ASYMPTOTIC STABILITY IN A CHEMOTAXIS-COMPETITION SYSTEM WITH INDIRECT SIGNAL PRODUCTION

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Abstract. This paper deals with a fully parabolic inter-species chemotaxis-competition system with indirect signal production
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(d_u \nabla u + \chi u \nabla w) + \mu_1 u(1 - u - a_1 v), \\
\frac{\partial v}{\partial t} &= d_v \Delta v + \mu_2 v(1 - v - a_2 u), \\
\frac{\partial w}{\partial t} &= d_w \Delta w - \lambda w + \alpha v,
\end{align*}
\]
where \( R^N (N \geq 1) \) is a smooth bounded domain, \( d_u > 0, d_v > 0 \) and \( d_w > 0 \) are the diffusion coefficients, \( \chi \in R \) is the chemotactic coefficient, \( \mu_1 > 0 \) and \( \mu_2 > 0 \) are the population growth rates, \( a_1 > 0, a_2 > 0 \) denote the strength coefficients of competition, and \( \lambda \) and \( \alpha \) describe the rates of signal degradation and production, respectively. Global boundedness of solutions to the above system with \( \chi > 0 \) was established by Tello and Wrzosek in [J. Math. Anal. Appl. 459 (2018) 1233-1250]. The main purpose of the paper is further to give the long-time asymptotic behavior of global bounded solutions, which could not be derived in the previous work.

1. Introduction. In this paper, we consider the following fully parabolic inter-species chemotaxis-competition system (see [28]):
\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(d_u \nabla u + \chi u \nabla w) + \mu_1 u(1 - u - a_1 v), \\
\frac{\partial v}{\partial t} &= d_v \Delta v + \mu_2 v(1 - v - a_2 u), \\
\frac{\partial w}{\partial t} &= d_w \Delta w - \lambda w + \alpha v, \\
\frac{\partial u}{\partial \nu} &= 0, \\
\frac{\partial v}{\partial \nu} &= 0, \\
\frac{\partial w}{\partial \nu} &= 0,
\end{align*}
\]
where \( R^N (N \geq 1) \) is a smooth bounded domain, \( d_u > 0, d_v > 0 \) and \( d_w > 0 \) are the diffusion coefficients, \( \chi \in R \) is the chemotactic coefficient, \( \mu_1 > 0 \) and \( \mu_2 > 0 \) are the population growth rates, \( a_1 > 0, a_2 > 0 \) denote the strength coefficients of competition, and \( \lambda \) and \( \alpha \) describe the rates of signal degradation and production.

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respectively. Moreover, we assume that the initial data \((u_0, v_0, w_0) \in (W^{1,p}(\Omega))^3\), \(p > N\) are nonnegative functions.

System (1) is an extension of the classical Patlak-Keller-Segel chemotaxis model \([15, 16, 22]\) and the interspecies Lotka-Volterra competition system \([20, 29, 21]\) in which individuals belonging to both competing population are assumed to disperse randomly in the region which they jointly occupy. The biased movement is referred to as chemoattraction (i.e. \(\chi < 0\)) if the cells move toward the increasing signal concentration, while it is called chemorepulsion (i.e. \(\chi > 0\)) whenever the cells move away from the increasing signal concentration. Moreover, when \(\chi > 0\) in \([28]\), individuals of the first species try to avoid encounters with competitors by means of chemorepulsion—a chemosensory reaction to the scent of rivals. In this context, the first species with density denoted by \(u\) of the signaling chemical secreted by the individuals in the second species, mathematically represented through its density \(v\) and \(w\) of the signaling chemical secreted by the individuals in the unit disk \(\Omega := B_1(0) \subset \mathbb{R}^2\), where \(\delta \geq 0\) and \(\tau > 0\) are given parameters and \(\mu(t) := \frac{1}{\pi \delta} \int_{\Omega} w(x, t) dx\). They showed the global existence of classical solution and found a critical mass \(m = 8\pi \delta\). Later, Hu and Tao \([13]\) studied the boundedness and large time behavior for a parabolic-parabolic ODE chemotaxis-growth system with indirect signal production. Moreover, Ding and Wang \([7]\) investigated the boundedness in the quasilinear fully parabolic chemotaxis model with indirect signal production. Wang \([30]\) studied the boundedness for the quasilinear parabolic-parabolic ODE chemotaxis-growth system with indirect signal production, and applied the results into the quasilinear attraction-repulsion chemotaxis model with logistic source. Fuest \([8]\) analyzed a parabolic-parabolic-ODE chemotaxis system with indirect signal consumption and derived that the solution \((u, v, w)\) is globally bounded and converges to a spatially constant equilibrium when either \(n \leq 2\) or \(\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{38}\). Furthermore, Xing and Zheng \([33]\) studied the boundedness and long-time behavior for the quasilinear parabolic-parabolic-ODE chemotaxis system with indirect signal consumption.

In order to understand the development of two-competing species system (1), let us mention some previous contributions in this direction. In recent years, the following two-species chemotaxis models with an incompressible fluid

\[
\begin{cases}
(n_1)_t + u \cdot \nabla n_1 = \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) + \mu_1 n_1(1 - n_1 - a_1 n_2), & \text{in } \Omega \times (0, \infty), \\
(n_2)_t + u \cdot \nabla n_2 = \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2(1 - a_2 n_1 - n_2), & \text{in } \Omega \times (0, \infty), \\
c_t + u \cdot \nabla c = \Delta c - (\alpha_1 n_1 + \alpha_2 n_2)c, & \text{in } \Omega \times (0, \infty), \\
u_t + \kappa(\nabla \cdot u)u = \Delta u + \nabla P + (\beta_1 n_1 + \beta_2 n_2)\nabla \phi, & \text{in } \Omega \times (0, \infty), \\
\nabla \cdot u = 0, & \text{in } \Omega \times (0, \infty),
\end{cases}
\]

have been studied by some authors, where \(\kappa \in \{0, 1\}\), the parameters \(\chi_i, \mu_i, a_i, \alpha_i, \beta_i, (i = 1, 2)\) are positive and \(\Omega \subset \mathbb{R}^N (N = 2, 3)\) is a bounded domain with
smooth boundary $\partial \Omega$. When $\kappa = 1$, system (3) is called as two-species chemotaxis-
Navier-Stokes model. In the two-dimensional setting, Hirata et al. [10] derived
global existence, boundedness and stabilization of classical solutions for (3). Moreover,
global existence of weak solutions, eventual smoothness and stabilization are
studied under the three-dimensional case in [11]. When $\kappa = 0$, system (3) is called
as two-species chemotaxis-Stokes model. In the three-dimensional setting, Cao et al.
[5] studied the global existence and asymptotic behavior of classical solutions for (3)
provided that $\frac{\mu_1}{\lambda_1} + \frac{\mu_2}{\lambda_2}$ is sufficiently large. Jin and Xiang [14] further gave the
explicit rates of convergence for any supposedly given global bounded classical solution.
Zheng et al. [35] investigated the boundedness and convergence rates for the
attraction-repulsion chemotaxis-fluid system. Recently, when $-(\alpha_1 n_1 + \alpha_2 n_2)c$ is
replaced with $-c + \alpha_1 n_1 + \alpha_2 n_2$ in (3), Cao et al. [6] studied the global boundedness
and stabilization of classical solutions to (3) in three-dimensional case provided that
$\mu_1$ and $\mu_2$ are sufficiently large. Moreover, Zheng et al. [38] considered the global
asymptotic stability for two-species chemotaxis-competition-fluid system with two
signals.

