On the non-local hydrodynamic-type system and its soliton-like solutions

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Abstract
We consider a hydrodynamic system of balance equations for mass and momentum. This system is closed by the dynamic equation of state, taking into account the effects of spatio-temporal non-localities. Using group theory reduction, we obtain a system of ODEs, describing a set of approximate traveling wave solutions to the source system. The factorized system, containing a small parameter, proves to be Hamiltonian when the parameter is zero. Using Melnikov’s method, we show that the factorized system possesses, in general, a one-parameter family of homoclinic loops, corresponding to the approximate soliton-like solutions of the source system.

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1. Introduction
In natural science, there exists a number of examples of the creation and stable evolution of so-called coherent states, or spatio-temporal patterns [1–3]. The location and specification of such patterns within the confines of adequate mathematical models are quite difficult, because they are usually described by nonlinear partial differential equations (PDEs). As exceptions, the completely integrable nonlinear PDEs should be mentioned [4–6]. Unfortunately, the number of equations which fall under this class is scarce. And, what is possibly more important, coherent structures occur very often in open dissipative systems which cannot be Hamiltonian in principle. In such cases, only a few approaches, the symmetry-based methods [7, 8] among them, could serve as alternatives to numerical modeling. It is worth noting that in recent years additionally, and seemingly independently of the group methods, the so-called ansatz-based approach is widely used [9–12]. It allows for solutions with pre-set properties (e.g. soliton-like, kink-like or periodic stationary structures or traveling waves (TW)) to be obtained, but we believe that they are less universal than approaches based on self-similarity methods, supported by qualitative analysis. This is because in the latter case one can analyze not only...
particular self-similar solutions, which fortunately can be expressed analytically, but also the whole family with the given symmetry.

Let us now focus our attention on a very important factor which is intensively used in this paper. Dissipative models of real processes very often turn out to be close (to some extent) to Hamiltonian models. Such a situation occurs for example in cases when dissipation, relaxation, heterogeneity and similar effects which ruin the regularity of the model, display a small-parameter description. In such a case, while searching for dissipative structures, not only qualitative analysis can be applied, but also one can use the whole power of Hamilton’s method. Apart from this, to research a system with small parameters, the approximate decomposition can be used, and physically justified small modules, regardless of whether or not they break the symmetry of an unperturbed model, may be taken into consideration.

This paper is concerned with the search for wave structures (particularly the soliton-like regimes) in a hydrodynamic model, taking into account the effects of spatio-temporal non-localities. Using the aforementioned methods, the criteria for the existence of the wave patterns will be formulated in a wide set of parameter values in those cases where the model under consideration is in some way equivalent to a Hamiltonian one, and in the case, when dissipative effects (associated with relaxation) and the mass forces are taken into account. Let us note that approaches based on the similarity methods very often enable one to obtain solutions serving as intermediate asymptotics [13–15], attracting nearby solutions, regardless of their symmetry.

The paper is organized as follows. In section 2, a hydrodynamic-type model of a non-local medium is formulated, a group-theory reduction to the system of ODEs describing a family of approximately invariant TW solutions is performed, and the existence of one-parametric family of periodic and soliton-like solutions is shown when the temporal relaxation and mass forces are ignored. In section 3, based on a generalization of Melnikov’s method, and the essential use of the previously obtained exact solution, the existence of soliton-like regimes is demonstrated when small mass force and relaxation effects are incorporated.

2. Nonlocal hydrodynamic-type system and the reduced system describing traveling wave solutions

As is well known, the internal structure of a medium influences the evolution of nonlinear waves during the high-rate processes [16–18]. The presence of the internal structure is manifested in fragmentation of smooth initial perturbations and intensification of shock fronts in heterogeneous media [19, 20], soliton features of P-waves in block-hierarchic geophysical media [21], and many other effects. But there does not exist a conventional and universal model describing structured media in a wide range of the parameters values. In the following paragraphs, we will introduce a modeling system, taking into account the non-local effects connecting with a structure. The models studied in this work apply when the ratio of a characteristic size $d$ of elements of the structure to the characteristic wavelength $\lambda$ of the wave pack is much less than unity and therefore the continual approach is still valid, but it is not so small that we can ignore the presence of the internal structure. As has been shown in a number of papers (see e.g. [22, 23]), in the long wave approximation the balance equations for mass and momentum retain their classical form, which in the one-dimensional case can be written as follows (cf with [24]):

$$\begin{cases}
u_t + p_x = \frac{3}{\rho}, \\
\rho_t + \rho^2 u_x = 0.
\end{cases}$$

