Automorphisms of Non-Singular Nilpotent Lie Algebras

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Abstract. For a real, non-singular, 2-step nilpotent Lie algebra \( n \), the group \( \text{Aut}(n)/\text{Aut}_0(n) \), where \( \text{Aut}_0(n) \) is the group of automorphisms which act trivially on the center, is the direct product of a compact group with the 1-dimensional group of dilations. Maximality of some automorphism groups of \( n \) follows and is related to how close is \( n \) to being of Heisenberg type. For example, at least when the dimension of the center is two, \( \dim \text{Aut}(n) \) is maximal if and only if \( n \) is of Heisenberg type. The connection with fat distributions is discussed.

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1. Introduction

A 2-step nilpotent real Lie algebra \( n \) with center \( z \) is called non-singular [E], or said to satisfy hypothesis \( (H) \) [M], or be the Lie algebra of a Métivier group [MS], if \( \text{ad} \, x : n \to z \) is onto for any \( x \not\in z \). Equivalently, the bracket defines a vector-valued antisymmetric form

\[
[ , ] : v \times v \to z,
\]

\( v = n/z \), such that the 2-forms \( \lambda([u, v]) \) on \( v \) are non-degenerate for all \( \lambda \in z^* \), \( \lambda \neq 0 \). Here we shall call such algebras fat for short, since they are the symbols of fat distributions (as opposite to ”flat”, or integrable, ones [Mo]), which motivate the questions.

Let \( m = \dim(z) \). While for \( m = 1 \) there is only one fat algebra up to isomorphisms, for \( m \geq 2 \) there is an uncountable number of isomorphism classes and for \( m \geq 3 \) they form a wild set.

In this paper we study the size of groups of automorphisms of \( n \). \( \text{Aut}(n) \) itself is the semidirect product of the group \( G(n) \) of graded automorphisms of \( n = v \oplus z \) with the abelian group \( \text{Hom}(v, z) \), times the group of dilations \((t, t^2)\). Hence, we concentrate on \( G = G(n) \).

We prove that there is an exact sequence

\[
1 \to G_0 \to G \to O(m)
\]
where $G_0$ is the subgroup of $G$ of elements that act trivially on the center. In other words, there are positive-definite inner products on $\mathfrak{z}$ which are invariant under all of $\text{Aut}(\mathfrak{n})$.

If a metric $g$ is also given on $\mathfrak{v}$, as in the case of the nilpotentization of a subriemannian structure, we also consider the subgroups $K_0$, $K$, of graded automorphisms that leave $g$ invariant, which define a compatible exact sequence

$$1 \to K_0 \to K \to O(m).$$

Next, we compute the terms in this sequence and the images $G/G_0$ and $K/K_0$, proving that the exactness of

$$1 \to \text{Lie}(K_0) \to \text{Lie}(K) \to \mathfrak{so}(m) \to 1$$

is equivalent to $\mathfrak{n}$ being of Heisenberg type, while the exactness of

$$1 \to \text{Lie}(G_0) \to \text{Lie}(G) \to \mathfrak{so}(m) \to 1$$

is strictly more general. As to $G_0(\mathfrak{n})$, we describe it in detail for the case $m = 2$, leading a proof that, at least in that case, $\dim \text{Aut}(\mathfrak{n})$ is maximal if and only if $\mathfrak{n}$ is of Heisenberg type.

In the last section we explain the connection with the Equivalence Problem for fat subriemannian distributions.

Algebras of Heisenberg type are defined as follows [K]. If $\mathfrak{v}$ is a real unitary module over the Clifford algebra $\text{Cl}(\mathfrak{z})$ associated to a quadratic form on $\mathfrak{z}$, the identity

$$<z, [u, v]>_{\mathfrak{z}} = <z \cdot u, v>_{\mathfrak{v}}$$

with $z \in \mathfrak{z} \subset \text{Cl}(\mathfrak{z})$, $u, v \in \mathfrak{v}$, defines a fat $[\cdot, \cdot] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$. Alternatively, they are characterized by possessing a positive-definite metric such that the operator $z \cdot$ defined by the above equation satisfies $z \cdot (z \cdot v) = -|z|^2 v$.

It follows from Adam’s theorem on frames on spheres [H] that for any fat algebra there is an Heisenberg type algebra with the same $\dim \mathfrak{z}$ and $\dim \mathfrak{v}$. That these were, in some sense, the most symmetric, was expected from the properties of their sublaplacians [BTV] [CGN] [GV] [K], but we found no explicit statements in this regard.

Related properties of the automorphism groups of nilpotent Lie groups are studied in [P] and [MS].

2. Automorphisms of fat algebras

Let $\mathfrak{n}$ be a 2-step Lie algebra with center $\mathfrak{z}$ and let $\mathfrak{v} = \mathfrak{n}/\mathfrak{z}$, so that

$$\mathfrak{n} \cong \mathfrak{v} \oplus \mathfrak{z}$$

and the Lie algebra structure is encoded into the map

$$[\cdot, \cdot] : \Lambda^2 \mathfrak{v} \to \mathfrak{z}.$$
Let $n = \dim \mathfrak{v}$ and $m = \dim \mathfrak{z}$. Relative to a basis compatible with (1), the bracket becomes an $\mathbb{R}^m$-valued antisymmetric form on $\mathbb{R}^n$ and an automorphism is a matrix of the form
\[
\begin{pmatrix}
a & 0 \\
c & b
\end{pmatrix}, \quad a \in \text{GL}(n), \ b \in \text{GL}(m), \ c \in \mathbb{R}^{n \times m}
\]
such that
\[b([u, v]) = [au, av].\]
\text{Aut}(\mathfrak{n})\] always contains the normal subgroup $\mathfrak{D}(\mathfrak{n})$ of dilations and translations
\[
\begin{pmatrix}
tI_n & 0 \\
c & t^2I_m
\end{pmatrix}, \quad t \in \mathbb{R}^*, \ c \in \mathbb{R}^{n \times m}.
\]
Let

