Analytic Classification of Generic Unfoldings of Antiholomorphic Parabolic Fixed Points of Codimension 1

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Abstract

We classify generic unfoldings of germs of antiholomorphic diffeomorphisms with a parabolic point of codimension 1 (i.e. a double fixed point) under conjugacy. These generic unfoldings depend on one real parameter. The classification is done by assigning to each such germ a weak and a strong modulus, which are unfoldings of the modulus assigned to the antiholomorphic parabolic point. The weak and the strong moduli are unfoldings of the Écalle-Voronin modulus of the second iterate of the germ which is a real unfolding of a holomorphic parabolic point. A preparation of the unfolding allows to identify one real analytic canonical parameter and any conjugacy between two prepared generic unfoldings preserves the canonical parameter. We also solve the realisation problem by giving necessary and sufficient conditions for a strong modulus to be realized. This is done simultaneously with solving the problem of the existence of an antiholomorphic square root to a germ of generic analytic unfolding of a holomorphic parabolic germ. As a second application we establish the condition for the existence of a real analytic invariant curve.

Keywords — Discrete dynamical systems, antiholomorphic dynamics, parabolic fixed point, classification, unfoldings, modulus of analytic classification

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1 Introduction

In this paper, we are interested in unfoldings of antiholomorphic germs with a parabolic fixed point of codimension 1, i.e. of multiplicity 2. Such an antiholomorphic germ $f_0: (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ has the form

$$f_0(z) = z + \frac{1}{2}z^2 + \left(\frac{1}{4} - \frac{b}{2}\right)z^4 + o(z^3),$$

in some local coordinate and for some invariant $b \in \mathbb{R}$.

This study is part of a large program to understand the local dynamics of singularities in low-dimensional complex dynamical systems, with particular emphasis on 1-resonant singularities (i.e. all resonance relations among the eigenvalues/multipliers are consequences of a single one). The simplest case is that of the parabolic point (multiple fixed point) of a holomorphic germ $f_0: (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. In that case one classifies the germs under conjugacy, i.e. local analytic changes of coordinate: the classification has been given by Écalle and Voronin, and to each germ is associated its classifying object, its Écalle-Voronin modulus (see [15], [3], or [6], or [7]).

Antiholomorphic dynamics has been mainly studied in the context of Julia sets of unicritical antiholomorphic polynomials $\bar{z}^d + c$, where the connectedness loci are given as multicorns: see [11], [5], [8], [10]. There it can be seen that the boundary points of hyperbolic components of odd period consist only of parabolic parameter values: this boundary is composed of real analytic arcs corresponding to generic codimension 1 bifurcations of antiholomorphic parabolic points linked at higher codimension isolated points. These references generalize some tools from the holomorphic case: Fatou coordinates, Écalle cylinders.

The analytic classification of germs of antiholomorphic diffeomorphisms $f_0$ with a parabolic fixed point of arbitrary codimension $k$ is given in [4]. For such a germ $f_0$, the holomorphic germ $g_0 = f_0 \circ f_0$ has a parabolic fixed point of the same codimension $k$. Hence it is no surprise that a modulus of classification is given by the Écalle-Voronin modulus of $g_0$, namely a modulus composed of the codimension, the formal invariant $b$, and $2k$ horn maps. However not all Écalle-Voronin moduli are realisable: indeed $k$ horn maps determine the $k$ other horn maps. Moreover, the formal invariant $b$ is real.

In dimension 1, the 1-resonant singularities occur as the coalescence of fixed points and/or periodic orbits. Hence it is natural to embed the diffeomorphism in a family of diffeomorphisms, called an unfolding separating the singularities into simple ones. Then each singularity is organizing rigidly the dynamics in its neighbourhood and the local “models” may not match globally. It is the limit of this mismatch that produces the Écalle-Voronin modulus. This is why it is natural to study generic unfoldings of 1-resonant singularities. The program has
been performed for holomorphic parabolic points in codimension $k$ ([9], [2] and [14]), and for resonant diffeomorphisms ([1] and [13]). In this paper we consider the analytic classification of generic unfoldings of codimension 1 antiholomorphic parabolic points.

The condition for a fixed point to be parabolic is $|f'_0(0)| = 1$, a condition of real codimension 1. Hence it is natural to unfold $f_0$ with one real parameter $\varepsilon$. An unfolding $f_\varepsilon$ will either have two fixed points, or a periodic orbit of period 2, or a parabolic fixed point. In the case of two fixed points, the dynamics near each fixed point is very simple: one is attractive and the other one is repulsive, so they are both locally linearisable. On the other hand, when we have a periodic orbit, the dynamics can be very complicated when the multiplier lies on the unit circle.

For a generic unfolding $f_\varepsilon$ of an antiholomorphic parabolic germ $f_0$ we are able to identify a canonical parameter, which is a real analytic invariant. Then an equivalence between two generic unfoldings will preserve the canonical parameter. A weak modulus of classification is given by an unfolding of the modulus of $f_0$. However, when we only work with real values of the parameters, we are not able to prove that the conjugacy between two unfoldings depends real analytically on the parameter: this is why we only speak of weak modulus of classification. To remedy, we need to extend $f_\varepsilon$ antiholomorphically to complex values of the parameter and to introduce a strong modulus of classification. Then $g_\varepsilon = f_\varepsilon \circ f_\varepsilon$ depends holomorphically on $\varepsilon$ and we are able to prove the existence of a conjugacy depending real analytically on the parameter. This is done using the tools developed for the holomorphic case.

After a classification problem is solved the natural question to ask is the modulus space. We have been able to give the strong modulus set (without a topology) for generic unfoldings of an antiholomorphic parabolic germ of codimension 1, i.e. necessary and sufficient conditions for a modulus to be realized. The proof uses a detour. Indeed the problem is solved for unfoldings $g_\varepsilon$ of a parabolic germ (see [2]). Hence it suffices to solve the problem of the existence of an antiholomorphic unfolding $f_\varepsilon$ depending antiholomorphically on $\varepsilon$ such that $g_\varepsilon = f_\varepsilon \circ f_\varepsilon$ (which we call an antiholomorphic square root), and this problem is easy to solve.

The idea of the necessary condition for the realisation is the following: the description of the modulus is done in the parameter $\hat{\varepsilon}$ belonging to the universal covering punctured at 0 of the complexified canonical parameter: we take $\arg \hat{\varepsilon} \in (-\pi + \delta, 3\pi - \delta)$ for some $\delta \in (0, \frac{\pi}{2})$. Hence, when $\arg \varepsilon \in (-\pi + \delta, \pi - \delta)$, we have two different unfoldings of the modulus at $\varepsilon = 0$. An obvious necessary condition is that these two different unfoldings describe the same dynamics. This condition is called the compatibility condition, and it turns out to also be sufficient. Hence to obtain the realisation in the antiholomorphic case it suffices to combine the compatibility condition and to solve the problem of finding a necessary and sufficient condition for the extraction of an antiholomorphic square root.
of a holomorphic parabolic unfolding \( g_{\varepsilon} \). By this we mean a germ depending antiholomorphically on the complex parameter \( \varepsilon \) and satisfying \( g_{\varepsilon} = f_{\varepsilon} \circ f_{\varepsilon} \).

The study of the unfoldings of \( f_{0} \) can help shed some light on why there are obstructions to certain simple geometric behaviours. For instance, it is proved in [4] that under a condition of infinite codimension, \( f_{0} \) will have an invariant real analytic curve. What is the obstruction? For a generic unfolding \( f_{\varepsilon} \) of \( f_{0} \), there will be some values of the parameter for which \( f_{\varepsilon} \) has two simple fixed points. The linearisation near each fixed point has one invariant analytic curve. There is no reason for these local invariant curves to match globally and this is why the mismatch is the generic situation in the unfolding. If the mismatch persists at the limit when \( \varepsilon \to 0 \), then we can expect that \( f_{0} \) has no real analytic invariant curve.

The second example, already mentioned above of a rare geometric phenomenon is the existence of an antiholomorphic square root for a holomorphic parabolic germ \( g_{0} \). Such a germ is said to have an antiholomorphic square root if there exists an antiholomorphic parabolic germ \( f_{0} \) such that \( f_{0} \circ f_{0} = g_{0} \). The existence of such a germ is a phenomenon of infinite codimension. Why? When we unfold \( g_{0} \) by a parameter \( \varepsilon \), each fixed point of \( g_{\varepsilon} = f_{\varepsilon} \circ f_{\varepsilon} \) has a first return map that describes the dynamics locally. For a generic unfolding of \( g_{0} \), the first return maps are independent. However the existence of \( f_{\varepsilon} \) forces the first return maps to be conjugate for parameter values for which \( f_{\varepsilon} \) has a periodic orbit. Since we can have very complicated dynamics of very diverse types for many values of the parameter, we see that this is a very strong condition, which explains the very high codimension.

The paper is organized as follows.

After a brief section of preliminaries on antiholomorphic parabolic germs, we define in Section 3 generic unfoldings \( f_{\varepsilon} \) of antiholomorphic parabolic germs \( f_{0} \) of codimension 1. We also determine their canonical parameter \( \varepsilon \) and we give a prepared form for generic unfoldings.

In Section 4, we prove the existence of Fatou coordinates for \( f_{\varepsilon} \). The Fatou coordinates are what constitute the sectorial normalisation, i.e. almost unique changes of coordinates defined on portions of the domain that conjugate \( f_{\varepsilon} \) to the “normal form”.

In Section 5, we describe the space of orbits of \( f_{\varepsilon} \) for the different values of \( \varepsilon \), using the transition functions between the Fatou coordinates.

In Section 6, we define the weak modulus of classification and we prove a weak version of the Classification Theorem, where we miss the analytic dependence of the conjugacy on the parameter.

In Section 7, we give the strong version of the Classification Theorem, which requires extending \( f_{\varepsilon} \) antiholomorphically to complex values of the parameter and defining a strong modulus of classification.
Lastly, in Section 8, we apply the results to discuss the geometric interpretation of the modulus. We find a necessary and sufficient condition to extract an antiholomorphic square root of a holomorphic unfolding of a parabolic point. We use it to give the modulus set.

2 Preliminaries

2.1 Notation

For the whole paper, we will use the following notation:

- $\sigma(z) = \overline{z}$ is the complex conjugation;
- $\tau(w) = \frac{1}{\overline{w}}$ is the antiholomorphic inversion;
- $T_C(Z) = Z + C$ is the translation by $C \in \mathbb{C}$;
- $L_c(w) = cw$ is the linear transformation with multiplier $c \in \mathbb{C}$;

2.2 Antiholomorphic Parabolic Fixed Points

A function $f: U \to \mathbb{C}$ defined on a domain $U \subseteq \mathbb{C}$ is antiholomorphic if $\frac{\partial f}{\partial z} \equiv 0$ on $U$. From this definition, together with the chain rule, it follows that antiholomorphy is an intrinsic property of $f$ under holomorphic changes of variable. Equivalently, $f: z \mapsto f(z)$ is antiholomorphic if $f \circ \sigma: z \mapsto f(\overline{z})$ is holomorphic, therefore $f(z)$ expands in a power series in terms of $\overline{z}$.

Let $f: (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a germ of antiholomorphic diffeomorphism (in the $\zeta$-variable) that fixes the origin. Recall that 0 is a parabolic fixed point if it is an isolated fixed point and if $\left| \frac{\partial f}{\partial \zeta}(0) \right| = 1$. It is proved in [4] that there exists a polynomial change of coordinate $z = p(\zeta)$ such that $f$ takes the form

$$\tilde{f}(z) = z + \frac{1}{2} z^{k+1} + \left( \frac{k+1}{8} - \frac{b}{2} \right) z^{2k+1} + o(\overline{z}^{2k+1}), \quad (1)$$

for some numbers $k \in \mathbb{N}^*$, $b \in \mathbb{R}$, respectively called the codimension and the formal invariant of $f$. The codimension is linked to the multiplicity of the fixed point: a fixed point of codimension $k$ has multiplicity $k + 1$. We concentrate on the codimension 1 case, so we assume that $k = 1$ for the rest of the paper. We further suppose that $f$ is always in a coordinate such that

$$f(z) = z + \frac{1}{2} z^2 + \left( \frac{1}{4} - \frac{b}{2} \right) z^3 + o(z^4). \quad (2)$$
The formal invariant allows us to define a formal normal form, namely $\sigma \circ v^1_2$, the composition of the time-$\frac{1}{2}$ map of the vector field

$$\dot{z} = v(t) = \frac{z^2}{1 + bz}$$

with the complex conjugation. In other words, there exists a formal change of coordinate $h$ such that $h \circ f \circ h^{-1} = \sigma \circ v^1_2$.

The local dynamics of $f$ at its fixed point is also described in [4]. Let us briefly review the important features. First, recall that the composite $f \circ f$ is a germ of holomorphic diffeomorphism with a holomorphic parabolic fixed point at the origin. The dynamics of $f \circ f$ is embedded in the dynamics of $f$; this has several consequences, for instance the codimension and the formal invariant of $f \circ f$ are the same as the ones of $f$. It is well known that the dynamics of $f \circ f$ is described using the so-called Écalle horn maps. In codimension 1, the space of orbits of $f \circ f$ is a quotient of the disjoint union of 2 spheres by the identification coming from a pair of germs $(\psi^0, \psi^\infty)$ of normalized diffeomorphisms, where $\psi^0$ sends 0 to 0 and $\psi^\infty$ sends $\infty$ to $\infty$, as in Figure 1, and by further identification of 0 and $\infty$. This orbit space depends only on the equivalence class $[\psi^0, \psi^\infty]$, where the equivalence relation is given by

$$(\psi^0, \psi^\infty) \sim_C (\psi^0', \psi^\infty') \iff \exists C \in \mathbb{C}, \psi^0, \psi^\infty = L_C \circ \psi^0, \psi^\infty' \circ L_C.$$  \hspace{1cm} (4)

![Figure 1: Space of orbits of $f \circ f$ in codimension 1.](image)

In the antiholomorphic case, the fact that the dynamics of $f \circ f$ is embedded in the dynamics of $f$ allows us to define the involution $L_{-1} \circ \tau: w \mapsto -\frac{1}{w}$ on the space of orbits of $f \circ f$. We then obtain the space of orbits of $f$ by quotienting the space of orbits of $f \circ f$ by $L_{-1} \circ \tau$. We are left with two projective spaces and a class of germs of diffeomorphisms $[\psi]$ as in Figure 2. The equivalence class is given by

$$\psi \sim_R \psi' \iff \exists R \in \mathbb{R}, \psi = L_R \circ \psi' \circ L_{-R}.$$  \hspace{1cm} (5)
Moreover, if $\psi$ is a representative of $[\psi]$, then $(\psi^0, \psi^\infty)$ defined by

\[
\begin{align*}
\psi^\infty & := \psi, \\
\psi^0 & := L_{-1} \circ \tau \circ \psi^\infty \circ \tau \circ L_{-1},
\end{align*}
\]

is a representative $[\psi^0, \psi^\infty]$ (the horn maps of $f \circ f$).

