ON BEHAVIOR OF PAIRS OF TEICHMÜLLER GEODESIC RAYS

MASANORI AMANO

Abstract. In this paper, we obtain the explicit limit value of the Teichmüller distance between two Teichmüller geodesic rays which are determined by Jenkins-Strebel differentials having a common end point on the augmented Teichmüller space. Furthermore, we also obtain a condition under which these two rays are asymptotic. This is similar to a result of Farb and Masur.

1. Introduction

Let \( T(X) \) be the Teichmüller space of an analytic finite Riemann surface \( X \). Each Teichmüller geodesic ray on \( T(X) \) is determined by a holomorphic quadratic differential on an initial point. We are interested in the behavior of two geodesic rays near the boundary of the Teichmüller space.

An interesting question is when the Teichmüller distance between the given two Teichmüller geodesic rays are bounded, or diverge. This question is answered completely by Ivanov [Iva01], Lenzhen and Masur [LM10] and Masur [Mas75], [Mas80]. The details are following. First, Masur showed that if the measured foliations of the given rays are Jenkins-Strebel and topologically equivalent, then the rays are bounded ([Mas75]). Masur also showed that if the measured foliations are uniquely ergodic and topologically equivalent with the condition that the sets of their critical trajectories do not contain closed loops, then the rays are asymptotic (and so bounded) ([Mas80]). Ivanov showed that if the measured foliations are absolutely continuous, then the rays are bounded, and if the measured foliations have non-zero intersection number, then the rays are divergent ([Iva01]). Finally, Lenzhen and Masur showed that if the measured foliations are not absolutely continuous, or the measured foliations are not topologically equivalent and have zero intersection number, then the rays are divergent ([LM10]).

On the other hand, Farb and Masur [FM10] considered two Jenkins-Strebel rays in the moduli space given by modularly equivalent holomorphic quadratic differentials. They showed that under some conditions of initial points of two rays, the limit value of the distance between the rays in the moduli space is the distance between the end points of the rays in the boundary of the moduli space. If in addition, the end points of the given rays are coincide, then they are asymptotic.
In this paper, we consider two Teichmüller geodesic rays \( r, r' \) on the Teichmüller space \( T(X) \) starting at \([Y, f], [Y', f']\) with the conditions that their holomorphic quadratic differentials \( q, q' \) are Jenkins-Strebel and the measured foliations \( f^{-1}_r(H(q)), f'^{-1}_r(H(q')) \) are topologically equivalent. Furthermore, we assume that the end points of \( r, r' \) on the augmented Teichmüller space are coincide. Under the above assumption, we determine the limit value of the Teichmüller distance between two points \( r(t), r'(t) \) on the given rays \( r, r' \) respectively (Theorem 3.1). Furthermore, we show that if two rays are modularly equivalent, then they are asymptotic (Corollary 3.5). This is similar to a theorem of Farb and Masur [FM10]. In Theorem 3.1 the limit value depends on the modulus of each annulus which is determined by the holomorphic quadratic differentials on the initial points of given rays. Therefore, this value depends on the choice of the initial points. We consider the minimum of the limit value of the Teichmüller distance between two rays when we shift the initial points of given rays. We show that the minimum is represented by the detour metric between the end points on the Gardiner-Masu boundary of \( T(X) \) (Proposition 4.15).

2. Background

2.1. Teichmüller spaces. Let \( X \) be a Riemann surface of type \((g, n)\) with \(3g - 3 + n > 0\). The Teichmüller space \( T(X) \) of \( X \) is the set of all equivalence classes \([Y, f]\) of pairs of a Riemann surface \( Y \) and a quasiconformal mapping \( f : X \to Y \). Two pairs \((Y_1, f_1)\) and \((Y_2, f_2)\) are equivalent if there is a conformal mapping \( h : Y_1 \to Y_2 \) such that \( h \circ f_1 \) is homotopic to \( f_2 \). The Teichmüller space \( T(X) \) has a complete distance, called the Teichmüller distance \( d_{T(X)} \). For any \( p_1 = [Y_1, f_1], p_2 = [Y_2, f_2] \in T(X) \), the distance is defined by

\[
d_{T(X)}(p_1, p_2) = \frac{1}{2} \inf_h \log K(h),
\]

where \( h \) ranges over all quasiconformal mappings \( h : Y_1 \to Y_2 \) such that \( h \circ f_1 \) is homotopic to \( f_2 \), and \( K(h) \) means the maximal quasiconformal dilatation of \( h \).

2.2. Holomorphic quadratic differentials. In this section, we refer to [Sti84] in detail.

A holomorphic quadratic differential \( q \) on \( X \) is a tensor which is represented locally by \( q = q(z)dz^2 \) where \( q(z) \) is a holomorphic function of the local coordinate \( z = x + iy \) on \( X \). We allow holomorphic quadratic differentials to have simple poles at the punctures of \( X \). For each holomorphic quadratic differential \( q = q(z)dz^2 \), \(|q|\) is defined locally by the differential 2-form \(|q(z)|dzdy\) on each coordinate \( z \). The norm of \( q \) is defined by \( \|q\| = \int_X |q(z)|dzdy \). We treat holomorphic quadratic differentials whose norm are finite. A holomorphic quadratic differential \( q \) is called of unit norm if it satisfies \( \|q\| = 1 \). A critical point of \( q \) is a zero of \( q \) or a puncture of \( X \). Each non-zero holomorphic quadratic differential has finitely many critical points. In a neighborhood of non-critical points, there exists a local coordinate \( \zeta \) on \( X \) such that \( q = d\zeta^2 \). Indeed, let \( p_0 \) be a non-critical point, then \( q^2 = q(z)^2dz \) has a single valued holomorphic branch in some neighborhood \( U \) around \( p_0 \). The new coordinate
Euclidean horizontal segment in $q$ consists of non-critical points of $q$. $p$ is defined for any $\zeta_n$ at $U_z$. A smooth path which satisfies $q(\gamma(t))\dot{\gamma}(t)^2 > 0$. We notice that a horizontal trajectory of $q$ consists of non-critical points of $q$. A horizontal trajectory of $q$ is represented by a Euclidean horizontal segment in $q$-coordinates. All horizontal trajectories of $q$ are classified by the following three types. A horizontal trajectory $\gamma$ of $q$ is critical if it joins critical points of $q$, closed if it is a closed path, recurrent otherwise. The recurrent trajectory is dense on a subsurface of $X$ which is surrounded by critical trajectories. Let $\Gamma_q$ be the set of all critical points and critical trajectories of $q$. A component of $X - \Gamma_q$ is an annulus which is swept out by closed trajectories of $q$ such that they are homotopic to each other, or a minimal domain which consists of infinitely many recurrent trajectories of $q$. If all components of $X - \Gamma_q$ are annuli, we call $q$ a Jenkins-Strebel differential.

2.3. Measured foliations. A foliation $F = \{(U_j, z_j)\}_j$ with singularities on $X$ is given by the pair of an open covering $\{U_j\}_j$ of $X - \{\text{finite distinguished points}\}$ and each local coordinate $z_j$ on $U_j$ such that the equation $z_k = \pm z_j + c$ where $c \in \mathbb{C}$ holds if $U_j \cap U_k \neq \emptyset$. These distinguished points and punctures of $X$ are called singularities of $F$. They are $p$-pronged singularities for $p \geq 1$, but the case of $p = 1, 2$ is attained only at the punctures of $X$. A maximal horizontal segment with respect to $\{z_j\}_j$ is called a leaf of $F$. A transverse measure $\mu$ of $F$ is given by a measure on the set of all transversal arcs of leaves of $F$ such that if transversal arcs $\alpha$ and $\beta$ are moved to each other by the homotopy and each orbit of which is contained in a single leaf of $F$, then $\mu(\alpha) = \mu(\beta)$. The pair $(F, \mu)$ is called a measured foliation on $X$. Let $\mathcal{S}$ be the set of all homotopy classes of non-trivial and non-peripheral simple closed curves on $X$. For any measured foliation $(F, \mu)$ and any $\alpha \in \mathcal{S}$, we can define the intersection number

$$i((F, \mu), \alpha) = \inf_{\alpha' \in \alpha} \int_{\alpha'} d\mu,$$

where $\alpha'$ ranges over all simple closed curves in $\alpha$. Two pairs $(F_1, \mu_1)$ and $(F_2, \mu_2)$ are equivalent if the equation

$$i((F_1, \mu_1), \alpha) = i((F_2, \mu_2), \alpha)$$

holds for any $\alpha \in \mathcal{S}$. We denote by $\mathcal{MF}(X)$ the set of all equivalence classes of measured foliations on $X$. The set $\mathcal{MF}(X)$ has the week-topology which is induced by intersection number functions in $\mathbb{R}_{\geq 0}^S$. We denote by $[F, \mu]$ the equivalence class of $(F, \mu)$. We consider the space of measured foliations $\mathcal{MF}(Y)$ on any other Riemann surface $Y$ of the same type as $X$. For any homeomorphism $f : X \to Y$, there exists a homeomorphism $f_* : \mathcal{MF}(X) \to \mathcal{MF}(Y)$ which is defined by $f_*([F, \mu]) = [f(F), \mu \circ f^{-1}] \in \mathcal{MF}(Y)$ for any $[F, \mu] \in \mathcal{MF}(X)$. After this, we denote by $\mu$ as the equivalence class of a measured foliation $[F, \mu] \in \mathcal{MF}(X)$. We
There is a one-to-one correspondence between the set of all holomorphic quadratic differentials of finite norm on $X$ of $H$. Trajectories which sweep out in $H$ foliation $H$ the metric $|M_q|$ non-zero holomorphic quadratic differential $2.4$. Measured foliations and holomorphic quadratic differentials. That is, it has the zero intersection number with all measured foliations. Of course, for any $\mu$, the set $\mu \in MF(X)$, the equation $i(f_*(\mu), f_*(\nu)) = i(\mu, \nu)$ holds.

**Remark.** The set $MF(X)$ contains the zero-measured foliation, denoted by 0, that is, it has the zero intersection number with all measured foliations. Of course, for any $\mu \in MF(X)$, we regard the measured foliation $0\mu$ as 0.

2.4. Measured foliations and holomorphic quadratic differentials. For any non-zero holomorphic quadratic differential $q$ on $X$, we can define the measured foliation $H(q) \in MF(X)$ consists of all horizontal trajectories of $q$ as leaves and $|dy|$ as a transverse measure where $z = x + iy$ is the $q$-coordinate. The singularities of $H(q)$ are the critical points of $q$.

**Remark.** There is a one-to-one correspondence between the set of all holomorphic quadratic differentials of finite norm on $X$ and $MF(X)$, we refer to [HM79].

