On prescribing scalar curvature on bundles and applications

Leonardo Francisco Cavenaghi and Llohan Dallagnol Sperança

Abstract. In this paper we discuss the problem of prescribing scalar curvature on some fiber bundles and Riemannian submersions. We provide computable conditions to smooth functions on the total spaces of these bundles to be realized as scalar curvature functions for some Riemannian submersions metrics. We apply the results to exotic spheres and tori bundles.

Keywords. Fiber Bundles, Compact Structure Group, Exotic manifolds, Prescribed Scalar Curvature, Tori symmetry.

1. Introduction

As examplified by The Bonnet–Meyers Theorem, The Differentiable Sphere Theorem ([Bre10]), The Poincaré Conjecture ([Per02],[Per03a],[Per03b]), a well known application of geometry consists of its own usage to the understanding of manifolds as topological spaces. However, the converse question, given a class of smooth manifolds, which are the admissible geometries on this class? remains unsolved for almost every manifold. Moreover, there are the very interesting examples of exotic spheres $\Sigma^n$, firstly introduced By J. Milnor in [Mil56]. These manifolds are homeomorphic to the standard sphere $S^n$ but not diffeomorphic to it. More generally, there are other examples of exotic manifolds such as exotic tori, exotic $\mathbb{R}^4$, among others.

Concerning exotic spheres, Gromoll–Meyer constructed the first exotic sphere with a metric of non-negative sectional curvature (see [GM74]); Wilhelm constructed metrics of positive Ricci curvature and almost non-negative sectional curvature in every exotic sphere of dimension 7 (see [Wil01]); later
Grove–Ziller and Goette–Kerin–Shankar built metrics of non-negative sectional curvature on these examples (see \cite{GZ00, GKS20}). Grove–Verdiani–Wilking–Ziller showed that some exotic spheres do not support metrics of non-negative sectional curvature with lots of symmetries (see \cite{GVWZ06}); and Grove–Verdiani–Ziller build an exotic unitary tangent space (see \cite{GVZ11}) with positive sectional curvature.

Nash (\cite{Nas79}), Poor (\cite{Poo75}), Searle–Wilhelm (\cite{SW15}), Wraith, Joachim and Crowley (\cite{Wra97, Wra07, JW08} and \cite{CW17a, CW17b}) proved the existence of metrics of positive Ricci curvature on some exotic manifolds. In \cite{CS18, CS19}, the authors built metrics of positive Ricci curvature on several exotic manifolds and bundles with fibers and/or bases of exotic manifolds. On the other hand, it is now known is there exists an exotic sphere with a metric of positive sectional curvature and Hitchin proved that there are exotic spheres that do not even admit metrics of positive scalar curvature (see \cite{Hit74}).

Considering the aforementioned facts it is natural to ask

**Question 1.** To which extend does the smooth structure determines the geometry?

More particularly,

**Question 2.** Do exotic manifolds admit similar geometries to their “classical” counterpart?

On the other hand, the problem on which functions can be realized as the scalar curvature for some Riemannian metric on a closed connected manifold had great development in the seminal work of Kazdan and Warner (see e.g \cite{KW75a, KW75b, KW75c}). In this paper we approach natural generalizations of this problem, namely:

(i) Given a fiber bundle $F \hookrightarrow M \rightarrow B$ with compact total space and structure group and a smooth function $f : M \rightarrow \mathbb{R}$, can $f$ be realized as the scalar curvature of some Riemannian submersion metric on $M$?

(ii) Given a smooth manifold $P$ with an effective $G$-action by a compact and connected Lie group $G$ and a smooth $G$-invariant function $f : P \rightarrow \mathbb{R}$, can $f$ be realized as the scalar curvature of some $G$-invariant metric on $P$?

Our first result solves (i) and (ii) to the case where $P$ is the total space of a $G$-principal bundle: item 3 in \cite{A} and some Riemannian submersion: Theorems \cite{E, F}

**Theorem A.** Let $F \hookrightarrow M \rightarrow B$ be a fiber bundle where $M, F, B$ and the structure group $G$ are compact. Assume that:

1. A principal orbit of $G$ on $F$ has finite fundamental group,
2. $F$ carries a $G$-invariant metric such that $\text{Ric}_{F^{reg}/G} \geq 1$.

Then
1. There is $\lambda \in (0,1]$, depending only on the geometry of the fiber, such that any smooth function $f : M \to \mathbb{R}$ satisfying $\frac{\min_{p \in B} f}{\max_{p \in B} f} \leq \lambda$ is the scalar curvature for some Riemannian metric on $M$, except maybe if $f = \text{constant} \geq 0$;

2. If $F$ has constant scalar curvature, then any smooth function $f : M \to \mathbb{R}$ is the scalar curvature of some Riemannian metric on $M$, except maybe if $f = \text{constant} \geq 0$;

3. To the case of principal bundles $G \hookrightarrow P \to B$ any smooth function $f$ on its total space (except possibly if $f = \text{constant} \geq 0$) can be realized as the scalar curvature of some $G$-invariant Riemannian submersion metric on $P$.

Theorem A is related to the Question 2 in the following manner. Eells and Kuiper in [EK62] computed the number of 7 (respectively 15)-exotic spheres that are realized as total spaces of sphere bundles. Therefore, by setting $G = O(n + 1)$, $n = 7, 15, F = S^n$, Theorem A enlightens Question 2:

Theorem B (Grou–Rigas). 16 (resp. 4.096) from the 28 (resp. 16.256) diffeomorphisms classes of the 7-dimensional (resp. 15)-exotic spheres, are such that any smooth real function $f : \Sigma^7 \to \mathbb{R}$ (resp. $f : \Sigma^{15} \to \mathbb{R}$) is the scalar curvature for some Riemannian metric on $\Sigma^7$ (resp. $\Sigma^{15}$), except maybe if $f = \text{constant} \geq 0$.

Theorem A is a natural generalization of the following Theorem in [BG] (and Theorema A in [Rig76]), from which Theorem B also follows:

Theorem C (Grou–Rigas). Let $F \hookrightarrow M \xrightarrow{\pi} B$ be a fiber bundle with $M, F, B$ and structure group $G$ compact. Assume that:

1. $F$ carries a $G$-invariant metric with positive sectional curvature;
2. The action of $G$ ($\dim G \geq 2$) on $F$ has only one orbit type.

Then there exists $\lambda \in (0,1)$ such that any smooth function $f : M \to \mathbb{R}$ satisfying $\frac{\min f}{\max f} \leq \lambda$ is the scalar curvature for some Riemannian metric on $M$, except maybe if $f = \text{constant} \geq 0$.

Theorem D below generalizes Theorem B to several classes of bundles over exotic spheres and some connected sums. Moreover, in Section 2 we also provide large classes of bundles to which Theorem A implies similar conclusions to the ones of Theorem D.

We adopt the following convention: $F \hookrightarrow P \times_G F \to B$ denotes the associated bundle to the $G$-principal bundle $G \hookrightarrow P \to B$ with fiber $F$.

