I. INTRODUCTION

Among rich variety of transport phenomena in magnetic materials the special attention is now focused on the topological Hall effect (THE). THE is the appearance of an additional transverse voltage due to itinerant carrier exchange interaction with chiral spin textures, such as magnetic skyrmions. THE has been extensively studied experimentally and has proved itself as an indicator of a non-zero chirality of the sample magnetization. The observation of THE has been reported for various systems exhibiting different chiral ordering of spins: skyrmion crystals, antiferromagnets (AFM) spin glasses and disordered arrays of magnetic skyrmions. Naturally, an appropriate microscopic theory of THE has to take into account the particular type of chiral spin ordering. In the case of a regular non-collinear spin structure with periodic or quasi-periodic spin arrangement, such as AFM lattices or skyrmion crystal, THE can be described in terms of an effective mean magnetic field. The meanfield approach is usually justified within the adiabatic approximation typical for systems with strong exchange coupling between the magnetization and the electrons carrying the current. Deviations from the condition of adiabaticity in THE due to long-ranged spin textures has been recently discussed in Ref.35. Another type of the chiral magnetization profiles studied experimentally is a disordered array of localized small spin textures spatially separated from each other. In this case a carrier moves freely most of the time, and the presence of localized magnetization vortices affects its trajectory by occasional scattering. The mean field approach is not adequate in this case as there is no regular long-range chiral spin structure which can be described by a homogeneous effective magnetic field. Instead, THE is driven by an asymmetric scattering of the carriers on individual spin textures being sensitive to their particular magnetization profile. The important feature of the individual scattering regime is that the properties of THE strongly depend on whether the carrier spin-flip processes are activated or not corresponding to so-called weak coupling regime and adiabatic regime, respectively. The transverse electric response arises from the spin Hall effect in the adiabatic regime and from the charge Hall effect in the weak coupling regime. Thus, the complete theory of THE for the irregular dilute chiral systems requires an accurate treatment of carrier scattering on a single chiral spin texture.

In this paper we develop the theory of the topological Hall effect in the diffusive regime for the dilute systems of localized magnetic textures. The approach is based on the consideration of asymmetric carrier scattering on individual spin textures. Our theory is applicable to disordered arrays of chiral spin textures with both electron spin subbands populated when THE can be generated both by charge and spin transverse currents. The paper is organized as follows: in section II the kinetic theory of THE is described accounting for the carrier scattering on host impurities and non-collinear spin textures, in section III the properties of the exchange asymmetric scattering are discussed, section IV covers the dependence of THE on material and spin texture parameters, we also describe the crossover between charge and spin Hall regimes of THE driven by the suppression of spin-flip scattering; in section V we summarize our results.

II. KINETIC THEORY

Let us consider two-dimensional degenerate electron gas (2DEG) described by the Hamiltonian:

\[ \mathcal{H} = \frac{p^2}{2m} - \alpha_0 \mathbf{S}(\mathbf{r}) \cdot \mathbf{\sigma} + \sum_i u(\mathbf{r} - \mathbf{r}_i) \] (1)

where the first term describes the electron free motion with an effective in-plane mass \( m \), the second term represents the electrons exchange interaction with a magnetic...
texture described by a static spin field \( S(\mathbf{r}) \), where \( \alpha_0 \) is an exchange coupling constant, \( \sigma \) is the vector of Pauli matrices, the last term describes scattering on host non-magnetic impurities located at \( \mathbf{r}_i \). The topological Hall effect appears when \( S(\mathbf{r}) \) has a non-collinear structure characterized by a non-zero spin chirality.

We consider the case, when the spin field \( S(\mathbf{r}) \) consists of two contributions:

\[
S(\mathbf{r}) = S_0 \mathbf{e}_z + \sum_j \delta S(\mathbf{r} - \mathbf{r}_j). \tag{2}
\]

The first term is a background homogeneous field directed perpendicular to the 2DEG plane leading to the Zeeman spin splitting \( \Delta = \alpha_0 S_0 \). We assume ferromagnetic exchange \( (\Delta > 0) \) and that the Fermi energy exceeds the spin splitting so that both spin subbands are populated. The second contribution describes localized non-collinear spin textures \( \delta S \) of a few nanometer size located at \( \mathbf{r}_j \) and causing an additional elastic scattering of the carriers. While magnetic skyrmions are the typical example of a localized non-collinear spin texture, our consideration covers much wider class of chiral spin textures, not necessarily having a non-zero topological charge. The feature of the chiral spin structures is that for a given incident electron flux there is a difference in scattering rates to the left and to the right, eventually leading to the Hall effect.

We consider the classic transport regime \( (k_F \ell \gg 1) \) where \( k_F = \sqrt{2mE_F}/\hbar \) is Fermi wave-vector, \( \ell \) is the mean free path) on the basis of the Boltzmann kinetic equation:

\[
e E \cdot \frac{\partial f_s(\mathbf{p})}{\partial \mathbf{p}} = \text{St}[f_s(\mathbf{p})],
\]

\[
\text{St}[f_s(\mathbf{p})] = \sum_{p',s'} \left( \mathcal{W}^{s's'}_{pp'} f_s(\mathbf{p}') - \mathcal{W}^{s's'}_{pp'} f_s(\mathbf{p}) \right), \tag{3}
\]

where \( f_s(\mathbf{p}) \) is the distribution function, \( \mathbf{p} \) is 2D momentum and \( s = \pm 1/2 \) is the carrier spin projection on the axis normal to the motion plane, \( \mathbf{E} \) is an in-plane electric field, \( \mathcal{W}^{s's'}_{pp'} \) is the elastic scattering rate from \((p', s')\) to \((p, s)\) state, and \( e \) is the electron charge. We solve Eq. (3) in linear approximation with respect to \( \mathbf{E} \).

Expressing the scattering rate \( \mathcal{W}^{s's'}_{pp'} \) in the form of the Fermi Golden Rule we assume that it has two contributions:

\[
\mathcal{W}^{s's'}_{pp'} = \frac{2\pi}{\hbar} \left| \int \alpha_s \mathcal{E}_{\mathbf{pp'}} \delta(\mathbf{r} - \mathbf{r}_j) \right| \delta \left( \varepsilon_{p}^{\prime} - \varepsilon_{p'} \right), \tag{4}
\]

where the first term in brackets describes the electron spin-independent scattering on non-magnetic impurities, the second term is driven by the scattering on chiral spin textures; interference effects between the two types of scatterers are neglected. Here \( n_s, n_{sk} \) are the sheet densities of impurities and localized magnetic textures, respectively, \( \mathcal{E}_{\mathbf{pp'}} \) is Fourier transform of the non-magnetic impurity potential \( u(\mathbf{r}) \) from Eq. (1), \( T^{s's'}_{pp'} \) is the exact \( T \) matrix of electron scattering on the spin texture, and the delta-function ensures energy conservation in the elastic scattering, the energy spectrum is \( \varepsilon_p^\prime = p'^2/2m - s\Delta \).

