Research Article

Finite-Time Stabilization for $p$-Norm Stochastic Nonlinear Systems with Output Constraints

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This paper investigates the finite-time stability problem of $p$-norm stochastic nonlinear systems subject to output constraint. To cope with the constraint on system output, a tan-type barrier Lyapunov function (BLF) is constructed. By using the constructed BLF and the backstepping technique, a new control algorithm is proposed with a continuous state-feedback controller being designed, which guarantees not only that the requirement of output constraint is always achieved but also that the origin of the system is finite-time stable. This result is demonstrated by both the rigorous analysis and the simulation example.

1. Introduction

During the past decades, the control problem of nonlinear systems has long been a hot topic, and many control design approaches have been proposed for various kinds of nonlinear systems, such as adaptive fuzzy control [1, 2], output tracking control [3, 4], $H_{\infty}$ control [5, 6], and sliding mode control [7–10]. Due to their important roles in many science and industry applications, the stochastic nonlinear systems have attracted much interest in recent years. With the development of stochastic theory, various control design strategies have been developed for types of stochastic nonlinear systems by the backstepping technique, see [11–14], for examples. Especially, some works have considered $p$-norm stochastic nonlinear systems, which are inherently nonlinear due to the fractional powers of such systems being not identically equal to one. It should be noted that the inherent nonlinearities cause the stability and control design problems, which are not very easy to be solved [15]. Luckily, the issues have been well studied for $p$-norm stochastic nonlinear systems with different structures by the adding a power integrator technique in the existing literatures. For instance, Li et al. [16] have considered the adaptive state-feedback stabilization for $p$-norm stochastic nonlinear systems; the output-feedback control has been addressed for $p$-norm stochastic nonlinear systems with time-varying delays in [17]; Zhao et al. [18] have proposed a neural tracking control algorithm for $p$-norm switched stochastic nonlinear systems. More latest studies can be found in [19–21] and the references within.

However, most of the abovementioned works about $p$-norm stochastic nonlinear systems did not take the output constraint into consideration. As it is well known, many actual systems are subject to output constraint due to the consideration of the system performance and operation safety [22, 23]. For this reason, the constrained control issue of nonlinear systems has drawn attention from many scholars. Tee et al. [24] have first proposed the notion of the barrier Lyapunov function (BLF) and consequently have developed a control design strategy for a class of strict-feedback deterministic nonlinear systems with output constraints. After then, with the aid of BLFs, control design schemes have been presented for many deterministic nonlinear systems with different types of constraints, including stability control for nonlinear systems with time-varying or asymmetric output constraints [25, 26], adaptive control for
nonlinear systems with full-state constrains [27], and sliding mode control for nonlinear systems with output constraints [28–30]. Moreover, since the finite-time control possesses some inherent advantages [31–33], techniques for the finite-time stabilization under output/state constraints have also been developed, respectively, for strict-feedback nonlinear systems [34], norm nonlinear systems [35–37], and switched nonlinear systems [38]. On the basis of these results, the constrained control schemes for some classes of stochastic nonlinear systems have also been proposed. Jin [39] has constructed an adaptive tracking controller for a class of output-constrained stochastic nonlinear systems in strict-feedback form. Later, the adaptive control problem and the finite-time control problem have been, respectively, addressed for stochastic nonlinear systems with full-state constraints in [40, 41]. Furthermore, the adaptive neural network or fuzzy constrained control problems have attracted some attention [42–46]. Nevertheless, the stochastic nonlinear systems with output constraints considered in most of the existing related works are in the strict-feedback form, rather than in p-normal form. On the contrary, the existing research has mainly focused on the adaptive control problem but did not take the finite-time stabilization into account.

Motivated by the above discussions, we will investigate the problem of the finite-time stabilization for a class of p-norm stochastic nonlinear systems with output constraints and unknown time-varying parameters. First of all, a BLF-based control strategy will be developed by the backstepping approach. Secondly, applying stochastic Lyapunov theorems and Itô’s formula, the constructed state-feedback controller is rigorously proved to be able to ensure the achievement of the output constraint and the finite-time stability of the considered systems simultaneously. Finally, the main result of this paper will be further demonstrated by a simulation example.

2. Problem and Preliminaries

2.1. Problem Statement. The following class of stochastic nonlinear systems are considered:

\[ \begin{align*}
\dot{x}_i & = \theta_i(t)x_i^p_i \, dt + f_i(x_i) dt + g_i^T(x_i) d\omega, \\
\dot{x}_n & = \theta_n(t)x_n^p \, dt + f_n(x_n) dt + g_n^T(x_n) d\omega, \\
y & = x_1,
\end{align*} \]

where \( \omega \) is a \( N \)-dimension standard Wiener process; \( x = (x_1, \ldots, x_N)^T \in \mathbb{R}^n, u \in \mathbb{R} \) and \( y \in \mathbb{R} \) are system state, control input, and output, respectively; \( \theta_i(t) \) is the time-varying parameter; the nonlinear functions \( f_i: \mathbb{R} \to \mathbb{R} \) and \( g_i: \mathbb{R} \to \mathbb{R}^n \) are continuous and satisfy \( f_i(0) = g_i(0) = 0 \); and the fractional powers \( q_j \)'s meet the requirement \( q_j \in \mathbb{R}^{2m-1} \). The output \( y \) is required to satisfy

\[ y \in \prod_{i=1}^{n} = \{ y(t) \in \mathbb{R}, |y(t)| < b \} = \{ x_1 \in \mathbb{R}, |x_1(t)| < b \}, \]

where \( b \) is a known positive constant.

This paper aims to design a continuous state-feedback controller for system (1), which can ensure that the origin of the closed-loop system is finite-time stable in probability and the requirement of the output constraint is achieved.

2.2. Preliminaries

Notations 1. For \( k = 1, \ldots, n \), let \( g_k^T(x_k) = (g_1^T(x_1), \ldots, g_n^T(x_n)) \) and \( \Pi_k = \{ x_k \in \mathbb{R}^n, |x_1(t)| < b \} \). For any \( \varsigma \in \mathbb{R} \) and \( \theta > 0 \), denote \( \phi(\varsigma) = |\varsigma|^\theta \cdot = |\varsigma|^\theta \text{sgn}(\varsigma) \).

Consider the following stochastic system:

\[ dx = f(x)dt + g(x)d\omega, \]

where \( f(x) \) and \( g(x) \) are continuous satisfying \( f(0) = 0 \) and \( g(0) = 0 \).

Definition 1 (see [13]). For any given \( V(x) \in C^2(\mathbb{R}^n) \), associated with system (1), the second-order differential operator is defined as follows:

\[ \mathcal{L}V = \frac{\partial V}{\partial x} f(x) + \frac{1}{2} \text{tr} \left( g^T(x) \frac{\partial^2 V}{\partial x^2} g(x) \right). \]

Definition 2 (see [24]). Suppose that \( \Pi \) is an open set containing the origin and \( V: \Pi \to \mathbb{R} \) is positive definite and continuously differentiable. Then, for system \( \dot{x} = g(x), V(x(t)) \) is called a BLF if for each solution \( x(t) \) starting from \( x(t_0) \in \Pi, V(x(t)) \to \infty \), as \( x(t) \to \partial \Pi \), \( V(x(t)) \leq \tau \) for all \( t \geq t_0 \) and for some \( \tau \in \mathbb{R}^+ \).

Assumption 1 (see [36]). For \( \forall i = 1, \ldots, n \), there exist known positive constants \( \theta_i \) and \( \beta_i \) such that \( \theta_i \leq \theta_i(t) \leq \beta_i \).

