Stable sheaves with twisted sections and the Vafa–Witten equations on smooth projective surfaces

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Abstract

This article describes a Hitchin–Kobayashi style correspondence for the Vafa–Witten equations on smooth projective surfaces. This is an equivalence between a suitable notion of stability for a pair \((E, \varphi)\), where \(E\) is a locally-free sheaf over a surface \(X\) and \(\varphi\) is a section of \(\text{End}(E) \otimes K_X\); and the existence of a solution to certain gauge-theoretic equations, the Vafa–Witten equations, for a Hermitian metric on \(E\). It turns out to be a special case of results obtained by Álvarez-Cónsul and García-Prada on the quiver vortex equation. In this article, we give an alternative proof which uses a Mehta–Ramanathan style argument originally developed by Donaldson for the Hermitian–Einstein problem, as it relates the subject with the Hitchin equations on Riemann surfaces, and surely indicates a similar proof of the existence of a solution under the assumption of stability for the Donaldson–Thomas instanton equations described in [T1] on smooth projective threefolds; and more broadly that for the quiver vortex equation on higher dimensional smooth projective varieties.

1 Introduction

In this article, we consider a set of gauge-theoretic equations on smooth projective surfaces, introduced by Vafa and Witten [VW] in the study of S-duality conjecture for \(N = 4\) supersymmetric Yang–Mills theory originally on closed four-manifolds, and recently discussed also by Haydys [Ha] and [W] in the context of “categorification” of Khovanov homology.

The equation can be seen as a higher-dimensional analogue of the Hitchin equation on compact Riemann surfaces [Hi]. The Hitchin equation is an equation for a pair consisting of a holomorphic structure on a vector bundle \(E\) over a Riemann surface \(\Sigma\), and a holomorphic section \(\Phi\) of the associated
bundle $\text{End}(E) \otimes K_\Sigma$, where $K_\Sigma$ is the canonical bundle of $\Sigma$. Simpson [S1] generalized it to higher dimensions for a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a torsion-free sheaf on a projective variety $X$, and $\theta$ is a section of $\text{End}(\mathcal{E}) \otimes \Omega^1_X$. The Vafa–Witten equation can be seen as an analogue of the Hitchin equations for surfaces, but in a different way of that pursued by Simpson mentioned above, since it takes up a section of $\text{End}(E) \otimes K_X$ as an extra field, which is just the same as in the Hitchin case, rather than that of $\text{End}(E) \otimes \Omega^1_X$ as in the Simpson case. Also, the Donaldson–Thomas instanton equation on compact Kähler threefolds, described in [T1], can be seen as a three-dimensional counterpart of the Hitchin equation in the same way as the Vafa–Witten equations. More broadly, these equations can be seen as special cases of those studied by Álvarez-Cónsul and García-Prada [AG] as the case of a twisted quiver bundle with one vertex and one arrow, whose head and tail coincide, and with twisting sheaf the anti-canonical bundle.

The Vafa–Witten equations. Let us describe the equation in the original form first. Let $X$ be a closed, oriented, smooth Riemannian four-manifold with Riemannian metric $g$, and let $P \to X$ be a principal $G$-bundle over $X$ with $G$ being a compact Lie group. We denote by $\mathcal{A}_P$ the set of all connections of $P$, and by $\Omega^+(X, \mathfrak{g}_P)$ the set of self-dual two-forms valued in the adjoint bundle $\mathfrak{g}_P$ of $P$. We consider the following equations for a triple $(A, B, \Gamma) \in \mathcal{A}_P \times \Omega^+(X, \mathfrak{g}_P) \times \Omega^0(X, \mathfrak{g}_P)$.

$$d_A \Gamma + d^*_A B = 0,$$

$$F^+_A + [B.B] + [B, \Gamma] = 0,$$

where $F^+_A$ is the self-dual part of the curvature of $A$, and $[B.B] \in \Omega^+(X, \mathfrak{g}_P)$ (See [M, §A.1], or [T2, §2] for its definition). We call these equations the Vafa–Witten equations. The above equations (1.1) and (1.2) with a gauge fixing condition form an elliptic system with the index being always zero.

Mares studied analytic aspects of the Vafa–Witten equations in his Ph.D thesis [M]. He also described the equations on compact Kähler surfaces, and discussed a relation between the existence of a solution to the equations and a stability of vector bundles as mentioned below.

The equations on compact Kähler surfaces. Let $X$ be a compact Kähler surface, and let $E$ a Hermitian vector bundle of rank $r$ over $X$. On a compact Kähler surface, the Vafa–Witten equations (1.1) and (1.2) reduce
to the following (see [M, Chap.7] for the detail).

\[ \bar{\partial}_A \varphi = 0, \]
\[ F_A^{0,2} = 0, \quad F_A^{1,1} \wedge \omega + [\varphi, \bar{\varphi}] = \frac{i\lambda(E)}{2} Id_E \omega^2, \]

where \( \varphi \in \Omega^{2,0}(X, \text{End}(E)) \), and \( \lambda(E) = 2\pi c_1(E) \cdot [\omega] / r [\omega]^2 \).

