On Sketching the $q$ to $p$ Norms

Aditya Krishnan $^\ast$  Sidhanth Mohanty $^\dagger$  David P. Woodruff $^\ddagger$

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Abstract

We initiate the study of data dimensionality reduction, or sketching, for the $q \to p$ norms. Given an $n \times d$ matrix $A$, the $q \to p$ norm, denoted $\|A\|_{q \to p} = \sup_{x \in \mathbb{R}^d \setminus \{\vec{0}\}} \frac{\|Ax\|_p}{\|x\|_q}$, is a natural generalization of several matrix and vector norms studied in the data stream and sketching models, with applications to datamining, hardness of approximation, and oblivious routing. We say a distribution $S$ on random matrices $L \in \mathbb{R}^{nd \to R^k}$ is a $(k, \alpha)$-sketching family if from $L(A)$, one can approximate $\|A\|_{q \to p}$ up to a factor $\alpha$ with constant probability. We provide upper and lower bounds on the sketching dimension $k$ for every $p, q \in [1, \infty]$, and in a number of cases our bounds are tight. While we mostly focus on constant $\alpha$, we also consider large approximation factors $\alpha$, as well as other variants of the problem such as when $A$ has low rank.

$^1$Computer Science Department, Carnegie Mellon University, Pittsburgh, PA, USA. arkrishn@andrew.cmu.edu
$^2$Computer Science Department, Carnegie Mellon University, Pittsburgh, PA, USA. sidhanthm96@gmail.com
$^3$Computer Science Department, Carnegie Mellon University, Pittsburgh, PA, USA. dwoodruf@cs.cmu.edu. The author would like to acknowledge the support by the National Science Foundation under Grant No. CCF-1815840.
1 Introduction

Data dimensionality reduction, or sketching, is a powerful technique by which one compresses a large dimensional object to a much smaller representation, while preserving important structural information. Motivated by applications in streaming and numerical linear algebra, the object is often a vector $x \in \mathbb{R}^n$ or a matrix $A \in \mathbb{R}^{n \times d}$. One of the most common forms of sketching is oblivious sketching, whereby one chooses a random matrix $L$ from some distribution $S$, and compresses $x$ to $Lx$ or $A$ to $L(A)$. The latter quantity $L(A)$ denotes a linear map from $\mathbb{R}^{nd}$, interpreting $A$ as an $nd$-dimensional vector, to an often much lower dimensional space, say $\mathbb{R}^k$ for a value $k \ll nd$.

Sketching has numerous applications. For example, in the data stream model, one sees additive updates $x_i \leftarrow x_i + \Delta$, where the update indicates that $x_i$ should change from its old value by an additive $\Delta$. Given a sketch $L \cdot x$, one can update it by replacing it with $L \cdot x + \Delta \cdot L_{x,i}$, where $L_{x,i}$ denotes the $i$-th column of $L$. Thus, it is easy to maintain a sketch of a vector evolving in the streaming model. Similarly, in the matrix setting, given an update $A_{i,j} \leftarrow A_{i,j} + \Delta$, one can update $L(A)$ to $L(A) + \Delta L(e_{ij})$, where $e_{ij}$ denotes the matrix with a single one in the $(i,j)$-th position, and is otherwise 0. If $L$ is oblivious, that is, sampled from a distribution independent of $x$ (or $A$ in the matrix case), then one can create $L$ without having to see the entire stream in advance. Other applications include distributed computing, whereby a vector or matrix is partitioned across multiple servers. For instance, server 1 might have a vector $x^1$ and server 2 a vector $x^2$. Given the sketches $Lx^1$ and $Lx^2$, by linearity one can combine them, using $L(x^1 + x^2) = Lx^1 + Lx^2$. In these applications it is important that the number $k$ of rows of $L$ is small, since it is proportional to the memory required of the data stream algorithm, or the communication in a distributed protocol. Here $k$ is referred to as the sketching dimension.

Sketching vector norms is fairly well understood, and we have tight bounds up to logarithmic factors for estimating the $\ell_p$-norms $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ for every $p \in [1, \infty]$; for a sample of such work, see [AMS96, BYJKS02, IW05, Ind06, KNW10, KNPW11] for work in the related data stream context, and [PW12, ANPW13, LW13] for work specifically in the sketching model. Recently, there is work [BBC+17] characterizing the sketching complexity of any symmetric norm on a vector $x$. A number of works have also looked at sketching matrix norms. In particular, the Schatten $p$-norms $\|A\|_p = \left(\sum_{i=1}^{\text{rank}(A)} \sigma_i(A)^p\right)^{1/p}$ have gained considerable attention. They have proven to be considerably harder to approximate than the vector $p$-norms, and understanding their complexity has led to important algorithmic and lower bound techniques. A body of work has focused on understanding the complexity of estimating matrix norms in the data stream model with 1-pass over the stream $[A^{+13}, LW16a]$, as well as with multiple passes [BCK+16], the sketching model [LNW14, LW17], statistical models [KV16, KO17], as well as the general RAM model [MNS+18, UCS16]. Dimensionality reduction in these norms also has applications in quantum computing [Win05, HMS11], and are studied in nearest neighbor search data structures [And10].

1.1 Our Contributions

We consider the sketching complexity of a new family of norms, namely, the $p \rightarrow q$ norms of a matrix. A common quantity that arises in various applications is the amount by which a linear
map $A$ “stretches” vectors. One way to measure this quantity is the maximum singular value of $A$, which can be written as $\sup_{\|x\|_2=1} \|Ax\|_2$, and is just the Schatten-$\infty$ norm, defined above. In this work we consider a different way of measuring this stretch, which considerably generalizes the operator norm.

For a linear operator $A$ from a normed space $\mathcal{X}$ to a normed space $\mathcal{Y}$, we define $\|A\|_{\mathcal{X} \to \mathcal{Y}}$ as $\sup_{\|x\|_X=1} \|Ax\|_Y$. Of specific interest to us is the case where $\mathcal{X} = \ell_q^d$ and $\mathcal{Y} = \ell_p^n$, and we denote the corresponding norm of such an operator by $\|A\|_{q\rightarrow p}$. Our objective is to study the sketching complexity of approximating this norm.

**Definition 1.1** ($\langle k, \alpha \rangle$-sketching family). Let $S$ be a distribution over linear functions from $\mathbb{R}^{n \times d}$ to $\mathbb{R}^k$ and $f$ a function from $\mathbb{R}^k$ to $\mathbb{R}$. We call $(S, f)$ a $\langle k, \alpha \rangle$-sketching family for the $q \rightarrow p$ norm if for all $A \in \mathbb{R}^{n \times d}$, $\Pr_{L \sim S} [f(L(A)) \in (1/\alpha, \alpha) \|A\|_{q\rightarrow p}] \geq \frac{5}{6}$.

We provide upper and lower bounds on $k$. The details of the specific results we have are described in Section 1.3.

### 1.2 Motivation

This problem is well-studied in mathematics when $p = q$ as it simply corresponds to $p$-matrix norm estimation\(^1\). An intriguing question is whether one can preserve $\|Ax\|_p$ in a lower-dimensional sketch space, given that the vectors $x$ come from the unit ball of a smaller norm.

Apart from being mathematically interesting, this problem has a number of applications. The operator norm is a special case when $p = q = 2$. The operator norm can be accurately estimated by any subspace embedding for $\ell_2$, discussed in detail in [CW13]. The dual of this norm is also the Schatten-1 norm, which has received considerable attention in the streaming model [LW16a, BCK+16]. The $q \rightarrow p$ norm problem is a natural generalization of the operator norm problem, and when $p < 2$, may be more appropriate in the context of robust statistics, where it is known that the $p$ norm for $p < 2$ is less sensitive to outliers, see, e.g., Chapter 3 of [Woo14] for a survey on robust regression, and [SWZ17] for recent work on $\ell_1$-low rank approximation.

The $2 \rightarrow q$ norms arise in the hardness of approximation literature and an algorithm for some instances of the problem was used to break the Khot-Vishnoi Unique Games candidate hard instance [KV15]. Work by [BBH+12] gives an algorithm running in time $\exp(n^{2/p})$ for approximating $2 \rightarrow p$ norms for all $p \geq 4$. These algorithms give a constant factor approximation when promised the $2 \rightarrow p$ norm is in a certain range (depending on the operator norm) rather than providing a general estimate of the $2 \rightarrow p$ norm. This same paper also discusses assumptions on the the NP-hardness and ETH hardness of approximating $2 \rightarrow p$ norms. The work of [BH15] extends that of [BBH+12] to all $p \geq 2$. The work of [BV11] gives a PTAS for computing $\|A\|_{q\rightarrow p}$ if $1 \leq p \leq q$ and $A$ has non-negative entries, and gives an application of this to the oblivious routing problem where congestion is measured using the $\ell_p$ norm. The paper also shows that it is hard to approximate $\|A\|_{q\rightarrow p}$ within a constant factor for general $A$, and general $p$ and $q$. Sketching may allow, for example, for reducing the original problem to a smaller instance of the same problem, which although may still involve exhaustive search, could give a faster concrete running time.