Figure 2: Space of orbits of $f$ in codimension 1.

3 Generic Unfoldings of Antiholomorphic Parabolic Germs

3.1 Real parameters

Note that it does not make sense in the context of iterations of antiholomorphic functions to speak of analytic dependence on complex parameters since this notion is not invariant under composition. What does make sense however is to speak of real analytic dependence on real parameters. And, indeed, a fixed point $z = 0$ of $f$ is multiple as soon as $|f'(0)| = 1$, which is generically a condition of real codimension 1. Hence we should expect a real unfolding parameter to be transversal to this condition.

Since the “natural” parameters are real, we will use mix anti-analyticity, which we introduce in the following definition. Note that the even iterates will be mix analytic.

**Definition 3.1.1.** Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{C}^m$ be domains and $f: U \times V \to \mathbb{C}$ be a function. We say $f$ is mix analytic (resp. mix anti-analytic) if for every $(t^*, z^*) \in U \times V$, there exist an $n$-dimensional rectangle $R = R^n(t^*, r) \subseteq U$ and a polydisc $P = D^m(z^*, \rho) \subseteq V$, where $r \in \mathbb{R}^n_{>0}$, $\rho \in \mathbb{R}^m_{>0}$, such that $f$ has a convergent power series expansion in $R \times P$ of the form

\[
f(t, z) = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^m} \alpha_{\alpha, \beta}(t - t^*)^\alpha(z - z^*)^\beta,
\]
or, respectively, of the form

\[ f(t, z) = \sum_{\alpha \in \mathbb{N}^n} \sum_{\beta \in \mathbb{N}^m} \alpha_{\alpha, \beta} (t - t^*)^\alpha (\overline{z} - \overline{z}^*)^\beta. \]

**Lemma 3.1.2.** Let \( U \subseteq \mathbb{R}^n \), \( V \subseteq \mathbb{C} \) and \( W \subseteq \mathbb{C} \) be domains. Let \( F: U \times V \to \mathbb{C} \) and \( G: U \times W \to \mathbb{C} \) be functions. Whenever the composition is possible,

1. if \( F \) and \( G \) are mix analytic, then so is \( F \circ (id \times G) \);
2. if \( F \) is mix analytic and \( G \) is mix anti-analytic, then \( F \circ (id \times G) \) and \( G \circ (id \times F) \) are mix anti-analytic;
3. if \( F \) and \( G \) are mix anti-analytic, then \( F \circ (id \times G) \) is mix analytic.

**Proof.** The proof is a simple computation of derivatives. Indeed, we can complexify \( t \in U \), show that the composition is holomorphic for 1 and 3 or holomorphic in \( t \) and antiholomorphic in \( z \) for 2, and then restrict \( t \) to the reals. \( \square \)

**Weierstrass Preparation Theorem 3.1.3** (Real analytic version). Let \( g: (\mathbb{R}^N, 0) \times (\mathbb{C}, 0) \to (\mathbb{C}, 0); (t, z) \mapsto g(t, z) \) be a germ of mix analytic function. If

\[ \frac{\partial^kg}{\partial z^k}(0, 0) = 0 \quad \text{and} \quad \frac{\partial^n g}{\partial z^n}(0, 0) \neq 0 \quad (k = 1, \ldots, n - 1), \]

then there exists a polynomial \( P_t(z) = z^n + a_{n-1}(t)z^{n-1} + \cdots + a_0(t) \), where \( a_j \) is real analytic, \( a_j(0) = 0 \), and a germ of mix analytic function \( h: (\mathbb{R}^N, 0) \times (\mathbb{C}, 0) \to \mathbb{C} \) with \( h(0, 0) \neq 0 \) such that

\[ g(t, z) = P_t(z)h(t, z). \]

Moreover, if \( g(t, z) = \overline{g(t, \overline{z})} \), then \( a_j \) is real valued.

**Proof.** The proof is identical to the proof of [12]. The fact that the \( a_j \)'s are real analytic follows from the fact that \( g \) is mix analytic and the Cauchy’s formula in the \( z \) variable.

For the second part, we have \( g(t, z) = P_t(z)h(t, z) \) as in the statement of the theorem. If \( \varphi(t) \) is a zero of \( P_t \), then \( \varphi(t) \) is also a zero with the same multiplicity, since \( \frac{\partial^j g}{\partial z^j}(t, z) = \frac{\partial^j g}{\partial z^j}(t, \overline{z}) \). If \( \alpha_1(t), \alpha_2(t), \ldots, \alpha_{\ell}(t), \overline{\alpha_j(t)} \) are the complex roots of \( P_t \) and \( \beta_j(t) \), \( j = 1, \ldots, \ell' \) are its real roots, then we have

\[ P_t(z) = (z - \alpha_1)^{q_1}(z - \overline{\alpha_1})^{q_1} \cdots (z - \alpha_{\ell})^{q_\ell}(z - \overline{\alpha_{\ell}})^{q_\ell}(z - \beta_1)^{m_1} \cdots (z - \beta_{\ell'})^{m_{\ell'}}. \]

For \( x \) real, the polynomial \( (x - \alpha_j)(x - \overline{\alpha_j}) \) is real valued, so \( x \mapsto P_t(x) \) is real valued. It follows that its coefficients \( a_j(t) \) are real for every \( t \). \( \square \)
3.2 Generic Unfoldings

We are now ready to define unfoldings of an antiholomorphic parabolic fixed point and the notion of equivalence of such unfoldings. As discussed above it is natural to work with one real parameter.

**Definition 3.2.1.**
1. Let \( f_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) be a germ of antiholomorphic diffeomorphism with a parabolic fixed point (antiholomorphic parabolic germ for short) of codimension 1. An *unfolding* of \( f_0 \) is a germ of mix anti-analytic diffeomorphism \( f : (\mathbb{R}, 0) \times (\mathbb{C}, 0) \to (\mathbb{C}, 0), (\varepsilon, z) \mapsto f(\varepsilon, z) = f_\varepsilon(z) \).

2. We can of course suppose that \( f_0 \) is in the form (2). Then the unfolding has the form
   \[
   f_\varepsilon(z) = f_0(z) + \sum_{j \geq 0} a_j(\varepsilon)z^j,
   \]
   with \( a_j(0) = 0 \). We say that the unfolding is *generic* if
   \[
   \frac{\partial \Re(a_0)}{\partial \varepsilon} \bigg|_{(\varepsilon,z)=(0,0)} \neq 0.
   \]

The equivalence of two families will be defined in terms of mix analyticity.

**Definition 3.2.2.** Let \( f_{1,\eta} \) and \( f_{2,\varepsilon} \) be two generic unfoldings of antiholomorphic parabolic germs of codimension 1. We say they are *equivalent* if there exists an open interval \( I \ni 0 \), a disc \( D(0,r) \), \( r > 0 \), and a mix analytic diffeomorphism \( H : I \times D(0,r) \to \mathbb{R} \times \mathbb{C} \) such that

1. \( H(0,0) = (0,0) \);
2. \( H(\eta, z) = (\beta(\eta), h_\eta(z)) \) with \( \beta \) real analytic and \( h_\eta \) mix analytic;
3. \( f_{2,\beta(\eta)} = h_\eta \circ f_{1,\eta} \circ h_\eta^{-1} \).

The main goal of the paper is to describe the equivalence classes.

3.3 Canonical Parameter

In the holomorphic case, it is proved in [9] that a generic unfolding of a parabolic fixed point has exactly one canonical complex parameter. In the same way, a generic unfolding in the antiholomorphic case will also have exactly one real canonical parameter.

**Lemma 3.3.1.** Let \( f_0 \) be an antiholomorphic parabolic germ of codimension 1 and let \( f_\varepsilon \) be a generic unfolding of \( f_0 \). Then there exists a change of coordinate and parameter \((z, \varepsilon) \mapsto (Z, \eta)\) such that the fixed points are located at \( Z^2 = \eta \) whenever they exist.
Proof. Suppose that $f_\varepsilon$ has the form (6). Let $z = x + iy$ and set $F(x, y, \varepsilon) = f_\varepsilon(z) - z$ and $F_1 = \Re F, \ F_2 = \Im F$. We find

$$F_1(x, y, \varepsilon) = \Re a_0(\varepsilon) + O(\varepsilon)O(|x, y|) + (1 + O(\varepsilon)) \frac{x^2 - y^2}{2} + O(|x, y|^3),$$
$$F_2(x, y, \varepsilon) = \Im a_0(\varepsilon) - 2y + O(\varepsilon)O(|x, y|) - (1 + O(\varepsilon))xy + O(|x, y|^3).$$

Since $\frac{\partial F_2}{\partial y}(0) = -2$, by the Implicit Function Theorem, there exists a real analytic function $m$ such that $F_2 = 0$ if and only if $y = m(x, \varepsilon) = O(|x, \varepsilon|^2)$.

We make the change of variable $z = z_1 + im(z_1, \varepsilon)$, which sends the real axis in $z_1$-space to $y = m(x, \varepsilon)$ in $z$-space. Let $f_{1, \varepsilon}$ be the expression of $f_\varepsilon$ in the new variable $z_1$. Let us now consider the corresponding equations (8) in the new variable $z_1$. For $y_1 = 0$, the first equation has the form $F_1(x_1, \varepsilon) = x_1^2(1 + O(\varepsilon) + O(x_1)) + O(\varepsilon)x_1 + O(\varepsilon)$, and we have $\frac{\partial F_1}{\partial x_1}(0) = 0$ and $\frac{\partial^2 F_1}{\partial x_1^2}(0) = 1$.

By the Weierstrass Preparation Theorem 3.1.3, there exists a polynomial $P_\varepsilon(x_1) = x_1^2 + \alpha_1(\varepsilon)x_1 + \alpha_0(\varepsilon)$, with $\alpha_0(\varepsilon)$ and $\alpha_1(\varepsilon)$ real and $\alpha_0'(0) \neq 0$, such that $F_1(x_1, \varepsilon) = 0$, if and only if $P_\varepsilon(x_1) = 0$.

A translation $Z = z_1 + \frac{1}{2}\alpha_1(\varepsilon)$ brings the fixed points to $Z^2 = \eta$, where $\eta = \frac{1}{4}\alpha_1^2(\varepsilon) - \alpha_0(\varepsilon)$. \hfill \Box

3.3.1 Normal Form

We consider the vector field

$$\dot{z} = v_\varepsilon(z) = \frac{z^2 - \varepsilon}{1 + b(\varepsilon)z}$$

and the time-$t$ map $v_\varepsilon^t$, where $b : (\mathbb{R}, 0) \to \mathbb{R}$ is a germ of real analytic function. Then $\sigma \circ v_\varepsilon^\frac{1}{2}$ is a generic unfolding of an antiholomorphic parabolic germ of codimension 1. It will be the natural “model” (normal form) to which we will compare any generic unfolding $f_\eta$: unique new parameter $\varepsilon$ and function $b(\varepsilon)$ will be found so that the multipliers of $f_\varepsilon^{\circ 2}$ at its fixed points be the same as those of $(\sigma \circ v_\varepsilon^{\frac{1}{2}})^{\circ 2}$ at its fixed points.

Since $\sigma \circ v_\varepsilon = v_\varepsilon \circ \sigma$, it follows that $\sigma \circ v_\varepsilon^{\frac{1}{2}} = v_\varepsilon^{\frac{1}{2}} \circ \sigma$. In particular, we have $(\sigma \circ v_\varepsilon^{\frac{1}{2}})^{\circ 2} = v_\varepsilon^{1}$, which correspond to the normal form in the holomorphic case for $\varepsilon$ real.

The multipliers of $v_\varepsilon$ are given by

$$\mu_\pm = \pm \frac{2\sqrt{\varepsilon}}{1 \pm b(\varepsilon)\sqrt{\varepsilon}}$$
and the multipliers of \( v^1_\varepsilon \) are \( \lambda_\pm = \exp(\mu_\pm) \). It follows that

\[
\varepsilon := \left( \frac{1}{\log(\lambda_+)} - \frac{1}{\log(\lambda_-)} \right)^{-2}, \quad b(\varepsilon) := \frac{1}{\log(\lambda_+)} + \frac{1}{\log(\lambda_-)}. \tag{10}
\]

Since the multipliers are preserved by analytic changes of coordinate, we see that \( b \) and \( \varepsilon \) are invariant. In particular, that gives us the hint on how to find the canonical parameter \( \varepsilon \) of an arbitrary antiholomorphic parabolic germ.

### 3.3.2 Canonical Parameter and Prepared Form

The formula (10) allows to define the canonical parameter and the formal invariant of the unfolding. These are the formal part of the modulus of classification that we will define in Section 6. The analytic part of the modulus will consist in a measure of the obstruction to a conjugacy of an unfolding to its model.

**Theorem 3.3.2** (Canonical Parameter). Let \( \tilde{f}_0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) be an antiholomorphic parabolic germ of codimension 1 with formal invariant \( b_0 \in \mathbb{R} \). Let \( \tilde{f}_\eta \) be a generic unfolding depending on the real parameter \( \eta \) with fixed points at \( z^2 = \eta \). Let \( \tilde{g}_\eta = \tilde{f}_0 \circ \tilde{f}_\eta \). We set

\[
\varepsilon := \left( \frac{1}{\log(\tilde{g}_\eta'(\sqrt{\eta}))} - \frac{1}{\log(\tilde{g}_\eta'(-\sqrt{\eta}))} \right)^{-2}, \quad b(\varepsilon) := \frac{1}{\log(\tilde{g}_\eta'(\sqrt{\eta})}) + \frac{1}{\log(\tilde{g}_\eta'(-\sqrt{\eta}))}, \tag{11, 12}
\]

where the logarithm is the principal branch. Then we have that \( \varepsilon \) and \( b \)

1. are real-valued;
2. are invariant under changes of coordinate;
3. can be continued into real analytic functions, in particular \( \varepsilon(0) = 0 \) and \( b(0) = b_0 \).

The parameter \( \varepsilon \) is called the canonical parameter and the function \( b(\varepsilon) \) the formal invariant of the family.

