Coherent superposition of orthogonal Hermite–Gauss modes

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A R T I C L E I N F O

Keywords:
Coherent superposition
Courant–Snyder theory
Beam propagation

A B S T R A C T

The coherent superposition of orthogonal modes can result in phase dependent transverse offsets, shifts of the focus position, variations of the Rayleigh length and a reduction of the beam quality factor of the coherent sum of modes in comparison to the incoherent sum. Relations for first and second order moments, the beam quality and the Rayleigh length for the superposition of Hermite–Gauss modes are derived. The Courant–Snyder formalism, which was originally developed in the context of charged particle optics, is applied to propagate an arbitrary coherent sum of orthogonal modes through a lens system. Relations of generating and observable optical functions are highlighted. In the last part of the report the elegant Hermite–Gauss solution is interpreted in terms of generating and observable functions and the solution is decomposed into a sum of standard Hermite–Gauss modes.

1. Introduction

The treatment of optical and quantum mechanical problems within the framework of the Courant–Snyder theory promises elegant and simplified solutions for many propagation, imaging and matching problems. The Courant–Snyder theory [1] was originally developed in the field of accelerator physics and thus is naturally applicable to classical charged particle optics. It can, however, also be favorably applied to the description of laser modes, because it not only describes the development of the transverse beam size through linear optical systems, but it also relates the beam size development to the development of the Gouy phase [2]. Thus, the field profile of a known mode composition can be determined at each point of an optical system by simple matrix multiplications. Due to the known similarities of the paraxial Helmholtz equation and the Schrödinger equation, analogue statements hold for a class of quantum mechanical systems.

In a broad sense, beams can be formed by ensembles of particles which are moving into a predominant direction, as for example electrons or photons, or by directed wave fields, like electromagnetic waves. A finite intensity and a localization in space, such that an average position and a transverse rms width can be defined at each point of the optical system under consideration, are characteristics of beams, besides their directivity. Based on these simple properties an rms envelope can be defined and the Courant–Snyder theory can be applied.

Classical, incoherent beams are widely discussed in terms of their phase space distribution, which is however insufficient for the treatment of coherent or partially coherent beams. In the laser community, the Wigner distribution was introduced for the description of coherent light by Bastiaans [3,4], while Kim proposed the Fourier transform of the cross-spectral density for the brightness definition of synchrotron radiation [5]. Kim noted that his equation resembles the quasi probability distribution which Wigner had introduced in the context of statistical mechanics and which had already been rediscovered by several authors in connection with optical problems. For an expedient review of the Wigner distribution and its relations to quantum mechanics and optics see I. V. Bazarov [6]. Hereinafter, the application of the cross-spectral density or the Wigner distribution for the description and the propagation of partially coherent beams with reference to charged particle optics advanced, especially in the field of synchrotron radiation and FEL physics [6–8].

While the phase space distribution is strictly positive, the Wigner distribution can be locally negative, but it is still positive and normalized in the complete integral over the phase space coordinates. An important property of the Wigner distribution is, that the marginal distributions, i.e., the projections of the distribution onto both phase space coordinates are equal to the marginal distributions of the classical phase space [6,9,10]. Due to this identity of the marginal distributions, an rms ellipse can be associated with the Wigner distribution, which is identical to the rms phase space ellipse associated to the phase space density of arbitrary particle distributions [11]. Despite the local negativity of the Wigner distribution, it behaves thus with respect to its rms properties just as the classical phase space distribution. It can be mapped through an optical system with the same matrices as the phase space and, just like point like particles move in phase space on concentric ellipses with a phase advance that is described by the Courant–Snyder theory (cf. Fig. 2 in [2]), also structures of the Wigner distribution move in the same way on such ellipses.

The area of the rms phase space ellipse connects the beam divergence with the beam size and is thus a measure of the beam quality. In
charged particle optics the beam quality is hence described by the beam emittance, which is directly proportional to the area of the rms phase space ellipse. A beam with smaller emittance can be focused to smaller beam sizes and the beam divergence remains smaller than this is the case for a beam with larger emittance. Naturally, the lower limit of the emittance follows Heisenberg’s uncertainty principle. A related beam quality factor, the M-square parameter, which is also proportional to the phase space area and thus to the emittance, is employed in laser physics and light optics.

The concept of an emittance as conserved quantity of motion, with a lower limit following the Heisenberg relation, and the Courant–Snyder formalism has recently also been applied to describe the manipulation of quantum mechanical vortex particles and the evolution of a wave packet in phase space [12].

Despite its solid foundation, the application of the rms envelope and the Courant–Snyder formalism is, however, not in all cases obvious. Especially interference effects, which are negligible in classical accelerator physics, are suspected to lead to deviations from the rms propagation characteristics. This is however not the case as will be discussed below. Also fully or partially coherent beams follow the standard propagation characteristics [13]. However, when describing a beam as coherent sum of basis modes, it will be necessary to clearly distinguish the beam parameters of the coherent sum, which are connected to observable beam sizes, and the parameters of the basis modes, which are not directly observable. Identifying the observable parameters and their relations to the generating parameters is of fundamental importance, but often not carefully attended in present literature on the propagation of laser beams.