$$
G = G(\mathfrak{n}) = \begin{cases}
\begin{pmatrix}a & 0 \\
0 & b
\end{pmatrix}, & a \in \text{SL}(n), \ b \in \text{GL}(m), \ b([u, v]) = [au, av].
\end{cases}
$$

Then $\text{Aut}(\mathfrak{n})$ is the semidirect product of $G(\mathfrak{n})$ with $\mathfrak{D}(\mathfrak{n})$. Let

$$
G_0 = G_0(\mathfrak{n}) = \begin{cases}
\begin{pmatrix}a & 0 \\
0 & I_m
\end{pmatrix}, & a \in \text{SL}(n), \ [au, av] = [u, v],
\end{cases}
$$

the subgroup of automorphisms that act trivially on the center. These are Lie groups, $G_0$ is normal in $G$, and the quotient group

$$
G/G_0
$$

can be identified with the group of $b \in \text{GL}(\mathfrak{z})$ such that $b([u, v]) = [au, av]$ for some $a \in \text{SL}(\mathfrak{v})$. Obviously,

$$
\dim \text{Aut}(\mathfrak{n}) = nm + 1 + \dim(G/G_0) + \dim(G_0).
$$

\textbf{Theorem 2.1.} Let $\mathfrak{n}$ be a fat algebra with center $\mathfrak{z}$. Then there is a positive definite metric on $\mathfrak{z}$ invariant under $G(\mathfrak{n})$.

\textbf{Proof.} Fix arbitrary positive inner products on $\mathfrak{v}$ and $\mathfrak{z}$. For $z \in \mathfrak{z}$, $u, v \in \mathfrak{v}$

$$
(T_z u, v)_{\mathfrak{v}} = (z, [u, v])_{\mathfrak{z}}
$$

defines a linear map $z \mapsto T_z$ from $\mathfrak{z}$ to $\text{End}(\mathfrak{v})$. Clearly,

$$
\mathfrak{n} \text{ fat } \iff T_z \in \text{GL}(\mathfrak{v}) \ \forall z \neq 0.
$$

Hence the hypothesis insures that the Pfaffian

$$
P(z) = \det(T_z)
$$
is non-zero on $\mathfrak{z} \setminus \{0\}$. This is a homogeneous polynomial of degree $n$, so it satisfies

$$
k\|z\|^n \leq |P(z)| \leq K\|z\|^n
$$

(3)
where $k, K$ are the minimum and maximum values of $|P|$ on the unit sphere, which are positive.

Let now $g_{a,b} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \text{Aut}(n)$. Then

$$T_{b^*z} = a^*T_za$$

because $(T_{b^*z}u, v)_v = (b^*z, [u, v])_z = (z, b([u, v]))_z = (T_zau, av)_v = (a^*T_zau, v)_v$. Consequently

$$P(b^*z) = (\det a)^2P(z).$$

In particular, if $g \in G$ then $P(b^*z) = P(z)$. This implies

$$k\|b^*z\|^n \leq |P(b^*z)| = |P(z)| \leq K\|z\|^n$$

for all $z$, therefore $\|b\| \leq \sqrt{K/k}$. The group of $b \in \text{GL}(\mathfrak{z})$ such that $g_{a,b} \in \text{Aut}(n)$ for some $a \in \text{SL}(\mathfrak{v})$, is therefore bounded in $\text{End}(\mathbb{R}^m)$. Its closure is a compact Lie subgroup of $\text{GL}(\mathfrak{z})$, necessarily contained in $\text{O}(\mathfrak{z})$ for some positive definite metric.

From now on $\mathfrak{z}$ will be assumed endowed with such invariant metric. If a metric $g$ on $\mathfrak{v}$ is also fixed, as in the case of the nilpotentization of a subriemannian structure, define the groups

$$K = K(n, g) = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \in \text{SO}(\mathfrak{v}), b \in \text{O}(\mathfrak{z}), [au, av] = b[u, v] \}$$

$$K_0 = K_0(n, g) = \{ \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, a \in \text{SO}(\mathfrak{v}), [au, av] = [u, v] \}.$$

Let $\mathfrak{g}, \mathfrak{g}_0, \mathfrak{k}, \mathfrak{k}_0$ be the Lie algebras of $G, G_0, K, K_0$ respectively. Then there is the commutative diagram with exact rows

$$
\begin{array}{c}
0 \rightarrow \mathfrak{g}_0 \rightarrow \mathfrak{g} \rightarrow \mathfrak{so}(m) \\
\uparrow \quad \uparrow \quad \uparrow \\
0 \rightarrow \mathfrak{k}_0 \rightarrow \mathfrak{k} \rightarrow \mathfrak{so}(m)
\end{array}
$$

where the vertical arrows are the inclusions. Below we prove that the bottom sequence extends to

$$0 \rightarrow \mathfrak{k}_0 \rightarrow \mathfrak{k} \rightarrow \mathfrak{so}(m) \rightarrow 0$$

if and only if $\mathfrak{n}$ is of Heisenberg type. This is not the case for the top one: the condition that

$$0 \rightarrow \mathfrak{g}_0 \rightarrow \mathfrak{g} \rightarrow \mathfrak{so}(m) \rightarrow 0$$

is exact defines a class of fat algebras strictly larger than Heisenberg type algebras. We describe it in the next section for $m = 2$.

