Continuous Ordinary Differential Equations and Infinite Time Turing Machines

Olivier Bournez
Ecole Polytechnique, LIX, 91128 Palaiseau Cedex, France
bournez@lix.polytechnique.fr

Sabrina Ouazzani
Ecole Polytechnique, LIX, 91128 Palaiseau Cedex, France
sabrina@lix.polytechnique.fr

Abstract

We consider Continuous Ordinary Differential Equations (C\text{0}-ODE) \( y' = f(y) \), where \( f : \mathbb{R}^d \to \mathbb{R}^d \) is a continuous function. They are known to always have solutions for a given initial condition \( y(0) = y_0 \), these solutions being possibly non unique. We restrict to our attention to a class of continuous functions, that we denote by \( C^\text{\#0}-ODE \): they always admit unique greedy solutions, i.e. going in greedy way in some fixed direction. We prove that they can be seen as models of computation over the ordinals (Infinite Time Turing Machines, ITTM) and conversely in a very strong sense. In particular, for such \( C^\text{\#0}-ODE \), to a greedy trajectory can be associated some ordinal corresponding to some time of computation, and conversely models of computation over the ordinals can be associated to some \( C^\text{\#0}-ODE \). In particular, analysing reachability for one or the other concept with respect to greedy trajectories has the same hardness. This also brings new perspectives on analysis in Mathematics, by providing ways to translate results for ITTMs to \( C^\text{\#0}-ODEs \). This also extends some recent results about the relations between ordinary differential equations and Turing machines, and more generally with (generalized) computability theory.

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Recent years have seen a renewal of interest for models of computation over the ordinals: See e.g. \[18, 34, 35, 12, 23\]. Beyond the study of the inherent properties of such models, recent developments tend to use them to prove results in other fields. In line with this trend, this paper is motivated by the idea that thanks to models of computation over the ordinals, new facts in mathematical analysis can be established. It also extends some recent results relating analog and digital models to polynomial ordinary differential equations (see \[9, 4, 3, 7, 8, 16\]), to a really wider settings.

Here we consider Continuous Ordinary Differential Equations ($C_0$-ODE). That is to say $y' = f(y)$, where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a (everywhere defined) continuous function. An Initial Value Problem (IVP), also called a Cauchy’s Problem, is obtained by adding some initial condition, and is of the form

$$y' = f(y), \quad y(t_0) = y_0. \tag{1}$$

As expected, we call trajectory any solution of the problem, that is to say, any derivable function $\xi : I \subset \mathbb{R} \to \mathbb{R}^d$, where $I$ is some interval containing $t_0$ satisfying $\xi(t_0) = y_0$, and $\xi'(t) = f(\xi(t))$ on its domain. The solution is said to be maximal, if $I$ is maximal (for inclusion) with this property. Such IVPs are known to always have solutions, but possibly non unique, by Peano-Arzelà’s Theorem: see e.g. \[13\].

> **Remark 1.** When in addition $f$ is locally Lipschitz (in particular if it is $C^1$ or smooth, i.e. $C^\infty$) then unicity is guaranteed. This is the famous Cauchy-Lipschitz theorem.

Given $y_0$, we denote by $\mathcal{R}_f^*(y_0)$ the set of points that can be reached by a solution of IVP \[ \text{(1)} : \mathcal{R}_f^*(y_0) = \{ y \mid \exists t \geq t_0, \xi(t) = y \text{ for } \xi \text{ solution of } \text{(1)} \}. \] Kneser’s theorem (cf \[19, \text{Theorem II.4.1}\]) states that it is necessarily a continuum, i.e. a closed connected set. It is known that among solutions of a $C_0$-ODE there always exists trajectories such that $(t, \xi(t))$ live in $fr(\mathcal{R}_f(t_0, y_0))$ where $fr(D) = \overline{D} = D$ is the frontier, and $\mathcal{R}_f(t_0, y_0) = \{(t, y) : t \geq t_0, \xi(t) = y \text{ for } \xi \text{ solution of } \text{(1)}\}$. This is sometimes called the Fukuhara property following \[17\] that established first their existence. See \[19, \text{Exercice I.4.4}, 20, 15, 24, \text{chapter 10}\].
Figure 1 Some solutions of the IVP \( y' = f_\rightarrow(y), \ y(0) = 0 \), considered in Example 2. Namely \( \chi_{-0.5,0.3} \) in blue, \( \chi_{-0.1,0.2} \) in red, and \( \chi_{-0.9,0.5} \) in green.

Example 2. A very famous example of \( C_0 \)-ODE demonstrating that non-unicity of solutions holds when Lipschitz hypothesis is not assumed is the following: Consider \( f_\rightarrow(y) = 3y^{2/3} \) for \( y \neq 0 \), and \( f_\rightarrow(0) = 0 \). It holds that this is indeed a continuous function from \( \mathbb{R} \) to \( \mathbb{R} \) (but not derivable, nor Lipschitz in 0). The solutions (Figure 1) over \( \mathbb{R} \) of the Initial Value Problem \( y' = f_\rightarrow(y), \ y(0) = 0 \) are the functions \( \chi_{-a,b} \) with \( a, b \in \mathbb{R}^{\geq 0} \cup \{\pm \infty\} \), where

\[
\chi_{-a,b}(t) = \begin{cases} 
0 & \text{if } -a \leq t \leq b \\
(t + a)^3 & \text{if } t < -a \\
(t - b)^3 & \text{if } t > b
\end{cases}
\]

We have \( \mathcal{R}_{f_\rightarrow}^*(y) = [y, +\infty) \) for all \( y \in \mathbb{R} \). Furthermore, \( \mathcal{R}_{f_\rightarrow}(0,0) = \{(t,y) : 0 \leq t, 0 \leq y \leq t^3 \} \), \( \mathcal{R}_{f_\rightarrow}(0,y) = \{(t, (t + b)^3) : 0 \leq t \} \), \( \mathcal{R}_{f_\rightarrow}(0,-y) = \{(t, (t - b)^3) : 0 \leq t \leq b \} \cup \{(t,y) : b \leq t, 0 \leq y \leq (t - b)^3 \} \) where \( b = y^{1/3} \) for \( y > 0 \). (See Figures 2, 3, 4).

\footnote{We use the above notation for conciseness reasons: For example, \( \chi_{-\infty, +\infty} : \mathbb{R} \to \mathbb{R} \) denotes the null function. Details about this ODE, can be found in appendix.}
Figure 2 \( \mathcal{R}_{f^-}(0,0) \) for the \( C_0 \)-ODE \( y' = f^-(y) \) considered in Example 2 in blue. The trajectory starting from \((0,0)\) in (dark) blue is living in \( fr(\mathcal{R}_{f^-}(0,0)) \) and is greedy. Actually, in every point \((t,y) \in fr(\mathcal{R}_{f^-}(0,0))\) there is a unique greedy trajectory starting from this point.

Figure 3 \( \mathcal{R}_{f^-}(0,-1) \) for the \( C_0 \)-ODE \( y' = f^-(y) \) considered in Example 2 in blue. The unique greedy trajectory starting from \((0,-1)\) in dark blue is living in \( fr(\mathcal{R}_{f^-}(0,-1)) \). \( \mathcal{R}_{f^-}(0,y) \) has a similar shape for \( y < 0 \).
Remark 3. In this context, the concept of frontier is often blunt\footnote{For example, the frontier of a $d' < d$-manifold is itself.} and we propose in some sense to restrict to trajectories living in “edges” of $fr(R_{f}(t_{0}, y_{0}))$, by considering greedy trajectories: Assume some direction $v \neq 0$, $v \in \mathbb{R}^{d}$ is fixed (we assume $v = (1, \ldots, 1)$ by default). Given $y \in \mathbb{R}^{d}$, let $H(y)$ (respectively: $H^{-}(y)$) denote the $y$-affine hyperplane (resp. negative half-plane): this is $\{ y' : y' \cdot v = (\text{resp.} <) y \cdot v \}$ where $\cdot$ is scalar product.

Definition 4 (Greedy trajectory). We say that a solution $\xi$ of IVP (1) is $v$-greedy if for all $t_{0}$ for all $t \geq t_{0}$, there exists some $\epsilon > 0$ such that there is no other solution $\bar{\xi}$ with $\xi(t) = \bar{\xi}(t)$, and $\bar{\xi}(t')$ reaching $H^{-}(\xi(t))$ for $t \leq t' \leq t + \epsilon$.

In other words, $\xi$ always selects locally locally an evolution in a greedy way in direction $v$. We say that some dynamics $f : \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is greedy iff for all $t_{0}, y_{0}$, there is a unique greedy trajectory solution of IVP (1). We write $C_{g}^{0}$-ODE for the class of $f$ which are greedy.

Remark 5. Notice that if the dynamics is such that unicity of trajectories hold locally (for example if it is $C^{1}$ or smooth), then it is greedy by definition.
Example 6. Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(y_1, y_2) = (f_1(y_1), f_2(y_2))$. We have $R_{f_{\mathbb{R}^2}}((y_1, y_2)) = \{ (y_1', y_2') : y_1 \leq y_1', y_2 \leq y_2' \}$. (See Figures 5 and 6). $f_{\mathbb{R}^2}$ and $f_{\mathbb{R}^1}$ are $C^0_0$-ODE. The vast majority of the examples from the literature are.

Remark 7. $f$ is not greedy in $y_0$ means that for all $\delta > 0$, and for any $\xi$ solution of IVP (1), one can find some other solution $\tilde{\xi}$ of (1), with $\tilde{\xi}(t') \cdot v < \xi(t') \cdot v$ for some $t_0 \leq t' \leq t_0 + \delta$. However, general $C_0$-ODE can be very pathological, and nowhere greedy functions can be built with some efforts.

Theorem 8 (Lavrentieff’s theorem [25]). There exists some $C_0$-ODE on $[0, 1]^2$ such there are

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3 A similar more pedagogical case, is discussed in [19] page 18.


Infinite Time Turing machines (ITTM) are a computational model introduced in 2000 by Hamkins and Lewis in [18], generalizing the computation process of Turing Machines (TM) to ordinal time. Our main theorem is then the following: $C_0^1$-ODEs and ITTMs are basically equivalent. Formally:

- **Theorem 9 ($C_0^1$-ODE simulates ITTMs).** Consider some ITTM machine $M$. Let ordinal $\lambda_M$ be the sup of the halting time of computations of $M$ over some input. For any $\mu < \lambda_M$, there is some $C_0^1$-ODE dynamics $f$ whose greedy solutions simulates $M$ up to time $\mu$.

  Dynamics $f$ can even be assumed with compact support $\mathbf{4}$.

  And conversely:

- **Theorem 10 (ITTMs simulate $C_0^1$-ODE).** Consider some $C_0^1$-ODE $f$ with compact support $\mathbf{5}$.

  For any time $t \in \mathbb{R}$, there exists some ITTM $M$ that is able to compute the positions reached by the greedy trajectory solution of IVP $\mathbf{1}$ up to time $t$.

This rather unexpected relation between models opens the way to many statements. A direct first consequence is the following:

- **Corollary 11.** Determining whether two points/configurations are reachable has same complexity for both models when considering greedy trajectories.

More generally results about ITTMs can possibly be transfered to $C_0^1$-ODEs and conversely: this includes gaps in halting time, possibility of deciding whether a real encodes a well-order using an ODE, ...and conversely this opens the possibility to apply results from analysis to the framework of ITTMs. This also demonstrates that the model of ITTMs is a very natural model, not restricted to the framework of set theory or computability theory, than can be opened to the many fields in experimental sciences, where $C_0$-ODEs appear.

We also prove the following (($\omega_1$ denotes the least non-countable ordinal):

- **Theorem 12.** $C_0^1$-ODEs functions $f$ can be stratified in a strict hierarchy indexed by countable ordinals $\alpha$.

  The proposed hierarchy, inspired by $\mathbf{21}$, is defined in a Cantor-Bendixson way, in terms of some derivative operators on sets of functions: The level of a function corresponds to the (possibly transfinite) number of times the derivative must be applied on a function before getting to the emptyset. We also characterize the first levels of this hierarchy. In particular, we have: $|f| = 1 \ (\mathcal{R}_T^g(\cdot)$ the function that maps points to the associated greedy trajectory, and continuity is with respect to thickness distance, see below):

- **Proposition 13.** $|f| = 1$ iff $\mathcal{R}_T^g(\cdot)$ is continuous.

All of this open ways to relate the translation from ITTMs to $\!$greedy dynamics, by relating $\lambda_M$ to the corresponding level $\alpha_f$ of $f$, and conversely, even if this remains to be fully explored.

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4 There exists some compact (i.e. bounded and closed) $K$ such that $f(y) = 0$ whenever $y \notin K$.

