A brief remark on Unruh effect and causality

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Abstract. Unruh effect states that the vacuum of a quantum field theory on Minkovski space-time looks like a thermal state for an eternal uniformly accelerated observer. In this talk we present an adaptation for a non eternal observer and discuss some problems that may occur with respect to causality. Specifically we show that our adaptation to a non eternal observer, inspired by the thermal-time hypothesis of Connes and Rovelli, is preserved from violating causality by Heisenberg uncertainty relation.

Introduction

The Unruh effect [18] states that an observer in Minkovski spacetime $M$ moving along an infinite worldline with constant acceleration $a$ and interacting -all along his lifetime- with the vacuum of a quantum field theory on $M$ sees this vacuum as a thermal equilibrium state with temperature

$$T_U = \frac{h a}{2 \pi k_b c}.$$  

In Unruh’s original approach, $T_U$ is obtained by observing that the vacuum for a quantization scheme on all $M$ is not a pure state for an alternative (but as well defined) quantization prescription on the Rindler wedge $W$\(^1\). The latest is physically relevant for $W$ is the (whole and only) region of $M$ with whom an eternal uniformly accelerated observer can interact. For non-eternal observers $W$ has no particular signification and the comparison between the two quantization prescriptions is no longer significant. In this sense the eternity of the observer is a strong requirement to derive Unruh’s result, and an equally strong argument to question its validity [8].

Eternity of the observer is generally overcame by viewing $T_U$ as a limit for asymptotic states (see [12, 15] for applications in particle physics). However such a limit is not always meaningful. For example one can identify $T_U$ to Hawking temperature for an eternal black hole (see [2] for instance) but not for a Kerr black hole. Indeed the identification of the two temperatures relies on the identification of the proper time of an observer moving around the black hole,

\(^1\) in cartesian coordinates $W$ is the set of points such that $x > |t|$ where $x$ is the direction of the acceleration.
with acceleration given by the gravity surface, to the time defined by the Killing field which generates the horizon. And in Kerr spacetime, as pointed out in [20], this Killing field has spacelike orbits near infinity. Hence it cannot be identified to the proper time of any physical observer. In this framework the question of an Unruh effect for a non eternal observer becomes relevant independently of the asymptotic acceptation.

Motivated by completely different reasons (namely, the issue of time in quantum gravity and a possible solution known as the thermal time hypothesis [5]) we recently proposed in [14] an adaptation for an observer with finite lifetime \( T \). Our construction based on the algebraic approach to Unruh effect [1, 18] is mathematically coherent but its physical interpretation still not clear, essentially because we found a temperature \( T \) that depends on the lifetime \( T \). Besides the criticisms on classical Unruh effect that we will not adress here, we face the following question: How can one possibly dream about ascribing some physical meaning to a timelife dependent temperature \( T(T) \)? The point is that a well informed observer should be able to deduce its lifetime \( T = T^{-1}(T_0) \) from a measure \( T_0 \) of the temperature whatever happens to him after the instant of the measurement. This is of course not acceptable from a causal point of view. In this talk we will discuss the constraints imposed by Heisenberg time-energy inequality on the acceptable form of the function \( T(T) \). In particular we will see that [13] yields the highest temperature allowed by uncertainty relation.

In the first part we discuss the pertinence of questioning the lifetime of the observer, stressing the algebraic approach to the Unruh effect based on the KMS formulation of statistical mechanics. In the second part we investigate the case of a non eternal observer within the framework of the thermal time hypothesis. The third part shows how time-energy uncertainty allows to avoid causality violation.

1. Unruh effect outside the asymptotic limit

As mentioned in the introduction, the eternity of the observer is generally overcame by viewing \( T_U \) as a limit for asymptotic states. Namely one interprets (1) as follows: the temperature defined by, say, a two level quantum system \( S \) interacting with the vacuum tends to \( T_U \) when the time of interaction of \( S \) with the vacuum is sufficiently long. Otherwise the ratio of the populations of the two levels provides, at most, an indication on the duration of the accelerated period. In this scheme the vacuum is seen as a thermal bath at temperature \( T_U \) and the thermalization of the detector comes as an asymptotic phenomenon. However

-first the example of Kerr black holes suggests that the asymptotic interpretation is not always available.