Theorem D. Let $\Sigma^7$ and $\Sigma^8$ be any homotopy spheres of dimensions 7 and 8, respectively; $\Sigma^{10}$ be any homotopy 10-sphere which bounds a spin manifold; $\Sigma^{4m+1}, \Sigma^{8m+5}$ be Kervaire spheres of dimensions $4m+1, 8m+5$, respectively. Then, there are explicit $G$-manifolds $P$ such that any smooth function $f$ (except possibly if $f = \text{constant} \geq 0$) is the scalar curvature for some
Riemannian metric on the total space of the following bundles

\[ S^l \hookrightarrow P \times_G S^l \to \Sigma^l, \]
\[ S^l \hookrightarrow P \times_G S^l \to M^l \# \Sigma^l, \]
\[ S^{8r+k} \hookrightarrow P \times_G S^{8r+k} \to (M^{8r+k} \times N^{5-k}) \# \Sigma^{8r+5}, \]
for \( l = 7, 8, 10, 4m+1, k = 0, 1, \) where

(i) \( M^7 \) is any 3-sphere bundle over \( S^4 \),
(ii) \( M^8 \) is either a 3-sphere bundle over \( S^5 \) or a 4-sphere bundle over \( S^4 \),
(iii) \( M^{10} = M^8 \times S^2 \) with \( M^8 \) as in item (ii),
(iv) \( M^{10} \) is any 3-sphere bundle over \( S^7 \), 5-sphere bundle over \( S^5 \) or 6-sphere bundle over \( S^4 \),
(v) \( M^{4m+1} \# \Sigma^{4m+1} \) where

(a) \( S^{2m} \hookrightarrow M^{4m+1} \to S^{2m+1} \) is the sphere bundle associated to any multiple \( \mathbb{O}(2m+1) \to O(2m+2) \to S^{2m+1} \), the frame bundle of \( S^{2m+1} \)
(b) \( \mathbb{C}P^m \hookrightarrow M^{4m+1} \to S^{2m+1} \) is the \( \mathbb{C}P^m \)-bundle associated to any multiple of the bundle of unitary frames \( U(m) \hookrightarrow U(m+1) \to S^{2m+1} \)
(c) \( M^{4m+1} = \frac{U(m+2)}{SU(2) \times U(m)} \)
(d) \( N^{5-k} \) is any manifold with positive Ricci curvature and

(d.i) \( S^{4r+k-1} \hookrightarrow M^{8r+k} \to S^{4r+1} \) is the \( k \)-th suspension of the unitary tangent \( S^{4r-1} \hookrightarrow T_l S^{4r+1} \to S^{4r+1} \),
(d.ii) for \( k = 1 \), \( \mathbb{H}P^m \hookrightarrow M^{8m+1} \to S^{4m+1} \) is the \( \mathbb{H}P^m \)-bundle associated to any multiple of \( \text{Sp}(m) \hookrightarrow \text{Sp}(m+1) \to S^{4m+1} \)
(d.iii) for \( k = 0 \), \( M = \frac{\text{Sp}(m+2)}{\text{Sp}(2) \times \text{Sp}(m)} \)
(d.iv) for \( k = 1 \), \( M^{5m+1} \) is as in item (v).

In addition, there is \( \lambda \in (0, 1) \) such that \( f \) is also smooth as real valued function on (except maybe if \( f = \text{constant} \geq 0 \))

(i') \( P \times_G \Sigma^l \),
(ii') \( P \times_G M^l \# \Sigma^l \),
(iii') \( P \times_G (M^{8r+k} \times N^{5-k}) \# \Sigma^{8r+5} \)

and it satisfies \( \frac{\min f}{\max f} \leq \lambda \), then \( f \) is also the scalar curvature of some Riemannian metric on the following bundles

\[ \Sigma^l \hookrightarrow P \times_G \Sigma^l \to \Sigma^l, \]
\[ M^l \# \Sigma^l \hookrightarrow P \times_G M^l \# \Sigma^l \to M^l \# \Sigma^l, \]
\[ (M^{8r+k} \times N^{5-k}) \# \Sigma^{8r+5} \hookrightarrow P \times_G (M^{8r+k} \times N^{5-k}) \# \Sigma^{8r+5} \to (M^{8r+k} \times N^{5-k}) \# \Sigma^{8r+5}. \]

\(^1\)That is, a bundle whose transition function \( \alpha : S^{n-1} \to G \) is a multiple of \( \tau_{2m} : S^{2m} \to O(2m+1), \tau_{m} : S^{2m} \to U(m) \) or \( \tau_{m}^{\mathbb{H}} : S^{4m+2} \to \text{Sp}(m) \), for \( G = O(2m), U(m+1) \) or \( \text{Sp}(m) \), the transition functions of the orthonormal frame bundle and its reductions, respectively.
We also studied the problem of prescribing function as scalar curvature on Riemannian submersions with specific constraints on the geometry of the fibers that are different in nature to the ones in Theorem A. These are obtained via a straightforward application of Variational Methods for PDE’s. Namely, define the constants

\[ b_k := \frac{k + 1}{8k}, \quad c_k := \frac{(k + 1)^2}{8(k - 1)k}, \quad \theta := \frac{2(k - 1)}{k + 1}, \quad k \geq 2. \]

**Theorem E.** Let \( \pi : (F^k, g_F) \hookrightarrow (M^{n+k}, g) \rightarrow (B^n, h) \) be a Riemannian submersion where \( M \) is orientable closed and connected and \( F \) is compact and connected. Assume that:

1. \( \min_F \text{scal}_F \leq 0 \),
2. \( (F, g_F) \) is a minimal submanifold of \( (M, g) \),
3. If \( A \) denotes the Gray–O’Neill tensor of the submersion \( \pi \), then

\[
\max_M \left\{ 3 \sum_{i,j} |A_{e_i}e_j|^2 - 2 \sum_{i,r} |A_{e_i}v_r|^2 \right\} \leq 0,
\]

where \( \{e_i\} \) is a \( g \)-orthonormal basis for \( \mathcal{H} \) and \( \{v_i\} \) is a \( g \)-orthonormal basis for \( \mathcal{V} \).

Then any smooth function \( f \in C^\infty(M; \mathbb{R}) \) satisfying

\[
\frac{\lambda_1}{2 + \epsilon} - b_k \max_M (f - \text{scal}_g + \text{scal}_F) + c_k \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1} + b_k \min_M \delta A > 0,
\]

for some \( \epsilon > 0 \), where \( \lambda_1 \) denotes the first positive eigenvalue of \(-\Delta_B\), is the scalar curvature for some Riemannian submersion metric on \( M \).

If \( M \) is isometric to a product, once its fibers are totally geodesic submanifolds and \( A \equiv 0 \) one has

**Theorem F.** Let \( M = (B^n \times F^k, g_B \times g_F) \) be a closed and connected Riemannian manifold such that \( B \) is orientable. Suppose that \( \min_F \text{scal}_F \leq 0 \). Then any smooth function \( f \in C^\infty(M; \mathbb{R}) \) satisfying

\[
\frac{\lambda_1}{2 + \epsilon} - b_k \max_B (f - \text{scal}_B) + c_k \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1} > 0,
\]

for some \( \epsilon > 0 \), where \( \lambda_1 \) denotes the first positive eigenvalue of \(-\Delta_B\), is the scalar curvature for some Riemannian submersion metric on \( M \).

Theorems E and F are natural generalizations of the problem of prescribing constant scalar curvature via warped products, largely studied in [DD87], [EYTK96]. Under the hypothesis of positive Ricci curvature on \( B \) and assuming that \( M \) is isometric to a product, one derives a different condition:
Corollary G. Let $M^{n+k} = (B^n \times F^k, g_B \times g_F)$ be a closed and connected Riemannian manifold such that $B$ is orientable. Assume that $\text{Ric}(g_B) \geq (n-1)$ and that $\min_F \text{scal}_F \leq 0$. Then any smooth function $f : M \to \mathbb{R}$ such that

$$n \left( \frac{8k}{(2+\epsilon)(k+1)} + (n-1) \right) > \max_B f + \frac{c_k}{b_k} \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1},$$

for some $\epsilon > 0$, is the scalar curvature function for some Riemannian submersion metric on $M$.

Our approach to prove Theorems E, F and Corollary G follows closely the work [EYTK96]. In this, the authors deal essentially with the case in which the manifold $F$ has constant scalar curvature, $\text{scal}_F = c$, considering separately the cases $c > 0$, $c < 0$, $c = 0$. In this sense our results are natural generalizations of theirs.

In addition, we justify the relevance of our results by the fact that the presented hypotheses are easily verifiable, for instance:

Theorem H. For each $n \geq 1$, there exist infinitely many diffeomorphism types of $n$-tori bundles

$$T^n \hookrightarrow (M, g) \to B,$$

over closed simply-connected smooth $(n + 4)$-manifolds $B$ that realizes infinitely many spin and non-spin diffeomorphism types, such that any smooth function on $M$ satisfying

$$\frac{8n(n+4)}{(2+\epsilon)(n+1)} > \max_M (f - \text{scal}_g),$$

for some $\epsilon > 0$, is the scalar curvature for some Riemannian submersion metric on $M$.

If $M$ is a trivial bundle, then we give a partial answer to Question 1 to a class of smooth manifolds with positive Ricci curvature and tori symmetry.

