Abandoning Monomorphisms:
Partial Maps, Fractions and Factorizations✩

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Abstract
For a composition-closed and pullback-stable class $S$ of morphisms in a category $C$ containing all isomorphisms, we form the category $\text{Span}(C, S)$ of $S$-spans $(s, f)$ in $C$ with first ‘leg’ $s$ lying in $S$, and give an alternative construction of its quotient category $C[S^{-1}]$ of $S$-fractions. Instead of trying to turn $S$-morphisms ‘directly’ into isomorphisms, we turn them separately into retractions and into sections, in a universal manner. Without confining $S$ to be a class of monomorphisms of $C$, the second of these two quotient processes leads us to the category $\text{Par}(C, S)$ of $S$-partial maps in $C$. Under mild additional hypotheses on $S$, $\text{Par}(C, S)$ has a localization, which is a split restriction category, or even a split range category (in the sense of Cockett, Guo and Hofstra), but which is still large enough to admit $C[S^{-1}]$ as its quotient. The construction of the range category is part of a global adjunction between relatively stable factorization systems and split range categories.

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1. Introduction

The formation of the category $\mathcal{C}[\mathcal{S}^{-1}]$ of fractions with respect to a sufficiently well-behaved class $\mathcal{S}$ of morphisms in $\mathcal{C}$, as first given in [4], is a fundamental device in homotopy theory. The construction is characterized by its localizing functor $\mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$ which is universal with respect to the property of turning morphisms in $\mathcal{S}$ into isomorphisms. An existence proof for $\mathcal{C}[\mathcal{S}^{-1}]$ is only sketched by Gabriel and Zisman [4] (see their Lemma 1.2 on p. 7 whose “proof is left to the reader”); for more elaborate proofs, see [7] and [1]. A particular and delicate point is the question of the size of the “homs” of $\mathcal{C}[\mathcal{S}^{-1}]$, as these may be large even when those of $\mathcal{C}$ are all small.

Assuming $\mathcal{S}$ to contain all isomorphisms and to be closed under composition and stable under pullback in $\mathcal{C}$ throughout this paper and, thus, departing from the original array of applications for the construction, we take a stepwise approach to the formation of $\mathcal{C}[\mathcal{S}^{-1}]$. Hence, we consider separately the two processes of transforming every morphism in $\mathcal{S}$ into a retraction and into a section, before amalgamating them to obtain the category of fractions. Not surprisingly, when $\mathcal{S}$ happens to be a class of monomorphisms in $\mathcal{C}$, the transformation of $\mathcal{S}$-morphisms into retractions essentially suffices to reach $\mathcal{C}[\mathcal{S}^{-1}]$, simply because the transformation of $\mathcal{S}$-morphisms into sections comes almost for free when $\mathcal{S}$ is a class of monomorphisms: one just considers the $\mathcal{S}$-span category $\text{Span}(\mathcal{C}, \mathcal{S})$ whose morphisms $(s, f) : A \to B$ are (isomorphism classes of) spans $A \xrightarrow{s} D \xrightarrow{f} B$ of morphisms in $\mathcal{C}$ with $s \in \mathcal{S}$; composition with $(t, g) : B \to C$ proceeds as usual, via pullback:

![Diagram](https://via.placeholder.com/150)

Trivially now, the functor $\mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}]$, $f \mapsto (1, f)$, turns $\mathcal{S}$-morphisms into sections, since monomorphisms have trivial kernel pairs (in the diagram above, for $s = g = 1$ and $f = t \in \mathcal{S}$ one can take $t' = f' = 1$).

In the general case, without confining $\mathcal{S}$ to be a class of monomorphisms, as a first step we will still form the category $\text{Span}(\mathcal{C}, \mathcal{S})$ as above. Then, transforming $\mathcal{S}$-morphisms into retractions in a universal manner is fairly
easy, while trying to transform them into sections turns out to be considerably more complicated, because of the missing mono hypothesis on $S$. The latter problem leads us to one of the main points of this paper, the construction of the $S$-partial map category $\text{Par}(C, S)$ (see Section 3), while the former problem makes us form (for lack of a better name) the $S$-retractive span category $\text{Retr}(C, S)$ (see Section 2). In Section 4 we see how to amalgamate the two constructions to obtain the category $C[S^{-1}]$.

While the fact that $C \to \text{Par}(C, S)$ is universal with respect to turning $S$-morphisms into sections serves as our legitimation for having given the category its name, unfortunately the category may fail to be a restriction category, i.e., it may fail to enjoy a property identified by Cockett and Lack [3] as fundamental for $S$-partial map categories when $S$ is a class of monomorphisms. That is why, in Section 5, we elaborate on how to obtain the $S$-partial map restriction category $\text{RePar}(C, S)$ as a quotient category of $\text{Par}(C, S)$. Under a fairly mild additional hypothesis on $S$, which holds in particular under the weak left cancellation condition ($s \cdot t \in S \implies t \in S$), $\text{RePar}(C, S)$ is a localization of $\text{Par}(C, S)$ and makes $\text{Retr}(C, S) = C[S^{-1}]$ its quotient category.

Of course, this additional condition holds a fortiori when $S$ belongs to a relatively stable orthogonal factorization system $(P, S)$ of $C$, so that $P$ is stable under pullback along $S$-morphisms. In that case we can form, as a further localization of $\text{Par}(C, S)$, a range category in the sense of [2]. Range categories not only have a restriction structure on the domains of morphisms, but also a kind of dually behaved structure on their codomains. Hence, in Section 7 we present the construction of the $S$-partial map range category $\text{RaRePar}(C, S)$, thus completing the quotient constructions given in this paper.

In summary, for $S$ satisfying the general hypotheses one has the commutative diagram

$$
\begin{array}{cccccc}
C & \longrightarrow & \text{Span}(C, S) & \longrightarrow & \text{Par}(C, S) & \longrightarrow & \text{RePar}(C, S) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Retr}(C, S) & \longrightarrow & C[S^{-1}] & \longrightarrow & \text{Retr}(C, S) & = & C[S^{-1}]
\end{array}
$$

which flattens to

$$
C \to \text{Span}(C, S) \to \text{Par}(C, S) \to \text{RePar}(C, S) \to \text{Retr}(C, S) = C[S^{-1}]
$$
when \( S \) satisfies the weak left cancellation property, and it extends further to

\[
C \rightarrow \text{Span}(C, S) \rightarrow \text{Par}(C, S) \rightarrow \text{RePar}(C, S) \rightarrow \cdots.
\]

\[\rightarrow \text{RaRePar}(C, S) \rightarrow \text{Retr}(C, S) = C[S^{-1}]\]

when \( S \) belongs to an \( S \)-stable factorization system \((P, S)\) of \( C \). When \( S \) is a class of monomorphisms, the chain simplifies to

\[
C \rightarrow \text{Span}(C, S) = \text{Par}(C, S) = \text{RePar}(C, S) \rightarrow \text{Retr}(C, S) = C[S^{-1}],
\]

and one then also has \( \text{RePar}(C, S) = \text{RaRePar}(C, S) \), should \( S \) be part of an \( S \)-stable factorization system \((P, S)\).

Quite a different picture emerges when one puts additional constraints on \( S \) that are typically satisfied by classes of epimorphisms, not monomorphisms. In Sections 4 and 5 we show that, when \( C \) has finite products with all projections lying in \( S \), and if there is no strictly initial object in \( C \), then \( \text{Par}(C, S) \) is equivalent to the terminal category \( 1 \), and one has

\[
C \rightarrow \text{Span}(C, S) \rightarrow \text{Retr}(C, S) \rightarrow \text{Par}(C, S) = \text{RePar}(C, S) = C[S^{-1}] \cong 1.
\]

When \( S \) is part of an orthogonal factorization system \((P, S)\) in \( C \), such that \( P \)-morphisms are stable under pullback along \( S \)-morphisms, then the construction of the split range category \( \text{RaRePar}(C, S) \) lies at the heart of a global adjunction that is presented in Section 8. Extending techniques developed in [2, 3], we show that \( \text{RaRePar} \) may be considered a 2-functor that is left adjoint to the 2-functor which assigns to every split range category \( X \) the category \( \text{Total}(X) \) (which has same objects as \( X \), but its morphisms are only the so-called total morphisms of \( X \)); it is is known to always carry a factorization system of the type considered. Consequently, the category \( \text{RaRePar}(C, S) \) may be characterized amongst split range categories by a universal property.

**2. Span categories and their quotients**

Throughout this paper, we consider a class \( S \) of morphisms in a category \( C \) such that

- \( S \) contains all isomorphisms and is closed under composition, and
- pullbacks of \( S \)-morphisms along arbitrary morphisms exist in \( C \) and belong to \( S \).
In particular, we may consider \( S \) as a (non-full) subcategory of \( C \) with the same objects as \( C \). For objects \( A, B \) in \( C \), an \( S \)-span \((s, f)\) with domain \( A \) and codomain \( B \) is given by a pair of morphisms

\[
A \xleftarrow{s} D \xrightarrow{f} B
\]

with \( s \) in \( S \) and \( f \) in \( C \). These are the objects of the category

\[
\text{Span}(C, S)(A, B)
\]

whose morphisms \( x : (s, f) \longrightarrow (\tilde{s}, \tilde{f}) \) are given by \( C \)-morphisms \( x \) with \( \tilde{s} \cdot x = s \) and \( \tilde{f} \cdot x = f \), to be composed “vertically” as in \( C \).

Of course, isomorphisms in this category are given by isomorphisms in \( C \) making the above diagram commute. Notationally we will not distinguish between the pair \((s, f)\) and its isomorphism class in \( \text{Span}(C, S)(A, B) \).

The hypotheses on \( S \) guarantee that, when composing \((s, f) : A \longrightarrow B\) “horizontally” with an \( S \)-span \((t, g) : B \longrightarrow C\) via a (tacitly chosen) pullback \((t', f')\) of \((f, t)\) (see the first diagram in the Introduction), the composite span \((t, g) \cdot (s, f) := (s \cdot t', g \cdot f')\) is again an \( S \)-span. We denote the resulting category\(^1\) of isomorphism classes of \( S \)-spans by

\[
\text{Span}(C, S).
\]

Now we can consider a compatible relation on \( \text{Span}(C, S) \), that is: a relation for \( S \)-spans such that

- only \( S \)-spans with the same domain and codomain may be related;
- vertically isomorphic \( S \)-spans are related;

\(^1\)We remind the reader that \( \text{Span}(C, S) \) may, unlike \( C \), fail to have small hom-sets.
• horizontal composition from either side preserves the relation.

It is a routine exercise, and a fact used frequently in this paper, to show that the least equivalence relation for $S$-spans generated by a given compatible relation is again compatible.