As we mentioned above, this can be seen as a generalization of the Hitchin equation [Hi] to Kähler surfaces. In fact, the stability condition that we consider is an analogy of that to the Hitchin equation.

**Stability for pairs.** We consider a pair \((\mathcal{E}, \varphi)\) consisting of a torsion-free sheaf \(\mathcal{E}\) and a section \(\varphi\) of \(\text{End}(\mathcal{E}) \otimes K_X\), which satisfies a stability condition. The stability here is defined by a slope for \(\varphi\)-invariant subsheaves similar to the Hitchin equation case [Hi].

Let \(X\) be a compact Kähler surface, and let \(\mathcal{E}\) be a torsion-free sheaf on \(X\), and let \(\varphi\) be a section of \(\text{End}(\mathcal{E}) \otimes K_X\). A subsheaf \(\mathcal{F}\) of \(\mathcal{E}\) is said to be a \(\varphi\)-invariant if \(\varphi(\mathcal{F}) \subset \mathcal{F} \otimes K_X\). We define a slope \(\mu(\mathcal{F})\) of a coherent subsheaf \(\mathcal{F}\) of \(\mathcal{E}\) by

\[ \mu(\mathcal{F}) := \frac{1}{\text{rank}(\mathcal{F})} \int_X c_1(\text{det} \mathcal{F}) \wedge \omega. \]

**Definition 1.1.** A pair \((\mathcal{E}, \varphi)\) consisting of a torsion-free sheaf \(\mathcal{E}\) and a section \(\varphi\) of \(\text{End}(\mathcal{E}) \otimes K_X\) is called semi-stable if \(\mu(\mathcal{F}) \leq \mu(\mathcal{E})\) for any \(\varphi\)-invariant coherent subsheaf \(\mathcal{F}\) with \(\text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})\). A pair \((\mathcal{E}, \varphi)\) is called stable if \(\mu(\mathcal{F}) < \mu(\mathcal{E})\) for any \(\varphi\)-invariant coherent subsheaf \(\mathcal{F}\) with \(\text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})\).

**Definition 1.2.** A pair \((\mathcal{E}, \varphi)\) consisting of a torsion-free sheaf \(\mathcal{E}\) and a section \(\varphi\) of \(\text{End}(\mathcal{E}) \otimes K_X\) is said to be poly-stable if it is a direct sum of stable sheaves with the same slopes in the sense of Definition 1.1.

**The Hitchin–Kobayashi correspondence for the Vafa-Witten equations.** A correspondence we describe in this article is a one-to-one correspondence between the existence of a solution to the Vafa-Witten equations on a locally-free sheaf \(\mathcal{E}\) on a smooth projective surface \(X\) and the stability in the sense of Definition 1.1. This fits into the setting for the above mentioned twisted quiver bundles and the quiver vortex equation studied by Álvarez-Cónsul and García-Prada [AG] (see also [BGM]), and the correspondence turns out to be a special case of their results. We state it in our setting as follows.
Theorem 1.3 ([AG]). Let $X$ be a Kähler surface with Kähler form $\omega$. Let $(E, \varphi)$ be a pair consisting of a locally-free sheaf $E$ on $X$ and a section $\varphi \in \text{End}(E) \otimes K_X$, where $K_X$ is the canonical bundle of $X$. Then, $(E, \varphi)$ is poly-stable if and only if $E$ admits a unique Hermitian metric $h$ satisfying $F_h + \Lambda[\varphi, \bar{\varphi}h] = i\frac{\lambda(E)}{2} \text{Id}_E \omega$, where $F_h$ is the curvature form of $h$, and $\Lambda := (\Lambda \omega)^\ast$.

In this article, we give an alternative proof for the existence part of the above theorem in the case of smooth projective surfaces, stated below as Theorem 1.4, by using a Mehta–Ramanathan style theorem.

Theorem 1.4. Let $X$ be a smooth projective surface, and let $E$ be a holomorphic vector bundle on $X$. We take a holomorphic section $\varphi$ of $\text{End}(E) \otimes K_X$, where $K_X$ is the canonical bundle of $X$. We assume that $(E, \varphi)$ is poly-stable in the sense of Definition 1.2 Then there exists a unique Hermitian metric $h$ of $E$ such that the equation $F_h + \Lambda[\varphi, \bar{\varphi}h] = i\frac{\lambda(E)}{2} \text{Id}_E \omega$ is satisfied.

Our proof, given in the next section, also uses a Donaldson-type functional on the space of Hermitian metrics on $E$, which is a modification of that defined by Donaldson in [D] for solving the Hermitian–Einstein problem. As in the Hermitian–Einstein case, one main point is to obtain a lower bound for the functional. To achieve this we use a Mehta–Ramanathan style argument, in other words, we reduce the problem to the corresponding lower dimensional one; the Hitchin equation on compact Riemann surfaces in our case. This method was developed originally by Donaldson [D] to prove the existence of a Hermitian–Einstein metric on a holomorphic vector bundle over a smooth projective surface under the assumption of stability.

