Robust Learning of Fixed-Structure Bayesian Networks

Ilias Diakonikolas∗
University of Southern California
diakonik@usc.edu

Daniel M. Kane†
University of California, San Diego
dakane@cs.ucsd.edu

Alistair Stewart‡
University of Southern California
alistais@usc.edu

June 24, 2016

Abstract

We investigate the problem of learning Bayesian networks in an agnostic model where an \( \epsilon \)-fraction of the samples are adversarially corrupted. Our agnostic learning model is similar to – in fact, stronger than – Huber’s contamination model in robust statistics. In this work, we study the fully observable Bernoulli case where the structure of the network is given. Even in this basic setting, previous learning algorithms either run in exponential time or lose dimension-dependent factors in their error guarantees. We provide the first computationally efficient agnostic learning algorithm for this problem with dimension-independent error guarantees. Our algorithm has polynomial sample complexity, runs in polynomial time, and achieves error that scales nearly-linearly with the fraction of adversarially corrupted samples.

1 Introduction

1.1 Motivation and Background

Probabilistic graphical models [KF09] provide an appealing and unifying formalism to succinctly represent structured high-dimensional distributions. The general problem of inference in graphical models is of fundamental importance and arises in many applications across several scientific disciplines, see [WJ08] and references therein.

In this work, we study the problem of learning graphical models from data [Nea03, DSA11]. There are several variants of this general learning problem depending on: (i) the precise family of graphical models considered (e.g., directed, undirected), (ii) whether the data is fully or partially observable, and (iii) whether the structure of the underlying graph is known a priori or not (parameter estimation versus structure learning). This learning problem has been studied extensively along these axes during the past five decades, see, e.g., [CL68, Das97, AKN06, WRL06, AHHK12, SW12, LW12, BMS13, BGS14, Bre15] for a few references, resulting in a beautiful theory and a collection of algorithms in various settings.

∗Supported in part by a Marie Curie Career Integration grant.
†Supported in part by NSF Award CCF-1553288 (CAREER). Some of this work was performed while visiting the University of Edinburgh.
‡Supported in part by a Marie Curie Career Integration grant.
The main vulnerability of all these algorithmic techniques is that they crucially rely on the assumption that the samples are precisely generated by a graphical model in the given family. This simplifying assumption is inherent for known guarantees in the following sense: if there exists even a very small fraction of arbitrary outliers in the dataset, the performance of known algorithms can be totally compromised. It is an important research direction to explore the natural setting when the aforementioned assumption holds only in an approximate sense. Specifically, we propose the following family of questions:

**Question 1** (Robust Learning of Graphical Models). Let $\mathcal{P}$ be a family of graphical models describing distributions over $\mathbb{R}^d$. Suppose we are given a set of $N$ samples drawn from some unknown distribution $\widetilde{P}$ over $\mathbb{R}^d$, such that there exists $P \in \mathcal{P}$ that is $\epsilon$-close to $\widetilde{P}$, in total variation distance. Can we efficiently find a distribution $Q \in \mathcal{P}$ that is $f(\epsilon)$-close, in total variation distance, to $\widetilde{P}$? (Here, $f : \mathbb{R} \rightarrow \mathbb{R}_+$ can be any increasing function that satisfies $\lim_{x \rightarrow 0} f(x) = 0$.)

More specifically, we would like to design robust learning algorithms for Question 1 whose sample complexity, $N$, is close to the information-theoretic minimum, and whose computational complexity is polynomial in $N$. We emphasize that the crucial requirement is that the error guarantee of the algorithm is independent of the dimensionality $d$ of the problem.

Question 1 fits in the framework of robust statistics [HR09, HRRS86]. Classical estimators from this field can be classified into two categories: either (i) they are computationally efficient but incur an error that scales polynomially with the dimension $d$, or (ii) they are provably robust (in the aforementioned sense) but are hard to compute. In particular, essentially all known estimators in robust statistics (e.g., the Tukey depth [Tuk75]) have been shown [JP78, Ber06, HM13] to be intractable in the high-dimensional setting. We also note that the robustness requirement does not typically pose information-theoretic impediments for the learning problem. In most cases of interest (see, e.g, [CGR15b, CGR15a, DKK+16]), the sample complexity of robust learning is comparable to its (easier) non-robust variant. The challenge is to design computationally efficient algorithms.

**Related Work.** We start by noting that efficient robust estimators are known for various one-dimensional structured distributions (see, e.g., [DDS12, CDSS13, CDSS14a, CDSS14b, ADLS15]). However, the robust learning problem becomes surprisingly challenging in high dimensions. Very recently, there has been algorithmic progress on this front: Two recent papers [DKK+16, LRV16] give polynomial-time algorithms with improved error guarantees for certain “simple” high-dimensional structured distributions. Specifically, [LRV16] provides error guarantees that scale logarithmically with the dimension, while [DKK+16] obtains the first dimension-independent error guarantees. The results of [DKK+16] apply to simple distributions, including Bernoulli product distributions, Gaussians, and mixtures thereof (under some natural restrictions). We remark that the algorithmic approach of this work builds on the framework of [DKK+16] together with additional technical and conceptual ideas.

**1.2 Formal Setting and Our Results**

In this work, we study Question 1 in the context of Bayesian networks [JN07]. We focus on the fully observable case when the underlying network is given. In the non-agnostic setting, this learning problem is straightforward: the “empirical estimator” (which coincides with the maximum likelihood estimator) is known to be sample and computationally efficient [Das97]. In sharp contrast, even this most basic regime is surprisingly challenging in the agnostic setting. For example, the very special case of agnostically learning a Bernoulli product distribution (corresponding to a trivial
network with no edges) was analyzed only recently in [DKK+16]. To formally state our results, we will need some terminology:

**Definition 1** (Dimension-Independent Agnostic Proper Learning). Let $\mathcal{P}$ be a family of probability distributions on $\{0, 1\}^d$. A randomized algorithm $A^\mathcal{P}$ is an agnostic distribution learning algorithm for $\mathcal{P}$, if for any $\epsilon > 0$, and any probability distribution $\tilde{P} : \{0, 1\}^d \to \mathbb{R}_+$, on input $\epsilon$ and sample access to $\tilde{P}$, with probability $9/10$, algorithm $A^\mathcal{P}$ outputs a hypothesis $Q \in \mathcal{P}$ such that $d_{TV}(\tilde{P}, Q) \leq f(\text{OPT}) + \epsilon$, where $\text{OPT} \overset{\text{def}}{=} \inf_{P \in \mathcal{P}} d_{TV}(\tilde{P}, P)$, and $f : \mathbb{R} \to \mathbb{R}_+$ is monotone increasing and satisfies $\lim_{x \to 0} f(x) = 0$.

We note that the value of OPT in the above definition is assumed to be unknown to the learning algorithm. By standard results, see e.g., Theorem 6 in [CDSS14b], the agnostic learning problem can be reduced to its special case that $\text{OPT} \leq \epsilon$. We will henceforth work with this simplified definition without loss of generality.

We point out that our agnostic learning model subsumes Huber’s $\epsilon$-contamination model [Hub64], which prescribes that the noisy distribution $\tilde{P}$ is of the form $(1 - \epsilon)P + \epsilon R$, where $P \in \mathcal{P}$ and $R$ is some arbitrary distribution.

**Bayesian Networks.** Fix a directed acyclic graph, $G$, whose vertices are labelled $[d] \overset{\text{def}}{=} \{1, 2, \ldots, d\}$, so that all edges point from vertices with smaller index to vertices with larger index. A probability distribution $P$ on $\{0, 1\}^d$ is defined to be a Bayesian network (or Bayes net) with graph $G$ if for each $i \in [d]$, we have that $\Pr_{X \sim P} [X_i = 1 \mid X_1, \ldots, X_{i-1}]$ depends only on the values $X_j$, where $j$ is a parent of $i$ in $G$. Such a distribution $P$ can be specified by its conditional probability table, the vector of conditional probabilities of $X_i = 1$ conditioned on every possible combination of values of the coordinates of $X$ at the parents of $i$. In order to clarify this notation, we introduce the following terminology:

**Definition 2.** Let $S$ be the set $\{(i, a) : 1 \leq i \leq d, a \in \{0, 1\}^{\text{Parents}(i)}\}$. Let $m = |S|$. For $(i, a) \in S$, the parental configuration $\Pi_{i,a}$ is defined to be the event that $X_{\text{Parents}(i)} = a$. Once $G$ is fixed, we may associate to a Bayesian network $P$ the corresponding conditional probability table $p \in [0, 1]^S$ given by $p_{i,a} = \Pr_{X \sim P} [X_i = 1 \mid \Pi_{i,a}]$. Furthermore, note that $P$ is determined by $p$.

We will frequently index $p$ as a vector. That is, we will use the notation $p_k$, for $1 \leq k \leq m$, and the associated events $\Pi_k$, where each $k$ stands for an $(i, a) \in S$ lexicographically ordered.

**Our Results.** We give the first efficient agnostic learning algorithm for Bayesian networks with a known graph $G$. Our algorithm has polynomial sample complexity and running time, and provides an error guarantee that scales near-linearly with the fraction of adversarially corrupted samples, under the following natural restrictions: First, we assume that each parental configuration is reasonably likely. Intuitively, this assumption seems necessary because we will need to observe each configuration many times in order to learn the associated conditional probability to good accuracy. Second, we assume that each of the conditional probabilities are balanced, i.e., bounded away from 0 and 1. This assumption is needed for technical reasons. In particular, we will need this to show that having a good approximation to the conditional probability table will imply that the corresponding Bayesian network is close in variation distance.