We postpone the proof to introduce the prepared form. When comparing unfoldings, it will be useful to compare them when they are in their prepared form, since the prepared form depends on the canonical parameter. A prepared form for the holomorphic case was presented in [9].
Theorem 3.3.3 (Prepared Form). Under the the hypotheses of Theorem 3.3.2, there exists a mix analytic diffeomorphism \( H: (\eta, z) \mapsto (\varepsilon, m_\eta(z)) = (\varepsilon, z_1) \) that maps the family \( \{\tilde{f}_\eta\}_\eta \) to a family in the prepared form

\[
f_\varepsilon(z_1) = z_1 + (z_1^2 - \varepsilon)[B_0(\varepsilon) + B_1(\varepsilon)z_1 + (\varepsilon^2 - \varepsilon)Q(\varepsilon, z_1)],
\]

that satisfies

1. \( B_0(0) = \frac{1}{2} \) and \( B_0 \) and \( B_1 \) are real-valued;
2. \( \tau_\pm := \frac{\partial f_\eta}{\partial z_1}(\pm \sqrt{\eta}) \begin{cases} \in \mathbb{R}, & \text{if } \varepsilon > 0; \\ = \tau_\mp, & \text{if } \varepsilon < 0; \\ = 1, & \text{if } \varepsilon = 0; \end{cases} \)
3. \( \lambda_\pm := \frac{\partial g_\varepsilon}{\partial z_1}(\pm \sqrt{\varepsilon}) = \frac{\tau_\pm}{\tau_\mp}, \) where \( g_\varepsilon := f_\varepsilon \circ f_\varepsilon \).

### 3.3.3 Proof of Theorems 3.3.2 and 3.3.3

For the proof of both theorems, by Lemma 3.3.1 and the Weierstrass Division Theorem, we can suppose \( \tilde{f}_\eta \) has the form

\[
\tilde{f}_\eta(z) = z + (z^2 - \eta)(C_0(\eta) + C_1(\eta)z + (\eta^2 - \eta)R(\eta, z)),
\]

with \( C_0(0) = \frac{1}{2}, \ C_1(0) = \frac{1}{4} - \frac{b(0)}{2} \) and \( R \) a mix analytic function. With the same reasoning applied to \( \tilde{g}_\eta := \tilde{f}_\eta \circ \tilde{f}_\eta \), we find

\[
\tilde{g}_\eta(z) = z + (z^2 - \eta)(D_0(\eta) + D_1(\eta)z + (\eta^2 - \eta)Q(\eta, z)),
\]

with \( D_0(0) = 1, \ D_1(0) = 1 - b(0) \) and \( Q \) a mix analytic function.

Lastly, we set

\[
\tilde{\tau}_\pm := \tilde{f}_\eta'(\pm \sqrt{\eta}) = 1 \pm 2\sqrt{\eta}(C_0 \pm C_1\sqrt{\eta}).
\]

**Proof of Theorem 3.3.2.** We complexify \( \eta \). The fact that \( \varepsilon \) and \( b \) are invariant is because the multipliers of \( \tilde{g}_\eta \) are invariant. They are holomorphic for \( \eta \neq 0 \) since they are invariant under the permutation \( \sqrt{\eta} \mapsto -\sqrt{\eta} \). Then we see they are bounded around \( \eta = 0 \) using \( \lambda_\pm = 1 \pm D_0\sqrt{\eta} + O(\eta) \). The details are found in [9].

Lastly, \( \varepsilon \) and \( b \) are real-valued for \( \eta \) real since the multipliers satisfy

\[
\lambda_\pm = \begin{cases} |\tau_\pm|^2, & \text{if } \eta \geq 0; \\ \tau_\pm \tau_\mp, & \text{if } \eta < 0; \end{cases}
\]

which follows from the chain rule. \( \square \)
Proof of Theorem 3.3.3. Let \( z_1 = w_\eta(z) = z + (z^2 - \eta)(A_0(\eta) + A_1(\eta)z) \) be a change of coordinate, where

\[
A_0 = \frac{\sqrt{\tau_+} - \sqrt{\tau_-}}{4\sqrt{\eta}}, \quad A_1 = \frac{\sqrt{\tau_+} + \sqrt{\tau_-} - 2}{4\eta}.
\]

Let us complexify \( \eta \). For \( \eta \neq 0 \), \( A_0 \) and \( A_1 \) are analytic since they are invariant under the permutation \( \sqrt{\eta} \mapsto -\sqrt{\eta} \). Also we have that \( w'(\pm\sqrt{\eta}) = \sqrt{\hat{\tau} \pm} \) at \( z_1 = \pm\sqrt{\eta} \) satisfies point 2. Also, point 1 is a consequence of point 2, and point 3 follows from the chain rule and point 2.

We do a last change of coordinate and parameter, that will preserve points 1 to 3, so that the new family depends on its canonical parameter \( \varepsilon \). Let \( L(\eta, z_1) = (\varepsilon(\eta), c(\eta)z_1) = (\varepsilon, L_\eta(z_1)) \), where \( \varepsilon : \eta \to \varepsilon(\eta) \) is given by (11) and \( c(\eta) = \sqrt{\varepsilon(\eta)/\eta} \). Since \( \varepsilon(\eta) = \eta(1 + O(\eta)) \), we see that the function \( c \) is real analytic and real valued, so \( f_\varepsilon = L_\eta \circ f_\eta \circ L_\eta^{-1} \) still satisfies points 1 to 3. Calling \( z_2 = c(\eta)z_1 \), then we have

\[
f_\varepsilon(z_2) = \bar{z}_2 + (\bar{z}_2^2 - c^2\eta)(B_0(\varepsilon) + B_1(\varepsilon)\bar{z}_2 + (\bar{z}_2^2 - \varepsilon)Q(\varepsilon, \bar{z}_2)),
\]

for some real valued \( B_0 \) and \( B_1 \) and some remainder \( Q \). Since \( c^2\eta = \varepsilon \), \( f_\varepsilon \) is in prepared form.

\[\square\]

4 Fatou Coordinates

Let \( f_0 \) be an antiholomorphic parabolic germ of codimension 1. From now on, we only consider prepared generic unfoldings depending on the canonical parameter \( \varepsilon \), that is

\[
f_\varepsilon(z) = \bar{z} + (\bar{z}^2 - \varepsilon)(B_0(\varepsilon) + B_1(\varepsilon)\bar{z} + (\bar{z}^2 - \varepsilon)Q(\varepsilon, \bar{z})),
\]

(15)

where \( B_0 \) and \( B_1 \) are real valued.

We will work with a representative of the germ, also noted \( f_\varepsilon \), defined on \((-r', r') \times D(0, r)\), where \( r', r > 0 \) may be as small as we want.

4.1 Time coordinate

We will work in the time coordinate of the vector field (9). On a given simply connected domain in a disk \( D(0, r) \setminus \{\pm \sqrt{\varepsilon}\} \), it is defined up to a constant. Moreover it is ramified with period \( 2\pi ib(\varepsilon) \) when turning around the two singular points as soon as \( b(\varepsilon) \neq 0 \): see Figure 3. We will use two charts covering the two halves of \( D(0, r) \setminus \{\pm \sqrt{\varepsilon}\} \).
Proposition 4.1.1. We define

\[ Z^+_\varepsilon(z) = \int_z^z \frac{1 + \zeta b(\varepsilon)}{\zeta^2 - \varepsilon} \, d\zeta, \]

\[ Z^-_\varepsilon(z) = \int_{-r}^z \frac{1 + \zeta b(\varepsilon)}{\zeta^2 - \varepsilon} \, d\zeta. \]

The principal branches of \( Z^\pm_\varepsilon \) are defined by (16) on simply connected domains inside \( \mathbb{C} \setminus \{-\sqrt{\varepsilon}, \sqrt{\varepsilon}\} \) containing \( \pm r \).

Chosen in this way, these principal branches have the following properties:

1. They satisfy

\[ Z^+_\varepsilon(z) - Z^-_\varepsilon(z) = \begin{cases} 
  i\pi b(\varepsilon), & \text{if } \Im z > 0 \text{ and } |z| = r; \\
  -i\pi b(\varepsilon), & \text{if } \Im z < 0 \text{ and } |z| = r;
\end{cases} \]

(17)

2. They conjugate \( v^1 \) on \( T_1 \);

3. \( Z^\pm_\varepsilon(z) = \overline{Z^\pm_\varepsilon(z)} \).

Proof. These properties are shown by direct computation. Note that points 1 and 3 use the fact that \( b(\varepsilon) = b(\overline{\varepsilon}) \), which follows from Theorem 3.3.2.

If we look at \( v^1_\varepsilon \) on a disk \( D(0, r) \), then for \( \varepsilon \ll r \), the dynamics of \( v^1_\varepsilon \) is almost identical to the dynamics of \( v^1_0 \) near the boundary of \( D(0, r) \). Therefore, the time coordinate for \( \varepsilon \) small is similar to the time coordinate for \( \varepsilon = 0 \) close to the hole. See Figure 3.

To obtain the rest of the surface corresponding to the interior of \( D(0, r) \), the path of integration in (16) will have to turn around the ramification points \( \pm \sqrt{\varepsilon} \).

The surface of Figure 3 will repeat itself with periods given by

\[ \alpha^\pm_\varepsilon := \int_{\gamma^\pm_\varepsilon} \frac{1 + b(\varepsilon)\zeta}{\zeta^2 - \varepsilon} \, d\zeta = \pm \frac{i\pi}{\sqrt{\varepsilon}} + i\pi b(\varepsilon), \]

(18)
where $\gamma_{\pm}^\varepsilon$ is a small simple curve surrounding exactly the singular point $\pm \sqrt{\varepsilon}$. In particular, $T_{\alpha_{\pm}^\varepsilon}$ are covering transformations of the surface.

Let $\Sigma$ be the lift of $\sigma$ on the time coordinate. By point 2 of the Proposition 4.1.1, we see that $\Sigma$ is the analytic continuation of the complex conjugation on the chart of the principal branch obtained with the relation

$$\Sigma \circ T_{\alpha_{\pm}^\varepsilon} = T_{\alpha_{\pm}^\varepsilon} \circ \Sigma.$$

### 4.2 Translation Domains

We define charts in the time coordinate determined naturally by the dynamics of $g_\varepsilon = f_\varepsilon \circ f_\varepsilon$. Let $G_\varepsilon$ be the lift of $g_\varepsilon$ in the time coordinate.

In [9], it is proved that $|G_\varepsilon - T_1| \leq C \max\{r, r'\}$. Therefore, $G_\varepsilon$ is as close to $T_1$ as we want for $r, r'$ small enough. If we take a vertical line $\ell$ or, for any $\beta \in (0, \frac{\pi}{2})$, a slanted line $\ell$ making an angle in $(\beta, \pi - \beta)$ with the horizontal direction, then for $r, r'$ small enough, $\ell$ and $G_\varepsilon(\ell)$ will not intersect.

**Remark 4.2.1.** Occasionally we will suppose that $z = r$ belongs to the domain. This is legitimate by slightly restricting $r$.

**Definition 4.2.2.** Let $\ell$ be a vertical line with the distance between $\ell$ and the holes at least 2. We consider the vertical strip $B_\ell$ lying between $\ell$ and $G_\varepsilon(\ell)$, including its boundary. We define the translation domain by

$$U_\varepsilon = \{G_\varepsilon^{\circ n}(Z) \mid Z \in B_\ell\}.$$  

The translation domain is called a *Glutsyuk* (resp. *Lavaurs*) translation domain when $\varepsilon > 0$ (resp. $\varepsilon < 0$).

For $\varepsilon > 0$ (resp. $\varepsilon < 0$), the holes are aligned vertically (resp. horizontally). The strip $B_\ell$ is parallel (resp. transversal) to the line of the holes. See Figures 4a and 4b.

### 4.3 Existence of Fatou Coordinates

The sectorial normalization of $f_\varepsilon$ can easily be proved using the analogous theorem for $f_\varepsilon \circ f_\varepsilon$.

**Theorem 4.3.1.** Let $f_\varepsilon$ be a generic unfolding in prepared form of an antiholomorphic parabolic germ of codimension 1. Let $F_\varepsilon$ be the lift of $f_\varepsilon$ on the time coordinate.

1. (*Existence*) For every $\varepsilon > 0$ (resp. $\varepsilon < 0$), on every Glutsyuk (resp. Lavaurs) translation domain $U_\varepsilon^\pm$, there exists a diffeomorphism $\Phi_\varepsilon^\pm : U_\varepsilon^\pm \to \mathbb{C}$ such that

$$\Phi_\varepsilon^\pm \circ F_\varepsilon \circ (\Phi_\varepsilon^\pm)^{-1} = \Sigma \circ T_2.$$  

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2. (Uniqueness) If \( \tilde{\Phi}_\varepsilon^\pm \) is another diffeomorphism satisfying (19), then there exists a real constant \( R_\pm(\varepsilon) \) such that
\[
\tilde{\Phi}_\varepsilon^\pm \circ (\Phi_\varepsilon^\pm)^{-1} = T_{R_\pm(\varepsilon)}.
\]

3. For \( \varepsilon > 0 \), the Fatou coordinates on \( U_\varepsilon^\pm \) commute with \( T_{\alpha_\varepsilon^\pm} \), where \( \alpha_\varepsilon^\pm \) is the period (18).

4. (Dependence on parameter) Let
\[
Q_\varepsilon^\pm = \bigcup_{\varepsilon} \{ \varepsilon \} \times U_\varepsilon^\pm.
\]
Then \( Q_\varepsilon^\pm \) is open in \( \mathbb{R} \times \mathbb{C} \) and there exists a family of Fatou coordinates \( \{ \Phi_\varepsilon^\pm \}_\varepsilon \) continuous on \( Q_\varepsilon^\pm \) and mix analytic for \( \varepsilon \neq 0 \). The family is uniquely determined by
\[
\Phi_\varepsilon^\pm(X_\varepsilon) = C(\varepsilon),
\]
where \( X_\varepsilon \) is a base point, \( C \) is real-valued, and both \( X_\varepsilon \) and \( C \) are real analytic in \( \varepsilon \neq 0 \) with continuous limit at \( \varepsilon = 0 \).

