Electromagnetic field quantization in a magnetodielectric with external charges

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Abstract

The electromagnetic field inside a cubic cavity filled up with a linear magnetodielectric medium and in the presence of external charges is quantized by modelling the magnetodielectric medium with two independent quantum fields. Electric and magnetic polarization densities of the medium are defined in terms of the ladder operators of the medium and eigenmodes of the cavity. Maxwell and constitutive equations of the medium together with the equation of motion of the charged particles have been obtained from the Heisenberg equations using a minimal coupling scheme. Spontaneous emission of a two level atom embedded in a magnetodielectric medium is calculated in terms of electric and magnetic susceptibilities of the medium and the Green function of the cubic cavity as an application of the model.

Keywords: Field quantization, Magnetodielectric, Spontaneous emission, external charges

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1 Introduction

The usual approach to quantum optics in non dispersive dielectric media is to introduce the medium by its linear or nonlinear susceptibilities. In one proce-
dure, the macroscopic fields are used to build an effective Lagrangian density whose Euler-Lagrange equations are identical to the macroscopic Maxwell equations in the dielectric, where the constitutive equations couple the polarization field to the electric field [1]-[7]. However, attempts to add dispersion to this effective scheme have run into difficulties [7, 8]. The reason for this is that inclusion of dispersion leads to a temporally nonlocal relationship between the electric field and the displacement field and the effective Lagrangian in this case is also nonlocal in time and cannot be used directly in a quantization scheme. Another problem is the inclusion of losses into the system. It is well known that the dissipative nature of a medium is an immediate consequence of its dispersive character and vice versa according to the Kramers-Kronig relations[9]. This suggests that in order to quantize the electromagnetic field in a dielectric in a way that is consistent with the Kramers-Kronig relations, one has to introduce the medium into the formalism explicitly. This should be done in such a way that the interaction between light and matter will generate both dispersion and damping of the light field. In an attempt to overcome these problems, by taking the polarization of the medium as a dissipative quantum system and based on the Hopfield model of a dielectric [10][11], a canonical quantization of the electromagnetic field inside a dispersive and absorptive dielectric can be presented, where the polarization of the dielectric is modelled by a collection of interacting matter fields[12][13]. The absorptive character of the medium is modelled through the interaction of the matter fields with a reservoir consisting of a continuum of the Klein-Gordon fields. In this model, eigen-operators for coupled systems are calculated and the electromagnetic field has been expressed in terms of them. The dielectric function is derived in terms of a coupling function which couples the polarization of the medium to the reservoir.

One approach to study a quantum dissipative system tries to relate dissipation to the interaction between the system and a heat bath containing a collection of harmonic oscillators [14][26]. In this method, the whole system is composed of two parts, the main system and a bath which interacts with the main system and causes the dissipation of energy on it. In order to quantize the electromagnetic field in the presence of an absorptive magnetodielectric, it is reasonable to taking the electromagnetic field as the main quantum dissipative system and the medium as the heat bath. In this point of view the polarizability of the medium is a result of the properties of the heat bath and accordingly, the polarizability should be defined in terms of the dynamical variables of the medium. In this method, contrary to the
damped polarization model, polarizability and absorptivity of the medium are not independent of each other \cite{12, 13}. Furthermore, if the medium is magnetizable, as well as polarizable, the medium can be modelled with two independent collection of harmonic oscillators, so that one of these collections describes the electric properties and the other describes the magnetic properties of the medium. This scheme propose a consistent quantization of the electromagnetic field in the presence of an absorptive magnetodielectric \cite{27}.

In the present paper, we generalize the quantization scheme presented in \cite{27} to the case where there are some external charges in the medium. We take a cubic cavity filled up with a linear magnetodielectric, containing some external charges, as the medium. As a simple application of the method, we study the spontaneous decay of an initially excited two-level atom in a dispersing and absorbing magnetodielectric.

2 Quantum dynamics

Quantum electrodynamics in a linear magnetodielectric medium, can be accomplished by modelling the medium with two independent quantum fields that interact with the electromagnetic field. One of these quantum fields, namely E-quantum field, describes the polarizability character of the medium and interacts with displacement field through a minimal coupling term. The other quantum field, M-quantum field, describes magnetizability character of the medium and interacts with magnetic field through a dipole interaction term. The Heisenberg equations for electromagnetic field and the E- and M-quantum fields, give not only the Maxwell equations but also the constitutive equations of the medium and the equations of motion of external charges.

For this purpose let us consider an ideal rectangular cavity with sides $L_1$, $L_2$ and $L_3$, along the coordinate axis $x$, $y$ and $z$, respectively, filled up with a magnetodielectric. The vector potential of the electromagnetic field in coulomb gauge can be expanded as \cite{28}

$$\vec{A}(\vec{r}, t) = \sum_{\vec{n}} \sum_{\lambda=1}^{2} \sqrt{\frac{4\hbar}{\varepsilon_0 V \omega_{\vec{n}}}} [a_{\vec{n}\lambda}(t) + a_{\vec{n}\lambda}^\dagger(t)] \vec{u}_{\vec{n}\lambda}(\vec{r}),$$

(1)
where
\[ \vec{u}_{n\lambda}(\vec{r}) = e_x(\vec{n}, \lambda)f_1(\vec{n}, \vec{r})\hat{x} + e_y(\vec{n}, \lambda)f_2(\vec{n}, \vec{r})\hat{y} + e_z(\vec{n}, \lambda)f_3(\vec{n}, \vec{r})\hat{z}, \]
\[ f_1(\vec{n}, \vec{r}) = \cos \frac{n_1\pi x}{L_1} \sin \frac{n_2\pi y}{L_2} \sin \frac{n_3\pi z}{L_3}, \]
\[ f_2(\vec{n}, \vec{r}) = \sin \frac{n_1\pi x}{L_1} \cos \frac{n_2\pi y}{L_2} \sin \frac{n_3\pi z}{L_3}, \]
\[ f_3(\vec{n}, \vec{r}) = \sin \frac{n_1\pi x}{L_1} \sin \frac{n_2\pi y}{L_2} \cos \frac{n_3\pi z}{L_3}, \]
(2)

the mode vector \( \vec{n} \) is a triplet of natural numbers \((n_1, n_2, n_3)\), \( \sum n \) means \( \sum_{n_1, n_2, n_3=1}^{\infty} \), \( V = L_1L_2L_3 \) is the volume of the cavity, \( \varepsilon_0 \) is the permittivity of the vacuum, \( \omega_{n} = c\sqrt{\frac{n_1^2\pi^2}{L_1^2} + \frac{n_2^2\pi^2}{L_2^2} + \frac{n_3^2\pi^2}{L_3^2}} \) is the frequency corresponding to the mode \( \vec{n} \) and \( \vec{e}(\vec{n}, \lambda) \), \( (\lambda = 1, 2) \) are polarization vectors which satisfy
\[ \vec{e}(\vec{n}, \lambda) \cdot \vec{e}(\vec{n}, \lambda') = \delta_{\lambda\lambda'}, \]
\[ \vec{e}(\vec{n}, \lambda) \cdot \vec{k}_{\vec{n}} = 0, \]
(3)

where \( \vec{k}_{\vec{n}} = \frac{n_1\pi x}{L_1} + \frac{n_2\pi y}{L_2} + \frac{n_3\pi z}{L_3} \) is the wave vector. Operators \( a_{\vec{n}\lambda}(t) \) and \( a_{\vec{n}\lambda}^\dagger(t) \) are annihilation and creation operators of the electromagnetic field and satisfy the following equal time commutation rules
\[ [a_{\vec{n}\lambda}(t), a_{\vec{m}\lambda'}^\dagger(t)] = \delta_{\vec{n}, \vec{m}}\delta_{\lambda\lambda'}. \]
(4)

