On Some Invariants of Birkhoff Billiards Under Conjugacy

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Abstract—In the class of strictly convex smooth boundaries each of which has no strip around its boundary foliated by invariant curves, we prove that the Taylor coefficients of the “normalized” Mather’s $\beta$-function are invariant under $C^\infty$-conjugacies. In contrast, we prove that any two elliptic billiard maps are $C^0$-conjugate near their respective boundaries, and $C^\infty$-conjugate, near the boundary and away from a line passing through the center of the underlying ellipse. We also prove that, if the billiard maps corresponding to two ellipses are topologically conjugate, then the two ellipses are similar.

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1. INTRODUCTION

A billiard is a mathematical model of the dynamics of a confined massless particle without friction which reflects elastically off the boundary (without friction): the particle moves along a straight line with constant speed until it hits the boundary, then reflects off with a reflection angle equal to the angle of incidence and follows the reflected straight line. This seemingly simple and fascinating dynamical system was introduced by G.D. Birkhoff [4] in 1920. Since then, it has captured much of attention from both physicists and mathematicians. We refer the reader to [17] and references therein for more details on billiards. It is presently a very active and popular subject with many challenging open problems.

We shall deal in this paper with the discrete billiard dynamics defined as follows. Let $\Omega$ be a bounded strictly convex domain in $\mathbb{R}^2$ with $C^r$ boundary $\partial \Omega$, with $r \geq 3$1). The phase space $\mathcal{M}$ of the billiard map consists of unit vectors $(x,v)$ whose foot points $x$ are on $\partial \Omega$ and that have inward directions. The billiard ball map $f: \mathcal{M} \to \mathcal{M}$ takes $(x,v)$ to $(x',v')$, where $x'$ represents the point where the trajectory starting at $x$ with velocity $v$ hits the boundary $\partial \Omega$ again, and $v'$ is the reflected velocity, according to the standard reflection law: the angle of incidence is equal to the angle of reflection.

Assume that the boundary $\partial \Omega$ is of length $\ell$ parameterized by arc-length $s$ and let $\gamma: \mathbb{T} \to \mathbb{R}^2$ denote such a parameterization, with $\mathbb{T} := \mathbb{R}/\ell\mathbb{Z}$. Let $\theta$ be the angle between $v$ and the positive
Definition 2. A curve $\Gamma \subset \Omega$ is called a caustic for the billiard in the domain $\Omega$ if any billiard orbit having one segment tangent to $\Gamma$ has all its segments tangent to it.

The caustics we are interested in are smooth and convex, and are such that all their respective corresponding tangential billiard trajectories have the same rotation number which is then referred to as the rotation number of the caustic.

A notion associated to a caustic is the so-called Lazutkin invariant.

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3) See [14, 17] for details.
4) In the present paper, rationals are considered in reduced form.
5) Throughout this paper, by smooth we mean $C^\infty$-smooth.
6) A caustic may be disconnected, nonconvex or singular.
Definition 3. The Lazutkin invariant \( \mathcal{L}_\Gamma(\omega) \) associated to a caustic \( \Gamma \) of rotation number \( \omega \) is given by
\[
\mathcal{L}_\Gamma(\omega) := \text{dist}(A, P) + \text{dist}(P, B) - |\widehat{AB}|,
\]
where \( A, B \in \Gamma, p \in \partial\Omega \), the lines \( (PA) \) and \( (PB) \) are tangent to \( \Gamma \) at the points \( A \) and \( B \), respectively, and \( |\widehat{AB}| \) is the length of the dashed arc of the caustic joining \( A \) to \( B \), i.e., the one located on the same side of the line \( (AB) \) as \( P \). It is well known that the Lazutkin invariant is independent of \( P \) (see Fig. 1).

![Fig. 1. Lazutkin invariant.](image)

It is easy to check that in the circular billiard concentric circles are caustics foliating the corresponding disk punctured at the origin. The peculiarity of this property is justified by the following exciting result due to M. Bialy.

Theorem 1 ([2]). If the phase space of the billiard ball map is foliated by continuous invariant curves which are not null-homotopic, then it is a circular billiard.

In particular, the answer to Question 1 is affirmative whenever one of the domains is a circle, since the existence of a global foliation of the phase cylinder by homotopically nontrivial invariant curves is invariant under conjugacy.

In general, billiards in ellipses are locally integrable: a neighborhood of their boundary is foliated by caustics. Then Birkhoff asked whether there are other examples of (locally) integrable billiards. He actually conjectured:

Conjecture 1 (Birkhoff). Amongst all convex billiards, only those in ellipses (circles being a distinct special case) are integrable.

In this paper we are mainly concerned with the invariant of Birkhoff billiards under conjugacies when both of the domains are nonintegrable and with the existence of conjugacy (local and global) for elliptic billiards tables (which corresponds to the integrable case according to the Birkhoff conjecture). Before summarizing the results of this work, let us introduce the notion of “normalized” Mather’s \( \beta \)-function.

The Mather’s \( \beta \)-function (or Mather’s minimal average action) is a key function in the celebrated Aubry–Mather theory as it encodes several important properties of the dynamics. The Mather’s \( \beta \)-function can be defined for any exact area-preserving twist map, not necessarily a billiard map. Roughly speaking, it associates to any fixed rotation number (not only rational ones) the minimal average action of orbits with that rotation number (whose existence, inside a suitable interval, is ensured by the twist condition), and more precisely defined as follows.
Definition 4. The Mather’s $\beta$–function at $p/q$ is defined as

$$\beta\left(\frac{p}{q}\right) := -\frac{1}{q} \max\{\text{length}(\mathcal{O}) : \mathcal{O} \in \text{Per}(p/q)\},$$

where length($\mathcal{O}$) denotes the Euclidean length of the orbit $\mathcal{O}$ seen as a piecewise linear curve. Then, by strict convexity of $\beta$ (see [12]), $\beta$ is extended continuously to $\mathbb{R}$ by setting

$$\beta(\omega) := \lim_{n \to \infty} \beta\left(\frac{p_n}{q_n}\right),$$

for any sequence $\{p_n/q_n\}_n$ of rationals converging to $\omega$.