5 We believe this last hypothesis can be relaxed using covering techniques from $\mathbf{14}$. 
2 Preliminaries

2.1 Turing Machines and Infinite Time Turing Machines

We consider actually here the following variation of the Turing machine model, that we call output Turing machine (oTM): an output Turing machine has three tapes, called input, scratch and output tape. First two tapes are classical Turing machine read-write tapes. The output tape is a write only tape, initially blank, where the output head can still move right or left, can possibly write but not able to read what is in front of the head of the output tape. We also assume the following (denoted by \( \ast \)) property: cell number \( j \) of the output tape is visited at most \( 2^j \) times by the head of the output tape, for all \( j \).

Now an ITTM is a oTM with an additional special state called the limit state, or \( \text{lim} \) for short. At step 0, the machine is in its initial configuration. Configuration at step \( n + 1 \) is obtained from configuration at step \( n \) by a classical output tape Turing machine transition. If \( \alpha \) is a limit ordinal, then the configuration at step \( \alpha \) is defined as follows: The read-write heads and the output tape head are rewound to the origin cell and the state is put in the lim state. The output tape and scratch tape are erased. The input tape is changed to the limit of the output tape: that is to say, cell number \( j \) of input tape has a value that corresponds to the limit of the history of cell number \( j \) of output tape up to step \( \alpha \).

- **Remark 14.** As the output tape is write only, the only possibility is either that a cell is always constant, or eventually 1: consequently, this limit always exists.

We still assume the \( \ast \)-property: cell number \( j \) of the output tape is visited at most \( 2j \) times by the head of the output tape between two limit states for all \( j \). A machine halts if it falls in some special halting state. These are variations of classical models (see Appendix B).

- **Proposition 15.** oTMs are equivalent to classical Turing machines, and ITTMs are basically equivalent to the model of Hamkins and Lewis [18].

Contrary to TMs where all natural numbers can be execution times of programs, it is proved in [18] that there exist many countable ordinals that can be written but cannot be such execution times, on input 0. A sequence of consecutive non clockable ordinals is called a gap. There are many gaps of size at least \( \omega \) [18]. A snapshot of some (possibly Infinite-Time) TM corresponds to some instantaneous description of the input and scratch tapes of the machine. The output tape will often play a special role in our constructions.

2.2 Coding Tapes and Snapshots with Reals

In this document, \( \mathbb{R} \) is the set of reals studied in classical mathematics (e.g. in analysis). Cantor space is denoted by \( \mathcal{C} = 2^\omega \).

- **Remark 16.** We differ here from the usual convention done in the ITTM context, where \( \mathbb{R} \) is often taken as a synonym for Baire space \( \mathcal{N} \). But actually \( \mathcal{N} \) is homeomorphic to the set of irrational numbers (cf [26]), and we indeed focus on \( \mathbb{R} \) as considered in analysis.

Whenever \( w \in \mathcal{C} \), we write \( \langle w \rangle_\mathbb{R} \) for \( \langle w \rangle_\mathbb{R} = \sum_{i=0}^{\infty} \frac{2w(i)+1}{4^i} \): In other words, \( \langle w \rangle_\mathbb{R} \) is coding \( w \) as a real of \([0,1]\), coding 0 by a 1, and a 1 by a 3 in a radix 4 expansion.

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6. It writes necessarily a 1 from the assumption that the tape is write only and initially blank.
7. And there is no need to talk about limit sup as in [18] to guarantee the existence of a limit.
Remark 17. Considering radix 2 expansion could be thought more natural (i.e. \( \sum_{i=0}^{\infty} \frac{w(i)}{2^i} \)) but above coding will help for some constructions\(^8\), and avoid potential problems with non-unicity of the radix 2 expansion of some reals like 1 = 0.111....

An observation is that limits over \( C \) and limits over \( \mathbb{R} \) commute for this encoding.

Proposition 18 (Limits commute with \( < \cdot >_R \) encoding). Consider a sequence \((w_n)_n\) of elements of \( C \). Write \( w_n = w_{n,1}w_{n,2} \ldots \). Assume that \( w_i^* = \lim_{n \to \infty} w_{n,i} \) exists for all \( i \). Write \( w^* = w_1^*w_2^* \ldots \). Then \( <w^*>_R = \lim_{n \to \infty} <w_n>_R \) (and the limit exists).

This follows from the fact that the series is normally converging, and hence that limits can be permuted. But in order to be self content, we provide a direct proof:

Proof. First, for \( I < J \),

\[
\left| \sum_{i \in I} \frac{2w_i^* + 1}{4^i} - \sum_{j \in J} \frac{2w_j^* + 1}{4^i} \right| \leq \sum_{i \in I,J} \frac{3}{4^i}
\]

and hence \( \sum_{i \in I} \frac{2w_i^* + 1}{4^i} \) is a Cauchy sequence (since \( \sum_i \frac{3}{4^i} \) is converging, and hence Cauchy) and consequently, it has a limit \( <w^*>_R \) when \( I \) goes to \( \infty \).

Fix some \( \epsilon > 0 \).

As a consequence, there must exists some \( i_0 \) such that for all \( i < I \)

\[
<w^*>_R - \sum_{i \leq I} \frac{2w_i^* + 1}{4^i} \leq \frac{\epsilon}{3}.
\]

As \( \sum_i \frac{3}{4^i} \) is convergent with limit \( \ell = 1 \), there exists some \( i_1 \) such that for all \( i_1 \leq I \), \( \sum_{i > I} \frac{3}{4^i} \leq \frac{\epsilon}{3} \).

Then

\[
<w_n>_R - \sum_{i \leq I} \frac{2w_{n,i} + 1}{4^i} \leq \sum_{i > I} \frac{2w_{n,i} + 1}{4^i} \leq \sum_{i > I} \frac{3}{4^i} \leq \frac{\epsilon}{3}
\]

for all \( i_1 \leq I \) and for all \( n \).

Take \( I = \max(i_0, i_1) \).

There exists some \( n_0 \) such that for all \( n \geq n_0 \)

\[
\left| \sum_{i \in I} \frac{2w_i^* + 1}{4^i} - \sum_{i \in I} \frac{2w_{n,i} + 1}{4^i} \right| \leq \frac{\epsilon}{3}
\]

(by taking limits over the above finite sum)

We get for all \( n \geq n_0 \)

\[
\left| <w_n>_R - <w^*>_R \right| \leq \left| <w_n>_R - \sum_{i \leq I} \frac{2w_{n,i} + 1}{4^i} \right| + \left| \sum_{i \leq I} \frac{2w_{n,i} + 1}{4^i} - \sum_{i \leq I} \frac{2w_i^* + 1}{4^i} \right| + \sum_{i \in I} \frac{2w_i^* + 1}{4^i} - <w^*>_R \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]

\( ^8 \) Precisely: in all references to Definition 28.
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Each (input, scratch, output) tape can be seen as a word \( w \in \mathcal{C} \). For some reasons, we will encode the input tape and scratch tape + head position using the following encoding \( < \cdot >_{\mathcal{C} \times \mathcal{C}} \), and deal with the output tape separately. For a given (input, scratch) tape + head position \( T \), \( < T >_{\mathcal{C} \times \mathcal{C}} = (w_L, w_R) \) consists of the following: \( w_R \in \mathcal{C} \) (respectively: \( w_L \)) is the content of the tape at the right (resp. left) of the head seen as a word over \( \mathcal{C} \) (resp. but written from left to right). In other words, the head is in front of the first letter of \( w_R \), and \( w_L^R w_R \) is the tape, where \( w^R \) denotes reverse operation. Let \( \mathcal{S} = \mathcal{Q} \times \mathcal{C}^4 \) be the set of snapshots: \( (q, i_L, i_R, s_L, s_R) \) for a oTM encodes the fact that the machine is in state \( q \), that \( < I >_{\mathcal{C} \times \mathcal{C}} = (i_L, i_R) \) encodes the input tape \( I \), and \( < W >_{\mathcal{C} \times \mathcal{C}} = (s_L, s_R) \) encodes the scratch tape \( S \). This can in turn be encoded as an element of \( \mathbb{R}^5 \): We write \( < \cdot >_{\mathbb{R}^5} \) for the function that encodes a snapshot by 5 reals defined as: \( (q, i_L, i_R, s_L, s_R) >_{\mathbb{R}^5} = (q|Q|, < i_L >_{\mathbb{R}}, < i_R >_{\mathbb{R}}, < s_L >_{\mathbb{R}}, < s_R >_{\mathbb{R}}) \) assuming the set of internal states of the machine is \( Q = \{1, \ldots, |Q|\} \). Let \( \mathcal{S}_{\mathbb{R}^5} \) is defined as the subset of \( \mathbb{R}^5 \) that corresponds to encodings of snapshots.

Previous encodings are intentionally forgetting the output tape. If one wants to describe the output tape + head position, we can see it as an element of \( \mathcal{C} \times \mathbb{N} : (o, n) \) encodes the fact that the output tapes values \( o \) and that the output head is in front of cell number \( n \). This can be in turn considered as an element of \( \mathbb{R}^2 \) using encoding function \( < (o, n) >_{\mathbb{R}^2} = (o >_{\mathbb{R}}, n) \in \mathbb{R}^2 \).

2.3 Some geometric constructions and tools

We need to build and describe some particular \( \mathcal{C}_0 \)-ODE \( y' = f(y) \), i.e. some continuous functions \( f : \mathbb{R}^d \to \mathbb{R}^d \).

To help for the intuition, we will use several tools. A first one is the concept of discrete time ODE (dODE), and discrete time IVP (dIVP) when some initial condition is added: When \( f : \mathbb{R} \to \mathbb{R}^d \), let \( \partial f_{\to q}(p) \) denotes

\[
\partial f_{\to q}(p) = \frac{f(q) - f(p)}{q - p}
\]

for \( p \neq q \). This can be called a discrete derivative as we have \( f(q) = f(p) + \partial f_{\to q}(p)(q - p) \). A discrete time ODE is given by some strictly increasing sequence of reals \( (p_i) \), and an equation of type \( \partial y_{\to p_{i+1}}(p_i) = F(y(p_i)) \) for some function \( F : \mathbb{R}^d \to \mathbb{R}^d \). A solution of the equation is some function \( y : \mathbb{R} \to \mathbb{R}^d \) that satisfies the equation for all \( i \). We say that some dIVP is \( \mathcal{C}_0 \)-ODE implementable (respectively: on some domain \( D \)) if we can build some \( \mathcal{C}_0 \)-ODE whose solutions are solutions of the dIVP (resp. and solutions satisfying the initial condition remain in domain \( D \)).

- Remark 19. \( \partial f_{\to 0}(p) \) is often written \( \Delta r(q, p) \) in literature (sometimes \( f' \) as in [2]) but we think that in the context of this article, this notation helps for intuition.

- Remark 20. A key remark is that if some dIVP is \( \mathcal{C}_0 \)-ODE implementable on some compact domain \( D \), from mean value theorem, function \( F \) must remain bounded. Many of our constructions are deeply conceived with the guiding principle of keeping this true.

Another tool will be to build the functions using various simpler basic blocks. Each block \( b \) will basically be given by some domain, i.e. some connected bounded subset \( D_b \subset \mathbb{R}^d \), and some continuous dynamics \( y' = f_b(y) \) over \( D \). Actually, we will use heavily the fact that

\[ \text{The theory of dODE is underdeveloped in literature, and we review some statements in Appendix (see also [2]).} \]
everything that we need can be encoded in reals of \([0,1]^{d'}\) for some \(d' < d\), but we will often play with coordinate systems: Each block comes with input(s) and output port. Formally, a \(d' < d\)-dimensional port \(\mathcal{P}\) in \(\mathbb{R}^d\) is given by some affine basis \((\mathbf{O}, \mathbf{e}_1, \ldots, \mathbf{e}_d)\) of \(\mathbb{R}^d\), and corresponds to all points of coordinates \((x_1, \ldots, x_{d'}, 0, \ldots, 0)\) for \(x_1, \ldots, x_{d'} \in [0,1]\) in this coordinate system. Every block \(b\) is built such that its ports \(\mathcal{P}\) are subsets of \(f^r(D_b)\), and with \(f_b\) constant on \(\mathcal{P}\). We will say that block \(b\) realizes function \(g : [0,1]^{d_1} \rightarrow [0,1]^{d_2}\) between \(d_1\)-dimensional input port \(\mathcal{I}\) and \(d_2\)-dimensional output port \(\mathcal{O}\), if \(C_\text{p-ODE} y' = f_b(y)\) is such that a trajectory starting in \((x_0, 0, \ldots, 0)\) in the coordinate system of \(\mathcal{I}\) eventually reach the point \((g(x), 0, \ldots, 0)\) in the coordinate system of \(\mathcal{O}\), for all \(x \in [0,1]^{d_1}\).

