NONLINEAR ELLIPTIC EQUATIONS AND INTRINSIC POTENTIALS OF WOLFF TYPE

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Abstract. We give necessary and sufficient conditions for the existence of weak (locally renormalized) solutions to the model equation

\[-\Delta_p u = \sigma u^q, \quad u > 0, \quad \text{on } \mathbb{R}^n,\]

in the case \(0 < q < p - 1\), where \(\sigma \geq 0\) is an arbitrary locally integrable function, or measure, and \(\Delta_p u = \text{div}(\nabla u |\nabla u|^{p-2})\) is the \(p\)-Laplace operator. Sharp global pointwise estimates and regularity properties of solutions are obtained as well. As a consequence, we characterize the solvability of the equation

\[-\Delta_p v = b \frac{|\nabla v|^p}{v} + \sigma, \quad v > 0, \quad \text{on } \mathbb{R}^n,\]

where \(b > 0\). These results are new even in the classical case \(p = 2\).

Our approach is based on the use of special nonlinear potentials of Wolff type adapted for “sublinear” problems, and related integral inequalities. It allows us to treat simultaneously several problems of this type, such as equations with general quasilinear operators \(\text{div} \mathcal{A}(x, \nabla u)\), fractional Laplacians \((-\Delta)^\alpha\), or fully nonlinear \(k\)-Hessian operators.

1. Introduction

In the present paper, we study elliptic equations of the type

\[
\begin{cases}
-\Delta_p u = \sigma u^q & \text{in } \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) = 0, & u > 0,
\end{cases}
\]

where \(0 < q < p - 1\), \(\Delta_p = \text{div}(\nabla u |\nabla u|^{p-2})\) is the \(p\)-Laplace operator, and \(\sigma \geq 0\) is an arbitrary locally integrable function, or locally finite Borel measure, \(\sigma \in \mathcal{M}^+(\mathbb{R}^n)\); if \(\sigma \in L^1_{\text{loc}}(\mathbb{R}^n)\) we write \(d\sigma = \sigma dx\).

Our main goal is to give necessary and sufficient conditions on \(\sigma\) for the existence of weak solutions to (1.1), understood in an appropriate renormalized sense. We also obtain matching upper and lower global pointwise bounds, and provide sharp \(W^{1,p}_{\text{loc}}\)-estimates of solutions. On our way, we identify key integral inequalities, and construct new nonlinear potentials of Wolff type that are intrinsic to a number of similar problems.

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In particular, our approach is applicable to general quasilinear \( A \)-Laplace operators \( \text{div} \ A(x, \nabla u) \), and fully nonlinear \( k \)-Hessian operators, as well as equations with the fractional Laplacian,

\[
\begin{cases}
(-\Delta)^{\alpha} u = \sigma u^q & \text{in } \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) = 0, & u > 0,
\end{cases}
\]

for \( 0 < q < 1 \) and \( 0 < \alpha < \frac{n}{2} \); this includes the range \( \alpha > 1 \) where the usual maximum principle is not available.

In the classical case \( p = 2 \), equation (1.1), or equivalently (1.2) with \( \alpha = 1 \), and \( 0 < q < 1 \), serves as a model sublinear elliptic problem. It is easy to see that it is equivalent to the integral equation \( u = N(u^q) \), where \( N \omega = (-\Delta)^{-1} \omega \) is the Newtonian potential of \( d \omega = u^q \sigma \) on \( \mathbb{R}^n \).

As we emphasize below, equation (1.1) with \( p = 2 \) and \( 0 < q < 1 \) is governed by the important integral inequality

\[
\left( \int_{\mathbb{R}^n} |\varphi|^q \sigma \right)^{\frac{1}{q}} \leq \kappa \left\| \Delta \varphi \right\|_{L^1(\mathbb{R}^n)},
\]

for all test functions \( \varphi \in C^2(\mathbb{R}^n) \) vanishing at infinity such that \( -\Delta \varphi \geq 0 \).

Inequality (1.3) represents the end-point case of the well-studied \((L^p, L^q)\) trace inequalities for \( p > 1 \). A comprehensive treatment of trace inequalities can be found in [Maz11].

More precisely, we will use a localized version of (1.3) where the measure \( \sigma \) is restricted to a ball \( B = B(x, r) \), and the corresponding best constant \( \kappa \) is denoted by \( \kappa(B) \). These constants are used as building blocks in our key tool, a nonlinear potential of Wolff type,

\[
K \sigma(x) = \int_0^\infty \frac{\kappa(B(x, r)) \frac{r}{n-2}}{r} \, dr, \quad x \in \mathbb{R}^n,
\]

which, together with the usual Newtonian potential \( N \sigma \), provides sharp global estimates of solutions in the case \( p = 2 \) and \( 0 < q < 1 \).

This work has been motivated by the results of Brezis and Kamin [BK92] who proved that (1.1), with \( p = 2 \) and \( 0 < q < 1 \), has a bounded solution \( u \) if and only if \( N \sigma \in L^\infty(\mathbb{R}^n) \); moreover, such a solution is unique, and there exists a constant \( c > 0 \) so that

\[
c^{-1} \left[ N \sigma(x) \right]^\frac{1}{1-q} \leq u(x) \leq c N \sigma(x), \quad x \in \mathbb{R}^n.
\]

As was pointed out in [BK92], both the lower and upper estimates of \( u \) are sharp in a sense. However, there is a substantial gap between them. We will be able to bridge this gap by using both \( N \sigma \) and \( K \sigma \), and extend these results to possibly unbounded solutions, as well as more general nonlinear equations.
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We will be referring to equation (1.1) with $1 < p < \infty$ and $0 < q < p - 1$, as well as other nonlinear problems where analogous phenomena occur in a natural way, as sublinear problems in general. One of the main features that distinguishes them from the “superlinear” case $q \geq p - 1$ is the absence of any smallness assumptions on $\sigma$.

Simultaneously with (1.1), we will be able to investigate the equation with singular natural growth in the gradient term,

\begin{equation}
\begin{cases}
-\Delta_p v = b \frac{\|\nabla v\|^p}{v} + \sigma & \text{in } \mathbb{R}^n, \\
\liminf_{x \to \infty} v(x) = 0, & v > 0,
\end{cases}
\end{equation}

where $\sigma \geq 0$ as above, and $b > 0$ is a constant that can be expressed in terms of $q$ in (1.1),

\begin{equation}
b = \frac{q(p - 1)}{p - 1 - q}, \quad 0 < q < p - 1.
\end{equation}

Equations (1.1) and (1.5) are formally related via the transformation

\begin{equation}
v = \frac{p - 1}{p - 1 - q} u^{\frac{p-1-q}{p-1}}.
\end{equation}

Actually, this relationship fails for some solutions $u$ and $v$ due to the occurrence of certain singular measures, as was first observed by Ferone and Murat [FM00] (see also [GM09]) who studied a similar phenomenon for a related class of equations. In general, a solution $v$ of (1.5) gives rise merely to a supersolution $u$ of (1.1). Nevertheless, a careful analysis allows us to give necessary and sufficient conditions for the existence of weak solutions to (1.5), and justify this transformation whenever possible (see Theorem 1.4 and the discussion in Sec. 7 below).

Equations of the type (1.1) and (1.5) have been extensively studied, mostly in bounded domains $\Omega \subset \mathbb{R}^n$, with $\sigma \in L^r(\Omega)$ for some $r > 1$, in [Kra64, BO86, BoOr96, ABL10, ABV10, AGP11]. Various existence and uniqueness results for solutions in certain Sobolev spaces, and further references, can be found there.

However, the precise conditions on $\sigma$ which ensure the existence of solutions are more subtle. In particular, $\sigma$ can be an $L^{1}_{\text{loc}}$-function, or a measure singular with respect to Lebesgue measure. (Notice that $\sigma$ must be absolutely continuous with respect to the $p$-capacity; see Lemma 3.6 below.) Analogues of our results for bounded domains $\Omega$ under minimal restrictions on $\sigma$ will be presented elsewhere.

We now introduce some elements of nonlinear potential theory that will be used throughout the paper. Originally, Wolff potentials were introduced in [HW83] in relation to the spectral synthesis problem in Sobolev spaces. The Wolff potential $W_{\alpha,p} \sigma$, where $\sigma \in M^+(\mathbb{R}^n)$, is defined, for $1 < p < \infty$...
and \(0 < \alpha < \frac{p}{p-1}\), by

\[
W_{\alpha,p}\sigma(x) = \int_0^\infty \left[ \frac{\sigma(B(x,r))}{r^{n-p}} \right]^\frac{1}{p-\alpha} \frac{r}{r} \, dr, \quad x \in \mathbb{R}^n.
\]

Here \(\sigma(B(x,r)) = \int_{B(x,r)} \sigma \, d\sigma\) for a ball \(B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}\).

In the context of quasilinear problems, Wolff potentials with \(\alpha = 1\) appeared in the fundamental work of Kilpeläinen and Malý [KM92, KM94]. A global version of one of their main theorems states that if \(U \geq 0\) is a solution to the equation

\[
\begin{cases}
-\Delta_p U = \sigma & \text{in } \mathbb{R}^n, \\
\liminf_{x \to \infty} U(x) = 0,
\end{cases}
\]

understood in a potential theoretic or renormalized sense (see [KKT09]), then there exists a constant \(K = K(n,p) > 0\) such that

\[
\frac{1}{K} W_{1,p}\sigma(x) \leq U(x) \leq K W_{1,p}\sigma(x), \quad x \in \mathbb{R}^n.
\]

Moreover, a solution \(U \geq 0\) to (1.9) exists if and only if \(1 < p < n\), and \(W_{1,p}\sigma \not\equiv +\infty\), or equivalently (see [PV08]),

\[
\int_1^\infty \left[ \frac{\sigma(B(0,r))}{r^{n-p}} \right]^\frac{1}{p-1} \frac{r}{r} < +\infty.
\]

It turns out that Wolff potentials alone are not enough to control solutions of (1.1). Along with \(W_{1,p}\sigma\), we will use intrinsic potentials of Wolff type associated with the localized weighted norm inequalities,

\[
\left( \int_B |\varphi|^q \, d\sigma \right)^\frac{1}{q} \leq \varpi(B) \|\Delta_p \varphi\|_{L^1(B)}^{\frac{1}{p-1}},
\]

for all test functions \(\varphi\) such that \(-\Delta_p \varphi \geq 0, \liminf_{x \to \infty} \varphi(x) = 0\). Here \(\varpi(B)\) denotes the best constant in (1.12) associated with the measure \(\sigma_B = \sigma|_B\) restricted to a ball \(B\).

We now introduce a new nonlinear potential \(K\sigma = K_{1,p,q}\sigma\) defined by

\[
K_{1,p,q}\sigma(x) = \int_0^\infty \left[ \frac{\varpi(B(x,r))}{r^{n-p}} \right]^\frac{1}{p-1-q} \frac{r}{r} \, dr, \quad x \in \mathbb{R}^n.
\]

As we will show below, \(K_{1,p,q}\sigma \not\equiv +\infty\) if and only if

\[
\int_1^\infty \left[ \frac{\varpi(B(0,r))}{r^{n-p}} \right]^\frac{1}{p-1-q} \frac{r}{r} < \infty.
\]

We state our main results for equation (1.1) in the form of the following theorems. Note that weak solutions \(u \in L^q_{\text{loc}}(d\sigma)\) are understood in the
renormalized, or potential theoretic sense (see Sec. 2 for definitions); if \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \), then they are the usual distributional solutions.

**Theorem 1.1.** Let \( 1 < p < n \), \( 0 < q < p - 1 \), and let \( \sigma \in M^+(\mathbb{R}^n) \).

(i) If both (1.11) and (1.14) hold, then there exists a minimal renormalized (\( p \)-superharmonic) solution \( u > 0 \) to (1.1) such that

\[
(1.15) \quad c^{-1} [K_{1,p,q} + (W_{1,p} \sigma)^{\frac{p-1}{p-1-q}}] \leq u \leq c [K_{1,p,q} + (W_{1,p} \sigma)^{\frac{p-1}{p-1-q}}],
\]

where \( c > 0 \) is a constant which depends only on \( p, q \), and \( n \).

(ii) Conversely, if there exists a nontrivial renormalized supersolution \( u \) to (1.1), then both (1.11) and (1.14) hold, and \( u \) is bounded below by the minimal solution of statement (i).

(iii) In the case \( p \geq n \) there are no nontrivial supersolutions on \( \mathbb{R}^n \).

We observe that neither of conditions (1.11) or (1.14) implies the other one. Condition (1.11) alone is not enough to ensure the existence of a global solution \( u \) even if all the local embedding constants \( \kappa(B(0, r)) \) in (1.14) are finite, unless \( \sigma \) is radially symmetric (see a counter example in Sec. 6 below).

In the next theorem, we characterize solutions with \( W^{1,p}_{\text{loc}} \)-regularity.

**Theorem 1.2.** Under the assumptions of Theorem 1.1, there exists a nontrivial distributional solution \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) to (1.1) if and only if (1.11) and (1.14) hold together with the local condition

\[
(1.16) \quad \int_B (W_{1,p} \sigma_B)^{(1+q)(p-1)} d\sigma < \infty,
\]

for all balls \( B = B(0, r) \) in \( \mathbb{R}^n \).

We remark that conditions (1.11), (1.14) and (1.16) are mutually independent.

Nonlinear elliptic PDE discussed above are studied in the general framework of nonlinear integral equations,

\[
(1.17) \quad u = W_{\alpha,p}(u^q d\sigma) \quad \text{in} \ \mathbb{R}^n, \quad u > 0,
\]

where \( 1 < p < \infty \), \( 0 < \alpha < \frac{n}{p} \). Here the special case \( \alpha = 1 \) corresponds to the \( p \)-Laplacian, whereas \( \alpha = \frac{2k}{k+1}, p = k+1 \) to the \( k \)-Hessian operator (see [TW99], [PV08]).

The special case \( p = 2 \) in (1.17) gives the fractional Laplace equation (1.2) in the equivalent integral form \( u = \frac{1}{c(\alpha,n)} I_{2\alpha}(u^q d\sigma) \), where \( I_{2\alpha} \mu \) is the Riesz potential of order \( 2\alpha \):

\[
I_{2\alpha} \mu = |x|^{2\alpha-n} * \mu = (n - 2\alpha) W_{\alpha,2\mu} = c(\alpha,n) (-\Delta)^{-\alpha} \mu,
\]

for \( \mu \in M^+(\mathbb{R}^n), 0 < 2\alpha < n \). In what follows, the normalization constant \( c(\alpha,n) \) will be dropped for the sake of convenience; in particular, \( I_{2\mu} = N_\mu \).
We will introduce in Sec. 4 a fractional version $K_{\alpha,p,q}$ of the intrinsic potential (1.13) for all $p > 1$, $0 < q < p-1$, $0 < \alpha p < n$, and in Sec. 5 deduce analogues of Theorem 1.1 and Theorem 1.2 for the $A$-Laplacians, $k$-Hessians and fractional Laplacians as a consequence of the general Theorem 4.8.

In particular, for the fractional Laplacian equation (1.2), let $\kappa(B)$ denote the least constant in the localized integral inequality
\begin{equation}
\|I_{2\alpha}\nu\|_{L^{q}(d\nu)} \leq \kappa(B) \nu(\mathbb{R}^{n}), \quad \forall \nu \in M^{+}(\mathbb{R}^{n}),
\end{equation}
where $0 < q < 1$. It is easy to see that the constant $\kappa(B)$ does not change if we restrict ourselves to absolutely continuous $\nu \in L_{1}^{+}(\mathbb{R}^{n})$.

We define the corresponding nonlinear potential of Wolff type by
\begin{equation}
K_{\alpha,2,q}\sigma(x) = \int_{0}^{\infty} \frac{[\kappa(B(x,r))]^{1/q} dr}{r}, \quad x \in \mathbb{R}^{n}.
\end{equation}
Conditions (1.11), (1.14) need to be replaced with
\begin{equation}
\int_{1}^{\infty} \frac{[\kappa(B(0,r))]^{1/q} dr}{r} + \int_{1}^{\infty} \frac{\sigma(B(0,r)) dr}{r} < \infty,
\end{equation}
which ensures that both $K_{\alpha,2,q}\sigma$ and $I_{2\alpha}\sigma$ are not identically infinite.

We state our main results for sublinear fractional Laplacian equations as follows.

**Theorem 1.3.** Let $0 < \alpha < \frac{n}{2}$, $0 < q < 1$, and $\sigma \in M^{+}(\mathbb{R}^{n})$.

(i) Suppose that (1.20) holds. Then there exists a minimal solution $u > 0$ to (1.2) such that $\liminf_{x \to \infty} u(x) = 0$, and
\begin{equation}
c^{-1} \left[ K_{\alpha,2,q}\sigma + (I_{2\alpha}\sigma)^{1-\eta} \right] \leq u \leq c \left[ K_{\alpha,2,q}\sigma + (I_{2\alpha}\sigma)^{1-\eta} \right],
\end{equation}
where $c > 0$ is a constant which depends only on $\alpha$, $q$, and $n$.

(ii) Conversely, if there exists a nontrivial supersolution $u$ to (1.2) then (1.20) holds, and $u$ satisfies the lower bound in (1.21).

It is worth observing that condition (1.20) characterizes the existence of $0 < u < \infty$ $d\sigma$-a.e. such that $u \geq I_{2\alpha}(u^{q}d\sigma)$ $d\sigma$-a.e. on $\mathbb{R}^{n}$, for $0 < q < 1$, which can be regarded as a sublinear version of Schur’s Lemma in this case.

We next turn to equation (1.5) treated via relation (1.7). The following theorem demonstrates that conditions (1.11) and (1.14) are necessary and sufficient for the solvability of this equation as well. In particular, if $u$ is a solution to (1.1) then $v$ is a solution to (1.5). The opposite implication fails to be true since $u$ is only a supersolution to (1.1). In order that $u$ be a genuine solution, one needs to impose extra restrictions on $v$ specified in statement (iii) of Theorem 1.4. These restrictions are sharp as is evident from simple examples (see details in Sec. 7).

**Theorem 1.4.** Let $1 < p < \infty$ and $0 < q < p-1$. Suppose $b > 0$ is defined by (1.6), and $\sigma \in M^{+}(\mathbb{R}^{n})$. 