Corollary I. For each $n \geq 5$ there exist infinitely many diffeomorphism types of closed simply-connected smooth $n$-manifolds $B$, realizing infinitely many spin and non-spin diffeomorphism types and with a $T^{n-4}$-invariant Riemannian metric with positive Ricci curvature, such that any smooth function $f : B \to \mathbb{R}$ satisfying

$$\frac{8(n-4)n}{(2+\epsilon)(n-3)} + n(n-1) > \max_B f$$

is the scalar curvature for some Riemannian metric on $B$.

Structure of the article. In Section 2 we prove Theorems A and D. Section 3 is divided in two subsections separating the proofs of Theorems E, F and Corollary G. In subsection 3.1 we prove Theorem F to well motivate the proof of Theorem E on the subsequent subsection. Finally, in subsection 3.3 we prove Theorem H and I. We also decided to include an appendix with
some needed formulae for the results on the paper. These are extracted from [GW09] up to errata.

2. Prescribing scalar curvature on fiber bundles with compact structure group and applications

In this section we prove Theorems A and D.

To the proof of Theorem A we make a simple and straight application of the following result by Kazdan-Warner:

**Theorem 2.1 (Kazdan-Warner).** Let \((M, g)\) be a compact manifold. Denote by \(\text{scal}_g\) the scalar curvature of \(g\). Let \(f \in C^\infty(M)\) be a smooth function on \(M\). If there exists a constant \(c > 0\) such that
\[
\min_p cf < \text{scal}_g(p) < \max_p cf, \quad \forall p \in M,
\]
there exists a Riemannian metric \(\tilde{g}\) on \(M\) such that \(\text{scal}_{\tilde{g}} = f\).

**Proof.** See [KW75a, Theorem A]. \(\square\)

**Proof of Theorem A.** According to the hypothesis 2 and Theorem A in [SW15], \(F\) carries a \(G\)-invariant Riemannian metric \(g_F\) with positive Ricci curvature. Given any Riemannian metric \(g_B\) on \(B\), consider on \(M\) the unique Riemannian submersion metric \(g\) such that its fibers are totally geodesic submanifolds isometric to \((F, g_F)\) (see [GW09, Proposition 2.7.1, p. 97]). We reinforce that fact that, to the case of principal bundles, this metric can be made \(G\)-invariant by defining a Kaluza-Klein \(G\)-invariant metric on the total space (see [CS18, Proposition 5.1, p. 27]).

Fix \(p \in M\) and let \(\{e_i\}_{i=1}^k\) be an orthonormal base for \(H_p\) and \(\{e_j\}_{j=k+1}^n\) be an orthonormal base for \(V_p\). Note that \(\{e_i\}_{i=1}^k \cup \{e^{-t} e_j\}_{j=k+1}^n\) is a \(g_t\)-orthonormal base to \(T_p M\). Using the formulae given by Proposition A.3 we obtain an expression for the scalar curvature of \(g_t\):
\[
\text{scal}_t(p) = \text{scal}_t^H(p) + 2e^{2t} \sum_{i,j} |A^*_t e_i e_j|^2 + e^{-2t} \text{scal}_F(p).
\]

Moreover,
\[
\text{scal}_t^H(p) = \text{scal}_B(p)(1 - e^{2t}) + e^{2t} \text{scal}_g^H(p).
\]

Denote by \(s_t := \min_{p \in B} \text{scal}_t(p)\) and by \(S_t := \max_{p \in B} \text{scal}_t(p)\). Then note that
\[
\lambda := \lim_{t \to -\infty} \frac{s_t}{S_t} = \frac{\min_{p \in B} \text{scal}_F}{\max_{p \in B} \text{scal}_F} \leq 1.
\]

Therefore, for each \(\lambda' < \lambda\) one can find \(t > 0\) such that
\[
\lambda' = \frac{\min f}{\max f} < \frac{s_{-t}}{S_{-t}} \leq 1,
\]
so the result follows from Theorem 2.1. The case where \( f = \text{constant} < 0 \) is guaranteed for Theorem C in [KW75a]. \qed

**Remark.** Apart from the case where the fiber bundle is a \( G \)-principal bundle, where the orbital distance metric is such that the the induced metric on the total space is a Riemannian submersion metric, there is no way to ensure that the metric realizing the prescribed function is a Riemannian submersion metric.

We now prove Theorem D and generalize it to several bundles. These constructions are deeply relied on the ones in [CS19].

Consider a compact connected principal bundle \( G ↪ P → M \) with a principal action \( \bullet \). Assume that there is another action on \( P \), which we denote by \( \ast \), that commutes with \( \bullet \). This makes \( P \) a \( G \times G \)-manifold. If one assumes that \( \ast \) is free, one gets a \( \ast \)-diagram of bundles:

\[
\begin{array}{c}
G \\
\bullet \\
\ast \ \\
M \\
\end{array}
\begin{array}{c}
P \\
\pi' \\
\pi \\
M' \\
M \\
\end{array}
\]

(7)

In (7), \( M \) is the quotient of \( P \) by the \( \bullet \)-action and \( M' \) is the quotient of \( P \) by the \( \ast \)-action.

Once \( \bullet \) and \( \ast \) commute, \( \bullet \) descends to an action on \( M' \) and \( \ast \) descends to an action on \( M \). We denote the orbit spaces of these actions by \( M'/\bullet \) and \( M/\ast \), respectively. Corollary 5.2 in [CS18] implies that one can choose a Riemannian metric \( g' \) on \( M' \) such that the orbit spaces \( M/\ast \) and \( M'/\bullet \) are isometric, as metric spaces. Furthermore, it can be shown that the orbits of the \( \ast \)-action on \( M \) have finite fundamental group if, and only if, the orbits of the \( \bullet \)-action does [CS18, Theorem 6.4]. This implies that if the \( G \)-manifold \( M \) satisfies the hypotheses of Theorem A, then \( M' \) also does.

The idea on considering diagrams like (7) is that \( M \) and \( M' \) can be taken as homeomorphic manifolds that are not diffeomorphic to each other. Moreover, one can consider the following associated bundles

1. The associated bundle \( M ↪ P \times_G M → M \) to \( \pi : P → M \),
2. The associated bundle \( M ↪ P \times_G M → M' \) to \( \pi' : P → M' \),

On Theorem D the manifolds \( M \) are always standard spheres with metrics of constant sectional curvature. Therefore, one obtains the following:

**Theorem 2.2.** Let \( M \) be a standard sphere with a metric of constant sectional curvature. Then any smooth function \( f : P \times_G M → \mathbb{R} \) (except maybe if \( f = \text{constant} \geq 0 \)) is the scalar curvature of some Riemannian metric on \( M ↪ P \times_G M → M \) if, and only if, it is the scalar curvature of some Riemannian metric on \( M ↪ P \times_G M → M' \).
On the other hand, any smooth function \( f : P \times_G M \to \mathbb{R} \) is a well defined function (possible not smooth) on \( f : P \times_G M' \to \mathbb{R} \). So we can conclude the following:

**Theorem 2.3.** Any smooth function \( f : P \times_G M \to \mathbb{R} \) that is also smooth as a real valued function on \( P \times_G M' \) (except maybe if \( f = \text{constant} \geq 0 \)) and satisfies

\[
\frac{\min_{P \times_G M'} f}{\max_{P \times_G M'} f} \leq \frac{\min_{M'} \text{scal}_{g'}}{\max_{M'} \text{scal}_{g'}}.
\]

is the scalar curvature of some Riemannian metrics on both \( P \times_G M \) and \( P \times_G M' \).

Theorem 2.2 and 2.3 finish the proof of Theorem D once the mentioned examples are constructed by the means of cross diagrams such as [7] in [CS18, Theorem A].

### 3. Prescribing scalar curvature on some Riemannian submersion and applicationss

#### 3.1. Proof of Theorem F

Once the proof of Theorem F is written, the proof of Theorem E follows similarly after small modifications. Let \( B^n \) be a closed smooth manifold and \( F \) be a compact manifold and \( f : B \times F \to \mathbb{R} \) be a smooth function. Theorem F is proved using Variational Methods to ensure the existence of positive solutions for the following PDE:

\[
\Delta_B u + \frac{k+1}{4k} (f - \text{scal}_B) u - \frac{k+1}{4k} u \frac{k-3}{k+1} \text{scal}_F = 0, \tag{8}
\]

Indeed, equation (8) is obtained from Lemma 1.