Definition 5. For a given domain $\Omega$, we define the “normalized” Mather’s $\beta$-function as the function $\lambda^{-3} \sqrt{\ell} (\beta + \ell \mathbf{id})$, where $\beta$ is the Mather’s $\beta$-function, $\lambda := \int_0^\ell \kappa(s)^{2/3} ds$ is the Lazutkin perimeter, $\kappa$ the curvature of $\partial \Omega$ and $\ell$ its perimeter.

The Taylor coefficients of the Mather’s $\beta$-function at 0 (see, e.g., [16] where they are computed) are related to the so-called Marvizi–Melrose invariants which were introduced by S. Marvizi and R. Melrose in [13] where they proved the following. For given $p,q \in \mathbb{N}$, denote by

$$L_{p,q} := \sup\{\text{length}(\mathcal{O}) : \mathcal{O} \in \text{Per}(p/q)\},$$

$$l_{p,q} := \inf\{\text{length}(\mathcal{O}) : \mathcal{O} \in \text{Per}(p/q)\},$$

and observe that the supremum and the infimum are attained as the set $\text{Per}(p/q)$ is compact.

Theorem 2 ([13]). For any positive integers $p$ and $k$ we have

$$\lim_{q \to \infty} q^k (L_{p,q} - l_{p,q}) = 0.$$ 

Moreover, $L_{p,q}$ has an asymptotic expansion as $q \to \infty$:

$$L_{p,q} \sim p\ell_0 + \sum_{k=1}^{\infty} \frac{\ell_{k,p}}{q^{2k}},$$

where $\ell_0 = \ell$ is the length of the boundary of the billiard table and $\ell_{k,p}$ are constants depending on the curvature of the boundary.

The collection $\{\ell_{k,1}\}_{k \geq 0}$ constitutes the Marvizi–Melrose spectral invariants (also called Marvizi–Melrose invariants).

In the case of nonintegrables domains, we prove that the Taylor coefficients of the “normalized” Mather’s $\beta$-function are invariant under smooth conjugacies (see Theorem 3). In contrast, we prove that any two elliptic billiard maps are $C^0$-conjugate near their respective boundaries, and $C^\infty$-conjugate near their respective boundaries and off a line passing through the origin. We also prove that the billiard maps corresponding to two ellipses are topologically conjugate only if the two ellipses are similar (i.e., they are the same up to a rescaling and an isometry) or, equivalently, have the same eccentricity (see Theorem 4). Finally, assuming the Birkhoff conjecture, it follows that the answer to Question 1 is affirmative whenever one of the domains is integrable (See Corollary 1).

It is an easy exercise to check that two ellipses are similar iff they have the same eccentricity.
Theorem 3. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two strictly convex domains with smooth boundaries. Let $\beta_j$ be the Mather’s $\beta$-function of $\Omega_j$, $\ell_j := \text{length}(\partial \Omega_j)$, $\lambda_j$ be the curvature of $\partial \Omega_j$ with arc-length parameter $s$ and $\kappa_j := \int_0^s \kappa_j(s) s^2/3 ds$. Assume that no strip in $\Omega_2 \times [0, \pi)$ containing the boundary $\partial \Omega_2 \times \{0\}$ is foliated by invariant curves and

$$f_{\Omega_1} \circ \tilde{h} = \tilde{h} \circ f_{\Omega_2},$$

for some diffeomorphism $\tilde{h} \in C^\infty(\mathbb{T} \times [0, \pi))$. Then $\ell_1 \ell_2 = 1$ and $\lambda_1^{-3} \sqrt{\ell_1} (\beta_1(\omega) + \ell_1 \omega)$ and $\lambda_2^{-3} \sqrt{\ell_2} (\beta_2(\omega) + \ell_2 \omega)$ have the same Taylor expansion at $\omega = 0$. In particular, $\lambda_1^{-3} \sqrt{\ell_1} \mathcal{L}_1(\omega)$ and $\lambda_2^{-3} \sqrt{\ell_2} \mathcal{L}_2(\omega)$ have the same Taylor expansion at $\omega = 0$, where $\mathcal{L}_j(\omega)$ is the Lazutkin invariant of $\Omega_j$ corresponding to the convex caustic of rotation number $\omega$ whenever it exists.

Remark 2. The invariants in Theorem 3 are different for ellipses with different eccentricities as it follows [16, Eq. (28)]. Indeed, let us denote the (incomplete) elliptic integral of the first kind by

$$F: [0, 2\pi) \times (0, 1) \ni (\varphi, k) \mapsto \int_0^\varphi \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

and the complete elliptic integral of the first kind by $K(k) := F(\pi/2, k)$. Denote by $E(k) := \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta$ the complete elliptic integral of second type.

Then we have for $0 < e \leq 1$:

$$\frac{d}{de} \left( \frac{\beta^{(5)}(0)}{\lambda^3} \right)(e) = \frac{(16e^3 - 16e^2 + 17e - 17) K(e)^2 - 16(e^2 - 2) K(e)E(e) - 15E(e)^2}{2(e - 1)e} < 0.\tag{2.1}$$

Denote by $\{\mathcal{I}_n^{j+1}\}_n$ the Marvizi–Melrose integral invariants of $\Omega_j$ (see [13]). We conjecture the following relation between the Taylor coefficients of the Mather’s $\beta$-function and the Marvizi–Melrose integral invariants $\{\mathcal{I}_n^{j+1}\}_n$.

Conjecture 2. For any $n \in \mathbb{N}$,

$$\frac{d^{2n+1} \beta_j}{d\omega^{2n+1}}(0) = (\mathcal{I}_3^j)^{n+2} \cdot \sum_{\sigma_1, \ldots, \sigma_n, 3\sigma_1 + \cdots + 3\sigma_n = n-1; 3\sigma_1 + \cdots + (2n+1)\sigma_n = 5(n-1)} r_n(\sigma_1, \ldots, \sigma_n) \cdot (\mathcal{I}_3^j)^{\sigma_1} \cdots (\mathcal{I}_3^{2n+1})^{\sigma_n},\tag{2.2}$$

where each $r_n(\sigma_1, \ldots, \sigma_n) \in \mathbb{Q}$ is a constant independent of the billiard table.