On the other hand, when $u = 0$ and $-(\alpha_1 n_1 + \alpha_2 n_2)c$ is replaced with $-c + \alpha_1 n_1 + \alpha_2 n_2$ in (3), the two-species chemotaxis-competition systems without fluid
have also been studied by some authors. The existence of nonconstant positive
steady states of (3) with $u = 0$ was derived in [31] under one dimensional case.
Lin et al. [19] proved that the solution of system (3) with $u = 0$ is global and
bounded for any $n \geq 2$ provided that $\Omega$ is convex. Bai and Winkler [1] studied
the large time behavior of global solutions to (3) with $u = 0$. Moreover, the global
existence, boundedness and large time behavior of solutions for parabolic-parabolic-
eliptic two-species chemotaxis model can be found in [27, 24, 3]. Furthermore,
the boundedness and large time behavior of solutions to two-species chemotaxis-
competition system with two signals are studied by some authors in [2, 34, 36].

As mentioned above works, the two-competing-species chemoattractant systems
have been studied by many authors. Recently, Tello and Wrzosek in [28] studied the
interspecies competition chemorepulsion system (1) with $\chi > 0$. They derived the
global existence, boundedness and linear stability analysis of the constant steady
state for system (1) with $a_1, a_2 \in (0, 1)$ when the strength of chemorepulsion is not
too high. However, to the best of our knowledge, the convergence rate of global
solutions to (1) remains open in the previous works. Therefore, our main purpose
in this paper is to further investigate the asymptotic stability of global bounded
solutions to system (1) according to different values of $a_1$ and $a_2$. For the simplicity,
we assume that the diffusion coefficients $d_u = d_v = d_w = 1$ throughout this paper.

Our main results in this paper are stated as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with smooth boundary
and the initial data $(u_0, v_0, w_0) \in (W^{1,p}(\Omega))^3$, $p > N$ are nonnegative functions.
Assume that $\chi \in \mathbb{R}$ and the parameters $\mu_1, \mu_2, a_1, a_2, \lambda, \alpha$ are positive. Then the
global bounded solution $(u, v, w)$ of (1) enjoys the following asymptotic properties.
(i) Suppose that $a_1, a_2 \in (0, 1)$ and
$$
\mu_1 > \frac{a_1^2 \chi^2 a_2^2}{4(1 - a_1 a_2)},
$$
then the global bounded solution $(u, v, w)$ of (1) exponentially converges to the coexistence state $(u^*, v^*, w^*)$, i.e. there
exist positive constants $C$ and $\gamma$ such that
$$
\|(u(\cdot, t), v(\cdot, t), w(\cdot, t)) - (u^*, v^*, w^*)\|_{L^\infty(\Omega)} \leq C e^{-\gamma t}
$$
for all $t > 0$, where
\[ u^* = \frac{1-a_1}{1-a_1a_2}, \quad v^* = \frac{1-a_2}{1-a_1a_2} \quad \text{and} \quad w^* = \frac{\alpha(1-a_2)}{\lambda(1-a_1a_2)}. \] 

(ii) Assume that $a_1 \in [1, \infty)$ and $a_2 \in (0, 1)$, then the global bounded solution $(u, v, w)$ of (1) algebraically converges to the constant steady state $(0, 1, \frac{2}{\lambda})$, i.e. there exist positive constants $C$ and $\zeta$ such that
\[ \|(u(\cdot, t), v(\cdot, t), w(\cdot, t)) - (0, 1, \frac{2}{\lambda})\|_{L^\infty(\Omega)} \leq C(t + 1)^{-\zeta} \]
for all $t > 0$.

(iii) Assume that $a_1 \in (0, 1)$, $a_2 \in [1, \infty)$ and $\mu_1 > \frac{\chi^2a^2}{\lambda(1-a_1)}$, then the global bounded solution $(u, v, w)$ of (1) algebraically converges to the constant steady state $(1, 0, 0)$, i.e. there exist positive constants $C$ and $\kappa$ such that
\[ \|(u(\cdot, t), v(\cdot, t), w(\cdot, t)) - (1, 0, 0)\|_{L^\infty(\Omega)} \leq C(t + 1)^{-\kappa} \]
for all $t > 0$.

**Remark 1.** Compared with the previous results in [28], we further show the exact convergence rate of global solutions for system (1) under the case $a_1, a_2 \in (0, 1)$. Furthermore, we also derive the exact convergence rates of global solutions for system (1) under the cases $0 < a_2 < 1 \leq a_1$ and $0 < a_1 < 1 \leq a_2$, respectively. However, it remains open for the convergence property of global solutions in the case $a_1 > 1$ and $a_2 > 1$.

**Remark 2.** When $\mu_1$ is enough large, it is not difficult to derive that the global bounded solutions of (1) converge to the constant steady state $(u^*, v^*, w^*)$ and $(1, 0, 0)$ under the cases $a_1, a_2 \in (0, 1)$ and $0 < a_1 < 1 \leq a_2$, respectively. However, when $0 < a_2 < 1 \leq a_1$, the global bounded solutions of (1) converge to the constant steady state $(0, 1, \frac{2}{\lambda})$ for any $\mu_1 > 0$. Moreover, we do not need any large restriction of $\mu_2$ in the study of long-time asymptotic stability. The methods used in this paper could be applied to the parabolic-parabolic-elliptic chemotaxis systems.

The rest of this paper is organized as follows. In Section 2, we show the existence of global bounded classical solution to system (1) and give some preliminary regularity estimates which are important for our main proofs. In Section 3, we shall study the asymptotic stabilization of global bounded solutions for system (1) by constructing different energy functionals according to different parameters $a_1$ and $a_2$ and prove Theorem 1.1.

2. **Preliminaries.** In this section, we provide more stronger regularity properties for any such bounded solution than those shown in [28], which are needed to achieve our desired rates of convergence in $L^\infty$-norm. To do this, we shall collect the $L^\infty$-boundedness of solutions for system (1) as follows.

