REPRESENTATION FORMULAS OF SOLUTIONS AND BIFURCATION SHEETS TO A NONLOCAL ALLEN-CAHN EQUATION

TATSUKI MORI
Graduate School of Engineering
Musashino University
Tokyo, 135-8181, Japan

KOUSUKE KUTO
Department of Applied Mathematics
Waseda University
Tokyo, 169-8555, Japan

TOHRU TSUJIKAWA
Faculty of Engineering
University of Miyazaki
Miyazaki, 889-2192, Japan

SHOJI YOTSUTANI*
Joint Research Center for Science and Technology
Ryukoku University
Seta, Otsu, 520-2194, Japan

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Abstract. We are interested in the Neumann problem of a 1D stationary Allen-Cahn equation with a nonlocal term. In our previous papers [4] and [5], we obtained a global bifurcation branch, and showed the existence and uniqueness of secondary bifurcation point. At this point, asymmetric solutions bifurcate from a branch of odd-symmetric solutions. In this paper, we give representation formulas of all solutions on the secondary bifurcation branch, and a bifurcation sheet which consists of bifurcation curves with heights.

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* Corresponding author: Shoji Yotsutani.
1. Introduction. We are interested in the following nonlinear Neumann problem with a nonlocal term

\[
\begin{cases}
-du_{xx} = (1 - u^2)\left(u - \frac{\mu}{2} \int_{-1}^{1} u \, dx\right), & x \in I := (-1, 1), \\
u_x(\pm 1) = 0, \\
u_x(x) \geq 0, & x \in I
\end{cases}
\]

as a continuation of Kuto-Mori-Tsujikawa-Yotsutani [4, 5], where \(d\) is a positive parameter and \(\mu\) is a nonnegative parameter.

Such type of nonlinear reaction-diffusion equations with nonlocal terms have been studied by many authors (see, e.g., [2], [3], [4], [5], [6], [7], [9], [8], [11], [10], [13], [14], and references therein).

Obviously, (1) has three constant solutions \(u = 0\) and \(u = \pm 1\). The reason why we focus on nondecreasing solutions is that any solution of (1) without the last condition can be constructed by connecting rescaled functions of monotone solutions. It is easy to see that if \(u(x)\) is a solution of (1), then \(-u(-x)\) is also a solution.

We employ a fundamental reduction of (1) to the Neumann problem of a class of the Allen-Cahn equations

\[
\begin{cases}
-du_{xx} = (1 - u^2)(u - \lambda), & x \in I, \\
u_x(\pm 1) = 0, \\
u_x(x) \geq 0, & x \in I
\end{cases}
\]

with the nonlocal constraint

\[
\lambda = \frac{\mu}{2} \int_{-1}^{1} u \, dx.
\]

Thus (1) is equivalent to the system of (2), (3).

We define a set \(S(\mu)\) of nonconstant solutions of the system of (2), (3) as

\[
S(\mu) := \{(\lambda, d, u) \in \mathbb{R} \times \mathbb{R}_+ \times C^2(\bar{I}) : u \text{ is a nonconstant solution of (2), (3)}\}
\]

where \(C^2(\bar{I}) := \{u \in C^2(\bar{I}) : u_x(\pm 1) = 0\}\).

We note that (1) with \(\mu = 0\) is referred as the Chafee-Infante problem [1]:

\[
\begin{cases}
-du_{xx} = (1 - u^2)u, & x \in I, \\
u_x(\pm 1) = 0, \\
u_x(x) \geq 0, & x \in I
\end{cases}
\]

Concerning (5), a detailed structure of the solution set

\[
\Gamma_0 := \{(d, u) \in \mathbb{R}_+ \times C^2(\bar{I}) : u \text{ is a nonconstant solution of (5)}\}
\]

is well known as follows (see, e.g., [1] and [12]): \(\Gamma_0\) bifurcates from the constant solution \(u = 0\) at \(d = 4/\pi^2\) and forms a curve parameterized as \(\Gamma_0 = \{(d, \phi^*(\cdot, d)) : 0 < d < 4/\pi^2\}\). Furthermore, each \(\phi^*(x, d)\) satisfies \(\phi^*(x, d) = -\phi^*(-x, d)\) for any \((x, d) \in \bar{I} \times (0, 4/\pi^2)\). Therefore, this odd symmetry of \(\phi^*(x, d)\) means that

\[
\Gamma_0 := \{(0, d, u) : (d, u) \in \Gamma_0\} \subset S(\mu) \text{ for all } \mu > 0.
\]

In Kuto-Mori-Tsujikawa-Yotsutani [4, 5], they showed the unique existence of the secondary bifurcation branch from \(\Gamma_0\) at the point \(d = \hat{d}(\mu)\) with \(0 < \mu < 1\), which
is obtained by
\[ \hat{d} (\mu) := \frac{1}{(1 + \hat{k}(\mu)^2 K(\hat{k}(\mu))^2}, \]  
(6)

where \( \hat{k}(\mu) \) is the unique solution of
\[ \frac{2}{3} \left\{ \frac{1}{2} + \frac{1}{(1 - k^2)^2} \left( -1 + k^2 + \frac{2E(k)}{K(k)} \right) \right\} = \frac{1}{\mu} \quad (0 < k < 1). \]  
(7)

Moreover, \( \hat{d}(\mu) \) is monotone in \( \mu \in (0, 1) \) and
\[ \hat{d}(\mu) \to \frac{4}{\pi^2} \quad \text{as } \mu \uparrow 1, \]  
(8)
\[ \hat{d}(\mu) \to 0 \quad \text{as } \mu \downarrow 0. \]  
(9)

For concerning (6) and (7), see (4.2), (4.4), (4.11) and (4.13) in [4]. We get (8) and (9) by lemma 4.7 in [4]. For instance, \( \hat{d}(1/2) = 0.2668 \cdots \) and \( \hat{d}(3/4) = 0.3336 \cdots \).

![Bifurcation diagram S(\mu)](image)

**Figure 1.** Bifurcation diagram \( S(\mu) \).

We see secondary bifurcation curves by Theorem 2.1 of [5].

