Finite-time consensus of nonlinear multi-agent systems via impulsive time window theory: a two-stage control strategy

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Abstract This paper concentrates on the finite-time consensus problem faced by nonlinear multi-agent systems (MASs) via impulsive time window theory with a two-stage control (TSC) strategy. The TSC strategy divides the whole control period into two parts: a variable impulsive control stage and a finite-time consensus control stage. Different from general single-stage control, TSC can dynamically adjust the time periods of impulsive control and finite-time control according to practical application requirements. Variable impulsive control is also discussed in this paper. Compared with the sampling based on traditional fixed impulsive theory, impulsive sampling in the TSC strategy occurs randomly within an impulsive time window and provides much more flexibility. In addition, a switching topology scheme is introduced in this paper to strengthen the stability of MASs. Finally, two numerical simulation examples (one leaderless case and one leader-following case) are used for the theoretical analysis.

Keywords Two-stage control (TSC) · Finite-time consensus · Impulsive time window theory · Multi-agent systems (MASs)

1 Introduction

In recent decades, a tremendous amount of research has focused on the distributed collaborative evolution of multi-agent systems (MASs) such as unmanned aerial vehicles (UAVs), sensor networks, flocking [1–5]. Recently, with intensive study in this field, the topic of consensus problems has gradually attracted the attention of researchers [6–9].

However, the high communication cost produced by continuous control was not considered in [6–9]. Therefore, impulsive control has adopted appropriate control protocols to overcome this shortcoming [10]. As crucial and efficient control theory techniques, impulsive control schemes have been studied in MASs. This type of strategy only allows each node to transmit information at impulsive instants, which dramatically reduces the network’s communication load. Recently, numer-
ous meaningful results regarding MASs consensus via impulsive control were extensively investigated in [11–14]. Guan et al. [13] proposed three types of impulsive protocols with a fixed topology, a switching topology, and external disturbances to reach consensus on continuous-time MAS networks. Liu et al. [14] presented a distributed impulsive consensus protocol with input saturation to guarantee asymptotically dynamic consensus.

The impulsive control methods proposed in [10–14] are performed under a fixed impulsive interval. In other words, the impulsive instants and impulsive intervals must be defined in advance, which makes the consensus condition stricter. Nevertheless, the information exchange between agents cannot precisely occur at fixed impulsive sampling times because of external interference and practical system constraints. For example, impulsive sampling should occur at a fixed time of $t_n$. Nevertheless, practical impulsivity may occur during an impulsive time window $(t_n - R, t_n + R)$ of radius $R$. In other words, the aforementioned conditions may lead to an incorrect impulsive consensus in the system. To obtain more flexible impulsive instants, some valuable methods for variable impulsive control with impulsive inputs constrained in impulsive time windows were proposed in [15, 16]. In recent years, the application of variable impulsive control in MASs has been discussed in [17–19]. Zhang et al. [17, 18] discussed the variable impulsive control with leader-following consensus for MASs with directed network topologies. Ma et al. [19] proposed a variable impulsive technique in an undirected interaction topology for the consensus of MASs.

Much research on impulsive control has been investigated. These studies have mainly concentrated on the final state of the given system and ignored the time required by the system to reach consensus. In other words, an MAS can enter an equilibrium state when $t \rightarrow \infty$. Compared with asymptotic convergence, finite-time consensus improves the convergence rate of the system. Therefore, the combination of impulsive control and finite-time consensus theory considers both communication cost and convergence rate parameters [20–23]. Li [22] presented a consensus method for MASs within finite time via impulsive control under disturbances. Tian and Li [23] discussed the finite-time consensus of second-order MASs with impulsive effects. However, the previous study considered only fixed impulsive control. Therefore, the combination of variable impulsive control and finite-time consensus relaxes the restriction of the consensus condition and improves the flexibility of the system.

Last but not least, many current studies focus on fixed topology networks [24–26]. However, the communication topologies between agents change due to external disturbances or link failures. Therefore, switching topology is also discussed in this paper. In addition, the current research on the combination of impulsive control and finite-time consensus adopts full-stage control, which means that both control methods act from the beginning of the process. However, due to the increasing complexity and diversification of the application scenarios of MASs, the convergence state of full-stage control is relatively singular with poor flexibility. Inspired by the related works, our paper concentrates on the finite-time consensus of nonlinear MASs via impulsive time window theory and a two-stage control (TSC) strategy. Knowledge about finite-time stability theory, algebraic graph theory, and impulsive differential equations is used to achieve MAS consensus within finite time. The primary contributions of this paper are as follows.

1. Unlike previous research, in this study, TSC is adopted for nonlinear MASs. TSC divides the whole control period into two parts: a variable impulsive control stage and a finite-time consensus control stage. This strategy can adjust the convergence period dynamically based on different practical application scenarios. Because impulsive control is a noncontinuous control strategy, the real-time update requirement is not strict. If the communication network conditions are not good, then the impulsive control period can be appropriately extended to reduce continuous control. On the other hand, if an MAS needs to obtain a faster convergence rate and the network conditions are good, the number of impulsive samples can be appropriately reduced. Therefore, the advantage of the proposed TSC technique in this paper is that it provides more options for practical applications.

2. Variable impulsive control technology is introduced in this paper. Compared with the fixed impulsive theory methods in existing studies, the variable impulsive strategy does not need to preset the impulsive sampling interval. Furthermore, impulsive instants can occur any time within the impulsive time window to increase the system flex-
isent by a degree matrix \( G \). The number of edges connected to each node is repre-
\[ \text{resented by a degree matrix } G. \]

(3) Due to the unreliability of an actual communication network, the communication links between agents may be interrupted. Therefore, this paper also considers switching topology criteria to improve the stability of MAS networks.

This paper proceeds as follows. Algebraic graph theory is introduced in Sect. 1. We formulate some related lemmas, definitions, and assumptions in Sect. 2. In Sect. 3, a rigorous mathematical proof is made for MASs in the leaderless case and leader-following case with switching topology to reach consensus states. Some numerical simulations are shown in Sect. 4. Lastly, the conclusion is discussed.

