Finite sample breakdown point of multivariate regression depth
median

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Abstract

Depth induced multivariate medians (multi-dimensional maximum depth estimators) in regression serve as robust alternatives to the traditional least squares and least absolute deviations estimators. The induced median ($\beta_{RD}^*$) from regression depth (RD) of Rousseeuw and Hubert (1999) (RH99) is one of the most prevailing estimators in regression.

The maximum regression depth median possesses outstanding robustness similar to the univariate location counterpart. Indeed, the $\beta_{RD}^*$ can, asymptotically, resist up to 33% contamination without breakdown, in contrast to the 0% for the traditional estimators (see Van Aelst and Rousseeuw, 2000) (VAR00). The results from VAR00 are pioneering and innovative, yet they are limited to regression symmetric populations and the $\epsilon$-contamination and maximum bias model.

With finite fixed sample size practice, the most prevailing measure of robustness for estimators is the finite sample breakdown point (FSBP) (Donoho (1982), Donoho and Huber (1983)). A lower bound (LB) of the FSBP for the $\beta_{RD}^*$, which is not sharp, was given in RH99 (in a corollary of a conjecture).

An exact FSBP (or even a sharper LB) for the $\beta_{RD}^*$ remained open in the last two decades. This article establishes a sharper lower and upper bounds of (and an exact) FSBP for the $\beta_{RD}^*$, revealing an intrinsic connection between the regression depth of $\beta_{RD}^*$ and its FSBP. This justifies the employment of the $\beta_{RD}^*$ as a robust alternative to the traditional estimators and demonstrating the necessity and the merit of using the FSBP in finite sample real practice instead of an asymptotic breakdown value.

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1 Introduction

The notion of depth in regression was introduced and investigated two decades ago. Regression depth (RD) of Rousseeuw and Hubert (1999) (RH99) is the most popular example in the literature.

One of the primary advantages of depth notion in regression is that it can be utilized to directly introduce a median-type maximum depth estimator (aka depth median), which can serve as a robust alternative to the traditional least squares and least absolute deviations estimators.

Robustness of the regression depth induced median ($\beta_{RD}^*$) was investigated two decades ago by Van Aelst and Rousseeuw (2000) (VAR00). It turns out that the regression median can, asymptotically, resist up to 33% contamination without breakdown, in contrast to the 0% for the traditional estimators.

The result in VAR00 was established for population (regression) symmetric distributions, under the $\epsilon$ contamination model and maximum bias framework in the asymptotic sense; it is not directly applicable to fixed finite sample practice.

In the finite sample scenario, the most prevailing robustness measure is the finite sample breakdown point (FSBP), introduced by Donoho and Huber (1983) and popularized and promoted by Rousseeuw (1984), Rousseeuw and Leroy (1987) (RL87) and Donoho and Gasko (1992), among others.

The FSBP of the $\beta_{RD}^*$ was briefly addressed in RH99, and one lower bound (LB) of the FSBP was given in Corollary of Conjecture 1 (CC1). Theorem 8 (TH8) of RH99 listed, albeit with no published proof available then, the limiting value of the FSBP, 1/3, for data from regression symmetric populations with a strictly positive density.

The LB in CC1 of RH99 depends on a generic LB of maximum RD value in part (a) of Conjecture 1 of RH99. It is approximately $1/(p+1)$ for a large $n$, and it can never approach the limit value 1/3 (for $p > 2$), which is the asymptotic breakdown value of the $\beta_{RD}^*$, in TH8 of RH99 and in VAR00. This implies that the LB in RH99 is not sharp.

A sharper LB of the FSBP (or the exact FSBP) for the $\beta_{RD}^*$ has remained open for the last two decades. This article tries to fill that gap, presenting the exact FSBP and sharp lower and upper bounds, complementing the complete robust spectrum of the $\beta_{RD}^*$. A sharper LB given in this article that utilizes the maximum RD value reveals an intrinsic connection between the maximum depth and the FSBP for the $\beta_{RD}^*$. The higher the depth value of the $\beta_{RD}^*$, the more robust the $\beta_{RD}^*$ is. The new LB can have the limiting value 1/3.

Furthermore, a sharp upper bound (UB) of the FSBP of the $\beta_{RD}^*$ given in this article indicates that in finite sample (small $n$s and large $p$s) cases the $\beta_{RD}^*$ can actually resist much less than the asymptotic result of 33% (1/3) contamination, which renders 1/3 invalid in finite sample practice. Findings here justify (i) the legitimacy of employing the $\beta_{RD}^*$ as an alternative to the traditional estimators and (ii) the necessity and merits of investigating the finite sample breakdown point of the $\beta_{RD}^*$. 

Throughout, we are concerned with the FSBP of the maximum depth estimator $\beta_{RD}^*$ for the multivariate candidate regression parameter $\beta = (\beta_1, \beta_2')' \in \mathbb{R}^p$ ($p \geq 2$) in the model:

$$y = (1, x')\beta + e,$$

(1)

where $'$ denotes the transpose of a vector, random vector $x = (x_1, \ldots, x_{p-1})'$ is in $\mathbb{R}^{p-1}$, and random variables $y$ and $e$ are in $\mathbb{R}^1$. The $\beta_1$ is the intercept term in the model (1).

Section 2 briefly reviews the history behind the notion of the breakdown point and introduces (i) two versions for the FSBP and (ii) the notion of regression depth and depth induced median. Section 3 establishes a general upper bound of the FSBP for any regression equivariant (see Def. in Section 3) estimator and sharp lower and upper bounds of (and the exact) FSBP for the regression depth median $\beta_{RD}^*$. Section 4 is devoted to the comparison of the sharpness of the LB of the FSBP for the $\beta_{RD}^*$ in RH99 and the one established in this article and reveals that the latter is sharper than the former. The article ends with some concluding remarks, including a discussion of the relevancy of the asymptotic breakdown point in finite samples, supported by substantial empirical evidence in Section 5.

2 Finite sample breakdown point and regression depth

2.1 Finite sample breakdown point (FSBP)

The notion of the breakdown point first appeared in Hodges (1967) and later was generalized by Hampel (1968, 1971). The finite sample versions of the breakdown point, including the addition breakdown point (ABP) and replacement breakdown point (RBP), were introduced in Donoho and Huber (1983) (DH83). They have become the most prevalent quantitative assessments of the global robustness of estimators, complementing the assessment of (i) local robustness of estimators captured by the influence function approach (see Hampel, et al. (1986)) and (ii) global robustness of estimators assessed by the asymptotic breakdown point via the maximum bias approach (see Hampel, et al. (1986) and Huber (1981)).

