ON METRIC SPACES WITH THE PROPERTIES OF DE GROOT AND NAGATA IN DIMENSION ONE

TARAS BANAKH, DUŠAN REPÓVŠ AND IHOR ZARICHNYI

Abstract. A metric space \((X, d)\) has the de Groot property \(GP_n\) if for any points \(x_0, x_1, \ldots, x_{n+2} \in X\) there are positive indices \(i, j, k \leq n + 2\) such that \(i \neq j\) and \(d(x_i, x_j) \leq d(x_0, x_k)\). If, in addition, \(k \in \{i, j\}\) then \(X\) is said to have the Nagata property \(NP_n\). It is known that a compact metrizable space \(X\) has dimension \(\dim(X) \leq n\) iff \(X\) has an admissible \(GP_n\)-metric iff \(X\) has an admissible \(NP_n\)-metric.

We prove that an embedding \(f : (0, 1) \rightarrow X\) of the interval \((0, 1) \subset \mathbb{R}\) into a locally connected metric space \(X\) with property \(GP_1\) (resp. \(NP_1\)) is open, provided \(f\) is an isometric embedding (resp. \(f\) has distortion \(\text{Dist}(f) = \|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}} < 2\)). This implies that the Euclidean metric cannot be extended from the interval \([-1, 1]\) to an admissible \(GP_1\)-metric on the triode \(T = [-1, 1] \cup [0, i]\). Another corollary says that a topologically homogeneous \(GP_1\)-space cannot contain an isometric copy of the interval \((0, 1)\) and a topological copy of the triode \(T\) simultaneously. Also we prove that a \(GP_1\)-metric space \(X\) containing an isometric copy of each compact \(NP_1\)-metric space has density \(\geq \epsilon\).

1. Introduction

In this paper we shall be interested in structural properties of metric spaces possessing the properties introduced by J. de Groot [5] and J. Nagata [10].

Let \(n\) be a non-negative integer. A metric \(d\) on \(X\) is said to have the de Groot property \(GP_n\) if for any \(n + 3\) points \(x_0, x_1, \ldots, x_{n+2} \in X\) there is a triplet of indices \(i, j, k \in \{1, \ldots, n+2\}\) such that

\[
d(x_i, x_j) \leq d(x_0, x_k) \quad \text{and} \quad i \neq j.
\]

If, in addition, \(k \in \{i, j\}\), then we say that the metric \(d\) has the Nagata property \(NP_n\) or that \(d\) is an \(NP_n\)-metric. It is clear that each \(NP_n\)-metric is also a \(GP_n\)-metric. In the Engelking’s monograph [4] the properties of Nagata and de Groot are denoted by \((\mu_4)\) and \((\mu_5')\), respectively. Those properties also are discussed in the Nagata’s book [11, V.3].

According to [5] and [10], for a separable metrizable space \(X\) the following conditions are equivalent:

- \(X\) has the covering dimension \(\dim(X) \leq n\);
- the topology of \(X\) is generated by an \(NP_n\)-metric on \(X\).

Date: November 19, 2009.

1991 Mathematics Subject Classification. 54E35, 54F45.

Key words and phrases. Nagata property, de Groot property, ultrametric space, distortion of an embedding.

This research was supported by Slovenian Research Agency grant P1-0292-0101, J1-9643-0101 and BI-UA/07-08-001.
the topology of $X$ is generated by a totally bounded $\text{GP}_n$-metric on $X$. In fact, the equivalence of the first two conditions hold for any metrizable space $X$. On the other hand, it is an open problem due to de Groot [5] if the existence of an admissible $\text{GP}_n$-metric on a (separable) space $X$ implies $\dim(X) \leq n$, see [4, p.231]. We recall that a metric $d$ on a topological space $X$ is said to be admissible if it generates the topology of $X$.

By [4, 4.2.D], a metric $d$ has the $\text{GP}_0$-property if and only if it has the $\text{NP}_0$-property if and only if the metric $d$ satisfies the strong triangle inequality $d(x_1, x_2) \leq \max\{d(x_0, x_1), d(x_0, x_2)\}$ for all points $x_0, x_1, x_2 \in X$. The latter means that $d$ is an ultrametric. Thus both $\text{NP}_n$-metric and $\text{GP}_n$-metric are higher dimensional analogs of ultrametric.

Due to efforts of many mathematicians the structure of ultrametric spaces is quite well understood. We shall recall two results: an Extension Theorem and a Universality Theorem.

**Extension Theorem 1.1.** Each admissible ultrametric defined on a closed subspace $A$ of an zero-dimensional compact metrizable space $X$ extends to an admissible ultrametric on $X$.

This theorem follows from its uniform version proved by Ellis in [2] or its “simultaneous” version proved by Tymchatyn and Zarichnyi [12]. The other theorem is due to A.Liman and V.Liman [6] and concerns universal ultrametric spaces. We define a (topological) metric space $X$ to be (topologically) homogeneous if for any two points $x, y \in X$ there is an isometry (a homeomorphism) $h : X \to X$ such that $h(x) = y$.

