Finding the coefficient of rolling friction using a pericycloidal pendulum

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Abstract. The paper presents a method of establishing the coefficient of rolling friction by the study of the damped motion of a pericycloidal pendulum. The pericycloid is the curve described by a point belonging to a mobile circle, which is in pure rolling around a fixed circle, accomplishing an interior contact. For a pendulum with known inertial characteristics, mass and moment of inertia, the equation of motion is found starting from the assumption of a linear dependence between the friction torque and the normal reaction force from contact point. Using an experimental set-up, the law obeyed by damped angular amplitude versus time is found for a pericycloidal pendulum. The experimental data are compared to the theoretical ones, by superposition; thus, the value of coefficient of rolling friction is obtained. The values are in a good agreement with the ones from technical literature.

1. Introduction

Three types of friction may occur between two solid bodies, but the rolling friction is the less wear generator and energy consumer. The main reason that explains this property is the fact that there is no relative motion in the theoretical contact point or curve, as compared to the cases of sliding and spinning friction.

Nevertheless, the rolling bearings are used in different mechanical assemblies where modern operating requirements impose greater angular velocities and in which the cumulated effect of rolling friction is considerable. For a correct estimation of the effect of rolling friction, the accurate knowledge of tribological parameters characteristic to this type of friction is necessary.

In this regard, a main issue is the manner in which the dependence between the rolling friction torque and the normal force is accepted [1-8]. In monographs on dynamics, based on the analogy to the dry friction sliding, the proportionality between the rolling friction torque and the normal force is accepted, the coefficient of rolling friction being the coefficient of proportionality. In recent works, based on elements of theory of elasticity [9] it reaches the conclusion that there is a power-law dependence and, also, it defines a coefficient of rolling friction. Regardless of the hypothesis involving the dependence between the two parameters (moment and normal force), the oscillatory motions proved to be a powerful instrument in establishing the characteristics of rolling friction.

Amid oscillatory motions, the most used are the cycloidal ones [10-13]. The coefficient of rolling friction was also found by means of an involute pendulum [14] or analyzed using a hypocycloidal motion [15].
The present paper employs a pendulum, whose mobile contact surface is an inner cylinder and the still contact surface is a sphere; therefore, accepting the hypothesis of plane-parallel motion of the pendulum, the points of the pendulum describe a family of pericycoids [16].

2. Theoretical considerations
The inner contact between a fixed circle \( \Gamma_0 \) of \( r \) radius and a mobile circle \( \Gamma \) of \( R \) radius is presented in figure 1. The contact between the two circles occurs in the point \( C \). The position of the mobile circle can be characterized using the stipulated position of a point of it and the orientation of the axes of a system attached to it.

![Figure 1. Finding the pure rolling condition](image)

To this end, two coordinate systems are defined: a fixed one with the origin \( O \) in the center of the immobile circle and with the versors \( i, j \) of the axes oriented as in figure 1, the \( k \) versor being normal to the plane of figure and oriented as to result a right coordinate system:

\[
k = i \times j
\]

and the second coordinate system has the origin \( O' \) in the center of mobile circle and the axes have the versors \( i', j' \), their orientation being set by the \( \varphi \) angle.

It is considered that, for the initial moment, the axes of the two systems have identical orientations, \( \varphi = 0 \). Assuming permanent contact between the two circles, it is obvious that the trajectory of the origin of the mobile circle is a circle of radius \( R - r \), with the center in the origin \( O \). In order to stipulate the position of the origin \( O' \), it is sufficient to know the angle \( \theta \) made by the line of the centers with the versor \( i \). Based on the above considerations, it results that the oscillatory system from figure 1 has two degrees of freedom, \((\theta, \varphi)\). Next, the hypothesis that pure rolling exists in the contact point is accepted. Two points overlay in \( C \) point: \( C_0 \) from the fixed circle and the point from the mobile circle, denoted \( C \). The pure rolling condition requires that, in the contact point, the relative velocity is zero.

