Gravitational lensing and frame dragging of light in the Kerr-Newman and the Kerr-Newman (anti) de Sitter black hole spacetimes

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Abstract
The null geodesics that describe photon orbits in the spacetime of a rotating electrically charged black hole (Kerr-Newman) are solved exactly including the contribution from the cosmological constant. We derive elegant closed form solutions for relativistic observables such as the deflection angle and frame dragging effect that a light ray experiences in the gravitational fields (i) of a Kerr-Newman black hole and (ii) of a Kerr-Newman-de Sitter black hole. We then solve the more involved problem of treating a Kerr-Newman black hole as a gravitational lens, i.e. a KN black hole along with a static source of light and a static observer both located far away but otherwise at arbitrary positions in space. For this model, we derive the analytic solutions of the lens equations in terms of Appell and Lauricella hypergeometric functions and the Weierstraß modular form. The exact solutions derived for null, spherical polar and non-polar orbits, are applied for the calculation of frame dragging for the orbit of a photon around the galactic centre, assuming that the latter is a Kerr-Newman black hole. We also apply the exact solution for the deflection angle of an equatorial light ray in the gravitational field of a Kerr-Newman black hole for the calculation of bending of light from the gravitational field of the galactic centre for various values of the Kerr parameter, electric charge and impact factor. In addition, we derive expressions for the Maxwell tensor components for a Zero-Angular-Momentum-Observer (ZAMO) in the Kerr-Newman-de Sitter spacetime.

1 Introduction
One of the most fundamental exact non-vacuum solutions of the gravitational field equations of general relativity is the Kerr-Newman black hole [1]. The

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Kerr-Newman (KN) exact solution describes the curved spacetime geometry surrounding a charged, rotating black hole and it solves the coupled system of differential equations for the gravitational and electromagnetic fields [1] (see also [2]).

The KN exact solution generalized the Kerr solution [3], which describes the curved spacetime geometry around a rotating black hole, to include a net electric charge carried by the black hole.

A more realistic description should include the cosmological constant [5], [6], [7], [8], [9], [10], [11]. Taking into account the contribution from the cosmological constant $\Lambda$, the generalization of the Kerr-Newman solution is described by the Kerr-Newman de Sitter (KNdS) metric element which in Boyer-Lindquist (BL) coordinates is given by [12], [13], [14]:

$$ds^2 = \frac{\Delta_{KN}^r}{\Xi^2 \rho^2} (c dt - a \sin^2 \theta d\phi)^2 - \frac{\rho^2}{\Delta_{KN}^r} dr^2 - \frac{\rho^2}{\Delta_{KN}^\theta} d\theta^2$$  \hspace{1cm} (1)

$$\Delta_{KN}^r := 1 + \frac{a^2}{r^2} + \frac{2GM}{c^2 r} + \frac{Ge^2}{c^4},$$  \hspace{1cm} (3)

$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$  \hspace{1cm} (4)

where $a$, $M$, $e$, denote the Kerr parameter, mass and electric charge of the black hole, respectively. Also $G$ denotes the gravitational constant of Newton and $c$ the speed of light. This is accompanied by a non-zero electromagnetic field $F = dA$, where the vector potential (in units $G = c = 1$) is [16], [14]:

$$A = -\frac{er}{\Xi(r^2 + a^2 \cos^2 \theta)} (dt - a \sin^2 \theta d\phi).$$  \hspace{1cm} (5)

As a consequence the 2-form of the electromagnetic field is computed to be:

$$F = -\frac{e[r^2 + a^2 \cos^2 \theta]}{\Xi(r^2 + a^2 \cos^2 \theta)^2} dr \wedge dt - \frac{2era^2 \cos \theta \sin \theta}{\Xi(r^2 + a^2 \cos^2 \theta)^2} d\theta \wedge dt$$

$$+ \frac{a \sin^2 \theta [r^2 + a^2 \cos^2 \theta]}{\Xi(r^2 + a^2 \cos^2 \theta)^2} dr \wedge d\phi + \frac{2era \cos \theta \sin \theta (r^2 + a^2)}{\Xi(r^2 + a^2 \cos^2 \theta)^2} d\theta \wedge d\phi.$$  \hspace{1cm} (6)

In appendix C, we compute the electric and magnetic fields for the Kerr-Newman-de Sitter spacetime as perceived by the Zero-Angular-Momentum-Observers (ZAMO) or else known as the locally-non-rotating frame [18].

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For the surrounding spacetime to represent a black hole, i.e. the singularity surrounded by the horizon, the electric charge and angular momentum $J$ must be restricted by the relation:

$$\frac{GM}{c^2} \geq \left[ \left( \frac{J}{Mc} \right)^2 + \frac{Ge^2}{c^4} \right]^{1/2} \Rightarrow \quad (7)$$

$$\frac{GM}{c^2} \geq \left[ a^2 + \frac{Ge^2}{c^4} \right]^{1/2} \Rightarrow \quad (8)$$

$$e^2 \leq GM^2 (1 - a'^2) \quad (9)$$

where in the last inequality $a' = \frac{a}{aGM/c^2}$ denotes a dimensionless Kerr parameter.

The KN(a)dS dynamical system of geodesics is a completely integrable system as was shown in [15], [12], [16], [17] and the geodesic differential equations take the form:

$$\int \frac{dr}{\sqrt{R'}} = \int \frac{d\theta}{\sqrt{\Theta'}}$$

$$\rho^2 \frac{d\phi}{d\lambda} = -\frac{\Xi^2}{\Delta_\phi \sin^2 \theta} (aE \sin^2 \theta - L) + \frac{a\Xi^2}{\Delta_{KN}} \left[ (r^2 + a^2)E - aL \right], \quad (11)$$

$$c\rho^2 \frac{dt}{d\lambda} = \frac{\Xi^2 (r^2 + a^2)(r^2 + a^2)E - aL}{\Delta_{KN} \Delta_\phi} - \frac{a\Xi^2 (aE \sin^2 \theta - L)}{\Delta_\phi}, \quad (12)$$

$$\rho^2 \frac{dr}{d\lambda} = \pm \sqrt{R'}, \quad (13)$$

$$\rho^2 \frac{d\theta}{d\lambda} = \pm \sqrt{\Theta'}, \quad (14)$$

where

$$R' = \Xi^2 \left[ (r^2 + a^2)E - aL \right]^2 - \Delta_{KN} (\mu^2 r^2 + Q + 2(\mu a E)^2), \quad (15)$$

$$\Theta' = \left[ Q + (L - aE)^2 \Xi^2 - \mu^2 a^2 \cos^2 \theta \right] \Delta_\phi - \frac{2(aE \sin^2 \theta - L)^2}{\sin^2 \theta} \quad (16)$$

Null geodesics are derived by setting $\mu = 0$. In the following we use geometrized units, $G = c = 1$, unless it is stipulated otherwise.

Despite the theoretical significance of Kerr-Newman and Kerr-Newman-(anti) de Sitter black holes, and their possible application to relativistic astrophysics, there is a scarcity of exact analytic results in the integration of the test particle and photon geodesic equations particularly in the latter case.

Most of the studies of photon trajectories in the KN and KN(a)dS spacetimes have focused in the investigation of the allowed parameter space and/or the analysis of selected orbits, see for instance [25] and [12], [26] without actually

1By solving the Hamilton-Jacobi equation by the method of separation of variables.
integrating the null geodesic equations. For the case of charged particle orbits in the KN spacetime, we refer the reader to the works of [4] and [29] (see also [27]).

It is therefore the purpose of this paper, to calculate for the first time, the exact solutions of the null geodesics in KN and KN(a)dS spacetimes and produce closed form solutions for relativistic observables such as: the deflection angle and frame dragging effect that light rays experience in the curved spacetime geometry of KN and KN(a)dS spacetimes. Furthermore, we treat analytically for the first time the more involved problem in which we consider the electrically charged, rotating (KN) black hole as a gravitational lens, i.e. a KN black hole along with a static source of light and a static observer both located far away but otherwise at arbitrary positions in space.

In addition, our contribution generalises in a non-trivial way our previous analytic results for the geodesy of Kerr and Kerr-(anti) de Sitter black holes (uncharged rotating black holes) [8], [30], which constitute a special case of the more general KN and KN(a)dS black holes and thus a comparison of the relativistic observables can be made. As we shall see in the main body of the paper, the electric charge of the black hole influences the geometry of spacetime and the corresponding relativistic observables such as the deflection angle and the frame dragging effect of light in a significant way which in principle is measurable. It is pleasing that the theory produced in this work is a complete theory for the propagation of light signals in the field of rotating charged black holes: all of its fundamental parameters enter the analytic solutions on an equal footing.

The resulting theory should also be of interest for the galactic centre studies given the strong experimental evidence we have from the observation of stellar orbits (in particular from the orbits of $S$-stars in the central arcsecond of the Milky Way) and flares that the Sagittarius A* region harbours a supermassive rotating black hole with mass of 4 million solar masses [21], [22]. It will help in measuring the fundamental properties of the black hole such as its mass, spin and electric charge, the cosmological parameters as well as in measuring novel relativistic effects such as the gravitational bending of light, and the corresponding Lense-Thirring effect at the strong field regime of General Relativity.

Although it is not the purpose of this paper to discuss how a net electric charge is accumulated inside the horizon of the black hole, we briefly mention recent attempts which address the issue of the formation of charged black holes. Indeed, we note at this point, that the authors in [35], have studied the effect of electric charge in compact stars assuming that the charge distribution is proportional to the mass density. They found solutions with a large positive net electric charge. From the local effect of the forces experienced on a single charged particle, they concluded that each individual charged particle is quickly ejected from the star. This is in turn produces a huge force imbalance, in which the gravitational force overwhelms the repulsive Coulomb and fluid pressure forces. In such a scenario the star collapses to form a charged black hole before all the charge leaves the system [35]. A mechanism for generating charge asymmetry that may be linked to the formation of a charged black hole has been suggested in [36].
For some phenomenological investigations of the electric charges of some astronomical bodies in Reissner-Nordström spacetimes and of toroidal configurations in Reissner-Nordström-(anti)-de Sitter black hole and naked singularity spacetimes see [37], [38], [39].

The material of this paper is organized as follows: In section 2 we investigate the propagation of a light signal on the sphere and in particular we derive the exact solution of polar spherical null geodesics in Kerr-Newman spacetime. The solution is expressed in terms of the Weierstraß elliptic function. The amount of frame dragging for such a photon orbit is four times the real period of the Weierstraß function which is expressed in terms of Gauß’s hypergeometric function, see eqn. (37). We subsequently apply our exact solutions for computing the amount of frame dragging that a polar null spherical geodesic experiences in the gravitational field of the SgrA* galactic black hole assuming that the latter is a Kerr-Newman black hole, for various sets of values for the spin and electric charge of the singularity. In section 2.1, we derive the exact solution for the Lense-Thirring effect for a spherical polar null orbit in KN-spacetime in the presence of the cosmological constant $\Lambda$. Our solution is expressed in terms of Appell’s hypergeometric function of two-variables $F_1$, see eqn. (40).

In section 3, we compute in closed analytic form the amount of Lense-Thirring precession that a spherical non-polar orbit (i.e. with impact parameter $\Phi \neq 0$) undergoes in Kerr-Newman spacetime, in terms of Appell’s hypergeometric function and Gauß’s ordinary hypergeometric function, see our eqn. (47). We also apply our exact formula (47) for the calculation of frame dragging for spherical non-polar orbits in the gravitational field of a charged rotating (KN) black hole for various sets of values of the physical parameters. Subsequently, in section 4 we derive the generalization of formula (47) in the case where the cosmological constant is present. The closed form analytic solution that computes the Lense-Thirring effect for a non-polar spherical photon orbit in KN(a)dS spacetime is expressed elegantly in terms of Lauricella’s hypergeometric function $F_D$, Appell’s $F_1$ and Gauß’s ordinary hypergeometric function $F$, see Eqn. (55). We also compute analytically the period in the polar $\theta$-coordinate in terms of generalized hypergeometric functions, eqn.(65).

In section 5, we solve for the first time in closed analytic form the important problem of the gravitational bending of light of an equatorial null unbound geodesic in the spacetime of a charged rotating KN black hole. Our exact solution for the deflection angle, involves Lauricella’s hypergeometric function of three variables, Eqns. (81), (85). Using a uniformization of the correspondence between the coefficients of the quartic radial polynomial-that is involved in the calculation of the gravitational light deflection-and its roots we provide a closed form compact analytic solution for the four roots in terms of Weierstraß elliptic function $\wp$ and its derivative $\wp'$. Subsequently, we apply thoroughly, our analytic solution for the deflection angle (85) for the calculation of the gravitational bending of light of an equatorial ray in the spacetime geometry of a KN black hole. The deflection angle of an

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Curved spacetime geometries surrounding non-spinning charged bodies or black holes.
equatorial light ray in KNdsS spacetime is calculated in section 6.

We then solve the more involved problem of treating a Kerr-Newman black hole as a gravitational lens, i.e. a KN black hole along with a static source of light and a static observer both located far away but otherwise at arbitrary positions in space. Again, for this model we give the analytic solutions of the lens equations in terms of Appell and Lauricella hypergeometric functions and the Weierstraß modular form, see eqns. (132), (133), (134). From constraints following from the condition that a photon escapes to infinity, an equation is derived which describes the boundary of the shadow of the electromagnetic rotating black hole on the observer’s image plane, see Equations (118). Some examples of solutions of the lens equations, for particular values of the physical parameters are worked out and exhibited along with the boundary of the shadow of the KN black hole on the observer’s image plane. We use section 9 for our conclusions.

