On Lambda Function and a Quantification of Torhorst Theorem

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Abstract

To any compact $K \subset \hat{\mathbb{C}}$ we associate a map $\lambda_K : \hat{\mathbb{C}} \rightarrow \mathbb{N} \cup \{\infty\}$ — the lambda function of $K$ — such that a planar continuum $K$ is locally connected if and only if $\Lambda_K(x) \equiv 0$. We establish basic methods of determining the lambda function $\lambda_K$ for specific compacta $K \subset \hat{\mathbb{C}}$, including a gluing lemma for lambda functions and some inequalities. One of these inequalities comes from an interplay between the topological difficulty of a planar compactum $K$ and that of a sub-compactum $L \subset K$, lying on the boundary of a component of $\hat{\mathbb{C}} \setminus K$. It generalizes and quantifies the result of Torhorst Theorem, a fundamental result from plane topology. We also find three conditions under which this inequality is actually an equality. Under one of these conditions, this equality provides a quantitative version for Whyburn’s Theorem, which is a partial converse to Torhorst Theorem.

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1 Lambda Function and Motivations

The newly developed theory of core decomposition with Peano quotient [10], for planar compacta, enables us to associate to each compact \( K \subset \hat{\mathbb{C}} \) a map \( \lambda_K : \hat{\mathbb{C}} \to \mathbb{N} \cup \{\infty\} \), called the \textit{lambda function of} \( K \). This function sends all points \( x \notin K \) to zero and may take a positive value for some \( x \in K \). It “quantifies” certain aspects of the topological structure of \( K \), that are more or less related to the property of being locally connected. In particular, a continuum \( K \subset \hat{\mathbb{C}} \) is locally connected if and only if \( \lambda_K(x) \equiv 0 \). On the other hand, if a continuum \( K \subset \hat{\mathbb{C}} \) is not locally connected at \( x \in K \) then \( \lambda_K(x) \geq 1 \); but
the converse is not necessarily true.

The quantification in terms of lambda function allows us to carry out a new analysis of the topology of $K$, by computing or estimating $\lambda_K(x)$ for specific choices of $x \in K$. In the current paper we will investigate into an interesting phenomenon that was firstly revealed in a fundamental result by Marie Torhorst, as one of the three highlights of [15]. This result is often referred to as Torhorst Theorem [16, p.106, (2.2)] and reads as follows.

**Theorem (Torhorst Theorem).** If $K \subset \hat{\mathbb{C}}$ is a locally connected continuum and if $U$ is a component of $\hat{\mathbb{C}} \setminus K$ then the boundary $\partial U$ is also a locally connected continuum.

We will obtain an inequality that includes Torhorst Theorem as a simple case. The inequality is about the lambda function $\lambda_K$, for planar compacta $K$, which is based on the core decomposition of $K$ with Peano quotient [10]. It provides a new concrete approach to quantify a specific aspect of the topology of $K$ and is closely related to the property of being locally connected. The core decomposition with Peano quotient is motivated by some open questions in [6], asking for extensions of two earlier models of polynomial Julia sets that had been developed in [3, 4]. Those models, briefly called BCO models, provide efficient ways (1) to describe the topology of unshielded compacta, like polynomial Julia sets, and (2) to obtain specific factor systems for polynomials restricted to the Julia set. The BCO models are special cases of a more general model for all planar compacta that works for any rational function restricted to its Julia set [10, 12].

Let us recall some basic notions and known results, so that we may conveniently present on this general model. This model then helps us to explain a primary motivation of our study, which is rooted in Caratheodory’s **Continuity Theorem**. See for instance [2] or [14, p.18].

**Theorem (Continuity Theorem).** A conformal homeomorphism $\varphi : \mathbb{D} \to \Omega \subset \hat{\mathbb{C}}$ of the open unit disk $\mathbb{D} = \{z : |z| < 1\}$ has a continuous extension $\varphi : \overline{\mathbb{D}} \to \overline{\Omega}$ if and only if the boundary $\partial \Omega$ is locally connected. (In other words, if $\partial \Omega$ is a Peano continuum.)

Firstly, a **Peano continuum** means the image of $[0, 1]$ under a continuous map. By
Hahn-Mazurkiewicz-Sierpiński Theorem [9, p.256, §50, II, Theorem 2], a continuum is locally connected if and only if it is a Peano continuum. Secondly, a Peano compactum is defined to be a compactum having locally connected components such that for any constant \( C > 0 \) at most finitely many of its components are of diameter greater than \( C \). Clearly, the Cantor ternary set is also a Peano compactum and a Peano continuum is just a Peano compactum that is connected. Concerning how such a definition arises from the discussions of BCO models, we refer to [10, Theorems 1-3].

Under mild assumptions, the Continuity Theorem can be extended to the case of conformal homeomorphism of infinitely connected circle domains [13, Theorems 4 and 5].

**Theorem (Extended Continuity Theorem).** Let \( \Omega \subset \hat{\mathbb{C}} \) be a domain having at most countably many non-degenerate boundary components \( P_n \) such that the sum of diameters \( \sum_n \operatorname{diam}(P_n) \) is finite. Let \( \varphi : D \to \Omega \) be a conformal homeomorphism from a circle domain \( D \) onto \( \Omega \). Under either of the following assumptions,

- the linear measure of \( \partial \Omega \setminus \bigcup_n P_n \) is \( \sigma \)-finite,
- the linear measure of \( \partial D \) is \( \sigma \)-finite,

the map \( \varphi \) has a continuous extension \( \bar{\varphi} : \bar{D} \to \bar{\Omega} \) if and only if \( \partial \Omega \) is a Peano compactum.

Given a compactum \( K \subset \hat{\mathbb{C}} \), there exists an upper semi-continuous decomposition of \( K \) into sub-continua, denoted as \( D_{PC}^K \), such that (1) the quotient space is a Peano compactum and (2) \( D_{PC}^K \) refines every other such decomposition of \( K \) [10, Theorem 7]. We will call \( D_{PC}^K \) the core decomposition of \( K \) with Peano quotient. The hyperspace \( D_{PC}^K \) under quotient topology is called the Peano model of \( K \). Every \( d \in D_{PC}^K \) is called an atom of \( K \), or an order one atom. Every atom of an order one atom is called an order two atom, and so on. Note that a compactum, such as the pseudo-arc or Cantor’s Teepee, may have a non-degenerate atom of order \( \infty \).

If the atoms of a planar compactum \( K \) are considered as the structural units of \( K \) then the model developed in [10, 12] may be summarized in a form that more or less mimics
the theory of atoms from physics.

**Theorem ([10, Theorem 7] and [12, Theorem 4]).** Every compactum $K \subset \hat{C}$ is made up of atoms; all its atoms are sub-continua of $K$ and they form an upper semi-continuous decomposition, with its quotient space being a Peano compactum, that refines every other such decomposition; moreover, for any rational $f : \hat{C} \to \hat{C}$ and any atom $d$ of $f^{-1}(K)$, the image $f(d)$ is an atom of $K$.

The results in the above theorem constitute a fundamental part of our study, which will be referred to as the **Theory of Atoms** for planar compacta. Using the hierarchy formed by atoms of atoms, we define the lambda function $\lambda_K : \hat{C} \to \mathbb{N} \cup \{\infty\}$ as follows.

**Definition (Lambda Function).** Given a compactum $K \subset \hat{C}$. Let $\lambda_K(x) = 0$ for $x \notin K$ and for any $x \in K$ such that $\{x\}$ is an atom of $K$. For the other points $x \in K$, let $\lambda_K(x) = m - 1$ where $m \geq 2$ is the smallest integer such that $\{x\}$ is an order $m$ atom of $K$. If such an integer $m$ does not exist, we put $\lambda_K(x) = \infty$.

When little is known about the topology of $K$, it is a difficult task to completely determine the lambda function $\lambda_K$. On the other hand, the level sets $\lambda_K^{-1}(n)$ are actually “computable” for specific choices of $K$. In such circumstances, the lambda function is very useful in describing certain aspects of the topology of $K$. For instance, one may directly check the following basic facts: (1) a compact $K \subset \hat{C}$ is a Peano compactum if and only if $\lambda_K(x) \equiv 0$; (2) if $K$ is the closed topologist’s sine curve we have $\lambda_K^{-1}(1) = \{ti : |t| \leq 1\}$ and $\lambda_K^{-1}(0) = \hat{C} \setminus \lambda_K^{-1}(1)$; and (3) if $K$ is the product of $[0, 1]$ with Cantor’s ternary set then $\lambda_K^{-1}(1) = K$ and $\lambda_K^{-1}(0) = \hat{C} \setminus K$.

The lambda function is a reasonable attempt, to find numerical scales that measure how far a planar compactum $K$ is from being a Peano compactum. This scale is interesting especially when $K$ is located on or coincides with the boundary of a planar domain $\Omega \subset \hat{C}$. Koebe’s conjecture claims that $\Omega$ always admits a conformal homeomorphism $\varphi : D \to \Omega$ of some circle domain $D$. Let us restrict ourselves to the case $D = \mathbb{D}$ and recall some
well known results that help to demonstrate how we relate such a scale to the boundary
behaviour of the conformal map $\varphi$.

The most fundamental one is by Fatou in 1906. See for instance [5, p.17,Theorem 2.1].

**Theorem (Fatou’s Theorem).** If $f : \mathbb{D} \to \mathbb{C}$ is analytic and bounded then the radial
limits $\lim_{r \to 1} f \left( r e^{i\theta} \right)$ exist for all $\theta \in [0, 2\pi)$, except possibly for a set of linear measure zero.

One may call the map $\zeta = e^{i\theta} \mapsto \lim_{r \to 1} \varphi \left( r e^{i\theta} \right)$ sending a point $\zeta \in \partial \mathbb{D}$ to the radial
limit $f(\zeta)$ the **boundary function** of $\varphi$, whenever this limit exists. Denote this boundary
function as $\varphi^b$. We also shortly call it the boundary of $\varphi$.

It is known that the prime end at $\zeta \in \partial \mathbb{D}$ is either of the first or of the second type,
if only the radial limit at $\zeta$ exists [5, p.177, Theorem 9.7]. Thus, in the special case that
the domain $\Omega$ has no prime end of the second type, the boundary function $\varphi^b$ and the
union $\hat{\varphi} = \varphi \cup \varphi^b$ are both continuous, if we remove the points $\zeta \in \partial \mathbb{D}$ at which the radial
limit does not exist. By Carathéodory’s Continuity Theorem, the map $\varphi^b$ is actually
defined well and continuous on the whole unit circle $\partial \mathbb{D}$ if and only if the boundary $\partial \Omega$
is a Peano continuum. When $\partial \Omega$ is not a Peano continuum, the boundary function $\varphi^b$
may be considered as a member of $L^\infty(\partial \mathbb{D}) \setminus C(\partial \mathbb{D})$. One may try to find two types of
quantities

- that describe how far such a boundary function is from being continuous, and
- that measure how far the boundary $\partial \Omega$ is from being a Peano continuum.