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with smooth boundary and the initial data $(u_0, v_0, w_0) \in (W^{1,p}(\Omega))^3$, $p > N$ are nonnegative functions. Assume that $\chi \in \mathbb{R}$ and the parameters $\mu_1, \mu_2, a_1, a_2, \lambda, \alpha$ are positive. Then system (1) possesses a unique nonnegative global classical solution $(u, v, w) \in (C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty)))^3$, which is uniformly bounded in $\Omega \times (0, \infty)$ in the sense that there exists a positive constant $C$ such that
\[ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all} \ t > 0. \]
Proof. The global boundedness result of Lemma 2.1 comes directly from Theorem 2.1 in [28]. Thus we delete the details.

Now, we shall derive the following more information on the regularity of solutions which follows from (5) in a straightforward manner utilizing cut-off arguments, standard parabolic regularity and the Neumann heat semigroup which follows from (5) in a straightforward manner utilizing cut-off arguments.

Lemma 2.2. Let \( (u, v, w) \) be a global bounded solution of (1). Then there exist \( \alpha \in (0, 1) \) and \( C > 0 \) such that
\[
||u||_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega \times [t, t+1])} + ||v||_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega \times [t, t+1])} + ||w||_{C^{2+\alpha, 1+\frac{\alpha}{2}}(\Omega \times [t, t+1])} \leq C
\]
for all \( t \geq 1 \). In particular, we have
\[
||w(\cdot, t)||_{W^{1, \infty}(\Omega)} + ||\Delta w(\cdot, t)||_{L^\infty(\Omega)} \leq C
\]
and
\[
||w(\cdot, t)||_{W^{1, \infty}(\Omega)} + ||v(\cdot, t)||_{W^{1, \infty}(\Omega)} \leq C
\]
for all \( t > 1 \).

Lemma 2.3. Assume that \( f : (1, \infty) \to \mathbb{R} \) is a uniformly continuous nonnegative function such that \( \int_1^\infty f(t)dt < \infty \). Then we have \( f(t) \to 0 \) as \( t \to \infty \).

Proof. The result of Lemma 2.3 has been proved in [1]. Here we delete the details of the proof.

3. Asymptotic stability. In this section, inspired in [12, 1], we shall study the asymptotic stabilization of global bounded solutions for system (1) by constructing different energy functionals according to different parameters \( a_1 \) and \( a_2 \).

3.1. Stabilization for \( a_1 \in (0, 1) \) and \( a_2 \in (0, 1) \). We first consider the convergence in the co-existence case when \( a_1 \in (0, 1) \) and \( a_2 \in (0, 1) \) by using energy functional methods. To do this, we define the following functions:
\[
E_1(t) := k_1 \int_\Omega \left( u - u^* - u^* \ln \frac{u}{u^*} \right) dx + k_2 \int_\Omega \left( v - v^* - v^* \ln \frac{v}{v^*} \right) dx + k_3 \frac{1}{2} \int_\Omega (w - w^*)^2 dx
\]
and
\[
F_1(t) := \int_\Omega (u - u^*)^2 dx + \int_\Omega (v - v^*)^2 dx + \int_\Omega (w - w^*)^2 dx
\]
for all \( t > 0 \) and some positive constants \( k_i, i = 1, 2, 3 \), where \( u^* = \frac{1-a_1}{1-a_1a_2} \), \( v^* = \frac{1-a_2}{1-a_1a_2} \) and \( w^* = \frac{\alpha(1-a_2)}{\alpha(1-a_1a_2)} \).

Lemma 3.1. Let \( a_1, a_2 \in (0, 1) \) and \( \mu_1 > \frac{\alpha^2 \chi^2 u^*_2}{4(1-a_1a_2)} \), and the assumptions of Lemma 2.1 hold, then there exist \( k_i > 0, i = 1, 2, 3 \) and \( \epsilon_1 > 0 \) such that the functions \( E_1 \) and \( F_1 \) satisfy
\[
E_1(t) \geq 0
\]
and
\[
E_1'(t) \leq -\epsilon_1 F_1(t) \text{ for all } t > 0.
\]
Proof. Define
\[
A_1(t) := k_1 \int_{\Omega} \left( u - u^* - u^* \ln \frac{u}{u^*} \right) dx,
\]
and
\[
B_1(t) := k_2 \int_{\Omega} \left( v - v^* - v^* \ln \frac{v}{v^*} \right) dx
\]
as well as
\[
C_1(t) := \frac{k_3}{2} \int_{\Omega} (w - w^*)^2 dx
\]
for all \( t > 0 \), then
\[
E_1(t) := A_1(t) + B_1(t) + C_1(t) \quad \text{for all} \quad t > 0.
\]

**Step 1.** We shall prove the nonnegativity of \( E_1(t) \). Let \( G_1(\xi) := \xi - u^* \ln \xi \) for all \( \xi > 0 \), it follows from second-order Taylor’s formula that for all \( x \in \Omega \) and \( t > 0 \), there exists \( \tau := \tau(x, t) \in (0, 1) \) such that
\[
G_1'(u) - G_1'(u^*) = \frac{u^*}{2(\tau u + (1 - \tau)u^*)^2} (u - u^*)^2 \geq 0,
\]
which implies that \( A_1(t) = k_1 \int_{\Omega} G_1(u) - G_1(u^*) dx \geq 0 \) for all \( t > 0 \). By a similar argument, we have \( B_1(t) \geq 0 \). Hence, it is easy to see that \( E_1(t) \geq 0 \) for all \( t > 0 \).

**Step 2.** We shall show that \( E_1'(t) \leq -\epsilon_1 F_1(t) \) with some \( \epsilon_1 > 0 \) for all \( t > 0 \). By a series of computations, we deduce from Young’s inequality that
\[
A_1'(t) = -k_1 \mu_1 \int_{\Omega} (u - u^*)^2 dx - k_1 \mu_1 a_1 \int_{\Omega} (u - u^*) (v - v^*) dx
\]
\[
- \frac{k_1 u^*}{2} \int_{\Omega} \left| \nabla u \right|^2 dx - k_1 \chi u^* \int_{\Omega} \left| \nabla u \right| \left| \nabla w \right| dx
\]
\[
\leq -\frac{k_1 \mu_1}{2} \int_{\Omega} (u - u^*)^2 dx + \frac{k_1 \mu_1 a_1^2}{2} \int_{\Omega} (v - v^*)^2 dx
\]
\[
+ \frac{k_1 \chi^2 u^2}{4} \int_{\Omega} \left| \nabla w \right|^2 dx,
\]
and
\[
B_1'(t) = -k_2 \mu_2 \int_{\Omega} (v - v^*)^2 dx - k_2 \mu_2 a_2 \int_{\Omega} (v - v^*) (u - u^*) dx
\]
\[
- \frac{k_2 v^*}{2} \int_{\Omega} \left| \nabla v \right|^2 dx
\]
\[
\leq -\frac{k_2 \mu_2}{2} \int_{\Omega} (v - v^*)^2 dx + \frac{k_2 \mu_2 a_2^2}{2} \int_{\Omega} (u - u^*)^2 dx.
\]
Proof. It follows from 3.1 and integration over (1) converges to the coexistence state as well as
\[ C_1'(t) = -k_3 \int_\Omega |\nabla w|^2 dx + k_3 \int_\Omega (w - w^*)(-\lambda w - \alpha v)dx \]
\[ = -k_3 \int_\Omega |\nabla w|^2 dx - \lambda k_3 \int_\Omega (w - w^*)^2 dx + a k_3 \int_\Omega (w - w^*)(v - v^*)dx \]  \hspace{1cm} (20)
\[ \leq -k_3 \int_\Omega |\nabla w|^2 dx - \lambda k_3 \int_\Omega (w - w^*)^2 dx + \frac{\alpha^2 k_3}{2\lambda} \int_\Omega (v - v^*)^2 dx \]
for all \( t > 0 \).

