Characteristic cycle of the exterior product of constructible sheaves

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Abstract

We show that the characteristic cycle of the exterior product of constructible complexes is the exterior product of the characteristic cycles of factors. This implies the compatibility of characteristic cycles with smooth pull-back which is a first step in the proof of the index formula.

The characteristic cycle of a constructible complex on a smooth scheme over a perfect field is defined as a cycle on the cotangent bundle \([11]\) supported on the singular support \([1]\). It is characterized by the Milnor formula \([11, (5.15)]\) for the vanishing cycles defined for morphisms to curves.

We prove a formula \((2.7)\) for the external product in Theorem \(2.2\). Theorem \(2.2\) implies the compatibility of characteristic cycles with smooth pull-back Corollary \(2.4\) which is a first step in the proof of the index formula \([11, \text{Theorem 7.13}]\). Note that Theorem \(2.2\) is proved without using the results in \([11]\) after Proposition 5.17 loc. cit. included. Corollary \(2.5\) corresponds to \([7, \text{Corollary 5.4.14}]\).

We briefly sketch the idea of proof of Theorem \(2.2\). First we show that the external product is micro-supported on the external product of the singular supports of the factors. We deduce this from projection formulas for nearby cycles over general base schemes in \([13]\) recalled in Section \(1\). The formula \((2.7)\) for characteristic cycle is deduced from the Thom-Sebastiani formula \([6]\) and a conductor formula \((2.9)\) for the additive convolution \([6, \text{Corollary 5.12}]\) which is an analogue for torsion coefficient of \([9, \text{Proposition (2.7.2.1)}]\).

Corollary \(2.4\) of Theorem \(2.2\) is a first step of the proof of index formula in general dimension \([11, \text{Theorem 7.13}]\). The formula \((2.9)\) for convolution on which the proof of Theorem \(2.2\) is based is essentially equivalent to the formula \((2.7)\) in the case where \(\dim X = \dim Y = 1\). This can be deduced from the special case of the index formula in dimension 2 proved earlier in \([12, \text{Theorem 3.19}]\).

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1 Nearby cycles and projection formulas

Let \( f : X \to S \) be a morphism of schemes. For the definition and properties of the vanishing topos \( X \times_S S \) and the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow{\psi_f} & & \downarrow{f} \\
X \times_S S & \xleftarrow{p_1} & S
\end{array}
\]

we refer to \([5], [6], [10]\).

Assume that \( f : X \to S \) is a morphism of finite type of noetherian schemes. Let \( \Lambda \) be a finite field of characteristic \( \ell \) invertible on \( S \) and let \( \mathcal{F} \) and \( \mathcal{G} \) be complexes bounded above of \( \Lambda \)-modules on \( X \) and on \( S \) respectively. A canonical morphism

\[
(1.2) \quad R\psi_f \mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} p_2^* \mathcal{G} \to R\psi_f (\mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} f^* \mathcal{G})
\]

on \( X \times_S S \) is defined as the adjoint of \( \psi_f^*(R\psi_f \mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} p_2^* \mathcal{G}) = \psi_f^* R\psi_f \mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} f^* \mathcal{G} \to \mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} f^* \mathcal{G} \)
induced by the adjunction \( \psi_f^* R\psi_f \mathcal{F} \to \mathcal{F} \), since \( \psi_f \) is of finite cohomological dimension by \([10\text{, Proposition 3.1]}\).

**Lemma 1.1** \((13\text{, Proposition 4]}\). Let \( f : X \to S \) be a morphism of finite type of noetherian schemes and let \( \mathcal{F} \) and \( \mathcal{G} \) be complexes bounded above of \( \Lambda \)-modules on \( X \) and on \( S \) respectively. We assume that the formation of \( R\psi_f \mathcal{F} \) commutes with finite base change.
Then, the canonical morphism

\[
(1.2) \quad R\psi_f \mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} p_2^* \mathcal{G} \to R\psi_f (\mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} f^* \mathcal{G})
\]

on \( X \times_S S \) is an isomorphism.

