Extreme Dilatonic Black Holes on a Torus

Kiyoshi Shiraishi*

Akita Junior College
Shimokitate-sakura, Akita-shi, Akita 010, Japan

(March 24, 2022)

Abstract

The interaction of maximally charged dilatonic black holes on $R^4 \times T^d$ is studied in the low-velocity limit. In particular, the scattering of two black holes on $R^4 \times S^1$ is investigated.

PACS number(s): 04.40.Nr, 04.50.+h, 04.70.Bw, 11.25.Mj
I. INTRODUCTION

In recent years, solitons in field theories have attracted much attention. Solitonic objects may play important roles in the unity of fundamental theories including string theories [1]. The relation between topology of the space and the solitonic objects is a key subject for exploring the ultimate theory of everything in higher dimensions. The mechanism of the realization of the physical three-dimensional space by compactification and the method of detecting the extra dimensions in our universe may be obtained by studying the solitonic objects in higher dimensions.

The exact solution for an arbitrary number of maximally charged dilatonic black holes has been found by several authors [2,3]. Static multi-soliton solutions, which saturate certain Bogomol’nyi-type bounds, have been studied in various theories [1]. There is no net static force among the solitonic objects described by such a multi-soliton solution. In the present paper, using the identification process, we show the multi-centered solution on the toroidally compactified spacetime, $R^4 \times T^d$ in Sec. II. The same method can be used for general toroidal compactifications in arbitrary dimensions. We find that the solution reduces to the multi-black hole solution on $R^4$ if the scale of the torus shrinks to zero.

Recently the present author has also studied the interaction of maximally charged dilatonic black holes at low velocities and obtained the moduli space metric for two such black holes explicitly [4,5]. In Sec. III in the present paper, we present the low-energy interaction of the multi-black hole system on $R^4 \times T^d$. We concentrate on the case with $d = 1$: This case arises in the five-dimensional Einstein-Maxwell-dilaton theory with Kaluza-Klein compactification on a circle. Two-body scattering is analyzed in this case in Sec. IV. The last section V is devoted to a brief conclusion.
II. THE EXACT MULTI-SOLITON SOLUTION ON $R^4 \times T^d$

The action for the effective field theory of string theory in $(d+4)$ dimensional spacetime, including $U(1)$ gauge field but discarding the antisymmetric field strength $H_{\mu\nu\lambda}$, is

$$S = \int d^{d+4}x \frac{\sqrt{-g}}{16\pi G} e^{-2\phi} \left[ R + 4(\nabla \phi)^2 - F^2 \right] + (\text{surface terms}) ,$$

where $R$ is the scalar curvature and $\phi$ stands for the dilaton. The vector field $A_\mu$ is related to the field strength $F_{\mu\nu}$ by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $\mu, \nu$ run over $0, 1, \ldots, d + 3$. $G$ is the Newton’s constant.

Rescaling the action as $g_{\mu\nu} \rightarrow e^{\frac{4}{d+2}\phi} g_{\mu\nu}$ yields an action for the Einstein-Maxwell-dilaton theory, which includes the Einstein-Hilbert action for gravitation:

$$S = \int d^{d+4}x \frac{\sqrt{-g}}{16\pi G} \left[ R - \frac{4}{d+2} (\nabla \phi)^2 - e^{-\frac{4}{d+2}\phi} F^2 \right] + (\text{surface terms}) .$$

We take the action of this form to study the solution and interaction of the solitonic objects derived from it in this paper.

The static multi-centered solution with flat asymptotic region $R^{d+4}$ takes the form $[3,4]$:

$$ds^2 = -V^{-\frac{1}{d+2}} dt^2 + V^{\frac{2}{d+2}} dx^2, \quad A_\mu dx^\mu = \frac{1}{\sqrt{2}} \left(1 - \frac{1}{V}\right) dt, \quad e^{-\frac{4}{d+2}\phi} = V^{\frac{4}{d+2}} ,$$

where

$$V = 1 + \sum_\alpha \frac{4\pi}{A_{d+2}} \frac{2Gm_\alpha}{(d+1)|x - x_\alpha|^{d+1}} ,$$

where $A_{d+2} = \frac{2\pi^{(d+3)/2}}{\Gamma((d+3)/2)}$. The solution describes the situation that the $\alpha$-th nonrotating, charged dilatonic black hole in the extreme limit with mass $m_\alpha$ is located at $x = x_\alpha$.

