Pattern Formation in Interface Depinning and Other Models: Erratically Moving Spatial Structures

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We study erratically moving spatial structures that are found in a driven interface in a random medium at the depinning threshold. We introduce a bond-disordered variant of the Sneppen model and study the effect of extremal dynamics on the morphology of the interface. We find evidence for the formation of a structure which moves along with the growth site. The time average of the structure, which is defined with respect to the active spot of growth, defines an activity-centered pattern. Extensive Monte Carlo simulations show that the pattern has a tail which decays slowly, as a power law. To understand this sort of pattern formation, we write down an approximate integral equation involving the local interface dynamics and long-ranged jumps of the growth spot. We clarify the nature of the approximation by considering a model for which the integral equation is exactly derivable from an extended master equation. Improvements to the equation are considered by adding a second coupled equation which provides a self-consistent description. The pattern, which defines a one-point correlation function, is shown to have a strong effect on ordinary space-fixed two-point correlation functions. Finally we present evidence that this sort of pattern formation is not confined to the interface problem, but is generic to situations in which the activity at successive time steps is correlated, as for instance in several other extremal models. We present numerical results for activity-centered patterns in the Bak-Sneppen model of evolution and the Zaitsev model of low-temperature creep.

I. INTRODUCTION

Driven interfaces in random media present several features of interest, both with regard to the morphology of the moving interface as well as the dynamics of the growth process. Experiments have been performed on several sorts of systems, ranging from fluid flow in porous media [32] to propagation of burning fronts [33]. These indicate that the disorder in the medium affects the properties of the interface in a crucial way. In particular, the large-distance scaling properties differ considerably from those of interfaces in uniform media. In both theoretical and experimental investigations, it is customary to characterize the spatial structure of the interface by its roughness. The main point of this paper is to show that there is sometimes an unusual sort of pattern formation in the system, which results in the interface acquiring a time-averaged shape. In such situations, this pattern provides an alternative characterization of interface morphology.

A customary measure of the roughness is provided by the exponent $\alpha$, defined by $W \sim L^\alpha$, where $W$ is the root mean squared width of the interface and $L$ is the size of the system. The experiments mentioned above, and others similar to these, report an anomalously large value of $\alpha$ — large compared to the predictions of existing theories for interface growth in nonrandom media [32]. It is recognised that the quenched nature of the disorder in the medium is responsible for this difference in scaling properties of the interface. Unlike thermal noise, which varies rapidly, a portion of the interface subject to a quenched-noise environment continues to experience the same forces until the growth process takes it forward to a new region. The pinning effect of quenched noise has a strong effect on the large scale properties of the interface.

Several theoretical models have been put forward to try to account for the effect of quenched disorder on the properties of the interface; [34] gives an account of some of the early work and the relationship to other problems involving pinning, while [35] is a a recent review. Amongst various proposals put forward to explain anomalous roughening are theories based on continuum equations with quenched disorder [36–38], the inclusion of noise with power-law amplitude [39–41] or long-ranged correlation [42–44], power-law distributions of pinning-center strengths [45], as well as a class of models with microscopic rules based on directed percolation [46–48] and models which relate the large-scale structure to the wetting properties of the invading fluid [49]. A number of these models base the explanation of anomalous roughening on the phenomenon of critical depinning, which is relevant to an interface just at the threshold of motion. In the opposite limit of large velocity, the interface encounters the disorder at any site only for a short time, suggesting that the quenched nature of disorder is not important in this limit, and the
FIG. 1. A schematic picture of the activity-centered pattern in an untilted \((m = 0)\) interface showing (a) the height gradient pattern as defined in Eq. 1, (b) the interface profile obtained by integrating the pattern in (a). The discontinuity at the origin in (a) leads to a cusp at the origin in (b), implying that the active site is most likely at the peak.

The interface behaves much as in a nonrandom medium \([14]\). However this is not true at low velocities of the interface near the depinning threshold. In particular, the limit of zero velocity is thought to be a dynamical critical point \([9]\) where the scaling properties of the interface are strongly affected by the disorder.

In a certain class of models, the quenched disorder enters as barriers of random strengths which impede interface motion. The formation of infinitely long directed percolating paths of these barriers is of special significance, as such paths can block the entire interface effectively \([13, 23, 24]\). The model proposed by Sneppen \([24]\) involves “extremal” dynamics: At each time step, the interface advances only along the weakest of the barriers. Extremal models were first introduced much earlier, in the context of invasion percolation for two-phase fluid flow in porous media, with a non-wetting fluid displacing a wetting one \([27, 28]\). The predictions of this invasion percolation model were borne out by experiments \([8]\).

Dynamical correlation functions involving the center of activity in invasion percolation obey scaling \([31, 32]\), and the process defines a self-organised critical phenomenon \([24]\): the interface organises itself to align along critical paths bordered by large barriers, without the necessity of tuning any external parameters. The model proposed by Sneppen \([24]\) is a modification of the invasion percolation model, incorporating surface tension effects which prevent very strong local convolutions of the interface, resulting in a self-affine, rather than self-similar, geometry.

An extremal model close to the Sneppen model was shown to result \([33]\) from a model of wetting-fluid invasion, by considering an interface advancing by merging meniscus arcs between adjacent pairs of pinning centers.

An interesting feature of extremal dynamics is that it induces strong spatial correlation in sites at which growth occurs \([34, 35]\) at successive time steps. In this paper we introduce and study a variant of the Sneppen model and show that the interface develops an interesting time-averaged structure as a result of correlations. The defining equation for the structure is

\[
\Psi(r) = \frac{1}{2}[(\nabla h(r + R(t)) - m)]
\]  

(1)

where \(R(t)\) is the position of the active site at time \(t\), \(h(r')\) denotes the height at the site \(r'\), \(\langle ... \rangle\) is a time average in the steady state, and \(m\) is the overall slope of the interface. The unusual point is that this structure \(\Psi(r)\) is not fixed in space, but moves with its center always at the growth site, which itself follows an erratic path. The moving origin is crucial to the definition, as time averages performed at a fixed point in space reveal no structure at all. This average structure defines a pattern, and we study its formation numerically and analytically.

Figure 1a depicts schematically the activity-centered pattern in height gradients with respect to the moving origin, for the untilted \((m = 0)\) interface. As can be seen, the interface develops an overall shape which is described by the height-gradient profile \(\Psi(r)\). The tail of the pattern falls slowly at large distances, as a power law. The corresponding height profile of the interface \(h(r)\) with respect to the active site is represented in Fig. 1b. The nature of the pattern is sensitive to tilt, and
Figure 2 shows the height-gradient pattern and height profile for a tilted interface.

We propose that this pattern is a simple way of characterising a new aspect of the morphology of the interface. Traditional ways of characterising the morphology involve, as has already been mentioned, determining the roughness exponent \( \alpha \). However such a definition does not hone in on the overall shape of the interface. In situations such as the one considered in this paper when the interface does develop a nontrivial structure, the pattern is a useful quantitative characterization. An important point about the time-averaged pattern is that Eq. 1 defines a one-point correlation function. As such, it would be expected to strongly influence the properties of customary two-point correlation functions. We verify this by numerically studying two-point correlations. Further we find that this sort of pattern formation is not restricted only to the Sneppen model, but also occurs in other extremal models, such as the Zaitsev model for low-temperature creep [37] and the Bak-Sneppen model [38] of biological evolution, albeit in other quantities.