Let \( \vec{F}(\vec{r}, t) \) be an arbitrary vector field, the transverse component \( \vec{F}^\perp(\vec{r}, t) \) and the longitudinal component \( \vec{F}^\parallel(\vec{r}, t) \) of \( \vec{F}(\vec{r}, t) \) are defined as
\[ \vec{F}^\perp(\vec{r}, t) = \vec{F}(\vec{r}, t) + \int_V d^3r' \vec{\nabla} G(\vec{r}, \vec{r}') \vec{\nabla} G(\vec{r}, \vec{r}'), \]
\[ \vec{F}^\parallel(\vec{r}, t) = -\int_V d^3r' \vec{\nabla} G(\vec{r}, \vec{r}') \vec{\nabla} G(\vec{r}, \vec{r}'), \]
(5)
(6)

where \( G(\vec{r}, \vec{r}') \) is the Green function of the cubic cavity
\[ G(\vec{r}, \vec{r}') = \frac{8}{V} \sum \frac{1}{|\vec{k}_{\vec{n}}|^2} \sin \frac{n_1\pi x}{L_1} \sin \frac{n_2\pi y}{L_2} \sin \frac{n_3\pi z}{L_3} \sin \frac{n_1\pi x'}{L_1} \sin \frac{n_2\pi y'}{L_2} \sin \frac{n_3\pi z'}{L_3}, \]
(7)

In the presence of external charges the displacement field is not purely transverse. The transverse component of the displacement field can be expanded
in terms of the cavity modes as

\[ \vec{D}^\perp(\vec{r}, t) = -i\varepsilon_0 \sum \sum \frac{2}{n} \sqrt{4\omega_\vec{n}_\lambda h} [a_{\vec{n}\lambda}^\dagger(t) - a_{\vec{n}\lambda}(t)]\vec{u}_{\vec{n}\lambda}(\vec{r}). \]  

(8)

The commutation relations between the components of the vector potential \( \vec{A} \) and the transverse components of the displacement field, \( \vec{D}^\perp \), can be obtained from (4) as follows

\[ [A_l(\vec{r}, t), -D^\perp_j(\vec{r}', t)] = i\hbar \delta^{\perp}_{lj}(\vec{r} - \vec{r}'), \]  

(9)

where \( \delta^{\perp}_{lj}(\vec{r} - \vec{r}') \) is the transverse delta function defined in terms of the eigenvector fields \( \vec{u}_{\vec{n}\lambda} \)

\[ \delta^{\perp}_{lj}(\vec{r} - \vec{r}') = \frac{8}{V} \sum \sum \frac{2}{n} u^l_{\vec{n}\lambda}(\vec{r})u^j_{\vec{n}\lambda}(\vec{r}') = \frac{8}{V} \sum \frac{\delta_{lj} - k_{\vec{n}l}k_{\vec{n}j}}{|k_{\vec{n}}|^2} f_l(\vec{n}, \vec{r})f_j(\vec{n}, \vec{r}'). \]  

(10)

The Hamiltonian of the electromagnetic field inside a magnetodielectric in the normal ordering form is

\[ H_F(t) = \int_V d^3r : \left[ \frac{\vec{D}^\perp \cdot \vec{D}^\perp}{2\varepsilon_0} + \frac{(\nabla \times \vec{A})^2}{2\mu_0} \right] : = \sum \sum \frac{2}{n} \sum \frac{\hbar \omega_{\vec{n}\lambda} a^\dagger_{\vec{n}\lambda}(t)a_{\vec{n}\lambda}(t)}, \]  

(11)

where \( \mu_0 \) is the magnetic permeability of the vacuum and \( : : \) denotes the normal ordering operator. By modelling the magnetodielectric medium with E- and M-quantum fields we can write the Hamiltonian of the medium as the sum of the Hamiltonians of the E- and M-quantum fields

\[ H_d = H_e + H_m \]

\[ H_e(t) = \sum \sum \int_{-\infty}^{+\infty} d^3 k \hbar \omega_\vec{k} b^\dagger_{\vec{n}\nu}(\vec{k}, t)b_{\vec{n}\nu}(\vec{k}, t) \]

\[ H_m(t) = \sum \sum \int_{-\infty}^{+\infty} d^3 k \hbar \omega_\vec{k} b^\dagger_{\vec{n}\nu}(\vec{k}, t)b_{\vec{n}\nu}(\vec{k}, t), \]  

(12)

where \( \omega_\vec{k} \) is dispersion relation of the medium. Quantum Dynamics of a dissipative harmonic oscillator interacting with an absorptive environment can
be investigated by modeling the environment with a continuum of harmonic oscillators [14]-[18]. In the case of quantization of electromagnetic field in the presence of a magnetodielectric, electromagnetic field is the main dissipative system and the medium play the role of the absorptive environment. The Hamiltonian (11) shows that electromagnetic field contains a numerable set of harmonic oscillators labeled by $\vec{n}, \lambda$. Therefore to each harmonic oscillator of the electromagnetic field labeled by $\vec{n}, \lambda$, a continuum of oscillators should be corresponded. In the present scheme, to each harmonic oscillator of the electromagnetic field labeled by $\vec{n}, \lambda$, we have corresponded two continuous set of harmonic oscillators defined by the ladder operators $d_{\vec{n}\nu}(\vec{k}, t)$, $d_{\vec{n}\nu}^\dagger(\vec{k}, t)$ and $b_{\vec{n}\nu}(\vec{k}, t)$, $b_{\vec{n}\nu}^\dagger(\vec{k}, t)$ which are to describe the electric and magnetic properties of the medium respectively and satisfy the equal time commutation relations

$$
[d_{\vec{n}\nu}(\vec{k}, t), d_{\vec{m}\nu'}^\dagger(\vec{k}', t)] = \delta_{\vec{n}, \vec{m}} \delta_{\nu\nu'} \delta(\vec{k} - \vec{k}'),
$$

$$
[b_{\vec{n}\nu}(\vec{k}, t), b_{\vec{m}\nu'}^\dagger(\vec{k}', t)] = \delta_{\vec{n}, \vec{m}} \delta_{\nu\nu'} \delta(\vec{k} - \vec{k}').
$$

(13)