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\(^1\)See, e.g., https://en.wikipedia.org/wiki/Matrix_norm
The $1 \to q$ norm turns out to be the maximum of the $q$-norm of the columns of $A$, which is related to the heavy hitters problems in data streams, e.g., the column with the largest $q$-norm may be the most significant or desirable in an application. Likewise, the $q \to \infty$ norms turn out to be the maximum of the $p$-norms of the rows of $A$, where $p$ is the dual norm to $q$, and therefore have similar heavy hitter applications. The $\infty \to q$ norm is maximized when $x \in \{-1,1\}^q$ and therefore includes the cut-norm as a special case, and is related to Grothendieck inequalities, see, e.g., [BdOFV10, NRV14, BRS17].

Our main motivation for studying the $p \to q$ norms comes from understanding and developing new techniques for this family of norms. Another family of norms that is well-studied in the data stream literature are the cascaded norms, which for an $n \times d$ matrix $A$ and parameters $p$ and $q$, are defined to be $(\sum_{i=1}^{n} \|A_{i,*}\|^q_p)^{1/q}$, where $A_{i,*}$ denotes the $i$-th row of $A$. That is, we compute the $q$-norm of the vector of $p$-norms of the rows of $A$. This problem originated in [CM05] and has applications to mining multi-graphs; the following sequence of work established tight bounds up to logarithmic factors for every $p,q \in [1,\infty]$ [JW09, AKO11]. This line of work led to very new techniques; one highlight is the use of Poincaré inequalities in proving information complexity lower bounds, which has then been studied in a number of followup works [AJP10, Jay13, AKR15].

### 1.3 Our Results

After establishing preliminary results and theorems in Section 2, we give our results for constant and large approximation factors. Our main theorem is as follows. Here $\ell_q^*$ is the dual norm of $\ell_q$, that is, $1/q^* + 1/q = 1$ (when $q = 1$, $q^* = \infty$, and vice versa).

**Theorem 1.2.** For all matrices $A \in \mathbb{R}^{n \times n}$ with rank $r$ and real values $p,q \in [1,\infty]$, the table below gives upper and lower bounds on $k$ for a $(k,\Theta(1))$-sketching family of various $q \to p$ norms.

| $q \to p$ Norm | $p^* \to q^*$ Norm | Upper Bound | Sec | Lower Bound | Sec |
|----------------|---------------------|-------------|-----|-------------|-----|
| $1 \to [1,2]$ | $[2,\infty] \to \infty$ | $O(n \log n)$ | 3.1 | $\Omega(n)$ | 4.2 |
| $1 \to [2,\infty]$ | $[1,2] \to \infty$ | $O(n^2 \frac{1}{\sqrt{r}} \log n)$ | 3.1 | $\Omega(n^2 \frac{1}{\sqrt{r}})$ | 4.3 |
| $[2,\infty] \to [1,2]$ | $[2,\infty] \to [1,2]$ | $O(n^2)$ | - | $O(n^2)$ | 4.4 |
| $2 \to [2,\infty]$ | $[1,2] \to 2$ | $O(\min\{n^{1-\frac{1}{p}} r^2 \log n, n^2\})$ | 3.2 | $\Omega(\min\{n^{1-\frac{1}{p}} r, n\})$ | 4.5 |
| $[1,2] \to [1,2]$ | $[2,\infty] \to [2,\infty]$ | $O(n^2)$ | - | $\Omega(\frac{n}{\log n})$ | 4.6 |
| $[1,2] \to [2,\infty]$ | $[1,2] \to [2,\infty]$ | $O(n^2)$ | - | $\Omega(\frac{n}{\log n})$ | 4.6 |

The constant factor hidden in Theorem 1.2 does not hold for all constants, the smallest constant it holds for varies depending on the specific values of $q,p$.

We also have several results for large approximation factors summarized in the theorem below.

**Theorem 1.3.** There exists a $\left(O\left(\frac{n^2}{\log n}\right),\alpha\right)$-sketching family for the $2 \to p$ and $\infty \to p$ norm and a $\left(O\left(\frac{n^2}{\log n}\right),\alpha\right)$-sketching family for the $q \to p$ norm for $q \geq 1$ and $1 \leq p \leq 2$.

Our algorithms combine several insights, which we illustrate here in the case of the $2 \to p$ norm for $p \geq 2$ and when the rank of $A$ is $r$: (1) we show by duality that $\|A\|_{2 \to p}$ is the same as $\|A^T\|_{p' \to 2}$, where $p'$ satisfies $\frac{1}{p'} + \frac{1}{p} = 1$ and is the dual norm to $p$. Although the proof is elementary, this plays several key roles in our argument, Next, we (2) use oblivious subspace
embeddings $S$ which provide constant factor approximations for all vectors simultaneously in an $r$-dimensional subspace of $\ell_2$, and enable us to say that with $Cr$ rows for a constant $C > 0$, we have $\|SA^T\|_{p^* \to 2} = \Theta(1)\|A^T\|_{p^* \to 2}$. Next, (3) we use that for a random Gaussian matrix $G \in \mathbb{R}^{Cr \times Cr}$, for a constant $C' > 0$, with appropriate variance, it has the property that simultaneously for all $x \in \mathbb{R}^{Cr}, \|Gx\|_1 = \Theta(1) \cdot \|x\|_2$. This is a special case of Dvoretzky’s theorem in functional analysis. Thus, instead of directly approximating $\|SA^T\|_{p^* \to 2}$, we can obtain a constant factor approximation by approximating $\|GSA^T\|_{p^* \to 1}$. This is another norm we do not know how to directly work with, so we apply duality (1) again, and argue this is the same as approximating $\|AS^TG^T\|_{\infty \to p}$. A key observation is now (4), that $\sup_{x \text{ s.t. } \|x\|_p=1} \|GSA^T\|_{p^* \to 1}$. We state a well known result that a Lipschitz function of a Gaussian vector is tightly concentrated $\ell$.

2 Preliminaries

In this section, we introduce the tools we use in this paper.

**Definition 2.1** (Total Variation Distance). Given two distributions $D$ and $D'$ over sample space $\Omega$ with density functions $p_D$ and $p_{D'}$, the total variation distance is defined in two equivalent ways as follows $d_{TV}(D, D') = \frac{1}{2}\|p_D - p_{D'}\|_1 = \sup \{|\text{Pr}_x \sim D[\mathcal{E}] - \text{Pr}_x \sim D'[\mathcal{E}]|\}$

The following result bounds the total variation distance between two multivariate Gaussians.

**Lemma 2.2.** [HP15, Lemma A4] Let $\lambda$ be the minimum eigenvalue of PSD matrix $\Sigma$, then $d_{TV}(\mathcal{N}(\mu, \Sigma), \mathcal{N}(\mu', \Sigma')) \leq \frac{C}{\sqrt{\lambda}}(\|\mu - \mu'\|_2 + \|\Sigma - \Sigma'\|_F)$ for an absolute constant $C$.

We state a well known result that a Lipschitz function of a Gaussian vector is tightly concentrated around its expectation, which is useful since $\ell_p$ norms are Lipschitz.

**Theorem 2.3.** [Tao12, Theorem 2.1.12] Let $X \sim \mathcal{N}(0, I_n)$ be a Gaussian random vector and let $f : \mathbb{R}^n \to \mathbb{R}$ be a 1-Lipschitz function. Then for some absolute constants $C, c > 0$, $\text{Pr}[|f(X) - \mathbb{E}[f(X)]| \geq \lambda] \leq C \exp(-c\lambda^2)$ Notice that this implies if $f$ is $t$-Lipschitz, then $\text{Pr}[|f(X) - \mathbb{E}[f(X)]| \geq \lambda] \leq C \exp(-c\lambda^2/t^2)$

It is possible to embed $\ell_p^r$ into $\ell_p^{O(n)}$ with constant distortion using a linear map when $p \in [1, 2)$, and we use the existence of such a linear map in our results.

**Lemma 2.4.** [Mat13, Theorem 2.5.1] For all $p \in [1, 2]$, there is an absolute constant $C_p$ such that for any $n$, there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^{Crn}$ such that $\|T(x)\|_p = (1 \pm \frac{1}{n}) \|x\|_2$. An important observation is that this implies for any linear map $A : \mathbb{R}^n \to \mathbb{R}^n$, we have $\|TA\|_{q \to p} = (1 \pm \frac{1}{n}) \|A\|_{q \to 2}$.

In the lemma below we make an important observation that highlights the connection between several $p \to q$ norms.

