Abstract. Let $S_n f$ be the $n$th partial sum of the Vilenkin-Fourier series of $f \in L^1(G)$. For $1 < p_- \leq p_+ < \infty$, we characterize all exponent $p(\cdot)$ such that if $f \in L^p(\cdot)(G)$, $S_n f$ converges to $f$ in $L^p(\cdot)(G)$.

1. Introduction

Let $\{p_i\}_{i \geq 0}$ be a sequence of integers, $p_i \geq 2$. Let $G = \prod_{i=0}^{\infty} \mathbb{Z}_{p_i}$ be the direct product of cyclic groups of order $p_i$, and $\mu$ the Haar measure on $G$ normalized by $\mu(G) = 1$. Each element of $G$ can be considered as a sequence $\{x_i\}$, with $0 \leq x_i < p_i$. Set $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. There is a well-known and natural measure preserving identification between group $G$ and closed interval $[0, 1]$. This identification consists in associating with each $\{x_i\} \in G$, $0 \leq x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1}$. If we disregard the countable set of $p_i$-rationals, this mapping is one-one, onto and measure-preserving.

For each $x = \{x_i\} \in G$, define $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $k = 0, 1, \ldots$. The set $\{\psi_n\}$ of characters of $G$ consists of all finite product of $\phi_k$, which we enumerate in the following manner. Express each nonnegative integer $n$ as a finite sum $n = \sum_{i=0}^{\infty} \alpha_k m_k$, with $0 \leq \alpha_k < p_k$, and define $\psi_n = \prod_{i=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\psi_n$ form a complete orthonormal system on $G$. For the case $p_i = 2$, $i = 0, 1, \ldots$, $G$ is the dyadic group, $\phi_k$ are Rademacher functions and $\psi_n$ are Walsh functions. In general, the system $\{\psi_n\}$ is a realization of the multiplicative Vilenkin system. In this paper, there is no restriction on the orders $\{p_i\}$.

For $f \in L^1(G)$, let $S_n f$, $n = 0, 1, \ldots$, be the $n$th partial sum of the Vilenkin-Fourier series of $f$. When the orders $p_i$ of cyclic groups are bounded Watari [16] showed that for $f \in L^p(G)$, $1 < p < \infty$,

$$\lim_{n \to \infty} \int_G |S_n f - f|^p d\mu = 0.$$  

Young [14], Schipp [11] and Simon [12] showed independently that results concerning mean convergence of partial sums of the Vilenkin-Fourier series are still valid even if the orders $p_i$ are unbounded.

Let $\{G_k\}$ be the sequence of subgroups of $G$ defined by $G_0 = G$, $G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} \mathbb{Z}_{p_i}$, $k = 1, 2, \ldots$.

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On the closed interval [0, 1], cosets of \( G_k \) are intervals of the form \([jm_k^{-1}, (j + 1)m_k^{-1}]\), \( j = 0, 1, \ldots, m_k - 1 \). By \( \mathcal{F} \) denote the set of generalized intervals. This set is the collection of all translations of intervals \([0, jm_k^{-1}], k = 0, 1, \ldots, j = 1, \ldots, p_k \). Note that a set \( I \) belongs to \( \mathcal{F} \) if (1) for some \( x \in G \) and \( k, I \subset x + G_k \), (ii) \( I \) is a union of cosets of \( G_{k+1} \), and (iii) if we consider \( x + G_k \) as a circle, \( I \) is an interval. Let \( \mathcal{F}_{-1} = \{ G \} \). For \( k = 0, 1, \ldots \), let \( \mathcal{F}_k \) be the collection of all \( I \in \mathcal{F} \) such that \( I \) is a proper subset of a coset of \( G_k \), and is a union of cosets of \( G_{k+1} \). The collections \( \mathcal{F}_k \) are disjoint, and \( \mathcal{F} = \bigcup_{k=-1}^{\infty} \mathcal{F}_k \).

We say that \( w \) is a weight function on \( G \) if \( w \) is measurable and \( 0 < w(x) < \infty \) a.e. Gosselin [5] (case sup_{n < \infty} \) and Young [15] (no restriction on the orders \( p_i \)) characterized all weight functions \( w \) such that if \( f \in L^p_w(G) \), \( 1 < p < \infty \), \( S_n f \) converges to \( f \) in \( L^p_w(G) \). Here \( L^p_w(G) \) denotes the space of measurable functions on \( G \) such that \( \| f \|_{p,w} = (\int_G |f|^p w \, d\mu)^{1/p} < \infty \).

**Definition 1.1.** (see [15]) (i) We say that \( w \) satisfies \( A_p(G), 1 < p < \infty \), condition if

\[
[w]_{A_p} = \sup_{I \in \mathcal{F}} \left( \frac{1}{\mu(I)} \int_I w \, d\mu \right) \left( \frac{1}{\mu(I)} \int_I w^{-1/(p-1)} \, d\mu \right)^{p-1} < \infty.
\]

(ii) We say that \( w \) satisfies \( A_1(G) \) condition if for every \( I \in \mathcal{F} \),

\[
[w]_{A_1} = \sup_{I \in \mathcal{F}} \frac{1}{\mu(I)} \int_I \left( w \left( \text{ess}\inf_I w(x) \right)^{-1} \right) \, d\mu < \infty.
\]

For the case where the orders of cyclic groups are bounded, Gosselin [5] defined \( A_p(G) \) condition, as the one where [11] condition holds for all \( I \) that are cosets of \( G_k \), \( k = 0, 1, 2, \ldots \). For this case \( A_p \) conditions, defined by Young and Gosselin, are equivalent (see [15]).