We remark that a parallel argument to that described in this article should give an alternative proof of the existence of a solution under the assumption of stability for the Donaldson–Thomas instanton equations described in [T1] on smooth projective threefolds; and more broadly that for the quiver vortex equation on higher dimensional smooth projective varieties.

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2 Existence of a solution under the assumption of stability

In Section 2.1, we introduce a functional on the space of Hermitian metrics of a holomorphic vector bundle over a compact Kähler surface, and lay out some properties of the functional such as its critical points and convexity. In Section 2.2, we mention the Hitchin equation on a compact Riemann surface and its generalization. We then give a proof of Theorem 1.4 using a Mehta–Ramanathan style theorem in Section 2.3.

2.1 Functional and its properties

Let $E$ be a holomorphic vector bundle over a compact Kähler surface $X$. We denote by $\text{Herm}^+(E)$ the set of all $C^\infty$ Hermitian metrics on $E$. Let $k, h \in \text{Herm}^+(E)$. We connect them by a smooth curve $h_t$ ($0 \leq t \leq 1$) so that $k = h_0$ and $h = h_1$. We denote by $F_{h_t} = F(h_t) \in A^{1,1}(\text{End}(E))$ the curvature of $h_t$. We put $v_t = h_t \partial_t h_t \in A^0(\text{End}(E))$. Let $\varphi$ be a holomorphic section of $\text{End}(E) \otimes K_X$, where $K_X$ is the canonical bundle of $X$. We define the following.

$$Q_1(h, k) := \log(\det(k^{-1}h)),$$

$$Q_2(h, k) := \sqrt{-1} \int_0^1 \text{tr} \left( v_t \cdot \left( F_{h_t} + \Lambda[\varphi, \bar{\varphi}^{h_t}] \right) \right) d\tau.$$

We then consider the following functional for pairs of Hermitian metrics $h$ and $k$.

$$D_{\varphi}(h, k) := \int_X Q_2(h, k) \wedge \omega - \frac{\lambda(E)}{2} \int_X Q_1(h, k) dV_g,$$

where $\lambda(E) = 2\pi c_1(E) \cdot [\omega]/r[\omega]^2$. This is a modification of the functional defined by Donaldson in [D] for solving the Hermitian–Einstein problem.

Firstly, we prove that $D_{\varphi}(h, k)$ does not depend upon the choice of a curve joining $k$ and $h$. As in the case of the Hermitian–Einstein metrics, one can prove the following.

**Proposition 2.1.** Let $h_t$ ($a \leq t \leq b$) be a differentiable curve in $\text{Herm}^+(E)$, and let $k$ be a fixed Hermitian metric of $E$. Then

$$\sqrt{-1} \int_a^b \text{tr} \left( v_t \cdot \left( F_{h_t} + \Lambda[\varphi, \bar{\varphi}^{h_t}] \right) \right) d\tau + Q_2(h_a, k) - Q_2(h_b, k)$$

lies in $\partial A^{0,1} + \bar{\partial} A^{1,0}$. 


Proof. Our proof goes in a similar way to that by Kobayashi [Ko, Chap. VI, Lem. 3.6] for the Hermitian–Einstein metrics except that we deal with the extra field $\varphi$. Let $\Delta$ be the domain in $\mathbb{R}^2$ defined by $\Delta := \{(t, s) : a \leq t \leq b, 0 \leq s \leq 1\}$, $h : \Delta \to \text{Herm}^+(E)$ a smooth map with $h(t, 0) = k, h(t, 1) = h_t$ for $a \leq t \leq b$, and $h(a, s)$ and $h(b, s)$ are the line segments from $k$ to $h_a$ and $k$ to $h_b$ respectively. Set $v = h^{-1}\partial_s h, w = h^{-1}\partial_t h, F = \bar{\partial}(h^{-1}\partial h)$, and $\Phi = \sqrt{-1}\text{tr}(h^{-1}\partial h(F + \Lambda[\varphi, \varphi^h]))$, where $\bar{d} = \frac{\partial}{\partial s}ds + \frac{\partial}{\partial t}dt$, the exterior derivative on $\Delta$. We then use the Stokes formula for the 1-form $\Phi$, namely,

$$
\int_{\Delta} \bar{d}\Phi = \int_{\partial\Delta} \Phi. \tag{2.2}
$$

The right-hand-side of (2.2) becomes

$$
\int_{\partial\Delta} \Phi = \sqrt{-1}\int_a^b \text{tr} \left( v_t \cdot (F h_t + \Lambda[\varphi, \varphi^h]) \right) dt + Q_2(h_a, k) - Q_2(h_b, k).
$$

Therefore we need to prove that $\bar{d}\Phi \in \partial A^{0,1} + \bar{\partial} A^{1,0}$. Put $M = F + \Lambda[\varphi, \varphi^h]$. From the definition of $\Phi$, we get

$$
\bar{d}\Phi = \sqrt{-1}\text{tr} \left( (\partial_s v - \partial_t w)M - w\partial_t M + v\partial_s M \right) ds \wedge dt.
$$