**Proposition 2.2.** Let $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$ be an algebra of Heisenberg type. There is a metric on $\mathfrak{z}$ such that $\mathfrak{g}/\mathfrak{g}_0 \cong \mathfrak{so}(m)$. 

Proof. There is an inner product in \( \mathfrak{v} \) such that the \( J_i = T_i \)'s satisfy the Canonical Anticommutation Relations
\[
J_w J_z + J_z J_w = -2 < z, w > I.
\]
For \( \|z\| = 1 \) let \( r_z \in O(\mathfrak{z}) \) be the reflection through the hyperplane orthogonal to \( z \) and \( J_z \in \text{SL}(\mathfrak{v}) \) be as above. Then
\[
g(J_z, -r_z) = \begin{pmatrix} J_z & 0 \\ 0 & -r_z \end{pmatrix} \in \text{Aut}(\mathfrak{n}).
\]
Indeed,
\[
(w, [J_z u, J_z v]) = (J_w, J_z u, J_z v) = (-J_z J_w u - 2(z, w)u, J_z v)
\]
\[
= -(J_z J_w u, J_z v) - 2(z, w)(u, J_z v) = (J_w u, J_z J_z v) + 2(z, w)(J_z u, v)
\]
\[
= -(J_w u, v) + 2(z, w)(J_z u, v) = (J_{-w + 2(z, w)z} u, v)
\]
\[
= (-w + 2(z, w)z, [u, v]) = (-r_z(u), [u, v])
\]
\[
= (w, -r_z([u, v])),
\]
so that
\[
-r_z([u, v]) = [J_z u, J_z v].
\]
The Lie group generated by the \(-r_z\) has finite index in \( O(\mathfrak{z}) \). \( \blacksquare \)

**Corollary 2.3.** Let \( \mathfrak{n} \) be a fat algebra with center of dimension \( m \). Then
\[
\dim(K/K_0) \leq \dim(G/G_0) \leq m(m - 1)/2
\]
with equality achieved for any Heisenberg type algebra of the same dimension with center of the same dimension.

Since \( \text{Aut}(\mathfrak{n})/\text{Aut}_0(\mathfrak{n}) = (G/G_0) \times (\text{dilations}) \), one obtains

**Corollary 2.4.** Let \( \mathfrak{n} \) be a fat algebra with center of dimension \( m \). Then
\[
\dim(\text{Aut}(\mathfrak{n})/\text{Aut}_0(\mathfrak{n})) \leq 1 + m(m - 1)/2,
\]
with equality achieved for any Heisenberg type algebra of the same dimension and with center of the same dimension.

A converse for Corollary 2.3 is

**Theorem 2.5.** If \( \mathfrak{n} \) is fat with center of dimension \( m \) and
\[
\dim(K/K_0) = m(m - 1)/2
\]
for some metric on \( \mathfrak{v} \), then \( \mathfrak{n} \) is of Heisenberg type.
Proof. The hypothesis implies that $\mathfrak{k}/\mathfrak{k}_0 = \mathfrak{g}/\mathfrak{g}_0 \cong \mathfrak{so}(m)$, so that $K/K_0$ acts transitively among the $|z| = 1$. For $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in this group, $-T_{bz} = aT_z a^{-1}$, hence $T_{bz}^2 = aT_z^2 a^{-1}$. Since $T_z$ is invertible, we can choose the metric such that $T_{z0}^2 = -I$ for any given $z_0$. Therefore $T_z^2 = -I$ for all $|z| = 1$, which implies the assertion. 

Maximal dimension means there are isomorphisms

$$\text{Lie}(K/K_0) = \text{Lie}(G/G_0) \cong \mathfrak{so}(m).$$

Therefore the simply connected covers are isomorphic: $\text{Spin}(m) \cong (G/G_0)e$. The induced homomorphism

$$\text{Spin}(m) \to (G/G_0)e$$

may or may not extend to a homomorphism

$$\text{Pin}(m) \to G/G_0.$$ 

If it does extend, it may or may not be injective, in which case it is an isomorphism. Therefore, among the algebras for which $\dim(G/G_0)$ is maximal, those for which $\text{Pin}(m) \cong G/G_0$ can be regarded as the most symmetric.

**Theorem 2.6.** Suppose $\mathfrak{n}$ is a 2-step graded algebra such that $\text{Aut}(\mathfrak{n})$ contains a copy of $\text{Pin}(m)$ inducing the standard action on $\mathfrak{z}$. Then $\mathfrak{n}$ is of Heisenberg type.

Proof. The assumption implies that there is a linear map $\mathfrak{z} \to \text{End}(\mathfrak{v})$, denoted by $z \mapsto J_z$ such that $J_z^2 = -|z|^2 I$ for all $z$ and

$$[J_z u, J_z v] = r_z([u, v])$$

for $u, v \in \mathfrak{v}$, $z \in \mathfrak{z}$, $|z| = 1$, where $r_z$ is the reflection in $\mathfrak{z}$ with respect of the line spanned by $z$. $\text{Pin}(m)$ is the group generated by the $J_z$’s with $||z|| = 1$, which acts linearly on $\mathfrak{v}$ and is compact. Fix a metric on $\mathfrak{v}$ invariant under it.

