COMBINATORIALLY FORMAL ARRANGEMENTS
ARE NOT DETERMINED BY THEIR POINTS AND LINES

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Abstract. An arrangement of hyperplanes is called formal, if the relations between the hyperplanes are generated by relations in codimension 2. Formality is not a combinatorial property, raising the question for a characterization for combinatorial formality. A sufficient condition for this is if the underlying matroid has no proper lift with the same points and lines. We present an example of a matroid with such a lift but no non-formal realization, thus showing that above condition is not necessary for combinatorial formality.

1. Introduction

Let $\mathbb{K}$ be a field. An arrangement $\mathcal{A}$ is a finite collection of linear subspaces of $V = \mathbb{K}^d$ of codimension 1. Each hyperplane $H \in \mathcal{A}$ is given as the kernel of a linear functional $\alpha_H \in V^*$ that is unique up to a scalar. Let $L(\mathcal{A})$ be the collection of all nonempty intersections of hyperplanes in $\mathcal{A}$. We require $V \in L(\mathcal{A})$ as well. The set $L(\mathcal{A})$ is ordered by reverse inclusion and ranked by $r(X) = \text{codim } X$ for $X \in L(\mathcal{A})$. In fact, $L(\mathcal{A})$ has the structure of a geometric lattice, called the lattice of flats. It contains the combinatorial data of the arrangement $\mathcal{A}$ and defines the underlying matroid $M(\mathcal{A})$. Two arrangements are called (combinatorially) isomorphic if their underlying matroids are equal up to isomorphism. Any property that is invariant under such an isomorphism is called combinatorial.

Consider the linear map $\Phi : \mathbb{K}\mathcal{A} := \bigoplus_{H \in \mathcal{A}} \mathbb{K}e_H \rightarrow V^*$ defined by $\Phi(e_H) = \alpha_H$. If $\ker \Phi$ is generated by its elements of weight at most three, i.e. vectors with 3 or fewer nonzero entries, $\mathcal{A}$ is called formal, see [FR86]. In [Yuz93], Yuzvinsky showed that formality is not combinatorial, so it is natural to ask whether matroids that admit only formal arrangements can be characterized intrinsically. A matroid is called taut if it is not a proper quotient of a matroid with the same points and lines, see Definition 2.8. An arrangement with an underlying taut matroid is necessarily formal. For a survey on this topic, see [Fal02, Ch. 3]. In loc. cit., Falk asked whether there is a non-taut matroid that only admits formal arrangements as realizations. In this paper we give such an example, thus showing the following.

**Theorem 1.1.** There is a realizable matroid $M$ that is not taut such that every realization of $M$ is formal.

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Let $E$ be a finite set. A matroid $M$ on the ground set $E$ is a collection $\mathcal{B}$ of subsets of $E$ subject to

(i) $\mathcal{B} \neq \emptyset$ and

(ii) for all $B, B' \in \mathcal{B}$ and every $f \in B' \setminus B$ there is an $e \in B \setminus B'$ such that $(B' \setminus \{f\}) \cup \{e\} \in \mathcal{B}$.

An element $B \in \mathcal{B}$ is called a basis or a base of $M$. Note that all bases have the same cardinality. Any subset of a base is an independent set of $M$. Subsets of $E$ that are not independent are dependent, the minimal dependent sets are called circuits.

The rank $\text{rk}(X)$ of a subset $X \subseteq E$ is the size of a maximal independent subset of $X$, and the rank of $M$ is defined by $\text{rk}(M) = \text{rk}(E)$. There is a notion of closure on $M$ sending subsets to their maximal supersets of the same rank, i.e.

$$\text{cl}(X) := \overline{X} := \{e \in E \mid \text{rk}(X) = \text{rk}(X \cup \{e\})\}.$$  

A set $X \subseteq E$ is called closed or a flat of $M$ if $X = \overline{X}$. The set $\mathcal{L} = \mathcal{L}(M)$ of all flats is partially ordered by inclusion. It has the structure of a geometric lattice and is called the lattice of flats.

Flats of rank one (resp. two) are called points (resp. lines) of $M$. An element $e \in E$ that is dependent on its own is called a loop, two dependent elements $\{i, j\}$ are called parallel. A matroid is called simple if it has no loops or parallel elements. A matroid is completely determined by its bases, circuits, rank function, closure or the lattice of flats.

For ease of notation we write $\mathcal{L}_k$ for the elements of $\mathcal{L}$ of rank $k$, and $\mathcal{L}_k^{\geq s}$ for flats of rank $k$ and cardinality greater than $s$. We call $\mathcal{L}_{\text{rk}(E)}$ the set of copoints of $M$. In fact, the collection of copoints contains enough information to uniquely define the matroid.

