Möbiubs Transformation and Einstein Velocity Addition in the Hyperbolic Geometry of Bolyai and Lobachevsky

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Abstract In this chapter, dedicated to the 60th Anniversary of Themistocles M. Rassias, Möbius transformation and Einstein velocity addition meet in the hyperbolic geometry of Bolyai and Lobachevsky. It turns out that Möbius addition that is extracted from Möbius transformation of the complex disc and Einstein addition from his special theory of relativity are isomorphic in the sense of gyrovector spaces.

1. Introduction

Einstein addition law of relativistically admissible velocities is isomorphic to Möbius addition that is extracted from the common Möbius transformation of the complex open unit disc. Accordingly, both Einstein addition and Möbius addition in the open unit ball of the Euclidean $n$-space possess the structure of a gyrovector space that forms a natural powerful generalization of the common vector space structure. Einstein and Möbius gyrovector spaces continue to attract research interest as novel algebraic settings for hyperbolic geometry, giving rise to the incorporation of Cartesian coordinates and vector algebra into the study of the hyperbolic geometry of Bolyai and Lobachevsky [68, 73]. Outstanding novel results and elegant compatibility with well-known results in hyperbolic geometry make the novel gyrovector space approach to analytic hyperbolic geometry [70] an obvious contender for augmenting the traditional way of studying hyperbolic geometry synthetically.

Professor Themistocles M. Rassias’ special predilection and contribution to the study of Möbius transformations is revealed in his work in the areas of Möbius transformations, including [23, 24, 25, 26] and [56, 61, 62], along with essential mathematical developments found, for instance, in [7, 12, 42, 45, 46, 47, 48]. The latter contain essential research on geometric transformations including Möbius transformations.

The initial purpose of this article, dedicated to the 60th Anniversary of Themistocles Rassias, is to extract Möbius addition in the ball $\mathbb{R}^n_0$ of the Euclidean $n$-space $\mathbb{R}^n$, $n \in \mathbb{N}$, from the Möbius transformation of the complex open unit disc, and to demonstrate the hyperbolic geometric isomorphism between the resulting Möbius addition and the famous Einstein velocity addition of special relativity theory. We will then see that
(1) Möbius addition in the ball $\mathbb{R}^n_c$ forms the algebraic setting for the Cartesian-Poincaré ball model of hyperbolic geometry, and

(2) Einstein addition in the ball $\mathbb{R}^n_c$ forms the algebraic setting for the Cartesian-Beltrami-Klein ball model of hyperbolic geometry, just as the common

(3) vector addition in the space $\mathbb{R}^n$ forms the algebraic setting for the standard Cartesian model of Euclidean geometry.

Remarkably, Items (1)–(3) enable Möbius addition in $\mathbb{R}^n_c$, Einstein addition in $\mathbb{R}^n_c$, and the standard vector addition in $\mathbb{R}^n$ to be studied comparatively, as in \cite{72}.

Counterintuitively, Einstein velocity addition law of relativistically admissible velocities is neither commutative nor associative. The breakdown of commutativity in Einstein addition seemed undesirable to Émile Borel in 1909. According to the historian of relativity physics Scott Walter, \cite[Sec. 10]{77}, the famous mathematician and a former doctoral student of Poincaré, Émile Borel (1871-1956) was renowned for his work on the theory of functions, in which a chair was created for him at the Sorbonne in 1909. In the years following his appointment he took up the study of relativity theory. Borel “fixed” the seemingly “defective” result that Einstein velocity addition law is noncommutative. According to Walter, Borel’s version of commutativized relativistic velocity addition involves a significant modification of Einstein’s relativistic velocity composition law.

Contrasting Borel, in this article we commutativize the Einstein velocity addition law by composing Einstein addition with an appropriate Thomas precession in a natural way suggested by analogies with the classical parallelogram addition law and supported experimentally by cosmological observations of stellar aberration.

Historically, the link between Einstein’s special theory of relativity and the non-Euclidean style was developed during the period 1908-1912 by Väisälä, Robb, Wilson and Lewis, and Borel \cite{77}. The subsequent development that followed 1912 appeared about 80 years later, in 2001, as the renowned historian Scott Walter describes in \cite{78}:

Over the years, there have been a handful of attempts to promote the non-Euclidean style for use in problem solving in relativity and electrodynamics, the failure of which to attract any substantial following, compounded by the absence of any positive results must give pause to anyone considering a similar undertaking. Until recently, no one was in a position to offer an improvement on the tools available since 1912. In his [2001] book, Ungar furnishes the crucial missing element from the panoply of the non-Euclidean style: an elegant nonassociative algebraic formalism that fully exploits the structure of Einstein’s law of velocity composition. The formalism relies on what the author calls the “missing link” between Einstein’s velocity addition formula and ordinary vector addition: Thomas precession . . .

Scott Walter, 2002 \cite{78}

Indeed, the special relativistic effect known as *Thomas precession* is mathematically abstracted into an operator called a *gyrator*, denoted “gyr”. The latter, in turn, justifies the prefix “gyro” that we extensively use in *gyrolanguage*, where we prefix a
gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and in nonassociative algebra. Thus, for instance, Einstein’s velocity addition is neither commutative nor associative, but it turns out to be both gyrocommutative and gyroassociative, giving rise to the algebraic structures known as gyrogroups and gyrovector spaces. Remarkably, the mere introduction of the gyrator turns Euclidean geometry, the geometry of classical mechanics, into hyperbolic geometry, the geometry of relativistic mechanics.

The breakdown of commutativity in Einstein velocity addition law seemed undesirable to the famous mathematician Émile Borel. Borel’s resulting attempt to “repair” the seemingly “defective” Einstein velocity addition in the years following 1912 is described by Walter in [77, p. 117]. Here, however, we see that there is no need to repair Einstein velocity addition law for being noncommutative since it suggestively gives rise to the gyroparallelogram law of gyrovector addition, which turns out to be commutative. The compatibility of the gyroparallelogram addition law of Einsteinian velocities with cosmological observations of stellar aberration is explained in [68, Chap. 13] and mentioned in [73, Sec. 10.2]. The extension of the gyroparallelogram addition law of \( k = 2 \) summands in \( \mathbb{R}^n \) to a corresponding \( k \)-dimensional gyroparallellepiped (gyroparallelotope) addition law of \( k > 2 \) summands is presented in this article and, with proof, in [68, Theorem 10.6].

2. MöBIUS ADDITION

The most general Möbius transformation of the complex open unit disc
\[
D = \{ z \in \mathbb{C} : |z| < 1 \}
\]
in the complex plane \( \mathbb{C} \) is given by the polar decomposition [1] [34],
\[
z \mapsto e^{i\theta} \frac{a + z}{1 + \overline{a}z} = e^{i\theta}(a \oplus_M z)
\]
Möbius addition \( \oplus_M \) in the disc is extracted from (2), allowing the generic Möbius transformation of the disc to be viewed as a Möbius left gyrotranslation
\[
z \mapsto a \oplus_M z = \frac{a + z}{1 + \overline{a}z}
\]
followed by a rotation. Here \( \theta \in \mathbb{R} \) is a real number, \( a, z \in D \), \( \overline{a} \) is the complex conjugate of \( a \), and \( \oplus_M \) represents Möbius addition in the disc.

Möbius addition \( a \oplus_M z \) and subtraction \( a \ominus_M z = a \ominus_M (-z) \) are found useful in the geometric viewpoint of complex analysis; see, for instance, [60, 64, 84] pp. 52–53, 56–57, 60], and the Schwarz-Pick Lemma in [21] Theorem 1.4, p. 64]. However, prior to the appearance of [63] in 2001 these were not considered ‘addition’ and ‘subtraction’ since it has gone unnoticed that, being gyrocommutative and gyroassociative, they share analogies with the common vector addition and subtraction, as we will see in the sequel.

Möbius addition \( \oplus_M \) is neither commutative nor associative. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of a gyrator
\[
\text{gyr} : D \times D \to \text{Aut}(D, \oplus_M)
\]
that generates gyroautomorphisms according to the equation
\[
\text{gyr}[a, b] = \frac{a \oplus_M b}{b \oplus_M a} = \frac{1 + ab}{1 + \overline{ab}} \in \text{Aut}(\mathbb{D}, \oplus_M)
\]
where \(\text{Aut}(\mathbb{D}, \oplus_M)\) is the automorphism group of the Möbius groupoid \((\mathbb{D}, \oplus_M)\). Here a groupoid is a nonempty set with a binary operation, and an automorphism of the groupoid \((\mathbb{D}, \oplus_M)\) is a bijective self-map \(f : \mathbb{D} \rightarrow \mathbb{D}\) of the set \(\mathbb{D}\) that respects its binary operation \(\oplus_M\), that is, \(f(a \oplus_M b) = f(a) \oplus_M f(b)\) for all \(a, b \in \mathbb{D}\). Being gyrations, the automorphisms \(\text{gyr}[a, b]\) are also called gyroautomorphisms.

The inverse of the automorphism \(\text{gyr}[a, b]\) is clearly \(\text{gyr}^{-1}[a, b] = \text{gyr}[b, a]\).

The gyration definition in (5) suggests the following gyrocommutative law of Möbius addition in the disc,
\[
a \oplus_M b = \text{gyr}[a, b](b \oplus_M a)
\]
The resulting gyrocommutative law (6) is not terribly surprising since it is generated by definition, but we are not finished.

Coincidentally, the gyroautomorphism \(\text{gyr}[a, b]\) that repairs in (7) the breakdown of commutativity, repairs the breakdown of associativity in \(\oplus_M\) as well, giving rise to the following left and right gyroassociative law of Möbius addition
\[
\begin{align*}
(a \oplus_M b \oplus_M z) &= (a \oplus_M b) \oplus_M \text{gyr}[a, b]z \\
(a \oplus_M b) \oplus_M z &= a \oplus_M (b \oplus_M \text{gyr}[b, a]z)
\end{align*}
\]
for all \(a, b, z \in \mathbb{D}\). Moreover, Möbius gyroautomorphisms possess their own rich structure obeying, for instance, the two elegant identities
\[
\begin{align*}
\text{gyr}[a \oplus_M b, b] &= \text{gyr}[a, b] \\
\text{gyr}[a, b \oplus_M a] &= \text{gyr}[a, b]
\end{align*}
\]
called the left and the right loop property.

In order to extend Möbius addition from the disc to the ball, we identify complex numbers of the complex plane \(\mathbb{C}\) with vectors of the Euclidean plane \(\mathbb{R}^2\) in the usual way,
\[
\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = u \in \mathbb{R}^2
\]
Then
\[
\bar{u}v + uv = 2u \cdot v
\]
\[
|u| = \|u\|
\]
give the inner product and the norm in \(\mathbb{R}^2\), so that Möbius addition in the disc \(\mathbb{D}\) of the complex plane \(\mathbb{C}\) becomes Möbius addition in the disc
\[
\mathbb{R}^2_{s=1} = \{ v \in \mathbb{R}^2 : \|v\| < s = 1 \}\]
of the Euclidean plane $\mathbb{R}^2$. Indeed,

$$D \ni u \oplus v := \frac{u + v}{1 + \bar{u}v}$$

$$= \frac{(1 + \bar{u}v)(u + v)}{(1 + \bar{u}v)(1 + uv)}$$

$$= \frac{(1 + \bar{u}v + u\bar{v} + |v|^2)u + (1 - |u|^2)v}{1 + \bar{u}v + u\bar{v} + |u|^2|v|^2}$$

$$= \frac{(1 + 2u \cdot v + \|v\|^2)u + (1 - \|u\|^2)v}{1 + 2u \cdot v + \|u\|^2\|v\|^2}$$

$$=: u \oplus_M v \in \mathbb{R}^2_{c=1}$$

for all $u, v \in D$ and all $u, v \in \mathbb{R}^2_{c=1}$. The last equation in (13) is a vector equation, so that its restriction to the ball of the Euclidean two-dimensional space is a mere artifact. Suggestively, we thus arrive at the following definition of Möbius addition in the ball of any real inner product space.

**Definition 1. (Möbius Addition in the Ball).** Let $\mathbb{V} = (\mathbb{V}, +, \cdot)$ be a real inner product space with a binary operation $+$ and a positive definite inner product $\cdot$ (37, p. 21); following [33], also known as Euclidean space) and let $\mathbb{V}_s$ be the $s$-ball of $\mathbb{V}$,

$$\mathbb{V}_s = \{ v \in \mathbb{V} : \|v\| < s \}$$

for any fixed $s > 0$. Möbius addition $\oplus_M$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$u \oplus_M v = \frac{(1 + \frac{2}{s}u \cdot v + \frac{1}{s^2}\|v\|^2)u + (1 - \frac{1}{s^2}\|u\|^2)v}{1 + \frac{2}{s}u \cdot v + \frac{1}{s^2}\|u\|^2\|v\|^2}$$

where $\cdot$ and $\|\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

In the limit of large $s$, $s \to \infty$, the ball $\mathbb{V}_s$ in Def. 1 expands to the whole of its space $\mathbb{V}$, and Möbius addition in $\mathbb{V}_s$ reduces to the vector addition, $+$, in $\mathbb{V}$. Accordingly, the right hand side of (15) is known as a Möbius translation [49, p. 129]. An earlier study of Möbius translation in several dimensions, using the notation $-u \oplus_M v := T_u v$, is found in [2] and in [3], where it is attributed to Poincaré. Both Ahlfors [2] and Ratcliffe [49], who studied the Möbius translation in several dimensions, did not call it a Möbius addition since it has gone unnoticed at the time that Möbius translation is regulated by algebraic laws analogous to those that regulate vector addition.

Möbius addition $\oplus_M$ in the open unit ball $\mathbb{V}_s$ of any real inner product space $\mathbb{V}$ is thus a most natural extension of Möbius addition in the open complex unit disc.

Like the Möbius disc groupoid $(\mathbb{D}, \oplus_M)$, the Möbius ball groupoid $(\mathbb{V}_s, \oplus_M)$ turns out to be a gyrocommutative gyrogroup, defined in Defs. 2 – 3 in Sec. 4, as one can check straightforwardly by computer algebra. Interestingly, the gyrocommutative law of Möbius addition was already known to Ahlfors [2, Eq. 39]. The accompanied gyroassociative law of Möbius addition, however, had gone unnoticed.
Möbius addition satisfies the gamma identity

\[
\gamma_{u \oplus_M v} = \gamma_u \gamma_v \sqrt{1 + \frac{2}{s^2} u \cdot v + \frac{1}{s^2} \|u\|^2 \|v\|^2}
\]

for all \(u, v \in V_s\), where \(\gamma_u\) is the gamma factor

\[
\gamma_u = \frac{1}{\sqrt{1 - \frac{\|u\|^2}{c^2}}}
\]

in the \(s\)-ball \(V_s\).