Furthermore, some calculations similar to [Ko, pp.199–200] show that $\partial_s v = -vw + h^{-1}\partial_s h, \partial_t w = -vw + h^{-1}\partial_t h, \partial_t M = \bar{\partial}D'v + \Lambda[\varphi, \partial_t \varphi^h], \partial_s M = \bar{\partial}D'w + \Lambda[\varphi, \partial_s \varphi^h]$, where $D = D' + \bar{\partial}(= D' + \partial)$ is the exterior covariant differentiation of the Hermitian connection defined by $h$. Using these, we get

$$
\bar{d}\Phi = \sqrt{-1}\text{tr} \left( \right. \left( vw - wv \right)F - w\bar{\partial}D'v + v\bar{\partial}D'w \left. \right) ds \wedge dt
$$

$$
+ \sqrt{-1}\text{tr} \left( \Lambda(v[\varphi, \partial_s \varphi^h] - w[\varphi, \partial_t \varphi^h] + (vw - wv)[\varphi, \varphi^h]) \right) ds \wedge dt. \tag{2.3}
$$

One can easily check that the second term of (2.3) vanishes because of $\partial_s \varphi^h = [\varphi^h, w], \partial_t \varphi^h = [\varphi^h, v]$, and the Jacobi identity. On the other hand, the first term of (2.3), which does not involve the extra field $\varphi$, is the same term in the Hermitian–Einstein case as in [Ko], and it becomes

$$
- \sqrt{-1}\text{tr} \left( vD'\bar{\partial}w + w\bar{\partial}D'v \right) ds \wedge dt.
$$

Hence, defining the $(0, 1)$-form $\alpha := \sqrt{-1}\text{tr} (v\bar{\partial}w)$, we get

$$
\bar{d}\Phi = - (\partial\alpha + \bar{\partial}\bar{\alpha} + \sqrt{-1}\bar{\partial}\partial\text{tr} (vw)) ds \wedge dt.
$$

Thus, $\bar{d}\Phi \in \partial A^{0,1} + \bar{\partial} A^{1,0}$. \qed
From Proposition 2.1, we deduce the following.

**Corollary 2.2.** Let \( h_t (a \leq t \leq b) \) be a piecewise differentiable closed curve in \( \text{Herm}^+(E) \) (namely, \( h_a = h_b \)). Put \( v_t = h_t^{-1} \partial_t h_t \). Then
\[
\sqrt{-1} \int_a^b \text{tr} \left( v_t \cdot \left( F_{h_t} + \Lambda [\varphi, \bar{\varphi}^{h_t}] \right) \right) dt
\]
lies in \( \partial A^{0,1} + \bar{\partial} A^{1,0} \).

Hence we obtain the following.

**Proposition 2.3.** \( D\varphi(h, k) \) does not depend on the choice of a curve joining \( k \) to \( h \).

We next fix a Hermitian metric \( k \) on \( E \), and define a functional \( D\varphi : \text{Herm}^+(E) \to \mathbb{R} \) by \( D\varphi(h) := D\varphi(k, h) \) for \( h \in \text{Herm}^+(E) \). Following [Ko, Chap.VI §3], one can prove the following two propositions. The first says that the critical points of the functional are solutions of the Vafa-Witten equations.

**Proposition 2.4.** Let \( k \) be a fixed Hermitian metrics on \( E \). Then \( h \) is a critical point of \( D\varphi(\cdot) := D\varphi(k, \cdot) \) if and only if \( h \) satisfies
\[
F_{h_t} + \Lambda [\varphi, \bar{\varphi}^{h_t}] = \sqrt{-1} \lambda(E) \frac{\Lambda(E)}{2} \text{Id}_E \omega.
\]

**Proof.** Let \( h_t (a \leq t \leq b) \) be a differentiable curve in \( \text{Herm}^+(E) \), which connects \( h \) and \( k \). Then, differentiating (2.1) with respect to \( t \), we get
\[
\frac{d}{dt} Q_2(h_t, k) = \sqrt{-1} \text{tr} (v_t \cdot (F_{h_t} + \Lambda [\varphi, \bar{\varphi}^{h_t}]))
\]
up to \( \partial A^{0,1} + \bar{\partial} A^{1,0} \). In addition, we have \( \partial_t Q_1(h_t, k) = \text{tr}(v_t) \). Hence we obtain
\[
\frac{d}{dt} D\varphi(h_t) = \sqrt{-1} \int_X \text{tr} (v_t \cdot \mu_\varphi(h_t)), \tag{2.4}
\]
where \( \mu_\varphi(h_t) := F_{h_t} \wedge \omega + [\varphi, \bar{\varphi}^{h_t}] - \sqrt{-1} \lambda(E) \frac{\Lambda(E)}{2} \text{Id}_E \omega^2 \). Thus the assertion holds.

The next proposition says that the functional \( D\varphi(\cdot) \) is convex.

