ON TWO CONJECTURES FOR CURVES ON $K3$ SURFACES

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We prove that the gonality among the smooth curves in a complete linear system on a $K3$ surface is constant except for the Donagi-Morrison example. This was proved by Ciliberto and Pareschi under the additional condition that the linear system is ample. The constancy was originally conjectured by Harris and Mumford.

As a consequence we prove that exceptional curves on $K3$ surfaces satisfy the Eisenbud-Lange-Martens-Schreyer conjecture and explicitly describe such curves. They turn out to be natural extensions of the Eisenbud-Lange-Martens-Schreyer examples of exceptional curves on $K3$ surfaces.

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1. Introduction

In connection with their work [7], Harris and Mumford conjectured that the gonality should be constant among the smooth curves in a linear system on a $K3$ surface. (The conjecture is unpublished.) Subsequently, Donagi and Morrison [4] pointed out the following counterexample:

The Donagi-Morrison example (cf. [4], (2.2)). Let $\pi : S \to P^2$ be a $K3$ surface of genus 2, i.e. a double cover of $P^2$ branched along a smooth sextic, and let $L := \pi^*O_{P^2}(3)$. The arithmetic genus of the curves in $|L|$ is 10. The smooth curves in the codimension one linear subspace $|\pi^*H^0O_{P^2}(3)| \subset |L|$ are bielliptic, whence with gonality 4. On the other hand the general curve in $|L|$ is isomorphic to a smooth plane sextic and therefore has gonality 5.

Ciliberto and Pareschi ([2], Thm. A) proved that this is indeed the only counterexample when $L$ is ample. The first aim of this note is to show that this result holds without the ampleness assumption. That is, we will prove:

Theorem 1. Let $S$ be a $K3$ surface and $L$ a globally generated line bundle on $S$. If the gonality of the smooth curves in $|L|$ is not constant, then $S$ and $L$ are as in the Donagi-Morrison example.
It has also been known that this result would follow from the Eisenbud-Lange-Martens-Schreyer conjecture on exceptional curves posed in [5] (see §4). (Recall that any smooth curve $C$ satisfies $\text{Cliff} C + 2 \leq \text{gon} C \leq \text{Cliff} C + 3$ and the curves for which $\text{gon} C = \text{Cliff} C + 3$ are conjectured to be very rare and called exceptional.)

In Thm. 4.3 in [5] an infinite series of examples of exceptional curves lying on $K3$ surfaces is constructed. The line bundles in these cases are not ample (cf. also Remark (c) on p. 36 in [2]), showing that there are interesting cases appearing when the line bundles are not ample.

We will consider a generalization of these examples:

"Generalized ELMS examples". Let $L$ be a line bundle on a $K3$ surface $S$ such that $L \sim 2D + \Gamma$ with $D$ and $\Gamma$ smooth curves satisfying $D^2 \geq 2$, $\Gamma^2 = -2$ and $\Gamma \cdot D = 1$. Assume furthermore that there is no line bundle $B$ on $S$ satisfying $0 \leq B^2 \leq D^2 - 1$ and $0 < B \cdot L - B^2 \leq D^2$.

Then $|L|$ is base point free and all the smooth curves in $|L|$ are exceptional, of genus $g = 2D^2 + 2 \geq 6$, Clifford index $c = D^2 - 1 = \frac{g-1}{2}$ and Clifford dimension $r = \frac{1}{2}D^2 + 1$. Moreover, for any smooth curve $C \in |L|$ the Clifford index is computed only by $O_C(D)$. (Recall that the Clifford dimension of a smooth curve is the minimal value of $\dim |A|$ where $A$ computes the Clifford index.)

We will prove the assertions in the example in Proposition 4.1. The examples in [5], Thm. 4.3, have $\text{Pic} S \cong \mathbb{Z}[D] \oplus \mathbb{Z}[\Gamma]$ with $D$ and $\Gamma$ as above, in which case the nonexistence of a divisor $B$ satisfying the conditions above can easily be verified.

As in [5], the curves in the “generalized ELMS examples” satisfy the Eisenbud-Lange-Martens-Schreyer conjecture.

The second main result of this note is:

**Theorem 2.** Let $C$ be a smooth exceptional curve on a $K3$ surface $S$. Then $C$ is either a smooth plane sextic belonging to the Donagi-Morrison example or $L := O_S(C)$ is as in the generalized ELMS examples.

In particular, $C$ satisfies the Eisenbud-Lange-Martens-Schreyer conjecture.

We remark that the proof of Theorem 1, as well as the assertions in the “generalized ELMS examples” (in Proposition 4.1) do not use the theorem of Green and Lazarsfeld [6] about constancy of the Clifford index (as in the case of Ciliberto and Pareschi’s paper, cf. [2], Rem. p. 32). The latter enters the picture only in the proof of Theorem 2.

We prove Theorem 1 by adding a suitable deformation-degeneration argument to the arguments of §1 and §2 in [2]. (We do not make use of §3 in [2].) We therefore use the same notation and conventions as in [2] and refer the reader to that paper for background material.

The note is organised as follows.

In Section 2 we obtain sharper versions of Lemma 2.2 and Proposition 2.3 in [2] and introduce an incidence variety, slightly different from the one considered in §3.
in [2], that we will need in the proof of Theorem 1.

In Section 3 we prove Theorem 1. The idea is as follows: Since Theorem 1 holds when \( L \) is ample, by [2], the ideal way to prove it would be to deform \((S, L)\) so as to

(i) keep the nonconstancy of the gonality among the smooth curves in \(|L|\), and
(ii) make \( L \) ample.

The condition (i) is easily preserved in a codimension two subspace of the moduli space: one just needs to keep the two line bundles \( M \) and \( N \) such that \( L \sim M + N \) coming from the instability of the well-known vector bundle considered in [2].