We say that two blocks \(b\) and \(b'\) can be combined if \(\hat{D}_b \cap \hat{D}_{b'} = \emptyset\), and the input port of \(b\) corresponds to the output port of \(b'\), and \(f_b\) and \(f_{b'}\) have same value over \(D_b \cap D_{b'}\). From the form of our blocks, and this last constraint, it will then always be possible to build some continuous function \(f\) over \(\mathbb{R}^d\) that coincides with \(f_b\) (respectively: \(f_{b'}\)) over \(D_b\) (resp. \(D_{b'}\)). As the blocks will be such that they guarantee that trajectories remain in considered blocks, this implies that the particular extension of \(f\) to the whole \(\mathbb{R}^d\) is not important, and we just need to know that this is feasible with a \(C_\text{p-ODE}\).  

A particular example of shape of domain \(D\) will play an key role is what we call toblerone: see Figure 7. We call block with a toblerone domain a block with a 1-dimensional port \((\mathbf{0}, 0, \ldots, 0, \mathbf{e}_1, \ldots, \mathbf{e}_d)\) and a \(d-1\)-dimensional port \((0, \ldots, 0, -1, \mathbf{e}_1, \ldots, \mathbf{e}_d)\), and a domain \(D\) that corresponds to \(\{(x_1, \ldots, x_d) : 0 \leq x_1 \leq 1, -x_2 \leq x_i \leq x_2, -1 \leq x_d \leq 0, \text{ for } 1 < i < d\}\), in some basis \((\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d)\) of \(\mathbb{R}^d\). Its direction is \(\mathbf{v} = (-1, 0, \ldots, 0)\). In all our examples, \(f\) will have constant value \(0\) on the 1-dimensional port.

Toblerones domains can be combined in order to share their 1-dimensional ports: See Figure 8. In particular, using dynamics \(f_{-1}\),

> **Lemma 21** (Toblerone path). One can define easily a toblerone path block with toblerone domain that realizes \(id_1 : [0,1] \rightarrow [0,1]^{d-1}\), \(x \mapsto (x, 0, \ldots, 0)\) between its 1-dimensional port and its \(d-1\)-dimensional port: cf Figure 8. We call direction of such a block the direction of the underlying toblerone domain.

---

10 And hence we will not detail how \(f\) is extended, leaving this as a tedious but easy exercice: For example using triangulations, and very classical tricks in analysis to get \(C^\infty\) functions that have particular values on some particular intervals/domains.
Figure 9 A toblerone path.

Proof. A toblerone path can be obtained for example as follows: Consider input port \(((0,\ldots,0),e_1,\ldots,e_d)\) and output port \(((0,\ldots,0,-1),e_1,\ldots,e_d)\), and with domain \(D_{\text{tobelrone}}\) as defined in page 11. Set dynamics

\[(x_1,x_2,\ldots,x_d)'=(0,\ldots,0,-f(-x_d))\]
on this domain.

Other blocks that can be defined rather easily in \(\mathbb{R}^d\) using smooth \(f\) include the following \((\lambda>0\text{ and }k>0)\):

- **Lemma 22** (Linear path). A linear path (Figure 10) block realizes identity \(id_{d-1}\) over \([0,1]^{d-1}\) between input port \(I=(O,e_1,\ldots,e_d)\) and \(O=(O',e_1,\ldots,e_d)\) with \(O'=O+\lambda e_d\).

![Figure 10 A linear path.](image)

Proof. A linear path can be obtained as follows: Consider input port \(((0,\ldots,0),e_1,\ldots,e_d)\) and output port \(((0,\ldots,0,1),e_1,\ldots,e_d)\): Set dynamics \((x_1,x_2,\ldots,x_d,z)'=(0,\ldots,0,1)\).

- **Lemma 23** (U-turn). A u-turn (Figure 11) block realizes identity \(id_{d-1}\) between input port \(I\) and \(O=(O',-e_1,e_2,\ldots,e_{d-1},-e_d)\) for \(O'=O-\lambda e_1\).

![Figure 11 A u-turn.](image)

Proof. A u-turn can be obtained as follows: Consider input port \(((1,0,\ldots,0),e_1,\ldots,e_d)\) and output port \((-1,0,\ldots,0,-e_1,e_2,\ldots,e_{d-1},-e_d)\): Set dynamics \((x_1,x_2,\ldots,x_d,z)'=\pi\cdot(-z+1,0,\ldots,0,x_1)\) for \(z \geq 1\). This implies \(x_1(t)=x_1(0)\cos(\pi\theta), x_2(t)=x_2(0),\ldots, x_{d-1}(t)=x_{d-1}(0), z(t)=1+x_1(0)\sin(\pi\theta)\) when entering this zone.

Then add zones on \(0\leq z \leq 1\) to distort speed correctly so that it works and remains continuous: Set \((x_1,\ldots,x_d,z)'=(0,0,\ldots,0,1-z+x_1z)\) for \(x_1>1\), so that \(z'=x_1\) when \(z=1\), \(z'=1\) when \(z=0\), and set \((x_1,\ldots,x_d,z)'=(0,0,\ldots,0,-(1-z)-x_1z)\) for \(x_1<-1\). Then \(z'=-x_1\) when \(z=1\), \(z'=-1\) when \(z=0\).
> **Lemma 24** (Dilation). A dilation block (Figures 12 and 13) realizes $g : (e_1, \ldots, e_{d-1}) \to (ke_1, e_2, \ldots, e_{d-1})$ between $I$ and $O = (O', e_1, \ldots, e_d)$ for $O' = O + \lambda e_1$.

![Figure 12](image1.png) A dilation (acting on $x$ of factor 2).

![Figure 13](image2.png) A dilation (acting on $y$ of factor $1/2$).

**Proof.** This has been done explicitly in [6].

Using continuous but non-smooth $f$, one can build the following:

> **Lemma 25** (Merge). A merge block (Figure 14) has two input ports $I_1 = (O_1, e_1, \ldots, e_d)$ and $I_2 = (O_2, e_1, \ldots, e_d)$ and some output port $O = (O', e_1, \ldots, e_d)$ with $O' = O_1 + \lambda e_d$, and realize $id_{d-1}$ both between input port $O_1$ and output port $O$ and between input port $O_2$ and output port $O$.

**Proof.** A merge of $d$-dimensional ports can only be realized in dimension $d + 2$.

We explain how to realize a merge of $d - 2$-dimensional ports in $\mathbb{R}^d$ ($d \geq 3$). Consider $d - 2$-dimensional ports $I_1 = (O_1, e_1, \ldots, e_d)$, $I_2 = (O_2, e_1, \ldots, e_d)$ and output port $O = (O', e_1, \ldots, e_d)$ with $O' = O_1 + \lambda e_d$, $\lambda > 0$. Assume without loss of generality that the coordinate system is chosen such that points of $I_1$ are of the form $(x_1, \ldots, x_{d-1}, 0, 0)$ and points $I_2$ are of the form $(x_1, \ldots, x_{d-1}, -1, 0)$: hence that the $d-1$ coordinate of $O_1$ (respectively $O_2$) is 0 (resp. $-1$). Then sets dynamic to $(x_1, \ldots, x_{d-1}, x_d)' = (0, \ldots, 0, f_{-\lambda}(x_{d-1}), 1)$. By construction, and by the properties of solutions of $f_{-\lambda}$, this is realizing a merge.

> **Remark 26.** A merge can be realized using some continuous $f$ in $\mathbb{R}^{d+1}$ (basically using dynamics $f_{-\lambda}$) but this is impossible using a smooth $f$.

![Figure 14](image3.png) A merge (symbolic view: this can exist only in dimension 4: See footnote 11).

---

11 From Cauchy-Lipchitz theorem, as unicity of solutions must necessarily be violated somewhere.
3 Simulating oTM using $C_0^I$-ODEs

3.1 In linear time

A first remark (already done in several papers, e.g. [6, 27, 32]) is the following: basically the successor function of a Turing machines corresponds to a piecewise affine function using previous encoding.

> **Proposition 27** (Piecewise affine functions and TMs). For any oTM $M$, there exist some (disconnected) piecewise affine function $F_{ct_M} : [0,1]^5 \rightarrow [0,1]^5$ such that for any snapshot $s \in S$, if we write $Next_M(s)$ for its successor according to the program of $M$, we have $<Next_M(s)>_{R^2} = F_{ct_M}(<s>_{R^2})$.

Proposition 27 follows from (an easy generalization of) the arguments of [6, Theorem 2.5], where it is proved that every deterministic (respectively: reversible) discrete one-tape Turing machine $M$ can be K-simulated (see terminology from this article) by iterations of a disconnected (resp: one to one) piecewise linear function $f : C \subset [0,1]^2 \rightarrow [0,1]^2$ with rational coefficients.

For completeness, we repeat and adapt here the proof.

The following concept has been formalized in [6] and can be generalized to our context as follows:

> **Definition 28** (Disconnected piecewise linear function). A function $f : [0,1]^5 \rightarrow [0,1]^5$ is disconnected piecewise linear if

1. there exist $n \in \mathbb{N}$ closed intervals $I_i = [a_i, b_i]$, with $a_i, b_i \in \mathbb{Q} \cap [0,1]$, and a finite set $Q$ of values of $[0,1]$
2. $f$ can be written

$$f : C = \bigcup_{q \in Q, n_1^i, n_2^i, n_3^i, n_4^i, n_5^i \in \{1,2,...,n\}} C_{n_1^i, n_2^i, n_3^i, n_4^i, n_5^i} \subset [0,1]^5 \rightarrow [0,1]^5$$

where, for all $q \in Q$, $n_1^i, n_2^i, n_3^i, n_4^i, n_5^i \in \{1,2,...,n\}$, $C_{n_1^i, n_2^i, n_3^i, n_4^i, n_5^i}$ is defined as

$$C_{n_1^i, n_2^i, n_3^i, n_4^i, n_5^i} = \{q\} \times I_{n_1^i} \times I_{n_2^i} \times I_{n_3^i} \times I_{n_4^i} \times I_{n_5^i}.$$

3. all the $I_i$ are at a strictly positive distance: there exists $\epsilon$, such that, for all $i \neq j$, $x \in I_i, y \in I_j \Rightarrow d(x,y) \geq \epsilon$.
4. on each $C_{n_1^i, n_2^i, n_3^i, n_4^i, n_5^i}$, $f$ is affine of type $f(q, x_1^i, x_2^i, x_3^i, x_4^i) = (q^+, \alpha_1^i + \beta_1^i x_1^i, \alpha_2^i + \beta_2^i x_2^i, \alpha_3^i + \beta_3^i x_3^i, \alpha_4^i + \beta_4^i x_4^i)$, where all of these constants $\alpha_k^i, \beta_k^i$, for $k \in \{i, s\}, j \in \{1,2\}$ are rational constants, and $\alpha_k^i \in \{1/4, 1/4\}$.

**Proof.** According to our encodings, each snapshot $(q, i_L, i_R, s_L, s_R)$ of a machine $M$ is encoded as $(q/Q, x_1^i = < i_L >_{R}, x_2^i = < i_R >_{R}, x_3^i = < s_L >_{R}, x_4^i = < s_R >_{R})$.

Let $Q \in [0,1]$ be the set of $q/n_q$ where $n_q$ is the number of internal states of $M$. Each $x_j^i$, for $k \in \{i, s\}, j \in \{1,2\}$ encodes its radix-$4$ expansion the content of the right/left part of input/scratch tape. Each such $x_j^i$ can be written as

$$x_j^i = \sum_{m=1}^{\infty} \frac{x_{j,m}}{4^m}$$

12 I.e. defined on some pieces at some positive distance, see Definition 28 in appendix for a formal definition.
with the $s_{k,m}^i \in \{1,3\}$.

We will denote $abc...$ the real number with radix-4 expansion $abc$, for $a,b,c \in \{1,3\}$.

Let $I_{1,1}, I_{2,1}, I_{1,2}, I_{2,2}, I_{1,3}, I_{2,3}$ be all the sets defined by:

- $I_{j,0}^k = \{t_j^k, \ell_j^k + 1/4| \text{ and } \ell_j^k = 0.a_j^k \}$
- or $I_{j,0}^k = \{\ell_j^k| \text{ and } \ell_j^k = 0.a_j^k \}$

for all the possible $a_j^k \in \{1,3\}$.

Call input tape $i$, and scratch tape $s$: The content of the tape at the left (respectively: right) of the head of tape $k$ is non-blank and the symbol on left (resp. in front) of the head of tape $k$ is $a_j^k$ in the first case when $k = 1$ (resp. $k = 2$), and blank in the second case.

In what follows, we will not make any more this distinction, and we will suppose, in the second case that $a_j^k = 0$.