(1)
Here \( u \) denotes mass velocity, \( p \) is pressure, \( \rho \) is density, \( t \) is time, \( \mathfrak{F} \) is mass force and \( x \) is mass (Lagrangian) coordinate, connected with the convenient spatial coordinate \( x_c \) in the following way:

\[
x = \int_{x_c}^{x} \rho(t, \xi) \, d\xi.
\]

Lower indices denote partial derivatives with respect to subsequent variables. Thus, the whole information about the presence of structure in this approximation is contained in a dynamic equation of state (DES), which should be incorporated to system (1) in order to make it closed.

Generally speaking, a DES for multi-component-structured media, manifesting the non-local features, takes the form of the integral equation \([25, 26]\), linking the generalized thermodynamical flow \( J \) and generalized thermodynamical force \( X \), causing this flow:

\[
J = \int_{-\infty}^{t} \int_{-\infty}^{t'} K(t, t'; x, x')X(t', x') \, dx' \, dt'.
\]  

(2)

Here \( K(t, t'; x, x') \) is a kernel, taking into account nonlocal effects. Function \( K \) can be calculated, in principle, by solving the dynamic problem of structure’s element interaction; however, such calculations are extremely difficult. Therefore, in practice one uses, as a role, some model kernels, describing well enough the main properties of the non-local effects and, in particular, the fact that these effects vanish rapidly as \(|t - t'|\) and \(|x - x'|\) grow. This property could be used in order to pass from the integral equation (2) to a pure differential equation.

One of the simplest state equations accounting for the effects of spatial nonlocality takes the form

\[
p = \hat{\sigma} \int_{-\infty}^{t} K_1(t, t') \left[ \int_{-\infty}^{+\infty} K_2(x, x') \rho^n(t', x') \, dx' \right] \, dt'.
\]  

(3)

where \( K_2(x, x') = \exp[-(x - x')^2/l^2] \) (cf with [18]). Using the fact that the function \( \exp[-(x - x')^2/l^2] \) extremely quickly approaches zero as \(|x - x'|\) grows, we substitute the function \( \rho^n(t', x') \) by the first three terms of its decomposition into the power series:

\[
\rho^n(t', x') = \rho^n(t', x) + [\rho^n(t', x)]_1 \frac{x' - x}{1!} + [\rho^n(t', x)]_2 \frac{(x' - x)^2}{2!} + O(|x - x'|^3),
\]

obtaining this way the following approximate flow-force relation:

\[
p = \hat{\sigma} \int_{-\infty}^{t} K_1(t, t') L(\rho, \rho_x, \rho_{xx}) \, dt',
\]  

(4)

where

\[
L(\rho, \rho_x, \rho_{xx}) = c_0 \rho^n(t', x) + c_2 [\rho^n(t', x)]_2x_x,
\]

\[
c_0 = l \int_{-\infty}^{+\infty} e^{-\tau} \, d\tau = l\sqrt{\pi}, \quad c_2 = \frac{l^3}{2} \int_{-\infty}^{+\infty} \tau^2 e^{-\tau} \, d\tau = \frac{l^3\sqrt{\pi}}{4}.
\]

Now we are going to introduce different functions \( K_1(t, t') \), responsible for the relaxing effects inside the elements of the internal structure. A medium with one relaxing component is usually described by the kernel

\[
K_1(t, t') = \tau^{-1} \exp \left[ -\frac{t - t'}{\tau} \right].
\]  

(5)

With such a kernel, the flow–force relation (4) can be expressed as a first-order PDE. In fact, inserting the kernel (5) into equation (4) and then differentiating it with respect to \( t \), one can obtain the following DES:

\[
\hat{\tau} p_t + p = \hat{\sigma} L(\rho, \rho_x, \rho_{xx}).
\]  

(6)
The parameter $\tau$ is called the time of relaxation. It plays an important role in describing the high-rate processes in media with the internal structure. Note that a DES containing relaxing terms is derived in [16] on the basis of the formalism of irreversible phenomenological thermodynamics, and in [27] on a purely empirical ground.

