An Energy Reducing Flow for Multiple-Valued Functions

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March 29, 2022

Abstract

By the method of discrete Morse flows, we construct an energy reducing multiple-valued function flow. The flow we get is Hölder continuous with respect to the $L^2$ norm. We also give another way of constructing flows in some special cases, where the flow we get behaves like ordinary heat flow.

*The author wants to thank his advisor, Professor Robert Hardt for introducing this subject to him and numerous fruitful, enjoyable discussions.
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1 Introduction

This work was originally motivated by a paper [CX], in which he constructed a mass reducing flow for integral currents. The ideal of his construction comes from Horihata and Kikuchi’s paper [HK] for nonlinear parabolic equations. More specifically, let $T_0$ be an integral current, and let $h > 0$ be given. He defines a step approximation sequence $\{T^k_h\}$ of integral currents with the same boundary as that of $T_0$ by choosing $T^k_h$ such that $T^k_h$ minimizes the functional

$$G(T) = G(T^k_{h-1}, T, h) = M(T)^2 + \frac{F_k(T^k_{h-1} - T)^2}{h},$$

where $T$ is an integral current with $\partial T = \partial T_0$. Then he constructed a $k + 1$ dimensional current $S_h$ by “connecting” those $T^k_h$. Finally he takes a weak limit $S$ of $S_h$ as $h \to 0$, slices $S$ with respect to $t$ to get an integral current at time $t$. The flow is Hölder continuous under the flat norm and reduces the mass of the initial integral current while keeping the boundary fixed. Later on, this same time discretization process was used in Haga, Hoshino and Kikuchi’s paper [HHK] to construct a harmonic map flow. This gives an alternative proof of the classical result due to J. Eells, Jr. and J.H.Sampson [ES] and to R.S.Hamilton [HRS].

Our work is trying to construct a what is so called “multiple-valued harmonic flow” by the similar time discretization method. One obvious obstacle here is that we do not have differential equations for multiple-valued functions. Therefore a lot of PDE methods can not be applied. Another thing that stands in the way is due to the lack of fundamental algebraic operations, for example, addition for multiple-valued functions. Hence we can not use linear interpolation to connect those multiple-valued function at different stages. All of them will be handled with great care.

In the first part of this paper, a basic description of multiple-valued functions and related results are given. Most of them can be found in [AF]. In the second part, we state and prove some theorems that we need for the construction later.

In the third part, an energy reducing flow of multiple-valued functions is constructed. Those properties of this flow are proved.

Finally, we look at the special case when $Q = Q_2(\mathbb{R})$ and the initial multiple-valued function $f_0$ satisfies $\eta(f_0) \equiv 0$, i.e symmetric. We will show that the flow will be symmetric all the time and each component separates immediately. A concrete example when the domain is two dimensional is also given.

2 Preliminaries

The theory of multiple-valued functions was developed in [AF]. It is the most natural framework for the regularity theory in geometric measure theory and promises a lot of future development and applications in other fields. Here we
introduce the basic notations and facts of multiple-valued functions. The readers are referred to [AF] for more details. We also use standard terminology in geometric measure theory, all of which can be found on page 669-671 of the treatise *Geometric Measure Theory* by H. Federer [FH].

The space \(Q = Q^n(R)\) consists of all the unordered \(Q\) points in \(R^n\), denoted by \(\sum_{i=1}^{Q} [p_i]\), \(p_i \in R^n\). We let \(\text{spt}(\sum_{i=1}^{Q} [p_i]) = \bigcup_{i=1}^{Q} \{p_i\}\).

We define a metric on \(Q\)

\[ \mathcal{G}: Q \rightarrow R \]

by setting for \(p_1, \ldots, p_Q, q_1, \ldots, q_Q \in R^n\),

\[ \mathcal{G}(\sum_{i=1}^{Q} [p_i], \sum_{i=1}^{Q} [q_i]) = \inf_{\sigma} \{\left(\sum_{i=1}^{Q} |p_i - q_{\sigma(i)}|^2\right)^{1/2} : \sigma \text{ is a permutation of } \{1, \ldots, Q\}\}. \]

We let \(|\sum_{i=1}^{Q} [p_i]|^2 = \sum_{i=1}^{Q} |p_i|^2 = \mathcal{G}^2(\sum_{i=1}^{Q} [p_i], Q[0])\).

\[ \zeta: O^*(n, 1) \times Q \rightarrow R^n \cap \{s : s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_Q\} \]

is defined for \(\pi \in O^*(n, 1), p \in Q\) by requiring \(-\infty < s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_Q < \infty\) and \(\pi q = \sum_{i=1}^{Q} [s_i]\). One can easily check that Lip(\(\zeta(\pi, \cdot)) = 1\) for each \(\pi \in O^*(n, 1)\).

\[ P \] is the positive integer characterized in the following manner: if

\[ b = \inf \{\arctan(1/n!), (\sin[2^{-1}\arctan(1/n!)])^{n-1}/2QQ!(n-1)\}, \]

and \(c\) denotes the unique positive integer for which

\[ 1/b \leq c < 1 + 1/b, \]

then

\[ P = 2^{-1}([4c(n-1) + 1]^{n} - [4c(n-1) - 1]^{n}). \]

**Theorem 2.1** ([AF], §1.2). *There exist \(\Pi_1, \Pi_2, \ldots, \Pi_p \in O^*(n, 1)\) such that*

1. \(\Pi_i(x) = x_i\) for each \(i = 1, \ldots, n\) and each \(x \in R^n\).
2. Lip(\(\xi_0) = 1\).
3. \(\xi: Q \rightarrow Q^*\) is a bilipschitzian homeomorphism with Lip(\(\xi) \leq P^{1/2}\) and Lip(\(\xi^{-1}|Q^*\) \(\leq 1/(n^{1/2}\sin(b/2))\)) corresponding to \(b\) as above, where

\[ \xi = \zeta(\Pi_1, \cdot) \ast \cdots \ast \zeta(\Pi_p, \cdot): Q \rightarrow Q^{PQ}, \]

\[ Q^* = \xi(Q) \]

*and*

\[ \xi_0 = \zeta(\Pi_1, \cdot) \ast \cdots \ast \zeta(\Pi_n, \cdot): Q \rightarrow R^{nQ}. \]
(Here we use the notation that whenever \( f : A \to B \) and \( g : A \to C \), we define
\[
(f \otimes g) : A \to B \times C, \quad (f \otimes g)(a) = (f(a), g(a)), \quad a \in A.
\]

**Remark 2.1.** Brian White showed that there is a modified bilipschitzian correspondence \( \xi : Q \to Q^* \subset \mathbb{R}^{PQ} \) such that for every \( p \in Q \), \( p \) has a small neighbourhood in \( Q \) such that \( \xi \) is an equidistance map over the neighbourhood.

The modification is to choose the orthogonal projections \( \Pi_1, \cdots, \Pi_P \) in \( \mathbb{R}^n \) as complete sets of coordinate projections corresponding to distinct orthonormal coordinate systems for \( \mathbb{R}^n \) and to compose the resulting map \( \xi \) there with proper scaling to get such a \( \xi \). It has some other useful properties that we will mention later. Moreover, we will use the modified \( \xi \) throughout the rest of this paper.

**Theorem 2.2** (\([AF]\), §1.2). Suppose \(-\infty < r(1) \leq r(2) \leq \cdots \leq r(Q) < \infty \) and \(-\infty < s(1) \leq s(2) \leq \cdots \leq s(Q) < \infty \). Then
\[
\sum_{i=1}^{Q} [r(i) - s(i)]^2 = \inf_{\sigma} \left\{ \sum_{i=1}^{Q} [r(i) - s(\sigma(i))]^2 : \sigma \text{ is a permutation of } \{1, \cdots, Q\} \right\}.
\]

**Remark 2.2.** This theorem says that the distance between two elements in \( Q_Q(\mathbb{R}) \) is obtained by matching those \( Q \)-tuples pairwise according to the ascending order.

**Theorem 2.3** (\([AF]\), §1.3). There exists an explicitly constructable, piecewise linear function
\[
\rho : \mathbb{R}^{PQ} \to \mathbb{R}^{PQ}
\]
such that \( \text{Lip}(\rho) < \infty \), \( \rho(\mathbb{R}^{PQ}) \subset Q^* \), and \( \rho(x) = x \) for each \( x \in Q^* \).

**Definition 2.1.** (a) \( f \) is a \( Q \)-valued function on some subset \( U \) of \( \mathbb{R}^m \) if it is a map
\[
f : U \subset \mathbb{R}^m \to Q
\]
(b) For a given smooth, compact embedded manifold \( N \) in \( \mathbb{R}^n \),
\[
\overline{Q(N)} = \{ \sum_{i=1}^{Q} [[p_i]], p_i \in N, i = 1, \cdots, Q \}
\]
(c) \( f \) is a \( Q \)-valued map from some subset \( U \) of \( \mathbb{R}^m \) into \( N \) if it is a map
\[
f : U \subset \mathbb{R}^m \to \overline{Q(N)} \subset Q
\]
(d) Similarly we can define \( \overline{Q(V)} \) for any vector space \( V \).

**Definition 2.2.** (a) \( f \) is called a \( Q \)-valued Lipschitz function(map) if there is a constant \( C > 0 \) such that
\[
G(f(x), f(y)) \leq C|x - y|, \quad x, y \in U.
\]
(b) $f$ is called affine if there are $A_1, \ldots, A_Q$ where each $A_i$ is an affine map from $\mathbb{R}^m$ to $\mathbb{R}^n$, such that

$$f(x) = \sum_{i=1}^{Q}[[A_i(x)]] .$$

(c) $f$ is called affinely approximatable at $x_0$ if there are affine maps $A_1, \ldots, A_Q$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ such that

$$\lim_{|x-x_0| \to 0} \frac{G(f(x), \sum_{i=1}^{Q}[[A_i(x)]])}{|x-x_0|} = 0.$$ 

(d) $f$ is strongly affinely approximatable at $x_0$ if (c) holds for $f$ at $x_0$ and $A_i = A_j$ if $A_i(x_0) = A_j(x_0)$.

Remark 2.3. (1) From [12], §1.4, if $f$ is a $Q$-valued Lipschitz function, then it is strongly affinely approximatable almost everywhere over its domain.

(2) If $f$ is affinely approximatable at $x_0$ with $\sum_{i=1}^{Q}[[A_i]]$ as its affine approximation, then obviously $f(x_0) = \sum_{i=1}^{Q}[[A_i(x_0)]]$ and $A_i(x) = A_i(x_0) + L_i(x-x_0)$ with $L_i \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$.