Stimulating and intriguing discussions on the notion of the breakdown point include Donoho (1982), Rousseeuw (1984), Rousseeuw and Leroy (1987), Lopuhaä and Rousseeuw (1991), Maronna and Yohai (1991), Lopuhaä (1992), Donoho and Gasko (1992), Tyler (1994), Müller (1995), Ghosh and Sengupta (1999), Davies (1987, 1990, 1993), Davies and Gather (2005), Maronna, et al. (2006), and Liu, et al. (2017), among others.

Some authors are favorable to the ABP in the discussion of the robustness property of estimators, whereas others prefer the RBP, which they believe is more simple, realistic and generally more applicable. Zuo (2001) presented some quantitative relationships between the two versions of the finite sample breakdown point, rendering the arguments on the preference (precedence) between the two versions void in many cases. Nevertheless, for a given estimator, sometimes one version can be more convenient for the derivation of a desired result. This is especially true for the FSBP of $\beta_{RD}^*$ as demonstrated in Proposition 3.3.

We now explore the definitions for the two versions of the FSBP. Let $X^n = \{X_1, \ldots, X_n\}$ be an uncontaminated sample of size $n$ in $\mathbb{R}^p$ (in the regression setting, $X^n$ is the $Z^n = \{(x_i', y_i)', i = 1, \ldots, n\}$ in $\mathbb{R}^p$).
A location estimator $T$ in $\mathbb{R}^p$ is called translation equivariant if $T(X^n + b) = T(X^n) + b$ for any $b \in \mathbb{R}^p$, where $X^n + b = \{X_1 + b, \ldots, X_n + b\}$. Translation equivariance is a preliminary and desirable property for any reasonable location estimator. In the regression setting, it becomes regression equivariance (see Section 3), the following definitions can be adapted directly for regression estimators.

**Definition 2.1** [DH83] The finite sample addition breakdown point of a translation equivariant estimator $T$ at $X^n$ in $\mathbb{R}^p$ is defined as

$$\text{ABP}(T, X^n) = \min \left\{ \frac{m}{n + m} : \sup_{Y^m} \| T(X^n \cup Y^m) - T(X^n) \| = \infty \right\},$$

where $Y^m$ denotes a dataset of size $m$ with arbitrary values and $X^n \cup Y^m$ denotes the contaminated sample by adjoining $Y^m$ to $X^n$, $\| \cdot \|$ stands for Euclidean norm.

It is apparent that the ABP of the univariate sample median is $1/2$, whereas that of the mean is $1/(n + 1)$.

**Definition 2.2** [DH83] The finite sample replacement breakdown point of a translation equivariant estimator $T$ at $X^n$ in $\mathbb{R}^p$ is defined as

$$\text{RBP}(T, X^n) = \min \left\{ \frac{m}{n} : \sup_{X^n_m} \| T(X^n_m) - T(X^n) \| = \infty \right\},$$

where $X^n_m$ denotes the corrupted sample from replacing $m$ points in $X^n$ with arbitrary $m$ values.

It is apparent that the RBP of the univariate sample median is $\lfloor(n + 1)/2\rfloor/n$, whereas that of the mean is $1/n$ (where $\lfloor \cdot \rfloor$ is the floor function).

In other words, the ABP and RBP of an estimator are respectively the minimum addition fraction and replacement fraction of the contamination which could drive the estimator beyond any bound.

### 2.2 Regression depth of Rousseeuw and Hubert (1999)

**Definition 2.3** For any $\beta \in \mathbb{R}^p$ and joint distribution $P$ of $(x', y)$ in (1), RH99 defined the regression depth of $\beta$, denoted by $\text{RD}(\beta; P)$, to be the minimum probability mass that needs to be passed when tilting (the hyperplane induced from) $\beta$ in any way until it is vertical.

The RD$(\beta; P)$ definition above is rather abstract and not easy to comprehend. Some characterizations, or equivalent definitions, were given in the literature, e.g. in Remark 5.1 of Zuo (2018) and in Lemma 2.1 of Zuo (2020), also see (3) below.

The maximum regression depth estimating functional $T^*_{RD}$ (also denoted by $\beta^*_{RD}$) is defined as

$$T^*_{RD}(P) = \arg\max_{\beta \in \mathbb{R}^p} \text{RD}(\beta; P).$$

(2)

If there are several $\beta$s that attain the maximum depth value on the right hand side (RHS) of (2), then the average of all those $\beta$s is taken.
We obtain the sample version of RD($\beta; P$) and $T_{RD}^*(P)$ by replacing $P$ with $P_n$, the latter is the empirical distribution based on a given sample $Z^n = \{(x'_i, y_i)', i = 1, \cdots, n\}$ in $\mathbb{R}^p$. (In the empirical case, the RD discussed originally in RH99 divided by $n$ is identical to definition 2.3 above).

The notions of breakdown point and regression depth seem unrelated and have nothing to do with each other. But in the next section, it is shown that they are actually closely connected through $T_{RD}^*(P_n)$.

3 Finite sample breakdown point of regression depth median

For a given $\beta = (\beta_1, \beta_2)' \in \mathbb{R}^p$, denoted (hereafter) by $H_\beta$ the unique hyperplane determined by $y = (1, x')\beta$. Denote the angle between the hyperplane $H_\beta$ and the horizontal hyperplane plane $H_h$ (determined by $y = 0$) by $\theta_\beta$ (hereafter consider only the acute one). That is, $\theta_\beta$ is the angle between the normal vector ($-\beta_2, 1)'$ and the normal vector ($0', 1)'$ in the ($x', y)'$-space. Hence,

$$\cos(\theta_\beta) = \frac{1}{\sqrt{\|\beta_2\|^2 + 1}}.$$  

Therefore, it is not hard to see that $|\tan(\theta_\beta)| = \|\beta_2\|$.

Tilting $\beta$ to a vertical position (some vertical hyperplane $H_v$) in Definition 2.3 means tilting $H_\beta$ along a hyperline $l_v(\beta)$ (the intersection hyperline of $H_\beta$ with the $H_v$) to $H_v$.

Let $\min_{f_r}(l_v(\beta), P_n)$ be the minimum of the two fractions of data points touched by tilting $H_\beta$ in the definition of $RD_{RH}$ to a vertical position $H_v$ along $l_v(\beta)$ in two ways (one way is by crossing the double wedge formed by two single wedges with an acute angle between $H_\beta$ and $H_v$ (the two shaded regions in Figure 1) and the other way is by passing through the double wedge formed by two single wedges with an obtuse angle between $H_\beta$ and $H_v$ (the other two regions in Figure 1)).

In other words, within the two-dimensional plane that is perpendicular to the horizontal hyperplane $H_h$ (vertical cross-section), one tilts $H_\beta$ in either a clockwise or counter-clockwise manner. Then it is readily seen that we have

**Proposition 3.1** For a given data set $Z^n$, the regression depth of $\beta$ defined in Definition 2.3 can be characterized as

$$RD(\beta; P_n) = \inf_{l_v(\beta)} \min_{f_r}(l_v(\beta), P_n),$$

where the infimum is taken over all possible $l_v(\beta)$s or equivalently all possible $H_v$s.

