Short-Time Asymptotics for Non-Self-Similar Stochastic Volatility Models

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\textbf{ABSTRACT}

We provide a short-time large deviation principle (LDP) for stochastic volatility models, where the volatility is expressed as a function of a Volterra process. This LDP does not require strict self-similarity assumptions on the Volterra process. For this reason, we are able to apply such an LDP to two notable examples of non-self-similar rough volatility models: models where the volatility is given as a function of a log-modulated fractional Brownian motion (Bayer, C., F. Harang, and P. Pigato. 2021. “Log-Modulated Rough Stochastic Volatility Models.” SIAM Journal on Financial Mathematics 12 (3): 1257–1284), and models where it is given as a function of a fractional Ornstein–Uhlenbeck (fOU) process (Gatheral, J., T. Jaisson, and M. Rosenbaum. 2018. “Volatility is Rough.” Quantitative Finance 18 (6): 933–949). In both cases, we derive consequences for short-maturity European option prices implied volatility surfaces and implied volatility skew. In the fOU case, we also discuss moderate deviations pricing and simulation results.

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\section{1. Introduction}

Recent years have seen wide interest in volatility modelling with Volterra processes in the quantitative finance community. This has been spurred by the success of rough volatility models, where volatility is a non-Markovian, fractional process (Gatheral, Jaisson, and Rosenbaum 2018). In many instances, in order to produce this type of dynamics, volatility is expressed as a function of a Volterra process, i.e., a suitable deterministic kernel integrated against a Brownian motion. In this context a very useful feature of such kernel and of the corresponding fractional process is self-similarity.

When looking at approximation formulas and asymptotics, self-similarity is usually key as it enables the translation of a small-noise result into a short-time one through space-time rescaling. This can then be used to price short-maturity options (see the discussion at the end of Section 3 in Gulisashvili 2018, 2020). Based on this procedure, several short-time formulas are available for rough volatility models, if the volatility process is expressed as a function of a fractional Brownian motion (fBM) (Forde and Zhang 2017), as a function...
of a Riemann–Liouville process (RLp), as in the rough Bergomi model (Bayer et al. 2019; Friz, Gassiat, and Pigato 2022; Fukasawa 2020), or as a solution to a fractional SDE, as in the fractional Heston model (Forde, Gerhold, and Smith 2021).

However, obtaining short-time approximation formulas is more difficult if volatility depends on a process which is not self-similar, such as the fractional Ornstein–Uhlenbeck (fOU) process (Garnier and Sølna 2017, 2018a, 2019, 2020a, 2020b; Horvath, Jacquier, and Lacombe 2019), or the log-modulated fBM (log-fBM) (Bayer, Harang, and Pigato 2021). In this paper, we address this issue, providing a short-time large deviation principle (LDP) for Volterra-driven stochastic volatilities, where the usual self-similarity assumption is replaced by a weaker scaling property for the kernel, that needs to hold only in the asymptotic sense (see conditions (K1) and (K2) below). We prove this general result starting from Cellupica and Pacchiarotti (2021), where a pathwise LDP for the log-price was proved when the volatility is function of a family of Volterra processes, and the price is solution to a scaled differential equation. Here, under suitable short-time asymptotic assumptions on the Volterra kernel, we prove a short-time LDP for the log-price process.

With our general result, we analyse more in-depth models with volatility given as a function of fOU or log-fBM, neither of which is self-similar. However, we note that both these processes can be seen as a perturbation of self-similar processes, so that our general result can be applied, assuming that the price process is a martingale and a moment condition on the price.

The first class of processes to which we apply our LDP are *log-modulated rough stochastic volatility models*, introduced in Bayer, Harang, and Pigato (2021) as a logarithmic perturbation of a more standard power-law Volterra stochastic volatility model, with volatility depending on a log-fBM. These models allow for the definition of a ‘true’, continuous volatility process with roughness (Hurst) parameter $H \in [0, 1]$ (including the ‘super-rough’ case $H = 0$), at the price of losing the self-similar structure of the power-law kernel. Differently from our LDP setting, however, in Bayer, Harang, and Pigato (2021) Edgeworth-type asymptotics are considered, meaning that log-moneyness is of the form $x\sqrt{t}$ ($t$ representing the time to maturity), while in order to observe a large deviations behaviour, we look here at a suitable log-moneyness regime (cf. Equation (16)). This regime is consistent with Forde–Zhang LDP for rough volatility (Forde and Zhang 2017) and the related large deviation results discussed below. When $H > 0$, we obtain a short-time LDP for the log-price process and consequent short-time option pricing, implied volatility and implied skew asymptotics. For this class of processes, the rate function only depends on the self-similar power-law kernel, while the speed depends also on the modulating logarithmic function. It is shown in Bayer, Harang, and Pigato (2021) that when $H = 0$ the implied volatility skew explodes as $t^{-1/2}$ (with a logarithmic correction), realizing the model-free bound in Lee (2005). Even though our proof only holds in the $H > 0$ case, the expression we obtain for skew asymptotics, computed for $H = 0$, is consistent with this model-free bound. We note that Baldi and Pacchiarotti (2022) have recently proved that in the $H = 0$ case, even if the log-modulated model is well-defined, an LDP cannot hold.

The second class of models to which we apply our LDP have a stochastic volatility given by a function of a *fractional Ornstein–Uhlenbeck* process. We find that, in short time, such a model behaves exactly as the analogous model, with volatility process given as the same function, computed on an fBM (i.e., the model studied in Forde and Zhang 2017). More precisely, we mean that the two models satisfy an LDP with the same speed and rate
function. It follows that also the short-time implied skew (computed as a suitable finite difference) is the same for fOU and fBM-driven stochastic volatility models. For small time scales, fOU is, in a sense, close to fBM (see Equation (14)), even though fOU is not self-similar. It is not uncommon when dealing with rough stochastic volatility, starting from the groundbreaking work of Gatheral, Jaisson, and Rosenbaum (2018), to consider at times fBM, at times fOU, depending on which is most convenient for the problem at hand. Our result can be seen as a justification for this type of procedure, as it shows that pricing vanilla options with one or the other volatility does not matter (too much) for short maturities. Moreover, the fOU process is the most standard choice for a stationary process with a fractional correlation structure. This is one of the reasons why it has been used as volatility process for option pricing and related issues (Garnier and Sølna 2017, 2018a, 2019, 2020a, 2020b; Horvath, Jacquier, and Lacombe 2019).

From our short-time LDP we formally derive the corresponding moderate deviation result, consistent with the one holding for self-similar rough volatility (Bayre et al. 2019). We provide numerical evidence for both these large and moderate deviations results and for the skew asymptotics. We investigate on simulations how the choice between fOU and fBM dynamics in the volatility affects volatility smiles and skews, how accurate are our approximations, and how they depend on the mean reversion parameter.

Background. In recent years, rough volatility has been widely used in option pricing, due to the great fit it provides to observed volatility surfaces (Bayer, Friz, and Gatheral 2016) and its ability to capture fundamental stylized facts of the implied volatility, notably the power-law explosion of the implied skew in short-time, which explodes as \( t^{H-1/2} \) under rough volatility (Alòs, León, and Vives 2007; Fukasawa 2011, 2017). Many authors have argued that \( H \) is actually positive but very close to 0 (Bayer, Friz, and Gatheral 2016; Fukasawa 2020), which would give an extreme skew explosion close to \( t^{-1/2} \). This is a model-free bound (Lee 2005), that can be reached pricing options using ‘singular’ local (or local-stochastic) volatility models (Friz, Pigato, and Seibel 2021; Pigato 2019), but it is hard to obtain with (rough) stochastic volatility, as in the limit \( H \to 0 \) the volatility process usually degenerates and can be defined only as a distribution, not as a genuine process (Forde et al. 2022; Forde, Gerhold, and Smith 2021; Hager and Neuman 2021; Neuman and Rosenbaum 2018). Moreover, one observes a skew-flattening phenomenon, as \( H \to 0 \), in some of these models. This was the main motivation, in Bayer, Harang, and Pigato (2021), to introduce the log-modulation of the power law kernel, allowing the corresponding stochastic volatility to be defined as a genuine process also in the \( H \to 0 \) limit. Technically, the logarithmic correction ensures that the variance remains finite even for \( H = 0 \), which in turn avoids the subsequent definition problems as \( H \to 0 \), as well as the skew-flattening problem. We refer to Bayer, Harang, and Pigato (2021) and references therein for a detailed discussion of the \( H \to 0 \) problem.

Volatility was already taken as an exponential function of a fOU process with \( H < 1/2 \) when rough volatility was first proposed by Gatheral, Jaisson, and Rosenbaum (2018). On the one hand, mostly because of the desired self-similarity property of the volatility process, the exponential of a fBM or of a RLP are often used for pricing options. On the other hand, it is argued, e.g., in Gatheral, Jaisson, and Rosenbaum (2018), that taking a fOU with small mean reversion is not very different from taking a fBM in the volatility, with the considerable advantage that fOU is a stationary process (Cheridito, Kawaguchi, and Maejima 2003), while fBM and RLP are not. For a thorough discussion on fOU-driven volatilities and
related implied volatilities we refer in particular to Garnier and Sølna (2017, 2020a), for the relation of fOU to fast mean-reverting Markov stochastic volatility we refer to Garnier and Sølna (2018a, 2019), for hedging under fOU volatility we refer to Garnier and Sølna (2020b), for portfolio optimization using fast mean-reverting fOU process with $H > 1/2$ we refer to Fouque and Hu (2018). A small-noise LDP under fOU volatility, with other related results, has been proved in Horvath, Jacquier, and Lacombe (2019), and LDP and moderate deviation principles for the rough Stein–Stein and other models, also in short time, have been discussed in Jacquier and Pannier (2022) (see Remark 4.4 for details).

In this paper we consider short-time pricing asymptotics, i.e., pricing short-maturity European options. This is a widely studied topic, as these short-maturity pricing formulas provide methods for fast calibration, a quantitative understanding of the impact of model parameters on generated implied volatility surfaces led to some widely used parametrizations of the volatility surface, and help in the choice of the most appropriate model to be fitted to data (Ait-Sahalia, Li, and Li 2020). Short-maturity approximations are also used to obtain starting points for calibration procedures, which are then based on numerical evaluations. They have applications also for hedging, trading and risk management.

