AVOIDING ALGEBRAIC INTEGERS OF BOUNDED HOUSE IN ORBITS OF RATIONAL FUNCTIONS OVER CYCLOTOMIC CLOSURES

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Abstract. Let \( k \) be a number field with cyclotomic closure \( k^{\text{cyc}} \), and let \( h \in k^{\text{cyc}}(x) \). For \( A \geq 1 \) a real number, we show that
\[
\{ \alpha \in k^{\text{cyc}} : h(\alpha) \in \mathbb{Z} \text{ has house at most } A \}
\]
is finite for many \( h \). We also show that for many such \( h \) the same result holds if \( h(\alpha) \) is replaced by orbits \( h(h(\cdots h(\alpha))) \). This generalizes a result proved by Ostafe that concerns avoiding roots of unity, which is the case \( A = 1 \).

1. Introduction

1.1. Rational functions and set avoidance. We begin with the following general definition.

Definition 1.1. Let \( F \) be a subfield of \( \mathbb{C} \), and \( P \) a subset of \( \mathbb{C} \). Let \( h \in F(x) \) be a rational function, and let \( h^n \) denote the function composition of \( h \) applied \( n \) times \((n = 0, 1, 2, \ldots)\).

- We say that \( h \) is \( P \)-avoiding (over \( F \)) if
  \[
  \# \{ \alpha \in F : h(\alpha) \in P \} < \infty.
  \]
- We say that \( h \) is strongly \( P \)-avoiding (over \( F \)) if
  \[
  \# \{ \alpha \in F : h^n(\alpha) \in P \text{ for some } n \geq 1 \} < \infty.
  \]

Let \( U \subseteq \mathbb{C} \) denote the set of roots of unity. This paper will be concerned with avoidance over the cyclotomic closure of a number field \( k \), which we denote by
\[
k^{\text{cyc}} := k(U).
\]
We say a rational function \( h(x) \) is special if \( h \) is conjugate, with respect to a Möbius transformation (i.e. via \( \text{PGL}_2(k) \)), to either \( \pm x^d \) or the Chebyshev polynomial \( T_d(x) \) defined by \( T_d(t + t^{-1}) = t^d + t^{-d} \).

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The question of $U$-avoidance has been examined by Dvornicich and Zannier [2] Corollary 1] and Ostafe [7], who prove the following results.

**Theorem ([2] Corollary 1).** A rational function $h \in k^{\text{cyc}}(x)$ is $U$-avoiding unless there exists a rational function $S \in k^{\text{cyc}}(x)$ and an integer $m$ such that $h(S(x)) = x^m$.

**Theorem ([7] Theorem 1.2).** Let $h = p/q \in k(x)$, where $p, q \in k[x]$. Assume $h$ is $U$-avoiding over $k^{\text{cyc}}$, and $\deg p > \deg q + 1$. Then $h$ is strongly $U$-avoiding unless $h$ is special.

In this paper we investigate a generalization of these results proposed by Ostafe (see [7, §4]). In order to state it, we need to define the following.

**Definition 1.2.** The house of an algebraic number $\alpha$, denoted $\alpha$, is the maximum value of $|\beta|$ across the $\mathbb{Q}$-Galois conjugates $\beta$ of $\alpha$.

For $A \geq 1$ a real number, let $P_A$ denote the set of algebraic integers $\alpha$ which have house at most $A$.

For example every algebraic integer has house at least 1, and by Kronecker’s theorem (the main result of [5], see also [4]) we have $P_1 = \mathbb{U}$.

We answer the following question.

**Question.** For $A \geq 1$ and $h \in k^{\text{cyc}}(x)$, under what conditions can one show that $h$ is (strongly) $P_A$-avoiding?

1.2. **Summary of results.** The degree of a nonconstant rational function $h$ with coefficients in some field $F$ is defined to be $[F(x) : F(h(x))]$. Consequently, note that $\deg(h_1 \circ h_2) = \deg h_1 \deg h_2$. If $h$ is written as a quotient of relatively prime polynomials $p/q$, then $\deg h = \max(\deg p, \deg q)$.

Our results on $P_A$-avoidance can be summarized as follows.

**Theorem 1.3.** Let $k$ be a number field, $A \geq 1$ and $\varepsilon > 0$. Let $h \in k^{\text{cyc}}(x)$ be a rational function.

- Then $h$ is $P_A$-avoiding unless there exists $S \in k^{\text{cyc}}(x)$ such that $h(S(x))$ equals a Laurent polynomial with $d$ terms, where $d \ll_{k, \varepsilon} A^{2+\varepsilon}$.