Notations \( \mathbb{R} \) is the set of real numbers. \( \mathbb{R}^{m} \) is a Euclidean space, \( \mathbb{R}^{m \times k} \) is the real matrix set of \( m \times k \). \( I_{m} \) is expressed as an identity matrix. \( \mathbb{N}^{+} \) is defined as a positive integer set. The symbol \( \| \cdot \| \) represents the Euclidean norm operation. \( \otimes \) denotes the Kronecker product. \( \lambda_{\max}(\mathcal{K}), \lambda_{\min}(\mathcal{K}) \), and \( \mathcal{K}^{T} \) define the largest and smallest eigenvalues and the transpose of any matrix \( \mathcal{K} \), respectively.

2 Preliminaries

2.1 Algebraic graph theory

In this section, algebraic graph theory information is provided. A directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \omega) \) describes the network structure of a MAS, where \( \mathcal{V} = (v_1, v_2, \ldots, v_N) \) is the set of agents in the network, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is a link set. Assume that \( \omega = [a_{ij}] \in \mathbb{R}^{N \times N} \) represents the relationships among all agents. If agent \( i \) can communicate with agent \( j \), then \( a_{ij} > 0 \); otherwise, \( a_{ij} = 0 \), and we let \( a_{ij} = 0 \). Moreover, \( \mathcal{G} \) is a directed balanced graph if \( a_{ij} = a_{ji} > 0 \). \( \mathcal{N}_{i} = \{ j \mid (v_i, v_j) \in \mathcal{E} \} \) represents all neighbors of agent \( i \).

The number of edges connected to each node is represented by a degree matrix \( D = \text{diag}(d_1, d_2, \ldots, d_N) \). Furthermore, the Laplacian matrix is expressed as \( \mathcal{L} = \mathcal{D} - \omega \), where \( \mathcal{L} = [l_{ij}]_{N \times N} \).

Since the topology graph that describes the agents varies over time, a directed switching topology is considered as follows. Here, we define graphs \( \mathcal{G}_g \), \( g = \{1, 2, 3, \ldots, \psi \}, \psi \in \mathbb{N}^+ \), where \( g \) represents all possible topologies. The switch signal is denoted as \( \sigma(t) : [t_0, +\infty) \rightarrow \mathcal{G} \), which is piecewise constant. Notably, \( \mathcal{G}_{\sigma(t)} \in \mathcal{G} \), \( \sigma \in \mathbb{R}^1 \), and \( \sigma(t) \) is fixed for \( t \in [t_n, t_{n+1}) \). Therefore, for each possible graph \( \mathcal{G}_{\sigma(t)} \), the aforementioned matrix is redefined as follows: \( \mathcal{L}_{\sigma(t)} = \left[ l_{ij}^{\sigma(t)} \right], \omega_{\sigma(t)} = \left[ a_{ij}^{\sigma(t)} \right] \)

and \( \mathcal{L}_{\sigma(t)} = \mathcal{G}_{\sigma(t)} - \mathcal{G}_{\sigma(t)}. \)

2.2 Problem statement

The classical model of a nonlinear MAS is considered in this section. We propose a dynamic equation with \( N \) agents, which is expressed by

\[ \dot{x}_i(t) = \mathcal{E} x_i(t) + \varphi_i(x_i(t)) + u_i(t), \quad i = 1, 2, \ldots, N, \]

where \( x(t) \in \mathbb{R}^{m} \) denotes the state vector, and \( x_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{im}(t)]^{T} \) is defined as the state of the MAS network. The control protocol of the network is indicated by \( u_i(t) \). \( \mathcal{E} \in \mathbb{R}^{m \times m} \) represents a constant matrix. A continuous nonlinear differentiable function is described by \( \varphi_i(x_i(t)) = [\varphi_{i1}(x_{i1}(t)), \varphi_{i2}(x_{i2}(t)), \ldots, \varphi_{im}(x_{im}(t))]^{T} \).

Assumption 1 There is a directed spanning tree in \( \mathcal{G}_{\sigma(t)} \), and a leader acts as the root node. Moreover, for all possible topologies \( \mathcal{G} \), the directed graph \( \mathcal{G}_{\sigma(t)} \in \mathcal{G} \) is strongly connected.

Assumption 2 Assume that \( \varphi(\cdot) \) is a continuous nonlinear function that satisfies the Lipschitz condition; then,

\[ \| \varphi(c) - \varphi(d) \| \leq \Gamma \| c - d \|, \quad \forall c, d \in \mathbb{R}^{m}, \]

where \( \Gamma \) is a known positive parameter.

Lemma 1 [25] Supposing that Assumption 1 holds, let the directed digraph \( \mathcal{G}_{\sigma(t)} \) be associated with \( \mathcal{L}_{\sigma(t)} \). Then, all eigenvalues of \( \mathcal{L}_{\sigma(t)} \) satisfy 0 = \( \lambda_1(\mathcal{L}_{\sigma(t)}) < \lambda_2(\mathcal{L}_{\sigma(t)}) \leq \ldots \leq \lambda_N(\mathcal{L}_{\sigma(t)}) \), \( \mathcal{L}_{\sigma(t)}I_N = 0 \), where \( I_N = [1, 1, \ldots, 1]^{T} \in \mathbb{R}^{N} \) is an eigenvector of \( \mathcal{L}_{\sigma(t)} \) associated with zero eigenvalue. Re \( \lambda_i(\mathcal{L}_{\sigma(t)}) > 0 \), \( i = 2, 3, \ldots, N. \)

Lemma 2 [27] If Assumption 1 holds, for any \( \gamma = [\gamma_1, \gamma_2, \ldots, \gamma_N]^{T} \), the following rule is satisfied:

\[ \gamma^{T} \mathcal{L}_{\sigma(t)} \gamma = \frac{1}{2} \sum_{i,j=1}^{N} a_{ij}^{\sigma(t)} (\gamma_i - \gamma_j)^2, \]
and $\mathcal{L}_{\sigma(t)}$ is positive semidefinite.

