QUILLEN’S THEOREM A AND THE WHITEHEAD THEOREM FOR BICATEGORIES

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ABSTRACT. We prove a bicategorical analogue of Quillen’s Theorem A. As an application, we deduce the well-known result that a pseudofunctor is a biequivalence if and only if it is essentially surjective on objects, essentially full on 1-cells, and fully faithful on 2-cells.

1. INTRODUCTION

Quillen’s Theorems A and B give conditions which imply that a functor of categories $F : C \to D$ induces a homotopy equivalence, respectively fibration, on geometric realizations of nerves [Qui73]. Bicategorical analogues of Quillen’s Theorem B have been discussed in [CHR15] and depend on a notion of fibration for bicategories; see, for example, [Bak, Buc14]. A biequivalence is not necessarily a fibration, and therefore a separate treatment of Theorem A is needed.

We prove a bicategorical Quillen Theorem A (Theorem 5.1) that generalizes to bicategories the essential algebraic content of the original result. Our proof of Theorem A depends on a lax slice bicategory and a version of terminal object we call inc-lax terminal for initial components (see Definition 4.2).

We describe the lax slice construction for a general lax functor of bicategories in Section 3 and prove that it is a bicategory. Our construction is similar but not identical to notions which have appeared in the literature, particularly [Buc14, Construction 4.2.1].

Section 4 is devoted to the definition and properties of lax terminal objects. Our notion of inc-lax terminal is stronger than standard notions of terminal object in a bicategory, but our proof shows that the slice bicategories over a biequivalence have inc-lax terminal objects.

We state and prove the Bicategorical Quillen Theorem A in Section 5. We show that, given a lax functor whose lax slices have the appropriate inc-lax terminal data, one can construct a reverse lax functor together with strong transformations between their composites and the respective identities. We show, moreover, that if the given lax functor is a pseudofunctor, then the reverse lax functor we construct is an inverse biequivalence.

Section 6 contains our main application, Theorem 6.1: the local characterization of biequivalences as those pseudofunctors which are essentially surjective on objects, essentially full on 1-cells, and fully faithful on 2-cells (see Definition 2.27). This is a bicategorical analogue of Whitehead’s theorem for topological spaces.
which says that a continuous function between CW complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence.

The corresponding result for 1-categories is the well-known statement that a functor (of 1-categories) is an equivalence if and only if it is essentially surjective on objects and fully faithful on morphisms. Proofs appear in many standard texts, e.g., [Mac98, Rie16]. Although the result for bicategories is also well-known, the authors do not know of a self-contained proof in the literature. Our naming of Theorem 6.1 as a Whitehead theorem follows the thesis of Schommer-Pries [SP09], which proves a Whitehead theorem for symmetric monoidal bicategories. The argument in [SP09] first replaces each symmetric monoidal bicategory with an equivalent 1-skeletal symmetric monoidal bicategory, and then proves the Whitehead theorem in the 1-skeletal case.

A proof of the general (non monoidal) Whitehead theorem can be extracted from [SP09], but our approach is different. Our bicategorical Quillen Theorem A uses a contractibility of fibers—in the form of inc-lax terminal objects—to construct an inverse pseudofunctor directly. Moreover, our proof of the Bicategorical Whitehead Theorem 6.1 does not depend on the Bicategorical Coherence Theorem [MP85, Str96], which asserts each bicategory $B$ is retract biequivalent to a 2-category $A$.

Our treatment also clarifies the role of choice in this result. Every instance of Whitehead’s Theorem—whether topological or algebraic—requires the axiom of choice. Our method of proof isolates its use to Proposition 4.7, which chooses the inc-lax terminal data for a pseudofunctor that is assumed to be essentially surjective, essentially full, and fully faithful. If one has constructions of these data by some other means, our application of the Bicategorical Quillen Theorem A yields a construction of an inverse biequivalence.

This work makes extensive use of pasting diagrams in bicategories, and was one of the motivations for the authors’ concurrent work [JY], which gives an elementary proof of a bicategorical pasting theorem. The pasting diagrams we use below are pasting diagrams in the sense defined there, and each has a unique composite by the Bicategorical Pasting Theorem of [JY].

2. BACKGROUND

This section contains the bicategorical background needed for our work. We include full details of the definitions since they will be used in subsequent sections.

**Definition 2.1.** A bicategory is a tuple $(B, 1, c, a, \ell, r)$ consisting of the following data.

(i) $B$ is equipped with a collection $\text{Ob}(B) = B_0$, whose elements are called objects in $B$. If $X \in B_0$, we also write $X \in B$.

(ii) For each pair of objects $X, Y \in B$, $B$ is equipped with a category $B(X, Y)$, called a hom category.

- Its objects are called 1-cells, and its morphisms are called 2-cells in $B$.
- Composition and identity morphisms in $B(X, Y)$ are called vertical composition and identity 2-cells, respectively.
- For a 1-cell $f$, its identity 2-cell is denoted by $1_f$.

(iii) For each object $X \in B$, $1_X : 1 \to B(X, X)$ is a functor, which we identify with the 1-cell $1_X(\ast) \in B(X, X)$, called the identity 1-cell of $X$. 

(iv) For each triple of objects \(X, Y, Z \in \mathcal{B}\),

\[ c_{XYZ} : \mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \to \mathcal{B}(X, Z) \]

is a functor, called the horizontal composition. For 1-cells \(f \in \mathcal{B}(X, Y)\) and \(g \in \mathcal{B}(Y, Z)\), and 2-cells \(\alpha \in \mathcal{B}(X, Y)\) and \(\beta \in \mathcal{B}(Y, Z)\), we use the notations

\[ c_{XYZ}(g, f) = gf \quad \text{and} \quad c_{XYZ}(\beta, \alpha) = \beta \star \alpha. \]

(v) For objects \(W, X, Y, Z \in \mathcal{B}\),

\[ a_{WXYZ} : c_{WZX} \circ \text{Id}_{\mathcal{B}(W,X)} \to c_{WYZ} \circ (\text{Id}_{\mathcal{B}(Y,Z)} \times c_{WXY}) \]

is a natural isomorphism, called the associator.

(vi) For each pair of objects \(X, Y \in \mathcal{B}\),

\[ c_{XY}(1_Y \times \text{Id}_{\mathcal{B}(X,Y)}) \xrightarrow{f_{XY}} \text{Id}_{\mathcal{B}(X,Y)} \xleftarrow{r_{XY}} c_{XY}(\text{Id}_{\mathcal{B}(X,Y)} \times 1_X) \]

are natural isomorphisms, called the left unitor and the right unitor, respectively.

The subscripts in \(c\) will often be omitted. The subscripts in \(a, \ell, r\) will often be used to denote their components. The above data is required to satisfy the following two axioms for 1-cells \(f \in \mathcal{B}(V, W)\), \(g \in \mathcal{B}(W, X)\), \(h \in \mathcal{B}(X, Y)\), and \(k \in \mathcal{B}(Y, Z)\).

**Unity Axiom:** The middle unity diagram

\[
\begin{array}{ccc}
(g1_W)f & \xrightarrow{a} & g(1_Wf) \\
& \searrow{r_{g1f}} & \downarrow{1_{g} \ell_{f}} \\
& & gf
\end{array}
\]

in \(\mathcal{B}(V, X)\) is commutative.

**Pentagon Axiom:** The diagram

\[
\begin{array}{ccc}
((kh)g)f & \xrightarrow{a_{k,h,g,f}} & (k(hg)f) \\
& \searrow{a_{k,h,g,f}} & \downarrow{1_{k} \ast a_{h,g,f}} \\
(k(h)g)f & \xrightarrow{a_{k,h,g,f}} & k((hg)f)
\end{array}
\]

in \(\mathcal{B}(V, Z)\) is commutative.

This finishes the definition of a bicategory.

