Admissible vectors and Radon-Nikodym theorems

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Abstract

Admissible vectors lead to frames or coherent states under the action of a group by means of square integrable representations. This work shows that admissible vectors can be seen as weights with central support on the (left) group von Neumann algebra. The analysis involves spatial and cocycle derivatives, noncommutative $L^p$-Fourier transforms and Radon-Nikodym theorems. Square integrability confine the weights in the predual of the algebra and everything may be written in terms of a (right selfdual) bounded element.

Keywords: locally compact group, unitary representation, admissible vector, von Neumann algebra, weight, Radon-Nikodym theorem, frame, coherent state

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1 Introduction

Admissible vectors lead to frames or coherent states under the action of a group. They fall into the fields of square integrable representations and reproducing kernel Hilbert spaces. Exhaustive criteria for the existence and characterization of admissible vectors have been established in the case the group von Neumann algebra is type I, using the Plancherel formula of the group [12]. Lie and discrete groups have also received special attention; see [14] and references therein. Techniques in von Neumann algebras, using central decompositions with respect to a trace in the semifinite case, have been used in the study of admissible vectors for unimodular separable groups [11, 3] and countable discrete groups [15, 3, 2]. The analysis in the general case (with type III summand) requires the use of weights (or Hilbert algebras), which generalize the notions of positive functional and trace [21, 20]. A study for arbitrary groups by means of convolution Hilbert algebras have been done in [13].

This work characterizes admissible vectors in terms of weights on the group von Neumann algebras. For each admissible vector there exists a weight with central support that satisfies certain inequalities with respect to the canonical weight associated to the convolution Hilbert algebra. These inequalities move directly to Connes’s spatial derivatives and are also equivalent to certain holomorphic extension properties of cocycle derivatives. The result is a noncommutative Radon-Nikodym theorem for the involved weights. See Theorem 1 below.

Square integrability is reflected in the fact that weights corresponding to admissible vectors are in the predual of the (left) group von Neumann algebra and, then, they are associated with positive elements of the Fourier algebra of the group. This implies that the spatial and cocycle derivatives can be expressed in terms of a (right selfdual) bounded element and its noncommutative $L^2$-Fourier transform. The Radon-Nikodym derivatives are then bounded operators (Theorem 2).

The involved weights may commute, that is, each one may be invariant under the modular automorphism group of the other. This is the case when the cocycle derivatives form an one-parameter group of unitary elements in the reduced von Neumann algebra, whose Stone’s generator is just the Radon-Nikodym derivative (Theorem 3). The reduced algebra is then semifinite (Corollary 4). If, moreover, the weights satisfy the KMS condition, the Radon-Nikodym derivative is in the centre of the algebra (Corollary 7).
The unimodular groups are those whose canonical weight is a trace, with trivial modular group. Any weight is then invariant with respect to the canonical weight and Theorem 4 is applicable. Furthermore, for unimodular groups even the spatial derivatives are bounded operators (Theorem 8) and KMS condition lead to admissible vectors described by traces (Corollary 9). For commutative groups, admissible vectors are associated with classical (Abelian) Radon-Nikodym derivatives at the side of the dual group (Theorem 10).

The content is organized as follows: Section 2 introduces the terminology and notation used along the work. Section 3 deals with admissible vectors for arbitrary locally compact groups. Section 4 restricts attention to commuting weights. Sections 5 and 6 treat the cases of unimodular and commutative groups, respectively. Section 7 remarks some connections of this work with previous ones on tracial conditions [11], dual integrable representations and brackets [2] and formal degrees [9].

2 Preliminaries

Let us begin by introducing some terminology and notation mainly borrowed from Folland [10], Strătilă [23] and Takesaki [26].

