An efficient iterative method for solving Zakharov-Kuznetsov Equation

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Abstract. In this paper, we apply new modified of Variational Iteration Method (VIM-II) which is a kind of analytical approximate method then, use it to solve Zakharov-Kuznetsov (ZK) equation that governs the behavior of the weakly nonlinear ion-acoustic waves in plasma. Two cases of this equation are considered and the results are compared with those that obtained by Adomian Decomposition Method (ADM) and Variational Homotopy Perturbation Method (VHPM). The results illustrate that proposed technique yields a very rapid convergence of the solution as well as low computational effort.

1. Introduction

The ZK equation [1] is one of two well-studied canonical two-dimensional extensions of the Korteweg-de Vries equation [2]. The second one is the Kadomtsev-Petviashvilli (KP) equation [3]. The ZK equation is known to follow from the exact equations of potential water waves by the symmetry-preserving truncation at a certain order in wave steepness. This equation, being formulated in terms of nonlinear normal variables, has long been recognized as an indispensable tool for theoretical analysis of surface wave dynamics. However, its potential as the basis for the numerical modeling of wave evolution has not been adequately explored.

The ZK equation is valid in anisotropic setting which is exactly the case for rotating fluids where the differential latitudinal dependence of the rotation rate causes anisotropy between the meridional and the longitudinal directions. Moreover, in contrast to the KP equation the ZK equation supports stable lump solitary waves. This makes the ZK equation a very attractive model equation for the study of vortices in geophysical flows. The ZK equation was first derived for describing weakly nonlinear ion-acoustic waves in strongly magnetized lossless plasma in two dimensions [1]. Traveling waves are very important because various phenomena in nature such as vibration and soliton or self-reinforcing solitary waves are described by them.

Recently, advances were reported on producing and developments of analytical approaches for solving nonlinear engineering problems. Some of these well-known methods include homotopy perturbation [4-6], variational approach [7], differential transformation [8], variational iteration [9], amplitude-frequency formulation [10] Adomian decomposition [11], and homotopy analysis method [12,13]. This influences researchers to consider these methods for finding analytical solution for ZK equation. Wazwaz [14] used extended tanh method for analytic treatment of the ZK equation, the modified ZK

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equation, and the generalized forms of these equations. Huang [15] applied the polynomial expansion method to solve the coupled ZK equations. Zhao et al. [16] obtained numbers of solitary waves, periodic waves and kink waves using the theory of bifurcations of dynamical systems for the modified ZK equation. Inc [17] solved nonlinear dispersive ZK equation using the Adomian decomposition method, and Biazar et al. [18] applied the homotopy perturbation method to solve the ZK equation. VIM-II is a powerful analytical technique which utilizes Laplace Transform to decrease computational efforts and increase accuracy of the classical Variational Iteration Method. The method has been used by some authors in wide variety of scientific and engineering applications to solve different types of differential equations [19-20]. The most sensible advantages of VIM-II are using Laplace Transform and choosing initial conditions simply and easily to solve linear and nonlinear equations. In this paper we apply VIM-II to find analytical solutions for nonlinear ZK \((m, n, k)\) equations in the following form:

\[
0 = \frac{\partial (u^m)}{\partial x} + \frac{\partial (u^n)}{\partial x} + \frac{\partial (u^k)}{\partial yy} + a(u^m)_{xx} + b(u^n)_{xxx} + c(u^k)_{yyy} , \quad m,n,k \neq 0
\]  

where \(a, b\) and \(c\) are arbitrary constants and \(m, n\) and \(k\) are integers.

The Effectiveness and convenience of the method is revealed in comparisons with the other solution techniques. The results reveal that these methods are very effective and convenient in predicting the solution of such problems, and it is predicted that the VIM-II can find a wide application in new engineering problems.

2. Basic concept of VIM-II

If we clarify the idea of the proposed method for solution of the nonlinear governing equation of Zakharov-Kuznetsov, the basic concept of Variational Iteration Method-II [21,22] is firstly treated. A general nonlinear equation of \(k^{th}\) order is considered at the following form:

\[
\frac{d^{k}u}{dt^{k}} + f(u, u', ..., u^{(k)}) = 0
\]  

The classical variational iteration algorithm is as follows:

\[
u_{n+1}(t) = v_{n}(t) + \int_{0}^{t} \lambda (v_{n}(\tau) + \tilde{f}(\tau))d\tau
\]  

where \(\lambda\) is a general Lagrange multiplier. We apply Laplace transform to identify the Lagrange multiplier [23]. By using Laplace transform, we have:

\[
s^kU(s) + \ell^k\{f(u, u', ..., u^{(k)})\} = 0
\]  

\[
U(s) = -\ell^k\{f(u, u', ..., u^{(k)})\}
\]