\[
v_{CC_0} = 0
\]
Since the velocity of the point \( C_0 \) is always zero, it results that for the existence of pure rolling in the contact point it is required that:

\[
v_C = 0
\]

and that means that the point \( C \) is the instantaneous center of rotation, \( CIR \). The relation (3) can be written as:

\[
v_{O'} + \vec{\omega} \times \overrightarrow{OC} = 0
\]

where

\[
\vec{\omega} = \dot{\theta}\mathbf{k}
\]

is the angular velocity of the mobile circle. Knowing that the trajectory of the \( O' \) point is circular, the velocity of this point is:

\[
v_{O'} = \dot{\phi}\mathbf{k} \times \overrightarrow{OO'}
\]

The versor \( \mathbf{u} \) is defined:

\[
\mathbf{u} = \begin{bmatrix} \cos \theta & \sin \theta \theta \end{bmatrix}^T
\]

and the relation (4) is written as:

\[
\dot{\theta}\mathbf{k} \times (R - r)\mathbf{u} + \dot{\phi}\mathbf{k} \times (-Ru) = 0
\]

and it may be expressed under the form:

\[
k \times \mathbf{u}\left[\dot{\theta}(R - r) - \dot{\phi}R\right] = 0
\]

The pure rolling condition becomes:

\[
\dot{\theta}(R - r) = \dot{\phi}R
\]

The relation (10) is integrated with respect to time, following the initial condition and the pure rolling condition expressed as function of displacement takes the form:

\[
\phi(\theta) = \frac{R - r}{R} \theta
\]

3. Establishing the motion of the center of mass of the pendulum

It is considered that the pendulum has a symmetry axis, chosen as the \( Ox' \) axis, and the center of gravity \( G \) is characterized in the mobile frame by the abscissa \( x_G' \). In order to specify the coordinates of the pendulum’s center of mass with respect to the fixed frame, the versor \( \mathbf{u} \) is used together to the versors of the mobile coordinate system:

\[
\mathbf{i}' = \begin{bmatrix} \cos \theta & \sin \theta \theta \end{bmatrix}^T
\]

\[
\mathbf{j}' = \begin{bmatrix} -\sin \theta & \cos \theta \theta \end{bmatrix}
\]

The position vector with respect to \( O \) point of the center of mass has the form:

\[
r_G = (R - r)\mathbf{u} + x_G' \mathbf{i}'
\]

The equation (13) is projected on the axes of the mobile frame and considering the relations (7), (11) and (12), it is obtained:
The curves (14) are parametric equations of a pericycloidal family which, according to the sign of the abscissa $x'_G$, can be common ($x'_G=0$), curate $x'_G<0$ or prolate $x'_G>0$ [16]. Beside the $x'_G$ parameter, the values of the radii $r$ and $R'$ are important, too. To illustrate this affirmation, in figure 2 there are presented the trajectories of some points from the axis of symmetry of the pendulum for two pair of values for $r$ and $R'$.

\[
\begin{align*}
  x_G(\varphi) &= x'_G \cos \left(\frac{R-r}{R} \theta\right) + (R-r) \cos \theta \\
  y_G(\varphi) &= x'_G \sin \left(\frac{R-r}{R} \theta\right) + (R-r) \sin \theta
\end{align*}
\] (14)

The theorem of motion of the center of mass and the moment of momentum theorem with respect to the center of mass are applied in order to find the motion of the pendulum [17]. The torsor of reaction forces in point $C$ consists in:

- the normal reaction $N$:

\[ N = -N u \] (16)

- the friction force $T$:

\[ T = -k \times u T \] (17)

- and the rolling friction torque:

\[ M_r = -M_r \text{sign}(\dot{\theta}) \] (18)
The only external force acting upon the pendulum is its weight:

\[ G = Mg \]  \tag{19}

where \( M \) is the mass of the pendulum and \( g \) is the gravitational acceleration.