We make use of two additional compatibilities between unitors. The statements and their proofs are bicategorical analogues of corresponding results for monoidal 1-categories; cf., [Kel64, Theorems 6 and 7].
Proposition 2.4. Suppose \( f \in B(X,Y) \) and \( g \in B(Y,Z) \) are 1-cells. Then the diagrams
\[
\begin{array}{c}
\xymatrix{
1_Z(gf) 
& 1_Z(f1_X) \\
gf 
& g1_f \\
\ell_g \ast f 
& r_g \\
\ell_g 
& 1_g \ast r_f \\
}
\end{array}
\]
in \( B(X,Z) \) are commutative.

Proposition 2.5. For each object \( X \) in \( B \), the equality
\[
\ell_{1_X} r_{1_X} : 1_X 1_X \cong 1_X
\]
holds in \( B(X,X) \).

2.1. Lax functors, transformations, and modifications.

Definition 2.6. Suppose \((B,1,c,a,\ell,r)\) and \((B',1',c',a',\ell',r')\) are bicategories. A lax functor
\[
(F,F^2,F^0) : B \rightarrow B'
\]
from \( B \) to \( B' \) is a triple consisting of the following data.

- \( F : B_0 \rightarrow B'_0 \) is a function on objects.
- For each pair of objects \( X,Y \) in \( B \), it is equipped with a functor
  \( F : B(X,Y) \rightarrow B'(FX,FY) \).
- For all objects \( X,Y,Z \) in \( B \), it is equipped with natural transformations
  \[
  \xymatrix{
  B(Y,Z) \times B(X,Y) 
  & B(X,Z) \\
  F \times F 
  & F^2 \\
  F(B(Y,FZ)) \times F'(FX,FY) 
  & F'(FX,FZ) \\
  \ar_{c} 
  & \ar_{c'} \\
  
  }
  \]
  with component 2-cells
  \[
  Fg \circ Ff \xrightarrow{F^2_g \ast f} F(gf) \quad \text{and} \quad F_{1_X} \xrightarrow{F^0_1} F_{1_X}.
  \]

The above data is required to make the following three diagrams commutative for all 1-cells \( f \in B(W,X) \), \( g \in B(X,Y) \), and \( h \in B(Y,Z) \).

Lax associativity:
\[
\xymatrix{
(Fh \circ Fg) 
& Fh \circ F(gf) \\
\ar_{a'} 
& \ar_{F_{gh,f}} \\
(Fh \circ Fg) \circ Ff 
& F((hg)f) \\
\ar_{F_{h,g} \ast Ff} 
& \ar_{F_a} \\
F(hg) \circ Ff 
& F((hg)f) \\
\ar_{F_{h,g} \ast Ff} 
& \ar^{F_{1_f}} \\
}
\]
Lax left and right unity:

\[ F_1 X \circ F f \xrightarrow{F_1 X \circ 1_{F f}} F (1_X \circ f) \]

\[ 1'_{FX} \circ F f \xrightarrow{F \ell} F f \]

\[ 1'_{FX} \circ F^0 f \xrightarrow{1'_{FX} \circ F^0_{FW}} F (f \circ 1_W) \]

\[ F f \circ 1'_{FW} \xrightarrow{r'} F f \]

This finishes the definition of a lax functor. Moreover:

- A lax functor is **unitary** (resp., **strictly unitary**) if each 2-cell \( F^0_X \) is an isomorphism (resp., identity).
- A **pseudofunctor** is a lax functor in which \( F^2 \) and \( F^0 \) are natural isomorphisms.
- A **strict functor** is a lax functor in which \( F^2 \) and \( F^0 \) are identity natural transformations.

We let \( \text{Id}_B \) denote the identity strict functor for a bicategory \( B \).

The next result defines the constant pseudofunctor at an object.

**Proposition 2.9.** Suppose \( X \) is an object in a bicategory \( B \), and \( A \) is another bicategory. Then there is a strictly unitary pseudofunctor

\[ \Delta_X : A \rightarrow B \]

defined as follows.

- \( \Delta_X \) sends each object of \( A \) to \( X \).
- For each pair of objects \( Y, Z \) in \( A \), the functor \( \Delta_X : A(Y, Z) \rightarrow B(X, X) \) sends
  - every 1-cell in \( A(Y, Z) \) to the identity 1-cell \( 1_X \) of \( X \);
  - every 2-cell in \( A(Y, Z) \) to the identity 2-cell \( 1_{1_X} \) of the identity 1-cell.

  For each object \( Y \) of \( A \), the lax unity constraint is

\[ (\Delta_X)^0_Y = 1_{1_X} : 1_X \rightarrow 1_X. \]

- For each pair of composable 1-cells \( (g, f) \) in \( A \), the lax functoriality constraint is

\[ (\Delta_X)^2_{g, f} = \ell_{1_X} : 1_X \rightarrow 1_X. \]

The proof that these data define a strictly unitary pseudofunctor is an exercise in the unity properties, notably Propositions 2.4 and 2.5, and we leave it to the reader.

**Definition 2.10.** Let \( (F, F^2, F^0) \) and \( (G, G^2, G^0) \) be lax functors \( B \rightarrow B' \). A **lax transformation** \( \alpha : F \rightarrow G \) consists of the following data.

- **Components:** It is equipped with a component 1-cell \( \alpha_X \in B'(FX, GX) \) for each object \( X \) in \( B \).
- **Lax naturality constraints:** For each pair of objects \( X, Y \) in \( B \), it is equipped with a natural transformation

\[ \alpha : \alpha_X^* G \rightarrow (\alpha_Y)_* F : B(X, Y) \rightarrow B'(FX, GY), \]
with a component 2-cell $\alpha_f : (Gf)\alpha_X \rightarrow \alpha_Y(Ff)$, as in the following diagram, for each 1-cell $f \in B(X,Y)$.

\[
\begin{array}{c}
\begin{array}{ccc}
FX & \xrightarrow{Ff} & FY \\
\downarrow \alpha_X & & \downarrow \alpha_Y \\
GX & \xrightarrow{Gf} & GY
\end{array}
\end{array}
\]

The above data is required to satisfy the following two pasting diagram equalities for all objects $X,Y,Z$ and 1-cells $f \in B(X,Y)$ and $g \in B(Y,Z)$.

**Lax unity:**

\[
\begin{array}{c}
\begin{array}{ccc}
FX & \xrightarrow{F1_X} & FX \\
\downarrow \alpha_X & & \downarrow \alpha_X \\
GX & \xrightarrow{1GX} & GX
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
FX & \xrightarrow{F1_X} & FX \\
\downarrow \alpha_X & & \downarrow \alpha_X \\
GX & \xrightarrow{1GX} & GX
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
FX & \xrightarrow{\ell} & GX \\
\downarrow \alpha_X & & \downarrow \alpha_X \\
GX & \xrightarrow{\ell} & GX
\end{array}
\end{array}
\]

**Lax naturality:**

\[
\begin{array}{c}
\begin{array}{ccc}
FX & \xrightarrow{F(gf)} & FZ \\
\downarrow \alpha_X & & \downarrow \alpha_Z \\
GX & \xrightarrow{G(gf)} & GZ
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
FX & \xrightarrow{F(gf)} & FZ \\
\downarrow \alpha_X & & \downarrow \alpha_Z \\
GX & \xrightarrow{G(gf)} & GZ
\end{array}
\end{array}
\]

This finishes the definition of a lax transformation. Moreover, a strong transformation is a lax transformation in which every component 2-cell $\alpha_f$ is an isomorphism.

We let $1_F$ denote the identity strong transformation on a lax functor $F$.

