Abstract

We study the geometrical and topological properties of the bulk (environment space) when we modify the geometry or topology of a brane-world. Through the characterization of a spherically symmetric space-time as a local brane-world immersed into six dimensional pseudo-Euclidean spaces, with different signatures of the bulk, we investigate the existence of a topological difference in the immersed brane-world. In particular the Schwarzschild’s brane-world and its Kruskal (or Fronsdal) brane-world extension are examined from point of view of the immersion formalism. We prove that there is a change of signature of the bulk when we consider a local isometric immersion and different topologies of a brane-world in that bulk.

1 Introduction

The immersion problem of space-times in spaces with higher dimension has been required in subjects linked to minimal class of the immersion, extrinsic gravity and theory of strings. Nowadays the immersion problem emerged with brane-world theory - a theory that lead us the idea of unification of fundamental interactions using extra dimensions. Such a model has been showing positive in the sense that we find perspectives and probably deep modifications in the physics, such as: unification in a TeV scale, quantum gravity in this scale and deviation of Newton’s law of gravity for small distances.

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A brane-world may be regarded as a space-time locally immersed in a higher dimensional space, the bulk, solution of higher dimensional Einstein’s equations. Furthermore, the immersed geometry is assumed to exhibit quantum fluctuations with respect to the extra dimensions at the TeV scale of energies. Finally, all gauge interactions belonging to the standard model must remain confined to the four-dimensional space-time. Contrasting with other higher dimensional theories, the extra dimensions may be large and even infinite, with the possibility of being observed by TeV accelerators. The integrability conditions of immersion relate the bulk geometry to the brane-world geometry. [1, 2]

On this context it is interesting to study the geometric and topological properties of the bulk (environment space) when we modify the geometry or topology of the brane-world. [3] Specifically, for example: we consider a brane-world immersed into bulk, if we modify the topology of the brane-world, what will happen with the signature of the bulk’s metric?

In the present paper we show that if we have \( Y : (M^n, g) \to (\bar{M}^\nu, \bar{g}) \) a local isometric immersion and a topology \( \tau'_\eta \) of \( Y(W^n) \), \( W^n \) a neighborhood of \( p \in M^n \), different of the induced topology \( \tau_\eta \) of the \( Y(W^n) \) and the determinants of the tensor metric \( g_{ij} \) and \( g'_{\mu\nu} \), (a coordinates transformation on \( g_{ij} \)), are not equal in sign at a point, then there is a change of signature of the bulk’s metric. Furthermore it is made an application that results from an example of general relativity in brane-worlds context. On the example we use the Schwarzschild space-time as a brane-world and we apply a change of topology via extension of Kruskal (or Fronsdal) metric obtaining in this form signature change of the six-dimensional flat bulk, where is immersed this brane-world.

2 The Schwarzschild’s Extension from Immersion Formalism

We are going to return to the following problem: Determine the physical space outside of an approximately spherical body with mass \( M \). The physical space is modeled through a 4-dimensional space-time, solution of Einstein equations, whose geometry is described with good approximation by Schwarzschild’s solution, representing the empty space-time with spherical symmetry outside of a body with spherical mass, where \( M = c^2 m G^{-1} \), \( c \) is the speed of light and \( G \) is the gravitational constant. [4]

We know that in spherical coordinates \( (t, r, \theta, \phi) \), the regions \( r = 0 \) and \( r = 2m \) are singular. When we remove the surface \( r = 2m \), the manifold
becomes separated in two disconnected components, one for \(2m < r < \infty\) and the other for \(0 < r < 2m\). Since we are dealing with the existence of the metric associated to a physical space, we require a connected space. Therefore, we define the following regions:

a) The exterior Schwarzschild space-time \((V_4, g)\):
\[ V_4 = P_2^2 \times S^2; \quad P_2^2 = \{(t, r) \in \mathbb{R}^2 | r > 2m\} \]

b) The Schwarzschild black hole \((B_4, g)\):
\[ B_4 = P_2^2 H \times S^2, \quad P_2^2 H = \{(t, r) \in \mathbb{R}^2 | 0 < r < 2m\} \]

In both cases, \(S^2\) is the sphere of radius \(r\) and the metric \(g\) is given by the usual metric of Schwarzschild. We know that \((B_4, g)\) and \((V_4, g)\) may be extensible for \(r = 2m\). The extension of \((V_4, g)\) was calculated by Kruskal but it was suggested by C. Fronsdal one year before.[8, 7]

Now we use the isometric immersion formalism to establish the extension of \((E, g) = ([P_2^2 \cup P_2^2 H] \times S^2, g),\) denoted by \((E', g') = (Q^2 \times S^2, g'),\) where \(Q^2\) is the Kruskal plane.