For notable results on short-maturity valuation formulas under Markovian stochastic volatility, we refer to Osajima (2015), and to Medvedev and Scaillet (2003, 2007) for Markovian stochastic volatility with jumps. Short-time skew and curvature under rough volatility have been discussed in Alòs and León (2017) and Fukasawa (2017). Short-maturity valuation formulas for European options and implied volatilities under rough stochastic volatility are given, e.g., in Bayer et al. (2019), El Euch et al. (2019), Forde and Zhang (2017), Friz, Gassiat, and Pigato (2021), Friz, Gassiat, and Pigato (2022), and Fukasawa (2020). Short-maturity local volatility under rough volatility is studied in Bourgey et al. (2023). Pathwise large and moderate deviation principles for rough stochastic volatility models are established in Cellupica and Pacchiarotti (2021), Catalini and Pacchiarotti (2023), Gulisashvili (2018, 2020, 2021, 2022), Gulisashvili, Viens, and Zhang (2018a, 2018b), Horvath, Jacquier, and Lacombe (2019), Jacquier, Pakkanen, and Stone (2018), and Jacquier and Pannier (2022).

Content of the paper. We consider in Section 2, an LDP for stochastic volatility models with volatility driven by general Volterra processes. In particular, in Section 2.2, we prove a short-time LDP for such models without relying on self-similarity. In Section 3, we see how these results provide short-time LDPs in relevant, non-self-similar examples such as the log-fBM and the fOU process. In Section 4, we derive practical consequences for option pricing and implied volatility, for volatility models where the volatility depends on log-fBM or fOU, at the large deviations regime. In the case of fOU, we also consider moderate deviations. A numerical study of the accuracy and dependence on relevant parameters of our results in the fOU case concludes the paper in Section 5.

2. Large Deviations for Volterra Stochastic Volatility Models

2.1. Small-Noise Large Deviations for the Log-Price

We are interested in stochastic volatility models with asset price dynamics described by

\[ dS_t = S_t \sigma (V_t) d(\tilde{\rho} \tilde{B}_t + \rho B_t), \quad 0 \leq t \leq T, \]  

(1)
where we set, without loss of generality, \( S_0 = 1 \) the initial price. The time horizon is \( T > 0 \), \( \bar{B} \) and \( B \) are two independent standard Brownian motions, \( \rho \in (-1, 1) \) is a correlation coefficient and \( \bar{\rho} = \sqrt{1 - \rho^2} \), so that \( \bar{B} = \bar{\rho}B + \rho B \) is a standard Brownian motion \( \rho \)-correlated with \( B \). We assume that the process \( V \) is a non-degenerate, continuous Volterra-type Gaussian process of the form

\[
V_t = \int_0^t K(t, s) \, dB_s, \quad 0 \leq t \leq T. \tag{2}
\]

Here, the kernel \( K \) is a measurable and square-integrable function on \([0, T]^2\), such that \( K(0, 0) = 0, K(t, s) = 0 \) for all \( 0 \leq t < s \leq T \) and

\[
\sup_{t \in [0, T]} \int_0^T K(t, s)^2 \, ds < \infty.
\]

One can verify that the covariance function of the process \( V \) defined as above is given by

\[
k(t, s) = E[V_t V_s] = \int_0^{t \wedge s} K(t, u)K(s, u) \, du, \quad t, s \in [0, T].
\]

We introduce now the modulus of continuity of the kernel \( K \), defined as

\[
M(\delta) = \sup_{\{t_1, t_2 \in [0, T]:|t_1 - t_2| \leq \delta\}} \int_0^T |K(t_1, s) - K(t_2, s)|^2 \, ds, \quad 0 \leq \delta \leq T.
\]

In order to ensure the continuity of the paths of \( V \), we assume that \( K \) satisfies the following condition.

(A1) There exist constants \( c > 0 \) and \( \vartheta > 0 \) such that \( M(\delta) \leq c \delta^\vartheta \) for all \( \delta \in [0, T] \).

Let us recall that the unique solution to Equation (1) is \((e^{X_t})_{t \in [0, T]}\), where the log-price process is defined by

\[
X_t = -\frac{1}{2} \int_0^t \sigma^2(V_s) \, ds + \bar{\rho} \int_0^t \sigma(V_s) \, dB_s + \rho \int_0^t \sigma(V_s) \, dB_s, \quad 0 \leq t \leq T.
\]

**Definition 2.1:** A modulus of continuity is an increasing function \( \omega : [0, +\infty) \rightarrow [0, +\infty) \) such that \( \omega(0) = 0 \) and \( \lim_{x \to 0} \omega(x) = 0 \). A function \( f \) defined on \( \mathbb{R} \) is called locally \( \omega \)-continuous, if for every \( \delta > 0 \) there exists a constant \( R(\delta) > 0 \) such that for all \( x, y \in [-\delta, \delta] \), inequality \( |f(x) - f(y)| \leq R(\delta) \omega(|x - y|) \) holds.

**Remark 2.1:** For instance, if \( \omega(x) = x^\vartheta \), \( \vartheta \in (0, 1) \), the function \( f \) is locally \( \vartheta \)-Hölder continuous. If \( \omega(x) = x \), the function \( f \) is locally Lipschitz continuous.

We consider the following assumptions on the volatility function \( \sigma \).

(S1) \( \sigma : \mathbb{R} \rightarrow (0, +\infty) \) is a locally \( \omega \)-continuous function for some modulus of continuity \( \omega \).
(Σ2) There exist constants \( \vartheta, M_1, M_2 > 0 \), such that \( \sigma(x) \leq M_1 + M_2 |x|^{\vartheta}, \ x \in \mathbb{R} \).

From now on, we denote by \( C([0, T]) \) (respectively \( C_0([0, T]) \)) the set of continuous functions on \([0, T] \) (respectively the set of continuous functions on \([0, T] \) starting at 0), endowed with the topology induced by the \( \text{sup} \)-norm.

Let \( \gamma : \mathbb{N} \to \mathbb{R}_+ \) be an infinitesimal, decreasing function, i.e., \( \gamma_n \downarrow 0 \), as \( n \to +\infty \). For every \( n \in \mathbb{N} \), we consider the following scaled version of Equation (1)

\[
\begin{align*}
\frac{dS^n_t}{S^n_t} &= \gamma_n \sigma(V^n_t) d(\bar{\rho}\bar{B}_t + \rho B_t), \quad 0 \leq t \leq T, \\
S^n_0 &= 1,
\end{align*}
\]

The log-price process \( X^n_t = \log S^n_t, 0 \leq t \leq T, \) in the scaled model is

\[
X^n_t = -\frac{1}{2} \gamma_n^2 \int_0^t \sigma^2(V^n_s) \, ds + \gamma_n \bar{\rho} \int_0^t \sigma(V^n_s) \, d\bar{B}_s + \gamma_n \rho \int_0^t \sigma(V^n_s) \, dB_s. \tag{4}
\]

Here the Brownian motion \( \bar{\rho}\bar{B} + \rho B \) is multiplied by a small-noise parameter \( \gamma_n \) and the Volterra process \( V^n \) is of the form

\[
V^n_t = \int_0^t K^n(t, s) \, dB_s, \quad 0 \leq t \leq T,
\]

where \( K^n \) is a suitable kernel. It can be verified that the covariance function of the process \( V^n \), for every \( n \in \mathbb{N} \), is given by

\[
k^n(t, s) = \int_0^{t \wedge s} K^n(t, u)K^n(s, u) \, du \quad \text{for } t, s \in [0, T].
\]

In the setting above, we are interested in an LDP for the family \( ((\gamma_n B, V^n))_{n \in \mathbb{N}} \) (we recall basic facts and notations on LDP in Appendix). Such an LDP holds under the following conditions on the covariance functions, as seen in Theorem 7.4 in Cellupica and Pacchiarotti (2021).

(K1) There exists an infinitesimal function \( \gamma_n \) and a kernel \( \hat{K} \) (regular enough to be the kernel of a continuous Volterra process) such that

\[
\lim_{n \to +\infty} \frac{K^n(t, s)}{\gamma_n} = \hat{K}(t, s) \tag{5}
\]

and

\[
\lim_{n \to +\infty} \int_0^T \frac{K^n(t, u)K^n(s, u) \, du}{\gamma_n^2} = \int_0^T \hat{K}(t, u)\hat{K}(s, u) \, du
\]

uniformly for \( t, s \in [0, T] \).

(K2) There exist constants \( \beta, M > 0 \), such that, for every \( n \geq n_0 \)

\[
\sup_{s,t \in [0, T], s \neq t} \frac{\int_0^T (K^n(t, u) - K^n(s, u))^2 \, du}{\gamma_n^2 |t - s|^{2\beta}} \leq M.
\]
**Theorem 2.1:** Let $\gamma : \mathbb{N} \to \mathbb{R}_+$ be an infinitesimal function. Suppose Assumptions (K1) and (K2) are fulfilled. Then $((\gamma_nB, V^n))_{n \in \mathbb{N}}$ satisfies an LDP on $C_0([0, T])$ with speed $\gamma_n^{-2}$ and good rate function

$$I_{(B, V)}(f, g) = \begin{cases} \frac{1}{2} \int_0^T \dot{f}(s)^2 \, ds & (f, g) \in \mathcal{H}_{(B, V)} \\ +\infty & (f, g) \notin \mathcal{H}_{(B, V)} \end{cases}$$

where

$$\mathcal{H}_{(B, V)} = \left\{ (f, g) \in C_0([0, T])^2 : f \in H_0^1[0, T], g(t) = \int_0^t \dot{K}(t, u) \dot{f}(u) \, du, \ 0 \leq t \leq T \right\},$$

where $\dot{K}$ is defined in Equation (5) and $H_0^1 = H_0^1[0, T]$ is the Cameron-Martin space.

If Assumptions $(\Sigma 1)$ and $(\Sigma 2)$ hold for the volatility function $\sigma (\cdot)$, we also have a sample path LDP for the family of processes $((X^n_t)_{t \in [0, T]}^n)_{n \in \mathbb{N}}$ and for the family of random variables $(X^n_t)_{n \in \mathbb{N}}$ (see Section 7 in Cellupica and Pacchiarotti 2021 for details). Let us denote $\hat{f}(t) = \int_0^t \dot{K}(t, u) \dot{f}(u) \, du$ for $f \in H_0^1$.