**Definition 2.13.** Suppose $\alpha, \beta : F \rightarrow G$ are lax transformations for lax functors $F, G : B \rightarrow B'$. A modification $\Gamma : \alpha \rightarrow \beta$ consists of a component 2-cell

\[
\Gamma_X : \alpha_X \rightarrow \beta_X
\]
in $B'(FX,GX)$ for each object $X$ in $B$, that satisfies the following modification axiom

\[(2.14) \quad a_X \left( \begin{array}{c} \beta_Y \\ a_f \\ \alpha_Y \end{array} \right) = a_X \left( \begin{array}{c} \beta_Y \\ \beta_f \\ \Gamma_X \end{array} \right) \]

for each 1-cell $f \in B(X,Y)$. A modification is invertible if each component 2-cell $\Gamma_X$ is an isomorphism.

2.2. Adjoint and invertible 1-cells. In this section we recall basic notions of internal adjunction, invertibility, and mates. We will need these for the constructions in following sections.

Definition 2.15. An internal adjunction in a bicategory $B$ is a quadruple $(f,g,\eta,\varepsilon)$ consisting of

- 1-cells $f : X \to Y$ and $g : Y \to X$;
- 2-cells $\eta : 1_X \to gf$ and $\varepsilon : fg \to 1_Y$.

These data are subject to the following two axioms, in the form of commutative triangles.

\[(2.16) \quad \]

If $(f,g,\eta,\varepsilon)$ is an adjunction with $f : X \to Y$, then the represented adjunctions given by pre- and post-composition induce isomorphisms of 2-cells; corresponding 2-cells under these isomorphisms are known as mates, and defined as follows.

Definition 2.17. Suppose $(f_0,g_0,\eta_0,\varepsilon_0)$ and $(f_1,g_1,\eta_1,\varepsilon_1)$ is a pair of adjunctions in $B$, with $f_0 : X_0 \to Y_0$ and $f_1 : X_1 \to Y_1$. Suppose moreover that $a : X_0 \to X_1$ and $b : Y_0 \to Y_1$ are 1-cells in $B$. The mate of a 2-cell $\omega : f_1a \to b f_0$ is given by the pasting diagram at left below. Likewise, the mate of a 2-cell $\nu : a g_0 \to g_1b$ is
given by the pasting diagram at right below.

The triangle identities imply that these define inverse bijections, and thus we have the following result.

**Lemma 2.18.** If \((f_0, g_0, \eta_0, \varepsilon_0)\) and \((f_1, g_1, \eta_1, \varepsilon_1)\) is a pair of adjunctions in \(B\), with \(f_0 : X_0 \to Y_0 \text{ and } f_1 : X_1 \to Y_1\), then taking mates establishes a bijection of 2-cells
\[
B(X_0, Y_1)(f_1 a, b f_0) \cong B(Y_0, X_1)(a g_0, g_1 b).
\]

for any 1-cells \(a : X_0 \to X_1\) and \(b : Y_0 \to Y_1\).

**Definition 2.19.** An adjunction \((f, g, \eta, \varepsilon)\) with \(f : X \to Y\) and \(g : Y \to X\) is called an internal equivalence or adjoint equivalence if \(\eta\) and \(\varepsilon\) are isomorphisms.

We say that \(f\) and \(g\) are members of an adjoint equivalence in this case, and we write \(X \simeq Y\) if such an equivalence exists. If \(f\) is a member of an adjoint equivalence, we let \(f^*\) denote an adjoint.

Since mates are formed by pasting with unit/counit and unitors, we have the following.

**Lemma 2.20.** If \((f, f^*)\) is an adjoint equivalence, then a 2-cell \(\theta : f s \to t\) is an isomorphism if and only if its mate \(\theta^* : s \to g t\) is an isomorphism.

**Definition 2.21.** A 1-cell \(f : X \to Y\) is said to be invertible or an equivalence if there exists a 1-cell \(g : Y \to X\) together with isomorphisms \(g f \cong 1_X\) and \(1_Y \cong f g\).

Clearly the 1-cells in an adjoint equivalence are invertible. The converse also holds, by a standard argument modifying one of the two isomorphisms in Definition 2.21.

**Proposition 2.22.** A 1-cell \(f : X \to Y\) in \(B\) is an equivalence if and only if it is a member of an adjoint equivalence.

2.3. Biequivalences. Now we give the definitions of invertible strong transformation and biequivalence.

**Definition 2.23.** Suppose that \(F\) and \(G\) are pseudofunctors of bicategories \(B \to C\) and \(\alpha : F \to G\) a strong transformation. We say that \(\alpha\) is invertible and write
\[
\alpha : F \xrightarrow{\cong} G
\]
if there is a strong transformation $\alpha^* : G \to F$ together with invertible modifications

$$\Theta : 1_F \cong \alpha^*\alpha \quad \text{and} \quad \Gamma : \alpha\alpha^* \cong 1_G.$$ 

**Remark 2.24.** The invertible strong transformations are the invertible 1-cells in a bicategory of pseudofunctors, strong transformations, and modifications. However, we will not need this infrastructure. Instead, we will make use of the following characterization result. \(\Box\)

**Proposition 2.25.** Suppose that $F$ and $G$ are pseudofunctors of bicategories $B \to C$ and suppose that $\alpha : F \to G$ is a strong transformation. Then $\alpha$ is invertible if and only if each $\alpha_X : F(X) \to G(X)$ is an invertible 1-cell in $C$.

**Proof.** One implication is immediate. For the other, suppose that $\alpha$ is a strong transformation and each component $\alpha_X$ is invertible. By Proposition 2.22 we may choose an adjoint inverse $\alpha_X^*$ for each component.

We will show that these components assemble to give a strong transformation $\alpha^* : G \to F$ together with invertible modifications $\eta : 1_F \cong \alpha^*\alpha$ and $\varepsilon : \alpha\alpha^* \cong 1_G$. We define the 2-cell aspect of $\alpha^*$ by taking component-wise mates of the 2-cells for $\alpha$. The transformation axioms for $\alpha^*$ follow from those of $\alpha$ by Lemma 2.18. Each mate of an isomorphism is again an isomorphism by Lemma 2.20, and therefore $\alpha^*$ is a strong transformation. The componentwise units and counits define the requisite invertible modifications to make $\alpha$ and $\alpha^*$ invertible strong transformations. \(\Box\)

**Definition 2.26.** A pseudofunctor $F : B \to C$ is a biequivalence if there exists a pseudofunctor $G : C \to B$ together with invertible strong transformations

$$\text{Id}_B \xrightarrow{\cong} GF \quad \text{and} \quad FG \xrightarrow{\cong} \text{Id}_C.$$ 

**Definition 2.27.** Suppose $F : B \to C$ is a lax functor of bicategories.

- We say that $F$ is essentially surjective if it is surjective on adjoint-equivalence classes of objects.
- We say that $F$ is essentially full if it is surjective on isomorphism classes of 1-cells.
- We say that $F$ is fully faithful if it is a bijection on 2-cells.

**Lemma 2.28.** If $F$ is a biequivalence, then each local functor

$$B(X,Y) \to C(FX,FY)$$

is essentially surjective and fully faithful. That is, $F$ is essentially full on 1-cells and fully faithful on 2-cells.

**Proof.** If $F$ and $G$ are inverse biequivalences, then one has local equivalences of categories

$$B(X,Y) \xrightarrow{\cong} B((GF)X,(GF)Y) \quad \text{and} \quad C(Z,W) \xrightarrow{\cong} C((FG)Z,(FG)W)$$

for $X, Y$ in $B$ and $Z, W$ in $C$. The composites $GF$ and $FG$ are fully faithful on 2-cells, and therefore by factoring the equivalences above one concludes that $F$ and $G$ are fully faithful on 2-cells. Moreover, given any 1-cell $h : FX \to FY$, the composite $GF$ is essentially full on 1-cells and therefore there is some $\tilde{h} : X \to Y$ such that $(GF)\tilde{h} \cong Gh$. But since $G$ is fully faithful, this implies that $F\tilde{h} \cong h$ in $B(FX,FY)$. Thus $F$ is essentially full on 1-cells. \(\Box\)
Remark 2.29. By the Whitehead Theorem for 1-categories, Lemma 2.28 implies that a biequivalence is a local equivalence of categories, but we will not make use of this conclusion.