If we want to extend the Huttner and Barnett model [12] to a magnetodielectric medium, we have to take a vector field $\vec{X}$ as electric polarization density of the medium and a heat bath $B$ interacting with $\vec{X}$ in order to take into account the absorption due to the dispersion induced from the electrical properties of the medium. Similarly, we must take a vector field $\vec{Y}$ as magnetic polarization density and a heat bath $\tilde{B}$ (independent of $B$) interacting with $\vec{Y}$ in order to take into account the absorption due to the dispersion induced from the magnetic properties of the medium. Following the Huttner and Barnett method, we find that a Fano diagonalization [11] process of the sum of the Hamiltonians related to the electric polarization $\vec{X}$, the heat bath $B$ and their interaction, lead to what we have named $H_e$ in Eq. (12). Also a Fano diagonalization process of the sum of the Hamiltonians related to the magnetic polarization $\vec{Y}$, the heat bath $\tilde{B}$ and their interaction, lead to what we have named $H_m$ in Eq. (12). If we use the Huttner and Barnett approach inside a rectangular cavity, the labels $\vec{n}$ and $\nu$ will appear in the expansion of the electric polarization field $\vec{X}$ in terms of the eigenmodes of the cavity. On the other hand, the heat bath $B$ interacting with the polarization field $\vec{X}$ consists of a continuous set of harmonic oscillators labelled by a continuous parameter $\omega$. It should be noted that the three labels $\vec{n}$, $\nu$ and $\omega$ remain in $H_e$ after the diagonalization process.

For a linear medium, the electric polarization density operator can be
written as a linear combination of ladder operators $d_{\vec{n}\nu}(\vec{k}, t)$ and $d^{\dagger}_{\vec{n}\nu}(\vec{k}, t)$

$$
\vec{P}(\vec{r}, t) = \sqrt{\frac{8}{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^{3}\vec{k} \left[ f(\omega_{\vec{k}}, \vec{r}) d_{\vec{n}\nu}(\vec{k}, t) + h.c. \right] \vec{v}_{\vec{n}\nu}(\vec{r})
$$

(14)

with

$$
\vec{v}_{\vec{n}\nu}(\vec{r}) = \vec{u}_{\vec{n}\nu}(\vec{r}), \quad \nu = 1, 2
$$

$$
\vec{v}_{\vec{n}3}(\vec{r}) = \vec{k}_{\vec{n}x} f_{1}(\vec{n}, \vec{r}) \hat{x} + \hat{k}_{\vec{n}y} f_{2}(\vec{n}, \vec{r}) \hat{y} + \hat{k}_{\vec{n}z} f_{3}(\vec{n}, \vec{r}) \hat{z},
$$

$$
\vec{k}_{\vec{n}} = \frac{\vec{k}}{|\vec{k}|}
$$

(15)

the function $f(\omega_{\vec{k}}, \vec{r})$ is the coupling function of electromanetic field and the E-quantum field. Also for a linearly magnetizable medium we can express the magnetic polarization density operator of the medium as a linear combination of the ladder operators of the M-quantum field \[27\],

$$
\vec{M}(\vec{r}, t) = \sqrt{\frac{8}{V}} \sum_{\vec{n}} \sum_{\nu=1}^{3} \int d^{3}\vec{k} \left[ g(\omega_{\vec{k}}, \vec{r}) b_{\vec{n}\nu}(\vec{k}, t) + h.c. \right] \vec{s}_{\vec{n}\nu}(\vec{r})
$$

(16)

where the function $g(\omega_{\vec{k}}, \vec{r})$ couples the electromagnetic field to the M-quantum field and

$$
\vec{s}_{\vec{n}\nu}(\vec{r}) = \frac{\nabla \times \vec{u}_{\vec{n}\nu}}{|\vec{k}_{\vec{n}}|}, \quad \nu = 1, 2,
$$

$$
\vec{s}_{\vec{n}3}(\vec{r}) = \hat{k}_{\vec{n}x} g_{1}(\vec{n}, \vec{r}) \hat{x} + \hat{k}_{\vec{n}y} g_{2}(\vec{n}, \vec{r}) \hat{y} + \hat{k}_{\vec{n}z} g_{3}(\vec{n}, \vec{r}) \hat{z},
$$

$$
g_{1}(\vec{n}, \vec{r}) = \sin \frac{n_{1}\pi x}{L_{1}} \cos \frac{n_{2}\pi y}{L_{2}} \cos \frac{n_{3}\pi z}{L_{3}},
$$

$$
g_{2}(\vec{n}, \vec{r}) = \cos \frac{n_{1}\pi x}{L_{1}} \sin \frac{n_{2}\pi y}{L_{2}} \cos \frac{n_{3}\pi z}{L_{3}},
$$

$$
g_{3}(\vec{n}, \vec{r}) = \cos \frac{n_{1}\pi x}{L_{1}} \cos \frac{n_{2}\pi y}{L_{2}} \sin \frac{n_{3}\pi z}{L_{3}}.
$$

(17)

Now let the total Hamiltonian, i.e., electromagnetic field, the E- and M-
quantum field and the external charges be like this

\[
\tilde{H}(t) = \int_V d^3r \left\{ \frac{[\tilde{D}^\perp(\tilde{r},t) - \tilde{P}^\perp(\tilde{r},t)]^2}{2\varepsilon_0} + \frac{(\nabla \times \tilde{A})(\tilde{r},t)^2}{2\mu_0} - \nabla \times \tilde{A}(\tilde{r},t) \cdot \tilde{M}(\tilde{r},t) \right\} +
\]

\[
H_e + H_m + \sum_{\alpha=1}^{N} \frac{[\tilde{P}_\alpha - q_\alpha \tilde{A}(\tilde{r}_\alpha,t)]^2}{2m_\alpha} + \frac{1}{2\varepsilon_0} \sum_{\alpha \neq \beta} q_\alpha q_\beta G(\tilde{r}_\alpha, \tilde{r}_\beta)
\]

\[-\frac{1}{\varepsilon_0} \sum_{\alpha=1}^{N} q_\alpha \int_V d^3r' G(\tilde{r}_\alpha, \tilde{r}') \nabla' \cdot \tilde{P}(\tilde{r}', t) \tag{18}\]

where \(q_\alpha, m_\alpha, \tilde{P}_\alpha\) and \(\tilde{r}_\alpha\) are charge, mass, linear momentum and position of the \(\alpha\)th particle respectively. The function \(G\) is the Green function given by (7). Using (9), the Heisenberg equation for \(\tilde{A}\) and \(\tilde{D}^\perp\) can be obtained as

\[
\frac{\partial \tilde{A}(\tilde{r},t)}{\partial t} = \frac{i}{\hbar} [\tilde{H}, \tilde{A}(\tilde{r},t)] = -\frac{\tilde{D}^\perp(\tilde{r},t) - \tilde{P}^\perp(\tilde{r},t)}{\varepsilon_0}, \tag{19}\]

\[
\frac{\partial \tilde{D}^\perp(\tilde{r},t)}{\partial t} = \frac{i}{\hbar} [\tilde{H}, \tilde{D}^\perp(\tilde{r},t)] = \frac{\nabla \times \nabla \times \tilde{A}(\tilde{r},t)}{\mu_0} - \nabla \times \tilde{M}^\perp(\tilde{r},t) - \tilde{J}^\perp(\tilde{r},t), \tag{20}\]

where

\[
\tilde{J}^\perp(\tilde{r},t) = \sum_{\alpha=1}^{N} \sum_{l=1}^{3} q_\alpha \frac{d\tilde{r}_\alpha}{dt} \delta_{il} (\tilde{r}_\alpha(t) - \tilde{r}) \tag{21}\]

