FIRST INTEGRALS OF ORDINARY LINEAR DIFFERENTIAL SYSTEMS

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Abstract

The spectral method for building first integrals of ordinary linear differential systems is elaborated. Using this method, we obtain bases of first integrals for linear differential systems with constant coefficients, for linear nonautonomous differential systems integrable in closed form (algebraic reducible systems, triangular systems, the Lappo-Danilevskii systems), and for reducible ordinary differential systems with respect to various transformation groups.

Key words: ordinary linear differential system, first integral, partial integral.

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Introduction

One of the most important problems of the general theory of differential systems is the problem of finding first integrals. In 1878 the French mathematician J.G. Darboux showed how general integrals of the first-order ordinary differential equations possessing sufficient invariant curves are constructed [1]. His investigation gave the classical problem (the Darboux problem) about building of first integrals by known partial integrals. The review of the literature and the current situation of the theory of integrals are given in the monographies [2 – 5].

Linear differential systems are of interest for mathematicians both per se and as a tool for studying nonlinear differential equations by means of the method of linearization. Note also that systems of linear differential equations play a broad and fundamental role in electrical, mechanical, chemical and aerospace engineering, communications, and signal processing. There exist many good works on linear differential systems (see, for example, [6 – 24]).

In this paper we study the Darboux problem of the existence of first integrals for main classes of linear ordinary differential systems. Using the method of partial integrals for polynomial differential systems [3, pp. 187 – 226; 28 – 31], we obtain the spectral method for building first integrals of linear differential systems [25 – 27].

At a later time, the spectral method has been applied to linear multidimensional differential systems [32 – 38] and to nonlinear Jacobi’s differential systems [39 – 44].

The material of this paper is made on the base of our articles [25 – 27; 36; 37; 45; 46].

To avoid ambiguity, we stipulate the following notions (using the article [47]).

Consider an ordinary differential system of n-th order

$$\frac{dx}{dt} = X(t, x),$$

(0.1)

where \( x \in \mathbb{R}^n, t \in \mathbb{R} \), the column vector \( \frac{dx}{dt} = \text{col}(\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}) \) are the coordinates of the vector function \( X(t, x) = \text{col}(X_1(t, x), \ldots, X_n(t, x)) \) are the continuously differentiable on a domain \( G \subset \mathbb{R}^{n+1} \) scalar functions \( X_i: G \to \mathbb{R}, i = 1, \ldots, n \).

We recall that by domain we mean open arcwise connected set.

A continuously differentiable scalar function \( F: G' \to \mathbb{R} \) is called a first integral on a domain \( G' \subset G \) of the ordinary differential system (0.1) if

\[ \mathcal{X} F(t, x) = 0 \quad \text{for all} \quad (t, x) \in G', \]

where the linear differential operator \( \mathcal{X}(t, x) = \partial_t + \sum_{i=1}^n X_i(t, x) \partial_{x_i} \) for all \( (t, x) \in G \) is the operator of differentiation by virtue of system (0.1).

A continuously differentiable scalar function \( F: G' \to \mathbb{R} \) is a first integral on a domain \( G' \subset G \) of the ordinary differential system (0.1) if and only if the function \( F: G' \to \mathbb{R} \) is constant along any solution \( x: t \to x(t) \) for all \( t \in J \) of system (0.1), where \( x: t \to x(t) \) is such that \( (t, x(t)) \in G' \) for all \( t \in J \subset \mathbb{R} \), i.e., \( F(t, x(t)) = C \) for all \( t \in J, C = \text{const} \).

A continuously differentiable scalar function \( w: G' \to \mathbb{R} \) is said to be a partial integral on a domain \( G' \subset G \) of the ordinary differential system (0.1) if

\[ \mathcal{X} w(t, x) = \Phi(t, x) \quad \text{for all} \quad (t, x) \in G', \]

where \( \Phi: G' \to \mathbb{R} \) is a scalar function such that \( \Phi(t, x)_{|w(t,x)=0} = 0 \) for all \( (t, x) \in G' \).

A continuously differentiable scalar function \( w: G' \to \mathbb{R} \) is a partial integral on a domain \( G' \subset G \) of the ordinary differential system (0.1) if and only if the function \( w: G' \to \mathbb{R} \) vanishes identically along any solution \( x: t \to x(t) \) for all \( t \in J \subset \mathbb{R} \) of system (0.1), where \( x: t \to x(t) \) is such that \( (t, x(t)) \in G' \) for all \( t \in J \), i.e., \( w(t, x(t)) = 0 \) for all \( t \in J \).
A set of the functionally independent on a domain $G' \subset G$ first integrals $F_l: G' \to \mathbb{R}$, $l = 1, \ldots, k$, of system (0.1) is called a basis of first integrals (or integral basis) on the domain $G'$ of system (0.1) if for any first integral $\Psi: G' \to \mathbb{R}$ of system (0.1), we have

$$\Psi(t, x) = \Phi(F_1(t, x), \ldots, F_k(t, x)) \quad \text{for all } (t, x) \in G',$$

where $\Phi$ is some continuously differentiable function on the range of the vector function $F: (t, x) \to (F_1(t, x), \ldots, F_k(t, x))$ for all $(t, x) \in G'$. The number $k$ is said to be the dimension of basis of first integrals on the domain $G'$ for system (0.1).

The ordinary differential system (0.1) on a neighbourhood of any point of the domain $G$ has a basis of first integrals of dimension $n$ [48, pp. 175 – 177; 49, pp. 256 – 263].

If the ordinary differential system (0.1) is autonomous (the vector function $X: x \to X(x)$ for all $x \in X \subset \mathbb{R}^n$ not depends on the independent variable $t$), then the ordinary differential system (0.1) in an $n$-dimensional integral basis has $n - 1$ functionally independent on the domain $X$ autonomous first integrals [50, pp. 161 – 169].

The paper is organized as follows.

In Section 1 the spectral method for building first integrals for linear homogeneous and nonhomogeneous differential systems with real constant coefficients is developed. Here we pay special attention to the construction of bases of real first integrals for these systems.

The Euler method [51, pp. 350 – 360; 52, pp. 93 – 101] and the matrix method [51, pp. 320 – 349; 53, pp. 166 – 172] are the main methods for finding solutions of linear homogeneous differential systems with constant coefficients. For building first integrals we know the method of integrable combinations [48, pp. 171 – 173] and the N.P. Erugin – N.A. Zboichik method [54, pp. 464 – 469; 55]. These approaches show only ways of constructing first integrals for linear differential systems, but they do not build first integrals in explicit form.

Using the method of partial integrals [3, pp. 187 – 226; 28 – 31] for polynomial differential systems, we obtain the spectral method of building first integrals in explicit form for linear differential systems with constant coefficients [26; 27; 45].

Note also that the results of Subsection 1.1.2 are co-ordinated with the articles [20 – 24]. The approach of finding autonomous first integrals with using the method of Jordan canonical forms was developed by M. Falcioni and J. Llibre [20]. The existence of common first integrals for two coupled third-order linear differential systems (which satisfy the Frobenius compatibility condition) is discussed in [21]. The problem of building rational first integrals for linear systems was considered by the French mathematician J.A. Weil in the paper [22]. The class of linear autonomous differential systems with no rational first integrals is determined by A. Nowicki [23]. First integrals are constructed using integrating factors in [24].

In Section 2 we investigate the problem of the existence of first integrals for linear nonautonomous differential systems integrable in closed form.

In the general theory of differential systems we know some results about reducibility of differential systems to some special forms. For example, any linear nonautonomous differential system is reducible to an algebraic reducible system or to a triangular system [56; 8, p. 33].

Three classes of linear nonautonomous differential systems integrable in closed form (algebraic reducible systems, triangular systems, the Lappo-Danilevsky systems) are considered. The regular method of building first integrals for these systems is elaborated [36; 46]. Also, the N.P. Erugin problem of the existence of autonomous first integrals [54, p. 469] for an nonautonomous homogeneous Lappo-Danilevskii differential system is solved [25].

In Section 3 we consider reducible linear nonautonomous differential systems with respect to various transformation groups (periodic, polynomial, orthogonal, Liapunov, exponential, and other groups) to the linear differential systems with constant coefficients.

The spectral method for building first integrals of reducible systems is elaborated.

The notion of reducible systems has been entered by A.M. Lyapunov [57]. The development of the theory of reducible systems is associated with the name of N.P. Erugin [58].

In addition, in this article some examples are given to illustrate the obtained results.
1. Integrals of ordinary linear differential system with constant coefficients

1.1. Linear homogeneous differential system

Consider an ordinary linear homogeneous differential system with constant coefficients
\[ \frac{dx}{dt} = Ax, \]  
where \( x = \text{col}(x_1, \ldots, x_n) \in \mathbb{R}^n \) and \( A = \|a_{ij}\| \) is a square constant matrix of order \( n \) with entries \( a_{ij} \in \mathbb{R}, \xi = 1, \ldots, n, j = 1, \ldots, n \). The differential system (1.1) is induced by the autonomous linear differential operator of first order
\[ \mathfrak{A}(x) = \sum_{\xi=1}^{n} A_{\xi} x \partial_{x_{\xi}} \]  
for all \( x \in \mathbb{R}^n \),

where the vectors \( A_{\xi} = (a_{\xi_1}, \ldots, a_{\xi_n}) \), \( \xi = 1, \ldots, n \).

The differential system (1.1) on a domain \( G \subset \mathbb{R}^{n+1} \) has a basis of first integrals of dimension \( n \). Moreover, the differential system (1.1) has also \( n-1 \) autonomous functionally independent first integrals (autonomous integral basis) on a domain \( X \subset \mathbb{R}^n \).

The aim of Section 1 is to build these bases of first integrals for system (1.1).

1.1.1. Linear partial integral

A complex-valued linear homogeneous function
\[ p: x \rightarrow \sum_{\xi=1}^{n} \nu_{\xi} x_{\xi} \]  
for all \( x \in \mathbb{R}^n \) (\( \nu_{\xi} \in \mathbb{C}, \xi = 1, \ldots, n \)) (1.2)
is a partial integral of system (1.1) if and only if
\[ \mathfrak{A}p(x) = \lambda p(x) \]  
for all \( x \in \mathbb{R}^n, \lambda \in \mathbb{C} \) (1.3)

The identity (1.3) is equivalent to the linear homogeneous system
\[ (B - \lambda E)\nu = 0, \]  
where the column vector \( \nu = \text{col}(\nu_1, \ldots, \nu_n) \in \mathbb{C}^n \), \( E \) is the \( n \times n \) identity matrix, and the matrix \( B \) is the transpose of the matrix \( A \).

The linear homogeneous system (1.4) has a nontrivial solution if and only if
\[ \det(B - \lambda E) = 0. \]  
(1.5)

We shall say that the equation (1.5) is the integral characteristic equation of system (1.1), and its roots are integral characteristic roots of system (1.1). Besides, a solution \( \nu \) of the linear system (1.4) is an eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda \).

Thus we have proved the following statement.

Lemma 1.1. The linear homogeneous function (1.2) is a partial integral of the differential system (1.1) if and only if the vector \( \nu \in \mathbb{C}^n \) is an eigenvector of the matrix \( B \).

Therefore a linear partial integral of the linear homogeneous differential system (1.1) is generated by an eigenvector of the matrix \( B \).

Lemma 1.1 is base for our spectral method of building first integrals of system (1.1). In addition, first integrals of the differential system (1.1) are building by eigenvectors and eigenvalues of the matrix \( B \) with using orders of elementary divisors.

1.1.2. Autonomous first integrals

Case of real eigenvalues. If the matrix \( B \) has two linearly independent real eigenvectors, then we can find an autonomous first integral of the differential system (1.1) by using following assertions (Theorem 1.1, Corollaries 1.1 and 1.2).
Theorem 1.1. Suppose \( \nu^1 \) and \( \nu^2 \) are real eigenvectors of the matrix \( B \) corresponding to the distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \) \((\lambda_1 \neq \lambda_2)\), respectively. Then the ordinary linear homogeneous differential system (1.1) has the autonomous first integral

\[
F: x \rightarrow [\nu^1 x]^{h_1} [\nu^2 x]^{h_2} \quad \text{for all } x \in \mathbb{X}, \quad (1.6)
\]

where \( \mathbb{X} \) is a domain of the domain of function \( DF \subset \mathbb{R}^n \), the numbers \( h_1 \) and \( h_2 \) are a real solution to the equation \( \lambda_1 h_1 + \lambda_2 h_2 = 0 \) with \( |h_1| + |h_2| \neq 0 \).

Proof. By Lemma 1.1, the linear functions

\[ p_k: x \rightarrow \nu^k x \quad \text{for all } x \in \mathbb{R}^n, \quad k = 1, k = 2, \]

are partial integrals of system (1.1). Hence,

\[ \mathfrak{A} \nu^k x = \lambda_k \nu^k x \quad \text{for all } x \in \mathbb{R}^n, \quad k = 1, k = 2. \]

Using these identities, we have

\[
\mathfrak{A} F(x) = [\nu^1 x]^{h_1 - 1} [\nu^2 x]^{h_2 - 1} (h_1 \operatorname{sgn}(\nu^1 x) [\nu^2 x] \mathfrak{A} \nu^1 x + h_2 \operatorname{sgn}(\nu^2 x) [\nu^1 x] \mathfrak{A} \nu^2 x) = \\
= (\lambda_1 h_1 + \lambda_2 h_2) F(x) \quad \text{for all } x \in \mathbb{X} \subset DF.
\]

If \( h_1 \) and \( h_2 \) are real numbers such that \( \lambda_1 h_1 + \lambda_2 h_2 = 0 \) with \( |h_1| + |h_2| \neq 0 \), then the scalar function (1.6) is an autonomous first integral of system (1.1).

Corollary 1.1. If \( \nu \) is a real eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda = 0 \), then the linear homogeneous function

\[
F: x \rightarrow \nu x \quad \text{for all } x \in \mathbb{R}^n \quad (1.7)
\]

is an autonomous first integral of the ordinary linear homogeneous differential system (1.1).

Indeed, suppose \( \nu \) is a real eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda = 0 \). Then, from the identity (1.3) it follows that \( \mathfrak{A} \nu x = 0 \) for all \( x \in \mathbb{R}^n \). This yields that the function (1.7) is an autonomous first integral on the space \( \mathbb{R}^n \) of system (1.1).

Corollary 1.2. Let \( \lambda \neq 0 \) be an eigenvalue of the matrix \( B \) corresponding to two real linearly independent eigenvectors \( \nu^1 \) and \( \nu^2 \). Then the ordinary linear homogeneous differential system (1.1) has the autonomous first integral

\[
F: x \rightarrow \frac{\nu^1 x}{\nu^2 x} \quad \text{for all } x \in \mathbb{X}, \quad (1.8)
\]

where \( \mathbb{X} \) is a domain from the set \( \{ x: \nu^2 x \neq 0 \} \subset \mathbb{R}^n \).

Let us remark that the first integrals (1.7) and (1.8) are algebraic. Note also that algebraicity of the first integral (1.6) depends on the numbers \( h_1 \) and \( h_2 \). For example, if the numbers \( h_1 \) and \( h_2 \) are rational, then the first integral (1.6) is algebraic. But this condition isn’t necessary for algebraicity of the first integral (1.6). At the same time we have

Property 1.1 (sufficient condition for algebraicity of basis of autonomous first integrals). If all eigenvalues of the matrix \( B \) are simple and rational, then the differential system (1.1) has a basis of autonomous algebraic first integrals.

Example 1.1. Consider the fourth-order ordinary linear differential system

\[
x(1) = x_1 - 2x_2 - x_4, \quad x(2) = 2x_2 + x_3 + x_4, \quad x(3) = 2x_1 - 4x_2 + 2x_3 - 2x_4.
\]

\[
\frac{dx_1}{dt} = x_1 - 2x_2 - x_4, \quad \frac{dx_2}{dt} = -x_1 + 4x_2 - x_3 + 2x_4,
\]

\[
\frac{dx_3}{dt} = 2x_2 + x_3 + x_4, \quad \frac{dx_4}{dt} = 2x_1 - 4x_2 + 2x_3 - 2x_4. \quad (1.9)
\]
We claim that the matrix
\[
B = \begin{pmatrix}
1 & -1 & 0 & 2 \\
-2 & 4 & 2 & -4 \\
0 & -1 & 1 & 2 \\
-1 & 2 & 1 & -2
\end{pmatrix}
\]
has the eigenvalues \( \lambda_1 = 0, \lambda_2 = \lambda_3 = 1, \) and \( \lambda_4 = 2. \) Indeed, the characteristic equation
\[
\begin{vmatrix}
1 - \lambda & -1 & 0 & 2 \\
-2 & 4 - \lambda & 2 & -4 \\
0 & -1 & 1 - \lambda & 2 \\
-1 & 2 & 1 & -2 - \lambda
\end{vmatrix} = 0 \iff \lambda(\lambda - 1)^2(\lambda - 2) = 0.
\]

The rank of the matrix \( B - \lambda_3 E \) is equal 2. Therefore the double eigenvalue \( \lambda_2 = 1 \) of the matrix \( B \) has \( \kappa = 4 - 2 = 2 \) simple elementary divisors \( \lambda - 1 \) and \( \lambda - 1. \)

The matrix \( B \) has four simple elementary divisors \( \lambda, \lambda - 1, \lambda - 1, \) and \( \lambda - 2. \)

The linear homogeneous system
\[
(B - \lambda_1 E) \colon \nu_1, \ldots, \nu_4 = 0 \iff \begin{cases}
\nu_1 - \nu_2 + 2\nu_4 = 0, \\
-2\nu_1 + 4\nu_2 + 2\nu_3 - 4\nu_4 = 0, \\
-\nu_2 + \nu_3 + 2\nu_4 = 0, \\
-\nu_1 + 2\nu_2 + \nu_3 - 2\nu_4 = 0
\end{cases} \iff \begin{cases}
\nu_1 = -\nu_4, \\
\nu_2 = \nu_4, \\
\nu_3 = -\nu_4.
\end{cases}
\]

Hence \( \nu^1 = (-1, 1, -1, 1) \) is a real eigenvector corresponding to the eigenvalue \( \lambda_1 = 0 \) of the matrix \( B. \)

The scalar function (by Corollary 1.1)
\[
F_1: x \mapsto -x_1 + x_2 - x_3 + x_4 \quad \text{for all} \quad x \in \mathbb{R}^4
\]
is an autonomous linear first integral of the ordinary differential system (1.9).

The linear homogeneous system
\[
(B - \lambda_2 E) \colon \nu_1, \ldots, \nu_4 = 0 \iff \begin{cases}
-\nu_2 + 2\nu_4 = 0, \\
-2\nu_1 + 3\nu_2 + 2\nu_3 - 4\nu_4 = 0, \\
-\nu_2 + 2\nu_4 = 0, \\
-\nu_1 + 2\nu_2 + \nu_3 - 3\nu_4 = 0
\end{cases} \iff \begin{cases}
\nu_1 = \nu_3 + \nu_4, \\
\nu_2 = 2\nu_4.
\end{cases}
\]

Hence \( \nu^2 = (2, 2, 1, 1) \) and \( \nu^3 = (1, 0, 1, 0) \) are two linearly independent real eigenvectors corresponding to the double eigenvalue \( \lambda_2 = 1 \) of the matrix \( B. \)

The scalar function (by Corollary 1.2)
\[
F_{23}: x \mapsto \frac{2x_1 + 2x_2 + x_3 + x_4}{x_1 + x_3} \quad \text{for all} \quad x \in \mathbb{R}^4
\]
where a domain \( X_1 \subset \{ x: x_1 + x_3 \neq 0 \}, \) is an autonomous first integral of system (1.9).

The linear homogeneous system
\[
(B - \lambda_4 E) \colon \nu_1, \ldots, \nu_4 = 0 \iff \begin{cases}
-\nu_1 - \nu_2 + 2\nu_4 = 0, \\
-2\nu_1 + 2\nu_2 + 2\nu_3 - 4\nu_4 = 0, \\
-\nu_2 - \nu_3 + 2\nu_4 = 0, \\
-\nu_1 + 2\nu_2 + \nu_3 - 4\nu_4 = 0
\end{cases} \iff \begin{cases}
\nu_1 = \nu_3, \\
\nu_2 = 2\nu_4.
\end{cases}
\]

Hence \( \nu^4 = (0, 2, 0, 1) \) is a real eigenvector corresponding to the eigenvalue \( \lambda_4 = 2 \) of the matrix \( B. \)
Using the linearly independent real eigenvectors $\nu^2$ and $\nu^4$ of the matrix $B$, we can build (by Theorem 1.1) an autonomous first integral of the differential system (1.9). Since the equation $h_2 + 2h_4 = 0$, we have, for example, $h_2 = 2$, $h_4 = -1$. The scalar function

$$F_{24} : x \to \frac{(2x_1 + 2x_2 + x_3 + x_4)^2}{2x_2 + x_4} \quad \text{for all } x \in \mathcal{X}_2,$$

where a domain $\mathcal{X}_2 \subset \{x : 2x_2 + x_4 \neq 0\}$, is an autonomous first integral of system (1.9).

The functionally independent first integrals $F_1$, $F_{23}$, and $F_{24}$ are an autonomous integral basis of system (1.9) on any domain $\mathcal{X}$ from the set $\{x : x_1 + x_3 \neq 0, 2x_2 + x_4 \neq 0\} \subset \mathbb{R}^4$.

**Case of complex eigenvalues.** Let the function (1.2) be a complex-valued partial integral of system (1.1). Then, from the identity (1.3) it follows that

$$\mathfrak{A} \Re p(x) = \lambda \Re p(x) - \bar{\lambda} \Im p(x) \quad \text{for all } x \in \mathbb{R}^n,$$

$$\mathfrak{A} \Im p(x) = \bar{\lambda} \Re p(x) + \lambda \Im p(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where the real numbers $\lambda = \Re \lambda$, $\bar{\lambda} = \Im \lambda$. Thus we have the following criteria for the existence of a complex-valued partial integral of system (1.1).

**Lemma 1.2.** The function (1.2) is a complex-valued partial integral of system (1.1) if and only if the system of identities (1.10) holds.

Using Lemma 1.2, we may establish the following propositions.

**Property 1.2.** If the system (1.1) has a complex-valued partial integral (1.2), then the complex conjugate function $\bar{p}$ is a complex-valued partial integral of system (1.1) and

$$\mathfrak{A} \bar{p}(x) = \bar{\bar{p}}(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where $\bar{\lambda}$ is the conjugate of the complex number $\lambda$ from the identity (1.3).

**Proof.** Using the system of identities (1.10), we get

$$\mathfrak{A} \bar{p}(x) = \mathfrak{A} \Re p(x) - i \mathfrak{A} \Im p(x) = \lambda \Re p(x) - \bar{\lambda} \Im p(x) - i (\bar{\lambda} \Re p(x) + \lambda \Im p(x)) =$$

$$= (\lambda - i \bar{\lambda})(\Re p(x) - i \Im p(x)) = \bar{\bar{p}}(x) \quad \text{for all } x \in \mathbb{R}^n. \blacksquare$$

Let $\mathbb{K}$ be either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$.

**Property 1.3.** The product $u_1 u_2$ of the polynomials $u_1 : \mathbb{R}^n \to \mathbb{K}$ and $u_2 : \mathbb{R}^n \to \mathbb{K}$ is a polynomial partial integral of system (1.1) if and only if the polynomials $u_1$ and $u_2$ are polynomial partial integrals of system (1.1).

**Proof.** From definition of partial integral and

$$\mathfrak{A} (u_1(x) u_2(x)) = u_2(x) \mathfrak{A} u_1(x) + u_1(x) \mathfrak{A} u_2(x) \quad \text{for all } x \in \mathbb{R}^n,$$

we get the assertion of Property 1.3. $\blacksquare$

**Property 1.4.** The system (1.1) has the complex-valued polynomial partial integral (1.2) if and only if the real polynomial

$$P : x \to \Re^2 p(x) + \Im^2 p(x) \quad \text{for all } x \in \mathbb{R}^n$$

is a partial integral of system (1.1). Moreover, the following identity holds

$$\mathfrak{A} P(x) = 2 \lambda P(x) \quad \text{for all } x \in \mathbb{R}^n,$$

where the number $\lambda = \lambda + i \bar{\lambda}$, $\bar{\lambda} = \Im \lambda$ is given by the identity (1.3).

**Proof.** Using Properties 1.2 and 1.3, and the identity

$$p(x) \bar{p}(x) = \Re^2 p(x) + \Im^2 p(x) \quad \text{for all } x \in \mathbb{R}^n,$$
we have the polynomial \( P \) is a real partial integral of system (1.1) if and only if the linear function (1.2) is a complex-valued partial integral of system (1.1).

By the system of identities (1.10), it follows that
\[
\mathfrak{A}(\text{Re}^2 p(x) + \text{Im}^2 p(x)) = 2 \text{Re} p(x) \mathfrak{A} \text{Re} p(x) + 2 \text{Im} p(x) \mathfrak{A} \text{Im} p(x) =
\]
\[
= 2 \text{Re} p(x)(\hat{\lambda} \text{Re} p(x) - \bar{\lambda} \text{Im} p(x)) + 2 \text{Im} p(x) (\bar{\lambda} \text{Re} p(x) + \hat{\lambda} \text{Im} p(x)) =
\]
\[
= 2 \hat{\lambda}(\text{Re}^2 p(x) + \text{Im}^2 p(x)) = 2 \hat{\lambda} P(x) \quad \text{for all} \ x \in \mathbb{R}^n. \quad \blacksquare
\]

**Property 1.5.** Let the function (1.2) be a complex-valued partial integral of system (1.1). Then the Lie derivative of the function
\[
\varphi: x \rightarrow \arctan \frac{\text{Im} p(x)}{\text{Re} p(x)} \quad \text{for all} \ x \in \mathcal{X} \subset \{x: \text{Re} p(x) \neq 0\}
\]
by virtue of system (1.1) is equal to
\[
\mathfrak{A} \varphi(x) = \bar{\lambda} \quad \text{for all} \ x \in \mathcal{X},
\]
where the number \( \lambda = \hat{\lambda} + i \bar{\lambda} \) (\( \hat{\lambda} = \text{Re} \lambda, \ \bar{\lambda} = \text{Im} \lambda \)) is given by the identity (1.3).

**Proof.** Using the identities (1.10), we obtain
\[
\mathfrak{A} \varphi(x) = \mathfrak{A} \arctan \frac{\text{Im} p(x)}{\text{Re} p(x)} = \frac{1}{1 + \frac{\text{Im}^2 p(x)}{\text{Re}^2 p(x)}} \frac{\text{Re} p(x) \mathfrak{A} \text{Im} p(x) - \text{Im} p(x) \mathfrak{A} \text{Re} p(x)}{\text{Re}^2 p(x)} =
\]
\[
= \frac{\text{Re} p(x)(\hat{\lambda} \text{Re} p(x) - \bar{\lambda} \text{Im} p(x)) - \text{Im} p(x)(\bar{\lambda} \text{Re} p(x) + \hat{\lambda} \text{Im} p(x))}{\text{Re}^2 p(x) + \text{Im}^2 p(x)} = \bar{\lambda} \quad \text{for all} \ x \in \mathcal{X}. \quad \blacksquare
\]

If the matrix \( B \) has an imaginary eigenvalue, then we can build an autonomous first integral of system (1.1) by using following assertions (Theorems 1.2, 1.3, and 1.4).

**Theorem 1.2.** Suppose \( \nu = \bar{\nu} + \bar{\nu} i \) (\( \text{Re} \nu = \bar{\nu}, \ \text{Im} \nu = \bar{\nu} \)) is an eigenvector of the matrix \( B \) corresponding to the imaginary eigenvalue \( \lambda = \hat{\lambda} + i \bar{\lambda} \) (\( \text{Re} \lambda = \hat{\lambda}, \ \text{Im} \lambda = \bar{\lambda} \neq 0 \)). Then the ordinary differential system (1.1) has the autonomous first integral
\[
F: x \rightarrow ((\bar{\nu} x)^2 + (\bar{\nu} x)^2) \exp\left(-2 \frac{\hat{\lambda}}{\lambda} \arctan \frac{\bar{\nu} x}{\bar{\nu} x}\right) \quad \text{for all} \ x \in \mathcal{X}, \quad (1.11)
\]
where \( \mathcal{X} \) is a domain from the set \( \{x: \nu x \neq 0\} \).

**Proof.** Taking into account Properties 1.4 and 1.5, we get
\[
\mathfrak{A} F(x) = \exp\left(-2 \frac{\hat{\lambda}}{\lambda} \arctan \frac{\bar{\nu} x}{\bar{\nu} x}\right) \mathfrak{A} \left((\bar{\nu} x)^2 + (\bar{\nu} x)^2\right) + ((\bar{\nu} x)^2 + (\bar{\nu} x)^2) \exp\left(-2 \frac{\hat{\lambda}}{\lambda} \arctan \frac{\bar{\nu} x}{\bar{\nu} x}\right) \
\]
\[
= \left(2 \hat{\lambda} - 2 \frac{\hat{\lambda}}{\lambda} \bar{\lambda}\right) \left((\bar{\nu} x)^2 + (\bar{\nu} x)^2\right) \exp\left(-2 \frac{\hat{\lambda}}{\lambda} \arctan \frac{\bar{\nu} x}{\bar{\nu} x}\right) = 0 \quad \text{for all} \ x \in \mathcal{X}.
\]
This implies that the function (1.11) is an autonomous first integral of system (1.1). \( \blacksquare \)

Transcendency of the first integral (1.11) of systems (1.1) depends on the imaginary eigenvalue \( \lambda = \hat{\lambda} + i \bar{\lambda} \) (if \( \hat{\lambda} = 0 \), then the first integral (1.11) of system (1.1) is algebraic).

The proof of Theorems 1.3 and 1.4 is similar to that one in Theorem 1.2.
Theorem 1.3. Let \( \nu^1 = \nu^1 + \bar{\nu}^1 i \) (\( \text{Re} \nu^1 = \bar{\nu}^1 \), \( \text{Im} \nu^1 = -\bar{\nu}^1 \)) be an eigenvector of the matrix \( B \) corresponding to the imaginary eigenvalue \( \lambda_1 = \lambda_1 + \bar{\lambda}_1 i \) (\( \text{Re} \lambda_1 = \bar{\lambda}_1 \), \( \text{Im} \lambda_1 = \bar{\lambda}_1 \neq 0 \)), \( \nu^2 \) be a real eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda_2 \neq 0 \). Then the ordinary differential system (1.1) has the autonomous first integral
\[
F: x \rightarrow \nu^2 x \exp \left( -\frac{\lambda_2}{\lambda_1} \arctan \left( \frac{\bar{\nu}^1 x}{\nu^1 x} \right) \right) \quad \text{for all } x \in \mathcal{X},
\]
where \( \mathcal{X} \) is a domain from the set \( \{ x : \nu^1 x \neq 0 \} \).

Theorem 1.4. Let \( \nu^1 = \nu^1 + \bar{\nu}^1 i \) and \( \nu^2 = \nu^2 + \bar{\nu}^2 i \) (\( \text{Re} \nu^1 = \nu^1 \), \( \text{Im} \nu^1 = \bar{\nu}^1 \), \( \tau = 1, 2 \)) be two linearly independent eigenvectors of the matrix \( B \) corresponding to the imaginary eigenvalues \( \lambda_1 = \lambda_1 + \bar{\lambda}_1 i \) and \( \lambda_2 = \lambda_2 + \bar{\lambda}_2 i \) (\( \text{Re} \lambda_1 = \lambda_1 \), \( \text{Im} \lambda_1 = \bar{\lambda}_1 \), \( \tau = 1, 2 \)) with \( \lambda_1 \neq \bar{\lambda}_2 \). Then the ordinary differential system (1.1) has the autonomous first integral
\[
F: x \rightarrow \lambda_1 \arctan \left( \frac{\bar{\nu}^2 x}{\nu^2 x} \right) - \lambda_2 \arctan \left( \frac{\bar{\nu}^1 x}{\nu^1 x} \right) \quad \text{for all } x \in \mathcal{X},
\]
where \( \mathcal{X} \) is a domain from the set \( \{ x : \nu^1 x \neq 0, \nu^2 x \neq 0 \} \).

Example 1.2. The autonomous differential system
\[
\frac{dx_1}{dt} = 2x_1 + x_2, \quad \frac{dx_2}{dt} = x_1 + 3x_2 - x_3, \quad \frac{dx_3}{dt} = -x_1 + 2x_2 + 3x_3 \quad (1.12)
\]
has the eigenvalues \( \lambda_1 = 3 + i \) and \( \lambda_2 = 2 \) corresponding to the eigenvectors \( \nu^1 = (1, i, -1) \) and \( \nu^2 = (3, -1, -1) \), respectively. The first integrals (by Theorem 1.2)
\[
F_1: x \rightarrow (x_1 - x_3)^2 + x_2^2 \exp \left( -6 \arctan \left( \frac{x_2}{x_1 - x_3} \right) \right) \quad \text{for all } x \in \mathcal{X}
\]
and (by Theorem 1.3)
\[
F_2: x \rightarrow (3x_1 - x_2 - x_3) \exp \left( -2 \arctan \left( \frac{x_2}{x_1 - x_3} \right) \right) \quad \text{for all } x \in \mathcal{X}
\]
are an autonomous integral basis of system (1.1) on any domain \( \mathcal{X} \subset \{ x : x_1 - x_3 \neq 0 \} \).

Example 1.3. The autonomous differential system
\[
\frac{dx_1}{dt} = -3x_1 + x_2 + 4x_3 + 2x_4, \quad \frac{dx_2}{dt} = 8x_1 - 3x_2 - 2x_3 + 6x_4, \quad \frac{dx_3}{dt} = -9x_1 + 3x_2 + 4x_3 - 4x_4, \quad \frac{dx_4}{dt} = 6x_1 - 3x_2 - 4x_3 + 2x_4 \quad (1.13)
\]
has the eigenvalues \( \lambda_1 = i \), \( \lambda_2 = 2i \) corresponding to the eigenvectors \( \nu^1 = (1 - i, -1 + 2i, 2i, 2) \), \( \nu^2 = (i, -1, i, 1 + 2i) \). The functionally independent first integrals (by Theorem 1.2)
\[
F_1: x \rightarrow (x_1 - x_3 + 2x_4)^2 + (-x_1 + 2x_2 + 2x_3)^2 \quad \text{for all } x \in \mathbb{R}^4,
\]
\[
F_2: x \rightarrow (-x_1 + 2x_4)^2 + (x_1 + x_3 + 2x_4)^2 \quad \text{for all } x \in \mathbb{R}^4,
\]
and (by Theorem 1.4)
\[
F_3: x \rightarrow \arctan \left( \frac{x_1 + x_3 + 2x_4}{-x_2 + x_4} \right) - 2 \arctan \left( \frac{x_1 + 2x_2 + 2x_3}{x_1 - x_2 + 2x_4} \right) \quad \text{for all } x \in \mathcal{X}
\]
are an autonomous integral basis of the ordinary differential system (1.13) on any domain \( \mathcal{X} \) from the set \( \{ x : x_1 - x_2 + 2x_4 \neq 0, x_2 - x_4 \neq 0 \} \subset \mathbb{R}^4 \).
Case of multiple elementary divisors.

**Definition 1.1.** Let \( \nu^0 \) be an eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda \) with the elementary divisor of multiplicity \( m \). A non-zero vector \( \nu \in \mathbb{C}^n \) is called a generalized eigenvector of order \( k \) if the vector \( \nu^k \) satisfying

\[
(B - \lambda E) \nu^k = k \nu^{k-1}, \quad k = 1, \ldots, m - 1. \tag{1.14}
\]

In this case we can build first integrals of system (1.1) by using following assertions.

**Theorem 1.5.** Let \( \lambda \) be the eigenvalue of the matrix \( B \) with elementary divisor of multiplicity \( m \) \((m \geq 2)\) corresponding to the real eigenvector \( \nu^0 \) and to the real 1-st order generalized eigenvector \( \nu^1 \). Then the system (1.1) has the autonomous first integral

\[
F: x \rightarrow \nu^0 x \exp\left( -\lambda \frac{\nu^1 x}{\nu^0 x} \right) \quad \text{for all } x \in \mathcal{X}, \quad \mathcal{X} \subset \{ x: \nu^0 x \neq 0 \}.
\]

**Proof.** Using the equalities (1.14) under the condition \( k = 1 \), we get

\[
\mathfrak{A} \nu^1 x = \lambda \nu^1 x + \nu^0 x \quad \text{for all } x \in \mathbb{R}^n.
\]

Then, the Lie derivative of the function \( F \) by virtue of system (1.1) is

\[
\mathfrak{A} F(x) = \left( \lambda - \lambda \frac{(\nu^1 x + \nu^0 x) \nu^0 x - \lambda \nu^0 x \nu^1 x}{(\nu^0 x)^2} \right) F(x) = 0 \quad \text{for all } x \in \mathcal{X}. \]

From Theorem 1.5, we have the following

**Corollary 1.3.** Let \( \lambda = \lambda \) \( \dagger \) \( \lambda i \) \((\Re \lambda = \lambda, \Im \lambda = \lambda \neq 0)\) be the complex eigenvalue of the matrix \( B \) with elementary divisor of multiplicity \( m \) \((m \geq 2)\) corresponding to the eigenvector \( \nu^0 = \nu^0 + \nu^0 i \) \((\Re \nu^0 = \nu^0, \Im \nu^0 = \nu^0)\) and to the 1-st order generalized eigenvector \( \nu^1 = \nu^1 + \nu^1 i \) \((\Re \nu^1 = \nu^1, \Im \nu^1 = \nu^1)\). Then the ordinary differential system (1.1) on a domain \( \mathcal{X} \subset \{ x: \nu^0 x \neq 0 \} \) has the autonomous first integrals

\[
F_1: x \rightarrow \left( \nu^0 x \right)^2 + (\nu^0 x)^2 \exp\left( -2 \frac{\lambda \alpha(x) - \lambda \beta(x)}{(\nu^0 x)^2 + (\nu^0 x)^2} \right) \quad \text{for all } x \in \mathcal{X}
\]

and

\[
F_2: x \rightarrow \arctan \frac{\nu^0 x}{(\nu^0 x)^2 + (\nu^0 x)^2} \quad \text{for all } x \in \mathcal{X},
\]

where the polynomials \( \alpha: x \rightarrow \nu^0 x \nu^1 x + \nu^0 x \nu^1 x, \beta: x \rightarrow \nu^0 x \nu^1 x - \nu^0 x \nu^1 x \) for all \( x \in \mathbb{R}^n \).

The proof of Theorem 1.6 is similar to that one in Theorem 1.5.

**Theorem 1.6.** Suppose \( \nu^0 \) and \( \nu^1 \) are a real eigenvector and a real 1-st order generalized eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda_1 = 0 \) with elementary divisor of multiplicity \( m \geq 2 \), and \( \nu^2 \) is a real eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda_2 \). Then the system (1.1) on a domain \( \mathcal{X} \) has the autonomous first integral

\[
F: x \rightarrow \nu^2 x \exp\left( -\lambda_2 \frac{\nu^1 x}{\nu^0 x} \right) \quad \text{for all } x \in \mathcal{X}, \quad \mathcal{X} \subset \{ x: \nu^0 x \neq 0 \}.
\]

If \( \lambda_2 = 0 \), then from Theorem 1.6, we have Corollary 1.1.

**Example 1.4.** The autonomous system of ordinary linear differential equations

\[
\frac{dx_1}{dt} = 4x_1 - 5x_2 + 2x_3, \quad \frac{dx_2}{dt} = 5x_1 - 7x_2 + 3x_3, \quad \frac{dx_3}{dt} = 6x_1 - 9x_2 + 4x_3 \tag{1.15}
\]

has the eigenvalue \( \lambda_1 = 0 \) with elementary divisor \( \lambda^2 \) of multiplicity 2 corresponding to the eigenvector \( \nu^1 = (1, -2, 1) \) and to the 1-st order generalized eigenvector \( \nu^2 = (0, -1, 1) \), and the simple eigenvalue \( \lambda_3 = 1 \) with elementary divisor \( \lambda - 1 \) corresponding to the eigenvector...
\( \nu^3 = (3, -3, 1) \). The scalar functions (by Theorem 1.5)

\[
F_1: x \to x_1 - 2x_2 + x_3 \quad \text{for all } x \in \mathbb{R}^3
\]

and (by Theorem 1.6)

\[
F_2: x \to (3x_1 - 3x_2 + x_3) \exp \frac{x_2 - x_3}{x_1 - 2x_2 + x_3} \quad \text{for all } x \in X
\]

are an autonomous integral basis of system (1.15) on a domain \( X \subset \{ x : x_1 - 2x_2 + x_3 \neq 0 \} \).