It is assumed that all required conditions in Laplace transform are zero. The inverse Laplace transform reads:

\[
u(t) = (-1)^{k-1} \int_{0}^{\infty} \frac{(s-t)^{k-1}}{(k-1)!} f(u(\tau), u'(\tau), ..., u^{(k)}(\tau))d\tau.
\]  

Hence, after identifying the Lagrange multiplier \(\lambda\), the variational iteration algorithm-II [21,22] is constructed as follows:
\[ u_{n+1}(t) = u_0(t) + (-1)^k \int_0^t \frac{(e-t)^{k-1}}{(k-1)!} f(u_n(e),u_n'(e),\ldots,u_n^{(k)}(e))de. \] (7)

The above equation is generally called as the variational iteration algorithm-II. In which, \( u_0(t) \) is the initial solution. The initial values are usually used for selecting the zeroth approximation \( u_0 \). With \( u_0 \) determined, then several approximations \( u_n \), \( n>0 \), follow immediately. Consequently, the exact solution could be obtained as follows:

\[ u(t) = \lim_{n \to \infty} u_n \] (8)

In the next section, VIM-II is applied to solve nonlinear ZK \((m, n, k)\) equation.

3. Implementation of VIM-II

In order to illustrate the effectiveness of the proposed method, we implement it to solve the ZK equation with different numerical examples.

3.1. Example 1

First, we consider the ZK \((2, 2, 2)\) equation [24]:

\[ u_t - (u^2)_x + \frac{1}{8} (u^2)_{xxx} + \frac{1}{8} (u^2)_{yxx} = 0 \] (9)

with initial value problem of:

\[ u(x, y, 0) = \frac{4}{3} \xi \sinh^2 \left( \frac{x+y}{2} \right) \] (10)

where \( \xi \) is an arbitrary constant. We assume \( \xi = 1 \). First, according to the method, by applying Laplace Transform to identify the Lagrange multiplier, we have:

\[ \lambda = 1 \] (11)

So, variational iteration algorithm-II is derived:

\[ u_{n+1} = u_0 + \int_0^t ( (u_n^2)_x - \frac{1}{8} (u_n^2)_{xxx} - \frac{1}{8} (u_n^2)_{yxx} )de \] (12)

We start with an initial approximation \( u_0(x, y, t) = \frac{4}{3} \sinh^2 \left( \frac{x+y}{2} \right) \) by the iteration formula (12), we can obtain the first components as follows:

\[ u_t(x, y, t) = \frac{4}{3} \sinh^2 \left( \frac{x+y}{2} \right) + \frac{22}{9} \sinh^4 \left( \frac{x+y}{2} \right) \cosh \left( \frac{x+y}{2} \right)t - \frac{1}{3} \cosh^4 \left( \frac{x+y}{2} \right)t \]

\[ = -\frac{8}{3} \sinh^2 \left( \frac{x+y}{2} \right) \cosh^2 \left( \frac{x+y}{2} \right)t - \frac{5}{9} \sinh^4 \left( \frac{x+y}{2} \right)t - \frac{2}{3} \sinh \left( \frac{x+y}{2} \right) \cosh^3 \left( \frac{x+y}{2} \right)t \] (13)

Proceeding in the same way, we can obtain the high order approximations. Comparison of exact solution [17] with the corresponding approximate solution of ZK \((2,2,2)\) for fixed values of \( y=0.1 \) and \( t=0.001 \) using two iterations \((n=2)\) is presented in Fig. 1. While in Fig. 2 the corresponding approximate solution is presented.
Figure 1. Comparison between the results of different solutions at y=0.1 and t=0.001

Figure 2. The surface shows the solution $u(x,y,t)$ for ZK(2,2,2) with y=0.1

3.2. Example 2

Consider the ZK (3.3.3) equation [25]:

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yxx} = 0$$

(14)

with initial condition:

$$u(x, y,0) = \frac{3}{2} \zeta \sinh\left(\frac{x + y}{6}\right)$$

(15)

As mentioned in the previous example we assume $\zeta = 1$. According to the method, the Lagrange multiplier can therefore be simply identified as:
\[ \lambda = 1 \]

and the following iteration formula can be obtained:

\[ u_{n+1} = u_0 - \int_0^1 [(u_n^3)_x + 2(u_n^3)_{xxx} + 2(u_n^3)_{yxx}] \, dx \]  

(17)

