A NOTE ON COHAMILTONIAN GRAPHS

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ABSTRACT. A connected graph \( G \) is said cohamiltonian if it has a bond with \(|E(G)| + |V(G)| + 2\) edges, such bond is said cohamiltonian. A cohamiltonian bond in a connected planar graph corresponds to a hamiltonian cycle of its dual. The problem to determine if a graph is cohamiltonian is NP-complete, as we establish here. We also prove a conjecture of Haidong Wu: that the \( n \)-dimensional hypercube \( Q_n \) is cohamiltonian. These results are based on the fact that \( B \) is a cohamiltonian bond of \( G \) if and only if the two connected components of \( G\backslash B \) are induced trees of \( G \), which is also proved here.

Key words: Cohamiltonian graph, bond, induced tree.

A bond in a graph is a minimal edge-cut. A bond in a graph \( G \) has at most \(|E(G)| − |V(G)| + 2\) edges. When the size of a bond is this number, we say that the bond is cohamiltonian. If \( G \) is a planar graph with dual graph \( G^* \), then each cohamiltonian bond of \( G \) is a hamiltonian cycle of \( G^* \). The problem of determining if a graph has a cohamiltonian bond is NP-complete as we prove in the next theorem:

**Theorem 1.** Let \( G \) be a graph. The following problems are NP-complete:

(a) Determining if \( G \) has a cohamiltonian bond

(b) Determining if \( G \) has a (not necessarily proper) 2-coloring such that each color class induces a tree in \( G \).

The \( n \)-dimensional hypercube \( Q_n \) may be defined as the graph with \( \{0, 1\}^n \) as vertex-set and such that two vertices are adjacent if and only if they differ by exactly one coordinate. We prove the following theorem, conjectured by Haidong Wu:

**Theorem 2.** For each positive integer \( n \), the \( n \)-dimensional hypercube has a cohamiltonian bond.

Theorems[1] and[2] are proved in the end of the paper. For proving Theorem[2] we establish the following criterion:

**Theorem 3.** A subset \( B \subseteq E(G) \) is a cohamiltonian bond of \( G \) if and only if \( aG\backslash B \) has two induced trees of \( G \) as connected components.

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Indeed:

\[ 2 \geq |V(T_1)| - |E(T_1)| + |V(T_2)| - |E(T_2)| = |V(G)| - |E(G)| + |B| \]

and the equality holds above if and only if \( T_1 \) and \( T_2 \) are trees and if and only if \( |B| = |E(G)| - |V(G)| + 2 \).

Suppose that \( B \) is a cohamiltonian bond of \( G \). By the minimality of \( B \) as edge-cut, \( G \setminus B \) has exactly two connected components \( T_1 \) and \( T_2 \), which are trees as we saw before. If, say, \( T_1 \) is not an induced tree of \( G \), then there is an edge \( e \in B \) with both end-vertices in \( T_1 \). This implies that \( B - e \) contradicts the minimality of \( B \) as edge-cut.

In other hand, suppose that, for \( B \subseteq E(G) \), \( G \setminus B \) has exactly two induced trees \( T_1 \) and \( T_2 \) as connected components. As we saw before, \( |B| = |E(G)| - |V(G)| + 2 \). We have to prove the minimality of \( B \) as edge-cut to check that \( B \) is a bond. If \( A \subseteq B \) is an edge-cut, then, as \( T_1 \) and \( T_2 \) are induced, each edge of \( B - A \) has an end-vertex in each one of \( T_1 \) and \( T_2 \), a contradiction. \( \square \)