**Lemma 2.5.** For any $p, q \geq 1$ and $d \times n$ matrix $A$, $\|A\|_{q \to p} = \|A^T\|_{p^* \to q^*}$. 

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Proof. Using the notation above for dual norms, we have

\[ \|A\|_{q \to p} = \sup \{ \|Ax\|_q : \|x\|_p \leq 1 \} \]

\[ = \sup \{ \sup \{ y^\top Ax : \|y\|_{q^*} \leq 1 \} : \|x\|_p \leq 1 \} \]

\[ = \sup \{ \|A^\top y\|_{p^*} : \|y\|_{q^*} \leq 1 \} \]

\[ = \|A^\top\|_{p^* \to q^*} \]

Throughout the paper, we make use of \(q^*\) to refer to \(q^{q-1}\) since \(\ell_{q^{q-1}}\) is the dual norm of \(\ell_q\).

We give a characterization of the \(1 \to p\) and \(\infty \to p\) norm of a matrix. The proofs can be found in Appendix A. For any \(d \times n\) matrix \(A\), we have

Lemma 2.6. \(\|A\|_{1 \to p} = \max_{i \in [n]} \{ \|A_{*,i}\|_p \} \).

Lemma 2.7. \(\|A\|_{\infty \to p} = \max_{x \in \{-1,1\}^n} \|Ax\|_p \).

We introduce the machinery of \(\epsilon\)-nets, a common tool in the study of random matrices (see [Ver10]) along with some relevant lemmas and defer the proofs to the full version’s Appendix.

Definition 2.8 (\(\epsilon\)-net). Let \(X\) be a normed space. For \(S \subseteq V\), we call a set \(N\) an \(\epsilon\)-net for \(S\) if for all \(v \in S\), there is \(v' \in N\) such that \(\|v - v'\|_X < \epsilon\).

For a linear operator \(A\), we show that to bound \(\|A\|_{X \to Y}\), it suffices to bound \(\|Ax\|_Y\) for \(x\) taken over an \(\epsilon\)-net of the unit ball in \(X\).

Lemma 2.9. Let \(X\) and \(Y\) be normed spaces and let \(A : X \to Y\) be a linear map. Suppose \(N\) is an \(\epsilon\)-net of the unit ball in \(X\), then \(\|A\|_{X \to Y} \leq \frac{1}{1 - \epsilon} \max_{v \in N} \|Av\|_Y \).

We also give a way to construct ‘small’ \(\epsilon\)-nets of unit balls.

Lemma 2.10. There is an \(\epsilon\)-net of the unit ball \(B\) in an \(n\)-dimensional normed space \(X\) with at most \(\left(\frac{1}{2\epsilon}\right)^n\) elements.

Another tool we use is subspace embeddings, which we define below.

Definition 2.11. An oblivious subspace embedding family (OSE family) is a distribution \(S\) over \(O(m) \times n\) matrices such that for any subspace \(K \subseteq \mathbb{R}^n\) of dimension \(m\), \(\Pr_{S \sim S}[\forall x \in K : \|Sx\|_2 = \Theta(1)\|x\|_2] \geq \frac{\epsilon^2}{12} \).

Lemma 2.12. [Sar06] There exist OSE families, where the matrices have dimension \(O(k) \times n\). Note that this means for any rank-\(k\) matrix \(A\), a randomly drawn \(S\) from such an oblivious subspace embedding family satisfies \(\|SAx\|_2 = \Theta(1)\|Ax\|_2\) simultaneously for all \(x\) with probability at least 99/100.

3 Sketching algorithms for constant factor approximations

3.1 Sketches for approximating \(\|A\|_{1 \to p}\)

We show how to use sketches for \(p\)-norms of vectors to come up with sketches for the \(1 \to p\) norm.
Lemma 3.1. Let \( x \) be an arbitrary vector in \( \mathbb{R}^n \). If \( S \) is a distribution over \( t \times n \) sketching matrices, and \( f : \mathbb{R}^t \rightarrow \mathbb{R} \) is a function such that \( \Pr_{S \sim S} [f(Sx) \in (\frac{1}{2} \|x\|_p, 2\|x\|_p)] \geq \frac{2}{3} \) then there is an \((O(nt \log n), 2)\)-sketching family \((S', g)\) for the \( 1 \rightarrow p \) norm of \( n \times n \) matrices.

Proof. Proof in Appendix B. \( \square \)

Given an \( n \)-dimensional vector \( x \), we have the following theorems from [KNW10] and [AKO11] respectively.

Theorem 3.2 (Efficient sketches for small norms). When \( p \in [1, 2] \), there is a function \( f \) and a distribution over sketching matrices \( F \) with \( O(1) \) rows such that for \( S \sim F \), \( f(Sx) \) is a constant factor approximation for \( \|x\|_p \) with probability at least 2/3.

Theorem 3.3 (Efficient sketches for large norms). When \( p > 2 \), there is a function \( f \) and a distribution over sketching matrices \( F \) with \( O(n^{1-2/p} \log n) \) rows such that for \( S \sim F \), \( f(Sx) \) is a constant factor approximation for \( \|x\|_p \) with probability at least 2/3.

Lemma 3.1 tells us the following as a corollary to Theorem 3.2 and Theorem 3.3.

Theorem 3.4. There is an \((O(n \log n), 2)\)-sketching family for the \( 1 \rightarrow p \) norm when \( p \in [1, 2] \) and a \((O(n^{2-2/p}) \log^2 n, 2)\)-sketching family for the \( 1 \rightarrow p \) norm when \( p \in (2, \infty] \).

3.2 Sketches for approximating \( \|A\|_{2 \rightarrow p} \) for \( p > 2 \)

We give a sketching algorithm for the \( 2 \rightarrow p \) norm of \( A \), whose number of measurements depends on the rank \( r \) of \( d \times n \) matrix \( A \).

Theorem 3.5. There is an \((O(n^{1-2/p} \log n), \Theta(1))\)-sketching family for the \( 2 \rightarrow p \) norm.

Proof. Observe that \( \|A\|_{2 \rightarrow p} \) is equal to \( \|A^T\|_{p \rightarrow 2} \) by Lemma 2.5 and let \( S \) be a \( Cr \times d \) matrix drawn from an oblivious subspace embedding family, which exists by Lemma 2.12. From Theorem 2.4, let \( G \) be a \( C \beta r \times C \) map such that for all \( x \), \( \|GSA^T x\|_1 = \Theta(1) \|SA^T x\|_2 \). Combining with the subspace embedding property, we get that \( \|GSA^T x\|_1 \geq \Theta(1) \|A^T x\|_2 \) for all \( x \), which is equivalent to saying \( \|GSA^T\|_{p \rightarrow 1} = \Theta(1) \|A\|_{2 \rightarrow p} \). Another application of Lemma 2.5 gives us that \( \|AS^T G^T\|_{\infty \rightarrow p} = \Theta(1) \|A\|_{2 \rightarrow p} \). Since \( AS^T G^T \) is \( n \times \beta r \), \( \|AS^T G^T\|_{\infty \rightarrow p} = \max_{x \in \{\pm 1\}^{\beta r}} \|AS^T G^T x\|_p \).

Our final ingredient is the existence of an \( O(n^{1-2/p} \log n \log(1/\delta)) \times n \) sketching matrix \( E \) and estimation function \( f \) such that for any \( x \), \( \Pr[f(E y) = \Theta(1) \|y\|_p] \geq 1 - \delta \) [And17] when \( p > 2 \). We set \( \delta = 2^{-2\beta r} \) and use a union bound over all \( 2^{\beta r} \) vectors in \( \{\pm 1\}^{\beta r} \) to conclude

\[
\Pr[\forall x \in \{\pm 1\}^{\beta r} : f(EAS^T G^T x) = \Theta(1) \|AS^T G^T x\|_q] \geq 1 - 2^{-\beta r}
\]

\[
\Pr \left[ \max_{x \in \{\pm 1\}^{\beta r}} f(EAS^T G^T x) = \Theta(1) \|AS^T G^T\|_{\infty \rightarrow q} \right] \geq 1 - 2^{-\beta r}
\]

Consequently, we get a sketch that consists of \( O(n^{1-2/p} \log n) \) measurements to get a \( \Theta(1) \) approximation to \( \|A\|_{2 \rightarrow p} \) with probability at least 0.99. \( \square \)
4 Sketching lower bounds for constant factor approximations

4.1 Lower Bound Techniques

The way we prove most of our lower bounds is by giving two distributions over \( n \times n \) matrices, \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), where matrices drawn from the two distributions have \( q \rightarrow p \) norm separated by a constant factor \( \kappa \) with high probability, which means a \((k, \sqrt{\kappa})\)-sketching family can distinguish between samples from the two distributions. We then show an upper bound on the variation distance between distributions of \( k \)-dimensional sketches of \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \). We then argue that if \( k \) is too small, then the total variation distance is too small to solve the distinguishing problem. We formalize this intuition in the following theorem.