Condition (ii) is not possible to achieve, but we will show that we can make \( L \) “almost ample”, in the sense that there is a unique rational curve \( \Gamma \) such that \( \Gamma \cdot L = 0 \). Moreover, we will show that \( H := L - \Gamma \) is globally generated and we will prove Theorem 1 by degenerating to the special curves \( C'' \cup \Gamma \) in the linear system \(|L|\), with \( C'' \in |H| \) smooth, and using the incidence variety from Section 2.

In Section 4 we prove the assertions in the “Generalized ELMS examples” in Proposition 4.1 and then we prove Theorem 2, which at this point is just a combination of Theorem 1 with the well-known theorem of Green and Lazarsfeld [6].

2. Some useful results

We will throughout this note use line bundles and divisor classes interchangeably. In particular, by the notation \( A \geq 0 \), where \( A \) is a line bundle or divisor class, we will mean that \( h^0(A) > 0 \), and we will write \( A > 0 \) if in addition \( A \) is nontrivial (or nonzero).

We first obtain some strengthenings of Lemma 2.2 and Prop. 2.3 in [2] in Lemma 2.1 and Proposition 2.1 below, respectively, as we will need these stronger versions in the proof of Theorem 1.

**Lemma 2.1.** Let \( L \) be a base point free line bundle on a K3 surface \( S \) and assume that \( L \sim M + N \) with \( h^0(M) \geq 2, h^0(N) \geq 2, M \cdot N = k \) and \( L^2 \geq 4k - 4 \).

Then either

(a) there is a smooth curve in \(|L|\) of gonality \( \leq k \); or
(b) \( M \sim N + \Gamma \) (possibly after interchanging \( M \) and \( N \)), for a smooth rational curve \( \Gamma \) such that \( \Gamma \cdot N = 1 \). In particular, \( L^2 = 4k - 2 \).

**Proof.** Among all the decompositions satisfying the conditions in the lemma, we pick one for which \( k \) is minimal, say \( L \sim M_0 + N_0 \) with \( M_0, N_0 = k_0 \leq k \). If \( k_0 = k \), we let \( M_0 = M \) and \( N_0 = N \). (Note that we have \( k_0 \geq 2 \) as \( L \) is globally generated, cf. [13], Lemma 3.7.)

If \( M_0 \sim N_0 \), then \( M_0 \) is nef, as \( L \) is. If it is not base point free, then \( M_0 \sim lE + \Gamma \), for \( l \geq 2 \), a smooth elliptic curve \( E \) and a smooth rational curve \( \Gamma \) such that \( E \cdot \Gamma = 1 \), by [13], §2.7. One then easily sees that \(|E|\) induces a pencil of degree \( \leq k_0 \) on all
the curves in \(|L|\) and we are in case (a). If \(M_0\) is base point free, we are in case (a) by [2], Lemma 2.2.

By symmetry we can therefore assume that \(M_0 \cdot L \geq N_0 \cdot L\) and \(h^0(N_0 - M_0) = 0\). We now show that either we are in case (a) or we can find a new decomposition \(L \sim M' + N'\) satisfying the following properties:

\[ M' \geq M_0, \ N' \leq N_0, \ M'.N' = k_0; \quad (2.1) \]
\[ M'^2 \geq N'^2 > 0; \quad (2.2) \]
\[ N' \text{ is globally generated with } h^0(N') \geq 2; \quad (2.3) \]
\[ h^1(M') = h^1(N') = 0; \quad (2.4) \]
\[ \text{the base divisor } \Delta' \text{ of } |M'| \text{ satisfies } \Delta'.L = 0. \quad (2.5) \]

If \(N_0\) is not nef, then there is a smooth rational curve \(\Gamma\) such that \(\Gamma.N_0 < 0\). Therefore \(\Gamma.M_0 > 0\) as \(L\) is nef, and \(h^0(M_0 + \Gamma) \geq h^0(M_0) \geq 2\), \(h^0(N_0 - \Gamma) = h^0(N_0) \geq 2\) and

\[(M_0 + \Gamma)(N_0 - \Gamma) = k_0 + \Gamma.N_0 - \Gamma.M_0 + 2 \leq k_0.\]

Hence, the minimality of \(k_0\) implies \(\Gamma.N_0 = -1\) and \(\Gamma.M_0 = 1\), so that \((M_0 + \Gamma)(N_0 - \Gamma) = k_0\). In particular, continuing the process, we reach a decomposition \(L \sim M' + N'\) satisfying (2.1) with \(N'\) nef. As above, if \(N'\) is not base point free, then \(N' \sim lE + \Gamma\), for \(l \geq 2\), a smooth elliptic curve \(E\) and a smooth rational curve \(\Gamma\) such that \(E.\Gamma = 1\). One then easily sees that \(|E|\) induces a pencil of degree \(\leq k_0\) on all the curves in \(|L|\) and we are in case (a). Otherwise (2.3) is satisfied.

If \(N'^2 = 0\), then \(N_0 \cdot L = k_0\), so that all the curves in \(|L|\) would carry a pencil of degree \(k_0\), and we are in case (a) again. Otherwise \(N'^2 > 0\), and as \(M'.L \geq M_0 \cdot L \geq N_0 \cdot L \geq N'.L\), we have \(M'^2 \geq N'^2 > 0\), so that (2.2) is satisfied. In particular, \(h^1(N') = 0\). Moreover, the above argument with \(M_0\) and \(N_0\) substituted by \(N'\) and \(M'\) respectively, shows that any \(\Delta > 0\) satisfying \(\Delta'^2 = -2\) and \(\Delta.M' < 0\), must satisfy \(\Delta.M' = -1\). Hence \(h^1(M') = 0\) by [10], Thm. 1, and (2.4) is satisfied.

Let now \(\Delta'\) be the (possibly zero) base divisor of \(|M'|\) and assume that \(\Delta'.L > 0\).

If \(h^1(M' - \Delta') > 0\), then by [13] we have \(M' - \Delta' \sim lE\) for a smooth elliptic curve \(E\) and an integer \(l \geq 2\). But then \(|E|\) is easily seen to induce a pencil of degree \(\leq k_0\) on the curves in \(|L|\), so that we are in case (a).

