Efficient Algorithms for High-Dimensional Convex Subspace Optimization via Strict Complementarity

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Abstract

We consider optimization problems in which the goal is find a $k$-dimensional subspace of $\mathbb{R}^n$, $k << n$, which minimizes a convex and smooth loss. Such problems generalize the fundamental task of principal component analysis (PCA) to include robust and sparse counterparts, and logistic PCA for binary data, among others. While this problem is not convex it admits natural algorithms with very efficient iterations and memory requirements, which is highly desired in high-dimensional regimes however, arguing about their fast convergence to a global optimal solution is difficult. On the other hand, there exists a simple convex relaxation for which convergence to the global optimum is straightforward, however corresponding algorithms are not efficient when the dimension is very large. In this work we present a natural deterministic sufficient condition so that the optimal solution to the convex relaxation is unique and is also the optimal solution to the original nonconvex problem. Mainly, we prove that under this condition, a natural highly-efficient nonconvex gradient method, which we refer to as gradient orthogonal iteration, when initialized with a “warm-start”, converges linearly for the nonconvex problem. We also establish similar results for the nonconvex projected gradient method, and the Frank-Wolfe method when applied to the convex relaxation. We conclude with empirical evidence on synthetic data which demonstrate the appeal of our approach.

1 Introduction

We consider the problem of finding a $k$-dimensional subspace of $\mathbb{R}^n$, $k << n$, which minimizes a given objective function, where we identify a subspace with its corresponding projection matrix. That is, we consider the following optimization problem:

$$\min f(X) \text{ subject to } X \in \mathcal{P}_{n,k} := \{QQ^\top | Q \in \mathbb{R}^{n \times k}, Q^\top Q = I\}. \quad (1)$$

Throughout this work and unless stated otherwise, we assume that $f(\cdot)$ is convex, $\beta$-smooth (gradient Lipschitz) and, for ease of presentation, we also assume that the gradient $\nabla f(\cdot)$ is a symmetric matrix over the space of $n \times n$ symmetric matrices $\mathcal{S}^n_1$.

Problems of interest that fall into this model include among other robust counterparts of PCA which are based on the smooth and convex Huber loss (see concrete examples in Section 3), logistic PCA [11], and sparse PCA [17].

In case the gradient is not a symmetric matrix at some point $X \in \mathcal{S}^n$, then denoting it by $\nabla_{\text{nonsym}} f(X)$, we can always take its symmetric counterpart $\nabla f(X) = \frac{1}{2} (\nabla_{\text{nonsym}} f(X) + (\nabla_{\text{nonsym}} f(X))^\top)$ and, unless stated otherwise, our derivations throughout this work will remain the same.
Motivated by high-dimensional problems, we are interested in highly efficient (in particular in terms of the dimension $n$) first-order methods for Problem (1). Moreover, we are interested in establishing, at least locally, fast convergence to the global minimizer despite the fact that Problem (1) is nonconvex. Our approach will not assume that $f(\cdot)$ admits a very specific structure (e.g., a linear or quadratic function), or will be based on an underlying statistical model, e.g., [16 19 12]. Instead, we will be interested in a deterministic condition that may hold for quite general $f(\cdot)$ (which is convex and smooth).

We will now briefly describe two natural dimension-efficient first-order methods for tackling Problem (1).

One natural gradient method for tackling Problem (1) is the nonconvex projected gradient method which follows the dynamics:

$$X_{t+1} \leftarrow \Pi_{\mathcal{P}_{n,k}} [X_t - \eta_t \nabla f(X_t)],$$

where $\Pi_{\mathcal{P}_{n,k}} [\cdot]$ denotes the Euclidean projection onto the set $\mathcal{P}_{n,k}$ (note that since this set is nonconvex, in general, the projection need not be unique). Given the gradient $\nabla f(X_t)$, the runtime to compute $X_{t+1}$ is dominated by the computation of the projection. It is not hard to show that the Euclidean projection is given by the projection matrix which corresponds to the span of the top $k$ eigenvectors of the matrix $X_t - \eta_t \nabla f(X_t)$. While accurate computation of this projection requires a (thin) singular value decomposition (SVD) of a $n \times n$ matrix, which amounts to $O(n^3)$ runtime, it can also be approximated up to sufficiently small error using fast iterative methods, such as the well-known orthogonal iteration method [8] (aka subspace iteration method [15]) whose every iteration takes in worst case only $O(kn^2)$ time. When the gradient $\nabla f(X_t)$ admits a favorable structure such as sparsity or low-rank factorization, this runtime could be significantly improved.

Another natural approach to tackle Problem (1) is to exploit the fact that each $X \in \mathcal{P}_{n,k}$ could be factored as $X = QQ^\top$, $Q \in \mathbb{R}^{n \times k}$ having orthonormal columns, and to apply gradient steps w.r.t. this factorization. This leads to the following dynamics, which we refer to as Gradient Orthogonal Iteration:

$$Z_{t+1} \leftarrow Q_t - \eta_t \frac{\partial f(Q_t Q_t^\top)}{\partial Q}|_{Q_t} = Q_t - \eta_t \nabla f(Q_t Q_t^\top)Q_t,$$

$$(Q_{t+1}, R_{t+1}) \leftarrow \text{QR-FACTORIZE}(Z_{t+1}),$$

where QR-FACTORIZE($\cdot$) denotes the QR factorization operation, i.e., in the above $Q_{t+1} \in \mathbb{R}^{n \times k}$ has orthonormal columns. This step is required so $Q_{t+1}$ has orthonormal columns (and so $Q_{t+1}Q_{t+1}^\top$ remains a projection matrix) for all $t \geq 1$.

As opposed to the Dynamics (2), which as discussed, an efficient implementation of will require to run a QR-based iterative method to compute the Euclidean projection onto $\mathcal{P}_{n,k}$ on each iteration, the Dynamics (3) only require a single QR factorization, and thus, given the gradient matrix $\nabla f(Q_t Q_t^\top)$, the next iterate $Q_{t+1}$ can be computed in overall $O(n^2k)$ time. As mentioned above, this runtime could be further significantly improved if the multiplication $\nabla f(Q_t Q_t)Q_t$ could be carried out faster than $O(n^2k)$ (for instance when the gradient is sparse or admits low-rank factorization), since all other operations require only $O(k^2n)$ time (e.g., the QR-factorization of $Z_{t+1}$). Obtaining provable guarantees on the fast convergence of the Dynamics (3) to a global optimal solution will be the main contribution of this work.

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2This method finds a $n \times k$ matrix $Q$ with orthonormal columns which approximately span the subspace spanned by the $k$ leading eigenvectors of a symmetric $n \times n$ matrix $A$, by repeatedly applying the iterations: $(Q, R) \leftarrow \text{QR-FACTORIZE}(AQ)$, where QR-FACTORIZE($\cdot$) returns the QR factorization.
While both Dynamics (2), (3), apply efficient iterations, since they are inherently nonconvex, arguing about their convergence to a global optimal solution of (1) is difficult in general. An alternative is to replace Problem (1) with a convex counterpart, for which, arguing about the convergence of first-order methods to a global optimal solution is well understood. Consider the convex set 
\[ F_{n,k} = \text{conv}(P_{n,k}), \]
where \( \text{conv}(\cdot) \) denotes the convex-hull operation. \( F_{n,k} \) is also called the Fantope and it is known to admit the following important characterization:
\[ F_{n,k} = \{ X \in S^n \mid I \succeq X \succeq 0, \text{Tr}(X) = k \}, \]
where \( A \succeq 0 \) denotes that \( A \) is a positive semidefinite (PSD) matrix, see for instance [14]. This leads to the convex problem:

\[
\min f(X) \quad \text{subject to} \quad X \in F_{n,k} = \{ X \in S^n \mid I \succeq X \succeq 0, \text{Tr}(X) = k \}. \tag{4}
\]

A well known first-order method applicable to (4) is the Frank-Wolfe method (aka conditional gradient) [9], which for the convex Problem (4) follows the dynamics:

\[
V_t \leftarrow \arg \min_{V \in P_{n,k}} \text{Tr}(V \nabla f(X_t)), \quad X_{t+1} \leftarrow (1 - \eta_t)X_t + \eta_t V_t, \quad \eta_t \in [0, 1]. \tag{5}
\]

It follows from Ky Fan’s maximum principle [5] that computing \( V_t \) amounts to computing the projection matrix onto the span of the \( k \) eigenvectors corresponding to the \( k \) smallest eigenvalue of \( \nabla f(X_t) \), and hence can be carried out efficiently using the orthogonal iterations method or similar methods, similarly to the computation of the projection in (2) discussed above. Note however, that the Frank-Wolfe iterates will not be, in general, low rank, and only yield a \( O(1/t) \) convergence rate [9]. The main goal of this work is thus to make progress on the following natural question:

\textit{When can the convex relaxation (4) be used to argue on the fast convergence of the nonconvex Dynamics (2), (3) to a global optimal solution of the nonconvex Problem (1)?}

### 1.1 The eigengap assumption and strict complementarity

We now turn to discuss our only non-completely standard assumption on Problems (1), (4), which will underly all of our contributions.

**Assumption 1 (Main assumption).** \( \text{An optimal solution } X^* \text{ to the convex Problem (4) is said to satisfy the eigen-gap assumption with parameter } \delta > 0, \text{ if } \lambda_{n-k}(\nabla f(X^*)) - \lambda_{n-k+1}(\nabla f(X^*)) \geq \delta. \)

A similar assumption has been considered in the recent works [7, 6, 3] on efficient first-order methods for convex relaxations of low-rank matrix optimization problems, where the underlying set is either the nuclear norm ball (matrices with bounded sum of singular values) or the set of PSD matrices with bounded trace, but not the Fantope.

**Theorem 1.** \( \text{If an optimal solution } X^* \text{ to Problem (4) satisfies Assumption 1 with some parameter } \delta > 0, \text{ then it has rank } k, i.e., } X^* \in P_{n,k}, \text{ and it is the unique optimal solution to both Problem (4) and Problem (1).} \)

\(^3\text{We note that one can also consider projection-based first-order methods for Problem (4), such as the projected gradient method, however in general, the projection onto the Fantope } F_{n,k} \text{ will not be a low-rank matrix and hence its computation will require an expensive SVD computation (see details in the sequel).} \)
To motivate Assumption 1 let us consider perhaps the simplest case in which \( f(\cdot) \) is a linear function, i.e., consider \( f(X) = f_{\text{lin}}(X) = \text{Tr}(XC) \), for some \( C \in \mathbb{S}^n \). For instance, the optimization problem underlying the task of PCA, in which one is interested in recovering the principal \( k \)-dimensional subspace given an empirical covariance matrix, takes exactly the form of (1) with \( f = f_{\text{lin}} \) and \( C \) being the minus empirical covariance matrix. For \( f = f_{\text{lin}} \), it is known that Problem (1) admits a unique optimal solution if and only if \( \lambda_{n-k}(C) - \lambda_{n-k+1}(C) > 0 \) (this follows from Ky Fan’s maximum principle [5]). Note that for PCA, with \(-C\) being the empirical covariance, the corresponding \( k \)-dimensional principle subspace is unique if and only if there is a gap between the \( k \) and \( k + 1 \) largest eigenvvalues of the covariance, which is in turn equivalent to \( \lambda_k(-C) - \lambda_{k+1}(-C) = \lambda_{n-k}(C) - \lambda_{n-k+1}(C) > 0. \) Of course it holds that \( \nabla f_{\text{lin}}(X) = C \). Thus, for linear \( f(\cdot) \), Assumption 1 is both a sufficient and necessary condition so that Problem (1) admits a unique optimal solution. Note that since the objective function is linear and \( \mathcal{F}_{n,k} = \text{conv}(\mathcal{P}_{n,k}) \), Assumption 1 is also both a sufficient and necessary condition so that Problem (1) admits a unique optimal solution and that this solution has rank \( k \).

For a general convex and smooth \( f(\cdot) \), Assumption 1 is not a necessary condition so that the consequences of Theorem 1 hold true, but it is a sufficient one.

