BOUND EDNESS OF CLASSICAL OPERATORS ON
REARRANGEMENT-INVARIANT SPACES

DAVID E. EDMUNDS, ZDENĚK MIHULA, VÍT MUSIL AND LUBOŠ PICK

Abstract. We study the behaviour on rearrangement-invariant spaces of such classical operators of interest in harmonic analysis as the Hardy-Littlewood maximal operator (including the fractional version), the Hilbert and Stieltjes transforms, and the Riesz potential. The focus is on sharpness questions, and we present characterisations of the optimal domain (or range) partner spaces when the range (domain) is fixed. When a rearrangement-invariant partner space exists at all, a complete characterisation of the situation is given. We illustrate the results with a variety of examples of sharp particular results involving customary function spaces.

1. Introduction

Given function spaces $X,Y$ and an operator $T$ that maps $X$ boundedly into $Y$, it is natural to ask whether there is a space bigger than $X$ that is also mapped boundedly by $T$ into $Y$, or a space smaller than $Y$ into which $T$ maps $X$ boundedly.

Such questions have attracted a great deal of attention in recent years, in particular in connection with embeddings of Sobolev spaces. By way of illustration we consider a particularly simple Sobolev embedding. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, let $p \in [1,n)$ and put $p^* = np/(n-p)$. It is classical that, in standard notation, the Sobolev space $W^{1,p}_0(\Omega)$ is embedded in $L^{p^*}(\Omega)$. Can $W^{1,p}_0(\Omega)$ be embedded in a space smaller than $L^{p^*}(\Omega)$? Is there a space larger than $W^{1,p}_0(\Omega)$ that can be embedded in $L^{p^*}(\Omega)$? To make such questions sensible the class of competing spaces must be specified. If we restrict ourselves to Lebesgue spaces as targets and domain spaces that are Sobolev spaces based on Lebesgue spaces, then the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is optimal in the sense that neither the domain nor the target space can be improved. This leaves open the question of optimality in classes of spaces wider than those involving the Lebesgue scale. If the class of admissible target spaces is taken to be that of rearrangement-invariant (r.i.) spaces, then the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is optimal in the sense that neither the domain nor the target space can be improved. This leaves open the question of optimality in classes of spaces wider than those involving the Lebesgue scale. If the class of admissible target spaces is taken to be that of rearrangement-invariant (r.i.) spaces, then the optimal range space turns out to be the Lorentz space $L^{p,q}(\Omega)$; there is a similar improvement of the domain space, involving a Sobolev space based on a Lorentz rather than a Lebesgue space.

The first results in this direction were obtained in [25] in connection with rearrangement-invariant quasinorms. Further extensions concerning rearrangement-invariant norms were added later in several papers, for instance [36, 37]. A comprehensive treatment of optimal Sobolev embeddings on Euclidean domains equipped with general measures having specific isoperimetric properties was given in [20].

Embeddings are not the only maps for which such questions are of interest and importance. The optimality of r.i. function spaces on which the Laplace transform $\mathcal{L}$ acts boundedly was studied in a recent paper [11]. A special case of the results obtained is that if $p \in (1,\infty)$ and $q \in [1,\infty]$, then $\mathcal{L}$ maps the Lorentz space $L^{p,q}(0,\infty)$ boundedly into $L^{p',q}(0,\infty)$, a fact which we denote by $\mathcal{L} : L^{p,q}(0,\infty) \to L^{p',q}(0,\infty)$. Moreover, both the domain and target spaces
are optimal: there is no r.i. space smaller than $L^{p/q}(0,\infty)$ into which $L$ maps $L^{p,q}(0,\infty)$, and there is no r.i. space larger than $L^{p,q}(0,\infty)$ mapped by $L$ into $L^{p,q}(0,\infty)$. Thus in particular $L: L^p(0,\infty) \to L^{p,q}(0,\infty)$ and the spaces involved form an optimal pair; if $p > 2$, there is no $q$ for which $L: L^p(0,\infty) \to L^q(0,\infty)$.

In the present paper we discuss such problems for classical operators of great interest in analysis and its applications, namely the Hilbert and Stieltjes transforms, the Riesz potential and various versions of the maximal operator. The action of these operators on specific classes of function spaces has been extensively studied over the several decades. Classical results are available for example in connection with familiar function spaces. The 1970s experienced a real boom of this theory involving weighted Lebesgue spaces and fundamental papers were written ([41, 52] for the Hardy–Littlewood maximal operator, [42] for singular and fractional integrals, [21, 44, 45] for the Hilbert transform). Later it become apparent that Lebesgue spaces are not sufficient for describing all the important situations and other function spaces were investigated. Classical Lorentz spaces which originated in 1950s and have been occurring occasionally later (see [8, 3]) became extremely fashionable in 1990s when the fundamental papers [13, 17, 14, 1, 12] appeared. Various important and deep results were obtained, see for example [13, 30, 15, 16, 19, 45, 21, 42]. The results naturally found their way into important monographs that are considered classic these days, see [54, 56, 31, 55, 39, 50, 24, 22].

On the other hand, surprisingly little attention has been paid to the sharpness of the results. In this paper we study the behaviour of these operators on rearrangement-invariant spaces, a class of function spaces that includes for example all Lebesgue, Lorentz, Orlicz, Lorentz-Zygmund spaces and more. Our focus is mainly on the optimality of function spaces.

We use the Hardy-Littlewood maximal operator $M$ to illustrate the results obtained and serve as an appetiser for the forthcoming attractions. Let $X$ be a rearrangement-invariant Banach function space over $\mathbb{R}^n$ with associate space $X'$; denote by $X'(0,\infty)$ the representation space of $X'$ and suppose that the function $\psi$ given by $\psi(t) = \chi(0,1)(t) \log(1/t)$ belongs to $X'(0,\infty)$. Let $Y'$ be the set of all $f$ such that

$$g(f) = \left\| \int_t^\infty f(s)s^{-1}\,ds \right\|_{X'(0,\infty)} < \infty.$$  

Endowed with the norm $g$, $Y'$ is an r.i. space with associate space $Y$ that not only has the property that $M: X \to Y$, but is also the optimal range space corresponding to $X$. If $\psi \notin X'(0,\infty)$, there is no r.i. space $Z$ over $\mathbb{R}^n$ such that $M: X \to Z$.

The situation turns out to be considerably more complicated in the case of the fractional maximal operator, another classical operator of harmonic analysis. The reason is that the appropriate analogue of the Riesz–Wiener–Herz inequality for the fractional maximal operator leads to an inevitable involvement of a supremum type operator, rather than just an integral mean. Supremum operators are not linear and in general are less manageable than their integral companions. However, using a fine analysis combining known and new techniques and various delicate estimates we are able to characterize the optimal range space for this operator as well. Since the general resulting condition is however naturally not so simple as in the case of the operator $M$, we include another, simpler characterization, available under a rather mild extra assumption. We also include an interesting and perhaps somewhat surprising result describing a vital link between optimality properties of a space and boundedness of a supremum operator on its associate space that leads to a self-explanatory characterization of the above-mentioned extra condition. This part of the paper is one of the most innovative ones.

We finally consider two other classical operators of harmonic analysis, namely the Hilbert transform and the Riesz potential. The importance of these operators is very well known,
and their properties have been deeply studied. Our contribution is the characterization of the optimality of the spaces involved. In case of the Hilbert Transform we use the Stieltjes Transform as the appropriate tool and obtain characterizations for it as well.

For each of the operators considered, we are also able to nail down the optimal domain partner when the range space is fixed, this task being in general slightly simpler than the converse one. To establish all this, a combination of new techniques developed here with those from [20, 25] and [36] is used.

We illustrate the results obtained with variety of nontrivial examples. For instance, we recover the well-known fact that

\[ M : L(\log L)^\alpha(Q) \to L(\log L)^{\alpha-1}(Q) \]

when \( \alpha \geq 1, \ Q \subset \mathbb{R}^n \) is a cube of finite measure and \( L(\log L)^\alpha(Q) \) is the classical Zygmund class defined as the collection of all measurable functions \( g \) on \( Q \) satisfying \( \int_Q |g(x)|((\log(1 + |g(x)|))^\alpha dx < \infty \), but we add the information that the range space cannot be improved in any way when the competing spaces are rearrangement invariant. Similar examples are even more interesting when the functions act on a set of unbounded measure, say, \( \mathbb{R}^n \). We will for example prove that if \( X \) is the space equipped with the norm \( \| f \|_X = \int_0^\infty f^*(t)w(t) \, dt \), where

\[ w(t) = (1 - \log t)^{\alpha_0} \chi_{(0,1)} + (1 + \log t)^{\alpha_\infty} \chi_{[1,\infty)} \]

and \( \alpha_0 \geq 1 \) and \( \alpha_\infty \in [-1,0] \), then the optimal (smallest possible) rearrangement-invariant range space \( Y \) such that

\[ M : X(\mathbb{R}^n) \to Y(\mathbb{R}^n) \]

is the space whose associate space has norm

\[ \|f\| = \sup_{0 < t < \infty} w(t)^{-1} \int_t^{\infty} f^*(s) \frac{ds}{s}, \ f \in \mathcal{M}_+(\mathbb{R}^n). \]

Such results have not been available before, and the latter norm cannot be identified with any customary known one.

We get analogous sets of examples for other operators, too. For example in the case of the fractional maximal operator we essentially improve some results from earlier papers such as [26, 27, 28, 47].

2. Preliminaries

In this section we collect all the background material that will be used in the paper. We start with the operation of the nonincreasing rearrangement of a measurable function.

Throughout this section, let \((R, \mu)\) be a \( \sigma \)-finite nonatomic measure space. We set

\[ \mathcal{M}(R, \mu) = \{ f : f \text{ is } \mu\text{-measurable function on } R \text{ with values in } [-\infty, \infty] \}, \]

\[ \mathcal{M}_0(R, \mu) = \{ f \in \mathcal{M}(R, \mu) : f \text{ is finite } \mu\text{-a.e. on } R \} \]

and

\[ \mathcal{M}_+(R, \mu) = \{ f \in \mathcal{M}(R, \mu) : f \geq 0 \}. \]

The nonincreasing rearrangement \( f^* : [0, \infty) \to [0, \infty] \) of a function \( f \in \mathcal{M}(R, \mu) \) is defined as

\[ f^*(t) = \inf \{ \lambda \in (0, \infty) : \{ s \in R : |f(s)| > \lambda \} \leq t \}, \ t \in [0, \infty). \]

The maximal nonincreasing rearrangement \( f^{**} : (0, \infty) \to [0, \infty] \) of a function \( f \in \mathcal{M}(R, \mu) \) is defined as

\[ f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds, \ t \in (0, \infty). \]

If \( |f| \leq |g| \mu\text{-a.e. in } R \), then \( f^* \leq g^* \). The operation \( f \mapsto f^* \) does not preserve sums or products of functions, and is known not to be subadditive. The lack of subadditivity of the operation
of taking the nonincreasing rearrangement is, up to some extent, compensated by the following fact (\[7\], Chapter 2, (3.10)): for every $t \in (0, \infty)$ and every $f, g \in \mathcal{M}(R, \mu)$, we have

\[
\int_0^t (f + g)^*(s) \, ds \leq \int_0^t f^*(s) \, ds + \int_0^t g^*(s) \, ds.
\]

This inequality can be also written in the form

\[
(f + g)^{**} \leq f^{**} + g^{**}.
\]

A fundamental result in the theory of Banach function spaces is the Hardy lemma (\[7\], Chapter 2, Proposition 3.6) which states that if two nonnegative measurable functions $f, g$ on $(0, \infty)$ satisfy

\[
\int_0^t f(s) \, ds \leq \int_0^t g(s) \, ds
\]

for all $t \in (0, \infty)$, then, for every nonnegative nonincreasing function $h$ on $(0, \infty)$, one has

\[
\int_0^\infty f(s)h(s) \, ds \leq \int_0^\infty g(s)h(s) \, ds.
\]

Another important property of rearrangements is the Hardy-Littlewood inequality (\[7\], Chapter 2, Theorem 2.2), which asserts that, if $f, g \in \mathcal{M}(R, \mu)$, then

\[
\int_R |fg| \, d\mu \leq \int_0^\infty f^*(t)g^*(t) \, dt.
\]

If $(R, \mu)$ and $(S, \nu)$ are two (possibly different) $\sigma$-finite measure spaces, we say that functions $f \in \mathcal{M}(R, \mu)$ and $g \in \mathcal{M}(S, \nu)$ are equimeasurable, and write $f \sim g$, if $f^* = g^*$ on $(0, \infty)$.

A functional $\varrho: \mathcal{M}_+(R, \mu) \to [0, \infty]$ is called a Banach function norm if, for all $f, g$ and $\{f_j\}_{j \in \mathbb{N}} \in \mathcal{M}_+(R, \mu)$, and every $\lambda \geq 0$, the following properties hold:

- (P1) $\varrho(f) = 0$ if and only if $f = 0$; $\varrho(\lambda f) = \lambda \varrho(f)$; $\varrho(f + g) \leq \varrho(f) + \varrho(g)$ (the norm axiom);
- (P2) $f \leq g$ a.e. implies $\varrho(f) \leq \varrho(g)$ (the lattice axiom);
- (P3) $f_j \overset{\mathcal{L}}{\sim} f$ a.e. implies $\varrho(f_j) \overset{\mathcal{L}}{\sim} \varrho(f)$ (the Fatou axiom);
- (P4) $\varrho(\chi_E) < \infty$ for every $E \subset R$ of finite measure (the nontriviality axiom);
- (P5) if $E$ is a subset of $R$ of finite measure, then $\int_E f \, d\mu \leq C_E \varrho(f)$ for some constant $C_E$, $0 < C_E < \infty$, depending on $E$ and $\varrho$ but independent of $f$ (the local embedding in $L^1$).

If, in addition, $\varrho$ satisfies

- (P6) $\varrho(f) = \varrho(g)$ whenever $f^* = g^*$ (the rearrangement-invariance axiom),

then we say that $\varrho$ is a rearrangement-invariant norm.

If $\varrho$ is a rearrangement-invariant norm, then the collection

$$X = X(\varrho) = \{ f \in \mathcal{M}(R, \mu) : \varrho(|f|) < \infty \}$$

is called a rearrangement-invariant space, sometimes we shortly write just an r.i. space, corresponding to the norm $\varrho$. We shall write $\|f\|_X$ instead of $\varrho(|f|)$. Note that the quantity $\|f\|_X$ is defined for every $f \in \mathcal{M}(R, \mu)$, and

$$f \in X \iff \|f\|_X < \infty.$$

With any rearrangement-invariant function norm $\varrho$ is associated another functional, $\varrho'$, defined for $g \in \mathcal{M}_+(R, \mu)$ as

$$\varrho'(g) = \sup \left\{ \int_R fg \, d\mu : f \in \mathcal{M}_+(R, \mu), \varrho(f) \leq 1 \right\}.$$
It turns out that $g'$ is also a rearrangement-invariant norm, which is called the *associate norm* of $\varrho$. Moreover, for every rearrangement-invariant norm $\varrho$ and every $f \in M_+(R, \mu)$, we have (see [4, Chapter 1, Theorem 2.9])

$$\varrho(f) = \sup \left\{ \int_R fg \, d\mu : g \in M_+(R, \mu), \, \varrho'(f) \leq 1 \right\}.$$  

If $\varrho$ is a rearrangement-invariant norm, $X = X(\varrho)$ is the rearrangement-invariant space determined by $\varrho$, and $\varrho'$ is the associate norm of $\varrho$, then the function space $X(\varrho')$ determined by $\varrho'$ is called the *associate space* of $X$ and is denoted by $X'$. We always have $(X')' = X$, and we shall write $X''$ instead of $(X')'$. Furthermore, the *Hölder inequality*

$$\int_R fg \, d\mu \leq \|f\|_X \|g\|_{X'},$$

holds for every $f, g \in M(R, \mu)$.

An important consequence of the Hardy lemma, which plays a crucial role in the theory of rearrangement-invariant spaces, is the *Hardy–Littlewood–Pólya principle* ([4, Chapter 2, Theorem 4.6]) which asserts that if two functions $f, g$ satisfy the so-called *Hardy–Littlewood–Pólya relation*, defined by

$$\int_0^t f^*(s) \, ds \leq \int_0^t g^*(s) \, ds, \quad t \in (0, \infty),$$

and sometimes denoted by $f \prec g$ in the literature, then $\|f\|_X \leq \|g\|_X$ provided that the underlying measure space is resonant. We note that throughout this paper we work solely on nonatomic measure spaces, which are resonant by [4, Chapter 2, Theorem 2.7].

For every rearrangement-invariant space $X$ over the measure space $(R, \mu)$ there exists a unique rearrangement-invariant space $X(0, \mu(R))$ over the interval $(0, \mu(R))$ endowed with the one-dimensional Lebesgue measure such that $\|f\|_X = \|f^*\|_{X(0, \infty)}$. This space is called the *representation space* of $X$. This follows from the Luxemburg representation theorem (see [4, Chapter 2, Theorem 4.10]). Throughout this paper, the representation space of a rearrangement-invariant space $X$ will be denoted by $X(0, \mu(R))$. It will be useful to notice that when $R = (0, \infty)$ and $\mu$ is the Lebesgue measure, then every $X$ over $(R, \mu)$ coincides with its representation space.

If $\varrho$ is a rearrangement-invariant norm and $X = X(\varrho)$ is the rearrangement-invariant space determined by $\varrho$, we define its *fundamental function*, $\varphi_X$, for every $t \in [0, \mu(R))$ by $\varphi_X(t) = \varrho(\chi_E)$, where $E \subset R$ is such that $\mu(E) = t$. The properties of rearrangement-invariant norms guarantee that the fundamental function is well defined. Moreover, one has

$$\varphi_X(t)\varphi_X(t) = t \quad \text{for every } t \in [0, \mu(R)).$$

Let $X$ and $Y$ be rearrangement-invariant spaces over $(0, \infty)$ and let $I : [0, \infty) \to [0, \infty)$ be a nondecreasing function. Then

$$\left\| \int_t^\infty \frac{f(s)}{I(s)} \, ds \right\|_{Y(0, \infty)} \leq C_1 \|f\|_{X(0, \infty)} \quad \text{for every } f \in M_+(0, \infty)$$

holds true with some positive constant $C_1$ if and only if

$$\left\| \int_t^\infty \frac{g(s)}{I(s)} \, ds \right\|_{Y(0, \infty)} \leq C_2 \|g\|_{X(0, \infty)} \quad \text{for every nonincreasing } g \in M_+(0, \infty)$$

is valid with some positive constant $C_2$. This result originated as a consequence ([20, Corollary 9.8]) of a more general principle established in [20, Theorem 9.5] in connection with sharp higher-order Sobolev-type embeddings and its extension to unbounded intervals was given in [15, Theorem 1.10].
An important corollary of the Hardy–Littlewood inequality \( \text{[23]} \) is the fact that if \( f \) is a nonincreasing function on \((0, \infty)\) and \( X \) is a rearrangement-invariant space over \((0, \infty)\), then in fact one has
\[
\|f\|_{X(0,\infty)} = \sup \left\{ \int_0^\infty g^*(t)f(t)\,dt : \|g\|_{X^*(0,\infty)} \leq 1 \right\}.
\]
In other words, for such \( f \), the supremum can be reduced to nonincreasing functions only without any loss of information. This fact has deep consequences and will be used in the proofs below.