Definition 2.3. If $f$ is affinely approximatable at $x_0$, then

- (a) $\sum_{i=1}^{Q}[[L_i]] \in Q(\text{Hom}(\mathbb{R}^m, \mathbb{R}^n))$, denoted by $Df(x_0)$ is defined as the differential of $f$ at $x_0$. We let $|Df(x_0)|^2 = \sum_{i=1}^{Q} |L_i|^2$, where $|L|$ is the Euclidean norm of the matrix associated with any $L \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$.
- (b) $\sum_{i=1}^{Q}[[L_i(v)]]$ is defined as the derivative of $f$ at $x_0$ in the direction $v$ and is denoted by $D_vf \in Q$. Let $|D_vf(x_0)|^2 = \sum_{i=1}^{Q} |L_i(v)|^2$.

Remark 2.4. The map $\xi$ mentioned before has the following properties

$$|\xi \circ f| = |f|, |D_v(\xi \circ f)| = |D_vf| .$$

Definition 2.4. (a) Suppose $A \subset \mathbb{R}^m$ is bounded and open and that $\partial A$ is an $m - 1$ dimensional submanifold of $\mathbb{R}^m$ of class 1. Whenever $V$ is a Euclidean vector space, $\mathcal{Y}_2(\mathbb{R}^m, V)$ and $\mathcal{Y}_2(A, V)$ are the real vector spaces of square summable functions whose distribution first derivatives are also square summable. $\partial \mathcal{Y}_2(\partial A, V)$ is the real vector space of all $\mathcal{H}^{m-1}$ measurable function $f : \partial A \to V$ such that

$$\int_{\partial A} |f|^2 d\mathcal{H}^{m-1} + \int_{z \in \partial A} |z|^{-m} \int_{x \in \partial A} |f(x+z) - f(z)|^2 d\mathcal{H}^{m-1} z < \infty$$

(b) Whenever $K \subset \mathbb{R}^m$ is $\mathcal{L}^m$ measurable with $\mathcal{L}^m(K \sim A) = 0$ and $f \in \mathcal{Y}_2(A, V)$ we set

$$\text{Dir}(f; K) = \int_K |Df|^2 d\mathcal{L}^m.$$
Additionally, we define for each $g \in \partial \mathcal{Y}_2(\partial A, V)$ and each $\mathcal{H}^{m-1}$ measurable set $L \subset \mathbb{R}^m$ with $\mathcal{H}^{m-1}(L \sim \partial A) = 0$,

$$\text{dir}(g; L) = \int_L |Dg|^2 d\mathcal{H}^{m-1}.$$ 

**Definition 2.5.** Assuming $f : \mathbb{R}^m \to V$ is locally $\mathcal{C}^m$ summable, we say $f$ is strictly defined if and only if whenever $x \in \mathbb{R}^m$, and there is some $y \in V$ for which

$$\lim_{r \downarrow 0} \int_{z \in B^m(x, r)} |f(z) - y| d\mathcal{L}^m z = 0,$$

then $f(x) = y$.

**Definition 2.6.** Whenever $f \in \mathcal{Y}_2(A, V)$ and $g : \partial A \to V$ we say that $f$ has boundary values $g$ if and only if there is $h \in \mathcal{Y}_2(\mathbb{R}^m, V)$ which is strictly defined such that

$$\mathcal{L}^m(\{x : f(x) \neq h(x)\}) = 0 = \mathcal{H}^{m-1}(\partial A \cap \{x : g(x) \neq h(x)\}).$$

**Definition 2.7.**

(a) We define

$$\mathcal{Y}_2(\mathbb{R}^m, Q) [\text{resp. } \mathcal{Y}_2(A, Q)]$$

to be the space of all functions $f : \mathbb{R}^m \to Q[\text{resp. } f : A \to Q]$ such that $\xi \circ f \in \mathcal{Y}_2(\mathbb{R}^m, R^{PQ})[\text{resp. } \xi \circ f \in \mathcal{Y}_2(A, R^{PQ})]$. We also define

$$\partial \mathcal{Y}_2(\partial A, Q)$$
as the space of all functions $g : \partial A \to Q$ such that $\xi \circ g \in \partial \mathcal{Y}_2(\partial A, R^{PQ})$.

(b) For each $f \in \mathcal{Y}_2(A, Q)$ and each $\mathcal{L}^m$ measurable set $K \subset \mathbb{R}^m$ which is $\mathcal{L}^m$ almost a subset of $A$, we define

$$\text{Dir}(f; K) = \text{Dir}(\xi_0 \circ f; K).$$

For each $g \in \partial \mathcal{Y}_2(\partial A, Q)$ and $\mathcal{H}^{m-1}$ measurable set $L \subset \mathbb{R}^m$ which is $\mathcal{H}^{m-1}$ almost a subset of $\partial A$, we set

$$\text{dir}(g; L) = \text{dir}(\xi_0 \circ g; L).$$

(c) Whenever $f \in \mathcal{Y}_2(\mathbb{R}^m, Q)[\text{resp. } f \in \mathcal{Y}_2(A, Q)]$ we say that $f$ is strictly defined if and only if $\xi \circ f$ is strictly defined. One notes that in case $f : \mathbb{R}^m \to R^{PQ}$ is locally $\mathcal{C}^m$ summable with $\text{im}(f) \subset Q^*$, $x \in \mathbb{R}^m$, $y \in R^{PQ}$, and

$$\lim_{r \downarrow 0} \int_{z \in B^m(x, r)} |f(z) - y| d\mathcal{L}^m z = 0,$$

then $y \in Q^*$ since $Q^*$ is closed.

(d) For $f \in \mathcal{Y}_2(A, Q), g \in \partial \mathcal{Y}_2(\partial A, Q)$ one says that $f$ has boundary values $g$ if and only if $\xi \circ f$ has boundary values $\xi \circ g$. 

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Definition 2.8. One says that \( f : A \to \mathbb{Q} \) is Dir minimizing if and only if \( f \in \mathcal{Y}(A, \mathbb{Q}) \) and, assuming \( f \) has boundary values \( g \in \partial \mathcal{Y}(\partial A, \mathbb{Q}) \), one has

\[
\text{Dir}(f; A) = \inf \{ \text{Dir}(h; A) : h \in \mathcal{Y}(A, \mathbb{Q}) \text{ has boundary values } g \}.
\]

Remark 2.5. All those definitions are easily extended to multiple-valued maps.

Theorem 2.4 \([\text{AF}], \S 2.2\). Suppose \( A \subset \mathbb{R}^m \) is bounded and open and \( \partial A \) is an \( m-1 \) dimensional submanifold of \( \mathbb{R}^m \) of class 1.

1. Let \( f \in \mathcal{Y}(A, \mathbb{Q}) \). Then
   (a) for \( \mathcal{L}^m \) almost all \( x \in A \), \( \text{apD}(\xi \circ f)(x) \) exists.
   (b) if \( x \in A \) and \( \text{apD}(\xi \circ f)(x) \) exists, then both \( \text{apD}(\xi \circ f)(x) \) and \( \text{apAf}(x) \) exist with \( |\text{apD}(\xi \circ f)(x)| = |\text{apAf}(x)| \).
   (c) whenever \( K \subset \mathbb{R}^m \) is \( \mathcal{L}^m \) measurable and is \( \mathcal{L}^m \) almost a subset of \( A \),
   \( \text{Dir}(f; K) = \int_A |\text{apAf}|^2 d\mathcal{L}^m \).

2. Let \( g \in \partial \mathcal{Y}(\partial A, \mathbb{Q}) \). Then there exists \( f \in \mathcal{Y}(A, \mathbb{Q}) \) such that
   (a) \( f \) has boundary values \( g \).
   (b) \( \text{Dir}(f; A) = \inf \{ \text{Dir}(h; A) : h \in \mathcal{Y}(A, \mathbb{Q}) \text{ has boundary values } g \} \).

Theorem 2.5 \([\text{AF}], \S 4.1.2\). Suppose \( V \) is a Euclidean vector space and \( A \subset \mathbb{R}^m \) is bounded and open such that \( \partial A \) is a compact \( m-1 \) dimensional submanifold of \( \mathbb{R}^m \) of class 1. Suppose \( K \) is a closed subset of \( V \), \( g \in \partial \mathcal{Y}(A, V) \), \( f_1, f_2, f_3, \ldots \in \mathcal{Y}(A, V) \) such that the following conditions hold:

(a) \( g(x) \in K \) for each \( x \in \partial A \),
(b) \( f_i(x) \in K \) for each \( x \in A \) and each \( i = 1, 2, 3, \ldots \),
(c) \( f \) has boundary values \( g \) for each \( i = 1, 2, 3, \ldots \),
(d) \( \sup_i \text{Dir}(f_i; A) < \infty \).

Then there exists a subsequence \( i_1, i_2, i_3, \ldots \) of \( 1, 2, 3, \ldots \) and \( f \in \mathcal{Y}(A, V) \) with the following properties:

1. \( 0 = \lim_{k \to \infty} \int_A |f - f_{i_k}|^2 d\mathcal{L}^m \),
2. \( \lim_{k \to \infty} \int_{x \in A} < \text{apD}f_{i_k}(x) - \text{apD}f(x), \phi(x) > d\mathcal{L}^m x = 0 \) \( V \) for each \( \phi \in \mathcal{D}(A, \mathbb{R}^m) \),
3. \( f(x) \in K \) for each \( x \in A \),
4. \( f \) has boundary values \( g \),
5. \( \lim_{k \to \infty} \text{Dir}(f_{i_k}; A) = \text{Dir}(f; A) + \lim_{k \to \infty} \text{Dir}(f_{i_k} - f; A) \in \mathbb{R} \).

Remark 2.6. It is easy to see that this theorem can be easily extended to the case when \( A \) is a cylinder of the form \([0, \infty) \times B^m_1 \), where \( B^m_1 = \{ x : x \in \mathbb{R}^m, |x| \leq 1 \} \).

Definition 2.9. Define

\[
(+) : \mathcal{Y}(A, \mathbb{Q}) \times \mathcal{Y}(A, \mathbb{R}^n) \to \mathcal{Y}(A, \mathbb{Q})
\]

by setting

\[
f(+)\phi(x) = (+)(f, \phi)(x) = \tau(-\phi(x)) f(x)
\]

for each \( f \in \mathcal{Y}(A, \mathbb{Q}), \phi \in \mathcal{Y}(A, \mathbb{R}^n), x \in A \). Here \( \tau \) is the translation operator

\[
\tau(y) : \mathbb{R}^n \to \mathbb{R}^n, \tau(y)(x) = x - y, \text{ for } x \in \mathbb{R}^n.
\]

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Theorem 2.6 ([AF], §2.3).

\[
\text{Dir}(f(+)\phi; A) = \text{Dir}(f; A) + 2Q \int_A < D(\eta \circ f), D\phi > d\mathcal{L}^m + Q\text{Dir}(\phi; A)
\]

whenever \( f \in \mathcal{Y}_2(A, \mathbb{Q}), \phi \in \mathcal{Y}_2(A, \mathbb{R}^n) \).