If \(h^1(M' - \Delta') = 0\), then \(M'.\Delta' = \frac{1}{2} \Delta'^2 < 0\) by Riemann-Roch, as \(h^0(M' - \Delta') = h^0(M')\) and \(h^1(M') = 0\). Moreover, \(N'.\Delta' \geq -M'.\Delta' + 1\), by assumption. Hence

\[(M' - \Delta')(N' + \Delta') = M'.N' - \Delta'.N' + \Delta'.M' - \Delta'^2 < k_0,\]

a contradiction on the minimality of \(k_0\).

Therefore, (2.5) is proved.

Now we set \(R' := M' - N'\). Then the condition \(L^2 \geq 4k_0 - 4\) is equivalent to \(R'^2 \geq -4\). We have showed above that \(h^2(R') = 0\).
Let now $D \in |N'|_s$ (the locus of smooth curves in $|N'|$, with notation as in [2]) and consider $O_D(M')$.

We now claim that

$$O_D(M')$$ is base point free if and only if $R'$ is not a smooth rational curve
satisfying $R'.N' = 1$ (in which case $L^2 = 4k_0 - 2$, so that $k = k_0$). (2.6)

If $R'^2 = -4$, then $L^2 = 4k_0 - 4 = N'^2 + M'^2 + 2k_0$, whence $N'^2 + M'^2 = 2k_0 - 4$, and it follows that $N'^2 \leq k_0 - 2$, since $N'^2 \leq M'^2$. Therefore $\deg O_D(M') = k_0 \geq N'^2 + 2 = 2g(D)$ and $O_D(M')$ is base point free.

If $R'^2 \geq -2$, then $R > 0$ by Riemann-Roch and the fact that $h^2(R') = 0$. We have

$$\deg O_D(M') = M'.N' = (N' + R').N' = N'^2 + R'.N' = 2g(D) - 2 + R'.N',$$

so $O_D(M')$ is base point free if $R'.N' \geq 2$. If $R'.N' \leq 1$, we must have $h^0(R') = 1$, since $D$ is not rational (as $N'^2 > 0$), whence $R'^2 = -2$. We will now show that $R'$ is irreducible with $R'.N' = 1$ and that $O_D(M')$ is not base point free.

We have $R'.L = 2R'.N' - 2$, whence $R'.N' = 1$ and $R'.L = 0$ by the nefness of $L$. So there has to exist a smooth rational curve $\Gamma \leq R'$ such that $\Gamma.N' = 1$. Now $2N' + \Gamma \leq L$, and since $h^0(2N' + \Gamma) = \frac{1}{2}(2N' + \Gamma)^2 + 2 \geq h^0(L)$, we must have $R' = \Gamma$. Furthermore, $O_D(M') \simeq \omega_D(x)$, where $x = D \cap R'$, so that $O_D(M')$ is not base point free. Hence (2.6) is proved.

By Lemma 2.2 in [2] and the conditions (2.1)-(2.6), we are therefore in case (a) unless $R' \sim M' - N'$ is a smooth rational curve and $R'.N' = 1$. In this case $k_0 = k$ so that $M_0 = M$ and $N_0 = N$. We have $(M - N)^2 = -2$, so that $M - N > 0$ by Riemann-Roch, and since $M - N \leq M' - N' = R'$, we have $M = M'$ and $N = N'$ and we are in case (b).

**Proposition 2.1.** Keep the same hypotheses and notation as in Prop. 2.3 in [2].

If we are in case (b) of [2], Prop. 2.3, then all the smooth curves in $|L|$ have gonality $d$ and Clifford index $d - 3$, so they are exceptional.

If we are in case (c) of [2], Prop. 2.3, with $\rho(g, d, 1) < 0$, then the following additional conditions hold:

- (c6) $M.L \geq N.L$ and $h^0(N - M) = 0$ unless $M \sim N$;
- (c7) $M$ is not of the form $M \sim N + \Delta$, with $\Delta$ a smooth rational curve such that
  $\Delta.N = 1$ (and $\Delta$ is the base divisor of $|M|$);
- (c8) for any smooth, irreducible $D \in |N|$, we have that $O_D(M)$ is base point free.

If, furthermore, the gonality among the smooth curves in $|L|$ is not constant, then

- (c9) the general $C' \in |L|$ satisfies $\text{Cliff } C = \text{Cliff } C' = \text{Cliff } O_{C'}(N) = d - 2$ and
  $\text{gon } C' = d + 1$ (whence is exceptional);
- (c10) $L^2 \geq 4d - 2$ and $h^0(M - N) > 0$;
- (c11) $M^2 > 0$ and $N^2 > 0$. 
Proof. Assume we are in (b) of [2], Prop. 2.3. Then, for any smooth \( C' \in |L| \), one easily sees that \( \mathcal{O}_{C'}(N) \) contributes to the Clifford index of \( C' \), as \( h^0(N) = h^0(L - N) \geq 2 \), so that

\[
Cliff\, C' \leq Cliff\, \mathcal{O}_{C'}(N) = \deg \mathcal{O}_{C'}(N) - 2(h^0(\mathcal{O}_{C'}(N)) - 1) \\
\leq L.N - 2h^0(N) - 2(\frac{1}{2}N^2 + 1) \\
= L.N - N^2 - 2 = N.(N + \Delta) - 2 = c_2(E_{C,A}) - 1 - 2 \\
= d - 3 = \text{gon} C - 3 \leq \text{gon} C' - 3.
\]

Since \( \text{Cliff}\, C' = \text{gon} C' - 2 \) or \( \text{gon} C' - 3 \) by [3], Thm. 2.3, we must have \( \text{Cliff}\, C' = \text{gon} C' - 3 = \text{gon} C - 3 = d - 3 \), so that all \( C' \in |L| \) have the same gonality \( d \) and the same Clifford index \( d - 3 \). Hence they are all exceptional.