-second, there is a distinction between the fact that the vacuum seen by an eternal accelerated observer is a thermal bath at temperature \( T_U \), and the fact that an interacting detector takes an infinite time to be in equilibrium with it. The first statement is a property of the vacuum state seen from an uniformly accelerated point of view, the second statement is a property of the detector. The distinction is somewhat clearer in the algebraic approach to Unruh effect developed in [1, 18]. Analyzing the geometrical properties of \( W \), Bisognano and Wichman observed that the vacuum state satisfies with respect to the proper-time translation of a uniformly accelerated observer \( O \) the same properties as an equilibrium thermal state at temperature \( T_U \) (the so called KMS conditions, see below). What is relevant here is that the edge of \( W \) is the causal horizon of \( O \). The vacuum being a thermal state appears as a geometrical property of \( W \) and does not rely on the nature of the detector. This point has been vividly discussed in the literature. It is true that the full physical treatment of the question cannot be limited to the observation that the vacuum is formally a thermal state. It also requires a process to measure the temperature, that is to say a precise coupling of a detector \( S \).
to the field. However in the same manner than a Gibbs state is an equilibrium state whether or not there is effectively a thermometer to measure its temperature, we believe that the vacuum being KMS can be in some meaningful and is not only a formal mathematical statement.

To make this idea clearer let us recall a few words about the invoked KMS condition. Given a statistical system with algebra of observables $\mathcal{A}$ and Hamiltonian $H$, a state $\omega$ is said to be KMS with parameter $\beta$ if it satisfies

$$\omega(\alpha_t(a)b) = \omega(b\alpha_t(\frac{i}{\beta}a)) \quad a, b \in \mathcal{A}$$

where $\alpha_t(a) = e^{iHt}ae^{-iHt}$ is the time translation, extended to complex variables. It has been shown (see [9] for a complete presentation of the subject) that for a system with a finite number of degrees of freedom, being KMS with parameter $\beta$ is equivalent to being a Gibbs equilibrium state at temperature $\beta^{-1}$. Moreover contrary to Gibbs definition KMS properties are still meaningful at the thermodynamical limit. Therefore given a system with a well known time evolution $\alpha_t$ one defines an equilibrium state in the following way:

An equilibrium state at temperature $\beta^{-1}$ is a state that satisfies the KMS condition -with parameter $\beta$- with respect to the time evolution $\alpha_t$.

In other terms, given a temperature and a time evolution, the KMS condition allows to characterize thermal equilibrium states,

$$\{ \text{time} \rightarrow \text{thermal state.} \}$$

Unruh temperature is obtained by noting that the vacuum state satisfies the KMS conditions -with parameter $T_U^{-1}$- with respect to the time evolution $\alpha_t$ corresponding to the proper time translation of the uniformly accelerated observer $O$.

Now within the KMS approach this is not so clear that $T_U$ has a meaning only as an asymptotic limit. Indeed eq.(2) is true for any value of the parameter $t$ and not only in the asymptotic limit. This suggests that at any moment of its lifetime $O$ sees the vacuum as a thermal state. In principle there should be no obstruction to talk about an “instant temperature” that would deliver an ideal detector. In other words within the algebraic framework of Bisognano-Wichman there should be no reason to identify the lifetime $T$ of the observer (which is infinite) with the duration of the measurement $L$ of the temperature (which may be finite). Hence, at least formally, it is quite natural to ask whether an observer with a finite time of interaction also sees the vacuum as a thermal state. Of course, practically the distinction between $T$ and $L$ is spurious since nobody knows what such an ideal detector could be. It is difficult to imagine a system that would be more sensitive to the vacuum than a two-level system $S$ coupled to the vacuum, in which case one precisely has $L = +\infty = T$. It remains that such a system $S$, interacting with the vacuum for a finite period of time only, should keep a trace of the thermal nature of the wedge horizon.

2. Finite lifetime observer

Several adaptations of Unruh effect have been proposed for an observer with a finite lifetime (see [14, 16, 17, 3, 13] for the most recent). By this one often means that the detector does interact with the vacuum only for a finite period of time. The result generally depends on

as well, one asks that the function $z \mapsto \omega(b\alpha_z(a))$ be analytic in the strip $0 < \text{Im} z < \beta$ for any $a, b \in \mathcal{A}$.
the nature of the coupling with the vacuum as well as on the shape of its switching on/off (as explained by J. Louko in his talk). What seems to be commonly admit is that the vacuum is still thermal at temperature \( T_U \) but the detector has no time to reach the thermal equilibrium. Note that the non-eternity of the observer leads to some non-trivial causality problems: Schlicht recently showed\[17\] that the rate of excitation after a finite period of interaction \( \mathcal{L} \) was independent of events occurring at a date \( \tau > \mathcal{L} \) only within a regularization of the Green function that is not the one used in the eternal case.

