Abstract

We address the issue of potential superluminal propagation of gravitational waves in backgrounds neighbouring the previously suggested bounce [1] in beyond Horndeski theory. We find that the bouncing solution lies right at the boundary of the region where the gravitational waves propagate at speed exceeding that of light, i.e. that solution suffers superluminality problem. We suggest a novel version of a completely stable bouncing model where both scalar and tensor perturbations remain safely subluminal not only on the solution itself but also in its neighbourhood.

1 Introduction

Recent studies have shown that Horndeski theories [2] and their extensions [3, 4] offer a remarkable framework for tackling various cosmological issues such as late time accelerated expansion of the Universe, or, notably, initial singularity problem (for a review see, e.g., Ref. [5]). A particularly attractive feature of these scalar-tensor theories is their ability to allow for the Null Energy Condition (NEC) violation 1 with no catastrophic consequences for the stability of the solutions, as reviewed, e.g., in Ref. [6]. For this reason, Horndeski and beyond Horndeski theories have been extensively used for constructing spatially flat FLRW cosmological scenarios with non-standard

1In fact, when gravity is modified, the NEC is replaced with the Null Convergence Condition (NCC) [8].
dynamics, e.g., the bouncing Universe and the Universe starting off with Genesis (see Refs. [5, 7] and references therein).

However, bouncing and Genesis models in unextended Horndeski theories still have severe problems related to stability. Namely, in these theories, the absence of ghosts and gradient instabilities at all times in a bouncing or Genesis cosmology can be achieved at a price of potential strong coupling and/or geodesic incompleteness; examples are given, e.g., in Refs. [9, 10, 11] and the no-go theorem is proven in Refs. [9, 10]. What saved the day was beyond Horndeski theories where geodesically complete bouncing and Genesis solutions were constructed; these solutions are stable during entire evolution without the risk of strong coupling [12, 13, 14, 15, 1, 16, 17]. Similar construction [17, 18] was given in the context of more general DHOST theories [19, 20].

Another issue to worry about in modified gravities, including Horndeski theories and their extensions, is the danger of superluminality. This issue has been discussed from various viewpoints in Refs. [21, 22, 23, 24, 25, 26] and references therein. It is generally accepted that superluminal propagation in any background is an undesirable feature that signals that the theory cannot descend, as low energy effective field theory, from any UV-complete, Lorentz-covariant theory 2. In practice (and in this paper in particular), one often does not pretend to define the Lagrangian in the entire phase space; one usually keeps only those terms in the Lagrangian that are sufficient for obtaining the solution and analyzing its stability (i.e., terms that do not vanish on the pertinent solution and its close neighbourhood). In that case, the absence of superluminality is required for perturbations about the cosmological solution of interest and in its vicinity.

It is worth noting that the superluminality and stability problems are not directly related to each other: superluminality may occur in a vicinity of a stable part of solution (like in Refs. [23, 26, 28]). So, the superluminality issue requires separate analysis in any cosmological model in (beyond) Horndeski theory.

Recent examples of stable bouncing and Genesis solutions given in Refs. [1, 16] have been constructed in such a way that there are no superluminal modes about the solution during entire evolution. However, the superluminality issue has not been analyzed even in the vicinity of these solutions. In this paper we fill this gap. Namely, we calculate the speed of perturbations about the homogeneous and isotropic background in the phase space around the completely stable bouncing solution of Ref. [1] and show that in the vicinity of this solution, there exists a domain of the phase space where tensor modes are superluminal. So, the bouncing solution of Ref. [1] does suffer superluminality problem. This drawback can be cured, however. We modify the Lagrangian of Ref. [1] in such a way that the bouncing solution still exists and is stable throughout the entire evolution, while both scalar and tensor sectors are safely subluminal at all times in a vicinity of the solution. It is worth noting that there are examples of beyond Horndeski and DHOST models with stable bounces where the speed of tensor modes is identically equal to 1 [17, 18].

2In this paper, as well as in most other papers where the (super)luminality issue is discussed, it is assumed that ordinary light propagates in metric $g_{\mu\nu}$ (entering the action (1)). This point is not entirely trivial, because the light cones are different for metric $g_{\mu\nu}$ and for $\tilde{g}_{\mu\nu}$ related to $g_{\mu\nu}$ by disformal transformation [27] $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \Gamma(\pi, X)\partial_{\mu}\pi\partial_{\nu}\pi$. Therefore, the notion of Lorentz covariance needs qualification: it refers here to the theory where light feels the metric $g_{\mu\nu}$. 
This paper is organized as follows. In Sec. 2.1 we introduce the Lagrangian of beyond Horndeski theory considered in Ref. [1] and revisit the stability and subluminality conditions. In Sec. 2.2 we study the bouncing solution of Ref. [1] and show that the gravitational waves exhibit superluminal propagation in the vicinity of the solution. In Sec. 3 we construct a modified version of a completely stable bouncing model where both scalar and tensor perturbations remain safely subluminal not only on the solution itself but also in its neighbourhood. We conclude in Sec. 4.

