POINCARÉ SERIES, EXPOSANTS OF AFFINE LIE ALGEBRAS, AND MCKAY-SLODOWY
CORRESPONDENCE

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ABSTRACT. Let $N$ be a normal subgroup of a finite group $G$ and $V$ be a fixed finite-dimensional $G$-module. The Poincaré series for the multiplicities of induced modules and restriction modules in the tensor algebra $T(V) = \oplus_{k \geq 0} V^\otimes k$ are studied in connection with the McKay-Slodowy correspondence. In particular, it is shown that the closed formulas for the Poincaré series associated with the distinguished pairs of subgroups of $SU_2$ give rise to the exponents of all untwisted and twisted affine Lie algebras except $A_{2n}^1$.

1. Introduction

McKay correspondence establishes a far-reaching one-to-one map between subgroups of the special unitary group $SU_2$ and affine Dynkin diagrams of simply laced types [17]. Since its introduction, numerous deep connections and applications have been found in combinatorics, algebraic geometry, representation theory and mathematical physics. For example, McKay’s observation corresponds to the classification of the minimal resolutions of the singularity of the action of a finite subgroup $G$ of $SU_2$ on $C^2$ in terms of simply laced Dynkin diagrams [10][12][15][21][24].

Let $G$ be a finite group and $V$ a faithful $G$-module. The tensor algebra $T(V) = \oplus_{k \geq 0} V^\otimes k$ is naturally a $G$-module. Similar to the well-known result of Molien [3] on $G$-invariants of the symmetric algebra $S(V)$, the Poincaré series $m_V(t)$ of $G$-invariants in the tensor algebra $T(V)$ are rational functions in terms of irreducible characters of $G$. In particular, when $G$ is a finite subgroup of $SU_2$, the Poincaré series $m_V(t)$ provides a conceptual interpretation of the exponents of the affine Lie algebra in simply laced types [1]. It is natural to expect similar explanation of the exponents for other types of affine Lie algebras, in particular twisted affine Lie algebras.

In the classic work [20] Slodowy studied the minimal resolution of the singularity of the action of the quotient $G/N$ of finite subgroups of $SU_2$ on the quotient space $C^2/N$, which generalizes the minimal resolution of the singularity of $C^2/G$, where $G$ is a finite subgroup of $SU_2$. Slodowy discovered that the minimal resolution of singularity of $C^2/N$ by the action of $G/N$ are in one-to-one correspondence to all affine Dynkin diagrams. Algebraically, Slodowy found that the pairs of finite groups $N, G$ of $SU_2$, where $N$ is a normal subgroup of $G$, essentially realize all affine Dynkin diagrams.

To be more specific, let $G$ be a finite group with a normal subgroup $N$. The Grothendieck group $K(G)$ is spanned by the irreducible complex modules $\rho_i, i \in I_G$. Similarly $K(N)$ is spanned by the irreducible complex modules $\phi_i, i \in I_N$. The induction functor $\text{Ind} : R(N) \to R(G)$ and the restriction functor $\text{Res} : R(G) \to R(N)$ are defined as usual and we will denote the images $\text{Ind}(\phi) := \hat{\phi}$ and $\text{Res}(\rho) := \hat{\rho}$ respectively.

It turns out that the dimension of the span $\{\hat{\rho}_i | i \in I_G\}$ agrees with that of the span $\{\hat{\phi}_i | i \in I_N\}$. We pare down the set $\{\hat{\rho}_i | i \in I_G\}$ to a basis $\{\hat{\rho}_i | i \in \hat{I}\}$ for the subspace span$(\hat{\rho}_i)$, and similarly we also pare down a corresponding basis $\{\hat{\phi}_i | i \in \hat{I}\}$ for the subspace span$(\hat{\phi}_i)$, then $|\hat{I}| = |\hat{I}|$ is equal to the common dimension of the two spans.

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Let $V$ be a fixed finite-dimensional $G$-module, and we also denote $	ext{Res} V = \bar{V} \in R(N)$. Clearly $\bar{V} \otimes \text{span}(\hat{\phi}_i) \subset \text{span}(\hat{\phi}_i)$ and $V \otimes \text{span}(\hat{\phi}_i) \subset \text{span}(\hat{\phi}_i)$. Therefore the tensor decompositions

$$ (1.1) \quad \bar{V} \otimes \hat{\rho}_j = \bigoplus_{i \in 1} a_{ij} \bar{\phi}_i \quad \text{and} \quad V \otimes \hat{\phi}_j = \bigoplus_{i \in 1} b_{ij} \hat{\phi}_i $$

give rise to two integral matrices $\bar{A} = (a_{ij})$ and $\bar{B} = (b_{ij})$ of the same size respectively. The corresponding representation graph is the digraph $R_V(\bar{G})$ (resp. $R_V(\hat{N})$) with vertices indexed by $\bar{1}$ (resp. $\hat{1}$), where $i$ is joined to $j$ by $\max(a_{ij}, a_{ji})$ (resp. $\max(b_{ij}, b_{ji})$) edges with an arrow pointing to $i$ if $a_{ij} > 1$ (resp. $b_{ij} > 1$).

When the pair of subgroups are subgroups of $SU_2$, Slodowy [20] observed that the representation graphs $R_V(\bar{G})$ and $R_V(\hat{N})$ are in fact the affine Dynkin diagrams (his original statement missed a couple of groups, cf. [9]). Of course, when $N = 1$, the graphs $R_V(G)$ are of simply-laced types according to the McKay correspondence. For nontrivial $N$, the explicit correspondence goes as follows. Let $C_n$ be the cyclic group of order $n$, $D_n$ the binary dihedral group of order $4n$, $T$ the binary tetrahedron group of order 24 and $O$ the binary octahedral group of order 48. For $(G, N) = (D_{2(n-1)}, D_{n-1}), (D_n, C_{2n}), (D_{2n}, C_{2n}), (O, T), (T, D_2), (D_2, C_2), (2n)$, the representation graph $R_V(\bar{G})$ is the twisted affine Dynkin diagram of type $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, $E_6^{(2)}$, $D_4^{(2)}$, $A_2^{(2)}$, respectively, and the representation graph $R_V(\hat{N})$ is the non-twisted multiply laced affine Dynkin diagram of type $B_4^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$, $A_1^{(1)}$, respectively. Moreover, $C_{\bar{A}} = 2I - \bar{A}$ and $C_{\bar{B}} = 2I - \bar{B}$ are the corresponding Cartan matrices of Dynkin diagrams $R_V(\bar{G})$ and $R_V(\hat{N})$, respectively.

To understand Slodowy’s idea and prepare for the later work, we provide a detailed exposition of the McKay-Slodowy correspondence in the first part.

The second part of the paper aims to generalize Benkart’s interpretation of the exponents to all affine Dynkin diagrams. Let $V$ be as above and for simplicity we will also write $\bar{V} = V$. For each $j \in 1$ (resp. $\hat{1}$), let $\bar{m}_j^i$ (resp. $\hat{m}_j^i$) be the multiplicity of $\bar{\rho}_j$ inside $V \otimes^k$ (resp. $\hat{\phi}_j$ inside $V \otimes^k$):

$$ m_j^i = \dim(\text{Hom}_N(\bar{\rho}_j, V \otimes^k)) \quad (\text{resp.} \quad \hat{m}_j^i = \dim(\text{Hom}_G(\hat{\phi}_j, V \otimes^k))). $$

Therefore the generating series

$$ (1.2) \quad m_j^i(t) = \sum_{k \geq 0} \bar{m}_j^i t^k \quad \text{and} \quad \hat{m}_j^i(t) = \sum_{k \geq 0} \hat{m}_j^i t^k $$

are the Poincaré series for the multiplicities of $\bar{\rho}_j$ and $\hat{\phi}_j$ in the tensor algebra $T(V) = \oplus_{k \geq 0} V \otimes^k$ respectively.

In the special case when $N = 1$ and $G$ is a finite group of $SU_2$, the Poincaré series $m_j^i(t) = \hat{m}_j^i(t)$ has been thoroughly studied by Benkart [11] where she gave a closed rational function formula of $m_j^i(t)$ in terms of the irreducible characters of $G$. Moreover, it turns out that the denominator also gives the detailed information about the spectrum of the Coxeter element, just as the Poincaré series of the $G$-invariants in the symmetric algebra $S(V)$ does [14][16][22][26]. Our main result will study the Poincaré series $\bar{m}_j^i(t)$ and $\hat{m}_j^i(t)$ for all the distinguished pairs of subgroups of $SU_2$ and provides a conceptual interpretation of the exponents for all non-simply laced and twisted affine Lie algebras.

This paper is organized as follows. In Section 2, after a brief review of McKay correspondence we provide a detailed exposition for the McKay-Slodowy correspondence to realize all affine Dynkin diagrams. To prepare for the later work, we clarify some of the missing points in the literature (e.g. our treatment of types $A_2^{(2)}$ and $A_2^{(2)}$) and emphasize the group theoretical construction using the induction and restriction functors as much as possible. In Section 3 the formulas of the
Poincaré series for the $N$-restriction of the irreducible $G$-modules and the induced modules of the irreducible $N$-modules in tensor algebra $T(V) = \oplus_{k \geq 0} V^\otimes k$ are given using the general theory. In Section 4, the Poincaré series $\hat{m}^i(t)$ and $\check{m}^i(t)$ are explicitly computed for the tensor algebra $T(V)$ for all the distinguished pairs of subgroups $N < G$ in $SU_2$, and finally we show that the closed formulas of the Poincaré series of the invariants in the tensor algebra $T(V)$ for the distinguished pairs of subgroups of $SU_2$ give rise to the exponents of all affine Lie algebras in both untwisted and twisted types except $A_{2n}^{(1)}$.

2. REALIZATIONS OF NON-SIMPLY LACED AFFINE DYNKIN DIAGRAMS

2.1. Simply-laced types.

The McKay correspondence establishes a fundamental relation between finite subgroups of $SU_2(\mathbb{C})$ and affine Dynkin diagrams of simply-laced types. We recall the algebraic version to prepare for further development.

2.1.1. Cyclic groups. For $n \geq 2$, let $C_n = \langle z^n = 1 \rangle$ be the cyclic group of order $n$. The canonical imbedding $\pi : C_n \rightarrow SU_2$ is given by $\pi(z) = \text{diag}(\theta_n^{-1}, \theta_n)$, where $\theta_n = e^{2\pi \sqrt{-1}/n}$, a primitive $n$th root of unity. The cyclic group $C_n$ has exactly $n$ complex irreducible modules $\xi_i (i = 0, 1, \ldots, n-1)$, which are all one-dimensional given by $\chi_{\xi_i}(z) = \theta_n^i (i = 0, 1, \ldots, n-1)$. Then $\pi \simeq \xi_1 \oplus \xi_{-1}$ and the multiplication rule $\xi_i \otimes \xi_k = \xi_{i+k}$ implies the fusion rule

$$\pi \otimes \xi_i = \xi_{i-1} \oplus \xi_{i+1}$$

which gives rise to the Dynkin diagram of type $A_{n-1}^{(1)}$.

$$\xi_{n-1} \xrightarrow{n > 2} \xi_0 \xrightarrow{n = 2} \xi_1$$

2.1.2. Binary dihedral groups. The binary dihedral group $D_n (n \geq 2)$ of order $4n$ is the group $\langle x, y | x^n = y^2 = -1, yxy^{-1} = x^{-1} \rangle$. The embedding $\pi$ of $D_n$ into $SU_2$ is given by

$$\pi(x) = \begin{pmatrix} \theta_{2n}^{-1} & 0 \\ 0 & \theta_{2n} \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$  

Clearly $\{ \pm 1, x^i (i = 1, \ldots, n-1), y, xy \}$ is a set of representatives of the conjugacy classes of $D_n$.

