ON WELL-CONDITIONED SPECTRAL COLLOCATION AND SPECTRAL METHODS BY THE INTEGRAL REFORMULATION

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Abstract. Well-conditioned spectral collocation and spectral methods have recently been proposed to solve differential equations. In this paper, we revisit the well-conditioned spectral collocation methods proposed in [T. A. Driscoll, J. Comput. Phys., 229 (2010), pp. 5980-5998] and [L.-L. Wang, M. D. Samson, and X. Zhao, SIAM J. Sci. Comput., 36 (2014), pp. A907–A929], and the ultraspherical spectral method proposed in [S. Olver and A. Townsend, SIAM Rev., 55 (2013), pp. 462–489] for an mth-order ordinary differential equation from the viewpoint of the integral reformulation. Moreover, we propose a Chebyshev spectral method for the integral reformulation. The well-conditioning of these methods is obvious by noting that the resulting linear operator is a compact perturbation of the identity. The adaptive QR approach for the ultraspherical spectral method still applies to the almost-banded infinite-dimensional system arising in the Chebyshev spectral method for the integral reformulation. Numerical examples are given to confirm the well-conditioning of the Chebyshev spectral method.

Key words. Spectral collocation method, ultraspherical spectral method, Chebyshev spectral method, integral reformulation, well-conditioning

AMS subject classifications. 65N35, 65L99, 33C45

1. Introduction. In recent decades, spectral collocation and spectral methods have been extensively used for solving differential equations because of their high order of accuracy; see, for example, [16, 39, 1, 2, 34] and the references therein. However, classical spectral collocation and spectral methods for differential equations lead to ill-conditioned matrices. For example, the condition number of the matrix in the rectangular spectral collocation method [8] for an mth-order differential operator grows like $O(N^{2m})$, where $N$ is the number of collocation points. Preconditioning techniques (see, for example, [5, 6, 25, 26, 21, 31, 32, 14]) and spectral integration techniques (see, for example, [19, 27, 7, 15, 40]) have been employed to improve the conditioning.

In this paper, from the viewpoint of the integral reformulation, we revisit recently proposed well-conditioned spectral collocation methods [7, 41] and the ultraspherical spectral method [29] for solving the mth-order linear ordinary differential equation

\begin{equation}
\frac{d^m}{dx^m} u(x) + \sum_{k=0}^{m-1} a^k(x) u^{(k)}(x) = f(x),
\end{equation}

together with $m$ linearly independent constraints

\begin{equation}
Bu = b.
\end{equation}

Here, $a^k(x)$, $u(x)$ and $f(x)$ are suitably smooth functions defined on $[-1,1]$ and $b$ denotes a constant $m$-vector.

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Define the integral operators:

$$
\partial_x^{-1}\phi(x) = \int_{-1}^{x} \phi(t)dt; \quad \partial_x^{-k}\phi(x) = \partial_x^{-1}\left(\partial_x^{-(k-1)}\phi(x)\right), \quad k \geq 2.
$$

Let \( v(x) = u^{(m)}(x) \). Employing (1.2), we can write

$$
u(x) = \partial_x^{-m}v(x) + \mathcal{X}(B\mathcal{X})^{-1}(b - B\partial_x^{-m}v(x)),
$$

where

$$
\mathcal{X} = \begin{bmatrix} 1 & x^1 & \cdots & x^{m-1} \end{bmatrix}.
$$

Here, \( \mathcal{X} \) can be replaced by any basis for the vector space of polynomials of degree less than \( m \). We use this \( \mathcal{X} \) for clarity. The problem (1.1)-(1.2) can be written as the following integral reformulation

$$
(1.3) \quad v(x) + \sum_{k=0}^{m-1} a^k(x)\partial_x^{k-m}v(x) - A(B\mathcal{X})^{-1}B\partial_x^{-m}v(x) = f(x) - A(B\mathcal{X})^{-1}b,
$$

where

$$
A = \begin{bmatrix} a^0(x) & a^0(x)x + a^1(x) & \cdots & \sum_{k=0}^{m-1} \frac{(m-1)!}{(m-1-k)!} a^k(x)x^{m-1-k} \end{bmatrix}.
$$

We rewrite (1.3) as a linear operator equation

$$
(1.4) \quad (I - K)v(x) = g(x),
$$

where \( I \) denotes the identity operator,

$$
Kv = -\sum_{k=0}^{m-1} a^k(x)\partial_x^{k-m}v(x) + A(B\mathcal{X})^{-1}B\partial_x^{-m}v(x),
$$

and

$$
g(x) = f(x) - A(B\mathcal{X})^{-1}b.
$$

If the linear operator \( B \) is bounded, then it is easy to show that \( K \) is a compact operator. As noted in [33, 19, 20, 36], the linear system in the Galerkin method [22] for (1.4) is well-conditioned if both \( \|I - K\| \) and \( \|(I - K)^{-1}\| \) are \( \mathcal{O}(1) \). [20, 36] The automatic spectral collocation method [7] for the problem (1.1)-(1.2) is a Chebyshev spectral collocation scheme based on the integral equation (1.3), which has been implemented in the Chebfun software system [9]. In this paper, we propose a Chebyshev spectral method for the integral reformulation (1.3).

