DICTIONARY LEARNING WITH ALMOST SURE ERROR CONSTRAINTS

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Abstract. A dictionary is a database of standard vectors, so that other vectors / signals are expressed as linear combinations of dictionary vectors, and the task of learning a dictionary for a given data is to find a good dictionary so that the representation of data points has desirable features. Dictionary learning and the related matrix factorization methods have gained significant prominence recently due to their applications in wide variety of fields like machine learning, signal processing, statistics etc. In this article we study the dictionary learning problem for achieving desirable features in the representation of a given data with almost sure recovery constraints. We impose the constraint that every sample is reconstructed properly to within a predefined threshold. This problem formulation is more challenging than the conventional dictionary learning, which is done by minimizing a regularised cost function. We make use of the duality results for linear inverse problems to obtain an equivalent reformulation in the form of a convex-concave min-max problem. The resulting min-max problem is then solved using gradient descent-ascent like algorithms.

1. Introduction

Signals have almost always been expressed as a linear combination of a standard database / collection of vectors. For instance, audio signals have historically been studied by expressing them as linear combination of Fourier basis, wavelets [Mal99], [Dau92] etc. It turns out that expressing signals in a well chosen basis helps in the study of the underlying characteristics of the signal than it is in its natural representation. In fact, it is often the case that signals in their natural representation are elements of a very high dimensional vector space even though there exists some low dimensional characteristics that could be exploited for its effective representation.

Almost all of natural signals are driven / outcomes of processes that are inherently low dimensional. Therefore, there is a lot of redundancy in data that is typically encountered in practise. Moreover, in certain applications [EV13],[SEC†14] data is concentrated around low dimensional subspaces or some such sort of clusters. Since every standard basis that is used in classical signal processing techniques like a Fourier basis, wavelets, DCT etc., is orthonormal, it turns out that most of the basis vectors do not contribute much due to the very nature of orthonormality. Therefore, representation using such bases does not maximally exploit the redundancy in the data. Later Frame theoretic ideas [DGM86], [CK12] have shown that by relaxing the orthonormality constraint and allowing the standard database to have more vectors than the effective dimension of the data, it allows us to exploit the redundancy in the data better than the standard orthonormal bases. However, it is natural to ask the question that given the data to be represented or analysed, what is the best database of vectors one has to use? The recent advent of
Dictionary Learning techniques [AEB06] [OF96] [OF97] [MBPS10] [MBP11] is an attempt to accomplish this task. In these techniques, a collection of vectors called atoms that constitute a database referred to as the dictionary is learned from the data and for the data with a desired objective. It has been successfully shown [AEB06] [WMM+10] [FBD09] [MES07] that learning a dictionary that is adapted to the data often outperforms the classical techniques by a considerable margin in a plethora of signal processing applications. For a brief overview of dictionary learning techniques and their application, see [TF11].

One of primary feature that is central to the success of modern day signal processing techniques is sparsity. Sparsity based techniques have been successfully implemented in tasks like signal compression [D+06] [CW08] [CT06], denoising [DLZS11] [FLN12], clustering [RSS10] etc. In particular, the inception and success of Compressed Sensing [D+06] [CW08] is noteworthy. Even though signals might not be sparse in their natural representation, they can be approximated reasonably well by a sparse linear combination of the atoms in some dictionary. For instance, natural images admit a reasonably approximate sparse representation in 2-d Discrete Cosine Transform (DCT) basis, typically with only less than five percent of the coefficients being non-zero. Given the fact that such hidden sparsity is common in signals encountered in the current age of big data, the use of sparsity based signal processing techniques has become more compelling. Therefore, for a given data, it is desirable to learn a dictionary that allows the possibility of sparse representation of the signals without losing much information.

Let $x$ be the signal that admits a reasonably approximate sparse representation $f_x$ (that is unknown and needs to be computed) in a dictionary $D = \{d(1), d(2), \ldots, d(K)\}$. We compute the sparse representation by solving the following convex optimization problem

\begin{equation}
\min_{f} \|f\|_1 \quad \text{subject to} \quad \|x - Df\|_2 \leq \epsilon,
\end{equation}

where $\epsilon$ is a positive real number that signifies the permissible error in approximation. Sometimes, in applications like denoising etc., $\epsilon$ is the bound on the noise. Ideally one would want to minimize the $\ell_0$-pseudo norm $\|f\|_0 := |\{i : f_i \neq 0\}|$, however, it leads to intractability of (1) in problems of large dimension. Fortunately, its convex relaxation works reasonably well in most of the practical setting. Given a data set $(x_t)_t$, we would want to use a “good” dictionary $D$ that offers better sparse representation of the data. The primary goal of this article is to learn such a dictionary for a given data set. We do that by solving the following problem