(i) If \( u \) is a renormalized solution to (1.1) then \( v \) defined by (1.7) is a renormalized solution to (1.5). Consequently, if both (1.11) and (1.14) hold, then (1.5) has a renormalized solution \( v \) which satisfies both the lower bound

\[
 v \geq c^{-1} \left[ (K_{1,p,q}\sigma)^{\frac{p-1-q}{p-1}} + W_{1,p}\sigma \right],
\]

and the upper bound

\[
 v \leq c \left[ (K_{1,p,q}\sigma)^{\frac{p-1-q}{p-1}} + W_{1,p}\sigma \right],
\]

where \( c > 0 \) depends only on \( p, q, \) and \( n \).

(ii) If (1.5) has a renormalized solution \( v > 0 \), then for every ball \( B \) and \( w_B = \frac{|Dv|^p}{v} \chi_B \), we have

\[
 ||v||_{L^\frac{q(p-1)}{p-q} \sigma} < \infty,
\]

Moreover, \( v \) satisfies the lower bound (1.22), and \( u \) defined by (1.7) is a renormalized supersolution to (1.1); consequently, both (1.11) and (1.14) hold.

(iii) Furthermore, if \( v \) satisfies a strong-type version of (1.24),

\[
 ||v||_{L^\frac{q(p-1)}{p-q} \sigma} < \infty,
\]

for every ball \( B \), then \( u \) is actually a renormalized solution to (1.1).

In conclusion, we remark that we have stated our results for minimal “ground state” solutions which vanish at infinity, but they have obvious analogues for solutions such that \( \liminf_{x \to \infty} u = c \) where \( c > 0 \), as discussed in [BK92] in the case \( p = 2 \).

The brief contents of the paper are as follows. In Sec. 2 we introduce basic definitions, notations, and preliminary results concerning quasilinear equations and nonlinear potentials. In Sec. 3, we obtain some useful estimates involving Wolff potentials \( W_{\alpha,p}\sigma \). The corresponding nonlinear integral equations (1.17) and properties of the intrinsic Wolff potentials \( K_{\alpha,p,q}\sigma \) are studied in Sec. 4. In Sec. 5, we prove our main theorems regarding equation (1.1), and discuss briefly more general quasilinear and fully nonlinear equations. In Sec. 6, we give a counter example which demonstrates that merely the finiteness of the embedding constants \( \varkappa(\sigma_B) \) is not enough for the existence of a global solution to (1.1), even if \( W_{1,p}\sigma < \infty \) a.e. Sec. 7 is devoted to equations with singular gradient terms (1.5).

2. Preliminaries

Let \( \Omega \) be an open set in \( \mathbb{R}^n \), we denote by \( M^+(\Omega) \) the class of all nonnegative locally finite Borel measures on \( \Omega \). We denote the \( \sigma \)-measure of a measurable set \( E \subset \Omega \) by \( \sigma(E) = |E|_\sigma = \int_E d\sigma \).

For \( p > 0 \) and \( \sigma \in M^+(\Omega) \), we denote by \( L^p(\Omega, d\sigma) \) (\( L^p_{\text{loc}}(\Omega, d\sigma) \), respectively) the space of measurable functions \( f \) such that \( |f|^p \) is integrable.
(locally integrable) with respect to \( \sigma \). For \( f \in L^p(\Omega, d\sigma) \), we set
\[
||f||_{L^p(\Omega, d\sigma)} = \left( \int_{\Omega} |f|^p \, d\sigma \right)^{\frac{1}{p}}.
\]
When \( d\sigma = dx \), we write \( L^p(\Omega) \) (respectively \( L^p_{\text{loc}}(\Omega) \)), and denote Lebesgue measure of \( E \subseteq \mathbb{R}^n \) by \( |E| \).

The Sobolev space \( W^{1,p}(\Omega) \) (\( W^{1,p}_{\text{loc}}(\Omega) \), respectively) is the space of all functions \( u \) such that \( u \in L^p(\Omega) \) and \( |\nabla u| \in L^p(\Omega) \) (\( u \in L^p_{\text{loc}}(\Omega) \) and \( |\nabla u| \in L^p_{\text{loc}}(\Omega) \), respectively).

Let \( L^1_{0,p}(\Omega) \) (\( 1 < p < n \)) denote the homogeneous Sobolev (Dirichlet) space, i.e., the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( ||u||_{1,p} = ||\nabla u||_{L^p(\Omega)} \) (see, e.g., [MZ97], Sec. 1.3.4).

The dual Sobolev space \( L^{-1,p'(\Omega)} = L^1_{-1,p'}(\Omega)^* \) is the space of distributions \( \nu \in D'(\Omega) \) such that
\[
||\nu||_{-1,p'} = \sup \frac{|\langle u, \nu \rangle|}{||u||_{1,p}} < +\infty,
\]
where the supremum is taken over all \( u \in L^1_{0,p}(\Omega), u \neq 0 \).

We will need Wolff's inequality [HW83] (see also [AH96], Sec. 4.5) in the case \( \Omega = \mathbb{R}^n \) for \( \nu \in M^+(\mathbb{R}^n) \):
\[
(2.1) \quad c^{-1} ||\nu||_{-1,p'}^p \leq \int_{\mathbb{R}^n} \mathbf{W}_{1,p} \nu \, d\nu \leq c ||\nu||_{-1,p'}^p,
\]
where \( 1 < p < n \), and \( c \) is a positive constant which depends only on \( n \) and \( p \). There is a local version of Wolff’s inequality (see [AH96], Theorem 4.5.5):
\[
(2.2) \quad \nu \in M^+(\mathbb{R}^n) \cap W_{\text{loc}}^{-1,p'}(\mathbb{R}^n) \iff \int_B \mathbf{W}_{1,p} \nu_B \, d\nu_B < \infty, \quad \text{for all balls } B,
\]
where \( B = B(x, R) \), and \( \nu_B = \nu|_B \).

For \( u \in W^{1,p}_{\text{loc}}(\Omega) \), we define the \( p \)-Laplacian \( \Delta_p \) (\( 1 < p < \infty \)) in the distributional sense, i.e., for every \( \varphi \in C_0^\infty(\Omega) \),
\[
(2.3) \quad \langle \Delta_p u, \varphi \rangle = \langle \text{div}(|\nabla u|^{p-2} \nabla u, \varphi) \rangle = -\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx.
\]

We will extend the usual distributional definition of solutions \( u \) of \( -\Delta_p u = \mu \), where \( \mu \in W_{\text{loc}}^{-1,p'}(\Omega) \), to \( u \) not necessarily in \( W^{1,p}_{\text{loc}}(\Omega) \). We will understand solutions in the following potential-theoretic sense using \( p \)-super-harmonic functions, which is equivalent to the notion of locally renormalized solutions in terms of test functions (see [KKT09]).

A function \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is called \( p \)-harmonic if it satisfies the homogeneous equation \( \Delta_p u = 0 \). Every \( p \)-harmonic function has a continuous representative which coincides with \( u \) a.e. (see [HKM06]). Then \( p \)-super-harmonic functions are defined via a comparison principle. We say that \( u : \Omega \to (-\infty, \infty) \) is \( p \)-superharmonic if \( u \) is lower semicontinuous, is
not identically infinite in any component of \( \Omega \), and, whenever \( D \subseteq \Omega \) and \( h \in C(\overline{D}) \) is \( p \)-harmonic in \( D \) with \( h \leq u \) on \( \partial D \), then \( h \leq u \) in \( D \).

A \( p \)-superharmonic function \( u \) does not necessarily belong to \( W^{1,p}_{\text{loc}}(\Omega) \), but its truncations \( T_k(u) = \min(k, \max(u, -k)) \) do, for all \( k > 0 \). In addition, \( T_k(u) \) are supersolutions, i.e., \( -\text{div}(\nabla T_k(u)|^{|p-2}\nabla T_k(u)) \geq 0 \), in the distributional sense. The generalized gradient of a \( p \)-superharmonic function \( u \) defined by [HKM06]:

\[
Du = \lim_{k \to \infty} \nabla(T_k(u)).
\]

We note that every \( p \)-superharmonic function \( u \) has a quasicontinuous representative which coincides with \( u \) quasieverywhere (q.e.), i.e., everywhere except for a set of \( p \)-capacity zero (see [HKM06]). Here the \( p \)-capacity is defined, for compact sets \( E \subseteq \mathbb{R}^n \), by

\[
\text{cap}_p(E) = \inf \left\{ \|\nabla u\|_{L^p(\mathbb{R}^n)}^{p-1} : u \geq 1 \text{ on } E, \ u \in C_0^\infty(\mathbb{R}^n) \right\}.
\]

We will assume that \( u \) is always chosen to be quasicontinuous.

Let \( u \) be \( p \)-superharmonic, and let \( 1 \leq r < \frac{n}{n-1} \). Then \( |Du|^{p-1} \), and consequently \( |Du|^p \), belongs to \( L^r_{\text{loc}}(\Omega) \) [KM92]. This allows us to define a nonnegative distribution \( -\Delta_p u \) for each \( p \)-superharmonic function \( u \) by

\[
-\langle \Delta_p u, \varphi \rangle = \int_\Omega |Du|^{p-2}Du \cdot \nabla \varphi \, dx,
\]

for all \( \varphi \in C_0^\infty(\Omega) \). Then by the Riesz representation theorem there exists a unique measure \( \mu[u] \in M^+(\Omega) \) so that \( -\Delta_p u = \mu[u] \), where \( \mu[u] \) is called the Riesz measure of \( u \).

For \( \omega \in M^+(\Omega) \), consider the equation

\[
-\Delta_p u = \omega \quad \text{in } \Omega.
\]

Solutions to such equations with measure data are generally understood in the potential-theoretic sense (see [KM92], [KM94], [Kil02]).

**Definition 2.1.** For \( \omega \in M^+(\Omega) \), \( u \) is said to be a \((p\text{-superharmonic})\) solution to the equation

\[
-\Delta_p u = \omega \quad \text{in } \Omega
\]

if \( u \) is \( p \)-superharmonic in \( \Omega \), and \( \mu[u] = \omega \).

Thus, if \( \sigma \in M^+(\mathbb{R}^n) \), then \( u \geq 0 \) is a solution to the equation

\[
-\Delta_p u = \sigma u^q \quad \text{in } \Omega
\]

if \( u \) is \( p \)-superharmonic in \( \Omega \), \( u \in L^q_{\text{loc}}(\Omega, d\sigma) \), and \( d\mu[u] = u^q \, d\sigma \).

Alternatively, we will use the framework of locally renormalized solutions. This notion introduced by Bidaut-Véron [BiVe03], following the development of the theory of renormalized solutions in [DMM99], is well suited for our purposes. As was shown recently in [KKT09], for \( \omega \in M^+(\Omega) \) it coincides with the notion of a \( p \)-superharmonic solution in Definition 2.1.
In particular, a $p$-superharmonic function $u \geq 0$ satisfying (2.6) is a locally renormalized solution defined in terms of test functions (see [KKT09], Theorem 3.15). This means that, for all $\varphi \in C_0^\infty(\Omega)$ and $h \in W^{1,\infty}(\Omega)$ with $h'$ having compact support, we have
\begin{equation}
\int_\Omega |Du|^p h'(u) \varphi \, dx + \int_\Omega |Du|^{p-2} Du \cdot \nabla \varphi \, h(u) \, dx = \int_{\mathbb{R}^n} h(u) \varphi \, d\omega.
\end{equation}
The converse is also true, i.e., if $u$ is a locally renormalized solution to (2.6), then there exists a $p$-superharmonic representative $\tilde{u} = u$ a.e.

We will call such solutions of (2.8) with $d\omega = u^qd\sigma$ (locally) renormalized, $p$-superharmonic, or simply solutions, of (2.7).

**Definition 2.2.** A function $u \geq 0$ is called a (renormalized) supersolution to (2.7) if $u$ is $p$-superharmonic in $\Omega$, $u \in L^q_{\text{loc}}(\Omega, d\sigma)$, and
\begin{equation}
\int_\Omega |Du|^{p-2} Du \cdot \nabla \varphi \, dx \geq \int_\Omega u^q \varphi \, d\sigma, \quad \forall \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0.
\end{equation}

As we will show below, supersolutions to (1.1) in the sense of Definition 2.2 are closely related to supersolutions associated with the integral equation (1.17), i.e., $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ such that
\begin{equation}
\mu \geq W_{\alpha,p}(u^q d\sigma) \quad d\sigma\text{-a.e.},
\end{equation}
in the case $\alpha = 1$. The following weak continuity result will be used to prove the existence of $p$-superharmonic solutions to quasilinear equations.

**Theorem 2.3 ([TW02]).** Suppose $\{u_j\}$ are nonnegative $p$-superharmonic functions in an open set $\Omega$ such that $u_j \to u$ a.e., where $u$ is $p$-superharmonic in $\Omega$. Then $\mu[u_j]$ converges weakly to $\mu[u]$, i.e., for all $\varphi \in C_0^\infty(\Omega)$,
\begin{equation}
\lim_{j \to \infty} \int_\Omega \varphi \, d\mu[u_j] = \int_\Omega \varphi \, d\mu[u].
\end{equation}

The next theorem is concerned with pointwise estimates of nonnegative $p$-superharmonic functions in terms of Wolff potentials.

**Theorem 2.4 ([KM94]).** Let $1 < p < \infty$, and let $u$ be a $p$-superharmonic function in $\mathbb{R}^n$ with $\liminf_{x \to \infty} u = 0$.
(i) If $p < n$ and $\omega = \mu[u]$, then
\begin{equation}
\frac{1}{K} W_{1,\rho^p}(\omega(x)) \leq u(x) \leq K W_{1,\rho^p}(\omega), \quad x \in \mathbb{R}^n,
\end{equation}
where $K$ is a positive constant depending only on $n$ and $p$.
(ii) In the case $p \geq n$, it follows that $u \equiv 0$.

3. **Wolff potential estimates**

We start with some useful estimates for Wolff potentials. Throughout this paper we will assume that $\sigma \in M^+(\mathbb{R}^n)$, i.e., $\sigma$ is a locally finite Borel measure on $\mathbb{R}^n$, and $\sigma \neq 0$. 
Lemma 3.1. Suppose $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $\sigma \in M^+(\mathbb{R}^n)$. Let $s = \min \{1, p-1\}$. Then there exists a positive constant $c$ which depends only on $n$, $p$, and $\alpha$ such that, for all $x \in \mathbb{R}^n$ and $R > 0$,

$$c^{-1} \int_0^R \left( \frac{\sigma(B(x, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \leq \inf_{B(x, R)} W_{\alpha, p} \sigma \leq \left( \frac{1}{|B(x, R)|} \int_{B(x, R)} |W_{\alpha, p} \sigma(y)|^s \, dy \right)^{\frac{1}{s}}$$

\hspace{5cm} (3.1)

$$\leq c \int_0^\infty \left( \frac{\sigma(B(x, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}$$

Proof. Without loss of generality we can assume that $x = 0$. We first prove the last estimate in (3.1). Clearly,

$$\frac{1}{|B(0, R)|} \int_{B(0, R)} |W_{\alpha, p} \sigma(y)|^s \, dy \leq I_1 + I_2,$$

where

$$I_1 = \frac{1}{|B(0, R)|} \int_{B(0, R)} \left( \int_0^R \left( \frac{\sigma(B(y, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s \, dy,$$

$$I_2 = \frac{1}{|B(0, R)|} \int_{B(0, R)} \left( \int_0^\infty \left( \frac{\sigma(B(y, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s \, dy.$$

To estimate $I_2$, notice that since $B(y, r) \subset B(0, 2r)$ for $y \in B(0, R)$ and $r > R$, it follows

$$I_2 \leq \left( \int_0^\infty \left( \frac{\sigma(B(0, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s.$$

To estimate $I_1$, suppose first that $p \geq 2$ so that $s = 1$. Then using Fubini’s theorem and Jensen’s inequality we deduce

$$I_1 \leq \int_0^R \left( \frac{1}{|B(0, R)|} \int_{B(0, R)} \sigma(B(y, r)) \, dy \right)^{\frac{1}{p-1}} \frac{dr}{r^{\frac{n-\alpha p}{p-1}+1}}.$$

Using Fubini’s theorem again, we obtain

$$\int_{B(0, R)} \sigma(B(y, r)) \, dy \leq \int_{B(0, 2R)} |B(y, r)| \, d\sigma = |B(0, 1)| \, r^n \sigma(B(0, 2R)).$$

Hence, there is a constant $c = c(n, p, \alpha)$ such that

$$I_1 \leq cR^{-\frac{n}{p-1}} \sigma(B(0, 2R))^\frac{1}{p-1} \int_0^R \frac{\sigma(B(0, 2r))^{\frac{1}{p-1}} \, dr}{r^{\frac{n-\alpha p}{p-1}+1}} \leq c \int_0^\infty \left( \frac{\sigma(B(0, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}.$$
Notice that this is the same estimate we have deduced above for $I_2$ with $s = 1$.

Let us now estimate $I_1$ for $1 < p < 2$ and $s = p - 1$. In this case, we will use the following elementary inequality: for every $R > 0$,
\[
\left( \int_0^R \left( \frac{\phi(r)}{r^\gamma} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{p-1} \leq c(p, \gamma) \int_0^{2R} \phi(r) \frac{dr}{r^\gamma},
\]
where $\gamma > 0$, $1 < p < 2$, and $\phi$ is a non-decreasing function on $(0, \infty)$.

Applying the preceding inequality with $\phi(r) = \sigma(B(0, 2r))$ and $\gamma = n - \alpha p$, and estimating as in the case $p \geq 2$, using Fubini’s theorem again, we obtain:
\[
I_1 \leq \frac{c}{|B(0, R)|} \int_{B(0, R)} \int_0^{2R} \frac{\sigma(B(y, r))}{r^{n-\alpha p}} \frac{dr}{r} dy
\]
\[
\leq c R^{-n} \sigma(B(0, 2R)) \int_0^{2R} r^{\alpha p - 1} dr = c R^{-n+\alpha p} \sigma(B(0, 2R))
\]
\[
\leq c \left( \int_{R}^{\infty} \left( \frac{\sigma(B(0, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{p-1},
\]
where $c$ denotes different constants depending only on $n$, $p$, $\alpha$. Combining the estimates for $I_1$ and $I_2$, we arrive at
\[
\frac{1}{|B(0, R)|} \int_{B(0, R)} (W_{\alpha, p, \sigma})^s dy \leq c \left( \int_{R}^{\infty} \left( \frac{\sigma(B(0, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^{s}.
\]
Making the substitution $\rho = 2r$ in the integral on the right-hand side completes the proof of the upper estimate in (3.1).