From Theorem 1.6, we obtain

**Corollary 1.4.** Suppose \( \nu^0 \) and \( \nu^1 \) are a real eigenvector and a real 1-th order generalized eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda_1 = 0 \) with elementary divisor of multiplicity \( m \geq 2 \), and \( \nu^2 = \nu^* + \nu^2 i \) (\( \text{Re} \nu^2 = \nu^2, \text{Im} \nu^2 = \nu^2 \)) is an eigenvector of the matrix \( B \) corresponding to the complex eigenvalue \( \lambda_2 = \lambda_2 + i \lambda_2 \) (\( \text{Re} \lambda_2 = \lambda_2, \text{Im} \lambda_2 = \lambda_2 \neq 0 \)).

Then the system (1.1) on a domain \( X \) has the autonomous first integrals

\[
F_1: x \to \left( (\nu^2 x)^2 + (\nu^2 x)^2 \right) \exp \left( -2 \frac{\nu^1 x}{\nu^0 x} \right) \quad \text{for all } x \in X
\]

and

\[
F_2: x \to \arctan \frac{\nu^2 x}{\nu^2 x} - \tilde{\lambda}_2 \frac{\nu^1 x}{\nu^0 x} \quad \text{for all } x \in X, \quad X \subset \{ x : \nu^0 x \neq 0, \nu^2 x \neq 0 \}.
\]

**Theorem 1.7.** Suppose \( \nu^{01} \) and \( \nu^{11} \) are a real eigenvector and a real 1-th order generalized eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda_1 \) with elementary divisor of multiplicity \( m_1 \geq 2 \), and \( \nu^{02} \) and \( \nu^{12} \) are a real eigenvector and a real 1-th order generalized eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda_2 \) with elementary divisor of multiplicity \( m_2 \geq 2 \). Then the system (1.1) has the autonomous first integral

\[
F: x \to \frac{\nu^{11} x}{\nu^{01} x} - \frac{\nu^{12} x}{\nu^{02} x} \quad \text{for all } x \in X, \quad X \subset \{ x : \nu^{01} x \neq 0, \nu^{02} x \neq 0 \}.
\]

Indeed, the Lie derivative of the function \( F \) on a domain \( X \) by virtue of system (1.1) is

\[
\mathfrak{A} F(x) = \frac{(\lambda_1 \nu^{11} x + \nu^{01} x) \nu^{01} x - \lambda_1 \nu^{01} x \nu^{11} x}{(\nu^{01} x)^2} - \frac{(\lambda_2 \nu^{12} x + \nu^{02} x) \nu^{02} x - \lambda_2 \nu^{02} x \nu^{12} x}{(\nu^{02} x)^2} = 0.
\]

**Theorem 1.8.** Suppose \( \lambda \) is the eigenvalue with elementary divisor of multiplicity \( m \geq 2 \) of the matrix \( B \) corresponding to an eigenvector \( \nu^0 \) and to generalized eigenvectors \( \nu^k, k = 1, \ldots, m - 1 \). Then the system (1.1) on a domain \( X \subset \{ x : \nu^0 x \neq 0 \} \) has the functionally independent autonomous first integrals

\[
F_\zeta: x \to \Psi_\zeta(x) \quad \text{for all } x \in X, \quad \zeta = 2, \ldots, m - 1,
\]

where the functions \( \Psi_\zeta: X \to \mathbb{R}, \zeta = 2, \ldots, m - 1, \) are the solution to the system

\[
\nu^k x = \sum_{\tau=1}^{k} \binom{k-1}{\zeta-1} \Psi_\tau(x) \nu^{k-\tau} x \quad \text{for all } x \in X, \quad k = 1, \ldots, m - 1.
\]

Proof. Using the identities (1.3) and the equalities (1.14), we obtain

\[
\mathfrak{A} \nu^0 x = \lambda \nu^0 x \quad \text{for all } x \in \mathbb{R}^n,
\]

\[
\mathfrak{A} \nu^k x = \lambda \nu^k x + k \nu^{k-1} x \quad \text{for all } x \in \mathbb{R}^n, \quad k = 1, \ldots, m - 1.
\]

The system (1.17) has the determinant \( (\nu^0 x)^{m-1} \) such that \( (\nu^0 x)^{m-1} \neq 0 \) for all \( x \) from a domain \( X \subset \{ x : \nu^0 x \neq 0 \} \). Therefore there exists the solution \( \Psi_\tau, \tau = 1, \ldots, m - 1, \)
on the domain $\mathcal{X}$ of the functional system (1.17). Let us show that

$$
\mathfrak{A}\Psi_k(x) = \begin{cases} 
1 & \text{for all } x \in \mathcal{X}, \ k = 1, \\
0 & \text{for all } x \in \mathcal{X}, \ k = 2, \ldots, m - 1,
\end{cases}
$$

(1.19)

The proof of identities (1.19) is by induction on $m$.

For $m = 2$ and $m = 3$, the assertions (1.19) follows from the identities (1.18).

Assume that the identities (1.19) for $m = \varepsilon$ is true. Using the system of identities (1.18) and the system (1.17) for $m = \varepsilon + 1$, $m = \varepsilon$, we get

$$
\mathfrak{A}\nu^x = \lambda \sum_{\tau=1}^{\varepsilon} \left( \frac{-1}{\tau-1} \right) \Psi_\tau(x) \nu^{\varepsilon-\tau} x + (\varepsilon - 1) \sum_{\tau=1}^{\varepsilon-1} \left( \frac{-2}{\tau-1} \right) \Psi_\tau(x) \nu^{\varepsilon-\tau} x + \nu^{\varepsilon-1} x + \nu^0 x \mathfrak{A}\Psi_\varepsilon(x)
$$

for all $x \in \mathcal{X}$, $\mathcal{X} \subset \{ x : \nu^0 x \neq 0 \}$.

Now taking into account the system (1.17) with $k = \varepsilon - 1$ and $k = \varepsilon$, the identity (1.18) with $k = \varepsilon$, and $\nu^0 x \neq 0$ for all $x \in \mathcal{X}$, we have

$$
\mathfrak{A}\Psi_\varepsilon(x) = 0 \quad \text{for all } x \in \mathcal{X}.
$$

This implies that the identities (1.19) for $m = \varepsilon + 1$ are true. So by the principle of mathematical induction, the statement (1.19) is true for every natural number $m \geq 2$.

Thus the functions (1.16) are functionally independent first integrals of system (1.1).

Theorem 1.8 is true both for the case of the real eigenvalue $\lambda$ and for the case of the complex eigenvalue $\lambda$ (Im $\lambda \neq 0$). In the complex case, from the complex-valued first integrals (1.16) of system (1.1), we obtain the real-valued first integrals of system (1.1)

$$
F^1_\zeta : x \to \Re \Psi_\zeta(x), \quad F^2_\zeta : x \to \Im \Psi_\zeta(x) \quad \text{for all } x \in \mathcal{X}, \quad \zeta = 2, \ldots, m - 1,
$$

where $\mathcal{X}$ is a domain from the set $\{ x : (\nu^0 x)^2 + (\nu^0 x)^2 \neq 0 \} \subset \mathbb{R}^n$.

Example 1.5. Consider the linear autonomous differential system

$$
\frac{dx_1}{dt} = 4x_1 - x_2, \quad \frac{dx_2}{dt} = 3x_1 + x_2 - x_3, \quad \frac{dx_3}{dt} = x_1 + x_3.
$$

(1.20)

The matrix $B = \begin{bmatrix} 4 & 3 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ has one triple eigenvalue $\lambda_1 = 2$.

The rank of the matrix $B - \lambda_1 E$ is equal 2. Therefore the eigenvalue $\lambda_1 = 2$ has $\kappa_1 = 3 - 2 = 1$ elementary divisor $(\lambda - 2)^3$ of multiplicity 3.

The system $(B - \lambda_1 E) \colon \nu_1, \nu_2, \nu_3 = 0 \Leftrightarrow \begin{cases} 2\nu_1 + 3\nu_2 + \nu_3 = 0, \\
-\nu_1 - \nu_2 = 0, \\
-\nu_2 - \nu_3 = 0 \end{cases} \Leftrightarrow \begin{cases} \nu_1 = \nu_3, \\
\nu_2 = -\nu_3. \end{cases}$

Hence $\nu^0 = (1, -1, 1)$ is an eigenvector of $B$ corresponding to the eigenvalue $\lambda_1 = 2$.

The 1-st order generalized eigenvector $\nu^1$ of the matrix $B$ corresponding to the eigenvalue $\lambda_1 = 2$ is a solution of the system

$$
(B - \lambda_1 E) \colon \nu_1, \nu_2, \nu_3 = \colon \nu_1, \nu_2, \nu_3 = 0 \Leftrightarrow \begin{cases} 2\nu_1 + 3\nu_2 + \nu_3 = 1, \\
-\nu_1 - \nu_2 = -1, \\
-\nu_2 - \nu_3 = 1 \end{cases} \Leftrightarrow \begin{cases} \nu_1 = -\nu_2 + 1, \\
\nu_3 = -\nu_2 - 1. \end{cases}
$$

Hence $\nu^1 = (1, 0, -1)$ is a generalized eigenvector of the 1-st order of the matrix $B$ corresponding to the eigenvalue $\lambda_1 = 2$. 

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The 2-nd order generalized eigenvector \( \nu^2 \) of the matrix \( B \) corresponding to the eigenvalue \( \lambda_1 = 2 \) is a solution of the system

\[
(B - \lambda_1 E) \colon (\nu_1, \nu_2, \nu_3) = 2 \colon (1, 0, -1) \Leftrightarrow \begin{cases}
2\nu_1 + 3\nu_2 + \nu_3 = 2, \\
-\nu_1 - \nu_2 = 0, \\
-\nu_2 - \nu_3 = -2
\end{cases}
\]

Hence \( \nu^2 = (0, 0, 2) \) is a generalized eigenvector of the 2-nd order of the matrix \( B \) corresponding to the eigenvalue \( \lambda_1 = 2 \).

The functionally independent scalar functions (by Theorem 1.5)

\[
F_1: (x_1, x_2, x_3) \rightarrow (x_1 - x_2 + x_3) \exp\left( -2 \frac{x_1 - x_3}{x_1 - x_2 + x_3} \right) \quad \text{for all } (x_1, x_2, x_3) \in X,
\]

\[
F_2: (x_1, x_2, x_3) \rightarrow \frac{(x_1 - x_3)^2 - 2x_3(x_1 - x_2 + x_3)}{(x_1 - x_2 + x_3)^2} \quad \text{for all } (x_1, x_2, x_3) \in X \quad \text{(by Theorem 1.8)}
\]

are a basis of autonomous first integrals on a domain \( X \subset \{(x_1, x_2, x_3): x_1 - x_2 + x_3 \neq 0\} \) of the ordinary differential system (1.20).

**Example 1.6.** The sixth-order ordinary autonomous linear differential system

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 - 2x_2 + x_3 - 2x_6, & \frac{dx_2}{dt} &= 3x_2 - x_3 - x_5 + 2x_6, \\
\frac{dx_3}{dt} &= -x_1 + x_3 + 2x_4 + 2x_5, & \frac{dx_4}{dt} &= -x_1 + x_4 + x_5 + x_6, \\
\frac{dx_5}{dt} &= x_1 + x_2 + x_5, & \frac{dx_6}{dt} &= x_1 - x_2 + x_3 - x_4 - x_6
\end{align*}
\]

has the triple complex eigenvalue \( \lambda_1 = 1 + i \) with the elementary divisor \( (\lambda - 1 - i)^3 \) of multiplicity 3 corresponding to the eigenvector \( \nu^0 = (1, 1, 0, 0, i, 0) \) and to the generalized eigenvectors \( \nu^1 = (0, 1, 0, i, i, 1), \nu^2 = (0, 1, i, 0, i, 0) \). The functions

\[
F_1: x \rightarrow P(x) \exp(-2\varphi(x)) \quad \text{for all } x \in X \quad \text{(by Theorem 1.2)},
\]

\[
F_2: x \rightarrow P(x) \exp\left( -2 \frac{\alpha(x) - \beta(x)}{P(x)} \right) \quad \text{for all } x \in X \quad \text{(by Corollary 1.3)},
\]

\[
F_3: x \rightarrow \varphi(x) - \frac{\alpha(x) + \beta(x)}{P(x)} \quad \text{for all } x \in X \quad \text{(by Corollary 1.3)},
\]

\[
F_4: x \rightarrow \frac{\gamma(x)P(x) + \beta^2(x) - \alpha^2(x)}{P^2(x)} \quad \text{for all } x \in X \quad \text{(by Theorem 1.8)},
\]

\[
F_5: x \rightarrow \frac{\delta(x)P(x) - 2\alpha(x)\beta(x)}{P^2(x)} \quad \text{for all } x \in X \quad \text{(by Theorem 1.8)},
\]

where the polynomials

\[
P: x \rightarrow (x_1 + x_2)^2 + x_5^2, \quad \alpha: x \rightarrow (x_1 + x_2)(x_2 + x_6) + x_5(x_4 + x_5),
\]

\[
\beta: x \rightarrow (x_1 + x_2)(x_4 + x_5) - x_5(x_2 + x_6), \quad \gamma: x \rightarrow x_2(x_1 + x_2) + x_5(x_3 + x_5),
\]

\[
\delta: x \rightarrow (x_1 + x_2)(x_3 + x_5) - x_2x_5 \quad \text{for all } x \in \mathbb{R}^6,
\]

the scalar function \( \varphi: x \rightarrow \arctan \frac{x_5}{x_1 + x_2} \) for all \( x \in X \), are first integrals on a domain \( X \) from the set \( \{x: x_1 + x_2 \neq 0\} \subset \mathbb{R}^6 \) of the ordinary differential system (1.21).

The functionally independent first integrals \( F_1, \ldots, F_5 \) are an autonomous integral basis on a domain \( X \) of the ordinary differential system (1.21).
1.1.3. Nonautonomous first integrals

The ordinary differential system (1.1) is induced the nonautonomous linear differential operator of first order $\mathfrak{B}(t, x) = \partial_t + \mathfrak{A}(x)$ for all $(t, x) \in \mathbb{R}^{n+1}$.

Adding one nonautonomous first integral of system (1.1) to an autonomous integral basis of system (1.1), we can construct a basis of first integrals for system (1.1).

Such procedure every time can be carried out on the base of the following statements.

Theorem 1.9. Suppose $\nu$ is a real eigenvector of the matrix $B$ corresponding to the eigenvalue $\lambda$. Then the ordinary differential system (1.1) has the first integral

$$F: (t, x) \to \nu x \exp(-\lambda t) \text{ for all } (t, x) \in \mathbb{R}^{n+1}.$$ 

Indeed, the Lie derivative of the function $F$ by virtue of system (1.1) is

$$\mathfrak{B}F(t, x) = \partial_t F(t, x) + \mathfrak{A}F(t, x) = -\lambda F(t, x) + \lambda F(t, x) = 0 \text{ for all } (t, x) \in \mathbb{R}^{n+1}.$$

Example 1.7 (continuation of Example 1.1). By Theorem 1.9, using the eigenvalue $\lambda_2 = 1$ corresponding to the eigenvector $\nu^2 = (2, 2, 1, 1)$, we can build the first integral $F_2: (t, x) \to (2x_1 + 2x_2 + x_3 + x_4) e^{-t}$ for all $(t, x) \in \mathbb{R}^5$ of system (1.9).

The functions $F_1, F_{23}, F_{24}$ (constructed in Example 1.1), and the function $F_2$ are an integral basis of the autonomous differential system (1.9) on a domain $\mathbb{R} \times X$, where $X$ is a domain from the set $\{x: x_1 + x_3 \neq 0, \ 2x_2 + x_4 \neq 0\} \subset \mathbb{R}^4$.

Example 1.8 (continuation of Example 1.2). The autonomous differential system (1.12) has the eigenvector $\nu^2 = (3, -1, -1)$ corresponding to the eigenvalue $\lambda_2 = 2$.

By Theorem 1.9, the differential system (1.12) has the first integral

$$F_3: (t, x_1, x_2, x_3) \to (3x_1 - x_2 - x_3) e^{-2t} \text{ for all } (t, x_1, x_2, x_3) \in \mathbb{R}^4.$$

The functionally independent first integrals $F_1, F_2$ (constructed in Example 1.1), and $F_3$ are a basis of first integrals for the differential system (1.12) on a domain $\mathbb{R} \times X$, where $X$ is a domain from the set $\{(x_1, x_2, x_3): x_1 - x_3 \neq 0\}$.

Corollary 1.5. Let $\nu = \nu^* + \nu i \ (\text{Re} \nu = \nu^*, \ \text{Im} \nu = \nu)$ be an eigenvector of the matrix $B$ corresponding to the complex eigenvalue $\lambda = \lambda^* + \lambda i \ (\text{Re} \lambda = \lambda^*, \ \text{Im} \lambda = \lambda \neq 0)$. Then the ordinary differential system (1.1) has the first integrals

$$F_1: (t, x) \to ((\nu^* x)^2 + (\nu x)^2) \exp(-2\lambda^* t) \text{ for all } (t, x) \in \mathbb{R}^{n+1}$$

and

$$F_2: (t, x) \to \arctan \frac{\nu x}{\bar{\nu} x} - \lambda t \text{ for all } (t, x) \in \mathbb{R} \times X,$$

where $X$ is a domain from the set $\{x: \nu x \neq 0\} \subset \mathbb{R}^n$.

Proof. Using Properties 1.4 and 1.5, we get

$$\mathfrak{B} F_1(t, x) = \exp(-2\lambda^* t) \mathfrak{B} ((\nu^* x)^2 + (\nu x)^2) + ((\nu^* x)^2 + (\nu x)^2) \mathfrak{B} \exp(-2\lambda^* t) =$$

$$= 2\lambda^* ((\nu^* x)^2 + (\nu x)^2) \exp(-2\lambda^* t) + ((\nu^* x)^2 + (\nu x)^2) \partial_t \exp(-2\lambda^* t) = 0 \text{ for all } (t, x) \in \mathbb{R}^{n+1},$$

$$\mathfrak{B} F_2(t, x) = \mathfrak{B} \arctan \frac{\nu x}{\bar{\nu} x} - \mathfrak{B} (\lambda t) = \lambda - \partial_t (\lambda t) = 0 \text{ for all } (t, x) \in \mathbb{R} \times X.$$

Therefore the scalar functions $F_1: \mathbb{R}^{n+1} \to \mathbb{R}$ and $F_2: \mathbb{R} \times X \to \mathbb{R}$ are first integrals of the ordinary linear autonomous differential system (1.1).
Example 1.9 (continuation of Example 1.3). Using the eigenvalue $\lambda_1 = i$ corresponding to the eigenvector $\nu^1 = (1 - i, -1 + 2i, 2i, 2)$, we can construct the first integral

$$
F_4: (t, x) \rightarrow \arctan \frac{-x_1 + 2x_2 + 2x_3}{x_1 - x_2 + 2x_4} - t \quad \text{for all } (t, x) \in \mathbb{R} \times X_1 \quad \text{(by Corollary 1.5)}
$$

of the differential system (1.13) on a domain $X_1$ from the set $\{x: x_1 - x_2 + 2x_4 \neq 0\}$.

The functionally independent first integrals $F_1, F_2, F_3$ (constructed in Example 1.3), and the function $F_4$ are an integral basis of the differential system (1.13) on a domain $\mathbb{R} \times X$, where $X$ is a domain from the set $\{x: x_1 - x_2 + 2x_4 \neq 0, x_4 - x_2 \neq 0\}$.

Theorem 1.10. Let $\lambda$ be the eigenvalue with elementary divisor of multiplicity $m \geq 2$ of the matrix $B$ corresponding to a real eigenvector $\nu^0$ and to a real generalized eigenvector $\nu^1$ of the 1-st order. Then the system (1.1) has the first integral

$$
F: (t, x) \rightarrow \frac{\nu^1 x}{\nu^0 x} - t \quad \text{for all } (t, x) \in \mathbb{R} \times X, \quad X \subset \{x: \nu^0 x \neq 0\}. \quad (1.22)
$$

Indeed, the Lie derivative of $F$ on a domain $\mathbb{R} \times X$ by virtue of system (1.1) is

$$
\mathcal{L}_t F(t, x) + \mathcal{L}_x F(t, x) = -1 + \frac{(\lambda \nu^1 x + \nu^0 x) \nu^0 x - \nu^1 x \lambda \nu^0 x}{(\nu^0 x)^2} = 0. \quad \blacksquare
$$

Example 1.10 (continuation of Example 1.4). The system (1.15) has the eigenvalue $\lambda_1 = 0$ corresponding to the eigenvector $\nu^1 = (1, -2, 1)$ and to the 1-st order generalized eigenvector $\nu^2 = (0, -1, 1)$. By Theorem 1.10, the scalar function

$$
F_3: (t, x_1, x_2, x_3) \rightarrow \frac{x_3 - x_2}{x_1 - 2x_2 + x_3} - t \quad \text{for all } (t, x_1, x_2, x_3) \in \mathbb{R}^4
$$

is a first integral of the linear differential system (1.15).

The functionally independent first integrals $F_1, F_2$ (constructed in Example 1.4), and $F_3$ are an integral basis of the differential system (1.15) on a domain $\mathbb{R} \times X$, where $X$ is a domain from the set $\{(x_1, x_2, x_3): x_1 - 2x_2 + x_3 \neq 0\}$.

Example 1.11 (continuation of Example 1.5). The system (1.20) has the eigenvalue $\lambda_1 = 2$ corresponding to the eigenvector $\nu^0 = (1, -1, 1)$ and to the 1-st order generalized eigenvector $\nu^1 = (1, 0, -1)$. Using Theorem 1.10, we can build the first integral

$$
F_3: (t, x_1, x_2, x_3) \rightarrow \frac{x_1 - x_3}{x_1 - x_2 + x_3} - t \quad \text{for all } (t, x_1, x_2, x_3) \in \mathbb{R}^4.
$$

of the linear autonomous differential system (1.20)

The functionally independent first integrals $F_1, F_2$ (constructed in Example 1.5), and $F_3$ are an integral basis of the differential system (1.20) on a domain $\mathbb{R} \times X$, where $X$ is a domain from the set $\{(x_1, x_2, x_3): x_1 - x_2 + x_3 \neq 0\}$.

If $\lambda$ is a complex number, then the function (1.22) is a complex-valued first integral of system (1.1). In this case, from the complex-valued first integral (1.22) of system (1.1), we obtain two real-valued first integrals of system (1.1)

$$
F_1: (t, x) \rightarrow \frac{\tilde{\nu}^0 x \tilde{\nu}^1 x + \tilde{\nu}^0 x \tilde{\nu}^1 x}{(\tilde{\nu}^0 x)^2 + (\tilde{\nu}^0 x)^2} - t \quad \text{for all } (t, x) \in \mathbb{R} \times X
$$

and

$$
F_2: (t, x) \rightarrow \frac{\tilde{\nu}^0 x \tilde{\nu}^1 x - \tilde{\nu}^0 x \tilde{\nu}^1 x}{(\tilde{\nu}^0 x)^2 + (\tilde{\nu}^0 x)^2} \quad \text{for all } (t, x) \in \mathbb{R} \times X,
$$

where $X$ is a domain from the set $\{x: (\tilde{\nu}^0 x)^2 + (\tilde{\nu}^0 x)^2 \neq 0\}$. 

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Example 1.12 (continuation of Example 1.6). Using the eigenvector \( \nu^0 = (1, 1, 0, 0, i, 0) \) and 1-st order generalized eigenvector \( \nu^1 = (0, 1, 0, i, 1, 1) \) corresponding to the eigenvalue \( \lambda_1 = 1 + i \), we can construct the first integral of system (1.21)

\[
F_6: (t, x) \rightarrow \frac{(x_1 + x_2)(x_2 + x_6) + x_5(x_4 + x_5)}{(x_1 + x_2)^2 + x_5^2} - t \quad \text{for all } (t, x) \in \mathbb{R} \times X.
\]

The functionally independent first integrals \( F_1, \ldots, F_5 \) (constructed in Example 1.6), and \( F_6 \) are a basis of first integrals for the differential system (1.21) on a domain \( \mathbb{R} \times X \), where \( X \) is a domain from the set \( \{x : x_1 + x_2 \neq 0\} \).

Let us remark that using Theorems 1.8, 1.9, 1.10, and Corollary 1.5, we can always build an integral basis of system (1.1) on the base of nonautonomous first integrals.

Example 1.13. The fifth-order ordinary linear autonomous differential system

\[
\frac{dx_1}{dt} = 2(x_1 - 3x_2 + 2x_3 + x_5), \quad \frac{dx_2}{dt} = 2x_1 - x_2 + 2x_3 + 2x_4, \quad \frac{dx_3}{dt} = x_2 - x_3,
\]

\[
\frac{dx_4}{dt} = -3x_1 + 5x_2 - 4x_3 - x_4 - 2x_5, \quad \frac{dx_5}{dt} = 4x_2 - 2x_3 + x_4
\]

(1.23)

has the eigenvalues \( \lambda_1 = -1, \lambda_2 = 1 - i, \) and \( \lambda_3 = 1 + i \) with elementary divisors \( (\lambda + 1)^3, \lambda - 1 + i, \) and \( \lambda - 1 - i, \) respectively, corresponding to the eigenvectors \( \nu^{01} = (1, 0, 1, 1, 0), \)
\[
\nu^2 = (1, 1 + i, 0, 1 + i, 1 - i), \quad \nu^3 = (1, 1 - i, 0, 1 - i, 1 + i)
\]
and to the generalized eigenvectors \( \nu^{11} = (1, 1/2, 1, 1, 0), \) \( \nu^{21} = (0, 1, 1, 0, 0) \).

Using the eigenvector \( \nu^{01} = (1, 0, 1, 1, 0) \), the generalized eigenvector \( \nu^{11} = (1, 1/2, 1, 1, 0) \) of the 1-st order, and the generalized eigenvector \( \nu^{21} = (0, 1, 1, 0, 0) \) of the 2-nd order corresponding to the eigenvalue \( \lambda_1 = -1 \), we can build the first integrals of system (1.23):

\[
F_1: (t, x) \rightarrow (x_1 + x_3 + x_4)e^t \quad \text{for all } (t, x) \in \mathbb{R}^6 \quad \text{(by Theorem 1.9)},
\]

\[
F_2: (t, x) \rightarrow \frac{2x_1 + x_2 + 2x_3 + 2x_4}{2(x_1 + x_3 + x_4)} - t \quad \text{for all } (t, x) \in \mathbb{R} \times X_1 \quad \text{(by Theorem 1.10)},
\]

and (by Theorem 1.8)

\[
F_3: (t, x) \rightarrow \frac{4(x_2 + x_3)(x_1 + x_3 + x_4) - (2x_1 + x_2 + 2x_3 + 2x_4)^2}{4(x_1 + x_3 + x_4)^2} \quad \text{for all } (t, x) \in \mathbb{R} \times X_1,
\]

where \( X_1 \) is a domain from the set \( \{x : x_1 + x_3 + x_4 \neq 0\} \subset \mathbb{R}^5 \).

Using the eigenvector \( \nu^2 = (1, 1 + i, 0, 1 + i, 1 - i) \) corresponding to the complex eigenvalue \( \lambda_2 = 1 - i \), we can construct (by Corollary 1.5) the first integrals of system (1.23):

\[
F_4: (t, x) \rightarrow ((x_1 + x_2 + x_4 + x_5)^2 + (x_2 + x_4 - x_5)^2)e^{-2t} \quad \text{for all } (t, x) \in \mathbb{R}^6
\]

and

\[
F_5: (t, x) \rightarrow \arctan \frac{x_2 + x_4 - x_5}{x_1 + x_2 + x_4 + x_5} + t \quad \text{for all } (t, x) \in \mathbb{R} \times X_2,
\]

where \( X_2 \) is a domain from the set \( \{x : x_1 + x_2 + x_4 + x_5 \neq 0\} \subset \mathbb{R}^5 \).

The functionally independent first integrals \( F_1, \ldots, F_5 \) are an integral basis of the linear autonomous differential system (1.23) on a domain \( \mathbb{R} \times X \), where \( X \) is a domain from the set \( \{x : x_1 + x_3 + x_4 \neq 0, \quad x_1 + x_2 + x_4 + x_5 \neq 0\} \subset \mathbb{R}^5 \).
1.2. Linear nonhomogeneous differential system

Consider an ordinary linear nonhomogeneous differential system with constant coefficients

$$\frac{dx}{dt} = Ax + f(t),$$  \hspace{1cm} (1.24)

where \( f: t \to \text{col} (f_1(t), \ldots, f_n(t)) \) for all \( t \in J \) is a continuous function on an interval \( J \subset \mathbb{R} \). The differential system (1.1) is the corresponding homogeneous system of (1.24).

The system (1.24) is induced the linear differential operator of first order

$$\mathcal{C}(t, x) = \partial_t + \sum_{\xi=1}^{n} (A_\xi x + f_\xi(t)) \partial_{x_\xi} \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \quad A_\xi = (a_{\xi 1}, \ldots, a_{\xi n}), \quad \xi = 1, \ldots, n.

1.2.1. Case of simple elementary divisors

If the matrix \( B \) has simple structure, then we can build first integrals of the differential system (1.24) by using following assertions (Theorem 1.11 and Corollary 1.6).

**Theorem 1.11.** Let \( \nu \) be a real eigenvector of the matrix \( B \) corresponding to the eigenvalue \( \lambda \). Then a first integral of system (1.24) is the scalar function

$$F: (t, x) \to \nu x \exp(-\lambda t) - \int_{t_0}^{t} \nu f(\zeta) \exp(-\lambda \zeta) d\zeta \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \quad (1.25)$$

where \( t_0 \) is a fixed point from the interval \( J \).

**Proof.** Using Lemma 1.1, we get

$$\mathcal{C} F(t, x) = \partial_t F(t, x) + A F(t, x) + \sum_{\xi=1}^{n} f_\xi(t) \partial_{x_\xi} F(t, x) = -\lambda \nu x \exp(-\lambda t) - \nu f(t) \exp(-\lambda t) +$$

$$+ \lambda \nu x \exp(-\lambda t) + \sum_{\xi=1}^{n} \nu_\xi f_\xi(t) \exp(-\lambda t) = 0 \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n.

Therefore the function (1.25) is a first integral of the differential system (1.24).

**Corollary 1.6.** Suppose \( \nu = \nu^* + \nu^* i \) (Re \( \nu = \nu \), Im \( \nu = \nu^* \)) is an eigenvector of the matrix \( B \) corresponding to the complex eigenvalue \( \lambda = \lambda^* + \lambda^* i \) (Re \( \lambda = \lambda \), Im \( \lambda = \tilde{\lambda} \neq 0 \). Then first integrals of the differential system (1.24) are the scalar functions

$$F_{\theta}: (t, x) \to \alpha_\theta(t, x) - \int_{t_0}^{t} \alpha_\theta(\zeta, f(\zeta)) d\zeta \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \quad \theta = 1, 2, \quad (1.26)$$

where \( t_0 \) is a fixed point from the interval \( J \), the functions

$$\alpha_1: (t, x) \to \left( \nu x \cos \tilde{\lambda} t + \nu x \sin \tilde{\lambda} i \right) \exp(-\lambda t) \quad \text{for all} \quad (t, x) \in \mathbb{R}^{n+1},$$

$$\alpha_2: (t, x) \to \left( \nu x \cos \tilde{\lambda} t - \nu x \sin \tilde{\lambda} i \right) \exp(-\lambda t) \quad \text{for all} \quad (t, x) \in \mathbb{R}^{n+1}.

**Proof.** From Theorem 1.11 it follows that the function (1.25) is a complex-valued first integral of the differential system (1.24). Then, the real and imaginary parts of this function are the real-valued first integrals (1.26) of the differential system (1.24).

**Example 1.13.** Let us consider the linear nonhomogeneous differential system

$$\frac{dx_1}{dt} = 2x_1 + x_2 + 2e^{2t}, \quad \frac{dx_2}{dt} = x_1 + 3x_2 - x_3 + 10, \quad \frac{dx_3}{dt} = -x_1 + 2x_2 + 3x_3 + e^{3t}. \quad (1.27)$$

The differential system (1.12) is the corresponding homogeneous system of (1.27).
Using the eigenvectors \( \nu^1 = (1, i, -1) \), \( \nu^2 = (3, -1, -1) \) corresponding to the eigenvalues \( \lambda_1 = 3 + i \), \( \lambda_2 = 2 \), respectively, we can build (by Corollary 1.6 and Theorem 1.11) the functionally independent first integrals (an integral basis) of system (1.27)

\[
\begin{align*}
F_1: (t, x_1, x_2, x_3) &\rightarrow ((x_1 - x_3 + 1) \cos t + (x_2 + x_3) \sin t) e^{-3t} + (\cos t - \sin t) e^{-t} + \sin t, \\
F_2: (t, x_1, x_2, x_3) &\rightarrow ((x_2 + 3) \cos t + (x_3 - x_1 - 1) \sin t) e^{-3t} - (\cos t + \sin t) e^{-t} + \cos t, \\
F_3: (t, x_1, x_2, x_3) &\rightarrow (3x_1 - x_2 - x_3 - 5) e^{-2t} + e^t - 6t \quad \text{for all } (t, x_1, x_2, x_3) \in \mathbb{R}^4.
\end{align*}
\]

**Remark.** Under the conditions of Corollary 1.6, we have the scalar function

\[
F: (t, x) \rightarrow (\nu x)^2 + (\bar{\nu} x)^2 \exp(-2\nu t) - 2 \left( \alpha_1(t, x) \int_{t_0}^t \alpha_1(\zeta, f(\zeta)) d\zeta + \alpha_2(t, x) \int_{t_0}^t \alpha_2(\zeta, f(\zeta)) d\zeta \right) + \left( \int_{t_0}^t \alpha_1(\zeta, f(\zeta)) d\zeta \right)^2 + \left( \int_{t_0}^t \alpha_2(\zeta, f(\zeta)) d\zeta \right)^2
\]

is also a first integral on the domain \( J \times \mathbb{R}^n \) of the linear differential system (1.24).

### 1.2.2. Case of multiple elementary divisors

If the matrix \( B \) has multiple elementary divisors, then we can construct first integrals of system (1.24) by using following assertions (Theorem 1.12 and Corollary 1.7).

**Theorem 1.12.** Let \( \lambda \) be the eigenvalue with elementary divisor of multiplicity \( m \geq 2 \) of the matrix \( B \) corresponding to a real eigenvector \( \nu^0 \) and to real generalized eigenvectors \( \nu^k, k = 1, \ldots, m - 1 \). Then the system (1.24) has the functionally independent first integrals

\[
F_{k+1}: (t, x) \rightarrow \nu^k x \exp(-\lambda t) - \sum_{\tau=0}^{k-1} \binom{k}{\tau} t^{k-\tau} F_{\tau+1}(t, x) - C_k(t)
\]

for all \( (t, x) \in J \times \mathbb{R}^n \), \( k = 1, \ldots, m - 1 \),

where the integral \( F_1: (t, x) \rightarrow \nu^0 x \exp(-\lambda t) - C_0(t) \) for all \( (t, x) \in J \times \mathbb{R}^n \) (by Theorem 1.11), the scalar functions

\[
C_k: t \rightarrow \int_{t_0}^t (\nu^k f(\zeta) \exp(-\lambda \zeta) + k C_{k-1}(\zeta)) d\zeta \quad \text{for all } t \in J, \ k = 0, \ldots, m - 1, \ t_0 \in J.
\]

**Proof.** The proof of Theorem 1.12 is by induction on \( m \).

By the equalities (1.14), it follows that

\[
\mathcal{E} \left( \nu^e x \exp(-\lambda t) \right) = (\varepsilon \nu^{e-1} x + \nu^e f(t)) \exp(-\lambda t)
\]

for all \( (t, x) \in J \times \mathbb{R}^n \), \( \varepsilon = 1, \ldots, m - 1 \).

Let \( m = 2 \). Using the system of identities (1.29), we get

\[
\mathcal{E} F_2(t, x) = \mathcal{E} \left( \nu^1 x \exp(-\lambda t) - t F_1(t, x) - C_1(t) \right) = (\nu^0 x + \nu^1 f(t)) \exp(-\lambda t) - F_1(t, x) - (\nu^1 f(t) \exp(-\lambda t) + C_0(t)) = (\nu^0 x \exp(-\lambda t) - C_0(t)) - F_1(t, x) = 0 \quad \text{for all } (t, x) \in J \times \mathbb{R}^n.
\]

Therefore the function \( F_2: J \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a first integral of system (1.24).
Assume that the functions (1.28) for \( m = \mu \) are first integrals of system (1.24). Then, from the identity (1.29) for the function \( F_{\mu+1} \) it follows that

\[
\mathcal{C} F_{\mu+1}(t, x) = \mathcal{C} \left( \nu^\mu x \exp(-\lambda t) - \sum_{\tau=0}^{\mu-1} \binom{\mu}{\tau} t^{\mu-\tau} F_{\tau+1}(t, x) - C_\mu(t) \right) = (\mu \nu^{\mu-1} x + \nu^\mu f(t)) \exp(-\lambda t) - \\
- \mu \sum_{\tau=0}^{\mu-2} \binom{\mu-1}{\tau} t^{\mu-\tau-1} F_{\tau+1}(t, x) - \mu F_\mu(t, x) - (\nu^\mu f(t) \exp(-\lambda t) + \mu C_{\mu-1}(t)) = \\
= \mu \left( \nu^{\mu-1} x - \sum_{\tau=0}^{\mu-2} \binom{\mu-1}{\tau} t^{\mu-\tau-1} F_{\tau+1}(t, x) - C_{\mu-1}(t) \right) - \mu F_\mu(t, x) = 0 \text{ for all } (t, x) \in J \times \mathbb{R}^n.
\]

This implies that the scalar function \( F_{\mu+1} : J \times \mathbb{R}^n \rightarrow \mathbb{R} \) (for \( m = \mu + 1 \)) is a first integral of the linear nonhomogeneous differential system (1.24).

So by the principle of mathematical induction, the scalar functions (1.28) are first integrals of the differential system (1.24) for every natural number \( m \geq 2 \). \[\blacksquare\]

**Example 1.14.** Consider the linear nonhomogeneous differential system

\[
\begin{align*}
\frac{dx_1}{dt} &= 4x_1 - x_2 + e^{3t}, \quad \frac{dx_2}{dt} = 3x_1 + x_2 - x_3 + 8t, \quad \frac{dx_3}{dt} = x_1 + x_3 + 4. \quad (1.30)
\end{align*}
\]

The differential system (1.30) has the eigenvalue \( \lambda_1 = 2 \) corresponding to the eigenvector \( \nu^0 = (1, -1, 1) \) and to the generalized eigenvectors \( \nu^1 = (1, 0, -1) \), \( \nu^2 = (-2, 2, 0) \).