Further, let \( h : W \to X \) be a morphism of schemes and consider the commutative diagram

\[
\begin{array}{ccc}
W & \xleftarrow{h} & X \\
\downarrow{f} & & \downarrow{f} \\
W \times_S S & \xleftarrow{\psi_f} & X \times_S S
\end{array}
\]

of vanishing toposes. By \([10\text{, Proposition 3.1]}\), \( \psi_f \) is of finite cohomological dimension.
Let \( \mathcal{F} \) and \( \mathcal{G} \) be complexes of \( \Lambda \)-modules on \( X \) and on \( W \) respectively. We define a base change morphism

\[
(1.4) \quad (\psi_f^* R\psi_f \mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} p_2^* \mathcal{G}) \to Rf_* (p_2^* \mathcal{F} \otimes_{\Lambda}^{\mathbb{L}} p_1^* \mathcal{G})
\]
on $W \times_S S$ as the adjoint of the morphism $f^* (h^* R\Psi_f \mathcal{F} \otimes_{\Lambda}^L p_1^* \mathcal{G}) = f^* h^* R\Psi_f \mathcal{F} \otimes_{\Lambda}^L p_1^* \mathcal{G} \to p_2^* \mathcal{F} \otimes_{\Lambda}^L p_1^* \mathcal{G}$ defined as follows. We identify $f^* h^* R\Psi_f = h^* f^* R\Psi_{id}$ and $p_2^* = h^* p_2^* = h^* R\Psi_{id}$ by the isomorphism $p_2^* \to R\Psi_{id}$ [3 Proposition 4.7] defined as the adjoint of $\Psi_{id}^* p_2^* \to \text{id}$. Then, the morphism in question is induced by the adjunction $f^* f^* \to \text{id}$.

**Lemma 1.2** ([13 Proposition 5]). Let $f: X \to S$ be a morphism of finite type of noetherian schemes and $h: W \to X$ be a morphism of schemes. Let $\mathcal{F}$ and $\mathcal{G}$ be complexes bounded above of $\Lambda$-modules on $X$ and on $W$ respectively. We assume that the formation of $R\Psi_f \mathcal{F}$ commutes with finite base change. Then, the canonical morphism

\[ h^* R\Psi_f \mathcal{F} \otimes_{\Lambda}^L p_1^* \mathcal{G} \to Rf^* (p_2^* \mathcal{F} \otimes_{\Lambda}^L p_1^* \mathcal{G}) \]

on $W \times_S S$ is an isomorphism.

We recall an interpretation of local acyclicity in terms of vanishing topos.

**Proposition 1.3** ([11 Proposition 1.7]). Let $f: X \to S$ be a morphism of schemes. Then, for a complex $\mathcal{F} \in D^+(X)$ bounded below, the following conditions (1) and (2) are equivalent:

1. The morphism $f: X \to S$ is locally acyclic relatively to $\mathcal{F}$.
2. The formation of $R\Psi_f \mathcal{F}$ commutes with every finite base change $T \to S$ and the canonical morphism $p_1^* \mathcal{F} \to R\Psi_f \mathcal{F}$ is an isomorphism.
3. The canonical morphism $p_1^* \mathcal{F}_T \to R\Psi_{f_T} \mathcal{F}_T$ is an isomorphism for every finite morphism $T \to S$, the cartesian diagram

\[
\begin{array}{ccc}
X & \to & X_T \\
\downarrow f & & \downarrow f_T \\
S & \to & T
\end{array}
\]

and the pull-back $\mathcal{F}_T$ of $\mathcal{F}$ on $X_T$.

**Corollary 1.4.** Let $f: X \to S$ be a morphism of finite type of noetherian schemes and let $\mathcal{F}$ be a bounded complex of $\Lambda$-modules on $X$. Assume that $f: X \to S$ is locally acyclic relatively to $\mathcal{F}$.

1. Let $\mathcal{G}$ be a complex bounded above of $\Lambda$-modules on $S$. Then, the canonical morphism (1.2) induces an isomorphism

\[ p_1^* \mathcal{F} \otimes_{\Lambda}^L p_2^* \mathcal{G} \to R\Psi_f (\mathcal{F} \otimes_{\Lambda}^L f^* \mathcal{G}) \]

on $X \times_S S$.