The first type of quantities concern the asymptotic of $\varphi(z)$ for $z \in \mathbb{D}$ as $|z| \to 1$ and
the second type are closely related to the topology of the boundary $\partial \Omega$. To analyse the
interplay between the asymptotic of $\varphi(z)$ and the topology of $\partial \Omega$, one may turn to discuss
the interaction between these two types of quantities. The lambda function $\lambda_{\partial \Omega}$ gives rise
to one quantity of the second type. If it vanishes everywhere, or equivalently, when $\partial \Omega$ is
a Peano continuum, the boundary function $\varphi^b$ belongs to $C(\partial \mathbb{D})$. Among others, one may
ask very basic questions concerning the above interplay. Can we say something about $\varphi^b$
when \( \lambda_{\partial \Omega}(x) \leq 1 \) for all \( x \)? Can we say something about \( \lambda_{\partial \Omega} \) when \( \varphi^b \) has finitely many discontinuities on \( \partial D \)?

From the study of holomorphic dynamics, the Mandelbrot set \( \mathcal{M} \) is a well known continuum, which consists of all the parameters \( c \in \mathbb{C} \) such that the Julia set of the polynomial \( z^2 + c \) is connected. The local connectedness of \( \mathcal{M} \) is still an open question. The conjecture is YES. Due to many works concerning quadratic polynomials, we know that \( \mathcal{M} \) is locally connected at many of its points, such as the Misiurewicz points and those lying on the boundary of a hyperbolic component. From these known results arises a natural question: is it true that \( \lambda_{\mathcal{M}}(x) = 0 \) for all \( x \) at which \( \mathcal{M} \) is known to be locally connected? Another question of interest is, can we find some upper bound for the lambda function \( \lambda_{\mathcal{M}} \)? In particular, can we show that \( \lambda_{\mathcal{M}}(x) \leq 1 \) for all \( x \)? More studies on many questions of a similar nature can be expected. In the current paper, we try to build some of the basic tools that may be of some help in further analysis of the lambda function \( \lambda_K \) for specific planar compacta \( K \), including the Mandelbrot set.

## 2 Main Results

The major target of the current paper is to establish inequalities of lambda functions. The result of Torhorst Theorem will be extended and even be quantified by one of these inequalities, which says that the complexity (in terms of lambda function) of the topology of a compactum \( K \subset \hat{\mathbb{C}} \) controls that of any sub-compactum \( L \subset K \) that lies on the boundary of any component of \( \hat{\mathbb{C}} \setminus K \). Moreover, there are several special sub-cases, under which we actually have equalities in terms of lambda function. One of those equalities provides a quantitative generalization of a theorem by Whyburn [16, p.113, (4.4)], which is a partial converse of Torhorst Theorem.

Let us start from a closer look at the atoms of a planar compactum \( K \) and those of a sub-compactum \( L \) that lies on the boundary \( \partial U \) of an arbitrary component \( U \) of \( \hat{\mathbb{C}} \setminus K \). We will have.
**Theorem 1.** Given a compact $K \subset \mathring{C}$ and a component $U$ of $\mathring{C} \setminus K$. If $L \subset \partial U$ is a compactum then every atom of $L$ lies in a single atom of $K$. Particularly, every atom of $\partial U$ lies in a single atom of $K$.

The above theorem provides an intrinsic connection between the core decomposition of $K$ and that of $L$. This connection turns out to be useful when we compare the lambda functions $\lambda_K$ and $\lambda_L$. In deed, we can obtain the following.

**Theorem 2.** Given a compactum $K \subset \mathring{C}$ and a component $U$ of $\mathring{C} \setminus K$. If $L \subset \partial U$ is a compactum then $\lambda_L(x) \leq \lambda_K(x)$ for all $x \in \mathring{C}$.

Let $\mathcal{A}$ denote the family of the components of $\mathring{C} \setminus K$. The function defined as below is called the **envelope function** of $K$.

$$\tilde{\lambda}_K(x) := \sup_{L \subset \partial U, U \in \mathcal{A}} \lambda_L(x) \leq \lambda_K(x) \quad (x \in \mathring{C}). \tag{1}$$

By Theorem 1, we can infer that $\tilde{\lambda}_K(x) = \sup_{U \in \mathcal{A}} \lambda_{\partial U}(x)$ for all $x \in \mathring{C}$. By Theorem 2, we have $\tilde{\lambda}_K \leq \lambda_K$, which will be called the **lambda inequality**. The simplest case of the lambda inequality happens when $K$ is a Peano compactum. In such a case the lambda function $\lambda_K$ vanishes everywhere, indicating that $\lambda_L(x) \equiv 0$ for any compactum $L$ that is contained in the boundary of a complementary component $U$. That is to say, each compactum $L \subset \partial U$ is also a Peano compactum. In particular, the boundary $\partial U$ is a Peano compactum. If $K$ is even assumed to be a Peano continuum then all sub-continua of the boundary $\partial U$, including $\partial U$ itself, are locally connected. This is slightly stronger than the result of Torhorst Theorem.

In the above setting, the boundary $\partial K$ of the whole continuum $K$ may not be locally connected, even if it is a continuum. In such a case, we have $\lambda_K(x) \equiv 0$ and $\lambda_{\partial K}(x) \neq 0$. See for instance [11, Example 3.2] or Example 6.5 in section 5 of this paper. On the other hand, we can construct a continuum $K$ such that for certain points $x \in \partial K$ we have $\lambda_K(x) > \lambda_{\partial K}(x)$. See Example 6.6. From those examples, we see that the connection between the two lambda functions $\lambda_K$ and $\lambda_{\partial K}$ is not clear, for a general compactum $K \subset \mathring{C}$.
However, if $K$ satisfies additional properties then $\lambda_K \leq \lambda_{\partial K}$ holds everywhere. See part (iii) of Theorem 3. Moreover, for specific choices of $K$, we may have $\lambda_K(x) < \lambda_K(x)$ for all $x$ lying on the boundary $\partial U$ of some complementary component $U$. For instance, we may let $U$ be the complement of the unit square $\left\{ a + bi : 0 \leq a, b \leq 1 \right\}$ and let $K$ be the compactum consisting of $\partial U$ and an infinite sequence of squares (only the boundary) of side length $< 1$, centered at $0.5 + 0.5i$ and converging to $\partial U$. Then it is easy to check that $\lambda_K(x) = 1$ for all $x \in \partial U$ and that $\lambda_K(x) \equiv 0$. See Figure 1 for a simplified depiction of $K$. In the construction of such a compactum $K$, the complement $\hat{C} \setminus K$ has infinitely many components that are of diameter greater than $1 - \epsilon$ for some constant $\epsilon \in (0, 1)$. In order to exclude such a situation, we borrow from Whyburn the idea of $E$-continuum [16 p.113] and define a compactum $K \subset \hat{C}$ to be an $E$-compactum if for any number $\delta > 0$ it has at most finitely many complementary components of diameter greater than $\delta$. Unfortunately, the condition of $E$-compactum alone is still not sufficient to generate the lambda equality. See Example 6.3, in which we construct an $E$-compactum $K$ such that $\lambda_K(x) > \lambda_K(x)$ for all $x$ on a line segment.

Figure 1: An infinite sequence of small squares (the boundaries) converging to $\partial U$.

The theorem below gives three conditions under which the lambda equality holds.

**Theorem 3.** Given a compactum $K \subset \hat{C}$, the Lambda Equality $\lambda_K = \lambda_K$ holds if one of the following conditions is satisfied:

(i) $K$ is an $E$-compactum such that the envelope function $\lambda_K(x)$ vanishes everywhere;

(ii) $K$ is an $E$-compactum whose complementary components have disjoint closures.

(iii) $K$ is a partially unshielded compactum.
Remark. In (i) and (ii) of Theorem 3, the assumption that $K$ be an $E$-compactum cannot be removed. Actually, if $K$ is the difference of $[0,1] \times [0,i] \subset \hat{\mathbb{C}}$ and the countable union $\bigcup R_n$, where $R_n = \left( \frac{1}{3^n}, \frac{2}{3^n} \right) \times \left( \frac{1}{3}, \frac{2}{3} \right)$; and then consider $W = \hat{\mathbb{C}} \setminus ([0,1] \times [0,i])$, which satisfies $\partial W \cap \partial R_n = \emptyset$ for all $n \geq 1$. Clearly, we have $\partial R_n \cap \partial R_m = \emptyset$ for $n \neq m$. See Figure 2 for a simple depiction of $K$. Clearly, $K$ is a continuum, but not an $E$-continuum, that satisfies the conditions (i) and (ii) in Theorem 3 both. Moreover, the only non-degenerate atom is $\left[ \frac{1}{3}, \frac{2}{3} i \right]$.

Figure 2: An sequence of rectangles converging to the segment $\left[ \frac{1}{3}, \frac{2}{3} i \right]$.

There are other issues concerning Theorem 3 that are noteworthy. Firstly, the Lambda Equality under condition (i) implies the theorem below as a direct corollary which extends Whyburn’s Theorem [16, p.113, (4.4)], saying that an $E$-continuum is a Peano continuum if and only if the boundary of any of its complementary components is a Peano continuum.

**Theorem (Extended Whyburn’s Theorem).** An $E$-compactum is a Peano compactum if and only if the boundary of any of its complementary components is a Peano compactum.

Secondly, the Lambda Equality may not hold for an $E$-compactum $K \subset \hat{\mathbb{C}}$ that has finitely many complementary components. See Example 6.4 for the construction of such a compactum. Lastly, our motivation to discuss *partially unshielded compacta* also comes from a fundamental result in the study of analytic functions. See Remark 4.9.

In the above results, we address on how the lambda functions $\lambda_K, \lambda_L$ are related when $L$ lies on the boundary of a component of $\hat{\mathbb{C}} \setminus K$. There are other choices of the planar compacta $K \supset L$ so that the lambda functions $\lambda_K, \lambda_L$ are intrinsically related. A typical
situation happens, if the common part of $K \setminus L$ and $L$ is a finite set. This is quite an analogue to that of the gluing lemma for continuous maps. The aim is to find sufficient conditions under which we may form the lambda function $\lambda_K$ by “gluing together” $\lambda_{K \setminus L}$ and $\lambda_L$. Another typical situation of particular interest is: when $K$ is a renormalizable polynomial Julia set and $L$ the small Julia set. In such a case, the small Julia set $L$ has a system of connected neighborhoods, say $\{U_n\}$, that satisfy $U_n \supset U_{n+1}$ for all $n \geq 1$. Every $U_n$ is the union of finitely many puzzles at the same level; moreover, the intersection $U_n \cap (K \setminus U_n)$ is a finite set for all $n$. Similar discussions also works well for more general choices of $K \supset L$ such that $K \setminus L$ has countably many components, if each of these components has a single limit point on $L$ and if for any constant $C > 0$ only finitely many of them are of diameter greater than $C$. This is the case when $K$ is the Mandelbrot set $\mathcal{M}$ and $L$ is a baby. Many results are obtained, centering around the topology of $\mathcal{M}$. In particular, the smallest closed equivalence is obtained, whose restriction to $\mathbb{Q}/\mathbb{Z}$ is given by the equivalence $\sim_{\mathcal{M}}^\mathbb{Q}$. See for instance [7] and the references therein.

