The algebraic hyperstructure of elementary particles in physical theory

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Abstract

Algebraic hyperstructures represent a natural extension of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Algebraic hyperstructure theory has a multiplicity of applications to other disciplines. The main purpose of this paper is to provide examples of hyperstructures associated with elementary particles in physical theory.

1 Introduction

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced in 1934 by the French mathematician F. Marty \cite{11}. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic. One of the first books, dedicated especially to hypergroups, is “Prolegomena of Hypergroup Theory”, written by P. Corsini in 1993 \cite{3}. Another book on “Hyperstructures and Their Representations”, by T. Vougiouklis, was published one year later \cite{14}. On the other hand, algebraic hyperstructure theory has a multiplicity of applications to other disciplines: geometry, graphs and hypergraphs, binary relations,
lattices, groups, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, $C^*$-algebras, artificial intelligence, probabilities and so on. A recent book on these topics is “Applications of Hyperstructure Theory”, by P.Corsini and V. Leoreanu, published by Kluwer Academic Publishers in 2003 [4]. We mention here another important book for the applications in Geometry and for the clearness of the exposition, written by W. Prenowitz and J. Jantosciak [13]. Another monograph is devoted especially to the study of Hyperring Theory, written by Davvaz and Leoreanu-Fotea [5]. It begins with some basic results concerning ring theory and algebraic hyperstructures, which represent the most general algebraic context, in which the reality can be modelled. Several kinds of hyperrings are introduced and analyzed in this book. The volume ends with an outline of applications in Chemistry and Physics [6, 7], canalizing several special kinds of hyperstructures: e−hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

The main purpose of this paper is to provide examples of hyperstructures associated with elementary particles in physical theory.

2 General n-ary hyperstructures

Throughout this paper, the symbol $H_1, H_2, \ldots, H_n$ will denote $n$ nonempty sets. Where $P^*(\bigcup_{i=1}^{n} H_i)$ denotes the set of all non empty subsets of $\bigcup_{i=1}^{n} H_i$.

Definition 2.1. A general $n$-ary hyperstructure is $n$ non empty sets $H_1, H_2, \ldots, H_n$ together with a hyperoperation,

$$f : H_1 \times H_2 \times \ldots \times H_n \rightarrow P^*(\bigcup_{i=1}^{n} H_i)$$

$$(x_1, \ldots, x_n) \mapsto f(x_1, \ldots, x_n) \subseteq \bigcup_{i=1}^{n} H_i - \emptyset.$$ 

Let $f$ be a general $n$-ary hyperoperation on $H_1, H_2, \ldots, H_n$ and $A_i$, subsets of $H_i$ for all $i = 1, \ldots, n$. We define

$$f(A_1, \ldots, A_n) = \bigcup\{f(x_1, \ldots, x_n) | x_i \in A_i, i = 1, \ldots, n\}.$$ 

We denote by $H^n$ the cartesian product $H \times \ldots \times H$ where $H$ appears $n$ times. An element of $H^n$ will be denoted by $(x_1, \ldots, x_n)$ where $x_i \in H$ for any $i$ with
1 \leq i \leq n. In general, a mapping $f : H^n \rightarrow \mathcal{P}(H)$ is called an $n$-ary hyperoperation and $n$ is called the order of hyperoperation. A hyperalgebra $(H, f)$ is a non-empty set $H$ with one $n$-ary hyperoperations $f$.

**Remark 2.2.** A general hyperoperation $* : X \times Y \rightarrow \mathcal{P}(X \cup Y)$ yields a general hyperoperation 

$$\otimes : \mathcal{P}(X) \times \mathcal{P}(Y) \rightarrow \mathcal{P}(X \cup Y)$$

defined by $A \otimes B = \bigcup_{a \in A, b \in B} a * b$.

Conversely a general hyperoperation on $\mathcal{P}(X) \times \mathcal{P}(Y)$ yields a general hyperoperation on $X \times Y$, defined by $x * y = \{x\} \otimes \{y\}$.

In the above definition if $A \subseteq X$, $B \subseteq Y$, $x \in X$, $y \in Y$, then we define,

$$A * y = A * \{y\} = \bigcup_{a \in A} a * y, \quad x * B = \{x\} * B = \bigcup_{b \in B} x * b,$$

$$A \otimes B = \bigcup_{a \in A, b \in B} a * b.$$

**Remark 2.3.** If we let $X = Y = H$, then we obtain the hyperstructure theory.

### 3 Algebraic hyperstructures

In this subsection, we summarize the preliminary definitions and results required in the sequel.