An integral basis of system (1.30) is the functions (by Theorems 1.11 and 1.12)

\[
\begin{align*}
F_1 : (t, x) &\rightarrow (x_1 - x_2 + x_3 - 4t)e^{-2t} - e^t \text{ for all } (t, x) \in \mathbb{R}^4, \\
F_2 : (t, x) &\rightarrow (x_1 - x_3 + 2t - 1)e^{-2t} - t F_1(t, x) - 2e^t \text{ for all } (t, x) \in \mathbb{R}^4, \\
F_3 : (t, x) &\rightarrow 2(x_2 - x_1 + 3t + 2)e^{-2t} - t^2 F_1(t, x) - 2t F_2(t, x) - 2e^t \text{ for all } (t, x) \in \mathbb{R}^4.
\end{align*}
\]

**Corollary 1.7.** Let \( \lambda = \lambda + \lambda i \) (Re \( \lambda = \lambda \), Im \( \lambda = \lambda \neq 0 \)) be the complex eigenvalue of the matrix B with elementary divisor of multiplicity \( m \geq 2 \) corresponding to an eigenvector \( \nu^0 = \nu^0 + \nu^0 i \) (Re \( \nu^0 = \nu^0 \), Im \( \nu^0 = \nu^0 \)) and to generalized eigenvectors \( \nu^k = \nu^k + \nu^k i \) (Re \( \nu^k = \nu^k \), Im \( \nu^k = \nu^k \)), \( k = 1, \ldots, m-1 \). Then first integrals of the linear nonhomogeneous differential system (1.24) are the functions

\[
F_{\theta,k+1} : (t, x) \rightarrow \alpha_{\theta k}(t, x) - \sum_{\tau=0}^{k-1} \binom{k}{\tau} t^{k-\tau} F_{\theta,\tau+1}(t, x) - C_{\theta k}(t)
\]

for all \( (t, x) \in J \times \mathbb{R}^n \), \( k = 1, \ldots, m-1 \), \( \theta = 1, 2 \),

where the first integrals (by Corollary 1.6)

\[
F_{\theta 1} : (t, x) \rightarrow \alpha_{\theta 0}(t, x) - C_{\theta 0}(t) \text{ for all } (t, x) \in J \times \mathbb{R}^n, \quad \theta = 1, 2,
\]

the scalar functions

\[
\begin{align*}
\alpha_{1k} : (t, x) &\rightarrow (\nu^k x \cos \lambda t + \nu^k x \sin \lambda t) \exp(-\lambda t) \text{ for all } (t, x) \in \mathbb{R}^{n+1}, \ k = 0, \ldots, m-1, \\
\alpha_{2k} : (t, x) &\rightarrow (\nu^k x \cos \lambda t - \nu^k x \sin \lambda t) \exp(-\lambda t) \text{ for all } (t, x) \in \mathbb{R}^{n+1}, \ k = 0, \ldots, m-1, \\
C_{\theta k} : t &\rightarrow \int_{t_0}^{t} (\alpha_{\theta k}(\zeta, f(\zeta)) + k C_{\theta,\theta-1}(\zeta)) d\zeta \text{ for all } t \in J, \ k = 0, \ldots, m-1, \ \theta = 1, 2, \ t_0 \in J.
\end{align*}
\]
Proof. Formally using Theorem 1.12, we get the complex-valued first integrals (1.28) of the differential system (1.24). Then, the real and imaginary parts of the functions (1.28) are the real-valued first integrals (1.31) of the differential system (1.24).}

Example 1.15. The linear nonhomogeneous differential system

\[
\begin{align*}
\frac{dx_1}{dt} &= -x_3 + x_4 + 4a \cos t, \\
\frac{dx_2}{dt} &= x_1 + x_3 + 4a \sin t, \\
\frac{dx_3}{dt} &= -x_2 + x_3 - x_4 + bt, \\
\frac{dx_4}{dt} &= -x_1 + x_2 + x_3 - x_4 + \frac{c}{\sin^2 t},
\end{align*}
\]

(1.32)

where \(a, b,\) and \(c\) are some real numbers, has the complex eigenvalues \(\lambda_1 = i, \lambda_2 = -i\) with elementary divisors \((\lambda - i)^2, (\lambda + i)^2\), respectively.

The number \(\lambda_1 = i\) is the eigenvalue corresponding to the eigenvector \(\nu^0 = (1, i, 1, 0)\) and to the generalized eigenvector \(\nu^1 = (-1 + i, 0, 0, i)\) of the 1-st order.

Using the real numbers \(\lambda^1_1 = 0, \lambda^1_2 = 1\), the real vectors \(\nu^0 = (1, 0, 1, 0), \nu^0 = (0, 1, 0, 0), \nu^1 = (0, 1, 0, 1)\), and the scalar functions

\[
\begin{align*}
\alpha_{10}: (t, x) &\to \cos t (x_1 + x_3) + \sin t x_2, \\
\alpha_{20}: (t, x) &\to \cos t x_2 - \sin t (x_1 + x_3), \\
\alpha_{11}: (t, x) &\to -\cos t x_1 + \sin t (x_1 + x_4), \\
\alpha_{21}: (t, x) &\to \cos t (x_1 + x_4) + \sin t x_4
\end{align*}
\]

for all \((t, x) \in J_t \times \mathbb{R}^4, \ J_t = (\pi l; \pi l + 1)\) for all \(l \in \mathbb{Z},\)

\[
C_{10}(t) = \int (4a + bt \cos t) \ dt = 4at + b(\cos t + t \sin t), \quad C_{20}(t) = -\int bt \sin t \ dt = b(t \cos t - \sin t),
\]

\[
C_{11}(t) = \int \left( -4a \cos^2 t + 4a \cos t \sin t + \frac{c}{\sin t} + 4at + b(\cos t + t \sin t) \right) \ dt = \]

\[
= 2a(t^2 - t + \sin^2 t - \sin t \cos t) + b(2 \sin t - t \cos t) + c \ln \left| \tan \frac{t}{2} \right| \quad \text{for all } t \in J_t,
\]

\[
C_{21}(t) = \int \left( 4a \cos^2 t + \frac{c \cos t}{\sin^2 t} + 4a \cos t \sin t + b(t \cos t - \sin t) \right) \ dt = \]

\[
= 2a(t + \sin^2 t + t \cos t) + b(2 \cos t + t \sin t) - \frac{c}{\sin t} \quad \text{for all } t \in J_t,
\]

we can build (by Corollaries 1.6 and 1.7) the first integrals of system (1.32)

\[
\begin{align*}
F_{11}: (t, x) &\to \cos t (x_1 + x_3) + \sin t x_2 - 4at - b(\cos t + t \sin t) \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4, \\
F_{21}: (t, x) &\to \cos t x_2 - \sin t (x_1 + x_3) + b(\sin t - t \cos t) \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4, \\
F_{12}: (t, x) &\to -\cos t x_1 + \sin t (x_1 + x_4) - tF_{11}(t, x) - 2a(t^2 - t + \sin^2 t - t \cos t) - b(2 \sin t - t \cos t) - c \ln \left| \tan \frac{t}{2} \right| \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4, \\
F_{22}: (t, x) &\to \cos t (x_1 + x_4) + \sin t x_4 - tF_{21}(t, x) - 2a(t + \sin^2 t + t \cos t) - b(2 \cos t + t \sin t) + \frac{c}{\sin t} \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4.
\end{align*}
\]

The functionally independent first integrals \(F_{11}, F_{21}, F_{12},\) and \(F_{22}\) are a basis of first integrals for the differential system (1.32) on a domain \(J_t \times \mathbb{R}^4\).
2. Integrals of ordinary linear nonautonomous differential systems integrable in closed form

2.1. Algebraic reducible systems

Consider an ordinary real linear nonhomogeneous differential system of the $n$-th order
\[
\frac{dx}{dt} = A(t)x + f(t),
\] (2.1)
where $x = \text{col}(x_1, \ldots, x_n) \in \mathbb{R}^n$, the continuous on an interval $J \subset \mathbb{R}$ coefficient matrix $A: t \to A(t)$ for all $t \in J$ is diagonalizable by a constant similarity matrix [59, p. 61], the vector function $f: t \to \text{col}(f_1(t), \ldots, f_n(t))$ for all $t \in J$ is continuous.

The corresponding homogeneous system of the nonhomogeneous system (2.1) is
\[
\frac{dx}{dt} = A(t)x.
\] (2.2)

By [48, p. 186], the nonhomogeneous differential system (2.1) and the corresponding homogeneous differential system (2.2) are called algebraic reducible.

2.1.1. Partial integrals

A complex-valued linear homogeneous function
\[
p: x \to \nu x \quad \text{for all } x \in \mathbb{R}^n, \quad \nu \in \mathbb{C}^n,
\] (2.3)
is a partial integral of the algebraic reducible system (2.2) if and only if the identity holds
\[
\mathcal{A}p(x) = \lambda(t)p(x) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n,
\] (2.4)
where the linear differential operator $\mathcal{A}(t, x) = \partial_t + A(t)x\partial_x$ for all $(t, x) \in J \times \mathbb{R}^n$, the scalar function $\lambda: J \to \mathbb{R}$. The identity (2.4) is equivalent to the linear system
\[
(B(t) = \lambda(t)E)\nu = 0,
\] (2.5)
where $E$ is the $n \times n$ identity matrix, and the matrix $B: t \to B(t)$ for all $t \in J$ is the transpose of the matrix $A: t \to A(t)$ for all $t \in J$.

The following basic propositions (Lemmas 2.1 and 2.2) are base for the method of building first integrals of the algebraic reducible systems (2.1) and (2.2).

Lemma 2.1. Suppose $\nu$ is a real eigenvector of the matrix $B: t \to B(t)$ for all $t \in J$ corresponding to the eigenfunction $\lambda: t \to \lambda(t)$ for all $t \in J$. Then the linear function (2.3) is a partial integral of the algebraic reducible differential system (2.2).

Indeed, if $\nu \in \mathbb{R}^n$ is an eigenvector of the matrix $B: t \to B(t)$ for all $t \in J$ corresponding to the eigenfunction $\lambda: t \to \lambda(t)$ for all $t \in J$, then $\nu$ is a solution to the system (2.5). This implies that the identity (2.4) holds. Thus the linear function (2.3) is a partial integral of the algebraic reducible differential system (2.2).

Lemma 2.2. Let $\nu = \nu + \nu i$ ($\nu = \text{Re}\nu$, $\nu = \text{Im}\nu \neq 0$) be a complex eigenvector of the matrix $B: t \to B(t)$ for all $t \in J$ corresponding to the eigenfunction $\lambda: t \to \lambda(t) + \text{Re}\lambda(t)i$ for all $t \in J$ ($\lambda: t \to \text{Re}\lambda(t)$, $\lambda: t \to \text{Im}\lambda(t)$ for all $t \in J$), let the functions $P$ and $\psi$ be defined by $P: x \to (\nu x)^2 + (\nu x)^2$ for all $x \in \mathbb{R}^n$ and $\psi: x \to \text{arctan} \frac{\nu x}{\nu i}$ for all $x \in \mathcal{X}$, where $\mathcal{X}$ is a domain from the set $\{x: \nu x \neq 0\} \subset \mathbb{R}^n$. Then, we have
\[
\mathcal{A}P(x) = 2\lambda(t)P(x) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n \quad \text{and} \quad \mathcal{A}\psi(x) = \lambda(t) \quad \text{for all } (t, x) \in J \times \mathcal{X}.
\]

Proof. Formally using Lemma 2.1, we get the complex-valued function (2.3) is a partial integral of the differential system (2.2) and the following identity holds
\[ \mathfrak{A}(\dot{v}x + i \nu x) = (\dot{\lambda}(t) + i \lambda(t))(\dot{\nu}x + i \nu x) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n. \]

This complex identity is equivalent to the real system of identities

\[ \mathfrak{A} \dot{v}x = \dot{\lambda}(t) \dot{v}x - \lambda(t) \nu x, \quad \mathfrak{A} \nu x = \dot{\lambda}(t) \dot{\nu}x + \lambda(t) \dot{v}x \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n. \]

Using this system of identities, we obtain

\[ \mathfrak{A} \nu x = \dot{\lambda}(t) \dot{v}x - \lambda(t) \nu x, \quad \mathfrak{A} \dot{v}x = \dot{\lambda}(t) \dot{\nu}x + \lambda(t) \dot{v}x \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n. \]

and

\[ \mathfrak{A} \psi(x) = \frac{\dot{v}x(\dot{\lambda}(t) \dot{v}x + \lambda(t) \dot{\nu}x) - \nu x(\dot{\lambda}(t) \dot{v}x - \lambda(t) \dot{\nu}x)}{(\dot{\nu}x)^2 + (\dot{\nu}x)^2} = \dot{\lambda}(t) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n. \]

2.1.2. First integrals

Using Theorems 2.1 and 2.2, we can obtain first integrals of the algebraic reducible non-homogeneous differential system (2.1).

**Theorem 2.1.** Suppose \( \nu \) is a real eigenvector of the matrix \( B: t \to B(t) \) for all \( t \in J \) corresponding to the eigenfunction \( \lambda: t \to \lambda(t) \) for all \( t \in J \). Then the algebraic reducible differential system (2.1) has the first integral

\[ F: (t, x) \to \nu x \varphi(t) - \int_{t_0}^{t} \nu f(\zeta) \varphi(\zeta) d\zeta \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \]

where the exponential function

\[ \varphi: t \to \exp\left( -\int_{t_0}^{t} \lambda(\zeta) d\zeta \right) \quad \text{for all} \quad t \in J, \]

and \( t_0 \) is a fixed point from the interval \( J \).

**Proof.** From Lemma 2.1, we get the following

\[ \mathfrak{B} F(t, x) = \mathfrak{A} F(t, x) + f(t) \partial_x F(t, x) = \nu x \partial_t \varphi(t) - \partial_t \int_{t_0}^{t} \nu f(\zeta) \varphi(\zeta) d\zeta + \varphi(t) \mathfrak{A} \nu x + \]

\[ + f(t) \partial_x \nu x \varphi(t) = -\lambda(t) \nu x \varphi(t) - \nu f(t) \varphi(t) + \lambda(t) \nu x \varphi(t) + \nu f(t) \varphi(t) = 0 \]

for all \( (t, x) \in J \times \mathbb{R}^n, \)

where the linear differential operator \( \mathfrak{B}(t, x) = \partial_t + (A(t) x + f(t)) \partial_x \) for all \( (t, x) \in J \times \mathbb{R}^n \)

is the operator of differentiation by virtue of system (2.1).

Therefore the function (2.6) is a first integral of the algebraic reducible system (2.1). \( \blacksquare \)

**Example 2.1.** Let us consider the linear differential system of the third-order

\begin{align*}
\frac{dx_1}{dt} &= \alpha_3(t) x_1 + (\alpha_1(t) - \alpha_3(t))(x_3 - x_2) + f_1(t), \\
\frac{dx_2}{dt} &= (\alpha_2(t) - \alpha_3(t))(x_3 - x_1) + \alpha_3(t) x_2 + f_2(t), \\
\frac{dx_3}{dt} &= (\alpha_3(t) - \alpha_2(t)) x_1 + (\alpha_3(t) - \alpha_1(t)) x_2 + (\alpha_1(t) + \alpha_2(t) - \alpha_3(t)) x_3 + f_3(t),
\end{align*}

(2.7)

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where the scalar functions \( \alpha_\xi : J \to \mathbb{R} \) and \( f_\xi : J \to \mathbb{R}, \ \xi = 1, 2, 3, \) are continuous on an interval \( J \subset \mathbb{R} \). The coefficient matrix

\[
A : t \to \begin{pmatrix}
\alpha_3(t) & \alpha_3(t) - \alpha_1(t) & \alpha_1(t) - \alpha_3(t) \\
\alpha_3(t) - \alpha_2(t) & \alpha_3(t) & \alpha_2(t) - \alpha_3(t) \\
\alpha_3(t) - \alpha_2(t) & \alpha_3(t) - \alpha_1(t) & \alpha_1(t) + \alpha_2(t) - \alpha_3(t)
\end{pmatrix}
\]

such that the transposed matrix \( B : t \to A^T(t) \) for all \( t \in J \) has the real eigenvectors \( \nu^1 = (0, -1, 1), \ \nu^2 = (-1, 0, 1), \) and \( \nu^3 = (1, 1, -1) \) corresponding to the eigenfunctions \( \lambda_\xi : t \to \alpha_\xi(t) \) for all \( t \in J, \ \xi = 1, 2, 3, \) respectively.

Therefore the differential system (2.7) is algebraically reducible.

By Theorem 2.1, we can build the first integrals on the domain \( J \times \mathbb{R}^3 \) of system (2.7)

\[
F_1 : (t, x) \to (x_3 - x_2) \exp\left(-\int_{t_0}^{t} \alpha_1(\zeta) d\zeta\right) - \int_{t_0}^{t} (f_3(\zeta) - f_2(\zeta)) \exp\left(-\int_{\zeta_0}^{\zeta} \alpha_1(\theta) d\theta\right) d\zeta,
\]

\[
F_2 : (t, x) \to (x_3 - x_1) \exp\left(-\int_{t_0}^{t} \alpha_2(\zeta) d\zeta\right) - \int_{t_0}^{t} (f_3(\zeta) - f_1(\zeta)) \exp\left(-\int_{\zeta_0}^{\zeta} \alpha_2(\theta) d\theta\right) d\zeta,
\]

\[
F_3 : (t, x) \to (x_1 + x_2 - x_3) \exp\left(-\int_{t_0}^{t} \alpha_3(\zeta) d\zeta\right) - \int_{t_0}^{t} (f_1(\zeta) + f_2(\zeta) - f_3(\zeta)) \exp\left(-\int_{\zeta_0}^{\zeta} \alpha_3(\theta) d\theta\right) d\zeta,
\]

where \( t_0 \) and \( \zeta_0 \) are fixed points from the interval \( J \).

The functionally independent first integrals \( F_1, F_2, \) and \( F_3 \) are an integral basis on the domain \( J \times \mathbb{R}^3 \) of the algebraic reducible differential system (2.7).

**Theorem 2.2.** Let \( \nu = \nu + \nu i \) (\( \nu = \text{Re} \nu, \ \nu = \text{Im} \nu \neq 0 \)) be a complex eigenvector of the matrix \( B : t \to B(t) \) for all \( t \in J \) corresponding to the eigenfunction \( \lambda : t \to \lambda(t) + \lambda(t) i \) for all \( t \in J \) (\( \lambda : t \to \text{Re} \lambda(t), \ \lambda : t \to \text{Im} \lambda(t) \) for all \( t \in J \)). Then the algebraic reducible differential system (2.1) has the first integrals

\[
F_\tau : (t, x) \to \gamma_\tau(t, x) - \int_{t_0}^{t} \gamma_\tau(\zeta, f(\zeta)) d\zeta \quad \text{for all} \ (t, x) \in J \times \mathbb{R}^n, \ \tau = 1, \ \tau = 2, \tag{2.8}
\]

where \( t_0 \) is a fixed point from the interval \( J \), the scalar functions

\[
\gamma_1 : (t, x) \to \left( \nu x \cos \int_{t_0}^{t} \lambda(\zeta) d\zeta + \nu x \sin \int_{t_0}^{t} \lambda(\zeta) d\zeta \right) \exp\left(-\int_{t_0}^{t} \lambda(\zeta) d\zeta\right) \quad \text{for all} \ (t, x) \in J \times \mathbb{R}^n,
\]

\[
\gamma_2 : (t, x) \to \left( \nu x \cos \int_{t_0}^{t} \lambda(\zeta) d\zeta - \nu x \sin \int_{t_0}^{t} \lambda(\zeta) d\zeta \right) \exp\left(-\int_{t_0}^{t} \lambda(\zeta) d\zeta\right) \quad \text{for all} \ (t, x) \in J \times \mathbb{R}^n.
\]

**Proof.** Formally using Theorem 2.1, we have the complex-valued function (2.6) is a first integral of the algebraic reducible system (2.1). Then the real and imaginary parts of this complex-valued first integral are the real-valued first integrals (2.8) of system (2.1).
**Example 2.2.** The algebraic reducible differential system

\[
\frac{dx_1}{dt} = -\tanh tx_1 - \cosh(t (x_2 - x_3)) - 3,
\]

\[
\frac{dx_2}{dt} = \left(\frac{1}{t} - 1 - \tanh t\right)x_1 + \left(1 - \frac{1}{t} - \cosh t\right)x_2 + \cosh t x_3 + 2t^3 + \sin t - e^t,
\]

\[
\frac{dx_3}{dt} = \left(\frac{1}{t} - 1 - \tanh t - \cosh t\right)x_1 + \left(1 - \frac{1}{t} + \tanh t - \cosh t\right)x_2 + \left(\cosh t - \tanh t\right)x_3 + 2t^3 + \sin t - e^t - \sinh^2 t
\]

has the eigenvectors \( \nu^1 = (-1, 1, 0) \), \( \nu^2 = (1, i, -i) \), and \( \nu^3 = (1, -i, i) \) corresponding to the eigenfunctions \( \lambda_1: t \to 1 - \frac{1}{t} \), \( \lambda_2: t \to -\tanh t + \cosh ti \), and \( \lambda_3: t \to -\tanh t - \cosh ti \) for all \( t \in J \), respectively, where \( J \) is an interval from the set \( \{ t: t \neq 0 \} \).

Using the eigenvector \( \nu^1 = (-1, 1, 0) \) and the corresponding eigenfunction \( \lambda_1: t \to 1 - \frac{1}{t} \) for all \( t \in J \), we can build (by Theorem 2.1) the first integral

\[
F_1: (t, x) \to t e^{-t}(x_2 - x_1) + \frac{t^2}{2} + 2t^4 + 8t^3 + 24t^2 + 45t + 45 + \frac{1}{2} \cos t + \frac{1}{2} t(t \cos t + \sin t) e^{-t}
\]

on the domain \( J \times \mathbb{R}^3 \) of the algebraic reducible differential system (2.9).

Using the eigenvector \( \nu^2 = (1, i, -i) \) and the eigenfunction \( \lambda_2: t \to -\tanh t + \cosh ti \) for all \( t \in J \), we can build (by Theorem 2.2) the first integrals of system (2.9)

\[
F_2: (t, x) \to \cosh t \left(\cos(\sinh t) x_1 + \sin(\sinh t) (x_2 - x_3)\right) + \\
+ (\sinh^2 t - 2) \cos(\sinh t) + (3 - 2 \sinh t) \sin(\sinh t) \quad \text{for all } J \times \mathbb{R}^3,
\]

\[
F_3: (t, x) \to \cosh t \left(-\sin(\sinh t) x_1 + \cos(\sinh t) (x_2 - x_3)\right) + \\
+ (3 - 2 \sinh t) \cos(\sinh t) + (2 - \sinh^2 t) \sin(\sinh t) \quad \text{for all } J \times \mathbb{R}^3.
\]

The functionally independent first integrals \( F_1, F_2, \) and \( F_3 \) are an integral basis on the domain \( J \times \mathbb{R}^3 \) of the algebraic reducible differential system (2.9).

In the case of the homogeneous algebraic reducible differential system (2.2), we have the following statements (Corollary 2.1 and Theorem 2.3).

**Corollary 2.1.** Under the conditions of Theorem 2.1, we have the scalar function

\[
F: (t, x) \to \nu x \exp\left(-\int_{t_0}^{t} \lambda(\zeta) \, d\zeta\right) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n, \quad t_0 \in J,
\]

is a first integral of the homogeneous algebraic reducible differential system (2.2).

**Theorem 2.3.** Let the assumptions of Theorem 2.2 hold, then the scalar functions

\[
F_1: (t, x) \to \left((\nu^*)^2 + (\nu x)^2\right) \exp\left(-2 \int_{t_0}^{t} \lambda^*(\zeta) \, d\zeta\right) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n
\]

and

\[
F_2: (t, x) \to \arctan \frac{\nu x}{\nu^*} - \int_{t_0}^{t} \lambda^*(\zeta) \, d\zeta \quad \text{for all } (t, x) \in J \times X, \quad t_0 \in J, \quad X \subseteq \{ x: \nu^* x \neq 0 \},
\]

are first integrals of the homogeneous algebraic reducible differential system (2.2).
Proof. From Lemma 2.2, we get
\[
\mathcal{A}F_1(t, x) = \exp\left(-2 \int_{t_0}^{t} \lambda(\zeta) d\zeta\right) \mathcal{A}\left((\bar{\nu}x)^2 + (\bar{\nu}x)^2\right) \mathcal{A}\exp\left(-2 \int_{t_0}^{t} \lambda(\zeta) d\zeta\right) =
\]
\[
= 2\lambda(t)((\nu x)^2 + (\bar{\nu}x)^2) \exp\left(-2 \int_{t_0}^{t} \lambda(\zeta) d\zeta\right) - 2\lambda(t)((\nu x)^2 + (\bar{\nu}x)^2) \exp\left(-2 \int_{t_0}^{t} \lambda(\zeta) d\zeta\right) = 0
\]
for all \((t, x) \in J \times \mathbb{R}^n\),
\[
\mathcal{A}F_2(t, x) = \mathcal{A}\arctan \frac{\bar{\nu}x}{\nu x} - \mathcal{A}\int_{t_0}^{t} \tilde{\lambda}(\zeta) d\zeta = \tilde{\lambda}(t) - \tilde{\lambda}(t) = 0 \quad \text{for all} \quad (t, x) \in J \times \mathcal{X}.
\]

Example 2.3. Consider the algebraic reducible differential system
\[
\frac{dx_1}{dt} = \alpha_2(t) x_1 + \alpha_3(t) (x_3 - x_2), \quad \frac{dx_2}{dt} = (\alpha_2(t) - \alpha_1(t)) x_1 + (\alpha_1(t) - \alpha_3(t)) x_2 + \alpha_3(t) x_3,
\]
\[
\frac{dx_3}{dt} = (\alpha_2(t) - \alpha_1(t) - \alpha_3(t)) x_1 + (\alpha_1(t) - \alpha_2(t) - \alpha_3(t)) x_2 + (\alpha_2(t) + \alpha_3(t)) x_3,
\]
(2.10)
where the scalar functions \(\alpha_\xi: J \to \mathbb{R}, \; \xi = 1, 2, 3\), are continuous on an interval \(J \subset \mathbb{R}\).

The system (2.10) has the eigenvectors \(\nu^1 = (-1, 1, 0), \; \nu^2 = (1, i, -i)\), and \(\nu^3 = (1, -i, i)\) corresponding to the eigenfunctions \(\lambda_1: t \to \alpha_1(t)\) for all \(t \in J\), \(\lambda_2: t \to \alpha_2(t) + \alpha_3(t) i\) for all \(t \in J\), and \(\lambda_3: t \to \alpha_2(t) - \alpha_3(t) i\) for all \(t \in J\), respectively.

Using the eigenvector \(\nu^1 = (-1, 1, 0)\) and the corresponding eigenfunction \(\lambda_1: t \to \alpha_1(t)\) for all \(t \in J\), we can build (by Corollary 2.1) the first integral
\[
F_1: (t, x) \to (x_2 - x_1) \exp\left(-\int_{t_0}^{t} \alpha_1(\zeta) d\zeta\right) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^3
\]
of the algebraic reducible differential system (2.10).

Using the eigenvector \(\nu^2 = (1, i, -i)\) and the corresponding complex-valued eigenfunction \(\lambda_2: t \to \alpha_2(t) + \alpha_3(t) i\) for all \(t \in J\), we can construct (by Theorem 2.3) the first integrals of the algebraic reducible differential system (2.10)
\[
F_2: (t, x) \to (x_1^2 + (x_2 - x_3)^2) \exp\left(-2 \int_{t_0}^{t} \alpha_2(\zeta) d\zeta\right) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^3
\]
and
\[
F_3: (t, x) \to \arctan \frac{x_2 - x_3}{x_1} - \int_{t_0}^{t} \alpha_3(\zeta) d\zeta \quad \text{for all} \quad (t, x) \in J \times \mathcal{X},
\]
where \(\mathcal{X}\) is a domain from the set \(\{x: x_1 \neq 0\} \subset \mathbb{R}^3\).

The functionally independent first integrals \(F_1, F_2,\) and \(F_3\) are an integral basis on the domain \(J \times \mathcal{X}\) of the algebraic reducible differential system (2.10).
2.2. Triangular systems

Consider an triangular linear nonhomogeneous differential system of the \( n \)-th order

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j + f_i(t), \quad i = 1, \ldots, n, \tag{2.11}
\]

where \( a_{ij} : J \to \mathbb{R} \) and \( f_j : J \to \mathbb{R} \) are continuous functions on an interval \( J \subset \mathbb{R} \).

The corresponding homogeneous system of the nonhomogeneous system (2.11) is

\[
\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j, \quad i = 1, \ldots, n. \tag{2.12}
\]

By Perron's theorem [56], any linear differential system can be transformed to an upper triangular system by an orthogonal transformation\(^1\). A triangular system is integrated in closed form and its fundamental matrix can be chosen triangular [8, pp. 32 – 33].

Using Theorem 2.4 and Corollary 2.2, we can build integral bases for the triangular differential systems (2.11) and (2.12). We now state the main results of this subsection.

**Theorem 2.4.** First integrals of system (2.11) are the functions

\[
F_{\tau} : (t, x) \to x_{n+1-\tau} \varphi_{n+1-\tau}(t) - \sum_{\xi=1}^{\tau-1} A_{\tau \xi}(t) F_\xi(t, x) - B_{\tau}(t)
\]

for all \((t, x) \in J \times \mathbb{R}^n, \, \tau = 1, \ldots, n,\) where the scalar functions

\[
A_{\tau \xi} : t \to \int_{t_0}^{t} \left( \sum_{k=1}^{\tau-\xi} a_{n+1-\tau,n+1-k-\tau}(\zeta) \psi_{n+1-k-\tau}(\zeta) A_{\tau-k \xi}(\zeta) \right) \varphi_{n+1-k-\tau}(\zeta) d\zeta,
\]

\[
B_{\tau} : t \to \int_{t_0}^{t} \left( f_{n+1-\tau}(\zeta) + \sum_{\xi=1}^{\tau-1} a_{n+1-\tau,n+1-\tau+\xi}(\zeta) \psi_{n+1-\tau+\xi}(\zeta) B_{\tau-\xi}(\zeta) \right) \varphi_{n+1-\tau+\xi}(\zeta) d\zeta,
\]

\[
\varphi_{\tau} : t \to \exp \left( - \int_{t_0}^{t} a_{\tau \tau}(\zeta) d\zeta \right), \quad \psi_{\tau} : t \to \exp \int_{t_0}^{t} a_{\tau \tau}(\zeta) d\zeta \quad \text{for all} \ t \in J, \ \tau = 1, \ldots, n. \tag{2.15}
\]

**Proof.** The system (2.11) is induced the linear differential operator of first order

\[
\mathcal{L}(t, x) = \partial_t + \sum_{i=1}^{n} \left( \sum_{j=i}^{n} a_{ij}(t) x_j + f_i(t) \right) \partial_{x_i} \quad \text{for all} \ (t, x) \in J \times \mathbb{R}^n.
\]

If \( \tau = 1 \), then the scalar function \( F_1 : (t, x) \to x_n \varphi_n(t) - B_1(t) \) for all \((t, x) \in J \times \mathbb{R}^n\) is a first integral of the differential system (2.11):

\[
\mathcal{L} F_1(t, x) = x_n \partial_t \varphi_n(t) + (a_{nn}(t) x_n + f_n(t)) \partial_{x_n}(x_n \varphi_n(t)) - \partial_t B_1(t) = -a_{nn}(t) \varphi_n(t) x_n +
\]

\(^1\)The transformation \( x = U(t) y \) is called orthogonal if the matrix \( U : t \to U(t) \) for all \( t \in J \) is orthogonal, i.e., \( U(t)U^T(t) = U^T(t)U(t) = E \) for all \( t \in J \), where \( E \) is the identity matrix.
Let $\tau = 2$. Then the Lie derivative of function $F_2$ by virtue of system (2.11) is equal to

$$\mathcal{L} F_2(t, x) = \mathcal{L}(x_{n-1} \varphi_{n-1}(t) - A_{21}(t) F_1(t, x) - B_2(t)) =$$

$$= -a_{n-1,n-1}(t) \varphi_{n-1}(t) x_{n-1} + (a_{n-1,n-1}(t) x_{n-1} + a_{n-1,n}(t) x_n + f_{n-1}(t)) \varphi_{n-1}(t) -$$

$$- \frac{1}{t} \int_{t_0}^{t} a_{n-1,n}(\zeta) \psi_n(\zeta) \varphi_{n-1}(\zeta) d\zeta F_1(t, x) - \frac{1}{t} \int_{t_0}^{t} f_{n-1}(\zeta) + a_{n-1,n}(\zeta) \psi_n(\zeta) B_1(\zeta) \varphi_{n-1}(\zeta) d\zeta =$$

$$= (a_{n-1,n}(t) x_n + f_{n-1}(t)) \varphi_{n-1}(t) - a_{n-1,n}(t) \psi_n(t) \varphi_{n-1}(t) F_1(t, x) -$$

$$- (f_{n-1}(t) + a_{n-1,n}(t) \psi_n(t) B_1(t)) \varphi_{n-1}(t) =$$

$$= a_{n-1,n}(t) \psi_n(t) \varphi_{n-1}(t) (x_n \varphi_n(t) - B_1(t) - F_1(t, x)) = 0 \text{ for all } (t, x) \in J \times \mathbb{R}^n.$$

Therefore the function $F_2: J \times \mathbb{R}^n \to \mathbb{R}$ is a first integral of system (2.11).

Suppose the functions $F_\tau: J \times \mathbb{R}^n \to \mathbb{R}, \tau = 1, \ldots, \varepsilon - 1$, are first integrals of the triangular differential system (2.11). Then, for $\tau = \varepsilon$, we have

$$\mathcal{L} F_{\varepsilon}(t, x) = \mathcal{L} x_{n+1-\varepsilon} \varphi_{n+1-\varepsilon}(t) + x_{n+1-\varepsilon} \mathcal{L} \varphi_{n+1-\varepsilon}(t) -$$

$$- \sum_{\xi=1}^{\varepsilon-1} \mathcal{L} A_{\varepsilon \xi}(t) F_{\xi}(t, x) - \sum_{\xi=1}^{\varepsilon-1} A_{\varepsilon \xi}(t) \mathcal{L} F_{\xi}(t, x) - \mathcal{L} B_{\varepsilon}(t) =$$

$$= \left(\sum_{\xi=0}^{\varepsilon-1} a_{n+1-\varepsilon,n+1-\varepsilon+\xi}(t) x_{n+1-\varepsilon+\xi} + f_{n+1-\varepsilon}(t)\right) \varphi_{n+1-\varepsilon}(t) - a_{n+1-\varepsilon,n+1-\varepsilon}(t) x_{n+1-\varepsilon} \varphi_{n+1-\varepsilon}(t) -$$

$$- \sum_{\xi=1}^{\varepsilon-1} \left(\sum_{k=1}^{\varepsilon-k} a_{n+1-\varepsilon,n+1-\varepsilon+k}(t) \psi_{n+1-\varepsilon+k}(t) A_{\varepsilon-k \xi}(t)\right) \varphi_{n+1-\varepsilon}(t) F_{\xi}(t, x) -$$

$$- (f_{n+1-\varepsilon}(t) + \sum_{\xi=1}^{\varepsilon-1} a_{n+1-\varepsilon,n+1-\varepsilon+\xi}(t) \psi_{n+1-\varepsilon+\xi}(t) B_{\varepsilon-\xi}(t)) \varphi_{n+1-\varepsilon}(t) =$$

$$= \left(\sum_{\xi=1}^{\varepsilon-1} a_{n+1-\varepsilon,n+1-\varepsilon+\xi}(t) x_{n+1-\varepsilon+\xi} - \sum_{\xi=1}^{\varepsilon-1} a_{n+1-\varepsilon,n+1-\varepsilon+\xi}(t) \psi_{n+1-\varepsilon+\xi}(t) \sum_{k=1}^{\varepsilon-k} A_{\varepsilon-\xi k}(t) F_k(t, x) -$$

$$- \sum_{\xi=1}^{\varepsilon-1} a_{n+1-\varepsilon,n+1-\varepsilon+\xi}(t) \psi_{n+1-\varepsilon+\xi}(t) B_{\varepsilon-\xi}(t)) \varphi_{n+1-\varepsilon}(t) =$$

$$= \sum_{\xi=1}^{\varepsilon-1} a_{n+1-\varepsilon,n+1-\varepsilon+\xi}(t) \psi_{n+1-\varepsilon+\xi}(t) \left(x_{n+1-\varepsilon+\xi} \varphi_{n+1-\varepsilon-\xi}(t) - \sum_{k=1}^{\varepsilon-1} A_{\varepsilon-\xi k}(t) F_k(t, x) -$$

$$- B_{\varepsilon-\xi}(t) - A_{\varepsilon-\xi,\varepsilon-\xi}(t) F_{\varepsilon-\xi}(t, x)) \varphi_{n+1-\varepsilon}(t) = 0 \text{ for all } (t, x) \in J \times \mathbb{R}^n.$$

This yields that if $\tau = \varepsilon$, then $F_\varepsilon: J \times \mathbb{R}^n \to \mathbb{R}$ is a first integral of system (2.11).

Thus the functions (2.13) are functionally independent first integrals of system (2.11).
Example 2.4. The linear nonhomogeneous differential system

$$\frac{dx_1}{dt} = -\frac{1}{t} x_1 + \frac{\sqrt{2}}{2} (6 - e^{-t}) x_2 - \frac{\sqrt{2}}{2} (6 + e^{-t}) x_3 + 8t^5 + 4t + 2(t^3 + 3t^2 + 6t + 6)e^{-t},$$

$$\frac{dx_2}{dt} = \frac{1}{2} \left(1 + \frac{2}{t} - 5e^t\right) x_2 + \frac{1}{2} \left(1 - \frac{2}{t} + 5e^t\right) x_3 + 2\sqrt{2}(e^t - t^3), \tag{2.16}$$

$$\frac{dx_3}{dt} = \frac{1}{2} \left(1 - \frac{2}{t} - 5e^t\right) x_2 + \frac{1}{2} \left(1 + \frac{2}{t} + 5e^t\right) x_3 + 2\sqrt{2} e^t$$

can be reduced by the orthogonal transformation

$$x_1 = y_1, \quad x_2 = \frac{\sqrt{2}}{2} (y_2 - y_3), \quad x_3 = \frac{\sqrt{2}}{2} (y_2 + y_3)$$

to the triangular differential system

$$\frac{dy_1}{dt} = -\frac{1}{t} y_1 - e^{-t} y_2 - 6y_3 + 8t^5 + 4t + 2(t^3 + 3t^2 + 6t + 6)e^{-t},$$

$$\frac{dy_2}{dt} = y_2 + 5e^t y_3 - 2t^3 + 4e^t, \quad \frac{dy_3}{dt} = \frac{2}{t} y_3 + 2t^3. \tag{2.17}$$

By Theorem 2.4, using the functions

$$\varphi_1: t \rightarrow t, \quad \varphi_2: t \rightarrow e^{-t}, \quad \varphi_3: t \rightarrow \frac{1}{t^2}, \quad \psi_1: t \rightarrow \frac{1}{t}, \quad \psi_2: t \rightarrow e^t, \quad \psi_3: t \rightarrow t^2$$

for all \( t \in J, \)

$$A_{11}: t \rightarrow 1, \quad A_{21}: t \rightarrow \frac{5}{3} t^3, \quad A_{31}: t \rightarrow \frac{1}{6} (9 - 2t) t^4, \quad A_{32}: t \rightarrow -\frac{1}{2} t^2$$

for all \( t \in J, \)

$$B_1: t \rightarrow t^2, \quad B_2: t \rightarrow t^5 + 4t + 2(t^3 + 3t^2 + 6t + 6)e^{-t}, \quad B_3: t \rightarrow (t - 1) t^6$$

for all \( t \in J, \)

where \( J \) is an interval from the set \( \{ t: t \neq 0 \}, \) we can build the functionally independent first integrals on the domain \( J \times \mathbb{R}^3, \) of the triangular differential system (2.17):

$$F_1: (t, y) \rightarrow \frac{1}{t^2} y_3 - t^2, \quad F_2: (t, y) \rightarrow e^{-t} y_2 - \frac{5}{3} t y_3 + \frac{2}{3} t^5 - 4t - 2(t^3 + 3t^2 + 6t + 6)e^{-t},$$

$$F_3: (t, y) \rightarrow t y_1 + \frac{1}{2} t^2 e^{-t} y_2 + \frac{1}{2} (3 - t) t^2 y_3 - \frac{1}{2} t^3 (2t^4 + t^3 + 4) - t^2 (t^3 + 3t^2 + 6t + 6)e^{-t}. \tag{2.16}$$

Now using the inverse transformation \( y_1 = x_1, y_2 = \frac{\sqrt{2}}{2} (x_2 + x_3), y_3 = \frac{\sqrt{2}}{2} (x_3 - x_2), \) we obtain the first integrals of the differential system (2.16):

$$\widetilde{F}_1: (t, x) \rightarrow \frac{\sqrt{2}}{2t^2} (x_3 - x_2) - t^2$$

for all \( (t, x) \in J \times \mathbb{R}^3, \)

$$\widetilde{F}_2: (t, x) \rightarrow \frac{\sqrt{2}}{6} (5t + 3e^{-t}) x_2 - \frac{\sqrt{2}}{6} (5t - 3e^{-t}) x_3 + \frac{2}{3} t^5 - 4t - 2(t^3 + 3t^2 + 6t + 6)e^{-t},$$

$$\widetilde{F}_3: (t, x) \rightarrow tx_1 + \frac{\sqrt{2}}{4} t^2 (t - 3 + e^{-t}) x_2 + \frac{\sqrt{2}}{4} t^2 (3 - t + e^{-t}) x_3 - \frac{1}{2} t^3 (2t^4 + t^3 + 4) - t^2 (t^3 + 3t^2 + 6t + 6)e^{-t}$$

for all \( (t, x) \in J \times \mathbb{R}^3, \) \( J \subset \{ t: t \neq 0 \}. \)

The functionally independent first integrals \( \widetilde{F}_1, \widetilde{F}_2, \) and \( \widetilde{F}_3 \) are an integral basis on any domain \( J \times \mathbb{R}^3 \) of the linear differential system (2.16).
From Theorem 2.4 with \( f_\tau(t) = 0 \) for all \( t \in J, \tau = 1, \ldots, n \), we have the following

**Corollary 2.2.** First integrals of system (2.12) are the functions

\[
F_\tau : (t, x) \rightarrow x_{n+1-\tau} \varphi_{n+1-\tau}(t) - \sum_{\xi = 1}^{\tau-1} A_{\tau \xi}(t) F_\xi(t, x) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \quad \tau = 1, \ldots, n,
\]

where the functions \( A_{\tau \xi} : J \rightarrow \mathbb{R}, \xi = 1, \ldots, \tau, \tau = 1, \ldots, n \), and \( \varphi_\tau : J \rightarrow \mathbb{R}, \tau = 1, \ldots, n \), are given by the formulas (2.14) and (2.15), respectively.