The gluing lemma also indicates two inequalities in terms of lambda functions, $\lambda_K \geq \lambda_{K \setminus L}$ and $\lambda_K \geq \lambda_L$. Here, we note that for general planar compacta $K \supset L$, the inequality $\lambda_K \geq \lambda_L$ does not necessarily hold. For instance, we may set $K = [0, 1]^2$ and $L$ a compactum that is not Peano, then $\lambda_K \leq \lambda_L$ everywhere and $\lambda_K(x) < \lambda_L(x)$ for certain points $x \in L$.

**Theorem 4.** If $K \supset L$ are planar compacta such that $K \setminus L$ intersects $L$ at finitely many points then $\lambda_K(x) = \max \{\lambda_{K \setminus L}(x), \lambda_L(x)\}$ for all $x$.

Renaming $K \setminus L$ as $A$, the above result actually implies that $\lambda_K$ coincides with $\lambda_A$ on $A \setminus L$ and with $\lambda_L$ on $L \setminus A$, equals $\max \{\lambda_{K \setminus L}(x), \lambda_L(x)\}$ on $A \cap L$, and vanishes elsewhere. In such a setting, we have an immediate corollary.

**Theorem (Glueing Lemma for Lambda Functions).** If in addition $\lambda_A(x) = \lambda_L(x)$
for all \( x \in A \cap L \) then \( \lambda_{K} \) may be obtained by gluing together \( \lambda_{A} \) and \( \lambda_{L} \), in the sense that

\[
\lambda_{K}(x) = \begin{cases} 
\lambda_{A}(x) & x \in A \\
\lambda_{L}(x) & x \in L \\
0 & \text{otherwise}
\end{cases}
\]  

(2)

**Remark.** The formation in Equation (2) is similar to the one that is illustrated in the well known gluing lemma for continuous maps. See for instance [1, p.69, Theorem (4.6)].

Theorem 4 and Equation (2) provide us a handy tool in the study of planar compacta. In the meanwhile, one must be cautious about the cases when \( A \cap L \) has infinitely many components. In Theorem 5.1 we will extend the result of Theorem 4 to such a case, under additional assumptions. In Example 6.3, we construct two Peano compacta \( X, Y \) lying in the unit square such that \( X \cup Y \) is not a Peano compactum. In this example, the intersection \( X \cap Y \) has infinitely many components all but one of which are single points. The only non-degenerate component of \( X \cap Y \) is a segment.

The rest of this paper is arranged as follows. Section 3 proves Theorems 1 to 2. Section 4 proves Theorem 3 together with the definition of partially unshielded compactum. Section 5 proves Theorem 4 and continues to discuss a more general setting. See Theorem 5.1. Section 6 gives some examples. In particular, section 6.1 provides two examples of planar continua \( K \supset L \) that do not satisfy the assumptions in Theorem 4. Section 6.2 constructs an \( E \)-continuum \( K \) for which the Lambda Equality does not hold; section 6.3 gives a concrete continuum \( K \) such that \( \lambda_{K}(x) > \lambda_{\partial K}(x) \) for certain points \( x \in \partial K \). The continuum \( K \) given in section 6.3 contains a sub-continuum \( L \), such that \( \lambda_{L}(x) \equiv 0 \) and that \( \lambda_{\partial L}(x) = 1 \) for certain points \( x \in \partial L \). Finally, section 7 introduces a geometric graph \( \Lambda_{K} \) for any planar compactum \( K \). Such a graph is really useful, if one wants to represent the lambda function \( \lambda_{K} \) and to illustrate the relative locations of the level sets \( \lambda_{K}^{-1}(m) \) for \( m \geq 0 \).
Our target in this section is to prove Theorems 1 and 2, the latter of which obtains the lambda inequality. To meet this target, we study specific relations on a compactum $K \subset \hat{C}$, which are considered as subsets of the product $K \times K$.

Given a relation $R$ on $K$, we call $R[x] = \{ y \in K : (x, y) \in R \}$ the fiber of $R$ at $x$. A relation $R$ is closed if it is a closed subset of $K \times K$. We mostly consider relations $R$ that are reflexive and symmetric, so that for all $x, y \in K$ we have (1) $x \in R[x]$ and (2) $x \in R[y]$ if and only if $y \in R[x]$. For such a relation, the iterated relation $R^2$ is defined naturally, so that its fiber at $x \in K$ is $R^2[x] = \{ y \in K : \exists z \in K \text{ such that } (x, z), (z, y) \in R \}$.

Recall that the Schönflies relation on a planar compactum $K$ is a reflexive symmetric relation. Under this relation, two points $x \neq y$ are related provided that there are two disjoint simple closed curves $J_i \ni x_i$ such that $U \cap K$ has infinitely many components intersecting $J_1, J_2$ both. Here $U$ is the component of $\hat{C} \setminus (J_1 \cup J_2)$ with $\partial U = J_1 \cup J_2$.

Given a compactum $K \subset \hat{C}$, denote by $R_K$ the Schönflies relation on $K$ and by $\overline{R_K}$ the closure of $R_K$, as subsets of the product space $K \times K$. We also call $\overline{R_K}$ the closed Schönflies relation. Let $\mathcal{D}_K$ be the finest upper semi-continuous decompositions of $K$ into sub-continua that splits none of the fibers $R_K[x] = \{ y \in K : (x, y) \in K \}$. Then $\mathcal{D}_K$ coincides with $\mathcal{D}^{PC}_K$, the core decomposition of $K$ with Peano quotient [10, Theorem 7]. Therefore the elements of $\mathcal{D}_K$ are the (order 1) atoms of $K$.

By [10, Theorem 2], all the fibers $\overline{R_K[x]}$ are continua, although the compactness and connectedness of the fibers of $R_K$ remain open. Moreover, the closed Schönflies relation $\overline{R_K}$ has been characterized in [12, Theorem 1]. According to this characterizing, we have $(x, x) \in \overline{R_K}$ for all $x \in K$; on the other hand, a point $y \neq x$ belongs to $\overline{R_K[x]}$ if and only if for all $r$ smaller than the distance between $x$ and $y$ the difference $K \setminus (B_r(x) \cup B_r(y))$ of $K$ with the two open disks $B_r(x), B_r(y)$ has infinitely many components that intersect both $\partial B_r(x)$ and $\partial B_r(y)$. Actually, to show that $y \neq x$ belongs to the fiber $\overline{R_K[x]}$, we only need to verify that for small enough $r > 0$ the difference $K \setminus (B_r(x) \cup B_r(y))$ has
infinitely many components intersecting both $\partial B_r(x)$ and $\partial B_r(y)$.

Theorem 1 follows from the following preliminary lemma, which relates the fibers of $\overline{R_K^2}$ to those of $\overline{R_L}$, where $L$ is a compact subset of $K$ satisfying particular properties.

**Lemma 3.1.** Given a compactum $K \subset \mathring{\mathbb{C}}$ and a component $U$ of $\mathring{\mathbb{C}} \setminus K$. Then, for any compactum $L \subset \partial U$ and for any point $x \in L$, we have $\overline{R_L}[x] \subset \overline{R_K^2}[x]$.

**Proof.** To obtain the containment $\overline{R_L}[x] \subset \overline{R_K^2}[x]$ for any given $x \in L$, we may fix an arbitrary point $y \neq x$ lying in the fiber $\overline{R_L}[x]$ and consider for integers $n \geq 1$ the annulus $A_n = \mathring{\mathbb{C}} \setminus (B_{1/n}(x) \cup B_{1/n}(y))$. Here $B_{1/n}(x)$ and $B_{1/n}(y)$ are open disks with radius $1/n$ centered at $x$ and $y$, respectively. Clearly, we have $\overline{B_{1/n}(x)} \cap \overline{B_{1/n}(y)} = \emptyset$ for large $n$; in such cases the intersection $A_n \cap L$ has infinitely many components intersecting both $\partial B_{1/n}(x)$ and $\partial B_{1/n}(y)$. We may choose an infinite sequence $\{P_i\}$ of such components that converge to some continuum $P_\infty$ under Hausdorff metric.

Since $x, y \in L \subset \partial U$, we may choose an arc $\alpha_0 \subset U$ connecting a point on $\partial B_{1/n}(x)$ to one on $\partial B_{1/n}(y)$. This arc contains a sub-arc $\alpha \subset A_n$ that is irreducible between a point on $\partial B_{1/n}(x)$ and a point on $\partial B_{1/n}(y)$. Slightly thickening $\alpha$ to a topological disc $\alpha^* \subset A_n$, satisfying $\alpha^* \cap K = \emptyset$, we can infer that $A_n \setminus \alpha^*$ is homeomorphic to $[0,1]^2$. We will verify the following.

**Claim.** $P_\infty$ contains two points $u_n \in \partial B_{1/n}(x), v_n \in \partial B_{1/n}(y)$ with $v_n \in \overline{R_K^2}[u_n]$.

The flexibility of the large enough integers $n$ ensures that $\lim_n u_n = x$ and $\lim_n v_n = y$. Since $\overline{R_K^2}$ is a closed relation, we surely obtain $y \in \overline{R_K^2}[x]$. This completes our proof.

The remaining issue is to verify the above claim. For the sake of convenience, we may consider $\overline{A_n \setminus \alpha^*}$ as $[0,1]^2$, in which $l_1 = \overline{A_n \setminus \alpha^*} \cap \partial B_n(x)$ is represented as $[0,1] \times \{1\}$ and $l_2 = \overline{A_n \setminus \alpha^*} \cap \partial B_n(y)$ is represented as $[0,1] \times \{0\}$. For any $z$ in $P_\infty \cap (0,1)^2$ and any $r \in (0,1)$, let $W_r$ denote the open rectangle centered at $z$ with diameter $r$. Since $P_i \to P_\infty$ under Hausdorff distance we may assume that every $P_i$ intersects $W_r$ and lies in $[0,1]^2$, which from now on represents $\overline{A_n \setminus \alpha^*}$. See Figure 3.
we may find a point $b$ close to $a$, after $L$ we assume that $\alpha$ of ends at $b$, contained in distinct components of $(0,1)^2$. Therefore, every $P_i$ can be connected to the right side by an arc in $[0,1]^2$ that does not intersect $P_\infty$. Roughly, we also say that every $P_i$ is to the right of $P_\infty$. Moreover, rename the continua $P_i$ such that $P_i$ can be connected to the right side by an arc in $[0,1]^2$ that does not intersect $P_j$ for all $j \geq i+1$. In other words, every $P_i$ is to the right of $P_{i+1}$.