**Proposition 2.5.** Let \( k \) be a fixed Hermitian metric on \( E \), and let \( \tilde{h} \) be a critical point of \( D\varphi(\cdot) = D\varphi(k, \cdot) \). Then \( D\varphi(\cdot) \) attains an absolute minimum at \( \tilde{h} \).
Proof. Let \( h_t \) \((0 \leq t \leq 1)\) be a differential curve with \( h_0 = \tilde{h} \). Differentiating (2.4), we get

\[
\frac{d^2}{dt^2} D\phi(h_t) = \frac{d}{dt} \sqrt{-1} \int_X (v_t \cdot \mu(h_t)) = \sqrt{-1} \int_X ((\partial_t v_t) \cdot \mu + v_t \cdot \partial_t \mu). 
\]

Furthermore, since \( \partial_t \bar{\phi} = [\bar{\phi}, v] \), we obtain

\[
\partial_t \mu = \bar{\partial} D' v \wedge \omega + [\phi, [\bar{\phi} h_t, v]]. 
\]

As \( h_0 \) is a critical point of \( D\phi(h, k) \), we get

\[
\frac{d^2}{dt^2} D\phi(h_t) \bigg|_{t=0} = \sqrt{-1} \int_X \left( v_t \cdot \left( \bar{\partial} D' v_t \wedge \omega^2 + [\phi, [\bar{\phi} h_t, v_t]] \right) \right) \bigg|_{t=0} = \|D' v_t\|_{L^2}^2 \bigg|_{t=0} + \|[\bar{\phi} h_t, v_t]\|_{L^2}^2 \bigg|_{t=0}. 
\]

Hence, \( h_0 \) is at least a local minimum of \( D\phi \). We then consider an arbitrary element in \( \text{Herm}^+(E) \) and join it to \( h_0 \) by a geodesic \( h_t \). Since \( \partial_t v_t = 0 \) if \( h_t \) is a geodesic (see [Ko, p. 204]), from the same computation above, we obtain

\[
\frac{d^2}{dt^2} D\phi(h) = \|D' v_t\|_{L^2}^2 + \|[\bar{\phi} h_t, v_t]\|_{L^2}^2. 
\]

This implies Proposition 2.5. \( \square \)

As in the case of the Hermitian–Einstein problem [D, Prop.6 (iii)], the functional \( D\phi(h, k) \) is the integration of Bott–Chern forms as stated below. This can be proved for example by a similar argument to that by Bradlow–Gomez [BG, Appendix] for the Higgs bundle case, so we here omit the details.

**Proposition 2.6.** Let \( h, h' \in \text{Herm}^+(E) \). Then

\[
\sqrt{-1} \bar{\partial} \partial \left( Q_2(h, h') - \frac{\lambda(E)}{2} Q_1(h, h') \wedge \omega \right) = -\frac{1}{2} \text{tr} \left( \left( \sqrt{-1}(F_h + \Lambda[\phi, \phi^h]) - \frac{\lambda(E)}{2} I_X \omega \right)^2 \right) \\
+ \frac{1}{2} \text{tr} \left( \left( \sqrt{-1}(F_{h'} + \Lambda[\phi, \phi^{h'}]) - \frac{\lambda(E)}{2} I_X \omega \right)^2 \right).
\]
2.2 The Hitchin equation on compact Riemann surfaces

In this section, we briefly describe the Hitchin equation [Hi] on compact Riemann surfaces and its generalization.

The equations. Let $\Sigma$ be a compact Riemann surface, and let $E$ a Hermitian vector bundle of rank $r$ over $\Sigma$. We consider the following equations for a pair $(A, \Phi) \in \mathcal{A}_E \times \Omega^{1,0}(\Sigma, \text{End} (E))$, where $\mathcal{A}_E$ is the set of all connections on $E$.

\[
\bar{\partial}_A \Phi = 0, \quad F_A + [\Phi, \bar{\Phi}] = \sqrt{-1} \frac{\lambda(E)}{2} \text{Id}_E \omega,
\]

where $F_A$ is the curvature of $A$, and $\lambda(E) := 2\pi c_1(E)/r[\omega]$. The above equations are called the Hitchin equations.

Stability. For a pair $(E, \Phi)$ consisting of a holomorphic vector bundle $E$ over a compact Riemann surface $\Sigma$ and a holomorphic section $\Phi$ of $\text{End} (E) \otimes K_\Sigma$, (semi-)stability of it can be defined in the same style of Definition 1.1. Hitchin [Hi] proved the existence of a solution to the equations can be deduced from the assumption of stability (see also [S1]).

L-twisted Hitchin equations. There is a generalization of the Hitchin equations on a compact Riemann surface $\Sigma$ by Lin [L] (see also [GR]), in which one takes a line bundle $L$ on $\Sigma$ instead of $K_\Sigma$. Namely, one considers the following equations on $X$ for a pair $(A, \Phi)$ consisting of a connection $A$ and a section $\Phi$ of $\text{End} (E) \otimes L$.

\[
\bar{\partial}_A \Phi = 0, \quad \Lambda F_A + \sigma([\Phi, \bar{\Phi}]) - \frac{i\lambda(E)}{2} \text{Id}_E = 0,
\]

where $\sigma$ is the contraction of sections of $\text{End} (E) \otimes L \otimes L^*$ to those of $\text{End} (E)$. We call these the L-twisted Hitchin equations. This was further generalized by Álvarez-Cónsul and García-Prada [AG] (see also [BGM]).