Formally, we say that a Bayesian network is $c$-balanced, for some $c > 0$, if all coordinates of the corresponding conditional probability table are between $c$ and $1 - c$. Using this terminology, we state our main result:

3
Theorem 3 (Main Result). Let $\epsilon > 0$. Let $P$ be a $c$-balanced Bayesian network on $\{0,1\}^d$ with known graph $G$ with the property that each parental configuration of any node has probability at least $\Omega(\epsilon^{1/2} \log^{1/4}(1/\epsilon))$, i.e., $\min_{(i,a)\in S} \Pr_P[\Pi_{i,a}] \geq \Omega(\epsilon^{1/2} \log^{1/4}(1/\epsilon))$. Let $\tilde{P}$ be a distribution on $\{0,1\}^d$ with $d_{TV}(P, \tilde{P}) \leq \epsilon$. There is an algorithm that given $G$, $\epsilon$, and sample access to $\tilde{P}$, with probability $9/10$, outputs a Bayesian network $Q$ with $d_{TV}(P, Q) \leq \epsilon \sqrt{\log 1/\epsilon} / \min_{(i,a)\in S} \Pr_P[\Pi_{i,a}]$. The algorithm takes at most $O(dm^2/\epsilon^2)$ samples and runs in polynomial time.

We remark that the condition on the minimum probability of any parental configuration follows from the assumption of being $c$-balanced when the fan-in of all nodes is bounded (e.g., for the case of trees). Indeed, if every node has at most $f$ parents, then $\min_{(i,a)\in S} \Pr_P[\Pi_{i,a}] \geq c^f$. So, as long as $c^f \geq \Omega(\epsilon^{1/2} \log^{1/4}(1/\epsilon))$, this condition automatically holds. On the other hand, this condition is impossible if a node has $\Omega(\log(1/\epsilon))$ many parents, since then some parental configuration must have small probability.

1.3 Overview of Algorithmic Techniques

Our approach builds on the filtering-based framework of [DKK+16] which was used to obtain agnostic learning algorithms for various simple families, including that of learning a binary product distribution. The latter can be seen as a starting point for our agnostic algorithm in this paper. At a high level, both algorithms work as follows: To each sample $X$, we associate a vector $F(X)$ so that learning the mean of $F(X)$ to good accuracy is sufficient to recover the distribution. In the case of binary products, $F(X)$ is simply $X$, while in our case it will need to take into account additional information about conditional means. From this point, our algorithm will try to do one of two things: Either we show that the sample mean of $F(X)$ is close to the mean of the correct distribution – in which case we can already approximate the distribution in question – or we are able to produce a filter, i.e., an efficient method which throws away some of our samples, but is guaranteed to leave us with a distribution closer to the one we are trying to learn than what we had before. If we produce a filter, we then iterate this algorithm on those samples that pass the filter until we are left with a good approximation.

To achieve the aforementioned scheme, we create a matrix $M$ which is roughly the sample covariance matrix of $F(X)$. We show that if the errors in our distribution are sufficient to notably disrupt the sample mean of $F(X)$, there must be many erroneous samples that are all far from the mean in roughly the same direction. If this is the case, it can be detected as a large eigenvalue in the matrix $M$. Specifically, if $M$ has no large eigenvalue, then we show that our sample mean is sufficient. If, on the other hand, $M$ does have a large eigenvalue, this will correspond to some direction in which many of our sample values of $F(X)$ will be far from the mean. Then, concentration bounds on $F(X)$, will imply that almost all samples far from the mean are erroneous, and thus filtering them out will provide an improved distribution.

We next look into the major ingredients required for the application of this filter-based framework to Bayesian networks, and compare our new proof with that given for balanced binary products in [DKK+16] on a somewhat more technical level.

To begin with, in each case, we need a function $F$ so that learning the mean of $F(X)$ is sufficient to learn the distribution. In the case of binary products, $F(X) = X$ is sufficient, since the coordinate means determine the distribution. For Bayesian networks, the situation is somewhat more complicated. In particular, we need to learn all of the relevant conditional means. Now each sample, $X$, will give us information on some of the relevant conditional means, but not all of them. In order to “fill out” our vector, we will need to produce entries corresponding to conditional means
for which the condition failed to happen. We do this by filling such entries with an approximation to the relevant conditional mean.

Next, we need to consider the eigenvalues of $M$. It is easy to show that sampling error can be ignored if we take sufficiently many samples, so we can instead consider the matrix given by the actual covariances of the distribution in question. We break this matrix into three parts. One coming from the distribution we are trying to learn, one coming from the subtractive error (i.e., the points that have smaller probability under the noisy distribution), and one coming from the additive error. In both cases, the noise-free distribution has a diagonal covariance matrix. For binary products this is trivial, but for Bayesian networks the analogous result takes some additional technical work. Furthermore, the term coming from the subtractive error will have no large eigenvalues. This is because of concentration bounds. These will imply that the tails of $F(X)$ are not wide enough so that taking a small amount of mass away from them can introduce much error. For the binary product case, this is by standard Chernoff bounds, but for Bayesian networks, we must instead rely on martingale arguments and Azuma’s inequality. Together, this says that any large eigenvalues are due to the additive error, which can be large. Finally, we will need to reuse our concentration bounds to show that if our additive errors are reasonably frequently far from the mean in a known direction, then they can be reliably distinguished from good samples.

1.4 Organization

Section 2 contains the results specific to Bayesian networks that we need to specialize our filtering algorithm for this setting. Section 3 gives the details of the algorithm and its proof of correctness. In Section 4 we conclude and propose directions for future work.

2 Technical Preliminaries

The structure of this section is as follows: First, we need to be able to bound the total variation distance between two Bayes nets in terms of their conditional probability tables. Second, we define a function $F(x, q)$ – which takes a sample and returns an $m$-dimensional vector by filling out the coordinates corresponding to unobserved conditional means – and show some of its properties. Finally, we derive a concentration bound from Azuma’s inequality.

We note that a few technical proofs from this section have been deferred to Appendix A.

**Lemma 4.** Let $P$ and $Q$ be Bayesian networks with the same dependency graph $G$. In terms of the conditional probability tables $p$ and $q$ of $P$ and $Q$, we have:

$$d_{TV}(P, Q)^2 \leq 2 \sum_{k=1}^{m} \sqrt{\Pr_P[\Pi_k] \Pr_Q[\Pi_k]} \frac{(p_k - q_k)^2}{(p_k + q_k)(2 - p_k - q_k)}.$$  

The dependence of this bound on both $\Pr_P[\Pi_k]$ and $\Pr_Q[\Pi_k]$ is not ideal. One reason that we need to specify a minimum probability for $\Pr_P[\Pi_k]$ is to argue that $\Pr_Q[\Pi_k]$ is close to this and so get a bound containing only $\Pr_P[\Pi_k]$.

We can obtain a simpler expression for $c$-balanced binary Bayesian networks and those in which the minimum probability of any of the $\Pi_k$ is larger than the final error bound:

**Corollary 5.** Suppose that: (i) $\min_{k \in [m]} \Pr_P[\Pi_k] \geq 2\epsilon$, and (ii) $P$ or $Q$ is $c$-balanced, and (iii) $\frac{3}{7} \sqrt{\sum_k \Pr_P[\Pi_k] (p_k - q_k)^2} \leq \epsilon$. Then we have that $d_{TV}(P, Q) \leq \epsilon$.
Note that a given sample \( x \) from \( P \) only gives us information about a subset of the conditional probabilities. Specifically, \( x_i \) gives us information about \( p_{i,a} \) if and only if \( x \in \Pi_{i,a} \). To get an \( m \)-dimensional vector from \( x \), we need to specify all the other unobserved coordinates. We will do so by setting these coordinates to their conditional means or an approximation or guess for these means.

**Definition 6.** Let \( F(x, q) \) for \( \{0,1\}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^m \) be defined as follows: If \( x \in \Pi_{i,a} \), then \( F(x, q)_{i,a} = x_i \), otherwise \( F(x, q)_{i,a} = q_{i,a} \).

When \( q = p \), we have that the expectation of the \( (i,a) \)-th coordinate of \( F(X,p) \), for \( X \sim P \), is the same conditioned on either \( \Pi_{i,a} \) or \( \neg \Pi_{i,a} \). Using the conditional independence properties of Bayesian networks, we will show that the covariance of \( F(x, p) \) is diagonal. Our algorithm makes crucial use of this fact to detect whether or not the empirical conditional probability table of the noisy distribution is close to the true conditional probability table.

First, we note that \( F \) is invertible in the following sense:

**Lemma 7.** For all \( x \in \{0,1\}^d \), \( q \in [0,1]^m \) and \( 1 \leq j \leq k \), \( x_1, \ldots, x_j \) can be recovered from the vector of coordinates \( F(x, p)_{i,a} \) for all \( (i, a) \in S \) with \( j \leq i \), and these coordinates of \( F \) can be computed from only \( x_1, \ldots, x_j \).

The coordinates of \( F(X, p) \) have the following conditional independence property:

**Lemma 8.** For \( X \sim P \) and any \((i, a) \in S\), if we condition on either \( \Pi_{i,a} \) or \( \neg \Pi_{i,a} \), then \( F(X, p)_{i,a} \) is independent of \( F(X, p)_{j,a} \) for all \( (j, a') \in S \) with \( j \leq i \) and \( (j, a') \neq (i, a) \).

Now we can show our claim about the mean and covariance of \( F(X, p) \):

**Lemma 9.** For \( X \sim P \), we have \( \mathbb{E}(F(X, p)) = P \). The covariance matrix of \( F(X, p) \) satisfies \( \text{Cov}[F(X, p)] = \text{diag}(\mathbb{P}_{P}(S)_{k}(1 - p_k)) \).