For a compatible equivalence relation $\sim$ we denote the $\sim$-equivalence class of $(s, f)$ by $[s, f]_{\sim}$, or simply by $[s, f]$ when the context makes it clear which relation $\sim$ we are referring to, and we write

$$\text{Span}_{\sim}(C, S)$$

for the resulting quotient category $\text{Span}(C, S)/\sim$. We observe that the pair of functors

$$\Phi_{\sim} = \Phi : C \longrightarrow \text{Span}_{\sim}(C, S) \leftarrow S^{\text{op}} : \Psi = \Psi_{\sim}$$

$$(f : D \to B) \longmapsto [1_D, f] [s, 1_D] \longmapsto (A \leftarrow D : s)$$

satisfies the Beck-Chevalley property, in the following sense:

• $\Phi$ and $\Psi$ coincide on objects, so that $\Phi A = \Psi A$ for all objects $A$ in $C$, and

• whenever the square on the left is a pullback diagram in $C$ with $t \in S$,

$$\begin{array}{ccc}
P & \overset{f'}{\longrightarrow} & E \\
\downarrow t' & & \downarrow t \\
D & \overset{f}{\longrightarrow} & B \\
\end{array} \quad \begin{array}{ccc}
\Phi P & \overset{\Phi f'}{\longrightarrow} & \Phi E \\
\downarrow \psi t' & & \downarrow \psi t \\
\Phi D & \overset{\Phi f}{\longrightarrow} & \Phi B \\
\end{array}$$

then the square on the right commutes.

Furthermore, one sees immediately that $(\Phi, \Psi)$ is $\sim$-consistent, that is:

• whenever $(s, f) \sim (\tilde{s}, \tilde{f})$, then $\Phi f \cdot \Psi s = \Phi \tilde{f} \cdot \Psi \tilde{s}$.

Now it is easy to confirm that $(\Phi, \Psi)$ is universal amongst all pairs of functors

$$F : C \longrightarrow D \leftarrow S^{\text{op}} : G \quad \text{(*)}$$

which satisfy the Beck-Chevalley property and are $\sim$-consistent, i.e., amongst pairs of functors satisfying the properties of the last three bullet points above, with $\Phi, \Psi, \text{Span}_{\sim}(C, S)$ respectively traded for $F, G, D$ everywhere:

---

²Here we disregard the fact that one actually has $\Phi A = A = \Psi A$ for all objects $A$. 

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Proposition 1. For a compatible equivalence relation $\sim$ on $\text{Span}(\mathcal{C}, \mathcal{S})$, every $\sim$-consistent pair of functors $F, G$ satisfying the Beck-Chevalley property as above factors as $F = H\Phi$, $G = H\Psi$, with a uniquely determined functor $H$, as in

\[ \begin{array}{ccc} 
\mathcal{C} & \xrightarrow{\Phi} & \text{Span}_\sim(\mathcal{C}, \mathcal{S}) \xrightarrow{\Psi} \mathcal{S}^{\text{op}} \\
F & \downarrow H & G \\
\mathcal{D} & \xrightarrow{\Psi} & \mathcal{S}^{\text{op}} 
\end{array} \]

Proof. Every morphism $[s, f]$ in $\text{Span}_\sim(\mathcal{C}, \mathcal{S})$ may be factored as $[s, f] = \Phi f \cdot \Psi s$. Hence, any functor $H$ factoring as claimed must necessarily map $[s, f]$ to $F f \cdot G s$, and $\sim$-consistency allows us to define $H$ in this way. Trivially then, $H$ preserves the identity morphisms $[1,1]$, and the Beck-Chevalley property ensures the preservation of composition:

\[
H([t, g] \cdot [s, f]) = H[s \cdot t', g \cdot f'] = F(g \cdot f') \cdot G(s \cdot t') = F g \cdot F f' \cdot G t' \cdot G s = F g \cdot G t \cdot F f \cdot G s = H[t, g] \cdot H[s, f].
\]

\[\square\]

3. The $\mathcal{S}$-retractive span category $\text{Retr}(\mathcal{C}, \mathcal{S})$

Example 1. There is a preorder for $\mathcal{S}$-spans with the same domain $A$ and codomain $B$ defined by

\[
(s, f) \preceq (\tilde{s}, \tilde{f}) \iff \exists x : (s, f) \rightarrow (\tilde{s}, \tilde{f}), x \in \mathcal{S}.
\]

We call the least equivalence relation on all $\mathcal{S}$-spans containing the reflexive and transitive relation $\preceq$ the zig-zag relation and denote it by $\sim_z$. The compatibility of $\preceq$ makes $\sim_z$ compatible. Writing just $z$ instead of $\sim_z$ when $\sim_z$ is used as an index, we define:

Definition 1. We call the quotient category

\[ \text{Retr}(\mathcal{C}, \mathcal{S}) := \text{Span}_z(\mathcal{C}, \mathcal{S}) \]

the $\mathcal{S}$-retractive span category of $\mathcal{C}$. It comes with the functors

\[
\Phi_z : \mathcal{C} \rightarrow \text{Retr}(\mathcal{C}, \mathcal{S}) \xleftarrow{\imath} \mathcal{S}^{\text{op}} : \Psi_z \quad (**)
\]

\[
(f : D \rightarrow B) \mapsto [1_D, f]_z \quad [s, 1_D]_z \mapsto (A \leftarrow D : s).
\]
Obviously, \( \Phi_z s \cdot \Psi_z s = 1 \) for all \( s \in S \); indeed, since \( (s, s) \leq (1,1) \) one has
\[
[1, s]_z \cdot [s, 1]_z = [s, s]_z = [1, 1]_z = 1.
\]
In fact, the functors \( \Phi_z, \Psi_z \) are universal with this property, as we note next.

**Corollary 1.** Any pair of functors \( (F,G) \) as in \( *(\ast) \) satisfying the Beck-Chevalley property and the equalities \( Fs \cdot Gs = 1 \) \( (s \in S) \) factors uniquely through the pair \( (\Phi_z, \Psi_z) \) of \( (**\ast) \), as in Proposition 1.

**Proof.** After Proposition 1 we just need to confirm that the equality \( Fs \cdot Gs = 1 \) \( (s \in S) \) makes \( (F,G) \sim_z \)-consistent. But this is clear since, when \( (s,f) \leq (\tilde{s},\tilde{f}) \), so that \( \tilde{s} \cdot x = s \), \( \tilde{f} \cdot x = f \) for some \( x \in S \), we have
\[
Ff \cdot Gs = F\tilde{f} \cdot Fx \cdot Gx \cdot G\tilde{s} = F\tilde{f} \cdot G\tilde{s}.
\]
\[\square\]

**Remark 1.** (1) We can think of Corollary 1 as “going halfway” towards the construction of the category \( \mathcal{C}[S^{-1}] \) of fractions with respect to \( S \) (see [4, 1]). While we will return to this aspect in Section 5 below, let us point out immediately that, when \( S \) is a class of monomorphisms in \( \mathcal{C} \), the category \( \text{Retr}(\mathcal{C}, S) \) is actually isomorphic to the category \( \mathcal{C}[S^{-1}] \). This follows from the observation that, for a monomorphism \( s \) in \( \mathcal{C} \) and any pair of functors \( F,G \) as in \( * \) satisfying the Beck-Chevalley property, one has
\[
Gs \cdot Fs = 1,
\]
which in conjunction with Corollary 1 makes the map \( Fs \) an isomorphism with \( (Fs)^{-1} = Gs \). Indeed, for a monomorphism \( s \) the following square on the left is a pullback diagram, so that the Beck-Chevalley property makes the square on the right commute:
\[
\begin{array}{ccc}
D & \xrightarrow{1} & D \\
\downarrow_{1} \quad \quad \quad \quad \quad \quad \downarrow_{s} & \quad \quad \quad \quad \quad \quad \quad \downarrow_{pb} & \quad \quad \quad \quad \quad \quad \quad \downarrow_{s} \\
D & \xrightarrow{s} & A \\
\end{array}
\quad
\begin{array}{ccc}
FD & \xrightarrow{1} & FD \\
\downarrow_{1} \quad \quad \quad \quad \quad \quad \quad \downarrow_{Gs} & \quad \quad \quad \quad \quad \quad \quad \downarrow_{Gs} \\
FD & \xrightarrow{Fs} & FA \\
\end{array}
\]

In particular, for \( \sim \) as in Proposition 1, one always has \( \Psi s \cdot \Phi s = 1 \) when \( s \) is monic.
We must caution the reader that very often the category $\text{Retr}(\mathcal{C}, \mathcal{S})$ (and, consequently, also the fraction category $\mathcal{C}[\mathcal{S}^{-1}]$) turns out to be trivial: If $\mathcal{C}$ has an initial object $0$ and $\mathcal{S}$ contains all morphisms $!^A : 0 \to A$ ($A$ in $\mathcal{C}$), then $\text{Retr}(\mathcal{C}, \mathcal{S})$ is equivalent to the terminal category $1$, i.e., all hom-sets of $\text{Retr}(\mathcal{C}, \mathcal{S})$ are singletons.

Indeed, with the provision $!^A \in \mathcal{S}$ one has $(!^A, !^B) \leq (s, f)$, for all $\mathcal{S}$-spans $(s, f) : A \to B$. Note that when $0$ is strictly initial, so that for all $\mathcal{C}$ any morphism $C \to 0$ is an isomorphism, $!^A$ is a pullback of $0 \to 1$, for $1$ terminal in $\mathcal{C}$; hence, having $0 \to 1$ in $\mathcal{S}$ suffices to render $\text{Retr}(\mathcal{C}, \mathcal{S})$ trivial in this case.

4. The $\mathcal{S}$-partial map category $\text{Par}(\mathcal{C}, \mathcal{S})$

Our next goal is to force the last equality of Remark 1(1) to hold for all morphisms $s \in \mathcal{S}$, without the assumption that $s$ be monic, by a suitable choice of an equivalence relation for $\mathcal{S}$-spans. This equivalence relation will be induced by a certain relation for $\mathcal{S}$-cospans. These are isomorphism classes of pairs $\langle f, s \rangle$ of $\mathcal{C}$-morphisms

$$A \xrightarrow{f} D \xleftarrow{s} B$$

with $s \in \mathcal{S}$; $A$ is the domain and $B$ the codomain of such an $\mathcal{S}$-cospan. Like for $\mathcal{S}$-spans, isomorphisms of $\mathcal{S}$-cospans live in the category

$$\text{Cospan}(\mathcal{C}, \mathcal{S})(A, B),$$

which has “vertical” morphisms $v : \langle f, s \rangle \longrightarrow \langle \tilde{f}, \tilde{s} \rangle$ obeying $v \cdot f = \tilde{f}$, $v \cdot s = \tilde{s}$. We call a relation for $\mathcal{S}$-cospans compatible if

- only $\mathcal{S}$-cospans with the same domain and codomain may be related;
- vertically isomorphic $\mathcal{S}$-cospans are related;
- “horizontal whiskering” by pre-composition from either side preserves the relation, that is: whenever $\langle f, s \rangle, \langle g, t \rangle$ are related, then also $\langle f \cdot h, s \cdot r \rangle, \langle g \cdot h, t \cdot r \rangle$ are related, for all $\mathcal{C}$-morphisms $h$ and $\mathcal{S}$-morphisms $r$ such that the composites $f \cdot h$, $s \cdot r$ are defined.