**Theorem A** (Secondary bifurcation curves). For any fixed \( 0 < \mu < 1 \), it holds that
\[ S(\mu) = \tilde{T}_0 \cup C(\mu), \]
where \( C(\mu) \) is a smooth curve parameterized by \( \lambda \in (-\mu, \mu) \) as
\[ C(\mu) = C_-(\mu) \cup C_0(\mu) \cup C_+ (\mu), \]  
(10)

where
\[ C(\mu) := \{ (\lambda, \, d(\lambda; \mu), \, u(\cdot; \lambda, d(\lambda; \mu))) : -\mu < \lambda < \mu \}, \]  
(11)
\[ C_+(\mu) := \{ (\lambda, \, d(\lambda; \mu), \, u(\cdot; \lambda, d(\lambda; \mu))) : 0 < \lambda < \mu \}, \]  
(12)
\[ C_0(\mu) := \{ (0, \, d(0; \mu), \, u(\cdot; 0, d(0; \mu))) \}, \]  
(13)
\[ C_-(\mu) := \{ (-\lambda, \, d(-\lambda; \mu), \, -u(\cdot; -\lambda, d(-\lambda; \mu))) : 0 < \lambda < \mu \}. \]  
(14)
\( \mathcal{C}(\mu) \) crosses \( \tilde{\Gamma}_0 \) at the symmetry breaking bifurcation point \( (\lambda, d, u) = (0, \hat{d}(\mu), \phi^*(\cdot; d(\mu))) \in \tilde{\Gamma}_0, \) and reaches singular limits \( (\lambda, d) = (\pm \mu, 0) \), and

\[
\begin{align*}
\lim_{\lambda \uparrow \mu} u(x; \lambda, d(\lambda, \mu)) &= \begin{cases}
\zeta(\mu) & \text{for } x = -1, \\
1 & \text{for } x \in (-1, 1),
\end{cases} \\
\lim_{\lambda \downarrow -\mu} u(x; \lambda, d(\lambda, \mu)) &= \begin{cases}
-1 & \text{for } x \in [-1, 1), \\
-\zeta(\mu) & \text{for } x = 1,
\end{cases}
\end{align*}
\] (15)

with some \( \zeta(\mu) \in (-1, 1) \).

Theorem A implies that solutions \((d, u)\) belonging to \( \mathcal{C}_\pm(\mu) \) are parameterized by \( \lambda \) \((-\mu < \lambda < \mu)\).

Concerning the case \( \mu \geq 1 \), see Theorems 2.2-2.4 of [5]. There is no secondary bifurcation point in this case, and the secondary bifurcation occurs only in the case \( \mu \in (0, 1) \).

Thus, we focus on the case \( \mu \in (0, 1) \). We are interested in the direction of secondary bifurcation branch \( \mathcal{C}(\mu) \) near the bifurcation point, and a mathematical proof of monotonicity with respect to \( \lambda \) which gives the exact multiplicity of solutions of (1). Moreover, it is important to get information concerning stability of stationary solutions.

The aim of this paper is to obtain exact expressions of all solutions of (2), and a bifurcation sheet by using two parameters. The bifurcation sheet means a surface whose level curves coincide with bifurcation curves of (2) with (3). These are the first step to investigate precise profiles of bifurcation curves.

The contents of this paper is as follows. In Section 2, we state main results. We show theorems of representation formula of solutions of (2). This formula can exhibit not only exact expressions of all solutions of (2) but also detail information on bifurcation curves of (2), (3). Moreover, we show theorems about parameterization of all solutions of (2), representation formula of an integral, representation formula of a bifurcation sheet, bifurcation curves of (2), (3), and another expression of bifurcation curves.

In Section 3, as a preliminary, we give the definition for the elliptic functions and the complete elliptic integrals, fundamental properties of \( \mathcal{A}(p, h) \) appeared in main theorems, and related elementary facts. In Section 4, we give the proofs of main theorems Theorems 2.1, 2.3 and 2.6, whose proofs are not given in Section 2. In Section 5, we give proofs of Propositions 4.1, 4.2 and 4.3, which are essential to Theorems 2.1, 2.3 and 2.6. Finally, in Section 6, we give concluding remarks.

2. Main results. Our strategy to find solutions of the nonlocal problem (2), (3) is as follows. First, we parameterize all solutions of an auxiliary problem (2). Second, we select solutions which satisfies (3). This kind of method was initiated by Lou-Ni-Yotsutani [7] and developed by many authors (see e.g. [3], [4], [5], [9], [11], [10], [14] and references therein).

We recall results concerning (2). It is well known that the following proposition holds. See [1], or \( d_C(\lambda) \) with \( f(u, \lambda) := (1 - u^2)(u - \lambda), \ m(\lambda) = \lambda \) in p.2694 of [4].

**Proposition A** (Existence and uniqueness of solutions of (2)). Let \( \lambda \in (-1, 1) \) and \( d > 0 \). There exists a solution \( u \) of (2), if and only if \( (\lambda, d) \in \mathcal{G} \), where

\[
\mathcal{G} := \left\{ (\lambda, d) : -1 < \lambda < 1, \ 0 < d < \frac{4(1 - \lambda^2)}{\pi^2} \right\}.
\] (16)
Moreover, the solution \( u \) is unique. The solution \( u(x; \lambda, d) \) has properties
\[
-1 < u(x; \lambda, d) < 1, \quad (17)
\]
\[
u(x; -\lambda, d) = -u(-x; \lambda, d). \quad (18)
\]

Now, the main results of this paper are as follows:

We can obtain the following theorem by rearranging Proposition 1 in Kosugi-Morita-Yotsutani [3].

**Theorem 2.1 (Representation formula of solutions of (2)).** Let \((\lambda, d) \in \mathcal{G}\). The unique solution \(u(x; \lambda, d)\) is represented by
\[
u(x; \lambda, d) = \sqrt{\frac{\lambda^2 + 3}{3}} \cdot \frac{(\beta(1 - hs) - \alpha) \cdot sn \left( K(\sqrt{h}) \frac{x + 1}{2}, \sqrt{h} \right)^2 + \alpha}{1 - hs \cdot sn \left( K(\sqrt{h}) \frac{x + 1}{2}, \sqrt{h} \right)^2} + \frac{\lambda}{3}, \quad (19)
\]
\[
s := \frac{p}{1 + \sqrt{1 - hp^2}}, \quad (20)
\]
\[
\alpha := \alpha(p, h) = \frac{2 - (1 + h) p - \sqrt{1 - hp^2}}{\sqrt{(h^2 - h + 1) p^2 - 2 (h + 1) p + 2}}, \quad (21)
\]
\[
\beta := \beta(p, h) = \frac{-(1 - h) p + \sqrt{1 - hp^2}}{\sqrt{(h^2 - h + 1) p^2 - 2 (h + 1) p + 2}}, \quad (22)
\]

where \((p, h) = (p(\lambda, d), h(\lambda, d))\) is the unique solution of the following system of transcendental equation
\[
\left\{
\begin{array}{l}
\mathcal{A}(p, h) = \sin \left( 3 \arctan \left( \frac{\lambda}{\sqrt{3}} \right) \right), \\
\mathcal{E}(p, h) = \frac{\sqrt{3}}{2} \sqrt{\frac{d}{\lambda^2 + 3}},
\end{array}
\right. \quad (23)
\]

with respect to \((p, h)\), and \(\mathcal{A}(p, h)\) and \(\mathcal{E}(p, h)\) are defined by
\[
\mathcal{A}(p, h) := \frac{3 \sqrt{3} (1 - p) (1 - hp) \sqrt{1 - hp^2}}{\left( (h^2 - h + 1) p^2 - 2 (h + 1) p + 2 \right)^{3/2}}, \quad (24)
\]
\[
\mathcal{E}(p, h) := \frac{\sqrt{p (2 - (1 + h) p)}}{\sqrt{(h^2 - h + 1) p^2 - 2 (h + 1) p + 2}}. \quad (25)
\]

Here, \(sn(\cdot, \cdot)\) is Jacobi’s elliptic function, and \(K(\cdot)\) is complete elliptic integral of the first kind.

