Inhomogeneous higher-order summary statistics for linear network point processes

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Abstract. We introduce the notion of intensity reweighted moment pseudostationary point processes on linear networks. Based on arbitrary general regular linear network distances, we propose geometrically corrected versions of different higher-order summary statistics, including the inhomogeneous empty space function, the inhomogeneous nearest neighbour distance distribution function and the inhomogeneous $J$-function. We also discuss their non-parametric estimators. Through a simulation study, considering models with different types of spatial interaction, we study the performance of our proposed summary statistics. Finally, we make use of our methodology to analyse two datasets: motor vehicle traffic accidents and spider data.

Key words and phrases: Inhomogeneous empty space function, Inhomogeneous $J$-function, Inhomogeneous nearest neighbour distance distribution function, Linear network, Product density, Regular distance metric, Traffic accident data.

1. INTRODUCTION

Nowadays point patterns are sampled on a variety of different spatial domains (Baddeley et al.; 2015). In particular, point patterns on linear networks and their associated statistical analysis have gained a considerable amount of interest; Figure 1 illustrates two such datasets, which will be analysed in this paper. Non-parametric analyses of linear network point processes usually tend to have two main ingredients: intensity estimation (first order) and estimation of summary statistics, which indicate whether the underlying point process tends to have a clustering/aggregating or inhibiting/regular behaviour. In essence, such a summary statistic reflects different characteristics of the distribution of points around a point of the point process and/or an arbitrary location within the study region. Such analyses differ depending on whether one assumes that the underlying point process has a non-constant intensity function, which is referred to as inhomogeneity. Thusfar, attention has mainly been paid to non-parametric...
estimators for second-order summary statistics, such as $K$-functions and pair correlation functions. A wide review of different non-parametric estimators for first- and second-order summary statistics of point patterns on linear networks can also be found in Moradi (2018). Initially a few poorly performing kernel-based intensity estimators were proposed (Borruso; 2005, 2008; Xie and Yan; 2008). Later, other non-parametric kernel-based intensity estimators were defined (Okabe et al.; 2009; Okabe and Sugihara; 2012; McSwiggan et al.; 2017; Moradi et al.; 2018) which, although being statistically well-defined, tended to be computationally expensive on large networks. Moreover, Rakshit et al. (2019) proposed a fast kernel intensity estimator based on a two-dimensional convolution which can be computed rapidly even on large networks. With the aim of finding middle-ground between global and local smoothing, as well as an alternative to kernel estimation, Moradi et al. (2019) introduced their so-called resample-smoothing technique which they applied to Voronoi intensity estimators on arbitrary spaces. They showed that their estimation approach mostly performs better than kernel estimators, in terms of bias and standard error.

Regarding second-order summary statistics and their estimation, Okabe and Yamada (2001) considered an analogue of Ripley’s $K$-function for homogeneous linear network point processes, which was obtained by using the shortest-path distance instead of the Euclidean distance when measuring distance between points. However, this modification did not provide a well-defined $K$-function for linear network point processes since its behaviour depends on the topography of the network in question. As a remedy, Ang et al. (2012) introduced geometrically corrected second-order summary statistics which did not depend on the explicit geometry of the linear network under consideration and has a fixed known behaviour for Poisson processes. Hence, the geometrically corrected $K$-function and pair correlation function can be used e.g. for model selection, hypothesis testing and residual analyses. These summary statistics were later extended to the case of multitype and spatio-temporal point patterns by Baddeley et al. (2014) and Moradi and Mateu (2019). Surrounding theses papers, there appeared a discus-

![Fig 1: Left: Spider webs on a brick wall. Right: Motor vehicle traffic accidents in an area of Houston, US, during April, 1999.](image_url)
sion on explicit choices of distances to be used for point processes on linear networks – e.g., is the shortest-path distance always the canonical choice of metric? Taking this into account, Rakshit et al. (2017) redefined the $K$- and pair correlation functions of Ang et al. (2012) under very general assumptions on the distance/metric used. More specifically, they considered what they referred to as the family of regular distance metrics.

Although second-order summary statistics are invaluable tools to analyse interaction among points of a point process, the point process may show structure beyond pairwise interactions. For this reason, in the case of point processes in $\mathbb{R}^d$, $d \geq 1$, it is common to also study the (inhomogeneous) empty space function and the (inhomogeneous) nearest neighbour distance distribution function (Møller and Waagepetersen; 2004; Illian et al.; 2008; van Lieshout; 2011; Chiu et al.; 2013; Baddeley et al.; 2015). Moreover, a combination of these summary statistics is provided through the (inhomogeneous) $J$-function (van Lieshout and Baddeley; 1996; van Lieshout; 2011), which is a powerful quantifier of points’ tendency to cluster around or to inhibit each other. Although these summary statistics are well studied for spatial, spatio-temporal and marked point processes in $\mathbb{R}^d$, $d \geq 1$ (van Lieshout and Baddeley; 1996; van Lieshout; 2006, 2011; Cronie and van Lieshout; 2015, 2016), their linear network point process versions have not yet appeared in the literature. The reasons for this seem mainly to be related to theoretical challenges connected to the geometry of the linear network under consideration. In this paper we tackle this problem and propose geometrically corrected analogues of these summary statistics for point processes on linear networks, which are defined based on regular linear network distances (Rakshit et al.; 2017). Moreover, to do so we introduce the class of intensity reweighted moment pseudostationary (IRMPS) point processes, which in turn (perhaps less interestingly) yields a definition of stationarity for linear network point processes. To best connect our work to the existing literature on statistics for linear network point patterns, we carry out our numerical evaluations using the shortest-path distance as metric.

The paper is organised as follows. Section 2 provides a wide background of spatial point processes on linear networks. In Section 3 we review higher-order summary statistics and propose their geometrically corrected analogues for point patterns on linear networks. Section 4 is devoted to evaluating the performance of the geometrically corrected inhomogeneous linear $J$-function for a few models with different types of interaction. In Section 5 we apply the geometrically corrected inhomogeneous linear $J$-function to two real datasets. The paper ends with a discussion in Section 6.

2. PRELIMINARIES

Throughout, $\mathbb{R}^d$, $d \geq 1$, denotes the $d$-dimensional Euclidean space, $\| \cdot \|$ denotes the $d$-dimensional Euclidean norm, and all subsets under consideration will be Borel sets in the space in question. Moreover, $\int d_1 u$ will be used to denote integration with respect to arc length and $\int dx$ will be used to denote integration with respect to Lebesgue measure.
2.1 Linear networks

Linear networks are, among other things, convenient tools for approximating geometric graphs/spatial networks. The spatial statistical literature usually defines a linear network as a finite union of (non-disjoint) line segments (Ang et al.; 2012; Baddeley et al.; 2015; Rakshit et al.; 2017). More specifically, we define a linear network as a union

\[ L = \bigcup_{i=1}^{k} l_i, \quad 1 \leq k < \infty, \]

of \( k \) line segments \( l_i = [u_i, v_i] = \{ tu_i + (1-t)v_i : 0 \leq t \leq 1 \} \subseteq \mathbb{R}^2, \ u_i \neq v_i \in \mathbb{R}^2, \) with (arc) lengths \( |l_i| = \|u_i - v_i\| \in (0, \infty), \ i = 1, \ldots, k, \) which are such that any intersection \( l_i \cap l_j, \ j \neq i, \) is either empty or given by line segment end points. We here restrict ourselves to connected networks since disconnected ones may simply be represented as unions of connected ones.

The Borel sets on \( L \) are given by \( \mathcal{B}(L) = \{ A \cap L : A \subseteq \mathbb{R}^2 \} \) and they coincide with the \( \sigma \)-algebra generated by \( \tau_L = \{ A \cap L : A \text{ is an open subset of } \mathbb{R}^2 \}; \) recall that \( A \subseteq L \) will mean that \( A \) belongs to \( \mathcal{B}(L). \) We further endow \( L \) with the Borel measure \( |A| = \nu_L(A) = \int_A d1u, \ A \subseteq L, \) which represents integration with respect to arc length. Note that the total network length is given by \( |L| = \sum_{i=1}^{k} |l_i|. \)

Remark 2.1. One could, in principle, also allow \( k = \infty \) with the additional assumption of local finiteness, i.e. any compact \( A \subseteq \mathbb{R}^2 \) intersects at most a finite number of line segments, which excludes pathological cases. This would result in the total network length \( |L| = \infty \) and, as a consequence, one would allow networks which are isometric to \( \mathbb{R}. \)

Each linear network \( L \) also has a graph theoretical interpretation. The endpoints of each line segment are called nodes/vertices, and the degree of each node is the number of line segments (edges) which share that node. The boundary of \( L \) is the set of all nodes with degree one and is denoted by \( \partial L. \) See e.g. Eckardt and Mateu (2018) for further details on graph theoretical aspects of linear networks.

2.2 Linear network point processes

Heuristically, a point process is a generalised sample in which the points may be dependent and the total point count may be random. More formally, given some probability space \( (\Omega, \mathcal{F}, P) \), a (finite simple) point process \( X = \{ x_i \}_{i=1}^{N}, \ 0 \leq N < \infty, \) on a linear network \( L \) is a random element/variable in the measurable space \( N_f \) of point configurations \( x = \{ x_1, \ldots, x_n \} \subseteq L, \ 0 \leq n < \infty; \) the associated \( \sigma \)-algebra is generated by the cardinality mappings \( x \mapsto N(x \cap A) \in \{0,1,\ldots\}, \ A \subseteq L, \ x \in N_f, \) and coincides with the Borel \( \sigma \)-algebra generated by a certain metric on \( N_f \) (Daley and Vere-Jones; 2008).