Combining (18)-(20), we derive
\[ E_1'(t) = A_1'(t) + B_1'(t) + C_1'(t) \]
\[ \leq -\left( \frac{k_1 \mu_1}{2} - \frac{k_2 \mu_2 a_2^2}{2} \right) \int_\Omega (u - u^*)^2 dx \]
\[ - \left( \frac{k_3 \mu_2}{2} - \frac{k_1 \mu_1 a_1^2}{2} - \frac{\alpha^2 k_3}{2\lambda} \right) \int_\Omega (v - v^*)^2 dx - \frac{\lambda k_3}{2} \int_\Omega (w - w^*)^2 dx \]  \hspace{1cm} (21)
\[ - \left( k_3 - \frac{k_1 \chi^2 u^*}{4} \right) \int_\Omega |\nabla w|^2 dx \quad \text{for all } t > 0. \]

Due to the conditions that \( a_1, a_2 \in (0, 1) \) and \( \mu_1 > \frac{\alpha^2 \chi^2 a_1^2 a_2^2}{4(1 - a_1^2 a_2^2)} \), we can let \( k_4 \in \left[ \frac{\alpha^2 \chi^2 a_1^2 a_2^2}{4(1 - a_1^2 a_2^2)}, \frac{\alpha^2 \mu_2}{\mu_2^2 \mu_1} \right) \) and \( k_5 \in \left( \frac{\mu_1 a_1^2}{\mu_2 a_2^2}, \frac{\mu_2 a_2^2}{\mu_1 a_1^2} \right) \), then \( k_4 \mu_1 - k_2 \mu_2 a_2^2 > 0 \), \( k_2 \mu_2 - k_1 \mu_1 a_1^2 > 0 \), \( k_3 - k_4 \chi^2 u^* > 0 \), \( k_4 \mu_1 - k_2 \mu_2 a_2^2 > 0 \), \( k_2 \mu_2 - k_1 \mu_1 a_1^2 > 0 \), \( k_3 - k_4 \chi^2 u^* > 0 \). Let \( \epsilon_1 := \min \left\{ \frac{k_1 \mu_1}{2} - \frac{k_2 \mu_2 a_2^2}{2}, \frac{k_2 \mu_2}{2} - \frac{k_1 \mu_1 a_1^2}{2}, -\frac{\alpha^2 k_3}{2\lambda}, -\frac{\lambda k_3}{2} \right\} \), we have
\[ E_1'(t) \leq -\epsilon_1 F_1(t) \quad \text{for all } t > 0. \]
The proof of Lemma 3.1 is complete. \( \square \)

With the help of Lemma 3.1, we shall give the following large time behavior of global solutions for system (1).

**Lemma 3.2.** Let the assumptions of Lemma 3.1 hold. Then the global solution of (1) converges to the coexistence state \((u^*, v^*, w^*)\), i.e.
\[ ||u(\cdot, t) - u^*||_{L^\infty(\Omega)} + ||v(\cdot, t) - v^*||_{L^\infty(\Omega)} + ||w(\cdot, t) - w^*||_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \]  \hspace{1cm} (22)

**Proof.** It follows from 3.1 and integration over \((1, \infty)\) that
\[ \int_1^\infty F_1(t) dt \leq \frac{1}{\epsilon_1} E_1(1) < \infty. \]  \hspace{1cm} (23)
According to Lemma 2.2, we derive that \((u, v, w)\) is Hölder continuous uniformly with respect to \( t \geq 1 \), in \( \Omega \times [t, t+1] \), so we infer that \( F(t) \) is uniformly continuous in \([1, \infty)\). By Lemma 2.3, we obtain
\[ F_1(t) \to 0 \quad \text{as } t \to \infty, \]  \hspace{1cm} (24)
i.e.
\[ \int_\Omega (u - u^*)^2 dx + \int_\Omega (v - v^*)^2 dx + \int_\Omega (w - w^*)^2 dx \to 0 \quad \text{as } t \to \infty. \]  \hspace{1cm} (25)
By using the Gagliardo-Nirenberg inequality, we have
\[ ||u(\cdot, t) - u^*||_{L^\infty(\Omega)} \leq C_1 ||u(\cdot, t) - u^*||_{W^{1,\infty}(\Omega)} \]  
which implies that there exist \( C > 0 \) and \( \gamma > 0 \) such that
\[ ||u(\cdot, t) - u^*||_{L^\infty(\Omega)} \leq C e^{-\gamma t} \]  
for all \( t > 0 \).

**Proof.** We again use the function \( G_1(\xi) = \xi - u^* \ln \xi \) for all \( \xi > 0 \), which is given in the proof of Lemma 3.1. According to L’Hopital’s rule, we have
\[ \lim_{\xi \to u^*} \frac{G_1(\xi) - G_1(u^*)}{(\xi - u^*)^2} = \frac{1}{2u^*}. \]  
By Lemma 3.2, we have \( ||u(\cdot, t) - u^*||_{L^\infty(\Omega)} \to 0 \) as \( t \to \infty \), which implies that there exists \( t_1 > 0 \) such that
\[ \frac{1}{4u^*} \int_\Omega (u - u^*)^2 dx \leq \int_\Omega \left( u - u^* - u^* \ln \frac{u}{u^*} \right) dx \leq \frac{3}{4u^*} \int_\Omega (u - u^*)^2 dx \]  
and
\[ \frac{1}{4v^*} \int_\Omega (v - v^*)^2 dx \leq \int_\Omega \left( v - v^* - v^* \ln \frac{v}{v^*} \right) dx \leq \frac{3}{4v^*} \int_\Omega (v - v^*)^2 dx \]  
for all \( t > t_1 \).

