Quantum Mechanics on $S^n$ and Meron Solution

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Abstract

A particle in quantum mechanics on manifolds couples to the induced topological gauge field that characterises the possible inequivalent quantizations. For instance, the gauge potential induced on $S^2$ is that of a magnetic monopole located at the center of $S^2$. We find that the gauge potential induced on $S^3$ ($S^{2n+1}$) is that of a meron (generalized meron) also sitting at the center of $S^3$ ($S^{2n+1}$).

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In recent years, it has been noted in various situations that the topological gauge structure may be induced upon quantization on topologically nontrivial manifolds. This gauge structure characterises the inequivalent quantizations that are possible on these manifolds. Particularly clear results are obtained for quantum mechanics on $S^n$ with radius $r$ embedded in $\mathbb{R}^{n+1}$.

Let $x_\alpha$ ($\alpha = 1, 2, \ldots, n + 1$) be a homogenous coordinate of $\mathbb{R}^{n+1}$ and define the $S^n$ by the constraint

$$\sum_{\alpha=1}^{n+1} x_\alpha^2 - r^2 = 0.$$  \hfill (1)

We shall represent the observables by the use of the covariant form in $\mathbb{R}^{n+1}$ that should be projected on the $S^n$.

Then the fundamental algebra of observables on the $S^n$ is given by [2]

$$[x_\alpha, x_\beta] = 0,$$  \hfill (2)

$$[x_\lambda, G_{\alpha\beta}] = i (x_\alpha \delta_{\lambda\beta} - x_\beta \delta_{\lambda\alpha}),$$

$$[G_{\alpha\beta}, G_{\lambda\mu}] = i (\delta_{\alpha\lambda} G_{\beta\mu} - \delta_{\alpha\mu} G_{\beta\lambda} + \delta_{\beta\mu} G_{\alpha\lambda} - \delta_{\beta\lambda} G_{\alpha\mu}),$$

and the induced gauge potential that appears in this representation has the form for $n \geq 2$

$$\begin{cases}
A_j(x) = \frac{1}{r(r + x_{n+1})} \sum_{k=1}^{n} S_{jk} x_k, & (j, k = 1, 2, \ldots n) \\
A_{n+1}(x) = 0
\end{cases}, \hfill (3)$$

where $S_{jk}$'s are Hermitian matrices of $SO(n)$. These can be written in $SO(n+1)$ covariant form

$$A_\alpha = \frac{1}{r^2} \sum_{\beta=1}^{n+1} S_{\alpha\beta} x_\beta , \hfill (4)$$

with the constraint on the wave function

$$\frac{1}{r} \sum_{\alpha=1}^{n+1} x_\alpha \gamma_\alpha \psi = \pm \psi . \hfill (5)$$

Where $\gamma_\alpha$'s are the Hermitian matrices satisfying the Clifford algebra of the
order $n + 1$ and $S_{\alpha\beta}$’s are Hermitian matrices that satisfy the Lie algebra of $SO(n + 1)$ and can be written as

$$S_{\alpha\beta} = \frac{1}{4i} [\gamma_\alpha, \gamma_\beta].$$  \hfill (6)

The potentials induced in the cases $n = 1$ and $n = 2$ can be visualised as those resulting from a magnetic flux and a magnetic monopole respectively located at the center of the embedding space. The even $n$ cases for $n \geq 4$ were interpreted as the (generalized) instantons,[3][4] which are the habitants on $S^n$. On the other hand, there are no topological interpretations for the odd $n$ ($n \geq 3$) cases, the first question in this respect being “What is sitting at the center of $S^3, S^5, S^7, \ldots$?” In this note we shall see that the odd $n$ ($n \geq 3$) cases can similarly be considered to be the potentials generated by the (generalized) merons sitting at the center of the embedding space $R^{n+1}$.

Although the meron solutions have singularities, they exist in our picture outside of the physical $S^n$, i.e. at the center of $R^{n+1}$ wherein the former is embedded. Thus we can expect the meron effect on the surface. The authors of ref.[5] have explicitly obtained the generators satisfying the fundamental algebra in terms of the induced gauge potential and have discovered among others the zero size instanton for $n = 3$ at the center of $S^3$. On the other hand, in ref.[3][4] BPST instantons appear on $S^{2n}$, $n = 2, 3, \ldots$, embedded in $R^{2n+1}$.