Although the coordinates of \( F(X, p) \) are not independent, the first and second moments are the same as that of a product distribution of the marginal of each coordinate.

We will also need a suitable concentration inequality. Thanks to conditional independence properties, we can use Azuma’s inequality to show:

**Lemma 10.** For \( X \sim P \) and any unit vector \( v \in \mathbb{R}^d \), we have

\[
\text{Pr}[\|v \cdot (F(X, p) - q)\| \geq T + \|p - q\|_2] \leq 2 \exp(-T^2/2) .
\]

**Proof.** It follows from Lemmas 7 and 8 that \( \mathbb{E}_{X \sim P}[F(X, p)_{k-1} \mid F(X, p)_{1}, \ldots, F(X, p)_{k-1}] = p_k \) for all \( 1 \leq k \leq m \) (see Claim 27 in the appendix). Thus, the sequence \( \sum_{k=1}^d v_k(F(X, p)_k - p_k) \) for \( 1 \leq k \leq m \) is a martingale, and we can apply Azuma’s inequality. Note that the support size of \( v_k(F(X, p)_k - p_k) \) is \( v_k \) and thus we have

\[
\text{Pr}[\|v \cdot (F(X, p) - p)\| \geq T] \leq 2 \exp(-T^2/2\|v\|_2) = 2 \exp(-T^2/2) .
\]

Consider an \( x \in \{0,1\}^d \). If \( x \in \Pi_{i,a} \), then we have \( F(x, p)_{i,a} = F(x, q)_{i,a} = x_i \) and so

\[
\|(F(x, p)_{i,a} - p_{i,a}) - (F(x, q)_{i,a} - q_{i,a})\| = |p_{i,a} - q_{i,a}| .
\]

If \( x \notin \Pi_{i,a} \), then \( F(x, p)_{i,a} = p_{i,a} \) and \( F(x, q)_{i,a} = q_{i,a} \), hence

\[
\|(F(x, p)_{i,a} - p_{i,a}) - (F(x, q)_{i,a} - q_{i,a})\| = 0 .
\]

Thus, we have

\[
\|(F(x, p) - p) - (F(x, q) - q)\|_2 \leq \|p - q\|_2 .
\]

An application of the Cauchy-Schwarz inequality gives that \( |v \cdot (F(X, q) - q)| \geq T + \|p - q\|_2 \) implies that \( |v \cdot (F(X, p) - p)| \geq T \). The probability of the former holding for \( X \) must therefore be at most the probability that the latter holds for \( X \).
3 Efficient Agnostic Learning Algorithm

Our main result will follow by iterating the efficient procedure described in the following proposition:

**Proposition 11.** Let $\epsilon > 0$. Let $P$ be a Bayesian network on $\{0,1\}^d$ with known graph $G$ with the property that each parental configuration of any node has probability at least $\Omega(e^{1/2} \log^{1/4}(1/\epsilon))$, i.e., $\min_{k \in [m]} \Pr P[\Pi_k] \geq \Omega(e^{1/2} \log^{1/4}(1/\epsilon))$. Let $P$ be a distribution with $\tilde{P} = w_P P + w_E E - w_L L$ for distributions $E$ and $L$ with disjoint support and $w_P, w_E, w_L \geq 0$ and $w_E + w_L \leq 2\epsilon$. There is an algorithm that given $G$, $\epsilon$ and sample access to $\tilde{P}$, with probability $1 - 1/(100(2d + 1))$, either

1. outputs a Bayesian network $Q$ with $d_{TV}(P, Q) \leq \epsilon \sqrt{\log 1/\epsilon/ \min_k \Pr P[\Pi_k]}$, or

2. gives a filter $A$ such that $\tilde{P}$ conditioned on $A$ accepting can be written as $w_P' P + w_E' E' + w_L' L'$, where $E'$ and $L'$ are distributions with disjoint support, $w_E', w_L' \geq 0$ and $w_E' + w_L' \leq w_E + w_L - \epsilon/d$. Furthermore, the probability that $A$ rejects a random element of $\tilde{P}$ is at most $O((w_E + w_L) - (w_E' + w_L'))$.

The algorithm uses $\tilde{O}(m^2/\epsilon^2)$ samples and runs in polynomial time.

If this algorithm produces a filter, we will iterate it using all filters generated in previous iterations, until it outputs a distribution. By the equation, $\tilde{P} = w_P P + w_E E - w_L L$, we mean that the pmfs of these distributions satisfy $\tilde{P}(x) = w_P P(x) + w_E E(x) - w_L L(x)$ for all $x \in \{0,1\}^d$. We note that when we initially $d_{TV}(\tilde{P}, P) \leq \epsilon$, there are distributions $E$ and $L$ that satisfy this:

**Fact 12.** Let $P$ and $Q$ be probability distributions supported over a discrete set $D$. There exist probability distributions $E$ and $L$ over $D$ with disjoint supports, so that for all $x \in D$ it holds:

(i) $Q(x) = P(x) + d_{TV}(P, Q) \cdot E(x) - d_{TV}(P, Q) \cdot L(x)$, and

(ii) $d_{TV}(P, Q) \cdot L(x) \leq P(x)$.

**Proof.** For $x \in D$, we define $E(x) = \frac{\max\{Q(x) - P(x), 0\}}{d_{TV}(P, Q)}$, and $L(x) = \frac{\max\{P(x) - Q(x), 0\}}{d_{TV}(P, Q)}$. It is clear that $E$ and $L$ have disjoint supports and that they satisfy conditions (i) and (ii). $\square$

We will show that, if an iteration produces a filter, the filter will reject more samples from $E$ than from $P$. This property has the effect of decreasing $w_E$. However, any false positives result in an increase in $w_L$. Perhaps counterintuitively, this implies that the total variation distance between $\tilde{P}$ and $P$ can increase (slightly) from iteration to iteration. However, we are able to show that $w_L + w_E$ decreases with high probability. We note that Proposition 11 suffices to establish Theorem 3. We include the simple proof here for the sake of completeness.

**Proof of Theorem 3 assuming Proposition 11.** Consider the first iteration of the overall algorithm. By Fact 12 we can write $\tilde{P} = P + d_{TV}(P, \tilde{P}) \cdot E - d_{TV}(P, \tilde{P}) \cdot L$ for distributions $E$ and $L$ with disjoint support. Since we have that $d_{TV}(P, \tilde{P}) \leq \epsilon$, it follows that $w_E + w_L \leq 2\epsilon$.

If in each iteration before the $i$-th one we output a filter satisfying the second condition of Proposition 11 then at the $i$-th iteration, we have $w_E^{(i)} + w_L^{(i)} \leq 2\epsilon - (i - 1)\epsilon/d$. Since $w_E^{(i)}, w_L^{(i)} \geq 0$, this can only happen for $2d$ iterations. By a union bound, the algorithm succeeds with probability at least $99/100$ for the first $2d + 1$ iterations. Therefore, with at least this probability, it will output a Bayesian network satisfying the first condition.

Furthermore, after any number of these iterations the probability that a sample from our original distribution $\tilde{P}$ is accepted by all of our filters is at least $\exp(-O(w_E + w_L)) = \exp(O(\epsilon)) \geq 9/10$. Finally, note that each iteration draws $\tilde{O}(m^2/\epsilon^2)$ samples from $\tilde{P}$, and so the total sample complexity is $\tilde{O}(dm^2/\epsilon^2)$. $\square$
The main part of this section is devoted to the proof of Proposition 11 and is structured as follows: In Section 3.1, we describe the algorithm Filter-Known-Topology establishing Proposition 11 in tandem with an overview of its analysis. In Section 3.2, we present a number of useful structural lemmas. Sections 3.3 and 3.4 give the proof of correctness for the two regimes of the algorithm Filter-Known-Topology.

3.1 Algorithm Filter-Known-Topology

The algorithm establishing Proposition 11 is presented below:

Algorithm Filter-Known-Topology

1. Draw \( O(\log(m)/\epsilon^2) \) samples from \( \tilde{P} \) and compute the empirical estimate \( \alpha \) for \( \min_k \Pr_{\tilde{P}}[\Pi_k] \).
2. Draw \( N = \Theta(m^2 \log(m/\epsilon)/\epsilon^2) \) samples from \( \tilde{P} \) and compute the empirical conditional probability table \( q \).
3. Draw \( N \) samples \( s^1, \ldots, s^N \) from \( \tilde{P} \).
4. Compute the sample covariance of \( F(\tilde{P}, q) \) with zeroed diagonal from these samples, i.e., the matrix \( M = ([M_{i,j}]_{1 \leq i,j \leq d} \text{ with } M_{i,j} = \frac{1}{N} \sum_{\ell=1}^{N} (F(s^\ell, q)_i - q_i)(F(s^\ell, q)_j - q_j), i \neq j, \text{ and } M_{i,i} = 0. \)
5. Compute approximations for the eigenvalue of \( M \) with largest absolute value, \( \lambda^* \), and the associated eigenvector \( v^* \).
6. If \( |\lambda^*| \leq O(\epsilon \log(1/\epsilon)/\alpha) \), return \( q \).
7. Let \( \delta := 3\sqrt{\epsilon |\lambda^*|/\alpha} \).
8. Draw another \( N \) samples \( t^1, \ldots, t^N \) from \( \tilde{P} \).
9. Let \( T > 0 \) be a real number so that
   \[
   \frac{\# \{ i : \left| v \cdot (F(t^i, q) - q) \right| > T + \delta \}}{N} > 6 \exp(-T^2/2) + 5\epsilon/d.
   \]
10. Return the filter \( A \), where \( A(x) = \text{Reject} \) if and only if \( \left| v \cdot (F(x, q) - q) \right| > T + \delta \).