2.2.1 Product densities Throughout, we will assume that the product densities/intensity functions \( \rho^{(m)} \) of all orders \( m \geq 1 \) exist. Formally, they may be defined through Campbell formulas: for any non-negative measurable function
\( f(\cdot) \) on the product space \( L^m \),

\[
\mathbb{E} \left[ \sum_{x_1, \ldots, x_m \in X} f(x_1, \ldots, x_m) \right] = \int_{L^m} f(u_1, \ldots, u_m) \rho^{(m)}(u_1, \ldots, u_m) d_1 u_1 \cdots d_1 u_m.
\]

Here the notation \( \sum_{\neq} \) is used to indicate that the summation is taken over distinct \( m \)-tuples. Since \( X \) is simple, i.e. \( x_i \neq x_j \) for any \( i \neq j \), \( x_i, x_j \in X \), we interpret \( \rho^{(m)}(u_1, \ldots, u_m) d_1 u_1 \cdots d_1 u_m \) as the probability of jointly finding points of \( X \) in some infinitesimal disjoint neighbourhoods \( du_1, \ldots, du_m \subseteq L \) of \( u_1, \ldots, u_m \in L \), with sizes \( |du| = d_1 u_1, \ldots, |du_m| = d_1 u_m \).

In the particular case \( m = 1 \), the right hand side of equation (1) reduces to \( \int_L f(u) \rho(u) d_1 u \), and in particular \( \mathbb{E}[N(X \cap A)] = \int_A \rho(u) d_1 u, \ A \subseteq L \), where \( \rho(u) = \rho^{(1)}(u) \) is called the intensity function of \( X \). Whenever \( \rho(u) = \rho > 0, \ u \in L \), is constant, we say that \( X \) is homogeneous and otherwise \( X \) is called inhomogeneous.

### 2.2.2 Correlation functions

As with any joint probability structure and its relationship to its marginal probabilities, product densities are such that large/small values of \( \rho^{(m)}(u_1, \ldots, u_m) \) do not necessarily imply that there is strong/weak dependence between points of \( X \) located around \( u_1, \ldots, u_m \in L \). For instance, for Poisson processes, where the points are independent, we have \( \rho^{(m)}(u_1, \ldots, u_m) = \prod_{i=1}^m \rho(u_i) \) so any \( \rho(u_i) \) being large may imply that \( \rho^{(m)}(u_1, \ldots, u_m) \) is large. Instead, in order to study \( m \)-point dependencies it is more natural to consider so-called correlation functions \( g_m, m \geq 1 \) (which do not actually represent correlations):

\[
(2) \quad g_m(u_1, \ldots, u_m) = \frac{\rho^{(m)}(u_1, \ldots, u_m)}{\rho(u_1) \cdots \rho(u_m)}, \quad u_1, \ldots, u_m \in L.
\]

Note that \( g_1(\cdot) = \rho(\cdot) / \rho(\cdot) = 1 \). Clearly, for a Poisson process with intensity function \( \rho(\cdot) \) we have \( g_m(\cdot) = 1, m \geq 1 \), so we interpret \( g_m(u_1, \ldots, u_m) > 1 \) as clustering/attraction between points of \( X \) located around \( u_1, \ldots, u_m \in L \). Similarly, \( g_m(u_1, \ldots, u_m) < 1 \) indicates inhibition/regularity. There further exist recursively defined expansions of \( g_m, m \geq 1 \) (van Lieshout; 2011):

\[
(3) \quad g_m(u_1, \ldots, u_m) = \sum_{D_1, \ldots, D_j} \sum_{j=1}^m \xi_{N(D_1)}(\{u_j : j \in D_1\}) \cdots \xi_{N(D_j)}(\{u_j : j \in D_j\}),
\]

where the sum \( \sum_{D_1, \ldots, D_j} \) ranges over all partitions \( \{D_1, \ldots, D_j\} \) of \( \{1, \ldots, m\} \) into \( j \) non-empty and disjoint sets. For instance, \( g_2(u, v) - g_1(u) = (\xi_2(u, v) + 1) - 1 = \xi_2(u, v) \).

### 2.2.3 Reduced Palm distributions

A central tool in the study of a point process \( X \) is its family of reduced Palm distributions \( \{\mathbb{P}_u^l(X \in \cdot) : u \in L\} \). Heuristically, \( \mathbb{P}_u^l(X \in \cdot) \) represents the distribution of \( X \) conditionally on \( X \) having a point at \( u \) which is removed once the process is realised; there actually exists a well-defined point process \( X_u^l \) with distribution \( \mathbb{P}_u^l(X \in \cdot) \). Formally, the most convenient way
of defining \( \{P_u^L(X \in \cdot) : u \in L\} \) is as the family of regular conditional distributions satisfying the reduced Campbell-Mecke formula (Møller and Waagepetersen; 2004, Appendix C): For any non-negative and measurable mapping \( f \) on \( L \times N_f \),

\[
E \left[ \sum_{x \in X} f(x, X \setminus \{x\}) \right] = \int_L E[f(u, X_u^t)] \rho(u) d_1 u = \int_L E_u^t[f(u, X)] \rho(u) d_1 u,
\]

where \( E_u^t[\cdot] \) denotes expectation under \( P_u^L(X \in \cdot) \).

### 2.3 Second-order summary statistics

Recalling (2), the particular function

\[
g(u, v) = g_2(u, v) = \frac{\rho^{(2)}(u, v)}{\rho(u) \rho(v)}, \quad u, v \in L,
\]

which quantifies pairwise interactions in \( X \), is commonly referred to as the pair correlation function. In practice, however, it is often more convenient to work with cumulative versions, so-called \( K \)-functions. Statistical estimators of such functions may be considered in e.g. exploratory data analyses, hypothesis testing and residual analyses. Further details on \( K \)-functions and pair correlation functions, together with their estimators, can be found in Ang et al. (2012), Baddeley et al. (2015, Chapter 17) and Rakshit et al. (2017).

If we were to define the inhomogeneous \( K \)-function of \( X \) on \( L \) in accordance with its original definition (Baddeley et al.; 2000), we would define it as

\[
\bar{K}^{L}_{\text{inhom}}(r) = \frac{1}{|W|} E \left[ \sum_{x_1, x_2 \in X} \frac{1\{x_1 \in W\} 1\{x_2 \in b_L(x_1, r)\}}{\rho(x_1) \rho(x_2)} \right]
\]

\[
= \frac{1}{|W|} \int_W E_u^t \left[ \sum_{x \in X} \frac{1\{x \in b_L(u, r)\}}{\rho(x)} \right] d_1 u,
\]

for \( r \geq 0 \), some \( W \subseteq L, |W| > 0 \), and the \( r \)-ball \( b_L(u, r) = \{v \in L : d_L(u, v) \leq r\}, u \in L \), which is determined by a distance/metric \( d_L \) on \( L \); the second equality follows by (4). There are, however, a few questions which immediately appear here.

The first question is related to something as basic as what the ball \( b_L(u, r) \) looks like. Note further that the geometry of the ball \( b_L(u, r) \) may change with \( u \in L \) (there may e.g. exist \( u, v \in L, u \neq v \), such that \( |b_L(u, r)| \neq |b_L(v, r)| \)).

Secondly, for the Euclidean version \( \bar{K}^{\mathbb{R}^d}_{\text{inhom}}(r) \), which is obtained by replacing \( W \subseteq L \) by \( W \subseteq \mathbb{R}^d \) and letting \( X \) be a point process on \( \mathbb{R}^d \) in (6), Baddeley et al. (2000) assumed so-called second-order intensity reweighted stationarity (SOIRS), i.e. that the intensity function of \( X \subseteq \mathbb{R}^d \) is strictly positive and that the pair correlation function of \( X \) is translation invariant in the sense that \( g(u_1, u_2) = \bar{g}(u_1 - u_2), u_1, u_2 \in \mathbb{R}^d, \) for some function \( \bar{g} : \mathbb{R}^d \to [0, \infty) \). Note that in the linear network setting, SOIRS cannot be defined in an analogous way since the vector \( u_1 - u_2 \), and thereby \( \bar{g}(u_1 - u_2) \), does not make sense on a linear network (e.g., for most \( u_1, u_2 \in L \) we have \( u_1 - u_2 \notin L \)). In the Euclidean setting, under SOIRS and by the Campbell formula, we obtain that

\[
\bar{K}^{\mathbb{R}^d}_{\text{inhom}}(r) = \frac{1}{|W|} \int_W \int_{b_{\mathbb{R}^d}(u, r)} g(u_1, u_2) du_1 du_2 = \int_{b_{\mathbb{R}^d}(u, r)} \bar{g}(u) du,
\]
for any \( u \in \mathbb{R}^d \). Hence, \( \check{K}_{\text{inhom}}^R(r) \) does not depend on the choice of the (positively Lebesgue sized) set \( W \subseteq \mathbb{R}^2 \) in (6) and the reduced Palm expectation in the Euclidean version of (6) does not depend on \( u \in \mathbb{R}^d \). This last observation is highly important in statistical settings – essentially, we need the reduced Palm integrand to be constant in \( u \) since we then may carry out the estimation of the inhomogeneous \( K \)-function by averaging over points in \( X \cap W \), rather than requiring repeated samples of \( X \cap W \).

It turns out that the choice of metric \( d_L \) and an appropriate notion of SOIRS for linear network point processes are related to each other, and that the former gives rise to the latter. We next look closer at metric choices.