In Appendix A, we present the theory in which the roots of the particular quartic radial polynomial involved in the calculations of the photonic trajectories in the KN spacetime, are elegantly expressed in terms of the Weierstraß modular form φ and its derivative φ’. In the appendix 5.A of section 5 we derive the closed form analytic solution for the relativistic periastron advance of a non-circular equatorial timelike geodesic in the KN spacetime in terms of Appell’s hypergeometric function of two-variables F₁ and Gauß’s hypergeometric function F, eqn. (101). We applied our exact solution for the calculation of periastron advance for the observed orbits of S-stars in the central arcsecond of the Milky Way, assuming that the SgrA* galactic supermassive black hole is a Kerr-Newman black hole and that the S-stars can be treated as neutral test particles.

2 Spherical polar null geodesics in Kerr-Newman spacetime

Depending on whether or not the coordinate radius r is constant along a given geodesic, the corresponding particle orbit is characterized as spherical or non-spherical respectively. In this section, we will concentrate on spherical polar photon orbits with a vanishing cosmological constant, i.e. we solve exactly for the first time null spherical polar geodesics in the Kerr-Newman spacetime. The exact solution of the spherical timelike and null geodesics in the Kerr and Kerr -(anti) de Sitter spacetimes-i.e.when the electric charge of the rotating black hole vanishes-have been derived in references [17], [30].

Assuming a zero cosmological constant, \( r = r_f \), where \( r_f \) is a constant value setting \( \mu = 0 \) and using equations (11) – (14) we obtain

\[
\frac{d\phi}{d\theta} = \frac{\nabla^p}{\sqrt{\Theta}} - aE + \frac{l}{\sin^2 \theta} \frac{1}{\sqrt{\Theta}} 
\]

(17)
where $\Theta$ now is given by

$$\Theta = Q - \cos^2 \theta \left[-a^2 E^2 + \frac{L^2}{\sin^2 \theta}\right]$$  \hspace{1cm} (18)

and

$$\Delta^{KN} := r^2 + a^2 + e^2 - 2Mr.$$  \hspace{1cm} (19)

It is convenient to introduce the parameters:

$$\Phi := \frac{L}{E}, \quad Q := \frac{Q}{E^2}.$$  \hspace{1cm} (20)

Now by defining the variable $z := \cos^2 \theta$, the previous equation can be written as follows,

$$d\phi = \frac{1}{2} \frac{dz}{\sqrt{\alpha z^3 - z^2(\alpha + \beta) + Q z}} \times \left\{ \frac{aP}{\Delta^{KN}} - a + \frac{\Phi}{1 - z} \right\}$$  \hspace{1cm} (21)

where

$$\alpha := -a^2, \quad \beta := Q + \Phi^2.$$  \hspace{1cm} (22)

It has been shown that a necessary condition for an orbit to be polar (meaning to intersect the symmetry axis of the Kerr field) is the vanishing of the parameter $L$, i.e. $L = 0$ \[23\]. Assuming $\Phi = 0$, in equation (21), we can transform it into the Weierstraß form of an elliptic curve by the following substitution

$$z := -\xi + \frac{\alpha + \beta}{12} - \frac{\alpha}{4}.$$  \hspace{1cm} (23)

Thus we obtain the integral equation

$$\int d\phi = \int \frac{d\xi}{2} \frac{1}{\sqrt{4\xi^3 - g_2 \xi^2 - g_3}} \times \left\{ \frac{aP'}{\Delta^{KN}} - a \right\}$$  \hspace{1cm} (24)

and this orbit integral can be inverted by the Weierstraß modular Jacobi form

$$\xi = \wp \left( -\phi + \epsilon \frac{A}{A^2} \right)$$  \hspace{1cm} (25)

where $A := -\frac{1}{2} \left( \frac{aP'}{\Delta^{KN}} - a \right)$, $P' = (r^2 + a^2)$ and the Weierstraß invariants take the form

$$g_2 = \frac{1}{12} \left( \alpha + \beta \right)^2 - \frac{Q\alpha}{4}, \quad g_3 = \frac{1}{216} \left( \alpha + \beta \right)^3 - \frac{Q\alpha^2}{48} - \frac{Q\alpha\beta}{48}$$  \hspace{1cm} (26)

The parameter $\xi$ is introduced for later purposes. In terms of the original variables, the exact solution for the polar orbit of the photon ($\Phi = 0$) takes the form:

$$\varphi(-\phi + \epsilon) = \frac{\alpha''}{4} \cos^2 \theta - \frac{1}{12} (\alpha'' + \beta'') = \tilde{\xi} := \frac{\xi}{A^2}.$$  \hspace{1cm} (27)
The Weierstraß invariants are then given by

\[ g_2'' = \frac{1}{12} (\alpha'' + \beta'')^2 - \frac{Q'' \alpha''}{4} = \frac{1}{12} \left( -a^2 + Q \right)^2 + \frac{Q}{4a^2 A^4}, \]

\[ g_3'' = \frac{1}{432a^6 A^6} \left( -2a^6 - 3a^4 Q + 3a^2 Q^2 + 2Q^3 \right). \]

The analytic expressions for the three roots \( e_i, i = 1, 2, 3 \) of the cubic in Weierstraß form and invariants \( g_2', g_3' \), can be obtained either by applying the algorithm of Tartaglia and Cardano or directly from eqn. (27) using the triplet of roots of the original cubic \((z_1, z_2, z_3) = (-Q/a^2, 0, 1)\). Either procedure yields:

\[ e_1 = \frac{(a^2 + 2Q)(a^2 + e^2 + (-2 + r)r^2)}{3a^2(e^2 - 2r)^2}, \]

\[ e_2 = \frac{(a^2 - Q)(a^2 + e^2 + (-2 + r)r^2)}{3a^2(e^2 - 2r)^2}, \]

\[ e_3 = -\frac{(2a^2 + Q)(a^2 + e^2 + (-2 + r)r^2)}{3a^2(e^2 - 2r)^2}. \]

The roots are organized in the ascending order of magnitude: \( e_1 > e_2 > e_3 \). Since we are assuming spherical orbits, there are two conditions from the vanishing of the polynomial \( R(r) \) and its first derivative. Implementing these two conditions, expressions for the parameter \( \Phi \) and Carter’s constant \( Q \) are obtained:

\[ \Phi = \frac{a^2 + r^2}{a}, \quad Q = -\frac{r^4}{a^2}, \]

\[ \Phi = \frac{a^2 M + a^2 r + 2e^2 r - 3Mr^2 + r^3}{a(M - r)}, \quad Q = -r^2\left[ 4a^2(e^2 - Mr) + (2e^2 + r(-3M + r))^2 \right] / a^2 (M - r)^2 \]

However, only the second solution is physical. This is also the conditions for the photon to escape to infinity. Equations (35), for vanishing electric charge, reduce correctly to the ones obtained in the case of the Kerr black hole.

The two half-periods \( \omega \) and \( \omega' \) of the Weierstraß function are given by the following Abelian integrals (for discriminant \( \Delta^e > 0 \))

\[ \omega = \int_{e_1}^{+\infty} \frac{dt}{\sqrt{4t^3 - g_2' t^2 + g_3'}}, \quad \omega' = i \int_{-\infty}^{-e_3} \frac{dt}{\sqrt{-4t^3 + g_2' t^2 + g_3'}} \]

3Explicitly eqn (27) reads: \( \xi = \sqrt[3]{-1 + (\xi_0^2)} \), where \( \xi_0 \) is the initial value of \( \xi \).

4We also mention the following integral identity in the definition of the real half-period \( \omega \):

\[ \omega = \int_{e_3}^{+\infty} \frac{dt}{\sqrt{4t^3 - g_2' t^2 + g_3'}} = \int_{e_1}^{+\infty} \frac{dt}{\sqrt{4t^3 - g_2' t^2 + g_3'}}, \]

when all the branch points are real and the lattice rectangular. A similar integral identity holds in the definition of \( \omega' \).
Table 1: Predictions for frame dragging from the galactic black hole for a photonic spherical polar Kerr-Newman orbit, for the set of values for the Kerr parameter and electric charge: $a_{\text{Gal}} = 0.52 \frac{GM}{c^2}$, $e_{\text{Gal}} = 0.85 \sqrt{GM}$ and $a_{\text{Gal}} = 0.9939 \frac{GM}{c^2}$, $e_{\text{Gal}} = 0.11 \sqrt{GM}$.

A closed form expression for the real half-period $\omega$ of the Jacobi modular form $\wp$ is: $\omega = \frac{1}{\sqrt{e_1 - e_3}} \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{e_1 - e_3}{e_1 - e_3}\right)$, where $F(\alpha, \beta, \gamma, x)$ is the hypergeometric function of Gauss. Thus substituting the formulae for the roots in terms of the parameters and initial conditions, we obtain the analytic exact result for the half-period $\omega$:

$$\omega = \frac{1}{\sqrt{(a^2 + Q)(a^2 + e^2 + (e^2 - 2r)^2)}} \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{a^2}{a^2 + Q}\right)$$  \hspace{1cm} (37)

After a complete oscillation in latitude, the angle of longitude, which determines the amount of dragging for the spherical photon polar orbit in the general theory of relativity (GTR) for the KN black hole, increases by:

$$\Delta \phi_{pKN}^{GTR} = 4\omega.$$  \hspace{1cm} (38)

Assuming that the centre of the Milky Way is a rotating black hole with a net electric charge, i.e. the structure of the spacetime near the region SgrA*, is described by the Kerr-Newman geometry we determined the precise frame dragging (Lense-Thirring effect) of a null orbit with a spherical polar geometry. For the values of the spin of the galactic centre black hole we used values inferred by observations $[33], [34]$. Our results are displayed in Table 1.

### 2.1 Null spherical polar geodesics in Kerr-Newman black hole with the cosmological constant and Lense-Thirring effect

We now derive the closed form solution for the amount of frame dragging for a photonic spherical polar orbit in Kerr-Newman spacetime in the presence of the cosmological constant $\Lambda$, thus generalizing the results of the previous section.

The relevant differential equation is:

$$\frac{d\phi}{d\theta} = \frac{-\Xi^2 (a \sin^2 \theta - \Phi)}{\Delta \sin^2 \theta} + \frac{a \Xi^2}{\Delta \sin^2 \theta} \left[\Phi - \Xi^2 a^2 \sin^2 \theta - \Xi^2 \frac{\Phi}{\sin^2 \theta} + 2a \Phi \Xi^2\right]$$  \hspace{1cm} (39)
For $\Phi = 0$, and using the change of variables, $z = \cos^2 \theta$, $-\frac{1}{2} \frac{dz}{\sqrt{1-z}} = \text{sign}(\frac{\pi}{2} - \theta) d\theta$, after a complete oscillation in latitude, the angle of longitude $\Delta \phi_{pK\Lambda}^{GTR}$, which determines the amount of dragging for the spherical polar orbit, is given by

$$
\Delta \phi_{pK\Lambda}^{GTR} = 4 \int_{0}^{1} -\frac{1}{2} \frac{dz}{\sqrt{1-z}} \left\{ \frac{z^2 a}{(1 + \frac{4^2 a^2}{3} z) + z[Qa^2 \frac{4}{3} + a^2 \Xi^3]} \right. \\
+ \frac{a \Xi^2 (r^2 + a^2)}{\Delta_{K\Lambda}^{KN} \sqrt{Q + z[Qa^2 \frac{4}{3} + a^2 \Xi^3]}} \left. \right\}
$$

$$
= -\frac{\Xi^2 a}{2} \frac{1}{\sqrt{Q}} F_1 \left( \frac{1}{2}, 1, -\frac{a^2 \Lambda}{3}, -\frac{Qa^2 \frac{4}{3} + a^2 \Xi^3}{Q} \right) \frac{\Gamma^2(1/2)}{\Gamma(1)} + \\
\frac{a \Xi^2 (r^2 + a^2) 1}{2} \frac{1}{\Delta_{K\Lambda}^{KN} \sqrt{Q}} F_1 \left( \frac{1}{2}, 1, -\frac{Qa^2 \frac{4}{3} + a^2 \Xi^3}{Q} \right) \frac{\Gamma(1/2)}{\Gamma(1)}.
$$

(40)

For $\Lambda = 0$, $\Delta \phi_{pK\Lambda}^{GTR}$ reduces correctly to $\Delta \phi_{pK\Lambda}^{GTR}$. Indeed,

$$
\Delta \phi_{pK\Lambda}^{GTR} \rightarrow_{\Lambda=0} \Delta \phi_{pK\Lambda}^{GTR} = \frac{\pi}{2} F_1 \left( \frac{1}{2}, 1, -\frac{a^2}{Q} \right) \left\{ \frac{-a(e^2 - 2Mr)}{r^2 + a^2 + e^2 - 2Mr} \right\}.
$$

(41)

For zero electric charge, equation (41), correctly reduces to the analytic solution for frame dragging that a polar null spherical orbit experiences in Kerr spacetime [30].

