The fourth moment of Dirichlet $L$-functions along the critical line

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Abstract. For a positive integer $q \not\equiv 2 \pmod{4}$, this work considers the fourth moment of Dirichlet $L$-functions averaged over both $t \in [0, T]$ and primitive characters to modulus $q$. An asymptotic formula with a power saving from both $q$-aspect and $t$-aspect in the error term is obtained.

1. Introduction

Moments of families of $L$-functions have a wide range of applications, and their computation is counted as a central problem in number theory, which may go back to Hardy and Littlewood [14]. If we define

$$M_k(T) = \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt,$$

Hardy and Littlewood proved that $M_1(T) \sim T \log T$, and then Ingham (see [27]; Chapter VII) showed the fourth moment to be $M_2(T) \sim \frac{1}{2\pi^2}T(\log T)^4$. In general, it is conjectured that

$$M_k(T) \sim C_k T(\log T)^{k^2},$$

for some constant $C_k$, whose precise value was predicted by Keating and Snaith [17] by analogies with random matrix theory. Although higher moments have not yet been computed, Soundararajan [26] obtained almost sharp upper bounds on GRH that $M_k(T) \ll T(\log T)^{k^2+\varepsilon}$, and the $\varepsilon$ on the power of $\log T$ was then removed by Harper [11].

Conrey, Farmer, Keating, Rubinstein and Snaith [7] refined the conjecture (1.1), and predicted that

$$M_k(T) = TP_{k^2}(\log T) + O(T^{\frac{7}{8}+\varepsilon}),$$

where $P_{k^2}$ is a polynomial of degree $k^2$. For $k = 2$, the asymptotic formula has already been proved by Heath-Brown [12] except for the strength of the error. More precisely, Heath-Brown [12] proved that

$$\int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = TP_4(\log T) + O\left(T^{\frac{7}{8}+\varepsilon}\right).$$

To deduce all main terms as well as a power saving error term is a significant challenge, and it requires a difficult analysis on off-diagonal terms to distinguish lower-order main
terms. Some deep estimates on the divisor problem
\[ \sum_{n \leq x} d(n)d(n + f) \]
were explored to obtain the power saving in \[12\]. Further progresses on the fourth moment of the Riemann zeta-function were based on methods originating in the spectral theory of automorphic forms, in particular the Kuznetsov formula. Then, Zavorotnyi \[33\] improved the result to
\[ \int_0^T |\zeta(\frac{1}{2} + it)|^4 dt = TP_4(\log T) + O(T^{\frac{7}{4} + \varepsilon}). \]
Motohashi established a beautiful explicit formula for a smoothed version of the fourth moment of the Riemann zeta-function in terms of the cubes of the central values of certain automorphic L-functions (to see Theorem 4.2 of \[22\]). Based on this explicit formula, Ivić and Motohashi \[16\] were able to replace the factor \(T^{\varepsilon}\) in (1.2) by a suitable power of \(\log T\), and this is the best estimate to date. A generalization of Motohashi’s formula to the fourth moment of Dirichlet L-functions weighted by a non-archimedean test function has also been obtained by Blomer, Humphries, Khan, and Milinovich \[6\], which proceeds differently with some important applications.

To some extent, the fourth moment averaging over \(t\) for an individual Dirichlet L-function is a direct extension of the problem from the Riemann zeta-function. Recently, Topacogullari \[28\] considered this moment and proved that
\[ \int_1^T |L(\frac{1}{2} + it, \chi)|^4 dt = \int_1^T P_\chi(\log t) dt + O\left(q^{2 - 3\theta} T^{\frac{7}{4} + \theta + \varepsilon} + qT^{\frac{7}{4} + \varepsilon}\right), \]
where \(P_\chi\) is a polynomial of degree 4 with coefficients depending on \(q\), and where \(\theta = 7/64\) is the current best-known bound on the size of the Hecke eigenvalue of a Maass form, due to Kim and Sarnak \[18\]. With \(\theta = 7/64\), this asymptotic formula is non-trivial in the range \(q \ll T^{25/107 - \varepsilon}\).

The fourth moment of Dirichlet L-functions at the central value
\[ \frac{1}{\varphi^*(q)} \sum^*_{\chi \pmod{q}} |L\left(\frac{1}{2}, \chi\right)|^4 \]
has gotten a lot of attention. Here, the sum is over all primitive characters modulo \(q\), and \(\varphi^*(q)\) is the number of these primitive characters. Due to a conjecture for the moments of unitary style in \[7\], it is predicted that

**Conjecture 1.1.** For any \(q \not\equiv 2 \pmod{4}\), we have
\[ \frac{1}{\varphi^*(q)} \sum^*_{\chi \pmod{q}} |L\left(\frac{1}{2}, \chi\right)|^4 = \prod_{\rho \neq 1} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} P_4(\log q) + O\left(q^{\frac{7}{4} + \varepsilon}\right), \]
where \(P_4\) is a computable absolute polynomial of degree 4.

It was first proved by Heath-Brown \[13\] that
\[ \frac{1}{\varphi^*(q)} \sum^*_{\chi \pmod{q}} |L\left(\frac{1}{2}, \chi\right)|^4 = \frac{1}{2\pi^2} \prod_{\rho \neq 1} \frac{(1 - p^{-1})^3}{(1 + p^{-1})}(\log q)^4 + O\left(2^{\omega(q)} \frac{q}{\varphi^*(q)}(\log q)^3\right), \]
where $\omega(q)$ means the number of distinct prime factors of $q$. This asymptotic formula is non-trivial if $\omega(q)$ is not too large. Then Soundararajan [25] filled in this exception with a sharper error term, so the leading term of the asymptotic formula was proved completely.

In 2011, Young [32] made an important breakthrough and proved the asymptotic formula for prime moduli that

\begin{equation}
\frac{1}{\varphi^*(q)} \sum_{\chi \mod q} \left| L\left( \frac{1}{2}, \chi \right) \right|^4 = P_4(\log q) + O\left( q^{\frac{1}{2}+\frac{1}{40}+\varepsilon} \right).
\end{equation}

Then Blomer, Fouvry, Kowalski, Michel and Miličević [3, 4] improved on the error term in (1.5) to $p^{-1/20}$, with an average result of Hecke eigenvalues to remove $\theta$, as well as some new results on bilinear forms in Kloosterman sums; see also Fouvry, Kowalski, and Michel [9], Kowalski, Michel, and Sawin [19], and Shparlinski and Zhang [24].

By distinguishing the main terms in a special divisor sum function of type

$$D_q\left(s, \lambda, \frac{h}{l}, r\right) = \sum_{(n,q)=1}^{\sigma_3(n)} \frac{n^s}{n^r} e\left( \frac{n h}{l} \right)$$

with $\sigma_3(n) = \sum_{d|n} d^3$, the author [30] succeed in deducing the asymptotic formula for general moduli. It is proved in [30] that, for any integer $q \not\equiv 2 \pmod{4}$,

\begin{equation}
\frac{1}{\varphi^*(q)} \sum_{\chi \mod q} \left| L\left( \frac{1}{2}, \chi \right) \right|^4 = \prod_{\rho|q} \left( 1 - p^{-1}\right)^3 \frac{1}{(1 + p^{-1})} P_4(\log q) + O\left( q^{-\frac{1}{2}+\frac{1}{40}+\varepsilon} \right).
\end{equation}

In (1.6), there is also a considerable improvement on the error term, as a special case, it sharpens the error term to $p^{-1/14}$ for prime moduli. This is due to an application of recent progress on bilinear forms in Kloosterman sums by Kerr, Shparlinski, Wu, and Xi [20].

Actually, the fourth moment of Dirichlet $L$-functions, including both $q$-aspect and $t$-aspect, was the first to draw attention to, which may go back to Montgomery [21], who proved that

$$\sum_{\chi \mod q} \left| L\left( \frac{1}{2} + it, \chi \right) \right|^4 dt \ll \varphi(q) T(\log q T)^4.$$

For easy of notation, we will apply

$$T_1 = T + 1$$

in place of $T$ in the error term, avoiding the case $T \to 0$. According to the conjecture in [7], we may predict that

**Conjecture 1.2.** For any positive integer $q \equiv 2 \pmod{4}$ and $T > 0$, there exist computable constants $c_0, c_1, c_2, c_3, c_4$ that

$$\frac{1}{\varphi^*(q)} \sum_{\chi \mod q} \int_{0}^{T} \left| L\left( \frac{1}{2} + it, \chi \right) \right|^4 dt = \prod_{\rho|q} \left( 1 - p^{-1}\right)^3 \frac{1}{(1 + p^{-1})}$$

$$\times \sum_{j=0}^{4} c_j \int_{0}^{T} \frac{1}{2} \sum_{a=0,1} \left( \log q + \frac{1}{2} \Gamma'\left( \frac{1}{2} \right) - \Gamma'\left( \frac{1}{2} \right) \right) dt + O\left( T_1^{\frac{1}{4}+\varepsilon} q^{-\frac{1}{4}+\varepsilon} \right).$$
Conjecture 1.2 looks a little different from Conjecture 1.1 since a polynomial seems gone. By Stirling’s approximation
\[ \frac{\Gamma'(1/2+it+\frac{a}{2})}{\Gamma(1/2+it+\frac{a}{2})} = \log \frac{T}{2} + O\left(\frac{1}{T}\right), \]
it is easy to see that the main terms of the conjecture would evolve to
\[ T \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} P_4(\log T q) \]
for large $T$. However, this should not be applied for small $T$ since the error in using Stirling’s approximation would be large.

The leading term of the asymptotic formula has already been obtained by Rane [23] for some of $q$ and large $T$, he proved that
\[ \frac{1}{\phi^*(q)} \sum_{\chi \mod q}^* \int_0^T |L\left(\frac{1}{2} + it, \chi\right)|^4 \, dt \]
\[ = \frac{T}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q T)^4 + O\left(2^{\omega(q)} T (\log q T)^3 (\log \log 3q)^5\right). \]

In 2010, Bui and Heath-Brown [5] proved the leading term for all $q \equiv 2 \pmod 4$ and $T \geq 2$ by sharpening the error term in (1.7). To be specific, they proved that, for $q \equiv 2 \pmod 4$ and $T \geq 2$,
\[ \frac{1}{\phi^*(q)} \sum_{\chi \mod q}^* \int_0^T |L\left(\frac{1}{2} + it, \chi\right)|^4 \, dt \]
\[ = \left(1 + O\left(\frac{\omega(q)}{\log q \sqrt{\phi(q)}}\right)\right) \frac{T}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})} (\log q T)^4 + O\left(\frac{q}{\phi^*(q)} T (\log q T)^2\right). \]

After applying the approximate functional equation (see also Lemma 2.1), the leading term comes from the diagonal terms. But to distinguish other main terms, one should deduce an asymptotic formula for the off-diagonal terms. By extending the method of Heath-Brown [12], Wang [29] tried to distinguish all main terms, proving that
\[ \frac{1}{\phi^*(q)} \sum_{\chi \mod q}^* \int_0^T |L\left(\frac{1}{2} + it, \chi\right)|^4 \, dt = T \sum_{j=0}^4 a_j (\log q T)^j + O\left(\frac{q}{\phi^*(q)} \min\{q^{1/2} T^{1+\epsilon}, q^{1/2+\epsilon}\} \right) \]
with
\[ a_4 = \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-p^{-1})^3}{(1+p^{-1})}, \quad a_j \ll q^\epsilon \quad \text{for} \quad j = 0, 1, 2, 3. \]

This asymptotic formula is non-trivial only for large $T \gg q^{1+\epsilon}$, and it is hardly to distinguish an explicit dependence of the coefficients $a_j$ on $q$, for $0 \leq j \leq 3$.

This work is devoted to deducing the asymptotic formula in Conjecture 1.2, with an error term owning a powering saving from $q$-aspect and $t$-aspect simultaneously, so that the asymptotic formula would hold uniformly in all $T$ and $q$. 
Theorem 1.3. We have that Conjecture 1.2 holds but with an error term $\mathcal{E}(T, q)$, bounded uniformly in $T$ and $q$ that

\begin{equation}
\mathcal{E}(T, q) \ll T^{1 - \frac{1}{92x - 96}\varepsilon} q^{-\frac{1}{92x - 96}\varepsilon}.
\end{equation}

Moreover, we have

\begin{equation}
\mathcal{E}(T, q) \ll T^{1 + \varepsilon} q^\Delta,
\end{equation}

where we may take $\Delta$ freely among

\begin{equation}
T^{\frac{11}{11^2}} q^{-\frac{1}{11^2} + \frac{\varepsilon}{4}} \text{ and } T^{-\frac{11}{11^2}}.
\end{equation}

The bound in (1.8) is just a special form to gain the same power saving from both $t$-aspect and $q$-aspect, and it is obvious a direct result of (1.9). For prime moduli, we can have a much better bound on the error term.