2.3.1 General regular distance metrics

Ang et al. (2012) used the shortest-path distance as metric \( d_L \) and defined a linear network point process as being second-order intensity reweighted pseudostationary (SOIRPS) if

\[
g(u_1, u_2) = \check{g}(d_L(u_1, u_2)), \quad u_1, u_2 \in L,
\]

for some function \( \check{g} : [0, \infty) \to [0, \infty) \); note that Ang et al. (2012) referred to this as second-order reweighted pseudostationary (SORS).

Taking the work of Ang et al. (2012) a step further, Rakshit et al. (2017) extended the second-order analysis for linear network point processes to incorporate a larger class of metrics on linear networks, namely so-called regular distance metrics; the shortest-path distance belongs to this family. The motivation for doing so had partly to do with there being several metrics available which suit different applications and partly to do with the family of SOIRPS point processes being relatively small.

In accordance with Rakshit et al. (2017), we next give a brief account on regular distance metrics \( d_L \) – note that we make no explicit assumption that \( d_L \) metrises \( L \), i.e. that \( d_L \) generates the (subspace) topology \( \tau_L \) inherited from \( \mathbb{R}^2 \) (see Section 2.1). Recall that a (distance) metric on \( L \) is a function \( d_L : L \times L \to [0, \infty) \) satisfying i) \( d_L(u, u) = 0 \), ii) \( d_L(u, v) = d_L(v, u) \), and iii) \( d_L(u, v) \leq d_L(u, w) + d_L(v, w) \) for any \( u, v, w \in L \).

**Definition 2.1.** (Rakshit et al.; 2017, Definition 1) A regular distance metric on a linear network \( L \) is a metric \( d_L : L \times L \to [0, \infty) \) which further satisfies i) being a continuous function of \( u \) and \( v \), and ii) for any fixed \( u \in L \), the partial derivative \( \partial d_L(u, v) / \partial v \) exists and is nonzero everywhere except at a finite set of locations \( v \).

We next comment on integration over linear networks. Fix a point \( u \in L \). For an integrable real-valued function \( f \), we have the following change-of-variables formula (Rakshit et al.; 2017, Proposition 1):

\[
\int_L f(v) d_1v = \int_0^\infty \sum_{v \in L : d_L(u, v) = r} \frac{f(u)}{J_{d_L}(u, v)} dr,
\]

where \( J_{d_L}(u, v) = |\partial d_L(u, v) / \partial v| \) is the Jacobian; there may be a finite collection of fixed points \( u \in L \) such that (8) does not hold. If there is no \( u \in L \) such that \( d_L(v, u) = r \), the sum on the right hand side of equation (8) is 0. Extending
equation (8) to functions \( f : L^m \rightarrow \mathbb{R}, m \geq 1 \), we have

\[
\int_L \cdots \int_L f(u_1, \ldots, u_m) d_1 u_1 \cdots d_1 u_m = \int_0^\infty \sum_{u_1 \in L : d_L(u_1, u) = r_1} \frac{1}{J_{d_L}(u_1, u)} \cdots \\
\cdots \int_0^\infty \sum_{u_m \in L : d_L(u_m, u) = r_m} \frac{1}{J_{d_L}(u_m, u)} f(u_1, \ldots, u_m) dr_1 \cdots dr_m \\
= \int_0^\infty \cdots \int_0^\infty \sum_{u_1 \in L : d_L(u_1, u) = r_1} \cdots \sum_{u_m \in L : d_L(u_m, u) = r_m} \frac{f(u_1, \ldots, u_m)}{\prod_{i=1}^m J_{d_L}(u_i, u)} dr_1 \cdots dr_m.
\]

Let \( D(u) = \max\{d_L(u, v) : v \in L\} \) be the farthest reachable distance from \( u \) and \( c_L(u, r) \) be the number of points exactly \( r \) units away from \( u \) according to the choice \( d_L \). Then, for \( r \in (0, D(u)) \), consider

\[
\tilde{J}_{d_L}(u, r) = \left[ \frac{1}{c_L(u, r)} \sum_{v \in L : d_L(u, v) = r} \frac{1}{J_{d_L}(u, v)} \right]^{-1},
\]

which is the harmonic mean of \( J_{d_L}(u, v) \) at all locations \( v \) exactly \( r \) units away from \( u \) according to \( d_L \). Moreover, the harmonic mean is zero if any \( J_{d_L}(u, v) \) is zero. Defining

\[
w_{d_L}(u, r) = \frac{\tilde{J}_{d_L}(u, r)}{c_L(u, r)},
\]

and if the function \( f \) in (8) only depends on the distance \( d_L(u, v) \), we obtain

\[
\int_L f(v) dv = \int_L h(d_L(u, v)) w_{d_L}(u, d_L(u, v)) dv = \int_0^{D(u)} h(r) dr,
\]

where \( h : [0, \infty) \rightarrow \mathbb{R} \). Hence, the equation above can be extended to

\[
\int_L \cdots \int_L h(d_L(u_1, u), \ldots, d_L(u_m, u)) \prod_{i=1}^m w_{d_L}(u, d_L(u_i, u)) d_1 u_1 \cdots d_1 u_m = \\
= \int_0^{D(u)} \cdots \int_0^{D(u)} h(r_1, \ldots, r_m) dr_1 \cdots dr_m
\]

for \( h : [0, \infty)^m \rightarrow \mathbb{R} \).

2.3.2 Second-order intensity reweighted pseudostationarity and inhomogeneous linear network \( K \)-functions Rakshit et al. (2017) considered point processes for which the pair correlation function (5) only depends on the regular distance metric \( d_L \), i.e. \( g(u, v) = \tilde{g}(d_L(u, v)) \) for some function \( \tilde{g} : [0, \infty) \rightarrow [0, \infty) \), and called such a point process \( d_L \)-correlated; with their notation, \( \delta \)-correlated since they used the notation \( \delta \) for \( d_L \). This is exactly what Ang et al. (2012) called second-order reweighted pseudostationary when \( d_L \) is given by the shortest-path distance. For any inhomogeneous \( d_L \)-correlated point process \( X \) and any \( 0 \leq r < R \), where \( R = \min_{u \in L} D(u) \), Rakshit et al. (2017) proposed the following
geometrically corrected inhomogeneous K-function

\[
K_{\text{inhom}}^L(r) = \mathbb{E}_u^L \left[ \sum_{x \in X} \frac{1\{x \in b_L(u, r)\}}{\rho(x)} w_{dL}(u, d_L(u, x)) \right]
= \frac{1}{|L|} \mathbb{E} \left[ \sum_{x_1, x_2 \in X} \frac{1\{x_1 \in W\} 1\{x_2 \in b_L(x_1, r)\}}{\rho(x_1)\rho(x_2)} w_{dL}(x_1, d_L(x_1, x_2)) \right]
= \int_0^r \bar{g}(t) dt,
\]

for any \( u \in L \). Note that here \( \bar{g} : [0, \infty) \to [0, \infty) \). For a Poisson process on the linear network \( L \), which has pair correlation \( g(\cdot) = 1 \), we have \( K(r) = r \); values larger than \( r \) indicate clustering within (pairwise) inter-point distance \( r \) while values smaller than \( r \) instead reveal inhibition. Similarly, \( g(r) > 1 \) indicates a clustering behaviour between \( r \)-separated point pairs while \( g(r) < 1 \) points to inhibition.

### 3. Higher-Order Summary Statistics

To quantify degrees of dependence of any order higher than two between points in a stationary point process in \( \mathbb{R}^d \), three common and powerful tools are given by the empty space function, the nearest neighbour distance distribution function and the J-function (Bartlett; 1964; Paloheimo; 1971; Diggle; 1979; van Lieshout and Baddeley; 1996). In a seminal paper, van Lieshout (2011) finally extended these summary statistics to inhomogeneous point processes in \( \mathbb{R}^d \) and later Cronie and van Lieshout (2015, 2016) extended them further to spatio-temporal and marked inhomogeneous point processes in \( \mathbb{R}^d \). Our aim here is to study these higher order summary statistics and their non-parametric estimation in the context of linear network point processes.

#### 3.1 The general case

Given some arbitrary (complete and separable metric) space \( S \) and some locally finite Borel reference measure \( \nu_S \), consider a point process \( X \) in \( S \) with product densities \( \rho^{(m)} \) (with respect to products of \( \nu_S \)), \( m \geq 1 \), and \( \bar{\rho} = \inf_{u \in S} \rho(u) > 0 \). The summary statistics of van Lieshout (2011) may now be expressed as follows. Given the closed ball \( b_S(u, r) = \{ v \in S : d_S(u, v) \leq r \} \subseteq S \) with centre \( u \in S \) and radius \( r \geq 0 \), which is based on some metric \( d_S(\cdot, \cdot) \) on \( S \), let

\[
\limsup_{m \to \infty} \left( \frac{\bar{\rho}^m}{m!} \int_{b_S(u, r)^m} g_m(u_1, \ldots, u_m) \nu_S(du_1) \cdots \nu_S(du_n) \right)^{1/m} < 1
\]

and consider the inhomogeneous empty space function (at \( u \in S \))

\[
F_{\text{inhom}}^S(r; u) = 1 - \mathbb{E} \left[ \prod_{x \in X} \left( 1 - \frac{\bar{\rho}}{\rho(x)} 1\{x \in b_S(u, r)\} \right) \right],
\]

the inhomogeneous nearest neighbour distance distribution function (at \( u \in S \))

\[
H_{\text{inhom}}^S(r; u) = 1 - \mathbb{E}_u^I \left[ \prod_{x \in X} \left( 1 - \frac{\bar{\rho}}{\rho(x)} 1\{x \in b_S(u, r)\} \right) \right],
\]
and the inhomogeneous \( J \)-function (at \( u \in S \))

\begin{equation}
J_{\text{inhom}}^S(r; u) = 1 + \sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} \int_{b_S(u,r)^m} \xi_{m+1}(u, u_1, \ldots, u_m)\nu_S(du_1) \cdots \nu_S(du_m).
\end{equation}

Given \( X \) within some bounded \( W \subseteq S \), non-parametric estimators of (15) and (16) based on \( X \cap W \) are given by

\begin{equation}
\hat{F}_{\text{inhom}}^S(r; u, X, W) = \prod_{x \in X \cap W} \left( 1 - \frac{\bar{\rho}(x)}{\rho(x)} 1\{x \in b_S(u, r)\} \right), \quad u \in W,
\end{equation}

\begin{equation}
\hat{R}_{\text{inhom}}^S(r; u, X, W) = \prod_{x \in X \cap W \setminus \{u\}} \left( 1 - \frac{\bar{\rho}(x)}{\rho(x)} 1\{x \in b_S(u, r)\} \right), \quad u \in X \cap W,
\end{equation}

respectively; note that in practice we would plug in an estimate of \( \rho(\cdot) \) into these estimators. These are local empirical summary statistics.