3 Spherical non-polar null geodesics in Kerr-Newman spacetime

In this section and assuming vanishing cosmological constant, we shall derive for the first time the solution in closed analytic form for the amount of frame dragging that a spherical non-polar photonic orbit experiences in the gravitational field of the KN-black hole, thereby generalising the results of [30].

For this purpose we integrate the differential equation for the azimuth (39) (for $\Lambda = 0$) for $\theta$ from $\pi/2$ to a turning point of the polar polynomial. The roots $z_m, z_3$ (of $\Theta(\theta) = 0$) are expressed in terms of the integrals of motion and the cosmological constant by the expressions:

$$
z_{3,m} = \frac{Q + \Phi^2 \Xi^2 - H^2 \pm \sqrt{(Q + \Phi^2 \Xi^2 - H^2)^2 + 4H^4 Q}}{-2H^2}
$$

(42)

and

$$
H^2 := \frac{a^2 \Lambda}{3} [Q + (\Phi - a)^2 \Xi^2] + a^2 \Xi^2
$$

(43)
For $\Lambda = 0$, the turning points take the form:

$$z_m = \frac{a^2 - Q - \Phi^2 + \sqrt{4a^2Q + (-a^2 + Q + \Phi^2)^2}}{2a^2},$$

(44)

where the subscript “m” stands for “min/max”. Consequently, the change $\Delta \phi$ as $\theta$ goes through a quarter of a complete oscillation is:

$$\Delta \phi^{GTR}_{nP NK} = -\frac{1}{2} \int_0^{z_m} dz \Phi \frac{1}{z \sqrt{\alpha z^2 - z(\alpha + \beta) + Q}} - \frac{1}{2} A_{np}^{KN} \int_0^{z_m} dz \Phi \frac{1}{z \sqrt{\alpha z^2 - z(\alpha + \beta) + Q}} + \frac{\Phi}{|a|} \frac{1}{\sqrt{z_m - z_3}} \frac{1}{1 - z_m} \frac{\pi}{2} F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1 - z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right) +$$

$$+ \frac{1}{2} a^{KN} \frac{1}{|a|} \frac{A_{np}^{KN}}{\sqrt{z_m - z_3}} \frac{\pi}{2} F_1 \left( \frac{1}{2}, 1, 1, \frac{z_m}{z_m - z_3} \right).$$

(45)

where

$$A_{np}^{KN} = -a^2 \Phi - a (e^2 - 2Mr).$$

(46)

The total change in azimuth is:

$$\Delta \phi^{GTR}_{nP NK} := 4 \times \Delta \phi^{GTR}_{nP NK}.$$  

(47)

Orbits with $\Delta \phi^{GTR}_{nP NK} > 0$ are called *prograde* and orbits with $\Delta \phi^{GTR}_{nP NK} < 0$ are called *retrograde*. The differential equation relevant for the time integration is

$$\frac{dt}{d\theta} = \frac{(r^2 + a^2)((r^2 + a^2) - \Phi a)}{\Delta^{KN} \sqrt{\Theta}} + \frac{a \Phi}{\sqrt{\Theta}} \frac{a^2 \sin^2 \theta}{\sqrt{\Theta}}.$$

(48)

Integrating for $\theta$ from $\pi/2$ to a turning point we obtain

$$t = \left\{ \frac{(r^2 + a^2)((r^2 + a^2) - \Phi a)}{\Delta^{KN} \sqrt{\Theta}} + a \Phi \right\} \frac{1}{2|a|} \sqrt{\frac{1}{z_m - z_3}} \pi F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1 - z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right)$$

$$- \frac{a^2 (1 - z_m) z_m}{|a|} \frac{1}{\sqrt{z_m^2 - z_3^2}} \frac{1}{2} F_1 \left( \frac{1}{2}, -1, 1, \frac{1 - z_m}{1 - z_m}, \frac{z_m}{z_m - z_3} \right).$$

(49)

Assuming that the galactic centre region SgrA* harbours a supermassive Kerr-Newman black hole we computed the frame-dragging that a null non-polar spherical orbit experiences in a such a curved spacetime geometry. Our results are displayed in tables 2, 3.
We again integrate for the polar coordinate $\theta$ the Gauß terms of the multivariable Lauricella’s fourth hypergeometric function the KN-(a)dS spacetime. The relevant differential equation is equation (39) amount of frame-dragging that a non-polar spherical null orbit experiences in $z$ use the variable $\xi$

| Parameters    | Predicted Frame-dragging |
|---------------|---------------------------|
| $\Phi = 1$, $Q = 13.8317$, $r = 1.88356$ | $\Delta_{\text{KN}}^{\text{PNK}} = 9.05651 = 1.87 \times 10^6$ arcsec = 518.9° |
| $\Phi = -1$, $Q = 19.6421$, $r = 2.34727$ | $\Delta_{\text{KN}}^{\text{PNK}} = -4.51044 = -930345$ arcsec = -258.429° |
| $\Phi = -3$, $Q = 16.4623$, $r = 2.68908$ | $\Delta_{\text{KN}}^{\text{PNK}} = -4.88855 = -1.00834 \times 10^6$ arcsec = -280.093° |

Table 2: Predictions for frame-dragging from a galactic electrically charged rotating black hole for non-polar null spherical geodesics. The Kerr parameter is $a_{\text{Gal}} = 0.52\frac{GM_{\text{BH}}}{c^2}$ and the black hole’s electric charge: $e_{\text{Gal}} = 0.85\sqrt{G} M_{\text{BH}}$.

| Parameters    | Predicted Frame-dragging |
|---------------|---------------------------|
| $\Phi = 1$, $Q = 15.9624$, $r = 1.99709$ | $\Delta_{\text{KN}}^{\text{PNK}} = 10.8305 = 2.23396 \times 10^6$ arcsec = 620.541° |
| $\Phi = -1$, $Q = 25.7661$, $r = 2.72683$ | $\Delta_{\text{KN}}^{\text{PNK}} = -3.7207 = -767450$ arcsec = -213.181° |
| $\Phi = -3$, $Q = 25.7614$, $r = 3.22929$ | $\Delta_{\text{KN}}^{\text{PNK}} = -4.32172 = -89148$ arcsec = -247.616° |

Table 3: Predictions from a galactic electrically charged rotating black hole for non-polar null spherical geodesics. The Kerr parameter is $a_{\text{Gal}} = 0.9939\frac{GM_{\text{BH}}}{c^2}$ and the black hole’s electric charge: $e_{\text{Gal}} = 0.11\sqrt{G} M_{\text{BH}}$.

4 Frame-dragging for spherical non-polar null orbits in Kerr-Newman-(anti) de Sitter spacetime

In this section we shall derive the first solution in closed analytic form for the amount of frame-dragging that a non-polar spherical null orbit experiences in the KN-(a)dS spacetime. The relevant differential equation is equation (39). We again integrate for the polar coordinate $\theta$ from $\pi/2$ to a turning point and use the variable $z$. Indeed, we compute the relevant integrals in closed form in terms of the multivariable Lauricella’s fourth hypergeometric function $F_D$ and the Gauß’ hypergeometric function:

$$\int_{\pi/2}^{\theta_{\text{min}}} \frac{a \Xi^2}{\Delta_{\text{KN}}} \left[ (r^2 + a^2) - a \Phi \right] \frac{1}{\sqrt{H}} \, d\theta = \frac{a \Xi^2}{\Delta_{\text{KN}}} \left[ (r^2 + a^2) - a \Phi \right] \sqrt{\frac{1}{z_m - z_3}} \frac{\pi}{2} F_D \left( \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{z_m}{z_m - z_3} \right)$$

(50)

Exact integration of the other term in equation (39) yields the result:

$$\int_{\pi/2}^{\theta_{\text{min}}} - \frac{\Xi^2}{\Delta_{\theta}} \frac{(a \sin^2 \theta - \Phi)}{\sqrt{H}} \, d\theta$$

$$= \left\{ \frac{-\Xi^2 a}{2|H| (1 - \eta z_m)(1 - z_m)} \frac{1}{\sqrt{z_m^2(z_m - z_3)}} F_D \left( \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{z_m}{z_m - z_3} \right) \right\}$$

(51)
where the parameters beta of the Lauricella hypergeometric function are given by the vectors:

\[ \beta_4^A = \left( 1, 1, \frac{1}{2}, \frac{1}{2} \right), \quad \beta_5^A = \left( 1, \frac{1}{2}, \frac{1}{2} \right) \]  

(52)

and the variables by the tuples of numbers:

\[ z^{\alpha_1}_{\Lambda_0} = \left( \frac{\eta(-z_m)}{1 - \eta z_m}, \frac{-z_m}{1 - \eta z_m}, 1, \frac{z_m}{z_m - z_3} \right), \]  

(53)

\[ z^{\alpha_2}_{\Lambda_0} = \left( \frac{\eta(-z_m)}{1 - \eta z_m}, 1, \frac{z_m}{z_m - z_3} \right). \]  

(54)

Also \( \eta := -a^2 \Lambda \). Thus we get for the amount of frame dragging:

\[
\Delta \phi^{GTR}_{nPKN\Lambda} = 4 \left[ \frac{a\Xi^2}{|H|} \frac{(r^2 + a^2) - a\Phi}{\Xi^2 \Phi} \sqrt{\frac{1}{z_m - z_3}} F \left( \frac{1}{2}, 1, \frac{z_m - z_3}{z_m} \right) \right. \\
+ \frac{\Xi^2}{2|H|} \frac{z_m}{(1 - \eta z_m)(1 - z_m)} \sqrt{\frac{1}{z_m(z_m - z_3)}} F_D \left( \frac{1}{2}, \beta_4^A, \frac{3}{2}, z^{\alpha_1}_{\Lambda_0} \right) \\
- \frac{\Xi^2}{2|H|} \frac{z_m}{\sqrt{z_m(z_m - z_3)}} \frac{1}{1 - \eta z_m} F_D \left( \frac{1}{2}, \beta_3, \frac{3}{2}, z^{\alpha_2}_{\Lambda_0} \right) \left. \right] 
\]  

(55)

For \( \Phi = 0 \), equation (55), reduces correctly to the result of the previous section for spherical polar null geodesics in the presence of \( \Lambda \).

\[
\Delta \phi^{GTR}_{nPKN\Lambda} \xrightarrow{\Phi=0} 4 \left[ \frac{a\Xi^2}{|H|} \frac{(r^2 + a^2)}{\Xi^2 \Phi} \sqrt{\frac{H^2}{Q + H^2}} F \left( \frac{1}{2}, 1, \frac{1}{2}, \frac{H^2}{Q + H^2} \right) \right. \\
- \frac{\Xi^2}{|H|} \sqrt{\frac{H^2}{Q + H^2}} \frac{1}{1 + \frac{a^2\Lambda}{Q + H^2}} F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{a^2\Lambda}{Q + H^2} \right) \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma^2(1)} \left. \right] 
\]  

(56)

In going from eqn. (55) to eqn. (56), we used the property of Lauricella’s fourth hypergeometric function \( F_D \):

\[
F_D(\alpha, \beta, \beta', \beta'', \gamma, x, 1, z) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta')}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta')} \times 
F_1(\alpha, \beta, \beta', \gamma - \beta', x, z). \]  

(57)

In addition, in order to establish the equality of eqn. (56) and eqn. (50) we need the following property\(^3\) of Appell’s hypergeometric function \( F_1 \):

\[
F_1(\alpha, \beta, \beta', \gamma, x, y) = (1 - x)^{-\beta}(1 - y)^{-\beta'} F_1 \left( \gamma - \alpha, \beta, \beta', \gamma, \frac{x}{x - 1}, \frac{y}{y - 1} \right). \]  

(58)

\(^3\)Equation (58) is easily proved by using the integral representation of Appell’s function \( F_1 \) and performing the change of variables: \( u = 1 - v \) to the original variable of integration \( u \).
Furthermore, eqn. \([50]\) reduces correctly for \(\Lambda = 0\):

\[
\Delta \phi^{\text{GTR}}_{nPKN} \Phi = \Lambda = 0 \quad \frac{a}{\Delta KN} \left( \frac{r^2 + a^2}{|a|} \sqrt{\frac{a^2}{Q} + \frac{\pi}{2}} \binom{1}{2} \binom{1}{2}, \frac{a^2}{Q} + Q \right)
\]

\[
- \frac{a}{|a|} \sqrt{\frac{a^2}{Q} + \frac{\pi}{2}} \binom{1}{2} \binom{1}{2}, \frac{a^2}{Q} + Q \right)
\]

\[
\int \frac{a^2}{Q} + \frac{\pi}{2} \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{a^2}{Q} + Q \right) \sqrt{\frac{a^2}{Q} + \frac{\pi}{2}} \binom{1}{2} \binom{1}{2}, \frac{a^2}{Q} + Q \right)
\]

\[
= \Delta \phi^{\text{GTR}}_{pKN}.
\]

For the time integration we shall need the differential equation:

\[
\frac{dt}{d\theta} = \frac{a \Xi^2 (r^2 + a^2) (r^2 + a^2) - \Phi a}{\pm \Delta KN \sqrt{\Theta}} - \frac{a \Xi^2 (a \sin^2 \theta - \Phi)}{\pm \Delta \varphi \sqrt{\Theta}}
\]

(60)

