Application of the Worldline Path Integral Method to the Calculation of Inverse Mass Expansions *

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Abstract

Higher order coefficients of the inverse mass expansion of one–loop effective actions are obtained from a one–dimensional path integral representation. For the case of a massive scalar loop in the background of both a scalar potential and a (non Abelian) gauge field explicit results to $O(T^5)$ in the proper time parameter are presented.

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The inverse mass expansion of the one–loop effective action of a scalar particle with mass $m$ in the background of a (non Abelian) gauge field $A_\mu$ and a matrix valued scalar potential $V$ is usually obtained from the one–loop determinant of the corresponding fluctuation operator $M$, which (in Euclidean space) reads

$$M = -D^2 + m^2 + V(x) \quad \text{with} \quad D_\mu = \partial_\mu + igA_\mu(x). \quad (1)$$

by using the Schwinger proper time representation,

$$\Gamma_{\text{eff}}[A, V] = -\ln(\det M) = -\text{Tr}(\ln M) = \int_0^\infty \frac{dT}{T} \text{Tr} \, e^{-TM}, \quad (2)$$

and expanding in powers of the proper time parameter $T$. For the gauge case the coefficients of this expansion were calculated up to $O(T^5)$ by field theory methods in [1]. The operator trace is nothing but the diagonal element of the heat kernel for the operator $M$, integrated over space-time. Using standard (recursive and non–recursive) heat kernel methods [2] the coefficients have been computed up to $O(T^7)$ for the pure scalar case and $O(T^5)$ for the gauge case only recently [3].

The worldline approach to the inverse mass expansion makes use of progress in calculations of one–loop amplitudes by string inspired techniques [4, 5, 6]. The operator trace is written as a one-dimensional path integral over the space of closed loops in space-time with fixed proper time circumference $T$. The one–loop effective action takes the form

$$\Gamma[A, V] = \int_0^\infty \frac{dT}{T} e^{-m^2T} \text{tr} P \int_{x(0)=x(0)} Dx \exp \left[-\int_0^T d\tau \left(\frac{\dot{x}^2}{4} + ig \dot{x}^\mu A_\mu + V\right)\right]. \quad (3)$$

Here $P$ denotes path ordering and the operator trace has reduced to an ordinary matrix trace. After expansion of the interaction exponential the path integral can be evaluated by Wick contractions. The appropriate Green function is the Green function of the Laplacian on the circle with periodic boundary conditions [5]. Since the defining equation of this Green function has no solution, one is forced to introduce a ‘background charge’ $\rho$ on the worldline. A very convenient choice is a uniformly distributed background charge $\rho = 1/T$ (which distinguishes the present formalism from former path integral approaches in this context [7]), yielding the translationally invariant Green function

$$G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T}. \quad (4)$$

Due to the existence of the zero mode this Green function cannot be applied to the path integral as it stands. One has to introduce a loop center of mass $x_0$ and a relative coordinate $y$,

$$x^\mu(\tau) = x_0^\mu + y^\mu(\tau) \quad \text{with} \quad \int_0^T d\tau \, y^\mu(\tau) = 0, \quad (5)$$

and separate the integration over the center of mass from the path integral,
For the calculation of the heat kernel coefficients to a given order in due to the freedom of choosing the origin somewhere on the loop.

\[ \Gamma[A, V] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \mathcal{P} \int d^4 x_0 \int \mathcal{D} y \exp \left[ - \int_0^T d\tau \left( \frac{\dot{y}^2}{4} + ig \gamma^\mu A_\mu + V \right) \right]. \]  

(7)

If one chooses Fock-Schwinger gauge for the background gauge field with reference to the center of mass \( x_0 \) [6], it can be written as

\[ A_\mu(x_0 + y) = y^\rho \int_0^1 d\eta \, \eta \, F_{\rho\mu}(x_0 + \eta y) \]  

(8)

and the background fields \( F \) and \( V \) can be expanded covariantly:

\[ F_{\rho\mu}(x_0 + \eta y) = e^{\eta y D} F_{\rho\mu}(x_0), \quad V(x_0 + y) = e^{\eta y D} V(x_0). \]  

(9)

Using these formulas the expansion of the interaction exponentials in Eq. (3) in the proper time parameter \( T \) yields the following manifestly covariant form of the action:

\[ \Gamma[F, V] = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \text{tr} \int d^4 x_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \int \mathcal{D} y \exp \left[ - \int_0^T d\tau \frac{\dot{y}^2}{4} \right] \int_0^{\tau_2} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \]

\[ \prod_{j=1}^n \left[ e^{y(\tau_j) D(\eta) y(\tau_j)} + ig y^{\mu_1}(\tau_j) y^{\rho_1}(\tau_j) \int_0^1 d\eta \eta \, \eta \, e^{\eta y(\tau_j) D(\eta) F_{\rho_1\mu_1}(x_0)} \right]. \]  

(10)

The factor \( \frac{1}{n} \) arises upon eliminating of the first \( \tau \)-integration, which is possible due to the freedom of choosing the origin somewhere on the loop.