In contrast to Theorem 3, the following holds for elliptic billiards and, hence, according to the Birkhoff conjecture 1, for convex integrable billiards.

Theorem 4. Let $\Omega_j$ be an ellipse with axes $a_j$ and $b_j$, $a_j > b_j$, and eccentricity $e_j$ and $f_{\Omega_j}$ be the associated billiard map ($j = 1, 2$). Then

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8\footnote{Observe that $\kappa_j > 0$ by strictly convexity and, therefore, $\lambda_j > 0$.}
9\footnote{We denote by $e$ the eccentricity.}
10\footnote{The Marvizi–Melrose integral invariants are indeed known to be algebraically equivalent to the Taylor coefficients of the Mather’s $\alpha$-function at $-\ell$ and, hence, to the Taylor coefficients of the Mather’s $\beta$-function at 0 as well (see, e.g., [14, Theorem 3.2.25]). However, to our best knowledge, no explicit algebraic formula relating them is known. Note also that the Marvizi–Melrose integral invariants $\{\mathcal{I}_n^{j+1}\}_n$ are algebraically equivalent to the Marvizi–Melrose invariants $\{\ell_n^{j+1}\}_{n \geq 0}$ (see [13, Theorem 5.15]).}
(i) The billiard map $f_1$ and $f_2$ are $C^0$-conjugate near their respective boundaries, i.e., there exist a neighborhood $\mathcal{V}_j$ of $\partial\Omega_j \times \{0\}$ with $\mathcal{V}_j \supset \partial\Omega_j \times \{0\}$ $(j = 1, 2)$, and such that their restrictions $f_1|_{\mathcal{V}_1}$ and $f_2|_{\mathcal{V}_2}$ are $C^0$-conjugate. Moreover, the restrictions $f_1|_{\mathcal{V}_1}$ and $f_2|_{\mathcal{V}_2}$ are $C^\infty$-conjugate, with $\mathcal{V}_j := \mathcal{V}_j \cap (\Lambda_j^* \times [0, \pi])$ and where\(^{11}\)

$$\Lambda_j^* := \partial\Omega_j \setminus \{p_j^*, -p_j^*\}, \text{ where } p_j^* := (a_j \cos \arctan^{-1}(1 - e_j^2), -b_j \sin \arctan^{-1}(1 - e_j^2)).$$

(ii) If the billiard maps $f_1$ and $f_2$ are (globally) $C^0$-conjugate, then $\Omega_1$ and $\Omega_2$ are similar (or, equivalently, $e_1 = e_2$).

As a consequence, assuming the Birkhoff conjecture, we have:

**Corollary 1.** Let $f$ and $g$ be smooth Birkhoff billiard maps corresponding to two strictly convex domains $\Omega_f$ and $\Omega_g$. Assume that $f$ and $g$ are topologically conjugate and that one of the two domains is integrable. Then $\Omega_f$ and $\Omega_g$ are similar.

3. PROOFS

3.1. Proof of Theorem 3

Given $j = 1, 2$, pick a symplectic change of coordinates $\phi_j$ with an associated Cantor set $\mathcal{C}_j$ corresponding to $f_{\Omega_j}$ given by Theorem A.2. Let

$$f_j := \phi_j^{-1} \circ f_{\Omega_j} \circ \phi_j, \quad \text{and} \quad h = (h_1, h_2) := \phi_1^{-1} \circ \tilde{h} \circ \phi_2.$$ 

Let $\beta_j := \beta_{\Omega_j}$, the minimal average action (or Mather’s $\beta$-function) of $f_{\Omega_j}$.

Observe that (2.1) reads

$$f_1 \circ h = h \circ f_2. \quad (3.1)$$

Now, recalling that\(^{12}\) $\alpha'_1(-\ell_1) = \alpha'_2(-\ell_2) = 0$, we have

$$f_1(h(s, -\ell_2)) \overset{(3.1)}{=} h(f_2(s, -\ell_2)) \overset{(A.3)}{=} h(s, -\ell_2), \quad (3.2)$$

i.e., $f_{\Omega_1}(\phi_1(h(s, -\ell_2))) = \phi_1(h(s, -\ell_2))$. Thus, $\phi_1(h_1(s, -\ell_2), h_2(s, -\ell_2)) = \phi_1(h(s, -\ell_2)) \in \mathbb{T} \times \{0\}$ \((A.2)\), whence

$$h_2(s, -\ell_2) = -\ell_1, \quad \forall \ s \in \mathbb{T}. \quad (3.3)$$

Given $j = 1$ and $\omega \in \alpha'_2(\mathcal{C}_2)$, let us denote by $\Gamma_\omega^2 := \mathbb{T} \times \{\beta'_j(\omega)\}$ the KAM torus of frequency $\omega$ of $f_j$ and by $\mathcal{D}_\omega^2$ the domain of the cylinder $\mathbb{T} \times \mathbb{R}$ bounded by the closed curves $\{\theta = \beta'_j(\omega)\}$ and $\{\theta = -\ell_j\}$.

Combining (3.1) and (A.3), we find that, for any $\omega \in \alpha'_2(\mathcal{C}_2) \setminus \{0\}$, $h(\Gamma_\omega^2)$ is an invariant curve of Diophantine frequency $\omega$ for $f_1$. Thus, as $h(\Gamma_\omega^2)$ is Lagrangian\(^{13}\), it is a KAM curve of frequency $\omega$ of $f_1$. Therefore, by uniqueness\(^{14}\), for any $\omega \in \alpha'_2(\mathcal{C}_2) \setminus \{0\}$,

$$h(\Gamma_\omega^2) = \Gamma_\omega^1, \quad (3.4)$$

and, by continuity, for any $(s, \theta) \in \mathbb{T} \times \mathcal{C}_2$,

$$\alpha'_1(h_2(s, \theta)) = \alpha'_2(\theta). \quad (3.5)$$

\(^{11}\)We use $-p$ to denote the symmetric of the point $p$ w.r.t. the origin.