**Theorem 1.2.** ([15]) Let \( w \) be a weight function on \( G \). For \( 1 < p < \infty \), the following statements are equivalent:

(i) \( w \in A_p(G) \),

(ii) There is a constant \( C \), depending only on \( w \) and \( p \), such that for every \( f \in L^p_w(G) \), we have

\[
\int_G |S_n f|^p w \, d\mu \leq C \int_G |f|^p w \, d\mu,
\]

(iii) For every \( f \in L^p_w(G) \), we have

\[
\lim_{n \to \infty} \int_G |S_n f - f|^p w \, d\mu = 0.
\]

In this paper we characterize all exponents \( p(\cdot) \) such that if \( f \in L^{p(\cdot)}(G) \), then partial sums \( S_n f \) of the Vilenkin-Fourier series of \( f \in L^{p(\cdot)}(G) \) converge to \( f \) with \( L^{p(\cdot)} \)-norm. Now we give a definition of variable Lebesgue space. Let \( p(\cdot) : G \to [1, \infty) \) be a measurable function. The variable Lebesgue space \( L^{p(\cdot)}(G) \) is the set of all measurable functions \( f \) such that for some \( \lambda > 0 \),

\[
\rho_{p(\cdot)}(f/\lambda) = \int_G (|f(x)|/\lambda)^{p(x)} \, d\mu(x) < \infty.
\]

\( L^{p(\cdot)}(G) \) is a Banach function space equipped with the Luxemburg norm

\[
\| f \|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \}.
\]
We use the notations $p_- = \text{essinf}_{x \in G} p(x)$ and $p_+ = \text{esssup}_{x \in G} p(x)$. The function $p'(\cdot)$ denotes the conjugate exponent function of $p(\cdot)$, i.e., $1/p(x) + 1/p'(x) = 1$ ($x \in G$). In this paper the constants $C, c$ are absolute constants and may be different in different contexts and $\chi_A$ denotes the characteristic function of set $A$.

Very recently the convergence of partial sums of the Walsh-Fourier series in $L^{p'(\cdot)}([0,1))$ space was investigated by Jiao et al. [7]. We denote by $C^{\text{log}}_d$ the set of all functions $p(\cdot) : [0,1) \to (0,\infty)$, for which there exists a positive constant $C$ such that

$$|I|^{p_-(I) - p_+(I)} \leq C$$

for all dyadic interval $I = [k2^{-n}, (k+1)2^{-n})$ ($k, n \in \mathbb{N}, 0 \leq k < 2^n$), here $|I|$ denotes the Lebesgue measure of $I$. Note that this condition may interpreted as dyadic version of log-Hölder continuity condition of $p(\cdot)$ (or on dyadic group). The log-Hölder condition is a very common condition for solving various problems of harmonic analysis in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [1],[4]).

**Theorem 1.3.** ([7]) Let $p(\cdot) \in C^{\text{log}}_d$ with $1 < p_- \leq p_+ < \infty$. If $f \in L^{p(\cdot)}([0,1))$, then for partial sums $S_n f$ of the Walsh-Fourier series of $f \in L^{p(\cdot)}([0,1))$ we have

$$\sup_{n \in \mathbb{N}} \|S_n f\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}.$$

Since Walsh polynomials are dense in $L^{p(\cdot)}([0,1))$, Theorem 1.3 implies that $S_n f$ converges to the original function in $L^{p(\cdot)}([0,1))$-norm (for more details see [7]).

In order to extend techniques and results of constant exponent case to the setting of variable Lebesgue spaces, a central problem is to determine conditions on an exponent $p(\cdot)$ under which the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}$ (see monographs Cruz-Uribe and Fiorenza [1] and Diening et.al. [4]). We now define the Hardy-Littlewood maximal function that is appropriate for the study of Vilenkin-Fourier series. For $f \in L^1(G)$, let

$$Mf(x) = \sup_{x \in I, I \in \mathcal{F}} \frac{1}{\mu(I)} \int_I |f| d\mu.$$

This maximal function was introduced first by P. Simon in [13]. He showed that the maximal operator is bounded in $L^p(G)$, $1 < p < \infty$ and is of weak type $(1,1)$. Young [15] obtained the following analogue of Muckenhoupt’s theorem [10].

**Theorem 1.4.** Let $w$ be a weight function on $G$. For $1 < p < \infty$, the following two statements are equivalent:

(i) $w \in A_p(G)$,

(ii) There is a constant $C$, depending only on $w$ and $p$, such that for every $f \in L^p_w(G)$, we have

$$\int_G (Mf)^p w d\mu \leq C \int_G |f|^p w d\mu.$$

In case $p = 1$ the following two statements are also equivalent:

(iii) $w \in A_1(G)$,

(iv) There is a constant $C$, depending only on $w$, such that for every $f \in L^1(G)$

$$\int_{\{Mf > y\}} w d\mu \leq Cy^{-1} \int_G |f| w d\mu, \ \ y > 0.$$
Definition 1.5. We say that the exponent \( p(\cdot) \), \( 1 < p_- \leq p_+ < \infty \) satisfies the condition \( \mathcal{A}(G) \), if there is a constant \( C \) such that for every \( I \in \mathcal{I} \),

\[
\frac{1}{\mu(I)} \| \chi_I \|_{p(\cdot)} \| \chi_I \|_{p(\cdot)} \leq C.
\]

(1.2)

The condition (1.2) plays exactly the same role for averaging operators in variable Lebesgue spaces as the Muckenhoupt \( A_p \) conditions for weighted Lebesgue spaces (see [8], [9], for Euclidian setting). We show that the \( \mathcal{A}(G) \) condition is necessary and sufficient for the \( L^{p(\cdot)}(G) \)-boundedness of Hardy-Littlewood maximal function. One of the main result of the present paper is the following theorem.

Theorem 1.6. Assume for the exponent \( p(\cdot) \) we have \( 1 < p_- \leq p_+ < \infty \). Then the following two statements are equivalent:

(i) \( p(\cdot) \in \mathcal{A}(G) \),

(ii) There is a constant \( C \), depending only on \( p(\cdot) \) such that for every \( f \in L^{p(\cdot)}(G) \), we have

\[
\| Mf \|_{p(\cdot)} \leq C \| f \|_{p(\cdot)}.
\]

By the symmetry of the definition, \( p(\cdot) \in \mathcal{A}(G) \) if and only if \( p'(\cdot) \in \mathcal{A}(G) \) and from Theorem 1.6 we have that, even though, \( M \) is not a linear operator, the boundedness of \( M \) implies the "dual" inequality.