2. Let $h: W \to X$ be a morphism of schemes and let $\mathcal{G}$ be a complex bounded above of $\Lambda$-modules on $W$. Then, the canonical morphism (1.4) defines an isomorphism

\[ p_1^* h^* \mathcal{F} \otimes_{\Lambda}^L p_1^* \mathcal{G} \to Rf^* (p_2^* \mathcal{F} \otimes_{\Lambda}^L p_1^* \mathcal{G}) \]

on $W \times_S S$. 

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Proof. By the assumption of local acyclicity and Proposition 1.3 \( (1) \Rightarrow (2) \), the formation of \( R\Psi_f^*,F \) commutes with finite base change and the canonical morphism \( p_1^*F \to R\Psi_f^*,F \) is an isomorphism.

1. By Lemma 1.1 \( (1.2) \) induces an isomorphism \( 1.5 \).

2. By Lemma 1.2 and by the canonical isomorphism \( h \ p_1^* \to p_1^*h^* \), the right hand side of \( 1.4 \) is identified with that of \( 1.6 \). Thus, the assertion follows. \( \square \)

We briefly recall the definition of additive convolution from \([6, 4.1]\). Let \( k \) be a field and let \( A_1 = A^{(0)} \) and \( A_2 = A^{(0)} \) denote the henselizations of the affine line and of the affine plane at the origins. Let \( f: X \to A_1 \) and \( g: Y \to A_1 \) be morphisms of finite type. We regard the fiber product \( (X \times Y)_2 = (X \times Y) \times_{A_1 \times A_2} A_2 \) as a scheme over \( A_1 \) by the composition of the second projection and the morphism \( a: A_2 \to A_1 \) induced by the addition \( +: A^2 \to A^1 \). Morphisms of vanishing toposes

\[
X \leftarrow_{A_1} A_1 \leftarrow_{pr_1} (X \times Y)_2 \leftarrow_{A_2} A_2 \leftarrow_{pr_2} Y \leftarrow_{A_1} A_1
\]

are defined by projections and by \( a: A_2 \to A_1 \).

Let \( \Lambda \) be a finite field of characteristic invertible in \( k \). For bounded complexes \( F \) and \( G \) of \( \Lambda \)-modules on \( X \leftarrow_{A_1} A_1 \) and on \( Y \leftarrow_{A_1} A_1 \), let \( F \otimes G \) denote \( pr_1^*F \otimes pr_2^*G \) on \( (X \times Y)_2 \leftarrow_{A_2} A_2 \) and define the additive convolution \( F \ast G \) on \( (X \times Y)_2 \leftarrow_{A_1} A_1 \) by

\[
(1.7) \quad F \ast G = R\lambda'^*(F \otimes G).
\]

2 External products

For the definitions and basic properties of the singular support of a constructible complex on a smooth scheme over a perfect field, we refer to \([1]\) and \([11]\).

Let \( k \) be a field and let \( \Lambda \) be a finite field of characteristic invertible in \( k \).

Proposition 2.1. Let \( X \) and \( Y \) be smooth schemes over \( k \) and \( F \) and \( G \) be constructible complexes of \( \Lambda \)-modules on \( X \) and on \( Y \) respectively. Assume that \( F \) is micro-supported on a closed conical subset \( C \subset T^*X \). Then \( F \Box_{\Lambda} G = pr_1^*F \otimes \Lambda pr_2^*G \) is micro-supported on \( C \times T^*Y \subset T^*(X \times Y) \).

Proof. It suffices to show that, for morphisms \( a: W \to X, b: W \to Y, c: W \to Z \) of smooth schemes over \( k \) such that the pair \( h = (a,b): W \to X \times Y \) and \( c: W \to Z \) is \( C \times T^*Y \)-transversal, the morphism \( c: W \to Z \) is locally acyclic relatively to \( h^*F \). By \([11]\) Lemma 3.6.9, the pair of morphisms \( a: W \to X \) and \( f = (b,c): W \to Y \times Z \) is \( C \)-transversal. Since \( F \) is assumed micro-supported on \( C \), the morphism \( f = (b,c): W \to Y \times Z \) is locally acyclic relatively to \( a^*F \).