\(^{12}\)We refer the reader to [16] for explicit formula of the coefficients of the Taylor’s expansion of the $\alpha$ and $\beta$ functions at the boundary.

\(^{13}\)$h(\Gamma_\omega^2)$ being a curve in the two-dimensional symplectic manifold.

\(^{14}\)$\text{Invariant tori in the two-dimensional case are in one-to-one correspondence with the rotation numbers (see [9, Remark 2.3]).}$
Thus, as \( h: \mathbb{T} \times [-\ell, -\ell + \varepsilon] \rightarrow \mathbb{T} \times [-\ell, -\ell + \varepsilon] \) is a \( C^\infty \)-diffeomorphism (for some \( \varepsilon_1, \varepsilon_2 > 0 \)), we infer
\[
 h(\mathcal{D}_\omega^2) = \mathcal{D}_\omega^1 \quad \text{and} \quad h(\partial \mathcal{D}_\omega^2) = \partial \mathcal{D}_\omega^1.
\]
(3.6)
Set \( J h := \det \partial_{(s,\theta)}h \). We claim that
\[
 \partial^k_s J h(s, -\ell_2) = 0, \quad \forall \ s \in \mathbb{T}, \ \forall \ k \geq 1.
\]
(3.7)
Fix \( s \in \mathbb{T} \). Indeed, we are going to construct inductively, for a given \( k \geq 1 \), a sequence \( \{\omega^k_n\}_n \setminus 0 \) such that
\[
 \partial^k_s J h(s, \beta^k_s(\omega^k_n)) = 0, \quad \forall \ n \geq 1,
\]
(3.8)
which implies (3.7) by continuity. Pick a strictly decreasing sequence \( \{\omega^0_n\}_n \subset \alpha'_2(\mathcal{C}_2) \setminus \{0\} \) converging to \( 0 \). As no strip near the boundary of \( \Omega_2 \) is foliated by invariant curves, then, by Mather's connecting theorem (see [8, 11]), any two successive invariant curves of \( f_2 \) are (asymptotically) connected by some orbit, the invariant curves being naturally ordered via their respective rotation numbers. Let then \( \mathcal{M}_2 \) be the closure of the union of all invariant curves and Mather's connecting orbits of \( f_2 \); in particular, \( \mathcal{M}_2 \) contains the family of KAM tori of \( f_2 \). As \( f_1 \) and \( f_2 \) are symplectic, we have \( \det \partial_{(s,\theta)}f_1 = \det \partial_{(s,\theta)}f_2 \equiv 1 \) and, therefore, differentiating (3.1), we obtain, for any \((s,\theta) \in \mathbb{R}^2 \), \( J h(f_2(s,\theta)) = J h(s,\theta) \), i.e., \( J h \) is constant along orbits of \( f_2 \). Consequently, \( J h \) is constant on \( \mathcal{M}_2 \) by continuity. Then, for any \( n \in \mathbb{N} \), as \( J h(s, \beta^k_s(\omega^k_n)) = J h(s, \beta^k_s(\omega^k_{n+1})) \), by the mean value theorem, there exists \( \omega^0_{n+1} < \omega^1_n < \omega^0_n \) such that \( \partial^k_s J h(s, \beta^k_s(\omega^k_n)) = 0 \). Observe that, by construction, \( \{\omega^k_n\}_n \setminus 0 \), which proves (3.8) for \( k = 1 \). Moreover, we recover the setting we started with, with now \( \{\omega^k_n\}_n \setminus 0 \) playing the role of \( \{\omega^0_n\}_n \). Iterating the argument, we then obtain (3.8) and, in particular, (3.7).

Thus, for any \( \omega \in \alpha'_1(\mathcal{C}_1) \) and \( k \geq 1 \),
\[
 \ell_1(\beta^k_s(\omega) + \ell_1) = \int_{\partial \mathcal{D}_\omega^1} \theta ds = \int_{\partial \mathcal{D}_\omega^1} d\theta ds = \int_{\partial \mathcal{D}_\omega^1} dh(s,\theta) \quad \text{(by Stokes–Whitney’s Theorem)}
\]
\[
 = \int_{\partial \mathcal{D}_\omega^1} |J h(s,\theta)| d\theta ds \quad \text{(by making} (s',\theta') = h(s,\theta))
\]
\[
 = \int_{\partial \mathcal{D}_\omega^1} |J h(s, -\ell_2)| + O(\theta + \ell_2)^k |d\theta ds
\]
\[
 = (\beta^k_s(\omega) + \ell_2) \int_{\mathbb{T}} |J h(s, -\ell_2)| ds + O(\beta^k_s(\omega) + \ell_2)^{k+1}
\]
\[
 = (\beta^k_s(\omega) + \ell_2) \int_{\mathbb{T}} |J h(s, -\ell_2)| ds + O(\omega^{2(k+1)}_{k+1})
\]
(3.9)
We now compute \( \int_{\mathbb{T}} |J h(s, -\ell_2)| ds \). In fact, by Lemma 1, we have \( J h(s,\theta) \equiv h'_2(\theta) \). But
\[
(\theta + \ell_j)^{-1/2} \alpha'_j(\theta) = 2\sqrt{2} \lambda_{-3/2} + O(\theta + \ell_j).
\]