Corollary 1.7. Let for exponent \( p(\cdot) \) we have \( 1 < p_- \leq p_+ < \infty \). Then the maximal operator \( M \) is bounded on \( L^{p(\cdot)}(G) \) if and only if \( M \) is bounded on \( L^{p'(\cdot)}(G) \).

We prove the following theorem (in the Euclidean setting see [1], Theorem 4.37)

Theorem 1.8. Let for the exponent \( p(\cdot) \) we have \( 1 < p_- \leq p_+ < \infty \). Then the following statements are equivalent:

(i) Maximal operator \( M \) is bounded on \( L^{p(\cdot)}(G) \),

(ii) There exists \( r_0, 0 < r_0 < 1 \), such that if \( r_0 < r < 1 \), then maximal operator \( M \) is bounded on \( L^{rp(\cdot)}(G) \).

Hereafter, we will denote by \( \mathcal{S} \) a family of pairs of non-negative, measurable functions. Given \( p, 1 \leq p < \infty \) if for some \( w \in A_p(G) \) we write

\[
\int_G f(x)^p w(x) d\mu \leq C \int_G g(x)^p w(x) d\mu, \quad (f,g) \in \mathcal{S},
\]

then we mean that this inequality holds for all pairs \( (f,g) \in \mathcal{S} \) such that the left hand side is finite, and that the constant \( C \) may depend on \( p \) and \( [w]_{A_p} \). If we write

\[
\| f \|_{p(\cdot)} \leq C_{p(\cdot)} \| g \|_{p(\cdot)}, \quad (f,g) \in \mathcal{S},
\]

then we mean that this inequality holds for all pairs \( (f,g) \in \mathcal{S} \) such that the left-hand side is finite and the constant may depend on \( p(\cdot) \).

Using this convention we can state the Rubio de Francia extrapolation theorem in the following manner.

Theorem 1.9. Suppose for some \( p_0 \geq 1 \) the family \( \mathcal{S} \) is such that for all \( w \in A_1(G) \)

\[
\int_G f(x)^{p_0} w(x) d\mu \leq C \int_G g(x)^{p_0} w(x) d\mu, \quad (f,g) \in \mathcal{S}.
\]

If for the exponent \( p(\cdot) \), we have \( p_0 \leq p_- \leq p_+ < \infty \) and the maximal operator \( M \) is bounded on \( L^{[p(\cdot)/p_0]'}(G) \), then

\[
\| f \|_{p(\cdot)} \leq C_{p(\cdot)} \| g \|_{p(\cdot)}, \quad (f,g) \in \mathcal{S}.
\]
Firstly, the Theorem 1.9 was proved in [3] (Theorem 3.5) for variable exponent Lebesgue spaces on $\mathbb{R}^n$ and maximal operator $M$ defined on cubes (balls) in $\mathbb{R}^n$, with sides are parallel to the coordinate axes. In [2] the Rubio de Francia extrapolation theorem is proved for general Function spaces, using $A_1$ weights and maximal operator $M$ defined by any Muckenhoupt basis (see Definition 2.3 in [2]). By Theorem 1.4 the set of generalized intervals $\mathcal{F}$ is Muckenhoupt basis which implies that Theorem 1.9 is direct consequence of Corollary 1.7, Theorem 1.8, and Theorem 3.5 from [2].

Now, we can formulate the main result of the present paper.

**Theorem 1.10.** Let for exponent $p(\cdot)$ we have $1 < p_- \leq p_+ < \infty$. Then the following statements are equivalent:

(i) $p(\cdot) \in A(G)$,

(ii) There is a constant $C$, depending only on $p(\cdot)$, such that for partial sums $S_n f$ of the Vilenkin-Fourier series of $f \in \mathcal{L}^{p(\cdot)}(G)$ we have

$$\sup_{n \in \mathbb{N}} \|S_n f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

(iii) Partial sums $S_n f$ of the Vilenkin-Fourier series of $f \in \mathcal{L}^{p(\cdot)}(G)$ converge to the original function in $\mathcal{L}^{p(\cdot)}$-space.

2. Preliminaries

The fundamental properties of $A_p(G)$ weights were investigated by Gosselin [5] and later by Young [15] (in this paper there is no restriction on the orders $p_i$). We formulate some properties of these weights (see [15]).

Note that if $w \in A_p(G)$, then $L^p_w(G) \subset L^q_w(G)$. We also mention that if $w \in A_p(G), 1 \leq p < \infty$, and $p < q < \infty$ then $w \in A_q(G)$. A important property of $A_p(G)$ weights is the reverse Hölder inequality.

**Proposition 2.1.** ([15]) Let $w \in A_p(G), 1 < p < \infty$. Then there exist $s > 1$ and a constant $C$ such that for any $I \in \mathcal{F}$,

$$\left( \frac{1}{\mu(I)} \int_I w^s d\mu \right)^{1/s} \leq \frac{C}{\mu(I)} \int_I w d\mu.$$

**Definition 2.2.** Let $I_0 \in \mathcal{F}$. We say that a weight $w$ (i.e. a nonnegative locally integrable function) satisfies $A_\infty(I_0)$ condition if for any $\alpha > 0$ there exists $\delta \in (0, 1)$ such that for any generalized interval $I \subset I_0$ and for any measurable subset $E \subset I$, $\mu(E) \leq \alpha \mu(I)$ implies $w(E) \leq \beta w(I)$ (for any measurable set $A$, $w(A) = \int_A w d\mu$).

The following proposition is a consequence of the reverse Hölder inequality.

**Proposition 2.3.** ([15]) (i) Suppose $w \in A_p(G), 1 < p < \infty$. Then there exists $1 < s < p$ such that $w \in A_s(G)$. (ii) Suppose $w \in A_p(G), 1 < p < \infty$, then $w \in A_\infty(G)$.