By means of the definitions of \( \mathcal{E}_1(t) \) and \( \mathcal{F}_1(t) \) in (9) and (10), respectively, it follows from the right inequalities of (29) and (30) that there exists \( C_1 > 0 \) such that
\[ C_1 \mathcal{E}_1(t) \leq \mathcal{F}_1(t) \]  
for all \( t > t_1 \).  
By Lemma 3.2, we have
\[ \mathcal{E}_1(t) \leq -c \mathcal{F}_1(t) \leq -c C_1 \mathcal{E}_1(t) \]  
for all \( t > t_1 \), which implies that there exist \( C_2 > 0 \) and \( L > 0 \) such that
\[ \mathcal{E}_1(t) \leq C_2 e^{-L(t-t_1)} \]  
for all \( t > t_1 \).  
According to the left inequalities of (29) and (30), there exists a positive constant \( C_3 \) such that
\[ \int_\Omega (u(x,t) - u^*)^2 dx + \int_\Omega (v(x,t) - v^*)^2 dx + \int_\Omega (w(x,t) - w^*)^2 dx \leq C_3 \mathcal{E}_1(t) \leq C_2 C_3 e^{-L(t-t_1)} \]  
for all \( t > t_1 \).
By the Gagliardo-Nirenberg inequality and Lemma 2.2, there exist positive constants $C_4, C_5, C_6$ and $C_7$ such that
\[
\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \\
\leq C_4\|u(\cdot, t) - u^*\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N-2}} + C_5\|v(\cdot, t) - v^*\|_{W^{1,\infty}(\Omega)}^{\frac{N}{N-2}} + C_6\|w(\cdot, t) - w^*\|_{L^2(\Omega)}^{\frac{2}{2}} \\
+ C_7\left(\int_\Omega (u(x, t) - u^*)^2 dx + \int_\Omega (v(x, t) - v^*)^2 dx \right)^{\frac{1}{2}} \\
\leq C_7(C_2C_3)^\frac{1}{N-2} e^{-\frac{L(t-t_1)}{N-2}} \\
:= C e^{-\gamma(t-t_1)}
\]
for all $t > t_1$. The proof of Lemma 3.3 is complete. \qed

3.2. Stabilization for $a_1 \in [1, \infty)$ and $a_2 \in (0, 1)$. We shall consider the convergence of global solutions when $a_1 \in [1, \infty)$ and $a_2 \in (0, 1)$ by applying energy functional methods. To this end, we introduce the following functions:

\[
\mathcal{E}_2(t) := l_1 \int_\Omega u dx + l_2 \int_\Omega (v - 1 - \ln v) dx + \frac{l_3}{2} \int_\Omega \left(w - \frac{\alpha}{\lambda}\right)^2 dx \tag{36}
\]
and

\[
\mathcal{F}_2(t) := \int_\Omega u^2 dx + \int_\Omega (v - 1)^2 dx + \int_\Omega \left(w - \frac{\alpha}{\lambda}\right)^2 dx \tag{37}
\]
for all $t > 0$ and some positive constants $l_i$, $i = 1, 2, 3$.

**Lemma 3.4.** Let $a_1 \in [1, \infty)$ and $a_2 \in (0, 1)$ and the assumptions of Lemma 2.1 hold, then there exist $l_i > 0$, $i = 1, 2, 3$ and $\epsilon_2 > 0$ such that the functions $\mathcal{E}_2$ and $\mathcal{F}_2$ satisfy

\[
\mathcal{E}_2(t) \geq 0 \tag{38}
\]
and

\[
\mathcal{E}'_2(t) \leq -\epsilon_2\mathcal{F}_2(t) \quad \text{for all } t > 0. \tag{39}
\]

**Proof.** Define

\[
\mathcal{A}_2(t) := l_1 \int_\Omega u dx, \tag{40}
\]
and

\[
\mathcal{B}_2(t) := l_2 \int_\Omega (v - 1 - \ln v) dx, \tag{41}
\]
as well as

\[
\mathcal{C}_2(t) := \frac{l_3}{2} \int_\Omega \left(w - \frac{\alpha}{\lambda}\right)^2 dx \tag{42}
\]
for all $t > 0$, then

\[
\mathcal{E}_2(t) := \mathcal{A}_2(t) + \mathcal{B}_2(t) + \mathcal{C}_2(t) \quad \text{for all } t > 0. \tag{43}
\]
Now, we divide the proof into the following two steps.

**Step 1.** Similar to Step 1 in the proof of Lemma 3.1, it is easy to prove the nonnegativity of $E_2(t)$ for all $t > 0$.

**Step 2.** We shall show that $E'_2(t) \leq -\epsilon_2 F_2(t)$ with some $\epsilon_2 > 0$ for all $t > 0$. By a series of computations, we deduce from Young’s inequality that

$$A'_2(t) = l_1 \int_{\Omega} \mu_1 u(1 - u - a_1 v) \, dx$$

$$\leq l_1 \mu_1 \int_{\Omega} u(1 - u - v) \, dx$$

$$= -l_1 \mu_1 \int_{\Omega} u^2 \, dx - l_1 \mu_1 \int_{\Omega} u(v - 1) \, dx$$

$$\leq -l_1 \mu_1 \int_{\Omega} u^2 \, dx + \frac{l_1 \mu_1}{2} \int_{\Omega} (v - 1)^2 \, dx,$$

and

$$B'_2(t) = -l_2 \mu_2 \int_{\Omega} (v - 1)^2 \, dx - l_2 l_2 a_2 \int_{\Omega} u(v - 1) \, dx - l_2 \int_{\Omega} \frac{\nabla v}{v} \, dx$$

$$\leq -\frac{l_2 \mu_2}{2} \int_{\Omega} (v - 1)^2 \, dx + \frac{l_2 \mu_2 a_2^2}{2} \int_{\Omega} u^2 \, dx$$

as well as

$$C'_2(t) = -l_3 \int_{\Omega} |\nabla w|^2 \, dx - l_3 \lambda \int_{\Omega} (w - \frac{\alpha}{\lambda})^2 \, dx$$

$$+ \lambda l_3 \int_{\Omega} (w - \frac{\alpha}{\lambda})(v - 1) \, dx$$

$$\leq -\frac{l_3 \lambda}{2} \int_{\Omega} (w - \frac{\alpha}{\lambda})^2 \, dx + \frac{l_3 \alpha^2}{2 \lambda} \int_{\Omega} (v - 1)^2 \, dx$$

for all $t > 0$.

Combining (44)-(46), we obtain

$$E'_2(t) = A'_2(t) + B'_2(t) + C'_2(t)$$

$$\leq -\left( \frac{l_1 \mu_1}{2} - \frac{l_2 \mu_2 a_2^2}{2} \right) \int_{\Omega} u^2 \, dx$$

$$- \left( \frac{l_1 \mu_1}{2} - \frac{l_2 \mu_2}{2} - \frac{l_3 \alpha^2}{2 \lambda} \right) \int_{\Omega} (v - 1)^2 \, dx$$

$$- \frac{l_3 \lambda}{2} \int_{\Omega} (w - \frac{\alpha}{\lambda})^2 \, dx$$

for all $t > 0$.