The plan of this paper is as follows. In Section II we introduce our model and discuss the connection with the problem of directed percolation, well established from earlier studies. In Section III, we discuss the correlations in the location of successive growth sites, a concept central to this paper because of its connection with pattern formation. In Sections IV and V we define the pattern and present numerical results as well as an integral equation which provides an understanding of this sort of pattern formation. We define a model for which the equation is exact and discuss how the approximation can be improved. In Section VI, we discuss the issue of temporal correlations with a view to seeing how they affect the pattern. In Section VII, we present our results for an ordinary two-point correlation function in our model and show that activity-centered pattern formation has to be taken into account in order to understand some features in it. Section VIII deals with pattern formation in other extremal models and we conclude with a summary of our results in Section IX.

II. EXTREMAL MODEL OF INTERFACE DEPINNING

The extremal-model description of fluid-fluid interfaces in porous media is valid when the wetting is dominated by capillary forces, and thermal fluctuations are not important. In the extremal model, the interface advances along the weakest barrier just ahead of it. The appealing feature of the model is that it is self-organised critical; the dynamics, which involves searches for the global minimum at every step, automatically tunes the interface to a critical state at the depinning transition, without the necessity of fixing any external parameter.

In the model proposed by Sneppen [24], the random medium is modelled by a square lattice in which the sites are assigned random numbers \( f \in [0,1] \). The random numbers could signify, for instance, the pore sizes in a porous medium. The interface is a directed path on this lattice and grows only at that perimeter bond with the smallest value of the random number; after every such move, a local rearrangement process [39] ensures the absence of very large slopes.

Extremal dynamics has also been proposed to describe very different situations — for instance, the phenomenon of low-temperature dislocation creep [37,33], crack propagation [34] and biological evolution [38]. We will see in Section VIII that the sort of pattern formation we find in the interface model occurs in these models as well.

A. The Extremal Bond Model

We study a modified version of the Sneppen model in this paper. In this version, hereafter referred to as the Extremal Bond model (EBM), the interface is taken to be a directed path on a square lattice (Fig. 3), with tilted cylindrical boundary conditions [11] which ensure that the mean slope is preserved. To every bond \( k \) on the lattice is pre-assigned a fixed random number \( f_k \) drawn from the interval \([0,1]\). The interface grows at that bond, in front of the interface, which carries the smallest random number. The local growth rules are the following; if the chosen minimal bond has a positive (negative) slope, the sequence of links with negative (positive) slope just below (on the left) also advances, as illustrated in Fig. 3. This preserves the length of the interface as would happen in situations with very high surface tension. The local dynamics of interface adjustment is similar to that for the low-noise Toom interface in [12]; the models are different in that the growth site is picked by the extremal
rule in our case, while it is picked stochastically in the Toom interface model.

The EBM differs from the Sneppen model in that the length of the interface is a strict constant of the motion and the \( f_k \)'s are associated with bonds rather than sites. While this modification does not change the values of any of the large-distance scaling properties of the Sneppen model, it has a few advantages. The interface aligns along directed spanning paths in a percolation problem, just as in the Sneppen model. In our case, the corresponding percolation problem is the diode-resistor percolation problem \([4]\). On the square lattice, it is dual to the directed bond percolation problem \([2]\) which is relatively well-studied. Another advantage is that the problem of interface growth in the EBM is conceptually simplified by the existence of a known one-to-one correspondence between the growing interface and a system of hard-core particles moving on a ring. The two-dimensional problem hence reduces to an effectively one-dimensional one. This also facilitates the numerics. The correspondence between the interface and the hard-core particles is detailed below.

Positive slope links of the interface are represented by particles \((n_j = 1)\) and negative-slope links by holes \((n_j = 0)\); see Fig. 3. The difference in height of the interface between sites \(j_1\) and \(j_2\) is given by \(h_{j_2} - h_{j_1} = \sum_{j=j_1}^{j_2} (2n_j - 1)\). In front of each link of the interface is a bond with a random number \(f\) assigned to it. Correspondingly, the site \(j\) with the particle (hole) representing this link carries a random number \(f_j\). Just as for the interface, at each time step, activity is initiated at the site with the minimum \(f_j\). The update rules for the interface translate to the following dynamics for particles and holes. If the site with minimum \(f_j\) contains a particle (hole), it exchanges with the first hole (particle) to the left (right). All sites hopped over, including the two which exchange the particle and hole, are refreshed by assigning a new set of \(f_j\)'s. This corresponds to the fact that the updated portion of the interface moves ahead and meets a fresh set of \(f\)'s on the square lattice. Because the number of positive slope links (and hence also the number of negative slope links) is conserved for the interface, in the particle-hole terminology, this implies that the number of particles is conserved. Hence we can define a density \(\rho\) for particles on the one-dimensional lattice. This density determines the mean slope \(m = 2p - 1\) of the interface. An untilted interface corresponds to half-filling. The reference direction for determining tilt is the easy direction of directed bond percolation on the square lattice, which is along the 45° line. Tilt refers to any density away from 0.5, which implies a slope different from 45°. The interface advances in a direction perpendicular to the direction of tilt and this translates to a nonzero current of particles on the ring.

### B. Connection to Diode-Resistor Percolation and Directed Percolation

Extremal models of interface depinning make use of a correspondence to the Diode Resistor Percolation (DRP) and Directed Percolation (DP) problems to predict various properties of the interface. In view of this, it is useful to recall some facts about the DP and DRP processes.

In the directed percolation problem, bonds on a lattice are occupied with probability \(p\). At some critical value \(p_c\) an infinite directed path of occupied bonds (in which every step is taken rightward or upward) first forms along a definite direction: on a \(2 - d\) square lattice, this is along the 45° direction. For \(p > p_c\), the network of these infinite paths forms an infinite connected cluster. For directed bond percolation on a square lattice the value of \(p_c\) is known to be \(\approx 0.6446\) \([4]\). There are two distinct correlation lengths, \(\xi\) along the easy direction and \(\xi\) transverse to it, both of which diverge as \(p \rightarrow p_c\): \(\xi \sim (p - p_c)^{-\nu\parallel}\), \(\xi \sim (p - p_c)^{-\nu\perp}\) respectively. The values of these exponents are known to be \(\nu\parallel \approx 1.733\) and \(\nu\perp \approx 1.097\) \([4]\).

Suppose we have a single source point, and we ask which portion of the plane can be reached from it via occupied directed bonds. For \(p > p_c\), this region is contained within a cone with opening angle \(\theta = \arctan(m)\) where \(m\) is the slope of the edge of the cone with respect to the 45° direction; the opening angle depends on \(p\). This relation can be inverted to find the critical probability \(p_c(m)\) viz. the probability at which a connection first appears along the direction with slope \(m\neq 0\). Correlation lengths along and perpendicular to this direction have exponents \(\nu\parallel = 1\) and \(\nu\perp = 0.5\) \([4]\). We refer to the direction along 45° as untilted \((m = 0)\); any other slope is referred to as tilted.