It is well known fact that the class $A_\infty$ can be defined in many equivalent ways. In particular, $w \in A_\infty(I_0)$ if and only if there exist positive constants $C, \varepsilon$ such that for any interval $I \subset I_0$ and for any measurable subset $E \subset I$,

$$\frac{w(E)}{w(I)} \leq c \left( \frac{\mu(E)}{\mu(I)} \right)^{\varepsilon}.$$

**(2.1)**
Proposition 2.4. Given \( w \in A_1(G) \), if \( s_0 = 1 + \frac{1}{8|w|_{A_1}} \), then for \( 1 < s \leq s_0 \) and for almost every \( x \),
\[
M_s w(x) \leq 2Mw(x) \leq 2|w|_{A_1} w(x).
\]
(2.2)

This type of estimates are well known in Euclidian setting. For the sake of completeness we will give a proof for the Vilenkin group.

For proving the result we need modified form of the Calderón-Zygmund decomposition lemma (see [14], Lemma 2).

Lemma 2.5. Given an interval \( I \in \mathcal{F} \) and a function \( f \in L^1(G) \), then for \( t \geq |f|_I \), there exists a collection \( I_j \) of disjoint generalized intervals \( I_j \subset I \) such that
\[
t < \frac{1}{\mu(I_j)} \int_{I_j} |f|d\mu \leq 3t, \forall I_j,
\]
and for almost every \( x \in I \setminus \cup_j I_j \), \( |f(x)| \leq t \).

We need an inequality that is the reverse of the weak \((1,1)\) inequality for maximal operator \( M \).

Lemma 2.6. Given a function \( f \in L^1(G) \), for every interval \( I \in \mathcal{F} \) and \( t \geq |f|_I \),
\[
\mu(\{x \in I : Mf(x) > t\}) \geq \frac{1}{3t} \int_{\{x \in I : |f(x)| > t\}} |f(x)|d\mu.
\]

Proof. \( t \geq |f|_I \); if \( t \geq \|f\|_{L^\infty} \), then this result is true. Otherwise, by Lemma 2.5, let \( I_i \) be the Calderón-Zygmund intervals of \( f \) relative to \( I \) and \( t \). For every \( x \in I_i \)
\[
Mf(x) \geq \frac{1}{\mu(I_j)} \int_{I_j} |f|d\mu > t.
\]
Since \( |f(x)| \leq t \) for almost every \( x \in I \setminus \cup_i I_i \), we have
\[
\mu(\{x \in I : Mf(x) > t\}) \geq \sum_j \mu(I_j)
\]
\[
\geq \frac{1}{3t} \sum_j \int_{I_j} |f|d\mu \geq \frac{1}{3t} \int_{\{x \in I : |f(x)| > t\}} |f(x)|d\mu(x)
\]
\[
\square
\]

Proof of proposition 2.4.

Let \( \varepsilon = (8|w|_{A_1})^{-1} \), \( s_0 = 1 + \varepsilon \), and fix a interval \( I \) and \( x_0 \in I \). To prove the first inequality it is sufficient to show that
\[
\frac{1}{\mu(I)} \int_I w(x)^s d\mu \leq 2Mw(x_0)^s.
\]

We have that
\[
\frac{1}{\mu(I)} \int_I w(x)^s d\mu = \frac{1}{\mu(I)} \int_I w(x)^s w(x) d\mu(x)
\]
\[
= \varepsilon(\mu(I))^{-1} \int_0^\infty t^{\varepsilon-1}w(\{x \in I : w(x) > t\}) \, dt \\
= \varepsilon(\mu(I))^{-1} \int_0^{Mw(x_0)} t^{\varepsilon-1}w(\{x \in I : w(x) > t\}) \, dt \\
+ \varepsilon(\mu(I))^{-1} \int_0^\infty t^{\varepsilon-1}w(\{x \in I : w(x) > t\}) \, dt.
\]

For first term we have
\[
\varepsilon(\mu(I))^{-1} \int_0^{Mw(x_0)} t^{\varepsilon-1}w(\{x \in I : w(x) > t\}) \, dt \\
\leq \varepsilon(\mu(I))^{-1}w(I) \int_0^{Mw(x_0)} t^{\varepsilon-1} \, dt = \frac{1}{\mu(I)} \int_I w(y) \, d\mu(y) \cdot Mw(x_0)^\varepsilon \leq Mw(x_0)^{1+\varepsilon}.
\]

Use Lemma 2.6 we obtain
\[
\varepsilon(\mu(I))^{-1} \int_0^\infty t^{\varepsilon-1}w(\{x \in I : w(x) > t\}) \, dt \\
= \varepsilon(\mu(I))^{-1} \int_0^\infty \int_{\{x \in I : w(x) > t\}} d\mu(x) \, dt \\
\leq 3\varepsilon(\mu(I))^{-1} \int_0^\infty t^{\varepsilon} \mu(\{x \in I : Mw(x) > t\}) \, dt \\
= \frac{3\varepsilon}{1 + \varepsilon} \frac{1}{\mu(I)} \int_I Mw(x)^{1+\varepsilon} \, d\mu(x) \\
\leq \frac{3\varepsilon[w]_A^{1+\varepsilon}}{1 + \varepsilon} \frac{1}{\mu(I)} \int_I w(x)^{1+\varepsilon} \, d\mu(x).
\]

From above estimates we get
\[
\frac{1}{\mu(I)} \int_I w(x)^{1+\varepsilon} \, d\mu(x) \leq Mw(x_0)^{1+\varepsilon} + \frac{3\varepsilon[w]_A^{1+\varepsilon}}{1 + \varepsilon} \frac{1}{\mu(I)} \int_I w(x)^{1+\varepsilon} \, d\mu(x).
\]

Since for all \( x \geq 1, x^{1/8x} \leq 2 \), we have
\[
\frac{3\varepsilon[w]_A^{1+\varepsilon}}{1 + \varepsilon} \leq \frac{3}{8}[w]_{A_1}^{-1}[w]_{A_3}(8[w]_{A_1})^{-1} \leq 1
\]

and consequently the first inequality in (2.2) is valid. The second inequality in (2.2) is clear. \( \square \)

3. PROOF OF THEOREM 1.7

Given a generalized interval \( I \in \mathcal{F} \) define the averaging operator \( A_I \) by
\[
A_I f(x) = \frac{1}{\mu(I)} \int_I f \, d\mu \chi_I(x).
\]

**Proposition 3.1.** Given an exponent \( p(\cdot), 1 < p_- \leq p_+ < \infty \), there exists a constant \( C > 0 \) such that for any interval \( I \in \mathcal{F} \)
\[
\|A_I f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}
\]
if and only if \( p(\cdot) \in \mathcal{A}(G) \).
The proof of Proposition 3.1 is essentially the same as for averaging operator defined by cubes for Euclidean setting (see for example [1], Proposition 4.47).