**Lemma 1.** Let \((B, g_B)\) and \((F, g_F)\) be Riemannian manifolds and \( \phi : B \to \mathbb{R} \) be a smooth function. Denote by \( \tilde{g} \) the warped metric on \( B \times \exp F \). Then the scalar curvature of \( \tilde{g} \) has the following expression

\[
\tilde{\text{scal}} = \text{scal}_B + e^{-2\phi} \text{scal}_F - k(k-1) |\nabla \phi|^2 - 2k |\nabla \phi|^2 - 2k \Delta_B \phi. \tag{9}
\]

Given \( f \in C^\infty(B \times F; \mathbb{R}) \), consider the PDE

\[
\text{scal}_B + e^{-2\phi} \text{scal}_F - f = k(k-1) |\nabla \phi|^2 + 2k \{ |\nabla \phi|^2 + \Delta_B \phi \}. \tag{10}
\]

The solution \( \phi \) is such that \( f \) is the scalar curvature of the warped metric \( \tilde{g} = g_B + e^{2\phi} g_F \).

To obtain equation (9) we note that if \( \phi = \log \varphi \) then \( \nabla \phi = \frac{1}{\varphi} \nabla \varphi \). Hence,

\[
\Delta_B \phi = -\frac{1}{\varphi^2} |\nabla \varphi|^2 + \frac{1}{\varphi} \Delta_B \varphi.
\]
Once, $e^{-2\phi} = \varphi^{-2}$ then

$$k(k - 1)|\nabla \phi|^2 + 2ke^{-2\phi} \left\{ |\nabla \phi|^2 + \Delta_B \phi \right\} = \frac{k(k - 1)}{\varphi^2} |\nabla \varphi|^2 + 2k \left\{ \frac{1}{\varphi^2} |\nabla \varphi|^2 - \frac{1}{\varphi^2} |\nabla \varphi|^2 + \frac{1}{\varphi} \Delta_B \varphi \right\}.$$ 

Therefore,

$$\text{scal}_B + \varphi^{-2} \text{scal}_F - \text{f} = \frac{k(k - 1)}{\varphi^2} |\nabla \varphi|^2 + \frac{2k}{\varphi} \Delta_B \varphi. \quad (11)$$

By changing variable $\varphi = u^{\frac{2}{k+1}}$ we obtain

$$\nabla \varphi = \frac{2}{k + 1} u^{\frac{k}{k+1}} \nabla u, \quad \Delta_B \varphi = \left( 2 \frac{(1 - k)}{(k + 1)^2} u^{\frac{2k}{k+1}} |\nabla u|^2 + \frac{2}{k + 1} u^{\frac{k}{k+1}} \Delta_B u \right).$$

Consequently, substituting the above term on equation (11) we obtain

$$\text{scal}_B + u^{-\frac{4}{k+1}} \text{scal}_F - \text{f} = 4 \frac{k(k - 1)}{(k + 1)^2} u^{-\frac{4(k - 1)}{k+1}} |\nabla u|^2 + 2k u^{-\frac{2}{k+1}} \left( 2 \frac{(1 - k)}{(k + 1)^2} u^{\frac{2k}{k+1}} |\nabla u|^2 + \frac{2}{k + 1} u^{\frac{k}{k+1}} \Delta_B u \right).$$

Remark. To show that PDE (8) has a positive solution, assume that $f, \text{scal}_B, \text{scal}_F$ are continuous functions. Denote by $H^1(B)$ the Sobolev Space $W^{1,2}(B)$ and define the following functional $J : H^1(B) \times F \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_B |\nabla u|^2 - \left( \frac{k + 1}{8k} \right) \int_B \left( f - \text{scal}_B \right) u^2 + \frac{(k + 1)^2}{8(k - 1)k} \int_B \text{scal}_F u^{2(k-1)/(k+1)}, \quad \forall u \in H^1(B). \quad (12)$$

Setting

$$b_k := \frac{k + 1}{8k}, \quad c_k := \frac{(k + 1)^2}{8(k - 1)k}, \quad \theta := \frac{k - 1}{k + 1},$$

the functional (12) can be written as

$$J(u) = \frac{1}{2} \int_B |\nabla u|^2 - b_k \int_B \left( f - \text{scal}_B \right) u^2 + c_k \int_B u^\theta \text{scal}_F. \quad (13)$$

Remark. Observe that $0 < \theta \leq 2$. 

We obtain the desired solution \( u \) to equation (8) as a minimum for \( J \) on the set \( M := \{ u \in H^1(B) : u \geq \epsilon_0, \int_B u^\theta \geq 1 \} \). Precisely, Theorem \( \text{[F]} \) follows from the following Lemma.

**Lemma 2.** Let \( \epsilon_0 > 0 \) be arbitrarily small and define \( M := \{ u \in H^1(B) : u \geq \epsilon_0, \int_B u^\theta \geq 1 \} \). Assume that \( f, \text{scal}_B, \text{scal}_F \) are continuous functions and that \( \min_F \text{scal}_F \leq 0 \). Let \( \lambda_1 \) be the first positive eigenvalue of \(-\Delta_B\). Assume that there is \( \epsilon > 0 \) such that the following inequality holds

\[
\alpha := \frac{\lambda_1}{2 + \epsilon} - b_k \max_B ( f - \text{scal}_B ) + c_k \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1} > 0.
\] (14)

Then

1. There is a constant \( c_0 > 0 \) such that \( J\big|_M > c_0 \),
2. \( J\big|_M \) is coercive,
3. \( J\big|_M \) is weakly lower semi-continuous,
4. \( M \) is weakly closed.

**Remark.** The necessity of \( \epsilon \) on the hypothesis of Lemma \( \text{[2]} \) is justified to guarantee that \( J\big|_M \) is coercive.

We now prove Lemma \( \text{[2]} \).

**Proof of Lemma \( \text{[2]} \)**

1. It is easy to see that \( M \neq \emptyset \). Therefore, let \( u \in M \). Recall the classical Poincaré inequality

\[
\int_B |\nabla u|^2 \geq \lambda_1 \int_B u^2.
\] (15)

According to the continuity of \( f, \text{scal}_B, \text{scal}_F \) we have

\[
J(u) \geq \frac{1}{2} \int_B |\nabla u|^2 - b_k \max_B (f - \text{scal}_B) \int_B u^2 + c_k \min_F \text{scal}_F \int_B u^\theta.
\]

Once \( B \) is compact and \( \theta \leq 2 \) the Hölder inequality implies that there is a continuous immersion \( L^2(B) \hookrightarrow L^\theta(B) \). Moreover,

\[
\left( \int_B u^\theta \right)^{\frac{2}{2^\theta}} \leq (\text{vol}(B))^{\frac{2}{\theta} - 1} \int_B u^2.
\]

Since \( \int u^\theta \geq 1 \) and \( \theta \leq 2 \) one has

\[
\int_B u^\theta \leq (\text{vol}(B))^{\frac{2}{\theta} - 1} \int_B u^2.
\] (16)

Since we assumed that \( \min_F \text{scal}_F \leq 0 \) one has

\[
c_k \min_F \text{scal}_F \int_B u^\theta \geq c_k \min_F \text{scal}_F (\text{vol}(B))^{\frac{2}{\theta} - 1} \int_B u^2.
\] (17)

Therefore,

\[
J(u) \geq \frac{1}{2} \int_B |\nabla u|^2 - b_k \max_B (f - \text{scal}_B) \int_B u^2 + c_k \min_F \text{scal}_F (\text{vol}(B))^{\frac{2}{\theta} - 1} \int_B u^2.
\] (18)
According to equation (15)

\[
J(u) \geq \left( \frac{\lambda_1}{2} - b_k \max_B (f - \text{scal}_B) + c_k \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1} \right) \int_B u^2.
\]

(19)

By hypothesis,

\[
\frac{\lambda_1}{2} - b_k \max_B (f - \text{scal}_B) + c_k \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1} > 0.
\]

According to equation (16) and the definition of \( M \), there is \( c_0 > 0 \) such that \( J(u) \geq c_0, \forall u \in M \).

