Small data blow-up and upper estimate of lifespan for the weakly coupled system of nonlinear damped wave equations outside a ball

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Abstract. In this paper, we consider the initial-boundary value problem for the $k$-component system of semi-linear classical damped wave equations outside a ball. By applying a test function approach with a special choice of test functions, which approximates the harmonic function being subject to the Dirichlet boundary condition on $\partial \Omega$, simultaneously we have succeeded in proving the blow-up result in a finite time as well as in catching the sharp upper bound of lifespan estimates for solutions in two and higher spatial dimensions.

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1 Introduction

This paper is concerned with investigating upper bound of lifespan estimates for solutions to the following weakly coupled system of semi-linear classical damped wave equations in an exterior domain:

\[
\begin{align*}
\partial_t^2 u_1(t, x) - \Delta u_1(t, x) + \partial_t u_1(t, x) &= |u_1(t, x)|^{p_1}, & (t, x) &\in (0, T) \times \Omega, \\
\partial_t^2 u_2(t, x) - \Delta u_2(t, x) + \partial_t u_2(t, x) &= |u_1(t, x)|^{p_2}, & (t, x) &\in (0, T) \times \Omega, \\
& \vdots \\
\partial_t^2 u_k(t, x) - \Delta u_k(t, x) + \partial_t u_k(t, x) &= |u_{k-1}(t, x)|^{p_k}, & (t, x) &\in (0, T) \times \Omega, \\
u_1(t, x) &= 0, & (t, x) &\in (0, T) \times \partial \Omega, \ell = 1, 2, \ldots, k, \\
u_1(0, x) &= \varepsilon u_{0, \ell}(x), \quad \partial_t u_1(0, x) = \varepsilon u_{1, \ell}(x), & x &\in \Omega, \ell = 1, 2, \ldots, k,
\end{align*}
\]

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where \( k \geq 2, \; p_\ell > 1 \) with \( \ell = 1, 2, \cdots, k \) and \( T > 0 \). The domain \( \Omega \) is given by \( \Omega = \{ x \in \mathbb{R}^n : |x| > 1 \} \) with \( n \geq 3 \). \( u_t : (0, T) \times \Omega \rightarrow \mathbb{C} \) with \( \ell = 1, \cdots, k \) denotes an unknown function to the problem (1.1). The positive constant \( \epsilon \) presents the size of initial data. The given functions \( u_{0,\ell} \) and \( u_{1,\ell} \) with \( \ell = 1, 2, \cdots, k \) represent the shape of the initial data. \( \Delta \) denotes the Laplace operator in the open subset \( \Omega \) of \( \mathbb{R}^n \).

To get started, let us make some attention to the Cauchy problem of (1.1) in the whole space, i.e. \( \Omega = \mathbb{R}^n \). Concerning \( k = 1 \), the single classical semi-linear damped wave equation

\[
\begin{align*}
    u_{tt} - \Delta u + u_t &= |u|^p, \\
    u(0, x) &= \varepsilon u_0(x), \quad \partial_t u(0, x) = \varepsilon u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

(1.2)

Todorova-Yordanov [31] introduced the so-called Fujita exponent \( p_{\text{Fuj}}(n) = 1 + \frac{2}{n} \) (see more [5, 8, 33], the well-known Fujita exponent for the semi-linear heat equation), which classifies between the global (in time) existence of solutions to (1.2) for \( p > p_{\text{Fuj}}(n) \) and a blow-up result in the inverse case \( 1 < p < p_{\text{Fuj}}(n) \). Especially, the treatment of the critical case \( p = p_{\text{Fuj}}(n) \), verified by Zhang [34] later, is also to conclude nonexistence of global solutions to (1.2). When the blow-up phenomenon in finite time occurs, the maximal existence time of solutions to (1.2), which is the so-called lifespan, can be estimated by a series of previous works [16, 22, 12, 18, 7, 15] and references therein as follows:

\[
\text{LifeSpan}(u) \sim \begin{cases} 
    e^{-\frac{3(n-1)}{2n}p_{\text{Fuj}}^{-1}} & \text{if } 1 < p < p_{\text{Fuj}}(n), \\
    \exp(Ce^{-(p-1)}) & \text{if } p = p_{\text{Fuj}}(n).
\end{cases}
\]

Regarding \( k = 2 \) of (1.1) in \( \mathbb{R}^n \) the authors in [27, 20, 23] described the following critical curve in the \( p_1 - p_2 \) plane:

\[
\gamma_{\text{max}}(p_1, p_2) := \max\{p_1, p_2\} + 1 \frac{1}{p_1 p_2 - 1} = \frac{n}{2}.
\]

More specific, they proved that the global (in time) small data Sobolev solutions exist if \( \gamma_{\text{max}}(p_1, p_2) < n/2 \), meanwhile, every non-trivial local (in time) weak solution blows up in finite time if \( \gamma_{\text{max}}(p_1, p_2) \geq n/2 \). One point worth noticing in [23, 24] is that the sharp upper bound estimate for the lifespan of solutions in the subcritical case \( \gamma_{\text{max}}(p_1, p_2) > n/2 \) was given by

\[
\text{LifeSpan}(u_1, u_2) \leq Ce^{-\frac{1}{\gamma_{\text{max}}(p_1, p_2) - n/2}}.
\]

Quite recently, for the purpose of making the study of lifespan self-contained, Chen.Dao [4] have found out the sharp lifespan estimates in the critical case \( \gamma_{\text{max}}(p_1, p_2) = n/2 \), namely,

\[
\text{LifeSpan}(u_1, u_2) \sim \begin{cases} 
    \exp(Ce^{-(p_1-1)}) & \text{if } p_1 = p_2, \\
    \exp(Ce^{-(p_1, p_2 - p_{\text{Fuj}}(n))}) & \text{if } p_1 \neq p_2.
\end{cases}
\]

To demonstrate this, the authors have applied a suitable test function method linked to the technical estimates for nonlinear differential inequalities and have constructed polynomial-logarithmic type time-weighted Sobolev spaces as well in terms of deriving upper bound estimates and lower bound estimates for the lifespan, respectively. For the more general cases \( k \geq 3 \) of (1.1) in \( \mathbb{R}^n \), Takeda [32] obtained both the global existence and the finite time blow-up result for small solutions to establish the critical condition

\[
\gamma_{\text{max}} := \max\{\gamma_1, \gamma_2, \cdots, \gamma_k\} = \frac{n}{2}.
\]
for any $n \leq 3$, which was further extended by Narazaki [21] for any $n \geq 4$ thanks to using weighted Sobolev spaces. Here we note that the aforementioned parameters $\gamma_\ell$ with $\ell = 1, 2, \ldots, k$ are introduced as in (1.4). Not much later, Nishihara-Wakasugi [24] improved these results for any $n \geq 1$ by the application of a weighted energy method. One may see that the authors in the latter paper also showed the following lifespan estimates for solutions from both the above and the below:

$$ce^{-\frac{1}{\gamma_{\max}-n/2}} \leq \text{LifeSpan}(u_1, u_2, \cdots, u_k) \leq C e^{-\frac{1}{\gamma_{\max}-n/2}}$$

for any small number $\delta > 0$. In other words, they gave an almost optimal estimate for the lifespan but it seems to be far from lower bound one to upper bound one.

