Nilradicals of Einstein solvmanifolds

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Abstract

A Riemannian Einstein solvmanifold is called standard, if the orthogonal complement to the nilradical of its Lie algebra is abelian. No examples of nonstandard solvmanifolds are known. We show that the standardness of an Einstein metric solvable Lie algebra is completely detected by its nilradical and prove that many classes of nilpotent Lie algebras (Einstein nilradicals, algebras with less than four generators, free Lie algebras, some classes of two-step nilpotent ones) contain no nilradicals of nonstandard Einstein metric solvable Lie algebras. We also prove that there are no nonstandard Einstein metric solvable Lie algebras of dimension less than ten.

1 Introduction

The theory of Riemannian homogeneous spaces with an Einstein metric splits into three very different cases depending on the sign of the Einstein constant, the scalar curvature. Among them, only in the Ricci-flat case the picture is complete: by the result of [AK], every Ricci-flat homogeneous space is flat. Despite many remarkable existence, nonexistence and partial classification results (see [WZ, BWZ]), the positive scalar curvature case is far from being completely understood. One of the main open questions in the case of negative scalar curvature (which we are dealing with in this paper), is the Alekseevski Conjecture [A1] asserting that any Einstein homogeneous Riemannian space with negative scalar curvature admits a simply transitive solvable isometry group. This is equivalent to saying that any such space is a solvmanifold, a solvable Lie group with a left-invariant Riemannian metric satisfying the Einstein condition.

Another question arises from the fact that all the known examples of Einstein solvmanifolds are standard. This means that the metric solvable Lie algebra $g$ of such a solvmanifold has the following property: the orthogonal complement $a$ to the derived algebra of $g$ is abelian. Are there any nonstandard Einstein solvmanifolds?

In the paper [H], J. Heber gave a deep and detailed analysis of standard Einstein solvmanifolds and also showed that for several classes of solvmanifolds, the existence of an Einstein left invariant metric implies standardness. What is more, nonstandard solvmanifolds must be algebraically very different from the standard ones. In particular, a solvable Lie group cannot carry two Einstein left-invariant metrics, one of which is standard, and the other one is not. Moreover, standard Einstein solvmanifolds form an open and compact subspace in the moduli space of all Einstein solvmanifolds of the given dimension and fixed scalar curvature.

Further progress was made by D. Schueth in [S], for solvmanifolds with $\dim[a, a] = 1$. All the Einstein solvmanifolds of dimension up to 7 are standard, as it is shown by Yu. Nikolayevsky: for dimension up to 6, this follows from their classification obtained in [N1, NN], for Einstein solvmanifolds of dimension 7, the standardness is proved in [N2].

The problem of studying and classifying standard Einstein solvmanifolds (and standard Einstein metric solvable Lie algebras) was given a great deal of attention in the last decade (a very incomplete list of references is [GK, L1, L2, L3, LW, Ni, N1, P, T]).
This paper addresses the following question: which algebraic properties of a nilpotent Lie algebra $\mathfrak{n}$ force any Einstein metric solvable Lie algebra with the nilradical $\mathfrak{n}$ to be standard? The focus on the nilradical is motivated by the following two reasons. Firstly, as it is shown by J. Lauret [L1], in the standard case, the nilradical is the only thing one needs to know to reconstruct both the Lie algebra structure and the inner product of any of its Einstein metric solvable extension (see Section 2 for details). Secondly, if the nonstandard Einstein solvmanifolds do exist at all, a natural approach to finding one might be to start with a nilpotent Lie algebra, which will then serve as the nilradical for the Lie algebra of such a solvmanifold.

The idea of using the properties of the nilradical to study the standardness also appeared in the paper [N2], one of the main result of which is that if for a metric nilpotent Lie algebra, the symmetric operator $\text{id} - (\text{Tr} (\text{ric})/\text{Tr} (\text{ric}^2)) \text{ric}$ is positive, then any Einstein metric solvable extension of it is standard.

An endomorphism $A$ of a linear space is called semisimple, if its nilpotent part vanishes, is called real, if all its eigenvalues are real, and is called nonnegative (respectively, positive), if all its eigenvalues are real and nonnegative (respectively, positive). In the latter case, we write $A \geq 0$ (respectively, $A > 0$).

A derivation $\phi$ of a Lie algebra $\mathfrak{l}$ is called pre-Einstein, if it is real, semisimple, and $\text{Tr} (\phi \circ \psi) = \text{Tr} \psi$, for any $\psi \in \text{Der}(\mathfrak{l})$.

In Proposition 1 we show that every Lie algebra $\mathfrak{l}$ admits a unique pre-Einstein derivation $\phi_\mathfrak{l}$, up to a conjugation (it could well happen that $\phi_\mathfrak{l} = 0$, even for a nilpotent $\mathfrak{l}$).

For any $\psi \in \text{Der}(\mathfrak{l})$ we denote $\text{ad}_\psi$ the corresponding inner derivation of $\text{Der}(\mathfrak{l})$. If $\psi$ is semisimple and real, the same is true for $\text{ad}_\psi$.

The main technical result of the paper is the following sufficient condition for the standardness.

**Theorem 1.** Let $\mathfrak{n}$ be a nilpotent Lie algebra with $\phi$ a pre-Einstein derivation. Suppose that

$$\phi > 0 \quad \text{and} \quad \text{ad}_\phi \geq 0. \quad (1)$$

Then any Einstein metric solvable Lie algebra with the nilradical $\mathfrak{n}$ is standard.

This theorem (and most of the subsequent ones) can be slightly strengthened: given an Einstein metric solvable Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$, it suffices to choose $\mathfrak{n}$ anywhere between its derived algebra and its nilradical and to check the condition (1) (see the proof of Theorem 1 in Section 3).

It would be tempting to get rid of the second condition ($\text{ad}_\phi \geq 0$) in Theorem 1. Although we do not know how to avoid it completely, the following can be proved:

**Theorem 2.** Let $\mathfrak{n}$ be a nilpotent Lie algebra whose pre-Einstein derivation is positive. Then any split Einstein metric solvable Lie algebra with the nilradical $\mathfrak{n}$ is standard.

Recall that a Lie algebra $\mathfrak{l}$ is called split, if for any $X \in \mathfrak{l}$ there exists $Y \in \mathfrak{l}$ such that $\text{ad}_Y$ is the nilpotent part of $\text{ad}_X$. For example, any algebraic Lie algebra is split.

As the main consequence of Theorem 1, we show that for an Einstein metric solvable Lie algebra, it is the nilradical which detects the standardness:

**Theorem 3.** Let $\mathfrak{g}$ and $\mathfrak{g}_0$ be solvable Lie algebras with the same nilradical. If $\mathfrak{g}_0$ admits a standard Einstein inner product, then any Einstein inner product on $\mathfrak{g}$ is also standard.

The idea of this theorem is close to [H, Theorem 5.3] saying that a solvable Lie algebra cannot carry two Einstein inner products, one of which is standard, and another one is not. Theorem 3 gives the positive answer to the question asked in [N2].

A nilpotent Lie algebra which can be a nilradical of a standard Einstein metric solvable Lie algebra is called an Einstein nilradical. If $\mathfrak{n}$ is an Einstein nilradical, its pre-Einstein derivation is a positive
multiple of the Einstein derivation (see the proof of Theorem 3). Not every nilpotent Lie algebra is an 
Einstein nilradical. By [H], a necessary condition for that is to admit an \( N \)-gradation (which is defined 
by the Einstein derivation). This condition is, however, far from being sufficient, and a very delicate 
analysis (see the stratification constructed in [LW]) is required to decide whether a given \( N \)-graded 
ilpotent Lie algebra is indeed an Einstein nilradical (examples of those which are not can found in 
[LW, Ni, P]).

Each of the following nilpotent Lie algebras \( n \) is an Einstein nilradical (see [L4] for a complete up-
to-date list), hence by Theorem 3, any Einstein metric solvable Lie algebra with such a nilradical \( n \) is 
standard:

(i) \( n \) is abelian [A2];

(ii) \( n \) has a codimension one abelian ideal [L3];

(iii) \( \dim n \leq 6; \) [W, L3];

(iv) \( n \) is uniform two-step nilpotent (this class includes, for example, nilpotent Lie algebras of Heisen-
berg type, that is, those arising from representations of Clifford algebras) [GK];

(v) \( n \) is a direct sum of Einstein nilradicals [P];

(vi) \( n = t_n \), the algebra of strictly upper-triangular \( n \times n \) matrices [P].

The fact that any Einstein metric solvable Lie algebra with an abelian nilradical \( n \) is standard is 
independently proved in [N2].

In view of (i), one may wonder what happens in the other extremal case, when \( n \) is the least 
commutative, a filiform. The answer is given by the following theorem (recall that a filiform algebra is 
generated by two elements):

**Theorem 4.** Let \( n \) be a nilpotent Lie algebra generated by no more than three elements (that is, \( \dim n - \dim[n, n] \leq 3 \)). Then any Einstein metric solvable Lie algebra with the nilradical \( n \) is standard.

Another consequence of Theorem 1 is the following theorem:

**Theorem 5.** Any Einstein metric solvable Lie algebra with a free nilradical is standard.

Free Einstein nilradical were classified in [Ni]. It is shown that apart from the abelian and the 
two-step ones, only six other free Lie algebras can be Einstein nilradicals.

In the second part of the paper, we consider the applications of Theorem 1 and Theorem 3 to two-
step nilpotent nilradical. A two-step nilpotent Lie algebra \( n \) is said to be of type \( (p, q) \), if \( \dim n = p + q \) 
and \( \dim[n, n] = p \) (clearly, \( p \leq D := q(q - 1)/2 \)). Any such algebra is determined by a \( p \)-dimensional 
subspace \( W \) of \( q \times q \) skew-symmetric matrices, that is, by a point on the Grassmannian \( G(p, o(q)) \). Two 
algebras of type \( (p, q) \) are isomorphic if and only if they lie in the same orbit \( [W] \) of the action of the 
group \( \text{SL}(q) \) on \( G(p, o(q)) \).

Let \( \mathcal{O}(p, q) \) be the class of two-step nilpotent Lie algebras \( n \) of type \( (p, q) \), which have the following 
property: for any \( \psi \in \text{Der}(n) \), \( \text{Tr} \psi = 0 \Rightarrow \text{Tr} \psi|_{[n, n]} = 0 \) (an equivalent definition: the canonical 
derivation acting as multiplication by 2 on \( [n, n] \) and as the identity on some linear complement to \( [n, n] \) 
is pre-Einstein). The class \( \mathcal{O}(p, q) \) consist of the two-step nilpotent algebras whose derivation algebra 
has, in the sense, the nicest possible structure. Another natural motivation to single the class \( \mathcal{O}(p, q) \) 
out is that the Einstein nilradicals belonging to \( \mathcal{O}(p, q) \) are precisely those whose eigenvalue type is 
\((1 < 2; q, p)\) which is, perhaps, the most well-studied class of the Einstein nilradicals. We prove the 
following “general position” theorem:
Theorem 6. 1. No two-step nilpotent Lie algebra $n \in \mathfrak{S}(p, q)$ can be the nilradical of a nonstandard Einstein metric solvable Lie algebra. For all the pairs $(p, q)$ such that $1 < p < D - 1$, except when $q$ is odd and $p = 2$ or $p = D - 2$, the set $\{W \mid [W] \in \mathfrak{S}(p, q)\}$ contains an open and dense subset of $G(p, o(q))$. The set $\mathfrak{S} = \bigcup_{p,q} \mathfrak{S}(p, q)$ is closed under taking direct products and passing to the dual algebra.

2. An Einstein metric solvable Lie algebra with a two-step nilradical $n$ is standard in each of the following cases:

(a) $n$ is nonsingular;

(b) $n$ is of type $(p, q)$, $q \leq 4$;

(c) $n$ is of type $(D - 1, q)$ (in fact, such an $n$ is even an Einstein nilradical).

Recall that a two-step nilpotent Lie algebra $n$ is called nonsingular, if for every $X \in n \backslash [n,n]$, the map $\text{ad}_X : n \to [n,n]$ is surjective. As to the missing pairs $(p, q)$ in assertion 1, note that any two-step nilpotent algebra with $p = 1$ or $p = D$ is an Einstein nilradical. The case $p = D - 1$ is covered by (c) of assertion 2 (see Remark 3 in Section 6.2). As for almost all the pairs $(p, q)$, the action of $GL(q)$ on $G(p, o(q))$ has no open orbits [E1], the set $\mathfrak{S}(p, q)$ is of the second Baire category in the non-Hausdorff space of orbits $G(p, o(q))/SL(q)$, the space of isomorphism classes of algebras of type $(p, q)$.

Using Theorems 1, 3, 4 and 6 we prove the following theorem:

Theorem 7. Any Einstein metric solvable Lie algebra whose nilradical has dimension at most 7 is standard. In particular, any Einstein metric solvable Lie algebra of dimension less than ten is standard.

The paper is organized as follows. In Section 2 we provide the necessary background on Einstein solvmanifolds. In Section 3 we establish the existence and uniqueness of the pre-Einstein derivation (Proposition 1) and give the proof of Theorems 1 and 2. In Section 4 we consider the applications of Theorem 1 to Einstein nilradicals and to free nilradicals proving Theorems 3 and 5, respectively. The proof of Theorem 4 is given in Section 5. In Section 6 we consider the two-step nilradicals and prove Theorem 6. Finally, in Section 7 we establish Theorem 7.

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2 Preliminaries

Let $\mathfrak{g}$ be a Lie algebra, with $B$ the Killing form. The coboundary operator $\delta$ acts from the space of endomorphisms of (the linear space) $\mathfrak{g}$ to the space of $\mathfrak{g}$-valued two-forms on $\mathfrak{g}$ by the formula $\delta(A)(X, Y) := -A[X, Y] + [AX, Y] + [X, AY]$, where $A \in \text{End}(\mathfrak{g})$, $X, Y \in \mathfrak{g}$. From the definition, $\delta(\text{id})(X, Y) = [X, Y]$ and $\text{Ker} \delta = \text{Der}(\mathfrak{g})$, the algebra of derivations of $\mathfrak{g}$.