Assume now that we are in (c) of [2], Prop. 2.3. Note that \( \text{Cliff}\, C' = \text{gon} C' - 3 \) by [3], Thm. 2.3, we must have \( \text{Cliff}\, C' = \text{gon} C' - 3 \) or \( \text{gon} C' - 3 \) or \( \text{gon} C' - 3 < 0 \) by Riemann-Roch. Hence (2.7) splits, contradicting the fact that \( E \) splits. Hence we can without loss of generality assume \((c_5)\) in Prop. 2.3 of [2],

\[
0 \rightarrow M \rightarrow E_{C,A} \rightarrow N \rightarrow 0,
\]

(2.7) splits. Hence we can without loss of generality assume \((c_6)\) by symmetry.

To prove \((c_7)\), assume by contradiction that \( M \sim N + \Delta \), with \( \Delta \) a smooth rational curve such that \( \Delta.N = 1 \). Then \( h^1(\Delta) = h^1(M - N) = 0 \) by Riemann-Roch. Hence (2.7) splits, contradicting the fact that \( E_{C,A} \) is globally generated off a finite set, as \( \Delta \) is the base divisor of \( |M| \) (cf. Lemma 1.1(d) in [2]).

Next note that \((c_9)\) follows from \((c_7)\) exactly as in the proof of (2.6) above.

Now assume that the gonality among the smooth curves in \( |L| \) is not constant. Then \((c_{11})\) follows as otherwise \( M \) (or \( N \)) would cut out on every \( C' \in |L| \) a pencil of degree \( \leq d \).

As one easily sees that \( \mathcal{O}_{C'}(N) \) contributes to the Clifford index of any \( C' \in |L| \), we get \( \text{Cliff}\, C' \leq \text{Cliff}\, \mathcal{O}_{C'}(N) = d - 2 \), whence by [3], Thm. 2.3, \( \text{gon} C' \leq \text{Cliff}\, C' + 3 \leq d - 2 + 3 = d + 1 \), so that \((c_9)\) follows.

By Cor. 1.3 and Prop. 2.1 in [5] we have \( g(C') \geq 2\text{Cliff}\, C' + 4 = 2d \), whence \( L^2 \geq 4d - 2 \) and the rest of \((c_{10})\) follows using \((c_6)\) and Riemann-Roch. \( \square \)

As the last preparatory material for the proof of Theorem 1, we will now consider an incidence variety that is slightly different from the one in [2], §3.

Assume that we are in case (c) of [2], Prop. 2.3, with \( \rho(g, d, 1) < 0 \) (without the assumption that the gonality is not constant). Consider the incidence \( \mathcal{I}_{L,N,d} \subset |L| \times |N| \times \text{Hilb}^d(S) \) defined by

\[
\mathcal{I}_{L,N,d} := \{(C, D, Z) \mid Z \subset C \text{ and } Z \in |\mathcal{O}_D(M)|\},
\]

and let \( p^1_{L,N,d}, p^2_{L,N,d} \) and \( p^3_{L,N,d} \) be the projections.

Lemma 2.2. Assume that \( M \not\sim N \). Then...
(a) $\mathcal{I}_{L,N,d}$ is irreducible of dimension $\dim |L| + 1$;
(b) the projection $\mathcal{I}_{L,N,d} \to |L| \times \text{Hilb}^d(S)$ is bijective onto its image;
(c) if $C \in |L|$ lies in $\text{Im} \, p_{1,L,N,d}^1$, then $\text{gon} \, C = d$.

**Proof.** The Hodge index theorem, (c6) and the fact that $M \not\sim N$ imply $D^2 = N^2 < M \cdot N = d$. Therefore, two distinct $D_1, D_2 \in |N|$ cannot share the same $Z$ and (b) follows.

Consider the incidence $\mathcal{I}_{N,d} \subset |N| \times \text{Hilb}^d(S)$ given by $\mathcal{I}_{N,d} := \{(D, Z) \mid Z \in |O_D(M)|\}$. This is smooth, irreducible of dimension $\dim |N| + \dim |O_D(M)| = d$, using the fact that $h^1(O_D(M)) = 0$ for reasons of degree. For any $D \in |N|$ and any $Z \in |O_D(M)|$, we have

$$\dim |L \otimes I_Z| = \dim |L| - d + 1 > 0,$$

as can be computed from

$$0 \longrightarrow M \longrightarrow L \otimes I_Z \longrightarrow \omega_D \longrightarrow 0. \quad (2.9)$$

and the fact that $h^1(M) = 0$ by property (c3) in [2], Prop. 2.3. Therefore $\mathcal{I}_{N,d} = \text{Im}(p_{1,N,d}^2 \times p_{2,L,N,d}^3)$ and the dimension of any fiber $(p_{1,N,d}^2 \times p_{2,L,N,d}^3)^{-1}([D, Z])$ is $\dim |L \otimes I_Z| = \dim |L| - d + 1$. This proves (a) and the fact that $Z$ does not impose independent conditions on $L$ also implies (c).

### 3. Proof of Theorem 1

Let $L$ be a globally generated line bundle on a $K3$ surface $S$ and assume that the gonality of the smooth curves in $|L|$ is not constant. Let $d$ be the minimal gonality among the smooth curves in $|L|$ and let $C \in |L|$ be a smooth $d$-gonal curve. Then $\rho(g, d, 1) < 0$ by Brill-Noether theory, where $g = \frac{1}{2}L^2 + 1$ is the genus of $C$. Hence we are in case (c) of [2], Prop. 2.3, and the conditions (c1)-(c5) therein and (c6)-(c11) in Proposition 2.1 are satisfied. In particular, we have:

$$L \sim M + N, \ M^2 > 0, \ N^2 > 0, \ h^i(M) = h^i(N) = 0 \text{ for } i = 1, 2; \quad (3.1)$$

$N$ is globally generated;

the general $C' \in |L|$ satisfies $\text{Cliff} \, C = \text{Cliff} \, C' = \text{Cliff} \, O_{C'}(N) = d - 2$

and $\text{gon} \, C' = d + 1$ (whence is exceptional);

$$L^2 \geq 4d - 2 \text{ and } h^0(M - N) > 0. \quad (3.4)$$

Assume now, to get a contradiction, that we are not in the Donagi-Morrison example. We claim that

$$h^1(M - N) > 0, \quad (3.5)$$

$M$ and $N$ are linearly independent in $\text{Pic} \, S$.