3. The Lax Slice Bicategory

In this section we describe a bicategorical generalization of slice categories.

Definition 3.1. Given a lax functor $F : B \to C$ and an object $X \in C$, the lax slice bicategory $F \downarrow X$ consists of the following.

1. Objects are pairs $(A, f_A)$ where $A \in B$ and $FA \xrightarrow{f_A} X$ in $C$.
2. 1-cells $(A_0, f_0) \to (A_1, f_1)$ are pairs $(p, \theta_p)$ where $A_0 \xrightarrow{p} A_1$ in $B$ and $\theta_p : f_0 \Rightarrow f_1(Fp)$ in $C$. We depict this as a triangle.

3. 2-cells $(p_0, \theta_0) \to (p_1, \theta_1)$ are singletons $(\alpha)$ where $\alpha$ is a 2-cell $p_0 \Rightarrow p_1$ in $B$ such that $F\alpha$ satisfies the equality shown in the pasting diagram below, known as the ice cream cone condition with respect to $\theta_0$ and $\theta_1$.

We describe the additional data of $F \downarrow X$ and prove that it satisfies the bicategory axioms in Proposition 3.2.

Proposition 3.2. Given a lax functor $F : B \to C$ and an object $X \in C$, the lax slice $F \downarrow X$ is a bicategory.

Proof. The objects, 1-cells, and 2-cells of $F \downarrow X$ are defined above. We structure the rest of the proof as follows:

1. define identity 1-cells and 2-cells;
2. define horizontal and vertical composition for 1-cells and 2-cells;
3. verify each collection of 1-cells and 2-cells between a given pair of objects forms a category;
4. verify functoriality of horizontal composition;
5. define components of the associator and unitor;
6. verify that the associator and unitors are natural isomorphisms; and
7. verify the pentagon and unity axioms.
Step (1). The identity 1-cell for an object \((A, f_A)\) is \((1_A, r')\) where

\[
r' = (1_{f_A} \circ F^0) \circ r^{-1},
\]

shown in the pasting diagram below.

![Pasting Diagram](image)

Step (2). The horizontal composite of 1-cells

\[
(A_0, f_0) \rightarrow (A_1, f_1) \rightarrow (A_2, f_2)
\]

is \((p_1 p_0, \theta')\), where \(\theta'\) is given by the composite of the pasting diagram formed from \(\theta_0, \theta_1\), and \(F^2\) as shown below.

![Pasting Diagram](image)

Horizontal and vertical composites of 2-cells in \(F_{\downarrow}X\) are given by their composites in \(B\), as we now explain. Given 1-cells and 2-cells

\[
(A_0, f_0) \rightarrow (A_1, f_1) \rightarrow (A_2, f_2)
\]

the following equalities of pasting diagrams show that \(F(\alpha_1 \circ \alpha_0)\) satisfies the necessary condition for \(\alpha_1 \circ \alpha_0\) to define a 2-cell in \(F_{\downarrow}X\). The first equality follows by
naturality of $F^2$. The second follows by the conditions for $(\alpha_0)$ and $(\alpha_1)$ separately.

Likewise, given $\alpha$ and $\alpha'$ as below,

$$\text{(3.6)}$$

the composite $\alpha' \alpha$ satisfies the necessary condition to define a 2-cell

$$(\alpha' \alpha) : (p, \theta) \to (p'', \theta'')$$

because $F$ is functorial with respect to composition of 2-cells.

**Step (3).** Vertical composition in $F \downarrow X$ is strictly associative and unital because it is defined $B$. Therefore each collection of 1-cells and 2-cells between a given pair of objects forms a category.

**Step (4).** Likewise, because horizontal composition of 2-cells in $F \downarrow X$ is defined by the horizontal composites in $B$, and these are functorial, it follows that horizontal composition of 2-cells in $F \downarrow X$ is functorial.
**Step (5).** The remaining data to describe in \( F \downarrow X \) are the associator and two unitors. Consider a composable triple of 1-cells

\[
(A_0, f_0) \xrightarrow{(p_0, \theta_0)} (A_1, f_1) \xrightarrow{(p_1, \theta_1)} (A_2, f_2) \xrightarrow{(p_2, \theta_2)} (A_3, f_3).
\]

Lax associativity (2.7) for \( F \) gives an equality of pasting diagrams shown below.

\[
\begin{align*}
F(p_2(p_1p_0)) &= F((p_2p_1)p_0) \quad \leftarrow \quad F(p_2p_1) \quad \leftarrow \quad F((p_2p_1)p_0) \\
\end{align*}
\]

Combining these with the triangles

\[
\begin{align*}
FA_0 \xrightarrow{F\theta_0} FA_1 \xrightarrow{F\theta_1} FA_2 \xrightarrow{F\theta_2} FA_3 \\
\end{align*}
\]

shows that \( Fa_B \) satisfies the relevant ice cream cone condition and hence \( a_B \) defines a 2-cell

\[
(a_B) : ((p_2, \theta_2)(p_1, \theta_1))(p_0, \theta_0) \to (p_2, \theta_2)((p_1, \theta_1)(p_0, \theta_0)).
\]

in \( F \downarrow X \). Note that one must implicitly make use of associators to interpret pasting diagrams of three triangles; the component of \( a_C \) in (3.7) cancels with its inverse to form the composite in the target of \( (a_B) \).

The left and right unitors are defined similarly: the unitors \( r_B \) and \( \ell_B \) satisfy the appropriate ice cream cone conditions and therefore given a 1-cell \( (p, \theta) : (a_0, f_0) \to (a_1, f_1) \), we have 2-cells

\[
(r_B) : (p, \theta)(1_{A_0}, r') \to (p, \theta) \quad \text{and} \quad (\ell_B) : (1_{A_1}, r')(p, \theta) \to (p, \theta).
\]

**Step (6).** Naturality of the associator and unitors defined in the previous step is a consequence of the corresponding naturality in \( B \) and \( C \) together with naturality of \( F^0 \) and \( F^2 \). Moreover, each component is an isomorphism because a lax functor preserves invertibility of 2-cells.

**Step (7).** Because the associator and unitor are defined by the corresponding components in \( B \), it follows that they satisfy the unity and pentagon axioms, (2.2) and (2.3).
**Proposition 3.9.** Suppose $F : B \to C$ is a lax functor of bicategories. Given a 1-cell $u : X \to Y$, there is a strict functor $F \downarrow u : (F \downarrow X) \to (F \downarrow Y)$ induced by whiskering with $u$.

**Proof.** The assignment on 0-, 1- and 2-cells, respectively, is given by

$$(A, f_A) \mapsto (A, uf_A)$$

$$(p, \theta) \mapsto (p, a^{-1}_C \circ (1_u \star \theta))$$

$$(\alpha) \mapsto (\alpha).$$

where the associator $a_C$ is used to ensure that the target of the 2-cell $a^{-1}_C \circ (1_u \star \theta)$ is $(uf_A) \circ (Fp)$.

To show that $F \downarrow u$ is strictly unital, recall that the identity 1-cell of $(A, f_A)$ is $(1_A, r')$ where

$$r' = (1_{f_A} \star F^0) \circ r^{-1}$$

is shown in (3.3). Then, using the functoriality of $(1_u \star -)$, the 2-cell component of $(F \downarrow u)(1_A, r')$ is shown along the top and right of the diagram below. The right unity property from Proposition 2.4 together with naturality of $a_C$ shows that the diagram commutes and therefore $F \downarrow u$ is strictly unital.

![Diagram](image)

A similar calculation using the functoriality of whiskering and naturality of the associator shows that $F \downarrow u$ is strictly functorial with respect to horizontal composition. □

**Definition 3.10.** We call the strict functor $F \downarrow u$ constructed in Proposition 3.9 the **change-of-slice functor**.