Overview of Analysis. Recall that we write \( \tilde{P} = w_P \cdot P + w_E \cdot E - w_L \cdot L \) for disjoint distributions \( E \) and \( L \), where \( w_P + w_E - w_L = 1 \) and \( w_E, w_L \geq 0 \) with \( w_E + w_L \leq 2\epsilon \). Thus, \( |w_P - 1| \leq 2\epsilon \) and for all events \( B \), \( w_L \Pr_L[B] \leq w_P \Pr_P[B] \) and \( w_E \Pr_E[B] \leq \Pr_{\tilde{P}}[B] \).

The basic idea of the analysis is as follows: By taking enough samples, we can ensure that \( q \) is close to the true conditional probability table of \( \tilde{P} \). This will give us a good approximation to the actual mean vector, \( p \) (and thus a good approximation to the distribution \( P \)) as long as the conditional expectations of \( \tilde{P} \) are close to the conditional expectations of \( P \). In other words, we will be in good shape so long as the error between \( P \) and \( \tilde{P} \) does not introduce a large error in the conditional probability table.
Thinking more concretely about this error, we may split it into two parts: $w_L L$, the subtractive error, and $w_E E$ the additive error. Using concentration results for $P$, it can be shown that the subtractive errors cannot cause significant problems for the conditional probability table. It remains to consider additive errors. These clearly can produce notable problems in the conditional probability table, since any given sample can produce terms $\sqrt{d}$ away from the mean. If many of the additional terms line up in some direction, this can lead to a notable discrepancy.

However, if many of these errors line up in some direction (which is necessary in order to have a large impact on the mean), the effects will be reflected in the first two moments. More concretely, if for some unit vector $v$, the expectation of $v \cdot F(E, q)$ is very far from the expectation of $v \cdot F(P, q)$, this will force the variance of $v \cdot F(\tilde{P}, q)$ to be large. This implies two things: First, it tells us that if $v \cdot F(\tilde{P}, q)$ is small for all $v$ (a condition equivalent to $\|M\|_2$ being small), we know that $q$ is a good approximation to the true conditional probability table. On the other hand, if $\|M\|_2$ is large, we can find a large eigenvector, which corresponds to a unit vector $v$ so that $v \cdot F(\tilde{P}, q)$ has large variance. Given such a vector, we note that a reasonable fraction of this variance must be coming from samples taken from $E$ that have $v \cdot F(X, q)$ very far from the mean. On the other hand, using concentration bounds for $v \cdot F(P, q)$, we know that very few valid samples are this far from the mean. This discrepancy will allow us to create a filter in which we throw away any samples $X$ where $v \cdot F(X, q)$ is too far from $v \cdot q$. Our filter can be built in such a way so that the vast majority of the rejected samples came from $E$, thus decreasing the variation distance.

### 3.2 Setup and Structural Lemmas

In order to understand the second moment matrix, $M$, we will need to break down this matrix in terms of several related matrices defined below:

- Let $M_E$ be the matrix with zero diagonal and $(i, j)$ entry $E_{X \sim P}[(F(X, q)_i - q_i)(F(X, q)_j - q_j)]$ when $i \neq j$.
- Let $M_P$ be the matrix with zero diagonal and $(i, j)$ entry $E_{X \sim P}[(F(X, q)_i - q_i)(F(X, q)_j - q_j)]$ when $i \neq j$.
- Let $M_k$ be the matrix with $(i, j)$ entry $E_{X \sim L}[(F(X, q)_i - q_i)(F(X, q)_j - q_j)]$ for all $i, j$.
- Let $M_E$ be the matrix with $(i, j)$ entry $E_{X \sim E}[(F(X, q)_i - q_i)(F(X, q)_j - q_j)]$ for all $i, j$.

We begin our analysis with some basic bounds, showing that with high probability that $\alpha, q, \epsilon$, and $M$ are all appropriately close to the things that they are designed to estimate.

**Lemma 13.** With probability at least $1 - \frac{1}{200(2d+1)^2}$, $\alpha - \min_k Pr[\Pi_k] \leq \epsilon$, $\|q - E_{X \sim \tilde{P}}[F(X, q)]\|_2 \leq \epsilon$ and $\|M - M_{\tilde{P}}\|_2 \leq \epsilon$.

The proof of this lemma is given in Appendix B.1. We will henceforth assume that the conditions of Lemma 13 hold.

Our next step is to analyze the spectrum of $M$, and in particular show that $M$ is close in spectral norm to $w_E M_E$. To do this, we begin by considering the spectral norm of $M_P$. In particular, we claim that it is relatively small. This can be shown by a relatively straightforward computation, as we know exactly the second moments of $F(P, q)$ (see Appendix B.4 for the simple proof):

**Lemma 14.** $\|M_P\|_2 \leq \sum_k Pr[\Pi_k] (p_k - q_k)^2$. 


Next, we wish to bound the contribution to $\tilde{M}_P$ coming from the subtractive error. We show that this is small due to concentration bounds on $P$. The idea is that for any unit vector $v$ we have that $v \cdot F(P, q)$ is tightly concentrated around $q$. Since $w_L L$ can at worst consist of a small fraction of the tail of this distribution, the expectation of $v \cdot (F(L, q) - q)$ cannot be too large.

**Lemma 15.** $w_L \|M_L\|_2 \leq O(\epsilon \log (1/\epsilon) + \epsilon \|p - q\|^2_2)$.

**Proof.** For any event $A$, we have that $w_L \Pr_P[A] \leq w_P \Pr_P[A] \leq 2 \Pr_P[A]$. Thus, using Lemma 13, 14, and 15. Note that this is small due to concentration bounds on $P$, we obtain:

$$\Pr_{X \sim L} [v \cdot (F(X, q) - q)] \geq T + \|p - q\|_2 \leq (4/w_L) \exp(-T^2/2).$$

By definition, $\|M_L\|_2$ is the maximum over unit vectors $v$ of $v^T M_L v$. For any unit vector $v$, we have

$$w_L v^T M_L v = w_L \mathbb{E}_{X \sim L} [(v \cdot (F(X, q) - q))^2]$$

$$\leq 2w_L \int_0^\infty \Pr_{X \sim L} [v \cdot (F(X, q) - q)] \geq T \, T dT$$

$$\leq \int_0^{\|p - q\|_2^2 + |\ln(2/w_L)|} w_L T dT + \int_{\|p - q\|_2^2 + |\ln(2/w_L)|}^\infty \exp\left(-\frac{(T - \|p - q\|_2^2)}{2}\right) T dT$$

$$\leq w_L \|p - q\|^2_2 + w_L (\log(1/w_L) + 1)$$

$$\leq \epsilon \log(1/\epsilon) + \epsilon \|p - q\|^2_2. \quad \text{(since } w_L \leq 2\epsilon)$$

Finally, combining the above results, since $M_P$ and $M_L$ have little contribution to the spectral norm of $\tilde{M}_P$, most of it must come from $M_E$.

**Lemma 16.** $\|M - w_E M_E\|_2 \leq O(\epsilon \log (1/\epsilon) + \sum_k \Pr_P[\Pi_k] (p_k - q_k)^2)$.

**Proof.** Note that $M_P = w_P M_P + w_E M_E - w_L M_L$. By the triangle inequality

$$\|M - w_E M_E\|_2 \leq \|M - M_P\|_2 + w_P \|M_P\|_2 w_L \|M_L\|_2$$

$$\leq O(\epsilon) + O\left(\sum_k \Pr_P[\Pi_k] (p_k - q_k)^2 + \epsilon \log(1/\epsilon) + \epsilon \|p - q\|^2_2\right),$$

using Lemmas 13, 14, and 15. Note that $\epsilon \|p - q\|^2_2 \leq \sum_k \Pr_P[\Pi_k] (p_k - q_k)^2$, since we assumed that $\min_k \Pr_P[\Pi_k] \geq \epsilon$.
Lemma 18. \((\mathbb{E}_{X \sim P}[F(X, q)] - q)_k = \text{Pr}_P[\Pi_k](p_k - q_k)\).  

**Proof.** We have that when \(\Pi_k\) does not occur, \(F(X, q)_k = q_k\). Thus, we can write:

\[
\mathbb{E}_{X \sim P}[F(X, q)_k - q_k] = \text{Pr}_P[\Pi_k]\mathbb{E}_{X \sim P}[F(X, q)_k - q_k | \Pi_k] = \text{Pr}_P[\Pi_k](p_k - q_k),
\]
as desired. \(\square\)

### 3.3 The Case of Small Spectral Norm

Using the lemmas from Section 3.2, we can bound the variation distance between \(P\) and \(Q\), the Bayesian net with conditional probability table \(q\), in terms of \(\|M\|_2\):

**Lemma 19.** \(\sqrt{\sum_k \text{Pr}_P[\Pi_k]^2(p_k - q_k)^2} \leq 2\sqrt{\epsilon\|M\|_2} + O(\epsilon\sqrt{\log(1/\epsilon)})\).