Indeed, if $h^1(M - N) = 0$, then (2.7) splits, so that $E_{C,A} \simeq M \oplus N$ and $h^1(E_{C,A} \otimes E_{C,A}^*) = 0$ and we are in the Donagi-Morrison example by [2], Cor.
1.6, a contradiction. Moreover, if $M$ and $N$ are linearly dependent in Pic $S$, then $M \sim mB$ and $N \sim nB$ for a nef and big $B$ in Pic $S$ and positive integers $m$ and $n$, whence the contradiction $h^1(M - N) = 0$.

Let $f : \mathcal{S} \to \mathcal{U}$ denote the Kuranishi deformation of $S = S_0, 0 \in \mathcal{U}$. Then $\mathcal{U}$ is smooth of dimension 20, cf. [11] or [1], VIII, Thm. 7.3. Let now $\mathcal{V}' \subset \mathcal{U}$ be the submanifold to which both line bundles $L$ and $N$ lift. Because of (3.1) and (3.6), $\mathcal{V}'$ is smooth of dimension 18 by [11], Thm. 14. Again by [11], Thm. 14, there is a Zariski-open dense subset $\mathcal{V} \subset \mathcal{V}'$ such that for any $t \in \mathcal{V} - \{0\}$, we have that $S_t$ is a smooth $K$3 surface and Pic $S_t$ has rank two, where $S_t$ denotes the surface corresponding to $t \in \mathcal{V}$. Therefore, letting $L_t, N_t$ and $M_t := L_t - N_t$ denote the deformations of $L = L_0, N = N_0$ and $M = M_0$, we have

$$\text{Pic}_Q S_t \simeq Q[N_t] \oplus Q[L_t].$$  \hfill (3.7)

The next lemma shows that the “nonconstancy of gonality” is preserved by the deformation.

**Lemma 3.1.** Let $t \in \mathcal{V} - \{0\}$ be general. Then

(i) there is a smooth curve $C_t \in |L_t|$ with $\text{gon} C_t = d$;

(ii) the general $C_t' \in |L_t|$ satisfies $\text{Cliff} C_t = \text{Cliff} C_t' = \text{Cliff} O_{C_t}(N_t) = d - 2$ and $\text{gon} C_t' = d + 1$ (whence is exceptional).

**Proof.** If (i) does not hold, then, as $M_t . N_t = d$, we must have $(M_t - N_t)^2 = -2$, $(M_t - N_t). L_t = 0$ and $h^0(M_t - N_t) = 1$ by Lemma 2.1. Hence also $(M - N)^2 = -2$ and $(M - N). L = 0$, whence $h^0(M - N) = 1$, so that $h^1(M - N) = 0$ by Riemann-Roch, contradicting (3.5).

As in the proof of Proposition 2.1, one sees that $\text{Cliff} C_t' \leq \text{Cliff} O_{C_t'}(N') = d - 2$ for general $C_t' \in |L_t|$, and equality must hold and $\text{gon} C_t' = d + 1$ by [3], Thm. 2.3, as these hold for $t = 0$ by (3.3), proving (ii). \hfill $\square$

We will need the following technical lemma about divisors on $S_t$:

**Lemma 3.2.** Let $t \in \mathcal{V} - \{0\}$ be general. Then there is a unique smooth, rational curve $\Gamma_t \subset S_t$ such that $\Gamma_t.L_t = 0$. Furthermore,

(i) $\Gamma_t \sim_{\text{Q}} -2N_t + \frac{2(d+N_t^2)}{\Gamma_t}L_t$;

(ii) $M_t - \Gamma_t$ is globally generated, $(M_t - \Gamma_t)^2 > 0$ and $h^1(M_t - \Gamma_t) = 0$;

(iii) $L_t - \Gamma_t \sim N_t + (M_t - \Gamma_t)$ is the only decomposition satisfying $h^0(N_t) \geq 2$,

$\begin{align*}
h^0(M_t - \Gamma_t) &\geq 2 \quad \text{and} \quad N_t . (M_t - \Gamma_t) \leq d - 1 \quad (\text{in fact,} \quad N_t . (M_t - \Gamma_t) = d - 1); \\
M_t - N_t - \Gamma_t &> 0 \quad \text{and} \quad h^1(M_t - N_t - \Gamma_t) = 0.
\end{align*}$

**Proof.** By Thm. A in [2] and Lemma 3.1 we have that $L_t$ cannot be ample, so that there is a smooth, rational curve $\Gamma_t \subset S_t$ such that $\Gamma_t.L_t = 0$.

As $h^1(M_0) = 0$ by (3.1) and $N_0$ is nef by (3.2), then by [10], Thm. 1, we can only have $(\Gamma_t . N_t, \Gamma_t . M_t) = (0, 0)$ or $(1, -1)$. Writing $\Gamma_t \sim aN_t + bL_t$ with $a, b \in \text{Q}$
by (3.7) we obtain $-2 = \Gamma_1^2 = aN_t\Gamma_t$, whence $a = -2$, $N_t\Gamma_t = 1$ and (i) easily follows. This also proves that $\Gamma_t$ is unique.

Note that (3.4), (3.6) and the Hodge index theorem imply $N^2 < d$, so that 

$$
\frac{2(d+N^2)}{L^2} \leq 1.
$$

Hence $M_t - \Gamma_t \sim N_t + (1 - \frac{2(d+N^2)}{L^2})L_t$ is big and nef (using (3.1)). Moreover, any smooth elliptic curve $E_t \subset S_t$ satisfies $E_t.(M_t - \Gamma_t) \geq E_t.N_t \geq 2$ by [13] (or [9], Thm. 1.1) and Kawamata-Viehweg vanishing.