For an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$, define the mean curvature vector $H$ by $\langle H, X \rangle = \text{Tr} \text{ad}_X$ (clearly, $H$ is orthogonal to the nilradical of $\mathfrak{g}$). For $A \in \text{End}(\mathfrak{g})$, let $A^\ast$ be its metric adjoint and $S(A) := \frac{1}{2}(A + A^\ast)$ be the symmetric part of $A$. Define a symmetric form $\Xi$ on $\text{End}(\mathfrak{g})$ by

$$\Xi(A_1, A_2) = \langle \delta(A_1), \delta(A_2) \rangle = \sum_{i,j} \langle \delta(A_1)(E_i, E_j), \delta(A_2)(E_i, E_j) \rangle,$$

for an orthonormal basis $\{E_i\}$ for $\mathfrak{g}$. Then the Ricci operator $\text{ric}$ of the metric Lie algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ (the symmetric operator associated to the Ricci tensor) is implicitly defined by

$$\text{Tr} \left( \text{ric} + S(\text{ad}_H) + \frac{1}{2} B \right) \circ A = -\frac{1}{4} \Xi(A, \text{id}),$$

for any $A \in \text{End}(\mathfrak{g})$, or, in the expanded form, by

$$\text{Tr} \left( \text{ric} + \frac{1}{2} (\text{ad}_H + \text{ad}_H^\ast) + \frac{1}{2} B \right) \circ A = \frac{1}{4} \sum_{i,j} \langle A[E_i, E_j] - [AE_i, E_j] - [E_i, AE_j], [E_i, E_j] \rangle.$$
For an Einstein metric solvable Lie algebra of negative scalar curvature \( c \dim g \) this gives

\[
\text{Tr} \left( |c| \text{id}_g - S(\text{ad}_H) - \frac{1}{2} B \right) \circ A = -\frac{1}{4} \sum_{i,j} \langle A[E_i, E_j] - [AE_i, E_j] - [E_i, AE_j], [E_i, E_j] \rangle. \tag{5}
\]

If \((n, \langle \cdot, \cdot \rangle)\) is a nilpotent metric Lie algebra, then \( H = 0 \) and \( B = 0 \), so (4) gives

\[
\text{Tr} \left( \text{ric}_n \circ A \right) = \frac{1}{4} \Xi(A, \text{id}) = \frac{1}{4} \sum_{i,j} \langle A[E_i, E_j] - [AE_i, E_j] - [E_i, AE_j], [E_i, E_j] \rangle, \tag{6}
\]

for any \( A \in \text{End}(n) \).

\textbf{Definition 1.} [H] An inner product on a solvable Lie algebra \( g \) is called \textit{standard}, if the orthogonal complement to the derived algebra \([g, g]\) is abelian. A metric solvable Lie algebra \((g, \langle \cdot, \cdot \rangle)\) is called \textit{standard}, if the inner product \( \langle \cdot, \cdot \rangle \) is standard.

As it is proved in [AK], any Ricci-flat metric solvable Lie algebra is flat. By the result of [DM], any Einstein metric solvable unimodular Lie algebra is also flat. In what follows, we always assume \( g \) to be nonunimodular (\( H \neq 0 \)), with an inner product of a strictly negative scalar curvature \( c \dim g \).

Any standard Einstein metric solvable Lie algebra admits a rank-one reduction [H, Theorem 4.18]. This means that if \((g, \langle \cdot, \cdot \rangle)\) is such an algebra, with the nilradical \( n \) and the mean curvature vector \( H \), then the subalgebra \( g_1 = RH \oplus n \), with the induced inner product, is also Einstein and standard. What is more, the derivation \( \Phi = \text{ad}_H|_n : n \to n \) is symmetric with respect to the inner product, and all its eigenvalues belong to \( \alpha N \) for some constant \( \alpha > 0 \). This implies, in particular, that the nilradical \( n \) of a standard Einstein metric solvable Lie algebra admits an \( N \)-gradation defined by the eigenspaces of \( \Phi \). As it is proved in [L1, Theorem 3.7], a necessary and sufficient condition for a metric nilpotent algebra \((n, \langle \cdot, \cdot \rangle)\) to be nilradical is that it is a standard Einstein metric solvable Lie algebra is

\[
\text{ric}_n = c \text{id}_n + \Phi, \tag{7}
\]

where \( c \dim g < 0 \) is the scalar curvature of \((g, \langle \cdot, \cdot \rangle)\). This equation, in fact, defines \((g, \langle \cdot, \cdot \rangle)\) in the following sense: given a metric nilpotent Lie algebra whose Ricci operator satisfies (7), with some constant \( c < 0 \) and some \( \Phi \in \text{Der}(n) \), one can define \( g \) as a one-dimensional extension of \( n \) by \( \Phi \). For such an extension \( g = RH \oplus n \), \( \text{ad}_H|_n = \Phi \), and the inner product defined by \( \langle H, n \rangle = 0 \), \( \|H\|^2 = \text{Tr} \Phi \) (and coinciding with the existing one on \( n \)) is Einstein, with the scalar curvature \( c \dim g \). Following [L1] we call a nilpotent Lie algebra \( n \) which admits an inner product \( \langle \cdot, \cdot \rangle \) and a derivation \( \Phi \) satisfying (7) an \textit{Einstein nilradical}, the corresponding derivation \( \Phi \) is called an \textit{Einstein derivation}, and the inner product \( \langle \cdot, \cdot \rangle \) the \textit{nilsoliton metric}.

As it is proved in [L1, Theorem 3.5], a nilpotent Lie algebra admits no more than one nilsoliton metric, up to conjugation and scaling (and hence, an Einstein derivation, if it exists, is unique, up to conjugation and scaling). Equation (7), together with (6), implies that if \( n \) is an Einstein nilradical, with \( \Phi \) the Einstein derivation, then

\[
\text{Tr} (\Phi \circ \psi) = -c \text{Tr} \psi, \quad \text{for any } \psi \in \text{Der}(n). \tag{8}
\]

We use the following notational convention: for a linear space \( V \), \( \text{End}(V) \) is the Lie algebra of linear transformations of \( V \) (that is, \( \text{End}(V) = gl(V) \)), \( \oplus \) is the direct sum of linear spaces (even if the summands are Lie algebras). For an endomorphism \( A \) of a linear space \( V \), we denote \( A^S \) and \( A^N \) its semisimple and nilpotent parts respectively: \( A = A^S + A^N \), \( [A^S, A^N] = 0 \). \( A^S \) can be further decomposed as \( A^S = A^S + A^R \), the real and the imaginary part of \( A^S \), respectively. The operator \( A^R \) is defined as follows: if \( V_1, \ldots, V_m \) are the eigenspaces of \( A^S \) acting on \( V \), with the eigenvalues \( \lambda_1, \ldots, \lambda_m \), respectively, then \( A^R \) acts by multiplication by \( \mu \in \mathbb{R} \) on every subspace \( (\oplus_{k: \text{Re} \lambda_k = \mu} V_k) \cap V \). For any \( A \in \text{End}(V) \), the operators \( A, A^S, A^N, A^R \), and \( A^{iR} \) commute. If \( \psi \) is a derivation of a Lie algebra \( g \), then each of the \( \psi^S, \psi^N, \psi^R \), and \( \psi^{iR} \) is also a derivation.
3 Pre-Einstein Derivation. Proof of Theorems 1 and 2

In this section we prove Theorem 1 and Theorem 2 and consider some examples.

Definition 2. A derivation \( \phi \) of a Lie algebra \( l \) is called pre-Einstein, if it is real, semisimple, and

\[
\text{Tr} (\phi \circ \psi) = \text{Tr} \psi, \quad \text{for any } \psi \in \text{Der}(l).
\]

We start with the following proposition.

Proposition 1.

(a) An arbitrary Lie algebra \( l \) admits a pre-Einstein derivation \( \phi_l \).

(b) The derivation \( \phi_l \) is determined uniquely up to an automorphism of \( l \).

(c) All the eigenvalues of \( \phi_l \) are rational numbers.

Proof. (a) The algebra \( \text{Der}(l) \) is algebraic. Let \( \text{Der}(l) = s \oplus t \oplus n \) be its Mal'cev decomposition, where \( t \oplus n \) is the radical of \( \text{Der}(l) \), \( s \) is semisimple, \( n \) is the set of all nilpotent elements in \( t \oplus n \) (and is the nilradical of \( t \oplus n \)), \( t \) is a torus, an abelian subalgebra consisting of semisimple elements, and \([t,s]=0\).

With any \( \psi \in t \), \( \psi^R \) and \( \psi^R \) are also in \( t \). The subspaces \( t_e = \{ \psi^R \mid \psi \in t \} \) and \( t_s = \{ \psi^R \mid \psi \in t_s \} \) are the compact and the completely \( R \)-reducible tori (the elements of \( t_s \) are diagonal matrices in some basis for \( l \)), \( t_s \oplus t_e = t \).

The quadratic form \( b \) defined on \( \text{Der}(l) \) by \( b(\psi_1, \psi_2) = \text{Tr} (\psi_1 \circ \psi_2) \) is invariant \( b(\psi_1, [\psi_2,\psi_3]) = b([\psi_1,\psi_3],\psi_2) \). In general, \( b \) is degenerate, with \( \text{Ker} b = n \), so for any \( \psi \in n \), \( b(t, \psi) = \text{Tr} \psi = 0 \). As \( s \) is semisimple and \([t,s]=0\), we also have \( b(t, \psi) = \text{Tr} \psi = 0 \), for any \( \psi \in s \). Moreover, for any \( \psi \in t_s \), \( b(t_s, \psi) = \text{Tr} \psi = 0 \).

So to produce a pre-Einstein derivation for \( l \) it suffices to find an element \( \phi \in t_s \) which satisfies (9), with any \( \psi \in t_s \). Such a \( \phi \) indeed exists, as the restriction of \( b \) to \( t_s \) is nondegenerate (even definite) and is unique, when a particular torus \( t \) is chosen.

(b) The subalgebra \( s \oplus t \) is a maximal reducible subalgebra of \( \text{Der}(l) \). As by [Mo, Theorem 4.1], the maximal reducible subalgebras of \( \text{Der}(l) \) are conjugate by an inner automorphism of \( \text{Der}(l) \) (which is an automorphism of \( l \)), and then \( t \), the center of \( s \oplus t \), is defined uniquely, the uniqueness of \( \phi \), up to an automorphism, follows.

(c) The proof is similar to [H, Theorem 4.14]. Suppose \( \phi \) has eigenvalues \( \mu_i \), with multiplicities \( d_i \), respectively, \( i = 1, \ldots, m \). In a Euclidean space \( \mathbb{R}^m \) with a fixed orthonormal basis \( f_i \), consider all the vectors of the form \( f_1 + f_2 - f_k \) such that \( \mu_i + \mu_j - \mu_k = 0 \). In their linear span choose a basis \( v_\alpha \), \( \alpha = 1, \ldots, p \), consisting of vectors of the above form and introduce an \( m \times p \) matrix \( F \) whose vector-columns are the \( v_\alpha \)'s. Then any vector \( \nu = (\nu_1, \ldots, \nu_m)^t \in \mathbb{R}^m \) satisfying \( F^t \nu = 0 \) defines a derivation \( \psi = \psi(\nu) \) having the same eigenspaces as \( \phi \), but with the corresponding eigenvalues \( \nu_i \). From (9) we must have \( \sum d_i (\mu_i - 1) \nu_i = 0 \) for any such \( \nu \), which implies that the vector \( (d_1 (\mu_1 - 1), \ldots, d_m (\mu_m - 1)) \) belongs to the column space of \( F \). So there exists \( x \in \mathbb{R}^p \) such that \( \mu = 1_m + D^{-1}Fx \), where \( \mu = (\mu_1, \ldots, \mu_m)^t \), \( 1_m = (1, \ldots, 1)^t \in \mathbb{R}^m \), and \( D = \text{diag}(d_1, \ldots, d_m) \). As \( \phi \) by itself is a derivation, we have \( F^t \mu = 0 \), which implies \( F^t 1_m + F^t D^{-1}Fx = 0 \), so that \( x = -(F^t D^{-1}F)^{-1}1_p \), as \( F^t 1_m = 1_p \) and \( \text{rk} F = p \). Then \( \mu = 1_m - D^{-1}F(F^t D^{-1}F)^{-1}1_p \) and the claim follows, as all the entries of \( D \) and of \( F \) are integers.

The following easy lemma shows that in the definition of the standardness, the derived algebra can be replaced by the nilradical (in fact, by any ideal in between). Note that for a standard \( g \), the nilradical coincides with the derived algebra [H, Corollary 4.11].

Lemma 1. Suppose that for an Einstein metric solvable Lie algebra \((g, \langle \cdot, \cdot \rangle)\), the orthogonal complement \( a \) to the nilradical \( n \) is abelian. Then \( g \) is standard.
**Proof.** The proof is close to that of [S, Corollary 2.4]. By [H, Lemma 4.11], for any \( Y \in \mathfrak{a} \), the operator \( \text{ad}_Y^* \) is a derivation, which must be normal by [H, Lemma 2.1]. As for any \( X \in \mathfrak{n} \cap [\mathfrak{g}, \mathfrak{g}] \), \( \text{ad}_Y X = 0 \), the normality of \( \text{ad}_Y \) implies \( \text{ad}_Y^* \text{ad}_Y X = 0 \), so \( [Y, X] = 0 \). In particular, \( [H, X] = 0 \), so, again, \( \text{ad}_X \in \text{Der}(\mathfrak{g}) \) and \( \text{ad}_X \) is normal. As \( \text{ad}_X \) is nilpotent, this implies \( \text{ad}_X = 0 \), and so \( X = 0 \) (as the Ricci curvature in the direction of \( X \) is nonnegative, see e.g. [Mi, Lemma 2.1]).

The rest of the proof of Theorem 1 follows the lines of the proof of [H, Theorem 5.3]: given an Einstein metric solvable Lie algebra \( (\mathfrak{g}, \langle \cdot, \cdot \rangle) \), we compare two derivations of \( \mathfrak{n} \): the pre-Einstein derivation \( \phi_\mathfrak{n} \) and \( \text{ad}_{[H, \mathfrak{n}]} \), and show that in the assumptions (1) they must coincide (up to conjugation and scaling). Note that a similar approach was also used in the proof of [NN, Theorem 4].

We need the following technical lemma for references. The proof of assertion 1 is essentially contained in the proof of [H, Theorem 5.3], the proof of assertion 2 follows from [N2, Theorem 3] (compare [H, Lemma 4.1]).

**Lemma 2.** Let \((l, \langle \cdot, \cdot \rangle)\) be a metric Lie algebra and let \( \psi \in \text{Der}(l) \).

1. Suppose \( \psi \) is semisimple, real, with the eigenvalues \( \lambda_1 < \cdots < \lambda_m \) and the corresponding eigenspaces \( \mathfrak{n}_1, \ldots, \mathfrak{n}_m \). Define a symmetric operator \( \psi^+ \) as follows: for \( W_1 = \oplus_{j=1}^m \mathfrak{n}_j \cap (\oplus_{j=i+1}^m \mathfrak{n}_j)^\perp \), set \( \psi^+_l = \psi |_{W_l} = \lambda_l \text{id}_{W_l} \). Then \( \Xi(\psi^+, \text{id}) \leq 0 \), with the equality if and only if \( \psi^+ \) is a derivation.

2. For an arbitrary \( \psi \in \text{Der}(l) \), denote \( S = (\psi + \psi^*)/2 \) the symmetric part of \( \psi \). Then

\[
\text{Tr} (\text{ric} + \text{ad}_H + \frac{1}{2} B) \circ [\psi, \psi^*] = \frac{1}{2} \sum_{i,j} \|S[E_i, E_j] - [SE_i, E_j] - [E_i, SE_j]\|^2 \geq 0,
\]

where \{\(E_i\)\} is an orthonormal basis for \((l, \langle \cdot, \cdot \rangle)\). In particular, if \((l, \langle \cdot, \cdot \rangle)\) is an Einstein nilradical, with the Einstein derivation \( \Phi \), then

\[
\text{Tr} (\Phi \circ [\psi, \psi^*]) = \frac{1}{2} \sum_{i,j} \|S[E_i, E_j] - [SE_i, E_j] - [E_i, SE_j]\|^2 \geq 0.
\]

3. Let \( \phi_i \) be the pre-Einstein derivations for the Lie algebras \( l_i \), \( i = 1, 2 \). Then the pre-Einstein derivation \( \phi_i \) for the Lie algebra \( l \), the direct sum of the Lie algebras \( l_1 \) and \( l_2 \), is \( \phi_1 \oplus \phi_2 \).