Again integrating exactly for the polar angle \(\theta\), from \(\theta = \pi/2\) to a turning point and using the variable \(z\), yields the following results:

\[
\int_{\pi/2}^{\theta_{\min}/\max} \frac{a \Xi^2 \sin^2 \theta}{\pm \Delta \varphi \sqrt{\Theta}} d\theta = -a \Xi^2 \sqrt{\frac{1}{2}(1 - z_m) \frac{z_m}{m} \frac{1}{m}} F_D \left( \frac{1}{2}, \frac{3}{2}, 2, \frac{z_{\Lambda_0}^3}{z_m} \right)
\]

\[
= -a \Xi^2 \sqrt{\frac{1}{2}(1 - z_m) \frac{z_m}{m} \frac{1}{m}} \frac{\Gamma \left( \frac{3}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma^2(1)} \times
\]

\[
F_D \left( \frac{1}{2}, \beta_3^{10}, 1, z \beta_4^{11} \right),
\]

(61)

where we have defined the tuples for the beta parameters of Lauricella’s function as:

\[
\beta_4^{11} = \left( 1, -1, \frac{1}{2}, \frac{1}{2} \right), \beta_3^{10} = \left( 1, -1, \frac{1}{2} \right),
\]

(62)

and the variable tuple:

\[
z \beta_4^{11} = \left( \eta(z_m), -z_m, \frac{z_m}{m}, \frac{z_m}{z_m - z_3} \right).
\]

(63)

Likewise exact integration of the term \(\int_{\pi/2}^{\theta_{\min}/\max} \frac{a \Xi^2 \Phi}{\pm \Delta \varphi \sqrt{\Theta}} d\theta\) yields the analytic result:

\[
\int_{\pi/2}^{\theta_{\min}/\max} \frac{a \Xi^2 \Phi}{\pm \Delta \varphi \sqrt{\Theta}} d\theta = a \Xi^2 \Phi \frac{1}{|H|} \left( \frac{1}{1 - \eta z_m} \frac{1}{m} \frac{1}{z_m - z_3} \right) \frac{1}{2} F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, 1, \eta(z_m), \frac{z_m}{z_m - z_3} \right).
\]

(64)
Thus we obtain for the period in the $\theta$ coordinate the exact result:

$$t = 4 \times \left[ \frac{\Xi^2 (r^2 + a^2)}{|H|} \left[ \frac{1}{\Delta_{KN} \sqrt{R}} \right] \sqrt{\frac{1}{z_m - z_3}} \right] \frac{\pi F_1 \left( \frac{1}{2}, 1, \frac{1}{2}, 1, \frac{1}{\eta z_m}, \frac{z_m}{z_m - z_3} \right)}{2} + a \Xi^2 \Phi \frac{1}{|H|} \frac{1}{1 - \eta z_m} \sqrt{z_m - z_3} \frac{1}{2} F_1 \left( \frac{1}{2}, 1, 1, 1, \frac{\eta (z_m - z_3)}{z_m - z_3} \right)$$

$$- a^2 \Xi^2 \frac{1}{2 |H|} \frac{(1 - z_m) z_m}{(1 - \eta z_m) \sqrt{z_m - z_3}} \frac{1}{\sqrt{z_m - z_3}} \frac{1}{2} F_1 \left( \frac{1}{2}, \frac{3}{4}, \frac{3}{2}, z_{AB} \right)$$

(65)

5 Light deflection of an equatorial unbound Kerr-Newman orbit

In this section, we are going to calculate for the first time the exact analytic solution for the bending of light for an equatorial unbound null geodesic in the gravitational field of an electrically charged rotating black hole.

For equatorial geodesics the parameter $Q$ vanishes and the relevant differential equation for the exact computation is the following:

$$\frac{d\phi}{dr} = e^{2 (\Phi - a)} + \Phi r^2 - 2 Mr (\Phi - a)$$

$$= \frac{(r^2 + a^2) \sqrt{\Delta_{KN} R}}{\Delta_{KN} \sqrt{R}},$$

(66)

where the quartic radial polynomial has the form:

$$R = r^4 + r^2 (a^2 - \Phi^2) + 2 Mr (a^2 + \Phi^2 - 2 \Phi a) - e^2 (\Phi - a)^2.$$  

(67)

We shall compute the following integral applying the partial fractions technique:

$$\int d\phi = \int \frac{e^{2 (\Phi - a)} + \Phi r^2 - 2 Mr (\Phi - a)}{(r^2 + a^2) \sqrt{\Delta_{KN} R}} dr$$

$$= \int \frac{\Phi}{\sqrt{R}} dr + \int \frac{A_{eq^{KN}}^{KN}}{(r - r_+)} \sqrt{\Delta_{KN} R} dr + \int \frac{A_{eq^{KN}}^{KN}}{(r - r_-)} \sqrt{\Delta_{KN} R} dr$$

(68)

In order to calculate the deflection angle from the previous radial integral we need to integrate from the distance of closest approach (e.g., from the maximum positive root of the quartic) to infinity. Thus $\Delta_{KN}^{\phi^{GR}} = 2 \int_{\alpha}^{\infty}$. We manipulate a bit further the terms in equation (66). In particular:

$$\frac{\Phi r^2}{\Delta_{KN} \sqrt{R}} - \frac{2 Mr (\Phi - a)}{\Delta_{KN} \sqrt{R}} = \Phi \left[ \frac{1}{\sqrt{R}} - \frac{(a^2 + e^2 - 2 Mr)}{\Delta_{KN} \sqrt{R}} \right]$$

$$= \Phi \left[ \frac{1}{\sqrt{R}} - \frac{(a^2 + e^2 - 2 Mr)}{\Delta_{KN} \sqrt{R}} \right]$$

(69)

We organize all roots in ascending order of magnitude as follows,

$$\alpha_\mu > \alpha_\nu > \alpha_i > \alpha_\rho$$

(71)
where \( \alpha_\mu = \alpha_{\mu+1}, \alpha_\nu = \alpha_{\mu+2}, \alpha_\rho = \alpha_\mu \) and \( \alpha_i = \alpha_{\mu-i}, i = 1, 2, 3 \) and we have that \( \alpha_{\mu-1} \geq \alpha_{\mu-2} > \alpha_{\mu-3} \). By applying the transformation

\[
r' = \frac{\omega z \alpha_{\mu+2} - \alpha_{\mu+1}}{\omega z - 1}
\]

or equivalently

\[
z = \frac{\alpha_{\mu} - \alpha_{\mu+2}}{\alpha_{\mu} - \alpha_{\mu+1}} \left( \frac{r' - \alpha_{\mu+1}}{r' - \alpha_{\mu+2}} \right)
\]

where

\[
\omega := \frac{\alpha_{\mu} - \alpha_{\mu+1}}{\alpha_{\mu} - \alpha_{\mu+2}}
\]

and after a dimensionless variable \( r' \) through \( r = r'M_{BH} \) has been introduced, we can bring our radial integrals into the familiar integral representation of Lauricella’s \( F_D \) and Appell’s hypergeometric function \( F_1 \) of three and two variables respectively \(^6\). We denote the roots of the quartic (67) by \( \alpha, \beta, \gamma, \delta : \alpha > \beta > \gamma > \delta \), while we define:

\[
H^\pm = \sqrt{\omega(\alpha_{\mu+1} - \alpha_{\mu+2})(\alpha_{\mu+1} - \alpha_{\mu+1})} \sqrt{\alpha_{\mu+1} - \alpha_{\mu+1}} \sqrt{\alpha_{\mu+1} - \alpha_{\mu-3}}
\]

The radii of the (dimensionless) event \((r'_+)^7\) and Cauchy \((r'_-)\) horizons are given by:

\[
r'_\pm = 1 \pm \sqrt{1 - a^2 - \epsilon^2}
\]

\[
\Delta \phi_{CT}^{KN} = 2 \left[ -2A_+^{eq KN} \sqrt{\omega(\alpha_{\mu+1} - \alpha_{\mu+2})} F_D \left( \frac{1}{2}, 1, 1, \frac{1}{2} \right) \frac{2}{3} \frac{2}{3} \frac{2}{3} \omega^2 \kappa^2_{+}, \mu^2 \right]
\]

\[
+ \frac{A_+^{eq KN}}{H^+} \left[ -2A_+^{eq KN} \sqrt{\omega(\alpha_{\mu+1} - \alpha_{\mu+2})} F_D \left( \frac{3}{2}, 1, 1, \frac{1}{2} \right) \frac{1}{5} \frac{1}{5} \frac{1}{5} \omega^2 \kappa^2_{+}, \mu^2 \right] \frac{\Gamma(3/2) \Gamma(1)}{\Gamma(5/2)}
\]

\[
+ \frac{A_-^{eq KN}}{\omega(\alpha_{\mu+1} - \alpha_{\mu+2})} F_D \left( \frac{1}{2}, 1, 1, \frac{1}{2} \right) \frac{3}{1} \frac{3}{1} \frac{3}{1} \omega^2 \kappa^2_{-}, \mu^2 \right] \frac{\Gamma(3/2) \Gamma(1)}{\Gamma(5/2)}
\]

\[
+ 2\Phi \frac{\Gamma(1/2) \Gamma(1)}{\Gamma(3/2)} \left( \frac{1}{\sqrt{(\gamma - \alpha)(\delta - \alpha)}} \right) F_1 \left( \frac{1}{2}, 1, 1, \frac{1}{2} \frac{1}{2} \frac{1}{2} \omega^2 \mu^2 \right)
\]

\[
(74)\]

\[^6\text{See Appendix B for the integral representation that the Appell-Lauricella hypergeometric function admits. We also have the correspondence } \alpha_{\mu+1} = \alpha_\mu, \alpha_{\mu+2} = \beta, \alpha_{\mu-1} = r'_+, \alpha_{\mu-2}, \alpha_{\mu-3} = \gamma, \alpha_{\mu} = \delta.\]

\[^7\text{In the usual units: } r_\pm = \frac{GM_{BH}}{c^2} \pm \sqrt{\left( \frac{GM_{BH}}{c^2} \right)^2 - \left( \frac{GM_{BH}}{c^2} \right)^2}.\]
where

\[
\begin{align*}
\frac{1}{\omega} &= \frac{\alpha - \alpha + 2}{\alpha - \alpha + 1} = \frac{\delta - \beta}{\delta - \alpha}, \quad (78) \\
\kappa_{\pm}^2 &= \frac{\alpha + 2 - \alpha_{\pm - 1}}{\alpha_{\pm + 1} - \alpha_{\pm - 1}} = \frac{\beta - r_{\pm}'}{\alpha - r_{\pm}'}, \quad (79) \\
\mu_{\pm}^2 &= \frac{\alpha + 2 - \alpha_{-3}}{\alpha_{+1} - \alpha_{-3}} = \frac{\beta - \gamma}{\alpha - \gamma}. \quad (80)
\end{align*}
\]

An equivalent expression is the following:

\[
\Delta \phi_{eKN}^{GTR} = 2 \left[ -\frac{2A_{+}^{eqKN}}{H^+} \frac{\sqrt{\omega}(\alpha_{+1} - \alpha_{+2})}{H^+} F_D \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right) F_1 \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right) \right]
\]

\[
+ \frac{A_{+}^{eqKN}}{H^+} \frac{\sqrt{\omega}(\alpha_{+1} - \alpha_{+2})}{H^+} \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right) \left[ F_D \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right) \right] F_1 \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right) \right]
\]

\[
+ \frac{1}{\kappa_{\pm}^2} F_D \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right) \left[ F_1 \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right) \right] \right]
\]

\[
+ 2\Phi \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)} \left( \frac{1}{\sqrt{(\gamma - \alpha)(\delta - \alpha)}} \right) F_1 \left( \frac{1}{2}, \beta_a^{\pm}, \frac{3}{2}, \frac{1}{\kappa_{\pm}^2} \right), \quad (81)
\]

where

\[
\begin{align*}
A_{+}^{eqKN} &= \frac{e^2(\Phi - a)}{2\sqrt{1 - a^2 - e^2}} + \frac{-2a(1 + \sqrt{1 - a^2 - e^2}) + \Phi(a^2 + e^2)}{-2\sqrt{1 - a^2 - e^2}}, \quad (82) \\
A_{-}^{eqKN} &= \frac{e^2(\Phi - a)}{-2\sqrt{1 - a^2 - e^2}} + \frac{-2a(1 - \sqrt{1 - a^2 - e^2}) - \Phi(a^2 + e^2)}{-2\sqrt{1 - a^2 - e^2}}, \quad (83)
\end{align*}
\]

In going from (77) to (81) we used the identity proven in [30] (eqn.(52)). The tuples of numbers for the beta parameters of the generalized hypergeometric functions in Equation (81) are defined in Equation (53) of [31]. Also we defined:

\[
\begin{align*}
z_{\pm}^r &= \left( \frac{1}{\omega}, \kappa_{\pm}^2, \mu_{\pm}^2 \right), \\
z_{A}^r &= \left( \frac{1}{\omega}, \mu_{\pm}^2 \right). \quad (84)
\end{align*}
\]

The angle of deflection \(\delta\) of light rays from the gravitational field of a galactic electrically charged rotating black hole or a massive charged rotating star is
defined to be the deviation of $\Delta \phi_{eKN}^{CTR}$ from $\pi$

$$\delta_{eKN} := \Delta \phi_{eKN}^{CTR} - \pi.$$  \hspace{1cm} (85)

The four roots of the quartic $67$, in terms of which the variables of the generalized hypergeometric functions of Appell and Lauricella’s are written in Equation 81, can be expressed in a very elegant compact closed analytic form in terms of the Weierstraß function $\wp(x, g_2, g_3)$ and its derivative. Here, we present the formulae and their derivation is relegated to the appendix A.

$$\alpha = \frac{1}{2} \wp'(-x_1/2 + \omega) - \wp(x_1),$$  \hspace{1cm} (86)

$$\beta = \frac{1}{2} \wp'(-x_1/2 + \omega + \omega') - \wp(x_1),$$  \hspace{1cm} (87)

$$\gamma = \frac{1}{2} \wp'(-x_1/2 + \omega') - \wp(x_1),$$  \hspace{1cm} (88)

$$\delta = \frac{1}{2} \wp'(-x_1/2) - \wp(x_1),$$  \hspace{1cm} (89)

where the point $x_1$ is defined by the equation:

$$a^2 - \Phi^2 = -6\wp(x_1),$$  \hspace{1cm} (90)

and $\omega, \omega'$ denotes the half-periods of the elliptic function $\wp$. The equations

$$2(a - \Phi)^2 = 4\wp'(x_1), -3\wp^2(x_1) + g_2 = -e^2(\Phi - a)^2$$  \hspace{1cm} (91)

determine the Weierstraß invariants $(g_2, g_3)$ with the result:

$$g_2 = \frac{1}{12} (a^2 - \Phi^2)^2 - e^2(\Phi - a)^2,$$  \hspace{1cm} (92)

$$g_3 = -\frac{1}{216} (a^2 - \Phi^2)^3 - \frac{1}{4} (a - \Phi)^4 - e^2(\Phi - a)^2 \left(\frac{a^2 - \Phi^2}{6}\right).$$  \hspace{1cm} (93)

We are working with dimensionless parameters, effectively setting $M = 1$.

