I. INTRODUCTION

During the last decades there has been an increasing effort to develop reliable, large scale quantum information processors. Since such a device could utilize quantum properties like superpositions and entanglement its computing power could potentially surpass every conceivable classical device for certain problems \cite{1, 2} with potential applications in various fields of science and technology. At the moment there are several physical realizations developed in parallel, each with their own benefits and drawbacks. One of the most advanced platforms for quantum information processing is based on ion traps \cite{3}, where many elementary operations have already been experimentally demonstrated with high precision \cite{4–6}. First experimental implementations were realized over a decade ago \cite{9, 10}, based on theoretical proposals in \cite{11–13}. However due to recent efforts in theory \cite{14–17} and experiment \cite{18} the operation times and error rates have significantly reduced. These realizations leverage the idea of geometric phases first introduced by Berry \cite{19, 20} where the cyclic evolution of a quantum state results in the acquisition of a phase. In the present article, we want to investigate how dissipation affects the functionality of the geometric phase gate and how one could alter the existing protocols to compensate for these effects.

In Sec. II we first review the ideal isolated case, introduce our notation and then present our open system model in the context of a single trapped ion. In Sec. III we then show how dissipation leads to additional phases and in which way they can be connected to the conventional geometrical and dynamical phases. Furthermore we show which conditions the experimental protocol must satisfy in order to implement a phase gate and how the sensitivity of the gate against small experimental errors is altered compared to the case where the system is perfectly isolated from its environment. In IV we then apply our results to the two-qubit phase gate protocol proposed in \cite{14} and examine the impact on the fidelity. We close the article with a summary and an outlook.

II. MODEL

In this section we will first introduce a model of an isolated phase gate. Then we expand the model to account for dissipative effects.

A. Isolated case

We consider the ion trap as a quantum harmonic oscillator with mass $m$ and frequency $\omega$ which is driven by

\begin{equation}
|00\rangle \rightarrow |00\rangle, \quad |11\rangle \rightarrow |11\rangle,
|01\rangle \rightarrow e^{i\Phi}|01\rangle, \quad |10\rangle \rightarrow e^{i\Phi}|10\rangle.
\end{equation}

Two-qubit phase gates are important since they can be used to convert the separable state $1/2(|11\rangle + |10\rangle + |01\rangle + |00\rangle)$ into a maximally entangled state $1/2(|11\rangle + i|10\rangle + i|01\rangle + |00\rangle)$. First experimental implementations were realized over a decade ago \cite{9, 10}, based on theoretical proposals in \cite{11–13}. However due to recent efforts in theory \cite{14–17} and experiment \cite{18} the operation times and error rates have significantly reduced. These realizations leverage the idea of geometric phases first introduced by Berry \cite{19, 20} where the cyclic evolution of a quantum state results in the acquisition of a phase. In the present article, we want to investigate how dissipation affects the functionality of the geometric phase gate and how one could alter the existing protocols to compensate for these effects.

In Sec. II we first review the ideal isolated case, introduce our notation and then present our open system model in the context of a single trapped ion. In Sec. III we then show how dissipation leads to additional phases and in which way they can be connected to the conventional geometrical and dynamical phases. Furthermore we show which conditions the experimental protocol must satisfy in order to implement a phase gate and how the sensitivity of the gate against small experimental errors is altered compared to the case where the system is perfectly isolated from its environment. In IV we then apply our results to the two-qubit phase gate protocol proposed in \cite{14} and examine the impact on the fidelity. We close the article with a summary and an outlook.
an external force leading to the Hamiltonian

\[ H_{\text{isol}}(t) = \hbar \omega a^\dagger a + V(t). \] (2)

Here \( a (a^\dagger) \) is the annihilation (creation) operator for the vibrational mode satisfying bosonic commutation relations \([a, a^\dagger] = 1\). The potential \( V(t) \) arises from the externally applied force \( F(t) \). Since we want to implement the operation described in Eq. (1) we need to introduce state-dependent forces \( F_j \) and \( F_0 \) that depend on an internal (e.g. spin-) state of the ion in order to distinguish these states. An ion in the internal state \(|1\rangle\) will only experience \( F_1 \) and vice versa. In the following we will use the notation \( F_j \) with \( j = 0, 1 \) labeling the internal state of the ion. Furthermore, the external forces \( F_j(t) \) are assumed to be homogeneous over the extent of the motional state. This can be assumed, for example, for forces realized by lasers if the wavelength of the laser is much greater than the amplitude of oscillation. Under these circumstances the Hamiltonian can be written in the following form

\[ H(t) = \hbar \omega (a^\dagger a) + |0\rangle\langle 0| \otimes V_0(t) + |1\rangle\langle 1| \otimes V_1(t), \] (3)

\[ V_j(t) = F_j(t)x = f_j(t)(a + a^\dagger), \] (4)

with \( f_j(t) = \frac{\hbar}{2m\omega} F_j(t) \).