Thus, rewriting the second part of (3.13) as
\[
 \left( \frac{h_2(\theta) + \ell_1}{\theta + \ell_2} \right)^{1/2} (h_2(\theta) + \ell_1)^{-1/2} \alpha'_1(h_2(\theta)) = (\theta + \ell_2)^{-1/2} \alpha'_2(\theta),
\]
\[\text{[15]}\]
Here and thereafter, we denote by \( \partial_{(s,\theta)}h \) the Jacobian matrix of \( h \).
\[\text{[16]}\]
Observe that \( \alpha'_2(\mathcal{C}_2) \) may have isolated points (see [1]). However, such a sequence exists as the KAM tori accumulate on the boundary.
\[\text{[17]}\]
Recall that \( \beta^k_s(\omega) + \ell_2 = O(\omega^2) \) (see [16]).
\[\text{[18]}\]
See [16].
and using (3.3), we obtain
\[ J h(s, -\ell_2) \equiv h'_2(-\ell_2) = \left( \frac{\lambda_1}{\lambda_2} \right)^3. \]
Hence, for any \( k \geq 1 \),
\[ \lambda_1^{-3} \ell_1(\beta_1'(\omega) + \ell_1) = \lambda_2^{-3} (\beta_2'(\omega) + \ell_2) + O \left( \frac{\omega^{2(k+1)}}{k+1} \right), \]
and, therefore,
\[ \lambda_1^{-3} \ell_1(\beta_1(\omega) + \ell_1 \omega) = \lambda_2^{-3} (\beta_2(\omega) + \ell_2 \omega) + O \left( \frac{\omega^{2k+3}}{(k+1)^2} \right). \tag{3.10} \]
Moreover, by symmetry of conjugacy, we have
\[ \lambda_2^{-3} \ell_2(\beta_2(\omega) + \ell_2 \omega) = \lambda_1^{-3} (\beta_1(\omega) + \ell_1 \omega) + O \left( \frac{\omega^{2k+3}}{(k+1)^2} \right). \tag{3.11} \]
Now, combining (3.10) and (3.11), we obtain \( \ell_1 \ell_2 = 1 \). Then (3.10) can be rewritten as
\[ \lambda_1^{-3} \sqrt{\ell_1} (\beta_1(\omega) + \ell_1 \omega) = \lambda_2^{-3} \sqrt{\ell_2} (\beta_2(\omega) + \ell_2 \omega) + O \left( \frac{\omega^{2k+3}}{(k+1)^2} \right), \tag{3.12} \]
i.e., \( \lambda_1^{-3} \sqrt{\ell_1} (\beta_1(\omega) + \ell_1 \omega) \) and \( \lambda_2^{-3} \sqrt{\ell_2} (\beta_2(\omega) + \ell_2 \omega) \) have the same Taylor expansion. Now, if a convex caustic of rotation number \( \omega \) of \( \Omega_j \) exists, then by [14, Theorem 3.2.10.] its Lazutkin invariant is given by
\[ L_j(\omega) = \alpha(\beta'(\omega)) \overset{def}{=} \omega \beta'(\omega) - \beta(\omega). \]
Therefore, for \( \omega \) small enough, we obtain \( \lambda_1^{-3} \sqrt{\ell_1} L_1(\omega) = \lambda_2^{-3} \sqrt{\ell_2} L_2(\omega) + O \left( \frac{\omega^{2k+3}}{k+1} \right). \tag{3.13} \]

**Lemma 1.** Under the notations and assumptions above, there exists\(^{19}\) \( g_1 \in C^\infty(\mathbb{R}, \mathbb{R}) \) such that, for any \( (s, \theta) \in \mathbb{T} \times \mathcal{C}_2 \), with \( \theta \) sufficiently close to \(-\ell_2\),
\[ h(s, \theta) = (g_1(\theta) + s, h_2(\theta)), \quad \alpha'_1(h_2(\theta)) = \alpha'_2(\theta). \tag{3.14} \]

**Proof.** Let \( (s, \theta) \in \mathbb{T} \times \mathcal{C}_2 \setminus \{-\ell_2\} \). Then we have
\[ (h_1(s, \theta) + \alpha'_1(h_2(s, \theta)), h_2(s, \theta)) \overset{(A,3)}{=} f_1 \circ h(s, \theta) \overset{(2.1)}{=} h \circ f_2(s, \theta) \overset{(A,3)}{=} (h_1(s + \alpha'_2(\theta), \theta), h_2(s + \alpha'_2(\theta), \theta)). \tag{3.15} \]
Hence, \( h_2(s, \theta) = h_2(s + \alpha'_2(\theta), \theta) \), which implies that \( h_2 \) is independent of \( s \), i.e., \( h_2(s, \theta) = h_2(\theta) \), as \( \alpha'_2(\theta) \notin \mathbb{Q} \). We have also
\[ h_1(s, \theta) + \alpha'_1(h_2(\theta)) \overset{(3.14)}{=} h_1(s + \alpha'_2(\theta), \theta), \tag{3.16} \]
which, differentiated w.r.t \( s \), yields \( \partial_s h_1(s, \theta) = \partial_s h_1(s + \alpha'_2(\theta), \theta) \). Thus, as \( \alpha'_2(\theta) \notin \mathbb{Q} \), \( \partial_s h_1 \) is also independent of \( s \): \( \partial_s h_1(s, \theta) = g_2(\theta) \), so that \( h_1(s, \theta) = g_1(\theta) + g_2(\theta) \cdot s \), for some \( g_1, g_2 \in C^\infty(\mathbb{R}, \mathbb{R}) \). Now, plugging the expression of \( h_1 \) found into (3.15) and using (A.1), we get
\[ h(s, \theta) = (g_1(\theta) + g_2(\theta) \cdot s, h_2(\theta)), \quad \alpha'_1(h_2(\theta)) = g_2(\theta) \alpha'_2(\theta). \tag{3.16} \]
Now, combining (3.5) and (3.16) with the fact that\(^{20}\) \( \alpha'_2(\theta) \neq 0 \) for \( \theta + \ell_2 > 0 \) small enough yields \( g_2(\theta) = 1 \), for \( \theta \) near \(-\ell_2 \). Thus, as \( \mathcal{C}_2 \setminus \{-\ell_2\} \) accumulates at \(-\ell_2 \), by continuity we obtain \( g_2 \equiv 1 \), and (3.13) is proven. \( \square \)

---

\(^{19}\) Actually, \( g_1 \) is as smooth as \( h \).

\(^{20}\) See [16].
Remark 3. (i) Observe that, according to the proof, the conclusion in Lemma 1 still holds if \( \tilde{h} \) is assumed merely to be \( C^2(\mathbb{T} \times [0, \pi), \mathbb{R}) \).