\begin{equation}
\begin{aligned}
\min_{D, (f_t)_t} & \frac{1}{T} \sum_{t=1}^{T} \|f_t\|_1 \\
\text{subject to} & \quad D \in D, \ f_t \in \mathbb{R}^K \\
& \quad \|x_t - Df_t\|_2 \leq \epsilon_t \quad \text{for every } t = 1, 2, \ldots, T,
\end{aligned}
\end{equation}

where the set $D$ is some convex subset of $\mathbb{R}^{n \times K}$ for some positive integers $n$ and $K$, and is typically chosen to be such that every dictionary vector is of at most unit length.
Alternatively, for a given dictionary \(D\), the sparse representation can also be obtained by solving the following regularised formulation

\[
(3) \quad f'_\gamma \in \arg\min_f \left( \|f\|_1 + \gamma \|x - Df\|_2^2 \right),
\]

where \(\gamma > 0\) is a regularization parameter. Evidently, the objective function is a weighted cost of sparsity inducing \(\ell_1\)-penalty and the error in representation. This trade off is controlled by the regularization parameter \(\gamma\). For large values of \(\gamma\), the representation is more accurate but less sparse and vice versa. For a given data \((x_t)_t\), assuming that one has the knowledge of a good value of \(\gamma\), the dictionary learning is done conventionally by solving the following optimization problem

\[
(4) \quad \min_{(f_t), D \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^T \left( \|f_t\|_1 + \gamma \|x_t - Df_t\|_2^2 \right).
\]

It is a known fact that the problems (1) and (3) are equivalent, i.e., for a given value of \(\epsilon > 0\) \((\gamma > 0)\) there exists some \(\gamma(\epsilon) > 0\) \((\epsilon(\gamma) > 0)\) such that the problems (1) and (3) with parameters \(\epsilon\) and \(\gamma(\epsilon)\) \((\epsilon(\gamma)\) and \(\gamma)\) respectively, admit identical optimal solutions. This implies that, given a dictionary one could chose either (1) or (3) on convenience of implementation to obtain the sparse representation of the data. However, the corresponding dictionary learning problems (2) and (4) need not be equivalent. Almost all of notable recent work on dictionary learning has been concentrated towards solving (4). On the contrary, there is little work done to solve the dictionary learning problem (2) in a meaningful manner.

The primary goal of this article is to solve the dictionary learning problem (2) in situations where the knowledge of good values of \((\epsilon_t)_t\) are known. In fact, in many image processing applications like denoising, inpainting etc., where the image is corrupted by some noise, it is often the case that good statistical information of the noise is readily available. Since the values of \((\epsilon_t)_t\) that are to be used for such applications depend on the noise characteristics, good estimates of their values are available beforehand. Therefore, in situations like these, considering the dictionary learning problem in the formulation (2) is natural. Moreover, an alternate perspective to look at (2) is that we are putting a hard constraint on the permissible error, and then optimizing for sparsity. This is advantageous because, such a formulation provides the user the possibility to specify the maximum permissible error limit. Obviously, one would not want to lose too much of information only to obtain sparse representations, this is ensured by putting a hard constraint on the error like in (2).

There is a lot of work already done on the problem (4), we want to highlight that such techniques can’t be applied directly to solve (2). In the sparse coding problem (3), the regularization parameter \(\gamma\) controls the tradeoff between sparsity and the error terms. For a given value of \(\gamma\), if the optimal solution to (3) is \(f'_\gamma\), one does not know the value of the error \(\|x - Df'_\gamma\|_2\) incurred before actually solving the problem (3). If the specified error bound is \(\epsilon\), there is no way to chose \(\gamma\) apriori such that the error bound \(\|x - Df'_\gamma\|_2 \leq \epsilon\) is ensured. This is due to the fact that even though the problems (1) and (3) are equivalent, the relation \(\epsilon \mapsto \gamma(\epsilon)\) is not straight forward and unknown beforehand. In fact such a relation depends on the data point \(x\) and the dictionary \(D\). Furthermore, if the sparse representations \((f_t)_t\) are obtained by solving (3) by using the same value of \(\gamma\) for each \(t\), it is very likely
that the error constraint $\|x_t - Df_t\|_2 \leq \epsilon_t$ is not satisfied for all $t$. However, due to equivalence of (1) and (3) we know that there exists a $\gamma_t(D)$ for each $t$ such that the resulting sparse representation obtained from (3) with $\gamma_t(D)$ as the regularizer is identical to the one obtained via solving (1). Therefore, we can consider the following alternate dictionary learning problem

$$\text{(5)} \quad \min_{(f_t), D \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} \left( \|f_t\|_1 + \gamma_t(D) \|x_t - Df_t\|_2^2 \right).$$

The nature of the map $D \rightarrow \gamma_t(D)$ is not straightforward, in fact it is unknown, thereby making the minimization over $D$ really hard. It is to be noted that the only way the dictionary learning problem (2) can be solved by considering (5) is if we were to somehow know in the hindsight the optimal solution $D^\ast$ to (2) and use $\gamma_t(D^\ast)$ as regularizers for every $t$ in (5). It is then obvious that learning such regularization parameters is itself as hard as learning the optimal dictionary of (2).