Before we determine the evolution of a quantum state in this model we will simplify the equations more by switching to an interaction picture with respect to \( \hbar \omega t \). This equation can be solved by inserting a coherent state ansatz

\[ \rho = |j, z_j(t)\rangle \langle j, z_j(t)|, \] (7)

\[ |z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \]

where \( j \) represents the internal state and \( z_j \) the coherent state label of a particle in the internal state \(|j\rangle\). Inserting this ansatz into Eq. (6) leads to the following condition equation for the coherent state label \( z_j(t) \)

\[ \dot{z}_j = \frac{1}{i\hbar} \tilde{f}_j(t). \] (8)

In the setting of a phase gate we want \( f(t) \) to be part of some protocol which is switched on at a certain time and is completed some time \( T \) later. Therefore \( f \) shall only be non-zero in the interval \([0, T]\) and it shall be such that the motional state undergoes a cyclic evolution \( z_j(0) = z_j(T) \) whereas the internal degrees of freedom acquire a phase according to Eqs. (1). It is known that such a cyclic quantum evolution leads to the acquisition of a phase \( \varphi_j = \varphi_{\text{d},j} + \varphi_{\text{a},j} \), where the dynamical and geometrical phases satisfy \( \varphi_{\text{d},j} = -(1/\hbar)\langle j, z_j(t)|H(t)|j, z_j(t)\rangle \) and \( \varphi_{\text{a},j} = i\langle j, z_j(t)|\partial_t|j, z_j(t)\rangle \), respectively. The total phase acquired is equal to twice the area enclosed by the trajectory \( z_j(t) \) in the interaction picture (see Eq. (11) [10, 14]). In the following we will expand this model to include dissipation.

### B. Damped case

The damping of the motional state of the ion shall be described by a Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation [21, 22] of the following form

\[ \dot{\rho} = -i\hbar^{-1}[\tilde{H}(t), \rho] + \gamma (2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a). \] (9)

Note that this equation is already formulated in the interaction picture, because the chosen Lindblad terms are identical in interaction and Schrödinger picture.

This model is valid when the temperature of the environment is close to zero or at least much smaller than \( E_{\text{particle}}/k_B \), which means that any excited motional state will decay towards the ground state.

Remarkably, Eq. (9) can still be solved by a pure, coherent state \(|z\rangle\langle z|\) although it contains damping terms. Inserting the ansatz from Eq. (7) for a particle in the internal state \(|j\rangle\) into Eq. (9) leads to the following equation for the coherent state label \( z_j(t) \)

\[ \dot{z}_j + \gamma z_j = \frac{1}{i\hbar} \tilde{f}_j(t). \] (10)

We can see how in the absence of an external force the coherent state moves towards the ground state due to the loss of energy to the environment.

In the next section we want to carry out a detailed analysis of the phases to show the influence of the damping in more depth and compare it to the geometrical and dynamical phase of an isolated harmonic oscillator.

### III. CONSEQUENCES OF THE DAMPING

#### A. Consequences for the phase

In the following we want to investigate how the Lindblad terms in the time evolution equation affect the phase. We therefore consider a model which is in principle identical to (9) but slightly more general. The Hamiltonian shall be of the form

\[ H(t) = |0\rangle\langle 0| \otimes H_0(t) + |1\rangle\langle 1| \otimes H_1(t), \]

where \( H_0(t) \) and \( H_1(t) \) are the Hamiltonians for the internal and motional states, respectively.
where |0⟩, |1⟩ correspond to internal state of the ion and $H_0(t)$ and $H_1(t)$ act on the motional degree of freedom. We assume that the dissipation and decoherence is well described by a general GKSL master equation and thus arrive at the following model

$$\dot{\rho} = -\frac{i}{\hbar}[H(t), \rho] + \mathcal{L}[\rho],$$  \hspace{1cm} (11)

$$\mathcal{L}[\rho] = \sum_{l=1}^{N} L_l \rho L_l^\dagger - \frac{1}{2} \left( L_l^\dagger L_l \rho + \rho L_l^\dagger L_l \right).$$

Furthermore we assume that this model has a pure state solution $|\Psi_j⟩⟨\Psi_j|$ where the internal state $j$ remains unchanged during the evolution and investigate the consequences. Although these are very limiting assumptions we will see that the results can nevertheless be applied to the phase gate scenario mentioned before.

Under these assumptions we can repeat the argument proposed in [20] for a cyclic quantum evolution governed by a time-dependent Schrödinger equation. In our case, however the cyclic quantum evolution of a pure state is modified by a damping term. The details of computation can be found in the appendix A. We then get a new complex valued term $\xi$ in addition to the dynamical and geometrical phase

$$\varphi = \varphi_k + \varphi_d + \xi,$$ \hspace{1cm} (12)

$$\xi = \xi_L + i \eta,$$ \hspace{1cm} (13)

where the individual terms are defined as follows:

$$\dot{\varphi}_k = i \left( \langle \Psi_0 | \partial_t | \Psi_0 \rangle - \langle \Psi_1 | \partial_t | \Psi_1 \rangle \right),$$

$$\dot{\varphi}_d = -\frac{1}{\hbar} \left( \langle H_0(t) \rangle - \langle H_1(t) \rangle \right),$$ \hspace{1cm} (14)

$$\dot{\xi} = -\frac{i}{\hbar} \sum_{l=1}^{N} \langle \Psi_0 | L_l | \Psi_0 \rangle \langle \Psi_1 | L_l^\dagger | \Psi_1 \rangle$$

$$- \frac{1}{2} \left( \langle \Psi_0 | L_l^\dagger L_l | \Psi_0 \rangle + \langle \Psi_1 | L_l^\dagger L_l | \Psi_1 \rangle \right).$$