2. To prove coerciveness we observe that

\[
J(u) \geq \frac{1}{2} \int_B |\nabla u|^2 - \left( b_k \max_B (f - \text{scal}_B) - c_k \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1} \right) \int_B u^2.
\]

(20)

From equation (14) we get

\[
J(u) > \frac{1}{2} \int_B |\nabla u|^2 - \frac{\lambda_1}{2 + \epsilon} \int_B u^2.
\]

(21)

According to the Poincaré inequality (15)

\[
J(u) > \lambda_1 \left( \frac{1}{2} - \frac{1}{2 + \epsilon} \right) \int_B |\nabla u|^2
\]

(22)

\[
J(u) > \frac{\lambda_1 \epsilon}{2(2 + \epsilon)} \int_B |\nabla u|^2,
\]

(23)

from where it follows that \( J \) is coercive.

3. To see that \( J \) is weakly lower semi-continuous, note that according to Kondrachov’s Theorem, one has that \( H^1(B) \) compactly embeds into \( L^2(B) \). Let \( \{u_n\} \subset M \) weakly converging to \( u \in H^1(B) \). Then \( u_n \rightarrow u \in L^2(B) \). Once \( B \) is compact and and \( \text{scal}_B, \text{scal}_F, f \) are continuous functions, one has

\[
\int_B |\nabla u|^2 \leq \liminf_{n \to \infty} \int_B |\nabla u_n|^2
\]

and, according to the Dominated Convergence Theorem,

\[
\liminf_{n \to \infty} J(u_n) = \liminf_{n \to \infty} \left( \int_M |\nabla u_n|^2 - b_k \int_B (f - \text{scal}_B) u_n^2 + c_k \int_B u_n^\theta \text{scal}_F \right)
\]

\[
\geq \int_M |\nabla u|^2 - b_k \int_B (f - \text{scal}_B) u^2 + c_k \int_B u^\theta \text{scal}_F = J(u).
\]

4. This also follows from the compact embedding of \( H^1(B) \) into \( L^2(B) \), that admits a continuous immersion into \( L^\theta(B) \). More precisely, if \( \{u_n\} \subset M \) weakly converges to \( u \in H^1(B) \), then \( \{u_n\} \) strongly converges to \( u \) in \( L^\theta(B) \). Therefore,

\[
1 \leq \lim_{n \to \infty} \int_B u_n^\theta = \int_B u^\theta.
\]
Moreover, there exists a subsequence \( \{u_{n_j}\} \) almost everywhere pointwise converging to \( u \). Hence, \( u = \lim_{j \to \infty} u_{n_j} \geq \epsilon_0 \), what finishes the proof. \( \square \)

**Proof of Theorem E**

According to Lemma 2, it follows that \( J|_M \) has a minimal point on \( M \). Therefore, given any function \( v \in H^1(B) \), one has

\[
\int_B \left( \Delta_B u + \frac{k+1}{4k} (f - \text{scal}_B) u - \frac{k+1}{4k} \text{scal}_F u^{k-3/k+1} \right) v = 0. \tag{24}
\]

Hence, it follows that \( u \) is a weak solution to PDE (8). Assuming that \( f, \text{scal}_B, \text{scal}_F \) are smooth functions, according to classical theory to the regularity of elliptic PDE’s (see [Aub98, Theorem 3.58, p. 87]), the solution \( u \) is smooth indeed. \( \square \)

We finish this subsection by proving Lemma 1.

**Proof of Lemma 1**

Consider a \( g_B \)-orthonormal basis \( \{w_j\} \) and a \( g_F \)-orthonormal \( \{v_i\} \). Then the set \( \{w_j\} \cup \{\text{e}^{-\phi}v_i\} \) defines a \( \tilde{g} \)-orthonormal basis to \( T(B \times F) \).

According to the equations on Proposition A.2 one has

\[
\tilde{K}(\text{e}^{-\phi}v_i, \text{e}^{-\phi}v_j) = e^{-2\phi}K_F(v_i, v_j) - |\nabla \phi|^2,
\]

\[
\tilde{K}(w_j, \text{e}^{-\phi}v_i) = -d\phi(w_j)^2 - \text{Hess} \phi(w_j, w_j).
\]

Assume that \( \{w_j\} \) is such that \( d\phi(w_1) = |\nabla \phi|, d\phi(w_j) = 0, \forall j \geq 2 \). Then the scalar curvature of \( \tilde{g} \) is given by

\[
\tilde{\text{scal}} = \text{scal}_B + e^{-2\phi} \text{scal}_F - k(k-1) |\nabla \phi|^2 - 2k \left\{ |\nabla \phi|^2 + \Delta_B \phi \right\}.
\]

\( \square \)

### 3.2. Proof of Theorem F

We now prove Theorem F. In the following we consider General Vertical Warpings by smooth and basic functions (see the Appendix A for further details).

**Lemma 3.** Let \( F \hookrightarrow (M, g) \xrightarrow{\pi} \) be a Riemnian submersion. Let \( \tilde{g} \) be a general vertical warping metric on \( M \) via the function \( u^{\frac{k+1}{k+3}} \) where \( u : M \to \mathbb{R} \) is a basic function, i.e, it is constant along \( F \). Then the scalar curvature of \( \tilde{g} \) is given by

\[
\tilde{\text{scal}} = \text{scal}_g - \frac{4k}{k+1} u^{-1} \Delta_B u + (4 + 2(k-1)) \frac{2}{k+1} u^{-1} du(H)
\]

\[
+ \left( u^{-\frac{k+1}{k+3}} - 1 \right) \text{scal}_F + \left( 1 - u^{\frac{k+1}{k+3}} \right) \left( 3 \sum_{i,j} |A_{ei}e_j|^2 - 2 \sum_{i,r} |A_{ei}^* v_r|^2 \right).
\]

\( \tag{25} \)
Proof. According to the formulae on Proposition \[A.1\] let $T_i = e^{-\phi}v_i$, $v_i \in \mathcal{V}$ and $X \in \mathcal{H}$. Then,

$$\tilde{K}(e^{-\phi}v_1, e^{-\phi}v_2) = (e^{-2\phi} - 1)K_F(v_1, v_2) + K_g(v_1, v_2) - |\nabla\phi|^2 + d\phi(\sigma(v_1, v_1) + \sigma(v_2, v_2)), $$

$$\tilde{K}(X, e^{-\phi}v) = K_g(X, v) - (1 - e^{2\phi})|A_X^*v|^2 - \text{Hess} \phi(X, X) - d\phi(X)^2 + 2d\phi(X)g(S_Xv, v).$$

Take a $g$-orthonormal $\{e_i\}$ to $\mathcal{H}$. Then,

$$\sum_{i,j} \tilde{K}(e_i, e_j) = (1 - e^{2\phi})\text{scal}_B + 2e^{2\phi}\text{scal}^\mathcal{H},$$

$$\sum_{r,s} \tilde{K}(e^r v_r, e^s v_s) = (e^{-2\phi} - 1)\text{scal}_F + \text{scal}^V - k(k - 1)|\nabla\phi|^2 - 2(k - 1)d\phi(H)$$

$$+ 2 \sum_{i,r} \tilde{K}(e^r v_r, e_i) = 2 \sum_{i,r} K(e_i, v_r) - 2(1 - e^{2\phi}) \sum_{i,r} |A_{e_i}^* v_r|^2$$

$$- 2k(\Delta_B \phi + |\nabla\phi|^2) + 4 \sum_{r,i} d\phi(e_i)g(S_{e_i} v_r, v_r).$$

Assume that $d\phi(e_1) = |\nabla\phi|$, i.e., $e_1 = \nabla\phi$, $d\phi(e_i) = 0$, $i \geq 2$. One has,

$$4 \sum_{r,i} d\phi(e_i)g(\nabla_{e_i} v_r, v_r) = 4 \text{tr} S_{\nabla\phi} = 4d\phi(H).$$