Proof. 1 and 4. It is proved in [9] that we can construct Fatou coordinates of \( g_\varepsilon \) on translation domains. A Fatou coordinate \( \Phi_\varepsilon^\pm : U_\varepsilon^\pm \rightarrow \mathbb{C} \) on the translation domain \( U_\varepsilon^\pm \) is a change of coordinate that rectifies \( G_\varepsilon \) to the normal form: \( \Phi_\varepsilon^\pm \circ G_\varepsilon \circ (\Phi_\varepsilon^\pm)^{-1} = T_1 \). We can choose a family \( \{ \Phi_\varepsilon^\pm \}_\varepsilon \) that is continuous in \( (\varepsilon, Z) \) and mix analytic for \( \varepsilon \neq 0 \), it suffices for instance to choose Fatou coordinates with a fixed base point \( Z_0 \): \( \Phi_\varepsilon(Z_0) = 0 \). Note that on a Glutsyuk translation domain \( U_\varepsilon^\pm (\varepsilon > 0) \), Fatou coordinates commute with \( T_{\alpha_\varepsilon^\pm} \).

Let \( P_\varepsilon = \Phi_\varepsilon^\pm \circ F_\varepsilon \circ (\Phi_\varepsilon^\pm)^{-1} \). We have \( P_\varepsilon \circ P_\varepsilon = T_1 \), so \( P_\varepsilon \) and \( T_1 \) commute. We prove that \( \Sigma \circ P_\varepsilon \) is a translation.
In the Fatou coordinate of the Glutsyuk translation domain (ε > 0), we can describe a fundamental domain of the orbits of $G_ε$ on $U_{ε}^±$ by quotienting the space by $T_1$ and $T_{α_±}$. This results in a torus, where each point represents an orbit of $G_ε$ on $U_{ε}^±$. Since $Σ ∘ P_ε$ commutes with $T_{α_±}$ and $T_1$, it induces a holomorphic diffeomorphism of this torus on itself. Such a mapping must be a translation, so it follows that $P_ε$ is of the form $Σ ∘ T_{C(ε)}$, for some $C(ε)$.

Similarly, in the Fatou coordinate of the Lavaurs translation domain ($ε < 0$), a fundamental domain of the orbits of $G_ε$ on $U_{±ε}$ is obtained by the quotient of $T_1$, yielding the well-known Écalle cylinder. The cylinder is isomorphic to $S^2 \setminus \{0, ∞\} = C^*$. Each point of $C^*$ corresponds to an orbit of $G_ε$ on $U_{±ε}$, and we can fill the holes at 0 and ∞ by identifying them to the fixed points. Since $Σ ∘ P_ε$ commutes with $T_1$ and $f$ maps $±\sqrt{ε}$ to $±\sqrt{ε}$, $Σ ∘ P_ε$ induces a holomorphic diffeomorphism of the sphere on itself, and such a mapping must be a linear map. It follows that $P_ε$ is of the form $Σ ∘ T_{C(ε)}$ for some $C(ε)$.

In both cases, $P_ε$ is of the form $Σ ∘ T_{C(ε)}$, for some $C(ε) = 1/2 + iy(ε)$, $y \in R$, where $y$ is real analytic in $ε \neq 0$ with continuous limit at $ε = 0$. It follows that $T_{iy(ε)/2} ∘ Φ_±^ε$ is a Fatou coordinate of $f_ε$. That (20) determines the family $\{Φ_±^ε\}_ε$ follows from point 2 below.

2. Recall that the Fatou coordinates of $g_ε$ are unique up to translation. To preserve (19), the constant of translation must be real.

3. This is true for Fatou coordinates of $g_ε$ (see [9]).

Corollary 4.3.2. Each strip $B_ε$ is conformally equivalent to a sphere minus two points.

5 Space of Orbits

5.1 Transition Functions

Equation (17) from Proposition 4.1.1 allows us to define a transition function $T_{−iπb} = Z_{ε}^- ∘ (Z_{ε}^+)^{-1}$ on the connected component above the fundamental hole in $U_{ε}^+$ to the corresponding component in $U_{ε}^-$, see Figure 5. Since $T_{−iπb}$ commutes with $T_1$, it sends orbits of the holomorphic normal form $v_{ε}^1$ on orbits of $v_{ε}^1$.

We define the transition functions of $f_ε$ that identify orbits represented on both $U_{ε}^+$ and $U_{ε}^-$. Let $$(Φ_ε^+, Φ_ε^-)$ be a pair of Fatou coordinates of $f_ε$ on $U_{ε}^+$ and $U_{ε}^-$ respectively. Let $V_{ε}^± = Φ_ε^±(U_{ε}^±)$. We define the transition function of $f_ε$ on the connected component above the fundamental hole where the composition is defined by

$$Ψ_ε = Φ_ε^- ∘ T_{−iπb} ∘ (Φ_ε^+)^{-1}. \quad (21)$$
(a) Transition functions on a Glutsyuk domain

(b) Transition functions on a Lavaurs domain

Figure 5: Transition functions between time coordinates for $\varepsilon > 0$ (left) and $\varepsilon < 0$ (right).

Notation 5.1.2. We will note it $\Psi_\varepsilon^\infty$ for $\varepsilon \leq 0$, and $\Psi_\varepsilon^G$ for $\varepsilon > 0$.

For $\varepsilon \leq 0$, we define a transition function below the fundamental hole by

$$\Psi_\varepsilon^0 = \Phi_\varepsilon^- \circ T_{i\pi b} \circ (\Phi_\varepsilon^+)^{-1}.$$  

The most important property of the transition functions is that they commute with $T_1$, so that they map orbits of $g_\varepsilon$ on orbits of $g_\varepsilon$.

Since they are defined using the Fatou coordinates of $f_\varepsilon$, they are also compatible with the orbits of $f_\varepsilon$. For $\varepsilon \leq 0$, we have

$$\Psi_\varepsilon^\infty \circ \Sigma \circ T_{1/2} = \Sigma \circ T_{1/2} \circ \Psi_\varepsilon^0,$$  \hspace{1cm} (22)

and for $\varepsilon > 0$, if we define $\Psi_\varepsilon^U = \Psi_\varepsilon^G$ and $\Psi_\varepsilon^L = T_{\alpha^-} \circ \Psi_\varepsilon^G \circ T_{\alpha^+}$ then

$$\Psi_\varepsilon^U \circ \Sigma \circ T_{1/2} = \Sigma \circ T_{1/2} \circ \Psi_\varepsilon^L.$$  \hspace{1cm} (23)

(See Figure 6a.)

5.2 Analytic Properties of the Transition Functions and Normalization

For $\varepsilon \leq 0$, $\Psi_\varepsilon^0$ is determined by $\Psi_\varepsilon^\infty$ in (22). Therefore, we only to describe the analytic properties of $\Psi_\varepsilon^\infty$ and $\Psi_\varepsilon^G$. The properties are the same as in the holomorphic case, with the exception of Equation (26) below, which is specific to the antiholomorphic case.
Proposition 5.2.1.

1. The transition functions of $f_\varepsilon$ commute with $T_1$. They have a series expansion of the form

$$\Psi^\infty_\varepsilon(W) = W + c^\infty_0(\varepsilon) + \sum_{n=1}^{\infty} c^\infty_n(\varepsilon)e^{2i\pi nW},$$

for $\varepsilon \leq 0$, and

$$\Psi^G_\varepsilon(W) = W + c^G_0(\varepsilon) + \sum_{n \in \mathbb{Z}^*} c^G_n(\varepsilon)e^{2i\pi nW},$$

for $\varepsilon > 0$. The constant terms satisfy

$$\Im c^\infty_0(\varepsilon) = -i\pi b(\varepsilon) \quad \text{and} \quad \Im c^G_0(\varepsilon) = -i\pi b(\varepsilon).$$

2. Moreover, there exists a choice of Fatou coordinates $\Phi^\pm_\varepsilon$ for which the constant terms are purely imaginary:

$$c^\infty_0(\varepsilon) = -i\pi b(\varepsilon), \quad \text{for } \varepsilon \leq 0,$$

$$c^G_0(\varepsilon) = -i\pi b(\varepsilon), \quad \text{for } \varepsilon > 0.$$

Proof. Equations (24) and (25) follow from the fact that $\Psi^\infty_\varepsilon$ and $\Psi^G_\varepsilon$ commute with $T_1$. Furthermore, for $\varepsilon \leq 0$, $c^\infty_n(\varepsilon) = 0$ for $n < 0$ because $\Psi^\infty_\varepsilon(W) - W$ is bounded when $\Im W \to \infty$.

For the proof of Equation (26) for $c^\infty_0$, we need the well-known equation

$$c^\infty_0 - c^0_0 = -2i\pi b.$$
This equation will be a direct consequence of Lemme 5.4.1, see Remark 5.4.2 below. By comparing the constant terms of (22), we find \( c_0^\infty = \overline{c_0^0}. \) Now Equation (26) for \( c_0^\infty \) follows by combining (29) with \( c_0^\infty = \overline{c_0^0}. \)

For \( c_0^G \), it follows by comparing the constant terms of Equation (23).

It suffices to postcompose one Fatou coordinate with a real translation to obtain the normalizations (27) or (28).

\[ \square \]

**Definition 5.2.2.** A family of transition functions \( \{ \Psi_\varepsilon \}_\varepsilon \) is said to be normalized if it is continuous in \( (\varepsilon, Z) \) and real analytic for \( \varepsilon \neq 0 \) and if (27) or (28) is verified. The corresponding pairs of families of Fatou coordinates \( \{ \Phi_\varepsilon^\pm \}_\varepsilon \) are said to be normalized.

### 5.3 Geometry of the Space of Orbits for \( \varepsilon > 0 \)

**Theorem 5.3.1.** For \( \varepsilon > 0 \) the space of orbits \( O_\varepsilon \) of \( f_\varepsilon \) on \( D(0, r) \setminus \{ \pm \sqrt{\varepsilon} \} \) is described by two Klein bottles \( K_\varepsilon^\pm \), identified along Moebius strips \( M_\varepsilon^\pm \), by a map \( [\Psi_\varepsilon] \) induced by the transition function \( \Psi_\varepsilon \): see Figure 7.

We add two points \( \{ P_\varepsilon^\pm \} \) to \( O_\varepsilon \) corresponding to the fixed points \( \pm \sqrt{\varepsilon} \) with the following topology: the unique neighbourhood of \( P_\varepsilon^\pm \) is \( K_\varepsilon^\pm \).

![Figure 7: Klein bottles and transition function.](image)

**Proof.** We set \( S_\varepsilon^\pm = Z_\varepsilon^{-1}(U_\varepsilon^\pm) \), which yields a covering of \( D(0, r) \setminus \{ \pm \sqrt{\varepsilon} \} \): see Figure 6b. Note that some orbits of \( f_\varepsilon \) in \( D(0, r) \setminus \{ \pm \sqrt{\varepsilon} \} \), are covered by both charts.

Let \( V_\varepsilon^\pm = \Phi_\varepsilon^\pm(U_\varepsilon^\pm) \). On \( V_\varepsilon^\pm \), the region of intersection corresponds to horizontal strips. The transition function \( \Psi_\varepsilon \) maps orbits from a strip in \( V_\varepsilon^+ \) to the corresponding strip in \( V_\varepsilon^- \). Since the orbits repeat themselves according to the period \( \alpha_\varepsilon^\pm \) on \( V_\varepsilon^\pm \), the strips also repeat with associated transition functions. See Figure 6a.
To describe the space of orbits of \( f_\varepsilon \), we begin by taking a vertical strip of width 1 in both \( V^+_\varepsilon \) and \( V^-_\varepsilon \). Together, they intersect all the orbits of \( f_\varepsilon \) at least once, except the fixed points. We identify \( W \in V^\pm_\varepsilon \) with \( T_1(W) \) and \( T_{\alpha^\pm}(W) \), leaving us with two torii \([0, 1] \times [-\frac{\alpha^\pm}{2}, \frac{\alpha^\pm}{2}]\). Since the transition function commutes with both \( T_1 \) and \( T_{\alpha^\pm} \), it is well-defined on the torii. We identify together \( w \) and \( \Psi_\varepsilon(w) \), as they represent the same orbit of \( f_\varepsilon \). This is the space of orbits of \( f_\varepsilon \circ f_\varepsilon \).

Lastly, since \( \Sigma \circ T_1 \) is compatible with \( T_1, T_{\alpha^\pm} \) and \( \Psi_\varepsilon \), it induces a mapping on each torus. We identity \( w \) and \( \Sigma \circ T_1(w) \) to obtain the space of orbits of \( f_\varepsilon \).

Each torus becomes a Klein bottle, as in Figure 7.

5.4 Geometry of the Space of Orbits for \( \varepsilon < 0 \)

Note that the space of orbits of \( f_\varepsilon \) is a quotient of that of \( g_\varepsilon \). Hence we will describe both. This will need introducing a few notions.

Let us consider two strips \( B_\ell^\pm \) on each side of the fundamental hole, and let \( S^\pm \) be two spheres corresponding to the completion of the strips \( B_\ell^\pm \) with sides identified. Then each orbit of \( f_\varepsilon \) corresponds to at least one point of each sphere, with \( \infty \) corresponding to \( \sqrt{\varepsilon} \), and 0 to \( -\sqrt{\varepsilon} \). Let \( \psi_0, \infty_\varepsilon \) be the mapping induced by \( \Psi_0, \infty_\varepsilon \) between neighbourhoods of 0 and \( \infty \) on the two spheres.

![Figure 8: Point in \( C^- \) mapped to the point of its orbit in \( C^+ \) by the Lavaurs transition.](image)

**Lemma 5.4.1.** Let \( \varepsilon < 0 \).

1. There exists a global map called the Lavaurs transition that maps an orbit of \( f_\varepsilon \) in \( S^- \) to the same orbit in \( S^+ \): see Figure 8. It is a linear map \( L^\varepsilon \) on the spheres, and a translation \( T^\varepsilon \) in the Fatou coordinates. When the Fatou coordinates are normalized, then

\[
T^\varepsilon = T \frac{\varepsilon}{\sqrt{\varepsilon}}.
\]  

2. The linear map \( L_{-1} \circ \tau : S^\pm \rightarrow S^\pm; w \mapsto -\frac{1}{w} \) is the action of \( f_\varepsilon \) on \( S^\pm \) (see Figure 9); it maps an orbit of \( f_\varepsilon \) to the same orbit and the quotient \( \mathbb{P}_\pm := S^\pm / L_{-1} \circ \tau \) is a real projective plane;
3. The maps $\kappa^0,\infty = \ell^L \circ \psi^0,\infty$ are first return maps of $g_\varepsilon$ around $-\sqrt{\varepsilon}$ and $+\sqrt{\varepsilon}$ respectively and they are compatible with the orbits of $f_\varepsilon$, i.e. $(L_{-1} \circ \tau) \circ (\ell^L \circ \psi^0_\varepsilon) = (\ell^L \circ \psi^\infty_\varepsilon) \circ (L_{-1} \circ \tau)$.