**Universality Theorem 1.2.** For each cardinal $\kappa$ there is a (homogeneous) ultrametric space $LM_\kappa$ of weight $\kappa^\omega$ containing an isometric copy of each ultrametric space of weight $\leq \kappa$.

The universal space $LM_\kappa$ in Theorem 1.2 can be constructed as follows: take any Abelian group $G$ of size $|G| = \kappa$, let $\mathbb{Q}_+$ be the set of all positive rational numbers, and let $LM_\kappa$ be the space of all maps $f : \mathbb{Q}_+ \to G$ which are eventually zero, in the sense that $f(x)$ is zero for all sufficiently large rational numbers $x \in \mathbb{Q}_+$. The space $LM_\kappa$ endowed with the ultrametric $d(f, g) = \sup\{x \in \mathbb{Q}_+ : f(x) \neq g(x)\}$ (where $\sup\emptyset = 0$) has the structure of an Abelian group and therefore is metrically homogeneous.

It is natural to ask if these two theorems have analogues for $\text{GP}_n$ or $\text{NP}_n$-metrics. As we shall see later, the answer is negative already for $n = 1$. To construct a suitable counterexample we shall first study the structure of $\text{GP}_1$-spaces $X$ in a neighborhood of an isometrically embedded interval $(0, 1) \subset X$.

**Theorem 1.3.** If a $\text{GP}_1$-metric space $X$ is locally connected, then each subset $I \subset X$, isometric to an interval $(a, b) \subset \mathbb{R}$, is open in $X$.

This theorem will be proved in Section 2. Now we discuss some of its corollaries. By the triode we understand the subspace $T = [-1, 1] \cup [0, i]$ of the complex plane $\mathbb{C}$. By Nagata’s Theorem [10], the triode $T$ carries an admissible $\text{NP}_1$-metric. Nonetheless, such a metric cannot restrict to the Euclidean
metric on the interval \([-1, 1] \subset T\) because the interval \((-1, 1)\) is not open in the triode. Thus we obtain:

**Corollary 1.4.** The Euclidean metric on the interval \([-1, 1]\) has the Nagata property \(NP_1\) but cannot be extended to an admissible \(GP_1\)-metric on the triode \(T\).

Therefore, Extension Theorem 1.1 cannot be generalized to metric spaces with the property \(NP_n\) or \(GP_n\) for \(n \geq 1\). Next, we show that the same concernsUniversality Theorem 1.2 its homogeneous version cannot be generalized to higher dimensions.

**Corollary 1.5.** If a \(GP_1\)-metric space \(X\) contains both an isometric copy of the interval \([0, 1]\) and a topological copy of the triode \(T\), then \(X\) is not topologically homogeneous.

**Proof.** Let \([0, 1] \subset X\) be an isometric copy of the interval \([0, 1]\). Assuming that \(X\) is topologically homogeneous and \(X\) contains a topological copy of the triode \(T\), we can find a topological embedding \(f : T \to X\) such that \(f(0) = \frac{1}{2} \in [0, 1] \subset X\). Since the triode does not embed into the interval \([0, 1]\), the point \(1/2\) is not an interior point of the interval \((0, 1)\) in the locally connected subspace \(Y = [0, 1] \cup f(T)\) of the \(GP_1\)-space \(X\). This contradicts Theorem 1.3. \(\square\)

In spite of the negative result in Corollary 1.5, we do not know the answer to the following

**Problem 1.6.** Is it true that for each infinite cardinal \(\kappa\) there is a \(GP_1\)-metric space \(U\) of weight \(\kappa^{\omega}\) that contains an isometric copy of each \(NP_1\)-metric space \(X\) of weight \(\leq \kappa\)?

The weight \(\kappa^{\omega}\) in Problem 1.6 cannot be replaced by \(\kappa\) because of the following theorem that will be proved in Section 3.

**Theorem 1.7.** If a \(GP_1\)-metric space \(X\) contains an isometric copy of each compact \(NP_1\)-metric space, then \(X\) has density \(\text{dens}(X) \geq c\).

Now let us return to Theorem 1.3. It implies that no non-open arc \(I\) in a locally connected \(GP_1\)-metric space \((X, d)\) is isometric to an interval \((a, b) \subset \mathbb{R}\). We can ask how much the metric \(d\) restricted to \(I\) differs from the Euclidean metric on \(I\). We can measure this distance using the notion of the distortion.