**Proof.** For notational simplicity, let \(D = \text{diag}(\sqrt{\text{Pr}_P[\Pi_k]})\), so we have

\[
\sqrt{\sum_k \text{Pr}_P[\Pi_k](p_k - q_k)^2} = \|D(p - q)\|_2.
\]

Let \(\mu^P, \mu^\bar{P}, \mu^L\) and \(\mu^E\) be \(\mathbb{E}[F(X, q)]\) for \(X\) distributed as \(P, \bar{P}, L\) or \(E\) respectively. Then we have

\[
\mu^\bar{P} = w_P\mu^P - w_L\mu^L + w_E\mu^E.
\]

Note that, by Lemma 18 \(\mu^\bar{P} - q = D^2(p - q)\). By Lemma 13 \(\|q - \mu^\bar{P}\|_2 \leq \epsilon\). Then, by the triangle inequality, we obtain

\[
\|D^2(p - q)\|_2 = \|\mu^P - q\|_2 \leq \|\mu^P - \mu^\bar{P}\|_2 + \epsilon
\]

\[
\leq w_L\|\mu^L - \mu^P\|_2 + w_E\|\mu^E - \mu^P\|_2 + \epsilon
\]

\[
\leq w_L\|\mu^L - q\|_2 + w_E\|\mu^E - q\|_2 + O(\epsilon\|D^2(p - q)\|_2) + \epsilon
\]

\[
\leq w_L\sqrt{\|M\|_2} + w_E\sqrt{\|M\|_2} + O(\epsilon\sqrt{\|D(p - q)\|_2}) + \epsilon
\]

\[
\leq O\left(\epsilon\sqrt{\log(1/\epsilon)}\right) + 2\epsilon\sqrt{\|M\|_2} + O(\sqrt{\epsilon}\|D(p - q)\|_2)
\]

\[
\leq O\left(\epsilon\sqrt{\log(1/\epsilon)}\right) + (3/2)\sqrt{\epsilon\|M\|_2} + \|D^2(p - q)\|_2/4,
\]

where we used the assumption that \(\epsilon\) is smaller than a sufficiently small constant. Rearranging the inequality completes the proof. \(\square\)

**Corollary 20.** If \(\|M\|_2 \leq O(\epsilon\log(1/\epsilon)/\alpha)\), then \(d_{TV}(P, Q) = O(\epsilon\sqrt{\log(1/\epsilon)/\min_k \text{Pr}_P[\Pi_k]})\).

**Proof.** By Lemma 13 \(|\alpha - \min_k \text{Pr}_P[\Pi_k]| \leq \epsilon\). Since \(d_{TV}(\bar{P}, P) = \max\{w_E, w_L\} \leq 2\epsilon\), we have that \(|\alpha - \min_k \text{Pr}_P[\Pi_k]| \leq 3\epsilon\). Since by assumption \(\min_k \text{Pr}_P[\Pi_k] \geq \sqrt{\epsilon}\), we have that \(\alpha = \Theta(\min_k \text{Pr}_P[\Pi_k])\).

Therefore, we have

\[
\sqrt{\sum_k \text{Pr}_P[\Pi_k](p_k - q_k)^2} \leq \sqrt{\sum_k \text{Pr}_P[\Pi_k]^2(p_k - q_k)^2/\min_k \text{Pr}_P[\Pi_k]}
\]

\[
\leq (2\sqrt{\epsilon\|M\|_2} + O(\epsilon\sqrt{\log(1/\epsilon)}))/\sqrt{\min_k \text{Pr}_P[\Pi_k]}
\]

(by Lemma 19)

\[
\leq O(\epsilon\sqrt{\log(1/\epsilon)/\min_k \text{Pr}_P[\Pi_k]}).
\]

Now we can apply Corollary 5 with \(O(\epsilon\sqrt{\log(1/\epsilon)/\min_k \text{Pr}_P[\Pi_k]})\) in place of \(\epsilon\). Note that condition (i) follows from our assumption that \(\min_k \text{Pr}_P[\Pi_k] = \Omega(\epsilon^{1/2}\log^{1/4}(1/\epsilon))\). \(\square\)
3.4 The Case of Large Spectral Norm

Now we consider the case when $\|M\|_2 \geq C\epsilon \ln(1/\epsilon)/\alpha$. We begin by showing that $p$ and $q$ are not too far apart from each other.

Claim 21. $\|p - q\|_2 \leq \delta := 3\sqrt{\epsilon \|M\|_2 / \alpha}$.

Proof. By Lemma 19, we have that $\sqrt{\sum_k \mathbb{P}[\Pi_k]^2 (p_k - q_k)^2} \leq 2\sqrt{\epsilon \|M\|_2} + O(\epsilon \sqrt{\log(1/\epsilon)})$. For sufficiently large $C$, this last term is smaller than $\frac{1}{2}\sqrt{\epsilon \|M\|_2} \geq \epsilon C\ln(1/\epsilon)$. Then we have $\sqrt{\sum_k \mathbb{P}[\Pi_k]^2 (p_k - q_k)^2} \leq (5/2)\sqrt{\epsilon \|M\|_2}$. However, $\left(\min_k \mathbb{P}[\Pi_k]\right)^2 \sum_k (p_k - q_k)^2 \leq \sum_k \mathbb{P}[\Pi_k]^2 (p_k - q_k)^2$ and $\min_k \mathbb{P}[\Pi_k] = \alpha + O(\epsilon)$. \hfill \qed

Next we show that most of the variance of $v^* \cdot F(\tilde{P}, q)$ comes from the additive error.

Claim 22. $w_E v^*^T M_E v^* \geq \frac{1}{2} v^*^T M v^*$.

Proof. By Lemma 16 and Claim 21, we deduce

$$\|M - w_E M_E\|_2 \leq O \left( \epsilon \log(1/\epsilon) + \sum_k \mathbb{P}[\Pi_k] (p_k - q_k)^2 \right) \leq O \left( \epsilon \log(1/\epsilon) + \left( \epsilon / \min_k \mathbb{P}[\Pi_k] \right) \|M\|_2 \right).$$

By assumption $\min_k \mathbb{P}[\Pi_k] \geq C'\epsilon$ for sufficiently large $C'$, so we can bound this second term by $\|M\|_2$. For large enough $C$, the first term is bounded by $\|M\|_2/6$. Thus, $\|M - w_E M_E\|_2 \leq \|M\|_2/2$. Since $v^T M v^* = \|M\|_2$, we obtain $w_E v^*^T M_E v^* \geq \frac{1}{2} v^*^T M v^*$, as required. \hfill \qed

This will imply that the tails of $w_E E$ are reasonably thick. In particular, we show that there must be a threshold $T > 0$ satisfying a property similar to the one desired in Step 9 of our algorithm.

Lemma 23. There exists a $T \geq 0$ such that

$$\mathbb{P}[v^* \cdot (F(X, q) - q) \geq T + \delta] \geq T + \delta > 6 \exp(-T^2/2) + 6\epsilon/d.$$  

Proof. Suppose for the sake of contradiction that this does not hold. For all events $A$, it holds that $w_E \mathbb{P}[A] \leq \mathbb{P}[\tilde{P}][A]$. Thus, we have

$$w_E \mathbb{P}[v^* \cdot (F(X, q) - q) \geq T + \delta \leq 6 \exp(-T^2/2) + 6\epsilon/d].$$

Note that for any $x \in \{0, 1\}^d$, we have that

$$|v^* \cdot (F(x, q) - q)| \leq \|F(x, q) - q\|_2 \leq \sqrt{d}.$$
since $F(x, q)$ and $q$ differ on at most $d$ coordinates. We have the following sequence of inequalities:

$$
\left\| M \right\|_2 \ll w_E v^* T M v^* \\
= 2w_E \int_0^{\sqrt{d}} \Pr_{x \sim E} [|v^* \cdot (F(X, q) - q)| \geq T] \, T \, dT \\
\leq 2w_E \int_0^{\delta + 4 \sqrt{\ln(1/w_E)}} T \, dT + \int_0^{\sqrt{d}} (\exp(-(T - \delta)^2) T + (\epsilon T/d)) \, dT \\
\ll w_E \delta^2 + w_E \ln(1/w_E) + \epsilon \\
\ll (\epsilon^2/\alpha^2) \left\| M \right\|_2 + \alpha \left\| M \right\|_2/C.
$$

Since $\epsilon^2/\alpha^2 = O(\epsilon)$ and $\alpha \leq 1$, for sufficiently large $C$, this gives the desired contradiction. 

We will also need that the samples $t_i$ actually give us a good approximation to the tails of $v^* \cdot F(\tilde{P}, q)$. Fortunately, this happens with high probability using the DKW inequality [DKW56, Mas90]:

Lemma 24. With probability at least $1 - 1/(200(2d + 1))$, for all $T > 0$, we have

$$
\left| \left\{ i : |v^* \cdot (F(t_i, q) - q)| > T + \delta \right\} \right|_N - \Pr_{X \sim \tilde{P}} [|F(X, q) - q| \geq T + \delta] \leq \epsilon/d.
$$

We assume that this is the case. We then have that Step 9 will find a $T > 0$:

Corollary 25. There exists a $T > 0$ such that

$$
\left\{ i : |v^* \cdot (F(t_i, q) - q)| > T + \delta \right\} > 6 \exp(-T^2/2) + 5\epsilon/d.
$$

Finally, we consider the distribution $\tilde{P}$ conditioned on $A$ accepting. We claim that the significant majority of rejected samples came from $E$, leading to a cleaner distribution. In particular, this will complete the proof of Proposition 11:

Claim 26. We can write $\tilde{P}'$ which is $\tilde{P}$ conditioned on $A$ accepting as $\tilde{P}' = w'_{E'} P + w'_{E'} E' - w'_{L'} L'$, where $L'$ and $E'$ have distinct supports, $w'_{L'} > 0$ and $w'_{E'} + w'_{L'} \leq w_E + w_L - \epsilon/d$. Furthermore, the probability that $A$ rejects a random element of $\tilde{P}$ is at most $O((w_E + w_L) - (w'_{E'} + w'_{L'}))$.

Proof. Let $A_R$ be the event that $A$ rejects.