**Theorem 4.1.** Suppose \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are distributions over \( d \times n \) matrices such that

\[
(i) \quad \Pr_{D \sim \mathcal{D}_1}[\|D\|_{q \rightarrow p} < s] \geq 1 - \frac{1}{n} \quad \text{and} \quad \Pr_{D \sim \mathcal{D}_2}[\|D\|_{q \rightarrow p} > ks] \geq 1 - \frac{1}{n}
\]

\[
(ii) \quad \text{for any linear map } L : \mathbb{R}^{d \times n} \rightarrow \mathbb{R}^k, d_{TV}(L(\mathcal{D}_1), L(\mathcal{D}_2)) = O \left( \frac{k^a}{n^b} \right)
\]

for constants \( s, k, a, b, \) any \((k, \sqrt{\kappa})\)-sketching family for the \( q \rightarrow p \) norm must satisfy \( k = \Omega(n^{b/a}) \).

**Proof.** Let \( \mathcal{D} \) be the distribution over matrices given by sampling from \( \mathcal{D}_1 \) with probability \( \frac{1}{2} \) and drawing from \( \mathcal{D}_2 \) with probability \( \frac{1}{2} \). We shall fix a sketching operator \( L : \mathbb{R}^{d \times n} \rightarrow \mathbb{R}^k \) and consider \( A \) drawn from a distribution \( \mathcal{D} \). Suppose \( f(L(A)) \) lies in \((1/\kappa, \kappa)|A|_{q \rightarrow p}\) with probability at least \( 5/6 \). It suffices to show that \( k \) must be \( \Omega(n^{b/a}) \) since the theorem statement then follows from Yao’s minimax principle. We must have

\[
\Pr_{A \sim \mathcal{D}_1} \left[ f(L(A)) \in \left( \frac{1}{\sqrt{\kappa}}, \sqrt{\kappa} \right) \|A\|_{q \rightarrow p} \right] \geq \frac{2}{3}, \quad \Pr_{A \sim \mathcal{D}_2} \left[ f(L(A)) \in \left( \frac{1}{\sqrt{\kappa}}, \sqrt{\kappa} \right) \|A\|_{q \rightarrow p} \right] \geq \frac{2}{3}
\]

Thus, we have an algorithm that correctly distinguishes with probability at least \( \frac{2}{3} \) if \( A \) was drawn from \( \mathcal{D}_1 \) or \( \mathcal{D}_2 \) by checking if \( f(L(A)) \) is greater than or less than \( \sqrt{\kappa}s \). The existence of this distinguishing algorithm means the total variation distance between the distributions of \( L(\mathcal{D}_1) \) and \( L(\mathcal{D}_2) \) is at least \( \frac{1}{2} \). From the theorem’s hypothesis, we know of a constant \( C \) such that \( \frac{k^a}{n^b} \geq \frac{1}{2} \), which gives us the desired upper bound. \( \square \)

We also show an upper bound on the variation distance of sketches for two distributions that we use throughout this paper. Define \( \mathcal{G}_{1,d \times n} \) as the distribution over \( d \times n \) Gaussian matrices and \( \mathcal{G}_{2,d \times n}[\alpha] \) as the distribution given by drawing a Gaussian matrix and adding \( \alpha \mathbf{u} \), where \( \mathbf{u} \) is a \( d \)-dimensional Gaussian vector to a random column. We write \( \mathcal{G}_i \) instead of \( \mathcal{G}_{i,d \times n} \) when the dimensions of the random matrix are evident from context.

**Lemma 4.2.** Let \( L \) be a linear sketch from \( \mathbb{R}^{d \times n} \rightarrow \mathbb{R}^k \) and let \( \mathcal{H}_i \) be the distribution of \( L(x) \) where \( x \) is drawn from \( \mathcal{G}_i \). Then \( d_{TV}(\mathcal{H}_1, \mathcal{H}_2) \leq \frac{Ck^2}{n} \) for an absolute constant \( C \).

**Proof.** We can think of \( L \) as a \( k \times nd \) matrix that acts on a sample from \( \mathcal{G}_1 \) or \( \mathcal{G}_2 \) as though it were an \( nd \)-dimensional vector. Without loss of generality, we can assume that the rows of \( L \) are orthonormal, since one can always perform a change of basis in post-processing. Thus, the
distribution \(\mathcal{H}_1\) is the same as \(\mathcal{N}(0, I_k)\). For fixed \(i\) and \(G\) a \(d \times n\) matrix of unit Gaussians, the distribution of \(L(G + au e^T)\) is Gaussian with covariance \(\mathbb{E}[L(G + au e^T)L(G + au e^T)^T]\), equal to \(I + \alpha^2 L_B L_B^T\) where \(L_B\) is the submatrix given by columns of \(L\) indexed \((i - 1)d + 1, (i - 1)d + 2, \ldots, id\). Let \(\mathcal{H}_{2,i}\) be \(\mathcal{N}(0, I + \alpha^2 L_B L_B^T)\). \(\mathcal{H}_2\) is the distribution of picking a random \(i\) and drawing a matrix from \(\mathcal{N}(0, I + L_B L_B^T)\).

We now analyze the total variation distance between \(\mathcal{H}_1\) and \(\mathcal{H}_2\) and get the desired bound from a chain of inequalities.

\[
d_{TV}(\mathcal{H}_1, \mathcal{H}_2) = \frac{1}{2} \int_{x \in \mathbb{R}^k} |p_{\mathcal{H}_1}(x) - p_{\mathcal{H}_2}(x)| dx \\
\leq \frac{1}{2} \int_{x \in \mathbb{R}^k} \left| \sum_{i=1}^{n} \frac{1}{n} p_{\mathcal{H}_1}(x) - \frac{1}{n} p_{\mathcal{H}_2}(x) \right| dx \\
\leq \frac{1}{n} \sum_{i=1}^{n} \frac{1}{2} \int_{x \in \mathbb{R}^k} |p_{\mathcal{H}_1}(x) - p_{\mathcal{H}_2}(x)| dx \\
\leq \frac{1}{n} \sum_{i=1}^{n} d_{TV}(\mathcal{N}(0, I_k), \mathcal{H}_{2,i}) \\
\leq \frac{1}{n} \sum_{i=1}^{n} C \alpha^2 \|L_B L_B^T\|_F \\
\leq \frac{1}{n} \sum_{i=1}^{n} C \alpha^2 \|L_B\|_F^2 \\
\leq \frac{C \alpha^2}{n} \|L\|_F^2 = \frac{C \alpha^2 k}{n} \quad \text{[from Lemma 2.2]}
\]

\[\square\]

4.2 Lower bounds for approximating \(\|A\|_{1 \rightarrow p}\) for \(1 \leq p \leq 2\)

We follow the lower bound template given in Section 4.1.

**Lemma 4.3.** For any \(\kappa\), there exist values \(s_p\) such that with probability at least \(1 - 1/n\), \(\|G_1\|_{1 \rightarrow p} \leq s_p\) and \(\|G_2\|_{1 \rightarrow p} \geq \kappa s_p\) for \(1 \leq p \leq 2\), and \(G_1 \sim G_1\) and \(G_2 \sim G_2[\kappa]\).

**Proof.** Recall that from Section 3.1, we know that \(\|A\|_{1 \rightarrow p} = \max_{i \in [n]} \|A_{*,i}\|_p\) which means that it suffices to give bounds on the maximum \(\ell_p\) norm across columns of \(G_1\) and \(G_2\) respectively.

The \(\ell_p\) norm is \(\zeta_p\)-Lipschitz, where \(\zeta_p\) is equal to \(n^{1/p - 1/2}\) in the regime \(1 \leq p \leq 2\). For a given vector of standard Gaussians \(g\), the probability that \(\|g\|_p\) deviates from \(\mathbb{E}[\|g\|_p]\) by more than \(\beta \zeta_p \sqrt{\log n}\) is at most \(C'e^{-\beta^2 \zeta_p^2 \log n}\) from Theorem 2.3 where \(C'\) is the constant \(C\) from the theorem, which for large enough choice of \(\beta\) can be made smaller than \(1/n^2\). By a union bound over all columns, the probability that \(\|G_1\|_{1 \rightarrow p} exceeds \mathbb{E}[\|g\|_p] + \beta \zeta_p \sqrt{\log n}\) is at most \(1/n\). On the other hand, consider the perturbed column vector of \(G_2\), which we denote \(g'\). The probability that \(\|g'\|_2\) is smaller than \(\mathbb{E}[\|g'\|_p] - \beta \sqrt{1 + \kappa^2} \zeta_p \sqrt{\log n} = \sqrt{1 + \kappa^2} (\mathbb{E}[\|g\|_p] - \beta \zeta_p \sqrt{\log n})\) is at most \(1/n^2\) by appropriate choice of \(\beta\) and Theorem 2.3, from which a lower bound on \(\|G_2\|_{1 \rightarrow p}\) that holds with probability at least \(1 - \frac{1}{n^2}\) immediately follows.
Since $E[\|g\|_p]$ is $\Theta(n^{1/p})$ and the deviations from expectations in upper bounds on $\|G_1\|_{1\rightarrow p}$ and lower bounds on $\|G_2\|_{1\rightarrow p}$ are asymptotically less than the expectations.

