ASYMPTOTICS FOR SINGULAR SOLUTIONS OF CONFORMALLY INVARIANT FOURTH ORDER SYSTEMS IN THE PUNCTURED BALL

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Abstract. We study the local asymptotic behavior of singular solutions to a critical fourth order system in the punctured ball, which generalizes the constant $Q$-curvature equation. More precisely, we apply some asymptotic analysis techniques to study the growth properties of the Jacobi fields in the kernel of the linearization of our system about a blow-up limit solution. Indeed, we prove that near the isolated singularity solutions converge to the so-called Emden–Fowler solution, which are classified in [3]. In other terms, our main result extends to the case of fourth order strongly coupled systems the celebrated asymptotic classification due to L. A. Caffarelli et al. [9] and N. Korevaar et al. [53].

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2000 Mathematics Subject Classification. 35J60, 35B09, 35B33, 35R11.
Key words and phrases. Fourth Order Systems, Strongly Coupled Systems, Critical Exponent, Blow-up Analysis, Local Asymptotic Behavior.

Research supported in part by Fulbright Program G-1-00001, CNPq grant 305726/2017-0, and the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001.

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1. Description of the results

In this paper, we study the local behavior for positive singular solutions of the critical fourth order system,

\[ \Delta^2 u_i = c(n)|\mathcal{U}|^{2^* - 2} u_i \quad \text{in} \quad B_1^i \quad \text{for} \quad i \in I. \]

(1)

Here \( B_1^i := B_1^i(0) \setminus \{0\} \subset \mathbb{R}^n \) is the unit punctured ball, \( n \geq 5 \), \( \Delta^2 \) is the bi-Laplacian, \( \mathcal{U} = (u_i)_{i \in I} \) is called a p-map solution, where \( I := \{1, \ldots, p\} \) is the index set and

\[ c(n) = \frac{(n-4)(n^2-4)}{16} \]

is a normalizing constant. The coupling term \( f_i(\mathcal{U}) = c(n)|\mathcal{U}|^{2^* - 2} u_i \) in the right-hand side of (1) is called the Gross–Pitaevskii nonlinearity with associated potential \( F(\mathcal{U}) = (f_i(\mathcal{U}))_{i \in I} \), where \( 2^* = 2n/(n-4) \) is the critical Sobolev exponent of the embedding \( H^2(B_1^i) \hookrightarrow L^{2^*}(B_1^i) \).

In what follows, we say that \( \mathcal{U} \) is a singular solution of (1), if the origin is a non-removable singularity for \( |\mathcal{U}| \), that is, \( \liminf_{|x| \to 0} |\mathcal{U}|(x) = \infty \); otherwise, it is called a removable singularity and \( \mathcal{U} \) is a regular solution of (1). We also say that the p-map \( \mathcal{U} \) is strongly positive (nonnegative) when \( u_i > 0 \) \( (u_i \geq 0) \) and \( \mathcal{U} \) is superharmonic in case \( \Delta u_i > 0 \) for all \( i \in I \).

Our main result proves that solutions of (1) have a local asymptotic profile close to the origin given by the Emden–Fowler solutions, that is, satisfying the limit blow-up system,

\[ \Delta^2 u_i = c(n)|\mathcal{U}|^{2^* - 2} u_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \quad \text{for} \quad i \in I. \]

(2)

Theorem 1. Let \( \mathcal{U} \) be a strongly positive superharmonic solution of (1). Then, either the origin is a removable singularity, or there exist an Emden–Fowler solution \( \mathcal{U}_{a,T} \) of (2) and \( 0 < \beta^*_a < 1 \) such that

\[ \mathcal{U}(x) = (1 + O(|x|^{\beta^*_a}))\mathcal{U}_{a,T}(|x|) \quad \text{as} \quad |x| \to 0. \]

(3)

Once the convergence to a radial Emden–Fowler solution is established, we will use the arguments in [53, Section 7] to improve the decay of the remainder term in (3), by allowing deformed Emden–Fowler solutions in the expansion (see Definition 68). This family of solutions is parameterized by a vector \( x_0 \in \mathbb{R}^n \) in the following way

\[ \mathcal{U}_{a,0,x_0}(x) = \left| \frac{x}{|x|} - x_0|x| \right|^{4-n} \mathcal{U}_{a,0} \left( \frac{|x|}{|x| - x_0|x|} \right)^{-1}. \]

In this spirit, another version of the last theorem is the following refined asymptotics:

Theorem 1’. Let \( \mathcal{U} \) be a strongly positive superharmonic solution of (1). Then, either the origin is a removable singularity, or there exist a deformed Emden–Fowler solution \( \mathcal{U}_{a,T,0} \) of (2) and \( \beta^*_a > 1 \) such that

\[ \mathcal{U}(x) = (1 + O(|x|^{\beta^*_a}))\mathcal{U}_{a,T,0}(|x|) \quad \text{as} \quad |x| \to 0. \]

Notice that when \( p = 1 \), system (1) reduces to the following fourth order critical equation,

\[ \Delta^2 u = c(n)u^{2^* - 1} \quad \text{in} \quad B_1^1. \]

(4)

Consequently, the local models in the neighborhood of the isolated singularity are given by the solutions of the limit blow-up equation

\[ \Delta^2 u = c(n)u^{2^* - 1} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \]

(5)

More accurately, we have the scalar version of Theorems 1 and 1':
Corollary 2. Let $u$ be a positive superharmonic solution of (4). Then, either the origin is a removable singularity, or there exist an Emden–Fowler solution $u_{a,T}$ of (5) and $\beta_0^* > 0$ such that

$$u(x) = (1 + O(|x|^{\beta_0^*}))u_{a,T}(|x|) \quad \text{as} \quad |x| \to 0.$$ 

Moreover, one can find a deformed Emden–Fowler solution $u_{a,T,0}$ and $\beta_1^* > 0$ such that

$$u(x) = (1 + O(|x|^{\beta_1^*}))u_{a,T}(|x|) \quad \text{as} \quad |x| \to 0.$$ 

Remark 3. There are few remarks, we would like to quote:

(i) The additional condition $-\Delta u > 0$ cannot be dropped. In contrast with the limit case (5) (for more details, see [3, Proposition 4.4] and [87, Theorem 2.1]), in the punctured ball it is not possible to prove this superharmonicity requirement using the same iterative argument. This condition also have a geometric motivation, for more details see M. Gursky and A. Malchiodi [37] and F. Hang and P. Yang [43]. A similar cone condition is also necessary in the case of the $\sigma_k$-curvature [40].

(ii) The higher asymptotic expansion recently proved in [38, 39] is believed to be true in our fourth order setting. In fact, this refined asymptotic behavior is the first step in order to understand the moduli space of the Q-curvature equation on $\mathbb{S}^{n-1} \setminus \Lambda$, where $\Lambda$ is finite set (see [70]).

(iii) We remark that the same arguments used to prove the asymptotics also implies the existence of singular solutions to (1). In fact, this follows as by-product of our linear analysis and the implicit function theorem. Namely, we also refer to [48] for more existence results on singular equations with prescribed singular set of higher Hausdorff dimension.

Since our techniques rely on blow-up analysis, the first step to obtain the asymptotic behavior near the singularity is to classify the solutions of the limit equation, in both regular case and singular case. When $p = 1$, we summarize below the classification obtained by C. S. Lin [63, Theorem 1.5] and R. L. Frank and T. König [31, Theorem 1.3]:

**Theorem A.** Let $u$ be a positive solution of (5) and assume that

(i) the origin is a removable singularity. Then, there exist $x_0 \in \mathbb{R}^n$ and $\mu > 0$ such that $u$ is radially symmetric about $x_0$ and

$$u_{\mu,x_0}(x) = \left(\frac{2\mu}{1 + \mu^2|x - x_0|^2}\right)^{\frac{n-4}{2}}. \quad (6)$$

We call $u_{\mu,x_0}$ a fourth order spherical solution.

(ii) the origin is a non-removable singularity. Then, $u$ is radially symmetric about the origin. Moreover, there exist $a \in (0,a_0]$ and $T_a \in (0,T_0]$ such that

$$u_{a,T}(x) = |x|^{\frac{4-n}{2}}v_a(\ln |x|) + T_a. \quad (7)$$

Here $a_0 = [n(n-4)/(n^2-4)]^{n-4/8}$ and $v_a$ is the unique periodic bounded solution of the fourth order Cauchy problem,

$$\begin{cases}
 v_a^{(4)} - K_2v_a^{(2)} + K_0v_a = c(n)v_a^{\frac{n+4}{2}} \\
v_a(0) = a, \quad v_a^{(1)}(0) = 0, \quad v_a^{(2)}(0) = b, \quad v_a^{(3)}(0) = 0,
\end{cases} \quad (8)$$

where $K_2, K_0$ are constants depending only on the dimension, $T_a$ is the fundamental period of $v_a$ and $T_0 = T_{a_0}$. We call both $u_{a,T}$ and $v_{a,T}$ Emden–Fowler (or a Delaunay-type) solutions and $a \in (0,a_0)$ its Fowler parameter, which can be chosen satisfying $a = \min_{t>0} v_a(t)$.

Remark 4. The cases $a = 0$ and $a = a_0$ are the limit cases. We observe that these values correspond to the equilibrium solutions to (8). Indeed, when $a = 0$ gives rise to the spherical solution $v_{\text{sph}}(t) = (\cosh(t-t_0))^{(n-4)/2}$, whereas for $a = a_0$, the solution $v_{\text{cyl}}(t) = a_0$ is called the cylindrical solution.
On the limit blow-up case for \( p > 1 \), the present authors in [3, Theorems 1 and 2] used sliding techniques and ODE analysis to classify the solutions of (2), the results are summarized as follows

**Theorem B.** Let \( \mathcal{U} \) be a nonnegative solution of (1) and assume that
(i) the origin is a removable singularity. Then, \( \mathcal{U} \) is weakly positive and radially symmetric about some \( x_0 \in \mathbb{R}^n \). Moreover, there exist \( \Lambda \in \mathbb{S}^{p-1}_+ = \{ x \in \mathbb{S}^{p-1} : x_i > 0 \} \) and a fourth order spherical solution given by (6) such that
\[
\mathcal{U} = \Lambda u_{x_0,u}.
\]
(ii) the origin is a non-removable singularity. Then, \( \mathcal{U} \) is strongly positive, radially symmetric about the origin and decreasing. Moreover, there exist \( \Lambda^* \in \mathbb{S}^{p-1}_{+,*} = \{ x \in \mathbb{S}^{p-1} : x_i > 0 \} \) and an Emden–Fowler solution given by (7) such that
\[
\mathcal{U} = \Lambda^* u_{a,T}.
\]

For the geometric point of view, the works of R. Schoen and S.-T. Yau [80, 81] highlighted the importance of studying singular equations and describing their asymptotic behavior near their singular sets. Indeed, a positive solution \( u \) of (4) gives rise to a conformally flat metric \( \bar{g} = u^{4/(n-4)} \delta_0 \) such that \( \bar{g} \) has constant \( Q \)-curvature equals \( Q_{\bar{g}} = n(n^2 - 4)/8 \), where \( \delta_0 \) is the standard flat metric. We observe that (4) is a particular case of a more general equation arising in conformal geometry, namely the \( Q \)-curvature equation on \((M, g)\), where \( g \) is a smooth metric on \( B_1 \),
\[
P_g = c(n) u^{2n-2n-1} \quad \text{in} \quad (B_1^*, g).
\]
Here
\[
P_g = \Lambda^2 \bar{g} - \text{div}(a_n R_g g - b_n \text{Ric}_g) du + \frac{n-4}{2} Q_g u
\]
is the Paneitz–Branson operator (see [6, 7, 75]), where
\[
Q_g = -\frac{1}{2(n-1)} \Delta R_g - \frac{2}{(n-2)^2} |\text{Ric}_g|^2 + \frac{n^2 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2,
\]
and
\[
a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} \quad \text{and} \quad b_n = \frac{4}{n-2}.
\]
We emphasize that (9) naturally appears as the deformation law for the \( Q \)-curvature of two conformal metrics on a Riemannian manifold [29]. This operator satisfies the same conformal invariance enjoyed by the conformal Laplacian \( L_g \) associated to the Yamabe equation. In this fashion, the quantity \( Q_g \) plays the same role of the scalar curvature \( R_g \) in the conformal Laplacian, for this reason this quantity is called the \( Q \)-curvature (for more details on the \( Q \)-curvature problem and its applications, we refer the reader to [13, 14, 19, 41]). Let us consider the following nonlinear geometric operator,
\[
\mathcal{N}_g(u) = P_g u - c(n) u^{2n-1}.
\]

**Remark 5.** The geometrical interpretation for Theorem 1 is summarized using the conformal metric \( \bar{g} = u_{a,T}^{4/(n-4)} \delta_0 \) as follows:
(i) If \( a = 0 \), then \( \bar{g} = g_{\text{sph}} \) extends through to the whole \( B_1 \) as the round metric;
(ii) If \( a = a_0 \), then \( \bar{g} = g_{\text{cyl}} \) is the cylindrical metric;
(iii) If \( a \in (0, a_0) \), then \( \bar{g} = g_{a,T} \) is the Emden–Fowler metric, interpolating between the spherical and the cylindrical metrics.

In fact, \( \lim_{a \to 0} T_a = \infty \), in other words \( v_0 = v_{\text{sph}} \) is the limit of \( v_a(t) = (t + T_a/2) \) as \( a \to 0 \). In terms of conformal geometry, those metrics represent the evolution of a cylindrical metric to a singular metric, which forms a bead of spheres along an axis.
Moreover, one can find a deformed Emden–Fowler solution

In a milestone paper, L. A. Caffarelli et al. \[9\] developed a measure-theoretic version of the Alexandrov technique to prove that solutions of (11) defined in the punctured ball are radially symmetric. Moreover, they provided a classification for global singular solutions of (11) in the punctured space (see also \[12, 27, 30, 59\]) and obtained the local behavior in the neighborhood of the origin, proving that any singular solution converges to an Emden–Fowler.

Later, N. Korevaar et al. \[53, Theorem 1\] gave a more geometric approach for proving (3), which is based on the growth of the Jacobi fields for the linearized operator around an Emden–Fowler solution. In fact, they proved the following asymptotics:

**Theorem C.** Let \( u \) be a positive solution of (11). Then, there exist an Emden–Fowler solution \( u_{a,T} \) of the blow-up limit and \( \beta_0^* > 0 \) such that

\[
u(x) = (1 + \mathcal{O}(|x|^{\beta_0^*}))u_{a,0}(|x|) \quad \text{as} \quad |x| \to 0.
\]

Moreover, one can find a deformed Emden–Fowler solution \( u_{a,0,x_0} \) and \( \beta_1^* > 1 \) satisfying

\[
u(x) = (1 + \mathcal{O}(|x|^{\beta_1^*}))u_{a,0,x_0}(|x|) \quad \text{as} \quad |x| \to 0.
\]

We should mention that F. C. Marques \[69, Theorem 1\] extended those asymptotics for the inhomogeneous case, at least for lower dimensions. Namely, he showed that the Emden–Fowler solutions are still the local models near the origin

**Theorem D.** Let \( u \) be a positive solution of (10). Assume that \( 3 \leq n \leq 5 \). Then, there exist an Emden–Fowler solution \( u_{a,T} \) and \( \beta_0^* > 0 \) such that

\[
u(x) = (1 + \mathcal{O}(|x|^{\beta_0^*}))u_{a,0}(|x|) \quad \text{as} \quad |x| \to 0.
\]

Moreover, there exist a deformed Emden–Fowler solution \( u_{a,0,x_0} \) and \( \beta_1^* > 1 \) satisfying

\[
u(x) = (1 + \mathcal{O}(|x|^{\beta_1^*}))u_{a,0,x_0}(|x|) \quad \text{as} \quad |x| \to 0.
\]

For \( n \geq 3 \) and \( 2^* = 2n/(n-2) \), we consider a second order analogue to (1),

\[-\Delta u_i = \frac{n(n-2)}{4} |u|^{2^* - 2} u_i \quad \text{in} \quad B_1^* \quad \text{for} \quad i \in I.
\]

\(12\)
On this second order system, R. Caju et al. [10] classified the limit blow-up solutions of (12), obtaining a asymptotic classification. Moreover, they also proved the same results in the case of a non-flat background metric, at least for lower dimensions.

**Theorem E.** Let $\mathcal{U}$ be a nonnegative positive solution of (12). Then, either the origin is a removable singularity, or there exists $\beta_0 > 0$ such that

$$\mathcal{U}(x) = (1 + O(|x|^\beta_0))\mathcal{U}_{a,T}(x) \quad \text{as} \quad |x| \to 0,$$

where $\mathcal{U}_{a,T}$ satisfies the limit blow-up system,

$$-\Delta u_i = \frac{n(n-2)}{4}|\mathcal{U}|^{2^* - 2} u_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \quad \text{for} \quad i \in I. \quad (13)$$

This theorem is the vectorial case of some classical asymptotic results due to L. A. Caffarelli et al. [9, Theorem 1.2] and N. Korevaar et al. [53, Theorem 1]. For the classification for regular solutions of (13), we refer the reader to [23, 24].

On the scalar case of (1), T. Jin and J. Xiong [52, Theorems 1.1 and 1.2] used a Green identity for the poly-Laplacian and studied an integral equation equivalent to (5), proving asymptotic radial symmetry and sharp global estimates for singular solutions to (4):

**Theorem F.** Let $u$ be a positive superharmonic solution of (4). Then, there exist $C_1, C_2 > 0$ such that

$$C_1|x|^{\frac{4-n}{2}} \leq u(x) \leq C_2|x|^{\frac{4-n}{2}}. \quad (14)$$

Moreover, $u$ is radially symmetric and

$$u(x) = (1 + O(|x|))\bar{u}(x) \quad \text{as} \quad |x| \to 0,$$

where $\bar{u}(r) = \int_{\partial B_1} u(r\theta)d\theta$ is the spherical average of $u$.

**Remark 6.** The convergence to the average in Theorem F is obtained for higher order equations

$$(-\Delta)^k u = u^{\frac{n+2k}{n-2k}} \quad \text{in} \quad B_1^*. \quad (15)$$

where $n \geq 2k, \; k \in \mathbb{N}$ and $(-\Delta)^k$ denotes the poly-Laplacian operator, with additional positivity condition $(-\Delta)^ju > 0$ for all $j = 1, \ldots, k - 1$. We also emphasize that when $k$ is odd, (15) becomes an integral equation and a more involved analysis is required.

The strategy we will use to prove Theorem 1 is called asymptotic analysis. Roughly speaking, this is a combination of classification results, some apriori estimates and linear analysis. The first step is to show that the Jacobi fields (these are the elements in the kernel of the linearization of (1) around an Emden–Fowler solution, see Definition 68) satisfy some growth properties

**Proposition 7.** The following properties hold for the projected linearized operator (see Lemma 21):

(i) For $j = 0$, the homogeneous equation $\mathcal{L}_0^0(\Phi) = 0$ has a solutions basis with $2p$ elements, which are either bounded or at most linearly growing as $t \to \infty$;

(ii) For each $j \geq 1$, the homogeneous equation $\mathcal{L}_0^j(\Phi) = 0$ has a solutions basis with $4p$ elements, which are exponentially growing/decaying as $t \to \infty$.

Inspired by [10,40], we will apply some spectral analysis to prove the last proposition. The issue is that not all the Jacobi fields are generated by a variation of some parameter in the expression for Emden–Fowler solutions. In order to overcome this problem, we show that the spectrum of the linearized operator is purely absolutely continuous. More precisely, it is a union of bands separated by gaps, which are away from the origin. We should point out that fourth order Jacobi fields also appears in the study of stability and non-degeneracy properties for Willmore hypersurfaces (for more details, see [57,58,68,79]).
Next, we also need to show that solutions of (1) are radially symmetric and satisfy a upper and lower bound estimate near the isolated singularity, which is the enunciate of the following proposition

**Proposition 8.** Let $U$ be a strongly positive superharmonic solution of (1). Then, $U$ is radially symmetric about the origin. Moreover, either the origin is a removable singularity or there exist $C_1, C_2 > 0$ satisfying

$$C_1|x|^{\frac{4-n}{2}} \leq |U|(x) \leq C_2|x|^{\frac{4-n}{2}} \quad \text{for } 0 < |x| < 1/2.$$  

(16)

The main ingredients in the proof of the last proposition are the classification result for regular solution to (2) in Theorem A, some blow-up method and a removable singularity theorem based on the sign of the Pohozaev invariant associated to (1). The difficulties in our argument are numerous. One of those is obviously caused by the lack of maximum principle due to the fourth order operator in the left-hand side of (1), as well as the effects of the nonlinear coupling term in the right-hand side. In order to handle the problem with the lack of maximum principle, we will use a Green identity to convert (1) into an integral system. Then, we are able to use some techniques from [52] (see also [17, 51, 86]) to prove that singular solutions satisfy an upper and lower bound near the isolated singularity; these arguments rely on an integral form of the moving spheres technique.

Finally, we will use the Propositions 7 and 8 and some blow-up analysis to present the proof of Theorem 1, which is as consequence of the result

**Proposition 9.** Let $U$ be a strongly positive superharmonic solution of (1) satisfying (16). Then, there exist $\beta_0^* > 0$ and an Emden–Fowler solution $U_{a,T}$ such that

$$U(x) = (1 + O(|x|^\beta_0))U_{a,T}(|x|) \quad \text{as } |x| \to 0.$$  

The proof of Proposition 9 the L. Simon [2, 83] technique to show local convergence to the blow-up limit solution, which is based on the classification theorem for global singular solution. This technique arises in the context of minimal surfaces through the stability operator and exemplifies the deep connections between the fields of geometric analysis and nonlinear PDEs. It is not surprising, for problems arising in geometry, that the low-frequency Jacobi fields of the linearized operator have a explicitly geometric interpretation. For more relation with problems arising in geometry [54–56].

Besides their applications in conformal geometry, strongly coupled fourth order systems also appear in several important branches of mathematical physics. For instance, in hydrodynamics, for modeling the behavior of deep-water and Rogue waves in the ocean [26, 65]. As well as, in the Hartree–Fock theory for Bose–Einstein double condensates [1, 28].

Here a description of division for the rest of the paper. In Section 2, we introduce the cylindrical transformation, the Pohozaev-type invariant and the linearized operator around an Emden–Fowler solution. In section 3, we use linear analysis to prove Proposition 7. Namely, we use spectral analysis to understand the growth properties of the Jacobi fields. Moreover, as a consequence, we prove a existence theorem based on Fredholm theory and the inverse function theorem. In Section 4, we prove Proposition 8, by using the asymptotic symmetry and a upper bound estimate based on a moving sphere method. Furthermore, we obtain a removable singularity theorem which implies the lower bound estimate. Finally, in Section 5, we use the growth estimates to apply the L. Simon technique and proof Proposition 9, and consequently Theorem 1. We also use the existence result and some iterative method to obtain the refined asymptotics using the deformed Emden–Fowler solutions.
2. Preliminaries

The aim of this section is to introduce some basic background for developing our asymptotic methods.

2.1. Kelvin transform. The moving spheres technique we will use further is based on the fourth order Kelvin transform for a \( p \)-map. In order to define the Kelvin transform, we need to establish the concept of inversion about a sphere \( \partial B_\mu(x_0) \), which is a map \( I_{x_0,\mu} : \mathbb{R}^n \rightarrow \mathbb{R}^n \setminus \{x_0\} \) given by
\[
I_{x_0,\mu}(x) = x_0 + K_{x_0,\mu}(x-x_0),
\]
where \( K_{x_0,\mu}(x) = \mu / |x-x_0| \) (for more details, see [3, Section 2.7]). The next step is a generalization of the Kelvin transform for fourth order operators applied to \( p \)-maps.

**Definition 10.** For any \( \mathcal{U} \in C^4(\mathbb{R}^n,\mathbb{R}^p) \), let us consider the fourth order Kelvin transform about the sphere with center at \( x_0 \in \mathbb{R}^n \) and radius \( \mu > 0 \) defined by
\[
\mathcal{U}_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{n-4}\mathcal{U}(I_{x_0,\mu}(x)).
\]

**Proposition 11.** System (1) is invariant under the action of Kelvin transform, i.e., if \( \mathcal{U} \) is a regular solution of (1), then \( \mathcal{U}_{x_0,\mu} \) is a solution of
\[
\Delta^2(u_i)_{x_0,\mu} = c(n)|\mathcal{U}_{x_0,\mu}|^{2^{*^*}-2}(u_i)_{x_0,\mu} \quad \text{in} \quad \mathbb{R}^n \setminus \{x_0\} \quad \text{for} \quad i \in I,
\]
where \( \mathcal{U}_{x_0,\mu} = ((u_1)_{x_0,\mu}, \ldots, (u_p)_{x_0,\mu}) \).

**Proof.** It is a direct consequence of the formula
\[
\Delta^2_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{-(n+4)}\Delta^2(I_{x_0,\mu}(x)) \quad \text{in} \quad \mathbb{R}^n \setminus \{x_0\},
\]
which is obtained by a simple computation. \( \square \)

2.2. Cylindrical transformation. Here the goal is to convert singular solutions of (1) into a regular solutions in a cylinder. Then, the local behavior of singular solutions to (1) near the origin reduces to the asymptotic global behavior for a tempered solutions of a fourth order ODE defined in a cylinder. More accurately, they are distributional solutions with test functions taken in the Schwartz space, which is the function space of all infinitely differentiable functions that are rapidly decreasing as \( t \to 0 \) along with all partial derivatives; the elements in this space also have a well-defined Fourier transform. For this matter, let us use the so-called cylindrical transformation (see also [3]). First, \( \tilde{\mathcal{C}} = (0,1) \times S^{n-1} \) is a piece of a cylinder and \( \Delta^2_{\text{ sph}} \) is the bi-Laplacian in spherical coordinates,
\[
\Delta^2_{\text{ sph}} = \partial_r^{(4)} + \frac{2(n-1)}{r} \partial_r^{(3)} + \frac{(n-1)(n-3)}{r^2} \partial_r^{(2)} - \frac{(n-1)(n-3)}{r^3} \partial_r,
\]
\[
+ \frac{1}{r^4} \Delta_\sigma + \frac{2(n-3)}{r^3} \partial_\sigma \Delta_\sigma - \frac{2(n-4)}{r^4} \Delta_\sigma,
\]
where \( \Delta_\sigma \) denotes the Laplace–Beltrami operator in \( S^{n-1} \). Then, we can rewrite (1) as the nonautonomous nonlinear equation,
\[
\Delta^2_{\text{ sph}} u_i = |\mathcal{V}|^{2^{*^*}-2} u_i \quad \text{in} \quad \tilde{\mathcal{C}} \quad \text{for} \quad i \in I.
\]

In addition, we apply the Emden–Fowler change of variables (or logarithm coordinates) given by \( V(t,\theta) = r^2 \mathcal{U}(r,\sigma) \), where \( r = |x|, \ t = - \ln r, \ \theta = x/|x| \) and \( \gamma = (n-4)/2 \), which sends the problem to the half-cylinder \( \mathcal{C} = (0,\infty) \times S^{n-1} \). Using this coordinate system and performing a lengthy computation (see [36, 88]), we arrive at the following fourth order nonlinear PDE system in the cylinder,
\[
\Delta^2_{\text{ cyl}} v_i = c(n)|\mathcal{V}|^{2^{*^*}-2} v_i \quad \text{in} \quad \mathcal{C} \quad \text{for} \quad i \in I.
\]
Along this lines, let us consider the cylindrical transformation as follows

\[ \Delta_{cyl}^2 = \partial_t^{(4)} - K_2 \partial_t^{(2)} + K_0 + \Delta_\theta^2 + \left(2 \partial_t^{(2)} - J_1 \right) \Delta_\theta, \tag{18} \]

where \(K_0, K_2, J_1\) are constants depending only in the dimension, which are defined by

\[ K_0 = \frac{n^2(n - 4)^2}{16}, \quad K_2 = \frac{n^2 - 4n + 8}{2} \quad \text{and} \quad J_1 = \frac{n(n - 4)}{2}. \]

Furthermore, the superharmonicity condition \(-\Delta u_i > 0\) is equivalent to

\[ \partial_t^{(2)} v_i + 2 \partial_t v_i + \sqrt{K} - \Delta_\theta v_i > 0. \]

Along this lines, let us consider the cylindrical transformation as follows

\[ \mathfrak{F}: C_\infty^c(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p) \to C_\infty^c(\mathbb{C}, \mathbb{R}^p) \quad \text{given by} \quad \mathfrak{F}(U) = r^n U(r, \sigma). \]

**Remark 12.** Although we know that the asymptotic symmetry result and the lower and upper bound estimates hold in the higher order setting, it is not trivial to compute the expressions for coefficients to the ODE on cylinder that depends on the order of the equation; this is an obvious difficulty to prove asymptotics for an analogue poly-Laplacian equation.

