Functional limit theorems for the maxima of perturbed random walks and divergent perpetuities in the $M_1$-topology

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Abstract
Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$ be a sequence of i.i.d. two-dimensional random vectors. In the earlier article Iksanov and Pilipenko (2014) weak convergence in the $J_1$-topology on the Skorokhod space of $n^{-1/2} \max_{0 \leq k \leq n} (\xi_1 + \ldots + \xi_k + \eta_1)$ was proved under the assumption that contributions of $\max_{0 \leq k \leq n} (\xi_1 + \ldots + \xi_k)$ and $\max_{1 \leq k \leq n} \eta_k$ to the limit are comparable and that $n^{-1/2} (\xi_1 + \ldots + \xi_{[n]} + \eta_{[n]})$ is attracted to a Brownian motion. In the present paper, we continue this line of research and investigate a more complicated situation when $\xi_1 + \ldots + \xi_{[n]}$, properly normalized without centering, is attracted to a centered stable Lévy process, a process with jumps. As a consequence, weak convergence normally holds in the $M_1$-topology. We also provide sufficient conditions for the $J_1$-convergence. For completeness, less interesting situations are discussed when one of the sequences $\max_{0 \leq k \leq n} (\xi_1 + \ldots + \xi_k)$ and $\max_{1 \leq k \leq n} \eta_k$ dominates the other.

An application of our main results to divergent perpetuities with positive entries is given.

Key words: functional limit theorem; $J_1$-topology; $M_1$-topology; perpetuity; perturbed random walk

1 Introduction and results
Let $(\xi_k, \eta_k)_{k \in \mathbb{N}}$ be a sequence of i.i.d. two-dimensional random vectors with generic copy $(\xi, \eta)$. No condition is imposed on the dependence structure between $\xi$ and $\eta$. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Further, let $(S_n)_{n \in \mathbb{N}_0}$ be the zero-delayed ordinary random walk with increments $\xi_n$ for $n \in \mathbb{N}$, i.e., $S_0 = 0$ and $S_n = \xi_1 + \ldots + \xi_n$, $n \in \mathbb{N}$. Then define its perturbed variant $(T_n)_{n \in \mathbb{N}}$, that we call perturbed random walk, by

$$T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}.$$ (1)

Recently it has become a rather popular object of research, see the recent book [9] for a survey and [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. It is worth noting that sometimes in the literature the term ‘perturbed random walk’ was used to denote random sequences other than those defined in (1). See, for instance, [5, 6, 12, 13, 14, 21, 27] and Section 6 in [7].

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Denote by $D := D[0, \infty)$ the Skorokhod space of real-valued right-continuous functions which are defined on $[0, \infty)$ and have finite limits from the left at each positive point. Throughout the paper we assume that $D$ is equipped with either the $J_1$-topology or the $M_1$-topology. We refer to [3] 13 and [26] for comprehensive accounts of the $J_1$- and the $M_1$-topologies, respectively. We write $\mathcal{M}_p(A)$ for the set of Radon point measures on a ‘nice’ space $A$. The $\mathcal{M}_p(A)$ is endowed with vague convergence. More information on these can be found in [22]. Throughout the paper $\xrightleftharpoons{\mathbb{D}}$ and $\xrightarrow{\mathbb{M}_1}$ will mean weak convergence on the Skorokhod space $D$ when endowed with the $J_1$-topology and the $M_1$-topology, respectively. The notation $\Rightarrow$ without superscript is normally followed by a specification of the topology and the space involved. Finally, we write $\xrightarrow{v}$ and $\xrightarrow{p}$ to denote vague convergence and convergence in probability, respectively.

In the present paper we are interested in weak convergence on $D$ of $\max_{0 \leq k \leq \lfloor n \rfloor} (S_k + \eta_{k+1})$, properly normalized without centering, as $n \to \infty$. It should not come as a surprise that the maxima exhibit three types of different behaviors depending on the asymptotic interplay of $A_n := \max_{0 \leq k \leq n} S_k$ and $B_n := \max_{1 \leq k \leq n+1} \eta_k$, namely, on whether (I) $A_n$ dominates $B_n$; (II) $A_n$ is dominated by $B_n$; (III) $A_n$ and $B_n$ are comparable.

Relying essentially upon findings in [11] three functional limit theorems for the maxima of perturbed random walks, properly rescaled without centering, were proved in [9] under the assumption that $\mathbb{E} \xi^2 < \infty$. Throughout the remainder of the paragraph we assume that the most interesting alternative (III) prevails. The situation treated in [11] was relatively simple because the limit process for $S_{\lfloor n \rfloor}/n^{1/2}$ was a Brownian motion, a process with continuous paths. As a consequence, the convergence took place in the $J_1$-topology on $D$, and, more surprisingly, the contributions of $(S_k)$ and $(\eta_j)$ turned out asymptotically independent, despite the possible strong dependence of $\xi$ and $\eta$. In the present paper we treat a more delicate case where the distribution of $\xi$ belongs to the domain of attraction of an $\alpha$-stable distribution, $\alpha \in (0, 2)$, so that the limit process for $S_{\lfloor n \rfloor}$, properly normalized, is an $\alpha$-stable Lévy process. We shall show that the presence of jumps in the latter process destroys dramatically an idyllic picture pertaining to the Brownian motion scenario: the convergence typically holds in the weaker $M_1$-topology on $D$, and the aforementioned asymptotic independence only occurs in some exceptional cases where $\xi$ and $\eta$ are themselves asymptotically independent in an appropriate sense.

Throughout the paper we assume that, as $x \to \infty$,

$$\mathbb{P}\{|\xi| > x\} \sim x^{-\alpha} \ell(x)$$

(2)

and that

$$\mathbb{P}\{\xi > x\} \sim c_1 \mathbb{P}\{|\xi| > x\}, \quad \mathbb{P}\{-\xi > x\} \sim c_2 \mathbb{P}\{|\xi| > x\}$$

(3)

for some $\alpha \in (0, 2)$, some $\ell$ slowly varying at $\infty$, some nonnegative $c_1$ and $c_2$ summing up to one. The assumptions mean that the distribution of $\xi$ belongs to the domain of attraction of an $\alpha$-stable distribution. To ensure weak convergence of $S_{\lfloor n \rfloor}$ without centering we assume that $\mathbb{E} \xi = 0$ when $\alpha \in (1, 2)$ and that the distribution of $\xi$ is symmetric when $\alpha = 1$. Then the classical Skorokhod theorem (Theorem 2.7 in [24]) tells us that

$$\frac{S_{\lfloor n \rfloor}}{a(n)} \xrightarrow{\mathbb{D}} S_\alpha(\cdot), \quad n \to \infty,$$

(4)

where $a(x)$ is a positive function satisfying $\lim_{x \to \infty} x \mathbb{P}\{|\xi| > a(x)\} = 1$ and $S_\alpha := (S_\alpha(t))_{t \geq 0}$ is an $\alpha$-stable Lévy process with the characteristic function

$$\mathbb{E}\exp(izS_\alpha(1)) = \exp(|z|^\alpha(\Gamma(2 - \alpha)/(\alpha - 1))(\cos(\pi \alpha/2) - i(c_1 - c_2)\sin(\pi \alpha/2)\text{sign } z)), \quad z \in \mathbb{R}$$
Analogously, Remark 1.3 \( \xi \) is the case whenever \( LEB \) with mean measure \( x \to \infty \), taking values in \((0, \infty)\). Theorem 1.1.