Theorem 1.4. For prime $p \geq 3$, we have that Conjecture 1.2 holds but with an error term $\mathcal{E}(T, p)$, bounded uniformly in $T$ and $p$ that

\begin{equation}
\mathcal{E}(T, p) \ll T^{1 - \frac{1}{11p} + \varepsilon} p^{-\frac{1}{11p} + \varepsilon}.
\end{equation}

Moreover, we have

\begin{equation}
\mathcal{E}(T, p) \ll T^{1 + \varepsilon} p^\Delta_1,
\end{equation}

where we may take $\Delta_1$ freely among

\begin{equation}
T^{\frac{11}{11^2}} p^{-\frac{1}{11^2}} \text{ and } \max \left\{ T^{\frac{1}{2}}, T^{-\frac{1}{11^2}} p^{-\frac{1}{11^2}} \right\}.
\end{equation}

When $T$ is small, considering the moment with a smooth function on $t$ could make the average essentially easy, as well as a considerable power saving from $q$-aspect in the error term. However, this hardly has any application here. Actually, the cost of last removing the smooth function will be too large to reserve any saving from $q$-aspect if $T$ is small with respect to $q$. A more feasible way is to extend the treatment at the central point by regarding $t$ as a parameter. Since the power saving from $q$-aspect is small at the central point, there is little room for expenditure in $q$-aspect when we treat the large $T$ case. That is to say, to cover all the range, it is important to gain power saving from $t$-aspect while costing nothing in $q$-aspect.

1.1. Sketch of the proof of Theorems 1.3 and 1.4. We split the averaging over $t$ into two parts, according to the size of $t$ that $t \leq q^{\varepsilon_0}$ and $t > q^{\varepsilon_0}$ for some small $\varepsilon_0 > 0$, and handle them with different methods.

For small $t$, the cost of applying a smooth function is large, and it is hardly to expect any remarkable power saving from the averaging over $t$. Thus, we treat the integrand directly, and pay our main attention to the saving from $q$-aspect. Thinking of $t$ as a parameter, we may extend the treatment at the central point in [30] to get the following result.
Theorem 1.5. For \( q \not\equiv 2 \pmod{4} \) and \( 0 \leq t \leq T \), we have

\[
\left(1.12\right) \frac{1}{\varphi(n)} \sum_{\chi \pmod{\varphi(n)}} \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 = \prod_{\text{prime } p} \frac{(1 - p^{-1})^3}{(1 + p^{-1})^3}
\]

\[
\times \frac{4}{d_0 q} \sum_{\varphi(d) \mu \left(\frac{q}{d}\right)} \sum_{\chi_0 \chi_1 \chi_2 \chi_3} \left( \log \frac{q}{\pi} + \frac{1}{2} \Gamma' \left(\frac{1}{2} \right) + \frac{1}{2} \Gamma' \left(\frac{1}{2} \right) \right)^j + O \left( T_1^{\frac{1}{7} + \varepsilon} q^{\frac{1}{7} + \frac{1}{7} + \varepsilon} \right),
\]

where \( \theta \) denotes the exponent towards the Ramanujan–Petersson conjecture.

The error term in (1.12) is non-trivial for \( T << q^{\frac{1}{7} + \frac{1}{7} \theta} \), but it is weaker than the main terms only for \( T << q^{\frac{1}{7} + \frac{1}{7} \theta} \). The best known value of \( \theta \) is \( 7/64 \), proved by Kim and Sarnak [18]. Thanks to Blomer, Fouvry, Kowalski, Michel and Miličević [3, 4], we may remove the dependence on the Ramanujan-Petersson conjecture and take \( \theta = 0 \) for prime moduli.

When \( t \) is large, we appeal to a weighted function \( \Phi(t) \) to force \( m \) and \( n \) to be close to each other. After some technical treatments, we may transform the problem essentially to a quadratic divisor problem

\[
\left(1.13\right) \frac{1}{\varphi(n)} \sum_{d \mid q} \varphi(d) \mu \left(\frac{q}{d}\right) \sum_{\chi \pmod{\varphi(n)}} F \left( \frac{h m_1 m_2 n_1 n_2}{H M_1 M_2 N_1 N_2} \right)
\]

for a compact support function \( F \). A divisor problem as in (1.13) but without the co-prime condition \( (m_1 m_2 n_1 n_2, q) = 1 \) has been well studied. By the delta method, Duke, Friedlander, and Iwaniec [8] provided an asymptotic formula for a remarkable range of \( h \). Bettin, Bui, Li, and Radziwiłł [1] introduced a different way to treat the divisor problem, which works specially for small \( h \) and provides a sharp error term. We extend the way to adapt the co-prime condition. We will obtain an asymptotic formula for (1.13) with a remarkable power saving from \( t \)-aspect but not costing \( q \)-aspect. The treatment would also borrow some technologies from Young [31] and Bettin, Chandee and Radziwiłł [2], etc.

Theorem 1.6. For \( q \not\equiv 2 \pmod{4} \) and \( T \gg q^\varepsilon \), we have

\[
\left(1.14\right) \frac{1}{\varphi(n)} \sum_{\chi \pmod{\varphi(n)}} \int_0^{2T} \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 \, dt
\]

\[
= \prod_{\text{prime } p} \frac{(1 - p^{-1})^3}{(1 + p^{-1})^3} \sum_{j=0}^4 c_j \int_0^{2T} \left( \log \frac{tq}{2\pi} \right)^j \, dt + O \left( \max \left\{ T^{\frac{15}{16}} (q/q_0)^{\frac{1}{16}}, T^{\frac{1}{4}} \right\} T^e q^\varepsilon \right),
\]

where \( q_0 = \max \{ d : d \mid q^\varepsilon, d < q^\varepsilon \} \) with \( q^\varepsilon = \prod_{\text{prime } p} p \).

The error term in (1.14) provides a power saving not less than \( T^{-\frac{1}{16} + \varepsilon} q^\varepsilon \) for all moduli, in particular, a power saving \( T^{-\frac{1}{16} + \varepsilon} q^{-\frac{1}{16} + \varepsilon} \) for prime moduli with large \( T \).

Theorems 1.3 and 1.4 are direct results of Theorems 1.5 and 1.6, and Theorems 1.5 and 1.6 are based on two shifted moments in next section.
1.2. **Two shifted moments.** Let $\Phi(t)$ be a smooth, nonnegative function with support contained in $[T/2, 4T]$, satisfying $\Phi(t) \ll T^{-j}$ for all $j = 0, 1, 2, \ldots$, where $T^{\frac{1}{4}+\varepsilon} \ll T_0 \ll T$. We have chosen to compute two shifted fourth moments of Dirichlet $L$-functions, which include the parameters $\alpha, \beta, \gamma, \delta$ or a weighted function $\Phi(t)$, and doing so allows for a clearer structure of the main terms. The first one is given by

$$M(\alpha, \beta, \gamma, \delta, t) = \frac{1}{\varphi^*(q)} \sum_\chi \frac{L\left(\frac{1}{2} + it + \alpha, \chi\right)L\left(\frac{1}{2} + it + \beta, \chi\right)L\left(\frac{1}{2} - it + \gamma, \chi\right)L\left(\frac{1}{2} - it + \delta, \chi\right)}{L\left(\frac{1}{2}, \chi\right)},$$

This moment does not contain the averaging over $t$, and we consider its asymptotic formula when $t$ is small. The second shifted moment is defined via

$$M(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{\varphi^*(q)} \sum_\chi \int_{\mathbb{R}} \Phi(t) \frac{L\left(\frac{1}{2} + it + \alpha, \chi\right)L\left(\frac{1}{2} + it + \beta, \chi\right)L\left(\frac{1}{2} - it + \gamma, \chi\right)L\left(\frac{1}{2} - it + \delta, \chi\right)}{L\left(\frac{1}{2}, \chi\right)} dt.$$

To present asymptotic formulae for these two shifted moments, we should introduce some notations for convenience. Let

$$Z_q(\alpha, \beta, \gamma, \delta) = \frac{\zeta_q(1 + \alpha + \gamma)\zeta_q(1 + \alpha + \delta)\zeta_q(1 + \beta + \gamma)\zeta_q(1 + \beta + \delta)}{\zeta_q(2 + \alpha + \beta + \gamma + \delta)},$$

$$X_{\alpha, \gamma}(q, t, a) = \left(\frac{q}{\pi}\right)^{-\alpha-\gamma} \frac{\Gamma\left(\frac{1}{2} - \alpha - it + \frac{a}{2}\right)\Gamma\left(\frac{1}{2} - \gamma + it + \frac{a}{2}\right)}{\Gamma\left(\frac{1}{2} + \alpha - it + \frac{a}{2}\right)\Gamma\left(\frac{1}{2} + \gamma + it + \frac{a}{2}\right)},$$

and

$$X_{\alpha, \beta, \gamma, \delta}(q, t, a) = X_{\alpha, \gamma}(q, t, a)X_{\beta, \delta}(q, t, a)$$

with $a = 0, 1$. Obviously, $Z_q(\alpha, \beta, \gamma, \delta)$ and $X_{\alpha, \beta, \gamma, \delta}(q, t, a)$ are symmetric with respect to the parameters $\alpha, \beta$ and also symmetric with respect to $\gamma, \delta$.

**Theorem 1.7.** For $q \equiv 2 \pmod{4}$, $t \asymp T \geq 0$, and $\alpha, \beta, \gamma, \delta \ll (\log T/\log q)^{-1}$, we have

$$M(\alpha, \beta, \gamma, \delta, t) = Z_q(\alpha, \beta, \gamma, \delta) + Z_q(-\gamma, -\delta, -\alpha, -\beta) \left(\frac{1}{2} \sum_{a=0,1} X_{\alpha, \beta, \gamma, \delta}(q, t, a)\right)$$

$$+ Z_q(\beta, -\gamma, -\delta, -\alpha) \left(\frac{1}{2} \sum_{a=0,1} X_{\alpha, \gamma}(q, t, a)\right) + Z_q(\alpha, -\gamma, -\delta, -\beta) \left(\frac{1}{2} \sum_{a=0,1} X_{\beta, \gamma}(q, t, a)\right)$$

$$+ Z_q(\beta, -\delta, -\gamma, -\alpha) \left(\frac{1}{2} \sum_{a=0,1} X_{\alpha, \delta}(q, t, a)\right) + Z_q(\alpha, -\delta, -\gamma, -\beta) \left(\frac{1}{2} \sum_{a=0,1} X_{\beta, \delta}(q, t, a)\right)$$

$$+ O\left(T_{\frac{1}{4}+\varepsilon} q^{-\frac{1}{2}} T^{\frac{1}{2}+\varepsilon}\right).$$
This theorem can be seen as an extension of Theorem 1.3 in [30]. By regarding $t$ as a parameter, we may deduce Theorem 1.7 following the treatment of [30, Theorem 1.3] step by step. Differences will come from the ratio of gamma factors, as well as the factor $\left(\frac{w}{n}\right)^{-it}$ in the asymptotic functional equation (see Lemma 2.1). These differences will not bring any essential changes in calculating the main terms, and will contribute at most a factor $T_1^{1+\epsilon}$ to the error term, which we will specify in Section 5.

**Theorem 1.8.** For $q \not\equiv 2 \pmod{4}$, $T \gg q^\epsilon$, and $\alpha, \beta, \gamma, \delta \ll (\log T_1 q)^{-1}$, we have

\[
M(\alpha, \beta, \gamma, \delta, \Phi) = Z_q(\alpha, \beta, \gamma, \delta) \int_R \Phi(t)dt + Z_q(-\gamma, -\delta, -\alpha, -\beta) \int_R \Phi(t) \left(\frac{tq}{2\pi}\right)^{-\alpha-\beta-\gamma-\delta} dt \\
+ Z_q(\beta, -\gamma, -\delta, -\alpha) \int_R \Phi(t) \left(\frac{tq}{2\pi}\right)^{-\alpha-\gamma} dt + Z_q(\alpha, -\gamma, -\delta, -\beta) \int_R \Phi(t) \left(\frac{tq}{2\pi}\right)^{-\beta-\gamma} dt \\
+ Z_q(\beta, -\delta, -\gamma, -\alpha) \int_R \Phi(t) \left(\frac{tq}{2\pi}\right)^{-\alpha-\delta} dt + Z_q(\alpha, -\delta, -\gamma, -\beta) \int_R \Phi(t) \left(\frac{tq}{2\pi}\right)^{-\beta-\delta} dt \\
+ O\left(T^{\frac{1}{4}+\epsilon}(q/q_0)^{-\frac{1}{4}+\epsilon}(T/T_0)^3 + T^\epsilon q^\epsilon(T/T_0)\right),
\]

where $q_0 = \max\{d : d \mid q^\ast, d < q^\ast \}$ with $q^\ast = \prod_{p \mid q} p$.

1.3. Proof of Theorem 1.5 and Theorem 1.6 from the shifted moments. For main terms of the asymptotic formulae in Theorems 1.7 and 1.8, the symmetry implies that all poles cancel out to form the holomorphy with respect to the shift parameters, which has been proved in a more general setting in Lemma 2.5.5 of [7]. Thus, taking the limit as all shifts go to zero in Theorem 1.7 gives Theorem 1.5.