Let 

$$C = \bigcup_{q \in Q, \ell_1, \ell_2, \ell_3, \ell_4} Q \times I_{1,\ell_1} \times I_{2,\ell_2} \times I_{1,\ell_3} \times I_{2,\ell_4}$$

Function $\text{Fct}_M$ will be defined as piecewise linear on $C$.

Assume that $(q/|Q|, x_1, x_2, x_1', x_2') \in C$ encodes the current snapshot of $M$ at time $t$. Call $\Delta x^k_j = x^k_j - \ell^k_j$ for $k \in \{i, s\}, j \in \{1,2\}$.

On $Q \times I_{1,\ell_1} \times I_{2,\ell_2} \times I_{1,\ell_3} \times I_{2,\ell_4}$, we define $\text{Fct}_M$ such that $\text{Fct}_M(q/|Q|, x_1, x_2, x_1', x_2') = (q'/|Q|, x_1', x_2', x_1'', x_2'')$ with $\Delta x^k_j = x^k_j - \ell^k_j$ defined as follows, where $\ell^k_j = 0.a_j^k$, $a_j^k \in \{1,3\}$ (to ease the description, let us assume that the letters $\{0,1\}$ of the alphabet of machine $M$ are renamed respectively $\{1,3\}$):

- $q'$ is the new state given by the program of $M$ when $M$ is in state $q$ and its input and scratch heads read respectively $a_1^k$ and $a_2^k$.
- $x_1' = x_1^k + \Delta x_1^k$ if the program of $M$ says not to move the head of tape $k$ and write symbol $a$ on tape $k$ when $M$ is in state $q$ and its input and scratch heads read respectively $a_1^k$ and $a_3^k$.
- $x_1' = x_1^k/4 + a_1^k/4, x_2' = 4\Delta x_1^k$ if the program of $M$ says to move right the head of tape $k$ when $M$ is in state $q$ and its input and scratch heads read respectively $a_2^k$ and $a_3^k$.
- $x_1' = 4x_1^k - 4a_1^k, x_2' = a_1^k/4 + \Delta x_2^k/4$ if the program of $M$ says to move left the head of tape $k$ when $M$ is in state $q$ and its input and scratch heads read respectively $a_2^k$ and $a_2^k$.

It can be checked that, in any case, $\text{Fct}_M$ is such that $\text{Fct}_M(q/|Q|, x_1, x_2, x_1', x_2')$ encodes the snapshot $M$ at time $t+1$ whenever $(q/|Q|, x_1, x_2, x_1', x_2')$ encodes the snapshot $M$ at time $t$. Furthermore, function $\text{Fct}_M$ is a disconnected piecewise linear function with rational coefficients.

Proposition 27 can be reformulated as follows. Let $\text{Snapshot}_{M,s}(t)$ denotes the snapshot of machine $M$ at time $t$ if started at time 0 in snapshot $s$. Then $y_s(t) = \langle \text{Snapshot}_{M,s}(t) \rangle_{t \geq 0}$ is solution of the following dODE: $\dot{y}_s(t)_{t \rightarrow t+1}(t) = \dot{c}_M(y_s(t))$ where $\dot{c}_M(y) = \text{Fct}_M(y) - y$.

Then, following [10], dilation, linear path, $u$-turn, merge blocks can be easily combined, and from the specific form (see Definition 28) of the functions $\text{Fct}_M$ of Proposition 27, we can then build some $C^\infty$-ODE that simulates the iterations of $\text{Fct}_M$, and hence the evolution of $M$: see [10] for detailed (and improved) constructions, and Figures 15 and 16 for illustrations. One gets:
**Figure 15** Combining two linear paths, two u-turns, two dilations in order to iterate some affine function: here we iterate (the restriction to some domain of) \((x, y) \mapsto (2x + 0.2, y/2 - 0.3)\).

**Figure 16** Combining linear paths, u-turns, dilations in order to iterate some (disconnected) piecewise affine function (Definition 28) over \([0,1]^2\). Working over \(\mathbb{R}^d\) this is possible using same principle to iterate \(\text{Fct}_M\) for any Turing machine \(M\). In the general case, some merge blocks may be required. As observed in [6], using reversible Turing machines, merge blocks can be avoided.

**Proposition 29.** This dODE is \(C_0\)-ODE implementable over \(\mathbb{R}^7\) on some compact domain.

**Remark 30.** Actually, observing that Turing machines can be simulated by reversible Turing machines (see [29], chapter 5), one can avoid merge blocks and simulating a Turing machine can even be done with smooth \(f\) (and even over \(\mathbb{R}^3\)) as detailed in [6].

If one wants to cover also the dynamics of the output tape using our encoding, this is not feasible with a piecewise affine function, but the following clearly holds from definitions:

**Proposition 31 (Covering output tape Turing machines).** For any oTM \(M\) started in snapshot \(s\) at time 0, if we write \(n_s(t)\) for the head position at time \(t\), and \(w_s(t) \in \mathcal{C}\) for the content of the output tape at time \(t\), we have

\[
\begin{align*}
\dot{y}_s(t)_{\rightarrow t+1}(t) & = \hat{c}_M(y_s(t)) \\
\dot{n}_s(t)_{\rightarrow t+1}(t) & = \text{headmove}(y_s(t)) \\
\dot{c} < w_s(t+1)_{\rightarrow t+1}(t) & = 2 \cdot \text{headwrite}(y_s(t)) \cdot 4^{-n_s(t)}
\end{align*}
\]

where function \(\text{headmove}(\cdot)\) and \(\text{headwrite}(\cdot)\) are piecewise constant on \(\mathcal{S}_{\mathbb{R}^3}\) with values in \([-1,1]\) and \([0,1]\), encoding whether the head of the output moves right/left or writes.
Proposition 34. $dODE(\mathcal{Q})$ is $C_0$-ODE implementable over $\mathbb{R}^9$.

Proof. From Proposition 32 we know that there is some $C_0$-ODE $y' = f(y)$ that implements the dynamics of $y_s(\cdot)$.

Using classical techniques to build smooth functions that value some particular value on some domains, and from the form of the involved functions, one can build smooth functions $\text{headmove}(\cdot)$ and $\text{headwrite}(\cdot)$ from $\mathbb{R}^3 \rightarrow \mathbb{R}$ so that

\[
(y_s(t), n(t), w(t))' = (f(y), \text{headmove}(y_s(t)), \text{headwrite}(y_s(t)) \cdot 4^{-n(t)})
\]

\[
= F(y_s(t), n(t), w(t))
\]

This basically consists of setting correctly the values of these functions $\text{headmove}(\cdot)$ and $\text{headwrite}(\cdot)$ so that their integral over $[t, t+1]$, for $t$ integer, values precisely the correct values of $\text{headmove}(\cdot)$ and $\text{headwrite}(\cdot)$ for each subdomain $Q \times I_1 e_1 \times I_2 e_2 \times I_1' e_1' \times I_2' e_2'$ involved in the proof of Proposition 27.

We think one can easily be convinced this is feasible, and that providing all details of these functions would not really help.

Observe also that all these functions can be built with compact support, as we care of their value only on a compact domain.

3.2 In bounded time

A key remark for this article is that by using a suitable change of variable, we can guarantee to simulate previous dODE but in a finite time! The trick is to consider some increasing function $e(t) : \mathbb{R} \rightarrow \mathbb{R}$ and to consider the following alternative encoding:

\[
\begin{align*}
\text{Snapshot}_{M,s}(t) &= \langle \text{Snapshot}_{M,s}(e(t)) \rangle_{\mathbb{R}^3} \pi_\gamma(t)^2 \\
\pi_x(t) &= 4^{-n_s(e(t))} \\
\pi_y(t) &= < w_s(e(t)) >_{\mathbb{R}}
\end{align*}
\]

Remark 33. This is not the first time that the idea of a change of variable on time are considered (see e.g. [1][28][5] or related parts in survey [4]) in order to map an infinite time to a bounded time. However, the particular construction here is original and is based on both a change of time variable + space variable in order to get to $C_0$-ODEs [13].

Let $\mathbf{y}_s(t) = \langle \text{Snapshot}_{M,s}(e(t)), \pi_x(e(t)), \pi_y(e(t)) \rangle$.

Proposition 34 (Petard’s construction). Then $\mathbf{y}_s(t)$ is solution of $dODE$

\[
\frac{d\mathbf{y}_s(e(t))}{e(t+1)-e(t)} = \frac{1}{e(t+1)-e(t)}F(y_s(e(t)))
\]

\[\text{where}\]

\[13\text{To satisfy the constraint mentioned in Remark 20}\]
\[ F(s, z, w) = \left( \frac{\partial M(s)}{\partial s} \left[ -\frac{255}{32} \text{headmove} \left( \frac{s}{z} \right) + \frac{257}{32} \right] z^2 + s \left[ -\frac{255}{32} \text{headmove} \left( \frac{s}{z} \right) + \frac{225}{32} \right] \right). \]

**Proof.** From Propositions 67, 68, and observing that \(-\frac{15}{8} a + \frac{9}{8}\) values 1/4 - 1 and 4 - 1 for \(a \in \{-1, 1\}\), and that \(-\frac{255}{32} a + \frac{32}{32}\) values respectively 1/16 - 1 and 16 - 1 for \(a \in \{-1, 1\}\):

\[
\begin{align*}
\partial_{\text{output}} s(t)_{e(t+1)}(e(t)) &= \frac{1}{e(t+1) - e(t)} \left[ \frac{\partial M(s(t))}{\partial s} \right] \frac{\text{headmove} \left( \frac{s(t)}{s(t)} \right)}{s(t)} + 9 \frac{s(t)}{s(t)} \\
\partial_{\text{output}} s(t)_{e(t+1)}(e(t))^2 &= \frac{1}{e(t+1) - e(t)} \left[ -\frac{255}{32} \text{headmove} \left( \frac{s(t)}{s(t)} \right) \right] + 225 \frac{s(t)}{s(t)} \\
\partial_{\text{output}} s(t)_{e(t+1)}(e(t)) &= \frac{2}{e(t+1) - e(t)} \text{headmove} \left( \frac{s(t)}{s(t)} \right) \cdot \frac{s(t)}{s(t)}
\end{align*}
\]

Then, using also Proposition 64

\[
\begin{align*}
\partial_{\text{output}} s(t)_{e(t+1)}(e(t)) &= \frac{1}{e(t+1) - e(t)} \left[ \frac{\partial M(s(t))}{\partial s} \right] \frac{\text{headmove} \left( \frac{s(t)}{s(t)} \right)}{s(t)} + 9 \frac{s(t)}{s(t)} \\
&+ \frac{\text{headmove} \left( \frac{s(t)}{s(t)} \right)}{s(t)} \cdot \frac{s(t)}{s(t)}
\end{align*}
\]

A key remark is then the following:

\textbf{Proposition 35 (Analysis of Petard’s construction).} Assume \((e(t+1) - e(t))/s(t)\) remains upper bounded\(^{14}\). Then the above dynamics is living in some domain whose shape is a tablerone, and is simulating the whole computation of \(oTM M\) for all time \(t \in \mathbb{N}\) but doing so in some finite time \(e^* \in \mathbb{R}\): it converges to \((0, \ldots, 0, < o^* > \in \mathbb{R}) \in \mathbb{R}^7\) at \(e^*\) where \(o^* = \lim_{t \to \infty} \text{output}_{s(t)}\) denotes the limit cell by cell of the output tape of \(M\).

\(^{14}\) In relation with Remark 20.
Proof. The fact that this lives in a toblorone domain comes from the equations, observing that all encodings take values in \([0, 1]^d\) for some \(d\), hence in some bounded domain. From the \(\ast\)-property, necessarily \(\tau(t)\) will go to infinity. As \(\ast > \mathbb{R}^d\) take values in \([0, 1]^d\), that means that both \(\nu(t)\) and \(\tau(t)\) will go to 0. Now, \(\nu(t)\) will converge, using Proposition 18 to the encoding of \(\lim_{t \to +x} \nu(t)\) at time \(\lim_{t \to +x} e(t)\). But, now if \((e(t + 1) - e(t))/\tau(t)\) remains bounded by \(K\), from the \(\ast\)-property, \(e^* = \lim_{t \to +x} e(t)\) is finite: indeed, \(e(t) = \sum_{j=0}^{t-1} e(j + 1) - e(j) \leq K \sum_{j=0}^{t-1} \frac{1}{3^j} \leq K\) is some bounded increasing sequence, hence converging, using Lemma 36.