We start with an arbitrary initial approximation \( u_0(x, y, t) = \frac{3}{2} \sinh(\frac{x+y}{6}) \) that satisfies the initial condition and by using the iteration formula (17), we have the following successive approximation:

\[
\begin{align*}
    u_1(x, y, t) &= \frac{3}{2} \sinh(\frac{x+y}{6}) - \frac{885}{512} \sinh^2(\frac{x+y}{6}) \cosh(\frac{x+y}{6}) t - \frac{5}{256} \sinh(\frac{x+y}{6}) \cosh^3(\frac{x+y}{6}) t \\
    &\quad - \frac{7}{1024} \sinh^3(\frac{x+y}{6}) t - \frac{3}{256} \cosh^3(\frac{x+y}{6}) t
\end{align*}
\]

(18)

Proceeding in the same way, we can obtain the high order approximations. Comparison of exact solution [17] with approximate solution of ZK (3,3,3) for fixed values of \( y = 0.1 \) and \( t = 0.001 \) using two iterations (\( n = 2 \)) is presented in Fig. 3. While in Fig. 4 the corresponding approximate solution is presented.

Figure 3. Comparison between the results of different solutions at \( y = 0.1 \) and \( t = 0.001 \).
Figure 4. The surface shows the solution $u(x,y,t)$ for ZK(3,3,3) with $y=0.1$

4. Conclusion
A novel approximate numerical solution for the ZK (m,n,k) equation is obtained by using VIM-II. Some illustrative examples are given to show the validity and simplicity of this method and results show that these schemes provide excellent approximations to the solution of this nonlinear problem with high accuracy. It is worth pointing out that this method presents a rapid convergence for the solutions.

5. References

[1] Zakharov V E, Kuznetsov E A, 1974 Soviet Phys. 39 285
[2] Korteweg D D, De Vries D G, 1895 Phil. Mag. 39 (240) 422
[3] Kadomtsev B B, Petviashvili V I, 1970 Phys. Dokl. 15 539
[4] Baramia H, Ghasemi E, Soleimanikutanaei S, Barari A, Ganji D D, 2011 J.of Porous Media 14 (6) 545
[5] Saravi M, Hermaan M, Ebarahimi khah H, 2013 J. Theo. Appl. Phys. 7 1
[6] Rostamiyan Y, Fereidoon A, Davoudabadi M R, Yaghoobi H, Ganji D D, 2010 Math. Comp. Appl. 15 (5) 816
[7] Bagheri S, Nikkar A, Ghaffarzadeh H, 2014 Lat. Am. J. Sol. Str. 11 (1) 157
[8] Yaghoobi H, Torabi M, 2011 Int. Comm. He. Mas. Trans. 38 (6) 815
[9] Nikkar A, Bagheri S, Saravi M, 2014 Lat. Am. J. Sol. Str. 11 (2) 320
[10] Zhang H L, 2008 Int. J. Non. Sci. Num. Sim. 9 (3) 297
[11] Sheikholeslami M, Ganji D D, Ashorynejad H R, Rokni H B, 2012 Appl. Math. Mech. 33 (1) 25
[12] He J H, 2004 Appl. Math. Comput. 156 (2) 527
[13] Khan Y, Taghipour R, Fallahian M, Nikkar A, 2012 Neu. Compu. Appl. doi 10.1007/s00521-012-1077-0
[14] Wazwaz A M, 2008 Comm. in Non. Sci. Num. Sim. 13 1039
[15] Huang W, 2006 Chaos Soli. Fractal. 29(2) 365
[16] Zhao X, Zhou H, Tang Y, Jia H, 2006 App. Math. Compu. 181 (1) 634
[17] Inc M, 2007 Chaos Soli. Fractal. 33 (5) 1783
[18] Biazar J, Badpeima F, Azimi F, 2009 Com. Math. with App. 58 (11) 2391
[19] Nikkar A, Mighani Z, Saghebian S M, Nojabaei S B, Daie M, 2012 Res. J. App. Sci. Eng. Tech. 5 (1) 296

[20] S. Ghasempour, Vahidi J, Nikkar A, Mighani M, 2012 Res. J. App. Sci. Eng. Tech. 5 (1) 339

[21] He J H, 2012 Abst. Appl. Analys. Article ID 916793, doi:10.1155/2012/916793.

[22] He J H, 2011 Phys. Let. A. 375 (38) 3362

[23] Hesameddin E, Latifizadeh H, 2009 Int. J. Non. Sci. Num. Sim. 10 (10) (2009) 1365

[24] Hesam S, Nazemi A, Haghbin A, 2010 Int. J. Eng. Nat. Sci. 4 (4) 235

[25] Matinfar M, Ghasemi M, Saeidy M, 2012 Iran. J. Sci. Tech. 36 351