We get, as in the proof of Theorem 2.1, that if $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \text{Aut}(\mathfrak{n})$, then

$$T_{b^* z} = a^* T_z a.$$ 

In particular:

$$T_{r_z(z)} = J_z J_x.$$ 

If $x = z$, we get $T_z = -J_z T_z J_z$, thus $T_z J_z = -J_z^{-1} T_z = J_z T_z$. If $x \perp z$, we get $T_z = J_x T_z J_x$, thus $T_z J_x = J_x^{-1} T_z = -J_x T_z$. It follows that $T_z^2$ commutes with $J_z$ and with $J_w$, $w \perp z$. 


Now, let $z \in \mathfrak{z}$ and $w \perp z$. Let $R_w(t)$ the $2t$-rotation from $z$ towards $w$. Then $R_w(t) = r_z r_{w(t)}$, with $w(t) = \cos(t)z + \sin(t)w$. It follows that

$$
\begin{pmatrix}
  J_z J_{w(t)} & 0 \\
  0 & R_w(t)
\end{pmatrix}
$$

is an orthogonal automorphism and, therefore, satisfies

$$T_{R_w(t)z} = (J_z J_{w(t)})^t T_z (J_z J_{w(t)}).$$

Since $(J_z J_{w(t)})^t = (J_z J_{w(t)})^{-1}$,

$$T_{R_w(t)z}^2 = (J_z J_{w(t)})^t T_z^2 (J_z J_{w(t)}) = J_{w(t)} J_z T_z^2 J_z J_{w(t)}.$$

Since $T_z^2$ commutes with $J_z$ and $J_w$,

$$T_{R_w(t)z}^2 = T_z^2 J_{w(t)} J_z J_{w(t)} = -T_z^2 J_{w(t)} J_{w(t)}.$$

But $J_{w(t)}^2 = -I$, so that (4) implies that

$$T_{R_w(t)z}^2 = T_z^2.$$

For all $z' \in \mathfrak{z}$ we can choose $w \in \mathfrak{z}$, $t \in \mathbb{R}$ such that $R_w(t)z = z'$, so we get

$$T_{z'}^2 = T_z^2,$$

for all $z' \in \mathfrak{z}$, $|z'| = 1$.

The antisymmetry of the bracket implies that $T_z$ is skew-symmetric. Rescaling the scalar product on $\mathfrak{v}$ we obtain that $T_z^2 = -I$, so $T_{z'}^2 = -I$ for all $z' \in \mathfrak{z}$, $|z'| = 1$. Therefore $\mathfrak{n}$ is of Heisenberg type. \hfill $\blacksquare$

### 3. The case of center of dimension 2

In this section we compute the groups $G, G_0, G/G_0$ in the case $m = 2$. The various types are parametrized by pairs

$$(c, r) \in (\mathbb{U}^\ell / \text{SL}(2, \mathbb{R})) \times \mathbb{Z}_+^\ell$$

where $\mathbb{U}$ is the upper-half plane and $2\ell = 2 \sum r_j = \dim \mathfrak{m} - 2$. As a corollary we conclude that $\text{Aut}(\mathfrak{n})$ is maximal if and only if $\mathfrak{n}$ is of Heisenberg type. These are complex Heisenberg algebras of various dimensions regarded as real Lie algebras.

First we recall the normal form for fat algebras with $m = 2$ deduced from [LT]. Given $c = a + bi \in \mathbb{C}$, let

$$Z(c) = \begin{pmatrix}
  a & b \\
  -b & a
\end{pmatrix}.$$

If $r \in \mathbb{Z}_+$, set

$$A(c, r) = \begin{pmatrix}
  Z(c) & Z(c) & \cdots \\
  I_2 & Z(c) & \cdots \\
  \vdots & \vdots & \ddots \\
  I_2 & Z(c)
\end{pmatrix}$$

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a $2r \times 2r$-matrix. If $c = (c_1, \ldots, c_\ell) \in \mathbb{C}^\ell$ and $r = (r_1, \ldots, r_\ell) \in \mathbb{N}_+^\ell$, set

$$A(c, r) = \begin{pmatrix} A(c_1, r_1) & A(c_2, r_2) & \cdots & A(c_\ell, r_\ell) \\ A(c_2, r_2) & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A(c_\ell, r_\ell) & \cdots & A(c_1, r_1) & \end{pmatrix}$$

which is a $2s \times 2s$ matrix, $s = r_1 + \ldots + r_\ell$.

Let now $\phi, \psi(c, r)$ be the 2-forms on $\mathbb{R}^{4s}$ whose matrices in the standard basis are

$$[\phi] = \begin{pmatrix} 0 & -I_{2s} \\ I_{2s} & 0 \end{pmatrix}, \quad [\psi(c, r)] = \begin{pmatrix} 0 & A(c, r) \\ -A^t(c, r) & 0 \end{pmatrix}.$$

Then

$$[u, v](c, r) = (\phi(u, v), \psi(c, r)(u, v)) = <u, [\phi]v > e_1 + <u, [\psi(c, r)]v > e_2$$

is an $\mathbb{R}^2$-valued antisymmetric 2-form on $\mathbb{R}^{4s}$. Let

$$n_{(c, r)} = \mathbb{R}^{4s} \oplus \mathbb{R}^2$$

be the corresponding Lie algebra.