**Theorem 2.2:** Under Assumptions $(\Sigma 1)$, $(\Sigma 2)$, (K1) and (K2), we have: (i) the family of processes $((X^n_t)_{t \in [0, T]}^n)_{n \in \mathbb{N}}$ satisfies an LDP with speed $\gamma_n^{-2}$ and good rate function

$$I_{X^n}(x) = \begin{cases} \inf_{f \in H_0^1[0, T]} \left[ \frac{1}{2} \| f \|_{H_0^1[0, T]}^2 + \frac{1}{2} \int_0^T \left( \ddot{x}(t) - \rho \sigma (\hat{f}(t)) \ddot{f}(t) \right)^2 \, dt \right] & x \in H_0^1[0, T] \\ +\infty & x \notin H_0^1[0, T] \end{cases}$$

(ii) the family of random variables $(X^n_t)_{n \in \mathbb{N}}$ satisfies an LDP with speed $\gamma_n^{-2}$ and good rate function

$$I_{X^n}(y) = \inf_{f \in H_0^1[0, T]} \left[ \frac{1}{2} \| f \|_{H_0^1[0, T]}^2 + \frac{1}{2} \int_0^T \left( y - \int_0^T \rho \sigma (\hat{f}(t)) \ddot{f}(t) \, dt \right)^2 \, dt \right], \ y \in \mathbb{R}.$$

**Remark 2.2:** From Theorem 4.8 in Forde and Zhang (2017), it follows that $I_{X^n}(0) = 0$, where

$$\begin{cases} \inf_{y \geq x} I_{X^n}(y) = \inf_{y > x} I_{X^n}(y) = I_{X^n}(x) \quad \text{for } x > 0, \\ \inf_{y \leq x} I_{X^n}(y) = \inf_{y < x} I_{X^n}(y) = I_{X^n}(x) \quad \text{for } x < 0. \end{cases}$$

### 2.2. Short-time Large Deviations for the Log-Price

It is well known that if the Volterra process is self-similar we can pass from small-noise to short-time regime (see the discussion at the end of Section 3 in Gulisashvili 2020). However, in general, this is not possible if the process is not self-similar. In this section, we obtain a short-time LDP that does not rely on the self-similarity assumption, by using the results of the previous section.
Let $\varepsilon : \mathbb{N} \to \mathbb{R}_+$ be a sequence decreasing to zero, i.e., $\varepsilon_n \downarrow 0$ as $n \to +\infty$. For every $n \in \mathbb{N}$ and $t \in [0, T]$, if $V$ is a Volterra process as in (2) we have

$$V_{\varepsilon_n t} = \int_0^{\varepsilon_n t} K(\varepsilon_n t, s) \, dB_s \xrightarrow{\text{law}} \int_0^t \sqrt{\varepsilon_n} K(\varepsilon_n t, \varepsilon_n s) \, dB_s = \int_0^t K^n(t, s) \, dB_s = V^n_t,$$  

with $K^n(t, s) = \sqrt{\varepsilon_n} K(\varepsilon_n t, \varepsilon_n s)$. Therefore, for every $n \in \mathbb{N}$ and $t \in [0, T]$, if $X$ is as in (3), we have

$$X_{\varepsilon_n t} = -\frac{1}{2} \int_0^{\varepsilon_n t} \sigma^2(V_s) \, ds + \tilde{\rho} \int_0^{\varepsilon_n t} \sigma(V_s) \, dB_s + \rho \int_0^{\varepsilon_n t} \sigma(V_s) \, dB_s \xrightarrow{\text{law}} -\frac{1}{2} \int_0^t \sigma^2(V^n_s) \, ds + \sqrt{\varepsilon_n} \tilde{\rho} \int_0^t \sigma(V^n_s) \, dB_s + \sqrt{\varepsilon_n} \rho \int_0^t \sigma(V^n_s) \, dB_s.$$

Define $V^n_t = V_{\varepsilon_n t}$ and suppose the family of processes $(V^n)_{n \in \mathbb{N}}$ satisfies an LDP with speed $\gamma_n^{-2}$ (depending on $\varepsilon_n$). Suppose furthermore that the family $((\gamma_n B, V^n))_{n \in \mathbb{N}}$ satisfies an LDP with speed $\gamma_n^{-2}$ (for details on this topic see Section 7 and in particular Theorem 7.4 in Cellupica and Pacchiarotti 2021) and let $X^n$ be the process defined in (4). If we consider the processes, defined on the same space, we have

$$X^n_t - \gamma_n \varepsilon_n^{-1/2} X_{\varepsilon_n t} = \frac{1}{2} (\gamma_n \varepsilon_n^{1/2} - \gamma_n^2) \int_0^t \sigma^2(V^n_s) \, ds.$$

Let us recall that two families $(Z^n)_{n \in \mathbb{N}}$ and $(\tilde{Z}^n)_{n \in \mathbb{N}}$ of random variables are exponentially equivalent (at the speed $v_n$, with $v_n \to \infty$ as $n \to \infty$) if for any $\delta > 0$,

$$\limsup_{n \to +\infty} \frac{1}{v_n} \log P(|\tilde{Z}^n - Z^n| > \delta) = -\infty.$$

As far as the LDP is concerned, exponentially equivalent families are indistinguishable. See Theorem 4.2.13 in Dembo and Zeitouni (1998).

**Theorem 2.3:** Under Assumptions $(\Sigma 1)$, $(\Sigma 2)$, $(K1)$ and $(K2)$, the two families $((X^n_t)_{t \in [0, T]})_{n \in \mathbb{N}}$ and $((\gamma_n \varepsilon_n^{-1/2} X_{\varepsilon_n t})_{t \in [0, T]})_{n \in \mathbb{N}}$ are exponentially equivalent and therefore satisfy the same LDP. In particular,

1. the family $((\gamma_n \varepsilon_n^{-1/2} X_{\varepsilon_n t})_{t \in [0, T]})_{n \in \mathbb{N}}$ satisfies an LDP with speed $\gamma_n^{-2}$ and good rate function given by (6);
2. the family of random variables $(\gamma_n \varepsilon_n^{-1/2} X_{\varepsilon_n T})_{n \in \mathbb{N}}$ satisfies an LDP with speed $\gamma_n^{-2}$ and good rate function given by (7).

**Proof:** We have

$$|X^n_t - \gamma_n \varepsilon_n^{-1/2} X_{\varepsilon_n t}| = \frac{1}{2} |\gamma_n \varepsilon_n^{1/2} - \gamma_n^2| \int_0^t \sigma^2(V^n_s) \, ds = \delta_n \int_0^t \sigma^2(V^n_s) \, ds,$$

where $\delta_n = \frac{1}{2} |\gamma_n \varepsilon_n^{1/2} - \gamma_n^2|$ and $\delta_n \to 0$. The family $(V^n)_{n \in \mathbb{N}}$ satisfies an LDP with a good rate function. Then, it is exponentially tight (at the inverse speed $\gamma_n^2$). Therefore
for every $R > 0$, there exists a compact set $C_R$ (of equi-bounded functions) such that
\[
\limsup_{n \to +\infty} \gamma^2_n \log \mathbb{P}(V^n \in C_R) \leq -R,
\]
with $(\cdot)^c$ indicating the complementary set. Thus, for every $\eta > 0$,
\[
\limsup_{n \to +\infty} \gamma^2_n \log \mathbb{P}\left(\sup_{t \in [0,T]} \int_0^t \sigma^2(V^n_s) \, ds > \eta / \delta_n, V^n \in C_R\right)
\]
\[
\leq \limsup_{n \to +\infty} \gamma^2_n \log \mathbb{P}\left(\sup_{t \in [0,T]} |X^n_t - \sqrt{\gamma_n} X_{\gamma_n t}| > \eta\right)
\]
\[
\leq \limsup_{n \to +\infty} \gamma^2_n \log \mathbb{P}\left(\sup_{t \in [0,T]} \int_0^t \sigma^2(V^n_s) \, ds > \eta / \delta_n, V^n \in C_R\right)
\]
\[
+ \limsup_{n \to +\infty} \gamma^2_n \log \mathbb{P}(V^n \in C_R^c) = -\infty,
\]
since the set $\{\sup_{t \in [0,T]} \int_0^t \sigma^2(V^n_s) \, ds > \delta / \delta_n, V^n \in C_R\}$ is eventually empty. \[\blacksquare\]

3. Applications

In this section, we consider some (non-self-similar) Volterra processes that satisfy assumption (A1) and such that the corresponding family $(V^n)_{n \in \mathbb{N}}$ defined by Equation (8) satisfies conditions (K1) and (K2). We also suppose that assumptions $(\Sigma 1)$ and $(\Sigma 2)$ on the volatility function are satisfied and $T = 1$. From Theorem 2.3, we obtain a short-time LDP for the corresponding log-price processes.

3.1. Log-fractional Brownian Motion and Modulated Models

Let us consider the kernel, for $0 \leq s \leq t \leq 1$,
\[
K(t, s) = C(t - s)^{H-1/2}(-\log(t - s))^{-p},
\]
where $0 \leq H \leq 1/2$, $p > 1$ and $C > 0$ is a constant. The corresponding Volterra process $V$ essentially amounts to the log-fBM introduced in Bayer, Harang, and Pigato (2021). There, an additional cutoff of the logarithm function was introduced in order to normalize the variance of the volatility at time one, but we can avoid here this complication as it does not affect our analysis in any way, since we only consider short-time asymptotics.

Condition (A1) for this kernel was proved in Bayer, Harang, and Pigato (2021) with $\vartheta = 2H$. Note that $\kappa(t, s) = C(t - s)^{H-1/2}$ is the well known kernel of the RLP, which also satisfies Assumption (A1) with $\vartheta = 2H$.

For $n$ large enough, we set
\[
K^n(t, s) = C\varepsilon_n^H(t - s)^{H-1/2}(-\log(\varepsilon_n(t - s)))^{-p}.
\]

Let us verify that conditions (K1) and (K2) are satisfied for $H > 0$. No small-time LDP can be verified in the case $H = 0$, as shown in Section 5.4 in Baldi and Pacchiarotti (2022).

(K1) For $s, t \in [0, 1], s < t$, since we can suppose $\varepsilon_n < 1$, we have
\[
\frac{\log \varepsilon_n(t - s)}{\log \varepsilon_n} \geq 1,
\]
and therefore
\[ \kappa(t, u) \left( \frac{\log \varepsilon_n(t-u)}{\log \varepsilon_n} \right)^{-p} \leq \kappa(t, u). \]

Then, thanks to Lebesgue’s dominated convergence Theorem, for \( s, t \in [0, 1] \),
\[
\lim_{n \to \infty} \int_0^{s \wedge t} \kappa(t, u) \kappa(s, u) \left( \frac{\log \varepsilon_n(t-u)}{\log \varepsilon_n} \right)^{-p} \left( \frac{\log \varepsilon_n(s-u)}{\log \varepsilon_n} \right)^{-p} \, du
= \int_0^{s \wedge t} \kappa(t, u) \kappa(s, u) \, du,
\]
so that \( \lim_{n \to \infty} \frac{k_n^H(t,s)}{\varepsilon_n^H(-\log \varepsilon_n)^{-2p}} = k(t,s) \).