**Example 2.5.** The second-order linear homogeneous differential system

\[
\frac{dx_1}{dt} = (\cos^2 t a(t) - \cos t \sin t b(t) + \sin^2 t c(t)) x_1 + (\cos t \sin t(a(t) - c(t)) + \cos^2 t b(t) - 1) x_2,
\]

\[
\frac{dx_2}{dt} = (\cos t \sin t(a(t) - c(t)) - \sin^2 t b(t) + 1) x_1 + (\sin^2 t a(t) + \cos t \sin t b(t) + \cos^2 t c(t)) x_2,
\]

where \( a : J \rightarrow \mathbb{R}, b : J \rightarrow \mathbb{R}, \) and \( c : J \rightarrow \mathbb{R} \) are continuous functions on an interval \( J \subset \mathbb{R} \), can be reduced by the orthogonal transformation \( x_1 = \cos t y_1 - \sin t y_2, x_2 = \sin t y_1 + \cos t y_2 \) to the triangular differential system

\[
\frac{dy_1}{dt} = a(t)y_1 + b(t)y_2, \quad \frac{dy_2}{dt} = c(t)y_2.
\]

By Corollary 2.2, the scalar functions

\[
F_1 : (t, y_1, y_2) \rightarrow y_2 \exp - \int_{t_0}^{t} c(\zeta) d\zeta \quad \text{for all} \quad (t, y_1, y_2) \in J \times \mathbb{R}^2
\]

and

\[
F_2 : (t, y_1, y_2) \rightarrow y_1 \exp \left( - \int_{t_0}^{t} a(\zeta) d\zeta \right) - \int_{t_0}^{t} b(\zeta) \exp \left( \int_{\zeta_0}^{\zeta} (c(\theta) - a(\theta)) d\theta \right) d\zeta F_1(t, y_1, y_2)
\]

for all \( (t, y_1, y_2) \in J \times \mathbb{R}^2, \quad t_0, \zeta_0 \in J \),

are first integrals of the triangular differential system (2.19).

Using the inverse transformation \( y_1 = \cos t x_1 + \sin t x_2, \ y_2 = - \sin t x_1 + \cos t x_2 \), we get the integral basis of the linear differential system (2.18):

\[
\bar{F}_1 : (t, x_1, x_2) \rightarrow (\cos t x_2 - \sin t x_1) \exp - \int_{t_0}^{t} c(\zeta) d\zeta \quad \text{for all} \quad (t, x_1, x_2) \in J \times \mathbb{R}^2,
\]

\[
\bar{F}_2 : (t, x_1, x_2) \rightarrow (\cos t x_1 + \sin t x_2) \exp \left( - \int_{t_0}^{t} a(\zeta) d\zeta \right) - \int_{t_0}^{t} b(\zeta) \exp \left( \int_{\zeta_0}^{\zeta} (c(\theta) - a(\theta)) d\theta \right) d\zeta \bar{F}_1(t, x_1, x_2) \quad \text{for all} \quad (t, x_1, x_2) \in J \times \mathbb{R}^2.
\]
2.3. The Lappo-Danilevskii systems

2.3.1. Linear homogeneous differential system

Consider an nonautonomous linear homogeneous differential system

\[
\frac{dx}{dt} = A(t) x, \quad x \in \mathbb{R}^n, \quad A: t \to \sum_{j=1}^{m} \alpha_j(t) A_j \quad \text{for all } t \in J,
\]

(2.20)

where continuous functions \( \alpha_j: J \to \mathbb{R} \), \( j = 1, \ldots, m \), are linearly independent on an interval \( J \subset \mathbb{R} \), and \( A_j, \ j = 1, \ldots, m \), are a commuting family of real constant \( n \times n \) matrices:

\[
A_j A_k = A_k A_j, \quad j = 1, \ldots, m, \quad k = 1, \ldots, m.
\]

The differential system (2.20) is a Lappo-Danilevskii system [8, pp. 34 – 36], i.e. the coefficient matrix \( A \) of system (2.20) is commutative with its integral [15]:

\[
A(t) \int_{t_0}^{t} A(\tau) \, d\tau = \int_{t_0}^{t} A(\tau) \, d\tau \, A(t) \quad \text{for all } t \in J, \quad t_0 \in J.
\]

In this subsection we study the problem of building first integrals for the Lappo-Danilevskii differential system (2.20). Using the approaches of constructing first integrals for linear partial differential systems [25], we also solve the N.P. Erugin problem of the existence of autonomous first integrals [54, p. 469] for the Lappo-Danilevskii system (2.20).

**Partial integrals.** A complex-valued linear homogeneous function

\[
p: x \to \nu x \quad \text{for all } x \in \mathbb{R}^n, \quad \nu \in \mathbb{C}^n,
\]

(2.21)

is a partial integral of the system (2.20) if and only if

\[
\mathfrak{A} p(x) = \lambda(t) p(x) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n,
\]

(2.22)

where the linear differential operator \( \mathfrak{A}(t, x) = \partial_t + A(t) x \partial_x \) for all \( (t, x) \in J \times \mathbb{R}^n \), the function \( \lambda: J \to \mathbb{C} \). The identity (2.22) is equivalent to the linear homogeneous system

\[
\left( \sum_{j=1}^{m} \alpha_j(t) B_j - \lambda(t) E \right) \nu = 0 \quad \text{for all } t \in J,
\]

(2.23)

where \( E \) is the identity matrix of order \( n \), the matrices \( B_j \) are the transpose of the matrices \( A_j, \ j = 1, \ldots, m \), respectively. Since \( A_j, \ j = 1, \ldots, m \), are a commuting family of matrices, we see that there exists a relation [60, pp. 203 – 207] between common eigenvectors and eigenvalues of the matrices \( B_j, \ j = 1, \ldots, m \).

The following basic statements (Lemmas 2.3 and 2.4) are base for the method of building first integrals of the Lappo-Danilevskii differential system (2.20). The proof of Lemmas 2.3 and 2.4 is similar to that one in Lemmas 2.1 and 2.2.

**Lemma 2.3.** Suppose \( \nu \) is a common real eigenvector of the matrices \( B_j \) corresponding to the eigenvalues \( \lambda_j, \ j = 1, \ldots, m \). Then the function (2.21) is a linear partial integral of system (2.20), where the scalar function \( \lambda: J \to \mathbb{R} \) in the identity (2.22) is given by

\[
\lambda: t \to \sum_{j=1}^{m} \lambda_j \alpha_j(t) \quad \text{for all } t \in J.
\]

**Proof.** If \( \nu \) is a common real eigenvector of the matrices \( B_j \) corresponding to the eigenvalues \( \lambda_j, \ j = 1, \ldots, m \), then \( \nu \) is a common real eigenvector of the matrix \( B: t \to \sum_{j=1}^{m} \alpha_j(t) B_j \) for all \( t \in J \) corresponding to the eigenfunction \( \lambda: t \to \sum_{j=1}^{m} \lambda_j \alpha_j(t) \) for all \( t \in J \).
Therefore $\nu$ is a solution to the functional system (2.23). This yields that the identity (2.22) is satisfied. Consequently the linear function (2.21) is a partial integral of the Lappo-Danilevskii homogeneous differential system (2.20).

**Lemma 2.4.** Suppose $\nu = \nu^* + \nu \imath \ (\nu^* = \text{Re } \nu, \ \nu = \text{Im } \nu \neq 0)$ is a common complex eigenvector of the matrices $B_j$ corresponding to the eigenvalues $\lambda_j^* = \lambda_j^* + \lambda_j^* \imath \ (\lambda_j^* = \text{Re } \lambda_j^*, \ \lambda_j^* = \text{Im } \lambda_j^*)$, $j = 1, \ldots, m$. Then the Lie derivatives of the scalar functions
\[ P: x \to (\nu x)^2 + (\nu x)^2 \quad \text{for all } x \in \mathbb{R}^n \quad \text{and} \quad \psi: x \to \arctan \frac{\nu x}{\nu x} \quad \text{for all } x \in X \]
by virtue of system (2.20) are equal to
\[ \mathfrak{A} P(x) = 2 \sum_{j=1}^m \lambda_j \alpha_j(t) P(x) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n \]
and
\[ \mathfrak{A} \psi(x) = \sum_{j=1}^m \tilde{\lambda}_j \alpha_j(t) \quad \text{for all } (t, x) \in J \times X, \]
where $X$ is a domain from the set $\{x: \nu x \neq 0\} \subset \mathbb{R}^n$.

*Proof.* Formally using Lemma 2.3, we obtain the complex-valued function (2.21) is a partial integral of the differential system (2.20) and the following identity holds
\[ \mathfrak{A} (\nu x + \nu \imath x) = (\nu x + \nu \imath x) (\nu x + \nu \imath x) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n. \]
This complex identity is equivalent to the real system of identities
\[ \mathfrak{A} \nu x = \lambda(t) \nu x - \tilde{\lambda}(t) \nu x, \quad \mathfrak{A} \nu x = \lambda(t) \nu x + \tilde{\lambda}(t) \nu x \quad \text{for all } (t, x) \in J \times \mathbb{R}^n, \]
where the scalar functions $\lambda: t \to \sum_{j=1}^m \lambda_j \alpha_j(t)$ for all $t \in J$, $\tilde{\lambda}: t \to \sum_{j=1}^m \tilde{\lambda}_j \alpha_j(t)$ for all $t \in J$.

Using this system of identities, we have
\[ \mathfrak{A} P(x) = \mathfrak{A} ((\nu x)^2 + (\nu x)^2) = 2 \nu x \mathfrak{A} \nu x + 2 \nu x \mathfrak{A} \nu x = 2 \nu x (\lambda(t) \nu x - \tilde{\lambda}(t) \nu x) + 2 \nu x (\lambda(t) \nu x + \tilde{\lambda}(t) \nu x) = 2 \lambda(t) P(x) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n, \]
\[ \mathfrak{A} \psi(x) = \nu x (\lambda(t) \nu x - \tilde{\lambda}(t) \nu x) - \nu x (\lambda(t) \nu x + \tilde{\lambda}(t) \nu x) = \tilde{\lambda}(t) \quad \text{for all } (t, x) \in J \times X. \]

*Nonautonomous first integrals.* Using Theorems 2.5, 2.6, and 2.7, we can obtain first integrals of the the Lappo-Danilevskii homogeneous differential system (2.20).

**Theorem 2.5.** Let the assumptions of Lemma 2.3 hold, then the scalar function
\[ F: (t, x) \to \nu x \exp \left( - \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) d\tau \right) \quad \text{for all } (t, x) \in J \times \mathbb{R}^n, \quad t_0 \in J, \quad (2.24) \]
is a first integral on the domain $J \times \mathbb{R}^n$ of the Lappo-Danilevskii system (2.20).

*Proof.* From Lemma 2.3, we get
\[ \mathfrak{A} F(t, x) = \exp \left( - \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) d\tau \right) \mathfrak{A} \nu x + \nu x \mathfrak{A} \exp \left( - \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) d\tau \right) = \]
\[ = \sum_{j=1}^m \lambda_j \alpha_j(t) F(t, x) - F(t, x) \partial_t \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) d\tau = 0 \quad \text{for all } (t, x) \in J \times \mathbb{R}^n. \]
Example 2.6. The linear homogeneous differential system
\[
\frac{dx_1}{dt} = (t + 2 \sin t) x_1 + \sin t x_2, \quad \frac{dx_2}{dt} = \sin t x_1 + t x_2
\]
has the coefficient matrix
\[
A : t \to \begin{bmatrix} t + 2 \sin t & \sin t \\ \sin t & t \end{bmatrix}
\]
such that \( A(t) = t A_1 + \sin t A_2 \) for all \( t \in \mathbb{R} \), where the constant matrices
\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.
\]
Since the matrices \( A_1 \) and \( A_2 \) are commutative, we see that the system (2.25) is a Lappo-Danilevskii system.

The matrices \( B_1 = A_1^2 = A_1 \) and \( B_2 = A_2^2 = A_2 \) have the common real eigenvectors
\[
\nu^1 = (1 - \sqrt{2}, 1) \quad \text{and} \quad \nu^2 = (1 + \sqrt{2}, 1)
\]
respectively, corresponding to the eigenvalues \( \lambda^1 = 1, \lambda^2 = 1 - \sqrt{2} \) and \( \lambda^2 = 1, \lambda^2 = 1 + \sqrt{2} \), respectively.

Using the functions \( \alpha_1 : t \to t \) for all \( t \in \mathbb{R} \) and \( \alpha_2 : t \to \sin t \) for all \( t \in \mathbb{R} \), we can build (by Theorem 2.5) the first integrals of the Lappo-Danilevskii system (2.25)
\[
F_1 : (t, x_1, x_2) \to ((1 - \sqrt{2}) x_1 + x_2) \exp \left( -\frac{t^2}{2} + (1 - \sqrt{2}) \cos t \right)
\]
for all \( (t, x_1, x_2) \in \mathbb{R}^3 \) and
\[
F_2 : (t, x_1, x_2) \to ((1 + \sqrt{2}) x_1 + x_2) \exp \left( -\frac{t^2}{2} + (1 + \sqrt{2}) \cos t \right)
\]
for all \( (t, x_1, x_2) \in \mathbb{R}^3 \).

The functionally independent first integrals \( F_1 \) and \( F_2 \) are an integral basis of the Lappo-Danilevskii differential system (2.25) on space \( \mathbb{R}^3 \).

Theorem 2.6. Let the assumptions of Lemma 2.4 hold, then first integrals of the Lappo-Danilevskii differential system (2.20) are the scalar functions
\[
F_1 : (t, x) \to (\nu x)^2 + (\bar{\nu} x)^2 \exp \left( -2 \int_{t_0}^{t} \lambda^j \alpha_j(\tau) d\tau \right)
\]
for all \( (t, x) \in J \times \mathbb{R}^n \) and
\[
F_2 : (t, x) \to \arctan \frac{\nu x}{\bar{\nu} x} - \int_{t_0}^{t} \lambda^j \alpha_j(\tau) d\tau
\]
for all \( (t, x) \in J \times \mathcal{X} \),

where \( t_0 \) is a fixed point from the interval \( J \), and a domain \( \mathcal{X} \) from the set \( \{ x : \nu x \neq 0 \} \).

Proof. Taking into account Lemma 2.4, we obtain
\[
\mathfrak{A} F_1(t, x) = \exp \left( -2 \int_{t_0}^{t} \sum_{j=1}^{m} \lambda^j \alpha_j(\tau) d\tau \right) \mathfrak{A} ((\nu x)^2 + (\bar{\nu} x)^2) +
\]
\[
+ ((\nu x)^2 + (\bar{\nu} x)^2) \mathfrak{A} \exp \left( -2 \int_{t_0}^{t} \sum_{j=1}^{m} \lambda^j \alpha_j(\tau) d\tau \right) = 2 \sum_{j=1}^{m} \lambda^j \alpha_j(t) F_1(t, x) +
\]
\[
+ ((\nu x)^2 + (\bar{\nu} x)^2) \partial_t \exp \left( -2 \int_{t_0}^{t} \sum_{j=1}^{m} \lambda^j \alpha_j(\tau) d\tau \right) = 0 \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n,
\]

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we can construct the integral basis of the Lappo-Danilevskii differential system (2.27) of the matrices and the scalar functions on any domain \( \Omega \)

\[ \mathfrak{A} F_2(t, x) = \mathfrak{A} \arctan \frac{\nu x}{\nu x} - \mathfrak{A} \int_{t_0}^{t} \sum_{j=1}^{m} \lambda_j \alpha_j(\tau) d\tau = \]

\[ = \sum_{j=1}^{m} \lambda_j \alpha_j(t) - \partial_t \int_{t_0}^{t} \sum_{j=1}^{m} \lambda_j \alpha_j(\tau) d\tau = 0 \quad \text{for all} \quad (t, x) \in J \times X. \]

Thus the functions \( F_1 \) and \( F_2 \) are first integrals of system (2.20).

**Example 2.7.** Consider the second order Lappo-Danilevskii differential system

\[ \frac{dx_1}{dt} = a \cos \omega t \, x_1 + b \sin \omega t \, x_2, \quad \frac{dx_2}{dt} = -d \sin \omega t \, x_1 + a \cos \omega t \, x_2, \quad (2.26) \]

where \( a, b, \) and \( d \) are positive constants, and \( \omega \) is an non-zero real number.

The matrices \( B_1 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \) and \( B_2 = \begin{bmatrix} 0 & -d \\ b & 0 \end{bmatrix} \) have the common complex eigenvectors \( \nu^1 = (\sqrt{a}, \sqrt{b} i) \) and \( \nu^2 = (\sqrt{d}, -\sqrt{b} i) \) associated with the eigenvalues \( \lambda_1 = a \lambda_2 = -\sqrt{bd} i \) and \( \lambda_2 = a, \lambda_2 = \sqrt{bd} i \), respectively.

Using the numbers \( \lambda_1^* = a, \lambda_2^* = 0, \lambda_1 = 0, \lambda_2 = -\sqrt{bd}, \) and the scalar functions \( \alpha_1 : t \to \cos \omega t, \alpha_2 : t \to \sin \omega t \) for all \( t \in \mathbb{R} \), we can find (by Theorem 2.6) the basis of first integrals for the Lappo-Danilevskii differential system (2.26)

\[ F_1: (t, x_1, x_2) \to (dx_1^2 + bx_2^2) \exp\left(-\frac{2a}{\omega} \sin \omega t\right) \quad \text{for all} \quad (t, x_1, x_2) \in \mathbb{R}^3, \]

\[ F_2: (t, x_1, x_2) \to \arctan \frac{\sqrt{b} x_2}{\sqrt{d} x_1} - \frac{\sqrt{b d}}{\omega} \cos \omega t \quad \text{for all} \quad (t, x_1, x_2) \in \Omega \]

on any domain \( \Omega \) from the set \( \{(t, x_1, x_2) : x_1 \neq 0\} \subset \mathbb{R}^3. \)

**Example 2.8.** The linear homogeneous differential system

\[ \frac{dx_1}{dt} = 3(tanh t + t^2) x_1 + tanh t (x_2 - x_3), \quad \frac{dx_2}{dt} = -2 tanh t x_1 + 3t^2 x_2, \quad \frac{dx_3}{dt} = tanh t (x_1 + x_2) + (tanh t + 3t^2) x_3 \quad (2.27) \]

is the Lappo-Danilevskii system of form (2.20) with the constant matrices

\[ B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} -2 & 2 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \]

and the scalar functions \( \alpha_1 : t \to tanh t + 3t^2 \) for all \( t \in \mathbb{R}, \alpha_2 : t \to - tanh t \) for all \( t \in \mathbb{R}. \)

Using the linearly independent common eigenvectors

\[ \nu^1 = (1, 0, -1), \quad \nu^2 = (i, i, 1), \quad \nu^3 = (-i, -i, 1) \]

of the matrices \( B_1, B_2 \) and the corresponding eigenvalues

\[ \lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_1^* = 1, \quad \lambda_2^* = i, \quad \lambda_3 = 1, \quad \lambda_3^* = -i, \]

we can construct the integral basis of the Lappo-Danilevskii differential system (2.27)
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\[ F_1: (t, x) \to \frac{1}{e^{t^3 \cosh^2 t}} (x_1 - x_3) \text{ for all } (t, x) \in \mathbb{R}^4 \text{ (by Theorem 2.5)}, \]

\[ F_2: (t, x) \to \frac{1}{e^{2t^3 \cosh^2 t}} (x_3^2 + (x_1 + x_2)^2) \text{ for all } (t, x) \in \mathbb{R}^4 \text{ (by Theorem 2.6)}, \]

\[ F_3: (t, x) \to \arctan \frac{x_1 + x_2}{x_3} + \ln \cosh t \text{ for all } (t, x) \in \Omega \text{ (by Theorem 2.6)} \]

on any domain \( \Omega \) from the set \( \{(t, x): x_3 \neq 0\} \subset \mathbb{R}^4 \).

**Lemma 2.5.** Let the following conditions hold:

(i) \( \nu^0 \) is a common eigenvector of \( B_j \) corresponding to the eigenvalues \( \lambda_j \), \( j = 1, \ldots, m \);

(ii) \( \nu^0, \theta = 1, \ldots, s - 1 \), are generalized eigenvectors of the matrix \( B_\zeta \) corresponding to the eigenvalue \( \lambda_\zeta \) with elementary divisor of multiplicity \( s \geq 2 \);

(iii) the linear differential system \( \frac{dx}{dt} = A_\zeta x \) has no the first integrals of the form

\[ F^\zeta_{\theta^0}: x \to a_j \Psi^\zeta_\theta(x) \text{ for all } x \in \mathcal{X}, \ j = 1, \ldots, m, \ j \neq \zeta, \ \theta = 1, \ldots, s - 1. \quad (2.28) \]

Then, we claim that

\[ a_\zeta \Psi^1_\theta(x) = 1 \text{ for all } x \in \mathcal{X}, \quad a_\zeta \Psi^\zeta_\theta(x) = 0 \text{ for all } x \in \mathcal{X}, \ \theta = 2, \ldots, s - 1, \quad (2.29) \]

and

\[ a_j \Psi^\zeta_\theta(x) = \mu^\zeta_\theta = \text{const} \text{ for all } x \in \mathcal{X}, \ j = 1, \ldots, m, \ j \neq \zeta, \ \theta = 1, \ldots, s - 1. \quad (2.30) \]

where the linear differential operators \( a_j(x) = A_j x \partial_x \) for all \( x \in \mathbb{R}^n \), the scalar functions \( \Psi^\zeta_\theta: \mathcal{X} \to \mathbb{R} \) are the solution to the functional system

\[ \nu^\theta x = \sum_{\rho=1}^{\theta} \theta \mu^{\rho-1}_\theta \Psi^\zeta_\rho(x) \nu^{\rho-\theta} x \text{ for all } x \in \mathcal{X}, \ \theta = 1, \ldots, s - 1, \ \mathcal{X} \subset \{x: \nu^0 x \neq 0\}. \quad (2.31) \]

**Proof.** From the proof of Theorem 1.8 it follows that the identities (2.29) for the linear differential system \( \frac{dx}{dt} = A_\zeta x \) are true. Since the linear differential operators of first order \( a_j, j = 1, \ldots, m \), are commutative and the system \( \frac{dx}{dt} = A_\zeta x \) hasn’t the first integrals (2.28), we see that the system of identities (2.30) is satisfied.

Thus there exist the functionally independent functions \( \Psi^\zeta_\theta: \mathcal{X} \to \mathbb{R}, \ \theta = 1, \ldots, s - 1 \) (the solution to the system (2.31)) and these functions satisfies (2.29) and (2.30).

**Theorem 2.7.** Under the conditions of Lemma 2.5, we get first integrals of the Lappo-Danilevskii differential system (2.20) are the scalar functions

\[ F^\theta_\zeta: (t, x) \to \Psi^\zeta_\theta(x) - \int_{t_0}^t \sum_{j=1}^m \mu^\zeta_\theta \alpha_j(\tau) d\tau \text{ for all } (t, x) \in J \times \mathcal{X}, \ \theta = 1, \ldots, s - 1, \quad (2.32) \]

where \( t_0 \) is a fixed point from the interval \( J \), a domain \( \mathcal{X} \) from the set \( \{x: \nu^0 x \neq 0\} \subset \mathbb{R}^n \).

**Proof.** From the identities (2.29) and (2.30), on the domain \( J \times \mathcal{X} \) we get the following

\[ A F^\theta_\zeta(t, x) = -\partial_t \int_{t_0}^t \sum_{j=1}^m \mu^\zeta_\theta \alpha_j(\tau) d\tau + \sum_{j=1}^m \alpha_j(t) a_j \Psi^\zeta_\theta(x) = 0, \ \theta = 1, \ldots, s - 1. \]

Therefore the functions (2.32) are functionally independent first integrals on the domain \( J \times \mathcal{X} \) of the Lappo-Danilevskii differential system (2.20).
Example 2.9. Lappo-Danilevskii differential systems with non-diagonal coefficient matrices of the second order are the linear differential systems of the form [58]

\[
\frac{dx_1}{dt} = (\alpha_1(t) + b_1 \alpha_2(t)) x_1 + \alpha_2(t) x_2, \quad \frac{dx_2}{dt} = b_2 \alpha_2(t) x_1 + \alpha_1(t) x_2, \tag{2.33}
\]

where the functions \( \alpha_1: J \to \mathbb{R} \) and \( \alpha_2: J \to \mathbb{R} \) are continuous, \( b_1 \) and \( b_2 \) are real numbers.

Consider the number \( D = b_1^2 + 4b_2 \). We have three possible cases.

Let \( D > 0 \). Then, using the common real eigenvectors \( \nu^1 = (1, -\lambda_2^2) \), \( \nu^2 = (1, -\lambda_1^2) \) and the corresponding eigenvalues \( \lambda_1^1 = 1, \lambda_2^2 = (b_1 - \sqrt{D})/2, \lambda_2^1 = 1, \lambda_2^2 = (b_1 + \sqrt{D})/2 \), we can build (by Theorem 2.5) the integral basis of system (2.33)

\[
F_k: (t, x_1, x_2) \to \left( x_1 - \lambda_3^{2-k} x_2 \right) \exp \left( -\int_{t_0}^t (\alpha_1(\tau) + \lambda_k^2 \alpha_2(\tau)) \, d\tau \right)
\]

for all \((t, x_1, x_2) \in J \times \mathbb{R}^2, \quad t_0 \in J, \quad k = 1, 2.\)

Let \( D < 0 \). Then, using the common complex eigenvector \( \nu^1 = (\lambda_1^1, 1) \) and the corresponding eigenvalues \( \lambda_1^1 = 1, \lambda_1^2 = \bar{\lambda} - \bar{\lambda} i \), where \( \bar{\lambda} = b_1/2, \bar{\lambda} = \sqrt{D}/2 \), we can construct (by Theorem 2.6) the basis of first integrals for system (2.33)

\[
F_1: (t, x_1, x_2) \to \left( \lambda^1 x_1 + x_2 \right) \exp \left( -2 \int_{t_0}^t (\alpha_1(\tau) + \lambda \alpha_2(\tau)) \, d\tau \right)
\]

for all \((t, x_1, x_2) \in J \times \mathbb{R}^2, \quad t_0 \in J,\)

\[
F_2: (t, x_1, x_2) \to \arctan \frac{\bar{\lambda} x_1}{\lambda x_1 + x_2} + \int_{t_0}^t \bar{\lambda} \alpha_2(\tau) \, d\tau
\]

for all \((t, x_1, x_2) \in J \times \mathcal{X}, \quad t_0 \in J,\)

where \( \mathcal{X} \) is a domain from the set \( \{ (x_1, x_2): \lambda_1^1 x_1 + x_2 \neq 0 \} \subset \mathbb{R}^2 \).

Let \( D = 0 \). Then, using the common real eigenvector \( \nu^0 = (\lambda_1^2, 1) \), the 1-st order real generalized eigenvector \( \nu^1 = (1, 0) \), and the corresponding eigenvalue \( \lambda_1^1 = b_1/2 \) with elementary divisor of multiplicity \( s = 2 \), we can find (by Theorems 2.5 and 2.7) the functionally independent first integrals of system (2.33)

\[
F_1: (t, x_1, x_2) \to \left( \lambda_1^2 x_1 + x_2 \right) \exp \left( -\int_{t_0}^t (\alpha_1(\tau) + \lambda_1^2 \alpha_2(\tau)) \, d\tau \right)
\]

for all \((t, x_1, x_2) \in J \times \mathbb{R}^2, \quad t_0 \in J,\)

\[
F_2: (t, x_1, x_2) \to \frac{x_1}{\lambda_1^2 x_1 + x_2} - \int_{t_0}^t \alpha_2(\tau) \, d\tau
\]

for all \((t, x_1, x_2) \in J \times \mathcal{X}, \quad t_0 \in J,\)

where \( \mathcal{X} \) is a domain from the set \( \{ (x_1, x_2): \lambda_1^2 x_1 + x_2 \neq 0 \} \subset \mathbb{R}^2 \).

In the complex case, from the complex-valued first integrals (2.32) of the Lappo-Danilevskii differential system (2.20), we obtain the real first integrals

\[
F_\theta^1: (t, x) \to \text{Re} \Psi_\theta(x) - \int_{t_0}^t \sum_{j=1}^m \text{Re} \mu_\theta^j \zeta \alpha_j(\tau) \, d\tau
\]

for all \((t, x) \in J \times \mathcal{X}, \quad \theta = 1, \ldots, s - 1,\)

\[
F_\theta^2: (t, x) \to \text{Im} \Psi_\theta(x) - \int_{t_0}^t \sum_{j=1}^m \text{Im} \mu_\theta^j \zeta \alpha_j(\tau) \, d\tau
\]

for all \((t, x) \in J \times \mathcal{X}, \quad \theta = 1, \ldots, s - 1.\)
Autonomous first integrals. The base of building autonomous first integrals for the Lappo-Danilevskii differential system (2.20) is the following basic proposition.

**Lemma 2.6.** A function \( F: \mathcal{X} \rightarrow \mathbb{R} \) is an autonomous first integral of the Lappo-Danilevskii differential system (2.20) if and only if this function is a first integral of the linear homogeneous system of partial differential equations

\[
a_j(x)y = 0, \quad j = 1, \ldots, m, \tag{2.34}
\]

where the linear differential operators of first order \( a_j(x) = A_j x \partial_x \) for all \( x \in \mathbb{R}^n \).

Proof [54, p. 470]. A function \( F: \mathcal{X} \rightarrow \mathbb{R} \) is an autonomous first integral of the Lappo-Danilevskii differential system (2.20) if and only if the following identity holds

\[
\sum_{j=1}^{m} \alpha_j(t) A_j x \partial_x F(x) = 0 \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n.
\]

Since the functions \( \alpha_j: J \rightarrow \mathbb{R}, \ j = 1, \ldots, m \), are linearly independent on the interval \( J \), we see that this identity is equivalent to the system of identities

\[
A_j x \partial_x F(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^n, \quad j = 1, \ldots, m.
\]

This yields that the function \( F: \mathcal{X} \rightarrow \mathbb{R} \) is a first integral of system (2.34).

By Theorems 2.8, 2.9, 2.10, and 2.11, we can construct autonomous first integrals of the Lappo-Danilevskii homogeneous differential system (2.20).

**Theorem 2.8.** Suppose \( \nu^k \) are common real eigenvectors of the matrices \( B_j \) corresponding to the eigenvalues \( \lambda^j_k, \ j = 1, \ldots, m, \ k = 1, \ldots, m+1 \). Then the Lappo-Danilevskii differential system (2.20) has the autonomous first integral

\[
F: x \rightarrow \prod_{k=1}^{m+1} \left| \nu^k x \right|^{h^j_k} \quad \text{for all} \quad x \in \mathcal{X}, \quad \mathcal{X} \subset D F, \tag{2.35}
\]

where the real numbers \( h^j_k, \ k = 1, \ldots, m+1 \), are an nontrivial solution to the system

\[
\sum_{k=1}^{m+1} \lambda^j_k h^j_k = 0, \quad j = 1, \ldots, m.
\]

Proof. First note that the Lappo-Danilevskii system (2.20) is induced the linear differential systems \( \frac{d x}{d t} = A_j x \) with the operators \( a_j(x) = A_j x \partial_x \) for all \( x \in \mathbb{R}^n, \ j = 1, \ldots, m \).

If \( \nu^k \) are common real eigenvectors of the matrices \( B_j \) corresponding to the eigenvalues \( \lambda^j_k, \ j = 1, \ldots, m, \ k = 1, \ldots, m+1 \), then the linear homogeneous functions (by Lemma 1.1) \( p_k: x \rightarrow \nu^k x \) for all \( x \in \mathbb{R}^n, \ k = 1, \ldots, m+1 \), are partial integrals of the linear autonomous differential systems \( \frac{d x}{d t} = A_j x, \ j = 1, \ldots, m \), and the following system of identities hold

\[
a_j p_k(x) = \lambda^j_k p_k(x) \quad \text{for all} \quad x \in \mathbb{R}^n, \quad j = 1, \ldots, m, \quad k = 1, \ldots, m+1. \tag{2.36}
\]

We obviously have

\[
a_j F(x) = \prod_{k=1}^{m+1} \left| p_k(x) \right|^{h^j_k} \sum_{k=1}^{m+1} \text{sgn} p_k(x) h^j_k \prod_{l=1, l \neq k}^{m+1} \left| p_l(x) \right| a_j p_k(x) \quad \text{for all} \quad x \in \mathcal{X}, \ j = 1, \ldots, m.
\]

Now taking into account the identities (2.36), we obtain

\[
a_j F(x) = \sum_{k=1}^{m+1} \lambda^j_k h^j_k F(x) \quad \text{for all} \quad x \in \mathcal{X}, \quad j = 1, \ldots, m.
\]
If the real numbers \( h_k, k = 1, \ldots, m + 1 \), are a nontrivial solution to the linear homogeneous system \( \sum_{k=1}^{m+1} \lambda_j^k h_k = 0, j = 1, \ldots, m \), then the scalar function (2.35) is a first integral for the linear homogeneous system of partial differential equations (2.34).

By Lemma 2.6, the function (2.35) is an autonomous first integral of system (2.20). □

**Example 2.10.** Consider the third-order Lappo-Danilevskii differential system

\[
\frac{dx_1}{dt} = e^t \cos e^t x_1 + 2 \sinh t (x_2 + x_3), \quad \frac{dx_2}{dt} = \sinh t x_1 + e^t \cos e^t x_3, \\
\frac{dx_3}{dt} = \sinh t x_1 + e^t \cos e^t x_2.
\]

(2.37)

The matrices \( B_1 = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \) and \( B_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \) have the common real eigenvectors \( \nu^1 = (1, -1, -1) \), \( \nu^2 = (0, 1, -1) \), and \( \nu^3 = (1, 1, 1) \) corresponding to the real eigenvalues \( \lambda_1^1 = -2, \lambda_1^2 = 1, \lambda_1^3 = 0 \), \( \lambda_2^1 = -1, \lambda_2^2 = 2, \lambda_2^3 = 1 \), respectively.

From the linear homogeneous system \(-2h_1 + 2h_3 = 0, h_1 - h_2 + h_3 = 0\), we have, for example, \( h_1 = 1, h_2 = 2, h_3 = 1 \). By Theorem 2.8, the scalar function

\[
F: (x_1, x_2, x_3) \rightarrow (x_1^2 - (x_2 - x_3)^2)(x_2 - x_3)^2 \quad \text{for all} \quad (x_1, x_2, x_3) \in \mathbb{R}^3
\]

is an autonomous first integral on space \( \mathbb{R}^3 \) of the Lappo-Danilevskii system (2.37).

By Theorem 2.5, using the functions \( \alpha_1: t \rightarrow \sinh t \) and \( \alpha_2: t \rightarrow e^t \cos e^t \) for all \( t \in \mathbb{R} \), we can build the nonautonomous first integrals of system (2.37)

\[
F_1: (t, x_1, x_2, x_3) \rightarrow e^{2 \cosh t \sinh t} (x_1 - x_2 - x_3), \quad F_2: (t, x_1, x_2, x_3) \rightarrow e^t (x_2 - x_3), \\
F_3: (t, x_1, x_2, x_3) \rightarrow e^{-2 \cosh t \sinh t} (x_1 + x_2 + x_3) \quad \text{for all} \quad (t, x_1, x_2, x_3) \in \mathbb{R}^4.
\]

Thus every set of the functionally independent first integrals \( \{F, F_1, F_2\}, \{F, F_1, F_3\}, \{F, F_2, F_3\} \), and \( \{F_1, F_2, F_3\} \) is an integral basis on the space \( \mathbb{R}^4 \) of system (2.37).

**Corollary 2.3.** Let \( \nu^k \) be real common eigenvectors of the matrices \( B_j \) corresponding to the eigenvalues \( \lambda_k^j, j = 1, \ldots, m, k = 1, \ldots, m + 1 \). Then an autonomous first integral of the Lappo-Danilevskii differential system (2.20) is the scalar function

\[
F_{12\ldots m(m+1)}: x \rightarrow \prod_{k=1}^{m} |\nu^k x|^{\Delta_k} |\nu^{m+1} x|^{-\Delta} \quad \text{for all} \quad x \in \mathcal{X},
\]

where the determinants \( \Delta_k, k = 1, \ldots, m \) are obtained by replacing the \( k \)-th column of the determinant \( \Delta = |\lambda_k| \) by \( \text{col}(\lambda_{m+1}^1, \ldots, \lambda_{m+1}^m) \), respectively.

**Example 2.11.** Consider the fourth order Lappo-Danilevskii differential system

\[
\frac{dx}{dt} = \alpha_1(t) A_1 x + \alpha_2(t) A_2 x, \quad x \in \mathbb{R}^4,
\]

(2.38)

with linearly independent continuous functions \( \alpha_1: J \rightarrow \mathbb{R}, \alpha_2: J \rightarrow \mathbb{R} \), and the commutative matrices

\[
A_1 = \begin{pmatrix} 0 & 6 & -1 & 3 \\ 1 & -7 & 1 & -5 \\ 2 & -6 & 3 & -3 \\ -2 & 10 & -2 & 8 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 4 & 0 & 2 \\ 1 & -4 & 1 & -3 \\ 0 & -4 & 1 & -2 \\ -2 & 6 & -2 & 5 \end{pmatrix}
\]
The matrices $B_1 = A_1^T$ and $B_2 = A_2^T$, where $T$ denotes the matrix transpose, have the linearly independent common real eigenvectors

$\nu^1 = (0, 2, 0, 1), \quad \nu^2 = (2, 2, 1, 1), \quad \nu^3 = (1, 0, 1, 0), \quad \nu^4 = (-1, 1, -1, 1)$

corresponding to the real eigenvalues

$\lambda_1^1 = -2, \quad \lambda_2^1 = 1, \quad \lambda_3^1 = 2, \quad \lambda_4^1 = 3, \quad \lambda_1^2 = -1, \quad \lambda_2^2 = 1, \quad \lambda_3^2 = 1, \quad \lambda_4^2 = 2$.

The determinants

$\triangle = \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} = -1, \quad \triangle_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1, \quad \triangle_{21} = \begin{vmatrix} -2 & 2 \\ -1 & 1 \end{vmatrix} = 0,$

$\triangle_{12} = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 1, \quad \triangle_{22} = \begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} = -1$.

By Corollary 2.3, the scalar functions

$F_{123}: x \rightarrow (2x_2 + x_4)(x_1 + x_3)$ for all $x \in \mathbb{R}^4$

and

$F_{124}: x \rightarrow \frac{(2x_2 + x_4)(-x_1 + x_2 - x_3 + x_4)}{2x_1 + 2x_2 + x_3 + x_4}$ for all $x \in \mathcal{X}$,

where $\mathcal{X}$ is a domain from the set $\{x: 2x_1 + 2x_2 + x_3 + x_4 \neq 0\} \subset \mathbb{R}^4$, are autonomous first integrals of the Lappo-Danilevskii differential system (2.38).