As in the case of the Vafa–Witten equations described in the previous section, a similar functional $D_{\Phi}(h, k)_\Sigma$ for a pair of Hermitian metrics $h$ and $k$ with properties such as Propositions 2.4 and 2.5 can be also defined for the L-twisted Hitchin equations as well.

Since the existence of a solution to the L-twisted Hitchin equations under the assumption of stability with $L$-valued operator was proved by Lin [L] (this was further generalized by Álvarez-Cónsul and García-Prada [AG], see also [BGM]), thus we have the following.
Proposition 2.7. Let $E$ be a holomorphic vector bundle on compact Riemann surface $\Sigma$, and let $L$ be a line bundle on $X$. Let $\Phi$ be a holomorphic section of $\text{End}(E) \otimes L$. If $(E, \Phi)$ is semi-stable with $L$-valued operator, then for any fixed Hermitian metric $k$ in $\text{Herm}^+(E)$, the set $\{D_\Phi(h, k)_\Sigma, h \in \text{Herm}^+(E)\}$ is bounded below.

We use this in the proof of Theorem 1.4.

2.3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. We follow a proof for the Hermitian–Einstein case, given by Donaldson [D] (see also [Ko, Chap.VI]). We consider the following evolution equation, which is the gradient flow for the functional $D_\varphi(h, k)$.

$$\partial_t h_t = - \left( i(\Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}]) - \frac{\lambda(E)}{2} h_t \right). \quad (2.7)$$

The above evolution equation (2.7) has a unique smooth solution for $0 \leq t < \infty$. One can see this by using a similar argument in [Ko, Chap.VI].

We then prove the following as in the Hermitian–Einstein case.

Proposition 2.8. Let $h_t (0 \leq t < \infty)$ be a 1-parameter family in $\text{Herm}^+(E)$, which satisfies (2.7). Then

(i) $\frac{d}{dt} D_\varphi(h_t, k) = - \left\| i(\Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}]) - \frac{\lambda(E)}{2} Id_E \right\|^2_{L^2} \leq 0.$

(ii) $\max_X \left\| i(\Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}]) - \frac{\lambda(E)}{2} Id_E \right\|^2$ is a monotone decreasing function of $t$.

(iii) If $D_\varphi(h_t, k)$ is bounded from below, then

$$\max_X \left\| i(\Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}]) - \frac{\lambda(E)}{2} Id_E \right\|^2 \to 0$$

as $t \to \infty$.

Proof. The proof here is a modification of that given in [Ko, Chap.VI §9, pp. 224–226]. Firstly, the above (i) is nothing but Proposition 2.4. Namely,
as $h_t$ satisfies (2.7), we get
\[
\frac{d}{dt} D_\varphi(h_t, k) = -\left( i\langle \Lambda F_{h_t} + [\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E, i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right)
\]
\[
= -\left\| i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right\|^2_{L^2}.
\]

To prove (ii), we define the operator $\Box' s := (iD''D's \land [\varphi, [\bar{\varphi}^{h_t}, s]])$. One can easily check that $\Box' v_t = \partial_t \left( i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right)$. Then, by using the evolution equation (2.7), we get
\[
(\partial_t + \Box') \left( i\langle \Lambda F_{h_t} + [\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right) = 0.
\]
We also have
\[
\Delta \left( i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right)^2 = -2 \left|\partial_t \left( i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right) \right|^2.
\]
Hence, we get
\[
(\partial_t + \Delta) \left( i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right)^2 = -2 \left|\partial_t \left( i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right) \right|^2 \leq 0.
\]
Thus, the maximum principle implies (ii).

Once (i) and (ii) are obtained, then (iii) follows from a similar argument as in [Ko, Chap.VI §9, pp 225–226]. Namely, by using a maximum principle argument, one bounds $\max_X \left| i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \right|^2$ by $\| i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \|_{L^2}^2$. One then deduces $\| i\langle \Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}] \rangle - \frac{\lambda(E)}{2} Id_E \|_{L^2}^2 \to 0$ as $t \to \infty$ from (ii).

We then prove that $D_\varphi(h, k)$ is bounded below under the assumption that $E$ is semi-stable.
Firstly, we recall that the notion of stability has the following generalization introduced by Simpson [S2, §3].

**Definition 2.9.** (a) Let \( W \) be a vector bundle on a compact Kähler manifold \( X \). A sheaf \( E \) together with a map \( \eta : E \rightarrow E \otimes W \) is called a sheaf with \( W \)-valued operator \( \eta \). (b) A torsion-free sheaf \( E \) with \( W \)-valued operator \( \eta \) is said to be semi-stable if \( \mu(F) \leq \mu(E) \) for any coherent subsheaves \( F \) of \( E \) with \( \text{rank}(F) < \text{rank}(E) \) and \( \eta(F) \subset F \otimes W \). It is called stable if the strict inequality holds in the definition of the semi-stability.