For the sake of argument, let us ignore the error constraint and simply consider the task of learning a dictionary by solving the following problem instead of (4).

$$\text{(6)} \quad \min_{(f_t), D \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} \left( \|f_t\|_1 + \gamma_t \|x_t - Df_t\|_2^2 \right).$$

Allowing independent regularizer for each $t$ gives the user, more freedom and control which is apparent in the formulation (2). It is clear that the performance of the dictionary learned this way for data analysis applications is critical to the regularization parameters $(\gamma_t)_t$ used to solve (6). Therefore, it is of high importance to know the right value of the regularization parameters that are best for the given data beforehand. One of the main challenges in learning a dictionary via this formulation or (4) is that, apriori we do not know such regularizer values. Typically, they are learned through cross validation techniques by solving multiple versions of the problem (4) for the data with different values of the regularizers. This means that when the data set is large, which is typical of the modern times, one has to solve the dictionary learning problem multiple times, thereby putting a high demand on computational requirements.

It is to be noted that none of the problems (2), (4) and (6) is jointly convex in arguments $(f_t)_t$ and $D$. However, all of them are convex with respect to each argument given that the other is held fixed. Due to this reason, dictionary learning problems (4) and (6) are solved by alternating the minimization over $(f_t)_t$ and $D$ iteratively. Therefore, the dictionary is updated from $D \rightarrow D'$ in the following manner

$$\text{(7)} \quad \begin{cases} f'_t \in \arg \min_{f_t} \left( \|f_t\|_1 + \gamma_t \|x_t - Df_t\|_2^2 \right) \quad \text{for every } t, \\ D' \in \arg \min_{D \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} \|x_t - Df'_t\|_2^2. \end{cases}$$

It is immediate that the dictionary update step is a Quadratic program which can be solved efficiently. In fact, based on co-ordinate descent methods to solve this optimization problem, in turns out that the alternating minimization to learn a good dictionary can be done online as in [MBPS10].

On the contrary, the dictionary learning problem (2) readily does not admit such an alternating minimization strategy. Once we fix $(f_t)_t$, the cost function in
remains constant for every dictionary such that \( \| x_t - D f_t \| \leq \epsilon_t \) for every \( t \). Therefore, there is no obvious way to update the dictionary variable. This is perhaps one of the primary reasons why dictionary learning is not done via the formulation (2) in the mainstream. In practise, where solving the dictionary learning problem (2) is necessary, it is still done by a slightly different but similar alternating minimization technique

\[
\begin{align*}
    f'_t &\in \left\{ \begin{array}{l}
                 \text{minimize} \\
                 \text{subject to}
               \end{array} \right. \| f_t \|_1, \\
    D' &\in \min_{D \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} \| x_t - D f'_t \|_2^2.
\end{align*}
\]

However, by replacing the convex problem of minimization over \((f_t)_t\) in (2) with its Lagrange dual, we get the following min-max-min problem equivalent to (2)

\[
\begin{align*}
    \min_{D} \max_{(\gamma_t)_{t}} \min_{(f_t)_{t}} \frac{1}{T} \sum_{t=1}^{T} \left( \| f_t \|_1 + \gamma_t \left( \| x_t - D f_t \|_2^2 - \epsilon_t^2 \right) \right) \\
    \text{subject to} \quad D \in \mathcal{D} \\
    \gamma_t > 0 \quad \text{for every} \quad t.
\end{align*}
\]

It is clear from (8) and (9) that the dictionary update (8) completely disregards the maximization over the dual variables \((\gamma_t)_t\), and treats them instead as constants. Therefore, there is no mathematical justification whatsoever on why dictionaries updated via (8) should eventually be optimal solutions to (2).