By the distortion of an injective map \(f : X \to Y\) between metric spaces \((X, d_X)\) and \((Y, d_Y)\) we understand the (finite or infinite) number

\[
\text{Dist}(f) = \|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}}
\]

where

\[
\|f\|_{\text{Lip}} = \sup_{x \neq x'} \frac{d_Y(f(x), f(x'))}{d_X(x, x')}
\]

is the Lipschitz constant of \(f\) (if \(|X| \leq 1\), then \(\|f\|_{\text{Lip}}\) is not defined, so we put \(\text{Dist}(f) = 1\)). The notion of distortion is widely used in studying the embeddability problems of metric spaces, see [1], [7], [8], [9].

It can be shown that an embedding \(f : X \to Y\) of a metric space \(X\) into a metric space \(Y\) has distortion \(\text{Dist}(f) = 1\) if and only if \(f\) is a similarity, which means that \(d_Y(f(x), f(x')) = \|f\|_{\text{Lip}} \cdot d_X(x, x')\) for all \(x, x' \in X\).

In terms on the distortion, Theorem 1.3 can be written as follows.
Corollary 1.8. Let $X$ be a locally connected metric space with property $\text{GP}_1$. Each embedding $f : (0, 1) \to X$ with distortion $\text{Dist}(f) = 1$ is open.

Proof. Let $f : (0, 1) \to X$ be an embedding with distortion $\text{Dist}(f) = 1$. Let $C = \|f\|_{\text{Lip}}$ and

$$g : (0, C) \to (0, 1), \quad g : t \mapsto t/C,$$

be the similarity mapping having the Lipschitz constant $\|g\|_{\text{Lip}} = 1/C$. It follows that the composition $f \circ g : (0, C) \to X$ has distortion

$$1 = \text{Dist}(f \circ g) = \|f \circ g\|_{\text{Lip}} \cdot \|(f \circ g)^{-1}\|_{\text{Lip}} = 1.$$

Since $\|f \circ g\|_{\text{Lip}} = 1$, we conclude that $\|(f \circ g)^{-1}\|_{\text{Lip}} = 1$ and hence $f \circ g : (0, C) \to X$ is an isometric embedding. By Theorem 1.8, the image $f \circ g((0, C)) = f((0, 1))$ is open in $X$. □

Problem 1.9. Can the equality $\text{Dist}(f) = 1$ in Corollary 1.8 be replaced by the inequality $\text{Dist}(f) < 2$?

This problem has an affirmative solution for metric spaces with the Nagata property $\text{NP}_1$. The following theorem can be easily derived from Proposition 4.1 and Corollary 5.2 proved at the end of the paper.

Theorem 1.10. Let $X$ be a locally connected metric space with property $\text{NP}_1$. Each embedding $f : (0, 1) \to X$ with distortion $\text{Dist}(f) < 2$ is open.

The inequality $\text{Dist}(f) < 2$ in this theorem is best possible because of the following simple example.

Example 1.11. On the triode $T = [-1, 1] \cup [0, i]$ consider the $\text{NP}_1$-metric

$$\rho(z, z') = \begin{cases} |z - z'| & \text{if } \text{sign}(\Re(z)) = \text{sign}(\Re(z')) , \\ \max\{\Re(z), |\Re(z')|, \Im(z), \Im(z')\} & \text{otherwise}. \end{cases}$$

It is easy to check that the identity embedding $f : [-1, 1] \to (T, \rho)$ has distortion $\text{Dist}(f) = 2$ but is not open.

In spite of Corollary 1.8 there is a hope that the following problem (related to an approximative extension of $\text{NP}_1$-metrics) has an affirmative solution.

Problem 1.12. Let $A$ be a closed subspace of a 1-dimensional space $X$. Is it true that for any a admissible $\text{NP}_1$-metric $d_A$ on $A$ there is an admissible $\text{NP}_1$-metric $d_X$ on $X$ such that the identity embedding $f : (A, d_A) \to (X, d_X)$ has distortion $\text{Dist}(f) \leq 2$?

2. Isometric arcs in $\text{GP}_1$-metric spaces

In this section we shall prove Theorem 1.3. A map $f : X \to Y$ between metric spaces is called non-expanding if its Lipschitz constant $\|f\|_{\text{Lip}} \leq 1$. For a point $x$ of a metric space $(X, d)$ and a subset $A \subset X$ we put $d(x, A) = \inf_{a \in A} d(x, a)$.

Lemma 2.1. Let $(X, d)$ be a $\text{GP}_1$-metric space containing an isometric copy of the closed interval $[0, 1]$ and let $V = \{x \in X : d(x, [0, 1]) < \frac{1}{4}d(x, \{0, 1\})\}$.

1. There is a non-expanding retraction $r : V \to (0, 1)$ such that

$$d(x, t) = \max\{|t - r(x)|, d(x, [0, 1])\}$$

for any $x \in V$, $t \in (0, 1)$.
(2) For any points \( x, y \in V \) with \( d(x, [0, 1]) \neq d(y, [0, 1]) \) we get

\[
d(x, y) \geq \max\{d(x, [0, 1]), d(y, [0, 1])\}.
\]

Proof. 1. Given any \( x \in V \), let \( D = d(x, [0, 1]) \) and consider the compact subset
\( D(x) = \{ t \in [0, 1] : d(x, t) = D \} \). We claim that \( D(x) \) is a closed subinterval
of \((0, 1)\) of length \( 2D \). Let \( a = \min D(x) \) and \( b = \max D(x) \).