The Lemma 3.2 shows that the condition \( p(\cdot) \in \mathcal{A}(G) \) is actually sufficient for modular inequality. Analogous estimate for the case \( L^p(\cdot)(\mathbb{R}^n) \) was obtained by Kopaliani [8]. The proof in [8] is based on some concepts from convex analysis. Lerner in [9] gave a different and simple proof. In this paper our approach is based on the adaptation of Lerner’s proof [9].

**Lemma 3.2.** Given exponent \( p(\cdot) \) such that \( 1 < p_- \leq p_+ < \infty \), suppose \( p(\cdot) \in \mathcal{A}(G) \). Let \( f \in L^p(\cdot)(G) \). If there exist a interval \( I \in \mathcal{F} \) and constants \( c_1, c_2 > 0 \) such that \( |f|_I \geq c_1 \) and \( \|f\|_{p(\cdot)} \leq c_2 \), where \( c_1, c_2 > 0 \), then there exists a constant \( c \) depending only on \( p(\cdot), c_1, c_2 \) such that

\[
\int_I |f(x)|^{p(x)} d\mu(x) \leq c \int_I |f(x)|^{p(x)} d\mu(x).
\]

**Proof.** Using the condition \( p_+ < \infty \) we may consider only the case \( c_1 = c_2 = 1 \). Since \( p_+ < \infty \), by continuity there exists \( \alpha > 0 \) such that

\[
\tag{3.1}
\int_I \alpha^{p'(y)-1} d\mu(y) = \int_Q |f(x)| d\mu(x).
\]

Since \( |f|_I \geq 1 \), we have \( \alpha \geq 1 \). By generalized Hölder inequality

\[
\int_I f(x) d\mu(x) \leq 2\|f\|_{p(\cdot)} \|\chi_I\|_{p'(\cdot)}
\]

we get \( \int_I \alpha^{p'(y)-1} d\mu(y) \leq 2\|\chi_I\|_{p'(\cdot)} \) and consequently,

\[
\tag{3.2}
\alpha \leq c/\|\chi_I\|_{p'(\cdot)}.
\]

Given this value \( \alpha \), we have that

\[
\tag{3.3}
\int_I |f(x)|^{p(x)} d\mu(x) = \int_I \left( \frac{1}{\mu(I)} \int_I \alpha^{p'(y)-1} d\mu(y) \right)^{p(x)} d\mu(x)
\]

\[
= \left( \frac{1}{\mu(I)} \right) \int_I \left( \frac{1}{\mu(I)} \int_I \alpha^{p'(y)-p'(x)} d\mu(y) \right)^{p(x)-1} d\mu(x) \int_I \alpha^{p'(y)} d\mu(y).
\]

For each \( x \in I \) partition \( I \) into \( E_1(x) = \{ y \in I : p'(y) > p'(x) \} \) and \( E_2(x) = I \setminus E_1(x) \). Using (3.3) and the estimate \( \alpha \geq 1 \), we obtain

\[
\int_I \alpha^{p'(y)-p'(x)} d\mu(y) = \int_{E_1(x)} \alpha^{p'(y)-p'(x)} d\mu(y) + \int_{E_2(x)} \alpha^{p'(y)-p'(x)} d\mu(y)
\]

\[
\leq c(\|\chi_I\|_{p'(\cdot)} p'(x) + \mu(I)).
\]

In view of \( p(\cdot) \in \mathcal{A}(G) \), we have

\[
\tag{3.4}
\frac{1}{\mu(I)} \int_I \left( \frac{1}{\mu(I)} \int_I \alpha^{p'(y)-p'(x)} d\mu(y) \right)^{p(x)-1} d\mu(x)
\]

\[
\leq c \frac{1}{\mu(I)} \int_I \left( \frac{1}{\mu(I)} \left( \|\chi_I\|_{p'(\cdot)} p'(x) + 1 \right) \right)^{p(x)-1} d\mu(x)
\]

\[
\leq c + c \frac{1}{\mu(I)} \int_I \left( \frac{1}{\mu(I)} \left( \|\chi_I\|_{p'(\cdot)} p'(x) \right) \right)^{p(x)-1} d\mu(x)
\]
Further,
\[ \int_I \alpha^{p^*(y)} d\mu(y) = 2\alpha \int_I |f(x)| d\mu(x) - \int_I \alpha^{p^*(y)} d\mu(y) \]
\[ \leq 2\alpha \int_{\{y \in I : 2\alpha|f(y)| > \alpha^{p^*(y)}\}} |f(y)| d\mu(y) \]
\[ \leq c \int_I |f(y)|^{p^*(y)} d\mu(y). \]

From (3.3), (3.4) and (3.5) we obtain desired estimate. \( \square \)

**Corollary 3.3.** Let \( 1 < p_- \leq p_+ < \infty \) and \( p^*(\cdot) \in A(G) \). Suppose that \( \xi_1 \leq t \leq \xi_2/\|\chi_I\|_{p^*} \), where \( \xi_1, \xi_2 > 0 \) and \( I \in \mathcal{F} \). Then \( \nu^p(x) \in A_\infty(I) \) with \( A_\infty \) constant depending only on \( p^*(\cdot), \xi_1, \xi_2 \).