To prove the lower estimate, notice that
\[
W_{\alpha, p, \sigma}(y) \geq \int_{2R}^{\infty} \left( \frac{\sigma(B(y, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = c \int_{R}^{\infty} \left( \frac{\sigma(B(y, 2\rho))}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}.
\]
Since $B(y, 2\rho) \supset B(0, \rho)$ for $y \in B(0, R)$ and $\rho > R$, there exists $c = c(n, p, \alpha) > 0$ such that
\[
\inf_{B(0, R)} W_{\alpha, p, \sigma} \geq c \int_{R}^{\infty} \left( \frac{\sigma(B(0, \rho))}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}.
\]

\[\square\]

**Corollary 3.2.** Suppose $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $\sigma \in M^+(\mathbb{R}^n)$.

(i) $W_{\alpha, p, \sigma} \not\equiv +\infty$ if and only if
\[
(3.2) \quad \int_{1}^{\infty} \left( \frac{\sigma(B(0, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.
\]

(ii) Condition (3.2) implies
\[
(3.3) \quad \int_{t}^{\infty} \left( \frac{\sigma(B(x, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty, \quad \forall x \in \mathbb{R}^n, \ t > 0.
\]
(iii) If (3.2) holds, then $W_{\alpha,p}\sigma \in L^s_{\text{loc}}(dx)$, where $s = \min(1,p-1)$, and

\begin{equation}
\liminf_{|x| \to \infty} W_{\alpha,p}\sigma(x) = 0.
\end{equation}

Proof. We first verify statement (ii). Suppose (3.2) holds. We may assume $x \neq 0$, since for $x = 0$ (3.3) is obvious. Clearly, $B(x,r) \subset B(0,2r)$ for $|x| < r$, and hence,

$$I_x := \int_{|x|}^{\infty} \left( \frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \leq \int_{|x|}^{\infty} \left( \frac{\sigma(B(0,2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

It follows that (3.3) holds for $t \geq |x|$. If $t < |x|$, then

$$\int_{t}^{\infty} \left( \frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} = \int_{|x|}^{|x|} \left( \frac{\sigma(B(x,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} + I_x < \infty,$$

since in the first integral $B(x,r) \subset B(0,2|x|)$. Thus, (3.3) holds for all $x$ and $t > 0$.

It remains to prove (3.4), since the other statements of Corollary 3.2 are immediate from (3.1) and (3.3). Suppose that (3.2) holds. For $R > 0$, let $A_R = \{ \frac{R}{2} < |x| < R \}$. Then by the upper estimate of Lemma 3.1 (with $x = 0$),

$$\inf_{|x| > R/2} W_{\alpha,p}\sigma(x) \leq \inf_{A_R} W_{\alpha,p}\sigma(x) \leq \left( \frac{1}{|A_R|} \int_{A_R} (W_{\alpha,p}\sigma)^s \ dx \right)^{\frac{1}{s}}$$

$$\leq c \int_{R}^{\infty} \left( \frac{\sigma(B(0,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r},$$

where $c$ does not depend on $R$. Since the right-hand side of the preceding inequality tends to zero as $R \to \infty$, we see that (3.4) holds. \hfill \Box

It is easy to see that if $\omega \in M^+(\mathbb{R}^n)$, and $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ is a weak solution to the equation $-\Delta_p u = \omega$, then $\omega \in W^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$. The converse statement is less obvious, and we were not able to find it in the literature. In the next lemma, for the sake of completeness, we give a proof in the case $\omega \geq 0$ using a series of Caccioppoli-type inequalities.

Lemma 3.3. Suppose $1 < p < n$, and $\omega \in M^+(\mathbb{R}^n) \cap W^{-1,p'}_{\text{loc}}(\mathbb{R}^n)$. If $u \geq 0$ is a $p$-superharmonic solution to the equation $-\Delta_p u = \omega$ in $\mathbb{R}^n$ such that $\liminf_{x \to \infty} u = 0$, then $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^1_{\text{loc}}(\mathbb{R}^n, d\omega)$.

Proof. Let us first show that $u \in L^1_{\text{loc}}(\mathbb{R}^n, d\omega)$ using Wolff’s inequality [HW83]. Fix a ball $B = B(0,R)$, $R > 0$. By Theorem 2.4, $u$ satisfies
the Wolff potential estimate (2.11). Hence,
\[ \int_B u \, d\omega \leq K \int_B \left( \int_0^R \frac{\omega(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \, d\omega(x) \]
\[ + K \int_B \int_R^\infty \left( \frac{\omega(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \, d\omega(x) := I + II. \]
Since \( B(x,r) \subset 2B = B(0,2R) \) for \( x \in B \) and \( r < R \), we obtain by (2.2),
\[ I \leq K \int_B \int_0^R \left( \frac{\omega(B(x,r) \cap 2B)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \, d\omega(x) \]
\[ \leq K \int_{\mathbb{R}^n} W_{1,p} \omega_{2B} \, d\omega_{2B} < \infty. \]

To estimate \( II \), notice that \( B(x,r) \subset B(0,2r) \), for \( r > R \) and \( x \in B \). Hence,
\[ II \leq K \omega(B) \int_R^\infty \left( \frac{\omega(B(0,2r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty \]
by Corollary 3.2.

We next show that \( u \in L^s_{\text{loc}}(\mathbb{R}^n, dx) \) for \( 0 < s \leq \frac{np}{n-p} \). Arguing as above, we use (2.11) and split the integral with respect to \( dr/r \) into two parts:
\[ \int_B u^s \, dx \leq c_s K^s \int_B \left( \int_0^R \left( \frac{\omega(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s \, dx \]
\[ + c_s K^s \int_B \left( \int_R^\infty \left( \frac{\omega(B(x,r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s \, dx := III + IV, \]
where \( c \) is a constant depending only on \( s \).

To estimate \( III \), notice that by (2.1) \( \omega_{2B} \in L^{-1,p'}(\mathbb{R}^n) \), and consequently there is a unique solution \( u_{2B} \in L^1_{\text{loc}}(\mathbb{R}^n) \) to the equation \(-\Delta u_{2B} = \omega_{2B} \) in \( \mathbb{R}^n \). Hence, by the Sobolev inequality, \( u_{2B} \in L^s_{\text{loc}}(\mathbb{R}^n) \) for \( 0 < s \leq \frac{np}{n-p} \). Clearly, \( u_{2B} \) is \( p \)-superharmonic, and satisfies (2.11) with \( \omega_{2B} \) in place of \( \omega \), i.e.,
\[ \int_0^\infty \left( \frac{\omega(B(x,r) \cap 2B)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \leq K u_{2B}(x). \]
Since \( B(x,r) \subset 2B \) for \( x \in B \) and \( r < R \), we estimate
\[ III \leq c \int_B \left( \int_0^R \left( \frac{\omega(B(x,r) \cap 2B)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s \, dx \leq c \int_B u_{2B}^s \, dx < \infty. \]

The estimate of \( IV \) is similar to that of \( II \):
\[ IV \leq c_s K |B| \left( \int_R^\infty \left( \frac{\omega(B(0,2r))}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right)^s < \infty \]
by Corollary 3.2. Thus, \( u \in L^s_{\text{loc}}(\mathbb{R}^n, dx) \) for \( s \leq \frac{np}{n-p} \).
We next show that there exists $0 < \beta \leq 1$ such that, for all balls $B$,

\begin{equation}
\int_B |Du|^p u^{\beta - 1} \, dx < \infty.
\end{equation}

Indeed, since $u$ is $p$-superharmonic, it is a locally renormalized solution to $-\Delta_p u = \omega$ as discussed in Sec. 2. Let $u_k = \min(u, k)$, where $k > 0$. Note that $u$, and hence $u_k$, is locally bounded below. Using $h(u) = u_k^\beta$ ($0 < \beta \leq 1$) in (2.8), and a cut-off function $\varphi \in C_0^\infty(B)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $\frac{1}{2} B$, we obtain

\begin{equation}
\int_{u \leq k} |Du|^p u^{\beta - 1} \varphi \, dx + \int_{\mathbb{R}^n} |Du|^{p - 2} Du \cdot \nabla \varphi u_k^\beta \, dx = \int_B u_k^\beta \varphi \, d\omega.
\end{equation}

As was shown above, $u \in L^1_{\text{loc}}(\mathbb{R}^n, d\omega)$, and hence the right-hand side is bounded by

\begin{equation}
\int_B u^\beta \varphi \, d\omega \leq \omega(B)^{1 - \beta} \left( \int_B u \, d\omega \right)^\beta < \infty,
\end{equation}

for $0 < \beta \leq 1$.

Since $u$ is $p$-superharmonic, we have $|Du| \in L^{r'(p - 1)}$ for $r' < \frac{n}{n - 1}$. By Hölder’s inequality with exponents $r'$ and $r > n$, we deduce from (3.6),

\begin{equation}
\int_{u \leq k} |Du|^p u^{\beta - 1} \varphi \, dx \leq c \left| |Du|^{p - 1} L^{r'(p - 1)}(B) \right| u^\beta L^{r}(B, dx)
\end{equation}

\begin{equation}
+ \omega(B)^{1 - \beta} \left( \int_B u \, d\omega \right)^\beta.
\end{equation}

If $\beta r = s \leq \frac{np}{n - p}$, where $r > n$ and $\beta \leq 1$, then the right-hand side of the preceding inequality is finite. Picking $r$ so that $r > n$ and is arbitrarily close to $n$, and passing to the limit as $k \to \infty$, we obtain (3.5) for $\beta = \beta_0$, provided

\begin{equation}
0 < \beta_0 < \frac{p}{n - p}, \quad \beta_0 \leq 1.
\end{equation}

In the case $\frac{p}{n - p} > 1$, i.e., for $p > \frac{n}{2}$, we can set $\beta_0 = 1$, which shows that in fact $Du \in L^p(\frac{1}{2} B, dx)$, for all $B = B(0, R)$. Hence, $Du = \nabla u$ in the distributional sense, and consequently $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$.

For $1 < p \leq \frac{n}{2}$, we fix $s$ so that $p < s \leq \frac{np}{n - p}$ which ensures that $u \in L^s_{\text{loc}}(\mathbb{R}^n, dx)$ as shown above. Applying Hölder’s inequality with exponents $p'$ and $p$, we obtain from (3.6) and (3.7),

\begin{equation}
\int_{u \leq k} |Du|^p u^{\beta - 1} \varphi \, dx \leq c \left( \int_B |Du|^{p u^{\beta_0 - 1}} \, dx \right)^\frac{1}{p} \left( \int_B u^{q(p + 1 - \beta_0)(p - 1)} \, dx \right)^\frac{1}{p}
\end{equation}

\begin{equation}
+ \omega(B)^{1 - \beta} \left( \int_B u \, d\omega \right)^\beta.
\end{equation}

Passing to the limit as $k \to \infty$, we deduce that (3.5) holds if $\beta \leq 1$ and

\begin{equation}
\beta p + (1 - \beta_0)(p - 1) \leq \beta p + p - 1 \leq s.
\end{equation}
In particular, (3.5) holds for $\beta = \beta_1 = \min \left(1, \frac{s-p+1}{p} \right)$.

If $\beta_1 = 1$, then $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ as above. In the case
$$\beta_1 = \frac{s-p+1}{p} < 1,$$
we set $\beta_j = \beta_1 + \frac{p-1}{p}\beta_{j-1}$, so that
$$\beta_j p + (1 - \beta_{j-1})(p-1) = s, \quad j \geq 2.$$ 

In other words,
$$\beta_j = \frac{s-p+1}{p} \left( \frac{p-1}{p} \right)^{j-1} \sum_{i=0}^{j-1} \left( \frac{p-1}{p} \right)^i, \quad j = 1, 2, \ldots,$$

Since
$$\lim_{j \to \infty} \beta_j = s - p + 1 > 1,$$
we can choose $J \geq 2$ so that $\beta_1 \leq \cdots \leq \beta_{J-1} < 1$, but $\beta_J \geq 1$.

If $\beta_J > 1$, then we will replace $\beta_J$ with $\beta_J = 1$. Clearly,
$$\beta_j p + (1 - \beta_{j-1})(p-1) = s, \quad j = 2, 3, \ldots, J-1; \quad \beta_J p + (1 - \beta_{J-1})(p-1) \leq s.$$ 

Arguing by induction, and using (3.5) with $\beta = \beta_j$, for $j = 2, 3, \ldots, J$, we estimate as above,
$$\int_{|u| \leq k} |Du|^p u^{\beta_j - 1} \varphi \, dx \leq C \left( \int_B |Du|^p u^{\beta_j - 1} \varphi \, dx \right)^{\frac{1}{p}} \times \left( \int_B u^{\beta_j p + (1 - \beta_{j-1})(p-1)} \varphi \, dx \right)^{\frac{1}{p}} + \omega(B)^{1-\beta} \left( \int_B u \, d\omega \right)^{\beta} < \infty.$$

Since $\beta_J = 1$ at the last step, we arrive at the estimate
$$\int_{|u| \leq k} |Du|^p \varphi \, dx \leq C_B < \infty,$$
where $C_B$ does not depend on $k$. Passing to the limit as $k \to \infty$, we conclude that $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n)$.

In the next theorem we obtain a lower bound for supersolutions of the integral equation (1.17).

**Theorem 3.4.** Let $1 < p < n$, $0 < q < p-1$, $0 < \alpha < \frac{p}{n}$, and $\sigma \in M^+(\mathbb{R}^n)$. Suppose $0 \leq u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ is a nontrivial solution of (2.10). Then the inequality

$$(3.9) \quad u \geq C \left( W_{\alpha,p}^{1,q} \right)^{\alpha - 1} \sigma \text{ a.e.}$$

holds, where $C$ is a positive constant depending only on $p, q,$ and $n$.

Before proving Theorem 3.4, we recall the following lemma.
Lemma 3.5. Let $1 < p < \infty$ and $0 < \alpha < \frac{n}{p}$. Then, for every $r > 0$,

$$W_{\alpha,p} \left[ (W_{\alpha,p}\sigma)^r \right] \geq c \frac{r}{r-1} (W_{\alpha,p}\sigma)^{\frac{r}{p-1}+1},$$

where $c = c_{n,p,\alpha}$ depends only on $n$, $p$, and $\alpha$.

Estimate (3.10) was proved in [CV14], Lemma 3.2, with the constant $C_{n,p,\alpha}$ on the right-hand side. Clearly, $\frac{r}{p-1} + 1 \leq \frac{r}{p-1}$, and hence, (3.10) follows with $c = e^{-1}C_{n,p,\alpha}$.

Proof of Theorem 3.4. Let $d\omega = u^q d\sigma$. Fix $x \in \mathbb{R}^n$ and pick $R > |x|$. Let $B = B(0,R)$, and let $d\sigma_B = \chi_B d\sigma$. Iterating (2.10), we obtain

$$u(x) \geq W_{\alpha,p} \left[ (W_{\alpha,p}\omega)^q \right] (x)$$

$$= \int_0^\infty \left( \frac{1}{t^{n-p}} \int_{B(x,t) \cap B} W_{1,p}\omega(z)^q d\sigma(z) \right) \frac{1}{t^{\frac{1}{p-1}}} dt.$$

We estimate,

$$W_{\alpha,p}\omega(z) = \int_0^\infty \left( \frac{\omega(B(z,s))}{s^{n-p}} \right)^{\frac{1}{p-1}} ds \geq c \int_0^\infty \left( \frac{\omega(B(z,2s))}{s^{n-p}} \right)^{\frac{1}{p-1}} ds,$$

where $c = c(n,p,\alpha) > 0$.

Notice that if $z \in B$ and $R \leq s$ then $B(z,2s) \supset B(0,s)$. Hence,

$$W_{\alpha,p}\omega(z) \geq c \int_0^\infty \left( \frac{\omega(B(0,s))}{s^{n-p}} \right)^{\frac{1}{p-1}} ds.$$

From this it follows,

$$u(x) \geq [c M(R)]^{\frac{q}{p-1}} W_{\alpha,p}\sigma_B(x),$$

where

$$M(R) = \int_R^\infty \left( \frac{\omega(B(0,s))}{s^{n-p}} \right)^{\frac{1}{p-1}} ds > 0.$$ Combining (2.10) with the preceding estimate, and using Lemma 3.5 with $r = q$ and $\sigma_B$ in place of $\sigma$, we obtain

$$u(x) \geq [c M(R)]^{\frac{q}{p-1}} W_{\alpha,p} \left[ (W_{\alpha,p}\sigma_B)^q \right] (x)$$

$$\geq c^{\frac{q}{p-1}} [c M(R)]^{\frac{q}{p-1}} [W_{\alpha,p}\sigma_B(x)]^{1+\frac{q}{p-1}}.$$

Iterating this procedure and using Lemma 3.5 with $r = q \sum_{k=0}^{j-1} \left( \frac{q}{p-1} \right)^k$, we deduce

$$u(x) \geq c^{\sum_{k=0}^{j-1} k \left( \frac{q}{p-1} \right)^k} \left( c M(R) \right)^{\frac{q}{p-1}} [W_{\alpha,p}\sigma_B(x)]^{\sum_{k=0}^{j-1} \left( \frac{q}{p-1} \right)^k}.$$
for all \( j = 2, 3, \ldots \). Since \( 0 < q < p - 1 \), obviously
\[
\sum_{k=1}^{\infty} k \left( \frac{q}{p-1} \right)^k < \infty.
\]
Letting \( j \to \infty \) in the preceding estimate we obtain
\[
u(x) \geq C \left[ W_{\alpha,p} \sigma_B(x) \right]^{\frac{p-1}{p-1-q}}, \quad B = B(0, R), \quad R > |x|,
\]
where \( C > 0 \) depends only on \( n, p, q, \) and \( \alpha \). Letting \( R \to \infty \) yields (3.9) for all \( x \in \mathbb{R}^n \).