**Definition 3.1.** Let $H$ be a non-empty set and let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of $H$.

(i) A hyperoperation on $H$ is a map $\otimes : H \times H \rightarrow \mathcal{P}^*(H)$ and the couple $(H, \otimes)$ is called a hypergroupoid. If $A$ and $B$ are non-empty subsets of $H$, then we denote

$$A \otimes B = \bigcup_{a \in A, b \in B} a \otimes b, \quad x \otimes A = \{x\} \otimes A \quad \text{and} \quad A \otimes x = A \otimes \{x\}.$$  

(ii) A hypergroupoid $(H, \otimes)$ is called a semihypergroup if for all $x, y, z$ of $H$ we have $(x \otimes y) \otimes z = x \otimes (y \otimes z)$, which means that

$$\bigcup_{u \in x \otimes y} u \otimes z = \bigcup_{v \in y \otimes z} x \otimes v.$$
We say that a semihypergroup \((H, \otimes)\) is a \textit{hypergroup} if for all \(x \in H\), we have \(x \otimes H = H \otimes x = H\). A hypergroupoid \((H, \otimes)\) is an \(H_v\)-\textit{group}, if for all \(x, y, z \in H\), the following conditions hold:

1. \(x \otimes (y \otimes z) \cap (x \otimes y) \otimes z \neq \emptyset\) (weak associativity),
2. \(x \otimes H = H \otimes x = H\) (reproduction).

\textbf{Definition 3.2.} Let \((L, \oplus)\) be a \(H_v\)-group, and \(K\) be a nonempty subset of \(L\). Then \(K\) is called a \(L_v\)-subgroup of \((L, \oplus)\) if \(a \oplus b \in P^*(K)\) for all \(a, b \in K\). That is to say, \(K\) is an \(H_v\)-subgroup of \((L, \oplus)\) if and only if \(K\) is closed under the binary hyperoperation on \(L\).

The concept of \(H_v\)-structures constitute a generalization of the well-known algebraic hyperstructures (hypergroup, hyperring, hypermodule, and so on) (for example you see [1]).

\section{Physical example (elementary particles)}

Since long time ago, one of the most important questions is that what is our universe made of? To answer this question many efforts have been yet done. Since 1897 when the electron was discovered by J. J. Thomson, the elementary particle physics was born. Nowadays, the biggest particle accelerator which is called the Large Hadron Collider (LHC) at CERN in France is being applied to find the last elementary particle (Higgs boson). The existence of this particle is necessary to understand our universe.

In particle physics, an elementary particle or fundamental particle is a particle which have no substructure, i.e. it is not known to be made up of smaller particles. If an elementary particle truly has no substructure, then it is one of the basic building blocks of the universe from which all other particles are made. To describe the elementary particles and the interacting forces between them, some different theories are proposed that their most important is the Standard Model [12]. The Standard Model (SM) of elementary particles has proved to be extremely successful during the past three decades. It has shown to be a well established theory. All predictions based on the SM have been experimentally verified and most of its parameters have been fixed. The only part of the SM that has not been directly experimentally verified yet is the Higgs sector. In the SM, the Quarks, Leptons and Gauge bosons are introduced as the elementary particles. The SM of particle physics contains six types of quarks, known as flavors: Up, Down, Charm, Strange, Bottom and Top plus their corresponding antiparticles. For every particle there
is a corresponding type of antiparticle, for every quark it is known as antiquark, that differs from the particle only in some of its properties (like electric charge) which have equal magnitude but opposite sign. Since the quarks are never found in isolation, therefore quarks combine to form composite particles which are called hadrons, see Ref.[1]. In particle physics, hadrons are composite particles made of quarks which are categorized into two families: baryons (made of three quarks) and mesons (made of one quark and one antiquark).