If $q$ has an annulus $A$ in $X - \Gamma_q$, then the restriction to $A$ of the measured foliation $H(q)$ can be written as $H(q)|_A = b_\gamma$ where $\gamma \in S$ corresponds to closed trajectories which sweep out in, and $b > 0$ is the height of $A$ with respect to the metric $|dy|$. If $q$ has a minimal domain $M$ in $X - \Gamma_q$, then the restriction to $M$ of the measured foliation $H(q)$ can be written as $H(q)|_M = \sum_{i=1}^{p} b_i e_i$ where $p > 0$ is bounded by the number which is determined by the topology of $M$, $b_i \geq 0$ for any $i = 1, \cdots, p$ and $\{e_i\}_{i=1}^{p}$ is the set of ergodic transverse measures which are projectively-distinct and pairwise having zero intersection number (that is, for any $i \neq i'$ and any $k \geq 0$, $e_{i'} \neq ke_i$ and $i(e_i, e_{i'}) = 0$). A transverse measure $e$ is called an *ergodic transverse measure* if it is non-zero and cannot be written as a sum of projectively-distinct and non-zero measured foliations. (For ergodicity, we refer to [KH95] for more informations.) The holomorphic quadratic differential $q$ has finitely many critical points, so the measured foliation $H(q)$ has finitely many these domains. Therefore, the measured foliation $H(q)$ can be written as

$$H(q) = \sum_{j=1}^{k} b_j G_j,$$

where $G_j$ is (a homotopy class of) a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number, and $b_j \geq 0$ if $G_j \in S$, $b_j > 0$ if $G_j$ is an ergodic measure. (For this notation, we refer the reader to [Iva92] for more details.) In particular, if $q$ is a Jenkins-Strebel differential, then we can write

$$H(q) = \sum_{j=1}^{k} b_j \gamma_j,$$
where $b_1, \cdots, b_k$ are positive real numbers and $\gamma_1, \cdots, \gamma_k$ are distinct simple closed curves such that $i(\gamma_j, \gamma_{j'}) = 0$ for any $j \neq j'$, and in this situation, we also call that $H(q)$ is Jenkins-Strebel. If $X - \Gamma_q$ has only one minimal domain and the measured foliation $H(q)$ is represented by $b$ where $b > 0$ and $e$ is an ergodic measure, and any other topologically equivalent measured foliation $\mu$ on $X$ is represented by $\mu = b' e$ where $b' > 0$, then $q$ and $H(q)$ are called uniquely ergodic. We come back to the general case and see that

$$i \left( \sum_{j=1}^{k} b_j G_j, \alpha \right) = \sum_{j=1}^{k} b_j i(G_j, \alpha)$$

for any $\alpha \in S$. For any $j = 1, \cdots, k$, we set

$$m_j = \frac{b_j}{i(G_j, V(q))},$$

where the measured foliation $V(q)$ is defined by $H(-q)$. If $G_j$ is a simple closed curve $\gamma_j \in S$, then $a_j := i(\gamma_j, V(q))$ means the infimum of the horizontal lengths of the simple closed curves in $\gamma_j$ with respect to the metric $|dx|$, and $m_j = \frac{b_j}{a_j}$ means a modulus of the annulus which is generated by $\gamma_j$, that is, the ratio of the height and the circumference of the annulus.

Let $q, q'$ be unit norm holomorphic quadratic differentials on $X$ and $H(q), H(q')$ be measured foliations in $\mathcal{MF}(X)$ constructed by $q, q'$ respectively.

**Definition 2.1.** The pair of holomorphic quadratic differentials $q, q'$ or measured foliations $H(q), H(q')$ is called topologically equivalent if there is a homeomorphism $\alpha : X - \Gamma_q \to X - \Gamma_{q'}$ which is homotopic to the identity such that the leaves of $H(q)$ are mapped to the leaves of $H(q')$. In this situation, we can write their measured foliations as $H(q) = \sum_{j=1}^{k} b_j G_j$, $H(q') = \sum_{j=1}^{k} b_j' G_j$ where $G_j$ is a simple closed curve or an ergodic measure for any $j = 1, \cdots, k$ such that these are projectively-distinct and pairwise having zero intersection number, and $b_j, b_j' > 0$ if $G_j \in S$. $b_j, b_j'$ are distinct simple closed curves such that these are projectively-distinct and pairwise having zero intersection number, and $b_j, b_j' > 0$ if $G_j$ is an ergodic measure. The pair of holomorphic quadratic differentials $q, q'$ or measured foliations $H(q), H(q')$ is called absolutely continuous if $q, q'$ are topologically equivalent and for $H(q) = \sum_{j=1}^{k} b_j G_j$, $H(q') = \sum_{j=1}^{k} b_j' G_j$, the set of subscripts of non-zero coefficients $b_j$ of $H(q)$ and the one of $H(q')$ are coincide. In this situation, we can regard as $b_j, b_j' > 0$ for any $j = 1, \cdots, k$.

**Remark.** If two holomorphic quadratic differentials $q, q'$ are Jenkins-Strebel and topologically equivalent, then they are absolutely continuous.

### 2.5. Teichmüller geodesic rays

Let $q$ be a unit norm holomorphic quadratic differential on $X$. A quasiconformal mapping $f$ on $X$ is called a Teichmüller mapping for $q$ if $f$ has the Beltrami coefficient $\frac{K(f) - 1}{K(f) + 1} \bar{q}$. The existence and uniqueness for Teichmüller mappings are the followings.

**Theorem 2.2.** (Teichmüller’s existence theorem) For any quasiconformal mapping $g : X \to Y$, there is a Teichmüller mapping $f$ which is homotopic to $g$.

**Theorem 2.3.** (Teichmüller’s uniqueness theorem) For any quasiconformal mapping $g : X \to Y$ which is homotopic to the Teichmüller mapping $f$, the inequality $K(f) \leq K(g)$ holds and the equality holds if and only if $f = g$. 


These facts are called the Teichmüller’s theorem. (We refer to [IT92] for details.) Therefore, a Teichmüller mapping is attained the supremum of the definition of the Teichmüller distance, i.e., for any \( p_1 = [Y_1, f_1], p_2 = [Y_2, f_2] \), there is the Teichmüller mapping \( h : Y_1 \to Y_2 \) which is homotopic to \( f_2 \circ f_1^{-1} \) such that 
\[
d_{T(X)}(p_1, p_2) = \frac{1}{2} \log K(h).
\]

For any \( p = [Y, f] \in T(X) \), let \( q \) be a unit norm holomorphic quadratic differential on \( Y \) and \( z = x + iy \) be a \( q \)-coordinate. For any \( t \geq 0 \), let \( Y_t \) be a Riemann surface determined by the local coordinate \( z_t = e^{-t} x + i e^t y \) and \( g_t : Y \to Y_t \) be the Teichmüller mapping which is determined by \( z \to z_t \). We assume that \( Y_0 = Y \), \( g_0 = id_Y \). By the Teichmüller’s theorem, this mapping \( r : [0, \infty) \to T(X) \) is the Teichmüller geodesic ray on \( T(X) \) starting at \( p \) and having the holomorphic quadratic differential \( q \). If the holomorphic quadratic differential \( q \) is of Jenkins-Strebel, the ray \( r \) is called a Jenkins-Strebel ray.

**Definition 2.4.** Let \( p = [Y, f], p' = [Y', f'] \in T(X), q, q' \) be unit norm holomorphic quadratic differentials on \( Y, Y' \) and \( r, r' \) be Teichmüller geodesic rays on \( T(X) \) starting at \( p, p' \) and having \( q, q' \) respectively. We denote by \( H(q) \in \mathcal{MF}(Y) \), \( H(q') \in \mathcal{MF}(Y') \) the measured foliations corresponding to \( q, q' \) respectively. We suppose that \( f^{-1}_r(H(q)), f'^{-1}_r(H(q')) \) are absolutely continuous, then the measured foliations are written as 
\[
f^{-1}_r(H(q)) = \sum_{j=1}^k b_j G_j, \quad f'^{-1}_r(H(q')) = \sum_{j=1}^k b'_j G_j, \quad H(q) = \sum_{j=1}^k b_j f_*(G_j) \quad \text{and} \quad H(q') = \sum_{j=1}^k b'_j f'_*(G_j)
\]
where \( b_1, \ldots, b_k, b'_1, \ldots, b'_k \) are positive real numbers and \( G_j \) is a simple closed curve or an ergodic measure for any \( j = 1, \ldots, k \) such that these are projectively-distinct and pairwise having zero intersection number. We set 
\[
m_j = n_{f_*(G_j), V(q)}, \quad m'_j = n_{f'_*(G_j), V(q')}
\]
for any \( j = 1, \ldots, k \). In this situation, the given rays \( r, r' \) are called modularly equivalent if there is \( \lambda > 0 \) such that \( m'_j = \lambda m_j \) for any \( j = 1, \ldots, k \).

**Definition 2.5.** The pair of Teichmüller geodesic rays \( r, r' \) on \( T(X) \) are called bounded if there is \( M > 0 \) such that 
\[
d_{T(X)}(r(t), r'(t)) < M \quad \text{for any} \quad t \geq 0,
\]
divergent if 
\[
d_{T(X)}(r(t), r'(t)) \to +\infty \quad \text{as} \quad t \to \infty,
\]
and asymptotic if there is a choice of initial points \( r(0), r'(0) \) such that 
\[
d_{T(X)}(r(t), r'(t)) \to 0 \quad \text{as} \quad t \to \infty,
\]
in other words, for the given rays \( r(t), r'(t) \), there is \( \alpha \in \mathbb{R} \) such that 
\[
d_{T(X)}(r(t), r'(t + \alpha)) \to 0 \quad \text{as} \quad t \to \infty.
\]

2.6. The end point of a Jenkins-Strebel ray. In this section, we refer to [Ab77], [HS07] and [IT92].

**Definition 2.6.** (Riemann surfaces with nodes) A connected Hausdorff space \( R \) is called a Riemann surface of type \((g, n)\) with nodes if \( R \) satisfies the following three conditions:

1. Any \( p \in R \) has a neighborhood which is homeomorphic to the unit disk \( \mathbb{D} \) or the set \( \{z_1, z_2 \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1, z_1 \cdot z_2 = 0\} \). (In the latter case, \( p \) is called a node of \( R \). We allow \( R \) to have finitely many nodes.)

2. Let \( p_1, \ldots, p_k \) be nodes of \( R \). We denote by \( R_1, \ldots, R_r \) the connected components of \( R \setminus \{p_1, \ldots, p_k\} \). For \( i = 1, \ldots, r \), each \( R_i \) is of type \((g_i, n_i)\) which satisfies 
\[
2g_i - 2 + n_i > 0, \quad n = \sum_{i=1}^r n_i - 2k, \quad \text{and} \quad g = \sum_{i=1}^r g_i - r + k + 1.
\]
Remark. The condition (2) means that we get a Riemann surface of type \((g,n)\) without nodes by opening each node of \(R\). All Riemann surfaces of type \((g,n)\) without nodes are included to this definition.