## 4. Lax Terminal Objects in Lax Slices

In this section we introduce a specialized notion of terminal object called inc-lax terminal and prove two key results. First, Proposition 4.7 proves that if a lax functor $F$ is essentially surjective, essentially full, and fully faithful, then the lax slices can be equipped with our specialized form of terminal object. Second, Proposition 4.8 proves that if $F$ is furthermore a pseudofunctor, then these terminal objects are preserved by change-of-slice functors. These are the two key properties of lax slices required for the construction of a reverse lax functor in Section 5.

Given an object $X$ of a bicategory $C$, recall Proposition 2.9 describes $\Delta_X$, the constant pseudofunctor at $X$. 
**Definition 4.1.** We say that $t \in C$ is **lax terminal** if there is a lax transformation $k : \text{Id}_C \to \Delta_t$. Such a transformation has component 1-cells $k_X : X \to t$ for $X \in C$ and 2-cells

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{\llap{k_X}} & \swarrow{k_u} & \searrow{k_Y} \\
\Delta_t & & \Delta_t \\
\end{array}
\]

satisfying the lax unity and lax naturality axioms.

**Definition 4.2.** Given lax functors $F, G : B \to C$, we say that a lax transformation $k : F \to G$ is **inc-lax** or **initial-component-lax** if each component $k_X : FX \to GX$ is initial in the category $C(FX, GX)$.

**Definition 4.3.** Suppose that $t \in C$ is a lax terminal object with lax transformation $k : \text{Id}_C \to \Delta_t$. We say $t$ is an **inc-lax terminal** object if $k$ is inc-lax and the component $k_t$ at $t$ is the identity 1-cell $1_t$.

**Explanation 4.4.** The universal property of initial 1-cells implies that, for a 1-cell $u : X \to Y$, the lax naturality constraint $k_u$ is equal to the composite of the left unitor with the universal 2-cell from each $k_X$ to the composite $k_Y u$, as shown below.

**Definition 4.5.** Suppose that $B$ and $C$ have inc-lax terminal objects $(t, k)$ and $(t', k')$, respectively. We say that a lax functor $F : B \to C$ **preserves initial components** if each composite

\[
FX \xrightarrow{Fk_X} Ft \xrightarrow{k'_{(Ft)}} t'
\]

is initial in $C(FX, t')$.

**Lemma 4.6.** Suppose that $F : B \to C$ preserves initial components. If $f : X \to t$ is any initial 1-cell in $B(X, t)$, then the composite

\[
FX \xrightarrow{Ff} Ft \xrightarrow{k'_{(Ft)}} t'
\]

is initial in $C(FX, t')$.

**Proof.** If $f$ is initial, then there is a unique isomorphism $f \cong k_X$. Therefore $Ff \cong Fk_X$ and hence their composites with $k'_{(Ft)}$ are isomorphic. Now

\[
(k'_{(Ft)}) \circ (Fk_X)
\]

is initial by hypothesis, and therefore the result follows. \qed
Now we show that, if $F$ is essentially surjective, essentially full, and fully faithful, then each lax slice $F \downarrow X$ has an inc-lax terminal object, and each change-of-slice functor $F \downarrow u$ preserves initial components. The first of these results requires the axiom of choice, and the second depends on the first.

**Proposition 4.7.** Suppose $F$ is a lax functor which is essentially surjective, essentially full, and fully faithful. Then for each $X \in C$ the lax slice $F \downarrow X$ has an inc-lax terminal object.

**Proof.** Since $F$ is essentially surjective on objects, there is a choice of object $\overline{X} \in B$ and invertible 1-cell

$$f_{\overline{X}} : F\overline{X} \to X$$

with adjoint inverse

$$f_{\overline{X}}^\dagger : X \to F\overline{X}.$$  

Therefore $(\overline{X}, f_{\overline{X}})$ is an object of $F \downarrow X$; we will show that it is an inc-lax terminal object. Given any other object $(A, f_A)$ in $F \downarrow X$, we have a composite

$$F A \xrightarrow{f_A} X \xrightarrow{f_{\overline{X}}^\dagger} F\overline{X}$$

in $C$. Since $F$ is essentially surjective on 1-cells, there is a choice of 1-cell $p_A$ together with a 2-cell isomorphism

$$\theta_A^f : f_{\overline{X}}^\dagger f_A \to F p_A$$

whose mate $\theta_A$ fills the triangle

$$\begin{array}{ccc}
FA & \xrightarrow{F p_A} & F\overline{X} \\
\downarrow f_A & \downarrow \theta_A & \downarrow f_{\overline{X}}^\dagger \\
X & \xrightarrow{f_{\overline{X}}^\dagger} & F\overline{X}
\end{array}$$

Note that $\theta_A$ is therefore also an isomorphism by Lemma 2.20. If $(A, f_A)$ is equal to the object $(\overline{X}, f_{\overline{X}})$, then we require the choice of $(p_{\overline{X}}, \theta_{\overline{X}})$ to be the identity 1-cell $(1_{\overline{X}}, r')$ described in (3.3).

Therefore $(p_A, \theta_A)$ defines a 1-cell $(A, f_A) \to (\overline{X}, f_{\overline{X}})$ in $F \downarrow X$ which is the identity 1-cell if $(A, f_A) = (\overline{X}, f_{\overline{X}})$. Now we show that $(p_A, \theta_A)$ is initial in the category of 1- and 2-cells $(A, f_A) \to (X, f_X)$. The universal property for initial 1-cells then implies that the components defined by $k_{(A, f_A)} = (p_A, \theta_A)$ assemble to form a lax transformation to the constant pseudofunctor at $(\overline{X}, f_{\overline{X}})$.

Given any other 1-cell $(q, \omega) : (A, f_A) \to (\overline{X}, f_{\overline{X}})$, we compose with $\theta_A^{-1}$ to obtain a 2-cell

$$\gamma' : f_{\overline{X}}(F p_A) \to f_{\overline{X}}(F q)$$
shown below.

\[
\begin{array}{ccc}
FA & \xrightarrow{Fq} & F\overline{X} \\
\downarrow F_{pA} & \searrow & \nearrow \omega \\
FX & \xrightarrow{\theta_A^{-1}} & FX \\
\downarrow f_X & \nearrow f_A & \nearrow f_X \\
X & \xrightarrow{\overline{f_X}} & X
\end{array}
\]

Since $f_X$ is an adjoint equivalence, this uniquely determines a 2-cell

\[\gamma : FpA \rightarrow Fq\]

such that $1_{\overline{f_X}} \times \gamma = \omega \theta_A^{-1}$. Therefore, because $F$ is fully faithful on 2-cells, we have a unique 2-cell

\[\overline{\gamma} : pA \rightarrow q\]

such that $F\overline{\gamma} = \gamma$ and hence satisfies the ice cream cone condition shown below.

\[
\begin{array}{ccc}
FA & \xrightarrow{Fq} & F\overline{X} \\
\downarrow F_{pA} & \searrow & \nearrow \omega \\
FX & \xrightarrow{\theta_A^{-1}} & FX \\
\downarrow f_X & \nearrow f_A & \nearrow f_X \\
X & \xrightarrow{\overline{f_X}} & X
\end{array}
\]

Therefore $(\overline{\gamma})$ is a 2-cell in $F \downarrow X$ from $(pA, \theta_A)$ to $(q, \omega)$. The diagram above, together with the invertibility of $\theta_A$ and the uniqueness of both $\gamma$ and $\overline{\gamma}$ implies that $(\overline{\gamma})$ is the unique such 2-cell in $F \downarrow X$. \hfill \square

**Proposition 4.8.** Suppose $F$ is a pseudofunctor which is essentially surjective, essentially full, and fully faithful. Then for each 1-cell $u : X \rightarrow Y$ in $\mathcal{C}$, the strict functor $F \downarrow u$ preserves initial components.