This convergence is actually uniform, since
\[
\frac{k_n^H(t,s)}{\varepsilon_n^H(-\log \varepsilon_n)^{-2p}} = C^2 \int_0^{s \wedge t} (t-u)^{H-1/2} (s-u)^{H-1/2} \times \left( 1 + \frac{\log(t-u)}{\log \varepsilon_n} \right)^{-p} \left( 1 + \frac{\log(s-u)}{\log \varepsilon_n} \right)^{-p} \, du,
\]
and therefore the sequence \( (k_n^H(\cdot, \cdot)/\varepsilon_n^H(-\log \varepsilon_n)^{-2p})_n \) is a monotone sequence of continuous functions converging pointwise to a continuous function. Then (K1) is proved (with \( \widetilde{K} = \kappa \) and \( \gamma_n = \varepsilon_n^H(-\log \varepsilon_n)^{-p} \)).

(K2) For every \( n \in \mathbb{N} \), \( s < t \), we have
\[
\int_0^t \left( \kappa(t, u) \left( \frac{\log \varepsilon_n(t-u)}{\log \varepsilon_n} \right)^{-p} - \kappa(s, u) \left( \frac{\log \varepsilon_n(s-u)}{\log \varepsilon_n} \right)^{-p} \right)^2 \, du
= \int_s^t \left( \kappa(t, u) \left( \frac{\log \varepsilon_n(t-u)}{\log \varepsilon_n} \right)^{-2p} \right) \, du + \int_0^s \kappa(t, u) \left( \frac{\log \varepsilon_n(t-u)}{\log \varepsilon_n} \right)^{-p} \, du
- \kappa(s, u) \left( \frac{\log \varepsilon_n(s-u)}{\log \varepsilon_n} \right)^{-p} \right)^2 \, du.
\]

Now, thanks to (10)
\[
\int_s^t \kappa(t, u)^2 \left( \frac{\log \varepsilon_n(t-u)}{\log \varepsilon_n} \right)^{-2p} \, du \leq \int_s^t \kappa(t, u)^2 \, du.
\]

Let us prove that
\[
\int_0^s \left( \kappa(t, u) \left( \frac{\log \varepsilon_n(t-u)}{\log \varepsilon_n} \right)^{-p} - \kappa(s, u) \left( \frac{\log \varepsilon_n(s-u)}{\log \varepsilon_n} \right)^{-p} \right)^2 \, du
\leq \int_0^s (\kappa(t, u) - \kappa(s, u))^2 \, du.
\]

The map \( x \to x^{H-1/2}(-\log x)^{-p} \) defines a decreasing function in a neighbourhood of \( x = 0 \) and \( x \to (-\log x)^{-p} \) an increasing function for \( x \in (0, 1) \). Then, for \( n \) large enough,
for $u \in (0, s)$, we have
\[
0 \leq (\varepsilon_n u)^{H-1/2} ((-\log(\varepsilon_n u))^{-p} - (\varepsilon_n (t - s + u))^{H-1/2} ((-\log(\varepsilon_n (t - s + u)))^{-p}
\]
\[
= (\varepsilon_n u)^{H-1/2} (-\log(\varepsilon_n (t - s + u)))^{-p} - (\varepsilon_n (t - s + u))^{H-1/2} ((-\log(\varepsilon_n (t - s + u)))^{-p}
\]
\[
=((\varepsilon_n u)^{H-1/2} - (\varepsilon_n (t - s + u))^{H-1/2}) ((-\log(\varepsilon_n (t - s + u)))^{-p}
\].

Therefore,
\[
\int_0^s \left( \kappa(t, u) \left( \frac{\log \varepsilon_n (t - u)}{\log \varepsilon_n} \right)^{-p} - \kappa(s, u) \left( \frac{\log \varepsilon_n (s - u)}{\log \varepsilon_n} \right)^{-p} \right)^2 du
\]
\[
= \frac{C^2}{\varepsilon_n^{2H} (-\log \varepsilon_n)^{-2p}} \int_0^s \left( (\varepsilon_n u)^{H-1/2} (-\log(\varepsilon_n (t - s + u)))^{-p}
\]
\[
-(\varepsilon_n u)^{H-1/2} (-\log(\varepsilon_n u))^{-p} \right)^2 du
\]
\[
\leq \frac{C^2}{\varepsilon_n^{2H} (-\log \varepsilon_n)^{-2p}} \int_0^s ((\varepsilon_n u)^{H-1/2} - (\varepsilon_n (t - s + u))^{H-1/2})^2
\]
\[
\times (-\log(\varepsilon_n (t - s + u)))^{-2p} du
\]
\[
= C^2 \int_0^s (u^{H-1/2} - (t - s + u)^{H-1/2})^2 \left( \frac{\log(\varepsilon_n (t - s + u))}{\log \varepsilon_n} \right)^{-2p} du
\]
\[
\leq C^2 \int_0^s (u^{H-1/2} - (t - s + u)^{H-1/2})^2 du = \int_0^s (\kappa(t, u) - \kappa(s, u))^2 du,
\]

Therefore, conditions (K1) and (K2) are verified with infinitesimal function $\gamma_n = \varepsilon_n^{H} (-\log \varepsilon_n)^{-p}$, limit kernel $\bar{K} = \kappa$, and $\beta = H$. A short-time LDP holds with inverse speed $\varepsilon_n^{2H} (-\log \varepsilon_n)^{-2p}$ and limit kernel $\kappa(t, s) = C(t - s)^{H-1/2}$.

The results proved that the log-fBM can be extended to a class of processes, that we refer to as modulated Volterra processes, defined, for $t \in [0, 1]$, as
\[
V_t = \int_0^t \kappa(t, s) L(t - s) \, dB_s.
\] (11)

Here, $\kappa$ is the kernel of a self-similar Volterra process of index $H > 0$, i.e.,
\[
\kappa(ct, cs) = c^{H-1/2} \kappa(t, s), \quad \text{for } c > 0,
\] (12)

that satisfies Assumption (A1), modulated by a slowly varying function $L$, i.e., a function such that
\[
\lim_{x \to 0^+} \frac{L(x\lambda)}{L(x)} = 1,
\]
for every $\lambda > 0$. Thanks to (8) and (12), we have
\[
V^n_t = \int_0^t \varepsilon_n^H \kappa(t, s) L(\varepsilon_n (t - s)) \, dB_s.
\]
First we note that here $K_n^n(t, s) = \varepsilon_n^H \kappa(t, s) L(\varepsilon_n (t - s))$ for $s, t \in [0, 1], s < t$ and

$$
\lim_{n \to \infty} \frac{K_n^n(t, s)}{L(\varepsilon_n) \varepsilon_n^H} = \lim_{n \to \infty} \frac{\kappa(t, s) L(\varepsilon_n (t - s))}{L(\varepsilon_n)} = \kappa(t, s).
$$

Note that the limit kernel is independent of $L$. For these processes, if assumptions (K1) and (K2) are satisfied, we have a short-time LDP with the same rate function as the self-similar process and inverse speed $L(\varepsilon_n)^2 \varepsilon_n^{2H}$. Therefore, the rate function does not depend on the modulating function $L$, but the speed of the LDP does.

### 3.2. Fractional Ornstein–Uhlenbeck Process

Let us recall that the Mandelbrot-Van Ness fBM $B^H$ is the centred continuous Gaussian process with covariance function

$$
\frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).
$$

This process is self-similar with exponent $H$ and admits a Volterra representation with kernel (see e.g., Nualart 2006)

$$
K_H(t, s) = c_H \left( \begin{pmatrix} t \\ s \end{pmatrix} \right)^{H-1/2} \left( \begin{pmatrix} t-s \\ s \end{pmatrix} \right)^{H-1/2} - \left( H - \frac{1}{2} \right) s^{1/2-H} \int_s^t u^{H-3/2} (u - s)^{H-1/2} \, du,
$$

(13)

where

$$
c_H = \left( \frac{2H \Gamma(3/2 - H)}{\Gamma(H + 1/2) \Gamma(2 - 2H)} \right)^{1/2}.
$$

For $H \in (0, 1)$ and $a > 0$, we consider the fOU process, solution to

$$
dV_t = -a V_t \, dt + dB_t^H,
$$

which is given explicitly, with initial condition $V_0 = 0$, by

$$
V_t = \int_0^t e^{-a(t-u)} \, dB_u^H, \quad t \geq 0.
$$

Here, the stochastic integral with respect to $B^H$ can be defined, by integration by parts and the stochastic Fubini theorem, as

$$
V_t = B_t^H - a \int_0^t e^{-a(t-u)} B_u^H \, du.
$$

(14)

We note from this equation that self-similarity for fOU is approximately inherited from the fBM, for small time scales. From (14) we obtain, for $V$, the Volterra representation

$$
V_t = \int_0^t K(t, s) \, dB_s,
$$
with

$$K(t, s) = K_H(t, s) - a \int_s^t e^{-a(t-u)} K_H(u, s) \, du,$$

and $K_H$ as above (see, e.g., Section 2 in Cellupica and Pacchiarotti (2021)). Condition (A1) for this process, with $\vartheta = \min\{2H, 1\}$, was established in Lemma 10 in Gulisashvili (2018). Here we have

$$K^n(t, s) = \varepsilon_n^H K_H(t, s) - a\varepsilon_n^{H+1} \int_s^t e^{-a\varepsilon_n^{H}(t-u)} K_H(u, s) \, du.$$

Let us verify that conditions (K1) and (K2) are satisfied.

(K1) It is enough to observe that

$$\left| \frac{K^n(t, s)}{\varepsilon_n^H} - K_H(t, s) \right| = a\varepsilon_n \int_s^t e^{-a\varepsilon_n^{H}(t-u)} K_H(u, s) \, du \leq C\varepsilon_n,$$

where $C > 0$ is a constant independent of $s, t \in [0, 1]$. Therefore

$$\lim_{n \to +\infty} \frac{K^n(t, s)}{\varepsilon_n^H} = K_H(t, s),$$

uniformly for $s, t \in [0, 1]$. Therefore also

$$\int_0^{t \wedge s} \hat{K}(t, u) \hat{K}(s, u) \, du = \lim_{n \to +\infty} \int_0^{t \wedge s} K^n(t, u) K^n(s, u) \, du \frac{\gamma_n^2}{\gamma_n^2}$$

uniformly for $t, s \in [0, T]$ and (K1) is proved (with $\hat{K} = K_H$ and $\gamma_n = \varepsilon_n^H$).