In the SM, gauge bosons consist of the photons ($\gamma$), gluons ($g$), $W^\pm$ and $Z$ bosons act as carriers of the fundamental forces of nature[2]. In fact, interactions between the particles are described by the exchange of gauge bosons. The third group of the elementary particles are leptons. There are six type of leptons including the electron($e$), electron neutrino($\nu_e$), muon($\mu$), muon neutrino($\nu_\mu$), tau($\tau$) and tau neutrino($\nu_\tau$). Every lepton has a corresponding antiparticle these antiparticles are known as antileptons. Leptons are an important part of the SM, especially the electrons which are one of the components of atoms. Since the leptons can be found freely in the universe and they are one of the important groups of the elementary particles, in this article we only concentrate on this group of particles.

4.1 Leptons

In the Standard Model, leptons still appear to be structureless. In this model, there are six flavors of leptons and six corresponding antiparticles. They form three generations [9] [10]. The first generation is the electronic leptons, comprising the electron($e$), electron neutrino($\nu_e$) and their corresponding antiparticles, i.e. positron($e^+$) and electron antineutrino ($\overline{\nu}_e$). The second generation is the muonic leptons, including muon($\mu$), muon neutrino($\nu_\mu$), antimuon($\mu^+$) and muon antineutrino($\overline{\nu}_\mu$). The third is the tauonic leptons, consist of tau($\tau$), tau neutrino($\nu_\tau$), antitau($\tau^+$) and tau antineutrino ($\overline{\nu}_\tau$). Therefore, there are 12 particles $\{e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau, e^+, \overline{\nu}_e, \mu^+, \overline{\nu}_\mu, \tau^+, \overline{\nu}_\tau\}$ in the leptons group. In the leptons group, the electron, muon and tau have the electric charge $Q = -1$ (the charge of a particle is expressed in unit of the electron charge) and the neutrinos are neutral. According to the definition of antiparticle, the electric charge of positron, antimuon and antitau is $Q = +1$ but the antineutrinos are neutral as well as neutrinos. The main difference between the neutrinos and antineutrinos is in the other quantum numbers such as leptonic numbers, see Refs. [9] [10]. In the SM, leptonic numbers are assigned to the members of every generation of leptons. Electron and electron neutrino have an electronic number of $L_e = 1$ while muon and muon neutrino have a muonic number of $L_\mu = 1$ and tau and tau neutrino have a tauonic number of $L_\tau = 1$. The antileptons have their respective generation’s leptonic numbers of $-1$. These numbers are classified in Table [10].
In every interaction the leptonic numbers are conserved. Conservation of the leptonic numbers means that the number of leptons of the same type remains the same when particles interact. This implies that leptons and antileptons must be created in pairs of a single generation. For example, the following processes are allowed under conservation of the leptonic numbers:

\[ e + \nu_e \rightarrow \{ e, \nu_e \} \text{ or } \mu + \nu_\mu \rightarrow \{ \mu, \nu_\mu \}, \]

where in the first interaction, the electronic numbers and in the second interaction the muonic numbers are conserved. In other interactions, outgoing particles might be different, therefore all leptonic numbers must be checked. In the following interactions both the electronic and muonic numbers are conserved:

\[ e + \nu_\mu \rightarrow \{ e, \nu_\mu \text{ or } \mu, \nu_e \} + \gamma \]

The only conditions required to occur a leptonic interaction are the conservation of the electric charges and the leptonic numbers. Conservation of the leptonic numbers means that the electronic, muonic and tauonic numbers must be conserved separately. Considering these conservation rules, for the electron-positron interaction the interacting modes are:

\[ e + e^+ \rightarrow \{ e, e^+ \text{ or } \mu, \mu^+ \text{ or } \tau, \tau^+ \} + \nu_e \text{ or } \nu_\mu \text{ or } \nu_\tau \text{ or } \nu_\mu + \nu_\tau \text{ or } \nu_\mu + \nu_\tau = L. \]

Other interactions between the members of the leptons group are shown in Table 2. To arrange this table we avoided writing the repeated symbols. For example in the productions of the electron-electron interaction we only write \( e + e \) instead of \( e + e \). All the interactions shown in Table 2 are in the first order. It means in the higher order other particles can be produced that we do not consider them. For example in the electron-electron scattering (Müller scattering) one or several photons might be appeared in productions of the interaction, i.e. \( e + e \rightarrow e + e + \gamma \) or in the electron-positron scattering (Bhabha scattering) we can have: \( e + e^+ \rightarrow e + e^+ + \gamma \).
There also exist other processes that we do not consider them in this work. For example: $e + e^+ \rightarrow \gamma + \gamma$, $e + e^+ \rightarrow W^- + W^+$, $\tau + \tau^+ \rightarrow Z^0 + Z^0$ and so on.