Assumption 1 is also tightly related to the convex Problem (4) through the concept of strict-complementarity which is a classical concept in constrained continuous optimization theory, see for instance [1]. A similar connection between an eigengap in the gradient at an optimal solution and strict complementarity has been already established in [3] for low-rank matrix optimization problems, where the underlying convex set is either the nuclear norm ball of matrices or the set of PSD matrices with unit trace. Now we establish a similar relationship for the convex relaxation (4) and the Fantope, which is slightly more involved. Let us write the Lagrangian of the convex Problem (4):

\[
L(X, Z_1, Z_2, s) = (X) - \langle Z_1, X \rangle - \langle Z_2, I - X \rangle - s(\text{Tr}(X) - k),
\]

where the dual matrix variables \( Z_1, Z_2 \) are constrained to be PSD, i.e., \( Z_1 \succeq 0, Z_2 \succeq 0 \).

The KKT conditions state that \( X^*, (Z_1^*, Z_2^*, s^*) \) are corresponding optimal primal-dual solutions if and only if the following conditions hold:

1. \( I \succeq X^* \succeq 0, \text{Tr}(X^*) = k, Z_1^* \succeq 0, Z_2^* \succeq 0, \)
2. \( \nabla f(X^*) = Z_1^* - Z_2^* + s^*I, \)
3. \( \langle Z_1^*, I - X^* \rangle = 0. \)

Condition 3 is known as complementarity. Since \( Z_1^*, Z_2^* \) are PSD and \( 0 \preceq X^* \preceq I \), this further implies that \( Z_1^*X^* = 0, Z_2^*(I - X^*) = 0 \), which in turn implies that

\[
\text{range}(X^*) \subseteq \text{nullspace}(Z_1^*) \land \text{range}(I - X^*) \subseteq \text{nullspace}(Z_2^*).
\]

**Definition 1.** A pair of primal-dual solutions \( X^*, (Z_1^*, Z_2^*, s^*) \) for Problem (4) is said to satisfy strict complementarity if

\[
\text{range}(X^*) = \text{nullspace}(Z_1^*) \lor \text{range}(I - X^*) = \text{nullspace}(Z_2^*),
\]

which is the same as,

\[
\text{rank}(Z_1^*) = n - \text{rank}(X^*) \lor \text{rank}(Z_2^*) = \text{rank}(X^*).
\]

**Theorem 2.** If an optimal solution \( X^* \) for Problem (4) with \( \text{rank}(X^*) = k \) satisfies strict complementarity for some corresponding dual solution, then \( \lambda_{n-k}(\nabla f(X^*)) - \lambda_{n-k+1}(\nabla f(X^*)) > 0. \) Conversely, if an optimal solution \( X^* \) for Problem (4) satisfies \( \lambda_{n-k}(\nabla f(X^*)) - \lambda_{n-k+1}(\nabla f(X^*)) > 0, \) then it satisfies strict complementarity for every corresponding dual solution.
Table 1: Summary of main algorithmic results.

| Algorithm                                      | Global conv. rate | Local conv. rate | Worst case runtime of single iteration | rank($X_t$) |
|------------------------------------------------|-------------------|------------------|----------------------------------------|-------------|
| Gradient Orthogonal Iteration (3)              | –                 | exp($-\Theta(\delta/\beta)t$) | $kn^2$                                 | $k$         |
| Proj. Grad. Descent on $P_{n,k}$ (2)           | –                 | exp($-\Theta(\delta/\beta)t$) | $k$-SVD                                | $k$         |
| Frank-Wolfe (5)                                | $k\beta/t$       | exp($-\Theta(\delta/\beta)t$) | $k$-SVD                                | $\min\{kt,n\}$ |

The proof is given in the appendix. Strict complementarity has played a central role in several recent works, both for establishing linear convergence rates for first-order methods, e.g., [20, 14, 10, 3], and improving the complexity of projected gradient methods, due to SVD computations, for low-rank matrix optimization problems, e.g., [7, 10].

1.2 Notation

Throughout this work we let $\|\cdot\|$ denote the Euclidean norm for vectors in $\mathbb{R}^n$ and the spectral norm (largest singular value) for matrices in $\mathbb{R}^{m \times n}$ or $\mathbb{S}^n$. We let $\|\cdot\|_F$ denote the Frobenius (Euclidean) norm for matrices. For a matrix $X \in \mathbb{S}^n$, we let $\lambda_i(X)$ denote the $i$th largest eigenvalue of $X$. We let $\langle \cdot, \cdot \rangle$ denote the standard inner-product for both spaces $\mathbb{R}^n$ and $\mathbb{S}^n$.

1.3 Main results

The main contribution of this work is the proof of the following theorem regarding the local linear convergence of the gradient orthogonal iteration Dynamics (3) to the global optimal solution of Problems (4) and (1).

**Theorem 3.** Suppose Assumption 1 holds true for some optimal solution $X^*$ for Problem (4) with some parameter $\delta > 0$. Let $G \geq \sup_{X \in F_{n,k}} \|\nabla f(X)\|$. Let $Q_1 = 1 \in \mathbb{R}^{n \times k}$, rank($Q_1$) = $k$ and consider the sequence $\{Q_t\}_{t \geq 1}$ generated by Dynamics (3) with fixed step-size $\eta_t = \eta = \frac{1}{4 \max\{\beta, G\}}$ for all $t \geq 1$, and when initialized with $Q_1 \in P_{n,k}$ such that

$$\|Q_1Q_1^T - X^*\|_F \leq \min\{1, \sqrt{\frac{\delta}{2(1 + \eta\beta)}}\}.$$  

Then, we have that

$$\forall t \geq 1: \quad f(Q_tQ_t^T) - f(X^*) \leq (f(Q_1Q_1^T) - f(X^*)) \exp\left(-\frac{\delta(t-1)}{40 \max\{\beta, G\}}\right).$$

We also prove the following two theorems regarding the local linear convergence of the projected gradient Dynamics (2) and the Frank-Wolfe Dynamics (5).

**Theorem 4.** Suppose Assumption 1 holds true for some optimal solution $X^*$ for Problem (4) with some parameter $\delta > 0$. Consider the sequence $\{X_t\}_{t \geq 1}$ generated by Dynamics (2) with a fixed step-size $\eta_t = \eta = 1/\beta$ for all $t \geq 1$, and when initialized with $X_1 \in F_{n,k}$ such that $\|X_1 - X^*\|_F \leq \frac{\delta}{4\beta}$. Then, for all $t \geq 1$ it holds that

1. rank($X_{t+1}$) = $k$, and thus, given $X_t, \nabla f(X_t)$, $X_{t+1}$ can be computed via a rank-$k$ SVD,
2. \( f(X_t) - f(X^*) = O(\exp(-\Theta(\delta/\beta)(t - 1))) \).

Note that the initialization requirement in Theorem 3 has poorer dependence on \( \delta \) than Theorem 4. It remains an open question if the initialization condition in Theorem 3 can be improved.

**Theorem 5.** Suppose Assumption 1 holds true for some optimal solution \( X^* \) for Problem 4 with some parameter \( \delta > 0 \). Consider the sequences \( \{(X_t, V_t)\}_{t \geq 1} \) generated by Dynamics 5 when \( \eta_t \) is chosen via line-search. Then, there exists \( T_0 = O\left(k(\beta/\delta)^3\right) \) such that,

\[
\forall t \geq T_0 : f(X_{t+1}) - f(X^*) \leq (f(X_t) - f(X^*)) \left(1 - \min\left\{ \frac{\delta}{12\beta}, \frac{1}{2} \right\}\right).
\]

Moreover, for all \( t \geq 1 \), the rank-\( k \) matrix \( V_t \) satisfies \( \|V_t - X^*\|_F^2 = O\left(\frac{\delta^2}{\delta^2} (f(X_t) - f(X^*)) \right) \).

### 1.4 What if Assumption 1 fails?

In case Assumption 1 does not hold or holds with negligible parameter \( \delta \), \( X^* \) need to longer be unique or of rank \( k \) and Theorems 3, 4, 5 are no longer relevant. Nevertheless, not all is lost, since by considering weaker versions of Assumption 1 which consider eigen-gaps in lower eigenvalues, we can still guarantee \( X^* \) has low rank, and that at least the projected gradient method, locally, will require only low-rank SVD to compute the projection onto the Fantope, while guaranteeing the standard convergence rate of \( O(1/t) \) (not linear rate as when Assumption 1 holds). This is captured in the following theorem, which is an additional main contribution of this work. While it is inspired by a similar result for the nuclear norm ball of matrices \( \| \), due to the more complex structure of the Fantope, its proof is substantially more involved.

**Theorem 6.** Let \( X^* \in F_{n,k} \) be some optimal solution to Problem 4 and let \( \mu_1 \geq \mu_2 \geq \ldots \mu_n \) denote the eigenvalues of \( -\nabla f(X^*) \). Let \( r \) be the smallest integer such that \( r \geq k \) and \( \mu_r - \mu_{r+1} > 0 \). Then, it holds that \( \text{rank}(X^*) \leq r \). Moreover, consider the projected gradient dynamics w.r.t. Problem 4 given by, \( X_{t+1} \leftarrow P_{F_{n,k}}[X_t - \beta^{-1}\nabla f(X_t)] \). For any \( r' \in \{r, \ldots, n - 1\} \), if \( \|X_1 - X^*\|_F \leq \frac{\mu_k - \mu_{r+1}}{4\beta} \), then it holds that,

1. For all \( t \geq 1 \), \( \text{rank}(X_{t+1}) \leq r' \), i.e., given \( X_t, \nabla f(X_t) \), \( X_{t+1} \) can be computed via a rank-\( r' \) SVD.

2. The sequence \( \{X_t\}_{t \geq 1} \) converges with the standard PGD rate: \( f(X_t) - f(X^*) = O(\beta\|X_1 - X^*\|_F^2/t) \).

**Remark 1.** Note that via the parameter \( r' \), Theorem 6 offers a flexible tradeoff between the radius of the ball in which PGD needs to be initialized in (increasing \( r' \) increases the radius), and the rank of the iterates which in turn, implies an upper-bound on the rank of SVD computations required for the projection, which controls the runtime of each iteration.

**Remark 2.** Theorem 6 may be in particular interesting when \( f(\cdot) \) is subspace-monotone in the sense that for any two subspaces \( S_1 \subseteq S_2 \subseteq \mathbb{R}^n \) and their corresponding projection matrices \( P_1, P_2 \in S^n \), it holds that \( f(P_2) \leq f(P_1) \). In this case, given an optimal solution \( X^* \) to the convex Problem 4 with eigen-decomposition \( X^* = \sum_{i=1}^{\min\{k, \ell\}} \lambda_i u_i u_i^\top \), when \( k < r << n \), using the projection matrix \( P^* = \sum_{i=1}^{\min\{k, \ell\}} u_i u_i^\top \) which satisfies \( f(P^*) \leq \min_{X \in P_{F_{n,k}}} f(X) \) may be of interest. For instance, it is not hard to show that \( f(\cdot) \) of the form \( f(X) = \sum_{i=1}^{m} g_i(\|q_i - Xq_i\|) \), where \( g_i(\cdot) \) is monotone non-decreasing and \( \{q_i\}_{i=1}^{m} \subseteq \mathbb{R}^n \), is subspace-monotone.
2 Analysis

2.1 Preliminaries

Lemma 1 (Euclidean projection onto the Fantope). Let $X \in \mathbb{R}^{n \times n}$ and consider its eigen decomposition $X = \sum_{i=1}^{n} \gamma_i u_i u_i^\top$. The Euclidean projection of $X$ onto the Fantope $F_{n,k}$ is given by:

$$\Pi_{F_{n,k}}[X] = \sum_{i=1}^{n} \gamma_i^+(\theta) u_i u_i^\top,$$

where $\gamma_i^+(\theta) = \min(\max(\gamma_i - \theta, 0), 1)$ and $\theta$ satisfies the equation $\sum_{i=1}^{n} \gamma_i^+(\theta) = k$. Moreover, for any $r \in \{k, \ldots, n-1\}$ it holds that $\text{rank}(\Pi_{F_{k}}(X)) \leq r$ if and only if $\sum_{i=r+1}^{n} \min(\gamma_i - r+1, 1) \geq k$.