The first two terms are identical to the ones found in [20] and therefore correspond to dynamical and geometrical phases which arise during a cyclic evolution of a quantum system. Since we have constructed relative phases they are expressed as the difference between the dynamical/geometrical phases of particles in the internal states 0 and 1. The last sum cannot be expressed in such a way and contains dissipative effects. In general it leads to real terms in the exponent which result in a loss of coherence. We can apply this equation to the damped harmonic oscillator if we set $L = \sqrt{\sigma} a$ and $|\Psi_j⟩ = |j, z_j⟩$. Furthermore we can identify $H_j(t)$ which corresponds to the Hamiltonian seen by a particle in the internal state $|j⟩$ as $\hbar \omega a^\dagger a + V_j(t)$ (see Eq. [9]). This means we can calculate the dynamical phase for a particle in the internal state $j$ with $|j, z_j⟩$ in the interaction picture by using Eqs. [10] and [5] as

$$\varphi_{a,j} = -\frac{1}{\hbar} \langle H_j(t) \rangle$$

$$= -\frac{1}{\hbar} \int_0^T \langle j, z_j(t) | e^{-\imath \omega \tau a^\dagger a} H(t) e^{\imath \omega \tau a^\dagger a} | j, z_j(t) \rangle \, dt$$

$$= -\frac{1}{\hbar} \int_0^T \langle j, z_j(t) | \hbar \omega a^\dagger a + \tilde{V}_j(t) | j, z_j(t) \rangle \, dt$$

$$= \int_0^T 2 \imath \omega \langle z_j(t) z_j^\dagger(t) \rangle - \omega |z_j(t)|^2 \, dt.$$ \hspace{1cm} (15)

For the geometric phase we arrive at

$$\varphi_{k,j} = i \left( \langle z_j(t) | e^{-\imath \omega \tau a^\dagger a} \partial_t e^{\imath \omega \tau a^\dagger a} | z_j(t) \rangle \right)$$

$$= \int_0^T -\imath \langle \dot{z}_j(t) z_j^\dagger(t) \rangle + \omega |z_j(t)|^2 \, dt,$$ \hspace{1cm} (16)

$$\int_0^T \imath \langle z_j(t) \dot{z}_j^\dagger(t) \rangle = \omega |z_j(t)|^2 \, dt,$$ \hspace{1cm} (17)

again with $|j, z_j⟩$ in the interaction picture. As we can see the dynamical and geometrical phase are remarkably similar for the harmonic oscillator. Furthermore we can combine these two phases for the total phase in the isolated ($\gamma = 0$) case $\varphi_{\text{isol}}$:

$$\varphi_{\text{isol}} = \langle \varphi_{d,0} - \varphi_{d,1} \rangle + \langle \varphi_{g,0} - \varphi_{g,1} \rangle$$

$$= \int_0^T \imath \langle \dot{z}_j(t) z_j^\dagger(t) \rangle - \dot{z}_1(t) z_1^\dagger(t) \rangle \, dt.$$ \hspace{1cm} (18)

For a cyclic evolution this reduces to the known result [10, 14]

$$\varphi_{\text{isol}} = 2(A_0 - A_1),$$ \hspace{1cm} (19)

where $A_j$ is the area enclosed by the cyclic evolution of $z_j$. This is shown in Fig. 1. From now on, we do not always write the time dependence of the coherent state labels explicitly in order to shorten the notation.

The influence of the dissipation is condensed in the term $\xi$

$$i \dot{\xi} = \gamma \left( z_0 z_1^\dagger - \frac{1}{2} (|z_0|^2 + |z_1|^2) \right)$$

$$= -\gamma |z_1 - z_0|^2 - i \gamma |z_1||z_0| \sin(\theta_1 - \theta_0),$$

where the phases $\theta_j$ are defined by $z_j = |z_j| e^{i\theta_j}$. Note that $\xi$ consists of a real as well as an imaginary part.

In summary, an initial state which is in a superposition of spin states

$$\rho(0) = \left( |a|^2 \begin{pmatrix} a b^* \\ |b|^2 \end{pmatrix} \otimes |z(0)⟩⟨z(0)| \right),$$
will be transformed into the following state after the cyclic evolution:

\[
\rho(T) = \left(ab^*e^{-i\varphi_{\text{init}}-i\xi} - ab^*e^{i\varphi_{\text{init}}+i\xi}\right) \otimes |z(0)\rangle\langle z(0)|.
\]

Since \(\xi\) is complex this shows that for \(\gamma \neq 0\) the damping results in an additional real term \(-\eta = \int_0^T -\gamma|z_1 - z_0|^2 dt\) in the exponent which does only depend on the damping strength \(\gamma\) and the amplitude of the path. This real term leads to a dephasing of the spin state by diminishing the off-diagonal elements of the density matrix. We will therefore refer to it as dephasing term from now on.