**Lemma 3** [28] Let $\gamma_1, \gamma_2, \ldots, \gamma_N \geq 0$, $0 < \zeta \leq 1$ and $u > 1$; then,
\[
N \sum_{i=1}^{N} \gamma_i^z \geq \left( \sum_{i=1}^{N} \gamma_i \right)^z \quad \text{and} \quad \sum_{i=1}^{N} \gamma_i^u \geq N^{1-u} \left( \sum_{i=1}^{N} \gamma_i \right)^u .
\]

**Lemma 4** [29] Assume that two constants $\xi \in (0, 1)$ and $\alpha > 0$ exist and that the continuous positive definite function $V : U \to \mathbb{R}$ satisfies:
\[
\dot{V}(x) \leq -\alpha V^\xi(x), \quad x \in \theta \setminus \{0\}.
\]

Then, the settling time that depends on the initial condition $x_0 \in U$ is expressed as the following inequality:
\[
T(x_0) \leq V_1^{1-\xi} (x_0),
\]
where $U$ and $\theta$ are the open neighborhoods of the origin, and $\theta \subseteq U = \mathbb{R}^m$. $V(x)$ is radially unbounded; hence, the origin is globally finite-time stable.

### 3 Main results

#### 3.1 Leaderless consensus

An effective TSC protocol without a leader is considered in this subsection as follows:

\[
u_i(t) = \begin{cases} b_n \sum_{n=1}^{\infty} \delta(t-t_n) \sum_{j \in A_i} a_{ij}^{\sigma(t)} (x_i(t)-x_j(t)), & t \in [0, T_1), \\ -\sigma x_i(t) - \nu \sum_{j \in A_i} a_{ij}^{\sigma(t)} \text{sign}|x_i(t)-x_j(t)|^\xi, & t \in [T_1, T_2), \end{cases}
\]

where $\delta(t-t_n)$ is an impulse function, and $b_n$ is an impulsive control gain. For simplicity, let $\text{sign}(y)|y|^\eta = \text{sign}(y)|y|^\eta$ and $\text{sign}(y_i) = [\text{sign}(y_{i1}), \text{sign}(y_{i2}), \ldots, \text{sign}(y_{im})]^T$. The impulsive sequence satisfies $0 < t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots$ and $\lim_{n \to \infty} t_n = +\infty$, and $\Pi = t_{n+1} - t_n$ is the maximum upper bound of the impulsive time window. We define $\Delta x_i(t_n) = x_i(t_n^+) - x_i(t_n^-)$ and assume that $x_i(t)$ is right-continuous; therefore, $x_i(t_n^+) = x_i(t_n^-) = \lim_{t \to t_n^+} x_i(t)$. The control parameter of the finite-time protocol is defined as $\sigma$, which plays an essential role in the smoothness of the consensus curve and convergence rate. $\nu > 0$ and $0 < \xi < 1$ are positive constants. A sign function is presented as $\text{sign}(y) = \begin{cases} -1, & \text{if } y < 0, \\ 0, & \text{if } y = 0, \\ 1, & \text{if } y > 0. \end{cases}$

For the first control stage, combining controller (3) and system (1), the impulsive control system can be defined as

\[
x_i(t) = \mathcal{E} x_i(t) + \varphi_i (x_i(t)), \quad t \neq t_n, \\
\Delta x_i(t_n) = b_n \sum_{j \in A_i} a_{ij}^{\sigma(t)} (x_i(t_n^-) - x_j(t_n^-)), \quad t = t_n, \\
x_i(t_n^+) = x_i(t_n^-), \quad n \in \mathbb{N}.
\]

**Definition 1** The average consensus of MASs (1) can be reached within finite time under condition (3) for any initial condition if
\[
\lim_{t \to T_2} \left\| x_i(t) - \frac{1}{N} \sum_{i=1}^{N} x_i(t) \right\| = 0, \quad i = 1, 2, \ldots, N.
\]

**Definition 2** The upper right-hand Dini derivative can be described as
\[
D^+ \varphi(x(t)) = \lim_{h \to 0^+} \sup_{t_n < t < t_n+h} \frac{1}{h} [\varphi(x(t+h)) - \varphi(x(t))].
\]

**Assumption 3** The first stage has the following rule
\[
t_{n+1}^l < t_n < t_{n+1}^u < t_{n+1}^l < t_{n+1} < t_{n+1}^u.
\]

**Remark** As shown in Fig. 1, the two endpoints of the $n$th variable impulsive time window are composed of $t_n^l$ and $t_n^u$. $t_n$ is the sampling time of the $n$th impulsive window and follows the criterion of Assumption 3. $T_1$ is the dividing point between the first controlling stage and the second controlling stage; $T_1$ simultaneously serves as the endpoint of the first controlling part and the start point of the second controlling part. Moreover, $T_1$ is independent of the initial states of the agents, and it is determined by the number of impulsive intervals set in advance.
\textbf{Assumption 4} Supposing that the switching graph $\mathcal{G}(t)$ is connected and balanced, let $\hat{x}(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t) = \frac{1}{N} (1_N \otimes I_m) x(t)$ be the average state of all agents. Then, one obtains

$$
\dot{\hat{x}}(t_n^+) = \frac{1}{N} (1_N \otimes I_m) x(t_n^+)
= \frac{1}{N} (1_N \otimes I_m) x(t_n^-)
- \frac{1}{N} (1_N \otimes I_m) b_n (1_N \otimes \mathcal{L}_\sigma(t) \otimes I_m) x(t_n^-)
= \hat{x}(t_n^-).
$$

(6)

If the graph is balanced, then $(1_N \otimes I_m) (b_n \mathcal{L}_\sigma(t) \otimes I_m) = b_n 1_N \mathcal{L}_\sigma(t) = 0$. Therefore, we denote $\epsilon_i(t) = x_i(t) - \hat{x}(t)$ as an error vector, where $\epsilon_i(t) = [\epsilon_{i1}(t), \epsilon_{i2}(t), \ldots, \epsilon_{in}(t)]^T$.