### 3.2 The Euclidean case

Besides providing the definitions of the summary statistics in (15), (16) and (17), for which the intuition will be clarified in a moment, van Lieshout (2011, Theorem 1) showed that under certain conditions, when \( S = \mathbb{R}^d \), \( b_S(u, r) = b_{\mathbb{R}^d}(u, r) = \{x \in \mathbb{R}^d : \|x - u\| \leq r\} \), \( u \in \mathbb{R}^d \), where \( \| \cdot \| \) is the Euclidean norm, the functions in Section 3.1 are almost everywhere constant as functions of \( u \in S = \mathbb{R}^d \). We may thus write \( H_{\text{inhom}}^{\mathbb{R}^d}(r) \) and \( F_{\text{inhom}}^{\mathbb{R}^d}(r) \) for (15) and (16) to emphasize this independence of the choice of \( u \in \mathbb{R}^d \). In addition, van Lieshout (2011, Theorem 1) also showed that (17) satisfies

\begin{equation}
J_{\text{inhom}}^{\mathbb{R}^d}(r; u) = J_{\text{inhom}}^{\mathbb{R}^d}(r) = \frac{1 - H_{\text{inhom}}^{\mathbb{R}^d}(r)}{1 - F_{\text{inhom}}^{\mathbb{R}^d}(r)}, \quad r \geq 0,
\end{equation}

for almost every \( u \in \mathbb{R}^d \) and \( F_{\text{inhom}}^{\mathbb{R}^d}(r) \neq 1 \), and truncating the sum in (17) at \( m = 1 \) here yields

\begin{equation}
J_{\text{inhom}}^{\mathbb{R}^d}(r) \approx 1 - \bar{\rho} \int_{b_{\mathbb{R}^d}(o,r)} \xi_2(o, u)du = 1 - \bar{\rho}(K_{\text{inhom}}^{\mathbb{R}^d}(r) - |b_{\mathbb{R}^d}(o, r)|),
\end{equation}

where e.g. \( o = (0, \ldots, 0) \in \mathbb{R}^d \) and we recall \( K_{\text{inhom}}^{\mathbb{R}^d}(r) \) from equation (7).

The main importance of (20) is that it is an inhomogeneous analogue/natural extension of the \( J \)-function of van Lieshout and Baddeley (1996) for stationary point processes:

\begin{equation}
J^{\mathbb{R}^d}(r) = \frac{1 - H^{\mathbb{R}^d}(r)}{1 - F^{\mathbb{R}^d}(r)} = \frac{1 - P^1(o)(X \cap b_{\mathbb{R}^d}(o, r) \neq \emptyset)}{1 - P(X \cap b_{\mathbb{R}^d}(o, r) \neq \emptyset)} = \frac{P^1(o)(X \cap b_{\mathbb{R}^d}(o, r) = \emptyset)}{P(X \cap b_{\mathbb{R}^d}(o, r) = \emptyset)}, \quad r \geq 0,
\end{equation}

where we recall that under stationarity \( P^1(o) \) is chosen to represent the family \( \{P^1_u(\cdot) : u \in \mathbb{R}^d\} \) since \( P^1_u \) is constant as a function of \( u \in \mathbb{R}^d \). We emphasise
that under stationarity, (20) reduces to (22); note that here $\rho(u) = \bar{\rho} = \rho > 0$ for any $u \in \mathbb{R}^d$. Noting that for a Poisson process $X$ we have $P^d_u(X \in \cdot) = P(X \in \cdot)$ for any $u$ by Slivniyak’s theorem (Chiu et al.; 2013), we see that for Poisson processes we have $J^{rd}_u(r) = 1$, $r \geq 0$; also $J^{rd}_{inhom}(r) = 1$ holds true for (inhomogeneous) Poisson processes. The interpretation is thus that we quantify how conditioning on there being a point of $X$ at some location increases/decreases the probability of seeing a further point within distance $r$. Hence, $J^{rd}_{inhom}(r) > 1$ indicate inhibition/regularity whereas these quantities being smaller than 1 indicate clustering/attraction between points of $X$ with inter-point distance at most $r$ – in the inhomogeneous case this should be understood in the sense of having scaled away the individual intensity contributions of the points of $X$.

One thing that has completely been left out of the discussion above is a discussion on the definition of stationarity. A point process $X = \{x_i\}_{i=1}^N$ in $\mathbb{R}^d$ being stationary means that its distribution satisfies $P(\{x + y : x \in X\} \in \cdot) = P(X \in \cdot)$ for any $y \in \mathbb{R}^d$. In other words, its distribution is invariant under a family $\mathcal{T}$ of transformations on the spatial domain, which here is given by the (Euclidean) family $\mathcal{T} = \mathcal{T}_{\mathbb{R}^d} = \{T_y : \mathbb{R}^d \to \mathbb{R}^d\}_{y \in \mathbb{R}^d}, T_y x = x + y, x \in \mathbb{R}^d,$ of translations/shifts. Note that stationarity is a very strong assumption which, among other things, implies homogeneity, i.e. that $X$ has a constant intensity. Moreover, we have mentioned that i) (15), (16) and (17) being constant in $u \in S = \mathbb{R}^d$, and ii) the relation (20) being satisfied, were proved under some conditions in van Lieshout (2011). More specifically, what van Lieshout (2011) assumed was so-called intensity reweighted moment stationarity (IRMS) for the point process $X$. We say that a point process $X = \{x_i\}_{i=1}^N$ in $\mathbb{R}^d$ with correlation functions $g_m, m \geq 1$, is IRMS if i) $\bar{\rho} = \inf_{u \in \mathbb{R}^d} \rho(u) > 0$, and ii) $g_m(u_1, \ldots , u_m) = g_m(T_y u_1, \ldots , T_y u_m) = g_m(u_1 + y, \ldots , u_m + y)$ for almost any $u_1, \ldots , u_m \in \mathbb{R}^d$, any $y \in \mathbb{R}^d$ and any $m \geq 1$. In fact, the original definition of van Lieshout (2011) states, equivalently, that the expansion terms $\xi_m, m \geq 1$, in (3) should be translation invariant in the above sense. In other words, all intensity reweighted factorial moments should be invariant under the transformations $T_{\mathbb{R}^d}$.

Note further that stationarity implies IRMS.

The proof of Theorem 1 in van Lieshout (2011) exploits that $X$ is IRMS to obtain that (15) and (16) are almost everywhere constant as functions of $u$. This in turn allows us to treat the distributions of (18) and (19) as constant with respect to $u$ and, as such, we may consider the estimators

$$\hat{F}_{inhom}^{rd}(r; X, W) = \frac{1}{N(I \cap W_{\Theta r})} \sum_{u \in I \cap W_{\Theta r}} \hat{F}_{inhom}^{rd}(r; u, X, W),$$

$$\hat{H}_{inhom}^{rd}(r; X, W) = \frac{1}{N(X \cap W_{\Theta r})} \sum_{u \in X \cap W_{\Theta r}} \hat{H}_{inhom}^{rd}(r; u, X, W),$$

where $I \subseteq W \subseteq \mathbb{R}^d$ is a fine grid and $W_{\Theta r}$ is the $r$-erosion of $W$.

### 3.3 The linear network case

In either of the two forms of distributional invariance appearing in the Euclidean setting above, we assume some form of distributional invariance with respect to a family of transformations. Hence, if we would like to consider different forms of distributional invariance for a point process $X$ on some linear
network \( L \), we would need to find a suitable family of transformations \( \mathcal{T} \), given
the distribution \( P(\cdot) = \mathbb{P}(X \in \cdot) \). Formally, this means working in the setting
of algebraic groups/geometric measure theory – to reproduce the proof of van Lieshout (2011, Theorem 1) in the linear network setting we would have to exploit Haar measure based arguments, where our reference measure \( \nu_L \) would be a Haar measure and the associated collection of transformations, \( \mathcal{T} \), would have a
group structure and act (transitively) on \( L \). However, such a structure seems (to the
best of our knowledge) to not be available; cf. Baddeley et al. (2017). As a consequence,
the independence of \( u \) in the summary statistics (15), (16) and (17) may not be attainable so when performing non-parametric estimation we may
not be able to justify the type of averaging over points of \( X \cap W, W \subseteq L \), that was considered in (23). This clearly poses a problem since in general we do not have access to repeated samples of \( X \cap W \).