The next lemma shows that if there exists a nontrivial supersolution to (1.17), then \( \sigma \) must be absolutely continuous with respect to the \((\alpha,p)\)-capacity defined for all \( E \subset \mathbb{R}^n \) by (see [AH96], Sec. 2.2)
\[
cap_{\alpha,p}(E) = \inf \left\{ ||f||_{L_p(\mathbb{R}^n)}^p : \int_E f \sigma \geq 1 \text{ on } E, \quad f \in L_p^+(\mathbb{R}^n) \right\}.
\]
As a consequence, if (1.1) has a nontrivial \( p \)-superharmonic supersolution, then \( \sigma \) is absolutely continuous with respect to the \( p \)-capacity defined by (2.4). Notice that \( \text{cap}_p(E) \approx \text{cap}_{1,p}(E) \) for compact sets \( E \).

**Lemma 3.6.** Let \( 1 < p < \infty, 0 < q < p - 1, 0 < \alpha < \frac{n}{p} \), and \( \sigma \in M^+(\mathbb{R}^n) \). Suppose there is a nontrivial solution \( u \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma) \) to inequality (2.10). Then there exists a constant \( C \) depending only on \( n, p, q, \alpha \) such that
\[
(3.11) \quad \sigma(E) \leq C \left[ \text{cap}_{\alpha,p}(E) \right]^{\frac{q}{p-1} \left( \int_E u^q d\sigma \right)^{\frac{p-1-q}{p-1}}},
\]
for all compact sets \( E \subset \mathbb{R}^n \).

**Proof.** Let \( d\omega = u^q d\sigma \). Then \( u \geq W_{\alpha,p} \omega \) \( \text{d}\sigma \text{-a.e.} \) By Theorem 1.11 in [Ver99],
\[
\int_E \frac{d\omega}{(W_{\alpha,p} \omega)^{p-1}} \leq C \text{cap}_{\alpha,p}(E),
\]
where \( C \) depends only on \( n, p, \) and \( \alpha \). Hence,
\[
(3.12) \quad \int_E u^{q-p+1} d\sigma \leq \int_E \frac{d\omega}{(W_{\alpha,p} \omega)^{p-1}} \leq C \text{cap}_{\alpha,p}(E).
\]
Note that \( q - p + 1 < 0 \). Using Hölder’s inequality with exponents \( r = \frac{p-1}{q} \) and \( r' = \frac{p-1}{p-1-q} \), we have
\[
\sigma(E) = \int_E u^{-\beta} u^\beta d\sigma \leq \left( \int_E u^{-\beta r} d\sigma \right)^{\frac{1}{r}} \left( \int_E u^{\beta r'} d\sigma \right)^{\frac{1}{r'}},
\]
where \( \beta = \frac{q(p-1-q)}{p-1} > 0 \). Then \( -\beta r = q - p + 1 \) and \( \beta r' = q \), and since \( u \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma) \), the preceding estimate implies (3.11). \( \square \)
4. Solutions of the nonlinear integral equation

4.1. Weighted norm inequalities and intrinsic potentials $K_{α,p,q}$. Let $1 < p < ∞$, $0 < q < p − 1$, and $0 < α < \frac{n}{p}$. Let $σ ∈ M^+(R^n)$. We denote by $κ$ the least constant in the weighted norm inequality

$$
(4.1) \quad ||W_{α,p}ν||_{L^q(R^n, dσ)} ≤ κ ν(\bar{R}^n)^{\frac{1}{p - 1}}, \quad ∀ν ∈ M^+(R^n).
$$

We will also need a localized version of (4.1) for $σ_E = σ|_E$, where $E$ is a Borel subset of $R^n$, and $κ(E)$ is the least constant in

$$
(4.2) \quad ||W_{α,p}ν||_{L^q(dσ|_E)} ≤ κ(E) ν(\bar{R}^n)^{\frac{1}{p - 1}}, \quad ∀ν ∈ M^+(R^n).
$$

In applications, it will be enough to use $κ(E)$ where $E = B$ is a ball, or the intersection of two balls.

We define the intrinsic potential of Wolff type in terms of $κ(B(x,s))$, the least constant in (4.2) with $E = B(x,s)$:

$$
(4.3) \quad K_{α,p,q}σ(x) = \int_0^∞ \left[ \frac{κ(B(x,s))^{q(p-1)}}{s^{n-αp}} \right]^{\frac{1}{p - 1}} ds, \quad x ∈ R^n.
$$

Remark 4.1. Notice that, for $α = 1$, in the definition of $K_{1,p,q}σ$ we can use either the constant $κ(B(x,s))$ in (1.12), or $κ(B(x,s))$ in (4.2) with $E = B(x,s)$, since by Theorem 2.4, for all $E$,

$$
(4.4) \quad \frac{1}{K} κ(E) ≤ κ(E) ≤ K κ(E),
$$

where $K$ is the constant in (2.11) which depends only on $p$ and $n$.

The proof of the following key lemma is based on Vitali’s covering lemma, and weak-type maximal function inequalities.

Lemma 4.2. Let $1 < p < ∞$, $0 < q < p − 1$, and $0 < α < \frac{n}{p}$.

(i) Suppose $0 ≤ u ∈ L^q_{loc}(R^n, dσ)$ is a nontrivial solution of (2.10). Then, for every ball $E = B$, (4.2) holds with

$$
(4.5) \quad κ(B) ≤ c(n,p,q,α) \left( \int_B u^q dσ \right)^{\frac{p-1-q}{q(p-1)}}.
$$

(ii) If in statement (i) we have $u ∈ L^q(R^n, dσ)$, then (4.1) holds with

$$
(4.6) \quad κ ≤ c(n,p,q,α) \left( \int_{R^n} u^q dσ \right)^{\frac{p-1-q}{q(p-1)}}.
$$

Proof. Let $dω = u^q dσ ∈ M^+(R^n)$. For $ν ∈ M^+(R^n)$, consider the maximal function

$$
(4.7) \quad M^p_ω(y) = \sup_{ρ>0} \left[ \frac{ν(B(y, \frac{ρ}{4}))}{ω(B(y, ρ))} \right], \quad y ∈ R^n,
$$
Thus \( E_t = \{ y \in \mathbb{R}^n : M_\omega^t(y) > t \} \), \( t > 0 \).

Suppose \( E_t \neq \emptyset \). Then, for every \( y \in E_t \), there exists a ball \( B(y, \rho_y) \) such that

\[
\frac{\nu(B(y, \rho_y))}{\omega(B(y, \rho_y))} > t.
\]

Thus \( E_t \subseteq \bigcup_{y \in E_t} B(y, \frac{\rho_y}{5}) \), and hence for any compact subset \( E \) of \( E_t \) there exists a \( k \in \mathbb{N} \) such that

\[
E \subseteq \bigcup_{j=1}^{k} B\left( y_j, \frac{\rho_{y_j}}{5} \right).
\]

Applying Vitali’s covering lemma, we find disjoint balls \( \left\{ B\left( y_{j_l}, \frac{\rho_{y_{j_l}}}{5} \right) \right\}_{l=1}^{m} \) such that

\[
E \subseteq \bigcup_{l=1}^{m} B\left( y_{j_l}, \rho_{y_{j_l}} \right).
\]

Consequently,

\[
\omega(E) \leq \sum_{l=1}^{m} \omega\left( B\left( y_{j_l}, \rho_{y_{j_l}} \right) \right) \leq \frac{1}{t} \sum_{l=1}^{m} \nu\left( B\left( y_{j_l}, \frac{\rho_{y_{j_l}}}{5} \right) \right) \leq \frac{1}{t} \nu(\mathbb{R}^n).
\]

Therefore,

\[
\sup_{t > 0} t \omega(E_t) := ||M_\omega^t||_{L^1,\infty(d\omega)} \leq \nu(\mathbb{R}^n).
\]

Clearly, for any \( y \in \mathbb{R}^n \) such that \( M_\omega^t(y) < \infty \), we have

\[
W_{\alpha,p}^t(y) = \int_0^\infty \left( \frac{\nu(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}
\]

\[
= 5 \int_0^\infty \left( \frac{\nu(B(y,\frac{s}{5}))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}
\]

\[
= 5 \int_0^\infty \left( \frac{\nu(B(y,\frac{s}{5}))}{\omega(B(y,s))} \right) \left( \frac{\omega(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}
\]

\[
\leq 5 \int_0^\infty \left( M_\omega^t(y) \right)^{\frac{1}{p-1}} W_{\alpha,p}^t(y) \leq 5 \int_0^\infty \left( M_\omega^t(y) \right)^{\frac{1}{p-1}} u(y).
\]

Note that if \( \nu(B(y,\frac{s}{5})) > 0 \) but \( \omega(B(y,s)) = 0 \) for some \( s > 0 \) then \( M_\omega^t(y) = \infty \). However, by (4.8) it follows that the set of such \( y \in B \) has \( \omega \)-measure zero, and consequently \( \sigma \)-measure zero, since by Lemma 3.1 we have \( \inf_B u > 0 \).

Hence,

\[
||W_{\alpha,p}^t||_{L^1(d\sigma_B)} \leq c \int_B \left( M_\omega^t \right)^{\frac{q}{p-t}} u^q d\sigma = c \int_B \left( M_\omega^t \right)^{\frac{q}{p-t}} d\omega.
\]
To complete our estimates, we invoke the well-known inequality
\[ ||f||_{L^r(X,\omega)} \leq C(r,\omega(X))^{1-r} ||f||_{L^1,\infty(X,\omega)}, \]
where \( 0 < r < 1, \) and \( \omega \) is a finite measure on \( X. \) Applying the preceding
inequality with \( r = \frac{q}{p-1} \) and \( f = M_{\omega}^\nu, \) we estimate
\[ ||W_{\alpha,p}\nu||_{L^q(d\sigma_B)}^q \leq c\omega(B)^{1 - \frac{q}{p-1}} ||M_{\omega}^\nu||_{L^1,\infty}(d\omega) \leq c\omega(B)^{1 - \frac{q}{p-1}} \nu(\mathbb{R}^n)^{\frac{q}{p-1}}, \]
where \( c \) depends only on \( n, p, q, \alpha. \) This proves statement (i) of Lemma 4.2.

If \( u \in L^q(\mathbb{R}^n, \sigma), \) then statement (ii) follows (i) with \( B = B(x, R) \) by
letting \( R \to \infty. \) \( \square \)

We will need a converse estimate to (4.2) for subsolutions \( u_B \) of equation
(1.17) with \( \sigma_B \) in place of \( \sigma, \) for a ball \( B. \)

**Corollary 4.3.** Let \( 1 < p < \infty, \) \( 0 < q < p - 1, \) and \( 0 < \alpha < \frac{n}{p}. \) Let
\( \sigma \in M^+(\mathbb{R}^n). \) Suppose \( u_B \in L^q(\mathbb{R}^n, d\sigma_B) \) is a subsolution associated with
\( \sigma_B, \) i.e., \( 0 \leq u_B \leq W_{\alpha,p}(u_B^q d\sigma_B) \) \( d\sigma_B \)-a.e. Then, for every ball \( B, \)
\[ (\int_B u_B^q d\sigma)^{\frac{p-1}{p-q}} \leq \kappa(B). \]

**Proof.** Without loss of generality we may assume \( \kappa(B) < \infty. \) Then using
(4.2) with \( d\nu = u_B^q d\sigma_B, \) we obtain
\[ \int_B u_B^q d\sigma \leq \int_B [W_{\alpha,p}(u_B^q d\sigma_B)]^q d\sigma \leq \kappa(B)^q \left( \int_B u_B^q d\sigma \right)^{\frac{q}{p-1}}, \]
which yields (4.9). \( \square \)

**4.2. Solutions in** \( L^q(\mathbb{R}^n, d\sigma). \) The next theorem is concerned with the ex-
istence of global solutions \( u \in L^q(\mathbb{R}^n, d\sigma) \) to (1.17).

**Theorem 4.4.** Let \( \sigma \in M^+(\mathbb{R}^n). \) Then equation (1.17) has a solution
\( u \in L^q(\mathbb{R}^n, d\sigma) \) if and only if there exists a constant \( \kappa > 0 \) such that (4.1)
holds.

**Proof.** The necessity of (4.1) follows from Lemma 4.2. To prove its suf-
fi ciency, we first show that (4.1) implies
\[ \int_{\mathbb{R}^n} (W_{\alpha,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma < \infty. \]
Indeed, fix a ball \( B = B(x, R). \) Applying (4.1) with \( d\nu = d\sigma_B \) we obtain
\[ \int_{\mathbb{R}^n} (W_{\alpha,p}\sigma_B)^q d\sigma \leq \kappa^q \sigma(B)^{\frac{q}{p-1}} < \infty. \]
Letting \( v_0 = (W_{\alpha,p} \sigma_B)^q \) where \( v_0 \in L^1(\mathbb{R}^n, \nu), \) and using \( d\nu = v_0 \, d\sigma \) in (4.1) we obtain

\[
\int_{\mathbb{R}^n} [W_{\alpha,p}(v_0 \, d\sigma)]^q \, d\sigma \leq \kappa^q \left( \int_{\mathbb{R}^n} v_0 \, d\sigma \right)^{\frac{q^2}{p-1}} < \infty.
\]

By Lemma 3.5 with \( r = q, \) we have

\[
[W_{\alpha,p}(v_0 \, d\sigma)]^q = [W_{\alpha,p}(W_{\alpha,p} \sigma_B)^q \, d\sigma)]^q
\]

\[
\geq [W_{\alpha,p}(W_{\alpha,p} \sigma_B)^q \, d\sigma_B)]^q \geq c_{p-1}^{q^2} (W_{\alpha,p} \sigma_B)^q(\frac{q^2}{p-1}+1).
\]

Let \( v_1 = c_{p-1}^{q^2} (W_{\alpha,p} \sigma_B)^q(\frac{q^2}{p-1}+1). \) Then \( v_1 \in L^1(\mathbb{R}^n, \nu), \) and

\[
\int_{\mathbb{R}^n} v_1 \, d\sigma \leq \kappa^q \left( \int_{\mathbb{R}^n} v_0 \, d\sigma \right)^{\frac{q^2}{p-1}}.
\]

Applying again (4.1) with \( d\nu = v_1 \, d\sigma \) we obtain

\[
\int_{\mathbb{R}^n} [W_{\alpha,p}(v_1 \, d\sigma)]^q \, d\sigma \leq \kappa^q \left( \int_{\mathbb{R}^n} v_1 \, d\sigma \right)^{\frac{q^2}{p-1}}
\]

\[
\leq \kappa^{q(1+\frac{q}{p-1})} \left( \int_{\mathbb{R}^n} v_0 \, d\sigma \right)^{\frac{q^2}{(p-1)^2}} < \infty.
\]

By Lemma 3.5 with \( r = q(\frac{q}{p-1} + 1), \) we estimate

\[
[W_{\alpha,p}(v_1 \, d\sigma)]^q = \left[ W_{\alpha,p}(c_{p-1}^{q^2} (W_{\alpha,p} \sigma_B)^q(\frac{q^2}{p-1}+1) \, d\sigma) \right]^q
\]

\[
\geq c_{p-1}^{q^2}(1+2\frac{q}{p-1}) (W_{\alpha,p} \sigma_B)^q(\frac{q^2}{(p-1)^2} + \frac{q}{p-1}+1).
\]

Setting

\[
v_2 = c_{p-1}^{q^2}(1+2\frac{q}{p-1}) (W_{\alpha,p} \sigma_B)^q(\frac{q^2}{(p-1)^2} + \frac{q}{p-1}+1),
\]

we obtain

\[
\int_{\mathbb{R}^n} v_2 \, d\sigma \leq \kappa^{q(1+\frac{q}{p-1})} \left( \int_{\mathbb{R}^n} v_0 \, d\sigma \right)^{\frac{q^2}{(p-1)^2}} < \infty.
\]

Arguing by induction and letting

\[
v_j = c_{p-1}^{q^2} \sum_{k=1}^{j} k(\frac{q}{p-1})^{k-1} (W_{\alpha,p} \sigma_B)^q \sum_{k=0}^{j} (\frac{q}{p-1})^k,
\]

we obtain

\[
\int_{\mathbb{R}^n} v_j \, d\sigma \leq \kappa^{q(\sum_{k=0}^{j-1}(\frac{q}{p-1})^k)} \left( \int_{\mathbb{R}^n} v_0 \, d\sigma \right)^{(\frac{q}{p-1})^j} < \infty.
\]

By Fatou’s lemma,

\[
\int_{\mathbb{R}^n} \liminf_{j \to \infty} v_j \, d\sigma \leq \liminf_{j \to \infty} \int_{\mathbb{R}^n} v_j \, d\sigma \leq \kappa^{q(p-1)} < \infty.
\]
Thus,
\begin{equation}
\frac{c}{\rho^{1/q}} \sum_{k=1}^{\infty} k \left( \frac{q}{p-1} \right)^{k-1} \int_{\mathbb{R}^n} (W_{\alpha,p} \sigma_B)^{\frac{q(p-1)}{p-1-q}} d\sigma \leq \kappa \left( \frac{q(p-1)}{p-1-q} \right) < \infty.
\end{equation}

Since \( c \) and \( \kappa \) in (4.11) are independent of \( B = B(x,R) \), letting \( R \to \infty \) and using the Monotone Convergence Theorem we deduce (4.10).

Next, we let \( u_0 = c_0 (W_{\alpha,p} \sigma) \), where \( c_0 > 0 \) is a small constant to be chosen later on, and construct a sequence \( u_j \) as follows:
\begin{equation}
\label{eq:4.12}
u_{j+1} = W_{\alpha,p}(u_j^q d\sigma), \quad j = 0, 1, 2, \ldots.
\end{equation}

Applying Lemma 3.5, we estimate
\begin{equation}
\nonumber
u_1 = W_{\alpha,p}(u_0^q d\sigma) = c_0^{\frac{q}{q-1}} W_{\alpha,p} \left( (W_{\alpha,p} \sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma \right)
\geq c_0^{\frac{q}{q-1}} \frac{q}{q-1} (W_{\alpha,p} \sigma)^{\frac{p-1-q}{p-1}},
\end{equation}
where \( c \) is the constant in (4.2). Choosing \( c_0 \) so that \( c_0^{\frac{q}{q-1}} c^{\frac{q}{q-1}} \geq c_0 \), we obtain \( u_1 \geq u_0 \).

