A HOMOLOGICAL NERVE THEOREM FOR OPEN COVERS

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Abstract. In this note we show that a particular homological nerve theorem, which was originally proved for a finite cover of a simplicial complex by subcomplexes, also holds for an open cover of an arbitrary topological space. The motivation for this is to affirmatively answer a question about the homology groups of Vietoris metric thickenings.

Given a cover \( \mathcal{U} = \{ U_i \}_{i \in I} \) of a topological space \( X \), the nerve of \( \mathcal{U} \), which we denote \( N(\mathcal{U}) \), is a simplicial complex whose vertex set is \( I \) and whose simplices are the finite subsets \( \sigma \subset I \) such that the intersection \( \bigcap_{i \in \sigma} U_i \) is nonempty. There are many nerve theorems, each of which relate a space \( X \) with \( N(\mathcal{U}) \), but vary on the assumptions placed on \( X \) and \( \mathcal{U} \), as well as the conclusions drawn.

One of the earliest examples of a nerve theorem is due to Borsuk [3]. Borsuk proved that if \( X \) is a finite-dimensional compact metric space and \( A \) is a finite cover of \( X \) by closed subsets of \( X \) such that intersection of any subset of \( A \) is an absolute retract, then \( X \) and \( N(A) \) have the same homotopy type.

Another early example of a nerve theorem is contained in the work of Leray in [8] and [9]. His work implies that if \( X \) is a finite simplicial complex and \( A \) is a finite cover of \( X \) by subcomplexes such that the intersection of any subset of \( A \) has trivial cohomology, then \( H^n(X) \cong H^n(N(A)) \) for all \( n \). The analogous nerve theorem for homology can be found in [4] for example, in which K. Brown notes that the theorem is "essentially due to Leray".

A sharper form of this homological nerve theorem is proven and used by R. Meshulam [11], which relaxes the condition that intersections of finite subsets of the cover \( A \) are homologically trivial, but only shows that \( H_j(X) \cong H_j(N(A)) \) for \( j \leq n \) for a particular \( n \).

The purpose of this note is to give a proof of the following theorem, which shows that the homological nerve theorem in [11] holds for the case where \( X \) is an arbitrary topological space and \( \mathcal{U} \) is an open cover of \( X \).

**Theorem 1.** Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be an open cover of a topological space \( X \), and let \( N \) be the nerve of this cover. Fix an integer \( k \in \mathbb{N} \). If \( H_j(\bigcap_{i \in \sigma} U_i) = 0 \) for all \( \sigma \in N^{(k)} \) and \( j \in \{ 0, \ldots, k - \dim \sigma \} \), then

1. \( H_j(X) \cong H_j(N) \) for all \( j \in \{ 0, \ldots, k \} \)
2. If \( H_{k+1}(N) \neq 0 \) then \( H_{k+1}(X) \neq 0 \).

In [11], Meshulam assumes that \( X \) is a finite simplicial complex, \( \mathcal{U} \) is a finite cover of \( X \) by subcomplexes, and takes homology to have coefficients in a field. By

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making minor adjustments to Meshulam’s proof, namely by using the homology—
and not cohomology—spectral sequence of a cover, we may drop all of these as-
sumptions, allowing $X$ to be an arbitrary space, $\mathcal{U}$ to be an open cover of $X$, and
the theorem holds for homology with arbitrary coefficient groups. The motivation
for considering such a generalization is to affirmatively answer a question posed by
Adams, Frick, and Virk [2].

**Question.** If $\mathcal{U}$ is a uniformly bounded open cover of a separable metric space
$X$, then do the Vietoris metric thickening $V^m(\mathcal{U})$ and the Vietoris complex $V(\mathcal{U})$
have the same homology groups?