Inserting the kernel

$$K_1(t, t') = a_1 \exp\left[-\frac{t-t'}{\tau_1}\right] + a_2 \exp\left[-\frac{t-t''}{\tau_2}\right]$$

into the integral equation (4), and differentiating it twice with respect to $t$, we can obtain, after some algebraic manipulation, the following DES:

$$h p_{tt} + \tau p_t + p = \alpha L[\rho, \rho_x, \rho_{xx}] + \mu L_\nu[\rho, \rho_x, \rho_{xx}],$$

(7)

where

$$h = \tau_1 \tau_2, \quad \tau = \tau_1 + \tau_2, \quad \alpha = \hat{\sigma} (a_1 \tau_1 + a_2 \tau_2), \quad \mu = \hat{\sigma} (a_1 + a_2) \tau_1 \tau_2.$$ 

The above equations can be interpreted as those, describing the medium with two relaxing components, which are characterized by the times of relaxations $\tau_1$ and $\tau_2$.

It is worth noting that a DES formally identical with (7) can be obtained from equation (3), using in the temporal part of the kernel function

$$K_1(t, t') = \exp\left[-\frac{t-t'}{\tau_1}\right] \cos\left[\frac{t-t'}{\tau_2} + \varphi_0\right],$$

which contains, besides the exponentially decaying, the oscillating term. In fact, as in the previous case, differentiating equation (4) twice with respect to time, we can obtain the equation, formally identical to equation (7):

$$h \tilde{p}_{tt} + \tilde{\tau} p_t + \tilde{p} = \tilde{\alpha} L[\tilde{\rho}, \tilde{\rho}_x, \tilde{\rho}_{xx}] + \tilde{\mu} L_\nu[\tilde{\rho}, \tilde{\rho}_x, \tilde{\rho}_{xx}],$$

(8)

with

$$\tilde{h} = \frac{\tau_1}{2}, \quad \tilde{\tau} = \gamma, \quad \tilde{\alpha} = \tilde{\hat{\sigma}} \cos \varphi_0, \quad \tilde{\mu} = \frac{\gamma}{2 \tau_2} (\tau_1 \cos \varphi_0 - \tau_2 \sin \varphi_0),$$

where $\gamma = 2 \tau_1 \tau_2 / (\tau_1^2 + \tau_2^2)$. In order to maintain the physical meaning of pressure and density, we introduce the variables $\tilde{p} = p - p_0$, $\tilde{\rho} = \rho - \rho_0$, where $p_0 > 0$ and $\rho_0 > 0$ are some constant equilibrium values of the parameters.

The difference between (7) and (8) arises from the fact that the coefficients of these formally coinciding equations belong to distinct domains of the parameter space. For example, the coefficients $\tilde{h}$ and $\tilde{\tau}$ from (7) satisfy the relation

$$4h < \tau^2$$

provided that $\tau_1 \neq \tau_2$, while for $\tilde{h}$ and $\tilde{\tau}$ the opposite inequality holds. As is shown in [28, 29], the properties of TW solutions to a non-local hydrodynamic-type model depend in an essential way on the values of the parameters.

It is worth noting that DESs very similar to (7) have been obtained in [30, 31] on the basis of the phenomenological non-equilibrium thermodynamics formalism, and in [19] on a purely empirical ground. We prefer to present the approach based on formula (2) and the modeling kernels of non-localities, since it is less cumbersome than that based on the thermodynamical formalism.

