Ortho-symmetric modules, Gorenstein algebras and derived equivalences

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ABSTRACT. A new homological symmetry condition is exhibited that extends and unifies several recently defined and widely used concepts. Applications include general constructions of tilting modules and derived equivalences, and characterisations of Gorenstein properties of endomorphism rings.

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1. INTRODUCTION

In representation theory and homological algebra of finite-dimensional algebras, and beyond, two kinds of conditions play a crucial role: Cohomological rigidity or orthogonality conditions single out modules with particularly good properties and assign important subcategories to particular modules. Generators and cogenerators and generalisations thereof are used to describe categories by certain modules and to define and compute homological dimensions. Examples of both kinds of conditions and combinations of these range from classical Morita theory over tilting and derived equivalences to higher Auslander Reiten theory, cluster tilting, maximal modifying modules in algebraic geometry, and to homological invariants such as dominant dimension and representation dimension.

Currently, particular focus in this context is on the following questions, and types of results: Suppose two modules share certain generating and rigidity properties.

(1) Are their endomorphism rings derived equivalent?
(2) What is the algebraic structure of these endomorphism rings?
(3) What is the structure of perpendicular categories associated with these modules?

These questions have been answered in special cases, attracting much attention. Typical assumptions involve restricting the homological conditions to degree one, that is, first extension groups; under such assumptions, general conjectures have been proven.

The aim of this article is to define and draw attention to a new homological symmetry condition (Definition [1.1]) and to demonstrate its usefulness by addressing the above questions.
Let $A$ be an algebra, finite dimensional over a field $k$. Let $M$ be an $A$-module. We denote by $\perp^n M$ (respectively, $M \perp^n$) the category of all $A$-modules $X$ such that $\text{Ext}_A^i(X,M) = 0$ (respectively, $\text{Ext}_A^i(M,X) = 0$) for all $1 \leq i \leq n$. The module $M$ is called $n$-rigid if $M \in \perp^n M$. By $\text{add}(M)$ we denote the full subcategory whose objects are the direct summands of finite direct sums of copies of $M$. The module $M$ is a generator, if each finitely generated projective $A$-module is in $\text{add}(M)$. It is a cogenerator if each finitely generated injective $A$-module is in $\text{add}(M)$.

**Definition 1.1.** Let $n \geq 1$ and $m \geq 0$ be integers. An $A$-module $M$ is $(n,m)$-ortho-symmetric if $\Lambda M$ is an $n$-rigid generator-cogenerator such that

$$\perp^n M \cap M^{\perp m} = \perp^m M \cap M^{\perp n}.$$ 

The $(n,0)$-ortho-symmetric modules are also called $n$-ortho-symmetric.

We are going to demonstrate the feasibility of this rather general new concept by addressing the above questions in the following way:

**Tilting and derived equivalences - Question (1).**
In Section 4 we will explicitly define tilting modules, and thus get derived equivalences between endomorphism rings of ortho-symmetric modules. Theorem 4.3 includes the following result. Here, a module $M$ is defined to be maximal with respect to a property if it is maximal with respect to properly increasing add$(M)$ and keeping this property.

**Theorem A.** Let $A$ be an algebra and let $M$ and $N$ be two $A$-modules. Suppose that $\Lambda M$ is maximal 1-ortho-symmetric and that $\Lambda N$ is 1-ortho-symmetric. Then:

1. The algebras $\text{End}_A(M)$ and $\text{End}_A(N)$ are derived equivalent if and only if $M$ and $N$ have the same number of indecomposable and non-isomorphic direct summands.
2. If $\Lambda N$ is maximal 1-ortho-symmetric, then $\text{End}_A(M)$ and $\text{End}_A(N)$ are derived equivalent.

Explicit examples of derived equivalences will be provided by a left and a right mutation of ortho-symmetric modules to be defined in Section 4; see Corollaries 4.6 and 4.7.

**Structure of algebras and of perpendicular categories - Questions (2) and (3).**

Here, our answer is in terms of Gorenstein conditions: An algebra $A$ is Gorenstein if the regular module $A$ as one-sided module has finite injective dimension. In this case, the left and right injective dimensions of $A$ are the same. If this dimension is less than or equal to a natural number $n$, then $A$ is called at most $n$-Gorenstein, in case of equality we say that $A$ is $n$-Gorenstein.

**Theorem B.** Let $A$ be an algebra, $M$ an $A$-module and $n$ a positive integer. Suppose that $M$ is an $n$-rigid generator-cogenerator. Let $\Lambda$ be the endomorphism algebra of $\Lambda M$.

If $\Lambda$ is at most $(n+2+m)$-Gorenstein with $0 \leq m \leq n$, then $\Lambda M$ is $(n,m)$-ortho-symmetric and the functor $\text{Hom}_A(M, -)$ induces an equivalence of Frobenius categories from $\perp^n M \cap M^{\perp n}$ to the category of $\Lambda$-projective $\Lambda$-modules.

Extended versions of this result will be proven in Section 3. A crucial role in the proofs is played by a certain subcategory of $\perp^n M \cap M^{\perp n}$, which is closed under applying $M$-relative (co)syzygy functors, see Definition 3.4. This subcategory is identified with the category of Gorenstein projective modules over the endomorphism ring of $M$ in Lemma 3.5 and it is shown to be equal to $\perp^n M \cap M^{\perp n}$ exactly if $M$ is $(n,m)$-ortho-symmetric, see Corollary 3.6.

In general, the converse of Theorem B may not be true. But, for some choices of $m$, the converse is true. In particular, when $m = 0$, the algebra $\Lambda$ in Theorem B is at most $(n+2)$-Gorenstein if and only if $\Lambda M$ is $n$-ortho-symmetric (Corollary 3.18). In Proposition 3.15 we characterise the endomorphism ring $\Lambda$ being $(n+m+2)$-Gorenstein in terms of $M$ being ortho-symmetric and satisfying additional conditions. There also is a characterisation of $\Lambda$ being $(n+3)$-Gorenstein, see Corollary 3.21.

In the final section we will construct and describe classes of examples, typically arising from self-injective algebras. For instance, over weakly 2-Calabi-Yau self-injective algebras, all 1-rigid generators are 1-ortho-symmetric (Lemma 5.2). One consequence of Theorem A is the following result, which extends [13, Proposition 2.5] from preprojective algebras of Dynkin type to arbitrary weakly 2-Calabi-Yau self-injective algebras.

**Corollary C.** Let $A$ be a weakly 2-Calabi-Yau self-injective algebra. Then all endomorphism algebras of maximal 1-rigid $A$-modules are derived equivalent.
Comparison with related concepts and results: The concept of ortho-symmetric modules generalises various definitions in the literature.
When specialising the second parameter in Definition 1.1 to 0, this concept specialises to n-precluster tilting objects in the sense of Iyama and Solberg [24] (note that their not yet written work is preceding ours). Assuming in addition the endomorphism rings to have finite global dimension, this specialises further to the familiar and widely used n-cluster tilting objects, or equivalently maximal n-orthogonal objects. Specialising both parameters to n yields another familiar concept, n-rigid generator-cogenerators.

Maximal n-ortho-symmetric modules generalize Iyama’s maximal n-orthogonal modules (see [21, 22]). In fact, endomorphism rings of the former have finite Gorenstein global dimension, while endomorphism rings of the latter have finite global dimension.

Corollary 3.18 which is a consequence of Theorem B recovers results observed in [24, 17]. (Note that this result is not contained in the published version [18] of Kong’s preprint [17].) Our methods are different from those used in [17, 24]. We strongly build upon the theory of add generator-cogenerators and provides the connection to Gorenstein conditions. In particular, a proof of this result is not contained in the published version [18] of Kong’s preprint [17].

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To illustrate the connection of these concepts with cluster algebras, recall the main results of [13]. There, the category of A-modules is considered, where A is a preprojective algebra of Dynkin type (which is an example of a weakly 2-Calabi-Yau self-injective algebra). Rigid (=1-rigid) modules are important, since they have open orbits in a module variety, and thus the closures are irreducible components, which provides a connection to semi-canonical bases. The first main result of [13] is that maximal 1-rigid modules coincide with maximal 1-orthogonal modules (and thus automatically are generator-cogenerators). The second main result of [13] shows that mutation of maximal 1-rigid modules categorifies the combinatorial mutation procedure underlying the definition and use of cluster algebras. Moreover, it is shown that the endomorphism rings of all maximal 1-rigid modules are derived equivalent.

Further connections and applications can be obtained for instance in the context of the Gorenstein Symmetry Conjecture, which states that an algebra has finite left injective dimension if and only if it has finite right injective dimension (see [3]). For endomorphism algebras of generator-cogenerators, this conjecture can be reformulated using our results (Lemma 3.13). It turns out that endomorphism algebras of almost ortho-symmetric modules (a generalisation of ortho-symmetric modules) satisfy the conjecture, see Corollaries 3.24 and 3.25.

Moreover, the results in Section 5 can be used to determine global and dominant dimensions, particularly in the context of representation dimension and its counterpart defined in [8].

The structure of this article is as follows: Section 2 sets up the theory of add(M)-split sequences for later use; this is one of the main tools for constructing derived equivalences. Section 3 concentrates on n-rigid generator-cogenerators and provides the connection to Gorenstein conditions. In particular, a proof of Theorem B is given. In Section 4 tilting modules and derived equivalences are constructed and a proof of (a stronger version of) Theorem A is given. The final Section 5 concentrates on self-injective algebras, for which examples of ortho-symmetric modules are provided, and Corollary C is proved.

2. Preliminaries

Throughout this paper, k is a fixed field. All categories and functors are k-categories and k-functors, respectively; algebras are finite-dimensional k-algebras, and modules are finitely generated left modules. Let C be a category. Given two morphisms f : X → Y and g : Y → Z in C, we denote the composition of f and g by fg, which is a morphism from X to Z, while we denote the composition of a functor F : C → D between categories C and D with a functor G : D → E between categories D and E by GF, which is a functor from C to E.

Let A be an algebra. We denote by A-mod the category of all A-modules, by A-proj (respectively, A-inj) the full subcategory of A-mod consisting of projective (respectively, injective) modules, by D the usual k-duality Hom_k(−, k), and by ν_A the Nakayama functor DHom_A(−, A) of A. Note that ν_A is an equivalence from A-proj to A-inj with the inverse ν_A^* = Hom_A(−, A)D. The global and dominant dimensions of A are denoted by gldim(A) and domdim(A), respectively. As usual, X^b(A-proj) is the bounded homotopy category of A-proj and D^b(A) is the bounded derived category of complexes over A-mod.
Let $M$ be an $A$-module. By $\text{add}(M)$ we denote the full subcategory of $A$-mod consisting of all direct summands of finite direct sums of copies of $M$. The number of indecomposable, non-isomorphic direct summands of $M$ is $\#(M)$. We denote by $\text{projdim}(M)$ and $\text{injdim}(M)$ the projective and injective dimensions of $M$, respectively.

2.1. Generators and cogenerators, and associated categories. A module $M$ is called a generator for $A$-mod if $\text{add}(A) \subseteq \text{add}(M)$: it is a cogenerator for $A$-mod if $\text{add}(D(A)) \subseteq \text{add}(M)$, and a generator-cogenerator if it is both a generator and a cogenerator for $A$-mod.

A homomorphism $f : M_0 \rightarrow X$ of $A$-module is called a right $\text{add}(M)$-approximation of $X$ if $M_0 \in \text{add}(M)$ and $\text{Hom}_A(M, f) : \text{Hom}_A(M, M_0) \rightarrow \text{Hom}_A(M, X)$ is surjective. Clearly, if $M$ is a generator, then $f$ is surjective. Dually, one can define left approximations of modules.

Let $X$ be a full subcategory of $A$-mod. For $n \in \mathbb{N}$, set

$$X^{\leq n} := \{ N \in A\text{-mod} \mid \text{Ext}_A^n(X, N) = 0 \text{ for all } X \in X \text{ and } 1 \leq i \leq n \}$$

and

$$^{\perp} X := \{ N \in A\text{-mod} \mid \text{Ext}_A^n(N, X) = 0 \text{ for all } X \in X \text{ and } 1 \leq i \leq n \}.$$

In this context, it is understood that $X^{\leq 0} = A\text{-mod} = \mathcal{X}$. Further, we define $X^{< n}$ (respectively, $X^{\leq n}$) to be the full subcategory of $A$-mod consisting of all those modules $N$ which admit a long exact sequence of $A$-modules

$$0 \rightarrow X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow N \rightarrow 0$$

(respectively,

$$0 \rightarrow N \rightarrow X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1} \rightarrow X_n \rightarrow 0$$)

such that $X_i \in X$ for all $0 \leq i \leq n$. When $X$ consists of one object $X$ only, then we write $X^{\leq n}$ and $X^{\leq n}$ for $X^{\leq n}$ and $X^{\leq n}$, respectively.

Suppose that $X$ contains $\text{add}(M)$. The $M$-relative stable category of $X$, denoted by $X^/[M]$, is defined to be the quotient category of $X$ modulo the full subcategory $\text{add}(M)$. More precisely, $X^/[M]$ has the same objects as $X$, but its morphism sets between $A$-modules $X$ and $Y$ are given by $X^/[M](X, Y) := \text{Hom}_A(X, Y)/M(A(X, Y))$, where $M(A(X, Y)) \subseteq \text{Hom}_A(X, Y)$ consists of homomorphisms factorising through modules from $\text{add}(M)$.

When $X = A\text{-mod}$ and $M = A$, then $X^/[M]$ is the stable module category of $A$, usually denoted by $A\text{-mod}$.

2.2. Relative syzygy and cosyzygy functors associated with generators and cogenerators. When $M$ is a generator, we first choose a minimal right $\text{add}(M)$-approximation $r_X : M_X \rightarrow X$ of $X$ with $M_X \in \text{add}(M)$, and then define $\Omega_M(X)$ to be the kernel of $r_X$. Since $M$ is a generator, the map $r_X$ is surjective and the sequence

$$0 \rightarrow \Omega_M(X) \rightarrow M_X \xrightarrow{r_X} X \rightarrow 0$$

is exact in $A\text{-mod}$. Up to isomorphism, $\Omega_M(X)$ is independent of the choice of $r_X$. Moreover, for any homomorphism $f : X \rightarrow Y$, there are two homomorphisms $g : M_X \rightarrow M_Y$ and $h : \Omega_M(X) \rightarrow \Omega_M(Y)$ such that there is a commutative diagram:

$$\begin{array}{ccc}
0 & \rightarrow & \Omega_M(X) \\
\downarrow{h} & & \downarrow{g} \\
0 & \rightarrow & \Omega_M(Y) \\
\downarrow{r_Y} & & \downarrow{r_Y} \\
0 & \rightarrow & M_Y \\
\end{array}$$

Further, if $f$ factorises through an object in $\text{add}(M)$, then $h$ factorises through $M_X$. So

$$\Omega_M : A\text{-mod}/[M] \rightarrow A\text{-mod}/[M],$$

sending $f$ to $h$, is a well-defined additive functor. This functor is called an $M$-relative syzygy functor. Inductively, for each $n \geq 1$, an $n$-th $M$-relative syzygy functor is defined by $\Omega^n_M(X) := \Omega_M(\Omega^{n-1}_M(X))$, where $\Omega^0_M(X) := X$. So there is a long exact sequence of $A$-modules

$$\cdots \rightarrow \Omega^n(X) \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0$$

with $M_i \in \text{add}(M)$ for $0 \leq i \leq n - 1$, which induces the following exact sequence

$$0 \rightarrow \text{Hom}_A(M, \Omega^n(X)) \rightarrow \text{Hom}_A(M, M_{n-1}) \rightarrow \cdots \rightarrow \text{Hom}_A(M, M_0) \rightarrow \text{Hom}_A(M, X) \rightarrow 0.$$
This provides the first $n$ terms of a minimal projective resolution of $\text{Hom}_A(M,X)$ as an $\text{End}_A(M)$-module. In other words,
\[ \Omega^n_{\text{End}_A(M)}(\text{Hom}_A(M,X)) = \text{Hom}_A(M,\Omega^n_M(X)). \]
The sequence $(\Dagger)$ is called a minimal right $n$-th $\text{add}(M)$-approximation sequence of $X$. Further, the $M$-resolution dimension of $X$ is defined by
\[ M\text{-resdim}(X) := \inf \{ n \in \mathbb{N} \mid \Omega^n_M(X) \in \text{add}(A,M) \}. \]
Equivalently, $M\text{-resdim}(X)$ equals the projective dimension of $\text{Hom}_A(M,X)$ as an $\text{End}_A(M)$-module. If $M = A$, then $\Omega^n_A$ is exactly the usual $n$-th syzygy functor of $A\text{-mod}$ and thus $M\text{-resdim}(X)$ is the projective dimension of $X$.

Dually, when $\underline{A}$ is a cogenerator, the minimal left $\text{add}(M)$-approximation of $X$
\[ 0 \longrightarrow X \xrightarrow{\delta} M^X \longrightarrow \Omega^M_{-1}(X) \longrightarrow 0 \]
can be used to define the $M$-relative cosyzygy functor
\[ \Omega^M_{-1} : A\text{-mod}/[M] \longrightarrow A\text{-mod}/[M], \]
and iteratively the $n$-th $M$-relative cosyzygy functor $\Omega^n_M$. Similarly, one can define minimal left $n$-th $\text{add}(M)$-approximation sequences as well as the $M$-coresolution dimension of $X$ by
\[ M\text{-coresdim}(X) := \inf \{ n \in \mathbb{N} \mid \Omega^{-n}_M(X) \in \text{add}(A,M) \}. \]
Equivalently, $M\text{-coresdim}(X)$ is equal to the projective dimension of $\text{Hom}_A(X,M)$ as an $\text{End}_A(M)^\text{op}$-module. If $M = D(A)$, then $\Omega^n_M$ is the usual $n$-th cosyzygy functor of $A\text{-mod}$. For simplicity, we shall write $\Omega^n_A$ for $\Omega^n_{D(A)}$.

So, when $M$ is a generator-cogenerator, we obtain the following two functors of the $M$-relative stable category of $A\text{-mod}$:
\[ \Omega^n_M : A\text{-mod}/[M] \longrightarrow A\text{-mod}/[M] \quad \text{and} \quad \Omega^n_M : A\text{-mod}/[M] \longrightarrow A\text{-mod}/[M]. \]

### 2.3 Relatively split sequences

Recall the definition of relatively split sequences in module categories (due to Hu and Xi, [15, Definition 3.1]):

**Definition 2.1.** An exact sequence of $A$-modules
\[ 0 \longrightarrow X \xrightarrow{\delta} M_0 \xrightarrow{g} Y \longrightarrow 0 \]
is called an $\text{add}(M)$-split sequence if $M_0 \in \text{add}(M)$, $f$ is a left $\text{add}(M)$-approximation of $X$ and $g$ is a right $\text{add}(M)$-approximation of $Y$.

The outer terms $X$ and $Y$ in the sequence $\delta$ are determined each other in the following way:
Let $X = X_0 \oplus \bigoplus_{i=1}^{t} X_i$ and $Y = Y_0 \oplus \bigoplus_{j=1}^{s} Y_j$, where $X_0, Y_0 \in \text{add}(M)$, and where $X_i$ and $Y_j$ are indecomposable and not in $\text{add}(M)$. Then $s = t$ and by a suitable reindexing, for each $1 \leq i \leq s$, there exists an $\text{add}(M)$-split sequence
\[ \delta_i : 0 \longrightarrow X_i \xrightarrow{f_i} M_i \xrightarrow{g_i} Y_i \longrightarrow 0. \]
Here, $f$ is left minimal with $X_0 = 0$ if and only if $g$ is right minimal with $Y_0 = 0$. In this case, the sequence $\delta$ is isomorphic to the direct sum of the sequences $\delta_i$ for all $1 \leq i \leq s$, and called a minimal $\text{add}(M)$-split sequence.

If $M$ is a generator-cogenerator, then $X \simeq \Omega_M(Y) \oplus X_0$ and $Y \simeq \Omega^M_{-1}(X) \oplus Y_0$ in $A\text{-mod}$, and thus there are isomorphisms in $A\text{-mod}/[M]$:
\[ \Omega^M_{-1} \Omega_M(Y) \simeq Y \quad \text{and} \quad X \simeq \Omega_M \Omega^M_{-1}(X). \]

An important property of $\text{add}(M)$-split sequences is that $\text{End}_A(X \oplus M)$ and $\text{End}_A(M \oplus Y)$ are derived equivalent via 1-tilting modules (see [15, Theorem 1.1]). Recall that two algebras $A$ and $\Gamma$ are derived equivalent if $\mathcal{D}^b(A)$ and $\mathcal{D}^b(\Gamma)$ are equivalent as triangulated categories (see [27]). One special kind of derived equivalences in representation theory of finite-dimensional algebras is provided by tilting modules (see [13, 9]).

**Definition 2.2.** An $A$-module $T$ is called $n$-tilting if
(1) $\text{projdim}(T) = n < \infty$;
(2) $\text{Ext}^j_T(T,T) = 0$ for all $j \geq 1$;
(3) there exists an exact sequence $0 \to \mathcal{A} \to T_0 \to T_1 \to \cdots \to T_n \to 0$ of $\mathcal{A}$-modules such that $T_j \in \text{add}(T)$ for all $0 \leq j \leq n$.

If $\mathcal{A}T$ satisfies the first two conditions, then $T$ is called partial $n$-tilting.