In the geometric point of view, this change of variables corresponds to the conformal diffeomorphism between the cylinder and the punctured space, \(\varphi : (\mathbb{C}, g_{cyl}) \to (\mathbb{R}^n \setminus \{0\}, \delta_0)\) defined by \(\varphi(t, \sigma) = e^{-t} \sigma\). Here \(g_{cyl} = dt^2 + d\sigma^2\) stands for the cylindrical metric and \(d\theta = e^{2t}(dt^2 + d\sigma^2)\) for its volume element obtained via the pullback \(\varphi^* \delta_0\), where \(\delta_0\) is the standard flat metric.

For rotationally symmetric \(p\)-maps solutions, (1) becomes the following ODE system

\[ v_i^{(4)} - K_2 v_i^{(2)} + K_0 v_i = c(n)|V|^{2^{*} - 2} v_i \quad \text{in} \quad (0, \infty). \tag{19} \]

**Remark 13.** The transformation \(\mathfrak{F}\) is a continuous bijection with respect to the Sobolev norms \(\| \cdot \|_{D^{2,2}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p)}\) and \(\| \cdot \|_{H^2(\mathbb{C}, \mathbb{R}^p)}\), respectively. Furthermore, this transformation sends singular solutions of (4) into solutions of (17) and by density, we get \(\mathfrak{F}: D^{2,2}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p) \to H^2(\mathbb{C}, \mathbb{R}^p)\).

2.3. **Pohozaev invariant.** In the next step, we define a type homological invariant associated to (1). This invariant is the main ingredient to provide a removable singularity theorem and one of the features for developing the convergence method in Section 4. The existence of a Pohozaev-type invariant is closely related to a conservation law to the Hamiltonian energy of the ODE system (19). Here we are based on [3, 53, 69, 76, 88].

Initially, we will define a energy which is conserved in time for all \(p\)-map solutions \(V\) of system (17) (see [31, 36, 86]).

**Definition 14.** For any \(V\) solution of (19), let us consider its Hamiltonian Energy given by

\[ \mathcal{H}(t, V) = \langle V^{(3)}(t), V^{(1)}(t) \rangle + \frac{1}{2} |V^{(2)}(t)|^2 + \frac{K_2}{2} |V^{(1)}(t)|^2 - \frac{K_0}{2} |V(t)|^2 + \tilde{c}(n)|V(t)|^{2^{*}}, \tag{20} \]

where \(\tilde{c}(n) = 2^{2^{*} - 1}c(n)\).

A standard computation shows that the Hamiltonian energy is invariant on the variable \(t\), that is, \(\partial_t \mathcal{H}(t, V) = 0\). Hence, we can integrate (20) over \(\theta\) to define another conserved quantity as follows

**Definition 15.** For any \(V\) solution of (17), let us define the cylindrical Pohozaev integral by

\[ \mathcal{P}_{cyl}(t, V) = \int_{S_t^n-1} \mathcal{H}(t, V)d\theta. \]

Here \(S_t^n = \{t\} \times S_n\) is the cylindrical ball with volume element given by \(d\theta = e^{2t}d\sigma\), where \(d\sigma\) is the volume element of the ball of radius \(r > 0\) in \(\mathbb{R}^n\).
Observe that by definition $\mathcal{P}$ also does not depend on $t$. Then, let us consider the cylindrical Pohozaev invariant $\mathcal{P}_{\text{cyl}}(V) := \mathcal{P}_{\text{cyl}}(t, V)$. Hence applying the inverse of cylindrical transformation, we recover the classical spherical Pohozaev integral defined by $\mathcal{P}_{\text{sph}}(r, U) := (\mathcal{P}_{\text{cyl}} \circ \mathfrak{S}^{-1})(t, V)$, which satisfies the following Pohozaev-type identity:

**Lemma 16.** Let $U, \bar{U} \in C^4(B_1^n, \mathbb{R}^p)$ and $0 < r_1 \leq r_2 < 1$ below

\[
\sum_{i=1}^p \int_{B_{r_1} \setminus B_{r_2}} (\Delta^2 u_i(x, \nabla \bar{u}_i) + \Delta^2 \bar{u}_i(x, \nabla u_i) - \gamma (\bar{u}_i \Delta^2 u_i + u_i \Delta^2 \bar{u}_i)) \, dx
\]

\[
= \sum_{i=1}^p \left[ \int_{\partial B_{r_2}} q(u_i, \bar{u}_i) \, d\sigma_{r_2} - \int_{\partial B_{r_1}} q(u_i, \bar{u}_i) \, d\sigma_{r_1} \right].
\]

Here

\[
q(u_i, \bar{u}_i) = \sum_{j=1}^3 \bar{l}_j(x, \nabla^2 u_i, \nabla^{4-j} \bar{u}_i) + \sum_{j=0}^3 \bar{l}_j^c(\nabla^j u_i, \nabla^{3-j} \bar{u}_i),
\]

where $\bar{l}$ is linear in each component. Moreover,

\[
\bar{l}_3^c(\nabla^3 u_i, \bar{u}_i) = \gamma \int_{\partial B_{r_2}} \bar{u}_i \partial_\nu \Delta u_i \, d\sigma_{r_2}.
\]

**Proof.** See the proof in [35, Proposition 3.3]. \qed

**Remark 17.** Using the last lemma, we present a explicit formula for the spherical Pohozaev integral

\[
\mathcal{P}_{\text{sph}}(r, U) = \sum_{i=1}^p \int_{\partial B_r} \left[ q(u_i, u_i) - \tilde{c}(n)r |U|^{2^*} \right] \, d\sigma_r.
\]

Consequently, using (21), we observe that $\mathcal{P}(r, U)$ does not depend on $r$. In other terms, we have the result

**Lemma 18.** Let $U$ be a solution of (1) and $0 < r_1 \leq r_2 < 1$. Then, $\mathcal{P}_{\text{sph}}(r_2, U) - \mathcal{P}_{\text{sph}}(r_1, U) = 0$.

**Proof.** Using (1), we have $\Delta^2 u_i - c(n)|U|^{2^*-2} u_i = 0$ in $B_1^n$ for any $i \in I$, which by multiplying by the factor $x \cdot \nabla u_i$ and integrating over $B_{r_2} \setminus B_{r_1}$ implies

\[
0 = \int_{B_{r_2} \setminus B_{r_1}} \left( \Delta^2 u_i - c(n)|U|^{2^*-2} u_i \right) (x \cdot \nabla u_i) \, dx := I_{1,i} - I_{2,i}.
\]

Using Lemma 16 in the left-hand side of the last, we conclude that $\mathcal{P}_{\text{sph}}(r_2, U) - \mathcal{P}_{\text{sph}}(r_1, U) = 0$ for all $i \in I$, which by summing over $i \in I$ finishes the proof. \qed

**Definition 19.** For any $U$ solution of (1), let us define the Spherical Pohozaev Invariant by

\[
\mathcal{P}_{\text{sph}}(r, U) := \mathcal{P}_{\text{sph}}(U).
\]

**Remark 20.** For easy reference, let us summarize the following properties:

(i) $\mathcal{P}_{\text{sph}}(U) = \omega_{n-1} \mathcal{P}_{\text{cyl}}(V)$, where $\omega_{n-1}$ is the $(n-1)$-dimensional Hausdorff measure of the unit sphere;

(ii) $\mathcal{P}_{\text{sph}}(u) = (\mathcal{P}_{\text{cyl}} \circ \mathfrak{S}^{-1})(v)$, where

\[
\mathcal{P}_{\text{cyl}}(v) = \int_{S_{r_1}} \left[ v^{(3)} v^{(1)} - \frac{1}{2} |v^{(2)}|^2 + \frac{K_2}{2} |v^{(1)}|^2 - \frac{K_0}{2} \left( v^2 + \tilde{c}(n)|v|^{2^*} \right) \right] \, d\theta.
\]

Hence, it follows by a direct computation that, if the regular solution is $U = \Lambda u_{x_0, \mu}$ for some $\Lambda \in \mathbb{S}^{n-1}$ and $u_{x_0, \mu}$ a spherical solution from Theorem A, we obtain that $\mathcal{P}_{\text{sph}}(U) = \mathcal{P}_{\text{sph}}(u_{x_0, \mu}) = 0$. 

Also, if the singular solution has the form \( U_0 = \Lambda u_{a,T} \) for some \( \Lambda \in S_{+,\ast}^{n-1} \) and \( u_{a,T} \) a Emden–Fowler solution from Theorem B and a direct computation, we get that \( P_{sph}(U_0) = P_{sph}(u_{a,T}) < 0 \).

2.4. Linearized operator. In this part of the paper, we use some arguments from \([5,72]\) to study the linearized operator at limit blow-up solution. The heuristic is that when the linearized operator is Fredholm, then its indicial roots determine the rate in which singular solutions of the nonlinear problem (1) converge to the Emden–Fowler solution. Here, we borrow some ideas from \([82,\text{Section 2}]\). In this fashion, we consider the following nonlinear operator acting on \( p \)-maps

\[
N(U) := \Delta^2 u_i - f_i(U).
\]

Using the cylindrical transformation and homogeneity of the Gross–Pitaevskii nonlinearity, we obtain

\[
N_{cyl}(\mathcal{V}) := \Delta^2_{cyl} v_i - f_i(\mathcal{V}).
\]  

(24)

In what follows, we will drop the subscript since we will often be using the cylindrical operator.

Lemma 21. The linearization of \( N : H^4(\mathcal{C},\mathbb{R}^p) \to L^2(\mathcal{C},\mathbb{R}^p) \) around an Emden–Fowler solution \( \mathcal{V}_{a,T} \) is given by \( \mathcal{L}^a(\Phi) = (\mathcal{L}^a_i(\Phi))_{i \in I} \). Here

\[
\mathcal{L}^a_i(\Phi) = \Delta^2_{cyl} \phi_i - n A_\beta(\Lambda, \Phi) v_{a,T}^{2* - 2} + \tilde{c}(n) \left( n v_{a,T}^{2* - 2} + 4 - n \right) \phi_i,
\]

(25)

where \( \tilde{c}(n) = c(n)(2^{*} - 1) \), or \( dN_{cyl}|_{\mathcal{V}_{a,T}}(\Phi) = \mathcal{L}^a(\Phi) \) is the Fréchet derivative of \( N \).

Proof. By definition, we have \( \mathcal{L}^a_i(\Phi) := \mathcal{L}_i(\mathcal{V}_{a,T})(\Phi) \), where

\[
\mathcal{L}_i(\mathcal{V}_{a,T})(\Phi) = \left. \frac{d}{dt} \right|_{t=0} N(\mathcal{V}_{a,T} + t\Phi)
\]

(26)

\[
= \Delta^2_{cyl} \phi_i - c(n) \left( (2^{*} - 2) |\mathcal{V}_{a,T}|^{2* - 4} \langle \mathcal{V}, \Phi \rangle v_i + |\mathcal{V}|^{2* - 2} \phi_i \right).
\]

In fact, writing \( f_i(\mathcal{V}) = c(n)|\mathcal{V}|^{2* - 2} v_i \) and using that \( f_i \) is \((2^{*} - 1)\)-homogeneous, we find

\[
N(\mathcal{V}_{a,T} + t\Phi) - N(\mathcal{V}_{a,T}) = \Delta^2_{cyl} \mathcal{V}_{a,T} + t \Delta^2_{cyl} \Phi - f_i(\mathcal{V}_{a,T} + t\Phi) - \Delta^2_{cyl} \mathcal{V}_{a,T} + f_i(\mathcal{V}_{a,T})
\]

\[
= t \Delta^2_{cyl} \Phi - f_i(\mathcal{V}_{a,T}) + f_i(\mathcal{V}_{a,T} + t\Phi)
\]

\[
= t \Delta^2_{cyl} \Phi - t c(n) \left( (2^{*} - 2) |\mathcal{V}_{a,T}|^{2* - 4} \langle \mathcal{V}, \Phi \rangle v_i + |\mathcal{V}|^{2* - 2} \phi_i \right) + \mathcal{O}(t^2),
\]

which by taking \( t \to 0 \) implies (26). Finally, using the classification in Theorem B, we can simplify (26) and obtain (25).

\[ \square \]

2.5. Jacobi fields. Unfortunately, the linearized is not Fredholm since it does not have closed range \([73,\text{Theorem 5.40}]\). In fact, this issue is caused by its non-trivial kernel. The elements in the kernel are called the Jacobi fields. For other construction involving fourth order Jacobi fields, we refer to \([5]\). Therefore, we need to introduce suitable Sobolev and Hölder spaces weighted spaces in which the linearized operator has a left inverse up to a discrete set of poles.

Definition 22. Given \( m,p,q \geq 1 \), \( \beta \in \mathbb{R} \) and \( \delta \in (0,1) \), let us define the weighted Sobolev and Hölder spaces in the cylinder given respectively by

\[
W^{m,\beta}_\delta(\mathcal{C},\mathbb{R}^p) = \{ e^{-\beta t} v : v \in W^{m,q}(\mathcal{C},\mathbb{R}^p) \}, \quad \text{with norm } \| v \|_{W^{m,\beta}_\delta(\mathcal{C},\mathbb{R}^p)} = \| e^{-\beta t} v \|_{W^{m,q}(\mathcal{C},\mathbb{R}^p)}.
\]

and

\[
C^{m,\delta}_{-\beta}(\mathcal{C},\mathbb{R}^p) = \{ e^{-\beta t} v : v \in C^{m,\delta}(\mathcal{C},\mathbb{R}^p) \}, \quad \text{with norm } \| v \|_{C^{m,\delta}_{-\beta}(\mathcal{C},\mathbb{R}^p)} = \| e^{-\beta t} v \|_{C^{m,\delta}(\mathcal{C},\mathbb{R}^p)}.
\]

Here we also denote the Hilbert space \( W^{m,2}_{-\beta}(\mathcal{C},\mathbb{R}^p) = H^{m}_{-\beta}(\mathcal{C},\mathbb{R}^p) \) and \( W^{m,q}(\mathcal{C}) = W^{m,q}(\mathcal{C},\mathbb{R}) \). Notice that when \( \beta = 0 \), we recover the classical Sobolev spaces of \( p \)-maps.
Remark 23. The spaces defined are the suitable functional spaces to obtain the asymptotic results we are searching for, since \( v \in W^{-m,q}_\beta(C) \) is equivalent to \( v \in W^{-m,q}(C) \) together with the decay \( v = O(e^{-\beta t}) \) as \( t \to \infty \). Additionally, by regularity theory, we can indistinguishably work with both the Sobolev or the Hölder spaces (see [3, Section 3]).

Definition 24. The Jacobi fields for \( \mathcal{L}^a : H^1_{-\beta}(C, \mathbb{R}^p) \to L^2_{-\beta}(C, \mathbb{R}^p) \), are the solutions \( \Phi \in L^2(C, \mathbb{R}^p) \) of the Jacobi fourth order system,

\[
\mathcal{L}^a(\Phi) = 0. \tag{27}
\]

2.6. Fourier eigenmodes. We study the kernel of linearized operator around an Emden–Fowler solution by decomposing into its Fourier eigenmodes. In the sequel, we fix the notation \( \mathbb{N} = \mathbb{Z}_+ \).

Initially, for each \( \theta \in \mathbb{S}^{n-1} \), let us consider \( \{\lambda_j, \chi_j(\theta)\}_{j \in \mathbb{N}} \) the eigendecomposition of the Laplace–Beltrami operator on with the normalized eigenfunctions,

\[
\Delta \chi_j(\theta) + \lambda_j \chi_j(\theta) = 0. \tag{28}
\]

Here the eigenfunctions \( \{\chi_j(\theta)\}_{j \in \mathbb{N}} \) are called spherical harmonics with associated sequence of eigenvalues \( \{\lambda_j\}_{j \in \mathbb{N}} \) given by \( \lambda_j = j(j + n - 2) \) counted with multiplicity \( m_j \), which are defined by

\[
m_0 = 1 \quad \text{and} \quad m_j = \frac{(2j + n - 2)(j + n - 3)!}{(n - 2)!j!}.
\]

In particular, we have \( \lambda_0 = 0, \lambda_1 = \cdots = \lambda_n = n - 1, \lambda_j \geq 2n \), if \( j > n \) and \( \lambda_j \leq \lambda_{j+1} \). Moreover, these eigenfunctions are the restrictions to \( \mathbb{S}^{n-1} \) of homogeneous harmonic polynomials in \( \mathbb{R}^n \). Here we denote by \( V_j \) the eigenspace generated by \( \chi_j(\theta) \). Using (28), it is easy to observe that the eigendata of the bi-Laplacian \( \Delta^2_\theta \) is given by \( \{\lambda_j^2, \chi_j(\theta)\}_{j \in \mathbb{N}} \).

2.6.1. Scalar case. When \( p = 1 \), the nonlinear operator (24) becomes

\[
\mathcal{N}(v) := \Delta^2_\theta v - c(n)v^{2^{**}-1} \quad \text{and} \quad \mathcal{L}^a(\phi) = \Delta^2_\theta \phi - \tilde{c}(n)v_{a,T}^{2^{**}-2}\phi.
\]

Furthermore, using the decomposition (18) in combination with (28) and (29), we get

\[
\mathcal{L}^a(\phi) = \partial_t^{(4)}\phi - K_2\partial_t^{(2)}\phi + K_0\phi + \Delta^2_\theta \phi + 2\partial_t^{(2)}\Delta_\theta \phi - J_1\Delta_\theta \phi - \tilde{c}(n)v_{a,T}^{2^{**}-2}\phi,
\]

which by projecting on the eigenspaces gives

\[
\mathcal{L}_j^a(\phi) = \phi^{(4)}(\phi^{(2)}(\phi + [K_0 + \lambda_j(\lambda_j + J_1) - \tilde{c}(n)v_{a,T}^{2^{**}-2}\phi]. \tag{30}
\]

Moreover, for any \( \phi \in L^2(\mathbb{S}^{n-1}) \), we write

\[
\phi(t, \theta) = \sum_{j=0}^{\infty} \phi_j(t)\chi_j(\theta), \quad \text{where} \quad \phi_j(t) = \int_{\mathbb{S}^{n-1}} \phi(t, \theta)\chi_j(\theta)d\theta.
\]

In other terms, \( \phi_j \) is the projection of \( \phi \) on the eigenspace \( V_j \). Hence, in order to understand the kernel of \( \mathcal{L}^a \), we consider the induced family of ODEs \( \mathcal{L}^a_j(\phi_j) = 0 \) for \( j \in \mathbb{N} \). In other terms, for each \( a \in (0, a_0) \), we have

\[
\ker(\mathcal{L}^a) = \bigcup_{j \in \mathbb{N}} \ker(\mathcal{L}^a_j) \quad \text{and} \quad \text{spec}(\mathcal{L}^a) = \bigcup_{j \in \mathbb{N}} \text{spec}(\mathcal{L}^a_j).
\]
Remark 25. For $p = 1$ and $j = 0, 1, \ldots, n$ some Jacobi fields are obtained by the variation of a two-parameter family of Emden–Fowler solutions. In particular, in the case $j = 0$, they are given by

$$\phi_{a,0}^+(t) = \partial_t |_{T=0} v_{a,T}(t) \quad \text{and} \quad \phi_{a,0}^-(t) = \partial_a |_{a=0} v_{a,T}(t).$$

However, the other two Jacobi fields cannot be directly constructed as variation of some family of solution to the limit equation. When $j = 1, \ldots, n$, the same construction can be performed, in this case we have that $\phi_{a,1}^+ = \cdots = \phi_{a,n}^+$ with exponential growth/decay. Explicitly, if $\{e_j\}_{j=1}^n$ be the standard basis of $\mathbb{R}^n$, then $\chi_j(\theta) = \langle e_j, \theta \rangle$ is the eigenfunction associated to $\lambda_j = n - 1$. Taking $x_0 = \tau e_j$ in (73) and (78) provides

$$v_{a,\tau e_j}(t, \theta) = v_a(t) + \tau e^{-t} \chi_j(\theta) \left( -v_a^{(1)} + \gamma v_a \right) + O(e^{-2t}) \quad \text{as} \quad t \to \infty,$$

which by differentiating with respect to $\tau$ implies

$$\phi_{a,j}^-(t, \theta) = e^{-t} \left( -v_a^{(1)} + \gamma v_a \right) + O(e^{-2t}) \quad \text{as} \quad t \to \infty.$$ 

Moreover, a direct computation gives $L^a_j(\phi_{a,j}^-) = 0$. Similarly, we can start differentiating the translation to obtain

$$\phi_{a,j}^+(t, \theta) = e^{-t} \left( v_a^{(1)} + \gamma v_a \right) + O(1) \quad \text{as} \quad t \to \infty.$$ 

Also notice that by Remark 71, whenever one has a parameter family of solutions to a nonlinear equation, the derivatives with respect to each parameter provides a solution to its linearized equation. Moreover, the low frequencies space on $S^{n-1}$ is spanned by the constant functions and the restrictions to $S^{n-1}$ of linear functions on $\mathbb{R}$.

2.6.2. System case. For $p > 1$ and $\Phi \in L^2(\mathcal{C}, \mathbb{R}^p)$, we write

$$\Phi(t, \theta) = \sum_{j=0}^\infty \Phi_j(t) \chi_j(\theta), \quad \text{where} \quad \Phi_j(t) = \int_{S^{n-1}} \Phi(t, \theta) \chi_j(\theta) d\theta. \quad (31)$$

Hence, for all $i \in I$ and $j \in \mathbb{N}$, we decompose (27) as

$$L^a_{i,j}(\Phi) = \phi_i^{(4)} - (K_2 + 2\lambda_j) \phi_i^{(2)} + \left[ K_0 + \lambda_j(\lambda_j + J_1) - \tilde{c}(n)v_{a,T}^{2s-2} \right] \phi_i - n\Lambda_i(\Lambda^*, \Phi)v_{a,T}^{2s-2}. \quad (32)$$

Hence, in order to understand the kernel of (32), we consider the induced equations $L^a_{i,j}(\phi_j) = 0$. Therefore, the study of the kernel of $L^a$ reduces to the solve infinitely many ODEs. In other terms, for each $a \in (0, a_0)$ we have that

$$\ker(L^a) = \bigcup_{i \in I} \bigcup_{j \in \mathbb{N}} \ker(L^a_{i,j}) \quad \text{and} \quad \text{spec}(L^a) = \bigcup_{i \in I} \bigcup_{j \in \mathbb{N}} \text{spec}(L^a_{i,j}).$$

In Fourier analysis, it is convenient to divide any $\Phi \in L^2(\mathcal{C}, \mathbb{R}^p)$ into its frequency modes by

$$\pi_0[\Phi](t, \theta) = \Phi_0(t) \chi_0(\theta), \quad \pi_1[\Phi](t, \theta) = \sum_{j=1}^{m_1} \Phi_j(t) \chi_j(\theta), \quad \text{and} \quad \pi_l[\Phi](t, \theta) = \sum_{j=m_l+1}^{m_{l+1}} \Phi_j(t) \chi_j(\theta).$$

In particular, the projections $\pi_0, \pi_1$ and $\sum_{l=2}^{\infty} \pi_l$ are called respectively the zero-frequency, low-frequency, and high-frequency modes.
2.7. Indicial roots. A general principle in nonlinear analysis states the asymptotic behavior of the linearized $L^a$ on the cylinder $C$ is directly related to its indicial roots, at least close the hyperbolic equilibrium points. Heuristically, this is a version of Poincaré–Bendixon theory, or, more generally, Hartman–Grobman [45]. Inspired by [74], we present the following definition:

**Definition 26.** A real number $\beta \in \mathbb{R}$ is called an indicial root of $L^a_{i,j} : H^4(C) \to L^2(C)$, if there exist a non-zero function $\phi \in C^4(C)$ and $\beta \prec \beta$ such that
\[
\lim_{t \to \infty} \|\phi(t,\theta)\|_{L^\infty(S^1)} > 0 \quad \text{and} \quad \lim_{t \to \infty} e^{-\beta t} L^a_{i,j} (\phi(t)e^{\beta t}) = 0.
\]
Denote by $\mathcal{I}^a_{i,j}$, the indicial roots of $L^a_{i,j}$. Moreover, if $\beta \in \mathbb{R}$ is called a indicial root of $L^a$ for all $i \in I$ and $j \in \mathbb{N}$, which is denoted by $\mathcal{I}^a = \bigcup_{i \in I} \bigcup_{j \in \mathbb{N}} \mathcal{I}^a_{i,j}$.

**Remark 27.** When the operator has constant coefficients the indicial roots are the solutions of the characteristic equation. However, if has periodic coefficients the indicial exponents are the real part of the eigenvalues of the monodromy matrix and are called Floquet exponents. We could also compare those with the definition of indicial roots for Fuchsian operators.

3. Linear Analysis

The objective of this section is to prove Proposition 7. In fact, we will show the linear stability of the linearized by studying its spectrum. Then, we are able to control the asymptotics to the global solutions by the growth of the Jacobi fields, which can be estimated by Floquet theory (or Bloch wave theory). Namely, we will show that $\text{spec}(L^a)$ is the disjoint union of nondegenerate intervals, which implies that $0 \in \mathcal{I}^a$ is isolated. The strategy is to use the integral decomposition in Proposition 30 to study the spectral bands of the Jacobi operator. We will do that applying the Fourier–Laplace transform together with some results from holomorphic functional analysis. For more details on this spectral analysis, see [71, 72] (see also [25, 78]). For complex numbers, we denote $\rho = \alpha + i \beta$ and $\mathbb{R} (\rho), \mathbb{I} (\rho)$ stands for its real and imaginary part, respectively.

