CLUSTERED COLOURING IN MINOR-CLOSED CLASSES

SERGEY NORIN*, ALEX SCOTT†, PAUL D. SEYMOUR‡, DAVID R. WOOD§

Received August 8, 2017
Revised July 30, 2018
Online First October 28, 2019

The clustered chromatic number of a class of graphs is the minimum integer \( k \) such that for some integer \( c \) every graph in the class is \( k \)-colourable with monochromatic components of size at most \( c \). We prove that for every graph \( H \), the clustered chromatic number of the class of \( H \)-minor-free graphs is tied to the tree-depth of \( H \). In particular, if \( H \) is connected with tree-depth \( t \), then every \( H \)-minor-free graph is \((2^{t+1} - 4)\)-colourable with monochromatic components of size at most \( c(H) \). This provides the first evidence for a conjecture of Ossona de Mendez, Oum and Wood (2016) about defective colouring of \( H \)-minor-free graphs. If \( t = 3 \), then we prove that 4 colours suffice, which is best possible. We also determine those minor-closed graph classes with clustered chromatic number 2. Finally, we develop a conjecture for the clustered chromatic number of an arbitrary minor-closed class.

1. Introduction

In a vertex-coloured graph, a monochromatic component is a connected component of the subgraph induced by all the vertices of one colour. A graph \( G \) is \( k \)-colourable with clustering \( c \) if each vertex can be assigned one of \( k \) colours such that each monochromatic component has at most \( c \) vertices. We shall consider such colourings, where the first priority is to minimise the number of colours, with small clustering as a secondary goal. With this

*Supported by NSERC grant 418520.
†Supported by a Leverhulme Trust Research Fellowship.
‡Supported by ONR grant N00014-14-1-0084 and NSF grant DMS-1265563.
§Supported by the Australian Research Council.

Mathematics Subject Classification (2010): 05C83; 05C15
viewpoint the following definition arises. The *clustered chromatic number* of a graph class $\mathcal{G}$, denoted by $\chi_*(\mathcal{G})$, is the minimum integer $k$ such that, for some integer $c$, every graph in $\mathcal{G}$ has a $k$-colouring with clustering $c$. See [24] for a survey on clustered graph colouring.

This paper studies clustered colouring in minor-closed classes of graphs. A graph $H$ is a *minor* of a graph $G$ if a graph isomorphic to $H$ can be obtained from some subgraph of $G$ by contracting edges. A class of graphs $\mathcal{M}$ is *minor-closed* if for every graph $G \in \mathcal{M}$ every minor of $G$ is in $\mathcal{M}$, and some graph is not in $\mathcal{M}$. For a graph $H$, let $\mathcal{M}_H$ be the class of $H$-minor-free graphs (that is, not containing $H$ as a minor). Note that we only consider simple finite graphs.

As a starting point, consider Hadwiger’s Conjecture, which states that every graph containing no $K_t$-minor is properly $(t-1)$-colourable. This conjecture is easy for $t \leq 4$, is equivalent to the 4-colour theorem for $t = 5$, is true for $t = 6$ [19], and is open for $t \geq 7$. The best known upper bound on the chromatic number is $O(t \sqrt{\log t})$, independently due to Kostochka [10,11] and Thomason [21,22]. This conjecture is widely considered to be one of the most important open problems in graph theory; see [20] for a survey.

Clustered colourings of $K_t$-minor-free graphs provide an avenue for attacking Hadwiger’s Conjecture. Kawarabayashi and Mohar [9] first proved an $O(t)$ upper bound on $\chi_*(\mathcal{M}_{K_t})$. In particular, they proved that every $K_t$-minor-free graph is $\lceil \frac{31}{2} t \rceil$-colourable with clustering $f(t)$, for some function $f$. The number of colours in this result was improved to $\lceil \frac{7t-3}{2} \rceil$ by Wood [23], to $4t-4$ by Edwards, Kang, Kim, Oum and Seymour [5], to $3t-3$ by Liu and Oum [13], and to $2t-2$ by Norin [15]. Thus $\chi_*(\mathcal{M}_{K_t}) \leq 2t-2$. See [8,7] for analogous results for graphs excluding odd minors. For all of these results, the function $f(t)$ is very large, often depending on constants from the Graph Minor Structure Theorem. Van den Heuvel and Wood [6] proved the first such result with $f(t)$ explicit. In particular, they proved that every $K_t$-minor-free graph is $(2t-2)$-colourable with clustering $\lceil \frac{t-2}{2} \rceil$. The result of Edwards et al. [5] mentioned below implies that $\chi_*(\mathcal{M}_{K_t}) \geq t-1$. Dvořák and Norin [4] have announced a proof that $\chi_*(\mathcal{M}_{K_t}) = t-1$.

Now consider the class $\mathcal{M}_H$ of $H$-minor-free graphs for an arbitrary graph $H$. The maximum chromatic number of a graph in $\mathcal{M}_H$ is at most $O(|V(H)| \sqrt{\log |V(H)|})$ and is at least $|V(H)| - 1$ (since $K_{|V(H)|-1}$ is $H$-minor-free), and Hadwiger’s Conjecture would imply that $|V(H)| - 1$ is the answer. However, for clustered colourings, fewer colours often suffice. For example, Dvořák and Norin [4] proved that graphs embeddable on any fixed surface are 4-colourable with bounded clustering, whereas the chromatic number is $\Theta(\sqrt{g})$ for surfaces of Euler genus $g$. Van den Heuvel and Wood [6]
proved that $K_{2,t}$-minor-free graphs are 3-colourable with clustering $t - 1$, and that $K_{3,t}$-minor-free graphs are 6-colourable with clustering $2t$. These results show that $\chi^*(\mathcal{M}_H)$ depends on the structure of $H$, unlike the usual chromatic number which only depends on $|V(H)|$.

At the heart of this paper is the following question: what property of $H$ determines $\chi^*(\mathcal{M}_H)$? The following definitions help to answer this question. Let $T$ be a rooted tree. The depth of $T$ is the maximum number of vertices on a root–to–leaf path in $T$. The closure of $T$ is obtained from $T$ by adding an edge between every ancestor and descendent in $T$. The connected tree-depth of a graph $H$, denoted by $\text{td}(H)$, is the minimum depth of a rooted tree $T$ such that $H$ is a subgraph of the closure of $T$. This definition is a variant of the more commonly used definition of the tree-depth of $H$, denoted by $\text{td}(H)$, which equals the maximum connected tree-depth of the connected components of $H$. See [14] for background on tree-depth. If $H$ is connected, then $\text{td}(H) = \overline{\text{td}}(H)$. In fact, $\text{td}(H) = \overline{\text{td}}(H)$ unless $H$ has two connected components $H_1$ and $H_2$ with $\text{td}(H_1) = \text{td}(H_2) = \text{td}(H)$, in which case $\overline{\text{td}}(H) = \text{td}(H) + 1$. We choose to work with connected tree-depth to avoid this distinction.

The following result is the primary contribution of this paper; it is proved in Section 2.

**Theorem 1.** For every graph $H$, $\chi^*(\mathcal{M}_H)$ is tied to the (connected) tree-depth of $H$. In particular,

$$\overline{\text{td}}(H) - 1 \leq \chi^*(\mathcal{M}_H) \leq 2\overline{\text{td}}(H) + 1 - 4.$$ 

The upper bound in Theorem 1 gives evidence for, and was inspired by, a conjecture of Ossona de Mendez, Oum and Wood [16], which we now introduce. A graph $G$ is $k$-colourable with defect $d$ if each vertex of $G$ can be assigned one of $k$ colours so that each vertex is adjacent to at most $d$ neighbours of the same colour; that is, each monochromatic component has maximum degree at most $d$. The defective chromatic number of a graph class $\mathcal{G}$, denoted by $\chi_\Delta(\mathcal{G})$, is the minimum integer $k$ such that, for some integer $d$, every graph in $\mathcal{G}$ is $k$-colourable with defect $d$. Every colouring of a graph with clustering $c$ has defect $c - 1$. Thus, the defective chromatic number of a graph class is at most its clustered chromatic number. Ossona de Mendez et al. [16] conjectured the following behaviour for the defective chromatic number of $\mathcal{M}_H$.

**Conjecture 2 ([16]).** For every graph $H$,

$$\chi_\Delta(\mathcal{M}_H) = \overline{\text{td}}(H) - 1.$$
Ossona de Mendez et al. [16] proved the lower bound, \( \chi(\mathcal{M}_H) \geq \overline{td}(H) - 1 \), in Conjecture 2. This follows from the observation that the closure of the rooted complete \( c \)-ary tree of depth \( k \) is not \((k-1)\)-colourable with clustering \( c \). The lower bound in Theorem 1 follows since \( \chi(\mathcal{M}_H) \leq \chi(\mathcal{M}_K_{s,t}) = \min\{s, t\} \) for every class. The upper bound in Conjecture 2 is known to hold in some special cases. Edwards et al. [5] proved it if \( H = K_t \); that is, \( \chi(\mathcal{M}_{K_t}) = t - 1 \), which can be thought of as a defective version of Hadwiger’s Conjecture. Ossona de Mendez et al. [16] proved the upper bound in Conjecture 2 if \( \overline{td}(H) \leq 3 \) or if \( H \) is a complete bipartite graph. In particular, \( \chi(\mathcal{M}_{K_{s,t}}) = \min\{s, t\} \).

Theorem 1 provides some evidence for Conjecture 2 by showing that \( \chi(\mathcal{M}_H) \) and \( \chi(\mathcal{M}_H) \) are bounded from above by some function of \( \overline{td}(H) \). This was previously not known to be true.

While it is conjectured that \( \chi(\mathcal{M}_H) = \overline{td}(H) - 1 \), the following lower bound, proved in Section 2.3, shows that \( \chi(\mathcal{M}_H) \) might be larger, thus providing some distinction between defective and clustered colourings.