Remark 3. Lemma 6 in particular implies that if $\text{rank}(X) \leq r$, then only the top $r$ component of the SVD of $X$ are needed to compute $\Pi_{F_{n,k}}[X]$, i.e., a rank-$r$ SVD of $X$. Moreover, given the rank-$(r+1)$ SVD, we can check the condition $\sum_{i=r+1}^{n} \min(\gamma_i - r+1, 1) \geq k$, to verify whether the projection has rank $\leq r$ or not.

Proof of Lemma 4. The first part of the lemma is a known fact, see for instance [17]. For the second part, let us prove that if $\sum_{i=r+1}^{n} \min(\gamma_i - r+1, 1) \geq k$, then $\theta$ must satisfy $\theta \geq \gamma_{r+1}$. Assume by way contradiction that $\theta < \gamma_{r+1}$. Then,

$$k = \sum_{i=1}^{n} \min(\max(\gamma_i - \theta, 0), 1) > \sum_{i=1}^{n} \min(\max(\gamma_i - \gamma_{r+1}, 0), 1) = \sum_{i=1}^{r} \min(\gamma_i - \gamma_{r+1}, 1),$$

which is a contradiction, so it must be that $\theta \geq \gamma_{r+1}$ and in that case Eq. 6 sets all the bottom $n - r$ components of the eigen-decomposition of $X$ to zero. Hence, $\text{rank}(\Pi_{F_{n,k}}[X]) \leq r$. The reversed direction holds from similar reasoning.

The following lemma which is central to our analysis connects between an optimal solution and the eigen-decomposition of its corresponding gradient

Lemma 2. Let $X^* \in F_{n,k}$ be an optimal solution to Problem 4 and write the eigen-decomposition of $-\nabla f(X^*)$ as $-\nabla f(X^*) = \sum_{i=1}^{n} \mu_i u_i u_i^\top$. Let $r$ be the smallest integer such that $r \geq k$ and $\mu_k - \mu_{k+1} > 0$. Then, for all $n \geq i \geq r+1$, $X^*$ is orthogonal to $u_i u_i^\top$, and $\text{rank}(X^*) \leq r$. In particular, if $r = k$, then $X^* \in P_{n,k}$ is the unique projection matrix onto the span of the $k$ leading eigenvectors of $-\nabla f(X^*)$.

Proof. Assume by contradiction that $X^*$ is not orthogonal $u_{r+1} u_{r+1}^\top, \ldots, u_n u_n^\top$. In this case, $\sum_{i=r+1}^{n} u_i^\top X^* u_i > 0$, and we can write,

$$\langle X^*, -\nabla f(X^*) \rangle = \sum_{i=1}^{r} \mu_i u_i^\top X^* u_i + \sum_{i=r+1}^{n} \mu_i u_i^\top X^* u_i,$$

$$< (a) \sum_{i=1}^{r} \mu_i u_i^\top X^* u_i + \mu_r \sum_{i=r+1}^{n} u_i^\top X^* u_i$$

$$= (b) \sum_{i=1}^{k-1} \mu_i u_i^\top X^* u_i + \mu_k \sum_{i=k}^{n} u_i^\top X^* u_i$$

$$= (c) \sum_{i=1}^{k-1} \mu_i u_i^\top X^* u_i + \mu_k \left( k - \sum_{i=1}^{k-1} u_i^\top X^* u_i \right),$$
where both (a) and (b) follow from the definition of $r$, and (c) follows since $\sum_{i=1}^n u_i^T X^T u_i = \text{Tr}(X^*\sum_{i=1}^n u_i u_i^T) = \text{Tr}(X^*I) = k$.

Let us denote the projection matrix onto the span of the top $k$ eigenvectors of $-\nabla f(X^*)$ by $P^* = \sum_{i=1}^k u_i u_i^T$, and note that $(P^*, -\nabla f(X^*)) = \sum_{i=1}^k \mu_i$. It follows that

$$
\langle P^* - X^*, \nabla f(X^*) \rangle = \langle X^* - P^*, -\nabla f(X^*) \rangle
$$

$$
< \sum_{i=1}^{k-1} \mu_i u_i^T X^T u_i + \mu_k \left( k - \sum_{i=1}^{k-1} u_i^T X^T u_i \right) - \sum_{i=1}^k \mu_i
$$

$$
= \sum_{i=1}^{k-1} \mu_i \left( u_i^T X^T u_i - 1 \right) + \mu_k \sum_{i=1}^{k-1} \left( 1 - u_i^T X^T u_i \right)
$$

$$
= \sum_{i=1}^{k} \left( 1 - u_i^T X^T u_i \right) (\mu_k - \mu_i) \leq 0,
$$

where the last inequality follows since for all $i$, $u_i^T X^T u_i \in [0,1]$.

Thus, we have that $X^*$ violates the first-order optimality condition which contradict that assumption that it is an optimal solution, and thus we have that $X^*$ must indeed be orthogonal to $u_{r+1}, u_{r+1}, \ldots, u_n u_n^T$.

An immediate consequence is that the eigenvectors of $X^*$ which correspond to non-zero eigenvalues must lie in $\text{span}\{u_1, \ldots, u_r\}$, and thus, it must be that $\text{rank}(X^*) \leq r$.

For the final part of the lemma, in case $r = k$, since for all $X \in F_{n,k}$, $\text{rank}(X) \geq k$, we have that $\text{rank}(X^*) = k$. In particular, $X^*$ is a projection matrix, i.e., $X \in P_{n,k}$. By the orthogonality result above, it follows that the eigenvectors of $X^*$ lie in $\text{span}\{u_1, \ldots, u_k\}$, which means that $X^*$ is indeed the projection matrix onto $\text{span}\{u_1, \ldots, u_k\}$, as stated in the lemma. Note that when $r = k$, this projection matrix is indeed unique (i.e., the subspace spanned by the top $k$ eigenvectors of $-\nabla f(X^*)$ is unique).

Now we can prove Theorem [1].

**Proof of Theorem [1]** We begin with the case that $X^*$ is an optimal solution to the convex relaxation [4] which satisfies Assumption [1] with some $\delta > 0$. It follows directly from Lemma [2] that $\text{rank}(X^*) = k$. From Lemma [2] it further follows that $X^*$ is the unique projection matrix onto the span of top $k$ eigenvectors of $-\nabla f(X^*)$, i.e., it is the unique matrix in $X \in P_{n,k}$ such that $\langle X, -\nabla f(X^*) \rangle = \sum_{i=1}^k \mu_i$, where we write the eigen-decomposition of $-\nabla f(X^*)$ as $-\nabla f(X^*) = \sum_{i=1}^k \mu_i u_i u_i^T$. From the von Neumann trace inequality it follows that for any matrix $X \in P_{n,k}$ it holds that $\langle X, -\nabla f(X^*) \rangle \leq \sum_{i=1}^n \lambda_i(X) \mu_i = \sum_{i=1}^k \lambda_i(X) \mu_i = \sum_{i=1}^k \mu_i$. Thus, we have that

$$
\forall X \in P_{n,k} \setminus \{X^*\} : \quad \langle X - X^*, \nabla f(X^*) \rangle = \langle X^* - X, -\nabla f(X^*) \rangle > 0.
$$

Since $F_{n,k} = \text{conv}\{P_{n,k}\}$, this further implies that,

$$
\forall X \in F_{n,k} \setminus \{X^*\} : \quad \langle X - X^*, \nabla f(X^*) \rangle > 0.
$$

Since $f(\cdot)$ is convex it further holds that,

$$
\forall X \in F_{n,k} \setminus \{X^*\} : \quad f(X^*) - f(X) \leq \langle X^* - X, \nabla f(X^*) \rangle < 0,
$$

and thus, we conclude that $X^*$ is indeed the unique optimal solution to Problem [4].
The following lemma shows that under Assumption 1, Problem (4) has a quadratic growth property. This property is crucial to obtaining our linear convergence rates.

**Lemma 3 (Quadratic Growth).** Let $X^* \in \mathcal{F}_{n,k}$ be an optimal solution to Problem (4) for which Assumption 1 holds with some parameter $\delta > 0$. Then,

$$\forall X \in \mathcal{F}_{n,k} : \|X - X^*\|_F^2 \leq \frac{2}{\delta} (f(X) - f(X^*)).$$

**Proof.** Let us write the eigen decomposition of the gradient $\nabla f(X^*)$ as $\nabla f(X^*) = \sum_{i=1}^{n} \lambda_i u_i u_i^\top$. For any $X \in \mathcal{F}_{n,k}$ it holds that:

$$f(X) - f(X^*) \geq \langle X - X^*, \nabla f(X^*) \rangle = \sum_{i=1}^{n} \lambda_i u_i^\top Xu_i - \sum_{i=n-k+1}^{n} \lambda_i$$

$$\geq (\lambda_{n-k+1} + \delta) \sum_{i=1}^{n-k} u_i^\top Xu_i - \sum_{i=n-k+1}^{n} \lambda_i$$

$$= (\lambda_{n-k+1} + \delta) \sum_{i=1}^{n-k} u_i^\top Xu_i - \sum_{i=n-k+1}^{n} \lambda_i$$

$$\geq (\lambda_{n-k+1} + \delta) \sum_{i=1}^{n-k} u_i^\top Xu_i - \lambda_{n-k+1} \sum_{i=n-k+1}^{n} (1 - u_i^\top Xu_i)$$

$$= \lambda_{n-k+1} \sum_{i=1}^{n} u_i^\top Xu_i - k\lambda_{n-k+1} + \delta \sum_{i=1}^{n-k} u_i^\top Xu_i,$$

where (a) follows from the convexity of $f(X)$, (b) follows from since according Lemma 2 and (c) follows from Assumption 1, and (d) follows from $X \preceq I$ which implies $u_i^\top Xu_i \leq u_i^\top u_i = 1$.

Using $\sum_{i=1}^{n} u_i^\top Xu_i = \text{Tr}(X \sum_{i=1}^{n} u_i u_i^\top) = \text{Tr}(XI) = k$ and Eq. (7) we have,

$$f(X) - f(X^*) \geq \delta \sum_{i=1}^{n-k} u_i^\top Xu_i = \delta \left( k - \sum_{i=n-k+1}^{n} u_i^\top Xu_i \right).$$

(8)

Also, using again the fact that $X^* = \sum_{i=n-k+1}^{n} u_i u_i^\top$, we have that

$$\|X - X^*\|_F^2 = \|X\|_F^2 + \|X^*\|_F^2 - 2 \sum_{i=n-k+1}^{n} u_i^\top Xu_i \leq 2 \left( k - \sum_{i=n-k+1}^{n} u_i^\top Xu_i \right),$$

(9)

where the last inequality follows from that for any $X \in \mathcal{F}_{n,k}$ it holds that $\|X\|_F^2 = \sum_{i=1}^{n} \lambda_i^2(X) \leq \sum_{i=1}^{n} \lambda_i(X) = k$. Combining Eq. (8) and (9) we have finally have,

$$f(X) - f(X^*) \geq \frac{\delta}{2} \|X - X^*\|_F^2.$$
Lemma 4. Let $X^\ast \in F_{n,k}$ be an optimal solution to Problem (4) which satisfies Assumption 1 with some parameter $\delta > 0$, and let $\eta > 0$. For any $X \in F_k$ which satisfies

$$
\|\mathbf{X} - \mathbf{X}^\ast\|_F \leq \frac{\eta \delta}{2(1 + \eta \beta)},
$$

it holds that $\text{rank}(\Pi_{F_{n,k}}(\mathbf{X} - \eta \nabla f(\mathbf{X}))) = k$.