In the first part of this paper the effect of the coherent superposition of orthogonal modes will be discussed in detail, and it will be shown how the Courant–Snyder formalism can be used to propagate an arbitrary coherent or incoherent sum of modes through an optical system. An important finding is, that the coherent superposition leads to modifications of the observable optical functions, which requires to distinguish between observable and generating functions. In the second part the elegant Hermite–Gauss solution is analyzed with respect to the generating and observable parameters. It will be shown that it can be described as a superposition of standard Hermite modes and that it describes a beam which follows the Courant–Snyder formalism in the usual way.

2. Coherent superposition of Hermite–Gauss modes

It is common practice to describe beams in light optics in terms of the Rayleigh length $L_R$, which is the distance from a beam waist over which the transverse beam size increases by a factor square root of 2. In the Courant–Snyder theory the more general $\beta$-function is employed. The structure of the $\beta$-function reflects in general the complexity of an optical system, in a free drift it simply develops as:

$$\beta(z) = \beta_0 \left(1 + \frac{z^2}{L_R^2}\right),$$  \hspace{1cm} (1)

where $z = 0$ is a focus position. The $\beta$-function at the focus, $\beta_0$, corresponds to the Rayleigh length of that focus.

The transverse rms size is given by

$$\sigma(z) = \sqrt{\epsilon \beta(z)},$$  \hspace{1cm} (2)

where $\epsilon$ denotes the beam emittance.

The $\beta$-function describes an optical system independent of the specific characteristics of a beam, i.e., the beam size of any beam which is properly matched to the initial conditions of an optical system is given by the Eq. (2), where only the beam emittance is characteristic for a specific beam. The emittance is a constant of motion in linear transport systems and connects the transverse rms size and the $\beta$-function. It is related to the beam quality factor $M^2$, which is commonly used in laser science, by the relation

$$\epsilon = \frac{M^2}{2\kappa}.$$  \hspace{1cm} (3)

with the wave-number $k = 2\pi/\lambda$. $\lambda$ denotes the wavelength of the radiation.

The Courant–Snyder formalism simplifies the propagation of laser modes through optical systems significantly, because the complex radius of curvature is replaced by the correlation straight of the Wigner distribution which can be traced through the system by a simple matrix multiplication. Moreover, the beam size variable $\omega$ is reinterpreted in terms of the $\beta$-function and the Gouy phase is identified as phase advance of the Courant–Snyder theory. The phase advance has a central role in the Courant–Snyder theory as it connects first and second order moments [1,2,14]. Furthermore, it represents a generalized criterion for the imaging condition between two points of an optical system, because it describes the rotation of a point on its phase-space ellipse as the beam propagates through the system. The Gouy phase takes of course the same central role in the wave description of beams as key parameter for the development of the field distribution in an optical system. The equivalence of Gouy and Courant–Snyder phase underlines the importance of this parameter and establishes another connection of ray optics and wave optics. For an in depth discussion of $\beta$-function and phase advance and their relations to standard laser parameters see [2].

The coherent superposition of orthogonal modes has been studied before by various authors, e.g., [15–17]. The approach in these publications is based on the abcd transport matrix in combination with a calculation of the intensity moments from the Wigner distribution or the coherence function. The results are often complex and, as Bisson pointed out [18], of limited accessibility. Below a different approach will be followed. It will be shown, that the second moment of any beam follows the rms envelope equation and thus, it can be traced through an optical system by means of the standard optical functions of the Courant–Snyder theory. Once this is shown, the only remaining task is to find the proper initial conditions of the beam. The coherent superposition leads to phase dependent shifts of the focus position, the generation of beam offsets and tilts, and variations of the Rayleigh length ($\beta$-function at the focus). Each member of the superposition of modes is described in a regular way by a set of standard beam parameters, incorporated in a single $\beta$-function [2]. The superposition, as a whole, represents a more complex beam that will also be described by a $\beta$–function [2], which differs, however, from the $\beta$-function of its individual elements. It is therefore mandatory to clearly distinguish the generating beam parameters, which characterize separate members of the superposition, from the resulting beam parameters that can be observed for the whole superposition. In the following, this is realized by adding an index $g$ to the parameters of the generating modes, while the observable parameters of the coherent sum carry no additional index.

The relation of generating to observable parameters is of fundamental importance for the interpretation of experimental results and for the comparison of experiments with theoretical predictions. The present work underlines hence the importance of these relations. Also general solutions of the Helmholtz equation, as the elegant Hermite Gauss solution, can be described as superposition of basis modes. Hence they also show a characteristic difference of generating and observable parameters, which should be taken into account when discussing the propagation characteristics of such solutions.

The paraxial wave equation can be solved in the form of a superposition of Hermite–Gaussian modes. The modes constitute a complete and orthogonal basis of solutions. The coherent superposition of modes leads to interference terms in the mathematical description, which are absent when an incoherent superposition is assumed but which are relevant for the beam characteristics of the wave. In the following the influence of the interference terms on the moments of the intensity distribution will be discussed. As usual the transverse position and the
size of the wave are described by the first direct and the second central moment of the intensity distribution. The calculations are in general straightforward but lengthy. Assistance by a symbolic computation program is highly appreciated. Only the main results are summarized, while intermediate results are omitted.