**Proof:**

“The minimum probability mass that needs to be passed when tilting (the hyperplane induced from) $\beta$ in any way until it is vertical” in Definition 2.3 means that (i) $H_v$ could be arbitrary, which is equivalent to the arbitrariness of the intersection hyperline $l_v(\beta)$ of the $H_\beta$ with the $H_v$, (ii) tilting $H_\beta$ along the hyperline $l_v(\beta)$ to the vertical position $H_v$ in either
2–D cross–section with the origin indicated

Figure 1: A two-dimensional vertical cross-section of a figure in $\mathbb{R}^p$. There are two ways to tilt $H_\beta$ to a vertical position $H_v$ (which does not necessarily contain the origin in the definition, it does in this figure) along hyperline $l_v(\beta)$ (which passes $(0, 2)$ in the figure). One way is crossing the two wedges each with an acute angle (the shaded double wedge), the other way is passing through the other two wedges each with an obtuse angle (the unshaded double wedge). That is, counter-clockwise or clockwise, tilting $H_\beta$ to $H_v$ along hyperline $l_v(\beta)$.

of two ways (clockwise or counter-clockwise, see Figure 1), (iii) obtaining the minimum of the two fractions of data points touched by tilting $H_\beta$ in two ways and (iv) calculating the overall minimum of the empirical probability mass w.r.t. all possible $H_v$s. In light of the discussions before Proposition 3.1, all (i), (ii), (iii) and (iv) are exactly captured in the RHS of (3).

For a given sample $Z^n$ in $\mathbb{R}^p$ ($Z^n$ and $P_n$ are used interchangeably hereafter), write

$$k^*(P_n) = \max_{\beta \in \mathbb{R}^p} n \text{RD}(\beta, P_n).$$

(4)

Remarks 3.1

(1) The RHS of (3) actually implicitly involves all possible $H_v$s and the fixed $H_\beta$.

(2) $n \text{RD}(\beta, P_n)$ in (4) is the least number of data points touched by tilting $H_\beta$ to a vertical position in light of (3). The $k^*(P_n)$ then is the maximum (w.r.t. $\beta$s) of the least number of data points touched by tilting a $H_\beta$ to a vertical position. That is the least number of data points touched by tilting $H_{\beta_m}$ in any way to a vertical position with $\beta_m$ attaining the maximum RD (i.e. $\beta_m$ is a RD maximizer).
3.1 A general upper bound of FSBP for the $T_{RD}^*$

A regression estimator $T$ is called regression equivariant (page 116 of RL87) if
\[
T\left(\{(x'_i, y_i + (1, x'_i)b)', i = 1, \ldots, n\}\right) = T\left(\{(x'_i, y_i)', i = 1, \ldots, n\}\right) + b, \ \forall \ b \in \mathbb{R}^p \tag{5}
\]

Regression equivariance is just as desirable as translation equivariance for a multivariate location estimator. Because of regression equivariance, we see often in the literature the phrase “without loss of generality, assume that $\beta = 0$ in the following discussion”. This statement is based on the application of [5].

We first establish a general ABP upper bound for any regression equivariant estimator (the RBP counter part has been given in the literature, e.g. page 125 of RL87).

**Proposition 3.2** For a given $Z^n$ in $\mathbb{R}^p$, any regression equivariant estimator $T$ in $\mathbb{R}^p$ satisfies
\[
\text{ABP}(T, Z^n) \leq \frac{m}{n + m},
\]
where $m = n - p + 1$ (throughout we assume that $n \geq p$).

**Proof:**

Assume that $u \in \mathbb{R}^p$ is a unit vector from the null space of the $(p - 1) \times p$ matrix $(w'_{m+1}, \ldots, w_n)'$, where $w'_i = (1, x'_i)$. Now construct two contaminated data sets:
\[
\begin{align*}
&\{(x'_1, y_1)', \ldots, (x'_m, y_m)', (x'_{m+1}, y_{m+1})', \ldots, (x'_n, y_n)', (x'_1, y_1 + w'_1 b)', \ldots, (x'_m, y_m + w'_m b)'\}, \\
&\{(x'_1, y_1 - w'_1 b)', \ldots, (x'_m, y_m - w'_m b)', (x'_{m+1}, y_{m+1})', \ldots, (x'_n, y_n), (x'_1, y_1), \ldots, (x'_m, y_m)'\},
\end{align*}
\]

where $b = \lambda u$, $\lambda > 0$. Denote the two contaminated data sets by $Z^n \cup Y^n_1$ and $Y^n_2 \cup Z^n$. It is not hard to see that
\[
\begin{align*}
Y^n_1 &= \{(x'_1, y_1 + w'_1 b)', \ldots, (x'_m, y_m + w'_m b)'\} \\
Y^n_2 &= \{(x'_1, y_1 - w'_1 b)', \ldots, (x'_m, y_m - w'_m b)'\}.
\end{align*}
\]

Notice that $w'_j b = 0$ for $m < j \leq n$. Subtracting the $y$ component of each point in $Z^n \cup Y^n_1$ (where $x'$ is the $x$ component of the point) by the inner product of the $w' = (1, x')$ with the vector $b$, we obtain $Y^n_2 \cup Z^n$. This, in conjunction with [5], immediately leads to
\[
\lambda = \|b\| = \|T(Z^n \cup Y^n_1) - T(Z^n \cup Y^n_2)\| = \|(T(Z^n \cup Y^n_1) - T(Z^n)) + (T(Z^n) - T(Z^n \cup Y^n_2))\| \leq 2 \sup_{Y^n} \|T(Z^n \cup Y^n) - T(Z^n)\|
\]

Letting $\lambda \to \infty$, we immediately obtain the desired result.  

Since $T_{RD}^*$ is regression equivariant (see Zuo (2018)), it follows immediately that
Corollary 3.1. For a given $Z^n$ in $\mathbb{R}^p$, we have

$$\text{ABP}(T_{RD}^*, Z^n) \leq \frac{n - p + 1}{2n - p + 1}. \quad (6)$$

This upper bound for $T_{RD}^*$ is not sharp, in the following we will establish a sharper one.

3.2 The exact ABP and a sharp RBP upper bound for the $T_{RD}^*$

We shall say $Z^n$ is in general position (IGP) when any $p$ of observations in $Z^n$ give a unique determination of $\beta$. In other words, any $(p-1)$ dimensional affine subspace of the space $(x', y)$ contains at most $p$ observations of $Z^n$. When the observations come from continuous distributions, the event ($Z^n$ being in general position) happens with probability one.