Note that for all $i \geq 1$ let $V_i$ be the unique component of $\hat{C} \setminus (P_{2i-1} \cup P_{2i+1} \cup l_i \cup l_2)$ whose boundary intersects each of $l_1$, $l_2$, $P_{2i-1}$ and $P_{2i+1}$. Clearly, we have $P_{2i} \subset V_i$. For the previously given point $z$ in $P_\infty \cap (0,1)^2$, we may choose for each $i \geq 1$ a point $a_i \in P_{2i} \cap W_r$ such that $\lim_{i \to \infty} a_i = z$. Since we also have $P_{2i} \cap W_r \neq \emptyset$ and $P_{2i} \subset L \subset \partial U$, we may find a point $b_i \in (W_r \cap V_i \cap U)$ for every $i \geq 1$. This point $b_i$ may be chosen very close to $a_i$. Check Figure 3 for relative locations of $a_i \in P_{2i+1}$ and $b_i \in (W_r \cap V_i \cap U)$.

Now, we may fix arcs $\alpha_i \subset U$ for each $i \geq 1$ that starts from a fixed point $b_0 \in U$ and ends at $b_i$. Let $c_i$ be the last point on $\alpha_i$ that leaves $\partial[0,1]^2$. Let $d_i$ be the first point on $\alpha_i$ after $c_i$ at which $\alpha_i$ intersects $\partial W_r$. Clearly, we have $c_i \in (l_1 \cup l_2)$. Let $\beta_i$ be the sub-arc of $\alpha_i$ from $c_i$ to $d_i$. Check Figure 3 for a rough depiction of two possible locations for $\beta_i$. Then $\beta_i$ and $\beta_j$ for $i \neq j$ are contained in distinct components of $(0,1)^2 \setminus W_r) \setminus L$. Since we assume that $L \subset K$ and that $K \cap U = \emptyset$, the arcs $\beta_i$ and $\beta_j$ for $i \neq j$ are actually contained in distinct components of $(0,1)^2 \setminus W_r) \setminus K$.

Let $x_n$ be the unique point on $l_1 \cap P_\infty$ such that the right piece of $l_1 \setminus \{x_n\}$ does not
intersect $P_\infty$. Let $y_n$ be the unique point on $l_2 \cap P_\infty$ such that the right piece of $l_2 \setminus \{y_n\}$ does not intersect $P_\infty$. The sequence $\{c_i\}$ then has a limit point in $\{x_n, y_n\}$. Assume with no loss of generality that $z_r = \lim_{i \to \infty} d_i$ for some point $z_r \in \partial W_r$. Since $\partial [0, 1]^2$ and $\partial W_r$ are two disjoint simple closed curves, from the choices of $x_n, y_n$ and $z_r$ we can infer that either $(x_n, z_r) \in R_K$ or $(y_n, z_r) \in R_K$. The flexibility of $r > 0$ then leads to the inclusion $z \in (\overline{R_K[x_n]} \cup \overline{R_K[y_n]})$.

Now consider the two closed sets $E_n = P_\infty \cap (\overline{R_K[x_n]} \cup l_1)$ and $F_n = P_\infty \cap (\overline{R_K[y_n]} \cup l_2)$, which satisfy $P_\infty = E_n \cup F_n$. From the connectedness of $P_\infty$ we see that $E_n \cap F_n \neq \emptyset$. Clearly, each point $w \in (E_n \cap F_n)$ necessarily falls into one of the following cases:

1. $w$ lies in $l_1 \subset \partial B_{1/n}(x)$ and belongs to $\overline{R_K[y_n]}$,
2. $w$ lies in $l_2 \subset \partial B_{1/n}(y)$ and belongs to $\overline{R_K[x_n]}$,
3. $w \notin (l_1 \cup l_2)$ but it lies in $\overline{R_K[x_n]} \cap \overline{R_K[y_n]} \cap (0, 1)^2$.

In case (1) we set $u_n = w, v_n = y_n$; in case (2) we set $u_n = x_n, v_n = w$; in case (3) we set $u_n = x_n, v_n = y_n$. Then, in cases (1) and (2) we have $v_n \in \overline{R_K[u_n]} \subset \overline{R_K^2[u_n]}$; and in case (3) we will have we will have $v_n \in \overline{R_K^2[u_n]}$. This verifies the claim.

With the help of Lemma 3.1, the proofs for Theorems 1 and 2 are not difficult, which may be summarized as follows.

**Proof for Theorem 1.** The later half of Theorem 1 follows from the former half, since $U$ is also a complementary component of $\partial U$. Therefore, we only consider the former half.

To this end, let $D_L^#$ consist of all those continua that are each a component of $d^* \cap L$ for some $d^* \in D_K$. Then $D_L^#$ is an upper semi-continuous decomposition of $L$. Since every fiber of $\overline{R_K^2}$ is entirely contained in a single element of $D_K$, the result of Lemma 3.1 implies that every fiber $\overline{R_L[z]}$ is entirely contained in a single element of $D_L^#$. This in turn implies that $D_L^#$ is refined by $D_L$. In other words, every atom of $L$, which is an element of $D_L$, is entirely contained in a single atom of $K$.
Proof for Theorem 2. To obtain \( \lambda_L(x) \leq \lambda_K(x) \) for all \( x \), we only need to consider the points \( x \in L \). With no loss of generality, we may assume that \( \lambda_K(x) = m - 1 \) for some integer \( m \geq 1 \). That is to say, there exist strictly decreasing continua \( d_1^* \supset d_2^* \supset \cdots \supset d_m^* = \{ x \} \) such that \( d_i^* \) is an atom of \( K \) and \( d_{i+1}^* \) an atom of \( d_i^* \) for \( 1 \leq i \leq m - 1 \). (It is possible that \( m = 1 \).) By Theorem 1, the atom of \( L \) containing \( x \), denoted as \( d_1 \), is a subset of \( d_1^* \). Since \( d_1 \subset d_1^* \) also satisfy the assumptions of Theorem 1, we can infer that the atom of \( d_1 \) containing \( x \), denoted as \( d_2 \), is a subset of \( d_2^* \). Repeating the same argument for \( m \) times, we obtain atoms \( d_i \) of \( L \) of order \( i \) for \( 1 \leq i \leq m \) such that \( d_i \subset d_i^* \) for all \( i \). Clearly, we have \( d_m = \{ x \} \) and hence \( \lambda_L(x) \leq m = \lambda_K(x) \). Since \( U \) is a component of \( \hat{C} \setminus K \) and since \( L \subset \partial U \), in the same way we can show that \( \lambda_L(x) \leq \lambda_{\partial U}(x) \) for all \( x \). From this we immediately have \( \sup_U \lambda_{\partial U} = \tilde{\lambda}_K \leq \lambda_K \). 

4 On Lambda Equalities

In this section we establish the two lambda equalities for \( E \)-compacta and then the one for partially unshielded compacta.

4.1 Two Lambda Equalities for \( E \)-compacta

The standing assumption of this sub-section is that \( K \subset \hat{C} \) is an \( E \)-compactum and that \( U_1, U_2, \ldots \) are the components of \( \hat{C} \setminus K \), whose diameters \( \delta(U_i) \) converge to zero. Torhorst Inequality requires that \( \lambda_K(x) \geq \sup_i \lambda_{\partial U_i}(x) \) for all \( x \in \hat{C} \) and for all \( i \geq 1 \). If \( x \) belongs to \( K^o \cup \left( \bigcup_i U_i \right) \) we have \( \lambda_K(x) = \tilde{\lambda}_K(x) = 0 \). So we only need to consider the points on \( \partial K \), which may not equal \( \bigcup_i \partial U_i \). Note that a point in the difference \( \partial K \setminus \bigcup_i \partial U_i \) is called a buried point of \( K \). It is known that the Julia set of a polynomial does not contain any buried points. However, a rational Julia set may contain buried points.

The following result concerns the fibers of \( \overline{R_K} \) and those of \( \overline{R_{\partial U_i}} \).

Lemma 4.1. If the fiber \( \overline{R_K}[x] \) contains a point \( y \neq x \) then \( y \in \overline{R_{\partial U_i}}[x] \) for some \( i \).
Proof. Let $\rho(x, y)$ be the spherical distance between $x$ and $y$. For each $n \geq 2$ let $B_n(x)$ and $B_n(y)$ be the open disks of radius $2^{-n}\rho(x, y)$ that are centered at $x$ and $y$ respectively. Let $A_n$ be the difference of $\hat{C}$ with $B_n(x) \cup B_n(y)$. Then $A_n$ is a topological annulus and $K \cap A_n$ has infinitely many components that intersect $\partial B_n(x)$ and $\partial B_n(y)$ both. Equivalently, the difference $A_n \setminus K$ has infinitely many components, say $\{P^n_j : j \geq 1\}$, that intersect $\partial B_n(x)$ and $\partial B_n(y)$ at the same time. Since the diameters of those $P^n_j$ are all greater than $\rho(x, y)/2$ and since we assume $K$ to be an $E$-compactum, there is an integer $i(n)$ such that $U_{i(n)}$ contains infinitely many of those $P^n_j$. Here one may verify that all those $P^n_j$ that are contained in $U_{i(n)}$ are each a component of $A_n \cap U_{i(n)}$.

Now, choose a subsequence $\{Q^n_k : k \geq 1\}$ of $\{P^n_j : j \geq 1\}$, with $Q^n_k \subset U_{i(n)}$, such that $Q^n_k$ converges under Hausdorff distance to a continuum $M_n$. Then $M_n$ is a subset of $\partial U_{i(n)}$ and intersects $\partial B_n(x)$ and $\partial B_n(y)$ both. Fixing any $a_n$ in $M_n \cap \partial B_n(x)$ and $b_n$ in $M_n \cap \partial B_n(y)$, we will have $(a_n, b_n) \in R_{\partial U_{i(n)}}$. Since $K$ is an $E$-compactum, we see that infinitely many $i(n)$ takes the same value, say 1. Therefore, we have two infinite sequences $\{c_n\} \subset \{a_n\}$ and $\{d_n\} \subset \{b_n\}$, with $c_n, d_n \in \partial U_1$, such that $(c_n, d_n) \in R_{\partial U_1}$ for all $n \geq 2$. Since $\lim_{n \to \infty} c_n = x$ and $\lim_{n \to \infty} d_n = y$, we readily have $(x, y) \in R_{\partial U_1}$, or equivalently $y \in R_{\partial U_1}[x]$. \hfill \Box

Now we are well prepared to prove parts (i) and (ii) of Theorem 3, whose results are respectively included in the following propositions.