Our solution to obtaining geometrically corrected summary statistics comes from combining expression (21) with (6) and (13). More precisely, in the Euclidean setting the truncation (21) contains the inhomogeneous \( K \)-function \( \bar{K}_L^{\text{inhom}}(r) \) in (7), but taking the discussion in Section 2.3 into consideration, since we are dealing with linear networks we should instead have the geometrically corrected \( K \)-function \( K_L^{\text{inhom}}(r) \) in (13) in the truncation. By looking closer at (15), (16) and (17), and revisiting the results and proofs in van Lieshout (2011), we arrive at the definitions below.

**Definition 3.1.** Given a point process \( X \) on a linear network \( L \), with product
densities \( \rho^{(m)}, m \geq 1 \), and \( \bar{\rho} = \inf_{u \in L} \rho(u) > 0 \), the inhomogeneous geometrically corrected linear empty space function, the linear nearest neighbour distance distribution function and the linear \( J \)-function at \( u \in L \) and \( r \geq 0 \), with respect to a regular distance metric \( d_L \), are given by

\[
F_{\text{inhom}}^L(r; u) = 1 - \mathbb{E}_{u} \left[ \prod_{x \in X} \left( 1 - \bar{\rho} 1\{x \in b_L(u, r)\} w_{d_L}(u, d_L(u, x)) \right) \right] \\
= - \sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} \int_{b_L(u, r)^m} g_m(u_1, \ldots, u_m) \prod_{i=1}^{m} w_{d_L}(u, d_L(u, u_i)) d_1 u_1 \cdots d_1 u_m, 
\]

(24)

\[
H_{\text{inhom}}^L(r; u) = 1 - \mathbb{E}_{u} \left[ \prod_{x \in X} \left( 1 - \bar{\rho} 1\{x \in b_L(u, r)\} w_{d_L}(u, d_L(u, x)) \right) \right] \\
= - \sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} K_{\text{inhom}}^{L,m}(r; u), 
\]

(25)

\[
J_{\text{inhom}}^L(r; u) = 1 + \sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} \times \\
\times \int_{b_L(u, r)^m} \xi_{m+1}(u, u_1, \ldots, u_m) \prod_{i=1}^{m} w_{d_L}(u, d_L(u, u_i)) d_1 u_1 \cdots d_1 u_m, 
\]

(26)
respectively, where
\[
K_{L_{\text{inhom}}}^{L,m}(r; u) = \mathbb{E}_u \left[ \sum_{x_1, \ldots, x_m \in X} \prod_{i=1}^m \frac{\mathbf{1}_{\{x_i \in b_L(u, r)\}} w_{d_L}(u, d_L(u, x_i))}{\rho(x_i)} \right] \\
= \int_{b_L(u, r)^m} g_{m+1}(u, u_1, \ldots, u_m) \prod_{i=1}^m w_{d_L}(u, d_L(u, u_i)) d_1 u_1 \cdots d_1 u_m;
\]

note that \( m = 1 \) gives us (13).

The expansion in (24) follows from an application of the Campbell formula, and the expansions in (25) follow from a combination of the Campbell formula and the Campbell-Mecke formula (van Lieshout; 2011; Cronie and van Lieshout; 2015).

The missing ingredient is still some form of distributional invariance, similar to IRMS in the Euclidean context, where the essence is that the correlation functions should only depend on the inter-point distances of the points. Intuitively, we would translate this idea to the linear network setting by letting the correlation functions only depend on the \( d_L \)-distances between the input points, i.e.

\[
g_m(u_1, \ldots, u_m) = \tilde{g}_m(\{d_L(u_i, u_j) : i, j = 1, \ldots, m, i \neq j\}), \quad u_1, \ldots, u_m \in L,
\]

for some family of functions \( \tilde{g}_m, m \geq 1; \) requiring this to hold for (only/at least) \( m \leq 2 \) would essentially yield the definition of second-order reweighted pseudostationarity in Rakshit et al. (2017). It turns out that this is not completely sufficient and we have to impose the slightly stronger condition that \( g_m(u_1, \ldots, u_m) \) is given by a function of the distances \( d_L(u_i), i = 1, \ldots, m, \) where \( u \in L \) is arbitrary. Note that if the metric \( d_L \) is independent of a chosen origin, the two concepts coincide. By additionally assuming homogeneity here we obtain an expression pertaining to the product densities and thereby a definition of moment stationarity for linear network point processes. If we here also assume that the moments characterise the whole distribution of the point process we obtain a definition of (pseudo)stationarity for linear network point processes. Note that we have chosen the names below in keeping with van Lieshout (2011) and Ang et al. (2012).

**Definition 3.2.** Let \( X \) be a point process on a linear network \( L \) and let \( d_L \) be a regular distance metric on \( L \). Given some \( k \geq 2 \), whenever the product densities \( \rho^{(m)}, 1 \leq m \leq k, \) exist, \( \bar{\rho} = \inf_{u \in L} \rho(u) > 0 \) and for any \( m \in \{2, \ldots, k\} \) the correlation function \( g_m : L^m \to [0, \infty) \) satisfies

\[
g_m(u_1, \ldots, u_m) = \tilde{g}_m(d_L(u, u_1), \ldots, d_L(u, u_m))
\]

for any \( u \in L \) and some function \( \tilde{g}_m : [0, \infty)^m \to [0, \infty) \), we say that \( X \) is \( k \)-th order intensity reweighted pseudostationary (with respect to \( d_L \)); when this holds for any order \( k \geq 2 \) we say that \( X \) is intensity reweighted moment pseudostationary (IRMPS).

If \( X \) is both homogeneous with \( \rho(u) = \bar{\rho} = \rho > 0, u \in L, \) and \( k \)-th order intensity reweighted pseudostationary, we say that it is \( k \)-th order pseudostationary; note that \( \rho^{(m)}(u_1, \ldots, u_m) = \bar{\rho}^{(m)}(d_L(u, u_1), \ldots, d_L(u, u_m)) \) for any \( u \in L \).
When this holds for any $k \geq 1$ we say that $X$ is moment pseudostationary. Finally, any moment pseudostationary $X$ such that the (factorial) moments completely and uniquely characterise its distribution may be referred to as (strongly) pseudostationary.

Note that the last condition in the definition above, i.e. that the moments completely and uniquely characterise the distribution of $X$, may be obtained by requiring that the conditions in Zessin (1983, Section 2) hold. It is worth mentioning that (to the best of our knowledge) our definition of pseudostationarity is the first version of some form of “strong stationarity” for linear network point processes to be provided in the literature.

Next, we state our main result, which is essentially a (geometrically corrected) linear network version of van Lieshout (2011, Theorem 1); its proof can be found in the Appendix.

Theorem 3.1. For any IRMPS point process $X$ on a linear network $L$, which also satisfies (14) with $S = L$, the summary statistics in Definition 3.1 satisfy

$$F_{\text{inhom}}^L(r; u) = F_{\text{inhom}}^L(r) = -\sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} \int_0^r \cdots \int_0^r \tilde{g}_m(t_1, \ldots, t_m) dt_1 \cdots dt_m,$$

$$H_{\text{inhom}}^L(r; u) = H_{\text{inhom}}^L(r) = -\sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} \int_0^r \cdots \int_0^r \tilde{g}_{m+1}(0, t_1, \ldots, t_m) dt_1 \cdots dt_m,$$

$$J_{\text{inhom}}^L(r; u) = J_{\text{inhom}}^L(r) = \frac{1 - H_{\text{inhom}}^L(r)}{1 - F_{\text{inhom}}^L(r)},$$

for (almost) any $u \in L$.

When $X$ is pseudostationary we immediately obtain the following corollary, upon noting that the intensity is constant.

Corollary 3.1. Let $X$ be pseudostationary with constant intensity $\rho > 0$. Then we obtain (cf. van Lieshout (2011, p. 186))

$$F_{\text{inhom}}^L(r) = F^L(r) = -\sum_{m=1}^{\infty} \frac{(-1)^m \bar{\rho}^m}{m!} \int_0^r \cdots \int_0^r \tilde{g}_m(t_1, \ldots, t_m) dt_1 \cdots dt_m,$$

$$H_{\text{inhom}}^L(r) = H^L(r) = -\sum_{m=1}^{\infty} \frac{(-1)^m \bar{\rho}^m}{m!} \int_0^r \cdots \int_0^r \tilde{g}_{m+1}(0, t_1, \ldots, t_m) dt_1 \cdots dt_m,$$

$$J_{\text{inhom}}^L(r) = J^L(r) = \frac{1 - H^L(r)}{1 - F^L(r)}.$$

It should further be noted that under homogeneity, and thereby under pseudostationarity, we have

$$F^L(r) = 1 - \mathbb{E} \left[ \prod_{x \in X} (1 - 1 \{ x \in b_L(u, r) \} w_{d_L}(u, d_L(u, x)) \} \right],$$

$$H^L(r) = 1 - \mathbb{E}_u^L \left[ \prod_{x \in X} (1 - 1 \{ x \in b_L(u, r) \} w_{d_L}(u, d_L(u, x)) \} \right].$$
for any arbitrary \( u \), since \( \rho(\cdot) \equiv \rho > 0 \).

We next look closer at how our summary statistics are affected by independent thinning. The proof of Lemma 3.1 can be found in the Appendix.