The triangle inequality implies that \( d(a, b) \leq d(a, x) + d(x, b) \leq 2D \). It follows
from \( D < \frac{1}{3}d(x, \{0, 1\}) \) that \( d(0, a) \geq d(0, x) - d(x, a) > 3D - D > D \) and similarly,
\( d(b, 1) > D \). Let us show that \([a, a + D] \subseteq D(x)\). Assuming the converse, we could
find a point \( x_1 \in (a, a + D) \setminus D(x) \). Then for the points
\[
x_0 = a, \ x_1, \ x_2 = x, \text{ and } x_3 = a - D
\]
we would get

\[
d(x_1, x_2) > D, \ d(x_1, x_3) = D + (x_1 - a) > D, \ d(x_2, x_3) > D \text{ and}
\[
d(x_0, x_3) = d(a, a - D) = D, \ d(x_0, x_2) = d(a, x) = D, \ d(x_0, x_1) = d(a, x_1) \leq D,
\]
which contradicts the \( \text{GP}_1 \)-property of the metric \( d \).

Thus \([a, a + D] \subseteq D(x)\). By analogy we can prove that \([b - D, b] \subseteq D(x)\).
Combined with \( b - a \leq 2D \), this implies that \([a, b] = [a, a + D] \cup [b - D, b] = D(x)\).
Assuming that \( b - a < 2D \), we could take \( x_0 \) to be the midpoint of the interval \([a, b]\)
and put \( x_1 = x, x_2 = x_0 - D, x_3 = x_0 + D \). Then

\[
\min\{d(x_1, x_2), d(x_1, x_3), d(x_2, x_3)\} > D = \max\{d(x_0, x_1), d(x_0, x_2), d(x_0, x_3)\},
\]
which contradicts the \( \text{GP}_1 \)-property of the metric \( d \).

Therefore, \( D(x) \) is a closed interval of length \( 2D \). Let \( r(x) \) be the midpoint
of this interval. Let us show that \( d(x, t) = \max\{t - r(x), 0\} \) for all \( t \in [0, 1] \).
This is obvious if \( t \in D(x) = [a, b] \). So assume that \( t \notin D(x) \). If \( t < a \), then
\( d(t, x) \leq d(t, a) + d(a, x) \leq a - t + D = r(x) - t \). On the other hand, \( b - t = d(t, b) \leq d(t, x) + d(x, b) = d(t, x) + D \) implies \( d(t, x) \geq b - t - D = r(x) - t \).
Therefore \( d(t, x) = r(x) - t = \max\{|r(x) - t|, 0\} \). The case \( t > b \) can be treated by
analogy.

Finally, we show that the map \( r : V \to (0, 1), \ r : x \mapsto r(x) \) is a non-expanding
retraction. It is clear that \( r(t) = t \) for any \( t \in (0, 1) \). Take any two points \( x, y \in V \).
Without loss of generality, \( r(y) \geq r(x) \). Let \( D_x = d(x, [0, 1]) \) and \( D_y = d(y, [0, 1]) \).
For the point \( t = r(x) - D_x = \min D(x) \) let us observe that

\[
r(y) - r(x) + D_x = r(y) - t \leq \max\{|r(y) - t|, D_y\} =
\[
d(t, y) \leq d(t, x) + d(x, y) = D_x + d(x, y)
\]
and hence \( |r(y) - r(x)| = r(y) - r(x) \leq d(x, y) \).

2. Take any two points \( x, y \in V \) with \( D_x = d(x, [0, 1]) \neq d(y, [0, 1]) = D_y \). We
need to prove that \( d(x, y) \geq \max\{D_x, D_y\} \). Without loss of generality, \( D_x < D_y \).
Assume conversely that \( d(x, y) < \min\{D_x, D_y\} = D_y \). Observe that

\[
d(r(x), 0) = D_x \geq d(x, 0) - d(x, y) - D_x > d(y, 0) - 2D_y > 3D_y - 2D_y = D_y
\]
and hence for any real \( a \) with \( \max\{D_x, d(x, y)\} < a < D_y \) the point \( x_1 = r(x) - a \in
(0, 1) \) is well-defined. By analogy we can prove that \( x_2 = r(x) + a \in (0, 1) \) is
well-defined.
So we can consider the 4 points: $x_0 = x, x_1 = r(x) - a, x_2 = r(x) + a, x_3 = y$, and derive a contradiction with the GP$_1$-property of the metric $d$ because:

$$\min\{d(x_1, x_2), d(x_1, x_3), d(x_2, x_3)\} \geq \min\{2a, D_y, D_y\} > \max\{a, a, d(x, y)\} \geq \max\{d(x_0, x_1), d(x_0, x_2), d(x_0, x_3)\}.$$

\[\square\]