(K2) For $s < t$ we have

$$\left| K^n(t, u) - K^n(s, u) \right|$$

$$\leq \varepsilon_n^H |K_H(t, u) - K_H(s, u)| + a\varepsilon_n^{H+1} \left| \int_u^t e^{-a\varepsilon_n^{H}(t-v)} K_H(v, u) \, dv - \int_u^s e^{-a\varepsilon_n^{H}(s-v)} K_H(v, u) \, dv \right|$$

$$\leq \varepsilon_n^H |K_H(t, u) - K_H(s, u)| + a\varepsilon_n^{H+1} \int_s^t e^{-a\varepsilon_n^{H}(t-v)} K_H(v, u) \, dv$$

$$+ a\varepsilon_n^{H+1} \left| \int_u^s \left( e^{-a\varepsilon_n^{H}(t-v)} - e^{-a\varepsilon_n^{H}(s-v)} \right) K_H(v, u) \, dv \right|$$

$$= \varepsilon_n^H |K_H(t, u) - K_H(s, u)| + a\varepsilon_n^{H+1} e^{-a\varepsilon_n^{H}(t-s)} \int_s^t e^{-a\varepsilon_n^{H}(s-v)} K_H(v, u) \, dv$$

$$+ a\varepsilon_n^{H+1} e^{-a\varepsilon_n^{H}(t-s)} - 1 \int_u^s e^{-a\varepsilon_n^{H}(s-v)} K_H(v, u) \, dv.$$
Therefore, denoting by $C > 0$ a constant (not depending on $s, t \in [0, 1]$), we have

$$\begin{align*}
\int_0^1 (K^n(t, u) - K^n(s, u))^2 \, du &
\leq \varepsilon_n^{2H} \int_0^1 (K_H(t, u) - K_H(s, u))^2 \, du \\
&+ \varepsilon_n^{2H+2} \int_0^1 du \left( \int_s^t e^{-ae_n(t-v)}K_H(v, u) \, dv \right)^2 \\
&+ C(t-s)^2 \varepsilon_n^{2H+4} \int_0^1 \left( \int_0^1 e^{-ae_n(t-v)}K_H(t, v) \, dv \right)^2 \\
&\leq \varepsilon_n^{2H} \int_0^1 (K_H(t, u) - K_H(s, u))^2 \, du \\
&+ \varepsilon_n^{2H+2} (t-s) \int_0^1 \int_0^1 K_H(v, u)^2 \, du \, dv \\
&+ C(t-s)^2 \varepsilon_n^{2H+4} \int_0^1 \int_0^1 K_H(v, u)^2 \, du \, dv \
&\leq C(t-s)^{2H+1} \varepsilon_n^{2H},
\end{align*}$$

since (see for example Lemma 8 in Gulisashvili 2018)

$$\sup_{s, t \in [0, T], s \neq t} \frac{\int_0^1 (K_H(t, u) - K_H(s, u))^2 \, du}{|t-s|^{2H}} \leq M.$$ 

Condition (K2) is verified with infinitesimal function $\gamma_n = \varepsilon_n^H$, $\beta = H$ and limit kernel $\hat{K} = K_H$. Therefore, the short-time asymptotic behaviour of the model with volatility given as a function of the fOU process is exactly the same as the one of the model with volatility given as a function of the fBM, meaning that they both satisfy LDPs where the speed and rate function are the same. Indeed, the rate function in (7) is the same that was found in Forde and Zhang (2017). This can be computed numerically as we do in Section 5.

4. Short-time Asymptotic Pricing and Implied Volatility

In this section, we discuss applications to option pricing and behaviour at short maturities of implied volatility for certain stochastic volatility models, using the LDP previously discussed. We denote

$$\begin{align*}
p(t, k) := E[(e^k - S_t)^+], \quad c(t, k) := E[(S_t - e^k)^+],
\end{align*}$$

the European put and call prices with maturity $t$ and log-moneyness $k$ (i.e., strike $e^k$, since $S_0 = 1$).

4.1. Large Deviations Pricing for Log-modulated Models

Let us consider the stochastic volatility model given by (1) and (11), i.e.,

$$\begin{align*}
\left\{ \begin{array}{l}
\text{d}S_t = S_t \sigma \left( V_t \right) \text{d}(\tilde{\rho} \tilde{B}_t + \rho B_t), \\
V_t = \int_0^t \kappa(t, s) L(t - s) \, dB_s,
\end{array} \right.
\end{align*}$$
with $\kappa$ kernel of a self-similar process, of exponent $H < 1/2$, that satisfies (A1), and $L$ slowly varying, such that $K(t, s) = \kappa(t, s)L(t - s)$ satisfies (A1), (K1), (K2). In particular, this holds true for the kernel in (9), that essentially is the kernel of the log-fBM in Bayer, Harang, and Pigato (2021), for $H \in [0, 1/2)$. Let

$$\Lambda(x) = I_{X_1}(x),$$

where $I_{X_1}$ is the rate function in (7).

Let us write $f_t \approx g_t$ if $\log(f_t) \sim \log(g_t)$ (see also Appendix).

**Theorem 4.1:** Let us assume that (A1), (K1), (K2), (A2), (A3) hold. If $x < 0$ and

$$k_t = xt^{-H+1/2}L(t)^{-1},$$

(16)

the short-time put price satisfies

$$p(t, k_t) = E[(e^{k_t} - S_t)^+] \approx \exp(-t^{-2H}L(t)^{-2}\Lambda(x)).$$

Let us now assume that the process $S$ is a martingale and there exist $p > 1, t > 0$ such that $E[|S_t|^p] < \infty$ (cf. Remark 4.2). If $x > 0$, $k_t$ is as in (16), we have

$$c(t, k_t) = E[(S_t - e^{k_t})^+] \approx \exp(-t^{-2H}L(t)^{-2}\Lambda(x)).$$

**Proof:** We just prove the call asymptotics (the least straightforward). From Theorems 2.2 and 2.3, following the computations in Section 3.1, we have that the family $(\varepsilon_n^{-H-1/2}L(\varepsilon_n)X_{\varepsilon_n})_{n \in \mathbb{N}}$ satisfies an LDP with inverse speed $\varepsilon_n^{2H}L(\varepsilon_n)^2$ and good rate function $I_{X_1}$ given by formula (7). Since $\inf_{y \geq x} I_{X_1}(y) = \inf_{y > x} I_{X_1}(y) = I_{X_1}(x)$ (see Remark 2.2) we have for $x > 0$

$$\lim_{n \rightarrow +\infty} \varepsilon_n^{2H}L(\varepsilon_n)^2 \log \mathbb{P}(\varepsilon_n^{-H-1/2}L(\varepsilon_n)X_{\varepsilon_n} > x) = -\Lambda(x),$$

for every sequence $\varepsilon_n \downarrow 0$. Therefore, setting $v_t = t^{H-1/2}L(t)$, so that $k_t = x/v_t$, we have

$$\lim_{t \rightarrow 0} t^{2H}L(t)^2 \log \mathbb{P}(X_t > k_t) = \lim_{t \rightarrow 0} t^{2H}L(t)^2 \log \mathbb{P}(v_tX_t > x) = -\Lambda(x),$$

i.e.,

$$\mathbb{P}(S_t > e^{k_t}) = \mathbb{P}(X_t > k_t) = \mathbb{P}(v_tX_t > x) \approx \exp(-t^{-2H}L(t)^{-2}\Lambda(x)).$$

(17)

Let us prove the upper bound. Let $t > 0$ be small enough such that $v_t \geq 1$ and fix $y > x$. We have

$$E[(S_t - \exp(k_t))^+] = E[(\exp(X_t) - \exp(k_t))^+]$$

$$= E[(\exp(X_t) - \exp(k_t))^+1_{\{v_tX_t \in (x,y]\}}]$$

$$+ E[(\exp(X_t) - \exp(k_t))^+1_{\{v_tX_t > y\}}]$$

$$\leq (e^{y/v_t} - e^{y/v_t})\mathbb{P}(v_tX_t > x) + E[\exp(X_t)^p]^{1/p}\mathbb{P}(v_tX_t > y)^{1/q}$$

$$\leq (e^{y} - e^{x})\mathbb{P}(v_tX_t > x) + E[\exp(X_t)^p]^{1/p}\mathbb{P}(v_tX_t > y)^{1/q},$$

where we have used Hölder’s inequality and the existence of $p > 1, t > 0$ such that $E[|S_t|^p] < \infty$. Moreover, $E[|S_t|^p]$ is uniformly bounded as $t \rightarrow 0$, using Doob’s maximal inequality for
the martingale $S$. Now from LDP (17) it follows
\[
\limsup_{t \to 0} t^{2H} L^2(t) \log E[(S_t - \exp(k_t))^+] \leq \max\left(-\Lambda(x), -\frac{\Lambda(y)}{q}\right)
\]
and we conclude by taking $y$ large enough (here we also use the goodness of the rate function, which implies that $\Lambda(y) \to \infty$ as $y \to \infty$).

Now let us look at the lower bound. We have
\[
E[(S_t - \exp(k_t))^+] \geq E((\exp(x_t) - \exp(k_t))1_{[v_t X_t > y]})
\]
\[
\geq (\exp(y/v_t) - \exp(x/v_t))P(v_t X_t > y)
\]
\[
\geq \exp(k_t) (\exp((y-x)/v_t) - 1)P(v_t X_t > y)
\]
\[
\geq \frac{y-x}{v_t} \exp(k_t) P(v_t X_t > y).
\]
Therefore
\[
t^{2H} L(t)^2 \log E[(S_t - \exp(k_t))^+] \geq t^{2H} L(t)^2 (k_t + \log(y-x) - \log v_t)
\]
\[
+ t^{2H} L(t)^2 \log P(v_t X_t > y)
\]
and the first summand goes to 0 as $t \to 0$. Therefore, for any $y > x$,
\[
\liminf_{t \to 0} t^{2H} L(t)^2 \log E[(S_t - \exp(k_t))^+] \geq \liminf_{t \to 0} t^{2H} L(t)^2 \log P(v_t X_t > y) \geq -\Lambda(y).
\]

By continuity of $\Lambda$ (Forde and Zhang 2017, Corollary 4.10) and the fact that the rate function is the same as for the self-similar process, this holds for $\Lambda(x)$ as well and the lower bound is proved.

The following implied volatility asymptotics is a consequence of the previous result and an application of Gao and Lee (2014). Let us denote with $\sim$ asymptotic equivalence ($f_t \sim g_t$ iff $f_t/g_t \to 1$).