Let $B = \text{End}_A(T)$. If $\mathcal{A}T$ is $n$-tilting, then $A$ and $B$ are derived equivalent, see for instance [14] Chapter III, Theorem 2.10 or [9] Theorem 2.1. The module $T_b$ also is $n$-tilting, and $\text{End}_{A^{\text{op}}}(T) \simeq A^{\text{op}}$ as algebras. Sometimes, in order to emphasise the algebra $B$, $T$ is called an $n$-tilting $A$-$B$-bimodule.

Note that Definition 2.3 can be replaced by the following condition: The module $\mathcal{A}A$ belongs to the smallest triangulated subcategory of $\mathcal{D}^b(A)$ containing $T$ and being closed under direct summands (for example, see [24] Theorem 6.4.1).

The following lemma will be crucial when proving Proposition 3.22, which in turn is our main tool for proving the Gorenstein Symmetry Conjecture for a class of algebras. The lemma is based on [24, Theorem 4.1(1)] that focuses on tilting complexes (a generalisation of tilting modules), while we need the analogous statement for partial tilting modules (another generalisation of tilting modules). For completeness, we include a proof.

**Lemma 2.3.** Let $\mathcal{A}A = P \oplus Q$ such that $P$ is indecomposable and not in $\text{add}(Q)$. Let $X$ be an indecomposable, non-projective $A$-module. If $X \oplus Q$ is a partial $n$-tilting $A$-module with $n \in \mathbb{N}$, then there is a long exact sequence of $A$-modules:

$$0 \to P \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to X \to 0$$

with $Q_i \in \text{add}(Q)$ for $0 \leq i \leq n-1$, and in particular, $X \oplus Q$ is $n$-tilting.

**Proof.** If $n = 0$, then Lemma 2.3 holds since $X \simeq P$. Suppose $n \geq 1$. Let $T := X \oplus Q$. Since $\mathcal{A}T$ is partial $n$-tilting, $	ext{projdim}(X) = n$ and $	ext{Ext}^j_A(X, T) = 0$ for all $j \geq 1$. Choose a minimal projective resolution of $X$:

$$0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0,$$

where each $f_i$ is a radical map, that is, it contains no identity map as direct summand. The first $(n+1)$-terms of this resolution define the complex

$$P^* : \quad 0 \to P_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \to 0$$

with $P_i$ in degree $-i$, which is isomorphic to $X$ in $\mathcal{D}^b(A)$. Thus, $P^*$ is self-orthogonal in $\mathcal{D}^b(A)$, that is, $\text{Hom}_{\mathcal{D}^b(A)}(P^*, P^*[m]) = 0$ for any $m \neq 0$. Since each term of $P^*$ is projective, $P^*$ is a self-orthogonal object in $\mathcal{X}^b(A$-$\text{proj})$. As $\text{Ext}^j_A(X, Q) = 0$ for all $j \geq 1$, applying $\text{Hom}_{\mathcal{A}}(-, Q)$ to the projective resolution of $X$ returns the long exact sequence

$$(\sharp) \quad 0 \to \text{Hom}_{\mathcal{A}}(X, Q) \xrightarrow{\langle f_0 \rangle} \text{Hom}_{\mathcal{A}}(P_0, Q) \xrightarrow{\langle f_1 \rangle} \cdots \to \text{Hom}_{\mathcal{A}}(P_{n-1}, Q) \xrightarrow{\langle f_n \rangle} \text{Hom}_{\mathcal{A}}(P_n, Q) \to 0.$$

Then $f_0$ being a radical map implies $\text{add}(P_n) \cap \text{add}(Q) = 0$. The assumption on $\mathcal{A}A$ then forces $P_n \in \text{add}(P)$; thus $P_n \simeq P^*[r]$ for some positive integer $r$.

**Claim:** $P_i \in \text{add}(Q)$ for $0 \leq i \leq n-1$.

This will be shown by induction on $i$. Vanishing of $\text{Hom}_{\mathcal{X}(A$-$\text{proj})}(P^*, P^*[n])$ means that, for any $A$-homomorphism $h : P_n \to P_0$, there are two maps $s : P_n \to P_1$ and $t : P_{n-1} \to P_0$ in $A$-mod such that $h = sf_1 + tf_0$. Both $f_1$ and $f_0$ are radical maps, and so is $h$. This forces $\text{add}(P_n) \cap \text{add}(P_0) = 0$. Since $P_n \simeq P^*[r]$, the assumption on $\mathcal{A}A$ implies $P_0 \in \text{add}(Q)$. Now, suppose that $P_i \in \text{add}(Q)$ for $0 \leq j \leq i-1$ and $1 \leq i \leq n-1$. Let $g_i : P_i \to P_j$ be an arbitrary $A$-homomorphism. Using the exact sequence $(\sharp)$, $g_i$ can be extended to a chain map from $P^*$ to $P^*[n-i]$:

$$\begin{array}{cccccccccccc}
0 & \to & P_0 & \to & P_1 & \to & \cdots & \to & P_{n-i} & \to & P_{n-1} & \to & P_n & \to & 0 \\
\downarrow 1 & & \downarrow s & & \downarrow s_1 & & \cdots & & \downarrow s_{n-1} & & \downarrow s_n & & \downarrow 0 & & \downarrow 1 \\
\cdots & \to & P_{i+1} & \to & P_i & \to & \cdots & \to & P_{j+1} & \to & P_j & \to & \cdots & \to & P_0 & \to & 0
\end{array}$$

Similarly, vanishing of $\text{Hom}_{\mathcal{X}(A$-$\text{proj})}(P^*, P^*[n-i])$ implies that $\text{add}(P_n) \cap \text{add}(P_i) = 0$, and further, $P_i \in \text{add}(Q)$.

**Claim:** $P_n \simeq P$.

By $(\sharp)$ and $AQ$ being projective, the sequence $0 \to \text{Ker}(f_i) \to P_i \to \text{Im}(f_i) \to 0$ for $0 \leq i \leq n-1$ is an add$(Q)$-split sequence in $A$-mod. By assumption, $X$ is indecomposable. Thus $P_n$ is indecomposable.
and isomorphic to $P$. So the exact sequence required in Lemma 2.3 exists and provides an add$(T)$-coresolution of $A\Lambda$ in Definition 2.2(3). Hence $\Lambda T$ is $n$-tilting. □

2.4. Gorenstein algebras and Gorenstein projective modules. An algebra $\Lambda$ is said to be Gorenstein (or Iwanaga-Gorenstein) if $\text{injdim}(\Lambda A) < \infty$ and $\text{injdim}(A \Lambda) < \infty$. In this case, $\text{injdim}(\Lambda A) = \text{injdim}(A \Lambda)$, and then $\Lambda$ is usually called $m$-Gorenstein, where $m := \text{injdim}(A \Lambda)$. If $m \leq m'$, then $\Lambda$ is also called at most $m'$-Gorenstein.

A complete projective resolution of $A$-module is an exact sequence of finitely generated projective $A$-modules:

$$P^* : \cdots \rightarrow P^{-3} \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow P^3 \rightarrow \cdots$$

such that the Hom-complex $\text{Hom}^*(P^*, A)$ is still exact. An $A$-module $X$ is called Gorenstein projective if there exists a complete projective resolution $P^*$ such that $X$ is equal to the image of the homomorphism $P^0 \rightarrow P^1$. The complex $P^*$ is called a complete projective resolution of $X$.

Note that $A$-modules are divided into three categories:

1. If $M$ is a generator, then the essential image of $\text{Hom}_A(M, \_)$ is equal to the full subcategory of $A$-modules consisting of reflexive modules.
2. If $M$ is a cogenerator, then the essential image of $\text{Hom}_A(\_, M)$ is equal to the full subcategory of $\Lambda$-modules consisting of reflexive modules.

3. If $M$ is a generator, then the essential image of $\text{Hom}_A(M, \_)$ is equal to the full subcategory of $A$-modules.

The category of all Gorenstein projective $A$-modules is denoted by $A$-gp. The category $A$-gp is known to naturally inherit an exact structure from $A$-mod, and it thus becomes a Frobenius category admitting add$(A)$ as the full subcategory of projective objects. Dually, one can define complete injective coresolutions and Gorenstein injective modules. For more details on Gorenstein algebras, see [11, Chapter 9-11].

The following well-known result explains the importance of reflexive modules in our context. It can be found, for instance, in [2, Lemma 3.1].

**Lemma 2.4.** Let $M$ be an $A$-module and let $\Lambda = \text{End}_A(M)$.

1. If $M$ is a generator, then the functor $\text{Hom}_A(M, \_): A$-mod $\rightarrow A$-mod is fully faithful. If moreover $M$ is a cogenerator, then the essential image of $\text{Hom}_A(M, \_)$ is equal to the full subcategory of $A$-modules consisting of reflexive modules.

2. If $M$ is a cogenerator, then the functor $\text{Hom}_A(\_, M): A$-mod $\rightarrow \Lambda$-mod is fully faithful. If moreover $M$ is a generator, then the essential image of $\text{Hom}_A(\_, M)$ is equal to the full subcategory of $\Lambda$-modules consisting of reflexive modules.

3. Rigid generator-cogenerators and Gorenstein algebras

3.1. Introduction. At the end of this Section we will prove Theorem B. We will proceed as follows: Given an $n$-rigid generator-cogenerator $M$, we first construct cotorsion pairs, where one category is $n$-orthogonal to $M$ and the other one collects modules with $M$-(co)-resolutions, see Lemma 3.2. Next we consider the intersection of a left perpendicular category to $M$ with a right perpendicular category, and its stable category modulo $M$. In Lemma 3.3 the relative (co-)syzygy is shown to provide an equivalence of categories

$$\Omega_M: \left(\text{Hom}^1(M \cap M^{\perp q})/[M] \right) \rightarrow \left(\text{Hom}^1(M \cap M^{\perp(q+1)})/[M] \right) = \Omega_M^{-1},$$

relating two such stable categories whose parameters differ by one. This allows to change the parameters defining the perpendicular categories.

Then we restrict to the subcategory

$$\mathcal{G}(\Lambda M) := \{X \in \Lambda M \cap M^{\perp n} \mid \Omega_M(X) \in \Lambda M \text{ and } \Omega_M^{-1}(X) \in M^{\perp n} \text{ for all } i \geq 1\} \subseteq A$$

This turns out to be a Frobenius category and its projective-injective objects are the objects add$(M)$. The category $\mathcal{G}(M)$ is equivalent to the category of Gorenstein projective $\Lambda$-modules (Lemma 3.5), where $\Lambda := \text{End}_A(M)$; this provides the first connection to Gorenstein homological algebra. Moreover, the category $\mathcal{G}(M)$ equals all of $\Lambda M \cap M^{\perp n}$ exactly when $M$ is $(n,m)$-ortho-symmetric (Corollary 3.6). Thus, ortho-symmetry in this context captures crucial information.

Next, higher Auslander-Reiten translation, as defined by Iyama, is used to construct new modules $M^+$ and $M^-$, which are shown to be rigid (Proposition 3.11). In Proposition 3.9 a tilting module is constructed, and it is shown that $\Lambda$ and the endomorphism rings of $M^+$ and of $M^-$ are derived equivalent.
Finally, Theorem [8] is proved. Moreover, partial converses are established: in Corollary 3.12 Λ being Gorenstein is characterised in terms of the $M$-(co)-resolution dimensions of $M^+$ and $M^-$. Λ being Gorenstein with a particular Gorenstein dimension is characterised in terms of the category $\mathcal{G}(\Lambda M)$ and further conditions (Proposition 3.15). For small parameter values such as $(n + 2)$- or $(n + 3)$-Gorenstein, easier characterisations are given in terms of intersections of left and right perpendicular categories (Corollaries 3.16, 3.18 and 3.21). There is also an upper bound for the global dimension of $\Lambda$ in similar terms (Corollary 3.17), illustrating again what information can be read off from intersections of perpendicular categories. Moreover, endomorphism algebras of almost ortho-symmetric modules are shown to satisfy the Gorenstein Symmetry Conjecture (Corollaries 3.22 and 3.25).

3.2. Notation and assumptions. Let $n$ be a fixed non-negative integer and let $A$ be a algebra. We say that an $A$-module $M$ is $n$-rigid if $\text{Ext}^i_A(M, M) = 0$ for $1 \leq i \leq n$. If, in addition, the direct sum $M \oplus N$ of $M$ with another $A$-module $N$ being $n$-rigid implies $N \in \text{add}(M)$, then $M$ is called maximal $n$-rigid. If $M$ is $m$-rigid for any positive integer $m$, then it is said to be self-orthogonal.

Throughout this section, we assume $\Lambda M$ to be a generator-cogenerator which is $n$-rigid and neither projective nor injective. Unless stated otherwise, $n$ is always assumed to be positive.

Let $\Lambda := \text{End}_A(M)$. Then $\Lambda$ is not self-injective and it has dominant dimension at least $n + 2$ (see, for instance, [27, Lemma 3]).

3.3. Cotorsion pairs. Recall the definition of cotorsion pairs, see for example [7, Chapter V, Definition 3.1]:

**Definition 3.1.** Let $\mathcal{Z}$ be a full subcategory of $A\text{-mod}$ closed under extensions. Let $X$ and $\mathcal{Y}$ be two full subcategories of $\mathcal{Z}$ closed under isomorphisms and direct summands. The ordered pair $(X, \mathcal{Y})$ is a cotorsion pair in $\mathcal{Z}$ if the following two conditions are satisfied:

1. $\text{Ext}^1_X(Y, X) = 0$ for every $X \in X$ and $Y \in \mathcal{Y}$.
2. For each $A$-module $Z \in \mathcal{Z}$, there exist short exact sequences in $A\text{-mod}$

\[
(a) \quad 0 \to Y \to X \to Z \to 0 \quad \text{and} \quad (b) \quad 0 \to Z \to Y' \to X' \to 0
\]

such that $X, X' \in X$ and $Y, Y' \in \mathcal{Y}$.

More generally, if (C1) and (C2)(a) hold, then $(X, \mathcal{Y})$ is called a left cotorsion pair in $\mathcal{Z}$. Dually, if (C1) and (C2)(b) hold, then $(\mathcal{Y}, X)$ is called a right cotorsion pair in $\mathcal{Z}$.

When $(X, \mathcal{Y})$ is a left (respectively, right) cotorsion pair in $\mathcal{Z}$, then $X = \perp \mathcal{Y} \cap \mathcal{Z}$ (respectively, $\mathcal{Y} = X^\perp \cap \mathcal{Z}$).

**Lemma 3.2.** (1) Let $1 \leq i \leq n$ and $X$ an $A$-module. Then $\Omega^i_M(X) \in M^{\perp i}$ and $\Omega^-_M(X) \in \perp^i M$.

(2) For any $A$-module $X$, there exist two exact sequences of $A$-modules:

\[
0 \to K_X \to \Omega^i_M(X) \oplus M_X \xrightarrow{[x, y]} X \to 0
\]

and

\[
0 \to X \xrightarrow{[x, y]} M^X \oplus \Omega^i_M(X) \to \Omega^{-n}_M(X) \to 0
\]

where $M_X, M^X \in \text{add}(M)$, $K_X \in M^{\leq n-1}$ and $C^X \in M_{\leq n-1}$, such that the homomorphisms

\[
\epsilon_X : \Omega^{-n}_M(X) \to X \quad \text{and} \quad \eta_X : X \to \Omega^i_M(X)
\]

are unique and natural for $X$ in the quotient category $A\text{-mod}/[M]$.

(3) The pairs $(\perp^i M, M^{\leq n-1})$ and $(M_{\leq n-1}, \perp^i M)$ are cotorsion pairs in $A\text{-mod}$ such that

\[
\perp^i M \cap M^{\leq n} = \text{add}(M) = M^{\perp n} \cap M \leq n.
\]

**Proof.** Claim (1) follows from the rigidity of $M$ and the properties of left or right approximations. In particular, (1) implies that $\Omega^i_M(X) \in M^{\perp n}$ and $\Omega^-_M(X) \in \perp^i M$.

(2) Only existence of the first exact sequence will be shown; existence of the second one can be proved dually.
In the following exact commutative diagram:

\[
\begin{array}{cccccccccc}
0 & \xrightarrow{} & \Omega_M^n(X) & \xrightarrow{} & M_{n-1}' & \xrightarrow{} & \ldots & \xrightarrow{} & M_1' & \xrightarrow{} & M_0' & \xrightarrow{} & \Omega_M^n(X) & \xrightarrow{} & 0 \\
\downarrow f_{n-1} & & \downarrow f_1 & & \downarrow f_0 & & & & \downarrow \varepsilon_X & & \\
0 & \xrightarrow{} & \Omega_M^n(X) & \xrightarrow{} & M_{n-1} & \xrightarrow{} & \ldots & \xrightarrow{} & M_1 & \xrightarrow{} & M_0 & \xrightarrow{} & \Omega_M^n(X) & \xrightarrow{} & 0 \\
\end{array}
\]

the second sequence is a minimal right \( n \)-th add\((M)\)-approximation of \( X \) and the first sequence is a minimal left \( n \)-th add\((M)\)-approximation of \( \Omega_M^n(X) \). This guarantees the existence of homomorphisms \( f_{n-1}, f_{n-2}, \ldots, f_1, f_0 \) and then \( \varepsilon_X \). Although \( \varepsilon_X \) may not be unique in \( A\text{-mod} \), it is unique in \( A\text{-mod}/[M] \).

Taking the mapping cone of the chain map \( (f_{n-1}, f_{n-2}, \ldots, f_0, \varepsilon_X) \) yields the following long exact sequence

\[
0 \longrightarrow M_{n-1}' \longrightarrow M_{n-2} \oplus M_{n-1} \longrightarrow \cdots \longrightarrow M_0' \oplus M_1 \longrightarrow \Omega_M^n(X) \oplus M_X \xrightarrow{\gamma_2} X \longrightarrow 0
\]

Now, let \( K_X \) be the kernel of the homomorphism \( \gamma_2 \). Then \( K_X \in M_{\leq n-1} \). The construction of \( \varepsilon_X \) shows that \( \varepsilon_X \) is unique and natural in the category \( A\text{-mod}/[M] \). This verifies the existence of the first sequence in (2).

(3) We shall show that \( (\Omega_M^n, M_{\leq n-1}) \) is a cotorsion pair in \( A\text{-mod} \); the second claim is dual.

Let \( U \in \Omega_M^n M \) and \( V \in M_{\leq n-1} \). Then there exists a long exact sequence

\[
0 \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow V \longrightarrow 0
\]

such that \( X_i \in \text{add}(M) \) for all \( 0 \leq i \leq n-1 \). Applying \( \text{Hom}_A(U, -) \) to this sequence gives Ext\(_A^1(U, V) = 0 \), and therefore \( \Omega_M^n M \subseteq M_{\leq n-1} \).

This verifies axiom \((C_1)\) in Definition 3.1 and also implies \( \Omega_M^n M \cap M_{\leq n} \subseteq \text{add}(M) \). Since \( M \) is \( n \)-rigid, there is an inclusion \( \text{add}(M) \subseteq \Omega_M^n M \cap M_{\leq n} \). Thus \( \Omega_M^n M \cap M_{\leq n} = \text{add}(M) \).

To show the axiom \((C_2)\) in Definition 3.1, note that \( \Omega_M^n(\Omega_M^n(X)) \in \Omega_M^n M \) by (1). As \( M \) is \( n \)-rigid, \( \Omega_M^n \oplus M \subseteq \text{add}(M) \).

The first exact sequence in (2) means that, for any \( A \)-module \( X \), there exists an exact sequence \( 0 \rightarrow V_X \rightarrow U_X \rightarrow X \rightarrow 0 \) such that \( V_X \in M_{\leq n-1} \) and \( U_X \in \Omega_M^n M \). It remains to show that there is another exact sequence \( 0 \rightarrow X \rightarrow V_X \rightarrow U_X \rightarrow 0 \) such that \( V_X \in \Omega_M^n M \) and \( U_X \in \Omega_M^n M \).

To check this, take an exact sequence \( 0 \rightarrow X \rightarrow I \xrightarrow{f} Y \rightarrow 0 \) of \( A \)-modules such that \( I \) is injective. Associated with \( Y \), there is an exact sequence \( 0 \rightarrow V_Y \rightarrow U_Y \xrightarrow{g} Y \rightarrow 0 \) such that \( V_Y \in \Omega_M^n M \) and \( U_Y \in \Omega_M^n M \). Now, taking the pull-back of \( f \) and \( g \) produces another two exact sequences

\[
0 \rightarrow X \rightarrow E \rightarrow U_Y \rightarrow 0 \quad \text{and} \quad 0 \rightarrow V_Y \rightarrow E \rightarrow I \rightarrow 0.
\]

It is sufficient to show \( E \in M_{\leq n-1} \). Actually, since \( I \in \text{add}(A M) \) and \( M \subseteq M_{\leq n} \subseteq [M_{\leq n-1}] \), we get \( \text{Ext}_A^1(I, V_Y) = 0 \). Thus \( E \cong V_Y \oplus I \in M_{\leq n-1} \).

Hence, the pair \((\Omega_M^n M, M_{\leq n-1})\) is a cotorsion pair in \( A\text{-mod} \). Dually, \((M_{\leq n-1}, M\Omega_M^n)\) is a cotorsion pair in \( A\text{-mod} \), too.