Consider two known isometric immersions of space-time \((E, g)\) into a pseudo Euclidean manifold of six dimensions, with different signatures:
- The Kasner immersion: [5]
\[ ds^2 = dY_1^2 + dY_2^2 - dY_3^2 - dY_4^2 - dY_5^2 - dY_6^2. \]

- The Fronsdal immersion: [7]
\[ ds^2 = dY'_1^2 - dY'_2^2 - dY'_3^2 - dY'_4^2 - dY'_5^2 - dY'_6^2, \]

Respectively given by (using \(2m \equiv 1\))

\[
\begin{align*}
Y_1 &= (1 - 1/r)^{1/2}\cos t \\
Y_2 &= (1 - 1/r)^{1/2}\sin t \\
Y_3 &= f(r), \quad (df/dr)^2 = \frac{1 + 4r^2}{4r^3(r-1)} \\
Y_4 &= rs\sin \theta \sin \phi \\
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\end{align*}
\]

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\begin{align*}
Y'_1 &= 2(1 - 1/r)^{1/2}\sinh(t/2) \\
Y'_2 &= 2(1 - 1/r)^{1/2}\cosh(t/2) \\
Y'_3 &= g(r), \quad (dg/dr)^2 = \frac{(r^2 + r + 1)}{r^3} \\
Y'_4 &= rs\sin \theta \sin \phi \\
Y'_5 &= rs\sin \theta \cos \phi \\
Y'_6 &= r\cos \theta
\end{align*}
\]

(1)

Notice that \(Y'_3\) is defined for \(r > 0\), while \(Y_3\) is defined only for \(r > 1\), suggesting the extension of \((E, g)\). In order to determine the metric \(g'\) (extension of \(g\), define the new coordinates \(u\) and \(v\) by:
- For \(r > 2m\),
\[
v = \frac{1}{4m} \left( \frac{r}{2m} \right)^{1/2} \exp \left( \frac{r}{4m} \right) Y'_1 \quad \text{and} \quad u = \frac{1}{4m} \left( \frac{r}{2m} \right)^{1/2} \exp \left( \frac{r}{4m} \right) Y'_2.\]

(2)
- For $0 < r < 2m$,

$$v = \frac{1}{4m} \left( \frac{-r}{2m} \right)^{1/2} \exp \left( \frac{r}{4m} \right) Y'_{1} \quad \text{and} \quad u = \frac{1}{4m} \left( \frac{-r}{2m} \right)^{1/2} \exp \left( \frac{r}{4m} \right) Y'_{2}, \quad (3)$$

where

$$u^2 - v^2 = \left( \frac{r}{2m} - 1 \right) \exp \left( \frac{r}{2m} \right) \iff Y'_{2}^2 - Y'_{1}^2 = 16m^2 \left( 1 - \frac{2m}{r} \right). \quad (4)$$

Now $r = r(Y'_{1}, Y'_{2})$ is implicitly defined by last equation, while $t = t(Y'_{1}, Y'_{2})$ is implicitly defined by

$$Y'_{1}/Y'_{2} = \tgh \left( \frac{t}{4m} \right). \quad (5)$$

Finally, the metric $g'$ in the new coordinates results

$$ds^2 = (32m^3/r) \exp(-r/2m) (dv^2 - du^2) - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (6)$$

Curiously this metric is exactly the same metric encountered by Kruskal. The $u$ and $v$ coordinates, $Q^2$ and all characteristics of Kruskal metric are given by $(E' = Q^2 \times S^2, g')$, without a singularity at $r = 2m$. We know that $(E, g)$ is disconnected because it is composed by two connected components. When we calculated the extension $(E', g')$ through the Fronsdal immersion we see that it is connected. [9]

In the following section, we will prove that the topology of $(E, g)$ is different from the topology of $(E', g')$. [9]

### 3 Schwarzschild’s Brane-world Topology

Let $(U_{\alpha}, \varphi_{\alpha})$ be a coordinate system on a point $p \in M^n$ of a differentiable manifold $M^n$. Generally speaking, the topology of a manifold $M^n$ is defined naturally through its open sets. If $A \subset M^n$, then $A$ is an open set of $M^n$ if $\varphi_{\alpha}(A \cap \varphi_{\alpha}^{-1}(U_{\alpha}))$ is an open set of $\mathbb{R}^n$, $\forall \alpha$. In other words, the atlas of $M^n$ determines its topology. [5]

The following theorem shows that the topology of $(E, g)$ is different from that of $(E', g')$.