The (semi-)stability of Definition 1.1 is included in this generalized notion of (semi-)stability by taking \( W = K_X \). This notion was further generalized to twisted quiver sheaves by Álvarez-Cónsul and García-Prada [AG] (see also [BGM]).

As mentioned in [S2, pp.37–38], we note that the arguments of Mehta–Ramanathan [MR1], [MR2] indicate the following.

**Proposition 2.10** ([S2], Prop. 3.6). If \( E \) is a torsion-free (semi-)stable sheaf on \( X \) with \( W \)-valued operator, then there exists a positive integer \( m \) such that, for a generic smooth curve \( D \subset X \) in a linear system \( |O_X(m)|, E|_D \) is a (semi-)stable sheaf with \( W|_D \)-valued operator.

From this, we get the following.

**Corollary 2.11.** Let \( (E, \varphi) \) be a torsion-free (semi-)stable pair on \( X \) in the sense of Definition 1.1. Then, there exists a positive integer \( m \) such that, for a generic smooth curve \( D \subset X \) in a linear system \( |O_X(m)|, (E|_D, \varphi|_D) \) is a (semi-)stable pair with \( (K_D \otimes O_D(-D)) \)-valued operator on \( D \).

**Proof.** This just follows from Proposition 2.10 and the adjunction formula. In fact, from Proposition 2.10, there exists a positive integer \( m \) such that, for a generic smooth curve \( D \subset X \) in \( |O_X(m)|, E|_D \) is a (semi-)stable sheaf with \( K_X|_D \)-valued operator. On the other hand, from the adjunction formula, we have \( K_X|_D = K_D \otimes O_D(-D) \). Hence, the assertion holds.

We now describe the behaviour of the functional \( D_\varphi(h, k) \) when it is restricted to a smooth curve \( D \subset X \). The notation we use is that \( D_\varphi(h, k)_X \) is the functional for the Vafa–Witten equations on \( E \), and \( D_\varphi(h, k)_D \) is that for the \( L \)-twisted Hitchin equations on \( E|_D \), where we take \( L = K_D \otimes O_D(-D) \).

**Proposition 2.12.** Let \( E \) be a holomorphic vector bundle on a smooth projective surface \( X \), and let \( \varphi \) be a holomorphic section of \( \text{End}(E) \otimes K_X \). Let
$D$ be a smooth curve of $X$ such that the line bundle $F$ defined by $D$ is ample. We use a positive closed $(1,1)$-form $\omega$ representing the Chern class of $F$ as a Kähler form for $X$. Then for a fixed Hermitian metric $k$ of $E$ and for all Hermitian metrics $h$ of $E$, we have

$$D_{\varphi}(h,k)_X \geq D_{\varphi}(h,k)_D - C \left( \max_X \left| i(\Lambda F_h + \ast[\varphi, \varphi^h]) - \frac{\lambda(E)}{2} \text{Id}_E \right|^2 \right) - C',$$

(2.8)

where $C$ and $C'$ are positive constants.

Proof. Firstly, we recall the Poincaré–Lelong formula as in [D, pp.12–13], [Ko, Chap.VI §10]. We follow notations in [Ko, Chap.VI §10]. Let $M$ be a smooth projective variety, and let $V$ be a closed hypersurface of $M$. Let $F$ be an ample line bundle on $M$ with a global holomorphic section $s$ such that $s^{-1}(0) = V$. We take a $C^\infty$ positive section $a$ of $F \otimes \bar{F}$. As $F$ is ample, we can take such $a$ so that $\omega := \frac{i}{2\pi} \partial \bar{\partial} \log a$ is positive. We use this as a Kähler form on $M$, and the restriction $\omega_V$ as a Kähler form on $V$. Put $f := |s|^2/a$.

Then, for all $(n-1, n-1)$-form $\eta$ on $M$, we have the following as current (see [Ko, Chap.VI §10]).

$$i \frac{1}{2\pi} \int_M (\log f) \partial \bar{\partial} \eta = \int_V \eta - \int_M \eta \wedge \omega.$$

(2.9)

We now set $\eta = Q_2(h,k) - \frac{\lambda(E)}{2} Q_1(h,k) \wedge \omega$ in (2.9), and get

$$D_{\varphi}(h,k)_X = \int_X \left( Q_2(h,k) - \frac{\lambda(E)}{2} Q_1(h,k) \wedge \omega \right) \wedge \omega$$

$$= \int_D \left( Q_2(h,k) - \frac{\lambda(E)}{2} Q_1(h,k) \wedge \omega \right)$$

$$- \frac{i}{2\pi} \int_X (\log f) \partial \bar{\partial} \left( Q_2(h,k) - \frac{\lambda(E)}{2} Q_1(h,k) \wedge \omega \right).$$

We then use Proposition 2.6 to obtain

$$D_{\varphi}(h,k)_X = D_{\varphi}(h,k)_D$$

$$+ \frac{1}{4\pi} \int_X (\log f) \left( \text{tr} \left( i(F_h + \Lambda[\varphi, \varphi^h]) - \frac{\lambda(E)}{2} \text{Id}_E \omega \right)^2 \right)$$

$$- \frac{1}{4\pi} \int_X (\log f) \left( \text{tr} \left( i(F_k + \Lambda[\varphi, \bar{\varphi}]) - \frac{\lambda(E)}{2} \text{Id}_E \omega \right)^2 \right).$$