Proof. Denote $Y^\ast = X^\ast - \eta \nabla f(X^\ast)$ and its eigenvalues in non-increasing order $\sigma_i = \lambda_i(Y^\ast), i = 1, \ldots, n$. Denote also $Y = X - \eta \nabla f(X)$ with its eigenvalues $\gamma_i = \lambda_i(Y), i = 1, \ldots, n$. Let us write the eigen-decomposition of $-\nabla f(X^\ast)$ as $-\nabla f(X^\ast) = \sum_{i=1}^n \mu_i u_i u_i^\top$. From Lemma 2 we have that under Assumption 1 it holds that $X^\ast = \sum_{i=1}^k u_i u_i^\top$. Thus, we can deduce that

$$
\sigma_i = \begin{cases} 
1 + \eta \mu_i & \text{if } i \in \{1, \ldots, k\}; \\
\eta \mu_i & \text{else.} 
\end{cases} \quad (10)
$$

From Lemma 1 we have that $\text{rank}(\Pi_{F_{n,k}}(Y)) = k$ if and only if $\sum_{i=1}^k \min(\gamma_i - \gamma_{k+1}, 1) \geq k$. Thus, a sufficient condition so that $\text{rank}(\Pi_{F_{n,k}}(Y)) = k$ is

$$
\gamma_k - \gamma_{k+1} \geq 1. \quad (11)
$$

By Weyl’s inequality for the eigenvalues and Eq. (10) we have,

$$
\gamma_k - \gamma_{k+1} = (\sigma_k - \sigma_{k+1}) + (\gamma_k - \sigma_k) + (\sigma_{k+1} - \gamma_{k+1}) \\
\geq 1 + \eta(\mu_k - \mu_{k+1}) - 2\|Y - Y^\ast\|_F \\
= 1 + \eta(\mu_k - \mu_{k+1}) - 2\|X - X^\ast - \eta \nabla f(X) + \eta \nabla f(X^\ast)\|_F \\
\geq 1 + \eta(\mu_k - \mu_{k+1}) - 2(1 + \eta \beta)\|X - X^\ast\|_F.
$$

Thus, we see that a sufficient condition so that $Y$ holds is that $X$ satisfies

$$
\|\mathbf{X} - \mathbf{X}^\ast\|_F \leq \frac{\eta \delta}{2(1 + \eta \beta)} \leq \frac{\eta(\mu_k - \mu_{k+1})}{2(1 + \eta \beta)},
$$

and so the lemma follows. \qed

We can also prove a more general version of Lemma 4 which in particular allows to relax Assumption 1. The following lemma offers a natural trade-off between the rank of the projected gradient mapping and the size of the ball around an optimal solution $X^\ast$ in which it is guaranteed to be upper-bounded.

Lemma 5. Let $X^\ast \in F_{n,k}$ be an optimal solution to Problem (4), and let $\mu_1 \geq \mu_2 \geq \ldots \mu_n$ denote the eigenvalues of $-\nabla f(X^\ast)$. Let $r$ be the smallest integer such that $r \geq k$ and $\mu_k > \mu_{r+1}$. Fix some $\eta > 0$. For any $X \in F_{n,k}$ which satisfies

$$
\|\mathbf{X} - \mathbf{X}^\ast\|_F \leq \frac{\eta(\mu_k - \mu_{r+1})}{2(1 + \eta \beta)}, \quad (12)
$$

it holds that $\text{rank}(\Pi_{F_k}(\mathbf{X} - \eta \nabla f(\mathbf{X}))) \leq r$.

More generally, for any $r' \in \{r, r+1, \ldots, n-1\}$ and for any $\eta > 0$, if $X \in F_{n,k}$ satisfies

$$
\|\mathbf{X} - \mathbf{X}^\ast\|_F \leq \frac{\eta(\mu_k - \mu_{r'+1})}{2(1 + \eta \beta)}, \quad (13)
$$

then $\text{rank}(\Pi_{F_k}(\mathbf{X} - \eta \nabla f(\mathbf{X}))) \leq r'$.
Proof. From Lemma \[2\] we have that \( r^* = \text{rank}(X^*) \leq r \). Denote \( Y^* = X^* - \eta \nabla f(X^*) \) and its eigenvalues \( \sigma_i = \lambda_i(Y^*), i = 1, \ldots, n \). Denote also \( Y = X - \eta \nabla f(X) \) with its eigenvalues \( \gamma_i = \lambda_i(Y), i = 1, \ldots, n \).

From the min-max principle for the eigenvalues, letting \( \mathcal{V} \subseteq \mathbb{R}^n \) denote some subspace of \( \mathbb{R}^n \), we have that for any \( i \in \{1, \ldots, r\} \),

\[
\sigma_i = \min_{\mathcal{V}: \dim(\mathcal{V}) = n-i+1} \max_{v \in \mathcal{V} \cap \|v\| = 1} v^T (X^* + (\eta(-\nabla f(X^*)))v.
\]

(14)

Let us write the eigen-decomposition of \( -\nabla f(X^*) \) as \( -\nabla f(X^*) = \sum_{i=1}^n \mu_i u_i u_i^T \). Note that in Eq. (14) we minimize over all the subspaces \( \mathcal{V} \) of dimension \( n-i+1 \), \( i \leq r \), and so,

\[
\mathcal{V} \cap \text{span}\{u_1, \ldots, u_r\} \neq \emptyset,
\]

(15)

otherwise the direct sum \( \mathcal{V} \oplus \text{span}\{u_1, \ldots, u_r\} \subseteq \mathbb{R}^n \) would have dimension \( n-i+1+r > n \).

Any unit vector \( v \in \mathcal{V} \) can be written as \( v = au + bw \) such that \( u \in \text{span}\{u_1, \ldots, u_r\} \), \( \|u\| = 1 \), \( w \in \text{span}\{u_{r+1}, \ldots, u_n\} \), \( \|w\| = 1 \), and \( a^2 + b^2 = 1 \). Thus, for any such unit vector \( v \), using Lemma \[2\] we have that,

\[
v^T (X^* + (\eta(-\nabla f(X^*)))v = a^2 u^T X^* u + a^2 \eta u^T (\nabla f(X^*))u + b^2 \eta w^T (\nabla f(X^*))w.
\]

(16)

Note that

\[
u^T (\nabla f(X^*))u \geq \mu_r \geq \mu_{r+1} \geq w^T (\nabla f(X^*))w.
\]

(17)

This implies that the inner maximum in (14) can only be obtained by vectors in \( \mathcal{V} \cap \text{span}\{u_1, \ldots, u_r\} \) (note (15) guarantees such vectors exist). Thus, plugging this observation into (14) we have that for any \( i \in \{1, \ldots, r\} \),

\[
\sigma_i = \min_{\mathcal{V}: \dim(\mathcal{V}) = n-i+1} \max_{v \in \mathcal{V} \cap \|v\| = 1} v^T (X^* + (\eta(-\nabla f(X^*)))v
\]

\[
\geq \min_{\mathcal{V}: \dim(\mathcal{V}) = n-i+1} \max_{v \in \mathcal{V} \cap \|v\| = 1} v^T X^* v + \eta \mu_r
\]

\[
= \min_{\mathcal{V}: \dim(\mathcal{V}) = n-i+1} \max_{v \in \mathcal{V} \cap \|v\| = 1} v^T X^* v + \eta \mu_r = \lambda_i(X^*) + \eta \mu_k,
\]

(18)

where (a) follows from the orthogonality of \( X^* \) to \( u_{r+1} u_{r+1}^T, \ldots, u_n u_n^T \) (see Lemma \[2\]), and (c) follows min-max principle for the eigenvalues, and since by definition \( \mu_r = \mu_k \).

Using the max-min principle for eigenvalues, we can write for any \( j \in \{r+1, \ldots, n\} \),

\[
\sigma_j = \max_{\mathcal{V}: \dim(\mathcal{V}) = j} \min_{v \in \mathcal{V} \cap \|v\| = 1} v^T (X^* + (\eta(-\nabla f(X^*)))v.
\]

(19)

This time we maximize over all subspaces of dimension \( j \), \( j \geq r + 1 \). Thus, it must hold that for each such subspace \( \mathcal{V} \),

\[
\mathcal{V} \cap \text{span}\{u_{r+1}, \ldots, u_n\} \neq \emptyset,
\]

otherwise the direct sum \( \mathcal{V} \oplus \text{span}\{u_{r+1}, \ldots, u_n\} \subseteq \mathbb{R}^n \) would have dimension of \( j+n-r > n \). Thus, using (16) and (17), we have that the inner minimum in (19) is obtained by
vectors in \( V \cap \text{span}\{u_{r+1}, \ldots, u_n\} \), which is not an empty set. Using this observation we have that for any \( j \in \{r+1, \ldots, n\} \),

\[
\sigma_j = \max_{V: \text{dim}(V) = j} \min_{v \in V \cap \text{span}\{u_{r+1}, \ldots, u_n\}, \|v\| = 1} \mathbf{v}^\top (X^* + \eta(-\nabla f(X^*))) \mathbf{v} \\
= \max_{(a): \text{dim}(V) = j} \min_{v \in V \cap \text{span}\{u_{r+1}, \ldots, u_n\}, \|v\| = 1} \mathbf{v}^\top (\eta(-\nabla f(X^*))) \mathbf{v} \\
= \max_{(b): \text{dim}(V) = j} \min_{v \in V, \|v\| = 1} \mathbf{v}^\top (\eta(-\nabla f(X^*))) \mathbf{v} = \eta \mu_j, \tag{20}
\]

where (a) follows since \( X^* \) is orthogonal to \( u_{r+1}^\top u_{r+1}, \ldots, u_n^\top u_n \) (see Lemma 2), and (b) follows since by the eigen decomposition of \(-\nabla f(X^*)\), restricting \( v \) to the intersection \( V \cap \text{span}\{u_{r+1}, \ldots, u_n\} \) does not increase the inner minimum.

From Lemma 1 we have the sufficient condition so that \( \text{rank}(\Pi_{F,n,k}(Y)) \leq r \):

\[
\sum_{i=1}^{r} \min(\gamma_i - \gamma_{r+1}, 1) \geq k \implies \text{rank}(\Pi_{F,n,k}(Y)) \leq r. \tag{21}
\]

By Weyl’s inequality we have that for any \( i \in \{1, \ldots, r\} \),

\[
\gamma_i - \gamma_{r+1} \geq \frac{\lambda_i(X^*) + \eta(\mu_i - \mu_{r+1}) - 2(1 + \eta \beta)\|X - X^*\|_F}{2} \geq \frac{\sigma_i - \sigma_{r+1} - 2\|X - \eta \nabla f(X) - X^* + \eta \nabla f(X^*)\|_F}{2} \geq \frac{\sigma_i - \sigma_{r+1} - 2(1 + \eta \beta)\|X - X^*\|_F}{2}. \tag{22}
\]

Thus, we have that

\[
\sum_{i=1}^{r} \min(\gamma_i - \gamma_{r+1}, 1) \geq \sum_{i=1}^{r} \min(\sigma_i - \sigma_{r+1} - 2(1 + \eta \beta)\|X - X^*\|_F, 1) \\
\geq \sum_{i=1}^{r} \min(\lambda_i(X^*) + \eta(\mu_i - \mu_{r+1}) - 2(1 + \eta \beta)\|X - X^*\|_F, 1) \\
\geq \sum_{i=1}^{r} \min(\lambda_i(X^*) + \eta(\mu_i - \mu_{r+1}) - 2(1 + \eta \beta)\|X - X^*\|_F, 1), \tag{23}
\]

where (a) follows from (22), and (c) follows from (18) and (20).

Thus, we indeed see that if

\[
\|X - X^*\|_F \leq \frac{\eta(\mu_r - \mu_{r+1})}{2(1 + \eta \beta)},
\]

then \( \sum_{i=1}^{r} \min(\gamma_i - \gamma_{r+1}, 1) \geq \sum_{i=1}^{r'} \lambda_i(X^*) = k \), which by (21) implies that indeed \( \text{rank}(\Pi_{F,n,k}(Y)) \leq r \), as needed.

For the second part of the lemma let us fix some \( r' \in \{r, ..., n - 1\} \). If we have that

\[
\|X - X^*\|_F \leq \frac{\eta(\mu_r - \mu_{r'+1})}{2(1 + \eta \beta)}, \tag{24}
\]

12
then similarly to (23), we will have that,
\[
\sum_{i=1}^{r'} \min(\gamma_i - \gamma_{r'+1}, 1) \geq \sum_{i=1}^{r} \min(\gamma_i - \gamma_{r'+1}, 1)
\geq \sum_{i=1}^{r} \min(\sigma_i - \sigma_{r'+1} - 2(1 + \eta \beta)\|X - X^*\|_F, 1)
\geq \sum_{i=1}^{r} \min(\lambda_i(X^*) + \eta(\mu_i - \mu_{r'+1}) - 2(1 + \eta \beta)\|X - X^*\|_F, 1)
\geq \sum_{i=1}^{r} \min(\lambda_i(X^*) + \eta(\mu_i - \mu_{r'+1}) - 2(1 + \eta \beta)\|X - X^*\|_F, 1)
\geq \sum_{i=1}^{r} \min(\lambda_i(X^*), 1) = \sum_{i=1}^{r^*} \lambda_i(X^*) = k,
\]
where (a) follows from the same reasoning as (22), (b) follows from (18) and (20), and (c) follows from (24).
Thus, by Lemma 4 we have that (24) indeed implies that rank(Π_{F}(Y)) \leq r', which proves the second part of the lemma.