**Definition 2.1.** Let $M, N$ be two matroids on the same ground set $E$. If any independent set of $M$ is independent in $N$, we call $M$ a weak map image of $N$ and write $M \prec N$. If $M$ is a weak map image of $N$ and further $\mathcal{L}(M) \subset \mathcal{L}(N)$, we call $M$ a quotient of $N$. Note that $\prec$ defines a partial order on the class of all matroids.

Let $X \subseteq E$. The deletion of $X$ from $M$ is the matroid $M - X$ on the ground set $E \setminus X$. Its independent sets are the independent sets of $M$ disjoint from $X$. The contraction of $X$ from $M$ is the matroid $M/X$ on $E \setminus X$. Its circuits are the minimal non-empty sets in $\{C \setminus X \mid C \in \mathcal{C}(M)\}$. A minor of $M$ is a matroid that arises as a sequence of deletions and contractions of $M$.

Sometimes the dependencies in $M$ can be realized as the linear dependencies of a set of vectors. Let $\text{rk}(M) = \ell$. If there is a set $A = \{v_1, \ldots, v_n\}$ of vectors of $\mathbb{K}^\ell$ such that $B \in \mathcal{B}$ if and only if $\{v_i \mid i \in B\}$ is a basis of $\mathbb{K}^\ell$, then $M$ is called $\mathbb{K}$-linear and $A$ is called a realization of $M$. Due to the next proposition, to show that a matroid $M$ is not realizable over a certain field $\mathbb{K}$, it suffices to find a minor of $M$ that is not realizable over $\mathbb{K}$.

**Proposition 2.2** ([Oxl92, Prop. 3.2.4]). If a matroid is realizable over a field $\mathbb{K}$, then all its minors are as well.

**Example 2.3.** Define $F_7$ as the matroid of rank 3 on $E = \{0, \ldots, 6\}$ with non-trivial lines

$$\mathcal{L}_2^{\geq 2}(F_7) = \{015, 024, 036, 123, 146, 256, 345\}$$

and define $F_7^-$ as the matroid of rank 3 on the same ground set $E$ with non-trivial lines

$$\mathcal{L}_2^{\geq 2}(F_7^-) = \mathcal{L}_2^{\geq 2}(F_7) \setminus \{345\}.$$
$F_7$ is called the \textit{Fano matroid} and $F_7^-$ is called the \textit{non-Fano matroid}. Pictures of the two matroids are given in Figure 1. In the pictures, every point is a point of the matroid, and three points are connected by a line segment if the three points are contained in a flat of rank two. Let $M \in \{F_7, F_7^-\}$ and let $A = (I_3 \mid X)$ be a representation of $M$ over a field $\mathbb{K}$, where $I_3$ is the $3 \times 3$ identity matrix and $X$ is a $3 \times 4$-matrix, such that the $i$-th column represents the element $i \in \{0, \ldots, 6\}$. Then

$$X = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1
\end{pmatrix}$$

and $M = F_7$ if and only if char($\mathbb{K}$) = 2 (cf. [Oxl92, Prop. 6.4.8]). Thus, the Fano matroid is only realizable over fields of characteristic two and the non-Fano matroid is only realizable over fields of characteristic different from two.

Let $V = \mathbb{K}^\ell$. A finite set $\mathcal{A} = \{H_1, \ldots, H_n\}$ with $H_1, \ldots, H_n$ (linear) hyperplanes in $V$ is called a \textit{(central) arrangement}. Choose linear forms $\alpha_i \in V^*$ such that ker($\alpha_i$) = $H_i$. Let $M = \mathcal{M}(\mathcal{A})$ be the $\mathbb{K}$-linear matroid realized by $(\alpha_1, \ldots, \alpha_n)$. It contains the combinatorial data of the arrangement $\mathcal{A}$. The lattice $L$ of $\mathcal{M}(\mathcal{A})$ is canonically isomorphic to the collection of all nonempty intersections of hyperplanes of $\mathcal{A}$. The rank of $\mathcal{A}$ is the codimension of the intersection of all its hyperplanes, i.e. $r(\mathcal{A}) = \text{codim}(\bigcap H)$. It coincides with the rank of the underlying matroid.