Proposition 3.3 For a given IGP $Z^n$ (or $P_n$) in $\mathbb{R}^p$, we have

(A) $\text{ABP}(T_{RD}^*, Z^n) = \frac{k^*(Z^n) - p + 1}{n + k^*(Z^n) - p + 1}, \quad (7)$

(B) $\text{RBP}(T_{RD}^*, Z^n) \leq \frac{k^*(Z^n) - p + 1}{n}. \quad (8)$

Proof:

(A)

(i) We claim that $m < k^*(Z^n) - p + 1$ contaminating points are not enough to breakdown $T_{RD}^*$ in the addition manner.

Assume, otherwise, that $m < k^*(Z^n) - p + 1$ contaminating points are enough to breakdown $T_{RD}^*$. That is,

$$\sup_{Y^m} \|T_{RD}^*(Z^n \cup Y^m)\| = \infty.$$  

Denote $\beta_{RD}^*(Z^n \cup Y^m)$ (slight notation abuse) as the maximizer of RD at $Z^n \cup Y^m$ which has the maximum norm among all the RD maximizers. There are at most finitely many RD maximizers for a fixed finite sample size $n + m$ (assume, without loss of generality, that RD maximizer contains at least $p$ sample points).

Notice that $T_{RD}^*(Z^n \cup Y^m)$ is not necessarily identical to $\beta_{RD}^*(Z^n \cup Y^m)$ in the non-unique maximizer case, and the RD of the former could be smaller than the latter and less than the maximum depth value due to the average. For example, in the case of the data set $\{(0,0)',(1,1)',(5,0)',(6,1)\}'$ in $\mathbb{R}^2$, the former is 0 whereas the latter is 1/2. See Figure [2]. This is one of the reasons we will treat $\beta_{RD}^*(Z^n \cup Y^m)$ in the sequel instead of $T_{RD}^*(Z^n \cup Y^m)$, in order to compensate for the weakness of the latter.

The other reason, which is the more important one, is

$$\sup_{Y^m} \|T_{RD}^*(Z^n \cup Y^m)\| = \infty \text{ if and only if } \sup_{Y^m} \|\beta_{RD}^*(Z^n \cup Y^m)\| = \infty.$$ 

The above implies that either
Figure 2: Four in general position points (represented by four filled circles) located in $\mathbb{R}^2$. Six lines formed, \{(0, 1), (−5, 1)(1, 0), (0, 0), (0, 1/6), (5/4, −1/4)\}, in (intercept, slope) form, each connecting two data points (unillustrated in the figure). Each line attains the maximum regression depth $1/2$. However, the average of these deepest lines, $T^{\ast}_{RD}$: \[ \left(-\frac{11}{4}, \frac{23}{12}\right) \] has regression depth $0$.

(I) $|\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})_1| \rightarrow \infty$ and $\|\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})_2\| \text{ is finite, or}$

(II) $\|\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})_2\| = |\tan(\theta_{\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})})| \rightarrow \infty$,

along a sequence of $(Y^m)_{j}$ as $j \rightarrow \infty$, where the subscript 2 on the LHS of (II) means that it is the non-intercept component of the $\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})$ (remember we write $\beta = (\beta_1, \beta_2)'$).

Case (I).

Assume that the hyperplane $H_{\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})}$ intersects the horizontal hyperplane $y = 0$ at the hyperline $l_v(\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j}))$ (when the two do not intersect then we assume that the hyperline exists at infinity, the arguments hearafter go through). Since the intercept term of $H_{\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})}$ approaches infinity and the $\|\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})_2\|$ is finite, the hyperplane no longer contains any points from $Z^n$. There are at most $m$ contaminating points from $(Y^m)_{j}$ on the hyperplane. Therefore it is readily seen that

\[ m + (p - 1) > m \geq (m + n)\text{RD}(\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j}), Z^n \cup (Y^m)_{j}) \]

\[ = k^\ast(Z^n \cup (Y^m)_{j}) \geq k^\ast(Z^n), \]  

where the first inequality is trivial and the second inequality follows from the facts that (i) one can tilt $H_{\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j})}$ along $l_v(\beta^{\ast}_{RD}(Z^n \cup (Y^m)_{j}))$ to a vertical position without touching
any points from \( Z^n \) and (ii) the definition of \( \text{RD} \) or Proposition 3.1. The third equality follows from (4) and the definition of \( \beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j) \). The last inequality follows from the fact that \((m + n)\text{RD}(\beta, Z^n \cup Y^m) \geq n \text{RD}(\beta, Z^n)\) in light of (3).

Case (II).

If, there exists a finite \( j \) such that \( \theta_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} = \pi/2 \), then at most \( m \) contaminating points from \((Y^m)_j\) are on the hyperplane \( H_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} \) which contains at most \( p - 1 \) points from \( Z^n \). The latter is due the fact that \( Z^n \) is IGP and the intersection hyperline between \( H_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} \) and the horizontal hyperplane \( y = 0 \) is a \((p - 2)\) dimensional subspace of the \( p \) dimensional space \((x', y)\). It is not hard to see that

\[
m + (p - 1) \geq (m + n)\text{RD}(\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j), Z^n \cup (Y^m)_j) = k^*(Z^n \cup (Y^m)_j) \geq k^*(Z^n),
\]

where the first inequality follows from the fact that the vertical hyperplane contains at most \( m + (p - 1) \) points from \( Z^n \cup (Y^m)_j \). The second equality follows from (4) and the definition of \( \beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j) \) above. The last inequality follows from the fact that \((m + n)\text{RD}(\beta, Z^n \cup Y^m) \geq n \text{RD}(\beta, Z^n)\) in light of (3).

Otherwise (i.e. \( \theta_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} < \pi/2 \), for any \( j \)), consider, without loss of generality (and in light of (3)) for any large \( j \) and \((Y^m)_j\) and \( H_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} \), a vertical hyperplane \( H_v \) (which depends on \( j \)) that contains no data points from \( Z^n \) and intersects with \( H_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} \) at \( l_v\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j) \).

Clearly, there exists a narrow vertical hyperstrip centered at \( H_v \) (with its two boundary hyperplanes parallel to \( H_v \)), and within the hyperstrip there are no data points from \( Z^n \).

Now, when tilting the hyperplane \( H_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} \) (which is already almost vertical for a large enough \( j \)) along \( l_v\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j) \) to its eventual vertical position of \( H_v \), it is readily apparent that

\[
m + p - 1 \geq (m + n) \min_{f_v}(l_v(\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j), P_{n+m}) \geq \text{RD}(\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j) = k^*(Z^n \cup (Y^m)_j) \geq k^*(Z^n),
\]

where \( P_{n+m} \) stands for the empirical distribution based on \( Z^n \cup (Y^m)_j \). The first inequality follows from the definition of \( \min_{f_v}(l_v(\beta, P_n) \) (since there is one way of tilting the hyperplane \( H_{\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)} \) along \( l_v(\beta_{\text{RD}}^{*}(Z^n \cup (Y^m)_j)) \) so that no original data points from \( Z^n \) (except at most \( p - 1 \) points form \( Z^n \) that are already on the hyperplane) are touched during the movement and the points it can touch are at most all the \( m \) contaminating points).