The gamma factor appears also in Einstein velocity addition of relativistically admissible velocities, and it is known in special relativity theory as the Lorentz gamma factor. The gamma factor \(\gamma_v\) is real if and only if \(v \in V_s\). Hence, the gamma identity (16) demonstrates that \(u, v \in V_s \Rightarrow u \oplus_M v \in V_s\) so that, indeed, Möbius addition \(\oplus_M\) is a binary operation in the ball \(V_s\).

3. Einstein Velocity Addition

Let \(c\) be any positive constant, let \((\mathbb{R}_c^n, +, \cdot)\) be the Euclidean \(n\)-space, and let

\[
R^n_c = \{v \in \mathbb{R}_c^n : \|v\| < c\}
\]

be the \(c\)-ball of all relativistically admissible velocities of material particles. It is the open ball of radius \(c\), centered at the origin of \(\mathbb{R}^n\), consisting of all vectors \(v\) in \(\mathbb{R}^n\) with magnitude \(\|v\| \) smaller than \(c\).

Einstein velocity addition in the \(c\)-ball of all relativistically admissible velocities is given by the equation [18], [40, p. 55], [50, Eq. 2.9.2], [63],

\[
u \oplus v = \frac{1}{1 + \frac{\|v\|}{c^2}} \left( u + \frac{1}{\gamma_u} v + \frac{1}{c^2} \frac{\gamma_v}{1 + \gamma_v} (u \cdot v) u \right)
\]

satisfying the gamma identity

\[
\gamma_{u \oplus v} = \gamma_u \gamma_v \left(1 + \frac{u \cdot v}{c^2}\right)
\]

for all \(u, v \in R^n_c\), where \(\gamma_u\) is the gamma factor (17),

\[
\gamma_u = \frac{1}{\sqrt{1 - \frac{\|u\|^2}{c^2}}}
\]

in the \(c\)-ball \(\mathbb{R}^n_c\).

In physical applications, \(\mathbb{R}^n = \mathbb{R}^3\) is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and \(\mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3\) is the \(c\)-ball of \(\mathbb{R}^3\) of all relativistically admissible, Einsteinian velocities. Furthermore, the constant \(c\) represents in physical applications the vacuum speed of light.

Einstein addition (19) of relativistically admissible velocities was introduced by Einstein in his 1905 paper [15] [16, p. 141] that founded the special theory of relativity. We may note here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in
the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version \( (19) \).

In full analogy with vector addition and subtraction, we use the abbreviation \( u \oplus v = u \oplus (-v) \) for Einstein subtraction, so that, for instance, \( v \ominus v = 0 \), \( \ominus v = 0 \ominus v = -v \) and, in particular,

\[
\ominus (u \oplus v) = \ominus u \ominus v
\]

and

\[
\ominus u \oplus (u \oplus v) = v
\]

for all \( u, v \) in the ball. Identity \( (22) \) is called the \textit{automorphic inverse property}, and Identity \( (23) \) is called the \textit{left cancellation law} of Einstein addition \[65, 68, 70\]. Einstein addition does not obey the immediate right counterpart of the left cancellation law \( (23) \) since, in general,

\[
\ominus (u \oplus v) \ominus v \neq u
\]

However, this seemingly lack of a right cancellation law will be repaired in \[47\], following the emergence of a second gyrogroup binary operation in Def. \[4\] below, which we introduce in order to capture analogies with classical results.

In the Newtonian limit of large \( c \), \( c \to \infty \), the ball \( \mathbb{R}^n_c \) expands to the whole of its space \( \mathbb{R}^n \), as we see from \( (18) \), and Einstein addition \( \oplus \) in \( \mathbb{R}^n_c \) reduces to the common vector addition \( + \) in \( \mathbb{R}^n \), as we see from \( (19) \) and \( (21) \).

Einstein addition is noncommutative. Indeed, \( \|u \oplus v\| = \|v \oplus u\| \), but, in general,

\[
u \oplus v \neq v \oplus u
\]

\( u, v \in \mathbb{R}^n_c \). Moreover, Einstein addition is also nonassociative since, in general,

\[
(u \oplus v) \oplus w \neq u \oplus (v \oplus w)
\]

\( u, v, w \in \mathbb{R}^n_c \).

It seems that following the breakdown of commutativity and associativity in Einstein addition some mathematical regularity has been lost in the transition from Newton velocity addition in \( \mathbb{R}^n \) to Einstein velocity addition \( (19) \) in \( \mathbb{R}^n_c \). This is, however, not the case since, as we will see in Sec. 4, the gyrator comes to the rescue \[43, 44, 63, 65, 68, 70, 78\]. Indeed, we will find in Sec. 4 that the mere introduction of gyrations endows the Einstein groupoid \( (\mathbb{R}^n_c, \ominus) \) with a grouplike rich structure \[67\] that we call a \textit{gyrocommutative gyrogroup}. Furthermore, we will find in Sec. 5 that Einstein gyrogroups admit scalar multiplication that turns them into Einstein gyrovector spaces. The latter, in turn, form the algebraic setting for the Cartesian-Beltrami-Klein ball model of hyperbolic geometry, just as Euclidean vector spaces \( \mathbb{R}^n \) form the algebraic setting for the standard Cartesian model of Euclidean geometry.

When the nonzero vectors \( u, v \in \mathbb{R}^n_c \subset \mathbb{R}^n \) are parallel in \( \mathbb{R}^n \), \( u \parallel v \), that is, \( u = \lambda v \) for some \( 0 \neq \lambda \in \mathbb{R} \), Einstein addition reduces to the Einstein addition of
parallel velocities [79, p. 50],

\[ u \oplus v = \frac{u + v}{1 + \frac{1}{c^2}||u||\cdot||v||}, \quad u \parallel v \]

which was confirmed experimentally by Fizeau’s 1851 experiment [39]. Owing to its simplicity, some books on special relativity present Einstein velocity addition in its restricted form (27) rather than its general form (19).

The restricted Einstein addition (27) is both commutative and associative. Accordingly, the restricted Einstein addition is a group operation, as Einstein noted in [15]; see [16, p. 142]. In contrast, Einstein made no remark about group properties of his addition law of velocities that need not be parallel. Indeed, the general Einstein addition (19) is not a group operation but, rather, a gyrocommutative gyrogroup operation, a structure that was discovered more than 80 years later, in 1988 [55], and is presented in Defs. 2–3 in Sec. 4.

4. Einstein Gyrogroups and Gyrations

A description of the 3-space rotation, which since 1926 [54] is named after Thomas, is found in Silberstein’s 1914 book [51]. In 1914 Thomas precession did not have a name, and Silberstein called it in his 1914 book a “certain space-rotation” [51, p. 169]. An early study of Thomas precession, made by the famous mathematician Émile Borel in 1913, is described in his 1914 book [6] and, more recently, in [52]. According to Belloni and Reina [5], Sommerfeld’s route to Thomas precession dates back to 1909. However, prior to Thomas’ discovery the relativistic peculiar 3-space rotation had a most uncertain physical status [77, p. 119]. The only knowledge Thomas had in 1925 about the peculiar relativistic gyroscopic precession [29] came from De Sitter’s formula describing the relativistic corrections for the motion of the moon, found in Eddington’s book [14], which was just published at that time [63, Sec. 1, Chap. 1].

The physical significance of the peculiar rotation in special relativity emerged in 1925 when Thomas relativistically re-computed the precessional frequency of the doublet separation in the fine structure of the atom, and thus rectified a missing factor of 1/2. This correction has come to be known as the Thomas half [9]. Thomas’ discovery of the relativistic precession of the electron spin on Christmas 1925 thus led to the understanding of the significance of the relativistic effect that became known as Thomas precession. Llewellyn Hilleth Thomas died in Raleigh, NC, on April 20, 1992. A paper [8] dedicated to the centenary of the birth of Llewellyn H. Thomas (1902–1992) describes the Bloch gyrovector of quantum information and computation.

For any \( u, v \in \mathbb{R}^n_c \), let \( \text{gyr}[u, v] : \mathbb{R}^n_c \to \mathbb{R}^n_c \) be the self-map of \( \mathbb{R}^n_c \) given in terms of Einstein addition \( \oplus \), (19), by the equation [55]

\[ \text{gyr}[u, v]w = \ominus(u \ominus v)\oplus\{u \ominus(v \ominus w)\} \]

for all \( w \in \mathbb{R}^n_c \). The self-map \( \text{gyr}[u, v] \) of \( \mathbb{R}^n_c \), which takes \( w \in \mathbb{R}^n_c \) into \( \ominus(u \ominus v)\oplus\{u \ominus(v \ominus w)\} \in \mathbb{R}^n_c \), is the gyration generated by \( u \) and \( v \). Being the mathematical abstraction of
the relativistic Thomas precession. The gyration has an interpretation in hyperbolic geometry\cite{76} as the negative hyperbolic triangle defect\cite{68} Theorem 8.55).

In the Newtonian limit, $c \to \infty$, Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the common vector addition $+$ in $\mathbb{R}^n$, which is associative. Accordingly, in this limit the gyration $\text{gyr}[u, v]$ in (28) reduces to the identity map of $\mathbb{R}^n$, called the trivial map. Hence, as expected, Thomas gyrations $\text{gyr}[u, v]$, $u, v \in \mathbb{R}^n_c$, vanish (that is, they become trivial) in the Newtonian limit.

It follows from the gyration equation (28) that gyrations measure the extent to which Einstein addition deviates from associativity, where associativity corresponds to trivial gyrations.

The gyration equation (28) can be manipulated (with the help of computer algebra) into the equation

$$\text{gyr}[u, v]w = w + \frac{Au + Bv}{D}$$

where

$$A = -\frac{1}{c^2} \left( \frac{\gamma_u^2}{\gamma_u + 1} \right) (\gamma_v - 1)(u \cdot w) + \frac{1}{c^2} \gamma_u \gamma_v (v \cdot w)$$

$$+ \frac{2}{c^4} \left( \frac{\gamma_u \gamma_v}{\gamma_u + 1} \right) (u \cdot v)(v \cdot w)$$

$$B = -\frac{1}{c^2} \frac{\gamma_v}{\gamma_v + 1} \left\{ \gamma_u (\gamma_v + 1)(u \cdot w) + (\gamma_u - 1)\gamma_v (v \cdot w) \right\}$$

$$D = \gamma_u \gamma_v \left( 1 + \frac{u \cdot v}{c^2} \right) + 1 = \gamma_u \oplus v + 1 > 1$$

for all $u, v, w \in \mathbb{R}^n_c$.

Allowing $w \in \mathbb{R}^n \supset \mathbb{R}^n_c$ in (29) – (30), that is, extending the domain of $w$ from $\mathbb{R}^n_c$ to $\mathbb{R}^n$, gyrations $\text{gyr}[u, v]$ are expendable to linear maps of $\mathbb{R}^n$ for all $u, v \in \mathbb{R}^n_c$.

In each of the three special cases when (i) $u = 0$, or (ii) $v = 0$, or (iii) $u$ and $v$ are parallel in $\mathbb{R}^n_c \subset \mathbb{R}^n$, $u \parallel v$, we have $Au + Bv = 0$ so that $\text{gyr}[u, v]$ is trivial,

$$\text{gyr}[0, v]w = w$$

$$\text{gyr}[u, 0]w = w$$

$$\text{gyr}[u, v]w = w, \quad u \parallel v$$

for all $u, v \in \mathbb{R}^n_c$, and all $w \in \mathbb{R}^n$.

It follows from (29) that

$$\text{gyr}[v, u](\text{gyr}[u, v]w) = w$$

for all $u, v \in \mathbb{R}^n_c$, $w \in \mathbb{R}^n$, so that gyrations are invertible linear maps of $\mathbb{R}^n$, the inverse of $\text{gyr}[u, v]$ being $\text{gyr}[v, u]$ for all $u, v \in \mathbb{R}^n_c$.

Gyrations keep the inner product of elements of the ball $\mathbb{R}^n_c$ invariant, that is,

$$\text{gyr}[u, v]|a \cdot \text{gyr}[u, v]|b = a \cdot b$$
for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_\mathcal{c}$. Hence, $\text{gyr}[\mathbf{u}, \mathbf{v}]$ is an isometry of $\mathbb{R}^n_\mathcal{c}$, keeping the norm of elements of the ball $\mathbb{R}^n_\mathcal{c}$ invariant,

$$\|\text{gyr}[\mathbf{u}, \mathbf{v}] \mathbf{w}\| = \|\mathbf{w}\|$$

Accordingly, $\text{gyr}[\mathbf{u}, \mathbf{v}]$ represents a rotation of the ball $\mathbb{R}^n_\mathcal{c}$ about its origin for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_\mathcal{c}$.

The invertible self-map $\text{gyr}[\mathbf{u}, \mathbf{v}]$ of $\mathbb{R}^n_\mathcal{c}$ respects Einstein addition in $\mathbb{R}^n_\mathcal{c}$,

$$\text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{a} \oplus \mathbf{b}) = \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a} \oplus \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_\mathcal{c}$, so that $\text{gyr}[\mathbf{u}, \mathbf{v}]$ is an automorphism of the Einstein groupoid $(\mathbb{R}^n_\mathcal{c}, \oplus)$. We recall that an automorphism of a groupoid $(\mathbb{R}^n_\mathcal{c}, \oplus)$ is a bijective self-map of the groupoid $\mathbb{R}^n_\mathcal{c}$ that respects its binary operation, that is, it satisfies (35). Under bijection composition the automorphisms of a groupoid $(\mathbb{R}^n_\mathcal{c}, \oplus)$ form a group known as the automorphism group, and denoted $\text{Aut}(\mathbb{R}^n_\mathcal{c}, \oplus)$. Being special automorphisms, gyrations $\text{gyr}[\mathbf{u}, \mathbf{v}] \in \text{Aut}(\mathbb{R}^n_\mathcal{c}, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_\mathcal{c}$, are also called gyroautomorphisms, $\text{gyr}$ being the gyroautomorphism generator called the gyrator.

The gyroautomorphisms $\text{gyr}[\mathbf{u}, \mathbf{v}]$ regulate Einstein addition in the ball $\mathbb{R}^n_\mathcal{c}$, giving rise to the following nonassociative algebraic laws that “repair” the breakdown of commutativity and associativity in Einstein addition:

$$\mathbf{u} \oplus \mathbf{v} = \text{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v} \oplus \mathbf{u})$$

Gyrocommutativity

$$\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$$

Left Gyroassociativity

$$\mathbf{(u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \text{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w})$$

Right Gyroassociativity

(36)

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_\mathcal{c}$. It is clear from the identities in (36) that the gyroautomorphisms $\text{gyr}[\mathbf{u}, \mathbf{v}]$ measure of the failure of commutativity and associativity in Einstein addition.