5 The algebraic hyperstructure of leptons

Contemporary investigations of hyperstructures and their applications yield many relationships and connections between various fields of mathematics.

Besides the motivation for investigation of hyperstructures coming from non-commutative algebra, geometrical structures and other mathematical fields there exist such physical phenomena as the nuclear fission. Nuclear fission occurs when a heavy nucleus, such as U235, splits, or fissions, into two smaller nuclei. As a result of this fission process we can get several dozens of different combinations of two medium-mass elements and several neutrons (as barium $Ba^{141}$ and krypton $Kr^{92}$ and 3 neutrons; strontium $Sr^{94}$, xenon $Xe^{142}$ 0 and 2 neutrons; lanthanum $La^{147}$, bromum $Br^{87}$ and 2 neutrons; $Sn^{132}$, $Mo^{101}$ and 3 neutrons and so on) . More precisely, the input of this reaction is always the same—the heavy uranium is bombarded with neutrons, but the result is in general different—there are about 90 different daughter nuclei that can be formed. The fission also results in the production of several neutrons, typically two or three. On the average about 2.5 neutrons are released per event. In any fission equation, there are many combinations of fission fragments, but they always satisfy the requirements of conservation of energy and charge.

Another typical example of the situation when the result of interaction between two particles is the whole set of particles is the interaction between a foton with certain energy and an electron. The result of this interaction is not deterministic. A photo-electric effect or Coulomb repulsion effect or changeover of foton onto a pair electron and positron can arise.

It is to be noted that a similar situation which occurs during uranium fission appears during several nuclear fission, too. The result depends on conditions. Although the input 2 particles are the same, the output can be variant. It can differ both in the number of arising particles and in their kind.

Another motivation for investigation of hyperstructures yields from technical processes as a time sequence of military car repairs with respect to its roadability consequences and its operational behavior. In this section we describe a certain construction of hyperstructures belonging to the important class of elementary particles. We find the algebraic hyperstructure of leptons.
Table 2: Interaction between leptons are shown.
Theorem 5.1. Let \( L = \{e, \nu_e, e^+, \bar{\nu}_e, \mu, \nu_\mu, \mu^+, \bar{\nu}_\mu, \tau, \nu_\tau, \tau^+, \bar{\nu}_\tau\} \). Then \( (L, \otimes) \) is an abelian \( H_v \)-group.

We summarize some results on associativity, in following lemmas.

Lemma 5.2. The list of \([A, B, C]\)’s that \([A, [B, C]] \not\subseteq [A, [B, C]]\) is:

\[
[e^+, \mu, e^+], [e^+, \tau, e^+], [e^+, \bar{\nu}_e, e^+], [\mu, e^+, e^+], [\mu^+, \bar{\nu}_e, e^+], [\tau, e^+, e^+], [\tau^+, \bar{\nu}_e, e^+], \\
[\bar{\nu}_e, e^+, e^+], [\bar{\nu}_e, e^+, \mu], [\bar{\nu}_e, e^+, \tau^+], [\bar{\nu}_e, e^+, \nu_\mu], [\bar{\nu}_e, e^+, \nu_\tau], [\nu_e, \nu_e, e^+], [\nu_e, \nu_e, \nu_\mu], [\nu_e, \nu_e, \nu_\tau], \\
[\nu_\mu, \nu_e, e^+], [\nu_\mu, \nu_e, \nu_\mu].
\]

Lemma 5.3. The list of \([A, B, C]\)’s that \([A, B, C] \not\subseteq A, [B, C]\) is:

\[
[e^+, \nu_e, e^+], [\mu, \bar{\nu}_e, e^+], [\tau^+, \bar{\nu}_e, e^+], [\bar{\nu}_e, e^+, e^+], [\bar{\nu}_e, e^+, \mu], [\bar{\nu}_e, e^+, \tau^+], [\bar{\nu}_e, e^+, \nu_\mu], [\bar{\nu}_e, e^+, \nu_\tau], [\nu_e, \nu_e, \nu_e], [\nu_\mu, \nu_e, \nu_e], [\nu_\mu, \nu_e, \nu_\mu], [\nu_\mu, \nu_e, \nu_\tau].
\]