**Proof.**

1. As for any \( i = 1, \ldots, m \), \( \bigoplus_{j=i}^m W_j = \bigoplus_{j=i}^m \mathfrak{n}_j \) and \( \{\mathfrak{n}_j, \mathfrak{n}_j\} \subset \mathfrak{V}_i \), with \( \lambda_k = \lambda_i + \lambda_j \) (or zero, if there is no such \( k \)), \( [W_i, W_j] \subset \bigoplus_{k=i}^m W_k := U_k \). The subspace \( U_k \) is invariant for \( \psi^+ \), and the restriction of \( \psi^+ \) to it is a symmetric operator whose eigenvalues are greater than or equal to \( \lambda_k = \lambda_i + \lambda_j \). So for any \( E_i \in W_i, E_j \in W_j \),

\[
\langle [\psi^+ E_i, E_j] + [E_i, \psi^+ E_j] - \psi^+[E_i, E_j], [E_i, E_j]\rangle = \langle (\lambda_i + \lambda_j) \text{id} - \psi^+ \rangle \delta_{i,j}[E_i, E_j], [E_i, E_j]\rangle \leq 0,
\]

with the equality only when \( \psi^+ \) is also a derivation.

2. The quadratic form \( \Xi \) defined by (2) on the linear space \text{End}(l) is positive semidefinite, with \( \text{Ker} \Xi = \text{Der}(l) \). Clearly, \( \Xi(A, \text{id}) = \Xi(A^*, \text{id}) \). Moreover, for any symmetric \( S \) and a skew-symmetric \( K \), we have

\[
\Xi(S, K) = -\Xi(KS, \text{id}) = \Xi(KS, \text{id}).
\]

To prove this, note that the expansion of \( \Xi(S, K) = \langle \delta(S), \delta(K) \rangle \) is a sum of nine terms, six of which vanish, so that

\[
\Xi(S, K) = \sum_{i,j} \langle [SE_i, E_j], [KE_i, E_j]\rangle + \langle [SE_i, E_j], [KE_i, E_j]\rangle + \langle [E_i, SE_j], [E_i, KE_j]\rangle.
\]

Indeed, for example, \( \sum_{i,j} \langle [SE_i, E_j], [KE_i, E_j]\rangle = \sum_{i,j} \langle \text{ad}^*_E S \text{ad}_E K \rangle = \sum_j \text{Tr} \text{ad}^*_E S \text{ad}_E K = 0 \), as the endomorphisms \( \text{ad}^*_E S \text{ad}_E \) are self-adjoint. Also, \( \sum_{i,j} \langle [SE_i, E_j], [E_i, KE_j]\rangle = 0 \), which can be seen if we take \( E_i \) as the eigenvectors of \( S \). The vanishing of the other four terms can be shown by a similar routine check. Then (12) follows from

\[
\sum_{i,j} \langle [SE_i, E_j], [KE_i, E_j]\rangle = - \sum_j \text{Tr} (K \text{ad}^*_E \text{ad}_E S) = - \sum_j \text{Tr} (\text{ad}^*_E \text{ad}_E K) = - \sum_{i,j} \langle [E_i, E_j], [KE_i, E_j]\rangle.
\]

(12) follows from
Now for $\psi \in \text{Der}(l)$, let $\psi = S + K$, with $S$ symmetric and $K$ skew-symmetric. Then $[\psi, \psi^*] = KS + (KS)^*$, so by (12), $\Xi([\psi, \psi^*], \text{id}) = 2\Xi(KS, \text{id}) = 2\Xi(S, K) = 2\Xi(S, S) = -2\Xi(S, S)$. This, together with (3), implies (10). Equation (11) then follows from (8) and the fact that $B = 0$ and $H = 0$.

3. Let $\mathfrak{d}_{ij}$, $i = 1, 2$, be the set of all $\psi \in \text{End}(l)$ such that $\psi(l_1) \subset l_1$, $\psi(l_1) \in \text{Der}(l_1)$, and $\psi|_{l_1} = 0$, for $j \neq i$. Clearly, $\mathfrak{d}_{ij} \subset \text{Der}(n)$. For $i \neq j$, let $\mathfrak{d}_{ij}$ be the set of all $\psi \in \text{End}(l)$ such that $\psi|_{l_1} \cap [l_1, l_1] = 0$ and $\psi(l_i)$ lies in the center of $l_i$. Then $\text{Der}(l) = \mathfrak{d}_{11} \oplus \mathfrak{d}_{22} \oplus \mathfrak{d}_{12} \oplus \mathfrak{d}_{21}$, as both sides vanish.

The claim now follows from assertion (b) of Proposition 1.

As to assertion 3, note that if the $l_i$'s are Einstein nilradicals, with $\Phi_i$ the Einstein derivations, then $l$ is also an Einstein nilradical whose Einstein derivation is $\Phi_1 \oplus \Phi_2$ [P, Theorem 4] (geometrically this says that the Riemannian product of two standard Einstein solvmanifolds of the same Ricci curvature is again a standard Einstein solvmanifold).

**Proof of Theorem 1.** Let $g$ be a solvable nonunimodular Lie algebra, with an Einstein inner product $\langle \cdot, \cdot \rangle$. Let $n$ be the nilradical of $g$, and let $a = n^\perp$. Denote $\lambda_1 \leq \ldots \leq \lambda_m$ the eigenvalues of the pre-Einstein derivation $\phi = \phi_\perp$ of $n$, and $n_1, \ldots, n_m$ the corresponding eigenspaces.

Suppose that $\phi$ satisfies (1), that is, $\lambda_i \geq 0$ and for any $\psi \in \text{Der}(n)$, $\psi(n_i) \subset n_{\phi(\psi)} \oplus n_{\phi(\psi)}^\perp$ (which is equivalent to $ad_\phi \geq 0$).

Define $F \in \text{End}(g)$ by $F|_n = \phi + \phi^*$ (as in assertion 1 of Lemma 2) and $F|_n = 0$. Then $F$ is symmetric and nonnegative. The eigenspaces of $F|_n$ are $n_i = \oplus_{j \neq i} n_j \cap (\oplus_{j \neq i} n_j)^\perp$ (as constructed in Lemma 2), with the corresponding eigenvalues $\lambda_i$ in the increasing order. Moreover, for any derivation $\psi$ of $n$, $\psi(n_i) \subset n_i \oplus n_i^\perp$, $n_i \cap n_i^\perp = 0$. It follows that $Tr(\psi \circ \phi) = Tr(\psi \circ \phi) = Tr \psi$, by definition of $\phi$. As $F$ is symmetric, we obtain that for any $\Psi \in \text{End}(g)$,

$$Tr (\Psi|_n \circ F) = Tr (\Psi \circ F) = Tr (\Psi|_n \circ \phi + \phi^*) = Tr (\Psi|_n \circ \phi) = Tr (\Psi \circ \phi) = Tr S(\Psi).$$

Also, $Tr F = Tr \phi$, and $Tr (F^2) = Tr (\phi^2) = Tr \phi$ (as $F$ and $\phi$ have the same nonzero eigenvalues).

Substituting $F$ as $A$ to (5) we find that the right-hand side is nonnegative. To see that, choose an orthonormal basis $\{E_k\}$ to be of the form $\{Y_1, \ldots, Y_s, E_1, \ldots, E_n\}$, where $E_s \in n$ are the eigenvectors of $F|_n$, and $Y_k \in a$.

Note that for any $E \in m_s$ and $Y \in a$, $ad_Y E \in \oplus_{s=1}^{m_s} m_k$, so $\langle F[E, Y], [F, E], Y \rangle = \langle [F, ad_Y E], E, ad_Y E \rangle \leq \gamma (\langle F - \lambda_1 id \rangle, \langle [F, ad_Y E], E, ad_Y E \rangle \geq 0. This implies that $\sum_{s=1}^{m_s} \langle F[E, Y], [F, E, Y] \rangle \geq 0$. This implies that $\sum_{s=1}^{m_s} \langle F[E, Y], [F, E, Y] \rangle \geq 0$ by Lemma 2. The remaining sum on the right-hand side is $\sum_{s=1}^{m_s} \langle F[Y, Y], [Y, Y] \rangle \geq 0$ as $F \geq 0$.

On the left-hand side of (5), the third summand vanishes as $F(g) \subset n$, so we get $|c|Tr F - Tr (S(ad_H)F) \leq 0$. Substituting $A = ad_H$ into (5) we obtain $|c| Tr S(ad_H)^2 / Tr S(ad_H)$, so

$$Tr (S(ad_H)^2) Tr F = Tr (S(ad_H) \circ F) Tr S(ad_H).$$

As $Tr F = Tr (F^2) = Tr \phi$ and $Tr (S(ad_H) \circ F) = Tr S(ad_H)$ from (13), the latter inequality gives $Tr (S(ad_H)^2) Tr F \leq (Tr (S(ad_H) \circ F))^2$. By the Cauchy-Shwartz inequality, all the inequalities above must be equalities. In particular, $\langle F[Y_k, Y_l], [Y_k, Y_l] \rangle = 0$, for all $Y_k, Y_l \in a$, which implies that $a$ is abelian, as $F|_n = \phi^\perp > 0$.

**Remark 1.** As it can be seen from the proof, we need the condition $ad_\phi \geq 0$ to be satisfied only on the subspace of $\text{Der}(n)$ spanned by $ad_{Y|n}$. More precisely, it is sufficient to require that for each $Y \perp n$, $ad_{Y|n}$ lies in the direct sum of the eigenspaces of $ad_\phi$ with nonnegative eigenvalues.
Proof of Theorem 2. Let \((\mathfrak{g}, \langle \cdot, \cdot \rangle)\) be an Einstein metric solvable Lie algebra with the nilradical \(\mathfrak{n}\). Suppose that \(\mathfrak{g}\) is split. Let \(\mathfrak{h} \subset \mathfrak{g}\) be a Cartan subalgebra. Then \(\mathfrak{g} = \mathfrak{h} + \mathfrak{n}\) (not a direct sum, in general), as \([\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}\). For any \(X \in \mathfrak{h}\), \((\text{ad}_X)\mathfrak{h} = 0\). As \(\mathfrak{g}\) is split, \((\text{ad}_X)^S\mathfrak{h} = \mathfrak{y}\) for some \(Y \in \mathfrak{g}\). Since \([Y, X] = 0, Y \in \mathfrak{h}\), so \(\mathfrak{h}\) is also split. All the nilpotent elements from \(\mathfrak{h}\) lie in \(\mathfrak{n}\) (as \(\mathfrak{n}\) is the maximal nilpotent ideal of \(\mathfrak{g}\)). It follows that \(\mathfrak{h}\) contains an abelian subalgebra \(\mathfrak{a}\) such that \(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}\) and \(\text{ad}_{\mathfrak{a}}\) is semisimple for any \(Y \in \mathfrak{a}\). Let now \(\phi\) be a pre-Einstein derivation for \(\mathfrak{n}\) and suppose that \(\phi > 0\). By conjugation, we can choose \(\phi\) to lie in the maximal reducible subalgebra of \(\text{Der}(\mathfrak{n})\) containing the torus \(\{\text{ad}_{\mathfrak{n}}|Y \in \mathfrak{a}\}\). As \(\phi\) lies in the center of that subalgebra (see the proof of Proposition 1), it commutes with all the \(\text{ad}_{\mathfrak{n}}|_{\mathfrak{y}}\), \(Y \in \mathfrak{a}\). As \(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}\), any vector \(Z \perp \mathfrak{n}\) differs from some \(Y \in \mathfrak{a}\) by a vector \(X \in \mathfrak{n}\), so \(\text{ad}_{\mathfrak{a}}Z = \text{ad}_{\mathfrak{a}}Y + \text{ad}_{\mathfrak{n}}X\). Now \(\text{ad}_{\mathfrak{a}}Y \in \text{Ker} \text{ad}_{\phi}\), as \(\text{ad}_{\mathfrak{a}}Y\) and \(\phi\) commute. If \(X = \sum_i X_i\), with \(\phi X_i = \lambda_i X_i\), then \(\text{ad}_{\phi}(\text{ad}_{\mathfrak{a}}X_i) = \lambda_i \text{ad}_{\mathfrak{a}}X_i\). Since \(\phi > 0\), it follows that \(\text{ad}_{\mathfrak{a}}X_i\) (and therefore \(\text{ad}_{\mathfrak{a}}Z\)) lies in the direct sum of the eigenspaces of \(\text{ad}_{\phi}\) with the nonnegative eigenvalues. By Remark 1, the proof can now be finished by the same arguments as in the proof of Theorem 1.

We end this section with two examples showing the limitations of Theorem 1. Of course, if a nilpotent algebra \(\mathfrak{n}\) has no positive derivations, Theorem 1 says nothing. Example 1 shows that the things could be worse: even when a positive derivation exists, the pre-Einstein derivation can be nonpositive. Example 2 shows that the first condition of Theorem 1 (\(\phi > 0\)) does not imply the second one (\(\text{ad}_{\phi} \geq 0\)).

Example 1. Any two-step nilpotent Lie algebra admits a positive derivation, the canonical derivation \(\Phi\) (\(\Phi\) acts as \(2i\delta\) on the derived algebra and as \(i\text{id}\) on its linear complement; see Section 6 for details). However, the pre-Einstein derivation of a two-step nilpotent Lie algebra can be nonpositive.

Let \(\mathfrak{b} = \oplus_{i=-1}^{1} \mathfrak{b}_i\), \(\mathfrak{m} = \oplus_{i=2}^{\infty} \mathfrak{m}_i\), \(\mathfrak{n} = \mathfrak{b} \oplus \mathfrak{m}\) be the linear space decompositions. Denote \(q_i = \dim \mathfrak{b}_i\), \(p_k = \dim \mathfrak{m}_k\). Define the structure of a two-step nilpotent Lie algebra on \(\mathfrak{n}\) by requiring that \([\mathfrak{b}_i, \mathfrak{b}_j] = \mathfrak{m}_{i+j}\) and \([\mathfrak{m}, \mathfrak{m}] = 0\). Moreover, suppose that every two-step nilpotent subalgebra \(\mathfrak{b}_i \oplus \mathfrak{m}_{2i} \subset \mathfrak{n}\), \(i = -1, 0, 1\), admits only one semisimple derivation, up to conjugation and scaling: the canonical derivation (by [E2, Proposition 3.4.3], for \(p, q\) large enough and satisfying \(2 < p < q(q - 1)/2 - 2\), a generic two-step nilpotent Lie algebra of type \((p, q)\) has that property).