Proposition 4.2. If $K$ is an $E$-compactum with $\tilde{\lambda}_K(x) \equiv 0$ then $\lambda_K(x) \equiv 0$.

Proof. The assumption $\tilde{\lambda}_K(x) \equiv 0$ implies that all the relations $R_{\partial U_i}$ are trivial, in the sense that the fibers $R_{\partial U_i}[x]$ are each a singleton for all $i$ and all $x \in \partial U_i$. Combing this with the conclusion of Lemma 4.1 we can infer that the fiber $R_K[x] = \{x\}$ for all $x \in K$. From this, we immediately see that every atom of $K$ is a singleton hence that $\lambda_K(x) \equiv 0$. \hfill \Box

Proposition 4.3. Given an $E$-compactum $K$. If $\partial U_i \cap \partial U_j = \emptyset$ for $i \neq j$ then $\lambda_K = \tilde{\lambda}_K$.
Proof. Let \( D_i \) denote the core decomposition of \( \partial U_i \). Since we assume that \( \partial U_i \cap \partial U_j = \emptyset \) for \( i \neq j \), the following collection

\[
D^*_K := \left( \bigcup_i D_i \right) \cup \left\{ \{x\} : x \in K \setminus \left( \bigcup_i \partial U_i \right) \right\}
\]

is a partition that divides \( K \) into sub-continua. It will suffice to show that \( D^*_K \) is the core decomposition of \( K \).

Recall that \( D_K \) is the finest monotone decomposition such that every fiber of \( R_K \) is contained in a single element of \( D_K \). By Lemma 3.1, we know that \( D_K \) is refined by \( D^*_K \). On the other hand, by Lemma 4.1 and the assumption, we see that \( \partial U_i \cap \partial U_j = \emptyset \) for \( i \neq j \) implies that every fiber of \( R_K \) is contained in a single element of \( D^*_K \). Therefore, we only need to verify that \( D^*_K \) is upper semi-continuous, which then indicates that \( D^*_K \) is a monotone decomposition hence is refined by \( D_K \).

In other words, we need to verify that the equivalence \( \sim \) determined by the partition \( D^*_K \) is closed as a subset of \( K \times K \). To this end, we consider an arbitrary sequence \( \{(x_n, y_n) : n \geq 1\} \) in \( K \times K \) with \( \lim_{n \to \infty} (x_n, y_n) = (x, y) \) such that \( x_n \sim y_n \) for all \( n \geq 1 \). There are two possibilities: either \( x = y \) or \( x \neq y \). In the first case, we have \( (x, y) = (x, x) \), which is surely an element of \( \sim \). In the second, the assumption that \( K \) is an \( E \)-compactum implies that there is some \( U_i \) such that \( \{x_n, y_n\} \subset \partial U_i \) for infinitely many \( n \geq 1 \). Consequently, the subset \( \{x, y\} \) is contained in a single element of \( \mathcal{D}_{\partial U_i} \), which is a sub-collection of \( D^*_K \). That is to say, we have \( x \sim y \). This ends our proof.  

The arguments in the above proof actually imply the following.

**Theorem 4.4.** Given an \( E \)-compactum \( K \). If \( \partial U_i \cap \partial U_j = \emptyset \) for \( i \neq j \) then every atom of \( K \) is either an atom of some \( \partial U_i \) or a singleton \( \{x\} \) with \( x \in K \setminus (\bigcup_i \partial U_i) \).

### 4.2 The Lambda Equality for Partially Unshielded Compacta

We start off by discussing the relationship between the atoms of a compactum \( L \) and those of a compactum \( K \), that is formed by the union of \( L \) with some (not all) components of
A special case for the choices of $K$ and $L$ is when $L = \partial K$.

Our discussion provides a very useful result, as stated in Proposition 4.6. To obtain that, we need the following result, which follows from \[10, \text{Lemma 3.3}\].

**Lemma 4.5.** Let $K \subset \mathbb{C}$ be a compact set and $U$ the region bounded by two parallel lines $L_1$ and $L_2$, such that $\partial U = L_1 \cup L_2$. If $\overline{U} \setminus K$ has at least $m \geq 2$ components intersecting both $L_1$ and $L_2$, then $\overline{U} \cap K$ has at least $m - 1$ components intersecting both $L_1$ and $L_2$.

On the other hand, if $\overline{U} \cap K$ has at least $m \geq 2$ components intersecting both $L_1$ and $L_2$, then $\overline{U} \setminus K$ has at least $m$ components intersecting both $L_1$ and $L_2$. Similarly, if $A \subset \mathbb{C}$ is a closed topological annulus, i.e. a set homeomorphic with $\{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$, and $K \subset \mathbb{C}$ a compactum then the following statements are equivalent: (1) $A \cap K$ has infinitely many components intersecting the two components of $\partial A$ both; (2) $A \setminus K$ has infinitely many components intersecting the two components of $\partial A$ both.

**Proposition 4.6.** Given a compactum $L \subset \mathbb{C}$ and a collection $\{U_\alpha : \alpha \in I\}$ of components of $\mathbb{C} \setminus L$. Let $K = L \cup (\cup_{\alpha \in I} U_\alpha)$. Then $R_K$ is a subset of $\{(z, z) : z \in K \setminus L\} \cup \overline{R_L}$. Consequently, every $d^* \in D_{PC}^K$ either is a singleton lying in $K \setminus L$ or is contained in a single element $d \in D_{PC}^L$.

**Proof.** Since $K = L \cup (\cup_{\alpha \in I} U_\alpha)$, every point $z \in (K \setminus L)$ lies in some $U_\alpha$. Thus the atom of $K$ containing $z$ is exactly the singleton $\{z\}$. From this it readily follows that every atom $d^* \in D_{PC}^K$ that intersects $L$ is a sub-continuum of $L$, hence that $\overline{R_K} = \{(z, z) : z \in K \setminus L\} \cup (\overline{R_K} \cap L^2)$. Therefore, we only need to show that $(\overline{R_K} \cap L^2) \subset \overline{R_L}$. Otherwise there is $(x, y) \in \overline{R_K} \cap L^2$ which does not belong to $\overline{R_L}$. According to the definition of $\overline{R_L}$, for any positive number $r$ smaller than the distance between $x$ and $y$, the difference $L \setminus (B_r(x) \cup B_r(y))$ has finitely many components intersecting the two circles $\partial B_r(x)$ and $\partial B_r(y)$ both. Let $A_r = \hat{\mathbb{C}} \setminus (B_r(x) \cup B_r(y))$. By applying Lemma 4.5, we can infer that $A_r \setminus L$ has at most finitely many components that intersect $\partial B_r(x)$ and $\partial B_r(y)$ both. As we assume that $K = L \cup (\cup_{\alpha \in I} U_\alpha)$, it is easy to see that every component of $A_r \setminus K$ is also a component of $A_r \setminus L$. Thus $A_r \setminus K$ has at most finitely many components that
intersect both \( \partial B_r(x) \) and \( \partial B_r(y) \). In other words, we have \((x, y) \notin \overline{R_K}\). This is absurd since we assume that \((x, y) \in \overline{R_K}\).

Then, we obtain a result that is slightly stronger than part (iii) of Theorem 3. To this end, we will focus on partially unshielded compacta, to be defined as follows. Typical examples of such a compactum is the union of a polynomial Julia set with an arbitrary collection of its bounded Fatou components. We agree that unshielded compacta are partially unshielded.

**Definition 4.7.** Let \( L \subset \hat{\mathbb{C}} \) be an unshielded compactum, which equals the boundary \( \partial U \) of one of its complementary components \( U \). A compactum \( K \) formed by the union of \( L \) with some complementary components of \( L \) other than \( U \) is called a partially unshielded compactum determined by \( L \).

We want to analyze the Lambda Equality for partially unshielded compacta. In our analysis, there are some basic facts, that are noteworthy, concerning an unshielded compactum \( L \) and any partially unshielded compactum \( K \) determined by \( L \). Firstly, every interior point of \( K \) lies in some complementary component of \( L \); secondly, every boundary point of \( K \) lies in \( L \). Thus we always have \( \partial K = L \); moreover, every atom of \( K \) intersects the interior \( K^\circ \) is a singleton. Therefore, in order to determine the core decomposition of \( K \) we only need to consider the core decomposition of \( \partial K \), which equals \( \partial U \).

**Theorem 4.8.** Let \( L \subset \hat{\mathbb{C}} \) be an unshielded compactum. Let \( K \) be a partially unshielded compactum determined by \( L \). Then every atom of \( L \) is also an atom of \( K \), from which we readily have \( \mathcal{D}_K = \mathcal{D}_L \cup \{ x \} : x \in K \setminus L \}. Consequently, the lambda equality \( \tilde{\lambda}_K = \lambda_K \) holds.

*Proof.* Since \( L \) is unshielded, it equals the boundary of one of its complementary components, say \( U \). By Lemma 3.1 we know that every atom of \( L \) is contained in a single atom of \( K \). On the other hand, by Lemma 4.6 we know that every atom of \( K \) that intersects \( L \) is contained in a single atom of \( L \). Combing these, we see that every atom of \( L \) is also
an atom of $K$. Since every singleton $\{x\}$ with $x \in K^o = K \setminus L$ is an atom of $K$, we immediately have $\mathcal{D}_K = \mathcal{D}_L \cup \{\{x\} : x \in K \setminus L\}$. From this follows the lambda equality $\hat{\lambda}_K = \lambda_K$. 

**Remark 4.9.** Theorem 4.8 gives a result that is stronger than part (iii) of Theorem 3. In particular, we know that the Lambda Equality $\lambda_K = \hat{\lambda}_K$ holds for each full compactum $K$. Therefore, a full compactum $K$ is a Peano compactum if and only if the boundary $\partial K$ is. In particular, if $G \subset \hat{\mathbb{C}}$ is a simply connected bounded domain then $\partial G$ is locally connected if and only if $K = \hat{\mathbb{C}} \setminus G$ is. This basic fact has been well known, see for instance the items (iii) and (iv) of [14, p.20, Theorem 2.1]. Now, it is extended to a quantitative version in Theorem 4.8 that applies to an arbitrary full continuum that may or may not be locally connected.

5 The Gluing Lemma for Lambda Functions

The results discussed in this section follow the philosophy of the well known gluing lemma in topology, for continuous maps. See for instance [1, p.69, Theorem (4.6)] for the simple case and [1, p.70, Theorem (4.8)] for the general setting. Firstly, we prove Theorem 4, which deals with the lambda functions for planar compacta $K \supset L$ such that $A = K \setminus L$ intersects $L$ at finitely many points $x_1, \ldots, x_n$.