For each \( a \in (0, \infty) \), let \( D_a \) denote the \textit{dilation operator} defined on every nonnegative measurable function \( f \) on \((0, \infty)\) by
\[
(D_a f)(t) = f(at), \quad t \in (0, \infty).
\]
Then the operator \( D_a \) is bounded on every rearrangement-invariant space over \((0, \infty)\) (hence in particular on the representation space of any rearrangement-invariant space over an arbitrary adequate measure space). More precisely, if \( X \) is any given rearrangement-invariant space over \((0, \infty)\) with respect to the one-dimensional Lebesgue measure, then we have
\[
\|D_a f\|_X \leq C\|f\|_X \text{ for every } f \in X,
\]
with some constant \( C \), \( 0 < C < \infty \), depending on \( X \) and \( a \), but independent of \( f \). For more details, see [3] Chapter 3, Proposition 5.11.

Among basic examples of function norms are those associated with the standard Lebesgue spaces \( L^p \). For \( p \in (0, \infty) \), we define the functional \( \varrho_p \) by
\[
\varrho_p(f) = \|f\|_{L^p} = \left( \int_R |f|^p \,d\mu \right)^{1/p}
\]
if \( 0 < p < \infty \),
\[
\varrho_p(f) = \|f\|_{L^p} = \sup_{t \in (0, \infty)} t^{1/p} f^*(t)
\]
if \( p = \infty \)
for \( f \in M_+(R, \mu) \). If \( p \in [1, \infty) \), then \( \varrho_p \) is a rearrangement-invariant function norm.

If \( 0 < p, q \leq \infty \), we define the functional \( \varrho_{p,q} \) by
\[
\varrho_{p,q}(f) = \|f\|_{L^{p,q}} = \left\| s^{1/q - 1/p} f^*(s) \right\|_q
\]
for \( f \in M_+(R, \mu) \). The set \( L^{p,q} \), defined as the collection of all \( f \in M(R, \mu) \) satisfying \( \varrho_{p,q}(|f|) < \infty \), is called a \textit{Lorentz space}. If either \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \) or \( p = q = 1 \) or \( p = q = \infty \), then \( \varrho_{p,q} \) is equivalent to a rearrangement-invariant function norm in the sense that there exists a rearrangement-invariant norm \( \sigma \) and a constant \( C \), \( 0 < C < \infty \), depending on \( p, q \) but independent of \( f \), such that
\[
C^{-1} \sigma(f) \leq \varrho_{p,q}(f) \leq C \sigma(f).
\]
As a consequence, \( L^{p,q} \) is considered to be a rearrangement-invariant space for these cases of \( p, q \) (see [3] Chapter 4]). If either \( 0 < p < 1 \) or \( p = 1 \) and \( q > 1 \), then \( L^{p,q} \) is a quasi-normed space. If \( p = \infty \) and \( q < \infty \), then \( L^{p,q} = \{0\} \). For every \( p \in [1, \infty) \), we have \( L^{p,p} = L^p \). Furthermore, if \( p, q, r \in (0, \infty) \) and \( q \leq r \), then the inclusion \( L^{p,q} \subseteq L^{p,r} \) holds.

If \( A = [\alpha_0, \alpha_\infty] \subseteq R^2 \) and \( t \in R \), then we shall use the notation \( A + t = [\alpha_0 + t, \alpha_\infty + t] \).

Let \( 0 < p, q \leq \infty \), \( A = [\alpha_0, \alpha_\infty] \subseteq R^2 \) and \( B = [\beta_0, \beta_\infty] \subseteq R^2 \). Then we define the functionals \( \varrho_{p,q;A} \) and \( \varrho_{p,q;A,B} \) on \( M_+(R, \mu) \) by
\[
\varrho_{p,q;A}(f) = \left\| \left( t^{1/p - 1/q} \ell^A(t) f^*(t) \right) \right\|_{L^q(0,\infty)}
\]
and
\[
\varrho_{p,q;A,B}(f) = \left\| \left( t^{1/p - 1/q} \ell^A(t) \ell^B(t) f^*(t) \right) \right\|_{L^q(0,\infty)},
\]
where
\[
\ell^A(t) = \begin{cases} (1 - \log t)^{\alpha_0} & \text{if } t \in (0, 1), \\ (1 + \log t)^{\alpha_\infty} & \text{if } t \in [1, \infty) \end{cases}
\]
and
\[
\ell^B(t) = \begin{cases} 
(1 + \log(1 - \log t))^{\beta_0} & \text{if } t \in (0, 1), \\
(1 + \log(1 + \log t))^{\beta_\infty} & \text{if } t \in [1, \infty).
\end{cases}
\]

The set \(L^{p,q}_\ell\), defined as the collection of all \(f \in \mathcal{M}(R, \mu)\) satisfying \(\varrho_{p,q,A}(|f|) < \infty\), is called a Lorentz–Zygmund space, and the set \(L^{p,q,A,B}\) defined as the collection of all \(f \in \mathcal{M}_+(R, \mu)\) satisfying \(\varrho_{p,q,A,B}(|f|) < \infty\), is called a generalized Lorentz–Zygmund space. The functions of the form \(\ell^A, \ell^B\) are called broken logarithmic functions. The spaces of this type proved to be quite useful since they provide a common roof for many customary spaces. These include not only Lebesgue spaces and Lorentz spaces, but also all types of exponential and logarithmic Zygmund classes, and also the spaces discovered independently by Maz’ya (in a somewhat implicit form involving capacity estimates [33, pp. 105 and 109]), Hansson [33] and Brézis–Weinberger [9], who used it to describe the sharp target space in a limiting Sobolev embedding (the spaces can be also traced in the works of Brudnyi [10] and, in a more general setting, Cwikel and Pustylnik [23]). One of the benefits of using broken logarithmic functions consists in the fact that the underlying measure space can be considered to have either finite or infinite measure. For the detailed study of generalized Lorentz–Zygmund spaces we refer the reader to [27, 28, 47, 49].

We further define the spaces \(L^{(p,q,A)}\) through the functionals \(\varrho_{(p,q,A)}\) given on \(\mathcal{M}_+(R, \mu)\) by
\[
\varrho_{(p,q,A)}(f) = \left\| t^{\frac{1}{p} - \frac{1}{q}} \ell^A(t) f^{**}(t) \right\|_{L^q(0,\infty)}
\]
and, in an analogous way, all the other spaces involving various levels of logarithms.

Let \(X\) and \(Y\) be rearrangement-invariant spaces over possibly different measure spaces \((R, \mu)\) and \((S, \nu)\), respectively, and let \(T\) be an operator defined on \(X\) with values in \(\mathcal{M}(S, \nu)\). We say that \(T\) is bounded from \(X\) to \(Y\), a fact which is denoted by \(T: X \rightarrow Y\), if there exists a positive constant \(C\) such that
\[
\|Tf\|_Y \leq C\|f\|_X \quad \text{for every } f \in X.
\]

In an important special case when \(T\) is the identity operator, we say that \(X\) is embedded into \(Y\) and write \(X \hookrightarrow Y\). If \(T'\) is another operator defined at least on \(Y'\) with values in \(\mathcal{M}(R, \mu)\) and such that
\[
\int_R (Tf) g \, d\mu = \int_S f(T'g) \, d\nu
\]
for every \(f \in X\) and \(g \in Y'\), then \(T': X \rightarrow Y\) is equivalent to \(T': Y' \rightarrow X'\).

Let \(P\) and \(Q\) be the integral operators defined by
\[
(Pf)(t) = \frac{1}{t} \int_0^t f(s) \, ds, \quad t \in (0, \infty),
\]
and
\[
(Qf)(t) = \int_t^\infty f(s) \frac{ds}{s}, \quad t \in (0, \infty),
\]
for those functions on \(f \in \mathcal{M}_0(0, \infty)\) for which the respective integrals have sense. As an interchange of integration shows,
\[
\int_0^\infty (Pf)(t) g(t) \, dt = \int_0^\infty f(t)(Qg)(t) \, dt,
\]
for all \(f\) and \(g\) for which the integrals make sense. Hence, the operators \(P\) and \(Q\) are formally adjoint with respect to the \(L^1\)-pairing and therefore satisfy a relation in the spirit of (2.8). As a consequence, one has the equivalence
\[
(2.9) \quad P: X \rightarrow Y \iff Q: Y' \rightarrow X'.
\]
for every pair of rearrangement-invariant spaces $X,Y$ over $(0,\infty)$ (with the same operator norm). Another important example is that when $(R,\mu)$ is arbitrary and both $T$ and $T'$ are identity operators. Then (2.8) is trivially satisfied and, as a consequence, one gets

\begin{equation}
X \hookrightarrow Y \iff Y' \hookrightarrow X'
\end{equation}

for every pair of rearrangement-invariant spaces $X,Y$, again with the same embedding constant (see [7, Chapter 1, Proposition 2.10]).

We will say that a rearrangement-invariant space $Y$ over $(S,\nu)$ is a range partner for a given rearrangement-invariant space $X$ over $(R,\mu)$ with respect to a sublinear operator $T$ if $T: X \rightarrow Y$. We say that $Y$ is the optimal range partner for $X$ if one has $Y \hookrightarrow Z$ for every range partner $Z$ for $X$ with respect to $T$. We analogously define a domain partner and the optimal domain partner, that is, the largest possible domain space.

Throughout the paper the convention that $1\infty = 0$, and $0\cdot\infty = 0$ is used without further explicit reference. We write $A \approx B$ when the ratio $A/B$ is bounded from below and from above by positive constants independent of appropriate quantities appearing in expressions $A$ and $B$.

3. The Hardy-Littlewood maximal operator

In this section, the relevant rearrangement-invariant spaces are considered over $\mathbb{R}^n$ endowed with the $n$-dimensional Lebesgue measure. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$.

The Hardy–Littlewood maximal operator, $M$, is defined for every locally integrable function $f$ on $\mathbb{R}^n$ and every $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$, whose edges are parallel to the coordinate axes of $\mathbb{R}^n$, that contain $x$.

The operator $M$ is merely sublinear, rather than linear, and it is clearly a contraction on $L^{\infty}$. On the other hand, $Mf$ is never integrable unless $f \equiv 0$. For every locally-integrable function $f$ on $\mathbb{R}^n$, one has $|f| \leq Mf$ almost everywhere. The most important information (for our purpose) concerning the operator $M$, now classical, states that there exist positive constants $c,c'$, depending only on $n$, such that

\begin{equation}
c(Mf)^+(t) \leq f^{**}(t) \leq c'(Mf)^+(t), \quad t \in (0,\infty),
\end{equation}

for every locally-integrable function $f$ on $\mathbb{R}^n$. The first inequality in (3.1) was established during the 1930s in works of R.M. Gabriel [29], F. Riesz [51] and N. Wiener [57], while the second was added later through the efforts of C. Herz [34] (for one dimension) and C. Bennett and R. Sharpley [6] (for higher dimensions). The result is summarized and proved in [7, Chapter 3, Theorem 3.8].

We shall now state the first principal result of this section, in which we characterize the optimal range partner to a given space with respect to the operator $M$.

**Theorem 3.1.** Let $X$ be a rearrangement-invariant space over $\mathbb{R}^n$ such that

\begin{equation}
\psi \in X'(0,\infty),
\end{equation}

where $\psi(t) = \chi_{(0,1)}(t) \log \frac{1}{t}, \ t \in (0,\infty)$. Define the functional $\sigma$ by

$$\sigma(f) = \left\| \int_0^\infty f^*(s) \frac{ds}{s} \right\|_{X'(0,\infty)}, \quad f \in M_+(\mathbb{R}^n).$$

Then $\sigma$ is a rearrangement-invariant norm and

\begin{equation}
M: X \rightarrow Y,
\end{equation}

for every pair of rearrangement-invariant spaces $X,Y$ over $(0,\infty)$ (with the same operator norm). Another important example is that when $(R,\mu)$ is arbitrary and both $T$ and $T'$ are identity operators. Then (2.8) is trivially satisfied and, as a consequence, one gets

\begin{equation}
X \hookrightarrow Y \iff Y' \hookrightarrow X'
\end{equation}

for every pair of rearrangement-invariant spaces $X,Y$, again with the same embedding constant (see [7, Chapter 1, Proposition 2.10]).

We will say that a rearrangement-invariant space $Y$ over $(S,\nu)$ is a range partner for a given rearrangement-invariant space $X$ over $(R,\mu)$ with respect to a sublinear operator $T$ if $T: X \rightarrow Y$. We say that $Y$ is the optimal range partner for $X$ if one has $Y \hookrightarrow Z$ for every range partner $Z$ for $X$ with respect to $T$. We analogously define a domain partner and the optimal domain partner, that is, the largest possible domain space.

Throughout the paper the convention that $1\infty = 0$, and $0\cdot\infty = 0$ is used without further explicit reference. We write $A \approx B$ when the ratio $A/B$ is bounded from below and from above by positive constants independent of appropriate quantities appearing in expressions $A$ and $B$.

3. The Hardy-Littlewood maximal operator

In this section, the relevant rearrangement-invariant spaces are considered over $\mathbb{R}^n$ endowed with the $n$-dimensional Lebesgue measure. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$.

The Hardy–Littlewood maximal operator, $M$, is defined for every locally integrable function $f$ on $\mathbb{R}^n$ and every $x \in \mathbb{R}^n$ by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is extended over all cubes $Q \subset \mathbb{R}^n$, whose edges are parallel to the coordinate axes of $\mathbb{R}^n$, that contain $x$.

The operator $M$ is merely sublinear, rather than linear, and it is clearly a contraction on $L^{\infty}$. On the other hand, $Mf$ is never integrable unless $f \equiv 0$. For every locally-integrable function $f$ on $\mathbb{R}^n$, one has $|f| \leq Mf$ almost everywhere. The most important information (for our purpose) concerning the operator $M$, now classical, states that there exist positive constants $c,c'$, depending only on $n$, such that

\begin{equation}
c(Mf)^+(t) \leq f^{**}(t) \leq c'(Mf)^+(t), \quad t \in (0,\infty),
\end{equation}

for every locally-integrable function $f$ on $\mathbb{R}^n$. The first inequality in (3.1) was established during the 1930s in works of R.M. Gabriel [29], F. Riesz [51] and N. Wiener [57], while the second was added later through the efforts of C. Herz [34] (for one dimension) and C. Bennett and R. Sharpley [6] (for higher dimensions). The result is summarized and proved in [7, Chapter 3, Theorem 3.8].

We shall now state the first principal result of this section, in which we characterize the optimal range partner to a given space with respect to the operator $M$.

**Theorem 3.1.** Let $X$ be a rearrangement-invariant space over $\mathbb{R}^n$ such that

\begin{equation}
\psi \in X'(0,\infty),
\end{equation}

where $\psi(t) = \chi_{(0,1)}(t) \log \frac{1}{t}, \ t \in (0,\infty)$. Define the functional $\sigma$ by

$$\sigma(f) = \left\| \int_0^\infty f^*(s) \frac{ds}{s} \right\|_{X'(0,\infty)}, \quad f \in M_+(\mathbb{R}^n).$$

Then $\sigma$ is a rearrangement-invariant norm and

\begin{equation}
M: X \rightarrow Y,
\end{equation}
where \( Y = Y(\sigma') \). Moreover, \( Y \) is the optimal (smallest) rearrangement-invariant space for which (3.3) holds.

Conversely, if (3.2) is not true, then there does not exist a rearrangement-invariant space \( Y \) for which (3.3) holds.

We now turn our attention to the question of optimal domain space when the target space is prescribed. This situation is considerably simpler than the reverse one as no associate norms need to be involved.

**Theorem 3.2.** Let \( Y \) be a rearrangement-invariant space over \( \mathbb{R}^n \) such that
\[
(3.4) \quad \psi \in Y(0, \infty),
\]
where \( \psi(t) = \min\{1, \frac{t}{\alpha}\} \) for \( t \in (0, \infty) \). Define the functional \( \varphi \) by
\[
\varphi(f) = \|f^*\|_{Y(0, \infty)}, \quad f \in \mathcal{M}_+(\mathbb{R}^n).
\]
Then \( \varphi \) is a rearrangement-invariant norm and (3.3) is satisfied, where \( X = X(\varphi) \). Moreover, \( X \) is the optimal (largest) rearrangement-invariant space for which (3.3) holds.

Conversely, if (3.4) is not true, then there does not exist a rearrangement-invariant space \( X \) for which (3.3) holds.

In our final result of this section we present a collection of nontrivial examples based on Lorentz-Zygmund spaces.

**Theorem 3.3.** Let \( p, q \in [1, \infty], \ A \in \mathbb{R}^2 \). Then
\[
(3.5a) \quad M : L^{p,q;A} \to \begin{cases} L^{1,1,1;A-1}, & p = 1, q = 1, \alpha_0 \geq 1, \alpha_\infty < -1, \\ Y, & p = 1, q = 1, \alpha_0 \geq 1, -1 \leq \alpha_\infty \leq 0, \\ L^{p,q;A}, & 1 < p < \infty \text{ or } p = \infty, 1 \leq q < \infty, \alpha_0 + \frac{1}{q} < 0 \text{ or } p = \infty, q = \infty, \alpha_0 \leq 0, \end{cases}
\]
where \( Y \) is the (unique) rearrangement-invariant space whose associate space \( Y' \) satisfies
\[
\|f\|_{Y'} = \sup_{0 < t < \infty} \int_t^\infty f^*(s) \frac{ds}{s}, \quad f \in \mathcal{M}_+(\mathbb{R}^n).
\]
These spaces are the optimal range partners with respect to \( M \).

We note that the space \( Y' \), given in terms of an operator-induced norm, cannot be expressed in terms of a Lorentz-Zygmund norm.

We shall now proceed to prove the stated results.

**Proof of Theorem 3.1.** The functional \( \sigma \) is obviously rearrangement invariant and, thanks to the Monotone Convergence Theorem, it satisfies the lattice axiom and the Fatou axiom. From (P1), only the triangle inequality needs proving. Let \( f, g \in M(\mathbb{R}^n) \). By the definition of the associate space, one has
\[
\left\| \int_t^\infty (f + g)^*(s) \frac{ds}{s} \right\|_{X'(0, \infty)} = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty h(t) \int_t^\infty (f + g)^*(s) \frac{ds}{s} dt.
\]
Since the function
\[
t \mapsto \int_t^\infty (f + g)^*(s) \frac{ds}{s}
\]
is nonincreasing on \((0, \infty)\), we in fact have (cf. (2.7))
\[
\left\| \int_t^\infty (f + g)^*(s) \frac{ds}{s} \right\|_{X'(0, \infty)} = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty h^*(t) \int_t^\infty (f + g)^*(s) \frac{ds}{s} dt.
\]
Thus, by the Fubini theorem,
\[ \left\| \int_t^\infty \frac{(f + g)^*(s)}{s} \, ds \right\|_{X'(0,\infty)} = \sup_{\|h\|_{X(0,\infty)} \leq 1} \int_0^\infty (f + g)^*(s)h^*(s) \, ds. \]

By (2.1) and the Hardy lemma, one has, for every constant \( h \),
\[ \int_0^\infty (f + g)^*(s)h^*(s) \, ds \leq \int_0^\infty f^*(s)h^*(s) \, ds + \int_0^\infty g^*(s)h^*(s) \, ds. \]

This estimate, combined with the preceding identity and the subadditivity of the supremum, finally yields
\[ \left\| \int_t^\infty \frac{(f + g)^*(s)}{s} \, ds \right\|_{X'(0,\infty)} \leq \left\| \int_t^\infty \frac{f^*(s)}{s} \, ds \right\|_{X'(0,\infty)} + \left\| \int_t^\infty \frac{g^*(s)}{s} \, ds \right\|_{X'(0,\infty)}, \]

establishing the triangle inequality for \( \sigma \).