Definition 2.7. (Augmented Teichmüller spaces) Let \(X\) be a Riemann surface of type \((g,n)\) without nodes which satisfies \(3g-3+n>0\). The augmented Teichmüller space \(\hat{T}(X)\) of \(X\) is the set of all equivalence classes \([R,f]\) of pairs of a Riemann surface \(R\) of type \((g,n)\) with nodes and a deformation \(f:X\to R\) which is a mapping such that it contracts some disjoint loops on \(X\) to points (the nodes of \(R\)) and is a homeomorphism except their loops on \(X\). Two pairs \((R_1,f_1)\) and \((R_2,f_2)\) are equivalent if there is a biholomorphic mapping \(h:R_1\to R_2\) such that \(f_2\) is homotopic to \(h\circ f_1\). Here, for Riemann surfaces with nodes \(R\) and \(S\), a homeomorphism \(f:R\to S\) is called biholomorphic if each restricted mapping of \(f\) which maps a component of \(R-\{\text{nodes of }R\}\) onto a component of \(S-\{\text{nodes of }S\}\) is biholomorphic. A topology on \(\hat{T}(X)\) is defined by the following neighborhoods.

For any compact neighborhood \(V\) of the set of nodes in \(R\) and any \(\varepsilon>0\), a neighborhood \(U_{V,\varepsilon}\) of a point \([R,f]\) is defined by

\[
U_{V,\varepsilon} = \{[S,g] \in \hat{T}(X) \mid \text{there is a deformation } h:S\to R \text{ which is } (1+\varepsilon)\text{-quasiconformal on } h^{-1}(R-V) \text{ such that } f \text{ is homotopic to } h \circ g\}.
\]

Let \(p = [Y,f] \in T(X)\) and \(q\) be a unit norm Jenkins-Strebel differential on \(Y\). We denote by \(r(t) = [Y_t,g_t \circ f]\) for any \(t \geq 0\) the Jenkins-Strebel ray on \(T(X)\) starting at \(p\) and having \(q\). We consider the end point of \(r\) in \(\hat{T}(X)\). We denote by \(H(q) = \sum_{j=1}^{k} b_j \gamma_j\) the measured foliation constructed by \(q\). The surface \(Y - \Gamma_q\) consists of annuli corresponding to \(\gamma_1, \cdots, \gamma_k\). By the \(q\)-coordinate \(z\), their annuli are represented by rectangles such that each rectangle identifies its vertical sides. Then, we denote by

\[
R_j(0) = \{z \in \mathbb{C} \mid 0 \leq \text{Re}z \leq a_j, \ 0 < \text{Im}z < b_j\}/(i\text{Im}z \sim a_j + i\text{Im}z)
\]

the annulus corresponding to \(\gamma_j\) where \(a_j = \text{Im}(\gamma_j,V(q))\) and \(\sim\) means the identification of vertical sides of the rectangle for any \(j = 1, \cdots, k\). Let \(m_j = \frac{b_j}{a_j}\) be a modulus of \(R_j(0)\) for any \(j = 1, \cdots, k\). Two horizontal sides of each \(R_j(0)\) are glued other ones by transformations of form \(z \mapsto \pm z + c\) where \(c \in \mathbb{C}\). The critical points of \(q\) are on the horizontal sides of each \(R_j(0)\). The Riemann surface \(Y\) is obtained by \(\overline{R_1(0)}, \cdots, \overline{R_k(0)}\) with these gluing mappings. Now, we view \(R_1(0), \cdots, R_k(0)\) as round annuli. First, for any \(j = 1, \cdots, k\), we cut \(R_j(0)\) at the half height:

\[
R_j^1(0) = \{z \in \mathbb{C} \mid 0 \leq \text{Re}z \leq a_j, \ 0 < \text{Im}z \leq \frac{b_j}{2}\}/(i\text{Im}z \sim a_j + i\text{Im}z),
\]

\[
R_j^2(0) = \{z \in \mathbb{C} \mid 0 \leq \text{Re}z \leq a_j, \ \frac{b_j}{2} \leq \text{Im}z < b_j\}/(i\text{Im}z \sim a_j + i\text{Im}z).
\]

The corresponding round annuli are defined by the following:

\[
A_j^1(0) = A_j^2(0) = \{w \in \mathbb{C} \mid \exp(-m_j\pi) \leq |w| < 1\}.
\]

The half surfaces \(R_j^1(0)\) is mapped to \(A_j^1(0)\) by the mapping \(z \mapsto w = \exp(2\pi \frac{z}{a_j})\), \(R_j^2(0)\) is mapped to \(A_j^2(0)\) by the mapping \(z \mapsto w = \exp(2\pi \frac{z-a_j}{a_j} - \frac{b_j}{2})\). We glue \(A_j^1(0)\) and \(A_j^2(0)\) at the line of these inner boundary \(|w| = \exp(-m_j\pi)\) by the mapping \(w \mapsto \frac{\exp(-2m_j\pi)w}{|w|}\) and we denote by \(A_j(0)\) the resulting surface. There is a
Similarly, we define \( R_j(0) = R_j^1(0) \cup R_j^2(0) \). This mapping is extended continuously to the mapping \( A_j(0) \) onto \( R_j(0) \). Then, we obtain the base surface \( Y \) after gluing \( A_1(0), \ldots, A_k(0) \) by the gluing mappings essentially the same as in the case of \( R_1(t), \ldots, R_k(t) \). We recall that the Teichmüller mapping \( g_t : Y \rightarrow Y_t \) is represented by \( z \mapsto z_t = e^{-t}x + ie^{ty} \) for any \( t \geq 0 \). We fix any \( t \geq 0 \). We denote by \( R_1(t), \ldots, R_k(t) \) the surfaces which are transformed from \( R_1(0), \ldots, R_k(0) \) by the mapping \( g_t \). For any \( j = 1, \ldots, k \), these are expressed by

\[
R_j(t) = \{ z_t \in \mathbb{C} \mid 0 \leq \text{Re}z_t \leq e^{-t}a_j, \ 0 < \text{Im}z_t < e^{t}b_j \}/(i\text{Im}z_t \sim e^{-t}a_j + i\text{Im}z_t). 
\]

Similarly, we define

\[
R_j^1(t) = \{ z_t \in \mathbb{C} \mid 0 \leq \text{Re}z_t \leq e^{-t}a_j, \ 0 < \text{Im}z_t \leq e^{t}b_j / (i\text{Im}z_t \sim e^{-t}a_j + i\text{Im}z_t), 
\]

\[
R_j^2(t) = \{ z_t \in \mathbb{C} \mid 0 \leq \text{Re}z_t \leq e^{-t}a_j, \ e^{t}b_j \leq \text{Im}z_t < e^{t}b_j / (i\text{Im}z_t \sim e^{-t}a_j + i\text{Im}z_t), 
\]

\[
A_j^1(t) = A_j^2(t) = \{ w_t \in \mathbb{C} \mid \exp(-e^{2t}m_j \pi) \leq |w_t| < 1 \}. 
\]

The mapping of \( R_j^1(t) \) onto \( A_j^1(t) \) is obtained by \( z_t \mapsto w_t = \exp(2\pi i \frac{e^{t}b_j}{a_j}) \), \( R_j^2(t) \) onto \( A_j^2(t) \) is obtained by \( z_t \mapsto w_t = \exp(2\pi i \frac{e^{t}b_j}{a_j}) \). The gluing mapping of \( A_j^1(t) \) onto \( A_j^2(t) \) is \( w_t \mapsto \exp(-2e^{2t}m_j \pi) \) in their inner boundary \( |w_t| = \exp(-e^{2t}m_j \pi) \), and we denote by \( A_j(t) \) the resulting surface. There is a biholomorphic mapping of \( A_j(t) \) onto \( R_j(t) \) which can be extended continuously to their closures. The surface \( Y_t \) is obtained by \( R_1(t), \ldots, R_k(t) \) with the gluing mappings the same as in the case of \( t = 0 \), that is, we use the coordinate \( z_t \) instead of \( z \). The surface \( Y_t \) is also obtained by \( A_1(t), \ldots, A_k(t) \) similarly. The Teichmüller mapping \( g_t \) is considered as the mapping of \( A_j^1(0) \) onto \( A_j^1(t) \) by \( w = re^{i\theta} \mapsto w_t = r\exp(2t)e^{i\theta} \) for any \( j = 1, \ldots, k \) and \( l = 1, 2 \). Therefore, the diagram of Figure 1 is commutative.

![Figure 1](image-url)
Remark. We notice that the moduli of $R_j(t)$ and $A_j(t)$ are equal to $e^{2t}m_j$ for any $t \geq 0$ and any $j = 1, \ldots, k$.

We consider the limit of $A_j(t)$ as $t \to \infty$ for any $j = 1, \ldots, k$. We set $A_j^1(\infty) = A_j^2(\infty) = \{w_\infty \in \mathbb{C} \mid 0 < |w_\infty| < 1\}$ and $\{pt\}$ is the set consists of an arbitrary point. The disjoint union $A_j(\infty) = A_j^1(\infty) \cup A_j^2(\infty) \cup \{pt\}$ becomes a complex cone by the following chart:

$$
\Phi : A_j(\infty) \to \{(w_1^j, w_2^j) \in \mathbb{C}^2 \mid |w_1^j| < 1, |w_2^j| < 1, w_1^j \cdot w_2^j = 0\},
$$

for any $j = 1, \ldots, k$. We denote by $Y_\infty$ the surface constructed by $A_1^1(\infty), \ldots, A_k^1(\infty)$ with the gluing mappings the same as in the case of $A_1^0, \ldots, A_k^0$ which are used the coordinate $w_\infty$ instead of $w$. The mapping $g_\infty : Y \to Y_\infty$ is constructed by the mapping of $A_j^1(0)$ onto $A_j^1(\infty) \cup \{pt\}$ by $w \to re^{i\theta} \to w_\infty = h_j(r)e^{i\theta}$ where $h_j : [\exp(-m_j), 1) \to (0, 1]$ is an arbitrary monotonously increasing diffeomorphism for any $j = 1, \ldots, k$ and $l = 1, 2$. The homotopy class of $g_\infty$ is independent of the choices of $h_j$ for any $j = 1, \ldots, k$. Then, we obtain $[Y_\infty, g_\infty \circ f]$ in $T(X)$ and denote it by $r(\infty)$. From the definition of a neighborhood of $[Y_\infty, g_\infty \circ f]$, the following proposition holds.

**Proposition 2.8.** ([HS07]) The Jenkins-Strebel ray $r(t) = [Y_t, g_t \circ f]$ on $T(X)$ starting at $p = [Y, f]$ and having the unit norm Jenkins-Strebel differential $q$ on $Y$ converges to a point $r(\infty) = [Y_\infty, g_\infty \circ f]$ in $T(X)$.

Remark. We can reconstruct the surface $Y_t$ from $Y_\infty$ for any $t \geq 0$. First, for any $j = 1, \ldots, k$ and $l = 1, 2$, we remove the punctured disk $\{w_\infty \in \mathbb{C} \mid 0 < |w_\infty| < \exp(-2t)\}$ from $A_j^l(\infty)$ of $Y_\infty$. Then, the interior of each resulting annulus is biholomorphic to the interior of $A_j(t)$. We obtain $Y_t$ by gluing $\{A_j(t)\}_{j=1,\ldots,k}$ with the suitable gluing mappings.