3 Extension of Luckhaus Lemma to Multiple-Valued Maps

Theorem 3.1 (Luckhaus Lemma, [SL], §2.6). Suppose \( N \) is an arbitrary compact subset of \( \mathbb{R}^n \), \( m \geq 2 \), and \( u, v \in W^{1,2}(\mathbb{S}^{m-1}; N) \). Then there is a constant \( C \) which depends only on \( m, n \) and \( N \) such that for each \( \epsilon \in (0,1) \)

\[
\int_{\mathbb{S}^{m-1} \times [0,\epsilon]} |Dw|^2 \leq C\epsilon \int_{\mathbb{S}^{m-1}} (|Du|^2 + |Dv|^2) + C\epsilon^{-1} \int_{\mathbb{S}^{m-1}} |u - v|^2,
\]

and

\[
\text{dist}^2(w(x, s), N) \leq C\epsilon^{1-m} \left( \int_{\mathbb{S}^{m-1}} |Du|^2 + |Dv|^2 \right)^{1/2} \left( \int_{\mathbb{S}^{m-1}} |u - v|^2 \right)^{1/2} + C\epsilon^{-m} \int_{\mathbb{S}^{m-1}} |u - v|^2
\]

for a.e.\((x, s) \in \mathbb{S}^{m-1} \times [0,\epsilon] \). Here \( D \) is the gradient on \( \mathbb{S}^{m-1} \) and \( \overline{D} \) is the gradient on the product space \( \mathbb{S}^{m-1} \times [0,\epsilon] \).

Theorem 3.2. Suppose \( N \) is an arbitrary smooth compact manifold embedded in \( \mathbb{R}^n \), \( m \geq 2 \), and \( u, v \in \mathcal{Y}_2(\mathbb{S}^{m-1}, \mathbb{Q}(N)) \). Then there is a constant \( C \) which depends only on \( m, n, Q \) and \( N \) such that for each \( \epsilon \in (0,1) \)

\[
\int_{\mathbb{S}^{m-1} \times [0,\epsilon]} |Dw|^2 \leq C\epsilon \int_{\mathbb{S}^{m-1}} (|Du|^2 + |Dv|^2) + C\epsilon^{-1} \int_{\mathbb{S}^{m-1}} \mathcal{G}^2(u, v),
\]

and

\[
\text{dist}^2(w(x, s), \mathbb{Q}(N)) \leq C\epsilon^{1-m} \left( \int_{\mathbb{S}^{m-1}} |Du|^2 + |Dv|^2 \right)^{1/2} \left( \int_{\mathbb{S}^{m-1}} \mathcal{G}^2(u, v) \right)^{1/2} + C\epsilon^{-m} \int_{\mathbb{S}^{m-1}} \mathcal{G}^2(u, v)
\]

for a.e.\((x, s) \in \mathbb{S}^{m-1} \times [0,\epsilon] \). Here \( D \) is the gradient on \( \mathbb{S}^{m-1} \) and \( \overline{D} \) is the gradient on the product space \( \mathbb{S}^{m-1} \times [0,\epsilon] \).
Proof. Apply Luckhaus lemma to the set $\xi \circ Q(N)$ and functions $\xi \circ u$, $\xi \circ v$, we get a function $\tilde{w} \in W^{1,2}(S^{m-1} \times [0, \varepsilon]; \mathbb{R}^p)$ such that $\tilde{w}$ agrees with $\xi \circ u$ in a neighbourhood of $S^{m-1} \times \{0\}$, $\tilde{w}$ agrees with $\xi \circ v$ in a neighbourhood of $S^{m-1} \times \{\varepsilon\}$.

$$\int_{S^{m-1} \times [0, \varepsilon]} |D \tilde{w}|^2 \leq \tilde{C} \varepsilon \int_{S^{m-1}} (|D(\xi \circ u)|^2 + |D(\xi \circ v)|^2) + \tilde{C} \varepsilon^{-1} \int_{S^{m-1}} |\xi \circ u - \xi \circ v|^2,$$

and $\text{dist}^2(\tilde{w}(x), s) \circ Q(N)) \leq \tilde{C} \varepsilon^{-1-m} \int_{S^{m-1}} (|D(\xi \circ u)|^2 + |D(\xi \circ v)|^2)^{1/2} \int_{S^{m-1}} |\xi \circ u - \xi \circ v|^2$.

for a.e. $(x, s) \in S^{m-1} \times [0, \varepsilon]$. Here $D$ is the gradient on $S^{m-1}$ and $\overline{D}$ is the gradient on the product space $S^{m-1} \times [0, \varepsilon]$.

Now we define $w = \xi^{-1} \circ \rho \circ \tilde{w} \in \mathcal{Y}_2(S^{m-1} \times [0, \varepsilon]; Q)$. It is easy to see that $w$ agrees with $u$ in a neighbourhood of $S^{m-1} \times \{0\}$ and $w$ agrees with $v$ in a neighbourhood of $S^{m-1} \times \{\varepsilon\}$.

The rest of the proof is obviously easy once we notice that Lip $\xi^{-1}$, Lip $\xi$, Lip $\rho$ are all finite, depending only on $m, n, Q$.

Corollary 3.1. Suppose $N$ is a smooth compact manifold embedded in $\mathbb{R}^n$, and $\Lambda > 0$. There are $\delta_0 = \delta_0(m, n, N, Q, \Lambda)$ and $C = C(m, n, N, Q, \Lambda)$ such that the following hold:

1. If we have $\varepsilon \in (0, 1)$ and if $u \in \mathcal{Y}_2(B^m_{\rho}(y); Q(N))$ with $\rho^{2-m} \int_{B^m_{\rho}(y)} |Du|^2 \leq \Lambda$, and $\varepsilon^{-2m} \rho^{-m} \int_{B^m_{\rho}(y)} |\xi \circ u - \lambda_{y, \rho}|^2 \leq \delta_0$, (where $\lambda_{y, \rho} = \int_{B^m_{\rho}(y)} \xi \circ u$) then there is $\sigma \in \left(\frac{\rho}{1+\varepsilon}, \rho\right)$ such that there is a map $w = w_\varepsilon \in \mathcal{Y}_2(B^m_{\rho}(y); Q(N))$ which agrees with $u$ in a neighborhood of $\partial B^m_{\rho}(y)$ and which satisfies

$$\sigma^{2-m} \int_{B^m_{\rho}(y)} |Dw|^2 \leq C \rho^{2-m} \int_{B^m_{\rho}(y)} |Du|^2 + \varepsilon^{-1} C \rho^{-m} \int_{B^m_{\rho}(y)} |\xi \circ u - \lambda_{y, \rho}|^2.$$  

2. If $\varepsilon \in (0, \delta_0]$, and if $u, v \in \mathcal{Y}_2(B^m_{\rho(1+\varepsilon)}(y); Q(N))$ satisfy the inequalities $\rho^{2-m} \int_{B^m_{\rho(1+\varepsilon)}(y)} |Du|^2 + |Dv|^2 \leq \Lambda$ and $\varepsilon^{-2m} \rho^{-m} \int_{B^m_{\rho(1+\varepsilon)}(y)} \mathcal{G}(u, v)^2 < \delta_0^2$, then there is $w \in \mathcal{Y}_2(B^m_{\rho(1+\varepsilon)}(y); Q(N))$ such that $w = u$ in a neighborhood of $\partial B^m_{\rho}(y)$, $w = v$ in a neighborhood of $\partial B^m_{\rho(1+\varepsilon)}(y)$ and $\rho^{2-m} \int_{B^m_{\rho(1+\varepsilon)}(y)} |Dw|^2 \leq C \rho^{2-m} \int_{B^m_{\rho(1+\varepsilon)}(y)} |Du|^2 + |Dv|^2 + C \varepsilon^{-2} \rho^{-m} \int_{B^m_{\rho(1+\varepsilon)}(y)} \mathcal{G}(u, v)^2.$

Proof. The proof is basically the same as in [SL], §2.7.
4 Compactness Theorem for Multi-Valued Minimizing Maps

Theorem 4.1 (Rellich Compactness Lemma, [SL], §1.3). Suppose \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^m \) and \( u_k \) is a sequence of \( W^{1,2}(\Omega) \) with \( \sup_k ||u_k||_{W^{1,2}(\Omega)} < \infty \). Then there is a subsequence \( u'_k \) and \( u \in W^{1,2}(\Omega) \) such that

\[
\begin{align*}
(a) & \ u'_k \rightharpoonup u \text{ weakly in } W^{1,2}(\Omega), \\
(b) & \ u'_k \rightarrow u \text{ strongly in } L^2(\Omega), \\
(c) & \int_{\Omega} |Du|^2 \leq \liminf_{k' \to \infty} \int_{\Omega} |Du'_k|^2.
\end{align*}
\]

Definition 4.1. Given a sequence \( u_k \), and \( u \in Y_2(\Omega, Q(N)) \), we say that,

\[
\begin{align*}
(a) & \ u_k \rightharpoonup u \text{ weakly in } Y_2(\Omega) \text{ if and only if } \xi \circ u_k \rightharpoonup \xi \circ u \text{ weakly in } W^{1,2}(\Omega), \\
(b) & \ u_k \rightarrow u \text{ strongly in } L^2(\Omega) \text{ if and only if } \xi \circ u_k \rightarrow \xi \circ u \text{ strongly in } L^2(\Omega).
\end{align*}
\]

Theorem 4.2. Suppose \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^m \) and \( u_k \) is a sequence of \( Y_2(\Omega, Q) \) with \( \sup_k ||u_k||_{Y_2(\Omega)} < \infty \). Then there is a subsequence \( u'_k \) and \( u \in Y_2(\Omega, Q) \) such that

\[
\begin{align*}
(a) & \ u'_k \rightharpoonup u \text{ weakly in } Y_2(\Omega), \\
(b) & \ u'_k \rightarrow u \text{ strongly in } L^2(\Omega), \\
(c) & \int_{\Omega} |Du|^2 \leq \liminf_{k' \to \infty} \int_{\Omega} |Du'_k|^2.
\end{align*}
\]

Proof. There is actually nothing to say, once we notice that the convergence of \( u_i \) is actually just equivalent to the convergence of \( \xi \circ u_i \). \( \square \)

Theorem 4.3. If \( \{u_j\} \) is a sequence of energy minimizing maps in \( Y_2(\Omega, Q(N)) \) with \( \sup_j ||u_j||_{Y_2(\Omega)} < \infty \) for each ball \( B^m_{\rho}(y) \subset \Omega \), then there is a subsequence \( \{u_{j'}\} \) and a minimizing map \( u \in Y_2(\Omega, Q(N)) \) such that \( u_{j'} \rightarrow u \) in \( Y_2(B^m_{\rho}(y), Q) \) on each ball \( B^m_{\rho}(y) \subset \Omega \).