The proof of Theorem 1.6 needs some narrative, but it is standard. Actually, we would obtain Theorem 1.6 by taking appropriate weighted functions $\Phi(t)$ in Theorem 1.8. Let $0 \leq \Phi_1(t) \leq 1$ be a weighted function supported on $[T, 2T]$, which is identical to unity when $T + T_0^{1+\epsilon} \leq t \leq 2T - T_0^{1+\epsilon}$; let $0 \leq \Phi_2(t) \leq 1$ be supported on $[T - T_0^{1+\epsilon}, 2T + T_0^{1+\epsilon}]$, which is identical to unity when $T \leq t \leq 2T$. It is obvious that

\[
M(0, 0, 0, 0, \Phi_1) \leq \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \sum_{n} \int_T^{2T} \left|L\left(\frac{1}{2} + it, \chi\right)\right|^4 dt \leq M(0, 0, 0, 0, \Phi_2).
\]

On the other hand, taking the limit as all shifts go to zero in Theorem 1.8 shows that, for $i = 1, 2$,

\[
M(0, 0, 0, 0, \Phi_i) = \prod_{p \nmid q} \frac{(1 - p^{-1})^3}{(1 + p^{-1})} \sum_{j=0}^4 c_j \int_T^{2T} \left(\log \frac{tq}{2\pi}\right)^j dt \\
+ O\left(T^{\frac{1}{4}+\epsilon}(q/q_0)^{-\frac{1}{4}+\epsilon}(T/T_0)^3 + T_0^{1+\epsilon} q^\epsilon\right).
\]

Inserting this into (1.18) and taking $T_0 = \max\{T^{\frac{1}{4}+\epsilon}(q/q_0)^{-\frac{1}{4}+\epsilon}, T^{\frac{1}{2}}\}$, we would establish Theorem 1.6.

The remainder is devoted to proving Theorems 1.7 and 1.8, where we may impose some restrictions on the shifts. More precisely, we assume that each of the shifts lies in a fixed annulus with inner and outer radii $\asymp (\log T_1 q)^{-1}$, which are separated enough so that $|\alpha \pm \beta| \gg (\log T_1 q)^{-1}$, etc. We initially prove the theorems with these restrictions in place.
Since every term in the asymptotic formulae are holomorphic, the maximum modulus principle extends our results to all shifts \( \ll (\log T_1 q)^{-1} \).

**Notation.** We use the common convention that \( \varepsilon \) denotes an arbitrarily small positive constant which may vary from line to line, and that notations \((a, b), [a, b]\) are the gcd and lcm of \( a \) and \( b \) respectively. The notation \( \sigma_{\alpha, \beta}(n) \) is defined via

\[
\sigma_{\alpha, \beta} = \sum_{d_1 d_2 = n} d_1^\alpha d_2^\beta.
\]

### 2. Background and auxiliary lemmas

#### 2.1. Dirichlet L-functions.

Let \( q \) be a positive integer and \( \chi \) be a primitive character modulo \( q \). The Dirichlet L-function \( L(s, \chi) \) is defined as

\[
L(s, \chi) = \sum_n \chi(n) n^{-s}
\]

for \( \text{Re}(s) > 1 \). Let

\[
a = \begin{cases} 
0, & \text{for } \chi(-1) = 1, \\
1, & \text{for } \chi(-1) = -1,
\end{cases}
\]

and let

\[
\Lambda(s, \chi) = \left(\frac{q}{\pi}\right)^{\frac{s}{2}} \Gamma\left(\frac{s + a}{2}\right) L(s, \chi).
\]

After extended to the whole plane, the Dirichlet L-function satisfies the following functional equation

\[
(2.1) \quad \Lambda(s, \chi) = i^{-a} q^{-\frac{s}{2}} \tau(\chi) \Lambda(1 - s, \chi^{-1}) \quad \text{with} \quad \tau(\chi) = \sum_{n (\text{mod } q)} \chi(n) e\left(\frac{n}{q}\right).
\]

#### 2.2. Approximate functional equation.

**Lemma 2.1** (Approximate functional equation). Let \( G(s) \) be an even entire function of exponential decay in any strip \( |\text{Re}(s)| < C \), satisfying \( G(0) = 1 \). For \( x > 0 \) and \( a = 0, 1 \), we define

\[
(2.2) \quad V_{\alpha, \beta, \gamma, \delta}(x, t, a) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} g_{\alpha, \beta, \gamma, \delta}(s, t, a) x^{-s} ds,
\]

where

\[
(2.3) \quad g_{\alpha, \beta, \gamma, \delta}(s, t, a) = \pi^{-2s} \frac{\Gamma\left(\frac{s + \alpha + \gamma + it + a}{2}\right) \Gamma\left(\frac{s + \beta + \gamma + it + a}{2}\right) \Gamma\left(\frac{s + \gamma + \delta - it + a}{2}\right) \Gamma\left(\frac{s + \delta + \gamma - it + a}{2}\right)}{\Gamma\left(\frac{s + \alpha + \gamma + it + a}{2}\right) \Gamma\left(\frac{s + \beta + \gamma + it + a}{2}\right) \Gamma\left(\frac{s + \gamma + \delta - it + a}{2}\right) \Gamma\left(\frac{s + \delta + \gamma - it + a}{2}\right)}.
\]

Furthermore, let

\[
(2.4) \quad \widetilde{V}_{\alpha, \beta, \gamma, \delta}(x, t, a) = X_{-\gamma, -\delta, -\alpha, -\beta}(q, t, a) V_{\alpha, \beta, \gamma, \delta}(x, t, a)
\]
with \(X_{-\gamma,-\delta,-\alpha,-\beta}(q, t, a)\) being defined as in (1.17). Then, for \(\chi(-1) = (-1)^a\), we have

\[
L\left(\frac{1}{2} + it + \alpha, \chi\right)L\left(\frac{1}{2} + it + \beta, \chi\right)L\left(\frac{1}{2} - it + \gamma, \overline{\chi}\right)L\left(\frac{1}{2} - it + \delta, \overline{\chi}\right)
\]

\[
= \sum_{m, n} \sigma_{\alpha, \beta}(m)\sigma_{\gamma, \delta}(n)X(m, n) \left(\frac{m}{n}\right)^{-it} V_{\alpha, \beta, \gamma, \delta}\left(\frac{mn}{q^2}, t, a\right)
\]

\[
+ \sum_{m, n} \sigma_{-\gamma, -\delta}(m)\sigma_{-\alpha, -\beta}(n)X(m, n) \left(\frac{m}{n}\right)^{-it} \overline{V}_{-\gamma, -\delta, -\alpha, -\beta}\left(\frac{mn}{q^2}, t, a\right).
\]

This approximate functional equation can be deduced standardly from the function equation of \(L(s, \chi)\); see also [15] and [32, Proposition 2.4]. The approximate functional equation holds for a general \(G\), and we will appeal to a special one.

**Definition 2.1** (Definition of \(G(s)\)). Let \(G(s) = P_{\alpha, \beta, \gamma, \delta}(s)\exp(s^2)\), where \(P_{\alpha, \beta, \gamma, \delta}(s)\) is an even polynomial in \(s\) satisfying the following common properties: it takes the value 1 at \(s = 0\); it is rational in the shifts \(\alpha, \beta, \gamma, \delta\); it is symmetric in the shifts; it is invariant under \(\alpha \to -\alpha, \beta \to -\beta\), etc.; it also takes zero at \(s = -\frac{\alpha + \gamma}{2}\) (as well as other points by symmetry).

### 2.3. Results due to Stirling’s approximation

We present some results about the ratios of gamma functions arising in the approximate functional equation.

**Lemma 2.2.** For \(t\) large, we have

\[
X_{\alpha, \beta, \gamma, \delta}(q, t, a) = \left(\frac{tq}{2\pi}\right)^{-\alpha - \beta - \gamma - \delta} \left(1 + O(t^{-1})\right),
\]

and for \(j \geq 0\),

\[
\frac{\partial^j}{\partial t^j} X_{\alpha, \beta, \gamma, \delta}(q, t, a) \ll t^{-j}.
\]

**Lemma 2.3.** For \(t\) large and \(s\) in any fixed vertical strip, we have

\[
g_{\alpha, \beta, \gamma, \delta}(s, t, a) = \left(\frac{t}{2\pi}\right)^{2s} \left(1 + O\left(t^{-1}(1 + |s|^2)\right)\right).
\]

Moreover, we have

\[
t^j \frac{\partial^j}{\partial t^j} V_{\alpha, \beta, \gamma, \delta}(x, t, a) \ll A, \left(1 + |x|/t\right)^{-A}
\]

for any fixed \(A > 0\) and \(j \geq 0\).

These two lemmas are well-known results, deduced from Stirling’s approximation standardly. To eliminate the difference between even and odd characters in the sum of \(M(\alpha, \beta, \gamma, \delta, \Phi)\), we appeal to the following lemma.

**Lemma 2.4.** For \(t\) large, we have

\[
V_{\alpha, \beta, \gamma, \delta}(x, t, 0) - V_{\alpha, \beta, \gamma, \delta}(x, t, 1) \ll t^{-1+\epsilon},
\]

\[
\overline{V}_{\alpha, \beta, \gamma, \delta}(x, t, 0) - \overline{V}_{\alpha, \beta, \gamma, \delta}(x, t, 1) \ll t^{-1+\epsilon}.
\]
Proof. Due to (2.9), we assume $x \ll t^{2+\varepsilon}$ in (2.10) and (2.11) since the estimates are
obvious otherwise. Recalling the definition of $V$ in (2.2), we rewrite that

$$V_{\alpha,\beta,\gamma,\delta}(x, t, 0) - V_{\alpha,\beta,\gamma,\delta}(x, t, 1) = \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} \left( g_{\alpha,\beta,\gamma,\delta}(s, t, 0) - g_{\alpha,\beta,\gamma,\delta}(s, t, 1) \right) x^{-s} ds.$$  

We move the integral to Re$(s) = -\varepsilon$ without encountering any poles, observing that
$g_{\alpha,\beta,\gamma,\delta}(s, t, 0) - g_{\alpha,\beta,\gamma,\delta}(s, t, 1)$ takes zeros at $s = 0$. On the new path, we can easily see
(2.10) from a result of (2.8) that

$$g_{\alpha,\beta,\gamma,\delta}(s, t, 0) - g_{\alpha,\beta,\gamma,\delta}(s, t, 1) \ll t^{-1}(1 + |s|^2).$$

By the definition of $\tilde{V}$ in (2.4), the estimate (2.11) is an immediate result of (2.6), (2.9)
and (2.10). This establishes the lemma. □

2.4. The number of primitive characters. Let $\varphi^*(q)$ denote the number of primitive characters modulo $q$. It is known that $\varphi^*(q)$ is a multiplicative function defined by

$$\varphi(p^m) = \begin{cases} p^{m-2}(p - 1)^2, & \text{for } m \geq 2, \\ p - 2, & \text{for } m = 1. \end{cases}$$

2.5. The orthogonality formula.

Lemma 2.5 (The orthogonality formula). For $(mn, q) = 1$, we have

$$(2.12) \quad \sum_{\chi \pmod{q}} \chi(n) = \sum_{d|q, m-n} \varphi(d) \mu(q/d).$$

Moreover

$$(2.13) \quad \sum_{\chi \pmod{q}} \chi(n) = \frac{1}{2} \sum_{d|q, m-n} \varphi(d) \mu(q/d) + \frac{(-1)^a}{2} \sum_{d|q, m+n} \varphi(d) \mu(q/d).$$

This orthogonality formula is well-known, and its proof may be refereed to [13] and [25].

2.6. Two partitions of unity. We appeal to a partition introduced in [1]. Let $f$ be a smooth function that

$$f(x) + f(1/x) = 1$$

for all $x \in \mathbb{R}$ and $f(x) \ll_j (1 + x)^{-j}$ for any fixed $j > 0$ and $x > 1$. Also it has the Mellin inversion

$$f(x) = \frac{1}{2\pi i} \int_{(\varepsilon)} \hat{f}(u)x^{-u} du,$$

where $\hat{f}(u)$ has a simple pole at $u = 0$ with residue 1, and satisfies

$$\hat{f}\left( \pm \frac{(\alpha - \beta)}{2} \right) = \hat{f}\left( \pm \frac{(\gamma - \delta)}{2} \right) = 0.$$
One may apply the identity
\begin{equation}
(2.14) \quad f\left(\frac{m_1}{m_2}\right) f\left(\frac{n_1}{n_2}\right) + f\left(\frac{m_2}{m_1}\right) f\left(\frac{n_1}{n_2}\right) + f\left(\frac{m_1}{m_2}\right) f\left(\frac{n_2}{n_1}\right) + f\left(\frac{m_2}{m_1}\right) f\left(\frac{n_2}{n_1}\right) = 1
\end{equation}
to partition unity into four roughly similar terms. Then in each term, there exists a comparison on the sizes of \(m_1, m_2\) and \(n_1, n_2\).

The second one is the dyadic partition. Let \(W(x)\) be a smooth non-negative function compactly supported on \([1, 2]\) such that
\[
\sum_M W\left(\frac{x}{M}\right) = 1,
\]
where \(M\) varies over a set of positive real numbers with \#\{\(M : X^{-1} \leq M \leq X\} \ll (\log X)\). The \(W\) function has the Mellin pair
\[
\left\{
\begin{aligned}
\hat{W}(u) &= \int_0^\infty W(x)x^{u-1}dx, \\
W(x) &= \frac{1}{2\pi i} \int_{(c_0)} \hat{W}(u)x^{-u}du.
\end{aligned}
\right.
\]

3. Initial treatment of the shifted moment

From this section, we start our proof of Theorem 1.8, which occupies next two sections. We assume \(T \gg q^\varepsilon\), a convention that holds throughout the proof of Theorem 1.8.