**Theorem:**

The topology of a gravitational field outside of a body with spherical symmetry is given by $\mathbb{R}^2 \times S^2$.

**Proof:**
By construction, \( E = [P_1^2 \cup P_{II}^2] \times S^2 \) and \( E' = Q^2 \times S^2 \). The topology of \( E \) is the Cartesian product topology of \([P_1^2 \cup P_{II}^2]\) by \( S^2 \), while that the topology of \( E' \) is the Cartesian product topology of \( Q^2 \) by \( S^2 \). The topology of \( S^2 \subset \mathbb{R}^3 \) is the usual topology induced by the topological space \((\tau_3, \mathbb{R}^3)\). On the other hand, the topologies of \([P_1^2 \cup P_{II}^2]\) \subset \mathbb{R}^2 \) and of \( Q^2 \subset \mathbb{R}^2 \), respectively \( \tau_p \) and \( \tau_q \), will be induced from \((\tau_2, \mathbb{R}^2)\). Since \( Q^2 \) is an extension of \([P_1^2 \cup P_{II}^2]\), we may define one isometric immersion,

\[
\psi : [P_1^2 \cup P_{II}^2] \rightarrow Q^2.
\]

Therefore, for an open set \( A \subset \mathbb{R}^2 \) given by

\[
A = \{(t, r) \in \mathbb{R}^2 | t^2 + (r - 2m)^2 < m^2 \text{ and } r > 0\},
\]

we have \( A \cap [P_1^2 \cup P_{II}^2] = A - \{(t, r) \in \mathbb{R}^2 | r = 2m\}. \) This is an open set of the topological space \([P_1^2 \cup P_{II}^2]\), composed of two connected components. Observe that open sets form a topological basis for the semi-plane \( t - r, r > 0 \). However, we have that \( \psi(A \cap [P_1^2 \cup P_{II}^2]) \) is given for an open set composed by four connected components. As the lines \( L_1 \) and \( L_2 \) defined for \( r = 2m \) from equation (2) we have that

\[
\{[\psi(A \cap [P_1^2 \cup P_{II}^2]) \cup L_1 \cup L_2]\} \cap D = B,
\]

where \( D \) is an open disk on \( \mathbb{R}^2 \) with center in the origin of \( Q^2 \). The set \( B \) is a plane disk, in the new coordinates \( r = r(Y_1', Y_2') \) and \( t = t(Y_1', Y_2') \). In this manner the topology of \( Q^2 \) is given by open sets of \( \mathbb{R}^2 \). Finally we have that the topology of \((E, g)\) is equal to \((\mathbb{R}^2 - \{(t, r) \in \mathbb{R}^2 | r = 2m\}) \times S^2\), clearly different of the topology of space-time \((E', g')\) that is \( \mathbb{R}^2 \times S^2 \).}

Next section we investigate changes in the bulk’s signature when we have changes in the topology of brane-world, under some conditions.

4 Change in the Bulk’s Signature

**Theorem:**

Let \((M^n, g)\) and \((\tilde{M}^D, \bar{g})\), \( D \geq n \), pseudo-Riemannian manifolds and consider \( Y : (M^n, g) \rightarrow (\tilde{M}^D, \tilde{g}) \) a local isometric immersion. Let \( \tau_\eta \) the topology of the \( Y(W^n) \) isometric immersed submanifold of \((\tilde{M}^D, \tilde{g})\), \( W^n \) a neighborhood of \( p \in (M^n, g) \). If we change topology of \( Y(W^n) \) to \( \tau'_\eta \), and if \( \det(g_{ij}) \), \( \det(g'_{\mu\nu}) \) differ in sign at a point, then there exist a change of signature of form assigned to \( \bar{g} \).

