The discrete Toda equation revisited: dual $\beta$-Grothendieck polynomials, ultradiscretization, and static solitons

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Received 30 October 2017, revised 9 February 2018
Accepted for publication 9 February 2018
Published 26 February 2018

Abstract
This paper presents a study of the discrete Toda equation that was introduced in 1977. In this paper, it is proved that the determinantal solution of the discrete Toda equation, obtained via the Lax formalism, is naturally related to the dual Grothendieck polynomials, a $K$-theoretic generalization of the Schur polynomials. A tropical permanent solution to the ultradiscrete Toda equation is also derived. The proposed method gives a tropical algebraic representation of the static solitons. Lastly, a new cellular automaton realization of the ultradiscrete Toda equation is proposed.

Keywords: discrete Toda equation, determinant solutions, dual Grothendieck polynomial, ultradiscrete systems, cellular automaton

(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Determinantal solution to the discrete Toda equation

The Toda equation

$$\frac{d}{dt^2} \log(1 + V_n(t)) = V_{n+1}(t) - 2V_n(t) + V_{n-1}(t)$$

was proposed as the model equation of dynamics of a one-dimensional lattice with the nearest neighbor interaction [18]. Today, the equation is known as a good example of an integrable

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1 There exist at least two ‘discrete Toda equations’, which are famous and well-investigated (to the best of our knowledge). See remark 1.1.
system owing to its rich structure. In this paper, we intend to study the discrete Toda equation\(^1\), which was originally given by Hirota \([3]\) as follows:

\[
\frac{u_n^{t-1} u_n^{t+1}}{(u_n^t)^2} = \frac{(1 - \delta^2 + \delta^2 u_{n-1}^t)(1 - \delta^2 + \delta^2 u_{n+1}^t)}{(1 - \delta^2 + \delta^2 u_{n}^t)^2}. \tag{1}
\]

Through the variable transformation

\[
u_n^t = \frac{1 - \delta^2 \tau_n^t + \delta^2 \tau_{n+1}^t}{\delta^2 (\tau_{n+1}^t)^2},
\]

we can obtain the bilinear equation

\[
(\tau_n^t)^2 + \tau_{n-1}^t \tau_{n+1}^t = \tau_{n-1}^t \tau_{n+1}^t \tag{2}
\]

from (1). Moreover, if we define two new variables given by

\[
a_n^t := \frac{\tau_{n+1}^t}{\tau_n^t}, \quad b_n^t := \frac{\tau_{n+2}^t}{\tau_{n+1}^t}, \tag{3}
\]

(2) can be rewritten as

\[
a_{n+1}^t + b_{n-1}^t = a_n^t + b_n^t, \quad a_{n+1}^t b_n^t = a_n^t b_{n+1}^t. \tag{4}
\]

One can recover (1) from (4) by using the following relation:

\[
u_n^t = \frac{1 - \delta^2 b_n^t}{\delta^2 a_n^t}. \tag{5}
\]

We consider (4) with the boundary condition

\[
a_0^t = a_{N+1}^t = 1, \quad b_0^t = b_N^t = 0, \tag{6}
\]

for some integer \(N > 0\). Let

\[
X' = \begin{pmatrix} a_1' & 1 & & & \\ a_2' & & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_N' \end{pmatrix}, \quad Y' = \begin{pmatrix} 1 & & & & \\ -b_1' & 1 & & & \\ & \ddots & \ddots & \ddots & \\ & & -b_N' & 1 & \end{pmatrix}
\]

be \(N \times N\) matrices. Equation (4) admits the discrete Lax formulation given as follows:

\[
X'^{t+1} Y' = Y'^{t+1} X'. \tag{7}
\]

By introducing \(L' := (Y')^{-1} X'\), we can rewrite (7) as

\[
Y'L' = L'^{t+1} Y' \quad \text{or} \quad X'L' = L'^{t+1} X'. \tag{8}
\]

Similar to other classical integrable systems (see, for example, \([4, 5, 9, 17]\)), one can construct solutions for arbitrary initial values (values at time \(t = 0\)) of the discrete Toda equation via the Lax representation (8). These solutions are expressed as a ratio of determinants, the elements of which are algebraic functions of the initial values. Hereafter, the term ‘determinantal solution’ means a solution of this type.

Remark 1.1. There exist at least two famous ‘discrete Toda equations’. One is (1), which we investigate in this study, and the other is expressed in the following bilinear form:
\[(\tau_n^t)^2 + \tau_{n-1}^{t+1} = \tau_n^{t+1} \tau_n^t - 1. \tag{9}\]

(Compare with (2)).

1.2. Determinantal solution and dual Grothendieck polynomials

Obtaining the determinantal solution to the discrete Toda equation, through the use of the Lax formulation (8), with an arbitrary (generic) initial condition is relatively straightforward. (See sections 2.1–2.3 for details). If all eigenvalues of the Lax matrix \(L^t\) degenerate to one value (as (27) in section 2.4), there exists a natural algebraic relation between these solutions and the dual Grothendieck polynomials, which are essentially \(K\)-theoretic analogues of the Schur polynomials [10]. Interestingly, the ‘other’ discrete Toda equation (9) corresponds to the conventional Schur polynomials (remark 2.4). (For recent developments concerning the Grothendieck polynomials and generalizations, see [2, 19], for example).

Recently, several researchers have reported interesting relations between ‘\(K\)-theoretic’ objects and classical integrable systems. Motegi and Sakai [13] discovered a remarkable relation between Grothendieck polynomials and determinantal solutions to certain integrable systems (TASEP). Very recently, Ikeda, Iwao, and Maeno [8] constructed a ring isomorphism between the quantum \(K\)-theory of the complex flag variety \(F_{\mathfrak{fl}}\) and the \(K\)-theory of the affine Grassmannian \(\text{Gr}_{SL_n}\) by using the relativistic Toda system. From this perspective, the discrete Toda equation (1) can be regarded as one of these ‘\(K\)-theoretic’ objects.