The desired theorem is immediate from Lemma 4.3, Lemma 4.2, and Theorem 4.1 using $D_1 = G_{1,n \times n}$, and $D_2 = G_2[k]$.

**Theorem 4.4.** Suppose $p \in [1, 2]$ and $(S, f)$ is a $(k, \sqrt{n})$-sketching family for the $1 \rightarrow p$ norm where $k$ is some constant, then $k = \Omega(n)$.

### 4.3 Lower bound for approximating $\|A\|_{1\rightarrow p}$ for $p > 2$

We follow the lower bound template given in Section 4.1.

Denote $E[\|g\|_p]$ as $\eta_p$. Let $G_1$ be the distribution over $n \times n$ matrices given by i.i.d. Gaussians, and $G_2[\alpha, \eta_p]$ be the distribution over $n \times n$ matrices given by taking a Gaussian matrix and adding $\alpha \eta_p$ to a random entry.

Since the proofs are very similar to those in Section 4.1 and Section 4.2. We defer them to Appendix C.1.

**Lemma 4.5.** For any $\kappa$, there exists $s_p$ such that with probability at least $1 - \frac{1}{n^\beta}$, $\|G_1\|_{1\rightarrow p} \leq s_p$ and $\|G_2\|_{1\rightarrow p} \geq \kappa s_p$, such that $G_1 \sim G_1$ and $G_2 \sim G_2[\kappa \eta_p \text{ for some absolute constant } C]$ and $p > 2$.

**Lemma 4.6.** Let $L$ be a linear sketch from $\mathbb{R}^{n \times n} \rightarrow \mathbb{R}^k$ and let $D_i$ be the distribution of $L(x)$ where $x$ is drawn from $G_i$. Then $d_{TV}(D_1, D_2) \leq \frac{Cn \eta_p \sqrt{n}}{n}$ for an absolute constant $C'$.

The theorem below immediately follows from Lemma 4.5, Lemma 4.6 and Theorem 4.1 using $D_1 = G_1$ and $D_2 = G_2[\kappa \eta_p]$.

**Theorem 4.7.** Suppose $(S, f)$ is a $(\kappa, \kappa)$-approximate sketching family for the $1 \rightarrow p$ norm for $p > 2$ and some constant $\kappa$, then $k = \Omega \left( \frac{n^2}{\kappa^2} \right)$. In particular, using the fact that $\eta_p$ is $\Theta(n^{1/p})$ for $p < \infty$ and $\Theta(\sqrt{\log n})$ when $p = \infty$ gives $k = \Omega \left( n^{2 - \frac{2}{p}} \right)$ when $p < \infty$ and $k = \Omega \left( \frac{n^2}{\log n} \right)$ when $p = \infty$.

### 4.4 Lower bound for approximating $\|A\|_{q\rightarrow p}$ when $q \geq 2$ and $p \leq 2$

We use the known lower bound of $\Omega(n^2)$ for sketching the $2 \rightarrow 2$ norm from [LW16b] to deduce a lower bound on sketching the $q \rightarrow p$ norm for $q \geq 2$ and $p \leq 2$.

**Theorem 4.8.** Suppose $q \geq 2$ and $p \leq 2$, and if $(S, f)$ is a $(\kappa(n), \gamma)$-approximate sketching family for the $q \rightarrow p$ norm where $\gamma$ is some constant, then $k(n) = \Omega(n^2)$.

**Proof.** We prove this by showing that if the hypothesis of the theorem statement holds, then the $2 \rightarrow 2$ norm can be sketched in $O(k)$ measurements.

Given an $n \times n$ matrix $A$ for which we want to sketch the $2 \rightarrow 2$ norm, note that by Theorem 2.4 there is a $Cn \times n$ matrix $L_1$ such that $\|L_1A\|_{2\rightarrow q^*} = \left( \frac{1}{\beta} \right) \|A\|_{2\rightarrow 2}$ for a constant $\beta$, and by Lemma 2.5 $\|L_1A\|_{2\rightarrow q^*} = \|AT_{L_1}^T\|_{q\rightarrow 2}$, and another application of Theorem 2.4 gives us another $Cn \times n$ matrix $L_2$ for which $\|L_2A^T L_1^T\|_{q\rightarrow p} = \left( \frac{1}{\beta^2} \right) \|A\|_{2\rightarrow 2}$, so we can sketch $A$ by drawing a random $L$ from $D$ and storing $L(L_2A^T L_1^T)$, which uses $k(Cn)$ measurements and serves as a sketch from which $f$ can be used to estimate...
\[ \|A\|_{2 \to 2} \text{ within a constant factor, which means from [LW16b], } k(Cn) \text{ must be } \Omega(n^2), \text{ which means } k(n) = \Omega(n^2/C^2) = \Omega(n^2). \]

\[ \square \]

## 4.5 Lower bounds for approximating \( \|A\|_{q \to p} \) for \( p, q \leq 2 \) and \( p, q \geq 2 \)

In this section, we show a lower bound on the sketching complexity of \( \|A\|_{q \to p} \) where \( A \) is a rank \( r \) matrix, when both \( p \) and \( q \) are at most 2. A corresponding lower bound for when \( p \) and \( q \) are at least 2 follows from Lemma 2.5. We achieve this by first showing a lower bound on the sketching complexity of \( \|A\|_{2 \to q} \) and then use Dvoretzky’s theorem along with the relation between the \( q \to p \) norm and the \( p^* \to q^* \) norm to deduce the result.

We show a lower bound for sketching the \( 2 \to q \) norm using the template from Section 4.1. We use distributions \( D_1 = G_{1,r \times n} \) and \( D_2[\alpha] = G_{2,r \times n}[\alpha \cdot \frac{d}{q^*}] \), as defined in Section 4.1 where \( d \) is \( \max\{n^{1/q}, \sqrt{r}\} \).

**Lemma 4.9.** There exist values \( s_q \) and \( t_q \) such that with high probability, \( \|G_1\|_{2 \to q} \leq s_q \) and \( \|G_2\|_{2 \to q} \geq C s_q \) for some absolute constant \( C \), for \( q > 2 \), and \( G_1 \sim D_1 \) and \( G_2 \sim D_2[\alpha] \).

**Proof.** Let \( N \) be a \( 1/3 \)-net of the Euclidean ball in \( \mathbb{R}^r \) with \( 7r \) elements, which exists by Lemma 2.10. For a fixed \( x \in N \), \( G_1x \) is distributed as an \( n \)-dimensional vector with independent Gaussians, whose \( q \)-norm is at most \( \beta_1 n^{1/q} \) for some constant \( \beta_1 \) in expectation and exceeds \( \beta_1 n^{1/q} + \beta_2 \sqrt{r} \) with probability at most \( \frac{1}{n} \) for appropriate constant \( \beta_2 \), which follows from the \( q \)-norm being 1-Lipschitz and Theorem 2.3. A union bound over all \( x \in N \) implies that with probability at least \( 1 - (7/8)^r \), \( \forall x \in N : \|G_1x\|_q \leq \beta_1 n^{1/q} + \beta_2 \sqrt{r} \).

Then by applying Lemma 2.9, we conclude that with probability at least \( 1 - (7/8)^r \), \( \|G_1\|_{2 \to q} \leq \frac{3}{2}(\beta_1 n^{1/q} + \beta_2 \sqrt{r}) \leq \frac{3}{2}(\beta_1 + \beta_2)d \). On the other hand, the perturbed row of \( G_2 \), called \( g' \) is distributed as \( \sqrt{1 + \alpha^2 d^2} g \) for a vector of i.i.d. Gaussians \( g \). If we take the unit vector \( u \) in the direction of \( g' \), then the entry of \( G_2u \) corresponding to the perturbed row is concentrated around \( \sqrt{1 + \alpha^2 d^2} \|g\|_2 = \sqrt{r + \alpha^2 d^2} \), which means \( \|G_2\|_{2 \to q} \geq (1 - o(1))\sqrt{r + \alpha^2 d^2} \geq 0.9ad \) with high probability. \[ \square \]

The theorem below immediately follows from Lemma 4.9, Lemma 4.2 and Theorem 4.1.

**Theorem 4.10.** Suppose \( q \geq 2 \) and \( (S,f) \) is a \( (k,\gamma) \)-sketching family for the \( 2 \to q \) norm of rank \( r \) matrices for some constant \( \gamma \). Then \( k = \Omega(nr/d^2) \).

**Theorem 4.11.** Suppose \( p, q \leq 2 \) and \( (S,f) \) is a \( (k,\gamma) \)-sketching family for the \( q \to p \) norm of rank \( r \) matrices for some constant \( \gamma \). Then \( k = \Omega(nr/d^2) \) where \( d = \max\{\sqrt{r}, n^{1/q^*}\} \).