Taking $\frac{l_1 \mu_1}{2} > \frac{\lambda \mu_1 (1 - a_2^2)}{a_2^2}$ and $\frac{l_3 \lambda}{2} \in \left( \frac{\mu_1}{\mu_2} + \frac{l_3 \alpha^2}{l_3 \lambda \mu_2}, \frac{\mu_1}{\mu_2} \frac{l_3 \alpha^2}{\mu_2 a_2^2} \right)$, then $\frac{l_1 \mu_1}{2} - \frac{l_2 \mu_2 a_2^2}{2} > 0$ and $\frac{l_2 \mu_2}{2} - \frac{l_3 \alpha^2}{2 \lambda} > 0$.

Let

$$\epsilon_2 := \min \left\{ \frac{l_1 \mu_1}{2} - \frac{l_2 \mu_2 a_2^2}{2}, \frac{l_2 \mu_2}{2} - \frac{l_1 \mu_1}{2} - \frac{l_3 \alpha^2}{2 \lambda}, \frac{l_3 \lambda}{2} \right\},$$

we have

$$E'_2(t) \leq -\epsilon_2 F_2(t)$$

for all $t > 0$.

The proof of Lemma 3.4 is complete.
Lemma 3.5. Let $a_1 \in [1, \infty)$, $a_2 \in (0, 1)$ and the assumptions of Lemma 2.1 hold. Then the global solution of (1) converges to the constant steady state $(0, 1, \frac{a}{\lambda})$, i.e.

$$
\| u(\cdot, t) \|_{L^\infty(\Omega)} + \| v(\cdot, t) - 1 \|_{L^\infty(\Omega)} + \| w(\cdot, t) - \frac{a}{\lambda} \|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.
$$  

(48)

Proof. By means of Lemma 3.4, it follows from the similar arguments as the proof of Lemma 3.2 that Lemma 3.5 holds.

In order to prove Theorem 1.1 (ii), it remains to give the convergence rates for global solutions in (1).

Lemma 3.6. Let $a_1 \in [1, \infty)$, $a_2 \in (0, 1)$ and the assumptions of Lemma 2.1 hold. Then there exist positive constants $C$ and $\varsigma$ such that

$$
\| u(\cdot, t) \|_{L^\infty(\Omega)} + \| v(\cdot, t) - 1 \|_{L^\infty(\Omega)} + \| w(\cdot, t) - \frac{a}{\lambda} \|_{L^\infty(\Omega)} \leq C(t - t_2)^{-\varsigma}
$$

for all $t > t_2$, where $t_2$ is some fixed time.

Proof. Let $G_2(\xi) := \xi - \ln \xi$ for all $\xi > 0$, according to L’Hopital’s rule, we obtain

$$
\lim_{\xi \to 1} \frac{G_2(\xi) - G_2(1)}{(\xi - 1)^2} = \frac{1}{2}.
$$  

(49)

By Lemma 3.5, we have $\| v(\cdot, t) - 1 \|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$, which implies that there exists $t_2 > 0$ such that

$$
\int_{\Omega} (v(x, t) - 1 - \ln v(x, t)) \, dx = \int_{\Omega} (G_2(v(x, t)) - G_2(1)) \, dx
$$

\leq \frac{3}{4} \int_{\Omega} (v(x, t) - 1)^2 \, dx
$$

(50)

and

$$
\int_{\Omega} (v(x, t) - 1 - \ln v(x, t)) \, dx = \int_{\Omega} (G_2(v(x, t)) - G_2(1)) \, dx
$$

\geq \frac{1}{4} \int_{\Omega} (v(x, t) - 1)^2 \, dx
$$

(51)

for all $t > t_2$. By means of the definitions of $\mathcal{E}_2$ and $\mathcal{F}_2$ in Lemma 3.4, it follows from (50) and Hölder’s inequality that there exist $C_1 > 0$ and $C_2 > 0$ such that

$$
\mathcal{E}_2(t) \leq l_1 \int_{\Omega} u \, dx + \frac{3l_2}{4} \int_{\Omega} (v - 1)^2 \, dx + \frac{l_3}{2} \int_{\Omega} (w(\cdot, t) - \frac{a}{\lambda})^2 \, dx
$$

$$
\leq l_1 |\Omega| \frac{1}{2} \left( \int_{\Omega} u^2 \, dx \right)^{\frac{1}{2}} + \frac{3l_2}{4} \left( \int_{\Omega} (v - 1)^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (v - 1)^2 \, dx \right)^{\frac{1}{2}}
$$

$$
+ \frac{l_3}{2} \left( \int_{\Omega} (w(\cdot, t) - \frac{a}{\lambda})^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (w(\cdot, t) - \frac{a}{\lambda})^2 \, dx \right)^{\frac{1}{2}}
$$

(52)

$$
\leq C_1 \left( \int_{\Omega} u^2 \, dx \right)^{\frac{1}{2}} + C_1 \left( \int_{\Omega} (v - 1)^2 \, dx \right)^{\frac{1}{2}}
$$

$$
+ C_1 \left( \int_{\Omega} (w(\cdot, t) - \frac{a}{\lambda})^2 \, dx \right)^{\frac{1}{2}}
$$

\leq C_2 \mathcal{F}_2^n(t) \text{ for all } t > t_2,
where we have used the boundedness properties of solution \((u, v, w)\) asserted by Lemma 2.1. Thus we have

\[ F_2(t) \geq C_3 E_2'(t) \quad \text{for all } t > t_2, \]

where \(C_3 := \frac{1}{c_2^2}\). Therefore, it follows from Lemma 3.4 that

\[ E_2'(t) \leq -\varepsilon_2 C_3 E_2(t) \quad \text{for all } t > t_2, \]

which implies

\[ E_2(t) \leq \frac{C_4}{t - t_2} \quad \text{for all } t > t_2, \]

with some positive constant \(C_4\). It follows from (51) and (55) that for all \(t > t_2\) there exist \(C_5 > 0\) and \(C_6 > 0\) such that

\[
\int_\Omega u^2 dx + \int_\Omega (v - 1)^2 dx + \int_\Omega (w(\cdot, t) - \frac{\alpha}{\lambda})^2 dx \leq C_5 E_2(t) \leq \frac{C_6}{t - t_2}.
\]

By using the Gagliardo-Nirenberg inequality, there exist positive constants \(C_7, C_8\) and \(C_9\) such that

\[
||u(\cdot, t)||_{L^\infty(\Omega)} \leq C_7 ||u(\cdot, t)||_{W^{1, \infty}(\Omega)} ||u(\cdot, t)||_{L^1(\Omega)},
\]

and

\[
||v(\cdot, t) - 1||_{L^\infty(\Omega)} \leq C_8 ||v(\cdot, t) - 1||_{W^{1, \infty}(\Omega)} ||v(\cdot, t) - 1||_{L^2(\Omega)},
\]

as well as

\[
||w(\cdot, t) - \frac{\alpha}{\lambda}||_{L^\infty(\Omega)} \leq C_9 ||w(\cdot, t) - \frac{\alpha}{\lambda}||_{W^{1, \infty}(\Omega)} ||w(\cdot, t) - \frac{\alpha}{\lambda}||_{L^2(\Omega)}.
\]