With Eq. (6), the Kronecker product is adopted to rewrite Eq. (4) into matrix form.

$$
\begin{align*}
\dot{\epsilon}(t) &= (1_N \otimes \Xi) \epsilon(t) + \varphi(x(t)) - \frac{1}{N} (1_N \otimes I_m) \varphi(x(t)), \\
\epsilon(t) &= [\epsilon_1^T(t), \epsilon_2^T(t), \ldots, \epsilon_N^T(t)]^T, \varphi(x(t)) = [\varphi_1^T(x_1(t)), \varphi_2^T(x_2(t)), \ldots, \varphi_N^T(x_N(t))]^T \in \mathbb{R}^{mN}.
\end{align*}
$$

(7)

where $\epsilon(t) = [\epsilon_1^T(t), \epsilon_2^T(t), \ldots, \epsilon_N^T(t)]^T$, $\varphi(x(t)) = [\varphi_1^T(x_1(t)), \varphi_2^T(x_2(t)), \ldots, \varphi_N^T(x_N(t))]^T \in \mathbb{R}^{mN}$.

\textbf{Theorem 1} Let Assumptions 1–4 hold and let $\lambda_2(\mathcal{L}_\sigma(t))$ represent the algebraic connectivity of the Laplacian matrix $\mathcal{L}_\sigma(t)$. Suppose that $\lambda_A$ and $\lambda_B$ describe the maximum eigenvalues of $(\Xi^T + \Xi)$ and $\Xi$, respectively; $0 < \rho_n < 1$ is the eigenvalue of $((b_n (I_m \otimes \mathcal{L}_\sigma(t)) + I_m N))$; $0 < \rho_n < 1$ is the eigenvalue of $(\mathcal{L}_\sigma(t))$; and $\lambda_B = 2 \Gamma$.

If there exists a constant $\zeta > 0$ such that $(\tau_{n+1}^l - \tau_n^l) (\lambda_A + 2 \Gamma) + \ln(\rho_n \zeta) \leq 0$, and the system satisfies the condition $2 \lambda_B + 2 \Gamma \leq 2 \sigma$, then the leaderless case of MASs can reach average consensus.

\textbf{Proof} In the first stage $t \in [t_0, T_1)$, the Lyapunov function is defined as

$$V(t) = \frac{1}{2} \epsilon^T(t) \epsilon(t).
$$

(8)

When $t \neq \tau_n$, we can obtain the derivation of the Lyapunov function as

\[D^+ V(t) = \frac{1}{2} \left( \epsilon^T(t) \dot{\epsilon}(t) + \dot{\epsilon}^T(t) \epsilon(t) \right) = \frac{1}{2} \left( \epsilon^T(t) \left( (I_N \otimes \Xi) \epsilon(t) + \varphi(x(t)) \right) - \frac{1}{N} (1_N \otimes I_m) \varphi(x(t)) + ((I_N \otimes \Xi) \epsilon(t) + \varphi(x(t)) - \frac{1}{N} (1_N \otimes I_m) \varphi(x(t))) \right) \]

\[+ \frac{1}{2} \left( (I_N \otimes \Xi^T) \epsilon(t) + (I_N \otimes \Xi) \epsilon(t) \right) \]

\[+ \frac{1}{2} \left( \lambda_A \epsilon^T(t) \epsilon(t) + 2 \epsilon^T(t) \varphi(x(t)) - \frac{1}{N} (1_N \otimes I_m) \varphi(x(t)) \right) \]

\[\leq \frac{1}{2} \left( \lambda_A \epsilon^T(t) \epsilon(t) + 2 \epsilon^T(t) \varphi(x(t)) - \frac{1}{N} (1_N \otimes I_m) \varphi(x(t)) \right).\]

(9)

According to Assumption 2,

$$\varphi(x(t)) - \varphi(\hat{x}(t)) \leq \Gamma (x(t) - \hat{x}(t)) \leq \Gamma \epsilon(t).$$

(10)

Then, we can easily obtain that

$$D^+ V(t) \leq \frac{1}{2} \left( \lambda_A \epsilon^T(t) \epsilon(t) + 2 \epsilon^T(t) \Gamma \epsilon(t) \right) \leq (\lambda_A + 2 \Gamma) \epsilon(t) \leq (\lambda_A + 2 \Gamma) V(t).$$

(11)

$t \in [\tau_n^l, \tau_{n+1}^l)$ leads to

$$V(\tau_{n+1}^l) \leq V(\tau_n^l) \exp(\lambda_A + 2 \Gamma)(\tau_{n+1}^l - \tau_n^l).$$

(12)

When $t = \tau_n$, we have

$$V(\tau_n^+) = \frac{1}{2} \epsilon^T(\tau_n^+) \epsilon(\tau_n^+).$$

\[= \frac{1}{2} \left( b_n((I_m \otimes \mathcal{L}_\sigma(t)) + I_m N) \right)^T \]

\[\times \epsilon^T(\tau_n^+) \left( b_n((I_m \otimes \mathcal{L}_\sigma(t)) + I_m N) \right) \epsilon(\tau_n^-) \leq \rho_n V(\tau_n^-).\]

(13)
When \( t \in [t_0, \tau_1^-) \), through (12), we have

\[
V(\tau_1^-) \leq V(t_0) \exp(\lambda_A + 2\Gamma)(\tau_1^- - t_0). \tag{14}
\]

By (12), we can subdivide the impulsive time window. For \( t \in [\tau_1^n, t_0^n) \cup [t_0^n, \tau_2^0) \cup [\tau_2^0, \tau_1^0] \), when \( n = 1, t \in [\tau_1^0, t_1) \cup [t_1, \tau_1^+] \cup [\tau_1^+, \tau_2^+ \cup \tau_2^+ - t_0) \).