3.1. Fourier–Laplace transform. In this section, following [72, Section 4], we introduce the Fourier–Laplace transform, which is the suitable transformation to invert the linearized operator since its invertibility properties. In fact, we can use the real parameter $\alpha = \Re z$ for $\rho \in \mathcal{R}^a$ in order to move the weight of the Sobolev space and invert this transform, up to some region in the complex plane. Before, we need to introduce some notation and tools.

**Definition 28.** Let $\bar{\beta} \in \mathbb{R}$ and $\Phi \in H^m_m(\hat{C}, \mathbb{R}^p)$, where $\hat{C} = \mathbb{R} \times S^{n-1}$ is the entire cylinder and $\Phi$ is extend to be zero outside $C$. We define the Fourier–Laplace transform $F^a : L^2(\hat{C}, \mathbb{R}^p) \to L^2(\hat{C}, \mathbb{R}^p)$, given by
\[
F^a (\Phi) (t, \theta, \rho) = \sum_{k \in \mathbb{Z}} e^{-ik \rho} \Phi (t+kT_a, \theta),
\]
where $\rho \in \mathcal{R}^a := \{ \alpha + i \beta \in \mathbb{C} : \beta < -\bar{\beta} T_a \} \subset \mathbb{C}$, for some $\bar{\beta} \in \mathbb{R}$. For the sake of simplicity, we fix the notation $\hat{\Phi} (t, \theta, \rho) := F^a (\Phi) (t, \theta, \rho)$.

Due to periodicity of the linearized operator, it makes to define the following space

**Definition 29.** For $C_a := [0, T_a] \times S^{n-1}$, let us define $L^2_a(C_a, \mathbb{R}^p)$, the space of Bohr $\alpha$-quasi-periodic $p$-maps. More precisely, it is the $L^2$-completion of $C^0_a(C_a, \mathbb{R}^p)$, where
\[
C^0_a(C_a, \mathbb{R}^p) := \{ w \in C^0_a(C_a, \mathbb{R}^p) : \Phi (T_a, \theta) = e^{i\alpha t} \Phi (0, \theta) \}.
\]

The main proposition of this appendix is that the Fourier–Laplace transform is well-defined and provides a direct integral decomposition of $L^2(C, \mathbb{R}^p)$.
Proposition 30. For any $a \in [0, a_0]$, it follows
\[ L^2(C, \mathbb{R}^p) = \int_{\alpha \in [0, 2\pi]} L^2_{\alpha}(C_a, \mathbb{R}^p) \, d\alpha. \]

Proof. We divide the proof in a sequence of claims:

Claim 1: The operator $F_a : L^2(C, \mathbb{R}^p) \rightarrow L^2_a(C_a, \mathbb{R}^p)$ is well-defined.

In fact, since $\Phi$ is analytic whenever $\Phi(t, \theta)$, that is, there exists an inversion branch on the complex plane.

As a matter of fact, for all $S$ where $\Phi(t, \theta, \rho)$ is analytic whenever $\Phi(t, \theta, \rho)$, we know that $|\Phi(t, \theta)| = O(e^{|\beta|t})$, which gives
\[ |F_a(\Phi)(t, \theta, \rho)| = \sum_{k \in \mathbb{Z}} e^{-i(\alpha + i\beta)k} |\Phi(t + kT_a, \theta)| \leq Ce^{|\beta|t} \sum_{k \in \mathbb{Z}} e^{(\alpha + \beta T_a)k}, \]
where we used that $\rho \in \mathcal{R}_a$ and all the exponents in the series are negative. Therefore, the last sum must converges uniformly in $\mathcal{R}_a$. We could rephrase this conclusion like $F_a : \mathcal{R}_a \rightarrow H^k_a(C, \mathbb{R}^p)$ is analytic whenever $\Phi \in H^k_a(C, \mathbb{R}^p)$.

In the next claim, we invert the Fourier–Laplace transform using a contour integral in some inversion branch on the complex plane.

Claim 2: Let $\Phi \in H^k_a(C, \mathbb{R}^p)$ and $\rho \in \mathcal{R}_a$. For each $t$ choose $\bar{t} \in [0, T_a)$ such that $t = \bar{t}$ mod $T_a$, that is, there exists $l \in \mathbb{Z}$ satisfying $t = \bar{t} + lT_a$. Then,
\[ \Phi(t, \theta) = \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} e^{i\beta t, \alpha + i\beta, \theta, \rho} \, d\alpha. \]

Indeed, since $z = \alpha + i\beta$ and $\rho \in \mathcal{R}_a$, for all $i \in I$, we obtain
\begin{align*}
\frac{1}{2\pi} \int_{\alpha=0}^{2\pi} e^{i\alpha \beta} \phi_i(t, \theta, z) \, d\alpha &= \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} e^{i\alpha \beta} \sum_{k \in \mathbb{Z}} e^{-i\alpha k} \phi_i(t + kT_a, \theta) \, d\alpha \\
&= \sum_{k \in \mathbb{Z}} \frac{1}{2\pi} \int_{\alpha=0}^{2\pi} e^{i(\alpha + i\beta)(l-k)} \phi_i(t + kT_a, \theta) \, d\alpha = \phi_i(t, \theta),
\end{align*}
which implies the proof of the claim.

In the sequel, we see how the change in the parameter $\alpha \in \mathbb{R}$ in this inversion influences the weight norm of the transformed function, which is a type of Parseval–Plancherel identity.

Claim 3: For each $\theta \in \mathbb{S}^{n-1}$ and $\alpha \in \mathbb{R}$, we have
\[ \left\| \hat{\Phi}(t, \theta, \rho) \right\|^2_{L^2(S_a, \mathbb{R}^p)} \simeq 2\pi \left\| \Phi(t, \theta) \right\|^2_{L^2_{\beta,T_a}(\mathbb{R}, \mathbb{R}^p)}, \quad (34) \]
where $S_a = [0, T_a] \times [0, 2\pi]$.

As a matter of fact, for all $i \in I$, it holds
\begin{align*}
\int_0^{T_a} \int_0^{2\pi} \left| \hat{\phi}_i(t, \theta, \rho) \right|^2 \, dt \, d\theta &= \int_0^{T_a} \int_0^{2\pi} \left( \sum_{k \in \mathbb{Z}} e^{-i\alpha k} \phi_i(t + kT_a, \theta) \right) \left( \sum_{k \in \mathbb{Z}} e^{-i\alpha k} \phi_i(t + kT_a, \theta) \right) \, dt \, d\theta \\
&= \int_0^{T_a} \int_0^{2\pi} \sum_{k \in \mathbb{Z}} \left( \sum_{l=-k}^{k} e^{i(l-k)\alpha} \phi_i(t + kT_a, \theta) \phi_i(t + kT_a, \theta) \right) \, dt \, d\theta \\
&= \int_0^{T_a} \int_0^{2\pi} \sum_{k \in \mathbb{Z}} \left( e^{2\beta k} \phi_i(t + kT_a, \theta) \right)^2 \, dt \, d\theta \\
&\simeq 2\pi \int_\mathbb{R} e^{2\beta t/T_a} \phi_i(t, \theta) \, dt.
\end{align*}
Next, we prove that \( \hat{\Phi} \) is a section of the flat bundle \( \mathbb{T}_a^n = \mathbb{S}_a^1 \times \mathbb{S}^{n-1} \) with holonomy \( \rho \in \mathbb{C} \) around the \( \mathbb{S}_a^1 \) loop, where we identify \( \mathbb{S}_a^1 = \mathbb{R}/T_a \mathbb{Z} \).

**Claim 4:** For each \( \theta \in \mathbb{S}^{n-1} \), we have
\[
\left\| \hat{\Phi}(t, \theta, \rho) \right\|^2_{L^2(\mathbb{S}_a, \mathbb{R}^p)} = 2\pi \left\| \hat{\Phi}(t, \theta) \right\|_{L^2(\mathbb{R}, \mathbb{R}^p)}.
\]
Indeed, by taking \( \beta = 0 \) in (34) and using (33), we get
\[
\Phi(t + T_a, \theta) = \mathcal{F}_a^{-1}\left( e^{i\rho \mathcal{F}_a(\Phi)} \right)(t, \theta),
\]
which clearly implies the claim.

Finally, the proof of the proposition is a consequence of Claims 2, 3 and 4. \( \square \)

### 3.2. Spectral analysis

Next, inspired by [71, Section 4.2], we study the geometric structure of the spectral of the linearized operator around an Emden–Fowler solution. The idea is to construct a the twisted operator, which is unitarily equivalent to the linearized operator and for which a Fredholm theory is available. In this direction, we introduce a suitable space in which the twisted operator is defined.

**Definition 31.** For each \( \alpha \in \mathbb{R} \) and \( k \in \mathbb{N} \), let us define the set of quasi-periodic \( p \)-maps \( \mathcal{H}^{n+4}_a([0, T_a], \mathbb{R}^p) \) to be the completion of the space of \( C^\infty([0, T_a], \mathbb{R}^p) \) under the \( \mathcal{H}^m \)-norm with boundary conditions given by
\[
\Phi(j)(T_a) = e^{iT_a \alpha} \Phi(j)(0) \quad \text{for} \quad j = 0, 1, \ldots, k - 1.
\]
We also denote by \( \mathcal{L}^a_{i,j,\alpha} \) the restriction of \( \mathcal{L}^a_{i,j} \) to \( \mathcal{H}^m_a([0, T_a]) \).

Initially, for any \( \rho \in \mathfrak{R}_a \) we use the inversion of the Fourier–Laplace transform to define \( \tilde{\mathcal{L}}^a(\rho) = \mathcal{F}_a \circ \mathcal{L}^a \circ \mathcal{F}_a^{-1} \), or equivalently \( \tilde{\mathcal{L}}^a(\hat{\Phi}) = \tilde{\mathcal{L}}^a(\Phi) \), which by (35) gives
\[
\tilde{\mathcal{L}}^a(\rho)(e^{i\rho \hat{\Phi}})(t, \theta, \rho) = e^{i\rho \tilde{\mathcal{L}}^a(\rho)(e^{i\rho \hat{\Phi}})(t, \theta)} \quad \text{and} \quad e^{-i\rho \tilde{\mathcal{L}}^a(\rho)}(e^{i\rho \hat{\Phi}}) = \tilde{\mathcal{L}}^a(\rho)(\hat{\Phi}).
\]
Using the last relation, we set \( \tilde{\mathcal{L}}^a: H^{m+4}_a(\mathbb{T}_a^n, \mathbb{R}^p) \to H^m(\mathbb{T}_a^n, \mathbb{R}^p) \), given by
\[
\tilde{\mathcal{L}}^a(\rho)(\hat{\Phi}) = e^{i\rho \mathcal{F}_a \circ \mathcal{L}^a \circ \mathcal{F}_a^{-1}}(e^{-i\rho \hat{\Phi}}).
\]

**Remark 32.** Notice that \( \tilde{\mathcal{L}}^a \) has the same coordinate expression as \( \mathcal{L}^a \), hence their Fourier eigenmodes decomposing \( \tilde{\mathcal{L}}^a \) and \( \tilde{\mathcal{L}}^a \) are also unitarily equivalent. Moreover, by Claim 3 of Proposition 30 one has that \( \mathcal{L}^a_{i,j,\alpha} \) coincides with the restriction of \( \tilde{\mathcal{L}}^a_{i,j}(\alpha) \) to \( [0, T_a] \). Furthermore, \( \mathcal{L}^a(\rho) \) acts on the same functional space for all \( \rho \in \mathbb{C} \).

This motivates the following definition:

**Definition 33.** For each \( a \in (0, a_0) \), \( j \in \mathbb{Z} \), \( i \in I \) and \( \alpha \in \mathbb{R} \), let us denote by \( \sigma_k(a, i, j, \alpha) \) the eigenvalues of \( \mathcal{L}^a_{i,j,\alpha} \). In addition, since for each \( a \in (0, a_0) \), \( j \in \mathbb{Z} \), \( i \in I \) one has \( \mathcal{L}^a_{i,j,0} = \mathcal{L}^a_{i,j,2\pi} \), it follows \( \sigma_k(a, i, j, \cdot) : \mathbb{S}_a^1 \to \mathbb{R} \). Therefore, let us define the \( k \)-th spectral band of \( \mathcal{L}^a_{i,j} \) by
\[
\mathfrak{B}_k(a, i, j) = \{ \sigma_k \in \mathbb{R} : \sigma_k = \sigma_k(a, i, j) \text{ for some } \alpha \in [0, 2\pi/T_a] \}.
\]

**Remark 34.** Notice that \( \text{spec}(\mathcal{L}^a) = \text{spec}(\tilde{\mathcal{L}}^a) = \bigcup_{i \in I} \bigcup_{j \in \mathbb{N}} \mathfrak{B}_k(a, i, j) \).

**Remark 35.** The eigenfunction \( \Phi_k \) corresponding to the eigenvalue \( \sigma_k(a, j, \alpha) \) satisfies
\[
\Phi(t + 2\pi/T_a) = e^{i\alpha \phi(t)} = e^{(2\pi - \alpha)j \Phi}(t) \quad \text{and} \quad \Phi(t + 2\pi) = e^{-i\alpha \hat{\Phi}(t)}.
\]
Furthermore, \( \sigma_k(a, j, 2\pi - \alpha) = \sigma_k(a, j, \alpha) \) since \( \mathcal{L}^a_j \) has real coefficients; thus we can restrict \( \sigma_k : \mathbb{S}_a^1 \to \mathbb{R} \) to the half-circle corresponding to \( \alpha \in [0, \pi] \).
Now we have conditions to enunciate and prove the most important result of this section:

**Proposition 36.** For any \( a \in [0, a_0] \), \( 0 \in \mathcal{I}_a \) is an isolated indicial root of \( \mathcal{L}^a \).

*Proof.* The proof will follow by estimating the end points of the spectral bands of \( \mathcal{L}^a \) and will be divided in some claims as follows:

**Claim 1:** For any \( a \in (0, a_0) \) and \( j, k \in \mathbb{N} \), the band \( \mathcal{B}_k(a, j) \) is a nondegenerate interval. In fact, each \( \mathcal{L}^a_{j} \) is a fourth order ordinary differential operator such that the ODE system \( \mathcal{L}_{a,j}\Phi = \sigma_k(a,j)\Phi \) has a 4-dimensional solution space. Suppose that \( \mathcal{B}_k(a, j) \) reduces to a single point, then \( \sigma_k \) would be constant on \([0, 2\pi]\) and \( \mathcal{L}_{a,j}\Phi = \sigma_k(a,j)\Phi \) would have an infinite dimensional solution space, which is contradiction.

**Claim 2:** For any \( a \in (0, a_0) \) and \( j, k \in \mathbb{N} \), we have the structure

\[
\mathcal{B}_{2k}(a, j) = [\sigma_{2k}(a, j, 0), \sigma_{2k}(a, j, \pi)] \quad \text{and} \quad \mathcal{B}_{2k+1}(a, j) = [\sigma_{2k+1}(a, j, \pi), \sigma_{2k+1}(a, j, 0)].
\]

This is a consequence of Floquet theory, since \( \mathcal{B}_{2k} \) are nondecreasing for any \( k \in \mathbb{Z} \) while \( \mathcal{B}_{2k+1} \) are all nonincreasing. Therefore, we have

\[
\sigma_0(a,j, 0) \leq \sigma_0(a,j, \pi) \leq \sigma_1(a,j, \pi) \leq \sigma_1(a,j, 0) \leq \ldots
\]

**Claim 3:** For any \( a \in (0, a_0) \) and \( j, k \in \mathbb{N} \), we have the lower bound

\[
\sigma_k(a,j, 0) > \sigma_0(a, 0, \alpha) + J_1 \lambda_j + \lambda_j^2.
\]  

(37)

As a matter of fact, we can relate \( \mathcal{B}_k(a, 0) \) to \( \mathcal{B}_k(a, j) \) since

\[
\mathcal{L}^a_j - \mathcal{L}^a_0 = -2\lambda_j \phi_j^{(2)} + J_1 \lambda_j + \lambda_j^2,
\]

which for \( \Phi \) an eigenvalue of \( \mathcal{L}^a_{j,\alpha} \) implies

\[
\sigma_k(a,j, \alpha)\Phi = \mathcal{L}^a_0 - 2\lambda_j \phi_j^{(2)} + (J_1 \lambda_j + \lambda_j^2) \Phi.
\]  

(38)

Using the decomposition \( \Phi = \sum_{t=0}^{L_0} c_t \Phi_t \), where \( \mathcal{L}^a_0\Phi_t = \sigma_t(a, 0, \alpha)\Phi_t \) we can reformulate (38) as

\[
\sum_{t \in \mathbb{N}} c_t \sigma_k(a,j, \alpha)\Phi_t = \sum_{t \in \mathbb{N}} c_t \left[ \sigma_t(a, 0, \alpha)\Phi_t - 2\lambda_j \phi_j^{(2)} + (J_1 \lambda_j + \lambda_j^2) \Phi_t \right],
\]

which provides

\[
2\lambda_j \phi_j^{(2)} = -[\sigma_k(a,j, \alpha) - \sigma_t(a, 0, \alpha) - J_1 \lambda_j - \lambda_j^2] \Phi_t.
\]

Finally, notice that the last equation admits quasi-periodic solutions if and only if

\[
\sigma_k(a,j, 0) > \sigma_0(a, 0, \alpha) + J_1 \lambda_j + \lambda_j^2,
\]

which proves the claim.

**Claim 4:** For any \( a \in (0, a_0) \) and \( j, k \in \mathbb{N} \), it follows that \( \mathcal{B}_k(a, j) \subset (0, \infty) \).

This is the most delicate part, thus we will separate the proof into some steps. First, by the classification in Theorem B, we are left to consider \( p = 1 \).

**Step 1:** For each \( a \in (0, a_0] \), we have

\[
\hat{c}(n) \left( \frac{1}{T_0} \int_0^{T_0} \text{e}_a^{2**}(t) \, dt \right)^{1-2/**} \leq \sigma_0(a, 0, 0) < 0,
\]

where \( \hat{c}(n) = c(n) - \check{c}(n) = -n (n^2 - 4)/2 < 0 \). Moreover, either \( \sigma_1(a, 0, 0) = 0 \) or \( \sigma_2(a, 0, 0) = 0 \).

In fact, in order to prove this we start by the upper bound. Using the Rayleigh quotient to of \( \mathcal{L}^a_0 \), we get

\[
\sigma_0(a, 0, 0) = \inf_{\phi \in H^1(T_0)} \frac{\int_0^{T_0} \phi \mathcal{L}^a_0 \phi \, dt}{\int_0^{T_0} \phi^2 \, dt}.
\]  

(39)
Since \( v_a \) is a periodic, it can taken as a test function in the right-hand side of (39); this provides
\[
\mathcal{L}_0^a v_a = v_a^{(4)} - K_2 v_a^{(2)} + K_0 v_a - \tilde{c}(n) v_{a,T}^{2*1} \\
= v_a^{(4)} - K_2 v_a^{(2)} + K_0 v_a - c(n) v_{a,T}^{2*1} + \tilde{c}(n) v_{a,T}^{2*1} \\
= \tilde{c}(n) v_{a,T}^{2*1},
\]
where we used that \( \mathcal{L}_0^0 \) and \( \tilde{\mathcal{L}}_0^0 \) have the same coordinate expression. Hence since \( \tilde{c}(n) \leq 0 \), the estimate (37) is a consequence of (39).

For proving the lower bound estimate, we observe that by (36) and the classification in Theorem A, one has the variational characterization
\[
v_a = \inf_{\phi \in H_0^1([0, T_a])} \frac{\int_0^{T_a} \left( |\phi^{(2)}|^2 - K_2 |\phi^{(1)}|^2 + K_0 |\phi|^2 \right) dt}{(\int_0^{T_a} \phi^{2*2} dt)^{2/2*2}}.
\]
Moreover, since \( v_a \) satisfies (8), we get
\[
\frac{\int_0^{T_a} \left( |\phi^{(2)}|^2 - K_2 |\phi^{(1)}|^2 + K_0 |\phi|^2 \right) dt}{(\int_0^{T_a} \phi^{2*2} dt)^{2/2*2}} \geq c(n) \left( \int_0^{T_a} v_a^{2*2} dt \right)^{1-2/2*}, \tag{40}
\]
for all \( \phi \in H_0^1([0, T_a]) \).

On the other hand, using the Hölder inequality, we get
\[
\int_0^{T_a} \phi^2 dt \leq T_a^{1-2/2*} \left( \int_0^{T_a} \phi^{2*2} dt \right)^{2/2*}, \tag{41}
\]
Then, for all \( \phi \in H_0^1([0, T_a]) \) a combination of (40) and (41) provides
\[
\int_0^{T_a} \phi \mathcal{L}_0^a \phi dt = \int_0^{T_a} \left( \phi^{(4)} - K_2 \phi^{(2)} + K_0 \phi - \tilde{c}(n) v_{a,T}^{2*1} \phi \right) dt \\
\geq c(n) \left( \int_0^{T_a} v_a^{2*2} dt \right)^{1-2/2*} \left( \int_0^{T_a} \phi^{2*2} dt \right)^{2/2*} - \tilde{c}(n) \int_0^{T_a} \phi^{2*2} dt \\
\geq T_a^{2/2*} \left[ c(n) \int_0^{T_a} \phi^2 dt \left( \int_0^{T_a} v_a^{2*2} dt \right)^{1-2/2*} - \tilde{c}(n) \int_0^{T_a} \phi^2 dt \left( \int_0^{T_a} \phi^{2*2} dt \right)^{1-2/2*} \right].
\]
In particular, taking \( \phi \in H_0^1([0, T_a]) \) satisfying \( \|\phi\|_{H_0^1([0, T_a])} = \|v_a\|_{H_0^1([0, T_a])} \), we obtain
\[
\int_0^{T_a} \phi \mathcal{L}_0^a \phi dt \geq c(n) T_a^{2/2*-1} \|\phi\|^2_{L^2([0, T_a])} \|v_a\|^{2*2-2}_{L^{2*2}([0, T_a])},
\]
which directly implies the lower bound estimate.

Finally, since \( \phi_{a,0}^+ = \partial_a v_a \) is a periodic solution of \( \mathcal{L}_0^a(\phi_{a,0}^+) = 0 \), we have that there is an eigenfunction with associated eigenvalue \( \alpha = 0 \) and subject to periodic boundary conditions \( \alpha = 0 \). Additionally, that the eigenfunction has two modal domains within the interval \([0, T_a]\) associated either to \( \sigma_1(a, 0) \) or to \( \sigma_2(a, 0) \).

In the remaining steps, we provide a more precise localization of the spectral bands of \( \mathcal{L}_a^a \):
Step 2: For any \( a \in (0, a_0) \), it follows that \( \mathcal{B}_k(a, 0) \subset (0, \infty) \) for each \( k \geq 3 \) and \( \mathcal{B}_k(a, 0) \subset [0, \infty) \) for each \( k \geq 2 \).

This a direct consequence of Claim 2 and Step 1.

Step 3: For any \( a \in (0, a_0) \) and \( j, k \in \mathbb{N} \), it follows that \( \mathcal{B}_k(a, j) \subset (0, \infty) \).

In fact, when \( j > n \) we have \( \lambda_j > 2n \), which by Claim 3 gives
\[
\sigma_k(a, j, 0) > \sigma_0(a, 0, 0) + n^3 \quad \text{for all} \quad k \in \mathbb{N}.
\]

On the other hand, since \( v_a \in (0, 1) \), by the lower bound, we have \( \sigma_0(a, 0, 0) \geq \bar{c}(n) \) and
\[
\sigma_k(a, j, 0) > \sigma_0(a, 0, 0) + n^3 \geq n^3 + \bar{c}(n) > 0.
\]

When \( 1 \leq j \leq n \), it follows from the geometric Jacobi fields constructed in Remark 25, since
\[
\mathcal{L}^a_j \left( \phi_{a,j}^{\pm} \right) = 0 \quad \text{and} \quad \phi_{a,j}^{\pm} = e^{\pm t \left( \pm v_a^{(1)}(1) + \gamma v_a \right)} + R_{\pm},
\]
where \( R_+ = O(1) \) and \( R_- = O(e^{-2t}) \) are positive, periodic solutions of \( \mathcal{L}^a_j \).

The last claim relates the spectral bands \( \mathcal{B}_k(a, j) \) and the indicial roots \( \mathcal{I}_j^a \).

Claim 5: The ODE \( \mathcal{L}_{a,j} \Phi = 0 \) admits a quasi-periodic solution, if and only if, for some \( k \in \mathbb{N}, \ 0 \in \mathcal{B}_k(a, j) \)

Indeed, we have that \( \Phi = \mathcal{F}^{-1}_a \left( e^{-i\alpha t} \mathcal{F} \right) \) solves \( \mathcal{L}_{a,j} \Phi = 0 \) since
\[
0 = \mathcal{L}^a_{j,a,j} \Phi = e^{i\alpha t} \mathcal{F}_a \left( \mathcal{L}^a_j \left( \mathcal{F}^{-1}_a \left( e^{-i\alpha t} \Phi \right) \right) \right) \quad \text{and} \quad \mathcal{L}^a_j \left( \mathcal{F}^{-1}_a \left( e^{-i\alpha t} \Phi \right) \right) = 0.
\]

Therefore, by Remark 35 it follows the claim. \( \square \)

3.3. Fredholm theory. Next, we need study the spectrum of the linearized operator. Indeed, our final goal is to conclude that \( \mathcal{L}^a \) is Fredholm, which will follow by showing that \( \mathcal{I}_a \subset \mathbb{R} \) is a discrete set. This is not a trivial statement, in fact we need to use the results about the Fourier–Laplace transform in Subsection 3.1 to find a left inverse for the linearized operator. In fact, we need to use some results of holomorphic functional analysis (see \([25, 78]\)).

Definition 37. Let \( H \) be a Hilbert space and \( \mathcal{L} : H \to H \) be a linear operator. We say that \( \mathcal{L} \) is Fredholm if \( \mathcal{L} \) is bounded and has a closed finite dimensional kernel and closed range.

In order to invert the twisted operator, we will use the analytic Fredholm theorem \([78, \text{Theorem 8.92}]\)

Theorem G. Let \( \mathcal{R} \subseteq \mathbb{C} \) be a domain and consider the map \( \mathcal{F} : \mathcal{R} \to \mathcal{L}(H) \) given by \( \mathcal{F}(\rho) : H \to H \). Then, either
(i) \( (\text{Id} - \mathcal{F}(\rho))^{-1} \) does not exist for all \( \rho \in \mathcal{R} \), or
(ii) \( (\text{Id} - \mathcal{F}(\rho))^{-1} \) exists for \( \rho \in \mathcal{R} \setminus \mathcal{D} \), where \( \mathcal{D} \subseteq \mathcal{R} \) is a discrete set. Moreover, the map \( \rho \mapsto (\text{Id} - \mathcal{F}(\rho))^{-1} \) is analytic and if \( \rho \in \mathcal{D} \), then \( \mathcal{F}(\rho) \phi = \phi \) has a finite-dimensional solution space.