Assumption 2. For \( i = 1, \ldots, n \), there are constants \( \mu \in (-[1 + \sum_{j=2}^n q_j^{2p-1}]/2)^{-1}, 0 \) and known nonnegative smooth functions \( \psi_i(x_i), \eta_i(x_i) \) such that

\[ \left| f_i(x_i) \right| \leq \psi_i(x_i) \prod_{j=1}^{i} x_j^{(1+2p)/(1+p)}, \]

\[ \left| g_i(x_i) \right| \leq \eta_i(x_i) \prod_{j=1}^{i} x_j^{(2n/(2+m))}, \]

for all \( t \geq 0 \), where \( \nu_i = 1, \nu_{i+1} = ((v_j + \mu)/q_j), j = 1, \ldots, n. \)

Remark 1. Note that condition (4) is borrowed from [34]. However, the systems considered in [34] are p-norm deterministic nonlinear systems, while we consider p-norm stochastic nonlinear systems with drift terms \( f_i \)'s and
diffusion terms $g_i$'s in this paper. In light of $\mu \in (-1 + \sum_{j=0}^{n} q_1 \cdots q_{j-1})^{-1}, 0)$, the value of $\mu$ is generally taken as $\mu = -(m/p)$ for simplicity, where $m$ and $p$ represent even and odd integers, respectively. Then, the value of each $v_j (j = 2, \ldots, n)$ can be obtained by applying $v_1 = 1$ and $\eta_{j+1} = ((v_j + \mu)/\eta_j > 0$. It can be also observed that both the denominator and numerator of each $v_j$ are odd.

**Lemma 1** (see [13]). Suppose that there exists a positive Lyapunov function $V \in C^2 (R^n)$, which satisfies $\lim_{x \to -\infty} V (x) = \infty$. If $\eta V$ is with respect to (3) and satisfies $\eta V \leq 0, \forall x \in R^n$, then system (3) has a solution for any initial value.

**Lemma 2** (see [13]). Suppose that system (3) admits a solution for each initial value. If there are $\kappa_{\infty}$ class functions $g_1 (\cdot)$ and $g_2 (\cdot)$, a positive $C^2$ Lyapunov function $V$, real numbers $c > 0$, and $0 < \gamma < 1$, such that

\[
\begin{align*}
V_1 (|x|) &\leq V (x) \leq V_2 (|x|), x \in R^n, \\
\eta V (x) &\leq -cV^\gamma (x), x \in R^n \setminus \{0\}. \tag{6}
\end{align*}
\]

For all $t \geq 0$, then the origin of system (3) is finite-time stable in probability.

**Lemma 3** (see [37]). Let $a, b, p, \rho$, and $\zeta$ be positive real numbers. For any $z_1, z_2 \in R$, we have

\[
\rho|z_1|^a |z_2|^b \leq \frac{a}{a + b} |z_1|^{a+b} + \frac{b}{a + b} \zeta^a |z_2|^{a+b}. \tag{7}
\]

**Lemma 4** (see [8]). Let $p \in (0, \infty)$; for any $c_i \in R, i = 1, \ldots, n, one has

\[
(|c_1| + \cdots + |c_n|)^p \leq d(|c_1|^p + \cdots + |c_n|^p), \tag{8}
\]

where $d = n^{p-1}$ if $p \geq 1$ and $d = 1$ if $0 < p < 1$.

**Lemma 5** (see [10]). Let $a,d \in R^+$ with $a \geq 1$. For any $c_1, c_2 \in R$, we have

\[
\begin{align*}
(i) \ &|c_1^d - c_2^d| \leq a \left(2^{a-2} + 2\right) |c_1 - c_2| (|c_1 - c_2|^{a-1} + |c_2|) \\
(ii) \ &|c_1^{(d/a)} - c_2^{(d/a)}| \leq 2 \left(1/2\right)^{a-1} \Omega [c_1]^{d} - |c_2|^{d} (1/2) \\
(iii) \ &|\Omega [c_1]^{1/a}| \leq |c_1|^{1/a} + |c_2|^{1/a} \leq 2 \left(1/2\right)^{1/a} (|c_1| + |c_2|). \\
\end{align*}
\]

3. Main Results

3.1. A Tan-Type BLF. Before carrying out the control design for system (1), we should handle the output constraint issue.

Firstly, we denote $\nu_\nu = max_{i \in \mathbb{N}} \{v_i\}$ and $c_i = (\nu_i / v_i) (i = 1, \ldots, n)$. Let $\sigma$ be a constant parameter satisfying $\sigma \geq \sigma_0$, where the value of $\sigma_0$ is chosen as below:

\[
\begin{align*}
(i) \ &If \ for \ all \ 2 \leq i \leq n, \ 1 \leq c_i \leq 2, \ then \ \sigma_0 = 2 \nu_0 \\
(ii) \ &If \ for \ all \ 2 \leq i \leq n, \ c_i \geq 2, \ then \ \sigma_0 = \nu_0. \\
\end{align*}
\]

Consequently, it is clear that $\sigma / v_i \geq 2$.

Then, a tan-type BLF can be constructed on $\Pi_1$ as follows:

\[
V_b (x_i) = \frac{2b^{4-\mu}}{(4\sigma - \mu))\pi \tan \left(\frac{\pi |x_i|^{4\sigma - \mu}}{2b^{4\sigma - \mu}}\right), \tag{9}
\]

where $\mu$ is given by Assumption 2 and $\sigma$ is defined as above.

It is not hard to obtain from the expression of $V_b (x_i)$ that

\[
\begin{align*}
\frac{\partial V_b}{\partial x_1} &\leq sec^2 \left(\frac{\pi |x_1|^{4-\mu}}{2b^{4\sigma - \mu}}\right) |x_1|^{4\sigma - 1} - G(x_1) |x_1|^{4\sigma - 1} \\
\frac{\partial^2 V_b}{\partial x_1^2} &\leq (4\sigma - \mu - 1)G(x_1) |x_1|^{4\sigma - 2} \\
&\leq |x_i|^{2(4\sigma - 1)} \tag{10}
\end{align*}
\]

Remark 2. It should be noted that the BLF is modified from [34], which is constructed by fully taking the advantage of the given nonlinear growth conditions. As stated in [34], the control strategy based on $V_b (x_i)$ is a universal method, which can handle stochastic systems with or without output constraints.

3.2. Controller Design and Stability Analysis. In what follows, a continuous state-feedback controller will be constructed, and the stability of system (1) under the designed controller will be thoroughly analysed. To this end, a theorem is presented to describe the main result.

**Theorem 1.** Suppose Assumptions 1-2 hold for system (1). For any constant $b > 0$, there is a continuous state-feedback controller such that

(i) The output of system (1) is kept in a given constrained set in the sense of probability, i.e., $P(|y(t)| < b) = 1$.

(ii) The origin of the closed-loop system is finite-time stable in probability.