This leads to the following result on \( \Omega_M^n M \) providing equivalences of additive categories.

**Lemma 3.3.** (1) The pair \((\Omega_M^n, \Omega_M^n)\) of additive functors

\[
\Omega_M^n : A\text{-mod}/[M] \longrightarrow A\text{-mod}/[M] \quad \text{and} \quad \Omega_M^n : A\text{-mod}/[M] \longrightarrow A\text{-mod}/[M]
\]

is an adjoint pair such that the diagram

\[
\begin{array}{ccc}
A\text{-mod}/[M] & \xrightarrow{\Omega_M^n} & \Omega_M^n M/[M] \\
\downarrow \sim & & \downarrow \sim \\
M^n/[M] & \xrightarrow{\Omega_M^n} & A\text{-mod}/[M].
\end{array}
\]

is commutative up to natural isomorphism.

(2) For any \( 0 \leq p, q \leq n-1 \), \( \Omega_M^n \) provide the following equivalences of additive categories:

\[
\Omega_M : \left( M^{n-p}(M \cap M^{\leq q})/[M] \right) \xrightarrow{\sim} \left( M^{n+1-p}(M \cap M^{n-q+1})/[M] \right) : \Omega_M^{-n}.
\]

**Proof.** (1) The pair \((\Omega_M^n, \Omega_M^n)\) of additive functors, together with the counit and unit adjunctions

\[
\varepsilon_X : \Omega_M^n \Omega_M^n(X) \longrightarrow X \quad \text{and} \quad \eta_X : X \longrightarrow \Omega_M^n \Omega_M^n(X)
\]
defined in Lemma 3.2(2) for any $A$-modules $X$ and $Y$, forms an adjoint pair. Lemma 3.2(1) implies $\Omega^i_M(X) \in M^{i,n}$ and $\Omega^i_M(X) \in \Lambda$ for each $X$. We have to show two equivalences

$$\Omega^i_M : \begin{array}{c} \Lambda \rightarrow M^{i,n} \rightarrow M^{i,n} \rightarrow \Lambda \end{array}$$

This is equivalent to showing that for $X \in \Lambda$ and $Y \in M^{i,n}$, the adjunction maps

$$\varepsilon_X : \Omega^i_M \Omega^n_M(X) \rightarrow X \quad \text{and} \quad \eta_Y : Y \rightarrow \Omega^i_M \Omega^n_M(Y)$$

are isomorphisms in $A$-mod/$|M|$.

To see this, let

$$(*) : 0 \rightarrow \Omega^i_M(X) \rightarrow M_{n-1} \rightarrow \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow X \rightarrow 0$$

be a minimal right $n$-th add($M$)-approximation sequence of $X$. Applying $\text{Hom}_A(M,-)$ to the sequence $(*)$ yields a long exact sequence of $\Lambda$-modules, where $\Lambda := \text{End}_A(M)$:

$$0 \rightarrow \text{Hom}_A(M, \Omega^i_M(X)) \rightarrow \text{Hom}_A(M, M_{n-1}) \rightarrow \cdots \rightarrow \text{Hom}_A(M, M_0) \rightarrow \text{Hom}_A(M, X) \rightarrow 0.$$ 

Since $X \in \Lambda$, applying $\text{Hom}_A(-,M)$ to the sequence $(*)$ gives another long exact sequence of $\Lambda^n$-modules:

$$0 \rightarrow \text{Hom}_A(X, M) \rightarrow \text{Hom}_A(M_0, M) \rightarrow \cdots \rightarrow \text{Hom}_A(M_{n-1}, M) \rightarrow \text{Hom}_A(\Omega^i_M(X), M) \rightarrow 0.$$ 

This implies that, for each $0 \leq i \leq n-1$, the sequence

$$0 \rightarrow \Omega^i_M(X) \rightarrow M_{i-1} \rightarrow \Omega^i_M(X) \rightarrow 0$$

is an add($M$)-split sequence. Consequently, the map $\varepsilon_X$ is an isomorphism in $A$-mod/$|M|$. Dually, $\eta_Y$ also is an isomorphism in $A$-mod/$|M|$.

(2) If $X \in \Lambda$ and $M^{i,n}$, then $\Omega^i_M(X) \in \Lambda$ and $\Omega^i_M(X) \in \Lambda$ for all $i \geq 1$. The following result collects several basic properties of the above category.

**Definition 3.4.** Let $A$ be a generator-cogenerator as before. Then

$$\mathcal{G}(A) \ := \ \{ \ X \in \Lambda \mid \Omega^i_M(X) \in \Lambda \quad \text{for all} \ i \geq 1 \} \subseteq A$$

The following result collects several basic properties of the above category.

**Lemma 3.5.** (1) The category $\mathcal{G}(M)$ is a Frobenius category. Its full subcategory of projective-injective objects equals add($M$).

(2) For any $X, Y \in \mathcal{G}(M)$, there are isomorphisms

$$\text{Ext}_A^i(X, Y) \simeq \text{Hom}_{\mathcal{G}(M)/|M|}(\Omega^i_M(X), Y) \simeq \text{Hom}_{\mathcal{G}(M)/|M|}(X, \Omega^i_M(Y))$$

for each $1 \leq i \leq n$.

(3) The functor $\text{Hom}_A(M, -)$ induces an equivalence of Frobenius categories:

$$\mathcal{G}(M) \xrightarrow{\sim} \Lambda\text{-gp}.$$ 

In particular, $\mathcal{G}(M)/|M| \xrightarrow{\sim} \Lambda\text{-gp}/|\Lambda|$ as triangulated categories.

**Proof.** (1) Note that, in general, the category $\Lambda$ is closed under taking $\Omega^{-1}_M$ and the category $M^{i,n}$ is closed under taking $\Omega_M$. Thus

$$\mathcal{G}(M) \ := \ \{ X \in A\text{-mod} \mid \Omega^i_M(X) \in \Lambda \quad \text{for all} \ i \in \mathbb{Z} \}.$$ 

Therefore, $\mathcal{G}(M)$ is closed under taking $\Omega_M$ and $\Omega^{-1}_M$.

**Claim.** The category $\mathcal{G}(M)$ is closed under extensions in $A$-mod.

In fact, for any $A$-module $X$ and for each $s \in \mathbb{N}$, there are equalities

$$\Omega^s_A(\text{Hom}_A(M, X)) = \text{Hom}_A(M, \Omega^s_M(X)) \quad \text{and} \quad \Omega^{-s}_A(\text{Hom}_A(X, M)) = \text{Hom}_A(\Omega^{-s}_M(X), M).$$

Combining this with Lemma 2.4, we see that if $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$ is an exact sequence such that $X_1 \in M^{i,1}$ and $X_3 \in M^{1,1}$, then there exists an exact sequence

$$0 \rightarrow \Omega^i_M(X_1) \rightarrow \Omega^i_M(X_2) \oplus M \rightarrow \Omega^i_M(X_3) \rightarrow 0.$$
with $M_i \in \text{add}(M)$ for each $i \in \mathbb{Z}$. Since $\frac{1}{m}M \cap M^{\perp n}$ is closed under extensions in $A$-mod, the category $\mathcal{G}(M)$ is also closed under extensions in $A$-mod.

Hence, $\mathcal{G}(M)$ naturally inherits an exact structure from $A$-mod and becomes a Frobenius category, whose full subcategory consisting of projective objects coincides with $\text{add}(M)$. By Lemma 3.4, $\mathcal{G}(M)/[M]$ is a triangulated category with shift functor $\Omega^1_M$.

(2) Note that

$$\text{Ext}_A^i(X, Y) \simeq \text{Ext}_A^{i-1}(\Omega M(X), Y) \simeq \cdots \simeq \text{Ext}_A^1(\Omega M^1(X), Y) \simeq \text{Hom}_{\mathcal{G}(M)/[M]}(\Omega M^1(X), Y),$$

$$\text{Ext}_A^i(X, Y) \simeq \text{Ext}_A^{i-1}(X, \Omega M^1(Y)) \simeq \cdots \simeq \text{Ext}_A^1(X, \Omega M^1(Y)) \simeq \text{Hom}_{\mathcal{G}(M)/[M]}(X, \Omega M^1(Y)).$$

(3) Recall that $\Lambda := \text{End}_A(M)$. By Lemma 2.4, the functor $\text{Hom}_A(M, -) : A$-mod $\rightarrow \Lambda$-mod is fully faithful.

**Claim.** The functor $\text{Hom}_A(M, -) : A$-mod $\rightarrow \Lambda$-mod restricts to a functor $F : \mathcal{G}(M) \rightarrow \Lambda$-gp.

In fact, given an arbitrary $A$-module $Z \in \mathcal{G}(M)$, we take a minimal right $\text{add}(M)$-approximation and a minimal left $\text{add}(M)$-approximation of $Z$ as follows:

$$\cdots \rightarrow N^{-3} \xrightarrow{g^{-3}} N^{-2} \xrightarrow{g^{-2}} N^{-1} \xrightarrow{g^{-1}} N^0 \xrightarrow{\pi} Z \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Z \xrightarrow{\lambda} N^1 \xrightarrow{g^1} N^2 \xrightarrow{g^2} N^3 \cdots$$

Since $\Omega^i_M(Z) \in \frac{1}{m}M \cap M^{\perp n} \subseteq \frac{1}{m}M$ for all $i \geq 0$, the induced complex

$$0 \rightarrow \text{Hom}_A(Z, M) \rightarrow \text{Hom}_A(N^0, M) \rightarrow \text{Hom}_A(N^1, M) \rightarrow \text{Hom}_A(N^2, M) \rightarrow \cdots$$

is exact. Similarly, since $\Omega^i_M(Z) \in \frac{1}{m}M \cap M^{\perp n} \subseteq \frac{1}{m}M$ for all $i \leq 0$, the induced complex

$$0 \rightarrow \text{Hom}_A(M, Z) \rightarrow \text{Hom}_A(M, N^1) \rightarrow \text{Hom}_A(M, N^2) \rightarrow \text{Hom}_A(M, N^3) \rightarrow \cdots$$

also is exact. Let

$$N^* := \cdots \rightarrow N^{-3} \xrightarrow{g^{-3}} N^{-2} \xrightarrow{g^{-2}} N^{-1} \xrightarrow{g^{-1}} N^0 \xrightarrow{\pi} N^1 \xrightarrow{\pi} N^2 \xrightarrow{\pi} N^3 \rightarrow \cdots$$

where $g^0 = \pi \lambda$. Then the image of the homomorphism $\text{Hom}_A(M, g_0) : \text{Hom}_A(M, N^0) \rightarrow \text{Hom}_A(M, N^1)$ is equal to $\text{Hom}_A(M, Z)$. Moreover, both Hom-complexes $\text{Hom}_A^*(M, N^*)$ and $\text{Hom}_A^*(N^*, M)$ are exact. Therefore, $\text{Hom}_A(Z, M) \in \Lambda$-gp. Restriction yields a fully faithful functor $F : \mathcal{G}(M) \rightarrow \Lambda$-gp.

It remains to show that $F$ is dense; then $F$ is an equivalence.

Let $Y$ be a Gorenstein projective $\Lambda$-module. Then $Y$ is reflexive. Since $M$ is a generator-cogenerator, it follows from Lemma 2.4 that there is a $A$-module $X$ such that $Y \simeq \text{Hom}_A(M, X)$.

**Claim.** $X \in \mathcal{G}(M)$.

**Proof.** Since $A$ and $Y$ is Gorenstein projective and $M$ is a generator, there exists an exact sequence of $A$-modules

$$M^* : \cdots \rightarrow M^{-3} \xrightarrow{f^{-3}} M^{-2} \xrightarrow{f^{-2}} M^{-1} \xrightarrow{f^{-1}} M^0 \xrightarrow{f^0} M^1 \xrightarrow{f^1} M^2 \xrightarrow{f^2} M^3 \rightarrow \cdots$$

with $M^i \in \text{add}(M)$ for all $i \in \mathbb{Z}$ such that $\text{Im}(f^0) = X$, and both Hom-complexes $\text{Hom}_A^*(M, M^*)$ and $\text{Hom}_A^*(M^*, M)$ are exact. Let $K^i := \text{Ker}(f^i)$. Then $X = K^1$ and the short exact sequence

$$0 \rightarrow K^1 \rightarrow M^1 \rightarrow K^{i+1} \rightarrow 0$$

induced from $M^*$ is actually an $\text{add}(M)$-split sequence. So $\Omega_M(K^{i+1}) \simeq K^i$ and $\Omega_M^{-1}(K^i) \simeq K^{i+1}$ in $A$-mod$/[M]$. It follows that $\Omega^m_M(K^{i-n}) \simeq K^i \simeq \Omega^m_M(K^{i+n})$ in $A$-mod$/[M]$. Since $M$ is $n$-rigid, Lemma 3.2 implies that $K^i \in M^{\perp n} \cap M^{\perp n}$ for each $i \in \mathbb{Z}$. In other words, $X \in \mathcal{G}(M)$. Hence $F$ is dense.

Given an exact sequence $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$ in $A$-mod with $X_i \in \mathcal{G}(M)$ for $1 \leq i \leq 3$, applying $\text{Hom}_A(M, -)$ to this sequence yields an exact sequence of $\Lambda$-modules:

$$0 \rightarrow \text{Hom}_A(M, X_1) \rightarrow \text{Hom}_A(M, X_2) \rightarrow \text{Hom}_A(M, X_3) \rightarrow 0.$$

Thus, (3) holds. □

For ortho-symmetric modules, $\mathcal{G}(M)$ can be described explicitly, which is one reason to choose ortho-symmetric as basic concept.

**Corollary 3.6.** Let $0 \leq m \leq n-1$. Then the following statements are equivalent:

1. $A$-mod is $(n, m)$-ortho-symmetric.
2. $\mathcal{G}(M) = \frac{1}{m}M \cap M^{\perp n}$.
3. $\mathcal{G}(M) = \frac{1}{m}M \cap M^{\perp n}$.

If any of these assertions holds, then $\mathcal{G}(M) = \frac{1}{n}M \cap M^{\perp n}$. 
Proof. For $1 \leq p, q \leq n$, set $^pM^q := \wed_p M \cap M^{q \cdot}$. By definition of $\mathcal{G}(M)$, there is an inclusion $\mathcal{G}(M) \subseteq ^nM^n$. Moreover, $\mathcal{G}(M)$ is closed under taking $\Omega_M$ and $\Omega_M^{-1}$ in $A$-mod. By Lemma 3.3(2), there are equivalences of additive categories:

$$(\ast) \quad n^m/[M] \xrightarrow{\Omega_M} (n-1)^{m+1}/[M] \xrightarrow{\Omega_M} \cdots \xrightarrow{\Omega_M} (m+1)^{m-1}/[M] \xrightarrow{\Omega_M} m^m/[M]$$

(1) implies both (2) and (3):

Suppose that (1) holds; then $^nM^m = ^mM^n$. Since $0 \leq m \leq n-1$, we even have $^nM^m = ^mM^n = ^mM^n$. Let $X \in ^nM^n$. Since $^mM^n$ is closed under taking $\Omega_M$ by Lemma 5.2(1), we get $\Omega_M(X) \in ^mM^n$. Note that $\Omega_M^mM^m(X) \in ^nM^n$ due to (1). This implies $\Omega_M^{-n}X \in ^mM^n$. Moreover, by Lemma 5.2(1), the category $^mM^n$ is closed under taking $\Omega_M^{-1}$ in $A$-mod. The above equivalences imply that $\Omega_M^{-i}X \in ^iM^i$ for all $1 \leq i \leq n-m$. In particular, $\Omega_M(X) \in ^1M^1$. So $^nM^n$ is closed under taking $\Omega_M$ in $A$-mod. Similarly, $^mM^n$ is closed under taking $\Omega_M^{-1}$ in $A$-mod. Thus $\mathcal{G}(M) = ^nM^n$.

(2) implies (1):

Suppose $\mathcal{G}(M) = ^nM^n$. Since $\mathcal{G}(M) \subseteq ^nM^n$ and $m < n$, it follows that $\mathcal{G}(M) = ^nM^n = ^mM^n$. Note that $\mathcal{G}(M)$ is closed under taking $\Omega_M$ in $A$-mod. Hence, because of (1), there is an inclusion $^nM^n \subseteq \mathcal{G}(M)$. Since $\mathcal{G}(M) = ^nM^n \subseteq ^mM^n$, we have $^nM^n = ^mM^n = ^mM^n$. Thus (1) holds. Dually, (3) also implies (1). □

3.5. Higher Auslander-Reiten translation and derived equivalences. Let $\tau : A$-mod$/[A] \xrightarrow{\approx} A$-mod$/[D(A)]$ and $\tau^{-} : A$-mod$/[D(A)] \xrightarrow{\approx} A$-mod$/[A]$ be the classical Auslander-Reiten translations. Iyama’s higher versions are defined by

$$\tau_{n+1} := \tau^nA \quad \text{and} \quad \tau_{n+1}^{-} := \tau^{-n}A.$$ 

Then there exist mutually inverse equivalences of additive categories:

$$\tau_{n+1} : \wed_nA/[A] \xrightarrow{\approx} D(A)$${}\wed_n$/[D(A)] \quad \text{and} \quad \tau_{n+1}^{-} : D(A)$${}\wed_n$/[D(A)] \xrightarrow{\approx} \wed_nA/[A].$$

The functors $\tau_{n+1}$ and $\tau_{n+1}^{-}$ are called the $(n+1)$-Auslander-Reiten translations. For more details and proofs, see [21] Section 1.4.1.

For $X \in ^nA$ and $Y \in D(A)$, set

$$X^+ := \tau_{n+1}^{-}X \oplus D(A) \quad \text{and} \quad Y^{-} := A \oplus \tau_{n+1}^{-}Y.$$ 

Then $X^+ \in D(A)$ and $Y^{-} \in \wed_nA$. Moreover, $\#(A \oplus X) = \#(X^+)$ and $\#(Y \oplus D(A)) = \#(Y^-)$.

**Lemma 3.7.** [21] Theorem 1.5] Let $X \in ^nA$ and $Y \in D(A)$-mod. For any $1 \leq i \leq n$, there exist functorial isomorphisms for any $A$-module $Z$:

$$\text{Ext}^{i+1}_A(Z, X \oplus D(A)) \xrightarrow{\approx} \text{Ext}^i_A(Z, \tau_{n+1}(X)) \quad \text{and} \quad \text{Hom}_A(Z, X \oplus D(A)) \xrightarrow{\approx} \text{Hom}_A(Z, \tau_{n+1}(X)).$$

In particular, $X^+ \in \wed_nA$ and $Y^{-} \in \wed_nA$. Moreover, $\#(A \oplus X) = \#(X^+)$ and $\#(Y \oplus D(A)) = \#(Y^-)$.

The following result will be used later.

**Lemma 3.8.** Let $0 \rightarrow X \rightarrow Y \xrightarrow{h} Z \rightarrow 0$ be an exact sequence of $A$-modules. Suppose that $Ah \in D(A)$ and the map $\text{Hom}_A(h, \tau_{n+1}(A)) : \text{Hom}_A(\tau_{n+1}(A), Y) \rightarrow \text{Hom}_A(\tau_{n+1}(A), Z)$ is surjective. Then there exists an exact sequence of $A$-modules:

$$0 \rightarrow \tau_{n+1}(X) \rightarrow \tau_{n+1}(Y) \oplus I \rightarrow \tau_{n+1}(Z) \rightarrow 0$$

where $I$ is injective.

**Proof.** For an $A$-module $N$, the transpose $\text{Tr}_A(N)$ is defined to be the cokernel of the homomorphism $\text{Hom}_A(\theta, A)$ induced from $\theta$, which appears in a minimal projective presentation of $N$:

$$\text{P}^1_N \xrightarrow{0} \text{P}^0_N \rightarrow N \rightarrow 0,$$

where $\text{P}^0_N$ and $\text{P}^1_N$ are projective. Let $(-)^* := \text{Hom}_A(-, A)$. Then there is an exact sequence of $A^{\text{op}}$-modules:

$$0 \rightarrow N^* \rightarrow (\text{P}^0_N)^* \rightarrow (\text{P}^1_N)^* \rightarrow \text{Tr}_A(N) \rightarrow 0$$

An exact sequence $0 \rightarrow X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \rightarrow 0$ in $A$-mod induces a long exact sequence of $A^{\text{op}}$-modules:

$$0 \rightarrow X_1^* \xrightarrow{f^*} X_2^* \xrightarrow{g^*} X_1^* \rightarrow \text{Tr}_A(X_3) \rightarrow \text{Tr}_A(X_2) \oplus Q \rightarrow \text{Tr}_A(X_1) \rightarrow 0$$
where $Q_A$ is projective. Since $\tau_A = DT\mathbf{r}$, there is the following exact sequence of $A$-modules:

$$0 \to \tau_A(X_1) \to \tau_A(X_2) \oplus I_0 \to \tau_A(X_3) \to v_A(X_1) \xrightarrow{v(\psi)} v_A(X_2) \xrightarrow{v(\rho)} v_A(X_3) \to 0,$$

where $I_0$ is injective and $v_A := D(-)^*$. Applying the functor $\Omega_A^n$ to the given exact sequence $0 \to X \to Y \xrightarrow{h} Z \to 0$ yields the short exact sequence of $A$-modules:

$$(*) \quad 0 \to \Omega_A^n(X) \xrightarrow{\psi} \Omega_A^n(Y) \oplus P \xrightarrow{\rho} \Omega_A^n(Z) \to 0,$$

where $P$ is projective and $\psi = \Omega_A^n(h)$ in the abelian group $\text{Hom}_A(\Omega_A^n(Y), \Omega_A^n(Z))$. Since $\tau_{n+1} = \tau A \Omega_A^n$ by definition, we then get a long exact sequence of the following form:

$$0 \to \tau_{n+1}(X) \to \tau_{n+1}(Y) \oplus I \to \tau_{n+1}(Z) \to v_A \Omega_A^n(X) \xrightarrow{v_A(\psi)} v_A \Omega_A^n(Y) \oplus v_A(P) \to v_A \Omega_A^n(Z) \to 0$$

where $I$ is injective.