**Corollary 4.1:** For model (1), let us assume that (A1), (K1), (K2), (Σ1), (Σ2) hold, that $S$ is a martingale and there exist $p > 1$, $t > 0$ such that $E[S_t^p] < \infty$. Then, with log-moneyness as in (16) and $x \neq 0$, the short-time asymptotics for implied volatility
\[
\sigma_{BS}(t, k_t) \to \frac{x}{\sqrt{2\Lambda(x)}} =: \Sigma_{LM}(x) \quad \text{as } t \to 0
\]
holds. As a consequence, with $k'_t = x't^{-H+1/2}L(t)^{-1}$, the finite difference implied volatility skew satisfies
\[
\frac{\sigma_{BS}(t, k_t) - \sigma_{BS}(t, k'_t)}{k_t - k'_t} \sim \frac{\Sigma_{LM}(x) - \Sigma_{LM}(x')}{x - x'} t^{H-1/2} L(t)
\]

**Remark 4.1:** When taking the kernel in (9) with $0 < H \leq 1/2$ we have
\[
L(t) \sim (-\log t)^{-p}
\]
in (19), and the finite difference skew at the LDP regime explodes as $t^{H-1/2}(-\log t)^{-p}$. We prove this for $H > 0$, because (K2) fails for $H = 0$. However, even for $H = 0$ the process is
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defined and the skew asymptotics (19) can be computed and is consistent with the ‘Gaussian’ result at the Edgeworth regime in Bayer, Harang, and Pigato (2021). It is also clear that

\[
\frac{\Sigma_{LM}(x) - \Sigma_{LM}(x')}{x - x'}
\]

is an approximation of \( \partial_x \Sigma_{LM}(0) \) for \( x, x' \) close to 0. Assuming \( \Lambda \) smooth and, as one expects, \( \Lambda(0) = 0 \) and \( \Lambda'(0) = 0 \), we have

\[
\frac{x}{\sqrt{2\Lambda(x)}} = \frac{1}{\sqrt{\Lambda''(0) + \frac{\Lambda'''(0)x}{3} + O(x^2)}} = \frac{1}{\sqrt{\Lambda''(0)}} \left( 1 - \frac{\Lambda'''(0)}{6\Lambda''(0)} x + O(x^2) \right)
\]

so that we can approximate the implied skew as

\[
\frac{\sigma_{BS}(t, k_t) - \sigma_{BS}(t, k'_t)}{k_t - k'_t} \approx \Sigma_{LM}'(0)t^{H-1/2}L(t) = -\frac{\Lambda'''(0)}{6\Lambda''(0)^{3/2}}t^{H-1/2}L(t).
\]

Note, however, that (19) and the asymptotics in Bayer, Harang, and Pigato (2021) are different mathematical results. In addition, besides providing the at-the-money behaviour, result (18) can also be used to compute the whole short-dated smile, including the wings, so it can be used for calibration and, for example, for tail risk hedging. Since, as noted at the end of Section 3.1, the rate function is the same as for the self-similar process and does not depend on the modulating function \( L \), it can be computed as explained in Section 5 for fOU.

**Proof:** We first prove Equation (18). Apply Corollary 7.1–Equation (7.2) in Gao and Lee (2014), along the lines of Appendix D in Friz, Gassiat, and Pigato (2021) or Corollary 4.15 in Forde and Zhang (2017). Then

\[
\sigma_{BS}^2(t, k_t) \sim -\frac{1}{t} \frac{k_t^2}{2 \log c(t, k_t)} \sim \frac{x^2}{2 \Lambda(x)}
\]

and taking the square root we get the result. Equation (19) follows easily from Equation (18).

### 4.2. Large Deviation Pricing Under Fractional Ornstein–Uhlenbeck Volatility

As a consequence of Theorems 2.2 and 2.3 and the computations in Section 3.2, we can derive asymptotic pricing formulas for European put and call options under the price dynamics in (1), with volatility driven by the process given in (14). In this case, we are considering the stochastic volatility dynamics

\[
\begin{cases}
\text{d}S_t = S_t \sigma(V_t) d(\rho \tilde{B}_t + \rho B_t), \\
\text{d}V_t = -a V_t \, dt + dB_t^H,
\end{cases}
\]

with \( S_0 = 1, \ V_0 = 0 \). Notice that this is written in differential form but \( V \) could also be written explicitly as in (14). With the same arguments used in the proof of Theorem 4.1,
we have
\[ \mathbb{P}(S_t > e^{xt/2-H}) = \mathbb{P}(X_t > xt^{1/2-H}) \approx \exp\{-t^{-2H}J(x)\}, \]
where \( J(x) = I_{X_1}(x) \). More explicitly, (7) reads
\[ J(x) = \inf_{f \in H^1_0[0,1]} \left[ \frac{1}{2} \|f\|^2_{H^1_0} + \frac{1}{2} \int_0^1 \int_0^1 \rho^2 \sigma^2 \hat{f}(t) \hat{f}^*(t) \, dt \right]. \quad (21) \]

We have the following theorem.

**Theorem 4.2:** Suppose \((\Sigma 1), (\Sigma 2)\) hold. If \( x < 0 \) and \( k_t = xt^H - 1/2 \), the put price in short-time satisfies
\[ p(t, k_t) = E[(e^{k_t} - S_t)^+] \approx \exp\{-t^{-2H}J(x)\}. \]
In addition, we now assume that the process \( S \) is a martingale and there exists \( p > 1, t > 0 \) such that \( E[S_t^p] < \infty \). If \( x > 0 \) and \( k_t = xt^H - 1/2 \), we have
\[ c(t, k_t) = E[(S_t - e^{k_t})^+] \approx \exp\{-t^{-2H}J(x)\}. \]

**Remark 4.2:** In both Theorems 4.1 and 4.2 the call price asymptotics holds under the assumption that the price process \( S \) is a martingale, along with a moment condition. In the diffusive case \((H = 1/2)\) several related results are available. In particular, martingality holds if \( \sigma \) has exponential growth and \( \rho < 0 \) (Jourdain 2004; Lions and Musiela 2007; Sin 1998). Note that the assumption of negative correlation is justified from a financial perspective.

In the rough case, martingality is known to hold when \( \sigma \) has linear growth and the driving process is the fBM (Forde and Zhang 2017). In Gassiat (2019), it is shown that for a class of rough volatility models with \( \sigma \) of exponential growth (that includes the rough Bergomi model) the stock price is a true martingale if and only if \( \rho \leq 0 \), while \( E[S_t^p] = +\infty \) for \( p > 1/(1 - \rho^2) \), for any \( t > 0 \).

Models where the volatility is a function \( \sigma \) of a Gaussian process are considered in Gulisashvili (2020). If \( \sigma \) grows faster than linearly, conditions for the explosion of moments are given both in the correlated and uncorrelated case.

For models (11) and (20), these are open questions. We expect the conditions for the call asymptotics in Theorems 4.1 and 4.2 to hold in case \( \rho < 0 \) and \( \sigma \) with exponential growth. In particular, martingality should definitely hold in the cases analogous to Gassiat (2019), but with fOU driver. Indeed, the distribution of the fOU process is more concentrated than the one of the fBM, because of the mean reversion property.

**Proof:** This follows from the classic argument that we spelled out in the proof of Theorem 4.1. The proof follows as in Appendix C, Proof of Corollary 4.13 in Forde and Zhang (2017).

Again, from this call and put price asymptotics, an application of Corollary 7.1 in Gao and Lee (2014) gives the following result.
Corollary 4.2: Under the assumptions of Theorem 4.2, writing \( k_t = xt^{1/2-H} \), we have, for \( x \in \mathbb{R} \setminus \{0\} \),

\[
\sigma_{BS}(t, k_t) \to \frac{|x|}{\sqrt{2J(x)}} =: \Sigma_{fOU}(x), \quad \text{as } t \to 0
\]  

(22)

As a consequence, the behaviour of the implied skew at the large deviations regime under fOU-driven volatility is as follows.

Corollary 4.3: Under the assumptions of Theorem 4.2, writing \( k_t = xt^{1/2-H} \), we have, for \( x \in \mathbb{R} \setminus \{0\} \),

\[
\sigma_{BS}(t, k_t) \to \frac{\Sigma_{fOU}(x)}{2x} t^{H-1/2}, \quad \text{as } t \to 0.
\]  

(23)

Remark 4.3 (On moderate deviations): Model (20) should satisfy a moderate deviation result analogous to the ones in Bayer et al. (2019) and Theorem 3.13 in Friz, Gassiat, and Pigato (2022). Let \( c(\cdot, \cdot) \) be as in (15), the price process \( S \) given in (20). Assume that \( J \) is \( n \in \mathbb{N} \) times continuously differentiable. Let \( H \in (0, 1/2) \), \( \beta > 0 \) and \( n \in \mathbb{N} \) such that \( \beta \in (2H n+1, 2H n] \). Set \( \ell_t = xt^{1/2-H+\beta} \). Then, we can formally compute the call asymptotics from Theorem 4.2, plugging \( \ell_t \) as log price instead of \( k_t \), so that we substitute \( x_t = xt^\beta \) to \( x \) in a Taylor expansion of \( J \) at 0 and get

\[
J(x_t) = \sum_{i=2}^{n} \frac{J^{(i)}(0)}{i!} x^i t^{\beta} + O(t^{(n+1)\beta}).
\]

Now, consider the speed \( t^{2H} \) in Theorem 4.2 and that \( t^{(n+1)\beta-2H} \to 0 \) if \( \beta \in (2H n+1, 2H n] \), recall from Bayer et al. (2019) and Forde and Zhang (2017) that \( J(0) = J'(0) = 0, J''(0) = 1/\sigma(0)^2 \) and we find that the call price should satisfy the following moderate deviations asymptotics, as \( t \to 0 \),

\[
\log c(t, \ell_t) = -\sum_{i=2}^{n} \frac{J^{(i)}(0)}{i!} x^i t^{\beta-2H} + O(t^{(n+1)\beta-2H}).
\]

We expect that a complete proof of this fact could be adapted from Proof of Theorem 3.13 in Friz, Gassiat, and Pigato (2022) or Proof of Theorem 3.4 in Bayer et al. (2019). Assuming this call price asymptotics holds true, the following implied volatility asymptotics can be derived using Corollary 7.1, Equation (7.2) in Gao and Lee (2014) and that \( J''(0) = 1/\sigma(0)^2 \)

\[
\sigma_{BS}^2(t, \ell_t) = \sum_{j=0}^{n-2} (-1)^j 2^j \sigma(0)^{2(j+1)} \left( \sum_{i=3}^{n} \frac{J^{(i)}(0)}{i!} x^i t^{j-i+2+\beta(2-2j)} \right) + O(t^{2H-2\beta}).
\]  

(24)

Remark 4.4 (On related results): A pathwise small-noise LDP under fOU volatility has been proved in Horvath, Jacquier, and Lacombe (2019), with different hypotheses in particular on the function \( \sigma \). From this LDP, a short-time result for a suitably renormalized process is also derived, with a time-scaling different from ours.
In Jacquier and Pannier (2022) asymptotic results are given for Volterra-driven volatility models, including large and moderate deviations, also in small-time. Hypothesis on the models are different from ours, for example $\sigma^2(x)$ is of linear growth, or alternatively a moment condition of type $\mathbb{E}[\sigma(V)^p] < \infty$ for any $p \geq 1$ holds. The rate function is given as an expression involving fractional derivatives of the minimizer. In particular, in Jacquier and Pannier (2022, Section 4.2.1) these results are applied to the rough Stein-Stein model, which is similar to (20), with the RL$p$ instead of the fBM, and with the specific choice of volatility function $\sigma^2(x) = x^2$. Analogous results should also hold with the fBM instead of the RL$p$ as driver of the volatility.

