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Analytic scaling analysis of high harmonic generation conversion efficiency

E. L. Falcão-Filho, V. M. Gkortsas, Ariel Gordon and Franz X. Kärtner

Abstract: Closed form expressions for the high harmonic generation (HHG) conversion efficiency are obtained for the plateau and cutoff regions. The presented formulas eliminate most of the computational complexity related to HHG simulations, and enable a detailed scaling analysis of HHG efficiency as a function of drive laser parameters and material properties. Moreover, in the total absence of any fitting procedure, the results show excellent agreement with experimental data reported in the literature. Thus, this paper opens new pathways for the global optimization problem of extreme ultraviolet (EUV) sources based on HHG.

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I. Introduction

For more than a decade, high harmonic generation (HHG) is pursued as a promising route towards a compact coherent short wavelength source in the XUV region [1-3]. Currently HHG is the only experimentally proven method for generating coherent XUV radiation and enables attosecond pulses [4]. Seeding of next generation free electron lasers in the XUV with high harmonics in combination with high gain harmonic generation is assumed to be a viable technique to transfer the coherence properties of the HHG seed to the hard x-ray range [5]. As a consequence, the laboratory concept of an HHG based XUV source is rapidly evolving to reality accelerated by advances in high power femtosecond laser systems and in HHG techniques, such as quasi-phase-matching HHG [6], two-color-HHG [7] and most recently cavity-based HHG [8]. In this context, quantitative scaling and optimization of HHG-based source characteristics is of key importance to accelerate the development and capabilities of this emerging research field.

In parallel to the experimental progress, a considerable effort has been devoted to the theoretical modeling and scaling of HHG [9-15]. Accurate quantitative simulation of the HHG efficiency including propagation effects is time-consuming, making a systematic parameter study impossible. Indeed, up to the present it is not entirely clear how the HHG efficiency scales for different experimental conditions, how far current results are from theoretical limits, and how to proceed to construct a maximally efficient HHG source for a particular range of harmonics. For example, over the last years, the role of the driving frequency, \( \omega_0 \), to HHG scaling has received great attention [16-19]. Dependences of HHG efficiency between \( \omega_0^5 \) and \( \omega_0^6 \) are being obtained from numerical simulations using the time dependent Schrödinger equation [16] including the single-atom response only. Preliminary experimental results
[17,18] are supporting these simulations. General scaling considerations concerning the scaling with drive frequency, including laser wavelength and pressure were presented recently [19], however, no expression for the HHG efficiency and its scaling as a function of all experimentally relevant parameters was presented. This paper gives the first, to our knowledge, closed form analytical expressions for HHG conversion efficiencies both for the plateau region and the cutoff region including both laser and material parameters. Also focusing conditions and related effects, such as intensity dependent phase mismatch, could be included, but is postponed for future work. For the purpose of this paper we consider plane wave geometry.

To this aim we make two simplifying assumptions. First, we use the single-active-electron (SAE) approximation, which is widely adopted. Second, multielectron effects are partially included using the recombination amplitude computed via the Hartree-Slater potential approach [20]. The HHG driven by a plane wave source includes besides the single-atom response the 1-D propagation effects due to absorption and phase mismatch caused by the neutral gas and plasma generation. Experimentally, HHG-setups either use free space focusing into a gas jet or cell or hollow fiber geometry. Considering a loose focusing regime [21] or hollow fiber geometry [6], the phase mismatch due to the Gouy-phase shift and the dipole phase is minimized or absent or are replaced by waveguide dispersion which can also be included in the 1-D model. Thus, even this simplistic model is expected to give upper bounds for the HHG efficiency, as reported in the literature [21-23]. In this paper, the discussion is also restricted to the adiabatic regime which holds for driver pulses as short as 4-optical-cycles excluding strong carrier envelope phase effects [11].