The moments are calculated for the projections of the two-dimensional transverse distribution onto the axis of the uncoupled coordinate system, which reduces the problem to the 1D case. The transverse coordinate is denoted by $x$, while $z$ denotes the longitudinal direction of predominant motion.

In terms of the generating $\beta$-function $\beta_k = \beta_k(z)$ Hermite-Gauss modes are given by [2]:

$$E_n = \frac{1}{\sqrt{2^n n!}} \left( \frac{k}{\sqrt{\beta_k}} \right)^{1/2} H_n \left( \sqrt{\frac{k}{\beta_k}} x \right) e^{-k^2 x^2 / 2} \left[ x^{2n} / (2n+1)! \right] / \sqrt{2^n} \delta_{n0},$$

where $n$ is the mode number, $\varphi_a$ is an arbitrary phase which subsumes also the term $kz = \omega t$, $H_a$ is a Hermite polynomial and the generating $a$-function is given by $a_n = \frac{1}{\sqrt{2^n n!}}$. Note, that Eq. (4) depends – besides mode number and phase – only on the $\beta$-function. In a free drift the phase advance $\int \frac{1}{\beta_k} \, dz$ can be expressed as atan$(\frac{z_x}{\beta_k})$, where $z_x$ is measured relative to the focus position, moreover the relation $a_n = -\frac{z_x}{\beta_k}$ holds.

The generating function of the Hermite polynomials is

$$H_n = (-1)^l \sum_{n=0}^l \frac{n!}{l! (n-l)!} (-1)^{n-l} H_n(2x^l),$$

where $l$, $m$ and $n$ are non-negative integers and the summation extends over all combinations of $l$ and $m$ for which $l + 2m = n$. Table 1 summarizes the first Hermite polynomials for further reference.

Hermite polynomials are orthogonal with respect to the exponential weight function

$$\int e^{-x^2} H_n(x) H_m(x) \, dx = \sqrt{\pi} 2^n n! \delta_{nm}. $$

Thus, the arguments of all polynomials and exponential amplitude terms in the coherent sum have to be equal to make use of the orthogonality condition. The modes are hence superimposed without relative transverse offset and with the same generating beta function, which implies that all modes reach a focus at the same position.

Eq. (4) is normalized such that the intensity $\int E_n E_n^* \, dx = 1$. Here $E_n$ is conjugate to $E_n$. Integrals span throughout the text from minus to plus infinity. In order to maintain the normalization when two modes with mode numbers $n$ and $m$ ($n \neq m$) are superimposed the relative intensity contributions of the two modes $a_n$ and $a_m$ need to be normalized such that $a_n^2 + a_m^2 = 1$.

The integrated intensity of the coherent sum $a_n E_n + a_m E_m$ reads then as

$$I = \int \left[ a_n E_n + a_m E_m \right] \left[ a_n E_n^* + a_m E_m^* \right] \, dx = \frac{a_n^2 E_n^2 - a_m^2 E_m^2 + a_n a_m (E_n E_m^* + E_m E_n^*) + a_m^2 E_m^2 E_n^2} \, dx,$$

where $a_n^2 E_n^2 \, dx + a_m^2 E_m^2 \, dx \neq 0$ follows from the orthogonality of the modes and ensures energy conservation. Note, that

$$\mathbb{R} \left( E_n E_m^* \right) = \mathbb{R} \left( E_n^* E_m \right) \text{ and } \mathbb{S} \left( E_n E_m^* \right) = -\mathbb{S} \left( E_n^* E_m \right),$$

so that the sum of both terms is real.

The orthogonality of the modes leads also to the condition that most combinations of the mode numbers $n$ and $m$ do not result in a contribution to the first and second moment of the field distribution. For the transverse position $\hat{x} = \int a_n E_n + a_m E_m \left[ a_n E_n^* + a_m E_m^* \right] \, dx$ the interference term $\int a_n a_m E_n E_m^* + E_m E_n^* \, dx$ is zero in all cases, except for $m = n + 1$. This is explained by the fact that $x H_a$ contains the same polynomial orders as $H_{a+1}$ (cf. Table 1). The mathematical structure of the integrands is thus similar to the square of a single mode and a kind of modified orthogonality condition is realized. Equally the interference term in the second moment is zero for all cases except for $m = n + 2$. Again, $x^2 H_a$ contains the same polynomial orders as $H_{a+2}$ (cf. Table 1) and the integrands are hence nonzero. The conditions read as

$$\int e^{-x^2} H_n(x) H_m(x) \, dx = \begin{cases} \sqrt{\pi} 2^n n! \, (n+1) & \text{for } m = n+1 \\ 0 & \text{else} \end{cases} \text{ for } m = n+1$$

and

$$\int e^{-x^2} H_n(x) H_m(x) \, x^2 \, dx = \begin{cases} \sqrt{\pi} 2^n n! \, (n+1)(n+2) & \text{for } m = n+2 \\ 0 & \text{else} \end{cases} \text{ for } m = n+2.$$

Thus, even if an infinite number of modes takes part, only combinations of two modes at a time lead to a contribution to the specific moment. As example consider the case of 4 modes with mode numbers $n = 1, 2, 5, 6$. The intensity relation $\sum a_n E_n \sum a_m E_m^*$ yields a sum of $d^2 = 16$ product terms, but only the products with mode number combinations $1–2$ and $5–6$ will contribute to the first moment. In this example no contribution exists of the form $m = n+2$ and all other product terms (combinations $1–5, 2–6, 3–5, 4–6$) will not contribute to the first or the second moment and can thus be ignored.