Lemma 38. For any \( m \in \mathbb{N}, \ a \in (0, a_0) \) and \( \beta \in \mathbb{R} \), the operator \( \mathcal{L}^a : H^m_{\beta}^4(\mathcal{C}, \mathbb{R}^p) \to H^m_{\beta}^4(\mathcal{C}, \mathbb{R}^p) \) is a bounded linear elliptic self-adjoint operator.

Proof. It follows since the principal symbol of \( \mathcal{L}^a \) in cylindrical coordinates is given by \( \partial_t^{(4)} + \Delta^2_a \). \( \square \)

The main result of this subsection states the invertibility of the linearized operator. We then use the analytic Fredholm theorem to prove the twisted operator is Fredholm away from a discrete set of poles the complex plane.

Proposition 39. For any \( m \in \mathbb{N} \) and \( \beta \notin \mathcal{I}^a \), the operator \( \mathcal{L}^a : H^m_{\beta}^4(\mathcal{C}, \mathbb{R}^p) \to H^m(\mathcal{C}, \mathbb{R}^p) \) is Fredholm.
Proof. In order to apply the analytic Fredholm theorem, we use the twisted operator from (36),
\[ \tilde{L}^{a}(\rho) : H^{m+4}(T_{a}^{n}, \mathbb{R}^{p}) \to H^{m}(T_{a}^{n}, \mathbb{R}^{p}) \] given by \[ \tilde{L}^{a}(\rho)(\hat{\Phi}) = e^{i\rho T_{a}t} \circ L^{a} \circ \mathcal{F}_{a}^{-1} \left( e^{-i\rho T_{a}t} \hat{\Phi} \right). \]

In what follows, we divide the proof in some claims:

Claim 1: For any \( \alpha \in (0, 2\pi) \), the operator \( L_{\alpha}^{a} \) is Fredholm.

For each \( \alpha \in (0, 2\pi) \), the operator \( L^{a}(\rho) \) is linear, bounded, elliptic and depends holomorphically on \( \rho \). Thus, by Theorem G is either never Fredholm or it is Fredholm for \( \rho \) outside a discrete set. We take \( z = \alpha \in (0, 2\pi) \) and suppose that there exists \( \hat{\Phi} \in H^{m+4}(T_{a}^{n}, \mathbb{R}^{p}) \) such that \( L^{a}(\rho)(\hat{\Phi}) = 0 \); thus \( L^{a}(\rho)(\Phi) \), where \( \Phi = \mathcal{F}_{a}^{-1}(e^{-i\rho T_{a}t} \hat{\Phi}) \). Then, \( \Phi \) is quasi-periodic and in particular \( \Phi \) is bounded. However, by Proposition 36 any bounded Jacobi field is a multiple of \( \Phi_{0}^{+} \), which is not quasi-periodic; thus \( L^{a}(\alpha) \) is injective. Finally, since this operator is formally self-adjoint, it follows that \( L^{a}(\rho) \) is Fredholm.

Claim 2: For any \( a \in (0, a_{0}) \) and \( \beta \in \mathfrak{J}^{a} \), there exists \( \mathcal{G}_{a} : H^{m}(T_{a}^{n}, \mathbb{R}^{p}) \to H^{m+4}(T_{a}^{n}, \mathbb{R}^{p}) \).

Using Claim 1, we can apply Theorem G to find a discrete set \( \mathfrak{D}_{a} \subset \mathfrak{N}_{a} \) and a meromorphic operator
\[ \tilde{G}^{a}(\rho) : H^{m}(T_{a}^{n}, \mathbb{R}^{p}) \to H^{m+4}(T_{a}^{n}, \mathbb{R}^{p}) \] such that \( \hat{\Phi} = \left( \tilde{G}^{a}(\rho) \circ \hat{L}^{a}(\rho) \right)(\hat{\Phi}) \),

for \( \rho \notin \mathfrak{D}_{a} \). In addition, notice \( \mathfrak{J}^{a} = \{ \beta \in \mathbb{R} : \beta = \Im(\rho) \text{ for some } \rho \in \mathfrak{D}_{a} \} \), which provides
\[ \mathcal{G}^{a}(\Phi) = \mathcal{F}_{a}^{-1} \left( e^{-i\rho T_{a}t} \left( \tilde{G}^{a}(e^{-i\rho T_{a}t}(\mathcal{F}_{a}(\Phi))) \right) \right). \]

Furthermore, by construction, we obtain that \( \hat{\Phi} = \mathcal{G}^{a}(\Phi) \in H^{m+4}_{-\Im(\rho)}(\mathcal{C}, \mathbb{R}^{p}) \), which by the Fredholm alternative concludes the proof of the claim.

\[ \square \]

Proposition 40. The set \( \mathfrak{J}^{a} \) is discrete.

\[ \text{Proof}. \text{ Note that each element in } \mathfrak{J}^{a} \text{ is the imaginary part of a pole to } \tilde{G}_{a}, \text{ which by the analytic Fredholm theory is a discrete subset of } \mathbb{C}. \text{ On the other hand, the operator } L^{a}(\rho) \text{ is unitarily equivalent to } L^{a}(\rho + 2\pi l) \text{ for each } l \in \mathbb{Z}; \text{ thus } \rho \text{ is a pole of } \mathcal{G}_{a}, \text{ if and only if } \rho + 2\pi l \text{ also is for any } l \in \mathbb{Z}. \text{ Therefore, } \mathcal{G}_{a} \text{ can only have finitely many poles in each horizontal strip}. \quad \square \]

Corollary 41. For any \( m \in \mathbb{N} \) and \( \beta \notin (0, 1) \), the operator \( L^{a} : H^{m+4}_{\beta}(\mathcal{C}, \mathbb{R}^{p}) \to H^{m}_{\beta}(\mathcal{C}, \mathbb{R}^{p}) \) is an isomorphism.

\[ \text{Proof}. \text{ In fact, it follows from the proof of Proposition 39 that } L^{a}(\rho) \text{ is injective for each } \rho \in \mathbb{C} \text{ with } -1 < \Im(\rho) < 0, \text{ which implies } L^{a} : H^{m+4}_{\beta}(\mathcal{C}, \mathbb{R}^{p}) \to H^{m}_{\beta}(\mathcal{C}, \mathbb{R}^{p}) \text{ is injective}. \text{ Finally, since by duality the operator } L^{a} : H^{m+4}_{0}(\mathcal{C}, \mathbb{R}^{p}) \to H^{m}_{0}(\mathcal{C}, \mathbb{R}^{p}) \text{ is formally self-adjoint; thus the surjectiveness holds}. \quad \square \]

3.4. Existence of singular solutions. In this section, we will prove existence of solutions to (1). We proceed by studying the spectral properties of the linearized operator at a Emden–Fowler solution. We remark that by the implicit function theorem, the existence of solutions to (1) is equivalent to the linearized operator \( L^{a} \) to be Fredholm. We already know that in some cases \( L^{a} \) does not satisfy this since its kernel is not closed. In order to overcome this issue, we introduce the following definition:

Definition 42. For each \( a,T \), let us consider the deficiency space generated by the Jacobi fields basis of the linearized operator,
\[ D_{a,j}(\mathcal{C}, \mathbb{R}^{p}) = \text{span}\{ \Phi^{+}_{a,j}, \Phi^{-}_{a,j}, \hat{\Phi}^{+}_{a,j}, \hat{\Phi}^{-}_{a,j} \}. \]
Remark 43. As a consequence of Proposition 36, we get
\[ D_{a,j}(C, \mathbb{R}^p) = \text{span}\{\Phi^+_{a,j}, \Phi^-_{a,j}\}. \]
Namely, any Jacobi fields with growth less than exponential is generated by the one obtained by variation of geometric parameters in the Emden–Fowler solution.

Now we can present the main result of the subsection:

Proposition 44. Let \( V_{a,T} \) be an Emden–Fowler solutions. Then,
(i) For \( \beta \in (\beta_{a,0}, \beta_{a,1}) \), the operator \( \mathcal{L}^a : C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \oplus D_{a,0}(C, \mathbb{R}^p) \to C^{0,\delta}_{-\beta}(C, \mathbb{R}^p) \) is a surjective Fredholm mapping with bounded left inverse, given by
\[ \mathcal{G}_0^a : C^{0,\delta}_{-\beta}(C, \mathbb{R}^p) \to C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \oplus D_{a,0}(C, \mathbb{R}^p). \]
(ii) For \( \beta \in (\beta_{a,1}, \beta_{a,2}) \), the operator \( \mathcal{L}^a : C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \oplus D_{a,0}(C, \mathbb{R}^p) \oplus D_{a,1}(C) \to C^{0,\delta}_{-\beta}(C, \mathbb{R}^p) \) is a surjective Fredholm mapping with bounded left inverse, given by
\[ \mathcal{G}_1^a : C^{0,\delta}_{-\beta}(C, \mathbb{R}^p) \to C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \oplus D_{a,0}(C, \mathbb{R}^p) \oplus D_{a,1}(C, \mathbb{R}^p). \]

Proof. We proceed as in Proposition 39. First, we decompose the linearized operator in Fourier modes and apply the Laplace–Fourier transform. Then, by conjugation, we defines a family of decomposed operator given by
\[ \mathcal{G}_j^a(\Phi) = \mathcal{F}_a^{-1}\left(e^{-i\beta t} \left( \tilde{\mathcal{G}}_j^a \left( e^{-i\beta t} (\mathcal{F}_a(\Phi)) \right) \right) \right); \]
this implies the proof of the proposition. \( \square \)

Remark 45. The necessity of adding the deficiency spaces \( D_{a,j}(C, \mathbb{R}^p) \) comes from a simple form of the linear regularity theorem from R. Mazzeo et al. [72, Lemma 4.18] and some ODE theory. In addition, note that if \( \beta = \pm \beta_{a,j} \), then \( \mathcal{L}^a \) does not have closed range. Moreover, we have Schauder estimates in the sense of weighted spaces. More precisely, if \( V \) is solution of the nonhomogeneous problem \( \mathcal{L}^a(V) = \Psi \), then \( V \in C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \) whenever \( \Psi \in C^{0,\delta}_{-\beta}(C, \mathbb{R}^p) \). More generally, it should be possible to find a inverse like
\[ \mathcal{G}_j^a : C^{0,\delta}_{-\beta}(C, \mathbb{R}^p) \to C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \bigoplus_{l=0}^j D_{a,l}(C, \mathbb{R}^p). \]

As a consequence of our results, we present the main result of the subsection:

Corollary 46. There exists at least one positive solution \( V \) of (17).

Another application is the following improved regularity theorem for solutions in cylindrical coordinates:

Corollary 47. Let \( V \) be solution of \( \mathcal{L}^a(V) = \Psi \). Assume that \( V \in C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \) and \( \Psi \in C^{0,\delta}_{-\beta}(C, \mathbb{R}^p) \). Hence, it holds
(i) If \( 0 < \tilde{\beta} < \beta < 1 \), then \( V \in C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \);
(ii) If \( 0 < \beta < 1 < \tilde{\beta} < \beta_{a,2} \), then \( V \in C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \oplus D_{a,1}(C, \mathbb{R}^p) \).

Proof. First, we use the left inverse operator \( \mathcal{G}_0^a \) in Proposition 44 to obtain
\[ \tilde{V} + e^{\Phi}_{a,0} = \mathcal{G}_0^a(\Psi) \in C^{4,\delta}_{-\beta}(C, \mathbb{R}^p) \oplus D_{a,1}(C, \mathbb{R}^p). \]
is also solution to $G_0^a(V) = \Psi$, which implies that $\hat{V} = V - \hat{V}$ satisfies $G_0^a(\hat{V}) = 0$. Then, $\hat{V}$ is exponentially decaying, that is, $\hat{V} \in C_{-1}^{4,\delta}(C, \mathbb{R}^p)$. Finally, $V \in C_{-\gamma_2}^{4,\delta}(C, \mathbb{R}^p)$ since $V = \hat{V} + \hat{V}$, which finishes the proof of (i). The proof of (ii) follows by the same argument, so that we omit it. □

3.5. Growth properties of the Jacobi fields. In this part, we will the spectral analysis developed before in order to determine the growth/decay rate in which the Jacobi fields growth. This is fundamental part of the convergence technique, we will perform in the next section.

We start making some considerations about the scalar case $p = 1$. By Theorem A, the operator (30) has periodic coefficients. In addition, by Proposition 36, we can use classical Floquet theory (or Boch wave theory) to study the asymptotic behavior of the Jacobi fields on the projection over $V_j$ (for more details, see [20, Theorem 5.1]). In order to apply Floquet theory, we transform the fourth order operator (30) into a first order operator on $\mathbb{R}^4$. Indeed, defining $X = (\phi, \phi^{(1)}, \phi^{(2)}, \phi^{(3)})$, we conclude that the fourth order equation $L_j^a(\phi) = 0$ is equivalent to the first order system $X'(t) = N_{a,j,n}(t)X(t)$. Here

$$N_{a,j,n}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -B_{j,n} & +C_{a,j,n}(t) \end{bmatrix},$$

where

$$B_{j,n} := K_2(n) + 2\lambda \quad \text{and} \quad C_{a,j,n}(t) = K_0(n) + \lambda_j(\lambda_j + J_1) - \hat{c}(n)v_{a,T}(t)^{2^* - 2}.$$ 

Notice that $N_{a,j,n}(t)$ is a $T_a$-periodic matrix. Consequently, the monodromy matrix is obtained by $M_{a,j,n}(t) = \exp \int_0^t A(\tau)d\tau$. Finally, we define the Floquet exponents, denoted by $\tilde{\mathcal{F}}_j^a$, as the complex frequencies associated to the eigenvectors of $M_{a,j,n}(t)$, which forms a four-dimensional basis for the kernel of $L_j^a$. Using Abel’s identity, we get that $N_{a,j,n}(t)$ is constant, which gives

$$\det(M_{a,j,n}(t)) = \exp \int_0^t \text{tr} N_{a,j,n}(\tau)d\tau = \exp \left( -\int_0^T C_{a,j,n}(\tau)d\tau \right) = 1.$$ 

Since $N_{a,j,n}(t)$ has real coefficients, all its eigenvalues are pairs of complex conjugated. Equivalently, $\tilde{\mathcal{F}}_j^a = \{ \pm \rho_{a,j}, \pm \tilde{\rho}_{a,j} \}$, where $\rho_{a,j} = \alpha_{a,j} + i\beta_{a,j}$ and $\tilde{\rho}_{a,j} = \tilde{\alpha}_{a,j} - i\beta_{a,j}$. Then, the set of indicial roots of $L_j^a$ are given by $\mathcal{F}_j^a = \{ -\beta_{a,j}, \beta_{a,j} \}$. Additionally, for any $\phi \in \ker(L_j^a)$, we have

$$\phi(t) = b_1 \phi_{a,j}^+(t) + b_2 \phi_{a,j}^-(t) + b_3 \tilde{\phi}_{a,j}^+(t) + b_4 \tilde{\phi}_{a,j}^-(t),$$

where $\phi_{a,j}^\pm(t) = e^{\pm \rho_{a,j} t}$ and $\tilde{\phi}_{a,j}^\pm(t) = e^{\pm \tilde{\rho}_{a,j} t}$. Hence, the exponential decay/growth rate is controlled by $|\beta_{a,j}|$. Therefore, the asymptotics properties of $L_j^a$ are obtained by study of $\mathcal{F}_j^a$. Let us remember that for the case of indicial roots, we are not considering the multiplicity, that is, $\beta_{a,2}$ stands for all the medium frequency ($j = 1$) Jacobi fields.

In the next lemma, we give some structure to the set of indicial roots of $L_j^a$.

Lemma 48. For any $a \in [0,a_0]$, it follows

(i) If $j = 0$, then $0 \not\in \mathcal{F}_j^a$.

(ii) If $j = 1$, then $\{-1, 1\} \subseteq \mathcal{F}_j^a$.

(iii) If $j > 1$, then $\min_{j=1} \mathcal{F}_j^a > 1$.

Moreover, $\mathcal{F}_j^a$ is a discrete set, namely,

$$\mathcal{F}_j^a = \{ \ldots, -\beta_{a,2}, -1, 0, 1, -\beta_{a,2}, \ldots \}.$$ 

In particular, the indicial root 0 is isolated.
The operator has the following expression
\[ a \]
are the solution of a fourth order characteristic equation. In this fashion, let us also introduce the following notation for the discriminant of indicial equation,
\[ D(a, n, j) := B^2_{j,n} - 4C_{a,j,n}. \]

We also divide each step into three cases with respect the Fourier eigenmodes, namely \( j = 0 \) (zero-frequency), \( j = 1 \) (low-frequency) and \( j > 1 \) (high-frequency).

**Step 1:** \( a = 0 \) (Spherical solution).

When \( a = 0 \), we have that \( v_{a,T} \equiv 0 \), thus the linearized operator becomes
\[ L^0_j(\phi) = \phi^{(4)} - (K_2 + 2\lambda_j)\phi^{(2)} + (K_0 + \lambda_j^2 + \lambda_jJ_1) \phi. \]
Therefore, we shall compute the roots of the fourth order characteristic equation
\[ \rho^4 - B_{j,n} \rho^2 - C(0, j, n) = 0, \]

**Zero-frequency:** \( j = 0, m_0 = 1 \) and \( \lambda_0 = 0 \).

The operator has the following expression
\[ L^0_0(\phi) = \phi^{(4)} - K_2\phi^{(2)} + K_0 \phi. \]
Notice that when \( D(a, n, j) > 0 \), then \( \beta_{a,j} = 0 \). More generally, the sign of \( D(a, n, j) \) controls the nature of the complex roots. It is straightforward to check that the indicial roots of this operator are
\[ \rho_{0,0} = \frac{n}{2} \quad \text{and} \quad \tilde{\rho}_{0,0} = \frac{n-4}{2}. \]

**Low-frequency:** \( j = 1, m_1 = n \) and \( \lambda_1 = \cdots \lambda_n = n - 1 \).
Here we obtain,
\[ L^0_1(\phi) = \phi^{(4)} - B_{1,n} \phi^{(2)} + C(0, 1, n) \phi, \]
and the indicial roots are given by
\[ \rho_{0,1} = \frac{1}{2}(n + 2) \quad \text{and} \quad \tilde{\rho}_{0,1} = \frac{1}{2}(n - 2). \]

**High-frequency:** \( j > 1, m_j > n \) and \( \lambda_j = \ell(n - 2 + \ell) \), for some \( \ell \in \mathbb{N} \).
Here we obtain,
\[ L^0_j(\phi) = \phi^{(4)} - B_{j,n} \phi^{(2)} + C(0, j, n) \phi, \]
and the indicial roots are given by
\[ \rho_{0,\ell} = \frac{1}{2} \left(2 + \sqrt{D(0, \ell, n)}\right) \quad \text{and} \quad \tilde{\rho}_{0,\ell} = \frac{1}{2} \left(2 - \sqrt{D(0, \ell, n)}\right), \]
where \( D(0, \ell, n) = n^2 - 4n + 4 + 4\ell(n + \ell - 2) \).

**Step 2:** \( a = a_0 \) (Cylindrical solution).

Since \( v_{a_0,T} \equiv a_0 = [n(n-4)/(n^2 - 4)]^{n - 4}/4 \), we will proceed identically as in the last step. First, we have
\[ L^0_j(\phi) = \phi^{(4)} - (K_2 + 2\lambda_j)\phi^{(2)} + \left(K_0 + \lambda_j^2 + \lambda_jJ_1 - \tilde{c}(n)a_0^{2^{n-2}}\right) \phi. \]
As before, we get

**Zero-frequency:** \( j = 0, m_0 = 1 \) and \( \lambda_0 = 0 \).

\[ \rho_{a_0,0} = \frac{1}{2} \sqrt{n^2 - 4n + 8 + \sqrt{D(a_0, 0, n)}} \quad \text{and} \quad \tilde{\rho}_{a_0,0} = \frac{1}{2} \sqrt{n^2 - 4n + 8 - \sqrt{D(a_0, 0, n)}}, \]
where $D(a_0, 0, n) = n^4 - 64n + 64$.

**Low-frequency:** $j = 1, m_1 = n$ and $\lambda_1 = \cdots \lambda_n = n - 1$.

$$\rho_{a,1} = \sqrt{\frac{n^2 + 2}{2}} \quad \text{and} \quad \bar{\rho}_{a,1} = 1.$$  

**High-frequency:** $j > 1$, $m_j > n$ and $\lambda_j = \ell(n - 2 + \ell)$, for some $\ell \in \mathbb{N}$.

$$\rho_{a,j} = \frac{1}{4} \sqrt{[(n + 2(\ell - 1))]^2 + \sqrt{D(a_0, \ell, n)}} \quad \text{and} \quad \bar{\rho}_{a,j} = \frac{1}{4} \sqrt{[(n + 2(\ell - 1))]^2 - \sqrt{D(a_0, \ell, n)}},$$

where $D(a_0, \ell, n) = n^4 + 64(\ell - 1)(n + \ell - 1)$.

**Step 3:** $a \in (0, a_0)$ (Emden–Fowler solution).

This is the most delicate case since $v_{a,T}$ is periodic, so the zeroth-order term in the operator $\mathcal{L}_j^a$ is also $T_a$-periodic. In this case, it is not possible to compute explicitly the Floquet exponents. Nonetheless, we are able to show that they are strictly greater than one when $j > 1$.

**Zero-frequency:** $j = 0$, $m_0 = 1$ and $\lambda_0 = 0$.

By Remark 25, it follows that $\hat{\phi}_{a,0}^+$ is bounded and $\hat{\phi}_{a,0}^-$ is linearly growing, then $0 \in \mathcal{I}_a$ with multiplicity 2.

**Low-frequency:** $j = 1$, $m_1 = n$ and $\lambda_1 = \cdots \lambda_n = n - 1$.

Again using Remark 25, it follows that $\hat{\phi}_{a,1}^+ = \cdots = \hat{\phi}_{a,n}^+$ is exponentially growing and $\hat{\phi}_{a,0}^-$ is exponentially decaying, then $\{-1, 1\} \subset \mathcal{I}_a$.

**High-frequency:** More generally, note that the indicial roots forms a increasing sequence,

$$\beta_{a,0} \leq \beta_{a,1} \leq \cdots \leq \beta_{a,j} \leq \beta_{a,j+1} \to \infty \quad \text{as} \quad j \to \infty,$$

which is a consequence of a comparison principle on $a \in (0, a_0)$ for the linearized operator.

**Lemma 49.** The indicial set $\mathcal{I}_a^j$ is a discrete. Moreover,

$$\mathcal{I}_a^j = \{-1, 1\} \cup \{ \lambda \alpha \}.$$

In particular, the indicial root 0 is isolated.

**Proof.** It follows direct by Proposition 36 in the last subsection.

Notice that Lemma 48 only provides exponential growth/decay for the Jacobi fields. Nevertheless, in order to apply the Leon Simon technique, we need something slightly stronger. Namely, for $j = 0$, we must show that the Jacobi fields are either periodic (bounded) or linearly growing. For the first two Jacobi fields $\phi_{a,0}^+$ and $\phi_{a,0}^-$, this follows because they arise as the variation of the necksize and translation parameters, respectively. The difficulty here is to show that the other two fields $\tilde{\phi}_{a,0}^+$, $\tilde{\phi}_{a,0}^-$ also satisfies those growth properties. In fact, we overcome this issue, observing that by the direct computation in Lemma 48, we know that $0 \in \mathcal{I}_a$ with multiplicity two. Next, we proceed as in [70, Proposition 4.14] and prove the following asymptotic expansion:

**Proposition 50.** Let $\psi \in C_0^\infty(\mathcal{C})$, $\beta \in (0, 1)$ and $\phi \in H^{4}_\beta(\mathcal{C})$ satisfying $\mathcal{L}_a(\phi) = \psi$. Then, $\phi$ has an asymptotic expansion $\phi = \sum_{j \in \mathbb{N}} \phi_j$ with $\mathcal{L}_a(\phi_j) = 0$ and $\phi_j \in H^{4}_\beta(\mathcal{C})$ for any $\beta < \beta_{a,j}$.

**Proof.** We divide the proof in some steps:

**Step 1:** For $\beta < 1$, it follows $\phi \in H^{m+4}_{-\beta}(\mathcal{C})$.

Indeed, take $\rho \in \mathcal{C}$ with $0 < \beta < \Im(\rho)$ and consider the transformed equation

$$\tilde{\mathcal{L}}^a(\rho)(\tilde{\phi}) = \tilde{\psi}, \quad (42)$$

where $\tilde{\phi} = e^{i\alpha}\phi$ and $\tilde{\psi} = e^{i\alpha}\psi$. By applying the inverse operator $\mathcal{G}^a(\rho)$ in both sides of (42), we get $\tilde{\phi} = \mathcal{L}^a(\rho)(\tilde{\psi})$. Then, since $\psi \in C_0^{\infty}(\mathcal{C})$, it follows that $\tilde{\psi}(\rho)$ is an entire function on $\rho$ and smooth on $(t, \theta)$. Notice that $\tilde{\phi}$ is analytic on the half-plane $\Im(\rho) > \beta$, because of the poles
of \( G^a(\rho) \) coincide with the zeros of \( \tilde{\psi} \). Finally, take \( \beta' \in (\beta, 1) \) and since \( G^a(\rho) \) has no poles in \( \Im(\rho) = (\beta', \beta) \), by the Cauchy formula, we can define the contour integral \( F_{a,1}^{-1} \) up to \( \Im(\rho) \).

**Step 2:** For each \( \beta \in (0, 1) \), there exist \( \beta'' \in (1, \beta_{a,2}) \), \( \phi'' \in H^{n+4}_{-\beta''} (C) \), and \( \phi' \in H^{n+4}_{-\beta} (C) \) with \( L^a(\phi') = 0 \) satisfying \( \phi = \phi' + \phi'' \).

Choose \( \beta'' \in (1, \beta_{a,2}) \) and \( \rho'' \) such that \( \Im(\rho'') = \beta'' \). Now let us define \( \tilde{\phi''} = \tilde{G}^a(\rho'') \). Finally, we apply the inverse \( F_{a,1}^{-1} \) in the two contours \( \Im(\rho) = \beta \) and \( \Im(\rho) = \beta'' \), which by periodicity does not take into account the vertical sides of the rectangle \([\beta, \beta''] \times [0, 2\pi] \subset \mathbb{C}\). In fact, \( \tilde{\phi} = \tilde{\phi''} = \tilde{G}^a(\rho) - \tilde{G}^a(\rho'') \) is the residue of a meromorphic function with pole at \(-i\).

We can continue this process shifting the contour integral to the other poles in the strip. \( \square \)

**Corollary 51.** Let \( \beta \in (0, 1) \), \( \psi \in C^\infty_0 (C) \cap L^2_\beta (C) \) and \( \phi \in H^4_\beta (C) \) satisfying \( L^a(\phi) = \psi \). Then, there exist \( \phi' \in H^4_\beta (C) \) and \( \phi'' \in D_{a,0} (C) \) such that \( \phi = \phi' + \phi'' \).