We can also see a new phase term \(i\varphi_L = \int_0^T -\gamma|z_1||z_0|\sin(\theta_1 - \theta_0) dt\) which depends on the relative position of \(z_0\) and \(z_1\). The integral over this term can vanish for sufficiently symmetrical \(z_j(t)\) (e.g. if \(z_j(t) = z_j(T - t)\) with \(j \in \{0, 1\}\) or if \(z_1\) or \(z_0\) is in the ground state during the entire operation. In Sec. IV we will see that the 2-qubit phase gates proposed in [14, 16] and realized in [10] do indeed have the latter property which means that even with damping the phases produced by those phase gates are still only determined by the respective areas. The dephasing term can however only vanish if \(f_1 = f_0\) which implies that there is no relative phase as well. We can also conclude that the dephasing is stronger for higher energies of the particle which means it is especially relevant for short operation times as we will see in Sec. IV A.

B. Consequences for the path

We have seen in the previous section how the damping results in additional phase terms. However, from Eq. [10] it is clear that the damping alters the path as well. Therefore, the paths which are closed in the isolated case are no longer closed in the damped case. It is a natural question to ask which forces \(\tilde{f}_j(t) = f_j(t)e^{i\omega t}\) can be used to achieve the cyclical evolution \(z_j(0) = z_j(T)\) in the damped case and whether some of those forces should be used preferably because they minimize the dephasing term. First we note that it is not possible to completely compensate the effects of the damping by applying some sophisticated force \(f_c\). This can be seen from Eq. [10] since the isolated dynamics of a coherent state are described by \(\dot{z}_j = 0\) such a force would need to satisfy \(\ddot{f} = e^{i\omega t}f_c(t)\) which is impossible for real \(f_c(t)\).

To determine the effects of a force \(f_j(t)\) on the path \(z_j(t)\) we have to solve Eq. [10]. This leads to the solution

\[
z_j(t) = z_j,\text{hom} + z_j,\text{inhom} = z_j(0)e^{-\gamma t} + \int_0^t \frac{i}{\hbar} \tilde{f}_j e^{-\gamma(t-\tau)} d\tau.
\]

We can therefore conclude that in order to achieve the cyclic dynamics \(z_j(0) = z_j(T)\) we need the forces to satisfy

\[
z_j(0) \left(e^{\gamma T} - 1\right) = \int_0^T f_j(\tau)e^{i\omega \tau}e^{\gamma \tau} d\tau.
\]

The equation shows explicitly that for \(\gamma \neq 0\) the condition depends on the initial state \(z_j(0)\). This means that in contrast to the undamped case where \(f_j\) would always lead to closed trajectories it now only works for a specific initial condition \(z_j(0)\). The fault tolerance of a quantum phase gate towards the initial motional state is therefore lost in the damped case.

An interesting observation at this point is that if we consider \(z_j(0) = 0\) we can derive forces \(f_d\) which return \(z_j\) to the ground state after time \(T\) in the damped case from the forces \(f_{\text{nd}}\) which accomplish this in the undamped case by using the formula

\[
f_d = f_{\text{nd}} \cdot e^{-\gamma t}.
\]

We will use this link between the damped and the undamped scenarios in the next section to generalize an already existing protocol for 2-qubit phase gate to account for dissipative effects.

For an experimental realization it is desirable to minimize the dephasing term for a given relative phase. To examine how this can be done we want to consider the case \(f_0(t) = 0\) and \(|z_j(0)| = 0\) for the sake of simplicity. This means that \(|z_0\rangle\) is the ground state at all times and we only need to discuss the dynamics of \(|z_1\rangle\). These simplifications are well justified because in an experimental setup the ground state can be prepared initially and an additional force \(f_0\) does not bring any benefits but just makes the computations more complex. We can then derive simple expression for the phase and dephasing terms.
after time \( T \) from Eq. \([19]\) if we write \( z_1 = r(t)e^{i\theta(t)} \)

\[
\int_0^T z_1^* \dot{z}_1 dt = \int_0^T re^{-i\theta} \left( \dot{r}e^{i\theta} + ir\dot{\theta}e^{i\theta} \right) dt \\
= \int_0^T r\dot{r}dt + i\int_0^T \dot{\theta}r^2 dt.
\]

Since \( z_1(0) = z_1(T) \) we can see that the first integral vanishes by integrating by parts and we are left with the following expression for the phase:

\[
\varphi_{\text{isol}} = \int_0^T \dot{\theta}r^2 dt, \quad \varphi_L = 0. \tag{23}
\]

In this case, \( \varphi_L = 0 \) because \( z_0 = 0 \) at all times. Whereas, for the dephasing term we find

\[
\eta = \gamma \int_0^T r^2 dt. \tag{24}
\]

We can see that the easiest way to minimize the dephasing for a given relative phase is to make \( \dot{\theta} \) as large as possible. This result also makes intuitive sense since if the path goes around the origin multiple times (large \( \dot{\theta} \)) it needs a smaller amplitude (which results in less dephasing) in order to sweep over the same area. Since \( \theta \) corresponds to the interaction picture it is affected by the frequency \( \omega \) of the harmonic oscillator.