The main contributions of this paper are in order. (I): Solving the Birkhoff interpolation problem (2.4) by the integral formulation (2.5) and deriving it from a right preconditioning rectangular spectral collocation method, we show that the well-conditioned spectral collocation method [41] is essentially a stable implementation of the spectral collocation discretization of the integral equation (1.3). (II): We show that the Chebyshev spectral method proposed in this work for the integral equation (1.3) leads to an almost-banded infinite-dimensional system (see (4.2)), which can be solved by the adaptive QR approach in [29, 30]. Because only Chebyshev series are
involved in the Chebyshev spectral method, no conversion operators $S_k$ (see §3) are required. We only need the multiplication operator $\mathcal{M}_0[a]$ (see §3), which represents multiplication of two Chebyshev series. Therefore, the Chebyshev spectral method is very easy to implement. We show that the Chebyshev spectral method for the integral equation (1.3) is a two-sided preconditioned version of the discrete operator equation (3.3) in the ultraspherical spectral method [29] for the differential problem (1.1)-(1.2). The Chebyshev spectral method is obviously well-conditioned and avoids the drawbacks mentioned in [29, §1.1.4]. Finally, for large bandwidth variable coefficients, iterative solvers can be employed to achieve computational efficiency because the involved matrix-vector product in the Chebyshev spectral method can be obtained in $O(n \log n)$ operations, where $n$ is the truncation parameter.

The rest of the paper is organized as follows. In §2, we derive the well-conditioned spectral collocation method [41] for (1.1)-(1.2) by utilizing a right preconditioning approach and show that it is a stable implementation of the spectral collocation discretization of the integral equation (1.3). In §3, we review the ultraspherical spectral method [29]. We also point out the relation (see (3.2)) between the multiplication operator $\mathcal{M}_0[a]$ representing multiplication of two Chebyshev series and the multiplication operator $\mathcal{M}_k[a]$ representing multiplication of two ultraspherical $(C^{(k)})$ series. In §4, utilizing the multiplication operator $\mathcal{M}_0[a]$ and the spectral integration operator [19], we present the Chebyshev spectral method and show the relation with the ultraspherical spectral method. Numerical examples confirming the well-conditioning of the Chebyshev spectral method are given in §5. We present brief concluding remarks in §6.

2. A well-conditioned spectral collocation method. Let \( \{x_j\}_{j=0}^N \) be a set of points satisfying
\[
-1 \leq x_0 < x_1 < \cdots < x_{N-1} < x_N \leq 1,
\]
and \( \{y_j\}_{j=0}^M \) be another set of distinct points satisfying
\[
-1 \leq y_0 < y_1 < \cdots < y_{M-1} < y_M \leq 1.
\]
The barycentric weights associated with the points \( \{x_j\}_{j=0}^N \) are defined by
\[
w_{j,x} = \prod_{n=0, n \neq j}^N (x_j - x_n)^{-1}, \quad j = 0, 1, \ldots, N.
\]
The barycentric resampling matrix [8], \( P^{x \rightarrow y} \in \mathbb{R}^{(M+1)\times(N+1)} \), which interpolates between the points \( \{x_j\}_{j=0}^N \) and \( \{y_j\}_{j=0}^M \), is defined by
\[
P^{x \rightarrow y} = [p_{ij}^{x \rightarrow y}]_{i=0,j=0}^{M,N},
\]
where
\[
p_{ij}^{x \rightarrow y} = \begin{cases} \frac{w_{j,x}}{y_i - x_j} \left( \sum_{l=0}^N \frac{w_{l,x}}{y_l - x_l} \right)^{-1}, & y_i \neq x_j, \\ 1, & y_i = x_j. \end{cases}
\]

**Lemma 2.1.** If \( N \geq M \), then \( P^{x \rightarrow y} P^{y \rightarrow x} = I_{M+1} \).
2.1. Pseudospectral differentiation matrices. The Lagrange interpolation basis polynomials of degree \( N \) associated with the points \( \{x_j\}_{j=0}^N \) are defined by
\[
\ell_j(x) = w_{j,x} \prod_{n=0, n \neq j}^N (x - x_n), \quad j = 0, 1, \ldots, N,
\]
where \( w_{j,x} \) is the barycentric weight (2.3). Define the pseudospectral differentiation matrices:
\[
D^{(k)}_{x \rightarrow y} = [k \ell_j(x_i)]_{i,j=0}^N, \quad D^{(k)}_{x \rightarrow x} = [k \ell_j(x)]_{i,j=0}^N,
\]
There hold
\[
D^{(k)}_{x \rightarrow y} = P_{x \rightarrow y} D^{(k)}_{x \rightarrow x}, \quad D^{(0)}_{x \rightarrow y} = P_{x \rightarrow y},
\]
and
\[
D^{(k)}_{x \rightarrow x} = \left(D^{(1)}_{x \rightarrow x}\right)^k, \quad k \geq 1.
\]
The matrix \( D^{(k)}_{x \rightarrow y} \) is called a rectangular \( k \)-th order differentiation matrix, which maps values of a polynomial defined on \( \{x_j\}_{j=0}^N \) to the values of its \( k \)-th order derivative on \( \{y_j\}_{j=0}^M \). Explicit formulae and recurrences for pseudospectral differentiation matrices can be found in, for example, [42, 43].