In this work, we concentrate on the study of the system of balance equations (1), closed by the DES (6):

$$\begin{cases} u_1 + p_{x} = 3/\rho, \\
\rho_1 + \rho^2 u_x = 0, \\
\hat{\tau} p_t + p = \frac{\beta}{v + 2} \rho_{x+2} + \sigma [\rho^{v+1} \rho_{xx} + (v + 1) \rho^v (\rho_x)^2], 
\end{cases}$$

(9)

where $v = n - 2$, $\beta = c_0 \hat{\sigma} (v + 2) \hat{\tau}$, $\sigma = c_2 \hat{\sigma} (v + 2) \hat{\tau}$. 

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In what follows, we put in (9) $\hat{\xi} = \epsilon \tau$, $3/\rho = \epsilon f(\rho)$ with $|\epsilon| \ll 1$. The introduction of a small parameter enables us to employ the approximate methods and to avoid difficulties arising from the fact that the external force, as well as the relaxing terms, can destroy the scaling symmetry of the problem.

It is easy to verify by the straightforward checking that for $\epsilon = 0$ the system (9) admits the following Lie symmetry group \cite{8, 7} generators:

$$
\hat{P}_0 = \frac{\partial}{\partial t}, \quad \hat{P}_1 = \frac{\partial}{\partial x}, \quad \hat{Z} = \frac{v + 3}{v + 1} \frac{\partial}{\partial t} + u \frac{\partial}{\partial u} + \frac{2}{v + 1} \frac{\partial}{\partial \rho} + \frac{4 + 2v}{v + 1} \frac{\partial}{\partial p}.
$$

(10)

For small $\epsilon$, the system (9) admits the operator

$$
\hat{X} = \hat{P}_0 + D \hat{P}_1 + \epsilon \xi \hat{Z}
$$

(11)

in approximate sense \cite{32}. The parameters $D$ and $\xi$ can be chosen arbitrarily, yet the relation $|\xi| = O(1)$ should take place. Let us note that the first equation of the system (9) admits the operator $\hat{Z}$ in a generally accepted sense when $f(\rho) = a \rho^{v+2}$. Nevertheless, the set (10) is not admitted by the system (9) because of the time derivative term contained in the DES.

A passage to the self-similar variables enabling us to factorize the system (9) is based on the operator (11). The characteristic system corresponding to this operator, up to $O(\epsilon^2)$, can be presented as follows:

$$
\left(1 + \frac{v + 3}{v + 1} \epsilon \xi t\right) dt = \frac{dx}{D} = \frac{du}{\epsilon \xi u} = \frac{d\rho}{\frac{2}{v + 1} \epsilon \xi \rho} = \frac{dp}{\frac{2(v + 1)}{v + 1} \epsilon \xi p}.
$$

(12)

Solving the system (12), and expressing the initial variables in terms of its first integrals, one can construct the following ansatz:

$$
u = (1 + \epsilon \xi t) U(\Omega), \quad p = \left(1 + \frac{2(2 + v)}{v + 1} \epsilon \xi t\right) \Pi(\Omega), \quad \rho = \left(1 + \frac{2}{v + 1} \epsilon \xi t\right) R(\Omega),
$$

(13)

where

$$
\Omega = x - Dt - \frac{v + 3}{v + 1} \epsilon \xi t^2 D / 2
$$

is new independent variable. It is now clear that the parameter $D$ can be determined with the velocity of the unperturbed TW (case $\epsilon$). The terms proportional to $\xi$ introduce the corrections, valid when $|\epsilon| \ll 1$.

The approximate self-similar reduction is performed in several steps. Inserting ansatz (13) into the second equation of the system (9), we obtain a first-order differential equation admitting the separation of variables:

$$
R^2 \dot{U} - D \dot{R} + \frac{2}{v + 1} \epsilon \xi R = O(\epsilon^2).
$$

Integration of this equation gives us the approximate first integral

$$
U(\Omega) = C_1 - \frac{D}{R} - \frac{2}{v + 1} \epsilon \xi \int \frac{d\Omega}{R(\Omega)} + O(\epsilon^2).
$$

(14)

In what follows, we assume that $C_1 = D/R_1$, where $0 < R_1 = \text{const}$. For $\epsilon = 0$ such a choice immediately leads to the asymptotics:

$$
\lim_{t \rightarrow +\infty} u(t, x) = 0, \quad \lim_{t \rightarrow +\infty} \rho(t, x) = R_1.
$$