Let \(\psi_0\) be the derivation of \(\mathfrak{n}\) acting by multiplying every vector from \(\mathfrak{b}_1\) by \(i\), and every vector from \(\mathfrak{m}_k\) by \(k\). For \(a, b \in \mathbb{R}\), define \(\psi_{a, b} = a\Phi + b\psi_0\in \text{Der}(\mathfrak{n})\). We claim that one of the derivations \(\psi_{a, b}\) is a pre-Einstein derivation for \(\mathfrak{n}\). Clearly, every \(\psi_{a, b}\) is semisimple and real. To check equation (9), it is sufficient to take only those \(\psi \in \text{Der}(\mathfrak{n})\) which commute with \(\phi\) and are semisimple. The restriction of any such \(\psi\) to every subalgebra \(\mathfrak{b}_i \oplus \mathfrak{m}_{2i}\) must be a semisimple derivation, hence must be proportional to the canonical derivation of \(\mathfrak{b}_i \oplus \mathfrak{m}_{2i}\). It follows that \(\psi = \psi_{a, b}\) for some \(a, b \in \mathbb{R}\) (in fact, \(T = \{\psi_{a, b} | a, b \in \mathbb{R}\}\) is a maximal torus of derivations of \(\mathfrak{n}\)). Now, a derivation \(\psi_{x, y} \in \mathfrak{T}\) is pre-Einstein, if \(\text{Tr} \psi_{x, y}\psi_{a, b} = \text{Tr} \psi_{a, b}\), for all \(a, b \in \mathbb{R}\). Solving this, we find that one of the eigenvalues of \(\psi_{x, y}\) is \(-y = (Q_1 + (8p - 2 + 3p - 1 + 4q - 1 + 9q)q_1 + (8p - 1 + 2p_0 + 2p_2 + 9q - 1 + 2q_0 + 18p - 2)p_1 - p_1 q_1)/Q_2\), where \(Q_1\) and \(Q_2\) are quadratic forms in the \(q_i\) and \(p_k\) with nonnegative coefficients, and \(Q_1\) does not contain \(p_1\) and \(q_1\). This expression can be made negative by choosing \(p_1\) and \(q_1\) large enough (the inequalities \(p_{i+j} \leq q_{i+j}\) and \(2 < p_{i+j} < \frac{1}{2} q_{i+j}\) are needed to have \([\mathfrak{b}_i, \mathfrak{b}_j] = \mathfrak{m}_{i+j}\) and for every \(\mathfrak{b}_i \oplus \mathfrak{m}_{2i}\) to be generic can be easily achieved by multiplying all the \(q_i\) and \(p_k\) by a large natural number).

Example 2. This example shows that the positivity of the pre-Einstein derivation \(\phi\) of a nilpotent Lie algebra does not imply the positivity of \(\text{ad}_{\phi}\). Let \(\mathfrak{n}\) be a two-step nilpotent Lie algebra attached to the graph \(\mathcal{G}_{n,n,0}\), \(n > 2\) (see [LW, Section 5]). The graph \(\mathcal{G}_{n,n,0}\) has \(2n + 2\) vertices \(v_i\), its edge set is \(E = \{e_{12}, e_{13}, ..., e_{n+2}, e_{2n+3}, ..., e_{2n+2}\}\), where \(e_{ij}\) is the edge joining \(v_i\) and \(v_j\). The algebra \(\mathfrak{n}\), as a linear space, is spanned by the \(v_i\)’s and the \(e_{ij}\)’s in \(E\), with the commutator relations given by \(j(\mathfrak{n}) = \text{Span}(E)\), and for \(i < j\), \([v_i, v_j] = e_{ij}\), if \(e_{ij} \in E\), and 0 otherwise. A direct computation shows that the pre-Einstein derivation \(\phi\) of \(\mathfrak{n}\) has the form \(\phi v_i = x_i v_i\), \(\phi e_{ij} = (x_i + x_j) e_{ij}\), up to conjugation, where \(x_1 = x_2 = 4(n + 2)d\), \(x_i = (n + 2)(n + 4)d\), \(i \geq 3\), and \(d = ((n + 4)^2 - 4)^{-1}\). It follows that \(\phi > 0\) (one can show that this is always true for the algebras attached to a graph). For \(\psi \in \text{Der}(\mathfrak{n})\),
which sends $e_3$ to $e_{12}$ and all the other $e_i$'s and $e_{ij}$'s to zero, \( \text{ad}_\phi \psi = (2x_1 - x_3)\psi \), with the coefficient $2x_1 - x_3 = (8 + 2n - n^2) d$ being negative for $n > 4$.

4 Applications of Theorem 1

In this section, we prove Theorem 3 and Theorem 5 applying Theorem 1 to Einstein nilradicals and to free nilpotent Lie algebras, respectively.

Proof of Theorem 3. To deduce Theorem 3 from Theorem 1, it suffices to show that for an Einstein nilradical the pre-Einstein derivation $\phi$ is (positively proportional to) the Einstein derivation and then to check that the conditions of Theorem 1 hold.

Let $n$ be an Einstein nilradical, with $\Phi$ the Einstein derivation. Then $\Phi$ is semisimple, real and satisfies (8), so $\phi = (-c)^{-1}\Phi$ is a pre-Einstein derivation by Definition 2. As $\Phi$ is positive, $\phi$ is also positive. It remains to check that $\text{ad}_\phi \geq 0$. Assume that there exists $\psi \in \text{Der}(n)$ belonging to an eigenspace of $\text{ad}_\phi$ with a negative eigenvalue. As $\phi = (-c)^{-1}\Phi$, we get $[\Phi,\psi] = \lambda\psi$, $\lambda < 0$. Substituting such a $\psi$ into (11) we obtain $X \text{Tr}(\psi\psi^*) \geq 0$, which implies $\psi = 0$.

Before giving the proof of Theorem 5, we recall some facts about the structure of free Lie algebras and their derivation algebras [B]. From among $p$-step nilpotent Lie algebras on $m \geq 2$ generators, a free Lie algebra $f(m,p)$ is the one having the maximal dimension ($f(m,p)$ is unique, up to an isomorphism). Given the generators $e_1,\ldots,e_m$, the algebra $f(m,p)$ is the linear span of all the $k$-folded brackets of the $e_i$'s, $k \leq p$, with the only relations between these brackets coming from the skew-symmetricity and the Jacobi identity. For every $k = 1,\ldots,p$, the subspace of $f(m,p)$ spanned by the $k$-folded brackets is the space of Lie polynomials $p(m,k)$. In particular, $p(m,1) = \text{Span}(e_1,\ldots,e_m)$. The direct sum decomposition $f(m,p) = \bigoplus_{k=1}^p p(m,k)$ is an $\mathbb{N}$-gradation. It corresponds to the canonical derivation $\Phi$ acting as a multiplication by $k$ on every $p(m,k)$.

For a free Lie algebra, any assignment of the images to the generators extends to a (unique) derivation: for any linear map $L : p(m,1) \to f(m,p)$, there exists a unique derivation whose restriction to $p(m,1)$ coincides with $L$. In particular, any endomorphism $L$ of $p(m,1)$ extends to a derivation $\rho(L) \in \text{Der}(f(m,p))$ (for example, $\rho(\text{id}) = \Phi$).

Proof of Theorem 5. The spaces $p(m,k)$ are invariant with respect to $\rho(L)$. For every $k = 1,\ldots,p$, let $\rho_k(L)$ be the restriction of $\rho(L)$ to the $p(m,k)$. Then $\rho_k$ is a representation of the Lie algebra $\mathfrak{gl}(m)$ on the space $p(m,k)$ of Lie polynomials.

To deduce Theorem 5 from Theorem 1, it suffices to show that the pre-Einstein derivation $\phi = \phi_{f(m,p)}$ of the free Lie algebra $f(m,p)$ is a positive multiple of $\Phi$. Then the condition $\phi > 0$ is clearly satisfied. The condition $\text{ad}_\phi \geq 0$ easily follows from the above, as for any $\psi \in \text{Der}(f(m,p))$ and any $s = 1,\ldots,p$, $\psi(p(m,s)) \subset \bigoplus_{k=s}^p p(m,k)$.

As the pre-Einstein derivation is unique, up to conjugation (assertion (b) of Proposition 1), to prove that $\phi_{f(m,p)}$ is indeed $\hat{c}\Phi$ (where $\hat{c} = \text{Tr} \Phi (\text{Tr} \Phi)^{-1} > 0$), it suffices to show that $\hat{c}\Phi$ satisfies (9). The proof is literally the same as that of Lemma 8 of [Ni]. As $\Phi$ is semisimple, the derivation $\text{ad}_\phi$ of $\text{Der}(f(m,p))$ is also semisimple. If $\psi \in \text{Der}(n)$ is an eigenvector of $\text{ad}_\phi$ with a nonzero eigenvalue, then $\text{Tr} \Phi \circ \psi = \text{Tr} \psi = 0$, and (9) is obviously satisfied. So it is sufficient to consider only those $\psi$ which commute with $\Phi$. For any such $\psi$, the spaces $p(m,k)$ are invariant, and moreover, $\psi = \rho(L)$ for some endomorphism $L$ of $p(m,1)$. As $\rho(\text{id}) = \Phi$, it suffices to prove the following: for any $L$ with $\text{Tr} L = 0$ and for any $k \leq p$, $\text{Tr} \rho_k(L) = 0$. This follows from the fact that $\rho_k|_{\mathfrak{sl}(m)}$ is a representation of the simple algebra $\mathfrak{sl}(m)$.

10
5 Nilradical having less than four generators

In this section, we establish Theorem 4 by proving a slightly more general fact: if \( n \) is a nilpotent Lie algebra generated by less than four elements, then any Einstein metric solvable Lie algebra \((g, \langle \cdot, \cdot \rangle)\) with \( g \supset n \supset [g, g] \) is standard.

**Proof.** Let \( n \) be a nilpotent Lie algebra, and \( C_0 = n, C_p = [C_{p-1}, n], (p \geq 1) \) be the descending central series for \( n \). Denote \( n^1 := n/C_1 \), with \( \pi : n \to n^1 \) the natural projection.

Every endomorphism \( A \) of \( n \) mapping \( C_1 \) to itself (in particular, any derivation of \( n \)) induces a well-defined endomorphism \( \pi(A) : n^1 \to n^1 \). In fact, \( \pi \) is a homomorphism of the associative algebras, so for any \( \psi_1, \psi_2 \in \operatorname{Der}(n) \), \( \pi(\psi_1 \psi_2) = \pi(\psi_1)\pi(\psi_2) \). In particular, if \( \psi_1 \) commutes with \( \psi_2 \), then \( \pi(\psi_1) \) and \( \pi(\psi_2) \) also commute, and if \( \psi \) is a nilpotent derivation, then \( \pi(\psi) \) is nilpotent. The converse is also true: if \( \pi(\psi)^n = 0 \), then \( \psi^n(n) \subset [n, n] \), and so \( \pi^n(n) \) is nilpotent and commutes with \( \pi(\psi) \). Furthermore, if for a derivation \( \psi \) all the eigenvalues of \( \pi(\psi) \) are real, then the same is true for \( \psi \) itself. Indeed, \( \psi^S \), the semisimple part of \( \psi \), is again a derivation, with the same eigenvalues as \( \psi \). As \( \psi^S \) is semisimple, the endomorphism \( \pi(\psi^S) \) is also semisimple. Moreover, \( \psi - \psi^S \) is nilpotent and commutes with \( \psi \), which implies that \( \pi(\psi) - \pi(\psi^S) \) is nilpotent and commutes with \( \pi(\psi) \). It follows that \( \pi(\psi^S) \) is the semisimple part of \( \pi(\psi) \). In particular, the eigenvalues of \( \pi(\psi^S) \) are the same as those of \( \pi(\psi) \). Assume all of them are real. Choose the vectors \( X_1, \ldots, X_d \in n \setminus C_1 \) such that \( \pi(X_i) \) is the basis of eigenvectors for \( \pi(\psi^S) \). Every term \( C_p \) of the descending central series is spanned by \( C_{p+1} \) and all the \((p+1)\)-folded brackets \( X_{i_1}X_{i_2} \cdots X_{i_{p+1}} = [\ldots [[X_{i_1}, X_{i_2}], X_{i_3}], \ldots, X_{i_{p+1}}] \), where \( i_j \in \{1, \ldots, d\} \). Choosing, for every \( p \geq 1 \), a linearly independent set of the \( X_{i_1}X_{i_2} \cdots X_{i_{p+1}} \)'s whose span complements \( C_{p+1} \) in \( C_p \) we get a basis for \( n \) having the property that every \( X_{i_1}X_{i_2} \cdots X_{i_{p+1}} \) from that basis is an eigenvector of \( \psi^S \) modulo \( C_{p+1} \), with a real eigenvalue. It follows that all the eigenvalues of \( \psi^S \) are real (the matrix of \( \psi^S \) is triangular, with a real diagonal).

Now let \((g, \langle \cdot, \cdot \rangle)\) be an Einstein metric solvable Lie algebra, \( n \) be a nilpotent ideal of \( g \) containing the derived algebra \([g, g]\), and let \( a \) be the orthogonal complement to \( n \) in \( g \).

Define a linear map \( \theta : a \to \operatorname{End}(n^1) \) by \( \theta(Y) = \pi(\operatorname{ad}_Y|_n) \). Then \( \theta(a) \) is a commuting family of endomorphisms of \( n^1 \). For every \( Y \in \operatorname{Ker} \theta \), \( \operatorname{ad}_Y \) is a nilpotent derivation of \( g \), hence \( Y \in \operatorname{Ker} B \), the kernel of the Killing form of \( g \). If \( \dim \operatorname{Range}(\theta) = 1 \), there is nothing to prove, as then, by the Lie’s Theorem, the nilradical of \( g \) has codimension 1 and the claim follows from Lemma 1.

Assume now that \( \dim n^1 \leq 3 \). Clearly \( \dim n^1 \neq 1 \). We will use [H, Lemma 4.7] saying that any Einstein inner product on \( g \) is standard, provided the index of the Killing form \( B \) of \( g \) is at most one.

Let \( \dim n^1 = 2 \). As \( \operatorname{Range}(\theta) \) is a commuting family of endomorphisms, \( \dim \operatorname{Range}(\theta) \leq 2 \), with the equality only when there exists \( Y \in a \) such that \( \theta(Y) = \pi(\operatorname{ad}_Y|_n) \). It follows that the Killing form \( B \) of \( g \) has rank at most 2 and there exists a vector \( Y \in a \) such that \( B(Y, Y) > 0 \). Then the index of \( B \) is at most one, hence \((g, \langle \cdot, \cdot \rangle)\) is standard.