3.1. Initial treatment. Using the approximate functional equation stated in Lemma 2.1, we break \(M(\alpha, \beta, \gamma, \delta, \Phi)\) into two terms that
\begin{equation}
(3.1) \quad M(\alpha, \beta, \gamma, \delta, \Phi) = A_1(\alpha, \beta, \gamma, \delta, \Phi) + A_{-1}(\gamma, \beta, \delta, \alpha, \beta, \Phi),
\end{equation}
where \(A_1\) is the contribution from the ‘first part’ of the approximate functional equation that
\[
A_1 = \frac{1}{\varphi(q) \chi (\mod q)} \sum_{m,n}^\star \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)\chi(m)\overline{\chi}(n)}{(mn)^{\frac{1}{2}}} \int_R \left(\frac{m}{n}\right)^{-it} V_{\alpha,\beta,\gamma,\delta}\left(\frac{mn}{q^2}, t, a\right) \Phi(t)dt,
\]
and \(A_{-1}\) is the ‘second part’ that
\[
A_{-1} = \frac{1}{\varphi(q) \chi (\mod q)} \sum_{m,n}^\star \frac{\sigma_{-\gamma,-\beta}(m)\sigma_{-\alpha,-\beta}(n)\chi(m)\overline{\chi}(n)}{(mn)^{\frac{1}{2}}} \int_R \left(\frac{m}{n}\right)^{-it} V_{-\gamma,-\beta,-\alpha,-\beta}\left(\frac{mn}{q^2}, t, a\right) \Phi(t)dt
\]
Our major focus is on the evaluation of \(A_1\), and the treatment of \(A_{-1}\) is identical.

Before applying the orthogonality formula of primitive characters, we first remove the dependence of \(V_{\alpha,\beta,\gamma,\delta}(x, t, a)\) on the parity of \(\chi\) by rewriting it into two parts as
\begin{equation}
(3.2) \quad V_{\alpha,\beta,\gamma,\delta}(x, t, a) = \frac{1}{2} \left( V_{\alpha,\beta,\gamma,\delta}(x, t, 0) + V_{\alpha,\beta,\gamma,\delta}(x, t, 1) \right)
\end{equation}
\[
+ \frac{\chi(-1)}{2} \left( V_{\alpha,\beta,\gamma,\delta}(x, t, 0) - V_{\alpha,\beta,\gamma,\delta}(x, t, 1) \right).
\]
Note that the second part would just contribute an error to $A_1$. To be specific, we insert (3.2) into $A_1$, and then the contribution of the second part is

$$
\frac{1}{2\varphi^*(q)} \sum_{\chi \mod q} \sum_{m,n} \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)\chi(-m)\overline{\chi}(n)}{(mn)^\frac{1}{2}} \times \int_R \left( \frac{m}{n} \right)^{-it} \left( V_{\alpha,\beta,\gamma,\delta} \left( \frac{mn}{q^2}, t, 0 \right) - V_{\alpha,\beta,\gamma,\delta} \left( \frac{mn}{q^2}, t, 1 \right) \right) \Phi(t) dt.
$$

The averaging over $t$-aspect forces $m$ and $n$ to be very close to each other. More precisely, integration by parts shows

$$
\int_R \left( \frac{m}{n} \right)^{-it} \left( V_{\alpha,\beta,\gamma,\delta} \left( \frac{mn}{q^2}, t, 0 \right) - V_{\alpha,\beta,\gamma,\delta} \left( \frac{mn}{q^2}, t, 1 \right) \right) \Phi(t) dt \ll_j \frac{T}{T_0 \log \frac{m}{n}}
$$

for any $j \geq 1$, which yields that the integral over $t$ is very small unless $\left| 1 - \frac{m}{n} \right| \ll T_0^{-1+\varepsilon}$. After applying the orthogonality formula (2.12) and the estimate (2.10), we find that (3.3) is bounded by

$$
\ll \frac{T^\varepsilon}{\varphi^*(q)} \sum_{d|q(m+n)} \varphi(d) \sum_{\substack{mn \leq (Tq)^{1+\varepsilon} \\left| 1 - \frac{m}{n} \right| \ll T_0^{1+\varepsilon}}} \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^\frac{1}{2}} + O(T^{-2020} q^{-2020})
$$

$$
\ll \frac{T^\varepsilon q^\varepsilon}{\varphi^*(q)} \sum_{d|q} \varphi(d) \frac{Tq}{T_0^{1-\varepsilon} \varphi(d)} \ll T^\varepsilon q^\varepsilon (T/T_0),
$$

which is the second error term of the asymptotic formula in Theorem 1.8.

Note that the first part of (3.2) has nothing to do with the parity of $\chi$, and we can just average all primitive characters in $A_1$ to evaluate its contribution. After applying the orthogonality formula to this part, we find that

$$
A_1(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{2} \sum_{\alpha=0,1} \frac{1}{\varphi^*(q)} \sum_{d|q} \varphi(d) \mu \left( \frac{q}{d} \right) \sum_{\substack{mn \equiv 1 \mod d \\text{and } n \equiv 0 \mod m}} \frac{\sigma_{\alpha,\beta}(m)\sigma_{\gamma,\delta}(n)}{(mn)^\frac{1}{2}} \times \int_R \left( \frac{m}{n} \right)^{-it} V_{\alpha,\beta,\gamma,\delta} \left( \frac{mn}{q^2}, t, a \right) \Phi(t) dt + O(T^\varepsilon q^\varepsilon (T/T_0)).
$$

It is easy to see that a similar expression holds for $A_{-1}$.

We break the sum in (3.4) into diagonal terms and off-diagonal terms, that is

$$
A_1(\alpha, \beta, \gamma, \delta, \Phi) = A_D(\alpha, \beta, \gamma, \delta, \Phi) + A_O(\alpha, \beta, \gamma, \delta, \Phi) + O(T^\varepsilon q^\varepsilon (T/T_0)),
$$

where

$$
A_D(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{2} \sum_{\alpha=0,1} \frac{1}{\varphi^*(q)} \sum_{d|q} \varphi(d) \mu \left( \frac{q}{d} \right) \sum_{(n,q)=1} \frac{\sigma_{\alpha,\beta}(n)\sigma_{\gamma,\delta}(n)}{n} \times \int_R V_{\alpha,\beta,\gamma,\delta} \left( \frac{n}{q^2}, t, a \right) \Phi(t) dt,
$$

$$
A_O(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{2} \sum_{\alpha=0,1} \frac{1}{\varphi^*(q)} \sum_{d|q} \varphi(d) \mu \left( \frac{q}{d} \right) \sum_{(n,q)=1} \frac{\sigma_{\alpha,\beta}(n)\sigma_{\gamma,\delta}(n)}{n} \times \int_R V_{\alpha,\beta,\gamma,\delta} \left( \frac{n}{q^2}, t, a \right) \Phi(t) dt.
$$
and

\[ A_0(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{2} \sum_{a=0,1} \frac{1}{\varphi^6(q)} \sum_{d \mid q} \varphi(d) \frac{\varphi(q)}{d} \sum_{\pm} \sum_{m-n=\pm h \neq 0} \frac{\sigma_{\alpha, \beta}(m)\sigma_{\gamma, \delta}(n)}{(mn)^2} \times \int_R \left( 1 \pm \frac{h}{n} \right)^{-it} V_{\alpha, \beta, \gamma, \delta} \left( \frac{mn}{q^2}, t, a \right) \Phi(t) dt. \]

Here the sum \( \sum_{m-n=\pm h \neq 0} \) is over positive integers \( m, n, \) and \( h. \) Also, we have

\[ A_{-1}(\gamma, -\delta, -\alpha, -\beta, \Phi) = A_{-D}(\gamma, -\delta, -\alpha, -\beta, \Phi) \]

\[ + A_{-O}(\gamma, -\delta, -\alpha, -\beta, \Phi) + O(T^\epsilon q^2(T/T_0)) \]

with similar expressions for \( A_{-D}(\gamma, -\delta, -\alpha, -\beta, \Phi) \) and \( A_{-O}(\gamma, -\delta, -\alpha, -\beta, \Phi). \)

3.2. **The diagonal terms.** For the diagonal terms, we insert the definition of \( V \) into (3.6) to see

\[ A_D(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{2} \sum_{a=0,1} \int_R \Phi(t) \frac{1}{2\pi i} \int_{(1)} \frac{G(s)}{s} q^{-s} g_{\alpha, \beta, \gamma, \delta}(s, t, a) \sum_{(n,q)=1} \frac{\sigma_{\alpha, \beta}(n)\sigma_{\gamma, \delta}(n)}{n^{1+2s}} ds dt. \]

Then, by the Ramanujan identity, the sum over \( n \) is

\[ \frac{\zeta_s(1 + \alpha + \gamma + 2s)\zeta_s(1 + \alpha + \delta + 2s)\zeta_s(1 + \beta + \gamma + 2s)\zeta_s(1 + \beta + \delta + 2s)}{\zeta_s(2 + \alpha + \beta + \gamma + \delta + 4s)}, \]

which has simple poles at \( 2s = -\alpha - \gamma, \) etc, while \( G(s) \) vanishes at these poles. Therefore, we can move the integral to \( \text{Re}(s) = -\frac{1}{2} + \epsilon, \) passing a pole at \( s = 0. \) By the estimate of \( g \) in (2.8), the integral on the new path is

\[ \ll q^{-\frac{1}{2}+\epsilon} \int_R \Gamma^{-\frac{1}{2}+\epsilon} \Phi(t) dt \ll T^{\frac{1}{2}+\epsilon} q^{-\frac{1}{2}+\epsilon}, \]

and residue at \( s = 0 \) is

\[ Z_q(\alpha, \beta, \gamma, \delta) \int_R \Phi(t) dt. \]

We summarize this calculation in the following:

**Lemma 3.1.** We have

\[ A_D(\alpha, \beta, \gamma, \delta, \Phi) = Z_q(\alpha, \beta, \gamma, \delta) \int_R \Phi(t) dt + O\left(T^{\frac{1}{2}+\epsilon} q^{-\frac{1}{2}+\epsilon}\right), \]

and similarly the contribution of the diagonal terms to \( A_{-1} \) is

\[ A_{-D}(\gamma, -\delta, -\alpha, -\beta, \Phi) = \frac{1}{2} Z_q(\gamma, -\delta, -\alpha, -\beta) \sum_{a=1} \int_R X_{a,\beta, \gamma, \delta}(q, t, a) \Phi(t) dt + O\left(T^{\frac{1}{2}+\epsilon} q^{-\frac{1}{2}+\epsilon}\right). \]
4. Off-diagonal terms and the proof of Theorem 1.8

4.1. A divisor problem. Our treatment of off-diagonal terms requires an estimate on a quadratic divisor problem.

**Lemma 4.1.** Let \( F(x_1, x_2, x_3, x_4, x_5) \) be a smooth function supported on \([1, 2]^5\) such that
\[
\frac{\partial F^{(j_1+j_2)}}{\partial x_i^{j_1} \partial x_i^{j_2}} \ll_{j_1,j_2} (T/T_0)^{j_1+j_2} q^e
\]
for any \( j_1, j_2 \geq 0 \) and \( i_1, i_2 = 1, 2, 3, 4, 5 \). With \( M_1, M_2, N_1, N_2, H \geq 1 \), we define
\[
S_d^\pm = \sum_{d|q, \frac{d}{m_1m_2-n_1n_2=\pm h \geq 1}} F\left( \frac{h}{H}, \frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{n_1}{N_1}, \frac{n_2}{N_2} \right),
\]
where the sum runs over positive integers \( m_1, m_2, n_1, n_2 \) and \( h \). Suppose that \( M_1 \leq M_2 T^e q^e \), \( N_1 \leq N_2 T^e q^e \) and \( H = o \left( (M_1M_2N_1N_2)^{\frac{1}{2}} \right) \). We have
\[
S_d^\pm = \sum_{d_1, d_2 \mid q} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} \sum_{m_1, n_1, d_1, d_2 \mid (m_1n_1, q)=1} \frac{k^2 h \Delta}{m_1n_1} \int_0^\infty F\left( \frac{kh\Delta}{H}, \frac{m_1}{M_1}, \frac{kh\Delta(x \pm 1)}{M_1m_2}, \frac{n_1}{N_1}, \frac{kh\Delta x}{n_1n_2} \right) dx + \mathcal{E},
\]
where \( k = (m_1, n_1) \), \( \Delta = [d, (d_1, d_2)] \), and
\[
\mathcal{E} \ll \frac{H}{d} N_1^{\frac{1}{2}} q_0^\frac{1}{2} (M_1 + N_1)(T/T_0)^2 T^e q^e.
\]
Here \( q_0 \) is defined as in Theorem 1.6 that \( q_0 = \max\{d : d \mid q^*, d < q^* \} \) with \( q^* = \prod_{p \mid q} p \).

**Remark.** In (4.1), the integral over \( x \) in \( S_d^\pm \) is actually on \( x > 1 \) since \( F \) is supported on \([1, 2]^5\). By making the change of variables \( x \to x + 1 \), we can obtain another form for \( S_d^\pm \) that
\[
S_d^- = \sum_{d_1, d_2 \mid q} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} \sum_{m_1, n_1, d \mid (m_1n_1, q)=1} \frac{k^2 h \Delta}{m_1n_1} \int_0^\infty F\left( \frac{kh\Delta}{H}, \frac{m_1}{M_1}, \frac{kh\Delta x}{m_1m_2}, \frac{n_1}{N_1}, \frac{kh\Delta(x + 1)}{n_1n_2} \right) dx + \mathcal{E}.
\]

This expression looks symmetrical with respect to the expression of \( S_d^\pm \).