By Corollary 25, Step 9 finds a $T > 0$ such that

$$
\left\{ i : |v^* \cdot (F(t_i, q) - q)| > T + \delta \right\} > 6 \exp(-T^2/2) + 5\epsilon/d.
$$

By Lemma 24, we have that

$$
\Pr_{X \sim \tilde{P}} [|F(X, q) - q| \geq T + \delta] > 6 \exp(-T^2/2) + 4\epsilon/d.
$$

On the other hand, by Lemma 10

$$
\Pr_{X \sim \tilde{P}} [|F(X, q) - q| \geq T + \delta] \leq 2 \exp(-T^2/2).
$$
Thus, we have $\Pr_{\bar{P}}[A_R] \geq 3 \Pr_P[A_R]$. Since

$$\Pr_P[A_R] \leq w_P \Pr_P[A_R] + w_E \Pr_P[A_R]$$

and $w_P \leq 1 + O(\epsilon)$, we must have

$$\Pr_P[A_R] \leq (2 + O(\epsilon))w_E \Pr_P[A_R]$$

and

$$\Pr_P[A_R] \leq (1/3 + O(\epsilon))w_E \Pr_P[A_R].$$

Then, we get that

$$\left(1 - \Pr_P[A_R]\right) \bar{P}'(x) = \left(1 - \Pr_P[A_R]\right) (\bar{P} \mid A_R)(x)$$

$$= w_P P(x) + w_E \left(1 - \Pr_E[A_R]\right) E(x) + w_L \left(1 - \Pr_L[A_R]\right) L(x) - w_P \Pr_P[A_R] P(x).$$

Thus, we have

$$w'_L = \frac{w_L (1 - \Pr_L[A_R]) - w_P \Pr_P[A_R]}{1 - \Pr_{\bar{P}}[A_R]}$$

$$\leq w_L + (1 + O(\epsilon)) \Pr_P[A_R] + O \left( \epsilon \Pr_P[A_R] \right)$$

$$\leq w_L + (1/3 + O(\epsilon)) w_E \Pr_E[A_R] + O \left( \epsilon \Pr_P[A_R] \right).$$

Also we have

$$w'_E = \frac{w_E (1 - \Pr_E[A_R])}{1 - \Pr_{\bar{P}}[A_R]}$$

$$\leq w_E \left(1 - \Pr_E[A_R]\right) + O \left( \epsilon \Pr_P[A_R] \right).$$

Thus,

$$w_L + w_E - w'_L - w'_E \geq (2/3 - O(\epsilon)) w_E \Pr_E[A_R] - O \left( \epsilon \Pr_P[A_R] \right)$$

$$\geq (1/3 - O(\epsilon)) \Pr_P[A_R] \geq \epsilon/d.$$  

Note that the penultimate inequality also gives that

$$\Pr_{\bar{P}}[A_R] \leq (3 + O(\epsilon)) (w_L + w_E - w'_L - w'_E).$$

This completes the proof.
4 Conclusions and Future Directions

In this paper we initiated the study of the efficient robust learning for graphical models. We described a computationally efficient algorithm for robustly learning Bayesian networks with a known topology, under some natural conditions on the conditional probability table. We believe that these conditions can be eliminated by a more careful application of our techniques. A challenging open problem that requires additional ideas is to generalize our results to the setting of structure learning, i.e., the case when the underlying directed graph is unknown.

This work is part of a broader agenda of systematically investigating the robust learnability of high-dimensional structured probability distributions. There is a wealth of natural probabilistic models that merit investigation in the robust setting, including undirected graphical models (e.g., Ising models), and graphical models with hidden variables (i.e., incorporating latent structure).

Acknowledgements. We are grateful to Daniel Hsu for suggesting the model of Bayes nets, and for pointing us to [Das97].

References

[ADLS15] J. Acharya, I. Diakonikolas, J. Li, and L. Schmidt. Sample-optimal density estimation in nearly-linear time. CoRR, abs/1506.00671, 2015.

[AHHK12] A. Anandkumar, D. J. Hsu, F. Huang, and S. Kakade. Learning mixtures of tree graphical models. In NIPS, pages 1061–1069, 2012.

[AKN06] P. Abbeel, D. Koller, and A. Y. Ng. Learning factor graphs in polynomial time and sample complexity. J. Mach. Learn. Res., 7:1743–1788, 2006.

[Ber06] T. Bernholt. Robust estimators are hard to compute. Technical report, University of Dortmund, Germany, 2006.

[BGS14] G. Bresler, D. Gamarnik, and D. Shah. Structure learning of antiferromagnetic ising models. In NIPS, pages 2852–2860, 2014.

[BMS13] G. Bresler, E. Mossel, and A. Sly. Reconstruction of markov random fields from samples: Some observations and algorithms. SIAM J. Comput., 42(2):563–578, 2013.

[Bre15] G. Bresler. Efficiently learning ising models on arbitrary graphs. In STOC, pages 771–782, 2015.

[CDSS13] S. Chan, I. Diakonikolas, R. Servedio, and X. Sun. Learning mixtures of structured distributions over discrete domains. In SODA, pages 1380–1394, 2013.

[CDSS14a] S. Chan, I. Diakonikolas, R. Servedio, and X. Sun. Efficient density estimation via piecewise polynomial approximation. In STOC, pages 604–613, 2014.

[CDSS14b] S. Chan, I. Diakonikolas, R. Servedio, and X. Sun. Near-optimal density estimation in near-linear time using variable-width histograms. In NIPS, pages 1844–1852, 2014.

[CGR15a] M. Chen, C. Gao, and Z. Ren. A general decision theory for huber’s $\epsilon$-contamination model. CoRR, abs/1511.04144, 2015.
[CGR15b] M. Chen, C. Gao, and Z. Ren. Robust covariance matrix estimation via matrix depth. *CoRR*, abs/1506.00691, 2015.

[CL68] C. Chow and C. Liu. Approximating discrete probability distributions with dependence trees. *IEEE Trans. Inf. Theor.*, 14(3):462–467, 1968.

[Das97] S. Dasgupta. The sample complexity of learning fixed-structure bayesian networks. *Machine Learning*, 29(2-3):165–180, 1997.

[DDS12] C. Daskalakis, I. Diakonikolas, and R.A. Servedio. Learning $k$-modal distributions via testing. In *SODA*, pages 1371–1385, 2012.

[DKK+16] I. Diakonikolas, G. Kamath, D. M. Kane, J. Li, A. Moitra, and A. Stewart. Robust estimators in high dimensions without the computational intractability. *CoRR*, abs/1604.06443, 2016.

[DKW56] A. Dvoretzky, J. Kiefer, and J. Wolfowitz. Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. *Ann. Mathematical Statistics*, 27(3):642–669, 1956.

[DSA11] R. Daly, Q. Shen, and S. Aitken. Learning bayesian networks: approaches and issues. *The Knowledge Engineering Review*, 26:99–157, 2011.

[HM13] M. Hardt and A. Moitra. Algorithms and hardness for robust subspace recovery. In *COLT 2013*, pages 354–375, 2013.

[HR09] P. J. Huber and E. M. Ronchetti. *Robust statistics*. Wiley New York, 2009.

[HRRS86] F. R. Hampel, E. M. Ronchetti, P. J. Rousseeuw, and W. A. Stahel. *Robust statistics. The approach based on influence functions*. Wiley New York, 1986.

[Hub64] P. J. Huber. Robust estimation of a location parameter. *Ann. Math. Statist.*, 35(1):73–101, 03 1964.

[JN07] F. V. Jensen and T. D. Nielsen. *Bayesian Networks and Decision Graphs*. Springer Publishing Company, Incorporated, 2nd edition, 2007.

[JP78] D. S. Johnson and F. P. Preparata. The densest hemisphere problem. *Theoretical Computer Science*, 6:93–107, 1978.

[KF09] D. Koller and N. Friedman. *Probabilistic Graphical Models: Principles and Techniques - Adaptive Computation and Machine Learning*. The MIT Press, 2009.

[LRV16] K. A. Lai, A. B. Rao, and S. Vempala. Agnostic estimation of mean and covariance. *CoRR*, abs/1604.06968, 2016.

[LW12] P. L. Loh and M. J. Wainwright. Structure estimation for discrete graphical models: Generalized covariance matrices and their inverses. In *NIPS*, pages 2096–2104, 2012.

[Mas90] P. Massart. The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. *Annals of Probability*, 18(3):1269–1283, 1990.