For \( t \in [\tau_1^0, t_1) \), we have

\[
V(t_1^-) \leq V(t_0) \exp(\lambda_A + 2\Gamma)(t_1 - t_0). \tag{15}
\]

When \( t = t_1 \), by inequality (13), one has

\[
V(t_1^+) \leq \rho_1 V(t_1^-). \tag{16}
\]

For \( t \in [t_1, \tau_1^+) \), it follows from (15) and (16) that

\[
V(\tau_1^+) \leq V(t_1^-) \exp(\lambda_A + 2\Gamma)(\tau_1^+ - t_1)
\leq \rho_1 V(t_0) \exp(\lambda_A + 2\Gamma)(\tau_1^+ - t_0).
\tag{17}
\]

Therefore, when \( t \in [\tau_1^0, \tau_1^+) \), the following can be derived from condition (17):

\[
V(\tau_1^+) \leq V(t_0) \exp(\lambda_A + 2\Gamma)(\tau_1^+ - t_0).
\tag{18}
\]

Similar to the above derivation process, when \( n = 2, \) for \( t \in [\tau_2^0, \tau_2^+) \),

\[
V(\tau_2^+) \leq \rho_1 \rho_2 V(t_0) \exp(\lambda_A + 2\Gamma)(\tau_2^+ - t_0). \tag{19}
\]

In general, for \( t \in [\tau_n^0, \tau_n^+) \), from the inequality

\[
\rho_n \exp(\lambda_A + 2\Gamma)(\tau_n^+ - \tau_n^-) \leq \frac{1}{c}. \tag{20}
\]

One can obtain

\[
V(t) \leq \rho_1 \rho_2 \times \ldots \times \rho_n V(t_0) \exp(\lambda_A + 2\Gamma)(t - t_0)
\leq V(t_0) \exp(\lambda_A + 2\Gamma)(t_1 - t_0) \rho_1
\exp(\lambda_A + 2\Gamma)(\tau_2^+ - t_1)
\times \ldots \times \rho_{n-1} \exp(\lambda_A + 2\Gamma)(\tau_n^+ - \tau_n^-)
\times \rho_n \exp(\lambda_A + 2\Gamma)(t - \tau_n^+)
\leq \frac{1}{c} \rho_n^\varepsilon V(t_0) \exp(\lambda_A + 2\Gamma)(t - \tau_n^+ + \tau_1^- - t_0).
\tag{21}
\]

Let \( c = \begin{cases} 1, & t \geq t_n, \\ 0, & t < t_n. \end{cases} \)

When \( t_n = T_1, \omega \in \mathbb{N}^+ \), inequality (21) further yields

\[
V(T_1) \leq \frac{1}{\xi^{\omega}} V(t_0) \exp(\lambda_A + 2\Gamma)(t_0 - t_0^\omega + \tau_1^- - t_0). \tag{22}
\]

where \( t_0^\omega \) is the number of impulsive sampling occurrences in the first stage, and \( T_1 \) is a random value in \([\tau_0^\omega, \tau_0^\omega + \tau_1^- - t_0]\).

We can set the size of \( \omega \) according to the actual application scenario, and the number of impulsive instants determines \( T_1 \). Furthermore, \( T_1 \) is the initial time of the finite-time stage and the time elapsed during the entire impulsive period. 

\( V(T_1) \) is the initial state of the finite-time control stage.

The error system for the finite-time control stage can be defined as follows:

\[
\hat{e}_i(t) = \Xi e_i(t) + \hat{\phi}_i(x_i(t), \hat{x}(t)) - \sigma e_i(t)
- \frac{\nu}{2} \sum_{j \in \mathcal{A}_j} a_{ij}^\sigma(t) \text{sign}[e_i(t) - e_j(t)]^\varepsilon, \tag{23}
\]

let \( \hat{\phi}_i(x_i(t), \hat{x}(t)) = \phi_i(x_i(t)) - \phi_i(\hat{x}(t)) \).

According to Theorem 1, for the second stage \( t \in [T_1, T_2] \), the Lyapunov function is expressed as \( V(t) = \frac{1}{2} e^T(t)e(t) \); then, one obtains

\[
D^+ V(t) = \sum_{i=1}^{N} \dot{e}_i(t) \hat{e}_i(t)
= \sum_{i=1}^{N} e_i(t) \left( \Xi e_i(t) + \hat{\phi}_i(x_i(t), \hat{x}(t)) - \sigma e_i(t)
- \frac{\nu}{2} \sum_{j \in \mathcal{A}_j} a_{ij}^\sigma(t) \text{sign}[e_i(t) - e_j(t)]^\varepsilon \right)
\leq -\frac{\nu}{2} \sum_{i=1}^{N} \sum_{j \in \mathcal{A}_j} \epsilon_i(t) a_{ij}^\sigma(t) \text{sign}[e_i(t) - e_j(t)]^\varepsilon
\leq -\frac{1}{2} \sum_{i=1}^{N} \sum_{j \in \mathcal{A}_j} \epsilon_i(t) a_{ij}^\sigma(t) \text{sign}[e_i(t) - e_j(t)]^\varepsilon.
\tag{24}
\]
By Lemmas 1, 2, and 3, one obtains

\[ D^+ V(t) \leq -\frac{1}{2} \nu \left( \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} (a_{ij}(t))^\frac{\nu}{2} (\epsilon_i(t) - \epsilon_j(t))^2 \right)^{\frac{1+\nu}{2}} \]

\[ \leq -\frac{1}{2} \nu \left( \frac{2\epsilon^T(t)\mathcal{L}_\sigma(t)\epsilon(t)}{1 - \frac{\nu}{2} \epsilon^T(t)\epsilon(t)} \right) \]

\[ \leq -\frac{1}{2} \nu \left( 4\lambda_2(\mathcal{L}_\sigma(t)) \right)^{\frac{1+\nu}{2}} V^{\frac{1+\nu}{2}}(t). \quad (25) \]

Thus, from Lemma 4, the time for MASs (1) to reach consensus is

\[ T_2 \leq T_1 + \frac{4}{\nu \left( 4\lambda_2(\mathcal{L}_\sigma(t)) \right)^{\frac{1+\nu}{2}} (1 - \xi)} V^{\frac{1+\nu}{2}}(T_1). \quad (26) \]

\( T_2 \) is the whole convergence time of the impulsive control stage and finite-time stage. It can be derived from the above inequality that \( V(t) \equiv 0 \) for all \( t \geq T_2 \). Thus, the leaderless case of the MASs exhibits global finite-time stability. The proof is completed. \( \square \)

**Remark 2** Different from most studies, this paper divides the control protocol into two stages. The TSC strategy is more flexible so that it can be adjusted in different scenes according to the actual application scenarios. For example, in a case with abundant communication resources, if the MASs need to reach consensus faster, then the number of impulsive occurrences can be reduced to achieve this goal. In another scenario, if the communication network conditions are not good, the requirement regarding the convergence speed of the system is not strict. Then, the impulsive control period can be appropriately extended. With full-stage control, the consensus of the MASs is not flexible. Thus, in general, the TSC strategy provides more options according to different scenarios.