Suppose that conditions \( \mathbb{2} \) and \( \mathbb{3} \) hold, that \( \mathbb{P}\{\eta > x\} \sim c\mathbb{P}\{|\xi| > x\} \) as \( x \to \infty \), for some \( c > 0 \), and that

\[
x \mathbb{P}\left\{ \left( \frac{\xi, \eta^+}{a(x)} \right) \in \cdot \right\} \Rightarrow \nu, \quad x \to \infty
\]

on \( \mathcal{M}_p(E) \). Then

\[
\max_{0 \leq k \leq n} \frac{S_k + \eta_{k+1}}{a(n)} \overset{\text{M}_1}{\Rightarrow} \sup_{0 \leq s \leq \cdot} S^{*}_a(s) \vee \sup_{\theta_k \leq \cdot} \left( S^{*}_a(\theta_k -) + j_k, n \to \infty, \right)
\]

where \((\theta_k, i_k, j_k)\) are the atoms of a Poisson random measure \( N^{(\nu)} \) and \( S^{*}_a \) is a copy of \( S_a \) whose Lévy-Itô representation is built upon the Poisson random measure \( \sum_k \varepsilon(\theta_k, i_k) \).

Under the additional assumption

\[
\nu\{(x, y) : 0 < y < x\} = 0,
\]

the convergence in \( \mathbb{6} \) holds in the \( J_1 \)-topology on \( D \).

Remark 1.2. It is perhaps worth stating explicitly that \( N^{(\nu)}(\cdot \times [\infty, +\infty) \times \cdot) = \sum_k \varepsilon(\theta_k, j_k) \) is a Poisson random measure on \([0, \infty) \times (0, \infty)\] with mean measure \( LEB \times \mu_c \), where \( \mu_c \) is a measure on \((0, \infty)\) defined by

\[
\mu_c((x, \infty]) = cx^{-\alpha}, \quad x > 0.
\]

Analogously, \( N^{(\nu)}(\cdot \times [0, \infty]) = \sum_k \varepsilon(\theta_k, i_k) \) is a Poisson random measure on \([0, \infty) \times \{0\})\) with mean measure \( LEB \times \nu^* \), where \( \nu^* \) is the Lévy measure of \( S_\alpha \) given by

\[
\nu^*((x, \infty]) = c_1 x^{-\alpha}, \quad \nu^*(-\infty, -x]) = c_2 x^{-\alpha}, \quad x > 0.
\]

Remark 1.3. Condition \( \mathbb{7} \) obviously holds if the measure \( \nu \) is concentrated on the axes. This is the case whenever \( \xi \) and \( \eta \) are independent and also in many cases when these are dependent. For instance, take \( \xi = |\log W| \) and \( \eta = |\log(1 - W)| \) satisfying \( \mathbb{3} \) for a random variable \( W \) taking values in \((0, 1)\) (details can be found in the proof of Theorem 1.1 in \([10]\)).
Suppose now that \( \eta = r \xi \) for some \( r > 0 \). Then the restriction of \( \nu \) to the first quadrant concentrates on the line \( y = rx \). Hence, condition (7) holds if, and only if, \( r \geq 1 \).

Let \( \rho \) be a positive random variable such that \( \mathbb{P}\{\rho > x\} \sim x^{-\alpha} \) as \( x \to \infty \) and \( \zeta \) a random variable which is independent of \( \rho \) and takes values in \([-\pi, \pi)\). Setting \( \xi = \rho \cos \zeta \) and \( \eta = \rho \sin \zeta \) we obtain

\[
\mathbb{P}\{\xi > x\} \sim (\mathbb{E}|\cos \zeta|^\alpha \mathbf{1}_{\{\zeta < \pi/2\}}) x^{-\alpha}, \quad \mathbb{P}\{-\xi > x\} \sim (\mathbb{E}|\cos \zeta|^\alpha \mathbf{1}_{\{\zeta > \pi/2\}}) x^{-\alpha}
\]
as \( x \to \infty \) by the Lebesgue dominated convergence theorem. Furthermore,

\[
x \mathbb{P}\left\{ \left( \frac{\xi, \eta}{a(x)} \right) \in \cdot \right\} \xrightarrow{\nu} x \to \infty,
\]

where \( a(x) = (\mathbb{E}|\cos \zeta|^\alpha)^{1/\alpha} x^{1/\alpha} \) and \( \nu \) is the image of the measure

\[
\frac{\alpha \mathbb{E}(\cos \zeta)^\alpha \mathbf{1}_{\{\zeta < \pi/2\}}}{\mathbb{E}|\cos \zeta|^\alpha} \mathbf{1}_{(r, \varphi) \in (0, \infty) \times [0, \pi]} r^{-\alpha-1} dr \mathbb{P}\{\zeta \in d\varphi\}
\]
under the mapping \( (r, \varphi) \to (r \cos \varphi, r \sin \varphi) \). Condition (7) is equivalent to \( \mathbb{P}\{\zeta \in (0, \pi/4)\} = 0 \).

Remark 1.4. Weak convergence of nondecreasing processes in the \( M_1 \)-topology is not as strong as it might appear. Actually, it is equivalent to weak convergence of finite-dimensional distributions just because a sequence of nondecreasing processes is always tight on \( D \) equipped with the \( M_1 \)-topology. This follows from the fact that the \( M_1 \)-oscillation

\[
\omega_3(f) := \sup_{t_1 \leq t_2 \leq 0 \leq t_2 - t_1 \leq \delta} M(f(t_1), f(t), f(t_2))
\]
of a nondecreasing function \( f \) equals zero, where \( M(x_1, x_2, x_3) := 0 \), if \( x_2 \in [x_1, x_3] \), and \( := \min(|x_2 - x_1|, |x_3 - x_2|) \), otherwise.