**Proof for Theorem 4.** Recall that the elements of $\mathcal{D}_A$ are called the (order 1) atoms of $A$, and those of $\mathcal{D}_L$ the (order 1) atoms of $L$. For $1 \leq i \leq n$, let $d_i^1 \in \mathcal{D}_A$ be the atoms of $A$ that contains $x_i$. Similarly, let $e_i^1 \in \mathcal{D}_L$ be the atoms of $L$ that contains $x_i$.

Let $K_1 = A_1 \cup L_1$, where $A_1 = \bigcup_{i=1}^{n} d_i^1$ and $L_1 = \bigcup_{i=1}^{n} e_i^1$.

We claim that the result holds for all points $x \notin K_1$. Indeed, let $\mathcal{E}_1$ consist of the components of $K_1$. Then the core decomposition $\mathcal{D}_K$ equals

$$\mathcal{D}_1 = (\mathcal{D}_A \setminus \{d_1^1, \ldots, d_n^1\}) \cup (\mathcal{D}_L \setminus \{e_1^1, \ldots, e_n^1\}) \cup \mathcal{E}_1.$$
This ensures that $\lambda_K(x) = \max \{\lambda_A(x), \lambda_L(x)\}$ for all $x \not\in K_1$. Moreover, for all $x \in A_1$ we have

$$\lambda_A(x) = \begin{cases} 0 & \{x\} \in \mathcal{D}_1 \\ 1 + \lambda_{A_1}(x) & \text{ otherwise} \end{cases}$$

Similarly, for all $x \in L_1$ we have

$$\lambda_L(x) = \begin{cases} 0 & \{x\} \in \mathcal{D}_1 \\ 1 + \lambda_{L_1}(x) & \text{ otherwise} \end{cases}$$

Therefore, it suffices to verify that $\lambda_{K_1}(x) = \max \{\lambda_{A_1}(x), \lambda_{L_1}(x)\}$. Since all the order 2 atoms of $A$ (respectively, $L$) that lie in $A_1$ (respectively, $L_1$) form the core decomposition of $A_1$ (respectively, $L_1$), we just repeat the above procedure again on $A_1, L_1$. This then produces two compacta $A_2 \subset A_1$ and $L_2 \subset L_1$ such that $\lambda_{K_1}(x) = \max \{\lambda_{A_1}(x), \lambda_{L_1}(x)\}$ for all $x \not\in K_2 = A_2 \cup L_2$. The same procedure may be carried out indefinitely, which gives rise to three decreasing sequences of compacta: (1) $A_1 \supset A_2 \supset \cdots$; (2) $L_1 \supset L_2 \supset \cdots$; and (3) $K_1 \supset K_2 \supset \cdots$, where $K_p = A_p \cup L_p$ for $p \geq 1$. For each $p \geq 1$ we have the following equations:

$$\lambda_{K_p}(x) = \begin{cases} 0 & \{x\} \in \mathcal{D}_{K_p} \\ 1 + \lambda_{K_{p+1}}(x) & \text{ otherwise} \end{cases} \quad (x \in K_{p+1}). \quad (3)$$

$$\lambda_{A_p}(x) = \begin{cases} 0 & \{x\} \in \mathcal{D}_{A_p} \\ 1 + \lambda_{A_{p+1}}(x) & \text{ otherwise} \end{cases} \quad (x \in A_{p+1}) \quad (4)$$

$$\lambda_{L_p}(x) = \begin{cases} 0 & \{x\} \in \mathcal{D}_{L_p} \\ 1 + \lambda_{L_{p+1}}(x) & \text{ otherwise} \end{cases} \quad (x \in L_{p+1}) \quad (5)$$

$$\lambda_{K_p}(x) = \max \{\lambda_{A_p}(x), \lambda_{L_p}(x)\} \quad (x \not\in K_{p+1}) \quad (6)$$

There are two possible cases. In the first, we have $K_p = K_{p+1}$ for some $p \geq 1$, indicating that $K_m = K_p$ for all $m \geq p$. In such a case, we have $\lambda_{K_p}(x) = \max \{\lambda_{A_p}(x), \lambda_{L_p}(x)\}$ for all $x \not\in K_{p+1} = K_p$ and $\lambda_{K_p}(x) = \max \{\lambda_{A_p}(x), \lambda_{L_p}(x)\}$ for all $x \in K_p$.

In the second case, we have $K_p \neq K_{p+1}$ for all $p \geq 1$. This implies that $\lambda_K(x) = \max \{\lambda_A(x), \lambda_L(x)\} = \infty$ for all $x$ in $K_\infty = \cap_p K_p$. Therefore, we further have $\lambda_K(x) = \max \{\lambda_A(x), \lambda_L(x)\}$ for all $x \not\in K_\infty$. This completes our proof. \qed
Secondly, we deal with the lambda functions of two planar compacta $K \supset L$ such that $K \setminus L$ has infinitely many components $P_1, P_2, \ldots$ satisfying the following properties:

(P1) for every $n \geq 1$ the closure $\overline{P_n}$ intersects $L$ at a single point $x_n$, and

(P2) for any constant $C > 0$ at most finitely many $\overline{P_n}$ are of diameter greater than $C$.

For the sake of convenience, we further assume that

(P3) $x_n \neq x_m$ for $n \neq m$.

This is the case, when $K$ is the Mandelbrot set $\mathcal{M}$ and $L$ a Baby Mandelbrot set. Other choices of $L$ include: (1) the closure of a hyperbolic component, (2) the closure of $\mathcal{P}_0$, which consists of all the parameters $c \in \mathcal{M}^o$ such that the core entropy of $z \mapsto z^2 + c$ is zero.

As an extension of Theorem 4 we obtain the following.

**Theorem 5.1.** Given two planar compacta $K \supset L$ that satisfy (P1) to (P3), we have

$$\lambda_K(x) = \begin{cases} 
\lambda_{\overline{P_n}}(x) & x \in P_n \text{ for some } n \\
\lambda_L(x) & x \in L \setminus \{x_n : n \in \mathbb{N}\} \\
\max \{\lambda_L(x_n), \lambda_{\overline{P_n}}(x_n)\} & x = x_n \text{ for some } x_n \\
0 & \text{otherwise}
\end{cases} \quad (7)$$

**Proof.** Since $\lambda_K(x) = 0$ for $x \notin K$, we only need to obtain the following equation for all $x \in K$

$$\lambda_K(x) = \begin{cases} 
\lambda_{\overline{P_n}}(x) & x \in P_n \text{ for some } n \quad \text{(case 1)} \\
\lambda_L(x) & x \in L \setminus \{x_n : n \geq 1\} \quad \text{(case 2)} \\
\max \{\lambda_L(x_n), \lambda_{\overline{P_n}}(x_n)\} & x = x_n \text{ for some } x_n \quad \text{(case 3)}
\end{cases} \quad (8)$$

In the rest part of our proof, for all $n, n_1 \geq 1$ denote by $d_n$ the atom of $\overline{P_n}$ that contains $x_n$, and by $e_{n_1}$ the atom of $L$ that contains $x_{n_1}$. Moreover, let $e'_{n_1}$ be the union of $e_{n_1}$ with all those $d_n$ that intersects $e_{n_1}$. Then

$$\mathcal{D}_K = (\mathcal{D}_L \setminus \{e_{n_1} : n_1 \geq 1\}) \cup \{e'_{n_1} : n_1 \geq 1\} \cup \left[ \bigcup_n (\overline{D_{P_n}} \setminus \{d_n\}) \right].$$
Clearly, we have \( \lambda_K(x) = \lambda_L(x) \) for all \( x \in L \) that is off \( \bigcup_{n_1} \epsilon_{n_1} \). Similarly, \( \lambda_K(x) = \lambda_{\overline{P}_n}(x) \) for all \( x \in (P_n \setminus d_n) \). Moreover, for any \( n_1 \geq 1 \) and any point \( x \in \epsilon'_{n_1} \), if \( \{ x \} = \epsilon'_{n_1} \) we have \( \lambda_K(x) = 0 \); otherwise, we have \( \lambda_K(x) = \lambda_{\epsilon'_{n_1}}(x) + 1 \).

For each \( n_1 \geq 1 \), let \( I_{n_1} \) be the collection of integers \( n \geq 1 \) such that \( x_n \in \epsilon'_{n_1} \). In the sequel, we only need to consider the continua \( \epsilon'_{n_1} \), that are non-degenerate. Clearly, the atoms of such a continuum must are divided into two families. Every atom in the first family is either an order two atom of some \( \overline{P}_n \) that is disjoint from \( \{ x_n \} \), or an order two atom of \( L \) that is disjoint from \( \{ x_n : n \geq 1 \} \). Such an atom is called an atom of pure type; every of the other atoms is called an atom of exceptional type or entangled type.

Each exceptional atom of \( \epsilon'_{n_1} \) is the union of an order two atom of \( L \), say \( d \), with the unique order two atom of \( \overline{P}_n \) containing \( x_n \) for all integer(s) \( n \in I_{n_1} \) with \( x_n \in d \). Every \( \epsilon'_{n_1} \) has at most countably many atoms of exceptional type. Such an atom will be generally denoted as \( \epsilon'_{n_1n_2}(n_2 \geq 1) \). Given a non-degenerate \( \epsilon'_{n_1} \) and a point \( x \in \epsilon'_{n_1} \), we shall have

\[
\lambda_K(x) = \begin{cases} 
\lambda_{\overline{P}_n}(x) & x \in P_n \text{ and it lies in an atom of pure type} \\
\lambda_L(x) & x \in L \text{ and it lies in an atom of pure type} \\
1 + \lambda_{\epsilon'_{n_1}}(x) & x \text{ lies in an atom of exceptional type} 
\end{cases}
\]  

(9)

Now, we only need to consider the exceptional atoms \( \epsilon'_{n_1n_2} \) of \( \epsilon'_{n_1} \), each of which is an order two atom of \( K \). And we only need to consider those \( \epsilon'_{n_1n_2} \) that are non-degenerate. Again, every of those \( \epsilon'_{n_1n_2} \) is a continuum that allows two types of atoms, the pure type and the exceptional type. The same arguments, that have been used in obtaining Equation (9), will lead us to the following equation, which holds for all \( x \in \epsilon'_{n_1n_2} \).

\[
\lambda_K(x) = \begin{cases} 
\lambda_{\overline{P}_n}(x) & x \in P_n \text{ and it lies in an atom of pure type} \\
\lambda_L(x) & x \in L \text{ and it lies in an atom of pure type} \\
2 + \lambda_{\epsilon'_{n_1n_2}}(x) & x \text{ lies in an atom of exceptional type} 
\end{cases}
\]  

(10)

Inductively, we can extend the above equations to atoms of \( K \) that are of higher and higher orders. All those atoms, regardless of their orders, will be either of pure type of
exceptional type. Given an order, there are at most countably many atoms of exceptional type, no two of which have common points. Clearly, our theorem holds for all points that are contained in at most finitely many exceptional atoms. In deed, such a point does not belong to
\[ \bigcup_{n_1n_2...n_p} e'_{n_1n_2...n_p} \]
for some \( p \geq 1 \) and hence falls into either (case 1) or (case 2) in Equation (8). Every other point \( x \in K \) necessarily lies in the exceptional atoms \( e'_{n_1n_2...n_p} \) for infinitely many \( p \). The decreasing sequence \( e'_{n_1n_2...n_p} \) converges to a continuum \( M_x \). There are three possibilities. If \( e'_{n_1n_2...n_p} \) is a single point for some \( p \geq 1 \) then \( x = x_n \) for some \( n \) and \( \lambda_K(x) = \max\{\lambda_L(x_n), \lambda_{P_n}(x_n)\} \) is finite. If \( M_x \) is a single point and \( e'_{n_1n_2...n_p} \neq M_x \) for all \( p \geq 1 \) then \( x = x_n \) for some \( n \) and \( \lambda_K(x) = \max\{\lambda_L(x_n), \lambda_{P_n}(x_n)\} \) is infinite. Otherwise, we have \( x \in (L \cup P_n) \) for some \( n \in I_{n_1} \) and \( \lambda_K(x) = \infty = \max\{\lambda_L(x), \lambda_{P_n}(x)\} \).