Now we consider the multi-black hole solution on a torus space. Although we can consider arbitrary dimensions, we concentrate on the case of $R^4 \times T^d$, with a physical interest. To

---

1Strictly speaking, there are naked singularities in the solution. Nevertheless, we use the term “black hole” because the extreme case may still have generic properties of black holes in terms of their classical dynamics.
derive the solution which globally has the topology of $R^4 \times T^d$ from the asymptotically
flat solution (3,4), we use the technique of identification. We denote the coordinate of $d$-
dimensional subspace as $\xi_i$ ($i = 1, 2, \ldots, d$). We obtain the torus space by identifying $\xi_i$ and
$\xi_i + L_i \ell_i$, where $\ell_i$ is an integer. $L_i$ is the circumference of the $i$-th direction of the torus. As
for the multi-centered solution, the copious images of a black hole at $x_\alpha$ must be located at
$x_\alpha + L$ with $L \equiv (L_1 \ell_1, L_2 \ell_2, \ldots, L_d \ell_d, 0, 0, 0)$ and have the same mass $m_\alpha$.

Consequently, on the space with toroidal compactification, the set of solutions for the
metric and the other fields takes the same form as Eq. (3), but $V$ has the form

$$V = 1 + \sum_\alpha \frac{2\tilde{G} m_\alpha}{r_\alpha} \sum_{\ell_1 = -\infty}^{\infty} \cdots \sum_{\ell_d = -\infty}^{\infty} \exp \left( -2\pi r_\alpha \frac{\ell_i^2}{L_i^2} \right) \prod_{i=1}^{d} \cos \ell_i \theta_{\alpha,i} , \quad (5)$$

where $r_\alpha = \sqrt{(x-x_\alpha)^2 + (y-y_\alpha)^2 + (z-z_\alpha)^2}$ and $\theta_{\alpha,i} = 2\pi(\xi_i - \xi_{\alpha,i})/L_i$. $\tilde{G}$ represents an
effective Newton constant defined by $\tilde{G} = G/V_T$ with $V_T = \prod_{i=1}^{d} L_i$.

In the limit of all $L_i \to 0$, $V$ is reduced to

$$V = 1 + \sum_\alpha \frac{2\tilde{G} m_\alpha}{r_\alpha} , \quad (6)$$

which is the same as the solution for the four dimensional case (3).

For $d = 1$, the sum over $\ell$ in Eq. (3) can be done analytically, and then $V$ is obtained as
follows:

$$V = 1 + \sum_\alpha \frac{2\tilde{G} m_\alpha}{r_\alpha} \frac{\sinh 2\pi r_\alpha/L}{\cosh 2\pi r_\alpha/L - \cos \theta_\alpha} . \quad (7)$$

The multi-soliton solution on a torus in Einstein-Maxwell-dilaton theory with an arbi-
trary dilaton coupling (3) can be constructed in the same manner.

### III. INTERACTIONS

The present author has applied the method of Ferrell and Eardley (3) to calculate the
interaction energy of the maximally charged dilatonic black holes at low velocities by making
use of the exact, static solution (3). The equations in detail for its derivation can be referred
to the previous work (4,5).
Since there are only two-body (velocity-dependent) forces in the multi-black hole system in this case\footnote{For a general dilaton coupling, the interaction has more complicate form \cite{4,5}.}, the general expression for the $O(v^2)$ lagrangian of the system of an arbitrary configuration of black holes can be easily obtained as: \cite{4,5}

\[
L_v^2 = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 + \sum_{\alpha\beta} \frac{4\pi G m_\alpha m_\beta}{A_{d+2}} \frac{|v_\alpha - v_\beta|^2}{2(d+1)|x_\alpha - x_\beta|^{d+1}}
\]

\[= \frac{1}{2} M V^2 + \sum_{\alpha\beta} \frac{m_\alpha m_\beta |v_\alpha - v_\beta|^2}{4M} \left( 1 + \frac{4\pi G}{A_{d+2}} \frac{2GM}{(d+1)|x_\alpha - x_\beta|^{d+1}} \right), \tag{8}
\]

where $v_\alpha$ is the velocity of the extreme dilatonic black hole with mass $m_\alpha$. The total mass $M$ is given by $M = \sum_\alpha m_\alpha$, and $V$ is the velocity of the center of mass; $V \equiv \sum_\alpha m_\alpha v_\alpha / M$. We have disregarded the constant term in the lagrangian, which is proportional to the total mass of the system.

Again we use the identification of the images to construct the interaction lagrangian of extreme dilatonic black holes on $R^4 \times T^d$. The images of a certain black hole have the same velocity as well as the same mass. We also note that the sum of the interaction lagrangian of the images yields an overall infinite factor, $\sum_{\ell_1 = -\infty}^\infty \cdots \sum_{\ell_d = -\infty}^\infty 1$. We have simply to omit this overall factor.