Remark 1.5. From a look at Theorem 2.1 underlying the proof of Theorem 1.1 it should be clear that a counterpart of Theorem 1.1 also holds when replacing the input vectors \( (\xi_k, \eta_k)_{k \in \mathbb{N}} \) with arrays \( (\xi_k^{(n)}, \eta_k^{(n)})_{k \in \mathbb{N}} \) for each \( n \in \mathbb{N} \). We however refrain from formulating such a generalization, for we are not aware of any potential applications of such a result.

Propositions 1.6 and 1.8 given next are concerned with the simpler situations (I) and (II), respectively.

**Proposition 1.6.** Suppose that conditions (2) and (3) hold and that

\[
\lim_{x \to \infty} \frac{\mathbb{P}\{\eta > x\}}{\mathbb{P}\{\zeta > x\}} = 0. \quad (8)
\]

Then

\[
\sup_{0 \leq s \leq t_1} J_1 \left( \frac{\max_{0 \leq k \leq [n]} (S_k + \eta_{k+1})}{a(n)} \right) \xrightarrow{J_1} \sup_{0 \leq s \leq t_1} S_\alpha(s), \quad n \to \infty. \quad (9)
\]

Remark 1.7. When \( \alpha \in (0, 1) \) and \( c_1 = 0 \), the right-hand side in (9) is the zero function because \( S_\alpha \) is then the negative of an \( \alpha \)-stable subordinator (recall that a subordinator is a nondecreasing Lévy process). In this setting there are two possibilities: either \( \sup_{k \geq 0} (S_k + \eta_{k+1}) < \infty \) a.s. or \( \sup_{k \geq 0} (S_k + \eta_{k+1}) = \infty \) a.s. Plainly, if the first alternative prevails, much more than (9) can be said, namely, \( \max_{0 \leq k \leq [n]} (S_k + \eta_{k+1})/r_n \) converges to the zero function in the \( J_1 \)-topology on \( D \) for any positive sequence \( (r_n) \) diverging to \( \infty \).
Now we intend to give examples showing that either of possibilities can hold. By Theorem 2.1 in [1], the supremum of \( S_k + \eta_{k+1} \) is finite a.s. if, and only if,
\[
\int_{(0,\infty)} \frac{x}{0} \mathbb{P}(-\xi > y)dy \mathbb{P}\{\xi \leq y\} < \infty \quad \text{and} \quad \int_{(0,\infty)} \frac{x}{0} \mathbb{P}(-\xi > y)dy \mathbb{P}\{\eta \leq y\} < \infty \quad (10)
\]
If \( \mathbb{P}\{-\xi > x\} \sim x^{-\alpha} \), \( \mathbb{P}\{\xi > x\} \sim x^{-\alpha}(\log x)^{-2} \) as \( x \to \infty \) and \( \mathbb{E}\{(\eta^+)\alpha < \infty \) hold, whereas if \( \mathbb{P}\{-\xi > x\} \sim x^{-\alpha} \), \( \mathbb{P}\{\xi > x\} \sim x^{-\alpha}(\log x)^{-1} \) as \( x \to \infty \), then the first integral in (10) diverges.

**Proposition 1.8.** Suppose that conditions (2) and (3) hold, that
\[
\lim_{x \to \infty} \frac{\mathbb{P}\{\eta > x\}}{\mathbb{P}\{|\xi| > x\}} = \infty
\]
and that \( \mathbb{P}\{\eta > x\} \) is regularly varying at \( \infty \) of index \( -\beta \) (necessarily \( \beta \in (0, \alpha) \)). Let \( b(x) \) be a positive function which satisfies \( \lim_{x \to \infty} x\mathbb{P}\{\eta > b(x)\} = 1 \). Then
\[
\max_{0 \leq k \leq [n]} \frac{(S_k + \eta_{k+1})}{b(n)} \xrightarrow{j} \sup_{\theta_k \leq \infty} j_k, \quad n \to \infty, \quad (12)
\]
where \((\theta_k, j_k)\) are the atoms of a Poisson random measure on \([0, \infty) \times (0, \infty)\) with mean measure \( \mathbb{L}\mathbb{E} \times \mu \), where \( \mu \) is a measure on \((0, \infty)\) defined by
\[
\mu((x, \infty)] = x^{-\beta}, \quad x > 0.
\]

Whenever the random series \( \sum_{k \geq 0} c_k T_{k+1} \) converges a.s., its sum is called *perpetuity* due to its occurrence in the realm of insurance and finance as a sum of discounted payment streams. When the random series diverges, it is natural to investigate weak convergence on \( D \) of its partial sums, properly rescaled, as the number of summands becomes large. Some results of this flavor can be found in [4] and [9] (in these works many references to earlier one-dimensional results can be found). Here, we prove functional limit theorems in the situations that remained untouched.

**Theorem 1.9.** In the settings of Theorem 1.1 and Propositions 1.6 and 1.8 functional limit theorems hold with \( \log \left( \sum_{k=0}^{[n]} c_k T_{k+1} \right) \) replacing \( \max_{0 \leq k \leq [n]} T_{k+1} \).

## 2 Proof of Theorem 1.1

For each \( n \in \mathbb{N} \), let \((x_i^{(n)}, y_i^{(n)}) \in \mathbb{N}^2\) be a sequence of \( \mathbb{R}^2 \)-valued vectors. Put \( S_0^{(n)} := 0 \),
\[
S_k^{(n)} := \sum_{i=1}^{k} x_i^{(n)}, \quad k \in \mathbb{N}, \quad T_k^{(n)} := S_k^{(n)} + y_k^{(n)}, \quad k \in \mathbb{N}_0
\]
and then define the piecewise constant functions
\[
f_n(t) := \sum_{k \geq 0} S_k^{(n)} \mathbbm{1}_{\frac{k}{n}, \frac{k+1}{n}}(t), \quad g_n(t) := \max_{0 \leq k \leq [nt]} T_k^{(n)}, \quad t \geq 0
\]
where \( \mathbbm{1}_A(x) = 1 \) if \( x \in A \) and \( = 0 \), otherwise.