**Proof.** The condition \( H = o \left( (M_1M_2N_1N_2)^{\frac{1}{2}} \right) \) implies that \( M_1M_2 \leq N_1N_2 \). We first consider the sum over the large variables \( m_2, n_2 \), where we rewrite the coprime condition \( (m_2n_2, q) = 1 \) in terms of the M"obius function that
\[
\sum_{m_2n_2 \mid (m_1m_2-n_1n_2=\pm h)} F\left( \frac{h}{H}, \frac{m_1}{M_1}, \frac{m_2}{M_2}, \frac{n_1}{N_1}, \frac{n_2}{N_2} \right) = \sum_{d_1, d_2 \mid q} \mu(d_1)\mu(d_2)
\times \sum_{d_1m_2-d_2n_2=\pm h} F\left( \frac{h}{H}, \frac{d_1m_2}{M_1}, \frac{d_2n_2}{N_1}, \frac{n_2}{N_2} \right).
\]
Let $d_{12} = (d_1, d_2)$. Applying (4.3) with variable changes $d_1 \to d_1d_{12}$, $d_2 \to d_2d_{12}$, we rewrite

$$S^\pm_d = \sum_{d_1d_2|d_{12}} \sum_{(d_1,d_2)=1} \mu(d_1d_{12})\mu(d_2d_{12})S^\pm_d(d_1, d_2, d_{12}),$$

where

$$S^\pm_d(d_1, d_2, d_{12}) = \sum_{d_1m_1m_2 - d_2n_1n_2 = \pm h/d_{12} (m_1, n_1, q) = 1, d|d_{12}} F\left(\frac{h}{H}, \frac{m_1}{M_1}, \frac{d_1d_2m_2}{M_2}, \frac{n_1}{N_1}, \frac{d_2d_{12}n_2}{N_2}\right).$$

By the definition of $q_0$, there must be $\min|d_1, d_2| \leq q_0$. Without loss of generality, we would focus ourselves on the evaluation of $S^\pm_d(d_1, d_2, d_{12})$ with $d_2 \leq q_0$, and the treatment of the other case is identical.

Since $(d_1, d_2) = 1$ and $(m_1n_1, q) = 1$, we have $(d_1m_1, d_2n_1) = (m_1, n_1) = k$. Now there is no restriction on the sum over $m_2$ and $n_2$ except for the identity $d_1m_2 - d_2n_2 = \pm h/d_{12}$, which we can rewrite as $m_2 = (\pm h/kd_{12})d_1m_1/k$ (mod $d_2n_1/k$) to eliminate the variable $n_2$. This yields that the sum over $m_2, n_2$ in $S^\pm_d(d_1, d_2, d_{12})$ is equal to

$$\sum_{m_2 = (\pm h/kd_{12})d_1m_1/k \text{ (mod } d_2n_1/k)} F\left(\frac{h}{H}, \frac{m_1}{M_1}, \frac{d_1d_2m_2}{M_2}, \frac{n_1}{N_1}, \frac{d_1d_2m_1m_2 \mp h}{n_1n_2}\right).$$

Observing that $d_{12}k | h$, we make the variable change $h \to d_{12}kh$, and then the condition $d | h$ in $S^\pm_d(d_1, d_2, d_{12})$ evolves into $d | d_{12}h$ as $(d, k) = 1$. We apply Possion’s summation formula to the sum over $m_2$ to see

$$(4.4) \quad S^\pm_d(d_1, d_2, d_{12}) = \sum_{d|d_{12}} \sum_{k \in \mathbb{Z}} e\left(\mp \frac{h}{d_{12}d_2n_1/k} \frac{d_1m_1}{d_2n_1/k}\right) \mathcal{F}_\pm(k, d_1d_2, h, m_1, n_1, l),$$

where

$$\mathcal{F}_\pm = \frac{k}{d_2n_1} \int_0^\infty \frac{d_{12}kh}{H} \cdot \frac{m_1}{M_1} \cdot \frac{d_1d_2d_1lx}{M_2} \cdot \frac{n_1}{N_1} \cdot \frac{d_1d_2m_1lx}{n_1n_2} \cdot e\left(\frac{kx}{d_2n_1}\right) dx$$

$$= \int_0^\infty \frac{d_{12}kh}{H} \cdot \frac{m_1}{M_1} \cdot \frac{d_1d_2d_1lx}{kM_2} \cdot \frac{n_1}{N_1} \cdot \frac{d_1d_2m_1lx}{kN_2} \cdot e(\frac{lx}{n_1n_2}) dx.$$

Since $F$ is supported on $[1, 2]^5$, the integral is actually on the range

$$x \leq \frac{kM_2}{d_1d_2d_1N_1} \leq \frac{kN_2}{d_1d_2d_1M_1}.$$

The contribution of the term $l = 0$ is

$$S^\pm_d(d_1, d_2, d_{12}) = \sum_{d|d_{12}} \int_0^\infty \frac{d_{12}kh}{H} \cdot \frac{m_1}{M_1} \cdot \frac{d_1d_2d_1lx}{kM_2} \cdot \frac{n_1}{N_1} \cdot \frac{d_1d_2m_1lx}{kN_2} \cdot e\left(\frac{d_{12}kh}{n_1n_2}\right) dx.$$
After a variable change $h \rightarrow h[d, d_{12}]/d_{12} = h\Delta/d_{12}$, this evolves into

$$
\sum_{m_1, n_1, h} \int_0^\infty F \left( \frac{kh\Delta}{H}, \frac{m_1}{M_1}, \frac{d_1 d_2 d_{12} n_1 x}{M_2}, \frac{n_1}{N_1}, \frac{d_1 d_2 d_{12} m_1 x}{N_2} \right) dx, + \frac{kh\Delta}{n_1 N_2}
$$

which would contribute to the main term.

For the terms $l \neq 0$, integrating by parts $j$ times shows

$$
\mathcal{F}_l(k, d_1, d_{12}, h, m_1, n_1, l) \ll T^\epsilon q^e \frac{1}{l^j} \left( \frac{d_1 d_2 d_{12} n_1}{kM_2} + \frac{d_1 d_2 d_{12} m_1}{kn_2} \right)^j (T/T_0)^j \frac{kM_2}{d_1 d_2 d_{12} n_1}
$$

for any fixed $j \geq 0$. This indicates that we can restrict the sum in (4.4) to $0 \leq |l| \leq L$ with

$$
L = \frac{d_1 d_2 d_{12} N_1}{kM_2} (T/T_0) T^\epsilon q^e.
$$

Thus, (4.4) evolves into

$$
S_4^+(d_1, d_2, d_{12}) = S_4^+(d_1, d_2, d_{12}) + \mathcal{E}'
$$

with

$$
(4.5) \quad \mathcal{E}' = \sum_{k \leq H/d} \sum_{d_1 d_{12} \mid \mu d_2 d_{12} \mid \mu_{d_1 d_2}} \mu(d_1 d_{12}) \mu(d_2 d_{12}) \int_0^\infty \sum_{m_1, n_1, h} \sum_{d_1 d_{12}} F \left( \frac{d_1 d_{12} k}{H}, \frac{m_1}{M_1}, \frac{d_1 d_{12} n_1 x}{M_2}, \frac{n_1}{N_1}, \frac{d_1 d_{12} m_1 x}{N_2} \right) \epsilon \left( \mp \frac{d_1 m_1}{d_2 n_1} \right) e(\pm l \overline{d_1 m_1}) e(lx) dx,
$$

where we have made variable changes $m_1 \rightarrow m_1 k$ and $n_1 \rightarrow n_1 k$. Note that $\frac{\partial F}{\partial m_1} \ll \frac{k}{M_1}(T/T_0) T^\epsilon q^e \ll m_1^{-1} (T/T_0) T^\epsilon q^e$ for $x = \frac{kM_2}{d_1 d_2 d_{12} N_1}$. After a summation by parts with the Weil bound for Kloosterman sums, we have

$$
\sum_{m_1} F \left( \frac{d_1 d_{12} k}{H}, \frac{m_1}{M_1}, \frac{d_1 d_{12} n_1 x}{M_2}, \frac{n_1}{N_1}, \frac{d_1 d_{12} m_1 x}{N_2} \right) e \left( \pm \frac{d_1 m_1}{d_2 n_1} \right) \ll (l, n_1 d_2) n_1^{\frac{1}{2}} d_2^{\frac{1}{2}} (1 + \frac{M_1}{N_1}) (T/T_0) T^\epsilon q^e.
$$

With this in (4.5), a direct calculation shows that

$$
\mathcal{E}' \ll T^\epsilon q^e \sum_{k \leq H/d} \sum_{d_1 d_{12} \mid \mu d_2 d_{12} \mid \mu_{d_1 d_2}} \sum_{h \leq H/d_{12}} \sum_{d_2 \mid d_{12} h} \sum_{0 < |l| \leq L} \sum_{d_1 d_{12}} (l, n_1 d_2) n_1^{\frac{1}{2}} d_2^{\frac{1}{2}} \left( 1 + \frac{M_1}{N_1} \right) (T/T_0) \frac{kM_2}{d_1 d_2 d_{12} N_1}
$$

$$
\ll \frac{H}{d} N_1^{\frac{1}{2}} q_0^e (M_1 + N_1) (T/T_0)^2 T^\epsilon q^e
$$

for $d_2 \leq q_0$. This gives the error term of (4.2).
When $d_1 \leq q_0$, an identical treatment shows that

$$S_d^+(d_1, d_2, d_{12}) = \sum_{m_1, n_1, h} \int_0^\infty F \left( \frac{kh\Delta}{H}, m_1, d_1 d_2 d_{12} n_1 x, \frac{n_1}{N_1}, \frac{d_1 d_2 d_{12} m_1 x}{k M_2} \pm \frac{kh\Delta}{n_1 N_2} \right) dx + \mathcal{E}.$$

The only difference is to eliminate the variable $m_2$ first, and then we should apply Possion’s summation formula to the sum over $n_2$ instead.

In conclusion, we sum $S_d^+(d_1, d_2, d_{12})$ over all possible values of $d_1, d_2,$ and $d_{12}$ to get

$$S_d^\pm = \sum_{d_1 d_2 \| d_{12} \|} \mu(d_1 d_{12}) \mu(d_2 d_{12})$$

$$\times \sum_{m_1, n_1, h} \int_0^\infty F \left( \frac{kh\Delta}{H}, m_1, d_1 d_2 d_{12} n_1 x, \frac{n_1}{N_1}, \frac{d_1 d_2 d_{12} m_1 x}{k M_2} \pm \frac{kh\Delta}{n_1 N_2} \right) dx + \mathcal{E}.$$

Making a variable change $x \to \frac{k^2 \Delta}{d_1 d_2 d_{12} m_1} (x \pm 1)$ in the integral, we have

$$S_d^\pm = \sum_{d_1 d_2 \| d_{12} \|} \mu(d_1 d_{12}) \mu(d_2 d_{12}) \sum_{m_1, n_1, h} \frac{k^2 \Delta}{m_1 n_1}$$

$$\times \int_0^\infty F \left( \frac{kh\Delta}{H}, m_1, \frac{kh\Delta(x)}{m_1 M_2}, \frac{n_1}{N_1}, \frac{kh\Delta x}{n_1 N_2} \right) dx + \mathcal{E}.$$

This would establish the lemma if we rewrite $d_1 d_{12}$ as $d_1$ and $d_2 d_{12}$ as $d_2$ in the sum. □

4.2. Evaluation of $A_O$ and $A_{-O}$. In this section, we produce asymptotic formulae for $A_O$ and $A_{-O}$. Before doing this, we present here a lemma required in following calculation.

**Lemma 4.2.** For any $s$, we have

$$\sum_{dq} \varphi(d) \mu \left( \frac{d}{d_1 d_2} \right) \sum_{d_1, d_2 \| q} \frac{\mu(d_1) \mu(d_2)}{[d_1, d_2]^\Delta} = \varphi^*(q) q^{-s} \prod_{p \mid q} \left( 1 - \frac{1}{p^{1-s}} \right),$$

where $\Delta = [d, (d_1, d_2)]$.