2.2. Pseudospectral integration matrices. Given \( \{y_j\}_{j=0}^M \) and \( b \) with \( m \geq 1 \), we consider the Birkhoff-type interpolation problem:
\[
\text{(2.4) Find } p(x) \in \mathbb{P}_{M+m} \text{ such that } \begin{cases} p^{(m)}(y_j) = u^{(m)}(y_j), & j = 0, \ldots, M, \\
Bp = b. \end{cases}
\]
Let \( \{\ell_{j,y}(x)\}_{j=0}^M \) be the Lagrange interpolation basis polynomials of degree \( M \) associated with the points \( \{y_j\}_{j=0}^M \). Then the Birkhoff-type interpolation polynomial takes the form
\[
p(x) = \sum_{j=0}^M u^{(m)}(y_j) \partial_x^{-m} \ell_{j,y}(x) + \sum_{i=0}^{m-1} \alpha_i x^i,
\]
where \( \alpha_i \) can be determined by the linear constraints \( Bp = [b_1 \ b_2 \ \cdots \ b_m]^T \). Obviously, the existence and uniqueness of the Birkhoff-type interpolation polynomial is equivalent to that of \( \{\alpha_i\}_{i=0}^{m-1} \). After obtaining \( \alpha_i \), we can rewrite (2.5) as
\[
p(x) = \sum_{j=0}^M u^{(m)}(y_j) B_{j,y}(x) + \sum_{j=1}^m b_j B_{M+j,y}(x).
\]

Now we give some examples for \( \{B_{j,y}\}_{j=0}^{M+m} \). The first-order Birkhoff-type interpolation problem takes the form:
\[
\text{Find } p(x) \in \mathbb{P}_{M+1} \text{ such that } \begin{cases} p'(y_j) = u'(y_j), & j = 0, 1, \ldots, M, \\
Bp = b_1. \end{cases}
\]
• Given \( Bp := ap(-1) + bp(1) \) with \( a + b \neq 0 \), we have
\[
B_{j,y}(x) = \partial_x^{-1}\ell_{j,y}(x) - \frac{b}{a + b} \int_{-1}^{1} \ell_{j,y}(x)dx, \quad j = 0, 1, \ldots, M,
\]
\[
B_{M+1,y}(x) = \frac{1}{a + b}.
\]

• Given \( Bp := \int_{-1}^{1} p(x)dx \), we have
\[
B_{j,y}(x) = \partial_x^{-1}\ell_{j,y}(x) - \frac{1}{2} \int_{-1}^{1} \partial_x^{-1}\ell_{j,y}(x)dx, \quad j = 0, 1, \ldots, M,
\]
\[
B_{M+1,y}(x) = \frac{1}{2}.
\]

The second-order Birkhoff-type interpolation problem takes the form:

Find \( p(x) \in \mathbb{P}_{M+2} \) such that \( \{ p''(y_j) = u''(y_j), \quad j = 0, 1, \ldots, M, \} \) \( Bp = b \).

• Given

\[
(2.6) \quad Bp = \left[ \begin{array}{c} ap(-1) + bp(1) \\ \int_{-1}^{1} p(x)dx \end{array} \right], \quad a \neq b,
\]

we have
\[
B_{j,y}(x) = \partial_x^{-2}\ell_{j,y}(x) - \frac{bx}{b - a} \int_{-1}^{1} \partial_x^{-1}\ell_{j,y}(x)dx 
+ \left( \frac{(a + b)x}{2(b - a)} - \frac{1}{2} \right) \int_{-1}^{1} \partial_x^{-2}\ell_{j,y}(x)dx, \quad j = 0, 1, \ldots, M,
\]
\[
B_{M+1,y}(x) = \frac{x}{b - a},
\]
\[
B_{M+2,y}(x) = \frac{1}{2} \frac{(a + b)x}{2(b - a)},
\]

and
\[
B'_{j,y}(x) = \partial_x^{-1}\ell_{j,y}(x) - \frac{b}{b - a} \int_{-1}^{1} \partial_x^{-1}\ell_{j,y}(x)dx 
+ \frac{a + b}{2(b - a)} \int_{-1}^{1} \partial_x^{-2}\ell_{j,y}(x)dx, \quad j = 0, 1, \ldots, M,
\]
\[
B'_{M+1,y}(x) = \frac{1}{b - a},
\]
\[
B'_{M+2,y}(x) = -\frac{a + b}{2(b - a)}.
\]

Let \( N = M + m \) and \( \{ x_i \}^N_{i=0} \) be the points as in (2.1). Define the \( m \)-th order pseudospectral integration matrix (PSIM) as:
\[
B^{(-m)}_{y \mapsto x} = [B_{j,y}(x_i)]^N_{i,j=0}.
\]
Define the matrices
\[ B_{y \to x}^{(k-m)} = \left[ B_j^{(k)}(x_i) \right]_{i,j=0}^N, \quad k \geq 1. \]

It is easy to show that
\[ B_{y \to x}^{(k-m)} = D^{(k)}_{x \to y} B_{y \to x}^{(-m)}, \quad k \geq 1. \]
The matrices \( B_{y \to x}^{(-m)} \) and \( B_{y \to x}^{(k-m)} \) can be computed stably even for thousands of collocation points; see [41, 11] for details.