**Theorem 3.** For each \( k \geq 2 \), there is a graph \( H_k \) with \( \overline{td}(H_k) = \overline{td}(H_k) = k \) such that
\[
\chi(\mathcal{M}_{H_k}) \geq 2k - 2.
\]

We conjecture an analogous upper bound:

**Conjecture 4.** For every graph \( H \),
\[
\chi(\mathcal{M}_H) \leq 2 \overline{td}(H) - 2.
\]

A further contribution of the paper is to precisely determine the minor-closed graph classes with clustered chromatic number 2. This result is introduced and proved in Conjecture 3. Section 4 studies clustered colourings of graph classes excluding so-called fat stars as a minor. This leads to a proof of Conjecture 4 in the \( \overline{td}(H) = 3 \) case. We conclude in Section 5 with a conjecture about the clustered chromatic number of an arbitrary minor-closed class that generalises Conjecture 4.

### 2. Tree-depth Bounds

The main goal of this section is to prove that \( \chi(\mathcal{M}_H) \) is bounded from above by some function of \( \overline{td}(H) \). We actually provide two proofs. The first proof depends on deep results from graph structure theory and gives no explicit bound on the clustering. The second proof is self-contained, but gives a worse upper bound on the number of colours. Both proofs have their own merits, so we include both.
Let $C\langle h,k \rangle$ be the closure of the rooted complete $k$-ary tree of depth $h$. (Here each non-leaf node has exactly $k$ children.)

If $r$ is a vertex in a connected graph $G$ and $V_i := \{v \in V(G) : \text{dist}_G(v,r) = i\}$ for $i \geq 0$, then $V_0, V_1, \ldots$ is called the BFS layering of $G$ starting at $r$.

### 2.1. First Proof

The first proof depends on the following Erdős-Pósa Theorem by Robertson and Seymour [18]. For a graph $H$ and integer $p \geq 1$, let $pH$ be the disjoint union of $p$ copies of $H$.

**Theorem 5** ([18]; see [17, Lemma 3.10]). For every non-empty graph $H$ with $c$ connected components and for all integers $p, w \geq 1$, for every graph $G$ with treewidth at most $w$ and containing no $pH$ minor, there is a set $X \subseteq V(G)$ of size at most $pwc$ such that $G - X$ has no $H$ minor.

The next lemma is the heart of our proof.

**Lemma 6.** For all integers $h, k, w \geq 1$, every $C\langle h,k \rangle$-minor-free graph $G$ of treewidth at most $w$ is $(2^h - 2)$-colourable with clustering $kw$.

**Proof.** We proceed by induction on $h \geq 1$, with $w$ and $k$ fixed. The case $h = 1$ is trivial since $C\langle 1,k \rangle$ is the 1-vertex graph, so only the empty graph has no $C\langle 1,k \rangle$ minor, and the empty graph is 0-colourable with clustering 0. Now assume that $h \geq 2$, the claim holds for $h-1$, and $G$ is a $C\langle h,k \rangle$-minor-free graph with treewidth at most $w$. Let $V_0, V_1, \ldots$ be the BFS layering of $G$ starting at some vertex $r$.

Fix $i \geq 1$. Then $G[V_i]$ contains no $kC\langle h-1,k \rangle$ as a minor, as otherwise contracting $V_0 \cup \cdots \cup V_{i-1}$ to a single vertex gives a $C\langle h,k \rangle$ minor (since every vertex in $V_i$ has a neighbour in $V_{i-1}$). Since $G$ has treewidth at most $w$, so does $G[V_i]$. By Theorem 5 with $H = C\langle h-1,k \rangle$ and $c = 1$, there is a set $X_i \subseteq V_i$ of size at most $kw$, such that $G[V_i \setminus X_i]$ has no $C\langle h-1,k \rangle$ minor. By induction, $G[V_i \setminus X_i]$ is $(2^{h-1} - 2)$-colourable with clustering $kw$. Use one new colour for $X_i$, thus $G[V_i]$ is $(2^{h-1} - 1)$-colourable with clustering $kw$.

Use disjoint sets of colours for even and odd $i$, and colour $r$ by one of the colours used for even $i$. No edge joins $V_i$ with $V_j$ for $j \geq i + 2$. Thus $G$ is $(2^h - 2)$-coloured with clustering $kw$. 

To drop the assumption of bounded treewidth, we use the following result of DeVos, Ding, Oporowski, Sanders, Reed, Seymour and Vertigan [3], the proof of which depends on the graph minor structure theorem.
Theorem 7 ([3]). For every graph $H$ there is an integer $w$ such that for every graph $G$ containing no $H$-minor, there is a partition $V_1, V_2$ of $V(G)$ such that $G[V_i]$ has treewidth at most $w$, for $i \in \{1, 2\}$.

Lemma 6 and Theorem 7 imply:

Lemma 8. For all integers $h, k \geq 1$, there is an integer $g(h, k)$, such that every $C\langle h, k \rangle$-minor-free graph $G$ is $(2^{h+1} - 4)$-colourable with clustering at most $g(h, k)$.

Fix a graph $H$. By definition, $H$ is a subgraph of $C\langle \text{td}(H), |V(H)| \rangle$. Thus every $H$-minor-free graph contains no $C(\text{td}(H), |V(H)|)$-minor. Hence, Lemma 8 implies

$$\chi_*(\mathcal{M}_H) \leq 2^{\text{td}(H)+1} - 4,$$

which is the upper bound in Theorem 1.

Note Theorem 26 below improves the $h=3$ case in Lemma 6, which leads to a small constant-factor improvement in Theorem 1 for $h \geq 3$.

2.2. Second Proof

We now present our second proof that $\chi_*(\mathcal{M}_H)$ is bounded from above by some function of $\text{td}(H)$. This proof is self-contained (not using Theorems 5 and 7).

Let $T$ be a rooted tree. Recall that the closure of $T$ is the graph $G$ with vertex set $V(T)$, where two vertices are adjacent in $G$ if one is an ancestor of the other in $T$. The weak closure of $T$ is the graph $G$ with vertex set $V(T)$, where two vertices are adjacent in $G$ if one is a leaf and the other is one of its ancestors. For $h, k \geq 1$, let $T\langle h, k \rangle$ be the rooted complete $k$-ary tree of depth $h$. Let $W\langle h, k \rangle$ be the weak closure of $T\langle h, k \rangle$.

Lemma 9. For $h, k \geq 2$, the graph $W\langle h, k \rangle$ contains $C\langle h, k - 1 \rangle$ as a minor.

Proof. Let $r$ be the root vertex. Colour $r$ blue. For each non-leaf vertex $v$, colour $k-1$ children of $v$ blue and colour the other child of $v$ red. Let $X$ be the set of blue vertices $v$ in $T\langle h, k \rangle$, such that every ancestor of $v$ is blue. Note that $X$ induces a copy of $T\langle h, k - 1 \rangle$ in $T\langle h, k \rangle$. Let $v$ be a non-leaf vertex in $X$. Let $w$ be the red child of $v$, and let $T_v$ be the subtree of $T\langle h, k \rangle$ rooted at $w$. Then every leaf of $T_v$ is adjacent in $W\langle h, k \rangle$ to $v$ and to every ancestor of $v$. Contract $T_v$ and the edge $vw$ into $v$. Now $v$ is adjacent to every ancestor of $v$ in $X$. Do this for each non-leaf vertex in $X$. Note that $T_u$ and $T_v$ are disjoint for distinct non-leaf vertices $u, v \in X$. Thus, we obtain $C\langle h, k - 1 \rangle$ as a minor of $W\langle h, k \rangle$.
A model of a graph \( H \) in a graph \( G \) is a collection \( \{J_x: x \in V(H)\} \) of pairwise disjoint subtrees of \( G \) such that for every \( xy \in E(H) \) there is an edge of \( G \) with one end in \( V(J_x) \) and the other end in \( V(J_y) \). Observe that a graph contains \( H \) as a minor if and only if it contains a model of \( H \).

**Lemma 10.** For \( h \geq 2 \) and \( k \geq 1 \), if a graph \( G \) contains \( W \langle h, 6k \rangle \) as a minor, then \( G \) contains subgraphs \( G' \) and \( G'' \), both containing \( W \langle h, k \rangle \) as a minor, such that \(|V(G') \cap V(G'')| \leq 1\).

**Proof.** Let \( \{J_x: x \in V(W \langle h, 6k \rangle)\} \) be a model of \( W \langle h, 6k \rangle \) in \( G \). Let \( r \) be the root vertex of \( W \langle h, 6k \rangle \). We may assume that for each leaf vertex \( x \) of \( T \langle h, 6k \rangle \), there is exactly one edge between \( J_x \) and \( J_r \).

Let \( Q \) be a tree obtained from \( J_r \) by splitting vertices, where:

- \( Q \) has maximum degree at most 3,
- \( J_r \) is a minor of \( Q \); let \( \{Q_v: v \in V(J_r)\} \) be the model of \( J_r \) in \( Q \), so each edge \( vw \) of \( J_r \) corresponds to an edge of \( Q \) between \( Q_v \) and \( Q_w \),
- there is a set \( L \) of leaf vertices in \( Q \), and a bijection \( \phi \) from \( L \) to the set of leaves of \( T \langle h, 6k \rangle \), such that for each leaf \( x \) of \( T \langle h, 6k \rangle \), if the edge between \( J_x \) and \( J_r \) in \( G \) is incident with vertex \( v \) in \( J_r \), then \( \phi^{-1}(x) \) is a vertex \( z \) in \( L \cap Q_v \), in which case we say \( x \) and \( z \) are associated.

Let \( L' \subseteq L \). Apply the following ‘propagation’ process in \( T \langle h, 6k \rangle \). Initially, say that the vertices in \( \phi(L') \) are alive with respect to \( L' \). For each parent vertex \( y \) of leaves in \( T \langle h, 6k \rangle \), if at least 2\( k \) of its 6\( k \) children are alive with respect to \( L' \), then \( y \) is also alive with respect to \( L' \). Now propagate up \( T \langle h, 6k \rangle \), so that a non-leaf vertex \( y \) of \( T \langle h, 6k \rangle \) is alive if and only if at least 2\( k \) of its children are alive with respect to \( L' \). Say \( L' \) is good if \( r \) is alive with respect to \( L' \).