- If $\deg h \gg_{k, A} 1$, then we can also assume $\deg S \leq 2$.

This theorem has an effective and more explicit form given as Theorem 2.5 and its corollaries.

Here is one amusing corollary of the above.

**Corollary 1.4.** Let $k$ be a number field, $A \geq 1$ and $\varepsilon > 0$. If $h$ has more than two poles, then $h$ is $P_A$-avoiding.

Using this result, we will deduce the following generalization of a result of Ostafe [7] Theorem 1.2], and give a simple proof using Theorem 2.5.

**Theorem 1.5.** Let $h = p/q \in k(x)$, where $p, q \in k[x]$. Let $A \geq 1$. Assume $h$ is $P_A$-avoiding over $k^{\text{cyc}}$, and $\deg p > \deg q + 1$. Then $h$ is strongly $P_A$-avoiding unless $h$ is special.
2. Full statement of results on \( P_A \)-avoidance

In order to state the full version of Theorem 1.3, we need to first state the following “Loxton theorem”.

**Theorem 2.1** (Loxton theorem, [2, Theorem L]). There exists a function \( L : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with the following property. For every number field \( k \), we can fix a real number \( B > 0 \) and a subset \( E \subseteq k \) of size at most \( [k : \mathbb{Q}] \) so that every algebraic integer \( \alpha \) in \( k^{cyc} \) can be written as

\[
\sum_{i=1}^{d} e_i \xi_i
\]

where \( e_i \in E, \xi_i \in \mathbb{U} \), and \( d \leq L(B \cdot \|\alpha\|) \).

In light of this, it will be convenient to make the following definition.

**Definition 2.2.** For every number field \( k \) we fix a pair \((B, E)\) (depending only on \( k \)) as above. We will call this the Loxton pair for \( k \). The Loxton function \( L \) will also remain fixed throughout the paper.

**Remark 2.3.** The exact nature of \( L \) is not important. However, it is possible to choose \( L(x) = O(x^{2+\varepsilon}) \). Moreover, in the case \( k = \mathbb{Q} \) one can select \( E = \{1\} \). See [6] for more details.

**Definition 2.4.** Let \( h \in k^{cyc}(x) \) and fix \((B, E)\) a Loxton pair for \( k \). Suppose that there exists a nonconstant \( S \in k^{cyc}(x) \), integers \( n_i \), roots of unity \( \beta_i \in \mathbb{U} \), and \( e_i \in E \) which satisfies

\[
\sum_{i=1}^{d} \beta_i e_i x^{n_i} = h(S(x)).
\]

In this case, we call the rational function \( \sum \beta_i e_i x^{n_i} \) a witness for \( h \).

If \( A \geq 1 \) is a real number, the witness is called \( A \)-short if \( d \leq L(AB) \).

Observe that if there exists a witness for \( h \), then \( h \) is seen to not be \( P_A \)-avoiding for sufficiently large \( A \), by simply selecting \( x \in \mathbb{U} \). We will prove the following sort of converse result.

**Theorem 2.5.** Let \( h(x) \in k^{cyc}(x) \) be nonconstant, and \( A \geq 1 \). Then \( h \) is \( P_A \)-avoiding unless there exists an \( A \)-short witness for \( h \).

According to Remark 2.3 above, the case \( k = \mathbb{Q} \) has a particularly nice phrasing.

**Corollary 2.6.** Let \( h(x) \in \mathbb{Q}^{cyc}(x) \) be nonconstant and \( A \geq 1 \). Then \( h \) is \( P_A \)-avoiding unless there exists \( S \in \mathbb{Q}^{cyc}(x) \) such that \( h(S(x)) \) is equal to a Laurent polynomial \( p \in \mathbb{Z}[\mathbb{U}][x] \) with \( |p(1)| \ll_{\varepsilon} A^{2+\varepsilon} \).

As stated, these results do not produce any bounds on the size of the degree of a witness. However, the following theorem shows that “most” \( h(x) \) are in fact \( P_A \)-avoiding.
**Theorem 2.7.** Let $k$ be a number field with Loxton pair $(B, E)$. Let $A \geq 1$ and let $h(x) \in k^{\text{cyc}}(x)$ be nonconstant. Suppose that

- $\deg h > 2016 \cdot 5^{L(AB)+1}$, or
- $h$ is a polynomial and $\deg h > (2L(AB) + 1)^2$.