2 Superluminality near the original bouncing solution

2.1 Stability and subluminality conditions in beyond Horndeski theory

Here we set up the notations and recall the general form of the stability conditions for cosmological solutions in (beyond) Horndeski theory [29, 14, 1]. Like in Ref. [1], it is sufficient for our purposes to consider a subclass of beyond Horndeski theory, whose Lagrangian reads

\[ \mathcal{L}(F,G,F_4) = F(\pi,X) - G_4(\pi,X)R + (2G_{4X}(\pi,X) - F_4(\pi,X)X) \left[ (\Box \pi)^2 - \pi_{;\mu\nu}\pi^{\mu\nu} \right] + 2F_4(\pi,X) \left[ \pi^{;\mu\nu}\pi^{;\nu\lambda} - \pi^{;\mu\lambda} \pi^{;\nu\nu} \right], \]  

where \( \pi \) is the scalar field, \( X = g_{\mu\nu}\pi^{;\mu\nu} \), \( \dot{\pi} = \partial_\mu \pi \), \( \nabla_\nu = \partial_\nu \), \( \Box = \nabla_\mu \nabla^\mu \), \( G_{4X} = \partial G_4/\partial X \). The functions \( F \) and \( G_4 \) are characteristic of the Horndeski theories, while non-vanishing \( F_4 \) extends the theory to beyond Horndeski type. In a cosmological context, we consider a homogeneous background scalar field \( \pi = \pi(t) \) and a spatially flat FLRW background metric

\[ ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j. \]  

The gravitational equations for the Lagrangian (1) read:

\[ \delta g^{00} : F - 2F_4 X + 6H^2 G_4 + 6HG_4 \dot{\pi} - 24H^2 X(G_{4X} + G_{4XX} X) + 12HG_{4\pi X} X \dot{\pi} \]
\[ + 6H^2 X^2(5F_4 + 2F_{4X} X) = 0, \]  

\[ \delta g^{ij} : F + 2(3H^2 + 2\dot{H}) G_4 - 12H^2 G_{4X} X - 8\dot{H} G_{4X} X - 8HG_{4\pi X} \ddot{\pi} - 16HG_{4XX} X \dddot{\pi} \]
\[ + 2(\ddot{\pi} + 2H \dot{\pi}) G_{4\pi} + 4XG_{4\pi X}(\ddot{\pi} - 2H \dot{\pi}) + 2XG_{4\pi X} + 2F_4 X(3H^2 X + 2\dot{H} X + 8H \ddot{\pi}) \]
\[ + 8HF_{4X} X^2 \ddot{\pi} + 4HF_{4\pi} X^2 \dddot{\pi} = 0, \]  

where \( H = \dot{a}/a \) is the Hubble parameter. The scalar field equation is a consequence of eqs. (3).

To find whether perturbations about the homogeneous and isotropic solution are stable and subluminal, one considers them at a linearized level. One adopts the ADM parametrization for the metric perturbations

\[ ds^2 = N^2 dt^2 - \gamma_{ij}(dx^i + N_i dt)(dx^j + N_j dt), \]  

where \( \gamma_{ij} \) is the perturbed metric and \( N \) is the lapse function.
where

\[ N = 1 + \alpha, \quad N_i = \partial_i \beta, \quad \gamma_{ij} = a^2(t) e^{2\zeta} \left( \delta_{ij} + h^T_{ij} + \frac{1}{2} h^T_{ik} h^T_{kj} \right). \]  

(5)

In eq. (5) \( \alpha, \beta \) and \( \zeta \) belong to the scalar sector of perturbations, while \( h^T_{ik} \) denote transverse traceless tensor modes (\( h^T_{ii} = 0, \partial_i h^T_{ij} = 0 \)). Below we utilize the unitary gauge approach (\( \delta \pi = 0 \)), where a dynamical DOF in the scalar sector is the curvature perturbation \( \zeta \). Upon integrating out non-dynamical \( \alpha \) and \( \beta \), one arrives at the unconstrained quadratic action for metric perturbations (see Refs. [29, 14, 1] for details)

\[ S = \int dt d^3x \ a^3 \left[ \frac{G_T}{8} \left( \dot{h}^T_{ij} \right)^2 - \frac{F_T}{8a^2} \left( \nabla h^T_{ij} \right)^2 + G_S \zeta^2 - F_S \frac{\left( \nabla \zeta \right)^2}{a^2} \right], \]  

(6)

with \( \left( \nabla \zeta \right)^2 = \delta^{ij} \partial_i \zeta \partial_j \zeta \), \( \triangle = \delta^{ij} \partial_i \partial_j \) and

\[ G_S = \frac{\Sigma G_T^2}{\Theta^2} + 3G_T, \]  

(7)

\[ F_S = \frac{1}{a} \frac{d\xi}{dt} - F_T, \]  

(8)

\[ \xi = a \left( G_T + D \dot{\pi} \right) G_T, \]  

(9)

while the coefficients \( G_T, F_T, D, \Theta \) and \( \Sigma \) are expressed in terms of the Lagrangian functions as follows:

\[ G_T = 2G_4 - 4G_{4X}X + 2F_4X^2, \quad F_T = 2G_4, \quad D = -2F_4X \dot{\pi}, \]  