By induction, we have \( u_j \leq u_{j+1} \) \((j = 0, 1, \ldots)\). Note that \( u_0 \in L^q(\mathbb{R}^n, d\sigma) \) by (4.10). Suppose that \( u_j \in L^q(\mathbb{R}^n, d\sigma) \), for some \( j \geq 0 \). Then, using (4.1) with \( d\nu = u_j^q d\sigma \), we obtain
\begin{equation}
\int_{\mathbb{R}^n} u_j^q \, d\sigma = \int_{\mathbb{R}^n} \left( W_{\alpha,p}(u_j^q d\sigma) \right)^q \, d\sigma
\leq \kappa \left( \int_{\mathbb{R}^n} u_j^q \, d\sigma \right)^{\frac{q}{p-1}} < \infty.
\end{equation}

Since \( u_j \leq u_{j+1} \), the preceding inequality yields, for all \( j = 0, 1, \ldots \),
\begin{equation}
\int_{\mathbb{R}^n} u_{j+1}^q \, d\sigma \leq \kappa^{\frac{p-1-q}{p-1}} < \infty.
\end{equation}
Passing to the limit as \( j \to \infty \) in (4.12), we conclude using the Monotone Convergence Theorem that \( u = \lim_{j \to \infty} u_j \) is a nontrivial solution of (1.17) such that \( u \in L^q(\mathbb{R}^n, d\sigma) \). \( \square \)

4.3. Solutions in \( L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \). In this subsection we prove the main theorem for general integral equations (1.17). We start with the following lemma.

Lemma 4.5. Suppose \( 0 \leq u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \) is a nontrivial solution of (2.10). Then, for all \( x \in \mathbb{R}^n \) and \( t > 0 \),
\begin{equation}
\sigma(B(x,t)) \left[ \int_t^{\infty} \left( \frac{\kappa(B(x,s))^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{p-1}{q-1}} \, ds \right]^{\frac{q}{p-1}} \leq c \int_{B(x,t)} u^q d\sigma,
\end{equation}
where \( c \) depends only on \( n, p, q, \) and \( \alpha \).
Proof. By Lemma 4.2, $\kappa(B(x,s)) < \infty$ for all $x \in \mathbb{R}^n$ and $s > 0$. Hence it is enough to prove (4.14) for $t$ large enough. Without loss of generality we can assume that $\sigma \neq 0$, and $\sigma(B(x,t)) > 0$. Let $d\omega = u^q d\sigma$. We estimate

$$
\int_{B(x,t)} u^q d\sigma \geq \int_{B(x,t)} (W_{\alpha,p}\omega)^q d\sigma
$$

$$
\geq \int_{B(x,t)} \left[ \int_t^{\infty} \left( \frac{\omega(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right]^q d\sigma(y).
$$

Since $B(y,2s) \supset B(x,s)$ if $s \geq t$ and $y \in B(x,t)$, it follows,

$$
\int_{B(x,t)} \left[ \int_t^{\infty} \left( \frac{\omega(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right]^q d\sigma(y)
$$

$$
= 2^{-\frac{(n-\alpha p)q}{p-1}} \int_{B(x,t)} \left[ \int_t^{\infty} \left( \frac{\omega(B(y,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right]^q d\sigma(y)
$$

$$
\geq 2^{-\frac{(n-\alpha p)q}{p-1}} \int_{B(x,t)} \left[ \int_t^{\infty} \left( \frac{\omega(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right]^q d\sigma(y)
$$

$$
\geq 2^{-\frac{(n-\alpha p)q}{p-1}} \sigma(B(x,t)) \left[ \int_t^{\infty} \left( \frac{\kappa(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{q(p-1)}{p-1-q}} ds \right]^q d\sigma(y)
$$

$$
\geq c \sigma(B(x,t)) \left[ \int_t^{\infty} \left( \frac{\kappa(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{q(p-1)}{p-1-q}} ds \right]^q
$$

where $c = c(n,p,q,\alpha)$; note that in the last line we used (4.5). Hence, (4.13) holds for all $x \in \mathbb{R}^n$ and $t > 0$. □

By picking $t$ in (4.13) large enough to ensure that $\sigma(B(x,t)) > 0$, we deduce the following corollary.

**Corollary 4.6.** Under the assumptions of Lemma 4.5, for all $x \in \mathbb{R}^n$ and $t > 0$,

$$
(4.14) \quad \int_t^{\infty} \left( \frac{\kappa(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1-q}} ds < \infty.
$$

The next lemma is an analogue of Corollary 3.2 for potentials $K_{\alpha,p,q}\sigma$.

**Lemma 4.7.** Let $1 < p < \infty$, $0 < q < p - 1$, and $0 < \alpha < \frac{q}{p}$. Let $\sigma \in M^+(\mathbb{R}^n)$. Suppose that (4.14) holds for $x = 0$ and $t = 1$, i.e.,

$$
(4.15) \quad \int_1^{\infty} \left( \frac{\kappa(B(0,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1-q}} ds < \infty.
$$

Then (4.14) holds for all $x \in \mathbb{R}^n$, $t > 0$, and $K_{\alpha,p,q}\sigma \in L^q_{loc}(\mathbb{R}^n,d\sigma)$. 
Proof. Notice that if \((4.15)\) holds, then obviously, for every \(t > 0\),

\[
\int_t^\infty \left( \frac{[\kappa(B(0, s))^{q(p-1)}]}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} < \infty.
\]

For a fixed \(x \in \mathbb{R}^n\), clearly \(B(x, s) \subset B(0, s + |x|)\), so that

\[
\int_t^\infty \left( \frac{[\kappa(B(x, s))^{q(p-1)}]}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq \int_t^\infty \left( \frac{[\kappa(B(0, s + |x|))^{q(p-1)}]}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} = \int_t^\infty \left( \frac{[\kappa(B(0, r))^{q(p-1)}]}{(r - |x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r - |x|}.
\]

If \(t > |x|\), then \(r - |x| > \frac{1}{2}r\) if \(r \geq t + |x|\). Hence,

\[
\int_{t+|x|}^\infty \left( \frac{[\kappa(B(0, r))^{q(p-1)}]}{(r - |x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r - |x|} \leq 2^{\frac{n-\alpha p}{p-1}+1} \int_{t+|x|}^\infty \left( \frac{[\kappa(B(0, r))^{q(p-1)}]}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.
\]

In the case \(t \leq |x|\),

\[
\int_{t+|x|}^\infty \left( \frac{[\kappa(B(0, r))^{q(p-1)}]}{(r - |x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r - |x|} = \int_{t+|x|}^{2|x|} \left( \frac{[\kappa(B(0, r))^{q(p-1)}]}{(r - |x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r - |x|} + \int_{2|x|}^\infty \left( \frac{[\kappa(B(0, r))^{q(p-1)}]}{(r - |x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r - |x|}
\]

\[
\leq \kappa(B(0, 2|x|)^{q(p-1)} \int_{t+|x|}^{2|x|} \left( \frac{1}{(r - |x|)^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r - |x|} + 2^{\frac{n-\alpha p}{p-1}+1} \int_{2|x|}^\infty \left( \frac{[\kappa(B(0, r))^{q(p-1)}]}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.
\]

Combining the preceding estimates proves \((4.14)\) for all \(x \in \mathbb{R}^n\) and \(t > 0\).
To show that $K_{\alpha,p,q}\sigma \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$, fix a ball $B(x,t)$ and let $B = B(x,2t)$. Splitting $K_{\alpha,p,q}\sigma$ into two parts, we estimate

$$I = \int_{B(x,t)} \left[ \int_0^t \left( \frac{\kappa(B(y,s))^{q(p-1)}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y),$$

$$II = \int_{B(x,t)} \left[ \int_t^\infty \left( \frac{\kappa(B(y,s))^{q(p-1)}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q d\sigma(y).$$

Clearly, in $II$ we have $B(y,s) \subset B(x,2s)$, and hence, by (4.14),

$$II \leq c\sigma(B(x,t)) \left[ \int_t^\infty \left( \frac{\kappa(B(x,2s))^{q(p-1)}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q < \infty.$$

In $I$, we have $B(y,s) \subset B$, and consequently,

$$I \leq \int_B (K_{\alpha,p,q}\sigma B)^q d\sigma.$$

Since $\kappa(B) < \infty$, by Theorem 4.4 with $\sigma_B$ in place of $\sigma$, the equation $u_B = W_{\alpha,p}(u_B^q d\sigma_B)$ has a solution $u_B$ such that $\int_B u_B^q d\sigma < \infty$. Hence by Lemma 4.2 with $\sigma_B$ in place of $\sigma$,

$$[\kappa(B(y,s) \cap B)]^{(q(p-1))} \leq c \int_{B(y,s)} u_B^q d\sigma_B,$$

where $c = c(p,q,\alpha,n)$. From this we obtain

$$\int_B (K_{\alpha,p,q}\sigma_B)^q d\sigma \leq c_{p,q}^q \int_B [W_{\alpha,p}(u_B^q d\sigma_B)]^q d\sigma = c_{p,q}^q \int_B u_B^q d\sigma < \infty.$$

This proves that both $I$ and $II$ are finite, i.e., $\int_B (K_{\alpha,p,q}\sigma)^q d\sigma < \infty$. \(\square\)

**Theorem 4.8.** Let $1 < p < \infty$, $0 < q < p - 1$, and $0 < \alpha < \frac{n}{p}$. Let $\sigma \in M^+(\mathbb{R}^n)$. Suppose that both (3.2) and (4.15) hold. Then there exists a solution $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to (1.17) such that $\liminf_{x \to \infty} u(x) = 0$, and $u$ satisfies the inequalities

$$C^{-1} \left[ K_{\alpha,p,q}\sigma + (W_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right] \leq u \leq C \left[ K_{\alpha,p,q}\sigma + (W_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}} \right],$$

where $C > 0$ is a constant which depends only on $n$, $p$, $q$, and $\alpha$.

The lower bound in (4.16) holds for any $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ which is a nontrivial solution of inequality (2.10).

**Proof.** Let $u_0 = c_0 (W_{\alpha,p}\sigma)^{\frac{p-1}{p-1-q}}$, where $c_0$ is a constant which will be chosen later. We construct a sequence $\{u_j\}$ as follows:

$$u_{j+1} = W_{\alpha,p}(u_j^q d\sigma), \quad j = 0, 1, 2, \ldots.$$
Choosing $c_0$ small enough and using Lemma 3.5 as in the proof of Theorem 4.4, we ensure that $u_j \leq u_{j+1}$.

We need to verify that $u_j$ are well defined, i.e., $u_j \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$. We set $d\omega_0 = u_0^q d\sigma$. Let us first show that, for all $x \in \mathbb{R}^n$ and $t > 0$,

$$
\omega_0(B(x, t)) = \int_{B(x, t)} u_0^q d\sigma \leq c \left[ \kappa(B(x, 2t)) \right]^{q(p-1)/(p-1-q)}
$$

(4.17)

$$
+ c \left( \int_t^{\infty} \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right)^{q(p-1)/(p-1-q)} \sigma(B(x, t)),
$$

where $c$ depends only on $n, p, q, \alpha$. We set $B = B(x, 2t)$, and denote by $B^c$ the complement of $B$ in $\mathbb{R}^n$. Clearly, for $y \in B(x, t)$ and $0 < r \leq t$, $B^c \cap B(y, r) = \emptyset$. Hence, for $y \in B(x, t)$,

$$
W_{\alpha, p\sigma B^c}(y) = \int_0^{\infty} \left( \frac{\sigma(B^c \cap B(y, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} dr
$$

$$
= \int_t^{\infty} \left( \frac{\sigma(B^c \cap B(y, r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} dr.
$$

If $r \geq t$, then $B(y, r) \subset B(x, 2r)$, and consequently

$$
W_{\alpha, p\sigma B^c}(y) \leq \int_t^{\infty} \left( \frac{\sigma(B^c \cap B(x, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} dr
$$

$$
\leq \int_t^{\infty} \left( \frac{\sigma(B(x, 2r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} dr \leq 2^{\frac{\alpha p}{p-1}} \int_t^{\infty} \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds.
$$

From this we deduce,

$$
\int_{B(x, t)} (W_{\alpha, p\sigma})^{q(p-1)/(p-1-q)} d\sigma \leq c \int_{B(x, t)} (W_{\alpha, p\sigma B^c})^{q(p-1)/(p-1-q)} d\sigma
$$

$$
+ c \int_{B(x, t)} (W_{\alpha, p\sigma B^c})^{q(p-1)/(p-1-q)} d\sigma \leq c \int_{B(x, t)} (W_{\alpha, p\sigma B^c})^{q(p-1)/(p-1-q)} d\sigma
$$

$$
+ c \left( \int_t^{\infty} \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right)^{q(p-1)/(p-1-q)} \sigma(B(x, t)).
$$

It follows from (4.15) that $\kappa(B) < \infty$. Using Theorem 4.4 with $\sigma_B$ in place of $\sigma$, we see that the equation $u_B = W_{\alpha, p\sigma_B}$ has a solution $u_B \in L^q(\mathbb{R}^n, d\sigma_B)$. By Theorem 3.4, $u_B \geq C (W_{\alpha, p\sigma_B})^{p-1/q}$. On the other hand, by Corollary 4.3,

$$
\int_{B(x, t)} u_0^q d\sigma \leq \left[ \kappa(B) \right]^{q(p-1)/(p-1-q)}.
$$

Combining the preceding estimates proves (4.17). In particular, this yields $u_0 \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$. 
We next estimate

\( A_0(x, t) := \int_t^\infty \left( \frac{\omega_0(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \),

in terms of the function

\( M(x, t) := \int_t^\infty \left( \frac{[\kappa(B(x, s))]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \)

(4.19)

\[ + \left( \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right)^{\frac{p-1-q}{p-1-q}} \]

By Lemma 4.7, \( M(x, t) < \infty \) for all \( x \in \mathbb{R}^n \) and \( t > 0 \). Let us show that

\( A_0(x, t) \leq c M(x, t) \),

for all \( x \in \mathbb{R}^n \), \( t > 0 \),

where \( c \) depends only on \( n, p, q, \alpha \).

Indeed, using (4.17), and making the substitution \( \rho = 2s \) in the first term, and replacing \( s \) by \( t \leq s \) in the lower limit of integration in the second term, we obtain

\[ A_0(x, t) \leq c \int_t^\infty \left( \frac{[\kappa(B(x, 2s))]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \]

(4.20)

\[ + c \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \left( \int_s^\infty \left( \frac{\sigma(B(x, \tau))}{\tau^{n-\alpha p}} \right)^{\frac{1}{p-1}} d\tau \right)^{\frac{p-1-q}{p-1-q}} ds \]

\[ \leq c \int_t^\infty \left( \frac{[\kappa(B(x, \rho))]^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} d\rho \]

(4.21)

\[ + c \left( \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right)^{\frac{p-1-q}{p-1-q}} = c M(x, t) \]

Now letting \( d\omega_j = u_j^q d\sigma \), for \( j = 1, 2, \ldots \), we will prove the estimate

\[ \omega_j(B(x, t)) \leq c \left[ [\kappa(B(x, t))]^{q} [\omega_{j-1}(B(x, 2t))]^{\frac{1}{p-1}} \right] \]

+ \( c \left( \int_t^\infty \left( \frac{\omega_{j-1}(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right)^{q} \sigma(B(x, t)) \)

(4.21)

where \( c \) depends only on \( n, p, q, \alpha \).
We have

\[ \omega_j(B(x,t)) = \int_{B(x,t)} (W_{\alpha,p}^{\omega_j-1})^q \, d\sigma \]

\[ = \int_{B(x,t)} \left[ \int_0^\infty \left( \frac{\omega_{j-1}(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q \, d\sigma(y) \]

\[ \leq c_q \int_{B(x,t)} \left[ \int_0^t \left( \frac{\omega_{j-1}(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q \, d\sigma(y) \]

\[ + c_q \int_{B(x,t)} \left[ \int_t^\infty \left( \frac{\omega_{j-1}(B(y,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q \, d\sigma(y) \]

\[ := c_q (I + II). \]

To estimate \( I \), notice that if \( y \in B(x,t) \) and \( 0 < s < t \), then \( B(y,s) \subset B = B(x,2t) \). Hence, by (4.2) with \( d\nu = \chi_B \, d\omega_{j-1} \), we have

\[ I \leq c \int_{B(x,t)} (W_{\alpha,p}^{\nu})^q \, d\sigma \leq c [\kappa(B(x,t))]^q [\omega_{j-1}(B(x,2t))]^{\frac{q}{p-1}}. \]

We now estimate \( II \). Since \( B(y,s) \subset B(x,2s) \) if \( y \in B(x,t) \) and \( s \geq t \), it follows that \( \omega_{j-1}(B(y,s)) \leq \omega_{j-1}(B(x,2s)) \) in \( II \), and consequently

\[ II \leq c \sigma(B(x,t)) \left[ \int_t^\infty \left( \frac{\omega_{j-1}(B(x,2s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q \]

\[ \leq c_1 \sigma(B(x,t)) \left[ \int_t^\infty \left( \frac{\omega_{j-1}(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^q. \]

Combining estimates \( I \) and \( II \), we obtain (4.21) for \( j = 1, 2, \ldots \).

We next estimate

\[ (4.22) \quad A_j(x,t) := \int_t^\infty \left( \frac{\omega_j(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s}, \]
for \( j = 1, 2, \ldots \). Using (4.21), and replacing the lower limit of integration \( s \) with \( t \leq s \) in the second term, we estimate

\[
A_j(x, t) \leq c \int_t^\infty \left( \frac{[\kappa(B(x, s))]^q [\omega_{j-1}(B(x, 2s))]^{\frac{q}{p-1}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds
\]

\[
+ c \int_t^\infty \left( \frac{[\omega_{j-1}(B(x, \tau))]^{\frac{q}{p-1}}}{\tau^{n-\alpha p}} \right)^{\frac{1}{p-1}} \sigma(B(x, s))^{\frac{q}{p-1}} ds
\]

\[
\leq c \int_t^\infty \left( \frac{[\kappa(B(x, s))]^q [\omega_{j-1}(B(x, 2s))]^{\frac{q}{p-1}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds
\]

\[
+ c [A_{j-1}(x, t)]^{\frac{q}{p-1}} \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds.
\]

Applying Hölder's inequality with exponents \( \frac{p-1}{p-1-q} \) and \( \frac{p-1}{q} \) in the first integral on the right-hand side, we obtain

\[
A_j(x, t) \leq c A_{j-1}(x, t)^{\frac{q}{p-1}} \left[ \int_t^\infty \left( \frac{[\kappa(B(x, s))]^{\frac{q(p-1)}{p-1}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds \right]^{\frac{p-1-q}{p-1}}
\]

\[
+ c [A_{j-1}(x, t)]^{\frac{q}{p-1}} \int_t^\infty \left( \frac{\sigma(B(x, s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} ds
\]

\[
\leq c [A_{j-1}(x, t)]^{\frac{q}{p-1}} [M(x, t)]^{\frac{p-1-q}{p-1}}
\]

with a different constant \( c \) depending only on \( n, p, q, \alpha \).