2. Derivation

The one-dimensional propagation equation commonly used for HHG [24] is

$$\partial_z E_h(z,t) = -\frac{1}{2\epsilon_0 c} e^{-i(\Delta k \cdot z)} \partial_t P - \frac{1}{2L_{abs}} E_h(z,t),$$

(1)

where $z$ is the position coordinate along the propagation direction, $\Delta k$ is the phase mismatch, $t$ is the retarded time appropriate for describing propagation at the speed of light in vacuum, $E_h$ is the electric field of the harmonics, and $P$ is the polarization induced in the medium. $L_{abs}$ is the absorption length, which later will become frequency dependent to take frequency dependent absorption into account. The driving field is assumed to be a top-hat pulse, represented by

$$E(t) = \begin{cases} E_0 \sin(\omega_0 t), & 0 < t < T \\ 0, & elsewhere \end{cases}$$

(2)

with time duration, $T = 2\pi N / \omega_0$, where $N$ means the number of optical cycles.

In the following, we use atomic units, where $4\pi\epsilon_0$, $\hbar$, the electron mass, $m_e$, and its charge, $e$, are set to unity, and the speed of light in vacuum equals the inverse fine structure constant $\alpha^{-1} \approx 137$. Defining $\rho$ as the density of atoms (number of atoms per atomic unit volume), Eq. (1) takes the form
\[
\partial_z E_h(z,t) = -(2\pi \alpha \rho) e^{-i(\Delta k \cdot z)} \dot{x} - \frac{\rho \sigma}{2} E_h(z,t),
\] 

where \(\sigma\) is the absorption cross section, \(x\) is the dipole moment of a single atom, and \(\dot{x} = \partial_t x\). With the Fourier transforms of the harmonic field \(E_h\) and the dipole velocity derived from the dipole acceleration \(\dot{x}\):

\[
\vec{E}_h(z,\omega) = \frac{1}{\sqrt{2\pi}} \int_0^T \! E_h(z,t) e^{i\omega t} dt,
\]

\[
\vec{v}(\omega) = \frac{1}{\sqrt{2\pi i \omega}} \int_0^T \! \dot{x}(t) e^{i\omega t} dt,
\]

over the finite pulse duration, Eq. (3) becomes

\[
\partial_z \bar{E}_h(z,\omega) = -(2\pi \alpha \rho) e^{-i(\Delta k \cdot z)} \vec{v}(\omega) - \frac{\rho \sigma(\omega)}{2} \bar{E}_h(z,\omega),
\]

where a frequency dependent absorption cross section is introduced. Note, that we assumed that \(\vec{v}(\omega)\) does not depend on \(z\), i.e. the changes in amplitude and phase of the driving pulse are small over the medium length \(L\). The results can also be applied to a weakly focused Gaussian beam for \(L << z_0\), where \(z_0\) is the Rayleigh length. In this case the solution to Eq. (6) is:

\[
\bar{E}_h(\omega) = -\frac{4\pi \alpha \bar{v}(\omega)}{\sigma(\omega)} g(\Delta k, L),
\]

where:

\[
g(\Delta k, L) = \frac{e^{i(\Delta k \cdot L)} - e^{-L/2L_{\text{abs}}}}{1 + 2i(\Delta k \cdot L_{\text{abs}})},
\]

and \(L_{\text{abs}} = 1/(\rho \sigma)\). If the propagation distance is long compared to the absorption length, in general \(L > 3L_{\text{abs}}\) is sufficient, and perfect phase matching conditions are satisfied, \(g(\Delta k, L) = 1\), Eq. (7) approaches its absorption limited value \(\bar{E}_h(\omega) = -4\pi \alpha \bar{v}(\omega)/\sigma(\omega)\).

The conversion efficiency into a given (odd) harmonic of \(\omega_0\), whose frequency is denoted by \(\Omega\), is given by

\[
\eta = \int_{\Omega-\omega_0}^{\Omega+\omega_0} \frac{\left| \bar{E}_h(\omega) \right|^2 d\omega}{\Omega^2 - \omega^2} \int_0^\infty \left| \bar{E}(\omega) \right|^2 d\omega,
\]

where \(\bar{E}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^T \! E(t) e^{i\omega t} dt\).