Turning back to our model (1.1), as far as there have been a lot of investigations in the study of the exterior problems for semi-linear classical damped wave equation. It is significant to recognize that the essential difference to the initial problem originates from the influence of reflection at the boundary and the lack of symmetric properties including scale-invariance, rotation-invariance and so on. Let us recall several previous literatures involving the exterior problem for the single equation, i.e. (1.1) with $k = 1$, as follows:

$$\begin{align*}
\partial_t^2 u(t, x) - \Delta u(t, x) + \partial_t u(t, x) &= |u(t, x)|^p, \quad (t, x) \in (0, T) \times \Omega, \\
u(t, x) &= 0, \quad (t, x) \in (0, T) \times \partial\Omega, \\
u(0, x) &= \epsilon u_0(x), \quad \partial_t u(0, x) = \epsilon u_1(x), \quad x \in \Omega.
\end{align*}$$

(1.3)

Particularly, Ikeda [9, 10, 11] and Ono [25] succeeded in proving the existence of global solutions to (1.3) if $p > p_{Fuj}(2)$ for $n = 2$ and $1 + \frac{2}{n+2} < p \leq 1 + \frac{2}{n+2}$ for $n = 3, 4, 5$. Meanwhile, one can find in the paper of Ogawa-Takeda [26] that the solutions to (1.3) cannot exist globally if $1 < p < p_{Fuj}(n)$ for any $n \geq 1$ by using Kaplan-Fujita method. Afterwards, the critical case $p = p_{Fuj}(n)$ was independently filled by Fino-Ibrahim-Wehbe [6] and Lai-Yin [17], where the finite time blow-up phenomena also occurs. To demonstrate these blow-up results, the authors employed a suitable test function method essentially, which was originally developed by Baras-Pierre [3] (see also [19, 34]). More precisely, the aid of the first eigenfunction of $-\Delta$ as well as the corresponding first eigenvalue over $\Omega$ was taken into consideration in [26, 6], whereas the employment of some properties on the Riemann-Liouville fractional derivative and the harmonic function in $\Omega$ came into play in [17]. However, their approaches seem to be difficult to apply directly in catching the lifespan estimates for solutions to (1.3) due to technical issues. More recently, Sobajima [28] (see also Ikeda-Sobajima [13]) established the following upper bound estimates for lifespan of solutions to (1.3):

$$\text{LifeSpan}(u_1) \leq \begin{cases} 
\left( e^{-\frac{1}{p-1}} \log (e^{-1}) \right)^{-\frac{n+1}{p-1}} & \text{if } 1 < p < 2 \text{ and } n = 2, \\
\exp \left( \frac{C}{e} \right) & \text{if } p = 2 \text{ and } n = 2, \\
\exp \left( C \epsilon^{-1} \right) & \text{if } 1 < p < p_{Fuj}(n) \text{ and } n \geq 3, \\
\exp \left( C \epsilon^{-(p-1)} \right) & \text{if } p = p_{Fuj}(n) \text{ and } n \geq 3.
\end{cases}$$

At first sight, these estimates are exactly the same as those for the corresponding Cauchy problem (1.2) in higher dimensions $n \geq 3$ but the difference comes from the case of $n = 2$, especially, the double exponential type appears in the critical exponent $p = 2$. This different behavior can be interpreted as the influence of recurrence of the Brownian motion in two-dimensional case. To the best of the authors’ knowledge, no work in terms of the study of the lifespan estimates for solutions to (1.1) exists in the literature so far even when this system consists two components, i.e. (1.1) with $k = 2$. Motivated strongly by [4] and [13], our main goal of this paper is to indicate not only the blow-up of solutions but
also the sharp upper bound of lifespan to (1.1) for any \( k \geq 2 \) in two and higher spatial dimensions. For this purpose, we appropriately determine the test function method, which approximates the harmonic function enjoying Dirichlet boundary condition on \( \partial \Omega \) for \( n = 2 \) and for any \( n \geq 3 \) individually, as well as the technical derivation of lifespan estimates modified from [14, 13, 4]. Nevertheless, as we can see later (Section 4.2), our strategy dealing with \( n \geq 3 \) does not work so well to explore the special case \( n = 2 \). Hence, considering \( n = 2 \) as an exceptional case of (1.1) we will discuss the treatment of this case in the other approach.

The structure of this paper is organized as follows: We state the main results including local well-posedness in the energy space, small data blow-up result and upper bound estimate for lifespan of solution in Section 2. Section 3 is to present the proof of local well-posedness result in the energy space. We give some of preliminary calculations of test functions in Section 4.1 that are used in the sequel. Finally, Section 4.2 is devoted to the proof of small data solution blow-up and upper bound estimate for lifespan of solution simultaneously.

**Notations:** We give the following notations which are used throughout this paper.

- We write \( f \leq g \) when there exists a constant \( C > 0 \) such that \( f \leq Cg \), and \( f \approx g \) when \( g \leq f \leq g \).
- Let us denote the matrix
  \[
P := \begin{pmatrix}
0 & 0 & \cdots & 0 & p_1 \\
p_2 & 0 & \cdots & 0 & 0 \\
0 & p_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & p_k & 0
\end{pmatrix}
\]
and the column vector
  \[
\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_k)^T := (P - I_k)^{-1}(1, 1, \cdots, 1)^T,
\]
for \( k \) times

where \( I_k \) and \((\alpha_1, \alpha_2, \cdots, \alpha_k)^T \) stand for the identity matrix and the transposition vector of vector \((\alpha_1, \alpha_2, \cdots, \alpha_k)\), respectively. We want to point out that due to the assumption \( p_\ell > 1 \) with \( \ell = 1, 2, \cdots, k \), it is obvious to recognize that

\[
det(P - I_k) = (-1)^{k+1} \left( \prod_{\ell=1}^{k} (p_\ell - 1) \right) \neq 0.
\]

So, the inverse matrix \((P - I_k)^{-1}\) exists. This means the above vector \( \gamma \) is well-defined. Then, we set the element \( \gamma_{\text{max}} := \max\{\gamma_1, \gamma_2, \cdots, \gamma_k\} \).

- Let \( m \in \mathbb{N} \cup \{0\} \) and \( p \in [1, \infty] \). We introduce the Sobolev space \( W^{m,p}(\Omega) \), which is a Banach space of measurable functions \( f: \Omega \to \mathbb{C} \) such that \( D^\alpha f \in L^p(\Omega) \) in the sense of distributions, for every multi-index \( \alpha \in (\mathbb{N} \cup \{0\})^n \) with \( |\alpha| \leq m \). The space \( W^{m,p}(\Omega) \) is equipped with the norm \( \|f\|_{W^{m,p}(\Omega)} \) given by
  \[
  \|f\|_{W^{m,p}(\Omega)} := \sum_{|\alpha| \leq m} \|D^\alpha f\|_{L^p(\Omega)}.
  \]

Here \( D := (\partial_{x_1}, \cdots, \partial_{x_n}) \) is a partial differential operator. Let \( W^{m,p}_0(\Omega) \) denote a closure of \( C_0^\infty(\Omega) \) in \( W^{m,p}(\Omega) \), where \( C_0^\infty(\Omega) \) is the space of functions in \( C^\infty(\Omega) \) which have a compact support in \( \Omega \).
For $m \in \mathbb{N} \cup \{0\}$, let $H^m(\Omega)$ denote $W^{m,2}(\Omega)$. Moreover, $H^m(\Omega)$ is equipped with the equivalent norm $\| \cdot \|_{H^m(\Omega)}$ given by

$$\|f\|_{H^m(\Omega)} := \left( \sum_{|\alpha| \leq m} \int_{\Omega} |D^\alpha f(x)|^2 \, dx \right)^{\frac{1}{2}}.$$ 

Then $H^m(\Omega)$ is a Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle_{H^m(\Omega)}$ given by

$$\langle f, g \rangle_{H^m(\Omega)} := \sum_{|\alpha| \leq m} \mathbb{R} \int_{\Omega} D^\alpha f(x) D^\alpha g(x) \, dx.$$ 

Furthermore, $H^m_0(\Omega) := W^{m,2}_0(\Omega)$.

The precise definition of the Laplace operator is as follows. We introduce a linear operator $\mathcal{A}$ with the domain $D(\mathcal{A})$ in $L^2(\Omega)$ defined by

$$D(\mathcal{A}) := \left\{ f \in H^1_0(\Omega) : \Delta f \in L^2(\Omega) \right\}, \quad \text{and} \quad \mathcal{A} f = \Delta f, \quad \text{for } f \in D(\mathcal{A}). \quad (1.5)$$

It is known that $\mathcal{A}$ is $m$-dissipative operator with the dense domain (see [2, Definition 2.2.2 and Proposition 2.6.2] for example) and we write $\mathcal{A}$ as $\Delta$ for short.

We define a Hilbert space $E$ as $E = E(\Omega) := (H^1_0(\Omega))^k \times (L^2(\Omega))^k$, which is called the energy space to the initial-boundary value problem (1.1).

Let $m \in \mathbb{N} \cup \{0\}$. For an interval $I \subset \mathbb{R}$, we introduce the space $C^m(I, X)$ of $m$-times continuously differentiable functions from $I$ to $X$ with respect to the topology in $X$.