**Proof.** Let \( I_0 \subset I \), where \( I_0, I \in \mathcal{F} \) and \( E \subset I_0 \) be any measurable subset with \( \mu(E) > \mu(I_0)/2 \). Define \( f = t\chi_E \). Then
\[ |f| = \frac{1}{\mu(I)} \int_I t\chi_E(x) d\mu(x) = t \frac{\mu(E)}{\mu(I)} \geq \frac{\xi_1}{2}, \]
\[ \|f\|_{p^*(\cdot)} = t\|\chi_E\|_{p^*(\cdot)} \leq \xi_2 \frac{\|\chi_I\|_{p^*(\cdot)}}{\|\chi_I\|_{p^*(\cdot)}} \leq \xi_2. \]

Therefore, \( f \) satisfies the hypotheses of Lemma 3.2 with \( c_1 = \xi_1/2, c_2 = \xi_2 \) and there exists a constant \( c \) depending only on \( p^*(\cdot), \xi_1, \xi_2 \) such that
\[ \frac{1}{2p^+} \int_{I_0} t^{p^*(\cdot)} d\mu(x) \leq c \int_E t^{p^*(\cdot)} d\mu(x), \]
which proves that \( \nu^p(x) \in A_\infty(I) \). \( \square \)

**Proof of Theorem 1.6.** The part \((ii) \Rightarrow (i)\) of Theorem 1.6 follows immediately from Proposition 3.1 and from the fact that \( |f|t\chi_I(x) \leq Mf(x) \) for any interval \( I \in \mathcal{F} \).

Implication \((i) \Rightarrow (ii)\). Suppose \( f \in L^{p^*(\cdot)}(G) \) and \( \|f\|_{p^*(\cdot)} \leq 1 \). It is sufficient to proof that there exists a positive constant \( C \) (independent of \( f \)) such that for any nonnegative function \( g \in L^{p^*(\cdot)}(G) \), with \( \|g\|_{p^*(\cdot)} \leq 1 \)
\[ \int_G Mf(x)g(x) d\mu(x) \leq C. \]

For each positive integer \( k \) set
\[ \Omega_k = \{ x \in G : Mf(x) > 3^k \}. \]

Note that
\[ \int_{G \setminus \Omega_k} Mf(x)g(x) d\mu(x) \leq C. \]
Define $D_k = \Omega_k \setminus \Omega_{k+1}$. Let $F_k$ be an arbitrary compact subset of $D_k$. We will prove that

\begin{equation}
\int_{\cup F_k} Mf(x)g(x)d\mu(x) \leq C.\tag{3.8}
\end{equation}

By simple limiting argument from (3.8) and from (3.7) we obtain (3.6).

Let $\mu(F_k) > 0$. There exists a finite collection of generalized intervals $I_\alpha, \alpha \in A_k$, $F_k \subset \cup_{\alpha \in A_k} I_\alpha$, such that $|f|I_\alpha > 3^k$, $\alpha \in A_k$ and for all fixed $\alpha$, there exists $x_\alpha \in I_\alpha$ such that $Mf(x_\alpha) \leq 3^{k+1}$. Note that if $I_{\alpha_1}$ and $I_{\alpha_2}$ belong to distinct $F_l$’s and are not disjoint ($\mu(I_{\alpha_1} \cap I_{\alpha_2}) > 0$) then one is a subset of the other. Consequently without loss of generality we may assume that in collection $I_\alpha, \alpha \in A_k$ if $\mu(I_{\alpha_1} \cap I_{\alpha_2}) > 0$ for some $\alpha_1$ and $\alpha_2$, then $I_{\alpha_1}$ and $I_{\alpha_2}$ belong to the same $F_l$’s (for some $l$). By Vitali covering lemma, we may select from collection $I_\alpha, \alpha \in A_k$ the finite collection of pairwise disjoint intervals $\{I^k_j\} j \in \{1, \ldots, N_k\}$ such that $F_k \subset \cup_j 3I^k_j$.

Without loss of generality we may assume that $\mu(F_k) > 0$ for all $k \geq 1$. Define the sets $E^k_1 = I^k_1 \cap F_k$, $E^k_2 = (3I^k_1 \setminus \cup_{k<j} 3I^k_j) \cap F_k$, $j > 1$. Note that the sets $E^k_j$ are pairwise disjoint and $\cup_j E^k_j = F_k$.

Define

$$Tg(x) = \sum_{k=1}^{\infty} \sum_j \left( \frac{1}{\mu(I^k_j)} \int_{E^k_j} gd\mu \right) \chi_{I^k_j}(x).$$

Using the above definition, we get

$$\int_{\cup_k F_k} (Mf)(x)g(x)d\mu(x) \leq 3^{k+1} \sum_{k=1}^{\infty} \sum_j \int_{E^k_j} gd\mu \leq 3 \sum_{k=1}^{\infty} \sum_j f_{I^k_j} \int_{E^k_j} gd\mu(x)$$

$$= 3 \int_G fTg \leq 6 \|f\|_{p(c)} \|Tg\|_{p'(c)},$$

and consequently for proving (3.8), it is sufficient to show that $\|Tg\|_{p'(c)} \leq C$.

Note that $I^k_j \subset \Omega_k = \cup_{l=0}^\infty D_{k+l}$ and hence $Tg = \sum_{l=0}^{\infty} T_ig$, where

$$T_ig(x) = \sum_{k=1}^{\infty} \sum_j a_{j,k}(g) \chi_{I^k_j \cap D_{k+l}}(x), \quad (l = 0, 1, \ldots)$$

where $a_{j,k}(g) = \frac{1}{\mu(I^k_j)} \int_{E^k_j} gd\mu$.

Let $I_1 = \{(j, k) : a_{j,k}(g) > 1\}$ and $I_2 = \{(j, k) : a_{j,k}(g) \leq 1\}$.