Two contributions can be distinguished in the square modulus of the \( T \)-matrix:

\[
|T^{s's'}_{pp'}|^2 = \frac{1}{\nu^2} \left( \mathcal{G}^{ss'}_{ss'}(\theta) + \mathcal{J}^{s's'}_{ss'}(\theta) \right), \tag{5}
\]

\[
\mathcal{G}^{ss'}_{ss'}(\theta) = \mathcal{G}^{ss}_{ss'}(-\theta) \quad \mathcal{J}^{s's'}_{ss'}(\theta) = -\mathcal{J}^{s's'}_{ss'}(-\theta).
\]

\( \mathcal{G}^{ss'}_{ss'}(\theta), \mathcal{J}^{s's'}_{ss'}(\theta) \) are dimensionless symmetric and asymmetric scattering rates, respectively, \( \theta \) is the scattering angle, \( \nu = m/2\pi\hbar^2 \) is 2D density of states (per one spin). In the introduced notation we omit the dependence of \( \mathcal{G}^{ss'}_{ss'}, \mathcal{J}^{s's'}_{ss'} \) on the scattering energy.

It is the asymmetric part \( \mathcal{J}^{s's'}_{ss'}(\theta) \) of an electron scattering on chiral spin textures that gives rise to the transversal current as the scattering rates to the left and to the right become unequal. The scattering asymmetry acts as an effective magnetic field, which sign can be either the same for both spin projections of an incident electron, hence leading to a charge Hall effect, or opposite for the opposite electron spin projections, leading to the spin Hall effect. The properties of \( \mathcal{J}^{s's'}_{ss'}(\theta) \) are discussed in Ref. and summarized in Section III.

In order to solve the kinetic equation (3) we write \( f_s(\mathbf{p}) = f_0^s + g_s(\mathbf{p}) \), where \( f_0^s \) and \( g_s \) are the equilibrium and non-equilibrium parts of the distribution function, respectively. The external electric field \( \mathbf{E} \) is directed along \( x \)-axis. The angular dependence of \( g_s(\mathbf{p}) \) can be expressed as a sum of two terms, even and odd with respect to the angle:

\[
g_s(\mathbf{p}) = g^+_s(\mathbf{p}) \cos \varphi + g^-_s(\mathbf{p}) \sin \varphi, \tag{6}
\]

where \( \varphi \) is the momentum \( p \) polar angle counted from \( x \) axis. After integrating the collision integral over \( \mathbf{p}' \) we arrive at the following system of equations:

\[
e E \begin{pmatrix} v_1 \frac{\partial f_0^s}{\partial \Omega_\uparrow} \\ \frac{\partial f^0_s}{\partial \Omega_\downarrow} \end{pmatrix} = \begin{pmatrix} -\tau_{\uparrow\uparrow}^{-1} \Omega_\uparrow \tau_{\downarrow\downarrow}^{-1} & -\Omega_\uparrow \tau_{\downarrow\downarrow}^{-1} \\ -\Omega_\downarrow \tau_{\downarrow\downarrow}^{-1} & -\Omega_\uparrow \tau_{\downarrow\downarrow}^{-1} \end{pmatrix} \begin{pmatrix} g^+_s \\ g^-_s \end{pmatrix}, \tag{7}
\]

Here we introduced the following parameters:

\[
\tau_{s\uparrow}^{-1} = \tau_{0\uparrow}^{-1} + \omega_s; \quad \tau_{s\downarrow}^{-1} = n_s \frac{2\pi}{\hbar} \nu \int \left| \mathcal{E}_{\mathbf{pp'}} \right|^2 (1 - \cos \theta) \frac{d\theta}{2\pi};
\]

\[
\omega_s = n_{sk} \frac{2\pi}{\hbar} \nu \int \left[ (1 - \cos \theta) \mathcal{G}_{ss}(\theta) + \mathcal{G}_{ss'} \frac{1}{\nu} \frac{d\theta}{2\pi} \right] ;
\]

\[
\tau_{ss'}^{-1} = n_{sk} \frac{2\pi}{\hbar} \nu \int \mathcal{G}_{ss'}(\theta) \cos \theta \frac{1}{\nu} \frac{d\theta}{2\pi};
\]

\[
\Omega_{ss'} = \frac{e B_{ss'}}{mc}; \quad B_{ss'} = (n_{sk} \phi_0) \int \mathcal{J}_{ss'}(\theta) \sin \theta d\theta, \tag{8}
\]
where $\tau_s$ is the total transport lifetime, $\tau_0$ is the transport lifetime for the scattering on non-magnetic impurities, $\omega_s^{-1}$ and $\tau_{ss}$ are the transport lifetimes for the scattering on chiral textures (here $s$ is the spin subband index opposite to $s$). The transverse Hall current due to the asymmetrical scattering is related to the parameter $\Omega_{ss'}$, which is analogous to the cyclotron frequency in the ordinary Hall effect; $B_{ss'}$ is the corresponding effective magnetic field, $\phi_0 = hc/|e|$ is the magnetic flux quantum, $c$ is the speed of light.

The coefficients $g_s^\mu (\mu = \pm)$ linearly depend on the electric field: $g_s^\mu = A_s^\mu E$, where $A_s^\mu$ can be obtained from Eq. (7) by inverting the collision integral matrix. The longitudinal $\sigma_{xx}^s$ and transverse $\sigma_{xy}^s$ conductivities within each spin subband $s$ are given by:

\[
\sigma_{xx}^s = e\nu \int d\varepsilon \sqrt{2m} A_{x}^s (\varepsilon), \\
\sigma_{xy}^s = e\nu \int d\varepsilon \sqrt{2m} A_{y}^s (\varepsilon), \\
\sigma_{xx} = \sigma_{xx}^+ + \sigma_{xx}^-,
\]

(9)

where the integration goes over the energy $\varepsilon$, $\hat{\rho}$ is the tensor of resistivity. The topological Hall effect is determined by $A_{x}$. Let us stress out that both spin-conserving and spin-flip scattering channels contain asymmetric parts $\Omega_{ss'}$ and thus contribute to the tensor of resistivity. The topological Hall effect is described by $\hat{A}_x$, and $\hat{A}_y$ contributes to $\hat{A}_x$. Moreover, as there exist different regimes of THE$^{[23]}$, it might be necessary to calculate the exact scattering rates $J_{ss'}$.