4. In the time coordinate, the first return map of $\sqrt{\varepsilon}$ (resp. $-\sqrt{\varepsilon}$) is $T_{-\alpha^+}^{-\alpha}$ (resp. $T_{\alpha^-}$) as seen on Figure 10b.

5. The first return maps can be written in the Fatou coordinate by

\begin{align*}
T^L \circ \Psi^\infty_\varepsilon &= \Phi^+_\varepsilon \circ T_{-\alpha^+} \circ (\Phi^+_\varepsilon)^{-1}, \tag{31} \\
T^L \circ \Psi^0_\varepsilon &= \Phi^+_\varepsilon \circ T_{\alpha^-} \circ (\Phi^+_\varepsilon)^{-1}. \tag{32}
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fatou_coordinate_diagram.png}
\caption{The space of orbits of $f_\varepsilon$ for $\varepsilon < 0$.}
\end{figure}

Remark 5.4.2. Comparing constant terms from Equations (31) and (32), we find the well known relation $c^\infty_0 - c^0_0 = -2i\pi b$ for $\varepsilon < 0$. It holds for any transition functions, not necessarily normalized.

The following theorem is a direct consequence of Lemma 5.4.1.

**Theorem 5.4.3.** The space of orbits $\mathcal{O}_\varepsilon$ of $f_\varepsilon$ is the quotient of a real projective plane by a diffeomorphism in the neighborhood of one point. This diffeomorphism corresponds to a first return map in the neighborhood of the periodic points. See Figure 9.

**Proof of Lemma 5.4.1.** The space of orbits of $g_\varepsilon$ can be described with a croissant, as in Figure 10a, and the return maps around each fixed point (see [9]). If we quotient by $f_\varepsilon$, we obtain the space of orbits of $f_\varepsilon$.

On a Fatou coordinate $W = \Phi^+_\varepsilon(Z)$ of $f_\varepsilon$, we take a strip of width 1 to the left of the fundamental hole. With the universal covering $E : W \mapsto \exp(-2i\pi W)$, we see that this strip is isomorphic to $\mathbb{C}^*$. We identify $\sqrt{\varepsilon}$ to $\infty$ and $-\sqrt{\varepsilon}$ to 0, so
that we now have a sphere. Each point represents an orbit of $g_\varepsilon$. On the sphere, $f_\varepsilon$ becomes $L_{-1} \circ \tau$.

In the $z$-coordinate, the sphere can be seen as a croissant $C^+$ going from $-\sqrt{\varepsilon}$ to $\sqrt{\varepsilon}$, see Figure 10a. We see that every orbit of $f_\varepsilon$ intersects the croissant. In the neighborhood of each fixed point $\pm \sqrt{\varepsilon}$, we can define a first return map $p^\pm$ in the croissant for the orbits of $g_\varepsilon$. In the time coordinate, the first return maps are $T_{-\alpha^\pm} + \varepsilon$ and $T_{\alpha^\pm} - \varepsilon$ (see Figure 10b), so we see that they are compatible with the orbits of $f_\varepsilon$ since $T_{-\alpha^\pm + \varepsilon} \circ F_\varepsilon = F_\varepsilon \circ T_{-\alpha^\pm - \varepsilon}$ and $-\alpha^\pm + \varepsilon = \alpha^\pm - \varepsilon$.

We consider a strip of width 1 in the Fatou coordinate of $\Phi^-_\varepsilon$ to the right of the fundamental hole, yielding a second croissant $C^-$ in the $z$-coordinate. We define a diffeomorphism that maps each point of the croissant $C^-$ to the first point of its forward orbit in the croissant $C^+$, see Figure 8. This map defines a global diffeomorphism from a sphere to another sphere, so it is a linear map $\ell^L$. In the Fatou coordinates, it is a translation; we call it the Lavaurs translation $T^L$.

Combining (21) and (31) and using the definition of $\alpha^\pm_\varepsilon$ (18), we obtain $T^L = \Phi^-_\varepsilon \circ T_{-\alpha^\pm_\varepsilon \sqrt{\varepsilon}} \circ (\Phi^+_\varepsilon)^{-1}$. Then because $T^L$ is a translation and $\Psi^\infty_\varepsilon$ is normalized (27), we obtain $T^L = T_{-\alpha^\pm_\varepsilon \sqrt{\varepsilon}}$.

Let $\psi^0,\infty_\varepsilon$ be the maps induced on the spheres by $\Psi^0,\infty_\varepsilon$. The space of orbits of $g_\varepsilon$ is obtained by identifying $w \in S^{-}$ with $\ell^L(w) \in S^{+}$, and $w \in S^{+}$ with $\psi^0,\infty_\varepsilon(w) \in S^{-}$. Since $\ell^L$ and $\psi^0,\infty_\varepsilon$ are compatible with the orbits of $f_\varepsilon$, $f_\varepsilon$ induces the global diffeomorphism $L_{-1} \circ \tau$ on the space of orbits of $g_\varepsilon$. We identify $w$ with $L_{-1} \circ \tau(w)$ to obtain the space of orbits of $f_\varepsilon$: this yields a real projective plane with identification of $w$ and $\psi^\infty_\varepsilon(w)$.

6 Weak Classification

We have all the tools to define the weak modulus of classification.

The goal is to prove the strong equivalence (Definition 3.2.2) of two families
with the same weak modulus. With the tools we have so far, we can only prove a weaker form of equivalence (see Definition 6.1.1 below). Indeed, Fatou coordinates, instrumental in the construction of the equivalence, are not analytic at $\varepsilon = 0$.

In Section 6.2, we will give a simple proof of weak equivalence and we will prove strong equivalence in Section 7.

### 6.1 Weak modulus of Classification

The space of orbits of $f_\varepsilon$ can be described by the codimension, the formal invariant and one transition function. These are the three parts of the weak modulus of classification.

**Definition 6.1.1.** Let $f_\eta$ be a generic unfolding of a parabolic antiholomorphic germ of codimension 1. Its *weak modulus of classification* is the triple $(\varepsilon, b, [\Psi_\varepsilon])$, where $\varepsilon$ is the canonical parameter of $f_\eta$, $b$ is the formal invariant (a real analytic function of $\varepsilon$) and $[\Psi_\varepsilon]$ is an equivalence class of normalized families of transition functions under the relation

$$\{\Psi_\varepsilon\}_\varepsilon \sim \{\Psi'_\varepsilon\}_\varepsilon \iff \varepsilon \neq 0 \text{ and continuous such that } \Psi'_\varepsilon = T_{C(\varepsilon)} \circ \Psi_\varepsilon \circ T_{-C(\varepsilon)}.$$  

### 6.2 Weak Classification Theorem

**Definition 6.2.1 (Weak Equivalence).** Let $f_{1,\eta}$ and $f_{2,\varepsilon}$ be two generic unfoldings of antiholomorphic parabolic germs of codimension 1. We say they are *weakly equivalent* if there exists an open interval $I \ni 0$, a disc $D(0, r)$, $r > 0$, and a mix analytic diffeomorphism $H: I \times D(0, r) \to \mathbb{R} \times \mathbb{C}$ such that

1. $H(0, 0) = (0, 0)$;
2. $H(\eta, z) = (\beta(\eta), h_\eta(z))$ with $\beta$ real analytic and $h_\eta$ continuous in $(\eta, z)$ and real analytic for $\eta \neq 0$;
3. $f_{2,\beta(\eta)} = h_\eta \circ f_{1,\eta} \circ h_\eta^{-1}$.

**Theorem 6.2.2 (Weak Classification).** Two generic unfoldings of antiholomorphic parabolic germs of codimension 1 are weakly equivalent if and only if they have the same weak modulus of classification.

**Proof.** One direction is obvious. Conversely, let $f_{1,\varepsilon}$ and $f_{2,\varepsilon}$ be the two families that, without loss of generality, we can suppose in prepared form, and let $\varepsilon$ be
their canonical parameter. Moreover, let us suppose that the two families have the same weak modulus \((\varepsilon, b, [\Psi_\varepsilon])\). Then \(\beta \equiv \text{id}\).

We use Fatou coordinates \(\Phi_{j, \varepsilon}^\pm\) of \(f_{j, \varepsilon}\) to construct the change of coordinate \(h_\varepsilon\) that conjugates \(f_{1, \varepsilon}\) to \(f_{2, \varepsilon}\). Since \(f_j\) have the same weak modulus, we can choose \(\Phi_j^\pm\) so that

\[
\Phi_{1, \varepsilon}^- \circ T_{-ib} \circ (\Phi_{1, \varepsilon}^+)^{-1} = \Psi_\varepsilon = \Phi_{2, \varepsilon}^- \circ T_{-ib} \circ (\Phi_{2, \varepsilon}^+)^{-1}.
\]

We can divide \(D(0, r) \setminus \{\pm \sqrt{\varepsilon}\}\) in two regions \(S_{\varepsilon}^\pm\) as in Figures 11 and 12. For \(z \in S_{\varepsilon}^\pm\), we define \(h_\varepsilon^\pm\) by

\[
h_{\varepsilon}^\pm(z) = (Z_{\varepsilon}^\pm)^{-1} \circ (\Phi_{1, \varepsilon}^\pm)^{-1} \circ \Phi_{2, \varepsilon}^\pm \circ Z_{\varepsilon}^\pm(z) .
\]

A direct computation shows that \((h_\varepsilon^\pm)^{-1} \circ f_{1, \varepsilon} \circ h_\varepsilon^\pm = f_{2, \varepsilon}\). So it only remains to prove that \(h_\varepsilon^+ = h_\varepsilon^-\) on \(S_{\varepsilon}^+ \cap S_{\varepsilon}^-\) in order to define \(h_\varepsilon\) on a \(D(0, r)\).

For \(\varepsilon > 0\), we have, using (17) and (21):

\[
h_{\varepsilon}^- \circ (h_{\varepsilon}^+)^{-1} = \left((Z_{\varepsilon}^-)^{-1} \circ (\Phi_{2, \varepsilon}^-)^{-1} \circ \Phi_{1, \varepsilon}^- \circ Z_{\varepsilon}^-ight)
\]

\[
\circ \left((Z_{\varepsilon}^+)^{-1} \circ (\Phi_{1, \varepsilon}^+)^{-1} \circ \Phi_{2, \varepsilon}^+ \circ Z_{\varepsilon}^+ight)
\]

\[
= (Z_{\varepsilon}^-)^{-1} \circ (\Phi_{2, \varepsilon}^-)^{-1} \circ \Psi_\varepsilon^G \circ \Phi_{2, \varepsilon}^+ \circ Z_{\varepsilon}^+
\]

\[
= (Z_{\varepsilon}^-)^{-1} \circ T_{-ib} \circ Z_{\varepsilon}^+
\]

\[
= \text{id}.
\]
For $\varepsilon < 0$, there are three cases. The intersection $S^+_\varepsilon \cap S^-_\varepsilon$ has three components: $I^+$ above $\sqrt{-\varepsilon}$, $I^-$ below $-\sqrt{-\varepsilon}$ and $I^L$ between $\sqrt{-\varepsilon}$ and $-\sqrt{-\varepsilon}$ (see Figure 11). For $I^\pm$, the computations are the same as in (34), but with $\Psi^{0,\infty}$. For $I^L$, the computation is similar, but we use the fact that $f_1$ and $f_2$ have the same Lavaurs translation (this is true as soon as $f_1$ and $f_2$ have the same formal invariant, see (30)).

We define

$$h_\varepsilon(z) = \begin{cases} h^+_\varepsilon(z), & \text{if } z \in R^+_\varepsilon; \\ h^-_\varepsilon(z), & \text{if } z \in R^-_\varepsilon; \end{cases}$$

then $h_\varepsilon$ is well-defined on $D(0,r) \setminus \{\pm \sqrt{-\varepsilon}\}$ by the above and we can analytically continue $h_\varepsilon$ on $\pm \sqrt{-\varepsilon}$, since it bounded.

7 Strong Classification

The Weak Equivalence Theorem 6.2.2 has a simple and direct proof. The main result of this section is that two families with same weak modulus of classification are strongly equivalent (see Definition 3.2.2). The proof is more involved and will necessitate to introduce the notion of strong modulus of classification.

Let $f_\varepsilon$ be an unfolding of an antiholomorphic parabolic germ in prepared form. In the proof of Theorem 6.2.2, we constructed a weak equivalence $h_\varepsilon$ using normalized Fatou coordinates of $f_\varepsilon$. The idea is to complexify $\varepsilon$ and to continue analytically the Fatou coordinates in a normalized way: they will be ramified in $\varepsilon$. The analytic extension of $h_\varepsilon$ will be shown to be holomorphic in $\varepsilon$ around $\varepsilon = 0$ in the nontrivial case when the transition maps are generically not translations, while in the trivial case, we will force $h_\varepsilon$ to be holomorphic using strongly normalized Fatou coordinates (defined below).

7.1 Complex Parameter

Recall that we work with a representative of the germ $f_\varepsilon$ defined on $(-r',r') \times D(0,r)$. We complexify $\varepsilon$ and we continue analytically $f_\varepsilon$ so that it is antiholomorphic in $\varepsilon$. Now

$$g_\varepsilon = f_\varepsilon \circ f_\varepsilon$$

is a full holomorphic unfolding of $g_0$.

Since $b$ is real-valued for real values of $\varepsilon$, it commutes with $\sigma$. It follows that $v^\varepsilon(z) = v_\varepsilon(\bar{z})$, for the vector field $v_\varepsilon(z) = \frac{z^2 - \varepsilon}{1+b(\varepsilon)z}$, and therefore the time-$t$ satisfies

$$\sigma \circ v^t_\varepsilon = v^t_\bar{\varepsilon} \circ \sigma.$$

The time coordinate is also defined for complex values of $\varepsilon$. However, the line of the holes, given by the period $\alpha^\pm_\varepsilon$, rotates when $\varepsilon$ varies around 0. See
Figure 14. Using the same definitions as in Proposition 4.1.1 we see that the two charts in time $Z^\pm$ cannot be defined uniformly in time. Therefore, we lift $\varepsilon$ on the universal covering of $D(0, r')^*$ and we work on a sector of the form (see Figure 13)

$$\Omega_\delta := \{ \hat{\varepsilon} = \rho e^{i\theta} \mid 0 < \rho < r', -\pi + \delta < \theta < 3\pi - \delta \}.$$  \hspace{1cm} (35)

Then $Z^\pm_{\hat{\varepsilon}}$ are well defined on $\Omega_\delta$.