**Proof.** For a matrix \( A \), from Lemma 2.5 we have that \( \|A\|_{2 \to q^*} = \|A^T\|_{q^* \to 2} \), and from Theorem 2.4, we know there is a \( C r \times r \) matrix \( L_1 \) such that \( \|L_1 A^T\|_{q^* \to p} = \Theta(1) \|A\|_{2 \to q^*} \). We can use \( (S,f) \) to sketch \( L_1 A^T \) to obtain an \( (O(k), \Theta(1)) \)-sketching family for the \( 2 \to q^* \) norm, whose lower bound from Theorem 4.10 gives us the desired lower bound. \[ \square \]
4.6 Lower bounds for approximating $\|A\|_{q\to p}$ for $1 \leq q \leq 2$ and $p \geq 2$

We prove the desired lower bound using the template from Section 4.1. Let $D_1$ be a distribution over $n \times n$ matrices where diagonal entries are Gaussians and off-diagonal entries are 0 and let $D_2[\alpha]$ be a distribution over $n \times n$ matrices where a matrix is drawn from $D_1$ and $\alpha \sqrt{\log n}$ is added to a random diagonal entry.

**Lemma 4.12.** There exists values $s_{p,q}$, $t_{p,q}$ and $\alpha$ such that with probability at least $1 - 1/n$, $\|G_1\|_{q\to p} \leq s_{p,q}$ and $\|G_2\|_{q\to p} \geq \kappa s_{p,q}$ for some desired constant factor $\kappa$ separation, such that $G_1 \sim D_1$ and $G_2 \sim D_2[\alpha]$.

We give the proof of Lemma 4.12 in Appendix C.2.

Without loss of generality, we can assume that any sketch of $G_1$ and $G_2$ acts on $\text{diag}(G_1)$ and $\text{diag}(G_2)$ respectively. **Lemma 4.6** gives an upper bound of $O(\sqrt{k \log n / \sqrt{n}})$ on the variation distance between $k$-dimensional sketches of these distributions. Thus, from the variation distance bound, Lemma 4.12 and Theorem 4.1, the desired theorem follows.

**Theorem 4.13.** Suppose $q \geq 2$ and $(S, f)$ is a $(k, \gamma)$-sketching family for the $\| \cdot \|_p$ norm of rank $r$ matrices for some constant $\gamma$, then $k = \Omega(n / \log n)$.

5 Sketching with large approximation factors

While our results primarily involve constant factor approximations, we give several preliminary results studying large approximation factors for sketching the important cases of the $2 \to q$ norm and $[1, \infty] \to [1, 2]$ norms. Our goal is, given an approximation factor $\alpha(n)$, to give upper and lower bounds on $k$ for a $(k, \alpha(n))$-sketching family for the respective norms. As a shorthand, we will refer to $\alpha(n)$ as $\alpha$.

5.1 Sketching upper bounds for large approximations of $\|A\|_{2\to q}$

It is sufficient to give a $(k, \alpha)$-sketching family for the $\infty \to q$ norm. To see why, given an input matrix $A \in \mathbb{R}^{n \times n}$, by **Lemma 2.5** we have that $\|A\|_{2\to q} = \|A^T\|_{q'\to 2}$. Using **Theorem 2.4**, there is a linear map such that this is equal within a constant factor of $\|GA^T\|_{q'\to 1} = \|AG^T\|_{\infty\to q}$.

**Theorem 5.1.** Given a matrix $A \in \mathbb{R}^{n \times n}$, there exists a $(O(\frac{\|A\|_{\infty}}{\alpha}), \alpha)$-sketching family given by $(S, f)$ for the $\infty \to q$ norm.

**Proof.** Let $B \in \mathbb{Z}^+$ be some positive integer to be chosen later. Let the columns of our sketch matrix $S$ be indexed by sets given by $\{B_i\}_{i=1}^{n/B}$ such that $B_i = ((i - 1)B, iB]$. For each column $v_{B_i}$, we define i.i.d random variables $\{\sigma_{ij}\}_{j=1}^B$ such that $\sigma_{ij} = 1$ with probability $\frac{1}{2}$ and $-1$ with probability $\frac{1}{2}$. Let the column $v_{B_i}$ be as follows:

$$v_{B_i}[j] = \begin{cases} 
\sigma_{ij} & \text{for } j \in [(i - 1)B, iB] \\
0 & \text{o/w}
\end{cases}$$
We define our linear map $L(A)$ to be $L(A) = AS$. Our function $f : \mathbb{R}^{n/B} \to \mathbb{R}$ simply optimizes over $\{-1,1\}^{n/B}$ and outputs $\|AS\|_{\infty \to q}$.

Since all $\sigma_{ij} \in \{-1,1\}$ we have that $f(L(A)) \leq \|A\|_{\infty \to q}$ since $Sx$ for $x \in \{-1,1\}^{n/B}$ has the property that $Sx \in \{-1,1\}^n$.

We now show a lower bound on $f(L(A))$. To do so, we let $T_i$ denote the column indices of $A$ such that the index is column $i$ in its respective block. We then notice that there exists $i \in [n/B]$ such that $\|A_{*,T_i}\|_{\infty \to q} \geq \frac{B}{\pi} \|A\|_{\infty \to q}$. We get this by applying the triangle inequality $\|A\|_{\infty \to q} \leq \sum_{i=1}^{n/B} \|A_{*,T_i}\|_{\infty \to q}$.

Let $i^*$ be the index that realizes this $n/B$-approximation to $\|A\|_{\infty \to q}$ and let $\{s_1\}_{i=1}^{n/B}$ be the assignment of signs that realizes the $\infty \to q$ norm of $A_{*,T_{i^*}}$.

$$f(L(A)) \geq \| \sum_{i=1}^{n/B} \sum_{j=1}^{B} s_i A_{*,B_{ij}} \|_q \geq \| \sum_{j=1}^{B} s_{i^*} A_{*,B_{i^*j}} \|_q + \sum_{j \neq i^*} \sum_{j=1}^{n/B} s_j A_{*,B_{ij}} \|_q$$

Notice that $z$ is symmetric around the origin and hence we get that $\|y + z + y - z\|_q \leq \|y + z\|_q + \|y - z\|_q$ which implies that $f(L(A)) \geq \|y + z\|_q \geq \Theta(1) \|y\|_q \geq \frac{B}{\pi} \|A\|_{\infty \to q}$ with probability at least $\frac{1}{4}$. Thus, we get an $O\left(\frac{\sqrt{B}}{\pi}\right)$ space sketch that gives us an $\alpha$-approximation by setting $B = n/\alpha$.

5.2 Sketching upper bounds for large approximations of $\|A\|_{q \to p}$ for $q \in [1, \infty]$ and $p \in [1, 2]$

We give a description of our sketch followed by the approximation factor. Towards the end of defining our sketch, let $B \in \mathbb{Z}^+$ be some positive integer to be chosen later. Let the rows of our sketch matrix $S$ be indexed by sets given by $\{B_i\}_{i=1}^{B}$ such that $B_i = ((i-1)B,iB]$. For each row $v_{B_i}$ we define i.i.d random variables $\{\sigma_{ij}\}_{j=1}^{B}$ such that $\sigma_{ij} = 1$ with probability $\frac{1}{2}$ and $-1$ with probability $\frac{1}{2}$. Let the row $v_{B_i}$ be as follows:

$$v_{B_i}[j] = \begin{cases} \sigma_{ij} & \text{for } j \in [(i-1)B,iB] \\ 0 & \text{ow} \end{cases}$$

Our algorithm simply outputs $\|SA\|_{q \to p}$. The proof of the theorem below can be found in Section D.

**Theorem 5.2.** Given a matrix $A \in \mathbb{R}^{n \times n}$, there exists an $(\hat{O}(\frac{n^2}{B^2}),\alpha)$-sketching family given by $(S,f)$ for the $q \to p$ norm for $p \in [1,2]$.

6 Further Directions

One interesting direction is to study the low-rank approximation problem with respect to the $q \to p$ norm. An important open question in the literature is to find input sparsity time low rank approximation algorithms with respect to the $2 \to 2$ norm, and a natural step might be to try this problem with for $q \to p$ norms for certain $q$ and $p$.
Another interesting problem would be to investigate algorithms for approximate nearest neighbors with respect to the $q \to p$ norm, in light of a question posed by [ANN+17] about what metric spaces admit efficient approximate nearest neighbor algorithms, with matrix norms mentioned as an object of interest.

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A Proofs from Section 2

Proof of Lemma 2.6. For any $x$ that is unit according to $\ell_1$,

$$\|Ax\|_p = \|A_{s,1}x_1 + A_{s,2}x_2 + \ldots + A_{s,n}x_n\|_p \leq \|A_{s,1}\|_p|x_1| + \|A_{s,2}\|_p|x_2| + \ldots + \|A_{s,n}\|_p|x_n| \leq \max_{i \in [n]}\{\|A_{s,i}\|_p\}$$

where the last inequality is because $|x_i|$ give a convex combination and is achieved for $x = e_{i^*}$ where $i^* = \arg \max_i\{\|A_{s,i}\|_p\}$.