To proceed, we have to recall the notation \( E = (-\infty, +\infty) \times [0, \infty) \setminus \{(0, 0)\} \). The proof of Theorem 1.1 is essentially based on the following deterministic result along with the continuous mapping theorem.
Theorem 2.1. Let \( f_0 \in D \) and \( \nu_0 = \sum_k \varepsilon(t_k, x_k, y_k) \) be a Radon measure on \([0, \infty) \times E\) satisfying \( \nu_0(\{0\} \times E) = 0 \) and \( t_k \neq t_j \) for \( k \neq j \). Suppose that

\[
\lim_{n \to \infty} f_n = f_0
\]

in the \( J_1 \)-topology on \( D \) and that

\[
\nu_n := \sum_{i \geq 1} \varepsilon_{(i/n, x_i^{(n)}, y_i^{(n)})} \mathbb{1}_{\{y_i^{(n)}>0\}} \to \nu_0, \quad n \to \infty
\]

on \( \mathcal{M}_p([0, \infty) \times E) \). Then

\[
\lim_{n \to \infty} g_n = g_0 := \sup_{0 \leq s \leq t} f_0(s) \vee \sup_{t_k \leq t} (f_0(t_k^{-}) + y_k)
\]

in the \( M_1 \)-topology on \( D \). This convergence holds in the \( J_1 \)-topology on \( D \) under the additional assumption

\[
\nu_0([0, \infty) \times \{(x, y) : 0 < y < x\}) = 0.
\]

Remark 2.2. Suppose that \( f_0 \) is continuous and that the set of points \((t_k, y_k)\) with \( y_k > 0\) is dense in \([0, \infty)\). Then

\[g_0(\cdot) = \sup_{t_k \leq \cdot} (f_0(t_k) + y_k).\]

Furthermore, condition (15) holds automatically, and condition (14) is equivalent to

\[
\sum_{i \geq 1} \varepsilon_{(i/n, y_i^{(n)})} \mathbb{1}_{\{y_i^{(n)}>0\}} \to \sum_k \varepsilon_{(t_k, y_k)} \quad n \to \infty.
\]

on \( \mathcal{M}_p([0, \infty) \times (0, \infty]) \). This is the setting of Theorem 1.3 in [11].

Remark 2.3. Here, we discuss the necessity of condition (15) for the \( J_1 \)-convergence. Suppose that in the setting of Theorem 2.1 there exists \( k \in \mathbb{N} \) such that \( 0 < y_k < x_k \), so that condition (15) does not hold. Now we give an example in which the \( J_1 \)-convergence in Theorem 2.1 fails to hold. With \( x_i^{(n)} = y_i^{(n)} = 0 \) for \( i \neq [n/2] \), \( v_{[n/2]}^{(n)} = 2 \) and \( y_{[n/2]}^{(n)} = 1 \) we have \( f_n(t) = 2 \mathbb{1}_{[1/2, \infty)}(t) \)

and

\[g_n(t) = \begin{cases} 0, & t < \frac{[n/2]-1}{n}, \\ 1, & \frac{[n/2]-1}{n} \leq t < \frac{[n/2]}{n}, \\ 2, & \frac{[n/2]}{n} \leq t. \end{cases} \]

Plainly, condition (14) holds with \( \nu_0 = \varepsilon_{(1/2, 2, 1)} \). Setting \( f_0(t) = g(t) := 2 \mathbb{1}_{[1/2, \infty)}(t) \) we conclude that \( \lim_{n \to \infty} f_n = f_0 \) in the \( J_1 \)-topology and \( \lim_{n \to \infty} g_n = g \) in the \( M_1 \)-topology. On the other hand, \( g_n \) has a jump of magnitude 1 at point \([n/2]/n\). Furthermore, this magnitude does not converge to 2, the size of the limit jump at point \(1/2\). This precludes the \( J_1 \)-convergence.

Lemma 2.4. Let \((h_n)_{n \in \mathbb{N}_0}\) be a sequence of nondecreasing functions in \( D \).

(a) \( \lim_{n \to \infty} h_n = h_0 \) in the \( M_1 \)-topology on \( D \) if, and only if, \( h_n(t) \) converges to \( h_0(t) \) for each \( t \) in a dense subset of continuity points of \( h_0 \) including zero.
Proof of Theorem 2.1. We start by showing that \(g_0 \in D\). Since \(g_0\) is nondecreasing, it has finite limits from the left on \((0, \infty)\). Using right-continuity of \(f_0\) we obtain

\[
g_0(t) = \lim_{\delta \to 0^+} g_0(t + \delta) = \lim_{\delta \to 0^+} \left( \sup_{0 \leq s \leq t} f_0(s) \vee \sup_{t \leq k \leq t + \delta} (f_0(t_k) - y_k) \right)
\]

for any \(t > 0\) which proves right-continuity of \(g_0\).

**Proof of the \(M_1\)-convergence.** Since \(f_0, g_0 \in D\), they have at most countably many discontinuities. Hence, the set

\[
K := \{T \geq 0 : \nu_0(\{T\} \times E) = 0; \quad T \text{ is a continuity point of } f_0 \text{ and continuity point of } g_0\}
\]

is dense in \([0, \infty)\). Since \(g_n\) is nondecreasing for each \(n \in \mathbb{N}\), according to Lemma 2.4 (a), it suffices to prove that

\[
\lim_{n \to \infty} g_n(T) = g_0(T)
\]

for all \(T \in K\). Observe that \(g_0(0) = 0\) as a consequence of \(f_n(0) = f_0(0) = 0\) and \(\nu_0(\{0\} \times E) = 0\). The last condition ensures that \(g_n(0) = \bar{y}_1^{(n)}\) converges to zero as \(n \to \infty\). This proves that relation (16) holds for \(T = 0\). Thus, in what follows we assume that \(T \in K\) and \(T > 0\).

Fix any such a \(T\). There exists a sequence \((\varepsilon_k)_{k \in \mathbb{N}}\) that vanishes as \(k \to \infty\) and such that its generic element denoted by \(\varepsilon\) is a continuity point of the nonincreasing function

\[
x \mapsto \nu_0([0, T] \times [-\infty, +\infty] \times (x, \infty]),
\]

so that \(\nu_0([0, T] \times [-\infty, +\infty] \times \{\varepsilon\}) = 0\). Put \(E_\varepsilon := [-\infty, +\infty] \times (\varepsilon, \infty)\). Condition (14) implies that \(\nu_0([0, T] \times E_\varepsilon) = \nu_n([0, T] \times E_\varepsilon) = m\) for large enough \(n\) and some \(m \in \mathbb{N}_0\), where the finiteness of \(m\) is secured by the fact that \(\nu_0\) is a Radon measure. The case \(m = 0\) is trivial. Hence, in what follows we assume that \(m \in \mathbb{N}\). Denote by \((\tilde{t}_1, \tilde{x}_i, \tilde{y}_i)_{1 \leq i \leq m}\) an enumeration of the points of \(\nu_0\) in \([0, T] \times E_\varepsilon\) with

\[
\tilde{t}_1 < \tilde{t}_2 < \ldots < \tilde{t}_m
\]

and by \((\tilde{t}_i^{(n)}, \bar{x}_i^{(n)}, \bar{y}_i^{(n)})_{1 \leq i \leq m}\) the analogous enumeration of the points \(\nu_n\) in \([0, T] \times E_\varepsilon\). Note that \(\tilde{t}_1 > 0\) in view of the assumption \(\nu_0(\{0\} \times E) = 0\), whereas the assumption \(t_k \neq t_j\) for \(k \neq j\) ensures that the inequalities in (17) are strict. Then

\[
\lim_{n \to \infty} \sum_{i=1}^{m} (|\tilde{t}_i^{(n)} - \tilde{t}_i| + |\bar{x}_i^{(n)} - \bar{x}_i| + |\bar{y}_i^{(n)} - \bar{y}_i|) = 0.
\]
Later on we shall need the following relation

\[ f_n(\bar{t}_i^{(n)} - 1/n) = f_n(\bar{t}_i^{(n)}) \rightarrow f_0(\bar{t}_i -), \quad n \rightarrow \infty \]  