The $n + 3$ irreducible modules are realized as follows. Let $\delta_i = \hat{\xi}_i$, the induced module of the $i$th irreducible module $\xi_i$ of the cyclic group $C_{2n} = \langle x \rangle$, a normal subgroup of index 2. The characters are determined by $\chi_{\delta_i}(x) = \theta_{2n}^i + \theta_{2n}^{-i}$ and $\chi_{\delta_i}(y) = 0$. Consequently $\delta_i = \delta_{2n-i}$ for $i = 0, 1, \ldots, n$. It is easy to see that $\delta_i$ are 2-dimensional irreducible $D_n$-modules for $1 \leq i \leq n - 1$, but $\delta_0$ and $\delta_n$ are decomposed into a sum of two irreducible one-dimensional modules:

$$\delta_0 = \delta_0^+ \oplus \delta_0^-,$$

$$\delta_n = \delta_n^+ \oplus \delta_n^-.$$
Then the irreducible components are determined by

\[ \chi_{\delta_0}(x) = 1, \quad \chi_{\delta_0}(y) = \epsilon, \]
\[ \chi_{\delta_1}(x) = -1, \quad \chi_{\delta_n}(y) = \epsilon \sqrt{(-1)^n}, \]

where \( \epsilon = \{\pm\} \) and we note that \( \chi_{\delta_0}(y^2) = \chi_{\delta_0}(x^n) = (-1)^n \), therefore \( \chi_{\delta_n}(y) = \epsilon \sqrt{(-1)^n} \). The fusion rule associated with the embedding \( \pi = \delta_1 \) is seen as: \( \pi \otimes \delta_0^i = \delta_1, \pi \otimes \delta_i = \delta_{i-1} \oplus \delta_{i+1} \) (1 \( \leq i \leq n - 1 \)), \( \pi \otimes \delta_n^i = \delta_{n-1} \). Therefore the representation graph \( \mathcal{R}_{\delta_1}(D_n) \) realizes the Dynkin diagram of type \( D_{n+2}^{(1)} \). For reference the character table of \( D_n \) is given in Table 1 (also see [5, 18, 25]), where \( |C_G(g)| \) is the cardinality of the conjugacy class containing \( g \) in \( G \).

\[
\begin{array}{cccccc}
\delta_0^- & \delta_0^+ & \delta_1 & \delta_2 & \cdots & \delta_n^- \\
\delta_0^+ & \delta_1 & \delta_2 & \cdots & \delta_n^- & \delta_n^+ \\
\end{array}
\]

Table 1

| \( y \) | \( |C_G(g)| \) | \( x \) | \( \cdots \) | \( x^{n-1} \) | \( y \) | \( yx \) |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | \( \cdots \) | 1 | \( \pm 1 \) | \( \pm 1 \) |
| 2 | 2(1)^i | \( \theta_{2n}^i + \theta_{2n}^{-i} \) | \( \cdots \) | \( \theta_{2n}^{(n-1)} + \theta_{2n}^{-(n-1)} \) | 0 | 0 |
| 1 | (-1)^n | -1 | \( \cdots \) | \( (-1)^{n-1} \) | \( \pm \sqrt{(-1)^n} \) | \( \mp \sqrt{(-1)^n} \) |

2.1.3. Binary tetrahedral group. The binary tetrahedral group \( T \) of order 24 is \( \langle x, y, z | x^2 = y^2 = z^3 = 1, yxy^{-1} = x^{-1}, zyz^{-1} = y^{-1}x, zyx^{-1} = x \rangle \), and the elements \( \pm 1, x, \pm z \) and \( \pm z^2 \) form a set of representatives of the seven conjugacy classes. The embedding \( \pi \) of \( T \) into \( SU_2 \) is given by

\[
\pi(x) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \pi(y) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \pi(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \theta_8^{-1} & \theta_8^{-1} \\ -\theta_8 & \theta_8 \end{pmatrix}.
\]

The binary dihedral group \( D_2 = \langle x, y \rangle \) is a normal subgroup of \( T \) of index 3. Though \( D_2 \) has 5 conjugacy classes, they generate only 3 conjugacy classes in \( T \). Therefore the irreducible \( D_2 \)-modules are induced to only three distinct \( T \)-modules such that \( \delta_2^+ = \hat{\delta}_0^- = \hat{\delta}_2^- \), more explicitly the induced characters obey that

\[
\begin{align*}
\chi_{\delta_0^+}(\pm 1) & = 3\chi_{\delta_0^+}(\pm 1) = 3 \quad (i = 0, 2), \quad \chi_{\delta_i^+}(\pm 1) = 3\chi_{\delta_i^+}(\pm 1) = \pm 6, \\
\chi_{\delta_i^+}(x) & = 3\chi_{\delta_i^+}(x) = 3, \quad \chi_{\delta_0^+}(x) = \chi_{\delta_0^+}(x) = -1, \quad \chi_{\delta_1^+}(x) = \chi_{\delta_1^+}(x) = 0, \\
\chi_{\delta_i^+}(\pm z) & = \chi_{\delta_i^+}(\pm z^2) = 0 \quad (i = 0, 2), \quad \chi_{\delta_i^+}(\pm z) = \chi_{\delta_i^+}(\pm z^2) = 0.
\end{align*}
\]

Explicitly these three induced \( T \)-modules decompose into seven irreducible \( T \)-modules as follows:

\[
\begin{align*}
\hat{\delta}_0^+ & = 2 \oplus \tau_0' \oplus \tau_0'', \\
\hat{\delta}_1 & = 3 \oplus \tau_0' \oplus \tau_0'', \\
\hat{\delta}_2 & = \hat{\delta}_2^- = \hat{\delta}_2 := \tau_2,
\end{align*}
\]
where $\tau_0$ is the trivial module and \{\tau_0, \tau_0', \tau_1', \tau_1''', \tau_2\} forms the complete set of irreducible $T$-characters. It follows from (2.2), (2.3) that $\pi \simeq \tau_1$ and

$$
\begin{align*}
\chi_{\tau_0'}(\pm 1) &= \chi_{\tau_0''}(\pm 1) = \chi_{\tau_0'}(x) = \chi_{\tau_0''}(x) = 1, \\
\chi_{\tau_0'}(\pm z) &= \chi_{\tau_0''}(\pm z^2) = \theta_3, \\
\chi_{\tau_0'}(\pm z^2) &= \chi_{\tau_0''}(\pm z) = \theta_5^2, \\
\chi_{\tau_1'}(1) &= \chi_{\tau_1''}(1) = \pm 2, \quad \chi_{\tau_1'}(x) = \chi_{\tau_1''}(x) = 0, \\
\chi_{\tau_1'}(\pm z) &= \chi_{\tau_1''}(\mp z^2) = \pm \theta_3, \\
\chi_{\tau_1'}(\pm z^2) &= \chi_{\tau_1''}(\mp z) = \mp \theta_5^2.
\end{align*}
$$

Then $\pi \otimes \tau_0^{(i)} = \tau_1^{(i)}, \pi \otimes \tau_1^{(i)} = \tau_0^{(i)} \oplus \tau_2 (0 \leq i \leq 2), \pi \otimes \tau_2 = \oplus_{i=0}^2 \tau_1^{(i)}$ give rise to the Dynkin diagram of type $E_6^{(1)}$. The character table of $T$ is given in Table 2 (also see [5, 25]).

\[
\begin{array}{cccccc}
\tau_0' & | & \\
\tau_1' & | & \\
\tau_0 & \tau_1 & \tau_2 & \tau_1'' & \tau_0'' \\
\end{array}
\]

| $\chi(g)$ \| $C_G(g)$ | 1 | -1 | $x$ | $z$ | $z^2$ | $-z$ | $-z^2$ |
|---|---|---|---|---|---|---|---|
| $\chi_{\tau_0}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_{\tau_0'}$ | 1 | 1 | 1 | $\theta_3$ | $\theta_5^2$ | $\theta_3$ | $\theta_5^2$ |
| $\chi_{\tau_0''}$ | 1 | 1 | 1 | $\theta_5^2$ | $\theta_3$ | $\theta_5^2$ | $\theta_3$ |
| $\chi_{\tau_1}$ | 2 | -2 | 0 | 1 | -1 | 1 | 1 |
| $\chi_{\tau_1'}$ | 2 | -2 | 0 | $\theta_3$ | $-\theta_5^2$ | $-\theta_3$ | $\theta_5^2$ |
| $\chi_{\tau_1''}$ | 2 | -2 | 0 | $\theta_5^2$ | $-\theta_3$ | $-\theta_5^2$ | $\theta_3$ |
| $\chi_{\tau_2}$ | 3 | 3 | -1 | 0 | 0 | 0 | 0 |

2.1.4. **Binary octahedral group.** Let $O = \langle u, y, z \rangle = \langle u^2 = y^2 = z^3 = -1, yuy^{-1} = u^{-1}, zuz^{-1} = z^{-1}u^{-1}, zyz^{-1} = u^2 \rangle$ be the binary octahedral group order 48. Then the subgroup $\langle u^2, y, z \rangle \simeq T$, the binary tetrahedral group $T$ of order 24 (see subsection 2.1.3), so the former is also denoted by $T$. The group $O$ is imbedded into $SU_2$ by letting $\pi(u) = diag(\theta_8, \theta_8^{-1})$ and $\pi(y), \pi(z)$ given by (2.1).

The subgroup $T$ has 7 conjugacy classes but generates 5 conjugacy classes in $O$, then the 7 irreducible $T$-modules $\tau_0^{(i)}, \tau_1^{(i)}, \tau_2^{(i)}$ induce $\tau_0, \tau_1, \tau_2$ (i = 0, 1, 2) into 5 different $O$-modules as $\hat{\tau}_1 = \hat{\tau}_1'', \hat{\tau}_0 = \hat{\tau}_0''$, and the induced characters are computed by

$$
\chi_{\hat{\tau}_k^{(i)}}(g) = \begin{cases} 
\chi_{\tau_k^{(i)}}(gh^{-1}g), & \forall g, h \in T \\
0, & \forall g \in O \setminus T
\end{cases}
$$

\[
\begin{array}{cccc}
\tau_0 & \tau_1 & \tau_2 & \tau_1'' & \tau_0'' \\
\end{array}
\]
for \( k = 0, 1, i = 0, 1, 2 \) and \( k = 2, i = 0 \). Therefore the induced modules decompose into irreducible \( O \)-modules as follows:

\[
\hat{\tau}_0 = \omega_0^+ + \omega_0^- , \quad \hat{\tau}_1 = \omega_1^+ + \omega_1^- , \quad \hat{\tau}_2 = \omega_2^+ + \omega_2^- ,
\]

\[
\hat{\tau}'_1 = \hat{\tau}' := \omega_3 , \quad \hat{\tau}_0' := \omega_4 .
\]

and the irreducible summands \( \{ \omega_i^{(\pm)} , \omega_3 , \omega_4 \mid i \in \{ 0, 1, 2 \} \} \) realize all irreducible \( O \)-modules.