Owing to the gyrocommutative law in (36), the gyrator is recognized as the familiar Thomas precession of special relativity theory. The gyrocommutative law was already known to Silberstein in 1914 [51] in the following sense. The Thomas precession generated by $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_\mathcal{c}$ is the unique rotation that takes $\mathbf{v} \oplus \mathbf{u}$ into $\mathbf{u} \oplus \mathbf{v}$ about an axis perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$ in $\mathbb{R}^n_\mathcal{c}$, thus giving rise to the gyrocommutative law. Obviously, Silberstein did not use the terms “Thomas precession” and “gyrocommutative law” since these terms have been coined later, respectively, following Thomas’ 1926 paper [54], and by the author in 1991 [57] following the discovery of the gyrocommutative and the gyroassociative laws of Einstein addition in [55]. Thus, contrasting the discovery before 1914 of what we presently call the gyrocommutative law of Einstein addition, the gyroassociative laws of Einstein addition, left and right, were discovered by the author about 75 years later, in 1988 [55].

Thomas precession has purely kinematical origin, as emphasized in [67], so that the presence of Thomas precession is not connected with the action of any force.
A most important and useful property of gyrations is the so called loop property (left and right),

\[
\begin{align*}
\text{Left Loop Property} & \quad \text{gyr}[u \oplus v, v] = \text{gyr}[u, v] \\
\text{Right Loop Property} & \quad \text{gyr}[u, v \oplus u] = \text{gyr}[u, v]
\end{align*}
\]

for all \(u, v \in \mathbb{R}^n\). The left loop property will prove useful in (46) below in solving a basic gyrogroup equation.

Identities (36) – (37) are the basic identities of the gyroalgebra of Einstein addition. They can be verified straightforwardly by computer algebra, as explained in [63, Sec. 8].

The grouplike groupoid \((\mathbb{R}^n_+, \oplus)\) that regulates Einstein addition, \(\oplus\), in the ball \(\mathbb{R}^n_+\) of the Euclidean \(n\)-space \(\mathbb{R}^n\) is a gyrocommutative gyrogroup called an Einstein gyrogroup. Einstein gyrogroups and gyrovector spaces are studied in [63 65 68 70]. Gyrogroupes are not peculiar to Einstein addition [69]. Rather, they are abound in the theory of groups [19 20 17], loops [27], quasigroup [28 35], and Lie groups [30 31 32].

Thus, the type of structure arising in the study of Einstein velocity addition (and Möbius addition) is of rather frequent occurrence and hence merits an axiomatic approach. Taking the key features of Einstein velocity addition law as axioms, and guided by analogies with groups, we are led to the following formal definition of gyrogroups.

**Definition 2. (Gyrogroups).** A groupoid is a non-empty set with a binary operation. A groupoid \((G, \oplus)\) is a gyrogroup if its binary operation satisfies the following axioms. In \(G\) there is at least one element, 0, called a left identity, satisfying

\[(G1) \quad 0 \oplus a = a\]

for all \(a \in G\). There is an element \(0 \in G\) satisfying axiom \((G1)\) such that for each \(a \in G\) there is an element \(\ominus a \in G\), called a left inverse of \(a\), satisfying

\[(G2) \quad \ominus a \oplus a = 0\]

Moreover, for any \(a, b, c \in G\) there exists a unique element \(\text{gyr}[a, b]c \in G\) such that the binary operation obeys the left gyroassociative law

\[(G3) \quad a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c.\]

The map \(\text{gyr}[a, b] : G \to G\) given by \(c \mapsto \text{gyr}[a, b]c\) is an automorphism of the groupoid \((G, \oplus)\), that is,

\[(G4) \quad \text{gyr}[a, b] \in \text{Aut}(G, \oplus),\]

and the automorphism \(\text{gyr}[a, b]\) of \(G\) is called the gyroautomorphism, or the gyration, of \(G\) generated by \(a, b \in G\). The operator \(\text{gyr} : G \times G \to \text{Aut}(G, \oplus)\) is called the gyrator of \(G\). Finally, the gyroautomorphism \(\text{gyr}[a, b]\) generated by any \(a, b \in G\) possesses the left loop property

\[(G5) \quad \text{gyr}[a, b] = \text{gyr}[a \oplus b, b].\]

The first pair of the gyrogroup axioms are like the group axioms. The last pair present the gyration axioms and the middle axiom links the two pairs.
As in group theory, we use the notation \( a \boxplus b = a \oplus (\ominus b) \) in gyrogroup theory as well.

In full analogy with groups, some gyrogroups are gyrocommutative according to the following definition.

**Definition 3. (Gyrocommutative Gyrogroups).** A gyrogroup \((G, \oplus)\) is gyrocommutative if its binary operation obeys the gyrocommutative law

\[
(G6) \quad a \oplus b = \text{gyr}[a, b](b \oplus a)
\]

for all \(a, b \in G\).

First gyrogroup properties are studied in [73, Chap. 1], and more gyrogroup theorems are studied in [63, 65, 68]. Thus, for instance, as in group theory, any gyrogroup possesses a unique identity element which is both left and right, and any element of a gyrogroup possesses a unique inverse.

In order to illustrate the power and elegance of the gyrogroup structure, we solve below the two basic gyrogroup equations (38) and (45).

Let us consider the gyrogroup equation

\[
(38) \quad a \oplus x = b
\]

in a gyrogroup \((G, \oplus)\) for the unknown \(x\). If \(x\) exists, then by the right gyroassociative law (36) and by (31), we have

\[
x = 0 \oplus x = (\ominus a \oplus a) \oplus x = \ominus a \oplus (\ominus a \oplus (\oplus a) x)
\]

noting that \(\text{gyr}[a, \ominus a]\) is trivial by (31).

Thus, if a solution to (38) exists, it must be given uniquely by

\[
(40) \quad x = \ominus a \oplus b
\]

Conversely, if \(x = \ominus a \oplus b\), then \(x\) is indeed a solution to (38) since by the left gyroassociative law and (31) we have

\[
a \oplus x = a \oplus (\ominus a \oplus b) = (a \oplus (\ominus a)) \oplus \text{gyr}[a, \ominus a] b
\]

\[
(41) \quad = b
\]

Substituting the solution (40) in its equation (38) and replacing \(a\) by \(\ominus a\) we recover the left cancellation law (23) for Einstein addition

\[
(42) \quad \ominus a \oplus (a \oplus b) = b
\]

The gyrogroup operation (or, addition) of any gyrogroup has an associated dual operation, called the gyrogroup cooperation (or, coaddition), which is defined below.
Definition 4. (The Gyrogroup Cooperation (Coaddition)). Let \((G, \oplus)\) be a gyrogroup with gyrogroup operation (or, addition) \(\oplus\). The gyrogroup cooperation (or, coaddition) \(\ominus\) is a second binary operation in \(G\) given by the equation
\[
a \ominus b = a \oplus \text{gyr}[a, \ominus b]b
\]
for all \(a, b \in G\).

Replacing \(b\) by \(\ominus b\) in (43) we have the cosubtraction identity
\[
a \ominus b := a \ominus (\ominus b) = a \ominus \text{gyr}[a, b]b
\]
for all \(a, b \in G\).

To motivate the introduction of the gyrogroup cooperation and to illustrate the use of the left loop property \((G5)\), we solve the equation
\[
x \oplus a = b
\]
for the unknown \(x\) in a gyrogroup \((G, \oplus)\).

Assuming that a solution \(x\) to (45) exists, we have the following chain of equations
\[
x = x \oplus 0
  = x \oplus (a \ominus a)
  = (x \ominus a) \oplus \text{gyr}[x, a](\ominus a)
  = (x \ominus a) \ominus \text{gyr}[x, a]a
  = (x \ominus a) \ominus \text{gyr}[x \ominus a, a]a
  = b \ominus \text{gyr}[b, a]a
  = b \ominus a
\]
where the gyrogroup cosubtraction, (44), which captures here an obvious analogy, comes into play. Hence, if a solution \(x\) to the gyrogroup equation (45) exists, it must be given uniquely by (46). One can show that the latter is indeed a solution to (45) \([68, \text{Sec. 2.4}]\).

The gyrogroup cooperation is introduced into gyrogroups in order to capture useful analogies between gyrogroups and groups, and to uncover duality symmetries with the gyrogroup operation. Thus, for instance, the gyrogroup cooperation uncovers the seemingly missing right counterpart of the left cancellation law (23), giving rise to the right cancellation law,
\[
(b \ominus a) \ominus a = b
\]
for all \(a, a\) in \(G\), which is obtained by substituting the result of (46) into (45).

Remarkably, the right cancellation law (47) can be dualized, giving rise to the dual right cancellation law
\[
(b \ominus a) \ominus a = b
\]

As an example, and for later reference, we note that it follows from the right cancellation law (47) that
\[
d = (b \ominus c) \ominus a \iff b \ominus c = d \ominus a
\]
for \(a, b, c, d\) in any gyrocommutative gyrogroup \((G, \oplus)\).
An elegant gyrocommutative gyrogroup identity that involves the gyrogroup cooperation, verified in [68, Theorem 3.12], is
\[(50)\]
\[a \oplus (b \oplus a) = a \boxplus (a \oplus b)\]

A gyrogroup cooperation is commutative if and only if the gyrogroup is gyrocommutative [65, Theorem 3.4] [68, Theorem 3.4]. Hence, in particular, Einstein coaddition is commutative. Indeed, Einstein coaddition, \(\boxplus\), in an Einstein gyrogroup \((\mathbb{R}_n^c, \oplus)\), defined in (43), can be written as [68, Eq. 3.195]
\[
\begin{align*}
\gamma u &\oplus v = \frac{\gamma u + \gamma v}{\gamma^2 + \gamma^2 + \gamma^2 (1 + \frac{u}{\gamma v})} - 1 (\gamma u + \gamma v) \\
&= \frac{\gamma u + \gamma v}{(\gamma u + \gamma v)^2 - (\gamma u + \gamma v) + 1} (\gamma u + \gamma v) \\
&= 2 \otimes \frac{\gamma u + \gamma v}{\gamma u + \gamma v} \\
&= 2 \otimes \frac{\gamma u + \gamma v}{2 + (\gamma u - 1) + (\gamma v - 1)}
\end{align*}
\]
\[\text{u, v } \in \mathbb{G}, \text{ demonstrating that it is commutative, as expected. The symbol } \otimes \text{ in (51) represents scalar multiplication so that, for instance, } 2 \otimes v = v \oplus v, \text{ for all } v \text{ in a gyrogroup } (G, \oplus), \text{ as explained in Sec. 5 below. It turns out that Einstein coaddition } \boxplus \text{ is more than just a commutative binary operation in the ball. Remarkably, it forms the (hyperbolic) gyroparallelgram addition law in the ball, illustrated in Fig. 6.}\

The extreme sides of (51) suggest that the application of Einstein coaddition to three summands is given by the following gyroparallellelepiped addition law
\[
\begin{align*}
\text{u } \boxplus \text{v } \boxplus \text{w} &:= 2 \otimes \frac{\gamma u + \gamma v + \gamma w}{2 + (\gamma u - 1) + (\gamma v - 1)} \\
\text{u, v, w } &\in \mathbb{G}, \text{ the ternary operation } \boxplus_3 \text{ being Einstein coaddition of order three.}
\end{align*}
\]

Einstein coaddition (52) of three summands is commutative and associative in the generalized sense that it is a symmetric function of the summands. The gyroparallellelepiped that results from the gyroparallellelepiped law (52) is studied in detail in [68, Secs. 10.9–10.12].

We may note that by (51) – (52) we have \(u \boxplus_3 v \boxplus_3 0 = u \boxplus v\), as expected. However, unexpectedly we have \(u \boxplus_3 v \boxplus_3 (\oplus v) \neq u\), in general.

The extension of (52) to the Einstein coaddition of \(k\) summands, \(k > 3\), is now straightforward, giving rise to the higher dimensional gyroparallelleotope law in \(\mathbb{R}_n^c\),
\[
\begin{align*}
\text{v}_1 \boxplus_k \text{v}_2 \boxplus_k \ldots \boxplus_k \text{v}_k &:= 2 \otimes \frac{\sum_{i=1}^{k} \gamma v_i \text{v}_i}{2 + \sum_{i=1}^{k} (\gamma v_i - 1)} \\
\text{v}_k &\in \mathbb{G}, k \in \mathbb{N}, \text{ where } \boxplus_k \text{ is a } k\text{-ary operation called Einstein coaddition of order } k.\
\end{align*}
\]
An interesting study of parallelotopes in Euclidean geometry is found in [10].
In the Euclidean limit $c \to \infty$, (i) gamma factors tend to 1, and (ii) the hyperbolic scalar multiplication, $\otimes$, of a gyrovector (see Sec. 6) by 2 tends to the common scalar multiplication of a vector by 2. Hence, in the Euclidean limit, the right-hand side of (53) tends to the vector sum $\sum_{i=1}^{k} v_i$ in $\mathbb{R}^n$, as expected.

5. EINSTEIN GYROVECTOR SPACES

Let $k \otimes v$ be the Einstein addition of $k$ copies of $v \in \mathbb{R}^n_c$, that is $k \otimes v = v \oplus v \ldots \oplus v$ ($k$ terms). Then, it follows from Einstein addition (19) and straightforward algebra that

$$k \otimes v = c \left( \frac{1 + \frac{\| v \|}{c}}{1 + \frac{\| v \|}{c}} \right)^k - \left( \frac{1 - \frac{\| v \|}{c}}{1 + \frac{\| v \|}{c}} \right)^k \frac{\| v \|}{c} v$$

The definition of scalar multiplication in an Einstein gyrovector space requires analytically continuing $k$ off the positive integers, thus obtaining the following definition [59].

**Definition 5.** An Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ is an Einstein gyrogroup $(\mathbb{R}^n_c, \oplus)$, $\mathbb{R}^n_c \subset \mathbb{R}^n$, with scalar multiplication $\otimes$ given by the equation

$$r \otimes v = c \left( \frac{1 + \frac{\| v \|}{c}}{1 + \frac{\| v \|}{c}} \right)^r - \left( \frac{1 - \frac{\| v \|}{c}}{1 + \frac{\| v \|}{c}} \right)^r \frac{\| v \|}{c} v$$

$$= c \tanh(r \tanh^{-1} \frac{\| v \|}{c}) \frac{\| v \|}{c} v$$

where $r$ is any real number, $r \in \mathbb{R}$, $v \in \mathbb{R}^n_c$, $v \neq 0$, and $r \otimes 0 = 0$, and with which we use the notation $v \otimes r = r \otimes v$.