By condition $p \in A(G)$ and Hölder inequality implies that for any interval $I \in F$,

$$\|\chi_{\lambda I}\|_{p(c)} \leq C \|\chi_I\|_{p(c)}.$$

We have

$$\alpha_{j,k}(g) \leq \frac{2}{\mu(I^k_j)} \|\chi_{E^k_j}\|_{p(c)} \|g\chi_{E^k_j}\|_{p'(c)} \leq \frac{2}{\mu(I^k_j)} \|\chi_{3I^k_j}\|_{p(c)}$$

$$\leq \frac{C}{\|\chi_{E^k_j}\|_{p(c)}} \leq \frac{C}{\|\chi_{I^k_j}\|_{p'(c)}}.$$

Let $(j, k) \in I_1$. Then by Corollary $\alpha_{j,k}(g)p'(x) \in A_\infty(I^k_j)$ and by Lemma (2.1)

$$\int_{I^k_j \cap D_{k+l}} \alpha_{j,k}(g)p'(x)d\mu(x) \leq C \left( \frac{\mu(I^k_j \cap D_{k+l})}{\mu(I^k_j)} \right) \int_{I^k_j} \alpha_{j,k}(g)p'(x)d\mu(x).$$
(3.9) \[ \leq C \left( \frac{\mu(I_j^k \cap D_{k+1})}{\mu(I_j^k)} \right)^\varepsilon \int_{E_j^k} g(x)^{p'(x)} d\mu(x). \]

If \((j, k) \in \mathcal{I}_2\), then we have
\[
\int_{I_j^k \cap D_{k+1}} \alpha_{j,k}(g)^{p'(x)} d\mu(x) \leq \int_{I_j^k \cap D_{k+1}} \alpha_{j,k}(g) d\mu
\]

(3.10) \[ = \frac{\mu(I_j^k \cap D_{k+1})}{\mu(I_j^k)} \int_{E_j^k} g(x) d\mu(x). \]

We need estimate \(\mu(I_j^k \cap D_{k+1})\) for \(l \geq 2\). Let \(x \in I_j^k\) and \(I \in \mathcal{F}\) be an arbitrary interval such that \(x \in I\). Observe that either \(I \subset 3I_j^k\) or \(I_j^k \subset 3I\). If the second inclusion holds, then \(3I \cap D_k \neq \emptyset\) and hence
\[ |f|_I \leq 3|f|_{3I} \leq 3 \cdot 3^{k+1} \leq 3^{k+l} \quad (l \geq 2). \]

Therefore, if \(|f|_I > 3^{k+l}\), then \(I \subset 3I_j^k\). From this and from weak type property of \(M\), we get
\[ \mu(I_j^k \cap D_{k+1}) \leq \mu\{x \in I_j^k : M(f \chi_{3I_j^k})(x) > 3^{k+l}\} \]

(3.11) \[ \leq \frac{C}{3^{k+l}} \int_{3I_j^k} |f| d\mu \leq C\frac{\mu(I_j^k)}{3^{k+l}} |f|_{3I_j^k} \leq C \frac{3^{k+1}}{3^{k+l}} \mu(I_j^k) \leq \frac{C}{3^l} \mu(I_j^k). \]

By estimates (3.8), (3.9), (3.11), when \(l \geq 2\) we obtain
\[
\int_G (T_l g(x))^{p'(x)} d\mu(x) = \sum_{k=1}^\infty \sum_{j} \int_{I_j^k \cap D_{k+1}} \alpha_{j,k}(g)^{p'(x)} d\mu(x)
\]

\[ \leq C3^{-\alpha} \sum_{(j,k) \in \mathcal{I}_1} \int_{E_j^k} g(x)^{p'(x)} d\mu(x) + C3^{-l} \sum_{(j,k) \in \mathcal{I}_2} \int_{E_j^k} g(x) d\mu(x)
\]

\[ \leq C3^{-\alpha} \left( \int_G g(x)^{p'(x)} d\mu(x) + \int_G g(x) d\mu(x) \right). \]

Where \(\alpha = \min\{1, \varepsilon\}\).

Using the fact that \(\|g\|_1 \leq 2\|\chi_G\|_{p'(\cdot)}\), and \(\int_G g(x)^{p'(x)} d\mu(x) \leq 1\) we obtain
\[ \|T_l g\|_{p'(\cdot)} \leq C3^{-\alpha/p'_+} \quad (l \geq 2). \]

For \(l = 0, 1\) if we use a trivial estimate \(\mu(I_j^k \cap D_{k+1}) \leq \mu(I_j^k)\), analogously will be obtained the estimate \(\|T_l g\|_{p'(\cdot)} \leq C\). Finally we obtain
\[ \|T g\|_{p'(\cdot)} \leq \sum_{l=0}^\infty \|T_l g\|_{p'(\cdot)} \leq C. \]

\(\square\)
4. Proof of Theorem 1.8

The implication (ii) \(\Rightarrow\) (i) is straightforward. Fix \(r_0, r_0 < r < 1\), and let \(s = 1/r\). By Hölder’s inequality, we have that \(Mf(x) \leq M(|f|^s)(x)^{1/s} = Msf(x)\). Note that

\[
\|f^s\|_{p(\cdot)} = \|f^s\|_{sp(\cdot)} ^{p(\cdot)} \\
and
\|Mf\|_{p(\cdot)} \leq M(|f|^s)_{r_{p(\cdot)}} \leq C\|f^s\|_{r_{p(\cdot)}} = C\|f\|_{p(\cdot)}.
\]

To prove that (i) \(\Rightarrow\) (ii), we first construct a \(A_1(G)\) weight using the Rubio de Francia iteration algorithm. Given \(h \in L^{p(\cdot)}(G)\), define

\[
R_h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{\|M^k\|_{L^{p(\cdot)}(G)}},
\]

where for \(k \geq 1\), \(M_k = M \circ M \circ \cdots \circ M\) denotes \(k\) iterations of the Maximal operator \(M\) and \(M^0f = |f|\). The function \(R_h(x)\) has the following properties:

(a) For all \(x \in G\), \(|h(x)| \leq R_h(x)\);
(b) \(R\) is bounded on \(L^{p(\cdot)}(G)\) and \(\|R_h\|_{p(\cdot)} \leq 2\|h\|_{p(\cdot)}\);
(c) \(R_h \in A_1(G)\) and \(\|R_h\|_{A_1} \leq 2\|M\|_{L^{p(\cdot)}(G)}\).