**Proof.** Since both sides of (4.6) are multiplicative functions on $q$, we just check the identity for prime power. If $q = p$ is a prime, the left-hand side of (4.6) is

$$(p - 1) \left( \frac{1}{p^s} - \frac{1}{p^{1+s}} \right) - \left( 1 - \frac{2}{p} + \frac{1}{p^{1+s}} \right) = (p - 2)p^{-s} \left( 1 - \frac{1}{p^{1-s}} \right),$$

and the identity holds obviously. If $q = p^m$ with $m \geq 2$, the left-hand side of (4.6) is

$$\varphi(p^m) \left( \frac{1}{p^{ms}} - \frac{1}{p^{1+ms}} \right) - \varphi(p^{m-1}) \left( \frac{1}{p^{ms}} - \frac{1}{p^{1+(m-1)s}} \right) = \varphi(p^{m-1})p^{-ms}(p - 1) \left( 1 - \frac{1}{p^{1-s}} \right),$$

which is equal to the right-hand side too. Combining these two cases would establish the lemma. □

We specify our asymptotic formulae for $A_O$ and $A_{-O}$ in the following:
Lemma 4.3. Let $A_O$ and $A_{-O}$ be defined as before. We have

\begin{equation}
A_O(\alpha, \beta, \gamma, \delta, \Phi) = M_{\alpha,\beta,\gamma,\delta}(\Phi) + M_{\beta,\alpha,\gamma,\delta}(\Phi) + M_{\alpha,\beta,\gamma,\delta}(\Phi) + M_{\beta,\alpha,\gamma,\delta}(\Phi) + O\left(T^{3+\epsilon}(q/q_0^2)^{-1/3+\epsilon}(T/T_0)^3 + T^\epsilon q^\epsilon\right),
\end{equation}

where

\begin{align*}
M_{\alpha,\beta,\gamma,\delta}(\Phi) &= \frac{\zeta_q(1 + \alpha - \beta)}{\zeta_q(2 + \alpha - \beta + \gamma - \delta)} \int_R \Phi(t) \left(\frac{tq}{2\pi}\right)^{-\beta - \delta} dt \\
&\quad \times \frac{1}{2\pi i} \int_{(\epsilon)} \frac{G(s)}{s} \zeta_q(1 - \beta - \delta + 2s) \zeta_q(1 + \alpha + \gamma + 2s) ds.
\end{align*}

Also,

\begin{equation}
A_{-O}(-\gamma, -\delta, -\alpha, -\beta, \Phi) = \tilde{M}_{-\gamma, -\delta, -\alpha, -\beta}(\Phi) + \tilde{M}_{-\delta, -\gamma, -\alpha, -\beta}(\Phi) + \tilde{M}_{-\beta, -\gamma, -\alpha, -\beta}(\Phi) + O\left(T^{3+\epsilon}(q/q_0^2)^{-1/3+\epsilon}(T/T_0)^3 + T^\epsilon q^\epsilon\right),
\end{equation}

where, for example,

\begin{align*}
\tilde{M}_{-\gamma, -\delta, -\alpha, -\beta}(\Phi) &= \frac{\zeta_q(1 + \alpha - \beta)}{\zeta_q(2 + \alpha - \beta + \gamma - \delta)} \int_R \Phi(t) \left(\frac{tq}{2\pi}\right)^{-\beta - \delta} dt \\
&\quad \times \frac{1}{2\pi i} \int_{(\epsilon)} \frac{G(s)}{s} \zeta_q(1 - \beta - \delta + 2s) \zeta_q(1 + \alpha + \gamma - 2s) ds.
\end{align*}

Proof. We focus ourself on the evaluation of $A_O$, and the treatment of $A_{-O}$ is identical. Recall the expression of $A_O$ in (3.7), and we rewrite it as

\begin{equation}
A_O(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{2} \sum_{a=0,1} \varphi^*(q) \sum_{d|q} \varphi(d) \mu\left(\frac{q}{d}\right) \sum_{m_1, m_2, n_1, n_2} \frac{1}{n_1^{\frac{1}{4}+\epsilon}} m_2^{\frac{1}{2}+\beta} n_1^{\frac{1}{4}+\gamma} n_2^{\frac{1}{4}+\delta} \\
\times \int_R \left(1 \pm \frac{h}{n_1 n_2}\right)^{-it} V_{\alpha, \beta, \gamma, \delta}\left(\frac{m_1 m_2 n_1 n_2}{q^2}, t, a\right) \Phi(t) dt.
\end{equation}

The estimate (2.9) yields that $V(x, t, a)$ decays rapidly in $x$ when $x > r^2$, that is to say, the sum over all $m_1 m_2 n_1 n_2 \gg (Tq)^{2+\epsilon}$ gives a negligible contribution $\ll T^{-2020} q^{-2020}$. Also, by integration by parts, there is

\begin{equation}
\int_R \left(1 \pm \frac{h}{n_1 n_2}\right)^{-it} V_{\alpha, \beta, \gamma, \delta}\left(\frac{m_1 m_2 n_1 n_2}{q^2}, t, a\right) \Phi(t) dt \ll \frac{T}{(hT_0/n_1 n_2)^j}
\end{equation}

for any fixed $j \geq 0$, which yields that the contribution of all the terms with $|h| \gg \sqrt{m_1 m_2 n_1 n_2 T_0^{-1} T^\epsilon q^\epsilon}$ is $O\left(T^{-2020} q^{-2020}\right)$. Hence, we have

\begin{equation}
A_O(\alpha, \beta, \gamma, \delta, \Phi) = \frac{1}{2} \sum_{a=0,1} \varphi^*(q) \sum_{d|q} \varphi(d) \mu\left(\frac{q}{d}\right) \sum_{m_1, m_2, n_1, n_2} \frac{1}{n_1^{\frac{1}{4}+\epsilon}} m_2^{\frac{1}{2}+\beta} n_1^{\frac{1}{4}+\gamma} n_2^{\frac{1}{4}+\delta} \\
\times \int_R \left(1 \pm \frac{h}{n_1 n_2}\right)^{-it} V_{\alpha, \beta, \gamma, \delta}\left(\frac{m_1 m_2 n_1 n_2}{q^2}, t, a\right) \Phi(t) dt + O\left(T^{-2020} q^{-2020}\right).
\end{equation}
Applying the first partition of unity (2.14), we rewrite $A_O$ as

$$(4.10) \quad A_O(\alpha, \beta, \gamma, \delta, \Phi) = A_{O,1} + A_{O,2} + A_{O,3} + A_{O,4} + O(T^{-2020} q^{-2020})$$

with obvious meanings. We will focus on $A_{O,1}$, contributed by $f \left( \frac{m_1}{m_2} \right) f \left( \frac{n_1}{n_2} \right)$, and the treatments for other three terms are identical. Recall that the factor $f \left( \frac{m_1}{m_2} \right) f \left( \frac{n_1}{n_2} \right)$ means that the sum in $A_{O,1}$ is actually over positive integers with $m_1 \leq m_2$ and $n_1 \leq n_2$.

We apply the dyadic partition of unity to the sums over $m_1, m_2, n_1, n_2,$ and $h$, and it follows that

$$A_{O,1} = \frac{1}{2} \sum_{a=0,1} \frac{1}{\varphi(q)} \sum_{d|q} \varphi(d) \mu \left( \frac{q}{d} \right) \times \sum_{q \leq (Tq)^{2e}} \left( S_{d,a}^+(M_1, M_2, N_1, N_2, H) + S_{d,a}^-(M_1, M_2, N_1, N_2, H) \right),$$

where

$$S_{d,a}^\pm(M_1, M_2, N_1, N_2, H) = \int_{\mathbb{R}} \sum_{m_1, m_2, n_1, n_2 = 1}^{M_1, M_2, N_1, N_2} \frac{1}{m_1^{\frac{1}{2}+\alpha} m_2^{\frac{1}{2}+\beta} n_1^{\frac{1}{2}+\gamma} n_2^{\frac{1}{2}+\delta}} V_{\alpha, \beta, \gamma, \delta} \left( \frac{m_1 m_2 n_1 n_2}{q^2}, t, a \right) \times (1 \pm \frac{h}{n_1 n_2})^{-it} f \left( \frac{m_1}{m_2} \right) f \left( \frac{n_1}{n_2} \right) W \left( \frac{H}{M_1} \right) W \left( \frac{M_1}{M_2} \right) W \left( \frac{n_1}{N_1} \right) W \left( \frac{n_2}{N_2} \right) \Phi(t) dt.$$

To estimate $S_{d,a}^\pm(M_1, M_2, N_1, N_2, H)$, we apply Lemma 4.1 with

$$F = \frac{1}{x_{1,3} x_{4,5}^{\frac{1}{2}+\alpha} x_{3,4}^{\frac{1}{2}+\beta} x_{4,5}^{\frac{1}{2}+\gamma} x_{5,1}^{\frac{1}{2}+\delta}} V_{\alpha, \beta, \gamma, \delta} \left( x_{2,3} x_{4,5} M_{1,2} N_{1,2} \right) \left( 1 \pm \frac{x_{1}}{x_{4,5} N_{1} N_{2}} \right)^{-it} \times f \left( \frac{x_2 M_1}{x_3 M_2} \right) f \left( \frac{x_4 N_1}{x_5 N_2} \right) W \left( x_1 \right) W \left( x_2 \right) W \left( x_3 \right) W \left( x_4 \right) W \left( x_5 \right).$$

It is easy to check the condition of Lemma 4.1 here. Then, it follows that

$$S_{d,a}^\pm(M_1, M_2, N_1, N_2, H) = M_{0,1}^\pm(d, a) + \mathcal{E}_0,$$

where

$$\mathcal{E}_0 \ll \frac{TE}{M_1^{\frac{1}{4}+\alpha} M_2^{\frac{1}{4}+\beta} N_1^{\frac{1}{4}+\gamma} N_2^{\frac{1}{4}+\delta}}.$$
with $E$ given by (4.2), and where

$$\mathcal{M}_0^+(d, a) = \sum_{d_1, d_2, q} \frac{\mu(d_1) \mu(d_2)}{[d_1, d_2]} \sum_{m_1, n_1, h \atop (m_1, n_1, d_2, q) = 1} \frac{k(h \Delta)^{-\beta - \delta}}{m_1^{1+\alpha - \beta} n_1^{1+\gamma - \delta}} \int_{R} \int_{0}^{\infty} (x \pm 1)^{-\frac{1}{T} - \frac{1}{2} - it} \, dx \, dt$$

$$\times V_{\alpha, \beta, \gamma, \delta} \left( \frac{k^2 h^2 \Delta^2 x(x \pm 1)}{q^2}, t, a \right) \left( \frac{1}{x} \right)^{-it} f \left( \frac{m_1^2}{k h \Delta(x \pm 1)} \right) f \left( \frac{n_1^2}{k h \Delta} \right) \Phi(t) dx dt$$

with $k = (m_1, n_1)$ and $\Delta = [d, (d_1, d_2)]$.

We come to the error term $E_0$ first, whose contribution to $A_{0,1}$ is bounded by

$$\ll \frac{T^{1+e}}{\varphi'(q)} \sum_{d_1, d_2, q} \varphi(d) (M_1 M_2 N_1 N_2)^{-\frac{1}{2}} \left( \frac{H}{d} N_1^{\frac{1}{2}} q_0^2 (M_1 + N_1) (T/T_0)^2 \right).$$

As $H \ll \sqrt{m_1 m_2 n_1 n_2 T_0^{-1} T^e q^e}$ and $M_1, N_1 \ll (M_1 M_2 N_1 N_2)^{\frac{1}{2}} T^e q^e$, it is bounded by

$$\ll \frac{1}{\varphi'(q)} (M_1 M_2 N_1 N_2)^{\frac{1}{2}} q_0^{\frac{1}{2}} (T/T_0)^3 T^e q^e \ll T^{\frac{1}{2}+e} (q/q_0^2)^{\frac{1}{2}+e} (T/T_0)^3.$$

In the summation of $\mathcal{M}_0^+(d, a)$ over $M_1, M_2, N_1, N_2$ and $H$, we may remove the conditions $M_1 \leq M_2 T^e q^e$, $N_1 \leq N_2 T^e q^e$, $M_1 M_2 N_1 N_2 \leq (T q)^{2+e}$, $H \ll \sqrt{m_1 m_2 n_1 n_2 T_0^{-1} T^e q^e}$ with a negligible error, by applying estimates of $f$ and $V$ and integration by parts on $t$ as before. After extending the summation over all $M_1, M_2, N_1, N_2$ and $H$, we remove the dyadic partition of unity to find

$$\mathcal{M}_1(d, a) = \sum_{M_1, M_2, N_1, N_2, H} \mathcal{M}_0^+(d, a)$$

$$= \sum_{d_1, d_2, q} \frac{\mu(d_1) \mu(d_2)}{[d_1, d_2]} \sum_{m_1, n_1, h \atop (m_1, n_1, d_2, q) = 1} \frac{k(h \Delta)^{-\beta - \delta}}{m_1^{1+\alpha - \beta} n_1^{1+\gamma - \delta}} \int_{R} \int_{0}^{\infty} (x \pm 1)^{-\frac{1}{T} - \frac{1}{2} - it} \, dx \, dt$$

$$\times V_{\alpha, \beta, \gamma, \delta} \left( \frac{k^2 h^2 \Delta^2 x(x \pm 1)}{q^2}, t, a \right) f \left( \frac{m_1^2}{k h \Delta(x \pm 1)} \right) f \left( \frac{n_1^2}{k h \Delta} \right) \Phi(t) dx dt.$$
Recalling the definition of $V$ and expressing $f$ in terms of its Mellin transform, we have

\begin{equation}
\mathcal{M}_1(d, a) = \mathcal{M}_1^+(d, a) + \mathcal{M}_1^-(d, a)
\end{equation}

\begin{align*}
\mathcal{M}_1(d, a) &= \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} \frac{1}{(2\pi i)^3} \int_{(c)} \int_{(c)} \int_{(1)} \int_{\mathbb{R}} \frac{G(s)}{s} g_{\alpha, \beta, \gamma, \delta}(s, t, a) \widehat{f}(u) \widehat{f}(v) q^{2s} \\
&\times \left\{ \sum_{\substack{m_1, n_1, h \in \mathbb{Z}^2 \setminus \{0\} \setminus \{d\} \setminus \{0\} \setminus \{d\}} \frac{k(h\Delta)^{-\beta-\delta-2s+u+v}}{m_1^{1+\alpha-\beta+2u} n_1^{1+\gamma-\delta+2v}} (J_+(s, u, v) + J_-(s, u, v)) \right\} \Phi(t) dt ds du dv
\end{align*}

with

\begin{align}
J_+(s, u, v) &= \int_0^{\infty} (x + 1)^{-\frac{1}{2} - \beta - s + u - it} x^{-\frac{1}{2} - \delta - s + v + it} dx, \\
J_-(s, u, v) &= \int_0^{\infty} (x + 1)^{-\frac{1}{2} - \delta - s + v + it} x^{-\frac{1}{2} - \beta - s + u - it} dx.
\end{align}