Arguing by induction, we see that \( A_j(x, t) < \infty \) for all \( x \in \mathbb{R}^n \) and \( t > 0 \). Moreover, \( A_{j-1}(x, t) \leq A_j(x, t) \), since \( \omega_{j-1} \leq \omega_j \). Hence, from the preceding estimate we deduce

(4.23) \( A_j(x, t) \leq C M(x, t), \quad j = 1, 2, \ldots, \forall x \in \mathbb{R}^n, \ t > 0 \)

with a constant \( C \) depending only on \( n, p, q, \alpha \). An immediate consequence of (4.23) is the estimate

\[
\omega_j(B(x, t)) \leq ct^{n-\alpha p} [M(x, t)]^{p-1}, \quad j = 1, 2, \ldots, \ x \in \mathbb{R}^n, \ t > 0,
\]

where \( c \) depends only on \( n, p, q, \alpha \). In particular, \( u_j \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma) \) for all \( j = 0, 1, 2, \ldots \).

Thus, by the Monotone Convergence Theorem, there exists a nontrivial solution to equation (1.17) given by

\[
u = \lim_{j \to \infty} u_j \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma).
\]
Moreover, by (4.23), we have

\[ \int_t^\infty \left( \frac{\int_{B(x,s)} u^q \, d\sigma}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq C M(x, t) \leq C M(x, 0), \]

where the constant \( C \) depends only on \( n, p, q, \alpha \), and \( M(x, t) \to M(x, 0) \) as \( t \to 0^+ \). Notice that

\[ M(x, 0) = K_{\alpha, p, q} \sigma(x) + [W_{\alpha, p} \sigma(x)]^{\frac{q-1}{p-1}}. \]

Letting \( t \to 0 \) in (4.24) yields

\[ u(x) = W_{\alpha, p}(u^q \, d\sigma)(x) = \int_0^\infty \left( \frac{\int_{B(x,s)} u^q \, d\sigma}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \leq C M(x, 0), \]

which proves the upper bound in (4.16).

Notice that by Lemma 4.2, \( \int_{B(x,s)} u^q \, d\sigma \geq c [\kappa(B(x,s))]^{\frac{q-1}{p-1}} \). Combined with Theorem 3.4, this yields the lower bound,

\[ u(x) \geq c M(x, 0), \]

for any nontrivial solution \( u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \) of \( u \geq W_{\alpha, p}(u^q \, d\sigma) \). In particular, \( M(x, 0) \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \). Moreover, by Corollary 3.2, we see that \( \liminf_{x \to \infty} u(x) = 0 \). This completes the proof of Theorem 4.8. \( \square \)

4.4. Solutions in \( L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma) \). In this section we will prove that the solution \( u \in L^q_{\text{loc}}(\mathbb{R}^n, \sigma) \) to (1.17) constructed in the proof of Theorem 4.8 actually has the property \( u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, \sigma) \) under the additional assumption

\[ \int_B (W_{\alpha, p} \sigma_B)^{(1+q)/(p-1)} \, d\sigma < \infty, \]

for all balls \( B \in \mathbb{R}^n \).

This condition is also necessary for \( u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, \sigma) \).

**Lemma 4.9.** Let \( 1 < p < \infty \), \( 0 < q < p - 1 \), and \( 0 < \alpha < \frac{n}{p} \). Let \( \sigma \in M^+(\mathbb{R}^n) \). Suppose that (3.2), (4.15), and (4.25) hold. Then \( W_{\alpha, p} \sigma \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, \sigma) \), and \( K_{\alpha, p, q} \sigma \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, \sigma) \).

**Proof.** Let \( x \in \mathbb{R}^n \) and \( t > 0 \). We need to show

\[ I_1 := \int_{B(x,t)} (W_{\alpha, p} \sigma)^{(1+q)/(p-1)} \, d\sigma < \infty, \]

\[ I_2 := \int_{B(x,t)} (K_{\alpha, p, q} \sigma)^{1+q} \, d\sigma < \infty. \]

To estimate \( I_1 \), we split \( W_{\alpha, p} \sigma \) into two integrals, and estimate them separately,

\[ I := \int_{B(x,t)} \left[ \int_0^t \left( \frac{\sigma(B(y,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^{(1+q)/(p-1)} \, d\sigma(y), \]
\[ II := \int_{B(x,t)} \left[ \int_t^\infty \left( \frac{\sigma(B(y,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^{\frac{1+q(p-1)}{p-1-q}} d\sigma(y). \]

We first estimate \( II \). If \( r \geq t \) and \( y \in B(x,t) \), then \( B(y,r) \subset B(x,2r) \), and hence, making the substitution \( s = 2r \), we get

\[ II \leq c\sigma(B(x,t)) \left[ \int_t^\infty \left( \frac{\sigma(B(x,s))}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{\frac{1+q(p-1)}{p-1-q}} < \infty, \]

by Corollary 3.2, where \( c = c(p,q,\alpha,n) \).

To estimate \( I \), notice that, if \( 0 < r < t \) and \( y \in B(x,t) \), then \( B(y,r) \subset B = B(x,2t) \). Hence,

\[ \int_0^t \left( \frac{\sigma(B(y,r))}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \leq W_{\alpha,p} \sigma_B(y), \]

which by (4.25) yields

\[ (4.26) \quad I \leq \int_B \left( W_{\alpha,p} \sigma_B \right)^{\frac{1+q(p-1)}{p-1-q}} d\sigma < \infty. \]

Thus, \( I_1 < \infty \).

We estimate \( I_2 \) in a similar way, splitting \( K_{\alpha,p,q} \sigma \) into two integrals,

\[ III := \int_{B(x,t)} \left[ \int_0^t \left( \frac{[\kappa(B(y,r))]^{\frac{q(p-1)}{p-1-q}}}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^{1+q} d\sigma(y), \]

\[ IV := \int_{B(x,t)} \left[ \int_t^\infty \left( \frac{[\kappa(B(y,r))]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{1+q} d\sigma(y). \]

To show that \( IV < \infty \), notice that \( [\kappa(B(y,r))] \subset [\kappa(B(x,2r))] \) if \( t \leq r \) and \( y \in B(x,t) \), which yields

\[ IV \leq c\sigma(B(x,t)) \left[ \int_t^\infty \left( \frac{[\kappa(B(x,s))]^{\frac{q(p-1)}{p-1-q}}}{s^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{ds}{s} \right]^{1+q} < \infty, \]

by Lemma 4.7, using as above the substitution \( s = 2r \).

Finally, we estimate \( III \). If \( r < t \) and \( y \in B(x,t) \), we have \( B(y,r) \subset B = B(x,2t) \). Then \( [\kappa(B(y,r))] = [\kappa(B(y,r) \cap B)] \), and consequently,

\[ III = \int_{B(x,t)} \left[ \int_0^t \left( \frac{[\kappa(B(y,r) \cap B)]^{\frac{q(p-1)}{p-1-q}}}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \right]^{1+q} d\sigma(y). \]
Since (4.25) holds, applying Theorem 3.5 in [CV14], with $\sigma_B$ in place of $\sigma$, we conclude that there exists a global solution $u_B \in L^{1+\theta}(\mathbb{R}^n, d\sigma_B)$ to the equation $u_B = W_{\alpha,p}(u^q_B d\sigma_B)$. By (4.16) with $\sigma_B$ in place of $\sigma$ we have

$$
\int_0^\infty \left( \frac{[\kappa(B(y,r) \cap B)]^q (r^{-1}- \theta)}{r^{n-\alpha p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} \leq C u_B(y), \quad y \in \mathbb{R}^n.
$$

Hence,

$$III \leq c \int_B (u_B)^{1+q} \ d\sigma < \infty.$$

Thus, both $II$ and $IV$ are finite, i.e., $I_2 < \infty$. \hfill \Box

**Theorem 4.10.** Suppose that (3.2), (4.15), and (4.25) hold. Then there exists a nontrivial solution $u \in L^{1+\theta}_{loc}(\mathbb{R}^n, d\sigma)$ to (1.17). Moreover, $u$ satisfies (4.16).

Conditions (3.2), (4.15), and (4.25) are necessary in order that a nontrivial solution $u \in L^{1+\theta}_{loc}(\mathbb{R}^n, d\sigma)$ to (1.17) exist.

**Proof.** By Theorem 4.8, there exists a nontrivial solution $u \in L^{\theta}_{loc}(\mathbb{R}^n, d\sigma)$ to the equation $u = W_{\alpha,p}(u^q d\sigma)$ such that (4.16) holds. The upper estimate in (4.16) actually yields $u \in L^{1+\theta}_{loc}(\mathbb{R}^n, d\sigma)$ by Lemma 4.9.

Conditions (3.2) and (4.15) are necessary for the existence of any nontrivial solution to (1.17) by Theorem 4.8. Condition (4.25) is necessary as well which follows from (3.9). \hfill \Box

5. PROOFS OF THEOREM 1.1, THEOREM 1.2, AND THEOREM 1.3

We will need the following version of the well-known comparison principle.

**Lemma 5.1.** Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Suppose that $\mu, \nu \in L^{-1,p}(\Omega)$, and $0 \leq \mu \leq \nu$. Suppose $u \in L^\theta_0(\Omega)$ and $v \in W^{1,p}(\Omega)$ are distributional solutions to the equations $-\Delta_p u = \mu$ and $-\Delta_p v = \nu$ in $\Omega$, respectively. Then $u \leq v$ a.e. in $\Omega$.

**Proof.** The proof is standard and relies on the use of the test function $\phi = u - \min\{u, v\} \in L^{1/p}_0(\Omega)$; see the proof of Lemma 3.22 in [HKM06]. \hfill \Box

The next version of the comparison principle is more delicate, and we provide a detailed proof.

**Lemma 5.2.** Suppose that $\Omega$ is a bounded open set in $\mathbb{R}^n$. Suppose that $\mu, \nu$ are nonnegative finite Borel measures on $\Omega$ such that $\mu \leq \nu$, where $\mu$ is absolutely continuous with respect to the $p$-capacity $\text{cap}_p(\cdot)$. If $u$ and $v$ are nonnegative $p$-superharmonic functions in $\Omega$ with Riesz measures $\mu$ and $\nu$, respectively, and $\min\{u, k\} \in L^1(\Omega)$ for all $k > 0$, then $u \leq v$ a.e.

**Proof.** Notice that $v_k = \min\{v, k\} \in W^{1,p}(\Omega)$ is $p$-superharmonic, and the corresponding Riesz measures $\nu_k = -\Delta_p v_k$ converge weakly to $\nu$ as $k \to \infty$.
(\cite{HKM06}, Sec. 7; \cite{KM92}). Let \( \mu_k = \chi_{\{v < k\}} \mu \), for \( k > 0 \). Then clearly, 
\( \nu_k|_{\{v < k\}} = \nu|_{\{v < k\}} \), and consequently \( \mu_k \leq \nu_k \).

For any \( \phi \in \mathcal{C}_0^\infty(\Omega) \), we have
\[
\left| \int_{\Omega} \phi \, d\mu_k - \int_{\Omega} \phi \, d\mu \right| = \left| \int_{v \geq k} \phi \, d\mu \right| \leq \int_{v \geq k} |\phi| \, d\mu \leq \max_{\Omega} |\phi| \{ \{v \geq k\} \} .
\]

We have \( \mu(\{v \geq k\}) \to \mu(\{v = \infty\}) \) as \( k \to \infty \). Since \( \mu \) is absolutely continuous with respect to \( \text{cap}_p(\cdot) \), and \( v \) is \( p \)-superharmonic, it follows that 
\( \mu(\{v = \infty\}) = 0 \), which yields \( \mu_k \to \mu \) weakly.

Let us denote by \( u_k \) the unique solution to the equation
\[
-\Delta_p u_k = \mu_k, \quad u_k \in L_0^{1,p}(\Omega),
\]
where \( \mu_k \in L^{-1,p'}(\Omega) \) since \( \mu_k \leq \mu \in L^{-1,p'}(\Omega) \). By Lemma 5.1, we have 
\( u_k \leq v_k \) for every \( k > 0 \), and \( u_k \leq u_j \) if \( k \leq j \). Passing to the limit as \( k \to \infty \), we obtain \( \bar{u} \leq v \), where \( \bar{u} = \lim_{k \to \infty} u_k \). Since \( \mu_k \to \mu \) weakly, it follows that \( \bar{u} \) is a \( p \)-superharmonic solution to the equation 
\( -\Delta_p \bar{u} = \mu \) where \( \min(\bar{u}, j) \in L_0^{1,p}(\Omega) \) for every \( j > 0 \). Since \( \mu \) is absolutely continuous with respect to the \( p \)-capacity \( \text{cap}_p(\cdot) \), and \( \min(\bar{u}, j) \in L_0^{1,p}(\Omega) \) for every \( k > 0 \), it follows by the uniqueness theorem (see \cite{Kil02}, and the references given there) that \( \bar{u} = u \) a.e., and consequently \( u \leq v \) a.e. \( \square \)

**Proof of Theorem 1.1.** Let \( 1 < p < n \). Suppose both (1.11) and (1.14) hold. Then by Theorem 4.8 there exists a nontrivial solution \( v \in L^q_0(\partial \sigma) \) of the equation
\[
(5.1) \quad v = K \mathbf{W}_{1,p}(v^q \, d\sigma) \quad \text{in } \mathbb{R}^n,
\]
where \( K \) is the constant in Theorem 2.4. By Theorem 3.4 (with \( K_p-1 \sigma \) in place of \( \sigma \)),
\[
v \geq C K^{-\frac{p-1}{p'-1-q}} (\mathbf{W}_{1,p} \sigma)^{-\frac{p-1}{p'-1-q}},
\]
where \( C \) is the constant in (3.9). We set
\[
w_0 = c_0 (\mathbf{W}_{1,p} \sigma)^{-\frac{p-1}{p'-1-q}}, \quad d\omega_0 = w_0^q \, d\sigma,
\]
where \( c_0 > 0 \) is a small constant to be determined later. In particular, we pick \( c_0 \leq C K^{-\frac{p-1}{p'-1-q}} \) so that
\[
w_0 \leq \frac{c_0}{C K^{-\frac{p-1}{p'-1-q}}} v \leq v.
\]
Clearly \( \omega_0 \) is a locally finite Borel measure since \( d\omega_0 \leq v^q \, d\sigma \) and \( v \in L^q_0(\partial \sigma) \). By Lemma 3.6 with \( \alpha = 1 \), \( \omega_0 \) is absolutely continuous with respect to \( \text{cap}_p(\cdot) \). Hence there exists a unique renormalized solution (see \cite{Kil02}) to the equation
\[
(5.2) \quad -\Delta_p u_1^k = \omega_0 \chi_{B(0,2^k)} \text{ in } B(0,2^k), \quad u_1^k = 0 \text{ on } \partial B(0,2^k),
\]

where \( \omega_0 \) is the solution to the problem
\[
(5.3) \quad -\Delta_p \omega_0 = -K \chi_{B(0,1)} \text{ in } B(0,1), \quad \omega_0|_{\partial B(0,1)} = 0.
\]
where $k = 0, 1, 2, \ldots$. Notice that the sequence $\{u^k_1\}$ is increasing by the comparison principle (Lemma 5.2). Moreover, by Theorem 2.4,

$$0 \leq u^k_1 \leq K W_{1,p}(\omega_1 \chi_{B(0,2^k)}) \leq K W_{1,p} \omega_0 \leq K W_{1,p}(v^q d\sigma) = v.$$  

Letting $u_1 = \lim_{k \to \infty} u^k_1$ and using the weak continuity of the $p$-Laplace operator (Theorem 2.3) and the Monotone Convergence Theorem, we see that $u_1$ is a $p$-superharmonic solution to the equation $-\Delta_p u_1 = \omega_0$ in $\mathbb{R}^n$. Since $u_1^k \leq v$, it follows that $u_1 \leq v$, and hence $\lim \inf_{|x| \to \infty} u_1(x) = 0$. By Theorem 2.4,

$$0 \leq u_1 \leq K W_{1,p} \omega_0 \leq K W_{1,p}(v^q d\sigma) = v.$$  

We deduce, using (3.10),

$$u_1 \geq \frac{1}{K} W_{1,p} \omega_0 = \frac{c_0^q}{K} W_{1,p} \left[ (W_{1,p} \sigma)^{q(p-1)} d\sigma \right]$$

$$\geq \frac{c_0^q}{K} K^{p-1-q} \left( W_{1,p} \sigma \right)^{p-1} = \frac{c_0^q}{K} K^{p-1-q} w_0.$$  

Hence, $c_0 \leq \min \left[ \left( \frac{c_0^q}{K} \right)^{p-1} K^{-1} \left( C K \right)^{p-1} \right]$, we have $v \geq u_1 \geq w_0$.