So we conclude that

$$\tilde{\text{scal}} = \text{scal}_g - 2(1 - e^{2\phi}) \sum_{i,r} |A_{e_i}^* v_r|^2 + (1 - e^{2\phi})(\text{scal}_B - \text{scal}^\mathcal{H}) + (e^{-2\phi} - 1)\text{scal}_F$$

$$- k(k - 1)|\nabla\phi|^2 - 2k(\Delta_B \phi + |\nabla\phi|^2) + (4 + 2(k - 1))d\phi(H).$$

Once equation (26) only differs to equation (10) by the terms

$$2(1 - e^{2\phi}) \sum_{i,r} |A_{e_i}^* v_r|^2 + (4 + 2(k - 1))d\phi(H),$$

by introducing the change of variables $\phi = \log \varphi$ and $\varphi = u^{\pm\frac{4}{k+1}}$, we conclude that

$$\tilde{\text{scal}} = \text{scal}_g - \frac{4k}{k + 1}u^{-1}\Delta_B u + (4 + 2(k - 1)) \frac{2}{k + 1}u^{-1}du(H)$$

$$+ \left(u^{-\frac{4}{k+1}} - 1\right)\text{scal}_F + \left(1 - u^{-\frac{4}{k+1}}\right) \left(\text{scal}_B - \text{scal}^\mathcal{H} - 2 \sum_{i,r} |A_{e_i}^* v_r|^2\right).$$

(27)

Lemma 4. Let $(F, g_F) \hookrightarrow (M, g) \rightarrow (B, h)$ be a Riemannian submersion with $M$ closed connected oriented and $(F, g_F)$ minimal. Let $\tilde{g}$ be a general vertical warping of $g$ by a smooth basic function $e^{2\phi}$. If $\Delta_M$ denotes the Laplace operator on the metric $\tilde{g}$, then the restriction of $\Delta_M$ to basic functions defines a strongly elliptic operator.
Proof. Let \( u : M \to \mathbb{R} \) be a basic function. Once \( M \) is compact, it is possible to choose a collection of open sets \( \{ U_n \} \subset M \) trivializing the submersion \( \pi \) in the following sense (see [Her60])

\[
U_n = B_n \times F, \quad B_n \subset B.
\]

Once \( u \) is a basic function,

\[
u|_{U_n}(p) = u(b, f) = u(b, f'), \quad \forall f, f' \in F, \; \forall b \in B_n.
\]

If \( \{ \psi_n \} \) denotes a unity partition on \( \{ B_n \} \), then

\[
u = \sum_n \psi_n u\quad \text{and there is a well defined injection}
\]

\[
\zeta : H^1(M) \to H^1(B)
\]

\[
u \mapsto v,
\]

where \( v = \sum_n v_n, \quad v_n(b) = \psi_n u(b, f), \quad \forall b \in B_n. \)

On the other hand, since \((F, g_F)\) is minimal, by identifying \( \zeta u = u \) one has \( \Delta_B u = \Delta_M u \) since \( \Delta_B u = \Delta_M - d\phi(H) \) and \( H \equiv 0 \) (see [GW09, Section 2.1.4, p.53]). Hence, once \( \Delta_B \) is strongly elliptic the result follows. □

Proof of Theorem \( \mathcal{E} \). Define

\[
\delta A := 3 \sum_{i,j} |A_{e_i e_j}|^2 - 2 \sum_{i,j} |A_{e_i v_r}|^2,
\]

where \( \{ e_i \} \) is a \( g \)-orthonormal basis to \( \mathcal{H} \) and \( \{ v_r \} \) is a \( g \)-orthonormal basis to \( \mathcal{V} \). According to Lemma \( \mathcal{D} \), given \( f : M \to \mathbb{R} \), we shall study the following elliptic problem

\[
\left( \frac{k + 1}{4k} \right) u(f - \text{scal}_g) = -\Delta_B u + \left( \frac{k + 1}{4k} \right) \left\{ \left( u^{\frac{k-3}{k+3}} - u \right) \text{scal}_F + \left( u - u^{\frac{k+5}{k+1}} \right) \delta A \right\}
\]

Recalling that \( \theta = \frac{2(k-1)}{k+1} \) and that \( \frac{k+1}{4k} = 2b_k \), define \( \gamma := \frac{2k+6}{k+1} \). Let \( \epsilon_0 > 0 \) be arbitrarily small and define

\[
\mathcal{M}_b := \{ u \in H^1(M) : u \geq \epsilon_0, \; u \text{ is basic and } \int_B u^\theta \geq 1 \}.
\]

Consider the following functional \( J \) defined on \( \mathcal{M}_b \):

\[
J(u) := \frac{1}{2} \int_B |\nabla u|^2 + \left( \frac{k + 1}{4k} \right) \int_B \left\{ -\frac{u^2}{2}(f - \text{scal}_g) + \frac{k + 1}{2(k-1)} u^{\frac{2(k-1)}{k+1}} - \frac{1}{2} u^2 \right\} \text{scal}_F + \left( \frac{k + 1}{4k} \right) \int_B \left( \frac{u^2}{2} - \frac{k + 1}{2k + 6} u^{\frac{2k+6}{k+1}} \right) \delta A.
\]
We conveniently rewrite $J$ as

$$J(u) = \frac{1}{2} \int_B |\nabla u|^2 + 2b_k \int_B \left\{ (\text{scal}_g + \delta A - \text{scal}_F - f) \frac{u^2}{2} + \text{scal}_F \theta^{-1} u^\theta - \delta A \gamma^{-1} u^\gamma \right\} \tag{32}$$

It is worth recalling that we are assuming that

1. $\max_B \delta A \leq 0$,
2. $\min_F \text{scal}_F \leq 0$.

Therefore,

$$J(u) \geq \frac{1}{2} \int_B |\nabla u|^2 + b_k \left( \min_M \text{scal}_g - \max_M f - \max_M \text{scal}_F + \min_M \delta A \right) \int_B u^2 + 2b_k \theta^{-1} \min_F \text{scal}_F (\text{vol}(B))^{2/\theta - 1} \int_B u^2 - 2b_k \max_M \delta A \gamma^{-1} \int_B u^\gamma, \tag{33}$$

where the penultimate term follows from equation (16).

According to the Hölder inequality applied to the continuous immersion $L^\gamma(B) \hookrightarrow L^2(B)$ one has

$$\left( \int_B u^2 \right)^{\frac{\gamma}{2}} \leq \text{vol}(B)^{\frac{\gamma}{2} - 1} \int_B u^\gamma. \tag{34}$$

Exploiting equation (33), the Poincaré inequality and the equation (34) imply that

$$J(u) \geq \left\{ \frac{\lambda_1}{2} - b_k \max_M (f - \text{scal}_g + \text{scal}_F) + c_k \min_M \text{scal}_F (\text{vol}(B))^{2/\theta - 1} + b_k \min_M \delta A \right\} \int_B u^2 - 2b_k \left\{ \gamma^{-1} \text{vol}(B)^{1-\frac{\gamma}{2}} \max_M \delta A \right\} \left( \int_B u^2 \right)^{\frac{\gamma}{2}}. \tag{35}$$

The proof of Theorem [E] is finished using equation (35) and a straightforward adaptation of Lemma [2] once realized that

$$-2b_k \gamma^{-1} \text{vol}(B)^{1-\frac{\gamma}{2}} \max_M \delta A \geq 0,$$

□

Proof of Corollary [G]. The proof of Corollary [G] follows easily once under the hypothesis Ric$(B) \geq (n - 1)$, the eigenvalue $\lambda_1$ of $-\Delta_B$ satisfies

$$\lambda_1 \geq n \tag{36}$$

with equality if, and only if, $B$ is isometric to the unit sphere (see [OBA62]).
Therefore, once $\text{Ric}(B) \geq (n - 1)$ implies that $\text{scal}_{g_B} \geq n(n - 1)$, it follows from equation \[1\] on Theorem \[E\] that a sufficient condition to the existence of solution is given by

$$\frac{n}{2 + \epsilon} > b_k \max_B f - b_k n(n - 1) + c_k \min_F \text{scal}_F \left(\text{vol}(B)\right)^{2/\theta - 1}, \quad \epsilon > 0. \quad (37)$$

In particular, if $\min_F \text{scal}_F = 0$ it reduces to

$$\frac{8kn}{(2 + \epsilon)(k + 1)} + n(n - 1) > \max_B f, \quad (38)$$

what finishes the proof. \[\square\]

3.3. Examples

We proceed by sketching some applications of Theorem \[E\]. The following constructions are obtained from Section 5.3 on [AB15].