1.3. Ultradiscretization

Another topic discussed in this paper is ultradiscretization. Let us introduce the transformations \(u_n^t = e^{U_n^t}\) and \(\delta = e^{-L^t}\) with a parameter \(\varepsilon > 0\) and a positive constant \(L > 0\) for (1). Then, by applying \(\varepsilon \log\) to both sides and taking the limit \(\varepsilon \to 0^+\), we obtain the ultradiscrete Toda equation [12] given by

\[U_{n+1}^t - 2U_n^t + U_{n-1}^t = \max[0, U_{n+1}^t - L] - 2 \max[0, U_n^t - L] + \max[0, U_{n-1}^t - L]. \tag{10}\]

In [12], Matsukidaira et al derived soliton solutions to (10) by ultradiscretizing those to the discrete Toda equation (1). Their solution is expressed by

\[U_n^t = T_{n+1}^t - 2T_n^t + T_{n-1}^t, \]

\[T_n^t = \max_{\mu \in \{0,1\}} \left[ \sum_{j=1}^N \mu_j S_j(t,n) - \sum_{1 \leq i < j \leq N} \mu_i \mu_j (P_i + \sigma_i \sigma_j Q_j) \right], \]

\[S_j(t,n) = P_j n - \sigma_j Q_j t + C_j, \]

\[0 \leq P_1 \leq P_2 \leq \cdots \leq P_N, \quad Q_j = \max[0, P_j - L], \quad \sigma_j \in \{-1, 1\}. \tag{11}\]

Here \(N\) and \(L\) are positive integers and \(P_j\) and \(C_j\) are arbitrary parameters. The operator \(\max_{\mu \in \{0,1\}} f(\mu_1, \mu_2, \ldots, \mu_N)\) denotes the maximum value amongst \(2^N\) possible values of \(f(\mu_1, \mu_2, \ldots, \mu_N)\) obtained by replacing each \(\mu_i\) by 0 or 1. It is known that the solution \(U_n^t\) defined by (11) possesses properties of solitary waves and soliton interactions [12]. It can be verified that \(U_n^t\) always takes non-negative values. On the other hand, Hirota proposed another solution to (10), which is called ‘the static soliton’ [6], and is expressed by
where \( C \) is a positive parameter and \( j_0 \leq j_1 \) are integers. One can verify that \( U_n' \) (12) may now take negative values. It is natural to expect the existence of solutions to the discrete Toda equation (4), the ultradiscretization of which yields the static solitons. (Here, we would like to note that Hirota [6] showed there are no time-independent solutions to the discrete Toda equation other than the trivial solution).

In this paper, we construct tropical permanent solutions for arbitrary initial values of the ultradiscrete Toda equation. Our solutions include as special cases (11) and (12). The solutions to (10) are expressed by the tropical tau functions \( T_n^t \) as

\[
U_n' = A_n' - B_n' + L, \quad A_n' = T_n' + T_{n+1}' - T_{n+2}' - T_n' - T_{n+1}' + T_{n+2}', \quad B_n' = T_n' + T_{n+1}' + T_{n+2}' - T_{n+1}' - T_{n+2}' + T_n'.
\]

(See section 3.2 for details). The new dependent variables \( A_n', B_n' \) represent the ultradiscretization of \( a_n, b_n \), which satisfy the evolution equation (30). A new cellular automaton realization of the system \( \{A_n', B_n'\} \) is proposed in section 3.3.

### 1.4. Organization of the paper

In section 2, we provide a determinantal solution to the discrete Toda equation (4) with the boundary condition (6) using the Lax formulation. Although this sequence of calculations is an established practice, we provide details in sections 2.1–2.3 in order to render this paper self-contained. In section 2.4, dual Grothendieck polynomials, as special solutions to the discrete Toda equation (1), are derived.

In section 3, the tropical permanent solution to the ultradiscrete Toda equation, obtained by ultradiscretizing the determinantal solution defined in the previous section, is provided. The solution realizes the behavior of the solution given in [6, 12]. A new cellular automaton realization of the system is proposed in section 3.3. A concrete example and concluding remarks are contained in section 4.

### 2. Solution to the discrete Toda equation

#### 2.1. Lax formulation and spectrum problem

A determinantal solution to the discrete Toda equation via the Lax formulation (8) is derived as follows. Let

\[
f(\lambda) := \det(\lambda E - L') = \lambda^N - I_1 \lambda^{N-1} + \cdots + (-1)^N I_N
\]

be the characteristic polynomial of \( L' \), which is \( r \)-invariant due to (8). We define the \( \mathbb{C} \)-algebra \( \mathcal{O} = \mathbb{C}[\lambda]/(f(\lambda)) \), which is \( N \)-dimensional, as a \( \mathbb{C} \)-vector space. At the same time, the spectral problem

\[
L' \psi' = \lambda \psi', \quad (\psi' \in \mathbb{C}^N),
\]

which is equivalent to

\[
(\lambda Y' - X') \psi' = 0
\]

is considered. We denote the \((i,j)\)th minor of the matrix \( \lambda Y' - X' \) by \( M_{ij} \) and define
Then it is easily verified that the vectors
\[ p := (\Delta_{N,1}, \Delta_{N,2}, \cdots, \Delta_{N,N})^T, \quad q := (\Delta_{1,1}, \Delta_{1,2}, \cdots, \Delta_{1,N})^T \]
possess the following properties:

(1) Property A
- The \(i\)th entry of \(p\), which is denoted by \(p_i\), is a monic polynomial of degree \((i - 1)\) in \(\lambda\).
- The \(i\)th entry of \(q\), which is denoted by \(q_i\), is of the form \(\lambda^{N-1} \times \text{polynomial of degree } (N - i)\) in \(\lambda^{-1}\).
- If \(\lambda\) is a root of \(f(\lambda)\), both \(p\) and \(q\) are eigenvectors of \(L^t\). In other words, \(p\) and \(q\) are solutions of the spectral problem (13).