If $\mathcal{U}$ is a cover of a space $X$, the Vietoris complex $V(\mathcal{U})$ is the simplicial complex
whose vertex set is $X$ and whose simplices are the finite subsets of $X$ contained in
some element of $\mathcal{U}$. If $X$ is a metric space and $\mathcal{U}$ is the collection of open subsets
of $X$ with diameter less than $r > 0$, then $V(\mathcal{U}) = VR(X;r)$ is the Vietoris-Rips complex. The Vietoris-Rips metric thickening $VR^m(X;r)$ was introduced in [11] and
later generalized by the Vietoris metric thickening $V^m(\mathcal{U})$ [2]. The Vietoris metric thickening $V^m(\mathcal{U})$ has the same underlying set as $V(\mathcal{U})$, but has a metric which
paves it a coarser topology than that of $V(\mathcal{U})$. See [2] for a precise definition. In
general, $V^m(\mathcal{U})$ is not a simplicial complex.

In [2], for any any $n \in \mathbb{N}$, the authors construct an open cover $\tilde{M}_\mathcal{U}$ of $V^m(\mathcal{U})$
that is good up to level $n$, that is, the intersection of any collection of at most $n$
sets from $\tilde{M}_\mathcal{U}$ is either empty or contractible. The authors of [2] remarked that
the above question could potentially be answered by using these covers in a Mayer-
Vietoris spectral sequence. The argument we present does just this, as we will
use Theorem 1 which is ultimately an application of the Mayer-Vietoris spectral
sequence.

**Answer to question.** Write $\mathcal{U} = \{U_i\}_{i \in \mathcal{I}}$ and let $\tilde{M}_\mathcal{U}$ be the open cover of $V^m(\mathcal{U})$
that is good up to level $n$, whose existence is guaranteed by [2]. Then $H_j(\bigcap_{i \in \sigma} U_i) \cong 0$ for all $\sigma \in N^{(n-1)}$ and $j \in \mathbb{N}$, in which case Theorem 1 then implies that
$H_j(V^m(\mathcal{U})) \cong H_j(N(\tilde{M}_\mathcal{U}))$ for all $j \leq n - 1$. It was shown in [2] that the $n$-skeleton
of $N(\tilde{M}_\mathcal{U})$ coincides with that of $N(\mathcal{U})$, hence $H_j(N(\tilde{M}_\mathcal{U})) \cong H_j(N(\mathcal{U})) \cong H_j(V(\mathcal{U}))$ for all $j \leq n - 1$, where the second isomorphism follows from Dowker
duality [5]. In total, we have that $H_j(V^m(\mathcal{U})) \cong H_j(V(\mathcal{U}))$ for all $j \leq n - 1$. Since
$n$ was arbitrary, we conclude that $V^m(\mathcal{U})$ and $V(\mathcal{U})$ have isomorphic homology groups. □

**Remark 2.** The condition that $X$ is separable was used in [2] to allow the authors
to apply a nerve theorem of Nagórkó [12] to conclude that $V^m(\mathcal{U})$ and $V(\mathcal{U})$ have
isomorphic homotopy groups. However, it was not used in our argument. Thus we
have shown that if $\mathcal{U}$ is a uniformly bounded open cover of a metric space $X$, then
$V^m(\mathcal{U})$ and $V(\mathcal{U})$ have isomorphic homology groups.

1. **MAYER-VIETORIS SPECTRAL SEQUENCE**

We give a brief description of the spectral sequence of a cover, which is sometimes
known as the Mayer-Vietoris spectral sequence. We use $S_n(X)$ to denote the group
of singular $n$-chains in a space $X$, and use $C_n(K)$ to denote the group of simplicial
$n$-chains in a simplicial complex $K$. For a simplicial complex $K$, we denote the
$n$-skeleton of $K$ by $K^{(n)}$ and the set of $n$-simplices of $K$ by $K_n$. Fix an open cover
\( \mathcal{U} = \{ U_i \}_{i \in \mathcal{I}} \) of \( X \) and to simplify the notation, let \( N = N(\mathcal{U}) \) be the nerve of \( \mathcal{U} \).