6 Some Examples

In this section we construct concrete examples that are related to some of our results, obtained in earlier sections. Usually, such an example provides hints or some evidence that particular assumptions in our theorems can not be removed or be further relaxed.

6.1 When Gluing Lemma does not work

In the following, we give specific choices of compacta \( A, B \subset \hat{C} \) that do not satisfy the conditions in Theorem 4. In such cases we can not infer \( \lambda_K(x) = \max\{\lambda_A(x), \lambda_B(x)\} \) for all \( x \), although we have \( \lambda_A(x) = \lambda_B(x) \) for all \( x \) in \( A \cap B \).

The first example provides simple planar compacta \( A, B \) with \( A \cap B \) uncountable.

**Example 6.1.** Let \( A = \{t + si : t \in K, 0 \leq s \leq 1\} \), where \( K \) is the Cantor ternary set. Let \( B = \{t + (1 + s)i : 0 \leq t \leq 1, s \in K\} \). Let \( A_1 = A \cup B \) and \( B_1 = (A + 1 + i) \cup (B + 1 - i) \). See the following Figure 4 for a simplified depiction of \( A, B, A_1, B_1 \).
Then $\lambda_A(x) = 1$ for all $x \in A$ and vanishes otherwise; similarly, $\lambda_B(x) = 1$ for all $x \in B$ and vanishes otherwise. That is to say, we have

$$
\lambda_A(x) = \begin{cases} 
1 & x \in A \\
0 & \text{otherwise}
\end{cases}
$$

and

$$
\lambda_B(x) = \begin{cases} 
1 & x \in B \\
0 & \text{otherwise.}
\end{cases}
$$

However, both $A \cap B$ and $A_1 \cap B_1$ are uncountable, thus the conditions in Theorem 4 are not satisfied. Moreover, we have

$$
\lambda_{A_1}(x) = \lambda_{A \cup B}(x) = \begin{cases} 
2 & x \in A \\
1 & B \setminus A \\
0 & \text{otherwise}
\end{cases}
$$

and

$$
\lambda_{A_1 \cup B_1}(x) = \begin{cases} 
\infty & x \in (A_1 \cup B_1) \\
0 & \text{otherwise.}
\end{cases}
$$

The second example gives planar compacta $A, B$ with $A \cap B$ countably infinite.

**Example 6.2.** Let $B$ be as given in Example 6.1. Let $A = \bigcup_{n \geq 0} A_n$. Here $A_0$ is the segment $\{si : 0 \leq s \leq 1\}$, $A_1$ is the continuum that consists of the line $\{1 + ti : 0 \leq t \leq 1\}$ and all those lines connecting $1 + i$ to $\frac{k}{k+1}$ for $k \geq 1$. For $n \geq 2$ the continuum $A_n$ is given by $\{2^{-n+1}t + si : t + si \in A_1\}$. See the following Figure 6. Then $A \cap B$ contains countably many points $i, 2^{-n} + i (n \geq 1)$. At each of those points the two lambda functions $\lambda_A, \lambda_B$ each take the value 1. Let $L_1 = \{t+i : t=0 \text{ or } 2^{-n} \text{ for some } n \geq 0\}$ and

$$
d = \{t + si : 0 \leq s \leq 1, t = 0 \text{ or } 2^{-n} \text{ for some } n \geq 0\} \cup \{t + i : 0 \leq t \leq 1\}.
$$

Then, $d$ is an atom of $A \cup B$ which is not locally connected; moreover, we have

$$
\lambda_A(x) = \begin{cases} 
1 & x \in L_1 \\
0 & \text{otherwise}
\end{cases}
$$

and

$$
\lambda_{A \cup B}(x) = \begin{cases} 
2 & x \in A_0 \\
1 & x \in (B \cup d) \setminus A_0 \\
0 & \text{otherwise.}
\end{cases}
$$
6.2 The Lambda Equality may fail for an E-compactum

This subsection constructs two concrete E-continua $K \subset \hat{\mathbb{C}}$. In the first one, we have $\lambda_K(x) - \tilde{\lambda}_K(x) = 1$ for $x$ lying on a segment. In the second one, we have $\lambda_K(x) = \infty$ and $\tilde{\lambda}_K(x) = 1$ for all $x$ lying on a sub-continuum of $K$. Of course, these continua satisfy none of the assumptions in part (i), (ii), (iii) of Theorem 3.

Example 6.3. Let $X$ denote the square $[1, 2] \times [0, i] \subset \hat{\mathbb{C}}$. Let $Y$ be an embedding of $[0, \infty)$ whose closure $\overline{Y}$ equals the union of $Y$ with $\partial X$. See the left part of Figure 6 for a simplified representation of $\overline{Y}$, which is depicted as blue. Let $f_1(z) = \frac{z}{2}$ and $f_2(z) = \frac{z+1}{2}$. Let $K_0 = \overline{Y}$. For all $n \geq 1$, let $K_n = f_1(K_{n-1}) \cup f_2(K_{n-1})$. Then $K_0, K_1, \ldots$ is an infinite sequence of continua converging to the segment $[0, i]$ under Hausdorff distance. Clearly,

$$K = \left( \bigcup_{n \geq 0} K_n \right) \cup [0, i]$$

is an E-continuum. See left part of Figure 6. Let $L_0 = \partial X$. For all $n \geq 1$, let $L_n = f_1(L_{n-1}) \cup f_2(L_{n-1})$. Then $L_0, L_1, \ldots$ is an infinite sequence of continua converging to
the segment \([0, i]\) under Hausdorff distance. Similarly, we see that
\[
L = \left( \bigcup_{n \geq 0} L_n \right) \cup [0, i]
\]
is also an \(E\)-continuum. See right part of Figure 6. Moreover, the continuum \(K\) has exactly one atom of order 1 that is not a singleton. This atom equals \(L\). Now it is routine to verify
\[
\lambda_K(x) = \begin{cases} 
1 & x \in L \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad \tilde{\lambda}(x) = \begin{cases} 
0 & x \in [0, i] \\
1 & x \in L \setminus [0, i] \\
0 & \text{otherwise}.
\end{cases}
\]
That is to say, we have \(\lambda_K(x) - \tilde{\lambda}_K(x) = \begin{cases} 
1 & x \in [0, i] \\
0 & \text{otherwise}.
\end{cases}\)

**Example 6.4.** Let \(C\) denote Cantor’s ternary set. Let \(U_1 \subset \hat{C}\) be the domain, not containing \(\infty\), whose boundary consists of two parts: one is the product \(C \times [0, i] = \{ t + si : 0 \leq t \leq 1, s \in C \}\) and the other is the boundary of \([0, \frac{4}{3}] \times [0, i]\). See the left part of Figure 7 for a simplified depiction of \(\partial U_1\). For \(2 \leq i \leq 4\) let \(U_i = f^i(U_1)\) be congruent copies of \(U_1\), where \(f(z) = iz\). See the left part of Figure 7. Then \(K = \bigcup_i \partial U_i\) is a continuum, whose complementary components are \(U_1, \ldots, U_4, U_5\). Here \(U_5\) is the one containing \(\infty\). Moreover, the only non-degenerate atom of \(K\) is
\[
d := C \times [0, i] \cup f_2(C \times [0, i]) \cup f_3(C \times [0, i]) \cup f_4(C \times [0, i]).
\]
Since the continuum \(d\) has a single atom, which is itself, we have
\[
\lambda_K(x) = \begin{cases} 
\infty & x \in d \\
0 & \text{otherwise}.
\end{cases}
\]
On the other hand, by the construction of \( U_1, \ldots, U_4 \) and \( U_5 \), we also have

\[
\tilde{\lambda}_K(x) = \begin{cases} 
1 & x \in d \\
0 & \text{otherwise}. 
\end{cases}
\]

Consequently, we have

\[
\lambda_K(x) - \tilde{\lambda}_K(x) = \begin{cases} 
\infty & x \in d \\
0 & \text{otherwise}. 
\end{cases}
\]

### 6.3 Comparing \( \lambda_K \) with \( \lambda_{\partial K} \) for a compactum \( K \subset \hat{\mathbb{C}} \)

In the following we give concrete examples of planar compacta \( K \), to illustrate the various situations that may happen, concerning the relations between \( \lambda_K \) and \( \lambda_{\partial K} \).

**Example 6.5.** Consider a spiral made of broken lines, as shown in the left of Figure 8, that converges to the boundary of the open square \( W = \{ t + si : 0 < t, s < 1 \} \subset \hat{\mathbb{C}} \).

We may thicken the spiral to an embedding \( h : [0, \infty) \times [0, 1] \rightarrow W \), of the unbounded strip \( U = [0, \infty) \times [0, 1] \). Such an embedding may be chosen appropriately, so that \( h(\partial U) \) consists of countably many segments. Then, we obtain a continuum \( K_0 = W \setminus h(U) \).

See the middle part of Figure 8. Clearly, the continuum \( K_0 \) is not locally connected on \( \partial W \); and it is locally connected at all the other points. Now, divide the thickened spiral \( h(U) \) into smaller and smaller quadruples, which are depicted in the right most part of Figure 8 as small rectangles. Let \( K \) be the union of \( K_0 \) with the newly added bars, used in the above mentioned division. Then \( K \) is locally connected everywhere hence is a Peano continuum. However, its boundary \( \partial K \) is not locally connected on \( \partial W \) and is locally connected.