(2) Property B
- The constant term of \(p_i\) is \((-1)^{i-1}a_1a_2' \cdots a_{i-1}'\).
- The coefficient of \(\lambda^{N-1}\) of \(q_i\) is \(b_1b_2' \cdots b_{i-1}'\).

Then, it can be directly proved that as a matrix over \(O\), the co-rank of \(\lambda Y^t - X^t\) is 1. Hence, the eigenvector of \(L\) must be unique up to a constant multiple. In terms of \(O\), we have the following lemma.

**Lemma 2.1.** Let \(\mathbb{C}[\lambda] \rightarrow O\) be the natural surjection and \(p, q \in O^N\) be the image of \(p, q \in \mathbb{C}[\lambda]^N\). Thus, there exists some \(F \in O\) such that
\[ q = -F \cdot p. \quad (15) \]
(The minus sign is used for convenience below).

Now, let us consider the inverse problem; for a given \(F \in O\), recover \(p\) and \(q\) with (15).

The answer is as follows. Equation (15) is rewritten as
\[
\left( \begin{array}{cccc}
* & * & \cdots & \beta_1 \\
* & \cdots & \beta_2 \\
& & \cdots & \\
\beta_N & & & \cdots & 1 \\
\end{array} \right)
\left( \begin{array}{c}
1 \\
\alpha_2 \\
\vdots \\
\alpha_N \\
\end{array} \right)
\equiv 0 \quad \text{mod } f(\lambda)
\]
(16)

Each \(*\) is a coefficient of \(p_i\) or \(q_i\) as a polynomial in \(\lambda\). Then we have
\[
\alpha_i = (-1)^{i-1}a_1a_2' \cdots a_{i-1}', \quad \beta_i = b_1b_2' \cdots b_{i-1}'.
\]
By applying Cramer’s rule to (16), one can express each entry of the matrix as a ratio of determinants. Let \(c: O \rightarrow \mathbb{C}^N\) be an arbitrary linear isomorphism. Then we have
\[\alpha_i = (-1)^{i-1} \left| \frac{e(\lambda_i^{(-1)}), e(\lambda_i), \ldots, e(\lambda_i^{N-1}), e(F\lambda_i), e(F\lambda_i^2), \ldots, e(F\lambda_i^{L-1})}{e(\lambda_i^{(-1)}), e(\lambda_i), \ldots, e(\lambda_i^{N-1}), e(F), e(F\lambda_i), \ldots, e(F\lambda_i^{L-2})} \right| \quad (18)\]

and

\[\beta_i = (-1)^{i-1} \left| \frac{e(\lambda_i^{(-1)}), e(\lambda_i), \ldots, e(\lambda_i^{N-2}), e(F), e(F\lambda_i), \ldots, e(F\lambda_i^{L-2})}{e(\lambda_i^{(-1)}), e(\lambda_i), \ldots, e(\lambda_i^{N-1}), e(F\lambda_i), e(F\lambda_i^2), \ldots, e(F\lambda_i^{L-2})} \right| \quad (19)\]

Note that these expressions are invariant under any exchange of \(c\). As long as the denominators are not 0, they recover values of \(d_n\) and \(b_n\).

**Remark 2.2.** The expressions (18) and (19) are also invariant under the transformation \(F \mapsto cF\) (\(c \in \mathbb{C}^\times\)). Thus, we may assume \(F\) to be an element of \(\mathcal{O}/\mathbb{C}^\times\) without loss of generality.

In the remainder of this section, we denote \(F = F_t\), \(p = p'\) and \(q = q'\), etc, to emphasize the \(t\)-dependencies of these quantities. From the discrete Lax equation (8) and (13), we have a spectral problem at time \(t + 1\) given by

\[L_t^{(t+1)}(X') = \lambda(X'), \quad L_t^{(t+1)}(Y') = \lambda(Y'). \quad (20)\]

Let \(p' := Y'p\) and \(q' = X'q\). From the concrete form of \(X'\) and \(Y'\) as well as property A, the pair \((p', q')\) also satisfies property A. Because any pair of vectors with property A uniquely restores the Lax matrix \(L\) with property B, we have

\[p^{t+1} = Y'p', \quad q^{t+1} = \theta X'q', \quad (\exists \theta \in \mathbb{C}^\times). \]

Substituting these relations into (15), we obtain

\[\theta X'q' = -F^{t+1}Y'p', \]

which implies

\[\theta |X'q'| = \theta |Y'|^{-1}X'q' = -F^{t+1}p'. \]

Through comparison with (15), we finally derive

\[F^{t+1} = \lambda \theta \cdot F^t. \quad (21)\]

Although this equation contains the unknown constant \(\theta\), expression (21) still determines the time evolution of \(L^t\) without ambiguity. See remark 2.2.