We computed using our exact analytic formula for the deflection angle, eqn. 67, the gravitational bending of light of an unbound equatorial ray in the gravitational field of a galactic KN-black hole, for different choices of the spin and electric charge of the black hole and the impact factor $\Phi$. We display our results in Tables 4, 5, 6 and Figures 1, 2, 3, 4, 5, 6, 7. The values for the Kerr parameter in Tables 4, 5, as we mentioned in the introduction, are in accordance with the central values reported for the spin of the Galactic centre black hole SgrA* from observation of near infrared periodic flares 33 and X-ray flares 34.

In Figures 3, 4 we display 3-d plots of the deflection angle $\delta_{eKN}$ as a function of the impact factor and the electric charge, for two fixed values of the Kerr parameter. From these plots we observe that the smaller the Kerr parameter
Figure 1: Plot of the deflection angle $\delta_{eKN}$ versus the electric charge $|e|$, for fixed parameters $\Phi = 10, a = 0.52$.

Figure 2: Plot of the deflection angle $\delta_{eKN}$ versus the impact factor $\Phi$, for fixed parameters $|e| = 0.85, a = 0.52$. 
## Predicted deflection

In figures 6, 7 we plot the deflection angle \( \delta \) versus the maximal root \( \alpha^{\pm} \) of the quartic for fixed Kerr (spin) parameter for two different values of the electric charge, for smaller values of the impact parameter \( \Phi \). We observe from the analysis that for a fixed small distance \( \alpha \) there is a strong dependence of the deflection angle on the electric charge the black hole carries: the larger the electric charge \( e \), the smaller \( \delta eKN \).

In figures 6, 7 we plot the deflection angle \( \delta eKN \) versus the maximal root \( \alpha^{\pm} \) of the quartic for fixed Kerr (spin) parameter for two different values of the electric charge. In figure 6 we fix the Kerr parameter to the value \( a = 0.52 \) and the two different choices for electric charge: \( e = 0.11 \), \( e = 0.85 \). We observe from the analysis that for a fixed small distance \( \alpha \) there is a strong dependence of the deflection angle on the electric charge the black hole carries: the larger the electric charge \( e \), the smaller \( \delta eKN \).

In figure 6 the values of the electric charge, for the SgrA* galactic black hole correspond to:

\[
e = 0.85 \sqrt{6.6743 \times 10^{-8} \cdot 4.06 \times 10^6 \times 1.9884 \times 10^{33} \text{esu}} = 1.77 \times 10^{36} \text{esu} \approx 5.94 \times 10^{26} \text{C},
\]

\[
e = 0.11 \sqrt{6.6743 \times 10^{-8} \cdot 4.06 \times 10^6 \times 1.9884 \times 10^{33} \text{esu}} = 2.29 \times 10^{35} \text{esu} \approx 7.65 \times 10^{25} \text{C}.
\]

Concerning the tentative values for the electric charge \( e \) we used in applying our exact solutions for the case of SgrA* black hole we note that their likelihood is debatable: There is an expectation that the electric charge trapped in the

| Parameters | Predicted deflection |
|------------|----------------------|
| \( a_{Gal} = 0.52, e = 0.4, \Phi = 5 \) | \( \delta_{eKN} = 1.7161 = 98.3254^\circ = 353971 \text{arcsec} \) |
| \( a_{Gal} = 0.52, e = 0.4, \Phi = 10 \) | \( \delta_{eKN} = 0.529162 = 30.3188^\circ = 109148 \text{arcsec} \) |
| \( a_{Gal} = 0.52, e = 0.4, \Phi = 15 \) | \( \delta_{eKN} = 0.317395 = 18.1854^\circ = 65467.3 \text{arcsec} \) |
| \( a_{Gal} = 0.52, e = 0.4, \Phi = 20 \) | \( \delta_{eKN} = 0.226997 = 13.006^\circ = 46821.5 \text{arcsec} \) |
| \( a_{Gal} = 0.52, e = 0.4, \Phi = 40 \) | \( \delta_{eKN} = 0.106251 = 6.08775^\circ = 21915.9 \text{arcsec} \) |

Table 4: Predictions for light deflection from a galactic electrically charged rotating black hole with Kerr parameter \( a_{Gal} = 0.52 \frac{GM_{BH}}{c^2} \) and the electric charge parameter \( e = 0.4 \). The values of the impact parameter \( \Phi \) are in units of \( \frac{GM_{BH}}{c^2} \).

| Parameters | Predicted deflection |
|------------|----------------------|
| \( a_{Gal} = 0.9939, e = 0.11, \Phi = 5 \) | \( \delta_{eKN} = 1.3125 = 75.2005^\circ = 270722 \text{arcsec} \) |
| \( a_{Gal} = 0.9939, e = 0.11, \Phi = 10 \) | \( \delta_{eKN} = 0.496749 = 28.4616^\circ = 102461.7 \text{arcsec} \) |
| \( a_{Gal} = 0.9939, e = 0.11, \Phi = 15 \) | \( \delta_{eKN} = 0.306428 = 17.5571^\circ = 63205.4 \text{arcsec} \) |
| \( a_{Gal} = 0.9939, e = 0.11, \Phi = 20 \) | \( \delta_{eKN} = 0.221555 = 12.6942^\circ = 45699.02 \text{arcsec} \) |
| \( a_{Gal} = 0.9939, e = 0.11, \Phi = 40 \) | \( \delta_{eKN} = 0.105111 = 6.02241^\circ = 21680.7 \text{arcsec} \) |

Table 5: Predictions for light deflection from a galactic electrically charged rotating black hole with Kerr parameter \( a_{Gal} = 0.9939 \frac{GM_{BH}}{c^2} \) and the electric charge parameter \( |e|/\sqrt{G M_{BH}} = 0.11 \). The values of the impact parameter \( \Phi \) are in units of \( \frac{GM_{BH}}{c^2} \).
Table 6: Predictions for light deflection from a galactic electrically charged rotating black hole with Kerr parameter $a_{Gal} = 0.26$, $e = 0.91$, $\Phi = 5^\circ$, $\delta_{eKN} = 1.57038 = 89.9759^\circ = 323913$ arcsec $a_{Gal} = 0.26$, $e = 0.91$, $\Phi = 10^\circ$, $\delta_{eKN} = 0.518656 = 29.7168^\circ = 106980$ arcsec $a_{Gal} = 0.26$, $e = 0.91$, $\Phi = 15^\circ$, $\delta_{eKN} = 0.313795 = 17.9791^\circ = 64724.8$ arcsec $a_{Gal} = 0.26$, $e = 0.91$, $\Phi = 20^\circ$, $\delta_{eKN} = 0.225192 = 12.9026^\circ = 46449.3$ arcsec $a_{Gal} = 0.26$, $e = 0.91$, $\Phi = 40^\circ$, $\delta_{eKN} = 0.105866 = 6.06567^\circ = 21836.4$ arcsec

Figure 3: Plot of the deflection angle $\delta_{eKN}$ as a function of the parameters $\Phi$, $e$ for fixed Kerr parameter $a = 0.52$. 
Figure 4: Plot of the deflection angle $\delta_{eKN}$ as a function of the parameters $\Phi, e$ for fixed Kerr parameter $a = 0.9939$.

Figure 5: Plot of the deflection angle $\delta_{eKN}$ versus the impact factor $\Phi$, for fixed parameters $|e| = 0.11, a = 0.9939$. 
Figure 6: The deflection angle $\delta_{KN}$ versus the root $\alpha$ for fixed Kerr parameter $a = 0.52$ for two different values of the electric charge. The values of the electric charge are in units of $\sqrt{GM_{BH}}$.

5.A Exact solution for the periastron advance in Kerr-Newman spacetime

In this appendix of the section we derive the closed form solution for the periastron advance for a timelike equatorial non-circular orbit and apply it to the computation of this relativistic effect for the observed orbits of $S-$stars in the central arcsecond of our Galaxy assuming that the galactic centre region Sagitarrius A* harbours a supermassive rotating black hole in which a net electric charge is present.

galactic nucleous will not likely reach so high values as the ones close to the extremal values predicted in (9) that allow the avoidance of a naked singularity. However, more precise statements on the electric charge’s magnitude of the galactic black hole or its upper bound will only be reached once the relativistic effects predicted in this work are measured and a comparison of the theory we developed with experimental data will take place.

---

8In this regard, we also mention that the author in [37], under the assumption that the curved geometry surrounding the massive object in the Galactic Centre is a Reissner-Nordström (RN) spacetime, obtained an upper bound of $e \lesssim 3.6 \times 10^{27}$ C. This upper bound does not distinguish yet between a RN black hole scenario and a RN naked singularity scenario.
Figure 7: The deflection angle $\delta_{eKN}$ versus the root $\alpha$ for a fixed Kerr parameter $a = 0.26$ for two different values of the electric charge. The values of the electric charge are in units of $\sqrt{GM_{BH}}$. 
charge has been trapped inside the event horizon $r_+$, to form a charged KN black hole.

The relevant differential equation is given by:

$$\frac{d\phi}{dr} = \frac{e^2(L - aE) + Lr^2 - 2Mr(L - aE)}{\pm(r^2 + a^2 + e^2 - 2Mr)\sqrt{R}},$$

where the quartic polynomial is given by

$$R = [(r^2 + a^2)E - aL]^2 - \Delta_{KN} [r^2 + (L - aE)^2]$$
$$= r^4(E^2 - 1) + 2Mr^3 + r^2(a^2E^2 - L^2 - e^2 - a^2)$$
$$+ 2Mr[L^2 + a^2E^2 - 2aLE] - e^2L^2 - E^2e^2a^2 + 2ae^2EL.$$

Using again the partial fractions technique and performing similar manipulations as in eqn. (70), we integrate from the periastron distance $r_P$ to the apoastron distance $r_A$:

$$\Delta\phi_{GR_{KN}}^{GT} = \int_{r_P}^{r_A} \frac{e^2(L - aE) + Lr^2 - 2Mr(L - aE)}{(r^2 + a^2 + e^2 - 2Mr)\sqrt{R}} dr$$

$$= \int_{r_P}^{r_A} \frac{L}{\sqrt{R}} dr + \int_{r_P}^{r_A} \frac{A_{teKN}^+}{(r-r_+)\sqrt{R}} dr + \int_{r_P}^{r_A} \frac{A_{teKN}^-}{(r-r_-)\sqrt{R}} dr,$$

where

$$A_{teKN}^\pm = \frac{\pm L(a^2 + e^2) \mp 2aEr'_\pm \mp e^2(L - aE)}{-2\sqrt{1 - a^2 - e^2}}.$$

Applying the transformation:

$$z = \frac{1}{\omega} \frac{r' - \alpha_{\mu+1}}{r' - \alpha_{\mu+2}} = \frac{\alpha - \gamma r' - \beta}{\alpha - \beta r' - \gamma}$$

and organizing the roots of the radial polynomial and the radii of the event and Cauchy horizon in the ascending order of magnitude

$$\alpha_\rho > \alpha_\sigma > \alpha_\nu > \alpha_i,$$

with the correspondence $\alpha_\rho = \alpha_\mu = \alpha, \alpha_\sigma = \alpha_{\mu+1} = \beta, \alpha_\nu = \alpha_{\mu+2} = \gamma, \alpha_i = \alpha_{\mu-i}, i = 1, 2, 3, \alpha_{\mu-1} = a_{\mu-2} = r'_\pm, \alpha_{\mu-3} = \delta$ we compute the exact analytic result in terms of Appell’s hypergeometric function $F_1$ and Gauß’s ordinary
hypergeometric function:

\[
\Delta \phi_{\text{GTR}} = \frac{2}{\sqrt{\omega L}} F\left(\frac{1}{2}, \frac{1}{2}, 1, \kappa'^2\right) \pi
\]