Let \( \dim n^1 = 3 \). If all the eigenvalues of all the \( \theta(Y) \)'s are real, then all the \( \operatorname{ad}_Y \)'s have only real eigenvalues, so \( g \) is completely solvable, hence standard (the index of \( B \) is zero). If one of the \( \theta(Y) \) has a nonreal complex eigenvalue, then \( \theta(Y) \) is semisimple for all \( Y \in a \), and there is a subspace \( a' \) of codimension 1 in \( a \) such that all the eigenvalues of \( \theta(Y) \) are real, when \( Y \in a' \). Then the index of the Killing form \( B \) of \( g \) is at most one, and again, any Einstein inner product on \( g \) is standard. \( \square \)

6 Two-step nilradical

In this section, we consider several classes of two-step nilpotent Lie algebras and show that neither of them contains the nilradical of a nonstandard Einstein metric solvable Lie algebra. We start with some preliminary facts, mostly following [E2].
6.1 Preliminaries

A two-step nilpotent Lie algebra \( n \) of dimension \( p + q \), is said to be of type \( (p, q) \), if its derived algebra \( m = [n, n] \) has dimension \( p \). Clearly, \( m \subset g(n) \), the center of \( n \), and \( 1 \leq p \leq D := \frac{1}{2}(q + 1) \).

Choose a subspace \( b \) complementary to \( m \) in \( n \) (in the presence of an inner product, we usually take \( b = m^\perp \)). The Lie bracket defines (and is defined by) a skew-symmetric bilinear map \( J : b \times b \to m \). For any \( f \in m^\ast \), the two-form \( J_f \in \Lambda^2 b \) is defined by \( J_f(X, Y) = f([X, Y]) \), for \( X, Y \in b \). If a particular basis \( \{ Z_k \} \) for \( m \) is chosen, we abbreviate \( J_{Z_k} \) to \( J_k \). In the presence of an inner product, we identify \( m \) with \( m^\ast \) and write \( J_Z \) for the skew-symmetric operator on \( b \) defined by \( \langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle \). \( X, Y \in b \), \( Z \in m \).

Choosing the bases \( \{ X_i \} \) for \( m \) and \( \{ Z_k \} \) for \( b \), we get a family of skew-symmetric \( q \times q \) matrices \( J_1, \ldots, J_p \), which are linearly independent, as \( m = [n, n] \). With respect to the choice of the bases, the matrices \( J_i \) are defined up to a simultaneous transformation \( J_k \to GJ_kG^\ast \), \( G \in \text{SL}(q) \) and linear combinations \( J_k \to \sum_r c_{kr} J_r \) with a nonsingular matrix \( (c_{kr}) \). Every choice of the basis \( \{ X_i \} \) defines a point \( W = \text{Span}(J_1, \ldots, J_p) \) on the Grassmannian \( G(p, o(q)) \) of \( p \)-planes in \( o(q) \). The isomorphism classes of two-step nilpotent Lie algebras of type \( (p, q) \) are in one-to-one correspondence with the orbits of the action of \( \text{SL}(q) \) on \( G(p, o(q)) \) defined by \( \text{Span}(J_1, \ldots, J_p) \to \text{Span}(GJ_1G^\ast, \ldots, GJ_pG^\ast) \), \( G \in \text{SL}(q) \).

The space \( X(p, q) \) of such orbits (the space of the isomorphism types of the two-step nilpotent Lie algebras of type \( (p, q) \)) is in general non-Hausdorff. The dimension count suggests that the stabilizer \( \text{SL}(q)_W \) of a generic \( W \in G(p, o(q)) \) (“the stabilizer in general position”) is finite. This is indeed the case, with a few exceptions, as it is proved in [E1].

The splitting \( n = b \oplus m \) of a two-step nilpotent Lie algebra \( n \) is a gradation, which corresponds to the canonical derivation \( \Phi \) defined by \( \Phi(X + Z) = X + 2Z \), for any \( X \in b \), \( Z \in m \). The algebra of derivations \( \text{Der}(n) \) of \( n = b \oplus m \) splits into a semidirect sum of the abelian ideal \( \mathfrak{J} = \{ \psi \in \text{Der}(n) | \psi(b) \subset m, \psi(m) = 0 \} \) consisting of nilpotent derivations and the subalgebra \( \mathfrak{G} = \mathfrak{R} \Phi \oplus o(q)_W \), where \( o(q)_W \) is the Lie algebra of the stabilizer \( \text{SL}(q)_W \) of \( W \in G(p, o(q)) \) [E2, Proposition 3.4.5]. Clearly, \( \text{Tr} \psi = 0 \) for every \( \psi \in o(q)_W \).

In the fixed bases for \( b \) and \( m \), any \( \psi \in \text{Der}(n) \) is represented by a matrix of the form \( \left( \begin{array}{c} F & 0 \\ U & 0 \end{array} \right) \), where \( U \) is an arbitrary \( p \times q \)-matrix (the set of the \( \left( \begin{array}{c} 0 & 0 \\ U & 0 \end{array} \right) \)'s is \( \mathfrak{J} \)), and the \( q \times q \)-matrix \( F \) and the \( p \times p \)-matrix \( M \) satisfy

\[
g(F)J_k := J_kF + F^TJ_k = \sum_{r=1}^p M_{kr}J_r. \tag{14}\]

For a given endomorphism \( F \), the corresponding \( M := M(F) \) is defined by equation (14) uniquely, as the \( J_r \)'s are linearly independent. With any \( \psi = \left( \begin{array}{c} F & 0 \\ U & 0 \end{array} \right) \in \text{Der}(n) \), each of the following is also a derivation:

\[
\left( \begin{array}{cc} 0 & 0 \\ U & 0 \end{array} \right) \cdot \left( \begin{array}{c} F & 0 \\ 0 & M \end{array} \right) = \left( \begin{array}{cc} F^S & 0 \\ 0 & M^S \end{array} \right), \quad \left( \begin{array}{cc} F & 0 \\ 0 & M \end{array} \right) = \left( \begin{array}{cc} F^N & 0 \\ 0 & M^N \end{array} \right), \quad \left( \begin{array}{cc} F^R & 0 \\ 0 & M^R \end{array} \right),
\]

where the superscripts \( S, N \) and \( R \) denote the semisimple part, the nilpotent part, and the real part, respectively. It is easy to see that \( M(F)^S = M(F)^S, M(F)^N = M(F)^N, \) and \( M(F)^R = M(F)^R \).

Let \( g \) be a solvable Lie algebra with the two-step nilradical \( n = b \oplus m \), and let \( a \) be a linear space complementary to \( n \). For every \( Y \in a \), consider the \( \mathfrak{G} \)-part of \( \text{ad}_Y |_n \) and define the endomorphisms \( F(Y) = \pi_b \circ \text{ad}_{Y |_b} : b \to b \) and \( M(Y) = M(F(Y)) = \text{ad}_{Y |_m} : m \to m \) (where \( \pi_b \) is the linear projection to \( b \)). Then both \( \{ F(Y) | Y \in a \} \) and \( \{ M(Y) | Y \in a \} \) are commuting families of endomorphisms, and the same is true for their semisimple parts and real parts.

Let \( n = b \oplus m \) be a two-step nilpotent Lie algebra whose isomorphism type is defined by \( [W] \), the \( \text{SL}(q) \)-orbit of the point \( W \in G(p, o(q)) \). Let \( W^\perp \in G(D - p, o(q)) \) be the \((D - p)\)-dimensional subspace of \( o(q) \) orthogonal to \( W \) with respect to the inner product \( \langle J, K \rangle = -\text{Tr} JK \) on \( o(q) \). The two-step nilpotent Lie algebra \( \overline{n} = b \oplus \overline{m} \), \( \dim \overline{m} = D - p \) defined by \( [W^\perp] \) is called dual to \( n \). It is easy to see that \( \overline{n} \) is well-defined and that \( \overline{n} \) is isomorphic to \( n \). Moreover, as \( \text{SL}(q)_W \) is isomorphic to \( \text{SL}(q)_{W^\perp} \), \( \text{SL}(q)_W \) consists of the matrices transposed to those from \( \text{SL}(q)_{W^\perp} \), to every \( \psi = \left( \begin{array}{c} F & 0 \\ 0 & M(F) \end{array} \right) \in \text{Der}(n) \), there corresponds the derivation \( \overline{\psi} = \left( \begin{array}{c} F^\ast & 0 \\ 0 & M(F) \end{array} \right) \in \text{Der}(\overline{n}) \).
6.2 Two-step nilradicals of class $O(p, q)$

In this section, we prove assertion 1 of Theorem 6 (Lemma 3 and Lemma 4).

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra of type $(p, q)$ with $\mathfrak{m} = [\mathfrak{n}, \mathfrak{n}]$. Consider two linear forms, $t$ and $t_1$, on $\text{Der}(\mathfrak{n})$ defined by $t(\psi) := \text{Tr} \psi$, $t_1(\psi) := \text{Tr} \psi|_{\mathfrak{m}}$.

**Definition 3.** $O(p, q)$ is the set of two-step nilpotent Lie algebras of type $(p, q)$ for which $\text{Ker} t \subset \text{Ker} t_1$.

We begin with the following easy observation.

**Lemma 3.** 1. A two-step nilpotent Lie algebra $\mathfrak{n}$ belongs to $O(p, q)$ if and only if its pre-Einstein derivation is a multiple of the canonical derivation $\Phi$:

$$\phi_{\mathfrak{n}} = \mu \Phi, \quad \text{where} \quad \mu = \text{Tr} \Phi/\text{Tr} \Phi^2 = (q + 2p)/(q + 4p).$$

2. Any Einstein metric solvable Lie algebra with the nilradical $\mathfrak{n} \in O(p, q)$ is standard.

**Proof.** 1. This follows from Definition 2 and from the uniqueness of the pre-Einstein derivation (assertion (b) of Proposition 1).

2. This follows from Theorem 1: both conditions $\Phi > 0$ and $\text{ad} \Phi \geq 0$ are clearly satisfied. $\Box$

The next lemma shows that the set $O(p, q)$ is “massive”:

**Lemma 4.** 1. If $\mathfrak{n} \in O(p, q)$, then $\mathfrak{n} \in O(D - p, q)$. If $\mathfrak{n}_i \in O(p_i, q_i)$ for $i = 1, 2$, then the direct sum of the Lie algebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ lies in $O(p_1 + p_2, q_1 + q_2)$.

2. For all the pairs $(p, q)$ such that $1 < p < D - 1$, except when $q$ is odd and $p = 2$ or $p = D - 2$, the set $\{ W \mid W \in O(p, q) \}$ contains an open and dense subset of $G(p, \mathfrak{o}(q))$.

**Remark 2.** Although the direct sum of a two-step Lie algebra $\mathfrak{n} \in O(p, q)$ and an abelian ideal $\mathbb{R}^m$ does not belong to $O(p, q + m)$, it is not difficult to see that any Einstein metric solvable Lie algebra having $\mathfrak{n} \oplus \mathbb{R}^m$ as the nilradical ($\mathfrak{n} \in O(p, q)$) is standard using Theorem 1 and assertion 3 of Lemma 2.

**Remark 3.** The pairs $(p, q)$ not covered by assertion 2 of Lemma 4 are $(1, q)$, $(D - 1, q)$, $(D, q)$, and $(2, 2k + 1)$, $(D - 2, 2k + 1)$. There is only one two-step nilpotent Lie algebra of type $(D, q)$, the free algebra. It is an Einstein nilradical by [GK, Proposition 2.9 (iii)]. If $p = 1$, the algebra $\mathfrak{n}$ is isomorphic to the direct sum of a Heisenberg algebra and an abelian ideal, hence is an Einstein nilradical. Every algebra of type $(D - 1, q)$ is an Einstein nilradical, as it is proved in Lemma 6 below. By Theorem 3, neither of the above algebras can be the nilradical of a nonstandard Einstein metric solvable Lie algebra.

In contrast, the two-step nilpotent Lie algebras of types $(2, 2k + 1)$ and $(D - 2, 2k + 1)$ are “the worst” from the point of view of Definition 3: the sets $O(2, 2k + 1)$ and $O(D - 2, 2k + 1)$ are empty. Indeed, from [LR, Theorem 5.1] it follows that any two odd-dimensional skew-symmetric matrices have two common complementary isotropic subspaces of nonequal dimensions. Therefore, for every two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{b} \oplus \mathfrak{m}$ of type $(2, 2k + 1)$, there is a decomposition $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$, $\dim \mathfrak{b}_1 = q$, such that $[\mathfrak{b}_1, \mathfrak{b}_1] = [\mathfrak{b}_2, \mathfrak{b}_2] = 0$ and $q_1 \neq q_2$. It follows that an endomorphism $\psi$ of $\mathfrak{n}$, acting as the identity on $\mathfrak{b}_1$, minus the identity on $\mathfrak{b}_2$, and zero on $\mathfrak{m}$, is a derivation. Then the trace of the derivation $\eta = (2k + 5)\psi + (q_2 - q_1)\Phi$ is zero, but $t_1(\eta) = \text{Tr} \eta|_{\mathfrak{m}} = 2(q_2 - q_1) \neq 0$. It follows that $O(2, 2k + 1) = \emptyset$, and hence $O(D - 2, 2k + 1) = \emptyset$ by assertion 1 of Lemma 4.

**Proof.** 1. For any $\psi = \left( \begin{smallmatrix} 0 & F \\ F^t & 0 \end{smallmatrix} \right) \in \text{Der}(\mathfrak{n})$, there exists (a unique) $\overline{\psi} = \left( \begin{smallmatrix} F \\ 0 \end{smallmatrix} \right) \in \text{Der}(\mathfrak{n})$ of the form $\overline{\psi} = \left( \begin{smallmatrix} F' \\ 0 \end{smallmatrix} \right)$. In particular, $W_{\mathbb{C}} \subset \mathfrak{o}(q)$ (with respect to the inner product $\langle J, K \rangle = -\text{Tr} JK$ on $\mathfrak{o}(q)$) is an invariant subspace of the linear operator $g(F')$ acting on $\mathfrak{o}(q)$. The subspace $W \subset \mathfrak{o}(q)$ is in general not an invariant subspace of $g(F')$. Define $\hat{g}(F') : W \to W$ by $\hat{g}(F') = \pi_W \circ g(F')|_W$, where $\pi_W$ is the orthogonal projection to $W$. Choosing an orthonormal basis $\{ J_k \}$ for $W$ we easily obtain that $\text{Tr} \hat{g}(F') = -\sum_{k=1}^{q^2} \text{Tr} \left( (J_k F' + F J_k) J_k \right) = -\sum_{k=1}^{q^2} \text{Tr} \left( (J_k F' + F J_k) J_k \right) = \text{Tr} g(F)|_W$. 

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As the trace of the linear operator \( g(F^i) \) (acting on the whole \( \mathfrak{o}(q) \)) equals \( \text{Tr} g(F^i) \mathbb{1} + \text{Tr} \hat{g}(F^i) \), we obtain \( \text{Tr} g(F^i) \mathbb{1} + \text{Tr} g(F^i) \mathbb{1} = \text{Tr} g(F^i) \). An easy computation shows that \( \text{Tr} g(F^i) = (q-1) \text{Tr} ~ F \).

As \( \text{Tr} g(F) \mathbb{1} = \text{Tr} M(F) \) and \( \text{Tr} g(F^i) \mathbb{1} = \text{Tr} \hat{M}(F^i) \), we find that \( \text{Tr} M(F) + \text{Tr} \hat{M}(F^i) = (q-1) \text{Tr} ~ F \), which implies \( \text{Tr} \psi + \text{Tr} \varphi = (q+1) \text{Tr} ~ F \).