**Proof of Theorem 1.3.** Let $X$ be locally connected GP$_1$-metric space and $I \subset X$ a subset isometric to an open interval $(a, b) \subset \mathbb{R}$. We need to check that each point $x_0 \in I$ is an interior point of $I$ in $X$. For a sufficiently small $\varepsilon > 0$ we can find an isometry $f : [0, 2\varepsilon] \to I \subset X$ such that $f(\varepsilon) = x_0$. Scaling the GP$_1$-metric $d$ of $X$ by a suitable constant, we can assume that $\varepsilon = \frac{1}{2}$.

We shall identify the interval $[0, 1]$ with a subinterval of $I$ and $1/2$ with the point $x_0$. Consider the neighborhood $V = \{x \in X : d(x, [0, 1]) < d(x, \{0, 1\})/3\}$ of $(0, 1)$ in $X$. By the local connectedness of $X$ at $x_0$, find a connected neighborhood $C(x_0) \subset V$ of the point $x_0 = 1/2$. We claim that $C(x_0) \subset I$. Otherwise there would exist a point $x_1 \in C(x_0) \setminus I$. Lemma 2.1(2) guarantees that the subset $D = \{x \in C(x_0) : d(x, [0, 1]) = d(x_1, [0, 1])\}$ is open-and-closed in $C(x_0)$, which implies that the neighborhood $C(x_0)$ is not connected and this is a contradiction. \[\square\]

### 3. Universal GP$_1$-spaces

In this section we study universal GP$_1$-spaces and prove Lemma 3.2 which implies Theorem 1.4 announced in the Introduction.

We shall need the following (probably known)

**Lemma 3.1.** Let $(X, d_X)$ be a NP$_1$-metric space and $(Y, d_Y)$ be an NP$_0$-metric space. Then the max-metric

$$d((x, y), (x', y')) = \max\{d_X(x, x'), d_Y(y, y')\}$$

on the product $X \times Y$ has the Nagata property NP$_1$.

**Proof.** Given any 4 points $(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y$, we need to find two distinct indices $i, j \in \{1, 2, 3\}$ such that

$$d((x_i, y_i), (x_j, y_j)) \leq \max\{d((x_0, y_0), (x_i, y_i)), d((x_0, x_j), (y_0, y_j))\}.$$

Since the metric on $X$ has the property NP$_1$, there are two distinct numbers $i, j \in \{1, 2, 3\}$ such that

$$d_X(x_i, x_j) \leq \max\{d_X(x_0, x_i), d_X(x_0, x_j)\}.$$

The NP$_0$-property of the metric space $Y$ ensures that

$$d_Y(y_i, y_j) \leq \max\{d_Y(y_0, y_i), d_Y(y_0, y_j)\}.$$

Combining these two inequalities, we conclude that

$$d((x_i, y_i), (x_j, y_j)) = \max\{d_X(x_i, x_j), d_Y(y_i, y_j)\} \leq \max\{d_X(x_0, x_i), d_X(x_0, x_j), d_Y(y_0, y_i), d_Y(y_0, y_j)\} = \max\{d((x_0, y_0), (x_i, y_i)), d((x_0, x_j), (y_0, y_j))\}.$$

\[\square\]
Lemma 2.1 implies that for a positive real number \( a \) the metric
\[
d((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}
\]
on the product \( \mathbb{I}_a = [-1, 1] \times \{0, a\} \subset \mathbb{R} \times \mathbb{R} \) has the Nagata property \( \text{NP}_1 \).

For a metric space \( X \) we shall write \( \mathbb{I}_a \hookrightarrow X \) if \( X \) contains an isometric copy of the space \( \mathbb{I}_a \).

**Lemma 3.2.** For any \( \text{GP}_1 \)-metric space \( X \) the set \( A = \{a \in (\frac{1}{16}, \frac{1}{8}) : \mathbb{I}_a \hookrightarrow X\} \) has cardinality \(|A| \leq \text{dens}(X)\).

**Proof.** For every \( a \in A \) fix an isometric embedding \( h_a : \mathbb{I}_a \hookrightarrow X \) and define a map \( f_a : \mathbb{I} \rightarrow X \) by letting \( f_a ((x, t)) = h_a(x, at) \) for \((x, t) \in \mathbb{I}_a \). The map \( f_a \) can be considered as an element of the function space \( C(\mathbb{I}_a, X) \) endowed with the sup-metric
\[
d(f, g) = \sup_{t \in \mathbb{I}_a} d(f(t), g(t)).
\]

By \([3, 3.4.16]\), the density of the function space \( C(\mathbb{I}_a, X) \) is equal to the density of \( X \). Now the assertion of the theorem will follow as soon as we check that the set \( \mathcal{F}_A = \{f_a : a \in A\} \) is discrete in \( C(\mathbb{I}_a, X) \). This will follow as soon as we show that \( d(f_a, f_b) \geq \frac{1}{32} \) for any numbers \( a \neq b \) in \( A \).