For the calculation of the heat kernel coefficients to a given order in \( T \) one has to perform the following steps:

- **Wick contractions**

  The sum in Eq. (10) is truncated and the exponentials are expanded. Then all possible Wick contractions have to be evaluated using the contraction rules

\[
\langle y^\mu(\tau_1) y^\nu(\tau_2) \rangle = -g^{\mu\nu} G_B(\tau_1, \tau_2) = -g^{\mu\nu}\left[ |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \right],
\]

\[
\langle \dot{y}^\mu(\tau_1) \dot{y}^\nu(\tau_2) \rangle = -g^{\mu\nu} \dot{G}_B(\tau_1, \tau_2) = -g^{\mu\nu}\left[ \text{sign}(\tau_1 - \tau_2) - \frac{2(\tau_1 - \tau_2)}{T} \right],
\]

\[
\langle \dot{y}^\mu(\tau_1) \dot{y}^\nu(\tau_2) \rangle = +g^{\mu\nu} \ddot{G}_B(\tau_1, \tau_2) = +g^{\mu\nu}\left[ 2\delta(\tau_1 - \tau_2) - \frac{2}{T} \right],
\]

(11)

where the dot denotes differentiation with respect to the first variable. From these rules one may alternatively derive contraction rules for exponentials, which can be used equivalently.
• **Integrations**

The polynomial \( \tau \)- and \( \eta \)-integrations are performed. After the expansion in the first step the \( \eta \)-integrations are trivial. The integrands of the \( \tau \)-integration involve the worldline Green function \( G_B \) and its derivatives. They are polynomial in the variables \( \tau_j \).

• **Basis reduction**

The number of terms will be drastically reduced by an appropriate basis reduction, which is essential for the practical use of the coefficients in numerical calculations. In the pure scalar case there is a unique basis with the property that no box operators \( \Box = \partial^2 \) occur. With the choice of the Green function Eq. (4) no self contractions exist, i.e. box operators do never occur and no partial integrations have to be performed to reduce the result. The only operation left is the identification of cyclic equivalent terms, which is trivial. The gauge case is much more complicated and requires a reduction algorithm, which is described below.

Finally one obtains the result in the form

\[
\Gamma[F, V] = \int_0^\infty \frac{dT}{T} [4\pi T]^{-d/2} e^{-m^2T} \sum_{n=1}^\infty \frac{(-T)^n}{n!} \int dx_0 \text{tr} O_n .
\] (12)

The factor \([4\pi T]^{-d/2}\) arises from the normalization of the free path integral. For the scalar case the calculation of the coefficients was done up to \( O_{11} \) \([8, 9]\).

In the following we describe the basis reduction algorithm for the gauge case as proposed by Müller \([10]\). It turns out that there is a minimal basis of invariants without box operators and partial integrations are still not necessary to reduce our results into this basis. However, besides the cyclic permutations one has to use the antisymmetry of the field strength tensor, the Bianchi identities and the exchange of covariant derivatives,

\[
D_\mu D_\nu X = D_\nu D_\mu X + ig [F_{\mu\nu}, X] , 
\] (13)

for a further reduction.

This is done by the following steps (for a more detailed discussion of the algorithm and a proof of minimality see \([10]\)):

• **Elimination of ‘middle’ derivatives**

The contractions of derivatives of a field strength tensor belong to different classes with respect to the contractions of the field strength tensor itself. This can be seen very easily in a diagrammatic picture, where the (cyclic) function \( \text{tr} \) is represented by a circle, the fields by points on the circle and the contractions between the fields by lines crossing the circle. In this picture the indices of a field strength tensor divide the circle into three sectors (left,middle,right). Consequently the derivatives fall into different classes, according to the sector, they are contracted with. The Bianchi identity involves all classes of derivatives and can be used to eliminate one of them. It is useful to take the symmetric choice and eliminate the
middle derivatives. This has to be done with all derivatives of a field strength tensor, which possibly requires exchange of derivatives.

- **Reduction of multiple contractions between factors**

Terms with multiple contractions are reduced by the following rules

\[
\text{tr}(...) F_{\mu\nu} \ldots D_{\mu} F_{\nu\kappa} \ldots = \frac{1}{2} \text{tr}(...) F_{\mu\nu} \ldots D_{\kappa} F_{\nu\mu} \ldots, \tag{14}
\]

\[
\text{tr}(...) D_{\mu} F_{\nu\kappa} \ldots D_{\nu} F_{\mu\lambda} \ldots = \frac{1}{2} \text{tr}(...) D_{\kappa} F_{\nu\mu} \ldots D_{\lambda} F_{\mu\nu} \ldots + \text{tr}(...) D_{\mu} F_{\nu\kappa} \ldots D_{\mu} F_{\nu\lambda} \ldots, \tag{15}
\]

\[
\text{tr}(...) F_{\mu\nu} \ldots D_{\mu} D_{\nu} X \ldots = \frac{1}{2} i g \, \text{tr}(...) F_{\mu\nu} \ldots [F_{\mu\nu}, X], \tag{16}
\]

which involve the Bianchi identity, the antisymmetry of the field strength tensor and the exchange of derivatives. The aim of these rules is to produce contractions of the field strength tensors among themselves whenever possible.