(15)

Inserting ansatz (13) into the first equation of the system (9), and using equation (14), we obtain the equation

$$
\Pi - D \left(\frac{D}{R^2} \dot{R} - \frac{2}{v} \epsilon \xi \dot{R} + \epsilon \xi \left(C_1 - \frac{D}{R}\right)\right) - \epsilon f(\rho) = O(\epsilon^2).
$$

(16)
Now we introduce a new function:
\[ G = \Pi + \frac{D^2}{\bar{R}}. \]  
(17)

Taking the derivative of (17) with respect to \( \Omega \) and employing (16), we obtain the equation
\[ \dot{G} = \epsilon \left[ f(R) - \xi \left( C_1 + \frac{1 - v}{1 + v} \cdot \frac{D}{\bar{R}} \right) \right] + O(\epsilon^2). \]  
(18)

Inserting ansatz (13) into the third equation, we get a second-order ODE. Excluding \( \Pi \) from this equation by means of formulae (16) and (17), and introducing new variable \( \bar{R} = Y \), we finally obtain a closed system which, up to \( O(\epsilon^2) \), takes the form
\[ \dot{\bar{R}} = Y, \]
\[ \sigma R^{v+2} \dot{Y} = G \bar{R} - \left[ D^2 + \frac{\beta}{v + 2} R^{v+3} + \sigma (v + 1) R^{v+1} Y^2 \right] - \tau \epsilon \frac{D^3}{\bar{R}} Y. \]  
(19)

In the following, we briefly summarize the results of analysis of the system (19) for \( \epsilon = 0 \), and formulate the conditions of the existence of periodic solutions as well as the homoclinic regimes, corresponding to soliton-like TW solutions of the initial system of PDEs.

Assuming that \( \epsilon = 0 \), we immediately obtain that \( G = G_1 = \text{const} \), and the system (19) reduces to
\[ \begin{cases} \frac{dR}{d\omega} = Y \\ \frac{dY}{d\omega} = (\sigma R^{v+2})^{-1} \left\{ G \bar{R} - \left[ D^2 + \frac{\beta}{v + 2} R^{v+3} + \sigma (v + 1) R^{v+1} Y^2 \right] \right\} \end{cases}, \]  
(20)

where \( \omega = x - Dt \). Incorporating the conditions (15), we express \( G_1 \) as follows:
\[ G_1 = \frac{D^2}{R_1} + \frac{\beta}{v + 2} R_1^{v+2}. \]  
(21)

Dividing the second equation of the system (20) by the first one, and introducing new variable \( Z = Y^2 = (dR/d\omega)^2 \), we obtain, after some algebraic manipulation, the linear inhomogeneous equation
\[ Z'(R) + 2[(v + 1) R]^{-1} Z(R) = \frac{2\left[ G_1 R - D^2 - \beta R^{v+3}/(v + 2) \right]}{\sigma R^{v+2}}. \]  
(22)

Solving this equation with respect to \( Z = Z(R) \) and next integrating the equation obtained after the substitution \( Z = (dR/d\omega)^2 \), we obtain the following quadrature:
\[ \omega - \omega_0 = \int \frac{\pm \sqrt{\sigma} R^{1+v} dR}{\sqrt{H_1 + 2 G_1 \left( \frac{R^{1+v}}{(1+v)} \right) - 2 D^2 \left( \frac{R^{1+v}}{(1+v)} \right) - \beta \left( \frac{R^{1+v}}{(1+v)} \right)}}. \]  
(23)

The direct analysis of formula (23) is rather difficult. To realize what sort of solutions we deal with, the methods of qualitative analysis can be applied to the dynamical system (20). It is evident that all isolated critical points of the system (20) are located on the horizontal axis \( OR \).