Let us generalize the translation domains of $g_\varepsilon$ to complex values of $\varepsilon$. Let $G_{\hat{\varepsilon}}$ be the lift of $g_\varepsilon$ on the time coordinate. The bound $|G_{\hat{\varepsilon}} - T_1| \leq C \max\{r, r'\}$ still holds for $\hat{\varepsilon} \in \Omega_\delta$, for some constant $C$ [9]. So we may take $r, r'$ small enough so that $G_{\hat{\varepsilon}}$ and $T_1$ are as close as needed. We take a line $\ell$ that is transversal to the line of the holes such that $\ell$ and $G_{\hat{\varepsilon}}(\ell)$ do not intersect each other and any of the holes (see Figure 14). Then we set $B_\ell$ to be region between $\ell$ and $G_{\hat{\varepsilon}}(\ell)$, including the boundary, and we define the translation domain

$$U_{\hat{\varepsilon}} = \{ G^{\circ n}_{\hat{\varepsilon}}(Z) \mid Z \in B_\ell \}.$$ 

Note that for $\arg \hat{\varepsilon} = \pi$, this corresponds to the Lavaurs translation domain defined in Definition 4.2.2.

We see that there are two non equivalent ways to choose $\ell$ for $\hat{\varepsilon}$ such that $-\pi + \delta < \arg \hat{\varepsilon} < \pi - \delta$ (see third row in Figure 14 for $\varepsilon > 0$), which explains why we choose $\Omega_\delta$ with an overlapping over these values.

**Lemma 7.1.1** ([9]). Let $\hat{\varepsilon}$ be the lift of $\varepsilon$ on the universal covering of $D(0, r')^*$. If $r, r'$ are small enough, then there exists a continuous family of translation domains $\{U_{\hat{\varepsilon}}\}_{\hat{\varepsilon} \in \Omega_\delta}$.

On the overlapping sector $-\pi + \delta < \arg \hat{\varepsilon} < \pi - \delta$, both $\hat{\varepsilon}$ and $e^{2i\pi \hat{\varepsilon}}$ project on the same point $\varepsilon$. However, if $\{U_{\hat{\varepsilon}}\}_{\hat{\varepsilon} \in \Omega_\delta}$ is a continuous family, then $U_{\hat{\varepsilon}} \neq U_{e^{2i\pi \hat{\varepsilon}}}$, as seen on Figure 14.

### 7.2 Fatou Coordinates and Transition Functions for a Complex Parameter

We define a complex conjugate on $\Omega_\delta$. In a small sector centered on the negative real axis, i.e. centered on $\arg \hat{\varepsilon} = \pi$, we define it as the reflection along this axis.
(a) Rotation of the holes when $\varepsilon$ goes around 0 clockwise

(b) Rotation of the holes when $\varepsilon$ goes around 0 counter-clockwise

Figure 14: Time coordinates and strips $B_\varepsilon$ when $\varepsilon$ rotates from $\mathbb{R}^-$. 
and then we anti-holomorphically continue it on $\Omega_\delta$. This gives us
\[
\arg \hat{\varepsilon} = 2\pi - \arg \hat{\varepsilon}.
\]

**Proposition 7.2.1 (Continuation of Fatou Coordinates).** Let $f_{\hat{\varepsilon}}$ be the family induced on $\Omega_\delta$ by $f_\varepsilon$ and let $F_{\hat{\varepsilon}}$ be the lift of $f_{\hat{\varepsilon}}$ on the time coordinate.

1. The set
\[
Q^\pm = \bigcup_{\hat{\varepsilon} \in \Omega_\delta} \{\hat{\varepsilon}\} \times U^\pm_{\hat{\varepsilon}}
\]
is a complex manifold of dimension 2, where $\{U^\pm_{\hat{\varepsilon}}\}_{\hat{\varepsilon} \in \Omega_\delta}$ is a continuous family of translation domains.

2. There exists a family $\Phi^\pm = \{\Phi^\pm_{\hat{\varepsilon}}\}_{\hat{\varepsilon}}$ of Fatou coordinates of $f_{\hat{\varepsilon}}$ such that
   - $\{\Phi^\pm_{\hat{\varepsilon}}\}_{\hat{\varepsilon}}$ satisfies
     \[
     \Phi^\pm_{\hat{\varepsilon}} \circ F_{\hat{\varepsilon}} \circ (\Phi^\pm_{\hat{\varepsilon}})^{-1} = \Sigma \circ T_1;
     \]
   - $\Phi^\pm$ is holomorphic on $Q^\pm$ and continuous at $\hat{\varepsilon} = 0$, i.e.
     \[
     \lim_{\hat{\varepsilon} \to 0} \Phi^\pm_{\hat{\varepsilon}}(\hat{\varepsilon}, \cdot) = \Phi^\pm_0(\cdot),
     \]
     where the convergence is uniform on compact sets and $\Phi^\pm_0$ is a Fatou coordinate of $f_0$ on $U^\pm_0$;
   - The family is uniquely determined by
     \[
     \Phi^\pm_{\hat{\varepsilon}}(X^\pm_{\hat{\varepsilon}}) + \Phi^\pm_{\hat{\varepsilon}}(X^\pm_{\hat{\varepsilon}}) = C^\pm(\hat{\varepsilon}),
     \]
     where $X^\pm_{\hat{\varepsilon}}$ is a base point, $C^\pm$ commutes with $\sigma$ and both $X^\pm_{\hat{\varepsilon}}$ and $C^\pm$ are holomorphic in $\hat{\varepsilon} \neq 0$ with continuous limit at $\hat{\varepsilon} = 0$.

**Proof.** The idea for the proof of Point 1 is that for $(\varepsilon_0, Z_0) \in Q^\pm$ with $Z_0 \in B_\ell$, there exists a bi-disk $D(\varepsilon_0, r_1) \times D(Z_0, r_2) \subset Q^\pm$ for some $r_1, r_2 > 0$ small enough. Then for other points $(\varepsilon_0, Z) \in Q^\pm$, there is a neighborhood of the form $D(\varepsilon_0, r'_1) \times G^\pm_{\delta_0}(D(Z_0, r'_2)) \subset Q^\pm$, for $r'_1, r'_2 > 0$ small enough and $n \in \mathbb{Z}$ and $Z_0 \in B_\ell$ such that $G^\pm_{\delta_0}(Z_0) = Z$. (More details in [9]).

For Point 2, we drop the upper indices $\pm$. It is known ([9]) that there exist Fatou coordinates $\{\tilde{\Phi}_{\hat{\varepsilon}}\}_{\hat{\varepsilon} \in \Omega_\delta}$ for $g_\varepsilon$ holomorphic in $\varepsilon$ with continuous limit at $\varepsilon = 0$. Let
\[
\tilde{P}_{\hat{\varepsilon}} = \tilde{\Phi}_{\hat{\varepsilon}} \circ F_{\hat{\varepsilon}} \circ (\tilde{\Phi}_{\hat{\varepsilon}})^{-1}.
\]
Then $\tilde{P}_{\hat{\varepsilon}} \circ \tilde{P}_{\hat{\varepsilon}} = T_1$, from which it follows that $\tilde{P}_{\hat{\varepsilon}} \circ T_1 = T_1 \circ \tilde{P}_{\hat{\varepsilon}}$. As in Theorem 4.3.1 it follows that $\tilde{P}_{\hat{\varepsilon}} = \Sigma \circ T_C(\varepsilon)$, with $C(\varepsilon) + \overline{C(\varepsilon)} = 1$ and $C$ depending analytically on $\varepsilon$ with continuous limit at $\varepsilon = 0$. We set $\Phi_{\hat{\varepsilon}} = T_{-\overline{C(\varepsilon)}} \circ \tilde{\Phi}_{\hat{\varepsilon}}$. 
To see that (38) determines \( \{ \Phi_{\hat{\varepsilon}} \} \), suppose \( \{ \Phi^\dagger_{\hat{\varepsilon}} \} \) is another family such that
\[
\overline{\Phi_\varepsilon(X_\varepsilon)} + \Phi_\varepsilon(X_\varepsilon) = \overline{\Phi^\dagger_\varepsilon(X_\varepsilon)} + \Phi^\dagger_\varepsilon(X_\varepsilon).
\]
By uniqueness of Fatou coordinates of \( g_\varepsilon \) up to translation, there exists \( D \) such that \( D(\hat{\varepsilon}) = \overline{D(\hat{\varepsilon})} \) and \( \Phi_\varepsilon = T_{D(\hat{\varepsilon})} \circ \Phi^\dagger_\varepsilon \). Substituting this in the previous equation yields \( D(\hat{\varepsilon}) = 0 \).

Proposition 7.2.2 (Continuation of the Transition Functions). Let \( \{ \Phi^\pm_{\hat{\varepsilon}} \} \) be two families of Fatou coordinates of \( f_\varepsilon \) on \( U^\pm_{\hat{\varepsilon}} \). We define the transition functions for \( \hat{\varepsilon} \in \Omega_\delta \) by
\[
\Psi^\infty_{\hat{\varepsilon}}(W) := \Phi^-_{\hat{\varepsilon}} \circ T_{-\pi b(\hat{\varepsilon})} \circ (\Phi^+_{\hat{\varepsilon}})(W),
\]
(40)
\[
\Psi^0_{\hat{\varepsilon}}(W) := \Phi^-_{\hat{\varepsilon}} \circ T_{\pi b(\hat{\varepsilon})} \circ (\Phi^+_{\hat{\varepsilon}})(W),
\]
(41)
where the composition is defined respectively above or below the fundamental hole. Then we have

1. \( \Psi^\infty_{\hat{\varepsilon}} \), for \( \arg\hat{\varepsilon} = \pi \), coincides with \( \Psi^\infty_{\varepsilon} \), for \( \varepsilon < 0 \);
2. \( \Psi^\infty_{\hat{\varepsilon}} \) commutes with \( T_1 \);
3. \( \Psi^\infty_{\hat{\varepsilon}} \) and \( \Psi^0_{\hat{\varepsilon}} \) satisfy
\[
\Sigma \circ T_1 \circ \Psi^\infty_{\hat{\varepsilon}} = \Psi^0_{\hat{\varepsilon}} \circ \Sigma \circ T_1; \tag{42}
\]
4. \( \Psi^\infty_{\hat{\varepsilon}} \) has the series expansion
\[
\Psi^\infty_{\hat{\varepsilon}}(W) = W + c^\infty_0(\hat{\varepsilon}) + \sum_{n=1}^{\infty} c^\infty_n(\hat{\varepsilon})e^{2\pi nW}; \tag{43}
\]
5. If \( \{ \Phi^\pm_{\hat{\varepsilon}} \} \) are families holomorphic in \( \hat{\varepsilon} \not= 0 \) with a limit as \( \hat{\varepsilon} \to 0 \), then \( \Psi^\infty_{\hat{\varepsilon}} \) is holomorphic in \( \hat{\varepsilon} \not= 0 \) and has a limit when \( \hat{\varepsilon} \to 0 \);
6. If we choose the base point \( X^+_{\hat{\varepsilon}} = Z^+_\varepsilon(r) = 0 \), then the family \( \{ \Phi^\pm_{\hat{\varepsilon}} \} \) determined by \( \Phi^\pm_{\hat{\varepsilon}}(X^\pm_{\varepsilon}) = 0 \) is a family of Fatou coordinates of \( f_\varepsilon \) depending analytically of \( \hat{\varepsilon} \not= 0 \) with continuous limit at \( \hat{\varepsilon} = 0 \). We can then further choose the second family of Fatou coordinates so that the transition functions be normalized, namely so that the constant term of \( c^\infty_0(\hat{\varepsilon}) \) in (43) is given by \( c^\infty_0(\hat{\varepsilon}) = -\pi b(\hat{\varepsilon}) \). Then \( \Phi^-_{\hat{\varepsilon}} \) also depends analytically of \( \hat{\varepsilon} \) with continuous limit at \( \hat{\varepsilon} = 0 \).
Proof. Point 1 follows from the fact that the Lavaurs translation domains $U_{\varepsilon}^\pm$ for $\varepsilon < 0$ coincide with the translation domains $U_{\varepsilon}^\pm$ for $\text{arg} \varepsilon = \pi$, and hence so do the Fatou coordinates and therefore, the transition functions.

Equation (42) follows directly from (37).

The proof of (43) is identical to the case of the Lavaurs transition function for $\varepsilon < 0$. See Proposition 5.2.1.

Point 5 is obvious.

Lastly, for point 6, we see that the families $\{\Phi_{\varepsilon}^\pm\}_{\varepsilon}$ both satisfy some condition of the form (38) with $C$ depending analytically on $\varepsilon$ with continuous limit at $\varepsilon = 0$.

Definition 7.2.3. We say that a pair of families of Fatou coordinates $\{\Phi_{\varepsilon}^\pm\}_{\varepsilon}$ is strongly normalized if they are chosen as described in point 6 of Proposition 7.2.2. When this is the case, the transition functions are said to be strongly normalized.

Continuation of the Lavaurs Translation. The Lavaurs translation was defined in Lemma 5.4.1. It can be extended for $\varepsilon \in \Omega_\delta$ in the following way. We consider two vertical strips in the Fatou coordinates, one on the left and one on the right of the fundamental hole, see Figure 14 for the strip on the left. In the $z$-coordinate, they correspond to two croissants, as in Figure 15. This allows us to define the Lavaurs translation for $\varepsilon \in \Omega_\delta$. Indeed, since each orbit intersects both croissants at least once, we can define it the same way we did for $\varepsilon < 0$.

Figure 15: Croissants for $\varepsilon = \frac{1}{4}$ and $\varepsilon = \frac{e^{i\pi/2}}{4}$ in the case of the normal form, with $b = 0$.

7.3 The strong modulus of analytic classification

Definition 7.3.1. Let $f_\eta$ be a generic unfolding of a parabolic antiholomorphic germ of codimension 1. Its strong modulus of classification is the triple $(\varepsilon, b, [\{\Psi_{\varepsilon}^\infty\}_{\varepsilon \in \Omega_\delta}])$, where $\varepsilon$ is the canonical parameter of $f_\eta$, $b$ is the formal invariant (a real analytic function of $\varepsilon$) and $[\{\Psi_{\varepsilon}^\infty\}_{\varepsilon \in \Omega_\delta}]$ is an equivalence class of
normalized families of transition functions under the relation \(\sim\)

\[
\{\Psi_\varepsilon^\infty\}_\varepsilon \sim \{\tilde{\Psi}_\varepsilon^\infty\}_\varepsilon \iff \exists C(\tilde{\varepsilon}) \text{ where } C \text{ is analytic for } \tilde{\varepsilon} \neq 0 \\
\text{with continuous limit at } \varepsilon = 0,
\]

\[
\tilde{\Psi}_\varepsilon^\infty \circ T_{C(\tilde{\varepsilon})} = C(\tilde{\varepsilon}) \circ \tilde{\Psi}_\varepsilon^\infty \circ T_{-C(\tilde{\varepsilon})}.
\]

### 7.4 The Strong Classification Theorem

Recall that the strong equivalence corresponds to Definition 3.2.2.