Let $(B, g_B)$ and $(F, g_F)$ be compact connected Riemannian manifolds. Fix $b \in B$ and assume that there is Lie group homomorphism $\rho : \pi_1(B, b) \to \text{Iso}(g_B)$. Let $\tilde{\pi} : \tilde{B} \to B$ be the projection of the universal covering of $B$. It is possible to define an action of $\pi_1(B, b)$ on $\tilde{M} := \tilde{B} \times F$ in the following manner

$$[\alpha] \cdot (\tilde{b}, f) := (\tilde{b} \cdot [\alpha], \rho(\alpha^{-1})f), \quad (39)$$

where $\tilde{b} \cdot [\alpha]$ denotes the Deck transformation associated to $\alpha$ applied to $\tilde{b} \in \tilde{B}$.

Denote by $M$ the orbit space according to the action \[39\] and let $\Pi : \tilde{M} \to M$ be the quotient map projection. Then there is a well defined fiber bundle

$$F \hookrightarrow M \xrightarrow{\pi} B$$

with projection $\pi$ defined as

$$\pi(\Pi(\tilde{b}, f)) := \tilde{\pi}(\tilde{b}).$$

The structure group of $\pi$ is precisely $\rho(\pi_1(B, b))$. The total space $M$ carries a Riemannian submersion metric $g$ such that its fibers are totally geodesic. Moreover, the horizontal distribution is integrable, meaning that $A \equiv 0$.

The previous construction furnishes lots of bundles such that the only restriction to a smooth function on its total space to be the scalar curvature function to some Riemannian submersion metric is an upper bound, depending only on the dimension of the base and the fiber, to the quantity $f - \text{scal}_g$. More precisely, the following Theorem of Corro–Galaz-García ([CGG16, Theorem A]) provides infinitely many examples:

**Theorem 3.1 (Corro–Galaz-García).** For each integer $n \geq 1$, the following hold:

(i) There exist infinitely many diffeomorphism types of closed simply-connected smooth $(n+4)$-manifolds $B$ with a $T^n$-invariant Riemannian metric with positive Ricci curvature.

(ii) The manifolds $B$ realize infinitely many spin and non-spin diffeomorphism types.
(iii) Each manifold $B$ supports a smooth, effective action of a torus $T^{n+2}$ extending the isometric $T^n$-action in item (i).

We can then simply take $F = T^n$ and $B$ as in Theorem 3.1 to obtain the following results as applications of Theorem E and Corollary G.

**Theorem 3.2.** For each integer $n \geq 1$, there exist infinitely many diffeomorphism types of closed simply-connected smooth $(n+4)$-manifolds $B$, realizing infinitely many spin and non-spin diffeomorphism types, such that any smooth function on the total space of the following bundles

$$T^n \hookrightarrow (M,g) \rightarrow B$$

satisfying

$$\frac{8n(n+4)}{(2+\epsilon)(n+1)} > \max_M (f - \text{scal}_g),$$

for some $\epsilon > 0$, is the scalar curvature for some Riemannian submersion metric on $M$.

To the case where $M$ is the product $B \times T^n$ one simply concludes

**Corollary 3.3.** For each $n \geq 5$ there exist infinitely many diffeomorphism types of closed simply-connected smooth $n$-manifolds $B$, realizing infinitely many spin and non-spin diffeomorphism types and with a $T^{n-4}$-invariant Riemannian metric with positive Ricci curvature, such that any smooth function $f : B \rightarrow \mathbb{R}$ satisfying

$$\frac{8(n-4)n}{(2+\epsilon)(n-3)} + n(n-1) > \max_B f$$

is the scalar curvature for some Riemannian metric on $B$.

**Acknowledgments**

The authors are thankful to Prof. Marcus Marrocos for useful comments on the analytical part of this paper and to Prof. Marcos Alexandrino for pointing out the procedure presented in subsection 3.3.

**Appendix A. General Vertical Warppings**

Let $F \hookrightarrow (M,g) \xrightarrow{\pi} B$ be a Riemannian submersion. It is possible to obtain new Riemannian submersions from $\pi$ by introducing some metric deformations changing the $g$ on vertical directions. More precisely, let $\phi : M \rightarrow \mathbb{R}$ be a smooth function. We define a new metric $\tilde{g}$ on $\pi$ in the following way

$$\tilde{g}(E,F) := g(E^H, F^H) + e^{2\phi} g(E^V, F^V), \quad \forall E, F \in T_p M, \forall p \in M.$$ 

Since this metric preserves the horizontal distribution, $\pi : (M, \tilde{g}) \rightarrow B$ remains a Riemannian submersion. Denote by $\tilde{\nabla}$, $\tilde{R}$ the Levi-Civita connection and the Riemann curvature tensor of $\tilde{g}$.
Proposition A.1. Let $F \hookrightarrow (M, g) \overset{\pi}{\rightarrow} B$ be a Riemannian submersion and $\tilde{g}$ be a general vertical warping of $g$ with respect to the function $e^{2\phi}$, $\phi \in C^\infty(M; \mathbb{R})$. Fix $p \in M$. Let $X, Y \in \mathcal{H}_p$, $V, V_i \in \mathcal{V}_p$, $i \in \{1, 2\}$. If $g(V_1, V_2) = 0$, the following formulae hold true for the sectional curvature $\tilde{K}$ of $\tilde{g}$:

\[
\tilde{K}(X, Y) = (1 - e^{2\phi})K_B(X, Y) + e^{2\phi}K(X, Y),
\]

\[
\tilde{K}(V_1, V_2) = (e^{2\phi} - e^{4\phi})K_F(V_1, V_2) + e^{4\phi}K_g(V_1, V_2)
\]
\[- e^{4\phi}|V_1|^2|V_2|^2|\nabla \phi|^2 + e^{4\phi}d\phi(\sigma(V_1, V_1))|V_2|^2 + e^{4\phi}d\phi(\sigma(V_2, V_2))|V_1|^2,
\]

\[
\tilde{K}(X, V) = K_g(X, V)e^{2\phi} - e^{2\phi} (1 - e^{2\phi}) |A_X^* V|^2 - \{\text{Hess } \phi(X, X) + d\phi(X)^2\} e^{2\phi}|V|^2 + 2e^{2\phi}d\phi(X)g(S_X V, V).
\]

Proof. See [GW09] Section 2.1.3, p. 52]

A.1. Warped products

Let $(B, g_B)$ and $(F, g_F)$ be Riemannian manifolds. Assume that $\pi : M = B \times F \rightarrow B$ is a trivial Riemannian submersion with the product metric on $M$. Let $\phi : B \rightarrow \mathbb{R}$ be a smooth function. Then the metric $\tilde{g} := g_B \times e^{2\phi}g_F$ is an example of general vertical warping known as warped product. The Riemannian manifold $(M, \tilde{g})$ is called warped product of $B$ and $F$, being usually denoted by $B \times e^{\phi} F$.