In the recent paper \[14\] we adapted the algebraic approach to the finite case and found a quite different result: the vacuum is still thermal but with a temperature depending on the lifetime of the observer. Let us summarize our argument. The starting point is the thermal time hypothesis of Connes and Rovelli developed in \[5\]. From the physical side this hypothesis is motivated by the issue of time in quantum gravity. From the mathematical side it comes from the observation that Von Neumann algebras are dynamical objects, namely to any Von Neumann algebra \( \mathcal{A} \) acting on an Hilbert space \( \mathcal{H} \) is associated a canonical (up to unitary) 1-parameter group of automorphisms

\[
s \in \mathbb{R} \mapsto \sigma_s \in \text{Aut}(\mathcal{A})
\]

built from a cyclic and separating vector \( \Omega \in \mathcal{H} \). Explicitly the modular group \( \sigma_s \) is

\[
\sigma_s(a) = \Delta^{is} a \Delta^{-is} \quad a \in \mathcal{A}, \ s \in \mathbb{R}
\]

where \( \Delta \) is given by the polar decomposition of Tomita’s operator \( S \) defined on \( \mathcal{A} \Omega \subset \mathcal{H} \) by \( Sa\Omega = a^*\Omega \). The remarkable point is that the state defined by \( \Omega \) is KMS with respect to \( \sigma_s \)

\[
\langle \Omega, \sigma_s(a) b \Omega \rangle = \langle \Omega, b \sigma_{-i} s(a) \Omega \rangle.
\]

Putting \( \alpha_s = \sigma_{\beta s} \) where \( \beta \) is a fixed constant, one is back to (2) as soon as

\[
t = -\beta s.
\]

In other terms, as a reformulation of the KMS-definition, one has\[9\]:

\[
\text{A thermal state at temperature } \beta^{-1} \text{ is a state whose associated modular group } \sigma_s \text{ coincides with the time flow } \alpha_t \text{ up to rescaling } (7).
\]

This definition is less tractable than Gibbs’s one. But it has the advantage to give one solution to the issue of time, simply by inverting the arrow of (3)

\[
\{ \begin{array}{l}
\text{thermal state} \\
\text{temperature}
\end{array} \quad \rightarrow \quad \text{time.} \tag{8} \]

Namely assuming that time flow is not known a priori but the system is in an equilibrium state \( \omega \) at temperature \( \beta^{-1} \), the thermal time hypothesis maintains that the flow of time is given by the modular flow associated to \( \omega \), the physical time \( t \) being related to the modular parameter \( s \)

\[3\] which in the present context can be stated as follows: assuming that covariance is preserved at the quantum level and that one is surrounded by a quantum superposition of states of the gravitational field, then one can a priori picks out any direction as the direction of time. How to combine this freedom at the quantum level with the locally unique intuition of physical flow of time at the classical level?
by (7). Bisognano-Wichman result is a nice illustration of the thermal time hypothesis. Taking for $\mathcal{A}$ the algebra of local observables on the wedge $W$ and for $\Omega$ the vacuum state, one has that the modular group coincides with the time flow of a uniformly accelerated observer $O$. So the line of universe of $O$ can be parameterized by its proper time $\tau$ or by the modular parameter $s$ and the ratio of the corresponding tangent vectors precisely yield Unruh temperature

$$\frac{ds}{d\tau} = \text{constant} = -T_U.$$  \hspace{1cm} (9)

A natural question is whether the same analysis is true for other regions of Minkowski spacetime. In [14] we have considered the modular group associated to the region causally connected to a uniformly accelerated observer $O$ with lifetime $T = 2\tau_0$, namely a diamond-shape region $D$ (\(|x| + |t| < C\), where $C$ is a function of $a$ and $\tau_0$). Assuming the field is conformally invariant, Hislop and Longo [10] have shown that the line of universe of $O$ is nothing but the orbit of one of its point under the action of the modular group. In other terms, as in the eternal case, the trajectory of $O$ can be parameterized by $O$’s proper time or by the modular parameter. But now the ratio of the two parameterizations is no longer constant since the proper time $\tau$ is bounded while the modular parameter $s$ is unbounded. Explicit computations detailed in [14] yield

$$\frac{ds}{d\tau} = T(\tau_0, \tau) = T_U \frac{\sinh a\tau_0}{\cosh a\tau_0 - \cosh a\tau}. \hspace{1cm} (10)$$