**Lemma 3.1.** Let \( X \) be IRMPS and let \( X_{th} \) be an independently thinned version of \( X \), generated through some measurable retention probability function \( \rho : L \to (0,1] \). The inhomogeneous geometrically corrected linear empty space and \( L \) nearest neighbour distance distribution functions for \( X_{th} \) are of the form

\[
F_{\text{inhom}}^{L,th}(r) = -\sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} \int_0^r \cdots \int_0^r g_m(t_1, \ldots, t_m) dt_1 \cdots dt_m,
\]

\[
H_{\text{inhom}}^{L,th}(r) = -\sum_{m=1}^{\infty} \frac{(-\bar{\rho})^m}{m!} \int_0^r \cdots \int_0^r g_{m+1}(0, t_1, \ldots, t_m) dt_1 \cdots dt_m,
\]

where \( \bar{\rho} = \inf_{u \in L} \rho(u) > 0 \). We further have that

\[
F_{\text{inhom}}^{L,th}(r) = 1 - \mathbb{E} \left[ \prod_{x \in X_{th}} \left( 1 - \frac{\bar{\rho} \mathbf{1}\{x \in b_L(u, r)\} w_{d_L}(u, d_L(u, x))}{\rho(x) \rho(x)} \right) \right]
\]

\[
= 1 - \mathbb{E} \left[ \prod_{x \in X} \left( 1 - \rho(x) \frac{\mathbf{1}\{x \in b_L(u, r)\} w_{d_L}(u, d_L(u, x))}{\rho(x)} \right) \right]
\]

and

\[
H_{\text{inhom}}^{L,th}(r) = 1 - \mathbb{E}_u^l \left[ \prod_{x \in X_{th}} \left( 1 - \frac{\bar{\rho} \mathbf{1}\{x \in b_L(u, r)\} w_{d_L}(u, d_L(u, x))}{\rho(x) \rho(x)} \right) \right]
\]

\[
= 1 - \mathbb{E}_u^l \left[ \prod_{x \in X} \left( 1 - \rho(x) \frac{\mathbf{1}\{x \in b_L(u, r)\} w_{d_L}(u, d_L(u, x))}{\rho(x)} \right) \right].
\]

**3.3.1 Poisson processes** Poisson processes serve many different purposes in spatial statistics and one of them is as benchmark for complete randomness.

The first thing we note is that for any Poisson process with intensity \( \rho(u) \), \( u \in L \), \( \bar{\rho} = \inf_{u \in L} \rho(u) > 0 \), we have \( \xi_m(\cdot) = 0 \) and \( g_m(\cdot) = 1 \), \( m \geq 2 \), since \( \rho^{(m)}(u_1, \ldots, u_m) = \rho(u_1) \cdots \rho(u_1) \). Hence, it is automatically IRMPS and the series expansions in the expressions for \( F_{\text{inhom}}^{L}(r; u) \) and \( H_{\text{inhom}}^{L}(r; u) \) are just the Taylor expansions of \( 1 - \exp\{-\bar{\rho}r\} \), so \( J_{\text{inhom}}^{L}(r; u) = 1 \). In particular, for a homogeneous Poisson process with constant intensity \( \rho > 0 \), \( F_{\text{inhom}}^{L}(r; u) = H_{\text{inhom}}^{L}(r; u) = 1 - \exp\{-\rho r\} \). Note further that for a linear network \( L \) which is isometric to \( \mathbb{R} \) (i.e. \( \mathbb{R} \) bent in a number of places), \( F_{\text{inhom}}^{L}(r; u) = H_{\text{inhom}}^{L}(r; u) = 1 - \exp\{-\rho r\} \neq 1 - \exp\{-2\rho r\} = F_{\text{inhom}}^{\mathbb{R}}(r; u) = H_{\text{inhom}}^{\mathbb{R}}(r; u) \) for a homogeneous (and thus pseudostationary) Poisson process.

**3.3.2 Log-Gaussian Cox processes** In the Euclidean context, log-Gaussian Cox processes (Møller et al.; 1998) are the most prominent clustering models. We next look closer at IRMPS log-Gaussian Cox processes.

Assume that there exists a well-defined covariance function which satisfies \( C(u_1, u_2) = C(d_L(u, u_1), d_L(u, u_2)) \in \mathbb{R} \), \( u_1, u_2 \in L \), for any \( u \in L \) and some
function $C$; note that the dependence associated to two locations is determined by the locations’ respective $d_L$-distances to some arbitrary point $u \in L$.

Let $X$ be a log-Gaussian Cox process with (a.s. locally finite) random intensity measure $\Gamma(A) = \int_A \Lambda(v) d_1 v = \int_A \exp\{Z(v)\} d_1 v$, $A \subseteq L$, where $Z$ is a Gaussian random field on $L$ with mean function $\mu(v) \in \mathbb{R}$, $v \in L$, and covariance function $C(\cdot)$.

Note first that by e.g. Chiu et al. (2013, Example 5.3),

$$\rho(m)(u_1, \ldots, u_m) = \mathbb{E}[\Lambda(u_1) \cdots \Lambda(u_m)] = \mathbb{E}\left[\exp\left\{\sum_i Z(u_i)\right\}\right] = \prod_i \rho(u_i) \prod_{1 \leq i < j \leq m} g(u_i, u_j),$$

whereby

$$g_m(u_1, \ldots, u_m) = \frac{\rho(m)(u_1, \ldots, u_m)}{\rho(u_1) \cdots \rho(u_m)} = \prod_{1 \leq i < j \leq m} g(u_i, u_j).$$

Now, due to the way the covariance function is defined,

$$\rho(v) = \rho(1)(v) = \mathbb{E}\left[\exp\{Z(v)\}\right] = \exp\left\{\mu(v) + C(d_L(u, v), d_L(u, v))/2\right\},$$

and

$$g(u_1, u_2) = \exp\{C(u_1, u_2)\} = \exp\{C(d_L(u, u_1), d_L(u, u_2))\},$$

so

$$g_m(u_1, \ldots, u_m) = \prod_{1 \leq i < j \leq m} \exp\{C(d_L(u, u_i), d_L(u, u_j))\} = \exp\left\{\sum_{1 \leq i < j \leq m} C(d_L(u, u_i), d_L(u, u_j))\right\},$$

and we see that the latter is a function of the form $\tilde{g}_m(d_L(u, u_1), \ldots, d_L(u, u_m))$, which by assumption does not change with $u$. Hence, $X$ is IRMPS.

### 3.4 Non-parametric estimation

We next turn to the non-parametric estimation of our newly defined summary statistics, based on an IRMPS point process $X$ on a linear network $L$; note that in practice $L$ is often a subnetwork of a larger network and the observed point pattern is assumed to be a realisation of $X$. In analogy with van Lieshout (2011); Cronie and van Lieshout (2015, 2016) we will focus on minus sampling estimators.

Recalling that the boundary $\partial L$ of a linear network $L$ is the set of all nodes with degree 1, given a regular distance $d_L$, we define the $r$-erosion of $L$ (or simply the $r$-reduced network), $r \geq 0$, as

$$L_{\ominus r} = \{u \in L : d_L(u, \partial L) \geq r\},$$

where $d_L(u, A) = \inf_{v \in A} d_L(u, v)$, $A \subseteq L$, $u \in L$. We further let $I$ be a set consisting of a large number of fixed points in $L$; this is analogous to a point grid in $\mathbb{R}^d$. 
For a given \( r \geq 0 \), we estimate \( \hat{F}^L_{\text{inhom}}(r) \), i.e. the inhomogeneous geometrically corrected linear empty space function evaluated at \( r \), by means of
\[
\hat{F}^L_{\text{inhom}}(r) = 1 - \frac{1}{N(I \cap L_{\bowtie r})} \sum_{u \in I \cap L_{\bowtie r}} \prod_{x \in X \cap b_L(u, r)} \left( 1 - \frac{\bar{\rho}}{\rho(x)} \omega_{dL(u, dL(u, x))} \right),
\]
and \( \hat{H}^L_{\text{inhom}}(r) \), i.e. the inhomogeneous geometrically corrected linear nearest neighbour distance distribution function at \( r \), by means of
\[
\hat{H}^L_{\text{inhom}}(r) = 1 - \frac{1}{N(X \cap L_{\bowtie r})} \sum_{u \in X \cap L_{\bowtie r}} \prod_{x \in X \setminus \{u\} \cap b_L(u, r)} \left( 1 - \frac{\bar{\rho}}{\rho(x)} \omega_{dL(u, dL(u, x))} \right).
\]

Having the estimators above at hand, we then estimate \( \hat{J}^L_{\text{inhom}}(r) \), i.e. the inhomogeneous geometrically corrected linear \( J \)-function at \( r \geq 0 \), by means of
\[
\hat{J}^L_{\text{inhom}}(r) = \frac{1 - \hat{H}^L_{\text{inhom}}(r)}{1 - \hat{F}^L_{\text{inhom}}(r)},
\]
provided that the denominator is non-zero. When we are working under the assumption of a pseudostationary process we alter the estimation by removing the ratios \( \bar{\rho}/\rho(x) \) in (29) and (30) since \( \rho(\cdot) \equiv \rho > 0 \).

Part of the motivation for considering a minus sampling estimation scheme here is that it yields (ratio) unbiasedness. The proof of Theorem 3.2 can be found in the Appendix.

**Theorem 3.2.** The estimator (29) is unbiased and (30) is ratio-unbiased in the sense that both its numerator and denominator are unbiased.