\(\diamondsuit\)

- **Lemma 36.** Let \(n(i)\) be some function \(\mathbb{N} \to \mathbb{N}\) such that \(n(i)\) values \(j\) at most \(2j\) times. Then \(\sum_{i=0}^{\infty} 4^{-n(i)} \leq 1.\)

Proof. We have

\[
\sum_{i=0}^{\infty} 4^{-n(i)} = \sum_{j=0}^{\infty} \sum_{i:n(i) = j} 4^{-j} \leq 2 \sum_{j=0}^{\infty} j 4^{-j} \leq 8/9 \leq 1.
\]

Indeed, by induction, \(\sum_{j=0}^{N} j 4^{-j} = 4/9 - (3N + 4)/9 \cdot 4^{-N}\), i.e. \(\sum_{j=0}^{\infty} j 4^{-j} = 4/9.\)

From the formula Proposition 67, \(\hat{\partial}(f \circ g) \rightarrow_q (p) = \hat{\partial} f \rightarrow_q (g(p)) \cdot \hat{\partial} g \rightarrow_q (p)\), writing \(\hat{\partial} t = e(t)\), dODE (3) can be reformulated (as both value \(\hat{\partial} \nu(t)_{\rightarrow t} \cdot \hat{\partial} \nu(t+1)\)) as:

\[
\hat{\partial} \nu(t)_{\rightarrow t+1} = \hat{\partial} \nu(t)_{\rightarrow t}.
\]

(4)

## 4 Climbing up the ordinal hierarchy

### 4.1 Petard basic block

The point is that this can be implemented using a \(C^0\)-ODE:

- **Proposition 37.** The dODE (3) is \(C^0\)-ODE implementable over \(\mathbb{R}^9\) over some toblorone domain, for some \(e(t)\) that keeps \((e(t + 1) - e(t))/\tau(t)\) upper bounded, and hence all previous conclusions hold.

Proof. From Proposition 32, there is some \(C^0\)-ODE \(y' = f(y)\) where \(y(t) = (y_s(t), n_s(t), w_s(t))\) with \(y_s(t)\) living in some compact domain \(K\) that implements the dODE (2). Furthermore, we can consider \(f\) null outside of the closure of \(D = \{(y, z, w) : y \in K, 0 < z, 0 \leq w \leq 1\}\).

- **Remark 38.** This comes from the way this function \(f\) is built, observing that the value of \(f\) outside of this domain is not important, and can be fixed to be 0 using classical techniques to build smooth functions that values a particular value on some particular domain.

As \(K\) is compact, hence bounded, there is some \(0 < M\) such that \(\|y\| \leq M/2\) for \(y \in K\). Write \(y = (y_1, \ldots, y_d)\). Consider \(\hat{\partial} y(t) = g(y(e(t)), n_s(t), w_s(t))\), where \(g(y, z, w) = (y 4^{-2z}, 4^{-z}, w)\), and let \(e(t) = \int_0^y \tau(u) du\). Then \(\hat{\partial} y(t)\) lives in \(D_2 = \{(y, z, w) : \frac{\|y\|}{2} \leq M, 0 < z \leq 1, 0 \leq w \leq 1\}\). Write \(f(y) = (f_1(y), \ldots, f_d(y))\). Setting \(h(y, z, w) = z\), we have:

\[
\hat{\partial} y(t) = \sum_{i=1}^d \frac{\partial g}{\partial y_i}(y(e(t))) \cdot y'_i(e(t)) \cdot e'(t)
\]

\[
= \sum_{i=1}^d \frac{\partial g}{\partial y_i}(y(e(t))) \cdot f_i(y(e(t))) \cdot h(y(t))
\]
We do not describe (also in figures) obvious machinery needed to set up correctly the input before realizing a petard.

4.2 Petard unit block

The previous constructions provide a way to build a petard which converge in finite time to $< \lim_{t \to \infty} \text{output}_{M,s}(t) >_{\mathbb{R}}$. This provides a petard’s block with a toblerone domain. By combining this block with a toblerone path as in Figure 18, one gets a petard unit block\footnote{We do not describe (also in figures) obvious machinery needed to set up correctly the input before realizing a petard.}. It has as input port the input port of the petard block, and as output port the output port of the toblerone path.

Such a block is basically computing the effect of $\lim_{t \to \infty} \text{output}_{M,s}(t)$ for some oTM, but corresponds to a $C^1_0$-ODE where unicity of solutions is lost: hence, we cannot be sure that the only trajectory is the one computing $\lim_{t \to \infty} \text{output}_{M,s}(t)$. However, this is possible to be precise about the behaviour of a petard unit block, and the key argument used in the reasoning is that the value of $\mathbf{F}$ on last coordinate is always non-negative from our hypotheses on $M$ (mainly: the output tape is write-only).

> Proposition 39 (Petard unit block’s behaviour). Trajectories starting from the point of coordinate $(< \text{Snapshot}_{M,s}(t) >_{\mathbb{R}}, 0, \ldots, 0)$ of the input port of some petard unit that reaches the output port does so in point of coordinates $(x^*, 0, \ldots, 0)$. Such possible $x^*$ are belonging to an interval of the form $[a, b]$ where $a = < \lim_{t \to \infty} \text{output}_{M,s}(t) >_{\mathbb{R}}$, $b = 1$ (Unless the program of $M$ has no instruction that writes on the output tape: Then $b = a = < \lim_{t \to \infty} \text{output}_{M,s}(t) >_{\mathbb{R}}$).
The one reaching $x^* = a$ is (the unique) $v$-greedy trajectory, where $v$ is the direction of the toblerone path.

Proof. As the dynamics in the interior of the domain of the toblerone path is smooth, from Cauchy-Lipschitz theorem, there is unicity of solutions as soon as a trajectory enters it, and a trajectory that reaches the output port of the petard unit $(x^*, 0, \ldots, 0)$ must necessarily have started from a point of coordinate $(x^*, 0, \ldots, 0)$ from the common port $C$ of the petard block and toblerone path. The problem is then to determine the possible such $x^*$. Now, the value of $F$ on last coordinate is always non-negative and this property can easily be maintained when extending the function to a continuous function in all previous constructions: Consequently, a trajectory evolving in the petard unit will necessarily have its last coordinate non-decreasing. From previous study, we know that at time $t$ it is in $\mathbb{w}_{sp}(t)$, so we know that $\sup_t \mathbb{w}_{sp}(t) = x^*$ is the infimum of the possible $x^*$.

(Unless the program of $M$ has no instruction that writes on the output tape) for continuity reasons, we cannot avoid that there must be strictly positive values for the last component of the dynamics arbitrary close to $C$. From Kneser’s theorem all $y^*$ with $x^* \leq y^* \leq 1$ can also be reached (Namely by some trajectory that first reach $C$ in the point of coordinate $x^*$, and then possibly come back one or several times to $C$, staying possibly arbitrarily close $^1$). A trajectory not reaching $a$ is by definition not $v$-greedy, where $v$ is the direction of the toblerone path, as the one reaching $a$ goes in a more greedy way in direction $v$ in $(x^*, 0, \ldots, 0)$.

4.3 Climbing the ordinal hierarchy

Then by connecting a petard unit block to another petard unit block by a linear path, providing the output of the first as input to the second one can then restart the computation of $oTM$ on this limit state. See Figure 19. First unit block will compute $< \lim_t output_{M,s}(t) >_{\mathbb{R}}$, that is to say the snapshot of $oTM$ at time $\omega$. Then second unit block will compute $< \lim_t output_{M,lim}output_{M,s}(t) >_{\mathbb{R}}$, i.e. snapshot of $oTM$ at time $\omega + \omega = \omega 2$. Using $k$ connected petard units, one can then simulate $M$ up to time $\omega k$, and so on. Actually, this is a rather amazing game: once we have a petard unit, as we did to iterate a piecewise affine function and simulate some arbitrary $oTM$, dilation, linear path, u-turn, merge blocks can be easily combined in order to iterate petard units. This provides a way to climb the ordinal hierarchy for ITTMs!

Remark 40. A key argument used here is that we focus on greedy trajectories. Otherwise we would have no guarantee on the behaviour of the composition of (two or more ) petard unit blocks: we would not have any guarantee on the input of the second petard unit, that could be distinct from $< \lim_t output_{M,s}(t) >_{\mathbb{R}}$.

---

$^1$This follows from continuity considerations adapting mathematical statements of [19] about (local) limits of trajectories. This fact being not crucial for our proof, we skip full justification.

$^2$This is represented for simplicity by a path in Figure 19 exactly connected to the input port of the second, but actually an addition of some constant term is needed to reencode correctly the output tape to the input tape of $M$. 

---
Now, petard units can be imbricated, and the above reasoning can be repeated: we basically only used the fact that any oTM can be $C^0_0$-ODE implemented on some compact domain to get that its limit can in turn be $C^0_0$-ODE implemented on some compact domain. Repeating the reasoning about the limits, we get limits of limits and so on. See Figure 20 for illustration. This basically proves Theorem 9.

Remark 41. The functions $f$ built are smooth but on a union of 1-manifolds.

5 Simulating $C^1_0$-ODEs using ITTMs

We need some concepts from computable analysis: cf [11][10][36]. Fix some standard notation of the set $Int^d$ ($Int$ for short) of all open $d$-dimensional cubes from $\mathbb{R}^d$ with edges parallel to the coordinate axes and with rational vertices. A name for a point $y \in \mathbb{R}^d$ is a list of $I_n \in Int$ with $\bigcap_{n=1}^{\infty} I_n = \{y\}$. A name for a continuous function $f : \mathbb{R}^d \to \mathbb{R}^k$ is a list of pairs of boxes $(I_n, J_n)$ with $I_n, J_n \in Int$ such that $f(I_n) \subset J_n$. A point or a function is computable if it has a computable name. Closed non-empty subsets $A \subseteq \mathbb{R}^d$ can be represented via the distance function $d_A : \mathbb{R}^d \to \mathbb{R}$, defined by $d_A(x) := \inf_{a \in A} \|x - a\|$. $A$ is said to be recursive if $d_A$ is computable, and a rec-name of $A$ is a name for $d_A$. $A$ is said to be co-recursively enumerable (co-r.e. for short) if $d_A$ is lower computable, i.e., one can list effectively all rational
greater than \(d_A(y)\). This is equivalent to say that it has a computable co-r.e.-name, i.e. a computable list of \(I_n \in \text{int}\) whose union is \(A^c\).

Given \(I \subseteq \mathbb{R} \times \mathbb{R}^d\), let \(R_f(I) = \bigcup_{(t,y) \in I} R_f(t,y)\). We assume some cylinder of security \(R = [t_0 - a, t_0 + a] \times B_r(y_0)\), for the IVP is fixed, i.e. \(a > 0, r > 0\), and \(\|f(y)\|\) is bounded by some \(M > 0\) for \(y \in R\) with \(Ma < r\). If \(f\) is bounded (in particular this holds if \(f\) has a compact support) by selecting \(M\) big enough, one can always find such a \(R\). Indeed:

\[\textbf{Lemma 42.} \ \text{Assume } f : \mathbb{R}^d \to \mathbb{R}^d \text{ is continuous. Consider } a > 0.\]

\[\text{Assume } f \text{ is bounded on } \mathbb{R}^d \text{ by } M \text{ (in particular this holds if } f \text{ has a compact support, as a continuous function over a compact has a maximum). Take } a = a \text{ and } r = Ma + 1.\]

\[\text{In the general case where } f \text{ is not necessarily bounded on the whole } \mathbb{R}^d \text{ take } r > 0, \text{ and consider } E_0 = [t_0 - a, t_0 + a] \times \overline{B_r(y_0)} \text{ for } a > 0 \text{ and } r > 0.\]

\[\text{As } f \text{ is continuous on compact } E_0, \text{ let } M > 0 \text{ be a bound for } \|f(t,y)\| \text{ on } E. \text{ Take } a = \min(a, r/(M + 1)).\]

\[\text{In any case, } E = [t_0 - a, t_0 + a] \times \overline{B_r(y_0)} \text{ is a cylinder of security for } f.\]

\[\textbf{Proof.} \ \text{From definitions, we always have } Ma < r \text{ on } E, \text{ and } f \text{ bounded by } M \text{ on } E. \]

We call safe zone the subset of points \((t,y) \in R\) with \(\|y - y_0\| \leq (Ma + r)/2|t-t_0|\). An \(\epsilon = 1/n\)-approximate solution is some continuous polygonal function \(\xi_n\) with \(\|\xi(t) - f(\xi(t))\| < \epsilon\) for all \(t \in [t_0 - a, t_0 + a]\). By considering a sufficiently small \(\epsilon = \epsilon(a,r,M)\), there always exists an \(\epsilon\)-approximate solution \(\xi_n\) with \(\xi_n(t_0) = y_0\) that remains in the safe zone: it can be built by taking \(\xi(t)\) constant rational on \(i\delta(n) \leq t \leq (i + 1)\delta(n)\) for some rational \(\delta(n) = \delta(n,a,r,M)\): call this an Euler path solution. More generally, still restricting \(\epsilon\) if needed, for a set \(S \ni (t_0,y_0)\) of diameter less than \(\epsilon\) included in the safe zone (we call such an \(S\) safe), we can guarantee that for all \((t,y) \in S\), there exists an \(\epsilon\)-approximate Euler solution \(\xi_n\) for \(\xi_n(t) = y\), that can be built with the same principle as an Euler path solution and remains in the safe zone. For a safe \(S\), let \(F_n(S)\) be the closed set of points of \(R\) reachable by some \(1/n\)-approximate Euler solution \(\xi_n\) with \(\xi_n(t) = y\), for some \((t,y) \in S\), possibly for some \(n \geq n_0\) that guarantees the solution to live in the safe zone. See appendix C.3 for explicit details about this paragraph, based on [31].