Now assume $L_t - \Gamma_t \sim A_t + B_t$ satisfies $h^0(A_t) \geq 2$, $h^0(B_t) \geq 2$ and $A_t.B_t \leq -d-1$. We have $\Gamma_t.(A_t + B_t) = 2$. Since $A_t.(B_t + \Gamma_t) \geq d$ and $B_t.(A_t + \Gamma_t) \geq d$ by Lemma 2.1 and (3.4), we can only have $A_t.B_t = d - 1$ and $\Gamma_t.A_t = \Gamma_t.B_t = 1$. Writing $A_t \sim xN_t + yL_t$ with $x, y \in \mathbb{Q}$ by (3.7) we therefore obtain $x = 1$. Moreover, from

$$
d = A_t.(B_t + \Gamma_t) = (N_t + yL_t).(-N_t + (1 - y)L_t)
$$

we obtain $2y(d + N^2) = y(1 - y)L^2$. Hence either $y = 0$ or $1 - y = \frac{2(d+N^2)}{L^2}$ and (iii) is proved.

Finally, note that $M_t - N_t - \Gamma_t \sim Q (1 - \frac{2(d+N^2)}{L^2})L_t$. Hence $h^1(M_t - N_t - \Gamma_t) = 0$ as $L_t$ is big and nef and $\frac{2(d+N^2)}{L^2} \leq 1$. Moreover, $h^0(M_t - N_t - \Gamma_t) > 0$ by Riemann-Roch, and by Lemma 3.1 combined with (c$_7$) in Proposition 2.1, we must have $M_t - N_t - \Gamma_t > 0$, proving (iv).

If now $(S_t, L_t)$ is as in the Donagi-Morrison example, then $L_t \sim 3B_t$ with $B_t^2 = 2$, and as this is preserved for $t = 0$, also $(S, L)$ is as in the Donagi-Morrison example, a contradiction.

To reach the desired contradiction, thus proving Theorem 1, we can therefore assume that the following additional conditions are satisfied:

$$
L \sim H + \Gamma, \text{ with } H \text{ globally generated and } \Gamma \text{ a smooth, rational curve such that } \Gamma.M = -1 \text{ and } \Gamma.N = 1; \quad (3.8)
$$

$$
M - \Gamma \text{ is globally generated and } h^1(M - \Gamma) = 0; \quad (3.9)
$$

$$
H \sim N + (M - \Gamma) \text{ is the only decomposition satisfying }
$$

$$
h^0(N) \geq 2, \quad h^0(M - \Gamma) \geq 2 \quad \text{and} \quad N.(M - \Gamma) \leq d - 1; \quad (3.10)
$$

$$
M - N - \Gamma \geq 0 \quad \text{and} \quad h^1(M - N - \Gamma) = 0. \quad (3.11)
$$

(In (3.10), note that in fact the given decomposition satisfies $N.(M - \Gamma) = d - 1$.)

Consider now the incidence $I_{L,N,d} \subset |L| \times |N| \times \text{Hilb}^d(S)$ defined in Section 2. By Lemma 2.2, we see that we would reach the desired contradiction, that is, that gon $C'' = d$ for general $C'' \in |L|$, if we show that

$$
\dim(p_{L,N,d}^{-1}(C'')) = 1 \text{ for general } C'' \in \text{Im} \, p_{L,N,d}'. \quad (3.12)
$$

We show (3.12) by showing that

$$
C'' \cup \Gamma \in \text{Im} \, p_{L,N,d} \text{ for general } C'' \in |H| \quad (3.13)
$$
and
\[ \dim(p_{L,N,d}^1)^{-1}(C'' \cup \Gamma) = 1 \text{ for general } C'' \in |H|. \tag{3.14} \]

(Recall that \(C''\) is smooth by (3.8).)

To this end we will need:

**Lemma 3.3.** All the smooth curves in \(|H|\) have gonality \(d - 1\), and for the general smooth \(C'' \in |H|\) we have

(i) \(\dim W_{d-1}^1(C'') = 0\) and

(ii) \(C''\) contains some \(W \in |O_D(M - \Gamma)|\) for some \(D \in |N|_s\) and \(|O_{C''}(W)|\) is a \(g_{d-1}'\).

**Proof.** By Prop. 2.3 in [2], Lemma 2.1 and (3.10), the minimal gonality of a smooth curve in \(|H|\) is \(d - 1\), as \(H^2 = L^2 - 2 \geq 4d - 4 = 4(d - 1)\) by (3.4). Hence, by Lemma 1.2 and Cor. 1.6 in [2] the first assertion follows from (3.11) by using the vector bundle \(N \oplus (M - \Gamma)\). Indeed, if the gonality of the smooth curves in \(|H|\) were not constant, then \((S, H)\) would be as in the Donagi-Morrison example, so that \(H \sim 3B\) for some \(B \in \text{Pic} S\) with \(B^2 = 2\), contradicting the fact that \(\Gamma.H = 2\).