Let $p \in [1, \infty]$. We denote by $p'$ the conjugate of $p$. Moreover, for an interval $I \subset \mathbb{R}$, we define the Lebesgue space $L^p(I, X)$ of all measurable functions from $I$ to $X$ endowed with the norm $\| \cdot \|_{L^p(I, X)}$ given by

$$\|u\|_{L^p(I, X)} := \left\| \|u(t, \cdot)\|_X \right\|_{L^p(I)}.$$ 

## 2 Main results

In this section, we state our two main results. The first one (Theorem 2.2) gives local well-posedness to the initial-boundary value problem (1.1) in the energy space $E$ in the case of $p_\ell \in [1, n/(n-2)]$ with $n \geq 3$ and $p_\ell \in [1, \infty)$ with $n = 1, 2$ for $\ell = 1, \ldots, k$. Here we say that well-posedness to (1.1) holds if existence, uniqueness of the solution and continuous dependence on the initial data are valid. The second one (Theorem 2.4) gives a small data blow-up result and an upper estimate of the lifespan of the small solution to the problem (1.1) when $\gamma_{\text{max}} \geq n/2$ and $n \geq 3$, “$\gamma_{\text{max}} > 1$ and $n = 2$” or “$\gamma_1 = \cdots = \gamma_k = 1$ and $n = 2$”.

### 2.1 Local well-posedness in the energy space

In order to state the local well-posedness result to the problem (1.1), we convert the original problem (1.1) into the following form:

$$\begin{cases}
\partial_t U(t, x) - \mathcal{B}U(t, x) = \mathcal{N}(U(t, x)), & (t, x) \in (0, T) \times \Omega, \\
U(t, x) = 0, & (t, x) \in (0, T) \times \partial \Omega, \\
U(0, x) = \varepsilon U_0(x), & x \in \Omega,
\end{cases} \quad (2.1)$$
where \( u = u(t, x) := (u_1(t,x), \cdots, u_k(t,x)) : (0, T) \times \Omega \to \mathbb{C}^k \) denotes the \( k \)-tuple of the unknown functions, \( U = U(t,x) := (u(t,x), \partial_t u(t,x)) = (u_1(t,x), \cdots, u_k(t,x), \partial_t u_1(t,x), \cdots, \partial_t u_k(t,x)) : (0, T) \times \Omega \to \mathbb{C}^{2k} \) is a new unknown function, \( u_0 = u_0(x) := (u_{0,1}(x), \cdots, u_{0,k}(x)) : \Omega \to \mathbb{C}^k \) denotes the \( k \)-tuple of the initial displacement and \( u_1 = u_1(x) := (u_{1,1}(x), \cdots, u_{1,k}(x)) : \Omega \to \mathbb{C}^k \) denotes the \( k \)-tuple of the initial velocity, and \( U_0 = U_0(x) := (u_0(x), u_1(x)) : \Omega \to \mathbb{C}^{2k} \) is a new given initial function. The linear operator \( B \) on the energy space \( E \) is defined by

\[
B := \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}
\]

with the dense domain \( D(B) := \{ (u, v) \in E : \Delta u \in (L^2(\Omega))^k, v \in (H_0^1(\Omega))^k \} \), where \( \Delta \) is the Laplace operator defined by (1.5). Moreover, the nonlinear mapping \( N : \mathbb{C}^k \to \mathbb{C}^k \) is given by

\[
N(U) := (0, \cdots, 0, -\partial_t u_1 + |u_k|^p + \cdots, -\partial_t u_k + |u_{k-1}|^p).
\]

We remark that the original problem (1.1) is equivalent to the problem (2.1) through the relation \( U = (u_1, \cdots, u_k, \partial_t u_1, \cdots, \partial_t u_k) \). The \( m \)-dissipativity of the operator \( B \) is known:

**Lemma 2.1** (\( m \)-dissipativity of \( B \)). The operator \( B \) defined by (2.2) is \( m \)-dissipative with the dense domain \( D(B) \) in \( E \). Moreover, \( B \) generates a contraction semigroup \( \{e^{tB}\}_{t \geq 0} \) on \( E \).

This lemma can be proved in the similar manner as the proof of [2, Proposition 2.6.9] and [2, Theorem 3.4.4].

For \( T \in (0, \infty) \), we define a solution space \( E(T) \) to the problem (2.1) as \( E(T) := L^\infty_t(0, T; E) \), which is a Banach space endowed with the norm \( \| \cdot \|_{L^\infty_t(0, T; E)} \). Next we introduce a notion of mild solution to the problem (2.1).

**Definition 2.1** (Mild solution). Let \( T \in (0, \infty) \) and \( U_0 \in E \). We say that a function \( U : (0, T) \times \Omega \to \mathbb{C}^k \) is a mild solution to (2.1) if \( U \) belongs to the class \( E(T) \) and it satisfies the following integral equation

\[
U(t) = e^tB U_0 + \int_0^t e^{(t-\tau)B} N(U(\tau)) d\tau,
\]

which is associated with (2.1), for any \( t \in [0, T) \).

**Definition 2.2** (Lifespan). We call the maximal existence time of the mild solution to (2.1) to be lifespan, which is denoted by \( T_e \), that is

\[
T_e := \sup \{ T \in (0, \infty) \mid \text{there exists a unique mild solution } U \text{ to (2.1) on } [0, T) \}.
\]

Next we state local well-posedness to the problem (2.1) in the energy space \( E \):

**Theorem 2.2** (Local well-posedness in the energy space). Let \( n \in \mathbb{N}, p_\ell \geq 1 \) for \( \ell = 1, \cdots, k, \varepsilon > 0 \) and \( U_0 \in E \). We assume that if \( n \geq 3 \), then \( p_\ell \leq n/(n-2) \) for \( \ell = 1, \cdots, k \). Then the initial-boundary value problem (1.1) is locally well-posed in the energy space \( E \). More precisely, the following statements hold:

- **Existence**: There exists a positive time \( T \) such that there exists a mild solution \( U \in E(T) \cap C([0, T); E) \) to the problem (2.1) on \( [0, T) \).

- **Uniqueness**: Let \( U \in E(T) \) be the solution to (2.1) obtained in the Existence part. Let \( T_1 \in (0, T) \) and \( V \in E(T) \) be another mild solution to (2.1) with the same initial data \( eU_0 \). Then the identity \( U(t) = V(t) \) holds for any \( t \in [0, T_1) \).
Continuous dependence on initial data: The flow map $E \to E(T, M)$, $eU_0 \mapsto U$ is Lipschitz continuous, where $U$ is the mild solution to (2.1) with initial data $eU_0$.

Blow-up alternative: If $T_\varepsilon < \infty$, then $\lim_{t \to T_\varepsilon} \|U(t)\|_E = \infty$.

2.2 Small data blow-up and upper estimate for lifespan

Before stating our main results, let us state the following definition of weak solutions to (1.1).

**Definition 2.3** (Weak solution). Let $p_\ell > 1$ with $\ell = 1, 2, \cdots, k$ and $T > 0$. A $k$-tuple of functions $(u_1, u_2, \cdots, u_k)$ is called a weak solution to (1.1) on $[0, T)$ if

$$(u_1, u_2, \cdots, u_k) \in \left( C([0, T), H^1_0(\Omega)) \cap C([0, T), L^2(\Omega)) \cap L^p_{loc}([0, T) \times \bar{\Omega}) \right)$$

and the following relations hold:

$$\int_0^T \int_\Omega |u_k(t, x)|^{p_k} \Phi(t, x) dxdt + \int_\Omega u_{0,1}(x) \Phi(0, x) dx$$

$$= \int_0^T \int_\Omega \left( \nabla u_1(t, x) \cdot \nabla \Phi(t, x) - \partial_t u_1(t, x) \partial_t \Phi(t, x) + \partial_i u_1(t, x) \Phi(t, x) \right) dxdt$$

(2.4)

and

$$\int_0^T \int_\Omega |u_{t}(t, x)|^{p_{t+1}} \Phi(t, x) dxdt + \int_\Omega u_{0,t+1}(x) \Phi(0, x) dx$$

$$= \int_0^T \int_\Omega \left( \nabla u_{t+1}(t, x) \cdot \nabla \Phi(t, x) - \partial_t u_{t+1}(t, x) \partial_t \Phi(t, x) + \partial_i u_{t+1}(t, x) \Phi(t, x) \right) dxdt$$

(2.5)

with $\ell = 1, 2, \cdots, k-1$, for any test function $\Phi = \Phi(t, x) \in C^3([0, T) \times \Omega)$ with supp$\Phi \subset [0, T) \times \bar{\Omega}$ such that $\Phi(t, \cdot)|_{\partial \Omega} = 0$.