**Claim.** $v_A(\phi)$ is injective.

**Proof:** Applying $v_A$ to the sequence $(*)$ returns a long exact sequence of $A$-modules:

$$D\text{Ext}_A^{n+1}(Y, A) \to D\text{Ext}_A^{n+1}(Y, A) \oplus P, A \xrightarrow{D\text{Ext}_A^{n+1}(Y, A)} v_A \Omega_A^n(Y) \oplus P \xrightarrow{v_A(\psi)} v_A \Omega_A^n(Z) \to 0,$$

which is isomorphic to the sequence

$$D\text{Ext}_A^{n+1}(Y, A) \to D\text{Ext}_A^{n+1}(Z, A) \to v_A \Omega_A^n(Y) \oplus v_A(P) \to v_A \Omega_A^n(Z) \to 0.$$

It remains to show that $D\text{Ext}_A^{n+1}(h, A)$ is surjective. Since $A A \in D(A A)^{1, n}$, Lemma 3.7 gives the commutative diagram:

$$\begin{array}{ccc}
D\text{Ext}_A^{n+1}(Y, A) & \xrightarrow{D\text{Ext}_A^{n+1}(h, A)} & D\text{Ext}_A^{n+1}(Z, A) \\
\downarrow \cong & & \downarrow \cong \\
\text{Hom}_A(\tau_{n+1}(A), Y) & \xrightarrow{\text{Hom}_A(\tau_{n+1}(h), A)} & \text{Hom}_A(\tau_{n+1}(A), Z).
\end{array}$$

The map $\text{Hom}_A(\tau_{n+1}(A), h) : \text{Hom}_A(\tau_{n+1}(A), Y) \to \text{Hom}_A(\tau_{n+1}(A), Z)$ being surjective implies that $\text{Hom}_A(\tau_{n+1}(A), h)$ (and thus also $D\text{Ext}_A^{n+1}(h, A)$) is surjective. Consequently, the map $v_A(\phi)$ is injective, providing the required exact sequence. □

**Proposition 3.9.** The $\Lambda$-module $D\text{Hom}_A(M^-, M)$ and the $\Lambda^\mu$-module $D\text{Hom}_A(M, M^\mu)$ are tilting modules of projective dimension $(n + 2)$. In particular, the algebras $\Lambda$, $\text{End}_A(M^-)$ and $\text{End}_A(M^\mu)$ are derived equivalent.

**Proof.** We only show that $D\text{Hom}_A(M^-, M)$ is an $(n + 2)$-tilting $\Lambda$-$\text{End}_A(M^-)$-bimodule. The other statement can be proved dually.

Choose a minimal injective coresolution of $A M$:

$$0 \to A M \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \xrightarrow{f} I_{n+1} \to \cdots$$

Let $X \in A \text{-mod}$ and let

$$g_X := \text{Hom}_A(X, f) : \text{Hom}_A(X, I_n) \to \text{Hom}_A(X, I_{n+1}).$$

**Claim.** $\text{Coker } (g_X) \simeq D\text{Hom}_A(\tau_{n+1}(M), X)$ as additive functors from the category of injective $A$-modules to the category of $B$-modules. Thus the left-exact sequence $(\dagger)$ is morphism to the sequence:

$$0 \to D(\text{Coker } (g_X)) \to D(\text{Hom}_A(X, I_{n+1})) \xrightarrow{D(g_X)} D(\text{Hom}_A(X, I_n)).$$

Note that $D\text{Hom}_A(X, -) \simeq \text{Hom}_A(v_A(-), X)$ as additive functors from the category of injective $A$-modules to the category of $B$-modules. Thus the left-exact sequence $(\dagger)$ is isomorphic to the sequence:

$$0 \to D(\text{Coker } (g_X)) \to \text{Hom}_A(v_A(I_{n+1}), X) \xrightarrow{h_X} \text{Hom}_A(v_A(I_n), X))$$

with $h_X := \text{Hom}_A(v_A(f), X)$, where $v_A(f) : v_A(I_n) \to v_A(I_{n+1})$ is a homomorphism of injective $A$-modules. It follows that

$$D(\text{Coker } (g_X)) \simeq \text{Hom}_A(\text{Coker } (v_A(f)), X)$$

as $B$-modules. Since $\tau_{n+1}(M) = \tau A \Omega_A^n(M) = \text{Coker } (v_A(f))$, there is an isomorphism $D(\text{Coker } (g_X)) \simeq \text{Hom}_A(\tau_{n+1}(M), X)$ in $B$-mod. Thus $\text{Coker } (g_X) \simeq D\text{Hom}_A(\tau_{n+1}(M), X)$ as $\text{End}_A(X)$-modules. In particular, $\text{Coker } (g_M) \simeq D\text{Hom}_A(\tau_{n+1}(M), M)$ as $A$-modules.
Since Ext$_{\Lambda}^{i}(M,M) = 0$ for all $1 \leq j \leq n$, applying Hom$_{\Lambda}(M,-)$ to the above minimal injective coresolution of $\Lambda M$, yields the long exact sequence of $\Lambda$-modules:

$$0 \longrightarrow \Lambda M \longrightarrow \text{Hom}_A(M, I_0) \longrightarrow \cdots \longrightarrow \text{Hom}_A(M, I_n) \xrightarrow{\text{gr}} \text{Hom}_A(M, I_{n+1}) \longrightarrow N \longrightarrow 0,$$

where $N := \text{DHom}_A(\tau_{n+1}(M), M)$. Let $T := \text{DHom}_A(M^-, M)$. Then $T = N \oplus \text{DHom}_A(A, M) \cong N \oplus \text{Hom}_A(M, D(A))$. Since $\Lambda$ is a cogenerator, the $\Lambda$-module $\text{Hom}_A(M, D(A))$ is projective. Hence, the sequence

$$(\dagger) \quad 0 \longrightarrow \Lambda M \longrightarrow \text{Hom}_A(M, I_0) \longrightarrow \cdots \longrightarrow \text{Hom}_A(M, I_n) \xrightarrow{\text{gr}} \text{Hom}_A(M, I_{n+1}) \longrightarrow N \longrightarrow 0$$

is a projective resolution of $\Lambda N$. Therefore, the $\Lambda$-module $T$ has projective dimension at most $n+2$. Since $\Lambda M$ is not injective, the canonical inclusion $\Lambda \rightarrow \text{Hom}_A(M, I_0)$ does not split. Thus the projective dimension of $T$ is exactly $n+2$. To show that $\Lambda T$ is a tilting module, it remains to prove Ext$_{\Lambda}^{i}(T, T) = 0$ for any $i \geq 1$. Actually, applying Hom$_{\Lambda}(-, \text{Hom}_A(M, D(A)))$ to the sequence $(\dagger)$ provides us with the following exact sequence (up to isomorphism)

$$0 \longrightarrow \text{Hom}_A(N, \text{Hom}_A(M, D(A))) \longrightarrow \text{Hom}_A(I_{n+1}, D(A)) \longrightarrow \cdots \longrightarrow \text{Hom}_A(I_0, D(A)) \longrightarrow \text{Hom}_A(M, D(A)) \longrightarrow 0.$$ 

This implies that Ext$_{\Lambda}^{i}(N, \text{Hom}_A(M, D(A))) = 0$ for any $i \geq 1$. Furthermore, applying Hom$_{\Lambda}(N, -)$ to the sequence $(\dagger)$ yields

$$\text{Ext}^{i}\Lambda(N, N) \cong \text{Ext}^{i+1}\Lambda(N, \Omega_{\Lambda}^{(N)}) \cong \cdots \cong \text{Ext}^{i+n+2}\Lambda(N, \Omega_{\Lambda}^{n+2}(N))$$

The projective dimension of $\Lambda N$ is $n+2$, hence Ext$_{\Lambda}^{i}(N, N) = 0$ for any $i \geq 1$, and therefore Ext$_{\Lambda}^{i}(T, T) = 0$. Thus $\Lambda T$ is a tilting module of projective dimension $n+2$. Since $\Lambda M$ is a cogenerator, there are isomorphisms of algebras

$$\text{End}_A(T) \cong \text{End}_A^{\sigma}(\text{Hom}_A(M^-, M)) \cong \text{End}_A(M^-)$$

by Lemma 2.2. Hence $T$ is an $(n+2)$-tilting $\Lambda$-End$_A(M^-)$-bimodule. In particular, the algebras $\Lambda$ and End$_A(M^-)$ are derived equivalent. □

**Lemma 3.10.** There are the following isomorphisms:

$$\Omega_{\Lambda}^{(n+2)}(\Lambda) \cong \text{DHom}_A(\tau_{n+1}(M), M) \quad \text{and} \quad \Omega_{\Lambda}^{(n+2)}(\Lambda) \cong \text{DHom}_A(M, \tau_{n+1}(M)).$$

*Proof.* When $\Lambda M$ is a generator-cogenerator, then the $\Lambda$-module Hom$_{\Lambda}(M, D(A))$ is projective-injective. So the exact sequence

$$(\dagger) \quad 0 \longrightarrow \Lambda \longrightarrow \text{Hom}_A(M, I_0) \longrightarrow \cdots \longrightarrow \text{Hom}_A(M, I_n) \xrightarrow{\text{gr}} \text{Hom}_A(M, I_{n+1}) \longrightarrow \text{DHom}_A(\tau_{n+1}(M), M) \longrightarrow 0$$

in the proof of Proposition 3.9 provides us with the following exact sequence of $\Lambda$-modules. Thus $\Omega_{\Lambda}^{(n+2)}(\Lambda) \cong \text{DHom}_A(\tau_{n+1}(M), M)$ as $\Lambda$-modules. Similarly, using $\Lambda^\sigma$-modules, $\Omega_{\Lambda}^{(n+2)}(\Lambda) \cong \text{DHom}_A(M, \tau_{n+1}(M))$ as $\Lambda^\sigma$-modules. □

3.6. When is the endomorphism algebra $\Lambda$ Gorenstein? To address this question, we are going to use the following tool:

**Proposition 3.11.** (1) Both $M^-$ and $M^+$ are n-rigid with $M^- \in \overset{\sim}{\Lambda}^n M$ and $M^+ \in M^{\sim n}$.

(2) injdim($\Lambda \Lambda$) = $n+2+M$-coresdim($M^-$).

(3) injdim($\Lambda M$) = $n+2+M$-resdim($M^+$).

*Proof.* Since $\Lambda M$ is an $n$-rigid generator-cogenerator, $M \in \overset{\sim}{\Lambda}^n \cap D(A) \overset{\sim}{\Lambda}^n$. By Lemma 3.7, $M \overset{\sim}{\Lambda}^n = \overset{\sim}{\Lambda}^n(M^+)$ and $\overset{\sim}{\Lambda}^n M = (M^-)^{\sim n}$. Now, (1) follows from the $n$-rigidity of $M$.

Since $\Lambda M$ is neither projective nor injective, we have $\tau_{n+1}(M) \neq 0$ and $\tau_{n+1}(M) \neq 0$. By Lemma 3.10

$$\text{injdim}(\Lambda \Lambda) = n+2+\text{injdim}(\Omega_{\Lambda}^{(n+2)}(\Lambda)) = n+2+\text{projdim}(\text{Hom}_{\Lambda}(\tau_{n+1}(M), M)).$$

Moreover, since $\Lambda M$ is a generator, the $\Lambda^\sigma$-module Hom$_{\Lambda}(A, M)$ is projective. Consequently,

$$\text{injdim}(\Lambda \Lambda) = n+2+\text{projdim}(\text{Hom}_{\Lambda}(M^-, M)\Lambda).$$

The equality projdim(Hom$_{\Lambda}(M^-, M)\Lambda) = M$-coresdim($M^-$) implies injdim($\Lambda \Lambda$) = $n+2+M$-coresdim($M^-$). This verifies (2). Similarly, we can prove (3). □

A consequence of Proposition 3.11 is a characterisation of $\Lambda$ being Gorenstein:

**Corollary 3.12.** The algebra $\Lambda$ is Gorenstein if and only if $M$-coresdim($M^-$) < $\infty$ and $M$-resdim($M^+$) < $\infty$. In this case, $M$-coresdim($M^-$) = $M$-resdim($M^+$).
As before, $\Lambda$ is an endomorphism ring of a generator-cogenerator. Proposition 3.11 and Corollary 3.12 allow to reformulate - and later on to prove in ortho-symmetric situations - in our setup, a celebrated open problem, the Gorenstein Symmetry Conjecture: An algebra has finite left injective dimension if and only if it has finite right injective dimension (see, for instance, [3 Conjectures]). By Proposition 3.11 the conjecture for $\Lambda$ can be reformulated in terms of (co)resolution dimension:

$$M\text{-coresdim}(M^-) < \infty \text{ if and only if } M\text{-resdim}(M^+) < \infty.$$  

This new form has the following equivalent characterisations, which will be used in Subsection 3.9 to show that the conjecture holds for a class of endomorphism algebras of generator-cogenerators.

Lemma 3.13. Let $\Lambda := \text{End}_A(M)$ as before. Then:

1. If $M\text{-coresdim}(M^-) < \infty$, then $M\text{-resdim}(M^+) < \infty$ if and only if $\text{Hom}_A(M-, M)$ is a tilting $\Lambda^\oplus$-module.
2. If $M\text{-resdim}(M^+) < \infty$, then $M\text{-coresdim}(M^-) < \infty$ if and only if $\text{Hom}_A(M, M^+)$ is a tilting $\Lambda$-module.
3. $M\text{-coresdim}(M^-) \leq 1$ if and only if $M\text{-resdim}(M^+) \leq 1$.

Proof. By Proposition 3.9 and Lemma 3.10 the $\Lambda$-module $D\text{Hom}_A(M-, M)$ is a tilting module and there is a long exact sequence of $\Lambda$-modules

$$0 \rightarrow \Lambda M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n \rightarrow D\text{Hom}_A(M-, M) \rightarrow 0,$$

with $E_i$ being projective-injective for $0 \leq i \leq n + 1$. In particular, $D\text{Hom}_A(M-, M)$ is self-orthogonal. Applying the dual $D$ to the above sequence returns the long exact sequence of $\Lambda^\oplus$-modules

$$0 \rightarrow \text{Hom}_A(M-, M) \rightarrow D(E_{n+1}) \rightarrow \cdots \rightarrow D(E_1) \rightarrow D(E_0) \rightarrow D(\Lambda) \rightarrow 0.$$  

This implies that $\text{Hom}_A(M-, M) = \Omega^{n+2}(D(\Lambda))$ and $D(\Lambda) = \Omega^{-n-2}(\text{Hom}_A(M-, M))$. For an arbitrary $\Lambda^\oplus$-module $Y$, let $\mathcal{Y}$ be the smallest triangulated subcategory of $\mathcal{D}(\Lambda)$, which is closed under direct summands and contains $Y$ and all projective-injective $A$-modules. Then $\mathcal{Y}(D(\Lambda)) = \mathcal{Y}(\text{Hom}_A(M-, M))$. Up to multiplicity and isomorphism, all projective-injective modules occur as direct summands of all tilting modules. Therefore, $D(\Lambda)$ is a tilting module if and only if so is $\text{Hom}_A(M-, M)$.

1. Suppose $M\text{-coresdim}(M^-) < \infty$. Then $\text{injdim}(\Lambda) < \infty$ by Proposition 3.11(2). Consequently, both $D(\Lambda)$ and $\text{Hom}_A(M-, M)$ are partial tilting $\Lambda^\oplus$-modules: In fact, $D(\Lambda)$ is a tilting module if and only if $\text{injdim}(\Lambda) < \infty$; equivalently, $M\text{-resdim}(M^+) < \infty$ by Proposition 3.11(3). Thus $\text{Hom}_A(M-, M)$ is a tilting module if and only if $M\text{-resdim}(M^+) < \infty$. This shows (1), while (2) is dual.

2. We only show necessity; sufficiency can be proved dually. Assume $M\text{-coresdim}(M^-) \leq 1$. By the proof of (1), $\text{Hom}_A(M-, M)_A$ is a partial $m$-tilting module with $m \leq 1$. Since $\#(\text{Hom}_A(M-, M)_A) = \#(\text{Hom}_A(M-, M)) = \#(\Lambda)$, this module even is $m$-tilting by [1 Corollary 2.6]. Thus $M\text{-resdim}(M^+) < \infty$ by (1). In this case, $\Lambda$ is a Gorenstein algebra, and hence $M\text{-resdim}(M^+) = M\text{-coresdim}(M^-) \leq 1$ by Corollary 3.12.  

Remarks on Subsections 3.5 and 3.6.

1. When $n = 0$, all the results in these two subsections make sense and still hold.
2. The tilting modules constructed in Proposition 3.9 are a special kind of the canonical tilting modules defined in [10 Section 3.4]. Connections between tilting modules and dominant dimensions are discussed in the preprint [10].

3. Corollary 3.12 may be compared with the following characterisation, derived from [4 Proposition 3.6], providing a different approach to the question when $\Lambda$ is Gorenstein: The algebra $\Lambda$ is Gorenstein if and only if $M$ is a $F_M$-cotilting $A$-module if and only if $M$ is an $F_M$-tilting $A$-module, where $F_M$ and $F_M$ are two additive subfunctors of $\text{Ext}^2_A(\Lambda, -)$ determined by $M$ (see [4] for details).

3.7. Proof of Theorem [3]. Suppose that $\Lambda$ is $(n + 2 + m)$-Gorenstein with $0 \leq m \leq n$.

Claim: $\mathcal{G}(M) = \mathcal{F}(M) = \mathcal{F}(M)$.

Proof: Since $m \leq n$, there are inclusions $\mathcal{F}(M) \subseteq 1 = \mathcal{F}(M) \subseteq \mathcal{F}(M) \subseteq \mathcal{F}(M)$. In order to show $\mathcal{F}(M) \subseteq \mathcal{F}(M)$, set

$$\Lambda := \{ Y \in \Lambda\text{-mod} \mid \text{Ext}^i_A(Y, \Lambda) = 0 \text{ for all } i \geq 1 \}.$$  

We are going to prove that $\text{Hom}_A(M, X) \in 1 = \text{Ext}^i_A(Y, \Lambda) = 0$ for any $X \in 1 = \mathcal{F}(M)$. A consequence of $\Lambda$ being a Gorenstein algebra is $\mathcal{F}(M) = \text{Ext}^i_A(Y, \Lambda)$ (see, for example, [11 Corollary 11.5.3]). This forces $\text{Hom}_A(M, X) \in \mathcal{F}(M)$. By Lemma 3.5(3), the functor $\text{Hom}_A(M, -)$ induces an equivalence from $\mathcal{G}(M)$ to $\text{Ext}^i_A(Y, \Lambda)$). Thus

$$\mathcal{G}(M) = \mathcal{F}(M) = \mathcal{F}(M) = \mathcal{F}(M).$$

It remains to show that $\text{Hom}_A(M, X) \in 1 = \Lambda$ since the algebra $\Lambda$ is at most $(n + 2 + m)$-Gorenstein, Proposition 3.11 implies that $M\text{-coresdim}(M^-) \leq m$. So there exists an exact sequence of $A$-modules:

$$0 \rightarrow \tau_{n+1}^\Lambda(M) \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_m \rightarrow 0.$$
with $M_i \in \text{add}(M)$ for $0 \leq i \leq m$ such that the induced sequence

$$0 \to \text{DHom}_A(\tau_{n+1}(M), M) \to \text{DHom}_A(M_0, M) \to \text{DHom}_A(M_1, M) \to \cdots \to \text{DHom}_A(M_m, M) \to 0$$

is a minimal injective coresolution of the $A$-module $\text{DHom}_A(\tau_{n+1}(M), M)$. Furthermore, let

$$0 \to A^M \to I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n \xrightarrow{f} I_{n+1} \to \cdots$$

be a minimal injective coresolution of $\Lambda^M$. Since $\Lambda^M$ is an $n$-rigid generator-cogenerator, there is an exact sequence of $A$-modules

$$0 \to \Lambda^M \to \text{Hom}_A(M, I_0) \to \cdots \to \text{Hom}_A(M, I_{n-1}) \to \text{Hom}_A(M, I_n + 1) \to \cdots$$

by Lemma 3.10, which gives the first $(n + 2)$ terms of a minimal injective coresolution of $\Lambda^M$. With this together with the above sequence (†), the module $\Lambda^M$ is seen to have the following minimal injective coresolution:

$$0 \to \Lambda^M \to \text{Hom}_A(M, I_0) \to \cdots \to \text{Hom}_A(M, I_{n-1}) \to \text{Hom}_A(M, I_n + 1) \to \cdots \to \text{DHom}_A(M, M) \to 0.$$ 

By Lemma 3.14, $\text{Hom}_A(\Lambda^M, Y), \text{Hom}_A(M, Z) \cong \text{Hom}_A(Y, Z)$ for any $A$-modules $Y$ and $Z$. If $Z \in \text{add}(\Lambda^M)$, then

$$\text{Hom}_A(\text{Hom}_A(M, Y), \text{Hom}_A(M, Z)) \cong \text{Hom}_A(\text{Hom}_A(M, Y), \text{Hom}_A(M, Z)) \cong \text{DHom}_A(\text{Hom}_A(M, Z), \text{Hom}_A(M, Y)) \cong \text{DHom}_A(Z, Y),$$

where the second isomorphism is implied by the following general result: For a projective module $P$ over an algebra $B$, there is a natural isomorphism $\text{DHom}_B(-, \text{Hom}_B(P)) \cong \text{DHom}_B(P, -)$. In order to finally show $\text{Hom}_A(M, X) \in \mathcal{I}$, applying $\text{Hom}_A(\text{Hom}_A(M, X), -)$ to the coreolution (‡) returns a bounded complex of $\text{End}_A(X)$-modules (up to isomorphism):

$$0 \to \text{Hom}_A(X, M) \to \text{Hom}_A(X, I_0) \to \cdots \to \text{Hom}_A(X, I_{n-1}) \to \text{DHom}_A(M_0, X) \to \cdots \to \text{DHom}_A(M_m, X) \to 0$$

where $g_x := \text{Hom}_A(X, f)$. **Subclaim:** This complex is exact. This implies $\text{Ext}_A^i(\text{Hom}_A(M, X), A) = 0$ for any $i \geq 0$, and therefore $\text{Hom}_A(M, X) \in \mathcal{I}$. **Proof of Subclaim.** The first part of the proof of Lemma 3.12 shows that $\text{Coker}(g_x) \cong \text{DHom}_A(\tau_{n+1}(M), X)$ as $\text{End}_A(X)$-modules. Since $X \in \mathcal{I}$, the sequence

$$0 \to \text{Hom}_A(X, M) \to \text{Hom}_A(X, I_0) \to \cdots \to \text{Hom}_A(X, I_{n-1}) \to \text{DHom}_A(M_0, X) \to \cdots \to \text{DHom}_A(M_m, X) \to 0$$

is exact. Since $X \in \mathcal{I}$ and $m \leq n$, the sequence

$$0 \to \text{Hom}_A(M_m, X) \to \text{Hom}_A(M_{m-1}, X) \to \cdots \to \text{Hom}_A(M_0, X) \to \text{Hom}_A(\tau_{n+1}(M), X) \to 0$$

also is exact, which gives rise to another exact sequence:

$$0 \to \text{DHom}_A(\tau_{n+1}(M), X) \to \text{DHom}_A(M_0, X) \to \cdots \to \text{DHom}_A(M_{m-1}, X) \to \text{DHom}_A(M_m, X) \to 0$$

by applying the duality D. So the complex is exact, as claimed.