2.7. **Extremal lengths.** Let $\rho$ be a locally $L^1$-measurable conformal metric on a Riemann surface $Y$, i.e., it is represented by the form $\rho = \rho(z)|dz|$ on any local coordinate $z$ of $Y$ where $\rho(z)$ is a non-negative measurable function of $z$ such that the equation $\rho(z)|dz| = \rho(w)|dw|$ holds for any local coordinates $z, w$ on a common neighborhood $U$ of $Y$. For any $\gamma \in S$, we define the $\rho$-length of $\gamma$ on $Y$ and $\rho$-area of $Y$ by

$$
l_\rho(\gamma) = \inf_{\gamma' \simeq \gamma} \int_\gamma \rho(z)|dz|,
$$

$$
A_\rho = \iint_Y \rho(z)^2 dx dy
$$

respectively. The **extremal length** $\text{Ext}_Y(\gamma)$ of $\gamma$ on $Y$ is defined by the following:

$$
\text{Ext}_Y(\gamma) = \sup_{\rho} \frac{l_\rho(\gamma)^2}{A_\rho},
$$

where $\rho$ ranges over all locally $L^1$-measurable conformal metrics on $Y$ such that $0 < A_\rho < \infty$.

For any $p = [Y, f] \in T(X)$ and any $\alpha \in S$ on $X$, we define
\section*{3. Proof of Theorem}

In this section, we prove Theorem \ref{thm:main} which is our main theorem.

\begin{theorem}
Let $p = [Y, f], p' = [Y', f'] \in T(X)$, $q, q'$ be unit norm Jenkins-Strebel differentials on $Y, Y'$ and $r, r'$ be Jenkins-Strebel rays on $T(X)$ starting at $p, p'$ and having $q, q'$ respectively. Let $r(\infty), r'(\infty)$ be the end points of $r, r'$ on the augmented Teichmüller space $T(X)$ respectively. We denote by $H(q) \in MF(Y)$, $H(q') \in MF(Y')$ the measured foliations corresponding to $q, q'$ respectively. Suppose that the measured foliations $f^{-1}_*(H(q)), f'^{-1}_*(H(q')) \in MF(X)$ are topologically equivalent. In this situation, we can write $f^{-1}_*(H(q)) = \sum_{j=1}^k b_j \gamma_j$, $f'^{-1}_*(H(q')) = \sum_{j=1}^k b'_j \gamma'_j$, $H(q) = \sum_{j=1}^k b_j f_* \gamma_j$ and $H(q') = \sum_{j=1}^k b'_j f'_* \gamma_j$ where $b_1, \ldots, b_k, b'_1, \ldots, b'_k$ are positive real numbers and $\gamma_1, \ldots, \gamma_k$ are distinct simple closed curves on $X$ such that $i(\gamma_j, \gamma'_j) = 0$ for any $j \neq j'$. We denote by $a_j = i(f_*(\gamma_j), V(q)), a'_j = i(f'_*(\gamma_j), V(q'))$ the circumferences of the annuli of $H(q), H(q')$ corresponding to $f_*(\gamma_j), f'_*(\gamma_j)$, and $m_j = \frac{b_j}{a_j}, m'_j = \frac{b'_j}{a'_j}$ the corresponding moduli respectively, for any $j = 1, \ldots, k$. If $r(\infty) = r'(\infty)$, then

\[
\lim_{t \to \infty} d_T(X)(r(t), r'(t)) = \frac{1}{2} \log \max_{j=1, \ldots, k} \left\{ \frac{m'_j}{m_j}, \frac{m_j}{m'_j} \right\}.
\]

We represent the Jenkins-Strebel rays by $r(t) = [Y_t, g_t \circ f], r'(t) = [Y'_t, g'_t \circ f']$ for any $t \geq 0$, and their end points by $r(\infty) = [Y_\infty, g_\infty \circ f], r'(\infty) = [Y'_\infty, g'_\infty \circ f']$ respectively. Since the measured foliations $f^{-1}_*(H(q)), f'^{-1}_*(H(q'))$ are topologically equivalent, there is a homeomorphism $\alpha : X \to X$ which is homotopic to the identity such that the leaves of $f^{-1}_*(H(q))$ are mapped to the leaves of $f'^{-1}_*(H(q'))$. Then, the mapping $f' \circ \alpha \circ f^{-1}$ which is homotopic to $f' \circ f^{-1}$ maps the leaves of $H(q)$ to the leaves of $H(q')$. We denote by $A_j(0), A'_j(0)$ the annuli corresponding to $f_*(\gamma_j), f'_*(\gamma_j)$ respectively, for any $j = 1, \ldots, k$. They are the same as in \ref{cor:main}. In this assumption, the annuli $A_j(0), A'_j(0)$ have the moduli $m_j, m'_j$ respectively,
for any \( j = 1, \cdots, k \). Then, the equations \( f'_j \circ \alpha \circ f^{-1}_j(f_j(\gamma_j)) = f'_j(\gamma_j), \quad f' \circ \alpha \circ f^{-1}(A_j(0)) = A'_j(0) \) hold for any \( j = 1, \cdots, k \). For any \( t \geq 0 \) and any \( j = 1, \cdots, k \), the annuli \( A_j(t), A'_j(t) \) can be determined the same as in \( \text{[2.6]} \) and we can see that \((g'_j \circ f') \circ \alpha \circ (g_t \circ f)^{-1}(A_j(t)) = A'_j(t)\).

In order to prove the equation of Theorem 3.1 we consider the upper and lower estimates of the limit supremum and limit infimum of the distance between given rays respectively.

3.1. **Upper estimate.** First, we show that

\[
\limsup_{t \to \infty} d_{T(X)}(r(t), r'(t)) \leq \frac{1}{2} \log \max_{j=1, \cdots, k} \left\{ \frac{m'_j}{m_j}, \frac{m'_j}{m'_j} \right\}.
\]

The idea of the following lemma comes from \( \text{[Gup11]} \).

**Lemma 3.2.** Let us choose \( 0 < \varepsilon < 1 \) arbitrary. Then, for any sufficiently large \( t \), there is a quasiconformal mapping \( F_t : Y_t \to Y'_t \) which is homotopic to \((g'_t \circ f') \circ (g_t \circ f)^{-1}\) such that the inequality \( \lim_{t \to \infty} K(F_t) < \max_{j=1, \cdots, k} \left\{ \frac{m'_j}{m_j}, \frac{m'_j}{m'_j} \right\} + \varepsilon \)

holds.

**Proof.** We set \( M_j = \frac{m'_j}{m_j} \) for any \( j = 1, \cdots, k \). By the equation \( r(\infty) = r'(\infty) \), there exists a biholomorphic mapping \( h : Y_{\infty} \to Y'_{\infty} \) such that \( h \circ g_{\infty} \circ f \) is homotopic to \( g'_{\infty} \circ f' \). From \( \text{[2.6]} \) we can write

\[
Y_{\infty} = \bigcup_{j=1}^k \overline{A^l_j(\infty) \cup A^l'_j(\infty)} \cup \bigcup_{j=1}^k \overline{A^l_j(\infty) \cup A^l'_j(\infty)},
\]

where \( A^l_j(\infty), A^l'_j(\infty) \) are the punctured disk \( \mathbb{D}^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \) for any \( j = 1, \cdots, k \) and \( l = 1, 2 \). Now, we fix any \( j = 1, \cdots, k \) and \( l = 1, 2 \). We set \( h^l_j = h|A^l_j(\infty) : A^l_j(\infty) \to h(A^l'_j(\infty)) \subset Y^l'_{\infty} \). Since \( h \) is a biholomorphic mapping, then we can set \( h^l_j(0) = 0 \) and \( \frac{dh^l_j(z)}{dz} \bigg|_{z=0} \neq 0 \). We describe \( h^l_j(z) = c^l_1 z + c^l_2 z^2 + \cdots + c^l_j z + \psi^l_j(z) \) where \( c^l_j \neq 0 \), \( -\pi < \arg c^l_j \leq \pi \) and \(-\pi \leq \arg c^l_2 < \pi \). For any \( t \geq 0 \), we set \( \delta^l_j(t) = \exp(-e^{2t}m_j \pi), \delta^l_j(t) = \exp(-e^{2t}m'_j \pi) \), then \( \delta'_j(t) = \delta^l_j(t)^M_j \).

After this, we assume that \( A^l_j(t) = \mathbb{D}^* - \mathbb{D}_{\delta^l_j(t)} = \{ z \in \mathbb{C} \mid |\delta^l_j(t) | \leq |z| < 1 \} \) and \( A^l'_j(t) = \mathbb{D}^* - \mathbb{D}_{\delta'_j(t)} = \{ z \in \mathbb{C} \mid |\delta'_j(t) | \leq |z| < 1 \} \) for any \( t \geq 0 \). For sufficiently large \( t \), we construct a quasiconformal mapping \( F_{j,t}^l : \mathbb{D}^* - \mathbb{D}_{\delta^l_j(t)} \to h^l_j(\mathbb{D}^*) - \mathbb{D}_{\delta'_j(t)} \). We consider the following three cases (1), (2) and (3).

(1) In the case of \( M_j > 1 \), we take \( X_j \) as

\[
X_j < \frac{\log M_j + \varepsilon - 1}{\log M_j} < 0.
\]

This is equivalent to

\[
M_j^{X_j} < \frac{\varepsilon}{M_j + \varepsilon - 1} < 1
\]
and
\[
\frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon.
\]

We take sufficiently large \( t \) such that the inequality \( \delta_j(t)^M_j < |c_j|^2 \delta_j(t)^M_j \) holds.

We set \( \Delta_j(t) = \delta_j(t)^M_j \). We construct \( F_{j,t}^1 \) by the following:
\[
F_{j,t}^1(z) = \begin{cases} 
P_{j,t}^1(z) & (\delta_j(t) \leq |z| \leq \Delta_j(t)) \\
Q_{j,t}^1(z) & (\Delta_j(t) \leq |z| \leq 2\Delta_j(t)) \\
h_{j,t}^1(z) & (2\Delta_j(t) \leq |z| < 1) 
\end{cases}
\]

(i) In \( \delta_j(t) \leq |z| \leq \Delta_j(t) \), we set
\[
P_{j,t}^1(z) = \Delta_j(t)^{-m_j z} \cdot e^t \cdot \frac{1}{1 - M_j^{X_j}} \cdot \frac{1}{\log \Delta_j(t) - \log \delta_j(t)} \cdot \frac{1}{|z|} \cdot \Delta_j(t)^{-M_j^{X_j}} \cdot z,
\]
which satisfies \( P_{j,t}^1(z) = \delta_j(t)^{M_j - 1} \cdot z \) on \( |z| = \delta_j(t) \), \( P_{j,t}^1(z) = c_j^2 \cdot z \) on \( |z| = \Delta_j(t) \). We can construct this function as follows. In Figure 2 the mapping \( \kappa_1 \) is a conformal mapping
\[
\kappa_1(z) = \frac{e^{-t}a_j}{2\pi i} \log z,
\]
the mapping \( \kappa_2 \) is a translation
\[
\kappa_2(z) = z - i \frac{e^t b_j M_j^{X_j}}{2},
\]
the mapping \( \kappa_3 \) is an affine transformation
\[
\kappa_3 : z = x + iy \mapsto x + \alpha_j^1(t)y + i\beta_j^1(t)y
\]
such that
\[
\alpha_j^1(t) = \frac{-\arg c_j}{e^{2\pi i M_j \pi (1 - M_j^{X_j})}}, \quad \beta_j^1(t) = \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} + \frac{\log |c_j|^2}{2\pi i M_j \pi },
\]
the mapping \( \kappa_4 \) is also a translation
\[
\kappa_4(z) = z + \frac{e^{-t}a_j}{2\pi} \arg c_j + i \left( \frac{e^t b_j M_j^{X_j}}{2} - \frac{e^{-t}a_j}{2\pi} \log |c_j|^2 \right),
\]
and the mapping \( \kappa_5 \) is a conformal mapping
\[
\kappa_5(z) = \exp \left( \frac{2\pi i}{e^{-t}a_j} z \right)
\]
which is the inverse of \( \kappa_1 \). The composition \( P_{j,t}^1 = \kappa_5 \circ \kappa_4 \circ \kappa_3 \circ \kappa_2 \circ \kappa_1 \) is the desired function.