For a general \(C'' \in |H|\), let \(|A''|\) be a \(g_{d-1}'\) on \(C''\). Then from (3.10), (3.11) and Prop. 2.3 in [2] we have \(E_{C'',A''} \simeq O_S(N) \oplus O_S(M - \Gamma)\) and from property (c8) in Proposition 2.1 we have that \(|O_D(M - \Gamma)|\) is base point free for any \(D \in |N|_s\). Pick a \(W \in |O_D(M - \Gamma)|\). From
\[
0 \rightarrow M - \Gamma \rightarrow H \otimes \mathcal{I}_W \rightarrow \omega_D \rightarrow 0
\]
and (3.9) we see that \(|H \otimes \mathcal{I}_W|\) is globally generated off \(W\). For general \(D\) and \(W\), the general member of \(|H \otimes \mathcal{I}_W|\) is smooth by Bertini (and the base point freeness of \(|O_D(M - \Gamma)|\)). Moreover, one easily computes that \(\dim |H \otimes \mathcal{I}_W| = \dim |H| - d + 2\), so that \(|O_{C''}(W)|\) is a \(g_{d-1}'\). Using the standard exact sequence involving \(E_{C'',A''}\),
\[
0 \rightarrow H^0(A'')^* \otimes O_S \rightarrow O_S(N) \oplus O_S(M - \Gamma) \rightarrow \omega_{C''} - A'' \rightarrow 0
\]
(cf. [2], (2), p. 17), one easily sees that, for any \(W \in |A''|\), one has \(h^0(N \otimes \mathcal{I}_W) = h^0(O_{C''}(N)(-A'')) = h^0(O_S) \oplus h^0(N - (M - \Gamma)) = 1\) and \(h^0(M - \Gamma \otimes \mathcal{I}_W) = h^0(O_{C''}(M - \Gamma)(-A'')) = h^0(M - \Gamma - N) + 1 \geq 2\), where we have used (3.9) and (3.11). Therefore, there is a \(D \in |N|\) containing \(W\). From what we saw above, for general \(C''\) and \(W\), this \(D\) is smooth. Moreover, there is an \(M' \in |M - \Gamma|\) containing \(W\) but not \(D\), so that \(W = D \cap M'\), whence \(W \in |O_D(M - \Gamma)|\). This proves (ii).

Consider the incidence \(\mathcal{I}_{H,N,d-1}\).

We have \(M - \Gamma \not\sim N\) since \(M - N - \Gamma\) is nontrivial by (3.11) and we have just seen that \(p_{H,N,d-1}^1\) is dominant, whence by Lemma 2.2(a) its fibers are one-dimensional, proving (i).

Now (3.13) follows from Lemma 3.3(ii).
Pick a general $C'' \in |H|$ satisfying (3.13). Then by Lemma 2.2(b), we have that 
\[ \dim(p_{L,N,d}^{-1}(C'' \cup \Gamma)) = \dim(\mathcal{P}_1 \cup \mathcal{P}_2), \]
where
\[ \mathcal{P}_1 = \left\{ Z \in \text{Hilb}^d(S) \mid Z \in |\mathcal{O}_D(M)| \text{ for some } D \in |N|, \text{ and } Z \subset C'' \right\} \]
and
\[ \mathcal{P}_2 = \left\{ Z \in \text{Hilb}^d(S) \mid Z \in |\mathcal{O}_D(M)| \text{ for some } D \in |N|, \right. \\
\left. Z = W + \{x\}, x = \Gamma \cap D \text{ and } W \subset C'' \right\}. \]

As $W \in |\mathcal{O}_D(M - \Gamma)|$, we have $\dim \mathcal{P}_2 = 1$ by Lemma 3.3.

To compute $\dim \mathcal{P}_1$, consider the incidence $\mathcal{I} \subset |H| \times |N| \times \text{Hilb}^d(S)$ defined by
\[ \mathcal{I} = \left\{(C'', D, Z) \mid Z \subset C'' \text{ and } Z \in |\mathcal{O}_D(M)| \right\}, \]
and let $q_1$, $q_2$ and $q_3$ be the projections. As in Lemma 2.2 the projection $\mathcal{I} \rightarrow |H| \times \text{Hilb}^d(S)$ is bijective onto its image, and as we can assume that $q_1$ is dominant, we have
\[ \dim \mathcal{P}_1 = \dim q_1^{-1}(C'') = \dim \mathcal{I} - \dim |H| \]
\[ = \dim \text{Im}(q_2 \times q_3) + \dim |H \otimes \mathcal{I}_Z| - \dim |H| \]
\[ = \dim \mathcal{I}_{N,d} + \dim |H \otimes \mathcal{I}_Z| - \dim |H| = d - (d - 1) = 1. \]

Here $\mathcal{I}_{N,d} := \{(D, Z) \mid Z \in |\mathcal{O}_D(M)| \} \subset |N| \times \text{Hilb}^d(S)$ is the incidence variety in the proof of Lemma 2.2 (where we showed that $\dim \mathcal{I}_{N,d} = d$) and $\dim |H \otimes \mathcal{I}_Z| = \dim |H| - (d - 1)$ is easily calculated from (2.9) tensored by $\mathcal{O}_S(-\Gamma)$, using Riemann-Roch and (3.9).

Hence (3.14) follows and Theorem 1 is proved.

Note that by [2], Thm. 3.1, we have the following consequence of Theorem 1:

**Theorem 3.1.** Let $S$ be a K3 surface and $L$ a globally generated line bundle on $S$, not as in the Donagi-Morrison example. Let $g$ be the genus and $d$ the gonality of the smooth curves in $|L|$.

If $\rho(d, g, 1) < 0$, then, for the general smooth $C \in |L|$, we have $\dim W_1^d(C) = 0$.

### 4. Proof of Theorem 2

We will first prove the assertions in the “generalized ELMS examples”.

**Proposition 4.1.** Let $L$ be a line bundle on a K3 surface $S$ such that $L \sim 2D + \Gamma$ with $D$ and $\Gamma$ smooth curves satisfying $D^2 \geq 2$, $\Gamma^2 = -2$ and $\Gamma \cdot D = 1$. Assume furthermore that there is no line bundle $B$ on $S$ satisfying $0 \leq B^2 \leq D^2 - 1$ and $0 < B \cdot L - B^2 \leq D^2$.

Then $|L|$ is base point free and all the smooth curves in $|L|$ are exceptional, of genus $g = 2D^2 + 2 \geq 6$, Clifford index $c = D^2 - 1 = \frac{g-4}{2}$ and Clifford dimension
Assume that $d := \text{gon}C \leq k$. Then $\rho(g, d, 1) < 0$, whence by [2], Prop. 2.3, there is a globally generated $N \in \text{Pic} S$ such that $h^0(N) \geq 2$, $h^0(L - N) \geq 2$, $h^1(N) = h^1(L - N) = 0$, $N.(L - N) \leq k$ and $(L - N)^2 \geq N^2 \geq 0$ (the latter by $(c_6)$ in Proposition 2.1 and by Riemann-Roch on $N$)

We want to show that $N \sim D$.