**Proof:**
This result basically came from linear algebra and from differentiable manifolds elementary theory. Suppose that \( \eta_1, \eta_2 \) are charts of \( Y(W^n) \) with intersecting domains \( U, V \). Then \( \eta_2 \circ \eta_1^{-1} \) is a diffeomorphism and so its domain \( \eta_1(U \cap V) \) must be open in \( \mathcal{R}^D \). Since \( (U \cap V) \) is a subset of \( U \) such that \( \eta_1(U \cap V) \) is open in \( \mathcal{R}^D \), then \( \eta_1 \mid_{(U \cap V)} \) is a chart of \( Y(W^n) \) with domain \( (U \cap V) \). These arguments are sufficient to conclude that the collection of coordinate domains of manifold \( Y(W^n) \) forms a basis for a topology on the set \( Y(W^n) \). The topology thus induced on the set \( Y(W^n) \) by its \( C^\infty \) structure is called \( \tau_\eta \) topology of the manifold \( Y(W^n) \). With this topology, a non-empty subset \( U \) of \( Y(W^n) \) is open iff each point of \( U \) has a coordinate neighborhood which lies in \( U \).  

We note that if \( Y : (M^n, g) \rightarrow (\bar{M}^D, \bar{g}) \) is a local isometric immersion, then \( \exists p \in W_n \subset M_n, W_n \) a neighborhood of \( p \), such that \( Y \mid_{W_n} \) is an embedding, and if \( Y(W_n) \) is assigned the metric tensor such that the induced map \( W_n \rightarrow Y(W_n) \) is an isometry, then \( Y(W_n, g) \) is a pseudo-Riemannian submanifold of \( (\bar{M}^D, \bar{g}) \). In other words we say that locally an isometric immersion is essentially a pseudo-Riemannian submanifold.

Denote \( \tau_\eta' \) the topology of \( Y(W^n, g) \) assigned to atlas \( A \) and another topology \( \tau_\eta'' \) of \( Y(W^n, g) \) assigned to new atlas \( A' \), both induced topologies on the set \( Y(W^n, g) \) by its \( C^\infty \) structure.

Let \( g_{ij} \) the components of metric tensor \( g \) on the differential quadratic form, \( g_{ij} dx^i dx^j \). By a real transformation this quadratic form at a point \( p \in W^n \) is represented by
\[
(dx^1)^2 + ... + (dx^r)^2 - (dx^{r+1})^2 - ... - (dx^n)^2,
\]
where \( S = 2r - n \) is the signature of the form.

Suppose it exists a transformation such that
\[
g'_{\mu\nu} = g_{ij} \frac{\partial x^i}{\partial x'^\mu} \frac{\partial x^j}{\partial x'^\nu},
\]
from the rule for multiplication of determinants we have
\[
det(g'_{\mu\nu}) = det(g_{ij})J^2,
\]
where \( J \) is the Jacobian of the transformation from \( g_{ij} \) to \( g'_{\mu\nu} \). Now suppose that \( det(g_{ij}) \) and \( det(g'_{\mu\nu}) \) differ in sign at a point, for instance at \( p \in (W^n, g) \). Thus we have an imaginary transformation from \( g_{ij} \) to \( g'_{\mu\nu} \) at \( p \in (W^n, g) \). 

For this transformation at \( p \) the form \( g_{\mu\nu} dx^\mu dx^\nu \) can be represented by
\[
(dx'^1)^2 + ... + (dx'^r)^2 - (dx'^{r+1})^2 - ... - (dx'^n)^2,
\]
where \( S' = 2r' - n \) is the signature of this form, clearly \( S \neq S' \).

We note that \( g \) is the induced metric of \( \bar{g} \), thus we have \( g(u,v) = \bar{g}(dY(u),dY(v)), \forall u,v \in T_p(W^n,\bar{g}) \). Using the facts before it is easy to see that exists a change of signature of form assigned to \( \bar{g} \) at point \( Y(p) \in Y(W^n,\bar{g}) \). We remember that for real transformations the signature of \( \bar{g} \) is invariant. △