The variables of the hypergeometric functions are given in terms of the roots of the quartic and the radii of the horizons by the expressions:

\[
\kappa^2 \pm := \frac{\alpha - \beta \mp - \gamma}{\alpha - \gamma \mp - \beta} \quad \kappa'^2 := \frac{\alpha - \beta \delta - \gamma}{\alpha - \gamma \delta - \beta}.
\]

The periapsis advance for an equatorial non-circular timelike geodesic in Kerr-Newman spacetime is defined as follows:

\[
\delta_{teKN} := \Delta \phi_{\text{GTR}} - 2\pi
\]

We applied our closed form formula \((101), (103)\) for calculating the relativistic periapsis advance for the observed orbits of \(S\)-stars in the central arcsecond of the Milky Way. By doing this exercise, we gain an appreciation of the effect of the electric charge of the rotating galactic black hole (we assume that the KN solution describes the curved spacetime geometry around SgrA*) on this observable. We also assume that the angular momentum axis of the orbit is co-aligned with the spin axis of the black hole and that the \(S\)-stars can be treated as neutral test particles. Indeed, we present the results of our computations in Tables 7, 8 that correspond to the orbits of the stars \(S2, S14\) respectively\(^9\). For fixed values of the parameters \(L, E, a\) we calculated the periapsis advance for different values of the electric charge. We observe the significant contribution of the electric charge on the phenomenon of periapsis advance in the theory of general relativity. Varying the electric charge in the range \((0.1 - 0.85)\sqrt{G \rho M_{BH}}\), we see that the effect due to the electric charge on the periapsis advance is of the order of 81.4 arcsec/rev. for the star \(S2\), with the observation: the larger the electric charge the smaller the periapsis advance. Similar results hold for the star \(S14\) where the effect due to the parameter \(e\) is computed to be of the

\[^9\text{The parameters are consistent with data for the periastron, apoastron distances and orbital period for the stars S2, S14 [40] (see also [41]).}\]
Table 7: Periastron precession for the star $S2$ in the central arcsecond of the galactic centre, using the exact formula (101), for three different values of the electric charge of the galactic black hole. The Kerr parameter is $a_{\text{Gal}} = 0.52 \frac{G M_{\text{BH}}}{c^2}$. We assume a central black hole mass $M_{\text{BH}} = 4.06 \times 10^6 M_\odot$.

| Parameters for the star $S2$ | Periapsis advance |
|-----------------------------|-------------------|
| $a = 0.52, e = 0.1, L = 75.4539876, E = 0.999979485$ | $\delta_{\text{KN}} = 676.5 \text{ arcsec/rev}$ |
| $a = 0.52, e = 0.33, L = 75.4539876, E = 0.999979485$ | $\delta_{\text{KN}} = 665.2 \text{ arcsec/rev}$ |
| $a = 0.52, e = 0.85, L = 75.4539876, E = 0.999979485$ | $\delta_{\text{KN}} = 595.1 \text{ arcsec/rev}$ |

Table 8: Periastron precession for the star $S14$ in the central arcsecond of the galactic centre, using the exact formula (101), for three different values of the electric charge of the galactic black hole. The Kerr parameter is $a_{\text{Gal}} = 0.52 \frac{G M_{\text{BH}}}{c^2}$. We assume a central black hole mass $M_{\text{BH}} = 4.06 \times 10^6 M_\odot$.

| Parameters for the star $S14$ | Periapsis advance |
|-----------------------------|-------------------|
| $a = 0.52, e = 0.11, L = 72.9456205, E = 0.999988863$ | $\delta_{\text{KN}} = 723.432 \text{ arcsec/rev}$ |
| $a = 0.52, e = 0.33, L = 72.9456205, E = 0.999988863$ | $\delta_{\text{KN}} = 711.595 \text{ arcsec/rev}$ |
| $a = 0.52, e = 0.85, L = 72.9456205, E = 0.999988863$ | $\delta_{\text{KN}} = 636.568 \text{ arcsec/rev}$ |

order of 87 arcsec/rev as $e$ varies in the range $(0.1 - 0.85)\sqrt{G M_{\text{BH}}}$. A more precise analysis would involve the calculation of relativistic periapsis advance for general timelike orbits, polar or inclined non-equatorial in the KN(a)dS spacetime, which is beyond the scope of the current publication. However, since the effect due to the parameter $e$ is significant already at this level of analysis and since we are entering an era of precision in observational astronomy the effect due to the electric charge of the KN spacetime singularity should be taken into account in the interpretations of future measurements for the relativistic effect of periapsis advance [42].

6 Exact solution for the deflection angle of unbound equatorial orbits in Kerr-Newman-de Sitter spacetime

Assume that $\Lambda > 0$. Then the relevant differential equation for the exact computation of the deflection angle of an equatorial ray in the field of an electrically charged rotating black hole with a positive cosmological constant is given by

$$d\phi = \Xi^2 (\Phi - a) \frac{dr}{\sqrt{R'}} + \frac{a \Xi^2}{\Delta_r^{KN}} \frac{[\Phi^2 + a^2] - a \Phi}{\sqrt{R'}}.$$

10Such an analysis will be a subject of a future publication
Using the partial fractions technique for the second term we write:

$$\frac{a\Xi^2 [r^2 + a^2] - a\Phi}{\Delta_{KN}^2 r'^2} = \frac{A^1}{r - r_{A}^+} + \frac{A^2}{r - r_{A}^-} + \frac{A^3}{r - r_{+}} + \frac{A^4}{r - r_{-}}$$  \hspace{1cm} (105)$$

where \( r_{A}^+, r_{\pm} \) are the four real roots of \( \Delta_{KN}^2 \). Thus one of the integrals we need to calculate is:

$$I_{A}^1 = \frac{1}{\sqrt{\Xi^2 (1 + \frac{4}{3}(\Phi - a)^2 )}} \int_{\alpha}^{r_{A}^+ / 2} \frac{A^1 dr}{(r - r_{A}^+)\sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}}$$  \hspace{1cm} (106)$$

Indeed, we compute in closed analytic form:

$$I_{A}^1 = -\frac{A^1}{\sqrt{\Xi^2 (1 + \frac{4}{3}(\Phi - a)^2 )}} \rho_1 \omega' H_{Ae}^+ F_D \left( \frac{1}{2}, \beta \beta; \frac{3}{2}, \frac{3}{2}; z_{A}^{e+}, \frac{3}{2} \right) \frac{\Gamma(1/2)}{\Gamma(3/2)} \hspace{1cm} (107)$$

The tuple of variables for the Lauricella’s fourth hypergeometric function \( F_D \) is defined in terms of the horizons and the radial roots of the Kerr-Newman-de Sitter black hole as follows:

$$z_{A}^{e+} := \left( \frac{r_{A}^+ - 2\alpha}{r_{A}^+ - 2\beta}, \frac{\beta - \gamma}{\alpha - \gamma}, \frac{\beta - \delta}{\alpha - \delta}, \frac{r_{A}^+ - 2\alpha}{r_{A}^+ - 2\beta} \right), \hspace{1cm} (108)$$

while we also define:

$$\rho_1 := \frac{r_{A}^+ - \beta}{r_{A}^+ - 2\beta} - \frac{2\alpha}{r_{A}^+ - \alpha} - \frac{2\alpha}{r_{A}^+ - \beta} \hspace{1cm} (109)$$

and

$$H_{Ae}^+ := \frac{\alpha - \beta}{|\beta - \alpha|} \frac{1}{r_{A}^+ - \beta} \frac{1}{\sqrt{\omega'(\gamma - \alpha)(\delta - \alpha)}} \hspace{1cm} (110)$$

In addition, from the first term we compute exactly in terms of Appell’s hypergeometric function:

$$\Xi^2 (\Phi - a) \int_{\alpha}^{r_{A}^+ / 2} \frac{dr}{\sqrt{(r - \alpha)(r - \beta)(r - \gamma)(r - \delta)}} = \frac{\Xi^2 (\Phi - a)}{\sqrt{\Xi^2 (1 + \frac{4}{3}(\Phi - a)^2 )}} \rho_1 \omega' \frac{\Gamma(1/2)}{\Gamma(3/2)} F_1 \left( \frac{1}{2}, \beta \beta; \frac{3}{2}, \frac{3}{2}; z_{A}^{e+}, \frac{3}{2} \right) \hspace{1cm} (111)$$
In total we get

\[
\Delta \phi_{CTR}^{KN_{\Lambda}} = \frac{\Xi^2(\Phi - a)}{\sqrt{\Xi^2(1 + \frac{A}{3}(\Phi - a)^2)}} \sqrt{\rho_1 \omega'(\gamma - \alpha)(\delta - \alpha)} F_1 \left( \frac{1}{2}, \beta_{\Lambda}^{A_3}, \frac{3}{2}, z_{\Lambda A+}^r \right) 
\]

where the tuple for the beta parameters of the Lauricella’s hypergeometric function is defined as follows:

\[
\beta_{\Lambda}^{A_3} := \left( -1, \frac{1}{2}, \frac{1}{2}, 1 \right). 
\]

In producing the analytic solution we applied the transformations:

\[
z = \frac{1}{\omega' r - \beta}, z \rightarrow \rho_1 z', 
\]

and

\[
\omega' := \frac{r_+^{\Lambda} - \alpha}{r_+^{\Lambda} - \beta}. 
\]

In addition we defined the tuple:

\[
z_{\Lambda A+}^r := \left( \frac{r_+^{\Lambda} - 2\alpha}{r_+^{\Lambda} - 2\beta}, \frac{r_+^{\Lambda} - \gamma}{r_+^{\Lambda} - \alpha}, \frac{r_+^{\Lambda} - 2\alpha - \beta - \delta}{r_+^{\Lambda} - 2\beta - \alpha - \delta} \right) 
\]

A complete phenomenological analysis of our exact solutions in the presence of the cosmological constant \( \Lambda \) will be a subject of a separate publication [48]. Nevertheless, it is evident from the closed form solutions we derived in this work that the cosmological constant does contribute to the gravitational bending of light in the KNdS spacetime.

7 The shadow of the electrically charged rotating (Kerr-Newman) black hole

The conditions for the spherical photon orbits in Kerr-Newman spacetime as implemented in section 2 yielded equations (35) for the parameter \( \Phi \) and Carter’s
constant $Q$. These are also the conditions for the photon to escape at infinity. When we treat the KN black hole as a gravitational lens, following the procedure developed in [8] for the case of a Kerr gravitational lens, and assuming large observer’s distance $r_O$ (i.e. $r_O \to \infty$) we derive simplified expressions relating the coordinates \((\alpha_i, \beta_i) = (-r_O^2 \sin \theta_O \frac{d \phi}{d r}|_{r=r_O}, r_O^2 \frac{d \theta}{d r}|_{r=r_O})\) on the observer’s image plane (see figure 8) to the integrals of motion:

\[
\Phi \simeq -\alpha_i \sin \theta_O, \quad Q \simeq \beta_i^2 + (\alpha_i^2 - a^2) \cos^2 \theta_O. \tag{117}
\]

By plugging equations (35) into equations (117), we derive the coordinates on the observer’s image plane at which the escaped photon will be detected for

\footnote{We also assume without loss of generality that $\phi_O = 0$.}
the case of a Kerr-Newman gravitational lens:

\[
x_i = \frac{1}{r_O \sin \theta_O} \left[ \frac{a^2(r + M) + 2e^2r - 3Mr^2 + r^3}{a(r - M)} \right],
\]

\[
y_i = \pm \left\{ -r^2 \left[ 4a^2(e^2 - Mr) + \left( 2r^2 + r(3M + r) \right)^2 \right] \\
- 2ra^2z_O \left[ r^3 - 3rM^2 + 2a^2M + 2e^2M \right] \\
- a^4 \left( r - M \right)^2 \right\}^{1/2} \left/ \left[ r_O^2 \sin^2 \theta_O a^2 (r - M)^2 \right]^{1/2} \right\}
\]

(118)

For zero electric charge, i.e for \( e = 0 \), equations (118) reduce correctly to the co-

ordinates on the observer’s image plane of an escaped photon from an uncharged

rotating (Kerr) black hole, eqns. (28) in [8]. Let us look at some examples of

how the shadow of the electromagnetic rotating black hole is perceived by an

observer at different polar positions \( \theta_O \) for different sets of values for the spin

and electric charge of the KN singularity [9]. Our results are displayed in Figures

9, 10, 11, 12, 13, 14, 15.

\[
a = 0.9939, \ e = 0.11, \ \theta_O = \frac{\pi}{3}
\]

![Figure 9: The boundary of the shadow of the charged rotating KN black hole for Kerr parameter \( a = 0.9939, \ e = 0.11 \) for an observer at polar position \( \theta_O = \pi/3 \).](image)

From our results we observe that the larger the electric charge (for fixed

black hole spin and polar position of the observer) the smaller the boundary of

the shadow of the KN black hole, see Figures 10-12 for an observer located at

\( \theta_O = \pi/3, \phi_O = 0 \). See also Figures 14-15 for an equatorial observer. When

\[12\] We note at this point that the shadow of the KN spacetime has also been studied in [13]. However, the author of [13] did not actually solved the KN lens equations and he also considered cases which violate [9].
Figure 10: The boundary of the shadow of the charged rotating KN black hole for Kerr parameter $a = 0.52, e = 0.85$ for an observer at polar position $\theta_O = \pi/3$.