In the diode-resistor percolation problem, every bond is occupied by a “diode” (a one-way connection) with a probability \(p\) or a “resistor” (a two-way connection) with a probability \(1 - p\). On a square lattice, the diodes all point up or right. Let us ask which regions of the plane are connected to a given source point. If \(p = 0\), a source point can reach the entire quadrant of which it is the left corner. As \(p\) decreases from 1 to \(p_c\), the opening angle \(\phi\) of the connected region increases from \(\pi/2\) to \(\pi\); beyond this, the entire plane can be reached from the source point. The edge of the connected region is bordered by diodes pointing rightward and upward, which prevent it from spreading leftward and downward (Fig. 4).

On the \(2 - d\) square lattice, DP and DRP are dual to each other \([3]\). The dual to a DRP configuration is constructed using the following rules. A diode in the DRP lattice is crossed by a diode in the dual lattice, whereas a resistor is crossed by an insulator (no connection) in the dual lattice.
Thus we recover the directed percolation problem on the dual lattice. The opening angle of the cones in the two problems are related by \( \phi + \phi' = \pi \).

In the EBM, the random medium is modelled by considering a square lattice with every bond assigned a random number \( f_k \) drawn from the interval \([0, 1]\). For a certain trial value \( f^* \), imagine occupying all bonds with \( f < f^* \) by resistors, and the rest with diodes. We thereby generate a DRP configuration with \( p = 1 - f^* \). When \( f^* \) takes on the value \( 1 - p_c \), an infinite connected path of diodes is formed. Such a path is called a ‘stopper’, and is significant for the dynamics of the EBM, as a moving interface with no overall tilt will align with such stoppers from time to time. When the interface aligns along a stopper, all the bonds in front of it are larger than \( f_c = 1 - p_c \) (Fig. 5a). Similarly a tilted interface with slope \( m \) aligns along the edge of the cone with the same slope and all the bonds in front of it are expected to have a value larger than \( f_c(m) = 1 - p_c(m) \). Since \( p_c(m) > p_c(0) \) for \( m \neq 0 \), \( f_c(m) < f_c(0) \) (Fig. 5b). Even when the interface is evolving between two stoppers, only a small fraction of its overall length actually is in between; the rest is still aligned with a stopper. The non-aligned fraction is expected to vanish in the thermodynamic limit. These expectations are confirmed by numerical studies of the EBM. As can be seen from the figures, the bonds in front of the interface are all mostly larger than a threshold value.

Consider an interface of slope \( m \) aligned along a critical DRP path of the same slope.

It then moves forward by puncturing the path at the site with the least value of \( f \), which for an infinite system is exactly \( 1 - p_c(m) \). On piercing through, a portion of the interface grows and fills out a loop of the infinite cluster while the rest of it remains pinned. However the interface motion within the loop is far from uniform. Just as the critical cluster at \( p_c(m) \) impedes the growth of the interface on length scales of the order of the size of the system, near-critical clusters impede its motion at length scales of the order of but smaller than the loop size. One can think of these clusters as forming a finer network of connections within the network formed by the critical cluster at \( p_c(m) \). While the interface is filling out a loop of the critical cluster, it encounters this finer mesh and as a result its motion is impeded temporarily. In what follows, we refer to these near-critical connections as ‘sub-stoppers’. A substopper can be characterized by the lowest value of \( f_k \) on it, say \( f^{ss} \), and also by the typical length scale \( l \) over which it provides for effective pinning of the interface. These are related through \( |f^{ss} - f_c(m)|^{-\nu_1} \sim l \).
III. CORRELATIONS IN THE ACTIVE-SITE MOTION

The above description of the evolution of the interface, contained as it is by networks of substoppers and stoppers, makes it clear that there are strong correlations between successive points of growth or forward motion. These correlations extend from small length scales up to scales of the order of the system size. Figure 6 shows the plot of the location of the active site for 10,000 time steps for both the tilted and untilted cases. It can be seen that there are jumps on all scales in the active site position. The figure corroborates the description of interface motion given above. Region (1) is a typical instance of the interface filling out a loop of size $l$. It shows that there are jumps of all sizes up to a length $l$, bearing out the substopper picture. Region (2) on the other hand marks an instance when the interface has aligned along a stopper $p_c(m)$ and hence there are jumps of all sizes up to the whole system, corresponding to an instance when the interface has been pinned by a critical cluster at $p_c$. Though the directionality of the active-site motion is evident in (b), this does not induce a net drift as mentioned in the text. The data displayed above is for $L = 1000$.

![Position of active site](image1)

**FIG. 6.** The location of the active site as a function of time for (a) untilted (b) tilted interface. The region marked (1) is a typical example of a window in time in which the activity is localised in a region. In (2) the activity ranges through the whole system, corresponding to an instance when the interface has been pinned by a critical cluster at $p_c(m)$. Though the directionality of the active-site motion is evident in (b), this does not induce a net drift as mentioned in the text. The data displayed above is for $L = 1000$.

![Position of active site](image2)

![Position of active site](image3)

**FIG. 7.** Monte Carlo results for the probability distribution of the jump of the active site for three different densities, $\rho = 0.5$ (plus sign), $\rho = 0.75$ (circles) and $\rho = 0.84375$ (triangles). If $\rho \neq 0.5$, $p(l)$ is not a symmetric function and $p(l)$ and $p(-l)$ (both of which are shown in the figure in the two $\rho \neq 0.5$ cases) do not coincide as they do for the symmetric case. While $p(l) > p(-l)$ for small $l$, the curves cross so that the situation is reversed for large jumps. The two curves asymptotically coincide with an asymptotic slope that differs from that for $\rho = 0.5$ We used $L = 65536$ and averaged over $3.10^9$ configurations.

More quantitatively, a measure of this long ranged motion of the active site is the probability distribution $p(l)$ that two consecutive locations of the active site are a distance $l$ apart. Figure 7 shows $p(l)$ for both the tilted and the untilted interface. In both cases, $p(l)$ decays as a power for large $l$: $p(l) \sim |l|^{-\pi}$.

In the untilted ($\rho = 1/2$) case, $p(l)$ is a purely symmetric function because of $r \rightarrow -r$ symmetry. We find $\pi = 2.25 \pm 0.05$, which compares well with earlier determined values of $\pi$ for the Sneppen model [34-36].

In the tilted case ($\rho \neq 1/2$), $p(l)$ is not a symmetric function (Fig. 7). As can be seen from the figure, there is a larger number of small jumps to the right, but more jumps of large magnitude to the left. It is convenient to separately analyse the even and odd parts $p_{\pm} \equiv (p(l) \pm p(-l))/2$ in order to find the exponents. We find that the even part, $p_{+}(l)$, decays asymptotically as $p_{+}(l) \sim |l|^{-\pi_{+}}$ with $\pi_{+} = 2.00 \pm 0.02$. The odd part $p_{-}(l)$ changes sign (as implied by the crossing of the curves in Fig. 7) and asymptotically follows $p_{-}(l) \sim |l|^{-\pi_{-}}$ with $\pi_{-} = 2.49 \pm 0.06$. We verified that the values of system size.
plotted as a function of \( l \) in the tilted case, the exponents can also be obtained exactly by using the 'backward-avalanche' technique introduced in \([45,35,36]\). The exponent \( \pi' \) is obtained from the range shown is larger \((\approx 2.9)\), the bending apparent in the lower right portion of the curve is consistent with an approach to the value \( \pi \).