Einstein gyrovector spaces are studied in [68, Sec. 6.18] and [70]. Einstein scalar multiplication does not distribute over Einstein addition, but it possesses other properties of vector spaces. For any positive integer $n$, and for all real numbers $r, r_1, r_2 \in \mathbb{R}$, and $v \in \mathbb{R}^n_c$, we have

$$n \otimes v = v \oplus \ldots \oplus v \quad \text{n terms}$$

$$(r_1 + r_2) \otimes v = r_1 \otimes v \oplus r_2 \otimes v \quad \text{Scalar Distributive Law}$$

$$(r_1, r_2) \otimes v = r_1 \otimes (r_2 \otimes v) \quad \text{Scalar Associative Law}$$

$$r \otimes (r_1 \otimes v \oplus r_2 \otimes v) = r \otimes (r_1 \otimes v) \oplus r \otimes (r_2 \otimes v) \quad \text{Monodistributive Law}$$

in any Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$.

Any Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ inherits the common inner product and the norm from its vector space $\mathbb{R}^n$. These turn out to be invariant under gyrations,
that is,

\[ \text{gyr}[a, b]u \cdot \text{gyr}[a, b]v = u \cdot v \]

(56)

\[ \|\text{gyr}[a, b]v\| = \|v\| \]

for all \(a, b, u, v \in \mathbb{R}_c^n\).

Unlike vector spaces, Einstein gyrovector spaces \((\mathbb{R}_c^n, \oplus, \otimes)\) do not possess the distributive law since, in general,

\[ r \otimes (u \oplus v) \neq r \otimes u \oplus r \otimes v \]

(57)

for \(r \in \mathbb{R}\) and \(u, v \in \mathbb{R}_c^n\). One might suppose that there is a price to pay in mathematical regularity when replacing ordinary vector addition with Einstein addition, but this is not the case as demonstrated in [63, 65, 68], and as noted by S. Walter in [78].

Owing to the breakdown of the distributive law in gyrovector spaces, the following gyrovector space identity, called the Two-Sum Identity [68, Theorem 6.7], proves useful:

\[ 2 \otimes (u \oplus v) = u \oplus (2 \otimes v \oplus u) \]

(58)

In full analogy with the common Euclidean distance function, Einstein addition admits the gyrodistance function

\[ d_{\oplus}(A, B) = \|A \oplus B\| \]

(59)

that obeys the gyrotriangle inequality [68, Theorem 6.9]

\[ d_{\oplus}(A, B) \leq d_{\oplus}(A, P) \oplus d_{\oplus}(P, B) \]

(60)

for any points \(A, B, P \in \mathbb{R}_c^n\) in an Einstein gyrovector space \((\mathbb{R}_c^n, \oplus, \otimes)\). The gyrodistance function is invariant under the group of motions of its Einstein gyrovector space, that is, under left gyrotranslations and rotations of the space [68, Sec. 4]. The gyrotriangle inequality (60) reduces to a corresponding gyrotriangle equality,

\[ d_{\oplus}(A, B) = d_{\oplus}(A, P) \oplus d_{\oplus}(P, B) \]

(61)

if and only if point \(P\) lies between points \(A\) and \(B\), that is, point \(P\) lies on the gyrosegment \(AB\), as shown in Fig. 2. Accordingly, the gyrodistance function is gyroadditive on gyrolines, as demonstrated in (61) and illustrated graphically in Fig. 2.

Furthermore, the Einstein gyrodistance function (59) in any \(n\)-dimensional Einstein gyrovector space \((\mathbb{R}_c^n, \oplus, \otimes)\) possesses a familiar Riemannian line element. It gives rise to the Riemannian line element \(ds_c^2\) of the Einstein gyrovector space with its gyrometric (59),

\[ ds_c^2 = \|(x + dx) \ominus x\|^2 \]

(62)

\[ = \frac{c^2}{c^2 - x^2} dx^2 + \frac{c^2}{(c^2 - x^2)^2} (x \cdot dx)^2 \]

\(x \in \mathbb{R}_c^n\), where \(dx^2 = dx \cdot dx\), as shown in [68, Theorem 7.6].

Remarkably, the Riemannian line element \(ds_c^2\) in (62) turns out to be the well-known line element that the Italian mathematician Eugenio Beltrami introduced in
1868 in order to study hyperbolic geometry by a Euclidean disc model, now known as the Beltrami-Klein disc\[38, p. 220\][4]. An English translation of his historically significant 1868 essay on the interpretation of non-Euclidean geometry is found in [53]. The significance of Beltrami’s 1868 essay rests on the generally known fact that it was the first to offer a concrete interpretation of hyperbolic geometry by interpreting “straight lines” as geodesics on a surface of a constant negative curvature. Beltrami, thus, constructed a Euclidean disc model of the hyperbolic plane\[38\][53], which now bears his name along with the name of Klein.

We have thus found that the Beltrami-Klein ball model of hyperbolic geometry is regulated algebraically by Einstein gyrovector spaces with their gyrodistance function \((59)\) and Riemannian line element \((62)\), just as the standard model of Euclidean geometry is regulated algebraically by vector spaces with their Euclidean distance function and the Riemannian line element \(ds^2 = dx^2\).

In full analogy with Euclidean geometry, the unique Einstein gyroline \(L_{AB}\), Fig. 1, that passes through two given points \(A\) and \(B\) in an Einstein gyrovector space \(\mathbb{R}^n_c = (\mathbb{R}^n_c, \oplus, \odot)\) is given by the parametric equation

\[
L_{AB}(t) = A \oplus (\ominus A \oplus B) \odot t
\]

with the parameter \(t \in \mathbb{R}\). The gyroline \(L_{AB}\) passes through the point \(A\) when \(t = 0\) and, owing to the left cancellation law \([23]\), it passes through the point \(B\) when \(t = 1\).
Einstein gyrolines in the ball $\mathbb{R}^n_c$ are chords of the ball, as shown in Fig. 1. These chords of the ball turn out to be the familiar geodesics of the Beltrami-Klein ball model of hyperbolic geometry [38]. Accordingly, Einstein gyrosegments are Euclidean segments, as shown in Fig. 2. The result that Einstein gyrosegments are Euclidean segments is well exploited in [72, 73] in the use of hyperbolic barycentric coordinates for the determination of various hyperbolic triangle centers. It enables one to determine points of intersection of gyrolines by common methods of linear algebra.

The gyromidpoint $M_{AB}$ of gyrosegment $AB$, shown in Fig. 2, is the unique point of the gyrosegment that satisfies the equation $d_\oplus(M_{AB}, A) = d_\oplus(M_{AB}, B)$. It is given by each of the following equations [70, Theorem 3.33],

$$M_{AB} = A \oplus (\ominus A \oplus B) \otimes \frac{1}{2} = \frac{\gamma_A A + \gamma_B B}{\gamma_A + \gamma_B} = \frac{1}{2} \otimes (A \mp B)$$  \hspace{1cm} (64)

in full analogy with Euclidean midpoints, shown in Fig. 5. One may note that the extreme right equation in (64) appears in (51) in an equivalent form.

The endpoints of a gyroline in an Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ are the points where the gyroline approaches the boundary of the ball $\mathbb{R}^n_c$. Following (63), the endpoints $E_A$ and $E_B$ of the gyroline $L_{AB}(t)$ in Fig. 1 are

$$E_A = \lim_{t \to -\infty} \{ A \oplus (\ominus A \oplus B) \otimes t \}$$

$$E_B = \lim_{t \to -\infty} \{ A \oplus (\ominus A \oplus B) \otimes t \}$$  \hspace{1cm} (65)

Explicit expressions for the gyroline endpoints in Einstein gyrovector spaces are presented in [162], p. 44.
6. VECTORS AND GYROVECTORS

Elements of a real inner product space $\mathcal{V} = (\mathcal{V}, +, \cdot)$, called points and denoted by capital italic letters, $A, B, P, Q$, etc, give rise to vectors in $\mathcal{V}$, denoted by bold roman lowercase letters $\mathbf{u}, \mathbf{v}$, etc. Any two ordered points $P, Q \in \mathcal{V}$ give rise to a unique rooted vector $\mathbf{v} \in \mathcal{V}$, rooted at the point $P$. It has a tail at the point $P$ and a head at the point $Q$, and it has the value $-P + Q$,

$$\mathbf{v} = -P + Q$$

The length of the rooted vector $\mathbf{v} = -P + Q$ is the distance between the points $P$ and $Q$, given by the equation

$$\|\mathbf{v}\| = \| -P + Q\|$$

Two rooted vectors $-P + Q$ and $-R + S$ are equivalent if they have the same value, that is,

$$-P + Q \sim -R + S$$

if and only if $-P + Q = -R + S$

The relation $\sim$ in (68) between rooted vectors is reflexive, symmetric and transitive, so that it is an equivalence relations that gives rise to equivalence classes of rooted vectors.

Two equivalent rooted vectors in a Euclidean vector plane are shown in Fig. 3. Being equivalent in Euclidean geometry, the two vectors in Fig. 3 are parallel and they possess equal lengths.

To liberate rooted vectors from their roots we define a vector to be an equivalence class of rooted vectors. The vector $-P + Q$ is thus a representative of all rooted vectors with value $-P + Q$. Accordingly, the two vectors in Fig. 3 are equal.

A point $P \in \mathcal{V}$ is identified with the vector $-O + P$, $O$ being the arbitrarily selected origin of the space $\mathcal{V}$. Hence, the algebra of vectors can be applied to points as well. Naturally, geometric and physical properties regulated by a vector space are independent of the choice of the origin.

Let $A, B, C \in \mathcal{V}$ be three non-collinear points, and let

$$u = -A + B$$
$$v = -A + C$$

be two vectors in $\mathcal{V}$ that possess the same tail, $A$. Furthermore, let $D$ be a point of $\mathcal{V}$ given by the parallelogram condition

$$D = B + C - A$$

The quadrangle (also known as a quadrilateral; see p. 52) $ABDC$ turns out to be a parallelogram in Euclidean geometry, Fig. 5, since its two diagonals, $AD$ and $BC$, intersect at their midpoints, that is,

$$\frac{1}{2}(A + D) = \frac{1}{2}(B + C)$$

Clearly, the midpoint equality (71) is equivalent to the parallelogram condition (70).
The Parallelogram Condition: 
\[ D = B + C - A \]
\[ M_{AD} = \frac{1}{2}(A + D) \]
\[ M_{BC} = \frac{1}{2}(A + C) \]
\[ M_{ABDC} = \frac{A + B + C + D}{4} \]
\[ M_{ABDC} = M_{AD} = M_{BC} \]

\[ (-A + B) + (-A + C) = -A + D \]

\[ u + v = w \]

Figure 5. The Euclidean parallelogram and its addition law in a Euclidean vector plane \((\mathbb{R}^2, +, \cdot)\). The diagonals \(AD\) and \(BC\) of parallelogram \(ABDC\) intersect each other at their midpoints. The midpoints of the diagonals \(AD\) and \(BC\) are, respectively, \(M_{AD}\) and \(M_{BC}\), each of which coincides with the parallelogram center \(M_{ABDC}\).

The vector addition of the vectors \(u\) and \(v\) that generate the parallelogram \(ABDC\), according to (69), gives the vector \(w\) by the parallelogram addition law, Fig. 5.

\[ w := -A + D = (-A + B) + (-A + C) = u + v \]

Here, by definition, \(w\) is the vector formed by the diagonal \(AD\) of the parallelogram \(ABDC\), as shown in Fig. 5.

Vectors in the space \(V\) are, thus, equivalence classes of ordered pairs of points, Fig. 3, which add according to the parallelogram law, Fig. 5.

Gyrovectors emerge in an Einstein gyrovector space \((V_c, \oplus, \otimes)\) in a way fully analogous to the way vectors emerge in the space \(V\), where \(V_c\) is the \(c\)-ball of the space \(V\), (14).

Elements of \(V_c\), called points and denoted by capital italic letters, \(A, B, P, Q\), etc, give rise to gyrovectors in \(V_c\), denoted by bold roman lowercase letters \(u, v\), etc. Any two ordered points \(P, Q \in V_c\) give rise to a unique rooted gyrovector \(v \in V_c\), rooted at the point \(P\). It has a tail at the point \(P\) and a head at the point \(Q\), and it has the value \(\ominus P \oplus Q\),

\[ v = \ominus P \oplus Q \]
The gyrolength of the rooted gyrovector \( v = \oplus P \oplus Q \) is the gyrodistance between the points \( P \) and \( Q \), given by the equation

\[
\|v\| = \|\oplus P \oplus Q\|
\]  

(74)

Two rooted gyrovectors \( \oplus P \oplus Q \) and \( \oplus R \oplus S \) are equivalent if they have the same value, that is,

\[
\oplus P \oplus Q \sim \oplus R \oplus S \quad \text{if and only if} \quad \oplus P \oplus Q = \oplus R \oplus S
\]  

(75)

The relation \( \sim \) in (75) between rooted gyrovectors is reflexive, symmetric and transitive, so that it is an equivalence relation that gives rise to equivalence classes of rooted gyrovectors.

Two equivalent rooted gyrovectors in an Einstein gyrovector plane are shown in Fig. 4. Being equivalent in hyperbolic geometry, the two gyrovectors in Fig. 4 possess equal gyrolengths.

To liberate rooted gyrovectors from their roots we define a gyrovector to be an equivalence class of rooted gyrovectors. The gyrovector \( \oplus P \oplus Q \) is thus a representative of all rooted gyrovectors with value \( \oplus P \oplus Q \). Accordingly, the two gyrovectors in Fig. 4 are equal.

A point \( P \) of a gyrovector space \( (V_c, \oplus, \otimes) \) is identified with the gyrovector \( \oplus O \oplus P \), \( O \) being the arbitrarily selected origin of the space \( V_c \). Hence, the algebra of gyrovectors can be applied to points as well. Naturally, geometric and physical properties regulated by a gyrovector space are independent of the choice of the origin.

Let \( A, B, C \in V_c \) be three non-gyrocollinear points of an Einstein gyrovector space \( (V_c, \oplus, \otimes) \), and let

\[
\begin{align*}
u & = \oplus A \oplus B \\
v & = \oplus A \oplus C
\end{align*}
\]  

(76)

be two gyrovectors in \( V \) that possess the same tail, \( A \). Furthermore, let \( D \) be a point of \( V_c \) given by the gyroparallelogram condition

\[
D = (B \boxplus C) \oplus A
\]  

(77)

Then, the gyroquadrangle \( ABDC \) is a gyroparallelogram in the Beltrami-Klein ball model of hyperbolic geometry in the sense that its two gyrodiagonals, \( AD \) and \( BC \), intersect at their gyromidpoints, that is,

\[
\frac{1}{2} \otimes (A \boxplus D) = \frac{1}{2} \otimes (B \boxplus C)
\]  

(78)

as illustrated in Fig. 5. Clearly by (79), the gyromidpoint equality (78) is equivalent to the gyroparallelogram condition (77).