Figure 11: The boundary of the shadow of the charged rotating KN black hole for Kerr parameter $a = 0.52, e = 0.4$ for an observer at polar position $\theta_O = \pi/3$. 
Figure 12: The boundary of the shadow of the charged rotating KN black hole for Kerr parameter $a = 0.52, \epsilon = 0.11$ for an observer at polar position $\theta_O = \frac{\pi}{3}$.

Figure 13: The boundary of the shadow of the charged rotating KN black hole for Kerr parameter $a = 0.9939, \epsilon = 0.11$ for an equatorial observer, i.e. located at polar position $\theta_O = \frac{\pi}{2}$. With the blue dot we exhibit the image solution of the KN lens equations, see Table 10.
Figure 14: The boundary of the shadow of the charged rotating KN black hole for Kerr parameter $a = 0.52$, $e = 0.85$, $\theta_O = \frac{\pi}{2}$ for an equatorial observer, i.e., located at polar position $\theta_O = \pi/2$. With the blue dot we exhibit the image solution of the KN lens equations, see Table 9.

Figure 15: The boundary of the shadow of the charged rotating KN black hole for Kerr parameter $a = 0.52$, $e = 0.11$, $\theta_O = \frac{\pi}{2}$ for an equatorial observer, i.e., located at polar position $\theta_O = \pi/2$. 
we compare the case of a fast-spinning KN black hole, Figure 9 with the corresponding uncharged rotating black hole, Figure 4, page 25 of [8], we also observe that the charged fast-spinning black hole has a slightly smaller boundary for its shadow. The significant deformation of the boundary of the shadow from circularity is present in both KN and Kerr fast spinning black holes.

8 Exact calculation of radial integrals for generic orbits in Kerr-Newman spacetime

We now perform the radial integration for generic orbits, i.e. orbits for which both $\Phi, Q$ differ from zero, assuming $\Lambda = 0$. The relevant differential equation for the second KN lens equation that determines the azimuth angle is:

$$\frac{d\phi}{dr} = \frac{a(-e^2 + 2Mr)}{\pm \Delta_{KN} \sqrt{R}} + \frac{-a^2 \Phi}{\pm \Delta_{KN} \sqrt{R}} + \frac{\Phi}{\pm \sin^2 \theta \sqrt{\Theta}} \frac{d\theta}{dr}. \quad (119)$$

Thus for the radial contribution we need to integrate the first two terms.

For an observer and a source located far away from the black hole, the relevant radial integrals can take the form:

$$\int_{-\alpha}^{r_s} \rightarrow - \int_{\alpha}^{r_s} + \int_{\alpha}^{\infty} \simeq 2 \int_{\alpha}^{\infty}$$

Thus we must compute the radial integral:

$$\Delta \phi_{GTR}^{\text{GT}} = \int_{\alpha}^{\infty} \frac{a(-e^2 + 2Mr) - a^2 \Phi}{\Delta_{KN} \sqrt{R}} dr \quad (121)$$

Using partial fractions we compute the previous integral in closed analytic form in terms of Lauricella’s and Appell’s generalized hypergeometric functions:
\[ \Delta \phi^{GTR}_{r \rightarrow 0 KN} = 2 \left[ -2A_{+}^{goKN} \sqrt{\omega}(\alpha_{\mu+1} - \alpha_{\mu+2}) F_D \left( \frac{1}{2}, \beta^{3}_{+} \frac{3}{2}, z'_{+} \right) ight. \\
+ \left. \frac{A_{-}^{goKN}}{\kappa_{+}^{2}} F_D \left( \frac{1}{2}, \beta^{3}_{-} \frac{3}{2}, z'_{-} \right) \right] \]

\[ \approx R_{2}^{KN}(x_i, y_i). \] (122)

where

\[ A_{\pm}^{goKN} = \frac{\pm 2Mar_{\pm} \mp a(e^2 + a\Phi)}{r_+ - r_-}. \] (123)

The radial polynomial now has the form:

\[ R(r) = r^4 + r^2(a^2 - \Phi^2 - Q) + 2M(Q + \Phi - a^2)r - e^2(Q + (\Phi - a)^2) - a^2 Q. \] (124)

Its four roots are solved in closed analytic form by the Weierstraß function and its derivative—see equations (86)-(89)—where the Weierstraß invariants are given now in terms of the spin, electric charge of the black hole, and the impact parameter and Carter’s constant, by the equations:

\[ g_2 = \frac{1}{12}(a^2 - \Phi^2 - Q)^2 - e^2(Q + (\Phi - a)^2) - a^2 Q, \] (125)

\[ g_3 = \frac{1}{216}(a^2 - \Phi^2 - Q)^3 - \frac{1}{4}(Q + (\Phi - a)^2)^2 \\
+ [-e^2(Q + (\Phi - a)^2) - a^2 Q] \left( \frac{a^2 - \Phi^2 - Q}{6} \right) \] (126)

while

\[ x_1 = \varphi^{-1} \left( - \frac{a^2 - \Phi^2 - Q}{6} \right). \] (127)

We have written the lens equations (10) and (119) in the form:

\[ R_{1}^{KN}(x_i, y_i) - A_1(x_i, y_i, x_S, y_S, m) = 0, \] (128)

\[ \Delta \phi(x_S, y_S, n) - R_{2}^{KN}(x_i, y_i) - A_2(x_i, y_i, x_S, y_S, m) = 0. \] (129)
In these equations \( n \) denotes the number of windings around the \( z \)-axis and \( m \) the number of turning points in the polar motion. The term \( R^{KN}_1(x_i, y_i) \) in closed analytic form is given by equation:

\[
2 \int_\alpha^\infty \frac{dr}{\sqrt{R}} = 2 \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)} \left( \frac{1}{\sqrt{(\gamma - \alpha)(\delta - \alpha)}} \right) F_1 \left( \frac{1}{2}, \beta^A_i, \frac{3}{2}, z^A_i \right) \equiv R^{KN}_1(x_i, y_i)
\]

and equations (38)–(39), (125)–(127). The angular parts of the lens equations:

\( A_1(x_i, y_i, x_S, y_S, m) \), \( A_2(x_i, y_i, x_S, y_S, m) \) are the same as in the uncharged case (the Kerr field) and have been computed analytically in [8] for the case of a light trajectory that encounters \( m \) turning points \( (m \geq 1) \) in the polar motion:

\[
\pm \int_{\theta_S}^{\theta_{\text{min}}/\text{max}} \pm \int_{\theta_{\text{max}}/\text{min}}^{\theta_{\text{max}}/\text{min}} \ldots \pm \int_{\theta_{\text{max}}/\text{min}}^{\theta_{\text{max}}/\text{min}} m-1 \text{ times}
\]

(131)

Combining the exact results for the radial integrals we have computed in this work with the angular integral computations of [8] for the Kerr black hole, which remain valid, as we said in the KN case we have that our exact solutions of the lens equations for a KN black hole are:

\[
R^{KN}_1(x_i, y_i) = A_1(x_i, y_i, x_S, y_S, m) \leftrightarrow \frac{2}{\sqrt{(\alpha - \gamma)(\alpha - \delta)}} \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)} F_1 \left( \frac{1}{2}, \beta^A_i, \frac{3}{2}, z^A_i \right)
\]

\[
= 2(m - 1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m - z_3)}} F_1 \left( \frac{1}{2}, \beta^A_i, \frac{3}{2}, z^A_i \right)
\]

\[
+ \frac{1}{2|a|} \sqrt{\frac{z_m(z_m - z_2)}{z_m(z_m - z_3)}} F_1 \left( \frac{1}{2}, \beta^A_i, \frac{3}{2}, z^A_i \right) \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)}
\]

\[
+ \frac{1}{2|a|} \sqrt{\frac{z_m(z_m - z_2)}{z_m(z_m - z_3)}} F_1 \left( \frac{1}{2}, \beta^A_i, \frac{3}{2}, z^A_i \right) \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)}
\]

\[
+ \frac{1}{2|a|} \sqrt{\frac{z_m(z_m - z_2)}{z_m(z_m - z_3)}} F_1 \left( \frac{1}{2}, \beta^A_i, \frac{3}{2}, z^A_i \right) \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)}
\]

\[
- \phi_S = R^{KN}_2(x_i, y_i) + A_2(x_i, y_i, x_S, y_S, m),
\]

\[
(132)
\]

\[
\xi_S = \varphi \left( 2\sigma_S \left[ R^{KN}_1(x_i, y_i) - 2(m - 1) \frac{1}{2|a|} \sqrt{\frac{z_m}{z_m(z_m - z_3)}} F_1 \left( \frac{1}{2}, \beta^A_i, \frac{3}{2}, z^A_i \right) \frac{\Gamma(1/2)\Gamma(1)}{\Gamma(3/2)} \right] + \cdots \right) + \epsilon
\]

(133)
where

\[ A_2(x_i, y_i, x_S, y_S, m) = 2(m-1) \times \left[ \frac{\Phi}{|a|} \sqrt{\frac{1}{z_m-z_3}} \frac{1}{1-z_m} \pi \frac{F_1\left(\frac{1}{2}, 1, 1, \frac{-z_m}{z_m-z_3}\right)}{2} \right] \]

\[ + \frac{\Phi}{2|a|} \sqrt{\frac{z_m-z_S}{z_m}} \frac{1}{\sqrt{z_m-z_3}} \frac{2}{1-z_m} \times F_D\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z_S^1\right) \]

\[ + [1-\text{sign}(\theta_S \circ \theta_{mS})] \frac{\Phi}{|a|} \sqrt{\frac{z_S z_S-z_m}{z_m}} \frac{1}{\sqrt{z_S-z_m}} \times F_D\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z_S^1\right) \]

\[ + [1-\text{sign}(\theta_O \circ \theta_{mO})] \frac{\Phi}{|a|} \sqrt{\frac{z_O z_O-z_m}{z_m}} \frac{1}{\sqrt{z_O-z_m}} \times F_D\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z_O^1\right) \]

\[ + [1-\text{sign}(\theta_{mO})] \left[ \frac{\Phi}{|a|} \sqrt{\frac{z_m-z_O}{z_m}} \frac{1}{\sqrt{z_m-z_3}} \frac{2}{1-z_m} \times F_D\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, z_O^1\right) \right] \]

(135)

and:

\[ \theta_{mO} := \text{Arccos}(\text{sign}(y_i)\sqrt{z_m}) = \text{Arccos}(\text{sign}(\beta_i)\sqrt{z_m}). \]  

(136)

\[ y_i \text{ is the possible position of the image and:} \]

\[ \theta_{mS} := \left\{ \begin{array}{ll} \theta_{mO}, & m \text{ odd} \\ \pi - \theta_{mO}, & m \text{ even} \end{array} \right. \]

(137)

Also \( \theta_1 \circ \theta_2 := \cos \theta_1 \cos \theta_2, \sigma_S := \text{sign} \theta_S \circ \theta_{mS} \) and \( \epsilon \) denotes a constant of integration. The Weierstraß invariants in equation (134) are defined in (26) and (22) [13].

A solution of the KN lens equations with \( m = 3 \) is presented in Table 9 for \( a = 0.52, e = 0.85 \). The solution as it appears on the observer’s image plane is exhibited along with the boundary of the shadow of the KN black hole in Fig.14.

Another solution of the KN lens equations for an electromagnetic fast spinning black hole (\( a = 0.9939, e = 0.11 \)), is presented in Table 10. The solution as it appears on the observer’s image plane is exhibited along with the boundary of the shadow of the KN black hole in Fig.15.