- **Final arrangement of indices and factors**

The indices still have to be ordered using the antisymmetry of the field strength tensor and the exchange of derivatives using e.g. a minimal contraction distance rule. After this the cyclic equivalent invariants are reduced as in the pure scalar case.

The reduction algorithm produces invariants with smaller number of derivatives and higher number of field strength tensors by exchange of derivatives. Therefore one has to start the reduction with the terms containing a maximum number of derivatives.

For a more specialized situation one can use existing additional symmetries, like the mirror symmetry [10], for a further reduction. For the general case Table 1 gives an overview of the number of invariants in the minimal basis to given order and number \( v \) of occurring scalar background fields.

| order | total | \( v = 0 \) | \( v = 1 \) | \( v = 2 \) | \( v = 3 \) | \( v = 4 \) | \( v = 5 \) | \( v = 6 \) |
|-------|-------|---|---|---|---|---|---|---|
| 1     | 1     | 0 | 1 |   |   |   |   |   |
| 2     | 2     | 1 | 0 | 1 |   |   |   |   |
| 3     | 5     | 2 | 1 | 1 | 1 |   |   |   |
| 4     | 18    | 7 | 5 | 4 | 1 | 1 |   |   |
| 5     | 105   | 36| 36| 23| 7 | 2 | 1 |   |
| 6     | 902   | 300| 329| 191| 63| 16| 2 | 1 |

Table 1: Number of basis invariants in different orders of the expansion.

The computerization of the calculation splits into two parts. The first consists of the expansion of the interaction exponentials and the Wick contractions. As
in the scalar case this is conveniently done with FORM [11] up to $O(T^6)$. The second part of the calculation consists of performing the $\tau$-integrals and the application of the basis algorithm to the raw coefficients. The coding of the basis algorithm requires a rule based system for symbolic manipulation with fast and flexible pattern matching. For this purpose we chose the new language ‘M’ developed by P. Overmann [12]. The reduction has been completed for the coefficients up to $O(T^5)$, the reduction of $O_6$ is currently done. Up to $O(T^5)$ the results have been checked to be equivalent with the result obtained from a modified non–recursive heat kernel method [13]. Additionally, a check with the results of [1] has been done up to $O(T^4)$.

The expression for $O_5$ is too large to be presented here and can be found in [9]. The results to $O(T^4)$ read (absorbing the coupling constant $g$ into the fields, $F_{\kappa\lambda\mu\nu} \equiv D_\kappa D_\lambda F_{\mu\nu}$ etc.):

\[
\begin{align*}
O_1 & = V, \\
O_2 & = V^2 + \frac{1}{6} F_{\kappa\lambda} F_{\lambda\kappa}, \\
O_3 & = V^3 + \frac{1}{2} V_\kappa V_\kappa + \frac{1}{2} V F_{\kappa\lambda} F_{\lambda\kappa} - \frac{2}{15} i F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\kappa} + \frac{1}{20} F_{\kappa\lambda\mu} F_{\kappa\mu\lambda}, \\
O_4 & = V^4 + 2 V V_\kappa V_\kappa + \frac{1}{3} V_\kappa V_\lambda V_\kappa + \frac{3}{5} V^2 F_{\kappa\lambda} F_{\lambda\kappa} + \frac{2}{5} V F_{\kappa\lambda} V F_{\lambda\kappa}
- \frac{4}{5} i F_{\kappa\lambda} V_\kappa V_\kappa - \frac{8}{15} i V F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\kappa} + \frac{1}{5} V F_{\kappa\lambda\mu} F_{\kappa\mu\lambda} - \frac{2}{15} F_{\kappa\lambda} F_{\lambda\mu} V_{\mu\kappa}
+ \frac{1}{3} F_{\kappa\lambda} F_{\mu\lambda\kappa} V_\mu + \frac{1}{3} F_{\kappa\lambda} V_\mu F_{\mu\lambda\kappa} + \frac{2}{35} F_{\kappa\lambda\mu} F_{\mu\nu} V_{\nu\mu} + \frac{4}{35} F_{\kappa\lambda\mu} F_{\mu\nu} V_{\nu\mu}
- \frac{1}{21} F_{\kappa\lambda} F_{\mu\nu} F_{\nu\mu} V_{\nu\mu} - \frac{8}{105} i F_{\kappa\lambda} F_{\mu\nu\kappa} F_{\nu\mu} - \frac{6}{35} i F_{\kappa\lambda} F_{\mu\nu\kappa} F_{\mu\nu} + \frac{11}{420} F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\kappa\nu\mu} + \frac{1}{70} F_{\kappa\lambda\mu\nu} F_{\lambda\kappa\nu\mu}.
\end{align*}
\]

In conclusion, we have obtained the inverse mass expansion of the one–loop effective action for a massive scalar particle in the background of a non Abelian gauge field and a matrix valued scalar potential up to $O(T^5)$, reduced to a minimal set of unique basis invariants. Within the framework of the worldline path integral method it is currently feasible to compute the coefficient $O_6$. Since the basis reduction of this coefficient has not been completed yet, the results will be presented in a forthcoming publication [13].
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