5 The Schwarzschild’s Brane-world Immersed into \((\bar{M}^6, \bar{g})\)

Randall and Sundrum have proposed an interesting scenario of extra non-compact dimensions in which four-dimensional gravity emerges as a low energy effective theory, to solve the hierarchy problems of the fundamental interactions.\([11]\) This proposal is based on the assumption that ordinary matter and its gauge interactions are confined within a four dimensional hypersurface, the physical brane, immersed in a five-dimensional space of constant curvature. In order to describe the real world, the Randall-Sundrum scenario has to satisfy all the existing tests of General Relativity, with base in the motion of material particles within the Schwarzschild’s brane-world that is given by four-dimensional geodesic equation. That is an excellent and famous model, but from the point of view of mathematics and of cosmological observations some constraints exist. The principal problem that appears on the mathematical model of the brane-worlds has origin on the choose of bulk, when we require a compatible immersion of the Schwarzschild’s brane-world into five-dimensional bulk of constant curvature.\([12, 13]\) An alternative bulk to Randall-Sundrum model is when we consider the case of six-dimensional flat bulk, where the Schwarzschild’s brane-world has a compatible immersion in \((\bar{M}^6, \bar{g})\).\([14]\)

According to section (2) to (5) we concluded that brane-world \((E, g)\) is disconnected because it is composed by two connected components. We have that the topology of \((E, g)\) is given by, \((R^2 - \{(t, r) \in R^2 | r = 2m\}) \times S^2\) that is different of the topology of the brane-world \((E', g')\) which is equal to \(R^2 \times S^2\), this latter is due to the theorem (1). For a trivial example of the theorem (2), suppose that space-time \((E, g)\) is immersed into a pseudo-Euclidean manifold of six dimensions \((M^6, \bar{g})\), where \(\bar{g}\) is given by Kasner and signature \(S = -2\), as we can see from section (2). Now from the new coordinates, suppose that \((E', g')\) is immersed into a pseudo-Euclidean manifold of six dimensions, \((\bar{M}^6, \bar{g})\), we must have a change of signature from \(S = -2\) to \(S' = -4\), for the \(\bar{g}\) metric that is given by Fronsdal, (section
We prove that beginning from a brane-world immersed in an environment space of higher dimensional (bulk) it is possible to change the bulk’s signature when we change the topology of brane-world. This result show us that we can have nature geometric and topological constraints which can to enjoin some link with physical properties of the space-time on brane-world context. For instance, the fact that we assume the brane-world \((E', g')\) to be connected, because disconnected components of the universe cannot interact by means of any signals and the observations are confined to the connected component wherein the observer is situated. In this case, some constraints appear in the bulk’s signature \((M^6, \bar{g})\), where we can to notice that \(\bar{g}\) metric must have only one time coordinate.

7 Acknowledgments

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b) The Schwarzschild black hole $(B_4, g)$:
$$B_4 = P_{II}^2 \times S^2; \quad P_{II}^2 = \{(t, r) \in \mathbb{R}^2 | 0 < r < 2m\}$$

In both cases, $S^2$ is the sphere of radius $r$ and the metric $g$ is given by the usual metric of Schwarzschild. We know that $(B_4, g)$ and $(V_4, g)$ may be extensible for $r = 2m$. The extension of $(V_4, g)$ was calculated by Kruskal but it was suggested by C. Fronsdal one year before. [3, 7]

Now we use the isometric immersion formalism to establish the extension of $(E, g) = ([P_I^2 \cup P_{II}^2] \times S^2, g)$, denoted by $(E', g') = (Q^2 \times S^2, g')$, where $Q^2$ is the Kruskal plane.

Consider two known isometric immersions of space-time $(E, g)$ into a pseudo Euclidean manifold of six dimensions, with different signatures:

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- The Fronsdal immersion: [7]
  $$ds^2 = dY_1'^2 - dY_2'^2 - dY_3'^2 - dY_4'^2 - dY_5'^2 - dY_6'^2,$$

Respectively given by (using $2m \equiv 1$)

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Y_2 &= (1 - 1/r)^{1/2} \sin \varphi \\
Y_3 &= f(r), \quad (df/dr)^2 = \frac{1 + 4r^2}{4r^2(r-1)} \\
Y_4 &= r \sin \theta \sin \varphi \\
Y_5 &= r \sin \theta \cos \varphi \\
Y_6 &= r \cos \theta
\end{align*}$$

and

$$\begin{align*}
Y_1' &= 2(1 - 1/r)^{1/2} \sinh(t/2) \\
Y_2' &= 2(1 - 1/r)^{1/2} \cosh(t/2) \\
Y_3' &= g(r), \quad (dg/dr)^2 = \frac{(r^2 + r + 1)}{r^2} \\
Y_4' &= r \sin \theta \sin \varphi \\
Y_5' &= r \sin \theta \cos \varphi \\
Y_6' &= r \cos \theta
\end{align*}$$