Clearly more mode combinations contribute to higher order moments, which are however beyond the scope of this paper.

In the following the influence of mode combinations of the case $m = n + 1$ and of the $m = n + 2$ case are discussed separately before a generalization to an arbitrary combination of modes is presented. The relative mode intensities $a_n$ are assumed to be normalized as $a_n^2 = 1$.

Table 1
| $H_n(x)$ | $n$ |
|---|---|
| $H_0(x) = 1$ | 0 |
| $H_1(x) = 2x$ | 1 |
| $H_2(x) = 2x^2 - 2$ | 2 |
| $H_3(x) = 2x^3 - 6x$ | 3 |
| $H_4(x) = 2x^4 - 12x^2 + 12$ | 4 |

As abbreviations

$$S_{0n} = \sum_{n=0}^{\infty} a_n^2 \quad S_{11} = \sum_{n=0}^{\infty} a_n a_{n+1} \sqrt{n+1} \quad S_{22} = \sum_{n=0}^{\infty} a_n a_{n+2} \sqrt{n+1} \quad (n+2)$$

are introduced, where $S_{0n}$ describes the incoherent part of the relations, while $S_{11}$ and $S_{22}$ correspond to the contributions of the $m = n+1$ and the $m = n+2$ case, respectively. The sums are positive and $S_{0n}$ is larger than $S_{11}$ or $S_{22}$.

3. The case $m = n + 1, S_2 = 0$

The first moment of coherently superimposed modes is given as

$$\hat{x} = \int \sum_{n=0}^{\infty} a_n E_n \sum_{n=0}^{\infty} a_n E_n^* \, dx = S_{11} \gamma \cos \left( \frac{\Delta \phi}{2} \right) \sin \left( \frac{\Delta \phi}{2} \right) \frac{z_x}{\beta_k},$$

where $\Delta \phi = \varphi_n - \varphi_{n+1}$ denotes the phase difference between the first and the subsequent mode. Here it is assumed that the phase difference for all mode combinations in the same group, which is not necessarily the case, but leads to simplified equations. Coherence requires also that the phase difference is constant over a sufficient time interval, which is only possible if both modes have the same frequency as $\varphi_n$ subsumes the term $kz = \omega t$. Note, that at no phase difference, offset and angle are simultaneously zero.

While Eq. (11) leaves the direct second moment unchanged, the central second moment

$$\hat{x^2} = \int \sum_{n=0}^{\infty} a_n E_n \sum_{n=0}^{\infty} a_n E_n^* x^2 \, dx \neq \hat{x}^2$$

and thus, the transverse rms size $\sigma = \sqrt{\langle x^2 \rangle}$ is modified.
leads to the far field diffraction angle is proportional to \( z \). Thus, the beam size minimum is not reached at \( z = 0 \).

Instead the beam size minimum is reached at \( z = -\beta_0 \left( \frac{S_{m} \sin \delta \Delta \phi_1}{S_{m} - S_{0}^2 \sin^2 \left( \frac{\Delta \phi_1}{2} \right)} \right) \). (15)

Rewriting Eq. (14) in terms of the shifted focus position \( z = z_k - \bar{z}_k \) leads to

\[
\sigma^2 = \frac{\beta_0}{2k} \left\{ S_m \left( S_m - S_0^2 \right) \right\} + \left[ S_m - S_0^2 \sin^2 \left( \frac{\Delta \phi_1}{2} \right) \right] \left( \frac{\Delta \phi_1}{2} \right)^2 .
\]

which is symmetric with respect to the position \( z = 0 \), but still not in the standard form Eq. (1). Thus the \( \beta \)-function of the coherent sum of the modes differs from the generating \( \beta \)-function \( \beta_0 \). Or, in other words, the Rayleigh length of the coherent sum of the modes differs from the Rayleigh length of the individual modes. Eq. (16) has the form

\[
\sigma^2 = \frac{\beta_0}{2k} \left( A + B \frac{z^2}{\beta_0} \right)
\]

with the parameters

\[
A = \frac{S_m(S_m - S_0^2)}{S_m - S_0^2 \sin^2 \left( \frac{\Delta \phi_1}{2} \right)},
\]

\[
B = \frac{S_m - S_0^2 \sin^2 \left( \frac{\Delta \phi_1}{2} \right)}{\beta_0}.
\]

While the transverse rms size at the focus is proportional to \( \sqrt{A} \), the far field diffraction angle is proportional to \( \sqrt{B} \), which leads to the relations:

\[
\beta_0 = \frac{\beta_0}{2k} \sqrt{A} = \frac{\beta_0}{2k} \sqrt{\frac{S_m(S_m - S_0^2)}{S_m - S_0^2 \sin^2 \left( \frac{\Delta \phi_1}{2} \right)}}
\]

and

\[
\epsilon = \frac{1}{2} \sqrt{AB} = \frac{1}{2k} \sqrt{S_m(S_m - S_0^2)}.
\]

With these relations, Eq. (16) is transformed into the standard form

\[
\sigma^2 = \epsilon \beta_0 \left( 1 + \frac{z^2}{\beta_0} \right)
\]

The interference term in the superposition of two modes with \( m = n + 1 \) leads thus to a phase dependent transverse offset, but also to a shift of the focus position, and a variation of the transverse size, the emittance and the \( \beta \)-function.