Proof of Lemma 2.7. For any $x$ such that there is a coordinate $x_j$ that is strictly between $1$ or $-1$, let $\epsilon = \min\{1 - x_j, x_j + 1\}$, consider

$$\|Ax\|_p = \|A_{s,j}x_j + \sum_{i \neq j} A_{s,i}x_i\|_p \leq \left(\frac{1 + x_j}{2}\right)\|A_{s,j} + \sum_{i \neq j} A_{s,i}x_i\|_p + \left(\frac{1 - x_j}{2}\right)\|A_{s,j} + \sum_{i \neq j} A_{s,i}x_i\|_p$$

where the inequality is due to the triangle inequality. Since $\|Ax\|_p$ is at most a convex combination of the $p$-norms after replacing $x_j$ with $1$ or $-1$, we can make $x_j$ one of $1$ or $-1$ without decreasing the $p$-norm.

Proof of Lemma 2.9. Pick $x^*$ on the unit ball such that $\|Ax^*\|_Y = \|A\|_{\mathcal{X} \to Y}$. There is $x \in N$ such that $\|x^* - x\|_\mathcal{X} < \epsilon$, which means

$$\|A(x^* - x)\|_Y \leq \|A\|_{\mathcal{X} \to Y}\|x - x^*\|_\mathcal{X} < \epsilon\|A\|_{\mathcal{X} \to Y}$$

On the other hand,

$$\|A(x^* - x)\|_Y \geq \|Ax^*\|_Y - \|Ax\|_Y \geq \|A\|_{\mathcal{X} \to Y} - \|Ax\|_Y$$

and hence

$$\|A\|_{\mathcal{X} \to Y} - \|Ax\|_Y < \epsilon\|A\|_{\mathcal{X} \to Y}$$

$$\|A\|_{\mathcal{X} \to Y} < \frac{\|Ax\|_Y}{1 - \epsilon} \leq \frac{1}{1 - \epsilon} \max_{x \in N}\|Ax\|_Y$$

Proof of Lemma 2.10. For $x$ in a normed space $\mathcal{X}$, we use the notation $B_X(r)$ to denote $\{y : \|x - y\|_\mathcal{X} < r\}$, the ball of radius $r$ around $x$.

Start with an empty set $N$ and while there is a point $x$ in the unit ball $B$ that has distance at least $\epsilon$ to every element in $N$, pick $x$ and add it to $N$. This process terminates when every $x \in B$ has distance less than $\epsilon$ to some element in $N$, thereby terminating with $N$ as an $\epsilon$-net. We claim that the size of $N$ meets the desired bound.
By construction, any \( y \) and \( y' \) in \( N \) are at least \( \epsilon \) apart, which means \( B = \{ B_x(\epsilon/2) : x \in N \} \) is a collection of disjoint sets and note that

\[
\bigcup_{S \in B} S \subseteq B_0(1 + \epsilon/2)
\]

By disjointness

\[
\text{Vol} \left( \bigcup_{S \in B} S \right) = \sum_{S \in B} \text{Vol}(S) = |N|\text{Vol}(B_0(\epsilon/2))
\]

where \( \text{Vol}(S) \) is the volume of \( S \) according to the Lebesgue measure.

And thus, we obtain

\[
|N| = \frac{\text{Vol} \left( \bigcup_{S \in B} S \right)}{\text{Vol}(B_0(\epsilon/2))} \leq \frac{\text{Vol}(B_0(1 + \epsilon/2))}{\text{Vol}(B_0(\epsilon/2))} = \left( \frac{1 + \epsilon/2}{\epsilon/2} \right)^n = \left( \frac{2 + \epsilon}{\epsilon} \right)^n
\]

which concludes the proof.

\[\square\]

**B Missing proofs from Section 3**

**Proof of Lemma 3.1.** Draw \( c \log n \) matrices \( S_1, S_2, \ldots, S_{c \log n} \) from \( D \) independently where \( c \) is a constant to be determined later. We define

\[
S := \begin{bmatrix}
S_1 \\
S_2 \\
\vdots \\
S_{c \log n}
\end{bmatrix}
\]

\[
g(Sx) := \text{median}\{f(S_1x), f(S_2x), \ldots, f(S_{c \log n}x)\}
\]

Let’s analyze the probability that \( g(Sx) \) falls outside \( L_x = \left( \frac{1}{2} \|x\|_p, 2\|x\|_p \right) \). In order for that to happen, more than half of \( f(S_1x), \ldots, f(S_{c \log n}x) \) must lie outside \( L_x \), and this happens to each \( f(S_i x) \) with probability at most \( \frac{1}{2} \). Using Hoeffding’s inequality, we know

\[
\Pr[|g(Sx)| \notin L] \leq 2 \exp \left( -\frac{c \log n}{72} \right)
\]

which for appropriate choice of \( c \) can be bounded by \( \frac{1}{n^2} \).

For a matrix \( A \) with \( n \) columns, a union bound tells us that for all \( i \), \( g(SA_{x,i}) \) falls in \( L_{A_{x,i}} \) with probability at least \( 1 - \frac{1}{n} \). Combined with Lemma 2.6, it follows that \( h(SA) := \max_i g(SA_{x,i}) \) is a 2-approximation to \( \|A\|_{1 \to p} \) with probability at least \( 1 - \frac{1}{n} \). \[\square\]
C Missing Proofs from Section 4

C.1 Missing Proofs from Section 4.3

Proof of Lemma 4.5. We denote $Ck$ as $a$ and set the exact value of $a$ in the end of the proof. For a fixed pair $i, j$ let us denote the perturbation term $a\eta_p e_i e_j^\top$ as $E_{ij}$. Recall that from Section 3.1, we know that $\|A\|_{1\rightarrow p} = \max_{i\in[n]} \|A_{*,i}\|_p$ which means that it suffices to give bounds on the maximum $\ell_p$ norm across columns of $G_1$ and $G_2$ respectively.

Since the $\ell_p$ norm is $1$-Lipschitz for any $p \geq 2$, we can apply Theorem 2.3 to show concentration around the expectation for $\|G_{s,i}\|_p$ for any column $i$ of a matrix $G$ of i.i.d Gaussian entries. Hence we have that for any column $i$, and some positive constant $\lambda$

$$\Pr[\|G_{s,i}\|_p \geq \lambda \mathbb{E}[\|G_{s,i}\|_p]] \leq C \exp(-c\lambda^2 \mathbb{E}[\|G_{s,i}\|_p^2])$$

Letting $g$ be an $n$-dimensional vector of i.i.d Gaussians, since we know $\mathbb{E}[\|g\|_p] = \Omega(\sqrt{\log n})$, there exists appropriate constant $\beta$ such that for any column $i$ of $G_1$ we have that $\|(G_1)_{*,i}\|_p$ is less than $\beta \mathbb{E}[\|g\|_p]$ with probability at least $1 - \frac{1}{n}$. By a union bound over all columns, the probability that $\|G_1\|_{1\rightarrow p} \leq \beta \mathbb{E}[\|g\|_p]$ is at least $1 - \frac{1}{n}$.

For a matrix $G_2 = G + E_{ij}$ drawn from $G_2[\alpha, \eta_p]$, we know that the perturbed column $j$ has norm at least $\alpha \eta_p - \|G_{s,i}\|_p$, which satisfies $(\alpha - \beta)\mathbb{E}[\|g\|_p] \leq \|G_2\|_{1\rightarrow p}$. Setting $\alpha \geq (\kappa + 1)\beta$ gives us the desired result.

Proof of Lemma 4.6. Recall perturbation term $a\eta_p e_i e_j^\top$ was referred to as $E_{ij}$. Just as in Lemma 4.2, we can think of $L$ as a $k \times n^2$ matrix that acts on a sample from $G_1$ or $G_2[\alpha]$ as though it were an $n^2$-dimensional vector. Without loss of generality, we can assume that the rows of $L$ are orthonormal, since as before we can always perform a change of basis in post-processing. Thus, the distribution $D_1$ is the same as $\mathcal{N}(0, I_k)$. For fixed $i, j$, the distribution of $L(G + E_{ij})$ is Gaussian with mean vector $L(E_{ij})$ (the $ij^{th}$ column of the $k \times n^2$ matrix $L$ scaled by $a\eta_p$) and covariance $I_k$ because of the following.

$$\mathsf{Cov}(L(G + E_{ij})) = \mathbb{E}(L(G + E_{ij}) - EL(G + E_{ij}))^\top (L(G + E_{ij}) - EL(G + E_{ij}))$$

$$= \mathbb{E}(L(G) - EL(G))^\top (L(G) - EL(G))$$

$$= \mathsf{Cov}_{G \sim \mathcal{N}(0, I_k)}(G) = I_k$$

Thus, $D_2$ is the distribution of picking a random $i, j$ and drawing a matrix from $\mathcal{N}(L(E_{ij}), I_k)$.