**Remark 4.5 (On applications):** As mentioned in the introduction, short-time asymptotic approximations to the implied volatility surface are used for model calibration, pricing and other applications. They give information on option prices with short maturity, with low computational burden. This helps for example in the creation of delta-hedging strategies that are sensitive to short-term moves in the underlying and in general in trading and risk management. Efficient and accurate methods for calibrating fOU-driven volatility models are relevant, for example, because these volatility models are used for computing option prices and implied volatilities (Garnier and Sølna 2017, 2020a) and for hedging (Garnier and Sølna 2020b). Furthermore, Garnier and Sølna (2018a) compare the price impact of fast mean-reverting Markov stochastic volatility models with the price impact of mean-reverting rough volatility models (see also Garnier and Sølna 2019). In Fouque and Hu (2018), a model with both return and volatility driven by a fast mean-reverting fOU process are used for portfolio optimization, in the $H > 1/2$ regime.

**5. Numerical Experiments**

In this section, we test the accuracy of short-time pricing formulas (22), and (24) and of the implied skew asymptotics (23). We do so for a stochastic volatility model with asset price dynamics given by (1), with both fBM-driven volatility (i.e., $V = B^H$ is the fBM) and fOU-driven volatility (i.e., $V = V^H$ is the fOU process, as in (20)). Recall that both fBM and fOU models lead to the same rate function.

For numerical experiments with log-fBM volatility, we refer to Bayer, Harang, and Pigato (2021). In particular, the discussion in Remark 4.1 on the at-the-money implied skew for log-modulated models is consistent with the numerical evaluations of at-the-money skews in Bayer, Harang, and Pigato (2021, Section 7).

From Section 3.2, we have the Volterra representation of the fBM

$$B^H_r = \int_0^t K_H(r, s) \, dB_s,$$

where $K_H$ is the kernel in (13), and the Volterra representation of the fOU process

$$V^H_r = \int_0^r K(r, s) \, ds = \int_0^r \left( K_H(r, s) - a \int_s^r e^{-a(r-u)} K_H(u, s) \, du \right) \, dB_s. \quad (25)$$

To evaluate the quality of approximations (22), (23) and (24), we first simulate Monte Carlo call prices under both these models, from which we then recover Black-Scholes implied
volatilities. In both cases we consider a volatility function $\sigma(\cdot)$, depending on positive parameters $\sigma_0, \eta$ given by

$$\sigma(x) = \sigma_0 \exp\left(\frac{\eta}{2} x\right).$$ (26)

To compute these prices under our stochastic volatility dynamics, we need to simulate the asset price at the fixed time horizon $t > 0$. Hence we consider a time-grid $t_k = k \frac{t}{N}$, $k = 0, \ldots, N$, and on this grid the random vector $(V_{t_1}, \ldots, V_{t_N}, B_{t_1}, \ldots, B_{t_N})$, first with $V = B^H$ and then $V = V^H$. In both cases, it is a multivariate Gaussian vector with zero mean and known covariance matrix, that can be computed from the Volterra representation of the processes. The whole vector can be simulated using a Cholesky factorization of this covariance matrix. We then use this vector to construct an approximate sample of the log-asset price

$$X_t = -\frac{1}{2} \int_0^t \sigma^2(V_s) \, ds + \rho \int_0^t \sigma(V_s) \, dB_s + \tilde{\rho} \int_0^t \sigma(V_s) \, d\tilde{B}_s$$

by using a forward Euler scheme on the same time-grid

$$X^N_t = -\frac{t}{2N} \sum_{k=0}^{N-1} \sigma^2(V_{t_k}) + \sum_{k=0}^{N-1} \sigma(V_{t_k}) \left(\rho(B_{t_{k+1}} - B_{t_k}) + \tilde{\rho} \left(\tilde{B}_{t_{k+1}} - B_{t_k}\right)\right).$$

We produce $M$ i.i.d. approximate Monte Carlo samples $(X^N_{t,m}, V^m_T)_{1 \leq m \leq M}$, that we use to evaluate call option prices by standard sample average. Then, we compute the corresponding implied volatilities $\sigma_{BS}(t, k)$ by Brent’s method (see Atkinson 2008; Press et al. 2007), where $t$ is the maturity and $k$ the log-moneyness.

Note that Theorem 4.2, Corollaries 4.2 and 4.3 do not apply to the model above, because $\sigma(\cdot)$ does not satisfy the polynomial growth condition ($\Sigma 2$). However, also in in the self-similar case, large deviations pricing results were first obtained under linear growth conditions in Forde and Zhang (2017) and then the conditions were weakened by Bayer et al. (2020) and Gulisashvili (2018) to include exponential growth. Therefore, we chose here to test our result on the exponential volatility in (26), for which our result should hold as well. This choice is more realistic, being analogous to the rough Bergomi model, and being the volatility function considered e.g., in Garnier and Sølna (2020b).

To evaluate the accuracy of large deviations approximation (22), we follow the choice in Friz, Gassiat, and Pigato (2022) and use as model parameters $H = 0.3$, $\rho = -0.7$, $\sigma_0 = 0.2$, $\eta = 1.5$, and as mean reversion parameter in fOU we take $\alpha = 1$ or $\alpha = 2$. These parameters are similar to the ones estimated on empirical volatility surfaces, as for example in Bayer, Friz, and Gatheral (2016). We take a rough, but not ‘extremely rough’ (0.3 instead of 0.1) Hurst parameter $H$, motivated by the recent study El Amrani and Guyon (2023).

We simulate $M = 10^6$ Monte Carlo samples using $N = 500$ discretization points. We estimate call option prices $E[(S_0 e^{X_t} - S_0 e^{kT})^+]$, where $k_t = xt^{1/2-H}$, and the corresponding implied volatility $\sigma_{BS}(t, k_t)$.

Then, we need to compute $\Sigma_{fOU}$. The rate function $J$ in (21) can be approximated numerically using the Ritz method, as described in detail in Gelfand and Fomin (2000, Section 40), Forde and Zhang (2017), and Friz, Gassiat, and Pigato (2022, Remark 4.3 and Section 5.1). The rate function is obtained through numerical optimization on a fixed, finite
Figure 1. Implied volatility smile with fBM-driven stochastic volatility, fOU-driven stochastic volatility with $a = 1$, fOU-driven stochastic volatility with $a = 2$ and large deviation approximation (22). Model parameters: $H = 0.3$, $\rho = -0.7$, $\sigma_0 = 0.2$ and $\eta = 1.5$. Monte Carlo parameters: $10^6$ trajectories and 500 time-steps. Recall that $k_t = xt^{1/2-H}$. The rate function is computed with the Ritz method with $N = 5$ Fourier basis function.

In Figure 1, we show for each model how, as the maturity $t$ becomes smaller, $\sigma_{BS}(t, k_t)$ gets closer to the asymptotic limit in (22), where $k_t = xt^{1/2-H}$. We recall that $t$ is the option maturity and we numerically evaluate $\sigma_{BS}(t, k_t)$ for $t \in \{0.05, 0.1, 0.2, 0.3, 0.5\}$ and $x \in [-0.2, 0.2]$ for 50 equidistant points. The fact that, even for very small maturities, the short-time limit is not reached, can be explained by the fact that the error is of order $t^{2H}$ (as shown in the self-similar case in Friz, Gassiat, and Pigato (2022)), which vanishes as $t \to 0$, albeit slowly, since $H < 1/2$.

In Figure 2, for each fixed maturity, we compare the implied volatility smiles produced by each model (fOU vs fBM-driven volatilities), in order to observe the influence of the magnitude of the mean reversion parameter $a$ on the volatility smiles. In particular, we note that implied volatilities generated by fOU-driven models seem to fall between the

number of coefficients associated to a basis of the Cameron-Martin space $H^1_0$. We take as basis the Fourier basis, i.e., $\{\hat{e}_i\}_{i \in \mathbb{N}}$ with

$$
\hat{e}_1(s) = 1, \quad \hat{e}_{2n}(s) = \sqrt{2} \cos(2\pi ns), \quad \hat{e}_{2n+1} = \sqrt{2} \sin(2\pi ns), \quad n \in \mathbb{N} \setminus \{0\},
$$

that we truncate to $N = 5$ (larger values of $N$ did not seem to improve the computation) and use the more explicit representation of the rate function $J$ in (21) given in Bayer et al. (2019, Proposition 5.1), Forde and Zhang (2017), and Friz, Gassiat, and Pigato (2022, Section 5.1).
Figure 2. Implied volatility smile with fBM-driven stochastic volatility, fOU-driven stochastic volatility with \( a = 1 \), fOU-driven stochastic volatility with \( a = 2 \) and large deviation approximation (22). Model parameters: \( H = 0.3 \), \( \rho = -0.7 \), \( \sigma_0 = 0.2 \) and \( \eta = 1.5 \). Monte Carlo parameters: \( 10^6 \) trajectories and 500 time-steps. We plot each model fixing the time horizon and varying \( x \), where \( k_t = x t^{1/2 - H} \). Rate function is computed with the Ritz method with \( N = 5 \) Fourier basis function.

Implied volatilities generated by fBM-driven models and the asymptotic smile, indicating convergence also if polynomial growth of \( \sigma(\cdot) \) is not satisfied in this example.

We test now the moderate deviation asymptotics in Remark 4.3. In order to do so, let us recall an expansion to the fourth order of the rate function that allows us to use the second order moderate deviation\(^4\). We denote now \( K_H f(t) = \int_0^t K_H(t, s)f(s) \, ds \) and with \( \overline{K_H} \) the adjoint of \( K_H \) in \( L^2[0, 1] \), so that \( \overline{K_H} f(u) = \int_0^1 K_H(t, u)f(t) \, dt \), where again \( K_H \) is the fBM kernel in (13).