Proof. The proof is basically the same as in [SL], §2.9. \( \square \)

5 Step minimizing sequences

We assume that \( f_0 \in Y_2(B^m_1, Q) \), \( f \) is strictly defined, and

\[
\text{Dir}(f_0; B^m_1) < \infty, ||\xi \circ f_0||_{L^2} < \infty
\]

Define

\[ M = \{u \in Y_2(B^m_1, Q), \text{ strictly defined, } u = f_0 \text{ on } \partial B^m_1 \} \]
Given an $h > 0$, we define inductively, for $k = 1, 2, 3, \ldots$, a multiple-valued sobolev function $f_k^h$ such that $f_0^h = f_0$ and $f_k^h$ minimizes the functional (step minimizing with respect to $h$)

$$G(g) = G(f_{k-1}^h, g, h) = \text{Dir}(g; B_1^m) + \frac{1}{h/2^k} ||\xi \circ f_{k-1}^h - \xi \circ g||_{L^2}^2$$

where $g \in \mathcal{M}$.

**Proposition 5.1.** $f_k^h$ is defined for each positive integer $k$ and each $h > 0$.

**Proof.** Let

$$L_k = \inf \{ G(f_{k-1}^h, g, h) : g \in \mathcal{M} \}, k = 1, 2, 3, \ldots$$

Since $G(f_0^h, f_0, h) = \text{Dir}(f_0; B_1^m) < \infty, L_1 < \infty.$

Let $\{g_i\} \subset \mathcal{M}$ be a sequence such that

$$\lim_{i \to \infty} G(f_0^h, g_i, h) = L_1,$$

then $\sup_i G(f_0^h, g_i, h) < \infty$.

It is easy to see

$$\text{Dir}(g_i; B_1^m) \leq G(f_0^h, g_i, h) \leq \sup_i G(f_0^h, g_i, h) < \infty,$$

By Theorem 2.5, there is a subsequence, still denoted as $g_i$ such that

$$g_i \rightharpoonup g \in \mathcal{Y}_2,$$

which gives

$$\int_{B_1^m} |\xi \circ f_0^h - \xi \circ g|^2 dx = \int_{B_1^m} \lim_{i \to \infty} |\xi \circ f_0^h - \xi \circ g_i|^2 dx \leq \liminf_{i \to \infty} \int_{B_1^m} |\xi \circ f_0^h - \xi \circ g_i|^2 dx \ (\text{by Fatou lemma})$$

and

$$\text{Dir}(g; B_1^m) \leq \liminf_{i \to \infty} \text{Dir}(g_i; B_1^m) < \infty$$

with $g = f_0^h$ on $\partial B_1^m$. By re-define $g$ at some points if necessary, we may assume $g \in \mathcal{M}$. Therefore

$$G(f_0^h, g, h) = \text{Dir}(g; B_1^m) + \frac{1}{h/2^k} \int_{B_1^m} |\xi \circ f_0^h - \xi \circ g|^2 dx$$

$$\leq \liminf_{i \to \infty} \text{Dir}(g_i; B_1^m) + \frac{1}{h/2^k} \int_{B_1^m} |\xi \circ f_0^h - \xi \circ g_i|^2 dx$$

$$\leq \liminf_{i \to \infty} G(f_0^h, g_i, h) = L_1$$
Hence $G(f^0_h, g, h) = L_1$. We just let $f^1_h$ to be $g$.

Now we assume that $f^k_h$ exists and $\text{Dir}(f^k_h; B^m_1) < \infty$.

$$L_{k+1} = \inf \{G(f^k_h, g, h) : g \in \mathcal{M} \} \leq G(f^k_h, f^k_h, h) = \text{Dir}(f^k_h; B^m_1) < \infty.$$ 

Let $\{g_i\} \subset \mathcal{M}$ be a sequence that

$$\lim_{i \to \infty} G(f^k_h, g_i, h) = L_{k+1},$$ 

then $\sup_i G(f^k_h, g_i, h) < \infty$.

It is easy to see $\text{Dir}(g_i; B^m_1) \leq G(f^k_h, g_i, h) \leq \sup_i G(f^k_h, g_i, h) < \infty$.

By Theorem 2.5, there is a subsequence, still denoted as $g_i$, such that $g_i \rightharpoonup g \in \mathcal{Y}_2$, which gives

$$\int_{B^m_1} |\xi \circ f^k_h - \xi \circ g|^2 dx = \int_{B^m_1} \lim_{i \to \infty} |\xi \circ f^k_h - \xi \circ g_i|^2 dx$$

$$\leq \liminf_{i \to \infty} \int_{B^m_1} |\xi \circ f^k_h - \xi \circ g_i|^2 dx \quad \text{(by Fatou lemma)}$$

and

$$\text{Dir}(g; B^m_1) \leq \liminf_{i \to \infty} \text{Dir}(g_i; B^m_1) < \infty$$

with $g = f^0_h$ on $\partial B^m_1$. By re-define $g$ at some points if necessary, we may assume $g \in \mathcal{M}$. Therefore

$$G(f^k_h, g, h) = \text{Dir}(g; B^m_1) + \frac{\int_{B^m_1} |\xi \circ f^k_h - \xi \circ g|^2 dx}{h/2^{k+1}}$$

$$\leq \liminf_{i \to \infty} \text{Dir}(g_i; B^m_1) + \frac{\liminf_{i \to \infty} \int_{B^m_1} |\xi \circ f^k_h - \xi \circ g_i|^2 dx}{h/2^{k+1}}$$

$$\leq \liminf_{i \to \infty} G(f^k_h, g_i, h) = L_{k+1}$$

Hence $G(f^k_h, g, h) = L_{k+1}$. We just let $f^{k+1}_h$ to be $g$.

Repeat the process, we will prove the proposition.

**Proposition 5.2.** The energy of the sequence $\{f^k_h\}$ is non-increasing,

$$\text{Dir}(f^k_h; B^m_1) \leq \text{Dir}(f^{k-1}_h; B^m_1) \leq \text{Dir}(f^0_h; B^m_1).$$

We also have the following estimate:

$$||\xi \circ f^{k-1}_h - \xi \circ f^k_h||^2_{L^2} \leq \frac{h}{2^k} (\text{Dir}(f^{k-1}_h; B^m_1) - \text{Dir}(f^k_h; B^m_1)).$$
Proof. The proof is quite straightforward once we notice that
\[ G(f_h^{k-1}, f_h^k, h) \leq G(f_h^{k-1}, f_h^{-1}, h) = \text{Dir}(f_h^{k-1}; B_1^m). \]

\[ \square \]

6 Construction of the flow

Fix \( h > 0 \), we will construct a multiple-valued function \( F_h \) on the cylinder \([0, \infty) \times B_1^m\) such that \( F_h(0, x) = f_0(x), x \in B_1^m\), and \( F_h(t, x) = f_0(x), t \in [0, \infty), x \in \partial B_1^m\).

When \( t \in [(i - 1)h, ih], i = 1, 2, 3, \ldots \),

\[ F_h(t, x) := f_h^{i-1}(x), \text{ if } t \in [(i - 1)h, ih - \frac{h}{2^i}] \]

\[ F_h(t, x) := \xi^{-1} \circ \rho \circ \left[ \frac{ih - t}{h/2^i} \xi \circ f_h^{i-1} + \frac{t - (ih - \frac{h}{2^i})}{h/2^i} \xi \circ f_0 \right] \text{ if } t \in [ih - \frac{h}{2^i}, ih] \]

It is easy to see that the function \( F_h \) is well-defined by using the fact that \( \rho \circ \xi = \xi \).

As for the boundary date of \( F_h \), we only have to check the boundary date on \([0, \infty) \times \partial B_1^m\). Take \( t \in [ih - h/2^i, ih], x \in \partial B_1^m\), since \( f_h^{i-1}(x) = f_h^i(x) = f_0(x) \),

\[ F_h(t, x) = \xi^{-1} \circ \rho \circ \left[ \frac{ih - t}{h/2^i} \xi \circ f_0 + \frac{t - (ih - \frac{h}{2^i})}{h/2^i} \xi \circ f_0 \right] = \xi^{-1} \circ \rho \circ \xi \circ f_0 = f_0. \]

Obviously, \( F_h \in \mathcal{Y}_2([0, \infty) \times B_1^m, \mathbb{Q}) \).

Denoting by \( F_k \) the function \( F_{1/2^k} \). We will show that for any \( T > 0 \),

\[ \sup_k \text{Dir}(F_k; [0, T] \times B_1^m) < \infty. \]

Choose positive integer \( N \) such that \((N - 1)h \leq T < Nh\), where \( h = 1/2^k \).