A simple example is the addition of only two modes. The interference term gets maximal when both modes contribute with the same intensity, the relevant equations reduce then to

\[
\tilde{z}_k = -\beta_0 \frac{\sin \Delta \phi_1}{3 + \cos (\Delta \phi_1)}
\]

\[
\beta_0 = \frac{\beta_0}{2k} \left( \frac{\Delta \phi_1}{2} \right)^2
\]

and

\[
\epsilon = \frac{\sqrt{2}}{2k} (n + 1).
\]

Fig. 1 shows the phase dependence of the focus position and of the minimal \( \beta \)-function.

The shift of the focus position [Eq. (15)] is on the order of the generating \( \beta \)-function at the focus, i.e., on the order of the Rayleigh length and the \( \beta \)-function varies between \( \sqrt{2} \) and \( \sqrt{2} \) times the generating \( \beta \)-function.

The emittance should be compared to the emittance of the incoherent addition, which is determined solely by \( S_m \) and thus yields \( \epsilon_{in} \approx \frac{n}{\beta_0} \) for the case under consideration. The coherent addition leads hence to an emittance reduction by a factor \( \sqrt{2} \).

Finally the factor \( \sqrt{\frac{A}{B} \beta_0} \) can be approximated by \( \sigma \approx \sqrt{\epsilon_{in} \beta_0} \) in Eq. (20) to see that the transverse offset near the focus can become roughly as large as the rms beam size.

4. The case \( m = n + 2, S_1 = 0 \)

Another mode combination which influences the second order moment is the case \( m = n + 2 \). While in the previous case the offset, i.e., the second term of Eq. (12) was not zero, the offset is zero for \( m = n + 2 \),
but the first term of Eq. (12) is modified. The transverse rms size is given in this case as:

\[
\sigma^2 = \frac{\beta_{m}}{2k} \left[ S_{in} \left( 1 + \frac{z_{g}^2}{\beta_{g}^2} \right) + S_{z} \left( 1 - \frac{z_{g}^2}{\beta_{g}^2} \right) + 2 \sin \left( \frac{4\phi_{1}}{k} \right) \frac{z_{g}}{\beta_{g}} \right].
\]

(24)

where \(4\phi_{z}\) denotes the phase difference between the two modes.

Again, the focus is shifted due to the linear term in \(z_{g}\). The minimum transverse size is reached at:

\[
\tilde{z}_{g} = \beta_{g} \frac{S_{z}}{S_{in} - S_{2} \cos \left( \frac{4\phi_{1}}{k} \right)}.
\]

(25)

Introducing \(z = z_{g} - \tilde{z}_{g}\) into Eq. (24) leads to \(\sigma^2 = \frac{\beta_{m}}{2k} \left( A + B \frac{z_{g}^2}{\beta_{g}} \right)\) with

\[
A = \frac{S_{2}^2 - S_{z}^2}{S_{in} - S_{2} \cos \left( \frac{4\phi_{1}}{k} \right)} \quad \text{and} \quad B = S_{in} - S_{2} \cos \left( \frac{4\phi_{1}}{k} \right)
\]

(26)

and thus to

\[
\beta_{ii} = \beta_{m} \frac{S_{in} - S_{2} \cos \left( \frac{4\phi_{1}}{k} \right)}{S_{in} - S_{2} \cos \left( \frac{4\phi_{2}}{k} \right)} \quad \text{and} \quad \epsilon = \frac{1}{2k} \sqrt{S_{in}^2 - S_{2}^2}.
\]

(27)

The emittance is reduced in comparison to the incoherent addition, as is immediately visible from Eq. (28). The effects on the focus position and on the \(\beta\)-function for the simple example of two modes with equal intensity are of similar magnitude as discussed above for the \(m = n + 1\) case.

5. The general case

The generalization is now straightforward and follows the steps outlined above. In order to simplify the equations, the phase difference \(\Delta\phi_{1}\) between two successive modes and \(\Delta\phi_{2}\) between one and the next but one mode where introduced above, assuming already that the phase difference is the same for all relevant mode combinations in the sums.
for the different cases. In the general case, i.e., when arbitrary modes are superimposed, this requires that $\Delta \psi_2 = 2 \Delta \phi_1$.

With this assumption the focus shift, $\beta$-function and emittance are found as

$$\tilde{z}_k = \frac{k \pi}{2} \frac{(2S_k - S_k^2)}{S_k - S_k \cos \Delta \phi_1 + S_k^2 \sin^2 \left(\frac{\Delta \phi_1}{2}\right)} \sin \Delta \phi_1$$

(29)

$$\beta_i = \beta_0 \frac{\sqrt{S_{i}^2 - S_{i} \cos \Delta \phi_1 + S_{i}^2 \sin^2 \left(\frac{\Delta \phi_1}{2}\right)}}{S_{i} - S_{i} \cos \Delta \phi_1 + S_{i}^2 \sin^2 \left(\frac{\Delta \phi_1}{2}\right)}$$

(30)

$$\varepsilon = \frac{1}{2} \frac{1}{\gamma^2} \left[2S_{i}^2 - S_{i} \cos \Delta \phi_1 + S_{i}^2 \sin^2 \left(\frac{\Delta \phi_1}{2}\right)\right]$$

(31)

The emittance Eq. (31) is in all cases (cf. Eq. (19) (23) (28)) independent of the phase. The phase influences thus only the local beam properties, but not the global beam quality.

