Fatou’s Lemma in Its Classic Form and Lebesgue’s Convergence
Theorems for Varying Measures with Applications to MDPs

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Abstract

The classic Fatou lemma states that the lower limit of a sequence of integrals of functions
is greater or equal than the integral of the lower limit. It is known that Fatou’s lemma for a
sequence of weakly converging measures states a weaker inequality because the integral of the
lower limit is replaced with integral of the lower limit in two parameters, where the second
parameter is the argument of the functions. This paper provides sufficient conditions when
Fatou’s lemma holds in its classic form for a sequence of weakly converging measures. The
functions can take both positive and negative values. The paper also provides similar results for
sequences of setwise converging measures. It also provides Lebesgue’s and monotone convergence
theorem for sequences of weakly and setwise converging measures. The obtained results are used
to prove broad sufficient conditions for the validity of optimality equations for average-costs
Markov decision processes.

1 Introduction

For a sequence of nonnegative measurable functions \( \{f_n\}_{n=1,2,...} \), Fatou’s lemma states the inequality
\[
\int_S \liminf_{n \to \infty} f_n(s) \mu(ds) \leq \liminf_{n \to \infty} \int_S f_n(s) \mu(ds).
\] (1.1)

Many problems in probability theory and its applications deal with sequences of probabilities or
measures converging in some sense rather than with a single probability or measure \( \mu \). Examples
of areas of applications include limit theorems [2], [15], [21, Chapter III], continuity properties of
stochastic processes [16], and stochastic control [5, 8, 10, 14].

If a sequence of measures \( \{\mu_n\}_{n=1,2,...} \) converging setwise to a measure \( \mu \) is considered instead
of a single measure \( \mu \), then equality (1.1) holds with the measure \( \mu \) in its right-hand side replaced
with the measures \( \mu_n \) [18, p. 231]. However, for a sequence of measures \( \{\mu_n\}_{n=1,2,...} \) converging
weakly to a measure \( \mu \), the weaker inequality
\[
\int_S \liminf_{n \to \infty, s' \to s} f_n(s') \mu(ds) \leq \liminf_{n \to \infty} \int_S f_n(s) \mu_n(ds).
\] (1.2)

holds. Studies of Fatou’s lemma for weakly converging probabilities were started by Serfozo [20]
and continued in [4] [6]. For a sequence of measures converging in total variation, Feinberg et al. [9]
obtained the uniform Fatou’s lemma, which is a more general fact than Fatou’s lemma.

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This paper describes sufficient conditions ensuring that Fatou’s lemma holds in its classic form for a sequence of weakly converging measures. In other words, we provide sufficient conditions under which the double lower limit in the left-hand side of (1.2) can be replaced with the standard lower limit for \( n \to \infty \). We consider the sequence of functions that can take both positive and negative values. In addition to the results for weakly converging measures, we provide parallel results for setwise converging measures. We also investigate the validity of Lebesgue’s and monotone convergence theorems for sequences of weakly and setwise converging measures. The results are applied to Markov decision processes with long-term average costs per unit time, for which we provide general conditions for the validity of optimality equations.

Section 2 describes the three types of convergence of measures: weak convergence, setwise convergence, and convergence in total variation. Section 3 describes conditions under which the double lower limit in the left-hand side of (1.2) can be replaced with the standard lower limit for sequences of weakly converging measures. In other words, we provide sufficient conditions under which the double lower limit in the left-hand side of (1.2) is equal to the standard lower limit. Section 4 describes sufficient conditions for the validity of Fatou’s lemma in its classic form for a sequence of weakly converging measures. This section also provides results for sequences of measures converging setwise. Sections 5 and 6 describe Lebesgue’s and monotone convergence theorems for weakly and setwise converging measures. Sections 7 deals with applications to Markov decision processes (MDPs).

2 Known Formulations of Fatou’s Lemmas for Varying Measures

Let \((\mathbb{S}, \Sigma)\) be a measurable space, \(\mathcal{M}(\mathbb{S})\) be the family of all nonnegative finite measures on \((\mathbb{S}, \Sigma)\), and \(\mathcal{P}(\mathbb{S})\) be the family of all probability measures on \((\mathbb{S}, \Sigma)\). When \(\mathbb{S}\) is a metric space, we always consider \(\Sigma := \mathcal{B}(\mathbb{S})\), where \(\mathcal{B}(\mathbb{S})\) is the Borel \(\sigma\)-field on \(\mathbb{S}\). Let \(\mathbb{R}\) be the real line and \(\overline{\mathbb{R}} := [-\infty, +\infty]\). We denote by \(I\{A\}\) the indicator of the event \(A\).

Throughout this paper, we deal with integrals of functions that can take both positive and negative values. An integral \(\int_{\mathbb{S}} f(s)\mu(ds)\) of a measurable \(\overline{\mathbb{R}}\)-valued function \(f\) on \(\mathbb{S}\) with respect to a measure \(\mu\) is defined if

\[
\min\{\int_{\mathbb{S}} f^+(s)\mu(ds), \int_{\mathbb{S}} f^-(s)\mu(ds)\} < +\infty,
\]

where \(f^+(s) = \max\{f(s), 0\}, f^-(s) = -\min\{f(s), 0\}, s \in \mathbb{S}\). If (2.1) holds then the integral is defined as

\[
\int_{\mathbb{S}} f(s)\mu(ds) = \int_{\mathbb{S}} f^+(s)\mu(ds) - \int_{\mathbb{S}} f^-(s)\mu(ds).
\]

All the integrals in the assumptions of the following lemmas, theorems, and corollaries are assumed to be defined. For \(\mu \in \mathcal{M}(\mathbb{S})\) consider the vector space \(L^1(\mathbb{S}; \mu)\) of all measurable functions \(f : \mathbb{S} \mapsto \overline{\mathbb{R}}\), whose absolute values have finite integrals, that is, \(\int_{\mathbb{S}} |f(s)|\mu(ds) < +\infty\).

We recall the definitions of the following three types of convergence of measures: weak convergence, setwise convergence, and convergence in total variation.

**Definition 2.1 (Weak convergence).** A sequence of measures \(\{\mu_n\}_{n=1,2,...}\) on a metric space \(\mathbb{S}\) converges weakly to a finite measure \(\mu\) on \(\mathbb{S}\) if, for each bounded continuous function \(f\) on \(\mathbb{S}\),

\[
\int_{\mathbb{S}} f(s)\mu_n(ds) \to \int_{\mathbb{S}} f(s)\mu(ds) \quad \text{as} \quad n \to \infty.
\]
Remark 2.2. Definition 2.1 implies that $\mu_n(S) \to \mu(S) \in \mathbb{R}$ as $n \to \infty$. Therefore, if the sequence $\{\mu_n\}_{n=1,2,...}$ converges weakly to $\mu \in \mathcal{M}(S)$, then there exists $N = 1,2,...$ such that $\{\mu_n\}_{n=N,N+1,...} \subset \mathcal{M}(S)$.

**Definition 2.3** (Setwise convergence). A sequence of measures $\{\mu_n\}_{n=1,2,...}$ on a measurable space $(S, \Sigma)$ converges setwise to a measure $\mu$ on $(S, \Sigma)$ if for each $C \in \Sigma$

$$\mu_n(C) \to \mu(C) \quad \text{as } n \to \infty.$$ 

**Remark 2.4.** A sequence of measures $\{\mu_n\}_{n=1,2,...}$ converges setwise to a finite measure $\mu$ as $n \to \infty$ if and only if (2.2) holds for each bounded measurable function $f$ on $S$.

**Definition 2.5** (Convergence in total variation). A sequence of finite measures $\{\mu_n\}_{n=1,2,...}$ on a measurable space $(S, \Sigma)$ converges in total variation to a measure $\mu$ on $(S, \Sigma)$ if

$$\sup \left\{ \left| \int_S f(s) \mu_n(ds) - \int_S f(s) \mu(ds) \right| : f : S \mapsto [-1,1] \text{ is measurable} \right\} \to 0 \quad \text{as } n \to \infty.$$ 

**Remark 2.6.** As follows from Definitions 2.1, 2.3 and 2.5 if a sequence of finite measures $\{\mu_n\}_{n=1,2,...}$ on a measurable space $(S, \Sigma)$ converges in total variation to a measure $\mu$ on $(S, \Sigma)$, then $\{\mu_n\}_{n=1,2,...}$ converges setwise to $\mu$ as $n \to \infty$ and the measure $\mu$ is finite. The latter follows from $|\mu_n(S) - \mu(S)| < +\infty$ when $n \geq N$ for some $N = 1,2,...$. Furthermore, if a sequence of measures $\{\mu_n\}_{n=1,2,...}$ on a metric space $S$ converges setwise to a finite measure $\mu$ on $S$, then $\{\mu_n\}_{n=1,2,...}$ converges weakly to $\mu$ as $n \to \infty$. As follows from Definition 2.5 and Remark 2.4, convergence in total variation can be viewed as uniform setwise convergence. As follows from [8, Theorem 2.5(iv)] and Definition 2.1, convergence in total variation can also be viewed as uniform weak convergence if $S$ is a metric space.

Recall the following definitions of the uniform and asymptotic uniform integrability of sequences of functions.

**Definition 2.7.** The sequence $\{f_n\}_{n=1,2,...}$ of measurable $\overline{\mathbb{R}}$-valued functions is called

- **uniform integrable (u.i.) with regard to (w.r.t.)** a sequence of measures $\{\mu_n\}_{n=1,2,...}$ if

  $$\lim_{K \to +\infty} \sup_{n=1,2,...} \int_S |f_n(s)| I\{s \in S : |f_n(s)| \geq K\} \mu_n(ds) = 0; \quad (2.3)$$

- **asymptotically uniform integrable (a.u.i.) w.r.t.** a sequence of measures $\{\mu_n\}_{n=1,2,...}$ if

  $$\lim_{K \to +\infty} \limsup_{n \to \infty} \int_S |f_n(s)| I\{s \in S : |f_n(s)| \geq K\} \mu_n(ds) = 0. \quad (2.4)$$

If $\mu_n = \mu \in \mathcal{M}(S)$ for each $n = 1,2,...$, then an (a.)u.i. w.r.t. $\{\mu_n\}_{n=1,2,...}$ sequence $\{f_n\}_{n=1,2,...}$ is called (a.)u.i. For a single finite measure $\mu$, the definition of an a.u.i. sequence of functions (random variables in the case of a probability measure $\mu$) coincides with the corresponding definition broadly used in the literature; see, e.g., [22, p. 17]. Also, for a single fixed finite measure, the definition of a u.i. sequence of functions is consistent with the classic definition of a family $\mathcal{H}$ of u.i. functions. We say that a function $f$ is (a.)u.i. w.r.t. $\{\mu_n\}_{n=1,2,...}$ if the sequence $\{f,f,...\}$ is (a.)u.i. w.r.t. $\{\mu_n\}_{n=1,2,...}$. A function $f$ is u.i. w.r.t. a family $\mathcal{N}$ of measures if

$$\lim_{K \to +\infty} \sup_{\mu \in \mathcal{N}} \int_S |f(s)| I\{s \in S : |f(s)| \geq K\} \mu(ds) = 0.$$
Theorem 2.8 (Equivalence of u.i. and a.u.i.; Feinberg et al. [4 Theorem 2.2]). Let \((S, \Sigma)\) be a measurable space, \(\{\mu_n\}_{n=1,2,...} \subset \mathcal{M}(S)\), and \(\{f_n\}_{n=1,2,...} \subset \mathcal{R}(S)\) be a sequence of measurable \(\mathbb{R}\)-valued functions on \(S\). Then there exists \(N = 1, 2, \ldots\) such that \(\{f_n\}_{n=N,N+1,...} \) is u.i. w.r.t. \(\{\mu_n\}_{n=N,N+1,...}\) iff \(\{f_n\}_{n=1,2,...} \) is a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,...}\).