**Proof.** For $(A, f_A) \in F \downarrow X$, let $(pA, \theta_A)$ denote the initial 1-cell from $(A, f_A)$ to the inc-lax terminal object

\[(\overline{X}, f_X) \in F \downarrow X.\]

Let $(\overline{u}, \theta_{\overline{u}})$ denote the initial 1-cell from

\[(F \downarrow u)(\overline{X}, f_X) = (\overline{X}, uf_X)\]

to the inc-lax terminal object

\[(\overline{Y}, f_{\overline{Y}}) \in F \downarrow Y.\]

We must show that the composite of $(\overline{u}, \theta_{\overline{u}})$ with $(F \downarrow u)(pA, \theta_A)$ is initial. This composite is given by $(\overline{u}pA, \theta')$, where $\theta'$ is the 2-cell determined by the pasting
The argument in Proposition 4.7 shows that \( \theta_A \) and \( \theta_U \) are isomorphisms. Since \( F \) is a pseudofunctor by hypothesis, the 2-cells \( F^2 \) are isomorphisms and hence \( \theta' \) is an isomorphism. Then, as in the proof of Proposition 4.7, composition with the inverse of \( \theta' \) shows that \((\Pi p_A, \theta')\) is initial. \( \square \)

5. Quillen Theorem A for Bicategories

In this section we explain how to construct a reverse lax functor \( G \). We assume only that \( F \) is lax functor, that its lax slices are equipped with inc-lax terminal objects, and that these are preserved by change-of-slice. The end of Section 4 explains how, with the axiom of choice, one can choose such data when \( F \) is an essentially surjective, essentially full, and fully faithful pseudofunctor. However, if one has a constructive method for obtaining these data in practice, then Theorem 5.1 gives a construction of \( G \) which does not depend on choice. In Section 6 we show that, under the hypotheses of the Bicategorical Whitehead Theorem 6.1, the \( G \) constructed here is an inverse biequivalence for \( F \).

**Theorem 5.1** (Bicategorical Quillen Theorem A). Suppose \( F : B \to C \) is a lax functor of bicategories and suppose the following:

1. For each \( X \in C \), the lax slice bicategory \( F \downarrow X \) has an inc-lax terminal object \((X, f_X)\). Let \( k^X \) denote the inc-lax transformation \( \text{Id}_{F\downarrow X} \to \Delta_{(X, f_X)} \).
2. For each \( u : X \to Y \) in \( C \), the induced functor \( F\downarrow u \) preserves initial components (Definition 4.5).

Then there is a lax functor \( G : C \to B \) together with lax transformations

\[ \eta : \text{Id}_B \to GF \quad \text{and} \quad \epsilon : FG \to \text{Id}_C. \]

The proof is structured as follows:

1. **Definition 5.2:** define the data for \( G = (G, G^2, G^0) \):
   (a) define \( G \) as an assignment on 0-, 1-, and 2-cells;
   (b) define the components of \( G^0 \) and \( G^2 \)
2. **Proposition 5.9:** Show that \( G \) defines a lax functor:
   (a) show that \( G \) is functorial with respect to 2-cells;
   (b) show that \( G^2 \) and \( G^0 \) are natural with respect to 2-cells;
   (c) verify the lax associativity axiom (2.7)
   (d) verify the left and right unity axioms (2.8).
3. Establish the existence of \( \eta \) and \( \epsilon \):
(a) define the components of $\eta$ and $\varepsilon$;
(b) verify the 2-cell components of $\eta$ and $\varepsilon$ are natural with respect to 2-cells;
(c) verify the unity axiom (2.11) for $\eta$ and $\varepsilon$;
(d) verify the horizontal naturality axiom (2.12) for $\eta$ and $\varepsilon$.

**Definition 5.2.** Suppose $F: B \to C$ is a lax functor satisfying the assumptions of Theorem 5.1.

**Step (1a).** We define an assignment on cells $G : C \to B$ as follows.

- For each object $X$ in $C$, the slice $F \downarrow X$ has an inc-lax terminal object $(\overline{X}, f_X)$.
  Define $G X = \overline{X}$.

- For each 1-cell $u : X \to Y$ in $C$, we have $(\overline{X}, u f_X) \in F \downarrow Y$, and inc-lax terminal object $(\overline{Y}, f_Y) \in F \downarrow Y$. The component of $k^Y$ at $(\overline{X}, u f_X)$ is an initial 1-cell

$$((\overline{\gamma}, \theta_0) : (\overline{X}, u f_X) \to (\overline{Y}, f_Y)).$$

Define $G u = \overline{\gamma}$.

- Given a 2-cell $\gamma : u_0 \to u_1$ in $C$, we have 1-cells in $F \downarrow Y$ given by $(\overline{u_0}, \theta_0)$ and $(\overline{u_1}, \theta_1)$, the components of $k^Y$. Pasting the latter of these with $\gamma$ yields a 1-cell $(\overline{u_1}, \theta_1 (\gamma \star f_X))$ shown in the pasting diagram below.

$$\begin{array}{ccc}
F X & \xrightarrow{F u} & F Y \\
\downarrow f_X & & \downarrow f_Y \\
X & \xrightarrow{\gamma} & Y \\
\gamma \star \theta_1
\end{array}$$

Since $(\overline{u_0}, \theta_0)$ is initial by construction and

$$(\overline{u_1}, \theta_1 (\gamma \star f_X))$$

is another 1-cell in $F \downarrow Y$ with source $(\overline{X}, u_0 f_X)$ and target $(\overline{Y}, f_Y)$, there is a unique 2-cell $(\gamma)$ in $F \downarrow Y$ such that $F \gamma$ satisfies the ice cream cone condition shown below.

$$\begin{array}{ccc}
F X & \xrightarrow{F \gamma} & F Y \\
\downarrow f_X & & \downarrow f_Y \\
X & \xrightarrow{\theta_0} & Y \\
\theta_0 \star \gamma
\end{array}$$

Define $G \gamma = \overline{\gamma}$.

**Step (1b).** Next we define the components of the lax constraints $G^0$ and $G^2$. 

\[ (\text{Diagram and text continue here}) \]
Following the definition of $G$ for $Y = X$ and $u = 1_X$, we obtain a 1-cell
\[ G1_X = 1_X : X \to X \]
together with $\theta_{1_X}$ filling the triangle below.

\[ (\text{Diagram}) \]

Composing $\theta_{1_X}$ with the left unitor $\ell$ we obtain a 1-cell in $F \downarrow X$
\[ (\ell_{\overline{X}} \circ \theta_{1_X}) : (\overline{X}, f_\overline{X}) \to (X, f_X). \]

By the unit condition for inc-lax terminal objects, the identity 1-cell for
\( (\overline{X}, f_\overline{X}) \) is initial and hence we have a unique 2-cell
\[ 1_{GX} = 1_{\overline{X}} : \overline{X} \to 1_X = G1_X \]
satisfying the ice cream cone condition for
\[ (\ell_{\overline{X}} \circ \theta_{1_X}) \quad \text{and} \quad (1_{\overline{X}}, r'). \]

We define $G^0_X$ to be this 2-cell.