Then $h$ is $P_A$ avoiding unless it has an $A$-short witness $h(S(x))$ for which $\deg S \leq 2$.

**Remark 2.8.** In fact, in the polynomial case $h \in k^{\text{cyc}}[x]$ the $A$-short witness can be assumed to be of the form $h(ax + b + cx^{-1})$. (See Theorem 3.3.)

3. **Background**

In addition to Theorem 2.1, we will need several other auxiliary results, which we reproduce here.

3.1. **Tools from arithmetic geometry.** In what follows, fix $k$ a number field, and $G_{\text{mult}} = \text{Spec } k[x, x^{-1}]$ as usual. By a torison coset of $G^d_{\text{mult}}$, we mean a translate $\beta \cdot T$; here $\beta$ is a torsion point of $G^d_{\text{mult}}$ and $T$ a subtorus of $G^d_{\text{mult}}$ (a connected algebraic subgroup).

**Theorem 3.1** ([2, Torsion Points Theorem]). Let $V$ be an algebraic subvariety of $G^d_{\text{mult}}$. The Zariski closure of the torsion points of $G^d_{\text{mult}}$ also contained in $V$ is a finite union of torsion cosets of $G^d_{\text{mult}}$.

We also use a special case of [2, Theorem 1].

**Theorem 3.2.** Let $k$ be a number field. Let $V/k$ be an affine variety irreducible over $k^{\text{cyc}}$ and let

$$\pi : V \to G^r_{\text{mult}}$$

be a morphism of finite degree, defined over $k$. Assume that the set of torsion points in $\pi(V(k^{\text{cyc}}))$ are Zariski-dense in $G^r_{\text{mult}}$.

Then there exists an isogeny $\mu : G^r_{\text{mult}} \to G^r_{\text{mult}}$ and a birational map $\rho : G^r_{\text{mult}} \dashrightarrow V$ defined over $k^{\text{cyc}}$ such that the diagram

$$\begin{array}{ccc}
G^r_{\text{mult}} & \xrightarrow{\mu} & V \\
\downarrow{\pi} & & \downarrow{\pi} \\
G^r_{\text{mult}}
\end{array}$$

commutes (over $k^{\text{cyc}}$).

**Proof.** We define the set

$$J = \{ \eta \in V(k^{\text{cyc}}) : \pi(\eta) \text{ is a torsion point of } G^r_{\text{mult}} \}$$

Thus $\pi(J)$ consists of the torsion points of $\pi(V(k^{\text{cyc}}))$. By hypothesis, $\pi(J)$ is Zariski dense. Since $\pi$ is of finite degree, it follows that $J$ is Zariski dense in $V$ as well. Then we can apply [2, Theorem 1], where the torsion coset in question is the entire $T = G^r_{\text{mult}}$. \qed
3.2. Results on compositions of rational functions. We reproduce the following results of Fuchs and Zannier, in 2012.

**Theorem 3.3** ([3]). Let $F$ be a field of characteristic zero. Let $p, q, h \in F(x)$ be rational functions with $p = h \circ q$. Denote by $\ell$ the sum of the number of terms in the numerator and denominator of $p$.

- Assume $q$ is not of the form $\lambda(ax^n + bx^{-n})$ for $a, b \in F$, $\lambda \in \text{PGL}(F)$, and $n \in \mathbb{Z}_{>0}$. Then
  \[ \deg h \leq 2016 \cdot 5^\ell. \]

- Suppose that $p, q \in k[x, x^{-1}]$ are Laurent polynomials and $h \in k[x]$ is a polynomial. Assume $q$ is not of the form $ax^n + b + cx^{-n}$ for $a, b, c \in F$ and $n \in \mathbb{Z}_{>0}$. Then
  \[ \deg h \leq 2(2\ell - 1)(\ell - 1). \]

The following formulation with iterated $h$'s will also be useful.

**Corollary 3.4** ([3]). Let $F$ be any field of characteristic zero. Let $q \in F(x)$ be non-constant, and $h \in F(x)$ with $\deg h = d \geq 3$ and not special. Then for any integer $n \geq 3$, the rational function $h^n \circ q$ has at least
\[ \log_5 \left( \frac{d^{n-2}}{2016} \right) \]
terms when written as a quotient of polynomials.

3.3. Estimates on sizes of orbits. Finally, we will use the following results, which are based off of results in [7, §2.3].

**Lemma 3.5.** Let $k$ be a number field and let $h = p/q \in k(x)$ be a rational function. Assume that $h = p/q$ with $\deg p > \deg q + 1$.