Qualitatively, the behaviour of this function may be understood thus. If the active site jumps a distance \( l \) at the previous instant, one can think of the interface as pinned by a DP cluster with loops of average linear dimension \( \sim l \). Most of the interface would then be pinned while a portion of it fills out a loop of linear dimension \( \sim l \). This would imply that on an average, any jump smaller than \( l \) is equally likely. On length scales larger than \( l \) the motion is like the original problem and hence the jump probability decays as a power. This line of argument would imply that \( p(l'|l) \) should be a scaling function of \( l'/l \). Figure 8b bears out this expectation.

Another equivalent way of understanding the function \( p(l'|l) \) is using the 'backward-avalanche' technique introduced in \([15]\). A backward-avalanche is defined as follows. If at time \( s + S \) a random number \( f_k \) is picked as the minimum, the magnitude of the backward avalanche is \( S \) if at time \( s \) the random number picked was larger than \( f_k \) but for time \( s + r \) for \( r < S \) the value of the random number picked was smaller. That is, one goes back in time to the first instant when the random number picked is larger than the present one and this interval of time is the magnitude of the backward-avalanche initiated at the present instant. If the active site hopped a distance \( l \) at the previous instant, one can think of the interface as pinned by a DP cluster with loops of average linear dimension \( \sim l \). Most of the interface would then be pinned while a portion of it fills out a loop of linear dimension \( \sim l \). This would imply that on an average, any jump smaller than \( l \) is equally likely. On length scales larger than \( l \) the motion is like the original problem and hence the jump probability decays as a power. This line of argument would imply that \( p(l'|l) \) should be a scaling function of \( l'/l \). Figure 8b bears out this expectation.

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In the untilted case, the height profile is an even function (Fig. 1a) and \( \Psi(r) \) is an odd function (Fig. 10a) with a strong density inhomogeneity clearly. This pattern is linked to the formation of a structure in terms of densities as

\[
\Psi(r) = \langle n(r + R(t)) \rangle - \rho. \tag{2}
\]

Here \( n(r) \) is the density at site \( r \) and the angular brackets denote a time average in the steady state. The defining Eq. (1) for height gradients can be written

\[
\Psi(r) = \langle \rho \rangle - \rho + R(t).
\]

As discussed in the previous section, the motion of the active spot in this model is very correlated. One can then ask whether this motion reorganizes the shape of the interface (or equivalently the arrangements of particles and holes in the particle model corresponding to the EBM) in any specific way. Below we will show that the correlated motion of the growth-spot in this model leads to the formation of a pattern in height gradients (or equivalently the density of particles) of the interface.

**IV. PATTERN FORMATION IN THE INTERFACE DEPINNING MODEL**

**A. Definition**

We studied \( \Psi(r) \) by Monte Carlo simulation in the particle-hole representation. We studied systems of size \( L \) ranging from \( 2^{10} \) to \( 2^{16} \). The system was allowed to evolve through \( 10^6 - 10^7 \) configurations before measurements were made. In order to speed up the algorithm to locate the site with the minimum \( f \), we used a logarithmic-bin search procedure. Steady state averages were computed using \( \sim 10^8 \) configurations. While this number of configurations was averaged over to get an accurate estimate of the decay exponent, even an average over about 100 configurations indicates the presence of a strong density inhomogeneity clearly.

In the untilted case, the height profile is an even function (Fig. 1a) and \( \Psi(r) \) is an odd function (Fig. 10a) with a strong indication of the correlation in successive jumps of the active site. As Fig. 9 indicates, the function falls off as a slow power. This is an indication of the long-term memory in the system, as the function would decay exponentially if successive jumps had not been correlated.

**B. Numerical results**

Another indication of the correlation in the jumps of the active site is the following quantity which is the analog of an avalanche, for jumps (Fig. 9). Let the avalanche be initiated at an instant \( s \) by a jump of magnitude \( l \) of the active spot. The “jump-avalanche” lasts as long as consecutive jumps in the active site are less than \( l \). That is, the avalanche is of duration \( S \) if at time \( s \) the active site hopped a length \( l \), and for \( S \) consecutive instants after that, the hops in the active site are smaller than \( l \) till at the \((S + 1)\)th instant, it hopped a distance greater than \( l \). The analog of this quantity for random numbers was first defined for models in the non-wetting invasion percolation regime \( \mathbb{P} \) and later also measured for the Sneppen model \( \mathbb{P} \). The jump-avalanche is a strong indication of the correlation in successive jumps of the active site.
The even part \( \Psi_\theta \) where the odd part decays as \( \Psi_{-\theta} \) more likely to contain a particle. Given the dynamics, values of \( \rho \) occur; such a region is thus more likely to contain small regions to the left are more often refreshed, making the active site likely to move in this direction and find itself amidst a particle cluster. The directionality of the active site motion is evident in Fig 6. This leads to densities higher than \( \rho \) on both sides of the active site (Fig. 10b) for \( \rho \) sufficiently different from 0.5. That particles are picked more often than would be expected on the basis of the nominal density is evident from the fact that the value of the density at the active-spot is larger than \( \rho \). For the density pattern in Fig. 10b, which corresponds to \( \rho = 0.75 \), the value of the density at the origin is \( \Psi(0) = 0.84 \).

In the next section we develop an approximate theoretical description for the exponents describing the decay of the pattern in terms of the exponent \( \pi \) which describes the long-ranged hops of the active site.

V. THE INTEGRAL EQUATION FOR PATTERN FORMATION

As discussed in the previous section, a pattern in height gradients or densities is formed in the interface depinning model. It is related to the motion of the active site which rearranges the particles and holes in its wake. For example, if the active site were to remain stationary at a given site for a certain length of time, the dynamics is such that it would build up a pile-up of particles to the left of the site and a pile-up of holes to the right \( i.e. \) it would create a density shock around itself. In this section, we write down an integral equation which provides a description of the pattern in terms of \( p(l) \), the probability that the active site hops a length \( l \) in consecutive instants, as well as the local dynamics of interface readjustment.

FIG. 10. Density profiles in the (a) untilted (\( \rho = 0.5 \)) and (b) tilted (\( \rho = 0.75 \)) cases. Note that in (b), the value at the origin is larger than 0.75, indicating that the active site is more than nominally likely to have a particle, because of the bootstrap effect discussed in the text. While (a) is an odd function of \( r \), (b) has no parity and the even and odd parts can be studied separately. The inset shows the odd part of the profile in the untilted (filled circles) and tilted (open circles) cases. Both decay as power laws though with differing exponents. We used \( L = 16384 \) and averaged over \( 10^8 \) configurations.

decaying asymptotically as a power law \( |r|^{-\theta} \) with \( \theta = 0.90 \pm 0.03 \) (Fig. 10, inset). In the tilted case as there is no \( r \rightarrow -r \) symmetry, \( \Psi(r) \) does not have a definite parity (Fig. 10b). It is useful to separately analyse the even and odd functions \( \Psi_{\pm}(r) = (\Psi(r) \pm \Psi(-r))/2 \). The odd part decays as \( \Psi_{-\theta}(r) \sim |r|^{-\theta_-} \) with \( \theta_+ = 1.04 \pm 0.05 \) (Fig. 10, inset). The even part \( \Psi_+(r) \approx -b(L) + a |r|^{-\theta_+} \) where \( \theta_+ = 0.46 \pm 0.05 \) and \( b(L) \rightarrow 0 \) as the lattice size \( L \rightarrow \infty \) (Fig. 11).