(10)

\[ \Theta = 2G_4H - 8HG_{4X}X - 8HG_{4XX}X^2 + G_{4\pi} \dot{\pi} + 2G_{4\pi X}X \dot{\pi} + 10HF_4X^2 + 4HF_{4X}X^3, \]  

(11)

\[ \Sigma = F_X X + 2F_{XX}X^2 - 6H^2G_4 + 42H^2G_{4X}X + 96H^2G_{4XX}X^2 + 24H^2G_{4XX}X^3 \]  

- \( 6H G_{4\pi} \dot{\pi} - 30HG_{4\pi X}X \dot{\pi} - 12HG_{4\pi XX}X^2 \dot{\pi} - 90H^2F_4X^2 - 78H^2F_{4X}X^3 \)  

- \( 12H^2F_{4XX}X^4 \). \]  

The stability conditions for a cosmological solution immediately follow from the expression for the action (6):

\[ G_T, F_T > \epsilon > 0, \quad G_S, F_S > \epsilon > 0, \]  

(13)

where \( \epsilon \) is a positive constant which ensures that there is no naive strong coupling, i.e. \( G_{S,T} \not\to 0 \) and/or \( F_{S,T} \not\to 0 \) at any time including asymptotics \( t \to \pm \infty \) (see Refs. [14, 1] for discussion). The absence of superluminal propagation at any time is expressed in terms of sound speeds of scalar and tensor modes as follows:

\[ c^2_T = \frac{F_T}{G_T} \leq 1, \quad c^2_S = \frac{F_S}{G_S} \leq 1. \]  

(14)

As we discussed in Sec. 1, the minimal subluminality requirement is that these inequalities are satisfied for a background solution of interest and nearby solutions to eqs. (3) as well.
2.2 Original bouncing solution and superluminality

The bouncing model of Ref. [1] complies with the stability conditions (13) at all times, thus, we claimed that it is completely stable. Also, the inequalities (14) are satisfied at all times for the bouncing solution itself. On top of that, the solution has simple asymptotics as \( t \to \pm \infty \): the beyond Horndeski theory boils down to GR + conventional massless scalar field long before and long after the bouncing epoch. The latter property is possible only provided \( \Theta \) in eq. (9) crosses zero at some moment of time, see Ref. [1] for a detailed discussion. This is the so-called \( \gamma \)-crossing phenomenon \(^3\), which is completely harmless [30] and corresponds to a transition between the branches of solutions for the Hubble parameter \( H \) in eq. (3a) [26].

However, a dangerous feature of the bouncing solution of Ref. [1] is that gravitational waves propagate strictly at the speed of light throughout the entire evolution \(^4\), i.e., \( c^2 T(T) = 1 \) for all \( t \). The latter feature is due to a deliberate choice of \( G_T = F_T = 1 \) for all \( t \) on the solution, which simplified the construction procedure. This property may or may not signal that the tensor perturbations are superluminal in a vicinity of the bouncing solution, which is the case if the solution happens to be just at the boundary (in phase space) between regions with subluminal and superluminal gravitational waves. Let us study this issue.

We begin with a brief description of the solution of Ref. [1]. It has the following Hubble parameter, with the bounce at \( t = 0 \),

\[
H(t) = \frac{t}{3(\tau^2 + t^2)}, \quad a(t) = (\tau^2 + t^2)^{\frac{1}{6}}, \quad (15)
\]

where the parameter \( \tau \) regulates the duration of the bouncing epoch. The background scalar field was chosen as \( \pi(t) = t \), which is always possible to achieve on a single solution by field redefinition. Hence, in the FLRW background (2) we have \( X = g^{\mu \nu} \pi_\mu \pi_\nu = 1 \) on this solution. The explicit bouncing solution of Ref. [1] has been obtained by making use of the so-called reconstruction procedure: for a chosen evolution of the Hubble parameter (15), we found an explicit Lagrangian of the form (1) by making use of the stability conditions (13) and background gravitational equations (3) (see Ref. [1] for a detailed description). Namely, we take the Ansatz for the Lagrangian functions in eq. (6) in terms of powers of \( X \):

\[
F(\pi, X) = f_0(\pi) + f_1(\pi) \cdot X + f_2(\pi) \cdot X^2 \quad (16a)
\]

\[
G_4(\pi, X) = \frac{1}{2} + g_{40}(\pi) + g_{41}(\pi) \cdot X, \quad (16b)
\]

\[
F_4(\pi, X) = f_{40}(\pi) + f_{41}(\pi) \cdot X. \quad (16c)
\]

Note that this Ansatz defines the Lagrangian only on the solution \( X = 1 \) and its close vicinity. The following steps amount to (i) choosing the functions \( f_{4i}(\pi), g_{4i}(\pi) \) \( (i = 0, 1) \) and \( f_2(\pi) \) in such a way that \( G_T = F_T = 1 \) and the stability conditions (13) hold; (ii) the remaining functions

---

\(^3\)The name of the phenomenon originates from Refs. [30, 26], where the coefficient \( \Theta \) is denoted by \( \gamma \).