Let \( G \) be a graph. A cut \((S, T)\) of \( G \) is said cohamiltonian if \( G[S] \) and \( G[T] \) are trees. If \( G[S] \) and \( G[T] \) are forests with two connected components each, we say that \((S, T)\) is a quasi-cohamiltonian cut. By Theorem 3 \( B \) is a cohamiltonian bond of \( G \) if and only if there is a cohamiltonian cut \((S, T)\) of \( G \) such that \( B \) is the set of edges of \( G \) crossing \((S, T)\). We use the symbol “\( \Delta \)” for the operation of symmetric difference of sets. For \( X \subseteq V(G) \), we denote \((S, T) \Delta X := (S \Delta X, T \Delta X)\). If \((S, T)\) is a cohamiltonian cut of \( G \) and \( I \) and \( J \) are disjoint pairs of vertices of \( G \) we say that \((I, J)\) is a quartet of \((S, T)\) if:

1. \((Q1)\) \(|S \cap I| = |S \cap J| = |T \cap I| = |T \cap J| = 1\).
2. \((Q2)\) \((S, T) \Delta I\) is cohamiltonian cut of \( G \),
3. \((Q3)\) \((S, T) \Delta J\) is a quasi-cohamiltonian cut of \( G \)
4. \((Q4)\) Each connected component of \( G[S \Delta J] \cup G[T \Delta J] \) has exactly one vertex of \( I \cup J \).

The cartesian product \( G \times H \) of two graphs \( G \) and \( H \) is defined as the graph with vertex set \( V(G) \times V(H) \) and edge-set as the union of \( \{(u, w)(v, w) : uv \in E(G) \text{ and } w \in V(H)\} \) and \( \{(w, u)(w, v), uv \in E(H) \text{ and } w \in V(G)\} \). Note that \( Q_{n+1} \cong Q_n \times Q_1 \). We will use this recursion to prove Theorem 2 using the next result.

**Theorem 4.** If \( G \) has a cohamiltonian cut with a quartet, then so has \( G \times Q_1 \).

Proof. Let us denote \((X, Y) + (Z, W) := (X \cup Z, Y \cup W)\) and \((X, Y) = (Y, X)\). Recall that \( V(Q_1) = \{0, 1\} \). For each \( i \in \{0, 1\} \) and each graph \( G \), set \( X \) or vertex \( v \), we denote \( G_i = G \times Q_1[[i]] \), \( X_i = X \times \{i\} \) and \( v_i := (v, i) \). Moreover, let \( H := G \times Q_1 \). The next claim is easy to check:
(1) Suppose that \(|i, j| = \{0, 1\}\) and \(X, Y, Z \subseteq V(G)\) induce trees in \(G\). If \(G[X]\) and \(G[Y]\) are connected components of \(G[X \cup Y]\) and \(|X \cap Z| = |Y \cap Z| = 1\), then \(H[X_i \cup Z_j]\) and \(H[X_i \cup Y_j \cup Z_j]\) are trees.

Suppose that \((I, J)\) is a quartet for a cohamiltonian cut \(P = (S, T)\) of \(G\). We shall prove that \(Q := (P_0 + P_1)\Delta I_0 = (P_0\Delta I_0) + P_1\) is a cohamiltonian cut of \(H\) with \((J_1, I_0)\) as a quartet. By (1) for \(X = S\Delta I\) and \(Z = T, (S_0\Delta I_0) \cup T_1\) induces a tree in \(H\). The same holds for \((T_0\Delta I_0) \cup S_1\). Thus, \(Q\) is a cohamiltonian cut of \(H\). It is left to prove that \((Q1), (Q2), (Q3)\) and \((Q4)\) hold for \((J_1, I_0)\). It is straightforward to check that \(Q\Delta I_0 = P_0 + P_1\) is an anti-cohamiltonian cut of \(H\) and, therefore, \((Q2)\) holds. It is also easy to check \((Q1)\) and \((Q4)\). It is left to prove that \(Q\Delta I_0\) is a cohamiltonian cut of \(H\) for \((Q3)\) to hold. As \((I, J)\) is a quartet for \((S, T)\), by \((Q4)\), \(S\Delta J\) has a partition into sets \(X\) with \(X \cap I = \{x\}\), \(Y \cap J = \{y\}\), \(X \cap J = Y \cap I = \emptyset\) and \(G[X]\) and \(G[Y]\) as connected components of \(G[S\Delta J]\). Note that \(X \cap (T\Delta I) = \{x\}\) and \(Y \cap (T\Delta I) = \{y\}\). By (1) for \(Z = T\Delta I, (S_1\Delta J_1) \cup (T_0\Delta I_0) = X_1 \cup Y_1 \cup (T_0\Delta I_0)\) is a tree. Analogously, \((T_1\Delta J_1) \cup (S_0\Delta I_0)\) is also a tree. Therefore, \(Q\Delta I_0\) is a hamiltonian partition of \(H\) and \((Q3)\) holds.