They are determined by the solutions of the algebraic equation
\[ P(R) = \frac{\beta}{v + 2} R^{v+3} - G_1 R + D^2 = 0. \]  
(24)
As can be easily seen, one of the roots of equation (24) coincides with \( R_1 \). The location of the second real root depends on the relations between the parameters. If \( v + 3 > 1 \), and \( D^2 \) satisfies the inequality

\[
D^2 > D^2_{cr} = \beta R_1^{v+3}.
\]

then there exists the second critical point \( A_2(R_2, 0) \) with \( R_2 > R_1 \) (for details see [33]). The analysis of linearization matrices of the system (20) shows that the critical point \( A_1(R_1, 0) \) is a saddle, while the critical point \( A_2(R_2, 0) \) is a center. Thus, the system (20) has only such critical points, which are characteristics of a Hamiltonian system. This circumstance suggests that there could exist a Hamiltonian system equivalent to the system (20). Introducing a new independent variable \( T \), obeying the equation \( \frac{d}{dT} = 2 \sigma R^{2(v+1)} \frac{d}{d\omega} \), one can write down the equivalent Hamiltonian system

\[
\frac{dR}{dT} = 2 \sigma Y R^{2(v+1)} = \frac{\partial H}{\partial Y},
\]

\[
\frac{dY}{dT} = 2R^v \left\{ GR - \left[ D^2 + \frac{\beta}{v + 2} R^{v+3} + \sigma (v + 1) R^{v+1} Y^2 \right] \right\} = - \frac{\partial H}{\partial R},
\]

based on the Hamiltonian function [33]:

\[
H = 2D^2 \frac{R^{v+1}}{v + 1} + \frac{\beta}{(v + 2)^2} R^{2(v+2)} + \sigma Y R^{2(v+1)} - 2G_1 \frac{R^{v+2}}{v + 2}.
\]

Remark. Let us note that the Hamiltonian system (26) can be obtained by inserting the TW ansatz \( u(t, x) = U(\omega), \quad \rho(t, x) = R(\omega) \) into the system

\[
u_t + \left\{ \frac{\beta}{v + 2} \rho^{v+2} + \sigma [\rho^{v+1} \rho_x + (v + 1) \rho^v (\rho_x)^2] \right\}_x = 0,
\]

\[
\rho_t + \rho^2 u_x = 0,
\]

which is equivalent to the system (9) when \( \epsilon = 0 \). In connection with this, a question arises whether or not the system (28) is also Hamiltonian. Looking for the Hamiltonian function in the form \( H = \Psi(u, \rho, \rho_x) \), one easily obtains that such formulation is possible merely for the physically irrelevant values of the parameters. However, the system (28) happens to possess a non-local Hamiltonian function [34], somewhat similar to that obtained in the study of the celebrated Camassa–Holm equation.

A study of the system (26) is shown in [33]. The main result obtained can be formulated in the form of the following statement.

**Theorem** ([33]). If \( v > -2 \) and \( D^2 > \beta R_1^{v+3} \), then the system (20) possesses a one-parameter family of periodic solutions, localized around the critical point \( A_2(R_2, 0) \) in a bounded set \( M \). The boundary of this set is formed by the homoclinic intersection of separatrices of the saddle point \( A_1(R_1, 0) \).

Thus, the unperturbed source system (9) possesses periodic and soliton-like invariant solutions. Let us note in conclusion that for some special case the integral on the RHS of formula (23) or, what is the same, on the RHS of the formula

\[
T - T_0 = \pm \int \frac{dR}{2 \sqrt{\sigma R^{1+v}} \sqrt{H_1 + 2G_1 \frac{R^{1+v}}{(1+v)} - 2D^2 \frac{R^{1+v}}{(1+v)} - \beta \frac{R^{1+v}}{(1+v)^2}}}.
\]
can be calculated explicitly:

\[ T = \pm \frac{1}{7\sqrt{2}} \left\{ 7 \ln \left[ \frac{R - 1}{3 - R + \sqrt{7 - 2R - R^2}} \right] \\
+ 2\sqrt{7} \ln \left[ \frac{7 - R + \sqrt{7 + 2R - R^2}}{R} \right] \right\} \]  

(30)

This solution corresponds to the following values of the parameters: \( D = 1 = R \), \( \sigma = 1 \), \( \beta = 1/2 \), \( \nu = 0 \), \( G_1 = 5/4 \), and \( T_0 = 0 \).