Using Ascoli/Arzela’s theorem as main ingredient the following holds:

\[\textbf{Proposition 43.} \ \text{Assume } S = \bigcap_n S_n, \text{ with every } S_n \text{ closed and safe. Then } \mathcal{R}_f(S) \cap R = \bigcap_n F_n(S_n). \text{ In particular, when } S \text{ is closed and safe, } \mathcal{R}_f(S) \cap R = \bigcap_n F_n(S).\]

We rely on explicit details about paragraph page 23 based on [31] developed in Appendix C.3.

\[\textbf{Proof.} \ \text{The inclusion } \mathcal{R}_f(S) \cap R \subseteq \mathcal{R}(S) = \bigcap_n F_n(S_n) \text{ is Lemma 73 (based on [31] Page 155), stated in appendix).}\]

\[\text{The other inclusion is mostly based on some applications of Peano-Arzela’s Theorem. Namely:}\]

\[\text{For all } n, \text{ (sufficiently big), one can build some } 1/n\text{-approximate solution } \xi_n \text{ that starts from some point } (t,y) \in S \text{ by Lemma 72 and remains in the safe zone. This family of solutions is uniformly bounded and equicontinuous. From Ascoli/Arzela’s theorem, one converging subsequence can be extracted from the sequence } (\xi_n)_n. \text{ This converging subsequence is a solution of the ODE, and it starts from a point } (t,y) \in S: \text{ Furthermore, this converging subsequence belongs to } \mathcal{R}(S). \text{ Consequently } \mathcal{R}_f(S) \text{ is non-empty, as well as } \mathcal{R}(S).\]

\[\text{Consider any point } (t^*,y^*) \text{ of } \mathcal{R}(S). \text{ For all } n, \text{ one can build some } 1/n\text{-approximate solution } \xi_n \text{ that starts from some point } (t,y) \in S \text{ and reaches } (t^*,y^*) \text{ by Lemma 72. This}\]
family of solutions is uniformly bounded and equicontinuous. From Ascoli/Arzela’s theorem, one converging subsequence can be extracted from the sequence \((t_n)\). This converging subsequence is a solution of the ODE, and starts from a point \((t, y) \in S\) and reaches \((t^*, y^*)\). This proves that \((t^*, y^*) \in R_{\xi}(S)\).

Observing that \(F_n(S_n)\) can be chosen with a rational description effectively computable:

> **Proposition 44 (Generalization/Reformulation of [31] [14])**. A co-r.e.-name of \(R_{\xi}(S) \cap R\) can be computed effectively from a rec-name of \(S\) and a name of \(f\).

**Proof.** The second assertion follows from the first, by considering \(S = S_n\) for all \(n\). From Proposition 43 we know that \(R_{\xi}(S) \cap R = \bigcap_{n} F_n(S_n)\). From results from [11] [33] about computability of co-r.e. sets, we just need to prove that one can effectively produce the involved \(F_n(S_n)\) from a rec-name of \(S\) and from a name of \(f\).

But this is clear from the definitions of the funnels (see appendix C.3 where we repeat their construction explicitly, and see [31] Section 4 which discusses explicitly how funnels \(F_n(S_n)\) can be represented (Lemma 74 repeated below)).

> **Remark 45.** The proof involves (Remark 70) the modulus of continuity of \(f\) that may be non-computable. However, this is not problematic, as we know that for \(n\) sufficiently big, we will produce a correct approximation of \(R_{\xi}(S) \cap R\) (even if one is not able to tell computably at which rank this will be correct).

> **Remark 46.** Notice that the approximation of \(R_{\xi}(S) \cap R\) by \(\bigcap_{n} F_n(S_n)\) is built to be always an over-approximation that converges to the correct value. The value may be non-computable.

> **Remark 47.** If the solution of IVP [1] is unique, we know that this trajectory is isolated in \(R_{\xi}(S) \cap R\). As an isolated point in some co-r.e. set is computable [11] Theorem 3.6], one deduces that \(\xi\) must then be computable. We find back the results from [31] and [14].

> **Remark 48.** This theorem is actually a generalization of both [31] and [14]. We believe it can be extended to deal with more general functions (e.g. not everywhere defined) generalizing the covering technique from [11].

Using tools from computable analysis to compute the underlying operations required to determine the unique \(v\)-greedy trajectory from the set of reachable states, one gets:

> **Proposition 49.** One can build some ITTM \(M\) that given the name of \(C_0\)-ODE \(f\), \(t_0\), and \(y_0\) outputs \(t^* = t^*(t_0, y_0), y^* = y^*(t_0, y_0)\), with \(y^* = \xi(t^*)\) for some \(t_0 < t^*\) for the unique greedy \(\xi\) trajectory solution of \([1]\) on \([t_0, t^*]\).

**Proof.** Consider some function \(\xi : [t_0, t_0 + a] \rightarrow \mathbb{R}^d\), that takes values in the safe zone, considered as a subset \((t, \xi(t))\) of \(R\), and a subset \(\mathcal{R}\) of the safe zone of \(R\), intended to correspond to reachable states, and function

\[
g(\xi, \mathcal{R}, t^*) = \sum_{i=0}^{\infty} \max(0, c_i) \cdot 2^{-i-1},
\]

where \((t_i, y_i)\) is some enumeration of rationals of \(B_r(y_0), i =< i_1, i_2 >\) where \(< \cdot, \cdot >\) is some computable bijection entre \(\mathbb{N}^2\) and \(\mathbb{N}\), and \(c_i\) is 1 if all the following conditions hold: \(t_0 < t_i < t_{i_2} < t^*, (t_i, y_{i_1}) \in \xi, (t_{i_2}, y_{i_2}) \in R_{\xi}(t_{i_1}, y_{i_1}), (t_{i_2}, y'_{i_2}) \in \xi, y_{i_2} = v < y'_{i_2} \cdot v\).
We have \( g(\xi, R) = 0 \iff (\forall i)c_i = 0 \) iff trajectory \( \xi \) is "globally" \( \nu \)-greedy among the possibilities given by reachable sets \( R \) on \([t_0, t^*]\) (for continuity reasons): In other words, \( \xi \) always selects a possibility of evolution in a greedy way in direction \( \nu \) over \([t_0, t^*]\).

Function \( g \) is computable.

Assume we know that for some \( t_0 < t^* \) where this holds. Given such a \( t^* \), one can get this trajectory as \( g^{-1}_1(R_\ell(t_0, y_0)) \) where \( g_1(\xi) = g(\xi, R_\ell(t_0, y_0)) \). As \( R_\ell(t_0, y_0) \) is co-recursively enumerable, and from the fact that the inverse image by a computable function of a co-recursively enumerable set is a co-recursively enumerable set \([30] \) page 4, \([33] \) bottom of page 138 we get that this trajectory \( \xi \) corresponds to a co-recursively enumerable subset of \( R \). As it is unique, it must be isolated on \( R \) (for the sup norm). As an isolated point in some co-r.e. set is computable \([11] \) Theorem 3.6], one deduces that \( \xi \) must then be computable when restricted to \([t_0, t^*]\).

The point is that this however requires to be able to start from a suitable \( t^* \) for which this holds. Determining if a given \( t^* \) is a correct guess is basically determining if the corresponding (computable analysis) approximation as above is converging for that \( t^* \). This is not feasible by a classical Turing machine, but this remains arithmetical (decidable with oracle the halting problem of Turing machines).

Recall that ITTM can decide arithmetical predicate as then can take limits: see \([18] \). Consequently, this is feasible by an ITTM: it tries in turn all rational \( t_\alpha \) for simplicity). Using that guess, it outputs \( t^* \) and the corresponding \( y^* \).

Theorem 10 follows from the following strategy, using classical ITTM machinery: at successor stage \( \alpha + 1 \), apply the ITTM \( M_{\text{succ}} \) of the proof of Proposition 49 to the state \( (t_\alpha, y_\alpha) \) computed at stage \( \alpha \) to get \( (t_{\alpha+1} = t^*(t_\alpha, y_\alpha), y_{\alpha+1} = y^*(t_\alpha, y_\alpha)) \) (and the machine is able to compute the position reached by the trajectory up to time \( t_{\alpha+1} \) by induction).

At limit stage \( \beta = \lim \alpha \) compute \( t^* = \sup t_\alpha \), \( y^* = \lim y_\alpha \) (and the machine is able to compute the position reached by the trajectory up to time \( t_{\alpha+1} \) by transfinite induction).

Necessarily there must be some ordinal \( \alpha \) with \( t_\alpha > t \): Otherwise, the sequence \( t_\alpha \) would be upper bounded increasing sequence: hence, it must have a limit \( t^* = \sup t_\alpha \). For continuity reasons of trajectories, the trajectory must be in \( y^* = \lim y_\alpha \) at that time. We get a contradiction: consider ordinal \( \alpha^* = \sup \alpha \). At successor time \( \alpha^* + 1 \) the machine must have produced through \( M_{\text{succ}} \) some \( t_{\alpha^*+1} > t^* \); a contradiction with the definition of \( t^* = \sup t_\alpha \).

6 The hierarchy of \( C^1_0 \)-ODE functions

We now state that the class of \( \nu \)greedy functions can be stratified in a Cantor-Bendixson way. We only sketch here the ideas, based on a generalization of Kechrin-Woodin’s hierarchy for derivable functions \([21] \). Given some closed \( P \subseteq R \subseteq \mathbb{R}^d \), \( \epsilon > 0 \), consider the following defined by transfinite induction, where \( \alpha \) is an ordinal, and \( \lambda \) a limit ordinal (assume \( t_0 = 0 \) for simplicity).

\[
\begin{align*}
P_{\ell,\tau}^0 &= R_\ell(0, y_0) \subseteq R \\
P_{\ell,\tau}^{\nu + 1} &= \{P_{\ell,\tau}^{\nu} \} \subseteq P_{\ell,\tau}^{\nu} \\
P_{\ell,\tau}^{\lambda} &= \bigcap_{\alpha < \lambda} P_{\ell,\tau}^{\alpha} 
\end{align*}
\]

where, given \( P, P_{\ell,\tau} \subseteq \{y \in P : \text{For every open neighborhood } U \text{ of } y, \text{there are rational points } \mathbf{p} < \mathbf{q}, \mathbf{r} < s \in U, \text{with } [\mathbf{p}, \mathbf{q}] \cap [\mathbf{r}, \mathbf{s}] \neq \emptyset, d_{\text{thickness}}(R_\ell^\mathbf{y}([\mathbf{p}, \mathbf{q}]), R_\ell^\mathbf{y}([\mathbf{p}, \mathbf{q}])) \geq \epsilon), \text{where } R_\ell^\mathbf{y}(I) \text{ corresponds to the points } (t, y) \in R_\ell(I) \text{ with } y \cdot \nu \text{ minimal with this property,}
\]
and \( d_{\text{thickness}} \) is the natural distance that measures the distance between two set of trajectories in the cylinder of security \( R \):

\[
d_{\text{thickness}}(R, R') = \sup_{t, (y, y') \in R, (y', y'') \in R'} \| y - y' \|.
\]

\begin{itemize}
  \item **Lemma 50.** When \( \alpha \leq \beta, \ P_{\varepsilon}^{\beta} \subseteq P_{\varepsilon}^{\alpha} \) and when \( \epsilon \leq \epsilon' \Rightarrow P_{\varepsilon}^{\alpha} \subseteq P_{\varepsilon}^{\alpha'} \). If \( P \) is closed, then \((P')_f, \epsilon \) is closed as well.
  \item **Proof.** From definitions, and from the fact that all conditions in the definition are closed.

  \begin{itemize}
    \item **Lemma 51.** For every \( \epsilon \in \mathbb{Q}, \epsilon > 0 \), there is some ordinal \( \alpha_f(\epsilon) \equiv \alpha(\epsilon) < \omega_1 \) such that \( P_{\varepsilon}^{\alpha} = P_{\varepsilon}^{\alpha_f(\epsilon)} = \text{def} \ P_{\varepsilon}^{\alpha_f} \) for all \( \alpha \geq \alpha(\epsilon) \).
    \item **Proof.** Assume this is not true: That would mean that we can find some sequence indexed by \( \alpha \) of distinct rational points in \( P_{\varepsilon}^{\alpha} - P_{\varepsilon}^{\alpha+1} \). This is impossible as the set of rational numbers is countable.