The main difficulty in solving (2) is that the dictionary variable does not appear directly in the objective function. It affects the feasibility of a candidate \( f_t \), and thereby affecting the cost indirectly. This makes it impossible to solve (2) through obvious alternating minimization techniques. We have considered a slightly more general problem formulation to (2) in this article, and have solved it. We go about doing this by replacing (1) with an equivalent convex-concave min-max formulation provided in [SC19] that pushes the dictionary variable to the cost function. This allows us to update the dictionary in a meaningful manner that minimizes the objective function of (2) in each iteration. Learning a dictionary to solve (2) by methods provided in this article is not only mathematically justified but outperforms the conventional techniques like (8) significantly.

### 2. The dictionary learning problem and its solution

Let \( n \) be a positive integer, \( \mathbb{H}_n \) be an \( n \)-dimensional Hilbert space equipped with an inner product \( \langle \cdot, \cdot \rangle \) and its associated norm \( \| \cdot \| \). For every \( x \in \mathbb{H}_n \) and \( \epsilon > 0 \), let \( B(x, \epsilon) := \{ y \in \mathbb{H}_n : \| x - y \| < \epsilon \} \) and let \( B(x, \epsilon) := \{ y \in \mathbb{H}_n : \| x - y \| \leq \epsilon \} \).

Every vector \( x \in \mathbb{H}_n \) is encoded as a vector \( f(x) \) in \( \mathbb{R}^K \) via the encoder map \( f : \mathbb{H}_n \rightarrow \mathbb{R}^K \). The reconstruction of the encoded samples from the codes \( f(x) \) is done by taking the linear combination \( \sum_{i=1}^{K} f_i(x) d(i) \) with some standard database of vectors \( D := (d(1), d(2), \ldots, d(K)) \) referred to as the dictionary. For a given

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\(^1\)The choice of notation \( \gamma_t \) for dual variables is intentional. In fact the equivalence between (1) and (3) is immediate from the fact that for a given \( \epsilon > 0 \), \( \gamma(\epsilon) \) is precisely the optimal value of the dual variable in the Lagrange dual problem to (1).
encoder map \( f \), since every vector \( x \in \mathbb{H}_n \) is identified by its code \( f(x) \), we shall refer to \( f(x) \) also as the representation of \( x \) under the encoder \( f \). Our objective is to find dictionaries that offer representation of vectors in \( \mathbb{H}_n \) with desirable characteristics, eg., sparsity, ability to classify the data etc. We refer to the task of finding such a dictionary as the dictionary learning problem, and in short DLP. In view of this, let us introduce ourselves to the various quantities involved:

- **Cost function**: It is desirable for the codes to have certain characteristics like sparsity, minimum energy etc. This is achieved by considering an encoder that optimizes a certain cost function \( c : \mathbb{R}^K \rightarrow [0, +\infty] \). The particular characteristics present in the codes depend on the type of cost function chosen. For instance, choosing \( c(\cdot) = \| \cdot \|_1 \) induces sparsity in codes. With regards to the cost function \( c \), we shall assume the following:

  2.1 Assumption

  - **Positive Homogeneity**: For every \( \alpha \geq 0 \) and \( f \in \mathbb{R}^K \), we have \( c(\alpha f) = \alpha^r c(f) \), where \( r > 0 \) is the order of homogeneity.

  - **Pseudo-Convexity**: The sublevel set \( V_c := \{ f \in \mathbb{R}^K : c(f) \leq 1 \} \) is convex.

  - **Inf-Compactness**: The set \( V_c \) is compact.

  It is to be noted that a cost function that satisfies Assumption 2.1 with unit order oh homogeneity is a gauge function corresponding to the set \( V_c \), and whenever the set \( V_c \) is symmetric about the origin, the cost function is a norm. Some typical examples of the cost function that are used in practice are:

  - The \( \ell_1 \)-norm : \( \| \cdot \|_1 \), provides sparse representations [BJM11, Tib96, D06].

  - The \( \ell_2 \)-norm : \( \| \cdot \|_2 \), provides unique representation and group sparsity [MVDGB08].

  - Gauge function corresponding to any user specified compact convex set containing the origin.

  In many applications, the cost function is obtained by adding a small penalty function to the actual objective function in order to obtain some other desirable features. For instance, in the basis pursuit denoising problem, it is customary to add a small \( \ell_2 \)-penalty to the \( \ell_1 \)-cost in order to enforce uniqueness of the optimal solution. We observe that by considering a generic definition of cost function \( c(\cdot) \) as discussed, such adjustments to the actual objective function are easily incorporated.