**Proof.** The proof contains three parts. First of all, the design procedure of the controller is explicitly displayed. Then, the system output is proved to be kept in the given constrained set with probability one. In the last part, the finite-time stability of system (1) is rigorously analysed. \(\square\)

3.2.1. Part I: Design Procedure

**Step 1.** Let $c_i = |x_i|^\sigma$, and choose the Lyapunov function $V_1 (x_i) = V_b (x_i)$. Then, we can directly get from Definition 1 that
\[ \ell V_1 \leq \theta_1(t)G(x_1)[x_1]^{4\sigma - \mu - 1}x_2^\mu + \frac{4\sigma - \mu - 1}{2} \]
\[ + \bar{\theta}_1G(x_1)|x_1|^{4\sigma - \mu - 1} \psi_1 |x_1|^{1+\mu} + \frac{\pi(4\sigma - \mu)}{2b^{4\sigma - \mu}} \]
\[ \cdot |x_1|^{2(4\sigma - \mu - 1)}G(x_1) \tan \left( \frac{\eta |x_1|^{4\sigma - \mu}}{2b^{4\sigma - \mu}} \right) \eta^2 |x_1|^{1+\mu} \]
\[ \leq \bar{\theta}_1(t)G(x_1)(x_1[(4\sigma - \mu - 1)\sigma + \theta_1(t) \cdot \bar{G}(x_1)[x_1]^\sigma, \]
\[ \therefore (12) \]

Substituting (12) into (13) yields
\[ \ell V_1(x_1) \leq -nG(x_1)x_1^\sigma + \theta_1(t)G(x_1)[x_1]^{(4\sigma - \mu - 1)/\sigma} \cdot (x_2^\mu - \xi_2^\mu) \]
\[ = -\frac{1}{2}G(x_1)x_1^\sigma - \left( n - \frac{1}{2} \right) G(x_1)x_1^\sigma + \theta_1(t) \cdot \frac{1}{2} \cdot (x_1^{2\mu} - \xi_2^{2\mu}) \]
\[ \leq \frac{1}{2}G(x_1)x_1^\sigma - \left( n - \frac{1}{2} \right) x_1^\sigma + \theta_1(t) \cdot \frac{1}{2} \cdot (x_1^{2\mu} - \xi_2^{2\mu}) \]
\[ \cdot G(x_1)[x_1]^{((4\sigma - \mu - 1)/\sigma)}(x_2^\mu - \xi_2^\mu) + H_1(x_1)G(x_1)x_1^\sigma, \]
\[ \therefore (13) \]

Step 2. We denote \( \xi_2 = [x_2]^{\sigma/\nu_2} - [\xi_2]^{\sigma/\nu_2} \) and define the positive Lyapunov function \( V_2 \) on \( \Pi_2 \) as \( V_2 = V_1 + \Psi_2 \) with
\[ \Psi_2 = \int_{\xi_2}^{x_2} \left[ \frac{r(\sigma/\nu_2)}{\xi_2^{\sigma/\nu_2}} - \frac{r(\sigma/\nu_2)}{\xi_2^{\sigma/\nu_2}} \right] \frac{r(\sigma/\nu_2)}{\xi_2^{\sigma/\nu_2}} \] dr.
\[ \therefore (14) \]

Since \( \frac{d[\xi_2]^{\sigma/\nu_2}}{dx_1} - \frac{[\xi_2]^{\sigma/\nu_2}}{dx_1} = -(\frac{d[\xi_2]^{\sigma/\nu_2}}{dx_1})x_1^{\sigma/\nu_2} - \frac{[\xi_2]^{\sigma/\nu_2}}{dx_1} = \frac{d[\xi_2]^{\sigma/\nu_2}}{dx_1} \] is valid, one can obtain
\[ \therefore (15) \]

Using Definition 1 again, we have
\[ \therefore (16) \]
where $\xi_3$ is the virtual controller required to be designed later.

In the following, each term in the right hand of (16) will be estimated by its upper bound.

Firstly, applying $0 < (\nu_2/\sigma) \leq 1, 0 < ((1 + \mu)/\sigma) \leq 1$ and Lemma 5, it is easily obtained that

$$x_2 - \xi_2 \leq \left| f_2(x_2) - \xi_2 \right| + \left| \xi_2 \right| x_2 - \xi_2 \leq 2 \left| \xi_2 \right|,$$

(17)

$$x_2^\theta \leq \xi_2 + \left| \xi_2 \right| \left( 1 + \nu_2/\sigma \right) \leq \xi_2 + \lambda_1 \left( 1 + \nu_2/\sigma \right) \xi_2 \leq 2 \xi_2 \left( 1 + \nu_2/\sigma \right),$$

(18)

$$x_2^\theta - \xi_2^\theta \leq 2 \left( 1 + \nu_2/\sigma \right) \left| \xi_2 \right| \left( 1 + \nu_2/\sigma \right) \leq 2 \xi_2 \left( 1 + \nu_2/\sigma \right).$$

(19)

It can be deduced from (19) and Lemma 3 that

$$\bar{\theta}_1(t)G(x_1) \left| x_1 \right|^{4a-\nu_2-1} \leq 2 \bar{\Psi}_1 G(x_1) \left| x_1 \right|^{(4a-\nu_2-1)/\sigma} \leq \frac{1}{8\xi^2} + H_{21}(x_2) \xi^2,$$

(20)

where $H_{21}(x_2) \geq (\left( \mu + 1 \right)/4\sigma)(2\bar{\Psi}_1 G(x_1))^{(4a/(\nu_2+1))} (4\sigma/(8(4\sigma - \mu - 1)))^{(4a/(\nu_2+1))} (4\sigma/(8(4\sigma - \mu - 1)))^{(4a/(\nu_2+1))} \geq 0$ is a $C^2$ function.

Secondly, one can obtain from Assumption 1 and Lemma 4 that

$$\| f_2(x_2) \| \leq \bar{\Psi}_2(x_2) \left( \left| x_1 \right|^{(2\nu_2+2)/\sigma} + \left| x_2 \right|^{(2\nu_2+2)/\sigma} \right) \leq \bar{\Psi}_2(x_2) \left( \left| \xi_1 \right|^{(2\nu_2+2)/\sigma} + \lambda_1 \left| \xi_1 \right|^{(2\nu_2+2)/\sigma} \right) \leq \bar{\Psi}_2(x_2) \left( \left| \xi_1 \right|^{(2\nu_2+2)/\sigma} + \lambda_1 \left| \xi_1 \right|^{(2\nu_2+2)/\sigma} \right),$$

(21)

$$\| g_2(x_2) \| \leq \eta_2(x_2) \left( \left| x_1 \right|^{2\nu_2+2/2\sigma} + \left| x_2 \right|^{2\nu_2+2/2\sigma} \right) \leq \eta_2(x_2) \left( \left| \xi_1 \right|^{2\nu_2+2/2\sigma} + \lambda_1 \left| \xi_1 \right|^{2\nu_2+2/2\sigma} \right) \leq \eta_2(x_2) \left( \left| \xi_1 \right|^{2\nu_2+2/2\sigma} + \lambda_1 \left| \xi_1 \right|^{2\nu_2+2/2\sigma} \right),$$

(22)

where $\bar{\Psi}_2(x_2) \geq \bar{\Psi}_2(x_2) (1 + \lambda_1^{(2\nu_2+2)/\sigma})$ and $\eta_2(x_2) \geq \eta_2(x_2) (1 + \lambda_1^{(2\nu_2+2)/\sigma})$ are nonnegative smooth functions.