### 3.4.1 Intensity estimation
In practice, the true intensity function is unknown so in order to exploit the estimators (29) and (30) we need to estimate the intensity function \( \rho(\cdot) \) in advance and then plug this estimate into (29) and (30). Obtaining good estimates for intensity functions of point processes on linear networks has been a challenging task due to geometrical complexities and unique methodological problems. Nevertheless, there have been a few particularly interesting proposals, including diffusion based kernel estimation (McSwiggan et al.; 2017), an edge-corrected classical kernel-based intensity estimator (Moradi et al.; 2018), resample-smoothed Voronoi estimation (Moradi et al.; 2019) and fast kernel smoothing using two-dimensional convolutions (Rakshit et al.; 2019). Although Moradi et al. (2019) showed that their approach in general generates better intensity estimates than kernel-based approaches, we have observed that (31) generally performs better numerically when it is combined with a kernel estimator. Regarding the associated bandwidth selection, one would expect that the estimator of Cronie and van Lieshout (2018) or Poisson likelihood cross-validation (Baddeley et al.; 2015) would be the best choice. It seems, however, that Scott’s rule of thumb (Scott; 2015; Rakshit et al.; 2019) in general yields more stable outputs of (31). In our numerical evaluations we make use of the fast kernel estimator of Rakshit et al. (2019) which we combine with Scott’s rule of thumb.
4. NUMERICAL EVALUATION

We next numerically evaluate the performance of the estimator of $J_{\text{inhom}}^L(r)$. For this purpose, we simulate point patterns from three different models with different types of spatial interaction – spatial randomness, regularity and clustering. For each model we make use of two linear networks: the network of a Chicago crime dataset and the network of a dataset on spider locations, which are both publicly available in the R package spatstat (Baddeley and Turner; 2005; Baddeley et al.; 2015).

Here we provide some general information about both of the networks:

- The Chicago linear network has 503 segments and 338 nodes, where 44 of them have degree 1, thus forming the boundary of the network. The total length of the network is 31150.21 feet and its maximum node degree is 5. The network is embedded in the window $W = [0.3894, 1281.9863] \times [153.1035, 1276.5602]$.
- The linear network for the data on spider locations is constructed by 156 nodes and 203 segments. It has a total length of 20218.75 millimetres and a maximum node degree of 3. There are 31 nodes with degree 1. The network is embedded in the window $W = [0, 1125]^2$.

For each model, and each network, we first generate one realisation and then estimate the intensity in accordance with the recommendations in Section 3.4.1. We next estimate $J_{\text{inhom}}^L$ based on that realisation, together with pointwise critical envelopes which are computed based on 99 realisations of a Poisson process with the obtained intensity estimate as intensity function. We do so to get an indication of how well our $J$-function estimator can reveal deviations from a Poisson process behaviour and what type of interaction the underlying model possesses. Throughout, following most of the previous literature on analysis of spatial interaction on linear networks, we let $d_L$ be given by the shortest-path distance.

It is important to note that in neither the log-Gaussian Cox process example nor the thinned simple sequential inhibition example below we have actually verified that the models are indeed IRMPS under $d_L$. They are models which, based one our general understanding, should exhibit clustering and inhibition, and we here want to see if our $J$-function estimator manages to capture these expected behaviours.

4.1 Poisson process

We here consider an inhomogeneous Poisson process $X$ with intensity function $\rho(x, y) = 0.005|\sin(x)|$; the expected number of points on the Chicago network is 101.9, and on the spider location network it is 62.3. A single realisation on each network is shown in the top row of Figure 2. The bottom row of Figure 2 shows the inhomogeneous linear $J$-function estimates for each realisation together with the corresponding pointwise critical envelopes based on 99 simulations from a Poisson process with the estimated intensity as intensity function. For each of the networks it can be seen that the $J$-function estimate of the simulated pattern stays around the mean of the $J$-function estimates for the envelope processes, and it entirely remains within the envelope.
4.2 Thinned simple sequential point process

We now consider a scenario where there is inhibition between the points. Initially we generate a realisation of a simple sequential inhibition point process with a total point count of 300 and inhibition distance 0.001$L$; this results in an inhibition distance of 46 feet for the Chicago network, and 30 mm for the spider location network. We then thin each pattern based on the constant retention probability $p(x, y) = 0.3$, $(x, y) \in L$. This results in a thinned simple sequential process with intensity function $\rho(x, y) = 0.3(300/|L|)$, with a total expected number of points of 90; $|L|$ is the total length of the network. The top row of Figure 3 shows two realisations of this process: on the Chicago network in the left panel and on the spider location network in the right panel. The corresponding
estimated inhomogeneous linear $J$-functions for each realisation is displayed on the bottom row of Figure 3. A critical envelope (grey area) based on 99 simulations from a Poisson process with the estimated intensity as intensity function is displayed together with each estimated $J$-functions. We see that it properly identifies an inhibitory behaviour between the points.

It should be noted that the model here in fact is homogeneous so it may be argued that we should instead use the homogeneous estimator where we set $\tilde{\rho}/\rho(x) = 1$ in (29), (30) and (31). However, since we in practice do not actually know whether a point pattern comes from a process which is homogeneous or not, we here want to see how well (31) captures spatial interaction under the (incorrect) assumption that the underlying process is inhomogeneous.

4.3 Log-Gaussian Cox process

In this section we first generate a realisation of a log-Gaussian random field on the window $W = [x_{\text{min}}, x_{\text{max}}] \times [y_{\text{min}}, y_{\text{max}}]$ and then evaluate it only at locations on the network $L \subseteq W$ in question. We then use this extracted realisation to simulate a realisation of an inhomogeneous Poisson process on the network. The driving Gaussian random field on $W$ has mean function $(x_1, y_1) \mapsto \log 0.002 + (x_1 - (x_{\text{max}} - x_{\text{min}}))/|L|$ and covariance function $((x_1, y_1), (x_2, y_2)) \mapsto \exp(-\| (x_1, y_1) - (x_2, y_2) \|)$, $(x_1, y_1), (x_2, y_2) \in W$. Hereby, the intensity is given by $\rho(x, y) = 0.002 \exp([-((x_1 - (x_{\text{max}} - x_{\text{min}}))/|L| + 0.5], (x, y) \in L$. The expected number of points on the Chicago network is 101.1, and for the spiders network it is 64.2. The top row of Figure 4 shows two realisations of such a model on the Chicago network (left) and the spiders network (right). The corresponding estimated inhomogeneous linear $J$-function for each realisation is exhibited on the bottom row of Figure 4. A critical envelope (grey area) based on 99 realisations from a Poisson process with the estimated intensity as intensity functions is displayed together with each estimated $J$-function. From Figure 4 we can see that the $J$-function estimate stays below the envelope for small and moderate interaction ranges $r$, thus indicating a clustering behaviour for the underlying model.

5. DATA ANALYSIS

We next apply the inhomogeneous linear $J$-function estimator to the two real datasets in Figure 1: a) a point pattern of motor vehicle traffic accidents in an area of Houston, US, which was previously studied in Moradi et al. (2019), b) the spider dataset which represents the locations of webs made by spiders in the mortar spaces of a brick wall – this dataset has also previously been studied in Ang et al. (2012). As in the case of the numerical evaluations, we here let $d_L$ be given by the shortest-path distance.

5.1 Houston motor vehicle traffic accidents

The right panel in Figure 1 shows the locations of 249 traffic accidents in an area of Houston, US, during April, 1999. The linear network $L$ has a total length of 708301.7 feet, and has 187 nodes with a maximum node degree of 4, and 253 line segments. For further details, see Levine (2006, 2009) and Moradi et al. (2019). Moradi et al. (2019) studied intensity estimation on this dataset, using their resample-smoothed Voronoi intensity estimator. Figure 5 shows that the
Fig 3: Top row: Realisations of independently thinned simple sequential inhibition processes with the intensity function $\rho(x, y) = 0.3(300/|L|)$; the unthinned processes have inhibition distance 0.001$|L|$; this results in an inhibition distance of 46 feet for the Chicago network (left) and 30 mm for the spider location network (right). The corresponding inhomogeneous linear $J$-function estimates for each realisation, together with pointwise critical envelopes (grey area) based on 99 simulations of inhomogeneous Poisson processes with the estimated intensities of the realisations in the top row as intensities. The solid lines are the estimated $J$-functions for the observed patterns and the dashed lines represent the theoretical linear $J$-function value for Poisson processes. Each $J$-function estimate plot is shown below its corresponding realisation.

estimated inhomogeneous linear $J$-function is almost entirely inside the pointwise critical envelope area which has been computed based on 99 simulations of a Poisson process with intensity given by the estimated intensity (which is obtained in analogy with the numerical evaluation section). Since the estimated $J$-function stays within the envelopes (except at the very end) there seem to be no clear indications of clustering/inhibition.
Fig 4: Top row: Realisations of log Gaussian Cox process models on the Chicago network (left) and on the spiders network (right). Bottom row: The corresponding inhomogeneous linear $J$-functions for each realisation together with pointwise critical envelopes (grey area) based on 99 simulations of inhomogeneous Poisson processes with the estimated intensities of the realisations in the top row as intensities. The solid lines are the estimated $J$-functions for the observed patterns and the dashed lines represent the theoretical linear $J$-function value for Poisson processes. Each $J$-function plot is displayed below its corresponding realisation.

5.2 Spiders data

The left panel in Figure 1 shows the locations of 48 webs of the urban wall spider Oecobius navus on the mortar lines of a brick wall. This dataset was recorded by Voss (1999) and it is stored in the R package spatstat (Baddeley and Turner; 2005; Baddeley et al.; 2015). It has previously been studied by Ang et al. (2012) through second-order summary statistics. The right panel of Figure 5 shows the estimated inhomogeneous linear $J$-function for this dataset together with a pointwise critical envelope based on 99 simulations of a Poisson process with the estimated intensity as intensity function (which is obtained in analogy with the numerical evaluation section). The estimated $J$-function is fully inside the envelope and does not indicate any deviations from being Poisson – this is in keeping with Ang et al. (2012).
6. DISCUSSION

Methods to statistically analyse point patterns on linear networks/graphs have become increasingly important, as the amount of available linear network point process data has had a steady increase in the last couple of years. Besides univariate analyses, which are carried out by finding intensity estimates for the data, higher-order analyses which detect spatial interaction, i.e. clustering or inhibition, are central in the (non-parametric) statistical analysis of linear network point processes.