- **Feasible dictionary set**: We reconstruct the samples by taking a linear combination with the dictionary vectors. Therefore, if we allow the dictionary vectors to have arbitrary lengths, every vector can be written as a linear combination with coefficients that are arbitrarily small. Thus, we consider an upper bound on the length of each dictionary vector and for simplicity, we have chosen this upper bound to be unity. Thus, the following is the set of feasible dictionaries:

\[
\mathcal{D} := \{ D = (d(1) \ d(2) \ \cdots \ d(K)) : \|d(i)\| \leq 1 \text{ for all } i = 1, 2, \ldots, K \}.
\]

- **The error constraint and the error threshold**: We would like to encode samples such that the reconstruction is similar to the original samples. Ideally, we would want to obtain exact reconstruction. However, permitting a small amount of error in the reconstruction allows us to encode samples to obtain other desirable

\[\text{The properties enlisted in the Assumption 2.1 are in force throughout the article.}\]
features like sparse representation etc. It must then be obvious that the amount of permissible error in reconstruction shouldn’t be too large. In view of this, we define the error function as the norm of the error in reconstruction: \( x - Df(x) \). We consider the following error constraint:
\[
\|x - Df(x)\| \leq \epsilon(x) + \delta c(f(x)),
\]
where \( \epsilon \geq 0 \) is the error threshold function, and \( \delta \geq 0 \) is a design parameter and can be identically set to zero. However, when \( \delta > 0 \) it brings in the effect of regularization. Some relevant examples of the error threshold function include:
- A constant function, where \( \epsilon(x) = \epsilon \) for some \( \epsilon \geq 0 \).
- An SNR type function, where \( \epsilon(x) = \epsilon \|x\| \) for some \( \epsilon \geq 0 \).

2.1. The coding problem. The central task in representing a given data optimally, is the coding problem. In broad generality, the coding problem is the task of encoding a vector \( x \in \mathbb{R}^n \) as another vector \( f \in \mathbb{R}^K \). The encoding is done so as to minimize a given objective function \( c: \mathbb{R}^K \rightarrow [0, +\infty] \) with the constraint that the error in reconstruction: \( x - Df \) is within the limits for a given dictionary \( D \in \mathbb{D} \). Formally, we have the following problem:
\[
(11)
\begin{align*}
\text{minimize} & \quad c \\
\text{subject to} & \quad \{c(f) \leq c, \|x - Df\| \leq \epsilon + \delta c^{1/r},
\end{align*}
\]
where \( \epsilon \) and \( \delta \) are some fixed non-negative real numbers. For \( \delta = 0 \), we see that the feasible collection of \( f \) is independent from the variable \( c \). As a consequence we see that for every feasible \( f \in \mathbb{R}^K \), the minimization over the variable \( c \) is achieved for \( c = c(f) \). Thus the coding problem reduces to the following more familiar formulation.
\[
\begin{align*}
\text{minimize} & \quad c(f) \\
\text{subject to} & \quad \|x - Df\| \leq \epsilon.
\end{align*}
\]

It might be surprising at first to see the rather unusual formulation of the coding problem (11). Our formulation (11) makes way for the possibility of \( \delta \) being strictly positive rather than just zero and by allowing \( \delta \) to take a positive but sufficiently small value, we obtain several advantages:
- The coding problem is always strictly feasible, which is easily seen by considering \( c = \frac{1}{\epsilon} \|x\| \) and \( f = 0 \). This is a crucial feature in the initial stages of learning an optimal dictionary, essentially when the data lies in a subspace of lower dimension \( m \), such that \( m, K \ll n \).
- A small positive value of \( \delta \) amounts to having an effect of regularization in the problem. Thus, if one wants to harvest the advantages that come from regularization, a carefully chosen positive value of \( \delta \) can be used. Such a value in principle, needs to be learned separately. However, in the context of sparse representation, a small value can be chosen by the user depending on the maximum signal loss that can be tolerated.
- Considering \( \delta > 0 \) in the coding problem leads to a useful fixed point characterization of the optimal dictionary. Such characterizations also lead to simple online algorithms that learn optimal dictionary.
An example of the coding problem which is of practical relevance is the classical Basis Pursuit Denoising problem [EA06], [CW08], that arises in various applications, and in particular compressed sensing.

\[
\begin{align*}
\text{minimize} & \quad \|f\|_1 \\
\text{subject to} & \quad \|x - Df\| \leq \epsilon.
\end{align*}
\]

We emphasise that the constraints in the coding problem are convex and the cost function is convex-continuous and coercive.\(^3\) Therefore, from the Weierstrass theorem we conclude that whenever the coding problem is feasible, it admits an optimal solution. To this end, let us define the following

**Definition 2.2.** Let \( x \in \mathbb{H}_n, D \in \mathcal{D} \), and let \( \epsilon, \delta \geq 0 \).

- Whenever, the coding problem (11) is feasible, let \( C(D, x, \epsilon) \) denote the optimal value achieved and let \( F(D, x, \epsilon) \) be the set of optimal codes.
- If the coding problem is in feasible, let \( C(D, x, \epsilon) := +\infty \) and \( F(D, x, \epsilon) := \emptyset \).