**Corollary 52.** The following properties hold for the projected scalar linearized operator:

(i) Assume \( j = 0 \), then the homogeneous equation \( L^a_0(\phi) = 0 \) has a solutions basis with two elements, which are either bounded or at most linearly growing as \( t \to \infty \);

(ii) Assume \( j \geq 1 \), then the homogeneous equation \( L^a_j(\phi) = 0 \) has a solutions basis with four elements, which are exponentially growing/decaying as \( t \to \infty \).

**Proof.** For (i), we use Corollary 51. Notice that (ii) follows directly from Lemma 48. \( \square \)

In the case \( p > 1 \), we can use a similar strategy as before to study the solutions of (27). In the scalar case, we have constructed a Jacobi fields basis with four elements (two in the zero-frequency case). For the system we must find a basis with \( 4p \) elements (\( 2p \) in the zero-frequency case), sharing the same growth properties in Corollary 52.

**Proof of Proposition 7.** Initially, note that by Theorem B there exist \( a \in [0, a_0] \), \( T \in [0, T_a] \) and \( \Lambda^* \in \mathbb{S}_p^{-1} \) only provides \( p + 1 \) families of solutions given by

\[
T \mapsto \Lambda^* v_a(t + T), \quad a \mapsto \Lambda^* v_{a,T}(t), \quad \text{and} \quad \theta \mapsto \Lambda^*(\theta)v_{a,T}(t),
\]

which by differentiation gives rise to some of elements of the basis.

\[
\Lambda^* \partial_T \big|_{T=0} v_{a,T}(t), \quad \Lambda^* \frac{1}{a} \partial_a \big|_{a=0} v_{a,T}(t), \quad \text{and} \quad \partial_\theta \Lambda^*(\theta)v_{a,T}(t),
\]

for \( i = 1, \ldots, p - 1 \). In order to construct all the Jacobi fields basis, let us consider \( \{e_i\}_{i \in I} \subset \mathbb{S}_p^{-1} \) a linearly independent set in \( \mathbb{R}^p \) with \( e_1 = \Lambda^* \), which gives four families of \( p \) Jacobi fields,

\[
\Phi_{a,j,i}^+ = e_i \phi_{a,j}^+, \quad \Phi_{a,j,i}^- = e_i \phi_{a,j}^-, \quad \tilde{\Phi}_{a,j,i}^+ = e_i \tilde{\phi}_{a,j}^+, \quad \text{and} \quad \tilde{\Phi}_{a,j,i}^- = e_i \tilde{\phi}_{a,j}^-.
\]

Then, using Theorem B, it is easy to check that \( B_0 = \bigcup_{i \in I} \{\Phi_{a,j,i}^\pm, \tilde{\Phi}_{a,j,i}^\pm\} \) is a basis to the kernel of \( L^a_j \) with \( 4p \) elements for each \( j \geq 1 \), and with \( 2p \) elements when \( j = 0 \). \( \square \)

4. **Qualitative properties and apriori estimates**

This section is devoted to prove Proposition 8. More precisely, we show that solutions of (1) are asymptotic radially symmetric and satisfies a upper and lower bound estimate near to the isolated singularity. Our strategy is to convert our system of differential equations into a system of non-local integral equations. Subsequently, we use the Kelvin transform to perform a moving sphere technique and the Pohozaev invariant to obtain the aforementioned qualitative properties. Here we are inspired in some techniques from [52] (see also [17, 51, 86]). The main difference to the proofs in [52] is that we need to deal with many components of our system, coupled by the Gross–Pitaevskii nonlinearity.
4.1. Integral representation formulas. For $n \geq 3$, it is well-known the following expression for the Green function of the laplacian on the unit ball,

$$G_1(x, y) = \frac{1}{(n-2)\omega_{n-1}} \left( |x - y|^{2-n} - \frac{x}{|x|} - \frac{x}{|x|} \right).$$

Here $\omega_{n-1}$ is the surface area of the unit sphere in $\mathbb{R}^n$. It also holds for any $u \in C^2(B_1) \cap C(\bar{B}_1)$ the decomposition,

$$u(x) = - \int_{B_1} G_1(x, y) \Delta u(y) dy + \int_{\partial B_1} H_1(x, y) u(y) d\sigma_y,$$

where

$$H_1(x, y) = - \partial_v G_1(x, y) = \frac{1}{\omega_{n-1}|x - y|^n} \quad \text{for} \ x \in B_1 \ \text{and} \ y \in \partial B_1,$$

where $v_y$ is the outward normal vector at $y$.

Similarly, in the case of the bi-Laplacian operator with $n \geq 5$ for any $u \in C^4(B_1) \cap C^2(\bar{B}_1)$, it follows

$$u(x) = \int_{B_1} G_2(x, y) \Delta^2 u(y) dy + \int_{\partial B_1} H_1(x, y) u(y) d\sigma_y - \int_{\partial B_1} H_2(x, y) \Delta u(y) d\sigma_y,$$

where

$$G_2(x, y) = \int_{B_1 \times B_1} G_1(x, y_1) G_1(y_1, y_2) dy_1 dy_2$$

and

$$H_2(x, y) = \int_{B_1 \times B_1} G_1(x, y_1) G_1(y_1, y_2) H_1(y_2, y) dy_1 dy_2.$$

By a direct computation, we have

$$G_2(x, y) = C(n, 2)|x - y|^{4-n} - A(x, y),$$

where $C(n, 2) = \frac{\Gamma(n-4)}{2^{n-1} \pi^{n/2}}$, $A : B_1 \times B_1 \to \mathbb{R}$ is a smooth map and $H_i(x, y) \geq 0$ for $i = 1, 2$.

In the following lemma, we will employ some ideas due to L. Caffarelli et al. [9] and L. Sun and J. Xiong [85].

**Lemma 53.** Let $U \in C^4(\bar{B}_1^*, \mathbb{R}^p) \cap L^1(B_1, \mathbb{R}^p)$ be a strongly positive solution of

$$\Delta^2 u_i = f_i^s(U) \quad \text{in} \ B_1^* \quad \text{for} \ i \in I,$$

where $f_i^s(U) := |U|^{s-1} u_i$ for $s > 2^* / 2$. Then, $|x|^{-\alpha} u_i \in L^1(B_1)$ for every $\alpha < n - 4s/(s - 1)$ and

$$u_i(x) = \int_{B_1} G_2(x, y) \Delta^2 u_i(y) dy + \int_{\partial B_1} H_1(x, y) u_i(y) d\sigma_y - \int_{\partial B_1} H_2(x, y) \Delta u_i(y) d\sigma_y,$$

for $i \in I$.

**Proof.** Let $\zeta \in C^\infty(\mathbb{R})$ such that $\zeta(t) = 0$ for $t \leq 1$, $\zeta(t) = 1$ for $t \geq 2$ and $0 \leq \zeta \leq 1$ for $1 \leq t \leq 2$. For small $\varepsilon > 0$, plugging $\zeta(\varepsilon^{-1}|x|)|x|^{4-\alpha}$ into (44), using integration by parts, it holds

$$\int_{B_1} f_i^s(U) \zeta (\varepsilon^{-1}|x|)|x|^{4-\alpha} dx = \int_{B_1} u_i \Delta^2 \left( \zeta (\varepsilon^{-1}|x|)|x|^{4-\alpha} \right) dx + \int_{\partial B_1} G(u_i) d\sigma_g,$$

where $G(u_i)$ involves $u_i$ and its derivatives up to third order. Taking $\alpha = \alpha_0 = 0$ in (45), we have that $\int_{B_1} u_i^s |x|^{4(\alpha_1 - 4s)} dx < \infty$ as $\varepsilon \to 0$, since by hypothesis $u_i \in L^1(B_1)$ for all $i \in I$. Moreover, by Hölder inequality, if $0 < \alpha_1 < [n(s-1) - 4]/s$, we have

$$\int_{B_1} u_i |x|^{-\alpha_1} dx = \int_{B_1} u_i |x|^{4s} |x|^{-\alpha_1 - 4s} dx \leq \left( \int_{B_1} u_i^s |x|^4 dx \right)^{1/s} \left( \int_{B_1} |x|^{4+\alpha_1 - 4s} dx \right)^{(s-1)/s} < \infty.$$
Next, using that $\alpha = \alpha_1$ in (45) and taking the limit $\varepsilon \to 0$, we obtain $\int_{B_1} u^2|x|^{4-\alpha_1} \, dx < \infty$. By the same argument, we find $\int_{B_1} u|x|^{-\alpha_2} \, dx < \infty$, if $\alpha_2 < [n(s - 1) - 4]\{s^{-1} + s^{-2}\}$. Iterating this procedure, it follows

$$\int_{B_1} u_i|x|^{-\alpha_k} \, dx < \infty \quad \text{and} \quad \int_{B_1} u_i^2|x|^{4-\alpha_k} \, dx < \infty,$$

if $0 < \alpha_k := [n(s - 1) - 4]\{\sum_{i=1}^k s^{-i}\}$ for $k \in \mathbb{N}$, which proves the first statement.

Next, let us consider

$$\hat{u}_i(x) = \int_{B_1} G_2(x,y)\Delta^2 u_i(y) \, dy + \int_{\partial B_1} H_1(x,y)u_i(y) \, d\sigma_y - \int_{\partial B_1} H_2(x,y)\Delta u_i(y) \, d\sigma_y \quad \text{for} \quad i \in I.$$ 

and define $w_i = u_i - \hat{u}_i$. Thus, $\Delta^2 w_i = 0$ in $B_1^n$ since $u_i^* \in L^1(B_1)$ and $\hat{u}_i \in L^{2**/2}(B_1) \cap L^1(B_1)$. Furthermore, since the Riesz potential $|x|^{4-n}$ is of weak type $(1, 2**/2)$ (see [34, Chapter 9]), for every $\varepsilon > 0$, we can choose $\alpha > 0$ such that $\int_{B_2^\alpha} \hat{f}_i(U(x)) \, dx < \varepsilon$, which implies that for all $M > 1$,

$$\mathcal{L}^n \left( \{x \in B_1 : |\hat{u}_i(x)| > M \} \right) \leq \mathcal{L}^n \left( \left\{ x \in B_1 : \int_{B_2^\alpha} G_2(x,y)f_i(U(y)) \, dy > \mu/2 \right\} \right) \leq c(n)\varepsilon M^{-2**/2}.$$ 

Hence, $w_i \in L^{2**/2}(B_1) \cap L^1(B_1)$ for all $i \in I$ and for every $\varepsilon > 0$ there exists $\eta > 0$ such that for all sufficiently large $\mu$, it holds

$$\mathcal{L}^n \left( \{x \in B_1 : |w_i(x)| > M \} \right) \leq \mathcal{L}^n \left( \{x \in B_1 : |u_i(x)| > M/2 \} \right) + \mathcal{L}^n \left( \{x \in B_1 : |\hat{u}_i(x)| > M/2 \} \right) \leq \varepsilon M^{-2**/2}.$$ 

Using the Bôcher theorem for 4/s functions from [32], we obtain that $\Delta^2 w(x) = 0$ in $B_1$. Finally, since $w_i = \Delta w_i = 0$ on $\partial B_1$, by the maximum principle, we have $w_i \equiv 0$ and thus $u_i = \hat{u}_i$, which gives the proof of the second part. \hfill \Box

**Remark 54.** If $U \in C^2(B_1^n, \mathbb{R}^p)$ is a nonnegative superharmonic $p$-map, then $U \in L^1(B_1/2, \mathbb{R}^p)$. Indeed, for $0 < r < 1$, we have

$$\left( r^{n-1} \hat{u}_i^{(1)} \right) \leq 0,$$

where $\hat{u}_i(r) = \int_{\partial B_1} u_i(r\theta) d\theta$, which integration implies $\hat{u}_i(r) \leq C \left( r^{2-n} + 1 \right)$.

In the next lemma, we use the Green identity to convert (1) into an integral system.

**Lemma 55.** Let $U$ be a strongly positive superharmonic solution of (1). Then, there exists $r_0 > 0$ such that

$$u_i(x) = \int_{B_{r_0}} |x - y|^{4-n} f_i(U(y)) \, dy + h_i(x) \quad \text{for} \quad i \in I,$$

(46)

where $h_i > 0$ satisfies $\Delta^2 h_i = 0$ in $B_{r_0}$. Moreover, one can find a constant $C(\bar{r}) > 0$ such that $\|\nabla \ln h_i\|_{C^0(B_{\bar{r}})} \leq C(\bar{r})$ for all $i \in I$ and $\bar{r} < r_0$.

**Proof.** Using that $-\Delta u_i > 0$ in $B_1^n$ and $u_i > 0$ in $\bar{B}_1$, it follows from the maximum principle that $c_{1i} := \inf_{B_1} u_i = \min_{\partial B_1} u_i > 0$. In addition, by Lemma 53, we have $f_i(U) \in L^1(B_1)$, which implies that there exists $r_0 < 1/4$ such that

$$\int_{B_r} |A(x,y)| f_i(U) \, dy \leq \frac{c_{1i}}{2} \quad \text{for} \quad x \in B_{r_0}.$$ 


where $A(x, y)$ is given by (43). Hence, for $x \in B_{r_0}$, we get
\[
h_i(x) = -\int_{B_r} A(x, y) f_i(U) dy + \int_{B_r \setminus B_x} G_2(x, y) f_i(U) dy
+ \int_{\partial B_1} H_1(x, y) u_i(y) d\sigma_y - \int_{B_1} H_2(x, y) \Delta u_i(y) d\sigma_y
\geq -\frac{c_{ii}}{2} + \int_{\partial B_1} H_1(x, y) u_i(y) d\sigma_y
\geq -\frac{c_{ii}}{2} + \inf_{B_1} u = \frac{c_{ii}}{2}.
\]
By hypothesis, $h_i \in C^\infty(B_{r_0})$ for all $i \in I$, which gives that $|\nabla h_i| \leq C_i(\tilde{r})$ in $B_{\tilde{r}}$ for all $\tilde{r} < \eta$ and $i \in I$, where $C_i(\tilde{r}) > 0$ depends only on $n, \eta, \tilde{r}$ and in the $L^1$ norm of $f_i(U)$. Consequently,
\[
\|\nabla \ln h_i\|_{C^0(B_{\tilde{r}})} \leq 2\frac{C(\tilde{r})}{c_{ii}} \quad \text{for} \quad i \in I,
\]
which finishes the proof. \hfill \Box

4.2. Moving spheres technique and the upper bound estimate. The objective of this part is to prove the upper bound estimate in Proposition 8. In this fashion, we use the moving spheres technique together with a classical blow-up argument. First, we have the following result from [52, Lemma 3.1], which proof we include for the sake of completeness.

**Lemma A.** Suppose $h \in C^1(B_2)$ is positive and
\[
|\nabla \ln h| \leq C_0 \quad \text{in} \quad B_{3/2},
\]
for some $C_0 > 0$. Then, there exists $0 < r_0 < 1/2$, depending only on $n \in \mathbb{N}$ and $C_0 > 0$, such that for every $x \in B_1$ and $0 < \mu \leq r_0$, it holds $h_{x, \mu}(y) \leq h(y)$ for $|y - x| \geq \mu$ and $y \in B_{3/2}$.

**Proof.** For any $x \in B_1$, we have
\[
\frac{d}{dr}(r^\gamma(h(x + r\theta))) = r^{\gamma - 1}h(x + r\theta) \left(\gamma - r\frac{\nabla h \cdot \theta}{h}\right) \geq r^{\gamma - 1}h(x + r\theta) \left(\gamma - C_0 r\right) > 0.
\]
Hence, for $0 < r < \tilde{r} := \min \left\{\frac{1}{2}, \frac{\mu}{C_0}\right\}$ and $\theta \in S^{n-1}$. For any $y \in B_\tilde{r}(x)$ and $0 < \mu < |y - x| \leq \tilde{r}$, let us consider
\[
\theta = \frac{y - x}{|y - x|}, \quad r_1 = |y - x| \quad \text{and} \quad r_2 = \frac{\mu^2}{|y - x|^2}r_1.
\]
Additionally, from (48), it follows that $r_2^\gamma h(x + r_2\theta) < r_1^\gamma h(x + r_1\theta)$, which provides $h_{x, \mu}(y) \leq h(y)$, for $0 < \mu < |y - x| \leq \tilde{r}$. On the other hand, we get
\[
h_{x, \mu}(y) = \left(\frac{\mu}{|y - x|}\right)^{n-4} h(I_{x, \mu}(y)) \leq \left(\frac{\mu}{\tilde{r}}\right)^{n-4} \max_{B_{3/2}(x)} h \leq e^{\frac{3}{2}C_0} \left(\frac{\mu}{\tilde{r}}\right)^{n-4} \inf_{B_{3/2}(x)} h \leq h(y),
\]
for $|y - x| \geq \tilde{r}$. Finally, choosing $\mu \leq r_0$ with $e^{\frac{3}{2}C_0} \left(\frac{\mu}{\tilde{r}}\right)^{n-4} \leq 1$, the proof of the lemma follows. \hfill \Box

The following lemma is the first step to apply the moving spheres method.

**Lemma 56.** Let $U$ be a positive solution of (46). For any $x \in B_1$, $z \in B_2 \setminus \{0\} \cup B_\mu(x)$ and $\mu < 1$, it holds that $u_i(z) - (u_i)_{x, \mu}(z) > 0$ for $i \in I$. \hfill \Box
Proof. Let $\mathcal{U}$ be a positive solution of (46). Replacing $u_i(x)$ by $r^\gamma u_i(rx)$ for $r = 1/2$, we may consider the equation defined in $B^*_2$ for convenience,

$$u_i(x) = \int_{B^*_2} |x - y|^{4-n} f_i(\mathcal{U}(y))dy + h_i(x) \quad \text{in} \quad B^*_2$$

(49)
such that $u_i \in C(B^*_2) \cap L^{n+4-4\gamma}(B_2)$ and $|\nabla \ln h_i| \leq C_0$ in $B_{3/2}$. Extending $u_i$ to be identically 0 outside $B_2$, we have

$$u_i(x) = \int_{\mathbb{R}^n} |x - y|^{4-n} f_i(\mathcal{U}(y))dy + h_i(x) \quad \text{in} \quad B^*_2.$$ 

Using the identities in [61, page 162], one has

$$\left(\frac{\mu}{|z - x|}\right)^{n-4} \int_{|y - x| > \mu} |I_{x,\mu}(z) - y|^{n-4} f_i(\mathcal{U}(y))dy = \int_{|y - x| \leq \mu} |z - y|^{n-4} f_i(\mathcal{U}(z))dy$$

and

$$\left(\frac{\mu}{|z - x|}\right)^{n-4} \int_{|y - x| \leq \mu} |I_{x,\mu}(z) - y|^{n-4} f_i(\mathcal{U}(y))dy = \int_{|y - x| > \mu} |z - y|^{n-4} f_i(\mathcal{U}(y))dy,$$

(51)

which gives

$$(u_i)_{x,\mu}(z) = \int_{\mathbb{R}^n} |z - y|^{n-4} f_i(\mathcal{U}(y))dy + (h_i)_{x,\mu}(z) \quad \text{for} \quad z \in I_{x,\mu}(B_2).$$

Consequently, for any $x \in B_1$ and $\mu < 1$ we have, for $z \in B^*_2 \cup B_\mu(x)$, it holds

$$u_i(z) - (u_i)_{x,\mu}(z) = \int_{|y - x| > \mu} E(x, y, \mu, z) [f_i(\mathcal{U}) - f_i(\mathcal{U}_{x,\mu})]dy + [(h_i)_{x,\mu}(z) - h_i(z)],$$

where

$$E(x, y, \mu, z) = |z - y|^{4-n} - \left(\frac{|z - x|}{\mu}\right)^{4-n} |I_{x,\mu}(z) - y|^{4-n}$$

will be used to estimate the difference between a $p$-map $\mathcal{U}$ and its Kelvin transform $\mathcal{U}_{x_0,\mu}$. Finally, it is straightforward to check that $E(x, y, z, \mu) > 0$ for all $|z - x| > \mu > 0$, $|y - x| > \mu > 0$, which concludes the proof. \hfill $\square$

Now we use Lemma 56 will be important in the prove of our main estimate. We use a contradiction argument.

**Proposition 57.** Let $\mathcal{U} \in C(B^*_2, \mathbb{R}^p) \cap L^{2n-1}(B_2, \mathbb{R}^p)$ be a positive solution of (49). Suppose that $h_i \in C^1(B_2)$ is a positive function satisfying (47) for any $i \in I$. Then,

$$\limsup_{|x| \to 0} |x|^{-\gamma} |\mathcal{U}(x)| < \infty.$$ 

**Proof.** Let us assume by contradiction that there exist $i \in I$ and $\{x_k\}_{k=1}^\infty \subset B_2$ such that $\lim_{k \to \infty} |x_k| = 0$ and $|x_k|^\gamma u_i(x_k) \to \infty$ as $k \to \infty$. For $|x - x_k| \leq 1/2|x_k|$, let us define

$$\tilde{u}_{ki}(x) := \left(\frac{|x_k|}{2} - |x - x_k|\right)^\gamma u_i(x).$$

Hence, since $u_i$ is positive and continuous in $\bar{B}_{|x_k|/2}(x_k)$, there exists a maximum point $\tilde{x}_k \in \bar{B}_{|x_k|/2}(x_k)$ of $\tilde{u}_{ki}$, that is,

$$\tilde{u}_{ki}(\tilde{x}_k) = \max_{|x - x_k| \leq |x_k|/2} \tilde{u}_{ki}(x) > 0.$$
Taking $2\mu_k := |x_k|/2 - |\bar{x}_k - x_k| > 0$, we get
\[ 0 < 2\mu_k \leq \frac{|x_k|}{2} \quad \text{and} \quad \frac{|x_k|}{2} - |x - x_k| \geq \mu_k \quad \text{for} \quad |x - \bar{x}_k| \leq \mu_k. \tag{52} \]
Moreover, it follows $2\gamma u_i(\bar{x}_k) \geq u_i(x)$ for $|x - \bar{x}_k| \leq \mu_k$ and
\[ (2\mu_k)^\gamma u_i(\bar{x}_k) = \bar{u}_k u_i(\bar{x}_k) \geq \bar{u}_k u_i(x_k) = 2^{-\gamma}|x_k|^\gamma u_i(x_k) \to \infty \quad \text{as} \quad k \to \infty. \tag{53} \]
Defining
\[ w_{ki}(y) = u_i(\bar{x}_k)^{-1}u\left(\bar{x}_k + yu_i(\bar{x}_k)^{-\gamma-1}\right) \quad \text{and} \quad h_{ki}(y) = u_i(\bar{x}_k)^{-1}h_i\left(\bar{x}_k + yu_i(\bar{x}_k)^{-\gamma-1}\right) \quad \text{in} \quad \Omega_k, \]
where
\[ \Omega_k = \left\{ y \in \mathbb{R}^n : \bar{x}_k + yu_i(\bar{x}_k)^{-\gamma-1} \in B^*_2 \right\}. \]
Now, we extend $w_{ki}$ to be zero outside of $\Omega_k$, which provides
\[ w_{ki}(y) = \int_{\mathbb{R}^n} f_i(W_k)|y - x|^{4-n}dx + h_{ki}(y) \quad \text{for} \quad y \in \Omega_k \tag{54} \]
and $w_{ki}(0) = 1$ for $k \in \mathbb{N}$, where $W_k = e_1 w_{ki}$. Moreover, from (52) and (53), it follows
\[ \|h_{ki}\|_{C^1(\Omega_k)} \to 0 \quad \text{and} \quad w_{ki}(y) \leq 2^\gamma \quad \text{in} \quad B_{R_{ki}}, \]
where $R_{ki} := \mu_k u_i(\bar{x}_k)^{-\gamma-1} \to \infty$ as $k \to \infty$. Using the regularity results in [61], one can find $w_{0i} > 0$ such that $w_{ki} \to w_{0i}$ as $k \to \infty$ in $C^{\alpha}_{\text{loc}}(\mathbb{R}^n)$, where $w_{0i} > 0$ satisfies
\[ w_{0i}(y) = c(n) \int_{\mathbb{R}^n} |y - x|^{4-n}w_{0i}^{2*-1}dx \quad \text{in} \quad \mathbb{R}^n. \]
or, equivalently $\Delta^2 w_{0i} = f_i(w_0)$ in $\mathbb{R}^n$. Furthermore, by construction, we have $w_{0i}(0) = 1$, which by B implies that there exist $\mu > 0$ and $y_0 \in \mathbb{R}^n$ such that
\[ w_{0i}(y) = \left(\frac{2\mu}{1 + \mu^2|y - y_0|^2}\right)^\gamma. \tag{55} \]
In the next claim, we use the last classification formula and apply the moving spheres technique.

**Claim 1:** For any $\mu > 0$, it holds that $(w_{0i})_{x, \mu}(y) \leq w_{0i}(y)$ for $|y - x| \geq \mu$.

Indeed, for a fixed $\mu_0 > 0$, we have $0 < \mu_0 < R_{k}/10$ when $k \gg 1$. We also consider
\[ \hat{\Omega}_k := \left\{ y \in \mathbb{R}^n : \bar{x}_k + yu(\bar{x}_k)^{-\gamma-1} \in B^*_2 \right\} \subset \subset \Omega_k. \]
In this direction, let us divide the proof of the claim into three step as follows

**Step 1:** For $k \gg 1$, it holds that $(w_{ki})_{x, \mu_0}(y) \leq w_{ki}(y)$ for $y \in \hat{\Omega}_k$ such that $|y| \geq \mu_0$.

In fact, by Lemma A, there exists $\tilde{r} > 0$ such that for all $0 < \mu \leq \tilde{r}$ and $\bar{x} \in B_{1/100}$,
\[ \left(\frac{\mu}{|y|}\right)^{n-4} h_{ki}(I_{0, \mu}(y) + \bar{x}) \leq h_{ki}(y + \bar{x}) \quad \text{for} \quad |y| \geq \mu \quad \text{and} \quad y \in B_{149/100}. \tag{56} \]

Let $k \gg 1$ be sufficiently large such that $\mu_0 u(\bar{x}_k)^{(4-n)-1} < \tilde{r}$. Hence, for any $0 < \mu \leq \mu_0$, it holds
\[ (h_{ki})_{x, \mu}(y) \leq h_{ki}(y) \quad \text{in} \quad \hat{\Omega}_k \setminus B_{\mu}, \tag{57} \]
which by passing to the limit as $k \to \infty$ concludes the proof of Step 1.

**Step 2:** There exists $\mu_1 > 0$, independent of $k$, such that $(w_{ki})_{x, \mu}(y) \leq w_{ki}(y)$ in $\hat{\Omega}_k \setminus B_{\mu}$ for $0 < \mu < \mu_1$.