It is easy to see that the least equivalence relation for $\mathcal{S}$-cospans containing a given compatible relation is again compatible.
Example 2. Like for $\mathcal{S}$-spans, there is a preorder for $\mathcal{S}$-cospans with the same domain $A$ and codomain $B$ given by

$$\langle f, s \rangle \preceq \langle \tilde{f}, \tilde{s} \rangle \iff \exists v : \langle f, s \rangle \rightarrow \langle \tilde{f}, \tilde{s} \rangle, \ v \in \mathcal{S}.$$  

The preorder is obviously compatible.

Every $\mathcal{S}$-cospan $\langle f, s \rangle$ gives, via pullback, the $\mathcal{S}$-span $(s', f') = \text{pb}(f, s)$. In fact, for objects $A, B$ in $\mathcal{C}$ one has a functor

$$\text{pb} : \text{Cospan}(\mathcal{C}, \mathcal{S})(A, B) \rightarrow \text{Span}(\mathcal{C}, \mathcal{S})(A, B)$$

which assigns to $v : \langle f, s \rangle \rightarrow \langle g, t \rangle$ the canonical morphism $v^* : (s', f') \rightarrow (t', g')$, i.e., the unique $\mathcal{C}$-morphism $v^*$ rendering the diagram

```
\begin{tikzpicture}
  \node (s) at (0,0) {$s$};
  \node (t) at (1,0) {$t$};
  \node (s') at (0,-1) {$s'$};
  \node (t') at (1,-1) {$t'$};
  \node (f) at (0.5,1) {$f$};
  \node (g) at (0.5,-1) {$g$};
  \node (f') at (1.5,1) {$f'$};
  \node (g') at (1.5,-1) {$g'$};
  \draw[->] (s) to (f);
  \draw[->] (t) to (g);
  \draw[->] (s') to (t');
  \draw[->] (s') to (s);
  \draw[->] (t') to (t);
  \draw[->] (s') to (f');
  \draw[->] (t') to (g');
  \draw[->] (f') to (g');
  \draw[->] (f) to (g);
  \draw[->] (f) to (s);
  \draw[->] (g) to (t);
  \draw[->] (s) to (t);
  \node (v) at (0.5,-0.5) {$v$};
  \draw[->] (v) to (f);
  \draw[->] (v) to (s);
end{tikzpicture}
```

commutative. We call $v^*$ the $\mathcal{S}$-span morphism induced by the $\mathcal{S}$-span morphism $v$; we will return to this terminology in Section 6.

Given a compatible $\mathcal{S}$-cospans relation, one wishes to consider a pair of $\mathcal{S}$-spans as related when they arise as the pullbacks of a pair of related $\mathcal{S}$-cospans. Unfortunately, the relation for $\mathcal{S}$-spans thus obtained may not even inherit reflexivity from the $\mathcal{S}$-cospans relation. However, after enlarging this relation, by allowing ‘horizontal whiskering’ via post-composition from either side in $\mathcal{C}$, we obtain a well-behaved relation for $\mathcal{S}$-spans, as follows.

**Definition 2.** Let $\mathcal{U}$ be a compatible $\mathcal{S}$-cospans relation. The $\mathcal{S}$-span companion relation $\approx$ induced by $\mathcal{U}$ is defined as follows:

$$\langle s, f \rangle \approx \langle t, g \rangle \text{ if, and only if, there exist morphisms } u \text{ in } \mathcal{S}, k \text{ in } \mathcal{C}, \text{ and } \mathcal{S}\text{-cospans } \langle f, s \rangle, \langle g, t \rangle \text{ such that } \langle f, s \rangle \mathcal{U} \langle g, t \rangle \text{ and, for some pullback diagrams}$$
one obtains the commutative diagram

Remark 2. The $S$-span companion relation $\approx$ as just defined is reflexive. Indeed, given an $S$-span $(s, f): A \rightarrow B$, one has the commutative diagram on the left and the trivial pullback diagram on the right:

Hence, with $\langle f, 1 \rangle \parallel \langle f, 1 \rangle$ by reflexivity of $\parallel$, one concludes $(s, f) \approx (s, f)$.

Proposition 2. Let $\approx$ be the $S$-span companion relation induced by a compatible relation $\parallel$ for $S$-cospans. Then $\approx$ is compatible, and it is symmetric when $\parallel$ is symmetric.

Proof. That isomorphic $S$-spans are $\approx$-related may be shown similarly to Remark 2 and $\approx$ trivially inherits symmetry from $\parallel$. To prove the compatibility of $\approx$, we consider $(s, f) \approx (t, g)$ and first show $(r, h) \cdot (s, f) \approx (r, h) \cdot (t, g)$, for all $(r, h)$ post-composable with $(s, f), (t, g)$. By hypothesis one has morphisms $\hat{s}, \hat{t}, u$ in $S$ and $\hat{f}, \hat{g}, k$ in $C$ such that $\langle \hat{f}, \hat{s} \rangle \parallel \langle \hat{g}, \hat{t} \rangle$ and, for the two pullback diagrams below on the right, the diagram on the left commutes.
The equalities \( k \cdot \hat{f} = f \) and \( k \cdot \hat{g} = g \) produce the following commutative diagrams, in which the squares are pullbacks (here \( x^*(y) \) denotes a pullback of \( y \) along \( x \)):

With the pullback diagrams on the right, it is easy to see that the following diagram on the left commutes:

From \( \langle \hat{f}, \hat{s} \rangle \Uparrow \langle \hat{g}, \hat{t} \rangle \), using invariance of \( \Uparrow \) under horizontal whiskering, we now obtain \( \langle \hat{f}, \hat{s} \cdot k^*(r) \rangle \Uparrow \langle \hat{g}, \hat{t} \cdot k^*(r) \rangle \). So, \( (s \cdot f^*(r), h \cdot r^*(f)) \approx (t \cdot g^*(r), h \cdot r^*(g)) \), as desired.

The proof that \( \approx \) is also preserved by pre-composition (rather than post-composition) in \( \text{Span}(C, S) \) proceeds very similarly.

Let us now return to Example 2 and apply Proposition 2 to the case that the \( S \)-cospan relation \( \Uparrow \) is the preorder \( \preceq \) on \( S \)-cospans. We denote its companion (or “associated”) relation for \( S \)-spans by \( \approx \) and let \( \sim \) denote the equivalence relation generated by \( \approx \); it is given by the symmetric and transitive hull of the compatible relation \( \approx \), and \( \sim \) is therefore compatible as well. Writing simply \( a \) when \( \sim \) is used as an index, we define:
Definition 3. We call the quotient category
\[ \text{Par}(\mathcal{C}, \mathcal{S}) := \text{Span}_a(\mathcal{C}, \mathcal{S}) \]
the $\mathcal{S}$-partial map category of $\mathcal{C}$. It comes with the functors
\[ \Phi_a : \mathcal{C} \rightarrow \text{Par}(\mathcal{C}, \mathcal{S}) \quad \Psi_a : \mathcal{S}^{\text{op}} \rightarrow \mathcal{C} \]
\[ (f : D \rightarrow B) \mapsto [1_D, f]_a \quad [s, 1_D]_a \leftarrow (A \leftarrow D : s). \]

Here is the key property of these functors:

**Lemma 1.** $\Psi_a s \cdot \Phi_a s = 1$, for every $s \in \mathcal{S}$.

**Proof.** Trivially $(1, 1) \leq (s, s)$. Consequently, for the kernel pair $(u, v)$ of $s$, $(u, v) \approx_a (1, 1)$ follows, so that $\Phi_a v \cdot \Psi_a u = 1$. Since, by the Beck-Chevalley property, $\Psi_a s \cdot \Phi_a s = \Phi_a v \cdot \Psi_a u$, this completes the proof. \qed

We can now prove that $(\Phi_a, \Psi_a)$ is universal with respect to the identity shown in Lemma 1:

**Theorem 1.** Any pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$, $G : \mathcal{S}^{\text{op}} \rightarrow \mathcal{D}$ satisfying the Beck-Chevalley property and the identity $G_s \cdot F s = 1$ ($s \in \mathcal{S}$) factors as $F = H \Phi_a$, $G = H \Psi_a$, with a uniquely determined functor $H$, as in

\[ \begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Phi_a} & \text{Par}(\mathcal{C}, \mathcal{S}) \\
\downarrow F & & \downarrow \Psi_a \\
\mathcal{D} & \xleftarrow{H} & \mathcal{S}^{\text{op}} \\
\end{array} \]

**Proof.** After Proposition 1 it suffices to show that the pair $(F, G)$ is necessarily $\sim_a$-consistent. Hence we consider $(s, f) \approx_a (g, t)$ and obtain (as in the proof of Proposition 2) the set of commutative diagrams

where now $\langle \tilde{f}, \tilde{s} \rangle \leq \langle \tilde{g}, \tilde{f} \rangle$. This gives us, in addition, a commutative diagram
with $v \in S$. By hypothesis then, $Gv \cdot Fv = 1$. Furthermore, the above pullback squares and the Beck-Chevalley property give us $F \hat{f} \cdot G \hat{s} = G \hat{s} \cdot F \hat{f}$ and $F \hat{g} \cdot G \hat{t} = G \hat{t} \cdot F \hat{g}$. Applying $F$ to $v \cdot \hat{f} = \hat{g}$ and $G$ to $v \cdot \hat{s} = \hat{t}$ we then obtain

$$
Ff \cdot Fs = Fk \cdot F\hat{f} \cdot Gs \cdot Gu = Fk \cdot Gs \cdot F\hat{f} \cdot Gu \\
= Fk \cdot Gs \cdot Fv \cdot F\hat{f} \cdot Gu = Fk \cdot Gt \cdot F\hat{g} \cdot Gu \\
= Fk \cdot F\hat{g} \cdot Gt \cdot Gu \\
= Fg \cdot Gt.
$$

Let us note immediately that our effort in considering the relation $\sim_a$ pays off only when $S$ is not restricted to containing only monomorphisms of $C$. Indeed, otherwise our construction returns just the category $\text{Span}(C, S)$, as studied earlier (see, for example, [5]):

**Corollary 2.** When $S$ is a class of monomorphisms, $(s, f) \sim_a (t, g)$ just means that the two $S$-spans are isomorphic. In other words, if $S$ contains only monomorphisms, $\text{Par}(C, S) = \text{Span}(C, S)$ is the $S$-span category.