**Definition 2.3.** Let \( D \in \mathcal{D} \) and let \( \epsilon, \delta \geq 0 \). A vector \( x \in \mathbb{H}_n \) is said to be \((D, \epsilon, \delta)\)-encodable if \( C(D, x, \epsilon) < +\infty \).

**Definition 2.4.** For \( x \in \mathbb{H}_n \) and \( D \in \mathcal{D} \), let \( H(D, x) \subset V_c \) be defined as

\[
H(D, x) := (C(D, x, \epsilon))^{-1/\epsilon} \cdot F(D, x, \epsilon).
\]

From the definitions it is immediate that for a given dictionary \( D, x \in \mathbb{H}_n \) is \((D, \epsilon, \delta)\)-encodable if and only if the corresponding coding problem is feasible. If so, every \((D, \epsilon, \delta)\)-encodable vector \( x \) is thus encoded as an element \( f^* \in F(D, x, \epsilon) \) while incurring a cost of \( C(D, x, \epsilon) \). So naturally, the dictionary learning problem is to find a dictionary \( D^* \) such that the average encoding cost is minimised. It should be noted that, both the encoding cost \( C(D, x, \epsilon) \) and the set of codes \( F(D, x, \epsilon) \) are specific to a given cost function \( c(\cdot) \), even though it is not specified in their notation.

2.2. The dictionary learning problem. Let \( P \) be a probability distribution on \( \mathbb{H}_n \) and \( X \) be a \( P \)-distributed random variable. Let \( c : \mathbb{R}^K \rightarrow \mathbb{R}_+ \) be a given cost function that satisfies Assumption 2.1, \( \epsilon : \mathbb{H}_n \rightarrow \mathbb{R}_+ \) be a given error threshold function and \( \delta \) be a non-negative real number. Our objective is to find a dictionary that facilitates optimal encoding of the data, which are the samples drawn from \( P \). We know that the cost incurred to encode the random variable \( X \) using the dictionary \( D \in \mathcal{D} \), is given by \( C(D, X, \epsilon(X)) \). Therefore, we consider the following dictionary learning problem:

\[
\text{minimize}_{D \in \mathcal{D}} \quad \mathbb{E}_P \left[ C(D, X, \epsilon(X)) \right].
\]

For a large integer \( T \), let \( [X : T] := (x_t)_{t=1}^T \) be a collection of samples drawn from the distribution \( P \). Let us consider the dictionary learning problem for the sampled

\[^3\text{Recall that a continuous function } c \text{ defined over an unbounded set } U \text{ is said to be coercive in the context of an optimization problem, if :}
\]

\[
\lim_{\|u\| \to \infty} c(u) = +\infty (-\infty),
\]

in the context of minimization (maximization) of \( c \) and the limit is considered from within the set \( U \).
data, given by:

\[
\text{(14)} \quad \minimize_{D \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} C(D, x_t, \epsilon(x_t)) .
\]

In principle, one would want to solve (13). However, in most practical situations, the knowledge of the entire distribution \( P \) is unknown. Often what is available is either a large collection of samples drawn from \( P \) or a sequence (likely an iid sequence) of \( P \)-distributed samples. Therefore, we shall discuss the practically more relevant dictionary learning problem (14) in this article.

For the special case of \( \delta = 0 \), the dictionary learning problem (14) can be restated using the definition of the coding cost \( C(D, x_t, \epsilon(x_t)) \) in the more conventional form as:

\[
\text{(15)} \quad \minimize_{D, (f_t)} \frac{1}{T} \sum_{t=1}^{T} c(f_t)
\]

subject to:

\[
\begin{align*}
D & \in \mathcal{D}, \\
& f_t \in \mathbb{R}^K, \text{ for all } t = 1, 2, \ldots, T, \\
\|x_t - Df_t\| & \leq \epsilon(x_t) \text{ for all } t = 1, 2, \ldots, T.
\end{align*}
\]

2.3. Main results. For \( x \in \mathbb{H}_n \) such that \( \|x\| \leq \epsilon(x) \), we immediately see that the pair \( \mathbb{R}_+ \times \mathbb{R}^K \ni (c^*, f^*) := (0, 0) \) is feasible for (11). Moreover, since \( c(f) > 0 \) for every \( f \neq 0 \) we conclude that \( C(D, x, \epsilon(x)) = 0 \) for every \( D \in \mathcal{D} \). As a result, such samples do not play any role in the optimization over the dictionary variable \( D \). Therefore, in the context of dictionary learning problem (14), we shall consider that \( \|x_t\| > \epsilon(x_t) \) for all \( t = 1, 2, \ldots, T \). Because if otherwise, such samples can be conveniently ignored without any effect on the dictionary learning problem.