Additionally, note that $(\sigma/\nu_2) \geq 2$. Then, as stated in (36), there exist $C^2$ functions $M_{21}(x_2) \geq 0$, $K_{21}(x_2) \geq 0$, and $M_{21}(x_2) \geq 0$, such that

$$\frac{\partial^2 \xi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) \leq \frac{\partial^2 \xi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) + \sigma \left( \sigma - 1 \right) \lambda_1 \xi_2 \left( x_2^\theta - \xi_2^\theta \right) \leq K_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma},$$

(23)

$$\frac{\partial^2 \xi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) \leq \frac{\partial^2 \xi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) + \sigma \left( \sigma - 1 \right) \lambda_1 \xi_2 \left( x_2^\theta - \xi_2^\theta \right) \leq K_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma},$$

(24)

$$\frac{\partial^2 \Psi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) \leq \frac{\partial^2 \Psi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) \leq M_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma} \leq M_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma},$$

(25)

$$\frac{\partial^2 \Psi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) \leq \frac{\partial^2 \Psi_2}{\partial x_2^\theta} \left( x_2^\theta - \xi_2^\theta \right) \leq M_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma} \leq M_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma} \leq M_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma},$$

(26)

Then, using (18), (21), (25), Assumptions 1-2, and Lemma 3, we can infer

$$\frac{\partial^2 \Psi_2}{\partial x_2^\theta} \left( \theta_1(t) x_2^\theta + f_1(x_1) \right) \leq M_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma} \leq M_{21}(x_2) \left| \xi_2 \right|^{1-2/\sigma},$$

(27)
where
\[ H_{22}(\mathbf{x}_2) \geq 3\sigma + \frac{1}{4\sigma}(\bar{\mathbf{g}}_1^T\bar{\mathbf{M}}_{21})^{(4\sigma/(3\sigma + 1))} \left( \frac{4\sigma}{32(\sigma - 1)} \right)^{(3\sigma/(\sigma - 1)) - \frac{(\sigma - 1)/(3\sigma + 1)}{}} + \frac{3\sigma - \mu}{4\sigma}(\bar{\mathbf{g}}_1^T\bar{\mathbf{M}}_{21})^{\frac{3\sigma}{(3\sigma - \mu)}} \]
\[ \cdot \left( \frac{4\sigma}{32(\sigma + \mu)} \right)^{(3\sigma/(\sigma - 1)) - \frac{(\sigma - 1)/(3\sigma + 1)}{}} \]
\[ + \frac{3\sigma - \mu}{4\sigma}(\bar{\mathbf{g}}_1^T\bar{\mathbf{M}}_{21})^{\frac{3\sigma}{(3\sigma - \mu)}} \]
\[ \cdot \left( \frac{4\sigma}{16(\sigma + \mu)} \right)^{(3\sigma/(\sigma - 1)) - \frac{(\sigma - 1)/(3\sigma + 1)}{}} , \]
(28)

is a nonnegative $C^2$ function.

Moreover, from (19), Assumption 2, and Lemma 3, it can be deduced that
\[ \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2} f_2(\mathbf{x}_2) \leq \left| \frac{\xi_2}{\zeta_2} \right|^\sigma \left[ \psi_2(\left| \frac{\xi_2}{\zeta_2} \right|)^{\sigma} + \zeta_2 \left( \frac{\psi_2}{\zeta_2} \right)^\sigma \right] \]
\[ \leq \frac{1}{8} \mathbf{H}_2(\mathbf{x}_2) \mathbf{K}_2^4, \]
(29)

where
\[ H_{22}(\mathbf{x}_2) \geq \psi_2 + ((4\sigma - \mu - \nu_2)/4\sigma) \left| \psi_2 \right|^{\sigma \rho} \left| \frac{\psi_2}{\zeta_2} \right|^\sigma \left( \frac{\psi_2}{\zeta_2} \right)^\sigma \geq 0 \]

is a $C^2$ function.

On the contrary, it is noted that
\[ \frac{1}{2} tr \left[ \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} f_2(\mathbf{x}_2) \right] = \frac{1}{2} \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} \left| g_1 \right|^2 + \frac{1}{2} \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} g_2^T \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} g_2 \]
\[ + \frac{1}{2} \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} \left| g_2 \right|^2 . \]
(30)

Then, applying (22), (26), Assumption 2, and Lemma 3, there clearly exist nonnegative $C^2$ functions
\[ H_{241}(\mathbf{x}_2), H_{242}(\mathbf{x}_2), \text{ and } H_{243}(\mathbf{x}_2) \text{ such that} \]
\[ \frac{1}{2} \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} \left| g_1 \right|^2 \leq \frac{4\sigma - \mu - \nu_2}{\sigma} \mathbf{K}_2^1 \left| \frac{\xi_2}{\zeta_2} \right|^{\sigma \rho} \left( \frac{\psi_2}{\zeta_2} \right)^\sigma \eta_2^1 \left| \xi_2 \right|^{\sigma \rho} \]
\[ + \left| \frac{4\sigma - \mu - \nu_2}{\sigma} \right|^{\sigma \rho} \left| \frac{\psi_2}{\zeta_2} \right|^\sigma \eta_2^1 \left| \xi_2 \right|^{\sigma \rho} \]
\[ \cdot \left| \frac{M_2^1}{\xi_2} \right|^{\sigma \rho} \left| \xi_2 \right|^{\sigma \rho} \eta_2^1 \left| \xi_2 \right|^{\sigma \rho} . \]
(31)

\[ \frac{1}{2} \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} \left| g_2 \right|^2 \leq \eta_1 \left| \frac{\xi_2}{\zeta_2} \right|^\sigma \left( \frac{\psi_2}{\zeta_2} \right)^\sigma \zeta_2^{\sigma / 2} \left| \frac{\psi_2}{\zeta_2} \right|^{\sigma / 2} \zeta_2^{\sigma / 2} \left| \epsilon_2 \right|^{\sigma / 2} \]
\[ + \epsilon_2^{\sigma / 2} \left| \frac{\psi_2}{\zeta_2} \right|^{\sigma / 2} \left| \epsilon_2 \right|^{\sigma / 2} \left| \frac{\psi_2}{\zeta_2} \right|^{\sigma / 2} \]
\[ \leq \frac{1}{24} \mathbf{K}_4 + H_{241}(\mathbf{x}_2) \mathbf{K}_4, \]
(32)

Substituting equations (31)–(33) into (30), one obtains
\[ \frac{1}{2} tr \left[ \frac{\partial^2 \psi_2}{\partial \mathbf{x}_2^T} f_2(\mathbf{x}_2) \right] \leq \frac{1}{8} \mathbf{K}_4 + H_{24}(\mathbf{x}_2) \mathbf{K}_4, \]
(34)

where
\[ H_{24}(\mathbf{x}_2) = H_{241}(\mathbf{x}_2) + H_{242}(\mathbf{x}_2) + H_{243}(\mathbf{x}_2) \geq 0 \]

is a $C^2$ function.

Let
\[ H_{3}(\mathbf{x}_2) = H_{24}(\mathbf{x}_2) + H_{23}(\mathbf{x}_2) + H_{22}(\mathbf{x}_2) + H_{24}(\mathbf{x}_2) \geq 0. \]

Design the virtual controller $\xi_1$ as
\[ \xi_1 = -\lambda_2(\mathbf{x}_2) \left[ \frac{\xi_2}{\zeta_2} \right] \left( \frac{\psi_2}{\zeta_2} \right)^{\sigma / 2} \]
\[ = \left( n - 1 + H_{24}(\mathbf{x}_2) \right)^\lambda_2 > 0. \]
(35)

Substituting (20), (27), (29), (34), and (35) into (16), one can obtain
\[ eV_2 \leq -\frac{1}{2} G(\mathbf{x}_2) \mathbf{K}_4^4 - (n - 1) \left( \frac{\xi_1}{\zeta_2} + \frac{\xi_2}{\zeta_2} \right) + \theta_2(t) \left[ \frac{\xi_2}{\zeta_2} \right] \]
\[ \cdot \left( \frac{\psi_2}{\zeta_2} \right)^{\sigma / 2} \]
(36)