As for (P4), let \( E \subset \mathbb{R}^n \) be a set of finite measure. We need to prove that
\[ \left\| \int_0^\infty \frac{\chi_E^*(-) \, ds}{s} \right\|_{X'(0,\infty)} < \infty. \]

Since \( \chi_E^* = \chi_{(0,|E|)} \), this amounts to showing the finiteness of the quantity
\[ \left\| \chi_{(0,|E|)}(t) \int_t^{|E|} \frac{ds}{s} \right\|_{X'(0,\infty)} = \left\| \chi_{(0,|E|)}(t) \log \frac{|E|}{t} \right\|_{X'(0,\infty)}. \]

As \( D_{|E|}(\chi_{(0,|E|)}(t) \log \frac{|E|}{t}) = \chi_{(0,1)}(t) \log \frac{1}{t} \), and the dilation operator \( D_{|E|} \) is bounded on \( X'(0,\infty) \), we obtain that \( \left\| \chi_{(0,|E|)}(t) \log \frac{|E|}{t} \right\|_{X'(0,\infty)} \) is finite if and only if \( |E| \leq 2 \), which, however, is guaranteed by the assumption. This shows (P4).

Finally, to verify (P5), let \( f \in M_+(\mathbb{R}^n) \) and let \( E \subset \mathbb{R}^n \) be of finite measure. Then, by the monotonicity of \( f^* \), we obtain
\[ \sigma(f) \geq \left\| \int_t^{|E|} f^*(s) \frac{ds}{s} \right\|_{X'(0,\infty)} \geq \left\| f^*(2t) \right\|_{X'(0,\infty)} = \| f^*(2t) \|_{X'(0,\infty)} \log 2. \]

Since \( X' \) itself is a rearrangement-invariant space, it satisfies (P5). In other words, there is a constant \( C_E \), \( 0 < C_E < \infty \), independent of \( f \), such that
\[ \int_E f \, d\mu \leq C_E \| f \|_{X'}. \]

By the rearrangement invariance of the space \( X' \) and the boundedness of the dilation operator on \( X'(0,\infty) \), we finally get from the preceding estimates that
\[ \int_E f \, d\mu \leq C_E \| f^* \|_{X'(0,\infty)} \leq \tilde{C}_E \| f^*(2t) \|_{X'(0,\infty)} \leq \frac{\tilde{C}_E}{\log 2} \sigma(f) \]

for some constant \( \tilde{C}_E \), \( 0 < \tilde{C}_E < \infty \), independent of \( f \). This shows that \( \sigma \) satisfies (P5) and, altogether, that \( \sigma \) is a rearrangement-invariant norm.

We shall now show that \( M : X \rightarrow Y \). Recall that
\[ \left\| \int_t^\infty g^*(s) \frac{ds}{s} \right\|_{Y'(0,\infty)} = \| g \|_{Y'(0,\infty)} \quad \text{for every } g \in M_+(0, \infty). \]
The next step is getting rid of the star in the last inequality, which can be done thanks to the equivalence of (2.5) and (2.6). We conclude that there exists a positive constant $C$ such that

$$
\left\| \int_t^\infty g(s) \, ds \right\|_{X'(0,\infty)} \leq C \|g\|_{Y'(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

We emphasize that this step (the fall of a star) is quite deep and that it does not follow from the Hardy–Littlewood inequality (as it might deceptively appear) because the integration takes place far away from zero. Once the inequality is unrestricted to monotone functions, we are entitled to apply the standard argument using associate spaces. Using (2.9), we get

$$
\left\| \frac{1}{t} \int_0^t g(s) \, ds \right\|_{Y(0,\infty)} \leq C \|g\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty),
$$

with the constant $C$ undamaged. Now we need our star back, but this time that is achieved easily. We just restrict the last inequality to the cone of nonincreasing functions and obtain

$$
\left\| \frac{1}{t} \int_0^t g^*(s) \, ds \right\|_{Y(0,\infty)} \leq C \|g^*\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

Applying the rearrangement invariance of the space $X$ and using the correspondence between a rearrangement-invariant space and its representation space, we readily see that this can be rewritten as

$$
\left\| \frac{1}{t} \int_0^t f^*(s) \, ds \right\|_{Y(0,\infty)} \leq C \|f\|_X \quad \text{for every } f \in \mathcal{M}(\mathbb{R}^n).
$$

By the first inequality in (3.1), we obtain that there exists a positive constant $C'$ such that

$$
\|(Mf)^*\|_{Y(0,\infty)} \leq C' \|f\|_X \quad \text{for every } f \in L_{\text{loc}}^1(\mathbb{R}^n).
$$

Finally, the rearrangement invariance of the space $Y$ yields

$$
\|Mf\|_Y \leq C' \|f\|_X \quad \text{for every } f \in L_{\text{loc}}^1(\mathbb{R}^n).
$$

In other words, $M: X \to Y$.

We shall now establish the optimality property of $Y$. To this end, assume that, for some rearrangement-invariant space $Z$ over $\mathbb{R}^n$, we have $M: X \to Z$. This means that there exists a positive constant $C$ such that for every $f \in L_{\text{loc}}^1(\mathbb{R}^n)$ the inequality

$$
\|Mf\|_Z \leq C \|f\|_X
$$

holds. Translated to the world of rearrangements, this reads

$$
\|(Mf)^*\|_{Z(0,\infty)} \leq C \|f^*\|_{X(0,\infty)}.
$$

Using the second inequality in (3.1), we get

$$
\left\| \frac{1}{t} \int_0^t f^*(s) \, ds \right\|_{Z(0,\infty)} \leq C' \|f^*\|_{X(0,\infty)} \quad \text{for every } f \in L_{\text{loc}}^1(\mathbb{R}^n),
$$

with some positive constant $C'$. A special case of the Hardy–Littlewood inequality together with (P2) for $Z$ now yields

$$
\left\| \frac{1}{t} \int_0^t g(s) \, ds \right\|_{Z(0,\infty)} \leq \left\| \frac{1}{t} \int_0^t g^*(s) \, ds \right\|_{Z(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

Thus, since $\|g\|_{X(0,\infty)} = \|g^*\|_{X(0,\infty)}$, we have

$$
\left\| \frac{1}{t} \int_0^t g(s) \, ds \right\|_{Z(0,\infty)} \leq C' \|g\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$
By (2.10), this is nothing else than
\[ \left\| \int_t^\infty g(s) \frac{ds}{s} \right\|_{L^q(0,\infty)} \leq C^q \|g^*\|_{L^q(0,\infty)} \quad \text{for every } g \in M_+(0, \infty). \]
Restricting this inequality to nonincreasing functions, we get
\[ \left\| \int_t^\infty g^*(s) \frac{ds}{s} \right\|_{L^q(0,\infty)} \leq C^q \|g^*\|_{L^q(0,\infty)} \quad \text{for every } g \in M_+(0, \infty). \]
By the definition of \( Y^\prime \) and by the rearrangement invariance of \( L^q(0,\infty) \), this can be rewritten as
\[ \|g\|_{Y^\prime} \leq C^q \|g^*\|_{Z^\prime} \quad \text{for every } g \in M_+(\mathbb{R}^n). \]
In other words, we have established the embedding \( Z^\prime \hookrightarrow Y^\prime \), which is, due to (2.10), equivalent to \( Y \hookrightarrow Z \). This shows that \( Y \) is indeed the optimal range partner for \( X \) with respect to \( M \).
Finally, assume that \( \psi \notin X^\prime(0,\infty) \) and suppose that \( M : X \rightarrow Y \) for some \( Y \). Then, following the same line of argument as above, we obtain that
\[ \|Qg^*\|_{Y^\prime(0,\infty)} \leq C \|g\|_{Y^\prime(0,\infty)}, \quad g \in M_+(0, \infty), \]
with some \( C, 0 < C < \infty, \) independent of \( g \). Inserting, in particular, \( g = \chi_{(0,1)} \), we obtain that the right side of the last inequality is finite, since \( Y^\prime \) is a rearrangement-invariant space, and, as such, it must obey the axiom (P4). The left side is however infinite, because we have
\[ \|Q\chi_{(0,1)}^*\|_{X^\prime(0,\infty)} = \|\psi\|_{X^\prime(0,\infty)} = \infty. \]
This is absurd, hence there is no such \( Y \). The proof is complete.

\begin{proof}[Proof of Theorem 3.2] The functional \( g \) obviously obeys (P1), (P2), (P3) and (P6). In particular, the triangle inequality follows immediately from the triangle inequality for \( Y(0,\infty) \) and (2.11). Thanks to the boundedness of the dilation operator on \( Y(0,\infty) \), (P4) is equivalent to \( \chi_{(0,1)}^* \in Y(0,\infty) \), which is however guaranteed by the assumption of the theorem, since \( \chi_{(0,1)}^* = \psi \). Finally, (P5) follows easily from the chain
\[ \rho(g) \geq \|g^*\chi_{(0,1)}\|_{Y(0,\infty)} \geq \|g^*(|E|)\|_Y \geq |E|\|\chi_{(0,1)}\|_{Y(0,\infty)} \int_E g(x) \, dx, \]
where \( E \subset \mathbb{R}^n \) is an arbitrary set of finite measure and \( g \in M_+(\mathbb{R}^n) \). We used the monotonicity of \( g^* \) and the Hardy–Littlewood inequality. The operator \( M \) is obviously bounded from \( X \) to \( Y \) thanks to (3.1). The optimality of \( X \) follows from the following simple argument. Suppose that \( M : Z \rightarrow Y \) for some rearrangement-invariant space \( Z \). Then \( \|Mf\|_Y \leq C \|f\|_Z \) for some \( C > 0 \) and all \( f \in Z \). Therefore, by (3.1) once again, we have \( \|f^*\|_Y \leq C \|f\|_Z \), which, however, is nothing else than the embedding \( Z \hookrightarrow X \). Finally, if \( \psi \notin Y(0,\infty) \) then there is no domain partner for \( Y \) with respect to \( M \), because if there was one, say, \( X \), then one would have in particular \( \|\chi_{(0,1)}^*\|_{Y(0,\infty)} \leq C \|\chi_{(0,1)}\|_{X(0,\infty)} \), but the right-hand side is finite due to (P4) for \( X \) and the left-hand side is equal to infinity since \( \psi \notin Y(0,\infty) \). The proof is complete.
\end{proof}

\begin{proof}[Proof of Theorem 3.3] We first recall that if for a rearrangement-invariant space \( X \) one has \( M : X \rightarrow X \), then automatically \( X \) is the optimal range (and domain) partner for itself with respect to \( M \). This immediately follows from the inequality \( f^{**} \geq f^* \) combined with (3.1).
Now [47, Theorem 3.8] together with (3.1) implies that \( M : L^{p,q;A} \rightarrow L^{p,q;A} \) when either \( 1 < p < \infty \) or \( p = \infty \), \( 1 \leq q < \infty \) and \( a_0 + \frac{1}{q} < 0 \) or \( p = \infty \), \( q = \infty \) and \( a_0 \leq 0 \). This proves the assertion in all cases except (3.5a) and (3.5b).

Assume now that $p = 1$, $q = 1$, $\alpha_0 \geq 1$ and $\alpha_{\infty} \leq 0$. By \cite[Theorem 7.1]{47}, $L^{1,1;\mathbb{A}}$ is equivalent to a rearrangement-invariant space. Moreover, by \cite[Theorem 6.6]{47}, $(L^{1,1;\mathbb{A}})' = L^{\infty,\infty;\mathbb{A}^{-1}}$. Thus, one has

$$\|\psi\|_{X'(0,\infty)} \approx \sup_{0<t<1} (1 - \log t)^{1-\alpha_0}(t) < \infty,$$

since $\alpha_0 \geq 1$. In other words, $\psi \in X'(0,\infty)$. Consequently, by Theorem 3.1, the optimal range partner $Y$ for $L^{1,1;\mathbb{A}}$ with respect to $M$ satisfies

$$f \approx \frac{\|f\|_{Y'}}{\|f\|_{L^\infty,\infty;\mathbb{A}^{-1}}}.$$

This establishes (3.5b).

It remains to prove (3.5a). To do this we have to show that, for this choice of parameters, the space $Y$ whose associate space has norm given by (3.3) coincides with $L^{1,1;\mathbb{A}^{-1}}$. We have

$$\|f\|_{Y'} = \sup_{0<t<\infty} \ell^\mathbb{A}_{-1}(t) \int_{t}^{\infty} f^*(s) \frac{ds}{s} \approx \sup_{0<t<\infty} \ell^\mathbb{A}_{-1}(t) \int_{t}^{\infty} f^*(s) \ell^\mathbb{A}+1(s) \frac{ds}{s} \leq \left( \sup_{0<s<\infty} f^*(s) \ell^\mathbb{A}+1(s) \right) \left( \sup_{0<t<\infty} \ell^\mathbb{A}_{-1}(t) \int_{t}^{\infty} \ell^\mathbb{A}+1(s) \frac{ds}{s} \right).$$

and, conversely,

$$\|f\|_{Y'} \geq \max \left\{ \sup_{0<t<1} (1 - \log t)^{-\alpha_0}(t) \int_{t}^{\sqrt{t}} f^*(s) \frac{ds}{s}, \sup_{1<t<\infty} (1 + \log t)^{-\alpha_{\infty}}(t) \int_{t}^{t^2} f^*(s) \frac{ds}{s} \right\} \geq \max \left\{ \sup_{0<t<1} (1 - \log t)^{-\alpha_0}(t) f^*(\sqrt{t}) \log(t^{-\frac{1}{2}}), \sup_{1<t<\infty} (1 + \log t)^{-\alpha_{\infty}}(t) f^*(t^2) \log t \right\} \approx \max \left\{ \sup_{0<t<1} (1 - \log t)^{1-\alpha_0}(\sqrt{t}) f^*(\sqrt{t}), \sup_{1<t<\infty} (1 + \log t)^{1-\alpha_{\infty}}(t^2) f^*(t^2) \right\} \approx \max \left\{ \sup_{0<t<1} (1 - \log t)^{1-\alpha_0}(t) f^*(t), \sup_{1<t<\infty} (1 + \log t)^{1-\alpha_{\infty}}(t) f^*(t) \right\} \approx \|f\|_{L^{\infty,\infty;\mathbb{A}^{-1}}}.$$

Therefore, $Y' = L^{\infty,\infty;\mathbb{A}^{-1}}$, and, finally, by \cite[Theorem 6.2]{47}, we get $Y = L^{1,1;\mathbb{A}^{-1}}$, as desired.

\section{4. The fractional maximal operator}

In this section we shall treat the fractional maximal operator $M_\gamma$, defined for a fixed $\gamma \in (0, n)$ and for every locally integrable function on $\mathbb{R}^n$ by

$$M_\gamma f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-n/\gamma}} \int_Q |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

The operator $M_\gamma$ can be defined in the same way also for $\gamma = 0$, in which case it coincides with the Hardy–Littlewood maximal operator, and constitutes thereby its natural generalization. The two types of operators nevertheless have to be treated separately because their behaviour in cases $\gamma = 0$ and $\gamma > 0$ is, rather surprisingly, substantially different, and, in the fractional case, a new approach involving a specific supremum operator is needed for the study of the optimal action of the operator on function spaces. Since the supremum operator is not linear, the use of
techniques based on associate norms and spaces is somewhat limited, and a certain care has to be exercised.

The result of [18, Theorem 1.1] shows that there exists a positive constant \( C \) depending only on \( \gamma \) and \( n \) such that, for every \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), one has

\[
(M_\gamma f)^*(t) \leq C \sup_{t \leq s < \infty} \frac{s^n}{s} f^{**}(s) \quad \text{for every } t \in (0, \infty),
\]

and, conversely, for every nonincreasing function \( g \) on \((0, \infty)\) there exists some \( f_0 \in L^1_{\text{loc}}(\mathbb{R}^n) \) such that \( f_0^* = g \) almost everywhere on \((0, \infty)\) and

\[
(M_\gamma f_0)^*(t) \geq c \sup_{t \leq s < \infty} \frac{1}{s^n} g^{**}(s) \quad \text{for every } t \in (0, \infty),
\]

where, again, \( c \) is some positive constant which depends only on \( \gamma \) and \( n \). For \( \gamma = 0 \), the combination of (4.2) and (4.1) coincides with (3.1), since the function \( g^{**} \) is nonincreasing on \((0, \infty)\) for any \( g \).

**Theorem 4.1.** Let \( X \) be a rearrangement-invariant space over \( \mathbb{R}^n \). Let \( \gamma \in (0, n) \) and assume that

\[
\inf_{1 \leq t < \infty} \frac{\varphi_X(t)^{\frac{n}{n-\gamma}}}{t} > 0.
\]

Define the functional \( \sigma \) by

\[
\sigma(f) = \sup_{h \sim f, h \geq 0} \left\| \int_1^\infty h(s) s^{\frac{n}{n-\gamma}-1} ds \right\|_{X'(0, \infty)}, \quad f \in M_+(\mathbb{R}^n),
\]

where the supremum is taken over all \( h \in M_+(\mathbb{R}^n) \) equimeasurable with \( f \). Then \( \sigma \) is a rearrangement-invariant norm and

\[
M_\gamma : X \to Y,
\]

where \( Y = Y(\sigma') \). Moreover, \( Y \) is the optimal (smallest) rearrangement-invariant space for which (4.5) holds.

Conversely, if (1.3) is not true, then there does not exist a rearrangement-invariant space \( Y \) for which (4.5) holds.