By formula (3.194.3) of [10] and the relationship between beta functions gamma functions, we have

\begin{align*}
J_+(s, u, v) &= \frac{\Gamma\left(\frac{1}{2} - \delta - s + u - it, \beta + \delta + 2s - u - v\right)}{\Gamma\left(\frac{1}{2} + \beta + s - u + it\right)} \\
&= \frac{\Gamma\left(\frac{1}{2} - \delta - s + v + it, \beta + \delta + 2s - u - v\right)}{\Gamma\left(\frac{1}{2} + \beta + s - u + it\right)}
\end{align*}

and

\begin{align*}
J_-(s, u, v) &= \frac{\Gamma\left(\frac{1}{2} - \beta - s + u - it, \beta + \delta + 2s - u - v\right)}{\Gamma\left(\frac{1}{2} + \delta + s - v - it\right)}
\end{align*}

By Stirling’s approximation,

\begin{align*}
\frac{\Gamma\left(\frac{1}{2} - \delta - s + v + it\right)}{\Gamma\left(\frac{1}{2} + \beta + s - u + it\right)} &= t^{\beta-\delta-2s+u+v} \exp\left(\frac{\pi i}{2} (-\beta - \delta - 2s + u + v)\right) \\
&\times \left(1 + O\left(\frac{1 + |s|^2 + |u|^2 + |v|^2}{t}\right)\right),
\end{align*}

and

\begin{align*}
\frac{\Gamma\left(\frac{1}{2} - \beta - s + v - it\right)}{\Gamma\left(\frac{1}{2} + \delta + s - u - it\right)} &= t^{\beta-\delta-2s+u+v} \exp\left(-\frac{\pi i}{2} (-\beta - \delta - 2s + u + v)\right) \\
&\times \left(1 + O\left(\frac{1 + |s|^2 + |u|^2 + |v|^2}{t}\right)\right).
\end{align*}
Thus, we have

\[ J_+(s, u, v) + J_-(s, u, v) = 2 \cos \left( \frac{\pi}{2} (\beta + \delta + 2s - u - v) \right) i^{\beta - \delta - 2s + u + v} \]

\[ \times \Gamma(\beta + \delta + 2s - u - v) \left( 1 + O \left( \frac{1 + |s|^2 + |u|^2 + |v|^2}{t} \right) \right), \]

where the contribution of the error \( O \left( \frac{1 + |s|^2 + |u|^2 + |v|^2}{t} \right) \) is less than the main term divided by \( T \), due to the rapid decay of \( G \) and \( \tilde{f} \) in \( s, u, \) and \( v \). Solely for notational convenience, we define

\[ z_1 = \beta + \delta + 2s - u - v, \quad z_2 = \alpha - \beta + 2u \quad \text{and} \quad z_3 = \gamma - \delta + 2v. \]

Then, the main term of the sum in the brace of (4.11) is equal to

\[ (t\Delta)^{-z_1} \Gamma(z_1) 2 \cos \left( \frac{\pi z_1}{2} \right) \sum_{h} \frac{1}{h^{z_1}} \sum_{m,n_1} \frac{k^{1-z_1}}{m_1^{1+z_2} n_1^{1+z_3}}, \]

Recalling that \( \Delta = [d, (d_1, d_2)] \) and \( k = (m_1, n_1) \), we may express the last sum over \( m_1, n_1 \) as an Euler product

\[ \prod_{p | q} \left( \sum_{j=0}^{\infty} \frac{p^{j(1-z_1)}}{p^{j(2+z_2+z_3)}} \right) \sum_{m,n \geq 0} \frac{1}{p^{m(1+z_2)+n(1+z_3)}} \]

\[ = \prod_{p | q} \left( 1 - \frac{1}{p^{1+z_1+z_2+z_3}} \right)^{-1} \left( \sum_{m,n \geq 0} \frac{1}{p^{m(1+z_2)+n(1+z_3)}} - \sum_{m,n \geq 1} \frac{1}{p^{m(1+z_2)+n(1+z_3)}} \right) \]

\[ = \prod_{p | q} \left( 1 - \frac{1}{p^{1+z_1+z_2+z_3}} \right)^{-1} \left( 1 - \frac{1}{p^{1+z_2}} \right)^{-1} \left( 1 - \frac{1}{p^{1+z_3}} \right)^{-1} \left( 1 - \frac{1}{p^{2+z_2+z_3}} \right), \]

which yields

\[ \sum_{m,n_1 \geq 0} \frac{k^{1-z_1}}{m_1^{1+z_2} n_1^{1+z_3}} = \frac{\zeta_q(1 + z_1 + z_2 + z_3) \zeta_q(1 + z_2) \zeta_q(1 + z_3)}{\zeta_q(2 + z_2 + z_3)}. \]

Moreover, the functional equation of the Riemann zeta-function indicates that

\[ \Gamma(z_1) 2 \cos \left( \frac{\pi z_1}{2} \right) \sum_{h} \frac{1}{h^{z_1}} = (2\pi)^{z_1} \zeta(1 - z_1). \]
Thus, we conclude that

\[(4.14)\]

\[
M_1(d, a) = \frac{1}{(2\pi i)^3} \sum_{d_1, d_2 \parallel q} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} \int(e) \int(e) \int(1) \int R \frac{G(s)}{s} g_{\alpha, \beta, \gamma, \delta}(s, t, a) \tilde{f}(u) \tilde{f}(v) \\
\times \frac{\zeta(1 - \beta - \delta - 2s + u + v) \zeta_q(1 + \alpha + \gamma + 2s + u + v) \zeta_q(1 + \alpha - \beta + 2u) \zeta_q(1 + \gamma - \delta + 2v)}{\zeta_q(2 + \alpha - \beta + \gamma - \delta + 2u + 2v)} \\
\times q^{2s} \left( \frac{2\pi}{t[d, (d_1, d_2)]} \right)^{\beta + \delta + 2s - u - v} \Phi(t)dt ds dv \left( 1 + O \left( \frac{1}{T} \right) \right).
\]

Now we come to deduce \(A_{\alpha, 1}\) from \(M_1(d, a)\). We shift the integration in \((4.14)\) over \(u\) and \(v\) towards \(\text{Re}(u) = -1/4 + \varepsilon/2\) and \(\text{Re}(v) = -1/4 + \varepsilon/2\). We collect poles from \(u = 0\) and \(v = 0\), and for the terms where only one of the two residues is taken we move the other integral to the \((-1/2 + \varepsilon)\)-line. We do not cross poles at \(u = -\alpha/\beta/2\) and \(v = -(\gamma - \delta)/2\) since we ensured that \(\tilde{f}(-\alpha/\beta/2) = \tilde{f}(-(\gamma - \delta)/2) = 0\). For the integral along the new lines and the residues at only one of \(u = 0\) and \(v = 0\), we move the line of integration over \(s\) to \(\frac{1}{2}\), and then a direct calculation with the estimate of \(g_{\alpha, \beta, \gamma, \delta}(s, t, a)\) in \((2.8)\) shows that all these are bounded by

\[(4.15)\]

\[
\ll T \sum_{d_1, d_2 \parallel q} \frac{1}{[d_1, d_2]} \left( \frac{q}{[d, (d_1, d_2)]} \right)^{\frac{1}{2}} (T[d, (d_1, d_2)])^{-\frac{1}{2} + \varepsilon}.
\]

By summing over \(d\), we find that its contribution to \(A_{\alpha, 1}\) is bounded by

\[
\ll T \int \frac{1}{\varphi'(q)} \sum_{d \parallel q} \varphi(d) \sum_{d_1, d_2 \parallel q} \frac{1}{[d_1, d_2]} \left( \frac{q}{[d, (d_1, d_2)]} \right)^{\frac{1}{2}} (T[d, (d_1, d_2)])^{-\frac{1}{2} + \varepsilon}
\]

\[
\ll T^{\frac{1}{2} + \varepsilon} q^{-\frac{1}{2} + \varepsilon},
\]

which is an acceptable error in the lemma.

For the residue at both \(u = 0\) and \(v = 0\), we move the line of the integral over \(s\) to \(\text{Re}(s) = \varepsilon\). After eliminating \(g_{\alpha, \beta, \gamma, \delta}(s, t, a)\) by \((2.8)\), we observe that it is equal to

\[
\frac{1}{2\pi i} \sum_{d_1, d_2 \parallel q} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} \int R(t) \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{G(s)}{s} t q^{2s} \left( \frac{2\pi}{t[d, (d_1, d_2)]} \right)^{\beta + \delta + 2s} \zeta_q(1 - \beta - \delta - 2s) \zeta_q(1 + \alpha + \gamma + 2s) \zeta_q(1 + \alpha - \beta) \zeta_q(1 + \gamma - \delta) ds dt,
\]

adding an error

\[
\ll \sum_{d_1, d_2 \parallel q} \frac{1}{[d_1, d_2]} \left( \frac{T q}{[d, (d_1, d_2)]} \right)^{2\varepsilon}
\]

whose contribution to \(A_{\alpha, 1}\) is bounded by \(\ll T^\varepsilon q^{2\varepsilon}\) and is acceptable in the lemma. An arrangement provides that the main contribution of the residue at both \(u = 0\) and \(v = 0\) to
\[ A_{O,1} \]
\[
\zeta_q(1 + \alpha - \beta)\zeta_q(1 + \gamma - \delta) \frac{1}{2\pi i} \int_R \Phi(t) \left( \frac{t}{2\pi} \right)^{-\delta} \int_{(\epsilon)} G(s) s^{-\beta+\delta} ds dt,
\]
where
\[
\mathcal{M}_{\alpha,\beta,\gamma,\delta}(s) = \zeta_q(1 - \beta - \delta - 2s)\zeta_q(1 + \alpha + \gamma + 2s)
\times \frac{q^{2s}}{\varphi^*(q)} \sum_{d|q} \varphi(d) \mu\left( \frac{q}{d} \right) \sum_{d_1,d_2|d} \frac{\mu(d_1)\mu(d_2)}{[d, (d_1, d_2)]^{\beta+\delta+2s}}.
\]

We execute the sum in (4.17) by applying Lemma 4.2. It then follows that
\[ \mathcal{M}_{\alpha,\beta,\gamma,\delta}(s) = \zeta_q(1 - \beta - \delta - 2s)\zeta_q(1 + \alpha + \gamma + 2s)q^{-\beta-\delta}. \]

Inserting this into (4.16) provides the main term of \( A_{O,1} \). In conclusion, we have
\[ A_{O,1} = \mathcal{M}_{\alpha,\beta,\gamma,\delta}(\Phi) + O\left( T^{\frac{1}{2}+e}(q/q_0)^{\frac{1}{2}+e}(T/T_0)^3 + T^e q^e \right). \]

There are similar expressions for \( A_{O,2}, A_{O,3}, \) and \( A_{O,4} \), and applying these into (4.10) gives (4.7) immediately. On the other hand, the proof of formula (4.8) would be identical after applying Stirling’s approximation (2.6) to \( X_{\alpha,\beta,\gamma,\delta}(q, t, a) \) at the beginning. \( \Box \)

4.3. **Assembling the main terms and proving the theorem.** In this section, we prove Theorem 1.8 by combining all main terms from off-diagonal terms and diagonal terms. We first deduce the main term for the off-diagonal terms from the asymptotic formulæ of \( A_O \) and \( A_{-O} \) stated in Lemma 4.3. Making the change of variables \( s \to -s \) in \( \overline{M}_{-\beta-\gamma-a}(\Phi) \) and then combining it with \( \mathcal{M}_{\beta,\gamma,\delta}(\Phi) \), we have
\[ \mathcal{M}_{\alpha,\beta,\gamma,\delta}(\Phi) + \overline{M}_{-\beta-\gamma-a}(\Phi) = Z_q(\alpha, -\delta, \gamma, -\beta, q) \int_R \Phi(t) \left( \frac{tq}{2\pi} \right)^{-\beta-\delta} dt + O(T^e q^e) \]
by the residue theorem, where the poles of the Riemann zeta-function are canceled by \( G(\frac{\alpha+\gamma}{2}) = 0 \), etc. After combining all the other terms of \( A_O \) and \( A_{-O} \) in the same way, we conclude that
\[ A_O(\alpha, \beta, \gamma, \delta, \Phi) + A_{-O}(\gamma, \delta, \alpha \beta, \Phi) \]
\[ = Z_q(\beta, -\gamma, -\delta, -\alpha) \int_R \Phi(t) \left( \frac{tq}{2\pi} \right)^{-\alpha-\gamma} dt + Z_q(\alpha, -\gamma, -\delta, -\beta) \int_R \Phi(t) \left( \frac{tq}{2\pi} \right)^{-\beta-\gamma} dt \]
\[ + Z_q(\beta, -\delta, -\gamma, -\alpha) \int_R \Phi(t) \left( \frac{tq}{2\pi} \right)^{-\alpha-\delta} dt + Z_q(\alpha, -\delta, -\gamma, -\beta) \int_R \Phi(t) \left( \frac{tq}{2\pi} \right)^{-\beta-\delta} dt \]
\[ + O\left( T^{\frac{1}{2}+e}(q/q_0)^{\frac{1}{2}+e}(T/T_0)^3 + T^e q^e \right). \]