**Theorem 2.9.** Suppose $\nu^k = \nu^k + \nu^k_\imath$ ($\nu^k_\imath = \text{Re } \nu^k, \nu^k = \text{Im } \nu^k$), $k = 1, \ldots, s, s \leq (m+1)/2$ (this set hasn’t complex conjugate vectors), and $\nu^\theta, \theta = s + 1, \ldots, m + 1 - s$, are common complex and real linearly independent eigenvectors of the matrices $B_j, j = 1, \ldots, m$. Then the Lappo-Danilevskii differential system (2.20) has the autonomous first integral

$F: x \rightarrow \prod_{k=1}^s (P_k(x))^{h_k} \exp(-2\bar{h}_k \varphi_k(x)) \prod_{\theta=s+1}^{m+1-s} |\nu^\theta x|^{h_\theta}$ for all $x \in \mathcal{X}, \mathcal{X} \subset \mathbb{R}^n, \quad (2.39)$

where the scalar functions $P_k: x \rightarrow (\nu^k)^2 + (\nu^k_\imath)^2$ for all $x \in \mathbb{R}^n$, $\varphi_k: x \rightarrow \arctan \frac{\nu^k_\imath x}{\nu^k x}$

for all $x \in \mathcal{X}$, the real numbers $h_k, \bar{h}_k, h_\theta$ are an nontrivial solution to the system

$2 \sum_{k=1}^s (\lambda_k^j h_k - \bar{\lambda}_k^j \bar{h}_k) + \sum_{\theta=s+1}^{m+1-s} \lambda_\theta^j h_\theta = 0, \quad j = 1, \ldots, m, \quad (2.40)$

the numbers $\lambda_k^j = \lambda_k^j + \lambda_k^j_\imath i$ ($\lambda_k^j_\imath = \text{Re } \lambda_k^j, \bar{\lambda}_k^j = \text{Im } \lambda_k^j$) and $\lambda_\theta^j$ are eigenvalues of the matrices $B_j$ corresponding to the eigenvectors $\nu^k$ and $\nu^\theta$, respectively.

**Proof.** By Lemma 1.1, Properties 1.4 and 1.5, the linear homogeneous functions

$p^\xi_k: x \rightarrow \nu^k x$ for all $x \in \mathbb{R}^n, \quad \xi = 1, \ldots, m + 1$,

are partial integrals of the linear differential systems $\frac{dx}{dt} = A_j x, \quad j = 1, \ldots, m$, and the following system of identities hold

$a_j^k P_k(x) = 2 \lambda_k^j P_k(x)$ for all $x \in \mathbb{R}^n, \quad j = 1, \ldots, m, \quad k = 1, \ldots, s,$

$a_j^k \varphi_k(x) = \bar{\lambda}_k^j$ for all $x \in \mathcal{X}, \quad j = 1, \ldots, m, \quad k = 1, \ldots, s, \quad (2.41)$

$a_j^\theta x = \lambda_\theta^j \nu^\theta x$ for all $x \in \mathbb{R}^n, \quad j = 1, \ldots, m, \quad \theta = s + 1, \ldots, m + 1 - s.$
Since the function $F: \mathcal{X} \rightarrow \mathbb{R}$ is given by (2.39), it follows that

$$a_j F(x) = \left( \prod_{k=1}^{s} (P_k(x))^* h_k^{-1} \exp(-2 h_k \varphi_k(x)) \right) \sum_{k=1}^{s} h_k \prod_{l=1, l \neq k}^{s} P_l(x) a_j P_k(x) +$$

$$+ \prod_{k=1}^{s} (P_k(x))^* h_k \exp(-2 h_k \varphi_k(x)) \sum_{k=1}^{s} a_j (\sum_{\theta=s+1}^{m+1-s} \text{sgn}(\nu^\theta x) h_\theta \prod_{l=s+1, l \neq \theta}^{m+1-s} |\nu^\theta x| a_j(\nu^\theta x))$$

for all $x \in \mathcal{X}$, $j = 1, \ldots, m$.

Using the system of identities (2.41), we get

$$a_j F(x) = \left( \sum_{k=1}^{s} \left( \lambda^j_k h_k - \tilde{\lambda}^j_k \tilde{h}_k \right) + \sum_{\theta=s+1}^{m+1-s} \lambda^j_\theta h_\theta \right) F(x) \text{ for all } x \in \mathcal{X}, j = 1, \ldots, m.$$

If the numbers $h_k^*, \tilde{h}_k^*$, $k = 1, \ldots, s$, and $h_\theta$, $\theta = s+1, \ldots, m+1-s$, are nontrivial real solutions to the system (2.40), then (by Lemma 2.6) the scalar function (2.39) is an autonomous first integral of the Lappo-Danilevskii differential system (2.20).

**Example 2.12.** Consider the fourth order Lappo-Danilevskii differential system

$$\frac{dx}{dt} = \alpha_1(t) A_1 x + \alpha_2(t) A_2 x, \quad x \in \mathbb{R}^4,$$

where linearly independent functions $\alpha_1: J \rightarrow \mathbb{R}$ and $\alpha_2: J \rightarrow \mathbb{R}$ are continuous, the matrices $A_1 = \begin{pmatrix} -1 & 2 & -2 & 0 \\ -6 & 5 & -4 & 2 \\ -3 & 2 & -2 & 1 \\ 2 & -1 & 2 & 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} -4 & 2 & 2 & 4 \\ -5 & 1 & 6 & 6 \\ 0 & -1 & 3 & 1 \\ -3 & 2 & 0 & 3 \end{pmatrix}$ commute.

The matrices $B_1 = A_1^T$ and $B_2 = A_2^T$, where $T$ denotes the matrix transpose, have the common eigenvectors $\nu^1 = (1+i, -i, 1+i, -1)$, $\nu^2 = (1-i, i, 1-i, 1)$, $\nu^3 = (1, -1, 2, 0)$, $\nu^4 = (-1, 0, 2, 2)$ corresponding to the eigenvalues $\lambda^1_1 = 1+i$, $\lambda^2_1 = 1-i$, $\lambda^3_3 = -1$, $\lambda^4_3 = i$, $\lambda^2_4 = -i$, $\lambda^3_4 = 1$, $\lambda^4_4 = 2$.

The linear homogeneous systems

$$2h_{11}^* - 2\tilde{h}_{11} - h_{31} = 0, \quad 2h_{11} + h_{31} = 0 \quad \text{and} \quad 2h_{12} - 2\tilde{h}_{12} + h_{42} = 0, \quad -2\tilde{h}_{12} + h_{42} = 0$$

have, for example, the solutions $h_{11} = 2, \tilde{h}_{11} = 1, h_{31} = 2$, and $h_{12} = 1, \tilde{h}_{12} = 2, h_{42} = 2$.

By Theorem 2.9, the scalar functions

$$F_1: x \rightarrow (x_1 - x_2 + 2x_3)((x_1 - x_3 - x_4)^2 + (x_1 - x_2 + x_3)^2) \exp(-\arctan\frac{x_1 - x_2 + x_3}{x_1 - x_3 - x_4})$$

and

$$F_2: x \rightarrow (-x_1 + 2x_3 + 2x_4)((x_1 - x_3 - x_4)^2 + (x_1 - x_2 + x_3)^2) \exp(-4\arctan\frac{x_1 - x_2 + x_3}{x_1 - x_3 - x_4})$$

are autonomous first integrals of the Lappo-Danilevskii differential system (2.42) on any domain $\mathcal{X}$ from the set $\{x: x_1 - x_3 - x_4 \neq 0\} \subset \mathbb{R}^4$. 

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Example 2.13 (continuation of Example 2.8). From the linear homogeneous system
\[ h_1 + 2h_2 = 0, \quad -h_1 - 2\hbar_2 = 0 \]
it follows that, for example, \( h_1 = -2, \ h_2 = \hbar_2 = 1 \).

Then, using the common eigenvectors \( \nu^1 = (1, 0, -1), \ \nu^2 = (i, i, 1) \) and the corresponding
eigenvalues \( \lambda_1^1 = \lambda_2^1 = 1, \ \lambda_1^2 = -1, \ \lambda_2^2 = i \), we can build (by Theorem 2.9) the autonomous
first integral of the Lappo-Danilevskii differential system (2.27)
\[ F: (x_1, x_2, x_3) \to \frac{(x_1 + x_2)^2 + x_3^2}{(x_1 - x_2)^2} \exp\left(-2\arctan\frac{x_1 + x_2}{x_3}\right) \quad \text{for all } (x_1, x_2, x_3) \in \mathcal{X}, \]
where \( \mathcal{X} \) is a domain from the set \( \{(x_1, x_2, x_3): x_1 \neq x_3, x_3 \neq 0\} \).

Theorem 2.10. Suppose \( \nu^\tau = \nu^* + \overline{\nu^\tau}i, \ \nu^{*+\tau} = \nu^* - \overline{\nu^\tau}i, \ \tau = 1, \ldots, s, \ s \leq m/2 \), \( \nu^{2s+1} = \nu^{*+2s+1} + \overline{\nu^{2s+1}}i \), and \( \nu, \ \theta = 2s + 2, \ldots, m + 1 \), are common complex and real linearly
independent eigenvectors of the matrices \( B_j, j = 1, \ldots, m \). Then the Lappo-Danilevskii diffe-
rential system (2.20) has the autonomous first integrals
\[ F_1: x \to \prod_{\tau=1}^s (P_\tau(x)) \left(\nu^* + \overline{\nu^\tau}i\right)^{h_\tau + \hbar_\tau^{*+\tau}} \exp\left(-2\left(h_\tau - \hbar_\tau^{*+\tau}\right)\varphi_\tau(x)\right) \]
(2.43)
\[ \cdot (P_{2s+1}(x)) \left(\nu^* + \overline{\nu^{2s+1}}i\right)^{h_{2s+1}} \exp\left(-2\hbar_{2s+1}\varphi_{2s+1}(x)\right) \prod_{\theta=2s+2}^{m+1} (\nu^\theta x)^{2h_\theta} \quad \text{for all } x \in \mathcal{X} \]
and
\[ F_2: x \to \prod_{\tau=1}^s (P_\tau(x)) \left(\nu^* + \overline{\nu^\tau}i\right)^{h_\tau + \hbar_\tau^{*+\tau}} \exp\left(2\left(h_\tau - \hbar_\tau^{*+\tau}\right)\varphi_\tau(x)\right) \]
(2.44)
\[ \cdot (P_{2s+1}(x)) \left(\nu^* + \overline{\nu^{2s+1}}i\right)^{h_{2s+1}} \exp\left(2\hbar_{2s+1}\varphi_{2s+1}(x)\right) \prod_{\theta=2s+2}^{m+1} (\nu^\theta x)^{2h_\theta} \quad \text{for all } x \in \mathcal{X}, \ \mathcal{X} \subset DF_1 \cap DF_2, \]
where the scalar functions
\[ P_\tau: x \to \left(\nu^\tau x\right)^2 + \left(\overline{\nu^\tau} x\right)^2, \ \varphi_\tau: x \to \arctan\frac{\overline{\nu^\tau} x}{\nu^\tau x} \quad \text{for all } x \in \mathcal{X}, \ \tau = 1, 2, \ldots, s, 2s + 1, \]
the numbers \( h_k = h_k^* + \hbar_k^i, \ k = 1, \ldots, m + 1 \), are an nontrivial solution to the system
\[ \sum_{k=1}^{m+1} \lambda_k^j h_k = 0, \quad j = 1, \ldots, m, \]
and \( \lambda_k^j \) are eigenvalues of the matrices \( B_j \) corresponding to the eigenvectors \( \nu^k \).

Proof. We form two complex-valued functions of real variables
\[ F: x \to \prod_{k=1}^{2s} (\nu^k x)^{h_k} (\nu^{*+k} x)^{h_{2s+1}} \prod_{\theta=2s+2}^{m+1} (\nu^\theta x)^{h_\theta} \quad \text{for all } x \in \mathcal{X} \]
and
\[ ** F: x \to \prod_{k=1}^{2s} (\nu^k x)^{l_k} (\nu^{*+k} x)^{l_{2s+1}} \prod_{\theta=2s+2}^{m+1} (\nu^\theta x)^{l_\theta} \quad \text{for all } x \in \mathcal{X}, \]
where \( h_k, \ l_k, \ k = 1, \ldots, m + 1 \), are some complex numbers, a domain \( \mathcal{X} \subset \mathbb{R}^n \).
We have

\[ a_j^* F(x) = \sum_{k=1}^{m+1} \lambda_k^j h_k^* F(x) \quad \text{for all } x \in \mathcal{X}, \quad j = 1, \ldots, m, \]

\[ a_j^{**} F(x) = \left( \sum_{k=1}^{2s} \lambda_k^j l_k + \lambda_{2s+1}^j l_{2s+1} + \sum_{\theta=2s+2}^{m+1} \lambda_\theta^j l_\theta \right)^{**} F(x) \quad \text{for all } x \in \mathcal{X}, \quad j = 1, \ldots, m. \]

Let \( h_k = h_k^* + \tilde{h}_k i, \ k = 1, \ldots, m + 1, \) be an nontrivial solution to the linear system

\[ \sum_{k=1}^{m+1} \lambda_k^j h_k = 0, \quad j = 1, \ldots, m. \]

Then \( l_k = \tilde{h}_{s+k} - \tilde{h}_{s+k} i, \ l_{s+k} = h_k - \tilde{h}_k i, \ k = 1, \ldots, s, \ l_{2s+1} = h_{2s+1} - \tilde{h}_{2s+1} i, \ l_\theta = \tilde{h}_\theta - \tilde{h}_\theta i, \]

\( \theta = 2s + 2, \ldots, m + 1 \) is an nontrivial solution to the linear system

\[ \sum_{k=1}^{2s} \lambda_k^j l_k + \lambda_{2s+1}^j l_{2s+1} + \sum_{\theta=2s+2}^{m+1} \lambda_\theta^j l_\theta = 0, \quad j = 1, \ldots, m, \]

and the functions \( \star: \mathcal{X} \to \mathbb{C}, \ \star^*: \mathcal{X} \to \mathbb{C} \) are autonomous first integrals of the system (2.20).

Since \( F_1(x) = \star^* F(x) \) \( \star F(x) \) for all \( x \in \mathcal{X} \) and \( F_2(x) = \star^* F_1(x) \star^{-i} (x) \) for all \( x \in \mathcal{X}, \) we see that the scalar functions (2.43) and (2.44) are autonomous first integrals (by Lemma 2.6) of the Lappo-Danilevskii differential system (2.20).

**Example 2.14.** The small oscillations of gyrocompass in the precession gyroscope theory are described by the following fourth-order linear differential system [61]

\[
\begin{align*}
\frac{dx_1}{dt} &= \omega_0 x_2 + \Omega(t)x_4, \\
\frac{dx_2}{dt} &= -\omega_0 x_1 + \Omega(t)x_3, \\
\frac{dx_3}{dt} &= -\Omega(t)x_2 - \omega_0 x_4, \\
\frac{dx_4}{dt} &= -\Omega(t)x_1 + \omega_0 x_3,
\end{align*}
\]

(2.45)

where a continuous on an interval \( J \subset \mathbb{R} \) function \( \Omega: J \to \mathbb{R} \) is the projection of the absolute angular velocity of the gyroscope’s sensitive element onto the direction of geocentric vertical line, the constant \( \omega_0 = \sqrt{g/R} \) \( (g \text{ is the acceleration due to gravity, } R \text{ is the Earth’s radius}). \)

The system (2.45) is a Lappo-Danilevskii differential system [8, pp. 37 – 38].

By Theorem 2.10, using the linearly independent common eigenvectors \( \nu^1 = (1, i, -1, i), \)

\( \nu^2 = (i, -i, i, 1), \) \( \nu^3 = (1, -i, -1, i), \) \( \nu^4 = (-i, -1, -i, 1) \) and the corresponding eigenvalues \( \lambda^1_1 = \lambda^1_2 = -\omega_0 i, \) \( \lambda^3_3 = \lambda^3_4 = \omega_0 i, \) \( \lambda^2_1 = -i, \) \( \lambda^2_2 = \lambda^2_3 = i, \) \( \lambda^3_4 = -i \) of the matrices

\[
B_1 = \begin{bmatrix}
0 & -\omega_0 & 0 & 0 \\
\omega_0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_0 \\
0 & 0 & -\omega_0 & 0 \\
\end{bmatrix}
\quad \text{and} \quad
B_2 = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

we can build the autonomous first integrals of the Lappo-Danilevskii differential system (2.45)

\[
F_1: x \to (x_1 - x_3)^2 + (x_2 + x_4)^2, \quad F_2: x \to (-x_2 + x_4)^2 + (x_1 + x_3)^2 \quad \text{for all } x \in \mathbb{R}^4.
\]

By Theorem 2.6, using the functions \( \alpha_1: t \to 1 \) for all \( t \in J, \) \( \alpha_2: t \to \Omega(t) \) for all \( t \in J, \)

the numbers \( \tilde{\lambda}_1^1 = \tilde{\lambda}_2^2 = 0, \) \( \tilde{\lambda}_1^1 = -\omega_0, \) \( \tilde{\lambda}_1^4 = -1, \) \( \tilde{\lambda}_2^1 = \tilde{\lambda}_2^2 = 0, \) \( \tilde{\lambda}_3^2 = -\omega_0, \) \( \tilde{\lambda}_2^3 = 1, \) and the common eigenvectors \( \nu^1 \) and \( \nu^2, \) we can construct the first integrals of system (2.45)
\[ F_3: (t, x) \rightarrow \arctan \frac{x_2 + x_4}{x_1 - x_3} + \int_{t_0}^{t} (\omega_0 + \Omega(\tau)) \, d\tau \text{ for all } (t, x) \in J \times \mathcal{X}_1, \mathcal{X}_1 \subset \{ x : x_1 - x_3 \neq 0 \}, \]

and

\[ F_4: (t, x) \rightarrow \arctan \frac{x_1 + x_3}{x_2 + x_4} + \int_{t_0}^{t} (\omega_0 - \Omega(\tau)) \, d\tau \text{ for all } (t, x) \in J \times \mathcal{X}_2, \mathcal{X}_2 \subset \{ x : x_2 - x_4 \neq 0 \}. \]

The functionally independent first integrals \( F_1, \ldots, F_4 \) are an integral basis of the Lappp-Danilevskii system (2.45) on any domain \( J \times \mathcal{X} \), where \( \mathcal{X} \subset \{ x : x_1 - x_3 \neq 0, \ x_2 - x_4 \neq 0 \} \).

**Example 2.15.** The Lappp-Danilevskii differential system

\[
\begin{align*}
\frac{dx_1}{dt} &= \alpha_1(t)x_1 + \alpha_2(t)x_3, & \frac{dx_2}{dt} &= -\alpha_1(t)x_2 + \alpha_2(t)x_4, \\
\frac{dx_3}{dt} &= -\alpha_2(t)x_1 + \alpha_1(t)x_3, & \frac{dx_4}{dt} &= -\alpha_2(t)x_2 - \alpha_1(t)x_4,
\end{align*}
\]

(2.46)

with lineally independent on an interval \( J \subset \mathbb{R} \) functions \( \alpha_1: J \to \mathbb{R} \) and \( \alpha_2: J \to \mathbb{R} \) such that the matrices \( B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \) and \( B_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \) have the lineally independent eigenvectors \( \nu^1 = (0, -i, 0, 1), \ \nu^2 = (0, i, 0, 1), \ \nu^3 = (-i, 0, 1, 0), \ \nu^4 = (i, 0, 1, 0) \) corresponding to the eigenvalues \( \lambda_1^1 = \lambda_2^2 = -1, \ \lambda_3^3 = \lambda_4^4 = 1, \ \lambda_1^2 = \lambda_2^3 = -i, \ \lambda_3^1 = \lambda_4^2 = i \).

From the linear system \( -h_1 - h_2 + h_3 = 0, \ -ih_1 + ih_2 - ih_3 = 0 \) it follows that, for example, \( h_1 = 0, \ h_2 = h_3 = 1 \). Then, by Theorem 2.10, the Lappp-Danilevskii differential system (2.46) has the autonomous first integrals

\[
F_1: x \rightarrow (x_1^2 + x_3^2)(x_2^2 + x_4^2) \quad \text{for all } x \in \mathbb{R}^4, \quad F_2: x \rightarrow \frac{x_1x_2 + x_3x_4}{x_1x_4 - x_2x_3} \quad \text{for all } x \in \mathcal{X},
\]

where \( \mathcal{X} \) is any domain from the set \( \{ x : x_1x_4 - x_2x_3 \neq 0 \} \subset \mathbb{R}^4 \).

**Example 2.16.** Consider the fifth-order Lappp-Danilevskii differential system

\[
\frac{dx}{dt} = \alpha_1(t)A_1x + \alpha_2(t)A_2x + \alpha_3(t)A_3x, \quad x \in \mathbb{R}^5,
\]

(2.47)

where linearly independent functions \( \alpha_j: J \to \mathbb{R}, \ j = 1, 2, 3 \) are continuous, the matrices

\[
B_1 = A_1^T = \begin{bmatrix} 4 & -2 & 2 & -7 & 11 \\ 0 & 2 & 2 & -7 & 7 \\ 0 & 4 & 0 & -2 & 2 \\ -4 & 8 & -8 & 6 & -2 \\ -4 & 4 & -4 & 4 & 0 \end{bmatrix}, \quad B_2 = A_2^T = \begin{bmatrix} 4 & 0 & 0 & -5 & 7 \\ 0 & 4 & 0 & -5 & 5 \\ 0 & 4 & 0 & 0 & 0 \\ -2 & 6 & -6 & 6 & -2 \\ -2 & 2 & -2 & 2 & 2 \end{bmatrix},
\]

\[
B_3 = A_3^T = \begin{bmatrix} 0 & 6 & -6 & -9 & 17 \\ 0 & 6 & -18 & 3 & -3 \\ -8 & 20 & -20 & 14 & -14 \\ -8 & 8 & -8 & 8 & -8 \end{bmatrix} \quad (T \text{ denotes the matrix transpose}).
\]

Using the linearly independent common eigenvectors \( \nu^1 = (1, 0, 0, i, i), \ \nu^2 = (1, 0, 0, -i, -i), \ \nu^3 = (1 + 2i, 1 + 2i, 2, 2, 0), \ \nu^4 = (1 - 2i, 1 - 2i, 2, 2, 0), \ \nu^5 = (0, 1, 1, 0, 0) \) of the matrices \( B_1, \)
such that \( \{ \), the numbers \( h \), the solution \( \Psi \) and \( B \) where the scalar functions \( \lambda \) corresponding to the eigenvalues \( \lambda \) \( \exp \) \( A \), is the domain \( X \), from the set \( \{ x : x_2 + x_3 \neq 0, x_1 + x_2 + 2x_3 + 2x_4 \neq 0, x_1 \neq 0 \} \subset \mathbb{R}^5 \).

**Theorem 2.11.** Let the following conditions hold:

(i) \( \nu^\theta \) are linearly independent common real eigenvectors of the matrices \( B_j \) corresponding to the eigenvalues \( \lambda_j \), \( l = 1, \ldots, r \), \( j = 1, \ldots, m \);

(ii) \( \nu^\theta \), \( \theta = 1, \ldots, s_1-1 \), are linearly independent real generalized eigenvectors of the matrix \( B_\zeta \) corresponding to the eigenvalue \( \lambda_1^{\zeta} \) with elementary divisor of multiplicity \( s_1 \), \( l = 1, \ldots, r \), such that \( \sum_{l=1}^r s_l \geq m + 1 \);

(iii) the linear differential system \( \frac{dx}{dt} = A_\zeta x \) has no the first integrals of the form

\[
F_j^{\zeta}: x \to a_j \Psi_j^{\zeta}(x) \text{ for all } x \in X, j = 1, \ldots, m, j \neq \zeta, \theta = 1, \ldots, s_1-1, l = 1, \ldots, r.
\]

Then autonomous first integrals of the Lappo-Danilevskii system (2.20) are the functions

\[
F: x \to \prod_{\xi=1}^k (\nu^{q_\xi} x)^{h_{q_\xi}} \exp \sum_{q=1}^{\varepsilon_\xi} h_{q_\xi} \Psi_{q_\xi}^{\xi}(x) \text{ for all } x \in X, X \subset DF, \tag{2.48}
\]

where the scalar functions \( \Psi_{q_\xi}^{\xi} : X \to \mathbb{R} \) are the solution to the functional system

\[
\nu^{q_\xi} x = \sum_{\rho=1}^q (\nu^{q_\xi})^{(q-1)} \Psi_{\rho_\xi}^{\xi}(x) \nu^{q_\rho_\xi-1} x \text{ for all } x \in X, q = 1, \ldots, \varepsilon_\xi, \xi = 1, \ldots, k, \tag{2.49}
\]

and \( \sum_{\xi=1}^k \varepsilon_\xi = m - k + 1, \varepsilon_\xi \leq s_\xi - 1, \xi = 1, \ldots, k, k \leq r \). Also, here

\[
a_j \Psi_{q_\xi}^{\xi}(x) = \mu^{j_\xi}_q \xi, \quad \mu^{j_\xi}_q = \text{const}, \quad j = 1, \ldots, m, q = 1, \ldots, \varepsilon_\xi, \xi = 1, \ldots, k, \tag{2.50}
\]

the numbers \( h_{q_\xi}, q = 0, \ldots, \varepsilon_\xi, \xi = 1, \ldots, k, \) are a nontrivial solution to the system

\[
\sum_{\xi=1}^k \left( \lambda^{j_\xi}_q h_{q_\xi} + \mu^{j_\xi}_q h_{q_\xi} \right) = 0, \quad j = 1, \ldots, m, \tag{2.51}
\]

**Proof.** The functional system (2.49) with any fixed index \( \xi \in \{1, \ldots, k\} \) has the determinant \( (\nu^{q_\xi} x)^{\xi_\varepsilon} \) for all \( x \in \mathbb{R}^n \) such that \( (\nu^{q_\xi} x)^{\xi_\varepsilon} \neq 0 \) for all \( x \in X \), where a domain \( X \subset \{ x : \nu^{q_\xi} x \neq 0, \xi = 1, \ldots, k \} \). Therefore for any fixed index \( \xi \in \{1, \ldots, k\} \) there exists the solution \( \Psi_{q_\xi}^{\xi} : X \to \mathbb{R}, q = 1, \ldots, \varepsilon_\xi, \) on the domain \( X \) of the functional system (2.49).
For any fixed index $\xi \in \{1, \ldots, k\}$ from the results of Lemma 2.5 it follows that the identities (2.50) are satisfied.

Consequently there exist $\sum_{\xi=1}^{k} \varepsilon_{\xi}$ functionally independent on the domain $\mathcal{X}$ scalar functions $\Psi_{q\xi} : \mathcal{X} \to \mathbb{R}$, $q = 1, \ldots, \varepsilon_{\xi}$, $\xi = 1, \ldots, k$, such that the identities (2.50) hold.

Since the functions $p_{\xi} : x \to \nu^{q\xi} x$ for all $x \in \mathbb{R}^{n}$, $\xi = 1, \ldots, k$, are partial integrals of the linear differential systems $\frac{dx}{dt} = A_{j} x$, $j = 1, \ldots, m$, we have

$$a_{j} F(x) = \left( \sum_{\xi=1}^{k} (\lambda_{\xi}^{j} h_{0\xi} + \sum_{q=1}^{\varepsilon_{\xi}} \mu_{q\xi}^{j} h_{q\xi}) \right) F(x) \quad \text{for all } x \in \mathcal{X}, \quad j = 1, \ldots, m.$$

If the numbers $h_{q\xi}$, $q = 0, \ldots, \varepsilon_{\xi}$, $\xi = 1, \ldots, k$, are an nontrivial solution to the linear homogeneous system (2.51), then the function (2.48) is a first integral of system (2.34).

By Lemma 2.6, the scalar function (2.48) is autonomous first integral on the domain $\mathcal{X}$ of the Lappo-Danilevskii differential system (2.20).

**Example 2.17.** Consider the fourth-order Lappo-Danilevskii differential system

$$\frac{dx}{dt} = \alpha_{1}(t) A_{1} x + \alpha_{2}(t) A_{2} x, \quad x \in \mathbb{R}^{4}, \quad (2.52)$$

where linearly independent functions $\alpha_{1} : J \to \mathbb{R}$ and $\alpha_{2} : J \to \mathbb{R}$ are continuous on an interval $J \subset \mathbb{R}$, the commuting matrices

$$A_{1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad A_{2} = \begin{bmatrix} 2 & 0 & -1 & 0 \\ -1 & 2 & 0 & 1 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & -3 & 1 \end{bmatrix}.$$

The matrix $B_{1} = A_{1}^{T}$, where $T$ denotes the matrix transpose, has the eigenvalue $\lambda_{1}^{j} = 1$ with elementary divisor $(\lambda_{1} - 1)^{4}$ corresponding to the eigenvector $\nu^{01} = (-1, 1, -1, 0)$ and to the generalized eigenvectors $\nu^{11} = (1, 0, -1, -1)$, $\nu^{21} = (1, -1, 3, 0)$, $\nu^{31} = (-3, 0, 9, 9)$.

From the functional system (2.49), we get the scalar functions

$$\Psi_{11} : x \to \frac{x_{1} - x_{3} - x_{4}}{-x_{1} + x_{2} - x_{3}} \quad \text{for all } x \in \mathcal{X},$$

$$\Psi_{21} : x \to \frac{-(x_{1} + x_{2} - x_{3})(x_{1} - x_{2} + x_{3}) - (x_{1} - x_{3} - x_{4})^{2}}{(-x_{1} + x_{2} - x_{3})^{2}} \quad \text{for all } x \in \mathcal{X},$$

$$\Psi_{31} : x \to \frac{1}{(-x_{1} + x_{2} - x_{3})^{3}} \left( ( -3x_{1} + 9x_{3} + 9x_{4})( -x_{1} + x_{2} - x_{3})^{2} - 3(-x_{1} + x_{2} - x_{3})(x_{1} - x_{3} - x_{4})(x_{1} - x_{2} + x_{3}) + 2(x_{1} - x_{3} - x_{4})^{3} \right) \quad \text{for all } x \in \mathcal{X},$$

where $\mathcal{X}$ is any domain from the space $\{ x : x_{1} - x_{2} + x_{3} \neq 0 \}$ of the space $\mathbb{R}^{4}$.

The Lappo-Danilevskii differential system (2.52) is induced on the space $\mathbb{R}^{4}$ the linear differential operators of first order

$$a_{1}(x) = x_{2} \partial_{x_{1}} + (2x_{2} - x_{3} - x_{4}) \partial_{x_{2}} + (x_{1} - x_{4}) \partial_{x_{3}} + (-x_{1} + 2x_{3} + 2x_{4}) \partial_{x_{4}},$$

and

$$a_{2}(x) = (2x_{1} - x_{3}) \partial_{x_{1}} + (-x_{1} + 2x_{2} + x_{4}) \partial_{x_{2}} + (-x_{1} + 3x_{3} + 2x_{4}) \partial_{x_{3}} + (x_{2} - 3x_{3} + x_{4}) \partial_{x_{4}}.$$
Using the identities
\[ a_1 \nu^{01} x = \nu^{01} x \text{ for all } x \in \mathbb{R}^4, \quad a_2 \nu^{01} x = 2 \nu^{01} x \text{ for all } x \in \mathbb{R}^4, \]
\[ a_1 \Psi_{11}^1(x) = 1, \quad a_1 \Psi_{21}^1(x) = 0, \quad a_1 \Psi_{31}^1(x) = 0 \text{ for all } x \in \mathcal{H}, \]
\[ a_2 \Psi_{11}^1(x) = -1, \quad a_2 \Psi_{21}^1(x) = 0, \quad a_2 \Psi_{31}^1(x) = 6 \text{ for all } x \in \mathcal{H}, \]
we obtain the linear homogeneous systems
\[ h_{11} + h_{21} = 0, \quad 2h_{11} - h_{21} = 0 \quad \text{and} \quad h_{12} + h_{22} = 0, \quad 2h_{12} - h_{22} + 6h_{32} = 0. \]
From these systems it follows that \( h_{11} = h_{21} = 0, \) \( h_{31} = 1, \) and \( h_{12} = 2, h_{22} = -2, h_{32} = -1. \)

Then, by Theorem 2.11, the scalar functions
\[ F_1: x \rightarrow \Psi_{21}^1(x) \quad \text{and} \quad F_2: x \rightarrow (-x_1 + x_2 - x_3)^2 \exp \left( -2 \Psi_{11}^1(x) - \Psi_{31}^1(x) \right) \]
are autonomous first integrals of the Lappo-Danilevskii differential system (2.52).

If the matrices \( B_j \) have some complex common eigenvectors \( \nu^{01} \) corresponding to the eigenvalues \( \lambda_j^\xi \) with elementary divisors \( s_i \), then the proof of Theorem 2.11 is also true.

Let the set \( V \) of \( m + 1 \) functions be given by
\[ V = \left\{ \nu^{0 \xi} x \text{ for all } x \in \mathbb{R}^n, \Psi_{q \xi}^\xi(x) \text{ for all } x \in \mathcal{H}: q = 1, \ldots, \varepsilon, \xi = 1, \ldots, k, \sum_{\xi=1}^k \varepsilon_\xi = m - k + 1 \right\}. \]

In the complex case, we shall have two logical possibilities:
1. Any function from the set \( V \) has the complex conjugate function in the set \( V \).
2. At least one function from the set \( V \) has not the complex conjugate function in the \( V \).

In these cases the system (2.20) has the following autonomous first integrals.

Case 1. The Lappo-Danilevskii differential system (2.20) has the autonomous first integral
\[ F: x \rightarrow \prod_{\xi=1}^{k_1} \left( (\nu^{0 \xi} x)^2 + (\nu^{0 \xi} x)^2 \right)^{h_{0 \xi}} \exp \left( -2 \nu^{0 \xi} x \arctan \frac{\nu^{0 \xi} x}{\nu^{0 \xi} x} \right) + \]
\[ + 2 \sum_{q=1}^{\varepsilon_\xi} \left( \nu^{0 \xi} \Psi_{q \xi}^\xi(x) - \bar{h}_{q \xi}^\xi \bar{\Psi}_{q \xi}^\xi(x) \right) \prod_{\rho=1}^{k_2} \nu^{0 \rho} x^{h_{0 \rho}} \exp \sum_{q=1}^{\varepsilon_\rho} h_{q \rho}^\xi \bar{\Psi}_{q \rho}^\xi(x) \quad \text{for all } x \in \mathcal{H} \subset DF, \]
where \( \nu^{0 \xi} \), \( \bar{h}_{q \xi}^\xi \), \( \bar{\Psi}_{q \xi}^\xi(x) \), \( k = \xi \) or \( k = \rho \), \( \xi = 1, \ldots, k_1 \), \( \rho = 1, \ldots, k_2 \), are a real nontrivial solution to the linear homogeneous system
\[ 2 \sum_{\xi=1}^{k_1} \left( \lambda_j^\xi h_{0 \xi} - \lambda_j^\xi \bar{h}_{0 \xi} \right) + \sum_{q=1}^{\varepsilon_\xi} \left( \nu^{0 \xi} \Psi_{q \xi}^\xi(x) - \nu^{0 \xi} \bar{\Psi}_{q \xi}^\xi(x) \right) \]
\[ + \sum_{\rho=1}^{k_2} \left( \lambda_j^\xi h_{0 \rho} + \sum_{q=1}^{\varepsilon_\rho} \nu^{0 \rho} \Psi_{q \rho}^\xi(x) \right) = 0, \quad j = 1, \ldots, m. \]

Here \( \nu^{0 \xi} = \nu^{0 \xi} + \bar{\nu}^{0 \xi} i \) are complex common eigenvectors of the matrices \( B_j \) corresponding to the eigenvalues \( \lambda_j^\xi = \lambda_j^\xi + \bar{\lambda}_j^\xi i \), \( j = 1, \ldots, m \), \( \xi = 1, \ldots, k_1 \), respectively; \( \nu^{0 \rho} \) are real common eigenvectors of \( B_j \) corresponding to the eigenvalues \( \lambda_j^\rho \), \( j = 1, \ldots, m \), \( \rho = 1, \ldots, k_2 \), respectively; the functions \( \Psi_{q \xi}^\xi = \Psi_{q \xi}^\xi + \bar{\Psi}_{q \xi}^\xi i \) and \( \Psi_{q \rho}^\xi \) are the solution to the functional system (2.49); the real numbers
\[ \mu^{0 \xi} = \text{Re} \left( a_j \Psi_{q \xi}^\xi(x) \right), \quad \bar{\mu}^{0 \xi} = \text{Im} \left( a_j \Psi_{q \xi}^\xi(x) \right), \quad \mu^{0 \rho} = a_j \Psi_{q \rho}^\xi(x) \quad \text{for all } x \in \mathcal{H}, \]
\[ q = 1, \ldots, \varepsilon_\xi, \quad k = \xi \quad \text{or} \quad k = \rho, \quad \xi = 1, \ldots, k_1, \quad \rho = 1, \ldots, k_2, \quad j = 1, \ldots, m; \]
the natural numbers $\varepsilon_\xi$ and $\varepsilon_\rho$ such that $2 \sum_{\xi=1}^{k_1} \varepsilon_\xi + \sum_{\rho=1}^{k_2} \varepsilon_\rho = m - 2k_1 - k_2 + 1$ with $2k_1 + k_2 \leq r$, $\varepsilon_\xi \leq s_\xi - 1$, $\xi = 1, \ldots, k_1$, $\varepsilon_\rho \leq s_\rho - 1$, $\rho = 1, \ldots, k_2$, where $k_1$ is an number of pairs of complex-conjugate common eigenvectors of the matrices $B_j$, and $k_2$ is an number of real common eigenvectors of the matrices $B_j$.

Example 2.18. Consider the sixth-order Lappo-Danilevskii differential system

$$\frac{dx}{dt} = \alpha_1(t)A_1 x + \alpha_2(t)A_2 x + \alpha_3(t)A_3 x, \quad x \in \mathbb{R}^6,$$

where $\alpha_j : J \to \mathbb{R}$, $j = 1, 2, 3$, are linearly independent continuous functions on an interval $J \subset \mathbb{R}$, the commuting matrices

$$B_1 = A_1^T = \begin{bmatrix} 3 & -1 & -3 & 3 & 5 & -2 \\ -4 & 3 & 5 & -6 & -5 & 5 \\ 4 & -3 & -5 & 4 & 8 & -4 \\ 1 & 0 & -1 & 4 & 3 & -3 \\ 0 & -2 & -2 & -1 & 3 & 1 \\ 2 & -3 & -4 & 5 & 6 & -2 \end{bmatrix}, \quad B_2 = A_2^T = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & -2 \\ -4 & 3 & 6 & -6 & -6 & 4 \\ 2 & 0 & -2 & 2 & 3 & -3 \\ 1 & 0 & -1 & 3 & 2 & -3 \\ -1 & 1 & 2 & -4 & -2 & 2 \\ 1 & -1 & -1 & 2 & 2 & -2 \end{bmatrix},$$

and $B_3 = A_3^T = \begin{bmatrix} -3 & 2 & 2 & -3 & -3 & 2 \\ -2 & -3 & -3 & 2 & 3 & -1 \\ -4 & 3 & 5 & -6 & -6 & 4 \\ -3 & 3 & 4 & -4 & -4 & 2 \\ 0 & -1 & 0 & -1 & -1 & 1 \\ -2 & 2 & 2 & -1 & -2 & 0 \end{bmatrix}$ ($T$ denotes the matrix transpose).