Moreover, it holds that
\[
(\lambda, d) \in \mathcal{G}_+ \iff (p(\lambda, d), h(\lambda, d)) \in \mathcal{G}_{p, h} \quad \text{and} \quad 0 < p(\lambda, d) < 1, \quad (26)
\]
\[
(\lambda, d) \in \mathcal{G}_0 \iff (p(\lambda, d), h(\lambda, d)) \in \mathcal{G}_{p, h} \quad \text{and} \quad p(\lambda, d) = 1, \quad (27)
\]
\[
(\lambda, d) \in \mathcal{G}_- \iff (p(\lambda, d), h(\lambda, d)) \in \mathcal{G}_{p, h} \quad \text{and} \quad p(\lambda, d) > 1, \quad (28)
\]
where

\[ G_+ := \{ (\lambda, d) : (\lambda, d) \in G \text{ and } \lambda > 0 \}, \]
\[ G_0 := \{ (0, d) : 0 < d < 4/\pi^2 \}, \]
\[ G_- := \{ (\lambda, d) : (\lambda, d) \in G \text{ and } \lambda < 0 \}. \]

In addition, it holds that

\[ d \downarrow 0 \Leftrightarrow h(\lambda, d) \uparrow 1 \]

for fixed \( \lambda \in (-1, 1) \).

For the special case \( \lambda = 0 \), we see the following representation formulas.

**Corollary 2.1** (Representation formulas of solutions of (2) with \( \lambda = 0 \)). Let \( \lambda = 0 \) and \( 0 < d < 4/\pi^2 \). The unique solution \( u(x; 0, d) \) is represented by

\[ u(x; 0, d) = -\frac{1 - \sqrt{1 - h}}{\sqrt{2 - h}} \cdot \frac{1 - (1 + \sqrt{1 - h}) \sn \left( \frac{K(\sqrt{h}) x + 1}{2}, \sqrt{h} \right)}{1 - (1 - \sqrt{1 - h}) \sn \left( \frac{K(\sqrt{h}) x + 1}{2}, \sqrt{h} \right)^2}, \]

where \( h \) is the unique solution of

\[ \frac{2}{(2 - h)K(\sqrt{h})} = d. \]

Moreover, the expression (33) is equivalent to

\[ u(x; 0, d) = \frac{(1 - \sqrt{1 - h}) \sn \left( K\left( \frac{1 - \sqrt{1 - h}}{1 + \sqrt{1 - h}} \right) x, \frac{1 - \sqrt{1 - h}}{1 + \sqrt{1 - h}} \right)}{\sqrt{2 - h}}. \]

We can obtain the following theorem by Theorem 2.1.

**Theorem 2.2** (Parameterization of all solutions of (2)). Let \( P \) be a mapping \( P : G \rightarrow G_{p,h} \) defined by

\[ P(\lambda, d) = (p(\lambda, d), h(\lambda, d)), \]

where \( p = p(\lambda, d), h = h(\lambda, d) \) are given in Theorem 2.1. Then \( P \) is bijection with

\[ P(G_+) = \{ (p, h) : 0 < p < 1, \ 0 < h < 1 \}, \]
\[ P(G_0) = \{ (1, h) : 0 < h < 1 \}, \]
\[ P(G_-) = \{ (p, h) : 1 < p < \frac{2}{1+h}, \ 0 < h < 1 \}, \]

where \( G_+, G_0 \) and \( G_- \) are defined by (29), (30) and (31), respectively. Moreover, let \( P^{-1} : G_{p,h} \rightarrow G \) be an inverse mapping of \( P \), then

\[ P^{-1}(p, h) = (\lambda(p, h), d(p, h)) \text{ for } (p, h) \in G_{p,h}, \]

where

\[ \lambda(p, h) := \sqrt{3} \tan \left( \frac{1}{3} \arcsin \left( \mathcal{A}(p, h) \right) \right), \]
\[ d(p, h) := 4 \cdot \frac{\mathcal{A}(p, h)^2}{\cos^2 \left( \frac{1}{3} \arcsin \left( \mathcal{A}(p, h) \right) \right)}. \]
\(A(p,h)\) and \(E(p,h)\) are defined by (24) and (25), respectively. In addition, \(\lambda = \lambda(p,h), d = d(p,h)\) and \(u = u(x; \lambda(p,h), d(p,h))\) satisfies (2) for \((p,h) \in G_{p,h}\).

Next, we get expressions of an integral of \(u\) by using \((\lambda,d)\) and \((p,h)\).

**Theorem 2.3** (Representation formula of an integral). Let \(u(x; \lambda, d)\) be the unique solution of (2), it holds that
\[
\frac{1}{2} \int_{-1}^{1} u(x; \lambda, d) dx = \sqrt{\frac{\lambda^2 + 3}{3}} \cdot \mathcal{M}(p(h), h(\lambda, d)) + \frac{\lambda}{3}
\]
for \((\lambda, d) \in G\), where
\[
\mathcal{M}(p, h) := \Pi \left( \frac{-hp}{1 + \frac{h}{\sqrt{1-hp^2}}}, \sqrt{\frac{h^2-1}{p^2-2(1+h)p+3}} \right)
\]
P = \(p(\lambda, d), h = h(\lambda, d)\) are given in Theorem 2.1. Here, \(\Pi(\cdot, \cdot)\) is the complete elliptic integral of the third kind.

Moreover, it holds that
\[
\frac{1}{2} \int_{-1}^{1} u(x; \lambda(p, h), d(p, h)) dx
= \frac{\mathcal{M}(p, h)}{\cos \left( \frac{1}{3} \arcsin(A(p, h)) \right)} + \frac{1}{3} \sqrt{3} \cdot \tan \left( \frac{1}{3} \arcsin(A(\mathcal{A}(p, h))) \right)
\]
for \((p, h) \in G_{p,h}\), where \(A(p, h), \lambda(p, h)\) and \(d(p, h)\) are defined by (24), (41) and (42), respectively.