The center of mass motion theorem has the form:

\[ Ma_G = N + T + G \]  \tag{20}

The moment of momentum theorem with respect to the center of mass is:

\[ J_G \dot{\theta} = \overrightarrow{GC} \times (N + T) + M_r \]  \tag{21}

where \( J_G \) represents the moment of inertia of the pendulum with respect to an axis normal to the plane of motion, passing through its center of mass \( G \). The vector \( \overrightarrow{GC} \) can be calculated using the relation:

\[ \overrightarrow{GC} = -x'_G \overrightarrow{i} - Ru \]  \tag{22}

Three scalar equations are now available, two from projections of relation (20) and one projection of equation (21). The unknowns of the problem are the components of the reaction torsor \( N, T \) and \( M_r \) and also the law of motion of the pendulum. An additional equation is needed to solve the problem and it is provided by the friction particularities from the contact point \( C \). By accepting the pure rolling hypothesis, the friction force remains unknown and the characterization of the rolling friction torque is required. Subsequently, it is accepted that the magnitude of the rolling friction torque is proportional to the magnitude of normal force \( N \):

\[ M_r = sN \]  \tag{23}

where \( s \) is the coefficient of rolling friction. Moreover, to ensure pure rolling in the contact point \( C \), the next condition must be fulfilled:

\[ |T| < \mu N \]  \tag{24}

where \( \mu \) is the dynamic coefficient of friction.

Given that in the center of mass motion theorem the unknowns are \( N, T \) and \( \theta \), the reactions \( N \) and \( T \) are solved as functions of \( \theta \):

\[ N(\theta) = M\dot{\theta}^2 \left[ 1 + \frac{R - r}{R^2} x'_G \cos \left( \frac{\theta}{R} \right) \right] (R - r) - M\ddot{\theta} \frac{R - r}{R} x'_G \sin \left( \frac{\theta}{R} \right) + Mg \cos \theta \]  \tag{25}

\[ T(\theta) = -M\dot{\theta} \left[ 1 + \frac{R - r}{R^2} x'_G \cos \left( \frac{\theta}{R} \right) \right] (R - r) - M\ddot{\theta}^2 \frac{R - r}{R} x'_G \sin \left( \frac{\theta}{R} \right) - Mg \sin \theta \]  \tag{26}

The expressions (22), (23), (25) and (26) are introduced in the equation (21) and it results a differential equation of second degree, of unknown \( \theta(t) \), having the form:
The resulting equation (27) is a non-linear one and it was integrated using the numerical procedure Runge-Kutta of fourth order, considering that, at the initial instant, the pendulum is at rest and $\theta_0$ is the angle of the axis with the vertical direction:

$$\begin{align*}
\dot{\theta} &= -(R-r)^{s \text{sign}(\dot{\theta})} \left( \sum_{g} \frac{r \sin(\theta)}{R} \right) + s \text{sign}(\dot{\theta}) \\
\frac{J_G}{M} &= \left\{ 2 \rho \left[ R \cos \left( \frac{r \theta}{R} \right) - \frac{s \text{sign}(\dot{\theta})}{2} \sin \left( \frac{r \theta}{R} \right) + \left( r \rho \dot{\theta} \right)^2 \right] + \frac{R-r}{R} \left( \rho \theta \right)^2 \right\} \sin \theta + s \text{sign}(\dot{\theta}) \cos \theta + \rho \sin \theta \\
+ &\left\{ 2 \rho \left[ R \cos \left( \frac{r \theta}{R} \right) - \frac{s \text{sign}(\dot{\theta})}{2} \sin \left( \frac{r \theta}{R} \right) + \left( r \rho \dot{\theta} \right)^2 \right] + \frac{R-r}{R} \left( \rho \theta \right)^2 \right\} \sin \theta + s \text{sign}(\dot{\theta}) \cos \theta + \rho \sin \theta = 0
\end{align*}$$

$$\left\{ \begin{array}{l}
\theta(0) = \theta_0 \\
\dot{\theta}(0) = 0
\end{array} \right.$$ (28)