Since \( F_h(t, x) = f_h^{i-1}(x), t \in [(i - 1)h, ih - \frac{h}{2^i}] \),

\[ \text{Energy of } F_h \text{ over } [(i - 1)h, ih - \frac{h}{2^i}] \times B_1^m = (h - \frac{h}{2^i}) \text{ Energy of } f_h^{i-1} \leq h \text{ Energy of } f_h^{i-1} \leq h \text{ Energy of } f_0 \]

Sum them up, we get

\[ \text{Energy of } F_h \text{ over } (\cup_i [(i - 1)h, ih - \frac{h}{2^i}] \times B_1^m) \cap [0, T] \times B_1^m \leq T \cdot \text{Energy of } f_0 \]
As for \( t \in [ih - \frac{h}{2}, ih] \), we have

\[
\int_{ih-h/2}^{ih} \left( \frac{\partial (\xi \circ F_h)}{\partial t} \right)^2 dt \leq \int_{ih-h/2}^{ih} \left( \frac{\partial (\xi \circ F_h)}{\partial x} \right)^2 dx
\]

Integrating of that gives

\[
\int_{ih}^{ih} \int_{B^m} \left( \frac{\partial (\xi \circ F_h)}{\partial t} \right)^2 dx \leq \int_{ih}^{ih} \int_{B^m} \left( \frac{\partial (\xi \circ F_h)}{\partial x} \right)^2 dx
\]

As for \( t \in [ih - \frac{h}{2}, ih] \), we have

\[
\frac{\partial (\xi \circ F_h)}{\partial x} \leq \left( \text{Lip} \right)^2 \left( \frac{ih-h/2}{h/2} \right)^2 \left( \frac{t-(ih-h/2)}{h/2} \right)^2 \left( \frac{\partial (\xi \circ F_h)}{\partial x} \right)^2
\]

Integrating of that gives

\[
\int_{ih}^{ih} \int_{B^m} \left( \frac{\partial (\xi \circ F_h)}{\partial x} \right)^2 dx
\]

As for \( t \in [ih - \frac{h}{2}, ih] \), we have

\[
\frac{\partial (\xi \circ F_h)}{\partial x} \leq \left( \text{Lip} \right)^2 \left( \frac{ih-h/2}{h/2} \right)^2 \left( \frac{t-(ih-h/2)}{h/2} \right)^2 \left( \frac{\partial (\xi \circ F_h)}{\partial x} \right)^2
\]

Energies of \( F_h \) over \( (U_i \cap [ih-h/2, ih] \times B^m \cap [0,T] \times B^m) \)
\[
\leq \sum_{i=1}^{\infty} \left[ (\text{Lip}\rho)^2 (\text{Dir}(f_{h}^{-1}; B_{1}^{m}) - \text{Dir}(f_{h}^{i}; B_{1}^{m})) + 2(\text{Lip}\rho)^2 \left( \frac{h}{3 \times 2^i} \right) 2\text{Dir}(f_{0}; B_{1}^{m}) \right] \\
\leq (\text{Lip}\rho)^2 \text{Dir}(f_{0}; B_{1}^{m}) + \frac{4}{3} (\text{Lip}\rho)^2 \text{Dir}(f_{0}; B_{1}^{m})(T + h) \\
\leq (\text{Lip}\rho)^2 \text{Dir}(f_{0}; B_{1}^{m}) + \frac{4}{3} (\text{Lip}\rho)^2 \text{Dir}(f_{0}; B_{1}^{m})(T + 1) < \infty
\]

In summary, we have
\[
\text{Dir}(F_{h}; [0, T] \times B_{1}^{m}) < \infty \text{ uniformly for } h = 1/2^k.
\]

Using Theorem 2.5, we have

**Theorem 6.1.** There exists a subsequence of \( F_{k} \) converging weakly in \( \mathcal{Y}_{2} \) to a multiple-valued function \( F \in \mathcal{Y}_{2}([0, \infty) \times B_{1}^{m}, \mathbb{Q}) \) such that
\[
F(0, x) = f_{0}(x), x \in B_{1}^{m},
\]
\[
F(t, x) = f_{0}(x), t \in [0, \infty), x \in \partial B_{1}^{m}.
\]

**Definition 6.1.** Denote
\[
F_{t}(\cdot) : B_{1}^{m} \rightarrow \mathbb{Q}, F_{t}(x) = F(t, x),
\]
for any \( t \in [0, \infty) \).

**Theorem 6.2.** For \( \mathcal{L}^1 \) almost every \( t > 0 \),
\[
\text{Dir}(F_{t}; B_{1}^{m}) \leq \text{Dir}(f_{0}; B_{1}^{m}).
\]

**Theorem 6.3.** The flow is Hölder continuous with respect to the \( \mathcal{L}^2 \) norm, i.e.,
\[
||\xi \circ F_{t} - \xi \circ F_{s}||_{\mathcal{L}^2} \leq \sqrt{s - t} \sqrt{\text{Dir}(f_{0}; B_{1}^{m})}
\]
for \( 0 \leq t < s \) for \( \mathcal{L}^1 \) almost every \( t, s \).

### 7 Proof of main theorems

**Lemma 7.1.** Suppose \( A_{i}, i = 1, 2, 3, \ldots \) are measurable sets in \( \mathbb{R} \), \( \sum_{i=1}^{\infty} \mathcal{L}^1(A_{i}) < \infty \), then
\[
\mathcal{L}^1(\lim A_{k}) = 0.
\]

*Proof.* This is a fundamental result whose proof can be found in almost any real analysis book.

**Lemma 7.2.** Suppose \( u_{i}, u \in \mathcal{Y}_{2}([0, \infty) \times B_{1}^{m}, \mathbb{R}^{PQ}) \), \( u_{i} \rightharpoonup u \) in \( \mathcal{Y}_{2} \), \( \sup_{i} ||u_{i}||_{\mathcal{Y}_{2}} < \infty \). Then for \( \mathcal{L}^1 \) almost every \( t > 0 \), there is a subsequence \( i_{k} \), such that \( u_{i_{k}}(t, \cdot), u(t, \cdot) \in \mathcal{Y}_{2}(B_{1}^{m}, \mathbb{R}^{PQ}) \) and
\[
u u_{i_{k}}(t, \cdot) \rightharpoonup u(t, \cdot) \text{ in } \mathcal{Y}_{2}(B_{1}^{m}, \mathbb{R}^{PQ}).
\]
Proof. It suffices to prove in the case that \( u = 0 \) and the domain is \([0, 1] \times B_1^m\).

Suppose
\[
\|u_i\|_{L^2_{2}}^2 = \int_0^1 dt \int_{B_1^m} |u_i|^2 + |\nabla u_i|^2 dx \leq M < \infty
\]
Since \( |\nabla u_i(t, \cdot)| \leq |\nabla u_i| \),
\[
\int_0^1 \lim |u_i(t, \cdot)|_{L^2_{2}}^2 dt \leq \lim \int_0^1 |u_i(t, \cdot)|_{L^2_{2}}^2 dt \leq \lim \int_0^1 dt \int_{B_1^m} |u_i|^2 + |\nabla u_i|^2 dx \leq M < \infty.
\]
Therefore, for \( L^1 \) almost every \( t > 0 \), \( \lim |u_i(t, \cdot)|_{L^2_{2}}^2 < \infty \).

By definition of \( \lim \inf \), for such \( t \), there is a subsequence \( i' \)(which may depend on \( t \)), such that
\[
\lim_{i' \to \infty} |u_{i'}(t, \cdot)|_{L^2_{2}}^2 < \infty
\]
By Theorem 4.2, there is a subsequence \( i'' \) such that
\[
\lim_{i'' \to \infty} u_{i''}(t, \cdot) \to h_t \in Y_2(B_1^m, \mathbb{R}^{pq})
\]
Hence \( u_{i''}(t, \cdot) \to h_t \) in \( L^2 \). We will show that \( h_t = 0 \).
\[
\int_0^1 \lim |u_{i''}(t, \cdot)|_{L^2_{2}}^2 dt \leq \lim \int_0^1 |u_{i''}(t, \cdot)|_{L^2_{2}}^2 dt = \lim |u_{i''}|_{L^2_{2}}^2 \to 0
\]
where the last limit comes from the assumption that \( u_{i''} \to 0 \) in \( Y_2 \).
So for \( L^1 \) almost every \( t > 0 \), \( \lim |u_{i''}(t, \cdot)|_{L^2_{2}}^2 = 0 \).

For such \( t \), there is a subsequence \( i''' \) such that
\[
\lim_{i''' \to \infty} u_{i'''}(t, \cdot) \to 0 \text{ in } L^2.
\]
This proves the lemma.

7.1 Proof of Theorem 6.2

Proof. Define
\[
A_k = \cup_{i=1}^{\infty} [ih - h/2^i, ih], h = 1/2^k
\]
\[
L^1(A_k) = \sum_{i=1}^{\infty} \frac{h}{2^i} = h = 1/2^k
\]
\[
\sum_{k=1}^{\infty} L^1(A_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = 1
\]
By Lemma 7.1, $\mathcal{L}^1(\overline{\lim A_k}) = 0$.

Apply Lemma 7.2 to $F_k$, there is a subset $B \subset [0, \infty)$, with $\mathcal{L}^1(B) = 0$, such that for any $t \notin B$, there is a subsequence (still denoted as $k$) such that

$$F_k(t, \cdot) \rightharpoonup F_i$$

in $\mathcal{Y}_2$.

Noticing

$$\overline{(\lim A_k)}^c = \lim (A_k^c) = \{ t : \text{there exists } n_0, \text{ when } k \geq n_0, x \in A_k^c \}$$

When $t \notin B \cup \overline{\lim A_k}$, after finite steps, $t \notin A_k$ for any $k$, i.e., after finite steps,

$$F_k(t, x) = f_h^{l-1}(x), \text{ for } h = 1/2^k, (l - 1)h \leq t < lh$$

Therefore

$$\text{Dir}(F_i; B_t^m) \leq \lim \text{Dir}(F_k(t, \cdot); B_t^m) = \lim \text{Dir}(f_h^{l-1}; B_t^m) \leq \text{Dir}(f_0; B_t^m)$$

for any $t \notin B \cup \overline{\lim A_k}$.

### 7.2 Proof of Theorem 6.3

**Proof.** Take $t, s \notin B \cup \overline{\lim A_k}$, $t < s$, there is a subsequence (still denoted as $k$) such that

$$F_k(t, \cdot) \rightharpoonup F_i, F_k(s, \cdot) \rightharpoonup F_s$$

in $\mathcal{L}^2$.

After some finite steps,

$$F_k(t, x) = f_h^{l-1}(x), \text{ for } h = 1/2^k, (l - 1)h \leq t < lh$$

$$F_k(s, x) = f_h^{l'-1}(x), \text{ for } h = 1/2^k, (l' - 1)h \leq s < lh.$$  

Therefore $\|\xi \circ F_k(t, \cdot) - \xi \circ F_k(s, \cdot)\|_{\mathcal{L}_2}^2 = \|\xi \circ f_h^{l-1} - \xi \circ f_h^{l'-1}\|_{\mathcal{L}_2}^2$ when $k$ is big enough.