**Inductive Step.** Suppose at step $i - 1$, there exist a $C^2$

Lyapunov function $V_{i - 1}: \Pi_{i - 1} \rightarrow R^+$, and a range of continuous virtual controllers $\xi_1, \xi_2, \ldots, \xi_i$ defined as
\[ \xi_1 = 0, \]
\[ \xi_1 = [x_1] \frac{\sigma}{\sigma} [x_1] \frac{\sigma}{\sigma}, \]
\[ \xi_2 = -\lambda_1(\mathbf{x}_2) \left[ \frac{\xi_2}{\zeta_2} \right] \left( \frac{\psi_2}{\zeta_2} \right)^{\sigma / 2}, \]
\[ \xi_2 = [x_2] \frac{\sigma}{\sigma} [x_2] \frac{\sigma}{\sigma}, \]
\[ \xi_3 = -\lambda_2(\mathbf{x}_2) \left[ \frac{\xi_2}{\zeta_2} \right] \left( \frac{\psi_2}{\zeta_2} \right)^{\sigma / 2}, \]
\[ \xi_3 = [x_3] \frac{\sigma}{\sigma} [x_3] \frac{\sigma}{\sigma}, \]
\[ \vdots \]
\[ \xi_i = -\lambda_{i - 1}(\mathbf{x}_{i - 1}) [\xi_{i - 1}] \left( \frac{\psi_2}{\zeta_2} \right)^{\sigma / 2}, \]
\[ \xi_i = [x_i] \frac{\sigma}{\sigma} [x_i] \frac{\sigma}{\sigma}, \]
with $\lambda_k(\mathbf{x}_k) > 0$, for $k = 1, \ldots, i - 1$, such that
\[ \ell V_{i-1} = \frac{1}{2} G(x_i) \xi_i^2 - (n + 2 - i) \]
\[ \cdot \sum_{i=1}^{i-1} \xi_i^2 + \theta_{i-1}(t)\xi_i \xi_{i-1} \xi_i^2 \]
\[ (4\sigma - \nu - \mu) \sigma \sigma \theta_i \]
\[ \xi_{i-1}^2 - \xi_{i+1}^2. \]
\[ (38) \]

Then, the following property can be inferred.

**Proposition 1.** Choose the \( i \)th Lyapunov function \( V_i: \Pi_i \rightarrow \mathbb{R}^+ \) as \( V_i = V_{i-1} + \Psi_i \) with
\[ \Psi_i = \int_{x_i}^{x_{i-1}} \left( [r \xi_i]^{\alpha \nu} - [\xi_i]^{\alpha \nu} \right) (4\sigma - \nu - \mu) \sigma \sigma \theta_i dr. \]
(39)

Then, \( V_i \) is \( C^2 \) on \( \Pi_i \) and there exists a virtual controller \( \xi_{i+1} \) such that
\[ \ell V_i \leq \frac{1}{2} G(x_i) \xi_i^2 - (n + 1 - i) \sum_{i=1}^{i-1} \xi_i^2 + \theta_{i-1}(t) \xi_i \xi_{i-1} \xi_i^2 \]
\[ (40) \]

where
\[ \xi_{i+1} = -\lambda_i(x_i) \xi_i \xi_i^2 with \lambda_i(x_i) = \left[ n - i + 1 + H_1(x_i) \right]^{1/\xi_i} > 0. \]
(41)

The proof of above Proposition 1 is provided in the Appendix.

**Step 3.** In light of the inductive step, when \( i = n \) and \( x_{n+1} = u \), Proposition 1 holds. Thus, we choose the overall Lyapunov function \( V_n \) as \( V_n = V_{n-1} + \Psi_n \) with
\[ \Psi_n = \int_{x_n}^{x_{n-1}} \left( [r \xi_n]^{\alpha \nu} - [\xi_n]^{\alpha \nu} \right) (4\sigma - \nu - \mu) \sigma \sigma \theta_i dr, \]
(42)

and define the virtual controller \( \xi_{n+1} \) as
\[ \xi_{n+1} = -\lambda_n(x_i) \xi_i \xi_i^2 with \lambda_n(x_i) = \left[ 1 + H_n(x_i) \right]^{1/\xi_i} > 0. \]
(43)

Then, \( V_n(x) \) is clearly a \( C^2 \) function on \( \Pi_n \), and it is easy to obtain that
\[ \ell V_n \leq \frac{1}{2} G(x_i) \xi_i^2 - \sum_{i=1}^{n} \xi_i^2 + \theta_n(t) \xi_n \xi_{n-1} \xi_i^2 \]
\[ (44) \]

Therefore, we can design
\[ u = \xi_{n+1} = -\lambda_n(x_i) \xi_i \xi_i^2 + \lambda_n(x_i) \xi_i \xi_{i-1} - \xi_{i+1}. \]
(45)

which results in
\[ \ell V_n \leq \frac{1}{2} G(x_i) \xi_i^2 - \sum_{i=1}^{n} \xi_i^2 \leq 0. \]
(46)

3.2.2. Part II: Verification of Keeping the Output Constraint. For any \( x(0) = (x_1(0))^T \in \Pi_n \), by Ito’s formula and (46), we can deduce
\[ 0 \leq EV_n(x(t)) = V_n(x(0)) + \int_0^t \ell V_n(x(t)) dt \]
\[ \leq V_n(x(0)) < \infty. \]
(47)

From \( V_n(x) = V_1(x_1) + \sum_{i=2}^{n} \Psi_i > 0 \) and (47), it can be further verified that
\[ 0 \leq EV_1(x_1(t)) \leq V_n(x_0) < \infty, \]
(48)

which indicates
\[ P[V_1(x_1(t)) < \infty] = 1. \]
(49)

Hence, \( P[|y(t)| < b] = P[|x_1(t)| < b] = 1 \). Then, Part I of Theorem 1 is proved.

3.2.3. Part III: Stability Analysis. Based on the definition of \( V_n \), we easily obtain the fact that \( V_n \) is radially unbounded. Combining the fact with (40) and Lemma 1 directly infers that system (1) has a solution \( x(t, x(0)) \) for any \( x(0) \in \Pi_n \).

On the contrary, by a simple calculation, one obtains
\[ V_n = V_1 + \sum_{l=2}^{n} \Psi_l \]
\[ = V_1 + \sum_{l=2}^{n} \int_{x_l}^{x_{l-1}} \left( [r \xi_l]^{\alpha \nu} - [\xi_l]^{\alpha \nu} \right) (4\sigma - \nu - \mu) \sigma \sigma \theta_i dr \]
\[ \leq \frac{2b^{4\sigma - \mu}}{(4\sigma - \mu)^n} \tan \left( \frac{n[x_1]^{4\sigma - \mu}}{2b^{4\sigma - \mu}} \right) + 2 \sum_{l=2}^{n} |\xi_l|^{(4\sigma - \mu)/\sigma}. \]
(50)

In addition, since \( 4\sigma - \mu > 1 \), it is easy to get that
\[ \frac{1}{2} (n|x_1|^{4\sigma - \mu} / 2b^{4\sigma - \mu}) < (n/2) \]
for all \( x_1 \in \Pi_1 \). Then, applying the characteristics of tangent functions, it is not difficult to infer that
\[
\tan\left(\frac{\pi |x_1|^{4\sigma - \mu}}{2^{4\sigma - \mu}}\right) \leq \frac{\pi}{2^{4\sigma - \mu}} |x_1|^{4\sigma - \mu} G(x_1) \tag{51}
\]

Now, let \( c = 2^{(4\sigma - \mu)/4} > 0 \) and \( 0 < y = (4\sigma/(4\sigma - \mu)) < 1 \). According to (44), (45), and Lemma 4, we obtain

\[
c \leq 2^{(4\sigma - \mu)/4} G(x_1) V_n^{4\sigma/(4\sigma - \mu)} 2^{4\sigma - \mu}
\]

\[
+ \frac{1}{2} \sum_{i=1}^{n} \zeta_i \leq 2 G(x_1) V_n \zeta_i + \frac{1}{2} \sum_{i=1}^{n} \zeta_i.
\tag{52}
\]

So, one further obtains

\[
\ell \leq \left(-\frac{1}{2} G(x_1) \zeta_i - \frac{1}{2} \sum_{i=1}^{n} \zeta_i\right) - \frac{1}{2} \sum_{i=1}^{n} \zeta_i \leq 0.
\tag{53}
\]

In other words, \( \ell \leq -cV_n \). Consequently, it is directly deduced from Lemma 2 that the origin of system (1) under controller (39) is finite-time stable in probability.