Fatou’s lemma (FL) for weakly converging probabilities was introduced in Serfoso [20] and generalized in [4, 6].

Theorem 2.9 (FL for weakly converging measures; [4 Theorem 2.4 and Corollary 2.7]). Let \(S\) be a metric space, \(\{\mu_n\}_{n=1,2,...} \subset \mathcal{M}(S)\), and \(\{f_n\}_{n=1,2,...} \subset \mathcal{R}(S)\) be a sequence of measurable \(\mathbb{R}\)-valued functions on \(S\). Assume that one of the following two conditions holds:

(i) \(\{f^-_n\}_{n=1,2,...} \) is a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,...}\);

(ii) there exists a sequence of measurable real-valued functions \(\{g_n\}_{n=1,2,...} \) on \(S\) such that \(f_n(s) \geq g_n(s)\) for all \(n = 1, 2, \ldots\) and \(s \in S\), and

\[-\infty < \int_S \liminf_{n \to \infty} g_n(s) \mu(ds) \leq \liminf_{n \to \infty} \int_S g_n(s) \mu_n(ds).\]  \hspace{1cm} (2.5)

Then inequality (1.2) holds.

Recall that FL for setwise converging measures is stated in Royden [18, p. 231] for nonnegative functions. FL for setwise converging probabilities is stated in Feinberg et al. [6 Theorem 4.1] for functions taking positive and negative values.

Theorem 2.10 (FL for setwise converging probabilities; [6 Theorem 4.1]). Let \((S, \Sigma)\) be a measurable space, a sequence of measures \(\{\mu_n\}_{n=1,2,...} \subset \mathcal{P}(S)\) converge setwise to \(\mu \in \mathcal{P}(S)\), and \(\{f_n\}_{n=1,2,...} \subset \mathcal{R}(S)\) be a sequence of measurable real-valued functions on \(S\). Then the inequality

\[\int_S \liminf_{n \to \infty} f_n(s) \mu(ds) \leq \liminf_{n \to \infty} \int_S f_n(s) \mu_n(ds)\]  \hspace{1cm} (2.6)

holds, if there exists a sequence of measurable real-valued functions \(\{g_n\}_{n=1,2,...} \) on \(S\) such that \(f_n(s) \geq g_n(s)\) for all \(n = 1, 2, \ldots\) and \(s \in S\), and

\[-\infty < \int_S \limsup_{n \to \infty} g_n(s) \mu(ds) \leq \limsup_{n \to \infty} \int_S g_n(s) \mu_n(ds),\]  \hspace{1cm} (2.7)

Under the condition that \(\{\mu_n\}_{n=1,2,...} \subset \mathcal{M}(S)\) converges in total variation to \(\mu \in \mathcal{M}(S)\), Feinberg et al. [4 Theorem 2.1] established uniform FL, which is a stronger statement than the classic FL.

Theorem 2.11 (Uniform FL for measures converging in total variation; [6 Theorem 2.1]). Let \((S, \Sigma)\) be a measurable space, a sequence of measures \(\{\mu_n\}_{n=1,2,...} \subset \mathcal{M}(S)\) converge in total variation to a measure \(\mu \in \mathcal{M}(S)\), and \(\{f_n, f\}_{n=1,2,...} \subset \mathcal{R}(S)\) be a sequence of measurable \(\mathbb{R}\)-valued functions on \(S\). Assume that \(f \in L^1(S;\mu)\) and \(f_n \in L^1(S;\mu_n)\) for each \(n = 1, 2, \ldots\). Then the inequality

\[\liminf_{n \to \infty} \inf_{C \in \Sigma} \left( \int_C f_n(s) \mu_n(ds) - \int_C f(s) \mu(ds) \right) \geq 0\]  \hspace{1cm} (2.8)

holds if and only if the following two statements hold:

(i) for each \(\varepsilon > 0\) \(\mu\{s \in S : f_n(s) \leq f(s) - \varepsilon\} \to 0\) as \(n \to \infty\), and, therefore, there exists a subsequence \(\{f_{n_k}\}_{k=1,2,...} \subset \{f_n\}_{n=1,2,...}\) such that \(f(s) \leq \liminf_{k \to \infty} f_{n_k}(s)\) for \(\mu\)-a.e. \(s \in S\);

(ii) \(\{f^-\}_{n=1,2,...} \) is a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,...}\).
3 Semi-Convergence Conditions for Sequences of Functions

Let \((S, \Sigma)\) be a measurable space, \(\mu\) be a measure on \((S, \Sigma)\), and \(\{f_n, f\}_{n=1,2,...}\) be a sequence of measurable \(\mathbb{R}\)-valued functions. In this subsection we establish the notions of lower and upper semi-convergence in measure \(\mu\) (see Definitions 3.5 and 3.6) for a sequences of functions \(\{f_n\}_{n=1,2,...}\) defined on a measurable space \(S\). Then, under the assumption that \(S\) is a metric space, we examine necessary and sufficient conditions for the following equalities (see Theorem 3.14, Corollary 3.17, and Example 3.16):

\[
\liminf_{n \to \infty, s' \to s} f_n(s') = \liminf_{n \to \infty} f_n(s), \tag{3.1}
\]

\[
\lim_{n \to \infty, s' \to s} f_n(s') = \lim_{n \to \infty} f_n(s), \tag{3.2}
\]

which improve the statements of FL and Lebesgue’s convergence theorem for weakly converging measures; see Theorem 4.11 and Corollary 5.1. For example, these equalities are important for approximating average-cost relative value functions for MDPs with weakly continuous transition probabilities by discounted relative value functions; see Section 7. For this purpose we introduce the notions of lower and upper semi-equicontinuous families of functions; see Definitions 3.9 and 3.10. Finally, we provide the sufficient conditions for lower semi-equicontinuity; see Definition 3.2 and Corollary 3.18.

Remark 3.1. Since

\[
\liminf_{n \to \infty, s' \to s} f_n(s') \leq \liminf_{n \to \infty} f_n(s), \tag{3.3}
\]

(3.1) is equivalent to the inequality

\[
\liminf_{n \to \infty} f_n(s) \leq \liminf_{n \to \infty, s' \to s} f_n(s'). \tag{3.4}
\]

To provide sufficient conditions for (3.1) we introduce the definitions of uniform semi-convergence.

Definition 3.2 (Uniform semi-convergence from below). A sequence of real-valued functions \(\{f_n\}_{n=1,2,...}\) on \(S\) semi-converges uniformly from below to a real-valued function \(f\) on \(S\) if for each \(\varepsilon > 0\) there exists \(N = 1, 2, \ldots\) such that

\[
f_n(s) > f(s) - \varepsilon, \tag{3.5}
\]

for each \(s \in S\) and \(n = N, N+1, \ldots\).

Definition 3.3 (Uniform semi-convergence from above). A sequence of real-valued functions \(\{f_n\}_{n=1,2,...}\) on \(S\) semi-converges uniformly from above to a real-valued function \(f\) on \(S\) if \(-\{f_n\}_{n=1,2,...}\) semi-converges uniformly from below to \(-f\) on \(S\).

Remark 3.4. A sequence \(\{f_n\}_{n=1,2,...}\) converges uniformly to \(f\) on \(S\) if and only if it uniformly semi-converges from below and from above.

Let us consider the following definitions of semi-convergence in measure.

Definition 3.5 (Lower semi-convergence in measure). A sequence of measurable \(\mathbb{R}\)-valued functions \(\{f_n\}_{n=1,2,...}\) lower semi-converges to a measurable real-valued function \(f\) in measure \(\mu\) if for each \(\varepsilon > 0\)

\[
\mu(\{s \in S : f_n(s) \leq f(s) - \varepsilon\}) \to 0 \text{ as } n \to \infty.
\]
**Definition 3.6** (Upper semi-convergence in measure). A sequence of measurable \( \mathbb{R} \)-valued functions \( \{f_n\}_{n=1,2,...} \) upper semi-converges to a measurable real-valued function \( f \) in measure \( \mu \) if \( \{-f_n\}_{n=1,2,...} \) lower semi-converges to \( -f \) in measure \( \mu \), that is, for each \( \varepsilon > 0 \)

\[
\mu(\{s \in S : f_n(s) \geq f(s) + \varepsilon\}) \to 0 \text{ as } n \to \infty.
\]

**Remark 3.7.** A sequence of measurable \( \mathbb{R} \)-valued functions \( \{f_n\}_{n=1,2,...} \) converges to a measurable real-valued function \( f \) in measure \( \mu \), that is, for each \( \varepsilon > 0 \)

\[
\mu(\{s \in S : |f_n(s) - f(s)| \geq \varepsilon\}) \to 0 \text{ as } n \to \infty,
\]

if and only if this sequence of functions both lower and upper semi-converges to \( f \) in measure \( \mu \).

**Remark 3.8.** Observe that if

\[f(s) \leq \liminf_{n \to \infty} f_n(s) \quad (f(s) \geq \limsup_{n \to \infty} f_n(s), \text{ or } f(s) = \lim_{n \to \infty} f_n(s)) \text{ respectively} \quad \text{for } \mu\text{-a.e. } s \in S,
\]

then \( \{f_n\}_{n=1,2,...} \) lower semi-converges (upper semi-converges, or converges respectively) to \( f \) in measure \( \mu \). Visa versa, Feinberg et al. [9, Lemma 3.1] implies that if \( \{f_n\}_{n=1,2,...} \) lower semi-converges (upper semi-converges, or converges respectively) to \( f \) in measure \( \mu \), then there exists a subsequence \( \{f_{n_k}\}_{k=1,2,...} \subset \{f_n\}_{n=1,2,...} \) such that

\[f(s) \leq \liminf_{k \to \infty} f_{n_k}(s) \quad (f(s) \geq \limsup_{k \to \infty} f_{n_k}(s), \text{ or } f(s) = \lim_{k \to \infty} f_{n_k}(s)) \text{ respectively} \quad \text{for } \mu\text{-a.e. } s \in S.
\]

Now let \( S \) be a metric space and \( B_\delta(s) \) be the open ball of radius \( \delta > 0 \) centered at \( s \in S \) in the space \( S \). We consider the notions of lower and upper semi-equicontinuity for a sequence of functions.

**Definition 3.9** (Lower semi-equicontinuity). A sequence \( \{f_n\}_{n=1,2,...} \) of real-valued functions on a metric space \( S \) is called lower semi-equicontinuous at the point \( s \in S \) if for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[f_n(s') > f_n(s) - \varepsilon \quad \text{for all } s' \in B_\delta(s) \text{ and for all } n = 1, 2, \ldots.
\]

The sequence \( \{f_n\}_{n=1,2,...} \) is called lower semi-equicontinuous (on \( S \)) if it is lower semi-equicontinuous at all \( s \in S \).

**Definition 3.10** (Upper semi-equicontinuity). A sequence \( \{f_n\}_{n=1,2,...} \) of real-valued functions on a metric space \( S \) is called upper semi-equicontinuous at the point \( s \in S \) (on \( S \)) if the sequence \( \{-f_n : n = 1, 2, \ldots\} \) is lower semi-equicontinuous at the point \( s \in S \) (on \( S \)).