To this end we first introduce some notation. For \( a \in A \) and \( i \in \{0, 1\} \) let
\[
I_a^i = f_a([-1, 1] \times \{i\}), \quad \partial I_a^i = f_a([-1, 1] \times \{i\}), \quad J_a^i = I_a^i \setminus \partial I_a^i, \quad c_a^i = f_a(\frac{3}{4}, i),
\]
and
\[
V_a^i = \{x \in X : d(x, I_a^i) \leq \frac{1}{3} d(x, \partial I_a^i)\}.
\]

By Lemma 2.1 there is a non-expanding retraction \( r_a^i : V_a^i \rightarrow J_a^i \) such that for every \( x \in V_a^i \) and \( t \in J_a^i \) we get
\[
d(x, t) = \max\{d(r_a^i(x), t), d(x, I_a^i)\}.
\]

Moreover, for any points \( x, y \in V_a^i \) with \( d(x, I_a^i) \neq d(y, I_a^i) \) we get
\[
d(x, y) \geq \max\{d(x, I_a^i), d(y, I_a^i)\}.
\]

To derive a contradiction, assume that \( d(f_a, f_b) < \varepsilon = \frac{1}{32} \) for some distinct numbers \( a, b \in A \). Observe that
\[
d(c_b^0, I_a^i) \leq d(c_b^0, c_a^0) + d(c_b^0, I_a^i) < \varepsilon + a < \frac{1}{32} + \frac{1}{8} = \frac{5}{32}
\]
while
\[
d(c_b^0, \partial I_a^i) \geq d(c_b^0, \partial I_a^i) - d(c_b^0, c_a^0) = \frac{1}{2} - \varepsilon = \frac{1}{2} - \frac{1}{32} = \frac{15}{32}.
\]

Consequently, \( d(c_b^0, I_a^i) < \frac{1}{4} d(c_b^0, \partial I_a^i) \) and hence \( c_b^0 \in V_a^1 \). We claim that \( d(c_b^0, I_a^i) = d(c_b^0, I_a^1) = a \). Otherwise, we may apply the formula (2) to derive a contradiction:
\[
d(c_b^0, I_a^1) \geq \max\{d(c_b^0, I_a^1), d(c_a^0, I_a^i)\} \geq d(c_a^0, I_a^1) = a > \varepsilon > d(f_a, f_b).
\]

Since the retraction \( r_a^1 : V_a^0 \rightarrow J_a^1 \) is non-expanding, we get
\[
d(r_a^1(c_b^0), c_a^0) = d(r_a^1(c_b^0), r_a^1(c_a^0)) \leq d(c_b^0, c_a^0) < \varepsilon < a.
\]

Now the formula (1) yields
\[
d(c_b^0, c_a^1) = \max\{d(r_a^1(c_b^0), c_a^0), d(c_b^0, I_a^1)\} = d(c_b^0, I_a^1) = a.
\]

By analogy we can prove that \( d(c_a^1, c_b^0) = b \), which contradicts \( d(c_b^0, c_a^1) = a \). \( \square \)
4. Obtuse arcs and embeddings with small distortion

In this section we shall introduce the notion of an obtuse arc and show that for each embedding \( f : [0, 1] \to X \) with \( \text{Dist}(f) < 2 \) the arc \( f([0, 1]) \) is obtuse. By a metric arc we understand a metric space that is homeomorphic to the unit interval \( I = [0, 1] \).

A metric arc \((I, d)\) is called obtuse if

- for any subarc \( J \subset I \) with endpoints \( a, b \) and any point \( z \in J \setminus \{a, b\} \) there are points \( x, y \in J \) with \( d(x, y) > \max\{d(z, x), d(z, y)\} \); and
- for any subarc \( J \subset I \) with endpoints \( a, b \) there is a point \( z \in J \) with \( d(a, b) > \max\{d(z, a), d(z, b)\} \).

In this case the metric \( d \) on \( I \) is called obtuse.

It is easy to see that each subinterval \([a, b]\) \( \subset \mathbb{R} \) endowed with the Euclidean metric is an obtuse arc. It can be shown that each continuously differentiable curve can be covered by finitely many obtuse subarcs.

**Proposition 4.1.** If an embedding \( f : I \to X \) of the unit interval \( I = [0, 1] \) into a metric space \((X, d_X)\) has distortion \( \text{Dist}(f) < 2 \), then the image \( I = f(I) \) is an obtuse arc in \( X \).