In general, the dependence of the encoding cost \( C(D, x, \epsilon) \) on the dictionary variable \( D \) is not immediately evident. Thereby, making the dictionary learning problem challenging. Moreover, the current formulation (11) of the coding problem, does not admit a straightforward alternating minimization scheme that alternates between the variables \((f_t)\) and \( D \) to update the dictionary. We replace the coding problem with its equivalent min-max formulation provided in [SC19], which we have reproduced here for completeness.

**Lemma 2.5.** Let \( D \in \mathcal{D}, \epsilon, \delta \geq 0 \) and let \( x \in \mathbb{H}_n \setminus B[0, \epsilon] \). Consider the coding problem (11) and the following inf-sup problem:

\[
\text{(16)} \quad \inf_{h, \eta} \sup_{\lambda} \left\{ \eta^r + \frac{1}{\eta} \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right) - \left( \delta \|\lambda\| + \langle \lambda, Dh \rangle \right) \right\}
\]

subject to:

\[
\begin{align*}
& h \in V_\epsilon, \\
& \eta > 0, \\
& \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0.
\end{align*}
\]

The optimal value in (16) is finite if and only if \( x \) is \((D, \epsilon, \delta)\)-encodable, in which case, it is equal to the encoding cost \( C(D, x, \epsilon) \).
Lemma 2.6. For every $x \in \mathbb{H}_n \setminus B[0, \epsilon(x)]$, let us define the function $J_x : \mathcal{D} \times \mathcal{V}_c \longrightarrow \mathbb{R}_+$ by

$$J_x(D, h) := \begin{cases} \inf_{\eta > 0} \sup_{\lambda} & \eta' + \frac{1}{\eta} \left( \langle \lambda, x \rangle - \epsilon(x) \|\lambda\| \right) - \left( \delta \|\lambda\| + \langle \lambda, Dh \rangle \right) \\ \text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0. \end{cases}$$

Then the following is true with regards to the function $J_t(D, h)$

- The map $\mathcal{D} \ni D \longmapsto J_x(D, h)$ is convex for every $h \in \mathcal{V}_c$.
- The map $\mathcal{V}_c \ni h \longmapsto J_x(D, h)$ is convex for every $D \in \mathcal{D}$.

2.3.1. Solution to the dictionary learning problem. For each $t = 1, 2, \ldots, T$, replacing the encoding cost $C(D, x_t, \epsilon(x_t))$ with the equivalent inf-sup problem (16) in the dictionary learning problem (14) gives us the following

$$\min_{D \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^T \begin{cases} \inf_{h_t, \eta_t} & \sup_{\lambda_t} \eta_t \left( \langle \lambda_t, x_t \rangle - \epsilon(x_t) \|\lambda_t\| \right) - \left( \delta \|\lambda_t\| + \langle \lambda_t, D h_t \rangle \right) \\ \text{subject to} & h_t \in \mathcal{V}_c \\ & \eta_t > 0 \\ & \langle \lambda_t, x_t \rangle - \epsilon \|\lambda_t\| > 0. \end{cases}$$

The optimization over variables $(h_t)_t$ is separable, and thus the minimization over them can be considered outside the summation. Doing so, and letting $J_t(D, h) := J(D, x_t, h)$ for every $t = 1, 2, \ldots, T$, we rewrite the dictionary learning problem (14) in the following simplified manner

$$\min_{D, (h_t)_t} \frac{1}{T} \sum_{t=1}^T J_t(D, h_t)$$

subject to

$$D \in \mathcal{D} \quad h_t \in \mathcal{V}_c \quad \text{for all } t = 1, 2, \ldots, T.$$  

This formulation gives us the ability to solve the dictionary learning problem through alternating the minimization over the variables $(h_t)_t$ and $D$ by keeping the other one fixed. Thus, we propose to solve the dictionary learning problem by going through the following optimization problems in order iteratively till a stopping criteria is satisfied.

$$\begin{cases} (h_t)_t \leftarrow \arg\min_{(h'_t)_t \in \mathcal{V}_c} \frac{1}{T} \sum_{t=1}^T J_t(D, h'_t) \\ D \leftarrow \arg\min_{D' \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^T J_t(D', h_t). \end{cases}$$

Since optimization over the variables $(h'_t)_t$ is separable, it is straightforward, and we conclude from Lemma 2.5 that $(h_t)_t$ is the optimal solution. Suppose that the user is provided with a black box that provides optimal codes $(f_t)_t$ by solving the coding problem (11). It is advantageous that the optimal solutions $(h_t)_t$ to (16) can be easily computed by simply re-scaling the optimal solutions $(f_t)_t$ that are put out by the black box and does not require to resolve the inf-sup problem of Lemma 2.5.
2.3.2. The dictionary update algorithm.