Fig. 2 displays the paths \( z_1(t) \) which are generated by forces of the form \( f_1 = e^{-\gamma t}\sin(\Omega t) \) and \( f_0 = 0 \) (see Eq. \([10]\)). The paths (a) and (b) correspond to the isolated case with \( \gamma = 0 \) whereas \( \gamma = 0.1 \) for the paths (c) and (d). We can see how the upper two paths are symmetrical with respect to the imaginary axis. As shown in \([14]\) this implies that the phase does not change (in first order) if the force is subjected to a homogeneous, small constant offset \( f \to f + \delta f \) in the \( \gamma = 0 \) case. In contrast the paths for \( \gamma \neq 0 \) are no longer symmetrical. However whether the path is symmetric or not depends on the force. In the next section we therefore want to investigate how the robustness can be maintained in the damped case by constructing forces differently compared to Eq. \([22]\).

\section{Consequences for the robustness}

At first we want to show how the condition for the robustness against small constant offsets of the force \( f \to f + \delta f \) reads in the dissipative case studied here. According to Eq. \([18]\) with \( z_0 = 0 \)

\[
\varphi_{\text{isol}} = \int_0^T \text{Im}(\dot{z}_1 z_1^*) dt \\
= -\frac{1}{\hbar} \int_0^T \text{Re}(z_1^* S f_1) dt,
\]

where we used Eq. \([10]\) and \( z_1^* = e^{-i\omega t} z_1 \) is the path in the Schrödinger picture. Therefore the offset to the phase in first order becomes

\[
\delta \varphi_{\text{isol}} = -\frac{\delta f}{\hbar} \int_0^T \text{Re}(z_1^* S) dt = 0.
\]

This result is identical to the one in the isolated case found in \([14]\). By inserting Eq. \([20]\), assuming \( z_1(0) = 0 \) and integrating by parts we can express this as a condition for the force

\[
0 = \int_0^T f(t) dt. \tag{25}
\]

Together with the condition for a cyclic evolution, Eq. \([21]\), we therefore have a set of conditions that for \( z_j(0) = 0 \) may be seen as orthogonality conditions for \( f(t) \)

\[
f(t) \perp \{ e^{\gamma t}\sin(\omega t), e^{\gamma t}\cos(\omega t), 1 \} =: \mathcal{C} \tag{26}
\]

This means that we can construct forces to suit our needs by a Gram-Schmidt procedure. It is useful to orthogonalize the set \( \mathcal{C} \) and then do one more orthogonalizing step for the arbitrary function \( g \) which will then become orthogonal to the set \( \mathcal{C} \). This method makes it possible to
construct a plethora of forces which will leave the phase unchanged under a small constant offset $\delta f$ and produce a cyclic evolution. By superposing many of such forces one can then ensure to meet further demands like e.g. $f(0) = f(T) = 0$.

We can conclude that it is possible to maintain the robustness of the phase gate against small constant offsets of the force $f \mapsto f + \delta f$ in the damped case. However as we have seen in the previous section (Eq. (21)) the gate loses its resistance against fluctuations in the initial motional state.

IV. APPLICATION TO 2-QUBIT PHASE GATES

In this section we want to show how the relations we found in the previous sections apply to two-qubit phase gates which have been realized in $^{[10, 18]}$. These two-qubit gates consist of two ions in a harmonic trap potential which experience a force that depends on the internal state ($\left| \uparrow \uparrow \right>$ or $\left| \downarrow \downarrow \right>$) of the ion. As shown in $^{[14]}$ the Hamiltonian of such a system can be written as

$$H_{\text{tot}} = H_+ + H_-, $$

$$H_\pm = \frac{p_\pm^2}{2} + \frac{1}{2} \Omega_\pm^2 x_\pm^2 + f_\pm x_\pm. $$ (27)

Here $H_+$ describes an oscillation of a stretch mode where the displacement from equilibrium position of the two ions are equal but in opposite directions and $H_-$ describes an oscillation of the center-of-mass mode where the displacement of the ions is identical.

Note that here we ignored a term in $H_{\text{tot}}$ which is proportional to the difference of the forces experienced by the two ions. This (purely time dependent) term will therefore lead to additional phases for certain configurations. We will however later (see Eq. (35)) present a way to construct forces which satisfy $\int_0^T f dt = 0$ so that this phase will vanish. A more detailed derivation of the Hamiltonian and discussion of the purely dependent term can be found in $^{[14]}$.