**Theorem 2.2.** Let \( L_B \) be the discretization of the linear operator \( B \). If for any \( p(x) \in \mathbb{P}_N \),
\[\begin{align*}
Bp &= L_B \begin{bmatrix} p(x_0) \\ \vdots \\ p(x_N) \end{bmatrix},
\end{align*}\]
then
\[\begin{bmatrix} D^{(m)}_{x \to y} \\ L_B \end{bmatrix} B_{y \to x}^{(-m)} = I_{N+1}.\]

**Proof.** The result follows from
\[\begin{align*}
D^{(m)}_{x \to y} B_{y \to x}^{(-m)} &= \begin{bmatrix} 0 \\ D^{(m-1)}_{x \to y} \end{bmatrix}, \\
L_B B_{y \to x}^{(-m)} &= \begin{bmatrix} 0 \\ I_m \end{bmatrix},
\end{align*}\]
and Lemma 2.1. \( \square \)

Now we give a concrete example for \( L_B \) satisfying (2.7). Let \( B \) be given by (2.6).
It is straightforward to discretize:
\[ L_B = \begin{bmatrix} a & 0 & \cdots & 0 & b \\ \omega_0 & \omega_1 & \cdots & \cdots & \omega_N \end{bmatrix}, \]
where \( \{\omega_j\}_{j=0}^N \) are Clenshaw-Curtis quadrature weights [18].

**2.3. The method.** Let \( \{x_j\}_{j=0}^N \) (with \( N = M+m \)) and \( \{y_j\}_{j=0}^M \) be the points as defined in (2.1) and (2.2), respectively. The rectangular spectral collocation method [8] for (1.1)-(1.2) finds a column vector \( u \) as an approximation of the vector of the exact solution \( u(x) \) at the points \( \{x_j\}_{j=0}^N \), i.e.,
\[ u \approx \begin{bmatrix} u(x_0) \\ u(x_1) \\ \cdots \\ u(x_N) \end{bmatrix}^T. \]
The collocation scheme for (1.1) is given by
\[ A_{M+1} u = f, \]
where
\[ A_{M+1} = D_{x \to y}^{(m)} + \text{diag}(a^{m-1}) D_{x \to y}^{(m-1)} + \cdots + \text{diag}(a^{1}) D_{x \to y}^{(1)} + \text{diag}(a^{0}) D_{x \to y}^{(0)}. \]
Here we use boldface letters to indicate a column vector obtained by discretizing at the points \( \{y_j\}_{j=0}^M \) except for the unknown \( u \). For example,
\[ a_0 = \begin{bmatrix} a^0(y_0) \\ a^0(y_1) \\ \cdots \\ a^0(y_M) \end{bmatrix}^T, \]
\[ f = \begin{bmatrix} f(y_0) \\ f(y_1) \\ \cdots \\ f(y_M) \end{bmatrix}^T. \]
Let \( L_B u = b \) be the discretization of the linear constraints (1.2) and satisfy the condition (2.7). The global collocation system is given by

\[
(Au = g)
\]

where

\[
A = \begin{bmatrix} A_{M+1} & L_B \end{bmatrix}, \quad g = \begin{bmatrix} f \ 0 \ b \end{bmatrix}.
\]

Consider the pseudospectral integration matrix \( B_{y \to x}^{(-m)} \) (2.2) as a right preconditioner for the linear system (2.8). We need to solve the right preconditioned linear system

\[
AB_{y \to x}^{(-m)} v = g.
\]

There hold

\[
A_{M+1} B_{y \to x}^{(-m)} \begin{bmatrix} I_{M+1} & 0 \end{bmatrix} = I_{M+1} + \text{diag}(\{a_{m-1}\}) \tilde{B}_{y \to y}^{(m-1-m)} + \cdots + \text{diag}(\{a_0\}) \tilde{B}_{y \to y}^{(0-m)},
\]

and

\[
A_{M+1} B_{y \to y}^{(-m)} \begin{bmatrix} 0 & I_m \end{bmatrix} = \text{diag}(\{a_{m-1}\}) \tilde{B}_{y \to y}^{(m-1-m)} + \cdots + \text{diag}(\{a_0\}) \tilde{B}_{y \to y}^{(0-m)},
\]

where, for \( k = 0, 1, \cdots, m-1, \)

\[
\tilde{B}_{y \to y}^{(k-m)} = \begin{bmatrix} B_{j \to y}^{(k)}(y_i) \end{bmatrix}_{i,j=0}^M, \quad \tilde{B}_{y \to y}^{(k-m)} = \begin{bmatrix} B_{j \to y}^{(k)}(y_i) \end{bmatrix}_{i=0,j=M+1}^{M,M+1}.
\]