III. ASYMMETRIC ELECTRON SCATTERING ON A CHIRAL SPIN TEXTURE

In this section we consider the features of asymmetric electron scattering on a single non-collinear magnetic texture. We express the scattering potential in the form:

\[
V(r) = -\alpha_0 \delta S(r) \cdot \sigma.
\]

(10)

Outside the localized spin texture of a characteristic diameter $a$ the magnetization is unperturbed so that $\delta S(r > a/2) \to 0$ and the scattering potential vanishes. The topological Hall effect appears due to the asymmetry in the electron scattering when the potential (10) can be characterized by a non-zero chirality. The details of the asymmetric scattering depend on the particular distribution of spins in the texture and its size as well as on the exchange interaction strength and the incident electron wavevector$^{[22]}$.

A. Chiral spin textures

To describe a non-collinear spin texture in 2D the following parametrization is commonly used:

\[
\delta S(r) = \begin{pmatrix}
\delta S_{\parallel}(r) \cos (z\phi + \gamma) \\
\delta S_{\parallel}(r) \sin (z\phi + \gamma) \\
\delta S_{z}(r)
\end{pmatrix}
\]

(11)

where $\phi$ is the polar angle, $\gamma$ is the polar vortex angle, and $\delta S_{\parallel}$ and $\delta S_z$ are the transverse and longitudinal components of the magnetic field, respectively.

\[\Lambda = \pi \left(1 - \frac{2r}{a}\right)\]

(12)

FIG. 1: The typical profiles $S_z(r) = S_0 \cos \Lambda(r)$ of non-collinear spin textures. Note that for $\eta = +1$, the $\delta S_z(r) < 0$ is negative.

where $r = (r, \phi)$ is the polar radius-vector, $r = 0$ corresponds to the center of the texture. The functions $\delta S_{\parallel}, \delta S_z \neq 0$ depend on the distance from the center $r$. The vorticity $\kappa$ describes the in-plane spin rotation with an initial phase $\gamma$. In what follows we consider that $\delta S_z$ is counted from the background magnetization $S_0$, whose sign we denote as $\eta = \text{sgn}(S_0)$. In the last section we will also consider the case when $S_0$ and $\delta S_z$ are independent.

Fig. 1 shows the profiles $S_z(r) = S_0 \cos \Lambda(r)$ for three examples of chiral spin textures with $\eta = +1$ (we assume that $\delta S_{\parallel} = S_0 - \delta S_z$). Two of them describe a magnetic skyrmion ($\Lambda_1(r) = \pi (1 - 2r/a), \Lambda_2(r) = \pi \sin^2 \left([\pi/2(1 + 2r/a)] \right)$). The skyrmion has opposite sign of spins in its center with respect to the background magnetization, which leads to the appearance of a nonzero topological charge called winding number $Q$. The nonzero $Q$ is particularly important for the thermal stability of skyrmions in ferromagnetic thin films$^{[10,14]}$. The third magnetization profile Fig. 1 ($\Lambda_3(r) = (2r/a) \pi (1 - 2r/a)$) corresponds to a non-collinear spin ring with the orientation of spins in the center parallel to $S_0$. Non-collinear rings have zero winding number, but they exhibit a similar topological Hall effect$^{[32,35]}$. Such spin textures can appear in a material with with spin-orbit interaction functionalized by magnetic impurities, in a vicinity of a defect or impurity$^{[33,35,37]}$. Let us notice that for the positive background spin orientation ($\eta = +1$) $\delta S_z$ is negative.

Substituting (11) into (10) we get for the scattering potential:

\[
V(r) = -\alpha_0 \left( e^{i\pi z\phi + i\gamma} \delta S_{\parallel}(r) \right) \left( -\delta S_z(r) \right).
\]

The potential $V(r)$ is a $2 \times 2$ matrix, which depends on a polar angle $\phi$ via the off-diagonal components. When
both functions $S_x, S_y$ are non-zero, the angular dependence of the potential leads to the appearance of the asymmetric part in electron scattering rates $\mathcal{J}_{ss'}(\theta) = -\mathcal{J}_{ss'}(-\theta)$, where $\theta$ is the scattering angle. The sign of $\mathcal{J}_{ss'}$ depends on $\kappa$, while the constant phase $\gamma$ plays no role in the scattering. The role of $\eta$ is more complicated; we further explicitly specify the dependence of $\mathcal{J}_{ss'}$ on $\eta$.

### B. Asymmetric scattering features

We consider the case when Fermi energy exceeds the background exchange splitting $E_F > \Delta/2$ so that both spin subbands are populated with electrons ($\Delta = a_0 S_0$).

The symmetry upon the time inversion allows us to present the asymmetric scattering rates $\mathcal{J}_{ss'}(\theta, \eta)$ introduced in Eq. (5) in the form (see the details in Appendix A):

$$
\begin{align*}
\mathcal{J}_{11}(\theta, \eta) &= \eta \Gamma_1(\theta) + \Pi(\theta), \\
\mathcal{J}_{1\bar{1}}(\theta, \eta) &= \eta \Gamma_1(\theta) - \Pi(\theta), \\
\mathcal{J}_{1\bar{1}}(\theta, \eta) &= \eta \Gamma_2(\theta),
\end{align*}
$$

where $\Gamma_{1\bar{1}}(\theta)$ and $\Pi(\theta)$ have no dependence on $\eta$. This representation is convenient for treating the topological charge and spin Hall effects independently. Indeed, the terms $\eta \Gamma_{1\bar{1}}$ describe the asymmetric scattering in the same transverse direction determined by the texture orientation $\eta$ and independent of an initial carrier spin state. These terms, therefore, lead to the charge Hall effect. On the contrary, the term $\Pi$ describes the scattering of spin up and spin down electron in the opposite transverse directions independent of $\eta$. This process leads to spin Hall effect, it is absent for spin-flip channels. Both $\Gamma_{1\bar{1}}$ and $\Pi$ change their sign upon $\kappa \rightarrow -\kappa$.