We can now easily prove Theorems 4 and 6 by proving the following unifying theorem.

**Theorem 7.** Let \{X_t\}_{t \geq 1} be a sequence produced by the projected gradient dynamics w.r.t. the convex Problem 4 with a fixed step-size \eta \in (0, 1/\beta):

\[X_{t+1} = \Pi_{F_n,k}(X_t - \eta \nabla f(X_t)).\]

Fix some optimal solution X^* and let \mu_1 \geq \mu_2 \geq ... \mu_n denote the eigenvalues of -\nabla f(X^*).
Let r be the smallest integer such that r \geq k and \mu_r - \mu_{r+1} > 0. Then rank(X^*) \leq r,
and if the initialization X_1 satisfies \|X_0 - X^*\|_F \leq \frac{\eta(\mu_k - \mu_{r+1})}{2(1 + \eta \beta)} then, for all t \geq 1,
rank(X_{t+1}) \leq r.

More generally, for every r' \in \{r, \ldots, n\}, if \|X_0 - X^*\|_F \leq \frac{\eta(\mu_k - \mu_{r'+1})}{2(1 + \eta \beta)} then, for all t \geq 1,
rank(X_{t+1}) \leq r'.

In particular, if r = k, i.e., Assumption 1 holds with some \delta > 0, and \|X_0 - X^*\|_F \leq \frac{\delta}{4\beta}, setting \eta = 1/\beta guarantees that for all t \geq 1, rank(X_{t+1}) = k, and the sequence \{X_t\}_{t \geq 1} converges linearly with rate:

\[\forall t \geq 1: \quad f(X_t) - f(X^*) \leq (f(X_1) - f(X^*)) \exp(-\Theta(\delta/\beta)(t-1)).\]

**Proof.** The fact that rank(X^*) \leq r follows directly from Lemma 2. We now continue to prove the claims regarding the PGD sequence \{X_t\}_{t \geq 1}. It is a well known fact that the distances of the iterates generated by the projected gradient method with step-size \eta \in (0, 1/\beta) to any optimal solution are monotone non-increasing, i.e., the sequence \{\|X_t - X^*\|_F\}_{t \geq 1} is monotone non-increasing, see for instance [2]. Thus, all results of the theorem regarding the rank of the iterates X_t, t \geq 1, follow immediately from this observation, the initializations listed in the theorem, and Lemma 5.

The linear convergence rate under Assumption 1 follows from the quadratic growth result — Lemma 3 and the known linear convergence rate of the projected gradient method for smooth functions that satisfy the quadratic growth property, see for instance [13].
2.3 Gradient Orthogonal Iteration Analysis

In this section we prove our main algorithmic result — the local linear convergence result of the gradient orthogonal iteration \( (3) \) given in Theorem 3. For convenience, we rewrite the Dynamics \( (3) \) as Algorithm 1 below which also introduces notation that will be helpful throughout the analysis. Throughout this section we also introduce the auxiliary sequence \( \{X_i\}_{i \geq 1} \subset F_{n,k} \) given by \( X_1 = Y_1 \) and \( X_{t+1} = \Pi_{F_{n,k}}[Y_t - \eta \nabla f(Y_t)] \).

At a very high-level our analysis of Algorithm 1 relies on the following two components:

1. With the help of Lemma 4 we can argue that, in the proximity of \( X^* \), \( \text{rank}(X_t) = k \), i.e., \( X_t \in P_{n,k} \). This in particular implies that \( X_t \) is the projection matrix onto the span of top \( k \) eigen-vectors of \( W_t \).

2. We view \( Q_t \) has the outcome of applying one iteration of the classical orthogonal iterations method to the matrix \( W_t \). Combined with the previous point, this allows us to argue that \( Y_t = Q_t Q_t^\top \) is sufficiently close to the exact projected gradient update \( X_t \), which drives the convergence.

Algorithm 1 Gradient Orthogonal Iteration

1: Input: \( Q_1 \in \mathbb{R}^{n \times k} \) such that \( Q_1^\top Q_1 = I \)
2: \( Y_1 \leftarrow Q_1 Q_1^\top \)
3: for \( t = 1 \ldots \) do
4: \( W_{t+1} = Y_t - \eta \nabla f(Y_t) \)
5: \( Q_{t+1} R_{t+1} = W_{t+1} Q_t \) (QR factorization)
6: \( Y_{t+1} = Q_{t+1} Q_{t+1}^\top \)
7: end for

The following key lemma establishes the connection between the sequence \( \{Y_t\}_{t \geq 1} \) produced by Algorithm 1 and the corresponding sequence of exact projected gradient steps \( \{X_t\}_{t \geq 1} \). The proof relies on extension of the classical orthogonal iteration method (see for instance [8, 15]).

Lemma 6. Fix some \( t \geq 1 \). Suppose that \( \eta < 1/G \), \( X_{t+1} \in P_{n,k} \), and \( \|X_{t+1} - Y_t\|_F < \sqrt{2} \). It holds that,

\[
\|X_{t+1} - Y_{t+1}\|_F^2 \leq \frac{1}{1 - \frac{\eta}{2\sqrt{2}}} \left( \frac{\eta G}{1 - \eta G} \right)^2 \|X_{t+1} - Y_t\|_F^2.
\]

Proof. Let us write the eigen decomposition of \( W_{t+1} = Y_t - \eta \nabla f(Y_t) \) as:

\[
W_{t+1} = V \Lambda V^\top = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix},
\]

where \( V_1 \in \mathbb{R}^{n \times k}, \Lambda_1 \in \mathbb{R}^{k \times k} \) correspond to the largest \( k \) eigenvalues.

The main part of the proof will be to prove that

\[
\|V_2^\top Q_{t+1}\|_F^2 \leq \frac{1}{\sigma_{\min}(V_1^\top Q_t)} \left( \frac{\eta G}{1 - \eta G} \right)^2 \|V_2^\top Q_t\|_F^2.
\]

4 orthogonal iterations compute a matrix \( Q \in \mathbb{R}^{n \times k} \) with orthonormal columns, such that \( QQ^\top \) is the projection matrix onto the span of top \( k \) eigenvectors of \( W_t \).
Note that by definition of $X_{t+1}$ we have that,

$$X_{t+1} = \arg \min_{X \in \mathcal{P}_{n,k}} ||X - W_{t+1}||^2_F = \arg \min_{X \in \mathcal{P}_{n,k}} ||X - W_{t+1}||^2_F$$

$$= \arg \max_{X, W_{t+1}} \langle X, W_{t+1} \rangle = V_1 V_1^T,$$

where (a) follows from the assumption of the lemma that $X_{t+1} \in \mathcal{P}_{n,k}$, and (b) follows since all matrices in $\mathcal{P}_{n,k}$ have the same Frobenius norm.

This further implies that

$$\sigma_{\text{min}}^2(V_1^T Q_t) = \lambda_k(V_1^T Q_t Q_1^T V_1)$$

$$= \sum_{i=1}^{k} \lambda_i(V_1^T Q_1 Q_1^T V_1) - \sum_{j=1}^{k-1} \lambda_j(V_1^T Q_1 Q_1^T V_1)$$

$$\geq \text{Tr}((V_1^T Q_1 Q_1^T V_1) - (k - 1) \lambda_1(V_1^T Q_1 Q_1^T V_1))$$

$$\geq \text{Tr}(X_{t+1} Y_t) - (k + 1) = \left(k - \frac{1}{2} ||X_t - Y_t||^2_F \right) - (k - 1)$$

$$= 1 - \frac{1}{2} ||X_{t+1} - Y_t||^2_F. \quad (25)$$

Thus, under the assumption of the lemma that $||X_{t+1} - Y_t||_F < \sqrt{2}$, we have that $(V_1^T Q_t)$ is invertible.

Since $(Q_{t+1}, R_{t+1})$ is the QR factorization of $W_{t+1}Q_t$, using the eigen-decomposition of $W_{t+1}$ we can write $Q_{t+1}R_{t+1} = V \Lambda V^T Q_t$. Multiplying both sides by $V^T$ we get

$$\begin{bmatrix} V_1^T Q_{t+1} \\ V_2^T Q_{t+1} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} V_1^T Q_t \\ V_2^T Q_t \end{bmatrix},$$

which leads to the two equations:

$$\Lambda_1 V_1^T Q_t = V_1^T Q_{t+1} R_{t+1}, \quad (26)$$

$$\Lambda_2 V_2^T Q_t = V_2^T Q_{t+1} R_{t+1}. \quad (27)$$

Under the assumption that $\eta < 1/G$, using Weyl’s inequality we have that $\lambda_k(W_{t+1}) \geq \lambda_k(Y_t) - \eta \lambda_1(V f(Y_t)) > 0$, and so $\Lambda_1$ is invertible. Since following $[23]$ we have that $\sigma_{\text{min}}^2(V_1^T Q_t) > 0$, it follows that $\text{rank}(\Lambda_1 V_1^T Q_t) = k$ and thus, from the equation $[26]$ we have that $V_1^T Q_{t+1}$ and $R_{t+1}$ are both invertible and we can write

$$R_{t+1} = (V_1^T Q_{t+1})^{-1} \Lambda_1 V_1^T Q_t.$$\[ \]

Multiplying equation $[27]$ on both sides from the right with $R_{t+1}^{-1}$, we get

$$V_2^T Q_{t+1} = \Lambda_2 V_2^T Q_t (V_1^T Q_t)^{-1} \Lambda_1^{-1} V_1^T Q_{t+1}.$$\[ \]

Now we can use this to bound $||V_2^T Q_{t+1}||^2_F$:

$$||V_2^T Q_{t+1}||^2_F = ||\Lambda_2 V_2^T Q_t (V_1^T Q_t)^{-1} \Lambda_1^{-1} V_1^T Q_{t+1}||^2_F$$

$$\leq_{(a)} ||(V_1^T Q_t)^{-1} \Lambda_1^{-1} V_1^T Q_{t+1}||^2_F ||\Lambda_2 V_2^T Q_t||^2_F$$

$$\leq_{(b)} ||(V_1^T Q_t)^{-1}||^2_F ||\Lambda_1^{-1}||^2_F ||V_1^T Q_{t+1}||^2_F ||\Lambda_2||^2_F ||V_2^T Q_t||^2_F$$

$$\leq_{(c)} \sigma_{\text{min}}^2(V_1^T Q_t) \left( \frac{\lambda_{k+1}(W_{t+1})}{\lambda_k(W_{t+1})} \right)^2,$$\[ \]

(28)

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where (a) and (b) follow from the inequalities $\|AB\|_F \leq \min\{\|A\|_F\|B\|_2, \|A\|_2\|B\|_F\}$, $\|AB\|_2 \leq \|A\|_2\|B\|_2$, and (c) follows from the eigen-decomposition of $W_{t+1}$ and by noting that since $V_1, Q_{t+1}$ both have orthonormal columns, it holds that $\|V_1^\top Q_{t+1}\|_2 \leq 1$.

We upper-bound $\lambda_{k+1}(W_{t+1})/\lambda_k(W_{t+1})$ by using Weyl’s inequality as follows:

$$\frac{\lambda_{k+1}(W_{t+1})}{\lambda_k(W_{t+1})} \leq \frac{\lambda_{k+1}(Y_t) + \eta \lambda_1(−\nabla f(Y_t))}{\lambda_k(Y_t) + \eta \lambda_n(−\nabla f(Y_t))} \leq \frac{\eta G}{1−\eta G},$$

(29)

where we have used the fact that $Y_t \in P_{n,k}$ and so $\lambda_k(Y_t) = 1, \lambda_{k+1}(Y_t) = 0$.