Let us consider (3). It follows from (43) that
\[
\lambda = \mu \cdot \frac{1}{2} \int_{-1}^{1} u(x; \lambda, d) dx
\]
\[
\Leftrightarrow \frac{1}{2} \lambda \int_{-1}^{1} u(x; \lambda, d) dx - \frac{1}{3} = \frac{1}{\mu} - \frac{1}{3}
\]
\[
\Leftrightarrow \left( \frac{1}{2} \lambda \int_{-1}^{1} u(x; \lambda, d) dx - \frac{1}{3} \right)^{-1} = \left( \frac{1}{\mu} - \frac{1}{3} \right)^{-1}
\]
\[
\Leftrightarrow \frac{\lambda}{\sqrt{\frac{\lambda^2}{3} + 1}} \cdot \frac{1}{\mathcal{M}(p(h), h(\lambda, d))} = \left( \frac{1}{\mu} - \frac{1}{3} \right)^{-1}
\]
Let us define \(\Xi(\lambda, d)\) by
\[
\Xi(\lambda, d) := \frac{\lambda}{\sqrt{\frac{\lambda^2}{3} + 1}} \cdot \frac{1}{\mathcal{M}(p(h), h(\lambda, d))} \quad \text{for} \ (\lambda, d) \in G.
\]
Let us call
\[ \{ (\lambda, d, \Xi(\lambda, d)) : (\lambda, d) \in \mathcal{G} \} \quad (51) \]
as a bifurcation sheet of (2), (3), since for each \( \mu \in (0, 1), (1/\mu - 1/3)^{-1} \)-level curve
of \( \Xi(\lambda, d) \) give the bifurcation curve of (2), (3).

It follows from (41) and (50) that
\[ \Xi(\lambda(p, h), d(p, h)) := \frac{\sqrt{3} \cdot \sin \left( \frac{1}{3} \text{Arcsin } (\mathcal{A}(p, h)) \right)}{\mathcal{M}(p, h)} \quad \text{for } (p, h) \in \mathcal{G}_{p, h}. \quad (52) \]

Thus we get the following theorem by (52) and the last part of Theorem 2.1.

**Theorem 2.4** (Representation formula of a bifurcation sheet). It holds that
\[ \left\{ \left( \lambda, d, \left( \frac{1}{2\lambda} \int_{-1}^{1} u(x; \lambda, d) dx - \frac{1}{3} \right)^{-1} \right) : (\lambda, d) \in \mathcal{G} \setminus \{0\} \times (0, 4/\pi^2) \right\} \]
= \{ (\lambda, d, \Xi(\lambda, d)) : (\lambda, d) \in \mathcal{G} \}
= \{ (\lambda(p, h), d(p, h), \Xi(\lambda(p, h), d(p, h))) : (p, h) \in \mathcal{G}_{p, h} \},
(53)

where \( \lambda(p, h), d(p, h), \Xi(\lambda, d) \) and \( \Xi(\lambda(p, h), d(p, h)) \) are defined by (41), (42), (50) and (52), respectively.

In particular,
\[ \left\{ \left( \lambda, d, \left( \frac{1}{2\lambda} \int_{-1}^{1} u(x; \lambda, d) dx - \frac{1}{3} \right)^{-1} \right) : (\lambda, d) \in \mathcal{G}, \lambda \in (0, 1) \right\} \]
= \{ (\lambda(p, h), d(p, h), \Xi(\lambda(p, h), d(p, h))) : (p, h) \in (0, 1) \times (0, 1) \}. \quad (54)

We obtain the following theorem from (46), (49), (50), (52) and Theorem 2.4.

**Theorem 2.5** (Bifurcation curves of (2), (3)). Suppose that \( \mu \in (0, 1) \). Let \( \mathcal{C}_+(\mu) \)
be defined in Theorem A. Then, it holds that
\[ \mathcal{C}_+(\mu) = \{ (\lambda, d, u(\cdot ; \lambda, d)) : \Xi(\lambda, d) = (1/\mu - 1/3)^{-1}, (\lambda, d) \in \mathcal{G}_+ \} \]
= \{ (\lambda(p, h), d(p, h), u(\cdot ; \lambda(p, h), d(p, h))) : \Xi(\lambda(p, h), d(p, h)) = (1/\mu - 1/3)^{-1}, (p, h) \in (0, 1) \times (0, 1) \}, \quad (55)

where \( u(\cdot ; \lambda, d), \Xi(\lambda, d), \mathcal{G}_+, \lambda(p, h), d(p, h) \) and \( \Xi(\lambda(p, h), d(p, h)) \) are defined by (19), (50), (29), (41), (42) and (52), respectively.

For the mathematical analysis of level sets of \( \Xi(\lambda, d) \) and \( \Xi(\lambda(p, h), d(p, h)) \),
\[ \sin((1/3) \cdot \text{Arcsin}(\mathcal{A}(p, h))) \] in \( \Xi(\lambda(p, h), d(p, h)) \) seems to cause troubles. Thus we give an expression without \( \sin((1/3) \cdot \text{Arcsin}(\mathcal{A}(p, h))) \) in the following theorem.

**Theorem 2.6.** Let \( \mu \in (0, 1) \). It holds that a level set of
\[ \Xi(\lambda(p, h), d(p, h)) = \left( \frac{1}{\mu} - \frac{1}{3} \right)^{-1} \quad \text{in } (p, h) \in (0, 1) \times (0, 1) \]
is equal to
\[ \frac{4}{9} \left( \frac{\mathcal{M}(p, h)}{1/\mu - 1/3} \right)^3 - \frac{\mathcal{M}(p, h)}{1/\mu - 3} + \frac{\mathcal{A}(p, h)}{\sqrt{3}} = 0 \quad \text{in } (p, h) \in (0, 1) \times (0, 1), \quad (57) \]
where \( \Xi(\lambda(p, h), d(p, h)) \) is defined by (52).
3. Preliminary. In this section, we give the definition for the elliptic functions and the complete elliptic integrals which are used in this paper. Let \( \text{sn}(x, k) \) and \( \text{cn}(x, k) \) be Jacobi’s elliptic functions. The following properties holds:

\[
\text{sn}^{-1}(z, k) = \int_{0}^{z} \frac{1}{\sqrt{1-k^2\xi^2}} \sqrt{1-\xi^2} \, d\xi \quad (-1 \leq z \leq 1, \ 0 < k < 1),
\]

\[
\text{sn}^2(x, k) + \text{cn}^2(x, k) = 1, \quad \text{cn}(0, k) = 1.
\]