![Figure 8: A Peano continuum K whose boundary is not locally connected.](image)
connected elsewhere. Therefore, we have

\[ \lambda_K(x) \equiv 0 \quad \text{and} \quad \lambda_{\partial K}(x) = \begin{cases} 1 & x \in \partial W \\ 0 & x \notin \partial W. \end{cases} \]

**Example 6.6.** Let the continuum \( K \) be defined as in Example 6.5. Let \( f_j(z) = \frac{z}{2} + \frac{j-1}{2}i \) for \( j = 1, 2 \). For any compact set \( X \subset \hat{\mathbb{C}} \), put \( \Phi(X) = f_1(X) \cup f_2(X) \). We will use the continuum \( K \) and the mapping \( \Phi \) to construct another continuum \( L \). See Figure 9.

![Figure 9](image.png)

Figure 9: Relative locations of \( \partial K + 1, \Phi^1(\partial K + 1) \) and \( \Phi^2(K + 1) \) in the continuum \( L \).

The above continuum \( L \) consists of the following parts:

1. the segment \([0, i]\);
2. a spiral converging to the boundary of \([0, 2] \times [0, i]\);
3. \( \partial K + 1 = \{z + 1 : z \in \partial K\} \);
4. \( \Phi^{2n}(K + 1) \) for all integers \( n \geq 1 \); and
5. \( \Phi^{2n-1}(\partial K + 1) \) for all integers \( n \geq 1 \).

On the one hand, we can check that \( L \) has a unique non-degenerate atom \( d \), which consists of the following parts:

1. the segment \([0, i]\);
2. the boundary of \([1, 2] \times [0, i]\), denoted as \( A \);
3. \( \Phi^{2n-1}(A) \) for all integers \( n \geq 1 \); and

4. the boundary of \([2^{-2n}, 2^{-2n+1}] \times [0, i] \) for all integers \( n \geq 1 \).

On the other hand, the boundary \( \partial L \) has a unique non-degenerate atom \( d^* \), which is the union of \( A \), the segment \([0, i] \), and \( \Phi^n(A) \) for all integers \( n \geq 1 \). See Figure 10 for a depiction of the atoms \( d \) and \( d^* \). Clearly, the atom \( d^* \) contains \( d \) and is a Peano continuum, while \( d \) is not locally connected on \([0, i] \) and is locally connected elsewhere.

Therefore, we can compute their lambda functions as follows:

\[
\lambda_L(x) = \begin{cases} 
2 & x \in [0, i] \\
1 & x \in d \setminus [0, i] \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad \lambda_{\partial L}(x) = \begin{cases} 
1 & x \in [0, i] \\
1 & x \in d^* \setminus [0, i] \\
0 & \text{otherwise}
\end{cases}
\]

From these we further infer that

\[
\lambda_L(x) - \lambda_{\partial L}(x) = \begin{cases} 
1 & x \in [0, i] \\
-1 & x \in d^* \setminus d \\
0 & \text{otherwise}
\end{cases}
\]

Remark 6.7. As far as we know, no compactum \( K \subset \hat{C} \) with \( \lambda_K \geq \lambda_{\partial K} \) has been found such that \( \lambda_K(x) > \lambda_{\partial K}(x) \) for at least one point \( x \in \partial K \).

6.4 On \( \lambda_{X \cup Y} \) and \( \lambda_{X \cap Y} \) for Peano compacta \( X, Y \subset \hat{C} \).

This subsection considers unions and intersections of specific Peano compacta in the plane.
Firstly, if $X$ and $Y$ are Peano continua in the plane with $X \cap Y \neq \emptyset$ then $X \cup Y$ is also a Peano continuum. However, the intersection $X \cap Y$ may not be locally connected, even when it is connected. See Example 6.8.

**Example 6.8.** Let $M$ be the union of $[0, 1] \times \{0\}$ with the vertical segments $\{0\} \times [0, 1]$ and $\{2^{-k}\} \times [0, 1]$ for integers $k \geq 0$. Then $M$ is a continuum and is not locally connected at points on $\{0\} \times (0, 1]$; moreover, we have

$$\lambda_M(x) = \begin{cases} 1 & x \in \{ti : 0 \leq t \leq 1\} \\ 0 & \text{otherwise} \end{cases}$$

We will construct two Peano continua $X$ and $Y$ satisfying $X \cap Y = M$. To this end, for all integers $k \geq 1$ we put

$$A_k = \bigcup_{j=1}^{2k-1} [0, 2^{-k+1}] \times \{j2^{-k}\}.$$  

Then $X = M \cup (\bigcup_k A_k)$ is a Peano continuum. See left part of Figure 11 for a rough approximate of $X$. Similarly, if for every $k \geq 1$ we set

$$B_k = \bigcup_{j=1}^{3k-1} [0, 2^{-k+1}] \times \{j3^{-k}\},$$

then $Y = M \cup \left(\bigcup_k B_k\right)$ is also a Peano continuum. See right part of Figure 11 for a rough approximate of $Y$. Moreover, we have $X \cap Y = M$.

Secondly, the union $X \cup Y$ of two Peano compacta $X, Y \subset \hat{\mathbb{C}}$ may not be a Peano compactum, although the intersection $X \cap Y$ is. Indeed, we will use special self-similar

![Figure 11: Two Peano continua that intersect at a non-locally connected continuum.](image-url)
sets, the so called fractal squares, to construct two such compacta. Here, given an integer \( n \geq 2 \), a fractal square is the attractor \( F \) of an iterated function system

\[
F_D := \left\{ f_d(x) = \frac{x + d}{n} : d \in D \right\}
\]

for some \( D \subset \{0, 1, \ldots, n - 1\}^2 \) which contains at least 2 and at most \( n^2 - 1 \) elements. For general theory on iterated function systems, we refer to [8].

**Example 6.9.** Let \( X \) be the fractal square determined by \( F_{D_X} \), where \( D_X = \{(i, 0) : i = 1, 2, 3\} \cup \{(0, 2)\} \). Let \( Y \) be the fractal square determined by \( F_{D_Y} \), where \( D_Y = \{(i, 0) : i = 1, 2, 3\} \cup \{(1, 2), (2, 2)\} \). See Figure 12 for relative locations of the small squares \( f_d([0, 1]^2) \) with \( d \in D_X \) and \( d \in D_Y \). Clearly, \( X \) and \( Y \) are Peano compacta, each of which contains the interval \([0, 1]\), such that \( X \cup Y \) contains all the segments \([0, 1] \times \left\{ \frac{2}{3^k} \right\} \) for \( k \geq 1 \).

Here, it is routine to check that \( X \cap Y \) is a Peano compactum having a countably infinite number of components. All but one of these components are single points and the only non-degenerate one is exactly the interval \([0, 1]\). On the other hand, for all \( k \geq 1 \) the horizontal strip \( \mathbb{R} \times (\frac{1}{3^k}, \frac{2}{3^k}) \) is disjoint from \( X \cup Y \). This implies that \( X \cup Y \) is not a Peano compactum. Moreover, we have

\[
\lambda_{X \cup Y}(x) = \begin{cases} 
1 & x \in [0, 1] \\
0 & \text{otherwise}. 
\end{cases}
\]

### 7 Appendix: The Lambda Tree for Planar Compacta

The definition of \( \lambda_K \) arises from the hierarchy structure for atoms of atoms, which may be represented by a geometric graph \( \Lambda_K \), called the lambda tree associated to \( K \). This
section is helpful for those who want to build some intuition about the lambda function \( \lambda_K \) for specific compacta \( K \subset \hat{\mathbb{C}} \).

The root of \( \Lambda_K \) is a node marked as \( K \), which branches out to atoms of order 1. Each atom of order 1, if it is not a singleton, branches out to its own atoms, each of which is an order 2 atom of \( K \).

This procedure is then repeated indefinitely, unless we reach an atom that is a singleton. That is to say, the branching process stops if we reach a degenerate atom. In this graph, two nodes are connected by an edge in \( \Lambda_K \) if and only if one is an atom of the other. Moreover, if \( \{x\} \) is a node and if the graph distance from the root to \( \{x\} \) is \( m \) then \( \lambda_K(x) = m - 1 \).

Given a compactum \( K \subset \hat{\mathbb{C}} \). Let \( V^1_K \) denote the set of order 1 atoms of \( K \). For \( n \geq 2 \), let \( V^n_K \) denote the set of order \( n \) atoms \( d \), except for those \( d \) such that \( d \) is a singleton and is also an order \( n - 1 \) atom. Set the set of nodes for \( \Lambda_K \) to be \( \mathcal{V}_K = \{ K \} \cup \left( \bigcup_{n \geq 1} \mathcal{V}^n_K \right) \). Then the set of edges \( \mathcal{E}_K \) has an element between two nodes if and only if one is an atom of the other.

We will equip the lambda tree \( \Lambda_K \) with a topology \( T_K \), that is quite different from the graph topology. This topology is the smallest one on \( \Lambda_K \) such that

- for the sub-tree consisting of the root and all the order-1 atoms of \( K \), all closed subsets of the cone over the Peano model of \( K \) are closed in \( T_K \);

- for an order-\( n \) atom \( d \) and the sub-tree with root \( d \) whose other vertices are just the order-1 atoms of \( d \), all closed subsets of the cone over the Peano model of \( d \) are closed in \( T_K \).

Checking a few simple examples will suffice to demonstrate how to compute the lambda tree \( \Lambda_K \) and why it is a helpful illustration of the above hierarchy structure. For instance, if \( K \) is Cantor’s ternary set then \( \Lambda_K \) is topologically equivalent to the cone over Cantor’s ternary set. See Figure 13.
If $K$ is a singleton then $\Lambda_K$ is a single point. If $K$ is Cantor’s Teepee, or an indecomposable continuum like the pseudo-arc then $\Lambda_K$ is an infinite path starting from the node.

If $K$ is the closed topologist’s sine curve the lambda tree $\Lambda_K$ is topologically equivalent to the union of two triangles intersecting at a common vertex, as depicted in the right part of Figure 14.

Since the core decomposition $D_K$ is totally determined by the topology of $K$, we see that the lambda tree is a topological invariant of planar compacta, in the sense that two homeomorphic compacta $K, L \subset \hat{C}$ have lambda trees that are equivalent as graphs and are homeomorphic with respect to the above topology. In such a case we say the two lambda trees are isomorphic.

However, it is possible for two planar compacta $K, L$ that are not homeomorphic to have isomorphic lambda trees. Indeed, by adding two horizontal bars to the left of the closed topologist’s sine curve, we may obtain three distinct compacta. See Figure 15. No two of these compacta are homeomorphic, but their lambda trees are isomorphic to one another. In particular, they share the same Peano model, which is equivalent to a $T-$shape or $Y-$shape. See the right most part of Figure 15 for a simple depiction of the
common lambda tree.

Figure 15: Three topologically different compacta that have isomorphic lambda trees.

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