Remark 3. Comparing with the existing results in most of the literatures, the paper mainly focuses on the finite-time stabilization, instead of the boundness of tracking error. Moreover, the proposed approach can be extend to the tracking control by introducing a coordinate transformation before constructing the BLF.

4. Simulation

In this section, we will provide the simulation results of the following example to illustrate the validity of the proposed strategy:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2^{7/5}, \\
\frac{dx_2}{dt} &= u + \frac{1}{2} x_1^{2} x_2^{1/5} dt + \frac{1}{8} (\sin x_2)^2 x_2^{3/5} dw, \\
y &= x_1,
\end{align*}
\tag{54}
\]

where \( q_1 = 7/5, q_2 = 1, \) and \( \mu = -4/11 \in (-5/12, 0) \). Then, \( v_1 = 1, v_2 = 5/11, \) and \( v_3 = 1/11 \). Since \( \delta_1 (t) = \delta_2 (t) = 1, \)

Assumption 1 is satisfied. And Assumption 2 is also satisfied with \( \psi_1 = \eta_1 = 0, \psi_2 = (1/2) x_1^2, \) and \( \eta_2 = (1/8) (\sin x_2)^2 \). Furthermore, note that \( c_1 = 1, c_2 = 11/5 > 2, \) and \( c_2 = 1 \). Thus, one can select \( \sigma = \sigma_0 = 1 \).

Now, let \( c_1 = [x_1]_1 \) and \( G(x_1) = \sec^2 (\pi |x_1|^{4\sigma/(4\sigma - \mu)}) \). In view of design procedure in Part I of the proof, we can design the virtual controller \( \xi_2 \) as

\[
\xi_2 = -2^{5\sigma} \zeta_1^{-5/11} = - \lambda_1 [\xi]^{-5/11}. \tag{55}
\]

Next, according to the design procedure, we denote \( \xi_2 = [x_2]^{-11/5} - [\xi_2]^{-11/5} \) and further obtain that...
\[ H_{21} = \frac{7}{44} \left( 2G(x_1) \right) \left( \frac{44}{296} \right)^{44/7}, \]
\[ H_{22} = \frac{37}{11} \left( 2G(x_1) \right) \left( \frac{37}{296} \right)^{44/7}, \]
\[ H_{23} = \frac{44}{44} \left( 2G(x_1) \right) \left( \frac{44}{56} \right)^{44/7}, \]
\[ H_{24} = \frac{44}{44} \left( 2G(x_1) \right) \left( \frac{44}{56} \right)^{44/7}. \]

Let \( H_3 = H_{21} + H_{22} + H_{23} + H_{24} \) and \( \lambda_2 = 1 + H_2 \); then, the controller \( u \) can be designed as
\[ u = -\lambda_2 \left( \frac{1}{11} \right)^{1/11} = -\lambda_2 \left( \frac{x_1}{11/5} - \left[ x_2 \right]^{11/5} - \left[ r-x \right]^{11/5} \right) \left( \frac{44}{96} \right)^{11/11}. \]

Finally, we suppose \( b = 1.5 \) and select some different initial states \( x(0) \)'s with each \( x(0) \) satisfying \( x(0) \in \Pi_2 \). The simulation results of system (48) are shown in Figures 1 and 2. Figure 1 curves trajectories of \( x_1(t) \) under different initial values, which illustrate that the output constraint is always not violated. Meanwhile, the trajectories of \( x_2(t) \) is given in Figure 2. It can be observed from the two figures that system (48) under controller (51) is finite-time stable.

### 5. Conclusion

In this paper, the stability issue is addressed for a class of \( p \)-norm stochastic systems with output constraints and unknown time-varying parameters. Using a tan-type BLF, the finite-time control strategy is proposed by the adding a power integrator technique. On this basis, the designed controller has been proved to ensure that the origin of the closed-loop system is finite-time stable in probability and the system output is kept in a pre-defined set. This conclusion has also been verified by the simulation results. It should be pointed out that the proposed approach is not applicable to the case of asymmetrical output constraints. In the future, we will try to modify the proposed method to be suitable for stochastic systems with asymmetrical output constraints or multi-input multi-output stochastic systems.

### Appendix

**Proof of Proposition 1.** For \( j, k = 1, \ldots, i - 1 \), one can get from the definition of \( \Psi_i \) that

\[ \frac{\partial \Psi_i}{\partial x_j} = \left[ \xi_j \right] \left( 4\sigma - \mu - v_i \right) \sigma, \]

\[ \frac{\partial \Psi_i}{\partial x_k} = \left[ \xi_k \right] \left( 4\sigma - \mu - v_i \right) \sigma \frac{\partial [ x_j^{\sigma/v_i} ]}{\partial x_j} - \frac{\partial [ x_j^{\sigma/v_i} ]}{\partial x_j} \int_{x_j}^{x_k} \left[ r \right]^{\alpha/v_i} - \left[ \xi_j \right]^{\alpha/v_i} \right] \left( 3\sigma - \mu - v_i \right) \sigma \, dr. \]

\[ \frac{\partial^2 \Psi_i}{\partial x_j \partial x_j} = \frac{\partial^2 \Psi_i}{\partial x_k \partial x_k} = \frac{\partial^2 \Psi_i}{\partial x_k \partial x_j} = \frac{\partial^2 \Psi_i}{\partial x_j \partial x_k} = \frac{4\sigma - \mu - v_i}{\sigma} \frac{\partial [ x_j^{\alpha/v_i} ]}{\partial x_j} \frac{\partial [ x_k^{\alpha/v_i} ]}{\partial x_k} - \frac{\partial [ x_j^{\alpha/v_i} ]}{\partial x_j} \frac{\partial [ x_k^{\alpha/v_i} ]}{\partial x_k} \int_{x_j}^{x_k} \left[ r \right]^{\alpha/v_i} - \left[ \xi_j \right]^{\alpha/v_i} \right] \left( 3\sigma - \mu - v_i \right) \sigma \, dr, \]

\[ \frac{\partial^3 \Psi_i}{\partial x_j \partial x_j^2} = \frac{4\sigma - \mu - v_i}{\sigma} \left[ x_j^{\alpha/v_i} \right] \left( 3\sigma - \mu - v_i \right) \sigma. \]
Then, we have
\[
\ell V_i = \ell V_{i-1} + \sum_{j=1}^{i-1} \frac{\partial \Psi}{\partial x_j} (\theta_j (t) x_{j+1}^i + f_j) + \frac{\partial \Psi}{\partial x_i} (\theta_i (t) x_{i+1}^i + f_i)
\]
\[
+ \frac{1}{2} \text{tr} \left[ \frac{\partial^2 \Psi}{\partial x_j^2} \frac{\partial^2 \Psi}{\partial x_i^2} \right].
\]
\[
\leq -\frac{1}{2} G(x_i) \zeta_i^4 - (n + 2 - i) \sum_{j=1}^{i-1} \zeta_j^4 + \theta_j (t) \zeta_j^4
\]
\[
\cdot \left[ \zeta_i^4 - \left( 4 - \mu - \gamma_{ij} \right) \sigma \left( x_{j+1}^i - \xi_{i+1}^4 \right) \right]
\]
\[
+ \sum_{j=1}^{i-1} \frac{\partial \Psi}{\partial x_j} (\theta_j (t) x_{j+1}^i + f_j) + \frac{\partial \Psi}{\partial x_i} (\theta_i (t) x_{i+1}^i + f_i)
\]
\[
+ \frac{1}{2} \text{tr} \left[ \frac{\partial^2 \Psi}{\partial x_j^2} \frac{\partial^2 \Psi}{\partial x_i^2} \right].
\]
\]
(A.2)

In the following, we introduce some sub-propositions to simplify the proof.