**Proof.** Because of Remark 1(1), the Theorem gives us the functor

$$
\text{Par}(C, S) \rightarrow \text{Span}(C, S), \quad [s, f]_a \mapsto [s, f]_\sim = (s, f),
$$

which is trivially inverse to $[s, f]_\sim \mapsto [s, f]_a$. \hfill \square

**Remark 3.** It is to be expected that the largest class $S$ possible, namely $S = \text{Mor}(C)$, will render $\text{Par}(C, S)$ trivial. Concretely, it is easy to see that, similarly to Remark 1(2), one has:

If $C$ has disjoint finite coproducts (so that the pullback of two distinct coproduct injections is given by the initial object), then $\text{Par}(C, \text{Mor}(C))$ is equivalent to the terminal category.

In fact, since for all spans $(s, f) : A \rightarrow B$ one has $\langle \nu_1, \nu_2 \rangle \preceq \langle f, 1_B \rangle$ (with coproduct injections $\nu_1, \nu_2$), the following diagrams show $[s, f]_a = [^{1_A}!^B]_a$. 

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But also for certain quite small classes \( S \) (whose morphisms are typically \textit{epic} in \( \mathcal{C} \)) will \( \text{Par}(\mathcal{C}, S) \) be trivial, as we show next. We call the category \( \mathcal{C} \) \textit{strictly connected} if for all objects \( A \) the hom-functor \( \mathcal{C}(A, -) : \mathcal{C} \rightarrow \text{Set} \) reflects strictly initial objects (see Remark \( \text{II}(2) \)). As \( \emptyset \) is strictly initial in \( \text{Set} \), this means that, for all objects \( A, B \) the hom-set \( \mathcal{C}(A, B) \) may be empty only if \( B \) is strictly initial. (Note however, that there is no existence assumption for a strictly initial object when \( \mathcal{C} \) is strictly connected!) Every pointed category is trivially strictly connected, but also non-pointed categories like \( \text{Set}, \text{Ord}, \text{Cat}, \text{Top}, \ldots \), (all with strict initial object \( \emptyset \)) are strictly connected.

\textbf{Theorem 2.} Let \( \mathcal{C} \) have a terminal object \( 1 \) and be strictly connected, and let the class \( S \) contain the morphisms \( !_A : A \rightarrow 1 \), for all objects \( A \) that are not strictly initial. Then all hom-sets of \( \text{Par}(\mathcal{C}, S) \) contain only either one or two morphisms; they are all singletons when \( \mathcal{C} \) has no strictly initial object, in which case \( \text{Par}(\mathcal{C}, S) \) is equivalent to the terminal category \( 1 \).

\textit{Proof.} Consider an \( S \)-span \((s : D \rightarrow A, f : D \rightarrow B)\). If \( B \) is strictly initial, \( f \) is necessarily an isomorphism, thus making \( D \cong 0 \) strictly initial as well. Hence, the \( S \)-spans \((s, f)\) and \((!_A, !_B)\) coincide. If \( B \) is not strictly initial, we have \( !_B \in S \), so its pullback along \( !_D \) exists and gives us the direct product \( D \times B \), with projections \( p_1, p_2 \), where \( p_1 : D \times B \rightarrow D \) is in \( S \) by pullback stability of \( S \). The \( S \)-cospan inequality \( \langle f, 1_B \rangle \leq \langle !_D, !_B \rangle \) and the commutative diagrams
show \([s, f]_a = [s \cdot p_1, p_2]_a\). Hence, it suffices to consider the \(\mathcal{S}\)-span \((s \cdot p_1, p_2)\), with \(B\) not strictly initial.

If \(D \times B \cong 0\) is strictly initial, the \(\mathcal{S}\)-span \((s \cdot p_1, p_2)\) must equal \((!^A, !^B)\). If \(D \times B\) is not strictly initial, \(D\) cannot be strictly initial either, and we have a morphism \(a : A \to D\), by the strict connectedness of \(\mathcal{C}\). Also, just as the product \(D \times B\) exists, so does the product \(A \times B\), with product projections \(\pi_1, \pi_2\), where \(\pi_1 \in \mathcal{S}\), and we can consider the diagrams

The morphism \(s \times 1_B\) shows that \(A \times B\) is, like \(D \times B\), not strictly initial, so that the morphisms \(!^{A \times B}, !^{D \times B}\) both lie in \(\mathcal{S}\). This gives the \(\mathcal{S}\)-cospan inequalities
\[
\langle 1_{A \times B}, s \times 1_B \rangle \leq \langle !^{A \times B}, !^{D \times B} \rangle \geq \langle a \times 1_B, !^{D \times B} \rangle,
\]
which then imply \([s \cdot p_1, p_2]_a = [\pi_1, \pi_2]_a\).

In summary: when \(B\) is strictly initial in \(\mathcal{C}\), \([!^A, !^B]_a\) is the only morphism \(A \to B\) in \(\text{Par}(\mathcal{C}, \mathcal{S})\); otherwise one may also have the morphism \([\pi_1, \pi_2]_a\), but no other.

**Remark 4.** (1) In every category \(\mathcal{C}\) with finite products there is a least class \(\mathcal{S}\) which satisfies our general hypotheses and contains all morphisms \(A \to 1\), for \(A\) not strictly initial in \(\mathcal{C}\); namely, the class \(\text{Proj}(\mathcal{C})\) of all morphisms that are either projections of a direct product that is not strictly initial, or that are isomorphisms of strictly initial objects. Hence, when \(\mathcal{C}\) is strictly connected, the assertion of Theorem 2 applies for \(\mathcal{S} = \text{Proj}(\mathcal{C})\).

(2) Theorem 2 leaves open the question whether, when \(\mathcal{C}\) is strictly connected and has finite products and a strictly initial object \(0\), the \(\text{Par}(\mathcal{C}, \mathcal{S})\)-morphisms
\[
0_{A, B} := ![A, !^B]_a, \; 1_{A, B} := [\pi_1, \pi_2]_a : A \to B
\]
are actually distinct. For \( \mathcal{S} = \text{Proj}(\mathcal{C}) \) it is not difficult to show that, if every object in \( \mathcal{C} \) is projective with respect to \( \text{Proj}(\mathcal{C}) \), then

\[
0_{A,B} = 1_{A,B} \iff A \times B \text{ strictly initial.}
\]

In particular, for \( \mathcal{C} = \text{Set} \) and \( \mathcal{S} = \text{Proj(\text{Set})} \), one has \( 0_{A,B} \neq 1_{A,B} \) for all non-empty sets \( A, B \).

5. The category \( \mathcal{C}[\mathcal{S}^{-1}] \) of fractions

It is now easy to construct the category \( \mathcal{C}[\mathcal{S}^{-1}] \) of fractions with respect to the class \( \mathcal{S} \) satisfying our general hypotheses (but not necessarily being a class of monomorphisms of \( \mathcal{C} \)), as a quotient category of \( \text{Par}(\mathcal{C}, \mathcal{S}) \). Recall (\[4, 1\]) that the category \( \mathcal{C}[\mathcal{S}^{-1}] \) is characterized by the admission of a localizing functor \( \mathcal{C} \to \mathcal{C}[\mathcal{S}^{-1}] \), universal with the property that it maps morphisms in \( \mathcal{S} \) to isomorphisms.

In order to construct such localizing functor we consider the least equivalence relation \( \sim_{az} \) for \( \mathcal{S} \)-spans containing both the zig-zag relation \( \sim_z \) (Example 1) and the equivalence relation \( \sim_a \) generated by the companion relation \( \approx_a \) that is associated with the preorder of \( \preceq \) of \( \mathcal{S} \)-cospans (see above Definition 3). As both generating relations are compatible, the relation \( \sim_{az} \) is compatible as well, and we can consider the pair of functors

\[
\Phi_{az} : \mathcal{C} \to \text{Span}_{az}(\mathcal{C}, \mathcal{S}) \leftarrow \mathcal{S}^{\text{op}} : \Psi_{az}
\]

(defined as in Proposition 1) which, by definition of \( \sim_{az} \), factors through both \( \text{Span}_a(\mathcal{C}, \mathcal{S}) \) and \( \text{Span}_a(\mathcal{C}, \mathcal{S}) \). For all \( s \in \mathcal{S} \), this makes \( \Phi_{az,s} \) by Lemma 1 and Example 1 both a section and a retraction, whence an isomorphism, with \( \Psi_{az,s} \) being its inverse.

**Theorem 3.** \( \text{Span}_{az}(\mathcal{C}, \mathcal{S}) \) is (a model of) the category \( \mathcal{C}[\mathcal{S}^{-1}] \) of fractions with respect to the class \( \mathcal{S} \), with localizing functor \( \Phi_{az} \).

**Proof.** It just remains to be shown that any functor \( F : \mathcal{C} \to \mathcal{D} \) which maps every \( s \in \mathcal{S} \) to an isomorphism factors uniquely through \( \Phi_{az} \). By Theorem 1 and Corollary 1 with \( G : \mathcal{S}^{\text{op}} \to \mathcal{D}, s \mapsto (Fs)^{-1} \), we obtain a pair \( (F, G) \) that is both \( \sim_{az} \) and \( \sim_z \)-consistent and therefore also \( \sim_{az} \)-consistent. Since it trivially satisfies the Beck-Chevalley property, Proposition 1 produces the unique factorization of \( F \) through \( \Phi_{az} \), given by

\[
\text{Span}_{az}(\mathcal{C}, \mathcal{S}) \to \mathcal{D}, \quad [s, f]_{az} \mapsto Ff \cdot (Fs)^{-1}.
\]

\( \square \)
Since by definition $C[S^{-1}]$ is a quotient category of both, $\text{Span}_2(C,S) = \text{Retr}(C,S)$ and $\text{Span}_a(C,S) = \text{Par}(C,S)$, from Remark 2 and Theorem 2 we derive:

**Corollary 3.** Under each of the following two conditions, $C[S^{-1}]$ is equivalent to the terminal category 1:

(a) $C$ has an initial object 0, and $S$ contains all morphisms with domain 0;

(b) $C$ is strictly connected and contains a terminal object 1, but does not contain a strictly initial object, and $S$ contains all morphisms with codomain 1.