Using the basic inequality

$$\left( \sum_{i=1}^{N} a_i \right)^2 \leq N \sum_{i=1}^{N} a_i^2$$

we have

$$\|\xi \circ F_k(t, \cdot) - \xi \circ F_k(s, \cdot)\|_{\mathcal{L}_2}^2 = \|\xi \circ f_h^{l-1} - \xi \circ f_h^{l'-1}\|_{\mathcal{L}_2}^2$$

$$\leq (l' - l) \sum_{i=1}^{l'-2} \|\xi \circ f_h^{i} - \xi \circ f_h^{i+1}\|_{\mathcal{L}_2}^2$$

$$\leq (l' - l) \sum_{i=1}^{l'-2} \frac{h}{2^{i+1}} \left( \text{Dir}(f_h^{i}; B_t^m) - \text{Dir}(f_h^{i+1}; B_t^m) \right)$$

$\leq (l' - l) h \sum_{i=1}^{l'-2} \left( \text{Dir}(f_h^{i}; B_t^m) - \text{Dir}(f_h^{i+1}; B_t^m) \right)$

$$\leq (l' - l) h \text{Dir}(f_0; B_t^m) \leq (s - t + h) \text{Dir}(f_0; B_t^m)$$

The theorem is proved once we let $k \to \infty$ in the above inequality. \qed

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8 Special cases of this flow

8.1 Further properties about minimizers $f^k_h$

In this section, we will look at the case when $n = 1$, namely, $Q = Q_1(\mathbb{R})$. In this case, we have, for any $S = \sum_{i=1}^Q [s_i]$, $W = \sum_{i=1}^Q [w_i]$ such that $-\infty < s_1 \leq s_2 \leq \cdots \leq s_Q < \infty$, $-\infty < w_1 \leq w_2 \leq \cdots \leq w_Q < \infty$,

$$|\xi(S) - \xi(W)|^2 = G^2(S,W) = \sum_{i=1}^Q |s_i - w_i|^2.$$ 

**Theorem 8.1.** The minimizer $f^k_h$ satisfies the following equation:

$$\int_{B_1^m} \phi, \frac{\eta(f^k_h) - \eta(f^{k-1}_h)}{h/2^k} > + < D\phi, D(\eta(f^k_h)) > d\mathcal{L}^m = 0,$$

for any $\phi \in C^1_0(B_1^m, \mathbb{R})$.

**Proof.** For simplicity, denote

$$f^k_h(x) = \sum_{i=1}^Q [f_i(x)], f^{k-1}_h(x) = \sum_{i=1}^Q [f^0_i(x)].$$

Take any smooth function $\phi \in C^1_0(B_1^m, \mathbb{R})$, let

$$u_t(x) = f^k_h(x) + tQ[\phi(x)] = \sum_{i=1}^Q [f_i(x) + t\phi(x)] \in \mathcal{M}.$$

Using Theorem 2.6, we have

$$\text{Dir}(u_t; B_1^m) = \text{Dir}(f^k_h; B_1^m) + 2Q \int_{B_1^m} < D(\eta \circ f^k_h), D(t\phi) > d\mathcal{L}^m + Q\text{Dir}(t\phi; B_1^m)$$

$$= \text{Dir}(f^k_h; B_1^m) + 2tQ \int_{B_1^m} < D(\eta \circ f^k_h), D\phi > d\mathcal{L}^m + t^2 Q\text{Dir}(\phi; B_1^m)$$

Fix any permutation $\sigma$ of $\{1, 2, \cdots, Q\}$,

$$\sum_{i=1}^Q |f_i(x) + t\phi(x) - f^0_{\sigma(i)}(x)|^2$$

$$= \sum_{i=1}^Q |f_i(x) - f^0_{\sigma(i)}(x)|^2 + t^2|\phi(x)|^2 + 2t < \phi(x), f_i(x) - f^0_{\sigma(i)}(x) >$$

$$= \sum_{i=1}^Q |f_i(x) - f^0_{\sigma(i)}(x)|^2 + t^2Q|\phi(x)|^2 + 2t < \phi(x), \sum_{i=1}^Q f_i(x) - \sum_{i=1}^Q f^0_{\sigma(i)}(x) >$$

$$= \sum_{i=1}^Q |f_i(x) - f^0_{\sigma(i)}(x)|^2 + t^2Q|\phi(x)|^2 + 2t < \phi(x), Q(\eta \circ f^k_h - \eta \circ f^{k-1}_h) >$$
\[ G^2(u_t, f_h^{k-1}) = G^2(f_h^k, f_h^{k-1}) + t^2 Q|\phi|^2 + 2t Q < \phi, \eta \circ f_h^k - \eta \circ f_h^{k-1} > \]

Hence
\[
G(f_h^{k-1}, u_t, h) = \text{Dir}(u_t; B^m_1) + \int_{B^m_1} G^2(u_t, f_h^{k-1}) d\mathcal{L}^m / (h/2^k)
\]
\[
= \text{Dir}(f_h^k; B^m_1) + 2t Q \int_{B^m_1} < D(\eta \circ f_h^k), D\phi > d\mathcal{L}^m + t^2 Q \text{Dir}(\phi; B^m_1) +
\]
\[
[ \int_{B^m_1} G^2(f_h^k, f_h^{k-1}) + t^2 Q|\phi|^2 + 2t Q < \phi, \eta \circ f_h^k - \eta \circ f_h^{k-1} > d\mathcal{L}^m ] / (h/2^k)
\]

Since \( u_0 = f_h^k \) and \( f_h^k \) minimizes the functional \( G(f_h^{k-1}, g, h) \),
\[
0 = \frac{dG(f_h^{k-1}, u_t, h)}{dt} |_{t=0} = 2Q \int_{B^m_1} < D(\eta \circ f_h^k), D\phi > d\mathcal{L}^m + 2Q \int_{B^m_1} < \phi, \eta \circ f_h^k - \eta \circ f_h^{k-1} > d\mathcal{L}^m / (h/2^k),
\]
which means
\[
\int_{B^m_1} < \phi, \frac{\eta(f_h^k) - \eta(f_h^{k-1})}{h/2^k} > + < D\phi, D(\eta(f_h^k)) > d\mathcal{L}^m = 0,
\]
for any \( \phi \in C^1_0(B^m_1, \mathbb{R}) \). \( \square \)

### 8.2 The case when the initial data is symmetric

In this section, \( Q \) will be two, i.e, \( Q = Q_2(\mathbb{R}) \).

Given \( g \in C^\infty(B^m_1; \mathbb{R}) \) such that \( g \) is nonnegative. Define
\[
f_0(x) = [[g(x)]] + [[-g(x)]] \in \mathcal{Y}_2(B^m_1; Q_2(\mathbb{R}))
\]

Now let’s consider the flow with \( f_0 \) being the initial data.

Fix \( h > 0 \), let
\[
f_h^0 = f_0.
\]

Therefore \( \eta(f_h^0) \equiv 0 \). By Theorem 8.1, \( \eta(f_h^1) \) satisfies the following equality:
\[
\int_{B^m_1} < \phi, \frac{\eta(f_h^1)}{h/2} > + < D\phi, D(\eta(f_h^1)) > d\mathcal{L}^m = 0,
\]
for any \( \phi \in C^1_0(B^m_1, \mathbb{R}) \).

That means \( \eta(f_h^1) \) is a weak solution of this boundary-value problem
\[
\begin{cases}
-\Delta u + \frac{u}{h/2} = 0 \quad \text{in } B^m_1 \\
u = 0 \quad \text{on } \partial B^m_1
\end{cases}
\]
From [EL], §6.2, there is a unique weak solution to it. Since obviously zero is a solution to the above problem, we get
\[ \eta(f^1_h) \equiv 0. \]
The same argument gives \( \eta(f^k_h) \equiv 0, \ k = 1, 2, \cdots \).
In spirit of \( \eta(f^1_h) \equiv 0 \), we can write \( f^1_h \) as
\[ f^1_h = [[f(x)]] + [\![ -f(x)\!]]. \]
where \( f \geq 0 \).
Since \( \xi \circ f^1_h = ( -f(x), f(x) ) \) and \( \xi \circ f^1_h \in \mathcal{Y}_2(B^m_1, \mathbb{R}^2) \),
\[ f \in \mathcal{Y}_2(B^m_1, \mathbb{R}). \]
Take any nonnegative function \( \phi \in C^1_{00}(B^m_1, \mathbb{R}) \), consider
\[ f_t(x) = [[f(x) + t\phi(x)]] + [\![ -f(x) - t\phi(x)\!]] \in \mathcal{M}. \]
We have
\[ \text{Dir}(f_t; B^m_1) = 2 \int_{B^m_1} |Df + tD\phi|^2 dx = 2 \int_{B^m_1} \|Df\|^2 + 2t[Df \cdot D\phi] dx \]
\[ \frac{1}{h/2} \int_{B^m_1} g^2(f^0_h, f_t) dx = \frac{4}{h} \int_{B^m_1} |f + t\phi - g|^2 dx \]
\[ = \frac{4}{h} \int_{B^m_1} \|f - g\|^2 + t^2\phi^2 + 2t\phi(f - g) dx \]
Therefore,
\[ 0 = \lim_{t \to 0} G(f^0_h, f_t, h) = 4 \int_{B^m_1} [Df \cdot D\phi + \frac{1}{h/2}\phi(f - g)] dx \]
for any nonnegative \( \phi \in C^1_{00}(B^m_1, \mathbb{R}) \). Because of the linearity of the above equation, we conclude that \( f \) is a weak solution of the following boundary-value problem:
\[ \begin{cases} -\Delta u + \frac{u}{h/2} = \frac{g}{h/2} & \text{in } B^m_1 \\ u = g & \text{on } \partial B^m_1 \end{cases} \]
By introducing \( \bar{u} = f - g \), we see \( \bar{u} \) is a weak solution of the following boundary-value problem:
\[ \begin{cases} -\Delta \bar{u} + \frac{\bar{u}}{h/2} = \Delta g & \text{in } B^m_1 \\ \bar{u} = 0 & \text{on } \partial B^m_1 \end{cases} \]
which has a unique weak solution \( \bar{u} \). Moreover by the regularity theorem in [EL], §6.3,
\[ \bar{u} \in C^\infty(B^m_1) \]
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Hence so is $f$. Hence $f$ actually is a smooth solution of the following PDE:

$$\frac{f - g}{h/2} = \Delta f.$$

Now let us denote:

$$f_h^k(x) = [[\tilde{f}_h^k(x)]] + [[-\tilde{f}_h^k(x)]], \quad \tilde{f}_h^k(x) \geq 0, \quad k = 0, 1, 2, \cdots.$$

From the previous argument, we know each $\tilde{f}_h^k$ is a smooth solution of the following PDE:

$$\frac{\tilde{f}_h^k - \tilde{f}_h^{k-1}}{h/2^{k-1}} = \Delta \tilde{f}_h^k$$

This gives the proof of this theorem:

**Theorem 8.2.** Suppose $f_0 = [[g]] + [[-g]] \in \mathcal{Y}_2(B_1^m, Q_2(\mathbb{R}))$, with nonnegative function $g \in C^\infty(B_1^m, \mathbb{R})$. Then $\eta(f_h^k) \equiv 0$ for $k = 0, 1, 2, \cdots$. Moreover, if we denote

$$f_h^k(x) = [[\tilde{f}_h^k(x)]] + [[-\tilde{f}_h^k(x)]], \quad \tilde{f}_h^k(x) \geq 0, \quad k = 0, 1, 2, \cdots.$$

then each $\tilde{f}_h^k \in C^\infty(B_1^m, \mathbb{R})$ and satisfies the following PDEs:

$$\frac{\tilde{f}_h^k - \tilde{f}_h^{k-1}}{h/2^{k-1}} = \Delta \tilde{f}_h^k$$

Next, we will show that $f_h^k$ has no branch points. Namely,

**Theorem 8.3.** With the same assumptions as the above theorem, if moreover, $g$ is not identically zero, then

$$\tilde{f}_h^k(x) > 0, \quad x \in \partial(B_1^m)^\circ, \quad k = 1, 2, \cdots.$$

In particular, $\tilde{f}_h^k(x) \neq 2[[0]], \quad \text{for any } x \in \partial(B_1^m)^\circ, \quad k = 1, 2, \cdots.$

**Proof.** From Theorem 8.2, $\tilde{f}_h^1$ satisfies the following PDE:

$$\frac{\tilde{f}_h^1 - \tilde{f}_h^0}{h/2} - \Delta \tilde{f}_h^1 = 0.$$

Consider this operator:

$$Lu = -\Delta u + \frac{u}{h/2}.$$

Let $\psi = \tilde{f}_h^1$. We have

$$L\psi = \frac{\tilde{f}_h^0}{h/2} \geq 0$$

By strong maximum principle of [EL], §6.4, we conclude that if $\psi$ attains a nonpositive minimum over $B_1^m$ at an interior point, then $\psi$ is constant within
Since $\psi \equiv 0$ on $\partial B_1^m$, either $\psi > 0$ in $(B_1^m)^\circ$ or $\psi \equiv 0$ in $B_1^m$. But if $\psi \equiv 0$ in $B_1^m$, then $f_h^0 = (h/2)L\psi \equiv 0$, which contradicts to our assumption about $g$. Therefore $\psi > 0$ in $(B_1^m)^\circ$,

which means $\tilde{f}_h^1(x) > 0, x \in (B_1^m)^\circ$.

Now we will use induction to prove the theorem. We assume that we have already showed that:

$$ \tilde{f}_h^k(x) > 0, x \in (B_1^m)^\circ $$

We will show that

$$ \tilde{f}_h^{k+1}(x) > 0, x \in (B_1^m)^\circ. $$

From Theorem 8.2, $\tilde{f}_h^{k+1}$ satisfies the following PDE:

$$ \frac{\tilde{f}_h^{k+1} - \tilde{f}_h^k}{h/2^{k+1}} - \Delta \tilde{f}_h^{k+1} = 0. $$

Consider the operator:

$$ Lu = -\Delta u + \frac{u}{h/2^{k+1}}, $$

and let

$$ \psi = \tilde{f}_h^{k+1}. $$

Since

$$ L\psi = \frac{\tilde{f}_h^k}{h/2^{k+1}} > 0 \text{ in } (B_1^m)^\circ, $$

the strong maximum principle tells either $\psi > 0$ in $(B_1^m)^\circ$ or $\psi \equiv 0$ in $B_1^m$. $\psi$ can not be identically zero because otherwise $f_h^k = (h/2^{k+1})L\psi \equiv 0$, a contradiction to our induction assumption. Therefore, $\psi > 0$ in $(B_1^m)^\circ$, i.e,

$$ \tilde{f}_h^{k+1} > 0 \text{ in } (B_1^m)^\circ. $$

\hfill \Box

9 Heat flow of multiple-valued functions

9.1 Construction of heat flow for single-valued functions by method of discrete Morse flow

We assume that $f_0 \in C^\infty(B_1^m, \mathbb{R})$, and

$$ \text{Dir}(f_0; B_1^m) < \infty, ||f_0||_{L^2} < \infty. $$

Define

$$ W = \{ u \in \mathcal{Y}_2(B_1^m, \mathbb{R}), \text{strictly defined, } u = f_0 \text{ on } \partial B_1^m \} $$

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Fix time $T$ and positive integer $N$, we let $h = T/N$. We define inductively, for $k = 0, 1, 2, \cdots, N$ a single-valued sobolev function $f_h^k$ such that $f_h^0 = f_0$ and $f_h^k$ minimizes the functional (step minimizing with respect to $h$)

$$G(g) = G(f_h^{k-1}, g, h) = \text{Dir}(g; B_1^m) + \frac{1}{h} ||f_h^{k-1} - g||_{L^2}^2$$

where $g \in W$.

Using the same method as in Proposition 5.1 and 5.2, we have:

**Proposition 9.1.** $f_h^k$ is defined for each positive integer $k$ and each $h = T/N > 0$.

**Proposition 9.2.** The energy of the sequence $\{f_h^k\}$ is non-increasing,

$$\text{Dir}(f_h^k; B_1^m) \leq \text{Dir}(f_h^{k-1}; B_1^m) \leq \text{Dir}(f_0; B_1^m).$$

We also have the following estimate:

$$||f_h^{k-1} - f_h^k||_{L^2} \leq h(\text{Dir}(f_h^{k-1}; B_1^m) - \text{Dir}(f_h^k; B_1^m)).$$

We can also prove the following result in a similar way as in Theorem 8.2,

**Theorem 9.1.** For any positive integer $N$, $f_h^i$ is smooth for $i = 0, 1, 2, \cdots, N$, and $h = T/N$. Moreover they satisfy the following PDEs:

$$\frac{f_h^{i+1} - 2f_h^i + f_h^i}{h} = \Delta f_h^{i+1}, \quad i = 0, 1, 2, \cdots, N - 1.$$

We define a function $F_N : [0, T] \times B_1^m \to \mathbb{R}$ as follows:

$$F_N(t, x) = \left[ \frac{(i + 1)h - t}{h} f_h^i(x) + \frac{t - ih}{h} f_h^{i+1}(x) \right], \quad \text{when } t \in [ih, (i+1)h], \quad x \in B_1^m,$$

where $i = 0, 1, \cdots, N - 1$.

When $t \in [ih, (i+1)h]$,

$$\frac{\partial F_N}{\partial t} = \frac{1}{h} (f_h^{i+1} - f_h^i),$$

$$D_x F_N = \left[ \frac{(i + 1)h - t}{h} Df_h^i + \frac{t - ih}{h} Df_h^{i+1} \right].$$

Therefore,

$$\int_{ih}^{(i+1)h} \int_{B_1^m} \left( \frac{\partial F_N}{\partial t} \right)^2 dt dx = \frac{1}{h} \int_{B_1^m} (f_h^i - f_h^{i+1})^2 dx$$

$$\leq \text{Dir}(f_h^i) - \text{Dir}(f_h^{i+1}),$$

and

$$\int_{ih}^{(i+1)h} \int_{B_1^m} |D_x F_N|^2 dt dx \leq 2 \int_{ih}^{(i+1)h} \left( \frac{(i + 1)h - t}{h} \right)^2 dt \int_{B_1^m} |Df_h^i|^2 dx$$

$$+ \int_{ih}^{(i+1)h} \left( \frac{t - ih}{h} \right)^2 dt \int_{B_1^m} |Df_h^{i+1}|^2 dx$$

$$= \frac{2h}{3} \text{Dir}(f_h^i) + \text{Dir}(f_h^{i+1}) \leq 4h \frac{3}{3} \text{Dir}(f_0)$$

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Integration over the domain $[0, T] \times B^n_1$ gives
\[
\text{Energy of } F_N \leq \sum_{i=0}^{N-1} (\text{Dir}(f_i^h) - \text{Dir}(f_{i+1}^h)) + \sum_{i=0}^{N-1} \frac{4h}{3} \text{Dir}(f_0) \leq \text{Dir}(f_0) + \frac{4T}{3} \text{Dir}(f_0) < \infty
\]

In spirit of Theorem 2.5, we have the following theorem,

**Theorem 9.2.** There is a subsequence of $F_N$, still denoted as $F_N$ for simplicity, such that

\[
F_N \rightharpoonup F \in \mathcal{Y}_2([0, T] \times B^n_1, \mathbb{R})
\]

for some single-valued sobolev function $F$. Moreover,

\[
F(0, x) = f_0(x), x \in B^n_1,
\]

\[
F(t, x) = f_0(x), t \in [0, T], x \in \partial B^n_1.
\]

Next we will show that $F \in C^\infty([0, T] \times B^n_1, \mathbb{R})$ and it solves the heat equation

\[
\frac{\partial F}{\partial t} = \Delta_x F.
\]

**Lemma 9.1.** For any fixed $h > 0$, $\sup_{x \in B^n_1} |f^n_h(x)|$ is non-increasing in $i$.

*Proof.* We put $\rho_{n-1} = \sup_{x \in B^n_1} |f^{n-1}_h(x)|$, and $v_n = \max\{|f^n_h|^2 - \rho^2_{n-1}, 0\}$. We will show that $v_n = 0$ a.e. in $B^n_1$.