Lemma 5.1 (Fourth order energy expansion): Let us assume that \( \sigma(\cdot) \) is continuously differentiable two times. Let \( J(x) \) be the energy function in (21). Then

\[
J(x) = \frac{J''(0)}{2} x^2 + \frac{J'''(0)}{3!} x^3 + \frac{J^{(4)}(0)}{4!} x^4 + O(x^5)
\] (27)

where

\[
J''(0) = \frac{1}{\sigma(0)^2}, \quad J'''(0) = -6 \frac{\rho \sigma'(0)}{\sigma(0)^4} \langle K_H 1, 1 \rangle,
\]
and

\begin{align*}
J^{(4)}(0) &= 12 \frac{\sigma'(0)^2}{\sigma(0)^6} \left\{ 9 \rho^2 \langle K_H 1, 1 \rangle^2 - \rho^2 \langle (K_H 1)^2, 1 \rangle - \langle (K_H 1)^2, 1 \rangle - 2 \rho^2 \langle K_H 1, K_H 1 \rangle \right\} \\
&\quad + 12 \frac{\sigma''(0)}{\sigma(0)^5} \rho^2 \langle (K_H 1)^2, 1 \rangle.
\end{align*}

Plugging (27) into (24) and fixing \( n = 4 \), from straightforward computations, we obtain the equivalent asymptotic formula

\[ \sigma(t, \ell_t) = \Sigma_{fOU}(0) + \Sigma'_{fOU}(0) x t^\beta + \frac{\Sigma''_{fOU}(0)}{2} x^2 t^{2\beta} + o(t^{2H-2\beta}) \] (28)

where

\begin{align*}
\Sigma_{fOU}(0) &= \sigma(0), \\
\Sigma'_{fOU}(0) &= \frac{\rho \sigma'(0) \langle K_H 1, 1 \rangle}{\sigma(0)}, \\
\frac{\Sigma''_{fOU}(0)}{2} &= \frac{\sigma'(0)^2}{\sigma(0)^3} \left\{ -3 \rho^2 \langle K_H 1, 1 \rangle^2 + \frac{\rho^2}{2} \langle (K_H 1)^2, 1 \rangle + \frac{1}{2} \langle (K_H 1)^2, 1 \rangle + \rho^2 \langle K_H 1, K_H 1 \rangle \right\} \\
&\quad + \frac{\sigma''(0) \rho^2}{\sigma(0)^2} \langle (K_H 1)^2, 1 \rangle.
\end{align*}

We plot in Figure 3 implied volatilities computed via Monte Carlo simulations and the corresponding approximation given in (28). We take again \( \sigma(\cdot) \) as in (26), with parameters \( H = 0.3, \rho = -0.7, \sigma_0 = 0.2, \eta = 0.2, \) and \( \beta = 0.125 \). We first note that we fix \( n = 4 \) in (24), and so we choose \( \beta \in \left( \frac{2H}{n+1}, \frac{2H}{n} \right] \), that is the interval \( (0.12, 0.15] \). With respect to our previous experiments, we also take the smaller vol of vol parameter \( \eta = 0.2 \), which is in line with the choices in Bayer et al. (2019) and Friz, Gassiat, and Pigato (2022). Indeed, the quality of the approximation deteriorates as \( \eta \) grows, and for larger \( \eta \) the asymptotic formula (28) is accurate on a smaller time interval.

In Figure 4, we compare, with \( k_t = xt^{H-1/2}, \, x > 0 \), the absolute value of the large deviations finite difference implied skew

\[ \Psi_t := \frac{|\sigma_{BS}(t, k_t) - \sigma_{BS}(t, -k_t)|}{2k_t} \] (29)

computed on fBm-driven and fOU-driven stochastic volatility models, with the asymptotic skew expected from Corollary 4.3, where we also use the approximation, as \( x \to 0 \),

\[ \frac{\Sigma_{fOU}(x) - \Sigma_{fOU}(-x)}{2x} \sim \Sigma'_{fOU}(0) = \frac{\rho \sigma'(0) \langle K_H 1, 1 \rangle}{\sigma(0)}. \]

We observe that, consistently with the smile slopes observed in Figure 2, larger mean reversion parameters \( a \) correspond to flatter smiles and smaller skews (in absolute value), smaller mean reversion parameters \( a \) correspond to steeper smiles and larger skews (in absolute value), fBm has a larger skew than fOU, and the asymptotic skew is even larger than the one generated from fBm, although very close to it. As maturity \( t \to 0 \), the difference between all these skews vanishes.
Figure 3. Moderate deviation implied volatilities with fBm-driven stochastic volatility (blue), with fOU-driven stochastic volatility with $a = 1$ (green) and fOU-driven stochastic volatility with $a = 2$ (light blue), $\ell_t = xt^{1/2-H+\beta}$ with $x = 0.3$ and $\beta = 0.125$. Model parameters: $H = 0.3$, $\rho = -0.7$, $\sigma_0 = 0.2$ and $\eta = 0.2$. Monte Carlo parameters: $10^7$ trajectories, 500 time-steps. We plot each model with fixed $x$ and varying maturity.

Figure 4. Implied skew in (29) with fBm-driven stochastic volatility, fOU-driven stochastic volatility with $a = 0.2$, $a = 1$, $a = 2$ and asymptotic skew from (23). Model parameters: $H = 0.3$, $\rho = -0.7$, $\sigma_0 = 0.2$ and $\eta = 1.5$. Monte Carlo parameters: $10^7$ trajectories and 500 time-steps. We take $x = 0.01$ (recall that $k_t = xt^{1/2-H}$). Log-plot on the left hand side, linear plot on the right hand side.

This could reflect the fact that larger mean reversion parameters $a$ give more concentrated volatility trajectories, with $V$ in (25) staying closer to 0 and therefore the stochastic volatility path $(\sigma_0 \exp(\frac{\eta}{2} V_t))_{t>0}$ staying closer to the spot-vol $\sigma_0$. This may produce flatter implied volatility surfaces and explain smiles and skews observed in Figures 2–4 corresponding to larger $a$’s. On the short end of the surface, however, all of these have to coincide due to our asymptotic results. Let us also mention that the discrepancy observed in Figure 2 on the level of the smile (regardless of the skew), between the asymptotic red line and all the simulated ‘positive maturity’ lines is likely due to a term-structure term of order $t^{2H}$, for which we refer the reader to Friz, Gassiat, and Pigato (2022) (large deviations setting) and El Euch et al. (2019) (central limit setting).
6. Conclusion

In this paper, we prove a short-time large deviation principle for stochastic volatility models, with volatility given as a function of a Volterra process. This result holds without strict self-similarity assumptions on the processes driving the model, and can therefore be applied to some notable examples of (non-self-similar) rough volatility models.

We first consider an application to the log-modulated rough stochastic volatility models introduced in Bayer, Harang, and Pigato (2021). We derive short-maturity asymptotics for European option prices and implied volatility surfaces. Our results on the implied skew at the large deviations regime are consistent with the results at the Edgeworth central-limit regime derived in Bayer, Harang, and Pigato (2021), but allow for valuation of options further from the money.

Then we consider models where volatility is given as a function of a fractional Ornstein–Uhlenbeck process, as e.g., in the seminal work Gatheral, Jaisson, and Rosenbaum (2018). In this case, we find that the limit short-maturity behaviour of option prices and implied volatilities, as well as the short time implied skew, is the same as the one of a model driven by a fractional Brownian motion. We investigate this fact numerically on simulation results, discussing also moderate deviations pricing and implied skew asymptotics.

Notes

1. Note that both fBM and RLp (as in Rough Bergomi) are non-stationary and give rise to non-stationary volatility processes
2. RLp is the stochastic process driving the volatility in the rough Bergomi model (Bayer, Friz, and Gatheral 2016)
3. Let us mention that, for other purposes, one could consider the stationary solution to the fractional SDE above (see for example Gatheral, Jaisson, and Rosenbaum 2018), explicitly given by \( \int_{-\infty}^{t} e^{-a(t-u)} dB_u^{H} \). However, we are interested in this paper in option valuation, so we take as volatility driver the process \( V_t \) above, with \( V_0 = 0 \), so that \( \sigma_0 = \sigma(V_0) = \sigma(0) \) is spot volatility in (20).
4. This expansion is given in Friz, Gassiat, and Pigato (2022, Lemma 6.1), where the kernel \( C(t-s)^{H-1/2} \) is used. However, in the proof of this result, the specific shape of the kernel is not used, but only self-similarity, and therefore it holds for \( k(t, s) \) in (13) as well.

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Appendix. The Large Deviations Principle

Large deviations give an asymptotic computation of small probabilities on an exponential scale (see e.g., Dembo and Zeitouni 1998 as a reference on this topic). We recall some basic definitions (see e.g., Section 1.2 in Dembo and Zeitouni 1998). Throughout this paper, a speed function is a sequence \( \{v_n : n \geq 1\} \) such that \( \lim_{n \to \infty} v_n = \infty \). A sequence of random variables \( \{Z_n : n \geq 1\} \), taking values on a topological space \( \mathcal{X} \), satisfies the large deviation principle (LDP) with rate function \( I \) and speed function \( v_n \) if \( I : \mathcal{X} \to [0, \infty] \) is a lower semicontinuous function,

\[
\liminf_{n \to \infty} \frac{1}{v_n} \log P(Z_n \in O) \geq - \inf_{x \in O} I(x)
\]

for all open sets \( O \), and

\[
\limsup_{n \to \infty} \frac{1}{v_n} \log P(Z_n \in C) \leq - \inf_{x \in C} I(x)
\]

for all closed sets \( C \). A rate function is said to be good if all its level sets \( \{x \in \mathcal{X} : I(x) \leq \eta\} : \eta \geq 0 \) are compact. Therefore, if an LDP holds, and \( \Gamma \) is a Borel set such that \( \inf_{x \in \Gamma} I(x) = \inf_{x \in \Gamma^o} I(x) \) (\( \Gamma^o \) and \( \Gamma \) are the interior and the closure of \( \Gamma \), respectively), then

\[
\lim_{n \to \infty} \frac{1}{v_n} \log P(Z_n \in \Gamma) = -I(\Gamma)
\]
where \( I(\Gamma) = \inf_{x \in \Gamma^o} I(x) = \inf_{x \in \Gamma} I(x) \). In this case we write
\[
P(Z_n \in \Gamma) \approx e^{-I(\Gamma)v_n}.
\]
Moreover \( \{Z_n : n \geq 1\} \) is exponentially tight with respect to the speed function \( v_n \) if, for all \( b > 0 \), there exists a compact \( K_b \subseteq \mathcal{X} \) such that
\[
\limsup_{n \to \infty} \frac{1}{v_n} \log P(Z_n \notin K_b) \leq -b.
\]
The concept of exponential tightness plays a crucial role in large deviations; in fact this condition is often required to establish that the LDP holds for a sequence of random variables taking values on an infinite dimensional topological space. In this paper, we refer to condition (8) and (9) in Section 2 in Macci and Pacchiarotti (2017) which yield the exponential tightness when the topological space \( \mathcal{X} \) of the continuous function is equipped with the topology of the uniform convergence.