**Remark 3.11.** We recall that a function \( f : S \mapsto \mathbb{R} \) is lower (upper) semi-continuous at a point \( s \in S \) if and only if the sequence \( \{f\} \) is lower (upper) semi-continuous at the point \( s \in S \). A function is lower (upper) semi-equicontinuous on \( S \) if it is lower (upper) semi-continuous at all the points \( s \in S \). As follows from Definitions 3.9 and 3.10 if a sequence of real-valued functions on a metric space \( S \) is lower (upper) semi-equicontinuous at the point \( s \in S \), then each function from this sequence is lower (upper) semi-continuous at the point \( s \).

Recall the definition of equicontinuity of a sequence of functions; see e.g. Royden [18, p. 177].

**Definition 3.12** (Equicontinuity). A sequence \( \{f_n\}_{n=1,2,...} \) of real-valued functions on a metric space \( S \) is called equicontinuous at the point \( s \in S \) (on \( S \)) if this sequence is both lower and upper semi-equicontinuous at the point \( s \in S \) (on \( S \)).
Theorem 3.14 states necessary and sufficient conditions for equality (3.1). This theorem and Corollary 3.17 generalize Feinberg and Liang [12, Lemma 3.3], where equicontinuity was considered.

**Lemma 3.13.** Let \( \{f_n\}_{n=1,2,...} \) be a pointwise nondecreasing sequence of lower semi-continuous \( \mathbb{R} \)-valued functions on a metric space \( S \). Then

\[
\liminf_{n \to \infty, s' \to s} f_n(s') = \lim_{n \to \infty} f_n(s), \quad s \in S.
\]

**Proof.** For each \( s \in S \),

\[
\liminf_{n \to \infty, s' \to s} f_n(s') = \sup \liminf_{k \to \infty} f_k(s') = \sup \liminf_{n \to \infty} f_n(s') = \sup f_n(s) = \lim_{n \to \infty} f_n(s),
\]

where the first equality follows from the definition of \( \liminf \), the third one follows from the lower semi-continuity of the function \( f_n \), and the second and the last equalities hold because the sequences \( \{f_n\}_{n=1,2,...} \) are pointwise nondecreasing. Hence, (3.6) holds. \( \square \)

**Theorem 3.14** (Necessary and sufficient conditions for (3.1)). Let \( \{f_n\}_{n=1,2,...} \) be a sequence of real-valued functions on a metric space \( S \), and let \( s \in S \). Then the following statements hold:

(i) if the sequence of functions \( \{f_n\}_{n=1,2,...} \) is lower semi-equicontinuous at \( s \), then each function \( f_n, n = 1,2,\ldots, \) is lower semi-continuous at \( s \) and equality (3.1) holds;

(ii) if \( \{f_n\}_{n=1,2,...} \) is the sequence of lower semi-continuous functions satisfying (3.1) and \( \{f_n(s)\}_{n=1,2,...} \) is a converging sequence, that is,

\[
\liminf_{n \to \infty} f_n(s) = \limsup_{n \to \infty} f_n(s),
\]

then the sequence \( \{f_n\}_{n=1,2,...} \) is lower semi-equicontinuous at \( s \).

**Remark 3.15.** Assumption (3.7) in Theorem 3.14(ii) is essential; see Example 3.16 below. Without this assumption, the remaining conditions of Theorem 3.14(ii) imply only the existence of a subsequence \( \{f_{n_k}\}_{k=1,2,...} \subset \{f_n\}_{n=1,2,...} \) such that \( \{f_{n_k}\}_{k=1,2,...} \) is lower semi-equicontinuous at \( s \). This is true because every subsequence \( \{f_{n_k}\}_{k=1,2,...} \) satisfying \( \lim_{k \to \infty} f_{n_k}(s) = \liminf_{n \to \infty} f_n(s) \) is lower semi-equicontinuous at \( s \) in view of Theorem 3.14(ii) since (3.7) holds for such subsequences. A counterexample, when the sequence \( \{f_n\}_{n=1,2,...} \) is not lower semi-equicontinuous at \( s \), is provided in Example 3.16.

**Example 3.16.** Consider \( S := [-1,1] \) endowed with the standard Euclidean metric and

\[
f_n(t) := \begin{cases} 
0, & \text{if } n = 2k - 1, \\
\max\{1 - n|t|, 0\}, & \text{if } n = 2k, \quad k = 1,2,\ldots, \quad t \in S.
\end{cases}
\]

Each function \( f_n, n = 1,2,\ldots, \) is nonnegative and continuous on \( S \). Equality (3.1) holds because

\[
0 \leq \liminf_{n \to \infty, s' \to 0} f_n(s') \leq \liminf_{n \to \infty} f_n(0) = f_{2k-1}(0) = 0, \quad k = 1,2,\ldots.
\]

Equality (3.7) does not hold because

\[
\limsup_{n \to \infty} f_n(0) = 1 > 0 = \liminf_{n \to \infty} f_n(0),
\]

where the first equality holds because \( f_{2k}(0) = 1 \) for each \( k = 1,2,\ldots, \) and the second equality holds because \( f_{2k-1}(0) = 0 \) for each \( k = 1,2,\ldots. \) The sequence of functions \( \{f_n\}_{n=1,2,...} \) is not lower semi-equicontinuous at \( s = 0 \) because \( f_{2k}(\frac{1}{2k}) = 0 < \frac{1}{2} = f_{2k}(0) - \frac{1}{2} \) for each \( k = 1,2,\ldots. \) Therefore, the conclusion of Theorem 3.14(ii) does not hold, and assumption (3.7) is essential. \( \square \)
Proof of Theorem 3.14. (i) According to Remark 3.11, the lower semi-continuity at \( s \) of each function \( f_n, n = 1, 2, \ldots \), follows from lower semi-equicontinuity of \( \{f_n\}_{n=1,2,\ldots} \) at \( s \). Thus, to prove statement (i) it is sufficient to verify (3.4), which is equivalent to (3.4) because of Remark 3.11.

Let us prove (3.4). Fix an arbitrary \( \varepsilon > 0 \). According to Definition 3.9 there exists \( \delta(\varepsilon) > 0 \) such that for each \( n = 1, 2, \ldots \) and \( s' \in B_{\delta(\varepsilon)}(s) \)

\[
f_n(s') \geq f_n(s) - \varepsilon.
\]

(3.8) implies

\[
\liminf_{n \to \infty, s' \to s} f_n(s') = \sup_{n \geq 1} \inf_{k \geq n, s' \in B_\delta(s)} f_k(s') \geq \inf_{k \geq n} f_k(s'),
\]

(3.9)

the inequality in (3.9) holds because \( \{\delta(\varepsilon)\} \subset \{\delta : \delta > 0\} \), and the inequality in (3.10) follows from (3.8) and (3.9). Then, inequality (3.4) follows from (3.10) since \( \varepsilon > 0 \) is arbitrary. Statement (i) is proved.

(ii) We prove statement (ii) by contradiction. Assume that the sequence of functions \( \{f_n\}_{n=1,2,\ldots} \) is not lower semi-equicontinuous at \( s \). Then there exist \( \varepsilon^* > 0 \), a sequence \( \{s_n\}_{n=1,2,\ldots} \) converging to \( s \), and a sequence \( \{n_k\}_{k=1,2,\ldots} \subset \{1,2,\ldots\} \) such that

\[
f_{n_k}(s_k) \leq f_{n_k}(s) - \varepsilon^*.
\]

(3.11)

If a sequence \( \{n_k\}_{k=1,2,\ldots} \) is bounded (by a positive integer \( C \)), then (3.11) contradicts to lower semi-continuity of each function \( f_n, n = 1, 2, \ldots, C \). Otherwise, without loss of generality, we may assume that the sequence \( \{n_k\}_{k=1,2,\ldots} \) is strictly increasing. Therefore, (3.11) and (3.7) imply that

\[
\liminf_{n \to \infty, s' \to s} f_n(s') \leq \lim_{n \to \infty} f_n(s) - \varepsilon^*.
\]

This is a contradiction to (3.11). Hence, the sequence of functions \( \{f_n\}_{n=1,2,\ldots} \) is lower semi-equicontinuous at \( s \).

Let us investigate necessary and sufficient conditions for equality (3.2).

Corollary 3.17. Let \( \{f_n\}_{n=1,2,\ldots} \) be a sequence of real-valued functions on a metric space \( S \) and \( s \in S \). If \( \{f_n(s)\}_{n=1,2,\ldots} \) is a convergent sequence, that is, \( 3.7 \) holds, then the sequence of functions \( \{f_n\}_{n=1,2,\ldots} \) is equicontinuous at \( s \) if and only if each function \( f_n, n = 1, 2, \ldots, \), is continuous at \( s \) and equality (3.2) holds.

Proof. Corollary 3.17 directly follows from Theorem 3.14 applied twice to the families \( \{f_n\}_{n=1,2,\ldots} \) and \( \{-f_n\}_{n=1,2,\ldots} \).

In the following corollary we establish sufficient conditions for lower semi-equicontinuity.

Corollary 3.18 (Sufficient conditions for lower semi-equicontinuity). Let \( S \) be a metric space and \( \{f_n\}_{n=1,2,\ldots} \) be a sequence of real-valued lower semi-continuous functions on \( S \) semi-converging uniformly from below to a real-valued lower semi-continuous function \( f \) on \( S \). If the sequence \( \{f_n\}_{n=1,2,\ldots} \) converges pointwise to \( f \) on \( S \), then \( \{f_n\}_{n=1,2,\ldots} \) is lower semi-equicontinuous on \( S \).

Proof. If inequality (3.3) holds for all \( s \in S \), then Remark 3.4 and Theorem 3.14(ii) imply that \( \{f_n\}_{n=1,2,\ldots} \) is lower semi-equicontinuous on \( S \) because the sequence of functions \( \{f_n\}_{n=1,2,\ldots} \) converges pointwise to \( f \) on \( S \). Therefore, to finish the proof, let us prove that (3.4) holds for each
s ∈ S. Indeed, the uniform semi-convergence from below of \( \{f_n\}_{n=1,2,...} \) to \( f \) on \( S \) implies that for an arbitrary \( \varepsilon > 0 \)

\[
\liminf_{n \to \infty, s' \to s} f_n(s') \geq f(s) - \varepsilon,
\]

for each \( s \in S \). Since \( \varepsilon > 0 \) is arbitrarily and \( f(s) = \lim_{n \to \infty} f_n(s), s \in S \), equality (3.12) follows from (3.4).

Let \( S \) be a compact metric space. The Ascoli theorem (see [14, p. 96] or [16, p. 179]) implies that a sequence of real-valued continuous functions \( \{f_n\}_{n=1,2,...} \) on \( S \) converges uniformly on \( S \) to a continuous real-valued function \( f \) on \( S \) if and only if \( \{f_n\}_{n=1,2,...} \) is equicontinuous and this sequence converges pointwise to \( f \) on \( S \). According to Corollary 3.18, a sequence of real-valued lower semi-continuous functions \( \{f_n\}_{n=1,2,...} \) on \( S \), converging pointwise to a real-valued lower semi-continuous function \( f \) on \( S \), is lower semi-equicontinuous on \( S \) if \( \{f_n\}_{n=1,2,...} \) semi-converges uniformly from below to \( f \) on \( S \). Example 3.19 illustrates that the converse statement to Corollary 3.18 does not hold in the general case; that is, there is a lower semi-equicontinuous sequence \( \{f_n\}_{n=1,2,...} \) of continuous functions on \( S \) converging pointwise to a lower semi-continuous function \( f \) such that \( \{f_n\}_{n=1,2,...} \) does not semi-converge uniformly from below to \( f \) on \( S \).