In order to evaluate the numerator in Eq. (9) we note that \(\dot{x}(t)\), the dipole acceleration of a single atom has the following property:

\[
\dot{x}(t + \pi/\omega) = -\beta \dot{x}(t),
\]
with $0 < \beta < 1$. The minus sign on the right hand side of Eq. (10) is due to the sign change in the driving field. Note, that $\beta$ accounts for the depletion of the ground state amplitudes during each half period defined by $|a(t + \pi/\omega)|^2 = \beta |a(t)|^2$ where $|a(t)|^2$ denotes the probability to find the atom in the ground state. Thus, $\beta = |a(\pi/\omega)|^2$ or, in other words:

$$
\beta = \exp \left[ - \int_{0}^{\pi/a_0} w(E(t)) \, dt \right],
$$

(11)

with the ionization rate $w(E)$ calculated by the Ammosov-Delone-Krainov formula [25]. Furthermore, only the first recombination event is taken into account, because quantum diffusion greatly reduces the contribution of multiple returns [10]. Under these assumptions and using Eq. (10) in Eq. (7) and substituting the result into Eq. (9) we obtain

$$
\eta = \frac{5 \omega_0^2 \alpha^2}{E_0^2 \Omega^2 \sigma^2(\Omega)} \left[ 1 - \beta^{4(N-1)} \right] \left| 1 + \beta \cdot e^{i \frac{\Omega}{\omega_0}} \right|^2 B(\Omega),
$$

(12)

where

$$
B(\Omega) = \left[ \int x(t) e^{-i \Omega t} \, dt \right]^2.
$$

(13)

The choice for the integration interval in (13) follows from the TSM [9, 10], showing that the dominant contribution to the high harmonics occurs in the interval $3\pi/2 a_0 < t < 5 \pi/2 a_0$. The high harmonics accounted for occur in $N - 1$ full cycles, which is the reason for the factor $N - 1$ instead of $N$ in the exponent of $\beta$ in Eq. (12). The last quarter of a cycle in the $N$-cycle pulse is neglected to keep the expression simple. Comparison with a full numerical simulation considering a Gaussian pulse at the end of the paper shows that this approximation even yields good results for a four-cycle non-flat-top pulses.

As discussed above, the SAE approximation is adopted, where the atoms are modeled by a single electron in an effective potential [Eq. (1) in Ref. 26]. In order to obtain a closed-form expression for the efficiency, we use the improved version of the TSM (ITSM) for $\tilde{x}(t)$ [26].

$$
\tilde{x}(t) = \frac{-1/2}{\omega_0^{3/2}} \sum_n \frac{a(t n \omega_0 / 2 I_p)}{E(t n \omega_0 / 2)} \left[ a(t)n \right] \sqrt{w(E(t n \omega_0 / 2))} \left| e^{-i S_n(t)} \right|,
$$

(14)

where $I_p$ is the ionization potential, $t_n$ is the birth time of an electron that returns to the origin at time $t$ and $n$ denotes the trajectory. $S_n(t) = S_n(t, t_n)$ is the semiclassical action and $a_{rec}(k_n) = \langle 0 | \hat{P}_k | V(r) \rangle k_n \rangle$ is the recombination amplitude, Eq. (7) of Ref. 20, which is obtained from the Hartree-Slater potential (see Ref. 27 for a more detailed explanation). The term $k_n(t)$ is the momentum upon return from of the $n$th trajectory which is related to the HHG frequency by $\Omega_n = I_p + k_n^2 / 2$. Notice, that in order to show the HHG scaling with
drive wavelength due to quantum diffusion, we pulled the factor $\omega_0^{3/2}$ in front leaving the denominator $[\omega_0(t - \theta n)/(2\pi)]^{3/2}$ dimensionless and largely invariant to drive wavelength.