**Theorem 7.4.1 (Strong Classification).** The following are equivalent:

1. Two generic unfoldings of antiholomorphic parabolic germs of codimension 1 are strongly equivalent.
2. They have the same weak modulus of classification.
3. They have the same strong modulus of classification.

**Proof.** We have seen that (1) implies (2) implies (3). Let us now show that (3) implies (1).

Let \(\{f_{j,\varepsilon}\}_\varepsilon\), \(j = 1, 2\), be two families unfolding antiholomorphic parabolic germs of codimension 1 with the same strong modulus. They induce families \(\{f_{j,\tilde{\varepsilon}}\}_\tilde{\varepsilon}\) with \(\tilde{\varepsilon} \in \Omega_3\).

We will give different arguments when the strong modulus is nontrivial and when it is trivial.

**The strong modulus is nontrivial.** This means that \(\Psi_{j,\tilde{\varepsilon}}^\infty\) is generically not a translation. We choose for each family \(\{f_{j,\varepsilon}\}_\varepsilon\) a pair of normalized Fatou coordinates (see Definition 5.2.2) so that the corresponding transition functions are equal: \(\Psi_{1,\tilde{\varepsilon}}^0 \equiv \Psi_{2,\tilde{\varepsilon}}^0\). Since the strong moduli are equal, the Lavaurs translations are equal.

We define a change of coordinate \(h_{\tilde{\varepsilon}}\) the same way we did for \(\varepsilon < 0\) in (33), namely by

\[
h_{\tilde{\varepsilon}}^\pm(z) = (Z_{\tilde{\varepsilon}}^\pm)^{-1} \circ (\Phi_{2,\tilde{\varepsilon}}^\pm)^{-1} \circ \Phi_{1,\tilde{\varepsilon}}^\pm \circ Z_{\tilde{\varepsilon}}^\pm(z),
\]

on two domains \(S_{\tilde{\varepsilon}}^\pm\) covering \(D(0, r)\). The domains are taken as in Figure 16. They are projections of domains \(R_{\tilde{\varepsilon}}^\pm\) as in Figure 17. As in Theorem 6.2.2 we can show that \(h_{\tilde{\varepsilon}}^\pm = h_{\tilde{\varepsilon}}^\mp\) on the intersection of their domains, yielding that \(h_{\tilde{\varepsilon}}\) is well defined. Moreover, \(f_{2,\tilde{\varepsilon}} = h_{\tilde{\varepsilon}}^1 \circ f_{1,\tilde{\varepsilon}} \circ h_{\tilde{\varepsilon}}^{-1}\). In particular this implies

\[
g_{2,\tilde{\varepsilon}} = h_{\tilde{\varepsilon}} \circ g_{1,\tilde{\varepsilon}} \circ h_{\tilde{\varepsilon}}^{-1}.
\]
Figure 16: Sectors $S^\pm_{\tilde{\varepsilon}}$ for $\arg \tilde{\varepsilon} = \pi$ (on the left), $\arg \tilde{\varepsilon} = 0$ (on the upper right) and $\arg \tilde{\varepsilon} = 2\pi$ (on the lower right).

Figure 17: Domains of definition of return maps. The domains $R^\pm_{\tilde{\varepsilon}}$ projecting on the sectors $S^\pm_{\tilde{\varepsilon}}$ of Figure 16.
We already know that \( h_{\varepsilon} \) is holomorphic in \( \varepsilon \) and we need to show that it is uniform in \( \varepsilon \). Recall that there is an overlapping region in \( \Omega_{\delta} \). On the sector \(-\pi + \delta < \arg \varepsilon < \pi - \delta\), both \( \varepsilon \) and \( e^{2i\pi \varepsilon} \) in \( \Omega_{\delta} \) project on the same point \( \varepsilon \in D(0, r')^* \). Let us prove that \( h_{\varepsilon} = h_{e^{2i\pi \varepsilon}} \) for \(-\pi + \delta < \arg \varepsilon < \pi - \delta\).

Let
\[
k_{\varepsilon} = (h_{e^{2i\pi \varepsilon}})^{-1} \circ h_{\varepsilon}.
\]
Hence for \( \varepsilon \) in the self-intersection of \( \Omega_{\delta} \) it follows from (45) that
\[
g_{1,\varepsilon} \circ k_{\varepsilon} = k_{\varepsilon} \circ g_{1,\varepsilon}.
\]
Since \( k_{\varepsilon} \) commutes with \( g_{1,\varepsilon} \), by Lemma 7.4.2 below there exists a rational number \( m \) such that \( k_{\varepsilon} = g_{1,\varepsilon}^m \). But \( \lim_{\varepsilon \to 0} k_{\varepsilon} = \text{id} \). It follows that \( m = 0 \) and \( k_{\varepsilon} = \text{id} \).

**The strong modulus is trivial.** Then all \( \Psi_{j,\varepsilon}^{0,\infty} \) are translations. In particular, for all normalized Fatou coordinates of \( \{\Phi_{j,\varepsilon}^\pm\}_\varepsilon \) of \( f_{j,\varepsilon} \) we have that \( \Psi_{1,\varepsilon}^{0,\infty} \equiv \Psi_{2,\varepsilon}^{0,\infty} \).

We choose strongly normalized Fatou coordinates (see Definition 7.2.3).

We use the change of coordinate \( h_{\varepsilon} \) defined in (44), which yields (45). For \(-\pi + \delta < \arg \varepsilon < \pi - \delta\), we define again for \( k_{\varepsilon} \) as in (46), which satisfies (47). Let \( K_\varepsilon^\pm \) the expression of \( k_{\varepsilon} \) in the Fatou coordinates. In these coordinates \( K_\varepsilon^\pm \) is a translation, that must commute with the transition maps. Moreover, \( K_\varepsilon^+(0) = 0 \) because of the strong normalization of the Fatou coordinates. Hence \( K_\varepsilon^+ = \text{id} \), which yields that \( k_{\varepsilon} = \text{id} \) on the corresponding domain in \( z \)-space, and then on the whole disk by analytic continuation.

**Lemma 7.4.2.** Let \( g_{\varepsilon} \) be a generic family unfolding a holomorphic parabolic germ, and \( \{(\Psi_{\varepsilon}^0, \Psi_{\varepsilon}^\infty)\}_{\varepsilon \in \Omega} \) be a family of transition functions of \( g_{\varepsilon} \). Let \( \Omega \) be a connected subset of parameter space with at least one accumulation point, and let \( \{h_{\varepsilon}\}_{\varepsilon \in \Omega} \) be an analytic family of diffeomorphisms over \( D(0, r) \) commuting with \( g_{\varepsilon} \). If at least one of \( \Psi_{\varepsilon}^0 \) or \( \Psi_{\varepsilon}^\infty \) is not a translation for at least one \( \varepsilon \in \Omega \), then there exists a rational number \( m \) such that \( h_{\varepsilon} = g_{\varepsilon}^m \) for all \( \varepsilon \in \Omega \).

**Remark 7.4.3.** If \( \Psi_{\varepsilon}^\infty \) or \( \Psi_{\varepsilon}^0 \) is not a translation for at least one \( \varepsilon \in \Omega \), then by the identity principal, it is not a translation over all \( \Omega \), except perhaps on a discrete set of points.

**Proof.** Let
\[
H_{\varepsilon}^\pm = \Phi_{\varepsilon}^+ \circ Z_{\varepsilon}^+ \circ h_{\varepsilon} \circ (Z_{\varepsilon}^-)^{-1} \circ (\Phi_{\varepsilon}^+)^{-1}.
\]
Then \( H_{\varepsilon}^\pm \) commutes with \( T_1 \) over \( U^\pm \), which means that \( H_{\varepsilon}^\pm \) is a translation \( T_{m_\varepsilon} \), where \( m_\varepsilon^\pm \) depends analytically on \( \varepsilon \in \Omega \). Moreover, we need have
\[
\Psi_{\varepsilon}^0 \circ T_{m_\varepsilon}^\pm = T_{m_{\varepsilon}} \circ \Psi_{\varepsilon}^0.
\]
From the form (43) of \( \Psi_{\varepsilon}^\infty \) and the similar form for \( \Psi_{\varepsilon}^0 \), it follows that \( m_{\varepsilon}^+ = m_{\varepsilon}^- := m_{\varepsilon} \). Moreover, if one of \( \Psi_{\varepsilon}^0 \) or \( \Psi_{\varepsilon}^\infty \) is not a translation, then \( m_{\varepsilon} \) is a rational number for almost all \( \varepsilon \in \Omega \), yielding that \( m_{\varepsilon} \) is constant.  

\[\square\]
8 Applications and Realisation

8.1 Antiholomorphic Square Root

**Theorem 8.1.1 (Antiholomorphic Square-Root).** Let \( g_\varepsilon \) be an unfolding of a holomorphic parabolic germ of codimension 1 depending holomorphically on the complex parameter \( \varepsilon \). There exists a family \( \{ f_\varepsilon \} \) depending anti-holomorphically of \( \varepsilon \) such that

\[
f_\varepsilon \circ f_\varepsilon = g_\varepsilon
\]

if and only if there exists families \( \{ \Psi^{0,\infty}_\hat{\varepsilon} \} \) of transition functions of \( g_\varepsilon \) such that for all \( \hat{\varepsilon} \in \Omega_\delta \)

\[
\Sigma \circ T_{\frac{1}{2}} \circ \Psi^\infty_{\hat{\varepsilon}} = \Psi^0_{\hat{\varepsilon}} \circ \Sigma \circ T_{\frac{1}{2}}.
\]

In particular, if \( \varepsilon \) is real, then \( f_\varepsilon \circ f_\varepsilon = g_\varepsilon \).

**Proof.** If \( f_\varepsilon \) exists, then we must have (49).

Now suppose (49) is true. We can suppose that \( \Psi^\infty_{\hat{\varepsilon}} \) is non-linear, since the linear case is trivial. Let \( \{ \Phi^\pm_{\hat{\varepsilon}} \} \) be normalized Fatou coordinates of \( g_\varepsilon \) associated to the transition functions. Let \( S^\pm_\hat{\varepsilon} \) be as in the proof of Theorem 7.4.1 (Section 7.4), see Figure 16. We define \( f^\pm_{\hat{\varepsilon}} \) by

\[
f^\pm_{\hat{\varepsilon}} = (Z^\pm_{\hat{\varepsilon}})^{-1} \circ (\Phi^\pm_{\hat{\varepsilon}})^{-1} \circ \Sigma \circ T_{\frac{1}{2}} \circ \Phi^\pm_{\hat{\varepsilon}} \circ Z^\pm_{\hat{\varepsilon}}.
\]

Then \( f^\pm_{\hat{\varepsilon}} \) is anti-holomorphic on \( S^\pm_\hat{\varepsilon} \). On the intersection of \( S^+_\hat{\varepsilon} \cap S^-_{\hat{\varepsilon}} \), we have that \( f^+_\hat{\varepsilon} \circ (f^-_{\hat{\varepsilon}})^{-1} = id \), because of (49). Indeed, the argument is the same as in the proof of Theorem 6.2.2.

Moreover, \( f_{\hat{\varepsilon}} \) is defined so that \( g_\varepsilon = f_{\hat{\varepsilon}} \circ f_{\hat{\varepsilon}} \). Let \( \gamma_\varepsilon = f^{-1}_{\hat{\varepsilon}+2\pi} \circ f_{\hat{\varepsilon}} \). When \( \arg \hat{\varepsilon} = 0 \), then \( g_\varepsilon = f\circ f_{\hat{\varepsilon}} = f_{\hat{\varepsilon}} \circ f_{\hat{\varepsilon}} \) and \( f_{\hat{\varepsilon}} = f^{-1}_{\hat{\varepsilon}+2\pi} \). Hence \( \gamma_\varepsilon \) commutes with \( g_\varepsilon \). Since \( \Psi^\infty_{\hat{\varepsilon}} \) is non-linear it follows by Lemma 7.4.2 that \( \gamma_\varepsilon = g_\varepsilon^{2m} \) for some rational \( m \), and since \( \gamma_\varepsilon \to id \) when \( \varepsilon \to 0 \), we have \( \gamma_\varepsilon = id \) for \( \arg \hat{\varepsilon} = 0 \). By the identity principle it follows that \( \gamma_\varepsilon = id \) for \( -\pi + \delta < \arg \hat{\varepsilon} < \pi - \delta \). We conclude that \( f_{\hat{\varepsilon}} \) is uniform in \( \varepsilon \). Since \( f_{\hat{\varepsilon}} \) is anti-holomorphic in \( \varepsilon \), it follows that it is real analytic in \( \varepsilon \) for \( \varepsilon \) real.

A necessary condition for a holomorphic parabolic germ \( g_0 \) to have an anti-holomorphic square root is that \( b \in \mathbb{R} \) and that there exist Fatou coordinates for which the transition functions \( \Psi^0_0 \) and \( \Psi^\infty_0 \) are linked by the equation

\[
\Psi^0_0 \circ \Sigma \circ T_{\frac{1}{2}} = \Sigma \circ T_{\frac{1}{2}} \circ \Psi^\infty_0.
\]

The latter is a condition of infinite codimension. Now that we have studied the unfolding we can explain this condition. Indeed, when we perturb \( g_0 \) to some \( g_\varepsilon \) with \( b(\varepsilon) \in \mathbb{R} \), we can see that the dynamics near the fixed points \( \pm \sqrt{\varepsilon} \) is given
by the first return maps. The interesting values of $\varepsilon$ are the ones for which the multiplier has modulus 1 of the form $\exp(2\pi i \beta)$. Indeed, in that case the singular points are generically nonlinearizable as soon as $\beta$ is either rational, or does not satisfy Bruno condition. When $\beta$ is rational and the fixed point is not linearizable, then the first return map has a formal invariant and a functional modulus. And when $\beta$ is irrational there may be, as described by Yoccoz and Perez-Marco, accumulation of periodic points or hedgehog dynamics in the neighborhood of the fixed point.