Here is an easy example for a class $S$ satisfying the preset general hypotheses but not trivializing $C[S^{-1}]$:

**Example 3.** In the category $\text{Ord}$ of preordered sets and their monotone (= order-preserving) maps, let $S$ be the class of fully faithful surjections $f : X \to Y$, i.e., of surjective maps $f$ with $(x \leq x' \iff f(x) \leq f(x'))$ for all $x, x' \in X$. Note that such maps are special equivalences of preordered sets, these being considered as small “thin” categories. We claim that $\text{Ord}[S^{-1}]$ is equivalent to the full reflective subcategory $\text{Pos}$ of $\text{Ord}$ of partially ordered sets and first show that the reflector $P : \text{Ord} \to \text{Pos}$ maps morphisms in $S$ to isomorphisms.

Indeed, with the axiom of choice granted, its surjectivity makes every $s : X \to Y$ in $S$ have a section $s'$ in $\text{Set}$ which, since $s$ is fully faithful, actually lives in $\text{Ord}$. Writing $(x \simeq \tilde{x} \iff x \leq \tilde{x}$ and $\tilde{x} \leq x)$ for all $x, \tilde{x} \in X$, so that the reflection of $X$ into $\text{Pos}$ may be taken to be the projection $p_X : X \to X/\simeq = PX$, from $s'(s(x)) \simeq x$ and $s(s'(y)) = y$ for all $x \in X, y \in Y$ we conclude that $Ps'$ is inverse to $Ps$ in $\text{Pos}$. Consequently, $P$ factors uniquely through $\Phi_{az}$, by the functor

$$\bar{P} : \text{Ord}[S^{-1}] \to \text{Pos}, \ [s,f]_{az} \mapsto Pf \cdot (Ps)^{-1} = P(f \cdot s').$$

We show that $\bar{P}$ is an equivalence of categories. Certainly, $\bar{P}$ is, like $P$, essentially surjective on objects. Noting that the reflection maps belong to $S$, for any monotone map $h : PX \to PY$ we have the monotone map $g := (p_Y)' \cdot h \cdot p_X : X \to Y$, so that $Pg \cdot p_X = p_Y \cdot g = p_Y \cdot (p_Y)' \cdot h \cdot p_X = h \cdot p_X$ and then $\bar{P}([1_X,g]_{az}) = Pg = h$ follows. Suppose that also $[s,f]_{az} : X \to Y$
satisfies $\bar{P}(\lfloor s, f \rfloor_{az}) = h$, so that $P(f \cdot s') = Pg$ and then $p_Y \cdot (f \cdot s') = p_Y \cdot g$. With $p_Y \in S$ one obtains the $S$-cospan inequalities

\[
\langle f \cdot s', 1_Y \rangle \leq \langle p_Y \cdot (f \cdot s'), p_Y \rangle = \langle p_Y \cdot g, p_Y \rangle \geq \langle g, 1_Y \rangle,
\]

which imply $[s, f]_a = [1_X, f \cdot s']_a = [1_X, g]_a$ and then $[s, f]_{az} = [1_X, g]_{az}$. This shows that $\bar{P}$ is fully faithful.

**Remark 5.** As a quotient category of $\text{Span}(C, S)$, in general the category $C[S^{-1}]$ may still fail to have small hom-sets. In fact, only few handy criteria are known that would guarantee its hom-sets to be small when $C$ has small hom-sets. One such criterion is the following (see, for example, [7], Theorem 19.3.1): With $C$ finitely complete, let $S$ be the class of morphisms mapped to isomorphisms by some functor $S : C \to B$ which preserves finite limits. If $S$ admits a so-called calculus of right fractions, then the hom-sets of $C[S^{-1}]$ are small. Moreover, the factorizing functor $\tilde{S} : C[S^{-1}] \to B$ with $\tilde{S} \Phi_{az} = S$ will not only be conservative (i.e., reflect isomorphisms), but also preserve finite limits (and, hence, be faithful).

6. The split restriction category $\text{RePar}(C, S)$

Cockett and Lack [3] show that the 2-category of categories $C$ equipped with a class $S$ of monomorphisms in $C$ satisfying our general hypotheses (together with functors and natural transformations compatible with the classes $S$) is 2-equivalent to the category of so-called split restriction categories (with functors and natural transformations compatible with the restriction structure). The 2-equivalence is furnished by $(C, S) \mapsto \text{Par}(C, S)$ which, when $S$ contains only monomorphisms, is the category ordinarily known as the category of $S$-partial maps in $C$ (see Corollary 2). However, without the mono constraint on $S$, while $\text{Par}(C, S)$ is characterized by the universal property given in Theorem [1], it remains unknown whether the category is a (split) restriction category; we suspect that it generally fails to be. Our goal is therefore to find a sufficiently large quotient category $\text{RePar}(C, S)$ of $\text{Par}(C, S)$ which is a (split) restriction category. For subsequent reference, let us first recall this notion in detail:

**Definition 4.** [3] A restriction structure on a category is an assignment

\[
\begin{align*}
& f : A \to B \\
& f : A \to A
\end{align*}
\]
of a morphism $\bar{f}$ to each morphism $f$, satisfying the following four conditions:

(R1) $f \cdot \bar{f} = f$ for all morphisms $f$;

(R2) $\bar{f} \cdot \bar{g} = \bar{g} \cdot \bar{f}$ whenever $\text{dom} f = \text{dom} g$;

(R3) $g \cdot \bar{f} = \bar{g} \cdot \bar{f}$ whenever $\text{dom} f = \text{dom} g$;

(R4) $\bar{g} \cdot f = f \cdot \bar{g} \cdot \bar{f}$ whenever $\text{cod} f = \text{dom} g$.

A category with a restriction structure is called a restriction category. A morphism $e$ such that $\bar{e} = e$ is called a restriction idempotent. A restriction idempotent $e$ is said to be split, if there are morphisms $m$ and $r$ such that $mr = e$ and $rm = 1$. One says that a restriction structure on a category is split if all the restriction idempotents are split.

For the construction of $\text{RePar}(\mathcal{C}, \mathcal{S})$, with the notation introduced at the beginning of Section 4, we consider the class

$\mathcal{S}^* := \text{closure under pullback of } \{v^* \mid v \text{ $\mathcal{S}$-cospan morphism, } v \in \mathcal{S}\}$

of $\mathcal{C}$-morphisms, consisting of all (existing) pullbacks in $\mathcal{C}$ of morphisms $v^*$ induced by $\mathcal{S}$-cospan morphisms $v$ with $v \in \mathcal{S}$; this, of course, is a pullback-stable collection of $\mathcal{C}$-morphisms containing all isomorphisms of $\mathcal{C}$.

**Remark 6.** (1) We note that, in general, $\mathcal{S}^*$ may not be comparable with $\mathcal{S}$ via inclusion; however, if $\mathcal{S}$ satisfies the weak left cancellation condition, so that $s \cdot t \in \mathcal{S}$ with $s \in \mathcal{S}$ implies $t \in \mathcal{S}$, then one has $\mathcal{S}^* \subseteq \mathcal{S}$. It turns out that under the provision of the weak left cancellation condition for $\mathcal{S}$, it suffices to define $\mathcal{S}^*$ as the closure of $\{v^* \mid v \text{ $\mathcal{S}$-cospan morphism, } v \in \mathcal{S}\}$ under pullback along $\mathcal{S}$-morphisms.

(2) When $\mathcal{S}$ is a class of monomorphisms, then $\mathcal{S}^*$ is the class of isomorphisms in $\mathcal{C}$. Indeed, when the morphism $v$ of the cube defining $v^*$ (at the beginning of Section 4) is monic, the pullback $(s', f')$ of $(f, s)$ serves also as a pullback for $(g, t)$, so that $v^*$ must be an isomorphism.

---

For all morphisms $f$, $\bar{f}$ is a restriction idempotent: consider $g = 1$ in (R3) and use $\bar{1} = 1$, from (R1).
Without imposing this cancellation condition we now modify the \( \leq \)-relation for \( S \)-spans of Example 1 and define the \( \leq^* \)-relation by
\[
(s, f) \leq^* (\tilde{s}, \tilde{f}) \iff \exists x : (s, f) \rightarrow (\tilde{s}, \tilde{f}), \ x \in S^*.
\]
Using the closure under pullback by \( S^* \) (along morphisms in \( S \) when \( S \) satisfies the weak left cancellation condition), one routinely proves that \( \leq^* \) is a compatible relation for \( S \)-spans. (We are, however, no longer being assured of its transitivity since \( S^* \) may fail to be closed under composition.) Hence, the least equivalence relation \( \sim^* \) containing \( \leq^* \) is also compatible. Writing just \( z^* \) when this modified zig-zag relation \( \sim^* \) is used as an index, we define:

**Definition 5.** The quotient category
\[
\text{RePar}(\mathcal{C}, S) := \text{Span}_{z^*} (\mathcal{C}, S)
\]

is called the \( S \)-partial map restriction category of \( \mathcal{C} \). It comes with the functors
\[
\Phi_{z^*} : \mathcal{C} \rightarrow \text{RePar}(\mathcal{C}, S) \quad \leftarrow \quad S^{\text{op}} : \Psi_{z^*} \\
(f : D \rightarrow B) \mapsto [1_D, f]_{z^*} \quad \quad [s, 1_D]_{z^*} \leftarrow (s : A \leftarrow D).
\]

Let us first confirm that \( \text{RePar}(\mathcal{C}, S) \) is indeed a quotient of \( \text{Par}(\mathcal{C}, S) = \text{Span}_a (\mathcal{C}, S) \):

**Lemma 2.** The relation \( \sim_a \) of Definition 3 is contained in \( \sim_{z^*} \).

*Proof.* Employing again the notation used in the proof of Theorem 1 when \( (s, f) \approx_a (t, g) \) we have an \( S \)-cospan morphism \( v : \langle \tilde{f}, \tilde{s} \rangle \rightarrow \langle \tilde{g}, \tilde{t} \rangle \) with \( v \in S \), which gives us the (vertical) \( S \)-span morphisms \( v^* : (\tilde{s}, \tilde{f}) \rightarrow (\tilde{t}, \tilde{g}) \) with \( v^* \in S^* \); consequently, \( (s, f) \sim_{z^*} (t, g) \). Since \( (s, f), (t, g) \) are obtained from \( (\tilde{s}, \tilde{f}), (\tilde{t}, \tilde{g}) \) by “horizontal whiskering”, \( (s, f) \sim_{z^*} (t, g) \) follows, by compatibility of \( \sim_{z^*} \).