is the transverse component of external current density. If we define transverse electrical field \(\tilde{E}^\perp\), induction \(\tilde{B}\) and magnetic field \(\tilde{H}\) as

\[
\tilde{E}^\perp = -\frac{\partial \tilde{A}}{\partial t}, \quad \tilde{B} = \nabla \times \tilde{A}, \quad \tilde{H} = \frac{\tilde{B}}{\mu_0} - \tilde{M}, \tag{22}\]

then (19) and (20) can be rewritten as

\[
\tilde{D}^\perp = \varepsilon_0 \tilde{E}^\perp + \tilde{P}^\perp, \quad \tilde{D}^\perp = \frac{\partial \tilde{D}^\perp}{\partial t} + \tilde{J}^\perp = \nabla \times \tilde{H}, \tag{23}\]

as expected. In the presence of external charges, the longitudinal components of the electrical and displacement field can be written as

\[
\tilde{E}^\parallel(\tilde{r},t) = -\frac{1}{\varepsilon_0} \sum_{\alpha=1}^{N} q_\alpha \tilde{A}_r G(\tilde{r}, \tilde{r}_\alpha(t)) - \frac{\tilde{P}_\parallel}{\varepsilon_0}, \tag{25}\]

\[
\tilde{D}^\parallel(\tilde{r},t) = \varepsilon_0 \tilde{E}^\parallel + \tilde{P}^\parallel.
\]
\[ \vec{D}(\vec{r}, t) = \varepsilon_0 \vec{E} + \vec{P} = - \sum_{\alpha=1}^{N} q_{\alpha} \vec{\nabla}_r G(\vec{r}, \vec{r}_\alpha(t)), \]  

where \( \vec{P} \) is the longitudinal component of electric polarization density. If we apply the Heisenberg equation to the ladder operators of the medium, we obtain the macroscopic constitutive equations of the medium which relate the electric and magnetic polarization densities to electric and magnetic field, respectively. Using commutation relations \( (13) \), we find from Heisenberg equations

\[ \dot{d}_{\vec{n}\nu}(\vec{k}, t) = \frac{i}{\hbar} [\hat{H}, d_{\vec{n}\nu}(\vec{k}, t)] = -\omega_{k} d_{\vec{n}\nu}(\vec{k}, t) + \frac{i}{\hbar} \sqrt{\frac{8}{V}} \int_{V} d^3r' f^*(\omega_{k}, \vec{r}') \vec{E}(\vec{r}', t) \cdot \vec{v}_{\vec{n}\nu}(\vec{r}'), \]  

\[ \dot{b}_{\vec{n}\nu}(\vec{k}, t) = \frac{i}{\hbar} [\hat{H}, b_{\vec{n}\nu}(\vec{k}, t)] = -\omega_{k} b_{\vec{n}\nu}(\vec{k}, t) + \frac{i}{\hbar} \sqrt{\frac{8}{V}} \int_{V} d^3r' g^*(\omega_{k}, \vec{r}') \vec{B}(\vec{r}', t) \cdot \vec{s}_{\vec{n}\nu}(\vec{r}'), \]  

with the following formal solutions

\[ d_{\vec{n}\nu}(\vec{k}, t) = d_{\vec{n}\nu}(\vec{k}, 0) e^{-i\omega_{k}t} + \frac{i}{\hbar} \sqrt{\frac{8}{V}} \int_{0}^{t} dt' e^{-i\omega_{k}(t-t')} \int_{V} d^3r' f^*(\omega_{k}, \vec{r}') \vec{E}(\vec{r}', t') \cdot \vec{v}_{\vec{n}\nu}(\vec{r}'), \]  

\[ b_{\vec{n}\nu}(\vec{k}, t) = b_{\vec{n}\nu}(\vec{k}, 0) e^{-i\omega_{k}t} + \frac{i}{\hbar} \sqrt{\frac{8}{V}} \int_{0}^{t} dt' e^{-i\omega_{k}(t-t')} \int_{V} d^3r' g^*(\omega_{k}, \vec{r}') \vec{B}(\vec{r}', t') \cdot \vec{s}_{\vec{n}\nu}(\vec{r}'). \]  

By substituting \( (29) \) in \( (14) \) and using the completeness relations

\[ \sum_{\vec{m}} \sum_{\nu=1}^{3} v_{\vec{m}\nu}^{\alpha}(\vec{r}) v_{\vec{m}\nu}^{\beta}(\vec{r}') = \sum_{\vec{m}} \sum_{\nu=1}^{3} s_{\vec{m}\nu}^{\alpha}(\vec{r}) s_{\vec{m}\nu}^{\beta}(\vec{r}') = \frac{V}{8} \delta_{\alpha\beta} \delta(\vec{r} - \vec{r}'), \]  

we can find the constitutive equation

\[ \vec{P}(\vec{r}, t) = \vec{P}_N(\vec{r}, t) + \varepsilon_0 \int_{0}^{t} dt' \chi_e(\vec{r}, |t| - t') \vec{E}(\vec{r}, \pm t'), \]  

where \( \vec{P}_N(\vec{r}, t) \) is the net electric polarization density.
for the medium which relates the electric polarization density to the total electric field, where the upper (lower) sign, corresponds to \( t > 0 \) \(( t < 0)\) respectively, and \( \vec{E} = \vec{E}^\perp + \vec{E}^\parallel \) is the total electrical field with \( \vec{E}^\perp \) and \( \vec{E}^\parallel \) defined in (22) and (25). The memory function

\[
\chi_e(\vec{r}, t) = \begin{cases} 
\frac{8\pi}{\hbar\varepsilon_0} \int_0^\infty d|\vec{k}| |\vec{k}|^2 |f(\omega, \vec{r})|^2 \sin \omega_{\vec{k}} t & t > 0 \\
0 & t \leq 0 \end{cases} 
\]  

(33)

is the electric susceptibility and in frequency domain satisfies the following relations

\[
Im[\chi_e(\vec{r}, \omega)] = \frac{4\pi^2}{\hbar\varepsilon_0} |f(\omega, \vec{r})|^2 \sum_i \frac{|\vec{k}_i|^2}{|\frac{d\omega}{dk}(|\vec{k}| = |\vec{k}_i|)|},
\]  

(34)

\[
Re[\chi_e(\vec{r}, \omega)] = \frac{8\pi}{\hbar\varepsilon_0} \int_0^\infty d|\vec{k}| |\vec{k}|^2 |f(\omega, \vec{r})|^2 \frac{\omega_{\vec{k}}}{\omega_{\vec{k}}^2 - \omega^2}
\]  

(35)

where

\[
\chi_e(\vec{r}, \omega) = \int_0^\infty dt \chi_e(\vec{r}, t)e^{i\omega t}
\]  