As matter of fact, since $w_{ki} \to w_{0i}$ as $k \to \infty$ in $C^0$-topology and $w_{0i}$ is given by (55) we get that there exists $C_0 > 0$ satisfying $w_{ki} \geq C_0 > 0$ on $B_1$ for $k \gg 1$. On the other hand, by (54)
and standard regularity results, it follows that $|\nabla^j w_{ki}| \leq C_0 < \infty$ on $B_1$ for $j = 1, 2, 3, 4$. Using Lemma A, there exists $r_0 > 0$, not depending on $k \gg 1$, such that for all $0 < \mu \leq r_0$, it holds that
\[ (w_{ki})_{x,\mu} (y) < w_{ki}(y) \quad \text{for} \quad 0 < \mu < |y| \leq r_0. \quad (58) \]
Again, since $w_{ki} \geq C_0 > 0$ on $B_1$ for $k \gg 1$, we obtain
\[ w_{ki}(x) \geq C_0^{2^* - 1} \int_{B_1} |x - y|^{4-n}dy \geq \frac{1}{C} (1 + |x|)^{4-n} \quad \text{in} \quad \Omega_k, \]
for some $C > 0$. Therefore, there exists $0 < \mu_1 \leq r_0$ sufficiently small such that for all $0 < \mu < \mu_1$, we have
\[ (w_{ki})_\mu (y) \leq \left( \frac{\mu_1}{|y|} \right)^{n-4} \max_{B_{r_0(y)}} w_{ki} \leq C \left( \frac{\mu_1}{|y|} \right)^{n-4} \leq w_{ki}(y) \quad \text{for} \quad y \in \Omega_k \quad \text{and} \quad |y| \geq r_0, \]
which in combination with (58) proves Step 2.

For $\mu_0$ fixed before, let us define,
\[ \mu_* := \sup \left\{ 0 < \mu \leq \mu_0 : (w_{ki})_{x,\mu} (y) \leq w_{ki}(y), \quad \text{for} \quad y \in \tilde{\Omega}_k \quad \text{such that} \quad |y - x_0| \geq \mu \quad \text{and} \quad 0 < \mu < \mu_0 \right\}. \]

**Step 3:** For $k \gg 1$, it holds that $\mu_* = \mu_0$.

Indeed, using (50), (51) and (57), it follows
\[ w_{ki}(y) - (w_{ki})_{0,\mu} (y) = \int_{\mathbb{R}^n \setminus B_\mu} E(0, y, z, \mu) \left[ w_{ki}(z)^{2^* - 1} - (w_{ki})_{0,\mu} (z)^{2^* - 1} \right] dz + \left[ h_{ki}(y) - (h_{ki})_{0,\mu} (y) \right] \]
\[ \geq \int_{\tilde{\Omega}_k \setminus B_\mu} E(0, y, z, \mu) \left[ w_{ki}(z)^{2^* - 1} - (w_{ki})_{0,\mu} (z)^{2^* - 1} \right] dz + J(\mu, w_{ki}, y), \]
for any $\mu_* \leq \mu \leq \mu_* + 1/2$ and $y \in \tilde{\Omega}_k$ with $|y| > \mu$, where
\[ J(\mu, w_{ki}, y) = \int_{\mathbb{R}^n \setminus \tilde{\Omega}_k} E(0, y, z, \mu) \left[ w_{ki}(z)^{2^* - 1} - (w_{ki})_{0,\mu} (z)^{2^* - 1} \right] dz \]
\[ = \int_{\Omega_k \setminus \tilde{\Omega}_k} E(0, y, z, \mu) \left[ w_{ki}(z)^{2^* - 1} - (w_{ki})_{0,\mu} (z)^{2^* - 1} \right] dz \]
\[ - \int_{\mathbb{R}^n \setminus \Omega_j} E(0, y, z, \mu) (w_{ki})_{0,\mu} (z)^{2^* - 1} dz. \]

For $z \in \mathbb{R}^n \setminus \tilde{\Omega}_k$ and $\mu_* \leq \mu \leq \mu_* + 1$, we obtain that $|z| \geq 1/2 u_i (\bar{x}_k)^{-\gamma - 1}$ and thus
\[ (w_{ki})_{0,\mu} (z) \leq \left( \frac{\mu}{|z|} \right)^{n-4} \max_{B_{\mu_* + 1}} w_{ki} \leq C u_i (\bar{x}_k)^{-2}. \]

In addition, since $u_i \geq C_0 > 0$ in $B_2 \setminus B_{1/2}$, by the definition of $w_{ki}$, we have
\[ w_{ki}(y) \geq \frac{C_0}{u_i (\bar{x}_k)} \quad \text{in} \quad \Omega_k \setminus \tilde{\Omega}_k. \]
We also observe that
\[ J(\mu, w_{ki}, y) \geq \frac{1}{2} \left( \frac{C_0}{u_i (\bar{x}_k)} \right)^{2^*-1} \int_{\Omega_k \setminus \tilde{\Omega}_k} E(0, y, z, \mu) dz - C \int_{\mathbb{R}^n \setminus \Omega_k} E(0, y, z, \mu) \left( \frac{\mu}{|z|} \right)^{n-4} dz \]
\[ \geq \begin{cases} \frac{1}{2} |y| C_0 u_i (\bar{x}_k)^{-1}, & \text{if} \quad \mu \leq |y| \leq \mu_* + 1 \\ \frac{1}{2} u_i (\bar{x}_k)^{-1}, & \text{if} \quad |y| > \mu_* + 1 \quad \text{and} \quad y \in \tilde{\Omega}_k. \end{cases} \quad (60) \]
Indeed, since \( E(0, y, z, \mu) = 0 \) for \(|y| = \mu\) and
\[
y \nabla_y \cdot E(0, y, z, \mu)|_{|y| = \mu} = (n - 4)|y - z|^{4 - n - 2}(|z|^2 - |y|^2) > 0,
\]
for \(|z| \geq \mu_s + 2\), and using the positivity and smoothness of \( E \), there exists \( 0 < \delta_1 \leq \delta_2 < \infty \) satisfying
\[
\delta_1|y - z|^{4 - n}(|y| - \mu) \leq E(0, y, z, \mu) \leq \delta_2|y - z|^{4 - n}(|y| - \mu), \tag{61}
\]
for \( \mu_s \leq \mu \leq |y| \leq \mu_s + 1\), \( \mu_s + 2 \leq |z| \leq R < \infty \). Furthermore, if \( R \gg 1 \), it follows that
\[
0 < C_0 \leq y \cdot \nabla_y (|y - z|^{n - 4}E(0, y, z, \mu)) \leq C_0 < \infty \text{ for all } |z| \geq \mu, \mu_s \leq \mu \leq |y| \leq \mu_s + 1.
\]
thus, (61) holds for \( \mu_s \leq \mu \leq |y| \leq \mu_s + 1 \) and \(|z| \geq R\). In addition, by the definition of \( E(0, y, z, \mu) \), there exists \( 0 < \delta_3 \leq 1 \) such that
\[
\delta_3|y - z|^{4 - n} \leq E(0, y, z, \mu) \leq |y - z|^{4 - n}, \tag{62}
\]
for \(|y| \geq \mu_s + 1 \) and \(|z| \geq \mu_s + 2\). Therefore, for \( k \gg 1 \), \( \mu \leq |y| \leq \mu_s + 1 \), we have
\[
J(\mu, w_{ki}, y) \geq \frac{1}{2} \left( \frac{C_0}{u_i(\bar{x}_k)} \right)^{2^{**} - 1} \int_{\bar{\Omega}_k \setminus \Omega_k} \delta_1|y - z|^{4 - n}(|y| - \mu)dz
\]
\[
- C \int_{\mathbb{R}^n \setminus \Omega_k} \delta_2|y - z|^{4 - n}(|y| - \mu) \left( \frac{\mu}{|z|} \right)^{n+4} dz
\]
\[
\geq \frac{1}{C_1}(|y| - \mu)u_i(\bar{x}_k)^{-1} - \frac{1}{C_2}(|y| - \mu)u_i(\bar{x}_k)^{-2^{**}}
\]
\[
\geq \frac{1}{2C_1}(|y| - \mu)u_i(\bar{x}_k)^{-1}.
\]
Similarly, for \(|y| \geq \mu_s + 1 \) and \( y \in \bar{\Omega}_k \), since \( u_i(\bar{x}_k) \to \infty \) as \( k \to \infty \), it follows that there exist \( C_1, C_2, C_3, C_4 > 0 \) satisfying
\[
J(\mu, w_{ki}, y) \geq \frac{1}{C_3}u_i(\bar{x}_k)^{-1} - \frac{1}{C_4}u_i(\bar{x}_k)^{-2^{**}} \geq \frac{1}{2C_3}u_i(\bar{x}_k)^{-1},
\]
which verifies (60).

Next, by (59) and (60), there exists \( \varepsilon_1 \in (0, 1/2) \), depending on \( k \in \mathbb{N} \), such that for \(|y| \geq \mu_s + 1\), it holds
\[
w_{ki}(y) - (w_{ki})_{0, \mu_+}(y) \geq \varepsilon_1 |y|^{4 - n} \text{ in } \Omega_k.
\]
Using the above inequality and the formula for \((w_{ki})_{0, \mu}\), there exists \( 0 < \varepsilon_2 < \varepsilon_1 \) such that for \(|y| \geq \mu_s + 1\), \( \mu_s \leq \mu \leq \mu_s + \varepsilon_2\), we have
\[
w_{ki}(y) - (w_{ki})_{0, \mu}(y) \geq \varepsilon_1 |y|^{4 - n} + \left[(w_{ki})_{0, \mu_+}(y) - (w_{ki})_{0, \mu}(y)\right] \geq \frac{\varepsilon_1}{2} |y|^{4 - n}. \tag{63}
\]
For \( \varepsilon \in (0, \varepsilon_3] \) chosen below, by (59) and (60), and using the inequality
\[
\left|(w_{ki})_{0, \mu}(z)^{2^{**} - 1} - (w_{ki})_{0, \mu}(z)^{2^{**} - 1}\right| \leq C(|z| - \mu),
\]
we get

\[
\begin{align*}
  w_{ki}(y) - (w_{ki})_{0,\mu}(y) & \geq \int_{|z| \leq \mu + 1} E(0, y, z, \mu) \left( w_{ki}(z) - (w_{ki})_{0,\mu}(z) \right) \, dz \\
  & \quad + \int_{\mu + 2 \leq |z| \leq \mu + 3} E(0, y, z, \mu) \left( w_{ki}(z) - (w_{ki})_{0,\mu}(z) \right) \, dz \\
  & \geq - C \int_{|z| \leq \mu + \varepsilon} E(0, y, z, \mu)(|z| - \mu) \, dz \\
  & \quad + \int_{\mu + \varepsilon \leq |z| \leq \mu + 1} E(0, y, z, \mu) \left( w_{ki}(z) - (w_{ki})_{0,\mu}(z) \right) \, dz \\
  & \quad + \int_{\mu + 2 \leq |z| \leq \mu + 3} E(0, y, z, \mu) \left( w_{ki}(z) - (w_{ki})_{0,\mu}(z) \right) \, dz,
\end{align*}
\]

for \( \mu_* \leq \mu \leq \mu_* + \varepsilon \) and \( \mu \leq |y| \leq \mu_* + 1 \). From (63), there exists \( \delta_5 > 0 \) such that for \( \mu_* + 2 \leq |z| \leq \mu_* + 3 \), it follows

\[
w_{ki}(z) - (w_{ki})_{0,\mu}(z) \geq \delta_5.
\]

In addition, since \( \|w_{ki}\|_{C^1(B_2)} \leq C \) (independent of \( k \)), there exists some constant \( C > 0 \), not depending on \( \varepsilon \), such that for \( \mu_* \leq \mu \leq \mu_* + \varepsilon \), we have

\[
\left| (w_{ki})_{0,\mu_*}(z) - (w_{ki})_{0,\mu}(z) \right| \leq C(\mu - \mu_*) \leq C\varepsilon,
\]

for \( \mu \leq |z| \leq \mu_* + 1 \). Also for \( \mu \leq |y| \leq \mu_* + 1 \), we find

\[
\begin{align*}
  \int_{\mu + \varepsilon \leq |z| \leq \mu + 1} E(0, y, z, \mu) \, dz & \leq \int_{\mu + \varepsilon \leq |z| \leq \mu + 1} \left| \left( y - z \right)^{4-n} - |I_{0,\mu}(y) - z|^{4-n} \right| \, dz \\
  & \quad + \int_{\mu + \varepsilon \leq |z| \leq \mu + 1} \left( \left( \frac{\mu}{y} \right)^{n-4} - 1 \right) \left| I_{0,\mu}(y) - z \right|^{n-4} \, dz \\
  & \leq C \left( \varepsilon^3 + |\ln \varepsilon| + 1 \right) (|y| - \mu)
\end{align*}
\]

and

\[
\begin{align*}
  \int_{\mu \leq |z| \leq \mu + \varepsilon} E(0, y, z, \mu)(|z| - \mu) \, dz & \leq \int_{\mu \leq |z| \leq \mu + \varepsilon} \left| \left( \frac{|z| - \mu}{y - z} \right)^{n-4} - \left( \frac{\mu}{|y|} \right)^{n-4} \right| \, dz \\
  & \quad + \varepsilon \int_{\mu \leq |z| \leq \mu + \varepsilon} \left( \left( \frac{\mu}{|y|} \right)^{n-4} - 1 \right) \left| I_{0,\mu}(y) - z \right|^{4-n} \, dz \\
  & \leq I + C\varepsilon (|y| - \mu),
\end{align*}
\]

where, for \( |y| \geq \mu + 10\varepsilon \), we have

\[
I = \left| \int_{\mu \leq |z| \leq \mu + \varepsilon} \left( \frac{|z| - \mu}{|y| - z} - \frac{|z| - \mu}{|I_{0,\mu}(y) - z|} \right) \, dz \right| \leq C\varepsilon \left( \varepsilon^3 + |\ln \varepsilon| + 1 \right) (|y| - \mu).
\]
Finally, using \((\alpha)\) is well-defined and positive. If \(\mu<0\) the convergence for singular solutions of \((w_{\mu})\) is finished, and Claim 1 as well. However, this is also a contradiction with \((\alpha)\) together with \((\alpha)\). Notice that this asymptotic approximation implies that solutions must become radially symmetric.

Step 1: We divide the proof of the claim in two steps as follows:

**Claim 1:**

Proposition 59. Asymptotic radial symmetry and the Harnack inequality. Corollary 58. Let \(\mathcal{U}\) be a strongly positive singular solution of \((1)\). Then, there exists \(C_2 > 0\) satisfying \(|\mathcal{U}(x)| \leq C_2 |x|^{-\gamma}\).

4.3. Asymptotic radial symmetry and the Harnack inequality. Here, we prove the convergence for singular solutions of \((1)\) to the spherical average, given by \(\bar{\mathcal{U}}(x) = \int_{\partial B_1} \mathcal{U}(r\theta)d\theta\). Notice that this asymptotic approximation implies that solutions must become radially symmetric close to the origin.

**Proposition 59.** Let \(\mathcal{U}\) be a strongly positive solution of \((1)\). Then, \(\mathcal{U}\) is radially symmetric and

\[|\mathcal{U}(x)| = (1 + O(|x|))|\bar{\mathcal{U}}(x)| \text{ as } |x| \to 0,\]

where \(\bar{\mathcal{U}}\) is the spherical average of \(\mathcal{U}\).

**Proof.** First, we will prove the following claim

**Claim 1:** There exists \(0 < \varepsilon < \min\{1/10, \bar{r}\}\) such that

\[|\mathcal{U}_{x,\mu}|(y) \leq |\mathcal{U}(y)| \text{ in } B_1(x) \setminus B_\mu(x),\]

for \(0 < \mu < |x| < \varepsilon\), where \(\bar{r}\) is such that \((56)\) holds for all \(0 < \mu \leq \bar{r}\). We divide the proof of the claim in two steps as follows:

**Step 1:** The critical parameter

\[\mu_*(x) := \sup\{0 < \mu_* \leq |x| : |\mathcal{U}_{x,\mu}|(y) \leq |\mathcal{U}(y)| \text{ for } y \in B_2 \setminus (B_\mu(x) \cup \{0\}) \text{ and } 0 < \mu < \mu_*\}\]

is well-defined and positive.
Indeed, using Lemma A, for every $x \in B_{r/10}^*$ one can find $0 < r_x < |x|$ such that for all $0 < \mu \leq r_x$, it follows $|U_{x,\mu}|(y) \leq |U|(y)$ for $0 < \mu < |y - x| \leq r_x$. Moreover, using (49), we get

$$|U|(x) \geq 4^{4-n} \int_{B_2} |f_i(U)|(y)dy := C_0 > 0,$$

(64)

which implies that there exists $0 < \mu_1 \leq r_x$ such that, for every $0 < \mu \leq \mu_1$, it holds

$$|U_{x,\mu}|(y) \leq |U|(y) \quad \text{for} \quad y \in B_2 \setminus \{B_{r_x}(x) \cup \{0\}\}.$$  

(65)

Hence, as a combination of (64) and (65), it follows the proof of Step 1. 

**Step 2:** There exists $\varepsilon > 0$ such that $\mu_* = |x|$ for all $|x| \leq \varepsilon$ and for every $\mu_* \leq \mu < |x| \leq \bar{r}$ and $y \in B_1$, we get for any $i \in I$,

$$u_i(y) - (u_i)_{x,\mu}(y) \geq \int_{B_1 \setminus B_\mu(x)} E(0, y, z, \mu) \left[ f_i(U(z)) - f_i(U_{x,\mu}(z)) \right] dz + J(\mu, u, y),$$

where

$$J(\mu, u, y) = \int_{B_2 \setminus B_1} E(x, y, z, \mu) \left[ f_i(U(z)) - f_i(U_{x,\mu}(z)) \right] dz - \int_{\mathbb{R}^n \setminus B_2} E(x, y, z, \mu)f_i(U_{x,\mu}(z))dz.$$  

In fact, for $y \in \mathbb{R}^n \setminus B_1$ and $\mu < |x| < \varepsilon < 1/10$, we have

$$I_{x,\mu}(y) = \left| x + \frac{\mu^2(y - x)}{|y - x|^2} \right| \geq |x| - \frac{10}{9}\mu^2 \geq |x| - \frac{10}{9}|x|^2 \geq \frac{8}{9}|x|.$$  

Using Proposition A, there exists $C > 0$ such that $|U|(I_{x,\mu}(y)) \leq C|x|^{-\gamma}$, which provides for all $y \in \mathbb{R}^n \setminus B_1$,

$$U_{x,\mu}(y) = \left( \frac{\mu}{|y - x|} \right)^{n-4} U(I_{x,\mu}(y)) \leq C \mu^{n-4}|x|^{-\gamma} \leq C|x|^\gamma \leq C \varepsilon^{\gamma}.$$  

By (64), we find $|U_{x,\mu}|(y) < |U|(y)$ for $y \in B_2 \setminus B_1$. In addition, by (60), (64) and (65), there exists $C_1 > 0$, independent of $x$, such that

$$J(\mu, |U|, y) \geq \int_{B_2 \setminus B_1} E(x, y, z, \mu) \left( C_0^{2^{*-1}} - C_0^{2^{*-1}\varepsilon^{n+4}} \right) dz$$

$$- C \int_{\mathbb{R}^n \setminus B_2} E(x, y, z, \mu) \left( |x - z|^{4-n}|x|^\gamma \right)^{2^{*-1}} dz$$

$$\geq \frac{1}{2} C_0^{2^{*-1}} \int_{B_2 \setminus B_1} E(x, y, z, \mu)dz - \varepsilon^{n+4} C \int_{\mathbb{R}^n \setminus B_2} E(x, y, z, \mu)|x - z|^{n+4}dz$$

$$\geq \frac{1}{2} C_0^{2^{*-1}} \int_{B_{19/10} \setminus B_{11/10}} E(0, y - x, z, \mu)dz - \varepsilon^{n+4} C \int_{\mathbb{R}^n \setminus B_{19/10}} E(0, y - x, z, \mu)|x - z|^{n+4}dz$$

$$\geq \frac{1}{C_1}(|y - x| - \mu),$$

for $y \in B_1 \setminus (B_{r_x}(x) \cup \{0\})$ and $0 < \varepsilon \ll 1$. Eventually, if $\mu_* < |x|$ by Lemma A we get a contradiction with the definition of $\mu_*$. Therefore, Step 2 is proved and so Claim 1.

Finally, for any $i \in I$, choose $0 < r_i < \varepsilon^2$ and $x_{1i}, x_{2i} \in \partial B_{r_i}$ satisfying

$$u_i(x_{1i}) = \max_{\partial B_{r_i}} u_i \quad \text{and} \quad u_i(x_{2i}) = \min_{\partial B_{r_i}} u_i.$$  

Now choosing

$$x_{3i} = x_{1i} + \frac{\varepsilon(x_{1i} - x_{2i})}{4|x_{1i} - x_{2i}|} \quad \text{and} \quad \mu = \sqrt{\frac{\varepsilon_i^2}{4}} \left( |x_{1i} - x_{2i}| + \frac{\varepsilon_i}{4} \right).$$
it follows from Claim 1 that \( (u_i)_{x_{2i}, \mu}(x_{2i}) \leq u_i(x_{2i}) \). Furthermore, we have
\[
(u_i)_{x_{3i}, \mu}(x_{2i}) = \left( \frac{\mu}{|x_{1i} - x_{2i}| + 4\varepsilon_i^{-1}} \right)^{n-4} u_i(x_{1i}) \\
= \left( \frac{1}{4 |x_{1i} - x_{2i}| \varepsilon_i^{-1} + 1} \right)^{n-4} u_i(x_{1i}) \\
\geq \left( \frac{1}{8r\varepsilon_i^{-1} + 1} \right)^{\gamma} u_i(x_{1i}),
\]
which implies
\[
\max_{\partial B_{r_i}} u_i \leq \left( 8r\varepsilon_i^{-1} + 1 \right)^{\gamma} \min_{\partial B_{r_i}} u_i,
\]
and this proves the proposition. □

As a consequence of the upper estimate, we prove the following Harnack inequality for solutions of (1), which scalar version can be found in [11, Theorem 3.6].

**Corollary 60.** Let \( U \) be a strongly positive solution of (1). Then, there exists \( C > 0 \) such that
\[
\max_{|x|=r} |U| \leq C \min_{|x|=r} |U| \quad \text{for} \quad 0 < r < 1/4.
\]
Moreover, \( |D^jU| \leq C|x|^{-j}|U| \) for \( j = 1, 2, 3, 4 \).

**Proof.** For any \( i \in I \), let us define \( \tilde{u}_i(y) = r^\gamma u_i(ry) \). Thus,
\[
\tilde{u}_i(y) = \int_{B_2} |y - z|^{4-n} f_i(\tilde{U})dz + \tilde{h}_i(y), \quad (66)
\]
where \( \tilde{h}_i(y) = r^\gamma h_i(ry) \). By Proposition 58, there exists \( C_2 > 0 \), such that \( \tilde{u}_i \leq C_2 \) in \( B_2 \setminus B_{1/10} \). Taking \( |x| = 1 \), let us consider
\[
(g_i)_x(y) = \int_{B_{2/\varepsilon_i}(x) \setminus B_{3/10}(x)} |y - z|^{4-n} f_i(\tilde{U})dz.
\]
Hence, for any \( y_1, y_2 \in B_{1/2}(x) \), we have
\[
(g_i)_x(y_1) \leq C \int_{B_{2/\varepsilon_i}(x) \setminus B_{3/10}(x)} |y - z|^{4-n} f_i(\tilde{U})dz \leq C(g_i)_x(y_2),
\]
which implies that \( g_i \) satisfies the Harnack inequality in \( B_{1/2}(x) \). On the other hand, \( h_i \) also satisfies the Harnack inequality in \( B_{1/2}(x) \) and
\[
\tilde{u}_i(y) = \int_{B_3(0)} |y - z|^{4-n} f_i(\tilde{U})dz + (g_i)_x(y) + \tilde{h}_i(y) \quad \text{in} \quad B_{1/2}(x).
\]
Now by [61, Theorem 2.3] we have that \( \sup_{B_{1/2}(x)} \tilde{u}_i \leq C \inf_{B_{1/2}(x)} \tilde{u}_i \), which by covering argument provides
\[
\sup_{1/2 \leq |y| \leq 3/2} \tilde{u}_i \leq C \inf_{1/2 \leq |y| \leq 3/2} \tilde{u}_i,
\]
and, by rescaling back to \( u_i \), the proof of the first part follows.

Next, for any fixed \( x \) and \( i \in I \), let \( r = |x| \) and \( \tilde{u}_i(y) = r^\gamma u_i(ry) \). Thus, \( \tilde{U}_i \) satisfies (66) and, by Proposition 58, it holds that \( \tilde{u}_i \leq C_2 \) in \( B_{3/2} \setminus B_{1/2} \). Finally, using the local estimates from [61, Section 2.1] and the smoothness of \( h_i \), one can find \( C > 0 \) satisfying \( |D^j \tilde{u}_i(x)| \leq C \) for \( |x| = 1 \) and \( j = 1, 2, 3, 4 \). Therefore, by scaling back to \( u_i \), the proof is finished. □
4.4. **Removable singularity classification and the lower bound estimate.** Next, we use the Pohozaev invariant and the Harnack inequality to provide a removable classification result, which implies the lower bound estimate in Proposition 8.

**Lemma 61.** Let $\mathcal{U} \in C(\mathbb{B}_2^*, \mathbb{R}^p) \cap L^{2**-1}(\mathbb{B}_2, \mathbb{R}^p)$ be a strongly positive solution of (49). Assume $h_i \in C^\infty(B_1)$ for all $i \in I$. If $\limsup_{|x| \to 0} |\mathcal{U}|(x) = \infty$, then $\liminf_{|x| \to 0} |\mathcal{U}|(x) = \infty$.

**Proof.** Let us consider a sequence $\{x_k\}_{k \in \mathbb{N}}$ satisfying $r_k = |x_k| \to 0$ and $u_i(x_k) \to \infty$ as $k \to \infty$. By the Harnack inequality, we have $\inf_{\partial B_{r_k}} u_i \to \infty$. Thus, we obtain $-\Delta(u_i - h_i) \geq 0$ in $\mathbb{B}_2^*$. Hence, since $h_i \in C^\infty(B_1)$ for all $i \in I$, it follows that $\min_{B_{r_j} \setminus B_{r_{j+1}}} u_i(x) \to \infty$ as $j \to \infty$. Therefore, we conclude

$$\min_{B_{r_k} \setminus B_{r_{k+1}}} (u_i - h_i) = \min_{\partial B_{r_k} \cap \partial B_{r_{k+1}}} (u_i - h_i),$$

which proves the lemma. \qed

**Lemma 62.** Let $\mathcal{U} \in C(\mathbb{B}_2^*, \mathbb{R}^p) \cap L^{2**-1}(\mathbb{B}_2, \mathbb{R}^p)$ be a strongly positive solution of (49). If $\lim_{|x| \to 0} |\mathcal{U}|(x) = 0$, then $|\mathcal{U}|$ can be extended as a continuous function through $B_1$.