Let \( k \in [0, 1) \) and \(-1 < \nu < 1\). The complete elliptic integrals of the first, second and third kinds are defined by

\[
K(k) := \int_{0}^{1} \frac{dt}{\sqrt{1-k^2t^2}} = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2 \theta}}
\]

\[
E(k) := \int_{0}^{1} \frac{1-k^2t^2}{\sqrt{1-t^2}} \, dt = \int_{0}^{\pi/2} \sqrt{1-k^2\sin^2 \theta} \, d\theta
\]

and

\[
\Pi(\nu, k) := \int_{0}^{1} \frac{dt}{(1+\nu t^2)\sqrt{1-k^2t^2}} = \int_{0}^{\pi/2} \frac{d\theta}{(1-\nu \sin^2 \theta)\sqrt{1-k^2\sin^2 \theta}}
\]

respectively. We see that \( K(k) \) is monotone increasing in \( k \),

\[
K(0) = \frac{\pi}{2}, \quad \lim_{k \to 1} K(k) = \infty
\]

and \( E(k) \) is monotone decreasing in \( k \),

\[
E(0) = \frac{\pi}{2}, \quad \lim_{k \to 1} E(k) = 1.
\]

We show graphs of complete elliptic integrals in Figure 2.

![Figure 2. complete elliptic integrals K(k), E(k) and Π(3/4, k).](image)

We show fundamental properties of \( \mathcal{A}(h, p) \).

**Lemma 3.1.** Let \( \mathcal{A}(p, h) \) be defined by (24). Then it holds that

\[
\mathcal{A}(0, h) = 1 \quad \text{for} \quad h \in [0, 1), \quad (60)
\]

\[
0 < \mathcal{A}(p, h) < 1 \quad \text{for} \quad (p, h) \in (0, 1) \times [0, 1), \quad (61)
\]

\[
\mathcal{A}(1, h) = 0 \quad \text{for} \quad h \in [0, 1), \quad (62)
\]

\[
-1 < \mathcal{A}(p, h) < 0 \quad \text{for} \quad (p, h) \in \mathcal{G}_{p, h} \cap \{(p, h) : p > 1\}, \quad (63)
\]

\[
\mathcal{A} \left( \frac{2}{1+h}, h \right) = -1 \quad \text{for} \quad h \in (0, 1), \quad (64)
\]

\[
\mathcal{A}_p(p, h) < 0 \quad \text{for} \quad (p, h) \in \mathcal{G}_{p, h}. \quad (65)
\]
Proof. We show (65). It holds that

$$A_p(p, h) = \frac{3\sqrt{3}p(2 - (1 + h)p)}{g_2(p, h)^{5/2} \sqrt{1 - hp^2}},$$  \hspace{1cm} (66)

where

$$g_1(p, h) := h(h^2 - 4h + 1)p^2 + 2h(h + 1)p - 2(h^2 + h - 1),$$ \hspace{1cm} (67)
$$g_2(p, h) := (h^2 - h + 1)p^2 - (2h + 2)p + 3.$$ \hspace{1cm} (68)

We have

$$g_1 \left( \frac{2}{1 + h}, h \right) = -2 \left( h^2 - h + 1 \right) \left( 1 - h \right)^2, \hspace{1cm} (69)$$
$$g_{1,p}(0, h) = 2h(h^2 - 4h + 1)p + h + 1, \hspace{1cm} (70)$$
$$g_{1,p}(0, h) = 2h(h + 1), \hspace{1cm} g_{1,p} \left( \frac{2}{1 + h}, h \right) = \frac{6h(1 - h)^2}{1 + h}, \hspace{1cm} (71)$$

which implies $g_{1,p}(p, h) > 0$ and $g_1(p, h) < 0$ for $(p, h) \in G_{p, h}$.

We get

$$g_2(p, h) \geq \begin{cases} 
\frac{(1 - 2h)(2 - h)}{h^2 - h + 1} > 0 & \text{for } 0 \leq h \leq 2 - \sqrt{3}, \\
\frac{3(1 - h)^2}{(1 + h)^2} > 0 & \text{for } 2 - \sqrt{3} < h < 1, 
\end{cases} \hspace{1cm} (72)$$

which implies

$$g_2(p, h) > 0 \hspace{1cm} \text{for} \hspace{1cm} (p, h) \in G_{p, h}. \hspace{1cm} (74)$$

Therefore, we obtain (65). It is easy to see (60)-(64).

We show the graph of $A(p, h)$ in Figures 3.

We will use the following elementary fact.

Lemma 3.2. It holds that

$$\frac{\lambda(9 - \lambda^2)}{(\lambda^2 + 3)^{3/2}} = \sin \left( 3 \arctan \left( \frac{\lambda}{\sqrt{3}} \right) \right) \hspace{1cm} (75)$$

for any $\lambda$.
Proof. Let us put
\[ z := \arctan \left( \frac{\lambda}{\sqrt{3}} \right). \]  
We get
\[ \lambda(9 - \lambda^2) \left( \frac{2}{\sqrt{3} \lambda^2 + 3} \right) = \tan z \left( 3 - \tan^2 z \right) = \cos^3 z \cdot \sin z \left( 3 - \frac{\sin^2 z}{\cos^2 z} \right) = \sin(3z). \]
Thus we obtain (75).