Probably the most studied properties of arrangements are freeness and asphericity. Let $S = S(V^*)$ be the symmetric algebra of $V^*$. The product $Q(\mathcal{A}) = \prod \alpha_i \in S$ is called the \textit{defining polynomial} of $\mathcal{A}$. Note that after chosing a basis $(e_1, \ldots, e_\ell)$ of $V$ and a dual basis $(x_1, \ldots, x_\ell)$ of $V^*$, we have $S \cong \mathbb{K}[x_1, \ldots, x_\ell]$. Let

$$\text{Der}(S) = \{ \theta : S \to S \mid \theta(fg) = f\theta(g) + g\theta(f) \text{ for all } f, g \in S \}$$

be the $S$-module of formal derivations of $S$. An arrangement is called \textit{free} if the $S$-module

$$D(\mathcal{A}) = \{ \theta \in \text{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S \}$$

is free. A complex arrangement is called \textit{aspherical} if the complement $\mathbb{C}^\ell \setminus \bigcup H$ is a $K(\pi, 1)$-space. Whether freeness and asphericity are combinatorial properties are important open problems in arrangement theory. A comprehensive summary about arrangement theory can be found in [OT92].
Definition 2.4. Let $K^A = \bigoplus_{H \in A} K e_H$ be the vector space with basis indexed by the hyperplanes in $A$ and define $\Phi : K^A \rightarrow V^*$ by $\Phi(e_H) = \alpha_H$ and linear extension. Let $F \subset \ker \Phi$ be the subspace generated by all elements of $\ker \Phi$ with at most three nonzero entries. Then $A$ is called formal if $F = \ker \Phi$.

The notion of formality first appeared in [FR86], where it was introduced as a necessary condition for asphericity. Later, Yuzvinsky showed that it is also necessary for free arrangements.

Theorem 2.5. Let $A$ be an arrangement.

(i) [FR86] If $A$ is aspherical, then it is formal.

(ii) [Yuz93] If $A$ is free, then it is formal.

Formality is not a combinatorial property. The first example in the literature is due to Yuzvinsky.

Example 2.6. [Yuz93, Ex. 2.2] Define $Q_0 = xyz(x + y + z)(2x + y + z)(2x + 3y + z)(2x + 3y + 4z)$ and define arrangements $A_1$ and $A_2$ by $Q(A_1) = Q_0 \cdot (3x + 5z)(3x + 4y + 5z)$ and $Q(A_2) = Q_0 \cdot (x + 3z)(x + 2y + 3z)$. Then the underlying matroids of $A_1$ and $A_2$ are the same, but $A_1$ is formal while $A_2$ is not.

Since formality is not combinatorial, it makes sense to ask for a property of the matroid such that each $K$-representation of it is formal.

Remark 2.7. Consider the map $\pi : V \rightarrow K^A$ defined by $\pi(x) = (\alpha_1(x), \ldots, \alpha_n(x))^T$. If $y = (y_1, \ldots, y_n) \in \ker \Phi$, consider the scalar product $\pi(x)y = \sum y_i \alpha_i(x) = \Phi(y)(x) = 0$, so $\text{Im} \pi = \ker \Phi^\perp$. Thus $\ker \Phi$ contains all the information of $A$ and $A$ can be reconstructed via

$$A \cong \{\ker \Phi^\perp \cap \{x_i = 0\} \mid i = 1, \ldots, n\}.$$ 

The same construction for $F$ yields $A_F := \{F^\perp \cap \{x_i = 0\} \mid i = 1, \ldots, n\}$, the formalization of $A$. Clearly, $r(A) \leq r(A_F)$ and $r(A) = r(A_F)$ if and only if $A$ is formal. Furthermore, it is easy to see that $M(A_F)$ is a quotient of $M(A_F)$ with the same points and lines.

Definition 2.8. [Fal02, Def. 3.5] A matroid $M$ is called taut if it is not a quotient of any matroid of higher rank with the same points and lines.

Because of Remark 2.7, a $K$-representation of a taut matroid is always formal, since its formalization cannot admit it as a proper quotient. This paper is dedicated to showing that the reverse implication is false, which answers a question raised by Falk in [Fal02]. To validate our claim, we use the theory established in [Cra70] about erections of matroids.

Definition 2.9. Let $M$ be a matroid on $E$ of rank $r > 1$. The truncation of $M$ is the (unique) matroid $T$ of rank $r - 1$ with $\mathcal{L}(T) = \mathcal{L}_{<r}(M) \cup E$. Thus, $T \prec M$. A matroid $N$ is an erection of $M$ if the truncation of $N$ is isomorphic to $M$. We further say $M$ is the trivial erection of itself.

Note that while the truncation is uniquely defined, there can be many erections of a matroid. Let $\mathcal{E}(M)$ be the collection of erections of $M$.