**Proof.** Let us consider the barrier functions from [61]. For any $i \in I$ and $\delta > 0$, we choose $0 < \eta \ll 1$ such that for any $i \in I$, it holds $u_i(x) \leq \delta |x|^{-\gamma}$ in $\mathbb{B}_2^*$. Fixing $\varepsilon > 0$, $\kappa \in (0, \gamma)$ and $M \gg 1$ to be chosen later, let us define

$$\psi_i(x) = \begin{cases} M|x|^{-\kappa} + \varepsilon |x|^{4-n-\kappa}, & \text{if } 0 < |x| < \eta \\ u_i(x), & \text{if } \eta < |x| < 2. \end{cases}$$

Notice that for every $0 < \kappa < n - 4$ and $0 < |x| < 2$, one can use a change a variables to find $C > 0$ such that

$$\int_{\mathbb{R}^n} |x - y|^{4-n} |y|^{-\kappa} dy = |x|^{4-n} \int_{\mathbb{R}^n} |x|^{-1} x - |x|^{-1} y |^{4-n} |y|^{-\kappa-4} dy$$

$$= |x|^{-\kappa+4} \int_{\mathbb{R}^n} |x|^{-1} x - z |^{4-n} |z|^{-\kappa-4} dz$$

$$\leq C \left( \frac{1}{n - 4 - \kappa} + \frac{1}{\kappa} + 1 \right) |x|^{-\kappa},$$

which gives for $0 < |x| < 2$ and $0 < \delta \ll 1$,

$$\int_{B_\eta} u_i^{2**-2}(y) \psi_i(y) |x - y|^{4-n} dy \leq \delta^{2**-2} \int_{\mathbb{R}^n} \psi_i(y) |x - y|^{n-4} |y|^{-4} dy$$

$$\leq C \delta^{2**-2} \psi_i(x)$$

$$\leq \frac{1}{2} \psi_i(x).$$
Moreover, for $0 < |x| < \eta$ and $\bar{x} = \eta x|\eta|^{-1}$, we get

$$\int_{B_2 \setminus B_{\eta}} u_i^{2^{**}-2}(y)\psi_i(y)|x-y|^{4-n}dy = \int_{B_2 \setminus B_{\eta}} |\bar{x} - y|^{n-4} u_i^{2^{**}-1}(y) |x-y|^{n-4}dy$$

$$\leq 2^{n-4} \int_{B_2 \setminus B_{\eta}} u_i^{2^{**}-1}(y) |x-y|^{n-4}dy$$

$$\leq 2^{n-4} u_i(\bar{x})$$

$$\leq 2^{n-4} \max_{\partial B_{\eta}} u_i$$

$$\leq 2^{n-4} \max_{\partial B_{\eta}} u_i.$$

The last inequality implies that for $0 < |x| < \tau$ and $M \geq \max_{\partial B_{\eta}} u_i$,

$$h_i(x) + \int_{B_2} u_i^{2^{**}-2}(y)\psi_i(y)|x-y|^{4-n}dy \leq h_i(x) + 2^{n-4} \max_{\partial B_{\eta}} u_i + \frac{1}{2} \psi_i(x) < \psi_i(x).$$

In the next claim, we show that $\psi_i$ can be taken indeed as barrier for any $u_i$.

**Claim 1:** For any $i \in I$, it holds that $u_i(x) \leq \psi_i(x)$ in $B_{\eta}^\star$.

Indeed, assume it does not hold. Since $u_i(x) \leq \delta|x|^{-\gamma}$ in $B_{\eta}^\star$, by the definition of $\psi_i$, there exists $\bar{\tau} \in (0, \eta)$, depending on $\varepsilon$, such that $\psi_i \geq u_i$ in $B_{\bar{\tau}}^\star$ and $\psi_i > u_i$ near $\partial B_{\eta}$. Let us consider,

$$\bar{\tau} := \inf \{ \tau > 1 : \tau \psi_i > u_i \text{ in } B_{\eta}^\star \}.$$

Then, we have that $\bar{\tau} \in (1, \infty)$ and there exists $\bar{x} \in B_{\eta} \setminus B_{\bar{\tau}}$ such that $\bar{\tau}\psi_i(\bar{x}) = u_i(\bar{x})$. Furthermore, for $0 < |x| < \bar{\tau}$, it follows

$$\bar{\tau}\psi_i(x) \geq \int_{B_2} u_i^{2^{**}-2}(y)\bar{\tau}\psi_i(y)|x-y|^{4-n}dy + \bar{\tau} h_i(x) \geq \int_{B_2} u_i^{2^{**}-2}(y)\bar{\tau}\psi_i(y)|x-y|^{4-n}dy + h_i(x),$$

which gives,

$$\bar{\tau}\psi_i(x) - u_i(x) \geq \int_{B_2} u_i^{2^{**}-2}(y)(\bar{\tau}\psi_i(y) - u_i(y))|x-y|^{4-n}dy.$$

Finally, by evaluating the last inequality at $\bar{x} \in B_{\eta} \setminus B_{\bar{\tau}}$, we get a contradiction and the claim is proved.

As a consequence of the claim, we get $u_i(x) \leq \psi_i(x) \leq M|x|^{-\kappa} + \varepsilon|x|^{4-n-\kappa}$ in $B_{\eta}^\star$, which implies that $u_i^{2^{**}-2} \in L^s(B_{\eta}^\star)$ for some $s > n/4$ and any $i \in I$ and, hence, using standard elliptic regularity the proof of the lemma follows. \(\square\)

**Lemma 63.** Let $U \subset C(B_{1}^\star) \cap L^{2^{**}-1}(B_2)$ be a strongly positive solution of (1). Assume $h_i \in C^\infty(B_1)$ for all $i \in \mathbb{N}$ is a positive function in $\mathbb{R}^n$ satisfying $\Delta^2 h_i = 0$ in $B_2$. If $\liminf_{|x| \to 0} |x|^\gamma |U|(x) = 0$, then $\lim_{|x| \to 0} |x|^\gamma |U|(x) = 0$.

**Proof.** Assume by contradiction that there exists $C > 0$ such that $\limsup_{|x| \to 0} |x|^\gamma |U|(x) = C > 0$; thus, from Lemma 61, we get that $\liminf_{|x| \to 0} |U|(x) = \infty$. Using the assumption and the Harnack inequality in Lemma 60, there exists $\{r_k\}_{k \in \mathbb{N}}$ such that $r_k \to 0$ and $r_k^\gamma u_i(r_k) \to 0$ as $k \to \infty$. As well as, $r_k$ is a local minimum point of $r^\gamma \bar{u}_i(r)$. Furthermore, let us define

$$\varphi_{ki}(y) = \frac{u_i(r_k \bar{y})}{u_i(r_k \bar{e}_n)},$$
which in combination (49) provides,
\[ \varphi_{ki}(y) = \int_{B_{2/r_k}^*} (r_k^\gamma u_k(r_k e_n))^2 r^2 \varphi_{ki}(\eta) r^2 |y - z|^{4-n} dz + h_{ki}(y) \text{ in } B_{2/r_k}^*, \]
where \( h_{ki}(y) = u(r_k e_n)^{-1} h_i(r_k y) \).

**Claim 1:** For any \( i \in I \), it follows that \( \lim_{k \to \infty} \varphi_{ki}(y) = 1/2(|y|^{4-n} + 1) \) in \( C^2_\text{loc}(\mathbb{R}^n \setminus \{0\}) \).

In fact, since \( u_i(r_k e_n) \to \infty \), we have that \( h_{ki}(r_k y) \to 0 \) as \( k \to \infty \) in \( C^0_\text{loc}(\mathbb{R}^n) \). Next, using the Harnack inequality, we obtain that \( r_k^\gamma u_k(r_k e_n) \to 0 \) and \( \varphi_{ki} \) is locally uniformly bounded in \( B_{2/r_k}^* \).

Hence,
\[ \lim_{k \to \infty} \int_{B_{\tau}} (r_k^\gamma u_k(r_k e_n))^{2r^2} \varphi_{ki}(y)^{2r^2 - 1} dy = 0 \text{ in } C^1_\text{loc}(\mathbb{R}^n \setminus \{0\}). \]

Thus, we have, for any \( \tau > 1 \), \( 0 < |y| < \tau \) and \( 0 < \varepsilon < |y|/100 \), which, up to subsequences, implies
\[ \lim_{k \to \infty} \int_{B_{\tau}} (r_k^\gamma u_k(r_k e_n))^{2r^2} \varphi_{ki}(z)^{2r^2 - 1} dy = A|y|^{4-n}, \]
for some \( A > 0 \). Moreover, since the left-hand side is locally uniformly bounded in \( C^{n+1}_\text{loc}(B_{\tau}) \), for any \( i \in I \), there exists \( g_i \in C^2(B_{\tau}) \) satisfying
\[ \lim_{k \to \infty} \int_{B_{2/r_k} \setminus B_{\tau}} (r_k^\gamma u_k(r_k e_n))^{2r^2 - 2} \varphi_{ki}(z)^{2r^2 - 1} dy = g_i(y) \geq 0 \text{ in } C^1_\text{loc}(B_{\tau}). \]

In addition, for any fixed \( R \gg 1 \) and \( y \in B_{\tau} \), we have
\[ \lim_{k \to \infty} \int_{|y| \leq R} (r_k^\gamma u_k(r_k e_n))^{2r^2 - 2} \varphi_{ki}(z)^{2r^2 - 1} dy = 0 \]
and for any \( y_1, y_2 \in B_{\tau} \), we obtain
\[ \int_{B_{2/r_k} \setminus B_R} (r_k^\gamma u_k(r_k e_n))^{2r^2 - 2} \varphi_{ki}(z)^{2r^2 - 1} dy_1 = \left( \frac{R + \tau}{R - \tau} \right)^{n-4} \int_{B_{2/r_k} \setminus B_R} (r_k^\gamma u_k(r_k e_n))^{2r^2 - 2} \varphi_{ki}(z)^{2r^2 - 1} dy_2. \]

Therefore, it follows
\[ g_i(y_1) \leq \left( \frac{R + \tau}{R - \tau} \right)^{n-4} g_i(y_2), \]
which by passing to the limit as \( R \to \infty \) and exchanging the roles of \( y_1 \) and \( y_2 \), we have \( g_i(y_2) = g_i(y_1) \). Thus, \( g_i(y) \equiv g_i(0) \) for all \( y \in B_{\tau} \) and \( i \in I \). Since \( \varphi_{ki} \) is locally uniformly bounded in \( B_{2/r_k}^* \), we have that it is locally uniformly bounded in \( C^{n+1}(B_{2/r_k}^*) \). Hence, up to subsequence, it follows that \( \varphi_{ki} \to \varphi_i \) as \( k \to \infty \) in \( C^1_\text{loc}(\mathbb{R}^n \setminus \{0\}) \), for some \( \varphi_i \), which gives
\[ \lim_{k \to \infty} \varphi_{ki}(y) = A|y|^{4-n} + \varphi_i(0) \text{ in } C^1_\text{loc}(\mathbb{R}^n \setminus \{0\}). \]
Using that $\varphi_{k}(e_{n}) = 1$ and 
\[
\frac{d}{dr} \left\{ r^{\gamma} \varphi_{k}(r) \right\} \bigg|_{r=1} = r_{k}^{-\gamma+1} u_{i}(r_{k}e_{n})^{-1} \frac{d}{dr} \left\{ r^{\gamma} \tilde{u}_{i}(r) \right\} \bigg|_{r=r_{k}} = 0,
\]
by taking the limit $k \to \infty$, it follows that $A = f(0) = 1/2$, which proves the claim.

In the next claim, we obtain some information about the limit of the Pohozaev invariant, which will be used to generate a contradiction.

**Claim 2:** \(\lim_{k \to \infty} \mathcal{P}(U, r_{k}) = 0\).

In fact, for any \(i \in I\), let us consider
\[
\tilde{u}_{i}(x) = \int_{B_{2}} f_{i}(U)|x - y|^{4-n}dy + h_{i}(x) \quad \text{in} \quad \mathbb{R}^{n} \setminus \{0\},
\]
which gives \(\tilde{u}_{i} = u_{i}\) in \(B_{2} \setminus \{0\}\), and
\[
\tilde{u}_{i}(x) = \int_{B_{2}} f_{i}(U)|x - y|^{4-n}dy + h_{i}(x) \quad \text{in} \quad \mathbb{R}^{n} \setminus \{0\}.
\]
Consequently, using that \(\Delta^{2}h_{i} = 0\) in \(B_{2}\) for any \(i \in I\), it follows \(\Delta^{2}\tilde{u}_{i} = f_{i}(U)\) in \(B_{2}^{+}\). On the other hand, we know that \(\mathcal{P}(U, r_{k})\) is a constant on \(r\). Moreover, since \(|D^{2}u_{i}(x)| \leq C\) near \(\partial B_{1}\) and \(r_{k}^{-\gamma}u(r_{k}e_{n}) = o(1)\) as \(k \to \infty\), we have \(|D^{2}u_{i}(x)| \leq C r_{j}^{-\gamma}u(r_{k}e_{n}) = o(1) r_{k}^{-\gamma - k}\) for all \(|x| = r_{k}\) and \(j = 1, 2, 3, 4\), which proves the second claim. Hence, using Claim 2, it holds that \(\mathcal{P}(U, r_{k}) = 0\) for \(k \in \mathbb{N}\). Hence, by (23), we get
\[
\sum_{i=1}^{p} \int_{\partial B_{1}} q(\varphi_{k}(x), \varphi_{k}(x)) \, dx + \tilde{c}(n) (r_{k}^{\gamma} u_{i}(r_{k}e_{n}))^{2^{*} - 2} \sum_{i=1}^{p} \int_{\partial B_{1}} |\varphi_{k}(x)|^{2^{*}} \, dx = 0.
\]
Next, sending \(k \to \infty\) and doing some manipulation, we obtain
\[
\int_{\partial B_{1}} q(|x|^{4-n} + 1, |x|^{4-n} + 1) \, dx = 0.
\]
By Theorem 2, we know that the spherical function \(U_{0,\mu}(x) = \left(\frac{2\mu}{1+|x|^{\gamma}}\right)^{\gamma}\) satisfies the limit blow-up system,
\[
\Delta^{2}U_{0,\mu} = c(n)f_{i}(U_{0,\mu}) \quad \text{in} \quad \mathbb{R}^{n},
\]
which implies that for any \(\mu > 0\), we get \(\mathcal{P}(U_{0,\mu}, 1) = \lim_{r \to \infty} \mathcal{P}(U_{0,\mu}, r) = 0\). Hence, if \(q\) is defined in (22), we find
\[
\sum_{i=1}^{p} \int_{\partial B_{1}} q(\mu^{-\gamma}(u_{i})_{0,\mu}, \mu^{-\gamma}(u_{i})_{0,\mu}) \, dx + \tilde{c}(n) \mu^{4-n} \sum_{i=1}^{p} \int_{\partial B_{1}} (u_{i})_{0,\mu}^{2^{*}} \, dx = 0,
\]
which by taking the limit as \(\mu \to 0\) gives,
\[
0 = \int_{\partial B_{1}} q(|x|^{4-n} + 1, |x|^{4-n} + 1) \, d\sigma - \int_{\partial B_{1}} q(|x|^{4-n}, |x|^{4-n}) \, d\sigma = (n - 4) \int_{\partial B_{1}} \partial_{\nu} \Delta (|x|^{4-n}) \, d\sigma \neq 0.
\]
This contradiction concludes the proof of the lower bound estimate. \(\square\)

Consequently, we have the removable singularity theorem and the lower bound estimate

**Corollary 64.** Let \(U\) be a strongly positive solution of (1). Then, \(\mathcal{P}(U) \leq 0\) and \(\mathcal{P}(U) = 0\) if and only if \(U\) has a removable singularity at the origin.

**Corollary 65.** Let \(U\) be a strongly positive singular solution of (1). Then, there exists \(C_{1} > 0\) such that \(C_{1}|x|^{-\gamma} \leq |U|(x)\).

**Proof of Proposition 8.** It is a consequence of Corollaries 58 and 65. \(\square\)
5. Local asymptotic behavior

In this section, our objective is to present the proof of Proposition 9, which as by-product gives the proof of Theorem 1 and Theorem 1’. For this matter, we use the growth properties in Proposition 7 and the estimates in Proposition 8.

Inspired by [40], we summarize the Leon Simon technique:
(a) There exist $C_1, C_2 > 0$ such that any solution of (17) satisfies the uniform estimate

$$C_1 \leq |\mathcal{V}(t, \theta)| \leq C_2;$$

(b) If $\tau_k \to \infty$ and $\mathcal{V}_k(t, \theta) := \mathcal{V}(t + \tau_k, \theta)$. Then, $\mathcal{V}_k$ converges uniformly on compact sets to a bounded solution $\mathcal{V}_\infty$ of (19);
(c) Any angular derivative $|\partial_\theta \mathcal{V}(t, \theta)|$ converges to 0 as $t \to \infty$;
(d) There exists $S > 0$ such that for any infinitesimal rotation $\partial_\theta$ of $\mathbb{S}^{n-1}$, and for any $\tau_k \to \infty$, if we set $A_k = \sup_{t \geq 0} |\partial_\theta \mathcal{V}(t, \theta)|$, and if $|\partial_\theta \mathcal{V}_k(\tau_k, \theta)| = A_k$ for some $(\tau_k, \theta_k) \in \mathcal{C}$, then $s_k \leq S$;
(e) $|\partial_\theta \mathcal{V}(t, \theta)|$ converges to 0 exponentially as $t \to \infty$, as well as $|\mathcal{V}(t, \theta) - \bar{\mathcal{V}}(t)|$, where $\bar{\mathcal{V}}$ is a spherical average of $\mathcal{V}$;
(f) There exists a bounded solution $\mathcal{V}_{a,T}$ of (19) and $\sigma > 0$ such that $\mathcal{V}(t, \theta)$ converges to $\mathcal{V}(t + \sigma)$ in an exponential rate as $t \to \infty$;

**Remark 66.** Using Theorem B and Propositions 7 and 8 most part of the steps above follows the same lines of [10, 53]. The main difference is caused the number of Jacobi fields to analyze; for the second order equation, we have two linearly independent Jacobi fields, one that grows unbounded and the other that is exponentially decreasing. In contrast, in our case we have four linearly independent Jacobi fields that behave in the same way, that is, two of them grow unbounded and the others are exponentially decaying. Although the strategy is similar, we include all the proofs here for the convenience of the reader.

5.1. Simple convergence. Here we prove a result that is equivalent to Proposition 9 but in cylindrical coordinates.

**Proposition 67.** Let $v$ be a positive singular solution of (17) satisfying (14). Then, there exists $\beta_0^* > 0$ and an Emden–Fowler solution $\mathcal{V}_{a,T}$ such that

$$\mathcal{V}(t) = (1 + \mathcal{O}(e^{\beta_0^* t}))\mathcal{V}_{a,T}(t) \quad \text{as} \quad t \to \infty. \quad (67)$$

**Proof.** Initially, by Remark 20, the origin is a non-removable singularity. Thus, using Corollary 64, we have that $\mathcal{P}(U) < 0$. Consider $\mathcal{V}(t, \theta) = e^{\gamma t}\mathcal{U}(e^t \theta)$ and $\{\tau_k\}_{k \in \mathbb{N}}$ such that $\tau_k \to \infty$ as $k \to \infty$. Let us define the sequence of translations,

$$\mathcal{V}_k(t, \theta) = \mathcal{V}(t + \tau_k, \theta) \quad \text{defined in} \quad (-\tau_k, \infty) \times \mathbb{S}^{n-1}.$$

Again, applying the sharp global estimates in Proposition 8, we get

$$C_1 \leq |\mathcal{V}_k(t, \theta)| \leq C_2. \quad (68)$$

By (68), we get that $\{\mathcal{V}_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $C_{loc}^4(\mathbb{R}, \mathbb{R}^p)$. Hence, by standard elliptic regularity, there exists $\mathcal{V}_\infty \in C_{loc}^4(\mathbb{R}, \mathbb{R}^p)$ such that, up to subsequence, $\mathcal{V}_k \to \mathcal{V}_\infty$. Moreover, $\mathcal{V}_\infty$ satisfies (17). Therefore, by Theorem B, $\mathcal{V}_\infty$ is an Emden–Fowler solutions, that is, there exist $\alpha \in (0, a_0)$ and $T_0 \in (0, T_0)$ such that $\mathcal{V}_\infty = \mathcal{V}_{a,T}$ and does not depend on the variable $\theta$.

**Claim 1:** The following elliptic estimates holds:
(i) $\mathcal{V}_k(t, \theta) = \mathcal{V}_k(t)(1 + o(1))$;
(ii) $\nabla \mathcal{V}_k(t, \theta) = -\mathcal{V}_k^{(1)}(t)(1 + o(1))$;
(iii) $\Delta \mathcal{V}_k(t, \theta) = \mathcal{V}_k^{(2)}(t)(1 + o(1))$;
(iv) $\nabla^{3/2} \mathcal{V}_k(t, \theta) = -\mathcal{V}_k^{(3)}(t)(1 + o(1))$. 
Indeed, if (i) is not valid, there would exist \( \tilde{\varepsilon} > 0 \) and \( \tau_k \to \infty, \theta_k \to \infty \) such that
\[
\left| \frac{V_k(\tau_k, \theta_k)}{V_k(\tau_k)} - 1 \right| \geq \tilde{\varepsilon},
\]
which is a contradiction since \( v_k \to v_\infty \) and \( v_\infty \) is radially symmetric. The same argument holds for (ii), (iii) and (iv). This estimates implies (b), that is, any angular derivative \( |\partial_\theta V_k| \) converges uniformly to zero.

**Claim 2:** The necksize of \( V_\infty \) does not depend on \( k \in \mathbb{N} \).

In fact, this is a consequence of the following identity
\[
\mathcal{P}(V_\infty) := \mathcal{P}(0, V_\infty) = \lim_{k \to \infty} \mathcal{P}(0, V_k) = \lim_{k \to \infty} \mathcal{P}(\tau_k, V) = \mathcal{P}(V).
\]
Consequently, we have that for each \( \{\tau_k\}_{k \in \mathbb{N}} \), the correspondent sequence \( v_k \) converges to \( v_{a,T} \) as \( k \to \infty \), where \( T \) does not depend on \( k \).

In the next claim, we prove (c), (d), (e) and (f).

**Claim 3:** There exist \( \sigma \in \mathbb{R} \) and \( \beta_0 > 0 \) such that
\[
|V_\sigma(t, \theta) - V_{a,T}(t)| \leq Ce^{\beta_0 t} \text{ in } (0, \infty) \times S^{n-1}.
\]

As a matter of fact, we divide the rescaling argument in three steps as following.

First, let \( T_a \in (0, T_0) \) be the period of \( v_{a,T} \) and define \( A_T = \sup_{t \geq 0} |\partial_\theta V_a| \). Since \( |\partial_\theta V_a| \) converges uniformly to zero as \( t \to \infty \), we have \( A_T < \infty \).

**Step 1:** For every \( c > 0 \), there exists an integer \( N > 0 \) such that, for any \( \tau > 0 \) either:
(i) \( A_T \leq ce^{-2\tau} \), or
(ii) \( A_T \) is attained at some point in \( I_N \times S^{n-1} \), where \( I_N = [0, NT_a] \).

Suppose that the claim is not true. Then, there exists \( C > 0 \) and \( \tau_k, \theta_k \to \infty \) such that \( |\partial_\theta V_a(s_k, \theta_k)| = A_{\tau_k} \) and \( A_{\tau_k} > Ce^{-2\tau_k} \) as \( k \to \infty \). We define \( \tilde{V}_k(t, \theta) = V_k(t + s_k, \theta) \) and \( \tilde{\Phi}_k = A_{-\tau_k}^{-1} \partial_\theta V_k \). In addition, we have that \( |\tilde{\Phi}_k| \leq 1 \) and satisfy the nonlinear system \( \mathcal{N}(\tilde{V}_k) = 0 \), which by differentiation with respect to \( \theta \) implies that \( \mathcal{L}^a(\tilde{\Phi}_k) = 0 \). Now, using standard elliptic regularity, we can extract a subsequence \( \{\tilde{\Phi}_k\}_{k \in \mathbb{N}} \) which converges in compact subsets to a nontrivial and bounded Jacobi field satisfying \( \mathcal{L}^a(\Phi) = 0 \). This is a contradiction since \( \Phi \) has no zero eigencomponent relative to \( \Delta_\theta \) and thus is unbounded. This proves Step 1.

Assuming that \( V_k(t, \theta) \) converges to \( V_{a}(t + T) \) as \( k \to \infty \), let us define
\[
W_k(t, \theta) = V_k(t, \theta) - V_{a}(t + T), \quad \eta_k = b \max_{I_N} |W_k| \quad \text{and} \quad \Phi_k = \eta_k V_k,
\]
where \( b > 0 \) will be chosen later and satisfies \( |\Phi_k| \leq b^{-1} \) in \( I_N \). Then, by Theorem B, it follows
\[
\Delta_{\text{cyl}}^2 W_k - \left[ f_i(V_k) - \Lambda^* v_{a,T}^{2s+1} \right] = 0, \quad (69)
\]
where
\[
|V_k|^{2s-2} v_{ki} - \Lambda^* v_{a,T}^{2s+1} = |V_k|^{2s-2} w_{ki} + \Lambda^* v_{a,T} \frac{|V_k|^{2s-2} - v_{a,T}^{2s+2}}{|V_k|^{2s-2} - v_{a,T}^{2s+2}} \sum_{j \in I} (v_{kj} + \Lambda^* v_{a,T} j).
\]

Multiplying (69) by \( \eta_k^{-1} \) and taking the limit as \( k \to \infty \) we get \( \mathcal{L}^a(\Phi^*) = 0 \), where \( \Phi^* = \lim_{k \to \infty} \Phi_k \).

**Step 2:** The Jacobi filed \( \Phi^* \) is bounded for all \( t \geq 0 \).

Using Proposition 7 and the Fourier decomposition (31), we get
\[
\Phi^* = b_1 \Phi_{a,0}^+ + b_2 \Phi_{a,0} - b_3 \Phi_{a,0}^+ + b_4 \Phi_{a,0} - + \bar{\Phi},
\]
where \( \bar{\Phi} \) is the projection on the subspace generated by the eigenfunctions associated to the nonzero eigenvalues of \( \Delta_\theta \). Using Proposition 7, we claim that \( \bar{\Phi} \) is bounded. Indeed, we need to verify that \( \partial_\theta \bar{\Phi} = \partial_\theta \Phi \) is bounded for \( t \geq 0 \). In this fashion, we have that \( \partial_\theta \Phi = \lim_{k \to \infty} \eta_k \partial_\theta V_k \).
Furthermore, if $\partial_{\theta}\Phi$ is zero the result follows. Then, we suppose that $\partial_{\theta}\Phi$ is nontrivial. In this case, if (i) of Step 1 happens, we get
\[
\sup_{t \geq 0} (\eta_k^{-1}|\partial_{\theta}\mathcal{V}_k|) \leq C\eta_k^{-1}e^{-2\tau_k} \leq C.
\]
On the other hand, if (ii) of Step 1 happens, since $\eta_k^{-1}|\partial_{\theta}\mathcal{V}_k|$ converges in the $C^4$-topology, we have
\[
\sup_{t \geq 0} (\eta_k^{-1}|\partial_{\theta}\mathcal{V}_k|) \leq \sup_{l_N} (\eta_k^{-1}|\partial_{\theta}\mathcal{V}_k|) \leq C.
\]
The last two inequalities implies the boundedness of $\tilde{\Phi}$.