For an edge \( vw \) of \( Q \) let \( L_{vw} \) be the set of vertices in \( L \) in the subtree of \( Q - vw \) containing \( v \), and let \( L_{uw} \) be the set of vertices in \( L \) in the subtree of \( Q - vw \) containing \( w \). Since \( L \) is the disjoint union of \( L_{uw} \) and \( L_{vw} \), every leaf vertex of \( T \langle h, 6k \rangle \) is in exactly one of \( \phi(L_{uw}) \) or \( \phi(L_{vw}) \). By induction, every vertex in \( T \langle h, 6k \rangle \) is alive with respect to \( L_{uw} \) or \( L_{vw} \) (possibly both). In particular, \( L_{uw} \) or \( L_{vw} \) is good (possibly both).

Suppose that both \( L_{uw} \) and \( L_{vw} \) are good. Then at least 2\( k \) children of \( r \) are alive with respect to \( L_{vw} \), and at least 2\( k \) children of \( r \) are alive with respect to \( L_{uw} \). Thus there are disjoint sets \( A \) and \( B \), each consisting of \( k \) children of \( r \), where every vertex in \( A \) is alive with respect to \( L_{uw} \), and every vertex in \( B \) is alive with respect to \( L_{uw} \). We now define a set of vertices, said to be chosen by \( v \), all of which are alive with respect to \( L_{uw} \). First, each vertex in \( A \) is chosen by \( v \). Then for each non-leaf vertex \( z \) chosen by
v, choose \( k \) children of \( z \) that are also alive with respect to \( L_{vw} \), and say they are \textit{chosen} by \( v \). Continue this process down to the leaves of \( T\langle h, 6k \rangle \). We now define the graph \( G' \), which is initially empty. For each vertex \( z \) chosen by \( v \), add the subgraph \( J_z \) to \( G' \). Furthermore, for each leaf vertex \( z \) of \( T\langle h, 6k \rangle \) chosen by \( v \) and for each ancestor \( y \) of \( z \) chosen by \( v \), add the edge in \( G \) between \( J_z \) and \( J_y \) to \( G' \). Define \( G'' \) analogously with respect to \( B \) and \( L_{vw} \). At this point, \( G' \) and \( G'' \) are disjoint.

The edge \( vw \) in \( Q \) either corresponds to an edge or a vertex of \( J_r \). First suppose that \( vw \) corresponds to an edge \( ab \) of \( J_r \), where \( v \) is in \( Q_a \) and \( w \) is in \( Q_b \). Let \( J_r^1 \) be the subtree of \( J_r - ab \) containing \( a \). Add \( J_r^1 \) to \( G' \), plus the edge in \( G \) between \( J_r^1 \) and \( J_z \) for each leaf \( z \) of \( T\langle h, 6k \rangle \) chosen by \( v \). Similarly, let \( J_r^2 \) be the subtree of \( J_r - ab \) containing \( b \), and add \( J_r^2 \) to \( G'' \), plus the edge in \( G \) between \( J_r^2 \) and \( J_z \) for each leaf \( z \) of \( T\langle h, 6k \rangle \) chosen by \( w \). Observe that \( G' \) and \( G'' \) are disjoint, and they both contain \( W\langle h, k \rangle \) as a minor, as desired.

Now consider the case in which \( vw \) corresponds to a vertex \( z \) in \( J_r \); that is, \( v \) and \( w \) are both in \( Q_z \). Let \( J_r^1 \) be the subtree of \( J_r \) corresponding to the subtree of \( Q - vw \) containing \( v \) (which includes \( z \)). Add \( J_r^1 \) to \( G' \), plus the edge in \( G \) between \( J_r^1 \) and \( J_z \) for each leaf \( z \) of \( T\langle h, 6k \rangle \) chosen by \( v \). Similarly, let \( J_r^2 \) be the subtree of \( J_r \) corresponding to the subtree of \( Q - vw \) containing \( w \) (which includes \( z \)). Add \( J_r^2 \) to \( G'' \), plus the edge in \( G \) between \( J_r^2 \) and \( J_z \) for each leaf \( z \) of \( T\langle h, 6k \rangle \) chosen by \( w \). Observe that both \( G' \) and \( G'' \) contain \( W\langle h, k \rangle \) as a minor, and \( V(G_1) \cap V(G_2) = \{ z \} \), as desired.

We may therefore assume that for each edge \( vw \) of \( Q \), exactly one of \( L_{vw} \) and \( L_{vw} \) is good. Orient \( vw \) towards \( v \) if \( L_{vw} \) is good, and towards \( w \) if \( L_{vw} \) is good. Since at most one leaf of \( T\langle h, 6k \rangle \) is associated with each leaf of \( Q \), each edge incident with a leaf of \( Q \) is oriented away from the leaf. Since \( Q \) is a tree, \( Q \) contains a sink vertex \( v \), which is therefore not a leaf. Let \( w_1, w_2 \) and possibly \( w_3 \) be the neighbours of \( v \) in \( Q \). Let \( L_i \) be the set of vertices in \( L \) in the subtree of \( Q - vw_i \) containing \( w_i \). Since \( vw_i \) is oriented towards \( v \), with respect to \( vw_i \), the set \( L_i \) is not good. Since no leaf of \( T\langle h, 6k \rangle \) is associated with \( v \), the sets \( \phi(L_1), \phi(L_2) \) and \( \phi(L_3) \) partition the leaves of \( T\langle h, 6k \rangle \). Since each non-leaf vertex \( y \) in \( T\langle h, 6k \rangle \) has \( 6k \) children, \( y \) is alive with respect to at least one of \( L_1, L_2 \) or \( L_3 \). In particular, at least one of \( L_1, L_2 \) or \( L_3 \) is good. This is a contradiction.

\textbf{Theorem 11.} Let \( f(h) := \frac{1}{6}(4^h - 4) \) for every \( h \geq 1 \). Then there is a function \( g: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \) such that for every \( k \geq 1 \), every graph either contains \( W\langle h, k \rangle \) as a minor or is \( f(h) \)-colourable with clustering \( g(h, k) \).
Proof. We proceed by induction on \( h \geq 1 \). In the base case, \( h = 1 \), since \( W(1, k) \) is the 1-vertex graph, the result holds with \( f(1) = g(1, k) = 0 \). Now assume that \( h \geq 2 \) and the result holds for \( h - 1 \) and all \( k \).

Let \( G \) be a graph, which we may assume is connected. Let \( V_0, V_1, \ldots \) be a BFS layering of \( G \).

Fix \( i \geq 1 \). Let \( s \) be the maximum integer such that \( G[V_i] \) contains \( s \) disjoint subgraphs \( G_1, \ldots, G_s \) each containing a \( W(h - 1, \max\{1, 6^{k-s}k\}) \) minor. First suppose that \( s \geq k \). Then \( G[V_i] \) contains \( k \) disjoint subgraphs each containing a \( W(h - 1, k) \) minor. Contracting \( V_0 \cup \cdots \cup V_{i-1} \) to a single vertex gives a \( W(h, k) \) minor (since every vertex in \( V_i \) has a neighbour in \( V_{i-1} \)), and we are done. Now assume that \( s \leq k - 1 \).

If \( s = 0 \), then \( G[V_i] \) contains no \( W(h - 1, 6^{k-s}k) \) minor. By induction, \( G[V_i] \) is \( f(h - 1) \)-colourable with clustering \( g(h - 1, 6^{k-s}k) \).

Now consider the case that \( s \in [1, k - 1] \). Apply Lemma 10 to \( G_j \) for each \( j \in [1, r] \). Thus \( G_j \) contains subgraphs \( G_j' \) and \( G_j'' \), both containing \( W(h - 1, 6^{k-s}k) \) as a minor, such that \( |V(G_j') \cap V(G_j'')| \leq 1 \). Let \( X := \bigcup_{j=1}^s (V(G_j') \cap V(G_j'')) \). Thus \( |X| \leq s \leq k - 1 \). Let \( A := G[V_i] - \bigcup_{j=1}^s V(G_j') \) and \( B := G[V_i] - \bigcup_{j=1}^s V(G_j'') \). By the maximality of \( s \), the subgraph \( A \) contains no \( W(h - 1, 6^{k-s}k) \) minor (as otherwise \( A, G_j', \ldots, G_j'' \) would give \( s+1 \) pairwise disjoint subgraphs satisfying the requirements). By induction, \( A \) is \( f(h - 1) \)-colourable with clustering \( g(h - 1, 6^{k}k) \) since \( 6^{k-s}k \leq 6^k k \). Similarly, \( B \) is \( f(h - 1) \)-colourable with clustering \( g(h - 1, 6^k k) \). By construction, each vertex in \( G[V_i] \) is in at least one of \( X, A \) or \( B \). Use one new colour for \( X \), which has size at most \( s \leq k - 1 \).

In both cases, \( G[V_i] \) is \( 2f(h - 1) + 1 \)-colourable with clustering \( \max\{g(h - 1, 6^k k), k - 1\} \). Use a different set of \( 2f(h - 1) + 1 \) colours for even \( i \) and for odd \( i \), and colour \( r \) by one of the colours used for even \( i \). No edge joins \( V_i \) with \( V_j \) for \( j \geq i + 2 \). Since \( f(h) = 4f(h - 1) + 2 \), \( G \) is \( f(h) \)-colourable with clustering \( g(h, k) := \max\{g(h - 1, 6^k k), k - 1\} \).

Note that the clustering function \( g(h, k) \) in Theorem 11 satisfies

\[
g(h, k) \leq k6^{k6^k \cdots},
\]

where the number of \( ks \) is \( h \).

Theorem 12. For every graph \( H \),

\[
\chi_*(\mathcal{M}_H) \leq \frac{1}{6}(4\overline{td}(H) - 4).
\]
Proof. Let $G$ be a graph not containing $H$ as a minor. By definition, $H$ is a subgraph of $C(\overline{td}(H), |V(H)|)$. Thus $G$ does not contain $C(\overline{td}(H), |V(H)|)$ as a minor. By Lemma 9, $G$ does not contain $W(\overline{td}(H), |V(H)|+1)$ as a minor. By Theorem 11, there is a constant $c = c(H)$, such that $G$ is $\frac{1}{6}(4\overline{td}(H) - 4)$-colourable with clustering at most $c$.

2.3. Lower Bound

We now prove Theorem 3, where $H_k := C(k,3)$, the closure of the complete ternary tree of depth $k$ (which has tree-depth and connected tree-depth $k$).

Lemma 13. $\chi_*(\mathcal{M}_{C(k,3)}) \geq 2k - 2$ for $k \geq 2$.