By (see Theorem 2.2)

\[
L_B B_{y \to x}^{(-m)} = \begin{bmatrix} 0 & I_m \end{bmatrix},
\]

we have

\[
v = \begin{bmatrix} v_{M+1} \\ b \end{bmatrix},
\]

and

\[
\tilde{A}_{M+1} v_{M+1} = \tilde{f},
\]

where

\[
\tilde{A}_{M+1} = I_{M+1} + \text{diag}(\{a_{m-1}\}) \tilde{B}_{y \to y}^{(m-1-m)} + \cdots + \text{diag}(\{a_0\}) \tilde{B}_{y \to y}^{(0-m)},
\]

and

\[
\tilde{f} = f - \left( \text{diag}(\{a_{m-1}\}) \tilde{B}_{y \to y}^{(m-1-m)} + \cdots + \text{diag}(\{a_0\}) \tilde{B}_{y \to y}^{(0-m)} \right) b.
\]

Obviously, \( v_{M+1} \) is an approximation of the vector

\[
v_{M+1} \approx \begin{bmatrix} u^{(m)}(y_0) & u^{(m)}(y_1) & \cdots & u^{(m)}(y_M) \end{bmatrix}^T.
\]
After solving (2.9), we obtain \( u \) by

\[
\begin{align*}
\begin{bmatrix} u \end{bmatrix} &= \mathbf{D}^{(m)}_{\mathbf{y} \rightarrow \mathbf{x}} \begin{bmatrix} \mathbf{v}_{M+1} \\ \mathbf{b} \end{bmatrix}.
\end{align*}
\]

By the above derivation, the linear system (2.9) is the same as that obtained by the following collocation scheme for the integral reformulation (1.3): (i) substitute \( v(x) = \sum_{j=0}^{M} u^{(m)}(y_j) \ell_j(x) \) into (1.3); (ii) collocate the resulting integral equation on points \( \{y_j\}_{j=0}^{M} \). It follows from this observation that the condition number of the coefficient matrix \( \tilde{\mathbf{A}}_{M+1} \) in (2.9) is independent of the number of collocation points. Numerical experiments showing this point are reported in [41, 11].

3. The ultraspherical spectral method. The ultraspherical spectral method [29] finds an infinite vector

\[
\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} u_0 & u_1 & \cdots \end{bmatrix}^T
\]

such that the Chebyshev expansion of the solution of (1.1)-(1.2) is given by

\[
u(x) = \sum_{j=0}^{\infty} u_j T_j(x), \quad x \in [-1, 1],
\]

where \( T_j(x) \) is the degree \( j \) Chebyshev polynomial [18, 28]. We review the details of this method in the following.

Note that the following recurrence relation

\[
\frac{d^k T_n}{dx^k} = \begin{cases} 2^{k-1} n(k-1)! C^{(k)}_{n-k}, & n \geq k, \\ 0, & 0 \leq n \leq k - 1, \end{cases}
\]

where \( C^{(k)}_j \) is the ultraspherical polynomial with an integer parameter \( k \geq 1 \) of degree \( j \) [18, 28]. Then the differentiation operator for the \( k \)th derivative is given by

\[
\mathbf{D}_k = 2^{k-1}(k-1)! \begin{bmatrix} 0 & k & k+1 & k+2 & \cdots \end{bmatrix}, \quad k \geq 1.
\]

Here \( \mathbf{0} \) in (3.1) denotes a \( k \)-dimensional zero row vector.

The conversion operator converting a vector of Chebyshev expansion coefficients to a vector of \( C^{(1)} \) expansion coefficients, denoted by \( \mathcal{S}_0 \), and the conversion operator converting a vector of \( C^{(k)} \) expansion coefficients to a vector of \( C^{(k+1)} \) expansion coefficients, denoted by \( \mathcal{S}_k \), are given by

\[
\mathcal{S}_0 = \begin{bmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots \\ \frac{1}{2} & \ddots & \ddots & \ddots \end{bmatrix}, \quad \mathcal{S}_k = \begin{bmatrix} 1 & 0 & -\frac{k}{k+2} & 0 & -\frac{k}{k+3} \\ \frac{k}{k+1} & 0 & \frac{k}{k+2} & 0 & \ddots \\ 0 & \frac{k}{k+2} & 0 & \ddots & \ddots \\ \frac{k}{k+3} & \ddots & \ddots & \ddots \end{bmatrix}, \quad k \geq 1.
\]
We also require the multiplication operator \( \mathcal{M}_0[a] \) that represents multiplication of two Chebyshev series, and the multiplication operator \( \mathcal{M}_k[a] \) that represents multiplication of two \( C^{(k)} \) series, i.e., if \( u \) is a vector of Chebyshev expansion coefficients of \( u(x) \), then \( \mathcal{M}_0[a]u \) returns the Chebyshev expansion coefficients of \( a(x)u(x) \), and \( \mathcal{M}_k[a]S_{k-1} \cdots S_0 u \) returns the \( C^{(k)} \) expansion coefficients of \( a(x)u(x) \). Suppose that \( a(x) \) has the Chebyshev expansion
\[
a(x) = \sum_{j=0}^{\infty} a_j T_j(x).
\]