(ii) Observe also that, if \( \Omega_1 \) and \( \Omega_2 \) are both ellipses and \( \tilde{h} \) is a \( C^2 \)-conjugacy near their respective boundaries of the associated billiard maps \( f_{\Omega_1} \) and \( f_{\Omega_2} \), then (3.4) and (3.5), and, therefore, (3.13) hold in some neighborhood \([-\ell, -\ell + \varepsilon_2]\) of \(-\ell\). Indeed, by the same argument, one gets that (3.4), (3.5) and (3.13) hold on \([-\ell, -\ell + \varepsilon_2) \setminus \mathbb{Q} \) and, therefore, by continuity they hold on \([-\ell, -\ell + \varepsilon_2) \).

3.2. Proof of Theorem 4

3.2.1. Proof of (i) in Theorem 4

We shall exhibit a \( C^\infty \)-conjugacy. Given \( j = 1, 2 \) let \( 0 < b_j \leq a_j \) such that

\[
\partial \Omega_j := \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a_j^2} + \frac{y^2}{b_j^2} = 1 \right\},
\]

so that \( e_j := \sqrt{1 - (b_j/a_j)^2} \). Let \( c_j := \sqrt{a_j^2 - b_j^2} \) be its semifocal distance, and \( \mathcal{F}_j := (\pm c_j, 0) \) its two foci. Consider the family of confocal elliptic caustics

\[
\partial \Omega_{j, \lambda} := \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a_j^2 - \lambda^2} + \frac{y^2}{b_j^2 - \lambda^2} = 1 \right\}, \quad \lambda \in [0, b).
\]

Consider the following auxiliary functions:

\[
k_j : [0, b_j] \ni \lambda \longmapsto \sqrt{\frac{a_j^2 - b_j^2}{a_j^2 - \lambda^2}} \in (e_j, 1), \quad \delta_j : [0, b_j] \ni \lambda \longmapsto 2F(\arcsin(\lambda/b_j), k_j(\lambda)) \in [0, \infty),
\]

\[
h_{j,1} : [0, 2\pi) \times [0, \pi) \ni (\phi, \theta) \longmapsto (a_j \cos \phi, b_j \sin \phi, \theta) \in \partial \Omega_j \times [0, \pi),
\]

\[
h_{j,2} : [0, 2\pi) \times [0, \tilde{\theta}_j) \ni (\phi, \theta) \longmapsto (\lambda, t) := \left( g_j(\phi, \theta), F(\phi, k_j(\phi, \theta)) \right) \in [0, b_j) \times [0, \infty),
\]

for some sufficiently small \( 0 < \tilde{\theta}_j < \pi \), where the maps \( F \) and \( K \) are the ones introduced in Remark 2 and

\[
g_j(\phi, \theta) := \frac{|a_j^2 \sin \phi + b_j^2 \cos \phi| \sin \theta}{\sqrt{a_j^2 \sin^2 \phi + b_j^2 \cos^2 \phi}}.
\]

(3.17)

It is obvious that each of the maps \( k_j, \omega_j, \) and \( h_{j,1} \) is a \( C^\infty \)-diffeomorphism (onto its respective codomains). The map \( h_{j,2} \) is a homeomorphism (onto its codomain) and its restriction \( h_{j,2}^* : \mathcal{T}^* \times [0, \tilde{\theta}_j) \longrightarrow [0, b_j) \times [0, \infty) \) is a \( C^\infty \)-diffeomorphism (onto its codomain), where \( \mathcal{T}^* := [0, 2\pi) \setminus \{ -\arctan^{-1}(b_j^2/a_j^2), \pi - \arctan^{-1}(b_j^2/a_j^2) \}. \)

In particular, for \( \{ l \} = \{ 1, 2 \} \setminus \{ j \}, \) the following map is a \( C^\infty \)-diffeomorphism as well:

\[
h_{j,3} : [0, b_l) \times [0, \infty) \ni (\lambda, t) \longmapsto (\delta_{j}^{-1} \circ \delta_l(\lambda), t) \in [0, b_j) \times [0, \infty).
\]

Let

\[
h_j := h_{j,1} \circ h_{j,2}^{-1} \circ h_{j,3} \quad \text{and} \quad h := h_1 \circ h_2^{-1}.
\]

\footnote{Indeed, \( Jh_{j,2}(\phi, \theta) = -|a_j^2 \sin(\phi) + b_j^2 \cos(\phi)| \left( 1 + \frac{(a_j^2 - b_j^2) \sin^2(\phi)}{a_j^2 \sin^2(\phi) + b_j^2 \cos^2(\phi)} \right)^{-1/2} \), which vanishes only on \( \{(\phi, \theta) \in [0, 2\pi) \times [0, \pi) : \tan \phi = -b_j^2/a_j^2 \) or \( \theta = \pi/2 \} \), where \( Jh_{j,2} \) denotes the Jacobian of \( h_{j,2} \).}
Then $h$ is a homeomorphism from a neighborhood $\mathcal{V}_2$ of $\partial \Omega_2 \times \{0\}$ onto a neighborhood $\mathcal{V}_1$ of $\partial \Omega_1 \times \{0\}$, with $\mathcal{V}_j \supset \partial \Omega_j \times \{0\}$ ($j = 1, 2$), and its restriction $h^*$ to the neighborhood $\tilde{\mathcal{V}}_2$ of $\partial \Omega_2 \times \{0\}$ is a $C^\infty$-diffeomorphism onto the neighborhood $\tilde{\mathcal{V}}_1$ of $\partial \Omega_1 \times \{0\}$, where $\tilde{\mathcal{V}}_j := \mathcal{V}_j \cap (\Lambda_j^* \times [0, \pi])$.

Then we claim that
\[ f_1 \circ h = h \circ f_2 \quad \text{on} \quad \mathcal{V}_2, \tag{3.18} \]
which would prove $(i)$ in Theorem 4. The proof of (3.18) follows easily a result proven in [5] (see also [10, Proposition 16]), which can be reformulated as follows. The main difference with the aforementioned result is that we provide the explicit value of $(\lambda, t)$, the “action-angle” coordinates (see Appendix B for the computation of $\lambda$).