If $m = n$, then $\Lambda^M$ is $(n, n)$-orthosymmetric since $\Lambda^M$ is an $n$-rigid generator-cogenerator. For $0 \leq m \leq n - 1$, $\Lambda^M$ is $(n, m)$-orthosymmetric by Corollary 3.6. This shows the first assertion of Theorem 3.8. The second assertion of Theorem 3.8 follows from $\mathcal{M} = \mathcal{I} \cap \mathcal{J}$ and Lemma 3.15. $\square$

A Gorenstein algebra has finite global dimension if and only if each Gorenstein projective module is projective. An easy consequence of Theorem 3.8 is the following observation.

**Corollary 3.14.** If $\Lambda$ has global dimension at most $2n + 2$, then $\mathcal{I} \cap \mathcal{J} = \text{add}(M)$.

**3.8. Characterisations.** Theorem 3.8 assumes the endomorphism ring to be Gorenstein and derives orthosymmetry and further conditions. In this subsection we are asking for converse statements, characterising Gorenstein properties (for specific Gorenstein parameters) in terms of ortho-symmetry. The next result generalises Theorem 3.8 and provides some necessary and sufficient conditions for $\Lambda$ to be Gorenstein, which can be used to show a converse of Theorem 3.8 for some small values of $m$.

**Proposition 3.15.** Let $0 \leq m \leq n$. Set $\mathcal{I} = \{0\} = M_{-1}$. Then the following statements are equivalent.

1. The algebra $\Lambda$ is at most $(n + 2 + m)$-Gorenstein.
2. $\mathcal{M} = \mathcal{I} \cap \mathcal{J}$ and $\text{Ext}_A^i(\Lambda, \Omega^m_{\mathcal{M}} \Omega^m_{\mathcal{M}}(X)) = 0$ for all $X \in \mathcal{I}$ and $n - m + 2 \leq i \leq n$.
3. $\mathcal{M} = \mathcal{I} \cap \mathcal{J}$ and $\text{Ext}_A^i(\Omega^m_{\mathcal{M}} \Omega^m_{\mathcal{M}}(X), M) = 0$ for all $X \in \mathcal{I}$ and $n - m + 2 \leq i \leq n$.
4. $\mathcal{M} = \mathcal{I} \cap \mathcal{J}$ and $(\mathcal{M}, \mathcal{I})$ is a left cotorsion pair in $\mathcal{M}$.
5. $\mathcal{M} = \mathcal{I} \cap \mathcal{J}$ and $(\mathcal{M}_{\leq m}, \mathcal{M})$ is a right cotorsion pair in $\mathcal{I}$.
Proof. We only show that (1) is equivalent to (2) and (4), respectively; the equivalence of (1), (3) and (5) is dual.

The proof starts with a sequence of reformulations of $\Lambda$ being at most $(n+2+m)$-Gorenstein:

This is equivalent to the following statement: For any $\Lambda$-module $N$, the $(n+2+m)$-syzygy $\Lambda$-module $\Omega^i_{\Lambda}(N)$ is Gorenstein projective. Clearly, $\Omega^i_{\Lambda}(N) \simeq \text{Hom}_{\Lambda}(M,Y)$ for some $\Lambda$-module $Y$, and $\Omega^i_{\Lambda}(\text{Hom}_{\Lambda}(M,Y)) = \text{Hom}_A(M, \Omega^i_{\Lambda}(Y))$. So $\Omega^i_{\Lambda}(n+2+m)(N) \simeq \text{Hom}_{\Lambda}(M, \Omega^i_{\Lambda}(n+2+m)(Y))$ as $\Lambda$-modules.

Thus $\Lambda$ is at most $(n+2+m)$-Gorenstein if and only if $\text{Hom}_{\Lambda}(M, \Omega^i_{\Lambda}(n+2+m)(Z)) \simeq \Lambda$-gp for any $\Lambda$-module $Z$. Equivalently, $\Omega^i_{\Lambda}(n+2+m)(Z) \in \mathcal{S}(\Lambda)$ by Lemma \[5.3\] and Corollary \[2.2\].

Since $\Omega^i_{\Lambda}(Z) \in \mathcal{M}^{-n}$ by Lemma \[5.3\], the algebra $\Lambda$ is at most $(n+2+m)$-Gorenstein if and only if $\Omega^i_{\Lambda}(X) \in \mathcal{M}$ for any $\Lambda$-module $X \in \mathcal{M}^{-n}$. It is this condition that we are going to verify. It holds for the case $m=0$ by Theorem \[3.1\] and Corollary \[3.6\], thus (1) and (2) are equivalent when $m=0$.

In the following, we assume that $m \geq 1$. Since $A$ is an $m$-rigid generator-cogenerator, by Lemma \[5.2\] there exists, for given $X$, an exact sequence of $A$-modules

$$0 \rightarrow K_X \rightarrow E_X \rightarrow X \rightarrow 0$$

such that $K_X \in \mathcal{M}^{m-1}$ and $E_X := \Omega^i_{\Lambda} \Omega^m_{\Lambda}(X) \oplus M_X \in \mathcal{M}^n$, where $M_X \in \text{add}(A)$. Since $M$ is $n$-rigid and $1 \leq m \leq n$, $K_X \in \mathcal{M}^{n-m+1}$ implies $E_X \in \mathcal{M}^{n-m+1}$. Since $\Omega^i_{\Lambda}(X) \in \mathcal{M}^m$, there are isomorphisms $\Omega^i_{\Lambda} \Omega^m_{\Lambda}(M_X) \simeq \Omega^i_{\Lambda}(M_X)$ as $A$-modules.

Consequently, $\Omega^i_{\Lambda}(X) \in \mathcal{M}$ if and only if $\Omega^i_{\Lambda}(E_X) \in \mathcal{M}$.

Claim. $\Omega^i_{\Lambda}(M)$ is $\mathcal{S}(\Lambda)$ and only if $E_X \in \mathcal{M}^n$.

In fact, by Lemma \[5.3\], there is an equivalence $\mathcal{M}^n \rightarrow \mathcal{M}^n$ induced by the adjoint pair $(\Omega^m_{\Lambda} \circ \Omega^i_{\Lambda}, \Omega^m_{\Lambda})$. Moreover, $\mathcal{S}(\Lambda)$ is a Frobenius category closed under taking $\Omega^i_{\Lambda}$ and $\Omega^m_{\Lambda}$ by Lemma \[5.3\].

Now, the claim follows from $E_X \in \mathcal{M}^n$.

Hence $\Lambda$ is at most $(n+2+m)$-Gorenstein if and only if $E_X \in \mathcal{M}$ for each $X \in \mathcal{M}^n$.

Suppose that (1) holds. Then $\Lambda$ is $(m,n)$-orthosymmetric such that $\mathcal{M}^n \simeq \mathcal{M}^{-n}$ by the proof of Theorem \[3.1\]. Moreover, $E_X \in \mathcal{M}$ is $\mathcal{M}^{-n}$ for all $n$. So (2) holds. Since $K_X \in \mathcal{M}^{n-1}$ and $X, E_X \in \mathcal{M}^{n-1}$, the sequence $0 \rightarrow K_X \rightarrow E_X \rightarrow X \rightarrow 0$ has all terms in $\mathcal{M}^{-n}$. This means that $(\mathcal{M}, \mathcal{M}^{n-1} \in \mathcal{M}^{n-1})$ is a left cotorsion pair in $\mathcal{M}^{n-1}$. Thus (4) also holds.

Suppose that (2) holds. Since $E_X \in \mathcal{M}^n \rightarrow \mathcal{M}^{n-1}$ by construction, the conditions in (2) imply that $E_X \in \mathcal{M}$.

Taking $m=1$ in Proposition \[3.1\] gives the following result.

Corollary 3.16. The algebra $\Lambda$ is at most $(n+3)$-Gorenstein if and only if $\mathcal{M}^n \simeq \mathcal{M}^{-n}$ in $\mathcal{M}$. This is also equivalent to $\Lambda$ being $(n,1)$-orthosymmetric when $n \geq 2$.

Another consequence is an upper bound for global dimension in terms of ortho-symmetry.

Corollary 3.17. The following statements are equivalent:

(1) $\text{gldim}(\text{End}_{\Lambda}(M)) \leq n+3$.
(2) $\mathcal{M}^{n+1} = \text{add}(M)$.
(3) $\mathcal{M}^{n} = \text{add}(M)$.
(4) $\mathcal{M}^{n} = \mathcal{M}^{n}$ for any $1 \leq p, q$ with $p+q = n+1$.

Proof. A Gorenstein algebra $\Lambda$ has finite global dimension if and only if $\Lambda$-gp $= \text{add}(\Lambda)$, and the equivalence of (1) and (3) follows from Corollary \[3.1\] and Lemma \[3.1\].

Suppose $n \geq 2$. Let $\mathcal{M}^n := \mathcal{M}^{n}$ for any $1 \leq p, q \leq n$. By Lemma \[3.2\], there is a series of equivalences of additive categories:

$$\mathcal{M}^n / [M] \xrightarrow{\Omega^m_{\Lambda}} \mathcal{M}^{n-1} / [M] \xrightarrow{\Omega^m_{\Lambda}} \cdots \xrightarrow{\Omega^m_{\Lambda}} \mathcal{M}^q / [M] \xrightarrow{\Omega^m_{\Lambda}} \mathcal{M}^{q+1} / [M].$$
It follows that (2), (3) and (4) are equivalent. □

Before focusing on the case \( m = 0 \), we recall the definition of maximal \( n \)-orthogonal modules. Following [21, Definition 2.2], \( _{\Lambda}M \) is maximal \( n \)-orthogonal if \( M^{\perp n} = \text{add}(\Lambda M) = \langle \Lambda M \rangle \). Note that \( _{\Lambda}M \) is maximal \( n \)-orthogonal if and only if \( \Lambda \) has global dimension exactly \( n + 2 \). For a proof, see, for example, [21, Proposition 2.2.2].

Replacing global dimension by Gorenstein global dimension, we obtain the following result, which corresponds to the case \( m = 0 \). This result was obtained in [24] by use of relative cotilting theory; another proof can be found in [17]. Here, we combine some results in this Section to provide a simple proof.

**Corollary 3.18.** The following statements are equivalent:

1. \( \text{add}(M) = \text{add}(M^+) \).
2. \( \text{add}(M) = \text{add}(M^+) \).
3. \( _{\Lambda}M \) is \( n \)-ortho-symmetric.
4. \( \text{End}_{\Lambda}(M) \) is \((n + 2)\)-Gorenstein.

**Proof.** Since \( _{\Lambda}M \) is an \( n \)-rigid generator-cogenerator, \( A \oplus D(A) \in \text{add}(M) \) and \( M \in \langle n \rangle A \cap D(A) \langle n \rangle \). The equivalence of (1) and (2) follows from the equivalence of additive categories

\[
\langle n \rangle A /[\Lambda] \xrightarrow{\cong} D(A) \langle n \rangle /[D(A)]
\]

induced by \( \tau_{n+1} \) and \( \tau_{n+1}^- \). Proposition 3.11 (2) and (3) implies that (4) is equivalent to (1) plus (2). Thus (1), (2) and (4) are equivalent. Clearly, either (1) or (2) implies (3) by Lemma 3.7. It remains to show that (3) implies (4).

In fact, by Proposition 3.11 if \( \Lambda \) is \( m \)-Gorenstein, then \( m \geq n + 2 \). So \( \Lambda \) is exactly \((n + 2)\)-Gorenstein whenever it is at most \((n + 2)\)-Gorenstein. The latter is equivalent to (3) by Proposition 3.15 and Corollary 3.6. Thus (3) implies (4). □

At this point, the definition of \( n \)-ortho-symmetric modules can be extended:

**Definition 3.19.** The \( A \)-module \( M \) is \( n \)-ortho-symmetric if any one of the equivalent conditions (1) – (4) in Corollary 3.18 is satisfied.

If, in addition, any indecomposable \( A \)-module \( X \) is isomorphic to a direct summand of \( M \) whenever \( M \oplus X \) is \( n \)-ortho-symmetric, then \( M \) is called maximal \( n \)-ortho-symmetric.

In the forthcoming article [24], \( n \)-ortho-symmetric modules are called \((n + 1)\)-precluster tilting modules, regarding them as a generalisation of \((n + 1)\)-cluster tilting modules which are exactly the maximal \( n \)-orthogonal modules in [21]. We prefer the more general term \( \text{ortho-symmetric} \) that indicates the left-right-orthogonal equality in Definition 3.11.

**Remarks on Corollary 3.18**

1. When \( n = 0 \), the equivalence among (1), (2) and (4) in Corollary 3.18 still holds. This special case was first studied in [4] and further explored in [18, 21, 24]. To unify this case, we say that \( M \) is \( 0 \)-ortho-symmetric.
2. By Corollary 3.18 and [21, Proposition 2.2.2], the module \( _{\Lambda}M \) is maximal \( n \)-orthogonal if and only if it is \( n \)-ortho-symmetric and \( \text{End}_{\Lambda}(M) \) has finite global dimension. If an \( n \)-orthogonal \( A \)-module is maximal \( n \)-rigid, then it is maximal \( n \)-ortho-symmetric. But the converse of this statement is not true in general (see Section 5 for counterexamples).

A consequence of Corollary 3.18 is the following practical criterion.

**Corollary 3.20.** If \( \text{add}(M^-) = \text{add}(M^+) \), then \( M \oplus M^- \) is \( n \)-ortho-symmetric.

**Proof.** Let \( V := M \oplus M^- \). Since \( _{\Lambda}M \) is a generator-cogenerator, so is \( V \). Suppose \( \text{add}(M^-) = \text{add}(M^+) \). By Proposition 3.11 (1), the module \( V \) is \( n \)-rigid. Note that

\[
\text{add}(V^+) = \text{add}(\tau_{n+1}(V) \oplus D(A)) = \text{add}(\tau_{n+1}(M) \oplus \tau_{n+1}(M) \oplus D(A)) = \text{add}(\tau_{n+1}(M) \oplus M) = \text{add}(M \oplus M^+) = \text{add}(M \oplus M^-) = \text{add}(V).
\]

Thus \( V \) is \( n \)-ortho-symmetric by Corollary 3.18. □

**Corollary 3.21.** Suppose that \( n \geq 2 \). Then the following statements are equivalent:

1. \( \langle n \rangle M \cap M^{\perp n} = \langle 1 \rangle M \cap M^{\perp n} \subseteq M^{\perp n} \).
2. \( M \)-coresdim\((M^-) = 1 \).
3. \( M \)-resdim\((M^+) = 1 \).
4. \( \text{End}_{\Lambda}(M) \) is \((n + 3)\)-Gorenstein.
Proof. Equivalence of (2), (3) and (4): The equivalence of (2) and (3) follows from Lemma 3.13(3) and Corollary 3.12. Moreover, by Proposition 3.11 (4) is equivalent to (2) combined with (3).

Equivalence of (1) and (4): If $\Lambda M \cap M^{1-n} = \Lambda M \cap M^{-n}$, then $\mathscr{G}(\Lambda M) = \Lambda M \cap M^{-n}$ by Corollary 3.6 since $n \geq 2$. Assume moreover $\Lambda M \cap M^{1-n} = M^{1-n}$. Then $\mathscr{G}(\Lambda M) = M^{1-n} = \Lambda M$, again by Corollary 3.6 and thus $\Lambda$ is $(n-2)$-Gorenstein by Corollary 3.18. Theorem B yields a contradiction to (4), which thus implies (1).

Conversely, suppose that (1) holds. By Corollary 3.16, the algebra $\Lambda$ is at most $(n+3)$-Gorenstein. By (1), $\Lambda M \neq M^{1-n}$. Thus $\Lambda$ is not $(n+2)$-Gorenstein by Corollary 3.18, this implies (4). □

3.9. Gorenstein Symmetry Conjecture. In this subsection, we shall introduce a class of endomorphism algebras of generator-cogenerators satisfying the Gorenstein Symmetry Conjecture. The following result plays a crucial role.

Proposition 3.22. Let $V$ be a basic $n$-rigid $A$-module with $n \geq 0$. Suppose that $V = M \oplus X$ such that $M$ is $n$-ortho-symmetric and $X$ is indecomposable with $X \not\in \text{add}(M)$ and $X \not\cong \tau_{n+1}(X)$. Let $m$ be a positive integer and let $B := \text{End}_A(V)$. Then the following statements are equivalent:

1. $\text{injdim}(\text{add}(B)) = n+2+m$.
2. $\text{injdim}(B) = n+2+m$.
3. There is a long exact sequence of $A$-modules
   $$0 \to \tau_{n+1}(X) \to M_0 \to M_1 \to \cdots \to M_{m-1} \to X \to 0$$
   with $M_i \in \text{add}(M)$ for $0 \leq i \leq m-1$, which induces the exact sequence of $B^+$-modules
   $$0 \to \text{Hom}_A(X, V) \to \text{Hom}_A(M_{m-1}, V) \to \cdots \to \text{Hom}_A(M_0, V) \to \text{Hom}_A(\tau_{n+1}(X), V) \to 0.$$
4. There is a long exact sequence of $A$-modules
   $$0 \to X \to M_0' \to M_1' \to \cdots \to M_{m-1}' \to \tau_{n+1}(X) \to 0$$
   with $M_i' \in \text{add}(M)$ for $0 \leq i \leq m-1$, which induces the exact sequence of $B$-modules
   $$0 \to \text{Hom}_A(V, X) \to \text{Hom}_A(V, M_0') \to \cdots \to \text{Hom}_A(V, M_{m-1}') \to \text{Hom}_A(V, \tau_{n+1}(X)) \to 0.$$

Proof. Since Morita equivalences of algebras preserve injective dimensions and exact sequences of modules, we may assume $A$ to be a basic algebra. As $\Lambda M$ is basic and $n$-ortho-symmetric, $M^- \simeq M \simeq M^+$ by Corollary 3.18. Hence, $V^- \simeq M \oplus \tau_{n+1}(X)$ and $V^+ \simeq M \oplus \tau_{n+1}(X)$. Since $X \not\in \text{add}(M)$ and $X \not\cong \tau_{n+1}(X)$,
   $$\text{add}(V) \cap \text{add}(V^-) = \text{add}(M) = \text{add}(V) \cap \text{add}(V^+).$$
This implies $V^- \text{-coresdim}(V^-) = V^- \text{-coresdim}(\tau_{n+1}(X))$ and $V^- \text{-resdim}(V^+) = V^- \text{-resdim}(\tau_{n+1}(X))$. If (3) holds, then $V^- \text{-coresdim}(\tau_{n+1}(X)) = m$, which shows (1) by Proposition 3.11(2). Dually, (4) implies (2).