The second inequality above follows (3) and the third equality follows from the introduction of \( \beta_{\text{RD}}^{*}(Z^n \cup Y^m) \) at the beginning and from (4). Finally, the fourth inequality comes from \((m + n)\text{RD}(\beta, Z^n \cup Y^m) \geq n \text{RD}(\beta, Z^n)\) in light of (3).
All the inequalities in (9), (10) and (11) lead to a contradiction. Thus, \( m < k^\ast(Z^n) - p + 1 \) contaminating points are not enough to break down \( T_{RD}^\ast \).

(ii) We claim that \( m = k^\ast(Z^n) - p + 1 \) points are enough to break down \( T_{RD}^\ast \) in the addition manner.

Let \( l_h \) be a hyperline in the (p-1)-dimensional \( x \) space \( (y = 0, \) the subspace of \( (x', y)' \) space in \( \mathbb{R}^p \) that contains \( p - 1 \) points of \( x' \)'s and \( H_h \) be the corresponding vertical hyperplane that intercepts with the horizontal hyperplane \( y = 0 \) at the hyperline \( l_h \).

Place \( m \) contaminating points in \( Y^m \) at a point \( Z = (x', y)' \) on the hyperplane \( H_h \), where \( y \) could be arbitrarily large and the position of \( Z \) is described below. Denote the resulting data set as \( Z^{n+m} := Z^n \cup Y^m \). Call the \( \beta \) corresponding to \( H_h \) by \( \beta_c \). It is readily seen that \( (n + m)RD(\beta_c, Z^{n+m}) = k^\ast(Z^n) \).

If we can show that \( \beta_c \) attains the maximum RD with respect to (w.r.t.) \( Z^{n+m} \), then we have added \( m \) points to break down \( T_{RD}^\ast \), which is the average of all maximizers of RD w.r.t. \( Z^{n+m} \). Equivalently, we have to show that for any given \( \beta \in \mathbb{R}^p \), \( RD(\beta, Z^{n+m}) \leq k^\ast(Z^n)/(n + m) \).

Denote by \( H_\beta \) the unique hyperplane determined by \( \beta \). Consider two cases in the sequel:

(i) \( Z \in H_\beta \) (ii) \( Z \notin H_\beta \).

(i) \( Z \in H_\beta \). Assume that \( H_\beta \) intersects with the horizontal hyperplane \( (y = 0) \) at hyperline \( l(\beta) \). Apparently, \( l(\beta) \) contains at most \( p - 1 \) points from \( Z^n \). Furthermore since \( H_\beta \) is almost vertical (\( y \) can be arbitrarily large), there exists a narrow vertical hyperstrip centered at \( l(\beta) \) (with its two parallel boundary hyperplanes parallel to \( H_v \), the vertical hyperplane that intercepts \( y = 0 \) at \( l(\beta) \)) within the hyperstrip there are no data points from \( Z^n \) except those with \( x \)-component on \( l(\beta) \). Tilting \( H_\beta \) to the position of \( H_v \), it touches at most \( m + (p - 1) \) points from \( Z^{n+m} \), that is, \( RD(\beta, Z^{n+m}) \leq k^\ast(Z^n)/(n + m) \).

(ii) \( Z \notin H_\beta \). Without loss of generality (w.l.o.g.), assume that \( H_\beta \) contains \( p \) points from \( Z^n \) (after all we only care about \( H_\beta \)'s that contain most points from \( Z^n \) so that \( \beta \) can have as large as possible RD value w.r.t. \( Z^{n+m} \) that is no greater than \( k^\ast(Z^n)/(n + m) \)). Let \( H_{v_0} \) be the vertical hyperplane that intersects with the horizontal hyperplane \( (y = 0) \) at \( l_{v_0} \) such that when tilting \( H_\beta \) along \( l_{v_0} \) to \( H_{v_0} \) in one (assume, w.l.o.g., that it is counter-clockwise, see Figure (3)) of two ways, the number of points in \( Z^n \) touched is exactly \( nRD(\beta, Z^n) := k \leq k^\ast(Z^n) \).

Note that there are only finitely many \( H_\beta \)'s considered above. Put all \( \beta \)'s that with counter-clockwise tilting above and touched \( nRD(\beta, Z^n) \) points of \( Z^n \) into a group called \( G_{c-cw} \). Those \( \beta \)'s that with clockwise tilting above and touched \( nRD(\beta, Z^n) \) points of \( Z^n \) into a group called \( G_{cw} \).

Now we show that for any \( \beta \) in \( G_{c-cw} \), \( RD(\beta, Z^{n+m}) \leq k^\ast(Z^n)/(n + m) \). (Treatments for \( \beta \) in \( G_{cw} \) are similar and thus skipped).

Tilting \( H_\beta \) along \( l_{v_0} \) to vertical position \( H_{v_0} \), call the region on \( H_{v_0} \) touched during the tilting process as \( R(\beta, l_{v_0}, \beta_\ast) \). Since there are finitely many \( H_\beta \)'s hence finitely many
Figure 3: A two-dimensional vertical cross-section of a figure in $\mathbb{R}^p$. Assume that tilting $H_\beta$ to a vertical position $H_{v_0}$ along hyperline $l_{v_0}(\beta)$ crossing the shaded double wedge touches the minimum fraction of data points in $Z^n$ among all possible $l$'s. That is, the fraction is exactly $\text{RD}(\beta, Z^n)$. Note that the $y$ component of $Z$ controls its vertical position and $x$ component of $Z$ can control the horizontal position of $H_{\beta_0}$ (also called $H_h$).

$R(\beta, l_{v_0}, \beta_c)$s. Consequently, there is at least one point $Z$ (with $y$ component arbitrarily large) on $H_{\beta_0}$ that does not lie in the union of all $R(\beta, l_{v_0}, \beta_c)$s. Place $m$ contaminating points at $Z$ so that when tilting $H_\beta$ along $l_{v_0}$ counter-clockwise to $H_{v_0}$, it does not touch $Z$, see Figure (3).