Clearly \( \omega_0^+ \) is the trivial module and \( \pi_1 = \omega_1^+ \) (resp. \( \omega_1^- \)), and we record other character values in the following table.

Therefore, \( \pi \otimes \omega_0^+ = \omega_1^+ , \pi \otimes \omega_1^+ \simeq \omega_0^+ + \omega_0^- , \pi \otimes \omega_2^+ \simeq \omega_1^+ + \omega_3 , \pi \otimes \omega_3 = \omega_2^+ + \omega_4 , \pi \otimes \omega_4 = \omega_3 , \) and the Dynkin diagram of type \( E_7^{(1)} \) is realized by \( T \)-irreducible modules. The character table of \( O \) is given in Table 3 (also see [3, 23]).

**Table 3**

| \( \chi \setminus \mathbb{C}_G(g) \) | 1 | \( -1 \) | \( u \) | \(-u\) | \( y \) | \( uy \) | \( z \) | \( -z \) |
|--------------------------------------|---|--------|---|-----|---|-----|---|-----|
| \( \chi_{\omega_0^+} \)             | 1 | 1      | 1 | 1   | 1 | 1   | 1 | 1   |
| \( \chi_{\omega_1^+} \)             | 2 | -2     | \( \sqrt{2} \) | -\( \sqrt{2} \) | 0 | 0   | 1 | -1  |
| \( \chi_{\omega_2^+} \)             | 4 | -4     | 0  | 0   | 0 | 0   | -1| 1   |
| \( \chi_{\omega_3} \)               | 2 | 2      | 0  | 0   | 2 | 0   | -1| -1  |
| \( \chi_{\omega_4} \)               | 2 | 2      | -1 | -1  | -1| 1   | 0 | 0   |
| \( \chi_{\omega_5^-} \)             | 2 | -2     | -\( \sqrt{2} \) | \( \sqrt{2} \) | 0 | 0   | 1 | -1  |
| \( \chi_{\omega_6^-} \)             | 1 | 1      | -1 | -1  | 1 | 1   | 1 | 1   |

2.2. Realizations of \( A_{2n-1}^{(2)} \) and \( B_n^{(1)} \) by the pair \( (D_{2(n-1)}, D_{n-1}) \).

We now start to describe the McKay-Slodowy correspondence, which covers the affine Dynkin diagrams of non-simply laced types. The main idea is to use simple modules either as induced \( G \)-modules from the irreducible \( N \)-modules or \( N \)-restrictions of the irreducible \( G \)-modules.

For fixed \( n \geq 3 \), let \( D_{2(n-1)} = \langle x, y \rangle \) be the binary dihedral group of order \( 8(n-1) \), where \( x^2 = y^2 = -1, xy = yx = x^{-1} \). So \( \langle x^2, y \rangle = D_{n-1} \triangleleft D_{2(n-1)} \) and \( |D_{2(n-1)} : D_{n-1}| = 2 \). As explained in subsection 2.1.2, there are \( 2n+1 \) irreducible \( D_{2(n-1)} \)-modules: \( \delta_0^{\pm}, \delta_i, \delta_{2(n-1)}^{\pm} \) (\( 1 \leq i \leq 2n-3 \)), and \( n+2 \) irreducible \( D_{n-1} \)-modules: \( \delta_{i}^{\pm}, \delta', \delta_{n-1}^{\pm} \) (\( 1 \leq i \leq n-2 \)).

First we consider restriction of the irreducible \( D_{2(n-1)} \)-modules to the subgroup \( D_{n-1} \). Note that \( \chi_{\delta_{i}}(x) = \theta_{4(n-1)}^{i} \) for \( 1 \leq i \leq 2n-3 \), \( \chi_{\delta_0^{\pm}}(y) = \chi_{\delta_0^{\pm}}(yx) = \pm 1 \) and \( \chi_{\delta_{2(n-1)}}(y) = \chi_{\delta_{2(n-1)}}(yx) = \pm 1 \) (subsubsection 2.1.2). Therefore
there are only \( n + 1 \) restrictions and they satisfy the following relations:

\[
\begin{align*}
\delta^+_0 &= \delta^+_{2(n-1)} = \delta^-_0, \\
\delta_i &= \delta^+_{2(n-1)-i} = \delta^-_i \quad (i = 1, 2, \ldots, n-2), \\
\delta^-_{n-1} &= \delta^+_n + \delta^-_{n-1}.
\end{align*}
\]

On the other hand, the induced characters of \( \chi_{\delta'_i} \) can be written as

\[
\chi_{\delta'_i} = \sum_{i=0}^{2(n-1)} (\chi_{\delta'_i}, \chi_{\delta'_k}) G \chi_{\delta_i} = \sum_{i=0}^{2(n-1)} (\chi_{\delta_i}, \chi_{\delta'_k}) H \chi_{\delta_i},
\]

where \((\chi_{\delta_i}, \chi_{\delta'_k})_G = (\chi_{\delta'_i}, \chi_{\delta'_k})_H\) by the Frobenius reciprocity (see [11]). In view of (2.5)–(2.6) the equation (2.7) implies that:

\[
\begin{align*}
\delta^+_0 &= \delta^+_0 \oplus \delta^+_2, \\
\delta^-_i &= \delta^-_i \oplus \delta^+_{2(n-1)-i} \quad (i = 1, 2, \ldots, n-2), \\
\delta^+_n &\oplus \delta^-_{n-1} = \delta^-_n.
\end{align*}
\]

Using the imbedding \( \pi = \delta'_1 \) for the set \( \{ \delta^+_0, \delta^-_i | 1 \leq i \leq n-1 \} \) and \( \pi = \delta^+_1 \) for the set \( \{ \delta^+_0, \delta^+_i, \delta^+_i | 1 \leq i \leq n-2 \} \), the fusion rules are easily computed as follows (by using those of \( D^{(1)}_{2n} \) or \( D^{(1)}_{n+1} \)):

\[
\begin{align*}
\delta'_1 \otimes \delta^+_0 &= \delta_1, \\
\delta'_1 \otimes \delta^-_1 &= \delta'_1 \oplus \delta^-_2, \\
\delta'_i \otimes \delta^-_i &= \delta^-_{i-1} \oplus \delta^-_{i+1} \quad (2 \leq i \leq n-2), \\
\delta'_1 \otimes \delta^-_{n-1} &= 2\delta^-_{n-2}, \\
\delta'_1 \otimes \delta^+_0 &= \delta'_1 \otimes \delta^+_2 = \delta'_1, \\
\delta'_1 \otimes \delta^-_1 &= \delta^+_1 \oplus \delta^-_1 \oplus \delta^+_1, \\
\delta'_1 \otimes \delta^+_i &= \delta^-_{i-1} \oplus \delta^-_{i+1} \quad (2 \leq i \leq n-3), \\
\delta'_1 \otimes \delta^-_{n-2} &= \delta^-_{n-3} \oplus 2\delta^+_n, \\
\delta'_1 \otimes \delta^+_n &= \delta^-_{n-1} = \delta^+_n.
\end{align*}
\]

The corresponding Dynkin diagrams are \( A^{(2)}_{2n-1} \) (i.e. \( R_{\delta'_1}(\hat{D}_{2(n-1)}) \)) and \( B^{(1)}_n \) (i.e. \( R_{\delta^+_1}(\hat{D}_{n-1}) \)) respectively. The numbers inside the nodes are the degrees of the characters.
2.3. Realizations of $D_{n+1}^{(2)}$ and $C_n^{(1)}$ by the pair $(D_n,C_{2n})$. The cyclic group $C_{2n} = \langle x \rangle$ is a normal subgroup of $D_n = \langle x, y \rangle$ with index 2. The restriction $\chi_{\tilde{\delta}}(x) = \theta_{2n}^i + \theta_{2n}^{-i} = \chi_{\xi_i}(x) + \chi_{\xi_{2n-i}}(x)$, where $\chi_{\xi_i}$ are irreducible characters of $C_{2n}$, then $\tilde{\delta}_i = \xi_i + \xi_{2n-i}$ for $1 \leq i \leq n - 1$. Dually one has that $\hat{\delta}_i = \delta_i$ for $1 \leq i \leq n - 1$. In this way, one obtains that

$$
\tilde{\delta}_0^\pm = \tilde{\xi}_0, \quad \hat{\delta}_1 = \xi_i \oplus \xi_{2n-i} (i = 1, 2, \ldots, n - 1), \quad \tilde{\delta}_n^\pm = \xi_n, \\
\hat{\xi}_0 = \delta_0^+ \oplus \delta_0^-, \quad \hat{\xi}_i = \tilde{\xi}_{2n-i} = \delta_i (i = 1, 2, \ldots, n - 1), \quad \hat{\xi}_n = \delta_n^+ \oplus \delta_n^-.
$$

Note that the embedding in this case is taken as either $\xi_1 \oplus \xi_{-1} = \tilde{\delta}_1$ or $\tilde{\xi}_1 = \delta_1$. If follows from the fusion rules of $C_{2n}$ and $D_n$ modules that

$$
(\xi_1 \oplus \xi_{-1}) \otimes \tilde{\delta}_0^+ = \delta_1, \quad \delta_1 \otimes \tilde{\xi}_0 = 2\hat{\xi}_1, \\
(\xi_1 \oplus \xi_{-1}) \otimes \delta_1 = 2\delta_0^+ \oplus \tilde{\delta}_2, \quad \delta_1 \otimes \hat{\xi}_1 = \tilde{\xi}_0 \oplus \hat{\xi}_2, \\
(\xi_1 \oplus \xi_{-1}) \otimes \delta_i = \delta_{i-1} \oplus \delta_{i+1}, \quad \delta_1 \otimes \hat{\xi}_i = \tilde{\xi}_{i-1} \oplus \tilde{\xi}_{i+1} (2 \leq i \leq n - 2), \\
(\xi_1 \oplus \xi_{-1}) \otimes \tilde{\delta}_{n-1} = \delta_{n-2} \oplus 2\delta_n^+, \quad \delta_1 \otimes \hat{\xi}_{n-1} = \tilde{\xi}_{n-2} \oplus \hat{\xi}_n, \\
(\xi_1 \oplus \xi_{-1}) \otimes \tilde{\delta}_n^+ = \delta_{n-1}, \quad \delta_1 \otimes \hat{\xi}_n = 2\hat{\xi}_{n-1}.
$$

Therefore the representation graph $\mathcal{R}_{(\xi_1, \xi_{-1})}(D_n)$ realizes the Dynkin diagram $D_{n+1}^{(2)}$ and $\mathcal{R}_{\delta_i}(C_{2n})$ realizes the Dynkin diagram $C_n^{(1)}$, which are depicted as follows.