(19)

for \( i = 1, \ldots, m \). To prove it, fix any \( i = 1, \ldots, m \) and assume that \( \bar{t}_i \) is a discontinuity point of \( f_0 \). Then condition (13) in combination with \( f_n(\bar{t}_i^{(n)} -) - f_n(\bar{t}_i^{(n)}) = \bar{x}_i^{(n)} \rightarrow \bar{x}_i \neq 0 \) as \( n \rightarrow \infty \) entails \( \bar{x}_i = f_0(\bar{t}_i) - f_0(\bar{t}_i-) \) and (19) (see the proof of Proposition 2.1 on p. 337 in [13]). If \( \bar{t}_i \) is a continuity point of \( f_0 \), (19) holds trivially. Arguing similarly, we also obtain

\[ \lim_{n \rightarrow \infty} \max_{t \in [0, t_i^{(n)} - 2/n]} f_n(t) = \sup_{t \in [0, \bar{t}_i]} f_0(t) \]  

(20)

for \( i = 1, \ldots, m \).

We first work with functions \( g_{n, \varepsilon} \) and \( g_{0, \varepsilon} \) which are counterparts of \( g_n \) and \( g_0 \) based on the restrictions of \( \nu_n \) and \( \nu_0 \) to \([0, T] \times E_\varepsilon \). Put

\[ A_{n,T} := \{ j \in \mathbb{N}_0 : 0 \leq j \leq [nT], \quad (j + 1)/n \neq \bar{t}_i^{(n)} \text{ for } i = 1, \ldots, m \}. \]

Now we are ready to write a basic decomposition

\[ g_{n,\varepsilon}(T) := \max_{0 \leq i \leq [nT]} \left( x_i^{(n)} + \ldots + y_i^{(n)} + \bar{y}_{i+1} \mathbb{1}_{\{y_{i+1}^{(n)} > \varepsilon\}} \right) \]

\[ = \max_{i \in A_{n,T}} f_n(i/n) \vee \max_{1 \leq k \leq m} \left( f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)} \right) \]

\[ = \max_{0 \leq i \leq [nT]} f_n(i/n) \vee \max_{1 \leq k \leq m} \left( f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)} \right) \]

\[ = \max_{t \in [0, T]} f_n(t) \vee \max_{1 \leq k \leq m} \left( f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)} \right), \]  

(21)

the third equality following from the fact that, for integer \( i \in [0, [nT]] \) such that \( i/n = \bar{t}_k^{(n)} - 1/n \) for some \( k = 1, \ldots, m \), we have \( f_n(i/n) < \max_{1 \leq k \leq m} \left( f_n(\bar{t}_k^{(n)} - 1/n) + \bar{y}_k^{(n)} \right) \) because all the \( \bar{y}_k^{(n)} \) are positive.

It is convenient to state the following known result as a lemma, for it will be used twice in the subsequent proof.

**Lemma 2.5.** Let \( s_0 \) be a continuity point of \( f_0 \) and \((s_n)_{n \in \mathbb{N}}\) a sequence of positive numbers converging to \( s_0 \) as \( n \rightarrow \infty \). Then \( \lim_{n \rightarrow \infty} \max_{t \in [0, s_n]} f_n(t) = \max_{t \in [0, s_0]} f_0(t) \).

**Proof.** We first observe that \( \sup_{t \in [0, s_0]} f_0(t) = \max_{t \in [0, s_0]} f_0(t) \) because \( s_0 \) is a continuity point of \( f_0 \) (hence, of the supremum). It is well-known (and easily checked) that (13) entails

\[ \lim_{n \rightarrow \infty} \sup_{0 \leq t \leq s_n} f_n(t) = \sup_{0 \leq t \leq s_0} f_0(t) \]  

(22)

in the \( J_1 \)-topology on \( D \). In particular, \( \lim_{n \rightarrow \infty} \sup_{t \in [0, s_n]} f_n(t) = \max_{t \in [0, s_0]} f_0(t) \). \hfill \Box

Recalling that \( T \) is a continuity point of \( f_0 \) and using Lemma 2.5 with \( s_n = T \) for all \( n \in \mathbb{N}_0 \) we infer

\[ \lim_{n \rightarrow \infty} \max_{t \in [0, T]} f_n(t) = \sup_{t \in [0, T]} f_0(t) \]
and thereupon
\[
\lim_{n \to \infty} g_{n, \varepsilon}(T) = \sup_{s \in [0, T]} f_0(s) \vee \sup_{t_k \leq T} (f_0(\bar{t}_k -) + \bar{y}_k) := g_{0, \varepsilon}(T) \tag{23}
\]

having utilized (18) and (19) for the second supremum.

Further, we claim that
\[
\sup_{t \geq 0} |g_0(t) - g_{0, \varepsilon}(t)| \leq \varepsilon \quad \text{and} \quad \sup_{t \geq 0} |g_n(t) - g_{0, \varepsilon}(t)| \leq \varepsilon. \tag{24}
\]

We only prove the first inequality, the proof of the second being analogous and simpler. Write
\[
|g_0(t) - g_{0, \varepsilon}(t)| = g_0(t) - g_{0, \varepsilon}(t) = \sup_{s \in [0, t]} f_0(s) \vee \sup_{t_k \leq t} (f_0(\bar{t}_k -) + y_k) - \sup_{s \in [0, t]} f_0(s) \vee \sup_{t_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k)
\]
for all \( t \geq 0 \). There are two possibilities: either
\[
\sup_{t_k \leq t} (f_0(\bar{t}_k -) + y_k) = \sup_{t_k \leq t, t_k \neq \bar{t}_k} (f_0(\bar{t}_k -) + y_k) \vee \sup_{t_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k) = \sup_{t_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k)
\]
in which case \( |g_0(t) - g_{0, \varepsilon}(t)| = 0 \) for all \( t \geq 0 \), i.e., the first inequality in (24) holds, or
\[
\sup_{t_k \leq t} (f_0(\bar{t}_k -) + y_k) = \sup_{t_k \leq t, t_k \neq \bar{t}_k} (f_0(\bar{t}_k -) + y_k).
\]

Observe that
\[
\sup_{t_k \leq t, t_k \neq \bar{t}_k} (f_0(\bar{t}_k -) + y_k) \leq \sup_{s \in [0, t]} f_0(s) + \varepsilon
\]
as a consequence of \( y_k \leq \varepsilon \) for all \( k \in \mathbb{N} \) such that \( t_k \neq \bar{t}_k \), and that
\[
\sup_{s \in [0, t]} f_0(s) \vee \sup_{\bar{t}_k \leq t} (f_0(\bar{t}_k -) + \bar{y}_k) \geq \sup_{s \in [0, t]} f_0(s).
\]

Hence, the first inequality in (24) holds in this case, too.