Then there exists a real number $T > 0$ and an integer $D$ (depending only on $h$) with the following properties. For $\alpha$ an algebraic number:

- If $|h^n(\alpha)| \leq A$ for some $n \geq 1$, then
  \[ |h^j(\alpha)| \leq \max(T, A) \quad \text{for } j = 0, \ldots, n - 1. \]

- If $h^n(\alpha)$ is an algebraic integer, then $ Dh^j(\alpha)$ is an algebraic integer for $j = 0, 1, \ldots, n - 1$.

**Proof.** Suppose that $h^n(\alpha) = \gamma$.

First, since $\deg p - \deg q > 1$ we can pick $0 \neq c \in \overline{\mathbb{Q}}$ (depending only on $h$) such that
\[ h(x) = c^{-1} \cdot \tilde{h}(cx) \]
and moreover $\tilde{h}$ is “monic” in the sense that $\tilde{h} = \tilde{p}/\tilde{q}$ and
\[
\begin{align*}
\tilde{p}(x) & = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \\
\tilde{q}(x) & = x^e + b_{e-1}x^{e-1} + \cdots + b_0.
\end{align*}
\]
(It is possible that \(c \notin k\); in this case we enlarge \(k\) to contain \(c\).) Now, for any \(j = 0, \ldots, n\) we have
\[
h^j(x) = c^{-1} \cdot \tilde{h}^j(cx).
\]
In particular, \(\tilde{h}^j(c\alpha) = c\gamma\).

The first part now follows readily [7, Corollary 2.7], applied to \(cA\), \(c\alpha\) and \(\tilde{h}\).

We proceed to the second part. Assume \(\gamma\) is an algebraic integer. Note that by replacing the value of \(n\), it suffices just to show that \(D\alpha\) is an algebraic integer.

Let \(\nu\) be an arbitrary finite place of \(k\). Then [7, Corollary 2.5] now implies that if \(\|c\alpha\|_\nu > \max\{1, \|a_i\|_\nu, \|b_i\|_\nu\}\) then
\[
\|\tilde{h}^j(c\alpha)\|_\nu = 0, 1, 2, \ldots
\]
is strictly increasing. Thus, in particular we must have
\[
\|c\alpha\|_\nu \leq \max(\|1\|_\nu, \|a_i\|_\nu, \|b_i\|_\nu, \|c\gamma\|_\nu)
\]
or else we contradict the fact that \(\tilde{h}^j(c\alpha) = c\gamma\).

Select \(D\) to be an integer for which \(Dc^{-1}, Dc^{-1}a_i, Dc^{-1}b_i\) are all algebraic integers. Multiplying the previous display by \(Dc^{-1}\), we obtain
\[
\|D\alpha\|_\nu \leq \max(\|Dc^{-1}\|_\nu, \|Dc^{-1}a_i\|_\nu, \|Dc^{-1}b_i\|_\nu, \|D\gamma\|_\nu) 
\]
\[
\leq 1.
\]
Since this is true for every finite place \(\nu\), it follows that \(D\alpha\) is an integer. Moreover, since \(D\) depends only on \(c, a_i, b_i\) and not on \(\gamma\), it follows that \(D\) depends only on \(h\), which is what we wanted to prove. \(\square\)

## 4. Proof of results on \(P_A\)-avoidance

**Proof of Theorem 2.5.** Assume \(h\) is not \(P_A\)-avoiding, so \(h(k^{\text{cyc}})\) contains infinitely elements of \(P_A\). By Theorem 2.1 and the pigeonhole principle, we can fix \(d \leq L(AB)\) and \(e_i \in E\) such that there exist infinitely many \(y \in k^{\text{cyc}}\) and \(\xi_1, \ldots, \xi_d \in U\) which obey
\[
h(y) = \sum_{i=1}^{d} e_i \xi_i.
\]

Take \(G_{\text{mult}}^{d+1}\) equipped with coordinates \((x_1, \ldots, x_d, y)\). Letting \(h = p/q\) for \(p, q \in k^{\text{cyc}}[x]\), consider the subvariety
\[
V \subseteq G_{\text{mult}}^{d+1}
\]
deefined by the equation
\[
p(y) = q(y) \sum_{i=1}^{d} e_i x_i.
\]
Moreover, for brevity let \( U_d \) denote the set of torsion points of \( \mathbb{G}^d_{\text{mult}} \). Let \( \Pi : V \to \mathbb{G}^d_{\text{mult}} \) be the projection onto the first \( d \) coordinates. We now consider the following iterative procedure. Initially, let \( W_0 = V, \beta_0 = 1 \in \mathbb{G}^d_{\text{mult}}, T_0 = \mathbb{G}^d_{\text{mult}} \) so the torsion coset \( \beta_0 T_0 \) is all of \( \mathbb{G}^d_{\text{mult}} \). So we have \( \Pi(W_0) \subseteq \beta_0 T_0 \) and \( \#(\Pi(W_0) \cap U_d) = \infty \). Then we recursively do the following procedure for \( i = 0, 1, 2, \ldots \).