Example 3.19. Define \( S := [0,1] \) endowed with the standard Euclidean metric, \( f(s) := I\{s \neq 0\} \), and for \( s \in S \)

\[
f_n(s) := \begin{cases} ns, & \text{if } s \in [0, \frac{1}{n}], \\ 1, & \text{otherwise.} \end{cases}
\]

Then the functions \( f_n, n = 1,2,\ldots \), are continuous on \( S \), the function \( f \) is lower semi-continuous on \( S \), and \( f_n \) converge pointwise to \( f \) on \( S \). In addition, the sequence of functions \( \{f_n\}_{n=1,2,...} \) is lower semi-equicontinuous because for each \( \varepsilon > 0 \) and \( s \in S \), (i) if \( s > 0 \), then there exists \( \delta(s,\varepsilon) = \min\{s - 1/(\lfloor\frac{1}{s}\rfloor + 1), \varepsilon/\lfloor\frac{1}{s}\rfloor\} \) such that \( f_n(s') \geq f_n(s) - \varepsilon \) for all \( n = 1,2,\ldots \) and \( s' \in B_{\delta(s,\varepsilon)}(s) \); and (ii) if \( s = 0 \), then \( f_n(s') \geq 0 = f_n(0) \) for all \( n = 1,2,\ldots \) and \( s' \in S \). The uniform semi-convergence from below of \( \{f_n\}_{n=1,2,...} \) to \( f \) does not hold because \( f_n(\frac{1}{n(n+1)}) = \frac{1}{n(n+1)} \leq 1 - \frac{1}{2} = f(\frac{1}{n(n+1)}) - \frac{1}{2} \) for each \( n = 1,2,\ldots \), that is, the converse statement to Corollary 3.18 does not hold.

4 Fatou’s Lemmas in the Classic Form for Varying Measures

In this section we establish Fatou’s lemmas in their classic form for varying measures. Some of the results, including Theorem 4.1, deal with more general situations. This section consists of two subsections dealing with weakly and setwise converging measures, respectively.

4.1 Fatou’s lemmas in the classic form for weakly converging measures

The main result of this subsection has the following formulation.

Theorem 4.1 (FL for weakly converging measures). Let \( S \) be a metric space, the sequence of measures \( \{\mu_n\}_{n=1,2,...} \) converge weakly to \( \mu \in \mathcal{M}(S) \), \( \{f_n\}_{n=1,2,...} \) be a lower semi-equicontinuous sequence of real-valued functions on \( S \), and \( f \) be a measurable real-valued function on \( S \). If the following conditions hold:

(i) the sequence \( \{f_n\}_{n=1,2,...} \) lower semi-converges to \( f \) in measure \( \mu \);

(ii) either \( \{f_n^-\}_{n=1,2,...} \) is a.u.i. w.r.t. \( \{\mu_n\}_{n=1,2,...} \) or Assumption (ii) of Theorem 2.9 holds,
then
\[ \int_{\mathbb{S}} f(s) \mu(ds) \leq \liminf_{n \to \infty} \int_{\mathbb{S}} f_n(s) \mu_n(ds). \]  
(4.1)

We recall that asymptotic uniform integrability of \( \{f_n\}_{n=1,2,...} \) w.r.t. \( \{\mu_n\}_{n=1,2,...} \) neither implies nor is implied by Assumption (ii) of Theorem 2.9. \[ \text{Examples 3.1 and 3.2}. \]

**Proof of Theorem 4.1.** Consider a subsequence \( \{f_{n_k}\}_{k=1,2,...} \subset \{f_n\}_{n=1,2,...} \) such that
\[ \lim_{k \to \infty} \int_{\mathbb{S}} f_{n_k}(s) \mu_n(ds) = \liminf_{n \to \infty} \int_{\mathbb{S}} f_n(s) \mu_n(ds). \]  
(4.2)

Assumption (i) implies that \( \mu(\{s \in \mathbb{S} : f_{n_k}(s) \leq f(s) - \varepsilon\}) \to 0 \) as \( k \to \infty \) for each \( \varepsilon > 0 \). Therefore, according to Remark 3.8, there exists a subsequence \( \{f_{k_j}\}_{j=1,2,...} \subset \{f_{n_k}\}_{k=1,2,...} \) such that \( f(s) \leq \liminf_{j \to \infty} f_{k_j}(s) \) for \( \mu \)-a.e. \( s \in \mathbb{S} \). Thus, Theorem 3.14(i) implies that
\[ f(s) \leq \liminf_{j \to \infty, s' \to s} f_{k_j}(s'), \]  
for \( \mu \)-a.e. \( s \in \mathbb{S} \) and, therefore,
\[ \int_{\mathbb{S}} f(s) \mu(ds) \leq \int_{\mathbb{S}} \liminf_{j \to \infty, s' \to s} f_{k_j}(s') \mu(ds). \]  
(4.3)

Theorem 2.9 applied to \( \{f_{k_j}\}_{j=1,2,...} \), implies
\[ \int_{\mathbb{S}} \liminf_{j \to \infty, s' \to s} f_{k_j}(s') \mu(ds) \leq \liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) \mu_{k_j}(ds). \]  
(4.4)

Hence, \[ \text{(4.1)} \] directly follows from \[ \text{(4.3)}, \text{ (4.4)}, \text{ and (4.2)}. \] \( \square \)

The following corollary establish that the setwise convergence in Theorem 2.10 can be substituted by the weak convergence if the integrands form a lower semi-equicontinuous sequence of functions.

**Corollary 4.2** (FL for weakly converging measures). Let \( \mathbb{S} \) be a metric space, a sequence of measures \( \{\mu_n\}_{n=1,2,...} \) converge weakly to \( \mu \in \mathcal{M}(\mathbb{S}) \), \( \{f_n\}_{n=1,2,...} \) be a lower semi-equicontinuous sequence of real-valued functions on \( \mathbb{S} \). If assumption (ii) of Theorem 4.1 holds then inequality 2.6 holds.

**Proof.** Inequality 2.6 directly follows from Theorem 4.1 and Remark 3.8. \( \square \)

The following example illustrates that Theorem 4.1 can provide a more exact lower bound for the limit of integrals than Theorem 2.9.

**Example 4.3.** Let \( \mathbb{S} := [0, 2] \). We endow \( \mathbb{S} \) with the following metric:
\[
\rho(s_1, s_2) = \text{I}\{s_1 \in [0, 1]\} \text{I}\{s_2 \in [0, 1]\} |s_1 - s_2| + \left(1 - \text{I}\{s_1 \in [0, 1]\}\text{I}\{s_2 \in [0, 1]\}\right) \text{I}\{s_1 \neq s_2\}.
\]

To see that \( \rho \) is a metric, note that for \( s_1, s_2 \in \mathbb{S} \) (i) \( \rho(s_1, s_2) \in [0,1] \); (ii) \( \rho(s_1, s_2) = 0 \) iff \( s_1 = s_2 \); (iii) \( \rho(s_1, s_2) \) is symmetric in \( s_1 \) and \( s_2 \); and (iv) for \( s_1 \neq s_2 \) and \( s_3 \in \mathbb{S} \), the triangle inequality holds because
\[
\rho(s_1, s_2) = |s_1 - s_2| \leq |s_1 - s_3| + |s_3 - s_2| = \rho(s_1, s_3) + \rho(s_3, s_2)
\]  
if \( s_1, s_2, s_3 \in \mathbb{S} \),
$s_3 \in [0,1]$, and $\rho(s_1, s_2) \leq 1 \leq \rho(s_1, s_3) + \rho(s_3, s_2)$ otherwise. Let $\mu$ be Lebesgue measure on $\mathbb{S}$ and 
\{\mu_n\}_{n=1,2,\ldots} \subset \mathcal{M}(\mathbb{S})$ be defined as

$$
\mu_n(C) := \sum_{k=0}^{n-1} \frac{1}{n} I\{\frac{k}{n} \in C\} + \mu(C \cap [1, 2]), \quad C \in \Sigma, \quad n = 1, 2, \ldots.
$$

Then $\mu_n$ converge weakly to $\mu$ (see Billingsley [2] p. 15, Example 2.2) and $\mu_n$ does not converge setwise to $\mu$ because $\mu_n([0,1] \setminus \mathbb{Q}) = 0 \neq 1 = \mu(\mathbb{Q})$, where $\mathbb{Q}$ is the set of all rational numbers in $[0,1]$. Define $f \equiv 1$ and $f_n(s) = 1 - I\{s \in (1 + \frac{1}{2^k}, 1 + \frac{2}{2^k})\}$, where $k = \lfloor \log_2 n \rfloor$, $f = n - 2^k$, $s \in \mathbb{S}$, and $n = 1, 2, \ldots$.

Since the subspace $[1,2] \subset \mathbb{S}$ is endowed with the discrete metric, every sequence of functions on $[1,2]$ is equicontinuous. Since $f_n(s) = 1$ for $n = 1, 2, \ldots$ and $s \in [0,1]$, the sequence $\{f_n\}_{n=1,2,\ldots}$ is equicontinuous and, thus, lower semi-equicontinuous on $\mathbb{S}$. In addition, (2.5) holds and $\{f_n\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$ because $f_n$ is nonnegative for $n = 1, 2, \ldots$. Since $\mu(\{s \in \mathbb{S} : f_n(s) < f(s)\}) = \frac{1}{2^{\lfloor \log_2 n \rfloor}} \to 0$ as $n \to \infty$, condition (i) from Theorem 4.1 holds. In view of Theorem 4.1,

$$
\liminf_{n \to \infty} \int_\mathbb{S} f_n(s) \mu_n(ds) = \liminf_{n \to \infty} \left( \int_0^1 f_n(s) \mu_n(ds) + \int_1^2 f_n(s) \mu_n(ds) \right) \\
= \liminf_{n \to \infty} \left( 1 + 1 - \frac{1}{2^{\lfloor \log_2 n \rfloor}} \right) = 2 \geq \int_\mathbb{S} f(s) \mu(ds) = 2.
$$

As follows from Theorem 3.14(i),

$$
\liminf_{n \to \infty, s' \to s} f_n(s') = \liminf_{n \to \infty} f_n(s) = 1 - I\{s \in [1,2]\}, \quad s \in \mathbb{S}.
$$

In view of Theorem 2.9 (1.2) and (2.6) imply

$$
2 = \liminf_{n \to \infty} \int_\mathbb{S} f_n(s) \mu_n(ds) \geq \int_\mathbb{S} \liminf_{n \to \infty, s' \to s} f_n(s') \mu(ds) = \int_\mathbb{S} \liminf_{n \to \infty} f_n(s) \mu(ds) = 1.
$$

Therefore, Theorem 4.1 provides a more exact lower bound (4.1) for the limit of integrals than (1.2) and (2.6) for weakly converging measures and lower semi-equicontinuous sequences of functions. □

4.2 Fatou’s lemmas for setwise converging measures

In this subsection we provide the counterparts to theorems from Subsection 4.1 for a sequence of measures $\{\mu_n\}_{n=1,2,\ldots}$ converging setwise to a measure $\mu$. The following theorem establishes the counterpart to Fatou’s lemma (Theorem 2.9) for setwise converging measures.

**Theorem 4.4 (FL for setwise converging measures).** Let $(\mathbb{S}, \Sigma)$ be a measurable space, a sequence of measures $\{\mu_n\}_{n=1,2,\ldots}$ converge setwise to a measure $\mu \in \mathcal{M}(\mathbb{S})$, and $\{f_n\}_{n=1,2,\ldots}$ be a sequence of $\mathbb{R}$-valued measurable functions on $\mathbb{S}$. If the sequence $\{f_n\}_{n=1,2,\ldots}$ lower semi-converges to a real-valued function $f$ in measure $\mu$ and $\{f_n\}_{n=1,2,\ldots}$ is a.u.i. w.r.t. $\{\mu_n\}_{n=1,2,\ldots}$, then inequality (4.1) holds.