The expression for the functional \( \sigma \) in Theorem 4.1 is somewhat implicit. Our next result however shows that it can be considerably simplified at a relatively low cost. We shall need a supremum operator. For a fixed \( \alpha \geq 0 \), define the operator \( T_\alpha \) on \( M(0, \infty) \) by

\[
T_\alpha f(t) = t^{-\alpha} \sup_{t \leq s < \infty} s^\alpha f^*(s), \quad t \in (0, \infty).
\]

**Theorem 4.2.** Let \( 0 < \gamma < n \) and let \( X \) be a rearrangement-invariant space over \( \mathbb{R}^n \). Assume that

\[
T_\frac{\gamma}{n} : X(0, \infty) \to X(0, \infty).
\]

Define the functional \( \tau \) by

\[
\tau(f) = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty f^*(s)(PT_\frac{\gamma}{n} h)(s) s^{\frac{n}{n-\gamma}} ds.
\]

Then \( \tau \) is a rearrangement-invariant norm such that

\[
M_\gamma : X \to Y,
\]
where \( Y = Y(\tau') \), and \( Y \) is the optimal (smallest) rearrangement-invariant space for which (4.5) holds. Moreover, \( \tau \) is equivalent to the functional

(4.7) \[ f \mapsto \left\| \int_{t}^{\infty} f^*(s) s^{\frac{n}{n-\gamma}} ds \right\|_{X'(0, \infty)}, \quad f \in \mathcal{M}_+(\mathbb{R}^n). \]

**Remark 4.3.** The assumption (4.6) of Theorem 4.2 is natural in view of the fact that the classical endpoint mapping properties for the fractional maximal operator \( M_{\frac{n}{n-\gamma}} \) are of the form

\[ M_{\frac{n}{n-\gamma}} : L^1 \rightarrow L_{\frac{n}{n-\gamma}, \infty}^{\frac{n}{n-\gamma}} \quad \text{and} \quad M_{\frac{n}{n-\gamma}} : L_{\infty}^{\frac{n}{n-\gamma}} \rightarrow L^\infty, \]

while those of \( T_{\frac{n}{n-\gamma}} \) are (cf. [36, 37, 32])

\[ T_{\frac{n}{n-\gamma}} : L^1 \rightarrow L^1 \quad \text{and} \quad T_{\frac{n}{n-\gamma}} : L_{\infty}^{\frac{n}{n-\gamma}} \rightarrow L_{\infty}^{\frac{n}{n-\gamma}}. \]

On the other hand, (4.6) is strictly stronger than (4.3). Indeed, assume that (4.6) is satisfied. Then, in particular, there exists a positive constant, \( K \), such that for every \( a \geq 1 \) one has

\[ \| T_{\frac{n}{n-\gamma}} \chi_{(0,a)} \|_{X(0, \infty)} \leq K \| \chi_{(0,a)} \|_{X(0, \infty)}. \]

Since

\[ T_{\frac{n}{n-\gamma}} \chi_{(0,a)}(t) = \chi_{(0,a)}(t) a^{-\frac{n}{n-\gamma}} t^{-\frac{n}{n-\gamma}} \quad \text{for } t \in (0, \infty), \]

we in fact have

\[ \| \chi_{(0,a)}(t) a^{-\frac{n}{n-\gamma}} t^{-\frac{n}{n-\gamma}} \|_{X(0, \infty)} \leq K \varphi_X(a). \]

Consequently,

\[ \varphi_X(a) a^{-\frac{n}{n-\gamma}} \geq K^{-1} \| \chi_{(0,a)}(t) t^{-\frac{n}{n-\gamma}} \|_{X(0, \infty)} \geq K^{-1} \| \chi_{(0,1)}(t) t^{-\frac{n}{n-\gamma}} \|_{X(0, \infty)}. \]

Hence

\[ \inf_{1 \leq a < \infty} \varphi_X(a) a^{-\frac{n}{n-\gamma}} \geq K^{-1} \| \chi_{(0,1)}(t) t^{-\frac{n}{n-\gamma}} \|_{X(0, \infty)} > 0, \]

and (4.3) follows. This shows the implication (4.6) \( \Rightarrow \) (4.3). The fact that this implication cannot be reversed follows on considering \( X = L_{\frac{n}{n-\gamma}, q} \) with \( q \in [1, \infty) \). Every such space obviously satisfies (4.3), but it follows from [32, Theorem 3.2] that the operator \( T_{\frac{n}{n-\gamma}} \) is not bounded on it, hence (4.6) does not hold.

For the optimal domain for the fractional maximal operator, we have the following result. Its proof is analogous to that of Theorem 3.2 and therefore is omitted.

**Theorem 4.4.** Let \( 0 < \gamma < n \) and let \( Y \) be a rearrangement-invariant space over \( \mathbb{R}^n \) such that

(4.8) \[ \psi \in Y(0, \infty), \]

where \( \psi(t) = (1 + t)^{-\frac{n}{n-\gamma}} \), \( t \in (0, \infty) \). Define the functional \( \sigma \) by

\[ \sigma(f) = \left\| t^{\frac{n}{n-\gamma}} f^*(t) \right\|_{Y'(0, \infty)}, \quad f \in \mathcal{M}_+(\mathbb{R}^n). \]

Then \( \sigma \) is a rearrangement-invariant norm and

(4.9) \[ M_{\gamma} : X \rightarrow Y, \]

where \( X = X(\sigma) \). Moreover, \( X \) is the optimal (largest) rearrangement-invariant space for which (4.9) holds.

Conversely, if (4.8) is not true, then there does not exist a rearrangement-invariant space \( X \) for which (4.9) holds.
Our next aim is to present an array of results concerning the optimal range partners for Lorentz-Zygmund spaces of the form $L^{p,q;\mathbb{A}}$ with respect to $M_\gamma$. Mapping properties of $M_\gamma$ on Lorentz–Zygmund spaces were studied in \cite{26}, where the following results were established:

\[
M_\gamma : L^{p,q;\mathbb{A}} \to \begin{cases}
L^{n^{-1};\mathbb{A}^{-1}} & p = 1, q = 1, \alpha_0 \geq 0, \alpha_\infty < 0, \\
L^{n^{-\infty};\mathbb{A}} & p = 1, q = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \\
L^{n^{-q};\mathbb{A}} & 1 < p < \frac{n}{\gamma}, 1 \leq q \leq \infty, \\
L^{n;\mathbb{A}^{-1}} & p = \frac{n}{\gamma}, 1 \leq q \leq \infty, \quad \alpha_0 < 0, \alpha_\infty > 0.
\end{cases}
\]

Our result concerning optimal range spaces for Lorentz–Zygmund spaces reads as follow.

**Theorem 4.5.** Let $\gamma \in (0,n)$, $p,q \in [1,\infty]$, $\mathbb{A} \in \mathbb{R}^2$. Then

\[(4.10a)\]

\[
M_\gamma : L^{p,q;\mathbb{A}} \to \begin{cases}
Y_1 & p = 1, q = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \\
L^{n^{-q};\mathbb{A}} & 1 < p < \frac{n}{\gamma}, \\
Y_2 & p = \frac{n}{\gamma}, 1 \leq q < \infty, \quad \alpha_\infty \geq 0 \text{ or } \alpha_\infty > 0
\end{cases}
\]

where $Y_1$ and $Y_2$ are (unique) rearrangement-invariant spaces whose associate spaces, $Y_1'$ and $Y_2'$, satisfy

\[
\|f\|_{Y_1'} = \sup_{0 < t < \infty} e^{-\mathbb{A}t} \int_t^\infty f^*(s)s^{\alpha_\infty - 1} ds, \quad f \in \mathcal{M}_+(\mathbb{R}^n),
\]

and

\[
\|f\|_{Y_2'} = \sup_{h \in \mathfrak{F}} \left\| t^{n^{-q} - 1} e^{-\mathbb{A}t} \int_t^\infty h(s)s^{\alpha_\infty - 1} ds \right\|_{L^p(0,\infty)}, \quad f \in \mathcal{M}_+(\mathbb{R}^n),
\]

respectively. In particular, in the case $\mathbb{A} = [0,0]$, we have $Y_1 = L^{n^{-\infty};\gamma}$ and $Y_2 = L^\infty$.

Moreover, these spaces are the optimal range partners with respect to $M_\gamma$.

Again, there is no simpler way of characterizing the spaces $Y_1'$ and $Y_2'$.

**Remark 4.6.** We note that the range spaces in Theorem 4.5 essentially improve those from \cite{26} when $p = q = 1$, $\alpha_0 \geq 0$, $\alpha_\infty \leq 0$ and $|\alpha_0| + |\alpha_\infty| > 0$, and also when $p = \frac{n}{\gamma}$, $1 \leq q < \infty$, $\alpha_0 < 0$ and $\alpha_\infty > 0$. It is also worth noting that the spaces $L^{n^{-1};\mathbb{A}^{-1}}$ and $L^{n^{-\infty};\mathbb{A}}$ are not comparable in the sense that neither of them is contained in the other (see \cite{47} for details).

Our next aim is to describe in more detail the relation between Theorems 4.1 and 4.2. Theorem 4.2 asserts, among other statements, that in the particular cases when \(4.6\) is satisfied, the functionals \(4.7\) and $\sigma$ from \(4.4\) are equivalent. We shall now point out an interesting fact that the converse is also true, namely if \(4.6\) is not satisfied, then the functional in \(4.7\) is not equivalent to $\sigma$ from \(4.4\). That, in fact, means that it is essentially smaller than $\sigma$. This is achieved through the following result, which is definitely of independent interest and maybe even a little surprising.

**Theorem 4.7.** Assume that $X$ is a rearrangement-invariant space over $\mathbb{R}^n$ and $\gamma \in (0,n)$. Then the following statements are equivalent:

(a) $T_\gamma : X(0,\infty) \to X(0,\infty)$,

(b) there exists a positive constant $C$ such that,

\[
\sup_{h \geq 0} \left\| \int_0^\infty h(s)s^{\alpha_\infty - 1} ds \right\|_{X'(0,\infty)} \leq C \left\| \int_0^\infty f^*(s)s^{\alpha_\infty - 1} ds \right\|_{X'(0,\infty)} \quad \text{for every } f \in \mathcal{M}_+(\mathbb{R}^n).
\]
Remark 4.8. We note that, since \( f \sim f^* \), the converse inequality in (4.11), namely
\[
\left\| \int_1^\infty f^*(s)s^{\frac{n-1}{n}} ds \right\|_{X'(0,\infty)} \leq \sup_{h \sim f} \left\| \int_1^\infty h(s)s^{\frac{n-1}{n}} ds \right\|_{X'(0,\infty)},
\]
is trivial. In other words, if (4.11) is true, then the two quantities are in fact equivalent.

In the proof of Theorem 4.7 we shall need the following auxiliary result of independent interest.

Lemma 4.9. Assume that \( I : (0, \infty) \to (0, \infty) \) is a nondecreasing function satisfying
\[
\int_0^t \frac{ds}{I(s)} \approx \int_t^{2t} \frac{ds}{I(s)} \quad \text{for every } t \in (0, \infty).
\]
Let \( N \in \mathbb{N}, 0 < t_1 < \cdots < t_N < \infty \) and \( a_1, \ldots, a_N > 0 \). Let
\[
u = \sum_{i=1}^N a_i \chi_{(0,t_i)}
\]
and let \( X \) be a rearrangement-invariant space. Then
\[
\left\| \int_t^\infty \frac{u(s)}{I(s)} ds \right\|_{X(0,\infty)} \approx \|v\|_{X(0,\infty)},
\]
where
\[
v = \sum_{i=1}^N a_i \frac{t_i}{I(t_i)} \chi_{(0,t_i)}.
\]

Proof. First, we have
\[
\int_t^\infty \frac{u(s)}{I(s)} ds = \sum_{i=1}^N \int_t^{a_i \chi_{(0,t_i)}} \frac{ds}{I(s)} \approx \sum_{i=1}^N a_i \chi_{(0,t_i)}(t) \int_t^{a_i \chi_{(0,t_i)}} \frac{ds}{I(s)} \quad \text{for every } t \in (0, \infty).
\]
By (4.12),
\[
\int_0^{t_i} \frac{ds}{I(s)} \geq \int_t^{t_i} \frac{ds}{I(s)} \geq \int_0^{t_i} \frac{ds}{I(s)} \approx \int_0^{t_i} \frac{ds}{I(s)} \quad \text{for every } t \in (0, \frac{t_i}{2}),
\]
whence
\[
\sum_{i=1}^N a_i \chi_{(0,t_i)}(t) \int_t^{a_i \chi_{(0,t_i)}} \frac{ds}{I(s)} \geq \sum_{i=1}^N a_i \chi_{(0,t_i)}(t) \int_t^{a_i \chi_{(0,t_i)}} \frac{ds}{I(s)} \approx \sum_{i=1}^N a_i \chi_{(0,t_i)}(t) \int_0^{t_i} \frac{ds}{I(s)} \quad \text{for every } t \in (0, \infty).
\]
Therefore, due to the boundedness of the dilation operator on \( X(0,\infty) \), we have
\[
\left\| \int_t^\infty \frac{u(s)}{I(s)} ds \right\|_{X(0,\infty)} \approx \left\| \sum_{i=1}^N a_i \frac{t_i}{I(t_i)} \chi_{(0,t_i)} \right\|_{X(0,\infty)} \approx \left\| \sum_{i=1}^N a_i \frac{t_i}{I(t_i)} \chi_{(0,t_i)} \right\|_{X(0,\infty)} = \|v\|_{X(0,\infty)} \geq \left\| \sum_{i=1}^N a_i \chi_{(0,t_i)}(t) \int_t^{a_i \chi_{(0,t_i)}} \frac{ds}{I(s)} \right\|_{X(0,\infty)} \approx \left\| \sum_{i=1}^N a_i \chi_{(0,t_i)}(t) \int_0^{t_i} \frac{ds}{I(s)} \right\|_{X(0,\infty)} \geq \left\| \int_t^\infty \frac{u(s)}{I(s)} ds \right\|_{X(0,\infty)}.
\]
\[ \square \]
We shall also need a variant of the result obtained in \[34\] Theorem 3.9 on the interval \((0, \infty)\). Here we present a more general claim with a shorter and more comprehensive proof.

In the following lemma, we work with the so-called quasiconcave functions instead of power functions. Recall that a nonnegative function \(\varphi\) defined on \([0, \infty)\) is said to be quasiconcave provided that \(\varphi\) is nondecreasing on \([0, \infty)\), \(\frac{\varphi(t)}{t}\) is nonincreasing on \((0, \infty)\) and \(\varphi(0) = 0\). It follows that \(\varphi\) is absolutely continuous except perhaps at the origin and

\[
\varphi(t) - \varphi(s) \leq \int_s^t \frac{\varphi(r)}{r} \, dr \quad \text{for } s, t \in (0, \infty).
\]

(4.13)

See, \[38\] Chapter II, Lemma 1.1.

**Lemma 4.10.** Let \(\varphi\) be a quasiconcave function. Then there exists a constant \(C > 0\) such that

\[
\int_0^x \sup_{t \leq s < \infty} \varphi(s)f(s) \, dt \leq C \int_0^x (\varphi f)^*(t) \, dt
\]

(4.14)

for every \(x \in (0, \infty)\) and every nonincreasing \(f \in \mathcal{M}_+(0, \infty)\).

Furthermore, if \(X\) is a rearrangement-invariant space over \((0, \infty)\), then

\[
\left\| \sup_{t \leq s < \infty} \varphi(s)f(s) \right\|_{X(0,\infty)} \leq C \left\| \varphi f \right\|_{X(0,\infty)}
\]

(4.15)

for every nonincreasing \(f \in \mathcal{M}_+(0, \infty)\).

**Proof.** Let \(f \in \mathcal{M}_+(0, \infty)\) be a nonincreasing function and fix \(x \in (0, \infty)\). We split the supremum into three parts, namely

\[
\int_0^x \sup_{t \leq s < \infty} \varphi(s)f(s) \, dt \leq \int_0^x \sup_{t \leq s \leq x} \varphi(s)f(s) \, dt + \int_0^x \sup_{x < s < \infty} \varphi(s)f(s) \, dt
\]

\[
\leq \int_0^x \sup_{t \leq s \leq x} \left[ \varphi(s) - \varphi(t) \right] f(s) \, dt
\]

\[
+ \int_0^x \varphi(t) \sup_{x < s < \infty} f(s) \, dt + x \sup_{x < s < \infty} \varphi(s)f(s)
\]

\[
= I + II + III.
\]

By (4.11) and the Hardy–Littlewood inequality, we have

\[
I \leq \int_0^x \sup_{t \leq s \leq x} \left( \int_t^s \frac{\varphi(r)}{r} \, dr \right) f(s) \, dt \leq \int_0^x \sup_{t \leq s \leq x} \int_t^s \frac{\varphi(r)}{r} f(r) \, dr \, dt
\]

\[
= \int_0^x \int_t^x \frac{\varphi(r)}{r} f(r) \, dr \, dt = \int_0^x \int_0^r \frac{\varphi(r)}{r} f(r) \, dr \, dr
\]

\[
= \int_0^x \varphi(r) f(r) \, dr \leq \int_0^x (\varphi f)^*(t) \, dt.
\]

The second term is obviously estimated by the right-hand side of (4.11). Let us consider the third term. Observe that, by (4.13),

\[
\varphi(2t) - \varphi(t) \leq \int_t^{2t} \frac{\varphi(r)}{r} \, dr \leq \varphi(t) \quad \text{for } t \in (0, \infty),
\]

since \(\varphi(t)/t\) is nonincreasing, whence \(\varphi(2t) \leq 2\varphi(t)\) for \(t \in (0, \infty)\). Using this and the fact that \(\varphi\) is nondecreasing, we get

(4.16)

\[
\varphi(t) \leq 2\varphi(t/2) \leq \frac{4}{t} \int_{t/2}^t \varphi(r) \, dr \leq \frac{4}{t} \int_0^t \varphi(r) \, dr \quad \text{for } t \in (0, \infty).
\]
Using (4.16) we obtain

\[
III = x \sup_{x \leq s < \infty} \varphi(s)f(s) \lesssim x \sup_{x \leq s < \infty} \left( \frac{1}{s} \int_{0}^{s} \varphi(r) \, dr \right) f(s) \\
\leq x \sup_{x \leq s < \infty} \frac{1}{s} \int_{0}^{s} \varphi(r) f(r) \, dr \leq x \sup_{x \leq s < \infty} \frac{1}{s} \int_{0}^{s} (\varphi f)^{(s)}(t) \, dt \\
= \int_{0}^{x} (\varphi f)^{(s)}(t) \, dt,
\]

where in the second inequality we used that \( f \) is nonincreasing and the third one is due to the Hardy–Littlewood inequality. The combination of these three estimates gives (4.14). The inequality (4.15) then follows from (4.14) by the Hardy-Littlewood-Pólya principle. \( \square \)

**Proof of Theorem 4.7.** Assume first that (a) is true. Then the associate norm of the optimal r.i. range partner space for \( X \) with respect to \( M \), is equivalent to (4.4) by Theorem 4.1. We note that the assumption (4.3) of this theorem is satisfied since it follows from (a), as was pointed out in Remark 4.3. Combining these two facts, we immediately obtain (b) (see also Remark 4.8).

The converse implication is considerably more involved. Suppose that (b) holds. Then the functional

\[
(4.17) \quad g \mapsto \left\| \int_{t}^{\infty} g^*(s) s^{-\gamma} \, ds \right\|_{X'(0, \infty)}
\]

is equivalent to \( \sigma \) from (4.3), which in turn is known to be a rearrangement-invariant norm thanks to Theorem 4.4. We note that (4.3) is indeed satisfied because it follows from the proof of Theorem 4.4 that it holds if and only if \( \sigma(u) < \infty \) for every nonnegative simple function \( u \), which can be readily verified here thanks to (b). Hence the collection

\[
Y(0, \infty) = \left\{ g \in \mathcal{M}(0, \infty), \left\| \int_{t}^{\infty} g^*(s) s^{-\gamma} \, ds \right\|_{X'(0, \infty)} < \infty \right\},
\]

endowed with the functional

\[
\|g\|_{Y(0, \infty)} = \left\| \int_{t}^{\infty} g^*(s) s^{-\gamma} \, ds \right\|_{X'(0, \infty)}
\]

is equivalent to a rearrangement-invariant space. Define the operator \( H \) on \( \mathcal{M}(0, \infty) \) by

\[
Hg(t) = \int_{t}^{\infty} |g(s)| s^{-\gamma} \, ds, \quad t \in (0, \infty).
\]

Then we have

\[
\|Hg\|_{X'(0, \infty)} = \|g\|_{Y(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
\]

Therefore, using also the equivalence of (2.5) and (2.6), it clearly follows that

\[
(4.18) \quad H : Y(0, \infty) \rightarrow X'(0, \infty)
\]

and that \( Y(0, \infty) \) is the optimal (largest possible) rearrangement-invariant space rendering (4.18) true (in other words, it is the optimal rearrangement-invariant domain partner space for \( X'(0, \infty) \) with respect to the operator \( H \)).