Let $\eta \in C_0^\infty(B^n_1)$ be a nonnegative function. It is easy to see that for any sufficiently small $\epsilon > 0$,

\[
\frac{1}{h \epsilon} \int_{B^n_1} |f^n_h - f^{n-1}_h|^2 - |(1 - \epsilon \eta)f^n_h - f^{n-1}_h|^2 \, dx = \frac{1}{h} \int_{B^n_1} \eta \{(1 - \epsilon \eta)|f^n_h|^2 - |f^{n-1}_h|^2 + |f^n_h - f^{n-1}_h|^2\} \, dx := I
\]

and hence

\[
0 \geq \frac{G(f^n_{h-1}, f^n_{h}, h) - G(f^{n-1}_{h-1}, (1 - \epsilon \eta)f^n_{h}, h)}{\epsilon} = \frac{\text{Dir}(f^n_{h}) - \text{Dir}((1 - \epsilon \eta)f^n_{h}) + I}{\epsilon}
\]

\[
= \frac{1}{\epsilon} \int_{B^n_1} \{(2\epsilon \eta - \epsilon^2 \eta^2)|Df^n_{h}|^2 - \epsilon^2 |D\eta|^2 |f^n_{h}|^2 + \epsilon(1 - \eta \eta)D[(f^n_{h})^2] \cdot D\eta\} \, dx + I
\]

Let $\epsilon \to 0$,

\[
0 \geq \int_{B^n_1} \{2\eta |Df^n_{h}|^2 + D[(f^n_{h})^2] \cdot D\eta\} \, dx + \lim_{\epsilon \to 0} I
\]

\[
\geq \int_{B^n_1} D[(f^n_{h})^2] \cdot D\eta \, dx + \frac{1}{h} \int_{B^n_1} \eta(|f^n_{h}|^2 - |f^{n-1}_h|^2) \, dx
\]

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Setting $\eta = v_n$, the last inequality gives us that
\[
0 \geq \frac{1}{h} \int_{\{x : |f_n^h(x)| > \rho_{n-1}\}} (|f_n^h|^2 - \rho_{n-1}^2)(|f_n^h|^2 - |f_n^{n-1}|^2)dx \\
\geq \frac{1}{h} \int_{\{x : |f_n^h(x)| > \rho_{n-1}\}} (|f_n^h|^2 - \rho_{n-1}^2)^2 dx,
\]
which proves $|f_n^h| \leq \rho_{n-1}$ a.e. in $B_1^n$. \qed

**Lemma 9.2.** $\{F_n\}$ is uniformly bounded in $[0, T] \times B_1^n$.

**Proof.** For any positive integer $n$, let $h = T/n$,
\[
F_n(t, x) = \frac{(i+1)h - t}{h} f_i^h(x) + \frac{t - ih}{h} f_{i+1}^h(x), \text{ when } ih \leq t \leq (i+1)h.
\]
Hence,
\[
|F_n(t, x)| \leq \frac{(i+1)h - t}{h} |f_i^h(x)| + \frac{t - ih}{h} |f_{i+1}^h(x)| \\
\leq \frac{(i+1)h - t}{h} \sup_{x \in B_1^n} |f_i^h(x)| + \frac{t - ih}{h} \sup_{x \in B_1^n} |f_{i+1}^h(x)| \\
\leq \sup_{x \in B_1^n} |f_0^h(x)| < \infty,
\]
where the third inequality comes from Lemma 9.1 and the last inequality follows from the smoothness of $f_0$. \qed

Applying Theorem K in [KK] gives us

**Lemma 9.3.** There exist constants $C > 0$ and $\mu > 0$ such that
\[
|F_n(t, x) - F_n(t', y)| \leq C(|t - t'|^\mu + |x - y|^{2\mu}), \forall (t, x), (t', y) \in [0, T] \times B_1^n,
\]
i.e. $\{F_n\}$ is uniformly equicontinuous.

Since each $f_i^h$ is smooth, each $F_n$ is smooth. By using the Arzela-Ascoli theorem, we have

**Theorem 9.3.** There is a subsequence $\{F_{k_j}\}$ of $\{F_n\}$ which uniformly converges to $F$, for $F : [0, T] \times B_1^n \to \mathbb{R}$ as in Theorem 9.2. Therefore $F$ is continuous.

**Proof.** The Arzela-Ascoli theorem guarantees that there are a continuous function $H : [0, T] \times B_1^n \to \mathbb{R}$ and a subsequence $\{F_{k_j}\}$ such that $F_{k_j}$ uniformly converges to $H$. Since the $\mathcal{Y}_2$ converges implies $L^2$ convergence in some subsequence, we know $F = H$. \qed
Remark 9.1. We will also just use the original sequence \( \{F_n\} \) without referring to any subsequence for simplicity.

**Theorem 9.4.** \( F \) is smooth and satisfies the heat equation

\[
\frac{\partial F}{\partial t} = \Delta_x F.
\]

**Proof.** Summing up the PDEs in Theorem 9.1 gives us that

\[
f_j^h - f_0^h = h \sum_{i=1}^{j-1} \Delta f_{i+1}^h, \quad j = 1, 2, \ldots, n
\]

Fix \( n \), let \( h = T/n \). For any fixed \( t \in [0, T] \), \( x \in B_{1}^m \), choose integer \( l \) such that \( lh \leq t < (l+1)h \), then

\[
F_n(t, x) = \frac{(l+1)h - t}{h} f_l^h(x) + \frac{t - lh}{h} f_{l+1}^h(x),
\]

\[
\Delta_x F_n(t, x) = \frac{(l+1)h - t}{h} \Delta f_l^h(x) + \frac{t - lh}{h} \Delta f_{l+1}^h(x).
\]

Therefore,

\[
F_n(t, x) - F_n(0, x) = \frac{(l+1)h - t}{h} f_l^h(x) + \frac{t - lh}{h} f_{l+1}^h(x) - f_0^h(x)
\]

\[
= \frac{(l+1)h - t}{h} (f_l^h(x) - f_0^h(x)) + \frac{h}{h} \sum_{i=0}^{l-1} \Delta f_{i+1}^h + (t - lh) \Delta f_{l+1}^h
\]

Choose \( \phi \in C_0^\infty(B_1^m; \mathbb{R}) \), do the integration by parts twice, we have

\[
\int_{B_1^m} (F_n(t, x) - F_n(0, x))\phi(x)dx = \sum_{i=0}^{l-1} \int_{B_1^m} f_{i+1}^h \Delta \phi dx + (t - lh) \int_{B_1^m} f_{l+1}^h \Delta \phi dx
\]

Letting \( n \to \infty \) gives

\[
\int_{B_1^m} (F(t, x) - F(0, x))\phi(x)dx = \int_0^t \int_{B_1^m} F(s, x) \Delta \phi(x)dxdx,
\]

\[
\int_{B_1^m} \frac{\partial F}{\partial t} \phi(x)dx = \int_{B_1^m} F(t, x) \Delta \phi(x)dx = -\int_{B_1^m} \nabla_x F(t, x) \cdot \nabla \phi dx
\]

In a world, \( F \) is a weak solution of the heat equation

\[
\frac{\partial F}{\partial t} = \Delta_x F.
\]
Since \( F(0, x) = f_0 \) is smooth, \( F \) must be the unique smooth solution of the heat equation:
\[
\frac{\partial F}{\partial t} = \Delta_x F, \quad (t, x) \in [0, T] \times B^m_1,
\]
\[
F(0, x) = f_0(x), \quad x \in B^m_1,
\]
\[
F(t, x) = f_0(x), \quad t \in [0, T], \quad x \in \partial B^m_1.
\]

9.2 Heat flow of multiple-valued functions

In this section, we use a modified sequence of functional to construct a flow for multiple-valued functions which is a generalization of ordinary heat flow.

Suppose \( f_0 = [[g]] + [[-g]] \in \mathcal{Y}_2(B^m_1, \mathbb{Q}_2(\mathbb{R})) \), with nonnegative, nonconstant function \( g \in C^\infty(B^m_1, \mathbb{R}) \). The functionals are
\[
G(g) = G(f^k_{-1}, g, h) = \text{Dir}(g; B^m_1) + \frac{1}{h} ||\xi \circ f^k_{-1} - \xi \circ g||^2_2
\]
where \( g \in \mathcal{M} \).

Using the same argument in Theorem 8.3, we know that the flow instantly separates. If we abuse our notation by denoting \( f^k_h = [[f^k_h]] + [[-f^k_h]] \), for nonnegative function \( f^k_h \in \mathcal{Y}_2(B^m_1, \mathbb{R}) \), then each \( f^k_h \) minimizes the functional
\[
G(g) = G(f^k_{-1}, g, h) = \text{Dir}(g; B^m_1) + \frac{1}{h} ||f^k_{-1} - g||^2_2
\]
where \( g \in \mathcal{W} \).

We define the discrete flow as follows:
\[
F^d_n(t, x) = \left\{ \begin{array}{ll}
[[\frac{(i+1)h-t}{h}f^i_h(x) + \frac{t-ih}{h}f^{i+1}_h(x)]] & \text{for } ih \leq t < (i+1)h, \quad h = T/n.
\end{array} \right.
\]

for \( ih \leq t < (i+1)h \), \( h = T/n \).

The above section tells that the limit multiple-valued function \( F : [0, T] \times B^m_1 \rightarrow \mathbb{Q}_2(\mathbb{R}) \) is smooth, and can be written as
\[
F(t, x) = [[F_1(t, x)]] + [[-F_1(t, x)]]
\]
for some nonnegative smooth function \( F_1 : [0, T] \times B^m_1 \rightarrow \mathbb{R} \) which satisfies the ordinary heat equation.

References

[AF] Frederick J. Almgren, Jr. Q-valued functions minimizing Dirichlet’s integral and the regularity of area-minimizing rectifiable currents up to codimension 2, Princeton University
Ivan Blank and Penelope Smith, Convergence of Rothe’s method for fully nonlinear parabolic equations, Journal of Geometric Analysis, Vol 15, Number 3, 2005

Xiaoxi Cheng, A mass reducing flow for integral currents, Indiana University Mathematics Journal, Vol. 42, No. 2(1993), 425-444

J. Eells, Jr. and J.H.Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. 86, 1964, 109-160

Lawrence C. Evans, Partial Differential Equations, American Mathematical Society, 1998

Herbert Federer, Geometric measure theory, Springer-Verlag, New York, 1969

Jun-ichi Haga, Keisuke Hoshino, Norio Kikuchi, Construction of harmonic map flows through the method of discrete Morse flows, preprint

K. Horihata and N. Kikuchi, A construction of solutions satisfying a Caccioppoli inequality for nonlinear parabolic equations associated to a variational functional of harmonic type, preprint

Richard S. Hamilton, Harmonic maps of manifold with boundary, Lecture notes in Math, 471, Springer, Berlin, 1975

Robert Hardt, Singularities of harmonic maps, Bulletin of the American Mathematical Society, Vol 34, No. 1, 1997, 15-34

Norio Kikuchi and Jozef Kacur, Convergence of Rothe’s method in Hölder spaces, Applications of Mathematics, No. 5, 2003, 353-365

Norio Kikuchi, Hölder estimates of solutions to difference partial differential equations of elliptic-parabolic type, Journal of Geometric Analysis, Vol 11, Number 1, 2001

Fang-Hua Lin, Gradient estimates and blow-up analysis for stationary harmonic maps, Annals of Math, 2nd Ser., Vol. 149, No. 3, 1999, 785-829

Leon Simon, Theorems on regularity and singularity of energy minimizing maps, Birkhäuser, 1996

Richard Schoen, Karen Uhlenbeck, A regularity theory for harmonic maps, Journal of Differential Geometry, 17, 1982, 307-335

Brian White, A new proof of the compactness theorem for integral currents, Comment. Math. Helvetici. 64, 1989, 207-220

Wei Zhu, A regularity theory for multiple-valued Dirichlet minimizing maps, preprint

William P. Ziemer, Weakly differentiable functions, Springer-Verlag, 1989