The density profiles of Fig. 10 correspond to the height patterns shown in Figs. 1 and 2. Qualitatively, the reason for this time-averaged structure of the height profile can be seen as follows. In the untilted case, on average, the active site is located at the peak (Fig. 1a) where \( f_k \)'s which have not been sampled earlier are most likely to occur; such a region is thus more likely to contain small values of \( f \). In the tilted case there is, in addition, a bootstrap effect at work. If \( \rho \geq 0.5 \), the active site is more likely to contain a particle. Given the dynamics,

FIG. 11. A log-log plot of the Fourier transform of the even part of the density profile in the tilted (\( \rho = 0.75 \)) case, for the data appearing in Fig. 10. At low \( q \) the function decays as a power \( \simeq q^{-\phi} \) with \( \phi \simeq 0.54 \) implying that at large \( r \), the behaviour is a power-law with decay \( r^{-(1-\phi)} \).
If the pattern is centered at \( R(t) \) at time \( t \), the dynamics causes two changes at the next instant: (i) A short-ranged readjustment of the interface changes the profile near \( R(t) \). The average density change at site \( R(t) + r \) is modelled through a density-increment function \( \Phi(r) \) which is short-ranged. (ii) The active site jumps a distance \( l = R(t+1) - R(t) \). Since the pattern is centered at the active site, the result of (i) followed by (ii) is that the average profile reproduces itself, except that it is centered at the shifted site \( R(t) + l \). Both effects are incorporated into the integral equation

\[
\Psi(r) = \int_{-\infty}^{+\infty} [\Psi(r - l) + \Phi(r - l)]p(l)dt .
\] (3)

This equation can be solved using Fourier transforms. In particular, the long-distance behaviour of the pattern \( \Psi(r) \) is related to the decay of \( p(l) \) for large \( l \), resulting in a scaling relation for the exponent \( \theta \) in terms of the exponent \( \pi \). The details of the analysis are given in section V B below.

However, this equation provides only an approximate description. To understand the nature of the approximation made, we define a "Lévy-flight model", which is similar in spirit to the models of \([20,34]\). As in these models, there is no explicit quenched disorder, but the effect of disorder is modelled by a long-ranged jump probability distribution. For this model, we show that the integral equation (3) holds exactly. Further, we can gain considerable insight into the mechanism of activity-centered pattern formation by studying the effect of different decays of the jump function \( p(l) \) within the Lévy-flight model.

A. Derivation of the Integral Equation for the Lévy-flight model

The Lévy-flight model is defined as follows. Consider a one-dimensional lattice with particles and holes. We assume that a jump probability distribution \( p(l) \) is specified a priori: if at \( l = 0 \) the active site is located at a site \( r \), at the next instant, it can lie a distance \( l \) away (i.e. at \( l + r \)) with a probability \( p(l) \). Evidently there are no temporal correlations in this model, since at every instant, the jump length is chosen afresh from the distribution \( p(l) \). Spatial correlations in the active-site motion are however built in by hand since \( p(l) \) is given. Once a particle or hole is picked for update, the local rules are assumed to be the same as in the Extremal Bond Model i.e. the particle (or hole) exchanges position with the nearest hole (or particle) to the left (or right).

We now derive an integral equation starting from the master equation for the Lévy flight model. We first define a configuration \( i \) to be a set of integers \( \{n_i(r)\}, R_i \) \( r = 1...N \) where \( N \) is the size of the lattice and \( R \) is the current position of the active site. The variables \( n(r) \) can take the values 0 or 1 depending on whether the site \( r \) is occupied by a particle or is empty and \( R \) can take any value between 1 and \( N \). The total number of states is, therefore, \( N_T = N \times C_N \) where \( N_p \) is the number of particles. In the usual manner, we characterise the steady state by a column vector \( |P \rangle \). The entries of this column vector are the probabilities \( P_i \) for the configuration \( i \) where \( i = 1...N_T \). To obtain the steady state, we need to solve the master equation \( d|P\rangle/dt = W|P\rangle \) where \( W \) is an \( N_T \times N_T \) matrix connecting the different states. The dynamics that connects different states is the following: the particle (hole) at \( R \) is exchanged with the nearest hole (particle) to the left (right) as specified earlier. Subsequently the active site now hops from \( R \) to \( R + l \) with a probability \( p(l) \). The diagonal elements of the matrix are \( W_{ii} = -1 \). The off-diagonal elements are given by \( W_{ij} = p(l) \) if configuration \( i \) is connected to configuration \( j \) by an elementary update and a jump of the active site of length \( l \) and by \( W_{ij} = 0 \) otherwise. Since the probability \( p(l) \) is normalised, \( \sum_{l=1}^{\infty} p(l) = 1 \) the sum \( \sum_{l} W_{ij} \) for each column of the matrix adds up to 0. This is a requirement for any stochastic matrix.

Every configuration \( i \) can go to \( N \) other configurations. Each of these differ from \( i \) in the positions of the particle and hole exchanged in the elementary update move, and also in the position of the active site. Similarly there are \( N \) configurations which feed into any configuration \( i \). The construction of these configurations is very similar to that carried out in the case of the low-noise Toom-interface and related models \([49]\). However, unlike in that case, here the transition probabilities are not all the same. As a result the steady state here is very different from the product-measure steady state found in \([49]\) and is difficult to characterise for general \( p(l) \).

However we are able to characterise one aspect of the steady state by defining a quantity \( \Psi \) in the following manner:

\[
\Psi(r) = \sum_{i=1}^{N_T} P_i n_i(r + R_i) - \rho \] (4)

That is, we add up the densities at a site \( r \) for each configuration that occurs in the steady state, having first shifted the origin separately in each configuration so that the site \( r \) for every state is one which lies \( r \) away from the active site. From this is subtracted the average density \( \rho \). The above equation is equivalent to Eq. 2 with the time average in the latter being replaced by a weighted sum over configurations here.

From examination of the \( W \) matrix, it is clear that in the case that \( p(l) \) is a constant independent of \( l \) (as in ordinary stochastic processes), all the \( P_i \)'s are equal. As a result, from the definition, the ACP vanishes. However, while this is a sufficient condition it is not necessary. There are instances (see Section VIII) when special symmetry considerations rule out any pattern formation.
However, the presence of an ACP definitely implies the presence of a nontrivial $p(l)$; a pattern indicates correlations in the active-site motion.

We will now derive Eq. 3 for the Lévy-flight model. Note that the probability $P_i = \sum_l p(l) \sum_{j \rightarrow i} P_j$ where the prime on the summation implies that only those configurations $j$ are considered in which the active site is $l$ sites away from its position in $i$. Further $j$ should transform to $i$ when the active-site is updated. This equation follows from the master equation. The sum is over all $l$’s.

