Multimomentum Maps in General Relativity

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Abstract

The properties of multimomentum maps on null hypersurfaces, and their relation with the constraint analysis of General Relativity, are described. Unlike the case of spacelike hypersurfaces, some constraints which are second class in the Hamiltonian formalism turn out to contribute to the multimomentum map.

1 Introduction

In order to quantize gravity, a very long-time effort has been produced by physicists over the last fifty years. Since the perturbative approach fails to produce a renormalizable theory, it seemed more viable to proceed with the analysis of canonical gravity, which leads to a non-perturbative approach to quantum gravity. The canonical quantization of field theories follows Dirac’s prescription to translate the Poisson brackets among the canonical variables into commutators of the operators corresponding to these variables, and a special treatment is reserved to those systems and fields which are constrained. Within this framework Dirac, Bergmann, Arnowitt-Deser-Misner, Isham and, more recently, Ashtekar, pursued the aim of building a canonical formalism for General Relativity.

The canonical approach has been successful, but it faces two important problems. First, to obtain a Hamiltonian formulation it is necessary to break

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manifest covariance. The other problem is that one has to deal with an infinite number of degrees of freedom when a field theory is considered.

In recent work [1], the authors have studied a multisymplectic version of General Relativity. In this approach, field theories can be treated as an extension of the usual symplectic treatment of classical mechanics. Here, instead of working with an infinite number of degrees of freedom, as is usually done with the symplectic approach, the whole theory is constructed on a 1-jet bundle, whose local coordinates are spacetime coordinates, the fields and their first derivatives.

It has been shown, in particular, how the classical multisymplectic analysis of the constraints is equivalent to the constraint analysis given by Ashtekar from a canonical point of view [1]. Since the constraints were studied on a spacelike hypersurface, in this paper, to complete our previous investigations, null hypersurfaces are considered. In the Hamiltonian formulation of General Relativity, the constraint analysis on null hypersurfaces plays an important role since such surfaces provide a natural framework for the study of gravitational radiation in asymptotically flat space-times [2-7]. Moreover, in a null canonical formalism, the physical degrees of freedom and the observables of the theory may be picked out more easily [5,6]. Therefore it appears very interesting to extend the constraint analysis of [1] to null hypersurfaces and find out whether equivalent results exist. This may also provide further insight into the techniques for dealing with second-class constraints [8].

In section 2, null tetrads are defined according to [5]. In section 3, constraints are studied for a self-dual action. Concluding remarks are presented in section 4.

2 Null Tetrads

In this paper we are only concerned with the local treatment of the problem on null hypersurfaces. Thus, many problems arising from the possible null-cone singularities are left aside. To obtain a consistent 3+1 description
of all the Einstein equations, one can introduce the null tetrad [5]

\[ e^0 = \frac{1}{N} \left( \frac{\partial}{\partial t} - N^i \frac{\partial}{\partial x^i} \right) \]

(1)

and

\[ e^k = \left( v^i_k + \alpha^i_k \frac{N^i}{N} \right) \frac{\partial}{\partial x^i} - \alpha^i_k \frac{\partial}{\partial t} . \]

(2)

The duals to these vectors are

\[ \theta^0 = (N + \alpha_i N^i) dt + \alpha_i dx^i \]

(3)

and

\[ \theta^k = v^i_k (N^i dt + dx^i) , \]

(4)

where

\[ v^i_k v^j_i = \delta^j_k \]

(5)

and

\[ \alpha^i = v^i_k \alpha_k . \]

(6)

The Minkowski metric is given by

\[ \eta_{\hat{a}\hat{b}} = \eta^{\hat{a}\hat{b}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \]

(7)

The tetrad labels \( \hat{a}, \hat{b}, \hat{c} \) range from 0 through 3, while the indices \( \hat{k}, \hat{l} \) range from 1 through 3. Analogous notation is used for the spacetime indices \( a, b, ..., i, j, \) etc.

