_a (1 + f_0(a))$ (we identify $a$ with $i$) and $s_{a,n} := (p_{a,a})^z$ (here we identify $n$ with $j$). From this assertion it follows that the first sum in (25) vanishes as $n \to \infty$.

A similar argument shows that the second sum in (25) behaves in the same way. As a result we obtain (see (11) $\lim_{n \to \infty} H_{\tilde{\mu}}(\tilde{\alpha}|\hat{\tilde{\alpha}}_{-n}^n(T)) = \gamma H_{\mu}(T)$. Now it remains to use (23) and Abramov’s formula according to which $h_{\tilde{\mu}}(\tilde{T}) = \gamma h_{\mu}(T)$.

**Theorem 2.** Let $\tilde{\alpha}_n(\tilde{\omega})$ be the atom of the partition $\vee_t^{n-1} \tilde{T}^i \tilde{\alpha}$ that contains $\tilde{\omega}$. Then

$$\lim_{n \to \infty} \frac{1}{n} \log \tilde{\mu}(\tilde{\alpha}_n(\tilde{\omega})) = h_{\tilde{\mu}}(\tilde{T}),$$

**Proof.** Examination of a standard proof of the SMB theorem for finite partition (see, e.g., [1]) shows that this proof applies to countable infinite partitions, including partitions of infinite entropy, provided that the following condition is satisfied (we state it for our case, using the above notation). For $\tilde{\omega} \in \tilde{\Omega}$, let

$$g_n(\tilde{\omega}) := -\log \tilde{\mu}(\tilde{\alpha}_n(\tilde{\omega})|\hat{\tilde{\alpha}}_n^n(\tilde{\omega})),$$

where $\tilde{C}_0(\tilde{\omega})$, $\tilde{C}_n(\tilde{\omega})$ are the atoms of $\tilde{\alpha}$ and $\hat{\tilde{\alpha}}_{-n}^n(T)$, respectively, containing $\tilde{\omega}$. It is required that

$$\int_{\tilde{\Omega}} \sup_n g_n(\tilde{\omega}) \tilde{\mu}(d\tilde{\omega}) < \infty. \quad (26)$$

We establish (26) using some formulas from the above proof of Theorem
Let \( \tilde{\omega} \in \tilde{C}_b \cap \tilde{T}^{-1}\tilde{C}_a \) for some \( a, b \in A \). From (13), (19) we see that
\[
\tilde{\mu}(\tilde{C}_0(\tilde{\omega})|\tilde{C}_n(\tilde{\omega})) = \begin{cases} 
   p_{a,b}, & \tilde{C}_n(\tilde{\omega}) \subset V G_n, \\
   1, & \tilde{C}_n(\tilde{\omega}) \subset G_n \setminus V G_n, \quad b = a, \\
   0, & \tilde{C}_n(\tilde{\omega}) \subset G_n \setminus V G_n, \quad b \neq a, \\
   \frac{p_{a,b}}{1 + f_0(a)}, & \tilde{C}_n(\tilde{\omega}) \subset B_n, \quad b \neq a, \\
   \frac{p_{a,a} + f_0(a)}{1 + f_0(a)}, & \tilde{C}_n(\tilde{\omega}) \subset B_n, \quad b = a.
\end{cases}
\] (27)

It is evident that if \( b \neq a \), then
\[ \tilde{\mu}(\{ \tilde{\omega} \in \tilde{C}_b \cap \tilde{T}^{-1}\tilde{C}_a : \tilde{C}_n(\tilde{\omega}) \subset G_n \setminus V G_n \}) = 0. \]

Hence, on a set of \( \tilde{\mu} \)-measure 1,
\[
g_n(\tilde{\omega}) \leq - \log p_{a,b} - \log \frac{p_{a,b}}{1 + f_0(a)} - \log \frac{p_{a,a} + f_0(a)}{1 + f_0(a)}.
\]

Since the right hand side of this inequality does not depend on \( n \), one can replace its left hand side by \( \sup_n g_n(\tilde{\omega}) \). From this we obtain (see (11))
\[
\int_\Omega \sup_n g_n(\tilde{\omega}) \tilde{\mu}(d\tilde{\omega}) \leq - \sum_{a \in A} \sum_{b \in A} (\pi_a p_{a,b} \log p_{a,b} + \pi_a p_{a,b} \log p_{a,b}) \\
+ \sum_{a \in A} \pi_a \log(1 + f_0(a)) - \sum_{a \in A} \pi_a \log \frac{p_{a,a} + f_0(a)}{1 + f_0(a)} \\
\leq 2 h_\mu(T) + \int_\Omega f(\omega) \mu(d\omega) + \log 2 < \infty.
\]

3 An example

In this section we construct a Markov chain with an infinite countable state set \( A \), transition probabilities \( p_{a,b}, a, b \in A \), and stationary probabilities \( \pi_a, a \in A \), that satisfy conditions (3).

We take \( A = \mathbb{N} \) and \( a = i \). Let \( p_{i,j} \) be of the form \( p_{i,1} = p_i, \quad p_{i,i+1} = q_i \) and \( p_{i,j} = 0 \) for \( j \in \mathbb{N} \setminus \{1, i + 1\} \), where \( p_i, q_i \) such that \( p_i + q_i = 1 \) are to be picked up.
Denote the stationary distribution by \( \pi \) and the transition matrix by \( \mathcal{P} \). By solving the equation \( \pi \mathcal{P} = \pi \) we obtain

\[
\pi_1 = Q^{-1}, \quad \pi_n = Q^{-1} \Pi_{i=1}^{n-1} q_i, \quad Q = 1 + \sum_{n=2}^{\infty} q_i.
\]

Hence \( Q \) must be finite.

If we put

\[
q_1 := \frac{1}{2 \log^2 2}, \quad q_n := \frac{1}{(n + 1) \log^2 (n + 1) \Pi_{i=1}^{n-1} q_i}, \quad n \geq 2,
\]

it is easy to show by induction that

\[
\Pi_{i=1}^{n-1} q_i = \frac{1}{n \log^2 n},
\]

so that \( Q < \infty \).

On the other hand,

\[
-\pi_n \log \pi_n = -Q^{-1} \Pi_{i=1}^{n-1} q_i \log(Q^{-1} \Pi_{i=1}^{n-1} q_i)
\]

so that

\[
- \sum_{n=1}^{\infty} \pi_n \log \pi_n = Q^{-1} \sum_{n=2}^{\infty} \frac{1}{n \log n} + S,
\]

where \( S \) is the sum of a converging series. So we see that the first condition in (3) is satisfied.

Finally, the structure of the transition matrix \( \mathcal{P} \) implies that

\[
- \sum_{k,l \in \mathbb{N}} \pi_k p_{k,l} \log p_{k,l} = - \sum_{k \in \mathbb{N}} \pi_k (p_{k,1} \log p_{k,1} + p_{k,k+1} \log p_{k,k+1})
\]

\[
\leq \log 2 \sum_{k \in \mathbb{N}} \pi_k < \infty.
\]

Thus the second condition in (3) is also satisfied.

Remark 2. 1. It is easy to understand that the stationary Markov chain with the just constructed parameters induces an ergodic shift \( T \) in its sample space.

2. Clearly there exists an unbounded function \( f_0 : A \rightarrow \mathbb{N} \) satisfying (3).
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