Let \( Z' \to Z \) be any finite morphism and we consider the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{a'} & W' = W \times_Z Z' & \xrightarrow{c'} & Z' \\
\downarrow{h'} & & \downarrow{\nu'} & & \downarrow{\nu'} \\
X \times Y & \xrightarrow{\nu} & Y & \xrightarrow{q} & Y \times Z'
\end{array}
\]
By Proposition 1.3 (3) $\Rightarrow$ (1), it suffices to show that the canonical morphism

\[(2.1) \quad p_1^*h^\ast(\mathcal{F} \boxtimes \mathcal{G}) \to R\Psi_c h^\ast(\mathcal{F} \boxtimes \mathcal{G})\]

on $W' \times_{Z'} Z'$ is an isomorphism. For the second term in (2.1), we have a canonical isomorphism

\[(2.2) \quad R^r_* R\Psi_{f'}(a^{**} \mathcal{F} \otimes f'^* q^* \mathcal{G}) \to R\Psi_c h^\ast(\mathcal{F} \boxtimes \mathcal{G}).\]

For the first term in (2.2), we apply Corollary 1.4.1 to $f' : W' \to Y \times Z'$ and to $a^{**} \mathcal{F}$ on $W'$ and $q^* \mathcal{G}$ on $Y \times Z'$. Since $f' : W' \to Y \times Z'$ is locally acyclic relatively to $a^{**} \mathcal{F}$, the assumption of Corollary 1.4.1 is satisfied and we obtain a canonical isomorphism

\[(2.3) \quad p_1^* a^{**} \mathcal{F} \otimes p_2^* q^* \mathcal{G} \to R\Psi_{f'}(a^{**} \mathcal{F} \otimes f'^* q^* \mathcal{G}).\]

Further we apply Corollary 1.4.2 to $f' : W' \to Y \times Z'$ and $r : Y \times Z' \to Z'$ and to $q^* \mathcal{G}$ on $Y \times Z'$ and $a^{**} \mathcal{F}$ on $W'$. By the generic local acyclicity [3, Corollaire 2.16], the second projection $r : Y \times Z' \to Z'$ is locally acyclic relatively to $q^* \mathcal{G}$. Hence the assumption of Corollary 1.4.2 is satisfied and we obtain a canonical isomorphism

\[(2.4) \quad p_1^* a^{**} \mathcal{F} \otimes p_2^* f'^* q^* \mathcal{G} \to R^r_* (p_1^* a^{**} \mathcal{F} \otimes p_2^* q^* \mathcal{G})\]

on $W' \times_{Z'} Z'$. Thus, (2.2) - (2.4) give an isomorphism

\[(2.5) \quad p_1^* h^\ast(\mathcal{F} \boxtimes \mathcal{G}) = p_1^* a^{**} \mathcal{F} \otimes p_2^* b'^* \mathcal{G} \to R\Psi_c h^\ast(\mathcal{F} \boxtimes \mathcal{G})\]

and the assertion follows. \(\square\)

For linear combinations $A = \sum a_m \cdot C_a$ and $A' = \sum a'_{m'} \cdot C_{a'}$ of irreducible components of closed conical subsets $C = \bigcup a \cdot C_a \subset T^*X$ and $C' = \bigcup a' \cdot C_{a'} \subset T^*Y$ of cotangent bundles, the external product $A \boxtimes A'$ is defined by

\[(2.6) \quad A \boxtimes A' = \sum_{a,a'} m_a m_{a'} \cdot C_a \times C_{a'}\]

as a linear combination supported on $C \times C' \subset T^*X \times T^*Y = T^*(X \times Y)$.

**Theorem 2.2.** Let $X$ and $Y$ be smooth schemes over a perfect field $k$ and $\mathcal{F}$ and $\mathcal{G}$ be constructible complexes of $\Lambda$-modules.

1. Assume that $\mathcal{F}$ and $\mathcal{G}$ are micro-supported on closed conical subsets $C \subset T^*X$ and on $C' \subset T^*Y$ respectively. Then $\mathcal{F} \boxtimes_{\Lambda}^L \mathcal{G}$ is micro-supported on $C \times C' \subset T^*X \times T^*Y = T^*(X \times Y)$.