In Euclidean domains, the most popular tools to carry out analyses of spatial interaction are second-order summary statistics such as inhomogeneous $K$-functions (Baddeley et al.; 2000). However, when the spatial domain is given by a linear network there immediately arise challenges due the spatially varying local geometry of the network. Early proposals of $K$-functions for linear networks did not take this into consideration, which resulted in erroneous spatial interaction estimates. This issue was finally solved by Ang et al. (2012) for the case where the imposed distance/metric on the network was given by the shortest-path distance – a chosen distance is used to define balls which determine whether two points are within interaction range of one another. These ideas were later extended to a broader class of metrics by Rakshit et al. (2017), so-called regular distance metrics, with the argument being that the shortest-path distance need not be the canonical distance for a given set of data on a given network. These $K$-functions are referred to as geometrically corrected $K$-functions.

It may be that second-order summary statistics are insufficient to analyse spatial interaction, because the interactions may be more intricate than pairwise interactions. The inhomogeneous nearest neighbour distance distribution function,
the empty space function and the $J$-function (van Lieshout; 2011) have proven themselves to be powerful higher-order summary statistics which may be used to analyse interaction in spatial point processes in Euclidean domains. Hence, one would hope that these could be extended straightforwardly to the linear network context. However, these summary statistics rely on a form of translation invariance of all the (factorial) moments of the underlying point process, which is referred to as intensity reweighted moment stationarity. To make such an extension possible, one would have to define a family of (transitive) transformations on the network in question but this seems unattainable for general networks. We here find a solution to these issues, which consists of i) proposing a new form of (factorial) moment invariance, which we refer to as intensity reweighted moment pseudostationarity (IRMPS), and ii) defining geometrically corrected versions of the above higher-order summary statistics, based on regular distance metrics. As a by-product, we obtain a definition of (pseudo)stationarity for linear network point processes as well as geometrically corrected summary statistics for such point processes. With these new summary statistics at hand, we proceed by studying some of their properties and defining non-parametric estimators for them, which we show are (ratio)unbiased when the true intensity function is assumed to be known. We finally evaluate the estimators of our summary statistics numerically, based on simulated data, and use them to analyse two sets of actual linear network point pattern data.

We believe that our new tools may be valuable as alternatives/complements to second-order summary statistics such as $K$-functions. Moreover, our proposed ideas open up for a significant amount of future research. E.g., it would be interesting to characterise which classes of models are IRMPS. In addition, extensions to spatio-temporal and marked point processes on linear networks are also very interesting (cf. Cronie and van Lieshout (2015, 2016)), given the growing amount of available datasets.

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Appendices

APPENDIX A: PROOFS

Proof of Theorem 3.1. We start with $H^L\text{inhom}(r; u)$ and note that we only need to show that $K^{L,m}\text{inhom}(r; u)$ does not depend on $u \in L$ for an arbitrary $m$
In the function of Daley and Vere-Jones (2008, equation (11.3.2)), we find that 1
functionals of $X^m$ under thinning since

\[ H_{\text{inhom}}^{L}(r; u) \]

\[ = \int_{b_{L}(u,r)^m} g_{m+1}(u, u_1, \ldots, u_m) \prod_{i=1}^{m} w_{d_{L}}(u, d_{L}(u, u_i)) d_{1} u_{1} \cdots d_{1} u_{m} \]

\[ = \int_{b_{L}(u,r)^m} \bar{g}_{m+1}(d_{L}(u, u), d_{L}(u, u_1), \ldots, d_{L}(u, u_m)) \times \]
\[ \times \prod_{i=1}^{m} w_{d_{L}}(u, d_{L}(u, u_i)) d_{1} u_{1} \cdots d_{1} u_{m} \]

\[ = \int_{0}^{r} \cdots \int_{0}^{r} g_{m+1}(0, t_1, \ldots, t_m) dt_1 \cdots dt_m. \]

Taking the above equality into account and recalling (25) we see that $H_{\text{inhom}}^{L}(r; u)$
do not depend on $u \in L$ for IRMPS point processes. Turning to $F_{\text{inhom}}^{L}(r; u)$, we have that

\[ F_{\text{inhom}}^{L}(r; u) = \]
\[ 1 - E \left[ \prod_{x \in X} \left( 1 - \bar{\rho} \mathbf{1}\{x \in b_{L}(u, r)\} w_{d_{L}}(u, d_{L}(u, x)) \right) \right] \]
\[ = - \sum_{m=1}^{\infty} \frac{(-\bar{\rho})^{m}}{m!} \int_{b_{L}(u,r)^m} g_{m}(u_1, \ldots, u_m) \prod_{i=1}^{m} w_{d_{L}}(u, d_{L}(u, u_i)) d_{1} u_{1} \cdots d_{1} u_{m}, \]
\[ = - \sum_{m=1}^{\infty} \frac{(-\bar{\rho})^{m}}{m!} \int_{b_{L}(u,r)^m} \bar{g}_{m}(d_{L}(u, u_1), \ldots, d_{L}(u, u_m)) \times \]
\[ \times \prod_{i=1}^{m} w_{d_{L}}(u, d_{L}(u, u_i)) d_{1} u_{1} \cdots d_{1} u_{m} \]
\[ = - \sum_{m=1}^{\infty} \frac{(-\bar{\rho})^{m}}{m!} \int_{0}^{r} \cdots \int_{0}^{r} g_{m}(t_1, \ldots, t_m) dt_1 \cdots dt_m. \]

Since both $H_{\text{inhom}}^{L}(r; u)$ and $F_{\text{inhom}}^{L}(r; u)$ do not depend on $u$, we finally conclude
that $J_{\text{inhom}}^{L}(r; u) = J_{\text{inhom}}^{L}(r) = (1 - H_{\text{inhom}}^{L}(r))/(1 - F_{\text{inhom}}^{L}(r))$, following the
steps in the proof of van Lieshout (2011, Theorem 1).

**Proof of Lemma 3.1.** The correlation functions $g_m, m \geq 1$, are invariant
under thinning since $\rho_{th}^{(m)}(u_1, \ldots, u_m) = \rho^{(m)}(u_1, \ldots, u_m) \prod_{i=1}^{m} p(u_i)$ where $\rho_{th}^{(m)}$
$m \geq 1$, are the product densities of $X_{th}$.

Next, we have that $1 - F_{\text{inhom}}^{L}(r)$ and $1 - H_{\text{inhom}}^{L}(r)$ coincide with the generating
functionals of $X$ and the reduced Palm process $X_{th}^{1}$, respectively, when evaluated
in the function $h(x) = 1 - \bar{\rho} \mathbf{1}\{x \in b_{L}(u, r)\} w_{d_{L}}(u, d_{L}(u, x)) / \rho(x)$. Exploiting
Daley and Vere-Jones (2008, equation (11.3.2)), we find that $1 - F_{\text{inhom}}^{L,th}(r)$ and $1 - H_{\text{inhom}}^{L,th}(r)$ are given by the same generating functionals, but instead evaluated in
the function $x \mapsto 1 - p(x) + p(x) h(x)$. Hence, they may alternatively be expressed
as the indicated expectations of products.
**Proof of Theorem 3.2.** We first start with $\hat{F}_{\text{inhom}}^L(r)$ and note that

\[
\mathbb{E}\left[ \hat{F}_{\text{inhom}}^L(r) \right] = 1 - \frac{1}{N(I \cap L_{\oplus r})} \sum_{u \in I \cap L_{\oplus r}} \mathbb{E} \left[ \prod_{x \in X \cap b_L(u,r)} \left( 1 - \frac{\bar{\rho}w_{d_L}(u,d_L(u,x))}{\rho(x)} \right) \right]
\]

\[
= 1 - \frac{1}{N(I \cap L_{\oplus r})} \sum_{u \in I \cap L_{\oplus r}} \left( 1 - F_{\text{inhom}}^L(r; u) \right)
\]

\[
= \frac{1}{N(I \cap L_{\oplus r})} \sum_{u \in I \cap L_{\oplus r}} F_{\text{inhom}}^L(r; u)
\]

by Definition 3.1. By Theorem 3.1 all of the summands in the last sum are equal (a.e.) and this yields that the whole expression equals $F_{\text{inhom}}^L(r)$, since $X$ is IRMPS.

Turning to $\hat{H}_{\text{inhom}}^L(r)$, by (4) and Theorem 3.1 we have

\[
\mathbb{E} \left[ \sum_{u \in X \cap L_{\oplus r}} \prod_{x \in X \setminus \{u\} \cap b_L(u,r)} \left( 1 - \frac{\bar{\rho}}{\rho(x)} w_{d_L}(u,d_L(u,x)) \right) \right] =
\int_{L_{\oplus r}} \mathbb{E}_u \left[ \prod_{x \in X \setminus \{u\} \cap b_L(u,r)} \left( 1 - \frac{\bar{\rho}}{\rho(x)} w_{d_L}(u,d_L(u,x)) \right) \right] \rho(u) \, d_1 u
\]

\[
= \int_{L_{\oplus r}} H_{\text{inhom}}^L(r; u) \rho(u) \, d_1 u = H_{\text{inhom}}^L(r) \int_{L_{\oplus r}} \rho(u) \, d_1 u.
\]

The expectation of the denominator of (30) is $\int_{L_{\oplus r}} \rho(u) \, d_1 u$, which yields the ratio-unbiasedness.

\[\square\]

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