Notice that $Y_3'$ is defined for $r > 0$, while $Y_3$ is defined only for $r > 1$, suggesting the extension of $(E, g)$. In order to determine the metric $g'$ (extension of $g$), define the new coordinates $u$ and $v$ by:

- For $r > 2m$,
  $$v = \frac{1}{4m} \left(\frac{r}{2m}\right)^{1/2} \exp\left(\frac{r}{4m}\right)Y_1' \quad \text{and} \quad u = \frac{1}{4m} \left(\frac{r}{2m}\right)^{1/2} \exp\left(\frac{r}{4m}\right)Y_2'.$$ (2)
- For $0 < r < 2m$,

$$v = \frac{1}{4m} \left( \frac{-r}{2m} \right)^{1/2} \exp(\frac{r}{4m}) Y_1'$$

and

$$u = \frac{1}{4m} \left( \frac{-r}{2m} \right)^{1/2} \exp(\frac{r}{4m}) Y_2',$$

(3)

where

$$u^2 - v^2 = \left( \frac{r}{2m} - 1 \right) \exp(\frac{r}{2m}) \iff Y_2'^2 - Y_1'^2 = 16m^2 \left( 1 - \frac{2m}{r} \right).$$

(4)

Now $r = r(Y_1', Y_2')$ is implicitly defined by last equation, while $t = t(Y_1', Y_2')$ is implicitly defined by

$$Y_1'/Y_2' = \tgh(\frac{t}{4m}).$$

(5)

Finally, the metric $g'$ in the new coordinates results

$$ds^2 = (32m^3/r) \exp(-r/2m)(dv^2 - du^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(6)

Curiously this metric is exactly the same metric encountered by Kruskal. The $u$ and $v$ coordinates, $Q^2$ and all characteristics of Kruskal metric are given by $(E' = Q^2 \times S^2, g')$, without a singularity at $r = 2m$. We know that $(E, g)$ is disconnected because it is composed by two connected components. When we calculated the extension $(E', g')$ through the Fronsdal immersion we see that it is connected. [9]

In the following section, we will prove that the topology of $(E, g)$ is different from the topology of $(E', g')$.[9]

### 3 Schwarzschild’s Brane-world Topology

Let $(U_\alpha, \varphi_\alpha)$ be a coordinate system on a point $p \in M^n$ of a differentiable manifold $M^n$. Generally speaking, the topology of a manifold $M^n$ is defined naturally through its open sets. If $A \subset M^n$, then $A$ is an open set of $M^n$ if

$$\varphi_\alpha(A \cap \varphi_\alpha^{-1}(U_\alpha))$$

is an open set of $\mathbb{R}^n$, $\forall \alpha$. In other words, the atlas of $M^n$ determines its topology. [5]

The following theorem shows that the topology of $(E, g)$ is different from that of $(E', g')$.

**Theorem:**

The topology of a gravitational field outside of a body with spherical symmetry is given by $\mathbb{R}^2 \times S^2$.

**Proof:**
By construction, $E = [P^2_I \cup P^2_{II}] \times S^2$ and $E' = Q^2 \times S^2$. The topology of $E$ is the Cartesian product topology of $[P^2_I \cup P^2_{II}]$ by $S^2$, while that the topology of $E'$ is the Cartesian product topology of $Q^2$ by $S^2$. The topology of $S^2 \subset \mathbb{R}^3$ is the usual topology induced by the topological space $(\tau_3, \mathbb{R}^3)$. On the other hand, the topologies of $[P^2_I \cup P^2_{II}] \subset \mathbb{R}^2$ and of $Q^2 \subset \mathbb{R}^2$, respectively $\tau_p$ and $\tau_q$, will be induced from $(\tau_2, \mathbb{R}^2)$. Since $Q^2$ is an extension of $[P^2_I \cup P^2_{II}]$, we may define one isometric immersion,

$$\psi : [P^2_I \cup P^2_{II}] \longrightarrow Q^2.$$ 

Therefore, for an open set $A \subset \mathbb{R}^2$ given by

$$A = \{(t, r) \in \mathbb{R}^2 | t^2 + (r - 2m)^2 < m^2 \text{ and } r > 0 \}.$$