Then \( \mathcal{M}_0[a] \) can be written as [29]:
\[
\mathcal{M}_0[a] = \frac{1}{2} \begin{bmatrix}
2a_0 & a_1 & a_2 & a_3 & \cdots \\
a_1 & 2a_0 & a_1 & a_2 & \cdots \\
a_2 & a_1 & 2a_0 & a_1 & \cdots \\
a_3 & a_2 & a_1 & 2a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots \\
a_1 & a_2 & a_3 & a_4 & \cdots \\
a_2 & a_3 & a_4 & a_5 & \cdots \\
a_3 & a_4 & a_5 & a_6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}.
\]

It follows from
\[
\mathcal{M}_k[a]S_{k-1} = S_{k-1}\mathcal{M}_{k-1}[a]
\]
that
\[
\mathcal{M}_k[a]S_{k-1}S_{k-2} \cdots S_0 = S_{k-1}S_{k-2} \cdots S_0\mathcal{M}_0[a].
\]

The explicit formula for the entries of \( \mathcal{M}_k[a] \) with \( k \geq 1 \) is given in [29], which uses the formula given in [3]. These multiplication operators \( \mathcal{M}_k[a] \) with \( k \geq 0 \) look dense; however, if \( a(x) \) is approximated by a truncation of its Chebyshev or \( C^{(k)} \) series, then \( \mathcal{M}_k[a] \) is banded.

Combining the differentiation, conversion and multiplication operators yields
\[
\mathcal{L}u = S_{m-1}S_{m-2} \cdots S_0 f,
\]
where
\[
\mathcal{L} := D_m + \sum_{k=1}^{m-1} S_{m-1}S_{m-2} \cdots S_k \mathcal{M}_k[a^k] D_k + S_{m-1}S_{m-2} \cdots S_0 \mathcal{M}_0[a^0],
\]
\( u \) and \( f \) are column vectors of Chebyshev expansion coefficients of \( u(x) \) and \( f(x) \), respectively. Assume that the linear constraint operator \( \mathcal{B} \) in (1.2) is given in terms of the Chebyshev coefficients of \( u \), i.e.,
\[
\mathcal{B}u = b.
\]

For example, for Dirichlet boundary conditions
\[
\mathcal{B} = \begin{bmatrix}
T_0(-1) & T_1(-1) & T_2(-1) & \cdots \\
T_0(1) & T_1(1) & T_2(1) & \cdots \\
\end{bmatrix} = \begin{bmatrix}
1 & -1 & 1 & -1 & \cdots \\
1 & 1 & 1 & 1 & \cdots \\
\end{bmatrix},
\]
and for Neumann conditions
\[ B = \begin{bmatrix} T_0'(1) & T_1'(1) & T_2'(1) & \cdots \end{bmatrix} \begin{bmatrix} 0 & 1 & -4 & \cdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & \cdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 4 & \cdots \end{bmatrix}. \]

Let \( P_n \) be the projection operator given by
\[ P_n = \begin{bmatrix} I_n & 0 \end{bmatrix}. \]

We obtain the following linear system
\[ \mathbf{A}_n \mathbf{u}_n := \left[ \frac{B P_n^T}{P_{n-m} \mathcal{L} P_n^T} \right] P_n \mathbf{u} = \left[ \mathbf{b} \right]. \]

The solution \( u(x) \) of (1.1)-(1.2) is then approximated by the \( n \)-term Chebyshev series:
\[ u(x) \approx \sum_{j=0}^{n-1} u_j T_j(x). \]

The condition number of the matrix \( \mathbf{A}_n \) grows like \( O(n) \). Define the diagonal preconditioner
\[ \mathcal{R} = \frac{1}{2^{m-1}(m-1)!} \text{diag} \left\{ 1, \frac{1}{m}, \frac{1}{m+1}, \ldots \right\}. \]

It was proved in [29] that
\[ \mathcal{C} = \begin{bmatrix} B & \mathcal{L} \end{bmatrix} \mathcal{R} - \mathcal{I} \]

is a compact operator. Therefore, the preconditioner \( \mathcal{R} \) results in a linear system with a bounded condition number.

**4. A Chebyshev spectral method for the integral reformulation.** In this section, we propose a Chebyshev spectral method for the integral reformulation (1.3). We refer to [4, 17, 13, 44] for Chebyshev spectral methods for differential, integral and integro-differential equations.