4. Experimental device and results

The experimental device assembled for exemplifying the methodology of finding the coefficient of rolling friction is presented in figure 3. On a fixed aluminum bar 1, having two cylindrical holes, two identical parts 2 with conical positioning surface, were mounted. On these two parts, two identical bearing balls 3 of diameter 25 mm were placed. The assembly of the two balls materializes the immobile cylinder. The mobile cylinder is an aluminum tube. The employment of the two balls has the advantage that eliminates the possibility of spin motion of the mobile body, motion probable occurring when a single ball is used. Supplementary, the axis of the mobile cylinder maintains its orientation due to the use of the two balls.

![Experimental device](image)

Figure 3. Experimental device

The proper running of the device requires identical loads in the ball-cylinder contacts and this condition is fulfilled by accurately controlling the horizontality of the axis of oscillation. Considering that the balls 4 and the supporting parts are precisely made, the horizontality of the axis of oscillation...
depends on the accuracy of ensuring horizontality of the surface of the rod on top of which the parts are positioned. To test the horizontality of the rod, it was oriented using a bearing ball: when the ball is in equilibrium, the horizontality is adequate. The alternative applied for the oscillating body, specifically the use of a cylinder allows for an accurate control of the position of the mass center, ensuring permanently $x_G = \theta$. In order to establish the motion of the pendulum, a laser stick was attached to it. When the cylinder oscillates, the laser beam generates a spot that moves on a ruler. The motion of the spot with respect to the scale is video recorded and afterwards the motion is analyzed using software and the instants when the spot reaches the maximum amplitude in both directions are found. The variation of amplitude with time for the cylinder from figure 3 is presented in figure 4. For further study, the value reached by the spot at equilibrium position is necessary. From physical considerations, the two curves represented in figure 4 should be symmetrical with respect to the straight line $x = x_0$. Based on this observation, $x_0$ is found from the condition that the function:

$$H(x_0) = \sum_{k=1}^{n} (x_k - x_0)^2$$  \hspace{1cm} (29)

must have a minimum in $x_0$, where $n$ is the number of experimental points. In figure 5, there are represented the experimental data and the straight line $x = x_0$ with $x_0$ found using the optimization condition. The fact that the optimum value was established for $x_0$ is confirmed by the perfect superposition of the two branches from figure 5, with respect to the line $x = x_0$. The distance between the ruler and the contact line is $L = 0.98$ m. The amplitude of oscillation initiation of the pendulum is:

$$\theta_0 = \arcsin \frac{\max(x - x_0)}{L} \approx 4.8^\circ$$

The displacement of the contact point with respect to the initial point corresponding to this rotation is $r\theta_0 \approx 2mm$, that can be neglected as compared to the amplitude of the spot $\approx 85mm$. Thus, the inclination angle of the pendulum with respect to the vertical for the extreme positions of the pendulum is found with the relation:

$$\psi_k = \arcsin \frac{x_k - x_0}{L}$$  \hspace{1cm} (30)

![Figure 4. Experimental data](image1)

![Figure 5. Founding the equilibrium position $x_0$ of the pendulum](image2)

The $\psi_k = \psi_k(t_k)$ relation should be correlated to the solution of differential equation (27), integrated for the data corresponding to the experimental set-up. Therefore, considering the density of aluminum $\rho = 2700kg/m^3$, the radii $R_{int} = 0.128/2m$, $R_{ext} = 0.136/2m$, the height $H = 0.105m$ the mass of the cylinder is calculated $M = 0.473kg$ and also the moment of inertia:
The laser stick has a mass \( m = 0.008 \text{ kg} \) and the center of mass situated at the distance \( d' = 0.075 \text{ m} \) with respect to the axis of the cylinder. The effect of the stick upon the change of position of the center of mass is insignificant:

\[
\Delta x_G = \frac{md'}{M + m} = 1.27(\text{mm})
\]

but it modifies the moment of inertia and the effect is calculated applying the Steiner theorem:

\[
\frac{J_z}{J_{G0}} = \frac{J_{G0} + md'^2}{J_{G0}} \approx 1.022
\]

The solution of equation (27) obtained for a coefficient of rolling friction \( s = 5 \cdot 10^{-6} \text{ m} \) is used for interpolation of the experimental data, as presented in figure 6.