To evaluate $B(\Omega)$, the saddle point method is used. All the terms in Eq. (13), using Eq. (14), are slowly varying except the phase containing the classical action. The stationary phase approximation will give the condition $\partial_t S_n(t) = \Omega$, implying the transition energy of the recolliding electron has to be equal to $\Omega$. This condition is fulfilled twice during each half cycle and is referred as short and long trajectories. As $\Omega$ increases approaching the cutoff frequency, $\Omega_{\text{cut}} = I_p + 3.17 U_p$, with the ponderomotive energy $U_p = (E_0/2\omega_0)^2$, the two trajectories merge. At this point of degeneracy (i.e. cutoff) $\partial_t^2 S_n(t) = 0$, and as a consequence an expansion of the action up to $3^{rd}$ order is necessary:

$$S_n(t) = S_n + \Omega \cdot (t - \tau_n) + \frac{\partial_t^2 S_n}{2} \left(t - \tau_n\right)^2 + \frac{\partial_t^3 S_n}{6} \left(t - \tau_n\right)^3,$$ (15)

where $\tau_n$ is the arrival time of each trajectory and $S_n = S(\tau_n, t, \Omega)$. Considering the expansion up to third order, as shown in Eq. (15), it is not possible to find a closed analytical formula for $B(\Omega)$. However, by focusing our analysis separately to the plateau region or cutoff region, where either the second or third order term is dominant a closed form expression is achieved.

Thus for the case of cutoff the total phase in the integrand of $B(\Omega)$ is given by $\phi = \bar{S}_{\text{cut}} - \Omega \cdot \tau_{\text{cut}} + \frac{\partial_t^2 S_n}{2} \left(t - \tau_{\text{cut}}\right)^2 / 6$, where the first two terms are constants and $B(\Omega)$ is reduced to an Airy function which can be evaluated numerically. The respective birth and arrival times are $t_{\text{b cut}} \approx 1.88/\omega_0$ and $\tau_{\text{cut}} \approx 5.97/\omega_0$. Accordingly, the final expression for the efficiency at the cutoff region can be written as:

$$\eta = 0.0236 \sqrt{\frac{2L_p}{\omega_0^5} \left| a_{\text{rel}} \right|^2 \left| g(L, L) \right|^2 \left[ 1 - \beta^{4(N-1)} \right] \frac{1 + \beta^2}{(1 - \beta^2)^N} \left[ 1 + \kappa_0 \omega \left( E(t_{\text{b cut}}) \right) \right]},$$ (16)

where $\kappa_0 \approx |a(t_{\text{b cut}})a(t_{\text{cut}})|^2$ accounts for the intra-cycle depletion of the ground state [28]. The efficiency at the cutoff region, given by Eq. (16), scales with a factor of $\omega_0^5$. A cubic dependence with $\omega_0$ is due to quantum diffusion. An additional factor of $\omega_0$ comes from the fact that we are considering the conversion efficiency into a single harmonic, and the bandwidth it occupies is $2\omega_0$. The fifth $\omega_0$ comes from the denominator in Eq. (9): The energy carried by a cycle of the driving laser field scales like its duration $2\pi/\omega_0$ at a given electric field amplitude. Note, that in Eq. (16) the factor $\omega_0^5 \left| E_0^5 \Omega_{\text{cut}}^2 \right| \sim U_p^{-5/2} (I_p + 3.17 U_p)^2 \sim U_p^{-9/2}$ for ponderomotive potentials large compared to the ionization potential. Thus by shifting the cutoff to shorter wavelength by increasing the ponderomotive potential via the laser wavelength has a price in efficiency that scales at constant field with $\sim U_p^{-9/2}$. This demonstrates how sensitive the conversion efficiency scales with drive wavelength, i.e. $\sim \lambda^3$, if cutoff extension is the goal to achieve.

In the plateau region, each harmonic has mainly contributions from two trajectories and, if the harmonic energy is not close to the cutoff, the third order term in Eq. (15) can be neglected. Then, the overall phase exhibits a quadratic dependence in $t$, and $B(\Omega)$ can be
expressed by the error function which are evaluated numerically. The final expression for the efficiency in the plateau region is

\[
\eta = 0.0107 \sqrt{\frac{2I_p \omega_0^5}{E_0^4 \Omega^2 \sigma^2}} \left| a_{rec} \right|^2 |g(\Delta k, L)|^2 \frac{1 - \beta^{4(N-1)}}{(1 - \beta^4)N} \left| 1 + \beta e^{i\left(\frac{1}{\omega_0} \right)} \right| ^2 
\]