**Proof.** The proof repeats several lines of the proofs of Theorems 4.1 and 2.9. Consider a subsequence $\{f_{n_k}\}_{k=1,2,\ldots} \subset \{f_n\}_{n=1,2,\ldots}$ such that

$$
\lim_{k \to \infty} \int_\mathbb{S} f_{n_k}(s) \mu_{n_k}(ds) = \liminf_{n \to \infty} \int_\mathbb{S} f_n(s) \mu_n(ds).
$$

(4.5)
Thus, \( \mu_K > 0 \). For this purpose we fix an arbitrary subsequence \( \{ f_{k_j} \}_{j=1,2,\ldots} \subset \{ f_{n_k} \}_{k=1,2,\ldots} \) such that \( f(s) \leq \liminf_{j \to \infty} f_{k_j}(s) \) for \( \mu \)-a.e. \( s \in \mathbb{S} \).

Thus,
\[
\int_{\mathbb{S}} f(s) \mu(ds) \leq \int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s) \mu(ds). \tag{4.6}
\]

Now we prove that
\[
\int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s) \mu(ds) \leq \lim_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) \mu_k(ds). \tag{4.7}
\]

For this purpose we fix an arbitrary \( K > 0 \). Then
\[
\liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) \mu_k(ds) \geq \liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) > -K \} \mu_k(ds) \tag{4.8}
\]
\[
+ \liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) \leq -K \} \mu_k(ds).
\]

The following inequality holds:
\[
\liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) > -K \} \mu_k(ds) \geq \int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s) \mu(ds). \tag{4.9}
\]

Indeed, if \( \mu(\mathbb{S}) = 0 \), then
\[
\liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) > -K \} \mu_k(ds) \geq -K \lim_{j \to \infty} \mu_k(\mathbb{S}) = 0 = \int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s) \mu(ds),
\]
where the first equality holds because the sequence \( \{ \mu_k \}_{n=1,2,\ldots} \) converges setwise to \( \mu \in \mathcal{M}(\mathbb{S}) \).

Otherwise, if \( \mu(\mathbb{S}) > 0 \), then Theorem 2.10 applied to \( \{ f_{k_j} \}_{n=1,2,\ldots} := \{ f_{k_{j+N}} \}_{n=1,2,\ldots} \), \( \tilde{g}_{k_j} \equiv -K \), \( \tilde{\mu}_{k_j}(C) := \frac{\mu_{n+x}(C)}{\tilde{\mu}_{k_{j+N}}(\mathbb{S})} \) and \( \tilde{\mu}(C) = \frac{\mu(C)}{\mu(\mathbb{S})} \), for each \( j = 1,2,\ldots \) and \( C \in \mathcal{B}(\mathbb{S}) \), where \( N = 1,2,\ldots \) is sufficiently large, implies
\[
\liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) > -K \} \mu_k(ds) \geq \int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) > -K \} \mu(ds). \tag{4.10}
\]

Here we note that the sequence \( \{ \tilde{\mu}_{k_j} \}_{j=1,2,\ldots} \subset \mathcal{P}(\mathbb{S}) \) converges setwise to \( \tilde{\mu} \in \mathcal{P}(\mathbb{S}) \). We remark also that
\[
f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) > -K \} \geq f_{k_j}(s), \tag{4.11}
\]
for each \( s \in \mathbb{S} \) because \( K > 0 \). Thus, (4.9) follows from (4.10) and (4.11).

Inequalities (4.8) and (4.9) imply
\[
\liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) \mu_k(ds) \geq \int_{\mathbb{S}} \liminf_{j \to \infty, s' \to s} f_{k_j}(s') \mu(ds) \tag{4.12}
\]
\[
+ \lim_{K \to +\infty} \liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s) I\{ s \in \mathbb{S} : f_{k_j}(s) \leq -K \} \mu_k(ds),
\]
which is equivalent to (4.7) because \( \{f_{k_j}\}_{j=1,2,...} \) is a.u.i. w.r.t. \( \{\mu_{k_j}\}_{j=1,2,...} \).

Hence, (4.1) directly follows from (4.6), (4.7), and (4.5).

The following corollary to Theorem 4.3 is the generalized version of Theorem 2.10.

**Corollary 4.5.** Let \((\mathbb{S}, \Sigma)\) be a measurable space, a sequence of measures \(\{\mu_n\}_{n=1,2,...}\) converge setwise to a measure \(\mu \in \mathcal{M}(\mathbb{S})\), and \(\{f_n\}_{n=1,2,...}\) be a sequence of \(\mathbb{R}\)-valued measurable functions on \(\mathbb{S}\). If the sequence \(\{f_n\}_{n=1,2,...}\) lower semi-converges to a real-valued function \(f\) in measure \(\mu\) and (2.7) holds, then inequality (4.4) holds.

**Proof.** Similarly to the proof of Theorem 4.4, consider a subsequence \(\{f_{n_k}\}_{k=1,2,...} \subset \{f_n\}_{n=1,2,...}\) such that

\[
\lim_{k \to \infty} \int_{\mathbb{S}} f_{n_k}(s)\mu_{n_k}(ds) = \liminf_{n \to \infty} \int_{\mathbb{S}} f_n(s)\mu_n(ds).
\]

(4.12)

Since the sequence \(\{f_n\}_{n=1,2,...}\) lower semi-converges to \(f\) in measure \(\mu\), we have that \(\mu(\{s \in \mathbb{S} : f_{n_k}(s) \leq f(s) - \varepsilon\}) \to 0\) as \(k \to \infty\) for each \(\varepsilon > 0\). Therefore, according to Remark 3.8, there exists a subsequence \(\{f_{k_j}\}_{j=1,2,...} \subset \{f_{n_k}\}_{k=1,2,...}\) such that \(f(s) \leq \liminf_{j \to \infty} f_{k_j}(s)\) for \(\mu\)-a.e. \(s \in \mathbb{S}\). Thus,

\[
\int_{\mathbb{S}} f(s)\mu(ds) \leq \int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s)\mu(ds).
\]

(4.13)

Now we prove that

\[
\int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s)\mu(ds) \leq \lim_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s)\mu_{k_j}(ds).
\]

(4.14)

Theorem 4.4 applied to the sequence \(\{f_{k_j} - g_{k_j}\}_{j=1,2,...}\), implies

\[
\int_{\mathbb{S}} \liminf_{j \to \infty} f_{k_j}(s)\mu(ds) - \int_{\mathbb{S}} \limsup_{j \to \infty} g_{k_j}(s)\mu(ds)
\]

\[
\leq \int_{\mathbb{S}} \liminf_{j \to \infty}(f_{k_j}(s) - g_{k_j}(s))\mu(ds)
\]

\[
\leq \liminf_{j \to \infty} \int_{\mathbb{S}} f_{k_j}(s)\mu_{k_j}(ds) - \limsup_{j \to \infty} \int_{\mathbb{S}} g_{k_j}(s)\mu_{k_j}(ds),
\]

where the first and third inequalities follow from the basic properties of infimums and supremums. Therefore, (4.14) directly follows from (2.7).

Hence, (4.1) directly follows from (4.13), (4.14), and (4.12).

Theorem 4.4 provides a more exact lower bound for a sequence of integrals than Theorem 2.10. This fact is illustrated in Example 4.6.

**Example 4.6** (cp. Feinberg et al. [9, Example 4.1]). Let \(\mathbb{S} = [0,1], \Sigma = \mathcal{B}([0,1])\), \(\mu\) be Lebesgue measure on \(\mathbb{S}\), and \(\mu_n\) be

\[
\mu_n(C) := \int_C 2\mathbf{I}\{s \in \mathbb{S} : \frac{2k}{2^n} < s < \frac{2k+1}{2^n}, k = 0, 1, 2, \ldots, 2^{n-1} - 1\}ds, \quad n = 1, 2, \ldots.
\]

Define \(f \equiv 1\) and \(f_n(s) = 1 - \mathbf{I}\{s \in \left[\frac{2k}{2^n}, \frac{j+1}{2^n}\right]\}\), where \(k = \lfloor \log_2 n \rfloor, j = n - 2^k, s \in \mathbb{S}\), and \(n = 1, 2, \ldots\). Then \(\mu_n\) converge setwise to \(\mu\), (2.7) holds and \(\{f_n\}_{n=1,2,...}\) is a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,...}\).
and the sequence \( \{f_n\}_{n=1,2,...} \) lower semi-converges to \( f \) in measure \( \mu \). In view of Theorem 4.4 and (2.6),

\[
1 = \liminf_{n \to \infty} \int_S f_n(s) \mu_n(ds) \geq \int_S f(s) \mu(ds) = 1 > 0 = \lim_{n \to \infty} \liminf_{n \to \infty} f_n(s) \mu(ds).
\]

Therefore, Theorem 4.4 provides a more exact lower bound for the limit of integrals than (2.6) for setwise converging measures.

5 Lebesgue’s Convergence Theorem for Varying Measures

In this section, we present Lebesgue’s convergence theorems for varying measures \( \mu_n \) and functions that are a.u.i. w.r.t. \( \{\mu_n\}_{n=1,2,...} \). This section is divided into two subsections dealing with Lebesgue’s convergence theorems for weakly converging measures (Subsection 5.1) and Lebesgue’s convergence theorems for setwise converging measures (Subsection 5.2).

5.1 Lebesgue’s convergence theorem for weakly converging measures

The following corollary directly follows from Theorem 2.9.

**Corollary 5.1** (Lebesgue’s convergence theorem for weakly converging measures). Let \( S \) be a metric space, \( \{\mu_n\}_{n=1,2,...} \) be a sequence of measures on \( S \) converging weakly to \( \mu \in \mathcal{M}(S) \), and \( \{f_n\}_{n=1,2,...} \) be an a.u.i. (see (2.4)) w.r.t. \( \{\mu_n\}_{n=1,2,...} \) sequence of measurable real-valued functions on \( S \) such that \( \lim_{n \to \infty} s' \to s f_n(s') \) exists for \( \mu \)-a.e. \( s \in S \), then

\[
\lim_{n \to \infty} \int_S f_n(s) \mu_n(ds) = \int_S \lim_{n \to \infty} s' \to s f_n(s) \mu(ds) = \int_S \lim_{n \to \infty} f_n(s) \mu(ds).
\]

**Proof.** Corollary 5.1 follows from Theorem 2.9 being applied to the sequences \( \{f_n\}_{n=1,2,...} \) and \( \{-f_n\}_{n=1,2,...} \). Of course, \( \lim_{n \to \infty} s' \to s f_n(s) = \lim_{n \to \infty} f_n(s) \) if the first limit exists.

The following corollary states the convergence theorem for weakly converging measures \( \mu_n \) and for an equicontinuous sequence of functions \( \{f_n\}_{n=1,2,...} \).

**Corollary 5.2** (Lebesgue’s convergence theorem for weakly converging measures). Let \( S \) be a metric space, the sequence of measures \( \{\mu_n\}_{n=1,2,...} \) converge weakly to \( \mu \in \mathcal{M}(S) \), \( \{f_n\}_{n=1,2,...} \) be a sequence of real-valued equicontinuous functions on \( S \), and \( f \) be a measurable real-valued function on \( S \). If the sequence \( \{f_n\}_{n=1,2,...} \) converges to \( f \) in measure \( \mu \) and is a.u.i. (see (2.4)) w.r.t. \( \{\mu_n\}_{n=1,2,...} \), then

\[
\lim_{n \to \infty} \int_S f_n(s) \mu_n(ds) = \int_S f(s) \mu(ds). \tag{5.1}
\]

**Proof.** Corollary 5.2 follows from Theorem 4.4 being applied to the sequences \( \{f_n\}_{n=1,2,...} \) and \( \{-f_n\}_{n=1,2,...} \).