For a simplex \( \sigma \subset \mathcal{I} \) in \( N \), let \( U_\sigma \) denote the intersection \( U_\sigma = \cap_{i \in \sigma} U_i \).

Given the open cover \( \mathcal{U} \) there is an associated double complex \( (A, \partial', \partial'') \). A double complex is a collection of modules \( \{ A_{p,q} \}_{p,q \in \mathbb{Z}} \) along with two collections of homomorphisms

\[ \partial' : A_{p,q} \to A_{p-1,q} \quad \partial'' : A_{p,q} \to A_{p,q-1} \]

which satisfy \( \partial' \partial' = \partial'' \partial' = 0 \) and \( \partial' \partial'' = \partial'' \partial' \). Note that some authors instead require that \( \partial'' \partial' = -\partial' \partial'' \). To define the double complex associated to \( \mathcal{U} \), set \( A_{p,q} = \bigoplus_{\sigma \in N_p} S_q(U_\sigma) \). The vertical differentials \( \partial'' : A_{p,q} \to A_{p,q-1} \) are induced by the boundary maps \( \partial : S_q(U_\sigma) \to S_{q-1}(U_\sigma) \), and the horizontal differentials \( \partial' : A_{p,q} \to A_{p-1,q} \) are defined as follows.

Fix a total order on the vertices \( \{ v_i \}_{i \in \mathcal{I}} \) of \( N \) so that each simplex of \( N \) has a unique representation \( \sigma = [v_0, \ldots, v_p] \) for which \( v_0 < \cdots < v_p \). Then if \( \sigma = [v_0, \ldots, v_p] \) is a \( p \)-simplex with \( v_0 < \cdots < v_p \), define \( \partial_j \sigma \) to be the \((p-1)\)-simplex \( \partial_j \sigma = [v_0, \ldots, \hat{v}_j, \ldots, v_p] \) in which \( \hat{v}_j \) denotes that the vertex \( v_j \) is omitted. Since \( U_\sigma \subset U_{\partial_j \sigma} \), we have that \( \partial_j \) defines an inclusion

\[ S_q(U_\sigma) \to S_q(U_{\partial_j \sigma}) \to \bigoplus_{\tau \in N_{p-1}} S_q(U_\tau) \]

for each \( \sigma \). These inclusions induce maps \( \delta_j : \bigoplus_{\sigma \in N_p} S_q(U_\sigma) \to \bigoplus_{\tau \in N_{p-1}} S_q(U_\tau) \).

We then define \( \partial' : A_{p,q} \to A_{p-1,q} \) by setting \( \partial' = \sum_{i=0}^p (-1)^i \delta_i \). One may check that \( \partial' \partial'' = 0 \) by expanding out

\[ \partial' \partial'' = \sum_{k=0}^p \sum_{j=0}^{p-1} (-1)^{k+j} \delta_j \delta_k \]

and using the relation \( \delta_j \delta_k = \delta_{k-1} \delta_j \) if \( j < k \). Note that \( \partial' \partial'' = \partial'' \partial' \).

Given the double complex \( A \), we may form the total complex \( \text{Tot} A \), whose degree \( n \) term is \( (\text{Tot} A)_n = \bigoplus_{p+q=n} A_{p,q} \). The differential \( \partial \) of \( \text{Tot} A \) is defined by setting \( \partial(c) = \partial'(c) + (-1)^p \partial''(c) \) for \( c \in A_{p,q} \), for all \( p,q \in \mathbb{N} \).

**Figure 1.** The double complex \( (A, \partial', \partial'') \)
There are two natural filtrations $F'$ and $F''$ of $\text{Tot} A$ which give rise to the spectral sequences $E'$ and $E''$ respectively. The first filtration is given by $F'_k(\text{Tot} A)_n = \bigoplus_{p \leq k} A_{p,n-p}$ and the second filtration is $F''_k(\text{Tot} A)_n = \bigoplus_{q \leq k} A_{n-q,q}$. For details on the spectral sequence associated to a filtered chain complex, see [10]. Since $A$ is a first quadrant double complex, the spectral sequences $E'$ and $E''$ both converge to filtrations of $H_\bullet(\text{Tot} A)$. We will use the second spectral sequence to show that $H_\bullet(\text{Tot} A) \cong H_\bullet(X)$, which we then compare with the first spectral sequence to prove Theorem [11].