\[
\times \left[ \frac{a(t_{b_1}) a(t_{a_1})}{\sin(\omega_0 \tau_{b_1})} \sqrt{|E(t_{b_1})|} \right] ^{3/2} \frac{e^{-i(\gamma_{j-1} - \delta_0)} - e^{-i(\gamma_{j-1} - \delta_0) - \frac{\pi}{2}}}{\sqrt{\hat{\omega}_i^2 \hat{S}_j}} \sin(\omega_0 \tau_{b_2}) \left[ |a(t_{a_1})| \sqrt{|E(t_{b_1})|} \right] ^{3/2} 
\]

where, \((t_{b_1}, t_{a_1})\), \((t_{b_2}, t_{a_2})\) and \(\gamma_{j-1} = S(t_{a_{j-1}}, t_{b_{j-1}})\) are the pairs of birth/arrival times and the corresponding semiclassical action for the short and the long trajectory of a particular harmonic, respectively. Equation (17) is valid for harmonic energies \(\Omega\) in the plateau region, satisfying the condition \(1 < (\Omega - I_p)/U_p < 3.1\). The upper limit is to keep the parabolic approximation of the classical action valid and the lower limit is related to approximate value for the error function used in \(B(\Omega)\) in Eq. (13).

Two interference mechanisms are built into Eq. (17). They are described by the last two terms. One is the interference between each half cycle, which under the condition of \(\beta \approx 1\) allows only odd harmonics, and the other is the interference between long and short trajectories. Notice, that intense pulses may break the symmetry due to substantial ionization between half-cycles, and then even harmonics can occur.

Comparing Eqs. (16) and (17), it is observed that both present the same term of \(\omega_0^5\) in the numerator and a term of \(E_0^4\) in the denominator. However, Eq. (17) contains the second derivative of the action, \(\hat{\omega}_i^2 \hat{S}_j\), which is related to the chirp of the attosecond pulses emitted in each recollision or in other words, is related to the temporal spreading of HHG frequencies during emission. Moreover, as \(\hat{\omega}_i^2 \hat{S}_j \propto E_0^2/\omega_0\), the overall efficiency exhibits an effective dependence proportional to \(\omega_0^5/E_0^4 \sim U_p^3\). In summary, the scaling of HHG efficiency with the driving frequency is \(\omega_0^5\) at the cutoff and \(\omega_0^6\) at the plateau region for fixed harmonic wavelength.

3. Discussion

Equations (16) and (17) provide closed-form expressions for the HHG conversion efficiency into a single harmonic \(\Omega\) at the cutoff frequency and in the plateau region, respectively. In the following, our predicted efficiencies are compared with experimental data in the literature. Experimental data are chosen where the corresponding Keldysh parameter \(\gamma = \sqrt{I_p/2U_p} << 1\), a prerequisite for validity of the TSM. This is the case for the experiments reported in Ref. 7 and Ref. 21. Figure 1 shows the values used for the recombination amplitude, \(a_{rec}(\Omega)\), and the absorption cross section, \(\sigma(\Omega)\), taken from Ref. 20 and Ref. 29, respectively.
Fig. 1. (a) Recombination amplitude, $a_{rec}(\Omega)$, and (b) absorption cross section, $\sigma(\Omega)$, for different noble gases from [20] and [29], respectively.

Figure 2 shows the prediction for HHG efficiency from Eq. (17) for the experimental situation in Ref. 21, where HHG was carried out in neon and argon with 35 fs long ($N=13$), 800 nm pulses. In neon under perfect phase matching and absorption limited conditions, $\left|g(\Delta k, L)\right|^2 = 1$, a maximum efficiency of $0.35 \times 10^{-6}$ is calculated for the 59th harmonic. Note, that the oscillations observed in Fig. 2 are related to the interference between long and short trajectories, which sensitively depend on pulse shape and might be different in the actual experiment, however, the maximum efficiency does not depend strongly on the pulse shape.