3. Approximate soliton-like solutions of the perturbed system

The purpose of this section is to analyze whether and when the perturbed system (19) possesses the homoclinic solution. Our analysis is based on the generalized Melnikov theory presented in [35]. The theory is based in an essential way on the knowledge of the homoclinic solution of the unperturbed system, which enables us to measure the distance between the stable and unstable saddle separatrices for small \( \epsilon \). For this reason, we specify the parameters in such a way that they fit the exact solution (30).

Since we are going to use the formalism developed in [35], we should pass in (19) to the independent variable \( T \), for which the unperturbed system becomes Hamiltonian. The standard representation for our system in this case will be as follows:

\[
\begin{align*}
\frac{dX}{dT} &= JD_X H(X, G) + \epsilon F(X, G), \\
\frac{dG}{dT} &= \epsilon L(X, G),
\end{align*}
\]

(31)

where \( X = (R, Y)^\nu \), \( DX = \left[ \frac{R}{\sqrt{7}}, \frac{\beta}{\sqrt{7}} \right]^\nu \), \( 0 < \epsilon \ll 1 \),

\( F(X, G) = (0, -2D^3Y/R)^\nu \),

\( L(X, G) = 2R^2 \left[ f(R) - \xi \left( C_1 + \frac{1 - \nu}{1 + \nu} \cdot \frac{D}{R} \right) \right] \),

while

\[ J = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \]

(\cdot, \cdot)^\nu means the transposition. In what follows, we put \( f(R) = a R^2 + b \).

So we want to measure the distance between the saddle separatrices of the perturbed system (31). The problem we deal with differs from the classical one [36, 37], since in the unperturbed case the Hamiltonian (27) depends on an extra variable \( G \), which plays the role of a parameter. We assume that \( G \) belongs to an open set \( U \), containing \( G_1 = D^2/R_1 + \beta R_1^{\nu+2}/(\nu + 2) \).

Following [35], we introduce the set

\[ \mathcal{M} = \{(X, G) \in \mathbb{R}^2 \times \mathbb{R}^1 : X = \gamma(G), \ \text{where} \ \gamma(G) \ \text{solves} \ (32) \} \]

\[ D_X H(\gamma(G), G) = 0 \quad \text{and} \quad \det[D_X H(\gamma(G), G)] \neq 0, \forall G \in U \].

We denote by \( W^s(\mathcal{M}) \) and \( W^u(\mathcal{M}) \) the \((1+1)\)-dimensional stable and unstable manifolds of \( \mathcal{M} \). It is obvious that the manifolds \( W^s(\mathcal{M}) \) and \( W^u(\mathcal{M}) \) coincide as \( \epsilon = 0 \) (in this case \( W^s(\mathcal{M}) = W^u(\mathcal{M}) = \Gamma \)). For sufficiently small \( \epsilon \), \( \mathcal{M} \) is transformed into the locally invariant manifold \( W^s(\mathcal{M}) \). Up to \( O(\epsilon^2) \), the dynamics on \( W^s(\mathcal{M}) \) is governed by the equation

\[ \dot{G} = \epsilon L(\gamma(G), G). \]
For $b = 2\xi - a$, $W^s(M)$ possesses the equilibrium point $(1, 0, 5/4)$. The analysis of the linear part of equation (33) shows that this point is stable as $a > \xi/2$ and unstable as $a < \xi/2$.

For nonzero $\epsilon$, the invariant manifolds $W^s(M)$ and $W^u(M)$ are transformed into the (locally) invariant manifolds $W^s(\epsilon)M$ and $W^u(\epsilon)M$, respectively, which do not coincide. Yet under certain conditions, the manifolds can still have the points of intersection. To state these conditions, a distance separating $W^s(\epsilon)M$ and $W^u(\epsilon)M$ is measured in the vicinity of a point $p \in \Gamma_1$, lying in the section $G_1 = 5/4$.