Let us now construct a sequence $u_j (j = 1, 2, \ldots)$ of functions which are $p$-superharmonic in $\mathbb{R}^n$, $u_j \in L^q_{\text{loc}} (d\sigma)$, so that

$$\begin{align*}
-\Delta_p u_j &= \sigma u^q_{j-1} \quad \text{in } \mathbb{R}^n, \quad j = 2, 3, \ldots, \\
c_j (W_{1,p} \sigma)^{p-1} \leq u_j &\leq v, \\
0 \leq u_{j-1} &\leq u_j, \\
\lim \inf_{|x| \to \infty} u_j(x) &= 0.
\end{align*}$$

(5.3)

Here $c_1 = c_0^{q(p-1)} K^{-1}$, and

$$c_j = \left( \frac{c_0^q}{K} \right)^{j(p-1)} \left( K^{-1} \right)^{j} \left( \frac{q(p-1)}{p-1} \right)^j c_0 \left( \frac{q(p-1)}{p-1} \right)^j, \quad j = 2, 3, \ldots.$$

Suppose that $u_1, \ldots, u_{j-1}$ have been constructed. Let $d\omega_{j-1} = u^q_{j-1} d\sigma$. Then $\omega_{j-1} \in M^+(\mathbb{R}^n)$, since $u_{j-1} \leq v$, where $v \in L^q_{\text{loc}}(d\sigma)$, and $\omega_{j-1}$ is absolutely continuous with respect to the $p$-capacity. Applying Lemma 5.2 again, we see that there exists a renormalized solution $u^k_j$ to the equation

$$-\Delta_p u^k_j = \omega_{j-1} \chi_{B(0,2^k)} \quad \text{in } B(0,2^k), \quad u^k_j = 0 \quad \text{on } \partial B(0,2^k).$$

Arguing by induction, let $u^k_{j-1}$ be the unique solution of the equation

$$-\Delta_p u^k_{j-1} = \omega_{j-2} \chi_{B(0,2^k)} \quad \text{in } B(0,2^k), \quad u^k_{j-1} = 0 \quad \text{on } \partial B(0,2^k).$$

Since $u_{j-2} \leq u_{j-1}$, by Lemma 5.2, we deduce $u^k_j \geq u^k_{j-1}$. Using Theorem 2.4, we have

$$0 \leq u^k_j \leq K W_{1,p} \left[ \omega_{j-1} \chi_{B(0,2^k)} \right] \leq K W_{1,p}(v^q d\sigma) = v.$$


Letting $u_j = \lim_{k \to \infty} u_j^k$ and using again the weak continuity of the $p$-Laplacian and the Monotone Convergence Theorem, we deduce that $u_j$ is a solution to the equation $-\Delta_p u_j = \sigma u_j^q$ on $\mathbb{R}^n$.

Moreover, $u_j \leq v$ since $u_j^k \leq v$ and hence $\liminf_{x \to \infty} u_j(x) = 0$. Furthermore, we have $u_{j-1} \leq u_j$ since $u_{j-1}^k \leq u_j^k$, for all $k \geq 1$. On the other hand, applying Theorem 2.4 and Lemma 3.5, and arguing by induction, we obtain

$$u_j \geq \frac{1}{K} W_{1,p}(u_{j-1}^q d\sigma) \geq \frac{1}{K} W_{1,p} \left[ c_{j-1}^q (W_{1,p} \sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma \right]$$

$$\geq c_{p-1-q}^{\frac{q}{p-1-q}} K^{-1} (W_{1,p} \sigma)^{\frac{p-1}{p-1-q}} = c_j (W_{1,p} \sigma)^{\frac{p-1}{p-1-q}}.$$  

Letting $u = \lim_{j \to \infty} u_j$ and using Theorem 2.3 together with the Monotone Convergence Theorem, we see that $u$ is a solution to the equation $-\Delta_p u = \sigma u^q$ on $\mathbb{R}^n$. Hence, by Theorem 2.4, $u \geq \frac{1}{K} W_{1,p}(w^q d\sigma)$. Applying Theorem 4.8, we deduce the lower bound in (1.15). The upper bound follows from $u \leq v$ and Theorem 4.8. We also have $\liminf_{x \to \infty} u(x) = 0$ since $u \leq v$, and $(1.15)$ follows by Corollary 3.2. Notice that by Remark 4.1 we can use here the potentials $K_{1,p,q} \sigma$ defined either in terms of $\kappa(B)$, or $\kappa(B)$ in the case $\alpha = 1$, since they are equivalent.

Let us now prove the minimality of $u$. Suppose $w \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ is any nontrivial $p$-superharmonic solution to (1.1). Let $d\nu = w^q d\sigma$. Then by Theorem 2.4, $w \geq \frac{1}{K} W_{1,p}(w^q d\sigma)$. Hence, by Lemma 3.6 with $\alpha = 1$, $\nu$ is absolutely continuous with respect to the $p$-capacity. By Theorem 3.4 with $K^{1-p} \sigma$ in place of $\sigma$,

$$w \geq C K^{\frac{1-p}{p-1-q}} (W_{1,p} \sigma)^{\frac{p-1}{p-1-q}}.$$ 

Note that by the choice of $c_0$ above, we have $\omega_0 \leq \nu$. Therefore, by Lemma 5.2, the function $u_j^k$ defined by (5.2) satisfies the inequality $u_j^k \leq w$ in $B(0, \delta^k)$ for every $k > 0$, and consequently $u_j = \lim_{k \to \infty} u_j^k \leq w$. Repeating this argument by induction, we obtain $u_j \leq w$ for every $j = 1, 2, \ldots$. It follows that $\lim_{j \to \infty} u_j = u \leq w$, which proves the minimality of $u$. This completes the proof of statement (i) of Theorem 1.1.

To prove statement (ii), suppose that $u$ is a supersolution of (1.1). Then by Theorem 2.4, $u \geq \frac{1}{K} W_{1,p}(w^q d\sigma)$. Hence, by Theorem 4.8, both (1.11) and (1.14) hold.

Statement (iii) is an immediate consequence of Theorem 2.4 (ii).

We now are in a position to give a characterization of $W^{1,p}_{\text{loc}}$-solutions of (1.1) stated in Theorem 1.2. We remark that a global analogue of condition (1.16), as was shown earlier by the authors [CV14], is necessary and sufficient for the existence of a finite energy solution $u \in L^1_{0,p}(\mathbb{R}^n)$ to (1.1).

**Proof of Theorem 1.2.** By Theorem 1.1, if both (1.11) and (1.14) hold, then there exists a $p$-superharmonic solution $u$ to (1.1) such that (1.15) holds.
Moreover, by (1.10), there is a constant $K > 0$ such that

$$u \geq \frac{1}{K} W_{1,p}(u^q d\sigma).$$

(5.4)

Suppose that additionally (1.16) holds for all balls $B$. Applying Theorem 4.10, we see that there exists a solution $v \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to the integral equation (1.17) such that (4.16) holds with $\alpha = 1$. Hence, there exists a constant $c > 0$ such that

$$c^{-1} u(x) \leq v(x) \leq c u(x) \quad d\sigma - \text{a.e.}$$

Consequently, $u \in L^{1+q}_{\text{loc}}(\mathbb{R}^n, d\sigma)$, and

$$\int_B W_{1,p}(u^q d\sigma_B) u^q d\sigma \leq C \int_B u^{1+q} d\sigma < \infty,$$

for every ball $B$. By a local version of Wolff’s inequality (2.2), we see that $u^q d\sigma \in W_{1+q}^{-1,p'}(\mathbb{R}^n)$. Applying Lemma 3.3, we conclude that $u \in W_{1,p}^1(\mathbb{R}^n)$.

Conversely, if there exists a nontrivial solution $u \in W_{1,p}^1(\mathbb{R}^n)$ to (1.1), then clearly a quasi-continuous representative of $u$ is a $p$-superharmonic solution, and $u^q d\sigma \in W_{1+q}^{-1,p'}(\mathbb{R}^n)$. It follows from (1.10) that

$$u \leq K W_{1,p}(u^q d\sigma).$$

By Wolff’s inequality (2.1), for every ball $B$,

$$\int_B u^{1+q} d\sigma \leq K \int_B W_{1,p}(u^q d\sigma_B) u^q d\sigma_B \leq C K \|u^q d\sigma_B\|_{L^{1+q}^{-1,p'}(\mathbb{R}^n)} < \infty.$$  

By Theorem 3.4, estimate (3.9) holds. Combining these estimates, we obtain that (1.16) holds for all balls $B$. By Theorem 1.1, both (1.11) and (1.14) hold as well, which completes the proof of Theorem 1.2.

**Proof of Theorem 1.3.** We remark that (1.2) is understood in the sense

$$u = I_{2\alpha}(u^q d\sigma) \quad \text{in } \mathbb{R}^n, \quad u \geq 0.$$  

Since $I_{2\alpha}(u^q d\sigma) = W_{\alpha,2}(u^q d\sigma)$, Theorem 1.3 is a special case of Theorem 4.8 with $p = 2$.  

**Remark 5.3.** (1) Direct analogues of our main theorems hold for the more general quasilinear $\mathcal{A}$-Laplace operator $\text{div } \mathcal{A}(x, \nabla u)$ in place of $\Delta p$:

$$(5.5) \quad -\text{div } \mathcal{A}(x, \nabla u) = \sigma u^q \quad \text{in } \mathbb{R}^n, \quad \liminf_{x \to \infty} u = 0,$$

under the standard monotonicity and boundedness assumptions on $\mathcal{A}$ which guarantee that the Wolff potential estimates (1.10) hold (see, e.g., [KM94], [KuMi14], [TW02], [PV08]).

(2) Similar results hold for the fully nonlinear $k$-Hessian operator $F_k$ ($k = 1, 2, \ldots, n$) defined by

$$F_k[u] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

(5.6)
where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the Hessian matrix $D^2u$ on $\mathbb{R}^n$. In other words, $F_k[u]$ is the sum of the $k \times k$ principal minors of $D^2u$, which coincides with the Laplacian $F_1[u] = \Delta u$ if $k = 1$.

Local Wolff potential estimates for the equation $F_k[u] = \mu$, where $\mu \in M^+(\mathbb{R}^n)$, in this case are due to Labutin [Lab02] (see also [TW02]); global estimates analogous to (1.10) can be found in [PV08]. The corresponding “sublinear” equation can be written in the form

\begin{equation}
F_k[u] = \sigma \, |u|^q \quad \text{in } \mathbb{R}^n, \quad \limsup_{x \to \infty} u = 0,
\end{equation}

where $0 < q < k$, and $u \leq 0$ is a $k$-convex function.

Similar equations in the supercritical case $q > k$ were considered in [PV08], and in the critical case $q = k$, in [JV10]. Intrinsic nonlinear potentials of the type $K_{\alpha,p,q} \sigma$ do not play a role there. However, the reduction of both (5.5) and (5.7) to (1.17) is carried over as in the case of the $p$-Laplacian treated above. See details in [PV08], [JV10], [JV12], [CV14].

### 6. Example

Suppose $0 < q < 1$, $n \geq 2$, and $0 < \alpha < \frac{n}{2}$. In this section we construct $\sigma \in M^+(\mathbb{R}^n)$ such that $\kappa(B(0, R)) < \infty$ for every $R > 0$, and the equation

\begin{equation}
\begin{cases}
( - \Delta )^\alpha u = \sigma \\
\liminf_{x \to \infty} u(x) = 0,
\end{cases}
\end{equation}

has a weak solution, but the equation

\begin{equation}
\begin{cases}
( - \Delta )^\alpha u = \sigma \, u^q \\
\liminf_{x \to \infty} u(x) = 0,
\end{cases}
\end{equation}

has no weak solutions. The condition $\kappa(B(0, R)) < \infty$ ensures that locally, for $\sigma B(0,R)$, in place of $\sigma$, weak solutions exist.

In other words, we need to construct a measure $\sigma$ such that $I_{2\alpha}\sigma < \infty$ a.e., that is,

\begin{equation}
\int_1^\infty \sigma(B(0,R)) \frac{dR}{R^{n-2\alpha}} < \infty,
\end{equation}

and $\kappa(B(0, R)) < \infty$ for every $R > 0$, but

\begin{equation}
\int_1^\infty \frac{[\kappa(B(0,R))]^{\frac{q}{2q-2\alpha}}}{R^{n-2\alpha}} \frac{dR}{R} = \infty.
\end{equation}

This requires $\kappa(B(0,R))^{\frac{q}{2q-2\alpha}}$ to grow much faster than $\sigma(B(0,R))$ as $R \to \infty$.

**Lemma 6.1.** Let $0 < q < 1$ and $0 < 2\alpha < n$. If

\begin{equation}
||I_{2\alpha}\nu||_{L^{q/(n\nu)}} \leq \kappa(\sigma) \nu(\mathbb{R}^n), \quad \forall \nu \in M^+(\mathbb{R}^n),
\end{equation}

then

\begin{equation}
\mathcal{K}(\sigma) := \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{(n-2\alpha)q}} \leq \kappa(\sigma)^q.
\end{equation}
Hence, (6.5) follows from the preceding estimate and Lemma 6.2.

Lemma 6.2. Let \( 0 < q < 1 \) and \( 0 < 2\alpha < n \). If \( d\sigma = \sigma(|x|)dx \) is radially symmetric then condition (6.4) is equivalent to \( 1_{2\alpha} \sigma \in L^{\frac{1}{n-1}}(d\sigma) \), and hence is not only necessary, but also sufficient for (6.3). Moreover, there exists \( c = c(q, \alpha, n) > 0 \) such that the least constant \( \kappa(\sigma) \) in (6.3) satisfies

\[
K(\sigma) \leq \kappa(\sigma)^q \leq cK(\sigma).
\]

Corollary 6.3. Let \( \sigma_{R, \gamma} = \chi_{B(0,R)}|x|^{-\gamma} \), where \( 0 \leq \gamma < n - q(n - 2\alpha) \) and \( R > 0 \). Then

\[
\frac{\omega_n}{n - \gamma - q(n - 2\alpha)} \leq \frac{\kappa(\sigma_{R, \gamma})^q}{R^{n-\gamma-q(n-2\alpha)}} \leq \frac{c}{n - \gamma - q(n - 2\alpha)},
\]

where \( c = c(q, \alpha, n) \), and \( \omega_n = |S^{n-1}| \) is the surface area of the unit sphere.

Proof. Letting \( x = 0 \) in (6.4) we have

\[
K(\sigma_{R, \gamma}) = \int_{|y|<R} \frac{|y|^{-\gamma}}{|y|^{q(n-2\alpha)}} dy = \omega_n \int_0^R r^{-\gamma-q(n-2\alpha)+n-1} dr
= \frac{\omega_n}{n - \gamma - q(n - 2\alpha)} R^{n-\gamma-q(n-2\alpha)}.
\]

Hence, (6.5) follows from the preceding estimate and Lemma 6.2. \( \square \)

Let

\[
\sigma = \sum_{k=1}^{\infty} c_k \sigma_{k, \gamma_k}(x + x_k),
\]

where \( |x_k| = k, \gamma_k = n - q(n - 2\alpha) - \epsilon_k \), and \( c_k, \epsilon_k \) are picked so that \( \sum_{k=1}^{\infty} c_k < \infty \), and \( \epsilon_k \to 0 \) fast enough; it suffices to set

\[
c_k = \frac{1}{k^2}, \quad \epsilon_k = \frac{1}{k^{n+2}}.
\]

Let \( R > 0 \). Clearly,

\[
\sigma(B(0,R)) \leq \sum_{k=1}^{\infty} c_k \sigma_{k, \gamma_k}(B(x_k, R)) \leq \sum_{k=1}^{\infty} c_k \sigma_{k, \gamma_k}(B(0,R)).
\]

Here

\[
\sigma_{k, \gamma_k}(B(0,R)) = \omega_n \int_0^{\min(k,R)} r^{-\gamma_k+n-1} dr
\]
\[
= \frac{\omega_n}{n - \gamma_k} \min(k,R)^{n-\gamma_k} \leq \frac{\omega_n}{q(n-2\alpha)} \min(k,R)^{q(n-2\alpha)+\epsilon_k}.
\]
Hence, for $R \geq 1$

$$
\sigma(B(0, R)) \leq \frac{\omega_n}{q(n-2\alpha)} \sum_{k=1}^N c_k k^q(n-2\alpha) + \frac{\omega_n}{q(n-2\alpha)} R^{q(n-2\alpha) + \epsilon N} \sum_{k=N}^\infty c_k.
$$

Picking $N$ large enough so that $\epsilon_N < (1-q)(n-2\alpha)$, we obtain (6.1).

Using Corollary 6.3, we will show that $\kappa(B(0, R)) < \infty$ for every $R > 0$, since $\epsilon_k > 0$, and consequently $\gamma_k$ is below the critical exponent $n-q(n-2\alpha)$.

Indeed, since $\kappa(\sigma)$ is obviously invariant under translations,

$$
(6.7) \quad \kappa(B(0, R))^q \leq \sum_{k=1}^\infty c_k \kappa(\chi_{B(x_k, R)} \sigma_{k, \gamma_k})^q.
$$

If $k > 2R$, then $|x - x_k| < R$, $|x| < k$ and $|x_k| = k$ yield $k > |x| > k/2$. Consequently, $\chi_{B(x_k, R)} \sigma_{k, \gamma_k}(x) \approx \frac{c}{k^{\gamma_k}} B(x_k, R)$. It follows that, for $\nu \in M^+(\mathbb{R}^n)$,

$$
||I_{2\alpha}^\nu||_{L^q(\chi_{B(x_k, R)} \sigma_{k, \gamma_k})} \leq \frac{c}{k^{\gamma_k}} ||I_{2\alpha}^\nu||_{L^q(\chi_{B(x_k, R)} \sigma_{k, \gamma_k})} \leq \frac{c}{k^{\gamma_k}} \kappa(\chi_{B(x_k, R)})^q \nu(\mathbb{R}^n)^q.
$$

Corollary 6.3 with $\gamma = 0$, yields $\kappa(\chi_{B(x_k, R)})^q \approx R^{n-q(n-2\alpha)}$. Hence,

$$
\kappa(\chi_{B(x_k, R)} \sigma_{k, \gamma_k})^q \leq \frac{c}{k^{\gamma_k}} R^{n-q(n-2\alpha)}.
$$

From this and (6.7) we deduce

$$
\kappa(B(0, R))^q \leq \sum_{1 \leq k \leq 2R} c_k [\kappa(\sigma_{k, \gamma_k})]^q + cR^{n-q(n-2\alpha)} \sum_{k>2R} c_k/k^{\gamma_k} < \infty.
$$

Note that each term in the first sum is finite by Corollary 6.3 since $0 < \gamma_k < n - q(n - 2\alpha)$ is below the critical exponent.