(36)

is the Fourier transform of the electric susceptibility and \(|\vec{k}_i|\)s are the roots of the algebraic equation \( \omega = \omega_{\vec{k}} \equiv \omega(|\vec{k}|) \). A feature of the present quantization method is it’s flexibility in choosing a dispersion relation and a coupling function such that they satisfy the relation (34). In other words, the dispersion relation and the coupling function \( f(\omega_{\vec{k}}, \vec{r}) \) are two free parameters of this theory up to the constraint relation (34). Various choices of \( \omega_{\vec{k}} \) and \( f(\omega_{\vec{k}}, \vec{r}) \) satisfying (34) do not change the commutation relations between electromagnetic field operators and lead to equivalent expressions for the field operators and physical observables. As can be seen from the relation (34), in order to have a finite value for susceptibility for any frequency \( \omega \), we should assume that the denominator of the fraction in (34) is non zero. so we assume that the dispersion relation is a monotonic function of \(|\vec{k}|\). For simplicity and simple calculations we take a linear dispersion relation \( \omega = c|\vec{k}| \) which leads to the following relation for the electric susceptibility:

\[
\chi_e(\vec{r}, t) = \begin{cases} 
\frac{8\pi}{\hbar c^2\varepsilon_0} \int_0^\infty d\omega \omega^2 |f(\omega, \vec{r})|^2 \sin \omega t & t > 0 \\
0 & t \leq 0 \end{cases} 
\]  

(37)
The operator \( \vec{P}_N(\vec{r}, t) \) in (32), is the noise electric polarization density

\[
\vec{P}_N(\vec{r}, t) = \sqrt{\frac{8}{V}} \sum_n \sum_{\nu=1}^3 \int d^3k [f(\omega_k, \vec{r})d_{\vec{n}\nu}(\vec{k}, 0)e^{-\omega_k t} + h.c.] \vec{e}_{\vec{n}\nu}(\vec{r}).
\]  
(38)

Similarly by substituting (30) in (16), we obtain the following expression for \( \vec{M}(\vec{r}, t) \)

\[
\vec{M}(\vec{r}, t) = \vec{M}_N(\vec{r}, t) + \frac{1}{\mu_0} \int_0^{\vert t \vert} dt' \chi_m(\vec{r}, \vert t \vert - t')  \vec{B}(\vec{r}, \pm t'),
\]  
(39)

where \( \chi_m \) is the magnetic susceptibility of the medium

\[
\chi_m(\vec{r}, t) = \frac{8\pi\mu_0}{hc^3} \int_0^\infty d\omega \omega^2 |g(\omega, \vec{r})|^2 \sin \omega t, \quad t > 0,
\]

\[
\chi_m(\vec{r}, t) = 0, \quad t \leq 0.
\]  
(40)

and operator \( \vec{M}_N(\vec{r}, t) \) is the noise magnetic polarization density

\[
\vec{M}_N(\vec{r}, t) = \sqrt{\frac{8}{V}} \sum_n \sum_{\nu=1}^3 \int d^3k [g(\omega_k, \vec{r})b_{\vec{n}\nu}(\vec{k}, 0)e^{-\omega_k t} + H.C] \vec{s}_{\vec{n}\nu}(\vec{r}).
\]  
(41)

If we are given a definite \( \chi_e(t) \) and \( \chi_m(t) \), then we can inverse the relations (37) and (43) and obtain the corresponding coupling functions \( f \) and \( g \) as

\[
|f(\omega, \vec{r})|^2 = \frac{hc^3 \varepsilon_0}{4\pi^2\omega^2} \int_0^\infty dt \chi_e(\vec{r}, t) \sin \omega t, \quad \omega > 0,
\]

\[
|f(\omega, \vec{r})|^2 = 0, \quad \omega = 0.
\]  
(42)

\[
|g(\omega, \vec{r})|^2 = \frac{hc^3}{4\pi^2\mu_0\omega^2} \int_0^\infty dt \chi_m(\vec{r}, t) \sin \omega t, \quad \omega > 0,
\]

\[
|g(\omega, \vec{r})|^2 = 0, \quad \omega = 0.
\]  
(43)

Therefore the constitutive equations (23), (32) and (41) together with the Maxwell equations, can be obtained directly from the Heisenberg equations applied to the electromagnetic field and the quantum fields \( E \) and \( M \). It is clear from (38) and (41) that the explicit forms of the noise polarization densities are known. Also because the coupling functions \( f, g \) are common factors in the noise densities \( \vec{P}_N, \vec{M}_N \), and susceptibilities \( \chi_e, \chi_m \), it is clear
that the strength of the noise fields are dependent on the strength of $\chi_e, \chi_m$, which describe the dissipative character of a magnetodielectric medium. Specialy when the medium tends to a non absorbing one, the noise polarizations tend to zero and this quantization scheme is reduced to the usual quantization in this medium [27].

Finally using the total Hamiltoian (18) and the commutation relations

$$[\vec{r}_\alpha, \vec{p}_\beta] = i\hbar \delta_{\alpha\beta} I$$  \hspace{1cm} (44)

it is easy to show that the Heisenberg equations of motion for charged particles are

$$m_\alpha \dddot{\vec{r}}_\alpha = q_\alpha \vec{E}(\vec{r}_\alpha, t) + \frac{q_\alpha}{2} [\vec{r}_\alpha \times \vec{B}(\vec{r}_\alpha, t) + \vec{B}(\vec{r}_\alpha, t) \times \dot{\vec{r}}_\alpha] \hspace{1cm} \alpha = 1, 2, ..., N$$  \hspace{1cm} (45)

3 Spontaneous emission of an excited two-level atom in the presence of a magnetodielectric medium

In this section as a simple application we use the quantization scheme in the previous section to calculate the spontaneous decay rate of an initially excited two-level atom embedded in a magnetodielectric. For this purpose, let us consider a one electron atom with position $\vec{R}$ which interacts with the electromagnetic field in the presence of a magnetodielectric medium. If we restrict our attention to the electric-dipole approximation [29, 30], then the Hamiltonian (18) can be approximated as

$$H = H_0 + H',$$

$$H_0 = H_F + H_e + H_m + H_{at},$$

$$H' = \int_V d^3r' \left[ -\frac{\vec{D}_{\perp} \cdot \vec{E}}{\varepsilon_0} - \vec{\nabla} \times \vec{A} \cdot \vec{M} \right] - e\vec{r} \cdot \vec{E}_D(\vec{R}, t),$$

(46)
where
\[
H_{at} = \frac{\vec{p}^2}{2m} + \frac{e}{\varepsilon_0} \sum_{q \neq e} q \alpha G(\vec{r}, \vec{r}_\alpha),
\]
and \(e, m, \vec{r}\) and \(\vec{p}\) are charge, mass, position and momentum of the atom electron. In Hamiltonian (46) we have ignored from \(\frac{e^2}{2m} \vec{A}(\vec{R}, t)\) and \(\frac{1}{2\varepsilon_0} \int_V d^3r \vec{F}^2(\vec{r}, t)\), because these terms do not affect the decay rate or level shifts in the dipole approximation and therefore will not be of concern here. For reality we assume that the atom is localized in a free space region \(V_0\) without polarization density. Then for a two-level atom with upper state \(|2\rangle\), lower state \(|1\rangle\) and transition frequency \(\omega_0\), the Hamiltonian (46) can be written as 

\[
\tilde{H} = \hbar \omega_0 \sigma^\dagger \sigma + H_F + H_e + H_m + H_{at}
+ \int_\Omega d^3r' \left[ -\vec{D}^\bot \cdot \vec{F} \varepsilon_0 - \vec{\nabla} \times \vec{A} \cdot \vec{M} \right] + \frac{e}{\varepsilon_0} (\vec{r}_{12} \sigma + \vec{r}_{12}^\dagger \sigma^\dagger) \cdot \vec{D}^\bot(\vec{R}, t)
\]