**Proposition A.1.** For \( i = 3, \ldots, n \), there exist smooth nonnegative functions \( \overline{\psi}_i (\cdot) \) and \( \overline{\eta}_i (\cdot) \) such that
\[
\left| f_i (x_i) \right| \leq \overline{\psi}_i (x_i) \left[ \left| \zeta_i^4 \right| (\gamma_{i+1}^4)^{\sigma} + \sum_{j=2}^{i} \left[ \left| \zeta_j^4 \right| (\gamma_{i+1}^4)^{\sigma} + \lambda_{j-1} (\gamma_{i+1}^4)^{\gamma_j (\gamma_{i+1}^4)^{\sigma}} \right] \right] \leq \overline{\psi}_i (x_i) \sum_{j=1}^{i} \left| \zeta_j^4 \right| (\gamma_{i+1}^4)^{\sigma},
\]
\[
\left\| g_i (x_i) \right\| \leq \overline{\eta}_i (x_i) \left[ \left| \zeta_i^4 \right| (2 \gamma_{i+1}^{2+\sigma}) + \sum_{j=2}^{i} \left[ \left| \zeta_j^4 \right| (2 \gamma_{i+1}^{2+\sigma}) + \lambda_{j-1} (2 \gamma_{i+1}^{2+\sigma}) \right] \right] \leq \overline{\eta}_i (x_i) \sum_{j=1}^{i} \left| \zeta_j^4 \right| (2 \gamma_{i+1}^{2+\sigma}),
\]
where \( \overline{\psi}_i (\cdot) \geq \sum_{j=i+1}^{i} (1 + \lambda_{j-1} (\gamma_{i+1}^4)^{\gamma_j (\gamma_{i+1}^4)^{\sigma}}) \), \( \overline{\eta}_i (\cdot) \geq \sum_{j=i+1}^{i} (1 + \lambda_{j-1} (2 \gamma_{i+1}^{2+\sigma})) \), both \( \overline{\psi}_i (\cdot) \) and \( \overline{\eta}_i (\cdot) \) are nonnegative smooth functions.

**Proposition A.2.** For \( i = 3, \ldots, n \), there exist nonnegative \( C^2 \) functions \( H_i (\cdot) \) such that
\[
\theta_{i-1} (t) \left[ \zeta_{i-1}^4 \left( 4 - \mu - \gamma_{i-1} \right) \sigma \left( x_{i-1}^i - \xi_{i-1}^i \right) \right] \leq \frac{1}{8} \zeta_{i-1}^4 + H_i (x_i) \zeta_i^4.
\]
(A.6)

**Proof.** Since \( (\gamma_{i-1}^4 + \mu)^{\sigma} \leq 1 \), it can be gotten from Lemma 5 that
\[
\zeta_{i-1}^4 - \xi_{i-1}^4 \leq 2^{1-(\gamma_{i-1}^4)^{\sigma}} \left[ \left| x_{i-1}^i \right|^{\sigma_{i-1}} \left[ \left| \zeta_i^4 \right| (\gamma_{i-1}^4)^{\sigma} \right] \right] \leq 2 \zeta_i^4 (\gamma_{i-1}^4)^{\sigma}.
\]
\]
(A.7)

From (A.7) and Lemma 3, one can verify that
\[
\left| f_i (x_i) \right| \leq \overline{\psi}_i (x_i) \left[ \left| \zeta_i^4 \right| (\gamma_{i+1}^4)^{\sigma} + \sum_{j=2}^{i} \left[ \left| \zeta_j^4 \right| (\gamma_{i+1}^4)^{\sigma} + \lambda_{j-1} (\gamma_{i+1}^4)^{\gamma_j (\gamma_{i+1}^4)^{\sigma}} \right] \right] \leq \overline{\psi}_i (x_i) \sum_{j=1}^{i} \left| \zeta_j^4 \right| (\gamma_{i+1}^4)^{\sigma},
\]
\[
\left\| g_i (x_i) \right\| \leq \overline{\eta}_i (x_i) \left[ \left| \zeta_i^4 \right| (2 \gamma_{i+1}^{2+\sigma}) + \sum_{j=2}^{i} \left[ \left| \zeta_j^4 \right| (2 \gamma_{i+1}^{2+\sigma}) + \lambda_{j-1} (2 \gamma_{i+1}^{2+\sigma}) \right] \right] \leq \overline{\eta}_i (x_i) \sum_{j=1}^{i} \left| \zeta_j^4 \right| (2 \gamma_{i+1}^{2+\sigma}),
\]
where \( \overline{\psi}_i (\cdot) \geq \sum_{j=i+1}^{i} (1 + \lambda_{j-1} (\gamma_{i+1}^4)^{\gamma_j (\gamma_{i+1}^4)^{\sigma}}) \), \( \overline{\eta}_i (\cdot) \geq \sum_{j=i+1}^{i} (1 + \lambda_{j-1} (2 \gamma_{i+1}^{2+\sigma})) \), both \( \overline{\psi}_i (\cdot) \) and \( \overline{\eta}_i (\cdot) \) are nonnegative smooth functions.

**Proof.** Since \( (\gamma_{i-1}^4 + \mu)^{\sigma} \leq 1 \), it can be gotten from Lemma 5 that
\[
\theta_{i-1} (t) \left[ \zeta_{i-1}^4 \left( 4 - \mu - \gamma_{i-1} \right) \sigma \left( x_{i-1}^i - \xi_{i-1}^i \right) \right] \leq \frac{1}{8} \zeta_{i-1}^4 + H_i (x_i) \zeta_i^4.
\]
(A.8)

**Proposition A.3.** For \( i = 3, \ldots, n \), there exist nonnegative \( C^2 \) functions \( H_i (\cdot) \) such that
\[
\sum_{j=1}^{i-1} \frac{\partial}{\partial x_j} (\theta_j (t) x_j^i + f_j) \leq \frac{1}{8} \sum_{j=4}^{i-1} \zeta_j^4 + H_i (x_i) \zeta_i^4.
\]
\]
(A.9)

**Proof.** For \( i = 3, \ldots, n, j = 1, \ldots, i-1 \), there are nonnegative \( C^2 \) functions \( M_{ij} (\cdot), M_{ij} (\cdot) \) such that
Lemma 3, there exist nonnegative functions where

\[ \Psi \leq \frac{\lambda_{i-1}(x_{i-1})}{\sigma} \left[ \frac{1}{\lambda_i} \right] (\psi/\lambda) \leq \frac{1}{8(i-1)} \sum_{l=1}^{i-1} \xi_l^4 + H_{12}(x_i) \xi_i^4, \]

where \( H_{12}(\cdot) \) are nonnegative C\(^2\) functions. Let \( H_{12}(\cdot) = \sum_{i=1}^{i-1} H_{12}(\cdot) \), which directly verify (A.9).