Let us show now that (6.2) holds. By Lemma 6.1,

$$
\kappa(B(0, R))^q \geq \mathcal{K}(\sigma_{B(0, R)}) = \sup_{x \in \mathbb{R}^n} \sum_{k=1}^\infty c_k \int_{|y| < R} \frac{\sigma_{k, \gamma_k}(y + x_k)}{|x - y|^{q(n-2\alpha)}} dy
$$

$$
\geq \sup_{k \geq 1} c_k \int_{|y| < R} \frac{\sigma_{k, \gamma_k}(y + x_k)}{|x_k + y|^{q(n-2\alpha)}} dy = \sup_{k \geq 1} c_k \int_{|z - x_k| < R} \frac{\sigma_{k, \gamma_k}(z)}{|z|^{q(n-2\alpha)}} dz
$$

$$
\geq \sup_{k \geq 1} c_k \int_{|z - x_k| < R, |z| < \gamma_k + q(n-2\alpha)} \frac{dz}{|z|^{n-\epsilon_k}} = \sup_{k \geq 1} c_k \int_{|z - x_k| < R, |z| < \gamma_k} \frac{dz}{|z|^{n-\epsilon_k}}.
$$

If $k \leq \frac{R}{2}$, then $B(0, k) \subset B(x_k, R)$. Hence, for $R > 2$,

$$
\kappa(B(0, R))^q \geq \sup_{1 \leq k \leq \frac{R}{2}} c_k \int_{|z| < \frac{R}{2}} \frac{dz}{|z|^{n-\epsilon_k}} \geq \omega_n \sup_{1 \leq k \leq \frac{R}{2}} \frac{c_k}{\epsilon_k}
$$

$$
\geq \omega_n \sup_{\frac{R}{4} \leq k \leq \frac{R}{2}} \frac{c_k}{\epsilon_k} \geq \omega_n 4^{-n} R^n.
$$
Since $\frac{n}{1-q} > n - 2\alpha$, the preceding estimate yields (6.2) as claimed.

7. Equations with singular gradient terms

In this section, we investigate the relationship between (1.1) and (1.5), and prove Theorem 1.4 using the framework of (locally) renormalized solutions. We will show that transformation (1.7) sends a solution $u$ of (1.1) to a solution $v$ of (1.5), but in the opposite direction, a solution $v$ of (1.5) generally gives rise merely to a supersolution $u$ of (1.1). Note that $u$ is a genuine solution only under some additional assumptions on $v$ as is clear from the following example.

For $0 < q < 1$, $p = 2$, $n \geq 3$, and $\sigma = 0$, obviously, $v = c|x|^{(1-q)(2-n)}$ is a weak solution of (1.5) for an appropriate $c > 0$, but the corresponding $u$, which is a constant multiple of $|x|^{2-n}$, is only superharmonic, and not harmonic. Thus, in this case $v$ satisfies (1.24), but not (1.25).

Proof of Theorem 1.4. To prove (i), suppose $u$ is a $p$-superharmonic solution to equation (1.1). Let $\gamma = \frac{p-1-q}{p-1}$, $v = \frac{1}{\gamma} u^\gamma$ and

$$u_k = \min\left(u, (\gamma k)^{\frac{1}{\gamma}}\right), \quad v_k = \min(v, k), \quad k = 1, 2, \ldots.$$  

Notice that $v$ is $p$-superharmonic since $x \mapsto x^\gamma$ is concave and increasing. We have

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla \phi \, dx = \int_{\mathbb{R}^n} u^q \phi \, d\sigma, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n).$$

By Theorem 3.15 in [KKT09], $u$ is a (locally) renormalized solution to (1.1). Therefore,

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla (h(u) \phi) \, dx = \int_{\mathbb{R}^n} u^q h(u) \phi \, d\sigma, \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^n) \text{ and } h \in W^{1,\infty}(\mathbb{R}) \text{ with } h' \text{ having compact support.}$$

Suppose $\phi \in C_0^\infty(\mathbb{R}^n)$ and $h(u) = \frac{1}{u_k}$. Then

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla \left(\frac{\phi}{u_k}\right) \, dx = \int_{\mathbb{R}^n} u^q \frac{\phi}{u_k} \, d\sigma.$$  

Consequently,

$$\int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla \frac{1}{u_k} \, dx$$

$$= \int_{\mathbb{R}^n} u^q \frac{\phi}{u_k} \, d\sigma + q \int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla u_k \frac{\phi}{u_k^{1+q}} \, dx.$$  

Notice that $Du = (\gamma v)^{\frac{1}{\gamma}-1} Dv$, and so

$$|Du|^{p-1} = (\gamma v)^{\frac{2}{\gamma}} |Dv|^{p-1}.$$  

Since $u$ is $p$-superharmonic,

$$\gamma v^\frac{2}{\gamma} |Dv|^{p-1} \in L^1_{\text{loc}}(\mathbb{R}^n).$$
From this it follows,
\[
\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \frac{(\gamma v)^{\frac{q}{2}}}{v_k(\gamma v_k)^{\frac{q}{2}}} \, dx
\]
(7.5)
\[
= \int_{\mathbb{R}^n} (\gamma v)^{\frac{q}{2}} \frac{\phi}{v_k(\gamma v_k)^{\frac{q}{2}}} \, d\sigma + b \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \frac{(\gamma v)^{\frac{q}{2}} \phi}{v_k(\gamma v_k)^{\frac{q}{2}}} \, dx.
\]
Let \( E = \text{supp}(\phi) \); then \( v_1 \geq \delta_E > 0 \) a.e., and hence q.e., since \( v_1 \) is a positive superharmonic function. Notice that the sequence \( \{v_k\} \) is increasing, so that \( v_k \geq \delta_E > 0 \) q.e. Consequently,
\[
|Dv|^{p-2} Dv \cdot \nabla \phi \frac{(\gamma v)^{\frac{q}{2}}}{v_k(\gamma v_k)^{\frac{q}{2}}} \leq \frac{||\nabla \phi||_{L^\infty(\mathbb{R}^n)}}{(\gamma \delta_E)^{\frac{q}{2}}} |Dv|^{p-1}(\gamma v)^{\frac{q}{2}} \text{ on } E.
\]
Using (7.4) and the Dominated Convergence Theorem, we obtain
\[
(7.6) \quad \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \frac{(\gamma v)^{\frac{q}{2}}}{v_k(\gamma v_k)^{\frac{q}{2}}} \, dx \rightarrow \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \, dx,
\]
as \( k \rightarrow \infty \), where the right-hand side is obviously finite.
Assuming temporarily that \( \phi \geq 0 \), we obtain from (7.5),
\[
0 \leq b \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \frac{(\gamma v)^{\frac{q}{2}}}{v_k(\gamma v_k)^{\frac{q}{2}}} \, dx
\]
(7.7)
\[
\leq \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi \frac{(\gamma v)^{\frac{q}{2}}}{v_k(\gamma v_k)^{\frac{q}{2}}} \, dx \leq C,
\]
where by (7.6), \( C \) does not depend on \( k \). Clearly,
\[
0 \leq |Dv|^{p-2} Dv \cdot \nabla v_k \frac{(\gamma v)^{\frac{q}{2}} \phi}{v_k(\gamma v_k)^{\frac{q}{2}}} \leq |Dv|^{p-2} Dv \cdot \nabla v_{k+1} \frac{(\gamma v)^{\frac{q}{2}} \phi}{v_{k+1}(\gamma v_{k+1})^{\frac{q}{2}}}.
\]
Thus, using the Monotone Convergence Theorem and (7.7), we deduce
\[
\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \frac{(\gamma v)^{\frac{q}{2}} \phi}{v_k(\gamma v_k)^{\frac{q}{2}}} \, dx \rightarrow \int_{\mathbb{R}^n} \frac{|Dv|^p \phi}{v} \, dx \leq \frac{C}{b}.
\]
as \( k \rightarrow \infty \). Hence,
\[
(7.8) \quad \frac{|Dv|^p}{v} \in L^1_{\text{loc}}(\mathbb{R}^n, dx).
\]
Notice that, for all \( \phi \in C^0_0(\mathbb{R}^n) \),
\[
(\gamma v)^{\frac{q}{2}} \frac{||\phi||_{L^\infty}}{(\gamma v_k)^{\frac{q}{2}}} \leq \frac{||\phi||_{L^\infty}}{(\delta_E)^{\frac{q}{2}}} (\gamma v)^{\frac{q}{2}} \text{ q.e. on } E = \text{supp}(\phi).
\]
Since \( \sigma \) is absolutely continuous with respect to the \( p \)-capacity, it follows that the preceding inequality holds on \( E \) d\( \sigma \)-a.e. Using the Dominated
Convergence Theorem and the fact that \((\gamma v)^{\frac{q}{p}} = u^q \in L^1_{\text{loc}}(\mathbb{R}^n, d\sigma)\), we obtain, for all \(\phi \in C_0^\infty(\mathbb{R}^n),\)

\[
\int_{\mathbb{R}^n} (\gamma v)^{\frac{q}{p}} \frac{\phi}{(\gamma v_k)^{\frac{q}{p}}} d\sigma \to \int_{\mathbb{R}^n} \phi d\sigma.
\]

Clearly,

\[
|Dv|^{p-2} Dv \cdot \nabla v_k \frac{(\gamma v)^{\frac{q}{p}} \phi}{v_k (\gamma v_k)^{\frac{q}{p}}} \leq \frac{|Dv|^p \phi}{v}.
\]

Using (7.8) and the Dominated Convergence Theorem again, we obtain

\[
\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla v_k \frac{(\gamma v)^{\frac{q}{p}} \phi}{v_k (\gamma v_k)^{\frac{q}{p}}} dx \to \int_{\mathbb{R}^n} \frac{|Dv|^p \phi}{v} dx \quad \text{as} \quad k \to \infty.
\]

Therefore, letting \(k \to \infty\) in (7.5), we deduce

\[
\int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi dx = b \int_{\mathbb{R}^n} \frac{|Dv|^p \phi}{v} dx + \int_{\mathbb{R}^n} \phi d\sigma, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n).
\]

Thus, \(v\) is a \(p\)-superharmonic (and hence locally renormalized) solution to (1.5). Moreover, if both (1.11) and (1.14) hold, then by Theorem 1.1 the minimal solution \(u\) satisfies (1.15), and consequently \(v\) satisfies both (1.22) and (1.23).

To prove (ii), suppose \(v\) is a \(p\)-superharmonic solution to (1.5). Let \(\omega_k = -\Delta_p v_k\). Then \(v_k \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) is \(p\)-superharmonic, and

\[
-\Delta_p v_k = b \frac{\nabla v_k |^p}{v_k} + \sigma \chi_{v<k} + \tilde{\omega}_k,
\]

where \(\tilde{\omega}_k\) is a nonnegative measure in \(\mathbb{R}^n\) supported on \(\{v = k\}\).

We have

\[
u_k = (\gamma v_k)^{\frac{1}{\gamma}} \quad \text{and} \quad u_k \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n),
\]

since \(v_k \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) and \(\frac{1}{\gamma} = \frac{p-1}{p-1-q} > 1\). Let \(\mu_k = -\Delta_p u_k\). Then it follows,

\[
\mu_k = -\Delta_p u_k = -\Delta_p v_k (\gamma v_k)^{\frac{q}{p}} - b \frac{|\nabla v_k|^p}{v_k} (\gamma v_k)^{\frac{q}{p}} \geq 0.
\]

Indeed, for any \(\phi \in C_0^\infty(\mathbb{R}^n),\)

\[
\int_{\mathbb{R}^n} \phi (\gamma v_k)^{\frac{q}{p}} d\omega_k = \int_{\mathbb{R}^n} \nabla (\phi (\gamma v_k)^{\frac{q}{p}}) \cdot \nabla v_k |\nabla v_k|^{p-2} dx
\]

\[
= \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{p}} \nabla \phi \cdot \nabla v_k |\nabla v_k|^{p-2} dx + b \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{p}} \phi \frac{|\nabla v_k|^p}{v_k} dx.
\]
Let $\phi$ and $v \in R$ in (7.12). Hence, (7.13) follows that where in the last expression we used (7.9). From the preceding estimates it follows that $\langle \phi, \mu_k \rangle \geq 0$ if $\phi \geq 0$, and consequently $u_k$ is $p$-superharmonic. Clearly, $u = (\gamma v)^{\frac{q}{p}} < +\infty$-a.e., and $u = \lim_{k \to +\infty} u_k$ is $p$-superharmonic in $R^n$ as the limit of the increasing sequence of $p$-superharmonic functions $u_k$.

Since $v$ is a $p$-superharmonic solution of the equation (1.5), it follows that $v$ is a locally renormalized solution (see [KKT09]). Then, for all $\phi \in C_0^\infty(R^n)$ and $h \in W^{1,\infty}(R)$ with $h'$ having compact support, we obtain

$$\int_{R^n} |Dv|^{p-2} Dv \cdot \nabla (h(v) \phi) \, dx = b \int_{R^n} \frac{|Dv|^p}{v^p} h(v) \phi \, dx$$

(7.11)

$$+ \int_{R^n} h(v) \phi \, d\sigma.$$ 

Let $\phi \in C_0^\infty(R^n)$, $\phi \geq 0$. For $k > 0$, set $h(v) = (\gamma v_k)^{\frac{q}{p}}$. Then

$$\int_{R^n} |Dv|^{p-2} Dv \cdot \nabla ((\gamma v_k)^{\frac{q}{p}} \phi) \, dx = b \int_{R^n} \frac{|Dv|^p}{v^p} (\gamma v_k)^{\frac{q}{p}} \phi \, dx + \int_{R^n} (\gamma v_k)^{\frac{q}{p}} \phi \, d\sigma,$$

which yields

$$\int_{R^n} |Dv|^{p-2} Dv \cdot \nabla (\gamma v_k)^{\frac{q}{p}} \phi \, dx = b \int_{R^n} \frac{|Dv|^p}{v^p} (\gamma v_k)^{\frac{q}{p}} \phi \, dx + \int_{R^n} (\gamma v_k)^{\frac{q}{p}} \phi \, d\sigma.$$ 

Hence,

$$\int_{R^n} |Dv|^{p-2} Dv \cdot \nabla (\gamma v_k)^{\frac{q}{p}} \phi \, dx = b \int_{v > k} \frac{|Dv|^p}{v^p} (\gamma v_k)^{\frac{q}{p}} \phi \, dx$$

(7.12)

$$+ \int_{R^n} (\gamma v_k)^{\frac{q}{p}} \phi \, d\sigma.$$ 

Consequently,

$$\int_{R^n} |Dv|^{p-2} Dv \cdot \nabla (\gamma v_k)^{\frac{q}{p}} \phi \, dx = b \gamma^\frac{q}{p} \frac{(p-1)}{q} \int_{v > k} \frac{|Dv|^p}{v^p} \phi \, dx$$

(7.13)

$$+ \int_{R^n} (\gamma v_k)^{\frac{q}{p}} \phi \, d\sigma.$$
Therefore,

\[ \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi (\gamma v_k)^{\frac{q}{2}} \, dx \geq \int_{\mathbb{R}^n} (\gamma v_k)^{\frac{q}{2}} \phi \, d\sigma. \]

(7.14)

Note that \( Du = (\gamma v)^{\frac{q}{p-1-q}} Dv \), so that \( |Du|^{p-1} = (\gamma v)^{\frac{q}{2}} |Dv|^{p-1} \), and

\[ \left| |Dv|^{p-2} Dv \cdot \nabla \phi (\gamma v_k)^{\frac{q}{2}} \right| \leq |\nabla \phi| |Dv|^{p-1} (\gamma v)^{\frac{q}{2}} \leq \|\nabla \phi\|_{L^\infty(\mathbb{R}^n)} |Du|^{p-1}. \]

(7.15)

Notice that \( |Du|^{p-1} \in L^1_{\text{loc}}(\mathbb{R}^n, dx) \). Using the Dominated Convergence Theorem, we obtain

\[ \int_{\mathbb{R}^n} |Dv|^{p-2} Dv \cdot \nabla \phi (\gamma v_k)^{\frac{q}{2}} \, dx \to \int_{\mathbb{R}^n} |Du|^{p-2} Dv \cdot \nabla \phi (\gamma v_k)^{\frac{q}{2}} \, dx \]

\[ = \int_{\mathbb{R}^n} |Du|^{p-2} Dv \cdot \nabla \phi \, dx. \]

From this and (7.13), we have

\[ b \gamma^{\frac{q}{2}} \int_{v > k} \frac{|Dv|^p}{v} \phi \, dx \leq k^{-\frac{q(p-1)}{p-1-q}} C(u, \phi) < \infty. \]

(7.16)

Therefore,

\[ \|v\|_{L^{\frac{p}{p-1-q}}(\mathbb{R}^n)} \to \infty (\phi \frac{|Du|^p}{v} \, dx) < \infty. \]

Using (7.14) and the Monotone Convergence Theorem, we deduce

\[ \int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla \phi \, dx \geq \int_{\mathbb{R}^n} u^q \phi \, d\sigma, \quad \forall \phi \in C^\infty_0(\mathbb{R}^n), \quad \phi \geq 0. \]

Moreover, \( u \) is \( p \)-superharmonic on \( \mathbb{R}^n \). This means that \( u \) is a supersolution of (1.1) in the (locally) renormalized sense (see [KKT09]). By Theorem (1.1), \( u \) satisfies the lower bound in (1.15), and consequently \( v \) satisfies (1.22).

It remains to prove (iii). Suppose additionally that

\[ \int_{B} \frac{|Dv|^p}{v} \, dx < \infty, \]

for every ball \( B \). Then

\[ \int_{v > k} \frac{|Dv|^p}{v} (\gamma v_k)^{\frac{q}{2}} \phi \, dx \to 0 \]

by the Dominated Convergence Theorem. Letting \( k \to \infty \) in (7.12), we deduce

\[ \int_{\mathbb{R}^n} |Du|^{p-2} Du \cdot \nabla \phi \, dx = \int_{\mathbb{R}^n} u^q \phi \, d\sigma, \quad \forall \phi \in C^\infty_0(\mathbb{R}^n). \]

Thus, \( u \) is a \( p \)-superharmonic, and hence a locally renormalized, solution to (1.1). This completes the proof of Theorem 1.4. \( \square \)
As a corollary of Theorem 1.4, one can characterize the existence of finite energy solutions \( v \in L^{1,p}_0(\mathbb{R}^n) \) to (1.5). It is easy to see that such solutions exist if and only if \( b < 1 \), and \( \sigma \in L^{-1,\dot{p}'}(\mathbb{R}^n) \), i.e., \( \int_{\mathbb{R}^n} (W_{1,p} \sigma) \, d\sigma < \infty \).

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