(48)

where \(\Omega = V - V_0\), is the region containing the magnetodielectric, \(\sigma = |1\rangle\langle 2|\) and \(\sigma^\dagger = |2\rangle\langle 1|\) are Pauli operators of the two-level atom and \(\vec{r}_{12} = \langle 1|\vec{r}|2\rangle\) is its transition dipole momentum. It is remarkable that the expansions (14) and (16) for electric and magnetic polarization densities are independent of the shape of the magnetodielectric and depend only on the shape of the cavity contained the magnetodielectric. The shape of the medium affects the coupling functions \(f(\omega, \vec{r})\) and \(g(\omega, \vec{r})\) and the time dependence of the ladder operators of the medium, such that for a given medium with definite susceptibilities \(\chi_e\) and \(\chi_m\), the dynamics of the total system leads to the correct constitutive equations (32) and (39) where the coupling functions are given by (42) and (43). This is just as the expansion (1) for the vector potential of the vacuum field where the presence or absence of the external charges do not change the form of the expansion and the presence of the external charges merely affect the time dependence of the ladder operators \(a_{\vec{p}, \lambda}, \ a_{\vec{p}, \lambda}^\dagger\).

To study the spontaneous decay of an initially excited atom we may look for the wave function of the total system in the framework of the Weisskopf-
Wigner theory\[29, 31\],

\[ |ψ(t)⟩ = c(t)|2⟩|0⟩_F|0⟩_e|0⟩_m + \sum_{\tilde{n}} \sum_{\lambda=1}^2 F_{\tilde{n}\lambda}(t)|\tilde{n}, \lambda⟩_F|0⟩_e|0⟩_m \]

\[ + \sum_{\tilde{m}} \sum_{\nu=1}^3 \int d^3\tilde{k} D_{\tilde{m}\nu}(\tilde{k}, t)|1⟩|0⟩_F|\tilde{m}, \tilde{k}, \nu⟩_e|0⟩_m \]

\[ + \sum_{\tilde{m}} \sum_{\nu=1}^3 \int d^3\tilde{k} M_{\tilde{m}\nu}(\tilde{k}, t)|1⟩|0⟩_F|\tilde{m}, \tilde{k}, \nu⟩_m, \quad (49) \]

where $|0⟩_F, |0⟩_e$ and $|0⟩_m$ are vacuum state of electromagnetic field and E- and M-quantum fields, respectively. The coefficients $c(t), F_{\tilde{n}\lambda}(t), D_{\tilde{m}\nu}(\tilde{k}, t)$ and $M_{\tilde{m}\nu}(\tilde{k}, t)$ are to be specified by Schrödinger equation

\[ i\hbar \frac{\partial |ψ(t)⟩}{\partial t} = H|ψ(t)⟩, \quad (50) \]

for initial conditions $c(0) = 1, F_{\tilde{n}\lambda}(0) = D_{\tilde{m}\nu}(\tilde{k}, 0) = M_{\tilde{m}\nu}(\tilde{k}, 0) = 0$. Substituting $|ψ(t)⟩$ from (49) in (50) and using the expansions (8), (14), (16) and applying the rotating wave approximation(RWA) \[29, 30\], we find the following coupled differential equations for the coefficients of the wave function (49)

\[ i\hbar \dot{M}_{\tilde{m}\nu}(\tilde{k}, t) = \hbar \omega_{\tilde{k}} M_{\tilde{m}\nu}(\tilde{k}, t) - \sum_{\tilde{n}} \sum_{\lambda=1}^2 [L_{\tilde{n}\lambda}^{\tilde{m}\nu}(\omega_{\tilde{k}})]^* F_{\tilde{n}\lambda}(t), \quad (51) \]

\[ i\hbar \dot{D}_{\tilde{m}\nu}(\tilde{k}, t) = \hbar \omega_{\tilde{k}} D_{\tilde{m}\nu}(\tilde{k}, t) + \sum_{\tilde{n}} \sum_{\lambda=1}^2 [Q_{\tilde{n}\lambda}^{\tilde{m}\nu}(\omega_{\tilde{k}})]^* F_{\tilde{n}\lambda}(t) + \quad (52) \]

\[ i\hbar \dot{F}_{\tilde{n}\lambda}(t) = \hbar \omega_{\tilde{n}} F_{\tilde{n}\lambda}(t) - \sum_{\tilde{m}} \sum_{\nu=1}^3 \int d^3\tilde{k} L_{\tilde{n}\lambda}^{\tilde{m}\nu}(\omega_{\tilde{k}}) M_{\tilde{m}\nu}(\tilde{k}, t) + \]

\[ \sum_{\tilde{m}} \sum_{\nu=1}^3 \int d^3\tilde{k} Q_{\tilde{n}\lambda}^{\tilde{m}\nu}(\omega_{\tilde{k}}) D_{\tilde{m}\nu}(\tilde{k}, t) + \left[ ie \frac{4\hbar\omega_{\tilde{n}}}{\varepsilon_0 V} \vec{r}_{12} \cdot \vec{u}_{\tilde{n}\lambda}(\vec{R}) \right] c(t), \quad (53) \]

\[ i\hbar c(t) = \hbar \omega_0 c(t) - \left[ ie \sum_{\tilde{n}} \sum_{\lambda=1}^2 \frac{4\hbar\omega_{\tilde{n}}}{\varepsilon_0 V} \vec{r}_{12} \cdot \vec{u}_{\tilde{n}\lambda}(\vec{R}) \right] F_{\tilde{n}\lambda}(t), \quad (54) \]
where

\[
Q_{n\lambda}^{m\nu}(\omega, \vec{k}) = i \sqrt{\frac{32 \hbar \omega}{\epsilon_0 V^2}} \int_{V-V_0} d^3 r' f(\omega, \vec{r}') \vec{u}_{n\lambda}(\vec{r}') \cdot \vec{v}_{m\nu}(\vec{r}'),
\]

\[
L_{n\lambda}^{m\nu}(\omega, \vec{k}) = \sqrt{\frac{32 \hbar \mu_0 \omega}{V^2}} \int_{V-V_0} d^3 r' g(\omega, \vec{r}') \vec{s}_{n\lambda}(\vec{r}') \cdot \vec{s}_{m\nu}(\vec{r}').
\]

(55)

One can solve these coupled differential equations by Laplace transformation. Let \( \tilde{f}(\rho) \) denotes the Laplace transformation of \( f(t) \), then taking the Laplace transformation of equations (51)-(53) and then their combination and using the completeness relations (31) we find

\[
[i \hbar \rho - \hbar \omega_{ni}] \tilde{F}_{n\lambda}(\rho) = \hbar \omega_{ni} \sum_{\vec{n}' \lambda'} 2 \sum_{\lambda' = 1} W_{n\lambda'}^{n\lambda}(\rho) + \left[ i e \sqrt{\frac{4 \hbar \omega}{\epsilon_0 V}} r_{12} \cdot \vec{u}_{n\lambda}(\vec{R}) \right] \tilde{c}(\rho),
\]