**Proof.** We need to show that the metric
\[
\rho(t, t') = d_X(f(t), f(t'))
\]
on \( I \), induced by the embedding \( f \), is obtuse. It follows that
\[
\left(\|f^{-1}\|_{\text{Lip}}\right)^{-1} \cdot |x - y| \leq \rho(x, y) \leq \|f\|_{\text{Lip}} \cdot |x - y|.
\]
Now we establish the two conditions of the definition of an obtuse arc.

1) Take any subinterval \([a, b]\) \( \subset I \) and a point \( z \in (a, b) \). Let \( x, y \in (a, b) \) be any two points such that \( z \) is the midpoint of the interval \((x, y)\). Then
\[
\max\{\rho(x, z), \rho(y, z)\} \leq \|f\|_{\text{Lip}} \cdot \max\{|x - z|, |y - z|\} = \|f\|_{\text{Lip}} \cdot |x - y|/2 \leq \frac{1}{2} \|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}} \cdot \rho(x, y) < \frac{1}{2} \cdot 2 \cdot \rho(x, y) < \rho(x, y).
\]

2) By analogy we can prove that for any subinterval \([a, b]\) \( \subset I \) the midpoint \( z \) of \([a, b]\) satisfies the inequality \( \max\{\rho(x, z), \rho(y, z)\} < \rho(x, y) \). \( \square \)

5. Obtuse arcs in \( \text{NP}_1 \)-metric spaces

In this section we study the structure of an \( \text{NP}_1 \)-metric space \( X \) in a neighborhood of an obtuse arc \( I \subset X \).

**Proposition 5.1.** Let \((X, d)\) be an \( \text{NP}_1 \)-metric space, \( I \subset X \) be an obtuse arc with endpoints \( a, b \) in \( X \) and let \( V = \{x \in X : d(x, I) < d(x, \{a, b\})\} \).

1) For every point \( x \in V \setminus I \) the set \( D(x) = \{t \in I : d(x, t) = d(x, I)\} \) is the finite union of closed subintervals of \( I \) each of which has diameter \( > d(x, I) \).
2) For any points \( x, y \in V \) with \( d(x, I) \neq d(y, I) \) we get
\[
d(x, y) \geq \max\{d(x, I), d(y, I)\}.
\]
Proof. 1. Given a point \( x \in V \setminus I \) put \( D = d(x, I) \) and consider the family \( I \) of maximal non-generate subintervals in the closed subset
\[
D(x) = \{ t \in I : d(t, x) = D \} \subseteq (a, b) = I \setminus \{ a, b \}.
\]

We claim that each maximal subinterval \([a_1, b_1] \in I \) has diameter \( \text{diam}[a_1, b_1] > D \). Assuming conversely that \( \text{diam}([a_1, b_1]) \leq D \), and using the second condition of the definition of an obtuse metric, we can find a point \( x_0 \in (a_1, b_1) \) such that \( D \geq d(a_1, b_1) > \max\{d(a_1, x_0), d(b_1, x_0)\} \). The maximality of the subinterval \([a_1, b_1] \in D(x) \subseteq (a, b) \) implies the existence of points \( x_1 \in (a_1) \setminus D(x) \) and \( x_2 \in (b_1, b) \setminus D(x) \) such that \( \max\{d(x_1, x_0), d(x_2, x_0)\} < \min\{D, d(x_1, x_2)\} \). Now we see that the quadruple of points \( x_0, x_1, x_2, x_3 = x \) witnesses that the metric \( d \) on \( X \) fails to have the Nagata property \( \text{NP}_1 \) because
\[
d(x_1, x_2) > \max\{d(x_1, x_0), d(x_2, x_0)\},
\]
\[
d(x_1, x_3) > D \geq \max\{d(x_0, x_1), d(x_0, x_3)\}
\]
and
\[
d(x_2, x_3) > D \geq \max\{d(x_0, x_2), d(x_0, x_3)\}.
\]

Taking into account that any two distinct maximal subintervals in the family \( I \) are disjoint and have diameter \( > D \), we conclude that the family \( I \) is finite. It remains to show that \( D(x) = \bigcup I \). Assuming the converse, we could find a point \( x_0 \in D(x) \setminus \bigcup I \) and a neighborhood \((a_1, b_1) \in I \setminus \bigcup I \) of the point \( x_0 \) in \( I \setminus \{a, b\} \) such that \( \text{diam}[a_1, b_1] < D \). The intersection \((a_1, b_1) \cap D(x) \) contains a non-degenerate subinterval and hence is nowhere dense in \((a_1, b_1) \). The obtuse property of the metric \( d \) guarantees the existence of two points \( x_1, x_2 \in (a_1, b_1) \) such that \( d(x_1, x_2) > \max\{d(x_1, x_0), d(x_2, x_0)\} \). Since \( D(x) \cap (a_1, b_1) \) is nowhere dense we can additionally assume that \( x_1, x_2 \notin D(x) \). Then for the quadruple of the points \( x_0, x_1, x_2, x_3 = x \) we get
\[
d(x_1, x_2) > \max\{d(x_1, x_0), d(x_2, x_0)\},
\]
\[
d(x_1, x_3) = d(x_1, x) > D = \max\{d(x_1, x_0), d(x_0, x_3)\}, \text{ and}
\]
\[
d(x_2, x_3) = d(x_2, x) > D = \max\{d(x_3, x_0), d(x_2, x_0)\},
\]

witnessing the failure of the Nagata property \( \text{NP}_1 \) for the metric \( d \).