Using the eigenvalue $\lambda_1^1 = 1 + 2i$ with elementary divisor $(\lambda^3 - 1 - 2i)^3$ of the matrix $B_1$ and the corresponding complex eigenvector $\nu^{01} = (1, 0, 1 + i, 1, i, 1)$, the generalized eigenvector of the 1-st order $\nu^{11} = (1, 1 + i, 0, 0, i, i)$, the generalized eigenvector of the 2-nd order $\nu^{21} = (2 + 2i, 0, 2 + 2i, 0, 2i, 2i)$, we can construct the scalar functions

$$\Psi_{11}^* : x \to \frac{(x_1 + x_2)(x_1 + x_3 + x_4 + x_6) + (x_3 + x_5)(x_2 + x_5 + x_6)}{(x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2} \quad \text{for all } x \in \mathbb{R},$$

$$\Psi_{11}^\circ : x \to \frac{(x_1 + x_3 + x_4 + x_6)(x_2 + x_5 + x_6) - (x_1 + x_2)(x_3 + x_5)}{(x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2} \quad \text{for all } x \in \mathbb{R},$$

$$\Psi_{21}^* : x \to \frac{1}{(x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2} \left( 2((x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2) \cdot \left( (x_1 + x_3)(x_1 + x_3 + x_4 + x_6) + (x_3 + x_5)(x_1 + x_3 + x_5 + x_6) - (x_1 + x_2)(x_1 + x_3 + x_4 + x_6) + (x_3 + x_5)(x_2 + x_5 + x_6) \right)^2 + \left( (x_1 + x_3 + x_4 + x_6)(x_2 + x_5 + x_6) - (x_1 + x_2)(x_3 + x_5) \right)^2 \right),$$

$$\Psi_{21}^\circ : x \to \frac{2}{(x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2} \left( ((x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2) \cdot \left( (x_1 + x_3 + x_4 + x_6)(x_1 + x_3 + x_5 + x_6) - (x_1 + x_3)(x_3 + x_5) \right) + \left( (x_1 + x_2)(x_1 + x_3 + x_4 + x_6) + (x_3 + x_5)(x_2 + x_5 + x_6) \right) \right)^2 \right) \quad \text{for all } x \in \mathbb{R},$$

where $\mathbb{R}$ is any domain from the set $\{ x \in \mathbb{R}^6 : x \neq 0 \}$ of the space $\mathbb{R}^6$.
Then, the Lappo-Danilevskii differential system (2.53) has the autonomous first integrals

\[ F_1 : x \to P(x) \exp \left( -4 \arctan \frac{x_3 + x_5}{x_1 + x_3 + x_4 + x_6} + 6 \tilde{\Psi}_1(x) + 2 \tilde{\Psi}_1(x) \right), \]

\[ F_2 : x \to P^2(x) \exp \left( -2 \arctan \frac{x_3 + x_5}{x_1 + x_3 + x_4 + x_6} + \Psi_1(x) - \tilde{\Psi}_1(x) \right), \]

and

\[ F_3 : x \to 2 \tilde{\Psi}_1(x) - 2 \Psi_2(x) - \tilde{\Psi}_2(x) \quad \text{for all } x \in \mathcal{X}, \]

where the polynomial \( P : x \to (x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2 \) for all \( x \in \mathbb{R}^6 \).

Case 2A. A common complex eigenvector of the matrices \( B_j, \ j = 1, \ldots, m, \) hasn’t the complex conjugate vector. Autonomous first integrals of the system (2.20) are the functions

\[ F_1 : x \to \prod_{\xi = 1}^{k_1} (P_\xi(x))^{\tilde{h}_{0\xi} + \tilde{h}_{0,(k_1 + \xi)}} \exp \left( -2 \left( \tilde{h}_{0\xi} - \tilde{h}_{0,(k_1 + \xi)} \right) \varphi_\xi(x) + \right. \]

\[ + \left. 2 \sum_{q=1}^{\varepsilon_\xi} \left( \tilde{h}_{q\xi} + \tilde{h}_{q,(k_1 + \xi)} \right) \tilde{\Psi}_{q\xi}(x) + \left( \tilde{h}_{q,(k_1 + \xi)} - \tilde{h}_{q\xi} \right) \tilde{\Psi}_{q\xi}(x) \right) \cdot \left( P_{2k_1+1}(x) \right)^{\tilde{h}_{0,(2k_1+1)}} \exp \left( -2 \tilde{h}_{0,(2k_1+1)} \varphi_{2k_1+1}(x) \right) \prod_{\rho=1}^{k_2} (\nu^0_{\rho} x)^{2 \nu^0_{\rho} \exp \left( 2 \sum_{q=1}^{\varepsilon_\rho} \tilde{h}_{q\rho} \tilde{\Psi}_{q\rho}(x) \right)} \]

and

\[ F_2 : x \to \prod_{\xi = 1}^{k_1} (P_\xi(x))^{\tilde{h}_{0\xi} + \tilde{h}_{0,(k_1 + \xi)}} \exp \left( 2 \left( \tilde{h}_{0\xi} - \tilde{h}_{0,(k_1 + \xi)} \right) \varphi_\xi(x) + \right. \]

\[ + \left. 2 \sum_{q=1}^{\varepsilon_\xi} \left( \tilde{h}_{q\xi} + \tilde{h}_{q,(k_1 + \xi)} \right) \tilde{\Psi}_{q\xi}(x) + \left( \tilde{h}_{q\xi} - \tilde{h}_{q,(k_1 + \xi)} \right) \tilde{\Psi}_{q\xi}(x) \right) \cdot \left( P_{2k_1+1}(x) \right)^{\tilde{h}_{0,(2k_1+1)}} \exp \left( 2 \tilde{h}_{0,(2k_1+1)} \varphi_{2k_1+1}(x) \right) \prod_{\rho=1}^{k_2} (\nu^0_{\rho} x)^{2 \nu^0_{\rho} \exp \left( 2 \sum_{q=1}^{\varepsilon_\rho} \tilde{h}_{q\rho} \tilde{\Psi}_{q\rho}(x) \right)} \]

for all \( x \in \mathcal{X}, \mathcal{X} \subset DF_1 \cap DF_2, \]

where the polynomials \( P_\xi : x \to (\nu^{0\xi} x)^2 + (\tilde{\nu}^{0\xi} x)^2 \) for all \( x \in \mathbb{R}^n, \) the scalar functions \( \varphi_\xi : x \to \arctan \frac{\tilde{\nu}^{0\xi x}}{\nu^{0\xi x}} \) for all \( x \in \mathcal{X}, \xi = 1, \ldots, k_1, \xi = 2k_1 + 1, \) the numbers \( h_{q\xi} = \tilde{h}_{q\xi} + \tilde{h}_{q\xi} i, \)

\( h_{q\rho} = \tilde{h}_{q\rho} + \tilde{h}_{q\rho} i, \quad q = 0, \ldots, \varepsilon_k, \quad k = \xi \) or \( k = \rho, \quad \xi = 1, \ldots, 2k_1 + 1, \quad \rho = 1, \ldots, k_2, \)

are an nontrivial solution to the linear homogeneous system

\[ \sum_{\xi = 1}^{2k_1} \left( \lambda^j_\xi h_{0\xi} + \sum_{q=1}^{\varepsilon_\xi} \mu^j_{q\xi} h_{q\xi} \right) + \lambda^j_{2k_1+1} h_{0,(2k_1+1)} + \sum_{\rho=1}^{k_2} \left( \lambda^j_{\rho} h_{0\rho} + \sum_{q=1}^{\varepsilon_\rho} \mu^j_{q\rho} h_{q\rho} \right) = 0, \quad j = 1, \ldots, m. \]

Here \( \nu^{0\xi} = \nu^{0\xi} + \tilde{\nu}^{0\xi} i, \quad \nu^{0,(k_1 + \xi)} = \frac{\tilde{\nu}^{0\xi}}{\nu^{0\xi}}, \quad \nu^{0,(2k_1+1)} = \nu^{0,(2k_1+1)} + \tilde{\nu}^{0,(2k_1+1)} i \)

are complex common eigenvectors of the matrices \( B_j \) corresponding to the eigenvalues

\( \lambda^j_\xi = \lambda^j_\xi i, \quad \lambda^j_{k_1+\xi} = \lambda^j_\xi \) \( i, \quad \xi = 1, \ldots, k_1, \) and \( \lambda^j_{2k_1+1} = \lambda^j_{2k_1+1} + \lambda^j_{2k_1+1} i, \quad j = 1, \ldots, m; \)

\( \nu^{0\rho} \) are real common eigenvectors of the matrices \( B_j \) corresponding to the real eigenvalues.
\[ \lambda_\rho^j, \ j = 1, \ldots, m, \ \rho = 1, \ldots, k_2, \ \text{respectively}; \ \text{the functions } \Psi^{\xi}_{q\xi} = \Psi_{q\xi}^{\xi} + \Psi_{q\rho}^{\xi} \ i \text{ and } \Psi_{q\rho}^{\xi} \text{ are} \]

the solution to the functional system (2.49); the numbers

\[
\mu^{ij}_{q\xi} = a_j \Psi_{q\xi}^{\xi}(x), \quad \mu^{ij}_{q\rho} = \text{Re} \mu^{ij}_{q\xi}, \quad \tilde{\mu}^{ij}_{q\xi} = \text{Im} \mu^{ij}_{q\xi}, \quad \mu^{ij}_{q\rho} = a_j \Psi_{q\rho}^{\xi}(x) \quad \text{for all } x \in \mathcal{X},
\]

the natural numbers \( \varepsilon_\xi \) and \( \varepsilon_\rho \) such that \( 2 \sum_{\xi=1}^{k_1} \varepsilon_\xi + \sum_{\rho=1}^{k_2} \varepsilon_\rho = m - 2k_1 - k_2 \) with \( 2k_1 + 1 + k_2 \leq r \).

\( \varepsilon_\xi \leq s_\xi - 1, \ \xi = 1, \ldots, k_1, \ \varepsilon_\rho \leq s_\rho - 1, \ \rho = 1, \ldots, k_2, \) where \( k_1 \) is an number of complex common eigenvectors (this set hasn’t complex conjugate vectors) of the matrices \( B_j \), and \( k_2 \) is an number of real common eigenvectors of the matrices \( B_j \).

**Example 2.19.** Consider the sixth-order Lappo-Danilevskii differential system

\[
\frac{dx}{dt} = \sum_{j=1}^{4} \alpha_j(t) A_j x, \quad x \in \mathbb{R}^6, \quad t \in J \subset \mathbb{R}, \quad (2.54)
\]

with linearly independent continuous functions \( \alpha_j : J \to \mathbb{R}, \ j = 1, \ldots, 4, \) and the matrices

\[
A_1 = \begin{bmatrix}
1 & -2 & 0 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 \\
0 & 3 & -2 & 0 & 2 \\
0 & 4 & 0 & 2 & 2 \\
-1 & 3 & -2 & 2 & 1 \\
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0 & 2 & 0 & 1 & 0 \\
-1 & 3 & 0 & 1 & 0 \\
-1 & 3 & 1 & 0 & 0 \\
2 & 2 & 0 & 1 & 0 \\
3 & 3 & 2 & 1 & 1 \\
\end{bmatrix}, \quad A_3 = \begin{bmatrix}
3 & 0 & 0 & 1 & 1 \\
-1 & 2 & 0 & 0 & 1 \\
-2 & 1 & 2 & 0 & 1 \\
1 & 2 & 1 & 1 & 1 \\
2 & 1 & 1 & 0 & 1 \\
\end{bmatrix}, \quad \text{and} \quad A_4 = \begin{bmatrix}
1 & -2 & 4 & 0 & 0 & 2 \\
-2 & 1 & -4 & 0 & 0 & 4 \\
-3 & 2 & -7 & 0 & 0 & 5 \\
3 & -4 & 10 & 2 & 7 & 7 \\
3 & -2 & 9 & 0 & 7 & 5 \\
1 & 4 & -5 & 0 & -4 & 2 \\
\end{bmatrix},
\]

The system (2.54) has the eigenvalue \( \lambda^1_1 = 1 + i \) with elementary divisor \( (\lambda^1 - 1 - i)^2 \) and the eigenvalue \( \lambda^2_2 = 2i \) with elementary divisor \( \lambda^1 - 2i \).

Since the eigenvalue \( \lambda^1_1 = 1 + i \) corresponding to the eigenvector \( \nu^{01} = (1, 1 + i, 0, 0, i, i) \) of the matrices \( B_j = A_j^T, \ j = 1, \ldots, 4 \) (\( T \) denotes the matrix transpose) and to the 1-st order generalized eigenvector \( \nu^{11} = (1 + i, 0, 1 + i, 0, i, i) \) of the matrix \( B_1 \), we have

\[
\Psi^{11}_1 : x \to \frac{(x_1 + x_2)(x_1 + x_2) + (x_2 + x_5 + x_6)(x_1 + x_3 + x_4 + x_5 + x_6)}{P_1(x)} \quad \text{for all } x \in \mathcal{X},
\]

where \( \mathcal{X} \) is a domain from the set \( \{ x : x_1 + x_2 \neq 0, \ x_1 + x_3 + x_4 + x_5 \neq 0 \} \) of the space \( \mathbb{R}^6 \), the polynomial \( \Psi^{11}_1 : x \to (x_1 + x_2)^2 + (x_2 + x_5 + x_6)^2 \) for all \( x \in \mathbb{R}^6 \).

Using the common eigenvector \( \nu^{02} = (1, 0, 1, i, 1, i) \) of the matrices \( B_1, \ldots, B_4 \) corresponding to the simple eigenvalue \( \lambda^2_2 = 2i \), we can build autonomous first integrals of the Lappo-Danilevskii differential system (2.54)

\[
F_1 : x \to P_1(x) (P_2(x))^2 \exp(-10y_1(x) + 8\Psi^{11}_1(x) + 6\Psi^{11}_1(x)) \quad \text{for all } x \in \mathcal{X},
\]

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where the polynomial $P_2: x \rightarrow (x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2$ for all $x \in \mathbb{R}^6$, the scalar functions $\varphi_1: x \rightarrow \arctan \frac{x_2 + x_5 + x_6}{x_1 + x_2}$, $\varphi_2: x \rightarrow \arctan \frac{x_3 + x_5}{x_1 + x_3 + x_4 + x_6}$ for all $x \in \mathcal{X}$.

Case 2B. Suppose a function $\Psi^{\beta, \gamma}_\xi$, $\gamma \in \{1, \ldots, k_1\}$, $\beta \in \{1, \ldots, \varepsilon, \gamma\}$ hasn’t the complex conjugate function. Then the system (2.20) has the autonomous first integrals

$$F_1: x \rightarrow \prod_{\xi=1}^{k_1} (P_\xi(x))^{h_{0\xi} + h_{0,(k_1+\xi)}} \exp \left( -2 (\tilde{h}_{0\xi} - \tilde{h}_{0,(k_1+\xi)}) \varphi_\xi(x) + \right.$$

$$+ 2 \sum_{q=1}^{\varepsilon} (1 - \delta_{q\xi} \delta_{\gamma\xi}) \left( (h_{q\xi} + h_{q,(k_1+\xi)}) \Psi^{\delta, \gamma}_{q\xi}(x) + (\tilde{h}_{q,(k_1+\xi)} - \tilde{h}_{q\xi}) \tilde{\Psi}^{\delta, \gamma}_{q\xi}(x) \right) +$$

$$+ 2 \left( h_{\beta\gamma} \Psi^{\delta, \gamma}_{\beta\gamma}(x) - \tilde{h}_{\beta\gamma} \tilde{\Psi}^{\delta, \gamma}_{\beta\gamma}(x) \right) \prod_{\rho=1}^{k_2} (p_{0\rho} x)^{2 \bar{h}_{0\rho}} \exp \left( 2 \sum_{q=1}^{\varepsilon} h_{q\rho} \Psi^{\delta, \gamma}_{q\rho}(x) \right)$$

for all $x \in \mathcal{X}$, and

$$F_2: x \rightarrow \prod_{\xi=1}^{k_1} (P_\xi(x))^{\bar{h}_{0\xi} + \bar{h}_{0,(k_1+\xi)}} \exp \left( 2 (\bar{h}_{0\xi} - \bar{h}_{0,(k_1+\xi)}) \varphi_\xi(x) + \right.$$

$$+ 2 \sum_{q=1}^{\varepsilon} (1 - \delta_{q\xi} \delta_{\gamma\xi}) \left( (\bar{h}_{q\xi} + \bar{h}_{q,(k_1+\xi)}) \Psi^{\delta, \gamma}_{q\xi}(x) + (\bar{h}_{q\xi} - \bar{h}_{q,(k_1+\xi)}) \bar{\Psi}^{\delta, \gamma}_{q\xi}(x) \right) +$$

$$+ 2 \left( \bar{h}_{\beta\gamma} \Psi^{\delta, \gamma}_{\beta\gamma}(x) + \bar{h}_{\beta\gamma} \tilde{\Psi}^{\delta, \gamma}_{\beta\gamma}(x) \right) \prod_{\rho=1}^{k_2} (\bar{p}_{0\rho} x)^{2 \bar{h}_{0\rho}} \exp \left( 2 \sum_{q=1}^{\varepsilon} \bar{h}_{q\rho} \Psi^{\delta, \gamma}_{q\rho}(x) \right)$$

for all $x \in \mathcal{X}$, where $\mathcal{X}$ is a domain from the set $DF_1 \cap DF_2$, the polynomials $P_\xi: x \rightarrow (h_{0\xi} x)^{2} + (\bar{h}_{0\xi} x)^{2}$ for all $x \in \mathbb{R}^6$, the scalar functions $\varphi_\xi: x \rightarrow \arctan \frac{\rho_{0\xi} x}{\bar{\rho}_{0\xi} x}$ for all $x \in \mathcal{X}$, $\xi = 1, \ldots, k_1$, the numbers $h_{q\xi} = h_{q\xi} + \bar{h}_{q\xi} i$, $h_{q\rho} = h_{q\rho} + \bar{h}_{q\rho} i$, $q = 0, \ldots, \varepsilon, k$, $k = \xi$ or $k = \rho$, $\xi = 1, \ldots, 2k_1$, $\rho = 1, \ldots, k_2$, are an nontrivial solution to the linear homogeneous system

$$\sum_{\xi=1}^{2k_1} \left( \lambda_{\xi} j_{0\xi} + \sum_{q=1}^{\varepsilon} m_{j_{q\xi}} h_{q\xi} \right) - \mu_{j_{\beta(\xi+k_1+\gamma)}}^{j_{\beta(\xi+k_1+\gamma)}} + \sum_{\rho=1}^{k_2} \left( \mu_{j_{\rho\xi}}^{j_{\rho\xi}} + \sum_{q=1}^{\varepsilon} m_{j_{q\xi}} \bar{h}_{q\rho} \right) = 0, \quad j = 1, \ldots, m,$$

and $\delta$ is the Kronecker symbol.

Here $\rho_{0\xi} = \rho_{0\xi}^* + \bar{\rho}_{0\xi} i$, $p_{0,(k_1+\xi)} = \bar{\rho}_{0\xi}$ are complex common eigenvectors of the matrices $B_j$ corresponding to the eigenvalues $\lambda_{\xi}^j = \lambda_{\xi}^j + \lambda_{\xi}^j i$, $\lambda_{j\xi+\xi}^j = \lambda_{\xi}^j$, $j = 1, \ldots, m$, $\xi = 1, \ldots, k_1$, respectively; $\rho_{0\rho}$ are real common eigenvectors of the matrices $B_j$ corresponding to the eigenvalues $\lambda_{\rho\rho}^j$, $j = 1, \ldots, m$, $\rho = 1, \ldots, k_2$, respectively; the scalar functions $\Psi^{\delta, \gamma}_{q\xi} = \Psi_{q\xi}^\delta + \tilde{\Psi}_{q\xi}^\gamma i$ and $\Psi_{q\rho}^{j\gamma}$ are the solution to the system (2.49); the numbers

$$\mu_{j_{q\xi}}^{j_{\gamma\xi}} = a_{j\xi} \Psi_{q\xi}(x), \quad m_{j_{q\xi}}^{j_{\gamma\xi}} = \text{Re} \mu_{j_{q\xi}}^{j_{\gamma\xi}}, \quad \bar{\rho}_{q\xi} = \text{Im} \mu_{j_{q\xi}}^{j_{\gamma\xi}}, \quad \mu_{q\rho}^{j\gamma} = a_{j\rho} \Psi_{q\rho}(x)$$

for all $x \in \mathcal{X}$, $q = 1, \ldots, \varepsilon, k$, $k = \xi$ or $k = \rho$, $\xi = 1, \ldots, 2k_1$, $\rho = 1, \ldots, k_2$, $j = 1, \ldots, m$;
the natural numbers $\varepsilon_\xi$ and $\varepsilon_\rho$ such that $2 \sum_{\xi=1}^{k_1} \varepsilon_\xi + \sum_{\rho=1}^{k_2} \varepsilon_\rho = m - 2k_1 - k_2 + 2$ with $2k_1 + k_2 \leq r$, $\varepsilon_\xi \leq s_\xi - 1$, $\xi = 1, \ldots, k_1$, $\varepsilon_\rho \leq s_\rho - 1$, $\rho = 1, \ldots, k_2$, where $k_1$ is an number of complex common eigenvectors (this set hasn’t complex conjugate vectors) of the matrices $B_j$, and $k_2$ is an number of real common eigenvectors of the matrices $B_j$.

Example 2.20. Consider the sixth-order Lappo-Danilevskii differential system

$$\frac{dx}{dt} = \alpha_1(t)$$

| 3 -4 4 1 0 2 | 0 -4 2 1 -1 1 | $x, x \in \mathbb{R}^6$, |
| -1 3 -3 0 -2 -3 | 1 3 0 0 1 -1 |
| -3 5 -5 -1 -2 -4 | 0 6 -2 -1 2 -1 |
| 3 -6 4 4 -1 5 | 2 -6 2 3 -4 2 |
| 5 -5 8 3 3 6 | 1 -6 3 2 -2 2 |
| -2 5 -4 -3 1 -2 | -2 4 -3 -3 2 -2 |

where linearly independent functions $\alpha_1: J \to \mathbb{R}$ and $\alpha_2: J \to \mathbb{R}$ are continuous.

Using the complex eigenvalue $\lambda_1 = 1 + 2i$ with elementary divisor $(\lambda^1 - 1 - 2i)^3$ of the matrix $B_1 = A_1^T$ ($T$ denotes the matrix transpose) corresponding to the common eigenvector $\nu^{01} = (1, 0, 1+i, 1, i, 1)$, to the 1-st order generalized eigenvector $\nu^{11} = (1, 1+i, 0, 0, 0, i)$, and to the 2-nd order generalized eigenvector $\nu^{21} = (2 + 2i, 0, 2 + 2i, 0, 2i, 2i)$, we can construct the autonomous first integrals of this Lappo-Danilevskii differential system

$$F_1: x \to P(x) \exp(-\varphi(x) - \tilde{\Psi}_{11}(x)),$$

$$F_2: x \to P(x) \exp(-2\varphi(x) + 2\tilde{\Psi}_{11}(x)),$$

$$F_3: x \to P^2(x) \exp(-2\varphi(x) - \tilde{\Psi}_{11}(x)),$$

and $F_4: x \to \tilde{\Psi}_{21}(x)$ for all $x \in \mathcal{X}$, where the polynomial $P: x \to (x_1 + x_3 + x_4 + x_5)^2 + (x_3 + x_6)^2$ for all $x \in \mathbb{R}^6$, the scalar functions $\varphi: x \to \arctan\frac{x_3 + x_5}{x_1 + x_3 + x_4 + x_6}$ for all $x \in \mathcal{X}$,

$$\tilde{\Psi}_{11}^{11}: x \to \frac{(x_1 + x_2)(x_1 + x_3 + x_4 + x_6) + (x_3 + x_5)(x_2 + x_5 + x_6)}{(x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2}$$

for all $x \in \mathcal{X}$,

$$\Psi_{21}^{11}: x \to \frac{1}{(x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2} \left(2\left((x_1 + x_3 + x_4 + x_6)^2 + (x_3 + x_5)^2\right) \cdot \left((x_1 + x_3)(x_1 + x_3 + x_4 + x_6) + (x_3 + x_5)(x_1 + x_3 + x_5 + x_6)\right) - (x_1 + x_2)(x_1 + x_3 + x_4 + x_6) + (x_3 + x_5)(x_2 + x_5 + x_6)^2 \right) \cdot \left((x_1 + x_3 + x_4 + x_6)(x_2 + x_5 + x_6) - (x_1 + x_2)(x_3 + x_5)^2\right)$$

for all $x \in \mathcal{X}$,

where $\mathcal{X}$ is any domain from the set $\{x: x_1 + x_3 + x_4 + x_6 \neq 0\}$ of the space $\mathbb{R}^6$. 

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2.3.2. Linear nonhomogeneous differential system

Consider an nonhomogeneous Lappo-Danilevskii differential system

\[
\frac{dx}{dt} = \sum_{j=1}^{m} \alpha_j(t) A_j x + f(t), \quad x \in \mathbb{R}^n, \quad t \in J, \quad f \in C(J),
\]  

(2.55)

where linearly independent on an interval $J \subset \mathbb{R}$ functions $\alpha_j : J \to \mathbb{R}$ are continuous, real constant $n \times n$ matrices $A_j$ such that $A_j A_k = A_k A_j$, $j = 1, \ldots, m$, $k = 1, \ldots, m$.

The corresponding homogeneous differential system of system (2.55) is equal to

\[
\frac{dx}{dt} = \sum_{j=1}^{m} \lambda_j \alpha_j(t) x, \quad x \in \mathbb{R}^n, \quad t \in J,
\]

where $\lambda_j$ are the eigenvalues of the constant matrices $A_j$. The corresponding homogeneous differential system of system (2.55) is the system (2.20).

Using Theorems 2.12, 2.13, and 2.14, we can build first integrals of the nonhomogeneous Lappo-Danilevskii differential system (2.55).

**Theorem 2.12.** Suppose that the conditions of Lemma 2.3 hold. Then the Lappo-Danilevskii differential system (2.55) has the first integral

\[
F : (t, x) \to \nu x \varphi(t) - \int_{t_0}^{t} \nu f(\tau) \varphi(\tau) d\tau \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n,
\]

(2.56)

where $t_0$ is a fixed point from the interval $J$, the exponential function

\[
\varphi : t \to \exp \left( - \int_{t_0}^{t} \sum_{j=1}^{m} \lambda_j \alpha_j(\tau) d\tau \right) \quad \text{for all} \quad t \in J.
\]

(2.57)

**Proof.** By Lemma 2.3, it follows that the Lie derivative of the function (2.56) by virtue of the Lappo-Danilevskii differential system (2.55) is equal to

\[
\mathfrak{B} F(t, x) = \nu x \partial_t \varphi(t) - \partial_t \int_{t_0}^{t} \nu f(\tau) \varphi(\tau) d\tau + (\mathfrak{A} \nu x + f(t) \partial_x \nu x) \varphi(t) = 0 \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n,
\]

where the linear differential operator $\mathfrak{B}(t, x) = \mathfrak{A}(t, x) + f(t) \partial_t$ for all $(t, x) \in J \times \mathbb{R}^n$ is induced by the Lappo-Danilevskii differential system (2.55).

**Example 2.21.** The second-order Lappo-Danilevskii differential system

\[
\frac{dx_1}{dt} = -3 \left( \frac{1}{t} + t^2 \right) x_1 - \frac{2}{t} x_2 + t \sin t + 2 t e^t e^{-t^3},
\]

\[
\frac{dx_2}{dt} = 4 \frac{t}{t} x_1 + 3 \left( \frac{1}{t} - t^2 \right) x_2 + t \left( \cosh t + 6 \arctan \frac{t}{3} \right) e^{-t^3}
\]

(2.58)

such that the coefficient matrix $A(t) = 3t^2 A_1 + \left( \frac{1}{t} + 6t^2 \right) A_2$ for all $t \in J \subset \{ t : t \neq 0 \}$, where the constant matrices $A_1 = 5 4 4 3$, and $A_2 = -3 -2 4 3$.

The matrices $B_1 = A_1^T$ and $B_2 = A_2^T$, where $T$ denotes the matrix transpose, have the eigenvalues $\lambda_1^1 = 1$, $\lambda_2^1 = -3$, and $\lambda_1^2 = -1$, $\lambda_2^2 = 1$ corresponding to the linearly independent common real eigenvectors $\nu^1 = (2, 1)$, $\nu^2 = (1, 1)$.

Using the scalar functions

\[
\alpha_1 : t \to 3t^2 \quad \text{for all} \quad t \in J, \quad \alpha_2 : t \to \frac{1}{t} + 6t^2 \quad \text{for all} \quad t \in J,
\]

\[
f_1 : t \to t \left( \sin t + 2t^2 \right) e^{-t^3}, \quad f_2 : t \to t \left( \cosh t + 6 \arctan \frac{t}{3} \right) e^{-t^3}
\]

for all $t \in J$,
\[ \varphi_1(t) = \exp\left( - \int (\lambda_1^2 \alpha_1(t) + \lambda_2^2 \alpha_2(t)) \, dt \right) = \exp\int \left( \frac{1}{t} + 3t^2 \right) dt = t \exp^3 \text{ for all } t \in J, \]

\[ \varphi_2(t) = \exp\left( - \int (\lambda_2^2 \alpha_1(t) + \lambda_2^2 \alpha_2(t)) \, dt \right) = \exp\int \left( 3t^2 - \frac{1}{t} \right) dt = \frac{1}{t} \exp^3 \text{ for all } t \in J, \]

\[ I_1(t) = \int (2f_1(t) + f_2(t)) \varphi_1(t) \, dt = \int \left( 2t^2 \sin t + 4t^3 e^{-t^2} + t^2 \cosh t + 6t^2 \arctan \frac{t}{3} \right) \, dt = -3t^2 + \]

\[ + 4t \sin t - 2(t^2 - 2) \cos t + 2(t^2 - 1)e^{-t^2} - 2t \cosh t + (t^2 + 2) \sinh t + 2t^3 \arctan \frac{t}{3} + 27 \ln(t^2 + 9), \]

\[ I_2(t) = \int (f_1(t) + f_2(t)) \varphi_2(t) \, dt = \int \left( \sin t + 2te^{-t^2} + \cosh t + 6 \arctan \frac{t}{3} \right) \, dt = \]

\[ = - \cos t + e^{-t^2} + \sinh t + 6t \arctan \frac{t}{3} - 9 \ln(t^2 + 9) \text{ for all } t \in J, \]

we can build (by Theorem 2.12) first integrals of the Lappo-Danilevskii system (2.58)

\[ F_1: (t, x_1, x_2) \rightarrow t e^{\alpha_1} (2x_1 + x_2) + 3t^2 - 4t \sin t + 2(t^2 - 2) \cos t - 2(t^2 - 1)e^{\alpha_2} + \]

\[ + 2t \cosh t - (t^2 + 2) \sinh t - 2t^3 \arctan \frac{t}{3} - 27 \ln(t^2 + 9) \text{ for all } (t, x_1, x_2) \in J \times \mathbb{R}^2 \]

and

\[ F_2: (t, x_1, x_2) \rightarrow \frac{1}{t} e^{\alpha_1} (x_1 + x_2) + \cos t - e^{\alpha_2} - \sinh t - 6t \arctan \frac{t}{3} + 9 \ln(t^2 + 9) \]

for all \((t, x_1, x_2) \in J \times \mathbb{R}^2\).

The functionally independent first integrals \(F_1\) and \(F_2\) are an integral basis of the Lappo-Danilevskii differential system (2.58) on any domain \(J \times \mathbb{R}^2\).

**Theorem 2.13.** Suppose that the conditions of Lemma 2.4 are satisfied. Then the Lappo-Danilevskii differential system (2.55) has the first integrals

\[ F_\rho: (t, x) \rightarrow \gamma_\rho(t, x) - \int_{t_0}^t \gamma_\rho(t, f(t)) \, dt \text{ for all } (t, x) \in J \times \mathbb{R}^n, \quad \rho = 1, 2, \quad (2.59) \]

where \(t_0\) is a fixed point from the interval \(J\), the scalar functions

\[ \gamma_1: (t, x) \rightarrow \left( \hat{\nu} x \cos \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) \, d\tau + \hat{\nu} x \sin \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) \, d\tau \right) \exp \left( - \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) \, d\tau \right), \]

\[ \gamma_2: (t, x) \rightarrow \left( \hat{\nu} x \cos \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) \, d\tau - \hat{\nu} x \sin \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) \, d\tau \right) \exp \left( - \int_{t_0}^t \sum_{j=1}^m \lambda_j \alpha_j(\tau) \, d\tau \right). \]

**Proof.** The idea of the proof of Theorem 2.13 is analogous to the proof of Corollary 1.6 or Theorem 2.2. Formally using Theorem 2.12, we obtain the complex-valued function (2.56) is a first integral of system (2.55). Then the real and imaginary parts of this complex-valued first integral are the real first integrals (2.59) of the Lappo-Danilevskii system (2.55). \(\Box\)

**Example 2.22.** Consider the second-order Egorin system of differential equations (right-hand side of system satisfy the Cauchy — Riemann equations) [62; 48, pp. 152 – 153]

\[ \frac{dx_1}{dt} = \alpha_1(t) x_1 + \alpha_2(t) x_2 + f_1(t), \quad \frac{dx_2}{dt} = - \alpha_2(t) x_1 + \alpha_1(t) x_2 + f_2(t), \quad (2.60) \]

where functions \(\alpha_j: J \rightarrow \mathbb{R}\) and \(f_j: J \rightarrow \mathbb{R}, \ j = 1, 2\) are continuous on an interval \(J \subset \mathbb{R}\).
Since \( \nu^1 = (1, i) \) and \( \nu^2 = (1, -i) \) are common eigenvectors of the matrices \( B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( B_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) corresponding to the eigenvalues \( \lambda_1^1 = 1, \lambda_1^2 = -i \), and \( \lambda_2^1 = 1, \lambda_2^2 = i \), respectively, we see that the Erugin system (2.60) has the functionally independent first integrals

where the numbers \( \mu_\lambda \) for all \( (t, x, \tau) \leq J \times \mathbb{R}^2 \), where \( \lambda \) is a fixed point from the interval \( J \), the scalar functions on the domain \( J \times \mathbb{R}^2 \);

For example, if \( \alpha_1: t \to t^2 \), \( \alpha_2: t \to 2t \), \( f_1: t \to t \), \( f_2: t \to 2t^2 \) for all \( t \in J \setminus \{ t: t \neq 0 \} \), then the Erugin system (2.60) has the functionally independent first integrals

\[
F_1: (t, x_1, x_2) \rightarrow \frac{1}{t}(x_1 \cos t^2 - x_2 \sin t^2) - \cos t^2 - \int_0^{t} \cos \tau^2 d\tau \quad \text{for all} \quad (t, x_1, x_2) \in J \times \mathbb{R}^2
\]

and

\[
F_2: (t, x_1, x_2) \rightarrow \frac{1}{t}(x_1 \sin t^2 + x_2 \cos t^2) - \sin t^2 - \int_0^{t} \sin \tau^2 d\tau \quad \text{for all} \quad (t, x_1, x_2) \in J \times \mathbb{R}^2.
\]

Note that these first integrals of the Erugin system (2.60) are nonelementary functions.

**Lemma 2.7.** Under the conditions of Lemma 2.5, we have

\[
a_j \nu^\theta x = \sum_{\rho=0}^{\theta} \binom{\theta}{\rho} \mu_\rho^{j} \nu^{\theta-\rho} x \quad \text{for all} \quad x \in \mathcal{X}, \quad j = 1, \ldots, m, \quad \theta = 1, \ldots, s - 1, \quad (2.61)
\]

where the numbers \( \mu_0^{j} = \lambda^j \), \( \mu_0^{j} = a_j \Psi_\theta^j(x) \), \( j = 1, \ldots, m, \theta = 1, \ldots, s - 1 \).

**Proof.** The proof of Lemma 2.7 is by induction on \( s \).

Let \( s = 2 \). Using the functional system (2.31), we get

\[
\nu^1 x = \Psi_1^j(x) \nu^0 x \quad \text{for all} \quad x \in \mathcal{X}.
\]

Then, taking into account Lemmas 2.3 and 2.5, we obtain

\[
a_j \nu^1 x = \mu_0^{j} \nu^1 x + \mu_1^{j} \nu^0 x \quad \text{for all} \quad x \in \mathcal{X}, \quad j = 1, \ldots, m.
\]

Therefore the system of identities (2.61) for \( s = 2 \) is true.

Suppose that the assertion of Lemma 2.7 is valid for \( s = \varepsilon \). Then, from the functional system (2.31) with \( s = \varepsilon + 1 \) and \( \theta = \varepsilon \), we have

\[
a_j \nu^\varepsilon x = \sum_{\rho=1}^{\varepsilon} \binom{\varepsilon - 1}{\rho - 1} \mu_\rho^{j} \nu^{\varepsilon - \rho} x + \sum_{\rho=1}^{\varepsilon} \binom{\varepsilon - 1}{\rho - 1} \Psi_\rho^j(x) \sum_{\beta=0}^{\varepsilon - \rho} \binom{\varepsilon - \rho}{\beta} \mu_\beta^{j} \nu^{\varepsilon - \rho - \beta} x \quad \text{for all} \quad x \in \mathcal{X}, \quad j = 1, \ldots, m.
\]

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By the induction hypothesis, so that

\[
\sum_{\rho=1}^{\varepsilon} \sum_{\delta=0}^{\varepsilon-1} \mu_{\rho}^j \nu^{\varepsilon-\rho} x + \sum_{\rho=1}^{\varepsilon-1} \sum_{\delta=0}^{\varepsilon-\delta} \mu_{\rho}^j \nu^{\varepsilon-\delta} x = \sum_{\rho=1}^{\varepsilon} \mu_{\rho}^j \nu^{\varepsilon-\rho} x
\]

for all \(x \in \mathcal{X}, \ j = 1, \ldots, m\).

Consequently the statement (2.61) for \(s = \varepsilon + 1\) is true. Thus by the principle of induction, the system of identities (2.61) is true for every natural number \(s \geq 2\).

**Theorem 2.14.** Suppose the system (2.55) satisfies the conditions of Lemma 2.5. Then the Lappo-Danilevskii system (2.55) on the domain \(J \times \mathcal{X}\) has the first integrals

\[
F_{\theta}: (t, x) \rightarrow \nu^\theta x \varphi(t) - \sum_{\rho=1}^{\theta} K_{\rho-1}(t) F_{\rho-1}(t, x) - C_{\theta}(t), \quad \theta = 0, \ldots, s - 1,
\]

where \(\mathcal{X}\) is a domain from the set \(\{x: \nu^\theta x \neq 0\} \subset \mathbb{R}^n\), the exponential function \(\varphi: J \rightarrow \mathbb{R}\) is given by the formula (2.57), the scalar functions

\[
K_{\rho-1}: t \rightarrow \int_{t_0}^{t} \left( \sum_{\delta=0}^{\theta-\rho} \mu_{\rho}^j \nu^{\varepsilon-\rho} x + \sum_{\delta=0}^{\varepsilon-\rho} \mu_{\rho}^j \nu^{\varepsilon-\delta} x \right) \, dt, \quad \rho = 1, \ldots, \varepsilon, \ \theta = 1, \ldots, s - 1,
\]

\[
C_{\theta}: t \rightarrow \int_{t_0}^{t} \left( \nu^\theta f(\tau) \varphi(\tau) + \sum_{\rho=1}^{\theta} \mu_{\rho}^j \nu^{\varepsilon-\rho} x \right) \, dt, \quad \theta = 0, \ldots, s - 1,
\]

\[
\mu_{\theta}^j: t \rightarrow \mu_{\theta}^j \alpha_j(t) \quad \text{for all } t \in J \ (\mu_{\theta}^j = \mu_{\theta}^j \psi_\theta(x), \ j = 1, \ldots, m), \ \theta = 1, \ldots, s - 1, \ t_0 \in J.
\]

**Proof.** The proof is by induction on \(s\). The case \(s = 1\) was considered in Theorem 2.12. Suppose \(s = 2\). Using Theorem 2.12 and Lemma 2.7, we obtain

\[
\forall F_1(t, x) = \mu_{\varepsilon}^j(t) \left( \nu^\varepsilon x \varphi(t) - C_0(t) - F_0(t, x) \right) = 0 \quad \text{for all } (t, x) \in J \times \mathcal{X}.
\]

Therefore the scalar function \(F_1: J \times \mathcal{X} \rightarrow \mathbb{R}\) is a first integral of system (2.55).

Suppose that the assertion of Theorem 2.14 is valid for \(s = \varepsilon\), i.e., the scalar functions \(F_\theta: J \times \mathcal{X} \rightarrow \mathbb{R}, \ \theta = 1, \ldots, \varepsilon - 1\), are first integrals of the Lappo-Danilevskii system (2.55). Then, from Lemma 2.7, we get on the domain \(J \times \mathcal{X}\)

\[
\forall F_\varepsilon(t, x) = \sum_{\rho=1}^{\varepsilon} \mu_{\rho}^j(t) \left( \nu^{\varepsilon-\rho} x \varphi(t) - \sum_{\eta=1}^{\varepsilon-\rho} K_{\eta-1}(t) F_{\eta-1}(t, x) - C_{\varepsilon-\rho}(t) \right) = 0.
\]

Consequently if \(s = \varepsilon + 1\), then the scalar function \(F_\varepsilon: J \times \mathcal{X} \rightarrow \mathbb{R}\) is a first integral on the domain \(J \times \mathcal{X}\) of the Lappo-Danilevskii differential system (2.55).