Which of the two contributions to the topological Hall effect (charge or spin) dominate strongly depends on whether the spin-flip processes are activated or not. Away from the threshold $E_F > \Delta/2$ the rate of the spin-flip scattering is controlled by the adiabatic parameter $\lambda_a = (a_0 S_0/h) \tau_a$, where $\tau_a = a/v_F$ is an electron time of flight through the texture of diameter $a$ with Fermi velocity $v_F = \sqrt{2E_F/m}$.

In the case of $\lambda_a \leq 1$ the spin-flip processes are effective, the asymmetric scattering arises from the interference between double spin-flip and single spin-conserving scattering events (so-called spin-chirality driven mechanism\cite{22130159}). This process is sensitive to the spin chirality $\chi$ defined for any three spins $\delta S_1, \delta S_2, \delta S_3$ forming the spin texture as $\chi = \text{sgn}(\delta S_1 \cdot [\delta S_2 \times \delta S_3])$. The non-zero chirality of the spin texture in the weak coupling regime leads to the charge Hall effect. The spin chirality based contribution is described by $\Gamma_{1\bar{1}}$. At $\lambda_a \leq 1$ these terms dominate $\Gamma_{1\bar{1}} \gg \Pi$, with spin-flip scattering prevailing $\Gamma_2 = 2\Gamma_1$.

In the opposite case of large adiabatic parameter $\lambda_a > 1$ the spin flip processes are suppressed in accordance with the adiabatic theorem. In this regime the scattering asymmetry is due to the Berry phase acquired by the wave-function of electron moving through a non-collinear spin field. The hallmark of this mechanism is that the sign of the effective magnetic field associated with the Berry phase appears to be opposite for spin up and spin down electrons, thus leading to the spin Hall effect\cite{22130159}. This adiabatic contribution to the Hall response is, therefore, described by $\Pi$. At $\lambda_a \gg 1$ the spin Hall effect dominates $\Pi \gg \Gamma_{1\bar{1}}$, and the charge Hall effect appears only due to nonzero carrier spin polarization $P_s$.

The interplay between charge and spin topological Hall effects leads to a few nontrivial features discussed in the following section.

### IV. TOPOLOGICAL HALL EFFECT

In this section we discuss the topological contribution to the Hall resistivity $\rho_{yx}^T$ in the diffusive regime for different systems.

#### A. Dilute array of chiral spin textures

Let us consider a two-dimensional film containing spatially localized chiral spin textures such as magnetic skyrmions or non-collinear magnetic rings (see Fig. 1). We assume that all the textures have the same vorticity $\kappa$, and the orientation $\eta = \text{sgn}(S_0) = +1$ is fixed, being determined by the background magnetization $S_0$. We consider the dilute regime, when the scattering rate on spin textures is much smaller than that on non-magnetic impurities $\omega_s \tau_0 \ll 1, \Omega_{ss'} \tau_0 \ll 1$, so the transport lifetime is given by $\tau_s = \tau_0$. Solving the system (7) for $A_1$ in the lowest order in $(\Omega_{ss'} \tau_0)$ we express the topological

![FIG. 2: The dependence of $\rho_{yx}^T$ on magnetic skyrmion diameter $ka$ for $A_1$ profile, and the crossover between charge and spin topological Hall effect. The parameters $P_s = 0.4$, $n_{sk} = 2 \times 10^{11}$ cm$^{-2}$.](image-url)
Hall resistivity $\rho_{yx}^T$ as a sum of two contributions:

$$\rho_{yx}^T = \rho_c + \rho_a,$$

$$\rho_c = \frac{1}{nee} \left( \phi_0 n_{sk} \right) \int_0^{2\pi} (\Gamma_1 + \Gamma_2) \sin \theta d\theta;$$

$$\rho_a = P_s \frac{1}{nee} \left( \phi_0 n_{sk} \right) \int_0^{2\pi} \Pi \sin \theta d\theta. \tag{14}$$

The term $\rho_c$ describes the charge transverse current (charge Hall effect) generated due to carrier asymmetric scattering due to spin-independent terms $\Gamma_{1,2}$ (see Eq. 13). The term $\rho_a$ describes the transverse spin current (spin Hall effect) driven by the spin-dependent contribution to the asymmetric scattering $\Pi$ (Eq. 13). The spin current does not lead to a charge separation unless there is an unequal number of spin up and spin down carriers in the system. Therefore, this contribution to the Hall resistivity is proportional to the carrier spin polarized $n_{sk} = 2 \times 10^{11} \text{ cm}^{-2}$. In Eq. 14 the notation $n$ stands for 2DEG sheet density.

The relative importance of the two contributions $\rho_a$ and $\rho_c$ in the appearance of the transverse charge current depends on the texture diameter $a$ or the Fermi level $E_F$ as discussed in the following sections.

1. Crossover between charge and spin Hall effect

Let us trace the dependence of $\rho_{yx}^T$ on the spin texture diameter $a$. We assume that the Fermi energy $E_F$ substantially exceeds the exchange spin splitting so that both spin subbands are populated and the spin polarization of the carriers is far below 100%: $P_s = \Delta/2E_F \ll 1$. In this case the rate of spin-flip processes are fully controlled by $\lambda_a$. The adiabatic parameter can be expressed as $\lambda_a = P_s(ka)$, where $k = (2E_Fm/\hbar)^{1/2}$. When spin-flip scattering is activated the topological Hall effect is dominated by the transverse charge currents ($\rho_c \gg \rho_a$ at $\lambda_a \leq 1$), due to the spin-chirality driven mechanism. In the opposite case $\rho_a \gg \rho_c$, the transverse current appears due to spin Hall effect induced by the adiabatic Berry phase mechanism. This regime is set when the spin-flip channels are suppressed at $\lambda_a \gg 1$. With the change of the texture diameter $a$ from small to large values a crossover occurs between the charge Hall and spin Hall dominated regimes. In addition, the change of the texture size affects the wave parameter $ka$, which determines the properties of the scattering related to the electron relation between the electron wavelength and the scatterer size. With these two consequences of the texture size variation acting simultaneously, the topological Hall resistivity $\rho_{yx}^T$ necessarily exhibits a non-monotonic dependence on $ka$.