Lemma 5.4. The list of \([A, B, C]\)’s that \([A, B, C] \neq [A, [B, C]]\) is:

\[
[e^+, \mu, e^+], [e^+, \tau, e^+], [e^+, \bar{\nu}_e, e^+], [e^+, \nu_\tau, e^+], [\mu, e^+, e^+], [\mu^+, \bar{\nu}_e, e^+], [\mu^+, \nu_\mu, e^+], [\nu_\mu, e^+, e^+], [\nu_\mu, \nu_\mu, e^+], [\nu_\mu, \nu_\mu, \nu_\mu], [\nu_\tau, e^+, e^+], [\nu_\tau, \nu_\tau, e^+].
\]

Remark 5.5. In general, the hyperoperation ” \( \otimes \) ” is not associative.

Theorem 5.6. If \((L, \otimes)\) is above \( H_v \)-group, then the following statements hold.

1) There is not any \( H_v \)-subgroups of order 5, 7, 8, 9, 10, and, 11 for \((L, \otimes)\);

2) All the 1—dimensional \( H_v \)-subgroups of \( L \) are:

\[
\begin{align*}
L_1 &= \{e\}, & L_1^1 &= \{e^+\}, & L_2 &= \{\mu\}, & L_4 &= \{\mu^+\}, & L_5 &= \{\tau\}, \\
L_6 &= \{\tau^+\}, & L_7 &= \{\bar{\nu}_e\}, & L_8 &= \{\bar{\nu}_\mu\}, & L_9 &= \{\bar{\nu}_\tau\}, & L_{10} &= \{\nu_e\}, \\
L_{11} &= \{\nu_\mu\}, & L_{12} &= \{\nu_\tau\}.
\end{align*}
\]
3) All the 2-dimensional $H_v$-subgroups of $L$ are:

$L_1^2 = \{e, \mu\}, \quad L_2^2 = \{e, \tau\}, \quad L_3^2 = \{e, \bar{\nu}\}, \quad L_4^2 = \{e, \bar{\nu}_\tau\},$

$L_5^2 = \{e, \nu\}, \quad L_6^2 = \{e^+, \mu^+\}, \quad L_7^2 = \{e^+, \tau^+\}, \quad L_8^2 = \{e^+, \nu_\mu\},$

$L_9^2 = \{e^+, \bar{\mu}\}, \quad L_{10}^2 = \{\mu, \tau\}, \quad L_{11}^2 = \{\mu, \bar{\nu}_e\}, \quad L_{12}^2 = \{\mu, \bar{\nu}\},$

$L_{13}^2 = \{\mu, \nu_\mu\}, \quad L_{14}^2 = \{\mu^+, \tau^+\}, \quad L_{15}^2 = \{\mu^+, \bar{\nu}_\mu\}, \quad L_{16}^2 = \{\mu^+, \nu_\tau\},$

$L_{17}^2 = \{\mu^+, \nu_\tau\}, \quad L_{18}^2 = \{\tau, \bar{\nu}_e\}, \quad L_{19}^2 = \{\tau, \bar{\nu}\}, \quad L_{20}^2 = \{\tau, \nu_\tau\},$

$L_{21}^2 = \{\tau^+, \bar{\nu}_e\}, \quad L_{22}^2 = \{\tau^+, \nu_\mu\}, \quad L_{23}^2 = \{\tau^+, \nu_\tau\}, \quad L_{24}^2 = \{\nu_\mu, \bar{\nu}\},$

$L_{25}^2 = \{\bar{\nu}_e, \bar{\nu}_\tau\}, \quad L_{26}^2 = \{\bar{\nu}_\mu, \bar{\nu}_\tau\}, \quad L_{27}^2 = \{\nu_e, \nu_\mu\}, \quad L_{28}^2 = \{\nu_e, \nu_\tau\},$

$L_{29}^2 = \{\nu_\mu, \nu_\tau\}.$

4) All the 3-dimensional $H_v$-subgroups of $L$ are:

$L_1^3 = \{e, \mu, \tau\}, \quad L_2^3 = \{e, \lambda, \bar{\nu}_e\}, \quad L_3^3 = \{e, \lambda, \nu_\mu\},$