Finally we get the following lagrangian of $O(v^2)$:

\[
\tilde{L}_v^2 = \frac{1}{2} M V^2 + \sum_{\alpha\beta} \frac{m_\alpha m_\beta |v_\alpha - v_\beta|^2}{4M} \left[ 1 + \frac{2\tilde{G} M}{\rho_{\alpha\beta}} \sum_{\ell_1 = -\infty}^\infty \cdots \sum_{\ell_d = -\infty}^\infty \exp \left( -2\pi \rho_{\alpha\beta} \sqrt{\sum_{i=1}^d \ell_i^2} \right) \prod_{i=1}^d \cos \ell_i \theta_{\alpha\beta,i} \right], \tag{9}
\]

where $\rho_{\alpha\beta} = \sqrt{(x_\alpha - x_\beta)^2 + (y_\alpha - y_\beta)^2 + (z_\alpha - z_\beta)^2}$ and $\theta_{\alpha\beta} = 2\pi (\xi_{\alpha,i} - \xi_{\beta,i}) / L_i$. \footnote{For a case of a general dilaton coupling, since the interaction contains many-body, velocity-dependent forces \cite{4,5}, the expression will be more complicated. The resemblance between Eq. (9) and Eq. (8) is a coincidence.} Note that we have not used any long-distance approximation. The dependence of the configuration is
exact at this order in \( v^2 \).

In the next section, we study the two-body scattering on \( R^4 \times S^1 \) using the low-energy interaction lagrangian.

IV. TWO-BODY SCATTERING ON \( R^4 \times S^1 \)

The slow motion of solitons can be described as geodesics on the space of parameters (moduli space) for static solutions. Therefore the calculation of the metric on moduli space is the most effective way to investigate the interaction of slowly moving solitons in classical field theory \[7\]. In this section, we consider the scattering of two maximally charged dilatonic black holes on \( R^4 \times S^1 \) using the metric of moduli space.

The \( O(v^2) \) lagrangian of the two-body system of the black holes with masses \( m_A \) and \( m_B \) is obtained from Eq. (9), that is

\[
\tilde{L}_{v^2} = \frac{1}{2}(m_A + m_B) V^2 + \frac{1}{2} \mu \left[ 1 + \frac{2\tilde{G}(m_A + m_B)}{\rho} \frac{\sinh 2\pi\rho/L}{\cosh 2\pi\rho/L - \cos \theta} \right] |\mathbf{v}_A - \mathbf{v}_B|^2, \tag{10}
\]

where \( \rho = \rho_{AB} \) and \( \theta = \theta_{AB} \). The reduced mass \( \mu \) is given by \( \mu = m_A m_B / (m_A + m_B) \).

Hereafter we assume that the black holes move in a (scattering) plane in the three-dimensional space. Thus the moduli space of this configuration is reduced to be a three-dimensional space parameterized by the mutual distance \( \rho \), the azimuthal angle \( \varphi \), and the angle difference on the extra circle \( \theta \).

For this two-body system, the metric on moduli space of relative motion can be read from Eq. (10) as

\[
ds_{MS}^2 = \gamma(\rho, \theta) \left( d\rho^2 + \rho^2 d\varphi^2 + \frac{L^2}{4\pi^2} d\theta^2 \right), \tag{11}
\]

with

\[
\gamma(\rho, \theta) = 1 + \frac{2\tilde{G}M}{\rho} \frac{\sinh 2\pi\rho/L}{\cosh 2\pi\rho/L - \cos \theta}, \tag{12}
\]

where \( M = m_A + m_B \).
We describe the two-body scattering using the coordinate where one of the extreme dilatonic black holes is located at the origin. We set a length scale such that $2\tilde{G}M = 1$ in the followings. The parameters used in the description of the scattering process is exhibited in FIG. 4. The effect of the extra dimensions becomes important when the typical, or minimum distance between the black holes $\rho_0$ and the compactification scale $L$ are the same order. Therefore, we choose $L = \pi$ in the following numerical calculations to see the effect well.

First we examine the case that $\theta$ is fixed to be zero everywhere. This situation satisfies the equation of motion for $\theta$ derived from the geodesic equation on the moduli space. The orbit of the projectile is determined by the geodesic equation derived from the line element (11, 12) by the variational method. The relation between the impact parameter $b$ and the scattering angle $\Theta$ is shown in FIG. 2 for $L = \pi$. Note that the horizontal axis indicates $1/b$. If $b$ is sufficiently large, the deflection is described by the Rutherford scattering [5]. For large $b$, the angle of deflection is well described by the expression for four-dimensional black holes [5,4]:

$$\Theta = 2 \arctan \frac{1}{2b}.$$  

(13)