2. Given two points \( x, y \in V \) with \( d(x, I) \neq d(y, I) \) we should prove that \( d(x, I) \geq \max\{d(x, I), d(y, I)\} \). Assume conversely, that \( d(x, I) < \max\{d(x, I), d(y, I)\} \). Without loss of generality \( d(x, I) < d(y, I) \). By the preceding item, the set
\[
D(x) = \{ z \in I : d(x, z) = d(x, I) \}
\]
contains two points \( x_1, x_2 \) with \( d(x_1, x_2) > d(x, I) \). Now we see that the quadruple of the points \( x_0 = x, x_1, x_2, x_3 = y \) satisfies the inequalities
\[
d(x_1, x_2) > d(x, I) = \max\{d(x_0, x_1), d(x_0, x_2)\},
\]
\[
d(x_1, x_3) \geq d(y, I) > \max\{d(x_0, x_1), d(x_0, x_3)\},
\]
\[
d(x_2, x_3) \geq d(y, I) > \max\{d(x_0, x_2), d(x_0, x_3)\},
\]

witnessing that the metric \( d \) fails to have the Nagata property \( \text{NP}_1 \).

\[\square\]

By an argument similar to that from Theorem \[1.3\] we apply Proposition \[5.1\] to prove the following

**Corollary 5.2.** Let \( X \) be a locally connected \( \text{NP}_1 \)-metric space \( X \) and \( I \subseteq X \) is an obtuse arc with endpoints \( a, b \). Then the set \( I \setminus \{ a, b \} \) is open in \( X \).
6. Acknowledgment

The authors express their sincere thanks to Michael Zarichnyi for fruitful discussions related to metric spaces with the Nagata property. We also acknowledge the referee for comments and suggestions.

References

[1] Y. Bartal, N. Linial, M. Mendel, A. Naor, On Some Low Distortion Metric Ramsey Problems, Discrete Comput. Geom. 33:1 (2005), 27–45.
[2] R. Ellis, Extending uniformly continuous pseudo-ultrametrics and uniform retracts, Proc. Amer. Math. Soc. 30:3 (1971), 599–602.
[3] R. Engelking, General Topology, Warsaw, PWN, 1977.
[4] R. Engelking, Theory of Dimensions, Finite and Infinite, Heldermann Verlag, 1995.
[5] J. de Groot, Non-archimedean metrics in topology, Proc. Amer. Math. Soc. 7 (1956), 948–953.
[6] A. Lemin, V. Lemin, On a universal ultrametric space, Topology Appl. 103:3 (2000), 339–345.
[7] J. Matoušek, Ramsey-like properties for bi-Lipschitz mappings of finite metric spaces, Comment. Math. Univ. Carolin. 33:3 (1992), 451–463.
[8] J. Matoušek, Note on bi-Lipschitz embeddings into normed spaces, Comment. Math. Univ. Carolin. 33:1 (1992), 51–55.
[9] M. Mendel, Metric Dichotomies, in: Limits of graphs in group theory and computer science (G. Arzhantseva, A. Valette eds.), EPFL Press, Lausanne, 2009; [arXiv:0710.1994].
[10] J. Nagata, On a special metric and dimension, Fund. Math. 55 (1964), 181–194.
[11] J. Nagata, Modern Dimension Theory, Sigma Series in Pure Math., 2. Heldermann Verlag, Berlin, 1983.
[12] E. D. Tymchatyn, M. Zarichnyi, A note on operators extending partial ultrametrics, Comment. Math. Univ. Carolin. 46:3 (2005), 515–524.

T. Banakh: Department of Mathematics, Ivan Franko National University of Lviv, Ukraine; and Instytut Matematyki, Uniwersytet Humanistyczno Przyrodniczy im. Jana Kochanowskiego w Kielcach, Poland
E-mail address: tbanakh@yahoo.com

D. Repovš: Faculty of Mathematics and Physics, and Faculty of Education, University of Ljubljana, P.O.B. 2964, Ljubljana, Slovenia
E-mail address: dusan.repovs@Guest.arnes.si

I. Zarichnyi: Department of Mathematics, Ivan Franko National University of Lviv, Ukraine
E-mail address: ihor.zarichnyj@gmail.com