![Diagram](image-url)
2.4. Realizations of $A_{2n}^{(2)}$ and $C_{n}^{(1)}$ by the pair $(D_{2n}, C_{2n})$. Fix $n \geq 2$, the binary dihedral group $D_{2n} = \langle x, y \rangle$ contains the normal cyclic group $C_{2n} = \langle x^2 \rangle$ with index 4. Similar to subsection 2.3, the general restriction $\chi_{\delta_i}(x^2) = \theta_{4n}^{2i} + \theta_{4n}^{-2i} = \theta_{2n}^i + \theta_{2n}^{-i}$ for $1 \leq i \leq 2n - 1$, i.e. $\delta_i = \xi_i + \xi_{2n-i}$, where $\xi_i$ are irreducible modules of $C_{2n}$. As for induction, $\hat{\xi}_i = \delta_i + \delta_{2n-i}$ due to $\langle x^2 \rangle \triangleleft \langle x \rangle \triangleleft D_{2n}$. Detailed restriction and induction relations are given as follows:

\[
\begin{align*}
\hat{\delta}_0^+ &= \delta_{2n} = \xi_0, & \delta_i &= \delta_{2n-i} = \xi_i \otimes \xi_{2n-i} (i = 1, 2, \ldots, n), \\
\hat{\delta}_0^- &= \delta_{2n}^- = \xi_0, & \hat{\delta}_i &= \hat{\delta}_{2n-i} = \hat{\delta}_i \otimes \delta_{2n-i} (i = 1, 2, \ldots, n).
\end{align*}
\]

Using the embeddings $\hat{\delta}_1 = \xi_1 \otimes \xi_{-1} : C_{2n} \hookrightarrow SU_2$ and $\delta_1 : D_{2n} \hookrightarrow SU_2$, one has the following fusion relations:

\[
\begin{align*}
(\xi_1 \otimes \xi_{-1}) \otimes \hat{\delta}_0^+ &= \delta_1, & \delta_1 \otimes \hat{\delta}_0^- &= 2\hat{\xi}_1, \\
(\xi_1 \otimes \xi_{-1}) \otimes \hat{\delta}_0^- &= 2\delta_0^+ \otimes \delta_2, & \delta_1 \otimes \hat{\delta}_1 &= \hat{\xi}_0 \otimes \hat{\xi}_2, \\
(\xi_1 \otimes \xi_{-1}) \otimes \hat{\delta}_i &= \delta_{i-1} \otimes \delta_{i+1}, & \delta_1 \otimes \hat{\delta}_i &= \hat{\xi}_{i-1} \otimes \hat{\xi}_{i+1} \quad (2 \leq i \leq n - 2), \\
(\xi_1 \otimes \xi_{-1}) \otimes \delta_{n-1} &= \hat{\delta}_{n+2} \otimes \hat{\delta}_n, & \delta_1 \otimes \hat{\delta}_{n-1} &= \hat{\xi}_{n-2} \otimes \hat{\xi}_n, \\
(\xi_1 \otimes \xi_{-1}) \otimes \delta_n &= 2\hat{\delta}_{n-1}^-, & \delta_1 \otimes \hat{\delta}_n &= 2\hat{\xi}_{n-1}.
\end{align*}
\]

Therefore the representation graphs $R_{(\xi_1, \xi_{-1})}(\hat{D}_{2n})$ and $R_{\delta_1}(\hat{C}_{2n})$ realize the twisted affine Dynkin diagram of type $A_{2n}^{(2)}$ and the non-simply laced affine Dynkin diagram of type $C_{n}^{(1)}$, respectively. The exact relations are shown in the following diagrams, where the numbers indicate the degrees of characters.

![Diagram](image)

2.5. Realizations of $E_{6}^{(2)}$ and $E_{4}^{(1)}$ by the pair $(O,T)$. As explained in subsection 2.1.4, when inducing up from the seven irreducible $T$-modules, there are only five different induced modules: $\hat{\tau}_0, \hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_1', \hat{\tau}_0'$. Correspondingly, when restricting down the eight irreducible $O$-modules, there are only five distinct $T$-modules: $\hat{\omega}_0, \hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3, \hat{\omega}_4$. 

![Diagram](image)
The induced modules decompose themselves into sum of irreducible modules as follows:

\[ \tilde{\tau}_0 = \omega_0^+ + \omega_0^- , \quad \tilde{\tau}_1 = \omega_1^+ + \omega_1^- , \quad \tilde{\tau}_2 = \omega_2^+ + \omega_2^- , \quad \tilde{\tau}'_1 = \tilde{\tau}'_1'' = \omega_3 , \quad \tilde{\tau}'_0 = \tilde{\tau}'_0'' = \omega_4 , \]

while the restriction also decomposes into irreducible modules:

\[ \hat{\omega}_0^+ = \hat{\omega}_0^- = \tau_0 , \quad \hat{\omega}_1^+ = \hat{\omega}_1^- = \tau_1 , \quad \hat{\omega}_2^+ = \hat{\omega}_2^- = \tau_2 , \quad \hat{\omega}_3 = \tau'_1 + \tau''_1 , \quad \hat{\omega}_4 = \tau'_0 + \tau''_0 . \]

Using the imbedding \( \tau_1 : T \hookrightarrow SU_2 \) and \( \omega_1^+ : O \hookrightarrow SU_2 \), the fusion products are given explicitly as follows,

\[ \begin{align*}
\tau_1 \otimes \hat{\omega}_0^+ &= \hat{\omega}_1^+ , \\
\tau_1 \otimes \hat{\omega}_1^+ &= \hat{\omega}_0^+ + \hat{\omega}_2^+ , \\
\tau_1 \otimes \hat{\omega}_2^+ &= \hat{\omega}_1^+ + \hat{\omega}_3 , \\
\tau_1 \otimes \hat{\omega}_3 &= 2\hat{\omega}_2^+ + \hat{\omega}_4 , \\
\tau_1 \otimes \hat{\omega}_4 &= \hat{\omega}_3 , \\
\end{align*} \]

\[ \begin{align*}
\omega_1^+ \otimes \tilde{\tau}_0 &= \tilde{\tau}_1 , \\
\omega_1^+ \otimes \tilde{\tau}_1 &= \tilde{\tau}_2 + 2\tilde{\tau}_1, \\
\omega_1^+ \otimes \tilde{\tau}_2 &= \tilde{\tau}_1 + 2\tilde{\tau}_1, \\
\omega_1^+ \otimes \tilde{\tau}'_1 &= \tilde{\tau}_2 + \tilde{\tau}_0, \\
\omega_1^+ \otimes \tilde{\tau}'_0 &= \tilde{\tau}'_1 , \\
\end{align*} \]

Therefore the representation graph \( R_{\tau_1}(\hat{O}) \) realizes the Dynkin diagram of type \( E_6^{(2)} \) and the representation graph \( R_{\omega_1^+}(\tilde{T}) \) realizes the Dynkin diagram of type \( F_4^{(1)} \).

2.6. Realizations of \( D_4^{(3)} \) and \( G_2^{(1)} \) by the pair \( (T,D_2) \). The binary dihedral group \( D_2 \) has five irreducible modules \( \delta_0^+, \delta_1, \delta_2^+ \). As \( D_2 \) is a normal subgroup of the binary tetrahedral group \( T \) (with index 3), the irreducible \( D_2 \)-modules are induced to only three distinct modules of \( T \) as follows:

\[ \hat{\delta}_0^+ = \tau_0 + \tau'_0 + \tau''_0 , \quad \hat{\delta}_1 = \tau_1 + \tau'_1 + \tau''_1 , \quad \hat{\delta}_2^+ = \hat{\delta}_2^- = \delta_0^- = \tau_2 , \]

where \( \{\tau_0^{(i)}, \tau_1^{(i)}, \tau_2 | i = 0, 1, 2\} \) form the complete set of irreducible \( T \)-modules.

Correspondingly, the restrictions of the 7 irreducible \( T \)-modules give rise to three distinct \( D_2 \)-modules:

\[ \tilde{\tau}_0 = \tilde{\tau}'_0 = \tilde{\tau}'_0'' = \delta_0^+ , \quad \tilde{\tau}_1 = \tilde{\tau}'_1 = \tilde{\tau}'_1'' = \delta_1 , \quad \tilde{\tau}_2 = \tilde{\tau}_2 = \delta_2^+ + \delta_0^- + \delta_2^- . \]

Using the embedding \( \delta_1 = \tilde{\tau}_1 \) and \( \tau_1 \), we get the fusion rules by those of \( E_6^{(1)} \) and \( D_4^{(1)} \) respectively as follows:

\[ \begin{align*}
\delta_1 \otimes \tilde{\tau}_0 &= \tilde{\tau}_1 , \\
\delta_1 \otimes \tilde{\tau}_1 &= \tilde{\tau}_0 + \tilde{\tau}_2 , \\
\delta_1 \otimes \tilde{\tau}_2 &= 3\tilde{\tau}_1 , \\
\tau_1 \otimes \tilde{\tau}_0 &= \tilde{\tau}_1 , \\
\tau_1 \otimes \tilde{\tau}_1 &= \tilde{\tau}_0 + \tilde{\tau}_2 , \\
\tau_1 \otimes \tilde{\tau}_2 &= 3\tilde{\tau}_1 , \\
\end{align*} \]

Thus the representation graphs \( R_{\delta_1}(\tilde{T}) \) and \( R_{\tau_1}(\hat{D}_2) \) realize the Dynkin diagrams \( D_4^{(3)} \) and \( G_2^{(1)} \) respectively.
The Dynkin diagram $A_1^{(1)}$ was realized by the McKay correspondence, this subsection gives another realization as a by-product of the McKay-Slodowy correspondence. The binary dihedral group $D_2$ has the cyclic group $C_2$ as a normal subgroup of index 4, so the induction of the two irreducible $C_2$ modules $\xi_i$ still gives two different modules of $D_2$, while the restriction of the five irreducible modules of $D_2$ is down to two different modules of $C_2$. The exact relations of the restriction and induction go as follows:

$$\hat{\delta}_0^+ = \hat{\delta}_2^+ = \xi_0,$$

$$\hat{\xi}_0 = \delta_0^+ \oplus \delta_0^- \oplus \delta_2^+ \oplus \delta_2^-,$$

$$\hat{\delta}_1 = 2\xi_1,$$

$$\hat{\xi}_1 = 2\delta_1,$$

where as usual the set of irreducible modules are $D_2^* = \{\delta_0^+, \delta_1, \delta_2^+\}$ and $C_2^* = \{\xi_0, \xi_1\}$.

The embeddings $2\xi_1 : C_2 \hookrightarrow SU_2$ and $\delta_1 : D_2 \hookrightarrow SU_2$ determine the following fusion rule (using those of $D_4^{(1)}$ and $A_1^{(1)}$):

$$2\xi_1 \otimes \delta_0^+ = \delta_1,$$

$$\delta_1 \otimes \hat{\xi}_0 = 2\hat{\xi}_1,$$

$$2\xi_1 \otimes \hat{\delta}_1 = 4\delta_0^+,$$

$$\delta_1 \otimes \hat{\xi}_1 = 2\hat{\xi}_0.$$

Subsequently the representation graphs $\mathcal{R}_{2\xi_1}(\tilde{D}_2)$ and $\mathcal{R}_{\delta_1}(\tilde{C}_2)$ realize the affine Dynkin diagram $A_2^{(2)}$ and the affine Dynkin diagram $A_1^{(1)}$, respectively.

**Remark 2.1.** In subsections 2.2, 2.3, 2.5 and 2.6, $\alpha_{\tilde{A}} = |G : N|^{-1}(\dim \hat{\phi}_i)_{i \in I}$ and $\alpha_{\tilde{B}} = (\dim \tilde{\rho}_i)_{i \in \hat{I}}$ are unique eigenvectors (up to constants) with zero eigenvalue for the Cartan matrices $C_{\tilde{A}}$ and $C_{\tilde{B}}$ respectively, while in subsection 2.4 and 2.7, $\alpha_{\tilde{A}}$ and $\alpha_{\tilde{B}}$ are eigenvectors with zero eigenvalue for $C_{\tilde{A}}$ and $C_{\tilde{B}}^T$ respectively. Here $\dim \tilde{\rho}_i (i \in \hat{I})$ and $|G : N|^{-1}\dim \hat{\phi}_i (i \in I)$ are two sets of relatively prime integers.