The proof of properties (a), (b), (c) are the same, as Euclidian setting (see [1], pp.157) and we omit it here. By property (c) and Proposition 2.4 there exists \(s_0 > 1\) such that for all \(s, 1 < s < s_0\),

\[
M_s(R_h)(x) = M_{s_0}(R_h)(x) \leq 4\|M\|_{L^{p(\cdot)}(G)} R_h(x).
\]

Let \(r_0 = 1/s_0\) and \(r_0 < r < 1\). Let \(s = 1/r\).

By properties (a) and (b) we have

\[
\|Mf\|_{r_{p(\cdot)}} = \|(Mf)^{1/s}\|_{p(\cdot)} ^{s} = \|M_s(|f|^s)\|_{p(\cdot)} ^{s} \leq \|M_s(R(|f|^s))\|_{p(\cdot)} ^{s} \\
\leq C\|M\|_{L^{p(\cdot)}(G)} \|R(|f|^s)\|_{p(\cdot)} ^{s} \leq C\|f\|_{r_{p(\cdot)}} ^{s} = C\|f\|_{r_{p(\cdot)}}.
\]

\(\square\)

5. Proof of Theorem 1.10

Since Vilenkin polynomials are dense in \(L^{p(\cdot)}(G)\) (\(1 \leq p_- \leq p_+ < \infty\)) the proof of equivalence of (ii) and (iii) is straightforward. The implications (i) \(\Rightarrow\) (ii) follows from Rubio de Francia extrapolation theorem (Theorem 1.9), if we use Young’s weighted estimates for partial sum \(S_n f\) of the Vilenkin-Fourier series (Theorem 1.2) and Theorem 1.6, Theorem 1.8.

Proof of (ii) \(\Rightarrow\) (i). Consider \(I \in F\). There is \(x \in G\) such that \(I\) is a proper subset of \(x + G_k\), and \(I\) is a union of cosets of \(G_{k+1}\). First consider the case \(\mu(I) \leq \mu(G_k)/2\). Take \(\alpha_k = [\mu(G_k)/2\mu(I)]\), where \([a]\) is the largest integer less than or equal to \(a\). We have \(\alpha_k \geq 1\). Let \(f \in L^{p(\cdot)}(G)\) be a nonnegative function with support in \(I\). We will bellow use the following estimate (see [15], pp.286-287): for \(x \in I\),

\[
\phi_k^{-(\alpha_k-1)/2}(x)S_{\alpha_k m_k}(f \phi_k^{-(\alpha_k-1)/2})(x) \geq \frac{1}{2\pi \mu(I)} \int_I f(t)d\mu(t) = \frac{1}{2\pi} A_I f(x).
\]

We have

\[
\|A_I f\|_{p(\cdot)} \leq C\|\phi_k^{-(\alpha_k-1)/2}S_{\alpha_k m_k}(f \phi_k^{-(\alpha_k-1)/2})\| \leq C\|f\|_{p(\cdot)}.
\]

From this estimate we obtain in standard way \([12]\) in case \(\mu(I) \leq \mu(G_k)/2\) (see Proposition 3.1).
Consider the case $\mu(I) > \mu(G_k)/2$. Note that every coset of $G_k$ is in $F_{k-1}$ and $\mu(G_k) \leq \mu(G_{k-1})/2$ and consequently (1.2) holds for all cosets of $G_k$. We have

$$\|\chi_I\|_{p(.)} \|\chi_{I}\|_{p'(.)} \leq \|\chi_{x+G_k}\|_{p(.)} \|\chi_{x+G_k}\|_{p'(.)} \leq C\mu(G_k) \leq C\mu(I).$$

□

REFERENCES

[1] D. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces. Foundations and Harmonic Analysis, Birkhäuser, Basel (2013).
[2] D. Cruz-Uribe, J.M. Martell and C. Pérez, Weights, extrapolation and theory of Rubio de Francia, Operator Theory: Advances and Applications, 2015, Birkhäuser/Springer Basel AG, Basel, 2011.
[3] D. Cruz-Uribe, A. Fiorenza, J.M. Martell and C. Pérez, The boundedness of classical operators on variable $L^p$ spaces, Ann. Acad.Sci. Fen. Math. 31(1): 239-264, 2006.
[4] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev spaces with variable exponents, Lecture Notes in Mathematics, 2017. Springer, Heidelberg (2011).
[5] J. A. Gosselin, A weighted norm inequality for Vilenkin-Fourier series, Proc. Amer. Math. Soc. 49 (1975), 349-353.
[6] Y. Jiao, D. Zhou, Z. Hao and W. Chen, Martingale hardy spaces with variable exponents. Banach J. Math. Anal., 10 (2016), no. 4, 750-770.
[7] Y. Jiao, F. Weisz, L. Wu, Lian, D. Zhou, Variable martingale Hardy spaces and their applications in Fourier analysis, Dissertationes Math. 550 (2020) pp. 1-67.
[8] T. Kopaliani. Infimal convolution and Muckenhoupt $A_{p(.)}$ condition in variable $L^p$ spaces. Arch. Math. (Basel), 89(2):185–192, 2007.
[9] A. K. Lerner, On some questions related to the maximal operator on variable $L^p$ spaces. Trans. Amer. Math. Soc., 362(8):4229-4242, 2010.
[10] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Amer. Math. Soc., 165(1972), 207-251.
[11] F. Schipp, On $L^p$-convergence of series with respect to product systems, Analysis Mathematica,2 (1976), 49-64.
[12] P. Simon,Verallgemeinerte Walsh-Fourierreihen,II, Acta Math.Acad. Sci. Hungar. 27 (1978), 41-64.
[13] P. Simon, On a maximal function, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 21 (1978), 41-44.
[14] W-S. Young, Mean convergence of generalized Walsh-Fourier series, Trans. Amer. Math. Soc. 218 (1976), 311-320.
[15] W-S. Young, Weighted norm inequalities for Vilenkin-Fourier series, Trans. Amer. Math. Soc. 340 (1993), 273-291.
[16] C. Watari, On generalized Walsh Fourier series Tohoku Math. J. (2) 10, 211–241 (1958).