A detailed analysis of gravitational lensing in the KN and KN(a)dS spacetimes will be a subject of a separate publication. We also leave for the future the exact analytic computation of the magnification factors for the KN and KN(a)dS spacetimes.

\[ ^{13}\text{In establishing (134) we used the fact that the sum of the second and third term on the right hand side of eq. (132) can be written as: } \int_{\xi_0}^{\xi} \omega \int_{\xi_0}^{\xi} \omega_s, \text{ where } (\xi_0, \xi) \text{ are extremal values of } \xi; \text{ thus, one can separate from it the expression } \sigma_s \int_{\xi_s}^{\xi} \propto \sigma_s \nu^{-1}(\xi). \]
### Table 9
Solution of the lens equations in the Kerr-Newman geometry and the predictions for the source and image positions for an observer at $\theta_O = \pi/2$, $\phi_O = 0$. The number of turning points in the polar variable is 3. The values for the Kerr parameter, impact factor are in units of $\frac{GM}{c^2}$, those of electric charge in units of $\sqrt{GM_{BH}}$ and of Carter’s constant in units of $\frac{G^2M^2}{c^4}$.

| Solution with Parameters: $a = 0.52, e = 0.85, Q = 16.51343, \Phi = -3.0120$ |
|---|
| $a_i \left( \frac{GM}{c^2} \right)$ | $3.0120$ |
| $\beta_i \left( \frac{GM}{c^2} \right)$ | $-4.06367$ |
| $x_i \left( \frac{2 \ GM}{r_O c^2} \right)$ | $1.506$ |
| $y_i \left( \frac{2 \ GM}{r_O c^2} \right)$ | $-2.031835$ |
| $m$ | $3$ |
| $z_S$ | $0.06859659416$ |
| $\theta_S$ | $74.82^\circ$ |
| $\Delta \phi (\text{rad})$ | $-8.39801$ |
| $\phi_S$ | $121.17^\circ$ |
| $\omega$ | $0.6211809022$ |
| $\omega'$ | $1.5336366498i$ |

### Table 10
Solution of the lens equations in the Kerr-Newman geometry and the predictions for the source and image positions for an observer at $\theta_O = \pi/2$, $\phi_O = 0$. The number of turning points in the polar variable is 3. The values for the Kerr parameter, impact factor are in units of $\frac{GM}{c^2}$, those of electric charge in units of $\sqrt{GM_{BH}}$ and of Carter’s constant in units of $\frac{G^2M^2}{c^4}$.

| Solution with Parameters: $a = 0.9939, e = 0.11, Q = 25.790421, \Phi = -3.0118023$ |
|---|
| $a_i \left( \frac{GM}{c^2} \right)$ | $3.0118023$ |
| $\beta_i \left( \frac{GM}{c^2} \right)$ | $-5.07843$ |
| $x_i \left( \frac{2 \ GM}{r_O c^2} \right)$ | $1.5059$ |
| $y_i \left( \frac{2 \ GM}{r_O c^2} \right)$ | $-2.53921$ |
| $m$ | $3$ |
| $z_S$ | $0.3110425032$ |
| $\theta_S$ | $56.10^\circ$ |
| $\Delta \phi (\text{rad})$ | $-7.92181$ |
| $\phi_S$ | $93.8862^\circ$ |
| $\omega$ | $0.5312062705$ |
| $\omega'$ | $1.1216689490i$ |

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9 Conclusions

We have solved in closed analytic form the null geodesic equations in Kerr-Newman and Kerr-Newman-(anti) de Sitter spacetimes. The analytic solutions were expressed elegantly in terms of generalized hypergeometric functions of Lauricella and Appell as well as the Weierstraß elliptic function.

We also solved the more involved problem of treating a Kerr-Newman black hole as a gravitational lens, i.e. a KN black hole along with a static source of light and a static observer both located far away but otherwise at arbitrary positions in space. Again, for this model we give the analytic solutions of the lens equation in terms of Appell and Lauricella hypergeometric functions and the Weierstraß modular form.

We applied our exact solutions for the calculation of the frame dragging effect for spherical polar and non-polar photon orbits in the gravitational field of a KN black hole. We also applied our exact solution for the gravitational bending of light that an equatorial unbound photon orbit experiences in the curved spacetime of a charged rotating black hole. We noted the significant dependence of the deflection angle on the electric charge of the spacetime singularity in regions of the parameter space. This result, in conjunction with our solution for the periapsis advance of a neutral test particle in an equatorial non-circular orbit in KN spacetime and its application to the observed orbits of S-stars, indicates that future measurements of the galactic centre black hole and its relativistic observables may constrain significantly or detect the electric charge of the galactic rotating black hole.

We also derived analytic expressions for the Maxwell tensor components for a ZAMO frame in the KNdS spacetime.

Future directions of research will include the application of the closed form analytic solution of the lens equations in the Kerr-Newman family of spacetimes to the important case of the SgrA* supermassive black hole.

Another interesting avenue of research will be the application of our exact solutions in the exciting field of $e^- - e^+$ pair creation by vacuum polarization around electromagnetic black holes and the theory of pulsars. Indeed, the Kerr-Newman electromagnetic field is such that $*F_{\mu\nu}F_{\mu\nu} \neq 0$ or in pulsar language $E \cdot B \neq 0$. It seems that the KN black hole is charged just like the neutron star in pulsar models [40].

This angle of research will have potentially very important applications for the gamma ray bursts from electromagnetic black holes [28] and pulsars as well as in the study of the vacuum structure of non-linear electrodynamics [47]. The study of the Faraday effect in the KN-(a)dS electromagnetic black hole will serve nicely towards such investigations. We aim to pursue this exciting interplay of relativistic astrophysics and non-linear electrodynamics in the future [48].

Thus the theory we developed in this work based on the exact solutions of null geodesics in the KNdS spacetime will help in exploring the strong field regime of General Relativity and the structure of Electrodynamics in curved spacetimes.
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A The roots of the quartic in terms of Weierstraß functions

We are going to use the addition theorem for the Weierstraß elliptic function to express the roots of the quartic \( P_4(x) = x^4 + ax^2 + bx + c \in \mathbb{C}[x] \) in terms of the Weierstraß functions following [44]. We first write \( x \) for a point of the cover \( \mathbb{C} \) and \( p = (x, y) \) for the corresponding point of the cubic determined by \( x = \wp(x) \) and \( y = \wp'(x) \). Then we study the intersections of the cubic \( y^2 = 4x^3 - g_2x - g_3 \) and the line \( y = ax + b : x_1, x_2, x_3 \) are the roots of

\[
F'(x) = 4x^3 - g_2x - g_3 - (ax + b)^2
= 4(x - x_1)(x - x_2)(x - x_3)
\] (138)

so

\[
4(x_1 - x_2)(x_1 - x_3) = F'(x_1) = 12x_1^2 - g_2 - 2a(ax_1 + b)
= 12x_1^2 - g_2 - 2ay_1.
\] (139)

and

\[
(x_2 - x_1) + (x_3 - x_1) = x_1 + x_2 + x_3 - 3x_1 = \frac{a^2}{4} - 3x_1.
\] (140)

We note that \( a \) is the slope of the line, so for distinct \( x_1, x_2, x_3 \) we have

\[
a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{y_1 - y_3}{x_1 - x_3}.
\] (141)

Now

\[
\begin{align*}
140^2 - 139^2 &= (x_2 + x_3 - 2x_1)^2 - 4x_1^2 + 4x_1x_3 + 4x_1x_2 - 4x_2x_3 \\
&= (x_2 - x_3)^2 = \left(\frac{a^2}{4} - 3x_3\right)^2 - 12x_1^2 + g_2 + 2ay_1 \\
&= \left(\frac{a}{2}\right)^4 - 6x_1\left(\frac{a}{2}\right)^2 + 4y_1\left(\frac{a}{2}\right) - 3x_1^2 + g_2.
\end{align*}
\] (142)

Thus

\[
X = \frac{a}{2} = \frac{1}{2}\frac{y_2 - y_1}{x_2 - x_1} = \frac{1}{2} \frac{\wp'(x_2) - \wp'(x_1)}{\wp(x_2) - \wp(x_1)}
\] (143)

\[
Y = x_3 - x_2 = \wp(-x_1 - x_2) - \wp(x_2) = \wp(x_1 + x_2) - \wp(x_2) \Rightarrow
\] (144)
\[ Y^2 = X^4 - 6\wp(x_1)X^2 + 4\wp'(x_1)X - 3\wp^2(x_1) + g_2 \equiv P_4(X). \] (145)

In the second equality of (144) we used the addition theorem \( x_1 + x_2 = -x_3 \) in \( \mathbb{C}/L \), \( L \) the period lattice, and the fact that the Weierstraß function is even. For fixed \( x_1 \), and variable \( x_2 \), \( P_4(X) = 0 \) only if \( Y = 0 \). This occurs for \( x_2 = -\frac{x_1}{2} \) to which may be added one of the three half-periods producing four roots of \( P_4(x) = 0 \) and these must be distinct, see equations (86)-(89).

**B Lauricella’s multivariable hypergeometric function \( F_D \)**

In this appendix B, we define Lauricella’s 4th hypergeometric function of \( m \)-variables and its integral representation:

\[
F_D(\alpha, \beta, \gamma, z) = \sum_{n_1, n_2, \ldots, n_m = 0}^{\infty} \frac{(\alpha)_{n_1+\cdots+n_m}(\beta_1)_{n_1} \cdots (\beta_m)_{n_m}}{(\gamma)_{n_1+\cdots+n_m}(1)_{n_1} \cdots (1)_{n_m}} z_1^{n_1} \cdots z_m^{n_m} \quad (146)
\]

where

\[
z = (z_1, \ldots, z_m), \quad \beta = (\beta_1, \ldots, \beta_m). \quad (147)
\]

The Pochhammer symbol \( (\alpha)_m = (\alpha, m) \) is defined by

\[
(\alpha)_m = \frac{\Gamma(\alpha + m)}{\Gamma(\alpha)} = \begin{cases} 1, & \text{if } m = 0 \\ \alpha(\alpha + 1) \cdots (\alpha + m - 1) & \text{if } m = 1, 2, 3 \end{cases} \quad (148)
\]

With the notations \( z^n := z_1^{n_1} \cdots z_m^{n_m} \), \( (\beta)_n := (\beta_1)_{n_1} \cdots (\beta_m)_{n_m} \), \( n! = n_1! \cdots n_m! \), \( |n| := n_1 + \cdots + n_m \) for \( m \)-tuples of numbers in (147) and of non-negative integers \( n = (n_1, \cdots, n_m) \) the Lauricella series \( F_D \) in compact form is

\[
F_D(\alpha, \beta, \gamma, z) := \sum_n (\alpha)_m (\beta)_n z^n (149)
\]

The series admits the following integral representation:

\[
F_D(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-z_1t)^{-\beta_1} \cdots (1-z_mt)^{-\beta_m} dt \quad (150)
\]

which is valid for \( \text{Re}(\alpha) > 0 \), \( \text{Re}(\gamma - \alpha) > 0 \). It converges absolutely inside the \( m \)-dimensional cuboid:

\[
|z_j| < 1 \quad (j = 1, \ldots, m). \quad (151)
\]
For \( m = 2 \) in the notation of Appell becomes the two variable hypergeometric function \( F_1(\alpha, \beta, \beta', \gamma, x, y) \) with integral representation:

\[
\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}(1-uy)^{-\beta'} \, du = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} F_1(\alpha, \beta, \beta', \gamma, x, y)
\]

(152)

**C Calculation of the Maxwell tensor components in the ZAMO frame for the KNdS spacetime**

The ZAMO basis vectors are determined by the transformation

\[
\hat{e}_0 = \left| g_{tt} - \Omega^2 g_{\phi\phi} \right|^{-1/2} \frac{\partial}{\partial t} + \Omega \left| g_{tt} - \Omega^2 g_{\phi\phi} \right|^{-1/2} \frac{\partial}{\partial \phi},
\]

(153)

\[
\hat{e}_\phi = \frac{1}{\sqrt{g_{\phi\phi}}} \frac{\partial}{\partial \phi},
\]

(154)

\[
\hat{e}_r = \left( \frac{\Delta^{KN}_r}{\rho^2} \right)^{1/2} \frac{\partial}{\partial r},
\]

(155)

\[
\hat{e}_\theta = \left( \frac{\Delta_\theta}{\rho^2} \right)^{1/2} \frac{\partial}{\partial \theta},
\]

(156)

where the angular velocity \( \Omega \) is given in the KNdS case by the expression:

\[
\Omega = \frac{-ac \sin^2 \theta [\Delta^{KN}_r - \Delta_\theta (r^2 + a^2)]}{\Xi^2 \rho^2 g_{\phi\phi}} = \frac{-ac [\Delta^{KN}_r - \Delta_\theta (r^2 + a^2)]}{(\Delta_\theta (r^2 + a^2)^2 - a^2 \sin^2 \theta \Delta^{KN}_r)}
\]

(157)

and the quantity \( g_{tt} - g_{\phi\phi}^2 = \frac{-e^2 \Delta^{KN}_r \Delta_\theta \sin^2 \theta}{\Xi^2} \). In addition, we compute for the lapse function \( \alpha_{ZAMO} \):

\[
\alpha_{ZAMO} := \left| g_{tt} - \Omega^2 g_{\phi\phi} \right|^{1/2} = \frac{e(\Delta^{KN}_r)^{1/2} \Delta_\theta^{1/2} \sin \theta}{\Xi^2 \sqrt{g_{\phi\phi}}}
\]

(158)

Equation (158), reduces correctly, assuming \( \Lambda = 0 \), to the lapse function derived in [19].

Now our analytic calculation for the electric (\( \vec{E} \)) and magnetic fields (\( \vec{B} \)) in the presence of the cosmological constant \( \Lambda \) in the ZAMO frame yields:

\[
E^r = \frac{(r^2 + a^2)e[-r^2 + a^2 \cos^2 \theta] \Delta_\theta^{1/2}}{\sqrt{(r^2 + a^2)^2 \Delta_\theta - a^2 \sin^2 \theta \Delta^N_r \rho^4}}
\]

(159)
Also it holds:

\[ E^\phi = B^\phi = 0 \]  

(163)

To the best of our knowledge equations (159)-(162) represent the first calculation of the Maxwell tensor components in the ZAMO frame for the case of the Kerr-Newman-de Sitter spacetime. Our solutions for the electric and magnetic fields, for zero cosmological constant, reduce correctly to the corresponding expressions for the Kerr-Newman spacetime and LNRF observers derived in [20].

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