The method is very responsive to the variation of the coefficient of rolling friction and to illustrate this, the experimental data and the solutions of equation (27), obtained for \( s = 4 \cdot 10^{-6} \text{ m} \) and \( s = 5 \cdot 10^{-6} \text{ m}, \) are represented in figure 7 and figure 8, respectively. Figure 6b presents a detail at the initiation of motion that proves, by the excellent concordance between experimental and theoretical period, the correctness of the theoretical model. To be mentioned that the experimental data from figure 6b were plotted for every 10 complete oscillations of the pendulum in order to make noticeable the effect of damping.

The normal force \( N \) and tangential force \( T \) variations are presented in figure 9 and figure 10 and the variation of the ratio \( T/N \) is finally represented in figure 11, with the purpose to check if the pure rolling condition is satisfied for the entire time interval of the motion. As it can be seen from figure 11, the maximum value of the considered ratio is approximately 0.12, a value too small for the dynamic sliding friction that is specified in literature to be around \( \mu = 0.5 \) [22].

At last, it is emphasized that it is incorrect to assimilate the described pendulum as a physical pendulum with the axis of oscillation passing through the contact points between the balls and cylinder.

\[
J_{G0} = \frac{\pi}{2} \left(R_{ext}^2 - R_{int}^2\right) \rho = 0.00205 \text{ kg} \cdot \text{m}^2
\]

\[
J_z = \frac{J_{G0} + md'^2}{J_{G0}} \approx 1.022
\]

**Figure 6.** Experimental data and theoretical curve for \( s = 5 \mu \text{m} \)
From figure 6b, it can be observed that the value of experimental period of oscillation is $T_{exp} = 0.604 \, \text{sec}$ . The period of the physical pendulum needs knowing the moment of inertia found with the relation:

$$J_{ph} = J_z + MR_{int}^2 = 0.00403 \, \text{kg} \cdot m^2$$

(34)

Next, the period of oscillation results:

$$T_{ph} = 2\pi \sqrt{\frac{J_{ph}}{MgR_{int}}} = 0.732 \, \text{sec}$$

(35)

This period is 21% greater than the period of the actual pendulum.
5. Conclusions
The paper presents the methodology and related device for finding the coefficient of rolling friction using a pendulum with pericycloidal motion. The pendulum is constructed using a mobile metallic cylinder supported by two immobile steel balls that have the axis of the centers in horizontal plane.

The differential equation of the oscillatory motion is deduced based on the assumption that proportionality between the rolling friction torque and normal force exists. Due to the dry friction existence, the differential equation of motion has a nonlinear character and requires a numerical technique for obtaining the solution. In order to obtain the value of the coefficient of rolling friction, the mobile cylinder is set into oscillatory pericycloidal motion and the spot generated by a laser–stick attached to the cylinder is filmed. The motion of the spot on a fixed ruler with known position permits - by analyzing the movie frame by frame - establishing the damping in time of the amplitude of the pendulum.

The value of the coefficient of rolling friction is found by imposing the condition that the experimental data are optimally interpolated by the solution of the equation of motion. Specifically, the experimental data are superposed over a lattice of solutions of the motion equation traced for a sequence of values of the coefficient of rolling friction; the curve that interpolates optimally the experimental values is chosen and, thus, the coefficient of rolling friction is precisely identified. The obtained results are in a very good agreement to the ones from technical literature.

As in previous works of the authors where oscillatory motions between other types of curves were involved, it is concluded that, despite the fact that the pure rolling condition is satisfied during the entire oscillation duration, the angular amplitude damping takes a quasi-linear aspect – as the theoretical considerations predict, only towards the last part of the oscillation, when the angular amplitude decreases under a certain value.

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