[Nea03] R. E. Neapolitan. *Learning Bayesian Networks*. Prentice-Hall, Inc., 2003.
A Omitted Proofs from Section 2

Lemma 4 Let $P$ and $Q$ be Bayesian networks with the same dependency graph $G$. In terms of the conditional probability tables $p$ and $q$ of $P$ and $Q$, we have:

$$d_H(P, Q)^2 \leq 2 \sum_{k=1}^{m} \sqrt{\Pr_P[\Pi_k] \Pr_Q[\Pi_k]} \frac{(p_k - q_k)^2}{(p_k + q_k)(2 - p_k - q_k)}.$$ 

Proof. Let $A$ and $B$ be two distributions on $\{0, 1\}^d$. Then we have:

$$1 - d_H^2(A, B) = \sum_{x \in \{0, 1\}^d} \sqrt{\Pr_A[x] \Pr_B[x]}.$$ (1)

For a fixed $i \in [d]$, the events $\Pi_{i,a}$ form a partition of $\{0, 1\}^d$. Dividing the sum above into this partition, we obtain

$$1 - d_H^2(A, B) = \sum_{a} \sum_{x \in \Pi_{i,a}} \sqrt{\Pr_A[x] \Pr_B[x]}$$

$$= \sum_{a} \sqrt{\Pr_A[\Pi_{i,a}] \Pr_B[\Pi_{i,a}]} \sum_{x \in \{0, 1\}^d} \sqrt{\Pr_A[x] \Pr_B[x]}$$

$$= \sum_{a} \sqrt{\Pr_A[\Pi_{i,a}] \Pr_B[\Pi_{i,a}]} (1 - d_H^2(A | \Pi_{i,a}, B | \Pi_{i,a})) .$$

Let $P_{\lesssim i}$ be the distribution over the first $i$ coordinates of $P$ and define $Q_{\lesssim i}$ similarly for $Q$. Let $P_i$ and $Q_i$ be the distribution of the $i$-th coordinate of $P$ and $Q$ respectively. We will apply the above to $P_{\lesssim i-1}$ and $P_{\lesssim i}$. First note that the $i$-th coordinate of $P_{\lesssim i} | \Pi_{i,a}$ and $Q_{\lesssim i} | \Pi_{i,a}$ is independent of the others, thus we may factorize the RHS of Equation (1) for these distributions, obtaining:

$$1 - d_H^2(P_{\lesssim i} | \Pi_{i,a}, Q_{\lesssim i} | \Pi_{i,a}) = (1 - d_H^2(P_{\lesssim i-1} | \Pi_{i,a}, Q_{\lesssim i-1} | \Pi_{i,a})) \cdot (1 - d_H^2(P_i | \Pi_{i,a}, Q_i | \Pi_{i,a})) .$$
Thus, we have:

\[
1 - d_H^2(P_{\leq i}, Q_{\leq i}) = \sum_a \sqrt{\Pr_P[\Pi_{i,a}] \Pr_Q[\Pi_{i,a}]} (1 - d_H^2(P_{\leq i} | \Pi_{i,a}, Q_{\leq i} | \Pi_{i,a})) \\
= \sum_a \sqrt{\Pr_P[\Pi_{i,a}] \Pr_Q[\Pi_{i,a}]} (1 - d_H^2(P_{\leq i-1} | \Pi_{i,a}, Q_{\leq i-1} | \Pi_{i,a})) (1 - d_H^2(P_i | \Pi_{i,a}, Q_i | \Pi_{i,a})) \\
= 1 - d_H^2(P_{\leq i-1}, Q_{\leq i-1}) - \sum_a \sqrt{\Pr_P[\Pi_{i,a}] \Pr_Q[\Pi_{i,a}]} (1 - d_H^2(P_{\leq i-1} | \Pi_{i,a}, Q_{\leq i-1} | \Pi_{i,a})) d_H^2(P_i | \Pi_{i,a}, Q_i | \Pi_{i,a}) \\
\geq 1 - d_H^2(P_{\leq i-1}, Q_{\leq i-1}) - \sum_a \sqrt{\Pr_P[\Pi_{i,a}] \Pr_Q[\Pi_{i,a}]} d_H^2(P_i | \Pi_{i,a}, Q_i | \Pi_{i,a}) .
\]

By induction on \(i\), we have

\[
d_H^2(P, Q) \leq \sum_{(i,a) \in S} \sqrt{\Pr_P[\Pi_{i,a}] \Pr_Q[\Pi_{i,a}]} d_H^2(P_i | \Pi_{i,a}, Q_i | \Pi_{i,a}) .
\]

Now observe that the \(P_i | \Pi_{i,a}\) and \(Q_i | \Pi_{i,a}\) are Bernoulli distributions with means \(p_{i,a}\) and \(q_{i,a}\).

For \(p, q \in [0, 1]\), we have:

\[
d_H^2(\text{Bernoulli}(p), \text{Bernoulli}(q)) = (\sqrt{p} - \sqrt{q})^2 + (\sqrt{1-p} - \sqrt{1-q})^2 \\
= \frac{(p-q)^2}{(\sqrt{p} + \sqrt{q})^2} + \frac{((1-p) - (1-q))^2}{(\sqrt{1-p} + \sqrt{1-q})^2} \\
= (p-q)^2 \cdot \frac{1}{(\sqrt{p} + \sqrt{q})^2} + \frac{1}{(\sqrt{1-p} + \sqrt{1-q})^2} \\
\leq (p-q)^2 \cdot \frac{1}{p+q} + \frac{1}{2-p-q} \\
= (p-q)^2 \cdot \frac{2}{(p+q)(2-p-q)} ,
\]

and thus

\[
d_H(P, Q)^2 \leq 2 \sum_k \sqrt{\Pr_P[\Pi_k] \Pr_Q[\Pi_k]} \frac{(p_k - q_k)^2}{(p_k + q_k)(2-p_k - q_k)} ,
\]

as required.

\[\square\]

**Corollary** Suppose that: (i) \(\min_{k \in [m]} \Pr_P[\Pi_k] \geq 2\epsilon\), and (ii) \(P\) or \(Q\) is \(c\)-balanced, and (iii) \(\frac{\sqrt{\sum_k \Pr_P[\Pi_k]} (p_k - q_k)^2}{2} \leq \epsilon\). Then we have that \(d_{TV}(P, Q) \leq \epsilon\).

**Proof.** When either \(P\) or \(Q\) is \(c\)-balanced the denominators in Lemma 4 satisfies \((p_k + q_k)(2-p_k - q_k) \geq c\) and so we have \(d_{TV}(p, q) \leq 2 \epsilon \sqrt{\sum_k \sqrt{\Pr_P[\Pi_k] \Pr_Q[\Pi_k]} (p_k - q_k)^2} .\) Thus, it suffices to show that \(\Pr_Q[\Pi_k] \leq 2 \Pr_P[\Pi_k]\) or indeed that \(\Pr_Q[\Pi_k] \leq \Pr_P[\Pi_k] + \epsilon\). This would follow from our conclusion that \(d_{TV}(P, Q) \leq \epsilon\).

We can show by induction on \(i\) that for all \(i\) and \(a\), \(\Pr_Q[\Pi_{i,a}] \leq \Pr_P[\Pi_{i,a}] + \epsilon\). Suppose we have that, for \(1 \leq i \leq i'\), \(\Pr_Q[\Pi_{i,a}] \leq \Pr_P[\Pi_{i,a}] + \epsilon\). Then we have that \(d_{TV}(P_{\leq i'-1}, Q_{\leq i'-1}) \leq \epsilon\). But for any \(a\), \(\Pi_{i',a}\) depends only on \(i < i'\). Thus, \(|\Pr_P[\Pi_{i',a}] - \Pr_Q[\Pi_{i',a}]| \leq d_{TV}(P_{\leq i'-1}, Q_{\leq i'-1}) \leq \epsilon\), and therefore \(\Pr_Q[\Pi_{i',a}] \leq \Pr_P[\Pi_{i,a}] + \epsilon\) for all \(a\).

\[\square\]
Lemma. For all $x \in \{0,1\}^d$, $q \in [0,1]^m$ and $1 \leq j \leq k$, $x_1, \ldots, x_j$ can be recovered from the vector of coordinates $F(x,p)_{i,a}$ for all $(i,a) \in S$ with $j \leq i$, and these coordinates of $F$ can be computed from only $x_1, \ldots, x_j$.

Proof. Note that whether or not $x \in \Pi_{i,a}$ depends only on $x_1, \ldots, x_{i-1}$. Thus by the definition of $F(x,p)$ to find $F(x,p)_{i,a}$, we need to know $x_i$ and whether $x \in \Pi_{i,a}$ and so $F(x,p)_{i,a}$ depends only on $x_1, \ldots, x_{i-1}$. Thus $F(x,p)_{i,a}$ for all $(i,a) \in S$ with $j \leq i$ is a function of $x_1, \ldots, x_j$.

We need to show that $x_1, \ldots, x_j$ is a function of $F(x,p)_{i,a}$ for all $(i,a) \in S$ with $j \leq i$. We have that $x_i = F(x,p)_{i,a}$ for the unique $a$ with $x \in \Pi_{i,a}$ which depends only on $x_1, \ldots, x_{i-1}$. Since $x_1$ has no parents, we have $x_i = F(x,p)_{i,a}'$ for the empty bitstring $a'$. By an easy induction, we can determine $x_1, \ldots, x_j$.

Lemma. For $X \sim P$ and any $(i,a) \in S$, if we condition on either $\Pi_{i,a}$ or $-\Pi_{i,a}$, then $F(X,p)_{i,a}$ is independent of $F(X,p)_{j,a'}$ for all $(j,a') \in S$ with $j \leq i$ and $(j,a') \neq (i,a)$.

Proof. Conditioned on $-\Pi_{i,a}$, $F(X,p)_{i,a}$ is constantly $p_{i,a}$ so is independent of all random variables. We consider the case when we condition on $\Pi_{i,a}$, when $F(X,p_{i,a}) = X_i$. Since $\Pi_{i,a}$ implies $-\Pi_{i,a}$ for all $a' \neq a$, in this case $X_i$ is independent of the constant $F(X,p)_{i,a'}$. So, we only need to consider the case when $j \leq i$. However, fixing $F(X,p)_{j,a'}$ for all $(j,a') \in S$ with $j < i$ is equivalent to fixing $X_1, \ldots, X_{i-1}$ and so

$$Pr[X_i = 1 \mid F(X,p)_{j,a'} \text{ for all } (j,a') \in S \text{ with } j < i] = Pr[X_i = 1 \mid X_1, \ldots, X_{i-1}] = Pr[X_i = 1 \mid \Pi_{i,a}]$$

for all combinations of $F(X,p)_{j,a'}$ for all $(j,a') \in S$ with $j < i$ which imply $\Pi_{i,a}$.

Lemma. For $X \sim P$, we have $E[F(X,p)] = p$. The covariance matrix of $F(X,p)$ satisfies $Cov[F(X,p)] = diag(Pr[P_{k,p}p_k(1-p_k)])$.