The Lemma shows that the assignment \( [s, f]_a \mapsto [s, f]_{z^*} \) describes a functor
\[
\Gamma : \text{Par}(\mathcal{C}, S) \rightarrow \text{RePar}(\mathcal{C}, S),
\]
uniquely determined by \( \Phi_{z^*} \Gamma = \Phi_a, \Psi_{z^*} \Gamma = \Psi_a \) (cp. Theorem 1). Consequently, \( \text{RePar}(\mathcal{C}, S) \) is a quotient category of \( \text{Par}(\mathcal{C}, S) \); but it is nothing new when \( S \) is a class of monomorphisms, as follows from Remark 6(2):
Corollary 4. (1) RePar(\(C, S\)) \(\cong\) Par(\(C, S\))/\(\sim\), with \(\sim\) induced by \(\Gamma\).

(2) If \(S\) is a class of monomorphisms, then

\[
\text{RePar}(\mathcal{C}, S) = \text{Par}(\mathcal{C}, S) = \text{Span}(\mathcal{C}, S).
\]

Without any additional condition on \(S\) one can prove:

Theorem 4. RePar(\(C, S\)) is a split restriction category, with its restriction structure defined by

\[
\boxed{[s, f]_{z^*} = [s, s]_{z^*}},
\]

for all \(S\)-spans \((s, f)\).

Proof. Trivially, \((s, f) \leq^* (t, g)\) implies \((s, s) \leq^* (t, t)\). Thus, writing just \([s, f]\) for \([s, f]_{z^*}\) in what follows, \([s, f] = [t, g]\) implies \([s, s] = [t, t]\). As a consequence, \((\ )\) is well-defined. We note that Lemmas \(1\) and \(2\) imply \([s, 1] \cdot [1, s] = 1\), a crucial identity when we check (R1-4) below. Since trivially \([s, s] = [1, s] \cdot [s, 1]\), the identity also shows that \([s, s]\), once recognized as a restriction idempotent, splits.

(R1) For every morphism \([s, f]\) one has

\[
[s, f] \cdot [s, f] = [s, f] \cdot [s, s] = [1, f] \cdot [s, 1] \cdot [1, s] \cdot [s, 1] = [s, f].
\]

(R2) For morphisms \([s, f]\) and \([t, g]\) with the same domain, we form the pullback square \(s \cdot t' = t \cdot s'\) in \(\mathcal{C}\) and obtain the needed equality below:

\[
[s, f] \cdot [t, g] = [s, s] \cdot [t, t] = [t \cdot s', s \cdot t'] \quad = [s \cdot t', t \cdot s'] = [t, t] \cdot [s, s] = [\overline{t, g}] \cdot [\overline{s, f}].
\]

(R3) With the same notation as in (R2), we have

\[
[t, g] \cdot [s, f] = [t, g] \cdot [s, s] = [s \cdot t', s \cdot t'] = [t, t] \cdot [s, s] = [\overline{t, g}] \cdot [\overline{s, f}].
\]

(R4) For morphisms \([s, f] : A \to B\) and \([t, g] : B \to C\), we form the pullback square \(t \cdot f' = f \cdot t'\) in \(\mathcal{C}\) and obtain the needed equality below:

\[
[t, g] \cdot [s, f] = [t, t] \cdot [s, f] = [s \cdot t', t \cdot f'] = [s \cdot t', f \cdot t'] = [1, f] \cdot [s \cdot t', t'] \quad = [1, f] \cdot [s, 1] \cdot [1, s] \cdot [s \cdot t', t'] \quad = [s, f] \cdot [s \cdot t', s \cdot t'] = [s, f] \cdot [\overline{t, g}] \cdot [\overline{s, f}].
\]

\(\Box\)
Remark 7. We note that, whilst the arguments used in the proof above to check (R1-4) for RePar(\(\mathcal{C}, \mathcal{S}\)) would work equally well for Par(\(\mathcal{C}, \mathcal{S}\)), the given argumentation that the restriction structure in question is well defined would not survive the trade of \(\sim_{\ast}\) for \(\sim_{a}\).

Next we show that the functor \(\Gamma\) of Corollary \([\boxed{4}]\) is a localizing functor, mapping the morphisms of the class \(\Phi_{a}(\mathcal{S}^{\ast})\) (with \(\Phi_{a} : \mathcal{C} \to \text{Par}(\mathcal{C}, \mathcal{S})\), \(f \mapsto [1, f]_{a}\)) to isomorphisms of \(\text{Par}(\mathcal{C}, \mathcal{S})\), provided that \(\mathcal{S}^{\ast} \subseteq \mathcal{S}\) (see Remark \([\boxed{6}]\):

**Theorem 5.** If \(\mathcal{S}^{\ast} \subseteq \mathcal{S}\), in particular if \(\mathcal{S}\) satisfies the weak left cancellation condition, then \(\text{RePar}(\mathcal{C}, \mathcal{S})\) is a localization of \(\text{Par}(\mathcal{C}, \mathcal{S})\):

\[
\text{RePar}(\mathcal{C}, \mathcal{S}) \cong \text{Par}(\mathcal{C}, \mathcal{S})[\Phi_{a}(\mathcal{S}^{\ast})^{-1}];
\]

also, \(\text{Retr}(\mathcal{C}, \mathcal{S})\) is then a quotient category of \(\text{RePar}(\mathcal{C}, \mathcal{S})\), and \(\text{Retr}(\mathcal{C}, \mathcal{S}) \cong \mathcal{C}[\mathcal{S}^{-1}]\).

**Proof.** As in Example \([\boxed{1}]\) for \(x \in \mathcal{S}^{\ast}\), since \(x \in \mathcal{S}\) by hypothesis, we have \((x, x) \in \text{Span}(\mathcal{C}, \mathcal{S})\) and \((x, x) \leq_{\ast} (1, 1)\), hence, \([x, x]_{z \ast} = 1\). This implies \(\Gamma[1, x]_{a} \cdot \Gamma[x, 1]_{a} = \Gamma[x, x]_{a} = 1\), and since \([x, 1]_{a} \cdot [1, x]_{a} = 1\) by Lemma \([\boxed{1}]\) we see that \(\Gamma \Phi_{a} x = \Gamma[1, x]_{a} = [1, x]_{z \ast}\) is an isomorphism, with inverse \(\Gamma[x, 1]_{a} = [x, 1]_{z \ast}\).

Now consider any functor \(F : \text{RePar}(\mathcal{C}, \mathcal{S}) \to \mathcal{D}\) mapping all \(\Phi_{a} x (x \in \mathcal{S}^{\ast})\) to isomorphisms. We must confirm that \(F\) factors as \(F \Gamma = F\), for a unique functor \(F' : \text{RePar}(\mathcal{C}, \mathcal{S}) \to \mathcal{D}\). But since \(\Gamma\) is bijective on objects and full, this assertion becomes obvious once we have shown that \(F'\) is well defined when (by necessity) putting \(F'[s, f]_{z \ast} := F[s, f]_{a}\) for all \(\mathcal{S}\)-spans \((s, f)\). Considering \((s, f) \leq_{\ast} (t, g)\), so that \(s = t \cdot x, f = g \cdot x\) for some \(x \in \mathcal{S}^{\ast}\), we first note that \([x, 1]_{a} \cdot [1, x]_{a} = 1\) implies \(F[1, x]_{a} \cdot F[x, 1]_{a} = 1\) since \(F[1, x]_{a}\) is an isomorphism; consequently,

\[
F[s, f]_{a} = F[1, f]_{a} \cdot F[s, 1]_{a} = F[1, g]_{a} \cdot F[1, x]_{a} \cdot F[x, 1]_{a} \cdot F[t, 1]_{a} = F[t, g]_{a}.
\]

Since \(\leq_{\ast}\) generates the equivalence relation \(\sim_{\ast}\), well-definedness of \(F'\) follows.

The additional statement on the existence of quotient functors and on \(\text{Retr}(\mathcal{C}, \mathcal{S})\) serving as a model for \(\mathcal{C}[\mathcal{S}^{-1}]\) follows from the following obvious inclusions of the relevant equivalence relations: \(\mathcal{S}^{\ast} \subseteq \mathcal{S}\) implies \(\sim_{\ast} \subseteq \sim_{z}\) which, by Lemma \([\boxed{2}]\) gives \(\sim_{a} = \sim_{a z}\). \(\square\)
Remark 8. There is an easy generalization of the main statement of Theorem 5: instead of $S^*$ one considers any pullback-stable subclass $T$ of $S$ which contains $S^*$. Rather than $\leq^*$ we may then consider the $S$-span relation

$$(s, f) \leq_T (\bar{s}, \bar{f}) \iff \exists x : (s, f) \rightarrow (\bar{s}, \bar{f}), x \in T,$$

and its generated equivalence relation, the $T$-zig-zag relation $\sim_{zT}$. Hence, when we write just $z_T$ when $\sim_{zT}$ is used as an index, an easy adaptation of the above proof then shows

$$\text{Span}_{z_T}(C, S) \cong \text{Par}(C, S)[\Phi_a(T)^{-1}].$$

Now, under the hypothesis $S^* \subseteq S$, the choice $T = S^*$ gives Theorem 5 while the choice $T = S$ returns Theorem 3, presenting $C[S^{-1}]$ as $\text{Par}(C, S)[\Phi_a(S)^{-1}]$.

7. The split range category $\text{RaRePar}(C, S)$

Range categories, as introduced by Cockett, Guo and Hofstra in [2], enhance the notion of restriction category, in the sense that, in addition to the restriction operator $\underline{(-)}$, they carry also a so-called range operator $\hat{(-)}$, which behaves somewhat dually to the restriction operator, as follows:

Definition 6. [2]. A range structure on a restriction category is an assignment

$$f : A \rightarrow B \quad \hat{f} : B \rightarrow B$$

of a morphism $\hat{f}$ to each morphism $f$, satisfying the following four conditions:

(RR1) $\hat{\bar{f}} = \hat{f}$ for all morphisms $f$;

(RR2) $\hat{f} \cdot f = f$ for all morphisms $f$;

(RR3) $\hat{g} \cdot \hat{f} = \hat{g} \cdot \hat{f}$ whenever $\text{dom}(f) = \text{dom}(g)$;

(RR4) $\hat{g} \cdot \hat{f} = \hat{g} \cdot \hat{f}$ whenever $\text{codom}(f) = \text{dom}(g)$.