- Consider the infinite set \( \beta_i^{-1} \Pi(W_i) \cap U_d \subseteq T_i \). By Theorem 3.1 applied to the subvariety \( T_i \), its Zariski closure consists of finitely many torsion cosets. Hence by pigeonhole principle, we map pick a particular torsion coset, say \( \beta_i T_{i+1} \), containing infinitely many elements of \( U_d \). Now set \( \beta_{i+1} = \beta_i \beta' \). Then we conclude that \( \beta_i T_{i+1} \) is the closure of some infinite subset of \( \Pi(W_i) \cap U_d \).
- Now consider the pre-image \( \pi^{-1}(\beta_{i+1} T_{i+1}) \). It is some closed subvariety of \( W_i \). Then by pigeonhole principle, we can set \( W_{i+1} \) to be any irreducible component of \( W_i \) such that \( \#(\Pi(W_{i+1}) \cap U_d) = \infty \).

From this we have constructed \( V = W_0 \supseteq W_1 \supseteq \cdots \) a decreasing sequence of subvarieties of \( V \), with \( W_i \) irreducible for \( i \geq 1 \). For dimension reasons, this sequence must eventually stabilize. Thus the torsion coset \( \beta_i T_i \) stabilizes too. So we conclude there exists
- an irreducible affine subvariety \( W \subseteq V \), and
- a particular torsion coset \( \beta T \subseteq \mathbb{G}^d_{\text{mult}} \), where \( \beta = (\beta_1, \ldots, \beta_d) \in U^d \) and \( T \) is a torus, and
- \( Z := \Pi(W) \cap U_d \) a set of torsion points of \( \mathbb{G}^d_{\text{mult}} \) such that

\[
\Pi(W) \subseteq \beta T, \quad Z = \beta T, \quad \text{and} \quad \#Z = \infty.
\]

Let \( r := \dim T \); note \( r \geq 1 \) since \( T \) contains the infinite set \( Z \). Also, let \( \beta = (\beta_1, \ldots, \beta_d) \).

We now wish to apply Theorem 3.2. Consider the composed map \( \pi : W \to \mathbb{G}^r_{\text{mult}} \) defined by taking \( \varphi \) as below:

\[
\begin{array}{ccc}
W & \xrightarrow{\varphi} & T \\
(x_1, \ldots, x_d, y) & \mapsto & (\beta_1^{-1} x_1, \ldots, \beta_d^{-1} x_d).
\end{array}
\]

From the fact that \( \overline{Z} = \beta \cdot T \), we conclude the set of torsion points in \( \pi(W) \) is Zariski dense in \( \mathbb{G}^r_{\text{mult}} \). Thus we can apply Theorem 3.2. This implies there is an isogeny \( \mu : \mathbb{G}^r_{\text{mult}} \to \mathbb{G}^r_{\text{mult}} \) and a birational map \( \rho : \mathbb{G}^r_{\text{mult}} \dashrightarrow W \) such that the diagram
commutes.

Assume

$$\rho(x) = (R_1(x), \ldots, R_d(x), R(x))$$

implying

$$\varphi(\rho(x)) = (\beta_1^{-1}R_1(x), \ldots, \beta_d^{-1}R_d(x), R(x))$$

for rational functions $R_1, \ldots, R_d, R$ (here $x \in \mathbb{G}^\text{r}_\text{mult}$). Now the right-hand side of $\varphi \circ \rho = \psi^{-1} \circ \mu$ is the composition of an isogeny and an isomorphism, thus (for instance by [1, Proposition 3.2.17]), we recover that $R_i(x) = \beta_i x^{v_i}$ for some vectors $v_i \in \mathbb{Z}^r$ which are linearly independent (and in particular nonzero).