The gyrovector addition of the gyrovectors \( u \) and \( v \) that generate the gyroparallelogram \( ABDC \) gives the gyrovector \( w \) by the gyroparallelogram addition law, Fig. 6

\[
w := \oplus A \oplus D = (\oplus A \oplus B) \boxplus (\oplus A \oplus C) =: u \boxplus v
\]  

(79)
The Gyroparallelogram Condition: \( D = (B \oplus C) \ominus A \)

\[
\begin{align*}
M_{AD} &= \frac{\gamma_A + \gamma_D}{\gamma_A + \gamma_D} = \frac{1}{2} \otimes (A \oplus D) \\
M_{BC} &= \frac{\gamma_B + \gamma_C}{\gamma_B + \gamma_C} = \frac{1}{2} \otimes (A \oplus C) \\
M_{ABDC} &= \frac{\gamma_A + \gamma_B + \gamma_C + \gamma_D}{\gamma_A + \gamma_B + \gamma_C + \gamma_D}
\end{align*}
\]

\( M_{ABDC} = M_{AD} = M_{BC} \)

\[ (\ominus A \oplus B) \oplus (\ominus A \oplus C) = \ominus A \oplus D \]

\[ u \oplus v = w \]

**Figure 6.** The Einstein gyroparallelogram and its addition law in an Einstein gyrovector plane \((\mathbb{R}_2, \oplus, \otimes)\). The gyrodiagonals \( AD \) and \( BC \) of gyroparallelogram \( ABDC \) intersect each other at their gyromidpoints. Detailed studies of the gyroparallelogram and its extension to higher dimensional gyroparallelepipeds are presented in [65, 68]. The gyroparallelogram addition law plays an important role in the gyrovector space approach to hyperbolic geometry, studied in [68, 70]. The gyromidpoints of the gyrodiagonals \( AD \) and \( BC \) are, respectively, \( M_{AD} \) and \( M_{BC} \), each of which coincides with the gyroparallelogram gyrocenter \( M_{ABDC} \). The analogies that this figure shares with Fig. 5 are obvious. Along these analogies there is a remarkable disanalogy. (i) Newton velocity addition, \(+\), and the parallelogram addition, \(+\), in Fig. 5 are identically the same binary operations in \( \mathbb{R}^n \). In contrast (ii) Einstein velocity addition, \( \oplus \), and its resulting gyroparallelogram addition, \( \ominus \), in this figure are two different binary operations in the ball \( \mathbb{R}_n \). This disanalogy raises the question as to whether uniform relativistic velocities in the Universe are added according to the noncommutative Einstein velocity addition, (19), or according to the commutative Einstein gyroparallelogram addition, \( \ominus \) in (43).

Here, by definition, \( w \) is the gyrovector formed by the gyrodiagonal \( AD \) of the gyroparallelogram \( ABDC \). The gyrovector identity in (79) is explained in (82) below.

Gyrovectors in the ball \( V_c \) are, thus, equivalence classes of ordered pairs of points, Fig. 4 which add according to the gyroparallelogram law, Fig. 6.
7. GYROPARALLELOGRAM – THE HYPERBOLIC PARALLELOGRAM

In Euclidean geometry a parallelogram is a quadrangle the two diagonals of which intersect at their midpoints. In full analogy, in hyperbolic geometry a gyroparallelogram is a gyroquadrangle the two gyrodiagonals of which intersect at their gyromidpoints, as shown in Fig. 6. Accordingly, if \( A, B \) and \( C \) are any three non-gyrocollinear points (that is, they do not lie on a gyroline) in an Einstein gyrovector space, and if a fourth point \( D \) is given by the gyroparallelogram condition

\[
D = (B \boxplus C) \boxdot A
\]

then the gyroquadrangle \( ABDC \) is a gyroparallelogram, as shown in Fig. 6.

Indeed, the two gyrodiagonals of gyroquadrangle \( ABDC \) are the gyrosegments \( AD \) and \( BC \), shown in Fig. 6 the gyromidpoints of which coincide, that is,

\[
\frac{1}{2} \Box (A \boxplus D) = \frac{1}{2} \Box (B \boxplus C)
\]

where, by \( 69 \), the result in \( 81 \) is equivalent to the gyroparallelogram condition \( 80 \).

Let \( ABC \) be a gyrotriangle in an Einstein gyrovector space \( (\mathbb{R}^n_c, \oplus, \odot) \) and let \( D \) be the point that augments gyrotriangle \( ABC \) into the gyroparallelogram \( ABDC \), as shown in Fig. 6. Then, \( D \) is determined uniquely by the gyroparallelogram condition \( 80 \), obeying the gyroparallelogram addition law \( 73 \) Theorem 5.5]

\[
(\ominus A \oplus B) \boxplus (\ominus A \oplus C) = (\ominus A \oplus D)
\]

shown in Fig. 6. In full analogy with the parallelogram addition law of vectors in Euclidean geometry, \( 72 \), the gyroparallelogram addition law \( 82 \) of gyrovectors in hyperbolic geometry can be written as

\[
\mathbf{u} \boxplus \mathbf{v} = \mathbf{w}
\]

where \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \) are the gyrovectors

\[
\mathbf{u} = \ominus A \oplus B
\]

\[
\mathbf{v} = \ominus A \oplus C
\]

\[
\mathbf{w} = \ominus A \oplus D
\]

which emanate from the point \( A \) \( 68 \) Chap. 5).

In his 1905 paper that founded the special theory of relativity \( 15 \), Einstein noted that his velocity addition does not satisfy the Euclidean parallelogram law:

“Das Gesetz vom Parallelogramm der Geschwindigkeiten gilt also nach unserer Theorie nur in erster Annäherung.”

A. Einstein \( 15 \)

[English translation: Thus the law of velocity parallelogram is valid according to our theory only to a first approximation.]

Indeed, Einstein velocity addition, \( \oplus \), is noncommutative and does not give rise to an exact “velocity parallelogram” in Euclidean geometry. However, as we see in Fig. 6 Einstein velocity coaddition, \( \boxplus \), which is commutative, does give rise to an exact “velocity gyroparallelogram” in hyperbolic geometry.
The Möbius Gyroline through the points $X$ and $B$

\[ A \oplus (A \oplus B) \otimes t \]

\[-\infty < t < \infty \]

\[ \oplus = \oplus_M \]

$A$, $t = 0$

$B$, $t = 1$

**Figure 7.** The unique gyroline $L_{AB}$ in a Möbius gyrovector space $(\mathbb{R}_c^n, \oplus, \otimes)$ through two given points $A$ and $B$. The case of the Möbius gyrovector plane, when $V_c = \mathbb{R}_c^2$ is the real open unit disc, is shown.

The breakdown of commutativity in Einstein velocity addition law seemed undesirable to the famous mathematician Émile Borel. Borel's resulting attempt to "repair" the seemingly "defective" Einstein velocity addition in the years following 1912 is described by Walter in [77, p. 117]. Here, however, we see that there is no need to repair Einstein velocity addition law for being noncommutative since, despite of being noncommutative, it gives rise to the gyroparallelogram law of gyrovector addition, which turns out to be commutative. The compatibility of the gyroparallelogram addition law of Einsteinian velocities with cosmological observations of stellar aberration is studied in [68, Chap. 13] and [73, Sec. 10.2]. The extension of the gyroparallelogram addition law of $k = 2$ summands into a higher dimensional gyroparallelootope addition law of $k > 2$ summands is mentioned in (51) – (53) and studied in [68, Theorem 10.6].

8. THE ISOMORPHISM BETWEEN MÖBIUS AND EINSTEIN ADDITION

Einstein addition, $\oplus = \oplus_E$, and Möbius addition, $\oplus_M$, admit the same scalar multiplication (55), $\otimes = \otimes_E = \otimes_M$. The isomorphism between $\oplus_E$ and $\oplus_M$ is given by the identities

\[ A \oplus_E B = 2 \otimes (\frac{1}{2} \otimes A \oplus_M \frac{1}{2} B) \]

\[ A \oplus_M B = \frac{1}{2} \otimes (2 \otimes A \oplus_E 2 \otimes B) \]

\[ A, B \in (\mathbb{R}_c^n, \oplus_E, \otimes_E) \]

\[ A, B \in (\mathbb{R}_c^n, \oplus_M, \otimes_M) \]

for all $A, B \in \mathbb{R}_c^n$. 

$AB$ that links the points $A$ and $B$ in $(\mathbb{R}_c^n, \oplus, \otimes)$, with one of its generic points $P$ and its gyromidpoint $M_{AB}$. The point $P$ lies between $A$ and $B$ and, hence, obeys the gyrotriangle equality, (61).

**Figure 8.** The gyrosegment $AB$ that links the points $A$ and $B$ in $(\mathbb{R}_c^n, \oplus, \otimes)$, with one of its generic points $P$ and its gyromidpoint $M_{AB}$. The point $P$ lies between $A$ and $B$ and, hence, obeys the gyrotriangle equality, (61).
The isomorphism in (85) is not trivial owing to the result that scalar multiplication, \( \otimes \), is non-distributive, that is, it does not distribute over gyrovector addition, \( \oplus \).

As examples of the use of the isomorphism (85) let \( A_e \in (\mathbb{R}_c^n, \oplus_E, \otimes_E) \) and \( A_m \in (\mathbb{R}_c^n, \oplus_M, \otimes_M) \) be points of an Einstein and a M"obius gyrovector space that are isomorphic to each other under the isomorphism (85). Then,

\[
A_e = 2 \otimes A_m
\]
\[
A_m = \frac{1}{2} \otimes A_e
\]

(86)

It follows from (86) that

\[
\gamma_{A_e} = \gamma_{2 \otimes A_m} = 2 \gamma^2_{A_m} - 1
\]
\[
\gamma_{A_m} A_e = \gamma_{2 \otimes A_m} (2 \otimes A_m) = 2 \gamma^2_{A_m} A_m
\]

(87)

More generally, for points \( A_{i,e}, A_{j,e} \in (\mathbb{R}_c^n, \oplus_E, \otimes_E) \) in an Einstein gyrovector space and their isomorphic image \( A_{i,m}, A_{j,m} \in (\mathbb{R}_c^n, \oplus_M, \otimes_M) \) in a corresponding M"obius gyrovector space, we have, \[72\] Eq. (2.278),

\[
\gamma_{ij,e} := \gamma_{\ominus_E A_{i,e} \oplus_E A_{j,e}} = 2 \gamma^2_{\ominus_M A_{i,m} \oplus_M A_{j,m}} - 1 =: 2 \gamma^2_{ij,m} - 1
\]

and, \[72\] Eq. (2.280),

\[
\sqrt{\gamma^2_{ij,e} - 1} = 2 \gamma_{ij,m} \sqrt{\gamma^2_{ij,m} - 1}
\]

(88)

Interestingly, in the following equation we see an elegant expression that remains invariant under the isomorphism (85) between Einstein and M"obius gyrovector spaces:

\[
\frac{\gamma_{ij,e} a_{ij,e}}{\sqrt{\gamma^2_{ij,e} - 1}} = \frac{\gamma_{ij,m} a_{ij,m}}{\sqrt{\gamma^2_{ij,m} - 1}}
\]

(90)

as one can readily check, where we use the notation

\[
a_{ij,e} = \ominus_E A_{i,e} \oplus_E A_{j,e}
\]
\[
a_{ij,m} = \ominus_M A_{i,m} \oplus_M A_{j,m}
\]

(91)

A study in detail of the isomorphism between Einstein and M"obius gyrovector spaces is found in \[68\] Sec. 6.21 and \[72\] Sec. 2.29.

Owing to the isomorphism between Einstein and M"obius addition in \( \mathbb{R}_c^n \), the triples \((\mathbb{R}_c^n, \oplus_M, \otimes)\) form M"obius gyrovector spaces just as the triples \((\mathbb{R}_c^n, \oplus_E, \otimes)\) form Einstein gyrovector spaces. We will now show in \[72\]–\[74\] below that the isomorphic image of an Einstein gyroline \( P_e(t) \) in an Einstein gyrovector space \((\mathbb{R}_c^n, \oplus_E, \otimes)\) is a M"obius gyroline \( P_m(t) \) in a corresponding M"obius gyrovector space \((\mathbb{R}_c^n, \oplus_M, \otimes)\).
Let
\[ P_e(t) = A_e \oplus_E (\ominus_E A_e \ominus_E B_e) \otimes t \]
(92)
\( t \in \mathbb{R} \), be the gyroline that passes through the distinct points \( A_e, B_e \in \mathbb{R}^n_c \) in an Einstein gyrovector space \((\mathbb{R}^n_c, \oplus_E, \otimes_E)\), shown in Fig. 1, p. 17 for \( n = 2 \). Furthermore, let \( A_m, B_m, P_m \in \mathbb{R}^n_c \) be the respective isomorphic images of the points \( A_e, B_e, P_e \in \mathbb{R}^n_c \) in (92) under the isomorphism expressed in (85)–(86). In the following chain of equations, which are numbered for subsequent explanation, we determine the isomorphic image of the Einstein gyroline (92) in the corresponding Möbius gyrovector space \((\mathbb{R}^n_c, \oplus_M, \otimes_M)\).

\[
\begin{align*}
2 \otimes P_m(t) &\overset{(1)}{=} 2 \otimes A_m \ominus_M (\ominus_M 2 \otimes A_m \otimes_M 2 \otimes B_m) \otimes t \\
&\overset{(2)}{=} 2 \otimes A_m \ominus_M (2 \otimes (-A_m) \otimes_M 2 \otimes B_m) \otimes t \\
&\overset{(3)}{=} 2 \otimes A_m \ominus_M 2 \otimes \{( -A_m \otimes_M B_m ) \} \otimes t \\
&\overset{(4)}{=} 2 \otimes A_m \ominus_M 2 \otimes \{( -A_m \otimes_M B_m ) \} \\
&\overset{(5)}{=} 2 \otimes \{ A_m \ominus_M (-A_m \otimes_M B_m ) \} \\
&\overset{(6)}{=} 2 \otimes \{ A_m \ominus_M (\ominus_M A_m \otimes_M B_m ) \} \otimes t
\end{align*}
\]
(93)
so that, finally, the two extreme sides of (93) give the equation
\[
P_m(t) = A_m \ominus_M (\ominus_M A_m \otimes_M B_m) \otimes t
\]
(94)

Derivation of the numbered equalities in (93) follows:

(1) This equation follows from (92) and (86), where the equations \( P_e = 2 \otimes P_m \), \( A_e = 2 \otimes A_m \) and \( B_e = 2 \otimes B_m \) that result from (86) are substituted into (92).