By the construction, the mapping \( P_{j,t}^1 \) is a quasiconformal mapping, and its dilatation is equal to \( K(\kappa_3) \). We see that \( \alpha_j^1(t) \rightarrow 0 \), \( \beta_j^1(t) \rightarrow \frac{M_j - M_j^{X_j}}{1 - M_j^{X_j}} > 1 \) as \( t \rightarrow \infty \). Then, the maximal dilatation of \( P_{j,t}^1 \) satisfies
Figure 2.

\begin{equation}
K(P_{j,t}^l) = \frac{|1 + \beta_j^l(t) - i\alpha_j^l(t)| + |1 - \beta_j^l(t) + i\alpha_j^l(t)|}{|1 + \beta_j^l(t) - i\alpha_j^l(t)| - |1 - \beta_j^l(t) + i\alpha_j^l(t)|} \to M_j - M_j^{X_j} \leq M_j + \varepsilon
\end{equation}

\begin{equation}
as t \to \infty.
\end{equation}

(ii) In \(\Delta_j(t) \leq |z| \leq 2\Delta_j(t)\), we set

\begin{equation}
Q_{j,t}^l(z) = c_j^l z + \phi_{\Delta_j(t)}(|z|)\psi_j^l(z),
\end{equation}

where \(\phi_{\Delta_j(t)} : [\Delta_j(t), 2\Delta_j(t)] \to [0, 1]\) is defined by

\begin{equation}
\phi_{\Delta_j(t)}(|z|) = \frac{|z|}{\Delta_j(t)} - 1.
\end{equation}

This function satisfies \(Q_{j,t}^l(z) = c_j^l z \text{ on } |z| = \Delta_j(t), Q_{j,t}^l(z) = h_j^l(z) \text{ on } |z| = 2\Delta_j(t)\).

We consider the partial derivatives of \(Q_{j,t}^l\),

\begin{equation}
\partial_z Q_{j,t}^l = c_j^l + \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} z^{-\frac{1}{2}} \psi_j^l(z) + \phi_{\Delta(t)}(|z|) \frac{d\psi_j^l(z)}{dz},
\end{equation}

\begin{equation}
\partial_z Q_{j,t}^l = \frac{1}{2\Delta_j(t)} z^{\frac{1}{2}} z^{-\frac{1}{2}} \psi_j^l(z).
\end{equation}

There is \(C > 0\) such that \(|\psi_j^l(z)| \leq C\Delta_j(t)^2\) for sufficiently large \(t\). We see that

\begin{equation}
\left| \frac{1}{2\Delta_j(t)} z^{\frac{1}{2}} z^{-\frac{1}{2}} \psi_j^l(z) \right| = \left| \frac{1}{2\Delta_j(t)} z^{-\frac{1}{2}} z^{\frac{1}{2}} \psi_j^l(z) \right| = \frac{|\psi_j^l(z)|}{2\Delta_j(t)} \leq \frac{C\Delta_j(t)}{2} \to 0
\end{equation}
as \( t \to 0 \). Then, \( |\partial_z Q_{j,t}^l| \to 0 \), \( |\partial_z Q_{j,t}^r| \to |c_j^t| \neq 0 \) as \( t \to 0 \). For sufficiently
large \( t \), \( \text{Jac } Q_{j,t}^l = |\partial_z Q_{j,t}^l|^2 - |\partial_z Q_{j,t}^r|^2 \neq 0 \). Hence, \( Q_{j,t}^l \) is a local homeomorphism.

We denote by \( D \) the closed set whose boundary is consists of two components
\( Q_{j,t}^l([|z| = \Delta_j(t)]) \) and \( \{ |w| = |c_j^t|\Delta_j(t) \} \) and \( Q_{j,t}^l([|z| = 2\Delta_j(t)]) \) is
\( h_j^l([|z| = 2\Delta_j(t)]) \), and its fundamental group is \( \pi_1(D) = \mathbb{Z} \). The equation \( Q_{j,t}^l([\Delta_j(t) \leq |z| \leq 2\Delta_j(t)]) = D \) holds because \( Q_{j,t}^l \) is a local homeomorphism with the above
boundary conditions. We regard the mapping \( Q_{j,t}^l : \{ \Delta_j(t) \leq |z| \leq 2\Delta_j(t) \} \to D \) as a
covering mapping. Let \( Q_{j,t}^l : \pi_1(\{ \Delta_j(t) \leq |z| \leq 2\Delta_j(t) \}) \to \pi_1(D) \) be the group
homomorphism induced by \( Q_{j,t}^l \). We see that \( Q_{j,t}^l(\pi_1(\{ \Delta_j(t) \leq |z| \leq 2\Delta_j(t) \})) = Z \circ \pi_1(D) \) because \( Q_{j,t}^l(z) = c_j^t z \) on \( |z| = \Delta_j(t) \). Then, the covering mapping \( Q_{j,t}^l \)
is a regular covering, and its covering transformation group is \( \mathbb{Z}/\mathbb{Z} = 1 \). Therefore, \( Q_{j,t}^l \) is a
homeomorphism. By the derivatives of \( Q_{j,t}^l \), for sufficiently large \( t \), it is a
quasiconformal mapping such that its dilatation holds \( K(Q_{j,t}^l) \to 1 \) as \( t \to \infty \).

(iii) In \( 2\Delta_j(t) \leq |z| < 1 \), \( F_{j,t}^l(z) = h_j^l(z) \) and \( K(h_j^l) = 1 \).

Therefore, for sufficiently large \( t \), we obtain the quasiconformal mapping \( F_{j,t}^l \) such that

\[
K(F_{j,t}^l) = \max\{ K(P_{j,t}^l), K(Q_{j,t}^l) \} \to \frac{M_j - M_j^{-X_j}}{1 - M_j^{X_j}} < M_j + \varepsilon
\]
as \( t \to \infty \).

(2) In the case of \( M_j < 1 \), we take \( X_j \) as

\[
X_j > \frac{\log \frac{M_j \varepsilon}{M_j - 1 + \varepsilon}}{\log M_j} > 2.
\]

This is equivalent to

\[
M_j^{X_j} < \frac{M_j \varepsilon}{M_j - 1 + \varepsilon} < M_j^2
\]

and

\[
1 - \frac{M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j + \varepsilon}.
\]

We take sufficiently large \( t \) such that the inequality \( \delta_j(t)^{M_j} < |c_j^t| \delta_j(t)^{M_j^{X_j}} \) holds.

We also set \( \Delta_j(t) = \delta_j(t)^{M_j^{X_j}} \), and also construct \( F_{j,t}^l \) following,

\[
F_{j,t}^l(z) = \begin{cases} 
     P_{j,t}^l(z) & (\delta_j(t) \leq |z| \leq \Delta_j(t)) \\
     Q_{j,t}^l(z) & (\Delta_j(t) \leq |z| \leq 2\Delta_j(t)) \\
     h_j^l(z) & (2\Delta_j(t) \leq |z| < 1)
\end{cases}
\]

The functions \( P_{j,t}^l, Q_{j,t}^l \) have the same notations as in the case of (1). The difference
is only the dilatation of \( P_{j,t}^l \). In this case, \( \delta_j^l(t) \to \frac{M_j - M_j^{-X_j}}{1 - M_j^{X_j}} < 1 \) as \( t \to \infty \).

Therefore,
The functions $P_{j,t}^l$, $Q_{j,t}^l$ hold. Since in the case of (1), for sufficiently large $t$, we obtain the quasiconformal mapping $F_{j,t}^l$ such that

$$K(F_{j,t}^l) = \max\{K(P_{j,t}^l), K(Q_{j,t}^l)\} \to \frac{1 - M_j^{X_j}}{M_j - M_j^{X_j}} < \frac{1}{M_j} + \varepsilon$$

as $t \to \infty$. Similarly as in the case of (1), for sufficiently large $t$, we obtain the quasiconformal mapping $F_{j,t}^l$ such that

$$K(F_{j,t}^l) = \max\{K(P_{j,t}^l), K(Q_{j,t}^l)\} \to \frac{1}{M_j} + \varepsilon$$

as $t \to \infty$.

(3) In the case of $M_j = 1$, we take sufficiently large $t$ such that the inequality $\delta_j(t) < |c_j|\delta_j(t)^\frac{1}{2}$ holds and set $\Delta_j(t) = \delta_j(t)^\frac{1}{2}$. We set

$$F_{j,t}^l(z) = \begin{cases} P_{j,t}^l(z) = c_j^l \left(1 - \frac{\log |z|}{\log \delta_j(t)} \right) z & (\delta_j(t) \leq |z| \leq \Delta_j(t)) \\ Q_{j,t}^l(z) & (\Delta_j(t) \leq |z| \leq 2\Delta_j(t)) \\ h_j^l(z) & (2\Delta_j(t) \leq |z| < 1) \end{cases}$$

The functions $P_{j,t}^l, Q_{j,t}^l$ are constructed similarly as in the case of (1). On $P_{j,t}^l$, the above notation is obtained by changing $M_j, M_j^{X_j}$ to $1, \frac{1}{2}$ respectively, in (i) of the case of (1). In this time, $\alpha_j^l(t) = \frac{2\arg c_j^l}{\varepsilon + M_j^l} \to 0$, $\beta_j^l(t) = 1 + \frac{2\log |c_j^l|}{\varepsilon + M_j^l} \to 1$ and then, $K(P_{j,t}^l) \to 1$ as $t \to \infty$. The function $Q_{j,t}^l$ also satisfying $K(Q_{j,t}^l) \to 1$ as $t \to \infty$. Therefore, for sufficiently large $t$, the quasiconformal mapping $F_{j,t}^l$ satisfies

$$K(F_{j,t}^l) = \max\{K(P_{j,t}^l), K(Q_{j,t}^l)\} \to 1$$

as $t \to \infty$.