We now analyze the total variation distance between $D_1$ and $D_2$ and get the desired bound from a chain of inequalities.

$$d_{TV}(D_1, D_2) = \frac{1}{2} \int_{x \in \mathbb{R}^k} |p_{D_1}(x) - p_{D_2}(x)| dx$$
\[
\frac{1}{2} \int_{x \in \mathbb{R}^k} \left| \sum_{i,j} \frac{1}{n^2} p_{D_i}(x) - \frac{1}{n^2} p_{N(L(E_{ij}), I_k)}(x) \right| dx
\]
\[
\leq \frac{1}{n^2} \sum_{i,j} \frac{1}{2} \int_{x \in \mathbb{R}^k} \left| p_{D_i}(x) - p_{N(L(E_{ij}), I_k)} \right| dx
\]
\[
= \frac{1}{n^2} \sum_{i,j} d_{TV}(D_{ij}, N(L(E_{ij}), I_k))
\]
\[
= \frac{1}{n^2} \sum_{i,j} d_{TV}(N(0, I_k), N(L(E_{ij}), I_k))
\]
\[
\leq \frac{1}{n^2} \sum_{i,j} C' \alpha \eta_p \| L_{*,ij} \|_2
\]
\[
= \frac{C' \alpha \eta_p}{n^2} \| L \|_{1,2}
\]
\[
\leq \frac{C' \alpha \eta_p}{n^2} \cdot n \| L \|_F = C' \alpha \eta_p \cdot \frac{\sqrt{k}}{n}
\]
[by Cauchy-Schwarz]

\[
\square
\]

C.2 Missing Proofs from Section 4.6

Proof of Lemma 4.12. We claim that for a diagonal matrix \( D \), \( \arg \max_{x \in \{e_i\}} \| Dx \|_p \) is achieved when \( x \) is one of the \( e_i \) standard basis vectors. To see this,
\[
\| Dx \|_p^p = \sum_{i=1}^n |d_{ii} x_i|^p = \sum_{i=1}^n |d_{ii}|^p (|x_i|^q)^{p/q} \leq \sum_{i=1}^n |d_{ii}|^p |x_i|^q \leq \max_i |d_{ii}|^p
\]
which is achieved by picking \( x = e_{i^*} \) where choice of \( i = i^* \) maximizes \( d_{ii} \).

Thus, to analyze the \( q \rightarrow p \) norm of \( G_1 \), it suffices to analyze \( \max_{x \in \{e_i\}} \| G_1 x \|_p \), which is the same as \( \| g \|_\infty \) where \( g \) is a vector of i.i.d. Gaussians. We can extract from the proof of Lemma 4.5 that \( \| g \|_\infty \) is upper bounded by \( \beta \sqrt{\log n} \) with probability at least \( 1 - \frac{1}{n} \).

On the other hand, if the perturbation is at index \((i, i)\) and we pick \( \alpha = \kappa(\beta + 1) \), then \( \| G_2 e_i \|_p \) is at least \( \kappa \beta \sqrt{\log n} \) with probability at least \( 1 - \frac{1}{n^2} \), implying the desired separation.

\[
\square
\]

D General approximation factors \( \alpha \)

D.1 Sketching Matrix Construction and Upper Bounds

Let us first define our sketch and then analyze its performance. For the sketch \( S \), we group the rows of \( A \) into \( n \) groups of size \( \alpha^2 \). We label the groups by \( B_1, \ldots, B_{\alpha^2} \) and let \( \sigma_{11}, \ldots, \sigma_{\alpha^2} \) be \( \pm 1 \) i.i.d random variables with equal probability for block \( B_i \). Notice then that the \( i \)-th row of \( S \) given by \((SA)_{i,*}\) is:
\[
(SA)_{i,*} \triangleq \sum_{j \in B_i} \sigma_{jj} A_{i,*}
\]
To analyze the performance of this sketch, we will need a helpful inequality describing the behavior of a random signed sums of reals.

**Theorem D.1. Khintchine’s Inequality [Haa81]**

Let \( \{x_i\}_{i=1}^n \in \mathbb{R} \) be reals and let \( \{s_i\}_{i=1}^n \) be i.i.d. \( \pm 1 \) random variables with equal probability and let \( 0 < t < \infty \), we then have:

\[
A_p \sqrt{\sum_{i=1}^{n} x_i^2} \leq \mathbb{E} \left| \sum_{i=1}^{n} s_i x_i \right|^{p/2} \leq B_p \sqrt{\sum_{i=1}^{n} x_i^2}
\]

For some constants \( A_p, B_p \) that only depend on \( p \).

Also recall that by Jensen’s inequality, we can relate two norms of a vector \( x \in \mathbb{R}^n \).

**Remark D.2.** For two positive reals, \( p \geq q > 1 \) and for a vector \( x \in \mathbb{R}^n \) we have that:

\[
\|x\|_p \leq n^{\frac{1}{q} - \frac{1}{p}} \|x\|_q
\]

We then have the following theorems describing the sketching complexity of the sketch \( S \) for \( 1 \leq p \leq 2 \) and for \( p > 2 \).

**Theorem D.3.** For any \( 1 \leq p \leq 2 \) and for the maximizer \( x \in \mathbb{R}^n \) of \( \|A\|_q \rightarrow p \) the sketch \( S \) defined earlier where each block \( B_i \) has size \( B \) has the property that

\[
\Theta(1) \frac{1}{B^{1-\frac{1}{p}}} \|SAx\|_p \leq \|Ax\|_p \leq \Theta(1) B^{\frac{1}{p} - \frac{1}{2}} \|SAx\|_p
\]

with probability at least \( \frac{100}{99} \).

**Proof.** Let us first show the first inequality in the theorem statement.

For some coordinate \( 1 \leq i \leq \frac{n}{B} \):

\[
\|(SAx)_i\|^p = \left| \sum_{j \in B_i} \sigma_j(Ax)_j \right|^p \leq \left( \sum_{j \in B_i} |(Ax)_j| \right)^p
\]

By **Remark D.2** relating \( \|\cdot\|_1 \) and \( \|\cdot\|_p \)

\[
\leq B^{p-1} \sum_{j \in B_i} |(Ax)_j|^p
\]

\[
\therefore \|(SAx)_i\|_p = \left( \sum_{i=1}^{n/B} |(SAx)_i|^p \right)^{1/p} \leq B^{1-\frac{1}{p}} \|Ax\|_p
\]

Notice that the first inequality holds irrespective of the vector \( x \), it holds for all vectors. Now let us show the second inequality of the theorem statement.
For some coordinate $1 \leq i \leq \frac{n}{B}$:

$$\left( \sum_{j \in B_i} (Ax)^j \right)^{1/p} \leq B^{\frac{1}{p} - \frac{1}{2}} \left( \sum_{j \in B_i} (Ax)^j \right)^{1/2}$$  \[1\]

By Remark D.2

$$\leq \Theta(1)B^{\frac{1}{p} - \frac{1}{2}} \frac{E}{} \sum_{j \in B_i} \sigma_j (Ax)^j \right|^{p^{1/p}}$$  \[2\]

By Khintchine’s Ineq.

\[\because \sum_{i=1}^{n/B} \sum_{j \in B_i} (Ax)^j \right| p = \|Ax\|^p_p \leq \Theta(1)B^{p(\frac{1}{p} - \frac{1}{2})} \|SAx\|_p^p\]

Notice that the second inequality of the theorem statement follows by Markov’s inequality.

Notice that the success probability of line [2] is constant for each block. To get constant success probability over the entire set of blocks, we construct $O(\log(n))$ i.i.d copies of each block $B_i$ given by $\{B_j\}_{i=1}^{O(\log(n))}$. We then pick $j$ such that it is the index realizing the quantity $\text{median}_{j \in O(\log(n))} \|S_j Ax\|_p$ where $S_j$ corresponds the sketch with the $j^{th}$ copy of the blocks. Then, by standard concentration bounds, we can get $1 - \frac{1}{n/B}$ success probability for each set of blocks $B_i$ and then union bound over the $\frac{n}{B}$ blocks giving us constant success probability.

\[\square\]

**Theorem D.4.** For any $p > 2$ and for the maximizer $x \in \mathbb{R}^n$ of $\|A\|_{q \rightarrow p}$ the sketch $S$ defined earlier where each block $B_i$ has size $B$ has the property that

$$\Theta(1) \frac{1}{B^{1 - \frac{1}{p}}} \|SAx\|_p \leq \|Ax\|_p \leq \Theta(1) \|SAx\|_p$$

The proof for Theorem D.4 is the same as that for Theorem D.3 except that there is no dilation while upper bounding the $\|Ax\|_p$ with the 2-norm in line [1] of the proof.

Notice that the above theorems imply that the sketch $S$ is a $\sqrt{B}$-approximation when $0 \leq p \leq 2$ and a $B^{1 - \frac{1}{p}}$-approximation when $p > 2$ because it states that the sketch is stretching $\|Ax\|_p^p$ by at most some factor and dilating it by at most some factor and hence the approximation ratio is simply the product of these factors.