On warped products the Gray–O’Neill tensor $A$ vanishes identically. Moreover, the second fundamental form of the fibers satisfy

\[
\tilde{\sigma}(T_1, T_2) = -e^{2\phi}g(T_1, T_2) \nabla \phi = -\tilde{g}(T_1, T_2).
\]

The following formulae for the sectional curvature of a warped product holds true

Proposition A.2. Consider a warped product $\pi : (B \times F, \tilde{g} = g_B \times e^{2\phi}g_F) \rightarrow M$ and let $\tilde{K}$ be the sectional curvature of $\tilde{g}$. Fix $(p, f) \in M \times F$ and take $X, Y \in \mathcal{H}_p$, $V, V_i \in \mathcal{V}_p$, $i \in \{1, 2\}$. Then

\[
\tilde{K}(X, Y) = K_B(X, Y);
\]

\[
\tilde{K}(V_1, V_2) = e^{2\phi}\left\{K_F(V_1, V_2) - e^{2\phi}|\nabla \phi|^2 (|V_1|^2|V_2|^2 - (V_1, V_2))^2\right\};
\]

\[
\tilde{K}(X, V) = -e^{2\phi}|V|^2 \left((\nabla \phi, X)^2 + \text{Hess } \phi(X, X)\right),
\]

\[
(46)
\]
A.2. Canonical deformation

Another very simple case of general vertical warping happens when one takes \( \phi(p) = t \in \mathbb{R}, \forall p \in M \). The metric \( \tilde{g} \) is usually known as canonical variation of \( g \). Let \( \tilde{g} = g\big|_B + e^{2t}g\big|_V \).

**Proposition A.3.** Let \( \pi : F \hookrightarrow (M, g) \rightarrow B \) be a Riemannian submersion with totally geodesic fibers. Let \( \tilde{K}, K_B, K_F \) denote the non-reduced sectional curvatures of \( \tilde{g} \), \( g_B \), \( g_F \), respectively, where \( \tilde{g} \) is the canonical variation of \( g \); \( g_B \) is the submersion metric on \( B \), and \( g_F \) the metric on \( F \). Then, if \( X, Y, Z \in \mathcal{H} \), and \( V, W \in \mathcal{V} \),

1. \( \tilde{K}(X, Y) = K_B(\pi_*X, \pi_*Y)(1 - e^{2t}) + e^{2t}K(X, Y) \),
2. \( \tilde{K}(X, V) = e^{4t}|A_X^*V|^2 \),
3. \( \tilde{K}(V, W) = e^{2t}K(V, W) \),
4. \( \tilde{R}(X, Y, Z, W) = e^{2t}g((\nabla_XA)_YZ, W) \).

References

[AB15] M.M. Alexandrino and R.G. Bettiol. *Lie Groups and Geometric Aspects of Isometric Actions*. Springer International Publishing, 2015.

[Aub98] T. Aubin. *Some Nonlinear Problems in Riemannian Geometry*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 1998.

[BG] Maria Alice Bozola Grou. Fibrados com curvatura não negativa. PhD thesis available in http://repositorio.unicamp.br/handle/REPOSIP/307229.

[Bre10] S. Brendle. *Ricci Flow and the Sphere Theorem*. Graduate studies in mathematics. American Mathematical Society, 2010.

[CGG16] D. Corro and Fernando Galaz-García. Positive ricci curvature on simply-connected manifolds with cohomogeneity-two torus actions. *arXiv: Differential Geometry*, 2016.

[CS18] Leonardo F Cavenaghi and Llohamm D Sperança. On the geometry of some equivariantly related manifolds. *International Mathematics Research Notices*, page rny268, 2018.

[CS19] L.F. Cavenaghi and L.D. Sperança. Positive ricci curvature on fiber bundles with compact structure group, 2019.

[CW17a] D. Crowley and D. J. Wraith. Intermediate curvatures and highly connected manifolds. *arXiv preprint arXiv:1704.07057*, 2017.

[CW17b] D. Crowley and D. J. Wraith. Positive Ricci curvature on highly connected manifolds. *Journal of Differential Geometry*, 106(2):187–243, 2017.

[DD87] F. Dobarro and E. Lami Dozo. Scalar curvature and warped products of riemann manifolds. *Transactions of the American Mathematical Society*, 303(1):161–168, 1987.
[EK62] J. Eells and N. Kuiper. An invariant of certain smooth manifolds. *Annali Mat. Pura e Appl.*, 60:413–443, 1962.

[EYTK96] Paul E. Ehrlich, Jung Yoon-Tae, and Seon-Bu Kim. Constant scalar curvatures on warped product manifolds. *Tsukuba J. Math.*, 20(1):239–256, 06 1996.

[GKS20] S. Goette, M. Kerin, and K. Shankar. Highly connected 7-manifolds and non-negative sectional curvature. *Annals of Mathematics*, 191(3):829–892, 2020.

[GM74] D. Gromoll and W. Meyer. An exotic sphere with nonnegative sectional curvature. *Annals of Mathematics*, pages 401–406, 1974.

[GVWZ06] Karsten Grove, Luigi Verdiani, Burkhard Wilking, and Wolfgang Ziller. Non-negative curvature obstructions in cohomogeneity one and the kervaire spheres. *Annali della Scuola Normale Superiore di Pisa–Classe di Scienze*, 5(2):159–170, 2006.

[GVZ11] K. Grove, L. Verdiani, and W. Ziller. An exotic $\mathbb{S}^4$ with positive curvature. *Geometric and Functional Analysis*, 21(3):499–524, 2011.

[GW09] D. Gromoll and G. Walshaw. *Metric Foliations and Curvature*. Birkhäuser Verlag, Basel, 2009.

[GZ00] K. Grove and W. Ziller. Curvature and symmetry of milnor spheres. *Annals of Mathematics*, 152:331–367, 2000.

[Her60] R. Hermann. A sufficient condition that a mapping of riemannian manifolds be a fibre bundle. 1960.

[Hit74] N. Hitchin. Harmonic spinors. *Advances in Mathematics*, 14(1):1–55, 1974.

[JW08] M. Joachim and D. J. Wraith. Exotic spheres and curvature. *Bulletin of the Mathematical Society*, 45:595–616, 2008.

[KW75a] Jerry L. Kazdan and F. W. Warner. A direct approach to the determination of gaussian and scalar curvature functions. *Inventiones mathematicae*, (28):227–230, 1975.

[KW75b] Jerry L. Kazdan and F. W. Warner. Existence and conformal deformation of metrics with prescribed gaussian and scalar curvatures. *Annals of Mathematics*, 101(2):317–331, 1975.

[KW75c] Jerry L. Kazdan and F. W. Warner. Scalar curvature and conformal deformation of riemannian structure. *J. Differential Geom.*, 10(1):113–134, 1975.

[Mil56] J. Milnor. On manifolds homeomorphic to the 7-sphere. *Annals of Mathematics*, 64:399–405, 1956.

[Nas79] J. Nash. Positive Ricci curvature on fibre bundles. *Journal of Differential Geometry*, 14(2):241–254, 1979.

[OBA62] Morio OBATA. Certain conditions for a riemannian manifold to be isometric with a sphere. *J. Math. Soc. Japan*, 14(3):333–340, 07 1962.

[Per02] Grisha Perelman. The entropy formula for the ricci flow and its geometric applications, 2002.
[Per03a] G. Perelman. Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, July 2003.

[Per03b] G. Perelman. Ricci flow with surgery on three-manifolds, March 2003.

[Poo75] W A Poor. Some exotic spheres with positive Ricci curvature. *Mathematische Annalen*, 216(3):245–252, 1975.

[Rig76] Alcibiades Rigas. Scalar curvatures on $o(m), g_2(m)$. *Proceedings of the American Mathematical Society*, 61(1):93–98, 1976.

[SW15] C. Searle and F. Wilhelm. How to lift positive Ricci curvature. *Geometry & Topology*, 19(3):1409–1475, 2015.

[Wil01] F. Wilhelm. Exotic spheres with lots of positive curvatures. *J. Geometric Anal.*, 11:161–186, 2001.

[Wra97] D. J. Wraith. Exotic spheres with positive Ricci curvature. *J. Differential Geom.*, 45(3):638–649, 1997.

[Wra07] D. J. Wraith. New connected sums with positive Ricci curvature. *Annals of Global Analysis and Geometry*, 32(4):343–360, 2007.

Leonardo Francisco Cavenaghi
Departamento de Matemática – Universidade Federal da Paraíba - Campus 1, 1º floor - Lot. Cidade Universitaria 58051-900, João Pessoa, PB, Brazil
e-mail: leonardofcavenaghi@gmail.com

Llohann Dallagnol Sperança
Instituto de Ciência e Tecnologia – Unifesp, Avenida Cesare Mansueto Giulio Lat-tes, 1201, 12247-014, São José dos Campos, SP, Brazil
e-mail: speranca@unifesp.br