Fig. 3(b) shows the calculated dependence of charge $\rho_c$, adiabatic $\rho_a$ and total $\rho_{yx}^T$ topological Hall resistivity on the skyrmion diameter $a$ for the magnetic skyrmion with magnetization spatial profile $\Lambda_1(r)$ shown in Fig. 1. For the calculation results shown in Fig. 3 the spin polarization was taken $P_s = 0.4$, and the skyrmion sheet density $n_{sk} = 2 \times 10^{11} \text{ cm}^{-2}$. The scattering rates $J_{sk'}$ were calculated using the phase function method. As can be seen in Fig. 2 for $\lambda_a \leq 1.8$ the charge contribution $\rho_c$ exceeds $\rho_a$, at that $\rho_{yx}^T$ is dominated by the purely charge current. For $\lambda_a \geq 4.5$ the adiabatic term prevails $\rho_a \gg \rho_c$ and $\rho_{yx}^T$ appears due to the spin current converted into the charge current. As was discussed above, indeed $\rho_{yx}^T$ appears to be non-monotonic when the crossover between $\rho_c$ and $\rho_a$ occurs in the range ($1.8 \lesssim \lambda_a \lesssim 4.5$). As the spin-flip processes are suppressed, firstly the charge contribution $\rho_c$ is decreased, and only later the adiabatic term $\rho_a$ starts to increase. This effect results in the appearance of local minimum for $\rho_{yx}^T$ in the crossover regime.

The behaviour of $\rho_{yx}^T$ in the crossover regime is highly sensitive to a particular magnetic texture profile. In Fig. 3 we present the dependence of $\rho_{yx}^T$ on $a$ for three different spin texture spatial profiles shown in Fig. 1. As can be seen in Fig. 3 the oscillating structure of $\rho_{yx}^T$ upon increasing $ka$ exhibits a significant variation even for two very similar skyrmion configurations $\Lambda_1$ and $\Lambda_2$. The strong dependence of $\rho_{yx}^T$ on $\Lambda(r)$ observed in the crossover regime is due to the significance of the interference in electron scattering as the wave parameter $ka \sim \pi$ (Ref. 13). The texture described by $\Lambda_2$ has larger spin gradients, so the adiabatic term activates at larger $ka$, and the magnitude of $\rho_{yx}^T$ for $\Lambda_2$ in the crossover regime is smaller than that of $\Lambda_1$.

We would also like to stress out that the topological Hall effect exists as well due to scattering on non-collinear spin rings having zero winding number (orange curve in Fig. 3). The transverse conductivity $\rho_{yx}^T$ due to scattering on non-collinear spin rings possesses all the features described above including the the existence of charge and...
spin Hall limiting regimes.

2. The magnitude of the topological Hall effect

The magnitude of THE for the dilute systems can be expressed in terms of the effective magnetic field $B_T$ introduced as:

$$\rho_{yx}^T = \frac{B_T}{nec}. \tag{15}$$

The field $B_T$ shows the magnitude of the external magnetic field applied to the sample, at which the ordinary Hall effect contribution to the transverse resistivity $\rho_{yx}^O$ becomes comparable with $\rho_{yx}^T$.

Usually in the THE estimates it is assumed that each skyrmion contributes via a magnetic flux quantum\[22\]. However, our analysis shows that such an estimate does not take into account the important features of the scattering. According to Eqs. (14,15), $\rho_{yx}^T$ and $B_T$ linearly depend on both the skyrmion sheet density $n_{sk}$ and the dimensionless scattering rates $\mathcal{J}_{xx'}$. Therefore, the actual magnitude of $B_T$ is renormalized differently depending on the scattering regime. The scattering rates $\mathcal{J}_{xx'}$ are small at $\lambda_0 \leq 1$, of the order of unity at an intermediate $\lambda_0 \sim 1$, and reach the order of tens at $\lambda_0 \gg 1$.

The magnitude of $B_T$ for $n_{sk} = 2 \times 10^{11}$ cm$^{-2}$ can be seen in Figs. 2-6. At $n_{sk}\phi_0 \approx 8$ T the value of $B_T$ in the intermediate regime is of the order of several kG; in the strong coupling regime ($\lambda_0 \gg 1$) it can go as high as several Tesla\[23\].

In the weak coupling regime $\rho_{yx}^T$ scales as $\Delta^3$, as the perturbation theory couples $\rho_{yx}^T$ with spin chirality and, therefore, THE requires the third order in the exchange interaction. In Fig. 4 the quantity $\rho_{yx}^T/\Delta^3$ is shown for two values of $ka = 2,3$. As can be seen from the figure, the scaling $\Delta^3$ holds up to $\Delta/2E_F \approx 0.2$, the deviation from the scaling relation indicates that the perturbation theory becomes invalid departing from the weak coupling regime.

Let us note, that although the asymmetrical scattering rates $\mathcal{J}_{xx'}$ are small in the weak coupling regime due to the strong dependence on the exchange interaction strength $\Delta$ ($\mathcal{J}_{xx'}$ is proportional to $(\Delta/2E_F)^3(ka)^2$ at $\lambda_0 \ll 1$), the magnitude of $B_T$ can be large due to higher spin textures sheet density. For example, for $n_{sk} = 5 \times 10^{12}$ cm$^{-2}$ and $\lambda_0 = 0.8$ ($ka = 2, P_s = 0.4$) one gets $B_T \approx 0.7$ T.

3. The sign of the topological Hall effect

In a real experiment when electron transport in a system with non-collinear spin textures is studied as a function of the external magnetic field $B_0$, it is often difficult to extract different contributions to the Hall effect. The total transverse resistivity $\rho_{yx}$ contains three contributions $\rho_{yx} = \rho_{yx}^O + \rho_{yx}^A + \rho_{yx}^T$, where $\rho_{yx}^O$, $\rho_{yx}^A$, $\rho_{yx}^T$ are attributed to the ordinary, anomalous and topological Hall effects respectively. Here we focus on the sign difference between $\rho_{yx}^T = (B_T/nec)$ and $\rho_{yx}^O = (B_0/nec)$, thus we should compare the signs of $B_T$ and $B_0$.

We assume, that the background magnetization $S_0$ is directed along the external magnetic field $B_0 > 0$. In general, there is now any fixed relation between the signs of the topological Hall resistivity $\rho_{yx}^T$, charge $\rho_c$ and adiabatic $\rho_a$ contributions as can be seen in Fig. 2 and Fig. 3. In these figures $\rho_a$ and $\rho_{yx}^T$ for $\lambda_3$ spin configuration change their signs upon increase of $ka$ in the crossover region. However, it is possible to specify the sign of $B_T$ in the limiting regimes, i.e. away from the threshold $E_F \gg \Delta/2$ and outside the adiabatic crossover $\lambda_0 \approx 1$.