Plugging (29) into (25) we indeed obtain

$$\|V_{t+1}^\top Q_t\|_F^2 \leq \frac{1}{\sigma_{\min}(V_1^\top Q_t)} \left( \frac{\eta G}{1−\eta G} \right)^2 \|V_t^\top Q_t\|_F^2.$$  

(30)

Now, for the final part of the proof, we note that $\|V_{t+1}^\top Q_t\|_F^2 = \text{Tr}(V_2^\top V_2^\top Y_{t+1}) = \text{Tr}((I−X_{t+1})Y_{t+1}) = k−\text{Tr}(X_{t+1}−Y_{t+1})^2$, and similarly, $\|V_t^\top Q_t\|_F^2 = k−\text{Tr}(X_{t+1}−Y_{t+1})^2$. Plugging these observations and (25) into (30), we obtain the lemma. \qed

Before continuing with the convergence analysis we need two auxiliary lemmas.

**Lemma 7.** Let $M \in S^n$ and let $X \in P_{n,k}$ be the projection matrix onto the span of the top $k$ eigenvectors of $M$. Then, for any $Z \in F_{n,k}$ it holds that

$$\langle X − Z, M \rangle ≤ \|Z − X\|_F^2 \|M\|_2.$$

**Proof.** Let us denote by $X_\perp$ the projection onto the orthogonal subspace, i.e., $X_\perp = I − X$.

It holds that

$$\langle X − Z, M \rangle = \langle X − Z, XM \rangle + \langle X − Z, X_\perp M \rangle = \langle X − Z, XM \rangle − \langle Z, X_\perp M \rangle.$$  

(31)

We consider each of the two terms on the RHS separately.

$$\langle X − Z, XM \rangle \leq \text{Tr}(X(X − Z)X) \cdot \lambda_1(M) \leq \text{Tr}(X(X − Z)X) \cdot \lambda_1(M)$$

(a)

$$= \langle X − Z, X \rangle \cdot \lambda_1(M) = (k − \langle X, X \rangle) \cdot \lambda_1(M) \leq \frac{1}{2} \|Z − X\|_F^2 \cdot \|M\|_2,$$  

(32)

where (a) holds since $X(X − Z)X$ is positive semidefinite.

$$\langle Z, X_\perp M \rangle = \text{Tr}(ZX_\perp M) = \text{Tr}(X_\perp ZX_\perp M)$$

(b)

$$≥ \text{Tr}(X_\perp ZX_\perp) \cdot \lambda_n(M) = \langle Z, X_\perp \rangle \cdot \lambda_n(M)$$

$$= \langle Z, I − X \rangle \cdot \lambda_n(M) = (k − \langle Z, X \rangle) \cdot \lambda_n(M) \geq −\frac{1}{2} \|Z − X\|_F^2 \cdot \|M\|_2,$$  

(33)

where (b) holds since $X_\perp ZX_\perp$ is positive semidefinite.

The lemma follows from plugging (32) and (33) into (31). \qed
Lemma 8. Fix some $t \geq 1$ and suppose $\eta < 1/\beta$. Then, it holds that,

$$\|X_{t+1} - Y_t\|_F^2 \leq \frac{\eta}{1 - \eta\beta} (f(Y_t) - f(X_{t+1})).$$

Proof. Define the following function

$$\phi(Z) := (Z, \nabla f(Y_t)) + \frac{1}{2\eta} \|Z - Y_t\|_F^2,$$

and note that it is $1/\eta$ strongly convex, and that by definition, $X_{t+1}$ is its minimizer over $F_{n,k}$. Thus,

$$\|X_{t+1} - Y_t\|_F^2 \leq 2\eta (\phi(Y_t) - \phi(X_{t+1}))$$

$$= 2\eta (Y_t - X_{t+1}, \nabla f(Y_t)) - \|X_{t+1} - Y_t\|_F^2.$$

Rearranging we get,

$$\|X_{t+1} - Y_t\|_F^2 \leq \eta(Y_t - X_{t+1}, \nabla f(Y_t))$$

$$\leq \eta(Y_t - X_{t+1}, \nabla f(X_{t+1})) + \eta(Y_t - X_{t+1}, \nabla f(Y_t) - \nabla f(X_{t+1}))$$

$$\leq \eta(Y_t - X_{t+1}, \nabla f(X_{t+1})) + \eta\beta\|X_{t+1} - Y_t\|_F^2$$

$$\leq \eta f(Y_t) - f(X_{t+1}) + \eta\beta\|X_{t+1} - Y_t\|_F^2,$$

where (a) follows from the $\beta$-smoothness of $f(\cdot)$, and (b) follows from the convexity of $f(\cdot)$. Rearranging, we get the lemma. \hfill \Box

The following lemma establishes the main step in the proof of the convergence rate of Algorithm 1.

Lemma 9. Let us denote $h_t = f(Y_t) - f(X^*)$ for all $t \geq 1$. Suppose that $\eta \leq \frac{1}{\max\{\beta, C_G\}}$ and suppose that $X_{t+1} \in P_{n,k}$ and that $\|X_{t+1} - Y_t\|_F \leq 1$. Denote the constants

$$C_0 = 2 \left( \frac{\eta G}{1 - \eta G} \right)^2, \quad C_1 = \frac{2(1 + \eta G)C_0}{1 - 2\eta\beta - 2C_0(1 + \eta G)}.$$

It holds that

$$h_{t+1} \leq \left( 1 - \frac{\eta\delta}{4(1 + C_1)} \right) h_t.$$

Proof. Using the $\beta$-smoothness of the $f(X)$, we have that for any $X \in F_{n,k}$ and $\eta \leq \frac{1}{\beta}$ it holds that

$$f(X) \leq f(Y_t) + (X - Y_t, \nabla f(Y_t)) + \frac{\beta}{2} \|X - Y_t\|_F^2$$

$$\leq f(Y_t) + (X - Y_t, \nabla f(Y_t)) + \frac{1}{2\eta} \|X - Y_t\|_F^2$$

$$\leq f(Y_t) + (X - Y_t, \nabla f(Y_t)) + \eta^{-1} \langle Y_t, Y_t - X \rangle$$

$$= f(Y_t) + \eta^{-1} \langle Y_t - X, Y_t - \eta\nabla f(Y_t) \rangle,$$  \quad (34)
where (a) follows since using the fact that $Y_t \in \mathcal{P}_{n,k}$, we have that for any $X \in \mathcal{F}_{n,k}$ it holds that $\|X\|_F^2 \leq k = \|Y_t\|_F^2 = \langle Y_t, Y_t \rangle$.

Since by the assumption of the lemma $X_{t+1} = \Pi_{\mathcal{F}_{n,k}}[Y_t - \eta \nabla f(Y_t)] \in \mathcal{P}_{n,k}$, using Lemma 2 we have that for all $Z \in \mathcal{F}_{n,k}$ it holds that $\langle X_{t+1} - Z, Y_t - \eta \nabla f(Y_t) \rangle \geq 0$. This implies that

$$\forall Z \in \mathcal{F}_{n,k}: \langle Y_{t+1}, Y_t - \eta \nabla f(Y_t) \rangle = \langle X_{t+1}, Y_t - \eta \nabla f(Y_t) \rangle - \langle X_{t+1} - Y_{t+1}, Y_t - \eta \nabla f(Y_t) \rangle \geq \langle Z, Y_t - \eta \nabla f(Y_t) \rangle - \langle X_{t+1} - Y_{t+1}, Y_t - \eta \nabla f(Y_t) \rangle \geq$$

$$\langle Z, Y_t - \eta \nabla f(Y_t) \rangle - \|X_{t+1} - Y_{t+1}\|^2 \|W_{t+1}\|_2,$$

where the last inequality is due to Lemma 7.

Setting $X = Y_{t+1}$ in (34) and plugging-in (35), we have that for any $Z \in \mathcal{F}_{n,k}$ it holds that

$$f(Y_{t+1}) \leq f(Y_t) + \eta^{-1} \left( \|Y_t - Z, Y_t - \eta \nabla f(Y_t)\| + \|X_{t+1} - Y_{t+1}\|^2 \|W_{t+1}\|_2 \right)$$

$$= f(Y_t) + \langle Z - Y_t, \nabla f(Y_t) \rangle + \frac{1}{2\eta} \|Z - Y_t\|_F^2 + \frac{1}{\eta} \|X_{t+1} - Y_{t+1}\|^2 \|W_{t+1}\|_2$$

$$\leq f(Y_t) + \langle Z - Y_t, \nabla f(Y_t) \rangle + \frac{1}{2\eta} \|Z - Y_t\|_F^2 + \frac{1 + \eta G}{\eta} \|X_{t+1} - Y_{t+1}\|^2_F,$$

where the last inequality is due to the following upper-bound on $\|W_{t+1}\|_2$:

$$\|W_{t+1}\|_2 = \|Y_t - \eta \nabla f(Y_t)\|_2 \leq \|Y_t\|_2 + \eta \|\nabla f(Y_t)\|_2 \leq 1 + \eta G.$$

In particular, setting $Z = (1 - \alpha)Y_t + \alpha X^*$ for some $\alpha \in [0, 1]$, we get that

$$f(Y_{t+1}) \leq f(Y_t) + \alpha \langle X^* - Y_t, \nabla f(Y_t) \rangle + \frac{\alpha^2}{2\eta} \|X^* - Y_t\|_F^2 + \frac{1 + \eta G}{\eta} \|X_{t+1} - Y_{t+1}\|^2_F.$$

Subtracting $f(X^*)$ from both sides, using the convexity of $f(\cdot)$, and Lemma 8 gives

$$h_{t+1} \leq \left( 1 - \alpha + \frac{\alpha^2}{\eta \delta} \right) h_t + \frac{1 + \eta G}{\eta} \|X_{t+1} - Y_{t+1}\|^2_F.$$

Setting $\alpha = \eta \delta / 2$ (note that since $\eta \leq 1 / G$, we have that $\alpha \in [0, 1]$), gives

$$h_{t+1} \leq \left( 1 - \frac{\eta \delta}{4} \right) h_t + \frac{1 + \eta G}{\eta} \|X_{t+1} - Y_{t+1}\|^2_F.$$  \hspace{1cm} (37)

We now continue to upper-bound the term $\|X_{t+1} - Y_{t+1}\|^2_F$. Using Lemma 8 we have that

$$\|X_{t+1} - Y_t\|^2_F \leq \frac{\eta}{1 - \eta \beta} (f(Y_t) - f(X_{t+1})).$$

Let us set $Z = X_{t+1}$ in (36) to obtain that

$$f(Y_{t+1}) \leq f(Y_t) + \langle X_{t+1} - Y_t, \nabla f(Y_t) \rangle + \frac{1}{2\eta} \|X_{t+1} - Y_t\|^2_F$$

$$+ \frac{1 + \eta G}{\eta} \|X_{t+1} - Y_{t+1}\|^2_F$$

$$\leq f(X_{t+1}) + \frac{1}{2\eta} \|X_{t+1} - Y_t\|^2_F + \frac{1 + \eta G}{\eta} \|X_{t+1} - Y_{t+1}\|^2_F,$$
where the last inequality is due to the convexity of $f(\cdot)$. Rearranging and using Lemma 6 along with the notation $C_0 = 2 \left( \frac{\eta G}{1 - \eta G} \right)^2$, we have that

$$f(X_{t+1}) \geq f(Y_{t+1}) - \frac{1}{\eta} \left( \frac{1}{2} + C_0(1 + \eta G) \right) \|X_{t+1} - Y_t\|_F^2.$$  

Plugging into (38) we obtain

$$\|X_{t+1} - Y_t\|_F^2 \leq \eta \left( f(Y_t) - f(Y_{t+1}) + \frac{1}{\eta} \left( \frac{1}{2} + C_0(1 + \eta G) \right) \|X_{t+1} - Y_t\|_F^2 \right),$$

and rearranging we obtain

$$\|X_{t+1} - Y_t\|_F^2 \leq \frac{1}{1 - \eta \beta} \left( f(Y_t) - f(Y_{t+1}) \right) \left( \frac{1}{1 - \eta \beta} + C_0(1 + \eta G) \right) \|X_{t+1} - Y_{t+1}\|_F^2.$$

Using Lemma 6 again we have,

$$\|X_{t+1} - Y_{t+1}\|_F^2 \leq \frac{2\eta C_0}{1 - 2\eta \beta - 2C_0(1 + \eta G)} (h_t - h_{t+1}).$$

Plugging back into (37) we obtain

$$h_{t+1} \leq \left( 1 - \frac{\eta \delta}{4} \right) h_t + \frac{2(1 + \eta G)C_0}{1 - 2\eta \beta - 2C_0(1 + \eta G)} (h_t - h_{t+1}).$$

Denoting $C_1 = \frac{2(1 + \eta G)C_0}{1 - 2\eta \beta - 2C_0(1 + \eta G)}$, we finally obtain

$$h_{t+1} \leq \frac{1}{1 + C_1} \left( 1 - \frac{\eta \delta}{4} + C_1 \right) h_t = \left( 1 - \frac{\eta \delta}{4(1 + C_1)} \right) h_t,$$

as required. The only thing left is to choose a feasible step size. We have to require:

$$1 - 2\eta \beta - 2C_0(1 + \eta G) > 0,$$

and the latter holds for any $\eta \leq \frac{1}{5 \max(\beta, G)}$.