Thus, by (1), (2), and (3), for sufficiently $t$, we construct the quasiconformal mapping $F_t : Y_t \to Y'_t$ by gluing $\{F_{j,t}^l\}_{t=1,2}$. For any case of (1), (2), and (3), each $h_j^l$ is homotopic to $(g_j^l \circ f') \circ (g_j \circ f)^{-1}$. Each $Q_{j,t}^l$ satisfies $K(Q_{j,t}^l) \to 1$ as $t \to \infty$ and the domain $\{\Delta_j(t) < |z| < 2\Delta_j(t)\}$ has the constant modulus for any $t$. Each $P_{j,t}^l$ produces the twist of angle $\arg c_j^l$ in the domain $\{\delta_j(t) < |z| < \Delta_j(t)\}$ and satisfies $|\arg c_j^1 + \arg c_j^2| < 2\pi$. Therefore, for sufficiently $t$, the mapping $F_t$ is homotopic to $(g_j^l \circ f') \circ (g_j \circ f)^{-1}$. We conclude that

$$\lim_{t \to \infty} K(F_t) = \lim_{t \to \infty} \max_{j=1,\ldots,k, l=1,2} K(F_{j,t}^l) < \max_{j=1,\ldots,k} \left\{ M_j, \frac{1}{M_j} \right\} + \varepsilon.$$ 

Therefore, by this lemma, for any sufficiently large $t$, the inequality

$$\limsup_{t \to \infty} d_{T(X)}(r(t), r'(t)) \leq \frac{1}{2} \log K(F_t) < \frac{1}{2} \log \left( \max_{j=1,\ldots,k} \left\{ \frac{m_j'}{m_j}, \frac{m_j}{m_j'} \right\} \right) + \varepsilon$$

holds. Since $\varepsilon$ is arbitrary, we are done.
3.2. **Lower estimate.** Next, we show that

\[
\liminf_{t \to \infty} d_T(X)(r(t), r'(t)) \geq \frac{1}{2} \log \max_{j=1, \ldots, k} \left\{ \frac{m'_j}{m_j} \right\}.
\]

We use the following theorems.

**Theorem 3.3.** ([Wal12]) Let \( r \) be a Teichmüller geodesic ray on \( T(X) \) starting at \( p = \{ Y, f \} \) and having the unit norm holomorphic quadratic differential \( q \) on \( Y \) with the corresponding measured foliation \( H(q) = \sum_{j=1}^k b_j f_*(G_j) \in \mathcal{MF}(Y) \) where \( G_j \) is a simple closed curve or an ergodic measure for any \( j = 1, \ldots, k \) such that these are projectively-distinct and pairwise having zero intersection number, and \( b_j > 0 \) if \( G_j \in \mathcal{S}, b_j \geq 0 \) if \( G_j \) is an ergodic measure. We set \( m_j = \frac{b_j}{\sum_{j=1}^k b_j} \) for any \( j = 1, \ldots, k \). If we write

\[
\mathcal{E}_r(\mu) = \left\{ \frac{1}{2} \sum_{j=1}^k m_j i(G_j, \mu)^2 \right\}^{\frac{1}{2}}
\]

for any \( \mu \in \mathcal{MF}(X) \), then, the equation

\[
\lim_{t \to \infty} e^{-2t} \text{Ext}_{r(t)}(\mu) = \sum_{j=1}^k m_j i(f_*(G_j), f_*(\mu))^2 = \mathcal{E}_r(\mu)^2
\]

holds.

**Theorem 3.4.** ([Wal12]) Let \( r, r' \) be Teichmüller geodesic rays on \( T(X) \) starting at \( p = \{ Y, f \}, p' = \{ Y', f' \} \) and having the unit norm holomorphic quadratic differentials \( q, q' \) on \( Y, Y' \), and \( H(q) \in \mathcal{MF}(Y), H(q') \in \mathcal{MF}(Y') \) be the corresponding measured foliations respectively. We set \( Z = \{ \mu \in \mathcal{MF}(X)^* \mid \mathcal{E}_r(\mu) = \mathcal{E}_{r'}(\mu) = 0 \} \). Suppose that \( H(q) = \sum_{j=1}^k b_j f_*(G_j) \), where \( b_1, \ldots, b_k \) are positive real numbers and \( G_j \) is a simple closed curve or an ergodic measure for any \( j = 1, \ldots, k \) such that these are projectively-distinct and pairwise having zero intersection number. If we can write \( H(q') = \sum_{j=1}^k b'_j f'_*(G_j) \) where \( b'_j \geq 0 \), and set \( m_j = \frac{b'_j}{\sum_{j=1}^k b'_j} \) for any \( j = 1, \ldots, k \), then,

\[
\sup_{\mu \in \mathcal{MF}(X)^*} \frac{\mathcal{E}_{r'}(\mu)^2}{\mathcal{E}_r(\mu)^2} = \max_{j=1, \ldots, k} \left\{ \frac{m'_j}{m_j} \right\}.
\]

Otherwise, the above supremum is \(+\infty\).

**Remark.** If \( f^{-1}_r(H(q)), f'^{-1}_r(H(q')) \) are absolutely continuous, then \( \sup \frac{\mathcal{E}_{r'}(\mu)^2}{\mathcal{E}_r(\mu)^2} \) and \( \sup \frac{\mathcal{E}_r(\mu)^2}{\mathcal{E}_{r'}(\mu)^2} \) are both finite.

We notice that the desired inequality (1) holds for the rays \( r, r' \) which satisfy that \( f^{-1}_r(H(q)), f'^{-1}_r(H(q')) \) are absolutely continuous. So, we do not require the conditions that the rays are Jenkins-Strebel and \( r(\infty) = r'(\infty) \). We write \( H(q) = \sum_{j=1}^k b_j f_*(G_j), H(q') = \sum_{j=1}^k b'_j f'_*(G_j) \) where \( b_j, b'_j > 0 \) and set \( m_j = \frac{b_j}{\sum_{j=1}^k b_j}, m'_j = \frac{b'_j}{\sum_{j=1}^k b'_j} \) for any \( j = 1, \ldots, k \). Therefore, in our case, by
Kerckhoff’s formula and the above two theorems,
\[
\liminf_{t \to \infty} d_{T(X)}(r(t), r'(t)) = \liminf_{t \to \infty} \frac{1}{2} \log \sup_{\mu \in \mathcal{MF}(X)} \frac{\operatorname{Ext}_{r(t)}(\mu)}{\operatorname{Ext}_{r'(t)}(\mu)} \\
\geq \frac{1}{2} \log \sup_{\mu \in \mathcal{MF}(X), \epsilon > 0} \liminf_{t \to \infty} e^{-2t} \operatorname{Ext}_{r(t)}(\mu) \\
= \frac{1}{2} \log \max_{j=1, \ldots, k} m'_j.
\]

This inequality comes from the fact that the limit of the supremum is greater than or equal to the supremum of the limit. Since the symmetry of the distance, the inequality (1) holds.

**Remark.** If \( f^{-1}_*(H(q)), f'^{-1}_*(H(q')) \) are not absolutely continuous, then
\[
\liminf_{t \to \infty} d_{T(X)}(r(t), r'(t)) = +\infty.
\]

**Proof of Theorem 3.1.** Since the upper and lower estimates, we obtain the desired equation.

**Corollary 3.5.** For any two Jenkins-Strebel rays \( r, r' \), they are asymptotic if and only if \( r, r' \) are modularly equivalent and \( r(\infty) = r'(\infty) \).

**Proof.** Under the assumption of Theorem 3.1 if in addition the given rays \( r, r' \) are modularly equivalent, then for \( \alpha = -\frac{1}{2} \log \lambda \),
\[
\lim_{t \to \infty} d_{T(X)}(r(t), r'(t + \alpha)) = \frac{1}{2} \log \max_{j=1, \ldots, k} \left\{ e^{2\alpha} m'_j, \frac{m_j}{e^{2\alpha} m'_j} \right\} = \frac{1}{2} \log 1 = 0.
\]

This means that the rays \( r, r' \) are asymptotic. Conversely, if two Jenkins-Strebel rays \( r, r' \) are asymptotic, we can set
\[
\lim_{t \to \infty} d_{T(X)}(r(t), r'(t)) = 0
\]
without loss of generality. From the above remark, \( f^{-1}_*(H(q)), f'^{-1}_*(H(q')) \) are absolutely continuous. By the inequality (1), \( m_j = m'_j \) for any \( j = 1, \ldots, k \). Therefore, the rays \( r, r' \) are modularly equivalent. Next, we show that the correspondence between the end points \( r(\infty), r'(\infty) \). The proof is similar as Proposition 2.8 ([HS07]). For any \( \epsilon > 0 \) and any compact neighborhood \( V \) of the set of nodes in \( Y(\infty) \), we recall the neighborhood \( U_{V, \epsilon}(r(\infty)) \) of \( r(\infty) \):
\[
U_{V, \epsilon}(r(\infty)) = \{ [S, g] \in \tilde{T}(X) \mid \text{there is a deformation } h : S \to Y_\infty \text{ which is } (1 + \epsilon)-\text{quasiconformal on } h^{-1}(Y_\infty - V) \text{ such that } g_\infty \circ f \sim h \circ g \}.
\]

We set \( V = V_1 \cup \cdots \cup V_k \) where \( V_j = V_j^1 \cup V_j^2 \cup \{pt\}, V_j^i = \{0 < |z| \leq \epsilon_j \} \subset A_j^i(\infty), 0 < \epsilon_j < 1 \) for any \( j = 1, \ldots, k \) and \( l = 1, 2 \). There exists \( T > 0 \) such that for any \( t > T \) and any \( j = 1, \ldots, k, \delta_j(t) < \epsilon_j \) and \( d_{T(X)}(r(t), r'(t)) < \frac{1}{2} \log(1+\epsilon), \text{i.e., there exists a } (1+\epsilon)-\text{quasiconformal mapping } \alpha_t : Y'_t \to Y_t \text{ such that } g_t \circ f \sim \alpha_t \circ g'_t \circ f' \). For
any $t > T$, we want to show that $r'(t) \in U_{V,\varepsilon}(r(\infty))$. We construct the deformation $h : Y'_t \to Y_\infty$ as follows. For any $j = 1, \cdots, k$ and $l = 1, 2$,

$$h_{|\alpha_t^{-1}(A^t_j)}(w) = \begin{cases} \alpha_t(w) & (w \in \alpha_t^{-1} (\{t_j \leq |z| < 1\})) \\ h_{j,t}(\alpha_t(w)) e^{t \text{arg} \alpha_t(w)} & (w \in \alpha_t^{-1} (\{t_j < |z| < \varepsilon_j\})) \\ p^\ell & (w \in \alpha_t^{-1} (\{|z| = \varepsilon_j(t)\})) \end{cases}$$

where $h_{j,t} : (\delta_j(t), t_j) \to (0, t_j)$ is an arbitrary monotonously increasing diffeomorphism. The mapping $h_{|\alpha_t^{-1}(A^t_j)}$ is equal to the $(1 + \varepsilon)$-quasiconformal mapping $\alpha_t$ on $h_{|\alpha_t^{-1}(A^t_j)}^{-1}(A^t_j(\infty) - V^t_j)$. By $h \sim g_\infty \circ g_t^{-1} \circ \alpha_t$, we obtain $h \circ g'_t \circ f' \sim g_\infty \circ f$. This means that $r'(t) \in U_{V,\varepsilon}(r(\infty))$ for any $t > T$. Since $\varepsilon$ and $V$ are arbitrary, we conclude that $r(\infty) = r'(\infty)$. \hfill $\square$

### 4. The detour metric

In this section, we obtain the minimum value of the equation of Theorem 3.1\ref{3.1} when we shift the initial points of the given two rays. This value is represented by the detour metric between the end points of the rays on the Gardiner-Masur boundary of $T(X)$.