Given a pair of composable arrows $u : X \to Y$ and $v : Y \to Z$ in $C$, we have initial 1-cells $(\overline{u}, \theta_{\overline{u}})$ and $(\overline{v}, \theta_{\overline{v}})$ shown below.

\[ (\text{Diagram}) \]

Pasting these together and composing with $F^2_{\overline{u}, \overline{v}}$, we obtain a 1-cell in $F \downarrow Z$
\[ (\overline{v} \circ \overline{u}, \theta') : (\overline{X}, v(uf_\overline{X})) \to (Z, f_Z). \]
where $\theta'$ is given by the following pasting diagram.

$$\begin{align*}
\xymatrix{ & F(\overline{v} \circ \overline{u}) \ar[dr] & \\
FZ \ar[ur] & & F\overline{v} \circ \overline{u} \\
F\overline{X} \ar[ur]^F & & F\overline{Y} \ar[ur]^F \\
\downarrow^{F(\overline{v} \circ \overline{u})} & \downarrow^{F(\overline{v} \circ \overline{u})} & \downarrow^{F(\overline{v} \circ \overline{u})} \\
X \ar[ur]^u & & Y \ar[u]^{\overline{v}} \\
\downarrow^u & & \downarrow^{\overline{u}} \ar[ur]_\theta \\
Z \ar[ur]^Z & & \ar[ur]^Z \\
\downarrow^v & & \downarrow^{\overline{v}} \ar[ur]_\theta \ar[u]\
\end{align*}$$

(5.7)

Now by definition, $(\overline{u}, \theta_{\overline{u}}) = k^1_{(\overline{X}, \overline{u} \circ \overline{X})}$. Therefore by hypothesis (2) the composite $(\overline{v} \circ \overline{u}, \theta')$ is an initial 1-cell $(\overline{X}, v(u \circ \overline{f}_X)) \longrightarrow (\overline{Z}, f_{\overline{Z}})$. We also have the component of $k^2$ at $(\overline{X}, (vu) \circ \overline{f}_X)$. This is an initial 1-cell

$$(\overline{v} \circ \overline{u}, \theta_{\overline{v} \circ \overline{u}}) : (\overline{X}, (vu) \circ \overline{f}_X) \longrightarrow (\overline{Z}, f_{\overline{Z}})$$

where $\theta_{\overline{v} \circ \overline{u}}$ is displayed below.

$$\begin{align*}
\xymatrix{ & F\overline{X} \ar[dr] & \\
F\overline{Z} \ar[ur] & & \ar[ur] \\
\overline{F}\overline{X} \ar[ur]^F & & \ar[ur]^F \\
\downarrow^{F(\overline{v} \circ \overline{u})} & \downarrow^{F(\overline{v} \circ \overline{u})} & \downarrow^{F(\overline{v} \circ \overline{u})} \\
\overline{X} \ar[ur]^u & & \overline{Y} \ar[ur]^u \\
\downarrow^u & & \downarrow^{\overline{u}} \ar[ur]_\theta \\
\overline{Z} \ar[ur]^Z & & \ar[ur]^Z \\
\downarrow^{\overline{v}} & & \downarrow^{\overline{v}} \ar[ur]_\theta \ar[u]\
\end{align*}$$

(5.8)

Composing $\theta_{\overline{v} \circ \overline{u}}$ with the associator

$$a^{-1}_C : v(uf_X) \longrightarrow (uv)f_X$$

yields another 1-cell

$$(\overline{v} \circ \overline{u}, a^{-1}_C \theta_{\overline{v} \circ \overline{u}}) : (\overline{X}, v(u \circ \overline{f}_X)) \longrightarrow (\overline{Z}, f_{\overline{Z}}),$$

and therefore there is a unique 2-cell in $B$

$$(Gv) \circ (Gu) = \overline{v} \circ \overline{u} \longrightarrow \overline{vu} = G(vu)$$

whose image under $F$ satisfies the ice cream cone condition for the triangles (5.7) and (5.8). We define $G_{v,u}^2$ to be this 2-cell.

**Proposition 5.9.** Under the hypotheses of Theorem 5.1, the assignment on cells defined above specifies a lax functor $G : B \longrightarrow C$.

**Proof.** **Step (2a).** To verify that $G$ defines a functor $C(X,Y) \longrightarrow B(GX,GY)$ for each $X$ and $Y$, first note that when $\gamma = 1_u$, then $1_\gamma$ satisfies the ice cream cone condition above, and hence by uniqueness of 2-cells out of an initial 1-cell, we have

$$G1_u = (1_\overline{u}) \circ 1_{(\overline{u})} = 1_{G\overline{u}}.$$
Now we turn to functoriality with respect to vertical composition of 2-cells. Consider a pair of composable 2-cells

\[ u_0 \xrightarrow{\gamma} u_1 \xrightarrow{\delta} u_2 \]

between 1-cells \( u_0, u_1, u_2 \in C(X, Y) \). We will show that the chosen lift \( G(\delta \gamma) = \overline{\delta \gamma} \) is equal to the composite \((G\delta) \circ (G\gamma) = \overline{\delta} \circ \overline{\gamma}\).

To do this, we note that \((\overline{u_0}, \theta_0)\) is an initial 1-cell and therefore we simply need to observe that \(\overline{\delta} \circ \overline{\gamma}\) satisfies the ice cream cone condition for \(\overline{\delta \gamma}\). Then the uniqueness of 2-cells from \((\overline{u_0}, \theta_0)\) to \((\overline{Y}, f_{\overline{Y}})\) will imply the result. This is done by the four pasting diagrams below. The first equality follows by functoriality of \(F\): we have \((F\delta)(F\gamma) = F(\overline{\delta} \circ \overline{\gamma})\). The next two equalities follow by the conditions for \(\overline{\gamma}\) and \(\overline{\delta}\) individually.

Since \(\overline{\delta \gamma}\) is the unique 2-cell satisfying this condition, we must have \(\overline{\delta \gamma} = \overline{\delta} \circ \overline{\gamma}\). Therefore the definition of \(G\) is functorial with respect to vertical composition of 2-cells.

**Step (2b).** Naturality of \(G^0\) is vacuous. Naturality of \(G^2\) follows because \((\overline{\delta} \circ \overline{u}, \theta')\) shown in (5.7) is initial. Therefore given \(\gamma : u_0 \longrightarrow u_1\) and \(\delta : v_0 \longrightarrow v_1\), the two composites

\[ (\overline{u_0 \circ u_0'}, \theta_0') \longrightarrow (\overline{u_1 u_1'}, \theta_{\overline{u_1 u_1'}}) \]

are equal.
Step (2c). Now we need to verify the lax associativity axiom (2.7) and two lax unity axioms (2.8) for \( G \). We show that each of the 2-cells involved is the projection to \( B \) of a 2-cell in a lax slice category, and that each composite in the diagrams is a 2-cell whose source is initial. Thus we conclude in each diagram that the two relevant composites are equal.

First, let us consider the lax associativity hexagon (2.7) for \( G^2 \) and the associators. Given a composable triple

\[
\begin{array}{ccc}
W & \xrightarrow{s} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{u} & Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{v} & Z
\end{array}
\]

we need to show that the following diagram commutes

\[
(Gv)((Gu)(Gs)) \xrightarrow{1 \ast G^2} (Gv)(G(us))
\]

where \( a_B \) and \( a_C \) denote the associators in \( B \) and \( C \) respectively. To do this, we observe that this entire diagram is the projection to \( B \) of the following diagram in \( F \downarrow Z \), where we use two key details from the description in Proposition 3.2:

- The horizontal composition of 2-cells in \( F \downarrow Z \) (namely, the whiskering of 2-cells by 1-cells) is given by horizontal composition in \( B \).
- The associator in \( F \downarrow Z \) is given by \( (a_B) \).

Now the 1-cells \( (\pi, \theta_{\pi}) \), \( (\pi, \theta_{\pi}) \), and \( (\xi, \theta_{\xi}) \) are defined to be components of \( k^X \), \( k^Y \), and \( k^Z \), respectively. We have \( (F \downarrow u)(\pi, \theta_{\pi}) = (\pi, 1_u \ast \theta_{1u}) \) and therefore

\[
(\pi, 1_u \ast \theta_{1u})(\xi, 1_v \ast (1_u \ast \theta_{1u}))
\]

is initial by hypothesis (2) and Lemma 4.6. The strict functor \( F \downarrow v \) sends this composite to

\[
(\pi, 1_v \ast \theta_{1v})(\xi, 1_v \ast (1_u \ast \theta_{1u}))
\]
so the upper-left corner of the hexagon is initial by hypothesis (2) and Lemma 4.6 again. Since \( a_B \) is an isomorphism, this implies that

\[
\left( (\pi, \theta) (\pi, 1_v \ast \theta) \right) (\pi, 1_v u \ast \theta)
\]

is also an initial 1-cell. Therefore the two composites around the diagram are equal and consequently their projections to \( B \) are equal.