2. We have

\[(2.7) \quad CC(\mathcal{F} \boxtimes_{\Lambda}^L \mathcal{G}) = CC(\mathcal{F}) \boxtimes CC(\mathcal{G}).\]

3. We have

\[(2.8) \quad SS(\mathcal{F} \boxtimes_{\Lambda}^L \mathcal{G}) = SS(\mathcal{F}) \boxtimes SS(\mathcal{G}).\]

We will deduce the assertion 2 from the following multiplicativity of the Artin conductor under the convolution. This is an analogue for torsion coefficient of that for $\text{Q}_c$-coefficient due to Laumon [9, Proposition (2.7.2.1)].
Lemma 2.3 ([6 Corollary 5.12]). Let $\mathcal{K}$ and $\mathcal{L}$ be constructible complexes of $\Lambda$-modules on the strict localization $A_1 = A^1_{(0)}$. Then, for the Artin conductor, we have

$$-a_0(\mathcal{K} \ast \mathcal{L}) = (-a_0 \mathcal{K}) \cdot (-a_0 \mathcal{L}).$$

Proof of Theorem 2.22. 1. By Proposition 2.11, the external product $\mathcal{F} \boxtimes_{\Lambda} \mathcal{G}$ is micro-supported on the intersection $(C \times T^*Y) \cap (T^*X \times C') = C \times C'$.

2. Write the singular supports $C = SS(\mathcal{F}) = \bigcup_a C_a$ and $C' = SS(\mathcal{G}) = \bigcup_{a'} C'_{a'}$ as the unions of irreducible components and set $CC(\mathcal{F}) = \bigcup_a m_a C_a$ and $CC(\mathcal{G}) = \bigcup_{a'} m_{a'} C'_{a'}$. Then, by 1, we have $CC(\mathcal{F} \boxtimes \mathcal{G}) = \bigcup_{a,a'} m_a C_a \times C'_{a'}$ for some integers $m_a$. It suffices to show $m_{b,\nu} = m_b \cdot m_{\nu'}$ for each pair of irreducible components $C_b$ and $C'_{\nu'}$.

After shrinking $X$, we may take a morphism $f : X \to A^1$ such that $f$ has an isolated characteristic point $u$, that $f(u) = 0$ and that the section $df$ meets only $C_b$. Similarly, after shrinking $Y$, we may take a morphism $g : Y \to A^1$ such that $g$ has an isolated characteristic point $v$, that $g(v) = 0$ and that the section $dg$ meets only $C'_{\nu'}$. Let $h : X \times Y \to A^1$ denote the morphism defined by the sum $f + g$. Since $dh = df + dg$, the morphism $h$ has an isolated characteristic point $(u, v)$ with respect to $C \times C' = \bigcup_{a,a'} C_a \times C'_{a'}$ and that the section $dh$ meets only $C_b \times C'_{\nu'}$. Further, we have $(C_b \times C'_{\nu'}, dh)_{T^*(X \times Y), (u, v)} = (C_b, df)_{T^*X, u} \cdot (C'_{\nu'}, dg)_{T^*Y, v} \neq 0$. Thus, by the Milnor formula [11 (5.15)], it suffices to show

$$- \dim \text{tot} \phi_{(u,v)}(\mathcal{F} \boxtimes \mathcal{G}, h) = (- \dim \text{tot} \phi_u(\mathcal{F}, f)) \cdot (- \dim \text{tot} \phi_v(\mathcal{G}, g)).$$

We canonically identify $u \times_{A^1} A^1$ with the strict localization $A^1_{(0)}$. Then the total dimension $\dim \text{tot} \phi_{(u,v)}(\mathcal{F}, f)$ equals the Artin conductor $a_0((R\Psi_f \mathcal{F})|_{u \times_{A^1} A^1})$ and similarly for the other terms. By [6 Theorem 4.5 (4.5.1)], we have an isomorphism

$$R\Psi_f \mathcal{F} \ast R\Psi_g \mathcal{G} \to R\Psi_h(\mathcal{F} \boxtimes \mathcal{G}).$$

The left hand side is the additive convolution [17]. Thus, we obtain the equality (2.10) by applying Lemma 2.3 to $\mathcal{K} = (R\Psi_f \mathcal{F})|_{u \times_{A^1} A^1}$ and $\mathcal{L} = (R\Psi_g \mathcal{G})|_{v \times_{A^1} A^1}$.