Henceforth, it follows from Lemma 2.2 and (56)-(59) that

\[
||u(\cdot, t)||_{L^\infty(\Omega)} + ||v(\cdot, t) - 1||_{L^\infty(\Omega)} + ||w(\cdot, t) - \frac{\alpha}{\lambda}||_{L^\infty(\Omega)} \\
\leq C_{10} \left(||u(\cdot, t)||_{L^1(\Omega)} + ||v(\cdot, t) - 1||_{L^2(\Omega)} + ||w(\cdot, t) - \frac{\alpha}{\lambda}||_{L^2(\Omega)}\right)^{\frac{1}{\alpha + \lambda}} \\
\leq C_{10} C_6^\frac{1}{\alpha + \lambda} (t - t_2)^{-\frac{1}{\alpha + \lambda}} =: C(t - t_2)^{-\zeta}
\]

for all \(t > t_2\), where \(C_{10} > 0\), \(C := C_{10} C_6^\frac{1}{\alpha + \lambda}\) and \(\zeta = \frac{1}{\alpha + \lambda}\). The proof of Lemma 3.6 is complete.

3.3. **Stabilization for** \(a_1 \in (0, 1)\) **and** \(a_2 \in [1, \infty)\). In this case, we shall show that the solution \((u, v, w)\) converges at least algebraically to \((1, 0, 0)\). We can construct the energy functions:

\[
E_3(t) := m_1 \int_\Omega (u - 1 - \ln u) \, dx + m_2 \int_\Omega vdx + \frac{m_3}{2} \int_\Omega w^2 dx
\]

and

\[
F_3(t) := \int_\Omega (u - 1)^2 dx + \int_\Omega v^2 dx + \int_\Omega w^2 dx
\]

for all \(t > 0\) and some positive constants \(m_i, i = 1, 2, 3\).
Lemma 3.7. Let $a_1 \in (0, 1)$, $a_2 \in [1, \infty)$, $\mu_1 > \frac{\chi^2 a_2}{4m(1-a_1)}$ and the assumptions of Lemma 2.1 hold, then there exist $m_i > 0$, $i = 1, 2, 3$ and $\epsilon_3 > 0$ such that the functions $E_3$ and $F_3$ satisfy

$$E_3(t) \geq 0$$

and

$$E_3'(t) \leq -\epsilon_3 F_3(t) \quad \text{for all } t > 0.$$  

Proof. Define

$$A_3(t) := m_1 \int_{\Omega} (u - 1 - \ln u) \, dx,$$

and

$$B_3(t) := m_2 \int_{\Omega} v \, dx,$$

as well as

$$C_3(t) := \frac{m_3}{2} \int_{\Omega} w^2 \, dx$$

for all $t > 0$, then

$$E_3(t) := A_3(t) + B_3(t) + C_3(t) \quad \text{for all } t > 0.$$  

Next, we divide the proof into the following two steps.

Step 1. Similar to Step 1 in the proof of Lemma 3.1, we can prove the nonnegativity of $E_3(t)$ for all $t > 0$.

Step 2. We shall show that $E_3'(t) \leq -\epsilon_3 F_3(t)$ with some $\epsilon_3 > 0$ for all $t > 0$. By a series of computations, we deduce from Young’s inequality that

$$A_3'(t) = -m_1 \mu_1 \int_{\Omega} \left( u - 1 \right)^2 \, dx - m_1 \mu_1 a_1 \int_{\Omega} (u - 1)v \, dx$$

$$- m_1 \int_{\Omega} \left( |\nabla u|^2 \right) \frac{u}{u} \, dx - m_1 \int_{\Omega} \left( \frac{1}{u} \nabla u \cdot \nabla w \right) \, dx$$

$$\leq - \frac{m_1 \mu_1}{2} \int_{\Omega} \left( u - 1 \right)^2 \, dx + \frac{m_1 \mu_1 a_1^2}{2} \int_{\Omega} v^2 \, dx + \frac{m_1 \chi^2}{4} \int_{\Omega} |\nabla w|^2 \, dx,$$

and

$$B_3'(t) = m_2 \int_{\Omega} \mu_2 v(1 - v - a_2 u) \, dx$$

$$\leq m_2 \mu_2 \int_{\Omega} v(1 - v - u) \, dx$$

$$= -m_2 \mu_2 \int_{\Omega} v^2 \, dx - m_2 \mu_2 \int_{\Omega} v(u - 1) \, dx$$

$$\leq - \frac{m_2 \mu_2}{2} \int_{\Omega} v^2 \, dx + \frac{m_2 \mu_2}{2} \int_{\Omega} (u - 1)^2 \, dx,$$

as well as

$$C_3'(t) = -m_3 \int_{\Omega} |\nabla w|^2 \, dx - m_3 \lambda \int_{\Omega} w^2 \, dx + \alpha m_3 \int_{\Omega} w v \, dx$$

$$\leq -m_3 \int_{\Omega} |\nabla w|^2 \, dx - \frac{\lambda m_3}{2} \int_{\Omega} w^2 \, dx + \frac{m_3 \alpha^2}{2\lambda} \int_{\Omega} v^2 \, dx$$

for all $t > 0$. 


Combining (69)-(71), we derive
\[ E'_3(t) = A'_3(t) + B'_3(t) + C'_3(t) \]
\[ \leq - \left( \frac{m_1 \mu_1}{2} - \frac{m_2 \mu_2}{2} \right) \int_{\Omega} (u - 1)^2 dx \]
\[ - \left( \frac{m_2 \mu_2}{2} - \frac{m_1 \mu_1 a_1^2}{2} - \frac{m_3 \alpha^2}{2\lambda} \right) \int_{\Omega} v^2 dx \]
\[ - \frac{\lambda m_3}{2} \int_{\Omega} w^2 dx - \left( m_3 - 4 \alpha^2 \frac{1}{4} \right) \int_{\Omega} |\nabla w|^2 dx \quad \text{for all } t > 0. \]

(72)

Thanks to the conditions that \( a_1 \in (0, 1) \), \( a_2 \in [1, \infty) \) and \( \mu_1 > \frac{\lambda^2 \alpha^2}{4(1 - \alpha^2)} \), we can let \( \frac{m_3}{m_1} \in \left( \frac{\lambda^2 \alpha^2}{4}, \frac{\lambda^2 \alpha^2}{4(1 - \alpha^2)} \right) \) and \( \frac{m_2}{m_1} \in \left( \frac{\mu_1 \alpha^2}{\mu_2}, \frac{\mu_1 \alpha^2}{\mu_2} \right) \) such that \( \frac{m_3 \mu_1}{2} - \frac{m_2 \mu_2}{2} > 0 \), \( \frac{m_2 \mu_2}{2} - \frac{m_1 \mu_1 a_1^2}{2} - \frac{m_3 \alpha^2}{2\lambda} > 0 \) and \( \frac{m_3 \alpha^2}{4} > 0 \).