In case of $S^3$, the authors of ref.[2] have obtained an induced gauge field, without argument on topological interpretations. Their gauge field can be written modulo gauge transformations (in $\gamma_5$-diagonal representation) as follows,

$$A_\mu = \frac{1}{r^2} S_{\mu\nu} x_\nu = \frac{1}{r^2} \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & -\sigma_{\mu\nu} \end{pmatrix} x_\nu ,$$  \hfill (7)

where $S_{\mu\nu}$’s ($\mu, \nu = 1, 2, 3, 4$) are the generators of the $SO(4)$ and

$$\sigma_{ij} = \sigma_{\bar{i}\bar{j}} = \frac{1}{2} \epsilon_{ijk} \tau_k, \ \sigma_{44} = -\sigma_{44} = \frac{1}{2} \tau_i \ (\tau_i : \text{Pauli matrices}).$$  \hfill (8)

Introducing the four component spinor

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} ,$$  \hfill (9)
where
\[ \psi_R = \frac{1 \pm \gamma_5}{2} \psi, \]
the constraint (5) reduces to
\[ \psi_L = \pm g^{-1}(x) \psi_R, \]
where
\[ g(x) = \frac{(x_4 - i \vec{x} \cdot \vec{\tau})}{r} \left( r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2} \right). \]  
Thus our wave functions are essentially two component spinors and we can rewrite (7) as
\[ A_\mu = \frac{1}{r^2} \sigma_{\mu \nu} x_\nu, \]
or
\[ A_\mu = \frac{1}{r^2} \tilde{\sigma}_{\mu \nu} x_\nu, \]
which can be considered as SU(2) Yang-Mills (Y-M) gauge fields. Note that there is a case of the trivial gauge field modulo gauge transformations
\[ A_\mu = 0, \]
We shall clarify that we can regard these configurations as topologically nontrivial objects in the embedding space \( R^4 \). These gauge fields can be considered as SU(2) gauge fields in \( R^4 \) space and they satisfy the Y-M equation in the embedding space.
Recall that the field equation
\[ D_{\mu} f_{\mu \nu} = \partial_{\mu} f_{\mu \nu} - i [a_{\mu}, f_{\mu \nu}] = 0, \]
for the 4-dimensional Euclidean SU(2) Y-M theory has instanton solutions, the instanton number of which is
\[ Q = \frac{1}{16\pi^2} \int \tr (f_{\mu \nu} \tilde{f}_{\mu \nu}) d^4x = 0, \pm 1, \pm 2, \ldots, \]
due to \( \Pi_3(SU(2)) = Z \). The \( Q = 0 \) case corresponds to the trivial solution with
\[ a_\mu = 0, \]
the single instanton solution \((Q = 1)\) can be written as
\[
a_\mu = \frac{2}{r^2 + \lambda^2} \sigma_{\mu \nu} x_\nu, \tag{18}
\]
and we also have the multi-instanton solutions with \(Q = m\).

In addition to these, there is a solution of Y-M equation called “meron”
\[
a_\mu = \frac{1}{r^2} \sigma_{\mu \nu} x_\nu, \tag{19}
\]
which has half an instanton number.[6]

Let us now consider each cases of induced gauge fields separately. First we consider the case of \((14) A_\mu = 0\). This is a trivial configuration, however, we note that we are concerned with the \(SU(2)\) gauge fields and in addition to the small gauge transformations there are large gauge transformations, that can change the instanton number. The large gauge transformation with the winding number 1 is
\[
g(x) = \frac{(x_4 - i \vec{x} \cdot \vec{\tau})}{r},
\]
while those with the winding number \(m\) are
\[
g^m (x) (m = \pm 1, \pm 2, \ldots ).
\]
Thus we can have the induced gauge fields due to large gauge transformation
\[
A_\mu = ig^{-1}(x) \partial_\mu g(x), \tag{20}
\]
which could be considered as the \(\lambda \to 0\) limit of an instanton or two merons sitting at \(r = 0\). This is the configuration found in ref.[5] and the instanton number inside the \(S^3\) is
\[
Q_{\text{inside}} = 1. \tag{21}
\]

Next we consider the case of \((12)\)
\[
A_\mu = \frac{1}{r^2} \sigma_{\mu \nu} x_\nu.
\]

\[5\text{Note that (13) can be considered as the case of } m = -1 \text{ in (27).}\]
In fact this is the meron solution inferred in (19), the instanton density of which is \( \frac{1}{2} \delta (r) \). This field is specified by the instanton number

\[
Q_{\text{inside}} = \frac{1}{2} .
\]  