3. We may assume $\mathcal{F}$ and $\mathcal{G}$ are perverse sheaves. Then, since the singular support is the support of the characteristic cycle by [11 Proposition 3.19.2], the assertion follows from 2. \qed

Corollary 2.4. Let $h : W \to X$ be a smooth morphism of smooth schemes over a perfect field $k$. Then, for a constructible complex $\mathcal{F}$ of $\Lambda$-modules on $X$, we have

$$CC h^* \mathcal{F} = h^! CC \mathcal{F}.$$

For the definition of the notation $h^! CC \mathcal{F}$, we refer to [11 Definition 5.16].

Proof. Since the assertion is étale local on $W$, we may assume $W = X \times A^m$ for an integer $m \geq 0$. Hence it follows from Theorem 2.22 2. \qed

In the above proof of Theorem 2.22 2, we deduced (2.7) from the multiplicativity of the Artin conductor under the convolution [6 Corollary 5.12]. Conversely, [6 Corollary 5.12] is an immediate consequence of (2.7) where $\dim X = \dim Y = 1$. This crucial case can be deduced from the index formula proved earlier in [12 Theorem 3.19] and is essentially
equivalent to [8, Exemples 2.3.8 (a)]. Corollary 2.4 is a first step of the proof of the index formula [11, Theorem 7.13] in general dimension. These logical implications are summarized in the diagram:

\[
\begin{align*}
\text{(index formula in dim. 2)} \\
& \Downarrow \\
\text{(multiplicativity of the Artin conductor)} \quad \Downarrow \\
& \text{(compatibility with smooth pull-back)} \quad \Rightarrow \\
& \text{(index formula in dim. 2)}.
\end{align*}
\]

**Corollary 2.5.** Let \( F \) and \( G \) be constructible complexes of \( \Lambda \)-modules on a smooth scheme \( X \) over \( k \). Assume that the intersection \( SS(F) \cap SS(G) \subset T^*X \) of the singular supports is a subset of the 0-section \( T^*_X \subset T^*X \). Then, the canonical morphism

\[
(2.13) \quad G \otimes^L R\text{Hom}_X(F, \Lambda) \to R\text{Hom}_X(F, G)
\]

is an isomorphism.

**Proof.** Set \( C = SS(F) \) and \( C' = SS(G) \). The assumption \( C \cap C' \subset T^*_X \) implies that the diagonal \( \delta: X \to X \times X \) is \( C' \times C \)-transversal. Since \( SSD_X \subset SSF \) by [11, Corollary 2.27], the external product \( G \boxtimes D_X F \) is micro-supported on \( C' \times C \subset T^*X \times T^*X = T^*(X \times X) \) by Theorem 2.2.1. Since the canonical morphism \( G \boxtimes D_X F \to R\text{Hom}_{X \times X}(pr^*_X F, pr^*_Y G) \) is an isomorphism by [4, (3.1.1)], the diagonal \( \delta: X \to X \times X \) is \( R\text{Hom}_{X \times X}(pr^*_X F, pr^*_Y G) \)-transversal by [11, Proposition 5.6]. Thus, the assertion follows from [11] Proposition 5.3.2.

Recall that a closed subset of a vector bundle said to be *conical* if it is stable under the action of the multiplicative group \( G_m \).

**Definition 2.6.** Let \( f: X \to Y \) be a morphism of smooth schemes over \( k \). Let \( C \subset T^*X \) and \( C' \subset T^*Y \) be closed conical subsets of the cotangent bundles.

For \( x \in X \), we say that \( f: X \to Y \) is \((C, C')\)-transversal if for \( \omega \in T^*Y \times_Y y \) at \( y = f(x) \in Y \), the conditions \( \omega \in C', f^*\omega \in C \) imply \( \omega = 0 \). We say \( f: X \to Y \) is \((C, C')\)-transversal if \( f: X \to Y \) is \((C, C')\)-transversal at every \( x \in X \). Or equivalently, if the intersection \( df^{-1}(C) \cap f^*C' \) of the inverse image of \( C \) by \( df: X \times_Y T^*Y \to T^*X \) and \( f^*C' = X \times_Y C' \) in \( X \times_Y T^*Y \) is a subset of the 0-section.