(1) implies both (2) and (3).

Suppose that (1) holds. Then $V^- \text{-coresdim}(\tau_{n+1}(X)) = m$ by Proposition 3.11(2). Let $N := \text{Hom}_A(V^-, V)$. Then $N^+ \simeq \text{Hom}_A(M(V) \oplus \text{Hom}_A(\tau_{n+1}(X), V))$. Since $X$ is indecomposable with $X \not\in \text{add}(M)$ and $X \not\cong \tau_{n+1}(X)$, the $B^+$-module $\text{Hom}_A(\tau_{n+1}(X), V)$ is indecomposable and not projective. Moreover, the proof of Lemma 3.13(1) shows that $N^+$ is partial $m$-tilting. Since $B^+ = \text{Hom}_A(M(V) \oplus \text{Hom}_A(\tau_{n+1}(X), V))$, Lemma 2.3 forces $N^+$ to be $m$-tilting. Now, (2) follows from Lemma 3.13(1), while (3) is a consequence of Lemma 2.3 and Lemma 2.4(2). Dually, (2) implies both (1) and (4). □

A slight generalisation of ortho-symmetric modules is the following definition.

Definition 3.23. Let $A V$ be an $n$-rigid generator-cogenerator with $n \geq 0$. Then $V$ is called almost $n$-ortho-symmetric if $V = M \oplus X$ such that $M$ is $n$-ortho-symmetric and $X$ is indecomposable.

Here, $X$ may be chosen to be zero; thus, ortho-symmetric modules are almost ortho-symmetric. When $A$ is self-injective, it is understood that $A V$ is $n$-ortho-symmetric for any $n \geq 0$. In this case, $\text{End}_A(A \oplus X)$ is almost $n$-ortho-symmetric for any indecomposable $A$-module $X$.

The endomorphism rings of almost ortho-symmetric modules satisfy the Gorenstein Symmetry Conjecture. This is a combinatorial consequence of Corollary 3.18 and Proposition 3.22.

Corollary 3.24. Let $V$ be an almost $n$-ortho-symmetric $A$-module with $B := \text{End}_A(V)$. Then:
   $$\text{injdim}(B) < \infty \text{ if and only if } \text{injdim}(B) < \infty.$$ 

So, $B$ satisfies the Gorenstein Symmetry Conjecture.

When restricting to self-injective algebras, the following result is a more precise expression of Corollary 3.24.
Corollary 3.25. Let $A$ be a self-injective algebra and $X$ an indecomposable and non-projective $A$-module. Set $B := \text{End}_A(A \oplus X)$. Then the following statements are equivalent:

1. $\text{injdim}(\mu B) < \infty$.
2. $\text{injdim}(B) < \infty$.
3. The $A$-module $A \oplus X$ is $(\text{injdim}(\mu B) - 2)$-ortho-symmetric.

In particular, $\text{gldim}(B) = 1 < \infty$ if and only if the $A$-module $A \oplus X$ is maximal $(1 - 2)$-orthogonal for some natural number $l \geq 2$.

Proof. The equivalence of (1) and (2) follows from Corollary 3.24. Clearly, (3) implies (1) by Corollary 3.18. It remains to show that (1) implies (3).

Suppose $s := \text{injdim}(\mu B) < \infty$. Then $2 \leq \text{domdim}(B) \leq s$. Let $n$ be a non-negative integer such that $\text{domdim}(B) = n + 2$. Then $\tau X$ is $n$-rigid by [26] Lemma 3. Set $V := A \oplus X$ and $m := V - \text{coresdim}(\tau_{n+1}(X))$. Then $s = n + 2 + m$ by Proposition 3.11(2). Note that $m = 0$ if and only if $X \simeq \tau_{n+1}(X)$. In this case, $\text{injdim}(\mu B) = n + 2$ and $A \oplus X$ is $n$-ortho-symmetric by Corollary 3.18.

Claim: $m = 0$.

Assume $m \geq 1$. Then $\tau_{n+1}(X) \not\cong X$. Since $A$ is self-injective, $\A A$ is $n$-ortho-symmetric. By Proposition 3.22 there is a long exact sequence of $A$-modules

$$0 \to \tau_{n+1}(X) \to P_0 \to P_1 \to \cdots \to P_{m-1} \to X \to 0$$

with $P_i \in \text{add}(\A A)$ for $0 \leq i \leq m - 1$, inducing a minimal projective resolution of $\text{Hom}_A(\tau_{n+1}(X), V)_B$

$$0 \to \text{Hom}_A(X, V) \to \text{Hom}_A(P_m-1, V) \to \cdots \to \text{Hom}_A(P_0, V) \to \text{Hom}_A(\tau_{n+1}(X), V) \to 0.$$

Applying the duality $D$ to this resolution yields the minimal injective coresolution of the $B$-module $D\text{Hom}_A(\tau_{n+1}(X), V)$

$$0 \to D\text{Hom}_A(\tau_{n+1}(X), V) \to D\text{Hom}_A(P_0, V) \to \cdots \to D\text{Hom}_A(P_{m-1}, V) \to D\text{Hom}_A(X, V) \to 0.$$

Since $D\text{Hom}_A(P, V) \simeq \text{Hom}_A(V, \nu_A(P))$ for any projective $A$-module $P$, there exists a minimal injective coresolution of the following form:

$$0 \to D\text{Hom}_A(\tau_{n+1}(X), V) \to \text{Hom}_A(V, \nu_A(P_0)) \to \cdots \to \text{Hom}_A(V, \nu_A(P_{m-1})) \to D\text{Hom}_A(X, V) \to 0$$

where $\nu_A$ is the Nakayama functor of $A$. Choose a minimal injective coresolution of $\A V$:

$$0 \to \A V \to I_0 \to \cdots \to I_n \to I_{n+1} \to \cdots$$

Since $\A V$ is $n$-rigid, the proof of Proposition 3.9 provides us with the long exact sequence of $B$-modules

$$0 \to \mu B \to \text{Hom}_A(V, I_0) \to \cdots \to \text{Hom}_A(V, I_{n+1}) \to D\text{Hom}_A(\tau_{n+1}(X), V) \to 0.$$

Both $\text{Hom}_A(V, I_j)$ for $0 \leq j \leq n + 1$ and $\text{Hom}_A(V, \nu_A(P_i))$ for $0 \leq i \leq m - 1$ are projective-injective. It follows that $\text{domdim}(B) = n + 2 + m \geq n + 2$, a contradiction. This shows $m = 0$, and thus (3) holds.

The last assertion in Corollary 3.25 is due to the fact that an ortho-symmetric module is maximal orthogonal if and only if its endomorphism algebra has finite global dimension. □

4. ORTHO-SYMMETRIC MODULES AND DERIVED EQUVALENCES

4.1. Introduction. Theorem A states the derived equivalence of the endomorphism rings of two ortho-symmetric modules, provided both are maximal ortho-symmetric, or one of them is so, and the other one has the same number of non-isomorphic indecomposable summands. In this Section, we will prove Theorem A in the stronger form of Theorem 4.3. In order to construct tilting modules providing these derived equivalences, we use again approximation sequences, which at the same time allow us to construct mutations of ortho-symmetric modules.

Lemmas 4.1 and 4.2 provide the basic tools, exhibiting tilting modules and showing that certain modules are rigid or even ortho-symmetric. Then Theorem 4.3 and several related results in the form of corollaries can be proved. Finally, Proposition 4.8 discusses a symmetry property, connecting left and right mutations.

4.2. Approximations, tilting modules and vanishing of cohomology. Throughout this section, let $A$ be an algebra, $M$ an $A$-module and $n$ a positive integer.

First of all, we shall construct (partial) 1-tilting modules over endomorphism algebras from ortho-symmetric modules, by taking right or left approximations of modules.
Lemma 4.1. Let 0 → K → M₀ → X → 0 be an exact sequence of A-modules such that g is a right add(M)-approximation of X with M₀ ∈ add(M). Set V := K ⊕ M and Λ := Endₜ(V).

(1) If the A-module X is 1-rigid, then the Λ-module Homₜ(A,V,X) is a partial 1-tilting Λ-module such that Endₜ(Homₜ(A,V,X)) ≃ Endₜ(X) as algebras.

(2) Suppose that M is n-ortho-symmetric, X is n-rigid and X ∈ M⁻¹(n⁻¹). Then:

(a) The module A/X is n-rigid.

(b) If add(X ⊕ M) = add(τₜ(A/X) ⊕ M), then A/V is n-ortho-symmetric.

Proof. (1) Suppose that X is 1-rigid.

Claim. Homₜ(A,K,g) : Homₜ(A,K,M₀) → Homₜ(A,K,X) is surjective.

In fact, since Extₜ¹(A,K,X) = 0, applying Homₜ(−,X) to the sequence 0 → K → M₀ → X → 0, returns an exact sequence 0 → Homₜ(A,K,X) → Homₜ(A,M₀,X) → Homₜ(A,K,X) → 0.

This implies that any homomorphism from K to X factorises through M₀. Because M₀ ∈ add(M) and g is a right add(M)-approximation of X, any homomorphism from K to X can be written as a composition of a homomorphism K → M₀ with g. In other words, the map Homₜ(A,K,g) : Homₜ(A,K,M₀) → Homₜ(A,K,X) is surjective. Using again that the map g is a right add(M)-approximation of X, the map g* := Homₜ(A,V,g) : Homₜ(A,V,M₀) → Homₜ(A,V,X)

is surjective. Since K,M₀ ∈ add(V), the exact sequence of A-modules 0 → Homₜ(A,V,K) → Homₜ(A,V,M₀) → Homₜ(A,V,X) → 0 is a projective resolution of L := Homₜ(A,V,X). Applying Homₜ(A,−) to this resolution, produces the following exact commutative diagram:

0 → Homₜ(A,X,X) → Homₜ(A,M₀,X) → Homₜ(A,K,X) → Extₜ¹(A,X,X) → 0

| → | → | → |

0 → Homₜ(A,L,L) → Homₜ(A,M₀,L) → Homₜ(A,K,L) → Extₜ¹(A,L,L) → 0

Thus Endₜ(A,X) ≃ Endₜ(A) and Extₜ¹(A,L,L) = 0.

(2) Since A/M is at least 1-rigid and g is a right add(M)-approximation of X, Extₜ¹(A,M,K) vanishes. Since M is n-rigid and X ∈ M⁻¹(n⁻¹), we obtain K ∈ M⁻¹n. As M is n-ortho-symmetric, 1/nM = M⁻¹, and hence K ∈ 1/nM.

To show that A/V is n-rigid, it suffices to show that K ∈ 1/nM.

Claim. K is 1-rigid.

In fact, applying Homₜ(A,K,−) to the sequence 0 → K → M₀ → X → 0, yields the exact sequence 0 → Homₜ(A,K,K) → Homₜ(A,K,M₀) → Homₜ(A,K,X) → Extₜ¹(A,K,K) → Extₜ¹(A,K,M₀).

By assumption, the module A/X is at least 1-rigid. By (1), the map Homₜ(A,g) is surjective. Since Extₜ¹(A,M₀) = 0 and M₀ ∈ add(M), we obtain Extₜ¹(A,K,K) = 0.

Claim. Extₜ¹(A,K,K) = 0 for 2 ≤ i ≤ n.

Proof. K ∈ 1/nM and M₀ ∈ add(M) implies Extₜ⁻¹(A,K,K) ∼ Extₜ(A,K,K) for all 2 ≤ i ≤ n. Moreover, since X ∈ M⁻¹(n⁻¹) and M₀ ∈ add(M), there are isomorphisms Extₜ⁻¹(A,K,K) ∼ Extₜ(A,X,X) for 2 ≤ i ≤ n and there is an injection from Extₜ⁻¹(A,K,K) into Extₜ(A,X,X).

Thus Extₜ¹(A,K,K) ∼ Extₜ¹(A,X,X) for all 2 ≤ j ≤ n − 1 and Extₜ¹(A,K,K) ∼ Extₜ¹(A,X,X). As A/X is n-rigid by assumption, the module A/K is n-rigid, too. This finishes the proof of (a).

To show part (b), it suffices to check that add(A/V) = add(τₜ(A/X) ⊕ D(A)). From A/M being n-ortho-symmetric, it follows that A ∈ D(A)⁻¹/n and τₜ(A) ∈ add(M) by Corollary 3.18. Since g : M₀ → X is a right add(M)-approximation of X, the map Homₜ(A,τₜ(A),g) : Homₜ(A,τₜ(A),M₀) → Homₜ(A,τₜ(A),X)

is surjective. By Lemma 3.3 there exists a short exact sequence of A-modules:

0 → τₜ(A,K) → τₜ(A,M₀) → τₜ(A,X) → 0

where A is injective. Since K ∈ 1/nM ⊆ 1/nA, there are isomorphisms Extₜ¹(A,M,τₜ(A,K)) ∼ DExtₜ⁻¹(A,M,τₜ(A,K)) = 0 for each 1 ≤ i ≤ n by Lemma 3.7. This implies τₜ(A,K) ∈ M⁻¹/nM ⊆ M⁻¹. Since A/M is n-ortho-symmetric, add(M) = add(τₜ(M₀) ⊕ D(A)) by Corollary 3.18. Thus τₜ(M₀) ⊕ I ∈ add(M) and h is a right add(M)-approximation of τₜ(A,X). Recall that g is a right add(M)-approximation of X. The equality add(X ⊕ M)
add(\tau_{n+1}(X) \oplus M) implies add(K \oplus M) = add(\tau_{n+1}(K) \oplus M). Note that add(M) = add(\tau_{n+1}(M) \oplus D(A))$. Thus add(AV) = add(\tau_{n+1}(V) \oplus D(A)). Combining this with (1), AV is n-ortho-symmetric by Corollary 3.18.

The following result is dual.

**Lemma 4.2.** Let $0 \to X \xrightarrow{f} M_0 \to C \to 0$ be an exact sequence of $A$-modules such that $f$ is a left add($M$)-approximation of $X$ with $M_0 \subset add(M)$. Define $U := M \oplus C$ and $\Gamma := End_A(U)$.

1. If the $A$-module $X$ is 1-rigid, then the $\Gamma$-module $Hom_A(X, U)$ is a partial 1-tilting $\Gamma^\ast$-module such that $End_{\Gamma^\ast}(\text{Hom}_A(X, U)) \simeq End_A(X)^\ast$ as algebras.

2. Suppose that $M$ is $n$-ortho-symmetric, $X$ is $n$-rigid and $X \in \frac{1}{n-1}M$. Then:
   
   (a) The module $A\,U$ is $n$-rigid.
   
   (b) If add($X \oplus M$) = add(\tau_{n+1}(X) \oplus M), then $A\,U$ is $n$-ortho-symmetric.

Now, the main result on constructing tilting modules can be stated. This result is a stronger version of Theorem 4.3, which also generalises Iyama’s result [22 Corollary 5.3.3(1)] from maximal 1-orthogonal modules to maximal 1-ortho-symmetric modules.

**Theorem 4.3.** Let $M$ be a maximal $n$-ortho-symmetric $A$-module and let $N$ be an $n$-ortho-symmetric $A$-module. Suppose that Ext$_i^A(M, N) = 0$ for all $1 \leq i \leq n - 1$. Then:

1. The module $Hom_A(M, N)$ is a partial 1-tilting $End_A(M)$-module. In particular, #(A,M) \leq #(A,N). Here, equality holds if and only if $Hom_A(M, N)$ is a 1-tilting $End_A(M)$-$End_A(N)$-bimodule.

2. If $A\,N$ is maximal $n$-ortho-symmetric, then $Hom_A(M, N)$ is a 1-tilting $End_A(M)$-$End_A(N)$-bimodule. In this case, $End_A(M)$ and $End_A(N)$ are derived equivalent.

**Proof.** Let $g : M_0 \to N$ be a minimal right add($M$)-approximation of $N$, where $M_0 \subset add(M)$. Since $A\,M$ is a generator, the map $g$ is surjective. Let $K := \text{Ker}(g)$. Then there is an exact sequence of $A$-modules:

$$0 \to K \to M_0 \xrightarrow{g} N \to 0$$

The modules $A\,M$ and $A\,N$ are $n$-ortho-symmetric with $N \in M^{\perp_{n-1}}$. By Corollary 3.18(2), add($N \oplus M$) = add(\tau_{n+1}(N) \oplus M). It follows from Lemma 4.1(2) that $K \oplus M$ is $n$-ortho-symmetric. Since $A\,M$ is maximal $n$-ortho-symmetric, $K \in add(M)$ and add($K \oplus M$) = add($M$). Let $\Lambda := End_A(A\,M)$. Then $\Lambda$ is Morita equivalent to $End_A(K \oplus M)$. Now, the first part of (1) is a consequence of Lemma 4.1(1). Observe that $End_A(Hom_A(M, N)) \simeq End_A(N)$ as algebras and that

$$#(A\,N) = #(A\,Hom_A(M, N)) \leq #(A\,\Lambda) = #(A\,M).$$

A partial 1-tilting module $T$ over an algebra $B$ is 1-tilting if and only if $(\#gT) = #(gB)$ (see, for instance, [11 Corollary 2.6]). Thus $Hom_A(M, N)$ is a 1-tilting $A$-module if and only if $(\#(A\,Hom_A(M, N)) = #(A\,\Lambda)$; equivalently, $(#(A\,N) = #(A\,M)$.

Assume in addition that $A\,N$ is maximal $n$-ortho-symmetric. Let $f : M \to N_0$ be a minimal left add($N$)-approximation of $M$ with $N_0 \subset add(N)$. Since $A\,N$ is a cogenerator, the map $f$ is injective. By assumption, $M$ is $n$-ortho-symmetric with $M \in \frac{1}{n-1}N$. Corollary 3.18(1) and Lemma 4.1(2)(b) imply that $N \oplus \text{Coker}(f)$ is $n$-ortho-symmetric. Since $A\,N$ is maximal $n$-ortho-symmetric, $\text{Coker}(f) \subset add(N)$. Note that $A\,M$ is at least 1-rigid. Applying $Hom_A(M^{-\ast})$ to the exact sequence

$$0 \to M \to N_0 \xrightarrow{g} \text{Coker}(f) \to 0,$$

returns another exact sequence of $A$-modules:

$$0 \to Hom_A(M, M) \to Hom_A(M, N_0) \to Hom_A(M, \text{Coker}(f)) \to 0.$$

It follows that $Hom_A(M, N)$ is a 1-tilting $A$-module, and therefore $End_A(M)$ and $End_A(N)$ are derived equivalent by [19 Theorem 2.1]. This shows (2). □

**Corollary 4.4.** Let $A$ be an algebra. Suppose that there exists a maximal 1-ortho-orthogonal $A$-module. Then each maximal 1-ortho-orthogonal $A$-module is maximal 1-ortho-orthogonal.

**Proof.** By Corollary 3.18 and [21 Proposition 2.2.2], a 1-ortho-orthogonal $A$-module $M$ is maximal 1-ortho-orthogonal if and only if $End_A(M)$ has finite global dimension. Derived equivalence preserves finiteness of global dimension of algebras (see, for example, [28 Lemma 2.1]). Now, Corollary 4.4 follows from Theorem 4.3. □
4.3. Mutations of ortho-symmetric modules. Next, we introduce mutations of modules from the point of view of approximations. Motivation comes from mutations of rigid modules over preprojective algebras of Dynkin type (see [13]) and mutations of modifying modules over normal, singular $d$-Calabi-Yau rings (see [23, 25]). Mutations here will be used to construct new ortho-symmetric modules from given ones, and further to establish derived equivalences between their endomorphism algebras.

Let $M$ be a basic $A$-module such that $\lambda M = N \oplus X$. Furthermore, let $f : X \to N^0$ and $g : N_0 \to X$ be minimal left and right add$(N)$-approximations of $X$, respectively, where $N^0, N_0 \in$ add$(N)$. Consider the following two exact sequences of $A$-modules:

$$X \xrightarrow{f} N^0 \to \text{Coker}(f) \to 0 \quad \text{and} \quad 0 \to \text{Ker}(g) \to N_0 \xrightarrow{g} X.$$ 

When $N$ is a cogenerator, then $f$ is injective. Dually, when $N$ is a generator, then $g$ is surjective.

**Definition 4.5.** Using the notations just introduced, set

$$\mu_X^-(M) := N \oplus \text{Coker}(f) \quad \text{and} \quad \mu_X^+(M) := \text{Ker}(g) \oplus N,$$

and call them the left mutation and right mutation of $M$ at $X$, respectively.

Applying Lemma 4.1 to mutations of ortho-symmetric modules leads to the following result.

**Corollary 4.6.** Let $M$ be a basic $n$-rigid $A$-module. Suppose that $\lambda M = N \oplus X$ such that $\lambda N$ is $n$-ortho-symmetric and $\tau_{n+1}(X) \simeq X$ as $A$-modules. Let $g : N_0 \to X$ be a minimal right add$(N)$-approximation of $X$. Then:

1. The module $\text{Hom}_A(\mu_X^+(M), M)$ is a 1-tilting $\text{End}_A(\mu_X^+(M))$-$\text{End}_A(M)$-bimodule. Thus $\text{End}_A(M)$ and $\text{End}_A(\mu_X^+(M))$ are derived equivalent.
2. The $A$-module $\mu_X^+(M)$ is $n$-ortho-symmetric.