**Step (2d).** Next we consider the lax unity axioms (2.8) for a 1-cell \( u : X \to Y \). We use subscripts \( B \) or \( C \) to denote the respective unitors. As with the lax associativity axiom, the necessary diagrams are projections to \( B \) of diagrams in \( F \downarrow Y \), each of whose source 1-cell is initial. Therefore the diagrams in \( F \downarrow Y \) commute and hence their projections to \( B \) commute.

\[
\begin{array}{ccc}
(G u)(G 1_X) \xrightarrow{G^2} G(u 1_X) & & (G 1_Y)(G u) \xrightarrow{G^2} G(1_Y u) \\
G r_C \downarrow & & \downarrow G r_C \\
(1_G)(G u) \xrightarrow{G^0 1} G u & & G(1_G) \xrightarrow{1} G u \\
\end{array}
\]

This completes the proof that \( G \) is a lax functor \( C \to B \). 

**Proof of Theorem 5.1.** Now we turn to the transformations

\[
\eta : \text{Id}_B \to GF \quad \text{and} \quad \epsilon : FG \to \text{Id}_C.
\]

**Step (3a).** The components of \( \epsilon \) are already defined in the construction of \( G \): given an object \( X \), we define \( \epsilon_X = f^\pi_X \) the 1-cell part of the inc-lax terminal object \((X, f^\pi_X)\). For a 1-cell \( u \), we define \( \epsilon_u = \theta \pi \), the 2-cell part of the initial 1-cell

\[
(\pi, \theta) : (X, u f^\pi_X) \to (Y, f^\pi_Y).
\]

To define the components of \( \eta \), suppose \( A \) and \( B \) are objects of \( B \) and suppose \( p : A \to B \) is a 1-cell between them. Then \((A, 1_A)\) defines an object of \( F \downarrow FA \). Therefore there is an initial 1-cell

\[
([A], \theta [A]) : (A, 1_A) \to (FA, f^\pi_{FA})
\]

to the inc-lax terminal object in \( F \downarrow FA \). We define

\[
\eta_A = [A] : A \to FA = G(FA).
\]

Given a 1-cell \( p : A \to B \) in \( B \) we have two different 1-cells in \( F \downarrow FB \)

\[
(A, 1_FA(Fp)) \to (FB, f^\pi_{FB}).
\]

One of these is the composite

\[
[FP, \theta_{FP}] \circ (F \downarrow FP)(\eta_A, \theta_{\eta_A}),
\]

and note that this is initial by hypothesis (2) and Lemma 4.6. The other 1-cell is the composite

\[
(\eta_B, \theta_{\eta_B}) \circ (p, \nu),
\]
where \( \nu \) denotes a composite of unitors. The 2-cell components of the composites (5.15) and (5.16) are given, respectively, by the two pasting diagrams below.

\[
\begin{array}{c}
\text{(5.17)} \\
\begin{tikzcd}
FA \arrow{r}{\eta_A} \arrow[swap]{d}{1_{FA}} & F(FA) \arrow{d}{\theta_{\eta_A}} \arrow{r}{F(\eta_A)} & F(Fp) \arrow{d}[swap]{\eta_p} \\
Fp \arrow[swap]{u}{\theta_p} \arrow[swap]{r}{\eta_p} & F(FA) \arrow{u}{\theta_{\eta_A}} \arrow[swap]{r}{F(Fp)} & F(Fp) \arrow{u}{\eta_p}
\end{tikzcd}
\end{array}
\quad
\begin{array}{c}
\begin{tikzcd}
FA \arrow{r}{\eta_B} \arrow[swap]{d}{1_{FA}} & F(FA) \arrow{d}{\theta_{\eta_B}} \arrow{r}{F(\eta_B)} & F(FB) \arrow{d}[swap]{\eta_p} \\
Fp \arrow[swap]{u}{\nu} \arrow[swap]{r}{\theta_p} & F(FA) \arrow{u}{\theta_{\eta_B}} \arrow[swap]{r}{F(FB)} & F(FB) \arrow{u}{\eta_p}
\end{tikzcd}
\end{array}
\]

Since the diagram at left in (5.17) corresponds to an initial 1-cell, we therefore have a unique 2-cell \( (Fp, \eta_A) \) in \( B \) whose image under \( F \) satisfies the ice cream cone condition with respect to the two outermost triangles in (5.17). We take \( \eta_p \) to be this 2-cell.

**Step (3b).** Naturality of the components \( \varepsilon_u \) with respect to 2-cells \( \gamma : u_0 \to u_1 \) is precisely the condition in (5.4) defining \( G\gamma = \overline{\gamma} \). Naturality of the components \( \eta_p \) with respect to 2-cells \( \omega : p_0 \to p_1 \) follows because the source 1-cell shown at left in (5.17) is initial.

**Steps (3c) and (3d).** The lax transformation axioms for \( \varepsilon \) and \( \eta \) follow immediately from the inc-lax terminal conditions for \( k^X \); the unit axiom follows from the unit condition for \( k^X \), and the 2-cell axiom follows from uniqueness of 2-cells out of an initial 1-cell.

\[\square\]

### 6. The Whitehead Theorem for Bicategories

In this section we apply the bicategorical Quillen Theorem A (5.1) to prove the Bicategorical Whitehead Theorem.

**Theorem 6.1** (Whitehead Theorem for Bicategories). A pseudofunctor of bicategories \( F : B \to C \) is a biequivalence if and only if \( F \) is

1. essentially surjective on objects;  
2. essentially full on 1-cells; and  
3. fully faithful on 2-cells.

**Proof.** One implication is immediate: if \( F \) is a biequivalence with inverse \( G \), then the internal equivalence \( FG \simeq \text{Id}_C \) implies that \( F \) is essentially surjective on objects. Lemma 2.28 proves that \( F \) is essentially full on 1-cells and fully faithful on 2-cells.

If \( F \) is essentially surjective, essentially full, and fully faithful, then Propositions 4.7 and 4.8 show that the lax slices have inc-lax terminal objects and that the strict functors \( F \downarrow u \) preserve initial components. Therefore we apply Theorem 5.1 to obtain \( G : C \to B \) together with \( \varepsilon \) and \( \eta \).

Moreover, the proof of Proposition 4.7 shows that the components \( \varepsilon_X = f^X_\chi \) and \( \varepsilon_u = \theta_u \) are invertible. Likewise, if the constraints \( F^0 \) and \( F^2 \) are invertible then the ice cream cone conditions for \( F(G^0) \) and \( F(G^2) \), together with invertibility of the \( \theta_u \), imply that \( F(G^0) \) and \( F(G^2) \) are invertible. Thus \( G^0 \) and \( G^2 \) are invertible.
because $F$ is fully faithful on 2-cells and therefore reflects isomorphisms. Therefore $G$ is a pseudofunctor.

Likewise in the construction of $\eta_A$ via Proposition 4.7, we note that $\theta_{\eta_A}$ and $f_{\eta A}$ are both invertible, so $F\eta_A$ is invertible. The assumption that $F$ is essentially surjective on 1-cells and fully faithful on 2-cells implies that $F$ reflects invertibility of 1-cells, and therefore $\eta_A$ is invertible. Similarly, the construction of $\eta_p$ under these hypotheses implies that $F(\eta_p)$ is invertible and hence $\eta_p$ is invertible.

Now $\eta$ and $\varepsilon$ are strong transformations with invertible components. Therefore by Proposition 2.25 we conclude that $\eta$ and $\varepsilon$ are invertible strong transformations. Thus $F$ and $G$ are inverse biequivalences. \hfill $\square$

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