Let us consider the weak coupling regime ($\lambda_0 \leq 1$), in which the charge current contribution to THE dominates ($\rho_c \gg \rho_a$). In this regime, the effective magnetic field is proportional to the chirality of the spin texture $B_T \propto \delta S_1 \cdot [\delta S_2 \times \delta S_3]$. For $\lambda = +1$, $\eta = +1$ the sign of the mixed vector product of any three spins $\delta S_1, \delta S_2, \delta S_3$ forming the skyrmion is negative and $B_T < 0$ due to $\delta S_z < 0$ (see Fig. 1), thus the sign of $B_T$ appears to be opposite to $B_0$. In the adiabatic regime ($\rho_a \gg \rho_c$) the electrons with positive spin projection (co-aligned with $S_0$) retain the same type of scattering asymmetry as for small $\lambda_0$. As these electrons constitute the majority at a positive spin polarization ($P_s > 0$), the effective magnetic field is also negative $B_T < 0$.

We conclude, that for $\lambda = +1$, $\eta = +1$ configurations, the topological field $B_T$ usually has the opposite sign to the sign of the external field $B_0$. For chiral spin configurations with negative vorticity $\lambda < 0$ the fields $B_T$ and $B_0$ have the same sign. However, in the crossover regime $\lambda_0 \sim 1$ and near the threshold $E_F \approx \Delta/2$ there is no any fixed relation between $B_0$ and $B_T$ signs.
4. Effect of the Fermi energy variation

The dependence of $\rho_{yx}^{T}$ on the variation of the Fermi energy $E_F$ exhibits a number of distinctive features. At $E_F < \Delta/2$ only one spin subband is occupied and spin polarization is $P_s = 1$. We start the analysis from the threshold $E_F \geq \Delta/2$, when the electrons start populating the second spin subband. In further consideration we keep $\Delta$ and a constant changing only the Fermi energy $E_F$, which we express through the dimensionless parameter $2E_F/\Delta = P_s^{-1}$. We also introduce a dimensionless spin texture diameter $\beta_{ex} = \sqrt{m\Delta/\hbar^2a}$, it is independent of $E_F$. Fig. 5 shows the dependence of $\rho_{yx}^{T}$ on $2E_F/\Delta$ and $P_s$ calculated for the $\Lambda_1$ skyrmion configuration for three different values of $\beta_{ex}$. As can be seen from the figure, $\rho_{yx}^{T}$ depends non-monotonically on $2E_F/\Delta$, with a maximum near the threshold and decreasing at a larger $E_F$. The suppression of $\rho_{yx}^{T}$ at a large $E_F$ is due to destructive scattering interference at $P_s \ll 1$ and $ka \gg 1$ (see Ref.[23]). The magnitude of $\rho_{yx}^{T}$ near the threshold is controlled by $\beta_{ex}$. As the spin-chirality driven mechanism relevant for a small skyrmion size does not work at $E_F < \Delta/2$ (there is no spin flip processes below the spin down subband edge), the decrease of $\beta_{ex}$ suppresses $\rho_{yx}^{T}$ at $E_F = \Delta/2$. These features of $\rho_{yx}^{T}$ are specific to THE and can be possibly used to distinguish it from AHE and OHE contributions.

The variation of the Fermi energy affects the asymmetric part of the scattering cross-section simultaneously through $ka$ and $P_s$ factors and, therefore, gives rise to a number of interesting features in the transverse resistivity behaviour. We demonstrate these peculiarities in Fig. 6 where the dependence of $\rho_{yx}^{T}$, $\rho_{c}$, and $\rho_{a}$ on $2E_F/\Delta$ is plotted for $\beta_{ex} = 3$ and $\beta_{ex} = 6$. The variation of $E_F$ directly affects the adiabatic parameter, which can be expressed as $\lambda_a = \beta_{ex}\sqrt{P_s}$. For $\beta_{ex} = 3$ (Fig. 6a) the adiabatic term $\rho_{a}$ is negative when far from the threshold. This is due to the complex scattering pattern typical for the intermediate range of the adiabatic parameter values ($1 \leq \lambda_a \leq 2$). We have already encountered this effect considering the behaviour of THE in the crossover regime: $\rho_{a}$ is negative in Fig. 2 for the same range of $\lambda_a$ as in Fig. 6a. For $\beta_{ex} = 6$ (Fig. 6b) $\lambda_a$ is larger and the interference in the carrier scattering manifests itself through the oscillation of $\rho_{c}, \rho_{a}$ magnitudes superimposed on the global suppression upon increasing of $E_F$. The same oscillating peculiarities of transverse response can be seen in Fig. 2 in the range $4 < \lambda_a < 5$.

Let us finally comment on the scattering rates behaviour in the vicinity of the threshold $E_F \approx \Delta/2$. Since the spin down and spin flip scattering channels are absent below the threshold $E_F < \Delta/2$, we conclude that at $E_F \approx \Delta/2$ the following relations are fulfilled: $I_2 \approx 0$, and $I_1(\theta) \approx \Pi(\theta)$. At that, only spin up scattering channel is activated with $J_{i\uparrow} \approx 2\Pi(\theta)$ (i.e. $\rho_{a} \approx \rho_{c}$).

B. Dense array of skyrmions

In this section we apply the diffusive theory of THE to the case when the dominating scattering mechanism changes from scattering on non-magnetic impurities to scattering on magnetic textures. This transition takes place, for example, in ferromagnetic films in the vicinity
of phase transition\textsuperscript{[22]}, when the sheet density of thermally activated chiral magnetic fluctuations increases\textsuperscript{[19,20,21,23]}. The exchange interaction between free carriers and localized magnetic moments is typically strong in these systems so that the adiabatic approximation for THE is applicable. At that, the spin-flip scattering channels are completely suppressed ($\Omega_{\uparrow \downarrow} = \tau_{\uparrow \downarrow}^{-1} = 0$), and THE originates solely from the spin Hall effect ($\rho_s \gg \rho_c$).

The switching of the dominant scattering mechanism affects the spin dependent scattering time $\tau_s$ (Eq.\textsuperscript{[5]}. In the dilute regime of low skyrmion density $\omega_s \tau_0 \ll 1$ considered in the previous section, the total transport scattering time $\tau_s$ is independent of the carrier spin, being determined by scattering on host non-magnetic impurities $\tau_s = \tau_0$. Increasing the skyrmion sheet density turns the system into the dense skyrmionic regime $\omega_s \tau_0 \gg 1$, when the total transport scattering time is determined solely by the magnetic skyrmions and, hence, depends on the carrier spin state $\tau_s = \omega_s^{-1}$. This transition affects both the longitudinal and transverse resistivities. Even in the dense skyrmionic regime the magnitude of $\Omega_{ss} \tau_s = \Omega_{ss} / \omega_s$ remains small, as the symmetric scattering is more effective than the asymmetric one\textsuperscript{[33]}. Thus, we can still solve the kinetic equation in the lowest order in $\Omega_{ss} \tau_s$ as described in the previous section.