4. Proof of Theorems 2.1, 2.3, 2.6 and Cor. 2.1. We prepare the following propositions to prove Theorems 2.1 and 2.3.

Proposition 4.1. Let \((\lambda, d) \in \mathcal{G}\). The unique solution \(u(x; \lambda, d)\) of (2) is represented by
\[
u(x; \lambda, d) = \sqrt{\frac{\lambda^2 + 3}{3}} \cdot \frac{(\beta(1 - hs) - \alpha) \cdot \sin \left( K(\sqrt{h}) \frac{x + 1}{2}, \sqrt{h} \right)}{1 - hs \cdot \sin \left( K(\sqrt{h}) \frac{x + 1}{2}, \sqrt{h} \right)} + \lambda, \quad (77)
\]
where \((s, h) = (s(\lambda, d), h(\lambda, d))\) is the unique solution of the following system of transcendental equation
\[
\begin{cases}
A(s, h) = \frac{2}{3\sqrt{3}} \cdot \sin \left( 3\arctan \left( \frac{\lambda}{\sqrt{3}} \right) \right), \\
E(s, h) = \frac{\sqrt{3}}{2} \sqrt{\lambda^2 + 3}, \\
(s, h) \in (0, 1) \times (0, 1)
\end{cases}
\]  
with respect to \((s, h)\), and \(A(s, h)\) and \(E(s, h)\) defined by
\[
A(s, h) := \frac{2(hs^2 - 2sh + 1)(hs^2 - 2s + 1)(1 - hs^2)}{\left( \sqrt{3h^2 s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(h + 1)s + 3} \right)^2},
\]
\[
E(s, h) := \frac{\sqrt{2s(1 - s)(1 - sh)K(\sqrt{h})}}{\sqrt{3h^2 s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(1 + h)s + 3}}.
\]
Moreover, it holds that
\[
(\lambda, d) \in \mathcal{G}_+ \iff (s(\lambda, d), h(\lambda, d)) \in \{(s, h) : 0 < s < \sigma(h), \ 0 < h < 1\},
\]
\[
(\lambda, d) \in \mathcal{G}_0 \iff (s(\lambda, d), h(\lambda, d)) \in \{(s, h) : s = \sigma(h), \ 0 < h < 1\},
\]
\[
(\lambda, d) \in \mathcal{G}_- \iff (s(\lambda, d), h(\lambda, d)) \in \{(s, h) : \sigma(h) < s < 1, \ 0 < h < 1\},
\]
where
\[
\sigma(h) := \frac{1}{1 + \sqrt{1 - h}},
\]
\( \mathcal{G}_+, \mathcal{G}_0 \) and \( \mathcal{G}_- \) are defined by (29), (30) and (31), respectively.

In addition, it holds that
\[
d \downarrow 0 \Leftrightarrow h(\lambda, d) \uparrow 1 \quad (87)
\]
for fixed \( \lambda \in (-1, 1) \).

**Proposition 4.2.** Let \( u(x; \lambda, d) \) be the unique solution of (2), it holds that
\[
\frac{1}{2} \int_{-1}^{1} u(x; \lambda, d) \, dx = \frac{\sqrt{\lambda^2 + 3}}{3} \cdot \mathcal{M}(\lambda, d, h(\lambda, d)) + \frac{\lambda}{3} \quad (88)
\]
for \((\lambda, d) \in \mathcal{G}\), where
\[
\mathcal{M}(s, h) := \frac{(hs^2 - 2(1 + h)s + 3) + 4(1 - s)(1 - sh) \cdot \Pi(-sh, \sqrt{h})}{\sqrt{3h^2s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(1 + h)s + 3}} \quad (89)
\]
s = \( s(\lambda, d) \), \( h = h(\lambda, d) \) are given in Proposition 4.1. Here, \( \Pi(\cdot, \cdot) \) is the complete elliptic integral of the third kind.

Moreover, it holds that
\[
\frac{1}{2} \int_{-1}^{1} u(x; \lambda(s, h), d(s, h)) \, dx
\]
\[
= \frac{\mathcal{M}(s, h)}{\cos \left( \frac{1}{3} \arcsin \left( \frac{3\sqrt{3}}{2} A(s, h) \right) \right)} + \frac{1}{3} \cdot \sqrt{3} \cdot \tan \left( \frac{1}{3} \arcsin \left( \frac{3\sqrt{3}}{2} A(s, h) \right) \right) \cos \left( \frac{1}{3} \arcsin \left( \frac{3\sqrt{3}}{2} A(s, h) \right) \right) \quad (90)
\]
for \((s, h) \in (0, 1) \times (0, 1) \), where
\[
\lambda(s, h) := \sqrt{3} \tan \left( \frac{1}{3} \arcsin \left( \frac{3\sqrt{3}}{2} A(s, h) \right) \right) \quad (91)
\]
\[
d(s, h) := 4 \cdot \frac{E(s, h)^2}{\cos^2 \left( \frac{1}{3} \arcsin \left( \frac{3\sqrt{3}}{2} A(s, h) \right) \right)} \quad (92)
\]
and \( A(s, h) \) is defined by (81).

**Proposition 4.3.** Let \( \mathcal{A}(s, h) \), \( \mathcal{E}(s, h) \), \( \mathcal{M}(s, h) \), \( \alpha(s, h) \) and \( \beta(s, h) \) are defined by (81), (82), (89), (78) and (79), respectively. It holds that
\[
\frac{3\sqrt{3}}{2} \cdot \mathcal{A}(s, h) \mid_{s=\frac{p}{1+\sqrt{1-h^2}}} = \mathcal{A}_p(h), \quad (93)
\]
\[
\mathcal{E}(s, h) \mid_{s=\frac{p}{1+\sqrt{1-h^2}}} = \mathcal{E}_p(h), \quad (94)
\]
\[
\mathcal{M}(s, h) \mid_{s=\frac{p}{1+\sqrt{1-h^2}}} = \mathcal{M}_p(h), \quad (95)
\]
\[
\alpha(s, h) \mid_{s=\frac{p}{1+\sqrt{1-h^2}}} = \alpha_p(h), \quad (96)
\]
\[
\beta(s, h) \mid_{s=\frac{p}{1+\sqrt{1-h^2}}} = \beta_p(h), \quad (97)
\]
where \( \mathcal{A}_p(h) \), \( \mathcal{E}_p(h) \), \( \mathcal{M}_p(h) \), \( \alpha_p(h) \) and \( \beta_p(h) \) and defined by (24), (25), (44), (21) and (22), respectively.
Moreover, it holds that
\[
\{(p, h) \in G_{p, h} : M(p, h) > 0\} = (0, 1) \times (0, 1),
\]
\[
\{(p, h) \in G_{p, h} : M(p, h) = 0\} = \{1\} \times (0, 1),
\]
\[
\{(p, h) \in G_{p, h} : M(p, h) < 0\} = G_{p, h} \cap \{(p, h) : p > 1\}.
\]

Before we give proofs of Propositions 4.1, 4.2 and 4.3, we give proofs of Theorems 2.1 and 2.3.

Proofs of Theorems 2.1 and 2.3. Let us introduce a bijection
\[
(s, h) \in (0, 1) \times (0, 1) \rightarrow (p, h) \in G_{p, h}
\]
as
\[
p = 2 \left(hs + \frac{1}{s}\right)^{-1}.
\]
We see that Theorems 2.1 and 2.3 follow from Proposition 4.1, 4.2 and 4.3.

We will give proofs of Proposition 4.1, 4.2 and 4.3 in the subsequent sections.

Now, we give a proof of Theorem 2.6.