Up to $O(\epsilon)$, this distance is equal to the Melnikov integral ([35], chapter IV):

$$M^{G_1}(a, \xi) = \int_{\Gamma_1} \left[ (D_X H, F(X, G)) + (D_X H, (D_G J D_X H) \int L(X, G) dT) \right] \times (X^{G_1}(T), G_1, T)^2 dT,$$

(34)

where $X^{G_1}(T) = (R^{G_1}(T), Y^{G_1}(T))$ is the homoclinic trajectory of the unperturbed system on the $G = G_1$ level corresponding to the hyperbolic fixed point of the vector field on $M_\epsilon$. The dependence on $(X^{G_1}(T), G_1, T)$ means, in other words, that the integration is carried out along the unperturbed trajectory $\Gamma$, described by formula (30).

Performing simple but tedious calculations, we finally obtain the following result:

$$M^{G_1}(a, \xi) = 0.262789 - 1.60194a + 7.18141(\xi + 4a)$$

(35)

(for the details see [38]). So, for $\epsilon$ sufficiently small, stable and unstable manifolds intersect near the values of $(a, \xi)$, satisfying the equation

$$\xi = -0.03659 - 3.7769a.$$  

(36)

The result obtained was verified by direct numerical simulation, confirming that the values of the parameters $(a, \xi)$ for which the manifolds $W^s(\epsilon)M$ and $W^u(\epsilon)M$ intersect lie reasonably close to the curve (36). An example of such intersection is shown in figure 1.

It is obvious, that the homoclinic solution of the system (19) corresponds to the soliton-like solution to the initial system (9). Temporal evolution of this solution is shown in figure 2, together with the soliton-like solution of the unperturbed system. Due to the smallness of the parameter $\epsilon$, the solutions almost coincide when $t = 0$. Yet for sufficiently large $t$, they become separated from each other.
4. Concluding remarks

Thus, we have analyzed a set of approximate TW solutions to the system (9), which can be applied to the description of the long nonlinear waves propagation in media with a mesoscale internal structure. The dynamic equation of state, closing the system of balance of the mass and momentum equation, and containing the information about the structure, is obtained in this work on the basis of the integral flow–force relation (2) and the modeling kernels of non-locality. Let us note that similar equations of state are obtained elsewhere on the basis of the methods of phenomenological irreversible thermodynamics [16, 30, 31], as well as on a purely empirical ground [27, 19]. The presented approach, being easier to handle, effectively gives the same equations of state as the more sophisticated ones.

The main goal of this work was to show that the system (9) possesses a set of approximate soliton-like solutions. In this connection, we would like to note that the presence of the relaxing term in a DES provides dissipation into the source system of PDEs, and this feature is inherited by the factorized system (19). As is well known, the homoclinic loop can appear among the solutions of a non-Hamiltonian dynamical system only ‘accidentally’, e.g., as a result of a limit cycle interaction with the nearby saddle point, and this can occur only for the specific values of the parameters. As is shown in this work, the appearance of the approximate soliton-like solution takes place along the one-parameter sub-manifold, belonging to the two-dimension set of parameters, providing that the factorized system is close to the Hamiltonian one. Using, in this case, the generalized Melnikov method, we have stated the conditions of the saddle separatrices intersection in the presence of a small mass force and when the temporal non-locality is taken into account. From the general considerations presented in [35], intersections of stable and unstable saddle separatrices are not transversal for the given type of perturbation and this is backed by the results of our numerical experiments that enabled us to reveal the loci of the saddle separatrix intersection with no signs of a ‘homoclinic blow-up’ observed. In view of this, it is interesting to investigate, what types of temporal non-locality and external force would lead to the transversal intersection and, as a consequence, to drastic changes in the phase portrait of the reduced system. Our preliminary analysis shows that the transversal intersection can take place in the case of periodic mass force and the DES (7), describing the structured media with two relaxing processes.
As mentioned earlier, the self-similar solutions (including the approximate ones) very often play the role of the intermediate asymptotics [13–15, 39]. We did not go into exploring the stability and attractive features of the approximate soliton-like solutions, since these issues require a special treatment going far beyond the scope of this work.

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