(2) Follows from Item (1) since the unary operations \( \ominus_E \) and \( - \) are identically the same in Einstein gyrovector spaces, and since \( -2 \otimes A_m = 2 \otimes (-A_m) \).

(3) Follows from Item (2) by the first identity in (85) applied to the second binary operation \( \ominus_E \) in Item (2).

(4) Follows from Item (3) by the scalar associative law of gyrovector spaces.

(5) Follows from Item (4) by the first identity in (85) applied to the remaining binary operation \( \ominus_M \) in Item (4).

(6) Follows from Item (5) since the unary operations \( \ominus_M \) and \( - \) are identically the same in Möbius gyrovector spaces.

A Möbius gyroline in a Möbius gyrovector plane \((\mathbb{R}^2_c, \oplus, \otimes)\) is shown in Fig. 7. Interestingly, a Möbius gyroline that does not pass through the center of the disc \( \mathbb{R}^2_c \) is a circular arc that approaches the boundary of the disc orthogonally. This feature of the Möbius gyroline indicates that Möbius gyrovector spaces form the
algebraic setting for the Poincaré ball model of hyperbolic geometry. The link between Einstein and Möbius gyrovector spaces and differential geometry is presented in [66].

As in (59)–(60), but now with \(\oplus = \oplus_M\), Möbius addition \(\oplus\) admits the gyrodistance function

\[
d_{\oplus}(A, B) = \|\ominus A \oplus B\|
\]

that obeys the gyrotriangle inequality [68, Theorem 6.9]

\[
d_{\oplus}(A, B) \leq d_{\oplus}(A, P) \oplus d_{\oplus}(P, B)
\]

for any \(A, B, P \in \mathbb{R}^n\) in a Möbius gyrovector space \((\mathbb{R}^n, \oplus, \otimes)\). Möbius gyrodistance function is invariant under the group of motions of its Möbius gyrovector space, that is, under left gyrotranslations and rotations of the space [68, Sec. 4]. The gyrotriangle inequality (96) reduces to a corresponding gyrotriangle equality,

\[
d_{\oplus}(A, B) = d_{\oplus}(A, P) \oplus d_{\oplus}(P, B)
\]

if and only if point \(P\) lies between points \(A\) and \(B\), that is, point \(P\) lies on the gyrosegment \(AB\), as shown in Fig. 8. Accordingly, the gyrodistance function is gyroadditive on gyrolines, as demonstrated in (97) and illustrated graphically in Fig. 8.

The one-to-one relationship between Möbius gyrodistance function (95) and the standard Poincaré distance function in the Poincaré ball model of hyperbolic geometry is presented in [68, Sec. 6.17].

Einstein coaddition, \(\boxplus = \boxplus_E\), in the ball, defined in (43), is commutative as shown in (51). Its importance stems from analogies with classical results that it captures. In particular, it proves useful in solving the gyrogroup equation (45), in the determination of gyromidpoints in (64), and in the formulation of the gyroparallelogram addition law in (82) and in Fig. 6.

9. MöBIUS COADDITION

We now wish to determine Möbius coaddition in the ball \(\mathbb{R}^n_c\) by means of the isomorphism between Möbius and Einstein gyrovector spaces. Let \(u_e, v_e, w_e \in (\mathbb{R}^n_c, \oplus_E, \otimes)\) be three elements of an Einstein gyrovector space such that

\[
w_e = u_e \boxplus_E v_e
\]

and let \(u_m, v_m, w_m \in (\mathbb{R}^n_c, \oplus_M, \otimes)\) be the corresponding elements of the corresponding Möbius gyrovector space. Then,

\[
w_m = u_m \boxplus_M v_m
\]

where Möbius coaddition \(\boxplus_M\) in \((\mathbb{R}^n_c, \oplus_M, \otimes)\) is determined from Einstein coaddition \(\boxplus_E\) in the following chain of equations, which are numbered for subsequent
explanation.

\[
u_m \boxplus_M v_m \underbrace{\longrightarrow}_{(1)} w_m
\]

\[
\underbrace{\frac{1}{2} \otimes w_e}_{(2)}
\]

\[
\underbrace{\frac{1}{2} \otimes (u_e \boxplus_E v_e)}_{(3)}
\]

\[
\underbrace{\frac{1}{2} \otimes \left\{ 2 \otimes \frac{\gamma u_e u_e + \gamma v_e v_e}{\gamma u_e + \gamma v_e} \right\}}_{(4)}
\]

\[
\underbrace{\gamma u_e + \gamma v_e}_{(5)}
\]

\[
\underbrace{\gamma u_e + \gamma v_e}_{(6)}
\]

\[
\underbrace{\frac{2 \gamma^2 u_m + 2 \gamma^2 v_m}{2 \gamma^2 u_m - 1 + 2 \gamma^2 v_m - 1}}_{(7)}
\]

\[
(100)
\]

Derivation of the numbered equalities in (100) follows:

(1) The equation in Item (1) is (99).

(2) The equation in Item (2) follows from the isomorphism (86) between \( w_m \) in a Möbius gyrovector space \((\mathbb{R}^n_c, \oplus_M, \otimes)\) and its isomorphic image \( w_e \) in the isomorphic Einstein gyrovector space \((\mathbb{R}^n_c, \oplus_E, \otimes)\).

(3) Follows from (2) by assumption (98).

(4) Follows from (3) by (51).

(5) Follows from (4) by the scalar associative law of gyrovector spaces, Sec. 5.

(6) Follows from (5) by (87).

Hence, by (100), Möbius coaddition \( \boxplus_M \) in a Möbius gyrovector space \((\mathbb{R}^n_c, \oplus_M, \otimes)\) is given by the equation

\[
u \boxplus_M v = \frac{\gamma^2 u + \gamma^2 v}{\gamma^2 u + \gamma^2 v - 1}
\]

for all \( u, v \in \mathbb{R}^n_c \).

In order to extend (100) from Möbius coaddition of order two to any order \( k \), \( k > 2 \), we rewrite (53) in the form

\[
w_e := v_{1,e} \boxplus_{E,k} v_{2,e} \boxplus_{E,k} \ldots \boxplus_{E,k} v_{k,e} = 2 \otimes \frac{\sum_{i=1}^{k} \gamma v_{i,e} v_{i,e}}{2 + \sum_{i=1}^{k} (\gamma v_{i,e} - 1)}
\]

where \( v_{i,e} \in (\mathbb{R}^n_c, \oplus_E, \otimes), i = 1, \ldots, k, \) are \( k \) elements of an Einstein gyrovector space and where \( w_e \in (\mathbb{R}^n_c, \oplus_E, \otimes) \) is their cosum, \( \boxplus_{E,k} \) being the Einstein \( k \)-ary cooperation, that is, the Einstein cooperation of order \( k \).
Let $v_{i,m}, i = 1, \ldots, k$, and $w_m$ be the respective isomorphic images of $v_{i,e}$, and $w_e$ in the corresponding Möbius gyrovector space $(\mathbb{R}_c^n, \oplus_M, \otimes)$, under isomorphism $\Theta$. Then,

\[
(103) \quad w_m = v_{1,m} \Box_{M,k} v_{2,m} \Box_{M,k} \cdots \Box_{M,k} v_{k,m}
\]

where Möbius coaddition of order $k$, $\Box_{M,k}$, is to be determined in the chain of equations below, which are numbered for subsequent interpretation:

\[
\begin{align*}
(1) \quad v_{1,m} \Box_{M,k} v_{2,m} \Box_{M,k} \cdots \Box_{M,k} v_{k,m} & \Longrightarrow w_m \\
(2) \quad \frac{1}{2} \otimes w_e \\
(3) \quad \frac{1}{2} \otimes (v_{1,e} \Box_{E,k} v_{2,e} \Box_{E,k} \cdots \Box_{E,k} v_{k,e}) \\
(4) \quad \frac{1}{2} \otimes \left\{ \frac{2 \otimes \sum_{i=1}^{k} \gamma v_{i,e} v_{i,e}^2}{2 + \sum_{i=1}^{k} (\gamma v_{i,e} - 1)} \right\} \\
(5) \quad \frac{\sum_{i=1}^{k} \gamma v_{i,e} v_{i,e}}{2 + \sum_{i=1}^{k} (\gamma v_{i,e} - 1)} \\
(6) \quad \frac{2 \sum_{i=1}^{k} \gamma v_{i,e} v_{i,e}^2}{2 + \sum_{i=1}^{k} (\gamma v_{i,e} - 1)} \\
\end{align*}
\]

Derivation of the numbered equalities in (100) follows:

1. The equation in Item (1) is (103).
2. The equation in Item (2) follows from the isomorphism $\Theta$ between $w_m$ in a Möbius gyrovector space $(\mathbb{R}_c^n, \oplus_M, \otimes)$ and its isomorphic image $w_e$ in the isomorphic Einstein gyrovector space $(\mathbb{R}_c^n, \oplus_E, \otimes)$.
3. Follows from (2) by the assumption in (102).
4. Follows from (3) by the equation in (102).
5. Follows from (4) by the scalar associative law of gyrovector spaces, Sec. 5.
6. Follows from (5) by (87).

Hence, by (104), Möbius coaddition of order $k$, $\Box_{M,k}$ in a Möbius gyrovector space $(\mathbb{R}_c^n, \oplus_M, \otimes)$ is given by the equation

\[
(105) \quad v_{1,m} \Box_{M,k} v_{2,m} \Box_{M,k} \cdots \Box_{M,k} v_{k,m} = \frac{\sum_{i=1}^{k} \gamma v_{i,e}^2 v_{i,m}}{1 + \sum_{i=1}^{k} (\gamma v_{i,m}^2 - 1)}
\]

for all $v_{i,m} \in (\mathbb{R}_c^n, \oplus_M, \otimes), i = 1, \ldots, k.$
10. Möbius double-gyroline

**Theorem 6.** Let $A, B \in \mathbb{R}^n_c$ be any two distinct points of a Möbius gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$, and let

\[ L_{AB}(t) = A \oplus (\ominus A \oplus B) \otimes t \]

for $t \in \mathbb{R}$, be the gyroline that passes through these points. Then,

\[ 2 \otimes L_{AB}(t) = A \oplus L_{AB}(2t) \]

**Proof.** Let

\[ F_1(t) = (\ominus A \oplus B) \otimes t \]
\[ F_2(t) = 2 \otimes F_1(t) \]

so that we have, by the scalar associative law of gyrovector spaces,

\[ F_2(t) = 2 \otimes F_1(t) \]
\[ = 2 \otimes (\ominus A \oplus B) \otimes t \]
\[ = (\ominus A \oplus B) \otimes (2t) \]
\[ = F_1(2t) \]

Hence, by (108) – (109), (107) can be written equivalently as

\[ 2 \otimes (A \oplus F_1(t)) = A \oplus (A \oplus F_1(2t)) = A \oplus (A \oplus F_2(t)) \]

so that instead of verifying (107) we can, equivalently, verify (110).

The proof of (110) is given by the following chain of equations, which are numbered for subsequent derivation.

\[ 2 \otimes (A \oplus F_1(t)) \overset{(1)}{=} A \oplus (2 \otimes F_1 \oplus A) \]
\[ \overset{(2)}{=} A \oplus (F_2 \oplus A) \]
\[ \overset{(3)}{=} A \oplus (A \oplus F_2) \]

as desired.

Derivation of the numbered equalities in (111) follows:

1. Follows from the Two-Sum Identity [58].
2. Follows from (108).
3. Follows from [50].

We may remark that in the Euclidean limit, when the radius $c$ of the ball $\mathbb{R}_c^n$ tends to $\infty$, the ball expands to the whole of its Euclidean $n$-space $\mathbb{R}^n$, both Möbius addition $\oplus$ and coaddition $\boxplus$ in the ball $\mathbb{R}_c^n$ reduce to the common vector addition.
Figure 9. $A$ and $B$ are any two given distinct points of a Möbius gyrovector space $(\mathbb{R}^n_\circ, \oplus, \otimes)$. The gyroline that passes through the points $A, B \in \mathbb{R}^n_\circ$ is $L_{AB}(t)$, $-\infty < t < \infty$, and its corresponding double-gyroline is $2 \otimes L_{AB}(t)$, so that it passes through the points $2 \otimes A, 2 \otimes B \in \mathbb{R}^n_\circ$. The latter turns out to be the Euclidean straight line in the ball that passes through the points $2 \otimes A$ and $2 \otimes B$. Furthermore, the double-gyroline $2 \otimes L_{AB}(t)$, parametrized by $t$, is identical with the cogyrotranslation by $A$, $A \oplus L_{AB}(2t)$, of its gyroline, parametrized by $2t$, as shown here for $n = 2$.

Thus, we see once again that in order to capture analogies with classical results, both gyrogroup operation and cooperation must be considered.

Theorem \textbf{6} suggests the following definition:

**Definition 7. (Möbius double-gyroline).** Let $A, B \in \mathbb{R}^n_\circ$ be two distinct points of a Möbius gyrovector space $(\mathbb{R}^n_\circ, \oplus, \otimes)$, and let

$$L_{AB}(t) = A \oplus (\ominus A \oplus B) \otimes t$$

$t \in \mathbb{R}$, be the gyroline that passes through these points. The Möbius double-gyroline $P_{AB}(t)$ of gyroline $L_{AB}(t)$ is the curve given by the equation

$$P_{AB}(t) = 2 \otimes L_{AB}(t)$$

$t \in \mathbb{R}$. 

+ in the space $\mathbb{R}^n_\circ$, and Identity (107) of Theorem 6 in the ball $\mathbb{R}^n_\circ$ reduces to the trivial identity in $\mathbb{R}^n$,

$$2[A + (-A + B)t] = A + [A + (-A + B)2t]$$

(112)
Following Def. 7, the gyrovector space identity (107) of Theorem 6 states that the double-gyroline of a given gyroline that passes through a point \( A \) in a Möbius gyrovector space coincides with the cogyrotranslation by \( A \) of the gyroline.

Remarkably, the double-gyroline of a given gyroline in a Möbius gyrovector space turns out to be the supporting chord of the gyroline, as shown in Fig. 9 and studied in Sec. 11.

Identity (107) of Theorem 6 can be written, equivalently, as

\[
L_{AB}(t) = \frac{1}{2} \otimes (A \boxplus L_{AB}(2t))
\]

Let \( P(t) \) be a generic point on a gyroline \( L_{AB}(t) \) for some value \( t \) of the gyroline parameter \( t \), so that \( P(0) = A \) and \( P(1) = A \). Then, (115) implies the equation

\[
P(t) = \frac{1}{2} \otimes P(0) \boxplus P(2t)
\]

\( t \in \mathbb{R} \). Equation (116), in turn, demonstrates that any point \( P(t) \) of a gyroline \( L_{AB}(t) \) is the midpoint of the points \( P(0) = A \) and \( P(2t) \) of the gyroline, as explained in (64), p. 18.