Proof of Theorem 2.6. It follows from (52) and (56) that
\[
\sin \left(\frac{1}{3} \arcsin (\mathcal{A}(p, h))\right) = \frac{\mathcal{M}(p, h)}{\sqrt{3} \left(\frac{1}{\mu} - \frac{1}{3}\right)},
\]
which implies
\[
\frac{1}{3} \arcsin (\mathcal{A}(p, h)) = \arcsin \left(\frac{\mathcal{M}(p, h)}{\sqrt{3} \left(\frac{1}{\mu} - \frac{1}{3}\right)}\right).
\]
Hence we get
\[
\mathcal{A}(p, h) = \sin \left(3 \arcsin \left(\frac{\mathcal{M}(p, h)}{\sqrt{3} \left(\frac{1}{\mu} - \frac{1}{3}\right)}\right)\right).
\]
Therefore, we obtain (57) by the triple angle formula.

Proof of Corollary 2.1. We obtain (33) with (34) by putting \(p = 1\) in (23) with (19) in view of (27) and (30).

We will show (35)

It is well known that we have
\[
u(x; 0, d) = \frac{\sqrt{2}}{\sqrt{1 + k^2}} \sin(K(k)x, k),
\]
where
\[
d = \frac{1}{(1 + k^2)K(k)^2}
\]
for \(k \in (0, 1)\) (see, e.g., (4.21) of [4]).

By taking
\[
k = \frac{1 - \sqrt{1 - h}}{1 + \sqrt{1 - h}}
\]
in (106) with (107), we obtain (33) with (34) in view of the identity
\[ K(k) = \frac{1}{1+k} K \left( \frac{2\sqrt{k}}{1+k} \right), \]  
(109)
which is called as Gauss’s transformation. In fact, it follows from the change of variable
\[ \sin \theta = \frac{2}{1+k} \frac{\sin \psi}{1+R} \]
with
\[ R := \sqrt{1-k^2 \sin^2 \psi}, \quad k_1 := \frac{2\sqrt{k}}{1+k} \]
that
\[ 1 - \sin^2 \theta = \frac{2+2R - 4\sin^2 \psi/(1+k)}{(1+R)^2}, \]
\[ 1 - k^2 \sin^2 \theta = \frac{2+2R - 4k \sin^2 \psi/(1+k)}{(1+R)^2}, \]
\[ \cos \theta d\theta = \frac{2}{1+k} \frac{\cos \psi d\psi}{(1+R)R}, \]
and
\[ K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \int_0^{\pi/2} \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta} \sqrt{1-k^2 \sin^2 \theta}} \]
\[ = \int_0^{\pi/2} \frac{1+R}{\sqrt{(2+2R - 4\sin^2 \psi/(1+k)) (2+2R - 4\sin^2 \psi/(1+k))}} \frac{2\cos \psi d\psi}{1+k \cdot R} \]
\[ = \frac{1}{1+k} \int_0^{\pi/2} \frac{1+R \cos \psi d\psi}{\sqrt{(1-\sin^2 \psi)(1+R)^2}} = \frac{1}{1+k} \int_0^{\pi/2} \frac{d\psi}{R} = \frac{1}{1+k} K(k_1). \]

5. Proofs of Proposition 4.1, 4.2 and 4.3. To prove Propositions 4.1 and 4.2, we employ representation formulas obtained in Proposition 1 and its proof in [3].

Lemma 5.1. Let \( E > 0 \) and \( A \) be constants. There exists a solution of
\[ \begin{cases} 
E^2 U_{XX} - U^3 + U - A = 0 & \text{in } (0,1), \\
U_X(0) = U_X(1) = 0, \\
U_X(X) > 0 & \text{in } (0,1), 
\end{cases} \]
if and only if the following system of a transcendental equation with respect to \((s,h)\)
\[ \begin{cases} 
\mathcal{A}(s,h) = A, \\
\mathcal{E}(s,h) = E, \\
0 < s < 1, \quad 0 < h < 1, 
\end{cases} \]
(110)
has a solution, where \( \mathcal{E}(s,h) \) and \( \mathcal{A}(s,h) \) are defined by (81) and (82) respectively. Moreover, the solution is unique.
Let \((s,h)\) be the unique solution of (110), then \(U(X)\) is represented by
\[
U(X; s, h) = \frac{\beta (1 - hs) \text{sn}^2(K(\sqrt{h})X, \sqrt{h}) + \alpha \cdot \text{cn}^2(K(\sqrt{h})X, \sqrt{h})}{(1 - hs) \text{sn}^2(K(\sqrt{h})X, \sqrt{h}) + \text{cn}^2(K(\sqrt{h})X, \sqrt{h})},
\] (111)
where \(\alpha\) and \(\beta\) are defined by (78) and (79).
Moreover, it holds that
\[
\int_0^1 U(X) dX = \mathcal{M}(h, s),
\]
where \(\mathcal{M}(h, s)\) is defined (89).

It is easy to see that the following lemma holds.

**Lemma 5.2.** Let \(\lambda \in (-1, 1)\). Let us put
\[
u(x) := \sqrt{\frac{\lambda^2 + 3}{3}} \cdot U(X) + \frac{\lambda}{3}, \quad X := \frac{x + 1}{2}.
\] (112)
Then \(\nu(x)\) is a solution of (2) if and only if \(U(X)\) is solution of
\[
\begin{cases}
\frac{\sqrt{3}}{2} \sum_{1/4}^{d/4} U_{XX} - U^3 + U - \frac{2}{3\sqrt{3}} \cdot \frac{\lambda(9 - \lambda^2)}{(\lambda^2 + 3)^{3/2}} = 0 \text{ in } (0, 1), \\
U_X(0) = U_X(1) = 0, \\
U_X(X) > 0 \text{ in } (0, 1).
\end{cases}
\] (113)
Moreover, it holds that
\[
\int_{-1}^1 \nu(x) dx = \sqrt{\frac{\lambda^2 + 3}{3}} \cdot \int_0^1 U(X) dX + \frac{\lambda}{3}.
\] (114)

We obtain the following lemmas by combining the above proposition and results by Kosugi-Morita-Yotsutani [3].

**Lemma 5.3.** Let \(\lambda \in (-1, 1)\). There exists a solution \(\nu(x)\) of (2), if and only if the following system of a transcendental equation with respect to \((s, h)\)
\[
\begin{cases}
\mathcal{A}(s, h) = \frac{2}{3\sqrt{3}} \cdot \frac{\lambda(9 - \lambda^2)}{(\lambda^2 + 3)^{3/2}} \\
\mathcal{E}(s, h) = \sqrt{3} \cdot \sqrt{\frac{d}{\lambda^2 + 3}},
\end{cases}
\] (115)
has a solution \((s, h)\), where \(\mathcal{E}(s, h)\) and \(\mathcal{A}(s, h)\) are defined by (81) and (82) respectively. The solution is unique.
For the unique solution \((s, h) = (s(\lambda, d), h(\lambda, d))\), \(\nu(x)\) is represented by (112) with (111), and (114) holds.