Representing \( v(x) \) and \( \partial_x^{-k} v(x) \) by Chebyshev series
\[ v(x) = \sum_{j=0}^{\infty} v_j T_j(x), \quad \partial_x^{-k} v(x) = \sum_{j=0}^{\infty} v_j^{(-k)} T_j(x), \]
we have (see [19])
\[ v^{(-k)} = Q^k v, \]
where
\[ v = \begin{bmatrix} v_0 & v_1 & \cdots \end{bmatrix}^T, \quad v^{(-k)} = \begin{bmatrix} v_0^{(-k)} & v_1^{(-k)} & \cdots \end{bmatrix}^T, \]
and, for \( i, j = 0, 1, \ldots, \infty, \)
\[ Q = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{3} & \frac{1}{8} & -\frac{1}{15} & \frac{1}{24} & \cdots & \frac{(-1)^{j+1}}{(j-1)(j+1)} & \cdots \end{bmatrix} \]

\[ \begin{bmatrix} 0 & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & \cdots & \frac{(-1)^{j+1}}{(j-1)(j+1)} & \cdots \end{bmatrix}. \]
It is easy to show that

\[(4.1) \quad D_k Q^k = S_{k-1} D_{k-1} Q^{k-1} = \cdots = S_1 S_{k-2} \cdots S_0.\]

In the rest of this paper, we represent \(A\) and \(X\) in terms of the Chebyshev coefficients of their elements. For example,

\[
X = \begin{bmatrix} x^0 & x^1 & \cdots & x^{m-1} \end{bmatrix}.
\]

We list first five \(x^j\) below

\[
\begin{align*}
x^0 &= \begin{bmatrix} 1 & 0 & \cdots \end{bmatrix}^T, \\
x^1 &= \begin{bmatrix} 0 & 1 & 0 & \cdots \end{bmatrix}^T, \\
x^2 &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \end{bmatrix}^T, \\
x^3 &= \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \cdots \end{bmatrix}^T, \\
x^4 &= \begin{bmatrix} \frac{3}{8} & 0 & \frac{1}{2} & 0 & \frac{1}{8} & 0 & \cdots \end{bmatrix}^T.
\end{align*}
\]

The integral equation (1.3) can be rewritten as

\[(4.2) \quad \tilde{L} v = f - A(X) \cdot (B) - b,
\]

where

\[
\tilde{L} := I + \sum_{k=0}^{m-1} \mathcal{M}_0 [a^k] Q^{m-k} - A(X) \cdot \cdot \cdot BQ^m.
\]

The equations (3.3) and (4.2) are related as follows. Substituting

\[
u = Q^m v + X(BX)^{-1}(b - BQ^m v)
\]

into (3.3), and utilizing (3.2), (4.1), and

\[
LX = S_{m-1} S_{m-2} \cdots S_0 A,
\]

we obtain

\[
S_{m-1} S_{m-2} \cdots S_0 \tilde{L} v = S_{m-1} S_{m-2} \cdots S_0 (f - A(X)^{-1} b),
\]

which can also be obtained by multiplying (4.2) by \(S_{m-1} S_{m-2} \cdots S_0\) from the left.

By truncating (4.2), we obtain the following linear system

\[(4.3) \quad \tilde{A}_n v_n := P_n \tilde{L} P_n^T v_n = P_n (f - A(X)^{-1} b).\]

After solving (4.3), we obtain

\[
u_n = P_n u \approx P_n Q^m P_n^T v_n + P_n X(BX)^{-1} (b - BQ^m P_n^T v_n).
\]

The solution \(u(x)\) of (1.1)-(1.2) is then approximated by the \(n\)-term Chebyshev series:

\[
u(x) \approx \sum_{j=0}^{n-1} u_j T_j(x).
\]
The Chebyshev spectral scheme converges at the same rate as $P_n^T P_n v$ converges to $v$. The well-conditioning of $\tilde{A}_n$ is obvious. Assume that the variable coefficients $a^k(x)$ can be accurately approximated by low-order polynomials. The almost banded structure of $\tilde{A}_n$ follows from the almost banded structure of the spectral integral operator $Q$ and the banded structure of the multiplication operator $M_0[a^k]$. The number of dense rows of $\tilde{A}_n$ is about $\max(j, m)$, where $j$ is the number of the Chebyshev coefficients needed to resolve the variable coefficients $a^k(x)$. The adaptive QR approach [29, 30] still applies to (4.2). Since $M_0[a^k]$ is a Toeplitz plus an almost Hankel operator and $Q$ is an almost banded operator, then the matrix-vector product for the matrix $\tilde{A}_n$ and an $n$-vector can be obtained in $O(n \log n)$ operators. Therefore, if $j$ is large, iterative solvers can be employed to achieve computational efficiency.

5. Numerical experiments. In this section, computational results are reported for three examples (all from [29]) to test the ultraspherical spectral (US) method, the diagonally preconditioned ultraspherical spectral (P-US) method, and the Chebyshev spectral (CS) method. All computations are performed with MATLAB and the Chebfun software system [9].