### 2.2. The determinantal formula for the tau function

From (21) and remark 2.2, one may identify \(F = F^t\) with \(\lambda^t F^0\) without a problem. Hereafter, we assume any root of \(f(\lambda)\) to be non-zero. Let \(M_\lambda : \mathbb{C}^N \rightarrow \mathbb{C}^N\) represent the \(\mathbb{C}\)-linear map

\[\mathbb{C}^N \xrightarrow{e} \mathcal{O} \xrightarrow{\lambda} \mathcal{O} \xrightarrow{e} \mathbb{C}^N.\]

By assumption, this is invertible. Let \(D = \det M_\lambda \neq 0\). The numerator of (18) can be rewritten as

\[\left| e(\lambda_i^{(-1)}), e(\lambda_i^{(-1)}), \ldots, e(\lambda_i^{N-1}), e(F\lambda_i), e(F\lambda_i^2), \ldots, e(F\lambda_i^{L-1}) \right| = D \cdot \left| e(\lambda_i^{(-1)}), e(\lambda_i^{(-1)}), \ldots, e(\lambda_i^{N-2}), e(F), e(F\lambda_i), \ldots, e(F\lambda_i^{L-2}) \right|.\]

By putting
\( \tau_n^c := |c(\lambda_{n-1}), c(\lambda_n), \ldots, c(\lambda_N), c(F^0\lambda'), c(F^0\lambda^{t+1}), \ldots, c(F^0\lambda^{t+n-2})|, \)

we obtain

\[
\begin{align*}
\alpha_n^t &= \frac{x_n^t + 1}{\tau_n} \lambda_n^t + 1, \\
\beta_n^t &= -\frac{x_n^t + 1}{\tau_n} \lambda_n^t + 1
\end{align*}
\]

from (17).

2.3. Double Casorati determinant

By choosing a specific isomorphism \( \mathbf{c} : \mathcal{O} \to \mathbb{C}^N \), one can derive an explicit formula for the tau function. One typical example is the double Casorati determinant formula, which is given below.

Assume that all eigenvalues of \( f(\lambda) \) are distinct. Thus,

\[
f(\lambda) = \prod_{i=1}^N (\lambda - \lambda_i), \quad (i \neq j \Rightarrow \lambda_i \neq \lambda_j).
\]

By the Chinese remainder theorem, the following map represents a linear isomorphism.

\[
\mathbf{c} : \mathcal{O} \to \mathbb{C}^N; \quad \varphi(\lambda) \mod f(\lambda) \mapsto (\varphi(\lambda_1), \ldots, \varphi(\lambda_N)).
\]

For this \( \mathbf{c} \), the tau function \( \tau_n^c \) can be expressed as

\[
\tau_n^c = \begin{vmatrix}
\lambda_1^{t-1} & \lambda_2^t & \ldots & \lambda_N^{t-1} & f_1 & f_1 & \ldots & f_1 & f_1 \\
\lambda_2^{t-1} & \lambda_2^t & \ldots & \lambda_N^{t-1} & f_2 & f_2 & \ldots & f_2 & f_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_N^{t-1} & \lambda_N^t & \ldots & \lambda_N^{t-1} & f_N & f_N & \ldots & f_N & f_N
\end{vmatrix},
\]

where \( \mathbf{c}(F^0) = (f_1, \ldots, f_N)^T \). By applying the Laplace expansion along the columns between the 1st and \( (N - n + 1) \)th positions, we obtain

\[
\tau_n^c = \sum_{\{i_1, \ldots, i_n\} \cup \{j_1, \ldots, j_{n-1}\} = [N]} (-1)^{\varepsilon} \prod_{i \in j} \lambda_i^{t-1} \cdot \prod_{i \in i_1} f_i \cdot \prod_{\{i \in j \} \subset j} (\lambda_{i_1} - \lambda_{i_2}) \cdot \prod_{\{i \in j \} \subset j} (\lambda_{i_{n-1}} - \lambda_{i_n}),
\]

where \( \varepsilon = i_1 + \cdots + i_n + 1 + \frac{(2N-n+2)(n-1)}{2} \).

2.4. The dual stable Grothendieck polynomials

In some special cases, the tau function is naturally related to the dual stable Grothendieck polynomial. Let us consider the most degenerate case, where

\[
f(\lambda) = (\lambda - \gamma)^N, \quad (\gamma \neq 0).
\]

(This assumption is valid in section 2.4 only). The \( \mathbb{C} \)-algebra \( \mathcal{O} \), therefore, is expressed as

\[
\mathcal{O} = \mathbb{C}[\lambda]/((\lambda - \gamma)^N) = \mathbb{C}[\mu]/(\mu^N),
\]

where \( \mu := \lambda - \gamma \). The \( \mathbb{C} \)-valued functions \( c_0, \ldots, c_{N-1} \) over \( \mathcal{O} \) can be defined so that they satisfy

\[
\begin{align*}
\end{align*}
\]
Moreover, we introduce the polynomial ring over \( \mathcal{O} \) denoted by \( \mathbb{C}[\mathcal{O}] = \mathbb{C}[c_0, \cdots, c_{N-1}] \).

Next, we set \( \beta := \gamma^{-1} \). Let
\[
\tau_n = |c(\lambda^{n-1}), c(\lambda^n), \cdots, c(\lambda^{N-1}), c(F\lambda), c(F\lambda^2), \cdots, c(F\lambda^{n-1})|
\]
be the \( n \)-th tau function corresponding to \( F \in \mathcal{O} \), which can be rewritten as follows:
\[
\tau_n = D^{n-1} \left| c(1), c(\lambda), \cdots, c(\lambda^{N-n}), c(F(\gamma - \lambda)^{n-2}), \cdots, c(F\lambda^{n-1}), c(F) \right|
\]
\[
= D^{n-1} \left| c(1), c(\mu + \gamma), \cdots, c((\mu + \gamma)^{N-n}), c(F(\gamma(\mu + \gamma))^{n-2}), \cdots, c(F(\mu - \gamma(\mu + \gamma)), c(F) \right|
\]
\[
= D^{n-1} \beta^{\frac{(n-1)(n-2)}{2}} \left| c(1), c(\mu), \cdots, c(\mu^{N-n}), c(F(\mu - \gamma(\mu + \gamma)), \cdots, c(F(\mu - \gamma(\mu + \gamma))^{n-2}) \right|.
\]