Fig. 2 shows as example the propagation of a coherent sum of modes through a periodical optical system. The calculations follow the standard procedures of the Courant-Snyder formalism. The only difference is, that two sets of optical functions are traced through the system, i.e., one set of generating functions and one set of observable functions. In the incoherent case these two sets are identical.

An initial $\beta$-function of $\beta_0 = 10$ cm is chosen, the $\alpha$-function at a focus is zero. The third optical function $\gamma$ is related to the rms beam divergence $\sigma'$ by $\sigma'(z) = \sqrt{\gamma}(z)$. The three optical functions are connected by the condition $\beta \gamma - \alpha' = 1$, thus at a focus the condition $\gamma$ is $\frac{1}{\beta_0}$. For the phase advance a value of $140^\circ$ between two lenses is chosen, which defines together with the periodicity condition the optical system. The phase advance is a free parameter which determines the distance between the lenses and the shift of the imaging points from one period to the next. The calculation of the optical functions $\alpha$, $\beta$ and $\gamma$ is derived from the abcd-matrix of drift and thin lens elements as [14]

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \frac{1}{2 \beta_0} \begin{pmatrix} a^2 + 2ab & b^2 & 0 \\ ac - 2bd & ad + bc - bd & 0 \\ c^2 - 2ac & -2bd & d^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix}.$$  

(32)

where the index $i$ indicates the initial values and the phase advance is given, as usual, by $\phi = \int \frac{1}{2} dz$. $a$, $b$, $c$ and $d$ are the elements of the abcd matrix, for details see [2]. The second panel of Fig. 2 displays the development of the $\beta$-function and the phase advance modulo $\pi$.

The beam in this illustrating example is generated by a superposition of three consecutive modes, $\gamma = 2$, $3$, $4$. Each mode contributes with the same intensity. The curvature of the phase front of the Hermite–Gauss modes is expressed by the generating $\alpha$-function and the Gouy phase is expressed by the generating phase advance as described in [2]. A wavelength of 800 nm is assumed and the first mode starts with $\phi = 0$ at $z_g = 0$. The other modes are shifted by $-0.4 \pi$ and $-0.8 \pi$ relative to the first mode. The start parameter of the generating $\beta$-function is determined by Eq. (30). At the focus position of the generating functions $z_k = 0$ the standard relations $a_k = 0$ and $\gamma_k = 1$ are held. These parameters are in a first step traced to the position $z = 0$, so that both sets of optical functions refer to the same position. The top panel of Fig. 2 shows the development of the generating $\beta$-function and the generating phase advance which corresponds to the phase of the first mode. While the observable $\beta$-function is periodic, the generating $\beta$-function is not periodic, i.e., while the observable $\beta$-function is properly matched the generating function is not matched.

Only the generating functions are used to track the modes through the system, but the observable $\beta$-function describes the rms beam size and the corresponding phase advance determines the imaging condition as demonstrated in the lower two rows of Fig. 2. The initial transverse intensity distribution is imaged whenever the phase advance is a multiple of $180^\circ$. Thus, the transverse intensity distribution has the same shape and the transverse offset is, relative to the beam size, the same. For better comparison the plots of the intensity distribution are reproduced with scaled coordinates in the lowest row.

6. The elegant Hermite-Gauss functions

Siegmam [19] established the so-called elegant Hermite–Gauss functions as a symmetrized solution of the paraxial Helmholtz equation by introducing a complex argument into the polynomial part of the Hermite–Gauss field description. Elegant Hermite–Gauss solutions are useful to treat several theoretical problems, and are hence a relevant example for a generalized solution of the paraxial Helmholtz equation. Besides in depth studies of more mathematical properties, e.g. [20,21], also propagation properties of the elegant Hermite–Gauss solutions have been studied [22–24].

Beams described by the elegant Hermite–Gauss solution change their transverse shape as they propagate and thus they are not simple modes. They also do not form an orthogonal basis with respect to the transverse coordinate, but rather a biorthogonal set of functions with a corresponding conjugate set.

While being mathematically elegant, the interpretation of the complex solution in terms of physical quantities is not straightforward. In the following, the elegant solution will be discussed in terms of the generating and the observable $\beta$-function of a coherent sum of modes.

Furthermore, the decomposition of the elegant solution is presented.