Proof. Fix an integer $c$. We now recursively define graphs $G_k$ (depending on $c$), and show by induction on $k$ that $G_k$ has no $(2k - 3)$-colouring with clustering $c$, and $C(k,3)$ is not a minor of $G_k$.

For the base case $k=2$, let $G_2$ be the path on $c+1$ vertices. Then $G_2$ has no $C(2,3) = K_{1,3}$ minor, and $G_2$ has no 1-colouring with clustering $c$.

Assume $G_{k-1}$ is defined for some $k \geq 3$, that $G_{k-1}$ has no $(2k-5)$-colouring with clustering $c$, and $C(k-1,3)$ is not a minor of $G_{k-1}$. As illustrated in Figure 1, let $G_k$ be obtained from a path $(v_1, \ldots, v_{c+1})$ as follows: for $i \in \{1, \ldots, c\}$ add $2c-1$ pairwise disjoint copies of $G_{k-1}$ complete to $\{v_i, v_{i+1}\}$.

![Figure 1. Construction of $G_k$](image)

Suppose that $G_k$ has a $(2k-3)$-colouring with clustering $c$. Then $v_i$ and $v_{i+1}$ receive distinct colours for some $i \in \{1, \ldots, c\}$. Consider the $2c-1$ copies of $G_{k-1}$ complete to $\{v_i, v_{i+1}\}$. At most $c-1$ such copies contain a vertex
assigned the same colour as $v_i$, and at most $c-1$ such copies contain a vertex assigned the same colour as $v_{i+1}$. Thus some copy avoids both colours. Hence $G_{k-1}$ is $(2k-5)$-coloured with clustering $c$, which is a contradiction. Therefore $G_k$ has no $(2k-3)$-colouring with clustering $c$.

It remains to show that $C(k,3)$ is not a minor of $G_k$. Suppose that $G_k$ contains a model $\{J_x : x \in V(C(k,3))\}$ of $C(k,3)$. Let $r$ be the root vertex in $C(k,3)$. Choose the $C(k,3)$-model to minimise $\sum_{x \in V(C(k,3))} |V(J_x)|$. Since $\{v_1, \ldots, v_{c+1}\}$ induces a connected dominating subgraph in $G_k$, by the minimality of the model, $J_r$ is a connected subgraph of $(v_1, \ldots, v_{c+1})$. Say $J_r = (v_i, \ldots, v_j)$. Note that $C(k,3)-r$ consists of three pairwise disjoint copies of $C(k-1,3)$. The model $X$ of one such copy avoids $v_{i-1}$ and $v_{j+1}$ (if these vertices are defined). Since $C(k-1,3)$ is connected, $X$ is contained in a component of $G_k - \{v_{i-1}, \ldots, v_{j+1}\}$ and is adjacent to $(v_i, \ldots, v_j)$. Each such component is a copy of $G_{k-1}$. Thus $C(k-1,3)$ is a minor of $G_{k-1}$, which is a contradiction. Thus $C(k,3)$ is not a minor of $G_k$.

3. 2-Colouring with Bounded Clustering

This section considers the following question: which minor-closed graph classes have clustered chromatic number 2? To answer this question we introduce three classes of graphs that are not 2-colourable with bounded clustering, as illustrated in Figure 2.

The first example is the $n$-fan, which is the graph obtained from the $n$-vertex path by adding one dominant vertex. If the $n$-fan is 2-colourable with clustering $c$, then the underlying path contains at most $c-1$ vertices of the same colour as the dominant vertex, implying that the other colour has at most $c$ monochromatic components each with at most $c$ vertices, and $n \leq c^2 + c - 1$. That is, if $n \geq c^2 + c$, then the $n$-fan is not 2-colourable with clustering $c$.

The second example is the $n$-fat star, which is the graph obtained from the $n$-star (the star with $n$ leaves) as follows: for each edge $vw$ in the $n$-star, add $n$ degree-2 vertices adjacent to $v$ and $w$. Note that the $n$-fat star is $C(3,n)$. Suppose that the $n$-fat star has a 2-colouring with clustering $c \leq n$. Deleting the dominant vertex in the $n$-fat star gives $n$ disjoint $n$-stars. Since $n \geq c$, in at least one of these $n$-stars, no vertex receives the same colour as the dominant vertex, implying there is a monochromatic component on $n+1 \geq c+1$ vertices. Thus, for $n \geq c$ there is no 2-colouring of the $n$-fat star with clustering $c$.

The third example is the $n$-fat path, which is the graph obtained from the $n$-vertex path as follows: for each edge $vw$ of the $n$-vertex path, add $n$
degree-2 vertices adjacent to \(v\) and \(w\). If \(n \geq 2c - 1\), then in every 2-colouring of the \(n\)-fat path with clustering \(c\), adjacent vertices in the underlying path receive the same colour, implying that the underlying path is contained in a monochromatic component with more than \(c\) vertices. Thus, for \(n \geq 2c - 1\) there is no 2-colouring of the \(n\)-fat path with clustering \(c\).

These three examples all need three colours in a colouring with bounded clustering. The main result of this section is the following converse result.

**Theorem 14.** Let \(\mathcal{G}\) be a minor-closed graph class. Then \(\chi^\ast(\mathcal{G}) \leq 2\) if and only if for some integer \(k \geq 2\), the \(k\)-fan, the \(k\)-fat path, and the \(k\)-fat star are not in \(\mathcal{G}\).

Lemma 24 below shows that every graph containing no \(k\)-fan minor, no \(k\)-fat path minor, and no \(k\)-fat star minor is 2-colourable with clustering \(f(k)\) for some explicit function \(f\). Along with the above discussion, this implies Theorem 14. We assume \(k \geq 2\) for the remainder of this section.

The following definition is a key to the proof. For an \(h\)-vertex graph \(H\) with vertex set \(\{v_1, \ldots, v_h\}\), a \(k\)-strong \(H\)-model in a graph \(G\) consists of \(h\) pairwise disjoint connected subgraphs \(X_1, \ldots, X_h\) in \(G\), such that for each edge \(v_iv_j\) of \(H\) there are at least \(k\) vertices in \(V(G) \setminus \bigcup_{i=1}^h V(X_i)\) adjacent to both \(X_i\) and \(X_j\). Note that a vertex in \(V(G) \setminus \bigcup_{i=1}^h V(X_i)\) might count towards this set of \(k\) vertices for distinct edges of \(H\). This definition leads to the following sufficient condition for a graph to contain a \(k\)-fat star or \(k\)-fat path

**Lemma 15.** If a graph \(G\) contains a \(k(k + 1)\)-strong \(H\)-model for some connected graph \(H\) with \(k^k\) edges, then \(G\) contains a \(k\)-fat star or a \(k\)-fat path as a minor.

**Proof.** Use the notation introduced in the definition of \(k\)-strong \(H\)-model. Since \(H\) is connected with \(k^k\) edges, \(H\) contains a \(k\)-vertex path or a \(k\)-leaf star as a subgraph. Suppose that \((v_1, \ldots, v_k)\) is a \(k\)-vertex path in \(H\). For
$i=1,2,\ldots,k-1$, let $N_i$ be a set of $k+1$ vertices in
\[
\left( V(G) \setminus \bigcup_{j=1}^{h} V(X_j) \right) \setminus \bigcup_{j=1}^{i-1} N_j,
\]
each of which is adjacent to both $X_i$ and $X_{i+1}$. Such a set exists since $X_i$ and $X_{i+1}$ have at least $k(k+1)$ common neighbours in $V(G) \setminus \bigcup_{j=1}^{h} V(X_j)$. For $i \in [1,k-1]$, contract one vertex of $N_i$ into $X_i$. Then contract each of $X_1,\ldots,X_h$ into a single vertex. We obtain the $k$-fat path as a minor in $G$. The case of a $k$-leaf star is analogous.

\begin{lemma}
If a connected graph $G$ contains a $(k+2c-2)$-strong $H$-model, for some graph $H$ with $c$ connected components, then $G$ contains a $k$-strong $H'$-model for some connected graph $H'$ with $|E(H')|=|E(H)|$.
\end{lemma}

\begin{proof}
We proceed by induction on $c \geq 1$. The case $c=1$ is vacuous. Assume $c \geq 2$, and the result holds for $c-1$. Let $H_1,\ldots,H_c$ be the components of $H$. We may assume that $H$ has no isolated vertices. Say $X_1,\ldots,X_h$ is a $(k+2c-2)$-strong $H$-model in $G$. For each edge $v_iv_j$ in $H$, let $N_{ij}$ be a set of $k+2c-2$ common neighbours of $X_i$ and $X_j$. For each component $H_a$ of $H$, note that $(\bigcup_{v_i \in V(H_a)} V(X_i)) \cup (\bigcup_{v_i,v_j \in E(H_a)} N_{ij})$ induces a connected subgraph in $G$, which we denote by $G_a$. Since $G$ is connected, there is a path $P$ between $G_a$ and $G_b$, for some distinct $a,b \in [1,c]$, such that no internal vertex of $P$ is in $G_1 \cup \cdots \cup G_c$. Note that $P$ might be a single vertex. For some edge $v_iv_j$ in $H_a$ and some edge $v_jv_{j'}$ in $H_b$, without loss of generality, $P$ joins some vertex $x$ in $V(X_i) \cup N_{ii'}$ and some vertex $y$ in $V(X_j) \cup N_{jj'}$. Let $H'$ be the graph obtained from $H$ by identifying $v_i$ and $v_j$ into a new vertex $v_0$. Now $H'$ has $c-1$ components and $|E(H')|=|E(H)|$. Define $X_0 := X_i \cup X_j \cup P$. If $x \notin V(X_i)$, then add the edge between $x$ and $X_i$ to $X_0$. Similarly, if $y \notin V(X_j)$, then add the edge between $y$ and $X_j$ to $X_0$. Remove $x$ and/or $y$ from $N_{\alpha\beta}$ for each edge $v_{\alpha}v_{\beta}$ of $H'$. Now $|N_{\alpha\beta}| \geq k+2(c-1)-2$. We obtain a $(k+2(c-1)-2)$-strong $H'$-model in $G$. By induction, $G$ contains a $k$-strong $H''$-model for some connected graph $H''$ with $|E(H'')|=|E(H)|$.
\end{proof}

\begin{lemma}
If a connected graph $G$ contains a $3k^k$-strong $H$-model for some graph $H$ with at least $k^k$ edges, then $G$ contains a $k$-fat star or a $k$-fat path as a minor.
\end{lemma}

\begin{proof}
We may assume that $H$ has exactly $k^k$ edges and has no isolated vertices. Say $H$ has $c$ connected components. Then $c \leq k^k$ and $3k^k \geq k^2+k+2c-2$. Hence $G$ contains a $(k^2+k+2c-2)$-strong $H$-model. The result then follows from Lemmas 15 and 16.
\end{proof}
Lemma 18. Let $G$ be a connected graph such that $\deg_G(v) \geq 2\ell k$ for some non-cut-vertex $v$ and integers $k, \ell \geq 1$. Then $G$ contains a $k$-fan as a minor, or $G$ contains a connected subgraph $X$ and $v$ has $\ell$ neighbours not in $X$ and all adjacent to $X$ (thus contracting $X$ gives a $K_{2,\ell}$ minor).