Define $M_{(c, r)} \in \text{End}(v)$ by

$$\phi(M_{(c, r)}u, v) = \psi_{(c, r)}(u, v),$$

whose matrix is

$$[M_{(c, r)}] = \begin{pmatrix} -A^t_{(c, r)} & 0 \\ 0 & -A_{(c, r)} \end{pmatrix}.$$ 

then we have

$$[u, v]_{(c, r)} = \phi(u, v)e_1 + \phi(M_{(c, r)}u, v)e_2, \quad \text{for } u, v \in \mathbb{R}^{4s}. \quad \text{(6)}$$

One can deduce [LT]

**Proposition 3.1.**

(a) Every fat algebra with center of dimension 2 is isomorphic to some $n_{(c, r)}$ with $c \in U^\ell$.

(b) Two of these are isomorphic if and only if the $r$’s coincide up to permutations and the $c$’s differ by some M"obius transformation acting componentwise.

(c) $n_{(c, r)}$ is of Heisenberg type if and only if $c = (c, \ldots, c)$ and $r = (1, \ldots, 1)$

Let now

$$\hat{n} = n_{(c, r)}$$

be fat and let $G = G(n)$, etc. We denote $\hat{n}$ the algebra obtained by replacing the matrices $A(c, r)$ by their semisimple parts and setting all $c_j = \sqrt{-1}$. The resulting $\hat{A}(c, r)$ consists of blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal and $\hat{n}$ is isomorphic to the Heisenberg type algebra $n_{((i, \ldots, i), (1, \ldots, 1))}$. The correspondence

$$n \mapsto \hat{n}$$

is functorial and seems extendable inductively to fat algebras of any dimension, although here we will maintain the assumption $m = 2$. 

Lemma 3.2. \( G_0(n) \subseteq G_0(\hat{n}) \) and \( \dim \text{Aut}(n) \leq \dim \text{Aut}(\hat{n}) \).

Proof. Let \( \phi, \psi, M_{c,r} \in \text{End}(v) \) be as above, so that
\[
\phi(M_{c,r}u, v) = \psi_{(c,r)}(u, v).
\]
By formula (6), \( g \in G_0(n_{c,r}) \) if and only if
\[
\phi(u, v) = \phi(gu, gv), \quad \phi(M_{c,r}u, v) = \phi(M_{c,r}gu, gv) = \phi(g^{-1}M_{c,r}gu, v),
\]
i.e., if and only if \( g \in \text{Sp}(\phi) \) and commutes with \( M_{c,r} \). In particular it commutes with the semisimple part \( M_{c,r} \). This is conjugate to a matrix having blocks
\[
Z(c) = \begin{pmatrix}
\Re(c) & \Im(c) \\
-\Im(c) & \Re(c)
\end{pmatrix}
\]
for various \( c \in \mathbb{C} \) along the diagonal, and zeros elsewhere. Every matrix commuting with such a matrix will surely commute with that having all \( c = 1 \). It follows that \( g \) also preserves \( \phi(M_{c,r}u, v) \) and, therefore, it is an automorphism of \( \hat{n} \) as well. Thus,
\[
G_0(n) \subseteq G_0(\hat{n}).
\]

From Corollary 2.3, \( \dim(G(n)/G_0(n)) \leq \dim(G(\hat{n})/G_0(\hat{n})) \), and therefore
\[
\dim G(n) = \dim(G(n)/G_0(n)) + \dim G_0(n) \leq \dim(G(\hat{n})/G_0(\hat{n})) + \dim G_0(\hat{n}) = \dim G(\hat{n}).
\]
Formula (2) implies \( \dim \text{Aut}(n) \leq \dim \text{Aut}(\hat{n}) \), as claimed. \( \blacksquare \)

Next we will describe \( g_0(n_{c,r}) \) for \( c \in U \) and \( r \in N_+ \), i.e., the case when the matrices \( A \) consist of a single block. Since \( c \) is \( \text{SL}(2, \mathbb{C}) \)-conjugate to \( i \), it is enough to take \( c = i \). Define the \( 2 \times 2 \)-matrices
\[
1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]
and let \( M_r(\mathbb{R}(1,i)) \) and \( M_r(\mathbb{R}(x,y)) \) denote the real vector spaces of \( r \times r \) matrices with coefficients in the span of \( 1, i \) and \( x, y \) respectively. Then the vector space
\[
\mathcal{R}(r) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, D \in M_r(\mathbb{R}(1,i)), B, C \in M_r(\mathbb{R}(x,y)) \right\},
\]
is a actually a matrix algebra.