If the forces on the two ions take the form $F_j = F(t)\sigma_i^j$ one can derive the following values for $f_\pm$

$$f_+(P) = f_-(A) = 0, $$

$$f_-(\uparrow\uparrow) = f_+(\downarrow\downarrow) = -2F/\sqrt{2m}, $$

$$f_-(\downarrow\uparrow) = f_+(\uparrow\downarrow) = 2F/\sqrt{2m}, $$ (28)

where $P \in \{\uparrow\uparrow, \downarrow\downarrow\}$ denotes parallel and $A \in \{\uparrow\downarrow, \down\uparrow\}$ anti-parallel spin combinations. We can bring these equations in the same form as in the previous section by introducing creation and annihilation operators for the stretch and center-of-mass mode $a_\pm, a^\dagger_\pm$ and switching to an interaction picture

$$|\Psi_I\rangle = e^{-iH_0 t^/h}\Psi$$

$$H_0 = \hbar \Omega_+ (a_+^\dagger a_+ + \frac{1}{2}) + \hbar \Omega_- (a_-^\dagger a_- + \frac{1}{2}). $$ (29)

The Hamiltonian then reduces to

$$\tilde{V} = \tilde{f}_+ a_+ + \tilde{f}_- a_+^\dagger + \tilde{f}_+ a_- + \tilde{f}_- a_-^\dagger$$

$$= \tilde{V}_+ + \tilde{V}_-.$$ (30)

To model the damping we can introduce two Lindblad terms similar to Eq. (7). We assume identical damping rates $\gamma$ since all degrees of freedom couple to the same bath which we assume to have a flat spectral density on the relevant frequency scale. We then find in the interaction picture

$$\rho = \mathcal{L}_+ [\rho] + \mathcal{L}_- [\rho], $$ (31)

$$\mathcal{L}_\pm := -i \frac{|\tilde{V}_\pm|^2}{\hbar} |\tilde{\rho}| + \gamma \left(2a_\pm^\dagger a_\pm - a_\pm a_\pm^\dagger \rho - \rho a_\pm^\dagger a_\pm \right).$$

We can see that the $\mathcal{L}_\pm$ only act on one of the two modes and are of the same form as the right hand side of Eq. (7). We can therefore solve Eq. (31) with an ansatz similar to Eq. (7)

$$\rho(t) = |j, z_\uparrow^j z_\downarrow^j\rangle \langle j, z_\uparrow^j z_\downarrow^j|,$$

where $j \in \{\uparrow\uparrow, \uparrow\downarrow, \down\uparrow, \down\down\}$ now corresponds to four possible internal states. This ansatz describes a pure quantum state where the internal degrees of freedom are given by $j$ and oscillation in the stretch and center-of-mass mode is described by $z_\uparrow^j$ and $z_\downarrow^j$ respectively. The resulting equations for the motional state are

$$z_\uparrow^j + \gamma z_\downarrow^j = \frac{1}{i\hbar} \tilde{f}_\pm(j). $$ (32)

Furthermore we can observe that Eq. (31) is of the form of Eq. (11) which means that we can apply the results from Sec. III A to calculate the phase and dephasing which arise during a cyclic evolution:

$$\varphi(T) = \varphi_{\text{isol}} + \xi, $$ (33)

where

$$\varphi_{\text{isol}} = 2(A_+^1 + A_-^1 - A_+^0 - A_-^0), $$

$$\xi = \gamma \int_0^T d_+ (\tau) + d_- (\tau) d\tau, $$

$$d_\pm = i|z_\uparrow^0 - z_\downarrow^0|^2 + |z_\uparrow^0|^2 |z_\downarrow^0|^2 \sin(\theta_\pm^0 - \theta_\pm^0). $$

We see that $\varphi_{\text{isol}} = \varphi_0 + \varphi_1$ is proportional to the areas swept in the stretch and center-of-mass modes of oscillation $A_+$ and $A_-$ respectively, and it is identical to the phase of the isolated evolution ($\gamma = 0$). Because of the symmetries of the forces described in Eq. (28) $\varphi_{\text{isol}}$ is only nonzero if 0 is a parallel and 1 an antiparallel spin combination or vice versa. The $\xi$ term originates from the Lindblad operators and since $\text{Im}(\xi) \neq 0$ it will result to dephasing equivalently to the one ion case discussed in Sec. III A. However, if the evolution starts in the ground state either $z_\downarrow^0$ or $z_\uparrow^1$ will remain in the ground state for the entire operation because of Eq. (28). Therefore $\text{Re}(\xi)$
will always be zero which means that the phase depends only on the difference of the swept phase space areas like in the undamped case studied in [14].

Since the equations (32) for the evolution of $z$ are identical to the one ion case, Eq. (22) still holds true and we can easily generalize the force $F_{\text{nd}}$ constructed in [14] to the damped oscillator:

$$F(t) = \kappa e^{-\gamma t} F_{\text{nd}}.$$  

According to Eq. (20) this means for the damped path

$$z_d = \kappa e^{-\gamma t/2} z_{\text{nd}}.$$  

Here we introduced two correction factors $\kappa$ and $\exp(-\gamma t/2)$ to the original force for the undamped case. $\kappa$ is a constant to compensate for the smaller area due to the damping and it therefore ensures that the phase (which corresponds to the area) stays the same. The exponential factor ensures that $z$ returns to the ground state after time $T$.