  \begin{itemize}
    \item **Theorem 52.** A \( C_0 \)-ODE \( f \) is !greedy if and only if \( \forall \epsilon \in \mathbb{Q}, \epsilon > 0 \exists \alpha < \omega_1 \text{ with } P_{\varepsilon}^{\alpha} = \emptyset \).
      \item This follows from arguments similar to [21] Fact 3.1.
      \item More precisely, we have:
      \end{itemize}

  \begin{itemize}
    \item **Proposition 53.** Let \( f \) be some continuous function.
      \item \( f \) is !greedy \( \iff \forall \epsilon \in \mathbb{Q}^+ (P_{\varepsilon}^{\emptyset} = \emptyset) \)
      \item \( \forall \epsilon \in \mathbb{Q}^+ \exists \alpha < \omega_1 (P_{\varepsilon}^{\alpha} = \emptyset) \)
      \item \( \forall \alpha < \omega_1 (P_{\varepsilon}^{\alpha} = \emptyset) \)
      \item \( \exists \alpha < \omega_1 (P_{\varepsilon}^{\alpha} = \emptyset) \)
      \item \( \exists \alpha < \omega_1 (P_{\varepsilon}^{\alpha} = \emptyset) \)
    \end{itemize}

  \begin{itemize}
    \item **Proof.** \( \Rightarrow \). Considering \( P = P_{\varepsilon}^{\emptyset} \), by contradiction, we just need to prove the following.
      \item **Lemma 54.** Let \( f \) be some continuous function and assume \( P \neq \emptyset \) is closed and \( \forall \in P, \ f \) is !greedy in \( y \). Then \( P_{\varepsilon}^{f, \epsilon} \) is nowhere dense in \( P \), and thus \( P_{\varepsilon}^{f, \epsilon} \subseteq P \) for all \( \epsilon \in \mathbb{Q}^+ \).
      \item Given \( p = (p_1, \ldots, p_d) \), \( q = (q_1, \ldots, q_d) \in \mathbb{R}^d \), we write \( p < q \) for \( p_i < q_i \) for \( 1 \leq i \leq d \), and \( [p, q] \) (respectively: \( (p, q) \)) for \( \{(x_1, \ldots, x_d) : p_i < x_i < q_i, \ (\text{resp. } p_i < x_i < q_i), \ 1 \leq i \leq d\} \) and we write \( p - q < \epsilon \) for \( p_i - q_i < \epsilon \) for \( 1 \leq i \leq d \).
      \item **Proof.** If \( P_{\varepsilon}^{f, \epsilon} \) is not nowhere dense in \( P \), there is an open \( I \) with \( \emptyset \neq I \cap P \subseteq P_{\varepsilon}^{f, \epsilon} \). Consider for each \( n \in \mathbb{N} \),
        \[
        E_n = \{ y : \exists p, q, r, s \text{ s.t. } x \in (p, q) \cap (r, s) \land q - p < 1/n \land s - r < 1/n \land d_{\text{thickness}}(R_y^{f, \epsilon}([p, q]), R_y^{f, \epsilon}([r, s])) > \epsilon/2 \}.
        \]
      \item Then \( E_n \) is an open subset in \( P \). We claim that \( E_n \) is dense in \( I \cap P \). Then by the Baire Category Theorem, \( \bigcap_n E_n \neq \emptyset \). But if \( y \in \bigcap_n E_n \) then \( f \) is not !greedy in \( y \) and we have a contradiction.
      \item It remains to prove our claim that \( E_n \) is dense in \( I \cap P \). Let \( J \) be an open interval with \( \emptyset \neq J \cap P \subseteq I \cap P \). Choose \( y \in J \cap P \). Then \( y \in P_{\varepsilon}^{f, \epsilon} \), so there are \( p, q, r, s \in J \) with \( q - p < (1/n), s - r < (1/n), \)
        \[
        [p, q] \cap [r, s] \neq \emptyset \text{ and } d_{\text{thickness}}(R_y^{f, \epsilon}([p, q]), R_y^{f, \epsilon}([r, s])) > \epsilon/2
        \]
From continuity of \( f \), we can actually change \( p, q, r, s \) slightly if necessary to assume that \((p, q) \cap (r, s) \cap P \neq \emptyset \); this comes from the form of the involved funnels, described in Section C.3. Let then \( z \in (p, q) \cap (r, s) \cap P \). Then clearly \( z \in J \cap E_n \) and \( E_n \) is dense in \( I \cap P \).

\( \Leftarrow \). Assume \( f \) is not !greedy and let \( y \) such that \( f \) is not !greedy in \( y \). Then there is some \( \epsilon > 0 \) such that for each \( n > 0 \) there are points \( p \leq y \leq q, r \leq y \leq s \) with \( q - p < 1/n \) and \( s - r < 1/n \) and \( d_{\text{thickness}}(R^y(\{p, q\}), R^y(\{r, s\})) > \epsilon \). Then it can be checked by transfinite induction on \( \alpha \) that \( y \in P^\alpha_\epsilon \) for all \( \alpha \) and therefore \( P^\infty_\epsilon \neq \emptyset \).

Considering the supremum of all such \( \alpha \) leads to the following definition

**Definition 55** (Rank of \( C^0_\epsilon \)-ODE functions). Let \( |f| \) be the least ordinal \( \alpha \) such that \( \forall \epsilon \in \mathbb{Q}^+, P^\alpha_\epsilon = \emptyset \).

**Proposition 56.** The union over \( \alpha \) of the functions with \( |f| = \alpha \) is the class of \( C^0_\epsilon \)-ODE functions. The hierarchy is strict: for each countable ordinal \( \alpha \geq 1 \) there is some \( C^1_\epsilon \)-ODE \( f \) with \( |f| = \alpha \).

**Proof.** The first item follows clearly from Theorem 52 and previous remarks.

The strictness of the hierarchy is actually demonstrated from our constructions: when in our constructions we build a petard from a dynamics iterating a petard of level \( \alpha \) we get a dynamic of level \( \alpha + 1 \).

To get a petard of level \( \beta = \lim \alpha \), build as above petards of level \( \alpha \) for each \( \alpha \), and imbricate all these in a unique petard.

Notice that this argument is still in spirit of [21].

Consequently, the dynamics \( f_\rightarrow \), from Example 2, is of rank 1, and the dynamics of a single petard is of rank 2.

Indeed:

Let \( F : \mathbb{R}^d \rightarrow \mathcal{P}(\mathbb{R}^d) \) that maps a point \( y \) to the unique greedy trajectory solution of IVP \( (1) \), where a trajectory \( \xi \) is considered as subset \( \{(t, \xi(t)) : t \in [t_0 - \delta, t_0 + \delta]\} \) over cylinder of security \( R \).

We say that \( F \) is continuous at \( y \) with respect to \( d_{\text{thickness}} \) if for all \( \epsilon > 0 \) there exists some \( \delta > 0 \) such that \( \|y - y'\| < \delta \) implies \( d_{\text{thickness}}(F(y), F(y')) < \epsilon \).

We state the following:

**Lemma 57.** Let \( f \) be \( C^0_\epsilon \)-ODE. Then

\[
\{y_0 : F \text{ is discontinuous at } y \text{(with respect to } d_{\text{thickness}}\text{)}\} = \bigcup_{\epsilon \in \mathbb{Q}^+} P^1_\epsilon, F = \bigcup_{n} P^1_{1/n}, F.
\]

**Proof.** Let \( F \) be discontinuous at \( y \). Then for some \( \epsilon > 0 \), for all \( \delta = 1/n > 0 \), there is some \( y_n \) with \( \|y_n - y\| < 1/n \) and \( d_{\text{thickness}}(F(y_n), F(y)) \geq \epsilon \). Let \( U \) be a neighborhood of \( y \). Let \( p < y < q \) be such that \( p, q \in U \) and \( d_{\text{thickness}}(\mathcal{R}^y(\{p, q\}), F(y)) \leq \epsilon/2 \). Then for some small enough \( n \), \( y_n \in (p, q) \), and thus we can find \( r < y_n < s, r, s \in (p, q) \), so that

\[
d_{\text{thickness}}(\mathcal{R}^y(\{r, s\}), F(y_n)) < \epsilon/4.
\]

Then

\[
d_{\text{thickness}}(\mathcal{R}^y(\{p, q\}), \mathcal{R}^y(\{r, s\})) \geq \epsilon/4
\]

so that we have shown that \( y \in P^1_{\epsilon/4} \).
Conversely, assume that \( y \in P_\epsilon^1 \). If \( F \) was continuous at \( y \), we could find a neighborhood \( U \) of \( y \) such that if \( x \in U \), then

\[
d_{\text{thickness}}(F(x), F(y)) < \epsilon/4.
\]

and thus for any \( x, z \in U \), \( d_{\text{thickness}}(F(x), F(z)) < \epsilon/2 \). But since \( y \in P_\epsilon^1 \), there are \( p, q, r, s \in U \) with \( p < q \), \( r < s \) in \( U \) and

\[
d_{\text{thickness}}(R^+([p, q]), R^+([r, s])) \geq \epsilon.
\]

We reach a contradiction from the definitions of these sets, considering any \( x \in [p, q] \), and \( z \in [r, s] \).

This opens ways to relate the halted time of ITTMs to the associated \( C^0_0 \)-ODE, even if this section remains to be fully explored.

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Proof of Proposition 15

Lemma 58. Assuming the output tape cannot be read is not a limitation in terms of computational power for Turing machines.

Proof. The machine can always maintain on scratch tape in parallel to the computation the information about the cells of the output tape that have been already written, so that it can emulates the reading of the output tape.

Notice also that the program of the machine can be made in such a way that the output tape produced at limit steps contains (using classical encodings for infinite time Turing machines, e.g. by taking a cell over two) the initial input tape, so that the input is never forgotten when a limit is taken.

We also assume the following property: every cell number $j$ of the output tape is visited at most $2j$ times by the output tape Turing machine. This is called the $\ast$-property.

Lemma 59. Assuming the $\ast$-property is not a limitation in terms of computational power for Turing machines.

Proof. This consists of simulating on the scratch tape the work to be done by the head on the output tape (in addition to work to be done on the scratch tape). While any simulation step is done, the output head systematically goes right. When by this simulation one realizes that something has to be written on the output tape, one does a loop that moves the output head left up to the correct position, writes (necessarily a 1 over a 0 from the model) and then the simulation continues. By doing so, a given cell at position $j$ is visited at most twice the number of 1 to be written on the output tape at some position $j' \leq j$: indeed, for each such cell $j'$, it will be visited at most once while the output head going right (corresponding to a simulation), and at most once while coming back left to $j'$.

Remark 60. This $\ast$-property is highly used in our simulations. In particular in Proposition 35 through Lemma 36.
C Repeating proofs from literature

C.1 About the $C_0$-ODE discussed in Example 2

We discuss here the $C_0$-ODE

$$y' = f_\cdot(y), \quad y(0) = 0$$

where $f_\cdot$ is described in Remark 2. This is mostly following and slightly adapted from [22, page 172-173].

- **Lemma 61.** Function $f_\cdot$ is indeed a continuous function from $\mathbb{R}$ to $\mathbb{R}$ (but not derivable, nor Lipschitz in 0). It is locally Lipschitz in $y$, for $y \neq 0$.

  When $a$ is some real, let

  $$y_a(t) = \begin{cases} 0 & \text{for } t < a \\ (t-a)^3 & \text{for } t \geq a \end{cases}$$

  The trajectory starting from 0 of $y' = f_\cdot(y)$ are given by the following.

- **Lemma 62.** The constant zero function is a solution. $y_{a}(\cdot)$ is a solution for all $a \geq 0$.

  **Proof.** The constant zero function is a solution because its derivative is everywhere zero.

  For each real $a$, one can check by differentiation that $y'_a(t) = 0$ whenever $t \leq a$ and $y'_a(t) = 3(t-a)^2 = 3y_a(t)^{2/3}$ for $t > a$. When $a \geq 0$, $y_a(0) = 0$, so $y_a(0)$ is a solution of the initial value problem. When $a < 0$, we instead have $y_a(0) = -a^3 > 0$ so in this case $y_a(t)$ is not a solution.

  We now show that

- **Lemma 63.** These are the only solutions.