It remains to note that
\[
|g_n(T) - g_0(T)| \leq |g_n(T) - g_{n, \varepsilon}(T)| + |g_{n, \varepsilon}(T) - g_{0, \varepsilon}(T)| + |g_{0, \varepsilon}(T) - g_0(T)|
\]
and then first let \( n \) tend to \( \infty \) and use (23), and then let \( \varepsilon \) go to zero through the sequence \((\varepsilon_k)\).

This shows that \( \lim_{n \to \infty} g_n(T) = g_0(T) \). The proof of the \( M_1 \)-convergence is complete.

**Proof of the \( J_1 \)-Convergence.** We intend to prove that whenever \( \bar{s} \) is a discontinuity point of \( g_{0, \varepsilon} \) there is a sequence \((s_n)_{n \in \mathbb{N}}\) converging to \( \bar{s} \) for which
\[
\lim_{n \to \infty} g_{n, \varepsilon}(s_n) = g_{0, \varepsilon}(\bar{s}) \quad \text{and} \quad \lim_{n \to \infty} g_{n, \varepsilon}(s_n-) = g_{0, \varepsilon}(\bar{s}-). \tag{25}
\]

Now we explain that (25) entails
\[
\lim_{n \to \infty} g_n = g_0 \tag{26}
\]
in the \( J_1 \)-topology on \( D \), which is the desired result. From the first part of the proof we know that \( \lim_{n \to \infty} g_{n, \varepsilon} = g_{0, \varepsilon} \) in the \( M_1 \)-topology on \( D \). Thus, if (25) holds, we conclude that
\[
\lim_{n \to \infty} g_{n, \varepsilon} = g_{0, \varepsilon} \tag{27}
\]
in the $J_1$-topology on $D$ by Lemma 2.4(b). Let $r \in [0, T]$ be a continuity point of $g_0$, where $T \in K$ (see the first part of the proof for the definition of $K$). In order to prove (26) it suffices to show that $\lim_{n \to \infty} g_n = g_0$ in the $J_1$-topology on $D[0, r]$ or, equivalently, that $\lim_{n \to \infty} \rho(g_n, g_0) = 0$ where $\rho$ is the standard Skorokhod metric on $[0, r]$. Since $r$ is also a continuity point of $g_0$, relation (27) ensures that $\lim_{n \to \infty} \rho(g_n, g_0, \varepsilon) = 0$. We proceed by writing

$$
\rho(g_n, g_0) \leq \rho(g_n, g_n, \varepsilon) + \rho(g_n, g_0, \varepsilon) + \rho(g_0, g_0, \varepsilon) \leq \sup_{0 \leq t \leq r} |g_n(t) - g_n,\varepsilon(t)| + \rho(g_n, g_0, \varepsilon)
$$

having utilized the fact that $\rho$ is dominated by the uniform metric on $[0, r]$ for the penultimate inequality and (24) for the last. Now sending $n \to \infty$ and then letting $\varepsilon$ approach zero through the sequence $(\varepsilon_k)$ proves $\lim_{n \to \infty} \rho(g_n, g_0) = 0$ and thereupon (26).

Passing to the proof of (25) we consider two cases.

**Case 1:** $\bar{s}$ is a discontinuity point of $g_0,\varepsilon$ and a continuity point of $f_0$.

We claim that $\bar{s} = t_k$ for some $k = 1, \ldots, m$. Indeed, if $g_0,\varepsilon(\bar{s}) = \sup_{t \in [0, \varepsilon]} f_0(t)$, then $\bar{s}$ is a continuity point of $g_0,\varepsilon$, a contradiction. Thus, we must have $g_0,\varepsilon(\bar{s}) = \max_{1 \leq j \leq n} f_0(\bar{t}_j) + \bar{y}_j$. The points $\bar{t}_1, \ldots, \bar{t}_m$ are the only discontinuities of $x \mapsto \max_{1 \leq j \leq n} f_0(\bar{t}_j) + \bar{y}_j$ on $[0, \infty)$. Therefore, $\bar{s} = t_k$ for some $k = 1, \ldots, m$, as claimed.

With this $k$, set $s_n = t_k(n) - 1/n$. Analogously to (21) we obtain

$$
g_n,\varepsilon(t_k(n) - 1/n) = \max_{t \in [0, t_k(n) - 2/n]} f_n(t) \vee \max_{1 \leq j \leq k} (f_n(t_j(n)) - 1/n) + \bar{y}_j(n) \quad (28)
$$

and

$$
g_n,\varepsilon((t_k(n) - 1/n) -) = g_n,\varepsilon((t_k(n) - 2/n) -) = \max_{t \in [0, t_k(n) - 2/n]} f_n(t) \vee \max_{1 \leq j \leq k - 1} (f_n(t_j(n)) - 1/n) + \bar{y}_j(n) \quad (29)
$$

We shall now show

$$
\lim_{n \to \infty} g_n,\varepsilon(t_k(n) - 1/n) = \sup_{t \in [0, t_k]} f_0(t) \vee \max_{t_j \leq t_k} (f_0(\bar{t}_j) + \bar{y}_j) = g_0,\varepsilon(t_k)
$$

and

$$
\lim_{n \to \infty} g_n,\varepsilon((t_k(n) - 1/n) -) = \sup_{t \in [0, t_k]} f_0(t) \vee \max_{t_j < t_k} (f_0(\bar{t}_j) + \bar{y}_j) = g_0,\varepsilon(t_k -).
$$

Indeed, while the limit relations

$$
\lim_{n \to \infty} \max_{1 \leq j \leq k} (f_n(t_j(n)) - 1/n) + \bar{y}_j(n) = \max_{t_j \leq t_k} (f_0(\bar{t}_j) + \bar{y}_j) \quad (30)
$$

and

$$
\lim_{n \to \infty} \max_{1 \leq j \leq k - 1} (f_n(t_j(n)) - 1/n) + \bar{y}_j(n) = \max_{t_j < t_k} (f_0(\bar{t}_j) + \bar{y}_j) \quad (31)
$$

are secured by (18) and (19), the limit relation

$$
\lim_{n \to \infty} \max_{t \in [0, t_k(n) - 2/n]} f_n(t) = \sup_{t \in [0, t_k]} f_0(t)
$$

holds in view of Lemma 2.5 with $s_n = t_k(n) - 1/n$ for $n \in \mathbb{N}$ and $s_0 = t_k$. Thus, formula (25) has been proved in Case 1.
CASE 2: \( \bar{s} \) is a discontinuity point of both \( g_0, \epsilon \) and \( f_0 \).