We can now substitute this in the RHS of Eq. 4.

$$\Psi(r) = \sum_l p_l \sum_i n_i(r + R_i) \sum'_{j \rightarrow i} P_j - \rho$$

The density at site $r$ in the configuration $i$ is multiplied by the sum of the probabilities of those configurations that lead to it after a local update; a local exchange of a particle and hole, and a subsequent jump of the active-site of length $l$. Hence we can write

$$n_i(r + R_i) = n_j(r + R_i - l) + \phi_j(r + R_i - l)$$

Here $\phi(r)$ is a short-ranged function (this is related to the ‘density-increment’ function appearing in Eq. 3) which depends on the local update rules. It is defined by Eq. 3 and is the difference between the two configurations $i$ and $j$ which are related by an update. This function is the same for this model as for the Bond model (Fig. 12) since it depends only on the local update rules. In the EBM, it is a short-ranged function whose range is determined by the average length of particle and hole hops. Later in this section we will comment on circumstances in which this function can become long-ranged.

Substituting Eq. 3 in Eq. 4, we find that the right hand side can be rewritten as

$$\Psi(r) = \sum_l p_l \sum_i |n_i(r + R_i) + \phi_i(r + R_i - l) - \rho| P_i$$

Using the definition of $\Psi(r)$ again, we finally obtain the integral equation 3 for $\Psi(r)$ in terms of $p(l)$,

$$\Psi(r) = \sum_l p_l [\Psi(r - l) + \Phi(r - l)]$$

Here we have defined an averaged function $\Phi(r) = \sum_j P_j \phi_j(r)$. From particle conservation, it follows that $\int \Phi(r)dr = 0$. For the symmetric case of half-filling, $N/2$ particles and $N/2$ holes, the function $\Phi$ is strictly an odd function. However, this does not hold when the number of particles is not equal to the number of holes (Fig. 12 b).

Thus we have been able to show that the integral equation (2) is valid for the Lévy-flight model. From the nature of the model, it is clear that while spatial correlations in the active site motion are built in by hand, there are no temporal correlations in the length of subsequent hops of the active site. Therefore the integral equation is exact only in such a case. In the next section, however, we carry over some of the predictions of the integral equation to the EBM and find that in some cases, it tallies quite well with numerical results. In Section VI, we discuss briefly how to generalise Eq. 3 to include temporal correlations such as are present in the EBM.

**B. Analysis of the integral equation**

We now investigate the predictions of Eq. 3 for $\Psi(r)$ in terms of a given active site hopping probability $p(l)$ and the short-ranged readjustment function $\phi(r)$. We solve the equation using shorter transforms. Defining $\hat{\Psi}(q) = \int e^{2\pi inq} \Psi(r)dr$ etc we find

$$\hat{\Psi}(q) = \frac{\hat{\phi}(q)(\hat{\rho}(q))}{1 - \hat{\rho}(q)}.$$
Since we are mainly interested in the $q \to 0$ behaviour of Eq. 3, we do not need the full functional form of $\Phi(q)$ but only the leading order behaviour.

Given a function $p(l)$ the integral equation predicts a corresponding $\Psi(r)$. The large-distance behaviour of the ACP thus depends on whether $p(l)$ is short or long-ranged. We consider now three different cases for the function $p(l)$ and solve for $\Psi(q)$ using equation 3. We substantiate the predictions of the equation by numerically simulating the Lévy flight model.

**Case 1:** Consider first the case of an infinite-ranged $p(l)$. The simplest case is when $p(l) = \frac{1}{N}$ where $N$ is the number of sites in the lattice. This case corresponds to usual stochastic processes. In this case $\hat{p}(q = 0) = 1$ and $\hat{p}(q \neq 0) = 0$. This implies that $\Psi(q) = 0$ for $q \neq 0$ and is therefore a very short-ranged function in space.

**Case 2:** Consider now the case when $p(l)$ is a short-ranged function. We will consider the case when it is symmetric and hence for low $q$ $\hat{p}(q) \sim q^2$. Substituting this in Eq. 3 we find that $\Psi(q) \sim \text{sgn}(q)/q$. This implies that $\Psi(r) \sim \text{sgn}(r)$. The particles and holes separate out completely and the active site is located at the boundary between the two. This is easy to understand if we consider the limiting case when the active site is totally stationary. In this case, the action of the dynamics is to move all the particles from the right of the active site to its left. Eventually, this leads to a total separation of particles and holes. This picture is modified only slightly when the active site executes a localised motion about any lattice site, and hence for a short-ranged $p(l)$ (Fig. 13).

**Case 3:** We now come to the case of interest for the interface depinning model i.e. when $p(l)$ decays as a slow power law. If $p_+(l) = 1/|l|^\pi_+$, Eq. 3 predicts that $\Psi(r) \sim \text{sgn}(r)|r|^{\theta_-}$. The exponents $\pi$ and $\theta_-$ are related by the scaling relation $\theta_- + \pi_+ = 3$. This is to be compared with the numerical estimate $\theta_- + \pi_+ = 3.15$ for the Extremal Bond Model in the untitled case. Consider now the case when none of the functions $p(l)$, $\Psi(r)$ and $\Phi(r)$ has a definite parity. This is relevant for the tilted interface in the Sznepken model. Since $\Phi(r)$ is short-ranged, $\hat{\Phi}(q) \approx i\phi_1 q + \phi_2 q^2$ as $q \to 0$. There is no $\phi_0$ term, as the elementary step of hopping a particle or hole conserves particle number, implying $\int \Phi(r) dr = 0$. The $q \to 0$ behaviour of $\hat{p}(q)$ is determined by the asymptotic power law decays of the even and odd parts $p_\pm(l)$ as $|l| \to \infty$. Thus we have $p_+(q) \approx 1 - A|q|^{\pi_+-1}$. We might have expected $\hat{p}_-(q) \approx Bq + C \text{sgn}(q) |q|^{\pi_--1}$, but in fact the mean velocity $\int |p_\pm(l)| dl$ of the active site vanishes (as mentioned in Section III) implying $B = 0$. Thus the integral equation predicts that to leading order, both $\Psi_+(r)$ and $\Psi_-(r)$ decay as powers $\sim |r|^{-\theta_\pm}$, with

$$\theta_+ + 2\pi_+ - \pi_- = 3$$

and

$$\theta_- + 3.$$

The prediction $\pi_+ + \theta_- = 3$ compares quite well with the numerically determined values 3.04 for $\Psi_-(r)$ in the tilted case. For $\Psi_+(r)$, however, the numerically determined value of $\theta_+ + 2\pi_+ - \pi_-(\approx 1.97)$ deviates substantially from the predicted value 3. The likely reason behind the discrepancy is explained in the next section.

**C. Drawbacks of the Approximation**

The integral equation (2) is exact for the Lévy flight model, but only approximate for the EBM of interface depinning. Here we briefly run over the nature of the approximations made, and possible directions for improvement.

One sort of approximation is the neglect of correlations between lengths of successive jumps; as we saw in Section III, these correlations are very marked. The extension of the integral equation to include such correlations is discussed in Section VI B.