Tilting $H_\beta$, counter-clockwise along $l_{v_0}$ to $H_{v_0}$, the number of the data points in $Z^{n+m}$ touched is $k$ (since only points in $Z^n$ originally lie in the two wedges passed might be touched). On the other hand, when tilting it in the clockwise way, the number of points touched in $Z^{n+m}$, denoted by $q$, is at least $p + m = k^*(Z^n) + 1$. Now we have

$$\text{RD}(\beta, Z^{n+m}) = \inf_{l_{v_0}(\beta)} \min_{f_r} (l_{v_0}(\beta), Z^{n+m}) \leq \min_{f_r} (l_{v_0}(\beta), Z^{n+m})$$

$$= \min\{k, q\}/(n + m) = k/(n + m) \leq k^*(Z^n)/(n + m),$$

where the first equality follows from Proposition 3.1, the second inequality is trivial, the third equality follows from above discussions on the number of points touched by tilting $H_\beta$ in two ways and from the definition of $\min_{f_r}$ given in (or before) Proposition 3.1, the fourth equality is trivial, so is the last inequality. We complete the proof of part (A).

(B)

We claim that $m = k^*(Z^n) - p + 1$ points are enough to break down $T^*_{RD}$ in the
replacement manner.

The proof of this part is similar to that of (ii) above. We employ the same notations. Let \( l_h \) and \( H_h \) be the same as before.

Move (or replace) \( m \) points from \( Z^n := \{(x'_i, y_i), i = 1, 2, \cdots, n\} \) to (by) a point \( Z = (x', y)' \) on the hyperplane \( H_h \), where \( y \) could be arbitrarily large and the position of \( Z \) is described below. Denote the contaminated data set as \( Z^n_m \). Call the \( \beta \) determined by \( H_h \) as \( \beta_c \). It is readily seen that \( n \text{RD}(\beta_c, Z^n_m) = k^*(Z^n) \).

If we can show that \( \beta_c \) attains the maximum RD with respect to \( Z^n_m \), then we have replaced \( m \) points to break down \( T^*_R \), which is the average of all maximizers of RD.

Equivalently, we have to show that for any given \( \beta \in \mathbb{R}^p \), \( \text{RD}(\beta, Z^n_m) \leq k^*(Z^n)/n \).

Denote by \( H_\beta \) the unique hyperplane determined by \( \beta \). Consider two cases in the sequel:

(i) \( Z \in H_\beta \). Assume that \( H_\beta \) intersects with the horizontal hyperplane \((y = 0)\) at hyperline \( l(\beta) \). Apparently, \( l(\beta) \) contains at most \( p - 1 \) points from \( Z^n \). Furthermore since \( H_\beta \) is almost vertical \((y \) can be arbitrarily large), there exists a narrow vertical hyperstrip centered at \( l(\beta) \) (with its two parallel boundary hyperplanes parallel to \( H_v \), the vertical hyperplane that intercepts \( y = 0 \) at \( l(\beta) \)), within the hyperstrip there are no data points from \( Z^n \) except those with \( x \)-component on \( l(\beta) \). Tilting \( H_\beta \) to the position of \( H_v \), it touches at most \( m + (p - 1) \) points from \( Z^n_m \), that is, \( \text{RD}(\beta, Z^n_m) \leq k^*(Z^n)/n \).

(ii) \( Z \not\in H_\beta \). Without loss of generality (w.l.o.g.), assume that \( H_\beta \) contains \( p \) points from \( Z^n \) (after all we only care about \( H_\beta \)s that contain most points from \( Z^n \) so that \( \beta \) can have as large as possible RD value w.r.t. \( Z^n_m \) that is no greater than \( k^*(Z^n)/n \)). Let \( H_{v_0} \) be the vertical hyperplane that intersects with the horizontal hyperplane \((y = 0)\) at \( l_{v_0} \) such that when tilting \( H_\beta \) along \( l_{v_0} \) to \( H_{v_0} \) in one (assume, w.l.o.g., that it is counter-clockwise, see Figure 3) of two ways, the number of points in \( Z^n \) touched is exactly \( n \text{RD}(\beta, Z^n) := k \leq k^*(Z^n) \).

Note that there are only finitely many \( H_\beta \)s considered above. Put all \( \beta \)s that with counter-clockwise tilting above and touched \( n \text{RD}(\beta, Z^n) \) points of \( Z^n \) into a group called \( G_{c-cw} \). Those \( \beta \)s that with clockwise tilting above and touched \( n \text{RD}(\beta, Z^n) \) points of \( Z^n \) into a group called \( G_{cw} \).

Now we show that for any \( \beta \) in \( G_{c-cw} \), \( \text{RD}(\beta, Z^n_m) \leq k^*(Z^n)/n \). (Treatment for \( \beta \) in \( G_{cw} \) is similar and thus skipped).

Tilting \( H_\beta \) along \( l_{v_0} \) to vertical position \( H_{v_0} \), call the region on \( H_{\beta_c} \) touched during the tilting process as \( R(\beta, l_{v_0}, \beta_c) \). Since there are finitely many \( H_\beta \)s hence finitely many \( R(\beta, l_{v_0}, \beta_c) \).s. there is at least one point \( Z \) (with \( y \) component arbitrarily large) on \( H_{\beta_c} \) that does not lie in the union of all \( R(\beta, l_{v_0}, \beta_c) \)s. Place \( m \) contaminating points at \( Z \) so that when tilting \( H_\beta \) along \( l_{v_0} \) counter-clockwise to \( H_{v_0} \), it does not touch \( Z \), see Figure 4.

Tilting \( H_\beta \), counter-clockwise along \( l_{v_0} \) to \( H_{v_0} \), the number of the data points in \( Z^n_m \) touched at most \( k \) (since points in \( Z^n \) originally lie in the two wedges passed might be moved
to (or replaced by) the point $Z$). On the other hand, when tilting it in the clockwise way, the number of points touched in $Z^n_m$, denoted by $q$, which is at least $p + m = k^*(Z^n) + 1$. Now we have

$$
RD(\beta, Z^n_m) = \inf_{l_v(\beta)} \min_{f_r} (l_v(\beta), Z^n_m) \leq \min_{f_r} (l_{v_0}(\beta), Z^n_m) \leq \min\{k, q\}/n = k/n \leq k^*(Z^n)/n,
$$

(15)

where the first equality follows from Proposition 3.1, the second inequality is trivial, the third inequality follows from above discussions on the number of points touched by tilting $H_\beta$ in two ways and from the definition of $\min_{f_r}$ given in (or before) Proposition 3.1, the fourth equality is trivial, so is the last inequality.

We complete the proof of part (B) and consequently that of the Proposition. ■

Remarks 3.2

(1) We conjecture that a sharp lower bound of RBP will be the same as the upper bound in (B). The ABP in (A) could be used for the comparison of sharpness with a LB of RBP given in RH99 whereas (A) and (B) could be employed to check the relevance of the asymptotic breakdown point $1/3$ (given in VAR00) in the finite sample practice. After the establishment of the lower bound in (A), this author learned that the same lower bound was obtained by Van Aelst, et al (2002) in their proof of limiting breakdown value $1/3$ for samples from regression symmetric distribution with a positive density.