\(^4\)Let us note that a setup suggested within the EFT approach in Ref. [13] may be also problematic due to strictly luminal propagation of gravitational waves.
$f_0(\pi)$ and $f_1(\pi)$ are found from the two independent background gravitational equations (3). An additional requirement imposed on the Lagrangian functions in Ref. [1] was that the beyond Horndeski theory (1) reduces to GR + conventional massless scalar field in both asymptotic past and asymptotic future. This implies that $F(\pi, X) \rightarrow X/(3\pi^2)$ (the field $\phi \propto \log \pi$ is a canonical massless scalar field), $G_4(\pi, X) \rightarrow 1/2$ and $F_4(\pi, X) \rightarrow 0$ as $\pi = t \rightarrow \pm \infty$ (hereafter we set $M_{Pl}^2/8\pi = 1$). Clearly, there remains substantial arbitrariness in the choice of the functions $f_0(\pi), \ldots, f_{41}(\pi)$; one example is given in Appendix A.

Let us now turn to the propagation speeds of perturbations about the homogeneous backgrounds in the vicinity of the original solution. We parametrize the phase space of homogeneous solutions by $\pi$ and $\dot{\pi}$, then the original solution is a line $\dot{\pi} = 1$, $\pi \in (-\infty, +\infty)$, and its vicinity is a strip along this line. The background equation (3a) is used to determine $H$ in terms of $\pi$ and $\dot{\pi}$. Note that eq. (3a) is a quadratic equation for $H$, so there are regions in phase space $(\pi, \dot{\pi})$ where the solution does not exist. The values of $\dot{H}$ and $\ddot{\pi}$ are obtained, for given $\pi$, $\dot{\pi}$, from eq. (3b)

![Figure 1: Variance $(1 - c_T^2(\pi, \dot{\pi}))$ of the speed squared of tensor modes in the phase space $(\pi, \dot{\pi})$ for the original model with bouncing solution [1]. Dashed lines are lines of constant $(1 - c_T^2(\pi, \dot{\pi}))$. White region is the one where the solution does not exist, see text. A black horizontal line corresponds to the original bounce with $\pi(t) = t$ (hence, $\dot{\pi} = 1$) and $c_T^2 = 1$. Negative values of $(1 - c_T^2(\pi, \dot{\pi}))$ (hatched region) mean superluminal propagation. The original solution lies right on the verge of the domain with superluminal tensor modes.](image)
Figure 2: [color online] Scalar sound speed squared $c^2_S(\pi, \dot{\pi})$ in the phase space $(\pi, \dot{\pi})$ for the original model with bouncing solution [1]. Dashed lines are lines of constant $c^2_S(\pi, \dot{\pi})$. As in Fig. 1, there are no solutions in the white region. Negative values of $c^2_S(\pi, \dot{\pi})$ (hatched region) mean gradient instability. For small $\pi$, the solution manages to safely pass between the white forbidden region and the hatched one with $c^2_S < 0$.

and time derivative of eq. (3a). We plug $H$, $\dot{H}$ and $\ddot{\pi}$ in eqs. (7) – (14) and obtain the desired propagation speeds $c^2_T(\pi, \dot{\pi})$ and $c^2_S(\pi, \dot{\pi})$.

The explicit expressions for $c^2_T(\pi, \dot{\pi})$ and $c^2_S(\pi, \dot{\pi})$ are cumbersome, and we do not give them here. The speed squared of gravitational waves $c^2_T(\pi, \dot{\pi})$ as a function of phase space points is shown in Fig. 1. We see that there are regions (with $\dot{\pi} < 1$) in the immediate vicinity of the bouncing solution $\dot{\pi} = 1$, where the propagation of gravitational waves is superluminal. Thus, the original model of Ref [1] is unsatisfactory, as it does not obey the requirement of the absence of superluminality.

We present for completeness the sound speed of scalar perturbations in Fig. 2. There is no superluminality in a vicinity of the original solution $\dot{\pi} = 1$.

Thus, choosing the Ansatz (16) and requiring $G_T = F_T = 1$ on the solution $\pi(t) = t$ is not a good idea. In the next Section we discuss ways of avoiding the superluminality problem and suggest a modified version of the stable bouncing solution, which has in its vicinity safely subluminal perturbations in both tensor and scalar sector.
3 Subluminal bouncing solution

Superficially, one can think of two approaches to avoid superluminality in the tensor sector. The first one amounts to changing the Ansatz (16c) for the Lagrangian so that the function $F_4$ involves a quadratic contribution $f_{42}(\pi) \cdot X^2$. Then it might be possible to still take $G_T = F_T = 1$ on the solution $X = 1$, while making $G_T < 1$, and hence $c_T^2 < 1$, in a vicinity of the solution by adjusting the new function $f_{42}(\pi)$. This is conceivable, since $f_{42}(\pi)$ contributes to $G_T$ but not to $F_T$, see eqs. (10). It appears, however, that this approach requires fine-tuning of the functions in the Ansatz. Moreover, according to eqs. (7) and (8), both $G_S$ and $F_S$ would involve $f_{42}(\pi)$, hence, one would have to take care of the subluminal propagation and absence of instabilities in the scalar sector (note that originally the scalar modes were automatically subluminal).