As noted in Case 3 of Section VB for tilted interfaces ($\rho \neq 0.5$), the pattern is not well represented by Eq. (2) even qualitatively. The reason for this is that the $\rho \neq 0.5$ pattern in the EBM is formed due to a feedback effect which is missing in the Lévy flight model. If $\rho > 0.5$, then within EBM dynamics, particles (or holes if $\rho < 0.5$) are picked more often than just $\rho N_{tr}^{\text{try}}$ times in $N_{tr}^{\text{try}}$ tries. This condition is not incorporated in Eq. 3 at all. To test how important this effect is, we simulated the Lévy flight model with the further constraint that if a site chosen for growth does not contain a particle, it is discarded and the search is continued till a site containing a particle is found. This leads to a pattern that much more

![Diagram](image.png)
FIG. 14. The activity-centered pattern $\Psi(r)$ for a Lévy Flight Model with the further constraint that only sites containing particles are chosen. The pattern generated with these rules resembles $\Psi_{1}(r)$ in the EBM more closely than if the rules were implemented without the constraint. This fits in with our conjecture as to why the Integral equation does not describe the even part of the pattern.

The reason for this is that in this case, the mean squared distance $\langle R^{2}(t) \rangle$ covered by the active site in time $t$ is finite. Hence for a large system, this is like a short-ranged motion and leads to particle hole segregation. In the process, as particle clusters build up, the density-increment function $\Phi$ to be short-ranged. In-
very close to the largest possible). A strong dependence of the pattern on the configurations averaged over is evident in Fig. 15. The pattern gets more squashed as the jumps in the active site leading to the configuration get larger. This implies that the pattern ceases to exist at stoppers and builds up again as the interface pierces through. The actual dynamics of pattern collapse and build-up is an interesting subject for further study.

It should be recalled that long jumps occur much more infrequently than short jumps (Fig. 7): the probability of occurrence of a stopper is thus very low. The time average in the definition of the pattern (Eq. 1) is dominated by configurations between stoppers (when the active site is moving within the loops of the directed percolation network), rather than those at stoppers (when the active site is on the backbone of the network).

B. A Hierarchy of Integral Equations

The integral equation 3 can be modified to include correlations in time. To do this, we enlarge the definition of a configuration \( i \) to include the active site at the previous instant as well. Now the off-diagonal elements of the transition matrix elements \( W_{ij} = p(l'|l) \) if configuration \( j \) has resulted as a consequence of a jump of the active site of length \( l \) and is connected to configuration \( i \) by an elementary update and a jump of the active site of length \( l' \). The function \( p(l'|l) \) is just the conditional jump probability already introduced in Section III.

Following the same procedure as before, we get an equation for \( \Psi_{l'} \), the pattern resulting from an active site hop of length \( l' \):

\[
\Psi_{l'}(r) = \sum_{l} p(l'|l)[\Psi_{l}(r-l') + \Phi_{l}(r-l')] .
\]

Keeping correlations up to one time step back gives the pattern a non-trivial dependence on the jump length \( l' \).

The integral equation \( 3 \) gives only a trivial dependence of \( \Psi_{l'}(r) \) on \( l' \).

This procedure can be further generalised by going back one more step in time and keeping the location of the active site two instants back.

This leads to an equation of the following sort:

\[
\Psi_{l_1,l_2}(r) = \sum_{l_2} p(l|l_1|l_2)[\Psi_{l_1,l_2}(r-l) + \Phi_{l_1,l_2}(r-l)]
\]

where \( p(l|l_1|l_2) \) is the conditional probability that the active site hops a length \( l \) given that at the previous instant it hopped a distance \( l_1 \) and in the instant before that, a length \( l_2 \). Similarly, \( \Psi_{l_1,l_2}(r) \) is the pattern formed when averaged over configurations that result after two consecutive jumps of \( l_1 \) and \( l_2 \) respectively.

Here \( \sum_{l_1} \Psi_{l_1,l_2}(r) = \Psi_l(r) \).

Keeping the time sequence of jumps leads to an infinite hierarchy of equations.

\[
\Psi_{l}(r) = \sum_{l'} \sum_{l'} p(l'|l)[\Psi_{l'}(r-l) + \Phi_{l'}(r-l)]
\]

\[
\Psi_{l_1,l_2}(r) = \sum_{l_1} p(l|l_1|l_2)[\Psi_{l_1,l_2}(r-l) + \Phi_{l_1,l_2}(r-l)]
\]

\[
\Psi_{l_1,l_2} = \ldots
\]

The integral equation \( 3 \) corresponds to curtailing this hierarchy at the first step by assuming that \( \Psi_{l}(r-l') = \Psi_l(r-l) \) and \( p(l'|l) = p(l) \).

VII. TWO-POINT CORRELATION FUNCTIONS
IN THE INTERFACE DEPINNING MODEL

The activity-centred pattern \( \Psi_l(r) \) defined in Eq. 1 is a one-point correlation function, defined with respect to an erratically-moving origin. As a result, it has a strong effect on the customary, space-time averaged two-point correlation function

\[
C(\Delta r) \equiv \{\langle n(r')n(r' + \Delta r) \rangle \} - \rho^2
\]

Here \( r' \) is a fixed site on the lattice, \( \langle \ldots \rangle \) stands for a time average and \( \{ \ldots \} \) stands for an average over all sites \( r' \).

Numerical results for \( C(\Delta r) \) for a tilted interface show that it saturates at a value \( C_{sat} \) which decreases with
increasing size $L$ \(^1\) (Fig 16). This happens because subtracting out the quantity $\rho^2$ from the correlation function is not correct, since the presence of the ACP causes a density inhomogeneity in the medium which cannot be accounted for by subtracting out a constant quantity. To account for the presence of the pattern we consider the correlation function

$$\Gamma(r, \Delta r) = \langle \delta n(r + R(t)) \delta n(r + \Delta r + R(t)) \rangle$$

where $R(t)$ is the location of the active site, $r$ is the distance from the active site and $\delta n(r + R(t)) \equiv n(r + R(t)) - \rho - \Psi(r)$ is the fluctuation around the average ACP. A reasonable expectation is that the $\delta n$’s are independent for large separations $\Delta r$, i.e.

$$\Gamma(r, \Delta r) = \langle \delta n(r + R(t)) \delta n(r + R(t) + \Delta r) \rangle \to 0$$

as $\Delta r \to \infty$. Now consider averaging the function $\Gamma(r, \Delta r)$ over $r$. On performing a space average over the right hand side of Eq. \ref{eq:gamma}, we note that

$$\langle \langle \delta n(r + R(t)) \delta n(r + \Delta r + R(t)) \rangle \rangle = \langle \langle \delta n(r) \delta n(r + \Delta r) \rangle \rangle.$$ 

Thus this implies that $\langle \langle n(r) n(r + \Delta r) \rangle \rangle - \langle \langle \rho + \Psi(r) \rangle \langle \rho + \Psi(r + \Delta r) \rangle \rangle$ approaches zero as $\Delta r \to \infty$. This predicts the saturation value of the correlation function

$$C^{sat} = \langle \langle \rho + \Psi(r) \rangle \langle \rho + \Psi(r + \Delta r) \rangle \rangle - \rho^2.$$ 

To test this, we subtracted this estimate of the saturation value from $C(\Delta r)$ and found that the saturation effect is in fact suppressed strongly (Fig. 16), supporting our interpretation. It is possible that another slightly different definition of the pattern would eliminate the slight shoulder which remains in Fig. 16, after subtraction of $C^{sat}$.