We however claim a considerably less obvious fact, namely that \( X'(0, \infty) \) is also the smallest possible rearrangement-invariant space in (4.18), that is, it is the the optimal rearrangement-invariant range partner space for \( Y(0, \infty) \) with respect to \( H \).

We know that \( H \) is bounded from \( Y(0, \infty) \) to \( X'(0, \infty) \). Therefore we are entitled to denote the optimal rearrangement-invariant range partner for \( Y(0, \infty) \) with respect to \( H \) by \( Y_R(0, \infty) \).
Denote further by $Y_{RD}(0, \infty)$ the optimal rearrangement-invariant domain partner for $Y_R(0, \infty)$ with respect to $H$. Then, using the same reasoning as above, we obtain that
\[
\|g\|_{Y_{RD}(0, \infty)} \approx \left\| \int_0^\infty g^*(s)s^{\frac{-1}{p}} \, ds \right\|_{Y_R(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
\]

Only an easy observation is needed to realize that once a space is the optimal domain partner of some space, then it is necessarily also the optimal domain partner to its own optimal range partner. Indeed, knowing that $Y(0, \infty)$ is optimal in $H : Y(0, \infty) \to Y'(0, \infty)$, assume that $H : Z(0, \infty) \to Y_R(0, \infty)$. By optimality of $Y_R(0, \infty)$ in $H : Y(0, \infty) \to Y_R(0, \infty)$, one necessarily has $Y_R(0, \infty) \to X'(0, \infty)$. Thus, $H : Z(0, \infty) \to X'(0, \infty)$. But, by optimality of $Y(0, \infty)$ in $H : Y(0, \infty) \to X'(0, \infty)$, it follows that $Z(0, \infty) \to Y(0, \infty)$.

Consequently, $Y(0, \infty) = Y_{RD}(0, \infty)$, that is,
\[
\left\| \int_0^\infty g^*(s)s^{\frac{-1}{p}} \, ds \right\|_{X'(0, \infty)} \approx \left\| \int_0^\infty g^*(s)s^{\frac{-1}{p}} \, ds \right\|_{Y_R(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
\]

Assume that $u = \sum_{i=1}^N b_i \chi_{(0,s_i)}$ for some $N \in \mathbb{N}$, $0 < s_1 < \cdots < s_N < \infty$ and $b_1, \ldots, b_N > 0$. Let further

\[
(4.19) \quad v = \sum_{i=1}^N b_i s_i \chi_{(0,s_i)},
\]

where $I(t) = t^{1-\frac{n}{p}}$, $t \in (0, \infty)$. Note that the function $I$ satisfies the assumptions of Lemma 1.9. Therefore we are entitled to use the lemma, whence we get
\[
\|v\|_{X'(0, \infty)} \approx \left\| \int_0^\infty u(s)s^{\frac{-1}{p}} \, ds \right\|_{X'(0, \infty)} \approx \left\| \int_0^\infty u(s)s^{\frac{-1}{p}} \, ds \right\|_{Y_R(0, \infty)} \approx \|v\|_{Y_R(0, \infty)}.
\]

Now, if $f \in \mathcal{M}(\mathbb{R}^n)$, then there is a sequence $\{v_n\}$ of nonnegative simple functions in the form of (4.19) satisfying $v_n \nearrow f^*$. By the Fatou property and the computations above, we get $X'(0, \infty) = Y_{RD}(0, \infty)$. This proves that $X'(0, \infty)$ is indeed the optimal range space in (4.18).

We next claim that
\[
(4.20) \quad \|g\|_{X(0, \infty)} \approx \left\| t^{\frac{n}{p}} g^*(t) \right\|_{Y'(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
\]

Indeed, by the definition of the associate norm, the Fubini theorem and the Hölder inequality, one has, for every $g \in \mathcal{M}(0, \infty)$,
\[
\left\| t^{\frac{n}{p}} g^*(t) \right\|_{Y'(0, \infty)} = \sup_{\|h\|_{Y(0, \infty)} \leq 1} \int_0^\infty |h(t)|t^{\frac{n}{p}} \left( \int_0^t g^*(s) \, ds \right) \, dt
\]
\[
= \sup_{\|h\|_{Y(0, \infty)} \leq 1} \int_0^\infty g^*(s) \left( \int_s^\infty |h(t)|t^{\frac{n}{p}} \, dt \right) \, ds
\]
\[
\leq \sup_{\|h\|_{Y(0, \infty)} \leq 1} \|g\|_{X(0, \infty)} \left( \int_s^\infty |h(t)|t^{\frac{n}{p}} \, dt \right) \left\| t^{\frac{n}{p}} \right\|_{X'(0, \infty)}.
\]

Now, the equivalence of (2.5) and (2.6) implies that
\[
\sup_{\|h\|_{Y(0, \infty)} \leq 1} \left\| \int_s^\infty |h(t)|t^{\frac{n}{p}} \, dt \right\|_{X'(0, \infty)} \leq C \sup_{\|h\|_{Y(0, \infty)} \leq 1} \left( \int_s^\infty |h^*(t)|t^{\frac{n}{p}} \, dt \right) \left\| t^{\frac{n}{p}} \right\|_{X'(0, \infty)} = C.
\]

It might be instructive to note that while this estimate, of course, follows from (b), the validity of (b) is in fact not necessary in order to get it. Altogether, combining the estimates, we get
\[
(4.21) \quad \left\| t^{\frac{n}{p}} g^*(t) \right\|_{Y'(0, \infty)} \leq C \|g\|_{X(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
\]
In order to prove (4.20), we now need to show the converse inequality to (4.21). Denote
\[ \|g\|_{Z(0,\infty)} = \left\| \int_0^\infty g^{**}(t) \right\|_{Y'(0,\infty)}, \quad g \in \mathcal{M}(0,\infty). \]
The functional \( g \mapsto \|g\|_{Z(0,\infty)} \) is a rearrangement-invariant norm. To see this, only (P4) needs proof, since everything else is readily verified. Applying standard techniques, (P4) reduces to (4.22)
\[ t^{\frac{n}{2}} \chi_{[1,\infty)}(t) \in Y'(0,\infty). \]
But, using the equivalence of (2.5) and (2.6) once again, we get
\[ \|t^{\frac{n}{2}} \chi_{[1,\infty)}(t)\|_{Y'(0,\infty)} = \frac{1}{\|X(0,1)\|_{X'(0,\infty)} \|f\|_{Y'(0,\infty)} \leq 1} \left( \int_1^\infty |f(t)|t^{\frac{n}{2}} \chi_{[1,\infty)}(t) dt \right) \]
\[ \leq \frac{1}{\|X(0,1)\|_{X'(0,\infty)} \|f\|_{Y'(0,\infty)} \leq 1} \left( \int s^\infty |f(t)|t^{\frac{n}{2}} \chi_{[1,\infty)}(t) dt \right) \]
\[ \leq \frac{1}{\|X(0,1)\|_{X'(0,\infty)} \|f\|_{Y'(0,\infty)} \leq 1} \left( \int s^\infty \|f\|_{Y'(0,\infty)} dt \right) \]
\[ \leq \frac{1}{\|X(0,1)\|_{X'(0,\infty)} \|f\|_{Y'(0,\infty)} \leq 1} < \infty. \]

We define the operator \( H' \) by
\[ H'g(t) = t^{\frac{n}{2}} \int_0^t |g(s)| ds, \quad g \in \mathcal{M}(0,\infty). \]
Then
\[ H': Z(0,\infty) \to Y'(0,\infty), \]
since, by the Hardy–Littlewood inequality,
(4.23)\[ \|H'g\|_{Y'(0,\infty)} \leq \|H'g^*\|_{Y'(0,\infty)} = \|g\|_{Z(0,\infty)} \quad \text{for every} \; g \in \mathcal{M}(0,\infty). \]
We also have
(4.24)\[ H: Y(0,\infty) \to Z'(0,\infty), \]
since, by the Fubini theorem, the Hölder inequality and (4.23), one has
\[ \|Hg\|_{Z'(0,\infty)} = \sup_{\|f\|_{Z(0,\infty)} \leq 1} \int_0^\infty f(t)Hg(t) dt = \sup_{\|f\|_{Z(0,\infty)} \leq 1} \int_0^\infty |f(t)|Hg(t) dt \]
\[ = \sup_{\|f\|_{Z(0,\infty)} \leq 1} \int_0^\infty H'f(t)|g(t)| dt \leq \|g\|_{Y(0,\infty)} \sup_{\|f\|_{Z(0,\infty)} \leq 1} \|H'f\|_{Y'(0,\infty)} \]
\[ \leq \|g\|_{Y(0,\infty)}. \]
But, as we know, \( Z'(0,\infty) \) is the optimal (smallest) rearrangement-invariant target partner for \( Y(0,\infty) \) with respect to \( H \). Consequently, it must be contained in \( Z'(0,\infty) \). By (2.10), this means that \( Z(0,\infty) \) is continuously embedded into \( X(0,\infty) \). In other words, there exists a positive constant, \( C' \), such that
(4.25)\[ \|g\|_{X(0,\infty)} \leq C'\|g\|_{Z(0,\infty)} = C'\left\| t^{\frac{n}{2}} g^{**}(t) \right\|_{Y'(0,\infty)} \quad \text{for every} \; g \in \mathcal{M}(0,\infty); \]
hence (4.20) follows from the combination of (4.21) and (4.25).
Now we know that $X(0, \infty) = Z(0, \infty)$, so in order to prove (a) it suffices to show that $T^*_\gamma : Z(0, \infty) \to Z(0, \infty)$. In other words, we claim that there exists a positive constant $C$ such that

$$
(4.26) \quad \left\| t^\gamma (T^*_\gamma g)^{**}(t) \right\|_{Y'(0, \infty)} \leq C \left\| t^\gamma g^{**}(t) \right\|_{Y'(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
$$

We first recall that there exists a positive constant $K$ depending only on $n$ and $\gamma$ such that

$$
(4.27) \quad (T^*_\gamma g)^{**}(t) \leq KT^*_\gamma (g^{**})(t) \quad \text{for every } g \in \mathcal{M}(0, \infty) \text{ and } t \in (0, \infty).
$$

Indeed, this follows from [44, Lemma 4.1], where a more general assertion is stated and proved.

Next, it follows from Lemma 4.10 that

$$
\left\| \sup_{t \leq s < \infty} s^\gamma g^s(s) \right\|_{Y'(0, \infty)} \leq C \left\| t^\gamma g^*(t) \right\|_{Y'(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
$$

In particular, since $g^{**}$ is also nonincreasing, we have

$$
(4.28) \quad \left\| \sup_{t \leq s < \infty} s^\gamma g^{**}(s) \right\|_{Y'(0, \infty)} \leq C \left\| t^\gamma g^{**}(t) \right\|_{Y'(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty).
$$

Thus, combining (4.27) and (4.28), we get

$$
\left\| t^\gamma (T^*_\gamma g)^{**}(t) \right\|_{Y'(0, \infty)} \leq K \left\| t^\gamma T^*_\gamma (g^{**})(t) \right\|_{Y'(0, \infty)} = K \left\| \sup_{t \leq s < \infty} s^\gamma g^{**}(s) \right\|_{Y'(0, \infty)} \leq KC \left\| t^\gamma g^{**}(t) \right\|_{Y'(0, \infty)} \quad \text{for every } g \in \mathcal{M}(0, \infty),
$$

proving (4.26). Hence (a) holds, as desired. The proof is complete. \qed

Let us now turn our attention to proofs of the main results.

**Proof of Theorem 4.4.** We begin by proving that $\sigma$ is a rearrangement-invariant norm. As in the proof of Theorem 3.1 only the triangle inequality and axioms (P4) and (P5) have to be verified. The triangle inequality follows by the same argument using measure-preserving transformations as in [36, Theorem 3.3].

We shall verify the validity of (P4). Let $E \subseteq \mathbb{R}$ be a measurable set with $|E| < \infty$ and let $h$ be such that $h \sim \chi_E$. We infer that there is a measurable set $F \subseteq \mathbb{R}$ such that $h = \chi_F$ and $|F| = |E|$. Assume moreover that $|E| \geq 1$. It follows from the regularity of the Lebesgue measure that there exists an open set $G \supseteq F$ such that $|G| \leq 2|F|$. Thus there are disjoint intervals $(a_k, b_k)$ satisfying $|F| \leq a_k$,

$$
F \subseteq (0, |F|) \cup \bigcup_k (a_k, b_k)
$$

and

$$
\sum_k (b_k - a_k) \leq 2|F|.
$$
Then we have
\[
\left\| \int_t^\infty h(s) \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \leq \left\| \int_t^\infty \left( \chi_{(0,|F|)}(s) + \sum_k \chi_{(a_k,b_k)}(s) \right) s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \\
\leq \left\| \int_t^\infty \chi_{(0,|F|)}(s) \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \\
+ \sum_k \left\| \int_t^\infty \chi_{(a_k,b_k)}(s) \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \\
\leq \frac{n}{\gamma} |F|^{\frac{\gamma}{n}} \left\| \chi_{(0,|F|)} \right\|_{X'(0,\infty)} \\
+ \sum_k \left\| \chi_{(0,a_k)}(t) \int_{a_k}^{b_k} \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \\
+ \sum_k \left\| \chi_{(a_k,b_k)}(t) \int_t^{b_k} \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)}.
\]

Let us observe that, due to (2.4), (4.3) is in fact equivalent to the existence of a constant \( C \) such that
\[
(4.29) \quad r^{\frac{\gamma}{n} - 1} \left\| \chi_{(0,r)} \right\|_{X'(0,\infty)} \leq C \quad \text{for } r \geq 1.
\]
Next, using the monotonicity of \( s^{\frac{\gamma}{n} - 1} \) and (4.29), we get (note that \( a_k \geq 1 \) is satisfied thanks to \( a_k \geq |F| \))
\[
\left\| \chi_{(0,a_k)}(t) \int_{a_k}^{b_k} \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \leq a_k^{\frac{\gamma}{n} - 1} \left\| \chi_{(0,a_k)} \right\|_{X'(0,\infty)} (b_k - a_k) \leq C(b_k - a_k).
\]

Note that \( C \) is independent of \( k \). Also,
\[
\left\| \chi_{(a_k,b_k)}(t) \int_t^{b_k} \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \leq \left\| \chi_{(a_k,b_k)}(t) \int_{a_k}^{b_k} \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \\
\leq a_k^{\frac{\gamma}{n} - 1} \left\| \chi_{(0,b_k-a_k)} \right\|_{X'(0,\infty)} (b_k - a_k) \\
\leq a_k^{\frac{\gamma}{n} - 1} \left\| \chi_{(0,a_k)} \right\|_{X'(0,\infty)} (b_k - a_k) \leq C(b_k - a_k),
\]
where we, once again, used the monotonicity, (2.4), (4.29) and
\[
b_k - a_k \leq |F| \leq a_k.
\]

Therefore
\[
\left\| \int_t^\infty h(s) \, s^{\frac{\gamma}{n} - 1} \, ds \right\|_{X'(0,\infty)} \leq \frac{n}{\gamma} |F|^{\frac{\gamma}{n}} \left\| \chi_{(0,|F|)} \right\|_{X'(0,\infty)} + 2C \sum_k (b_k - a_k) \\
\leq \frac{n}{\gamma} C|F| + 4C|F| = C_{n,\gamma}|E|.
\]
Taking the supremum over all such \( h \), we get
\[
(4.30) \quad \sigma(\chi_E) \leq C_{n,\gamma}|E|.
\]
If \( E \subset \mathbb{R}^n \) has \( |E| < 1 \), we get just \( \sigma(\chi_E) \leq C_{n,\gamma} \) by the monotonicity of \( \sigma \).
As for (P5), let $E$ be a measurable subset of $\mathbb{R}^n$ having finite measure and assume that $f \in L^1(E)$. Denote $r = |E|$ and set $h(s) = f^\ast(s-r)\chi_{(r,2r)}(s)$. Then $f \sim h$ and

$$
\sigma(f) \geq \left\| \int_t^\infty h(s) s^{\frac{n}{n-1}} \, ds \right\|_{X'(0,\infty)} = \left\| \int_t^\infty f^\ast(s-r)\chi_{(r,2r)}(s) s^{\frac{n}{n-1}} \, ds \right\|_{X'(0,\infty)} \\
\geq \left\| \chi_{(0,r)}(t) \int_r^{2r} f^\ast(s-r) s^{\frac{n}{n-1}} \, ds \right\|_{X'(0,\infty)} = \left\| \chi_{(0,r)} \right\|_{X'(0,\infty)} \int_r^{2r} f^\ast(s-r) s^{\frac{n}{n-1}} \, ds \\
\geq \left\| \chi_{(0,r)} \right\|_{X'(0,\infty)} (2r)^{\frac{n}{n-1}} \int_r^{2r} f^\ast(s-r) \, ds \\
\geq C_{n,\gamma,X} \|f\|_{L^1(E)},
$$

and (P5) follows.

We now claim that $M_\gamma : X \to Y$. Assume that $g \in \mathcal{M}_+(0,\infty)$. Define $f(x) = g(\omega_n|x|^n)$ for $x \in \mathbb{R}^n \setminus \{0\}$, where $\omega_n$ is the volume of the $n$-dimensional unit ball. Then $f$ is defined almost everywhere on $\mathbb{R}^n$ and one has $g \sim f$. Thus, by the definitions of $\sigma$ and $Y$, we get

$$
\left\| \int_t^\infty g(s)s^{\frac{n}{n-1}} \, ds \right\|_{X'(0,\infty)} \leq \sigma(f) = \|f\|_Y = \|g\|_{Y'(0,\infty)}.
$$

Since $g$ was arbitrary, we obtain by (2.8),

$$
\left\| t^{\frac{n}{n-1}} \int_0^t g(s) \, ds \right\|_{Y(0,\infty)} \leq \|g\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

Restricting this inequality to nonincreasing functions, we obtain that

$$
\left\| t^{\frac{n}{n-1}} g^\ast(t) \right\|_{Y(0,\infty)} \leq \|g^\ast\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

Applying Lemma 3.10, we get that there exists a positive constant $C$ such that

$$
\left\| \sup_{t \leq s < \infty} s^{\frac{n}{n-1}} g^\ast(s) \right\|_{Y(0,\infty)} \leq C \|g^\ast\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

Thus, by (4.1), one has

$$
\|M_\gamma f\|_Y \leq C \left\| \sup_{t \leq s < \infty} s^{\frac{n}{n-1}} f^\ast s^\ast(s) \right\|_{Y(0,\infty)} \leq C \|f^\ast\|_{X(0,\infty)} = C \|f\|_X \quad \text{for every } f \in X,
$$

whence $M_\gamma : X \to Y$.