### 3. Poincaré Series

#### 3.1. Poincaré series for a pair of finite groups $N \triangleleft G$.

Let $G$ be a finite group and $V$ be a $G$-module, the Poincaré series $m_V(t)$ for the $G$-invariants in the tensor algebra $T(V) = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ was given by Benkart \([\mathbb{1}]\) and she shows that the general Poincaré series for a $G$-irreducible module also admits a nice compact formula similar to the situation of the symmetric tensors.
In particular, when $G$ is a finite subgroup of $SU_2$, the Poincaré series can be computed via the McKay correspondence. Moreover, the exponents of the affine Lie algebras of simply laced type are realized via the Poincaré series $m_V(t)$, except when $G$ is a cyclic group of odd order, i.e. the Dynkin diagram $A_2^{(1)}$.

To understand the general case and recover the missing exponents for all affine Lie algebras, we consider the Poincaré series for certain modules that are both $N$-modules and $G$-modules for a given pair of finite groups $N \triangleleft G$ in view of the McKay-Slodowy correspondence. We will show that the Poincaré series for both $V$ and $V|_N$ in the tensor algebra $T(V)$ will provide an answer in the general situation. First we recall the following results.

**Lemma 3.1.** [11 Cor. 18.7.1] Let $\chi_{\phi_i}$ and $\chi_{\phi_j}$ be the characters of $N$ afforded by simple modules $\phi_i$ and $\phi_j$ respectively. Then the induced $G$-modules $\hat{\phi}_i \cong \hat{\phi}_j$ if and only if $\chi_{\phi_i}$ and $\chi_{\phi_j}$ are $G$-conjugate.

**Lemma 3.2.** [11 Cor. 18.7.5] Let $N$ be a normal subgroup of $G$. Then the number of nonisomorphic $G$-modules induced from the simple $N$-modules is equal to the number of conjugacy classes of $G$ contained in $N$.

**Lemma 3.3.** [11 Cor. 18.11.2] (Clifford’s theorem) Let $N$ be a normal subgroup of $G$ and let $\rho$ be a simple $G$-module. There exists a simple submodule $\phi$ of $\rho$ and an integer $e \geq 1$ such that $\rho \cong e(\oplus_{t \in T} \phi)$, where $H$ is the inertia group of $\phi$, $T$ is a left transversal for $H$ in $G$ and the conjugates $\rho_i$ $(t \in T)$ of $\phi$ are pairwise non-isomorphic simple $N$-modules.

With these preparations, we have the following result.

**Lemma 3.4.** Let $N \triangleleft G$, and let $\{\rho_i|i \in I_G\}$ (resp. $\{\phi_i|i \in I_N\}$) be the set of pairwise nonisomorphic $\mathbb{C}$-irreducible modules of $G$ (resp. $N$). Let $\{\hat{\rho}_i|i \in \hat{I}\}$ be the set of non-isomorphic $N$-restrictions of irreducible $G$-modules with $\hat{\rho}_i \cap \hat{\rho}_j = 0$ for $i,j \in \hat{I}$, let $\{\hat{\phi}_i|i \in \hat{I}\}$ be the set of non-isomorphic induced $G$-modules. Then $|\{\hat{\rho}_i|i \in \hat{I}\}| = |\{\hat{\phi}_i|i \in \hat{I}\}|$, and this common number is equal to $|\Upsilon(N)|$, where $\Upsilon(N) = \Upsilon \cap N$ and $\Upsilon$ is a fixed set of conjugacy class representatives of $G$.

**Proof.** We define a map $f : \hat{\rho}_i |i \in \hat{I}| \rightarrow \hat{\phi}_i |i \in \hat{I}|$ by

$$f(\hat{\rho}_i) = \hat{\phi}_i \quad \text{provided that} \quad \rho_i \preceq \hat{\phi}_k \quad \text{and} \quad \phi_k \preceq \hat{\rho}_i,$$

where $\preceq$ means isomorphic to a submodule. By Frobenius reciprocity $(\chi_{\rho_i}, \chi_{\phi_k})_G = (\chi_{\phi_k}, \chi_{\rho_i})_H$, so the map is given simply as $f(\hat{\rho}_i) = \hat{\phi}_i$ provided that $\phi_k \preceq \hat{\rho}_i$.

We show the map is well-defined. Suppose $\hat{\rho}_i \cong \hat{\rho}_j$ for some $i,j \in \hat{I}$, such that $f(\hat{\rho}_i) = \hat{\rho}_k$ and $f(\hat{\rho}_j) = \hat{\phi}_l$ with $\phi_k \preceq \hat{\rho}_i$ and $\hat{\phi}_l \preceq \hat{\rho}_j$ respectively. Then $\phi_k$ and $\hat{\phi}_l$ are $G$-conjugate by Lemma 3.3. Namely, $\chi_{\phi_k}$ and $\chi_{\phi_l}$ are $G$-conjugate. By Lemma 3.1, $\hat{\phi}_k \cong \hat{\phi}_l$.

We will verify $f$ is a bijective. The surjective is guaranteed by the definition of $f$. Let $\hat{\rho}_i \not\cong \hat{\rho}_j$ for any $i,j \in \hat{I}$, such that $f(\hat{\rho}_i) = \hat{\phi}_k$ and $f(\hat{\rho}_j) = \hat{\phi}_l$ with $\phi_k \preceq \hat{\rho}_i$ and $\phi_l \preceq \hat{\rho}_j$ respectively. Since $\hat{\rho}_i \cap \hat{\rho}_j = 0$, we have $\phi_k \not\cong \phi_l$. Thus $\chi_{\phi_k} \neq \chi_{\phi_l}$. $N$ is a normal subgroup of $G$ implies that $\chi_{\phi_k}$ and $\chi_{\phi_l}$ are not $G$-conjugate. By Lemma 3.1 there is $\hat{\phi}_k \not\cong \hat{\phi}_l$. Then $f$ is an injective.

By $f$ is a bijective and Lemma 3.2 we have $|\{\hat{\rho}_i|i \in \hat{I}\}| = |\{\hat{\phi}_i|i \in \hat{I}\}| = |\Upsilon(N)|$. This completes the proof.

Let $G$ be a finite group equipped with a faithful module. Steinberg [24] studied the decomposition of the tensor product of the faithful module and any irreducible module in terms of characters. We have an analogous result for a pair of finite groups $N \triangleleft G$ in view of Lemma 3.3.
Lemma 3.5. Let $N \leq G$ be a pair of finite groups. Let \{\hat{\rho}_i | i \in \hat{I}\} (resp. \{\hat{\phi}_i | i \in \hat{I}\}) be the set of $N$-restriction modules of irreducible $G$-modules (resp. induced $G$-modules of irreducible $N$-modules). Let \rho be a faithful $N$-module which is also the restriction of a faithful $G$-module with $d = \chi_{\rho}(1)$. Then,

1. The column vectors $(\chi_{\hat{\rho}_i}(g))$ and $(\chi_{\hat{\phi}_i}(g))$ are respectively the eigenvectors of the matrices $(\delta \rho_{ji} - a_{ij})$ and $(\delta \rho_{ji} - b_{ij})$

   with eigenvalue $d - \chi_{\rho}(g)$, where $g$ runs through $\Upsilon(N) = N \cap \Upsilon$, and $\Upsilon$ is a set of conjugacy class representatives of $G$.

2. In particular, the column vectors $(\hat{d}_i)$ and $(\hat{\hat{d}}_i)$ are eigenvectors with eigenvalue 0 respectively, where $\hat{d}_i = \chi_{\hat{\rho}_i}(1)$ and $\hat{\hat{d}}_i = \chi_{\hat{\phi}_i}(1)$.

Theorem 3.6. Let $N \leq G$ be a pair of finite groups and \{\hat{\rho}_i | i \in \hat{I}\} the set of complex $N$-restriction of irreducible $G$-modules. Let $\tilde{\Lambda}$ be the adjacency matrix of the representation graph $\mathcal{R}_V(G)$ and $\mathcal{M}_1$ the matrix $I - t \tilde{\Lambda}^T$ with the $i$th column replaced by $\hat{\delta} = (1, 0, \ldots, 0)^T \in \mathbb{R}^{\hat{I}}$. Assume that $V$ is a faithful (restriction) $N$-module such that $V \cong V^*$. Then the Poincaré series $\tilde{m}^i(t) = \sum_{k \geq 0} \tilde{m}_k^i t^k$ of the multiplicities of $\hat{\rho}_i$ in $T(V) = \oplus_{k \geq 0} V \otimes k$ is given by

$$
\tilde{m}^i(t) = \frac{\det(\mathcal{M}_i)}{\det(I - t \tilde{\Lambda}^T)} = \prod_{g \in \Upsilon(N)} \frac{\det(\mathcal{M}_i)}{(1 - \chi_V(g)t)},
$$

where $\chi_V$ is the character of $V$ and $\Upsilon(N) = N \cap \Upsilon$, and $\Upsilon$ is a fixed set of conjugacy representatives of $G$.

Proof. Note that $\tilde{m}_k^i = \dim(\text{Hom}_N(\hat{\rho}_i, V \otimes k))$ and $\hat{\rho}_i$ is trivial iff $i = 0$, using the argument of [1] Thm. 2.1 it follows that

$$
\tilde{m}^i(t) = \sum_{k \geq 0} \dim(\text{Hom}_N(\hat{\rho}_i, V \otimes k)) t^k
= \delta_{i,0} + t \sum_{k \geq 1} \dim(\text{Hom}_N(V \otimes \hat{\rho}_i, V \otimes (k-1))) t^{k-1}
= \delta_{i,0} + t \sum_{k \geq 1} \dim(\sum_j a_{ji} \text{Hom}_N(\hat{\rho}_j, V \otimes (k-1))) t^{k-1}
= \delta_{i,0} + t \sum_j a_{ji} \sum_{k \geq 0} \dim(\text{Hom}_N(\hat{\rho}_j, V \otimes k)) t^k
= \delta_{i,0} + t \sum_j a_{ji} \tilde{m}^j(t).
$$

Write $\mathbf{m} = (\tilde{m}^i(t))_{i \in \hat{I}}$, the column vector formed by Poincaré series, then (3.2) can be written as the matrix identity:

$$(I - t \tilde{\Lambda}^T) \mathbf{m} = \hat{\delta}$$. Then the first equality of (3.1) follows from Cramer’s rule.