Usually, if the two-point correlation function saturates as the separation between the two points is increased, the saturation value is associated with a nonzero value of the space-fixed average $\langle n(r') \rangle - \rho$. The unusual aspect here is that there is saturation even though $\langle n(r') \rangle = \rho$.

We studied the manner in which $C(\Delta r) - C^{sat}$ approaches zero as $\Delta r \to \infty$. In the tilted case, we found that the correlation function decays exponentially, by studying a model where the rules are the same except that only particles are picked. This corresponds to a case of extreme tilt and is similar to the case studied by \cite{[46]}. In the untitled case, $C(\Delta r) - C^{sat}$ decays as a power law $\sim r^{-\kappa}$ with $\kappa \approx 0.6$.

**VIII. PATTERNS IN OTHER MODELS**

In the Extremal Bond Model of interface depinning, we have described the activity-centered pattern in height gradients which forms as a result of correlated motion of the active site. However, this is not the only sort of pattern that is formed. There is pattern formation also in the value of the average random number $f$ at a site, as a function of the distance from the active site. In analogy with Eq. \ref{eq:gamma} we define this pattern in random numbers as

$$\Psi_f(r) = \langle f(r + R(t)) \rangle - \langle \langle f \rangle \rangle.$$ 

Here $f(r + R(t))$ is the random number at a distance $r$ from the active-site and the time average $\langle \langle .. \rangle \rangle$ and space

\(^1\)At much larger values of $r$, proportional to the system size $L$, $C(r)$ changes sign, as a result of the particle conservation sum rule $\int C(r) \, dr = 0$. 

**FIG. 16.** Two-point density-density correlation function for $L = 4096$ (open circles) and for $L = 16384$ (squares). The saturation value is reduced strongly (filled circles) on subtracting the contribution of the ACP. This lends support to the idea that the one-point correlation function defined by the ACP enters the definition of the connected part of the two-point correlation function. The data was obtained by averaging over $10^6$ configurations.
average {...} are performed over configurations in the steady state, as before.

Moreover, patterns are found in other extremal models as well. Figures 17, 18 and 19 show the $f$-patterns in the EBM, the Bak-Sneppen model of biological evolution [38] and the Zaitsev model of low-temperature creep [37] respectively. Numerically, it is difficult to directly extract the manner in which the patterns shown in Figs. 17-19 approach their asymptotic values, as fits to power-law decays are very sensitive to the assumed saturation value. We avoided this problem by studying the Fourier transforms of the functions, as the saturation value influences only the single Fourier mode at $q = 0$.

In all three cases, the Fourier transforms show evidence of power law behaviour as $q \to 0$, implying power law approaches of the $f$-patterns to their respective saturation values in real space. However, we have not developed an analytical description of the $f$-patterns in any of these models.

Pattern formation thus seems to be generic to extremal models. However there are instances when considerations of symmetry rule out the formation of a pattern. This is exemplified in the following model, similar in spirit to that considered in [50]. The rules of the dynamics are the same as in the EBM, random numbers are assigned to every site and the minimum is picked, except that there is no net current of particles as there was in the EBM. If the site picked is occupied by a particle, the particle exchanges place with the first hole to its left, and if the site is occupied by a hole, the hole too exchanges place with the first particle to its left. At half filling, there is no net current. This model is the most symmetric of those considered so far and there is no density pattern formed. However, as in the Bak-Sneppen and the Zaitsev models, there is a nontrivial pattern in random numbers in this model.

We emphasize that the feature of the dynamics which is responsible for activity-centered pattern formation is the existence of correlations in the motion of the active site. Extremal models constitute just one class in which there are such correlations. An example of another such class is models of certain types of reaction-diffusion systems, where the activity is quite constrained and correlated. In another physical context, it would seem that coherent
structures which form in turbulent flows\cite{51} may well be
-described by activity-centered patterns. In our definition
of the pattern, we take an average over all times, keeping
track of the moving structure. This is to be contrasted
with previously used methods to identify such moving
structures, based on the notion of conditional sampling,
namely, averaging only over those time zones in which the
activity is at a particular space-time location\cite{51}. Our
definition takes configurations at all times into account,
but requires a shift of the origin at every instant.

IX. SUMMARY AND CONCLUDING REMARKS

We have introduced a variant of the Sneppen model
of interface depinning — the Extremal Bond Model —
and have studied the effect of the dynamics of the growth
process on the shape of the interface. Our principal re-
sult is the observation that there is a nontrivial structure
which forms in the interface, and which moves along with
the active site. A simple time average of the height gra-
dients, measured in a frame of reference which moves
with the active site, defines the activity-centered pattern
which serves to quantify the structure. Our numerical
study shows that the pattern has a tail which decays as
a power law at large distances.

An understanding of the mechanism underlying
activity-centered pattern formation was obtained by
writing an integral equation which relates the pattern to
the probability distribution of active-site jump lengths.
The integral equation could be derived by writing an ex-
tended master equation for the Lévy flight model, making
it clear that the equation is exact only when there are no
temporal correlations between successive jump lengths.
In the extremal bond model, however, such correlations
are strong. We have shown that a correct description of
the pattern, involving temporal correlations, necessitates
keeping an infinite hierarchy of equations. Terminating
this hierarchy at the very first step results in our inte-
gral equation. It would be of interest to understand how
much one can better this description by keeping more
steps in the hierarchy.

The activity-centered pattern is a one-point function,
and so enters the definition of two-point correlation func-
tions. The physical point is that the density inhomog-
enity caused by the pattern must be taken into account,
by subtracting the relevant quantity from the density-
density correlation function. If this is not done, and the
square of the the nominal density \( \rho \) is subtracted instead,
the correlation function can exhibit a rather unusual sort
of finite size effect. This is an interesting point since the
space-fixed time-averaged density at a site is \( \rho \), and it
is only the density as defined in Eq. 2 with respect to
the moving active site which is different from the nom-
inal density. Yet the space-fixed two-point function is
affected by this “hidden” pattern.

The presence of the pattern clearly points to a non-
homogeneity in the interface: the region around the ac-
tive site looks very different on average from the region
far away from it. For instance, we expect there to be a
larger length of interface in a region of fixed size \( x \) around
the active site, than in a region opposite it. We mon-
tored mean squared fluctuations of the height around
the instantaneous average, in regions around and oppo-
site the active site, and found a pronounced difference
(factor \( \simeq 2 \), for both tilted and untilted interfaces with
\( x = 256, L = 4096 \)). This effect is smaller at stoppers,
in keeping with our finding that the pattern itself is sup-
pressed there. This excess length of interface associated
with the activity-centered pattern may provide a useful
way to identify the active region in experiment.

Finally, it was pointed out that activity-centered pat-
ttern formation may occur in a wide variety of other phys-
ical contexts, ranging from low-temperature creep of dis-
locations to structures in turbulent flows. We have pre-
sented numerical evidence for this sort of pattern forma-
tion in a number of other extremal models. But more
generally we expect activity-centered patterns to form
whenever there are strong correlations between successive locations of the active site.

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