**Proofs of Propositions 4.1 and 4.2.** We note that (115) is rewritten as
\[
\begin{cases}
\mathcal{A}(s, h) = \frac{2}{3\sqrt{3}} \cdot \sin \left(3 \text{Arcsin} \left(\frac{\lambda}{\sqrt{3}}\right)\right), \\
\mathcal{E}(s, h) = \frac{\sqrt{3}}{2} \cdot \sqrt{\frac{d}{\lambda^2 + 3}},
\end{cases}
\] (116)
for \(0 < s < 1, \quad 0 < h < 1\).
by Lemma 3.2. Thus Propositions 4.1 and 4.2 follow from Proposition A, Lemma 5.3 with (116).

We can obtain (87) by (59) and (116).

Proof of Proposition 4.3. Let

\[ s := \frac{p}{1 + \sqrt{1 - hp^2}}. \]  \hfill (117)

Then it holds that

\[ 2 \left( hs + \frac{1}{s} \right)^{-1} = p, \]  \hfill (118)

\[ \frac{1}{s} - hs = \frac{1}{s} + hs = \sqrt{1 - hp^2}. \]  \hfill (119)

First, we note that

\[ 3h^2s^4 - 4(h^2 + h)s^3 + (4h^2 + 2h + 4)s^2 - 4(h + 1)s + 3 \]

\[ = s \left( \frac{1}{s} + hs \right) \left( (h^2 - h + 1) \left( \frac{2}{\frac{1}{s} + hs} \right)^2 - 2(h + 1) \cdot \frac{2}{\frac{1}{s} + hs} + 3 \right) \]

\[ = s \left( \frac{1}{s} + hs \right) \left( (h^2 - h + 1) p^2 - (2h + 2)p + 3 \right). \]  \hfill (120)

We have

\[ 2(hs^2 - 2sh + 1)(hs^2 - 2s + 1)(1 - hs^2) \]

\[ = s^3 \left( \frac{1}{s} + hs \right)^3 \left( \frac{1}{s} + hs \right)^2 \cdot \left( \frac{1}{s} - hs \right) \cdot \frac{1}{s} + hs \]

\[ = s^3 \left( \frac{1}{s} + hs \right)^3 2(1 - p)(1 - hp)\sqrt{1 - hp^2}, \]  \hfill (121)

which implies (93).

We have

\[ 2s(1 - s)(1 - sh) = s^2 \left( \frac{1}{s} + hs \right) \left( 2 - (1 + h) \cdot \frac{2}{\frac{1}{s} + hs} \right) \]

\[ = s^2 \left( \frac{1}{s} + hs \right)^2 \frac{p}{2} \cdot (2 - (1 + h)p) \]  \hfill (122)

which implies (94).

We have

\[ 1 - hs^2 - 2(1 - h)s = s \left( \frac{1}{s} + hs \right) \left( \frac{1}{s} - hs \right) \left( \frac{1}{s} + hs \right) \]

\[ = s \left( \frac{1}{s} + hs \right) (- (1 - h)p + \sqrt{1 - hp^2}), \]  \hfill (123)
which implies (97).
We have
\[ 3hs^2 - 2(1 + h)s + 1 = -(1 - hs^2 - 2(1 - h)s + 2(1 + hs^2) - 4s 
= s \left( \frac{1}{s} + hs \right) \left( -(1 - h)p - \sqrt{1 - hp^2} + 2 - 2 \cdot \frac{2}{s + hs} \right) 
= s \left( \frac{1}{s} + hs \right) (2 - (1 + h)p - \sqrt{1 - hp^2}) \]
which implies (96).
We have
\[ -(hs^2 - 2(1 + h)s + 3) = -(3hs^2 - 2(1 + h)s + 1) - 2(1 - hs^2) 
= s \left( \frac{1}{s} + hs \right) \left( -(2 - (1 + h)p - \sqrt{1 - hp^2}) - 2 \cdot \frac{1}{s + hs} \right) 
= s \left( \frac{1}{s} + hs \right) ((1 + h)p - 2 - \sqrt{1 - hp^2}) \]
which implies (95).
We see that (98), (99), (100) are equivalent to
\[
\{(s, h) \in (0, 1) \times (0, 1) : \mathcal{M}(s, h) > 0\} = \{(s, h) : 0 < s < \sigma(h), 0 < h < 1\}, \\
\{(s, h) \in (0, 1) \times (0, 1) : \mathcal{M}(s, h) = 0\} = \{(s, h) : s = \sigma(h), 0 < h < 1\}, \\
\{(s, h) \in (0, 1) \times (0, 1) : \mathcal{M}(s, h) < 0\} = \{(s, h) : \sigma(h) < s < 1, 0 < h < 1\},
\]
where \(\sigma(h)\) is defined by (86). This fact is already proved in Lemma 3.3 of [3].

6. Concluding remarks. Figure 4 shows the bifurcation sheet \(\Xi(\lambda, d)\) defined by (50). Figure 5 shows the bifurcation sheet \(\Xi(\lambda(p, h), d(p, h))\) defined by (52) with \(p = 1 - P\) and \(h = 1 - H\) on \(PH\)-plane. These are drawn based on Theorem 2.4.

Figure 6 shows a level set of \(\Xi(\lambda, d) = (1/\mu - 1/3)^{-1}\). Figure 7 shows a level set of \(\Xi(\lambda(p, h), d(p, h)) = (1/\mu - 1/3)^{-1}\) with \(p = 1 - P\) and \(h = 1 - H\) on \(PH\)-plane. In these figures, we take \(\mu = 3/100, 1/4, 1/2, 3/4, 97/100\). These are drawn based on Theorems 2.4 and 2.5.

They suggest the direction of secondary bifurcation curves near the bifurcation point and monotonicity of this curves with respect to \(\lambda\), which gives important information concerning non-existence of saddle-node bifurcation and stability of bifurcating solutions after the secondary bifurcation point.

In a forthcoming paper, we will show from (57) in Theorem 2.6 that
\[ d(\lambda; \mu) = \hat{d}(\mu) - C(\mu)\lambda^2 + o(\lambda^2) \quad \text{as} \quad \lambda \to 0 \]
for each given \(\mu \in (0, 1)\), where \(C(\mu)\) is a positive constant depending on \(\mu\). Moreover, \(C(\mu)\) is exactly calculated by (57).

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E-mail address: t-mori@musashino-u.ac.jp
E-mail address: kuto@waseda.jp
E-mail address: tujikawa@cc.miyazaki-u.ac.jp
E-mail address: shoji@math.ryukoku.ac.jp