The elegant solution, indicated by the tilde, reads in terms of the generating $\beta$-function as

$$\tilde{E}_n = \sqrt{2 \pi n!} \left(\frac{k}{\pi \beta_0}\right)^{1/4} \left[1 + \frac{z^2}{\beta_0^2 n!}\right]^{1/4}$$

$$\times H_n \left(\frac{k}{2 \beta_0} \left(\frac{k}{\beta_0} + i z_2\right)\right) \left[1 - \frac{k^2}{2 \beta_0^2} \left(\frac{k}{\beta_0} + i z_2\right)^2 \right] \sqrt{\frac{2}{\pi \beta_0}}.$$

(33)

Eq. (33) is normalized analogous to the Hermite–Gauss modes, i.e.,

$$\int \tilde{E}_n^* \tilde{E}_m^* dx = 1.$$  

The first order moment of the elegant solution is zero. Calculating the rms beam size yields:

$$\sigma^2 = \frac{\beta_0}{\pi} \left(\frac{A + B \frac{z^2}{\beta_0^2}}{2} \right)$$

$$B = 2n + 1,$$

which leads to

$$\beta_0 = \beta_0 \sqrt{\frac{4n - 1}{4n - 1}},$$

$$\varepsilon = \frac{1}{2 \pi} \frac{1}{\sqrt{4n - 1} \sqrt{4n - 1}},$$

$$\varepsilon = \frac{1}{2 \pi} \frac{1}{\sqrt{4n - 1}\sqrt{4n - 1}}.$$  

(35)

(36)

The focus position is not shifted, i.e. $z_2 = z$.

Other than in the standard Hermite–Gauss solution, where the $\beta$-function at the focus, i.e., the Rayleigh length, is independent of the mode number, the Rayleigh length of the elegant solution scales inversely to $\sqrt{n}$ for $n > 1$, while the transverse beam size at the focus stays nearly constant. Since the $\beta$-function describes an optical system independent of the specific characteristics of a beam, one may say that the optics is not fixed in case of the elegant solution. As shown above, this is the result of a coherent superposition of basis modes and a specific property which needs to be taken into account when discussing the propagation of generalized solutions.

As example for the decomposition of the elegant solution the case $n = 2$ will be explicitly executed below. Since the elegant solution exhibits no offset it is to be expected that the case $m = n + 1$ does not appear in the sum of orthogonal modes. Moreover, the focus is not shifted, while the observable $\beta$-function at the focus is reduced in comparison to the generating $\beta$-function (cf. Eq. (35)). These conditions are reached at a phase difference of the modes $\Delta \phi_2 = 2 \pi$ (cf. Eqs. (25) and (27)). Since the phase enters with a factor $1/2$ in the exponential (cf. Eq. (4)) this corresponds to a change of sign.
7. Decomposition of the elegant solution

To simplify the notation, the normalization terms \( I_n = \frac{1}{\sqrt{2\pi n!}} \) for the Hermite–Gauss mode and \( \tilde{I}_n = \sqrt{\frac{2\pi n!}{\gamma n}} \) for the elegant solution are introduced, and a common factor \( \frac{k}{\beta_0} \sqrt{\frac{1}{2} \beta_0 + i z} \) is dropped. This factor has to be included in the final equations.

The elegant solution (Eq. (33)) now has the form:

\[
\hat{E}_n = \tilde{I}_n \left( \frac{1}{1 + \frac{2k}{\beta_0}} \right)^{1/4} H_n \left( \frac{k}{\beta_0 + \frac{2}{\beta_0} i z} \right) e^{i \left( \frac{1}{2} \beta g \left( \frac{z}{\beta_0} \right) \right)}
\]

(37)

while the Hermite–Gauss mode (Eq. (4)) reads as:

\[
E_n = I_n H_n \left( \frac{k}{\beta_0} \right) \left( \frac{1}{\beta_0 + \frac{2}{\beta_0} i z} \right) e^{-\frac{1}{2} \beta g \left( \frac{z}{\beta_0} \right)},
\]

(38)

where already \( z_\epsilon = z \) is used.

Introducing \( H_2 = 4x^2 - 2 \) into the elegant solution leads to

\[
\hat{E}_2 = \tilde{I}_2 \left( \frac{1}{1 + \frac{2k}{\beta_0}} \right)^{1/2} \left( \frac{2k}{\beta_0 + \frac{2}{\beta_0} i z} \right) e^{-\frac{1}{2} \beta g \left( \frac{z}{\beta_0} \right)}.
\]

(39)

which is transformed with the relation

\[
\frac{1}{1 + \frac{2k}{\beta_0}} = \frac{1}{\sqrt{1 + \frac{2k}{\beta_0}} e^{-i \beta g \left( \frac{z}{\beta_0} \right)}},
\]

into:

\[
\hat{E}_2 = \tilde{I}_2 \left( \frac{1}{1 + \frac{2k}{\beta_0}} \right) \left( \frac{k}{\beta_0} \right) x^2 - 2 \ e^{-\frac{1}{2} \beta g \left( \frac{z}{\beta_0} \right)} + \tilde{I}_2 \ e^{-i \beta g \left( \frac{z}{\beta_0} \right)} \left( \frac{1}{\beta_0 + \frac{2}{\beta_0} i z} \right) e^{\frac{1}{2} \beta g \left( \frac{z}{\beta_0} \right)}.
\]

(40)

where \( e^{-i \beta g \left( \frac{z}{\beta_0} \right)} - e^{-\frac{1}{2} \beta g \left( \frac{z}{\beta_0} \right)} \) = 0 has been added.