Taking
\[ \epsilon_3 := \min \left\{ \frac{m_1 \mu_1}{2} - \frac{m_2 \mu_2}{2}, \frac{m_2 \mu_2}{2}, \frac{m_1 \mu_1 a_1^2}{2}, \frac{m_3 \alpha^2}{2\lambda}, \frac{\lambda m_3}{2} \right\}, \]
we have
\[ E'_3(t) \leq -\epsilon_3 F_3(t) \quad \text{for all } t > 0. \]

The proof of Lemma 3.7 is complete.

Lemma 3.8. Let the assumptions of lemma 3.7 hold. Then the global solution of (1) converges to the constant steady state \((1, 0, 0)\), i.e.
\[ ||u(\cdot, t) - 1||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{L^\infty(\Omega)} + ||w(\cdot, t)||_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \]

Proof. By means of Lemma 3.7, it follows from the similar arguments as the proof of Lemma 3.2 that Lemma 3.8 holds.

Now we give the convergence rates for global solutions in (1) for \( a_1 \in (0, 1) \) and \( a_2 \in [1, \infty) \).

Lemma 3.9. Let the assumptions of Lemma 3.7 hold. Then there exist positive constants \( C \) and \( \kappa \) such that
\[ ||u(\cdot, t) - 1||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{L^\infty(\Omega)} + ||w(\cdot, t)||_{L^\infty(\Omega)} \leq C(t - t_3)^{-\kappa} \]
for all \( t > t_3 \), where \( t_3 \) is some fixed time.

Proof. Denote \( G_2(\xi) := \xi - \ln \xi \) for all \( \xi > 0 \), based on L’Hôpital’s rule, we obtain
\[ \lim_{\xi \to 1} \frac{G_2(\xi) - G_2(1)}{(\xi - 1)^2} = \frac{1}{2}. \]

(74)

By Lemma 3.8, we have \( ||u(\cdot, t) - 1||_{L^\infty(\Omega)} \to 0 \) as \( t \to \infty \), which implies that there exists \( t_3 > 0 \) such that
\[ \int_{\Omega} (u(x, t) - 1 - \ln u(x, t)) dx = \int_{\Omega} (G_2(u(x, t)) - G_2(1)) dx \]
\[ \leq \frac{3}{4} \int_{\Omega} (u(x, t) - 1)^2 dx \quad \text{for all } t > t_3. \]
and
\[ \int_{\Omega} (u(x,t) - 1 - \ln u(x,t)) \, dx = \int_{\Omega} (G_2(u(x,t)) - G_2(1)) \, dx \geq \frac{1}{4} \int_{\Omega} (u(x,t) - 1)^2 \, dx \] (76)
for all \( t > t_3 \). By means of the definitions of \( E_3 \) and \( F_3 \) in Lemma 3.7, it follows from (75) and Hölder’s inequality that there exist \( C_1 > 0 \) and \( C_2 > 0 \) such that
\[ E_3(t) \leq \frac{3m_1}{4} \int_{\Omega} (u - 1)^2 \, dx + m_2 \int_{\Omega} \nu \, dx + \frac{m_3}{2} \int_{\Omega} w^2 \, dx \]
\[ \leq \frac{3m_1}{4} \left( \int_{\Omega} (u - 1)^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} (v^2 \, dx) \right)^{\frac{1}{2}} + m_2 |\Omega|^{\frac{1}{2}} \left( \int_{\Omega} v^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq C_1 \left( \int_{\Omega} (u - 1)^2 \, dx \right)^{\frac{1}{2}} + C_1 \left( \int_{\Omega} v^2 \, dx \right)^{\frac{1}{2}} + C_1 \left( \int_{\Omega} w^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq C_2 F_3^2(t) \quad \text{for all } t > t_3, \]
where we have used the boundedness properties of solution \((u, v, w)\) asserted by Lemma 2.1. Thus we have
\[ F_3(t) \geq C_3 E_3^2(t) \quad \text{for all } t > t_3, \] (78)
where \( C_3 := \frac{1}{C_2^2} \). Therefore, it follows from Lemma 3.7 that
\[ E_3'(t) \leq -\epsilon_3 C_3 E_3^2(t) \quad \text{for all } t > t_3, \] (79)
which implies
\[ E_3(t) \leq \frac{C_4}{t - t_3} \quad \text{for all } t > t_3, \] (80)
with some positive constant \( C_4 \). It follows from (76) and (80) that for all \( t > t_3 \) there exist \( C_5 > 0 \) and \( C_6 > 0 \) such that
\[ \int_{\Omega} (u - 1)^2 \, dx + \int_{\Omega} \nu \, dx + \int_{\Omega} w^2 \, dx \leq C_5 E_3(t) \leq \frac{C_6}{t - t_3}. \] (81)
By using the Gagliardo-Nirenberg inequality, there exist positive constants \( C_7, C_8 \) and \( C_9 \) such that
\[ \|u(\cdot, t) - 1\|_{L^\infty(\Omega)} \leq C_7 \|u(\cdot, t) - 1\|_{\dot{W}^{1/2, \infty}(\Omega)} \|u(\cdot, t) - 1\|_{L^2(\Omega)}^{\frac{1}{2}}, \] (82)
and
\[ \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C_8 \|v(\cdot, t)\|_{\dot{W}^{1/2, \infty}(\Omega)} \|v(\cdot, t)\|_{L^1(\Omega)}^{1/2}, \] (83)
as well as
\[ \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C_9 \|w(\cdot, t)\|_{\dot{W}^{1/2, \infty}(\Omega)} \|w(\cdot, t)\|_{L^2(\Omega)}^{1/2}. \] (84)
Henceforth, it follows from Lemma 2.2 and (82)-(84) that
\[ ||u(\cdot, t) - I||_{L^\infty(\Omega)} + ||v(\cdot, t)||_{L^\infty(\Omega)} + ||w(\cdot, t)||_{L^\infty(\Omega)} \]
\[ \leq C_{10} \left(||u(\cdot, t) - 1||^2_{L^2(\Omega)} + ||v(\cdot, t)||_{L^1(\Gamma)} + ||w(\cdot, t)||^2_{L^2(\Omega)}\right)^{\frac{1}{4}} \]
\[ \leq C_{10} C_6^{\frac{1}{N+1}} (t - t_3)^{-\frac{1}{N}} \]
\[ \varepsilon = (t - t_3)^{-\kappa} \]

for all \( t > t_3 \), where \( C_{10} > 0 \), \( C := C_{10} C_6^{\frac{1}{N+1}} \) and \( \kappa = \frac{1}{N+1} \). The proof of Lemma 3.9 is complete.

Now we begin with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** The statements of Theorem 1.1 directly come from Lemma 3.3, Lemma 3.6 and Lemma 3.9.

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