Proof. Let $r$ be a vertex of $G - v$. For each $w \in N_G(v)$, let $P_w$ be a $wr$-path in $G - v$. If $|P_w \cap N_G(v)| \geq k$ for some $w \in N_G(v)$, then $G$ contains a $k$-fan minor. Now assume that $|P_w \cap N_G(v)| \leq k - 1$ for each $w \in N_G(v)$. Let $H$ be the digraph with vertex set $N_G(v)$, where $N_H^+(w) := V(P_w) \cap N_G(v)$ for each vertex $w$. Thus $H$ has maximum outdegree at most $k - 1$, and the underlying undirected graph of $H$ has average degree at most $2k - 2$. Since $|V(H)| \geq 2\ell k$, by Turán’s Theorem, $H$ contains a stable set $S$ of size $\ell$. Let $X := \bigcup\{P_w : w \in S\} - S$, which is connected since $S$ is stable. Each vertex in $S$ is adjacent to $v$ and to $X$, as desired.

Lemma 19. Let $G$ be a graph with distinct vertices $v_1, \ldots, v_k$, such that $C := G - \{v_1, \ldots, v_k\}$ is connected and $\deg_C(v_i) \geq k^3$ for each $i \in [1, k]$. Then $G$ contains a $k$-fan or $k$-fat star as a minor.

Proof. The idea of the proof is to attempt to build a $k$-fan model by constructing a subtree $X$ such that each $v_i$ is adjacent to a subset $S_i$ of $k$ leaves of $X$ (where the $S_i$ are disjoint). We construct $X$ and the $S_i$ by adding, one at a time, paths to some neighbour $w$ of some $v_i$ to increase the size of $S_i$. We always choose a neighbour at maximal distance from some root vertex, among all neighbours of all $v_i$ for which $S_i$ is not yet large enough: this ensures that later paths will not pass through the sets $S_i$ that have been previously constructed.

We now formalise this idea. Let $r$ be a vertex in $C$. Let $V_0, V_1, \ldots, V_n$ be a BFS layering of $C$ starting at $r$. Initialise $t := n$ and $X := \{r\}$ and $S_i := \emptyset$ for $i \in [1, k]$ and $S := \emptyset$. The following properties trivially hold:

1. $X$ is a (connected) subtree of $C$ rooted at $r$ with (non-root) leaf set $S$.
2. $S_i \cap S_j = \emptyset$ for distinct $i, j \in [1, k]$.
3. $S_i$ is a set of at most $k + 1$ neighbours of $v_i$ for $i \in [1, k]$ (and so $|S| \leq k(k + 1)$).
4. $|N_{C - V(X)}(v_i)| \geq k^3 - 1 - (k - 1)|S| > 0$ for $i \in [1, k]$.

Now execute the following algorithm, which maintains properties (0)–(4). Think of $V_i$ as the ‘current’ layer.

While $|S_i| \leq k$ for some $i \in [1, k]$ repeat the following: If $V_i \cap N_{C - V(X)}(v_i) = \emptyset$ for all $i \in [1, k]$ with $|S_i| \leq k$, then let $t := t - 1$. Properties (0)–(4) are trivially maintained. Otherwise, let $w$ be a vertex in $V_i \cap N_{C - V(X)}(v_i)$ for
some \( i \in [1,k] \) with \( |S_i| \leq k \). Since \( V_0, V_1, \ldots, V_n \) is a BFS layering of \( C \) rooted at \( r \) and \( r \) is in \( X \), there is a path \( P \) from \( w \) to \( X \) consisting of at most one vertex from each of \( V_0, \ldots, V_t \), and with no internal vertices in \( X \). By (0) and since \( w \notin S \), \( P \) avoids \( S \). By (1), the endpoint of \( P \) in \( X \) is not a leaf of \( X \). If \( P \) contains at least \( k \) vertices in \( N_C(v_j) \) for some \( j \in [1,k] \), then \( G \) contains a \( k \)-fan minor and we are done. Now assume that \( P \) contains at most \( k-1 \) vertices in \( N_C(v_j) \) for each \( j \in [1,k] \). Let \( S_i := S_i \cup \{w\} \) and \( S := S \cup \{w\} \) and \( X := X \cup P \). Now \( w \) is a leaf of \( X \), and property (1) is maintained. Properties (0), (2) and (3) are maintained by construction. Property (4) is maintained since \( |S| \) increases by 1 and \( P \) contains at most \( k-1 \) vertices in \( N_C(v_j) \) for each \( j \in [1,k] \).

The algorithm terminates when \( |S_i| = k + 1 \) for each \( i \in [1,k] \). Delete \( C - V(X) \). Contract \( X - S \) (which is connected by (1)) to a single vertex \( z \). Since \( S \) is the set of leaves of \( X \), each vertex in \( S_i \) is adjacent to both \( v_i \) and \( z \). Contract one edge between \( v_i \) and \( S_i \) for each \( i \in [1,k] \). We obtain the \( k \)-fat star as a minor.

**Lemma 20.** Let \( G \) be a bipartite graph with bipartition \( A, B \), such that at least \( p \) vertices in \( A \) have degree at least \( k|A| \), and every vertex in \( B \) has degree at least 2. Then \( G \) contains a \( k \)-strong \( H \)-model for some graph \( H \) with at least \( p/2 \) edges.

**Proof.** Let \( H \) be the graph with \( V(H) := A \) where \( vw \in E(H) \) whenever \( |N_G(v) \cap N_G(w)| \geq k \). Since every vertex in \( B \) has degree at least 2, every vertex in \( A \) with degree at least \( k|A| \) is incident with some edge in \( H \). Thus \( H \) has at least \( p/2 \) edges. By construction, \( G \) contains a \( k \)-strong \( H \)-model.

For the remainder of this section, let \( d := (k+2)k^k(18k^{2k+1}+1) \). A vertex \( v \) is high-degree if \( \deg(v) \geq d \), otherwise \( v \) is low-degree.

**Lemma 21.** If a 2-connected graph \( G \) has at least \( (k+2)k^k \) high-degree vertices, then \( G \) contains a \( k \)-fat path, a \( k \)-fat star, or a \( k \)-fan as a minor.

**Proof.** Let \( A \) be a set of exactly \( (k+2)k^k \) high-degree vertices in \( G \). Let \( C_1, \ldots, C_p \) be the components of \( G - A \). Say \( (v,C_j) \) is a heavy pair if \( v \in A \) and \( v \) has at least \( 6k^{k+1} \) neighbours in \( C_j \). Since \( 6k^{k+1} \geq k^3 \), by Lemma 19, if some \( C_j \) is in at least \( k \) heavy pairs, then \( G \) contains a \( k \)-fan or \( k \)-fat star as a minor, and we are done. Now assume that each \( C_j \) is in fewer than \( k \) heavy pairs. Let \( h \) be the total number of heavy pairs. Then there is a set \( P \) of at least \( h/k \) heavy pairs containing at most one heavy pair for each component \( C_j \). For each such heavy pair \( (v,C_j) \), by Lemma 18 with \( \ell = 3k^k \), \( G[V(C_j) \cup \{v\}] \) contains a \( k \)-fan as a minor (and we are done) or a \( K_{2,3k^k} \).
minor, where $G[\{v\}]$ is the subgraph corresponding to one of the vertices in the colour class of size 2 in $K_{2,3k^k}$. We obtain a $3k^k$-strong $H$-model for some graph $H$, where $|E(H)| = |P| \geq h/k$. If $h/k \geq k^k$, then we are done by Lemma 17. Now assume that $h < k^{k+1}$. In particular, the number of vertices in $A$ that are in a heavy pair is less than $k^k+1$. Let $A'$ be the set of vertices in $A$ in no heavy pair; thus $|A'| \geq 2k^k$. Let $H$ be the bipartite graph with bipartition $A,B$, where there is one vertex $w_j$ in $B$ for each component $C_j$, and $v \in A$ is adjacent to $w_j \in B$ if and only if $v$ is adjacent to some vertex in $C_j$. In $H$, every vertex in $A'$ has degree at least $(d - |A|)/6k^{k+1}$, which is at least $3k^k|A|$. (Note that $d$ is defined so that this property holds.) Since $G$ is 2-connected, each $C_j$ is adjacent to at least two vertices in $A$. Thus, every vertex in $B$ has degree at least 2 in $H$. By Lemma 20, $H$ contains a $3k^k$-strong model of a graph with at least $|A'|/2 \geq k^k$ edges. By Lemma 17 we are done.

Lemma 22. Let $V_0, V_1, \ldots$ be a BFS layering in a connected graph $G$. If $G[V_i \cup V_{i+1} \cup \cdots \cup V_{i+c}]$ contains a path on at least $k^{c+1}$ vertices for some $i,c \geq 0$, then $G$ contains a $k$-fan minor.