It is straightforward to see that

\[ g^{a\hat{b}} t_a t_{\hat{b}} = -\frac{2}{N^2} (\alpha_1 + \alpha_2 \alpha_3) . \]

(8)

This implies that the hypersurfaces \( t = \text{const.} \) are null if and only if \( \alpha_1 + \alpha_2 \alpha_3 = 0 \). By a particular choice of coordinates, it is always possible to set \( \alpha_2 = \alpha_3 = 0 \).
In many of the following equations the tetrad vectors appear in the combination
\[ \tilde{p}_{\hat{a}\hat{e}}^{ac} = \frac{e}{2} (e^a_b e^c_e - e^a_c e^c_b) \],
(9)
where \( e = N\nu \) with \( \nu = det(\nu^i_j) \).

3 The Self-Dual Action

In [5,7] the Hamiltonian formulation of a complex self-dual action on a null hypersurface in Lorentzian space-time was studied. The 3+1 decomposition was inserted into the Lagrangian, and the constraints were derived with the usual Dirac’s procedure. In this section the results of [5] are briefly summarized and then compared with the corresponding constraints obtained by the multimomentum map. Since these constraints correspond to the secondary constraints of the Hamiltonian formalism [8], the discussion is focused on them.

The complex self-dual part of the connection are the complex one-forms given by
\[ (+)\omega_{a}^{\hat{a}\hat{e}} = \frac{1}{2} \left( \omega_{a}^{\hat{a}\hat{e}} - \frac{i}{2} e^{\hat{a}\hat{e}}_{b\hat{d}} \omega_{a}^{b\hat{d}} \right) \].
(10)
Explicitly, one has
\[ (+)\omega_{a}^{0\hat{1}} = (+)\omega_{a}^{2\hat{3}} = \frac{1}{2} \left( \omega_{a}^{0\hat{1}} + \omega_{a}^{2\hat{3}} \right) \],
(11)
\[ (+)\omega_{a}^{2\hat{1}} = \omega_{a}^{2\hat{1}} \quad (+)\omega_{a}^{0\hat{3}} = \omega_{a}^{0\hat{3}} \quad (+)\omega_{a}^{\hat{0}\hat{2}} = (+)\omega_{a}^{i\hat{3}} = 0 \].
(12)
The curvature of a self-dual connection is equal to the self-dual part of the curvature:
\[ \Omega((+))\omega = (+)\Omega(\omega) \].
(13)
Thus, the complex self-dual action to be considered is [1]
\[ S_{SD} \equiv \frac{1}{2} \int_M d^4x \, e \, e^a_d e^b_h (+)\Omega_{ab}^{\hat{a}\hat{b}} \].
(14)
The tetrad vectors occur in the following equations in a combination which is the self-dual part of eq. (9). The 13 secondary constraints obtained in the Hamiltonian formalism in [5] after the 3+1 split, and written with the notation of the present paper, are as follows:

\[
\mathcal{H}_0 \equiv -\left(\frac{e}{N}\right)^2 V_2^i \left[ (+) \Omega_{ij} \hat{0}^1 V_3^j \right] + (+) \Omega_{ij} \hat{2}^1 V_1^j \approx 0 ,
\]

\[
\mathcal{H}_i \equiv \frac{e}{N} \left[ (+) \Omega_{ij} \hat{0}^1 V_1^j + (+) \Omega_{ij} \hat{0}^3 V_3^j \right] \approx 0 ,
\]

\[
\mathcal{G}_1 \equiv -\partial_i \left( \frac{e}{N} V_1^i \right) - 2 \frac{e}{N} (+) \omega_i \hat{0}^3 V_3^i \approx 0 ,
\]

\[
\mathcal{G}_2 \equiv - \frac{e}{N} (+) \omega_i \hat{0}^3 V_1^i \approx 0 ,
\]

\[
\mathcal{G}_3 \equiv -\partial_i \left( \frac{e}{N} V_3^i \right) + \frac{e}{N} (+) \omega_i \hat{2}^1 V_1^i + 2 \frac{e}{N} (+) \omega_i \hat{0}^1 V_3^i \approx 0 ,
\]