(2) RH99 also presented in their Corollary of Conjecture 1 (CC1) a LB of FSBP for the $T^*_RD$, it is in the replacement breakdown format,

$$
RBP(T^*_RD, Z^n) \geq \frac{1}{n} \left( \left\lfloor \frac{n}{p + 1} \right\rfloor - p + 1 \right),
$$

(16)

under the assumptions that (i) the (a) of Conjecture 1 of RH99 holds true and (ii) $x_i$ are IGP. The RHS of (16) does not fit the cases considered in Zuo (2001) and consequently it is difficult to convert it to a LB of ABP. This makes it difficult to compare the two LBs (in ABP and RBP format) of FSBP for the $T^*_RD$ directly. That is, it is not clear which one is sharper at this moment. We tackle this problem next. ■

4 Sharpness of LBs of FSBP for the $T^*_RD$

Let us focus on the LB of the RBP in RH99. The essential difference between this and the LB of the ABP in Proposition 3.3 is the former (RH99) employing a lower bound, $\lfloor n/(p + 1) \rfloor$, for the maximum depth value whereas the latter (Proposition 3.3) utilizing the maximum depth value, $k^*(Z^n)$, directly. Note that $\lfloor n/(p + 1) \rfloor \leq k^*(Z^n)$ (conjecture 1 of RH99).

Consequently, (i) the former, the RHS of (16), purely depends on $n$ and $p$ (this could be an advantage) whereas the latter, $(k^*(Z^n) - p + 1)/(n + k^*(Z^n) - p + 1)$, depends on
the configuration of $Z^n$ (Note that FSBPs dependent on the data are not rare, e.g. that of median absolute deviations (MAD), also see Huber (1984) Theorem 3.1 and Davis and Guther (2007)) (ii) for a fixed $p > 2$, as $n \rightarrow \infty$ the former approaches $1/(p+1)$ which can never be $1/3$ (the asymptotic breakdown value) while the latter can approach $1/3$ with data from the regression symmetric (see Rousseeuw and Struyf (2004) for definition) population.

(iii) when $p > 2$, the $T^*_{RD}$ can resist a much high fraction of contamination than the one given in the RHS of (16).

ABP and RBP, albeit employing different contamination schemes, are the fraction (or percentage) of contamination that can force an estimator beyond any bound (becoming useless). From the contamination fraction/percentage interpretation, ABP is usually slightly smaller than RBP for the same estimator. For example, in the sample mean case, it is $1/(n+1)$ v.s. $1/n$; in the sample median case, it is $n/(n+n)$ v.s. $((n+1)/2)/n$. Directly comparing ABP and RBP, in terms of their magnitude, is unfavorable (or unfair) to ABP. But, without a better measure, it is at least one approach.

To better appreciate the sharpness of the LB of the ABP in Proposition 3.3 and know better the quantitative difference between the LB of the ABP in Proposition 3.3 and the LB of the RBP in RH99 for the same $T^*_{RD}$, we carry out a small scale simulation study to calculate the average differences of $(k^*(Z^n) - p + 1)/(n + k^*(Z^n) - p + 1)$ (the LB of the ABP in Prop. 3.3) with $(\lceil n/p+1 \rceil - p + 1)/n$ (the LB of the RBP in RH99) in 1000 multivariate $N(0, I)$ samples for different small $n$s and $p$s; the results are given in Table 1.

| n  | 10  | 20  | 30  | 50  | 100 | 200 |
|----|-----|-----|-----|-----|-----|-----|
| p=2| -3.725 | -1.776 | -0.913 | -2.237 | -2.456 | -1.805 |
| p=3| 10.38 | 8.736 | 5.235 | 4.646 | 5.139 | 5.328 |
| p=5| 31.52 | 16.85 | 15.09 | 12.02 | 11.47 | 11.15 |

Table 1: Average differences (expressed in percentage points) between the LB of the ABP in Prop. 3.3 and the LB of the RBP in the CC1 of RH99, based on 1000 multivariate standard normal samples.

It is readily apparent from the table that the LB of the ABP in Proposition 3.3 is sharper than the LB of the RBP in RH99 because of all the positive entries (when $p > 2$) and as well as the negative ones (when $p = 2$) since all entries should be negative if the number of contaminating points $m$ is the same in the two contamination schemes.

The positive entries in the table imply that the $T^*_{RD}$ can resist much higher contamination percentages than what provided by the LB of the RBP in RH99. Note that when $n$ increases so does the difference (e.g. when $p = 2$ and $n = 500$, the entry in the table will be $-1.352$), the same is true w.r.t the dimension parameter $p$. The LB of the RBP in RH99 becomes negative and uninformative if $n \leq (p + 1)(p - 2)$ (this explains the unusually large entry in the case $p = 5, n = 10$).

The results in the table demonstrate the merit of the LB of the ABP in Proposition 3.3.
One question that might be raised for the results in the table is: are those results distribution-dependent? That is, if the underlying distribution of the samples changes, does the LB of the ABP in Proposition 3.3 still have any advantage over the one in RH99? To answer this question, we carried out a small scale simulation study and results are reported in table 2.

Average of (the LB of the ABP in Prop. 3.3 – the LB of the RBP in RH99) in 1000 samples

| p | n  | 10  | 20  | 30  | 50  | 100 | 200 |
|---|----|-----|-----|-----|-----|-----|-----|
| 2 | -3.687 | -1.089 | -0.929 | -2.261 | -2.604 | -2.042 |
| 3 | 10.45 | 8.742 | 5.214 | 4.572 | 4.888 | 4.993 |
| 5 | 31.36 | 16.92 | 16.09 | 11.85 | 11.30 | 10.78 |

Table 2: Average differences (expressed in percentage points) between the LB of the ABP in Proposition 3.3 and the LB of the RBP in RH99, based on 1000 contaminated multivariate normal samples.

Here we generated 1000 samples \( Z^{(n)} = \{ (x'_i, y'_j)^t, i = 1, \cdots, n, x'_i \in \mathbb{R}^{p-1} \} \) from the Gaussian distribution with the zero mean vector and 1 to \( p \) as its diagonal entries of the diagonal covariance matrix for various \( ns \) and \( ps \). Each sample is contaminated by 5% i.i.d. normal \( p \)-dimensional points with individual mean 10 and variance 0.1. Thus, we no longer have symmetric errors and a homoscedastic variance model.

Comparing the table entries in Tables 1 and 2, we conclude that the sharpness of the LB of the ABP in Proposition 3.3 over the LB of the RBP in RH99 almost does not depend on the underlying distributions overall (confirmed in the multivariate t-distribution case). However, \( k^*(Z^n) \) does depend on the configuration of points in \( Z^n \).

LBs in VAR00 are not applicable when one replaces \( P \) by \( P_n \) or a real data set \( Z^n \). This is because the regression symmetric assumption does not hold in practice for fixed sample size data sets, and the depth region \( \{ \beta : \text{RD}(\beta, Z^n) \geq \eta \} \) for \( 0 < \eta < 1/3 \) is no longer bounded even if \( x_i \) are IGP. The boundedness is also assumed in Lemma 3 and TH2 of VAR00.