It would also be possible to construct forces via the Gram-Schmidt process described in Sec. III C to maintain the resistance against small constant offsets of the force. Since these forces now have to produce closed paths in both modes there are more orthogonality conditions.

The paths of the resulting $z_\pm$ with and without damping and under the influence of different forces are shown in Fig. 3. The plots (a), (b) and (c) are in the interaction picture whereas the plot (d) is in the Schrödinger picture.

The parameters for trajectory (a) were chosen identically to [14]: $T = 0.8 \mu s$, $\omega/2\pi = 2$ MHz and $\gamma/\omega = 0$. The trajectory (b) corresponds to the same force and parameters as trajectory (a) but now with a damping $\gamma/\omega = 0.1$. We can see how the path is no longer closed and that the area decreased as well. In the plot (c) we used the same damping and parameters as in (b) but with the adjusted force (22). Now the paths are closed again and the area difference is identical to (a). The plot (d) shows the trajectory for a shorter operation time $T = 0.3\mu s$ in the Schrödinger picture. It is important to note that the ticks are different for this plot because the trajectory has a much greater amplitude. The intuitive reason for this is that the particle needs a large momentum to complete the loop in a shorter time, the trajectory is therefore stretched out in $p \propto \text{Im}(z^3)$ direction. This illustrates that shorter operation times come with the trade-off of more dephasing (see Sec. IV A and Fig. 4).

Figure 3. (color online) Figure (a), (b) and (c) show paths of $z_+(↑↓)$ (solid blue line) and $z_-(↑↓)$ (dashed red line) in the interaction picture. Figure (d) shows paths in the Schrödinger picture. Plot (a) corresponds to one of the cases studied by [14]. The damping was set to zero and the other parameters were chosen as $T = 0.8\mu s$, $\omega/(2\pi) = 2$ MHz. Plot (b) shows a path generated by the same force but in the presence of fairly strong damping $\gamma = 0.1\omega$. We can clearly see how the paths are no longer closed which would lead to a loss of fidelity. In plot (c) the parameters are identical to (b) but now the force (22) which accounts for the damping was used. We can see how both paths are now closed. Furthermore the difference of the areas (which corresponds to the phase) is identical to (a). Plot (d) shows the trajectory for a shorter operation time $T = 0.3\mu s$ in the Schrödinger picture. Note that the ticks are different for this plot because the trajectory has a much greater amplitude which leads to more dephasing. Because of that, we can conclude that shorter operation times come with the trade-off of more dephasing (see Fig. 4).
A. Influence of damping on the fidelity

The fidelity measures the overlap of the final state $\rho_f$ with the desired state $|\Psi_d\rangle$

$$\mathcal{F} = \langle \Psi_d | \rho_f | \Psi_d \rangle.$$ 

Since we want to implement a two qubit phase gate our desired state $|\Psi_d\rangle$ is

$$|\Psi_d\rangle = ae^{i\varphi_{\text{isol}}}|P\rangle + b|A\rangle,$$

with $|a|^2 + |b|^2 = 1$. $P$ and $A$ denote an arbitrary parallel and anti-parallel spin combination (e.g. $P = \uparrow \uparrow$ and $A = \downarrow \downarrow$). The final (spin-) state (in the basis $\{ |P\rangle, |A\rangle \}$) after the cyclic evolution is

$$\rho_f = \left( \begin{array}{cc} |a|^2 & ab e^{i\varphi_{\text{isol}} - \Gamma} \\ a^* b e^{-i\varphi_{\text{isol}} - \Gamma} & |b|^2 \end{array} \right),$$

where

$$\Gamma = \gamma \int_0^T |z_+(A)|^2 + |z_-(P)|^2 dt.$$ 

We can calculate the fidelity as

$$\mathcal{F} \geq \frac{1 + e^{-\Gamma}}{2}. \quad (36)$$ 

Figure 4 shows the maximal infidelity $1 - \mathcal{F}$ and phase difference $\Delta \varphi = 2(\varphi(A) - \varphi(P))$ for different values of $\gamma$ with constant $T = 0.8\mu s$ and the plots (c) and (d) show these values for different operation times and constant $\gamma/\omega = 10^{-4}$. The solid blue line in (b) and (d) represent the phase difference of trajectories which result from the force (34) which accounts for the damping and the dashed red lines represent the phase difference of trajectories which result from the original force.