Assume now that \( \text{Tr} \psi = 0 \). Then \( \text{Tr} \psi = (q+1) \text{Tr} ~ F \), which implies \( \text{Tr} ~ F = 0 \), as from \( n \in \mathcal{O}(p,q) \) it follows that \( \text{Tr} \psi = (2p/q + 1) \text{Tr} ~ F \) (and as \( 2p < q^2 \)). This proves the first statement of assertion 1.

The second statement follows directly from assertion 3 of Lemma 2 and assertion 1 of Lemma 3.

2. According to [E2, Proposition 3.4.3], for every pair \((p,q)\) such that \( 1 < p < D - 1 \), with few exceptions, there exists an open and dense \( \mathcal{O}(p,q) \subset \mathcal{G}(p,\mathfrak{o}(q)) \) such that for any \( W \in \mathcal{O}(p,q) \), the two-step nilpotent algebra \( \mathfrak{n} \) corresponding to \( W \) has the following property: any automorphism of \( \mathfrak{n} \) with determinant 1 has only eigenvalues whose module is 1.

The excepted pairs are \((p,q) = (3,4), (3,5), (3,6), (2, q)\) and the dual ones.

Clearly, if \( W \in \mathcal{O}(p,q) \), then \([W] \in \mathcal{O}(p,q)\), as for the corresponding two-step nilpotent Lie algebra \( \mathfrak{n} \), any derivation whose trace vanishes has zero real part. This proves the assertion for all the “generic” pairs. Moreover, as for the dual pairs \((p,q)\) and \((D - p, q)\), both the Grassmannians \( \mathcal{G}(p,\mathfrak{o}(q)) \) and \( \mathcal{G}(D - p,\mathfrak{o}(q)) \) and the spaces \( \mathcal{O}(p,q) \) and \( \mathcal{D}(D - p, q) \) are homeomorphic (assertion 1), it remains to consider only the following cases: \((p,q) = (3,4), (3,5), (3,6), (2,2k)\).

Let \( S \subset \mathfrak{o}(q)^n \), for a given pair \((p,q)\), be the set of linearly independent \( p \)-tuples of skew-symmetric \( q \times q \) matrices. The projection \( \text{Span}: S \rightarrow \mathcal{G}(p,\mathfrak{o}(q)) \) sends open and dense subsets to open and dense subsets. Note that to prove the assertion it suffices to show that the set \( \{ W : [W] \in \mathcal{O}(p,q) \} \subset \mathcal{G}(p,\mathfrak{o}(q)) \) (or its \( \text{Span}^{-1} \) preimage in \( S \)) has a nonempty interior. Indeed, by Definition 3, the fact that \([W] \in \mathcal{O}(p,q)\) for \( W = \text{Span}(J_1, \ldots, J_p) \) means that all the solutions of a certain system of linear equations (expressing the fact that \( \psi \) is a derivation whose trace is zero) satisfies another linear system, \( t_1(\psi) = 0 \). The entries of the matrix \( \mathcal{M} \) of that system are linear in the entries of the \( J_k \)'s. On an open and dense subset \( S' \subset S \), \( \text{rk} \mathcal{M} \) is maximal, and the condition that any solution satisfies \( t_1(\psi) = 0 \) is algebraic in the entries of \( \mathcal{M} \). If it is satisfied on some open subset of \( S' \), then it is satisfied on an open and dense subset of \( S' \), and hence of \( S \).

Case \((p,q) = (2,2n)\). Let \( \tilde{S} \subset S \subset \mathfrak{o}(2n)^2 \) be the set of the pairs \((J_1, J_2)\) such that \( \text{det} J_1 J_2 \neq 0 \), and the polynomial \( \text{det}(J_2 - x J_1) \) has no multiple roots for \( x \in \mathbb{C} \). Then \( S \) is open and dense in \( S \) (being a nonempty complement to the zero set of a certain system of polynomial equations). By (14), the fact that \([W] \in \mathcal{O}(2,2n)\) for \( W = \text{Span}(J_1, J_2) \) is equivalent to the following: any two matrices \( F \) and \( M \) satisfying the equations \( J_k F + F^t J_k = \sum_{k=1}^{n} \text{M}_{k} J_{r_k}, k = 1, 2 \), also satisfy the equation \( \text{Tr} ~ F = 2n \text{Tr} ~ M \). This, in turn, is equivalent to the existence of a pair of matrices \( K_1, K_2 \in \mathfrak{o}(2n) \) such that

\[
K_1 J_1 + K_2 J_2 = 2 I_{2n}, \quad \text{Tr} (K_r J_s) = 2n \delta_{r,s}.
\]

For any pair \((J_1, J_2) \in \tilde{S} \) the system (15) has a solution. Indeed, by [LR, Theorem 5.1], we can simultaneously reduce the matrices \( J_1, J_2 \) to the form \( J_1 = \begin{pmatrix} 0 & I_n^t \\ -I_n & 0 \end{pmatrix} \), \( J_2 = \begin{pmatrix} 0 & C^t \\ -C & 0 \end{pmatrix} \), where \( C \in \text{GL}(n) \). For \( K_r = \begin{pmatrix} 0 & T_r \\ -T_r & 0 \end{pmatrix} \), the system (15) is equivalent to \( T_1 = -2I_n - T_2 C^t \), \( [T_2, C^t] = 0 \), \( \text{Tr} ~ T_2 = 0 \), \( \text{Tr} (T_2 C^t) = -n \), \( \text{Tr} (T_2 C^t + 2C^t) = 0 \). If \( n > 2 \), we can easily find a complex solution taking \( T_2 \) diagonal in the Jordan basis for \( C^t \) and using the fact that all the eigenvalues of \( C^t \) are distinct. The real part of it gives a real solution to the system (15). If \( n = 2 \), take \( T_2 = (\lambda_1 - \lambda_2)^{-2}(2 \text{Tr} C^t)I_2 - 2C^t \), \( T_1 = -2I_2 - T_2 C^t \) (where \( \lambda_1 \neq \lambda_2 \) are the eigenvalues of \( C^t \)).

Case \((p,q) = (3,4)\). There exists a \( W \in \mathcal{G}(3,\mathfrak{o}(4)) \) such that the two-step nilpotent Lie algebra \( \mathfrak{n} \) defined by \( W \) is nonsingular, hence belongs to \( \mathcal{D}(3,4) \) by Lemma 5 below. The set of such \( W \)'s has a nonempty interior, so \( \{ W : [W] \in \mathcal{D}(3,4) \} \) contains an open and dense subset of \( \mathcal{G}(3,\mathfrak{o}(4)) \) (we prove a stronger fact in assertion 1 of Lemma 7).

Case \((p,q) = (3,5)\). As it follows from [E1, 5.4], the action of \( \text{SL}(5) \) on \( \mathcal{G}(3,\mathfrak{o}(5)) \) has open orbits, and there exist locally rigid two-step nilpotent algebras of type \((3,5)\). As \( \dim \text{SL}(5) = 24 \) and
so contains an open and dense subset of the Grassmannian $G$.

From (14) we find that for any derivation of the two-step nilpotent Lie algebra $\text{Der}(\pi)$ by $\{O \cap U\}$ the open set $U \cap O$ of every $X$ for which $\text{Tr}(X, Y) = 0$ and $\text{Tr}(X, X) = 0$ for all the matrices from $W$ that are symmetric, with $(3 \times 3$ skew-symmetric) matrices having a three-dimensional isotropic subspace complementary to $W$. On that subset, the fiber over a point $\pi(3,6)$ of the Grassmannian $G(3,6)$ is the subspace $V_L \subset \mathfrak{o}(6)$ of the $(6 \times 6$ skew-symmetric) matrices having $L$ as an isotropic subspace. Let $G(3, E) \subset G(3, 6) \times G(3, \mathfrak{o}(6))$ be the corresponding Grassmann bundle: the fiber over $L \in G(3, 6)$ is the Grassmannian $G(3, V_L)$. Both $G(3, E)$ and $G(3, \mathfrak{o}(6))$ are compact differentiable manifolds of dimension 36, and the projection on the second factor $\pi : G(3, E) \to G(3, \mathfrak{o}(6))$, $\pi(L, W) = W$, is a differentiable mapping. Take a point $(L, W) \in G(3, E)$ and choose a basis $e_i$ for $\mathbb{R}^6$ in such a way that $L = \text{Span}(e_1, e_2, e_3)$ and $W = \text{Span}(J_1, J_2, J_3)$, where $J_k = \begin{pmatrix} 0 & T_k & 0 \\ \ -T_k & C_k & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $k = 1, 2, 3$. It is not difficult to see that $\text{Ker} d\pi(L, W) = 0$, if there is no nonzero $3 \times 3$ matrix $S$ such that all the matrices $T_k S$ are symmetric, and that all the $J_k$ have a tree-dimensional isotropic subspace complementary to $L$, if there exists a $3 \times 3$ matrix $P$ such that $T_k^t P - P T_k = C_k$, for all $k = 1, 2, 3$. Both these conditions are satisfied, provided the corresponding $9 \times 9$ determinants $d_1, d_2$ (whose entries are linear in the entries of the $T_k$) do not vanish. A direct computation shows that $d_1 \neq 0$ and $d_2 \neq 0$, if we take say $T_1 = I_3, T_2 = \text{diag}(1, 2, 3)$, and $T_3$ skew-symmetric, with $(T_3)_{12} = 1, (T_3)_{13} = 2, (T_3)_{23} = 3$. It follows that the subset of the triples $(T_1, T_2, T_3)$, for which $d_1 \neq 0$ and $d_2 \neq 0$ is open and dense in $\mathfrak{g}(3)^3$. As the conditions $d_1, d_2 \neq 0$ do not depend of the choice of the basis for $W$, the subset of those $W \in G(3, V_L)$, for which $\text{Ker} d\pi(L, W) = 0$ and all the matrices from $W$ have a three-dimensional isotropic subspace complementary to $L$, is open and dense in the fiber $G(3, V_L)$. On that subset, $\pi$ is locally onto, so the Grassmannian $G(3, \mathfrak{o}(6))$ contains an open subset $U$ such that for any $W = \text{Span}(J_1, J_2, J_3) \in U$, all the $J_k$ have a pair of complementary three-dimensional isotropic subspaces.

By [E1, 5.4], there exists an open and dense subset $\mathcal{O} \subset G(3, \mathfrak{o}(6))$ such that the stabilizer subgroup $\text{SL}(6)_W$ of every $W \in \mathcal{O}$ has dimension one. For any such $W$, the Lie algebra $\mathfrak{sl}(6)_W$ is one-dimensional, so for every two-step nilpotent Lie algebra $\mathfrak{n}$ of type $(3,6)$ defined by some $[W]$, $W \in \mathcal{O}$, we get $\text{Der}(\mathfrak{n}) = 3 \oplus \mathfrak{R}\Phi \oplus \mathfrak{R}\psi_0$ for some derivation $\psi_0$ with $\text{Tr} \psi_0 = 0$. Both linear forms $t$ and $t_1$ from Definition 3 vanish on $3$, so $\mathfrak{n} \in \mathcal{O}(3,6)$ if and only if $t_1(\psi_0) = 0$. Now take $W = \text{Span}(J_1, J_2, J_3)$ from the open set $U \cap \mathcal{O}$. For any $\mathfrak{n} = b \oplus m$ defined by such a $W$, we have $b = b_1 \oplus b_2$, where $b_1$ and $b_2$ are three-dimensional isotropic subspaces of all the three $J_k$'s. Then a semisimple endomorphism $\psi$ defined by $\psi(X_1 + X_2 + Z) = X_1 - X_2, X_1 \in b_1, X_2 \in b_2, Z \in m$, is a derivation of $\mathfrak{n}$. As $\text{Tr} \psi = 0$, we can take $\psi_0 = \psi$. Since $\psi|_m = 0, t_1(\psi) = 0$, so $\mathfrak{n} \in \mathcal{O}(3,6)$. The claim follows, as $U \cap \mathcal{O}$ is a nonempty open subset of the Grassmannian $G(3, \mathfrak{o}(6))$. 

\[\square\]
6.3 Two-step nonsingular nilradicals

Assertion 2(a) of Theorem 6, which we prove in this section, follows from assertion 2 of Lemma 3 and Lemma 5 below.

Let \( \mathfrak{n} \) be a two-step nilpotent Lie algebra with the derived algebra \( \mathfrak{m} \). Algebra \( \mathfrak{n} \) is called nonsingular, if for any \( X \in \mathfrak{n} \setminus \mathfrak{m} \), the map \( \text{ad}_X : \mathfrak{n} \to \mathfrak{m} \) is surjective [E2]. The reason for the name comes from the following equivalent definition. Choose an arbitrary inner product on \( \mathfrak{n} \) and set \( \mathfrak{b} = \mathfrak{m}^\perp \). Then \( \mathfrak{n} \) is nonsingular if and only if for any nonzero \( Z \in \mathfrak{m} \), the operator \( Z \) is nonsingular. The nonsingular two-step nilpotent Lie algebras are one of “the most well behaved of all two-step nilpotent Lie algebras” [E2]. In particular, many interesting examples of Einstein nilradicals are two-step nonsingular, for instance, the Heisenberg-type algebras [GK].

Note that the nonsingularity condition imposes strong restrictions on the type. The obvious one is that \( q = \dim \mathfrak{b} \) must be even. A subtler dimension restriction comes from topology: for a (metric) nonsingular two-step nilpotent Lie algebra, the unit sphere \( S^{r-1} \subset \mathfrak{b} \) must admit \( p \) continuous and pointwise linearly independent vector fields defined (with respect to some basis \( \{ Z_i \} \) for \( \mathfrak{m} \)) by \( X \to J_kX \), for \( X \in S^{r-1} \). It follows from the Adams vector field Theorem, that \( p \leq \rho(q) - 1 \), where \( \rho(q) \) is the Radon-Hurwitz number.