We fix the linear isomorphism \( c : \mathcal{O} \to \mathbb{C}^N \) as
\[
c := c_0e_1 + c_1e_2 + \cdots + c_{N-1}e_N, \quad e_i = (0, \cdots, 1, \cdots, 0)^T.
\]
Thus, we have
\[
c(c^{\mu-1}) = e_i, \quad c(F(\mu - \gamma(\mu + \gamma))^{p-1}) = (\kappa_{p,1}, \cdots, \kappa_{p,N})^T,
\]
where
\[
\kappa_{p,q} = \sum_{i=0}^{\infty} \left( \frac{1}{i} \right) \gamma^p c_{q-p-1}(F) = \beta^{p-1} \sum_{i=0}^{\infty} \left( \frac{1}{i} \right)^{\beta} c_{q-p-1}(F).
\]

The tau function \( \tau_n \), as an element of \( \mathbb{C}[\mathcal{O}] \), can be expressed as
\[
\tau_n = D^{n-1} \beta^{\frac{(n-1)(n-2)}{2}} \left| e_1, e_2, \cdots, e_{N-n+1}, c(F), c(F(\mu - \gamma(\mu + \gamma))^{n-2}) \right|
\]
\[
= D^{n-1} \beta^{\frac{(n-1)(n-2)}{2}} \det(\kappa_{p,N-n+1+q})^{n-1}_{p,q=1}
\]
\[
= D^{n-1} \det \left( \sum_{i=0}^{\infty} \left( \frac{1}{i} \right)^{\beta} c_{N-n+1+q-p-1}(F) \right)^{1 \leq p, q \leq n-1}.
\]

Under natural identification\(^2\) \( c_i/c_0 \leftrightarrow h_i \), where \( h_i = h_i(x_1, x_2, \cdots) \) is the \( i \)-th complete symmetric polynomial in infinitely many variables \( x_1, x_2, \cdots \), the tau function \( \tau_n \) is proportional to
\[
\det \left( \sum_{i=0}^{\infty} \left( \frac{1}{i} \right)^{\beta} h_{N-n+1+q-p-1} \right)^{1 \leq p, q \leq n-1}.
\]

\(^2\)The identification \( c_i/c_0 \leftrightarrow h_i \) (or \( c_i/c_0 \leftrightarrow (-1)^i h_i \)) appears in [8] in order to relate the geometrical information of \( F_{\mu} \) to the symmetric polynomials. The origin of this technique dates back to Fulton’s historic work [1, Part III].

On the other hand, this identification can be understood in terms of the Boson–Fermion correspondence. See [8, section 6].
In accordance with [11, 16], one finds that this expression exactly coincides with the Jacobi–Trudi type formula for the dual $\beta$-Grothendieck polynomial

$$g^{(\beta)}_{R_{n-1}},$$

where $R_k$ is the Young diagram $R_k = ((N - k)k)$.

**Remark 2.3.** The dual $\beta$-Grothendieck polynomial reduces to the Schur polynomial when $\beta = 0$: $g^{(0)}_{R_k} = s_{R_k}$.

**Remark 2.4.** By a similar method, one can derive the tau function for the ‘other’ discrete Toda equation (9). In fact, it is described as

$$\tau_n = \left| b(1), b(\lambda), \ldots, b(\lambda^{N-n}), b(G), b(G\lambda), \ldots, b(G\lambda^{n-2}) \right|, \quad (28)$$

where $G \in O' := \mathbb{C}[\lambda]/(g(\lambda))$, $g(\lambda)$ is the characteristic polynomial of the Lax matrix of (9) (see, for example, [7]), and $b : O' \to \mathbb{C}^N$ is an arbitrary linear isomorphism. Note that (28) is invariant under the transformation $\lambda \mapsto \lambda + \gamma$, while (22) is not. This implies that one cannot derive the dual Grothendieck polynomial from this expression. In fact, the tau function (28) is naturally related to the determinant

$$\det(h_{N-n+1+q-p})_{1 \leq p, q \leq n-1},$$

which is the Jacobi–Trudi formula for the Schur polynomial $s_{R_{n-1}}$.

### 3. Solutions to the ultradiscrete Toda equation

#### 3.1. New ultradiscrete evolution equation

Equation (4) is equivalent to the following:

$$a_{n+1}^{t+1} = \frac{a_n^{t} + b_n^{t+1} - a_n'}{a_n' + b_n'}, \quad b_{n+1}^{t+1} = \frac{a_n^{t} + b_n^{t+1} - b_n'}{a_n' + b_n'}. \quad (29)$$

Setting $a_n' = e^{-\frac{A_n'}{\varepsilon}}$, $b_n' = e^{-\frac{B_n'}{\varepsilon}}$, and taking the limit $\varepsilon \to 0^+$, one derives the ultradiscrete evolution equation given by

$$\begin{cases}
A_{n+1}^{t+1} = \min[A_{n+1}^{t}, B_{n+1}^{t}] - \min[A_n^{t}, B_n^{t}] + A_n', \\
B_{n+1}^{t+1} = \min[A_{n+1}^{t}, B_{n+1}^{t}] - \min[A_n^{t}, B_n^{t}] + B_n', \\
A_0 = A_N = 0, \quad B_0 = B_N = +\infty. \quad (30)
\end{cases}$$

**Proposition 3.1.** Let $U_n^t := A_n^t - B_n^t + L$. ($U_0^t = U_N^t := -\infty$). Then, (30) refers to the ultradiscrete Toda equation (10).