Another way to deal with the superluminality threat is to abandon the requirement that $c_T^2 = 1$ on the bouncing solution. Let us follow this route, i.e. construct a model with the stable bouncing solution that has $c_T^2 < 1$. Then a small deviation from the solution (no matter with or without the external sources) will not have superluminal excitations too. At the same time we make sure that $c_T^2(t)|_{t \to \pm \infty} \to 1$ in accordance with GR form of the asymptotics. Needless to say, our construction involves the Lagrangian functions different from those in Sec. 2.2.

![Figure 3: The speed squared of tensor perturbations is non-negative, smaller than 1 for all times and asymptotically tends to 1 in both infinite past and future. Right panel shows the behaviour of the coefficient $G_T$ in eq. (18) with $u = 1/10$, $w = 1$ and $\tau = 10.$](image)

We still take the Hubble parameter in the form (15), a rolling scalar field $\pi(t) = t$ and now choose, instead of (16), the Ansatz (this is a matter of convenience)

$$F(\pi,X) = f_1(\pi) \cdot X + f_2(\pi) \cdot X^2 + f_3(\pi) \cdot X^3,$$

(17a)

$$G_4(\pi,X) = \frac{1}{2} + g_{40}(\pi) + g_{41}(\pi) \cdot X,$$

(17b)

$$F_4(\pi,X) = f_{40}(\pi) + f_{41}(\pi) \cdot X.$$

(17c)

In contrast to our previous construction we keep only $F_T = 1$, while taking

$$G_T = 1 + \frac{5w}{2} \text{sech} \left( \frac{t}{\tau} + u \right) - 2 \text{sech}^2 \left( \frac{t}{\tau} + u \right),$$

(18)
with an arbitrary parameters $u$ and $w$ (we use $t$ and $\pi$ interchangeably as long as the bouncing solution is discussed). The choice in eq. (18) makes $c^2 T(\pi) = \mathcal{F}_T / \mathcal{G}_T$ behave as shown in Fig. 3, where we also plot $\mathcal{G}_T$. We skip the further reconstruction steps, since they are basically the same as in Ref. [1], and give the results only.

The reconstructed Lagrangian functions (17) are shown in Fig. 4, while the analytical expressions are given in Appendix B. The asymptotic behavior of the functions as $t \to \pm \infty$ is as follows:

$$f_1(t) = \frac{1}{3t^2}, \quad f_0(t) = f_2(t) \propto \frac{1}{t^4}, \quad g_{40}(t) = g_{41}(t) \propto e^{-2|t|/\tau}, \quad f_{40}(t) = f_{41}(t) \propto e^{-|t|/\tau},$$

which indeed corresponds to the asymptotic Lagrangian (see eqs. (17) and (1))

$$\mathcal{L}_{t \to \pm \infty} = -\frac{1}{2} R + \frac{X}{3\pi^2},$$

and describes GR and a conventional massless scalar field $\phi = \sqrt{\frac{2}{3}} \log(\pi)$ in both distant past and future. We plot corresponding $\mathcal{G}_S$, $\mathcal{F}_S$ and $c_S^2$ as functions of $\pi = t$ in Fig. 5 in order to illustrate that the new bouncing solution does not involve ghost and gradient instabilities as well as superluminal modes in the scalar sector at all times.
Figure 5: The coefficients $G_S$ and $F_S$ (top panel) and $c_S^2$ (bottom panels). The sound speed squared for the scalar perturbations is non-negative for all times and asymptotically tends to 1 in both infinite past and future.

Figs. 6 and 7 illustrate that the superluminality issue is indeed resolved: the new bouncing solution is safely far away from the superluminal regions in $(\pi, \dot{\pi})$ plane for both tensor and scalar modes. A peculiar property of our set up is that the speed of tensor perturbations tends to 1 as $t \to \pm\infty$, and yet there is always a finite gap in the phase space between our solution $\dot{\pi} = 1$ and the line $c_T^2(\pi, \dot{\pi}) = 1$, see Fig. 6. Let us note that as it stands, the new solution is not separated from the superluminal domains by any kind of forbidden area, hence, these domains are in principle reachable. However, we recall that the reconstructed Lagrangian (17) is valid only in a vicinity of the solution with $X = 1$. Therefore, the would-be superluminal domains are away from the region of validity of our analysis, so we consider our construction healthy.

To end up this section, we discuss whether our new bouncing solution requires strong fine-tuning of the initial data. To this end, we stick to the reconstructed Lagrangian functions (17) as given in Appendix B, and solve the system of background gravitational equations (3) for $H$ and $\pi$ for different initial conditions. The resulting trajectories in the phase space $(\pi, \dot{\pi})$ are shown in Fig. 8. The shaded region in both figures (which is of the same nature as the white regions in Figs. 1, 2 and 7) is a forbidden domain, where eq. (3a), viewed as the quadratic equation for $H$, has negative discriminant. As pointed out in Ref. [26] and in Sec. 2.1, zero discriminant, i.e. the transition between the branches of solution for the Hubble parameter, occurs at $\gamma$-crossing. Since we have required that the beyond Horndeski theory (17) tends to GR + massless scalar field long before and long after the bouncing epoch, the $\gamma$-crossing has to take place at some moment of time. Hence, the bouncing trajectories have to touch the boundary of the forbidden domain,
Figure 6: [color online] Variance \((1 - c_T^2(\pi, \dot{\pi}))\) of the speed squared of tensor modes in the phase space \((\pi, \dot{\pi})\) for the new bouncing solution. Dashed lines are lines of constant \((1 - c_T^2(\pi, \dot{\pi}))\). A thick line in the upper part denotes the boundary of the (hatched) superluminal region. The black horizontal line at \(\dot{\pi} = 1\) corresponds to the new bouncing solution. The plot shows that there are no superluminal tensor perturbations in a vicinity of the bouncing solution.