The first term can now be replaced by \( \frac{1}{2} \tilde{I}_2 \hat{E}_2 \), while the second term \( e^{-i \beta g \left( \frac{z}{\beta_0} \right)} \) is replaced by \( \frac{1}{\sqrt{1 + \frac{2k}{\beta_0}}} e^{i \beta g \left( \frac{z}{\beta_0} \right)} \)

\[
\hat{E}_2 = \frac{1}{2} \left( \frac{\tilde{I}_2}{\tilde{I}_2} \right) \hat{E}_2 = 2 I_2 \left( 1 + \frac{z}{\beta_0} \right) e^{-\frac{1}{2} \beta g \left( \frac{z}{\beta_0} \right)} \left( \frac{1}{\beta_0 + \frac{2}{\beta_0} i z} \right).
\]

(41)

which leads with \( 1 + \frac{1}{2k} \frac{z}{\beta_0} = \sqrt{1 + \frac{2k}{\beta_0} e^{i \beta g \left( \frac{z}{\beta_0} \right)}} \) to

\[
\hat{E}_2 = \frac{1}{2} \left( \frac{\tilde{I}_2}{\tilde{I}_2} \right) E_2 - \tilde{I}_2 \tilde{E}_0
\]

(42)

The negative sign corresponds, as already mentioned, to a phase difference of \( 2\pi \). In the following the sign will be kept however.

The calculation of other orders works in the same way, but gets increasingly complex with increasing order. The results for the first five solutions are summarized as:

\[
\begin{align*}
\hat{E}_0 &= E_0 \\
\hat{E}_1 &= 2^{-1} \frac{\tilde{I}_1}{\tilde{I}_2} E_1 = E_1 \\
\hat{E}_2 &= 2^{-2} \left( \frac{\tilde{I}_1}{\tilde{I}_2} E_2 - 2 \frac{\tilde{I}_1}{\tilde{I}_2} E_0 \right) \\
\hat{E}_3 &= 2^{-2} \left( \frac{\tilde{I}_1}{\tilde{I}_2} E_3 - 6 \frac{\tilde{I}_1}{\tilde{I}_2} E_0 \right) \\
\hat{E}_4 &= 2^{-2} \left( \frac{\tilde{I}_1}{\tilde{I}_2} E_4 - 12 \frac{\tilde{I}_1}{\tilde{I}_2} E_2 + 12 \frac{\tilde{I}_1}{\tilde{I}_2} E_0 \right)
\end{align*}
\]

(43)

where the previously dropped factor should be considered as included on both sides.

Comparing Eq. (43) with the generating function of the Hermite polynomials, Table 1, reveals that the numerical coefficients in Eq. (43) follow the coefficients of the Hermite polynomials.

Without further proof it may hence be expected that the general solution can be written as

\[
\hat{E}_n = (-1)^n \sum_{m=2}^{n} \frac{m!}{\Gamma(m!)^{2} \left( \frac{1}{2} \right)^{n}} \frac{\tilde{I}_m}{\tilde{I}_n} E_m
\]

(44)

where the first part is given by the generating function of the Hermite polynomials (cf. Eq. (6)). Based on Eq. (43) the sums \( S_m \) and \( S_n \), as well as \( \beta \)-function and emittance can be calculated (Eqs. (27) and (28)). The results are of course identical to the \( \beta \)-function and emittance given by Eqs. (35) and (36).

The decomposition reveals a strong contribution of lower order modes to a field described by an higher order elegant Hermite–Gauss solution, which explains the relatively good beam quality of these fields even for high \( n \).

8. Conclusion

The coherent superposition of modes leads to variations of characteristic beam parameters, which makes it necessary to distinguish the generating parameters from the observable parameters of the coherent sum. In this report the general relations of generating optical beam parameters and observable beam parameters have been established which are of fundamental importance for the interpretation of experimental findings and for a comparison of experiment and theory. It could be shown, that the observable beam size of a coherent sum follows in all cases the standard rms envelope equation. It can hence be described by a beam quality factor and a \( \beta \)-function which develops in the standard way, independent of whether the beam is coherent or incoherent. The rms envelope equation and the Courant–Snyder theory are thus established as general framework for the description of beams, which is suitable to tackle propagation, matching and imaging problems. This statements holds, whether the underlying decomposition of modes is known or not.

The generating parameters follow, of course, also the standard transport relations, but with its own set of initial parameters. Modes, or coherent sums of modes, can thus be efficiently propagated through linear optical systems with simple methods, if the initial parameters are known. While the propagation of the generating functions allows the determination of the transverse field and intensity distribution, the observable parameters determine the imaging and matching conditions. The relations of the initial generating and observable parameters are derived for various cases in this report.

Finally, as an example for generalized solutions of the paraxial Helmholtz equation, the elegant Hermite–Gauss solution is interpreted as a coherent sum of standard Hermite–Gauss modes. Also in this case deviates the observable \( \beta \)-function (Rayleigh length) from the generating \( \beta \)-function. This needs to be taken into account, when discussing the propagation characteristics of these beams. While the \( \beta \)-function in case of the standard Hermite–Gauss modes is independent of the mode number (and equal to the generating \( \beta \)-function for pure modes) it decreases with increasing order of the solution in case of the elegant Hermite–Gauss solution. The decomposition reveals a strong contribution of lower order components in the higher order elegant solution, which explains the comparatively weak scaling of the beam quality with the mode number. The decomposition allows to interpret results obtained for the elegant solutions in terms of the standard solutions which is more easily accessible and yields a better insight into the physical properties.
Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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