#### 4.1. The horofunction boundary of metric spaces and the detour metric.

We recall the definition of the detour metric in the case of the general metric space, and refer the reader to \cite{Wall} for more details. First, we consider the horofunction compactification of a metric space. This is given by Gromov \cite{Gro81}.

Let $(X, d)$ be a metric space, $b$ be a base point of $X$. The distance $d$ on $X$ is called proper if any closed ball with respect to $d$ is compact, geodesic if for any two points in $X$, there exists a geodesic which joins them. Suppose that the distance $d$ has the two properties which are proper and geodesic. It is well known that any Teichmüller distance satisfies these conditions. We denote by $C(X)$ the set of all continuous functions of $X$ into $\mathbb{R}$ which is equipped with the topology of the uniform convergence on any compact set of $X$. We define a mapping $\psi : X \to C(X)$ by $z \mapsto \{\psi_z(x) := d(x, z) - d(b, z)\}_{x \in X}$.

**Theorem 4.1.** (\cite{Gro81}) The mapping $\psi$ is an embedding and the set $\psi(X)$ is relatively compact on $C(X)$.

By this theorem, the space $X$ is identified with $\psi(X)$. The closure $\overline{\psi(X)}$ is called the horofunction compactification of $X$. The boundary $X(\infty) = \psi(X) - \psi(X)$ is called the horofunction boundary of $X$. We call $\xi \in X(\infty)$ a horofunction. We can denote by $X \cup X(\infty)$ the horofunction compactification of $X$.

**Remark.** The topological space $X \cup X(\infty)$ satisfies the first countability axiom.

**Definition 4.2.** For any $\xi, \xi' \in X(\infty)$, we define the detour cost

$$H(\xi, \xi') = \sup_{W \ni x \in \text{W} \cap X} \inf_{d(b, x) + \xi'(x)}$$

where $W$ ranges over all neighborhoods of $\xi$ in $X \cup X(\infty)$.

There is another definition of the detour cost.
Definition 4.3. For any $\xi, \xi' \in X(\infty)$,

$$H(\xi, \xi') = \inf_{\gamma} \liminf_{t \to \infty} (d(b, \gamma(t)) + \xi'(\gamma(t))),$$

where $\gamma$ ranges over all paths $\gamma: \mathbb{R}_{\geq 0} \to X$ which converge to $\xi$.

Definition 4.4. Let $A \subset [0, +\infty)$ be an unbounded set which contains $0$. A mapping $r : A \to X$ is called a almost geodesic on $X$ if for any $\varepsilon > 0$, there exists $T \geq 0$ such that for any $s, t \in A$ with $T \leq s \leq t$,

$$|d(r(0), r(s)) + d(r(s), r(t)) - t| < \varepsilon.$$

Any geodesic is an almost geodesic. Rieffel proved that any almost geodesic on $X$ converges to a point on $X(\infty)$ ([Rie02]).

Proposition 4.5. ([Wal11]) For any $\xi, \xi', \xi'' \in X(\infty)$ and $x \in X$, the detour cost $H$ satisfies the following properties:

1. $H(\xi, \xi) = 0$ if and only if $\xi \in X_B(\infty)$,
2. $H(\xi, \xi') \geq 0$,
3. $H(\xi, \xi'') \leq H(\xi, \xi') + H(\xi', \xi'').$

By this proposition, for any $\xi, \xi' \in X_B(\infty)$, the symmetrization $H(\xi, \xi') + H(\xi', \xi)$ satisfies the axiom of the distance.

Definition 4.6. For any $\xi, \xi' \in X_B(\infty)$,

$$\delta(\xi, \xi') = H(\xi, \xi') + H(\xi', \xi)$$

is a (possibly $+\infty$-valued) distance on $X_B(\infty)$. This $\delta$ is called the detour metric for $(X, d)$ and the base point $b \in X$.

4.2. The Gardiner-Masur boundary of Teichmüller spaces. The Gardiner-Masur compactification and the Gardiner-Masur boundary of the Teichmüller space is induced by Gardner and Masur [GM91]. Liu and Su [LS12] show that the horofunction compactification of the Teichmüller space with the Teichmüller distance is the same as the Gardiner-Masur compactification of the same one. Let $T(X)$ be a Teichmüller space of $X$. We define a mapping $\tilde{\phi} : T(X) \to \mathbb{R}^S_{\geq 0}$ by $p \mapsto \{\text{Ext}_p^\#(\gamma)\}_{\gamma \in S}$, and denote by $\pi : \mathbb{R}^S_{\geq 0} - \{0\} \to P\mathbb{R}^S_{\geq 0}$ a natural projection.

Theorem 4.7. [GM91] The composition $\phi = \pi \circ \tilde{\phi} : T(X) \to P\mathbb{R}^S_{\geq 0}$ is an embedding and the closure $\bar{\phi}(T(X))$ is a compact set.

This closure is called the Gardiner-Masur compactification of $T(X)$ and we denote by $T(X)^{GM} = \bar{\phi}(T(X))$. The boundary $\partial GM(T(X)) = \bar{\phi}(T(X)) - \phi(T(X))$ is called the Gardiner-Masur boundary of $T(X)$.

We set the base point $b = [X, id] \in T(X)$. For any $p \in T(X)$ and any $\mu \in \mathcal{MF}(X)$, we define

$$\mathcal{E}_p(\mu) = \left\{ \frac{\text{Ext}_p(\mu)}{K_p} \right\}^\#,$$

where $K_p = e^{2d_r(b, p)}$. By the definition of the Gardiner-Masur compactification, the family $\{\mathcal{E}_p(\gamma)\}_{\gamma \in S}$ corresponds to $p \in T(X)$. 


Proposition 4.8. [Miy08] For any $\xi \in \partial_{GM} T(X)$, there exists a continuous function $E_\xi : \mathcal{MF}(X) \to \mathbb{R}_{\geq 0}$ which satisfies the following properties:

1. $E_\xi(t\mu) = tE_\xi(\mu)$ for any $t \geq 0$ and any $\mu \in \mathcal{MF}(X)$,
2. $\{E_\xi(\gamma)\}_{\gamma \in \mathcal{G}}$ corresponds to $\xi \in \partial_{GM} T(X)$,
3. If a sequence $\{x_n\} \subset T(X)$ converges to $\xi \in \partial_{GM} T(X)$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $t_0 > 0$ which does not depend on $\mathcal{MF}(X)$ such that $E_{x_{n_k}}$ converges to $t_0 E_\xi$ uniformly on any compact set of $\mathcal{MF}(X)$.

For any $p \in \overline{T(X)}^{GM}$, we define

$$Q(p) = \sup_{\nu \in \mathcal{MF}(X)^*} \frac{E_p(\nu)}{\text{Ext}_p^+(\nu)}$$

and for any $\mu \in \mathcal{MF}(X)$,

$$L_p(\mu) = \frac{E_p(\mu)}{Q(p)}$$

Proposition 4.9. [LS12] For any $\{p_n\} \subset \overline{T(X)}^{GM}$ and any $p \in \overline{T(X)}^{GM}$, $p_n$ converges to $p$ as $n \to \infty$ if and only if $L_{p_n}$ converges to $L_p$ uniformly on any compact set of $\mathcal{MF}(X)$ as $n \to \infty$.

For any $p \in \overline{T(X)}^{GM}$, we define a function $\psi_p : T(X) \to \mathbb{R}$ by

$$\psi_p(x) = \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{L_p(\mu)}{\text{Ext}_p^+(\mu)}$$

for any $x \in T(X)$.

Remark. By the definition and Kerckhoff’s formula, if $p \in T(X)$, then

$$\psi_p(x) = \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{E_p(\mu)}{\text{Ext}_p^+(\mu)} - \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{E_p(\mu)}{\text{Ext}_p^+(\mu)}$$

$$= \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\text{Ext}_p^+(\mu)}{\text{Ext}_p^+(\mu)} - \log \sup_{\mu \in \mathcal{MF}(X)^*} \frac{\text{Ext}_p^+(\mu)}{\text{Ext}_p^+(\mu)}$$

$$= d_{T(X)}(x,p) - d_{T(X)}(b,p)$$

for any $x \in X$. Thus, we can consider the horofunction compactification of $T(X)$ by the function $\psi_p$ for any $p \in T(X)$.

The following is deduced from Proposition 4.9.

Theorem 4.10. [LS12] We define a mapping $\psi : \overline{T(X)}^{GM} \to C(T(X))$ by $p \mapsto \psi_p$. Then $\psi$ is injective and continuous. In particular, $\overline{T(X)}^{GM}$ and $\psi(\overline{T(X)}^{GM})$ are homeomorphic. Furthermore, $\psi(\overline{T(X)}^{GM}) = \overline{\psi(T(X))}$.

Therefore, the horofunction compactification $\overline{T(X)}^{GM}$ of $T(X)$ can be identified with the Gardiner-Masur compactification $\overline{T(X)}^{GM}$. We denote by $T(X) \cup T(X)(\infty)$ the horofunction compactification of $T(X)$. Then, we can assume that $T(X)(\infty) = \partial_{GM} T(X)$. 


4.3. The detour metric for the Teichmüller distance. We consider the detour metric in the case of the Teichmüller space with the Teichmüller distance. Its representation is given by Walsh [Wal12]. We notice that there exist non-Busemann points on $\partial_{GM} T(X)$ if $3g - 3 + n > 1$. This result is proved by Miyachi [Miy11].

**Theorem 4.11.** (Wal12) For any point $p$ on $T(X)$ and any Busemann point $\xi$, there exists a unique Teichmüller geodesic ray on $T(X)$ starting at $p$ and converging to $\xi$.

By this theorem, we can assume that the set of Busemann points are consists of end points of all Teichmüller geodesic rays.

**Theorem 4.12.** (Wal12) Let $r$, $r'$ be Teichmüller geodesic rays on $T(X)$ converging to Busemann points $\xi$, $\xi'$ respectively. Then, the rays $r$, $r'$ are modularly equivalent if and only if $\xi = \xi'$.