The Hodge index theorem and the fact that $L \not\sim 2N$ yield $N^2 \leq k - 1$. If equality holds, then for the same reason we have $N.(L - N) = k$, whence $2k - 1 = N.L = N.(2D + \Gamma) \geq 2DN$. It follows that $D.N \leq k - 1$ and $N \sim D$ by the Hodge index theorem, as desired.

If $N^2 \leq k - 2 = D^2 - 1$, then the assumption on the nonexistence of $B$ implies $N.(L - N) = k$. Let now $F := D - N$. Then one easily computes $k = D.(L - D) = F.(F + \Gamma) + N.(L - N) = F.(F + \Gamma) + k$, whence $F.(F + \Gamma) = 0$. As $h^1(L - N) = 0$ we must have $\Gamma.(L - N) \geq -1$ by [10], Thm. 1, whence $\Gamma.N = 0$ or 1. As $1 = \Gamma.D = \Gamma.(F + N)$, we conclude that $\Gamma.F = F^2 = 0$ and $\Gamma.N = 1$. We then get

$$F.L = F.(2D + \Gamma) = 2DF = 2(N + F).F = 2N.F = (L - 2N - \Gamma).N = N.(L - N) - N^2 - \Gamma.N = k - N^2 - 1.$$ But then $0 < F.L \leq k - 1$, a contradiction on the nonexistence of $B$.

It follows that $L \sim D + (D + \Gamma)$ is the only decomposition satisfying $h^0(D) \geq 2$, $h^0(D + \Gamma) \geq 2$ and $D.(D + \Gamma) \leq k := D^2 + 1$. Therefore, we cannot be in case (c) of Prop. 2.3 in [2], by condition $(c_7)$ in Proposition 2.1. Hence we must be in case (b) and by Proposition 2.1 all the smooth curves in $|L|$ have gonality $k + 1$ and Clifford index $k - 2$, so they are exceptional.

From Thms. 3.6 and 3.7 in [5], the Clifford dimension of any smooth $C \in |L|$ is $\frac{1}{2}(k + 1)$ and only $\mathcal{O}_C(D)$ computes the Clifford index.

We now recall the conjecture in [5]:

**Conjecture (Eisenbud, Lange, Martens, Schreyer).** Let $C$ be a smooth curve of Clifford dimension $r \geq 3$. Then:

(a) $C$ has genus $g = 4r - 2$ and Clifford index $c = 2r - 3$,
(b) $C$ has a unique line bundle $A$ computing $c$ (and deg $A = g - 1$),
(c) $A^2 \simeq \omega_C$ and $A$ embeds $C$ as an arithmetically Cohen-Macaulay curve in $\mathbb{P}^r$, 

$r = \frac{1}{2}D^2 + 1$. Moreover, for any smooth curve $C \in |L|$ the Clifford index is computed only by $\mathcal{O}_C(D)$.
(d) $C$ is $2r$-gonal, and there is a one-dimensional family of pencils of degree $2r$, all of the form $|A-B|$, where $B$ is a divisor of $2r-3$ points of $C$.

In [5] the conjecture is proved for $r \leq 9$, and in general it is proved that if $C$ satisfies (a), then it also satisfies (b)-(d). We therefore see that the curves in the “generalized ELMS examples” satisfy the conjecture.

To prove Theorem 2, we use the well-known theorem of Green and Lazarsfeld. Let $C \in |L|$ be a smooth exceptional curve on a $K3$ surface, of genus $g$, Clifford index $c$ and gonality $c+3$, different from a smooth plane sextic in the Donagi-Morrison example. Then, by Theorem 1, all smooth curves in $|L|$ have the same gonality $d = c+3$. By Brill-Noether theory, $\rho(d-1, g, 1) = 2(d-1) - 2 - g < 0$, otherwise the curves would have carried a $g^1_d$. Hence $c < \lfloor \frac{2g-1}{2} \rfloor$. By [6] all the smooth curves in $|L|$ have Clifford index $c$ and there is a line bundle $N$ on $S$ such that $c = \text{Cliff}_C(N)$ and (see e.g. [12,9,8]) we also have that $|N|$ is base point free, $h^0(N) \geq 2$, $h^0(L-N) \geq 2$, $h^1(N) = h^1(L-N) = 0$ and $N.(L-N) = c+2$.

By Lemma 2.1 we must have (possibly after interchanging $N$ and $L-N$) that $L \sim 2N + \Gamma$ for a smooth rational curve $\Gamma$ satisfying $\Gamma.N = 1$. In particular $c = N^2 - 1$ and $N^2 > 0$. Therefore, the general element $D \in |N|$ is a smooth curve.

To show that we are in the “generalized ELMS examples” we have left to show that there is no line bundle $B$ on $S$ satisfying $0 \leq B^2 \leq N^2 - 1$ and $0 < B.L - B^2 \leq N^2$.

Assume such a $B$ exists. Then the numerical conditions imply $(L-B)^2 \geq N^2 + 3 > 0$ and $(L-B).L \geq 2N^2 + 3 > 0$, so that $h^0(L-B) \geq 2$ by Riemann-Roch. Similarly $h^0(B) \geq 2$ and one therefore easily sees that $O_C(B)$ contributes to the Clifford index of $C$, for any smooth $C \in |L|$. Hence

$$\text{Cliff}_C \leq \text{Cliff}_C(B) \leq B.L - 2(h^0(B) - 1) \leq B.L - B^2 - 2 \leq N^2 - 2 = c - 1,$$

a contradiction.

Thus, Theorem 2 is proved.

Remark 4.1. Note that the curves in the “generalized ELMS examples” have $\dim W^1_d(C) = 1$ and $\rho(d, g, 1) = 0$, where $d = \text{gon} C$ (cf. Theorem 3.1).

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