In argon, for an interaction length, $L=10$ cm, absorption length, $L_{abs}=6.8$ cm, and phase mismatch, $\Delta k = 0.0667$ cm$^{-1}$, $\left|g(\Delta k, L)\right|^2 = 0.261$, a maximum conversion efficiency into the 27th harmonic of $1.2 \times 10^{-3}$ is calculated from Eq. (17) as shown in Fig. 2(b). The measured maximum efficiencies stated in Ref. 21 are $\rho_c$ for Ne and $1.3 \times 10^{-3}$ for Ar, which compares very well with the maximum efficiency calculated and shown in Fig. 2(a) and Fig. 2(b) given the simplicity of the model.

Fig. 2. $\eta$ from Eq. (17), as function of $E_0$, driven by pulses of 11 cycles at 800nm. (a) For the 59th harmonic generate in neon under perfect phase match and absorption limited conditions. (b) For the 27th harmonic generate in argon under the conditions of $L=10$ cm, $L_{abs}=6.8$ cm and $\Delta k = 0.0667$ cm$^{-1}$.
A third case is taken from Ref. 7, which presents maximum measured efficiency values of $1 \times 10^{-5}$ (17th harmonic) and $1 \times 10^{-7}$ (35th–47th harmonic) for He, pumped by pulses of 27 fs at 400 nm (20 cycles) and 800 nm (10 cycles), respectively. The detailed experimental conditions are not quantified, however, we use the stated electric field strengths and assume perfect phase matching and absorption limited propagation. The laser intensity used was $5 \times 10^{14}$ W/cm$^2$ for 800 nm and $8 \times 10^{14}$ W/cm$^2$ for 400 nm. The calculated efficiencies from Eq. (17) are shown in Fig. 3 and are indeed $1 \times 10^{-5}$ and $1 \times 10^{-7}$ for the given harmonics. Notice, that no fitting procedure was used, just the direct application of Eq. (17).

![Figure 3](image1.png)

**Fig. 3.** Efficiency spectrum for HHG in He obtained from Eq. (17). (a) Driven by electric field $E_0 = 0.125$ a.u. and pulses of 10 cycles at 810 nm. (b) Driven by $E_0 = 0.151$ a.u. and pulses of 20 cycles at 420 nm.

As a final comparison, in Fig. 4, the results calculated from Eqs. (16) and (17) for a square pulse are shown together with those for a Gaussian pulse using the ITSM without the saddle point approximation, i.e. by solving the integrals in Eqs. (9) and (14) numerically. Again excellent agreement is obtained even for pulses as short as 4 cycles. For pulses below 4 cycles the HHG spectrum starts to show a strong dependence on the carrier envelope phase and the agreement with Eqs. (16) and (17) deteriorates in the cutoff region.

![Figure 4](image2.png)

**Fig. 4.** Comparison of Eqs. (16)-(17), which are derived considering a top-hat pulse, and the numerical simulation considering a Gaussian pulse. Efficiencies calculated considering Ne driven by electric field $E_0 = 0.105$ and $\lambda_0 = 800$ nm. (a) For pulses of $\tau_{FWHM} = 13$ cycles. (b) For pulses of $\tau_{FWHM} = 4$ cycles. The red lines and the stars represent respectively the values obtained using Eqs. (16) and (17), and the blue lines are the numerical simulation for Gaussian pulses.
Although most of the recent scaling discussions are focused on the drive frequency [16-19], other parameters may be equally important for maximizing HHG, such as the ionization level of the medium which determines the plasma dispersion and with it the phase matching [30]. The phase mismatch due to plasma generation is a function of the driving frequency and electric field, and critically determines the overall efficiency of the HHG process.

The phase mismatch due to plasma generation, as a function of the driving frequency and electric field, is given by

\[ \Delta k_{\text{plasma}} = -\Omega \omega_p^2 / (2c\omega_0^2) . \]  

Here, \( \omega_p \) is the plasma frequency, which is proportional to the electron density, \( \rho_e \). It is clear from Eq. (18) that the plasma contribution to phase mismatch increases for longer drive wavelength. The plasma generation can be reduced by lowering the field strength, \( E_0 \), which will have a direct impact on Eqs. (16) and (17) and therefore needs to be considered in a more general analysis.