Now from the method of building scalar functions (2.62) it follows that the Lappo-Danilevskii differential system (2.55) has the functionally independent first integrals (2.62).

**Example 2.23.** The third-order Lappo-Danilevskii differential system

\[
\frac{dx_1}{dt} = tx_1 - (t + 1)x_2 + (t + 1)x_3 + \frac{1}{2}, \quad \frac{dx_2}{dt} = (2t + 1)x_1 - x_2 + (2t + 1)x_3 - 2t^2,
\]

\[
\frac{dx_3}{dt} = -(t + 1)x_1 + (2t + 1)x_2 - (t + 2)x_3 + t
\]

\[
(2.63)
\]
such that the coefficient matrix $A(t) = A_1 + t A_2$ for all $t \in \mathbb{R}$, where the constant matrix

\[
A_1 = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & -1 \end{bmatrix}.
\]

The matrix $B_1 = A_1^T$, where $T$ denotes the matrix transpose, has triple eigenvalue $\lambda_1 = -1$. The rank of the matrix $B_1 - \lambda_1 E$ is equal 2. Therefore the eigenvalue $\lambda_1 = -1$ has $\kappa_1 = 3 - 2 = 1$ elementary divisor $(\lambda_1 + 1)^3$.

The matrices $B_1$ and $B_2 = A_2^T$ have the common real eigenvector $\nu_0 = (1, 0, 1)$ corresponding to the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0$. Also, the matrix $B_1$ has the 1-st order generalized eigenvector $\nu_1 = (0, 1, 0)$ and the 2-nd order generalized eigenvector $\nu_2 = (0, 2, 2)$ corresponding to the eigenvalue $\lambda_1 = -1$.

Now using the common eigenvector $\nu_0$ and the generalized eigenvectors $\nu_1$, $\nu_2$, we can build the real first integrals on the space $\mathbb{R}^3$, $\mathbb{R}_0$ for system (2.62)

\[
\Psi_1^1: x \rightarrow \frac{x_2}{x_1 + x_3}, \quad \Psi_1^2: x \rightarrow \frac{2(x_1 + x_3)(x_2 + x_3) - x_2^2}{(x_1 + x_3)^2} \quad \forall x \in \mathbb{X} \subset \{x: x_1 + x_3 \neq 0\}.
\]

The real numbers $\mu_1^1 = a_1 \Psi_1^1(x) = 1$, $\mu_1^2 = a_1 \Psi_1^2(x) = 0$, and $\mu_2^1 = a_2 \Psi_1^1(x) = 2$, $\mu_2^2 = a_2 \Psi_1^2(x) = 2$, where the linear differential operators of first order

\[
a_1(x) = (-x_2 + x_3) \partial x_1 + (x_1 - x_2 + x_3) \partial x_2 - (x_1 - x_2 + 2x_3) \partial x_3 \quad \forall x \in \mathbb{R}^3,
\]

\[
a_2(x) = (x_1 - 2x_2 + x_3) \partial x_1 + 2(x_1 + x_3) \partial x_2 - (x_1 - 2x_2 + x_3) \partial x_3 \quad \forall x \in \mathbb{R}^3.
\]

Using the scalar functions

\[
\alpha_1: t \rightarrow 1, \quad \alpha_2: t \rightarrow t, \quad f_1: t \rightarrow t - \frac{1}{2}, \quad f_2: t \rightarrow -2t^2, \quad f_3: t \rightarrow t, \quad \varphi: t \rightarrow e^t \quad \forall t \in \mathbb{R},
\]

\[
C_0: t \rightarrow (t - \frac{1}{2}) e^t, \quad \mu_1^1: t \rightarrow 2t + 1, \quad K_0^1: t \rightarrow t(t + 1), \quad C_1: t \rightarrow -\frac{1}{2} e^t, \quad \mu_2^1: t \rightarrow 2t,
\]

\[
K_0^2: t \rightarrow 2(t^2 + 2t + 2), \quad K_0^2: t \rightarrow 2t(t + 1), \quad C_2: t \rightarrow -(2t^2 - 3t + 4) e^t \quad \forall t \in \mathbb{R},
\]

we can build (by Theorem 2.14) the basis of first integrals on the space $\mathbb{R}^4$ for system (2.63)

\[
F_0: (t, x) \rightarrow (x_1 + x_3 - t + \frac{1}{2}) e^t, \quad F_1: (t, x) \rightarrow (x_2 + \frac{1}{2}) e^t - t(t + 1) F_0(t, x),
\]

\[
F_2: (t, x) \rightarrow (2x_2 + 2x_3 + 2t^2 - 3t + 4) e^t - t^2(t^2 + 2t + 2) F_0(t, x) - 2t(t + 1) F_1(t, x).
\]

The proof of Theorem 2.14 is true both for real eigenvectors (common and generalized) of the matrix $B^c$, and for complex eigenvectors (common and generalized) of the matrix $B^c$.

In the complex case, from the complex-valued first integrals (2.62) of the Lappo-Danilevskii differential system (2.55), we obtain the real first integrals

\[
F_0^1: (t, x) \rightarrow \text{Re} F_0(t, x), \quad F_0^2: (t, x) \rightarrow \text{Im} F_0(t, x) \quad \forall (t, x) \in J \times \mathbb{X}, \quad \theta = 0, \ldots, s - 1.
\]

**Example 2.24.** The fourth-order Lappo-Danilevskii differential system

\[
\begin{align*}
\frac{dx_1}{dt} &= \frac{1}{t} (x_1 - x_2) - x_4 + t \cos t, \\
\frac{dx_2}{dt} &= -x_1 + \frac{1}{t} x_2 - x_3 - t \sin t, \\
\frac{dx_3}{dt} &= (1 + \frac{1}{t}) x_2 + \frac{1}{t} x_3 + x_4 + t^2, \\
\frac{dx_4}{dt} &= (3 - \frac{1}{t}) x_1 + (2 - \frac{1}{t}) x_3 + \frac{1}{4} x_4 + 2t.
\end{align*}
\]

55
has the coefficient matrix of the form \( A(t) = A_1 + \frac{1}{t} A_2 \) for all \( t \in J \subset \{t : t \neq 0\} \), where the constant matrix \( A_1 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
3 & 0 & 2 & 0
\end{pmatrix} \) and \( A_2 = \begin{pmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
-1 & 0 & -1 & 1
\end{pmatrix}. \)

The matrix \( B_1 = A_1^T \), where \( T \) denotes the matrix transpose, has two double complex eigenvalues \( \lambda_1^1 = i \) and \( \lambda_2^1 = -i \). The rank of the matrix \( B_1 - \lambda_1^1 E \) is equal 3. Therefore the eigenvalue \( \lambda_1^1 = i \) has \( \nu_1 = 4 - 3 = 1 \) elementary divisor \((\lambda_1^1 - i)^2\).

Thus the matrix \( B_1 \) has two elementary divisors \((\lambda_1^1 - i)^2\) and \((\lambda_1^1 + i)^2\).

The matrices \( B_1 \) and \( B_2 = A_2^T \) have the common complex eigenvector \( \nu^0 = (1, -i, 1, 0) \) corresponding to the eigenvalues \( \lambda_1^1 = i \) and \( \lambda_2^1 = 1 \). Also, the matrix \( B_1 \) has the generalized eigenvector of the 1-st order \( \nu^1 = (-i, 1, 0, 1) \).

From the eigenvector \( \nu^0 \) and the generalized eigenvector \( \nu^1 \), we obtain the function

\[
\Psi_1^x : x \rightarrow -\frac{ix_1 + x_2 + x_4}{x_1 - ix_2 + x_3} \text{ for all } x \in \mathbb{C} \subset \{x : (x_1 + x_3)^2 + x_2^2 \neq 0\}.
\]

The numbers \( \mu_0^1 = i, \mu_0^1 = 1, \mu_1^1 = a_1 \Psi_1^x(x) = 1, \mu_2^1 = a_2 \Psi_1^x(x) = -1 \), where

\[
a_1(x) = -x_4 \partial_{x_1} - (x_1 + x_3) \partial_{x_2} + (x_2 + x_4) \partial_{x_3} + (3x_1 + 2x_3) \partial_{x_4} \text{ for all } x \in \mathbb{R}^4,
\]

\[
a_2(x) = (x_1 - x_2) \partial_{x_1} + x_2 \partial_{x_2} + (x_2 + x_3) \partial_{x_3} - (x_1 + x_3 - 4) \partial_{x_4} \text{ for all } x \in \mathbb{R}^4.
\]

Using the scalar functions

\[
\alpha_1 : t \rightarrow 1, \quad \alpha_2 : t \rightarrow \frac{1}{t}, \quad f_1 : t \rightarrow t \cos t, \quad f_2 : t \rightarrow -t \sin t, \quad f_3 : t \rightarrow t^2, \quad f_4 : t \rightarrow 2t,
\]

\[
\varphi : t \rightarrow \frac{1}{t} \left( \cos t - i \sin t \right), \quad C_0 : t \rightarrow t + \cos t + t \sin t - i(t \sin t - t \cos t), \quad \mu_1^1 : t \rightarrow 1 - \frac{1}{t},
\]

\[
K_0^1 : t \rightarrow t - \ln |t|, \quad C_1 : t \rightarrow -t + \frac{t^2}{2} + \cos t + 4 \sin t + \frac{1}{2} \cos 2t - t \cos t - \int \frac{\cos t}{t} \ dt + \int \frac{\sin t}{t} \ dt
\]

for all \( t \in J \),

we can build the integral basis of the Lappo-Danilevskii differential system (2.64)

\[
F_0^1 : (t, x) \rightarrow \frac{1}{t} \left( \cos t (x_1 + x_3) - \sin t x_2 \right) - t - \cos t - t \sin t \text{ for all } (t, x) \in J \times \mathbb{R}^4,
\]

\[
F_0^2 : (t, x) \rightarrow -\frac{1}{t} \left( \cos t x_2 + \sin t (x_1 + x_3) \right) + \sin t - t \cos t \text{ for all } (t, x) \in J \times \mathbb{R}^4,
\]

\[
F_1^1 : (t, x) \rightarrow \frac{1}{t} \left( \cos t (x_2 + x_4) - \sin t x_1 \right) - (t - \ln |t|) F_0^1(t, x) +
\]

\[
+ t - \frac{t^2}{2} - \cos t - 4 \sin t - \frac{1}{2} \cos 2t + t \cos t + \int \frac{\cos t}{t} \ dt \text{ for all } (t, x) \in J \times \mathbb{R}^4,
\]

\[
F_1^2 : (t, x) \rightarrow \frac{1}{t} \left( \cos t x_1 + \sin t (x_2 + x_4) \right) + (t - \ln |t|) F_0^2(t, x) +
\]

\[
+ 4 \cos t - \sin t - \frac{1}{2} \sin 2t + t \sin t + \int \frac{\sin t}{t} \ dt \text{ for all } (t, x) \in J \times \mathbb{R}^4.
\]
3. Integrals of reducible ordinary differential systems

3.1. Linear homogeneous differential system

Consider a system of \( n \) ordinary linear differential equations

\[
\frac{dx}{dt} = A(t)x,
\]

where \( x = \text{colon}(x_1, \ldots, x_n) \) from the arithmetical phase space \( \mathbb{R}^n \), the real \( n \times n \) coefficient matrix \( A: t \to A(t) \) for all \( t \in J \) is continuous on an interval \( J \subset \mathbb{R} \).

One of the most efficient methods for investigation of linear nonautonomous differential systems is the method of reducibility. In this method linear nonautonomous systems are reduced to linear systems with constant coefficients by various transformation groups [63; 8; 10].

The idea of this method is due to A.M. Lyapunov [57]. He studied linear periodic systems and showed that there exists a transformation which does not change the character of the growth of solutions and reduces a system with periodic coefficients to a system with constant coefficients. Lyapunov called systems having this condition reducible systems. The general theory of reducible systems was developed by N.P. Erugin in his article [58].

Later on, the notion of reducible system was given for systems of difference equations [64] and for multidimensional differential systems (see [65 – 69; 9, pp. 72 – 82, 242 – 246]).

Let \( G \) be a multiplicative group of real continuously differentiable on the interval \( J \) nonsingular matrices of order \( n \). The differential system (3.1) is called reducible with respect to the nonautonomous transformation group \( G \) if there exist a constant matrix \( B \) and a matrix \( g \in G \) such that the linear transformation \( y = g(t)x \) reduces the nonautonomous differential system (3.1) to the system with constant coefficients

\[
\frac{dy}{dt} = By, \quad y \in \mathbb{R}^n.
\]

In addition, the transformation matrix \( g \) such that

\[
\frac{dg(t)}{dt} =Bg(t) - g(t)A(t) \quad \text{for all} \quad t \in J.
\]

By Theorems 3.1 – 3.4, using eigenvectors and corresponding eigenvalues of the matrix \( C = B^T \), where \( T \) denotes the matrix transpose, and a transformation matrix \( g \in G \), we can construct first integrals of the reducible system (3.1). The following basic statements (Lemmas 3.1 and 3.2) are base for the method of building integral basis of system (3.1).

**Lemma 3.1.** Suppose the system (3.1) is reducible to the system (3.2) by a transformation matrix \( g \in G \), and \( \nu \) is a real eigenvector of the matrix \( C \) corresponding to the eigenvalue \( \lambda \). Then the linear function \( p: (t,x) \to \nu g(t)x \) for all \( (t,x) \in J \times \mathbb{R}^n \) is a partial integral of the reducible differential system (3.1) such that

\[
\mathfrak{A}p(t,x) = \lambda p(t,x) \quad \text{for all} \quad (t,x) \in J \times \mathbb{R}^n,
\]

where the linear differential operator \( \mathfrak{A}(t,x) = \partial_t + A(t)x \partial_x \) for all \( (t,x) \in J \times \mathbb{R}^n \).

Indeed, using the matrix identity (3.3), we get

\[
\mathfrak{A}p(t,x) = \partial_t p(t,x) + A(t)x \partial_x p(t,x) = \nu g'(t)x + A(t)x \nu g(t) = \\
= \nu(Bg(t) - g(t)A(t))x + A(t)x \nu g(t) = \nu B g(t)x + (A(t)x \nu g(t) - \nu g(t)A(t)x) = \\
= C \nu g(t)x + \lambda \nu g(t)x = \lambda p(t,x) \quad \text{for all} \quad (t,x) \in J \times \mathbb{R}^n.
\]

**Lemma 3.2.** Suppose the system (3.1) is reducible to the system with constant coefficients (3.2) by a transformation matrix \( g \in G \), and \( \nu = \nu^* + \nu^*i \) (\( \text{Re} \nu = \nu^* \), \( \text{Im} \nu = \nu^* \)) is a complex eigenvector of the matrix \( C \) corresponding to the eigenvalue \( \lambda = \lambda^* + \lambda^*i \) (\( \text{Re} \lambda = \lambda^* \), \( \text{Im} \lambda = \lambda^* \)).
Then the Lie derivatives of the scalar functions

\[ P: (t, x) \rightarrow (\nu g(t) x)^2 + (\tilde{\nu} g(t) x)^2 \] for all \((t, x) \in J \times \mathbb{R}^n\)

and

\[ \varphi: (t, x) \rightarrow \arctan \frac{\tilde{\nu} g(t) x}{\nu g(t) x} \] for all \((t, x) \in \Lambda, \ \Lambda \subset \{(t, x): t \in J, \ \nu g(t) x \neq 0\} \subset J \times \mathbb{R}^n,\]

by virtue of the reducible system (3.1) are equal to

\[ \mathfrak{A} P(t, x) = 2 \lambda P(t, x) \] for all \((t, x) \in J \times \mathbb{R}^n\) and \[ \mathfrak{A} \varphi(t, x) = \tilde{\lambda} \] for all \((t, x) \in \Lambda.\]

Proof. Formally using Lemma 3.1, we obtain the linear function \(p: (t, x) \rightarrow \nu g(t) x\) for all \((t, x) \in J \times \mathbb{R}^n\) is a complex-valued partial integral of the reducible linear differential system (3.1) and the following identity holds

\[ \mathfrak{A} (\nu g(t) x + i \tilde{\nu} g(t) x) = (\lambda + \tilde{\lambda} i) (\nu g(t) x + i \tilde{\nu} g(t) x) \] for all \((t, x) \in J \times \mathbb{R}^n.\)

This complex identity is equivalent to the real system of identities

\[ \mathfrak{A} \nu g(t) x = \nu g(t) x - \tilde{\lambda} \tilde{\nu} g(t) x \] for all \((t, x) \in J \times \mathbb{R}^n,\]

\[ \mathfrak{A} \tilde{\nu} g(t) x = \tilde{\nu} g(t) x + \lambda \nu g(t) x \] for all \((t, x) \in J \times \mathbb{R}^n.\]

Using this system of identities, we have

\[ \mathfrak{A} P(t, x) = \mathfrak{A} ((\nu g(t) x)^2 + (\tilde{\nu} g(t) x)^2) = 2 \nu g(t) x \mathfrak{A} \nu g(t) x + 2 \tilde{\nu} g(t) x \mathfrak{A} \tilde{\nu} g(t) x = \]

\[ = 2 \nu g(t) x (\nu g(t) x - \tilde{\lambda} \tilde{\nu} g(t) x) + 2 \tilde{\nu} g(t) x (\nu g(t) x + \lambda \nu g(t) x) = \]

\[ = 2 \lambda (\nu g(t) x)^2 + (\tilde{\nu} g(t) x)^2) = 2 \lambda P(t, x) \] for all \((t, x) \in J \times \mathbb{R}^n;

\[ \mathfrak{A} \varphi(t, x) = \mathfrak{A} \arctan \frac{\tilde{\nu} g(t) x}{\nu g(t) x} = \]

\[ = \frac{\nu g(t) x (\lambda \nu g(t) x + \tilde{\nu} g(t) x) - \tilde{\nu} g(t) x (\lambda \nu g(t) x - \tilde{\lambda} \tilde{\nu} g(t) x)}{(\nu g(t) x)^2 + (\tilde{\nu} g(t) x)^2)} = \tilde{\lambda} \] for all \((t, x) \in \Lambda.\]

Now we can state the Theorems 3.1 – 3.4 for building first integrals of system (3.1).

**Theorem 3.1.** Let the conditions of Lemma 3.1 be satisfied. Then the scalar function

\[ F: (t, x) \rightarrow \nu g(t) x \exp(-\lambda t) \] for all \((t, x) \in J \times \mathbb{R}^n\)

is a first integral on the domain \(J \times \mathbb{R}^n\) of the reducible differential system (3.1).

Proof. From Lemma 3.1, we get

\[ \mathfrak{A} F(t, x) = \mathfrak{A} (\nu g(t) x \exp(-\lambda t)) = \exp(-\lambda t) \mathfrak{A} \nu g(t) x + \nu g(t) x \mathfrak{A} \exp(-\lambda t) = \]

\[ = \lambda \nu g(t) x \exp(-\lambda t) + \nu g(t) x \partial_t \exp(-\lambda t) = 0 \] for all \((t, x) \in J \times \mathbb{R}^n.

Therefore the function \( F: J \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a first integral of the reducible system (3.1). ■

**Example 3.1.** The second-order nonautonomous differential system [8, p. 155]

\[
\frac{dx_1}{dt} = (t^2 + t + 2)x_1 - (t^3 + t^2 + t - 1)x_2, \quad \frac{dx_2}{dt} = (t + 1)x_1 - (t^2 + t - 1)x_2 \quad (3.4)
\]
is reducible to the system with constant coefficients \( \frac{dy_1}{dt} = y_1, \quad \frac{dy_2}{dt} = 2x_2 \) by the polynomial

group\(^1\) \( P(2) \) with the transformation matrix \( g(t) = \begin{vmatrix} -t & 1 + t^2 \\ 1 & -t \end{vmatrix} \) for all \( t \in \mathbb{R} \).

Using the real eigenvectors \( \nu^1 = (1, 0), \nu^2 = (0, 1) \) of the matrix \( C = \text{diag}(1, 2) \) and the corresponding eigenvalues \( \lambda_1 = 1, \lambda_2 = 2 \), we can build (by Theorem 3.1) the basis of first integrals on space \( \mathbb{R}^3 \) for the reducible differential system (3.4)

\[
F_1: (t, x_1, x_2) \rightarrow (-t x_1 + (1 + t^2) x_2) e^{-t}, \quad F_2: (t, x_1, x_2) \rightarrow (x_1 - t x_2) e^{-2t}.
\]

**Theorem 3.2.** Let the conditions of Lemma 3.2 be satisfied. Then the reducible differential system (3.1) has the first integrals

\[
F_1: (t, x) \rightarrow (g(t) x)^2 \exp(-2 \lambda t) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n
\]

and

\[
F_2: (t, x) \rightarrow \arctan\left(\frac{\nu g(t) x}{\nu g(t) x}\right) - \lambda t \quad \text{for all} \quad (t, x) \in \Lambda, \quad \Lambda \subset \{(t, x): t \in J, \nu g(t) x \neq 0\}.
\]

**Proof.** It follows from Lemma 3.2 that

\[
\mathfrak{A} F_1(t, x) = \exp(-2 \lambda t) \mathfrak{A} ((\nu g(t) x)^2 + (\nu g(t) x)^2) + (\nu g(t) x)^2 + (\nu g(t) x)^2) \mathfrak{A} \exp(-2 \lambda t) = 2 \lambda F_1(t, x) + (\nu g(t) x)^2 + (\nu g(t) x)^2) \partial_x \exp(-2 \lambda t) = 0 \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n,
\]

\[
\mathfrak{A} F_2(t, x) = \mathfrak{A} \arctan\left(\frac{\nu g(t) x}{\nu g(t) x}\right) - \mathfrak{A} (\lambda t) = \lambda - \partial_x (\lambda t) = 0 \quad \text{for all} \quad (t, x) \in \Lambda.
\]

Therefore the scalar functions \( F_1: J \times \mathbb{R}^n \to \mathbb{R} \) and \( F_2: \Lambda \to \mathbb{R} \) are first integrals of the reducible homogeneous differential system (3.1).

**Example 3.2.** The linear differential system [70, pp. 125 – 126]

\[
\begin{align*}
\frac{dx_1}{dt} &= -x_2 + \sqrt{2} \cos 2t x_3 + \sqrt{2} \sin 2t x_4, \\
\frac{dx_2}{dt} &= x_1 + \sqrt{2} \sin 2t x_3 - \sqrt{2} \cos 2t x_4, \\
\frac{dx_3}{dt} &= -\sqrt{2} \sin 2t x_1 + \sqrt{2} \cos 2t x_2 - x_4, \\
\frac{dx_4}{dt} &= \sqrt{2} \cos 2t x_1 + \sqrt{2} \sin 2t x_2 + x_3
\end{align*}
\]

(3.5)

is reducible by the group\(^2\) \( P_4(2\pi) \). Indeed, the \( 2\pi \)-periodic nondegenerate transformation

\[
y_1 = \cos t x_1 + \sin t x_2, \quad y_2 = -\sin t x_1 + \cos t x_2, \quad y_3 = \cos t x_3 + \sin t x_4, \quad y_4 = -\sin t x_3 + \cos t x_4
\]

reduces the system (3.5) to the linear system with constant coefficients

\[
\begin{align*}
\frac{dy_1}{dt} &= \sqrt{2} y_3, \\
\frac{dy_2}{dt} &= -\sqrt{2} y_4, \\
\frac{dy_3}{dt} &= \sqrt{2} y_2, \\
\frac{dy_4}{dt} &= \sqrt{2} y_1.
\end{align*}
\]

By Theorem 3.2, using the eigenvalues \( \lambda_1 = 1 + i, \lambda_2 = -1 + i \) and the corresponding complex eigenvectors \( \nu^1 = (1 + i, 1 - i, \sqrt{2}, \sqrt{2} i), \nu^2 = (1 + i, -1 + i, -\sqrt{2} i, -\sqrt{2}) \) of the matrix \( C \), we can build the basis of first integrals for the reducible system (3.5)

\(^1\) \( P(n) \) is the multiplicative group of \( n \times n \) polynomial matrices with nonzero constant determinants.

\(^2\) \( P_n(\omega) \) is the multiplicative group of \( \omega \)-periodic invertible continuously differentiable square matrices of order \( n \). At the same time \( P_n(\omega) \) is a sub-group of the Lyapunov group \( L(n) \).

\( L(n) \) is the multiplicative group of invertible continuously differentiable on \( T = (0; \infty) \) square matrices of order \( n \) such that these matrices and their inverse matrices are bounded on the interval \( T \).

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\( F_1: (t, x) \to \left( (\cos t - \sin t)x_1 + (\cos t + \sin t)x_2 + \sqrt{2}\cos tx_3 + \sqrt{2}\sin tx_4 \right)^2 + \\
+ \left( (\cos t + \sin t)x_1 + (\sin t - \cos t)x_2 - \sqrt{2}\sin tx_3 + \sqrt{2}\cos tx_4 \right)^2 e^{-2t} \) for all \((t, x) \in \mathbb{R}^5\), \\
\( F_2: (t, x) \to \arctan \frac{(\cos t + \sin t)x_1 + (\sin t - \cos t)x_2 - \sqrt{2}\sin tx_3 + \sqrt{2}\cos tx_4}{(\cos t - \sin t)x_1 + (\cos t + \sin t)x_2 + \sqrt{2}\cos tx_3 + \sqrt{2}\sin tx_4} - t \) for all \((t, x) \in \Lambda\), \\
\( F_3: (t, x) \to \left( (\cos t + \sin t)x_1 + (\sin t - \cos t)x_2 + \sqrt{2}\sin tx_3 - \sqrt{2}\cos tx_4 \right)^2 + \\
+ \left( (\cos t - \sin t)x_1 + (\cos t + \sin t)x_2 - \sqrt{2}\cos tx_3 - \sqrt{2}\sin tx_4 \right)^2 e^{2t} \) for all \((t, x) \in \mathbb{R}^5\), \\
\( F_4: (t, x) \to \arctan \frac{(\cos t - \sin t)x_1 + (\cos t + \sin t)x_2 - \sqrt{2}\cos tx_3 - \sqrt{2}\sin tx_4}{(\cos t + \sin t)x_1 + (\sin t - \cos t)x_2 + \sqrt{2}\sin tx_3 - \sqrt{2}\cos tx_4} - t \) for all \((t, x) \in \Lambda\),

where a domain \( \Lambda \subset \{(t, x): (\cos t - \sin t)x_1 + (\cos t + \sin t)x_2 + \sqrt{2}\cos tx_3 + \sqrt{2}\sin tx_4 \neq 0, (\cos t + \sin t)x_1 + (\sin t - \cos t)x_2 + \sqrt{2}\sin tx_3 - \sqrt{2}\cos tx_4 \neq 0\} \) of the space \( \mathbb{R}^5 \).

**Example 3.3.** The third-order Euler differential system

\[
\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = \frac{1}{t^3} x_1 - \frac{1}{t^2} x_2 - \frac{2}{t} x_3
\]  

(3.6)

is reducible to the linear system with constant coefficients

\[
\frac{dy_1}{d\tau} = y_2, \quad \frac{dy_2}{d\tau} = y_2 + y_3, \quad \frac{dy_3}{d\tau} = y_1 - y_2
\]

by the exponential group\(^1\) \(\text{Exp}(3)\) with the transformation

\[y_1 = x_1, \quad y_2 = tx_2, \quad y_3 = t^2x_3, \quad t = e^\tau.\]

The matrix \( C = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix}\) has the eigenvalues \( \lambda_1 = 1, \lambda_2 = i, \) and \( \lambda_3 = -i \) corresponding to the eigenvectors \( \nu^1 = (1, 1, 1), \nu^2 = (1, -1, i), \) and \( \nu^3 = (1, -1, -i). \)

An integral basis of the reducible system (3.6) is the scalar functions

\( F_1: (t, x) \to \frac{1}{t} x_1 + x_2 + tx_3 \) for all \((t, x) \in (0; +\infty) \times \mathbb{R}^3\) (by Theorem 3.1),

\( F_2: (t, x) \to (x_1 - tx_2)^2 + t^4x_3^2 \) for all \((t, x) \in (0; +\infty) \times \mathbb{R}^3\) (by Theorem 3.2),

and

\( F_3: (t, x) \to \arctan \frac{t^2x_3}{x_1 - tx_2} - \ln t \) for all \((t, x) \in \Lambda\) (by Theorem 3.2),

where \( \Lambda \) is any domain from the set \( \{(t, x): t > 0, x_1 - tx_2 \neq 0\}. \)

---

\(^1\) \(\text{Exp}(n)\) is the multiplicative group of invertible continuously differentiable on the interval \((0; +\infty)\) square matrices \(g\) of order \(n\) such that \(\lim_{t \to +\infty} \frac{1}{t} \|g^{\pm t}(t)\| = 0.\)

At the same time the Lyapunov group \(L(n)\) is a sub-group of the exponential group \(\text{Exp}(n).\)
Theorem 3.3. Suppose the system (3.1) is reducible to the system (3.2) by a transformation matrix \( g \in G \), and \( \lambda \) is an eigenvalue with elementary divisor of multiplicity \( m \geq 2 \) of the matrix \( C \) corresponding to a real eigenvector \( \nu^0 \) and to a real generalized eigenvector \( \nu^1 \) of the 1-st order. Then the reducible system (3.1) has the first integral

\[
F: (t, x) \rightarrow \frac{\nu^1 g(t) x}{\nu^0 g(t) x} - t \quad \text{for all} \quad (t, x) \in \Lambda, \quad \Lambda \subset \{(t, x): t \in J, \nu^0 g(t) x \neq 0\}. \tag{3.7}
\]

Proof. From the matrix identity (3.3) it follows that

\[
\mathfrak{A} \nu^1 g(t) x = \partial_t \nu^1 g(t) x + A(t) x \partial_x \nu^1 g(t) x = \nu^1 g'(t) x + A(t) x \nu^1 g(t) = \\
= \nu^1 (B g(t) - g(t) A(t)) x + A(t) x \nu^1 g(t) = C \nu^1 g(t) x = (\lambda \nu^1 + \nu^0) g(t) x = \\
= \lambda \nu^1 g(t) x + \nu^0 g(t) x \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n.
\]

Using this identity and Lemma 3.1, we get on the domain \( \Lambda \)

\[
\mathfrak{A} F(t, x) = 2 \frac{\nu^1 g(t) x}{\nu^0 g(t) x} - \mathfrak{A} t = \frac{(\lambda \nu^1 g(t) x + \nu^0 g(t) x) \nu^0 g(t) x - \lambda \nu^0 g(t) x \nu^1 g(t) x}{(\nu^0 g(t) x)^2} - 1 = 0.
\]

In the complex case, if \( \lambda \) is a complex eigenvalue, then from the complex-valued first integral (3.7) of the reducible differential system (3.1), we obtain the real first integrals

\[
F_1: (t, x) \rightarrow \frac{\tilde{\nu}^0 g(t) x \tilde{\nu}^1 g(t) x + \tilde{\nu}^0 g(t) x \tilde{\nu}^1 g(t) x}{(\tilde{\nu}^0 g(t) x)^2 + (\tilde{\nu}^0 g(t) x)^2} - t \quad \text{for all} \quad (t, x) \in \Lambda
\]

and

\[
F_2: (t, x) \rightarrow \frac{\tilde{\nu}^0 g(t) x \tilde{\nu}^1 g(t) x - \tilde{\nu}^0 g(t) x \tilde{\nu}^1 g(t) x}{(\tilde{\nu}^0 g(t) x)^2 + (\tilde{\nu}^0 g(t) x)^2} \quad \text{for all} \quad (t, x) \in \Lambda
\]

where \( \Lambda \) is any domain from the set \( \{(t, x): t \in J, (\tilde{\nu}^0 g(t) x)^2 + (\tilde{\nu}^0 g(t) x)^2 \neq 0\} \) of the space \( \mathbb{R}^{n+1} \), the real vectors \( \tilde{\nu}^k = \text{Re} \nu^k \), \( \tilde{\nu}^k = \text{Im} \nu^k \), \( k = 0, 1 \).

Example 3.4. Linear Hamiltonian systems of second order reducible by orthogonal transformation group\(^1\) are the linear differential systems of the form [8, pp. 142 – 143]

\[
\frac{dx}{dt} = IA(t)x, \quad x \in \mathbb{R}^2, \quad A: t \rightarrow \begin{bmatrix} \psi(t) + \beta(t) & \alpha(t) \\ \alpha(t) & \psi(t) - \beta(t) \end{bmatrix} \quad \text{for all} \quad t \in J, \tag{3.8}
\]

where the simplectic matrix \( I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), the functions \( \alpha: t \rightarrow a \cos 2\varphi(t) - b \sin 2\varphi(t), \)

\( \beta: t \rightarrow a \sin 2\varphi(t) + b \cos 2\varphi(t), \psi: t \rightarrow \varphi(t) + c \) for all \( t \in J \) are continuously differentiable on the interval \( J \subset \mathbb{R} \), and \( a, b, c \) are some real numbers.

The orthogonal transformation \( y_1 = \cos \varphi(t)x_1 - \sin \varphi(t)x_2, y_2 = \sin \varphi(t)x_1 + \cos \varphi(t)x_2 \) reduces the linear differential system (3.8) to the linear autonomous Hamiltonian system

\[
\frac{dy}{dt} = IB y \quad \text{with the constant matrix} \quad B = \begin{bmatrix} c + b & a \\ a & c - b \end{bmatrix}.
\]

Consider the real number \( D = a^2 + b^2 - c^2 \). We have three possible cases for building first integrals of the reducible linear Hamiltonian system (3.8).

\(^1\) \( O(n) \) is the multiplicative group of continuously differentiable on an interval \( T \subset \mathbb{R} \) square orthogonal matrices of order \( n \). At the same time if \( T = (0; +\infty) \), then the orthogonal group of transformations \( O(n) \) is a sub-group of the Lyapunov group \( L(n) \).
Let $D = 0$. Then, using the real eigenvector $\nu^0 = (c+b,a)$, the real 1-st order generalized eigenvector $\nu^1 = (c+b,a-1)$, and the corresponding double eigenvalue $\lambda_1 = 0$, we can build (by Theorems 3.1 and 3.3) the integral basis on a domain $\Lambda$ of system (3.8)

$$F_1: (t, x) \to ((c+b) \cos \varphi(t) + a \sin \varphi(t)) x_1 + (a \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2,$$

$$F_2: (t, x) \to \frac{((c+b) \cos \varphi(t) + (a-1) \sin \varphi(t)) x_1 + ((a-1) \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2}{((c+b) \cos \varphi(t) + a \sin \varphi(t)) x_1 + (a \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2} - t,$$

where $\Lambda$ is any domain from the set $M = \{(t,x): t \in J, (c+b) \cos \varphi(t) + a \sin \varphi(t)) x_1 + (a \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2 \neq 0\}$.

Let $D > 0$. Then, using the real eigenvectors $\nu^k = (c+b,a-\lambda_k)$, $k = 1,2$, and the corresponding real eigenvalues $\lambda_1 = \sqrt{D}$, $\lambda_2 = -\sqrt{D}$, we can construct (by Theorem 3.1) the basis of first integrals of the Hamiltonian system (3.8)

$$F_k: (t, x) \to (((c+b) \cos \varphi(t) + (a-\lambda_k) \sin \varphi(t)) x_1 + ((a-\lambda_k) \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2) e^{-\lambda_k t}$$

for all $(t,x) \in J \times \mathbb{R}^2, \ k = 1,2$.

Let $D < 0$. Then, using the eigenvectors $\nu^k = (c+b,a-\lambda_k)$, $k = 1,2$, and the corresponding complex eigenvalues $\lambda_1 = \sqrt{-D} i$, $\lambda_2 = -\sqrt{-D} i$, we can find (by Theorem 3.2) the functionally independent first integrals of the Hamiltonian system (3.8)

$$F_1: (t, x) \to (((c+b) \cos \varphi(t) + a \sin \varphi(t)) x_1 + (a \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2)^2 - D (\sin \varphi(t) x_1 + \cos \varphi(t) x_2)^2$$

for all $(t,x) \in J \times \mathbb{R}^2$,

$$F_2: (t, x) \to \arctan \frac{\sqrt{-D} (\sin \varphi(t) x_1 + \cos \varphi(t) x_2)}{((c+b) \cos \varphi(t)+a \sin \varphi(t)) x_1 + (a \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2} + \sqrt{-D} t$$

for all $(t,x) \in \Lambda \subset M, \ M \subset \mathbb{R}^3$.

**Theorem 3.4.** Suppose the system (3.1) is reducible to the system (3.2) by a transformation matrix $g \in G$, and $\lambda$ is the eigenvalue with elementary divisor of multiplicity $m \geq 2$ of the matrix $C$ corresponding to a real eigenvector $\nu^0$ and to real generalized eigenvectors $\nu^k$, $k = 1, \ldots, m - 1$. Then the system (3.1) has the functionally independent first integrals

$$F_{\xi}: (t, x) \to \Psi_{\xi}(t,x) \quad \text{for all} \ (t, x) \in \Lambda, \ \xi = 2, \ldots, m - 1,$$

where the functions $\Psi_{\xi}: \Lambda \to \mathbb{R}, \ \xi = 2, \ldots, m - 1$, are the solution to the functional system

$$\nu^k g(t) x = \sum_{\tau=1}^{k} \frac{(\nu^{-1}) \Psi_{\tau}(t,x) \nu^{k-\tau} g(t)x}{(c+b) \cos \varphi(t)+a \sin \varphi(t)) x_1 + (a \cos \varphi(t) - (c+b) \sin \varphi(t)) x_2} + \sqrt{-D} t$$

and $\Lambda$ is any domain from the set $\{(t,x): t \in J, \nu^0 g(t) x \neq 0\} \subset J \times \mathbb{R}^n$.

**Proof.** From the matrix identity (3.3) and the notion of generalized eigenvectors for matrix (Definition 1.1) it follows that

$$\mathfrak{A} \nu^k g(t) x = \partial_t \nu^k g(t) x + A(t) x \partial_x \nu^k g(t) x = \nu^k g(t) x + A(t) x \nu^k g(t) = \nu^k (B g(t) - g(t) A(t)) x + A(t) x \nu^k g(t) = \nu^k B g(t) x + C \nu^k g(t) x = (\lambda \nu^k + k \nu^{k-1}) g(t) x =$$

$$= \lambda \nu^k g(t) x + k \nu^{k-1} g(t) x \quad \text{for all} \ (t,x) \in J \times \mathbb{R}^n, \ k = 1, \ldots, m - 1.$$

Hence using Lemma 3.1, we obtain the system of identities
\[ \mathfrak{A} \nu^0 g(t) x = \lambda \nu^0 g(t) x \quad \text{for all } (t, x) \in J \times \mathbb{R}^n, \quad (3.11) \]

\[ \mathfrak{A} \nu^k g(t) x = \lambda \nu^k g(t) x + k \nu^{k-1} g(t) x \quad \text{for all } (t, x) \in J \times \mathbb{R}^n, \quad k = 1, \ldots, m - 1. \]

The functional system (3.10) has the determinant \((\nu^0 g(t) x)^{m-1} \neq 0\) for all \((t, x) \in \Lambda\), where a domain \(\Lambda \subset \{(t, x) : t \in J, \nu^0 g(t) x \neq 0\}\). Therefore there exists the solution \(\Psi, \tau = 1, \ldots, m - 1, \) on the domain \(\Lambda\) of the functional system (3.10). Let us show that

\[ \mathfrak{A} \Psi_k(t, x) = \begin{cases} 1 & \text{for all } (t, x) \in \Lambda, \ k = 1, \\ 0 & \text{for all } (t, x) \in \Lambda, \ k = 2, \ldots, m - 1, \end{cases} \quad (3.12) \]

The proof of identities (3.12) is by induction on \(m\).

For \(m = 2\) and \(m = 3\), the assertions (3.12) follows from the identities (3.11).