### 3.2 Leader-following consensus

A dynamic model of the leader node 0 is considered in this subsection:

\[ \dot{x}_0(t) = \mathcal{L} x_0(t) + \varphi_0(x_0(t)). \quad (27) \]

Supposing that \( x_0(t) \in \mathbb{R}^m \) represents the state of node 0, the switching interaction topology for a leader is defined by \( \mathcal{G}_0(t) \).

**Remark 3** In the leaderless case, each node in the network can communicate with other nodes, and the state values of all agents can eventually converge to an average value. In contrast, the leader-following consensus case can ignore the information of follower nodes. Namely, all follower nodes directly or indirectly communicate with the leader node. Finally, all agents will be in the same state as that of the leader node. Furthermore, we can choose different control schemes according to the requirements of practical applications. Therefore, the consensus of MASs with a leader is discussed in this subsection.

Consider an effective TSC protocol with a leader as follows:

\[ u_i(t) = \begin{cases} b_n \sum_{n=1}^{\infty} \delta(t - t_n) \bigg( \sum_{j \in \mathcal{N}_i} a_{ij}(t) (x_i(t) - x_j(t)) \\ + q_i (x_i(t) - x_0(t)) \bigg), t \in [t_0, T_1), \\ -\sigma(x_i(t) - x_0(t)) - \nu \left( \sum_{j \in \mathcal{N}_i} a_{ij}(t) \text{sign}(x_i(t)) \right. \\ \left. - x_j(t) \right)^\xi + q_i \text{sign}(x_i(t) - x_0(t))^\xi \bigg), t \in [T_1, T_2). \end{cases} \quad (28) \]

Let \( q_i > 0 \) represent the relationship between leader node 0 and a follower node, and let the remaining parameters be identical to those in the leaderless case.

**Definition 3** For any initial state, the consensus of MASs with a leader can be realized within finite time, and the following is satisfied:

\[ \lim_{t \to T_2} \| x_i(t) - x_0(t) \| = 0, \quad i = 1, 2, \ldots, N. \quad (29) \]

Let \( \xi(t) = x_i(t) - x_0(t) \), where \( \xi(t) = [\xi_{i1}(t), \xi_{i2}(t), \ldots, \xi_{im}(t)]^T \). Similar to (7), for the first control stage, we can rewrite the error system of condition (28) into a matrix form:

\[ \begin{cases} \dot{\xi}(t) = (IN \otimes \mathcal{L} \sigma(t)) \xi(t) + \dot{\varphi}(x(t), \bar{x}_0(t)), \quad t \neq t_n, \\ \Delta \xi(t_n) = b_n(I_m \otimes \mathcal{H}_i(t_n)) \xi(t_n^-), \quad t = t_n, \end{cases} \quad (30) \]

where \( \mathcal{H}_i(t) = \mathcal{L}_i(t) + \mathcal{D} \), which describes the structure of \( \mathcal{G}_i(t) \). \( \dot{\varphi}(x(t), \bar{x}_0(t)) = [\varphi_1^T(x_1(t)) - \varphi_0^T(x_0(t)), \varphi_2^T(x_2(t)) - \varphi_0^T(x_0(t)), \ldots, \varphi_N^T(x_N(t)) - \varphi_0^T(x_0(t))]^T \), where \( \bar{x}_0(t) = 1_N \otimes x_0(t) \in \mathbb{R}^{mn}, \xi(t) = [\xi_{11}(t), \xi_{12}(t), \ldots, \xi_{1m}(t)]^T, \) and \( \mathcal{D} = \text{diag}[q_1, q_2, \ldots, q_N] \) as a diagonal matrix.
Theorem 2 If Assumptions 1–4 hold, let \( \lambda_{\text{min}}(H_{\eta(t)}) \) represent the smallest eigenvalue of matrix \( H_{\eta(t)} \). Let \( \lambda_A \) and \( \lambda_B \) denote the maximum eigenvalues of \( ZX + \Sigma \) and \( \Sigma \), respectively. \( 0 < \theta < 1 \) represent the eigenvalue of \( [(b_n(I_m \otimes H_{\eta(t)}) + I_{mN})]^T [(b_n(I_m \otimes H_{\eta(t)}) + I_{mN})] \). Assuming that there is a known parameter \( \xi > 1 \) such that \( (\tau_{n+1} + \tau_n)(\lambda_A + 2\Gamma) + \ln(\theta_n \xi) \leq 0 \) and the system satisfies the condition \( 2\lambda_B + 2\Gamma \leq 2\sigma \), then the consensus of MASs with a leader can be achieved within finite time.

Proof The Lyapunov function is chosen as

\[
V(t) = \frac{1}{2} \xi^T(t)\xi(t).
\] (31)

The proof process for the first stage is similar to that of the leaderless case; then, for \( t \in [\tau_n, \tau_{n+1}] \), we can obtain

\[
V(t) \leq \frac{1}{\xi^{n-1}} V(t_0) \exp(\lambda_A + 2\Gamma)(t - \tau_n + \tau_{n+1} - t_0).
\] (32)

where \( c = \begin{cases} 1, & t \geq t_n, \\ 0, & t < t_n. \end{cases} \)

When \( t_0 = T_1, \omega \in \mathbb{N}^+ \), inequality (32) can further yield

\[
V(T_1) \leq \frac{1}{\xi_0} V(t_0) \exp(\lambda_A + 2\Gamma)(t_0 - \tau_n + \tau_{n+1} - t_0).
\] (33)

Compared with (23), the error equation of the second stage can be rewritten as

\[
\dot{\xi}_i(t) = Z\xi_i(t) + \varphi_i(x(t), x_0(t)) - \sigma \xi_i(t) - \nu \left( \sum_{j \in \mathcal{N}_i} a_{ij} \xi_j(t) \right) + q_i \cdot \text{sign}(|\dot{\xi}_i(t)|^\xi) \]