(22)

Thus in general

\[
A_\mu = 0 + (\text{large gauge trans.}) ,
\]  

(23)

can be expressed as

\[
A_\mu = ig^{-m}(x) \partial_\mu g^m(x) ,
\]  

(24)

and

\[
Q_{\text{inside}} = m .
\]  

(25)

On the other hand,

\[
A_\mu = \frac{1}{r^2} \sigma_{\mu \nu} x_\nu + (\text{large gauge trans.}) ,
\]  

(26)

reduces to

\[
A_\mu = g^{-m}(x) \frac{1}{r^2} \sigma_{\mu \nu} x_\nu g^m(x) + ig^{-m}(x) \partial_\mu g^m(x) ,
\]  

(27)

and

\[
Q_{\text{inside}} = \frac{1}{2} + m .
\]  

(28)

This argument can of course be generalized to \( S^{2n-1} \) \( (n \geq 2) \). We again have two cases with

\[
A_\mu = iG^{-m} \partial_\mu G^m ,
\]  

\[
Q_{\text{inside}} = m ,
\]  

\[
A_\mu = G^{-m} \frac{1}{r^2} \Sigma_{\mu \nu} x_\nu G^m + iG^{-m} \partial_\mu G^m ,
\]  

\[
Q_{\text{inside}} = \frac{1}{2} + m ,
\]  

\[
\Sigma_{\mu \nu} = \frac{1}{4i} \left( \frac{1 + \Gamma_{2n+1}}{2} \right) [\Gamma_\mu, \Gamma_\nu] .
\]  

(33)
$G$ is large gauge transformation that follows from nontriviality of $\Pi_{2n-1}(SO(2n))$.

$m = 0$ case of (31) is nothing but the (generalized) meron solution\[7\] of YM theory in $R^{2n}$.

To summarize, the induced gauge fields for quantum mechanics on $S^{2n-1}$ ($n \geq 2$) can be considered to be generated by merons (anti-merons) sitting at the center ($r = 0$) of $R^{2n}$ where our $S^{2n-1}$ is embedded. The gauge fields are specified by the instanton number $Q_{\text{inside}} = \frac{m^2}{2}$ ($m$: integer). For even $m$, $F_{\mu\nu} = 0$ on $S^{2n-1}$, but for odd $m$, $F_{\mu\nu} \neq 0$ even on $S^{2n-1}$.

To see the physical effect to a particle on $S^3$ of a meron at the center, let us consider a quantum mechanical phase factor

$$P \exp \left[ i \oint A_\mu dx^\mu \right],$$

(34)

for the closed loop. To be specific we consider the case $x_4 = 0$, i.e. the closed loop inside $S^2$ ($x_1^2 + x_2^2 + x_3^2 = r^2$). In the region $x_4 = 0$ the meron solution

$$A_\mu = \frac{1}{r^2} \sigma_{\mu\nu} x_\nu,$$

(35)

reduces to the Wu-Yang monopole

$$A_i = \frac{1}{2r^2} \epsilon_{ijk} x_j \tau_k,$$

(36)

with the magnetic charge $g = 1/2e$. Thus the phase factor is fixed by the magnetic flux going through the loop on $S^2$ that comes from the monopole.

A particle on $S^3$, moving in the region of $x_4 = 0$, is under the same effect as a particle on $S^2$ with a monopole of the magnetic charge $g = 1/2e$. On the other hand, for quantum mechanics on $S^3$ with an instanton(2-merons) at the center, no physical effect is expected on $S^3$ at least for $x_4 = 0$, since no magnetic flux pierces through $S^2$.

Recent investigations have revealed a close connection between the quantum mechanically induced gauge potentials on manifolds on one hand and the so called topological terms appearing in field theories on the other.\[8]\[9]\[10\]

Actually, the Hopf term in the $2+1$ dimensional $O(3)$ nonlinear sigma model in $R^2$-space has been shown\[8]\[9] to be an effect of quantum mechanics on $S^1$, while the Wess-Zumino term in the chiral model defined in $1+1$ dimensions can be understood\[10] as the Dirac monopole potential that is induced on $S^2$. It is interesting to note that $O(3)$ nonlinear sigma model on $S^2$ space\[8]
were shown to be described by quantum mechanics on $S^3$ where the merons studied in this note should play a role.

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