Let $E = E''$ be the second spectral sequence of the double complex $A$. The terms of the $E^1$ page are obtained by taking the homology of $A$ with respect to $\partial'$. To describe the $E^1$ page explicitly, we need the following proposition.

**Proposition 3.** Let $q \in \mathbb{N}$ and let $A_{\bullet,q}$ denote the chain complex

\[
\cdots \longrightarrow \bigoplus_{\sigma \in N_q} S_q(\sigma) \longrightarrow \bigoplus_{\sigma \in N_1} S_q(\sigma) \longrightarrow \bigoplus_{\sigma \in N_0} S_q(\sigma) \longrightarrow 0.
\]

Then $H_j(A_{\bullet,q}) \cong 0$ for all $j > 0$ and $H_0(A_{\bullet,q}) \cong S^q(X)$.

Here $S^q(X)$ denotes the group of singular $q$-chains in $X$ whose elements $\sum_{i=0}^m n_i \sigma_i$ satisfy the condition that each singular simplex $\sigma_i$ has image in an element of $\mathcal{W}$.

To prove Proposition 3, we provide a straightforward generalization of the proof in [11], pg. 166], which assumes that $X$ is a CW-complex and $\mathcal{W}$ is a cover of $X$ by subcomplexes. A similar proposition, but for cohomology, is proved by Frigerio and Maffei in [11]. A more general version of Proposition 3 is also proved by N. Ivanov [11].

**Proof.** We prove the proposition by giving an alternative characterization of the groups $\bigoplus_{\sigma \in N_q} S_q(U_\sigma)$. We begin by noting that $\bigoplus_{\sigma \in N_q} S_q(U_\sigma)$ has a basis $B$ consisting of pairs $(\sigma, f)$ where $\sigma$ is a $p$-simplex of $N$ and $f$ is a map $f : \Delta^q \rightarrow U_\sigma$. For any map $f : \Delta^q \rightarrow X$, let $N^f$ be the subcomplex of $N$ consisting of simplices $\sigma$ such that $\text{im}(f) \subset U_\sigma$. Then there is a bijection between $B$ and the set of pairs $(f, \sigma)$ where $f$ is an arbitrary map $f : \Delta^q \rightarrow X$ and $\sigma$ is a $p$-simplex of $N^f$. This is to say that for each $p$, there exists an isomorphism

\[
\bigoplus_{\sigma \in N_p} S_q(\sigma) \cong \bigoplus_{f : \Delta^q \rightarrow X} C_p(N^f).
\]

Moreover, these isomorphisms define an isomorphism of chain complexes

\[
\cdots \longrightarrow \bigoplus_{\sigma \in N_q} S_q(U_\sigma) \longrightarrow \bigoplus_{\sigma \in N_1} S_q(U_\sigma) \longrightarrow \bigoplus_{\sigma \in N_0} S_q(U_\sigma) \longrightarrow 0
\]

\[
\cdots \longrightarrow \bigoplus_{f : \Delta^q \rightarrow X} C_2(N^f) \longrightarrow \bigoplus_{f : \Delta^q \rightarrow X} C_1(N^f) \longrightarrow \bigoplus_{f : \Delta^q \rightarrow X} C_0(N^f) \longrightarrow 0
\]