\[
\chi^i = -2 \partial_j \frac{e}{N} V_2^i \left[ (+) \Omega_{ij} \right] V_1^j - 2 \frac{e}{N} (+) \omega_j \hat{0}^3 V_2^j \left[ i \ V_3^j \right] - 4 \frac{e}{N} (+) \omega_j \hat{0}^1 V_2^j \left[ i \ V_1^j \right] + 2 \frac{e}{N} (+) \omega_j \hat{0}^3 N^i \left[ j \ V_1^j \right] + \frac{e}{N} (+) \omega_0 \hat{0}^3 V_1^i \approx 0 ,
\]

\[
\phi_i \equiv -\frac{e}{N} \left[ (+) \Omega_{ij} \hat{0}^1 V_3^j + (+) \Omega_{ij} \hat{2}^1 V_1^j \right] \approx 0 .
\]

The irreducible second-class constraints turn out to be \( \mathcal{H}_0, \mathcal{G}_3, \chi^i, \phi, V_2^i \) and \( \phi, V_3^j \) [5]. Note that, following [5,7], we have set to zero all the \( \alpha \) parameters in the course of deriving eqs. (15)–(21).
Let us now discuss the constraints from our point of view. The multimomentum map is [9]

\[
I_{SN}^{\pm}[\xi, \lambda] = \int_{SN} \left[ \left( + \right) \tilde{p}^{ac} \left( \xi^b_a \left( + \right) \omega^b_{\ b} \right) - \left( D_a \left( + \right) \lambda \right)^{bd} + \left( + \right) \omega^b_{\ a \ b} \xi^b \right] d^3 x_c .
\] (22)

The constraint equations obtained from setting to zero this multimomentum map are then

\[
\int_{SN} \left( + \right) \lambda^0 i \left[ \partial_i \left( \frac{e}{N} V^i_1 \right) \right] d^3 x_0 = 0 ,
\] (23)

\[
\int_{SN} \left( + \right) \lambda^3 \left[ \partial_i \left( \frac{e}{N} V^i_3 \right) \right] d^3 x_0 = 0 ,
\] (24)

\[
\int_{SN} \left( + \right) \lambda^1 2 D_i \left( + \right) \tilde{p}^{0i}_{\ 12} d^3 x_0 = 0 ,
\] (25)

\[
\int_{SN} e V^i_2 \left[ \left( + \right) \Omega^0 i j V^j_3 + \left( + \right) \Omega^3 V^j_3 \right] \xi^0 d^3 x_0 - \int_{SN} \frac{2e}{N^3} \left[ \left( + \right) \Omega^0 i j \right] \xi^0 d^3 x_0 + \left( + \right) \Omega^0 i j \xi^0 d^3 x_0 = 0 ,
\] (26)

\[
\int_{SN} \frac{e}{N} \left[ \left( + \right) \Omega^0 i j \right] \xi^0 d^3 x_0 + \left( + \Omega^0 i j \right] \xi^0 d^3 x_0 = 0 .
\] (27)
On the other hand, the Euler-Lagrange equations resulting from the action (14) are

\[ e^b_e \left[ \Theta_{b d}^{(+) \hat{c} \hat{b}} - \frac{1}{2} e^d_a e^h_c \Theta_{b d}^{(+) \hat{c} \hat{b}} \right] = 0 , \]  

(28)

and

\[ D_b^{(+)} \tilde{p}^{ab} = 0 . \]  

(29)

The self-dual Einstein equations (28) can be written explicitly in the form

\[ (+) G_h^0 \equiv e^b_1 \Theta_{b h}^{(+) \hat{1} \hat{0}} + e^b_3 \Theta_{b h}^{(+) \hat{3} \hat{0}} = 0 , \]  

(30)

\[ (+) G_h^1 \equiv e^b_0 \Theta_{b h}^{(+) \hat{1} \hat{0}} + e^b_2 \Theta_{b h}^{(+) \hat{2} \hat{1}} = 0 , \]  

(31)

\[ (+) G_h^2 \equiv e^b_1 \Theta_{b h}^{(+) \hat{1} \hat{2}} + e^b_3 \Theta_{b h}^{(+) \hat{3} \hat{2}} = 0 , \]  