(34)

By Lemmas 1, 2, and 3, the proof process is identical to that of (24), and by taking the derivative of \( V(t) \), for simplicity, we have

\[
D^+ V(t) = \sum_{i=1}^{N} \xi_i(t) \dot{\xi}_i(t) = \sum_{i=1}^{N} \xi_i(t) \left( Z\xi_i(t) + \varphi_i(x(t), x_0(t)) - \sigma \xi_i(t) - \nu \sum_{j \in \mathcal{N}_i} a_{ij} \xi_j(t) \right) + q_i \cdot \text{sign}(|\dot{\xi}_i(t)|^\xi).
\]

Therefore, from Lemma 4, the settling time of the leader-following consensus problem is

\[
T_2 \leq T_1 + \frac{4}{\nu \left( 4\lambda_{\text{min}}(H_{\eta(t)}) \right)^{1+\xi}} V^{1+\xi}(T_1).
\] (36)

Then, this result shows that \( V(t) \equiv 0 \) for all \( t \geq T_2 \). Thus, MAS consensus for the leader-following case is reached.

\[ \square \]

Remark 4 If the impulsive interval \( \tau_n = \tau_{n+1} - t_n \) is a positive constant, we can say that the MASs based on fixed impulsive control can realize consensus within finite time. Supposing that Assumptions 1–4 are satisfied, Theorem 2 of this paper can be rewritten as the same consensus criterion in Theorem 1 of [11]; that is, \( (\tau_{n+1} - t_n)(\lambda_A + 2\Gamma) + \ln(\theta_n \xi) \leq 0 \). The proof process for the second control stage corresponds to Theorem 2.

Remark 5 Although the TSC problem has been discussed in [20], MAS consensus via variable impulsive control under switching topology was not considered. Additionally, different from that of the fixed-time impulsive control strategies in [20–22], and [23], the impulsive interval \( \tau_n = t_{n+1} - t_n \) is a fixed positive constant for any \( n \). For the variable impulsive control strategy, by Theorem 1 and Theorem 2, the impulsive time window can be adjusted within a maximum range \( \Pi = \tau_{n+1} - \tau_n \), where \( \Pi \) are the components of the free time window and impulsive time window. Consequently, the \( t_n \) impulsive instants can more flexibly occur within the maximum impulsive time window, which enables the system to handle more complex situations in practical systems, such as time delays and temporary interruptions of communication links.

4 Simulation

Two simulation examples with and without a leader are given in this section to prove the effectiveness and reliability of our control protocol in (3) and (28). Subsequently, we employ Chua’s circuit [30], which can be utilized to describe the dynamic behaviors of the MASs:

\[
\begin{align*}
\dot{x}_{11}(t) &= \mu_1(x_{12}(t)) - x_{11}(t) - \phi(x_{11}(t)), \\
\dot{x}_{12}(t) &= x_{11}(t) - x_{12}(t) + x_{13}(t), \\
\dot{x}_{13}(t) &= -\mu_2 x_{12}(t),
\end{align*}
\] (37)
where $x_i(t)$ denotes the state of the $i$th agent. $\mu_1$ and $\mu_2$ are known constants, and the function $\phi(\cdot)$ is shown as follows:

$$\phi(x_i(t)) = \pi_2 x_i(t) + 0.5(\pi_1 - \pi_2)(|x_{i1} + 1| - |x_{i1} - 1|),$$

where $\pi_1 < \pi_2 < 0$ are two known parameters.

From (1) and (38), we obtain

$$Z = \begin{bmatrix}
-\mu_1(1 + \pi_2) & \mu_1 & 0 \\
1 & -1 & 1 \\
0 & -\mu_2 & 0
\end{bmatrix},$$

$$\varphi_i(x_i(t)) = \begin{bmatrix}
-0.5\mu_1(\pi_1 - \pi_2)(|x_{i1} + 1| - |x_{i1} - 1|) \\
0 \\
0
\end{bmatrix}.$$ 

Let $\mu_1 = 8.21$, $\mu_2 = 13.145$, $\pi_1 = -1.053$, and $\pi_2 = -0.635$. Then, we can derive that $\lambda_A = 13.5264$, $\lambda_B = 4.289$, and $\Gamma = |\pi_1\mu_1| = 8.64513$.

**Example 1** In the leaderless consensus case, we consider three interaction topologies $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ with four nodes for the MASs, as indicated in Fig. 2. Then, as detailed in Fig. 2, the degree matrices of the topology graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are expressed as

$$\mathcal{D}_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \mathcal{D}_2 = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \mathcal{D}_3 = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$ 

Hence, by $\mathcal{L}_{n(t)} = \mathcal{L}_{n(t)} - \mathcal{L}_{o(t)}$, the Laplacian matrix of graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ can be attained as follows.

$$\mathcal{L}_1 = \begin{bmatrix}
1 & 0 & 0 & -1 \\
-1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}, \mathcal{L}_2 = \begin{bmatrix}
2 & -1 & 0 & -1 \\
-1 & 3 & -1 & -1 \\
0 & -1 & 2 & -1 \\
-1 & -1 & -1 & 3
\end{bmatrix}, \mathcal{L}_3 = \begin{bmatrix}
2 & -1 & -1 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}.$$ 

From Theorem 1 and the given known parameters, let $b_n = -0.4$, $\zeta = 1.03$, and the step length be 0.0001. By simple matrix computation, we have $\rho_n = 0.5200$. The maximum upper bound can be attained such that $t_{n+1} - t_n \leq \frac{\ln(\rho_n b_n)}{\lambda_A + 2\Gamma} = \frac{\ln(1.03(0.5200))}{30.81666}, n \geq 1$. For the convenience of analysis and calculation, we choose $\Pi = 0.02$. As illustrated in Fig. 3, impulsive instants can occur within the impulsive time window. Let $\sigma(t) : [t_0, +\infty)$ be presented the switch signal, $\sigma(t) = (n\text{mod}4), t \in [t_n, t_{n+1})$.