(56)

where we have applied the initial conditions \( F_{n\lambda}(0) = D_{n\mu}(\vec{k}, 0) = M_{n\mu}(\vec{k}, 0) = 0 \) and

\[
W_{n\lambda'}^{n\lambda}(\rho) = \frac{8}{V} \sqrt{\frac{\omega_{n\lambda'}}{\omega_{ni}}} \times
\left\{ \int_{\Omega} d^3 r \left[ Z_e(\rho, \vec{r}) \vec{u}_{n\lambda}(\vec{r}) \cdot \vec{u}_{n\lambda'}(\vec{r}) + Z_m(\rho, \vec{r}) \vec{s}_{n\lambda}(\vec{r}) \cdot \vec{s}_{n\lambda'}(\vec{r}) \right] \right\},
\]

(57)

\[
Z_e(\rho, \vec{r}) = \frac{1}{2\pi} \int_0^\infty d\omega \frac{\chi_{ei}(\omega, \vec{r})}{\rho - \omega}, \quad Z_m(\rho, \vec{r}) = \frac{1}{2\pi} \int_0^\infty d\omega \frac{\chi_{mi}(\omega, \vec{r})}{\rho - \omega}.
\]

(58)

Here \( \chi_{ei}(\omega, \vec{r}) \) and \( \chi_{mi}(\omega, \vec{r}) \) are the imaginary parts of electric and magnetic susceptibilities in frequency domain, respectively. Equation (56) is a complicated algebraic equation for \( \tilde{F}_{n\lambda}(\rho) \) and may be solved by iteration method. In the first step of an iteration method \( \tilde{F}_{n\lambda}(\rho) \) may be approximated by

\[
\tilde{F}_{n\lambda}(\rho) = \tilde{F}_{n\lambda}^{(0)}(\rho) + \sum_{\mu'} \sum_{\lambda' = 1} \frac{\omega_{n} W_{n\lambda'}^{\mu'\lambda}(\rho) \tilde{F}_{n\lambda'}^{(0)}(\rho)}{\rho - \omega_{n}[1 + W_{n\lambda'}^{\mu'\lambda}(\rho)]},
\]

(59)
where
\[ F_{n\lambda}^{(0)}(\rho) = \frac{\nu e^{\frac{4\nu}{\varepsilon_0 V}} \cdot \bar{u} \cdot \bar{u}^{\dagger}}{i\hbar \rho - \nu \omega \cdot \bar{R}} \cdot \check{c}(\rho), \] (60)
is the solution of (56) in the absence of medium, that is when \( W_{n\lambda}^{n'\lambda'} = 0 \).

Now combination of equation (59) and the Laplace transformations of the equation (54) yields
\[ \hbar (\check{c}(\rho) - c(0)) = \hbar \omega_0 \check{c}(\rho) + \left[ \bar{r}_{12} \cdot \left( G^{(0)}(\bar{R}, \bar{R}, \nu \rho) + \check{G}(\bar{R}, \bar{R}, \nu \rho) \right) \right] \cdot \bar{r}_{12} \cdot \check{c}(\rho), \] (61)
where
\[ G^{(0)}(\bar{R}, \bar{R}, \nu \rho) = \frac{4e^2}{\varepsilon_0 V} \sum_{n, n'} \sum_{\lambda, \lambda'} \frac{\nu \omega \cdot \bar{u} \cdot \bar{u}^{\dagger}(\bar{R})}{\nu \omega \cdot \nu \omega' \cdot \bar{u} \cdot \bar{u}^{\dagger}(\bar{R})}, \] (62)
\[ \check{G}(\bar{R}, \bar{R}, \nu \rho) = \frac{4e^2}{\varepsilon_0 V} \sum_{n, n'} \sum_{\lambda, \lambda'} \frac{\nu \omega \cdot \bar{u} \cdot \bar{u}^{\dagger}(\bar{R}) W_{n\lambda}^{n'\lambda'}(\nu \rho) \bar{u} \cdot \bar{u}^{\dagger}(\bar{R})}{\nu \omega \cdot \nu \omega' \cdot \bar{u} \cdot \bar{u}^{\dagger}(\bar{R}) \left( 1 + W_{n\lambda}^{n'\lambda'}(\nu \rho) \right) \left( \nu \omega \cdot \nu \omega' \right)}. \] (63)
The diadic \( G^{(0)} \) gives the contribution of the vacuum field for the spontaneous emission and level shift of the atom, that is, when there is no magnetodielectric, while the diadic \( \check{G} \) gives the spontaneous and frequency shift of the atom due to the presence of the medium. From definitions (57) and (58) it is obvious that \( Z_c(\nu \rho, \bar{r}) \), \( Z_c(\nu \rho, \bar{r}) \) and \( W_{n\lambda}^{n'\lambda'}(\nu \rho) \) are analytic functions in complex semi plane \( \text{Re}(\rho) > 0 \) and since the denominator in diadic \( \check{G} \) has no zero in semi plane \( \text{Re}(\rho) > 0 \) so \( \check{G}(\bar{R}, \bar{R}, \nu \rho) \) is analytic for \( \text{Re}(\rho) > 0 \). Taking the inverse Laplace transformation of (60) we find the following integro-differential equation
\[ \dot{c}(t) = -i\omega_0 c(t) - (\Gamma_0 + i\Delta_0) c(t) + \int_0^t K(t - t') c(t') dt', \] (64)
where
\[ K(t - t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{-i\omega(t - t')} \left[ \bar{r}_{12} \cdot \left( \check{G}(\bar{R}, \bar{R}, \omega + i\tau) \right) \right] \cdot \bar{r}_{12}, \] (65)
and \( \Gamma_0 \) and \( \Delta_0 \) are contribution of the vacuum field for the decay rate and the lamb shift of the two-level atom [31]. Here we restrict our attention to
the weak coupling regime, where the Markov approximation \cite{31,32} applies. That is to say, we may replace \( c(t') \) in the integrand in (61) by
\[
c(t') = c(t)e^{i\omega_0(t-t')},
\]
and approximate the time integral in (61) in very large times by
\[
\int_0^t dt' e^{i(\omega_0-\omega)(t-t')} \approx \left[ iP \frac{1}{\omega_0 - \omega} + \pi\delta(\omega_0 - \omega) \right],
\]
where \( P \) denotes the principal cauchy value. After some simple algebra and using the Kramers-Kronig relations for the diadic \( \tilde{G} \) we deduce
\[
\dot{c}(t) = -i\omega_0 c(t) - (\Gamma_0 + \Gamma + i\Delta_0 + i\Delta)c(t),
\]
where
\[
\Gamma = -\frac{1}{\hbar} \left[ \vec{r}_{12} \cdot Im\tilde{G}(\vec{R}, \vec{R}, \omega_0 + i0^+) \cdot \vec{r}_{12} \right],
\]
\[
\Delta = \frac{1}{\hbar} \left[ \vec{r}_{12} \cdot Re\tilde{G}(\vec{R}, \vec{R}, \omega_0 + i0^+) \cdot \vec{r}_{12} \right],
\]
are decay constant and the level shift due to the presence of the magnetodielectric, respectively.