which is indeed the case in Fig. 8. We have checked that every trajectory in Fig. 8 describes healthy bouncing solution. Since there is a whole set of these trajectories with different initial conditions, we conclude that no fine-tuning is involved in the solution construction.

4 Conclusion

We have extended the analysis of the bouncing solution [1] and studied the phase space around the solution. Our analysis has shown that, although the solution itself is free of pathological DOF and superluminal modes during entire evolution, there is a region with superluminal gravitational waves in its close neighborhood. Thus, the original model [1] suffers the superluminality problem. To cure this, we have suggested a new version of the bouncing scenario, where the propagation speed of the tensor modes is strictly smaller than the speed of light during and around the bouncing epoch. As a result, there are no superluminal regions of phase space close to the new solution.
Figure 7: [color online] Scalar sound speed squared $c^2_S(\pi, \dot{\pi})$ in the phase space $(\pi, \dot{\pi})$ for the new bouncing solution. The black horizontal line corresponds to the solution and is again safely away from both the white zone with no solutions and the pathological hatched zone. Right panel is a blow up of the bounce region.

Figure 8: [color online] Bouncing trajectories with different initial conditions in $(\pi, \dot{\pi})$ plane. There are no solutions inside the shaded region, while the trajectories touch the boundary of this region and exhibit required $\gamma$-crossing. The right panel is a blow up of the bounce region, with the same solutions as in the left panel.
By analysing the behaviour of the bouncing trajectories with different initial conditions we have also checked that our construction does not involve fine-tuning.

The novel solution might be considered as another step towards constructing application-oriented cosmologies. Complete stability and GR asymptotics of the new bouncing solution make it a promising candidate for future realistic models of the early Universe.

Acknowledgements

The authors are grateful to S. Dubovsky and R. Rattazzi for helpful discussions and to A. Vikman, G. Ye and A. Anabalon for useful correspondence. The work of S.M. on Sec. 2 has been supported by the Foundation for the Advancement of Theoretical Physics and Mathematics BASIS grant, the work on Sec. 3 has been supported by Russian Science Foundation grant 19-12-00393.

Appendix A

In this Appendix we give an example set of Lagrangian functions (16) for a healthy bouncing model suggested in Ref. [1]:

\[
g_{40}(\pi) = -g_{41}(\pi) = -\frac{w}{2} \text{sech}^2 \left( \frac{\pi}{\tau} + u \right),
\]

\[
f_{40}(\pi) = -f_{41}(\pi) + w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right),
\]

\[
f_{41}(\pi) = \frac{3w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right)}{2\pi \tau} \left[ \pi^2 \tanh \left( \frac{\pi}{\tau} + u \right) + \tau^2 \tanh \left( \frac{\pi}{\tau} \right) - \pi \tau \right],
\]

\[
f_0(\pi) = \frac{1}{3\tau (\pi^2 + \tau^2)^2} \left[ -\tau^3 + 3\tau (\pi^2 + \tau^2)^2 f_2(\pi) - 3\pi \tau^2 w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \tanh \left( \frac{\pi}{\tau} \right) \right.
\]

\[+3\pi^3 w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \tanh \left( \frac{\pi}{\tau} + u \right) + 6\pi \tau^2 w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \tanh \left( \frac{\pi}{\tau} + u \right) \left. \right],
\]

\[
f_1(\pi) = -\frac{1}{3\tau (\pi^2 + \tau^2)^2} \left[ \tau^3 + 6\tau (\pi^2 + \tau^2)^2 f_2(\pi) - 3\pi \tau^2 w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \tanh \left( \frac{\pi}{\tau} \right) \right.
\]

\[+3\pi^3 w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \tanh \left( \frac{\pi}{\tau} + u \right) + 6\pi \tau^2 w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \tanh \left( \frac{\pi}{\tau} + u \right) - \pi^2 \tau \left. \right],
\]

\[
f_2(\pi) = \frac{1}{12\tau (\pi^2 + \tau^2)^2} \left\{ \tau^3 + 3\tau^3 + 4\pi w \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \left[ -4\pi \tau + 9\tau^2 \tanh \left( \frac{\pi}{\tau} \right) \right. \right.
\]

\[+3(\pi^2 - 2\tau^2) \tanh \left( \frac{\pi}{\tau} + u \right) \right\},
\]

where \( u \) and \( w \) are numerical parameters. The bouncing solution is given by eq. (15) and has \( \dot{\pi} = 1 \). The plots in Sec. 2.2 are given for \( u = 1/10, \tau = 10, w = 1 \).
Appendix B