**Proof.** We set $\Delta_n := \max[0, U_n^t - L] - \max[0, U_{n-1}^t - L]$. Equation (30) is equivalent to

$$\Delta_n = A_n^{t+1} - A_n^{t}, \quad A_{n+1}^{t+1} - A_n^{t+1} = B_n^{t+1} - B_n^{t}. \quad (31)$$

Therefore, we have

3 Note the obvious relation $\max[X, Y] = -\min[-X, -Y]$. 


\[ U_{n+1} - 2U_n + U_{n-1} = A_n^{t+1} - 2A_n + A_n^{t-1} - B_n^{t+1} + 2B_n - B_n^{t-1} \]
\[ = A_n^{t+1} - 2A_n + A_n^{t-1} - (A_{n+1}^{t+1} - A_n) + (A_{n+1}^{t-1} - A_n^{t-1}) \]
\[ = \Delta_{n+1} - \Delta_n, \]

which implies the desired result. \(\square\)

**Remark 3.2.** Only when \(U_1^t \leq L\) can one recover \(A_n^t\) and \(B_n^t\) from \(U_n^t\), using the formula
\[ A_n^t = \sum_{k=1}^{n-1} (U_{n-k}^{t-1} - U_{n-k}^t) + C, \quad B_n^t = L - U_n^t + \sum_{k=1}^{n-1} (U_{n-k}^{t-1} - U_{n-k}^t) + C, \]

where \(C\) is an arbitrary number. In fact, we have \(A_n^t - A_n^{t+1} = \sum_{k=1}^{n-1} (U_{n-k}^{t+1} - 2U_{n-k}^t + U_{n-k}^{t-1}) = \sum_{k=1}^{n-1} (\Delta_{n-k+1} - \Delta_{n-k}) = \Delta_a - \Delta_1 = \Delta_0\), where the last equality follows from \(U_1^t \leq L \Rightarrow \Delta_1 = 0\).

### 3.2. Tropical permanent solution

In this section, we derive a tropical permanent solution to (30). Let \(P = (p_{ij})_{1 \leq i,j \leq N}\) \((p_{ij} \in \mathbb{R} \cup \{\infty\})\) be an \(N \times N\) matrix. The tropical permanent \(\text{TP}[P]\) is an element of \(\mathbb{R} \cup \{\infty\}\) defined by
\[ \text{TP}[P] := \min_{\sigma \in \mathcal{S}_N} \left[ p_{1,\sigma(1)} + p_{2,\sigma(2)} + \cdots + p_{N,\sigma(N)} \right]. \]

We start with the determinantal solution (25) to the discrete Toda equation, where we define the tropical permanent \(T_n^t\) associated with \(\tau_n^t\) by
\[
T_n^t := \text{TP} \left| \begin{array}{cccc}
(n-1)\Lambda_1 & n\Lambda_1 & \cdots & (N-1)\Lambda_1,
(n-1)\Lambda_2 & n\Lambda_2 & \cdots & (N-1)\Lambda_2,
\vdots & \vdots & \ddots & \vdots,
(n-1)\Lambda_N & n\Lambda_N & \cdots & (N-1)\Lambda_N,
F_1 + t\Lambda_1 & F_1 + (t+1)\Lambda_1 & \cdots & F_1 + (t+n-2)\Lambda_1,
F_2 + t\Lambda_2 & F_2 + (t+1)\Lambda_2 & \cdots & F_2 + (t+n-2)\Lambda_2,
\vdots & \vdots & \ddots & \vdots,
F_N + t\Lambda_N & F_N + (t+1)\Lambda_N & \cdots & F_N + (t+n-2)\Lambda_N
\end{array} \right|.
\]

(31)

**Proposition 3.3.** Let \(a_n^t = a_n^t(\varepsilon), b_n^t = b_n^t(\varepsilon)\) be real analytic functions of \(\varepsilon > 0\) with
\[ \varepsilon \ll 1 \quad \Rightarrow \quad a_n^t, b_n^t > 0 \]

and
\[ \lim_{\varepsilon \to 0^+} \varepsilon \log a_n^t = A_n^t, \quad \lim_{\varepsilon \to 0^+} \varepsilon \log b_n^t = B_n^t. \]

The following two claims, therefore, hold:
(1) All eigenvalues of the Lax matrix $L'$ are distinct and positive.

$$f(\lambda) = \prod_{i=1}^{N}(\lambda - \lambda_i), \quad \epsilon \ll 1 \Rightarrow 0 < \lambda_N < \cdots < \lambda_1.$$  

(2) Under the linear isomorphism (24), the image of $F^t \in \mathcal{O}$ (section 2.1) satisfies

$$\epsilon \ll 1 \Rightarrow (-1)^n f_n^t > 0, \quad \text{where} \quad c(F^t) = (f_1^t, \cdots, f_n^t)^T.$$  

Proof. (1) The distinctness and positivity of the eigenvalues are direct consequences of the fact that $L'$ is a totally non-negative and irreducible matrix (if $\epsilon \ll 1$). One may refer to the textbook [15, section 5]. (2) Comparing the 1st components on both sides of (15), we have $F^t = -\det(\lambda Y^t - X^t)_{1,1}$, where $M_{ij}$ is the $(i,j)$th minor of the matrix $M$. Using the Cauchy–Binet formula, we derive

$$F^t = -\det((\lambda E - X^t(Y^t)^{-1}) \cdot Y^t)_{1,1} = -\det(\lambda E - X^t(Y^t)^{-1})_{1,1} \cdot \det Y_{1,1} = -\det(\lambda E - L^t)_{1,1}.$$  

(The second equality follows from the fact that $Y^t$ is the lower triangle). Generally, it is known [15, section 5.3] that for any totally non-negative and irreducible matrix $A$, the principal minor $p(\lambda) = \det(\lambda E - A)_{1,1}$ satisfies $p(\lambda_1) > 0$, $p(\lambda_2) < 0$, $p(\lambda_3) > 0$, \ldots, where $0 < \lambda_N < \cdots < \lambda_1$ are eigenvalues of $A$. The claim naturally follows from the fact that $L'$ is totally non-negative and irreducible.