**Lemma 2.3** (Relation between mild solution and weak solution). We assume the same assumptions as in Theorem 2.2. Then $U = (u_1, \cdots, u_k)$ is a mild solution to (2.1) on $[0, T)$ in the sense of Definition 2.1 if and only if it is the weak solution to (1.1) on $[0, T)$ in the sense of Definition 2.3.

Our main results are read as follows:

**Theorem 2.4** (Small data blow-up and upper estimate of lifespan). Let $n \geq 2$. With $\ell = 1, 2, \cdots, k$, assume that the exponents $p_\ell > 1$ fulfill the condition

$$\max\{\gamma_1, \gamma_2, \cdots, \gamma_k\} \geq \frac{n}{2}$$

(2.6)

and the initial data $u_{0,\ell}, u_{1,\ell}$ satisfy

$$u_{0,\ell}(x) \Psi(x) \in L^1(\Omega), \quad u_{1,\ell}(x) \Psi(x) \in L^1(\Omega)$$

(2.7)
Lemma 3.1 (Sobolev embedding) from a suitable closed set in \( \Omega \)

In this section, we give a proof of local well-posedness in the energy space to the problem (2.2). Remark in investigating global existence results for solutions to (1.1) is to find out an appropriate upper bound in estimating the lifespan of solution to (1.1), where the function \( \Psi = \Psi \) for the lifespan of weak solutions to (3.1) equal.

\[
\text{LifeSpan}(u_1, u_2, \cdots, u_k) \leq \begin{cases} 
C \left( e^{-1} \log (e^{-1}) \right)^{(\max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| - n/2\})^{-1}} & \text{if } \max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| > n/2, \ n = 2,} \\
\exp \exp \left( C e^{-1} \right) & \text{if } \max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| = n/2, \ n = 2,} \\
\exp \left( C e^{-|p_1, p_2, \cdots, p_k| - 1} \right) & \text{if } \max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| > n/2, \ n \geq 3,} \\
\exp \left( C e^{-|p_1| - 1} \right) & \text{if } \max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| = n/2, \ n \geq 3,} \\
\exp \left( C e^{-|p_1| - 1} \right) & \text{if } \max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| > n/2, \ n \geq 3,} \\
\exp \left( C e^{-|p_1| - 1} \right) & \text{if } \max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| = n/2, \ n \geq 3,} \\
\exp \left( C e^{-|p_1| - 1} \right) & \text{if } \max\{|\gamma_1, \gamma_2, \cdots, \gamma_k| > n/2, \ n \geq 3,}
\end{cases}
\]

where \( C \) is a positive constant independent of \( \varepsilon \).

Remark 2.1. This remark is to underline that in the present paper we have succeeded not only in proving the local well-posedness in the energy space and the blow-up phenomenon of small data solutions but also in catching upper bound estimates for lifespan of solutions to (1.1). We will devote to our concern in investigating global existence results for solutions to (1.1) in a forthcoming paper.

Remark 2.2. As we can see in Theorem 2.4, an open problem, which should be recognized to explore in a further study, is to find out an appropriate upper bound in estimating the lifespan of solution to (1.1) in two dimension, where the critical case is of interest and the exponents \( p_1, p_2, \cdots, p_k \) are not necessarily equal.

### 3 Proof of Theorem 2.2

In this section, we give a proof of local well-posedness in the energy space to the problem (1.1) (Theorem 2.2). For \( U_0 \in E \) and \( T > 0 \), we introduce a nonlinear mapping \( \mathcal{F} \) defined by

\[
\mathcal{F}[U](t) := \varepsilon e^{\mathcal{B}} U_0 + \int_0^t e^{(t-\tau)\mathcal{B}} \mathcal{N}(U(\tau)) d\tau
\]

for \( t \in [0, T] \). To construct a local mild solution to (2.1), we will prove that \( \mathcal{F} \) is a contraction mapping from a suitable closed set in \( E(T) \) into itself. We recall the following Sobolev embedding:

**Lemma 3.1** (Sobolev embedding). Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) which has a Lipschitz continuous boundary. Let \( p \in [1, n] \) and \( q \in [p, np/(n-p)] \) Then the embedding \( W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \) holds.
For the proof of this lemma, see [1].

Now we give a proof of Theorem 2.2.

**Proof of Theorem 2.2.** We prove the Existence part only, since the other parts can be proved in a standard manner. Let \( M \geq 2\varepsilon\|U_0\|_E \). We take \( T > 0 \) such as

\[
T \leq \frac{1}{4} \min \left\{ 1, \frac{1}{2C_\ast \sum_{\ell=1}^{k} M^{p_{\ell}-1}} \right\},
\]

where \( C_\ast > 0 \) is defined in (3.2) and (3.3) and independent of \( T \). We introduce a closed ball \( E(T, M) \) at the origin with radius \( M \) in the Hilbert space \( E(T) \) given by

\[
E(T, M) := \left\{ U \in E(T) : \|U\|_{L^2_{\infty}(0,T;E)} \leq M \right\}
\]

with a metric \( d_T : E(T) \times E(T) \rightarrow \mathbb{R}_{\geq 0} \) defined by

\[
d_T(U_1, U_2) := \|U_1 - U_2\|_{L^2_{\infty}(0,T;E)}.\]

We prove that the nonlinear mapping \( J \) given by (3.1) is a contraction mapping from \( E(T, M) \) into itself. Let \( U \in E(T, M) \). By the definitions of the nonlinear function \( N \) and the energy space \( E \) and the Sobolev embedding \( H^1(\Omega) \hookrightarrow L^{2p}(\Omega) \) (Lemma 3.1) for \( \ell = 1, \cdots, k \), the estimates

\[
\|N(U)\|_E = \|(-\partial_t u_1 + |u_k|^{p_1}, \cdots, -\partial_t u_k + |u_{k-1}|^{p_k})\|_{(L^2(\Omega))^k}
\leq \sum_{\ell=1}^{k} \|\partial_t u_{\ell}\|_{L^2(\Omega)} + \sum_{\ell=1}^{k} \|u_{\ell-1}\|_{L^{2p_{\ell}}}^{p_{\ell}} \leq \|U\|_E + \sum_{\ell=1}^{k} \|U\|_{L^2(\Omega)} \leq M + C, \sum_{\ell=1}^{k} M^{p_{\ell}} \]

(3.2)

hold for some positive constant \( C \), independent of \( T \), where we set \( u_0 := u_k \). By Lemma 2.1 and the estimates (3.2)

\[
\|J(U(t))\|_E \leq \|e^{e^{p}U(t)}\|_E + \int_0^t \|e^{(t-\tau)^{p}N(U(\tau))}\|_E d\tau \leq \varepsilon\|U_0\|_E + T \sup_{t \in (0,T)} \|N(U(t))\|_E
\]

\[
\leq \frac{M}{2} + TM + C, T \sum_{\ell=1}^{k} M^{p_{\ell}} \leq \frac{M}{2} + \frac{M}{4} + \frac{M}{8} \leq M,
\]

which implies that the mapping \( J \) is well defined from \( E(T, M) \) to itself. Let \( U, V \in E(T, M) \). We set \( U := (u_1, \cdots, u_k, \partial_t u_1, \cdots, \partial_t u_k) \) and \( V := (v_1, \cdots, v_k, \partial_t v_1, \cdots, \partial_t v_k) \). By the definitions of the nonlinear function \( N \) and the energy space \( E \), the H"older inequality and the Sobolev embedding \( H^1(\Omega) \hookrightarrow L^{2p}(\Omega) \) (Lemma 3.1) for \( \ell = 1, \cdots, k \), the inequalities