5.2 Lebesgue’s convergence theorem for setwise converging measures

The following corollary directly follows from Theorem 4.4.

\[
\lim_{n \to \infty} \int_S f_n(s) \mu_n(ds) = \int_S f(s) \mu(ds).
\]

\[
\tag{5.1}
\]
Corollary 5.3 (Lebesgue’s convergence theorem for setwise converging measures). Let \((\Sigma, \Sigma)\) be a measurable space, a sequence of measures \(\{\mu_n\}_{n=1,2,...}\) converge setwise to a measure \(\mu \in \mathcal{M}(\Sigma)\), and \(\{f_n\}_{n=1,2,...}\) be a sequence of \(\mathbb{R}\)-valued measurable functions on \(\Sigma\). If the sequence \(\{f_n\}_{n=1,2,...}\) converges to a measurable real-valued function \(f\) in measure \(\mu\) and this sequence is a.u.i. (see \(2.4\)) w.r.t. \(\{\mu_n\}_{n=1,2,...}\), then (5.1) holds.

**Proof.** Corollary 5.3 follows from Theorem 4.4 being applied to the sequences of functions \(\{f_n\}_{n=1,2,...}\) and \(\{-f_n\}_{n=1,2,...}\). \(\square\)

6 Monotone Convergence Theorem for Varying Measures

In this section, we present monotone convergence theorems for varying measures \(\mu_n\) and functions that are a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,...}\). This section is divided into two subsections: monotone convergence theorems for weakly converging measures (Subsection 6.1) and monotone convergence theorems for setwise converging measures (Subsection 6.2).

6.1 Monotone convergence theorem for weakly converging measures

The following theorem is the main result in this subsection.

**Theorem 6.1** (Monotone convergence theorem for weakly converging measures). Let \(\Sigma\) be a metric space, \(\{\mu_n\}_{n=1,2,...}\) be a sequence of measures on \(\Sigma\) that converge weakly to \(\mu \in \mathcal{M}(\Sigma)\), \(\{f_n\}_{n=1,2,...}\) be a sequence of lower semi-continuous \(\mathbb{R}\)-valued functions on \(\Sigma\) such that \(f_n(s) \leq f_{n+1}(s)\) for each \(n = 1, 2, \ldots\) and \(s \in \Sigma\), and \(f(s) := \lim_{n \to \infty} f_n(s)\), \(s \in \Sigma\). If the following conditions hold:

(i) the function \(f\) is upper semi-continuous;

(ii) the functions \(f^-\) and \(f^+\) are a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,...}\);

then (5.1) holds.

The following example demonstrate the necessity of the condition (i) in Theorem 6.1.

**Example 6.2.** Consider \(\Sigma = [0, 1]\) endowed with the standard Euclidean metric, \(f(s) = I\{s \in (0, 1]\}, s \in \Sigma, f_n(s) = \min\{ns, 1\}, n = 1, 2, \ldots\) and \(s \in \Sigma\), and probability measures probability measures

\[
\mu_n(C) := \int_C n I\{s \in \left[0, \frac{1}{n}\right]\} \nu(ds), \quad \mu(C) := I\{0 \in C\}, \quad C \in \mathcal{B}(\Sigma), \quad n = 1, 2, \ldots, \tag{6.1}
\]

where \(\nu\) is Lebesgue measure on \(\Sigma\).

Then \(f_n(s) \uparrow f(s)\) for each \(s \in \Sigma\) as \(n \to \infty\) and the sequence of probability measures \(\mu_n\) converges weakly to \(\mu\). Since functions \(f_n\) and \(f\) are bounded, condition (ii) from Theorem 6.1 holds. The function \(f_n\) is continuous, and the function \(f\) is lower semi-continuous, but \(f\) is not upper semi-continuous. Since \(\int_\Sigma f_n(s) \mu_n(ds) = \frac{1}{2}, n = 1, 2, \ldots,\) and \(\int_\Sigma f(s) \mu(ds) = 0,\) formula (5.1) does not hold.

**Proof of Theorem 6.1** Since \(f^+\) is a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,...}\),

\[
\limsup_{n \to \infty} \int_\Sigma f_n(s) \mu_n(ds) \leq \limsup_{n \to \infty} \int_\Sigma f(s) \mu_n(ds) \leq \int_\Sigma \limsup_{s' \to s} f(s') \mu(ds) = \int_\Sigma f(s) \mu(ds), \tag{6.2}
\]
where the first inequality holds because the sequence \( \{f_n\}_{n=1,2,...} \) is nondecreasing, the second one follows from Theorem 2.9 being applied to the sequence \( \{-f\} \), and the equality follows from the upper semi-continuity of \( f \). Since \( f_1^- \) is a.u.i. w.r.t. \( \{\mu_n\}_{n=1,2,...} \) and the sequence \( \{f_n\}_{n=1,2,...} \) is nondecreasing, \( \{f_n^-\}_{n=1,2,...} \) is a.u.i. w.r.t. \( \{\mu_n\}_{n=1,2,...} \). Then Theorem 2.9 applied to the sequence \( \{f_n\}_{n=1,2,...} \) implies

\[
\int_{\mathbb{S}} \liminf_{n \to \infty, s' \to s} f_n(s') \mu(ds) \leq \liminf_{n \to \infty} \int_{\mathbb{S}} f_n(s) \mu_n(ds). \tag{6.3}
\]

According to Lemma 5.13

\[
\int_{\mathbb{S}} \liminf_{n \to \infty, s' \to s} f_n(s') \mu(ds) = \int_{\mathbb{S}} f(s) \mu(ds). \tag{6.4}
\]

Therefore, (6.2), (6.3), and (6.4) imply (5.1).

\[\square\]

**Corollary 6.3.** Let \( \mathbb{S} \) be a metric space, \( \{\mu_n\}_{n=1,2,...} \) be a sequence of measures on \( \mathbb{S} \) that converge weakly to \( \mu \in \mathcal{M}(\mathbb{S}) \), and \( \{f_n\}_{n=1,2,...} \) be a pointwise nondecreasing sequence of measurable \( \mathbb{R} \)-valued functions on \( \mathbb{S} \). Let \( f(s) := \lim_{n \to \infty} f_n(s) \) and \( f_n(s) := \liminf_{s' \to s} f_n(s') \), \( s \in \mathbb{S} \). If the following conditions hold:

(i) the function \( f \) is real-valued and upper semi-continuous;

(ii) \( \{f_n\}_{n=1,2,...} \) lower semi-converge to \( f \) in measure \( \mu \);

(iii) the functions \( f_1^- \) and \( f^+ \) are a.u.i. w.r.t. \( \{\mu_n\}_{n=1,2,...} \);

then (5.1) holds.

The following example demonstrates the necessity of the condition (i) from Corollary 6.3

**Example 6.4.** Consider \( \mathbb{S} = [0,1] \) endowed with the standard Euclidean metric, \( f(s) = 1 \),

\[
f_n(s) = \begin{cases} 1, & \text{if } s = 0, \\
\min\{ns, 1\}, & \text{if } s \in (0,1], \end{cases} \quad n = 1, 2, \ldots, \quad s \in \mathbb{S},
\]

and probability measures \( \mu_n, n = 1, 2, \ldots, \) and \( \mu \) defined in (6.1). Then \( f_n(s) = \min\{ns, 1\} \) and \( f_n(s) \uparrow f(s) \) for each \( s \in \mathbb{S} \) as \( n \to \infty \) and the sequence of probability measures \( \mu_n \) converges weakly to \( \mu \). Since functions \( f_1 \) and \( f \) are bounded, condition (ii) from Theorem 6.1 holds. Condition (i) from Corollary 6.3 does not hold because \( f_n(0) = 1 \) and \( f_n(0) = 0 \) for each \( n = 1, 2, \ldots \). Since \( \int_{\mathbb{S}} f_n(s) \mu_n(ds) = \frac{1}{2} \), \( n = 1, 2, \ldots \), and \( \int_{\mathbb{S}} f(s) \mu(ds) = 1 \), formula (5.1) does not hold.

\[\square\]

**Proof of Corollary 6.3.** Since the function \( f_{n_k} \) is lower semi-continuous, Theorem 6.1 implies

\[
\lim_{n \to \infty} \int_{\mathbb{S}} f_{n_k}(s) \mu_n(ds) = \int_{\mathbb{S}} \lim_{n \to \infty} f_{n_k}(s) \mu(ds). \tag{6.5}
\]

Condition (i) implies that there exists a subsequence \( \{f_{n_k}\}_{k=1,2,...} \) of the sequence \( \{f_n\}_{n=1,2,...} \) such that

\[
\liminf_{k \to \infty} f_{n_k}(s) \geq f(s) \quad \text{for } \mu\text{-a.e. } s \in \mathbb{S}. \tag{6.6}
\]
Since \( f_n(s) \leq f_n(s) \leq f(s), \ n = 1, 2, \ldots \) and \( s \in \mathbb{S} \), and the sequence \( \{f_n\}_{n=1,2,\ldots} \) is pointwise nondecreasing, (6.6) implies that

\[
f(s) = \lim_{n \to \infty} f_n(s) \quad \text{for } \mu\text{-a.e. } s \in \mathbb{S}.
\] (6.7)

Hence, (6.5) and (6.7) imply

\[
\lim_{n \to \infty} \int_{\mathbb{S}} f_n(s) \mu_n(ds) = \int_{\mathbb{S}} f(s) \mu(ds).
\] (6.8)

Since \( f_n(s) \leq f_n(s) \leq f(s), \ n = 1, 2, \ldots \) and \( s \in \mathbb{S} \),

\[
\lim_{n \to \infty} \int_{\mathbb{S}} f_n(s) \mu_n(ds) \leq \liminf_{n \to \infty} \int_{\mathbb{S}} f_n(s) \mu_n(ds) \leq \limsup_{n \to \infty} \int_{\mathbb{S}} f_n(s) \mu_n(ds) \leq \limsup_{n \to \infty} \int_{\mathbb{S}} f(s) \mu_n(ds).
\] (6.9)

Theorem 2.9 being applied to the sequence \( \{-f\} \) and the upper semi-continuity of \( f \) imply

\[
\limsup_{n \to \infty} \int_{\mathbb{S}} f(s) \mu_n(ds) \leq \int_{\mathbb{S}} f(s) \mu(ds).
\] (6.10)

Therefore, (6.8), (6.9), and (6.10) imply (5.1). \qed

6.2 Monotone convergence theorem for setwise converging measures

The following corollary is the counterpart to Theorem 6.1 for setwise converging measures.

Corollary 6.5 (Monotone convergence theorem for setwise converging measures). Let \((\mathbb{S}, \Sigma)\) be a measurable space, a sequence of measures \(\{\mu_n\}_{n=1,2,\ldots} \) converge setwise to a measure \(\mu \in \mathcal{M}(\mathbb{S})\), and \(\{f_n\}_{n=1,2,\ldots} \) be a pointwise nondecreasing sequence of measurable \(\mathbb{R}\)-valued functions on \(\mathbb{S}\). Let \(f(s) := \lim_{n \to \infty} f_n(s), \ s \in \mathbb{S}\). If the functions \(f^- \) and \(f^+ \) are a.u.i. w.r.t. \(\{\mu_n\}_{n=1,2,\ldots}\), then (5.1) holds.