we have $A \cap [P^2_I \cup P^2_{II}] = A - \{(t, r) \in \mathbb{R}^2 | r = 2m\}$. This is an open set of the topological space $[P^2_I \cup P^2_{II}]$, composed of two connected components. Observe that open sets form a topological basis for the semi-plane $t - r$, $r > 0$. However, we have that $\psi(A \cap [P^2_I \cup P^2_{II}])$ is given for an open set composed by four connected components. As the lines $L_1$ and $L_2$ defined for $r = 2m$ from equation (2) are on $Q^2$ we have that

$$\{(\psi(A \cap [P^2_I \cup P^2_{II}]) \cup L_1 \cup L_2) \cap D = B,$$

where $D$ is an open disk on $\mathbb{R}^2$ with center in the origin of $Q^2$. The set $B$ is a plane disk, in the new coordinates $r = r(Y'_1, Y'_2)$ and $t = t(Y'_1, Y'_2)$. In this manner the topology of $Q^2$ is given by open sets of $\mathbb{R}^2$. Finally we have that the topology of $(E, g)$ is equal to $(\mathbb{R}^2 - \{(t, r) \in \mathbb{R}^2 | r = 2m\}) \times S^2$, clearly different of the topology of space-time $(E', g')$ that is $\mathbb{R}^2 \times S^2. \triangle [9]

Next section we investigate changes in the bulk’s signature when we have changes in the topology of brane-world, under some conditions.

4 Change in the Bulk’s Signature

Theorem:

Let $(M^n, g)$ and $(\tilde{M}^D, \tilde{g})$, $D \geq n$, pseudo-Riemannian manifolds and consider $Y : (M^n, g) \longrightarrow (\tilde{M}^D, \tilde{g})$ a local isometric immersion. Let $\tau_\eta$ the topology of the $Y(W^n)$ isometric immersed submanifold of $(\tilde{M}^D, \tilde{g})$, $W^n$ a neighborhood of $p \in (M^n, g)$. If we change topology of $Y(W^n)$ to $\tau'_\eta$, and if $\det(g_{ij})$, $\det(g_{\mu\nu}')$ differ in sign at a point, then there exist a change of signature of form assigned to $\tilde{g}$.

Proof:
This result basically came from linear algebra and from differentiable manifolds elementary theory. Suppose that $\eta_1, \eta_2$ are charts of $Y(W^n)$ with intersecting domains $U, V$. Then $\eta_2 \circ \eta_1^{-1}$ is a diffeomorphism and so its domain $\eta_1(U \cap V)$ must be open in $\mathbb{R}^D$. Since $(U \cap V)$ is a subset of $U$ such that $\eta_1(U \cap V)$ is open in $\mathbb{R}^D$, then $\eta_1|_{(U \cap V)}$ is a chart of $Y(W^n)$ with domain $(U \cap V)$. These arguments are sufficient to conclude that the collection of coordinate domains of manifold $Y(W^n)$ forms a basis for a topology on the set $Y(W^n)$. The topology thus induced on the set $Y(W^n)$ by its $C^\infty$ structure is called $\tau_\eta$ topology of the manifold $Y(W^n)$. With this topology, a non-empty subset $U$ of $Y(W^n)$ is open iff each point of $U$ has a coordinate neighborhood which lies in $U$. [10]

We note that if

\[ Y : (M^n, g) \rightarrow (\bar{M}^D, \bar{g}) \]

is a local isometric immersion, then $\exists p \in W_n \subset M_n$, $W_n$ a neighborhood of $p$, such that $Y|_{W_n}$ is an embedding, and if $Y(W_n)$ is assigned the metric tensor such that the induced map $W_n \rightarrow Y(W_n)$ is an isometry, then $Y(W_n, g)$ is a pseudo-Riemannian submanifold of $(\bar{M}^D, \bar{g})$. In other words we say that locally an isometric immersion is essentially a pseudo-Riemannian submanifold.

Denote $\tau_\eta$ the topology of $Y(W^n, g)$ assigned to atlas $A$ and another topology $\tau'_\eta$ of $Y(W^n, g)$ assigned to new atlas $A'$, both induced topologies on the set $Y(W^n, g)$ by its $C^\infty$ structure.

Let $g_{ij}$ the components of metric tensor $g$ on the differential quadratic form, $g_{ij} dx^i dx^j$. By a real transformation this quadratic form at a point $p \in W^n$ is represented by

\[
(dx^1)^2 + \ldots + (dx^r)^2 - (dx^{r+1})^2 - \ldots - (dx^n)^2,
\]

where $S = 2r - n$ is the signature of the form.