We sum up from (3.1), (3.5) and (3.8) that
\[ M(\alpha, \beta, \gamma, \delta, \Phi) = A_D(\alpha, \beta, \gamma, \delta, \Phi) + A_{-D}(\gamma, -\delta, -\alpha, -\beta, \Phi) \]
\[ + A_O(\alpha, \beta, \gamma, \delta, \Phi) + A_{-O}(\gamma, -\delta, -\alpha, -\beta, \Phi) + O((T/T_0)T^e q^e). \]

Together with Lemma 3.1 and (4.18), this would establish Theorem 1.8.
5. Proof of Theorem 1.7

In this section, we sketch the proof of Theorem 1.7. We follow closely the argument in [30, Theorem 1.3] and keep track of the difference. The calculation of the main terms for Theorem 1.7 is identical to [30, Theorem 1.3] since \( t \) does not cause any essential difference here. We bound the quantities \( E_{M,N} \) and \( E_{M,N}^{-1} \) in [30, Theorem 3.1] by

\[
E_{M,N}, \ E_{M,N}^{-1} \ll T_1^{2+\epsilon} q^{-\frac{1}{2} + \theta + \epsilon} M^{-\frac{1}{2}} N^{\frac{1}{2}},
\]

where the extra factor \( T_1^{2+\epsilon} \) comes from the ratios of gamma factors in applying spectral large sieve inequalities. The proof of (5.1) is identical to [30, Section 9], and the necessary variation on the ratios of gamma factors is an exercise based on Stirling’s approximation.

Now, it remains to bound the quantity \( B_{M,N} \) in [30, (3.4)] with \( M \) and \( N \) far away from each other. After omitting all harmless parameters such as \( \alpha, \beta, \gamma, \delta \), and \( a \), we recall that

\[
B_{M,N} = \frac{1}{\varphi^*(q)} \sum_{d|q} \varphi(d) \mu\left(\frac{q}{d}\right) \sum_{(mn,q)=1, \ m \equiv m \mod d} \frac{d(m)d(n)}{m^{\frac{1}{2} + \epsilon}n^{\frac{1}{2} - \epsilon}} V\left(\frac{mn}{q^2}, t\right) W\left(\frac{m}{M}\right) W\left(\frac{n}{N}\right).
\]

It is easy to see the trivial bound

\[
B_{M,N} \ll q^{-1+\epsilon}(MN)^{\frac{1}{2}}.
\]

Together with \( MN \ll (T_1 q)^{2+\epsilon} \), this means that Theorem 1.7 is non-trivial only for \( T \ll q^{\frac{1}{2} - \frac{1}{\theta}} \).

We write \( T_1 = q^\tau \) with \( 0 \leq \tau \leq \frac{1}{3} - \frac{3}{4} \theta \). Let

\[
\eta = \frac{1}{14} - \frac{3}{4} \theta - \frac{1}{4} \tau, \quad M = q^\mu, \quad N = q^{\nu}.
\]

By (5.1) and (5.2), it remains to show

\[
B_{M,N} \ll q^{-\eta+\epsilon}
\]

for

\[
2 - 2\eta \leq \mu + \nu \leq 2 + 2\tau, \quad 1 - 2\theta - 2\eta - 4\tau \leq \nu - \mu.
\]

With

\[
W_i(x) = x^\mu W(x), \quad N = N_1 N_2, \quad N_1 = q^{\nu_1}, \quad N_2 = q^{\nu_2}, \quad \nu_1 \leq \nu_2,
\]

an evaluation identical to [30, Section 10] reduces the problem to bounding

\[
R(d, a) \ll q^{-\eta+\epsilon},
\]

where

\[
R(d, a) = \frac{N_2}{a\varphi^*(q) \sqrt{MN}} \sum_{(m,q)=1} d(m) W_i\left(\frac{m}{M}\right) \sum_{(n_1,q)=1} \sum_{h \neq 0} e\left(\frac{hm\bar{a}_1}{d}\right) W_i\left(\frac{n_1}{N_1}\right) W\left(\frac{h}{H}\right)
\]

is an analogue of \( R(d, a) \) in [30, (10.3)]. The difference is the \( W \) function in place of the \( W \) function, as a result, a longer range of the \( h \)-sum with

\[
H = adT_1 N_2^{-1} \ll T_1 q N_2^{-1}.
\]

Since the \( h \)-sum vanishes for \( N_2 > T_1 q \), we may assume that

\[
\nu_2 \leq 1 + \tau.
\]
Now we divide the region in (5.3) into several parts, according to

1. \( v - \mu \geq 1 + 2\eta + 4\tau; \)
2. \( 1 - 2\theta - 2\eta - 4\tau \leq v - \mu < 1 + 2\eta + 4\tau, \)
   - \( \frac{1}{2} - \theta - 2\eta - 3\tau < \nu_1 < \frac{1}{2} + 2\eta + \tau; \)
   - \( \nu_1 \geq \frac{1}{2} + 2\eta + \tau. \)

Then for each range, we prove that the estimate (5.4) holds.

5.1. **The range with** \( v - \mu \text{ large}. \) For the range with \( v - \mu \geq 1 + 2\eta + 4\tau, \) a summation by parts with the Weil bound shows

\[
R(d, a) \ll \frac{N_2H}{aq^{1+\varepsilon}} \left( \frac{N}{M} \right)^{\frac{1}{2}} \left( d^{\frac{1}{2}+\varepsilon} + N_1d^{-\frac{1}{2}} \right).
\]

With \( H = adT_1N_2^{-1}, \ d \leq q, \) and \( N_1 \ll N_2 \ll q^{1+\tau}, \) it follows that

\[
R(d, a) \ll q^{\frac{1}{2}+2\tau+\varepsilon} \left( \frac{N}{M} \right)^{-\frac{1}{2}} + q^{3\tau} \left( \frac{N}{M} \right)^{-\frac{1}{2}} \ll q^{-\eta+\varepsilon}.
\]

5.2. **The range with** \( v - \mu \text{ close to 1}. \) After combining \( m \) and \( h \) into a longer variable \( l = mh, \) we have

\[
R(d, a) \ll \frac{N_2q^\varepsilon}{aq \sqrt{MN}} \sum_{l \leq L} \left| \sum_{(n_1,q)=1} e \left( \frac{an_1l}{d} \right) W_l \left( \frac{n_1}{N_1} \right) \right|
\]

with

\[
L = MHq^\varepsilon \ll \frac{adT_1M}{N_2} q^\varepsilon.
\]

We bound this double sums with the following lemma; see also [30, Lemma 10.1] and [20, Theorem 2.4].

**Lemma 5.1.** Let \( q \) be a positive integer and \( (\alpha_k) \) be a sequence of complex numbers satisfying \( \alpha_k \ll k^\varepsilon. \) For any positive integers \( L, K, \) we have

\[
\left| \sum_{l \leq L} \sum_{k \leq K \atop (k,q)=1} \alpha_k e \left( \frac{alk}{q} \right) \right| \ll LKq^\varepsilon \cdot \Delta(L, K, q)
\]

uniformly in \( a \) with \( (a,q) = 1, \) where we may take the saving \( \Delta(L, K, q) \) freely among

\begin{align*}
(5.6a) \quad & L^{-\frac{1}{2}}K^{-\frac{1}{2}}q^\frac{1}{2} + L^{-\frac{1}{2}} + q^{-\frac{1}{2}} + K^{-\frac{1}{2}}, \\
(5.6b) \quad & L^{-\frac{1}{2}}K^{-1}q^\frac{1}{2} + K^{-1}q^\frac{1}{2} + L^{-\frac{1}{2}} + q^{-\frac{1}{2}} + K^{-\frac{1}{2}}.
\end{align*}

For the range with \( 1 - 2\theta - 2\eta - 4\tau \leq v - \mu < 1 + 2\eta + 4\tau \) and \( \frac{1}{2} - \theta - 2\eta - 3\tau < \nu_1 < \frac{1}{2} + 2\eta + \tau, \) we apply (5.6a) to have

\[
R(d, a) \ll \frac{N_2q^\varepsilon}{aq \sqrt{MN}} \left( L^{\frac{1}{2}}N_1^\frac{1}{2}d^\frac{1}{2} + L^\frac{1}{2}N_1 + LN_1d^{-\frac{1}{2}} + LN_1^{-\frac{1}{2}} \right).
\]
As $L \ll \frac{adT M}{N_2} q^e$ and $N_1 \ll q^{1+2\eta+\tau} = q^{\frac{9}{14}-\frac{3}{14}\eta-\frac{1}{14}\tau} \leq q$, an easy calculation shows
\[
R(d, a) \ll T_1^\frac{1}{2} q^{-\frac{1}{4}e} N_1^\frac{1}{2} + T_1 q^e \left( \frac{N}{M} \right)^{-\frac{2}{3}} N_1^\frac{1}{3} \ll q^{-\frac{1}{2}+\frac{1}{4}+\frac{1}{4}v_1+e} + q^{-\frac{1}{2}(\nu-\mu)+\frac{1}{4}v_1+e}.
\]
Since $\nu - \mu > 1 - 2\theta - 2\eta - 4\tau$ and $v_1 < \frac{1}{2} + 2\eta + \tau$, we have
\[
- \frac{1}{4} + \frac{1}{2} + \frac{1}{2} \tau + \frac{3}{2} v_1 \leq -\frac{1}{8} + \frac{1}{2} \eta + \frac{3}{4} \tau \leq -\eta \quad \text{for} \quad \eta \leq \frac{1}{12} - \frac{1}{2} \tau,
\]
\[
\tau - \frac{1}{2}(\nu - \mu) + \frac{1}{2} v_1 \leq -\frac{1}{4} + \eta + \frac{1}{2} \tau \leq -\eta \quad \text{for} \quad \eta \leq \frac{1}{12} - \frac{1}{4} \theta - \frac{1}{6} \eta,
\]
and thus (5.4) holds.

For the remaining range, we have
\[
1 - 2\theta - 2\eta - 4\tau \leq \nu - \mu < 1 + 2\eta + 4\tau,
\]
(5.7) \[
\frac{1}{2} + 2\eta + \tau \leq v_1 \leq \frac{1}{2} + \frac{1}{2} \tau + \frac{1}{4}(\nu - \mu) \leq \frac{3}{4} + \frac{3}{2} \eta + \frac{3}{4} \tau,
\]
(5.8) then by (5.6b),
\[
R(d, a) \ll \frac{N_2 q^e}{aq \sqrt{MN}} \left( L_1^\frac{3}{4} d_1^\frac{1}{2} + L d_1^\frac{1}{2} + L^\frac{1}{2} N_1 + LN_1 d_1^\frac{1}{2} + LN_1^\frac{1}{2} \right).
\]
After a simple calculation with $L \ll \frac{adT M}{N_2} q^e$ and $N_1 \ll q^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}} = q^{\frac{1}{14}-\frac{3}{14}\eta+\frac{1}{14}\tau} \leq q$, it follows that
\[
R(d, a) \ll T_1^\frac{1}{2} q^{-\frac{1}{4}e} N_1^\frac{1}{2} + T_1^\frac{1}{2} q^{-\frac{1}{4}e} N_1^\frac{1}{2} + T_1 q^e \left( \frac{N}{M} \right)^{-\frac{2}{3}} N_1^\frac{1}{3} \ll q^{\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\eta+e} + q^{-\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\eta+e} + q^{-\frac{1}{2}(\nu-\mu)+\frac{1}{4}v_1+e}.
\]
Then by (5.7) and (5.8), we have
\[
\frac{1}{4} + \frac{1}{2} \tau - \frac{1}{2} v_1 \leq \frac{1}{4} + \frac{1}{2} \tau - \frac{1}{4} \times \left( \frac{1}{2} + 2\eta + \tau \right) = -\tau,
\]
\[
- \frac{1}{2} + \frac{1}{2} \tau + \frac{1}{2} v_1 \leq -\frac{1}{8} + \frac{1}{4} \eta + \frac{3}{4} \tau \leq \tau \quad \text{for} \quad \eta \leq \frac{1}{10} - \tau,
\]
\[
\tau - \frac{1}{2}(\nu - \mu) + \frac{1}{2} v_1 \leq \frac{1}{4} + \frac{1}{2} \tau - \frac{1}{8}(\nu - \mu) \leq -\frac{1}{4} + \frac{3}{2} \theta + \frac{3}{4} \eta + \frac{1}{4} \tau \leq -\eta \quad \text{for} \quad \eta \leq \frac{1}{14} - \frac{3}{4} \theta - \frac{1}{6} \tau,
\]
and then (5.4) follows.

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