Proof. Since \(f_n \uparrow f\), (5.1) follows directly from Theorem 4.4 being applied to the sequences \(\{f_n\}_{n=1,2,\ldots}\) and \(\{-f_n\}_{n=1,2,\ldots}\). \qed

7 Applications to Markov Decision Processes

Consider a discrete-time MDP with a state space \(\mathbb{X}\), an action space \(\mathbb{A}\), one-step costs \(c\), and transition probabilities \(q\). Assume that \(\mathbb{X}\) and \(\mathbb{A}\) are Borel subsets of Polish (complete separable metric) spaces. Let \(c(x,a) : \mathbb{X} \times \mathbb{A} \mapsto \mathbb{R}\) be the one-step cost and \(q(B|x,a)\) be the transition kernel representing the probability that the next state is in \(B \in \mathcal{B} (\mathbb{X})\), given that the action \(a\) is chosen at the state \(x\). The cost function \(c\) is assumed to be measurable and bounded below.

The decision process proceeds as follows: at each time epoch \(t = 0, 1, \ldots\), the current state of the system, \(x\), is observed. A decision-maker chooses an action \(a\), the cost \(c(x,a)\) is accrued, and the system moves to the next state according to \(q(\cdot|x,a)\). Let \(H_t = (\mathbb{X} \times \mathbb{A})^t \times \mathbb{X}\) be the set of histories for \(t = 0, 1, \ldots\) A (randomized) decision rule at period \(t = 0, 1, \ldots\) is a regular transition probability \(\pi_t : H_t \mapsto \mathbb{A}\), that is, (i) \(\pi_t(\cdot|h_t)\) is a probability distribution on \(\mathbb{A}\), where \(h_t = (x_0,a_0,x_1,\ldots,a_{t-1},x_t)\), and (ii) for any measurable subset \(B \subset \mathbb{A}\), the function \(\pi_t(B|\cdot)\) is measurable on \(H_t\). A policy \(\pi\) is a sequence \((\pi_0,\pi_1,\ldots)\) of decision rules. Let \(\Pi\) be the set of all policies. A policy \(\pi\) is called non-randomized if each probability measure \(\pi_t(\cdot|h_t)\) is concentrated at
one point. A non-randomized policy is called stationary if all decisions depend only on the current state.

The Ionescu Tulcea theorem implies that an initial state \( x \) and a policy \( \pi \) define a unique probability \( P^x_\pi \) on the set of all trajectories \( \mathbb{H}_\infty = (X \times A)^\infty \) endowed with the product of \( \sigma \)-fields defined by Borel \( \sigma \)-fields of \( X \) and \( A \); see Bertsekas and Shreve \cite{BertsekasShreve1996} pp. 140–141] or Hernández-Lerma and Lasserre \cite{HernandezLermaLasserre1996} p. 178]. Let \( \mathbb{E}^x_\pi \) be an expectation w.r.t. \( P^x_\pi \).

For a finite-horizon \( N = 0, 1, \ldots \), let us define the expected total discounted costs,

\[
v^\pi_{N, \alpha} := \mathbb{E}^x_\pi \sum_{t=0}^{N-1} \alpha^t c(x_t, a_t), \quad x \in X, \tag{7.1}
\]

where \( \alpha \in [0, 1) \) is the discount factor and \( v^\pi_{0, \alpha}(x) = 0 \). When \( N = \infty \), equation (7.1) defines an infinite-horizon expected total discounted cost denoted by \( v^\pi_\alpha(x) \). Let \( v_\alpha := \inf_{\pi \in \Pi} v^\pi_\alpha(x), \ x \in X \). A policy \( \pi \) is called optimal for the discount factor \( \alpha \) if \( v^\pi_\alpha(x) = v_\alpha(x) \) for all \( x \in X \).

The average cost per unit time is defined as

\[
w^\pi_1(x) := \limsup_{N \to \infty} \frac{1}{N} v^\pi_{N, 1}(x), \quad x \in X.
\]

Define the optimal value function \( w_1(x) := \inf_{\pi \in \Pi} w^\pi_1(x), \ x \in X \). A policy \( \pi \) is called average-cost optimal if \( w^\pi_1(x) = w_1(x) \) for all \( x \in X \).

We remark that in general action sets may depend on current states, and usually the state-dependent sets \( A(x) \) are considered for all \( x \in X \). In our problem formulations \( A(x) = \mathbb{A} \) for all \( x \in X \). This problem formulation is simpler than a formulation with the sets \( A(x) \), and these two problem formulations are equivalent because we allow that \( c(x, a) = +\infty \) for some \( (x, a) \in X \times \mathbb{A} \). For example, we may set \( A(x) = \{ a \in \mathbb{A} : c(x, a) < +\infty \} \). For a formulation with the sets \( A(x) \), one may define \( c(x, a) = +\infty \) when \( a \in \mathbb{A} \setminus A(x) \) and use the action sets \( \mathbb{A} \) instead of \( A(x) \).

To establish the existence of the average-cost optimal policies via an optimality inequality for problems with compact action sets, Schäl \cite{Schal1989} considered two continuity conditions \( \mathbf{W} \) and \( \mathbf{S} \) for problems with weakly and setwise continuous transition probabilities, respectively. For setwise continuous transition probabilities, Hernández-Lerma \cite{HernandezLerma1996} generalized Assumption \( \mathbf{S} \) to Assumption \( \mathbf{S}^* \) to cover MDPs with possibly noncompact action sets. For the similar purpose, when transition probabilities are weakly continuous, Feinberg et al. \cite{FeinbergHammondSchal1986} generalized Assumption \( \mathbf{W} \) to Assumption \( \mathbf{W}^* \).

We recall that a function \( f : U \mapsto \mathbb{R} \) defined on a metric space \( U \) is called inf-compact (on \( U \)), if for every \( \lambda \in \mathbb{R} \) the level set \( \{ u \in U : f(u) \leq \lambda \} \) is compact. A subset of a metric space is also a metric space with respect to the same metric. For \( U \subset \mathbb{R} \), if the domain of \( f \) is narrowed to \( U \), then this function is called the restriction of \( f \) to \( U \).

**Definition 7.1** (Feinberg et al. \cite[Definition 1.1]{FeinbergHammondSchal1986}, Feinberg \cite[Definition 2.1]{Feinberg1989}). A function \( f : X \times \mathbb{A} \mapsto \mathbb{R} \) is called \( \mathcal{K} \)-inf-compact, if for every nonempty compact subset \( \mathcal{K} \) of \( X \) the restriction of \( f \) to \( \mathcal{K} \times \mathbb{A} \) is an inf-compact function.

**Assumption \( \mathbf{W}^* \)** (Feinberg et al. \cite{FeinbergHammondSchal1986}, Feinberg and Lewis \cite{FeinbergLewis1987}, or Feinberg \cite{Feinberg1989}).

(i) the function \( c \) is \( \mathcal{K} \)-inf-compact and bounded below;

(ii) the transition probability \( q(\cdot|x, a) \) is weakly continuous in \( (x, a) \in X \times \mathbb{A} \).

**Assumption \( \mathbf{S}^* \)** (Hernández-Lerma \cite{HernandezLerma1996}, or Hernández-Lerma and Lasserre \cite{HernandezLermaLasserre1996}).

(i) the function \( c \) is bounded below, lower semi-continuous, and the function \( c(x, a) \) is
inf-compact in $a \in A(x)$ for each $x \in X$;
(ii) the transition probability $q(\cdot|x,a)$ is setwise continuous in $a \in A$ for each $x \in X$.

We recall that Assumption $W^*$ (i) is stronger than Assumption $S^*$ (i); see [7, Lemmas 2.2, 2.3] and [17]. Let

$$m_\alpha := \inf_{x \in X} v_\alpha(x), \quad u_\alpha(x) := v_\alpha(x) - m_\alpha,$$

$$\underline{w} := \liminf_{\alpha \uparrow 1} (1 - \alpha) m_\alpha, \quad \bar{w} := \limsup_{\alpha \uparrow 1} (1 - \alpha) m_\alpha. \quad (7.2)$$

The function $u_\alpha$ is called the discounted relative value function. If either Assumption $W^*$ or Assumption $S^*$ holds, let us consider the following assumption.

**Assumption B.** (i) $w^* := \inf_{x \in X} w_1(x) < +\infty$; and (ii) $\sup_{\alpha \in [0,1)} u_\alpha(x) < +\infty$, $x \in X$.

As follows from Schäl [19, Lemma 1.2(a)], Assumption B(i) implies that $m_\alpha < +\infty$ for all $\alpha \in [0,1)$. Thus, all the quantities in (7.2) are defined.

It is known [5, 19] that, if a stationary policy $\phi$ satisfies the average-cost optimality inequality (ACOI)

$$w + u(x) \geq c(x, \phi(x)) + \int_X u(y) q(dy|x, \phi(x)), \quad x \in X, \quad (7.3)$$

for some nonnegative measurable function $u : X \to \mathbb{R}$, then the stationary policy $\phi$ is average-cost optimal. A nonnegative measurable function $u(x)$ satisfying inequality (7.3) with some stationary policy $\phi$ is called an average-cost relative value function. The following two theorems state the validity of the ACOI under Assumptions $W^*$ (or Assumption $S^*$) and B.

**Theorem 7.2** (Feinberg et al. [5, Corollary 2]). Let Assumptions $W^*$ and $B$ hold. For an arbitrary sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$, let

$$u(x) := \liminf_{n \to \infty, y \to x} u_{\alpha_n}(y), \quad x \in X. \quad (7.4)$$

Then there exists a stationary policy $\phi$ satisfying ACOI (7.3) with the function $u$ defined in (7.4). Therefore, $\phi$ is a stationary average-cost optimal policy. In addition,

$$w^{\phi}(x) = \underline{w} = \lim_{\alpha \uparrow 1} (1 - \alpha) v_\alpha(x) = \lim_{\alpha \uparrow 1} (1 - \alpha) m_\alpha = \bar{w} = w^*, \quad x \in X. \quad (7.5)$$

**Theorem 7.3** (Hernández-Lerma [13, Theorems 5.4.3 and 5.4.6]). Let Assumptions $S^*$ and $B$ hold. For an arbitrary sequence $\{\alpha_n \uparrow 1\}_{n=1,2,\ldots}$, let

$$u(x) := \liminf_{n \to \infty} u_{\alpha_n}(x), \quad x \in X. \quad (7.6)$$

Then there exists a stationary policy $\phi$ satisfying ACOI (7.3) with the function $u$ defined in (7.6). Therefore, $\phi$ is a stationary average-cost optimal policy. In addition, (7.5) holds.

The following corollary from Theorem 7.2 provides a sufficient condition for the validity of ACOI (7.3) with a relative value function $u$ defined in (7.6).
Corollary 7.4. Let Assumptions $W^*$ and $B$ hold and there exist a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$ of nonnegative discount factors such that the sequence of functions $\{u_{\alpha_n}\}_{n=1,2,...}$ is lower semi-equicontinuous. Then the conclusions of Theorem 7.2 hold for the function $u$ defined in (7.6) for this sequence $\{\alpha_n\}_{n=1,2,...}$.

Proof. Since the sequence of functions $\{u_{\alpha_n}\}_{n=1,2,...}$ is lower semi-equicontinuous, the functions $u$ defined in (7.4) and in (7.6) coincide in view of Lemma 3.14(i). □

In view of (7.5), $w^\phi(x)$ does not depend on $x$. If Assumptions $W^*$ (or Assumption $S^*$) and $B$ hold, let us define $w := w_1$; see (7.5) for other equalities for $w$.