SUBCASE 2.1: \( \bar{s} = \bar{t}_k \) for some \( k = 1, \ldots, m \). We intend to check that (25) holds with \( s_n = t_k^{(n)} - 1/n \). Using formulae (28) and (29) and recalling (20), (30) and (31) we infer

\[
\lim_{n \to \infty} g_{n, \epsilon}(\bar{t}_k^{(n)} - 1/n -) = \sup_{t \in [0, \bar{t}_k]} f_0(t) \lor \max_{\bar{t}_j < \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j) = g_{0, \epsilon}(\bar{t}_k -)
\]

and

\[
\lim_{n \to \infty} g_{n, \epsilon}(\bar{t}_k^{(n)} - 1/n) = \sup_{t \in [0, \bar{t}_k]} f_0(t) \lor \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j).
\]

Since

\[
f_0(\bar{t}_k) = f_0(\bar{t}_k -) + \bar{x}_k \leq f_0(\bar{t}_k) + \bar{y}_k.
\]

in view of (13), we conclude that

\[
\sup_{t \in [0, \bar{t}_k]} f_0(t) \lor \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j) = \sup_{t \in [0, \bar{t}_k]} f_0(t) \lor \max_{\bar{t}_j \leq \bar{t}_k} (f_0(\bar{t}_j -) + \bar{y}_j) = g_{0, \epsilon}(\bar{t}_k),
\]

thereby finishing the proof of (25) in this subcase.

SUBCASE 2.2: \( \bar{s} \notin \{\bar{t}_1, \ldots, \bar{t}_m\} \). Let \( r \) be a continuity point of \( f_0 \) satisfying \( r > \bar{s} \). Recall that (13) entails (22). Hence, there is a sequence \( (\lambda_n)_{n \in \mathbb{N}} \) of continuous strictly increasing functions of \([0, r] \) onto \([0, r] \) such that

\[
\lim_{n \to \infty} \sup_{t \in [0, r]} |\lambda_n(t) - t| = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup_{s \in [0, r]} \left( \sup_{t \in [0, \lambda_n(s)]} f_0(t) - \sup_{t \in [0, s]} f_n(t) \right) = 0.
\]

In particular, \( \lim_{n \to \infty} \sup_{t \in [0, s_n]} f_n(t) = \sup_{t \in [0, \bar{s}]} f_0(t) \) and \( \lim_{n \to \infty} \sup_{t \in [0, s_n]} f_n(t) = \sup_{t \in [0, s]} f_0(t) \), where \( s_n := \lambda_n(\bar{s}) \). We shall show that (25) holds with this choice of \( s_n \). To this end, it only remains to note that

\[
\lim_{n \to \infty} \max_{t_j^{(n)} < s_n} (f_n(t_j^{(n)}) + \bar{y}_j^{(n)}) = \lim_{n \to \infty} \max_{t_j^{(n)} \leq s_n} (f_n(t_j^{(n)}) + \bar{y}_j^{(n)}) = \max_{\bar{t}_j \leq \bar{s}} (f_0(\bar{t}_j -) + \bar{y}_j) = \max_{\bar{t}_j < \bar{s}} (f_0(\bar{t}_j -) + \bar{y}_j)
\]

as a consequence of \( \bar{s} \notin \{\bar{t}_1, \ldots, \bar{t}_m\} \) and (18). Therefore,

\[
\lim_{n \to \infty} g_{n, \epsilon}(s_n -) = \lim_{n \to \infty} \sup_{t \in [0, s_n]} f_n(t) \lor \max_{t_j^{(n)} < s_n} (f_n(t_j^{(n)}) + \bar{y}_j^{(n)}) = \sup_{t \in [0, \bar{s}]} f_0(t) \lor \max_{\bar{t}_j < \bar{s}} (f_0(\bar{t}_j -) + \bar{y}_j) = g_{0, \epsilon}(\bar{s} -)
\]

and

\[
\lim_{n \to \infty} g_{n, \epsilon}(s_n) = \lim_{n \to \infty} \sup_{t \in [0, s_n]} f_n(t) \lor \max_{t_j^{(n)} \leq s_n} (f_n(t_j^{(n)}) + \bar{y}_j^{(n)}) = \sup_{t \in [0, \bar{s}]} f_0(t) \lor \max_{\bar{t}_j \leq \bar{s}} (f_0(\bar{t}_j -) + \bar{y}_j) = g_{0, \epsilon}(\bar{s})
\]

which proves (25).

The proof of Theorem 2.1 is complete. \( \Box \)
Proof of Theorem 1.1. By Corollary 6.1 on p. 183 in [22], condition (5) entails
\[ \sum_{l \geq 1} \varepsilon(l/n, \xi_l/a(n), \eta_i^n/a(n)) \Rightarrow \sum_k \varepsilon(\theta_i, i, j_k), \quad n \to \infty \]
on the space topology on \( M_p([0, \infty) \times E) \) and thereupon
\[ \left( \sum_{l \geq 1} \mathbb{1}_{\{\xi_l \neq 0\}} \varepsilon(l/n, \xi_l/a(n)), \sum_{l \geq 1} \varepsilon(l/n, \xi_l/a(n), \eta_i^n/a(n)) \right) \Rightarrow \left( \sum_k \mathbb{1}_{\{i \neq 0\}} \varepsilon(\theta_i, i), \sum_k \varepsilon(\theta_i, i, j_k) \right) \]
(32) as \( n \to \infty \) on \( M_p([0, \infty) \times (-\infty, +\infty) \setminus \{0\}) \times M_p([0, \infty) \times E) \) because the first coordinates are just the restrictions of the second from \([0, \infty) \times (-\infty, +\infty) \setminus \{0\}\).