There are many cases that the medium is homogeneous inside the region \( \Omega \) and \( W_{\vec{n}\lambda}(i\rho) \) in the denominator of \( \tilde{G} \) in (63) is independent of polarization label \( \lambda \) and vector label \( \vec{n} \). For example for the box \( 0 < x < L_1, 0 < y < L_2, 0 < z < L_3 \) filled up uniformly with a homogeneous magnetodielectric, from the definition (57), we have
\[
W_{\vec{n}\lambda}(i\rho) \equiv W(i\rho) = \frac{1}{2} (Z_e(i\rho) + Z_m(i\rho)).
\]
Generally speaking for a piece wisely homogeneous medium with \( N \) homogeneous pieces in volumes \( \Omega_1, \Omega_2, \ldots, \Omega_N \), applying the definitions (\ref{17}) in (57), we note that \( W_{\vec{n}\lambda}(i\rho) \) in denominator of \( \tilde{G} \) can be approximated by
\[
W_{\vec{n}\lambda}(i\rho) \equiv W(i\rho) \simeq \sum_{i=1}^N \frac{\Omega_i}{V} \left( Z_e^{(i)}(i\rho) + Z_m^{(i)}(i\rho) \right),
\]
where \( Z_e^{(i)} \) and \( Z_m^{(i)} \) denote the quantities \( Z_e \) and \( Z_m \) for the \( i \)th piece respectively. Now we can substitute \( W_{\vec{n}\lambda}'(i\rho) \) from (57) in the numerator of \( \tilde{G} \)
given by (63) and do the summations over $\lambda, \lambda'$ by using the completeness relations

$$2 \sum_{\lambda=1}^{\lambda} u^i_{n\lambda}(\vec{R}) u^j_{n\lambda}(\vec{r}) = (\delta_{ij} - \hat{k}_n^{n' \lambda} k_n^{n' \lambda}) f_i(\vec{n}, \vec{R}) f_j(\vec{n}, \vec{r}),$$

$$2 \sum_{\lambda=1}^{\lambda} u^i_{n\lambda}(\vec{R}) s^j_{n\lambda}(\vec{r}) = 3 \sum_{\mu=1}^{\mu} \varepsilon_{j\mu} \hat{k}_n^{n\mu} f_i(\vec{n}, \vec{R}) g_j(\vec{n}, \vec{r}).$$

(72)

Now in the box frame we have

$$\tilde{G}_{\alpha\beta}(\vec{R}, \vec{r}, \vec{r} \rho) = \frac{2 e^2}{\varepsilon_0} \int \sum_{i=1}^{N} \sum_{\gamma=1}^{3} \{ Z_e^{(i)}(\rho) \eta_{\alpha\gamma}^{\varepsilon} \zeta_{\beta\gamma}^{\varepsilon} + Z_m^{(i)}(\rho) \eta_{\alpha\gamma}^{m} \zeta_{\beta\gamma}^{m} \},$$

(73)

where

$$\eta_{\alpha\gamma}^{\varepsilon}(\vec{R}, \vec{r}, \vec{r} \rho) = \frac{4}{V} \sum_{\vec{n}} \frac{\omega_{\vec{n}} \varepsilon_{\alpha\gamma}^{\varepsilon} f_{\alpha}(\vec{n}, \vec{R}) f_{\gamma}(\vec{n}, \vec{r}),$$

$$\zeta_{\beta\gamma}^{\varepsilon}(\vec{R}, \vec{r}, \vec{r} \rho) = \frac{4}{V} \sum_{\vec{n}} \frac{\omega_{\vec{n}} \varepsilon_{\beta\gamma}^{\varepsilon} f_{\beta}(\vec{n}, \vec{R}) f_{\gamma}(\vec{n}, \vec{r}),$$

$$\eta_{\alpha\gamma}^{m}(\vec{R}, \vec{r}, \vec{r} \rho) = \frac{4}{V} \sum_{\vec{n}} \sum_{\mu=1}^{\mu} \frac{\omega_{\vec{n}} \varepsilon_{\gamma\mu}^{m} \hat{k}_n^{n\mu} f_{\alpha}(\vec{n}, \vec{R}) g_{\gamma}(\vec{n}, \vec{r}),$$

$$\eta_{\beta\gamma}^{m}(\vec{R}, \vec{r}, \vec{r} \rho) = \frac{4}{V} \sum_{\vec{n}} \sum_{\mu=1}^{\mu} \frac{\omega_{\vec{n}} \varepsilon_{\gamma\mu}^{m} \hat{k}_n^{n\mu} f_{\beta}(\vec{n}, \vec{R}) g_{\gamma}(\vec{n}, \vec{r}).$$

(74)

It is remarkable to note that since we have assumed that the center of the atom is localized in a free region, then in the argument of the tensors $\eta$, $\zeta$, $\eta^m$ and $\zeta^m$ in (74), we have $\vec{R} \neq \vec{r}$ and therefore the recent series are all convergent. Inspection of the definitions (58) shows that the real and imaginary parts of $Z_e(\omega + i0^+)$ and $Z_m(\omega + i0^+)$ for real frequency $\omega$ are

$$Re[Z_e(\omega + i0^+)] = \frac{1}{2\pi} P \int_0^{\infty} d\omega' \frac{\chi_{ei}(\omega')}{\omega - \omega'} \equiv \alpha_e(\omega),$$

$$Im[Z_e(\omega + i0^+)] = -\frac{1}{2} \chi_{ei}(\omega) \vartheta(\omega) \equiv \gamma_e(\omega),$$

$$Re[Z_m(\omega + i0^+)] = \frac{1}{2\pi} P \int_0^{\infty} d\omega' \frac{\chi_{mi}(\omega')}{\omega - \omega'} \equiv \alpha_m(\omega),$$

$$Im[Z_m(\omega + i0^+)] = -\frac{1}{2} \chi_{mi}(\omega) \vartheta(\omega) \equiv \gamma_m(\omega).$$

(75)
where $\vartheta(\omega)$ is the step function and $P$ denotes the Cauchy principal value. Finally using (69), (73), (74), (75) and

$$\frac{1}{\omega + i0^+ - \omega'} = P \frac{1}{\omega - \omega'} - i\pi \delta(\omega - \omega'),$$

we find the spontaneous emission and the shift frequency of a two level atom due to the presence of a magnetodielectric. From (69) and (73) it is clear that the spontaneous emission and the lamb shift depend on the (i) position of the atom, (ii) orientation of the dipole of the atom, (iii) size and sort of the medium, (iv) relative position of the medium respect to the center of the atom, (v) the volume and the shape of the cavity. For a non rectangular cavity we must use a different set of eigenvectors and therefore the spontaneous emission depends also on the shape of the cavity.

4 Conclusion

The scheme introduced by the present authors to quantize the electromagnetic field in a magnetodielectric is generalized to the case where some external charges are present in the medium. The spontaneous emission of a two-level atom embedded in a magnetodielectric is calculated as an application of the model.

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