**Proof of Theorem 2.** The result is clear for \(n \leq 2\). By Theorem 3, it is enough to prove that each \(Q_n\) has a cohamiltonian cut with a quartet for \(n \geq 3\). By Theorem 4 and from the fact that \(Q_{n+1} \equiv Q_n \times Q_1\) for each \(n \geq 3\), it follows by induction that the result holds provided \(Q_3\) has a cohamiltonian cut with a quartet. This is presented in Figure 1. We define a cohamiltonian cut \((S, T)\) of \(Q_3\) setting \(S\) as the set of gray vertices and \(T\) the set of black vertices in Figure 1. For \(I = \{i_1, i_2\}\) and \(J := \{j_1, j_2\}\) as in Figure 1, \((I, J)\) is a quartet of \((S, T)\). This can be verified in the representations of \((S, T)\Delta J\) and \((S, T)\Delta I\) in Figures 2 and 3.

**Figure 1.** \((S, T)\)  
**Figure 2.** \((S, T)\Delta J\)  
**Figure 3.** \((S, T)\Delta I\)

**Proof of Theorem 3.** Garey, Johnson and Endre Tarjan [2] proved that the problem of determining if a planar graph \(G\) is hamiltonian is NP-complete and this problem reduces to problem (a). Moreover, problem (a) reduces to problem (b). So, it is left to prove that problem (c) is in the NP class. For this, we will reduce it to the feasibility of a system of quadratic inequalities on 0-1 integer variables, which was reduced by Fortet [1] to a 0-1 integer linear programming, a problem in the NP-class, as proved by Kannan and Monma [3]. We will treat each induced tree.
as a rooted tree whose vertices are distributed in levels. The level of a vertex in a rooted tree is its distance from the root. The possible levels are in $L := \{0, \ldots, |V(G)|\}$. Let $C$ be the set of colors. So, for each $(v, c, l) \in V(G) \times C \times L$, we define a 0-1 variable $x(v, c, l)$, which shall assume value one if and only if $v$ has color $c$ and is in level $l$ of the tree induced by color $c$. Now, it is easy to check that an attribution of values to the variables $x(v, c, l)$ corresponds to such a partition if and only if the following conditions are satisfied:

- Each vertex has a unique color and level:
  $$\forall v \in V(G) : \sum_{c \in C} \sum_{l \in L} x(v, l, c) = 1.$$

- Each color colors a unique vertex of level zero:
  $$\forall c \in C : \sum_{v \in V(G)} x(v, c, 0) = 1.$$

- For each $l \in L - \{0\}$, if $v$ has color $c$ and level $l$, then there is a unique edge $vu \in E(G)$ such that $u$ has color $c$ and level $l - 1$:
  $$\forall v \in V(G), \forall c \in C, \forall l \in L : x(v, c, l) \left(1 - \sum_{u \in \mathcal{N}_G(v)} x(u, c, l - 1)\right) = 0.$$

- If $v$ is a vertex of color $c$ and level $l$ and $u$ is a neighbor of $v$, then $c$ does no color $u$ or $u$ has level $l - 1$ or $l + 1$:
  $$\forall v \in V(G), \forall c \in C, \forall l_v \in L, \forall u \in \mathcal{N}_G(v) : x(v, c, l_v) \left(\sum_{l_u \in L - \{l_v \pm 1\}} x(u, c, l_u)\right) = 0.$$

\[\square\]

**References**

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