where the differential in the bottom complex is induced by the boundary maps $C_p(N^f) \rightarrow C_{p-1}(N^f)$ on simplicial $p$-chains. Observe that for each $f : \Delta^q \rightarrow X$, the complex $N^f$ consists of all finite subsets of the set $\{i \in \mathcal{I} : \text{im}(f) \subset U_i\}$. Hence $N^f$ is either empty or contractible. Hence $H_p(N^f) \cong 0$ for all $p > 0$ and the homology groups of the bottom chain complex (and hence the top as well) are zero at each position except $\bigoplus_{f : \Delta^q \rightarrow X} C_0(N^f)$. Here we note that $H_0(N^f)$ is either 0 or $\mathbb{Z}$, depending on whether $N^f$ is empty or not, which in turn depends on whether $f$ has image in some element of $\mathcal{W}$. Thus we can take the set of maps $f : \Delta^q \rightarrow X$
which have image in some element of \( \mathcal{W} \) to be a basis for \( \bigoplus_{f_i} H_0(N_f) \). This implies that \( \bigoplus_{f_i} H_0(N_f) \cong S^\mathcal{W}(X) \), completing the proof. \( \square \)

Using Proposition 3 we see that the \( E^1 \) page of the second spectral sequence is of the form

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots \\
0 & S^\mathcal{W}_2(X) & 0 & 0 & \ldots \\
& \downarrow & \downarrow & \downarrow & \\
0 & S^\mathcal{W}_1(X) & 0 & 0 & \ldots \\
& \downarrow & \downarrow & \downarrow & \\
0 & S^\mathcal{W}_0(X) & 0 & 0 & \ldots \\
& \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & \ldots \\
\end{array}
\]

and where the differentials are induced by \( \partial'' \). Hence the second spectral sequence collapses at the \( E^2 \) page. Let \( H_q^\mathcal{W}(X) \) denote the \( q \)-th homology group of the chain complex \( S^\mathcal{W}_\bullet(X) \), which we note is isomorphic to \( H_q(X) \) since \( \mathcal{W} \) is an open cover. Then we see that \( E^2_{p,q} \cong H_q^\mathcal{W}(X) \cong H_q(X) \) for all \( q \in \mathbb{N} \) and \( E^2_{p,q} \cong 0 \) if \( p > 0 \). Hence the homology of the total complex of \( A \) is isomorphic to the homology of \( X \), i.e. \( H_q(\text{Tot } A) \cong H_\bullet(X) \).

### 2. Homological nerve theorem

We are now ready to prove Theorem 1. We remind the reader that our proof is simply a modification of the proof in [11], in which we use the homology spectral sequence of the cover \( \mathcal{W} \), rather than the cohomology spectral sequence.

**Proof of Theorem 1.** Given an open cover \( \mathcal{W} \) of a space \( X \), let \( N \) be the nerve of \( \mathcal{W} \), let \( A \) be the double complex associated to \( \mathcal{W} \), and let \( E = E^1 \) be the first spectral sequence of the double complex \( A \). The terms of the \( E^1 \) page of the first spectral sequence are obtained by taking the homology of \( A \) with respect to \( \partial' \). Hence \( E^1 \) is

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots \\
0 & \bigoplus_{\sigma \in N_0} H_2(U_\sigma) & \bigoplus_{\sigma \in N_1} H_2(U_\sigma) & \bigoplus_{\sigma \in N_2} H_2(U_\sigma) & \ldots \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \bigoplus_{\sigma \in N_0} H_1(U_\sigma) & \bigoplus_{\sigma \in N_1} H_1(U_\sigma) & \bigoplus_{\sigma \in N_2} H_1(U_\sigma) & \ldots \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \bigoplus_{\sigma \in N_0} H_0(U_\sigma) & \bigoplus_{\sigma \in N_1} H_0(U_\sigma) & \bigoplus_{\sigma \in N_2} H_0(U_\sigma) & \ldots \\
& \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & \ldots \\
\end{array}
\]