11. Euclidean straight lines in Möbius gyrovector spaces

Euclidean straight lines (lines, in short) appear naturally in Einstein gyrovector space balls where they form gyrolines, as shown in Fig. 11. In this section we employ the isomorphism (85) between Einstein and Möbius gyrovector spaces for the task of expressing lines in Möbius gyrovector spaces.

Let \( A, B \in (\mathbb{R}^n_c, \oplus_{\mathbb{M}}, \otimes) \) be two distinct points of a Möbius gyrovector space. We know that the unique gyroline in an Einstein gyrovector spaces \((\mathbb{R}^n_c, \oplus_{\mathbb{E}}, \otimes)\) that passes through the points \( A, B \in \mathbb{R}^n_c \) is the set of point \( P_{AB}(t) \) given by

\[
P_{AB}(t) = A \oplus_{\mathbb{E}} (\ominus_{\mathbb{E}} A \oplus_{\mathbb{E}} B) \otimes t
\]

\( t \in \mathbb{R} \). It is the intersection of a line and the ball \( \mathbb{R}^n_c \), as shown in Fig. 11 for the disc \( \mathbb{R}^2_c \). This line passes through the point \( A \) when \( t = 0 \) and through the point \( B \) when \( t = 1 \).

Unlike Einstein gyrolines, which are line segments, Möbius gyrolines are Euclidean circular arcs that intersect the boundary of the ball \( \mathbb{R}^n_c \) orthogonally, as shown in Fig. 11 for the disc \( \mathbb{R}^2_c \). In order to accomplish the task we face in this section, in the following chain of equations (118) we express (117) in terms of Möbius addition \( \oplus_{\mathbb{M}} \) rather than Einstein addition \( \oplus_{\mathbb{E}} \), noting that both Einstein and Möbius scalar multiplication \( \otimes \) are identically the same, as remarked in Sec. 8. Starting from (117), we have the following chain of equations, which are numbered for subsequent derivation:
\[ P_{AB}(t) = A \oplus_E (\ominus_E A \ominus_E B) \otimes t \]

\[ \overset{1}{=} A \oplus_E ((-A) \ominus_E B) \otimes t \]
\[ \overset{2}{=} 2 \otimes \{ \frac{1}{2} \otimes A \ominus_M \frac{1}{2} \otimes ((-A) \ominus_E B) \otimes t \} \]
\[ \overset{3}{=} \frac{1}{2} \otimes A \ominus_M \{ ((-A) \ominus_E B) \otimes t \ominus_M \frac{1}{2} \otimes A \} \]
\[ \overset{4}{=} \frac{1}{2} \otimes A \ominus_M \{ 2 \otimes ((-\frac{1}{2} \otimes A) \ominus_M \frac{1}{2} B) \otimes t \ominus_M \frac{1}{2} \otimes A \} \]
\[ \overset{5}{=} \frac{1}{2} \otimes A \ominus_M \{ [(-\frac{1}{2} \otimes A) \ominus_M (B \ominus_M (-\frac{1}{2} \otimes A))] \otimes t \ominus_M \frac{1}{2} \otimes A \} \]
\[ \overset{6}{=} \frac{1}{2} \otimes A \ominus_M \{ \frac{1}{2} \otimes A \ominus_M [(-\frac{1}{2} \otimes A) \ominus_M (B \ominus_M (-\frac{1}{2} \otimes A))] \otimes t \} \]
\[ \overset{7}{=} \frac{1}{2} \otimes A \ominus_M \{ \frac{1}{2} \otimes A \ominus_M [\ominus_M \frac{1}{2} \otimes A \ominus_M (B \ominus_M \frac{1}{2} \otimes A)] \otimes t \} \]

(118)

Hence, by (118),

(119) \[ P_{AB}(t) = \frac{1}{2} \otimes A \ominus_M \{ \frac{1}{2} \otimes A \ominus_M [\ominus_M \frac{1}{2} \otimes A \ominus_M (B \ominus_M \frac{1}{2} \otimes A)] \otimes t \} \]

Derivation of the numbered equalities in (118) follows:

1. Follows from the result that \( \ominus_E A = -A \) (as well as \( \ominus_M A = -A \); see Item 7 below).
2. Follows from isomorphism (85) between \( \oplus_E \) and \( \oplus_M \), applying the isomorphism to the first \( \oplus_E \) in (1).
3. Follows from the Two-Sum Identity, (58).
4. Again, follows from isomorphism (85) between \( \oplus_E \) and \( \oplus_M \), as in Item 2, now applying the isomorphism to the remaining \( \oplus_E \) in (3).
5. Again, follows from the gyrogroup Two-Sum Identity, as in Item 3.
6. Follows from the gyrogroup identity (50),

(120) \[ A \oplus (B \oplus A) = A \ominus (A \oplus B) \]

7. Follows from the result that \( \ominus_M A = -A \) (as well as \( \ominus_E A = -A \); see Item 11 above).

In both (117) and (119), the set of points \( P_{AB}(t), t \in \mathbb{R} \), forms a line in the ball \( \mathbb{R}^n_+ \) of the Möbius gyrovector space \( (\mathbb{R}^n_+, \ominus_M, \otimes) \), where the points \( A \) and \( B \) lie. In (117) this line is expressed in terms of operations of Einstein gyrovector spaces while in (119) this line is expressed in terms of operations of Möbius gyrovector spaces, obtained by means of isomorphism (85) between Einstein and Möbius gyrovector spaces. By (119) we have the following theorem:

**Theorem 8.** Let \( A \) and \( B \) be two distinct points in a Möbius gyrovector space \( (\mathbb{R}^n_+, \ominus_M, \otimes) \). The unique line that passes through these points, Fig. 11, is given by
the equation
\[ P_{AB}(t) = \frac{1}{2} \otimes A \boxplus_M \left\{ \frac{1}{2} \otimes A \oplus_M (\ominus_M \frac{1}{2} \otimes A \oplus_M (B \ominus_M \frac{1}{2} \otimes A) \otimes t \right\} \]

Let \( A, B \in \mathbb{R}_c^n \) be any two distinct points in a Möbius gyrovector space \((\mathbb{R}_c^n, \oplus_M, \otimes_M)\), and let \( L_{\frac{1}{2} \otimes A, B \oplus \frac{1}{2} \otimes A}(t) \), \( t \in \mathbb{R} \), be the unique gyroline through the points \( \frac{1}{2} \otimes A \) and \( B \ominus \frac{1}{2} \otimes A \). Then, as shown in Fig. 7, the gyroline is given by the equation
\[ L_{\frac{1}{2} \otimes A, B \oplus \frac{1}{2} \otimes A}(t) = \frac{1}{2} \otimes A \oplus [\ominus \frac{1}{2} \otimes A \oplus (B \ominus \frac{1}{2} \otimes A)] \otimes t \]
so that (121) can be written as
\[ P_{AB}(t) = \frac{1}{2} \otimes A \boxplus L_{\frac{1}{2} \otimes A, B \oplus \frac{1}{2} \otimes A}(t) \]

The line \( P_{AB}(t) \) of Theorem 8 in (121) is recognized by means of (122) – (123) as the cogyrotranslation by \( \frac{1}{2} \otimes A \) of the Möbius gyroline \( L_{\frac{1}{2} \otimes A, B \oplus \frac{1}{2} \otimes A}(t) \).

As shown in Fig. 12, the line \( P_{AB}(t) \) is the supporting chord of the gyroline \( L_{\frac{1}{2} \otimes A, B \oplus \frac{1}{2} \otimes A}(t) \).

Let
\[ C = \frac{1}{2} \otimes A \]
\[ D = B \ominus \frac{1}{2} \otimes A \]

Then, by the scalar associative law of gyrovector spaces and by the right cancellation law (48), we have
\[ A = 2 \otimes C \]
\[ B = D \boxplus \frac{1}{2} \otimes A = D \boxplus C = C \boxplus D \]
so that (123) can be written as
\[ P_{2 \otimes C, C \boxplus D} = C \boxplus L_{CD}(t) \]
thus leading to the following theorem:

**Theorem 9.** Let \( C, D \in \mathbb{R}_c^n \) be two distinct points in a Möbius gyrovector space \((\mathbb{R}_c^n, \oplus_M, \otimes_M)\), and let
\[ L_{CD}(t) = C \oplus (C \boxplus D) \otimes t \]
\( t \in \mathbb{R} \), be the gyroline that passes through the points \( C \) and \( D \). Then, the supporting chord of gyroline \( L_{CD}(t) \) is the line given by the cogyrotranslation of the gyroline by \( C \),
\[ C \boxplus L_{CD}(t) \]

Furthermore, the supporting chord passes through the points \( P_1, P_2, P_3 \), Fig. 10,
where
\[ P_1 = C \boxplus C = 2 \otimes C \]
\[ P_2 = D \boxplus D = 2 \otimes D \]
\[ P_3 = C \boxplus D \]
Let \( Q = \ominus_{M} A \oplus_{M} B \) so that, by the gyrogroup left cancellation law \((22)\), \( A \oplus_{M} Q = B \). Restricting the line parameter \( t \in \mathbb{R} \) to \( t \geq 0 \), we obtain the Euclidean ray (ray, in short) \( P_{AB}(t), t \geq 0 \). It is the ray with edge \( A \) that contains the point \( A \oplus_{M} Q = B \), which is the right gyrotranslation of \( A \) by \( Q \). As such, it contains the sequence of all successive right gyrotranslation of \( A \) by \( Q \), that is, the sequence \( P_{0} = A, P_{1} = A \oplus_{M} Q, P_{2} = (A \oplus_{M} Q) \oplus_{M} Q, P_{3} = ((A \oplus_{M} Q) \oplus_{M} Q) \oplus_{M} Q, \) etc, as shown in Fig. 11.

The points \( P_{k}, k = 1, 2, 3, \ldots \) lie on the ray \( P_{AB}(t), t \geq 0 \), as shown in Fig. 11 and as observed in [63, Figs. 6.3-6.5]. Owing to the left loop property of gyrations in gyrogroup Axiom (G5), we have

\[
(130) \quad \text{gyr}[\ominus_{M} A \oplus_{M} B, P_{k}] = \text{gyr}[\ominus_{M} A, B]
\]

for all \( k = 0, 1, 2, 3, \ldots \). As an example, the proof of (130) for \( k = 0 \) and \( k = 1 \) follows:
By the left loop property of gyrations and by the gyrogroup left cancellation law\(^{(23)}\) we have in any gyrogroup \((G, \oplus)\),
\[
\text{gyr}[\ominus A \oplus B, P_0] = \text{gyr}[\ominus A \oplus B, A] \\
= \text{gyr}[\ominus A \oplus B, A \oplus (\ominus A \oplus B)] \\
= \text{gyr}[\ominus A \oplus B, B] \\
= \text{gyr}[\ominus A, B]
\]
(131)
and
\[
\text{gyr}[\ominus A \oplus B, P_1] = \text{gyr}[\ominus A \oplus B, A \oplus Q] \\
= \text{gyr}[\ominus A \oplus B, A \oplus (\ominus A \oplus B)] \\
= \text{gyr}[\ominus A \oplus B, B] \\
= \text{gyr}[\ominus A, B]
\]
(132)

The validity of (130) for all \(k = 0, 1, 2, 3, \ldots\) suggests the conjecture that (130) is valid not only for the points of the sequence \(\{P_0 = A, P_1 = B, P_2, P_3, \ldots\}\) that lie on the ray \(P_{AB}(t), t \geq 0\), as shown in Fig. 11 but for all the points of the ray,
Figure 12. The Euclidean straight line (line, in short) $P_{AB}(t)$, $-\infty < t < \infty$, (123), that passes through the point $A$ and $B$ in a Möbius gyrovector plane $(\mathbb{R}^2_+, \odot, \otimes)$ is shown. It is the supporting chord of the gyroline that passes through the points $\frac{1}{2}A$ and $B \odot \frac{1}{2}A$ in the Möbius gyrovector plane. The two endpoints of both the line and the gyroline, corresponding to $t \to \pm \infty$, are $E_A$ and $E_B$.

that is,

\begin{align}
(133) \quad \text{gyr} \left[ \odot \mu A \oplus \mu B, P_{AB}(t) \right] &= \text{gyr} \left[ \odot \mu A, B \right]
\end{align}

for all $t \geq 0$. Numerical experiments support the conjecture.

12. Euclidean Barycentric Coordinates

In order to set the stage for the introduction of hyperbolic barycentric coordinates, we present here the notion of Euclidean barycentric coordinates that dates back to Möbius’ 1827 book titled “Der Barycentrische Calcul” (The Barycentric Calculus). The word barycenter means center of gravity, but the book is entirely geometrical and, hence, called by Jeremy Gray [22], Möbius’s Geometrical Mechanics. The 1827 Möbius book is best remembered for introducing a new system of coordinates, the barycentric coordinates. The use of barycentric coordinates in Euclidean geometry is described in [73, 72, 80], and the historical contribution of Möbius’ barycentric coordinates to vector analysis is described in [13, pp. 48–50].
For any positive integer \( N \), let \( m_k \in \mathbb{R} \) be \( N \) given real numbers such that
\[
\sum_{k=1}^{N} m_k \neq 0
\]
and let \( A_k \in \mathbb{R}^n \) be \( N \) given points in the Euclidean \( n \)-space \( \mathbb{R}^n \), \( k = 1, \ldots, N \). Then, by obvious algebra, the equation
\[
\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ A_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ P \end{pmatrix}
\]
for the unknowns \( m_0 \in \mathbb{R} \) and \( P \in \mathbb{R}^n \) possesses the unique solution given by
\[
m_0 = \sum_{k=1}^{N} m_k
\]
and
\[
P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}
\]
satisfying for all \( X \in \mathbb{R}^n \),
\[
X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k}
\]

Following Möbius, view (137) as the representation of a point \( P \in \mathbb{R}^n \) in terms of its barycentric coordinates \( m_k, k = 1, \ldots, N \), with respect to the set of points
\[
S = \{ A_1, \ldots, A_N \}
\]
Identity (138), then, insures that the barycentric coordinate representation (137) of \( P \) with respect to the set \( S \) is covariant (or, invariant in form) in the following sense. The point \( P \) and the points of the set \( S \) of its barycentric coordinate representation vary together under translations. Indeed, a translation \( X + A_k \) of \( A_k \) by \( X \), \( k = 1, \ldots, N \), on the right-hand side of (138) results in the translation \( X + P \) of \( P \) by \( X \) on the left-hand side of (138).

In order to insure that barycentric coordinate representations with respect to a set \( S \) are unique, we require the set \( S \) to be pointwise independent.