We shall now prove the optimality of the space $Y$ in (4.5). Suppose that for some rearrangement-invariant space $Z$, one has $M_\gamma : X \to Z$. Let $g$ be a nonincreasing function in $\mathcal{M}_+(0,\infty)$. Then there exists a function $f_0 \in L^1(\mathbb{R}^n)$ such that $f_0 \sim g$ and (4.2) holds. Since $M_\gamma : X \to Z$, we have

$$
\|(M_\gamma f_0)^\ast\|_{Z(0,\infty)} \leq C \|f_0\|_{X(0,\infty)} = C \|g^\ast\|_{X(0,\infty)}.
$$

By (4.2), this yields

$$
\left\| \sup_{t \leq s < \infty} s^{\frac{n}{n-1}} g^\ast(s) \right\|_{Z(0,\infty)} \leq C \|g^\ast\|_{X(0,\infty)}.
$$
We emphasize that \(C\) does not depend on \(g\). The last estimate trivially implies
\[
\| t^{\frac{n}{\gamma}} g^{*}(t) \|_{Z(0,\infty)} \leq C \| g^{*} \|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_{+}(0,\infty).
\]

Therefore, by the Hardy–Littlewood inequality, we obtain
\[
\| t^{\frac{n}{\gamma}} P g(t) \|_{Z(0,\infty)} \leq C \| g \|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_{+}(0,\infty).
\]

By (4.3), this yields
\[
\left\| \int_{t}^{\infty} h(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)} \leq C \| h \|_{Z'(0,\infty)} \quad \text{for every } h \in \mathcal{M}_{+}(0,\infty).
\]

In particular, for every \( f \in \mathcal{M}_{+}(\mathbb{R}^{n}) \) and \( h \in \mathcal{M}_{+}(0,\infty) \) such that \( h \sim f \), one has
\[
(4.32) \quad \left\| \int_{t}^{\infty} h(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)} \leq C \| h \|_{Z'(0,\infty)} = C \| h^{*} \|_{Z'(0,\infty)} = C \| f^{*} \|_{Z'(0,\infty)} = C \| f \|_{Z'}.
\]

Consequently,
\[
\sigma(f) = \sup_{h \sim f \atop h \geq 0} \left\| \int_{t}^{\infty} h(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)} \leq C \| f \|_{Z'}.
\]

By the definition of \( Y \), this means that \( Z' \hookrightarrow Y' \), or \( Y \hookrightarrow Z \), proving the optimality of \( Y \) in (4.5).

Finally, assume that (4.13) is not true and assume that \( M_{\gamma}: X \rightarrow Y' \) for some rearrangement-invariant space \( Y \) over \( \mathbb{R}^{n} \). Then it follows from the above that
\[
(4.33) \quad \sup_{h \sim f \atop h \geq 0} \left\| \int_{t}^{\infty} h(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)} \leq C \| f \|_{Y'} \quad \text{for every } f \in Y'.
\]

Take any \( f \in \mathcal{M}_{+}(\mathbb{R}^{n}) \) satisfying \( f^{*} = \chi_{(0,1)} \) and let \( h = \chi_{(b,1+b)} \) for some fixed but arbitrary \( b \in (1,\infty) \). Then \( f \sim h \) and
\[
\left\| \int_{t}^{\infty} h(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)} = \left\| \int_{t}^{\infty} \chi_{(b,1+b)}(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)}
\]
\[
\geq \left\| \chi_{(0,b)}(t) \int_{b}^{1+b} s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)}
\]
\[
= \left\| \chi_{(0,b)} \right\|_{X'(0,\infty)} \int_{b}^{1+b} s^{\frac{2}{\gamma}-1} \, ds
\]
\[
\geq \frac{b}{\varphi_{X}(b)} (1+b)^{\frac{2}{\gamma}-1} \geq 2^{\frac{2}{\gamma}-1} \frac{b^{\frac{2}{\gamma}}}{\varphi_{X}(b)}.
\]

Since (4.3) is not satisfied, there exists a sequence \( b_{k} \rightarrow \infty \) such that
\[
\lim_{k \rightarrow \infty} \frac{b_{k}^{\frac{2}{\gamma}}}{\varphi_{X}(b_{k})} = \infty.
\]

This implies that
\[
\left\| \int_{t}^{\infty} h(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)} = \infty.
\]

Since \( \| f \|_{Y'} < \infty \) by (P4) for \( Y' \), this contradicts (4.33). The proof is complete. \( \square \)

**Proof of Theorem 4.2.** We shall first prove that \( \tau \) is equivalent to the functional in (4.7). By the definition of the associate space, we get
\[
\left\| \int_{t}^{\infty} f^{*}(s) s^{\frac{2}{\gamma}-1} \, ds \right\|_{X'(0,\infty)} = \sup_{\| h \|_{X(0,\infty)} \leq 1} \int_{0}^{\infty} h(t) \int_{t}^{\infty} f^{*}(s) s^{\frac{2}{\gamma}-1} \, ds \, dt.
\]
Since the function \( t \mapsto \int_t^\infty s^{\frac{1}{\gamma}} f(s) \, ds \) is obviously nonincreasing on \((0, \infty)\) regardless of \(f\), we in fact have, by the corollary of the Hardy–Littlewood inequality (see (2.7)),

\[
\left\| \int_t^\infty f(s) s^{\frac{1}{\gamma}} \, ds \right\|_{X'(0, \infty)} = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty h^*(t) \int_t^\infty f(s) s^{\frac{1}{\gamma}} \, ds \, dt.
\]

Thus, the Fubini theorem and the definition of \(P\) yield

\[
\left\| \int_t^\infty f(s) s^{\frac{1}{\gamma}} \, ds \right\|_{X'(0, \infty)} = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty f^*(s) (Ph^*)(s) s^{\frac{1}{\gamma}} \, ds.
\]

The trivial pointwise estimate \(h^* \leq T_\gamma h\) implies that \((Ph^*)(s) \leq (PT_\gamma h)(s)\) for every \(h\) and every \(s\). Hence, we obtain that

\[
(4.34) \quad \left\| \int_t^\infty f^*(s) s^{\frac{1}{\gamma}} \, ds \right\|_{X'(0, \infty)} \leq \tau(f).
\]

To prove the converse inequality, let \(K\) be the operator norm of \(T_\gamma\) on \(X(0, \infty)\). Then, by the definition of \(\tau\), the Fubini theorem, and the Hölder inequality, we have

\[
\tau(f) = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty f^*(s) (PT_\gamma h)(s) s^{\frac{1}{\gamma}} \, ds = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty (T_\gamma h)(t) \int_t^\infty f^*(s) s^{\frac{1}{\gamma}} \, ds \, dt
\]

\[
\leq \sup_{\|h\|_{X(0, \infty)} \leq 1} \left\| T_\gamma h \right\|_{X(0, \infty)} \left\| \int_t^\infty f^*(s) s^{\frac{1}{\gamma}} \, ds \right\|_{X'(0, \infty)}.
\]

By the definition of \(K\), we arrive at

\[
(4.35) \quad \tau(f) \leq K \left\| \int_t^\infty f^*(s) s^{\frac{1}{\gamma}} \, ds \right\|_{X'(0, \infty)},
\]

and the desired equivalence is established.

Now we shall prove that \(\tau\) is a rearrangement-invariant norm. We first note that the function \(s \mapsto s^{\frac{1}{\gamma}} (PT_\gamma h)(s)\) is always nonincreasing on \((0, \infty)\), regardless of \(h\). This follows from the easily verified fact that the expression \(s^{\frac{1}{\gamma}} (PT_\gamma h)(s)\) is a constant multiple of the integral mean over the interval \((0, s)\) of the obviously nonincreasing function \(t \mapsto \sup_{y<s} y^{\frac{1}{\gamma}} h^*(y)\) with respect to the measure \(d\mu(t) = t^{-\frac{1}{\gamma}} \, dt\). Therefore, (2.31) and Hardy’s lemma yield

\[
\tau(f + g) = \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty (f + g)^*(s) (PT_\gamma h)(s) s^{\frac{1}{\gamma}} \, ds
\]

\[
\leq \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty f^*(s) (PT_\gamma h)(s) s^{\frac{1}{\gamma}} \, ds + \sup_{\|h\|_{X(0, \infty)} \leq 1} \int_0^\infty g^*(s) (PT_\gamma h)(s) s^{\frac{1}{\gamma}} \, ds
\]

\[
= \tau(f) + \tau(g).
\]

All the other properties in (P1) as well as (P2), (P3) and (P6) are readily verified. We shall show (P4). Let \(E \subset (0, \infty)\) be of finite measure and denote \(a = |E|\). By (4.33), one has

\[
\tau(\chi_E) \leq K \left\| \int_t^\infty \chi_E(s) s^{\frac{1}{\gamma}} \, ds \right\|_{X'(0, \infty)} = K \left\| \chi(0, a)(t) \int_t^a s^{\frac{1}{\gamma}} \, ds \right\|_{X'(0, \infty)}
\]

\[
\leq \frac{K \gamma}{n} \frac{a^{\frac{1}{\gamma}}}{\gamma} \left\| \chi(0, a) \right\|_{X'(0, \infty)},
\]

and so

\[
\tau(\chi_E) \leq \frac{K \gamma}{n} \frac{a^{\frac{1}{\gamma}}}{\gamma} \left\| \chi(0, a)(t) \right\|_{X'(0, \infty)} < \infty
\]
by the property (P4) for $X'(0,\infty)$. It remains to verify (P5). Let $f \in \mathcal{M}(\mathbb{R}^n)$ and let $E \subset \mathbb{R}^n$ be of finite nonzero measure. Denote $a = |E|$. Then, by the monotonicity of the function $s \mapsto s^{\frac{\pi}{n}}(PT^*_a h)(s)$ on $(0,\infty)$, we have

$$
\tau(f) = \sup_{\|h\|_{X(0,\infty)} \leq 1} \int_0^\infty f^*(s)(PT^*_a h)(s)s^{\frac{\pi}{n}} \, ds \geq \sup_{\|h\|_{X(0,\infty)} \leq 1} \int_0^a f^*(s)(PT^*_a h)(s)s^{\frac{\pi}{n}} \, ds
$$

$$
\geq \sup_{\|h\|_{X(0,\infty)} \leq 1} a^{\frac{n}{\pi}}(PT^*_a h)(a) \int_0^a f^*(s) \, ds.
$$

Now let us take $h_0 = \frac{x(0,a)}{\|x(0,a)\|_{X(0,\infty)}}$. Then $\|h_0\|_{X(0,\infty)} = 1$, whence

$$
\sup_{\|h\|_{X(0,\infty)} \leq 1} a^{\frac{n}{\pi}}(PT^*_a h)(a) \geq a^{\frac{n}{\pi}}(PT^*_a h_0)(a) = \frac{a^{\frac{n}{\pi}}}{\|x(0,a)\|_{X(0,\infty)}} \int_0^a s^{\frac{\pi}{n}} \, ds = \frac{n}{n - \gamma} \frac{a^{\frac{n}{\pi}}}{\|x(0,a)\|_{X(0,\infty)}}.
$$

Altogether,

$$
\int_E f(x) \, dx \leq \int_0^a f^*(s) \, ds \leq \frac{n - \gamma}{n} a^{\frac{n}{\pi}} \|x(0,a)\|_{X(0,\infty)} \tau(f),
$$

and (P5) follows. We have shown that $\tau$ is a rearrangement-invariant norm. This entitles us to take $Y = Y'(\tau')$.

We now claim that $M_\gamma : X \to Y$. By (4.31) and since $\tau(f) = \|f\|_{Y'}$ for every $f \in \mathcal{M}_+(\mathbb{R}^n)$, we have

$$
(4.36) \quad \left\| \int_0^\infty f^*(s)s^{\frac{\pi}{n} - 1} \, ds \right\|_{X'(0,\infty)} \leq \|f\|_{Y'} \quad \text{for every } f \in \mathcal{M}_+(\mathbb{R}^n).
$$

Let $g \in \mathcal{M}_+(0,\infty)$ be nonincreasing. We define $f(x) = g(\omega_n |x|^n)$ for $x \in \mathbb{R}^n \setminus \{0\}$, where $\omega_n$ is the volume of the $n$-dimensional unit ball. Then $f$ is defined almost everywhere on $\mathbb{R}^n$ and one has $g \sim f$. Therefore, (4.36) implies that

$$
\left\| \int_0^\infty g(s)s^{\frac{\pi}{n} - 1} \, ds \right\|_{X'(0,\infty)} \leq \|g\|_{Y'(0,\infty)} \quad \text{for every nonincreasing } g \in \mathcal{M}_+(0,\infty).
$$

Using the equivalence of (2.24) and (2.6) with the (nondecreasing) function $I(s) = s^{1 - \frac{\pi}{2}}$, $s \in (0,\infty)$, we obtain that there exists a positive constant $C$ such that

$$
\left\| \int_0^\infty g(s)s^{\frac{\pi}{n} - 1} \, ds \right\|_{X'(0,\infty)} \leq C \|g\|_{Y'(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

By (2.8), this in turn gives

$$
\left\| t^{\frac{\pi}{n} - 1} \int_0^t g(s) \, ds \right\|_{Y(0,\infty)} \leq C \|g\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

Restricting this inequality to nonincreasing functions, we obtain that

$$
\left\| t^{\frac{\pi}{n} - 1} g^*(t) \right\|_{Y(0,\infty)} \leq C \|g^*\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$

Applying Lemma 4.10, we get that there exists a (possibly different) positive constant $C$ such that

$$
\left\| \sup_{t \leq s < \infty} s^{\frac{\pi}{n}} g^*(s) \right\|_{Y(0,\infty)} \leq C \|g^*\|_{X(0,\infty)} \quad \text{for every } g \in \mathcal{M}_+(0,\infty).
$$
Thus, by (4.31), one has
\[ \| M_{\gamma}f \|_Y \leq C \sup_{t \leq s \leq \infty} s^{n} f^*(s) \leq C \| f^* \|_{X(0,\infty)} = C \| f \|_X \quad \text{for every } f \in X, \]
whence \( M_{\gamma} : X \to Y \).

It remains to prove the optimality of the space \( Y \). Assume that \( M_{\gamma} : X \to Z \) for some rearrangement-invariant space \( Z \) over \( \mathbb{R}^n \). Then (4.32) holds thanks to the same argument as in the proof of Theorem 4.1, that is,
\[ \left\| \int_{\mathbb{R}^n} h(t)s^{n-1} \, ds \right\|_{X'(0,\infty)} \leq C \| f \|_{Z'} \quad \text{for every } h \sim f. \]

Since \( f^* \sim f \), this yields, in particular,
\[ \left\| \int_{t}^{\infty} f^*(s)s^{n-1} \, ds \right\|_{X'(0,\infty)} \leq C \| f \|_{Z'}. \]

This estimate combined with (4.35) yields
\[ \tau(f) \leq KC \| f \|_{Z'}, \quad f \in \mathcal{M}_{+}(\mathbb{R}^n). \]

As \( \tau(f) = \| f \|_{Y'} \), this means that \( Z' \sim Y' \), or \( Y \sim Z \), proving the optimality of \( Y \). The proof is complete. \( \square \)

**Proof of Theorem 4.3.** Note that \( L^{p,q,h} \) is equivalent to a rearrangement-invariant space under any of the assumptions thanks to [13, Theorem 7.1].

Let us first treat the cases when \( T_\alpha : L^{p,q,h}(0,\infty) \to L^{p,q,h}(0,\infty) \). To this end we have to investigate when there exists a positive constant \( C > 0 \) such that
\begin{equation}
\tag{4.37}
\left\| t^{-\frac{\alpha}{p}} \sup_{t \leq s < \infty} s^{\frac{n}{p}} f^*(s) \right\|_{L^{p,q,h}} \leq C \| f \|_{L^{p,q,h}} \quad \text{for every } f \in \mathcal{M}_{+}(\mathbb{R}^n),
\end{equation}

We first consider the case when \( q = \infty \). Then (4.37) reads as
\begin{equation}
\tag{4.38}
\sup_{0 < t < \infty} t^{-\frac{n}{p}} \ell^h(t) \sup_{t \leq s \leq \infty} s^{\frac{n}{p}} f^*(s) \leq C \sup_{0 < t < \infty} t^{-\frac{n}{p}} \ell^h(t) f^*(t).
\end{equation}

One has
\begin{align*}
\sup_{0 < t < \infty} t^{-\frac{n}{p}} \ell^h(t) \sup_{t \leq s \leq \infty} s^{\frac{n}{p}} f^*(s) & = \sup_{0 < t < \infty} t^{-\frac{n}{p}} \ell^h(t) \sup_{t \leq s \leq \infty} s^{\frac{n}{p}} f^*(s) s^{\frac{n}{p}} \ell^{-h}(s) \\
& \leq \left( \sup_{0 < t < \infty} t^{-\frac{n}{p}} \ell^h(t) f^*(t) \right) \left( \sup_{0 < t < \infty} t^{-\frac{n}{p}} \ell^h(t) \sup_{t \leq s \leq \infty} s^{\frac{n}{p}} \ell^{-h}(s) \right).
\end{align*}

Thus, (4.38) is obviously satisfied if \( s \mapsto s^{\frac{n}{p}} \ell^{-h}(s) \) is equivalent to a nonincreasing function. This happens precisely if either \( p < \frac{n}{2} \) or \( p = \frac{n}{2}, \alpha_0 \leq 0 \) and \( \alpha_\infty \geq 0 \). It is easy to see that in all the remaining cases, that is when either \( p > \frac{n}{2} \) or \( p = \frac{n}{2} \) and \( \alpha_0 > 0 \), or \( p = \frac{n}{q}, \alpha_0 \leq 0 \) and \( \alpha_\infty < 0 \), the inequality (4.38) is false as one can observe by plugging the function \( f^* = \chi_{(0,a)} \) into the inequality for \( a \in (0,1) \) or for \( a \in (1,\infty) \), respectively.

Now let us consider the case when \( q < \infty \). We recall that then (4.37) reads as
\begin{equation}
\tag{4.39}
\left( \int_{0}^{\infty} t^{-\frac{n}{p} + \frac{\alpha}{p} - 1} \ell^h(q) \, dt \right) \sup_{t \leq s < \infty} s^{\frac{n}{p}} f^*(s) \| f \|_{L^{p,q,h}} \leq C \left( \int_{0}^{\infty} f^*(t) s^{\frac{n}{p} - 1} \ell^h(t) \, dt \right)^{\frac{1}{q}}
\end{equation}

for some \( C > 0 \) and all \( f \in \mathcal{M}_{+}(\mathbb{R}^n) \). By [32, Theorem 3.2], (4.39) holds if and only if there exists a constant \( K \) such that, for every \( \tau \in (0,\infty) \),
\begin{equation}
\tag{4.40}
\tau^{\frac{n}{p} + \frac{\alpha}{p} - 1} \ell^h(q) \, dt \right) \leq K \left( \int_{0}^{\tau} t^{\frac{n}{p} - 1} \ell^h(t) \, dt \right)^{\frac{1}{q}}.
\end{equation}
Elementary calculation shows that (4.40) holds if and only if \(1 \leq p < \frac{n}{2}\). Adding all conditions together we infer that \(T_\gamma\) is bounded on r.i. space \(L^{p,q;\mathbb{h}}(0,\infty)\) if and only if one of the conditions (4.10a), (4.10b) or (4.10c) holds.