Proof. We proceed by induction on $c$. Let $P$ be a path in $G[V_i \cup V_{i+1} \cup \cdots \cup V_{i+c}]$ on $k^{c+1}$ vertices. First suppose that $P$ contains $k$ vertices $v_1, \ldots, v_k$ in $V_i$ (which must happen in the base case $c=0$). Each vertex $v_i$ has a neighbour in $V_{i-1}$. Thus, contracting $G[V_0 \cup \cdots \cup V_{i-1}]$ into a single vertex and contracting $P$ between $v_i$ and $v_{i+1}$ to an edge (for $i \in [1,k-1]$) gives a $k$-fan minor. Now assume that $P$ contains at most $k-1$ vertices in $V_i$ and $c \geq 1$. Thus $P-V_i$ has at least $k^{c+1}-(k-1)$ vertices and at most $k$ components. Thus, some component of $P-V_i$ has at least $\lceil (k^{c+1}-k+1)/k \rceil = k^c$ vertices and is contained in $G[V_{i+1} \cup V_{i+2} \cup \cdots \cup V_{i+c}]$. By induction, $G$ contains a $k$-fan minor.

Say a vertex $v$ in a coloured graph is properly coloured if no neighbour of $v$ gets the same colour as $v$.

Lemma 23. Let $G$ be a 2-connected graph containing no $k$-fan, $k$-fat star or $k$-fat path as a minor. Let $h$ be the number of high-degree vertices in $G$. Let $r$ be a vertex in $G$. Then $G$ is 2-colourable with clustering at most $dk^{3(k+2)k^k}$. Moreover, if $h = 0$, then we can additionally demand that $r$ is properly coloured.
Proof. Let $V_0, V_1, \ldots$ be the BFS layering of $G$ starting at $r$.

First suppose that $h = 0$. Colour each vertex $v \in V_i$ by $i \mod 2$. Then $r$ is properly coloured. Every monochromatic component is contained in some $V_i$. Suppose that some component $X$ of $G[V_i]$ has at least $d^k$ vertices. Thus $i \geq 1$. Since $G$ and thus $X$ has maximum degree at most $d$, $X$ contains a path of $k$ vertices. Contracting $G[V_0 \cup \cdots \cup V_{i-1}]$ into a single vertex gives a $k$-fan minor. This contradiction shows that the 2-colouring has clustering at most $d^k$.

Now assume that $h \geq 1$. By Lemma 21, $h \leq (k+2)k^k$. Colour all the high-degree vertices black. Let $I$ be the set of integers $i \geq 0$ such that $V_i$ contains a high-degree vertex. Colour all the low-degree vertices in $\bigcup \{V_i : i \in I\}$ white.

Let $V_i, V_{i+1}, \ldots, V_{i+c}$ be a maximal sequence of layers with no high-degree vertices, where $c \geq 0$. Thus $V_{i-1}$ is empty or contains a high-degree vertex. Similarly, $V_{i+c+1}$ is empty or contains a high-degree vertex. If $c$ is even, then colour $V_i \cup V_{i+2} \cup \cdots \cup V_{i+c}$ white and colour $V_{i+1} \cup V_{i+3} \cup \cdots \cup V_{i+c-1}$ black. If $c$ is odd, then colour $V_i \cup V_{i+2} \cup \cdots \cup V_{i+c-1}$ and $V_{i+c}$ white, and colour $V_{i+1} \cup V_{i+3} \cup \cdots \cup V_{i+c-2}$ black. Note that if $c \geq 2$, then at least one of $V_{i+1}, \ldots, V_{i+c-1}$ is black.

We show now that each black component $X$ has bounded size. If $X$ contains some high-degree vertex, then every vertex in $X$ is high-degree and $|X| \leq h \leq (k+2)k^k$. Now assume that $X$ contains no high-degree vertices. Say $X$ intersects $V_j$. Since each black layer is preceded by and followed by a white layer, $X$ is contained in $V_j$. Every vertex in $X$ has degree at most $d$ in $G$. Thus if $X$ has at least $d^k$ vertices, then $X$ contains a path of length $k$, and contracting $V_0 \cup \cdots \cup V_{j-1}$ to a single vertex gives a $k$-fan. Hence $X$ has at most $d^k$ vertices.

Finally, let $X$ be a white component. Then $X$ is contained within at most $3h \leq 3(k+2)k^k$ consecutive layers (since in the notation above, if all of $V_i, V_{i+1}, \ldots, V_{i+c}$ are white, then $c \leq 1$). Suppose that $|X| \geq d^{k(3(k+2)k^k)}$. Since $X$ has maximum degree at most $d$, $X$ contains a path of length $k^3(k+2)k^k$. Thus, Lemma 22 with $c + 1 = 3(k+2)k^k$ implies that $G$ contains a $k$-fan minor. Hence $|X| \leq d^{k^3(k+2)k^k}$.

We now complete the proof of Theorem 14.

Lemma 24. Let $G$ be a graph containing no $k$-fan, no $k$-fat path, and no $k$-fat star as a minor. Then $G$ is 2-colourable with clustering $kd^{k^3(k+2)k^k}$.

Proof. We may assume that $G$ is connected. Let $r$ be a vertex of $G$. If $B$ is a block of $G$ containing $r$, then consider $B$ to be rooted at $r$. If $B$ is a block of $G$ not containing $r$, then consider $B$ to be rooted at the unique vertex in
that separates $B$ from $r$. Say $(B,v)$ is a high-degree pair if $B$ is a block of $G$ and $v$ has high-degree in $B$. Note that one vertex might be in several high-degree pairs.

Suppose that some vertex $v$ is in at least $k$ high-degree pairs with blocks $B_1,\ldots,B_k$. Since $d \geq 2k(k+1)$, by Lemma 18 with $\ell = k+1$, for $i \in [k]$, there is a connected subgraph $X_i$ in $B_i - v$ and there is a set $N_i \subseteq N_{B_i}(v) \setminus V(X_i)$ of size $k+1$, such that each vertex in $N_i$ is adjacent to $X_i$. For $i \in [1,k]$, contract $X_i$ into a single vertex, and contract one edge between $v$ and $N_i$. We obtain a $k$-fat star as a minor. Now assume that each vertex is in fewer than $k$ high-degree pairs.

Colour each block $B$ in non-decreasing order of the distance in $G$ from $r$ to the root of $B$. Let $B$ be a block of $G$ rooted at $v$ (possibly equal to $r$). Then $v$ is already coloured in the parent block of $B$. Let $h_B$ be the number of high-degree pairs involving $B$. By Lemma 23, $B$ is $2$-colourable with clustering at most $d^{3k^2(k+2)k^k}$, such that if $h_B = 0$, then $v$ is properly coloured. Permute the colours in $B$ so that the colour assigned to $v$ matches the colour assigned to $v$ by the parent block. Then the monochromatic component containing $v$ is contained within the parent block of $B$ along with those blocks rooted at $v$ that form a high-degree pair with $v$. As shown above, there are at most $k$ such blocks. Thus, each monochromatic component has at most $kd^{3k^2(k+2)k^k}$ vertices.

4. Excluding a Fat Star

This section considers colourings of graphs excluding a fat star. We need the following more general lemma.

Lemma 25. For every planar graph $H$,

$$\chi_s(\mathcal{M}_H) \leq 2\chi_\Delta(\mathcal{M}_H).$$

Proof. The grid minor theorem of Robertson and Seymour [18] says that every graph in $\mathcal{M}_H$ has tree-width at most some function $w(H)$. (Chekuri and Chuzhoy [2] recently showed that $w$ can be taken to be polynomial in $|V(H)|$.) Alon, Ding, Oporowski, and Vertigan [1] observed that every graph with tree-width $w$ and maximum degree $\Delta$ is $2$-colourable with clustering $24w\Delta$. Let $k := \chi_\Delta(\mathcal{M}_H)$. That is, every $H$-minor-free graph $G$ is $k$-colourable with monochromatic components of maximum degree at most some function $d(H)$. Apply the above result of Alon et al. [1] to each monochromatic component. Thus $G$ is $2k$-colourable with clustering $24w(H)d(H)$. Hence $\chi_s(\mathcal{M}_H) \leq 2k$. 

\[\]
A variant of Lemma 25 holds for arbitrary graphs $H$ with “2” replaced by “3”. The proof uses a result of Liu and Oum [13] in place of the result of Alon et al. [1]; see [5,6].

**Theorem 26.** For $k \geq 3$, the clustered chromatic number of the class of graphs containing no $k$-fat star minor equals 4.

**Proof.** As illustrated in Figure 2, the $k$-fat star is planar. Ossona de Mendez et al. [16] proved that graphs containing no $k$-fat star minor are 2-colourable with defect $O(k^{13})$. Thus, Lemma 25 implies that the clustered chromatic number of the class of graphs containing no $k$-fat star is at most 4. To obtain a bound on the clustering, note that a result of Leaf and Seymour [12] implies that every graph containing no $k$-fat star minor has tree-width $O(k^{15})$. Since the 3-fat star is $C\langle 3,3 \rangle$, Lemma 13 implies that for $k \geq 3$, the clustered chromatic number of the class of graphs containing no $k$-fat star minor is at least 4.

Every graph $H$ with $\overline{td}(H) \leq 3$ is a subgraph of the $k$-fat star for some $k \leq |V(H)|$. Thus Theorem 26 implies Conjecture 4 in the case of connected tree-depth 3.

**Corollary 27.** For every graph $H$ with $\overline{td}(H) \leq 3$, 

$$\chi_\star(M_H) \leq 4.$$  

We can push this result further.

**Theorem 28.** For every graph $H$ with $td(H) \leq 3$, 

$$\chi_\star(M_H) \leq 5.$$  

**Proof.** Say $H$ has $p$ components. Each component of $H$ is a subgraph of the $k$-fat star for some $k \leq |V(H)|$. Let $H'$ consist of $p$ pairwise disjoint copies of the $k$-fat star. Let $G$ be an $H$-minor-free graph. Thus $G$ is also $H'$-minor-free. By the Grid Minor Theorem of Robertson and Seymour [18] and since $H'$ is planar, $G$ has treewidth at most $w = w(H')$. By Theorem 5, there is a set $X$ of at most $(p - 1)(w - 1)$ vertices in $G$, such that $G - X$ contains no $k$-fat star as a minor. By Theorem 26, $G - X$ is 4-colourable with clustering at most some function of $H$. Assign vertices in $X$ a fifth colour. Thus $G$ is 5-colourable with clustering at most some function of $H$. 