We can now prove Theorem 3.

**Proof of Theorem 3**. The theorem follows from Lemma 9, it only remains to prove that the necessary conditions hold for all $t \geq 1$, i.e., that for all $t \geq 1$, it holds that $\|X_{t+1} - Y_t\|_F \leq 1$, and $X_{t+1} \in P_{n,k}$, i.e., $\text{rank}(X_{t+1}) = k$. The proof is by induction. For the base case $t = 1$, we first note that using Lemma 8 we have that,

$$\|X_2 - Y_1\|_F^2 \leq \frac{\eta}{1 - \eta \beta} (f(Y_1) - f(X_2))$$

$$\leq \frac{\eta}{1 - \eta \beta} f(Y_1) - f(X^*)$$

$$\leq \frac{\eta}{1 - \eta \beta} \left( (Y_1 - X^*, \nabla f(X^*)) + \frac{\beta}{2} \|Y_1 - X^*\|_F^2 \right)$$

$$\leq \frac{\eta}{1 - \eta \beta} \left( G + \frac{\beta}{2} \|Y_1 - X^*\|_F^2 \right),$$

(39)
where (a) follows from the $\beta$-smoothness of $f(\cdot)$, and (b) follows from Lemma 7 and recalling that under Assumption 1 $X^*$ is the projection matrix onto the span of the top $k$ eigenvectors of $-\nabla f(X^*)$ (see Lemma 2).

Note that
\[
\eta \frac{1 - \eta \beta}{1 - \eta \beta} \left( G + \beta \right) \leq 1 \iff \eta \left( G + \frac{3\beta}{2} \right) \leq 1,
\]
which clearly holds for our choice of step-size $\eta = \frac{1}{3 \max(\beta, G)}$.

Thus, under the initialization assumption $\|Y_1 - X^*\|_F \leq 1$, it indeed follows that $\|Y_2 - X_1\|_F \leq 1$. Also, combining the initialization condition for $Y_1$ listed in the theorem, together with Lemma 3, which will be essential to proving the local linear convergence of Frank-Wolfe under Assumption 1.

In this section we prove Theorem 5. Our analysis extends the one in [6] which only considered the case $t = 1$ of the induction holds.

Suppose now the induction holds for all $i \in \{1, \ldots, t-1\}$, for some $t \geq 2$, and we will prove it for $t$. Using Lemma 8 we have that,
\[
\|X_{t+1} - Y_t\|_F^2 \leq \frac{\eta}{1 - \eta \beta} \left( f(Y_t) - f(X_t+1) \right) \leq \frac{\eta}{1 - \eta \beta} \left( f(Y_t) - f(X^*) \right)
\]
\[
\leq \left( a \right) \frac{\eta}{1 - \eta \beta} \left( f(Y_t) - f(X^*) \right)
\]
\[
\leq \left( b \right) \frac{\eta}{1 - \eta \beta} \left( G + \frac{\beta}{2} \right) \|Y_1 - X^*\|_F^2,
\]
where (a) follows by using the induction hypothesis for all $i \leq t$ together with Lemma 9 which guarantees that $f(Y_i) \leq f(Y_1)$, and (b) follows from the same steps as in (39).

Since we have already established that the RHS of (40) is upper-bounded by 1 in the base case of the induction, it follows that $\|X_{t+1} - Y_t\|_F \leq 1$.

Using the quadratic growth of $f(\cdot)$ (Lemma 3) we have that,
\[
\|Y_t - X^*\|_F^2 \leq \frac{2}{\delta} \left( f(Y_t) - f(X^*) \right) \leq \frac{2}{\delta} \left( f(Y_1) - f(X^*) \right)
\]
\[
\leq \left( b \right) \frac{2}{\delta} \frac{\eta}{1 - \eta \beta} \left( G + \frac{\beta}{2} \right) \|Y_1 - X^*\|_F^2,
\]
where again, (a) follows by using the induction hypothesis for all $i \leq t$ together with Lemma 9 which guarantees that $f(Y_i) \leq f(Y_1)$, and (b) follows from (39).

Thus, using the fact that for our choice of step-size, $\frac{\eta}{1 - \eta \beta} \left( G + \frac{\beta}{2} \right) \leq 1$, using the initialization assumption on $Y_1$, and invoking Lemma 4 it follows that indeed rank($X_{t+1}$) = $k$, i.e., $X_{t+1} \in P_{n,k}$, and thus the induction holds for step $t$ as well.

\section{Frank-Wolfe Analysis}

In this section we prove Theorem 5. Our analysis extends the one in [6] which only considered the case $k = 1$.

We begin with a lemma, whose proof is similar to the arguments used in the proof of Lemma 3 which will be essential to proving the local linear convergence of Frank-Wolfe under Assumption 1.

\begin{lemma}
Let $X \in F_{n,k}$ and assume that $\lambda_{n-k}(\nabla f(X)) - \lambda_{n-k+1}(\nabla f(X)) \geq \delta_X > 0$. Then, for $V \in \arg \min_{P \in P_{n,k}} (P, \nabla f(X))$ it holds that
\[
\langle X - V, \nabla f(X) \rangle \geq \frac{\delta_X}{2} \|X - V\|_F^2.
\]
\end{lemma}
Proof. The proof follows the same lines as the proof of Lemma 3 but replacing $X$ with $X$, and noting that $V$ is the (unique) projection matrix onto the span of the top $k$ eigenvectors of $-\nabla f(X)$, similarly to use of $X^*$ as the projection matrix onto the span of the top $k$ eigenvectors of $-\nabla f(X^*)$ in the proof of Lemma 3.

Algorithm 2 Frank-Wolfe for Problem (1)

1: $X_1 \leftarrow$ arbitrary point in $\mathcal{F}_{n,k}$
2: for $t = 1$... do
3: $V_t \leftarrow \arg \min_{V \in \mathcal{P}_{n,k}} \langle V, \nabla f(X_t) \rangle$
4: Choose step size $\eta_t \in [0,1]$ using one of the two options:
5: First Option : $\eta_t \leftarrow \arg \min_{\eta \in [0,1]} f((1-\eta)X_t + \eta V_t)$
6: Second Option : $\eta_t \leftarrow \arg \min_{\eta \in [0,1]} f(X_t) + \eta (V_t - X_t, \nabla f(X_t)) + \frac{\eta^2 \beta}{2}\|V_t - X_t\|_F$
7: $X_{t+1} \leftarrow (1-\eta_t)X_t + \eta_t V_t$
8: end for

Theorem 8 (Formal version of Theorem 5). Let $\{X_t\}_{t \geq 1}$ be a sequence produced by Algorithm 2 and denote $\forall t \geq 1$, $h_t := f(X_t) - f(X^*)$. Then

$$\forall t \geq 1 : \quad h_t = O(k\beta/t). \quad (41)$$

In addition, if Assumption 1 holds with parameter $\delta > 0$, then there exists $T_0 = O\left(k(\beta/\delta)^3\right)$ such that

$$\forall t \geq T : \quad h_{t+1} \leq h_t \left(1 - \min\left\{\frac{\delta}{12\beta}, \frac{1}{2}\right\}\right). \quad (42)$$

Finally, under Assumption 1, we have that,

$$\forall t \geq 1 : \quad \|V_t - X^*\|_F^2 = O\left(\frac{\beta^2}{\delta^3 h_t}\right). \quad (43)$$

Proof. Result (41) follows from standard convergence results for the Frank-Wolfe method with line-search [9], and the fact that the Euclidean diameter of the Fantope $\mathcal{F}_{n,k}$ is $\sqrt{2k}$.

For the second part, observe that under Assumption 1 using the the $\beta$-smoothness of $f(\cdot)$, the quadratic growth result (Lemma 3) and (41), we have that for all $t \geq 1$,

$$\|\nabla f(X_t) - \nabla f(X^*)\|_F \leq \beta\|X_t - X^*\| \leq \beta \sqrt{\frac{2ht}{\delta}} = O\left(\sqrt{\frac{k\beta^3}{12\delta}}\right).$$

Thus, for some $T_0 = O\left(k(\beta/\delta)^3\right)$ we have that,

$$\forall t \geq T_0 : \quad \|\nabla f(X_t) - \nabla f(X^*)\|_F \leq \frac{\delta}{3}.$$
Thus, for all $t \geq T_0$, $\lambda_{n-k+1} < \lambda_{n-k}$ and the matrix $V_t$ is uniquely defined and given by $V_t = \sum_{i=n-k+1}^n v_i v_i^T$. Using $X_{t+1} = (1 - \eta_t)X_t + \eta_t V_t$, the smoothness of $f(\cdot)$, and the fact that $\eta_t$ is chosen via line-search, we have that,

$$\forall \eta \in [0, 1] : f(X_{t+1}) \leq f(X_t) + \eta (V_t - X_t, \nabla f(X_t)) + \frac{\eta^2 \beta}{2} \|V_t - X_t\|_F^2.$$ 

Subtracting $f(X^*)$ from both sides and using Lemma[10] with gap $\delta_X = \delta/3$, we have that for all $t \geq T_0$,

$$\forall \eta \in [0, 1] : h_{t+1} \leq h_t + \frac{\eta}{2} \langle V_t - X_t, \nabla f(X_t) \rangle + \left(\frac{\eta^2 \beta}{2} \frac{\eta \delta}{12}\right) \|V_t - X_t\|_F^2$$

$$\leq (1 - \frac{\eta}{2})h_t + \left(\frac{\eta^2 \beta}{2} \frac{\eta \delta}{12}\right) \|V_t - X_t\|_F^2,$$

where the last inequality follows from the convexity of $f(\cdot)$.

Now, if $\frac{\delta}{6 \beta} \leq 1$, by setting $\eta = \frac{\delta}{6 \beta}$ we have that $h_{t+1} \leq (1 - \frac{\delta}{12 \beta})h_t$. Otherwise, $\delta > 6 \beta$ and so, setting $\eta = 1$, we get that $h_{t+1} \leq \frac{1}{2} h_t$, which proves Result (42).

Finally, for the third part of the lemma, recalling that $V_t$ and $X^*$ are the projection matrices onto the span of the top $k$ eigenvectors of $-\nabla f(X_t)$ and $-\nabla f(X^*)$, respectively, using the well known Davis-Kahan sin$\theta$ theorem (see for instance [15]), we have that for all $t \geq 1$,

$$\|V_t - X^*\|_F^2 \leq \frac{8 \|\nabla f(X_t) - \nabla f(X^*)\|_F^2}{(\lambda_{n-k}(\nabla f(X^*)))^2 - \lambda_{n-k+1}(\nabla f(X^*))^2} \leq \frac{8 \beta^2 \|X_t - X^*\|_F^2}{\delta^2} \leq \frac{16 \beta^2 h_t}{\delta^2},$$

where the last inequality follows from the quadratic growth result, Lemma[8] Thus, Result (43) follows.