Assume $\dim V = d$, $d - \chi_V(g)$ be all eigenvalues of the matrix $dI - \tilde{\Lambda}$, where $g$ runs over the set $\Upsilon(N) = \Upsilon \cap N$ and $\Upsilon$ is a fixed set of representatives of conjugacy classes of $G$ (see Lemma 3.5(1)). Therefore $\det(tI - \tilde{\Lambda}) = \prod_{g \in \Upsilon(N)} (t - \chi_V(g))$. Denote $n = |\hat{I}| = |\Upsilon(N)|$, then

$$
\det(I - t \tilde{\Lambda}^T) = t^n \det(t^{-1}I - \tilde{\Lambda}^T) = t^n \prod_{g \in \Upsilon(N)} (t^{-1} - \chi_V(g)) = \prod_{g \in \Upsilon(N)} (1 - \chi_V(g)t),
$$

which is the second identity in (3.1). \hfill \Box

The following result is obtained similarly as Theorem 3.6.
Theorem 3.7. Let \( N \trianglelefteq G \) be a pair of finite subgroups and \( \hat{\phi}_i (i \in \hat{I}) \) the induced \( G \)-module of an irreducible \( N \)-module \( \phi_i \). Let \( V \) be a finite-dimensional self-dual \( G \)-module \( V \cong V^* \). Let \( \tilde{B} \) be the adjacency matrix of the representation graph \( \mathcal{R}_V(N) \) and \( M_i^g \) the matrix \( I - tB^T \) with the \( i \)th column replaced by \( \delta = (1, 0, \ldots, 0)^T \in \mathbb{R}^{|I|} \). Then the Poincaré series \( \hat{m}_i(t) = \sum_{k \geq 0} \hat{m}_i^k t^k \) of the multiplicities of \( \hat{\phi}_i \) in \( T(V) = \oplus_{k \geq 0} V^\otimes k \) is given by

\[
\hat{m}_i(t) = \frac{\det(M_i^g)}{\det(I - tB^T)} = \frac{\det(M_i^g)}{\prod_{g \in \mathcal{Y}(N)} (1 - \chi_V(g)t)},
\]

where \( \chi_V \) is the character of \( V \), \( \mathcal{Y}(N) = N \cap \mathcal{Y} \), and \( \mathcal{Y} \) is a fixed set of conjugacy representatives of \( G \).

Remark 3.8. If \( G = N \), both the identities (3.1) and (3.3) coincide and specialize to the result [1, Thm 2.1].

3.2. Example \( A_4 \triangleleft S_4 \). The alternating group \( A_4 = \langle (123), (124) \rangle \cong K_4 \times C_3 \) and its four irreducible modules \( \phi_i \) can be lifted from the one-dimensional modules of the Klein subgroup \( K_4 \), where \( \dim(\phi_i) = 1 (i = 0, 1, 2) \) and \( \dim(\phi_3) = 3 \). Similarly the irreducible \( S_4 \)-modules \( \rho_i \) can from induced by those of \( A_4 \)-modules as follows:

\[
\hat{\phi}_0 = \rho_0^+ \oplus \rho_0^- \quad \hat{\phi}_1 = \hat{\phi}_2 = \rho_1 \quad \hat{\phi}_3 = \rho_2^+ \oplus \rho_2^-.
\]

Table 4 and 5 list the character tables for \( S_4 \) and \( A_4 \).

| \( \chi \backslash g \) | Character table of \( S_4 \) |
|--------------------------|---------------------------|
| \( C_G(g) \) | 1 | (12) | (123) | (1234) | (12)(34) | 3 |
| \( \chi_{\rho_0^+} \) | 1 | 1 | 1 | 1 | 1 |
| \( \chi_{\rho_0^-} \) | 1 | -1 | 1 | -1 | 1 |
| \( \chi_{\rho_1} \) | 2 | 0 | -1 | 0 | 2 |
| \( \chi_{\rho_3} \) | 3 | 1 | 0 | -1 | -1 |
| \( \chi_{\rho_2^-} \) | 3 | -1 | 0 | 1 | -1 |

| \( \chi \backslash g \) | Character table of \( A_4 \) |
|--------------------------|---------------------------|
| \( C_G(g) \) | 1 | (123) | (132) | (12)(34) | 3 |
| \( \chi_{\phi_0} \) | 1 | 1 | 1 | 1 |
| \( \chi_{\phi_1} \) | 1 | \( \theta_3 \) | \( \theta_3^2 \) | 1 |
| \( \chi_{\phi_2} \) | 1 | \( \theta_3^2 \) | \( \theta_3 \) | 1 |
| \( \chi_{\phi_3} \) | 3 | 0 | 0 | -1 |

On the other hand, the restriction of \( \rho_i \) is similarly given as follows,

\[
\hat{\rho}_0^+ = \hat{\rho}_0^- = \phi_0, \quad \hat{\rho}_1 = \hat{\phi}_1 \oplus \hat{\phi}_2, \quad \hat{\rho}_2^+ = \hat{\rho}_2^- = \phi_3.
\]

Therefore the \( A_4 \)-restrictions of \( \rho_i \) form the set \( \{ \hat{\rho}_0^+, \hat{\rho}_1, \hat{\rho}_2^+ \} \) while the inductions of \( A_4 \)-irreducible modules form the set \( \{ \hat{\phi}_0, \hat{\phi}_1, \hat{\phi}_3 \} \) with \( \hat{I} = \{ 0, 1, 2 \} \) and \( \hat{I} = \{ 0, 1, 3 \} \). Note that \( \phi_3 \) is a faithful \( A_4 \)-module and \( \hat{\rho}_2^+ = \phi_3 \), we have the following fusion rules:

\[
\phi_3 \otimes \hat{\rho}_0^+ = \hat{\rho}_2^+, \quad \rho_2^+ \otimes \hat{\phi}_0 = \phi_3,
\]

\[
\phi_3 \otimes \hat{\rho}_1 = 2\hat{\rho}_2^+, \quad \rho_2^+ \otimes \hat{\phi}_1 = \phi_3,
\]

\[
\phi_3 \otimes \hat{\rho}_2^+ = \hat{\rho}_0^+ \oplus 2\hat{\rho}_2^+ \oplus \hat{\rho}_1, \quad \rho_2^+ \otimes \hat{\phi}_3 = \hat{\phi}_0 \oplus 2\hat{\phi}_3 \oplus 2\hat{\phi}_1.
\]
The corresponding graphs $\mathcal{R}_{\phi_3}(\tilde{S}_4)$ and $\mathcal{R}_{\rho_2}(\tilde{A}_4)$ are shown below, where the numbers inside the nodes are the degrees of characters.

So the adjacency matrices of the representation graphs $\mathcal{R}_{\phi_3}(\tilde{S}_4)$ and $\mathcal{R}_{\rho_2}(\tilde{A}_4)$ are

$$\tilde{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

It follows from Theorem 3.6 and 3.7 that the pair $(S_4, A_4)$ gives rise to $\det(I - t\tilde{A}^T) = \det(I - t\tilde{B}^T) = \Pi_{g \in \Upsilon(N)}(1 - \chi_V(g)t) = (1 - 3t)(1 + t) = 1 - 2t - 3t^2$ for $V = \phi_3 = \tilde{\rho}_2$. Subsequently the Poincaré series of the restrictions of $A_4$-modules and the induced $S_4$-modules associated with $T(V) = \oplus_{k \geq 0}V^{\otimes k}$ are computed by

$$\hat{m}^0(t) = \hat{m}^0(t) = \frac{1 - 2t - 2t^2}{1 - 2t - 3t^2} = 1 + t^2 + 2t^3 + 7t^4 + 20t^5 + 61t^6 + 182t^7 + \cdots$$

$$\hat{m}^2(t) = \hat{m}^3(t) = \frac{t}{1 - 2t - 3t^2} = t + 2t^2 + 7t^3 + 20t^4 + 61t^5 + 182t^6 + 547t^7 + \cdots$$

$$\hat{m}^1(t) = 2\hat{m}^1(t) = \frac{2t^2}{1 - 2t - 3t^2} = 2t^2 + 4t^3 + 14t^4 + 40t^5 + 122t^6 + 364t^7 + \cdots.$$

### 4. Poincaré Series for SU2

If $N \triangleleft G$ is any pair of finite subgroups of SU2 and $V = \mathbb{C}^2$ in Section 2, then the Poincaré series $\hat{m}^0(t)$ and $\hat{m}^0(t)$ are $N$-invariants and $G$-invariants inside the tensor algebra $T(V)$.

**Theorem 4.1.** Let $N \triangleleft G$ be a pair of finite subgroups of SU2 and $V = \mathbb{C}^2$. Then the Poincaré series $\hat{m}^0(t)$ and $\hat{m}^0(t)$ are $N$-invariants $T(V)^N$ and G-invariants $T(V)^G$ in $T(V) = \oplus_{k \geq 0}V^{\otimes k}$ given by

$$\hat{m}^0(t) = \hat{m}^0(t) = \frac{\det(I - tA^T)}{\det(I - t\tilde{A}^T)} = \frac{\det(I - tB^T)}{\det(I - t\tilde{B}^T)} = \prod_{g \in \Upsilon(N)}(1 - \chi_V(g)t) = \prod_{g \in \Upsilon(N)}(1 - \chi_V(g)t),$$

where $\tilde{A}$ (resp. $\tilde{B}$) are the adjacency matrices of twisted (resp. untwisted nonsimply laced) affine Dynkin diagrams $\mathcal{R}_V(\tilde{G})$ (resp. $\mathcal{R}_V(\tilde{N})$), and $A$ and $B$ are the adjacency matrices of the finite Dynkin diagrams obtained by removing the special node corresponding to the trivial module. $\chi_V$ is the character of $V$ and $\Upsilon(N) = \Upsilon \cap N$, where $\Upsilon$ is a fixed set of conjugacy class representative of $G$.

Besides the special vertex, the Poincaré series for other vertices of restriction modules and induced modules of the Dynkin diagrams also have close relationship.
of the simply laced affine Lie algebras (except $A_1$).

**Remark 4.3**

**Theorem** (cf. [8]) that the generalized Poincaré series can be written as a quotient of the characteristic polynomials of finite and $N$-invariants in $T(V) = \oplus_{k \geq 0} V^k$ satisfy the following relation:

$$
\hat{m}^i(t) = \begin{cases} 
\hat{m}^i(t), & i' \text{ is a long root in } R_V(\hat{N}) \\
G : N \hat{m}^i(t), & i' \text{ is a short root in } R_V(\hat{N})
\end{cases}
$$

where $\hat{\phi}_\alpha = f(\hat{\psi}_\alpha)$, $f$ is the bijective in Lemma 5.4. In particular, for $(G, N) = (D_2, C_2), (D_{2n}, C_{2n}) (n \geq 2)$ and $V = \mathbb{C}^2$, one has that

$$
\hat{m}^i(t) = \begin{cases} 
\hat{m}^0(t), & i \text{ is the special vertex of } R_V(\hat{G}) \\
2\hat{m}^i(t), & i \text{ (resp. } i') \text{ in the finite Dynkin diagram of } R_V(\hat{G}) \text{ (resp. } R_V(\hat{N})).
\end{cases}
$$

**Remark 4.3.** For the pairs of the finite subgroups $N < G \leq SU_2$ in Section 2 and the natural $G$-module $V = \mathbb{C}^2$, the Poincaré series of the multiplicities of the trivial $N$-module inside $T(V) = \oplus_{k \geq 0} V^k$ coincides with the usual Poincaré series of $N$-invariants and $G$-invariants in $T(V) = \oplus_{k \geq 0} V^k$. In this sense Theorem 4.1 is analogous to the generalized Poincaré series $[\tilde{P}_{(G,N)}(t)]_0$ associated with the symmetric algebras $S(V)$ defined by Stekolschik [25], who proved a generalized Ebeling’s theorem (cf. [8]) that the generalized Poincaré series can be written as a quotient of the characteristic polynomials of finite and affine Coxeter transformations,

$$
[\tilde{P}_{(G,N)}(t)]_0 = \frac{\det M_0(t)}{\det M(t)},
$$

where $\det M_0(t) = \det(t^2I - C)$, $\det M(t) = \det(t^2I - C_a)$, $C$ and $C_a$ are the Coxeter transformation and its affine analog respectively.