Since the spin-flip processes are suppressed in the adiabatic regime, the spin up and spin down channels are uncoupled and contribute independently to the conductivity. Keeping only the leading terms with respect to $\Omega_{ss} \tau_s$ we get for the resistivity tensor:

$$\rho_{xx} = \frac{m}{ne^2\langle \tau \rangle}, \quad \rho_{yx}^T = \frac{2\pi}{ne} \frac{1}{\phi_0 n_{sk}} \int_0^{2\pi} \Pi \sin \theta d\theta,$$

$$\langle \tau \rangle = \frac{1}{2} \left[ (1 + P_s) \tau_1 + (1 - P_s) \tau_2 \right],$$

$$\Omega_s = \frac{1}{2} \left[ \frac{\tau_1^2}{\langle \tau \rangle^2} - (1 - P_s) \frac{\tau_2^2}{\langle \tau \rangle^2} \right]. \quad (16)$$

Here $P_s = (n_\uparrow - n_\downarrow)/(n_\uparrow + n_\downarrow)$ is the spin polarization of the 2D free carriers. We have also introduced an averaged scattering time $\langle \tau \rangle$. The introduced parameter $\Omega_s$ controls the conversion of the spin Hall to the charge Hall current.

In Fig.\textsuperscript{[7]} we plot the dependence of $\rho_{xx}$ and the spin/charge Hall factor $\Omega_s$ on the skyrmion sheet density $n_{sk}$ via the parameter $\omega_s \tau$ covering the transition between scattering on non-magnetic impurities and skyrmions. In the dilute regime $\langle \tau \rangle = \tau_0$, and $\rho_{xx}$ does not depend on $n_{sk}$. When $\tau_{ss}^{-1}$ exceeds $\tau_0$, the longitudinal resistivity $\rho_{xx} \propto \langle \tau \rangle^{-1}$ increases linearly with $n_{sk}$ as shown in Fig.\textsuperscript{[7]} From the experimental point of view this transition leads to the peak in resistivity upon increase of the temperature in the vicinity of the phase transition.

According to Eq.\textsuperscript{[16]} the topological Hall resistivity $\rho_{yx}^T$ is proportional to the skyrmion sheet density $n_{sk}$. The crossover in the dominating scattering mechanism affects $\rho_{yx}^T$ only via the $\Omega_s$-factor. In the dilute regime ($\omega_s \tau_0 \ll 1$) this parameter coincides with the carrier spin polarization $\Omega_s = P_s$ as the scattering time on host impurities $\tau_0$ is spin independent. However, in the dense regime ($\omega_s \tau_0 \gg 1$) the scattering time $\tau_s$ depends on the carrier spin, this dependence creates an additional spin imbalance favoring the conversion of spin to charge currents. As a result, the $\Omega_s$-factor is renormalized accounting for $\tau_\uparrow \neq \tau_\downarrow$.

Let us mention, that to describe dependence of the resistivities $\rho_{xx}, \rho_{yx}$ on temperature and external magnetic field in the vicinity of FM-transition one should specify a specific model of skyrmion/antiskyrmion creation adequate for the considered material system. The general expressions for $\rho_{xx}$ and $\rho_{yx}^T$ in the adiabatic regime\textsuperscript{[16]} are applicable for any spin-dependent scattering mechanisms, not necessarily due to skyrmions. We point out, that in the leading order with respect to $\Omega_{ss} \tau_s$, the effect of $\tau_s$ on $\rho_{yx}$ can be fully described by the replacement of the carriers spin polarization $P_s$ by an effective $\Omega_s$-factor, which accounts for $\tau_\uparrow \neq \tau_\downarrow$.

\section{C. Paramagnetic chiral systems}

In the previous sections we considered THE in a 2D magnetic layer with a background magnetization $S_0$ and local deviations forming chiral magnetic textures. Unlike anomalous Hall effect, THE does not necessarily require macroscopic spin polarization of the carriers in the sample. Therefore, THE is allowed in a system with no background magnetization provided it still has localized chiral spin textures. We will refer to this situation as to a chiral paramagnetic case. In the absence of a preferred magnetization direction the chiral spin textures with opposite orientations can be created in the same sample. A few special of features of $\rho_{yx}^T$ are expected in such a case.

One scenario leading to a chiral paramagnetic system with two independent spin orientations has been recently proposed on the basis of a magnetic polaron (BMP) in a semiconductor\textsuperscript{[32]} BMP is a collective state of a carrier
localized at an impurity center and other paramagnetic impurities lying within the localization radius. Due to the exchange interaction the carrier polarizes other magnetic impurities within its wavefunction radius; at that a non-collinear spin structure of a chiral BMP appears due to the spin-orbit splitting of the carrier band states. This effect can be viewed as a case of Dzyaloshinskii-Moriya interaction mediated by a localized electron state. There are two opposite orientations of the spin field forming chiral BMP connected by time inversion symmetry. We denote these two configurations by the orientation of spins in the center $\xi = \text{sgn}(S_z|_{r=0}) = \pm 1$. The example of the doublet of chiral BMPs with account for Dresselhaus spin-orbit interaction is shown in Fig. 8.

The presence of two spin textures with opposite orientation $\xi = \pm 1$ in the same layer modifies the expression for the charge contribution $\rho_c$ to THE [14]. Indeed, as $S_0 \approx 0$ the sign of the spin-chirality driven contributions $\Gamma_{1,2}$ to the carrier asymmetric scattering on the spin texture depends on its orientation and the contributions to $\rho_c$ from textures with $\xi = \pm 1$ have opposite sign. We arrive at the modified expression for $\rho_c$ accounting for both texture orientations $\xi = \pm 1$:

$$\rho_c = P_\xi \frac{1}{n_{ec}} \langle \phi_0 n_{sk} \rangle \int (\Gamma_1 + \Gamma_2) \sin \theta d\theta,$$

where $n_{\pm}$ are sheet densities of $\xi = \pm 1$ spin textures, respectively, $n_{sk} = n_+ + n_-$ is the total sheet density, $P_\xi$ is the polarization of the texture array in terms of their orientations. Here we consider the dilute regime with $\tau_s = \tau_0$. It follows from [17], that observation of THE in chiral paramagnetic systems is possible only when there is an imbalance in the texture orientations i.e. $P_\xi \neq 0$.