We are thus in a position to use Theorem 4.2 in these cases, hence the optimal range \(Y\) for the space \(L^{p,q;\mathbb{h}}\) with respect to \(M_\gamma\) satisfies

\[
\|f\|_{Y'} = \left\| \int_t^\infty f^*(s)s^{\frac{\gamma}{n} - 1}\,ds \right\|_{(L^{p,q;\mathbb{h}})'}.
\]

Now we have by [47, Theorems 6.2 and 6.6] that \((L^{p,q;\mathbb{h}})' = L^{p',q';\mathbb{h}}\), so we in fact get

\[
\|f\|_{Y'} = \left\| \int_t^\infty f^*(s)s^{\frac{\gamma}{n} - 1}\,ds \right\|_{L^{p',q';\mathbb{h}}}.
\]

that is,

\[
\|f\|_{Y'} = \left\| \int_t^\infty f^*(s)s^{\frac{\gamma}{n} - 1}\,ds \right\|_{L^{p',q';\mathbb{h}}}.
\]

When \(p = 1, q = 1, \alpha_0 \geq 0\) and \(\alpha_\infty \leq 0\), this establishes the assertion in the case (4.10c). In the particular case \(A = [0,0]\) we have

\[
\|f\|_{Y'} = \sup_{0 < t < \infty} \|\ell^{-\mathbb{h}}(t)\int_t^\infty f^*(s)s^{\frac{\gamma}{n} - 1}\,ds\|_{L^{p',q';\mathbb{h}}},
\]

to prove the assertion, our next step will be to simplify the expression for \(\|f\|_{Y'}\) if one of the conditions (4.10a) or (4.10b) holds. We start with the lower bound. One has, by the classical Hardy inequality (see e.g. [40]), that

\[
\|f\|_{Y'} \geq \int_t^\infty f^*(s)s^{\frac{\gamma}{n} - 1}\,ds \sup_{0 < t < \infty} \|t^{\frac{1}{\mathbb{h}} - \frac{q'}{q}}\ell^{-\mathbb{h}}(t)\|_{L^{p',q';\mathbb{h}}}.
\]

where \(c, c'\) are positive constants independent of \(f\) and \(r\) is such that \(\frac{1}{q} = \frac{1}{p} + \frac{\gamma}{n}\). We shall show however that the converse inequality holds as well. First let \(q = 1\). Then

\[
\|f\|_{Y'} = \sup_{0 < t < \infty} t^{\frac{1}{p} - \frac{q}{q'}}\ell^{-\mathbb{h}}(t)\int_t^\infty f^*(s)s^{\frac{1}{p} + \frac{\gamma}{n} - 1}\,ds \sup_{0 < t < \infty} t^{\frac{1}{p} - \frac{q}{q'}}\ell^{-\mathbb{h}}(t)g(t) \|_{L^{p',q';\mathbb{h}}(0,\infty)} ^{-1}
\]

Now assume that \(1 < q \leq \infty\). Then, by the classical Hardy inequality (see e.g. [40]), we get that there exists a positive constant \(C\) such that

\[
\left\| \int_t^\infty g(s)\,ds \right\|_{L^{p',q';\mathbb{h}}(0,\infty)} \leq C \left\| \int_t^\infty \ell^{-\mathbb{h}}(t)\ell^{-\mathbb{h}}(t)g(t) \right\|_{L^{p',q';\mathbb{h}}(0,\infty)}
\]

Given \(f \in \mathcal{M}\), we set \(g(t) = f^*(t)t^{\frac{\gamma}{n} - 1}, t \in (0,\infty)\), which leads to

\[
\|f\|_{Y'} \leq C\|f\|_{L^{p',q';\mathbb{h}}},
\]

hence, altogether, \(Y' = L^{p',q';\mathbb{h}}\). Since \(1 < r' < \infty\), we have, by [47, Theorems 6.2 and 6.6], that \(Y = L^{r',q';\mathbb{h}}\), establishing the assertion.
We shall now treat the case \(4.10\). The general formula follows directly by \((4.3)\) of Theorem 4.4 and the definition of the norm of \(L^{p,q,A}\). Note that since \(T_z\) is not bounded on \(L^{p,q,A}\) in this case, the supremum in \((4.3)\) is essential and cannot be avoided by setting \(h = f^*\) as follows from Theorem 4.7.

Let us now focus on the special case when \(A = [0, \infty]\). We denote the optimal partner for \(L^{p,q,A}\) with respect to \(M_\gamma\) by \(Y\). Our aim is to show that \(Y = L^\infty\) or, equivalently, that \(Y' = L^1\). We first notice that \(L^1\) is (up to equivalence) the only r.i. space whose fundamental function, denoted by \(\psi\), satisfies \(\psi(t) = t\). Indeed, assume that \(X\) has such a fundamental function. Then

\[
\|f\|_{\Lambda(X)} := \int_0^\infty f^*(t) \, dt = \|f\|_{L^1}
\]

and

\[
\|f\|_{M(X)} := \sup_{t \in (0, \infty)} \psi(t) t^\gamma(t) = \sup_{t \in (0, \infty)} \int_0^t f^*(s) \, ds = \|f\|_{L^1}.
\]

Consequently, by [7, Chapter 2, Theorem 5.13], we have \(\Lambda(X) = X = M(X)\), hence \(X = L^1\). Therefore, it is enough to verify that the fundamental function of \(Y'\), \(\varphi\), say, satisfies \(\varphi(t) \approx t\) for \(t \in (0, \infty)\). As for the proof of the lower bound, we make use of the same calculation as in \((4.31)\) with \(f = \chi_E\) and \(|E| = t\). We obtain

\[
\|\chi_E\|_{Y'} \geq C_{n,\gamma} t^{n,\gamma} \|\chi_{(0,0)}\|_{(L^{p,q,A}_{(0,\infty)})'}, \quad 0 < t < \infty,
\]

which, thanks to \((2.4)\), can be rewritten as

\[
\varphi(t) \geq C_{n,\gamma} \frac{t^{1+\frac{n}{2}}}{\|\chi_{(0,t)}\|_{L^{p,q,A}_{(0,\infty)}}}, \quad 0 < t < \infty,
\]

and the estimate then follows since the fundamental function of \(L^{p,q,A}\) is \(t^{n/2}\). To prove the converse inequality, let us use the same upper bound which appears in the proof of the validity of (P4) in the proof of Theorem 4.1. Observe that \((4.29)\) now holds on the whole of \((0, \infty)\) and hence we get \((4.30)\) also for all sets \(E\) with \(|E| < 1\). That gives the desired relation \(\varphi(t) \lesssim t\) for \(t \in (0, \infty)\).

\[\square\]

5. The Hilbert Transform

A very important example of a singular integral with odd kernel is the Hilbert transform, namely the operator \(H\), defined for appropriate functions on \(\mathbb{R}\) by

\[
Hf(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t} \, dt.
\]

This operator is defined for every function \(f : \mathbb{R} \to \mathbb{R}\) for which the integral converges almost everywhere. The Hilbert transform arises in the study of boundary values of the real and imaginary parts of analytic functions. It is a cornerstone of several important disciplines including real and complex analysis and the theory of PDEs. In this section we shall study its sharp boundedness properties on rearrangement-invariant spaces over \(\mathbb{R}\). A key technical background tool will be the Stieltjes transform, \(S\), which is defined for every nonnegative measurable function \(f\) on \((0, \infty)\) by

\[
(Sf)(t) = \frac{1}{t} \int_0^t f(s) \, ds + \int_t^\infty f(s) \frac{ds}{s}, \quad t \in (0, \infty).
\]

It might be useful to note that

\[
(5.1) \quad S = P + Q = P \circ Q = Q \circ P.
\]
Whenever we will say that the Hilbert transform is bounded from a function space $X$ to a function space $Y$, we implicitly assume that $H$ is well defined for every $f \in X$, that is, $f \in L^1_{\text{loc}}(\mathbb{R})$ and the limit in the definition of $Hf$ exists for a.e. $x \in \mathbb{R}$. Let us recall that, by [7, Chapter 3, Theorem 4.8], a sufficient condition for the existence of this limit, for a given $f \in L^1_{\text{loc}}(\mathbb{R})$, is
\begin{equation}
(Sf^*)(1) < \infty.
\end{equation}

Our main result in this section reads as follows.

**Theorem 5.1.** Let $X$ be a rearrangement-invariant space over $\mathbb{R}$ such that
\begin{equation}
\eta \in X'(0, \infty),
\end{equation}
where
\begin{equation}
\eta(t) = \chi_{(0,1]}(t)(1 - \log t) + \chi_{(1,\infty)}(t)\frac{1}{t}, \quad t \in (0, \infty).
\end{equation}
Define the functional $\sigma$ by
\begin{equation}
\sigma(f) = \|Sf^*\|_{X'(0,\infty)}, \quad f \in M_+(\mathbb{R}).
\end{equation}
Then $\sigma$ is a rearrangement-invariant norm and
\begin{equation}
H: X \to Y,
\end{equation}
where $Y = Y(\sigma')$. Moreover, $Y$ is the optimal (smallest) rearrangement-invariant space for which \((5.5)\) holds.

Conversely, if \((5.3)\) is not true, then there does not exist a rearrangement-invariant space $Y$ for which \((5.5)\) holds.

For the optimal domain, we have the following result. Again, the proof is analogous to the appropriate proofs above, and therefore omitted.

**Theorem 5.2.** Let $Y$ be a rearrangement-invariant space over $\mathbb{R}$ such that
\begin{equation}
\eta \in Y(0, \infty),
\end{equation}
where $\eta$ is the function from \((5.4)\). Define the functional $\sigma$ by
\begin{equation}
\sigma(f) = \|Sf^*\|_{Y(0,\infty)}, \quad f \in M_+(\mathbb{R}).
\end{equation}
Then $\sigma$ is a rearrangement-invariant norm and
\begin{equation}
H: X \to Y,
\end{equation}
where $X = X(\sigma)$. Moreover, $X$ is the optimal (biggest) rearrangement-invariant space for which \((5.7)\) holds.

Conversely, if \((5.6)\) is not true, then there does not exist a rearrangement-invariant space $X$ for which \((5.7)\) holds.

We provide several examples of the optimal range partners for Lorentz-Zygmund spaces with respect to the Hilbert transform. The proof is similar to that of Theorem 3.3 and therefore omitted.

**Theorem 5.3.** Assume that $p, q \in [1, \infty]$, $\mathbb{A} \in \mathbb{R}^2$. Then
\begin{equation}
H: L^{p,q;\mathbb{A}} \to \begin{cases}
L^{1;1;\mathbb{A}}_{1,1}, & p = 1, \quad q = 1, \quad \alpha_0 \geq 1, \quad \alpha_\infty < 0, \\
L^{p,q;\mathbb{A}}, & 1 < p < \infty, \\
Y, & p = \infty, \quad q = 1, \quad \alpha_0 < -1, \quad \alpha_\infty \geq 0 \text{ or } \frac{1}{q} < 0, \quad \alpha_\infty + \frac{1}{q} > 0, \\
L^{\infty;\infty;\mathbb{A}}_{\infty,\infty}, & p = \infty, \quad q = \infty, \quad \alpha_0 \leq 0, \quad \alpha_\infty > 1,
\end{cases}
\end{equation}
where $Y$ is defined by its associate space $Y'$ whose norm is given by
\[
\left\| \int_t^\infty f^*(s) \frac{ds}{s} \right\|_{L^1(1/\mu,-1)} , \ f \in M_+(\mathbb{R}).
\]

These spaces are the optimal range partners with respect to $H$.

At the end of this section, we aim to prove Theorem 5.1. We start with a lemma which recalls a well-known fact. We insert a short proof for the sake of completeness.

**Lemma 5.4.** Let $X$ and $Y$ be rearrangement-invariant Banach function spaces over $\mathbb{R}$. Assume that (5.2) is satisfied for every $f \in X$. Then the Hilbert transform $H$ is bounded from $X$ to $Y$ if and only if the Stieltjes transform $S$ is bounded from $X(0,\infty)$ to $Y(0,\infty)$.

**Proof.** Assume first that $H$ is bounded from $X$ to $Y$. Fix a function $f \in M_+(0,\infty)$ such that $(Sf^*)(1) < \infty$. Then, by a simple modification of [7, Chapter 3, Proposition 4.10], there exists a function $g \in M_+([1/\mu,-1])$, equimeasurable with $f$, such that
\[
(Sf^*)(t) \leq 2\pi (Hg)^*(t) \quad \text{for every } t \in (0,\infty).
\]
Thus, by the property (P2) of $Y$, we have
\[
\| (Sf^*) \|_{Y(0,\infty)} \leq 2\pi \| (Hg)^* \|_{Y(0,\infty)}.
\]
By the rearrangement invariance of $Y$, this turns into
\[
\| (Sf^*) \|_{Y'_{\infty}(0,\infty)} \leq 2\pi \| Hg \|_{Y(0,\infty)}.
\]
It follows from the boundedness of $H$ from $X$ to $Y$ that
\[
\| Hg \|_{Y(0,\infty)} \leq C \| g \|_{X(0,\infty)}
\]
for some constant $C$, $0 < C < \infty$, independent of $g$ (hence of $f$). We thus get, altogether, using also the definition of the representation space and the equimeasurability of $f$ and $g$, that
\[
\| (Sf^*) \|_{Y(0,\infty)} \leq 2C\pi \| g \|_{X(0,\infty)} = 2C\pi \| g^* \|_{X(0,\infty)} = 2C\pi \| f^* \|_{X(0,\infty)}.
\]
In other words, $S$ is bounded from $X(0,\infty)$ to $Y(0,\infty)$.

Conversely, assume that the Stieltjes transform is bounded from $X(0,\infty)$ to $Y(0,\infty)$. By an appropriate modification of [7, Chapter 3, Theorem 4.8], there exists a positive constant $C$ independent of $f$ such that
\[
(Hf)^*(t) \leq C(Sf^*)(t) \quad \text{for every } t \in (0,\infty).
\]
We then get, similarly as above,
\[
\| Hf \|_{Y(0,\infty)} = \| (Hf)^* \|_{Y(0,\infty)} \leq C \| (Sf^*) \|_{X(0,\infty)} \leq C' \| f^* \|_{X(0,\infty)} = C' \| f \|_{X(0,\infty)}
\]
for some suitable constant $C'$, proving that $H : X \to Y$. The proof is complete. $\square$

Our next step will be a characterization of the optimal range partner with respect to the Stieltjes transform.

**Theorem 5.5.** Let $X$ be a rearrangement-invariant Banach function space over $(0,\infty)$ such that
\[
\eta \in X'(0,\infty),
\]
where $\eta$ is the function from (5.4). Define the functional $\sigma$ by
\[
\sigma(f) = \| Sf^* \|_{X'(0,\infty)}, \ f \in M_+(0,\infty).
\]
Then $\sigma$ is a rearrangement-invariant norm and
\[
S : X \to Y,
\]
where \( Y = Y(\sigma') \). Moreover, \( Y \) is the optimal (smallest) rearrangement-invariant space for which (5.9) holds.

Conversely, if (5.8) is not true, then there does not exist a rearrangement-invariant space \( Y \) for which (5.9) holds.

**Proof.** Consider the functional \( \sigma(f) = \|Sf^*\|_{X'(0,\infty)}, f \in M_+(0,\infty) \). We shall prove that \( \sigma \) is a rearrangement-invariant norm. As in the proof of Theorem 3.1, the axioms (P2), (P3) and (P6) for \( \sigma \) are clearly satisfied. The verification of the triangle inequality is even easier than in the proof of Theorem 3.1. It follows from (5.1) that
\[
Sf^* = Qf^{**} \quad \text{for every } f \in M(0,\infty),
\]
which in conjunction with (2.2) immediately yields the triangle inequality for \( \sigma \). As usual, all other properties in (P1) are readily verified. Also the verification of (P5) is easy. In fact, it immediately follows from the analogous property of the functional \( \sigma \) from Theorem 3.1, because, by (5.10), one has \( Sf^* \geq Qf^* \). It only remains to verify the validity of (P4). To this end, let \( E \subset \mathbb{R} \) be a set of finite measure. We need to prove that \( \|S\chi_E^*\|_{X'} < \infty \). Calculation shows that this is equivalent to saying that \( \eta \in X' \), a fact guaranteed by the assumption. This shows (P4), and, consequently, it completes the proof of the fact that \( \sigma \) is a rearrangement-invariant Banach function norm.

We will now prove that \( S: X \to Y \). The operator \( S \) is self-adjoint with respect to the \( L^1 \)-pairing in the sense that
\[
\int_0^\infty (Sf)(t)g(t) \, dt = \int_0^\infty f(t)(Sg)(t) \, dt
\]
for every admissible \( f \) and \( g \). Hence, it suffices to prove that \( S: Y' \to X' \). That, however, follows trivially from the definition of \( Y' \).

The proof of optimality of the space \( Y \) as well as that of the nonexistence of a rearrangement-invariant range partner for \( X \) in case \( \eta \not\in X' \) are completely analogous to their counterparts from the proof of Theorem 3.1 and hence are omitted. \( \square \)

Finally, Theorem 5.1 immediately follows from Theorem 5.5 and Lemma 5.4.

6. The Riesz potential

**Definition 6.1.** Let \( 0 < \gamma < n \). Then the Riesz potential of order \( \gamma \), \( I_\gamma \), of a measurable function \( f \) on \( \mathbb{R}^n \) is defined by
\[
(I_\gamma f)(x) = \int_{\mathbb{R}^n} f(y)\phi(x-y) \, dy, \quad x \in \mathbb{R}^n,
\]
where
\[
\phi(y) = c(\gamma)|y|^{\gamma-n}, \quad c(\gamma) = \Gamma\left(\frac{n-\gamma}{2}\right)\left(\pi^{\frac{n}{2}}2^n\Gamma\left(\frac{\gamma}{2}\right)\right)^{-1}.
\]

We are going to make use of a special case of the O’Neil inequality. In its general form ([46, Lemma 1.5]), it states that, for the convolution of two measurable functions \( f, g \) on \( \mathbb{R}^n \), defined by
\[
(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y) \, dy, \quad x \in \mathbb{R}^n,
\]
we have
\[
(f * g)^{**}(t) \leq tf^{**}(t) + \int_0^\infty f^*(s)g^*(s) \, ds \quad \text{for every } t \in (0,\infty).
\]

With the particular choice
\[
g(x) = |x|^{\gamma-n}, \quad x \in \mathbb{R}^n,
\]
we obtain that
\[
(I_\gamma f)^*(t) \leq C \int_t^\infty f^{**}(s)s^{\frac{\gamma}{n} - 1} \, ds \quad \text{for every } t \in (0, \infty),
\]
with some positive constant \(C\), depending on \(\gamma\) and \(n\), but independent of \(f\) and \(t\).