**Lemma 5.** Any nonsingular two-step nilpotent Lie algebra of type \((p,q)\) belongs to \( \mathfrak{D}(p,q) \).

**Proof.** Let \( \mathfrak{n} \) be a nonsingular two-step nilpotent Lie algebra. To show that \( \mathfrak{n} \in \mathfrak{D}(p,q) \), it suffices to check that for any \( \psi \in \text{Der}(\mathfrak{n}) \) of the form \( \psi = \begin{pmatrix} F & 0 \\ 0 & M \end{pmatrix} \), with both \( F \) and \( M \) semisimple and real, the equation \( T \psi = 0 \) implies \( T M = 0 \) (or, equivalently, \( Tr F = 0 \)). Let \( \lambda_k \) be the eigenvalues of \( M \), with \( Z_k \) the corresponding eigenvectors. According to (14) we have \( J_kF + F^tJ_k = \lambda_kJ_k \), for all \( k = 1, \ldots, p \), so \( 2J_kF = \lambda_kJ_k + S_k \), where \( S_k \) are some symmetric matrices. It follows that \( 2F = \lambda_kI_q + J_k^{-1}S_k \), so \( q\lambda_k = 2Tr F \), for all \( k \). This implies \( Tr \psi = Tr F + Tr M = (1 + 2p/q)Tr F \). \( \square \)

**Example 3.** Note that for a two-step nilpotent Lie algebra \( \mathfrak{n} \) of type \((p,q)\), the fact that \( \mathfrak{n} \in \mathfrak{D}(p,q) \) and even a much stronger condition of nonsingularity does not guarantee that \( \mathfrak{n} \) is an Einstein nilradical, as the following example shows. Let \( \mathfrak{n} \) be a nonsingular two-step nilpotent Lie algebra of type \((2,2n)\). As it follows from Lemma 5 and assertion 1 of Lemma 3, if \( \mathfrak{n} \) is an Einstein nilradical, its eigenvalue type must be \((1 < 2; 2n, 2)\), so \( \mathfrak{n} \) is defined by an orbit \([W] \) of \( W = \text{Span}(J_1, J_2) \subset G(2, \mathfrak{o}(2n)) \) such that \( J_1^2 + J_2^2 = -J_{2n} \) and \( \det(xJ_1 + yJ_2) \neq 0 \) unless \( x = y = 0 \). It follows that the matrix \( K = J_2^{-1}J_1 \) is normal, hence semisimple. So for any \( a, b, c, d \in \mathbb{R} \) with \( ad - bc \neq 0 \), the matrix \( (aJ_1 + bJ_2)^{-1}(cJ_1 + dJ_2) = (aK + bI_{2n})^{-1}(cK + bI_{2n}) \) is also semisimple. This property is preserved under the action of \( GL(2n) \) on \( G(2, \mathfrak{o}(2n)) \) (which sends \( J \) to \( TJT^{-1} \) for \( T \in GL(2n) \)). Take now two skew-symmetric \( 8 \times 8 \) matrices of the form \( \tilde{J}_i = \begin{pmatrix} 0 & B_i \\ -B_i^t & 0 \end{pmatrix} \), where \( B_2 = I_4 \), \( B_1 = \begin{pmatrix} I_2^t \\ 0 \end{pmatrix} \), and \( J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Clearly, any nontrivial linear combination of the \( \tilde{J}_i \)'s is nonsingular, but \((J_2)^{-1}\tilde{J}_1\) is not semisimple.

6.4 Two-step nilradicals of types \((D - 1, q), (2, 5) \) and \((3, 4)\)

In this section, we consider two classes of two-step nilradicals whose type is not covered by assertion 1 of Lemma 4 and prove assertions 2(b) and 2(c) of Theorem 6.

**Lemma 6.** Any two-step nilpotent Lie algebra of type \((D - 1, q)\) is an Einstein nilradical.

As a consequence of Theorem 3, such an algebra cannot be the nilradical of any nonstandard Einstein metric solvable Lie algebra.

**Proof.** The isomorphism type of a two-step nilpotent Lie algebra \( \mathfrak{n} \) of type \((D - 1, q)\) is completely determined by the one-dimensional subspace in \( \mathfrak{o}(q) \) orthogonal to \( W \). What is more, if \( W^\perp = \mathbb{R}J \) for a nonzero \( J \in \mathfrak{o}(q) \), then the isomorphism type of \( \mathfrak{n} \) is determined by a single number \( d = \frac{1}{2} \text{rk} J > 0 \).
To prove that \( n \) is an Einstein nilradical, it suffices to produce an Einstein derivation \( \Phi \) and an inner product \( (\cdot, \cdot) \) on \( n \) (the nilsoliton inner product) in such a way that (7) is satisfied for some number \( c < 0 \). Choose a basis \( \{X_i\} \) for \( b \) in such a way that the matrix of \( J \) has the form \( J = (J_{2d}^0 0) \), where \( J_{2d} = (0 I_d 0) \). Introduce the inner product on \( b \) in such a way that the basis \( \{X_i\} \) is orthonormal.

Denote \( b_1 = \text{Span}(X_1, \ldots, X_{2d}) \), \( b_2 = b_1^\perp = \ker J \) and \( l = q - 2d = \dim b_2 \). By the result of [GK, Proposition 2.9], we can assume that \( l > 0 \).

Let \( F \) be an endomorphism of \( b \) whose eigenspaces are \( b_1 \) and \( b_2 \), with the corresponding eigenvalues \( \mu_1 \neq \mu_2 \), respectively. The operator \( g(F) \) acting on \( \mathfrak{o}(q) \) according to (14) is semisimple and real, with the eigenvalues \( 2\mu_1, \mu_1 + \mu_2, 2\mu_2 \) whose corresponding eigenspaces are

\[
L_{2\mu_1} = \left\{ \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}, \ K \in \mathfrak{o}(2d) \right\}, \quad L_{\mu_1 + \mu_2} = \left\{ \begin{pmatrix} 0 & T^t \\ -T & 0 \end{pmatrix} \right\}, \quad L_{2\mu_2} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}, \ N \in \mathfrak{o}(l) \right\}.
\]

Clearly, \( J \in L_{2\mu_1} \), so \( W^\perp = L_{2\mu_1}^\perp \oplus L_{\mu_1 + \mu_2} \oplus L_{2\mu_2} \), where \( L_{2\mu_1}^\perp = L_{2\mu_1} \cap J^\perp \) (recall that \( \perp \) refers to the inner product in \( \mathfrak{o}(q) \); the above decomposition of \( W^\perp \) is orthogonal with respect to that inner product).

Introduce an inner product on \( m \) in such a way that \( m \) admits an orthogonal decomposition \( m = m_{11} \oplus m_{12} \oplus m_{22} \) with the following property: for every \( Z \in m_{ij} \), the operator \( J_Z \) belongs to \( L_{\mu_1 + \mu_j} \).

Choose an orthonormal basis \( \{Z_1, \ldots, Z_{d-1}\} \) for \( m \) such that \( m_{11} = \text{Span}(Z_1, \ldots, Z_{d(2d-1)-1}) \), \( m_{12} = \text{Span}(Z_{d(2d-1)}, \ldots, Z_{d(2d-1)-1+2d}) \), \( m_{22} = \text{Span}(Z_{d(2d-1)+1+2d}, \ldots, Z_{d-1}) \). Equation (7) gives

\[
\sum_{k=1}^{D-1} J^k_{2\mu_1} = 2(c + \mu_1 + \mu_2)F, \quad \text{Tr} (J_i J^j_i) = 4(c + \mu_1 + \mu_j)\delta_{ir}, \quad \text{for } Z_r \in m_{ij}, \tag{16}
\]

where \( I_q \) is the identity matrix (we abbreviate \( J_{Z_k} \) to \( J_k \)). Now choose any bases \( \{J_i\} \) in each of the subspaces \( L_{2\mu_1}^\perp, L_{\mu_1 + \mu_2}, L_{2\mu_2} \) such that the second equation of (16) is satisfied. Then an easy computation shows that the first equation of (16) is equivalent to

\[
2d(c + \mu_1 + \mu_2) + (l - 1)(c + 2\mu_2) = -(c + \mu_2), \quad l(c + \mu_1 + \mu_2) + (2d - 1 - 1/d)(c + 2\mu_1) = -(c + \mu_1).
\]

Solving this we find, up to scaling:

\[
\mu_1 = (q^2 - q + 2)d - 2q, \quad \mu_2 = (q^2 - q + 2)d - 2q + 1, \quad -c = (2q^2 - 3q + 5)d - 4q + 2.
\]

To check that (16) is satisfied, it remains to show that \( \mu_1 < -c < \mu_1 + \mu_j \). As \( d \geq 1 \) and \( q = 2d + 1 > 2d \), this is indeed the case, with the only exception: when \( q = 3 \) and \( d = 1 \), \( -c = 2\mu_1 = 4 \). However, when \( d = 1 \), \( L_{2\mu_1} = 0 \), so it is sufficient to check that \( -c < \mu_1 + \mu_j \), which is true.

Remark 6. In view of Lemma 6, the fact that any two-step nilpotent Lie algebra of type \( (1, q) \) is an Einstein nilradical and [GK, Proposition 2.9 (iv)], one may wonder, whether a two-step nilpotent Lie algebra dual to an Einstein nilradical is an Einstein nilradical by itself. This is not true in general, as can be seen from the example of the two-step nilpotent Lie algebra \( n \) attached to the graph \( G_{2,2,0} \) [LW, Figure 1]. By [LW, Proposition 5.6], \( n \) is not an Einstein nilradical. The dual algebra \( \overline{n} \) is attached to the graph which has the same vertex set as \( G_{2,2,0} \) and whose edge set is the complement to that of \( G_{2,2,0} \). Applying [LW, Theorem 5.3] one can easily see that \( \overline{n} \) is an Einstein nilradical.

Lemma 7. 1. Any Einstein metric solvable Lie algebra whose nilradical is a two-step nilpotent Lie algebra of type \( (p, q) \), \( p \leq 4 \) is standard.

2. Any two-step nilpotent Lie algebra of type \( (2, 5) \) is an Einstein nilradical.

Proof. 1. If \( q \leq 3 \), then \( \dim n \leq 6 \) and the claim follows from [L3, Theorem 5.1] and [W, Theorem 3.1].

Any two-step nilpotent Lie algebra of type \( (p, 4) \) with \( p = 1, 2, 5, 6 \) is an Einstein nilradical (this follows from [L3, Theorem 5.1], [W, Theorem 3.1], Lemma 6, and [GK, Proposition 2.9(iii)], respectively), and the claim then follows from Theorem 3.
It remains to consider the cases $p = 3, 4$. Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be an Einstein metric solvable Lie algebra whose nilradical $\mathfrak{n}$ is two-step nilpotent of type $(p, 4)$, $p = 3, 4$. Let $\mathfrak{n} = \mathfrak{b} \oplus \mathfrak{m}$ be the orthogonal decomposition, with $\mathfrak{m} = [\mathfrak{n}, \mathfrak{n}]$, and let $\mathfrak{a} = \mathfrak{n}^4$.

For every $Y \in \mathfrak{a}$, define the operators $F(Y) : \mathfrak{b} \to \mathfrak{b}$ and $M(Y) : \mathfrak{m} \to \mathfrak{m}$ by $F(Y) = \pi_\mathfrak{b} \circ \text{ad}_Y|_\mathfrak{b}$ and $M(Y) = \text{ad}_Y|_\mathfrak{m}$, where $\pi_\mathfrak{b}$ is the orthogonal projection to $\mathfrak{b}$. Then both $\{F(Y)| Y \in \mathfrak{a}\}$ and $\{M(Y)| Y \in \mathfrak{a}\}$ are commuting families of operators, and the same is true for their semisimple parts $F(Y)^{S}, M(Y)^{S}$ (see Section 6.1). Note also that $M(Y)$ is completely determined by $F(Y)$ by (14). In particular, if $F(Y)^{S}$ is real, then $M(Y)^{S}$ is real, hence $B(Y, Y) \geq 0$ (where $B$ is the Killing form of $\mathfrak{g}$).

The linear space $F(Y)^{S}$ is a commuting family of semisimple endomorphisms of $\mathbb{R}^4 = \mathfrak{b}$. We have a direct sum decomposition $\mathfrak{b} = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_m$ on invariant subspaces, such that every $\mathfrak{b}_i$ is either one- or two-dimensional. When $\dim \mathfrak{b}_1 = 1$, $F(Y)^{S}|_{\mathfrak{b}_1} = i_1(Y) \text{id}_{|_{\mathfrak{b}_1}}$, for some linear form $i_1$ on $\mathfrak{a}$. When $\dim \mathfrak{b}_2 = 2$, we can choose a basis $X_1, X_2$ for $\mathfrak{b}_1$ such that $F(Y)^{S}|_{\mathfrak{b}_1} = c_1(Y) \text{id}_{|_{\mathfrak{b}_1}} + d_1(Y)J_1$ for some linear forms $c_1, d_1$ on $\mathfrak{a}$, with $J_1 : \mathfrak{b}_2 \to \mathfrak{b}_1$ defined by $J_1(X_1) = X_2^2$, $J_1(X_2) = -X_1^2$.

If all the $\mathfrak{b}_i$’s are of dimension one (all the eigenvalues of all the $F(Y)$’s are real), then $B(Y, Y) \geq 0$, for all $Y \in \mathfrak{a}$. If all the $\mathfrak{b}_i$’s except one are one-dimensional, then $\mathfrak{a}$ contains a subspace of codimension one such that for every $Y$ from that subspace all the eigenvalues of $F(Y)$ are real, so the index of $B$ is at most one. In both cases, the claim follows from [H, Lemma 4.7]. The same is true if we have two-dimensional subspaces $\mathfrak{b}_1, \mathfrak{b}_2$ and the linear forms $d_1, d_2$ are proportional. The only remaining case is therefore when $\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$, $\dim \mathfrak{b}_1 = \dim \mathfrak{b}_2 = 2$ and the forms $d_1, d_2$ are linearly independent.

As for a generic $Y \in \mathfrak{a}$, all the (complex) eigenvalues of $F(Y)$ are distinct, all the $F(Y)$’s are semisimple (and hence all the $M(Y)$’s are semisimple). Fix the basis $X_1^2, X_2^2, X_1, X_2$ for $\mathfrak{b}$. Then the matrix of $F(Y)$ has the form $F(Y) = \begin{pmatrix} c_1(Y)I_2 + d_1(Y)J_1 & 0 \\ 0 & c_2(Y)I_2 + d_2(Y)J_1 \end{pmatrix}$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The operators $g(F(Y)) : \mathfrak{o}(4) \to \mathfrak{o}(4)$ defined by (14) are commuting and semisimple and have four eigenspaces:

$$L_1 = \mathbb{R} \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad L_2 = \mathbb{R} \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}, \quad L_3 = \text{Span}(\begin{pmatrix} I_2 & 0 \\ 0 & J \end{pmatrix}), \quad L_4 = \text{Span}(\begin{pmatrix} 0 & I_{1,1} \\ -I_{1,1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{1,1}J \\ -I_{1,1}J & 0 \end{pmatrix}),$$

where $I_{1,1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, with the corresponding eigenvalues $2c_1(Y)$, $2c_2(Y)$, $c_1(Y) + c_2(Y) \pm i(d_2(Y) - d_1(Y))$, $c_1(Y) + c_2(Y) \pm i(d_2(Y) + d_1(Y))$ (the first two span a single two-dimensional eigenspace, when $c_1(Y) = c_2(Y)$).

Suppose $p = 3$. If $c_1(Y) = c_2(Y)$, any $g(F(\mathfrak{a}))$-invariant three-dimensional subspace of $\mathfrak{o}(4)$ is a direct sum of a one-dimensional subspace from $L \subset L_1 \oplus L_2$ and one of the spaces $L_3, L_4$. A direct check shows that if $L \not\subset L_1 \cup L_2$, then $\mathfrak{n} \in \mathcal{D}(3, 4)$, and the claim follows from Lemma 3 (in fact, such an $\mathfrak{n}$ is an Einstein nilradical). If $L = L_1$ or $L = L_2$, then $\mathfrak{n}$ is an Einstein nilradical, as it is shown below.

