Velocity addition in Special Relativity and in Newtonian Mechanics are isomorphic

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Abstract

In the one dimensional case, velocity addition in Special Relativity and in Newtonian Mechanics, respectively, are each a commutative group operation, and the two groups are isomorphic. There are infinitely many such isomorphisms, each indexed by one positive real parameter.

1. Velocity addition in Special Relativity

Let $c > 0$ be the velocity of light in vacuum. Then, as is well known, Angel, in the case of uniform motion along a straight line, the special relativistic addition of velocities is given by

$$(SR) \quad u \ast v = \frac{u + v}{1 + uv/c^2}, \quad u, v \in (-c, c)$$

thus the binary operation $\ast$ acts according to

$$\ast : (-c, c) \times (-c, c) \rightarrow (-c, c)$$

It follows immediately that

1) $\ast$ is associative and commutative
2) \( u \ast v \ast w = (u + v + w + uvw/c^2)/(1 + (uv + uw + vw)/c^2) \)

for \( u, v, w \in (-c, c) \)

3) \( u \ast 0 = 0 \ast u = u, \ u \in (-c, c) \)

4) \( u \ast (-u) = (-u) \ast u = 0, \ u \in (-c, c) \)

5) \( \partial/\partial u(u \ast v) = (1 - v^2/c^2)/(1 + uv/c^2)^2 > 0, \ u, v \in (-c, c) \)

6) \( \lim_{u,v \to c} u \ast v = c, \ \lim_{u,v \to -c} u \ast v = -c \)

Therefore

7) \(((-c, c), \ast)\) is a commutative group with the neutral element 0, while \(-u\) is the inverse element of \(u \in (-c, c)\)

2. Velocity addition in Newtonian Mechanics

As is well known, in the case of uniform motion along a straight line, the addition of velocities in Newtonian Mechanics is given by

\[
(NM) \quad x + y, \ x, y \in \mathbb{R}
\]

thus it is described by the usual additive group \((\mathbb{R}, +)\) of the real numbers, a group which is of course commutative, with the neutral element 0, while \(-x\) is the inverse element of \(x \in \mathbb{R}\).

3. Isomorphisms of the two groups

8) \(((-c, c), \ast)\) and \((\mathbb{R}, +)\) are isomorphic groups through the mappings

8.1) \( \alpha : (-c, c) \longrightarrow \mathbb{R}, \) where

\[\alpha(u) = k \ln((c + u)/(c - u)), \ u \in (-c, c)\]
and

8.2) \( \beta : \mathbb{R} \rightarrow (-c, c) \), where

\[ \beta(x) = \frac{c(e^{x/k} - 1)}{(e^{x/k} + 1)}, \quad x \in \mathbb{R} \]

with

8.3) \( k = c^2\alpha'(0) > 0 \)

**Proof of 8)**

Let us first find \( \alpha \). According to the standard definition of group homomorphism, we have

\( \alpha \) group homomorphism \( \iff \alpha(u \ast v) = \alpha(u) + \alpha(v) \), \( u, v \in (-c, c) \)

Thus it follows that

\[ \alpha(u \ast v) - \alpha(u) = \alpha(v), \quad u, v \in (-c, c) \]

and since the right hand term does not depend on \( u \), we conclude that neither does the left hand term. Consequently, assuming that \( \alpha \) has a derivative on its domain of definition \( (-c, c) \), we obtain

\[ \frac{d}{du} (\alpha(u \ast v) - \alpha(u)) = 0, \quad u, v \in (-c, c) \]

or in view of (SR) and 5), the relation follows

\[ \alpha'((u + v)/(1 + uv/c^2))((1 - v^2/c^2)/(1 + uv/c^2)) = \alpha'(u) \]

for \( u, v \in (-c, c) \)

Taking now \( u = 0 \), one obtains

\[ \alpha'(v)(1 - v^2/c^2) = \alpha'(0), \quad v \in (-c, c) \]
\[ \alpha'(v) = \frac{c^2\alpha'(0)}{c^2 - v^2}, \quad v \in (-c, c) \]

Thus, since \( \alpha(0) = 0 \) results form the fact that \( \alpha \) is assumed to be a group homomorphism, one obtains

\[ \alpha(u) = \alpha(0) + c^2\alpha'(0) \int_0^u \frac{dv}{c^2 - v^2} = \frac{c^2\alpha'(0)}{c^2 - v^2} = c^2\alpha'(0) \ln((c + u)/(c - u)), \quad u \in (-c, c) \]

in other words, (8.1) and (8.3). And since obviously the resulting \( \alpha \) in (8.1) is a bijective mapping, it follows that it is not only a group homomorphism, but also a group isomorphism. In this way, its inverse mapping \( \beta = \alpha^{-1} \) exists and it is also a group isomorphism. Finally, a simple computation based on (8.1) will then give (8.2).

4. Note

The special relativistic addition \( * \) of velocities in (SR) is in fact well defined not only for pairs of velocities

\[ (u, v) \in (-c, c) \times (-c, c) \]

but also for the larger set of pairs of velocities

\[ (u, v) \in [-c, c] \times [-c, c], \quad uv \neq -c^2 \]

This corresponds to the fact that in Special Relativity the velocity \( c \) of light in vacuum is supposed to be attainable.

On the other hand, the Newtonian addition \( + \) of velocities (NM) does of course only make sense physically for

\[ (x, y) \in \mathbb{R} \times \mathbb{R} \]

since infinite velocities are not supposed to be attainable physically.
As for the group isomorphisms $\alpha$ and $\beta$, they only generate mappings between pairs of velocities in

$$(−c, c) \times (−c, c) \xrightarrow{\alpha} \mathbb{R} \times \mathbb{R}$$

and

$$\mathbb{R} \times \mathbb{R} \xrightarrow{\beta} (−c, c) \times (−c, c)$$

thus they do not cover the cases of addition $u * v$ of special relativistic velocities $u = −c$ and $v < c$, or $−c < u$, and $v = c$.

Consequently, in spite of the group isomorphisms $\alpha$ and $\beta$, there is an essential difference between the addition of velocities in Special Relativity, and on the other hand, Newtonian Mechanics. Indeed, in the latter case, the addition $+$ is defined on the open set $\mathbb{R} \times \mathbb{R}$, while in the former case the addition $\ast$ is defined on the set

$$\{ (u, v) \mid −c \leq u, v \leq c, \ uv \neq −c^2 \}$$

which is neither open, nor closed.

5. The uniqueness of the velocity addition in Special Relativity

In Benz, it has recently be shown that under very general and mild conditions, the formula (SR) is uniquely determined, even if one starts with motions not along a straight line, but in arbitrary, thus possibly infinite dimensional pre-Hilbert spaces as well.

Reference

1. Angel, Roger B : Relativity, The Theory and its Philosophy. Pergamon, New York, 1980

2. Benz, W : A characterization of relativistic addition. Abh. Math.