We have interpreted (10) as a temperature by noting that for given $\tau_0$ and acceleration $a$, $T(\tau_0, \tau)$ is almost a constant for most of $O$’s lifetime and takes the value observed in the middle of its life, $T(\tau_0, \tau) \simeq T(\tau_0, 0)$ or, written as a function of $\tau_0$ and $a$,\footnote{The observer’s proper time $\tau$ is measured from $-\tau_0$ to $\tau_0$.}

$$T(a, \tau_0) = T_U \frac{\cosh a\tau_0 + 1}{\sinh a\tau_0}. \hspace{1cm} (11)$$

The main difference with other adaptations of Unruh effect to the finite case is the lifetime-dependence of $T$. In this framework the distinction between the lifetime of the observer and the duration of the measurement that we stressed in the precedent section yields a serious problem: if they are not equal, then an observer knowing his acceleration should be able to deduce from a measure $T_0$ his lifetime $T^{-1}(T_0)$. So the instant of his death would be fixed by the value $T_0$, whatever may happen to him after the instant of the measurement. One solution is to identify the duration of the measurement $L$ to the lifetime of the observer $T$, as in the asymptotic limit for the eternal case. Another solution, that only relies on the unquestionable observation that the duration of the measurement cannot exceed the lifetime of the observer,\footnote{The analogy Unruh/Hawking temperature as well as the modular group approach suggest that the origin of the thermalization of the vacuum relies more on the presence of an horizon than on the acceleration. For an inertial non eternal observer the edge of the diamond still acts like a causal horizon, hence $T$ should not vanish for $a = 0$, which is indeed the case. On the contrary an eternal inertial observer can interact with all of Minkovski spacetime and $W$ is no longer an horizon. Hence the vanishing of $T_U$ at $a = 0$. This questions the analysis of the diamond in terms of causal horizon[11].}

$$L \leq T,$$
3. Condition on the temperature from Heisenberg uncertainty relation

Let us summarize our discussion in a slightly more general context: assuming that the vacuum seen by a uniformly accelerated observer with finite lifetime $T$ looks like a thermal bath at temperature $T$, then either $T$ does not depend on $T$ and coherence with the eternal case yields $T = T_U$, or $T$ is a function of $T$ such that $\lim_{T \to \infty} T(T) = T_U$. In the first case finiteness of $T$ only modifies the possible duration of the measurement. Its main effect is that a detector $S$ may not have enough time to reach the thermal equilibrium. In the second case finiteness modifies also the temperature of the thermal bath. Hence, depending on the precision of the measurement, $T$ may be deduced from experimental data which is problematic with respect to causality. Does this forbid the temperature to be lifetime dependent? Or does this invalid the idea that the time of measurement does not necessarily equals the lifetime of the observer?

The answer to both question is negative, because time-energy uncertainty relation prevents a lifetime dependent temperature from being automatically ruled out by causal considerations. The simplest interpretation of Heinsenberg's fourth inequality states [4] that the time $\Delta t$ needed for a non-dissipative quantum system to evolve in a significant manner is as long as the uncertainty $\Delta E$ on the energy is small. The excited state of the quantum system $S$ used as Unruh detector can be distinguished from the ground energy level $E$ only if the latest is known with accuracy greater than the energy gap $k_b T$. Hence a significant evolution of $S$ requires a period of time not shorter than $\frac{h}{k_b T}$ (this lowest value being reached by a perfect detector of heat capacity 1). Since the time of measurement cannot exceed the lifetime of the observer, an observer with lifetime $T$ is not able to measure a temperature with accuracy greater than $\frac{h}{k_b T}$. Thus if

$$T(T) < \frac{h}{k_b T}$$

the Unruh observer has no time to measure $T$ precisely enough to be able to predict his lifetime. From this point of view a $T$-dependent temperature satisfying (12) is causally acceptable. This is not a necessary condition. One may expect $T(T)$ to be obtained from correction of $T_U$ in powers of $T^{-1}$. If the accuracy in temperature measurement is less than

$$\Delta T(T) = |T(T) - T_U|$$

then the observer is not able to affirm that what he is measuring is distinct from what he would measure if he was eternal. In other words he doesn’t know whether he may live forever or not. Hence

$$\Delta T(T) < \frac{h}{k_b T}$$

is another condition making a $T$-dependent temperature causally acceptable.