If $c_1(Y) \neq c_2(Y)$, then every three-dimensional $g(F(\mathfrak{a}))$-invariant subspace of $\mathfrak{o}(4)$ is a direct sum of one of the $L_1, L_2$ and one of the $L_3, L_4$. Without loss of generality (permuting the basis vectors and changing the sign of some of them, if necessary), we can take $W = L_1 \oplus L_2$. Then $\mathfrak{n}$ is defined by the commutator relations $[X_1, X_2] = Z_1, [X_1, X_3] = [X_2, X_4] = Z_2, [X_1, X_4] = [X_3, X_2] = Z_3$ and is an Einstein nilradical (the nilsoliton inner product is defined by taking the vectors $X_1, Z_0$ orthonormal).

Let now $p = 4$. There are two $g(F(\mathfrak{a}))$-invariant four-dimensional subspaces of $\mathfrak{o}(4)$: $L_1 \oplus L_2 \oplus L_3$ (choosing $L_4$ instead of $L_3$ gives an isomorphic algebra) and $L_3 \oplus L_4$. Using [P, Theorem 1] we find that in both cases, $\mathfrak{n}$ is an Einstein nilradical: in the first case, $\mathfrak{n}$ is isomorphic to the Lie algebra given by the commutator relations $[X_1, X_2] = \sqrt{2}Z_1, [X_3, X_4] = \sqrt{2}Z_2, [X_1, X_3] = [X_2, X_4] = Z_3, [X_1, X_4] = [X_2, X_3] = Z_4$; in the second one — by the commutator relations $[X_1, X_3] = Z_1, [X_1, X_4] = Z_2, [X_2, X_3] = Z_3, [X_2, X_4] = Z_4$ (the nilsoliton inner product in the both cases is defined by taking $X_1, Z_0$ orthonormal). The claim then follows from Theorem 3.

2. A two-step nilpotent Lie algebra $\mathfrak{n}$ of type $(2, 5)$ is determined by a point $W \in G(2, \mathfrak{o}(5))$. Let
W = \text{Span}(J_1, J_2)$, with $J_1, J_2 \in \mathfrak{o}(5)$ linearly independent. If $\text{Ker} J_1 \cap \text{Ker} J_2 \neq 0$, the algebra $\mathfrak{n}$ is decomposable: it is a direct sum of two-step nilpotent Lie algebra of a smaller dimension and an abelian ideal. Any such $\mathfrak{n}$ is an Einstein nilradical, as it follows from [W, Theorem 3.1], [L3, Theorem 5.1, Proposition 3.3].

Assume $\text{Ker} J_1 \cap \text{Ker} J_2 = 0$. Then by [LR, Theorem 5.1], $\mathfrak{n}$ is isomorphic to one of the two algebras given by the following $W = \text{Span}(J_1, J_2)$:

$$J_1 = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad J_1 = \begin{pmatrix} 0 & 0 & 0 \\ J & 0 & 0 \\ 0 & 0 & J \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0_1 & 0 \\ J & 0 & 0 \end{pmatrix}$$

where $J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Using [P, Theorem 1] we find that in the both cases, $\mathfrak{n}$ is an Einstein nilradical: in the first case, $\mathfrak{n}$ is isomorphic to the Lie algebra given by the commutator relations $[X_1, X_2] = 2Z_1$, $[X_1, X_5] = \sqrt{2}Z_2$, $[X_3, X_4] = 2Z_1$, in the second one — by the commutator relations $[X_1, X_4] = Z_1$, $[X_2, X_4] = Z_2$, $[X_2, X_3] = \sqrt{2}Z_1$, $[X_1, X_5] = \sqrt{2}Z_2$ (the nilsoliton inner product in the both cases is defined by taking $X_i, Z_k$ orthonormal).

As a consequence of Lemma 7 we obtain that any Einstein metric solvable Lie algebra with a two-step nilradical of dimension at most seven is standard.

### 7 Low-dimensional nilradicals. Proof of Theorem 7

The second assertion of Theorem 7 follows trivially from the first one (and Lemma 1). Let $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ be an Einstein metric solvable Lie algebra with the nilradical $\mathfrak{n}$, $\dim \mathfrak{n} \leq 7$. If $\dim \mathfrak{n} \leq 6$, the claim follows from the fact that any such $\mathfrak{n}$ is an Einstein nilradical ([W, Theorem 3.1], [L3, Theorem 5.1]) and Theorem 3. Suppose $\dim \mathfrak{n} = 7$. If $\mathfrak{n}$ is decomposable (is a direct sum of nonzero ideals), then each of the summands is an Einstein nilradical, hence such is $\mathfrak{n}$ [P, Theorem 4]. Denote $\mathfrak{m} = [\mathfrak{n}, \mathfrak{n}]$, $\dim \mathfrak{m} = p$. Then $\mathfrak{n}$ is generated by $7-p$ elements. If $p \geq 4$, the claim follows from Theorem 4. If $p = 0$ or $p = 1$, $\mathfrak{n}$ is an Einstein nilradical: in the first case, $\mathfrak{n}$ is abelian, in the second one, $\mathfrak{n}$ is a Heisenberg algebra (if it is indecomposable). We have two remaining cases to consider: $p = 2$ and $p = 3$. In each of these cases, if $\mathfrak{n}$ is two-step nilpotent, the claim follows from Lemma 7.

So it suffices to consider only those seven-dimensional nilpotent algebras, which are indecomposable and not two-step nilpotent, and whose derived algebra has dimension 2 or 3.

First suppose $p = 2$. Let $\mathfrak{m} = \text{Span}(Z_1, Z_2)$, with $Z_2$ in the center of $\mathfrak{n}$. Then $\mathfrak{z}(\mathfrak{n}) = \mathbb{R}Z_2$. Indeed, $Z_1$ is not in $\mathfrak{z}(\mathfrak{n})$ (as otherwise $\mathfrak{n}$ is two-step), and no vector from $\mathfrak{n} \setminus \mathfrak{m}$ is in $\mathfrak{z}(\mathfrak{n})$ (as otherwise the span of that vector is a direct summand, so $\mathfrak{n}$ is decomposable). Let $\mathfrak{b}$ be a subspace of $\mathfrak{n}$ complementary to $\mathfrak{m}$. Consider the operator $\text{ad}_{Z_1}$ (which is nonzero). Clearly, $\text{ad}_{Z_1}|\mathfrak{m} = 0$ and $\text{ad}_{Z_1}(\mathfrak{b}) \subset \mathfrak{m}$. What is more, $\text{ad}_{Z_1}(\mathfrak{b}) = \mathbb{R}Z_2$ (for otherwise $\mathfrak{n}$ is not nilpotent), so we get a nonzero one-form $\ell$ on $\mathfrak{b}$ such that $[Z_1, X] = \ell(X)Z_2$. Define the two-forms $\omega^1, \omega^2$ on $\mathfrak{b}$ by $[X, Y] = \omega^1(X, Y)Z_1 + \omega^2(X, Y)Z_2$. From the Jacobi identity, $\omega^1(X, Y) = 0$ for all $X, Y \in \text{Ker} \ell$, hence, $\omega^1 = \ell \wedge \ell$ for some $\ell \neq 0$ (note that $\omega^1 \neq 0$, as otherwise $Z_1 \notin \mathfrak{m}$), and in particular, $\text{rk} \omega^1 = 2$. Moreover, the intersection $\text{Ker} \omega^1 \cap \text{Ker} \omega^2$ is the set of generators whose bracket with any generator is zero, hence it lies in $\mathfrak{z}(\mathfrak{n})$, so $\text{Ker} \omega^1 \cap \text{Ker} \omega^2 = 0$. In particular, the forms $\omega^1$ and $\omega^2$ are not proportional. From [LR, Theorem 5.1], a pair of non-proportional $5 \times 5$ skew-symmetric matrices without a common kernel and such that one of them has rank two can be reduced by a choice of the basis to the form

$$J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0_1 & 0 \\ 0 & 0 & 0 & 0_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0_1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J \end{pmatrix},$$
where $J = \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$. Let $X_1, \ldots, X_5$ be the corresponding basis for $\mathfrak{b}$. As $\omega^1$ vanishes on any two vectors from $\ker l$, we obtain $l(X_i) = 0$, $i = 3, 4, 5$. Then $\mathfrak{n}$ is defined by the following commutator relations:

$$[X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2, \quad [X_4, X_5] = Z_2, \quad [X_1, Z_1] = aZ_2, \quad [X_2, Z_1] = bZ_2,$$

where $a$ and $b$ are not simultaneously zeros. If $b = 0$, then $a \neq 0$, so the vector $X_3 - Z_1/a$ lies in $\mathfrak{z}(\mathfrak{n})$, which is a contradiction. If $b \neq 0$, we can replace $X_1$ by $X_1 - (a/b)X_2$, which gives after scaling the following set of relations for $\mathfrak{n}$:

$$[X_1, X_2] = Z_1, \quad [X_1, X_3] = Z_2, \quad [X_4, X_5] = Z_2, \quad [X_2, Z_1] = 2Z_2. \quad \text{By [P, Theorem 1], we obtain that such an } \mathfrak{n} \text{ is an Einstein nilradical, as it is isomorphic to the (metric) Lie algebra given by the commutator relations } [X_1, X_2] = 2Z_1, \quad [X_1, X_3] = Z_2, \quad [X_4, X_5] = \sqrt{3}Z_2, \quad [X_2, Z_1] = \sqrt{3}Z_2, \text{ with the nilsoliton inner product defined by taking } X_i, Z_k \text{ orthonormal. The claim now follows from Theorem 3.}

Now suppose $p = 3$. We combine the approaches used in the proof of Theorem 2, Theorem 4, and Lemma 7. As it was shown in the proof of Theorem 2, there exists a subspace $\mathfrak{a} \subset \mathfrak{g}$ complementary to $\mathfrak{n}$ such that for any $Y_1, Y_2 \in \mathfrak{a}$, $\operatorname{ad}Y_1Y_2 = [\operatorname{ad}Y_1, \operatorname{ad}Y_2] = 0$, and the map $Y \rightarrow \operatorname{ad}Y^2$ is linear on $\mathfrak{a}$ (such an $\mathfrak{a}$ is the complement, in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, to $\mathfrak{h} \cap \mathfrak{n}$).

The set of operators $\operatorname{ad}Y^2|_{\mathfrak{n}}$ is a torus, a commuting family of semisimple derivations of $\mathfrak{n}$. As $\mathfrak{m} = [\mathfrak{n}, \mathfrak{n}]$ is a characteristic ideal, it is complemented in $\mathfrak{n}$ by a linear subspace $\mathfrak{b}$ invariant with respect to all of the $\operatorname{ad}Y^2$. 

As in the proof of Theorem 4, define a linear map $\theta : \mathfrak{a} \rightarrow \text{End}(\mathfrak{b})$ by $\theta(Y) = \pi_\mathfrak{b} \circ (\operatorname{ad}Y^2)|_{\mathfrak{n}}$ (where $\pi_\mathfrak{b}$ is the linear projection on $\mathfrak{b}$). Then $\theta(\mathfrak{a})$ is a commuting family of semisimple endomorphisms of $\mathfrak{b}$. If all the eigenvalues of $\theta(Y)$ are real, then all the eigenvalues of $\theta(Y)$ are real, and so $B(Y, Y) = \text{Tr}(\operatorname{ad}Y)^2 \geq 0$ (where $B$ is the Killing form of $\mathfrak{g}$). By [H, Lemma 4.7], the metric algebra $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is standard, provided $\mathfrak{a}$ contains a subspace $\mathfrak{a}'$ of codimension at most one such that $B(Y, Y) \geq 0$ for all $Y \in \mathfrak{a}'$. In particular, this will be the case, if $\mathfrak{a}$ contains a subspace $\mathfrak{a}'$ of codimension at most one such that all the eigenvalues of $\theta(Y)$ are real for all $Y \in \mathfrak{a}'$. The only possible way to avoid that for a commuting family of semisimple endomorphisms of the four-dimensional space $\mathfrak{b}$ is when $\mathfrak{b}$ splits into a direct sum of two two-dimensional subspaces $\mathfrak{b}_1, \mathfrak{b}_2$ invariant with respect to $\theta(Y)$ and corresponding to the complex eigenvalues $c_j(Y) \pm id_j(Y), j = 1, 2$, with the linear forms $d_1, d_2$ on $\mathfrak{a}$ linearly independent (see the proof of assertion 1 of Lemma 7). Choose the bases $X_1', X_2'$ for $\mathfrak{b}_j$ such that $X_1' + iX_2'$ is an eigenvector of $\theta(Y)$ with the eigenvalue $c_j(Y) + id_j(Y)$.

Then the weights for the torus $\{\text{ad}Y^2\mathfrak{n}\}$ are of the form $k_1c_1(Y) + k_2c_2(Y) + i(l_1d_1(Y) + l_2d_2(Y))$ for some integers $k_1, k_2, l_1, l_2$. As $d_1$ and $d_2$ are linearly independent, two such weights can only be equal if their $l_1$’s are equal and $l_2$’s are equal. The corresponding weight space has an even dimension, unless $l_1 = l_2 = 0$. As $\mathfrak{n}$ is assumed to be not two-step nilpotent and is generated by $\mathfrak{b}$, the space $[\mathfrak{b}, [\mathfrak{b}, \mathfrak{b}]]$ must be nonzero. Note that $[[\mathfrak{b}_1, \mathfrak{b}_2], \mathfrak{b}_1] = 0$, as otherwise the three-dimensional space $\mathfrak{m}$ would contain two even-dimensional weight spaces corresponding to the different (and not complex conjugate) weights. Similarly, $[[\mathfrak{b}_1, \mathfrak{b}_2], \mathfrak{b}_2] = 0$, and by the Jacobi identity, $[[\mathfrak{b}_1, \mathfrak{b}_1], \mathfrak{b}_2] = [\mathfrak{b}_2, \mathfrak{b}_2, \mathfrak{b}_1] = 0$. The only three-folded brackets which could be nonzero are therefore $[[\mathfrak{b}_1, \mathfrak{b}_1], \mathfrak{b}_1]$ or $[[\mathfrak{b}_2, \mathfrak{b}_2], \mathfrak{b}_1]$. Assume that say $[[\mathfrak{b}_1, \mathfrak{b}_1], \mathfrak{b}_1] \neq 0$. Then the vector $Z = X_1', X_2', Z_2, Z_2, Z_2$ is nonzero, and $\mathfrak{m}$ is a direct sum of the one-dimensional weight space $\mathfrak{R}Z$, and the two-dimensional weight space $[Z, \mathfrak{b}_1]$, with the weight $3c_1(Y) \pm d_1(Y)$. As no subspace from $\{\mathfrak{b}_1, \mathfrak{b}_2\}$ can have such weights, we obtain that $[\mathfrak{b}_1, \mathfrak{b}_2] = 0$. Since $\mathfrak{b}$ generates $\mathfrak{n}$, this implies that $\mathfrak{n}$ is decomposable: it is a direct sum of the subalgebras (in fact, the ideals) generated by the $\mathfrak{b}_j$’s, which is a contradiction.

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