Theorem 2.10. [Cra70, Thm. 9] Let $M$ be a matroid, then the set $\mathcal{E}(M)$ together with the relation $\prec$ from Definition 2.1 has the structure of a geometric lattice. Its minimal element is the trivial erection $M$. Define the free erection of $M$ as the maximal element of $\mathcal{E}(M)$.
Let $M$ be a matroid on $E$ and let $k \in \mathbb{N}$. A subset $X \subset E$ is \emph{$k$-closed} if it contains the closures of all its $k$-element subsets. We say $X$ \emph{spans} $M$ if $X = M$. The following theorem characterizes erections of $M$ by their copoints.

**Theorem 2.11.** [Cra70, Thm. 2] Let $M$ be a matroid of rank $r$ on $E$. A set $F$ of subsets (called blocks) of $E$ is the set of copoints of an erection of $M$ if and only if

(i) each block spans $M$;

(ii) each block is $(r - 1)$-closed;

(iii) each basis of $M$ is contained in a unique block.

3. Proof of Theorem 1.1

Let $M$ be the simple matroid on $E = \{0, \ldots, 12\}$ of rank 3 with the following nontrivial flats in rank 2:

\[
\mathcal{L}^2_2(M) = \left\{ \{0, 3, 9\}, \{0, 4, 7\}, \{0, 5, 6\}, \{8, 9, 10\}, \{7, 10, 11\}, \{1, 4, 9\}, \{1, 3, 7\}, \{1, 5, 8\}, \{6, 9, 11\}, \{6, 10, 12\}, \{2, 5, 9\}, \{2, 3, 6\}, \{2, 4, 8\}, \{7, 9, 12\}, \{8, 11, 12\} \right\}.
\]

For a picture of $M$ see Figure 2. Note that for $X = \{0, \ldots, 8\} \subset E$, $M$ contains the underlying matroid of Example 2.6 as a minor. For the subset $Y = \{6, \ldots, 12\}$ the non-Fano matroid $F_7^-$ is also a minor of $M$, see Figure 3.

A realization of $M$ over $\mathbb{Q}$ is given by

\[
A = \begin{pmatrix}
1 & 4 & 4 & 8 & 4 & 2 & 1 & 0 & 0 & 4 & 4 & 4 & 4 \\
1 & -2 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & -5 & -5 & 5 & 5 \\
1 & 5 & -10 & 10 & 4 & -1 & 0 & 0 & 1 & 6 & -6 & -6 & 6
\end{pmatrix},
\]

![Figure 2. The matroid $M$.](image)
A arrangement $M$ can be free (since its characteristic polynomial does not factor). Furthermore, as a complex

where the $i$-th column of $A$ belongs to the element $i \in E$. We mention that no realization of $M$ can be free (since its characteristic polynomial does not factor). Furthermore, as a complex arrangement $A$ is not aspherical, since it has a simple triangle (cf. [FR86, Cor. 3.3]). We have not verified whether other realizations of $M$ are not aspherical, yet we mention that there are realizations of $M$ that do not admit a simple triangle.

Next we define the matroid $N$ of rank 4 with the same points and lines as $M$. The non-trivial flats
of rank 3 are given by

$$\mathcal{L}_3(N) = \begin{cases}
\{0,1,3,4,7,9,12\}, & \{0,4,5,6,7\}, & \{0,4,7,10,11\}, & \{0,8,11,12\}, \\
\{0,2,3,5,6,9,11\}, & \{1,2,3,6,7\}, & \{1,3,7,10,11\}, & \{1,6,10,12\}, \\
\{1,2,4,5,8,9,10\}, & \{0,1,5,6,8\}, & \{0,5,6,10,12\}, & \{2,7,10,11\}, \\
\{6,7,8,9,10,11,12\}, & \{2,3,4,6,8\}, & \{2,3,6,10,12\}, & \{3,8,11,12\}, \\
\{0,3,8,9,10\}, & \{0,2,4,7,8\}, & \{1,5,8,11,12\}, & \{4,6,10,12\}, \\
\{1,4,6,9,11\}, & \{1,3,5,7,8\}, & \{2,4,8,11,12\}, & \{5,7,10,11\}, \\
\{2,5,7,9,12\} & & & 
\end{cases}.$$ 

Furthermore, $\mathcal{L}_3(N)$ also contains every three-element subset of $E$ that is not in $\mathcal{L}_2(N) = \mathcal{L}_2(M)$ or a subset of a flat in $\mathcal{L}_3^{>3}(N)$, i.e.

$$\mathcal{L}_3^{=3}(N) = \begin{cases}
\{0,1,10\}, & \{0,2,12\}, & \{3,4,10\}, & \{3,5,12\}, \\
\{0,2,10\}, & \{1,2,12\}, & \{3,5,10\}, & \{4,5,12\}, \\
\{0,1,11\}, & \{0,1,2\}, & \{3,4,11\}, & \{3,4,5\}, \\
\{1,2,11\}, & & & \{4,5,11\}
\end{cases}.$$ 

Note that $\mathcal{L}_3(N)$ satisfies the conditions from Theorem 2.11, so $N$ is an erection of $M$. This implies that $M$ is not taut. Next we show that $N$ is the only non-trivial erection of $M$.