In the proof of Corollary 7.1 on p. 218 in [22] it is shown that the convergence of the first coordinates in (32) implies \( S_{[1]}/a(n) \xrightarrow{a.s.} S_\alpha(\cdot) \) as \( n \to \infty \). Starting with full relation (32) exactly the same reasoning leads to the conclusion
\[ \left( \frac{S_{[1]}}{a(n)}, \sum_{l \geq 1} \varepsilon(l/n, \xi_l/a(n), \eta_i^n/a(n)) \right) \Rightarrow \left( S_\alpha^* \cdot \sum_k \varepsilon(\theta_i, i, j_k) \right), \quad n \to \infty \]
or, equivalently,
\[ \left( \frac{S_{[1]}}{a(n)}, \sum_{l \geq 1} \varepsilon(l/n, \xi_l/a(n), \eta_i^n/a(n)) \mathbb{1}_{\{\eta_i > 0\}} \right) \Rightarrow \left( S_\alpha^* \cdot \sum_k \varepsilon(\theta_i, i, j_k) \right), \quad n \to \infty \]
in the product topology on \( D \times M_p([0, \infty) \times E) \). By the Skorokhod representation theorem there are versions which converge a.s. Retaining the original notation for these versions we want to apply Theorem 2.1 with \( f_n(\cdot) = S_{[1]}/a(n) \), \( f_0 = S_\alpha^* \), \( \nu_n = \sum_{l \geq 1} \varepsilon(l/n, \xi_l/a(n), \eta_i^n/a(n)) \mathbb{1}_{\{\eta_i > 0\}} \) and \( \nu_0 = N(\nu) = \sum_k \varepsilon(\theta_i, i, j_k) \). We already know that conditions (13) and (14) are fulfilled a.s. It is obvious that \( N(\nu)(\{0\} \times E) = 0 \) a.s. In order to show that \( N(\nu) \) does not have clustered jumps a.s. i.e., \( \theta_k \neq \theta_j \) for \( k \neq j \) a.s., it suffices to check this property for \( N(\nu)([0, T] \times [-\infty, +\infty] \times (\delta, \infty)) \cap (-) \) with \( T > 0 \) and \( \varepsilon > 0 \) fixed. This is done on p. 223 in [22]. Hence Theorem 2.1 is indeed applicable with our choice of \( f_n \) and \( \nu_n \), and (6) follows.

3 Proofs of Propositions 1.6 and 1.8 and Theorem 1.9

Proof of Proposition 1.6. Fix any \( T > 0 \). Note that (5) entails
\[ \lim_{x \to \infty} x \mathbb{P}\{ \eta > \varepsilon a(x) \} = 0 \]
(33) for all \( \varepsilon > 0 \) because \( a(x) \) is regularly varying at \( \infty \) (of index 1/\( \alpha \)). Since, for all \( \varepsilon > 0 \),
\[ \mathbb{P}\{ \sup_{0 \leq s \leq T} \eta_{[ns]} + 1 > \varepsilon a(n) \} = 1 - \left( \mathbb{P}\{ \eta \leq \varepsilon a(n) \} \right)^{[nT] + 1} \leq ([nT] + 1) \mathbb{P}\{ \eta > \varepsilon a(n) \} \to 0 \]
as \( n \to \infty \) in view of (33), we infer
\[ \frac{\sup_{0 \leq s \leq T} \eta_{[ns]} + 1}{a(n)} \xrightarrow{a.s.} 0, \quad n \to \infty. \]
This in combination with (4) enables us to conclude that
\[
\frac{S_{[n]} + \eta_{[n]}+1}{a(n)} \xrightarrow{J_1} \mathcal{S}_a(\cdot), \quad n \to \infty
\]
by Slutsky’s lemma. Relation (9) now follows by the continuous mapping theorem because the supremum functional is continuous in the \(J_1\)-topology.

Proof of Proposition 1.8. To begin with, we note that \(\lim_{n \to \infty} (b(n)/a(n)) = \infty\) as a consequence of (11). Consequently,
\[
S_{[n]}/b(n) \xrightarrow{J_1} \Xi(\cdot)
\]
in view of (4), where \(\Xi(t) := 0\) for \(t \geq 0\). Further, according to Theorem 3.6 on p. 62 in combination with Corollary 6.1 on p. 183 in [22], regular variation of \(\mathbb{P}\{\eta > x\}\) ensures that
\[
\sum_{k \geq 0} \varepsilon(k/n, \eta_{k+1}/b(n)) \mathbb{1}_{\{\eta_{k+1} > 0\}} \Rightarrow N := \sum_k \varepsilon(\theta_k, j_k), \quad n \to \infty
\]
on \(\mathcal{M}_p([0, \infty) \times (0, \infty])\) and thereupon
\[
\left( S_{[n]}/b(n), \sum_{k \geq 0} \mathbb{1}_{\{\eta_{k+1} > 0\}} \varepsilon(k/n, \eta_{k+1}/b(n)) \right) \Rightarrow (\Xi(\cdot), N), \quad n \to \infty
\]
on \(D \times \mathcal{M}_p([0, \infty) \times (0, \infty])\) equipped with the product topology. Arguing as in the proof of Theorem 1.1 we obtain (12) by an application of Remark 2.2 with \(f_n(\cdot) = S_{[n]}/b(n)\), \(f_0 = \Xi\), \(\nu_n = \sum_{k \geq 0} \varepsilon(k/n, \eta_{k+1}/b(n)) \mathbb{1}_{\{\eta_{k+1} > 0\}}\) and \(\nu_0 = N\). The condition \(N((a,b) \times (0, \infty]) \geq 1\) a.s. whenever \(0 < a < b\) required in Remark 2.2 holds because \(\mu((0, \infty]) = \infty\).

Proof of Theorem 1.9. The limit relations of Theorem 1.1 and Propositions 1.6 and 1.8 can be written in a unified form as
\[
\max_{0 \leq k \leq [n]} T_{k+1}/c(n) \Rightarrow X(\cdot), \quad n \to \infty
\]
in the \(J_1\)- or the \(M_1\)-topology on \(D\). Using this limit relation together with the inequality
\[
\max_{0 \leq k \leq n} T_{k+1} \leq \log \left( \sum_{k=0}^{n} e^{T_{k+1}} \right) \leq \log(n+1) + \max_{0 \leq k \leq n} T_{k+1}
\]
and the fact that \(\lim_{n \to \infty} (\log n/c(n)) = 0\) we arrive at the desired conclusion
\[
\log \sum_{k=0}^{[n]} e^{T_{k+1}} / c(n) \Rightarrow X(\cdot), \quad n \to \infty
\]
in the \(J_1\)- or \(M_1\)-topology on \(D\).

Since the limit process \(X\) is nonnegative a.s., the result remains true on replacing \(\log\) with \(\log^+\).
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