Proof. Note that $E[F(X,p)]_k = Pr[P_{k,p}]p_k + (1 - Pr[P_{k,p}])p_k = p_k$ for all $1 \leq k \leq m$.

Next we show that for any $(i,a) \neq (j,a')$, that $E[(F(X,p)_{i,a} - p_{i,a})(F(X,p)_{j,a'} - p_{j,a'})] = 0$. We assume without loss of generality that $i \geq j$. On the one hand, if $i = j$ then $\Pi_{i,a}$ and $\Pi_{j,a'}$ cannot simultaneously hold. Therefore, with probability 1 either $F(X,p)_{i,a} = p_{i,a}$ or $F(X,p)_{j,a'} = p_{j,a'}$. Hence, $(F(X,p)_{i,a} - p_{i,a})(F(X,p)_{j,a'} - p_{j,a'})$ is 0 with probability 1, and thus has expectation 0. Otherwise, if $i > j$, we claim that even after conditioning on the value of $F(X,p)_{j,a'}$, that the expected value of $F(X,p)_{i,a}$ is $p_{i,a}$. In fact, we claim that even after conditioning on all of the values of $X_1, \ldots, X_{i-1}$ that the expectation of $F(X,p)_{i,a}$ is $p_{i,a}$. This is because subject to this conditioning either $\Pi_{i,a}$ holds, in which case, $F(X,p)_{i,a} = X_i$, which is 1 with probability $p_{i,a}$, or $\Pi_{i,a}$ doesn’t hold, and $F(X,p)$ is deterministically $p_{i,a}$. This completes the proof that $F(X,p)_{i,a}$ and $F(X,p)_{j,a'}$ are uncorrelated.

Finally, for any $(a,i) \in S$,

$$E[(F(X,p)_{i,a} - p_{i,a})^2] = Pr[P_{i,a}]E[(X_i - p_{i,a})^2 \mid \Pi_{i,a}] = Pr[P_{i,a}]p_{i,a}(1-p_{i,a})$$

This completes the proof.

Claim 27. We have that $E_{X \sim P}[F(X,p)_k \mid F(X,p)_1, \ldots, F(X,p)_{k-1}] = p_k$, for all $1 \leq k \leq m$.  

19
Proof. Firstly we claim that \( \mathbb{E}[F(X, p)_k \mid F(X, p)_1, \ldots, F(X, p)_{k-1}] = p_k \) for all \( 1 \leq k \leq m \).

Let \((i, a) \in S\) be such that \( F(X, p)_k = F(X, p)_{i,a} \). Then, by definition, \( F(X, p)_1, \ldots, F(X, p)_{k-1} \) includes \( F(X)_{j,a'} \) for all \((j, a') \in S\) with \( j < i \). By Lemma 7, these determine the value of the parents of \( X_i \), i.e., whether or not \( \Pi_{i,a} \) occurs. If \( F(X, p)_1, \ldots, F(X, p)_{k-1} \) imply \( \Pi_{i,a} \), then by Lemma 8,

\[
\mathbb{E}[F(X, p)_k \mid F(X, p)_1, \ldots, F(X, p)_{k-1}] = \mathbb{E}[F(X, p)_k \mid \Pi_k] = p_k ,
\]

and if they imply \( \neg \Pi_k \), then we similarly have:

\[
\mathbb{E}[F(X, p)_k \mid F(X, p)_1, \ldots, F(X, p)_{k-1}] = \mathbb{E}[F(X, p)_k \mid \neg \Pi_k] = p_k .
\]

We have that \( \mathbb{E}[F(X, p)_k \mid F(X, p)_1, \ldots, F(X, p)_{k-1}] = p_k \) for all combinations of \( F(X, p)_1, \ldots, F(X, p)_{k-1} \).

\( \square \)

B Omitted Proofs from Section 3.2

B.1 Omitted Proofs from Section 3.2

Lemma 13 With probability at least \( 1 - \frac{1}{200(2d+1)} \), \( \alpha - \min_k \mathbb{P}_{\tilde{p}}[\Pi_k] \leq \epsilon \), \( \|q - \mathbb{E}_{X \sim \tilde{p}}[F(X, q)]\|_2 \leq \epsilon \) and \( \|M - M_{\tilde{p}}\|_2 \leq \epsilon \).

Proof. By the Chernoff and union bounds, with probability at least \( 1 - \frac{1}{600(2d+1)} \), we have that our empirical estimates for \( \mathbb{P}_{\tilde{p}}[\Pi_k] \) are correct to within \( \epsilon \) using \( O(\log m/\epsilon^2) \) samples. This implies that \( \alpha - \min_k \mathbb{P}_{\tilde{p}}[\Pi_k] \leq \epsilon \).

Note that for fixed \( i, a \) we have that

\[
q_{i,a} - \mathbb{E}_{X \sim \tilde{p}}[F(X, q)]_{i,a} = \mathbb{P}_{\tilde{p}}[\Pi_{i,a}] \left( q_{i,a} - \mathbb{P}_{X \sim \tilde{p}}[X_i = 1 \mid \Pi_{i,a}] \right)
\]

and that the latter conditional probability does not depend on \( q \). For \( \tau > 0 \), suppose that we have \( \Omega(m^2 \mathbb{P}_{\tilde{p}}[\Pi_{i,a}]^2 \log(1/\tau)/\epsilon^2) \) samples for which \( \Pi_{i,a} \) holds. Then, by the Chernoff bound, it follows that

\[
\left| q_{i,a} - \mathbb{P}_{X \sim \tilde{p}}[X_i = 1 \mid \Pi_{i,a}] \right| \leq \frac{\epsilon}{m \mathbb{P}_{\tilde{p}}[\Pi_{i,a}]}
\]

with probability at least \( 1 - \tau \). Again by the Chernoff bound, this happens with probability at least \( 1 - \tau \) when we have \( O(m^2 \log(1/\tau)/\epsilon^2) \) samples from \( \tilde{P} \). By a union bound, we have \( \|q - \mathbb{E}_{X \sim \tilde{p}}[F(X, q)]\|_\infty \leq \epsilon/m \) with probability at least \( 1 - \frac{1}{6(2d+1)} \) with \( O(m^2 \log(m)/\epsilon^2) \) samples. In this case, we have \( \|q - \mathbb{E}_{X \sim \tilde{p}}[F(X, q)]\|_2 \leq \epsilon \) as required.

Now we assume that this holds and need to show that \( \|M - M_{\tilde{p}}\|_2 \leq \epsilon \). Let \( \mu_{\tilde{p}} = \mathbb{E}_{X \sim \tilde{p}}[F(X, q)] \).

When \( i = j \), \( M_{i,j} = (M_{\tilde{p}})_{i,j} = 0 \). Consider some unequal \( i, j \). Since \( \|\mu_{\tilde{p}} - q\|_\infty \leq \epsilon/m \), we have

\[
(M_{\tilde{p}})_{i,j} = \mathbb{E}_{X \sim \tilde{p}}[(F(X, q)_i - q_i)(F(X, q)_j - q_j)]
\]

\[
= \mathbb{E}_{X \sim \tilde{p}}[(F(X, q)_i - \mu_{i,\tilde{p}})(F(X, q)_j - \mu_{j,\tilde{p}})] + O(\epsilon/m) .
\]

Similarly, if \( S \) is the empirical distribution over the samples, we have

\[
M_{i,j} = \mathbb{E}_{X \sim S}[(F(X, q)_i - q_i)(F(X, q)_j - q_j)]
\]

\[
= \mathbb{E}_{X \sim S}[(F(X, q)_i - \mu_{i,\tilde{p}})(F(X, q)_j - \mu_{j,\tilde{p}})] + O(\epsilon/m) .
\]

20
Now note that this is a sum of independent centered random variables each with absolute value at most one. By Hoeffding’s inequality, it follows that with probability $2 \exp(-N\epsilon^2/2m^2)$ we have that $|M_{i,j} - (M_\tilde{P})_{i,j}| \leq \epsilon/m$. Since $N = \Omega(m^2 \log(m)/\epsilon^2)$, by a union bound, this holds for all $i, j$ with probability at least $1 - \frac{1}{600(2d+1)}$. Then, we have that $\|M - M_\tilde{P}\|_2 \leq \|M - M_\tilde{P}\|_F \leq \epsilon$.

**Lemma 14** $\|M_P\|_2 \leq \sum_k \Pr_P[\Pi_k](p_k - q_k)^2$.

**Proof.** Note that when $x \in \{0,1\}^d$ is such that $x \not\in \Pi_k$, we have that $F(x,q)_k = q_k$ and that $\Pr_P[\Pi_i \land \Pi_j] \leq \Pr_P[\Pi_i] \Pr_P[\Pi_j]$. By definition we have that $(M_P)_{i,i} = 0$ and, for $i \neq j$, we can write

$$(M_P)_{i,j} = \mathbb{E}_{X \sim P}[(F(X,q)_i - q_i)(F(X,q)_j - q_j)] = \Pr_P[\Pi_i \land \Pi_j](p_i - q_i)(p_j - q_j).$$

Either way, it holds that

$$(M_P)_{i,j}^2 \leq \Pr_P[\Pi_i] \Pr_P[\Pi_j](p_i - q_i)^2(p_j - q_j)^2.$$

Thus, we obtain

$$\|M_P\|_2^2 \leq \|M_P\|_F^2 = \sum_{i,j} (M_P)_{i,j}^2 \leq \sum_{i,j} \Pr_P[\Pi_i] \Pr_P[\Pi_j](p_i - q_i)^2(p_j - q_j)^2 = \left( \sum_k \Pr_P[\Pi_k](p_k - q_k)^2 \right)^2,$$

as desired. \qed