A restriction category equipped with a range structure is a range category; it is a split range category when it is split as a restriction category.
Our goal is now to find a sufficiently large quotient of $\text{RePar}(\mathcal{C}, S)$ which is a range category. To this end, throughout the rest of the paper, we assume that

*the class $S$ is part of a relatively stable orthogonal factorization system $(\mathcal{P}, S)$,*

so that, in addition to having $S$ being stable under pullback in $\mathcal{C}$, one has $\mathcal{P}$ being stable under pullback along $S$-morphisms. For every morphism $f$, we let

\[ f = s_f \cdot p_f \]

denote a (tacitly chosen) $(\mathcal{P}, S)$-factorization. As for every orthogonal factorization system, the general hypotheses on $S$ as listed in Section 2, now come for free, and $S$ is also weakly left cancellable (as defined in Remark 6). Consequently, for the pullback-stable class $S^*$ of Section 6, one has $S^* \subseteq S$. We denote by

\[ S^o \]

the least pullback-stable class $T$ with $S^* \subseteq T \subseteq S$ satisfying the additional $(\mathcal{P}, S)$-stability property

\[ \forall p, q \in \mathcal{P}, x \in S, y \in T (x \cdot q = p \cdot y \implies x \in T). \]

(Since this property, along with pullback stability, is stable under taking intersections and is trivially satisfied for $T = S$, there is such a class $S^o$.)

We can now define the desired quotient of $\text{Par}(\mathcal{C}, S)$ by choosing $T = S^o$ in Remark 8 and considering the zig-zag relation $\sim_{z^o}$, for which we write just $z^o$ when used as an index. It is the least equivalence relation containing the relation $\leq_{S^o}$, which we abbreviate as $\leq^o$.

**Definition 7.** We call

\[ \text{RaRePar}(\mathcal{C}, S) := \text{Span}_{z^o}(\mathcal{C}, S) \]

the $S$-partial map range category of $\mathcal{C}$.

Before confirming that this category is indeed a range category, we note that, since $S^* \subseteq S^o$, we have the functor

\[ \Lambda : \text{RePar}(\mathcal{C}, S) \to \text{RaRePar}(\mathcal{C}, S), \quad [s, f]_{S^*} \mapsto [s, f]_{z^o}. \]
Its induced equivalence relation presents its codomain as a quotient of its domain. Furthermore, with $\Gamma$ as defined before Corollary 4 from Remark 8 we obtain the first assertion of the following statement.

**Corollary 5.** (1) \( \text{RaRePar}(\mathcal{C}, \mathcal{S}) \cong \text{Par}(\mathcal{C}, \mathcal{S})[\Phi_\omega(\mathcal{S}^o)^{-1}] \) with localization,

\[ \Lambda \Gamma \colon \text{Par}(\mathcal{C}, \mathcal{S}) \to \text{RaRePar}(\mathcal{C}, \mathcal{S}). \]

(2) If $\mathcal{S}$ is a class of monomorphisms, then

\[ \text{RaRePar}(\mathcal{C}, \mathcal{S}) = \text{Par}(\mathcal{C}, \mathcal{S}) = \text{Span}(\mathcal{C}, \mathcal{S}). \]

**Proof.** (2) For $\mathcal{S}$ a class of monomorphisms, $\mathcal{S}^*$ is the class of isomorphisms in $\mathcal{C}$ (by Remark 8(2)), which trivially satisfies the additional property defining $\mathcal{S}^*$ (since $\mathcal{P}$, dually to $\mathcal{S}$, satisfies the weak right cancellation property, and $\mathcal{P} \cap \mathcal{S}$ is the class of isomorphisms). Consequently, also $\mathcal{S}^*$ is the class of isomorphisms in $\mathcal{C}$. \qed

As a quotient of the split restriction category $\text{RePar}(\mathcal{C}, \mathcal{S})$, $\text{RaRePar}(\mathcal{C}, \mathcal{S})$ is a split restriction category too, with its restriction structure given by

\[ [s, f]_{\mathcal{S}^o} = \Lambda [s, s]_{\mathcal{S}^*} = [s, s]_{\mathcal{S}^o} \]

for all $\mathcal{S}$-spans $(s, f)$. Now we show:

**Theorem 6.** $\text{RaRePar}(\mathcal{C}, \mathcal{S})$ is a split range category, with its range structure defined by

\[ [s, f]_{\mathcal{S}^o} = [s_f, s_f]_{\mathcal{S}^o} \]

for all $\mathcal{S}$-spans $(s, f)$, where $s_f$ belongs to the $(\mathcal{P}, \mathcal{S})$-factorization of $f = s_f \cdot p_f$.

**Proof.** To show that $\widehat{(-)}$ is well-defined, we consider $\mathcal{S}$-spans $(s, f), (t, g)$ with $(s, f) \leq^o (t, g)$, so that there exists a morphism $x \in \mathcal{S}^o$ with $s = t \cdot x$, $f = g \cdot x$. We have the diagonal morphism $d$ with $s_g \cdot d = s_f$ and $d \cdot p_f = p_g \cdot x$. By weak left cancellation, the first identity gives $d \in \mathcal{S}$, so that the second identity then implies $d \in \mathcal{S}_o$. Since $s_g \cdot d = s_f$, so that $(s_f, s_f) \leq^o (s_g, s_g)$, well-definedness of $\widehat{(-)}$ follows.

To now check (RR1-RR4), we write $[s, f]$ for $[s, f]_{\mathcal{S}^o}$.

(RR1) holds trivially since $[s, s]$ is a restriction idempotent, for all $s \in \mathcal{S}$. 26
(RR2) For an \( S \)-span \((s, f)\) with \((P, S)\)-factorization \(f = s_f \cdot pf\) and \((u, v)\) the kernel pair of \(s_f\), one has
\[
(s_f, s_f) \cdot (s, f) = (s_f, s_f) \cdot (1, s_f) \cdot (s, pf) = (u, s_f \cdot v) \cdot (s, pf) = (1, s_f) \cdot (u, v) \cdot (s, pf)
\]
in \(\text{Span}(C, S)\). Since \([u, v]_a = 1\) by Lemma \(\blacksquare\) also \([u, v]_{a^2} = 1\), and one concludes
\[
\widehat{[s, f]} \cdot [s, f] = [s_f, s_f] \cdot [s, f] = [1, s_f] \cdot [u, v] \cdot [s, pf] = [1, s_f] \cdot [s, pf] = [s, f].
\]

(RR3) For composable \( S \)-spans \((s, f), (t, g)\) we must show \([\hat{t}, g] \cdot [s, f] = [t, g] \cdot [s, f]\), where \([t, g] = [t, t]\). But the consecutive pullback diagrams
\[
\begin{array}{ccc}
& & s_f \\
& v' & \downarrow \phi_f \\
\hat{t} & \searrow & t \\
\downarrow \phi_f & & \downarrow t \\
& & s_f
\end{array}
\]
in \(C\) and the \(S\)-stability of \(P\) show
\[
[t, t] \cdot [s, f] = [t \cdot s'_f, t \cdot s'_f] = [s_f \cdot v', t \cdot s'_f] = [t, t] \cdot [s_f, s_f] = [t, t] \cdot [s, f].
\]

(RR4) Using the same notation as in (RR3) we just observe that the \(S\)-part of the \((P, S)\)-factorization of \(g \cdot s'_f\) serves also as the \(S\)-part of the \((P, S)\)-factorization of \(g \cdot s'_f \cdot p'_f\). But this observation implies immediately the desired equality \([\hat{t}, g][s, f] = [t, g][s, f]\). \(\square\)

The following chart summarizes our constructions under the provisions
of this section:
\[ S \subseteq \mathcal{C} \]
\[ \text{Span}(\mathcal{C}, S) \]
\[ \text{Span}_a(\mathcal{C}, S) \xrightarrow{\text{Par}(\mathcal{C}, S)} \text{Par}(\mathcal{C}, S) \]
\[ \text{Span}_{a^*}(\mathcal{C}, S) \xrightarrow{\text{RePar}(\mathcal{C}, S)} \text{Par}(\mathcal{C}, S)[\Phi_a(S^*)^{-1}] \]
\[ \text{Span}_{a^*}(\mathcal{C}, S) \xrightarrow{\text{RaRePar}(\mathcal{C}, S)} \text{Par}(\mathcal{C}, S)[\Phi_a(S^*)^{-1}] \]
\[ \text{Span}_a(\mathcal{C}, S) \xrightarrow{\text{Retr}(\mathcal{C}, S)} \text{Par}(\mathcal{C}, S)[\Phi_a(S)^{-1}] = \mathcal{C}[S^{-1}] = \text{Span}_{az}(\mathcal{C}, S) \]

8. Split range categories vs. relatively stable factorization systems

Extending some results obtained in \[3, 2\] we now provide a setting which presents \((\mathcal{C}, S) \mapsto \text{RaRePar}(\mathcal{C}, S)\) as the left adjoint to the formation of the category \(\text{Total}(\mathcal{X})\) for every split range category \(\mathcal{X}\). In particular, the category \(\text{RaRePar}(\mathcal{C}, S)\) will be characterized by a universal property.

Recall that, for a restriction category \(\mathcal{X}\) with restriction operator \((-)\), a morphism \(f\) in \(\mathcal{X}\) is called total if \(\bar{f} = 1\). As identity morphisms and compositions of total morphisms are total, one obtains the category \(\text{Total}(\mathcal{X})\), which has the same objects as \(\mathcal{X}\). Any functor \(F : \mathcal{X} \to \mathcal{Y}\) which preserves the restriction operations of the categories restricts to a functor \(F : \text{Total}(\mathcal{X}) \to \text{Total}(\mathcal{Y})\), and any (componentwise) total natural transformation \(\alpha : F \to G\) of such functors keeps this role under the passage to total categories.

Recall further that \(i\) in \(\mathcal{X}\) is a restriction isomorphism if, for some morphism \(i^-\), one has \(i^- \cdot i = \bar{i}\) and \(i \cdot i^- = \bar{i}^-\); such \(i^-\) is unique and called the restricted inverse of \(i\). We denote the class of restriction isomorphisms in \(\text{Total}(\mathcal{X})\) by \(\text{Relso}(\mathcal{X})\). Remarkably, as shown in Proposition 3.3 of \[3\], when \(\mathcal{X}\) is a split restriction category, the pullback \(j\) of \(i \in \text{Relso}(\mathcal{X})\) along any total morphism \(f\) exists in \(\text{Total}(\mathcal{X})\) and belongs to \(\text{Relso}(\mathcal{X})\) again: \(j\) is part
of the splitting of the restriction idempotent $\overline{i \cdot f} = j \cdot r$ where $r \cdot j = 1$, producing the pullback diagram

$$
\begin{array}{ccc}
\overline{i \cdot f} & \xrightarrow{\downarrow} & i \\
\downarrow \downarrow & & \downarrow \downarrow \\
\overline{j} & \xrightarrow{\downarrow} & \overline{i} \\
\end{array}
$$

If now $\mathcal{X}$ is a split range category with range operator $\widehat{(-)}$, then $\hat{f} = 1$ implies $\overline{i \cdot f} \cdot j = 1$. Hence, as Theorem 4.7 of [2] shows, the class $\text{RaSur}(\mathcal{X}) = \{f \mid \overline{f} = 1, \hat{f} = 1\}$ of range surjections in $\text{Total}(\mathcal{X})$ is stable under pullback along $\text{Relso}(\mathcal{X})$; moreover, $(\text{RaSur}(\mathcal{X}), \text{Relso}(\mathcal{X}))$ is an orthogonal factorization system of $\text{Total}\mathcal{X}$ where, as a class of sections, the class $\text{Relso}(\mathcal{X})$ is trivially a class of monomorphisms in $\text{Total}(\mathcal{X})$.