Here we give an explicit example of the Lagrangian functions (17) defining the beyond Horndeski theory (1), which admits a new completely stable bouncing solution with no superluminal modes in the vicinity of the solution:

\[ g_{40}(\pi) = -g_{41}(\pi) = -\frac{1}{2} \text{sech}^2 \left( \frac{\pi}{\tau} + u \right), \quad (27) \]

\[ f_{40}(\pi) = -f_{41}(\pi) + \frac{5w}{4} \text{sech} \left( \frac{\pi}{\tau} + u \right), \quad (28) \]

\[ f_{41}(\pi) = \frac{1}{(8\pi\tau)} \text{sech} \left( \frac{\pi}{\tau} + u \right) \left[ -25\pi\tau w + 8\pi\tau \text{sech} \left( \frac{\pi}{\tau} + u \right) \right. \]
\[ + 12\pi^2 \text{sech} \left( \frac{\pi}{\tau} + u \right) \text{tanh} \left( \frac{\pi}{\tau} + u \right) + 12\tau^2 \text{sech} \left( \frac{\pi}{\tau} + u \right) \text{tanh} \left( \frac{\pi}{\tau} \right) \left. \right], \quad (29) \]

\[ f_1(\pi) = \frac{1}{12\tau(\pi^2 + \tau^2)^2} \left[ 4\tau(\pi^2 - 3\tau^2) + 12\tau(\pi^2 + \tau^2)^2 f_3(\pi) \right. \]
\[ + 15w \text{sech} \left( \frac{\pi}{\tau} + u \right) \left( \tau(\pi^2 - 2\tau^2) + 2\pi(\pi^2 + \tau^2) \text{tanh} \left( \frac{\pi}{\tau} + u \right) \right) \]
\[ - 12 \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \left( \tau(\pi^2 - 2\tau^2) + \pi\tau^2 \text{tanh} \left( \frac{\pi}{\tau} + u \right) + \pi(3\pi^2 + 2\tau^2) \text{tanh} \left( \frac{\pi}{\tau} + u \right) \right) \left. \right], \quad (30) \]

\[ f_2(\pi) = \frac{1}{12\tau(\pi^2 + \tau^2)^2} \left[ 4\tau^3 - 24\tau(\pi^2 + \tau^2)^2 f_3(\pi) \right. \]
\[ - 5w \text{sech} \left( \frac{\pi}{\tau} + u \right) \left( \tau(\pi^2 - 2\tau^2) + 2\pi(\pi^2 + \tau^2) \text{tanh} \left( \frac{\pi}{\tau} + u \right) \right) \]
\[ + 4 \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \left[ 3\pi\tau^2 \text{tanh} \left( \frac{\pi}{\tau} \right) + (\pi^2 - 2\tau^2) \left( \tau + \pi \text{tanh} \left( \frac{\pi}{\tau} + u \right) \right) \right] \left. \right], \quad (31) \]

\[ f_3(\pi) = \frac{1}{24\tau(\pi^2 + \tau^2)^2} \left[ 6\tau^3 (-1 + \tau^2) \right. \]
\[ + 5w \text{sech} \left( \frac{\pi}{\tau} + u \right) \left( -16\pi^2 \tau - 3\tau^3 + 3\pi(\pi^2 + \tau^2) \text{tanh} \left( \frac{\pi}{\tau} + u \right) \right) \]
\[ + 4 \text{sech}^2 \left( \frac{\pi}{\tau} + u \right) \left[ 8\pi^2 \tau + 3\tau^3 + 12\pi\tau^2 \text{tanh} \left( \frac{\pi}{\tau} \right) + 6\pi(\pi^2 - \tau^2) \text{tanh} \left( \frac{\pi}{\tau} + u \right) \right] \left. \right]. \quad (32) \]

The bouncing solution is given by eq. (15) and has \( \dot{\pi} = 1 \). The plots in Sec. 3 are given for \( u = 1/10, \tau = 10, w = 1 \).
References

[1] S. Mironov, V. Rubakov and V. Volkova, Bounce beyond Horndeski with GR asymptotics and $\gamma$-crossing, JCAP 1810 (2018) no.10, 050 [arXiv:1807.08361 [hep-th]].

[2] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. 10, 363 (1974).

[3] M. Zumalacarregui and J. Garca-Bellido, Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian, Phys. Rev. D 89 (2014) 064046 [arXiv:1308.4685 [gr-qc]].

[4] J. Gleyzes, D. Langlois, F. Piazza and F. Vernizzi, Healthy theories beyond Horndeski, Phys. Rev. Lett. 114 (2015) no.21, 211101 [arXiv:1404.6495 [hep-th]].

[5] T. Kobayashi, Horndeski theory and beyond: a review, Rept. Prog. Phys. 82 (2019) no.8, 086901 [arXiv:1901.07183 [gr-qc]].

[6] V. A. Rubakov, The Null Energy Condition and its violation, Phys. Usp. 57 (2014) 128 [Usp. Fiz. Nauk 184 (2014) no.2, 137] [arXiv:1401.4024 [hep-th]].