V. SUMMARY & OUTLOOK

We examined how phase gates based on the geometrical phases of driven trapped ions behave under dissipation. We used forces which depend on an internal state of the trapped ion in order to construct relative phases which show up in the density operator. We then showed that in the special case of a cyclic GKSL-type evolution admitting pure state solutions the total phase will always have an additional third contribution beyond dynamical- and geometric phases due to dissipation

$$\varphi = \varphi_d + \varphi_g + \xi,$$

$$\xi = \varphi_L + \eta.$$ 

This third contribution will in general be complex and can be directly related to the Lindblad operators. Applied to the harmonic oscillator this means that the damping results in a new additional phase that can however vanish for certain special cases. More severely, dephasing occurs which cannot be avoided and depends on the amplitude of the oscillation and the damping strength. We applied our results obtained for a single trapped ion to a two-qubit phase gate proposed in [14] which is based on two trapped ions. We found that in the presence of damping the phase produced by the gate depends on the area swept in the interaction picture alone, if the ion is in the ground state at the beginning of the operation. However due to the dephasing the fidelity of the gate is reduced. This loss of fidelity is especially noticeable for large damping strengths or short operation times. Furthermore, our calculations show how due to the damping the phase gate no longer operates independently of the initial motional state. On the other hand it is possible to maintain the robustness against small constant offsets of the force $f \rightarrow f + \delta f$ in the damped case.
by constructing the force using a Gram-Schmidt procedure that also ensures that the force produces a cyclic evolution.

An extension of this work is to look at a derivation of the two trapped ion model and coupling to the surrounding heat baths from first principles even at non-zero temperature. Such a model will most likely not allow for an analytical solution any more and the question whether the resulting relative phase depends only on the phase space area alone remains open.

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Appendix A: Analyzing the phase of a dissipative time evolution

Since we assumed that the evolution can be described by a pure state we can construct two solutions to Eq. (11):

\[ \rho_{00}(t) = |0⟩⟨0| ⊗ |Ψ_0(t)⟩⟨Ψ_0(t)| \]
\[ \rho_{11}(t) = |1⟩⟨1| ⊗ |Ψ_1(t)⟩⟨Ψ_1(t)|. \]

In order to determine the relative phase between those two states we examine the evolution of the superposition which means that we have a density operator of the form

\[ \rho = \rho_{00} + \rho_{01} + (\rho_{01})^\dagger + \rho_{11} \]
\[ \rho_{01} = e^{iφ(t)}|Ψ_0⟩⟨Ψ_1|. \]  

(A1)

We already know that \( \rho_{00} \) and \( \rho_{11} \) solve the equation so after inserting \( \rho \) into Eq. (11) we are left with

\[ \dot{\rho}_{01} = -\frac{i}{\hbar}[H, \rho_{01}] + [\rho_{01}] \]
\[ L[\rho_{01}] = \sum_{i=1}^N L_i \rho_{01} L_i^\dagger - \frac{1}{2} \left(L_i^\dagger L_i \rho_{01} + \rho_{01} L_i^\dagger L_i \right). \]  

(A2)

In analogy to [20] we can use (A1) and calculate

\[ \dot{\rho}_{01} = -iφ + e^{-iφ(t)} \frac{d}{dt} |Ψ_0⟩⟨Ψ_1| \]
\[ -\dot{φ} = -i⟨Ψ_0| \left( \frac{d}{dt} |Ψ_0⟩⟨Ψ_1| \right) |Ψ_1⟩ + ie^{-iφ} ⟨Ψ_0|\rho_{01}|Ψ_1⟩ \]  

(A4)

(A5)

Since we want to determine \( \dot{φ} \) this leaves us with two terms to evaluate:

\[ ⟨Ψ_0| \left( \frac{d}{dt} |Ψ_0⟩⟨Ψ_1| \right) |Ψ_1⟩ = ⟨Ψ_0|\dot{Ψ}_0⟩ + ⟨Ψ_1|\dot{Ψ}_1⟩, \]

and

\[ e^{-iφ(t)} ⟨Ψ_0|\rho_{01}|Ψ_1⟩ = -\frac{ie^{-iφ}}{\hbar} \left( ⟨Ψ_0|H|Ψ_0⟩ + L[ρ_{01}] |Ψ_1⟩ \right) \]
\[ = -\frac{i}{\hbar} \left( ⟨Ψ_0|H|Ψ_0⟩ - ⟨Ψ_1|H|Ψ_1⟩ \right) \]
\[ + \sum_{i=1}^N ⟨Ψ_0|L_i|Ψ_0⟩ ⟨Ψ_1|L_i^\dagger|Ψ_1⟩ \]
\[ - \frac{1}{2} \left( ⟨Ψ_0|L_i^\dagger L_i|Ψ_0⟩ + ⟨Ψ_1|L_i^\dagger L_i|Ψ_1⟩ \right). \]

This leads to the final result

\[ -\frac{dφ}{dt} = -i \left( ⟨Ψ_0|\partial_t|Ψ_0⟩ - ⟨Ψ_1|\partial_t|Ψ_1⟩ \right) + \frac{1}{\hbar} (⟨H_0⟩ - ⟨H_1⟩) \]
\[ -i \sum_{i=1}^N ⟨Ψ_0|L_i|Ψ_0⟩ ⟨Ψ_1|L_i^\dagger|Ψ_1⟩ \]
\[ - \frac{1}{2} \left( ⟨Ψ_0|L_i^\dagger L_i|Ψ_0⟩ + ⟨Ψ_1|L_i^\dagger L_i|Ψ_1⟩ \right). \]

Here we also used that \( ⟨Ψ|Ψ⟩ \) is purely imaginary and therefore \( ⟨Ψ|Ψ⟩ = -⟨Ψ|Ψ⟩ \).

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