\[
\|N(U) - N(V)\|_E = \|(-\partial_t u_1 + \partial_t v_1 + (|u_k|^{p_1} - |v_k|^{p_1}), \cdots, -\partial_t u_k + \partial_t v_k + (|u_{k-1}|^{p_k} - |v_{k-1}|^{p_k}))\|_{(L^2(\Omega))^k}
\leq \sum_{\ell=1}^{k} \|\partial_t u_{\ell} - \partial_t v_{\ell}\|_{L^2(\Omega)} + \sum_{\ell=1}^{k} \|u_{\ell-1}\|_{L^{2p_{\ell}}(\Omega)}^{p_{\ell}} \leq \|U - V\|_E + C \sum_{\ell=1}^{k} \left( \|u_{\ell-1}\|_{L^{2p_{\ell}}(\Omega)} + \|v_{\ell-1}\|_{L^{2p_{\ell}}(\Omega)} \right) \|u_{\ell-1} - v_{\ell-1}\|_{L^{2p_{\ell}}(\Omega)}
\leq d_T(U, V) + C, \sum_{\ell=1}^{k} \left( \|U\|_{L^{2p_{\ell}}(\Omega)} + \|V\|_{L^{2p_{\ell}}(\Omega)} \right) \|U - V\|_E
\]

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\[
\leq d_T(U, V) + 2C_0 \sum_{\ell=1}^{k} M_{\ell}^{-1} d_T(U, V) \tag{3.3}
\]
hold, where we set \(v_0 := v_k\). By Lemma 3.1 and the estimates, the inequalities (3.3),
\[
d_T(\mathcal{J}[U], \mathcal{J}[V]) = \sup_{t \in [0, T]} \|\mathcal{J}[U](t) - \mathcal{J}[V](t)\|_E \leq \int_0^T \|e^{(t-\tau)\mathcal{B}}(\mathcal{N}(U(\tau)) - \mathcal{N}(V(\tau)))\|_E d\tau
\]
\[
\leq T \sup_{t \in [0, T]} \|\mathcal{N}(U(t)) - \mathcal{N}(V(t))\|_E \leq T d_T(U, V) + 2C_0 T \sum_{\ell=1}^{k} M_{\ell}^{-1} d_T(U, V)
\]
hold, which implies that the mapping \(\mathcal{J}\) is a contraction mapping. By the contraction mapping principle, we see that there exists a unique function \(U \in E(T, M)\) such that the identity \(\mathcal{J}[U](t) = U(t)\) holds for any \(t \in [0, T]\). By a standard argument, we can prove that \(U \in C([0, T]; E)\), which completes the proof of the theorem. \(\square\)

4 Proof of Theorem 2.4

4.1 Test function method

At first, let us introduce a test function \(\varphi = \varphi(\rho)\) having the following properties:

\[\varphi \in C_0^\infty([0, \infty)) \text{ and } \varphi(\rho) := \begin{cases} 1 & \text{if } \rho \in [0, 1/2], \\ \text{decreasing} & \text{if } \rho \in (1/2, 1), \\ 0 & \text{if } \rho \in [1, \infty). \end{cases}\]

Then, another test function \(\varphi^* = \varphi^*(\rho)\) is given by

\[\varphi^*(\rho) := \begin{cases} 0 & \text{if } \rho \in [0, 1/2), \\ \varphi(\rho) & \text{if } \rho \in [1/2, \infty). \end{cases}\]

Let \(R \in (0, \infty)\) be a large parameter. We introduce two test functions \(\phi_R = \phi_R(t, x)\) and \(\phi^*_R = \phi^*_R(t, x)\) as follows:

\[\phi_R(t, x) := \left(\varphi\left(\frac{t^2 + (|x| - 1)^4}{R^4}\right)\right)^{1+2} \text{ and } \phi^*_R(t, x) := \left(\varphi^*\left(\frac{t^2 + (|x| - 1)^4}{R^4}\right)\right)^{1+2}\]

with a positive constant \(\lambda\), which will be fixed later. In addition, we define two notations

\[Q_R := \left\{(t, x) \in (0, T) \times \Omega : t^2 + (|x| - 1)^4 < R^4\right\},\]

\[Q^*_R := \left\{(t, x) \in (0, T) \times \Omega : \frac{R^4}{2} < t^2 + (|x| - 1)^4 < R^4\right\}.

The following useful lemma comes into play in our proof in the next section.
Lemma 4.1. The following estimates hold for any \((t, x) \in Q_R:\)

\[
\begin{align*}
(i) \ \ & |\partial_t \phi_R(t, x)| \leq R^{-2}(\phi'_R(t, x))^{\frac{1}{n+1}}, \\
(ii) \ \ & |\partial^2_t \phi_R(t, x)| \leq R^{-4}(\phi'_R(t, x))^{\frac{1}{n+1}}, \\
(iii) \ \ & |\Delta \phi_R(t, x)| \leq R^{-2}(\phi'_R(t, x))^{\frac{1}{n+2}}.
\end{align*}
\]

Moreover, by taking \(\Psi = \Psi(x)\) as in (2.9) we have the further estimate as follows:

\[
(iv) \ \ & |\Delta(\Psi(x)\phi_R(t, x))| \leq R^{-2}\Psi(x)(\phi'_R(t, x))^{\frac{1}{n+2}}.
\]

Proof. First of all, a direct calculation leads to

\[
\partial_t \phi_R(t, x) = \frac{2(\lambda + 2)}{R^4} \left(\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) + \frac{t}{R^4}\right),
\]

\[
\partial^2_t \phi_R(t, x) = \frac{2(\lambda + 2)}{R^4} \left(\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) + \frac{t}{R^4}\right)^2
\]

\[
+ \frac{4(\lambda + 1)(\lambda + 2)}{R^8} \left(\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) + \frac{t}{R^4}\right)^2,
\]

and

\[
\nabla \phi_R(t, x) = \frac{4(\lambda + 2)}{R^4} \left(\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) + \frac{t}{R^4}\right)^{\frac{1}{n+1}}.
\]

\[
\Delta \phi_R(t, x) = \frac{12(\lambda + 2)}{R^4} \left(\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) + \frac{t}{R^4}\right)^{\frac{1}{n+1}}
\]

\[
+ \frac{16(\lambda + 1)(\lambda + 2)}{R^8} \left(\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) + \frac{t}{R^4}\right)^2,
\]

\[
+ \frac{16(\lambda + 2)}{R^8} \left(\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) + \frac{t}{R^4}\right)^2.
\]

Thanks to the auxiliary properties

\[
\varphi'\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) \neq 0, \ \ \varphi''\left(\frac{t^2 + \langle |x| - 1\rangle^4}{R^4}\right) \neq 0 \ \text{and} \ |x| - 1 \leq R
\]

for any \((t, x) \in Q'_R,\) we may conclude the estimates from (i) to (iii). Next, in order to verify the last estimate one observes that

\[
\Delta(\Psi(x)\phi_R(t, x)) = \Delta \Psi(x)\phi_R(t, x) + \nabla \Psi(x) \cdot \nabla \phi_R(t, x) + \Psi(x)\Delta \phi_R(t, x)
\]

\[
= \nabla \Psi(x) \cdot \nabla \phi_R(t, x) + \Psi(x)\Delta \phi_R(t, x).
\]

because of the fact \(\Delta \Psi(x) = 0.\) Noticing that

\[
\nabla \Psi(x) = \begin{cases} 
\frac{x}{|x|^2} & \text{if } n = 2, \\
(n-2)\frac{x}{|x|^n} & \text{if } n \geq 3,
\end{cases}
\]

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we obtain
\[ \nabla \Psi(x) \cdot \nabla \Phi_R(t, x) = \begin{cases} 
\frac{4(\lambda + 2)}{R^4} \cdot \frac{(|x| - 1)^2}{|x|} \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\frac{4}{n+1}} \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) & \text{if } n = 2, \\
\frac{4(n - 2)(\lambda + 2)}{R^4} \cdot \frac{(|x| - 1)^3}{|x|^{n-1}} \left( \varphi \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) \right)^{\frac{4}{n+1}} \varphi' \left( \frac{t^2 + (|x| - 1)^4}{R^4} \right) & \text{if } n \geq 3,
\end{cases} \]
which implies
\[ |\nabla \Psi(x) \cdot \nabla \Phi_R(t, x)| \lesssim R^{-2} \Psi(x) (\Phi_R^\ell(t, x))^{\frac{4}{n+1}} \]
by applying the elementary inequalities
\[ \begin{cases} 
1 - \frac{1}{|x|} \leq \log |x| & \text{if } n = 2, \\
| |x|^{n-1} + 1 \geq 2|x| & \text{if } n \geq 3,
\end{cases} \]
for any $|x| \geq 1$. Finally, using again the estimate (iii), it is obvious to obtain the estimate (iv). Hence, our proof is complete. \qed

In the following proof, we will utilize the test functions as well as the notations defined in Section 4.1.