5. A Conjecture about Clustered Colouring

We now formulate a conjecture about the clustered chromatic number of an arbitrary minor-closed class of graphs. Consider the following recursively defined class of graphs. Let \( \mathcal{X}_{1,c} := \{ P_{c+1}, K_{1,c} \} \). Here \( P_{c+1} \) is the path with \( c+1 \) vertices, and \( K_{1,c} \) is the star with \( c \) leaves. As illustrated in Figure 3, for \( k \geq 2 \), let \( \mathcal{X}_{k,c} \) be the set of graphs obtained by the following three operations. For the first two operations, consider an arbitrary graph \( G \in \mathcal{X}_{k-1,c} \).

- Let \( G' \) be the graph obtained from \( c \) disjoint copies of \( G \) by adding one dominant vertex. Then \( G' \) is in \( \mathcal{X}_{k,c} \).
- Let \( G^+ \) be the graph obtained from \( G \) as follows: for each \( k \)-clique \( D \) in \( G \), add a stable set of \( k(c-1)+1 \) vertices complete to \( D \). Then \( G^+ \) is in \( \mathcal{X}_{k,c} \).
- If \( k \geq 3 \) and \( G \in \mathcal{X}_{k-2,c} \), then let \( G^{++} \) be the graph obtained from \( G \) as follows: for each \( (k-1) \)-clique \( D \) in \( G \), add a path of \( (c^2-1)(k-1)+(c+1) \) vertices complete to \( D \). Then \( G^{++} \) is in \( \mathcal{X}_{k,c} \).

A vertex-coloured graph is \textit{rainbow} if every vertex receives a distinct colour.

\textbf{Lemma 29.} For every \( c \geq 1 \) and \( k \geq 2 \), for every graph \( G \in \mathcal{X}_{k,c} \), every colouring of \( G \) with clustering \( c \) contains a rainbow \( K_{k+1} \). In particular, no graph in \( \mathcal{X}_{k,c} \) is \( k \)-colourable with clustering \( c \).

\textbf{Proof.} We proceed by induction on \( k \geq 1 \). In the case \( k = 1 \), every colouring of \( P_{c+1} \) or \( K_{1,c} \) with clustering \( c \) contains an edge whose endpoints receive distinct colours, and we are done. Now assume the claim for \( k-1 \) and for \( k-2 \) (if \( k \geq 3 \)).
Let \( G \in \mathcal{X}_{k-1,c} \). Consider a colouring of \( G' \) with clustering \( c \). Say the dominant vertex \( v \) is blue. At most \( c-1 \) copies of \( G \) contain a blue vertex. Thus, some copy of \( G \) has no blue vertex. By induction, this copy of \( G \) contains a rainbow \( K_k \). With \( v \) we obtain a rainbow \( K_{k+1} \).

Now consider a colouring of \( G' \) with clustering \( c \). By induction, the copy of \( G \) in \( G' \) contains a clique \( w_1,\ldots,w_k \) receiving distinct colours. Let \( S \) be the set of \( k(c-1)+1 \) vertices adjacent to \( w_1,\ldots,w_k \) in \( G' \). At most \( c-1 \) vertices in \( S \) receive the same colour as \( w_i \). Thus some vertex in \( S \) receives a colour distinct from the colours assigned to \( w_1,\ldots,w_k \). Hence \( G' \) contains a rainbow \( K_{k+1} \).

Now suppose \( k \geq 3 \) and \( G \in \mathcal{X}_{k-2,c} \). Consider a colouring of \( G^{++} \) with clustering \( c \). By induction, the copy of \( G \) in \( G^{++} \) contains a clique \( w_1,\ldots,w_{k-1} \) receiving distinct colours. Let \( P \) be the path of \( (c^2-1)(k-1)+(c+1) \) vertices in \( G^{++} \) complete to \( w_1,\ldots,w_{k-1} \). Let \( X_i \) be the set of vertices in \( P \) assigned the same colour as \( w_i \), and let \( X := \bigcup_i X_i \). Thus \( |X_i| \leq c-1 \) and \( |X| \leq (c-1)(k-1) \). Hence \( P-X \) has at most \( (c-1)(k-1)+1 \) components, and \( |V(P-X)| \geq (c^2-1)(k-1)+(c+1)-(c-1)(k-1) = c((c-1)(k-1)+1)+1 \). Some component of \( P-X \) has at least \( c+1 \) vertices, and therefore contains a bichromatic edge \( xy \). Then \( \{w_1,\ldots,w_{k-1}\} \cup \{x,y\} \) induces a rainbow \( K_{k+1} \) in \( G^{++} \).

We conjecture that a minor-closed class that excludes every graph in \( \mathcal{X}_{k,c} \) for some \( c \) is \( k \)-colourable with bounded clustering. More precisely:

**Conjecture 30.** For every minor-closed class \( \mathcal{M} \) of graphs,

\[
\chi_*(\mathcal{M}) = \min\{k : \exists c \mathcal{M} \cap \mathcal{X}_{k,c} = \emptyset\}.
\]

Several comments about Conjecture 30 are in order:

- To prove the lower bound in Conjecture 30, let \( k \) be the minimum integer such that \( \mathcal{M} \cap \mathcal{X}_{k,c} = \emptyset \) for some integer \( c \). Thus, for every integer \( c \) some graph \( G \in \mathcal{X}_{k-1,c} \) is in \( \mathcal{M} \). By Lemma 29, \( G \) has no \( (k-1) \)-colouring with clustering \( c \). Thus \( \chi_*(\mathcal{M}) \geq k \).
- Note that the \( k=1 \) case of Conjecture 30 is trivial: a graph is 1-colourable with bounded clustering if and only if each component has bounded size, which holds if and only if every path has bounded length and every vertex has bounded degree.
- We note that Theorem 14 implies Conjecture 30 with \( k=2 \). If \( G = P_{c+1} \), then \( G' \) is contained in the \( c(c+1) \)-fan and \( G^+ \) is contained in the \( (2c-1) \)-fat path. If \( G = K_{1,c} \), then \( G' \) is the \( c \)-fat star and \( G^+ \) is contained in the \( (2c-1) \)-fat star. It follows that if a minor-closed class \( \mathcal{M} \) excludes every
graph in $\mathcal{X}_{2,c}$ for some $c$, then $\mathcal{M}$ excludes the $c(c+1)$-fan, the $(2c-1)$-fat path, and the $(2c-1)$-fat star. Then $\chi_{\ast}(\mathcal{M}) \leq 2$ by Theorem 14.

- We now relate Conjectures 4 and 30. Fix a graph $H$. Conjecture 30 says that the clustered chromatic number of $\mathcal{M}_H$ equals the minimum integer $k$ such that for some integer $c$, every graph in $\mathcal{X}_{k,c}$ contains $H$ as a minor. Let $k := \bar{t}_{d}(H) \geq 2$. An easy inductive argument shows that every graph in $\mathcal{X}_{2k-2,c}$ contains a $C_{k,c}$ minor. Thus, for a suitable value of $c$, every graph in $\mathcal{X}_{2k-2,c}$ contains $H$ as a minor. Hence, Conjecture 30 implies Conjecture 4.

- Consider the case of excluding the complete bipartite graph $K_{s,t}$ as a minor for $s \leq t$. Van den Heuvel and Wood [6] proved the lower bound, $\chi_{\ast}(\mathcal{M}_{K_{s,t}}) \geq s+1$ for $t \geq \max\{s,3\}$. Their construction is a special case of the construction above. We claim that Conjecture 30 asserts that $\chi_{\ast}(\mathcal{M}_{K_{s,t}}) = s+1$ for $t \geq \max\{s,3\}$. To see this, first note that an easy inductive argument shows that every graph in $\mathcal{X}_{s+1,t}$ contains a $K_{s,t}$ subgraph; thus $\mathcal{M}_{K_{s,t}} \cap \mathcal{X}_{s+1,t} = \emptyset$. Furthermore, another easy inductive argument shows that for all $s,c \geq 1$, there is a graph in $\mathcal{X}_{s,c}$ containing no $K_{s,\max\{s,3\}}$ minor. This implies that $\mathcal{M}_{K_{s,t}} \cap \mathcal{X}_{s,c} \neq \emptyset$ for all $t \geq \max\{s,3\}$. Together these observations show that $\min\{k: \exists c, \mathcal{M}_{K_{s,t}} \cap \mathcal{X}_{k,c} = \emptyset\} = s+1$ for $t \geq \max\{s,3\}$. That is, Conjecture 30 asserts that $\chi_{\ast}(\mathcal{M}_{K_{s,t}}) = s+1$ for $t \geq \max\{s,3\}$. Van den Heuvel and Wood [6] proved the upper bound, $\chi_{\ast}(\mathcal{M}_{K_{s,t}}) \leq 3s$ for $t \geq s$, which was improved to $2s+2$ by Dvořák and Norin [4].

### 6. An Alternative View

We conclude the paper with alternative versions of Conjectures 2 and 30 that shift the focus to characterising minimal minor-closed classes of given defective and clustered chromatic number.

We start with some definitions. Let $H$ and $G$ be two vertex-disjoint graphs, and let $S \subseteq V(G)$. Let $G'$ be obtained from $G \cup H$ by joining every vertex of $S$ to every vertex of $H$ by an edge. Then we say that $G'$ is obtained from $G$ by taking a join with $H$ along $S$. Let $\mathcal{H}$ be a class of graphs. We say that a graph $G'$ is an $\mathcal{H}$-decoration of a graph $G$, if $G'$ is obtained from $G$ by repeatedly taking joins with graphs in $\mathcal{H}$ along cliques of $G$. For a class of graphs $\mathcal{G}$, let $\mathcal{G} \land \mathcal{H}$ denote the class of all minors of $\mathcal{H}$-decorations of graphs in $\mathcal{G}$. One can routinely verify that the $\land$ operation is associative. The examples below show that it is not always commutative.

First, we introduce notation for some minor-closed classes that will serve as the basis for our constructions. Let $\mathcal{I}$ denote the class of graphs on at
most one vertex, let $\mathcal{O}$ denote the class of edgeless graphs, and let $\mathcal{P}$ denote the class of linear forests (that is, subgraphs of paths). Let $\mathcal{T}_d$ denote the class of all graphs of tree-depth at most $d$. Then $\mathcal{T}_1$ is a class of all edgeless graphs. It follows from the alternative definition of tree-depth given in [14, Section 6.1] that $\mathcal{T}_{d+1} = \mathcal{O} \wedge \mathcal{T}_d$.