## 3 Empirical Evidence

The goal of this section is to bring preliminary empirical evidence is support of our theoretical investigation. We focus on 1. demonstrating the plausibility of our main eigen-gap assumption, Assumption[1] on standard setups of robust subspace recovery, and 2. demonstrating the convergence of the nonconvex algorithms considered from simple initializations.

The first robust recovery model we consider is a spiked covariance model, in which we draw a uniformly-distributed projection matrix onto a $k$-dimensional subspace $P \in \mathcal{P}_{n,k}$, and we generate $m$ samples $q_1, \ldots, q_m \in \mathbb{R}^n$ such that for each $i \in [m]$ we set $q_i = Pz_i/\|Pz_i\|$ with probability $1 - p$, and $q_i = z_i$ with probability $p$, where $p \in (0, 0.5]$, and $z_1, \ldots, z_m$ are i.i.d. uniformly distributed unit vectors. The goal is to recover $P$ by minimizing the following objective function over $\mathcal{F}_{n,k}$:

$$f(X) = \sum_{i=1}^m \text{Huber}_\gamma(\|q_i - aXq_i\|), \quad \text{Huber}_\gamma(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \leq \gamma \\ \gamma(|x| - \frac{1}{2} \gamma) & \text{else.} \end{cases}$$

Here $a \in (0, 1]$ is a regularization parameter and we set it to slightly less than one.

The second model we consider is that of sparsely corrupted entries in which again we draw a uniformly-distributed projection matrix $P$. This time the data points $q_1, \ldots, q_m$ are generated by taking $q_i = Pz_i/\|Pz_i\|$ for each $i \in [m]$, where as before $z_1, \ldots, z_m$ are
i.i.d. uniformly distributed unit vectors, but for each \( i \in [m] \), with probability \( p \), we pick a uniformly distributed entry \( j \in [n] \) and set it to \(-1\) or \(+1\) (with equal probability). The goal is to recover \( P \) by minimizing the following objective function over \( F_{n,k} \):

\[
f(X) = \sum_{i=1}^{m} \sum_{j=1}^{n} \text{Huber}_{\gamma}([q_i]_j - [aXq_i]_j),
\]

For both models we set the Huber loss parameter to \( \gamma = 0.1 \). For the first model we set \( a = 0.9 \) and for the second \( a = 0.8 \). For a given projection matrix \( X \in P_{n,k} \), we measure the recovery error according to \( \|X - P\|_F^2 \). For both models we let \( X_{PCA} \in P_{n,k} \) denote the standard PCA solution, i.e., the projection matrix onto the span of the top \( k \) eigenvectors of the empirical covariance \( X = \frac{1}{m} \sum_{i=1}^{m} q_i q_i^T \). For both models we set \( n = 100 \), \( k = 10 \), and \( m = 500 \). For both models we use the projected gradient method to find a projection matrix \( X^* \in P_{n,k} \) which has negligible dual gap \((< 10^{-10})\).\(^5\) For this \( X^* \) we measure the corresponding eigen-gap \( \lambda_{n-k} (\nabla f(X^*)) - \lambda_{n-k+1} (\nabla f(X^*)) \) and the recovery error \( \|X^* - P\|_F \). The results are given in Table 2. For each set of parameters the results are the average of 20 i.i.d. experiments.

As can be seen, for both models the recovery error is significantly lower than that of naive PCA, which demonstrates the usefulness of the chosen models. We can also see that across all parameters, the eigen-gap Assumption \(^1\) indeed holds with substantial values of \( \delta \).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Noise prob. (}p\text{)} & 0.05 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 \\
\hline
\downarrow \text{Model 1: spiked covariance} \downarrow & \hline
\text{Eigen-gap (}\delta\text{)} & 3.21 & 2.87 & 2.36 & 2.04 & 1.501 & 1.03 \\
\text{\|X^* - P\|}_F & 0.0047 & 0.0075 & 0.012 & 0.016 & 0.022 & 0.0298 \\
\text{\|X_{PCA} - P\|}_F & 0.045 & 0.072 & 0.115 & 0.157 & 0.212 & 0.292 \\
\hline
\downarrow \text{Model 2: sparsely corrupted entries} \downarrow & \hline
\text{Eigen-gap (}\delta\text{)} & 5.72 & 5.49 & 5.15 & 4.81 & 4.38 & 3.79 \\
\text{\|X^* - P\|}_F & 0.049 & 0.067 & 0.097 & 0.111 & 0.134 & 0.148 \\
\text{\|X_{PCA} - P\|}_F & 0.148 & 0.199 & 0.291 & 0.335 & 0.401 & 0.439 \\
\hline
\end{array}
\]

Table 2: Recovery and eigen-gap results for the spiked covariance model and corrupted entries model with varying noise probabilities. Each result is the average of 20 i.i.d. experiments.

In a second experiment we fix for both models \( p = 0.1 \) and vary the dimension \( n \) (while keeping \( k, m \) fixed as before). The results are given in Table 3. In particular, we see that the eigen-gap \( \delta \) does not change substantially with the dimension.

We turn to demonstrate the empirical performance of the projected gradient method w.r.t. to the nonconvex set \( P_{n,k} \) (PGD), as given in [2], and and gradient orthogonal iteration (GOI), as given in [3], for the two models discussed above. We fix \( n = 100 \) and \( p = 0.1 \) (keeping all other parameters unchanged). For both methods we heuristically use the step-size \( \eta = 1/\lambda \), where \( \lambda = \lambda_1 (\sum_{i=1}^{m} q_i q_i^T) \), i.e., the largest eigenvalue of the (unnormalized) empirical covariance. We note that smaller values of \( \eta \) seem too conservative in practice from our experimentations. We initialize both methods with the \( k \)-PCA projection matrix \( X_{PCA} \). The results, which are the averages of 20 i.i.d. runs, for

\[\text{For } X \in F_{n,k} \text{ the dual gap is defined as } \text{dg}(X) = \langle X - V, \nabla f(X) \rangle, \text{ where } V \in \arg \min_{Z \in P_{n,k}} \langle Z, \nabla f(X) \rangle. \text{ Since } f(\cdot) \text{ is convex, we in particular have } f(X) - \min_{Y \in F_{n,k}} f(Y) \leq \text{dg}(X).\]
Table 3: Recovery and eigen-gap results for the spiked covariance model and corrupted entries model with varying dimension. Each result is the average of 20 i.i.d. experiments.

| dim. (n) | 100 | 200 | 300 | 400 |
|----------|-----|-----|-----|-----|
| Model 1: spiked covariance ↓ |
| Eigen-gap (δ) | 2.87 | 3.02 | 2.96 | 3.04 |
| ∥X∗ − Q∥_F | 0.0071 | 0.005 | 0.0043 | 0.0035 |
| ∥X_{PCA} − Q∥_F | 0.068 | 0.049 | 0.043 | 0.036 |
| ↓ Model 2: sparsely corrupted entries ↓ |
| Eigen-gap (δ) | 5.49 | 5.902 | 6.06 | 6.1 |
| ∥X∗ − Q∥_F | 0.067 | 0.0617 | 0.058 | 0.055 |
| ∥X_{PCA} − Q∥_F | 0.199 | 0.208 | 0.206 | 0.202 |

The spiked covariance model are given in Figure 1. The results for the sparsely corrupted entries model are very similar and given in Figure 2. Since PGD and GOI exhibit very similar convergence, we only plot the recovery error for PGD and also plot the distance between the iterates of the two methods as function of time. We can indeed observe the rapid convergence of the iterates produced by both methods.

Moreover, in order to verify the convergence of PGD to the global optimal solution (and not just a stationary point of the nonconvex Problem (1)), we verify using the procedure suggested in Remark 3 that on each iteration t, the projection step onto the Fantope \( \mathcal{F}_{n,k} \) is also of rank k, i.e., identical to the projection onto \( \mathcal{P}_{n,k} \). This means that the iterates of PGD w.r.t. the nonconvex Problem (1) and the iterates of PGD w.r.t. the convex relaxation (4), coincide. Indeed, for all random instances generated and for all iterations executed, we observe that the projection onto the Fantope is indeed of rank k. This suggests that the nonconvex PGD (and consequently also GOI) in particular converges to the global optimal solution of the convex relaxation (4).

Figure 1: Convergence of PGD and GOI for the spiked covariance model. Left panel shows recovery error of PGD. Right panel shows distance (in Frobenius norm) between iterates of PGD (\( X_t \)) and those of GOI (\( Y_t \)).
Figure 2: Convergence of PGD and GOI for the sparsely corrupted entries model. Left panel shows recovery error of PGD. Right panel shows distance (in Frobenius norm) between iterates of PGD ($X_t$) and those of GOI ($Y_t$).

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A Proof of Theorem 2

Proof. For a dual solution \((Z_1^*, Z_2^*, s^*)\) we denote \(r_1 = rank(Z_1^*)\) and \(r_2 = rank(Z_2^*)\). First observe that for any dual solution \((Z_1^*, Z_2^*, s^*)\), it holds that \(Z_1^*\) and \(Z_2^*\) are orthogonal to each other. This is true since, denoting by \(X^*\) the corresponding primal solution, we have that,

\[
(Z_1^*, Z_2^*) = \text{Tr}(Z_1^* Z_2^*) = \text{Tr}(Z_1^* (X^* + (I - X^*)Z_2^*))
= \text{Tr}(Z_1^* X^* Z_2^*) + \text{Tr}(Z_1^* (I - X^*) Z_2^*) = 0,
\]

where the last equality follows from the complementarity conditions \(Z_1^* X^* = 0\) and \((I - X^*) Z_2^* = 0\).

Let us write the eigen decompositions \(Z_1^* = \sum_{i=1}^{r_1} \rho_i u_i u_i^T\) and \(Z_2^* = \sum_{j=1}^{r_2} \mu_j v_j v_j^T\).
From the orthogonality of $Z_1^*$ and $Z_2^*$ established above, we get an orthonormal set of vectors \{u_1, \ldots, u_{r_1}, v_1, \ldots, v_{r_2}, w_1, \ldots, w_{n-r_1-r_2}\} and we can complete it to an orthonormal basis of $\mathbb{R}^n$:

$$B = \{u_1, \ldots, u_{r_1}, v_1, \ldots, v_{r_2}, w_1, \ldots, w_{n-r_1-r_2}\},$$

where $Z_1^* w_i = 0$ and $Z_2^* w_i = 0$ for any $i \in \{1, \ldots, n - r_1 - r_2\}$.

From the KKT conditions for Problem (4), we have that

$$\nabla f(X^*) = Z_1^* - Z_2^* + s^* I,$$

and so it follows that any $v \in B$ is an eigenvector of $\nabla f(X^*)$. Thus, we can write the eigenvalues of $\nabla f(X^*)$ in non-increasing order from left to right as:

$$\lambda_{n-k} (< s^* \text{ times}), \ldots, \lambda_{n-k+1} (> s^* - s^* - \mu_k) = \mu_k > 0.$$

For the first direction of the theorem, let us assume $X^*$ satisfies strict complementarity, so for some dual solution $(Z_1^*, Z_2^*, s^*)$ we have that $r_1 = n - k$ or $r_2 = k$.

Now, if $r_1 = n-k$, using (44) we get that $\lambda_{n-k}(< s^*$ and $\lambda_{n-k+1} (> s^*$, and so there is a gap of

$$\lambda_{n-k}(< s^* - s^* - \mu_k) = \mu_k > 0.$$

Otherwise, if $r_2 = k$, then using (44) we have that $\lambda_{n-k+1} (> s^* - \mu_k$ and $\lambda_{n-k} (> s^*$, and so there is a gap of

$$\lambda_{n-k} (> s^* - s^* - \mu_k) = \mu_k > 0.$$

In both cases we get a positive eigen-gap, which proves the first direction of the theorem.

For the reversed direction, let us assume $X^*$ satisfies the eigen-gap assumption, and recall that according to Theorem 1 it follows that $\text{rank}(X^*) = k$. Suppose by way of contradiction that there exists a dual solution $(Z_1^*, Z_2^*, s^*)$ for which $r_1 < n - k$ and $r_2 < k$. In this case we have from (44) that,

$$\lambda_{n-k}(< s^* = \lambda_{n-k+1} (> s^*),$$

which contradicts the existence of an eigen-gap and so, it must be that $r_1 = n - k$ or $r_2 = k$. \qed