**Lemma 2.** Let $(s, \theta) \in \mathcal{V}_j$. Set $(\phi, \theta) := h_{j,1}^{-1}(s, \theta)$, $(\lambda, t) := h_{j,2}(\phi, \theta) \text{ and } \delta_j(\lambda) := 2F(\arcsin(\lambda/b_j), k_j(\lambda))$. Then the segment joining $(s, \theta)$ and $f_j(s, \theta)$ is tangent to $\Omega_{j,\lambda}$ and
\[ f_j \circ h_{j,1} \circ h_{j,2}^{-1}(\lambda, t) = f_j(s, \theta) = h_{j,1} \circ h_{j,2}^{-1}(\lambda, t + \delta_j(\lambda)). \tag{3.19} \]

### 3.2.2. Proof of $(ii)$ in Theorem 4

We adopt the notations in Section 3.2.1. Given $j = 1, 2$ and two coprime integers $0 < m < n/2$, consider the function
\[ g_j: (-c_j^2, 0) \ni \xi \mapsto F(\arcsin \sqrt{\frac{b_j^2}{b_j^2 - \xi} \left(1 + \frac{\xi}{c_j^2}\right)} - \frac{2m}{n} \pi \left(1 + \frac{\xi}{c_j^2}\right)). \]
Then $f_j$ has a periodic orbit of rotation $m/n$ whose caustic is a hyperbola iff the equation
\[ g_j(\xi) = 0, \quad \xi \in (-c_j^2, 0) \tag{3.20} \]
has a solution. But we have
\[ g_j(-c_j^2) = \arcsin \frac{b_j}{a_j} - \frac{m}{n} \pi, \quad \text{and} \quad \lim_{\xi \to 0^-} g_j(\xi) = +\infty. \]

Hence, if
\[ \frac{m}{n} \geq \frac{1}{\pi} \arcsin \frac{b_j}{a_j}, \tag{3.21} \]
then (3.20) has a solution. In contrast, if
\[ \frac{m}{n} < \frac{1}{\pi} \arcsin \frac{b_j}{a_j}, \tag{3.22} \]
then (3.20) has no solution. Indeed, assuming (3.22), we have, for any $\xi \in (-c_j^2, 0), \quad g_j(\xi) > u_j(\xi) > 0, \]
where
\[ u_j(\xi) := F \left( \arcsin \sqrt{\frac{b_j^2}{b_j^2 - \xi} \left(1 + \frac{\xi}{c_j^2}\right)} - \frac{2}{\pi} \arcsin \frac{b_j}{a_j} F \left( \frac{\pi}{2} \sqrt{1 + \frac{\xi}{c_j^2}} \right) \right), \]
and, using Wolfram Mathematica [7], one gets
\[ \min \{ u_j(\xi) : a_j > b_j > 0, -a_j^2 + b_j^2 < \xi < 0 \} = 5.65246 \cdot 10^{-9}. \]

---

\[22\] See [15, p. 4569, Eq. (22)].
Now, assume $f_1$ and $f_2$ are (globally) $C^0$-conjugate. By contradiction, assume $e_1 > e_2$, i.e., $b_1/a_1 < b_2/a_2$. Then, for any two coprime integers $0 < m < n/2$ such that,

$$\frac{1}{\pi} \arcsin \frac{b_1}{a_1} \leq \frac{m}{n} < \frac{1}{\pi} \arcsin \frac{b_2}{a_2},$$

$f_1$ admits a periodic orbit of rotation $m/n$ whose caustic is a hyperbola, while $f_2$ does not admit such a periodic orbit, a contradiction. Thus, $e_1 = e_2$. □

**APPENDIX A. AUXILIARY FACTS ABOUT BILLIARD DYNAMICS**

**Theorem A.1 ([11, 12, 14]).** Let $f$ be a monotone twist map. Then:

(i) $\beta$ is strictly convex. In particular, it is continuous and admits a Legendre–Fenchel transform

$$\alpha(c) = \sup_{\omega \in \mathbb{R}} \omega \cdot c - \beta(\omega),$$

the so-called Mather’s $\alpha$ function. Moreover, $\alpha$ is convex and we have

$$\alpha'(\beta'(\omega)) = \omega,$$

at each point of differentiability $\omega$ of $\beta$.

(ii) $\beta$ is differentiable at any irrational.

(iii) $\beta$, and hence $\alpha$, is an invariant under any symplectic change of coordinates.

**Theorem A.2 ([9, 14]).** Let $\Omega \subset \mathbb{R}^2$ be a smooth strictly convex closed curve of length $\ell$ and $f$ its corresponding billiard map. Then there exists $\epsilon_0 > 0$ small, a Cantor set $\mathcal{C} \subseteq [-\ell, -\ell + \epsilon_0]$, with $-\ell \in \mathcal{C}$, a $C^\infty$-smooth (exact) symplectomorphism $\phi: \mathbb{T} \times [-\ell, -\ell + \epsilon_0) \ni (x_0, \theta_0) \mapsto (x_1, \theta_1) \in \phi(\mathbb{T} \times [-\ell, -\ell + \epsilon_0)) \subset \mathbb{T} \times [0, \pi)$ such that

$$\phi(\mathbb{T} \times \{-\ell\}) = \mathbb{T} \times \{0\}, \quad (A.2)$$

$$\phi^{-1} \circ f \circ \phi(x, \theta) = (x + \alpha'(\theta), \theta), \quad \forall (x, \theta) \in \mathbb{T} \times \mathcal{C}. \quad (A.3)$$

In particular, the KAM curve of rotation number$^{23}$ $\omega \in \alpha'(\mathcal{C})$ of $\phi^{-1} \circ f \circ \phi$ is given by the graph of the constant function $\mathbb{T} \ni x \mapsto \beta'(\omega)$.