Suppose now that the unfolding $g_\varepsilon$ is of the form $f_\varepsilon^{\circ 2}$ for some antiholomorphic parabolic unfolding $f_\varepsilon$. Then, when the product of the multipliers at the fixed points of $g_\varepsilon$ has modulus 1 and is different from 1, the two fixed points form a periodic orbit of period 2 of $f_\varepsilon$. This means that the first return maps in the neighborhood of the two fixed points must be conjugate for all values of $\varepsilon$. For instance, just considering the rational values of $\beta$, the first return maps must have the same codimension and the same formal invariants, as well as the same analytic parts of the modulus. This is obviously a very strong condition and it should be no surprise that this is not possible when (50) is not satisfied.

8.2 Realisation

A bonus of Theorem 8.1.1 is to provide a necessary and sufficient condition for a strong modulus

$$(\varepsilon, b, \{\Psi_\varepsilon^\infty\}_{\varepsilon \in \Omega_\delta}).$$

(51)

to be realised as the strong modulus of a generic family $f_\varepsilon$ unfolding an antiholomorphic parabolic germ. As a comparison, the modulus of a germ of generic analytic family unfolding a holomorphic parabolic point has the form

$$(\varepsilon, b, \{(\Psi_0^\varepsilon, \Psi_\varepsilon^\infty)\}_{\varepsilon \in \Omega_\delta}).$$

(52)

The realisation problem was solved for such germs (see [2]). Hence, taking a triple of the form (51), we extend it to a triple of the form (52) using (49). Then the triple (51) is realisable as an antiholomorphic germ of family if and only if the extended triple (52) is realisable as a holomorphic germ of family, which has a square root. The latter is the case if and only if the extended triple satisfies some compatibility condition. This condition states that the dynamics given by $(\Psi_0^\varepsilon, \Psi_\varepsilon^\infty)$ and by $(\Psi_{2i\pi}^0, \Psi_{2i\pi}^\infty)$ are conjugate. It takes a special form in the case of antiholomorphic germs of families that we now state.

The compatibility condition in the antiholomorphic case. We have seen that for $\varepsilon > 0$ the dynamics can be described by the Glutsyuk modulus $\Psi^G_\varepsilon$, where we bring the family to the normal form in the neighborhood of each fixed point.
point through the Glutsyuk Fatou coordinates and we compare the normalizations. This description can of course be extended in the whole overlapping part of the projection of \( \Omega_\delta \) in \( \varepsilon \)-space. In the Glutsyuk Fatou coordinates, the first return maps (or their inverses) are simply given by \( T_{-\alpha_\varepsilon^\pm} \), where \( \alpha_\varepsilon^\pm \) is defined as

\[
\alpha_\varepsilon^\pm := \int_{\gamma^\pm} \frac{1 + b(\varepsilon)b'}{\zeta^2 - \varepsilon} \, d\zeta = \frac{\pm i\pi}{\sqrt{\varepsilon}} + i\pi b(\varepsilon),
\]

(53)

where \( \gamma^\pm \) is a small loop surrounding \( \pm \sqrt{\varepsilon} \) in the positive direction. Note that

\[
\begin{cases}
\alpha_{\varepsilon,2\pi}^\pm = \alpha_{\varepsilon}^\mp, \\
\Sigma(\alpha_\varepsilon^\pm) = -\alpha_{\varepsilon}^\mp.
\end{cases}
\]

(54)

We consider a germ of generic analytic family \( g_\varepsilon \) unfolding a holomorphic parabolic germ and its modulus \( (\varepsilon, b, [\{ (\Psi_\varepsilon^0, \\ \Psi_\varepsilon^\infty) \}_\varepsilon \in \Omega_\delta]) \). Let us denote \( \tilde{\varepsilon} = \varepsilon e^{2i\pi} \). For arg \( \tilde{\varepsilon} \in (-\pi + \delta, \pi - \delta) \) we use the return maps \( \Psi_\varepsilon^0 \circ T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} \) and \( \Psi_\varepsilon^\infty \circ T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} \), while for arg \( \tilde{\varepsilon} \in (\pi + \delta, 3\pi - \delta) \) we use \( T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} \circ \Psi_\varepsilon^0 \) and \( T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} \circ \Psi_\varepsilon^\infty \).

**Theorem 8.2.1** (Compatibility Condition [2]). *We consider a germ of generic analytic family \( g_\varepsilon \) unfolding a holomorphic parabolic germ and its modulus \( (\varepsilon, b, [\{ (\Psi_\varepsilon^0, \\ \Psi_\varepsilon^\infty) \}_\varepsilon \in \Omega_\delta]) \). Let us denote \( \tilde{\varepsilon} = \varepsilon e^{2i\pi} \). For arg \( \tilde{\varepsilon} \in (-\pi + \delta, \pi - \delta) \), let \( H_\varepsilon^0, H_\varepsilon^\infty, H_\varepsilon^0, H_\varepsilon^\infty \) be defined as follows

\[
H_\varepsilon^0 \circ \Psi_\varepsilon^0 \circ T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} = T_{-\alpha_\varepsilon^0} \circ H_\varepsilon^0,
\]

\[
H_\varepsilon^\infty \circ \Psi_\varepsilon^\infty \circ T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} = T_{-\alpha_\varepsilon^\infty} \circ H_\varepsilon^\infty,
\]

\[
H_\varepsilon^0 \circ T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} \circ \Psi_\varepsilon^0 = T_{-\alpha_\varepsilon^0} \circ H_\varepsilon^0,
\]

\[
H_\varepsilon^\infty \circ T_{-\frac{\varepsilon}{\sqrt{\varepsilon}}} \circ \Psi_\varepsilon^\infty = T_{-\alpha_\varepsilon^\infty} \circ H_\varepsilon^\infty,
\]

(55)

and uniquely determined by

\[
\lim_{\Im W \to +\infty} (H_\varepsilon^\infty - \text{id}) = 0,
\]

\[
\lim_{\Im W \to +\infty} (H_\varepsilon^0 - \text{id}) = 0,
\]

\[
\lim_{\Im W \to -\infty} (H_\varepsilon^0 - \text{id}) = 0,
\]

\[
\lim_{\Im W \to -\infty} (H_\varepsilon^\infty - \text{id}) = 0.
\]

(56)
Figure 18: Domains of definition of return maps.
Then there exist $D_\varepsilon, D'_\varepsilon$ for $\arg \varepsilon \in (-\pi + \delta, \pi - \delta)$ such that the following compatibility condition is satisfied

$$H_\varepsilon^\infty \circ (H_0^0)^{-1} = T_{D_\varepsilon} \circ H_\varepsilon^0 \circ (H_\varepsilon^\infty)^{-1} \circ T_{D'_\varepsilon}. \quad (57)$$

**Theorem 8.2.2** (Realisation Theorem [2]). Let be given a family $(\varepsilon, b, \{(\Psi^\infty_\varepsilon, \Psi^0_\varepsilon)\}_{\varepsilon \in \Omega_\delta})$, where $b$ is an analytic function of $\varepsilon$ defined on the projection of $\Omega_\delta$ and such that $\Psi^{0,\infty}_\varepsilon$ depend analytically on $\hat{\varepsilon} \in \Omega_\delta$ with continuous limit when $\hat{\varepsilon} \to 0$. Let $H_\varepsilon^0, H_\varepsilon^\infty, H_\varepsilon^0$ be defined in (55) and satisfying the compatibility condition (57) together with (56). Then there exists a germ of generic analytic family $g_\varepsilon$ unfolding a holomorphic parabolic germ realizing this modulus.

In the case of a germ of parabolic family of the form $g_\varepsilon = f_{\varepsilon} \circ f_\varepsilon$ we have the additional condition (42), under which the compatibility condition takes a certain form. Indeed, we have

$$H_\varepsilon^0 = T_{\frac{\varepsilon}{\sqrt{\varepsilon}}} \circ T_{\frac{1}{2}} \circ \Sigma \circ H_\varepsilon^\infty \circ \Sigma \circ T_{-\frac{1}{2}} \circ T_{-\frac{i\pi}{\varepsilon}}, \quad (58)$$

$$H_\varepsilon^0 = T_{\frac{\varepsilon}{\sqrt{\varepsilon}}} \circ T_{\frac{1}{2}} \circ \Sigma \circ H_\varepsilon^\infty \circ \Sigma \circ T_{-\frac{1}{2}} \circ T_{-\frac{i\pi}{\varepsilon}}. \quad (59)$$

If we let $N_\varepsilon = T_{\frac{\varepsilon}{\sqrt{\varepsilon}}} \circ T_{\frac{1}{2}} \circ \Sigma$, the compatibility condition then takes the form: there exist $D_\varepsilon, D'_\varepsilon$ for $\arg \varepsilon \in (-\pi + \delta, \pi - \delta)$ such that

$$H_\varepsilon^\infty \circ N_\varepsilon \circ (H_\varepsilon^\infty)^{-1} \circ N_\varepsilon^{-1} = T_{D_\varepsilon} \circ N_\varepsilon \circ H_\varepsilon^\infty \circ N_\varepsilon^{-1} \circ (H_\varepsilon^\infty)^{-1} \circ T_{D'_\varepsilon}. \quad (60)$$

**Theorem 8.2.3** (Realisation Theorem in the anti-holomorphic case). Let be given a family $(\varepsilon, b, \{(\Psi^\infty_\varepsilon, \Psi^\infty_\varepsilon)\}_{\varepsilon \in \Omega_\delta})$, where $b$ is an analytic function of $\varepsilon$ defined on the projection of $\Omega_\delta$ and such that $\Psi^{\infty,\infty}_\varepsilon$ depends analytically on $\hat{\varepsilon} \in \Omega_\delta$ with continuous limit when $\hat{\varepsilon} \to 0$. Let $\Psi^\infty_\varepsilon$ be defined through (42). Let $H_\varepsilon^\infty$ be defined by

$$\begin{cases} H_\varepsilon^\infty \circ \Psi^\infty_\varepsilon \circ T_{-\frac{i\pi}{\varepsilon}} = T_{-\alpha_\varepsilon^\infty} \circ H_\varepsilon^\infty, & \text{if } \arg \varepsilon \in (-\pi + \delta, \pi - \delta); \\ H_\varepsilon^\infty \circ T_{-\frac{i\pi}{\varepsilon}} \circ \Psi_\varepsilon^\infty = T_{-\alpha_\varepsilon^\infty} \circ H_\varepsilon^\infty, & \text{if } \arg \varepsilon \in (\pi + \delta, 3\pi - \delta); \end{cases} \quad (61)$$

and satisfying

$$\lim_{3W \to +\infty} (H_\varepsilon^\infty - \id) = 0.$$

Suppose $H_\varepsilon^\infty$ satisfies the antiholomorphic compatibility condition, namely there exist $D_\varepsilon, D'_\varepsilon$ for $\arg \varepsilon \in (-\pi + \delta, \pi - \delta)$ such that

$$H_{\varepsilon_2^\infty} \circ N_\varepsilon \circ (H_{\varepsilon_2^\infty})^{-1} \circ N_\varepsilon^{-1} = T_{D_\varepsilon} \circ N_\varepsilon \circ H_\varepsilon^\infty \circ N_\varepsilon^{-1} \circ (H_\varepsilon^\infty)^{-1} \circ T_{D'_\varepsilon}. \quad (62)$$

Then there exists a germ of generic antiholomorphic family $f_\varepsilon$ unfolding an antiholomorphic parabolic germ and realizing this strong modulus.
Proof. We want to realize $f_{\varepsilon}$. We first realize $g_{\varepsilon}$ with modulus $(\varepsilon, b, \{(\Psi_{\varepsilon}^{\infty}, \Psi_{\varepsilon}^{0})\}_{\varepsilon \in \Omega_{d}})$, and then extract $f_{\varepsilon}$ as an antiholomorphic square root satisfying $g_{\varepsilon} := f_{\varepsilon} \circ f_{\varepsilon}$ using Theorem 8.1.1. Let us denote $\tilde{\varepsilon} = \hat{\varepsilon} e^{2\pi i}$. The functions $H^{0}_{\varepsilon}$ and $H^{0}_{\tilde{\varepsilon}}$, defined in (58) and (59) respectively, and $H^{\infty}_{\varepsilon}$ and $H^{\infty}_{\tilde{\varepsilon}}$, defined by (61), satisfy (55) and (57), so by Theorem 8.2.2 we can realize $g_{\varepsilon}$ and then take its antiholomorphic square root by Theorem 8.1.1.

Remark 8.2.4. For arg $\hat{\varepsilon} = 0$ and arg $\tilde{\varepsilon} = 2\pi$, the compatibility condition (60) becomes

$$H^{\infty}_{\varepsilon} \circ N_{\varepsilon} \circ (H^{\infty}_{\varepsilon})^{-1} \circ N_{\varepsilon}^{-1} = T_{D_{\varepsilon}} \circ N_{\varepsilon} \circ H_{\varepsilon}^{\infty} \circ N_{\varepsilon}^{-1} \circ (H_{\varepsilon}^{\infty})^{-1} \circ T_{D_{\varepsilon}}. \quad (63)$$

Indeed, in this case $\tilde{\varepsilon} = \hat{\varepsilon}$. This is a necessary condition for the realisation.

8.3 Germs of Families with an Invariant Real Analytic Curve

It was shown in [4] that an antiholomorphic parabolic germ preserves a real analytic curve if and only if $\Psi_{0}^{\infty}$ commutes with $T_{1/2}$, which is a condition of infinite codimension. This means that it is exceptional that an antiholomorphic parabolic germ preserves a real analytic curve. Nonetheless, the real axis is invariant in the formal normal form. Where is the obstruction?

Let us look at a generic unfolding. For $\varepsilon > 0$, the germ is analytically conjugate to the normal form in the neighborhood of each fixed point $\pm \sqrt{\varepsilon}$ or, equivalently conjugate to $\sigma$ composed with the time-$\frac{1}{2}$ of $\dot{z} = \lambda_{\pm}(z \mp \sqrt{\varepsilon})$, where $\lambda_{\pm} = \frac{2\sqrt{\varepsilon}}{1+b(\varepsilon)\sqrt{\varepsilon}}$. The flow lines of the vector field are of the form $\cos \theta y - \sin \theta (x \pm \sqrt{\varepsilon}) = 0$ for some $\theta \in [0, \pi)$. Among these flow lines exactly one is fixed by $\sigma$. In a generic unfolding there is no reason why these local invariant lines would match globally. If this mismatch persists till the limit, then we expect at the limit some $\Psi_{0}^{\infty}$ that does not commute with $T_{1/2}$. 

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