Dodecahedral topology fails to explain quadrupole–octupole alignment

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Abstract

The cosmic microwave background (CMB) quadrupole and octupole, as well as being weaker than expected, align suspiciously well with each other. The non-trivial spatial topology can explain the weakness. Might it also explain the alignment? The answer in the case of the Poincaré dodecahedral space is, as one might expect, a resounding no.

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I dedicate this article to my friend and collaborator Jesper Gundermann, whose untimely death on 10 June 2006 saddened all who knew him. His energetic enthusiasm and deep love of science brought joy to those of us lucky enough to work with him.

1. Introduction

Soon after the release of the first-year WMAP data [1], Tegmark et al [2] noted that the CMB quadrupole and octupole aligned with each other unusually well, at roughly the 98% level. Multipole vectors—discovered by Maxwell [3] in the nineteenth century, widely forgotten, then reintroduced by Copi et al [4]—provide a useful tool for analysing the alignment in greater detail. While exact confidence levels vary depending on what one measures, all researchers agree that the quadrupole–octupole alignment is unusual at roughly the 99% level or better [5–7]. The combination of the 1-in-100 alignment with the 1-in-600 overall weakness of the low-$\ell$ modes motivates one to seek a physical explanation.

Non-trivial spatial topology can explain the weakness of the low-$\ell$ modes. Might it also explain the quadrupole–octupole alignment? The present paper simulates the CMB in a Poincaré dodecahedral space [8–10] and checks the quadrupole–octupole alignment. Absolutely no correlation is found.
2. Simulating the space

We use the late Jesper Gundermann’s simulation \cite{11} of the CMB in the Poincaré dodecahedral space, for $\Omega_{\text{total}} = 1.028$, $\Omega_m = 0.26$, $h \approx 0.71$, and with modes through $k_{\text{max}} = 102$. This simplified simulation, while neglecting the Doppler contribution and the sound speed, turns out to be sufficient for our needs. As we will see in section 3, absolutely no quadrupole–octupole correlation is found. Section 4 then argues that the very precise flatness of the distribution in figure 1 justifies our conclusion that the dodecahedral topology fails to explain the quadrupole–octupole alignment, in spite of the severe simplifications in our model.

3. Measuring the alignment

For each simulated CMB sky, we use the polynomial method \cite{6} to compute the two quadrupole vectors $\{u_{2,1}, u_{2,2}\}$ and the three octupole vectors $\{u_{3,1}, u_{3,2}, u_{3,3}\}$. Following \cite{5}, we take the cross product $w_2 = u_{2,1} \times u_{2,2}$, which we normalize to obtain a unit vector $n_2 = w_2 / |w_2|$ orthogonal to the plane of the quadrupole. Similarly, we take the cross product of each of the three possible pairs of octupole vectors

$$w_{3,1} = u_{3,2} \times u_{3,3}, \quad w_{3,2} = u_{3,3} \times u_{3,1}, \quad w_{3,3} = u_{3,1} \times u_{3,2},$$

which we normalize to obtain unit vectors $n_{3,i} = w_{3,i} / |w_{3,i}|$ orthogonal to each of the three octupole planes. The three dot products $D_i = |n_2 \cdot n_{3,i}|$ then measure the extent to which the quadrupole plane does or does not align with each of the three octupole planes.

In a simply connected universe, one expects no correlation between the quadrupole vector $n_2$ and each octupole vector $n_{3,i}$. As $n_2$ and $n_{3,i}$ (for some fixed $i$) wander randomly over the 2-sphere, their dot product follows a flat distribution on the interval $[-1, +1]$ (this is a consequence of the wonderful fact that radial projection of a sphere onto a circumscribed cylinder via $(x, y, z) \mapsto \left( \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z \right)$ preserves area). Hence each $D_i = |n_2 \cdot n_{3,i}|$, being the absolute value of the dot product, follows a flat distribution on $[0, 1]$.

In the real universe the quadrupole aligns surprisingly well with the octupole, giving dot products $\{D_1, D_2, D_3\} = \{0.84, 0.87, 0.95\}$ for the DQ-corrected Tegmark (DQT) cleaning \cite{2}.

\footnote{Gundermann’s extensive correspondence with JW does not state the Hubble constant explicitly. Most likely, he used the commonly cited value of $h = 0.71$ or something very close to that.}
of the first-year WMAP data or \{0.85, 0.87, 0.93\} for the Lagrange internal linear combination (LILC) cleaning [12] of the same data.

The question of whether a multiconnected spatial topology might explain the observed quadrupole–octupole alignment may be rephrased more precisely as: does a given topology predict a flat distribution for each \(D_i\) or does it predict a distribution skewed towards the high end? For the Poincaré dodecahedral space, our simulations (recall section 2) yield a flat distribution (figure 1), implying that the dodecahedral topology does nothing to explain the quadrupole–octupole alignment.

To be fully rigorous we should point out that even though the individual dot products \(D_i\) follow the same flat distribution in the dodecahedral topology that they do in the simply connected model, it is nevertheless conceivable that their sum \(D_1 + D_2 + D_3\) might follow a slightly different distribution in the two cases, depending on the internal correlations among the three \(D_i\) in the dodecahedral case. In practice, however, our simulations find the observed sum to be unusual at roughly the 99% level regardless of whether we compare to the dodecahedral topology or a simply connected space.

4. Discussion

The present paper asks whether the dodecahedral topology might explain the quadrupole–octupole alignment. To obtain any hope of a positive answer, the distribution in figure 1 would need to be heavily skewed to the right. The fact that it is flat eliminates all hope that this topology might explain the alignment.

The distribution in figure 1 is not just approximately flat, but it is astonishingly flat. Thus, it is inconceivable that the ‘true’ distribution is heavily skewed to the right and that our model’s simplifications have somehow contrived to flatten it so perfectly.

In practical terms, the results of this paper are fully and easily reproducible: one may start with any plausible set of cosmological parameters and use any simulation (no matter how crude!) and confirm the basic results of the paper, namely that

- the distribution in figure 1 is very nearly flat and, therefore,
- the dodecahedral topology does nothing to explain the quadrupole–octupole alignment.

Aurich et al have recently posted a more comprehensive paper [13], based on more detailed simulations, that confirms these results for the dodecahedral space while finding analogous—but slightly different—results for a variety of other topologies.

5. Conclusion

The Poincaré dodecahedral space topology, while explaining the weakness of the low-\(\ell\) modes, completely fails to explain the quadrupole–octupole alignment. While this negative result might dash any hope for a topological explanation of the alignment, good scientific practice demands that one analyse a few other plausible topologies before reaching any firm conclusion about whether topology might play a role\(^2\).

The quadrupole–octupole alignment is most likely due not to topology but rather to foreground contamination and in particular to point sources, as Prunet et al argue in detail [15].\(^3\)

\(^2\) JW thanks a referee for pointing out a paper by Cresswell et al [14] that studies slab spaces and finds that they too fail to explain the alignment. The latest paper by Aurich et al [13] considers a broader selection of spaces and finds that even though the alignment properties differ from space to space, none can fully account for the observed alignment.

\(^3\) JW thanks a referee for pointing this out.
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