Proposition 3.4. Under the assumptions in proposition 3.3, we write $\Lambda_i = -\lim_{\epsilon \to 0^+} \epsilon \log \lambda_i$ and $F_n^t = -\lim_{\epsilon \to 0^+} \epsilon \log |f_n^t|$. Let $\tau_n^t$ be the tau function (25) and $T_n^t$ be the tropical permanent (31). If $\lambda_1, \cdots, \lambda_N$ satisfy the condition$^4$

$$-\lim_{\epsilon \to 0^+} \epsilon \log |\tau_n^t| = T_n^t,$$

for any $\lambda_i, \lambda_j (i \neq j)$, we obtain

$$-\lim_{\epsilon \to 0^+} \epsilon \log |\tau_n^t| = T_n^t.$$  

(This implies that the ultradiscretization of $\tau_n^t$ coincides with $T_n^t$).

Proof. From (26) and proposition 3.3 (2), we have

$$|\tau_n^t| = \sum_{N^k=N-k \atop \|k\|=n-1} \prod_{j \in J} \lambda_j^{k_j} \cdot \prod_{i \in I} (\lambda_i - \lambda_j) \cdot \prod_{\{i < j\} \subseteq I} (\lambda_i - \lambda_j).$$  

Thus, we can derive

$$-\lim_{\epsilon \to 0^+} \epsilon \log |\tau_n^t| = \min_{N^k=N-k \atop \|k\|=n-1} \left[ \sum_{k=1}^{N-n+1-k} (N-k)A_{jk} + \sum_{l=1}^{n-1} (F_{il} + (t + n - 1 - l)A_{jk}) \right].$$  

$^4$If we take $a_n, b_n$ generically, this condition holds automatically.
using the formula

\[ \lim_{\varepsilon \to 0^+} \varepsilon \log \prod_{i < j \in I} (\lambda_i - \lambda_j) = \sum_{k=1}^{N-n+1} (N - n + 1 - k)\Lambda_i. \]

(This formula follows from condition (32)). On the other hand, by applying the Laplace expansion for the tropical permanent along the rows between the 1st and \((N - n + 1)\)th positions [14], we obtain

\[
T_n' = \min_{|I| = n-1} \left[ \text{TP} \begin{bmatrix}
(n-1)\Lambda_1 & n\Lambda_1 & \cdots & (N-1)\Lambda_1 \\
(n-1)\Lambda_2 & n\Lambda_2 & \cdots & (N-1)\Lambda_2 \\
\vdots & \vdots & \ddots & \vdots \\
(n-1)\Lambda_{N-n+1} & n\Lambda_{N-n+1} & \cdots & (N-1)\Lambda_{N-n+1} \\
F_{i_1} + t\Lambda_{i_1} & F_{i_1} + (t+1)\Lambda_{i_1} & \cdots & F_{i_1} + (t+n-2)\Lambda_{i_1} \\
F_{i_2} + t\Lambda_{i_2} & F_{i_2} + (t+1)\Lambda_{i_2} & \cdots & F_{i_2} + (t+n-2)\Lambda_{i_2} \\
\vdots & \vdots & \ddots & \vdots \\
F_{i_{n-1}} + t\Lambda_{i_{n-1}} & F_{i_{n-1}} + (t+1)\Lambda_{i_{n-1}} & \cdots & F_{i_{n-1}} + (t+n-2)\Lambda_{i_{n-1}} \\
\end{bmatrix} \right. \\
+ \text{TP} \begin{bmatrix}
\vdots & \vdots & \ddots & \vdots \\
F_{i_{n-1}} + t\Lambda_{i_{n-1}} & F_{i_{n-1}} + (t+1)\Lambda_{i_{n-1}} & \cdots & F_{i_{n-1}} + (t+n-2)\Lambda_{i_{n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix} \\
\left. = \min_{|I| = n-1} \sum_{k=1}^{N-n+1} (N-k)\Lambda_k + \sum_{l=1}^{n-1} (F_{i_l} + (t+n-1 - l)\Lambda_{i_l}) \right].
\]

Here, we use the formula

\[
\text{TP} \begin{bmatrix}
\Lambda_1 & 2\Lambda_1 & \cdots & N\Lambda_1 \\
\Lambda_2 & 2\Lambda_2 & \cdots & N\Lambda_2 \\
\vdots & \vdots & \ddots & \vdots \\
\Lambda_N & 2\Lambda_N & \cdots & N\Lambda_N \\
\end{bmatrix} = \sum_{k=1}^{N} (N - k)\Lambda_k,
\]

where \(\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_N\). This completes the proof.

\[ \square \]

Remark 3.5. Under the assumptions in proposition 3.3,

\[
da'_n = \frac{|r'_n||r'_{n+1}|}{|r_n||r_{n+1}|}, \quad db'_n = \frac{|r'_n||r'_{n+1}|}{|r'_n+1||r'_{n+1}|},
\]

(33)

holds. Thus, the relationships between \(T_n'\) and \(A'_n, B'_n\) are given by

\[
A'_n = T_n' + T_{n+1}' - T_{n+1}' - T_{n+2}' - T_{n+1}' - T_{n+1}', \quad B'_n = T_n' + T_{n+1}' - T_{n+2}' - T_{n+1}' - T_{n+1}'.
\]

(34)

3.3. Cellular automaton realization

The ultradiscrete evolution equation (30) can be realized in the form of a cellular automaton as follows. Consider \(N\) cells numbered from 1 to \(N\) (see figure 1). At time \(t \in \mathbb{Z}\), the \(n\)th cell contains \(A'_n\) ‘kickers’ and \(B'_n\) ‘balls’. The state at time \(t+1\) is obtained according to the following rules. • A kicker kicks one ball, if it exists, to the neighboring cell on the left. • A
kicker who has no ball to kick moves to the neighboring cell on the right. Figure 1 illustrates a typical example. The solution to the ultradiscrete Toda equation (10) associated with the above scenario is given in figure 2.