Suppose it exists a transformation such that

\[ g'_{\mu\nu} = g_{ij} \left( \frac{\partial x^i}{\partial x'^\mu} \right) \left( \frac{\partial x^j}{\partial x'^\nu} \right), \]

from the rule for multiplication of determinants we have

\[ \det(g'_{\mu\nu}) = \det(g_{ij})J^2, \]

where $J$ is the Jacobian of the transformation from $g_{ij}$ to $g'_{\mu\nu}$. Now suppose that $\det(g_{ij})$ and $\det(g'_{\mu\nu})$ differ in sign at a point, for instance at $p \in (W^n, g)$. Thus we have an imaginary transformation from $g_{ij}$ to $g'_{\mu\nu}$ at $p \in (W^n, g)$. [2]

For this transformation at $p$ the form $g_{\mu\nu} dx^\mu dx^\nu$ can be represented by

\[
(dx'^1)^2 + \ldots + (dx'^r)^2 - (dx'^{r+1})^2 - \ldots - (dx'^n)^2,
\]
where $S' = 2r' - n$ is the signature of this form, clearly $S \neq S'$.

We note that $g$ is the induced metric of $\bar{g}$, thus we have $g(u, v) = \bar{g}(dY(u), dY(v)), \forall u, v \in T_P(W^n, g)$. Using the facts before it is easy to see that exists a change of signature of form assigned to $\bar{g}$ at point $Y(p) \in Y(W^n, g)$. We remember that for real transformations the signature of $\bar{g}$ is invariant. ∆

5 The Schwarzschild’s Brane-world Immersed into $(\bar{M}^6, \bar{g})$

Randall and Sundrum have proposed an interesting scenario of extra non-compact dimensions in which four-dimensional gravity emerges as a low energy effective theory, to solve the hierarchy problems of the fundamental interactions. [11] This proposal is based on the assumption that ordinary matter and its gauge interactions are confined within a four dimensional hypersurface, the physical brane, immersed in a five-dimensional space of constant curvature. In order to describe the real world, the Randall-Sundrum scenario has to satisfy all the existing tests of General Relativity, with base in the motion of material particles within the Schwarzschild’s brane-world that is given by four-dimensional geodesic equation. That is an excellent and famous model, but from the point of view of mathematics and of cosmological observations some constraints exist. The principal problem that appears on the mathematical model of the brane-worlds has origin on the choose of bulk, when we require a compatible immersion of the Schwarzschild’s brane-world into five-dimensional bulk of constant curvature. [12] [13] An alternative bulk to Randall-Sundrum model is when we consider the case of six-dimensional flat bulk, where the Schwarzschild’s brane-world has a compatible immersion in $(\bar{M}^6, \bar{g})$. [14]

According to section (2) to (5) we concluded that brane-world $(E, g)$ is disconnected because it is composed by two connected components. We have that the topology of $(E, g)$ is given by, $(R^2 - \{ (t, r) \in R^2 | r = 2m \}) \times S^2$ that is different of the topology of the brane-world $(E', g')$ which is equal to $R^2 \times S^2$, this latter is due to the theorem (1). For a trivial example of the theorem (2), suppose that space-time $(E, g)$ is immersed into a pseudo-Euclidean manifold of six dimensions $(\bar{M}^6, \bar{g})$, where $\bar{g}$ is given by Kasner and signature $S = -2$, as we can see from section (2). Now from the new coordinates, suppose that $(E', g')$ is immersed into a pseudo-Euclidean manifold of six dimensions, $(\bar{M}^6, \bar{g})$, we must have a change of signature from $S = -2$ to $S' = -4$, for the $\bar{g}$ metric that is given by Fronsdal, (section
We prove that beginning from a brane-world immersed in an environment space of higher dimensional (bulk) it is possible to change the bulk’s signature when we change the topology of brane-world. This result show us that we can have nature geometric and topological constraints which can to enjoin some link with physical properties of the space-time on brane-world context. For instance, the fact that we assume the brane-world $(E', g')$ to be connected, because disconnected components of the universe cannot interact by means of any signals and the observations are confined to the connected component wherein the observer is situated. In this case, some constraints appear in the bulk’s signature $(M^6, \bar{g})$, where we can to notice that $\bar{g}$ metric must have only one time coordinate.

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