### 4.2 Proof of Theorem 2.4

At first, we introduce the following test function:
\[ \Phi_R = \Phi_R(t, x) := \Psi(x) \phi_R(t, x), \]
which enjoys the conditions
\[ \Phi_R \in C^2([0, T) \times \Omega), \quad \text{supp} \Phi_R \subset [0, T) \times \bar{\Omega} \quad \text{and} \quad \Phi_R(t, \cdot)|_{\partial \Omega} = 0. \]

We define the following functionals with $\ell = 1, 2, \cdots, k - 1$:
\[ I_R[u_\ell] := \int_0^T \int_\Omega |u_\ell(t, x)|^{p_{\ell+1}} \Phi_R(t, x) \, dx \, dt = \int_{Q_R} |u_\ell(t, x)|^{p_{\ell+1}} \Phi_R(t, x) \, dx \, dt \]
and
\[ I_R[u_k] := \int_0^T \int_\Omega |u_k(t, x)|^{p_1} \Phi_R(t, x) \, dx \, dt = \int_{Q_R} |u_k(t, x)|^{p_1} \Phi_R(t, x) \, dx \, dt. \]

Let us assume that $(u_1, u_2, \cdots, u_k) = (u_1(t, x), u_2(t, x), \cdots, u_k(t, x))$ is a weak solution to (1.1) in the sense of Definition 2.3. By plugging $\Phi(t, x) = \Phi_R(t, x)$ into (2.4) and (2.5), taking integration by parts then we obtain
\[ I_R[u_k] + \varepsilon \int_\Omega (u_{0,1}(x) + u_{1,1}(x)) \Phi_R(0, x) \, dx \\
= \int_{Q_R} u_1(t, x) \left( \partial^2_t \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_t \Phi_R(t, x) \right) d(t, x), \quad (4.1) \]
and
\[
I_R[u_\ell] + \varepsilon \int_{\Omega} (u_{0,\ell+1}(x) + u_{1,\ell+1}(x))\Phi_R(0, x)\,dx \\
= \int_{Q_R} u_{\ell+1}(t, x) \left( \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_x \Phi_R(t, x) \right) \,dt, x
\]
with \( \ell = 1, 2, \cdots, k - 1 \), respectively. It is clear to observe that
\[
\lim_{R \to \infty} \int_{\Omega} (u_{0,\ell}(x) + u_{1,\ell}(x))\Phi_R(0, x)\,dx = \lim_{R \to \infty} \int_{\Omega} (u_{0,\ell}(x) + u_{1,\ell}(x))\Psi(x)\phi_R(0, x)\,dx \\
= \int_{\Omega} (u_{0,\ell}(x) + u_{1,\ell}(x))\Psi(x)\,dx
\]
for any \( \ell = 1, 2, \cdots, k \). Together with the conditions (2.7) and (2.8) for the initial data, we deduce that there exists a sufficiently large constant \( R_0 \) so that it holds
\[
\int_{\Omega} (u_{0,\ell}(x) + u_{1,\ell}(x))\Phi_R(0, x)\,dx \geq C^0_\ell
\]
for any \( R > R_0 \), where \( C^0_\ell \) are suitable positive, small constants with \( \ell = 1, 2, \cdots, k \). Thus, it entails immediately from (4.1) and (4.2) that
\[
I_R[u_k] + C^0_1 \varepsilon \leq \int_{Q_R} u_1(t, x) \left( \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_x \Phi_R(t, x) \right) \,dt, x
\]
and
\[
I_R[u_\ell] + C^0_{\ell+1} \varepsilon \leq \int_{Q_R} u_{\ell+1}(t, x) \left( \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_x \Phi_R(t, x) \right) \,dt, x
\]
with \( \ell = 1, 2, \cdots, k - 1 \). Employing Lemma 4.1 gives the following estimate:
\[
\left| \partial_t^2 \Phi_R(t, x) - \Delta \Phi_R(t, x) - \partial_x \Phi_R(t, x) \right| \\
= \left| \Psi(x) \partial_t^2 \phi_R(t, x) - \Delta (\Psi(x) \phi_R(t, x)) - \Psi(x) \partial_x \phi_R(t, x) \right| \\
\leq R^{-2} \Psi(x)(\phi^*_{\ell}(t, x))^{\frac{1}{p+2}}
\]
due to the relation \( 0 < \phi^*_{\ell}(t, x) < 1 \) for any \( R > R_0 \). Hence, one achieves
\[
I_R[u_k] + C^0_1 \varepsilon \leq R^{-2} \int_{Q_R} |u_1(t, x)| \Psi(x)(\phi^*_{\ell}(t, x))^{\frac{1}{p+2}} \,dt, x.
\]
The application of Hölder inequality implies
\[
I_R[u_k] + C^0_1 \varepsilon \leq R^{-2} \left( \int_{Q_R} \Psi(x)\,dx \right)^{\frac{1}{p+2}} \left( \int_{Q_R} |u_1(t, x)|^{p_1} \Psi(x)(\phi^*_{\ell}(t, x))^{\frac{p+1}{p+2}} \,dt, x \right)^{\frac{1}{p+2}} \\
\leq \Theta_{p_2}(R) \left( \int_{Q_R} |u_1(t, x)|^{p_2} \Psi(x)(\phi^*_{\ell}(t, x))^{\frac{p+1}{p+2}} \,dt, x \right)^{\frac{1}{p+2}},
\]

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where the function \( \Theta_{p_2} = \Theta_{p_2}(R) \) is defined by
\[
\Theta_{p_2}(R) = \begin{cases} 
R^2 - \frac{1}{R} \log R \left(1 - \frac{1}{R}ight) & \text{if } n = 2, \\
R^{n-\frac{n}{2}} & \text{if } n \geq 3.
\end{cases}
\]

Here we notice that to derive the previous inequality, we have used the following estimate:
\[
\int_{Q_0^2} \Psi(x) d(t, x) \leq \begin{cases} 
R^2(R + 1)^2 \log(R + 1) \approx R^4 \log R & \text{if } n = 2, \\
R^2(R + 1)^n \approx R^{n+2} & \text{if } n \geq 3,
\end{cases}
\]

for any \( R > R_0 \). Carrying out a similar way one finds
\[
I_R[u_1] + C_0^n \varepsilon \leq \Theta_{p_2}(R) \left( \int_{Q_0} |u_2(t, x)|^{p_1} \Psi(x) (\phi_R^*(t, x))^{p_2} d(t, x) \right)^{\frac{1}{p_1}},
\]
\[
I_R[u_2] + C_0^n \varepsilon \leq \Theta_{p_2}(R) \left( \int_{Q_0} |u_3(t, x)|^{p_1} \Psi(x) (\phi_R^*(t, x))^{p_2} d(t, x) \right)^{\frac{1}{p_1}},
\]
\[
\vdots
\]
\[
I_R[u_{k-1}] + C_0^n \varepsilon \leq \Theta_{p_2}(R) \left( \int_{Q_0} |u_k(t, x)|^{p_1} \Psi(x) (\phi_R^*(t, x))^{p_2} d(t, x) \right)^{\frac{1}{p_1}}.
\]