(2.10)
Note that the last term in the right hand side of (2.10) is bounded by a constant as the Hermitian metric $k$ is fixed. We then estimate the second term in the right hand side of (2.10). As in [Ko], we use the primitive decomposition to obtain $\text{End}(E)$-valued $(1,1)$-form $S$ with $F_h = (\Lambda F_h)\omega + S$ and $S \wedge \omega = 0$. Then we get
\[
\left( i(F_h + \Lambda[\varphi, \bar{\varphi}^h]) - \frac{\lambda(E)}{2} \text{Id}_E \omega \right)^2
= \left( i(\Lambda F_h + *[\varphi, \bar{\varphi}^h])\omega + iS - \frac{\lambda(E)}{2} \text{Id}_E \omega \right)^2
= \left( i(\Lambda F_h + *[\varphi, \bar{\varphi}^h]) - \frac{\lambda(E)}{2} \text{Id}_E \right)^2 \omega^2 - S \wedge S.
\]

Hence, we obtain
\[
(\log f) \left( \text{tr} \left( i(F_h + \Lambda[\varphi, \bar{\varphi}^h]) - \frac{\lambda(E)}{2} \text{Id}_E \omega \right)^2 \right)
= (\log f) \left( \text{tr} \left( i(\Lambda F_h + *[\varphi, \bar{\varphi}^h])\omega - \frac{\lambda(E)}{2} \text{Id}_E \omega \right)^2 - \text{tr} (S \wedge S) \right).
\]

As in [Ko, p. 233], $\text{tr} (S \wedge S) \geq 0$, and we can also assume that $f = |s|^2/a \leq 1$, thus,
\[
(\log f) \left( \text{tr} \left( i(F_h + \Lambda[\varphi, \bar{\varphi}^h]) - \frac{\lambda(E)}{2} \text{Id}_E \omega \right)^2 \right)
\geq (\log f) \left| i(\Lambda F_h + *[\varphi, \bar{\varphi}^h]) - \frac{\lambda(E)}{2} \text{Id}_E \right| \omega^2.
\]

The proposition follows from this and (2.10). \qed

We now take a smooth curve $D$ of $X$ in Corollary 2.11. Then, by Proposition 2.12, for a given Hermitian metric $k$ there exists positive constants $C$ and $C'$ such that
\[
D_\varphi(h_t, k)_X \geq D_\varphi(h_t, k)_D - C \left( \max_X \left| i(\Lambda F_{h_t} + *[\varphi, \bar{\varphi}^{h_t}]) - \frac{\lambda(E)}{2} \text{Id}_E \right|^2 \right) - C'.
\]

From Proposition 2.8 (i), we have $D_\varphi(h, k)_X \geq D_\varphi(h_t, k)_X$ for all $t > 0$. On the other hand, from Proposition 2.8 (ii), there exists $t_1 > 0$ such that
\[
\max_X \left| i(\Lambda F_{ht} + *[\varphi, \bar{\varphi}_{ht}]) - \frac{\lambda(E)}{2} \mathrm{Id}_E \right| < 1 \text{ for } t \geq t_1. \]
Thus, we get
\[
D_{\varphi}(h, k)_X \geq D_{\varphi}(h_t, k)_D - C - C'
\]
for \( t \geq t_1 \) by Proposition 2.12. From Proposition 2.7, \( D_{\varphi}(h_t, k)_D \) is bounded below, thus so is \( D_{\varphi}(h, k)_X \).

Once the lower bound of the functional is obtained, the argument originally implemented by Donaldson [D, §3] works for the Vafa–Witten case as we describe it below.

Firstly, the Uhlenbeck type weak compactness theorem for the Vafa–Witten equations by Mares [M, §3.3] applies to give an \( L^q \) limit along the flow away from a finite set of points. On the other hand, since we have the lower bound for the functional \( D_{\varphi}(-) \), \( C^0 \)-norm of \( i(\Lambda F_{ht} + *[\varphi, \bar{\varphi}_{ht}]) - \frac{\lambda(E)}{2} \mathrm{Id}_E \) converges to zero by Proposition 2.8 (iii). Hence the above \( L^q \) weak limit satisfies the Vafa–Witten equations. We then invoke the removal singularity theorem for the Vafa–Witten equation by Mares [M, §3.4] to obtain a bundle \( E' \) and a solution to the Vafa–Witten equations across the singular set. This bundle is semi-stable in the sense of Definition 1.1 as it admits a solution to the Vafa–Witten equations. One then constructs a non-zero holomorphic map between \( E \) and \( E' \). From the assumption that \( E \) is stable, this map is isomorphism. Hence, we obtain a solution to the Vafa–Witten equations on \( E \). The uniqueness of the solution follows from the convexity of the functional \( D_{\varphi}(-) \).

\[
\square
\]

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