**Lemma 2.7.** Let \( f: X \to Y \) be a morphism of smooth schemes over \( k \) and let \( \gamma: X \to X \times Y \) be the graph of \( f \). For closed conical subsets \( C \subset T^*X \) and \( C' \subset T^*Y \) the following conditions are equivalent:

1. \( f \) is \((C, C')\)-transversal.
2. \( \gamma \) is \( C \times C'\)-transversal.

Further, if the condition (2) is satisfied, the closed subset \( \gamma^0(C \times C') \subset T^*X \) equals the subset \( C + f^*C' \subset T^*X \) consisting of the sum \( \alpha + \beta \) of \( \alpha \in C \) and \( \beta \in f^*C' \).
Proof. The condition (1) is equivalent to the following condition:

\((1')\) For \(\beta \in f^*C'\), if \(df(\beta) \in C\), then we have \(\beta = 0\).

The condition (2) is equivalent to the following the condition:

\((2')\) For \(\alpha \in C\) and \(\beta \in f^*C'\), if \(\alpha + df(\beta) = 0\), then we have \(\alpha = 0\) and \(\beta = 0\).

Hence the conditions (1) and (2) are equivalent. Since \(\gamma^*(C \times C') = C \times f^*C'\), we obtain \(\gamma^o = C + f^*C'\). \(\square\)

For a separated morphism \(h: W \to X\) of finite type and for a constructible complex \(F\) of \(\Lambda\)-modules, a canonical morphism \(c_{h,F}: h^*F \otimes \mathbb{R}h!\Lambda \to \mathbb{R}h!F\) is defined in \([11, (8.13)]\) as the adjoint of the morphism \(\mathbb{R}h!(h^*F \otimes \mathbb{R}h!\Lambda) \to F \otimes \mathbb{R}h!\Lambda\) induced by the adjunction \(\mathbb{R}h!\mathbb{R}h!\Lambda \to \Lambda\).

**Proposition 2.8.** Let \(f: X \to Y\) be a morphism of smooth schemes over \(k\) and let \(\gamma: X \to X \times Y\) be the graph of \(f\). Let \(F\) be a constructible complex of \(\Lambda\)-modules on \(X\) and set \(C = SS(F)\).

Let \(G\) be a constructible complex of \(\Lambda\)-modules on \(Y\) and set \(C' = SS(G)\). Assume \(f\) is \((C, C')\)-transversal. Then, the canonical morphism

\[\gamma^*(F \boxtimes G) \otimes \mathbb{R}\gamma^! \Lambda \to \mathbb{R}\gamma^!(F \boxtimes G)\]

is an isomorphism and \(F \otimes f^*G = \gamma^*(F \boxtimes G)\) is micro-supported on \(C + f^*C' \subset T^*X\) consisting of the sum \(\alpha + \beta\) of \(\alpha \in C\) and \(\beta \in f^*C'\).

The morphism \((2.14)\) is the same as \([2, (5.3)]\).

**Proof.** The assumption that \(f\) is \((C, C')\)-transversal means that \(\gamma\) is \(C \times C'\)-transversal by Lemma 2.7. Since \(F \boxtimes G\) is micro-supported on \(C \times C'\) by Theorem 2.2.1, the morphism \(\gamma\) is \(F \boxtimes G\)-transversal by \([11, Proposition 5.6 (1) \Rightarrow (2)]\). Thus, the morphism \((2.14)\) is an isomorphism. Further, \(F \otimes f^*G = \gamma^*(F \boxtimes G)\) is micro-supported on \(\gamma^o(C \times C') = C + f^*C'\) by \([11, Lemma 2.11.4 (1) \Rightarrow (2)]\) and Lemma 2.7. \(\square\)

**Corollary 2.9.** Let the notation be as in Proposition 2.8 and assume that \(f\) is \(C\)-transversal. Then, for every constructible complex \(G\) of \(\Lambda\)-modules on \(Y\), the conclusion of Proposition 2.8 is satisfied.

The conclusion of Corollary 2.9 is shown to be equivalent to the local acyclicity of \(f\) in \([2, Theorem B.2]\).

**Proof.** Since \(f\) is \((C, T^*Y)\)-transversal, it is \((C, C')\)-transversal for any closed conical subset \(C' \subset T^*Y\). \(\square\)

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