Let us now choose the parameter \( \lambda \) fulfilling
\[
\lambda \geq \max_{1 \leq l \leq k} \frac{2}{p_l - 1} = \min_{1 \leq l \leq k} \frac{2}{p_l - 1}, \quad \text{i.e.} \quad \min_{1 \leq l \leq k} \frac{2}{p_l - 1} \lambda = \frac{2}{\lambda} \geq 1
\]
so that we may arrive at the following relations:
\[
I_R[u_k] + C_0^n \varepsilon \leq \Theta_{p_2}(R) \left( \int_{Q_0} |u_1(t, x)|^{p_2} \Psi(x) \phi_R^*(t, x) d(t, x) \right)^{\frac{1}{p_1}},
\]
\[
I_R[u_1] + C_0^n \varepsilon \leq \Theta_{p_2}(R) \left( \int_{Q_0} |u_2(t, x)|^{p_1} \Psi(x) \phi_R^*(t, x) d(t, x) \right)^{\frac{1}{p_1}},
\]
\[
I_R[u_2] + C_0^n \varepsilon \leq \Theta_{p_2}(R) \left( \int_{Q_0} |u_3(t, x)|^{p_1} \Psi(x) \phi_R^*(t, x) d(t, x) \right)^{\frac{1}{p_1}},
\]
\[
\vdots
\]
\[
I_R[u_{k-1}] + C_0^n \varepsilon \leq \Theta_{p_2}(R) \left( \int_{Q_0} |u_k(t, x)|^{p_1} \Psi(x) \phi_R^*(t, x) d(t, x) \right)^{\frac{1}{p_1}}. \tag{4.3}
\]

Without loss of generality, we assume that \( \gamma_k = \max\{\gamma_1, \gamma_2, \ldots, \gamma_k\} \). A straightforward calculation gives
\[
\gamma_k = \frac{1 + p_k + p_{k-1}p_k + \cdots + p_2p_3 \cdots p_k}{\prod_{l=1}^{k} p_l - 1}.
\]

From the condition (2.6), one gains
\[
\gamma_k \geq \frac{n}{2}, \quad \text{that is,} \quad \Gamma(n, p_1, p_2, \ldots, p_k) := \gamma_k - \frac{n}{2} \geq 0. \tag{4.4}
\]

Let us now separate our consideration into two cases as follows:
Case 1: $n \geq 3$. At first, let us devote our attention to the subcritical case $\Gamma(n, p_1, p_2, \cdots, p_k) > 0$. Now we plug the above chain of estimates into (4.3) successively to achieve

$$I_R[u_{k-1}] + C_k^0 \leq R^{n-\frac{n+2}{p_1}} (I_R[u_k])^{\frac{1}{p_1}} \quad \text{(since } \phi_R(t, x) \leq \phi_R(t, x) \text{ in } Q_R)$$

$$\leq R^{n-\frac{n+2}{p_1}} \left( \int_{Q_R} |u_{1}(t, x)|^{p_2} \Psi(x) \phi_R(t, x) d(t, x) \right)^{\frac{1}{p_2}}$$

$$= R^{n-\frac{n+2}{p_1}} \left( \int_{Q_R} |u_{1}(t, x)|^{p_2} \Psi(x) \phi_R(t, x) d(t, x) \right)^{\frac{1}{p_2}}$$

which is equivalent to

$$C_k^0 \leq R^{-2\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \right)^n (1 - \frac{1}{p_1})} I_R[u_{k-1}]^{\frac{1}{p_1}} - I_R[u_{k-1}]. \quad (4.5)$$

Thus, the employment of the elementary inequality

$$A y^s - y \leq A \frac{t^s}{y} \quad \text{for any } A > 0, y \geq 0 \text{ and } 0 < s < 1$$

follows immediately

$$R^{-2\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \right)^n (1 - \frac{1}{p_1})} I_R[u_{k-1}]^{\frac{1}{p_1}} - I_R[u_{k-1}]$$

$$\leq R^{-2\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \right)^n (1 - \frac{1}{p_1})} I_R[u_{k-1}] - I_R[u_{k-1}] \quad (4.6)$$

for all $R > R_0$. Linking (4.6) and (4.7) we deduce

$$C_k^0 \leq R^{-2\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \right)^n} \quad (4.8)$$

for all $R > R_0$. Because of the assumption $\Gamma(n, p_1, p_2, \cdots, p_k) > 0$, letting $R \to \sqrt{T_k}$ in (4.8) we obtain

$$T_k \leq C_k^{\Gamma(n, p_1, p_2, \cdots, p_k)^{-1}}. \quad (4.7)$$

This is the first estimate for lifespan of solutions we wanted to prove. The next step is to focus on the critical case $\Gamma(n, p_1, p_2, \cdots, p_k) = 0$, where there exist two exponents $p_{j_1} \neq p_{j_2}$ with $j_1, j_2 \in \{1, 2, \cdots, k\}$ and $j_1 \neq j_2$. After repeating an analogous manner to (4.14), one may arrive at the following estimate:

$$I_R[u_{k-1}] + C_k^0 \leq R^{-2\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k} \right)^n (1 - \frac{1}{p_1})}$$

$$\times \left( \int_{Q_R} |u_{k-1}(t, x)|^{p_1} \Psi(x) \phi_R(t, x) d(t, x) \right)^{\frac{1}{p_1}}$$

$$= \left( \int_{Q_R} |u_{k-1}(t, x)|^{p_1} \Psi(x) \phi_R^*(t, x) d(t, x) \right)^{\frac{1}{p_1}}, \quad (4.9)$$

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where we note that the assumption $\Gamma(n, p_1, p_2, \cdots, p_k) = 0$ is used. Let us now define the following auxiliary functionals:

$$h_{p_k} = h_{p_k}(r) = \int_{Q_k} |u_{k-1}(t, x)|^{p_k} \Psi(x) \phi_t^*(t, x) d(t, x) \quad \text{and} \quad \mathcal{H}_{p_k} = \mathcal{H}_{p_k}(R) = \int_0^R h_{p_k}(r)^{r-1} dr.$$

On the one hand, carrying out the change of variable $\rho = \frac{r^2 + (|x| - 1)^4}{r}$ one derives

$$\mathcal{H}_{p_k}(R) = \int_0^R \left( \int_0^{T_x} \int_{\Omega} |u_{k-1}(t, x)|^{p_k} \Psi(x) \phi_t^*(t, x) d(t, x) \right)^{r-1} dr$$

$$= \frac{1}{4} \int_{Q_k} |u_{k-1}(t, x)|^{p_k} \Psi(x) \left( \int_{R}^{\infty} (\phi^*(\rho))^{4+2} \rho^{1-1} d\rho \right) d(t, x)$$

$$\leq \frac{1}{4} \int_{Q_k} |u_{k-1}(t, x)|^{p_k} \Psi(x) \left( \int_{1/2}^{1} (\phi^*(\rho))^{4+2} \rho^{1-1} d\rho \right) d(t, x), \quad (4.10)$$

where the support condition for $\phi^*(\rho)$ is applied to (4.10). On the other hand, using the property $\phi^*(\rho) \equiv \phi(\rho)$ for any $\rho \in [1/2, 1]$ in (4.10) we have

$$\mathcal{H}_{p_k}(R) \leq \frac{1}{4} \int_{Q_k} |u_{k-1}(t, x)|^{p_k} \Psi(x) \sup_{\rho \in (0, R)} \left( \phi \left( \frac{r^2 + (|x| - 1)^4}{r^4} \right) \right)^{4+2} \left( \int_{1/2}^{1} \rho^{1-1} d\rho \right) d(t, x)$$

$$\leq \frac{\log 2}{4} \int_{Q_k} |u_{k-1}(t, x)|^{p_k} \Psi(x) \left( \phi \left( \frac{r^2 + (|x| - 1)^4}{R^4} \right) \right)^{4+2} d(t, x)$$

$$= \frac{\log 2}{4} \int_{Q_k} |u_{k-1}(t, x)|^{p_k} \Psi(x) \phi_t(t, x) d(t, x) d(t, x) = \frac{\log 2}{4} T_R|u_{k-1}|. \quad (4.11)$$

Moreover, it is obvious to recognize the relation

$$h_{p_k}(R) = RH'_{p_k}(R). \quad (4.12)$$

Hence, collecting all the obtained estimates from (4.9) to (4.12) we conclude

$$\frac{4}{\log 2} \mathcal{H}_{p_k}(R) + C_{k, k}^{0} \leq \left( RH'_{p_k}(R) \right)^{-p_1 p_2 \cdots p_k},$$

that is,

$$R^{-1} \leq C \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(R) + C_{k, k}^{0} \right)^{-p_1 p_2 \cdots p_k} \mathcal{H}'_{p_k}(R).$$