**Proof.** (1) Since $\lambda M$ is at least 1-rigid, $\text{Ext}_A^1(M, M) = 0$. Hence, $\text{Ext}_A^1(X, N) = 0$ because of $X, N \in$ add$(M)$. Let $K := \text{Ker}(g)$ and let $f : K \to N_0$ be the inclusion. Then $f$ is a left add$(N)$-approximation of $K$. So the sequence

$$0 \to K \xrightarrow{f} N_0 \xrightarrow{g} X \to 0$$

is an add$(N)$-split sequence. Since $X \notin$ add$(N)$ and $g$ is minimal, $f$ is also minimal and $\#(X) = \#(K)$. Recall that $\mu_X^+(M) = K \oplus N$. By Lemma 4.1, the $\text{End}_A(\mu_X^+(M))$-module $\text{Hom}_A(\mu_X^+(M), M)$ is a partial 1-tilting module. It even is 1-tilting since $\#(X) = \#(K)$. Thus (1) holds.

The existence of a derived equivalence between $\text{End}_A(M)$ and $\text{End}_A(\mu_X^+(M))$ also follows directly from [15 Theorem 1.1].

(2) Observe that $\lambda N$ is $n$-ortho-symmetric, $X$ is $n$-rigid and $X \in M^{\perp_n} \subseteq N^{\perp(n-1)}$. Since $\tau_{n+1}(X) \simeq X$ as $A$-modules, Lemma 4.1(2)(b) implies that the $A$-module $\mu_X^+(M)$ is $n$-ortho-symmetric. □

The following result is dual.

**Corollary 4.7.** Let $M$ be a basic $n$-rigid $A$-module. Suppose that $\lambda M = N \oplus X$ such that $\lambda N$ is $n$-ortho-symmetric and $\tau_{n+1}(X) \simeq X$ as $A$-modules. Let $f : X \to N^0$ be a minimal left add$(N)$-approximation of $X$. Then:

1. The module $\text{Hom}_A(M, \mu_X^-(M))$ is a 1-tilting $\text{End}_A(M)$-$\text{End}_A(\mu_X^-(M))$-bimodule. Thus $\text{End}_A(M)$ and $\text{End}_A(\mu_X^-(M))$ are derived equivalent.
2. The $A$-module $\mu_X^-(M)$ is $n$-ortho-symmetric.

Under certain conditions, left and right mutations of 1-ortho-symmetric modules behave in a symmetric form, as illustrated by the following fact. This result generalises [13 Corollary 5.8], whose proof relies heavily on extension groups of modules in exchange sequences being one-dimensional.

**Proposition 4.8.** Let $\lambda M$ be basic and maximal 1-ortho-symmetric. Suppose that $M = N \oplus X$ where $X$ is indecomposable, neither projective nor injective and such that $\tau_1(X) \simeq X$. Then there exists an exact sequence of $A$-modules

$$0 \to X \xrightarrow{f} N_1 \to N_0 \xrightarrow{g} X \to 0$$

with $N_0, N_1 \in$ add$(N)$ such that the sequences

$$0 \to X \xrightarrow{f} N_1 \to K \to 0 \quad \text{and} \quad 0 \to K \to N_0 \xrightarrow{g} X \to 0$$

are minimal add$(N)$-split sequences, where $K := \text{Ker}(g)$. In particular,

$$\mu_X^-(M) \simeq K \oplus N \quad \text{and} \quad \mu_X^-(N \oplus K) \simeq X \oplus N.$$
Proof.  As $A M$ is a generator-cogenerator and $A X$ is indecomposable and neither projective nor injective, $A N$ is a generator-cogenerator. Let $g : N_0 \to X$ be a minimal right add$(N)$-approximation of $X$ and let $K := \text{Ker } (g)$. Then $\mu^\tau_1(K) = K \oplus N$. Since $M$ is 1-ortho-symmetric and $\tau_2(X) \simeq X$, the $A$-module $N$ is 1-ortho-symmetric by Corollary 3.18(2). Moreover, since $M$ is 1-rigid, the proof of Corollary 4.6(1) shows that the sequence

$$0 \to K \to N_0 \xrightarrow{\mu} X \to 0$$

is a minimal add$(N)$-split sequence. In particular, $\#(X) = \#(K)$. Therefore $K$ is indecomposable and does not belong to add$(N)$. Consequently, $K$ is neither projective nor injective. Moreover, by Corollary 3.18(2), the module $\mu^\tau_1(K)$ is 1-ortho-symmetric. Since $N$ is basic and 1-ortho-symmetric, Corollary 3.18(2) yields that $\tau_2(K) \simeq K$ as $A$-modules.

Let $\mu : M_0 \to K$ be a minimal right add$(M)$-approximation of $K$ with $Y := \text{Ker } (\mu)$. Then there exists an exact sequence of $A$-modules:

$$0 \to Y \xrightarrow{\lambda} M_0 \xrightarrow{\mu} K \to 0,$$

where $\lambda$ is the canonical inclusion. Since $\mu$ is minimal, the map $\lambda$ is a radical homomorphism, that is, it contains no identity map as a direct summand. The sequence $\delta$ corresponds to an element

$$\delta \in \text{Ext}^1_\mathcal{D}(K,Y) \simeq \text{Hom}_{\mathcal{D}^b(A)}(K,Y[1])$$

where $\mathcal{D}^b(A)$ denotes the bounded derived category of $A$-mod. In other words, there is a distinguished triangle in $\mathcal{D}^b(A)$:

$$Y \xrightarrow{\lambda} M_0 \xrightarrow{\mu} K \xrightarrow{\delta} Y[1].$$

Since $\text{Hom}_{\mathcal{D}^b(A)}(K,N[1]) \simeq \text{Ext}^1_\mathcal{D}(K,N') = 0$ and $\lambda$ is a radical map, add$(Y) \cap \text{add}(N) = 0$. Recall that $K$ is 1-rigid and $\tau_2(K) \simeq K$ as $A$-modules. Since $A M$ is 1-ortho-symmetric, the module $Y \oplus M$ is 1-ortho-symmetric by Lemma 4.1(2)(b). This forces $Y \in \text{add}(A)$ because $A M$ is maximal 1-ortho-symmetric. As $M = N \oplus X$ and $X$ is indecomposable, $Y \simeq X^m$ for some $m \in \mathbb{N}$. Let $\Lambda := \text{End}_A(M)$. The proof of Lemma 4.1(1) shows that $\text{Hom}_A(M,K)$ is a partial 1-tilting $A$-module with a minimal projective resolution:

$$0 \to \text{Hom}_A(M,Y) \xrightarrow{\lambda^*} \text{Hom}_A(M,M_0) \xrightarrow{\mu^*} \text{Hom}_A(M,K) \to 0.$$  

Since $\lambda^*$ is a radical map and $\text{Hom}_A(M,K)$ is 1-rigid, we obtain

$$\text{add}_A(\text{Hom}_A(M,Y)) \cap \text{add}_A(\text{Hom}_A(M,M_0)) = 0.$$  

This implies that add$(A Y) \cap \text{add}(M_0) = 0$. Since $Y \simeq X^m$ and $M_0 \in \text{add}(M)$, we get $M_0 \in \text{add}(N)$. Since $\text{Ext}^1_\mathcal{D}(K,N) = 0$, the map $\lambda$ is a left add$(N)$-approximation of $Y$. Thus $\delta$ is a minimal add$(N)$-approximation sequence. Hence $Y$ is indecomposable and $Y \simeq X$. Now, set $N_1 := M_0$ and $f := \lambda$. Let $h$ be the composition of $\mu$ with the inclusion $K \to N_0$. Then there is an exact sequence of $A$-modules

$$0 \to X \xrightarrow{f} N_1 \xrightarrow{h} N_0 \xrightarrow{g} X \to 0$$

which satisfies all properties required. □

5. ORTHO-SYMMETRIC MODULES OVER SELF-INJECTIVE ALGEBRAS

5.1. Introduction.  In this Section, we first provide methods to construct ortho-symmetric modules over self-injective algebras. A rich source of ortho-symmetric modules is from self-injective or symmetric algebras and in particular from weakly Calabi-Yau self-injective algebras. Over weakly $(n + 1)$-Calabi-Yau self-injective algebras, rigid $n$-generators coincide with $n$-ortho-symmetric modules (Lemma 5.2). Over self-injective algebras many ortho-symmetric modules can be constructed as sums of $\Omega$-shifts of given modules (Lemma 5.3 and Corollary 5.4). Another construction uses tensor products (Lemma 5.5). Next we turn to comparing maximal ortho-symmetric modules with modules satisfying similar conditions. As we have noted already in Subsection 5.3 maximal orthogonal modules are both maximal rigid and ortho-symmetric and these two properties together imply maximal ortho-symmetric. Both inclusions are proper, as will be shown by considering explicit examples, in the context of symmetric Nakayama algebras. Here, certain examples of ortho-symmetric modules can be classified (Proposition 5.7). Under additional assumptions, however, it can be shown that maximal 1-rigid implies maximal 1-orthogonal (Proposition 5.8 and Corollary 5.9).
5.2. Classes of examples related to self-injective algebras. Recall the definition of weakly Calabi-Yau triangulated categories:

**Definition 5.1.** Let \( \mathcal{T} \) be a \( k \)-linear Hom-finite triangulated category with shift functor \([1]\). Then \( \mathcal{T} \) is said to be weakly \( m \)-Calabi-Yau if there are natural \( k \)-linear isomorphisms

\[
\text{Hom}_{\mathcal{T}}(Y, X[m]) \cong \text{DHom}_{\mathcal{T}}(X, Y)
\]

for any \( X, Y \in \mathcal{T} \). The least such \( m \) is called its weak Calabi-Yau dimension.

An important class of weakly Calabi-Yau triangulated categories is provided by the stable module categories of self-injective algebras (see [5][2][20]).

When \( A \) is a self-injective \( k \)-algebra, then the stable module category \( A\text{-mod} \) of \( A \) is a \( k \)-linear Hom-finite triangulated category; its shift functor is the cosyzygy functor \( \Omega_A^{-1} \) (see, for example, [13] Section 2.6). If the category \( A\text{-mod} \) is weakly \( m \)-Calabi-Yau, then the algebra \( A \) also is called weakly \( m \)-Calabi-Yau.

The stable category \( A\text{-mod} \) has a Serre duality \( \Omega_A \nu_A \) by [12] Proposition 1.2. Thus, \( A \) is weakly \( m \)-Calabi-Yau if and only if \( \Omega_A^m \) and \( \Omega_A \nu_A \) are naturally isomorphic as auto-equivalences of \( A\text{-mod} \). Equivalently, \( \Omega_A^{m+1} \nu_A \) is naturally isomorphic to the identity functor of \( A\text{-mod} \). In particular, if \( A \) is symmetric, then it is weakly \( m \)-Calabi-Yau if and only if \( \Omega_A^{m+1} \nu_A \) is naturally isomorphic to the identity functor of \( A\text{-mod} \).

Rigid generators over Calabi-Yau self-injective algebras coincide with ortho-symmetric modules:

**Lemma 5.2.** Let \( A \) be a weakly \( (n+1) \)-Calabi-Yau self-injective algebra. Then, for any \( A \)-module \( M \), there is equality \( \nu_A^m M = M^{1-n} \). In particular, if \( \nu_A^m M \) is an \( n \)-rigid generator, then it is \( n \)-ortho-symmetric.

**Proof.** Since the algebra \( A \) is weakly \( (n+1) \)-Calabi-Yau, there are isomorphisms \( \text{Ext}_A^i(M, Y) \simeq D\text{Ext}_A^{i-1}(Y, M) \) for any \( A \)-module \( Y \) and for any \( 1 \leq i \leq n \), and \( \tau_{n+1}(M) = \Omega_A^{m+2} \nu_A(M) \simeq M \) in \( A\text{-mod} \).

Thus \( \nu_A^m M = M^{1-n} \). □

For arbitrary self-injective algebras, the following construction can be used:

**Lemma 5.3.** Suppose that \( A \) is self-injective and that \( M \) is a basic \( A \)-module without any projective direct summands. Let \( q \) be a positive integer such that \( \Omega_A^{(n+2)q} \nu_A^q(M) \simeq M \) as \( A \)-modules. Then the \( A \)-module \( A \oplus \bigoplus_{j=0}^{q-1} \Omega_A^{(n+2)q} \nu_A^q(M) \) is \( n \)-ortho-symmetric if and only if \( \text{Ext}_A^{i+2q}(\nu_A^q(M), M) = 0 \) for all \( 1 \leq i \leq n \) and \( 0 \leq t \leq q-1 \).

**Proof.** Since \( A \) is self-injective, we obtain \( \tau = \Omega_A^2 \nu_A \). This yields \( F := \tau^q = \Omega_A^{n+2} \nu_A \). By assumption, \( F^q(M) \simeq M \). Let \( N := \bigoplus_{j=0}^{q-1} F^j(M) \). Then \( F(N) \simeq N \) as \( A \)-modules. By Corollary 3.18 the module \( A \oplus N \) is \( n \)-ortho-symmetric if and only if \( \text{Ext}_A^i(F^j(M), M) = 0 \) for all \( 1 \leq s \leq n \) and \( 0 \leq t \leq q-1 \). Furthermore, \( \text{Ext}_A^i(F^j(M), M) \simeq \text{Ext}_A^{i+2q}(\nu_A^q(M), M) \). □

**Lemma 5.4.** Suppose that \( A \) is a symmetric algebra and that \( M \) is a basic \( A \)-module without projective direct summands. Then:

1. The \( A \)-module \( A \oplus M \) is \( n \)-ortho-symmetric if and only if \( M \) is \( n \)-rigid and \( \Omega_A^{m+2}(M) \simeq M \).
2. Let \( m \) be a positive integer such that \( (m+2)q = n+2 \) for some integer \( q \). If \( M \) is \( n \)-ortho-symmetric, then the \( A \)-module \( A \oplus \bigoplus_{j=0}^{q-1} \Omega_A^{(m+2)q} \nu_A^q(M) \) is \( m \)-ortho-symmetric.

**Proof.** Since \( A \) is symmetric, it is self-injective and \( \nu_A \) is the identity functor. Moreover, \( \tau = \Omega_A^2 \) and \( \tau^- = \Omega_A^{-2} \). This implies \( \tau_{n+1} = \Omega_A^{m+2} \) and \( \tau_{n+1}^- = \Omega_A^{-(n+2)} \). Now, Corollary 5.4) follows from Corollary 3.18 Statement (2) is a consequence of Lemma 5.3 □

Ortho-symmetric modules also can be constructed by forming tensor products:

**Lemma 5.5.** Let \( A \) be an \( n \)-ortho-symmetric \( A \)-module and let \( B \) be a self-injective algebra. Then \( M \otimes_k B \) is an \( n \)-ortho-symmetric \( A \otimes_k B \)-module.

**Proof.** If \( A \) is projective, then \( A \) is self-injective. In this case, the algebra \( A \otimes_k B \) is self-injective and the \( A \otimes_k B \)-module \( M \otimes_k B \) is a projective generator, which is \( n \)-ortho-symmetric. So we now assume \( A \) not to be projective. Since \( A \) is a generator-cogenerator, \( M \otimes_k B \) as an \( A \otimes_k B \)-module is a non-projective generator-cogenerator. Let \( N := M \otimes_k B \) and \( \Gamma := \text{End}_{A \otimes_k B}(N) \). By Corollary 5.18 to show that \( N \) as an \( A \otimes_k B \)-module is \( n \)-ortho-symmetric, it is sufficient to prove that \( N \) is \( n \)-rigid and \( \Gamma \) is \((n+2)\)-Gorenstein.
If \( X_i \in A\text{-mod} \) and \( Y_i \in B\text{-mod} \) for \( i = 1, 2 \), then
\[
\text{Hom}_{A \otimes k}(X_1 \otimes_k Y_1, X_2 \otimes_k Y_2) \cong \text{Hom}_A(X_1, X_2) \otimes_k \text{Hom}_B(Y_1, Y_2).
\]
In particular, \( \Gamma \cong \text{End}_A(M) \otimes_k \text{End}_B(B) \cong \text{End}_A(M) \otimes_k B \) as algebras. Since \( _AM \) is \( n \)-ortho-symmetric, Corollary 5.13(4) implies that \( \text{End}_A(M) \) is \((n + 2)\)-Gorenstein. By assumption, the algebra \( B \) is self-injective. Consequently, the algebra \( \Gamma \) is \((n + 2)\)-Gorenstein. Moreover, since \( B \) is self-injective, it can be checked that, for each \( i \geq 1 \),
\[
\text{Ext}_{gA \otimes k}(N, N) \simeq \text{Ext}_{gA}(M, M) \otimes_k \text{End}_B(B) \cong \text{Ext}_{gA}(M, M) \otimes_k B.
\]
Since \( _AM \) is \( n \)-rigid, the \( A \otimes k \)-module \( N \) is \( n \)-rigid, too. Thus, it is \( n \)-ortho-symmetric. \( \square \)

5.3. Comparing concepts - specific examples over Nakayama algebras. Now we are going to discuss the inclusion relations noted in Subsection 5.3.

\[
\{ \text{maximal orthogonal modules} \} \subset \{ \text{maximal rigid, ortho-symmetric modules} \} \subset \{ \text{maximal ortho-symmetric modules} \}
\]

By specific examples, we will show that all inclusions are proper. To do so, we construct maximal 1-ortho-symmetric modules over Nakayama symmetric algebras.

Let \( \triangle_A \) be the cyclic quiver with set of vertices \( \{0, 1, \ldots, e \} \), and let \( k \triangle_A \) be the path algebra over the field \( k \). Define a quotient algebra of \( k \triangle_A \) as follows:

\[
A_{e,ae+1} := k \triangle_A / J_{ae+1}
\]

where \( J \) is the Jacobson radical of \( k \triangle_A \) and \( a \in \mathbb{N} \). Then \( A_{e,ae+1} \) is a Nakayama symmetric algebra, and conversely, each elementary Nakayama symmetric algebra over \( k \) is Morita equivalent to \( A_{e,ae+1} \) for some pair \((a, e)\) of natural numbers.

Let \( S_i \) be the simple \( A_{e,ae+1} \)-module corresponding to the vertex \( i \in \mathbb{Z}/e\mathbb{Z} \). For any \( 1 \leq t \leq ae + 1 \), there exists a unique indecomposable \( A_{e,ae+1} \)-module, denoted by \( L(i,t) \), which has \( S_i \) as its top and is of length \( t \). Particularly, \( L(1, i) = S_i \) and \( P_i := L(i, ae+1) \) is the projective cover of \( S_i \). Moreover, the socle of \( L(i,t) \) is \( L(i+t-1,1) \) and \( \Omega(L(i,t)) = L(i+t, ae+1 - i - t) \).

Let \( \text{dd}(i,t) \) be the maximal natural number such that \( L(i,t) \) is \( \text{dd}(i,t) \)-rigid. Note that \( \text{dd}(i,t) + 2 \) coincides with the dominant dimension of the algebra \( A_{e,ae+1} \otimes L(i,t) \) by [26] Lemma 3.

**Lemma 5.6.** Suppose that \( e \geq 1 \) and \( a \geq 2 \), and that \( 0 \leq i \leq e-1 \) and \( 1 \leq t \leq ae \). Then:

1. \( \Omega^2(L(i,t)) \simeq L(i,t) \) as \( A_{e,ae+1} \)-modules.
2. \( \text{dd}(i,t) = \begin{cases} 
 2e - 2, & \text{if } t = 1 \text{ or } t = ae; \\
 1, & \text{if } 2 \leq t \leq e-1 \text{ or } (a-1)e+2 \leq t \leq ae-1; \\
 0, & \text{if } e \leq t \leq (a-1)e+1.
\end{cases} \)
3. The \( A_{e,ae+1} \)-module \( A_{e,ae+1} \otimes L(i,t) \) is \((2e-2)\)-ortho-symmetric.

**Proof.** Observe that \( \Omega^2(L(i,t)) = L(i+1,t) \). This implies (1). Statement (2) was proved in [39]. Applying Corollary 5.3(1) to the module \( A_{e,ae+1} \otimes L(i,t) \), (3) is seen to follow from (1) and (2). \( \square \)

When considering symmetric Nakayama algebras \( A_{e,ae+1} \), Corollary 5.3(1) implies that 1-ortho-symmetric modules only can exist over the algebras \( A_{3q,3q+1} \). Therefore, we concentrate on modules over \( A_{3q,3q+1} \) with \( q \geq 1 \). For each \( A_{3q,3q+1} \)-module \( X \), define the orbit of \( X \) as follows: \( \Theta_X := \bigoplus_{j=0}^{2q-1} \Omega^j(X) \). By Lemma 5.6(1), \( \Theta_X = \Theta_{\Omega X} \).