**Theorem 4.13.** (Wal12) Let $\xi$, $\xi'$ be Busemann points and $r$, $r'$ be Teichmüller geodesic rays on $T(X)$ starting at $b = [X, id]$ and converging to $\xi$, $\xi'$ respectively. We denote by $q$, $q'$ the unit norm holomorphic quadratic differentials on $X$ corresponding to the given rays $r$, $r'$ respectively. If the measured foliations $H(q)$, $H(q') \in \mathcal{MF}(X)$ are absolutely continuous, then we can write $H(q) = \sum_{j=1}^{k} b_j G_j$, $H(q') = \sum_{j=1}^{k} b'_j G_j$ where $b_j$, $b'_j > 0$ and $G_j$ is a simple closed curve or an ergodic measure for any $j = 1, \ldots, k$ such that these are projectively-distinct and pairwise having zero intersection number. We set $m_j = \frac{b_j}{\varphi(G_j, V(q))}$, $m'_j = \frac{b'_j}{\varphi(G_j, V(q'))}$ for any $j = 1, \ldots, k$. Then the detour metric $\delta$ between $\xi$ and $\xi'$ is represented by

$$\delta(\xi, \xi') = \frac{1}{2} \log \max_{j=1,\ldots,k} m_j + \frac{1}{2} \log \max_{j=1,\ldots,k} m'_j.$$

If $H(q)$, $H(q')$ are not absolutely continuous, then $\delta(\xi, \xi') = +\infty$.

We combine Theorems 4.11, 4.12 and 4.13 to obtain the following.

**Proposition 4.14.** Let $r$, $r'$ be Teichmüller geodesic rays on $T(X)$ starting at $p = [Y, f]$, $p' = [Y', f']$ and having the unit norm holomorphic quadratic differentials $q$, $q'$ on $X$, $Y'$ and converging to Busemann points $\xi$, $\xi'$ respectively. If the measured foliations $f_*^{-1}(H(q))$, $f'^{-1}_*(H(q')) \in \mathcal{MF}(X)$ are absolutely continuous, then we can write $f_*^{-1}(H(q)) = \sum_{j=1}^{k} b_j G_j$, $f'^{-1}_*(H(q')) = \sum_{j=1}^{k} b'_j G_j$ where $b_j$, $b'_j > 0$, $G_j$ is a simple closed curve or an ergodic measure for any $j = 1, \ldots, k$ such that these are projectively-distinct and pairwise having zero intersection number. We set $m_j = \frac{b_j}{\varphi(f^*\phi, \varphi(G_j, V(q)))}$, $m'_j = \frac{b'_j}{\varphi(f'^*\phi, \varphi(G_j, V(q')))}$ for any $j = 1, \ldots, k$. In this situation, the detour metric between $\xi$ and $\xi'$ is also represented by

$$\delta(\xi, \xi') = \frac{1}{2} \log \max_{j=1,\ldots,k} m_j + \frac{1}{2} \log \max_{j=1,\ldots,k} m'_j.$$

If $f_*^{-1}(H(q))$, $f'^{-1}_*(H(q'))$ are not absolutely continuous, then $\delta(\xi, \xi') = +\infty$.

**Proof.** By Theorem 4.11 there exist two Teichmüller geodesic rays $s$, $s'$ starting at $b = [X, id]$ and having the unit norm holomorphic quadratic differentials $\varphi$, $\varphi'$ on $X$ and converging to Busemann points $\xi$, $\xi'$ respectively. By Theorem 4.12 the two
pairs \( r, s \) and \( r', s' \) are modularly equivalent respectively. If the measured foliations \( f_*^{-1}(H(q)), f_*^{-1}(H(q')) \) are absolutely continuous, then the measured foliations \( H(\varphi), H(\varphi') \in \mathcal{MF}(X) \) are also absolutely continuous, and can be written as \( H(\varphi) = \sum_{j=1}^{k} c_j G_j, H(\varphi') = \sum_{j=1}^{k} c'_j G_j \) where \( c_j, c'_j > 0 \) for any \( j = 1, \ldots, k \). Let \( n_j = \frac{c_j}{n(G_j, \varphi)}, n'_j = \frac{c'_j}{n(G_j, \varphi')} \) for any \( j = 1, \ldots, k \), then there are \( \lambda, \lambda' > 0 \) such that \( n_j = \lambda m_j, n'_j = \lambda' m'_j \) respectively. Therefore, by Theorem 4.13

\[
\delta(\xi, \xi') = \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{n'_j}{n_j} + \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{n_j}{n'_j} = \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{m'_j}{m_j} + \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{m_j}{m'_j}.
\]

If the measured foliations \( f_*^{-1}(H(q)), f_*^{-1}(H(q')) \) are not absolutely continuous, then \( H(\varphi), H(\varphi') \) are not also absolutely continuous, and we conclude that \( \delta(\xi, \xi') = +\infty \).

We suppose that two rays \( r, r' \) satisfy that \( f_*^{-1}(H(q)), f_*^{-1}(H(q')) \) are absolutely continuous. Let \( \xi, \xi' \) be Busemann points corresponding to the end points of \( r, r' \) respectively, then

\[
\liminf_{t \to \infty} d_{T(X)}(r(t), r'(t)) \geq \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{m'_j}{m_j} \left\{ \frac{m_j}{m_j} \right\}
\]

\[
\geq \frac{1}{2} \log \left( \max_{j=1, \ldots, k} \frac{m_j}{m'_j} \right) \left( \max_{j=1, \ldots, k} \frac{m'_j}{m_j} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{m'_j}{m_j} + \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{m_j}{m'_j} \right)
\]

\[
= \frac{1}{2} \delta(\xi, \xi').
\]

Now, we consider the minimum value of the equation of Theorem 4.1 when we shift the initial points of the given rays \( r, r' \).

Proposition 4.15. Under the assumption of Theorem 3.1, the minimum of the limit value of the distance between the given rays \( r(t), r'(t) \) when we shift the initial points \( r(0), r'(0) \) is given by \( \frac{1}{2} \delta(\xi, \xi') \) where \( \xi, \xi' \) are the end points of the rays \( r, r' \) on the Gardiner-Masur boundary of \( T(X) \) respectively.

Proof. By Theorem 3.1 and the inequality (2), we see that

\[
\lim_{t \to \infty} d_{T(X)}(r(t), r'(t)) = \frac{1}{2} \log \max_{j=1, \ldots, k} \frac{m'_j}{m_j} \frac{m_j}{m'_j} \left\{ \frac{m'_j}{m_j} \right\}
\]

\[
\geq \frac{1}{2} \delta(\xi, \xi').
\]

We notice that the detour metric is invariant when we shift the initial points of the rays \( r, r' \). The equality holds if we consider
\[ \beta = \frac{1}{4} \log \frac{\max_{j=1,\ldots,k} m_j}{\max_{j=1,\ldots,k} m_j'} \]

and the rays \( r(t), r'(t + \beta) \). In this situation, we compute that

\[
\max_{j=1,\ldots,k} e^{2\beta m_j'} m_j = \max_{j=1,\ldots,k} \left\{ \sqrt{\max_{j=1,\ldots,k} m_j' m_j}, \sqrt{\max_{j=1,\ldots,k} m_j m_j'} \right\} = \sqrt{\max_{j=1,\ldots,k} m_j' m_j} \cdot \sqrt{\max_{j=1,\ldots,k} m_j m_j'}
\]

and similarly,

\[
\max_{j=1,\ldots,k} \frac{m_j}{e^{2\beta m_j'}} = \sqrt{\max_{j=1,\ldots,k} m_j' m_j} \cdot \sqrt{\max_{j=1,\ldots,k} m_j m_j'}.
\]

Therefore, we conclude that

\[
\lim_{t \to \infty} d_{T(X)}(r(t), r'(t + \beta)) = \frac{1}{2} \log \max_{j=1,\ldots,k} \left\{ \frac{e^{2\beta m_j'} m_j}{m_j}, \frac{e^{2\beta m_j} m_j'}{m_j'} \right\} = \frac{1}{2} \left( \frac{1}{2} \log \max_{j=1,\ldots,k} m_j' m_j + \frac{1}{2} \log \max_{j=1,\ldots,k} m_j m_j' \right) = \frac{1}{2} \delta(\xi, \xi').
\]

We can also obtain the following.

**Proposition 4.16.** If two rays \( r, r' \) are asymptotic, then the end points satisfy that \( \xi = \xi' \) and the rays are modularly equivalent.

**Proof.** From the remark where is at the above of Corollary 3.5 the measured foliations \( f_*^{-1}(H(q)), f_*^{-1}(H(q')) \) are absolutely continuous. Then, this is immediately by the inequality (2) and Theorem 4.12.

\[ \square \]

5. The tables of the classification about the behavior of two Teichmüller rays

In this section, we give the tables of the classification of the conditions under which given two Teichmüller geodesic rays are bounded, diverge, or asymptotic.

Let \( r, r' \) be two Teichmüller geodesic rays on \( T(X) \) starting at \([Y, f], [Y', f']\) and having unit norm holomorphic quadratic differentials \( q, q' \) on \( Y, Y' \), and \( H(q), H(q') \) be measured foliations corresponding to \( q, q' \) respectively. We set \( H = f_*^{-1}(H(q)), H' = f_*^{-1}(H(q')) \). In the following tables, the notation “top.equi.” means topologically equivalent, “abs.conti.” means absolutely continuous, “J-S.”
means Jenkins-Strebel, “u.e.” means uniquely ergodic, and “mod.equ.i.” means modularly equivalent.

\[
H, H' : \left\{ \begin{array}{ll}
top.\text{equi. and} & \text{abs.conti.} \Rightarrow \text{bounded} \quad \text{[Iva01]} \\
 & \text{not abs.conti.} \Rightarrow \text{diverge} \quad \text{[LM10]}
\end{array} \right.
\]

\[
\text{not top.equi. and } i(H, H') \left\{ \begin{array}{ll}
\neq 0 & \Rightarrow \text{diverge} \quad \text{[Iva01]} \\
= 0 & \Rightarrow \text{diverge} \quad \text{[LM10]}
\end{array} \right.
\]

By the above table,

\[r, r' : \text{bounded} \iff H, H' : \text{abs.conti.} \]

If \(H, H'\) are absolutely continuous and Jenkins-Strebel, we denote by \(r(\infty), r'(\infty)\) the end points of the given rays \(r, r'\) on the augmented Teichmüller space \(\hat{T}(X)\) respectively.

\[
H, H' : \text{abs.conti. and} \left\{ \begin{array}{ll}
\text{J-S. and} & \begin{array}{l}
r, r' : \text{mod.equi. and } r(\infty) = r'(\infty) \\
\Rightarrow \text{asymptotic} \quad \text{(Cor 3.5)}
\end{array} \\
\text{otherwise} & \Rightarrow \text{bounded but not asymptotic} \quad \text{(Cor 3.5)}
\end{array} \right.
\]

\[
\text{not J-S. and} \left\{ \begin{array}{ll}
\text{u.e., and } \Gamma_q \text{ and } \Gamma_{q'} \text{ do not contain closed loops} & \Rightarrow \text{asymptotic} \quad \text{[Mas80]} \\
\text{otherwise} & \Rightarrow \text{unknown}
\end{array} \right.
\]

Let \(\xi, \xi'\) be the end points of the rays \(r, r'\) on the Gardiner-Masur boundary respectively.

\[r, r' : \text{asymptotic} \iff \xi = \xi' \iff r, r' : \text{mod.equi.} \quad \text{(Prop 4.16) [Wal12]} \]

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**Department of Mathematics Tokyo Institute of Technology 2-12-1 Ookayama, Meguroku, Tokyo 152-8551, Japan**

E-mail address: amano.m.ab@m.titech.ac.jp