Note that
\[
1^t = 1, \quad i^t = -i, \quad x^t = x, \quad y^t = y.
\]
Letting \( A^t \) denote the transpose or an \( \mathbb{R} \)-matrix and \( A^t \), \( A^* \) the transpose and conjugate transpose of \( \mathbb{R}[i, x, y] \)-matrices, one obtains
\[
A^t = A^*
\]
for \( A \in M_r(\mathbb{R}(1,i)) \) while
\[
A^t = A^t
\]
for \( A \in M_r(\mathbb{R}(x,y)) \).
With the notation

\[ J_1 = [\phi] \quad J_2 = [\psi(i,\ldots,i),(1,\ldots,1)] , \]

\[ g_0(\hat{n}) = \{ X \in \mathbb{R}^{4r \times 4r} : J_1 X + X^t J_1 = 0, J_2 X + X^t J_2 = 0 \} . \]

From [S] we know that

\[ g_0(\hat{n}) \cong \mathfrak{sp}(r, \mathbb{C})^\mathbb{R} \]

Changing basis,

\[ g_0(\hat{n}) = \{ X \in \mathcal{R}(r) : J_1 X + X^t J_1 = 0, J_2 X + X^t J_2 = 0 \} \]

where

\[ J_1 = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} , \quad J_2 = \begin{pmatrix} 0 & iI_r \\ iI_r & 0 \end{pmatrix} . \]

This gives an alternative description of this algebra:

\[ g_0(\hat{n}) = \left\{ \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : A \in M_r(\mathbb{R}(1,i)), B,C \in M_r(\mathbb{R}(x,y)), B^t = B, C^t = C \right\} \]

We now restrict our attention to matrices \( \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \) in \( g_0(\hat{n}) \) where \( A, B, C \) have the respective forms

\[
\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{pmatrix} = \begin{pmatrix} b_1 & \cdots & b_{r-1} & b_r \\ \vdots & \ddots & \ddots & 0 \\ b_{r-1} & \cdots & \ddots & \vdots \\ b_r & 0 & \cdots & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \cdots & 0 & c_1 \\ \vdots & \ddots & \ddots & c_2 \\ c_1 & c_2 & \cdots & c_r \end{pmatrix}
\]

with coefficients in \( \mathbb{R}^{2 \times 2} \). Let \( A_k = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix} \) having \( a_k = 1 \) and zero otherwise and \( A_k' \) the matrix of the same form but with \( a_k = i \) and zeros elsewhere. Similarly, let \( B_k \) (resp. \( C_k \)) the matrix \( \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \) (resp., \( \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \)) with \( b_k \) (resp. \( c_k \)) equal to \( x \) and zeros elsewhere, and \( B_k' \) (resp. \( C_k' \)) with \( b_k \) (resp. \( c_k \)) equal to \( y \) and zeros elsewhere.

**Theorem 3.3.** Let \( n = n_{(c,r)} \), \( (c,r) \in \mathbb{U} \times \mathbb{N} \), and regard \( g_0(\hat{n}) \) as a subalgebra of \( \mathfrak{gl}(n) \). Then,

1. \( g_0(\hat{n}) \) is the \( \mathbb{R} \)-span of \( A_i, A_i', B_i, B_i', C_i, C_i' \) for \( 1 \leq i \leq r \).
2. The semisimple part of \( g_0(\hat{n}) \) is the span of \( A_1, A_1', B_1, B_1', C_1, C_1' \).
3. The solvable radical is the span of \( A_i, A_i', B_i, B_i', C_i, C_i' \) with \( 1 < i \leq r \).

In particular, the \( \mathbb{R} \)-dimension the \( g_0(\hat{n}) \) is equal to \( 6r \) and the semisimple part of \( g_0(\hat{n}) \) is isomorphic to \( \mathfrak{sp}(1,\mathbb{C}) \).
Proof. It is enough to consider the case \( n = n_{(i,r)} \). Let \( T_2 = [\psi_{(i,r)}] \) and write \( T_2 = J_2 + N_2 \) where

\[
N_2 = \begin{pmatrix}
0 & N' \\
-N^t & 0
\end{pmatrix}, \quad \text{with } N = \begin{pmatrix}
0 & \cdots & 0 & 0 \\
1 & \ddots & 0 & 0 \\
& \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0
\end{pmatrix}.
\]

From Lemma 3.2, \( g_0(n) = \{ X \in g_0(\hat{n}) : T_2X + X^tT_2 = 0 \} \). As \( g_0(n) \subset g_0(\hat{n}) \) one obtains

\[
g_0(n) = \{ X \in g_0(\hat{n}) : N_2X + X^tN_2 = 0 \}.
\]

The conditions on \( \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in g_0(n) \) are, explicitly,

\[
\begin{alignat}{2}
0 &= NC - C^tN^t = NC - (NC)^t \quad & & \quad \text{(7)} \\
0 &= N^tA - AN^t \quad & & \quad \text{(8)} \\
0 &= N^tB - B^tN = N^tB - (N^tB)^t \quad & & \quad \text{(9)}
\end{alignat}
\]

For the first equation, note that \( NC \) symmetric if and only if \( c_{i,j+1} = c_{j,i+1} \) and \( c_{1,j} = 0 \) for \( i, j < n \). Since \( C \) is symmetric, \( c_{i,j+1} = c_{j,i+1} = c_{i+1,j} \) and \( c_{1,j} = 0 \) for \( i, j < n \). We conclude:

If \( i + j = k \leq r \), \( c_{i,j} = c_{i,k-i} = c_{i-1,k-i+1} = \cdots = c_{1,k-1} = 0 \)

If \( i + j = k > r \), \( c_{i,j} = c_{i,k-i} = c_{i+1,k-i-1} = \cdots = c_{r,k-i-r} = 0 \)

Thus, the strict upper antidiagonals are zero and each lower antidiagonal have all its elements equal.

For the second equation, note that \( N^t \) and \( A \) commute. This is equivalent to \( c_{i,j} = c_{i,s} \) when \( j - i = s - t \) and \( c_{1,1} = 0 \) for \( i > 1 \). The first condition implies that each diagonal have all its elements equal, while the second implies that the strict lower diagonals are zero.