**Proposition 3.1.** We have $\mathcal{E}(M) = \{M,N\}$.

**Proof.** Suppose $N' \neq M$ is an erection of $M$. Then, the copoints of $N'$ have to fulfil the conditions (i)--(iii) from Theorem 2.11. The 2-closed sets with respect to $M$ that span $M$ are precisely
\( L_3(N) \cup S \), where

\[
S = \left\{ \{0,1,12\}, \{3,4,12\}, \{0,1,2,10\}, \{3,4,5,10\}, \{0,2,11\}, \{3,5,11\}, \{0,1,2,11\}, \{3,4,5,11\}, \{1,2,10\}, \{4,5,10\}, \{0,1,2,12\}, \{3,4,5,12\} \right\}.
\]

We argue that no element of \( S \) can be a copoint of \( N' \), thus implying our statement. First assume that \( X \subseteq S \) is of cardinality 3. Then \( X \) is a basis of \( M \), thus by Theorem 2.11(iii) there is a unique block \( Z \in L_3(N) \) with \( X \subseteq Z \). So if \( X \) is a copoint of \( N' \), then \( Z \) is not. Now observe that for every choice of \( X \), there are bases \( B \) of \( M \) with \( B \subseteq Z \) that are not a subset of any other possible block in \( L_3(N) \cup S \). For completeness, we specify a base for each of the seven choices for \( X \):

- if \( X = \{0,1,12\} \) or \( X = \{3,4,12\} \), then \( Z = \{0,1,3,4,7,9,12\} \) and \( B = \{0,1,3\} \).
- if \( X = \{0,2,11\} \) or \( X = \{3,5,11\} \), then \( Z = \{0,2,3,5,6,9,11\} \) and \( B = \{0,2,3\} \).
- if \( X = \{1,2,10\} \) or \( X = \{4,5,10\} \), then \( Z = \{1,2,4,5,8,9,10\} \) and \( B = \{1,2,4\} \).
- if \( X = \{6,7,8\} \), then \( Z = \{6,7,8,9,10,11,12\} \) and \( B = \{7,8,9\} \).

Finally assume that \( Y \subseteq S \) is of cardinality 4. This case reduces to the first one since there always is a \( X \subseteq S \) with \( X \subseteq Y \), so with the same reasoning as before, \( Y \) is not a copoint of \( N' \). Thus \( N' = N \).

Since \( N \) is the only non-trivial erection of \( M \), if \( \mathcal{A} \) is a non-formal arrangement with \( M = \mathcal{M}(\mathcal{A}) \), then \( N \) must be a (potentially trivial) quotient of \( \mathcal{M}(\mathcal{A}_P) \). Hence, if \( N \) is not realizable, any arrangement realizing \( M \) must be formal.

**Proposition 3.2.** The matroid \( N \) is not realizable over any field \( \mathbb{K} \).

**Proof.** First, observe that the deletion \( N - \{0, \ldots, 5\} \) is the non-Fano matroid \( F_7^{-} \), so by Proposition 2.2 and Example 2.3, \( N \) is realizable only over fields of characteristic different from 2. Furthermore, it turns out that \( F_7 \) is a minor of \( N \) as well. To see this, consider the contraction \( P = N/\{6\} \) and consider parallel elements as a single point. The points of \( P \) then are

\[
L_1(P) = \{[0,5], [1], [2,3], [4], [7], [8], [9,11], [10,12]\}
\]

and the non-trivial lines of \( P \) are

\[
L_2^{>2}(P) = \left \{ \begin{array}{ll}
\{[0,5],[1],[8]\}, & \{[1],[2,3],[7]\}, \\
\{[0,5],[2,3],[9,11]\}, & \{[1],[4],[9,11]\}, \\
\{[0,5],[4],[7]\}, & \{[2,3],[4],[8]\}, \\
\{[7],[8],[9,11],[10,12]\} & \end{array} \right \}.
\]

Thus, \( F_7 = P - \{[10,12]\} \) is a minor of \( N \), so again by Proposition 2.2 and Example 2.3, \( N \) is not realizable over any characteristic.

**Corollary 3.3.** Every realization of the matroid \( M \) is formal.

This completes the proof of Theorem 1.1.
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