  **Proof.** Let $\xi$ be a solution of the initial value problem, and suppose that $\xi(\cdot)$ is not the constant zero function. Since $f_\cdot(y) = 3y^{2/3} \geq 0$ for all $y$, $\xi(t)$ must be non-decreasing that is $\xi(s) \leq \xi(t)$ for $s < t$. We must also have $\xi(0) = 0$, so $\xi(t) \geq 0$ for all $t$.

  Whenever $b > 0$, and $\xi(b) > 0$, the initial value problem

  $$y' = 3y^{2/3}, \quad y(b) = \xi(b)$$

  has at most one solution on $[b, +\infty)$, as the function is locally Lipschitz.

  But it has a solution on $[b, +\infty)$, namely the function $y_a(\cdot)$, where $(b-a)^3 = \xi(b)$, that is $a = b - (\xi(b))^{1/3}$. It follows from uniqueness that we get the same value $a$ for each starting point $(b, y(b))$ on the curve.

  Thus for some $a \geq 0$, we have $\xi(t) = y_a(t)$ whenever $\xi(t) > 0$. Since $\xi(\cdot)$ and $y_a(t)$ are continuous, non-decreasing, and map 0 to 0, we must have $\xi(t) = y_a(t)$ for all $t$.

C.2 Review of some statements from the theory of dODEs

- **Proposition 64** (Derivative of a product).

  $$\partial (f \cdot g)_{-q}(p) = \partial f_{-q}(p) \cdot g(p) + f(p) \cdot \partial g_{-q}(p)$$

  $$= \partial f_{-q}(p) \cdot g(q) + f(p) \cdot \partial g_{-q}(p)$$

  $$= \partial f_{-q}(p) \cdot g(p) + f(p) \cdot \partial g_{-q}(p) + (q-p)\partial f_{-q}(p) \cdot \partial g_{-q}(p)$$
Proof. The result follows from three equations
\[ f(q)g(q) - f(p)g(p) = (f(q) - f(p))g(p) + f(q)(g(q) - g(p)) \]
\[ f(q)g(q) - f(p)g(p) = (f(q) - f(p))g(q) + f(p)(g(q) - g(p)) \]
\[ f(q)g(q) - f(p)g(p) = (f(q) - f(p))g(q) + f(p)(g(q) - g(p)) + (q - p) \frac{f(q) - f(p)}{q - p} [g(q) - g(p)] \]

> **Proposition 65** (Derivative of \( 1/f(x) \)). Whenever \( p \neq q \), \( f(p) \neq 0 \), \( f(q) \neq 0 \):
\[
\dot{\frac{1}{f(q) - q}}(p) = -\frac{\dot{f}(q)}{f(q)^2}
\]
Proof. The result follows from:
\[
\frac{1}{f(q)} - \frac{1}{f(p)} = \frac{f(p) - f(q)}{f(p)f(q)}
\]

> **Proposition 66** (Derivative of a ratio). Whenever \( p \neq q \), \( g(p) \neq 0 \), \( g(q) \neq 0 \),
\[
\dot{\frac{f}{g}}_{q}(p) = \frac{\dot{f}(q)g(p) - f(q)\dot{g}(q)}{g(p)g(q)}
\]
Proof. The result follows from:
\[
\frac{f(q)}{g(q)} - \frac{f(p)}{g(p)} = \frac{f(q)g(p) - f(p)g(q)}{g(p)g(q)}
\]
\[
= \frac{g(p)[f(q) - f(p)] - f(p)[g(q) - g(p)]}{g(p)g(q)}
\]

> **Proposition 67** (Derivative of a (specific) composition). Whenever \( p \neq q \), \( g(p) \neq g(q) \),
\[
\dot{g}_{q}(p) \neq 0:
\]
\[
\dot{f}(g(q)) = \dot{g}(g(q)) \cdot \dot{g}(q)
\]
This can be reformulated as
\[
\dot{(f \circ g)}_{q}(p) = \dot{g}(g(q)) \cdot \dot{g}(q)
\]
Proof. The result follows from:
\[
\frac{f(g(q)) - f(g(p))}{g(q) - g(p)} = \frac{f(g(q)) - f(g(p))}{q - p}, \quad \frac{q - p}{g(q) - g(p)}
\]

> **Proposition 68** (Derivative of \( k^n \)). Let \( k > 0 \) be some constant, and \( p \neq q \).
\[
\dot{k(n)}_{q}(p) = k^n(p) \left[ \frac{k^n(q) - n(p) - 1}{q - p} \right]
\]
C.3 Details about the paragraph page 23

We reuse some assertions and computations from [31]. For completeness, we repeat here some part of this article (with some very slight modifications on notations, to deal with the case of initial data living in $S$).

C.3.1 Some geometrical properties of solutions: Repeating [31]

Consider a solution $y(t)$ of IVP (1), and assume for the rest of this section for simplicity that $t_0 = 0$.

As often in the context of ODEs, we consider each point $y(t) \in \mathbb{R}^d$ as point $(t, y_1(t), \ldots, y_d(t))$ of $\mathbb{R}^{d+1}$.

**Remark 69.** We denote on the coming subsections, to remain close to [31], $y(t)$ for a trajectory instead of $\xi(t)$.

By the $n$-dimensional mean value theorem, for $t_1$ and $t_2$ in $[0, a]$,\[ \|y(t_2) - y(t_1)\| < |t_2 - t_1| \sup_{0 \leq t \leq a} \|y'(t)\|. \]

Notice that here $y(t)$ may be a continuous and piecewise differentiable function such as an $\epsilon$-approximate polygonal solution. Note also that in $\mathbb{R}^d$ the tangent of the acute angle between the vector $p_{t_2}, y(t_2)$ and the $t$-axis is $\|y'(t)\|$. Thus, if it is assumed that $\|f(y(t))\| \leq M$ when $0 \leq t \leq a$, it follows that within any time interval $[t_0, t_0 + \delta]$ included in $[0, a]$\[ \|y(t_0 + \delta) - y(t_0)\| \leq M\delta, \tag{9} \]

In other words, the acute angle between the vector $(t_0 + \delta, y(t_0 + \delta)) - (t_0, y(t_0))$ and the $t$-axis is at most $\arctan M$.

On the cylinder of security $R$, $f$ is continuous and hence effectively uniformly continuous, i.e., there is a (nondecreasing and unbounded) function $N$ such that\[ \|f(y_2) - f(y_1)\| \leq \frac{1}{n} \text{ whenever } \|y_2 - y_1\| \leq \frac{Ma}{N(n)}. \]

**Remark 70.** Observe that this function $N$ is basically encoding the modulus of continuity of $f$.

Take then a division $\mathcal{D}_n$ of $[0, a]$ into $N(n)$ subintervals of equal length and denote $\delta(n) = \frac{a}{N(n)}$. Thus, assuming $y$ is in cylinder of security $R$, in each subinterval $[i\delta(n), (i + 1)\delta(n)]$,\[ \|y'(t) - y'(i\delta(n))\| \leq \frac{1}{n}. \]

Denoting now by $\angle(a, b)$ the angle between the vectors $a$ and $b$ it follows from the general inequality
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\[ |\sin \angle(a, b)| \leq \frac{\|a - b\|}{\|b\|} \]

(which is equal to \((a \cdot (a - b))^2 \geq 0\) that in \(\mathbb{R}^{n+1}\)

\[ \angle((1, y'(t)), (1, y'(i\delta(n)))) \leq \alpha_n \]

where

\[ \alpha_n = \arcsin \frac{1}{n}. \]

C.3.2 An explicit construction of a forward Funnel: Repeating [31]

Using the division \(D_n\) an approximation funnel \(F_n(S)\) for some solution \(y\) is constructed as follows. For each subinterval \([i\delta(n), (i+1)\delta(n)]\) denote by \(C_i(y_1)\) the truncated hypercone in \(\mathbb{R}^{n+1}\) with apex \((i\delta(n), y_1)\), axis direction \((1, f(y_1))\), semiangle \(\alpha_n\) and \(i\delta(n) \leq t \leq (i+1)\delta(n)\). I.e., \(C_i(y_1)\) consists of all points \((t, z)\) such that

\[ \angle((t - i\delta(n), z - y_1), (1, f(y_1))) \leq \alpha_n \] and \(i\delta(n) \leq t \leq (i+1)\delta(n)\).

The section of the approximation funnel \(F_n(S)\) where \(i\delta(n) \leq t \leq (i+1)\delta(n)\) is denoted by \((F_n(S))_i\) and is defined recursively by

\[ (F_n(S))_0 = C_0(y_0), \quad (t_0, y_0) \in S \]

\[ (F_n(S))_{i+1} = \bigcup_{(i+1)\delta(n) \leq (i+1)\delta(n) \in (F_n(S))_i} C_{i+1}(y), \quad (\text{for } i = 0, \ldots, N(n) - 2). \]

\(F_n(S)\) is then the union of \((F_n(S))_0, \ldots, (F_n(S))_{N(n) - 1}\).

We write \(F_n(S)\) for some approximation funnel built as above.

C.3.3 Trajectories remain in the safe zone: Repeating and adapting [31]

Points \((t, y)\) with \(\|y - y_0\| > r\) do not appear. This assumption will be valid for sufficiently small \(\alpha_n\).

\textbf{Lemma 71.} In a parallelepiped of security, points \((t, y)\) with \(\|y - y_0\| > r\) do not appear for sufficiently small \(\alpha_n\) if \(n \geq n_0\) for some \(n_0\).

Furthermore, all points that appear are in the safe zone.

\textbf{Proof.} To see this take a point \((t, y)\) in a truncated hypercone \(C_i(y_1)\) where \(\|y_1 - y_0\| \leq r\). Then

\[ \angle((t - i\delta(n), y_1), (1, f(y_1))) \leq \alpha_n \]

and

\[ \angle((1, 0), (1, f(y_1))) = \arctan \|f(y_1)\| \leq \arctan M. \]

By the general inequality \(\angle(a, b) \leq \angle(a, c) + \angle(c, b)\)

\[ \frac{\|y - y_1\|}{t - i\delta(n)} = \tan \angle((t - i\delta(n), y_1), (1, 0)) \leq \tan(\alpha_n + \arctan M). \]
Now, $\alpha_n$ is chosen to be so small that $\tan(\alpha_n + \arctan M) < (\frac{c}{2} + M)/2$. Going through the sections $(F_n(S))_0, \ldots, (F_n(S))_{N(n) - 1}$ one by one until $C_i(y_1)$ is reached, one sees that then

$$\|y - y_0\| \leq i\delta(n) \tan(\alpha_n + \arctan M) + (t - i\delta(n)) \tan(\alpha_n + \arctan M) \leq r.$$ 

Thus the whole approximation funnel $F_n(S)$ is included in the truncated parallelepiped $\|y - y_0\| \leq r$, $0 \leq t \leq a$.

In the sequel it will be assumed, by choosing a large enough value of $n$, say $n > n_0$, that this is the case.

It suffices to take

$$n_0 \geq \frac{1}{\sin(\arctan((\frac{c}{2} + M)/2) - \arctan M)}.$$ 

C.3.4 Points in the funnels are reachable by Euler path solutions

\begin{itemize}
  \item **Lemma 72** (proof in [31, Page 156-157]). Fix a closed safe subset $S \subset R$.

  Fix some $\epsilon > 0$. Take a sufficiently large $n_j$ and a point $(t', w_1) \in (F_{n_j}(S))_i$ for some $i$. There exists a polygonal (i.e. an Euler path solution) $\epsilon$-approximate solution $v(t)$ with $v(t') = w_1$ and $v(0) \in S$.

  **Proof.** This is by definition of the funnels, that are precisely made to maintain this property true: This is explicitly explained in [31, Page 156-157].

C.3.5 In a cylinder of security, solutions remains always in the funnels

\begin{itemize}
  \item **Lemma 73** (proof in [31, Page 155]). Consider $y(t)$ be a solution of the IVP (1) with $(t_0, y_0) \in S$. Then $(t, y(t))$ is in $C_i(y(i\delta(n)))$ for $n = n_0, n_0 + 1, \ldots$ and $i = 0, \ldots, N(n) - 1$ and $i\delta(n) \leq t \leq (i + 1)\delta(n)$. Consequently, $(t, y(t))$ is in $F_n(S)$ for all $n \geq n_0$ and $0 \leq t \leq a$.

  This follows from the $n$-dimensional mean value theorem, and the definition of the funnels that are precisely made to maintain this property true: See [31, Page 155] for an explicit proof.

C.3.6 The funnels $F_n(S)$ have computable parameterizations

\begin{itemize}
  \item **Lemma 74** ( [31, Section 4]). Each $F_n(S)$ can be described by finitely many computable parameters for $S \in \text{Int}$.

  Recall that $\text{Int}$ are open $d$-dimensional cubes from $\mathbb{R}^d$ with edges parallel to the coordinate axes and with rational vertices.