The operations used in Conjecture 30 can be described as follows.

- Adding a vertex adjacent to several copies of graphs in the class $\mathcal{G}$ (and taking all possible minors) produces the class $\mathcal{I} \wedge \mathcal{G}$.
- Adding stable sets complete to cliques in graphs in $\mathcal{G}$ produces the class $\mathcal{G} \wedge \mathcal{I}$.
- Adding paths complete to cliques in graphs in $\mathcal{G}$ produces the class $\mathcal{G} \wedge \mathcal{P}$.

Note that by definition $\mathcal{G} \wedge \mathcal{H}$ is a minor-closed class for any pair of minor-closed classes $\mathcal{G}$ and $\mathcal{H}$.

We next present an analogue of Lemma 29 using the notions introduced above. A class of graphs $\mathcal{G}$ is $k$-cluster rainbow (respectively, $k$-defect rainbow) if for every $c$ there exists $G \in \mathcal{G}$ such that every colouring of $G$ with clustering (respectively, defect) at most $c$ contains a rainbow clique of size $k$. For example, $\mathcal{I}$ is 1-cluster rainbow and 1-defect rainbow, $\mathcal{P}$ is 2-cluster rainbow, but not 2-defect rainbow. Note that if a class of graphs $\mathcal{G}$ is $k$-cluster rainbow, then clearly $\chi_\star(\mathcal{G}) \geq k$. Similarly, if $\mathcal{G}$ is $k$-defect rainbow, then $\chi_\Delta(\mathcal{G}) \geq k$.

The proof of the following lemma parallels the proof of Lemma 29; we present it for completeness.

**Lemma 31.** Let $\mathcal{G}, \mathcal{H}$ be graph classes, such that $\mathcal{G}$ is $k$-cluster rainbow and $\mathcal{H}$ is $\ell$-cluster rainbow. Then $\mathcal{G} \wedge \mathcal{H}$ is $(k + \ell)$-cluster rainbow.

**Proof.** Fix $c$, and let $G \in \mathcal{G}$ and $H \in \mathcal{H}$ be such that every colouring of $G$ with clustering at most $c$ contains a rainbow clique of size $k$, and every colouring of $H$ with clustering at most $c$ contains a rainbow clique of size $\ell$. Let $J$ be obtained from $G$ by taking a join of $G$ with $H$, $(c-1)k+1$ times along every clique $S$ of $G$. Then $J \in \mathcal{G} \wedge \mathcal{H}$ by definition. It remains to show that every colouring $\phi : V(J) \rightarrow C$ of $J$ for some set of colours $C$ with clustering at most $c$ contains a rainbow clique of size $k + \ell$. By the choice of $J$ there exists a clique $S$ in $G$ of size $k$, which is rainbow in $\phi$. Let $H_1, H_2, \ldots, H_r$ be copies of $H$ glued along $S$ to $G$. By the choice of $H_i$ for every $i$ there exists a clique $S_i$ of size $\ell$ in $H_i$ that is rainbow in $\phi$. Suppose for a contradiction that $S \cup S_i$ is not rainbow for any $i$. Then there exists $s \in S$ with a neighbour of the same colour in $S_i$ for at least $c$ choices of $i$. Thus $s$ belongs to a monochromatic component of size at least $c+1$ in $\phi$, a contradiction. \[\blacksquare\]
Note that an analogue of Lemma 31 also holds for defective colourings. The proof is identical.

Let \( \mathcal{G} \) be a graph class obtained by taking a wedge-product of \( v \) copies of \( \mathcal{I} \) and \( p \) copies of \( \mathcal{P} \) in some order such that \( v + 2p = k + 1 \). Then we say that \( \mathcal{G} \) is \( k \)-cluster critical. By Lemma 31 the clustered chromatic number of a \( k \)-cluster critical class is at least \( k + 1 \). (In fact, it is not difficult to see that equality holds.) For example, the class \( \mathcal{I} \wedge \mathcal{P} \) of minors of fans, the class \( \mathcal{I} \wedge \mathcal{I} \wedge \mathcal{I} \) of minors of fat stars, and the class \( \mathcal{P} \wedge \mathcal{I} \) of minors of fat paths are all possible 2-cluster critical classes. Thus, Theorem 14 is equivalent to the statement that \( \chi(\mathcal{G}) \leq 2 \) if and only if \( \mathcal{G} \) contains no 2-cluster critical class.

The discussion above implies that for all \( k \) and \( c \) every graph in \( \mathcal{X}_{k,c} \) is a member of some \( k \)-cluster critical class. Conversely, for all \( n, k \) there exists \( c \) such that for every graph \( G \in \mathcal{X}_{k,c} \) there exists a \( k \)-cluster critical class \( \mathcal{G} \) such that \( \mathcal{X}_{k,c} \) contains as minors all graphs in \( \mathcal{G} \) on at most \( n \) vertices. Thus Conjecture 30 can be reformulated as follows.

**Conjecture 32.** Let \( \mathcal{M} \) be a minor-closed class of graphs and \( k \geq 0 \) an integer. Then \( \chi(\mathcal{G}) \geq k + 1 \) if and only if \( \mathcal{G} \not\subset \mathcal{M} \) for some \( k \)-cluster critical class \( \mathcal{G} \).

Similarly, note that the \( k \)-term \( \wedge \)-product \( \wedge^k \mathcal{I} = \mathcal{I} \wedge \mathcal{I} \wedge \ldots \wedge \mathcal{I} \) is the class of minors of connected graphs of tree-depth \( k \) and therefore the following conjecture is equivalent to Conjecture 2.

**Conjecture 33.** Let \( \mathcal{M} \) be a minor-closed class of graphs and \( k \geq 0 \) an integer. Then \( \chi_{\Delta}(\mathcal{G}) \geq k + 1 \) if and only if \( \wedge^{k+1} \mathcal{I} \not\subset \mathcal{M} \).

**Acknowledgement.** This research was initiated at the 2017 Barbados Graph Theory Workshop held at the Bellairs Research Institute. Thanks to the workshop participants for creating a stimulating working environment. Thanks to the referees for several instructive comments.

**References**

[1] N. Alon, G. Ding, B. Oporowski and D. Vertigan: Partitioning into graphs with only small components, *J. Combin. Theory Ser. B* **87** (2003), 231–243.

[2] C. Chekuri and J. Chuzhoy: Polynomial bounds for the grid-minor theorem, *J. ACM* **63** (2016), 40.

[3] M. DeVos, G. Ding, B. Oporowski, D. P. Sanders, B. Reed, P. Seymour and D. Vertigan: Excluding any graph as a minor allows a low tree-width 2-coloring, *J. Combin. Theory Ser. B* **91** (2004), 25–41.

[4] Z. Dvořák and S. Norin: Islands in minor-closed classes. I. Bounded treewidth and separators, 2017, arXiv:1710.02727.
[5] K. Edwards, D. Y. Kang, J. Kim, S. Oum and P. Seymour: A relative of Hadwiger’s conjecture, SIAM J. Discrete Math. 29 (2015), 2385–2388.
[6] J. van den Heuvel and D. R. Wood: Improper colourings inspired by Hadwiger’s conjecture, J. London Math. Soc. 98 (2018), 129–148.
[7] D. Y. Kang and S. Oum: Improper coloring of graphs with no odd clique minor, Combin. Probab. Comput., 2019, arXiv:1612.05372.
[8] K. Kawarabayashi: A weakening of the odd Hadwiger’s conjecture, Combin. Probab. Comput. 17 (2008), 815–821.
[9] K. Kawarabayashi and B. Mohar: A relaxed Hadwiger’s conjecture for list colourings, J. Combin. Theory Ser. B 97 (2007), 647–651.
[10] A. V. Kostochka: The minimum Hadwiger number for graphs with a given mean degree of vertices, Metody Diskret. Analiz. 38 (1982), 37–58.
[11] A. V. Kostochka: Lower bound of the Hadwiger number of graphs by their average degree, Combinatorica 4 (1984), 307–316.
[12] A. Leaf and P. Seymour: Tree-width and planar minors, J. Comb. Theory, Ser. B 111 (2015), 38–53.
[13] C.-H. Liu and S. Oum: Partitioning $H$-minor free graphs into three subgraphs with no large components, J. Combin. Theory Ser. B 128 (2018) 114–133.
[14] J. Nešetřil and P. Ossona de Mendez: Sparsity, vol. 28 of Algorithms and Combinatorics, Springer, 2012.
[15] S. Norin: Conquering graphs of bounded treewidth, 2015, Unpublished manuscript.
[16] P. Ossona de Mendez, S. Oum and D. R. Wood: Defective colouring of graphs excluding a subgraph or minor, Combinatorica 39 (2019), 377–410.
[17] J.-F. Raymond and D. M. Thilikos: Recent techniques and results on the Erdős-Pósa property, Discrete Appl. Math. 231 (2017), 25–43.
[18] N. Robertson and P. Seymour: Graph minors. V. Excluding a planar graph, J. Combin. Theory Ser. B 41 (1986), 92–114.
[19] N. Robertson, P. Seymour and R. Thomas: Hadwiger’s conjecture for $K_6$-free graphs, Combinatorica 13 (1993), 279–361.
[20] P. Seymour: Hadwiger’s conjecture, in: John Forbes Nash Jr. and Michael Th. Rassias, eds., Open Problems in Mathematics, 417–437, Springer, 2015.
[21] A. Thomason: An extremal function for contractions of graphs, Math. Proc. Cambridge Philos. Soc. 95 (1984), 261–265.
[22] A. Thomason: The extremal function for complete minors, J. Combin. Theory Ser. B 81 (2001), 318–338.
[23] D. R. Wood: Contractibility and the Hadwiger conjecture, European J. Combin. 31 (2010), 2102–2109.
[24] D. R. Wood: Defective and clustered graph colouring, Electron. J. Combin., #DS23, 2018.

Sergey Norin

Department of Mathematics and Statistics
McGill University
Montréal, Canada
snorin@math.mcgill.ca

Alex Scott

Mathematical Institute,
University of Oxford,
Oxford, U.K.
scott@maths.ox.ac.uk