If $b$ is small and comparable to $L$, since the shape of the moduli space in the vicinity of the origin looks like the one of the five-dimensional black hole system, then the scattering angle $\Theta$ diverges when $b$ approaches a certain value $b_{\text{crit}} [4]$. The two dilatonic black holes coalesce for $b < b_{\text{crit}} [4]$. The critical value $b_{\text{crit}}$ is given by

$$b_{\text{crit}} = \frac{L}{\pi},$$  

(14)

in the case on $R^4 \times S^1$. For small $b$, the scattering angle is well approximated by the expression for five-dimensional black holes [4]:

$$\Theta = \frac{1 - \sqrt{1 - \frac{L}{b^2\pi}}} {\sqrt{1 - \frac{L}{b^2\pi}}} \pi.$$  

(15)

Next we consider the effect of the relative motion of black holes on the extra space $S^1$. We take $\rho_0 = 1$, and at this position we assume non-zero value of the gradient of $\theta$ along
the path, that is $|\nabla \theta|$. For simplicity, we set the initial value of $\theta (= \theta_{\text{init}})$ at this point equals to zero. We show an example of the trajectory of an extreme dilatonic black hole when $L = \pi$, $\rho_0 = 1$, and the initial value of $|\nabla \theta| = |\nabla \theta|_{\text{init}} = 4$ in FIG. 3. As one can see from the behavior in FIG. 3, the value of the angle $\varphi_0$ turns out to be very sensitive to the initial value of $|\nabla \theta|$, though the effect is suppressed if $\rho_0$ is much larger than $L$. This is due to the periodic nature of the interaction on $\theta$ and thus the motion is not merely a projection of the path of two-body scattering in the isotropic five dimensions. The periodicity of the effective force may bring about chaotic motion of black holes. More deep inspection will be needed and studied elsewhere.

V. CONCLUSION

We have shown the multi-soliton solution in Einstein-Maxwell-dilaton theory on $R^4 \times T^d$, and investigated the interaction of maximally charged dilatonic black holes on $R^4 \times T^d$ in the low velocity limit. The scattering of two black holes on $R^4 \times S^1$ has been studied numerically.

These features found in the case of $R^4 \times S^1$ may exist also in a general dimensional case. The effect of the extra dimensions can be found in the process of the “near-head-on” collision. The effect of the relative motion in the extra space is also observable in our three-dimensional world if the typical scale of the two-body system is the same order of the compactification scale.

We must bear in mind that our analyses are based on the low-velocity lagrangian. The terms of higher order in $v$ and the radiational back-reaction may become important if all the typical scales of the scattering process are very small. These corrections will be studied in the future work.

The true effective theory of string theory contains the antisymmetric tensor field $B_{\mu \nu}$ as well as the higher-order terms of the curvature and the field strength. The analysis on the theory including these will be of much importance if we pursue the connection between string theory and solitons [11, 12].
REFERENCES

[1] M. J. Duff, R. R. Khuri and J. X. Lu, Phys. Rep. 259, 213 (1995).

[2] G. W. Gibbons and K. Maeda, Nucl. Phys. B298, 741 (1988).
   D. Garfinkle, G. Horowitz and A. Strominger, Phys. Rev. D43, 3140 (1991); (E) D45, 3888 (1992).

[3] K. Shiraishi, J. Math. Phys. 34, 1480 (1993).

[4] K. Shiraishi, Nucl. Phys. B402, 399 (1993).

[5] K. Shiraishi, Int. J. Mod. Phys. D2, 59 (1993); preprint AJC-HEP-26, gr-qc/9507029.

[6] R. C. Ferrell and D. M. Eardley, Phys. Rev. Lett. 59, 1617 (1987).
   J. Traschen and R. Ferrell, Phys. Rev. D45, 2628 (1992).

[7] N. Manton, Phys. Lett. B110, 54 (1982); ibid. B154, 397 (1985).
   R. S. Ward, Phys. Lett. B158, 424 (1985).
   M. F. Atiyah and N. J. Hitchin, The Geometry and Dynamics of Magnetic Monopoles
   (Princeton University Press, Princeton, 1988).

[8] M. J. Duff and J. Rahmfeld, Phys. Lett. B345, 441 (1995).

[9] R. R. Khuri and R. C. Myers, preprint McGill/95-38, CERN-TH/95-213, hep-th/9507043.

[10] C. G. Callan, J. M. Maldacena and A. W. Peet, preprint PUPT-1565, hep-th/9510134.
FIGURES

FIG. 1. A schematic view of the two-body scattering process and an introduction of parameters.

FIG. 2. The scattering angle Θ versus the inverse of the impact parameter $b$, for $L = \pi$. Two curves described by Eq. (13) and Eq. (15) are also exhibited as dashed lines.

FIG. 3. An example of the trajectory of an extreme dilatonic black hole when $L = \pi$, $\rho_0 = 1$, and the initial value of $|\nabla \theta|_{init} = 4$, and $\theta_{init} = 0$. 

This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9511005v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9511005v1