Next, the initial position state values of the four agent nodes are randomly chosen in the interval $[-3, 3]$. The TSC convergence process is shown in Fig. 4. We choose 5 and 10 impulsive instances; then, $T_1 \in [0.1, 0.12]$ and $T_2 \in [0.2, 0.22]$ are random values. As demarcation points, the last impulsive instants of case 1 and case 2 can be described as $t_5 = T_1 = 0.1049s$ and $t_{10} = T_1 = 0.2013s$, respectively. Therefore, the results satisfy the condition range of $T_1$. For the finite-time control stage, from Theorem 1, let $\alpha = 40, \nu = 2$, and $\eta = 0.7$. As shown in Fig. 4, the settling time $T_2$ can be estimated to be approximately 0.2s and 0.25s, while the consensus error for each node can converge to zero for all $t \geq 0.2s$ and $t \geq 0.25s$. It can be said that MASs (1) in the leaderless case can achieve average consensus under control protocol (3) with switching topology. By comparing Figs. 4 and 5, we observe that the convergence rate can be dynamically adjusted by setting different impulsive sampling sizes. The convergence time in Fig. 5 may be shorter than that of Fig. 4, but the full-stage control process has only one convergence stage, and the number of impulsive samples cannot be changed.

In Fig. 6, purely variable impulsive control can alleviate the communication burden between the nodes. However, such asymptotic consensus is difficult to achieve in the real world, as shown in Fig. 6. Meanwhile, purely finite-time
continuous control can yield a faster convergence speed while also increasing the communication cost, as shown in Fig. 7. Therefore, the control scheme of the leaderless case proposed in this study is efficient and realistic, as shown in Fig. 4.

**Example 2** In the leader-following consensus case, three interaction topologies with a leader node and four follower nodes are shown in Fig. 8. The 0 node is a leader, and the remaining nodes are the follower agents. The degree matrices of the topology graphs $\bar{G}_1$, $\bar{G}_2$, and $\bar{G}_3$ are expressed as

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. $$

The relationship matrix between follower nodes and leader node is expressed as

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. $$

Then, through $H\sigma(t) = L\sigma(t) + D$, we can easily obtain the pinning gain matrices of graphs $\bar{G}_1$, $\bar{G}_2$, and $\bar{G}_3$ as follows:

$$H_1 = \begin{bmatrix} 2 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 3 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix}. $$

The given known parameters are identical to those above. By simple calculation, we can infer that $\theta_n = 0.9000$; then, the maximum upper bound of $\Pi = ^{\tau_{n+1}} - ^{\tau_n}$ can be derived as $\tau_{n+1} - \tau_n \leq \frac{\ln(\theta_n \xi)}{\lambda_A + 2\Gamma} = -\frac{\ln(1.03\theta_n)}{30.81666}$,
Fig. 8 Interaction topologies of $\bar{G}_1$, $\bar{G}_2$, and $\bar{G}_3$

Fig. 9 Relationships between $t_n$ and the impulsive time window for the leader-following cases

$n \geq 1$. For convenience, the maximum upper bound is defined as $\Pi = 0.0024$. As shown in Fig. 9, impulsive instants can occur within the impulsive time window.

The initial position state values of the leader agent 0 and four follower agent nodes are randomly selected from the interval $[-3, 3]$. In Fig. 10, the numbers of impulsive instants are set to 8 and 12, i.e., $T_1 \in [0.0192, 0.0216]$ and $T_1 \in [0.0288, 0.0312]$ are random values. The last impulsive instants of case 1 and case 2 can be expressed as $t_8 = T_1 = 0.0207s$ and $t_{12} = T_1 = 0.0299s$, respectively. Hence, the results satisfy the condition range of $T_1$. In the second control stage, from Theorem 2, let $\sigma = 40$, $\nu = 2$, and $\eta = 0.7$. From Fig. 10, the settling time $T_2$ can be estimated to be approximately $0.12s$. In other words, all follower nodes will follow the leader node for all $t \geq 0.12s$.

Overall, consensus can be reached for the leader-following case under protocol (28) with switching topology. As indicated in Fig. 12, the variable impulsive control strategy must converge to zero only when $t \to \infty$. Furthermore, purely finite-time continuous control can obtain a faster convergence speed while increasing the communication cost, as displayed in Fig. 13. By comparing Figs. 10 and 11, we observe that the full-stage control strategy has only one convergence state, and it cannot dynamically adjust the convergence rate like TSC. Therefore, the consensus technique illustrated in Fig. 10 in this investigation is necessary.

Remark 6 From Theorems 1 and 2, a smaller $\rho_n$ value helps obtain a larger $\Pi$. Through simple calculation, Fig. 4 yields a larger maximum upper bound value. In other words, Fig. 4
shows a greater impulsive interval than Fig. 10. A larger impulsive interval has a smaller convergence speed. Comparing Figs. 4 with 10, the consensus rate in Fig. 4 is smaller than that in Fig. 10 because $\rho_n < \theta_n$. In addition, if $\rho_n = 1$, the maximum upper bound becomes negative. Thus, the control protocol cannot affect the consensus of the MASs. Therefore, we should choose $0 < \rho_n < 1$ to obtain a suitable $\Pi$ range.

5 Conclusion

This paper concentrates on the finite-time consensus of nonlinear MASs via impulsive time window theory according to a TSC strategy. TSC divides the whole control period into two parts: a variable impulsive control stage and a finite-time consensus control stage. Unlike general single-stage control studies, this research provides a new investigation perspective for application scenarios under different communication network conditions. Variable impulsive control is also discussed in this paper. Compared with that of traditional fixed impulsive theory, impulsive sampling in the TSC strategy occurs randomly within the impulsive time window, which provides much more flexibility. To overcome the unreliability of an actual network, a switching topology scheme is introduced to improve the system’s stability. Finally, two numerical simulation examples (a leaderless case and a leader-following case) demonstrate the theoretical analysis. In future research, we will concentrate on studying the proposed multistage control strategy in actual applications.

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**Data availability statement** The data that support the findings of this study are available from the corresponding author upon reasonable request.

**Declarations**

**Conflict of interest** The authors declare that they have no conflicts of interest.

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