(32)

\[ (+) G_h^3 \equiv e^b_0 \Theta_{b h}^{(+) \hat{0} \hat{3}} + e^b_2 \Theta_{b h}^{(+) \hat{2} \hat{3}} = 0 . \]  

(33)

It is easy to show that the equations independent of time derivatives on a null hypersurface are the spatial components of eqs. (30) and (32), jointly with the equations

\[ D_i^{(+)} \tilde{p}^{0i}_{\hat{0} \hat{1}} = D_i^{(+)} \tilde{p}^{0i}_{\hat{0} \hat{3}} = 0 , \]  

(34)

\[ D_i^{(+)} \tilde{p}^{0i}_{\hat{1} \hat{2}} = 0 , \]  

(35)

which are equivalent to (23)–(25), and

\[ D_j^{(+)} \tilde{p}^{-ij}_{\hat{1} \hat{2}} = 0 . \]  

(36)

The comparison of eqs. (15)–(21) with eqs. (23)–(27) shows that eq. (23) corresponds to eq. (17), eq. (24) to eq. (19), eq. (25) to eq. (18), eq. (26) to eqs. (15) and (16), and eq. (27) to eq. (16).

Remarkably, the second-class constraints \( \mathcal{H}_0 \) and \( \mathcal{G}_3 \) are found to contribute to the multimomentum map (see section 4).
4 Concluding Remarks

This paper has considered the application of the multimomentum-map technique to study General Relativity as a constrained system on null hypersurfaces. Its contribution lies in relating different formalisms for such a constraint analysis. We have found that, on null hypersurfaces, the multimomentum map provides just a subset of the full set of constraints of the theory, while the other constraints turn out to be those particular Euler-Lagrange equations which are not of evolutionary type [9].

Although the multimomentum map is expected to yield only the secondary first-class constraints [8], we have found that some of the constraints which are second class in the Hamiltonian formalism occur also in the multimomentum map. The group-theoretical interpretation of this property seems to be that our analysis remains covariant in that it deals with the full diffeomorphism group of spacetime, say $\text{Diff}(M)$, jointly with the internal rotation group $O(3,1)$. Hence one incorporates some constraints which are instead ruled out if one breaks covariance, which amounts to taking subgroups of the ones just mentioned. In other words, only when $\text{Diff}(M)$ and $O(3,1)$ are replaced by their subgroups $\text{Diff}(S_N) \times \text{Diff}(\mathbb{R})$ and $O(3)$, the constraints (24) and (26) become second class and hence do not contribute to the multimomentum map.

In [10], the holomorphic multimomentum map has been studied to obtain a formulation of complex general relativity, and this appears to be another challenging field of research.

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References
[1] G. Esposito, G. Gionti and C. Stornaiolo, Nuovo Cimento B 110 (1995) 1137.

[2] J. N. Goldberg, Found. Phys. 14 (1984) 1211.

[3] R. Nagarajan and J. N. Goldberg, Phys. Rev. D 31 (1985) 1354.

[4] J. N. Goldberg, in Gravitation and Geometry, edited by W. Rindler and A. Trautman, Bibliopolis, Naples, 1987.

[5] J. N. Goldberg, D. C. Robinson and C. Soteriou, Class. Quantum Grav. 9 (1992) 1309.

[6] R. A. d’Inverno and J. A. Vickers, Class. Quantum Grav. 12 (1995) 753.

[7] J. N. Goldberg and C. Soteriou, Class. Quantum Grav. 12 (1995) 2779.

[8] M. J. Gotay, J. Isenberg, J. E. Marsden and R. Montgomery, Momentum Mappings and the Hamiltonian Structure of Classical Field Theories with Constraints, Springer, Berlin, in press.

[9] G. Esposito and C. Stornaiolo, Multimomentum Maps on Null Hypersurfaces (DSF preprint 97/4).

[10] G. Esposito and C. Stornaiolo, Class. Quantum Grav. 12 (1995) 1733.