Recall that the Coxeter transformation is the product of all simple reflections of the root system (similar for affine case), while the spectrum of the Coxeter transformation is closely related with that of the Cartan matrix [4,6]. Based on this, Benkart [1] showed that the Poincaré series for invariants of group $G \leq SU_2$ in $T(V)$ can be used to get the exponents and Coxeter number of the simply laced affine Lie algebras (except $A_{2n}^{(1)}$). We now show that all the remaining cases are recovered by the relative Poincaré series for the restriction and induction modules associated with the pairs of subgroups in view of the McKay-Slodowy correspondence.

The exponents and Coxeter numbers of the twisted and untwisted non-simply laced types are displayed in Table 6.

| Dynkin diagrams | Exponents | Coxeter number |
|-----------------|-----------|---------------|
| $A_1$           | 1         | 2             |
| $B_n$           | $1, 3, 5, \ldots, 2n - 1$ | $2n$         |
| $C_n$           | $1, 3, 5, \ldots, 2n - 1$ | $2n$         |
| $F_4$           | 1, 5, 7, 11 | 12           |
| $G_2$           | 1, 5      | 6             |
| $A_1^{(1)}$     | 0, 1      | 1             |
| $A_2^{(2)}$     | 0, 2      | 2             |
| $A_2^{(2)}$     | 0, 1, \ldots, $\ell$ | $\ell$ |
| $B_{2l+1}^{(1)}$ | 0, 1, \ldots, $\ell - 1$, $\ell$, $\ell + 1, \ldots, 2\ell$ | $2\ell$ |
| $B_{2l}^{(1)}$, $A_{2l-1}^{(2)}$ | $0, 2, \ldots, 2\ell - 2, 2\ell - 1, 2\ell, \ldots, 2(2\ell - 1)$ | $2(2\ell - 1)$ |
| $C_1^{(1)}$, $D_1^{(2)}$ | 0, 1, \ldots, $\ell$ | $\ell$ |
| $F_4^{(1)}$, $E_6^{(2)}$ | 0, 2, 3, 4, 6 | 6 |
| $G_2^{(1)}$, $D_4^{(3)}$ | 0, 1, 2 | 2 |
Lemma 4.4. [4, Thm. 2] Let $A = (A_{ij})_{l \times l}$ be a generalized Cartan matrix such that (1) $A_{ii} = 2$, (2) $A_{ij} = 0$ if and only if $A_{ji} = 0$, and (3) the primitive graph of $A$ has no odd cycles. Let $R$ be the Coxeter transformation of $A$. Then there are $l$ complex numbers $\eta_1, \ldots, \eta_l$ satisfying $\eta_j + \eta_{j+1} = 2 \pi \sqrt{-1}$ for $1 \leq j \leq l$, such that the spectrum of $R$ is $e^{\eta_1}, \ldots, e^{\eta_l}$ and the spectrum of $A$ is $4 \cosh^2(\eta_1/4), \ldots, 4 \cosh^2(\eta_l/4)$.

We have the following result to realize the exponents of affine Kac-Moody Lie algebras in relation with the McKay-Slodowy correspondence.

Theorem 4.5. Let $N < G \leq SU_2$ and $V = \mathbb{C}^2$. Let $\tilde{A}$ (resp. $A$) be the adjacency matrix of the nonsimply laced affine (resp. finite) Dynkin diagrams $\mathcal{R}_V(\tilde{G})$ (resp. of $\mathcal{R}_V(G)$). Let $\Delta$ (resp. $\Delta$) be the set of exponents $\tilde{m}_i$ (resp. $m_i$) of the affine Lie algebra (resp. finite simple Lie algebra) associated with the Dynkin diagram and $\tilde{h}$ (resp. $h$) be the affine (resp. finite) Coxeter number. Then the Poincaré series for $N$-invariants and $G$-invariants in $T(V) = \oplus_{k \geq 0} V^\otimes k$ are

$$m^0(t) = \frac{\det(I - tA^T)}{\det(I - t\tilde{A}^T)} = \prod_{g \in \Upsilon(N)} \left(1 - \chi_V(g)t\right) = \prod_{\tilde{m}_i \in \Delta} \left(1 - 2 \cos\left(\frac{\tilde{m}_i \pi}{\tilde{h}}\right)t\right),$$

where $\chi_V$ is the character of $V$, $\Upsilon(N) = N \cap \Upsilon$, and $\Upsilon$ is a fixed set of conjugacy representatives of $G$.

Proof. The affine Cartan matrix $C_{\tilde{A}} = 2I - \tilde{A}$ in view of the McKay-Slodowy correspondence (see Sect. 2). Note the duality of the exponents $\tilde{m}_i + \tilde{m}_{n-i+1} = \tilde{h}$ ($n = |\tilde{I}|$) implies that for each $i \in \tilde{I}$, the index set of $\tilde{\Delta}$

$$2\tilde{m}_i \pi \sqrt{-1} = \frac{2\tilde{m}_i \pi \sqrt{-1}}{\tilde{h}} + \frac{2\tilde{m}_{n-i+1} \pi \sqrt{-1}}{\tilde{h}} = 2\pi \sqrt{-1}.$$

It follows from Lemma 4.4 that the roots of the characteristic polynomial of Cartan matrix $C_{\tilde{A}}$ are

$$4 \cosh^2\left(\frac{\tilde{m}_1 \pi \sqrt{-1}}{2\tilde{h}}\right), 4 \cosh^2\left(\frac{\tilde{m}_2 \pi \sqrt{-1}}{2\tilde{h}}\right), \ldots, 4 \cosh^2\left(\frac{\tilde{m}_n \pi \sqrt{-1}}{2\tilde{h}}\right).$$

Therefore the roots of the characteristic polynomial of $\tilde{A} = 2I - C_{\tilde{A}}$ are

$$4 \cosh^2\left(\frac{m_1 \pi \sqrt{-1}}{2h}\right) - 2, 4 \cosh^2\left(\frac{m_2 \pi \sqrt{-1}}{2h}\right) - 2, \ldots, 4 \cosh^2\left(\frac{m_n \pi \sqrt{-1}}{2h}\right) - 2,$$

or

$$2 \cos\left(\frac{m_1 \pi}{h}\right), 2 \cos\left(\frac{m_2 \pi}{h}\right), \ldots, 2 \cos\left(\frac{m_n \pi}{h}\right).$$

Therefore

$$\det(I - t\tilde{A}^T) = \det(I - t\tilde{A}) = \prod_{\tilde{m}_i \in \Delta} \left(1 - 2 \cos\left(\frac{\tilde{m}_i \pi}{\tilde{h}}\right)t\right).$$

Similarly the adjacency matrix $A$ of the finite Dynkin diagram $\mathcal{R}_V(G)$ also has the property

$$\det(I - tA^T) = \prod_{m_i \in \Delta} \left(1 - 2 \cos\left(\frac{m_i \pi}{h}\right)t\right).$$

Remark 4.6. As a consequence of formula (4.2), we see that the character values $\chi_V(g)$ ($g \in \Upsilon(N)$) coincide with $2 \cos(\tilde{m}_i \pi/\tilde{h})$ ($i \in \tilde{I}$). In particular, when $N = 1$, the formula (4.2) give rise to the exponents of all affine Lie algebras except $A^{(1)}_{2n}$. \qed
4.1. Closed form of Poincaré series for $N$-invariants and $G$-invariants. In this section, we give closed-form expressions of the Poincaré series $\tilde{m}^0(t) = \tilde{m}^0(t)$ for $N < G \leq SU_2$ and $V \cong \mathbb{C}^2$ in $T(V) = \oplus_{k \geq 0} V^k$. The closed-form expressions of $\tilde{m}^0(t) = \tilde{m}^0(t)$ as the trivial $N$-module in $T(V)$ have been considered in [1], we revisit the closed-form expressions from the viewpoint of the adjacency matrices $A$ and $\tilde{A}$ of non-simply laced Dynkin diagrams. For the three pairs $(O, T), (T, D_2), (D_2, C_2)$, the Poincaré series $\tilde{m}^i(t)$ and $\tilde{m}^i(t)$ are given for $i$ running through $I$ and $\tilde{I}$, respectively. It is known that the Cartan matrices of finite Dynkin diagrams are closely related with the Chebyshev polynomials of the first and second kinds [11][13]. We also start by recalling some simple facts about Chebyshev polynomials (also see [19]).

The Chebyshev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are recursively defined by:

\begin{align}
T_0(t) &= 1, \quad T_1(t) = t, \quad T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t) \quad \text{for all } n \geq 1, \\
U_0(t) &= 1, \quad U_1(t) = 2t, \quad U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t) \quad \text{for all } n \geq 1.
\end{align}

The additive closed forms of the polynomials $T_n(t)$ and $U_n(t)$ are

\begin{align}
T_n(t) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} t^{n-2i}(t^2 - 1)^i = t^n \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} (1 - t^{-2})^i, \\
U_n(t) &= \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n}{i} (2t)^{n-2i}.
\end{align}

In addition, the multiplicative expressions of $T_n(t)$ and $U_n(t)$ are the following well known formulas:

\begin{align}
T_n(t) &= 2^{n-1} \prod_{i=1}^{n} \left( t - \cos \left( \frac{2i - 1}{2n} \pi \right) \right), \\
U_n(t) &= 2^n \prod_{i=1}^{n} \left( t - \cos \left( \frac{\pi i}{n+1} \right) \right).\end{align}

Moreover, $T_n(t)$ and $U_n(t)$ are related by

\begin{equation}
T_n(t) = U_n(t) - tU_{n-1}(t).
\end{equation}

4.1.1. The pair $(D_{2(n-1)}, D_{n-1})$. From Sect. [22] the imbedding of the subgroup $D_{n-1}$ is $\pi = \delta' \cong \mathbb{C}^2$, so the set $\Upsilon(D_{n-1}) = \{ \pm 1, (x^2)^i (i = 1, \ldots, n - 2), y \}$. Since $\chi_{\delta'}((x^2)^i) = \theta^2_{4(n-1)} + \theta^{2i}_{4(n-1)} = 2 \cos(\pi i/(n - 1))$ and $\chi_{\delta'}(y) = 0$, thus the denominator in formula [4.11] is

\begin{equation}
\det(I - t\tilde{A}^T) = \prod_{g \in \Upsilon(D_{n-1})} (1 - \chi_V(g)t) = (1 - 4t^2) \prod_{i=1}^{n-2} \left( 1 - 2 \cos \left( \frac{\pi i}{n - 1} \right) t \right).
\end{equation}

The pair $(D_{2(n-1)}, D_{n-1})$ realizes the the twisted affine Dynkin diagram of type $A_{2n-1}^{(2)}$. Removing the special vertex the finite Dynkin diagram of type $C_n$ is obtained. Let $A$ be the adjacency matrix of type $C_n$, then

$$A^T = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 1 & 0 & 2 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.$$
Set \( c_{n-1}(t) = \det(I - tA^T) \), then \( c_0(t) = 1, c_1(t) = 1 - 2t^2 \). Using cofactor expansion it is easy to get the recursive relation

\[
(4.11) \quad c_{n+1}(t) = c_n(t) - t^2 c_{n-1}(t) \quad \text{for } n \geq 1.
\]