**Proposition 2.7.** Consider the dictionary learning problem \((14)\) for the data \((x_t)_{t}\) such that \(\|x_t\| > \varepsilon(x_t)\) for all \(t = 1, 2, \ldots, T\). Let \((h_t)_{t}\) be given such that there exists a dictionary \(D \in \mathcal{D}\) for which the inclusion \(h_t \in H(D, x_t)\) holds for all \(t = 1, 2, \ldots, T\). Then the following inf-sup problem

\[
\inf_{D, (\eta_t)_{t}} \sup_{(\lambda_t)_{t}} \left\{ \frac{1}{T} \sum_{t=1}^{T} \eta_t \left( \langle \lambda_t, x_t \rangle - \varepsilon(x_t) \| \lambda_t \| \right) - \left( \delta \| \lambda_t \| + \langle \lambda_t, Dh_t \rangle \right) \right\}
\]

subject to

\[
D \in \mathcal{D},
\]

\[
\eta_t > 0, \text{ and } \langle \lambda_t, x_t \rangle - \varepsilon(x_t) \| \lambda_t \| > 0 \text{ for all } t = 1, 2, \ldots, T,
\]

admits a unique saddle point solution \((D', (\eta'_t)_{t}, (\lambda'_t)_{t})\).

Given a dictionary \(D \in \mathcal{D}\), let \(\Gamma : \mathcal{D} \rightarrow \mathcal{D}\) denote the algorithm that updates the dictionary variable to \(\Gamma(D)\). From the alternating minimization scheme, we have

\[
\text{argmin}_{D' \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} J_t(D', h_t)
\]

where \(h_t \in H(D, x_t)\) for all \(t = 1, 2, \ldots, T\).

Once the sequence \((h_t)_{t}\) is fixed, we see that by replacing \(J_t(D', h_t)\) with the inf-sup problem of its definition, the optimization over the variables \((\eta_t, \lambda_t)_{t}\) is separable over \(t\). Thus, the dictionary update problem \((19)\) reduces to the inf-sup problem of proposition 2.7. Consequently, the dictionary update is done by seeking a saddle point solution to the the inf-sup problem. This can be done efficiently by employing gradient descent-ascent type algorithms.

The discussion on solving the dictionary learning problem \((14)\) is summarized in the following algorithm.

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Algorithm 1: A procedure to obtain dictionaries optimal for (14)

Input: The data \( (x_t) \subset \mathbb{H}_m \), a positive integer \( K \), cost and error threshold functions \( c \) and \( \epsilon \) respectively, the regularizer \( \delta > 0 \), and an initial dictionary \( D_0 \).

Output: An optimal dictionary \( D^* \) that solves (14) and the corresponding representation vectors \( (f^*_t) \), for the data.

1. Remove irrelevant samples: For each \( t \in \{1, 2, \ldots, T\} \), if \( \|x_t\| \leq \epsilon(x) \), discard the sample \( x_t \).
2. Set \( D \leftarrow D_0 \).
3. Iterate till stopping criteria is met.
   Compute the unit codes: For each \( t \in \{1, 2, \ldots, T\} \), solve the coding problem (11) and normalise the codes to obtain \( h_t \in H(D, x) \).
   Compute the saddle point solution: \( (D^*, (\eta^*_t), (\lambda^*_t)) \) of Proposition 2.7 by iterating
   For \( t = 1, 2, \ldots, T \)
   \[
   \eta_t \leftarrow \eta_t + a \left( -r \eta_t^{-1} + \frac{\langle \lambda_t, x_t \rangle - \epsilon(x_t) \|\lambda_t\|}{\eta_t^2} \right)
   \]
   \[
   \lambda_t \leftarrow \lambda_t + a \left( \frac{1}{\eta_t} (x_t - D(\eta_t h_t)) - \frac{\epsilon(x_t) + \delta \eta}{\|\lambda_t\|} \lambda_t \right).
   \]
   For \( i = 1, 2, \ldots, K \)
   \[
   d(i) \leftarrow d(i) + a \left( \frac{1}{T} \sum_{t=1}^{T} h_t(i) \lambda_t \right).
   \]
   Update dictionary: \( D \leftarrow (d(1) \ d(2) \ \cdots \ d(K)) \).
4. Repeat
5. Output the dictionary and codes.

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