$L_4^3 = \{e, \nu, \bar{\nu}_e\}, \quad L_5^3 = \{e^+, \mu^+, \bar{\nu}_e\}, \quad L_6^3 = \{e^+, \mu^+, \nu_\mu\},$

$L_7^3 = \{e^+, \nu, \nu_\mu\}, \quad L_8^3 = \{\mu, \bar{\nu}_e\}, \quad L_9^3 = \{\mu, \bar{\nu}\},$

$L_{10}^3 = \{\mu, \nu_\mu\}, \quad L_{11}^3 = \{\mu^+, \bar{\nu}_e\}, \quad L_{12}^3 = \{\mu^+, \bar{\nu}\},$

$L_{13}^3 = \{\tau, \bar{\nu}_e\}, \quad L_{14}^3 = \{\tau, \nu_\mu\}, \quad L_{15}^3 = \{\nu_\mu, \bar{\nu}\},$

$L_{16}^3 = \{\nu_e, \nu_\mu, \nu_\tau\}.$

5) All the 4-dimensional $H_v$-subgroups of $L$ are:

$L_1^4 = \{e, \mu, \nu_\mu, \nu_\tau\}, \quad L_2^4 = \{e, \mu^+, \nu_\mu, \nu_\tau\}, \quad L_3^4 = \{e, \lambda, \nu_e, \lambda, \nu_\mu\},$

$L_4^4 = \{e, \lambda, \nu_\mu, \nu_\tau\}, \quad L_5^4 = \{e^+, \mu^+, \nu_\mu, \nu_\tau\}, \quad L_6^4 = \{\mu, \nu_\mu, \nu_\tau\},$

$L_7^4 = \{\mu, \nu_\mu, \nu_\tau\}, \quad L_8^4 = \{\nu_\mu, \nu_\tau\}.$

6) All the 6-dimensional $H_v$-subgroups of $L$ are:

$L_1^6 = \{e, \mu, \tau, \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau\}, \quad L_2^6 = \{e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau\},$

$L_3^6 = \{e, \mu, \tau, \nu_\mu, \nu_e, \nu_\mu\}, \quad L_4^6 = \{e, \mu^+, \nu_\mu, \nu_e, \nu_\tau\},$

$L_5^6 = \{e, \mu^+, \nu_\mu, \nu_e, \nu_\mu\}, \quad L_6^6 = \{e^+, \mu^+, \nu_\mu, \nu_e, \nu_\mu\},$

$L_7^6 = \{e^+, \mu^+, \nu_\mu, \nu_e, \nu_\mu\}, \quad L_8^6 = \{e^+, \mu^+, \nu_\mu, \nu_e, \nu_\mu\}.$

The result follows immediately from Theorem 5.6 and Definitions 3.1, 3.2.

**Conclusion 5.7.** If $(L, \otimes)$ is above $H_v$-group, then the following statements (the inclusion $H_v$-subgroups) hold:

(i) for 1-dimensional:

$L_1^1 \subset L_2^1, \quad L_1^1 \subset L_3^1, \quad L_1^1 \subset L_3^3, \quad L_1^1 \subset L_4^1, \quad L_1^1 \subset L_2^2, \quad L_1^1 \subset L_3^3, \quad L_1^1 \subset L_4^4, \quad L_1^1 \subset L_1^6,$
\[
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^,
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^,
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^,
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^,
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^,
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^,
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^,
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^.
\]

(i) for 2-dimensional:

\[
L_1^\subset L_2^\subset L_3^\subset L_4^\subset L_5^\subset L_6^\subset L_7^\subset L_8^\subset L_9^\subset L_{10}^\subset L_{11}^\subset L_{12}^.
\]
$$\begin{array}{l}
L_2^3 \subset L_3^3, \quad L_2^5 \subset L_4^3, \quad L_2^7 \subset L_6^3, \quad L_2^9 \subset L_8^3, \quad L_2^{11} \subset L_{10}^3, \quad L_3^3 \subset L_4^3, \\
L_2^3 \subset L_2^6, \quad L_2^5 \subset L_6^3, \quad L_2^7 \subset L_6^3, \quad L_2^9 \subset L_6^3, \quad L_2^{11} \subset L_6^3, \quad L_3^3 \subset L_6^3, \\
L_2^3 \subset L_2^8, \quad L_2^5 \subset L_8^3, \quad L_2^7 \subset L_8^3, \quad L_2^9 \subset L_8^3, \quad L_2^{11} \subset L_8^3, \quad L_3^3 \subset L_8^3, \\
L_2^3 \subset L_2^{10}, \quad L_2^5 \subset L_5^6, \quad L_2^7 \subset L_5^6, \quad L_2^9 \subset L_5^6, \quad L_2^{11} \subset L_5^6, \quad L_3^3 \subset L_5^6.
\end{array}$$