Taking integration of both the sides of the above estimate over $[R_0, \sqrt{T_R}]$ one finds

$$\log \sqrt{T_R} - \log R_0 \leq C \int_{R_0}^{T_x} \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(R) + C_{k, k}^{0} \right)^{-p_1 p_2 \cdots p_k} \mathcal{H}'_{p_k}(R) dR$$

$$\leq \frac{C \log 2}{4(1 - p_1 p_2 \cdots p_k)} \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(R) + C_{k, k}^{0} \right)^{1-p_1 p_2 \cdots p_k} \bigg|_{R_0}^{R = \sqrt{T_R}}$$

$$\leq \frac{C \log 2}{4(p_1 p_2 \cdots p_k - 1)} \left( \frac{4}{\log 2} \mathcal{H}_{p_k}(R_0) + C_{k, k}^{0} \right)^{1-p_1 p_2 \cdots p_k}$$

$$C \left( C_{k, k}^{0} \right)^{-p_1 p_2 \cdots p_k} \log 2 \leq \frac{4}{4(p_1 p_2 \cdots p_k - 1)} e^{1-p_1 p_2 \cdots p_k},$$
which yields
\[
T_e \leq \exp \left( \frac{C (C_k^0)^{1-p_1 p_2 \cdots p_k} \log 2}{4(p_1 p_2 \cdots p_k - 1)} e^{-\left( p_1 p_2 \cdots p_k - 1 \right)} + 2 \log R_0 \right).
\]
This is to indicate the next desired estimate for lifespan of solutions. Finally, we will give our verifi-
cation to the critical case \( \Gamma(n, p_1, p_2, \cdots, p_k) = 0 \) when \( p_1 = p_2 = \cdots = p_k \). It follows that
\[
p := p_1 = p_2 = \cdots = p_k = 1 + \frac{2}{n}.
\]
Obviously, one should realize that the following relation holds:
\[
\begin{align*}
\partial_t^2 (u_1 + u_2 + \cdots + u_k) - \Delta (u_1 + u_2 + \cdots + u_k) + \partial_t (u_1 + u_2 + \cdots + u_k) \\
= |u_1|^p + |u_2|^p + \cdots + |u_k|^p \geq C |u_1 + u_2 + \cdots + u_k|^p.
\end{align*}
\]
(4.13)
For this reason, we consider the treatment of the system (1.1) as that of the single inequality (4.13).
Then, following the same approach as in the paper [13] we may arrive at
\[
T_e \leq \exp \left( C e^{-p-1} \right).
\]
- **Case 2**: \( n = 2 \). Repeating an argument as we did in the case \( n \geq 3 \) we give the following estimates:
\[
I_R [u_{k-1}] + C_{\overline{k}}^0 e \leq R^{2 - \frac{4}{p_1}} (\log R)^{1 - \frac{1}{p_1}} \left( I_R [u_k] \right)^{\frac{1}{p_1}} \quad \text{(since } \phi_k^* (t, x) \leq \phi_k (t, x) \text{ in } Q_R \text{)}
\]
\[
\leq R^{2 - \frac{4}{p_1}} (\log R)^{1 - \frac{1}{p_1}} \left( \int_{Q_R} |u_1 (t, x)|^{p_2} \Psi(x) \phi_k^* (t, x) d(t, x) \right)^{\frac{1}{p_1^{\frac{1}{p_2}}}}
\]
\[
= R^{2 - \frac{4}{p_1}} (\log R)^{1 - \frac{1}{p_1^{\frac{1}{p_2}}}} \left( \int_{Q_R} |u_1 (t, x)|^{p_2} \Psi(x) \phi_k^* (t, x) d(t, x) \right)^{\frac{1}{p_1^{\frac{1}{p_2}}}}
\]
\[
\leq R^{2 - \frac{4}{p_1}} (\log R)^{1 - \frac{1}{p_1^{\frac{1}{p_2}}}} \left( I_R [u_1] \right)^{\frac{1}{p_1^{\frac{1}{p_2}}}}
\]
\[
\vdots
\]
\[
\leq R^{2 - \frac{4}{p_1}} \cdots \left( \log R \right)^{1 - \frac{1}{p_1^{\frac{1}{p_2}}} \cdots \frac{1}{p_1^{\frac{1}{p_k-1}}}} \left( \int_{Q_R} |u_{k-1} (t, x)|^{p_k} \Psi(x) \phi_k^* (t, x) d(t, x) \right)^{\frac{1}{p_1^{\frac{1}{p_2}}} \cdots \frac{1}{p_1^{\frac{1}{p_k}}}}
\]
\[
= R^{2 - \frac{4}{p_1}} \cdots \left( \log R \right)^{1 - \frac{1}{p_1^{\frac{1}{p_2}}} \cdots \frac{1}{p_1^{\frac{1}{p_k-1}}}} \left( \int_{Q_R} |u_{k-1} (t, x)|^{p_k} \Psi(x) \phi_k^* (t, x) d(t, x) \right)^{\frac{1}{p_1^{\frac{1}{p_2}}} \cdots \frac{1}{p_1^{\frac{1}{p_k}}}}.
\]
(4.14)
We pay attention that the subcritical case is equivalent to \( \Gamma(2, p_1, p_2, \cdots, p_k) > 0 \). Then, the immediate employment of Lemma 4.2 to (4.14) with the function \( \eta(t, x) = |u_{k-1} (t, x)|^{p_k} \Psi(x) \) and the parameters
\[
\omega = C_{\overline{k}}^0 e, \quad \sigma = 2 \Gamma(2, p_1, p_2, \cdots, p_k), \quad \mu = 1 \quad \text{and} \quad p = p_1 p_2 \cdots p_k
\]
leads to
\[
\sqrt{T_e} \leq C \left( e^{-1} \log (e^{-1}) \right)^{\frac{2}{1 + 2(p_1 p_2 \cdots p_k - 1)}},
\]
which implies what we wanted to show. When the critical case occurs, i.e. \( \Gamma(2, p_1, p_2, \cdots, p_k) = 0 \), with \( p_1 = p_2 = \cdots = p_k \), we have \( p_1 = p_2 = \cdots = p_k = 2 \). Therefore, we may apply analogous strategies to the case \( n \geq 3 \) as well as in the paper [13] to conclude

\[
T_e \leq \exp \exp \left( C e^{-1} \right).
\]

Summarizing, our proof is completed.

**Appendix**

**Lemma 4.2.** Let \( \omega > 0, C_0 > 0, R_1 > 0, \sigma \geq 0 \) and \( \mu \in \mathbb{R} \). We assume \( 0 \leq \eta = \eta(t, x) \in L^1_{\text{loc}}([0, T), L^1(\Omega)) \) for \( T > R_1 \) satisfying the following inequality:

\[
\omega + \int_{Q_n} \eta(t, x) \phi_R(t, x) \, dt \, dx \leq C_0 R^{-\frac{\mu}{p-1}} \left( \int_{Q_n} \eta(t, x) \phi_R'(t, x) \, dt \, dx \right)^{\frac{1}{\mu}}
\]

for any \( R \in [R_1, \sqrt{T}) \), where \( \phi_R(t, x), \phi_R'(t, x) \)) and \( Q_R \) are introduced as in Section 4.1. Then, they hold

\[
\sqrt{T} \leq \begin{cases} 
C \omega \frac{\mu}{p-1} \left( \log (\omega^{-1}) \right)^{\frac{1}{\mu}} & \text{if } \sigma > 0 \text{ and } \mu \in \mathbb{R}, \\
\exp \left( C \omega \frac{\mu}{p-1} \right) & \text{if } \sigma = 0 \text{ and } \mu < \frac{1}{p-1}, \\
\exp \exp \left( C \omega \frac{\mu}{p-1} \right) & \text{if } \sigma = 0 \text{ and } \mu = \frac{1}{p-1},
\end{cases}
\]

where \( C \) is a positive constant independent of \( \omega \).

**Proof.** We follow the proof of Lemma 2.2 in [13] with a minor modification to conclude the desired estimates above. \( \Box \)

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