Assume that the identities (3.12) for \(m = \varepsilon\) is true. Then, using the system of identities (3.11) and the identities (3.10) for \(m = \varepsilon + 1, m = \varepsilon\), we get

\[ \mathfrak{A} \nu^\varepsilon g(t) x = \lambda \sum_{\tau=1}^{\varepsilon} (\frac{\varepsilon-1}{\varepsilon-\tau}) \Psi_\tau(t, x) \nu^{\varepsilon-\tau} g(t) x + (\varepsilon - 1) \sum_{\tau=1}^{\varepsilon-1} (\frac{\varepsilon-2}{\varepsilon-\tau}) \Psi_\tau(t, x) \nu^{\varepsilon-\tau-1} g(t) x + \nu^{\varepsilon-1} g(t) x + \nu^0 g(t) x \mathfrak{A} \Psi_\varepsilon(t, x) \quad \text{for all } (t, x) \in \Lambda. \]

Now taking into account the system (3.10) with \(k = \varepsilon - 1\) and \(k = \varepsilon\), the identity (3.11) with \(k = \varepsilon\), and \(\nu^0 g(t) x \neq 0\) for all \((t, x) \in \Lambda\), we have

\[ \mathfrak{A} \Psi_\varepsilon(t, x) = 0 \quad \text{for all } (t, x) \in \Lambda. \]

This implies that the identities (3.12) for \(m = \varepsilon + 1\) are true. So by the principle of mathematical induction, the statement (3.12) is true for every natural number \(m \geq 2\).

Now from the method of building scalar functions (3.9) it follows that the reducible differential system (3.1) has the functionally independent first integrals (3.9).

The proof of Theorem 3.4 is true both for the case of the real eigenvalue \(\lambda\) and for the case of the complex eigenvalue \(\lambda\) (\(\text{Im} \lambda \neq 0\)). In the complex case, from the complex-valued first integrals (3.9) of system (3.1), we obtain the real first integrals of system (3.1)

\( F^1_\xi : (t, x) \to \text{Re} \Psi_\xi(t, x), \quad F^2_\xi : (t, x) \to \text{Im} \Psi_\xi(t, x) \quad \text{for all } (t, x) \in \Lambda, \quad \xi = 2, \ldots, m - 1, \)

where \(\Lambda\) is any domain from the set \(\{(t, x) : t \in J, \nu^0 g(t) x^2 + (\nu^0 g(t) x)^2 \neq 0\}\).

**Example 3.5.** The third-order Euler differential system

\[ \frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = \frac{1}{t^3} x_1 - \frac{1}{t^2} x_2, \quad (3.13) \]

is reducible to the linear differential system with constant coefficients

\[ \frac{dy_1}{d\tau} = y_2, \quad \frac{dy_2}{d\tau} = y_2 + y_3, \quad \frac{dy_3}{d\tau} = y_1 - y_2 + 2y_3 \]

by the exponential group \(\text{Exp}(3)\) with the transformation

\[ y_1 = x_1, \quad y_2 = tx_2, \quad y_3 = t^2 x_3, \quad t = e^\tau. \]

The matrix \(C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\) has the triple eigenvalue \(\lambda_1 = 1\) with the elementary divisor \((\lambda - 1)^3\) corresponding to the eigenvector \(\nu^0 = (1, -1, 1)\), to the generalized eigenvector of the 1-st order \(\nu^1 = (0, 0, 1)\), and to the generalized eigenvector of the 2-nd order \(\nu^2 = (0, 2, 0)\).
A basis of first integrals for the reducible system (3.13) is the scalar functions

\[ F_1: (t, x) \rightarrow \frac{1}{t} x_1 - x_2 + tx_3 \quad \text{for all} \quad (t, x) \in (0; + \infty) \times \mathbb{R}^3 \quad \text{(by Theorem 3.1),} \]

\[ F_2: (t, x) \rightarrow \frac{t^2 x_3}{x_1 - tx_2 + t^2 x_3} - \ln t \quad \text{for all} \quad (t, x) \in (0; + \infty) \times \mathbb{R}^3 \quad \text{(by Theorem 3.3),} \]

and (by Theorem 3.4)

\[ F_3: (t, x) \rightarrow \frac{2tx_2(x_1 - tx_2 + t^2 x_3) - t^4 x_3^2}{(x_1 - tx_2 + t^2 x_3)^2} \quad \text{for all} \quad (t, x) \in \Lambda, \]

where \( \Lambda \) is a domain from the set \( \{(t, x): t > 0, x_1 - tx_2 + t^2 x_3 \neq 0\} \).

### 3.2. Linear nonhomogeneous differential system

A real linear nonhomogeneous differential system of the \( n \)-th order

\[
\frac{dx}{dt} = A(t) x + f(t), \quad f \in C(J),
\]

such that the system (3.1) is the corresponding homogeneous system of system (3.14).

The system (3.14) is called reducible with respect to the nonautonomous transformation group \( G \) if there exist a constant matrix \( B \) and a matrix \( g \in G \) such that the linear transformation \( y = g(t)x \) reduces the system (3.14) to the system

\[
\frac{dy}{dt} = B y + g(t)f(t), \quad y \in \mathbb{R}^n.
\]

Let us remark that if the nonhomogeneous system (3.14) is reduced to the system (3.15), then homogeneous system (3.1) is reduced to the system with constant coefficients (3.2).

The reducible system (3.14) is induced the linear differential operator of first order

\[
\mathfrak{B}(t, x) = \partial_t + (A(t)x + f(t))\partial_x = \mathfrak{A}(t, x) + f(t)\partial_x \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n.
\]

#### 3.2.1. Case of simple elementary divisors

If the matrix \( C \) is diagonalizable, then using Theorem 3.5 and Corollary 3.1, we can build first integrals of the reducible nonhomogeneous differential system (3.14).

**Theorem 3.5.** Let the system (3.14) be reducible to the system (3.15) by a transformation matrix \( g \in G \), let \( \nu \) be a real eigenvector of the matrix \( C \) corresponding to the eigenvalue \( \lambda \). Then a first integral of the reducible system (3.14) is the scalar function

\[ F: (t, x) \rightarrow \nu g(t)x \exp(-\lambda t) - \int_{t_0}^{t} \nu g(\zeta)f(\zeta)\exp(-\lambda \zeta)\,d\zeta \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \quad (3.16) \]

where \( t_0 \) is a fixed point from the interval \( J \).

**Proof.** From Lemma 3.1 it follows that on the domain \( J \times \mathbb{R}^n \):

\[
\mathfrak{B} F(t, x) = \mathfrak{B} \nu g(t)x \exp(-\lambda t) - \mathfrak{B} \int_{t_0}^{t} \nu g(\zeta)f(\zeta)\exp(-\lambda \zeta)\,d\zeta = -\lambda \nu g(t)x \exp(-\lambda t) +
\]

\[
+ \mathfrak{A} \nu g(t)x \exp(-\lambda t) + f(t)\partial_x \nu g(t)x \exp(-\lambda t) - \partial_x \int_{t_0}^{t} \nu g(\zeta)f(\zeta)\exp(-\lambda \zeta)\,d\zeta = 0.
\]

Therefore the function (3.16) is a first integral of the reducible system (3.14).
Corollary 3.1. Let the system (3.14) be reducible to the system (3.15) by a transformation matrix \( g \in G \), and let \( \nu = \nu + \nu i \) (\( \text{Re} \nu = \nu, \text{Im} \nu = \nu i \)) be an eigenvector of the matrix \( C \) corresponding to the complex eigenvalue \( \lambda = \lambda + \lambda i \) (\( \text{Re} \lambda = \lambda, \text{Im} \lambda = \lambda i \)). Then first integrals of the reducible system (3.14) are the scalar functions

\[
F_\theta: (t, x) \rightarrow \alpha_\theta(t, x) - \int_{t_0}^{t} \alpha_\theta(\zeta, f(\zeta)) d\zeta \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \quad \theta = 1, 2,
\]

where \( t_0 \) is a fixed point from the interval \( J \), the functions

\[
\begin{align*}
\alpha_1: (t, x) &\rightarrow (v_2(t) x \cos \tilde{\lambda} t + v_1(t) x \sin \tilde{\lambda} t) \exp(-\tilde{\lambda} t) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \\
\alpha_2: (t, x) &\rightarrow (v_1(t) x \cos \tilde{\lambda} t + v_2(t) x \sin \tilde{\lambda} t) \exp(-\tilde{\lambda} t) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n.
\end{align*}
\]

Proof. Formally using Theorem 3.5, we obtain the complex-valued function (3.16) is a first integral of system (3.14). Then the real and imaginary parts of this complex-valued first integral are the real first integrals (3.17) of the reducible system (3.14). \( \blacksquare \)

Example 3.6. Consider the second-order linear nonhomogeneous differential system

\[
\begin{align*}
\frac{dx_1}{dt} &= (a \sin \omega t \cos \omega t)x_1 - (b + a \cos^2 \omega t)x_2 + f_1(t), \\
\frac{dx_2}{dt} &= (b + a \sin^2 \omega t)x_1 - (a \sin \omega t \cos \omega t)x_2 + f_2(t),
\end{align*}
\]

where \( a, b, \) and \( \omega \) are real numbers such that \( \omega \neq b \) and \( \omega \neq a + b \), the scalar functions \( f_1: J \rightarrow \mathbb{R} \) and \( f_2: J \rightarrow \mathbb{R} \) are continuous on an interval \( J \subset \mathbb{R} \).

The system (3.18) is reducible to the nonhomogeneous system with constant coefficients

\[
\frac{dy_1}{dt} = (\omega - a - b)y_2 + f_1(t) \cos \omega t + f_2(t) \sin \omega t, \quad \frac{dy_2}{dt} = (b - \omega)y_1 - f_1(t) \sin \omega t + f_2(t) \cos \omega t
\]

by the orthogonal group of transformations \( O(J, 2) \) with the transformation

\[
y_1 = \cos \omega t x_1 + \sin \omega t x_2, \quad y_2 = -\sin \omega t x_1 + \cos \omega t x_2
\]

Consider the real number \( D = (b - \omega)(\omega - a - b) \). We have two possible cases for building first integrals of the reducible linear differential system (3.18).

Let \( D > 0 \). Then, using the real eigenvectors \( \nu^k = (b - \omega, \lambda_k) \), \( k = 1, 2 \), of the matrix

\[
C = \begin{bmatrix} 0 & b - \omega \\ \omega - a - b & 0 \end{bmatrix},
\]

and the corresponding real eigenvalues \( \lambda_1 = \sqrt{D}, \lambda_2 = -\sqrt{D} \), we can build (by Theorem 3.5) the basis of first integrals for the reducible system (3.18)

\[
F_k: (t, x_1, x_2) \rightarrow \left( ((b - \omega) \cos \omega t - \lambda_k \sin \omega t)x_1 + (\lambda_k \cos \omega t + (b - \omega) \sin \omega t)x_2 \right) e^{-\lambda_k} - \int_{t_0}^{t} \left( ((b - \omega) \cos \omega \zeta - \lambda_k \sin \omega \zeta)f_1(\zeta) + (\lambda_k \cos \omega \zeta + (b - \omega) \sin \omega \zeta)f_2(\zeta) \right) e^{-\lambda_k} d\zeta
\]

for all \( (t, x_1, x_2) \in J \times \mathbb{R}^2 \), \( k = 1, 2 \).

\[\text{Note that the linear homogeneous differential system corresponding to the linear nonhomogeneous differential system (3.18) is a simplified differential system (has not nutations), which is describing free oscillations of gyroscopic pendulum in twin gyrocompass [17, pp. 528 – 529].}\]
Let $D < 0$. Then, using the eigenvectors $\nu^k = (\omega - b, \lambda_{3-k})$, $k = 1, 2$, and the corresponding complex eigenvalues $\lambda_1 = \sqrt{-D} i$, $\lambda_2 = -\sqrt{-D} i$, we can (by Corollary 3.1) construct the functionally independent first integrals of the reducible system (3.18)

$$F_\theta: (t, x_1, x_2) \rightarrow \alpha_\theta(t, x_1, x_2) - \int_{t_0}^{t} \alpha_\theta(\zeta, f_1(\zeta), f_2(\zeta)) d\zeta \text{ for all } (t, x_1, x_2) \in J \times \mathbb{R}^2, \theta = 1, 2,$$

where $t_0$ is a fixed point from the interval $J$, the scalar functions

$$\alpha_1: (t, x_1, x_2) \rightarrow (\omega - b) \cos \sqrt{-D} t \cos \omega t + \sqrt{-D} \sin \sqrt{-D} t \sin \omega t) x_1 + \left( (\omega - b) \cos \sqrt{-D} t \sin \omega t - \sqrt{-D} \sin \sqrt{-D} t \cos \omega t \right) x_2 \text{ for all } (t, x_1, x_2) \in J \times \mathbb{R}^2,$$

$$\alpha_2: (t, x_1, x_2) \rightarrow (\sqrt{-D} \cos \sqrt{-D} t \sin \omega t - (\omega - b) \sin \sqrt{-D} t \cos \omega t) x_1 - \left( \sqrt{-D} \cos \sqrt{-D} t \cos \omega t + (\omega - b) \sin \sqrt{-D} t \sin \omega t \right) x_2 \text{ for all } (t, x_1, x_2) \in J \times \mathbb{R}^2.$$

3.2.2. Case of multiple elementary divisors

If the matrix $C$ has multiple elementary divisors, then using Theorem 3.6 and Corollary 3.2, we can build first integrals of the reducible nonhomogeneous differential system (3.14).

**Theorem 3.6.** Let the system (3.14) be reducible to the system (3.15) by a transformation matrix $g \in G$, and let $\lambda$ be the eigenvalue with elementary divisor of multiplicity $m \geq 2$ of the matrix $C$ corresponding to a real eigenvector $\nu^0$ and to real generalized eigenvectors $\nu^k$, $k = 1, \ldots, m - 1$. Then the system (3.14) has the functionally independent first integrals

$$F_{k+1}: (t, x) \rightarrow \nu^k g(t) x \exp(-\lambda t) - \sum_{\tau=0}^{k-1} \binom{k}{\tau} t^{k-\tau} F_{\tau+1}(t, x) - C_k(t)$$

for all $(t, x) \in J \times \mathbb{R}^n$, $k = 1, \ldots, m - 1$,

where the first integral (by Theorem 3.5)

$$F_1: (t, x) \rightarrow \nu^0 g(t) x \exp(-\lambda t) - C_0(t) \text{ for all } (t, x) \in J \times \mathbb{R}^n,$$

the scalar functions

$$C_k: t \rightarrow \int_{t_0}^{t} (\nu^k g(\zeta) f(\zeta) \exp(-\lambda \zeta) + k C_{k-1}(\zeta)) d\zeta \text{ for all } t \in J, k = 0, \ldots, m - 1,$$

and $t_0$ is a fixed point from the interval $J \subset \mathbb{R}$.

**Proof.** The proof of Theorem 3.6 is by induction on $m$.

By the system of identities (3.11), it follows that

$$\mathfrak{B}(\nu^\varepsilon g(t) x \exp(-\lambda t)) = -\lambda \nu^\varepsilon g(t) x \exp(-\lambda t) + (\mathfrak{A} \nu^\varepsilon g(t) x + f(t) \partial_x \nu^\varepsilon g(t) x) \exp(-\lambda t) =$$

$$= -\lambda \nu^\varepsilon g(t) x \exp(-\lambda t) + (\lambda \nu^\varepsilon g(t) x + \varepsilon \nu^{\varepsilon-1} g(t) x) \exp(-\lambda t) + \nu^\varepsilon g(t) (f(t) \exp(-\lambda t) =$$

$$= (\varepsilon \nu^{\varepsilon-1} g(t) x + \nu^\varepsilon g(t) f(t)) \exp(-\lambda t) \text{ for all } (t, x) \in J \times \mathbb{R}^n, \varepsilon = 1, \ldots, m - 1.$$

Therefore, we have

$$\mathfrak{B}(\nu^\varepsilon g(t) x \exp(-\lambda t)) = (\varepsilon \nu^{\varepsilon-1} g(t) x + \nu^\varepsilon g(t) f(t)) \exp(-\lambda t)$$

for all $(t, x) \in J \times \mathbb{R}^n$, $\varepsilon = 1, \ldots, m - 1$. 

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Let \( m = 2. \) Using the identities (3.20) with \( \varepsilon = 1, \) we get

\[
\mathfrak{B} F_2(t, x) = \mathfrak{B} (\nu^1 g(t)x \exp (-\lambda t) - t F_1(t, x) - C_1(t)) =
\]

\[
= (\nu^0 g(t)x + \nu^1 g(t)f(t)) \exp(-\lambda t) - F_1(t, x) - (\nu^1 g(t)f(t) \exp(-\lambda t) + C_0(t)) =
\]

\[
= (\nu^0 g(t)x \exp (-\lambda t) - C_0(t)) - F_1(t, x) = 0 \text{ for all } (t, x) \in J \times \mathbb{R}^n.
\]

Hence the function \( F_2: J \times \mathbb{R}^n \to \mathbb{R} \) is a first integral of the reducible system (3.14).

Assume that the functions (3.19) for \( m = \mu \) are first integrals of system (3.14). Then, from the identities (3.20) for the function \( F_{\mu+1} \) it follows that

\[
\mathfrak{B} F_{\mu+1}(t, x) = \mathfrak{B} (\nu^\mu g(t)x \exp (-\lambda t) - \sum_{\tau=0}^{\mu-1} (\mu_\tau) t^{\mu-\tau} F_{\tau+1}(t, x) - C_\mu(t)) =
\]

\[
= (\mu \nu^{\mu-1} g(t)x + \nu^\mu g(t)f(t)) \exp(-\lambda t) - \mu \sum_{\tau=0}^{\mu-2} (\mu_\tau) t^{\mu-\tau-1} F_{\tau+1}(t, x) -
\]

\[
- \mu F_\mu(t, x) - (\nu^\mu g(t)f(t) \exp(-\lambda t) + \mu C_{\mu-1}(t)) =
\]

\[
= \mu \left( \nu^{\mu-1} g(t)x - \sum_{\tau=0}^{\mu-2} (\mu_\tau) t^{\mu-\tau-1} F_{\tau+1}(t, x) - C_{\mu-1}(t) \right) - \mu F_\mu(t, x) = 0 \text{ for all } (t, x) \in J \times \mathbb{R}^n.
\]

This implies that the scalar function \( F_{\mu+1}: J \times \mathbb{R}^n \to \mathbb{R} \) (for \( m = \mu + 1 \)) is a first integral of the reducible linear nonhomogeneous differential system (3.14).

So by the principle of mathematical induction, the scalar functions (3.19) are first integrals of the reducible system (3.14) for every natural number \( m \geq 2. \)

**Example 3.7.** Consider the linear nonhomogeneous differential system\(^1\)

\[
\frac{dx_1}{dt} = (\sigma + \psi(t)) x_2 + \delta (\alpha \sin \varphi(t) - \beta \cos \varphi(t)) x_3 + f_1(t),
\]

\[
\frac{dx_2}{dt} = - (\sigma + \psi(t)) x_1 + \delta (\alpha \cos \varphi(t) + \beta \sin \varphi(t)) x_3 + f_2(t),
\]

\[
\frac{dx_3}{dt} = \gamma (\beta \cos \varphi(t) - \alpha \sin \varphi(t)) x_1 - \gamma (\alpha \cos \varphi(t) + \beta \sin \varphi(t)) x_2 + f_3(t),
\]

where \( \psi: J \to \mathbb{R}, \) \( f_j: J \to \mathbb{R}, \) \( j = 1, 2, 3, \) are continuous functions on an interval \( J \subset \mathbb{R}, \)

the function \( \varphi: t \to \int_{t_0}^t \psi(\zeta) d\zeta \) for all \( t \in J, \) \( t_0 \in J, \) and \( \alpha, \beta, \gamma, \delta, \sigma \neq 0 \) are real numbers.

We claim that the differential system (3.21) is a reducible system by the orthogonal group of transformations \( O(J, 3). \) Indeed, the orthogonal transformation

\[
y_1 = \cos \varphi(t) x_1 - \sin \varphi(t) x_2, \quad y_2 = \sin \varphi(t) x_1 + \cos \varphi(t) x_2, \quad y_3 = x_3
\]

reduce the system (3.21) to the system with constant coefficients

\[
\frac{dy_1}{dt} = \sigma y_2 - \beta \delta y_3 + \cos \varphi(t) f_1(t) - \sin \varphi(t) f_2(t),
\]

\[
\frac{dy_2}{dt} = - \sigma y_1 + \alpha \delta y_3 + \sin \varphi(t) f_1(t) + \cos \varphi(t) f_2(t), \quad \frac{dy_3}{dt} = \beta \gamma y_1 - \alpha \gamma y_2 + f_3(t).
\]

\(^1\)Note that the linear homogeneous differential system corresponding to the nonhomogeneous system (3.21)

is the linearization [8, pp. 139–142] of differential equations, which are describing the motion of a symmetric balanced nonautonomous gyrostat with one point of attachment [71, pp. 219–226; 72].
The matrix $C = \begin{pmatrix} 0 & -\sigma \omega & \beta \gamma \\ \sigma & 0 & -\alpha \gamma \\ -\beta \delta & \alpha \delta & 0 \end{pmatrix}$ has the characteristic equation

$$\det(C - \lambda E) = 0 \iff \lambda(\lambda^2 + (\alpha^2 + \beta^2)\gamma \delta + \sigma^2) = 0.$$ 

Consider the real number $D = (\alpha^2 + \beta^2)\gamma \delta + \sigma^2$. We have three possible cases for building first integrals of the reducible linear differential system (3.21).

Let $D = 0$. The matrix $C$ has the eigenvalue $\lambda_1 = 0$ with the elementary divisor $\lambda^3$ corresponding to the real eigenvector $\nu^{10} = (\alpha \gamma, \beta \gamma, \sigma)$ and to the generalized eigenvectors $\nu^{11} = \left((\alpha + \beta) \frac{\gamma}{\sigma}, (\beta - \alpha) \frac{\gamma}{\sigma}, 1\right)$, $\nu^{12} = \left(\frac{2}{\sigma^2} \beta, -\frac{2}{\sigma^2} \alpha, \frac{2}{\sigma}\right)$. Using Theorems 3.5 and 3.6, we can construct the basis of first integrals for the reducible differential system (3.21)

$$F_1: (t, x) \rightarrow \gamma (\alpha \cos \varphi(t) + \beta \sin \varphi(t)) x_1 + \gamma (\beta \cos \varphi(t) - \alpha \sin \varphi(t)) x_2 + \sigma x_3 - C_0(t), \quad (3.22)$$

$$F_2: (t, x) \rightarrow \frac{\gamma}{\sigma} ((\alpha + \beta) \cos \varphi(t) + (\beta - \alpha) \sin \varphi(t)) x_1 +$$

$$+ \frac{\gamma}{\sigma} ((\beta - \alpha) \cos \varphi(t) - (\alpha + \beta) \sin \varphi(t)) x_2 + x_3 - t F_1(t, x) - C_1(t) \quad \text{for all } (t, x) \in J \times \mathbb{R}^3,$$

$$F_3: (t, x) \rightarrow \frac{2}{\sigma^2} (\beta \cos \varphi(t) - \alpha \sin \varphi(t)) x_1 - \frac{2}{\sigma^2} (\alpha \cos \varphi(t) + \beta \sin \varphi(t)) x_2 +$$

$$+ \frac{2}{\sigma} x_3 - t^2 F_1(t, x) - 2t F_2(t, x) - C_2(t) \quad \text{for all } (t, x) \in J \times \mathbb{R}^3,$$

where the scalar functions

$$C_0: t \rightarrow \int_{t_0}^{t} \left(\gamma (\alpha \cos \varphi(\zeta) + \beta \sin \varphi(\zeta)) f_1(\zeta) + \gamma (\beta \cos \varphi(\zeta) - \alpha \sin \varphi(\zeta)) f_2(\zeta) + \sigma f_3(\zeta)\right) d\zeta,$$

$$C_1: t \rightarrow \int_{t_0}^{t} \left(\frac{\gamma}{\sigma} ((\alpha + \beta) \cos \varphi(\zeta) + (\beta - \alpha) \sin \varphi(\zeta)) f_1(\zeta) +ight.$$

$$+ \frac{\gamma}{\sigma} ((\beta - \alpha) \cos \varphi(\zeta) - (\alpha + \beta) \sin \varphi(\zeta)) f_2(\zeta) + f_3(\zeta) + C_0(\zeta)\right) d\zeta \quad \text{for all } t \in J,$$

$$C_2: t \rightarrow \int_{t_0}^{t} \left(\frac{2}{\sigma^2} (\beta \cos \varphi(\zeta) - \alpha \sin \varphi(\zeta)) f_1(\zeta) - \frac{2}{\sigma^2} (\alpha \cos \varphi(\zeta) + \beta \sin \varphi(\zeta)) f_2(\zeta) +ight.$$

$$+ \frac{2}{\sigma} f_3(\zeta) + 2C_1(\zeta)\right) d\zeta \quad \text{for all } t \in J, \quad t_0 \in J.$$

Let $D < 0$. The matrix $C$ has the linearly independent eigenvectors $\nu^{10} = (\alpha \gamma, \beta \gamma, \sigma)$, $\nu^2 = (\sigma^2 + \gamma \delta \beta^2, -\alpha \beta \gamma \delta - \sigma \sqrt{-D}, \delta(-\alpha \sigma + \beta \sqrt{-D})), \nu^3 = (\sigma^2 + \gamma \delta \beta^2, -\alpha \beta \gamma \delta + \sigma \sqrt{-D}, -\delta(\alpha \sigma + \beta \sqrt{-D}))$ corresponding to the eigenvalues $\lambda_1 = 0, \lambda_2 = \sqrt{-D}, \lambda_3 = -\sqrt{-D}$. An integral basis of system (3.21) is the function (3.22) and the functions (by Theorem 3.5)

$$F_k: (t, x) \rightarrow \left(\left((\sigma^2 + \gamma \delta \beta^2) \cos \varphi(t) - (\alpha \beta \gamma \delta + \sigma \lambda_k) \sin \varphi(t)\right) x_1 -
$$

$$- \left((\alpha \beta \gamma \delta + \sigma \lambda_k) \cos \varphi(t) + (\sigma^2 + \gamma \delta \beta^2) \sin \varphi(t)\right) x_2 + \delta(\beta \lambda_k - \alpha \sigma)x_3\right) e^{-\lambda_k t}.$$
\[ - \int_{t_0}^{t} \left( (\sigma^2 + \gamma \delta^2) \cos \varphi(\zeta) - (\alpha \beta \gamma \delta + \sigma \lambda_k) \sin \varphi(\zeta) \right) f_1(\zeta) - \left( (\alpha \beta \gamma \delta + \sigma \lambda_k) \cos \varphi(\zeta) + (\alpha \gamma \delta^2 + \sigma \lambda_k) \cos \varphi(\zeta) \right) f_2(\zeta) + \left( \beta \lambda_k - \alpha \sigma \right) f_3(\zeta) \right) e^{-\lambda_k t} d\zeta \quad \forall (t, x) \in J \times \mathbb{R}^3, \quad k = 2, 3. \]

Let \( D > 0 \). The matrix \( C \) has the linearly independent eigenvectors \( \nu^1 = (\alpha \gamma, \beta \gamma, \sigma), \nu^2 = (-\sigma^2 - \gamma \delta^2, \alpha \beta \gamma \delta + \sigma \sqrt{D} i, \delta(\alpha \sigma - \beta \sqrt{D} i)), \nu^3 = (-\sigma^2 - \gamma \delta^2, \alpha \beta \gamma \delta - \sigma \sqrt{D} i, \delta(\alpha \sigma + \beta \sqrt{D} i)) \) corresponding to the eigenvalues \( \lambda_1 = 0, \lambda_2 = \sqrt{D} i, \) and \( \lambda_3 = -\sqrt{D} i. \) An integral basis of system (3.21) is the function (3.22) and the functions (by Corollary 3.1)

\[ F_k : (t, x_1, x_2, x_3) \rightarrow \alpha_k (t, x_1, x_2, x_3) - \int_{t_0}^{t} \alpha_k (\zeta, f_1(\zeta), f_2(\zeta), f_3(\zeta)) d\zeta, \quad k = 2, 3, \]

where \( t_0 \) is a fixed point from the interval \( J, \) the scalar functions

\[ \alpha_2 : (t, x) \rightarrow (-\sigma^2 - \gamma \delta^2 \cos \sqrt{D} t \cos \varphi(t) + (\alpha \beta \gamma \delta \cos \sqrt{D} t + \sigma \sqrt{D} \sin \sqrt{D} t) \sin \varphi(t) \right) x_1 + \]

\[ + \left( (\alpha \beta \gamma \delta \cos \sqrt{D} t + \sigma \sqrt{D} \sin \sqrt{D} t) \cos \varphi(t) + (\sigma^2 + \gamma \delta^2) \cos \sqrt{D} t \sin \varphi(t) \right) x_2 + \]

\[ + \delta(\alpha \sigma \cos \sqrt{D} t - \beta \sqrt{D} \sin \sqrt{D} t) x_3 \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^3, \]

\[ \alpha_3 : (t, x) \rightarrow ((\sigma^2 + \gamma \delta^2) \sin \sqrt{D} t \cos \varphi(t) + (\sigma \sqrt{D} \cos \sqrt{D} t - \alpha \beta \gamma \delta \sin \sqrt{D} t) \sin \varphi(t) \right) x_1 + \]

\[ + \left( (\sigma \sqrt{D} \cos \sqrt{D} t - \alpha \beta \gamma \delta \sin \sqrt{D} t) \cos \varphi(t) - (\sigma^2 + \gamma \delta^2) \sin \sqrt{D} t \sin \varphi(t) \right) x_2 - \]

\[ - \delta(\beta \sqrt{D} \cos \sqrt{D} t + \alpha \sigma \sin \sqrt{D} t) x_3 \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^3. \]

**Corollary 3.2.** Let the differential system (3.14) be reducible to the system (3.15) by a transformation matrix \( g \in G, \) and let \( \lambda = \lambda + \tilde{\lambda} i \) (Re \( \lambda = \lambda \), Im \( \lambda = \tilde{\lambda} \neq 0 \) be the complex eigenvalue of the matrix \( C \) with elementary divisor of multiplicity \( m \geq 2 \) corresponding to an complex eigenvector \( \nu^0 = \nu^0 + \tilde{\nu}^0 i \) (Re \( \nu^0 = \nu^0 \), Im \( \nu^0 = \tilde{\nu}^0 \) ) and to generalized eigenvectors \( \nu^k = \nu^k + \tilde{\nu}^k i \) (Re \( \nu^k = \nu^k \), Im \( \nu^k = \tilde{\nu}^k \), \( k = 1, \ldots, m - 1 \). Then first integrals of the reducible system (3.14) are the functions

\[ F_{\theta,k+1} : (t, x) \rightarrow \alpha_{\theta k} (t, x) - \sum_{\tau=0}^{k-1} \binom{k}{\tau} t^{k-\tau} F_{\theta,\tau+1} (t, x) - C_{\theta k} (t) \]

\begin{equation}
(3.23)
\end{equation}

for all \( (t, x) \in J \times \mathbb{R}^n, \quad k = 1, \ldots, m - 1, \quad \theta = 1, 2, \)

where the first integrals (by Corollary 3.1)

\[ F_{\theta 1} : (t, x) \rightarrow \alpha_{\theta 0} (t, x) - C_{\theta 0} (t) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \quad \theta = 1, 2, \]

the scalar functions

\[ \alpha_{1k} : (t, x) \rightarrow (\nu^k g(t) x \cos \tilde{\lambda} t + \tilde{\nu}^k g(t) x \sin \tilde{\lambda} t) \exp(-\tilde{\lambda} t) \quad \text{for all} \quad (t, x) \in J \times \mathbb{R}^n, \]

\[ \alpha_{2k} : (t, x) \rightarrow (\tilde{\nu}^k g(t) x \cos \tilde{\lambda} t + \nu^k g(t) x \sin \tilde{\lambda} t) \exp(-\tilde{\lambda} t) \quad \forall (t, x) \in J \times \mathbb{R}^n, \]

\[ C_{\theta k} : t \rightarrow \int_{t_0}^{t} (\alpha_{\theta k} (\zeta, f(\zeta)) + k C_{\theta, k-1} (\zeta)) d\zeta \quad \text{for all} \quad t \in J, \quad \theta = 1, 2, \quad k = 0, \ldots, m - 1, \quad t_0 \in J. \]
Proof. Since the functions (3.19) are complex-valued first integrals (by Theorem 3.6) of the reducible system (3.14), we see that the real and imaginary parts of the functions (3.19) are the real first integrals (3.23) of the reducible system (3.14). ■

Example 3.8. The linear nonhomogeneous Euler differential system

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2 + a, & \frac{dx_2}{dt} &= x_3 + \frac{b}{t^2} \ln t, & \frac{dx_3}{dt} &= x_4 + \frac{c}{t^3} \sin \ln t, \\
\frac{dx_4}{dt} &= \left(1 - \frac{3}{t^2} x_2 - \frac{6}{t} x_4 + \frac{d}{t^4 \cos \ln t}\right) (a, b, c, d \in \mathbb{R})
\end{align*}
\]

is reducible to the linear differential system with constant coefficients

\[
\begin{align*}
\frac{dy_1}{d\tau} &= y_2 + a e^{\tau}, & \frac{dy_2}{d\tau} &= y_2 + y_3 + b \tau, & \frac{dy_3}{d\tau} &= 2y_3 + y_4 + c \sin \tau, \\
\frac{dy_4}{d\tau} &= -y_1 - 3y_2 - 9y_3 - 3y_4 + \frac{d}{\cos \tau}
\end{align*}
\]

by the exponential group \(\text{Exp}(4)\) with the exponential transformation

\[
y_1 = x_1, \quad y_2 = tx_2, \quad y_3 = t^2 x_3, \quad y_4 = t^3 x_4, \quad t = e^\tau,
\]

The matrix \(C = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & -3 \\ 0 & 1 & 2 & -9 \\ 0 & 0 & 1 & -3 \end{pmatrix}\) has the complex eigenvalues \(\lambda_1 = -i\) and \(\lambda_2 = i\) with the multiple elementary divisors \((\lambda + i)^2\) and \((\lambda - i)^2\), respectively.

The eigenvalue \(\lambda_1 = -i\) of the matrix \(C\) is corresponding to the complex eigenvector \(\nu^0 = (1, 1 + 2i, 1 + 3i, i)\) and to the generalized eigenvector of the 1-st order \(\nu^1 = (-i, 2 + i, i, 0)\).

By Corollaries 3.1 and 3.2, using the numbers \(\lambda_1 = 0, \tilde{\lambda}_1 = -1, \) the linearly independent vectors \(\tilde{\nu}^0 = (1, 1, 0, 0), \tilde{\nu}^0 = (0, 2, 3, 1), \tilde{\nu}^1 = (0, 2, 0, 0), \) \(\tilde{\nu}^1 = (-1, 1, 1, 0)\), the functions

\[
\begin{align*}
\alpha_{10} &: (t, x) \to \cos \ln t \left(x_1 + tx_2 + t^2 x_3\right) - t \sin \ln t \left(2x_2 + 3tx_3 + t^2 x_4\right) \quad \text{for all} \quad (t, x) \in J_t \times \mathbb{R}^4, \\
\alpha_{20} &: (t, x) \to t \cos \ln t \left(2x_2 + 3tx_3 + t^2 x_4\right) + \sin \ln t \left(x_1 + tx_2 + t^2 x_3\right) \quad \text{for all} \quad (t, x) \in J_t \times \mathbb{R}^4, \\
\alpha_{11} &: (t, x) \to 2t \cos \ln t x_2 + \sin \ln t \left(x_1 - tx_2 - t^2 x_3\right) \quad \text{for all} \quad (t, x) \in J_t \times \mathbb{R}^4, \\
\alpha_{21} &: (t, x) \to \cos \ln t \left(-x_1 + tx_2 + t^2 x_3\right) + 2t \sin \ln t x_2 \quad \text{for all} \quad (t, x) \in J_t \times \mathbb{R}^4,
\end{align*}
\]

and

\[
\begin{align*}
C_{10}(t) &= \int \left((a + \frac{b}{t} \ln t + \frac{c}{t} \sin \ln t) \cos \ln t - \left(\frac{2b}{t^2} \ln t + \frac{3c}{t^2} \sin \ln t + \frac{d}{t^2 \cos \ln t}\right) t \sin \ln t\right) dt = \\
&= \frac{a}{2} \left(\cos \ln t + \sin \ln t\right) + \frac{b}{2} \left(\cos \ln t - 2 \sin \ln t + \left(2 \cos \ln t + \sin \ln t\right) \ln t\right) - \\
&\quad - \frac{c}{4} \left(6 \ln t - 2 \sin^2 \ln t - 3 \sin 2 \ln t\right) + d \ln \left|\cos \ln t\right| \quad \text{for all} \quad t \in J_t, \\
C_{20}(t) &= \int \left(\frac{2b}{t^2} \ln t + \frac{3c}{t^2} \sin \ln t + \frac{d}{t^2 \cos \ln t}\right) t \cos \ln t + \left(\frac{a}{t} + \frac{b}{t^2} \ln t + \frac{c}{t} \sin \ln t\right) \sin \ln t\right) dt = \\
&= \frac{a}{2} \left(\sin \ln t - \cos \ln t\right) + \frac{b}{2} \left(2 \cos \ln t + \sin \ln t + \left(2 \sin \ln t - \cos \ln t\right) \ln t\right) + \\
&\quad + \frac{c}{4} \left(2 \ln t + 6 \sin^2 \ln t - 2 \sin 2 \ln t\right) + d \ln t \quad \text{for all} \quad t \in J_t,
\end{align*}
\]

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where \( \beta(t) = \int t \tan t \, dt \), we can build the first integrals of the reducible system (3.24)

\[
F_{11} : (t, x) \rightarrow \cos \ln t (x_1 + tx_2 + t^2 x_3) - t \sin \ln t (2x_2 + 3tx_3 + t^2 x_4) -
\]

\[
- \frac{a}{2} (\cos \ln t + \sin \ln t) t - b(\cos \ln t - 2 \sin \ln t + (2 \cos \ln t + \sin \ln t) \ln t) +
\]

\[
+ \frac{c}{4} (6 \ln t - 2 \sin^2 \ln t - 3 \sin 2 \ln t) - d \ln |\cos \ln t| \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4,
\]

\[
F_{21} : (t, x) \rightarrow t \cos \ln t (2x_2 + 3tx_3 + t^2 x_4) + \sin \ln t (x_1 + tx_2 + t^2 x_3) -
\]

\[
- \frac{a}{2} (\sin \ln t - \cos \ln t) t - b(2 \cos \ln t + \sin \ln t + (2 \sin \ln t - \cos \ln t) \ln t) -
\]

\[
- \frac{c}{4} (2 \ln t + 6 \sin^2 \ln t - \sin 2 \ln t) - d \ln t \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4,
\]

\[
F_{12} : (t, x) \rightarrow 2t \cos \ln t x_2 + \sin \ln t (x_1 - tx_2 - t^2 x_3) - \ln t F_{11}(t, x) -
\]

\[
- \frac{a}{2} (2 \sin \ln t - \cos \ln t) t - b(6 \cos \ln t + \sin \ln t + 4 \ln t \sin \ln t) +
\]

\[
+ \frac{c}{8} (2 \ln t + 6 \ln^2 t + 3 \cos 2 \ln t - \sin 2 \ln t) - d(\ln t \ln |\cos \ln t| + \beta(\ln t)) \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4,
\]

\[
F_{22} : (t, x) \rightarrow \cos \ln t (-x_1 + tx_2 + t^2 x_3) + 2t \sin \ln t x_2 - \ln t F_{21}(t, x) +
\]

\[
+ \frac{a}{2} (2 \cos \ln t + \sin \ln t) t + b(\cos \ln t - 6 \sin \ln t + 4 \ln t \cos \ln t) -
\]

\[
- \frac{c}{8} (6 \ln t + 2 \ln^2 t + \cos 2 \ln t - 3 \sin 2 \ln t + 4 \sin^2 \ln t) - \frac{d}{2} \ln^2 t \quad \text{for all } (t, x) \in J_t \times \mathbb{R}^4
\]
on any domain \( J_t \times \mathbb{R}^4 \subset \mathbb{R}^5, l = 0, 1, 2, \ldots \), where the intervals

\[
J_0 = \left( 0; e^{\frac{\pi}{2}} \right), \quad J_s = \left( e^{\frac{\pi}{2} + \pi(s-1)}; e^{\frac{\pi}{2} + \pi s} \right), \quad s = 1, 2, \ldots .
\]

The functionally independent first integrals \( F_{11}, F_{21}, F_{12}, \) and \( F_{22} \) are an integral basis of the Euler differential system (3.24) on any domain \( J_t \times \mathbb{R}^4 \).
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