**APPENDIX B. COMPUTATION OF THE “ACTION-ANGLE” COORDINATES FOR ELLIPTIC BILLIARDS**

It is well known that the caustic parameter $\lambda$, together with its conjugate $t := F(\phi, k_j(\sqrt{\lambda}))$, is an action-angle coordinate for the billiard in the ellipse $\Omega_j^{24}$. We are only left to show that $\lambda = g_j(\phi, \theta)$ with $g_j$ defined as in (3.17) is indeed the parameter of the caustic corresponding to the billiard trajectory $\{f_j^n(h_{j,1}(\phi, \theta)) : n \in \mathbb{Z}\}$ starting at $h_{j,1}(\phi, \theta)$. Let$^{25}$ $q_\lambda(t) = (q_x^\lambda(t), q_y^\lambda(t)) = (c \cos \phi, b \sin \phi)$, where $c$ and $s$ are the Jacobi elliptic functions:

$$c_\lambda(t; k) := \cos(\am(t; k)) \quad \text{and} \quad s_\lambda(t; k) := \sin(\am(t; k)),$$

where $\am(t; k)$, called the amplitude of $t$, is given by $t = F(\phi, k)$ iff $\phi = \am(t; k)$. Then$^{26}$ the trajectory starting at $q_\lambda(t)$ and tangent to the caustic

$$\Omega_\lambda := \left\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1\right\}, \quad \lambda \in [0, b),$$

$^{23}$ By construction, each frequency $\omega \in \alpha'(\mathcal{C}) \setminus \{0\}$ is Diophantine, i.e., there exists $a, \tau > 0$ such that $|q \omega - p| \geq a/q^\tau$, for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

$^{24}$ See, e.g., [5] or [10, Proposition 16].

$^{25}$ For the sake of simplicity, we will drop the subscripts $j$.

$^{26}$ See [5] or [10, Proposition 16].
hits the boundary at \(q_\lambda(t + \delta_\lambda)\), where \(\delta_\lambda =: \delta(\lambda)\) is given in Lemma 3.19. Then the point of tangency \(p_\lambda(t)\) of the segment \([q_\lambda(t), q_\lambda(t + \delta_\lambda)]\) to the caustic \(\Omega_\lambda\) is given by \(p_\lambda(t) = (\sqrt{a^2 - \lambda^2} \cos \varphi, \sqrt{b^2 - \lambda^2} \sin \varphi)\), for some \(\varphi \in [0, 2\pi)\). Thus, the slope of the line \((q_\lambda(t), q_\lambda(t + \delta_\lambda))\) is given by

\[
- \sqrt{\frac{b^2 - \lambda^2}{a^2 - \lambda^2}} \cot \varphi, \text{ using its tangency property to } \Omega_\lambda \text{ at } p_\lambda(t);
\]

\[
\frac{b \sin \phi - \sqrt{b^2 - \lambda^2} \sin \varphi}{a \cos \phi - \sqrt{a^2 - \lambda^2} \cos \varphi}, \text{ using its points } q_\lambda(t) \text{ and } p_\lambda(t);
\]

\[
\tan \left(\theta - \arctan \frac{b \cos \phi}{a \sin \phi}\right) = \frac{a \tan \phi \tan \theta - b}{a \tan \phi + b \tan \theta}, \text{ using the fact that the slope of the tangent to } \Omega \text{ at } q_\lambda(t) \text{ is } \frac{b \cos \phi}{a \sin \phi} \text{ and that the angle between this tangent and the line } (q_\lambda(t), q_\lambda(t + \delta_\lambda)) \text{ is } \theta.
\]

Hence,

\[
- \sqrt{\frac{b^2 - \lambda^2}{a^2 - \lambda^2}} \cot \varphi = \frac{b \sin \phi - \sqrt{b^2 - \lambda^2} \sin \varphi}{a \cos \phi - \sqrt{a^2 - \lambda^2} \cos \varphi}, \quad \text{(B.1)}
\]

\[
- \sqrt{\frac{b^2 - \lambda^2}{a^2 - \lambda^2}} \cot \varphi = \frac{a \tan \phi \tan \theta - b}{a \tan \phi + b \tan \theta}. \quad \text{(B.2)}
\]

Now, (B.1) is equivalent to

\[
a \sqrt{b^2 - \lambda^2} \cos \phi \cos \varphi + b \sqrt{a^2 - \lambda^2} \sin \phi \sin \varphi = \sqrt{(a^2 - \lambda^2)(b^2 - \lambda^2)}. \quad \text{(B.3)}
\]

Relation (B.2) implies

\[
\frac{\cos^2 \varphi}{(a^2 - \lambda^2)(a \tan \phi \tan \theta - b)^2} = \frac{\sin^2 \varphi}{(b^2 - \lambda^2)(a \tan \phi + b \tan \theta)^2} = \frac{\cos^2 \varphi + \sin^2 \varphi}{(a^2 - \lambda^2)(a \tan \phi \tan \theta - b)^2 + (b^2 - \lambda^2)(a \tan \phi + b \tan \theta)^2} = \frac{1}{(a^2 - \lambda^2)(a \tan \phi \tan \theta - b)^2 + (b^2 - \lambda^2)(a \tan \phi + b \tan \theta)^2} \quad \text{(B.4)}
\]

so that

\[
\frac{\cos \varphi}{\sqrt{a^2 - \lambda^2}(a \tan \phi \tan \theta - b)} = \frac{- \sin \varphi}{\sqrt{b^2 - \lambda^2}(a \tan \phi + b \tan \theta)} \quad \text{(B.2)}
\]

\[
= \frac{\pm 1}{\sqrt{(a^2 - \lambda^2)(a \tan \phi \tan \theta - b)^2 + (b^2 - \lambda^2)(a \tan \phi + b \tan \theta)^2}}. \quad \text{(B.5)}
\]

Thus, plugging into (B.3) the expressions of \(\cos \varphi\) and \(\sin \varphi\) obtained from (B.5) and then squaring both sides of the relation obtained yields \(\lambda = g(\phi, \theta)\), which completes the computation.

\[\square\]

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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