Another example is given in figure 3, where a traveling soliton and a static soliton [6] interact.

4. Example and concluding remarks

4.1. Example

As seen above, one can construct the tropical permanent solution to the ultradiscrete Toda equation for any initial state. The method we use here can be referred to as the tropical inverse scattering method. In this section, we demonstrate the method used to construct the tropical permanent solution through an example.

Let us consider the initial state \((N = 8, L = 1)\)

\[
(A_0^1, A_0^2, \ldots, A_0^8) = (1, 1, 1, 0, 0, 0, 0), \quad (B_0^1, B_0^2, \ldots, B_0^7) = (2, 2, 2, 4, 1, 1, 1).
\]

Let \(q := e^{-\frac{1}{4}}\). We set the initial values

\[
(a_0^0, a_0^1, \ldots, a_0^8) = (q, 2q, 3q, 4q, 1, 2, 3, 4), \quad (b_0^0, b_0^1, \ldots, b_0^7) = (q^4, q^7, q^5, q, q, q, q, q, q)
\]

for the discrete Toda equation. The characteristic polynomial of the Lax matrix \(L^0\) is calculated as

\[f(\lambda) = \sum_{i=0}^{8} (-1)^i I_i \lambda^{8-i},\]

where

\[I_0 = 1, \quad I_1 = 10 + 13q + 3q^2 + q^3, \quad I_2 = 35 + 115q + 96q^2 + 24q^3 + 10q^4 + 8q^5 + 2q^6, \]

\[I_3 = 50 + 368q + 605q^2 + 360q^3 + 102q^4 + 63q^5 + 41q^6 + 8q^7 + \cdots, \quad I_8 = 576q^4.
\]

For sufficiently small \(q > 0\), the roots \(0 < \lambda_2 < \cdots < \lambda_1\) of \(f(\lambda)\) can be expanded as
\[ \lambda_1 = 4 + 4q - 4q^2 + \cdots, \quad \lambda_2 = 3 + \frac{9q^2}{2} - \frac{81q^3}{4} + \cdots, \]
\[ \lambda_3 = 2 - 2q^2 + 6q^3 + \cdots, \quad \lambda_4 = 1 - q + \frac{3q^2}{2} + \cdots, \]
\[ \lambda_5 = 4q + 4q^2 - 4q^3 + \cdots, \quad \lambda_6 = 3q + \frac{9}{2}q^3 + \cdots, \quad \lambda_7 = 2q + \cdots, \quad \lambda_8 = q + \cdots. \]

(Higher-order terms have been omitted owing to space limitations. Note that to execute all calculations below, many higher-order terms are required. For example, \( \lambda_1 \) must be calculated up to the \( q^{13} \) term, the coefficient of which is \( \frac{27560920906072627}{11337408} \). For each \( \lambda_i \), \( f_i = -\det(\lambda_i Y - X)_{1,1} \) is calculated as follows:

\[ \ldots 31.2 \ldots \]
\[ \ldots 4.11 \ldots \]
\[ \ldots 13.2 \ldots \]
\[ \ldots 22.11 \ldots \]
\[ \ldots 31.2 \ldots \]
\[ \ldots 4.11 \ldots \]
\[ \ldots 13.2 \ldots \]
\[ \ldots 2211 \ldots \]
Therefore, we have

\[(\Lambda_1, \cdots, \Lambda_8) = (0, 0, 0, 0, 1, 1, 1, 1), \quad (F_1, \cdots, F_8) = (13, 12, 11, 10, 6, 5, 4, 3).\]

Substituting these data values to the tropical permanent \(T_t^N(31)\), we obtain:

\[
f_1 = -\frac{32}{3}q^3 + \cdots, \quad f_2 = \frac{9}{2}q^2 + \cdots, \quad f_3 = -4q^1 + \cdots, \quad f_4 = 6q^0 + \cdots, \quad f_5 = -256q^0 + \cdots, \quad f_6 = 108q^5 + \cdots, \quad f_7 = -96q^4 + \cdots, \quad f_8 = 144q^3 + \cdots.
\]

Further, substituting them into \(A_t^N = T_t^N + T_t^{n+1} - T_{n+1}^N, B_t^n = T_t^n + T_t^{n+1} - T_{n+1}^N - T_{n+1}^{n+1}, \) and \(U_t^n = L + A_t^n - B_t^n = 1 + A_t^n - B_t^n\), we obtain the solution to the ultradiscrete Toda equation (10), as represented in figure 4.

4.2. Concluding remarks

The primary objective of this study was to understand the algebraic structure and positivity of the discrete Toda equation with boundary conditions. The solution itself is constructed in a straightforward manner. Under certain natural identifications, it is possible to obtain a family of special solutions that correspond to the dual \(\beta\)-Grothendieck polynomials, which are the \(K\)-theoretic analogues of the Schur polynomials. One can infer that the discrete Toda equation is of ‘Grothendieck’ polynomial type (\(\simeq K\)-theoretical), while the other discrete Toda equation is of the ‘Schur’ polynomial type. This result is expected to clarify the deeper structures of these discrete Toda equations.

The ultradiscrete analogues of the Toda equation were also studied. It was proved that the ultradiscrete Toda equation reduces to a new evolution equation, which is the ultradiscretization
of the Lax formulation. The tropical permanent solutions were also given. Our result shows the correspondence between the determinant solution to the discrete Toda equation and the tropical permanent solution to the ultradiscrete Toda equation. Moreover, the cellular automaton realization of the Toda equation was proposed. The proposed method has the inherent advantage of being applicable to arbitrary initial values. In particular, we have generalized Hirota’s formula for static solitons.

Acknowledgments

One of the authors (SI) is partially supported by KAKENHI (26800062). We would like to thank Editage (www.editage.jp) for the English language editing. We are grateful to the anonymous referees for their many helpful suggestions and comments.

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