where the differentials are induced by \( \partial' \). For each \( m \in \mathbb{N} \), there exists a surjective map \( g_m : E^1_{m,0} \to C_m(N) \) defined as follows. For each \( \sigma \in N \), let \( P_\sigma \) denote the set of path components of \( U_\sigma \), identify \( H_0(U_\sigma) \) with \( \bigoplus_{\sigma \in P_\sigma} \mathbb{Z} \), and let \( f_\sigma : H_0(U_\sigma) \to \mathbb{Z} \) be the map which sends \((n_\iota)_{\iota \in P_\sigma}\) to the sum \( \sum_{\iota \in P_\sigma} n_\iota \). Then for each \( m \in \mathbb{N} \), let \( g_m : \bigoplus_{\sigma \in N_m} H_0(U_\sigma) \to \bigoplus_{\sigma \in N_m} \mathbb{Z} \) be the map induced by the collection \( \{f_\sigma : \sigma \in \)
N_m\). It is not too difficult to check that the collection \( \{g_m : m \in \mathbb{N}\} \) defines a morphism of chain complexes, \( g : E^1_{k,0} \to C_\ast(N) \).

Under the assumption that \( \widetilde{H}_j(U_\sigma) \cong 0 \) for all \( \sigma \in N^{(k)} \) and \( j \in \{0, \ldots, k - \dim \sigma\} \), we see that for all \( m \leq k \), the \( m \)-th antidiagonal of the \( E^1 \) page contains only one nontrivial term, \( \bigoplus_{\sigma \in N_m} H_0(U_\sigma) \). Moreover, \( g_m : \bigoplus_{\sigma \in N_m} H_0(U_\sigma) \to C_\ast(N) \) is an isomorphism for \( m \leq k \). Then from the commutative diagram

\[
\begin{array}{cccccc}
E^1_{k+2,0} & \rightarrow & E^1_{k+1,0} & \rightarrow & E^1_{k,0} & \rightarrow & E^1_{k-1,0} & \rightarrow & \cdots \\
\downarrow g_{k+2} & & \downarrow g_{k+1} & & \downarrow \cong & & \downarrow \cong & & \\
C_{k+2}(N) & \rightarrow & C_{k+1}(N) & \rightarrow & C_k(N) & \rightarrow & C_{k-1}(N) & \rightarrow & \cdots
\end{array}
\]

we immediately see that \( E^2_{m,0} \cong H_m(N) \) for all \( m \leq k - 1 \). Using the fact that \( g_{k+1} \) is surjective and \( g_k, g_{k-1} \) are isomorphisms, it is also straightforward to see that \( g_k \) induces an isomorphism \( E^2_{k,0} \cong H_k(N) \) and \( g_{k+1} \) induces a surjection \( E^2_{k+1,0} \to H_{k+1}(N) \). Note that for \( m \leq k \), the \( m \)-th antidiagonal of the \( E^2 \) page contains only one nontrivial term, \( E^2_{m,0} \), and that \( E^2_{p,q} \cong E^\infty_{p,q} \) for \( p+q \leq k \). Hence for \( 0 \leq m \leq k \), \( H_m(Tot A) \cong H_m(N) \). Consequently, \( H_m(N) \cong H_m(Tot A) \cong H_m(N) \) for all \( m \leq k \). Lastly since there is a surjection \( E^2_{k+1,0} \to H_{k+1}(N) \), if \( H_{k+1}(N) \neq 0 \), then we must also have \( E^\infty_{k+1,0} \neq 0 \). Since the differentials entering and leaving the term \( E^r_{k+1,0} \) are zero homomorphisms for all \( r \geq 2 \), we have \( E^\infty_{k+1,0} \cong E^2_{k+1,0} \), which in turn implies that \( H_{k+1}(X) \cong H_{k+1}(Tot A) \neq 0 \).

Remark 4. The fact that \( \mathcal{U} \) is an open cover is only used for the isomorphism \( H_\ast(\mathcal{U}) \cong H_\ast(X) \) which is used to establish \( H_\ast(\mathcal{U}) \cong H_\ast(Tot A) \). Hence Theorem[1] holds slightly more generally for any space \( X \) and cover \( \mathcal{U} \) such that \( H_\ast(\mathcal{U}) \cong H_\ast(X) \), for example if \( \mathcal{U} \) is a collection of sets whose interiors cover \( X \).

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