**Definition 10. (Euclidean Pointwise Independence).** A set \( S \) of \( N \) points, \( S = \{ A_1, \ldots, A_N \} \), in \( \mathbb{R}^n \), \( n \geq 2 \), is pointwise independent if the \( N - 1 \) vectors \( -A_1 + A_k, k = 2, \ldots, N \), are linearly independent in \( \mathbb{R}^n \).

We are now in the position to present the formal definition of Euclidean barycentric coordinates, as motivated by mass and center of momentum velocity of Newtonian particle systems.

**Definition 11. (Barycentric Coordinates).** Let
\[
S = \{ A_1, \ldots, A_N \}
\]
be a pointwise independent set of $N$ points in $\mathbb{R}^n$. The real numbers $m_1, \ldots, m_N$, satisfying
\begin{equation}
\sum_{k=1}^{N} m_k \neq 0
\end{equation}
are barycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set $S$ if
\begin{equation}
P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}
\end{equation}

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates $(m_1, \ldots, m_N)$ of the point $P$ in (142) are equivalent to the barycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates $(m_1, \ldots, m_N)$ are also written as $(m_1: \ldots:m_N)$.

Barycentric coordinates that are normalized by the condition
\begin{equation}
\sum_{k=1}^{N} m_k = 1
\end{equation}
are called \textit{special barycentric coordinates}.

Equation (142) is said to be the (unique) barycentric coordinate representation of $P$ with respect to the set $S$.

\textbf{Theorem 12. (Covariance of Barycentric Coordinate Representations).}

Let
\begin{equation}
P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}
\end{equation}
be the barycentric coordinate representation of a point $P \in \mathbb{R}^n$ in a Euclidean $n$-space $\mathbb{R}^n$ with respect to a pointwise independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n$. The barycentric coordinate representation (144) is covariant, that is,
\begin{equation}
X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k}
\end{equation}
for all $X \in \mathbb{R}^n$, and
\begin{equation}
RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k}
\end{equation}
for all $R \in SO(n)$.

\textit{Proof.} The proof is immediate, noting that rotations $R \in SO(n)$ of $\mathbb{R}^n$ about its origin are linear maps of $\mathbb{R}^n$. $\square$

Following the vision of Felix Klein in his \textit{Erlangen Program} [41], it is owing to the covariance with respect to translations and rotations that barycentric coordinate representations possess geometric significance. Indeed, translations and rotations
in Euclidean geometry form the group of motions of the geometry, and according to Felix Klein’s Erlangen Program, a geometric property is a property that remains invariant in form under the motions of the geometry.

13. HYPERBOLIC BARYCENTRIC, GYROBARYCENTRIC, COORDINATES

Guided by analogies with Sec. 12 in this section we introduce barycentric coordinates into hyperbolic geometry [71, 72, 73].

Definition 13. (Hyperbolic Pointwise Independence). A set $S$ of $N$ points $S = \{A_1, \ldots, A_N\}$ in the ball $\mathbb{R}^n_s$, $n \geq 2$, is pointwise independent if the $N-1$ gyrovectors in $\mathbb{R}^n_s$, $\otimes A_1 \oplus A_k$, $k = 2, \ldots, N$, considered as vectors in $\mathbb{R}^n \supset \mathbb{R}^n_s$, are linearly independent.

We are now in the position to present the formal definition of gyrobarycentric coordinates, that is, hyperbolic barycentric coordinates, as motivated by the notions of relativistic mass and center of momentum velocity in Einstein’s special relativity theory. Gyrobarycentric coordinates, fully analogous to barycentric coordinates, thus emerge when Einstein’s relativistic mass meets the hyperbolic geometry of Bolyai and Lobachevsky [74].

Definition 14. (Gyrobarycentric Coordinates). Let

\begin{equation}
S = \{A_1, \ldots, A_N\}
\end{equation}

be a pointwise independent set of $N$ points in $\mathbb{R}^n_s$. The real numbers $m_1, \ldots, m_N$, satisfying

\begin{equation}
\sum_{k=1}^{N} m_k \gamma A_k > 0
\end{equation}

are gyrobarycentric coordinates of a point $P \in \mathbb{R}^n_s$ with respect to the set $S$ if

\begin{equation}
P = \frac{\sum_{k=1}^{N} m_k \gamma A_k A_k}{\sum_{k=1}^{N} m_k \gamma A_k}
\end{equation}

Gyrobarycentric coordinates are homogeneous in the sense that the gyrobarycentric coordinates $(m_1, \ldots, m_N)$ of the point $P$ in (149) are equivalent to the gyrobarycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates $(m_1, \ldots, m_N)$ are also written as $(m_1: \ldots: m_N)$.

Gyrobarycentric coordinates that are normalized by the condition

\begin{equation}
\sum_{k=1}^{N} m_k = 1
\end{equation}

are called special gyrobarycentric coordinates.

Equation (149) is said to be the gyrobarycentric coordinate representation of $P$ with respect to the set $S$. 
Finally, the constant of the gyrobarycentric coordinate representation of $P$ in (149) is $m_0 > 0$, given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{j,k=1 \atop j<k}^{N} m_j m_k (\gamma_{\oplus A_j \oplus A_k} - 1)}$$

**Theorem 15. (Gyrocovariance of Gyrobarycentric Coordinate Representations).** Let

(152a) \[ P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \]

be a gyrobarycentric coordinate representation of a point $P \in \mathbb{R}^n_s$ in an Ein-stein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with respect to a pointwise independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n_s$.

Then

(152b) \[ \gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0} \]

and

(152c) \[ \gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0} \]

where $m_0 > 0$ is the constant of the gyrobarycentric coordinate representation (152a) of $P$, given by

(152d) \[ m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{j,k=1 \atop j<k}^{N} m_j m_k (\gamma_{\oplus A_j \oplus A_k} - 1)} \]

Furthermore, the gyrobarycentric coordinate representation (152a) and its associated identities in (152b) – (152d) are gyrocovariant, that is,

(153a) \[ X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \]

(153b) \[ \gamma_{X \oplus A} = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}}{m_0} \]

(153c) \[ \gamma_{X \oplus P} (X \oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{m_0} \]

(153d) \[ m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{j,k=1 \atop j<k}^{N} m_j m_k (\gamma_{\oplus (X \oplus A_j) \oplus (X \oplus A_k)} - 1)} \]
for all $X \in \mathbb{R}^n_s$, and

\begin{align}
(154a) \quad RP &= \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} R A_k}{\sum_{k=1}^{N} m_k \gamma_{RA_k}} \\
(154b) \quad \gamma_{RP} &= \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}}{m_0} \\
(154c) \quad \gamma_{RP}(RP) &= \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}(RA_k)}{m_0} \\
(154d) \quad m_0 &= \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{j,k=1 \atop j<k}^{N} m_j m_k (\gamma_{\Xi(RA_j)\Xi(RA_k)} - 1)}
\end{align}

for all $R \in SO(n)$.

The proof of Theorem 15 is presented in [72, Theorem 4.4] and [73, Theorem 4.6].

**Theorem 16.** It is assumed in Theorem 15 that the point $P$ in (149) lies inside the ball $\mathbb{R}^n_s$, implying that $m_0^2 > 0$ and that the gamma factor $\gamma_P$ of $P$ is a real number. The constant $m_0$ of a gyrobarycentric coordinate representation (149) of a point $P$ determines whether $P$ lies inside the ball $\mathbb{R}^n_s$.

If the coefficients $m_k$, $k = 1, \ldots, N$, in the gyrobarycentric coordinate representation (149) of $P$ are all positive or all negative, then the point $P$ lies in the convex span of the points of the set $S$, that is, $P$ lies inside the $(N-1)$-gyrosimplex $A_1 \ldots A_N$. This gyrosimplex, in turn, lies inside the ball $\mathbb{R}^n_s$.

1. The point $P$ lies inside the $(N-1)$-gyrosimplex $A_1 \ldots A_N$ if and only if the coefficients $m_k$, $k = 1, \ldots, N$, of its gyrobarycentric coordinate representation (152a) are all positive or all negative. Clearly, in this case $m_0^2 > 0$.

Otherwise, when all the coefficients $m_k$ are nonzero but do not have the same sign, the location of $P$ has the following three possibilities that correspond to whether the gamma factor (152b) of $P$ is real, infinity, or imaginary:

2. The point $P$ does not lie inside the $(N-1)$-gyrosimplex $A_1 \ldots A_N$, but it lies inside the ball $\mathbb{R}^n_s$. In this case the gamma factor $\gamma_P$ of $P$ is a real number and, hence, $m_0^2 > 0$.

3. The point $P$ lies on the boundary of the ball $\mathbb{R}^n_s$ if and only if the gamma factor $\gamma_P$ of $P$ is undefined, $\gamma_P = \infty$, so that $m_0^2 = 0$.

4. The point $P \in \mathbb{R}^n_s$ does not lie in the ball $\mathbb{R}^n_s$ or on its boundary if and only if the gamma factor $\gamma_P$ of $P$ is purely imaginary, so that $m_0^2 < 0$.

Examples for the use of gyrobarycentric coordinates for the determination of several hyperbolic triangle centers are found in [72, 73].
Employing the technique of gyrobarycentric coordinate representations, we will now determine the endpoints \( E_A \) and \( E_B \) of a gyroline \( L_{AB}(t) \) in an Einstein gyrovector space \((\mathbb{R}^n_s, \oplus, \otimes)\), shown in Fig. 1, p. 17.

Let \( A_1, A_2 \in \mathbb{R}^n_s \) be two distinct points of an Einstein gyrovector space \((\mathbb{R}^n_c, \oplus, \otimes)\), and let \( P \) be a generic point on the gyroline, \((117)\),

\[
P_{12}(t) = A_1 \oplus (\ominus_A A_1 A_2) \otimes t
\]

\( t \in \mathbb{R} \), that passes through these two points. Furthermore, let

\[
P(t) = \frac{m_1 \gamma A_1 A_1 + m_2 \gamma A_2 A_2}{m_1 \gamma A_1 + m_2 \gamma A_2}
\]

be the gyrobarycentric coordinate representation of \( P \) with respect to the pointwise independent set \( S = \{A_1, A_2\} \), where the gyrobarycentric coordinates \( m_1 \) and \( m_2 \) are to be determined. Owing to the homogeneity of gyrobarycentric coordinates, we can select \( m_2 = -1 \), obtaining from \((156)\) the gyrobarycentric coordinate representation

\[
P(t) = \frac{m \gamma A_1 A_1 - \gamma A_2 A_2}{m \gamma A_1 - \gamma A_2}
\]

According to Def. \((13)\) of the gyrobarycentric coordinate representation of \( P \) in \((149)\) and its constant \( m_0 \) in \((151)\), the constant \( m_0 \) of the gyrobarycentric coordinate representation of \( P \) satisfies the equation

\[
m_0^2 = m_1^2 + m_2^2 + 2m_1 m_2 \gamma \ominus_{A_1 \ominus A_2} = m^2 + 1 + 2m \gamma_{12}
\]

where we use the convenient notation

\[
a_{ij} = \ominus A_i \ominus A_j
\]

\[
\gamma_{ij} = \gamma_{a_{ij}}
\]

\( i, j \in \mathbb{N} \).

As remarked in Item (3) of Remark \((16)\) the point \( P \) lies on the boundary of the ball \( \mathbb{R}^n_s \) if and only if \( m_0 = 0 \), that is by \((158)\), if and only if

\[
m^2 - 2m \gamma_{12} + 1 = 0
\]

The two solutions of \((160)\) are

\[
m = \gamma_{12} + \sqrt{\gamma_{12}^2 - 1}
\]

\[
m = \gamma_{12} - \sqrt{\gamma_{12}^2 - 1}
\]
The substitution into (157) of each of the two solutions (161) gives the two endpoints $E_{A_1}$ and $E_{A_2}$ of the gyroline $P_{12}(t)$ in (155),

$$E_{A_1} = \frac{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1}A_1 - \gamma_{A_2}A_2}{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1} - \gamma_{A_2}}$$

(162)

$$E_{A_2} = \frac{(\gamma_{12} - \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1}A_1 - \gamma_{A_2}A_2}{(\gamma_{12} - \sqrt{\gamma_{12}^2 - 1})\gamma_{A_1} - \gamma_{A_2}}$$

which are shown in Fig. 1, p. 17, for $A_1 = A$ and $A_2 = B$.

The expressions for $\oplus A_1 \oplus E_{A_1}$ and $\oplus A_1 \oplus E_{A_2}$ that follow from (162) by means of the gyrocovariance identity (153a) in Theorem 15 are particularly elegant. Indeed, by the gyrocovariance identity (153a) with $X = \ominus A_1$, applied to each of the two equations in (162), we have

$$\ominus A_1 \oplus E_{A_1} = \frac{\gamma_{12}a_{12}}{(\gamma_{12} + \sqrt{\gamma_{12}^2 - 1}) - \gamma_{12}}$$

(163)

$$\ominus A_1 \oplus E_{A_2} = \frac{\gamma_{12}a_{12}}{\sqrt{\gamma_{12}^2 - 1}}$$

where we use the notation (159), noting that $\ominus A_1 \oplus A_1 = 0$ and $\gamma_{\ominus A_1 \oplus A_1} = \gamma_0 = 1$.

The equations in (163) imply, by means of the left cancellation law (23),

$$E_{A_1} = A_1 \ominus \frac{\gamma_{12}a_{12}}{\sqrt{\gamma_{12}^2 - 1}}$$

(164)

$$E_{A_2} = A_1 \oplus \frac{\gamma_{12}a_{12}}{\sqrt{\gamma_{12}^2 - 1}}$$

Interestingly, (164) remains invariant in form under the isomorphism (83), as seen from (90). Accordingly, the equations in (164) with $\oplus = \oplus_E$ being Einstein addition are used in calculating the endpoints of an Einstein gyroline in Fig. 1, p. 17 and the same equations (164), but with $\oplus = \oplus_M$ being Möbius addition are used in calculating the endpoints of a Möbius gyroline in Fig. 7, p. 24.

In Fig. 9, p. 31 the points $A, B \in (\mathbb{R}_E^2, \oplus_E, \otimes)$ of a Möbius gyrovector plane are shown along with their respective isomorphic images $2 \oplus A, 2 \oplus B \in (\mathbb{R}_E^2, \oplus_M, \otimes)$ of an Einstein gyrovector plane, under the isomorphism (86). Indeed, as expected, Fig. 9 indicates that the endpoints $E_A$ and $E_B$ of

(1) the Möbius gyroline (a circular arc) through the points $A$ and $B$, and of
(2) the Einstein gyroline (a chord) through the points $2 \oplus A$ and $2 \oplus B$.
are coincident.

This Chapter appears in [75].

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