Example 1. Consider the linear differential equation

$$u'(x) + x^3 u(x) = 100 \sin(20,000x^2), \quad u(-1) = 0.$$ 

The exact solution is

$$u(x) = \exp\left(-\frac{x^4}{4}\right) \int_{-1}^{x} 100 \exp\left(\frac{t^4}{4}\right) \sin(20,000t^2) dt.$$ 

In Figure 1 we plot the sparsity patterns of the matrices $A_n$ and $\tilde{A}_n$ when $n = 50$ to show the almost banded structures. We report the condition numbers in Table 1 and observe that the condition number of the US method behaves like $O(n)$, while those of the P-US method and the CS method remain a constant. The computed oscillatory solution and its first-order derivative by the CS method are plotted in Figure 2. The $L^2$ norm errors for the computed first-order derivative approximating...
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### Table 1

**Comparison of condition numbers of the matrices for Example 1.**

| n  | US          | P-US        | CS          |
|----|-------------|-------------|-------------|
| 128| 2.4045e+02  | 3.9813      | 2.5955      |
| 256| 4.8312e+02  | 3.9864      | 2.5955      |
| 512| 9.6846e+03  | 3.9889      | 2.5955      |
| 1024| 1.9391e+04 | 3.9901      | 2.5955      |

the exact first-order derivative to machine precision and the computed solution by the CS method are

\[
\left( \int_{-1}^{1} (u'(x) - \tilde{u}'(x))^2 \, dx \right)^{\frac{1}{2}} = 0,
\]

and

\[
\left( \int_{-1}^{1} (u(x) - \tilde{u}(x))^2 \, dx \right)^{\frac{1}{2}} = 1.2602 \times 10^{-14},
\]

respectively.

![Fig. 2. The highly oscillatory solution (left) and its first-order derivative (right) computed by the Chebyshev spectral method for Example 1.](image-url)

**Example 2.** Consider the linear differential equation

\[
u'(x) + \frac{1}{ax^2 + 1} u(x) = 0, \quad u(-1) = 1.
\]

The exact solution is

\[
u(x) = \exp \left( -\arctan(\sqrt{ax}) + \arctan(\sqrt{a}) \right).
\]

We take \(a = 50,000\). The variable coefficient function can be approximated to roughly machine precision by a polynomial of degree 7315. The multiplication operator \(M_0[a^0]\) has a very large bandwidth, resulting in essentially dense linear systems.
We compare condition numbers (in Table 2), number of iterations (using Bi-CGSTAB in Matlab with TOL= $10^{-14}$, in Figure 3 (left)), and $L_2$ norm errors (in Figure 3 (right)) of US, P-US and CS. Observe from Table 2 that the condition number of the US method behaves like $O(n)$, while those of the P-US method and the CS method remain a constant even for $n$ up to 8192. As a result, the P-US method and the CS method only require several iterations to converge (see Figure 3 (left)), while the usual US scheme requires much more iterations with a degradation of accuracy as depicted in Figure 3 (right).

| $n$  | US       | P-US     | CS     |
|------|----------|----------|--------|
| 1024 | 1.7953e+03 | 3.3256   | 1.0912 |
| 2048 | 3.5925e+03 | 3.3261   | 1.0912 |
| 4096 | 7.1870e+03 | 3.3264   | 1.0912 |
| 8192 | 1.4376e+04 | 3.3265   | 1.0912 |

**Example 3.** Consider the high order differential equation

$$u^{(10)}(x) + \cosh(x)u^{(8)}(x) + x^2u^{(6)}(x) + x^4u^{(4)} + \cos(x)u^{(2)}(x) + x^2u(x) = 0,$$

with boundary conditions

$$u(\pm 1) = 0, \quad u'(\pm 1) = 1, \quad u^{(k)}(\pm 1) = 0, \quad 2 \leq k \leq 4.$$

In Figure 4 we plot the sparsity pattern of the matrix $\tilde{A}_n$ when $n = 100$ and the computed solution by the Chebyshev spectral method. We also note that the condition number of $\tilde{A}_n$ remains a constant (about $1.7444$) for different values of $n$. The computed solution is odd to about machine precision,

$$\left( \int_{-1}^{1} (u(x) - \tilde{u}(x))^2 \, dx \right)^{\frac{1}{2}} = 5.2171 \times 10^{-14}.$$
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6. Concluding remarks. We have revisited the well-conditioned spectral methods [7, 41] and the ultraspherical spectral method [29] from the viewpoint of the integral reformulation (1.3). We also proposed a Chebyshev spectral method for the integral reformulation (1.3), which preserves the almost banded structure, avoids the conversion operators $S_k$ and only needs the multiplication operators $M_0[a^k]$. Therefore, the Chebyshev spectral method is very easy to implement. The well-conditioning of these methods come from that of the integral operator. The integral reformulation approach can also be used to interpret the well-conditioning of the fractional spectral collocation methods [23, 10]. However, we have to mention that although it is independent of the discretization parameter, the condition number of the coefficient matrix may be very large. For example, see the singular perturbation problem with a tiny parameter [29, 11].

Newton iteration techniques and the tensor-product techniques [24, 38, 37, 12] can be used in the extensions of this work to nonlinear problems and high-dimensional problems, respectively. Recently, Shen, Wang and Xia [35] proposed a fast structured direct spectral method for differential equations with variable coefficients by employing the low-rank property of the coefficient matrix. The computational complexity of their method is nearly linear. Numerical experiments (including variable coefficients with steep gradients) show that the low-rank property still holds for the coefficient matrix in the Chebyshev spectral method proposed in this work. Theoretical explanation for this point is being investigated and will be reported elsewhere.

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