Consider the following equicontinuity condition (EC) on the discounted relative value functions.

Assumption EC. There exists a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$ of nonnegative discount factors such that

(i) the sequence of functions $\{u_{\alpha_n}\}_{n=1,2,...}$ is equicontinuous;

(ii) there exists a nonnegative measurable function $U(x)$, $x \in \mathbb{X}$, such that $U(x) \geq u_{\alpha_n}(x)$, $n = 1, 2, \ldots$, and $\int_{\mathbb{X}} U(y)q(dy|x,a) < +\infty$ for all $x \in \mathbb{X}$ and $a \in A$.

Under each of the Assumptions $W^*$ or $S^*$ and under Assumptions $B$ and EC, there exist a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$ of nonnegative discount factors and a stationary policy $\phi$ satisfying the average-cost optimality equation (ACOE)

$$w + u(x) = c(x,\phi(x)) + \int_{\mathbb{X}} u(y)q(dy|x,\phi(x)) = \min_{a \in A} \left[ c(x,a) + \int_{\mathbb{X}} u(y)q(dy|x,a) \right],$$

(7.7)

with $u$ defined in (7.4) for the sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$; see Feinberg and Liang [12] Theorem 3.2 for $W^*$ and Hernández-Lerma and Lasserre [14] Theorem 5.5.4 for $S^*$. In addition, since the left equation in (7.7) implies inequality (7.3), every stationary policy $\phi$ satisfying (7.7) is average-cost optimal. Observe that in this case the function $u$ is continuous (see [12] Theorem 3.2 for $W^*$ and [14] Theorem 5.5.4 for $S^*$) while under conditions of Theorems 7.2 and 7.3 the corresponding functions $u$ may not be continuous; see Examples 7.6, 7.8. The function $u$ described in Theorem 7.2 is lower semi-continuous and the function $u$ described in Theorem 7.3 is measurable; see [5] [14] for details. Below we provide more general conditions for the validity of the ACOE. In particular, under these conditions the relative value functions $u$ may not be continuous.

Now, we introduce Assumption LEC, which is weaker than Assumption EC. Indeed, Assumption EC(i) is obviously stronger than LEC(i). In view of the Ascoli theorem (see [14] p. 96 or [18] p. 179), EC(i) and the first claim in EC(ii) imply LEC(ii). The second claim in EC(ii) implies LEC(iii). It is shown in Theorem 7.6 that the ACOE holds under Assumptions $W^*$, $B$, and LEC.

Assumption LEC. There exists a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$ of nonnegative discount factors such that

(i) the sequence of functions $\{u_{\alpha_n}\}_{n=1,2,...}$ is lower semi-equicontinuous;

(ii) there exists a subsequence $\{\alpha_{n_k}\}_{k=1,2,...}$ of $\{\alpha_n\}_{n=1,2,...}$ such that $\lim_{k \to +\infty} u_{\alpha_{n_k}}(x)$ exists for each $x \in \mathbb{X}$;

(iii) for each $x \in \mathbb{X}$ and $a \in A$, the sequence $\{u_{\alpha_n}\}_{n=1,2,...}$ is a.u.i. w.r.t. $q(\cdot|x,a)$.

The following Assumption C is weaker than Assumption LEC. It is shown in Theorem 7.7 that the ACOE holds under Assumptions $S^*$, $B$, and C.

Assumption C. There exists a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$ of nonnegative discount factors such that conditions (ii,iii) from Assumption LEC hold.
The average-cost value is $w \in \mathbb{Q}$ transition probabilities are.

Example 7.6. Consider the sequence $\{a_n\}_{n=1,2,...}$ of non-negative discount factors. If Assumption LEC is satisfied for the sequence $\{a_n\}_{n=1,2,...}$, then there exist a stationary policy $\phi$ and a subsequence $\{a_{n_k}\}_{k=1,2,...}$ of $\{a_n\}_{n=1,2,...}$ such that the ACOEs (7.7) hold with the function $u(x)$ defined in (7.6), and

$$u(x) = \lim_{k \to \infty} u_{a_{n_k}}(x), \quad x \in \mathbb{X}. \quad (7.8)$$

Proof. Consider the subsequence $\{a_{n_k}\}_{k=1,2,...}$ of the sequence $\{a_n\}_{n=1,2,...}$ stated in Assumption LEC(ii). Then Assumption LEC(ii) implies (7.8).

Since Assumptions $W^*$ and $B$ hold and the sequence $\{a_{n_k}\}_{k=1,2,...}$ is lower semi-equicontinuous, then Corollary 7.4 implies that there exists a stationary policy $\phi$ satisfying the ACOI with $u$ defined in (7.8) for the sequence $\{a_{n_k}\}_{k=1,2,...}$,

$$w + u(x) \geq c(x, \phi(x)) + \int_{\mathbb{X}} u(y)q(dy|x, \phi(x)). \quad (7.9)$$

To prove the ACOEs, it remains to prove the opposite inequality to (7.9). According to Feinberg et al. [7] Theorem 2(iv)], for each $k = 1, 2, \ldots$ and $x \in \mathbb{X}$ the discounted-cost optimality equation is $v_{a_{n_k}}(x) = \min_{a \in \mathbb{A}} [c(x, a) + \alpha_{n_k} \int_{\mathbb{X}} v_{a_{n_k}}(y)q(dy|x, a)]$, which, by subtracting $m_\alpha$ from both sides and by replacing $\alpha_{n_k}$ with 1, implies that for all $a \in \mathbb{A}$

$$(1 - \alpha_{n_k}) m_{a_{n_k}} + u_{a_{n_k}}(x) \leq c(x, a) + \int_{\mathbb{X}} u_{a_{n_k}}(y)q(dy|x, a), \quad x \in \mathbb{X}. \quad (7.10)$$

Let $k \to \infty$. In view of (7.7), (7.8), and Fatou’s lemma (see, e.g., Theorem 4.1), (7.10) implies that for all $a \in \mathbb{A}$

$$w + u(x) \leq c(x, a) + \int_{\mathbb{X}} u(y)q(dy|x, a), \quad x \in \mathbb{X},$$

which implies

$$w + u(x) \leq \min_{a \in \mathbb{A}} [c(x, a) + \int_{\mathbb{X}} u(y)q(dy|x, a)] \leq c(x, \phi(x)) + \int_{\mathbb{X}} u(y)q(dy|x, \phi(x)), \quad x \in \mathbb{X}. \quad (7.11)$$

Thus, (7.9) and (7.11) imply (7.7). □

In the following example, Assumptions $W^*$, $B$, and LEC hold. Hence the ACOEs hold. However, Assumption EC does not hold. Therefore, Assumption LEC is more general than Assumption EC.

Example 7.6. Consider $\mathbb{X} = [0, 1]$ equipped with the Euclidean metric and $\mathbb{A} = \{a^{(1)}\}$. The transition probabilities are $q(0|x, a^{(1)}) = 1$ for all $x \in \mathbb{X}$. The cost function is $c(x, a^{(1)}) = I\{x \neq 0\}$, $x \in \mathbb{X}$. Then the discounted-cost value is $v_\alpha(x) = u_\alpha(x) = I\{x \neq 0\}$, $\alpha \in [0, 1]$ and $x \in \mathbb{X}$, and the average-cost value is $w = w_1(x) = 0$, $x \in \mathbb{X}$. It is straightforward to see that Assumptions $W^*$ and $B$ hold. In addition, since the function $u(x) = I\{x \neq 0\}$ is lower semi-continuous, but it is not continuous, the sequence of functions $\{u_{\alpha_n}\}_{n=1,2,...}$ is lower semi-equicontinuous, but it is not equicontinuous for each sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$. Therefore, Assumption LEC holds with $U \equiv 1$ and Assumption EC does not hold. The ACOE (7.7) holds with $w = 0$, $u(x) = I\{x \neq 0\}$, and $\phi(x) = u^{(1)}$, $x \in \mathbb{X}$. □

The following theorem establishes the validity of ACOEs under Assumptions $S^*$, $B$, and $C$. 21
**Theorem 7.7.** Let Assumptions $S^*$ and $B$ hold. Consider a sequence $\{\alpha_n \uparrow 1\}_{n=1,2,...}$ of nonnegative discount factors. If Assumption $C$ is satisfied for the sequence $\{\alpha_n\}_{n=1,2,...}$, then there exist a stationary policy $\phi$ and a subsequence $\{\alpha_{n_k}\}_{k=1,2,...}$ of $\{\alpha_n\}_{n=1,2,...}$ such that the ACOEs (7.7) hold with the function $u(x)$ defined in (7.8) for the subsequence $\{\alpha_{n_k}\}_{k=1,2,...}$.

**Proof.** Consider the subsequence $\{\alpha_{nk}\}_{k=1,2,...}$ of the sequence $\{\alpha_n\}_{n=1,2,...}$ whose existence is stated in Assumption $C$ and the function $u$ defined in (7.8) for the sequence $\{\alpha_{nk}\}_{k=1,2,...}$. According to Theorem 7.6 if Assumptions $S^*$ and $B$ hold, then (i) equalities in (7.3) hold; (ii) there exists a stationary policy $\phi$ satisfying the ACOI (7.9) with the function $u$ defined in (7.5) for the sequence $\{\alpha_{nk}\}_{k=1,2,...}$; and (iii) for each $k = 1, 2, \ldots$ and $x \in X$ the discounted-cost optimality equation is $v_{\alpha_{nk}}(x) = \min_{a \in A} [c(x,a) + \alpha \int_X v_{\alpha_{nk}}(y)q(dy|x,a)]$. Therefore, the same arguments as in the proof of Theorem 7.5 starting from (7.10) imply the validity of (7.7) with $u$ defined in (7.5) for the sequence $\{\alpha_{nk}\}_{k=1,2,...}$. \hfill $\square$

Observe that the MDP described in Example 7.6 also satisfies Assumptions $S^*$, $B$, and $C$. We provide Example 7.8 in which Assumptions $S^*$, $B$, and $C$ hold. Hence the ACOEs hold. However, Assumptions $W^*$, LEC, and EC do not hold.

**Example 7.8.** Let $X = [0, 1]$ and $A = \{a^{(1)}\}$. The transition probabilities are $q(0|x, a^{(1)}) = 1$ for all $x \in X$. The cost function is $c(x,a^{(1)}) = D(x)$, where $D$ is the Dirichlet function defined as

$$D(x) = \begin{cases} 0, & \text{if } x \text{ is rational,} \\ 1, & \text{if } x \text{ is irrational,} \end{cases} \quad x \in X.$$ 

Since there is only one available action, Assumption $S^*$ holds. The discounted-cost value is $v_{\alpha}(x) = u_{\alpha}(x) = D(x) = u(x)$, $\alpha \in [0, 1)$ and $x \in X$, and the average-cost value is $w = w_{1}(x) = 0$, $x \in X$. Then Assumptions $B$ and $C$ hold. Hence, the ACOEs (7.7) hold with $w = 0$, $u(x) = D(x)$, and $\phi(x) = a^{(1)}$, $x \in X$. Thus, the average-cost relative function $u$ is nor lower semi-continuous. However, since the function $c(x,a^{(1)}) = D(x)$ is not lower semi-continuous, Assumption $W^*$ does not hold. Since the functions $u_{\alpha}(x) = D(x)$ are not lower semi-continuous, Assumptions LEC and EC do not hold either. \hfill $\square$

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