In order to finish the proof of Step 2, we must show that $b_2 = b_4 = 0$. Indeed, since $\Phi_k = \eta_k^{-1}\mathcal{W}_k \to \Phi$ as $k \to \infty$, we obtain
\[
\mathcal{V}_k = \mathcal{V}_{a,T} + \eta_k \Phi^* + o(\eta_k)
\]
\[
= \mathcal{V}_{a,T} + \eta_k (b_1 \Phi_{a,0}^+ + b_2 \Phi_{a,0}^- + b_3 \Phi_{a,0}^+ + b_4 \Phi_{a,0}^- + \tilde{\Phi}) + o(\eta_k).
\]
On the other hand,
\[
\mathcal{P}(0, \mathcal{V}_k) = \mathcal{P}(\tau_k, \mathcal{V}) - \mathcal{P}(\mathcal{V}) + \mathcal{O}(e^{-2\tau_k}) = \mathcal{P}(\mathcal{T}, \mathcal{V}_a) + \mathcal{O}(e^{-2\tau_k}).
\]
Then, if either $b_2 \neq 0$ or $b_4 \neq 0$ we would have a contradiction, since $\eta_k^{-1}e^{-2\tau_k} = o(1)$ as $k \to \infty$ and the two sides of last equality would differ for sufficiently large $k$.

Let us define
\[
\mathcal{W}_\tau(t, \theta) = \mathcal{V}(t + \tau, \theta) - \mathcal{V}_a(t + T) \quad \text{and} \quad \eta(\tau) = b \max_{I_N} \mathcal{W}_\tau|,
\]
where $I_N$ is defined in Step 1 and $b > 0$ will again be chosen later. For a fixed $C_1 > 0$, we have the following:

**Step 3:** Assume that $N, b, \tau \gg 1$ and $0 < \eta \ll 1$. Then, there exists $|\delta| \leq C_1 \eta(\tau)$ such that
\[
2\eta(\tau + NT_a + \delta) \leq \eta(\tau). \quad (70)
\]
Suppose that (70) does not hold. Then, there would exist some $\tau_k \to \infty$ such that $\eta(\tau_k) \to 0$ and for $s > 0$ satisfying $|s| \leq C_1 \eta(\tau_k)$ we have $\eta(\tau_k + NT_a + s) > 1/2\eta(\tau_k)$. Similarly to the previous step, let us define $\Phi_k = \eta(\tau_k)^{-1}\mathcal{W}_\tau$; thus by Step 2, we can suppose that $\{\Phi_k\}_{k \in \mathbb{N}}$ converges to a bounded Jacobi Field $\Phi^*$, which provides
\[
\Phi^* = b_1 \Phi_{a,0}^+ + b_3 \Phi_{a,0}^+ + \tilde{\Phi}, \quad (71)
\]
where $\tilde{\Phi}$ has exponential decay. Since $|\phi| < b^{-1}$ on $I_N$, we get that $b_1$ and $b_3$ are uniformly bounded and independent of $\tau_k > 0$. Moreover, we know that $\Phi_{a,0}^+ = \partial_{a} \mathcal{V}_{a,T}$ is bounded and $\Phi_{a,0}^- = \partial_{\theta} \mathcal{V}_{a,T}$ is linearly growing, which by $\mathcal{V}_{\tau_k} = \eta(\tau_k)\Phi_k + o(\eta(\tau_k))$ and $\Phi_k(t - \eta(\tau_k)b_1, \theta) - \Phi_k(t, \theta) = o(1)$ as $\tau_k \to \infty$ gives
\[
|b_3 \Phi_{a,0}^+| \leq b^{-1} + |\tilde{\Phi}| \quad \text{on} \quad I_N. \quad (72)
\]
Setting $s_k = -\eta(\tau_k)b_1$, we can choose $C_1 > 0$ sufficiently large such that $|s_k| < |C_1 \eta(\tau_k)|$. Hence for $t \in [0, 2NT_0]$, we get
\[
\mathcal{W}_{\tau_k + s_k}(t, \theta) = \mathcal{V}(t + \tau_k) - \eta(\tau_k)b_1, \theta) - \mathcal{V}_{a,T}(t)
\]
\[
= \mathcal{V}_{\tau_k}(t - \eta(\tau_k)b_1, \theta) - \mathcal{V}_{a,T}(t - \eta(\tau_k)b_1) - \eta(\tau_k)b_1 \frac{\mathcal{V}_{a,T}(t - \eta(\tau_k)b_1) - \mathcal{V}_{a,T}(t)}{-\eta(\tau_k)b_1}
\]
\[
= \eta(\tau_k) \Phi_k(t - \eta(\tau_k)b_1, \theta) - \eta(\tau_k)b_1 \Phi_{a,0}^+ + o(\eta(\tau_k))
\]
\[
= \mathcal{W}_{\tau_k}(t, \theta) - \eta(\tau_k)b_1 \Phi_{a,0}^+ + o(\eta(\tau_k)).
\]
Therefore, by (71) we obtain
\[ W_{\tau_k+s_k} = \eta(\tau_k) \Phi + \eta(\tau_k)b_3 \Phi_{a,0}^+ + o(\eta(\tau_k)) \text{ on } [0, 2NT_a], \]
which implies
\[ \max_{I_N} |W_{\tau_k+s_k+NT_a}| = \max_{[NT_a, 2NT_a]} |W_{\tau_k+s_k}| \leq \eta_{N_k} \max_{I_N} (|\Phi| + |b_3 \Phi_{a,0}^+|) + o(\eta(\tau_k)). \]

Then, (72) combined with the fact that \( \Phi \) decreases exponentially in a fixed rate implies that one can choose \( N, b > 0 \) sufficiently large satisfying \( \max_{I_N} |w_{\tau_k+s_k+NT_a}| \leq 2^{-1}\eta(\tau_k) \), which is contradiction since it is equivalent to \( \eta(\tau + NT_a + s) \leq \eta(\tau) \).

In the next step, we use Step 3 to construct a sequence that converges to the correct translation parameter.

**Step 4:** There exists \( \sigma > 0 \) such that \( |W_\sigma| \) converges exponentially to 0 as \( t \to \infty \).

First, choose \( \tau_0, N \gg 1 \) such that Step 3 is satisfied and \( C_1\eta(\tau_0) \leq 2^{-1}NT_a \). Set \( s_0 = -\eta(\tau_0)b_1 \) as above; thus \( |s_0| \leq C_1\eta(\tau_0) \leq 2^{-1}NT_a \). Let us define inductively three sequences:
\[
\begin{align*}
\sigma_k &= \tau_0 + \sum_{i=0}^{k-1} s_i, \\
\tau_k &= \tau_{k-1} + s_{k-1} + NT_a = \sigma_k + kNT_a, \\
s_k &= -\eta(\tau_k)b_1.
\end{align*}
\]

Note that by induction, we have \( \eta(\tau_k) \leq 2^{-k}\eta(\tau_0) \) and \( |s_k| < 2^{-1}NT_a \). Then, there exists \( \sigma = \lim_{k \to \infty} \sigma_k \leq \tau_0 + NT_a \) and so \( \tau_k \to \infty \) as \( k \to \infty \). Now choose \( k \in \mathbb{N} \) such that \( t = kNT_a + [t] \) with \([t] \in I_N\) and write
\[
W_\sigma(t, \theta) = \mathcal{V}(t + \sigma, \theta) - \Lambda^* v_{a,T} = \mathcal{V}(t + \sigma, \theta) - \mathcal{V}(t + \sigma_k, \theta) + \mathcal{V}(t + \sigma_k, \theta) - \Lambda^* v_{a,T}(t).
\]

In addition, since \( \partial_\theta \mathcal{V} \) is uniformly bounded, we get
\[
\mathcal{V}(t + \sigma, \theta) - \mathcal{V}(t + \sigma_k, \theta) = \partial_\theta \mathcal{V}(t_0) \sum_{i=k}^{\infty} s_i = \mathcal{O}(2^{-k}),
\]
for some \( t_0 > 0 \) and \( \mathcal{V}(t + \sigma_k, \theta) - \mathcal{V}_{a,T}(t) = \mathcal{V}(\tau_k + [t], \theta) - \mathcal{V}_{a,T}([t]) = \mathcal{W}([t], \theta), \) which provides
\[
W_\sigma(t, \theta) = \mathcal{W}_{\tau_k}([t], \theta) + \mathcal{O}(2^{-k}).
\]

Therefore, using that \( b \max_{I_N} |W_{\tau_k}(t, \theta)| = \eta(\tau_k) \leq 2^{-k}\eta(\tau_0) \), we obtain that \( |W_\sigma(t, \theta)| = \mathcal{O}(2^{-k}) \) as \( k \to \infty \), or in terms of \( t = kNT_a + [t] \),
\[
|W_\sigma(t, \theta)| \leq C_1 e^{-\frac{\ln 2}{NT_a} t},
\]
which by taking \( \beta_0^* = -\ln 2/NT_a \) concludes the proof of (67).

**Proof of Proposition 9.** It follows by undoing the cylindrical transformation in (67). □

**Proof of Theorem 1.** Combine Propositions 8 and 9. □
5.2. Deformed Emden–Fowler solutions. In this part, we follow [53] to introduce the family of deformed Emden–Fowler solutions. Here, we call $\gamma = (n - 4)/2$ the Fowler rescaling exponent; this notation will be used throughout the paper. This $2n$-parameter family of solutions is constructed using the pullback of a composition of three conformal transformations described as follows.

First, take $p = 1$ and consider an Emden–Fowler solution with $T = 0$, given by $u_{a,0}(x) = |x|^{-\gamma}v_a(-\ln |x|)$; thus, using an inversion about the unit sphere, we obtain $\tilde{u}_{a,0}(x) = |x|^{-\gamma}v_a(\ln |x|)$. Second, we employ an Euclidean translation about $x_0 \in \mathbb{R}^n \setminus \{0\}$ to get $\tilde{u}_{a,0,x_0}(x) = |x|^2 - x_0|^{-\gamma}v_a(\ln |x|^2 - x_0)$. Finally, applying another inversion, we find

$$u_{a,0,x_0}(x) = |x|^{-\gamma}|\theta - x_0||^{-\gamma}v_a(-\ln |\theta - x_0| + \ln |\theta - x_0|),$$

where $\theta = x|x|^{-1}$. Moreover, in cylindrical coordinates, we have

$$v_{a,0,x_0}(t, \theta) = |\theta - x_0e^{-t}||^{-\gamma}v_a(t + \ln |\theta - x_0e^{-t}|).$$

Finally, taking a time translation $T \in (0, T_a)$, we construct the families $u_{a,T,x_0}$ and $v_{a,T,x_0}$.

**Definition 68.** For $p > 1$, let us define the deformed (vectorial) Emden–Fowler solution by

$$U_{a,T,x_0} = \Lambda^*u_{a,T,x_0} \quad \text{and} \quad V_{a,T,x_0} = \Lambda^*v_{a,T,x_0},$$

where $\Lambda^* \in \mathbb{S}^{p-1}_+$ and $u_{a,T,x_0}$ and $v_{a,T,x_0}$ are scalar Emden–Fowler solutions.

**Remark 69.** The parameters $x_0 \in \mathbb{R}^n$ and $T \in (0, T_a)$ correspond to conformal motions. In contrast, the so-called Fowler parameter $a \in (0, a_0)$ does not have a geometrical interpretation.

**Proposition 70.** For any $a \in (0, a_0)$ and $x_0 \in \mathbb{R}^n$, we have

$$U_{a,0,x_0}(x) = (1 + \mathcal{O}(|x|))U_{a,0}(|x|) \quad \text{as} \quad |x| \to 0.$$  

**Proof.** Initially, we take $p = 1$ and calculate the Taylor series of $u_{a,0,x_0}$ near $|x| = 0$,

$$|x||^{-\gamma} - x_0||^{-\gamma} = 1 + \gamma(x_0 \cdot x) + \mathcal{O}(|x|^2).$$

Similarly,

$$\ln |x||^{-\gamma} - x_0||^{-\gamma} = -(x_0 \cdot x) + \mathcal{O}(|x|^2),$$

which implies

$$v_a(-\ln |x| - (x_0 \cdot x) + \mathcal{O}(|x|^2)) = v_a(-\ln |x|) - v_a^{(1)}(-\ln |x|)(x_0 \cdot x) + \mathcal{O}(|x|^2).$$

Combining, (74) and (75), we obtain

$$u_{a,0,x_0}(x) = |x|^{-\gamma} \left[ v_a(-\ln |x|) + (x_0 \cdot x) \left( v_a^{(1)}(-\ln |x|) + \gamma v_a(-\ln |x|) \right) + \mathcal{O}(|x|^2) \right]$$

$$= u_{a,0}(x) + |x|^{-\gamma}(x_0 \cdot x) \left( -v_a^{(1)} + \gamma v_a \right) + \mathcal{O}(|x|^{-\gamma-2}),$$

which together with Theorem B implies

$$U_{a,0,x_0}(x) = U_{a,0}(x) + |x|^{-\gamma}(x_0 \cdot x) \left( -\mathcal{V}_a^{(1)} + \gamma \mathcal{V}_a \right) + \mathcal{O}(|x|^{-\gamma-2});$$

this concludes the proof of the Proposition. $\square$

**Remark 71.** Additionally, since the Jacobi fields and the indicial roots are not counted with multiplicity, these functions are all equal; thus, it is simpler to write $\Phi_{a,1}^+ = \cdots = \Phi_{a,n}^+$ and (76) can be reformulated as

$$U_{a,0,x_0}(x) = |x|^{-\gamma} \left[ v_a \left( -\ln |x| + |x| \left( \sum_{j=1}^n x_j \chi_j(\theta)\Phi_{a,j} \right) + \mathcal{O}(|x|^2) \right) \right]$$

$$= |x|^{-\gamma} \left[ v_a \left( -\ln |x| + (x_0 \cdot x)\Phi_{a,1}^+(-\ln |x|) + \mathcal{O}(|x|^2) \right) \right] \quad \text{as} \quad |x| \to 0.$$
In cylindrical coordinates, we can rewrite
\[ \mathcal{V}_{a,0,x_0}(t, \theta) = \mathcal{V}_a(t) + e^{-t} \langle \theta, a \rangle \left( -\nabla_a^\| + \gamma \mathcal{V}_a \right) + \mathcal{O}(e^{-2t}) \quad \text{as} \quad t \to \infty. \] (78)

Nevertheless, for the translation \( \bar{\mathcal{V}}_{a,x_0}(t, \theta) = \mathcal{V}_a(t - t_0, \theta) \) with \( t_0 = -\ln |x_0| \), we have
\[ \bar{\mathcal{V}}_{a,x_0}(t, \theta) = e^t \left( -\nabla_a^\| + \gamma \mathcal{V}_a \right) + \mathcal{O}(1) \quad \text{as} \quad t \to \infty. \]

Also, notice that when \( \langle a, \theta \rangle > 0 \), then \( |\mathcal{V}_{a,0,x_0}(t, \theta)| > |\mathcal{V}_a(t)| \) and \( |\bar{\mathcal{V}}_{a,0,x_0}(t, \theta)| > |\bar{\mathcal{V}}_a(t)| \). Moreover, the opposite inequality holds when \( \langle a, \theta \rangle < 0 \).

5.3. Improved convergence. In order to complete our analysis, we will present the proof of Theorem 1'. Our proof is based on the surjectiveness of the linearized operator stated in Proposition 4. In the next result, we provide a higher order expansion for solutions of \( (5) \) when \( p = 1 \), which can be stated as

**Proposition 72.** Let \( \mathcal{U} \) be a singular solution of \( (1) \). Then, for any \( x_0 \in \mathbb{R}^n \) there exists an Emden–Fowler solution \( \mathcal{V}_{a,T} \) such that
\[ \mathcal{U}(x) = \left| x \right|^{-\gamma} \left[ \mathcal{V}_a(-\ln |x| + T) + (x_0 \cdot x) \Phi_{a,1}^+(\ln |x| + T) + \mathcal{O}(\left| x \right|^{\beta^*_1}) \right] \quad \text{as} \quad |x| \to 0, \] (79)

where \( \beta^*_1 := \min\{2, \beta_{a,2}\} > 0 \).

**Proof.** Again, we start with \( p = 1 \). Using the asymptotics proved in Theorem 1, we deduce
\[ u(x) = \left| x \right|^{-\gamma} v_a(-\ln |x|) = \left| x \right|^{-\gamma} \left[ v_a(-\ln |x| + T_a) + w(-\ln |x|) \right], \]
where \( \phi \in C^4_{-\beta}(C) \) for some \( \beta > 0 \). Moreover, since \( v_a \) satisfies \( (8) \), that is, \( \mathcal{N}(v_a) = 0 \), we can expand \( \mathcal{N} \) in Taylor series around \( v_a,T \) to get \( \mathcal{L}^a(\phi) = \psi(\phi) \), where
\[ \psi(\phi) = (v_a + \phi)^{2s-1} - v_a^{2s-1} - \tilde{c}(n)v_a^{2s-2}\phi. \]

It is straightforward to see that if \( \phi \in C^m_{-\beta}(C) \), then \( \psi(\phi) \in C^m_{-2\beta}(C) \) for any \( m \in \mathbb{N} \). Now, we can run the iterative method. First, assume that \( \beta \in (0,1/2) \), then using Claim 1, we obtain \( \psi(\phi) \in C^0_{-2\beta}(C) \). In addition, by (i) of Corollary 47, we have \( \phi \in C^4_{-2\beta}(C) \) and \( \psi(\phi) \in C^4_{-2\beta}(C) \), which implies \( w \in C^4_{-\beta}(C) \). After some steps, we conclude that \( w \in C^4_{\beta'}(C) \) for some \( \beta' \in (1/2, 1) \). Therefore, \( \psi(\phi) \in C^4_{\beta'}(C) \) and by (ii) of Corollary 47, we find that \( \phi \in C^4_{-2\beta'}(C) \oplus D_{a,1}(C) \), which provides \( \phi \in C^4_{\beta'}(C) \) for \( \beta' = \min\{2, \beta_{a,2}\} \). In addition, we observe that \( \beta' > \beta \) is optimal, which by Theorem B implies \( (79) \). \( \square \)

**Proof of Theorem 1'.** It follows directly from Proposition 72, because the deformed Emden–Fowler solutions also satisfy the same estimate \( (79) \), which is a consequence of \( (77) \). \( \square \)

In conclusion, we have the following refined asymptotics:

**Corollary 73.** Let \( \mathcal{V} \) be a solution of \( (17) \) and \( \mathcal{V}_{a,T} \) an Emden–Fowler solution of \( (5) \). Then, there exists \( \beta^*_1 > 1 \) such that
\[ \left| \mathcal{V}(t, \theta) - \mathcal{V}_{a,T}(t) - \pi_0 \mathcal{V}(t, \theta) - \pi_1 \mathcal{V}(t, \theta) \right| \leq C e^{-\beta^*_1 t} \quad \text{for} \quad t > 0, \]
where \( \beta_{a,1} = \min\{2, \beta_{a,2}\} \).
6. Further problems to explore

In this section, we will propose some problems in the direction of Theorems 1 and 1'.

Geometric fourth order systems. First, notice that (1) is a particular case of the following fourth order geometric system,

\[ P_g u_i = c(n)|\mathcal{U}|^{2^{*^*}-2}u_i \text{ in } (B^*_1, g) \quad \text{for } i \in I. \quad (80) \]

Here \( B^*_1(0) \subset \mathbb{R}^n \) with \( n \geq 5 \), \( B^*_1 = B^*_1(0) \setminus \{0\} \) is the punctured ball, \( g \) is a smooth metric on \( B^*_1(0) \), \( \mathcal{U} = (u_i)_I \) is a \( p \)-map with Euclidean norm denoted by \( |\mathcal{U}| \) and \( I = \{1, \ldots, p\} \), \( f_i(\mathcal{U}) = c(n)|\mathcal{U}|^{2^{*^*}-2}u_i \) is the Gross–Pitaevskii nonlinear strong coupling term with \( 2^{*^*} = 2n/(n-4) \) the critical Sobolev exponent, \( c(n) \) is a normalizing constant and \( P_g \) is the fourth order operator given by

\[
P_g u_i = \Delta^2_g u_i - \sum_{j=1}^{p} \text{div} \left( \frac{(n-2)^2 + 4}{2(n-1)(n-2)} R_g g - \frac{4}{n-2} \text{Ric}_g \right) u_j + \frac{n-4}{2} \sum_{j=1}^{p} Q_{ij}(x) u_j,
\]

where \( Q_{ij}(x) \in \text{Sym}_p(\mathbb{R}) \), \( \text{Ric}_g \) is the Ricci tensor and \( R_g \) is the scalar curvature of \( (B^*_1, g) \). Notice that when \( p = 1 \) and \( Q_{11} = Q_g \), the \( Q \)-curvature, the system (80) is the celebrated singular \( Q \)-curvature equation (9). Then, the ultimately goal is to extend the asymptotics in Theorem 1 to the nonhomogeneous system (80). We would like to find some hypotheses on \( Q_{ij}(x) \in M^*_p(\mathbb{R}) \) and bounds on the dimension such that (3) still holds for solutions of the geometric system (80). This problem is inspired in [69, Theorem 1] and [10, Theorem 1.3].

Three-dimensional \( Q \)-curvature equation. It also makes sense to investigate the three-dimensional case of the \( Q \)-curvature problem. The difference is that the nonlinear coupling term in the right-hand-side has a negative power,

\[ P_g u = c_3 u^{-7} \text{ in } B^*_1. \quad (81) \]

The objective is to study the asymptotic behavior of solutions to (81) near the isolated singularity. For more details, see [42, 44, 49, 50].

Four-dimensional \( Q \)-curvature equation. Another research direction is the four-dimensional \( Q \)-curvature equation. It is well-known that due to the Trudinger–Moser inequality that the critical nonlinearity to be considered is the exponential,

\[ P_g u = c_4 e^{4u} \text{ in } B^*_1, \quad (82) \]

which is called the Liouville equation. The objective is to establish qualitative properties for solution to (82), such as asymptotic radial symmetry and local behavior. We refer to the papers [46, 47, 63, 66, 67] for some works when \( g \) is a conformally flat metric on \( B_1 \).

Higher order refined asymptotics. More generally, inspired by [38, 39] we believe that the following higher order asymptotics must hold for solutions of (17)

\[
\| \mathcal{V}(t, \theta) - \mathcal{V}_{a,T}(t) - \Lambda^* \sum_{l=1}^{m} \sum_{j=0}^{m-1} b_{ij}(t, \theta) t^j e^{-\beta_{a,j}} \| \leq C t^m e^{-\beta_{a,m} t} \quad \text{for } t > 0,
\]

where \(-\beta_{a,m}\) is the \( m \)-th indicial roots the powers \( t^j \) for \( j = 1, \ldots, m \) correspond to the higher algebraic multiplicity.
Higher order integral systems. Another interesting open question is whether is possible to extend our analysis to higher order systems in the punctured ball,

\((-\Delta_4)^k u_i = c(k,q,n) |\mathcal{U}|^{q_k-2} u_i\) in \(B_1^*\) for \(i \in I\),

or the more general integral system,

\(u_i(x) = c(k,q,n) \int_{B_1} |x - y|^{n-k} |\mathcal{U}(y)|^{q_k-2} u_i(y) dy + h_i(x)\) in \(B_1^*\) for \(i \in I\),

The first step to run the asymptotic analysis methods is to study the limit blow-up equation. Before answering this question a even more delicate analysis on the linearized operator is necessary.

Schrödinger-type fourth order systems. We are also interested in solutions \(\mathcal{U} = (u_1,u_2)\) to the strongly coupled fourth order system satisfying \(u_1,u_2 > 0\),

\[
\begin{cases}
    P_g u_1 = \mu_1 u_1^{s-1} + \beta u_1^{s/2-1} u_2^{s/2} & \text{in } B_1^*, \\
    P_g u_2 = \mu_2 u_2^{s-1} + \beta u_2^{s/2-1} u_1^{s/2} & \text{in } B_1^*,
\end{cases}
\]

where \(\mu_1,\mu_2,\beta > 0\) and \(s = 2^{*\ast}\) is the critical Sobolev of \(H^2\). This type of coupling system seems to be harder to tackle than the one in (80) since the behavior of the coupling term depends strongly on the dimension (see [18] in the references therein). The idea is to obtain similar classification and asymptotics for the local behavior of the solution to (83) in the neighborhood of the origin (for more details, see [21, 60, 90]). The first step would be to study the following flat system,

\[
\begin{cases}
    \Delta^2 u_1 = \mu_1 u_1^{s-1} + \beta u_1^{s/2-1} u_2^{s/2} & \text{in } B_1^*, \\
    \Delta^2 u_2 = \mu_2 u_2^{s-1} + \beta u_2^{s/2-1} u_1^{s/2} & \text{in } B_1^*,
\end{cases}
\]

As much as we know, this classification is not known even for second order system with \(p = 1\).

Sixth order Q-curvature system. One could ask whether our asymptotics techniques could be extend to the nonlinear critical sixth order system

\[\Delta^3 u_i = c(n) |\mathcal{U}|^{2^{*\ast\ast}-2} u_i\] in \((B_1^*,g)\) for \(i \in I\).

Here \(B_1^n(0) \subset \mathbb{R}^n\) with \(n \geq 7\), \(B_1^n = B_1^n(0) \setminus \{0\}\) is the punctured ball, \(g\) is a smooth metric on \(B_1^n(0)\), \(\mathcal{U} = (u_i)_I\) is \(p\)-map with Euclidean norm denoted by \(|\mathcal{U}|\) and \(I = \{1,\ldots,p\}\), \(f_i(\mathcal{U}) = |\mathcal{U}|^{2^{*\ast\ast}-2} u_i\) is the Gross-Pitaevskii nonlinear strong coupling term with \(2^{*\ast\ast} = 2n/(n-6)\) the critical Sobolev exponent of \(H^3\), \(c(n)\) is a normalizing constant and \(\Delta^3\) is the three-Laplacian. The initial technical difficulty would be to compute the coefficients \(K_0(n),\ldots,K_6(n)\) of the expansion of \(\Delta^3\) in cylindrical coordinates applied on radially symmetric functions, in other terms

\[\Delta^3_{cyl} = \sum_{j=0}^{6} K_j(n) \partial_t^{(j)} + L_\theta,\]

where \(L_\theta\) represents the terms with angular derivatives, which should also be expanded in terms of the Laplace–Beltrami for studying the Jacobi fields. At least for the scalar case, it is expected that same phenomena of simplification occur due to the conformal invariance and so \(K_j(n) = 0\) for some \(j = 1,2,3,4,5\). In fact, we always have that for the leader symbol \(K_6(n) \equiv 1\). This analysis could also be considered for the anomalous dimensions \(n < 6\) and \(n = 6\). In the direction of [29], we observe that for \(p = 1\) (84) is the flat case of a the \(Q^6\)-curvature equation, which is the sixth-order analogue of the \(Q\)-curvature. The problem is that even computing the expression of the associated Paneitz–Brason operator is an considerably hard problem.
After this paper was finished, we learned that J. Ratzkin [77] had obtained a similar result to Corollary 2.

Acknowledgments The paper was completed while the first-named author was visiting the Department of Mathematics at Princeton University, whose hospitality he gratefully acknowledges. He would also like to express his gratitude to Professor Fernando Marques.

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