Besides the single-atom response, the other major contribution to be considered in the wavelength scaling is the medium characteristics, such as, recombination amplitude and absorption cross section, represented by \( |a_{\text{rec}}(\Omega)|^2 / \sigma^2(\Omega) \). This quantities exhibit a strong wavelength dependence which can have an important role if cutoff extension is the goal. Thus, in order to illustrate the significance of that statement we consider absorption limited HHG in neon and ask what the optimum drive wavelength is to achieve maximum conversion efficiency in the cutoff region. Equation (16) is used to compute the HHG efficiency at cutoff. The result is displayed in Fig. 5(a) as a function of drive wavelength and cutoff energy \( (I_p + 3.17 U_p) \). A global maximum for Ne efficiency is clearly observed around \( \lambda_0 = 1.2 \mu m \) and \( \Omega_{\text{cut}} = 451 eV \). It is at first surprising that the maximum efficiency shifts for different driver wavelength \( \lambda_0 \), but the maximum efficiency itself does not strongly depend on the driver wavelength, as one may expect from the scaling of the single-atom response. This behavior is also reproduced using a Gaussian pulse and numerical evaluation of Eq. (9) using the ITSM at constant field amplitude \( E_0 \) while varying \( \omega_0 \), as shown in Fig. 5(b). The reason is that in the range from 30 to 800 eV for Ne, the recombination amplitude \( a_{\text{rec}} \) increases while the absorption cross section \( \sigma \) decreases, as shown in Fig. 1, compensating almost completely the reduction due to the front factor of \( U_p^{-9/2} \) scaling the single-atom response. In particular the absorption cross section decreases more than two orders of magnitude for that range. In Fig. 6, the same problem is considered but more constrains are imposed. In Fig. 6(a) the efficiency at cutoff for a gas cell of 5 mm length at 1 bar of pressure and perfect phase matching is assumed. In this case, the maximum value of \( 8 \times 10^{-6} \) was reached for the HHG efficiency at \( \lambda_0 = 0.6 \mu m \) and \( \Omega_{\text{cut}} = 148 eV \). In Fig. 6(b), the plasma and neutral atom phase mismatch is included to the problem. In this final case, the maximum value of \( 2.7 \times 10^{-7} \) was reached for the HHG efficiency at \( \lambda_0 = 0.8 \mu m \) and \( \Omega_{\text{cut}} = 107 eV \).
Fig. 5. (a) Neon HHG efficiency at the cutoff region, using Eq. (15), as a function of the driver wavelength, $\lambda_0$, and the cutoff energy, $\Omega_{\text{cutoff}}$. (b) Full spectrum obtained for Ne, considering a Gaussian pulse with $E_0 = 0.16$ a.u. for different driver wavelengths. For both cases, perfect phase matching $|g(\Delta k, L)|^2 = 1$ and a 5-cycle-driver-pulse were assumed.
4. Conclusion

In summary, closed form expressions for the HHG conversion efficiency using square shaped pulses are presented for the plateau and cutoff regions. It is shown that the computed efficiencies are also in good agreement with Gaussian shaped pulses. Based on these expressions, absolute theoretical HHG conversion efficiencies were calculated for different gases under different laser pumping conditions. The calculated efficiencies are in good agreement with experimental results obtained from different groups under different experimental conditions. The presented formulas simplify the HHG optimization problem considerably and enable a complete HHG scaling analysis. As an example the efficiency at cutoff for Neon was optimized for plane wave geometry and the global maximum on the Ne efficiency was obtained under various conditions. Provide that most of the performed analysis were considering neon, the formulas and concepts discussed in this paper can be applied for any atomic or molecular gas. For this purpose it is just necessary to use the respective gas properties such as dispersion, ionization potential, absorption cross-section and recombination...
amplitude. A more comprehensive analysis including other gases will be published in a forthcoming paper. Due to its generality and simplicity, the theory presented in this paper should have a significant impact on the development of HHG based EUV sources.

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