Equation (9) is analogous to equation (7): the condition \( N^tB \) symmetric is equivalent to each antidiagonal have all its elements equal and that the strict lower antidiagonals are zero.

From all this we conclude that the span of \( A_i, A'_i, B_i, B'_i, C_i, C'_i \) with \( 1 \leq i \leq r \) is \( g_0(n) \) and (1) follows.

(2) and (3) follow from (1) and the explicit presentation of the matrices \( A_i, A'_i, B_i, B'_i, C_i, C'_i \).

\[ \blacksquare \]

Corollary 3.4. (of the proof) Let \( n \) be fat. Then \( \dim(g_0(n)) \) is maximal if and only if \( n \) is of Heisenberg type.

Proof. Let \( (c, r) = ((c_1, \ldots, c_l), (r_1, \ldots, r_l)) \) be such that \( n = n_{(c,r)} \). We know that \( g_0(n) \subset g_0(\hat{n}) \). If \( c_i \neq c_j \) for some \( i, j \), then there is not intertwining operator between the blocks corresponding to these invariants, so \( g_0(n) \neq g_0(\hat{n}) \).
When \(c_1 = c_2 = \cdots = c_l\) we can consider \(c_j = i\) for all \(j\). Let \(r = \sum r_i\).

In this case if \((A,B,C) \in g_0(n)\) must satisfy the equations (7), (8), (9) but with \(N\) such that coefficients \(n_{j+1,j}\) are 0 or 1. Suppose now that \(g_0(n)\) is not of Heisenberg type, then some \(n_{j+1,j}\) is equal to 1. We assume that \(n_{21} = 1\) and let \(A \in M_r(\mathbb{R}(1,i))\) such that \(a_{12} = 1\) and 0 otherwise, then

\[X = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}\]

belongs to \(g_0(\hat{n})\) but is not in \(g_0(n)\).

It can be shown in general that the semisimple part of \(g_0(n)\) is isomorphic to \(\oplus_i sp(m_i, \mathbb{C})\), where \(m_i\) is the multiplicity of the pair \((c_i, r_i)\) in \((c, r)\).

In the case \(m = 2\), \(g/g_0\) is either 0 or isomorphic to \(so(2)\).

**Theorem 3.5.** \(g(n)/g_0(n) \cong so(2)\) if \(c_1 = \cdots = c_l\), and 0 otherwise.

**Proof.** \(g/g_0\) is a compact subalgebra of \(gl(2)\), hence of the form \(gs\) for some \(g \in SL(2, \mathbb{R})\) and it is nonzero if and only if there exists \(X \in sl(v)\) such that, in the notation of the proof of Theorem 3.3,

\[
\begin{pmatrix} X & 0 \\ 0 & g^{-1} \end{pmatrix}
\]

is a derivation of \(n\). For \(g = 1\), if \(T_1, T_2\) correspond to the standard basis of \(\mathfrak{z}\), the equations for \(X\) become

\[(a) \quad T_1 X + X^t T_1 = T_2, \quad (b) \quad T_2 X + X^t T_2 = -T_1 \]

In normal form, and for a single block \(A_{(i,r)}\),

\[T_1 = J_1 = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & iI_r + N \\ iI_r - N^t & 0 \end{pmatrix}.\]

We decompose

\[T_2 = J_2 + N_2, \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & iI_r \\ iI_r & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & N \\ -N^t & 0 \end{pmatrix}\]

and regard \(J_1, J_2, T_1, T_2, N_2\) as matrices with coefficients in \(\mathbb{R}^{2 \times 2}\). Note that \(J_1, J_2\) correspond to \(\hat{n}\), of Heisenberg type. Let

\[Y_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & s & 0 \\ 0 & 2i & 0 & 0 & 0 & s & 0 \\ 0 & 1 & 4i & 0 & 0 & s & 0 \\ 0 & 0 & 2i & 6i & 0 & s & 0 \\ 0 & 0 & 0 & 31 & 8i & s & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 & (n-2)1 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & 2(n-1)i \end{pmatrix}.\]
A straightforward calculation shows that

\[ X_0 = \begin{pmatrix} -Y^*_0 & 0 \\ 0 & -Y^*_0 + iI_r + N \end{pmatrix} \]

is a solution of (a), (b). We conclude that

\[ \begin{pmatrix} X_0 & 0 \\ 0 & i \end{pmatrix} \]

is a derivation of \( \mathfrak{n}_{(i,r)} \), which lies in \( \mathfrak{g}(\mathfrak{n}_{(i,r)}) \) but not in \( \mathfrak{g}_0(\mathfrak{n}_{(i,r)}) \).

For any \( c \in U \), \( \mathfrak{n}_{(c,r)} \cong \mathfrak{n}_{(i,r)} \), hence they have the same \( \mathfrak{g}/\mathfrak{g}_0 \) up to isomorphisms. In fact, for any \( g \in \text{SL}(2, \mathbb{R}) \), the algebra \( \mathfrak{n}_{(g_i,r)} \) has a derivation of the form

\[ \begin{pmatrix} X & 0 \\ 0 & gi^g \end{pmatrix}. \]

For a fixed \( g \), these \( X \) are unique modulo \( \mathfrak{g}_0 \) and come in normal form. Clearly, \( c \) determines the \( 2 \times 2 \) matrix \( gi^g \) and the complex number \( g \cdot i \).