Thus

$$\rho(x) = (\beta_1 x^{v_1}, \ldots, \beta_d x^{v_d}, R(x))$$

and we obtain an identity

$$h(R(x)) = \sum_{i=1}^d e_i = \beta_i x^{v_i}.$$

Since the $v_i$ were independent, it follows that one can specialize $x$ to a choice of the form $x = (x^{c_1}, \ldots, x^{c_r})$ for some integers $c_i \in \mathbb{Z}$ so that the terms $x^{v_i}$ are pairwise distinct. Thus we finally arrive at

$$h(S(x)) = \sum_{i=1}^d \beta_i e_i x^{n_i}$$

where $S$ is a rational function (defined by $S(x) := R(x^{n_r}, \ldots, x^{c_r})$), and the right-hand side is nonconstant in $x$. This is the desired $A$-short witness. □

**Proof of Theorem 2.7.** First, suppose $h(x) \in k_{\text{cyc}}(x)$. By Theorem 2.5 we thus have an identity

$$h(S(x)) = \sum_{i=1}^d \beta_i e_i x^{n_i}$$

where the right-hand side has at most $d \leq L(A \cdot B)$ terms.

First assume $S = \mu(ax^n + bx^{-n})$ for some $\mu \in \text{PGL}_2(k)$. Set now $\bar{S} = \mu(ax + bx^{-1})$, $\deg \bar{S} = 2$. We now see that

$$h(\bar{S}(x))$$

is a Loxton witness, establishing the theorem.

Otherwise Theorem 3.3 applies with $\ell = d + 1$, and we deduce that

$$\deg h \leq 2016 \cdot 5^{d+1}.$$ 

This implies one direction.
The case where \( h \in k^{\text{cyc}}[x] \) is identical, except we use the other part of Theorem \([3, 3]\) instead. (That \( S \) is a Laurent polynomial follows from the fact that it cannot have any nonzero poles, in light of the right-hand side having the same property.) \( \square \)

**Proof of Corollary [1, 4]**. Suppose for contradiction we can set \( h(S(x)) \) equal to a Laurent polynomial; extend this to an identity holding in \( \mathbb{C} \). Then \( h \circ S \) has at most one pole, namely \( x = 0 \). Moreover a rational function \( S \) omits at most one point in its range. Thus \( h \) has a pole in the range of \( S \) which is not \( S(0) \); this is absurd. \( \square \)

5. **Proof of results on strong \( P_A \)-avoidance**

We now prove Theorem \([1, 5]\) which we restate here for convenience of the reader.

**Theorem.** Let \( h = p/q \in k(x), \) where \( p, q \in k[x] \). Let \( A \geq 1 \). Assume \( h \) is \( P_A \)-avoiding over \( k^{\text{cyc}} \), and \( \deg p > \deg q + 1 \). Then \( h \) is strongly \( P_A \)-avoiding unless \( h \) is special.

**Proof of Theorem \([1, 5]\)**. Since \( h \) is given to be \( P_A \)-avoiding, it suffices to show that for a given \( \gamma \in P_A \), there are only finitely many \( \alpha \in k^{\text{cyc}} \) such that \( h^n(\alpha) = \gamma \) for some \( n \geq 1 \).

Assume for contradiction there are infinitely many. Then selecting \( T > 0 \) and \( D \in \mathbb{Z} \) by Lemma \([3, 5]\), we make the following claim.

**Claim.** For any integer \( N \), \( D \cdot h^N(x) \) is not weakly \( P_C \)-avoiding for

\[
C := D \cdot \max(T, A).
\]

To see this, observe that there are only finitely many solutions to \( h^n(\alpha) = \gamma \) for \( n \leq N \), hence there are infinitely many with \( n > N \). Then by applying Lemma \([3, 5]\) to such pairs \((\alpha, n)\) with \( n > N \), we discover infinitely many \( \alpha \) such that \( D \cdot h^N(\alpha) \) is an algebraic integer; moreover the house of \( D \cdot h^N(\alpha) \) is at most \( D \cdot \max(T, A) = C \). Thus, we have proved the claim.

Consequently, for every integer \( N \) there exists a \( C \)-short witness.

If \( h \) is not special and \( \deg h \geq 3 \), then take any

\[
N > 2 + \log_{\deg h} \left( 2016 \cdot 5^{L(BD \max(T, A))} \right)
\]

and note that we now have

\[
D \cdot h^N(S(x)) = \sum_{i=1}^{d} \beta_i x^{n_i}
\]

where

\[
d \leq L(BC) = L(BD \max(T, A)).\]

This contradicts Corollary \([3, 4]\). For \( \deg h = 2 \) one can apply the same proof with \( h \) replaced by \( h \circ h \). \( \square \)
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