[7] S. Mironov, V. Rubakov and V. Volkova, Cosmological scenarios with bounce and Genesis in Horndeski theory and beyond: An essay in honor of I.M. Khalatnikov on the occasion of his 100th birthday, JETP Vol. 156 (4) (2019) [arXiv:1906.12139 [hep-th]].

[8] F. J. Tipler, Energy conditions and spacetime singularities, Phys. Rev. D 17 (1978) 2521.

[9] M. Libanov, S. Mironov and V. Rubakov, Generalized Galileons: instabilities of bouncing and Genesis cosmologies and modified Genesis, JCAP 1608 (2016) no.08, 037 [arXiv:1605.05992 [hep-th]].

[10] T. Kobayashi, Generic instabilities of nonsingular cosmologies in Horndeski theory: A no-go theorem, Phys. Rev. D 94 (2016) no.4, 043511 [arXiv:1606.05831 [hep-th]].

[11] A. Ijjas and P. J. Steinhardt, Fully stable cosmological solutions with a non-singular classical bounce, Phys. Lett. B 764 (2017) 289 [arXiv:1609.01253 [gr-qc]].

[12] Y. Cai, Y. Wan, H. G. Li, T. Qiu and Y. S. Piao, The Effective Field Theory of nonsingular cosmology, JHEP 1701 (2017) 090 [arXiv:1610.03400 [gr-qc]].

[13] P. Creminelli, D. Pirtskhalava, L. Santoni and E. Trincherini, Stability of Geodesically Complete Cosmologies, JCAP 1611 (2016) no.11, 047 [arXiv:1610.04207 [hep-th]].

[14] R. Kolevatov, S. Mironov, N. Sukhov and V. Volkova, Cosmological bounce and Genesis beyond Horndeski, JCAP 1708 (2017) no.08, 038 [arXiv:1705.06626 [hep-th]].

[15] Y. Cai and Y. S. Piao, A covariant Lagrangian for stable nonsingular bounce, JHEP 1709 (2017) 027 [arXiv:1705.03401 [gr-qc]].
[16] S. Mironov, V. Rubakov and V. Volkova, Genesis with general relativity asymptotics in beyond Horndeski theory, Phys. Rev. D 100 (2019) no.8, 083521 [arXiv:1905.06249 [hep-th]].

[17] G. Ye and Y. S. Piao, Implication of GW170817 for cosmological bounces, Commun. Theor. Phys. 71 (2019) no.4, 427 [arXiv:1901.02202 [gr-qc]].

[18] G. Ye and Y. S. Piao, Bounce in general relativity and higher-order derivative operators, Phys. Rev. D 99 (2019) no.8, 084019 [arXiv:1901.08283 [gr-qc]].

[19] D. Langlois and K. Noui, Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability, JCAP 1602 (2016) 034 [arXiv:1510.06930 [gr-qc]].

[20] D. Langlois, Dark energy and modified gravity in degenerate higher-order scalartensor (DHOST) theories: A review, Int. J. Mod. Phys. D 28 (2019) no.05, 1942006 [arXiv:1811.06271 [gr-qc]].

[21] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis and R. Rattazzi, Causality, analyticity and an IR obstruction to UV completion, JHEP 0610 (2006) 014 [hep-th/0602178].

[22] E. Babichev, V. Mukhanov and A. Vikman, k-Essence, superluminal propagation, causality and emergent geometry, JHEP 0802 (2008) 101 [arXiv:0708.0561 [hep-th]].

[23] P. Creminelli, A. Nicolis and E. Trincherini, Galilean Genesis: An Alternative to inflation, JCAP 1011 (2010) 021 [arXiv:1007.0027 [hep-th]].

[24] P. Creminelli, K. Hinterbichler, J. Khoury, A. Nicolis and E. Trincherini, Subluminal Galilean Genesis, JHEP 1302 (2013) 006 [arXiv:1209.3768 [hep-th]].

[25] D. A. Easson, I. Sawicki and A. Vikman, When Matter Matters, JCAP 1307 (2013) 014 [arXiv:1304.3903 [hep-th]].

[26] D. A. Dobre, A. V. Frolov, J. T. G. Ghersi, S. Ramazanov and A. Vikman, Unbraiding the Bounce: Superluminality around the Corner, JCAP 1803 (2018) 020 [arXiv:1712.10272 [gr-qc]].

[27] J. D. Bekenstein, The Relation between physical and gravitational geometry, Phys. Rev. D 48 (1993) 3641 [gr-qc/9211017].

[28] A. Ijjas and P. J. Steinhardt, Classically stable nonsingular cosmological bounces, Phys. Rev. Lett. 117 (2016) no.12, 121304 [arXiv:1606.08880 [gr-qc]].

[29] T. Kobayashi, M. Yamaguchi and J. Yokoyama, Generalized G-inflation: Inflation with the most general second-order field equations, Prog. Theor. Phys. 126 (2011) 511 [arXiv:1105.5723 [hep-th]].

[30] A. Ijjas, Space-time slicing in Horndeski theories and its implications for non-singular bouncing solutions, JCAP 1802 (2018) no.02, 007 [arXiv:1710.05990 [gr-qc]].