Let us note, that for the positive texture polarization $P_\xi > 0$, the sign of $\rho_c$ is different to that of $\rho_c$ for magnetic skyrmions, or non-collinear rings in Fig. 1. Indeed, as we already mentioned, $\delta S_z < 0$ for the magnetic skyrmions case leading to $\rho_c > 0$. On the contrary, $\delta S_z > 0$ is positive for $\xi = +1$ shown in Fig. 8 so that $\rho_c < 0$.

V. SUMMARY

We have developed a theory of the topological Hall effect in a 2D system with randomly located chiral magnetic textures. We calculated THE resistivity $\rho^T_{yx}$ on the basis of Boltzmann kinetic equation accounting for the carrier scattering asymmetry on a localized chiral spin texture. We have shown, that $\rho^T_{yx}$ can be expressed as a sum of two contributions: $\rho^T_{yx} = \rho_c + \rho_a$. The first one $\rho_c$ describes the transverse charge current due to spin-independent asymmetric scattering, while the second one $\rho_a$ is spin-dependent and describes the spin Hall effect, contributing in its turn to the charge Hall current provided the free carriers are spin polarized. We have investigated the interplay between adiabatic and non-adiabatic regimes of free carriers scattering on spin textures. We predict the appearance of a local minimum in the dependence of $\rho^T_{yx}$ on a skyrmion size associated with the crossover from charge Hall dominating to spin Hall dominating regime. The non-monotonic features were also found for the dependence of THE on the carriers Fermi energy. We have obtained general expressions for longitudinal and transverse components of the resistivity tensor upon the transition from dilute to dense skyrmion array in the particularly important case of the adiabatic scattering. Finally, we have clarified the role of spin-independent charge contribution to THE in dilute magnetic systems with zero background magnetization, showing that the sign of $\rho_c$ is determined by the orientation of chiral spin texture in its center.

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Appendix A: Symmetry of scattering rates

In this Appendix we derive the relations [13] for the dimensionless asymmetric scattering rates $I_{s'\ell}(\theta, \eta)$. The background polarization $\eta = \text{sgn}(S_0) = \pm 1$, it also determines the orientation of spins inside the core of
a chiral spin texture. The starting point is the time-reversal invariance, which states that if we make a replacement \( S(r) \rightarrow -S(r) \), then the scattering rate from \((p', s') \rightarrow (p, s)\) with a scattering angle \(\theta = \varphi - \varphi'\) is equal to that of \((-p', s') \rightarrow (-p, s')\) with the scattering angle \(-\theta\) (\(s'\) denotes the carrier spin state opposite to \(s\)). The replacement \( S(r) \rightarrow -S(r) \) leads to \( \eta \rightarrow -\eta \), \( \kappa \rightarrow \kappa \), and \( \gamma \rightarrow \gamma + \pi \). Collecting these operations together we obtain:

\[
\mathcal{G}_{ss'}(\theta, \eta) + \mathcal{J}_{ss'}(\theta, \eta) = \mathcal{G}_{s's}(\theta, -\eta) + \mathcal{J}_{s's}(\theta, -\eta) \tag{A1}
\]

Taking into account that \( \mathcal{G}_{ss'}(\theta, \eta) = \mathcal{G}_{s's}(\theta, -\eta) \) and \( \mathcal{J}_{ss'}(\theta, \eta) = -\mathcal{J}_{s's}(\theta, -\eta) \) we get that symmetric \( \mathcal{G}_{ss'} \) and asymmetric \( \mathcal{J}_{ss'} \) rates should satisfy:

\[
\mathcal{G}_{ss'}(\theta, \eta) = \mathcal{G}_{s's}(\theta, -\eta),
\]
\[
\mathcal{J}_{ss'}(\theta, \eta) = -\mathcal{J}_{s's}(\theta, -\eta). \tag{A2}
\]

We further focus on \( \mathcal{J}_{ss'} \). The relations \((A2)\) couple the two scattering channels with the opposite spin orientations. For the spin-conserving channels we have:

\[
\mathcal{J}_{\uparrow\uparrow}(\theta, \eta) = -\mathcal{J}_{\downarrow\downarrow}(\theta, -\eta). \tag{A3}
\]

Let us introduce the symmetrized and antisymmetrized combinations of \( \mathcal{J}_{\uparrow\uparrow}(\theta, \eta), \mathcal{J}_{\uparrow\downarrow}(\theta, -\eta) \) with respect to \( \eta \):

\[
\Gamma(\theta, \eta) = \frac{1}{2} (\mathcal{J}_{\uparrow\uparrow}(\theta, \eta) - \mathcal{J}_{\uparrow\downarrow}(\theta, -\eta)),
\]
\[
\Pi(\theta) = \frac{1}{2} (\mathcal{J}_{\uparrow\uparrow}(\theta, \eta) + \mathcal{J}_{\uparrow\downarrow}(\theta, -\eta)). \tag{A4}
\]

Since the background polarization \( \eta = \pm 1 \) can take only two values, it is obvious that the function \( \Pi \) does not depend on \( \eta \), while \( \Gamma(-\eta) = -\Gamma(\eta) \). Let us extract the dependence of \( \Gamma \) on \( \eta \) in the explicit form: \( \Gamma(\eta, \theta) \equiv \eta \Gamma_{1}(\theta) \), where \( \Gamma_{1} \) depends only on \( \theta \) and on the energy of an incident electron. Expressing the rates of the spin-conserving channels and using the symmetry \((A3)\) we arrive at the relations \((13)\):

\[
\begin{align*}
\mathcal{J}_{\uparrow\uparrow}(\theta, \eta) &= \eta \Gamma_{1}(\theta) + \Pi(\theta), \\
\mathcal{J}_{\downarrow\downarrow}(\theta, \eta) &= \eta \Gamma_{1}(\theta) - \Pi(\theta). \tag{A5}
\end{align*}
\]

As for the spin-flip channels, there is an additional symmetry \( \mathcal{J}_{\uparrow\downarrow}(\theta, \eta) = \mathcal{J}_{\downarrow\uparrow}(\theta, \eta) \) as the Hamiltonian is hermitian, this symmetry leads to the absence of a spin Hall part:

\[
\mathcal{J}_{\uparrow\downarrow}(\theta, \eta) = \mathcal{J}_{\downarrow\uparrow}(\theta, \eta) = \eta \Gamma_{2}(\theta), \tag{A6}
\]

where \( \Gamma_{2}(\theta) \) does not depend on \( \eta \).