5 Concluding remarks

(I) The state of the art on the FSBP of the \( T_{RD}^p \).

No existing result in the literature covers or implies Proposition 3.3. The most closely related results include the same LB of ABP given in the proof of TH1 in Van Aelst, et al (2002) and the LB of RBP given in RH99 (but no exact FSBP or any upper bounds), Lemma 3 and TH2 in VAR00, but they are limited to the regression symmetric population distributions in an asymptotic bias sense, and those lower bounds (THs 4.1 and 4.2) in Mizera (2002) (M02) which are, maximally abstracted and general and also in an \( \epsilon \)-contamination and maximum bias framework. Furthermore, all of those LBs are not sharp (in light of Proposition 3.3).

(II) Finite sample versus asymptotic breakdown point, the merit of FSBP.
The asymptotic breakdown value or the limit of the finite sample breakdown value of $T_{RD}^*$, 1/3, was given in AVR00 (TH2), RH99 (TH8) and Van Aelst, et al (2002) (TH1), respectively.

Average of (the UB of RBP in Prop. 3.3 – 1/3 (asymptotic BP)) in 1000 samples

|   | 10   | 20   | 30   | 50   | 100   | 200   |
|---|------|------|------|------|-------|-------|
| p=2 | 2.447 | 5.997 | 7.617 | 9.105 | 10.664 | 11.975 |
| p=3 | -7.563 | -2.118 | 0.387 | 3.072 | 5.897  | 8.149  |
| p=5 | -20.523 | -13.053 | -9.190 | -5.083 | -0.941 | 1.927  |

Table 3: Average differences (expressed in percentage points) between the UB of RBP in Proposition 3.3 and the asymptotic breakdown value 1/3, based on 1000 multivariate standard normal samples.

This limiting result, 1/3, can be obtained directly from the (A) of Proposition 3.3 if samples come from the assumed distribution in above references, in this case $k^*(Z^n) = nRD(\beta_{RD}^*, Z^n)$ approaches $n/2$ as $n \to \infty$ (see TH6 and 7 of RH99). Namely, Proposition 3.3 recovers TH8 of RH99 and TH1 of Van Aelst, et al (2002). The $T_{RD}^*$ however, has to be used in finite sample practice, and the limit 1/3 is not that informative. For example, the latter implies that to break down $T_{RD}^*$, one has to use $n/3$ contaminating points. The (B) of Proposition 3.3 asserts that one just needs $k^*(Z^n) - p + 1$ contaminating points. The differences of $(k^*(Z^n) - p + 1)/n$ with 1/3 in finite sample cases (especially small sample sizes) are given in table 3 or Figure 2 below.

The table entries reveal once again the merit of the UB of the RBP for the FSBP of $T_{RD}^*$ because it is quite different than the asymptotic breakdown value (ABV) 1/3 in all cases considered. For example, in p=2 case, the UB of RBP indicates that $T_{RD}^*$ has a resistance rate for contamination much higher than the ABV 1/3; in p = 3 case, ABV ranges from overestimating the contamination rate for small n’s to underestimating the rate for $n > 20$; in p = 5 case, ABV overestimates the contamination rate for most of n’s considered. The ABV, 1/3, is irrelevant for these finite sample cases because it over-estimates systematically the FSBP of the $T_{RD}^*$ in small sample n’s and large p cases whereas it under-estimates the FSBP for large n’s with respect to the simulated data, the UB of the RBP in Proposition 3.3 increases when n increases for a fixed p and decreases as p increases for a fixed n.

Simulation results of the UB of RBP in Proposition 3.3 for the $T_{RD}^*$ in 1000 samples can also be displayed graphically in terms of their distributions such as in Figure 4.

Inspection of the figure reveals that (i) the UB of the RBP in Proposition 3.3 is always lower than the ABV 1/3 when $n \leq 20$ and $p > 2$, (ii) it decreases as $p$ increases for a fixed $n$ and increases as $n$ does for a fixed $p$ and (iii) outliers exist in various cases, including $p = 2, n = 20, 40, 50$; $p = 3, n = 20, 30, 40, 50$ and $p = 5, n = 20, 40, 50$. All these observations and results demonstrate the merit of the FSBP and the relevance of the bounds in Proposition 3.3 (and the irrelevance of the ABV 1/3) in finite sample practice.

(III) Justification of regression by the maximum depth estimator (median).
Figure 4: Boxplots for the UB of RBP in Proposition 3.3 for $T_{RD}^*$ based on 1000 multivariate standard normal samples for three $p$s and five $n$s.

Proposition 3.3 reveals the intrinsic connection between the breakdown point and the maximum depth value. This kind of connection was also discussed in M02.

This intrinsic connection clearly justifies employment of the maximum depth median as a robust alternative to the traditional regression (the least squares or the least absolute deviations) estimators since the former is much more robust both in a finite sample sense and in an asymptotic sense.

(IV) Location counterpart and other related results.

The location counterpart of RD and $\beta_{RD}^*$ are respectively halfspace depth (Tukey (1975)) and halfspace median (HM). The finite sample breakdown point of the latter has been investigated thoroughly in the literature, e.g. Donoho (1982), Donoho and Gasko (1992) (DG92), Chen (1995) (C95), Chen and Tyler (2000) (CT00), and Liu, et al. (2017) (LZW17).

In summary, the asymptotic breakdown point of the HM can be as high as 1/3 under symmetry and other assumptions (see, C95, CY00 and DG92), also see LZW17 (Proposition 2.10) where only $X^n$ is assumed to be IGP. The exact expression of the FSBP of HM is given in LZW17 under two assumptions (i) the $X^n$ is IGP and (ii) a special contaminating scheme: all contaminating points lie at the same site. It seems that the idea of the proof of Proposition 3.3 could be extended to establish the bounds of the FSBP for HM with a more regular and general contamination scheme.

The exact FSBP of the projection regression depth median, $T_{PRD}^*$, a major competitor of the $T_{RD}^*$, has been investigated and established in Zuo (2019). The asymptotic breakdown point of the $T_{PRD}^*$ reaches the highest possible value of 50%.

(V) Computation of regression median.

The computation of RD and $T_{RD}^*$ is challenging and has been discussed in RH99 briefly, in Rousseeuw and Struyf (1998), in Van Aelst, Rousseeuw, Hubert, and Struyf (2002), and in Liu and Zuo (2014). A R package “mrfDepth” has been developed by Segaert, Hubert, Rousseeuw, Raymaekers, and Vakili (2020). Like most other high breakdown point methods,
$T_{RD}$ has to be computed approximately, which might affect its actual finite sample breakdown value as pointed out in the literature.

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