(i) for 3-dimensional:

$$L_3^3 \subset L_1^1, \quad L_3^3 \subset L_2^1, \quad L_3^3 \subset L_3^1, \quad L_3^3 \subset L_2^1, \quad L_3^3 \subset L_3^1, \quad L_3^3 \subset L_3^1,$$
$$L_3^3 \subset L_4^1, \quad L_3^3 \subset L_4^1, \quad L_3^3 \subset L_4^1, \quad L_3^3 \subset L_4^1, \quad L_3^3 \subset L_4^1, \quad L_3^3 \subset L_4^1,$$
$$L_3^3 \subset L_5^1, \quad L_3^3 \subset L_5^1, \quad L_3^3 \subset L_5^1, \quad L_3^3 \subset L_5^1, \quad L_3^3 \subset L_5^1, \quad L_3^3 \subset L_5^1,$$
$$L_3^3 \subset L_6^1, \quad L_3^3 \subset L_6^1, \quad L_3^3 \subset L_6^1, \quad L_3^3 \subset L_6^1, \quad L_3^3 \subset L_6^1, \quad L_3^3 \subset L_6^1.$$

(ii) for 4-dimensional:

$$L_4^1 \subset L_2^4, \quad L_4^1 \subset L_3^4, \quad L_4^1 \subset L_4^4, \quad L_4^1 \subset L_5^4, \quad L_4^1 \subset L_6^4, \quad L_4^1 \subset L_7^4, \quad L_4^1 \subset L_8^4,$$
$$L_4^1 \subset L_2^6, \quad L_4^1 \subset L_3^6, \quad L_4^1 \subset L_4^6, \quad L_4^1 \subset L_5^6, \quad L_4^1 \subset L_6^6, \quad L_4^1 \subset L_7^6, \quad L_4^1 \subset L_8^6.$$

(iii) for 6-dimensional:

$$L_6^1 \subset L_1^6, \quad L_6^1 \subset L_2^6, \quad L_6^1 \subset L_3^6, \quad L_6^1 \subset L_4^6, \quad L_6^1 \subset L_5^6, \quad L_6^1 \subset L_6^6, \quad L_6^1 \subset L_7^6, \quad L_6^1 \subset L_8^6.$$
References

[1] M. Aguilar-Benitez et al., *Particles Data Group*, Phys. Lett. 170B (1986).

[2] H. Bhabha, *Proc. Roy. Soc.*, A154 (1936) 195.

[3] P. Corsini, *Prolegomena of hypergroup theory*, Second edition, Aviani editor, (1993).

[4] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, (2003).

[5] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, USA, (2007).

[6] B. Davvaz and A. Dehghan Nezhad, *Chemical examples in Hypergroups*, Ratio Mathematica-Numero 14 (2003), 71-74.

[7] B. Davvaz, A. Dehghan Nezhad, and A. Benvidi, *Chain reactions as experimental examples of ternary algebraic hyperstructures*, Communications in mathematical and in computer chemistry, (2011), pp. 00.

[8] A. Dehghan Nezhad and B. Davvaz, *Universal hyperdynamical systems*, Bulletin of the Korean Mathematical Society, 47 (2010), No. 3, pp. 513-528.

[9] D. Griffiths, *Introduction to Elementary Particles*, John Wiley & Sons, (1987).

[10] F. Halzen and A. Martin, *Quarks & Leptons: An Introductory Course in Modern Particle Physics*, John Wiley & Sons, (1984).

[11] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm (1934), 45-49.

[12] T. Muta, *Foundations of quantum chromodynamics. Second edition*, World Sci. Lect. Notes Phys. 57 (1998) 1.

[13] W. Prenowitz and J. Jantosciak, *Join Geometries*, Springer-Verlag, UTM., (1979).

[14] T. Vougiouklis, *Hyperstructures and their representations*, Hadronic Press, Florida, (1994).