**Proposition 5.7.** Let \( A = A_{3q,3q+1} \) with \( q \geq 1 \) and \( a \geq 2 \). Then:

1. Up to taking (arbitrary) syzygies, maximal 1-ortho-symmetric, basic \( A \)-modules are exactly
\[
A \oplus \Theta_{L(0,1)} \oplus \Theta_{L(0,2)} \quad \text{and} \quad A \oplus \Theta_{L(1,1)} \oplus \Theta_{L(0,2)}.
\]
They are connected by mutations.
2. The following \( A \)-module is maximal 1-rigid:
\[
A \oplus \Theta_{L(0,1)} \oplus \Theta_{L(0,2)} \oplus \bigoplus_{r=1}^{q-1} L(0,3r+2).
\]
Proof. (1) If $X$ is a 1-ortho-symmetric, basic $A$-module, then $X \cong A \oplus \bigoplus_{i=1}^{m} \theta_{L(i,t_i)}$ such that $\theta_{L(i,t_i)}$ are 1-rigid, where $0 \leq i \leq 3q-1$ and $1 \leq t_i \leq 3q$ for all $1 \leq s \leq m \in \mathbb{N}$. By Lemma 5.4(1), $\theta_{L(i,t_i)} = \theta_{\Omega^j(L(i,t_i))}$, and the sum of the lengths of $L(i,t_i)$ and $\Omega^3(L(i,t_i))$ is equal to $3q+1$. Therefore, $L(i,t_i)$ and $\Omega^3(L(i,t_i))$ are 1-rigid, we may assume that $1 \leq t_i \leq 3q-1$ by Lemma 5.4(2).

Claim. For any $1 \leq t \leq 3q-1$, if $\text{Ext}^1_A(\Omega^3(L(0,t)), L(0,t)) = 0$, then $t \leq 2$. In particular, if $\theta_{L(i,t)}$ is 1-rigid, then $t \leq 2$.

**Proof.** If $t \geq 3$, then

$$
\text{Ext}^1_A(\Omega^3(L(0,t)), L(0,t)) \cong \text{Hom}_A(\Omega^3(L(0,t)), L(0,t)) \cong \text{Hom}_A(L(2,t), L(0,t)) \neq 0.
$$

As $\theta_{L(i,t)}$ is 1-rigid, $1 \leq t_i \leq 2$ for each $1 \leq s \leq m$.

Claim. Both $\theta_{L(i,0)}$ and $\theta_{L(i,2)}$ are 1-rigid.

**Proof.** It suffices to check that

$$
\text{Ext}^1_A(\Omega^{3j}(L(0,t)), L(0,t)) \cong \text{Hom}_A(\Omega^{3j+1}(L(0,t)), L(0,t)) = 0
$$

for $1 \leq j \leq 2$ and for $1 \leq t \leq 2$. Actually, this can be read off from the following formulae on syzygies of $L(0,1)$ and $L(0,2)$:

1. $(\star) \quad \Omega^{3j+1}L(0,1) = \begin{cases} L(3p+1, b-1), & j = 2p \quad \text{for } 1 \leq p \leq q; \\ L(3p-1, 1), & j = 2p - 1 \quad \text{for } 1 \leq p \leq q; \end{cases}$

2. $(\star\star) \quad \Omega^{3j+1}L(0,2) = \begin{cases} L(3p+2, b-2), & j = 2p \quad \text{for } 1 \leq p \leq q; \\ L(3p-1, 2), & j = 2p - 1 \quad \text{for } 1 \leq p \leq q; \end{cases}$

where $b = 3q+1$.

Since both $\theta_{L(i,0)}$ and $\theta_{L(i,2)}$ are closed under taking $\Omega^3$, $A \oplus \theta_{L(i,0)}$ and $A \oplus \theta_{L(i,2)}$ are 1-ortho-symmetric by Corollary 5.4(1). As $\Omega^3(L(i,t)) = L(i+1, t)$ for each $i \in \mathbb{Z}/e\mathbb{Z}$, both $A \oplus \theta_{L(i,1)}$ and $A \oplus \theta_{L(i,2)}$ are 1-ortho-symmetric, too.

Next, we complete $M_0 := A \oplus \theta_{L(i,2)}$ to a maximal 1-ortho-symmetric module by adding other 1-ortho-symmetric $A$-modules. Before doing this, set

$$
M_0^{i+1} = \theta_{L(i,2)}^{i+1} := \{ Y \in A-\text{mod} | \text{Ext}^1_A(\theta_{L(i,2)}, Y) = 0 \}.
$$

By $(\star\star)$, this category coincides with the category of all $A$-modules $Y$ such that

$$
\text{Hom}_A(L(3p+2, b-2), Y) = 0 = \text{Hom}_A(L(3p-1, 2), Y)
$$

for all $1 \leq p \leq q$.

Let $L(i,t) \in \theta_{L(i,2)}^{i+1}$ with $1 \leq t \leq 3q$ such that $M_0 \oplus \theta_{L(i,t)}$ is basic and 1-ortho-symmetric. Note that $\Omega^3(L(i,t)) = L(i+3, t)$ and $\theta_{\Omega^3(L(i,t))} = \theta_{L(i,t+3)}$. So, we can choose $0 \leq i \leq 2$ to represent $\theta_{L(i,t)}$. Further, since $\theta_{L(i,t)}$ is 1-rigid and basic, $1 \leq t \leq 2$ and $L(i,t) \neq L(0,2)$. However, if $Y \in \{ L(1,2), L(2,1), L(2,2) \}$, then

$$
\text{Hom}_A(L(3q+2, b-2), Y) = \text{Hom}_A(L(2, b-2), Y) \neq 0.
$$

This implies that $L(i,t) = L(0,1)$ or $L(1,1)$. Therefore, both $M_1 := M_0 \oplus \theta_{L(i,0)}$ and $M_2 := M_0 \oplus \theta_{L(i,1)}$ are 1-ortho-symmetric. Since $\theta_{L(i,0)}$ and $\theta_{L(i,1)}$ are stable under taking $\Omega^3$, it follows from Corollary 5.4(2) that $M_1$ and $M_2$ are connected by mutations in the sense that

$$
\mu_{\theta_{L(i,0)}}(M_1) = M_2 \quad \text{and} \quad \mu_{\theta_{L(i,1)}}(M_2) = M_1.
$$

Moreover, since $\text{Ext}^1_A(L(0,1), L(1,1)) \neq 0$, the module $M_0 \oplus \theta_{L(i,0)} \oplus \theta_{L(i,1)}$ is not 1-ortho-symmetric. Hence $M_1$ and $M_2$ are maximal 1-ortho-symmetric, and also the only ones up to taking arbitrary syzygies. This shows (1).

(2) Recall that $M_1 = A \oplus \theta_{L(i,0)} \oplus \theta_{L(i,2)}$. Let $\mathcal{S}$ be the set of indecomposable, non-isomorphic and non-projective $A$-modules $Y$ such that $Y \in M_1^{i+1} \setminus \text{add}(M_1)$. We claim that

$$
(\mathcal{S}) \quad \mathcal{S} = \{ \Omega^3(L(0,3v+2)) | 0 \leq u \leq 2q-1 \text{ and } 1 \leq v \leq q-1 \}.
$$

Since $\Omega^3(\theta_{L(i,0)}) \cong \Omega^3(\theta_{L(i,1)})$ and $\Omega^3(\theta_{L(i,2)}) \cong \theta_{L(i,2)}$ for any $j \geq 1$, the category $M_1^{i+1}$ is closed under taking $\Omega^3$ in $A$-mod. Note that $\Omega^3(L(i,t)) = L(i+3, t)$. So, if $L(i,t)$ belongs to $M_1^{i+1}$, so does $L(i+3, t)$. If moreover $0 \leq i \leq 2$ and $1 \leq t \leq 2$, then $A \oplus \theta_{L(i,t)}$ is 1-ortho-symmetric by the proof of (1) and so is $M_1 \oplus \theta_{L(i,t)}$. This leads to $L(i,t) \in \{ L(0,1), L(0,2) \}$ by the proof of (1).

In order to show (2), it suffices to prove the following statement: Given a pair $(i,t)$ with $0 \leq i \leq 2$ and $3 \leq t \leq 3q-1$, the module $L(i,t) \in M_1^{i+1}$ if and only if $i = 0$ and $t = 3v+2$ for some integer $v$ with $1 \leq v \leq q-1$.

Clearly, $L(i,t) \in M_1^{i+1}$ if and only if for any $1 \leq p \leq q$,

$$
\text{Hom}_A(L(3p+1, b-1), L(i,t)) = 0 = \text{Hom}_A(L(3p-1, 1), L(i,t)) \quad \text{see (\star)}
$$

for $1 \leq q \leq 2q-1$ and $1 \leq v \leq q-1$.\]
and
\[ \text{Hom}_A(L(3p + 2, b - 2), L(i, t)) = 0 = \text{Hom}_A(L(3p - 1, 2), L(i, t)) \] (see (**)).

Hence, if \( i = 0 \), then \( L(1, t) \in M_1^{1,1} \) if and only if \( t = 3v + 2 \) for some integer \( v \) with \( 1 \leq v \leq q - 1 \). However, if \( i = 1, 2 \), then \( \text{Hom}_A(L(3q + 2, b - 2), L(i, t)) = \text{Hom}_A(L(2, b - 2), L(i, t)) \neq 0 \) and thus \( L(i, t) \notin M_1^{1,1} \) in this case. This verifies the above statement, and therefore (2) is true.

For any \( 1 \leq t, t' \leq 3q - 1 \), the Auslander Reiten formula shows
\[ \text{Ext}_A(L(0, t), L(0, t')) \simeq D \text{Hom}_A(L(0, t'), \Omega^2(L(0, t))) \simeq D \text{Hom}_A(L(0, t'), L(1, t)) = 0. \]

This implies that \( N := \bigoplus_{t=1}^{3q-1} L(0, 3r + 2) \) is 1-rigid. Since \( N \in M_1^{1,1} \) and \( M_1 \) is 1-ortho-symmetric, the \( A \)-module \( M_1 \oplus N \) is 1-rigid. To check this module is maximal 1-rigid, it is sufficient to show the following fact:

**Claim.** For any pair \( (u, v) \) of integers with \( 1 \leq u \leq 2q - 1 \) and \( 1 \leq v \leq q - 1 \), there exists another integer \( r \) with \( 1 \leq r \leq q - 1 \) such that
\[ \text{Ext}_A(L(0, 3v + 2), L(0, 3r + 2)) \neq 0 \]

or \[ \text{Ext}_A(L(0, 3r + 2), \Omega^3(L(0, 3v + 2))) \neq 0. \]

**Proof:** There are three cases:

(i) If \( u = 2d \) with \( 1 \leq d \leq q - 1 \), then
\[ \text{Ext}_A(L(0, 3d + 2), \Omega^d(L(0, 3v + 2))) \simeq \text{Hom}_A(L(3d + 2, b - 3d - 2), L(3d, 3v + 2)) \neq 0. \]

(ii) If \( u = 2d - 1 \) with \( 1 \leq d \leq q - 1 \), then
\[ \text{Ext}_A(\Omega^d(L(0, 3v + 2)), L(0, 3d + 2)) \simeq \text{Hom}_A(L(3d - 1, 3v + 2), L(0, 3d + 2)) \neq 0. \]

(iii) If \( u = 2q - 1 \), then
\[ \text{Ext}_A(L(0, 3v + 2), \Omega^{3q-3}(L(0, 3v + 2))) \simeq \text{Hom}_A(L(0, 3v + 2), L(3q - 2, 3v + 2)) \neq 0. \]

Thus \( M_1 \oplus N \) is maximal 1-rigid, finishing the proof of (2). \( \square. \)

In Proposition 5.2, the \( A \)-module \( M_1 := A \oplus \bar{\partial}_{L(0,1)} \oplus \bar{\partial}_{L(0,2)} \) is maximal 1-rigid only in the case \( A = A_{3,3a+1} \), and it is even maximal 1-ortho-orthogonal if and only if \( A = A_{3,7} \). In fact, when \( A = A_{3,3a+1} \), we have
\[ M_1 = A \oplus L(0, 1) \oplus (2, 3a) \oplus L(0, 2) \oplus L(0, 3a - 1). \]

If \( a \geq 3 \), then the following canonical sequence
\[ 0 \longrightarrow L(0, 3a - 4) \longrightarrow L(0, 3a - 1) \xrightarrow{f} L(0, 3) \longrightarrow 0, \]
implies that \( L(0, 3a - 4) \in M_1^{1,1} \), where \( f \) is a minimal right add(\( A \oplus M_1 \))-approximation. But \( L(0, 3a - 4) \) is not 1-rigid by Lemma 5.6(2).

### 5.4. Maximal 1-rigid and maximal 1-orthogonal modules

Finally, we provide a sufficient condition for maximal 1-rigid, 1-ortho-symmetric modules over self-injective algebras to be maximal 1-orthogonal. Recall that an algebra \( A \) is said to have no loops if \( \text{Ext}_A^1(S, S) = 0 \) for each simple \( A \)-module \( S \). By the no loops conjecture (proved by Igusa, [19]), if \( A \) has finite global dimension, then it has no loops.

**Proposition 5.8.** Let \( A \) be a self-injective algebra without loops. Let \( A_M \) be a generator which is maximal 1-rigid. Suppose that each indecomposable, non-projective direct summand \( X \) of \( M \) satisfies \( \Omega^1 \nu_A(X) \simeq X \) as \( A \)-modules. If the algebra \( \text{End}_A(M) \) has no loops, then \( \text{gldim} \text{End}_A(M) = 3 \). In this case, the \( A \)-module \( M \) is maximal 1-ortho-symmetric.

**Proof.** Without loss of generality, assume that \( A_M \) is basic. Let \( M = \bigoplus_{i=1}^t M_i \) be a decomposition of \( A_M \) into indecomposable direct summands. Since \( A \) is self-injective without loops, the \( A \)-module \( A \oplus S \) is 1-rigid for any simple \( A \)-module \( S \). However, since \( A_M \) is maximal 1-rigid, it is not projective.

Since \( A \) is self-injective, \( \tau = \Omega^2 \nu_A \) and \( \nu_A \Omega_A \simeq \Omega_A \nu_A \), which implies \( \tau_2 = \tau \nu_A \simeq \Omega_A \nu_A \). By assumption, each indecomposable, non-projective direct summand \( X \) of \( M \) satisfies \( \Omega^1 \nu_A(X) \simeq X \) as \( A \)-modules. It follows from Corollary 3.18 that \( A_M \) is 1-ortho-symmetric. Furthermore, since \( M \) is maximal 1-rigid by assumption, it is maximal 1-ortho-symmetric. Also, by Corollary 3.18 if \( M_i \) is non-projective, then
\[ M \setminus M_i := \bigoplus_{1 \leq j \leq t, j \neq i} M_j \]
is also 1-ortho-symmetric.

Let \( \Lambda := \text{End}_A(M) \). For each \( 1 \leq i \leq t \), denote by \( S_i \) the top of the projective \( \Lambda \)-module \( \text{Hom}_A(M, M_i) \). Let \( \theta_i : U_i \to M_i \) be a minimal right add(\( M \setminus M_i \))-approximation of \( M_i \), and let \( K_i := \text{Ker} (\theta_i) \). In particular, if \( M_i \) is non-projective, then \( \theta_i \) is surjective, and there is an exact sequence:
\[ 0 \longrightarrow K_i \longrightarrow U_i \xrightarrow{\theta_i} M_i \longrightarrow 0. \]
Now, suppose that $A$ has no loops. Then $\text{Ext}^1_A(S_i, S_i) = 0$ and therefore $S_i$ has the following minimal projective presentation:

$$\text{Hom}_A(M, U_i) \xrightarrow{\theta_i} \text{Hom}_A(M, M_i) \twoheadrightarrow S_i \twoheadrightarrow 0.$$ 

Moreover, the image of the map $\text{Hom}_A(M_i, \theta_i) : \text{Hom}_A(M_i, U_i) \rightarrow \text{Hom}_A(M_i, M_i)$ is equal to the radical of $\text{End}_A(M_i)$. Thus if $M_i$ is projective, then the image of $\theta_i$ equals the radical $\text{rad}(M_i)$ of $M_i$.

**Case 1.** Suppose that $M_i$ is not projective.

**Claim.** $\text{projdim}(S_i) = 3$.

**Proof:** Since $U_i \in \text{add}(M \setminus M_i)$ and $\theta_i$ is surjective, $K_i \neq 0$ and $\text{projdim}(S_i) \geq 2$. Because of $\Omega^1_A(S_i) = \text{Hom}_A(M, K_i)$, it is sufficient to show that $\text{projdim}(A \text{Hom}_A(M, K_i)) = 1$. Note that $A\text{Hom}_A(M, K_i)$ is a maximal 1-ortho-symmetric and $A\text{Hom}_A(M, K_i)$ is 1-ortho-symmetric. By Proposition 4.8 there exists a minimal $\text{add}(M \setminus M_i)$-split sequence:

$$0 \rightarrow M_i \rightarrow V_i \rightarrow K_i \rightarrow 0$$

such that $V_i \in \text{add}(M \setminus M_i)$. Since $A\text{Hom}_A(M, \theta_i)$ to this sequence results in a short exact sequence of $A$-modules:

$$0 \rightarrow \text{Hom}_A(M, M_i) \rightarrow \text{Hom}_A(M, V_i) \rightarrow \text{Hom}_A(M, K_i) \rightarrow 0.$$ 

This means that $\text{projdim}(A \text{Hom}_A(M, K_i)) = 1$ and therefore $\text{projdim}(S_i) = 3$. Furthermore, the simple $A$-module $S_i$ has a minimal projective resolution of the following form:

$$0 \rightarrow \text{Hom}_A(M, M_i) \rightarrow \text{Hom}_A(M, V_i) \rightarrow \text{Hom}_A(M, U_i) \rightarrow \text{Hom}_A(M, M_i) \rightarrow S_i \rightarrow 0.$$ 

**Case 2.** Suppose that $M_i$ is projective.

**Claim.** $\text{projdim}(S_i) \leq 2$.

**Proof:** Since $\text{Im}(\theta_i) = \text{rad}(M_i)$ in this case, there is an exact sequence of $A$-modules:

$$0 \rightarrow K_i \rightarrow U_i \xrightarrow{\theta_i} \text{rad}(M_i) \rightarrow 0.$$ 

Note that $\text{rad}(M_i) = \Omega^1_A(S_i)$ where $S$ is the top of $M_i$. Since $A$ is self-injective without loops, the simple $A$-module $S$ is 1-rigid and so is $\text{rad}(M_i)$. Recall that $\theta_i : U_i \rightarrow M_i$ is a minimal right $\text{add}(M \setminus M_i)$-approximation and that $M_i$ is projective. This implies that $\theta_i$ is a minimal right $\text{add}(M)$-approximation of $\text{rad}(M_i)$. As $M_i$ is 1-ortho-symmetric and $\text{rad}(M_i)$ is 1-rigid, Lemma 4.12(a) implies that $K_i \oplus M_i$ is 1-rigid. Since $M_i$ is maximal 1-rigid, $K_i \in \text{add}(A\text{Hom}_A(M, K_i))$. Thus the simple $A$-module $S_i$ has the minimal projective resolution:

$$0 \rightarrow \text{Hom}_A(M, K_i) \rightarrow \text{Hom}_A(M, U_i) \rightarrow \text{Hom}_A(M, M_i) \rightarrow S_i \rightarrow 0.$$ 

This yields $\text{projdim}(S_i) \leq 2$.

Hence $\text{gldim}(A) = 3$. Since $A\text{Hom}_A(M)$ is maximal 1-ortho-symmetric, Corollary 5.18 yields that $A\text{Hom}_A(M)$ is maximal 1-orthogonal. $\square$

When focussing on weakly 2-Calabi-Yau self-injective algebras, the following result is a slightly simplified variation on Proposition 5.8.

**Corollary 5.9.** Let $A$ be a weakly 2-Calabi-Yau self-injective algebra without loops. Let $A\text{Hom}_A(M)$ be a basic generator that is maximal 1-rigid. If $\text{End}_A(M)$ has no loops, then $\text{gldim(End}_A(M) = 3$. In this case, the $A\text{-mod}$ is maximal 1-orthogonal.

**Proof.** Since $A$ is weakly 2-Calabi-Yau, the functor $\Omega^1_A(M)$ is naturally isomorphic to the identity functor of $A\text{-mod}$. This implies that $\Omega^1_A(M)(X) \cong X$ for any indecomposable non-projective $A$-module $X$. Thus Corollary 5.9 follows from Proposition 5.8. $\square$

**Remarks on Corollary 5.9.**

1. When $A$ is a connected, weakly 2-Calabi-Yau self-injective algebra over an algebraically closed field $k$, then $A$ is Morita equivalent to a deformed preprojective algebra $P^f(\Delta)$ of a generalised Dynkin type $\Delta$ (see [5 Theorem 1.2]). But we don’t know when a deformed preprojective algebra $P^f(\Delta)$ of a generalised Dynkin type $\Delta$ is weakly 2-Calabi-Yau in general. It is the case for all preprojective algebras of generalised Dynkin type, but also for some deformed preprojective algebras of generalised Dynkin type. In this sense, Corollary 5.9 extends [13 Proposition 6.2], from preprojective algebras of Dynkin type to arbitrary weakly 2-Calabi-Yau self-injective algebras without loops, and thus also can be applied to some deformed preprojective algebras of generalised Dynkin type. For more information on this class of algebras, we refer to [5] [12].

2. All endomorphism algebras of maximal 1-rigid objects in a weakly 2-Calabi-Yau triangulated category are at most 1-Gorenstein (see, for example, [29 Proposition 4.6.1])). This implies that, if $A$ is a weakly 2-Calabi-Yau self-injective algebra and $A\text{Hom}_A(M)$ is a maximal 1-rigid generator, then the stable endomorphism algebra
End\(_4(M)\) of \(M\) is at most 1-Gorenstein. However, the algebra End\(_4(M)\) is at most 3-Gorenstein by Lemma 5.2 and Corollary 5.18.

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