As in [2], but without any restriction to monomorphisms, we form the (very large) 2-category

$$
\text{StableFactS}
$$

of relatively stable factorization systems. Its objects are triples $(C, P, S)$, where $C$ is a category equipped with an orthogonal factorization system $(P, S)$, such that $C$ has pullbacks along $S$-morphisms and $P$ is stable under them; its morphisms $F : (C, P, S) \to (D, Q, T)$ are functors $F : C \to D$ with $F(P) \subseteq Q$ and $F(S) \subseteq T$ which preserve pullbacks along $S$-morphisms; 2-cells are natural transformations whose naturality squares involving $S$-morphisms are pullback squares.

$$
\text{StRangeCats}
$$

denotes the (very large) 2-category of split range categories, with their range-preserving restriction functors and total natural transformations. Then, as in [2], we have the 2-functor

$$
\text{Total} : \text{StRangeCats} \longrightarrow \text{StableFactS}
$$
where \( \text{Total}(F) \) is the restriction of \( F \), which we may write simply as \( F \) again.

Our aim is to show that there is a left adjoint, that takes \((C, P, S)\) to \( \text{RaRePar}(C, S) \). (We write \( \text{RaRePar}(C, S) \) for \( \text{RaRePar}(C, P, S) \) since \( P \) is determined by \( C \) and \( S \).) For that, we first show (in extension of the notation of Section 7):

**Lemma 3.** For every functor \( F : (C, P, S) \to (D, Q, T) \) in \( \text{StableFactS} \) one has

\[
F(S) \subseteq (F(S^*))^0 \subseteq T^0,
\]

where \((F(S^*))^0 \) is the least pullback-stable class \( V \) in \( D \) with \( F(S^*) \subseteq V \subseteq T^0 \) satisfying the \((Q, T)\)-stability property.

**Proof.** Since \( F \) transforms pullbacks of \( S \)-morphisms into pullbacks of \( T \)-morphisms, for every morphism \( v \) of \( S \)-cospans one has (in the notation of Section 4) \( F(v^*) = (Fv)^* \). This implies \( F(S^*) \subseteq T^* \) and then \((F(S^*))^0 \subseteq T^0 \).

To prove the other inclusion claimed, for any class \( V \) as in the Lemma we form the class \( U = F^{-1}(V) \cap S \), which trivially satisfies \( S^* \subseteq U \subseteq S \), as well as the \((P, S)\)-stability property. Consequently, \( S^0 \subseteq U \), and then \( F(S^0) \subseteq F(U) \subseteq V \). With this last inclusion holding for all \( V \), \( F(S^0) \subseteq (F(S^*))^0 \) follows. \( \Box \)

As a consequence of Lemma 3, every \( F : (C, P, S) \to (D, Q, T) \) in \( \text{StableFactS} \) gives us the well-defined range-preserving restriction functor

\[
\text{RaRePar}(F) : \text{RaRePar}(C, S) \to \text{RaRePar}(D, T), \quad [s, f]_{x^0} \mapsto [F s, F f]_{x^0}.
\]

The resulting functor \( \text{RaRePar} : \text{StableFactS} \to \text{StRangeCats} \) is easily seen to be actually a 2-functor; it sends a 2-cell \( \alpha : F \Rightarrow G \) to the total natural transformation \([1, \alpha] : \text{RaRePar}(F) \Rightarrow \text{RaRePar}(G) \) whose component at \( A \) in \( C \) is defined by \([1, \alpha]_A = [1_{FA}, \alpha_A] \). We claim that \( \text{RaRePar} \) is left adjoint to \( \text{Total} \):

**Theorem 7.** There is a 2-adjunction

\[
\text{RaRePar} \dashv \text{Total} : \text{StRangeCats} \to \text{StableFactS}.
\]

**Proof.** To construct the unit \( \eta : \text{Id}_{\text{StableFactS}} \to \text{Total} \circ \text{RaRePar} \) at \((C, P, S)\) in \( \text{StableFactS} \), since in the notation of Sections 4 and 7 the functor

\[
C \xrightarrow{\Phi^*} \text{Par}(C, S) \xrightarrow{\Delta^*} \text{RaRePar}(C, S), \quad f \mapsto [1, f] = [1, f]_{x^0},
\]
has total values, we consider its restriction,

\[ \eta_{(C, P, S)} : (C, P, S) \rightarrow \text{Total} (\text{RaRePar}(C, S)). \]

First we show that \( \eta_{(C, P, S)} \) lives in \( \text{StableFactS} \). Certainly, for \( p \in P \), \([1, p]\) is total and \( \hat{[1, p]} = 1 \), so that \([1, p] \in \text{RaSur}(\text{RaRePar}(C, S)) \). Likewise, for \( s \in S \), one easily sees \([1, s] \in \text{Relso}(\text{RaRePar}(C, S)) \). Furthermore, given the left pullback square one obtains the pullback square on the right,

\[
\begin{array}{ccc}
{s'} & \xrightarrow{f'} & s \\
\downarrow & & \downarrow \\
{f} & & {s} \\
\end{array}
\]

living in the split restriction category \( \text{RaRePar}(C, S) \). But, as one easily confirms, the top row of that pullback square equals \([1, f']\), so that the right diagram is in fact the \( \eta_{(C, P, S)} \)-image of the left diagram.

For 1-cells \( F, G : (C, P, S) \rightarrow (D, Q, T) \) and a 2-cell \( \alpha : F \Rightarrow G \), we need to show the commutativity of the following diagram, both at the 1-cell and 2-cell levels.

\[
\begin{array}{ccc}
(C, P, S) & \xrightarrow{\eta_{(C, P, S)}} & \text{Total}(\text{RaRePar}(C, S)) \\
\downarrow F \Rightarrow \alpha & & \downarrow \text{Total}(\text{RaRePar}(F)) \Rightarrow \text{Total}(\text{RaRePar}(G)) \\
(D, Q, T) & \xrightarrow{\eta_{(D, Q, T)}} & \text{Total}(\text{RaRePar}(D, T)) \\
\end{array}
\]

Since for every morphism \( f \) in \( C \) one has

\[ \eta_{(D, Q, T)}(Ff) = [1, Ff] = \text{RaRePar}(F)([1, f]) = \text{RaRePar}(F)(\eta_{(C, P, S)}(f)) \]

thus showing commutativity at the 1-cell level:

\[ \eta_{(D, Q, T)} \circ F = \text{Total}(\text{RaRePar}(F)) \circ \eta_{(C, P, S)}. \]

At the 2-cell level, commutativity follows easily as well since, for all objects \( A \) in \( C \), one has

\[ \eta_{(D, Q, T)}(\alpha_A) = [1, \alpha_A] = [1, \alpha]_{\eta_{(C, P, S)}}(A). \]
Next we define the counit $\varepsilon : \text{RaRePar} \circ \text{Total} \to \text{Id}_{\text{SplitRangeCats}}$. For a split range category $\mathcal{X}$, since $\text{Relso}(\mathcal{X})$ is a collection of monomorphisms, one has $\text{RaRePar}(\text{Total}(\mathcal{X}), \text{Relso}(\mathcal{X})) = \text{Par}(\text{Total}(\mathcal{X}), \text{Relso}(\mathcal{X}))$ (see Corollary 5.2), and one may define the functor $\varepsilon_{\mathcal{X}} : \text{RaRePar}(\text{Total}(\mathcal{X})) \to \mathcal{X}$ as in Theorem 3.4 of [3], by simply taking $[s, f]$ to $f \cdot s^-$. To confirm that $\varepsilon$ is 2-natural, we consider 1-cells $H, K : \mathcal{X} \to \mathcal{Y}$ of split range categories and a 2-cell $\beta : H \Rightarrow K$ and show the commutativity of the following diagram at both, the 1-cell and 2-cell levels.

At the 1-cell level, for every morphism $[s, f]$ in $\text{RaRePar}(\text{Total}(\mathcal{X}))$, we have

$$\varepsilon_{\mathcal{Y}}(\text{RaRePar}(H)([s, f])) = \varepsilon_{\mathcal{Y}}([Hs, Hf]) = Hf \cdot (Hs)^- = Hf \cdot H(s^-) = H(f \cdot s^-) = H(\varepsilon_{\mathcal{X}}([s, f])).$$

At the 2-cell level, for every object $X$ in $\mathcal{X}$, we just note that

$$(\varepsilon_{\mathcal{Y}}[1, \beta])_X = \varepsilon_{\mathcal{Y}}([1, \beta_X]) = \beta_X = \beta_{\varepsilon_{\mathcal{X}}(X)} = (\beta_{\varepsilon_{\mathcal{X}}})_X.$$ 

Finally, since the composite functor

$$\text{Total}(\mathcal{X}) \xrightarrow{\eta_{\text{Total}(\mathcal{X})}} \text{Total}(\text{RaRePar}(\text{Total}(\mathcal{X}))) \xrightarrow{\text{Total}(\varepsilon_{\mathcal{X}})} \text{Total}(\mathcal{X})$$

is described by $f \mapsto [1, f] \mapsto f$, the first triangular identity for the adjunction holds trivially. For the second one, we see that the composite functor

$$\text{RaRePar}(C, \mathcal{P}, S) \xrightarrow{\text{RaRePar}(\eta_{(C, \mathcal{P}, S)})} \text{RaRePar}(\text{Total}(\text{RaRePar}(C, \mathcal{P}, S))) \cdots$$

is described by

$([s, f] \mapsto [[1, s], [1, f]]) \mapsto [1, f][s, 1] = [s, f],$

so that it maps identically as well. \hfill \Box
Remark 9. We note that the counit $\varepsilon_X$ at the split range category $X$ as described in the above proof is actually an isomorphism (see Theorem 3.4 of [3]), so that $\text{StRangeCats}$ may be considered as a full reflective subcategory of $\text{StableFactS}$.

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