In the case of an arbitrary fat \( \mathfrak{n}_{(c,r)} \), each block \( (c_k, r_k) \) determines a corresponding \( X_k \) such that

\[ \begin{pmatrix} X_k & 0 \\ 0 & gi^g_k \end{pmatrix} \]

is a derivation of \( \mathfrak{n}_{(c_k, r_k)} \). If \( n_{(c,r)} \) has a derivation in \( \mathfrak{g} \) that is not in \( \mathfrak{g}_0 \), then its must have one which is combination of such, acting on \( v \) as \( X_1 + X_2 + \cdots \). This forces all the \( gi^g_k \) to be the same and all the \( c_i \) to be the same. The reciprocal is clear.

In particular, all algebras \( \mathfrak{n}_{(c,r)} \) with \( c_1 = \ldots = c_\ell \) and \( r_1 > 1 \) maximize the dimension of \( \mathfrak{g}/\mathfrak{g}_0 \), but they are not Heisenberg type.

Lauret had pointed out to us that there were non Heisenberg type algebras such that \( \mathfrak{g}(\mathfrak{n})/\mathfrak{g}_0(\mathfrak{n}) \neq 0 \). Independently, Oscari proved that this holds whenever the \( c_i \)'s all agree.

### 4. Fat distributions

Let \( D \) be a smooth vector distribution on a smooth manifold \( M \), i.e., a subbundle of the tangent bundle \( T(M) \). Its nilpotentization, or symbol, is the bundle on \( M \) with fiber

\[ N^D(M)_p = \bigoplus_j D_p^{(j)} / D_p^{(j-1)} \]

where \( D_p^{(1)} = D_p \) and \( D_p^{(j+1)} = D_p^{(j)} + [\Gamma(D), \Gamma(D^j)]_p \). The Lie bracket in \( \Gamma(T(M)) \) induces a graded nilpotent Lie algebra structure on each fiber of \( N^D(M) \). If \( D^{(j)} = T(M) \) for some \( j \), \( D \) is called completely non-integrable. If \( D^{(2)} = T(M) \), the nilpotentization is 2-step, which in the notation of the previous section, is

\[ \mathfrak{n}_p = N^D(M)_p = D_p \oplus \frac{D_p + [\Gamma(D), \Gamma(D)]_p}{D_p} = \mathfrak{v}_p + 3_p, \]
It is also easy to see that $D$ is fat in the sense of Weinstein [Mo] if and only if $n_p = v + \mathfrak{z}$ is non-singular, i.e., fat in the sense defined in the section 1.

A subriemannian metric $g$ defined on $D$ determines a metric on $v$. On $\mathfrak{z}$ we put a metric $\sigma$ invariant under $G$. Let \{$\phi_1, ..., \phi_m; \psi_1, ..., \psi_n$\} be a coframe on $M$ such that
\[ D = \cap \ker \phi_i, \]
with \{\$\phi_1, ..., \phi_m$\} and \{\$\psi_1, ..., \psi_n$\} orthonormal with respect to $g + \sigma$. Define $T_z \in \text{End}(D)$ as before, by
\[ \sigma(z, [u, v]) = g(T_z u, v). \]

Then $D$ is fat if and only if $T_z$ is invertible for all non-zero $z \in \mathfrak{z}$. The structure equations for the coframe can be written
\[ d\phi_k \equiv \sum_i (T_k \psi_i) \wedge \psi_i \mod(\phi_\ell) \]
with the $T_k$'s having the property that any non-zero linear combination of them is invertible. This is deduced from the fact that if $u, v \in v$, then $d\phi[u, v] = -\phi([u, v])$, since $u(\phi(v)) = u(0) = 0$. The $d\psi$'s are essentially arbitrary.

Let now $M$ be a the simply connected Lie group with a fat Lie algebra $n$, $D$ the left-invariant distribution on $M$ such that $D_e = v$. For a left-invariant coframe, the structure equations take the form
\[ d\phi_k = \sum_i (J_k \psi_i) \wedge \psi_i, \quad d\psi_i = 0 \]
where $J_1, ..., J_m$ are anticommuting complex structures on $D$.

The results from the previous sections lead to consider fat distributions satisfying
\begin{equation}
(4.1)
\[ d\phi_k = \sum_i (J_k \psi_i) \wedge \psi_i \mod(\phi_\ell) \]
\end{equation}
where the $J_k$ are sections of $\text{End}(T(M)^*)$ satisfying the Canonical Commutation Relations
\[ J_i J_j + J_j J_i = -2\delta_{ij}. \]

The Equivalence Problem for these systems has been discussed for distributions with growth vector $(2n, 2n + 1), (4n, 4n + 3)$ and $(8, 15)$. In these cases $n$ is parabolic, i.e., isomorphic to the Iwasawa subalgebra of a real semisimple Lie algebra $\mathfrak{g}$ of real rank one. The Tanaka [T] subriemannian prolongation of such algebra is $\mathfrak{g}$, while in the non-parabolic case is just
\[ n + \mathfrak{r}(n) + \mathfrak{a}(n) \]
where $\mathfrak{a}(n)$ the 1-dimensional Lie algebra of dilations [Su]. In this case, Tanaka’s theorem implies that, in the notation of [Z], the first pseudo G-structure $P^0$ already carries a canonical frame.
As this paper was being written, E. van Erp pointed out to us his article [Er], where fat distributions are called polycontact and those satisfying (4.1) arise by imposing a compatible conformal structure.

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