Subsequently one has that

\[
(4.12) \quad c_{n-1}(t) = 2t^n T_n \left( \frac{t^{-1}}{2} \right),
\]

where \( T_n(t) \) is the Chebyshev polynomial of the first kind. Therefore we have that

\[
(4.13) \quad c_{n-1}(t) = 2t^n \left( \sum_{i=0}^{[n/2]} \binom{n}{2i} (1 - (\frac{t^{-1}}{2})^{-2})^i \right)
\]

meanwhile the multiplicative expression of \( T_n(t) \) gives that

\[
(4.14) \quad c_{n-1}(t) = 2t^n \left( 2^{n-1} \prod_{i=1}^{n} \left( \frac{t^{-1}}{2} - \cos \left( \frac{(2i-1)\pi}{2n} \right) \right) \right)
= \prod_{i=1}^{n} \left( 1 - 2\cos \left( \frac{(2i-1)\pi}{2n} \right) t \right).
\]

The formula (4.10) is also related with the Chebyshev polynomial. In fact, using the Laplace expansion and (4.12) we have that

\[
\det(I - t\tilde{A}^T) = c_{n-1}(t) - t^2 c_{n-3}(t)
= 2t^n \left( T_n \left( \frac{t^{-1}}{2} \right) - T_{n-2} \left( \frac{t^{-1}}{2} \right) \right).
\]

It follows from (4.9), (4.4) and (4.6) that

\[
T_n(t) - T_{n-2}(t) = tU_{n-1}(t) + tU_{n-3}(t) - 2U_{n-2}(t)
= (2t^2 - 2)U_{n-2}(t)
= (2t^2 - 2) \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} (-1)^i \binom{n-2-i}{i} (2t)^{n-2-2i}.
\]

Therefore

\[
\det(I - t\tilde{A}^T) = (1 - 4t^2) \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} (-1)^i \binom{n-2-i}{i} t^{2i}.
\]

Summarizing these, we have proved the following result.

**Theorem 4.7.** Associated with the pair \( D_{n-1} \cap D_{2(n-1)} \leq SU_2 \), the Poincaré series for \( D_{n-1} \)-invariants and \( D_{2(n-1)} \)-invariants in \( T(V) \) are given by

\[
(4.15) \quad \hat{m}^0(t) = \hat{m}^0(t) = \frac{\prod_{i=1}^{n} \left( 1 - 2\cos \left( \frac{(2i-1)\pi}{2n} \right) t \right)}{(1 - 4t^2) \prod_{i=1}^{n-2} \left( 1 - 2\cos \left( \frac{\pi}{n-\pi} t \right) \right)} = \frac{2^{1-n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{i} (1 - 4t^2)^i}{(1 - 4t^2) \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} (-1)^i \binom{n-2-i}{i} t^{2i}}.
\]
### 4.1.2. The pair \((D_n,C_{2n})\).

Recall Sect. 2.3, the imbedding of \(D_n\) is \(\pi = \xi_1 + \xi_{n-1} \cong \mathbb{C}^2\), so \(\Upsilon(C_{2n}) = \{ \pm 1, x^i (i = 1, \ldots, n-1) \}\). Then \(\chi_{\xi_1+\xi_{n-1}}(\pm 1) = \pm 2\) and \(\chi_{\xi_1+\xi_{n-1}}(x^i) = 2 \cos(\pi i/n)\) imply that

\[
\text{det}(I - tA^T) = \prod_{g \in \Upsilon(C_{2n})} (1 - \chi_V(g)t) = (1 - 4t^2)^{n-1} \left(1 - 2 \cos \left(\frac{\pi}{n} t\right)\right).
\]

As the pair \((D_n,C_{2n})\) realizes the twisted affine Dynkin diagram \(\text{D}^{(2)}_{n+1}\). The adjacency matrix \(A\) of the finite Dynkin diagram is \(B_n\) after removing the special vertex of \(\text{D}^{(2)}_{n+1}\). As \(B_n\) is the dual of \(C_n\), the denominator \(b_{n-1}(t) := \text{det}(I - tA^T) = c_{n-1}(t)\) given by the same formulas \((4.12), (4.13)\) and \((4.14)\).

Using the cofactor expansion and combining with \((4.12), (4.9)\) and \((4.4)\) it follows that

\[
\text{det}(I - tA^T) = b_{n-1}(t) - 2t^2b_{n-2}(t)
= 2t^n T_n \left(\frac{t-1}{2}\right) - 2t^2 \left(2^{n-1} T_{n-1} \left(\frac{t-1}{2}\right)\right)
= (t^{n-1} - 4t^{n+1}) U_{n-1} \left(\frac{t-1}{2}\right)
= (1 - 4t^2) \sum_{i=0}^{\left\lfloor (n-1)/2 \right\rfloor} (-1)^i \binom{n-1}{i} t^{2i},
\]

where we have used identity \((4.6)\). This then proves the following result.

**Theorem 4.8.** For the pair \(C_{2n} \subset D_n \leq \text{SU}_2\) and \(V = \mathbb{C}^2\), the Poincaré series for \(C_{2n}\)-invariants and \(D_n\)-invariants in \(T(V) = \oplus_{k \geq 0} V^\otimes k\) can be given by

\[
\tilde{m}_n^0(t) = \eta_0^0(t) = \frac{\prod_{i=1}^{n} \left(1 - 2 \cos \left(\frac{2i-1}{2n}\pi\right) t\right)}{(1 - 4t^2) \prod_{i=1}^{n-1} \left(1 - 2 \cos \left(\frac{\pi}{n} t\right)\right) t} = \frac{2^{(1-n)} \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{i} (1 - 4t^2)^i}{(1 - 4t^2) \sum_{i=0}^{\left\lfloor (n-1)/2 \right\rfloor} (-1)^i \binom{n-1-i}{i} t^{2i}}.
\]

**Remark 4.9.** The two-dimensional natural module of \(C_{2n}\) gives rise to the same Poincaré series of invariants for the two pairs of subgroups \((D_n,C_{2n})\) and \((D_{2n},C_{2n})\), so the equality \((4.17)\) gives the Poincaré series of invariants for \((D_{2n},C_{2n})\). We remark that \((4.17)\) is also given as a reduced form of the Poincaré series for \(C_{2n}\)-invariant in \(T(V)\) [Thm. 3.23].

### 4.1.3. The pairs \((O,T), (T,D_2)\) and \((D_2,C_2)\).

The Poincaré series \(\tilde{m}_n^0(t) = \eta_0^0(t)\) for these pairs of subgroups \(N \subset G \leq \text{SU}_2\) and \(V \cong \mathbb{C}^2\) can be computed using Theorems 4.1 or 4.5.
We list the Poincaré series $\hat{m}^0(t) = \hat{m}^0(t)$ for the pairs $(O, T)$, $(T, D_2)$, $(D_2, C_2)$ in order as follows.

\[
\hat{m}^0(t) = \hat{m}^0(t) = \frac{\det(1 - tA^T)}{\prod_{g \in \Gamma(T)}(1 - \chi_{\hat{V}}(g)t)} = \frac{(1 - 4\cos^2(\frac{\pi}{12})t^2)(1 - 4\cos^2(\frac{5\pi}{12})t^2)}{(1 - 2t)(1 + 2t)(1 - t)(1 + t)} = 1 - 4t^2 + t^4 + 2t^6 + 6t^8 + 22t^{10} + \cdots
\]

\[
\hat{m}^0(t) = \hat{m}^0(t) = \frac{\det(1 - tA^T)}{\prod_{g \in \Gamma(D_2)}(1 - \chi_{\hat{V}}(g)t)} = \frac{(1 - 4\cos^2(\frac{\pi}{12})t^2)}{(1 - 2t)(1 + 2t)} = 1 - 3t^2 + 1 + t^2 + 4t^4 + 16t^6 + 64t^8 + 256t^{10} + \cdots
\]

\[
\hat{m}^0(t) = \hat{m}^0(t) = \frac{\det(1 - tA^T)}{\prod_{g \in \Gamma(C_2)}(1 - \chi_{\hat{V}}(g)t)} = \frac{(1 - 2\cos(\frac{\pi}{3})t)}{(1 - 2t)(1 + 2t)} = 1 - 4t^2 + 1 + 4t^2 + 16t^4 + 64t^6 + 256t^8 + 1024t^{10} + \cdots
\]

If $i$ is not the special vertex of the Dynkin diagrams realized by $(O, T)$, $(T, D_2)$, $(D_2, C_2)$, the Poincaré series $\hat{m}^i(t)$ and $\hat{m}^i(t)$ can be directly worked out by applying Theorem 5.6 and 5.7. They also can be computed by combining the results of the series $\hat{m}^0(t) = \hat{m}^0(t)$ and the identity $\hat{m}^i(t) = \delta_{i,0} + t \sum a_{ji}\hat{m}^j(t)$ (resp. $\hat{m}^i(t) = \delta_{i,0} + t \sum b_{ji}\hat{m}^j(t)$). The Poincaré series of the restriction of the $T$-modules $\hat{\omega}_i$ and induced $O$-modules $\hat{r}_i$ in $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ are

\[
\hat{m}^1(t) = \hat{m}^1(t) = \frac{1}{t}(\hat{m}^0(t) - 1) = t + 2t^3 + 6t^5 + 22t^7 + 86t^9 + \cdots
\]

\[
\hat{m}^2(t) = \hat{m}^2(t) = \frac{1}{t}(\hat{m}^1(t) - \hat{m}^0(t)) = t^2 + 4t^4 + 16t^6 + 64t^8 + 256t^{10} + \cdots
\]

\[
\hat{m}^3(t) = 2\hat{m}^1(t) = \frac{1}{t}(\hat{m}^2(t) - \hat{m}^1(t)) = t^3 + 10t^5 + 42t^7 + 170t^9 + \cdots
\]

\[
\hat{m}^4(t) = 2\hat{m}^0(t) = t\hat{m}^3(t) = t^4 + 10t^6 + 42t^8 + 170t^{10} + \cdots
\]

The Poincaré series of the restriction of $D_2$-modules $\hat{r}_i$ and induced $T$-modules $\hat{\delta}_i$ in $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ are

\[
\hat{m}^1(t) = \hat{m}^1(t) = \frac{1}{t}(\hat{m}^0(t) - 1) = t + 4t^3 + 16t^5 + 64t^7 + 256t^9 + \cdots
\]

\[
\hat{m}^2(t) = 3\hat{m}^2(t) = 3t\hat{m}^1(t) = 3t^2 + 12t^4 + 48t^6 + 192t^8 + \cdots
\]

The Poincaré series of the restricted $C_2$-module $\hat{\delta}_1$ and the induced $D_2$-module $\hat{\xi}_1$ in $T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$ is

\[
2\hat{m}^1(t) = \hat{m}^1(t) = \frac{1}{t}(\hat{m}^0(t) - 1) = 4t + 16t^3 + 64t^5 + 256t^7 + 1024t^9 + \cdots
\]

We remark that Benkart also studied the non-simply laced untwisted types (such as $B_n^{(1)}$ and $C_n^{(1)}$) in [2].

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