Proposition A.4. For \( i = 3, \ldots, n, j = 1, \ldots, i-1 \), there exist nonnegative C\(^2\) functions \( H_{13}(\cdot) \) such that

\[ \frac{\partial^2 \psi}{\partial x_i} \psi_i \leq \sum_{i=1}^{i-1} \xi_l^4 + H_{13}(x_i) \xi_i^4. \]

Proof. For \( i = 3, \ldots, n \), according to Proposition A.1 and Lemma 3, there exist nonnegative C\(^2\) functions \( H_{13}(\cdot) \) such that

\[ \frac{\partial^2 \psi}{\partial x_i} \psi_i \leq \left( \frac{4\sigma - \mu}{\sigma} \right) \psi_i \sum_{j=1}^{i-1} \left| \xi_j \right| (\psi/\lambda)^\alpha \]

\[ \leq \frac{1}{8} \sum_{l=1}^{i-1} \xi_l^4 + H_{13}(x_i) \xi_i^4. \]

Proposition A.5. For \( i = 3, \ldots, n \), there exist nonnegative C\(^2\) functions \( H_{14}(\cdot) \) such that

\[ \frac{1}{2} \text{tr} \left[ \frac{\partial^2 \psi}{\partial x_i} \psi_i \right] \leq \frac{1}{8} \sum_{l=1}^{i-1} \xi_l^4 + H_{12}(x_i) \xi_i^4. \]

Proof. Note that

\[ \frac{\partial^2 \psi}{\partial x_i} \psi_i \leq \frac{\lambda_{i-1}(x_{i-1})}{\sigma} \left[ \frac{1}{\lambda_i} \right] (\psi/\lambda) \leq \frac{1}{8(i-1)} \sum_{l=1}^{i-1} \xi_l^4 + H_{12}(x_i) \xi_i^4, \]

where \( K_{ik}(\cdot) \geq 0 \) is a C\(^2\) function.

From equation (A.10) and Proposition A.1, it can be inferred that

\[ \frac{1}{2} \left\| \frac{\partial^2 \psi}{\partial x_i} g_l \right\|^2 \leq \frac{1}{32(i-1)} \sum_{l=1}^{i-1} \xi_l^4 + H_{12}(x_i) \xi_i^4, \]
where $H_{ikj} (\cdot) \geq 0$ is a $C^2$ function. 

Then, one gets
\[
1/2 \sum_{k=1}^{i-1} \frac{\partial^2 \Psi_i}{\partial x_k \partial x_j} \|g_j\| \leq \frac{1}{32} \sum_{i=1}^{i-1} \zeta_i^4 + H_{i41}(x_i) \zeta_i^4, \tag{A.18}
\]

where $H_{i41}(\cdot) = \sum_{k=1}^{i-1} H_{ikj}(\cdot)$. 

Besides, if $k \neq j, k > j$, there exist $C^2$ functions $K_{ikj}(\cdot) \geq 0$ such that
\[
\left| \frac{\partial^2 [\xi_i^{\sigma/n}]}{\partial x_k \partial x_j} \right| \leq \frac{\partial M_{ij}(x_i)}{\partial x_k} \left( \sum_{l=1}^{i-1} |\zeta_l| + |x_j|^{(\sigma/n)-1} \right) + \tilde{M}_{ij}(x_i) \sum_{l=1}^{i-1} |\zeta_l| + |x_k|^{(\sigma/n)-1} \right|.
\tag{A.19}
\]

Meanwhile, if $k \neq j, k < j$, we have
\[
\left| \frac{\partial^2 [\xi_j^{\sigma/n}]}{\partial x_k \partial x_j} \right| = \frac{\partial^2 [\xi_j^{\sigma/n}]}{\partial x_k \partial x_j} \leq \tilde{K}_{ikj}(x_i)
\tag{A.20}
\]

Hence, for $\forall 1 \leq k \neq j \leq i-1$, combining (A.19) with (A.20) yields
\[
\left| \frac{\partial^2 [\xi_i^{\sigma/n}]}{\partial x_k \partial x_j} \right| \leq K_{ikj}(x_i) \sum_{l=1}^{i-1} |\zeta_l| + |x_j|^{(\sigma/n)-1} + |x_k|^{(\sigma/n)-1}. \tag{A.21}
\]

Then, for $\forall 1 \leq k \neq j \leq i-1$, it can be deduced from (A.21) and Lemma 3 that
\[
1/2 \sum_{k=j+1}^{i-1} \frac{\partial^2 \Psi_i}{\partial x_k \partial x_j} \|g_j\| \leq 2^\sigma - \mu - v_i K_{ikj}(x_i) \left( \sum_{l=1}^{i-1} |\zeta_l| + |x_j|^{(\sigma/n)-1} + |x_k|^{(\sigma/n)-1} \right) \times |\zeta_i|^{(3\sigma - \mu - v_i)\eta_i} \times |x_i - x_i\| \times \frac{\eta_i}{\sum_{m=1}^{i-1} |\zeta_m|^{(2\eta_{i+1})/2\eta_i}}
\]

where $K_{ikj}(\cdot) \geq 0$ are $C^2$ functions. Hence, one has
\[
1/2 \sum_{k=j+1}^{i-1} \frac{\partial^2 \Psi_i}{\partial x_k \partial x_j} \|g_j\| \leq \frac{1}{32} \sum_{i=1}^{i-1} \zeta_i^4 + H_{i42}(x_i) \zeta_i^4, \tag{A.23}
\]

where $H_{i42}(\cdot) = \sum_{k=j+1}^{i-1} H_{ikj}(\cdot)$. Similarly, applying (A.10), Proposition A.1 and Lemma 3, we get
\[
\sum_{j=1}^{i-1} \frac{\partial^2 \Psi_i}{\partial x_i \partial x_j} \|g_j\| \leq \frac{1}{32} \sum_{i=1}^{i-1} \zeta_i^4 + H_{i43}(x_i) \zeta_i^4, \tag{A.24}
\]

where $H_{i43}(\cdot)$ and $H_{i44}(\cdot)$ are nonnegative $C^2$ functions. Substituting (A.18), (A.23)–(A.25) into (A.15) directly infers (A.14). Till now, the proof of Proposition A.5 is finished.

Combining Propositions A.1–A.5 with (A.2) yields
\[
eq \frac{1}{2} G(x_i) \zeta_i^4 - (n + 1 - i) \sum_{l=1}^{i-1} \zeta_l^4 + H_i(x_i) \zeta_i^4
\]

\[
+ \theta_i(t)[\zeta_i]^{(\theta_i - \mu - v_i)\eta_i} \lambda_{i+1} \theta_i(t)[\zeta_i]^{(\theta_i - \mu - \nu_i)\eta_i} \cdot (x_{i+1} - x_{i+1}), \tag{A.26}
\]

where $H_i(\cdot) = \sum_{r=1}^{i} H_{ir}(\cdot) \geq 0$ is a $C^2$ function.

Design
\[
\zeta_{i+1} = -\lambda_i(x_i)[\zeta_i]^{(\nu_{i+1})/\theta_i} \text{ with } \lambda_i(x_i)
\]

\[
= \left[ \frac{n - i + 1 + H_i(x_i)}{\theta_i} \right]^{1/\theta_i} > 0. \tag{A.27}
\]

Then, substituting the value of $\zeta_{i+1}$ into (A.27) yields that (40) holds. \(\square\)
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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