Strictly speaking $(\Delta t)^{-1} \geq \frac{\Delta E}{h}$ is the frequency of oscillation of the probability $P(b_m, t)$ of obtaining the eigenvalue $b_m$ in the measurement of a given observable $B$ not commuting with the Hamiltonian. So the above interpretation of time-energy relation may not be valid concerning the measurement of the energy during a transition between ground and excited states. Moreover the uncertainty relation is valid for non dissipative system, which put some constraints on the accelerating process of the Unruh observer. Both restrictions can be overcome by the following considerations: a quantum system $S$ coupled to the vacuum does not constitute by itself an Unruh thermometer; one also needs a process to measure the energy levels of $S$ (in the same manner that a column of mercury alone is not a thermometer; it requires gradations marks to be readable). To measure energy gap between quantum levels, one applies some
time-dependent perturbations in order to localize the resonances. A second interpretation of
time-energy uncertainty [4] indicates that a sinusoidal perturbation acting for a time \( T \) cannot
determine resonance with accuracy greater than \( \frac{\hbar}{T} \). Hence the following bounds for the time
dependent temperature

\[
T(T) < \frac{\hbar}{k_b T}, \quad \Delta T(T) < \frac{\hbar}{k_b T}.
\]  

(15)

An adaptation of Unruh effect to non eternal observer is causally acceptable only if it
satisfies one of the conditions (15) (which are stronger than (12) and (14)). Let us examine our
proposition (11) at this light. For small acceleration \( T_D \) is causally acceptable with respect to
first of condition (15) since

\[
\lim_{a \to 0} T_D(a, \tau_0) = \frac{2\hbar}{\pi k_b T}.
\]  

(16)

For large accelerations \( T_D \) no longer depends on the lifetime, \( \lim_{a \to +\infty} T_D(a, \tau_0) = T_U \), which does
not cause problems for causality. For intermediate acceleration the first condition (15) is not
satisfied for large \( \tau_0 \) (see fig. 1) so we have to check for the second one. Explicitly

\[
\Delta T(a, \tau_0) = T_D(a, \tau_0) - T_U = \frac{\hbar}{\pi k_b T} f(a\tau_0)
\]  

(17)

where \( f(x) = x(\cosh x + 1) - 1 \) is positive on \( \mathbb{R}^+ \) and bounded by 2 hence

\[
0 \leq \Delta T(a, \tau_0) \leq \frac{2\hbar}{\pi k_b T}
\]  

(18)

which satisfies the second of (15) for any lifetime \( T \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Plain line is the plot of \( T_D(\tau_0) \) for intermediate acceleration \( a = 3 \) (vertical axe in \( \frac{\hbar}{k_B} \) unit). Causally acceptable points are under the dashed line (which is the plot of \( \frac{\hbar}{k_B T} \)).}
\end{figure}

To finish let us underline that (16) and (18) do satisfy (15) by a factor \( \frac{2}{\pi} \sim 1 \). Most
often time-energy uncertainty deals more with size orders than with exact values (in literature
one often finds \( \gtrsim \) rather than \( \geq \)) hence the diamond’s temperature has the order of the
maximum value that one can canonically assign to a finite region of Minkovski spacetime.
4. Conclusion
Let us state our conclusion in a brief sentence: a non-eternal observer does not live long enough to realize that he is not eternal. This is quite paradoxical. Moreover our corrections to Unruh temperature are of poor interest since they are causally acceptable only if they are not physically detectable (independently of any causal consideration, it is also most likely that corrections are too small to be detected since $T_U$ itself is already extremely small: $T = 1K$ for $a = 10^{19}ms^{-2}$). In fact the notable point is that the good causal-behavior of $T_D$ allows - at least from a causal point of view - to talk of an Unruh temperature for non eternal observers. Hopefully this will be of some interest in a situation where the asymptotic limit is not available, like for the Kerr metric.

5. Appendix
The reader not familiar with algebraic quantum fields theory may wonder how uncertainty relations are implemented in this framework. Let us recall that [9] physical quantities are build by averaging fields (via suitable test functions) on a given region $\mathcal{R}$ of spacetime. This yields an algebra $\mathcal{A}$ of operators, interpreted as the algebra of local observables on $\mathcal{R}$. $\mathcal{A}$ is represented on an Hilbert space $\mathcal{H}$ carrying a unitary representation of the Poincare group and equipped with one particular state $\Omega$ (a ray in $\mathcal{H}$), the physical vacuum, which is invariant under the action of Poincare group. In our example we associate to an observer $O$ with lifetime $T$ the Hilbert space obtained by taking for $\mathcal{R}$ the region of Minkowski spacetime which is causally connected to $O$. This is in this Hilbert space that uncertainty-relations make sense. Note that up to Lorentz transformation there are only two possibilities: either $T$ is infinite and $\mathcal{R} = W$, or $T$ is finite and $\mathcal{R}$ is the diamond $D$.

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