This inequality is known to be sharp, but merely in a broader sense than, for example, the corresponding estimate for the Hardy–Littlewood maximal operator. This was firstly observed by O’Neil in the final remark of the paper [46], where it is pointed out that the inequality can be reversed when \(f, g\) are radially decreasing positive functions. Furthermore, by an appropriately modified argument from [27, Theorem 10.2(iii)], we get that, for every \(f \in M(\mathbb{R}^n)\) there exists a function \(g \in M(0, \infty)\) equimeasurable with \(f\) such that
\[
(I_\gamma g)^*(t) \geq c \int_t^\infty f^{**}(s)s^{\frac{\gamma}{n} - 1} \, ds \quad \text{for every } t \in (0, \infty),
\]
with some constant \(c, \, 0 < c < \infty\), depending on \(\gamma\) and \(n\), but independent of \(f\) and \(t\).

We shall now turn our attention to a weighted version of the Stieltjes transform, which plays a key role in the matter of optimal spaces for the Riesz potential.

**Definition 6.2.** Let \(\alpha \in (1, \infty)\). The weighted Stieltjes transform, \(S_\alpha\), is defined for every nonnegative measurable function \(f\) on \((0, \infty)\) by
\[
(S_\alpha f)(t) = t^{\frac{1}{\alpha} - 1} \int_0^t f(s) \, ds + \int_t^\infty f(s)s^{\frac{1}{\alpha} - 1} \, ds, \quad t \in (0, \infty).
\]

We note that, for every admissible \(f\) and \(t\), one has
\[
(S_\alpha f)(t) = c_\alpha \int_t^\infty (Pf)(s)s^{\frac{1}{\alpha} - 1} \, ds,
\]
where \(c_\alpha = \frac{\alpha - 1}{\alpha}\).

Our main result of this section reads as follows.

**Theorem 6.3.** Let \(\gamma \in (0, n)\) and let \(X\) be a rearrangement-invariant space over \(\mathbb{R}^n\) such that
\[
(6.1) \quad \xi^n_\gamma \in X'(0, \infty),
\]
where, for \(\alpha > 0\),
\[
(6.2) \quad \xi_\alpha(t) = (t + 1)^{\frac{1}{\alpha} - 1}, \quad t \in (0, \infty).
\]
Define the functional \(\sigma\) by
\[
\sigma(f) = \|S_\frac{n}{\gamma} f^*\|_{X'(0, \infty)}, \quad f \in M_+(\mathbb{R}^n).
\]
Then \(\sigma\) is a rearrangement-invariant norm and
\[
(6.3) \quad I_\gamma : X \to Y,
\]
where \(Y = Y(\sigma')\). Moreover, \(Y\) is the optimal (smallest) rearrangement-invariant space for which \((6.3)\) holds.

Conversely, if \((6.1)\) is not true, then there does not exist a rearrangement-invariant space \(Y\) for which \((6.3)\) holds.

As in the preceding sections, we also characterize optimal domains. We also omit the proof, since it is analogous, again, to that of Theorem 3.2.

**Theorem 6.4.** Let \(\gamma \in (0, n)\) and let \(Y\) be a rearrangement-invariant space over \(\mathbb{R}^n\) such that
\[
(6.4) \quad \xi^n_\gamma \in Y(0, \infty),
\]
where \( \xi_\alpha \) is the function from \([6.2]\). Define the functional \( \sigma \) by

\[
\sigma(f) = \left\| S_{\frac{n}{\gamma}} f^* \right\|_{Y(0, \infty)}, \ f \in M_+(\mathbb{R}^n). 
\]

Then \( \sigma \) is a rearrangement-invariant norm and

\[
(6.5) \quad I_\gamma : X \rightarrow Y,
\]

where \( X = X(\sigma) \). Moreover, \( X \) is the optimal (biggest) rearrangement-invariant space for which \((6.5)\) holds.

Conversely, if \((6.4)\) is not true, then there does not exist a rearrangement-invariant space \( X \) for which \((6.5)\) holds.

We use Theorem \([6.3]\) to provide several examples of the optimal range partners for Lorentz-Zygmund spaces with respect to the Riesz potential.

**Theorem 6.5.** Assume that \( \gamma \in (0, n) \), \( p, q \in [1, \infty] \), \( \Lambda \in \mathbb{R}^2 \). Then

\[
(6.6) \quad I_\gamma : L^{p,q;\Lambda} \rightarrow \begin{cases} 
Y_1 & p = 1, q = 1, \alpha_0 \geq 0, \alpha_\infty \leq 0, \\
L_{n^-\gamma}^{\frac{n}{\gamma}q;\Lambda} & 1 < p < \frac{n}{\gamma}, \\
L^{\infty,q;\Lambda-1} & p = \frac{n}{q}, 1 \leq q \leq \infty, \alpha_0 < \frac{1}{q'}, \alpha_\infty > \frac{1}{q'}, \\
L^{\infty,q;[-\frac{1}{q'},\alpha_\infty-1],[-1,0]} & p = \frac{n}{q}, 1 < q \leq \infty, \alpha_0 = \frac{1}{q'}, \alpha_\infty > \frac{1}{q'}, \\
L^{\infty,1;[-\frac{1}{q'},\alpha_\infty-1],[-1,0],[1,0]} & p = \frac{n}{q}, q = 1, \alpha_0 < 0, \alpha_\infty = 0, \\
L^{\infty} & p = \frac{n}{q}, q = 1, \alpha_0 = 0, \alpha_\infty = 0, \\
Y_3 & p = \frac{n}{q}, q = 1, \alpha_0 > 0, \alpha_\infty > 0, \\
Y_2 & p = \frac{n}{q}, 1 < q \leq \infty, \alpha_0 > \frac{1}{q'}, \alpha_\infty > \frac{1}{q'},
\end{cases}
\]

where

\[
\| f \|_{Y_2} = \| f \|_{L^\infty} + \| t^{-\frac{1}{q'}} e^{\alpha_\infty-1}(t) f^*(t) \|_{L^q(1, \infty)}, \\
\| f \|_{Y_3} = \| t^{-\frac{1}{q'}} e^{\alpha_\infty-1}(t) f^*(t) \|_{L^q(0, 1)},
\]

and \( Y_1 \) is defined by its associate space \( Y_1' \) whose norm is given by

\[
\| f \|_{Y_1'} = \sup_{0 < t < \infty} t^{-\frac{n}{q'}} \int_0^t f^*(s) s^{\frac{n}{q'}-1} ds, \ f \in M_+(\mathbb{R}^n).
\]

In particular, if \( \Lambda = [0, 0] \), we have \( Y_1 = L_{n^-\gamma}^{\frac{n}{\gamma}} \).

Moreover, these spaces are the optimal range partners with respect to \( I_\gamma \).

**Proof.** We note that \( L^{p,q;\Lambda} \) is equivalent to a rearrangement–invariant Banach function space due to \([17]\), Theorem 7.1) in all the cases.

Assume that \( p \in (1, \infty) \) and \( q \in (1, \infty] \). We need to check when \( \xi_\alpha \in X'(0, \infty) \) is satisfied, that is, when

\[
\int_0^\infty t^{\frac{n}{q'}-1} e^{-\alpha_0 q'} (t+1)^{\frac{n}{q'}-1} dt < \infty.
\]

It is easy to see that this integral is finite if and only if either

\[
p \in (1, \frac{n}{\gamma})
\]

or

\[
p = \frac{n}{\gamma} \text{ and } \alpha_\infty > \frac{1}{q'}.
\]
Henceforth, we assume one of the two conditions. By \([47, \text{Theorem 6.2}]\), the associate space of \(L^{p,q;A}\) is equivalent to \(L^{p',q';-A}\). By the classical weighted Hardy inequality, we get
\[
\left\| S^2_g \right\|_{p',q';-A} = \left\| t^{p' - \frac{1}{2}} e^{-A}(t) \int_t^\infty g^{**}(s)s^{-\frac{n-1}{2}} \, ds \right\|_{q'} \\
\leq \left\| t^{p' - \frac{1}{2}} e^{-A}(t)g^{**}(t)t^{\frac{1}{p'} - 1} \right\|_{q'} = \left\| t^{p' - \frac{1}{2}} e^{-A}(t)g^{**}(t) \right\|_{q'}
\]
which follows immediately from the estimate
\[
S^2_g(t) = \int_t^\infty g^{**}(s)s^{-\frac{n-1}{2}} \, ds = \int_t^\infty \frac{1}{s^{2 - \frac{n}{2}}} \int_t^s g^*(u) \, du \, ds \\
\geq \int_t^\infty g^*(u) \, du \int_t^\infty \frac{1}{s^{2 - \frac{n}{2}}} \, ds = \frac{n}{n - \gamma} t^{\frac{n}{2}} g^{**}(t).
\]
If \(p \in (1, \frac{2}{n})\), then \(p' \in (1, \frac{2}{n})\). By \([47, \text{Theorem 3.8}]\), \(L^{p',q';-A}\) is equivalent to \(L^{p,q;A}\). Hence \(Y\) is equivalent to \(L^{p,q;A}\), where \(r = \frac{np}{n - p} \in (\frac{n}{n - q}, \infty)\), by \([47, \text{Theorem 6.2}]\).

If \(p = \frac{n}{2}\), then \(r' = 1\). If \(q \in (1, \infty)\) (and hence \(q' \in (1, \infty)\)), we obtain \((6.6)\) for \(q \in (1, \infty)\) by virtue of \([47, \text{Theorem 6.7}]\). If \(q = \infty\) (and hence \(q' = 1\)), we combine \([47, \text{Theorem 3.8}]\) with \([47, \text{Theorem 6.6}]\) in order to prove \((6.6)\) for \(q = \infty\).

In the remaining cases the proof is analogous to that of \([47, \text{Theorem 6.5}]\). We omit the details. \(\square\)

We finally note that the result stated in \(\text{Theorem 6.3}\) follows in the usual way from the corresponding result for the weighted Stieltjes transform. Its proof is analogous to that of \(\text{Theorem 6.5}\).

**Theorem 6.6.** Let \(\alpha \in (1, \infty)\). Let \(X\) be a rearrangement-invariant Banach function space over \((0, \infty)\) such that
\[
(6.7) \quad \xi_\alpha \in X'(0, \infty),
\]
where \(\xi_\alpha\) is defined by \((6.2)\). Define the functional \(\sigma\) by
\[
\sigma(f) = \left\| S_\alpha f^* \right\|_{X'(0, \infty)}, \quad f \in \mathcal{M}_+(0, \infty).
\]
Then \(\sigma\) is a rearrangement-invariant norm and
\[
(6.8) \quad S_\alpha : X \to Y,
\]
where \(Y = Y(\sigma')\). Moreover, \(Y\) is the optimal (smallest) rearrangement-invariant space for which \((6.8)\) holds.

Conversely, if \((6.7)\) is not true, then there does not exist a rearrangement-invariant space \(Y\) for which \((6.8)\) holds.

**Acknowledgment.** We are greatly indebted to Lenka Slavíková for stimulating discussions about the subject.

**References**

[1] E. Agora, M. J. Carro, and J. Soria. Boundedness of the Hilbert transform on weighted Lorentz spaces. *J. Math. Anal. Appl.*, 395(1):218–229, 2012.

[2] M. A. Arínó and B. Muckenhoupt. Maximal functions on classical Lorentz spaces and Hardy’s inequality with weights for nonincreasing functions. *Trans. Amer. Math. Soc.*, 320(2):727–735, 1990.
[3] R. J. Bagby. Maximal functions and rearrangements: some new proofs. *Indiana Univ. Math. J.*, 32(6):879–891, 1983.

[4] R. J. Bagby and D. S. Kurtz. $L(\log L)$ spaces and weights for the strong maximal function. *J. Analyse Math.*, 44:21–31, 1984/85.

[5] R. J. Bagby and J. D. Parsons. Orlicz spaces and rearranged maximal functions. *Math. Nachr.*, 132:15–27, 1987.

[6] C. Bennett and R. Sharpley. Weak-type inequalities for $H^p$ and BMO. In *Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1*, Proc. Sympos. Pure Math., XXXV, Part, pages 201–229. Amer. Math. Soc., Providence, R.I., 1979.

[7] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[8] D. W. Boyd. The Hilbert transform on rearrangement-invariant spaces. *Canad. J. Math.*, 19:599–616, 1967.

[9] H. Brézis and S. Wainger. A note on limiting cases of Sobolev embeddings and convolution inequalities. *Comm. Partial Differential Equations*, 5(7):773–789, 1980.

[10] Ju. A. Brudny˘ı. Rational approximation and imbedding theorems. *Dokl. Akad. Nauk SSSR*, 247(2):269–272, 1979.

[11] E. Burianková, D. E. Edmunds, and L. Pick. Optimal function spaces for the Laplace transform. *Rev. Mat. Complut.*, 30(3):451–465, 2017.

[12] M. J. Carro and C. Ortiz-Caraballo. Boundedness of integral operators on decreasing functions. *Proc. Roy. Soc. Edinburgh Sect. A*, 145(4):725–744, 2015.

[13] M. J. Carro and J. Soria. Boundedness of some integral operators. *Canad. J. Math.*, 45(6):1155–1166, 1993.

[14] M. J. Carro and J. Soria. The Hardy-Littlewood maximal function and weighted Lorentz spaces. *J. London Math. Soc. (2)*, 55(1):146–158, 1997.

[15] C. Bennett and R. Sharpley. Interpolation of operators, volume 129 of Pure and Applied Mathematics. Academic Press, Inc., Boston, MA, 1988.

[16] A. Cianchi. Strong and weak type inequalities for some classical operators in Orlicz spaces. *J. London Math. Soc. (2)*, 60(1):187–202, 1999.

[17] A. Cianchi and D. E. Edmunds. On fractional integration in weighted Lorentz spaces. *Quart. J. Math. Oxford Ser. (2)*, 48(192):439–451, 1997.

[18] A. Cianchi, R. Kerman, B. Opic, and L. Pick. A sharp rearrangement inequality for the fractional maximal operator. *Studia Math.*, 138(3):277–284, 2000.

[19] A. Cianchi and V. Musil. Optimal domain spaces in Orlicz-Sobolev embeddings. *To appear in Indiana Univ. Math. J.*, 2019.

[20] A. Cianchi, L. Pick, and L. Slavíková. Higher-order Sobolev embeddings and isoperimetric inequalities. *Adv. Math.*, 273:568–650, 2015.

[21] R. R. Coifman and C. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.*, 51:241–250, 1974.

[22] D. V. Cruz-Uribe, J. M. Martell, and C. Pérez. Weights, extrapolation and the theory of Rubio de Francia, volume 215 of Operator Theory: Advances and Applications. Birkhäuser/Springer Basel AG, Basel, 2011.

[23] M. Cwikel and E. Pustylnik. Weak type interpolation near “endpoint” spaces. *J. Funct. Anal.*, 171(2):235–277, 2000.

[24] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of Lecture Notes in Mathematics. Springer, Heidelberg, 2011.

[25] D. E. Edmunds, R. Kerman, and L. Pick. Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms. *J. Funct. Anal.*, 170(2):307–355, 2000.
[26] D. E. Edmunds and B. Opic. Boundedness of fractional maximal operators between classical and weak-type Lorentz spaces. *Dissertationes Math. (Rozprawy Mat.)*, 410:50, 2002.

[27] W. D. Evans, B. Opic, and L. Pick. Interpolation of operators on scales of generalized Lorentz-Zygmund spaces. *Math. Nachr.*, 182:127–181, 1996.

[28] W. D. Evans, B. Opic, and L. Pick. Real interpolation with logarithmic functors. *J. Inequal. Appl.*, 7(2):187–269, 2002.

[29] R. M. Gabriel. An Additional Proof of a Maximal Theorem of Hardy and Littlewood. *J. London Math. Soc.*, 6(3):163–166, 1931.

[30] D. Gallardo. Orlicz spaces for which the Hardy-Littlewood maximal operator is bounded. *Publ. Mat.*, 32(2):261–266, 1988.

[31] J. García-Cuerva and J. L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[32] A. Gogatishvili, B. Opic, and L. Pick. Weighted inequalities for Hardy-type operators involving suprema. *Collect. Math.*, 57(3):227–255, 2006.

[33] K. Hansson. Imbedding theorems of Sobolev type in potential theory. *Math. Scand.*, 45(1):77–102, 1979.

[34] C. Hertz. The Hardy–Littlewood maximal theorem. *Symposium on Harmonic Analysis, University of Warwick*, 1968.

[35] R. Hunt, B. Muckenhoupt, and R. Wheeden. Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.*, 176:227–251, 1973.

[36] R. Kerman and L. Pick. Optimal Sobolev imbeddings. *Forum Math.*, 18(4):535–570, 2006.

[37] R. Kerman and L. Pick. Optimal Sobolev imbedding spaces. *Studia Math.*, 192(3):195–217, 2009.

[38] S. G. Kreĭn, Yu. I. Petunin, and E. M. Semënov. *Interpolation of linear operators*, volume 54 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, R.I., 1982.

[39] V. G. Maz’ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, augmented edition, 2011.

[40] B. Muckenhoupt. Hardy’s inequality with weights. *Studia Math.*, 44:31–38, 1972. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, I.

[41] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[42] B. Muckenhoupt and R. L. Wheeden. Weighted norm inequalities for singular and fractional integrals. *Trans. Amer. Math. Soc.*, 161:249–258, 1971.

[43] B. Muckenhoupt and R. L. Wheeden. Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform. *Studia Math.*, 55(3):279–294, 1976.

[44] V. Musil and R. Oľhava. Interpolation theorem for Marcinkiewicz spaces with applications to Lorentz gamma spaces. *Math. Nachr.*, 2018.

[45] Vít Musil. Fractional maximal operator in orlicz spaces. *J. Math. Anal. Appl.*, 474(1):94 – 115, 2019.

[46] R. O’Neil. Convolution operators and $L(p, q)$ spaces. *Duke Math. J.*, 30:129–142, 1963.

[47] B. Opic and L. Pick. On generalized Lorentz-Zygmund spaces. *Math. Inequal. Appl.*, 2(3):391–467, 1999.

[48] D. Peša. Equivalence of duals and down duals of certain operator-induced norms. Preprint, 2016.

[49] L. Pick, A. Kufner, O. John, and S. Fučík. *Function spaces. Vol. 1*, volume 14 of *De Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, extended edition, 2013.
[50] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[51] F. Riesz. Sur un théorème de maximum de MM. Hardy et Littlewood. *J. London Math. Soc.*, 7(1):10–13, 1932.

[52] E. T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.

[53] E. T. Sawyer. Boundedness of classical operators on classical Lorentz spaces. *Studia Math.*, 96(2):145–158, 1990.

[54] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[55] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[56] E. M. Stein and G. Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[57] N. Wiener. The ergodic theorem. *Duke Math. J.*, 5(1):1–18, 1939.

David E. Edmunds, Department of Mathematics, University of Sussex, Falmer, Brighton, BN1 9QH, UK

E-mail address: davideedmunds@aol.com

ORCiD: 0000-0003-2394-9385

Zdeněk Mihula, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: mihulaz@karlin.mff.cuni.cz

ORCiD: 0000-0001-6962-7635

Vít Musil, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: musil@karlin.mff.cuni.cz

ORCiD: 0000-0001-6083-227X

Luboš Pick, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: pick@karlin.mff.cuni.cz

ORCiD: 0000-0002-3584-1454