2.1 Group algebras

Let \( G \) be a locally compact group (lc group, for brevity) with fixed left Haar measure \( ds \). \( C_c(G) \) denotes the space of complex-valued continuous functions on \( G \) with compact support and \( L^p(G) \), \( 1 \leq p \leq \infty \), the usual \( L^p \)-spaces with respect to \( ds \). \( (\cdot|\cdot) \) and \( || \cdot || \) are, respectively, the inner product and norm in \( L^2(G) \). The modular function of \( G \) is denoted by \( \delta_G \). Recall that \( \delta_G \) is a continuous homomorphism from \( G \) into the multiplicative group \( \mathbb{R}^+ \) that satisfies

\[
d(st) = \delta_G(t) \, ds, \quad d(s^{-1}) = \delta_G(s^{-1}) \, ds, \quad s, t \in G.
\] (1)

The left and right regular representations \( \lambda \) and \( \rho \) of \( G \) on \( L^2(G) \) are defined by

\[
(\lambda(s)f)(t) := f(s^{-1}t), \quad s, t \in G, f \in L^2(G),
\] (2)

\[
(\rho(s)f)(t) := \delta_G^{1/2}(s)f(ts), \quad s, t \in G, f \in L^2(G).
\] (3)

The left and right group von Neumann algebras of \( G \) are denoted by \( \mathcal{L}(G) \) and \( \mathcal{R}(G) \), i.e.,

\[
\mathcal{L}(G) := \{ \lambda(s) : s \in G \}'' \quad \mathcal{R}(G) := \{ \rho(s) : s \in G \}''
\] (4)

(double commutant). One has \( \mathcal{L}(G)' = \mathcal{R}(G) \). See e.g. [26] Sect. VII.3.

The convolution product \( f * g \) and involutions \( f \mapsto f^\sharp \) and \( f \mapsto f^\flat \) are defined at first in \( C_c(G) \) by

\[
[f * g](s) := \int_G f(t)g(t^{-1}s) \, dt, \quad s \in G,
\]

\[
f^\sharp(s) := \delta_G(s^{-1})f(s^{-1}), \quad s \in G,
\]

\[
f^\flat(s) := \overline{f(s^{-1})}, \quad s \in G.
\] (5)

where the bar denotes complex conjugation.

The extension of the convolution product \( * \) and involution \( \sharp \) to \( L^1(G) \) leads to a structure of Banach \( * \)-algebra for \( L^1(G) \). The Banach dual space of \( L^1(G) \) is \( L^\infty(G) \). The symbol \( \langle \cdot, \cdot \rangle \) represents this duality:

\[
\langle f, \varphi \rangle := \int_G f(s)\varphi(s) \, ds, \quad f \in L^1(G), \varphi \in L^\infty(G).
\]

A function of positive type on \( G \) is a function \( \varphi \in L^\infty(G) \) that defines a positive linear functional on the \( C^* \)-algebra \( L^1(G) \), i.e., that satisfies \( \langle f^\sharp * f, \varphi \rangle \geq 0 \) for all \( f \in L^1(G) \). In what follows \( \mathcal{P}(G) \) denotes the set of continuous functions of positive type on \( G \).

Let \( A(G) \) be the Fourier algebra of \( G \). It is well known [17] Sect.2.3 [26] Sect.VII.3] that the following conditions are equivalent:

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(i) $\varphi \in A(G)^+ := \mathcal{P}(G) \cap A(G)$.
(ii) $\varphi = g \ast g^*$, $g \in L^2(G)$.
(iii) There exists a positive normal functional $\omega$ on $L(G)$ such that $\omega(\lambda(s)) = \varphi(s)$, for all $s \in G$.
Moreover, $\varphi \mapsto \omega$ is a bijection of $A(G)^+$ onto $L(G)^+$, the positive part of the predual of $L(G)$.
A vector $f \in L^2(G)$ is left (right) bounded if there exists a bounded operator $L_f$ ($R_f$) on $L^2(G)$ such that
$$L_f g := f \ast g \quad (R_f g := g \ast f), \quad g \in C_c(G).$$

### 2.2 Weights

A weight on a von Neumann algebra $M$ is a map $\omega : M^+ \to [0, \infty]$, defined at first on the positive cone $M^+$ of $M$, satisfying the following conditions:
$$\omega(A + B) = \omega(A) + \omega(B), \quad \omega(\alpha A) = \alpha \omega(A), \quad A, B \in M^+, \quad \alpha \geq 0,$$
where the convention $0(+\infty) = 0$ is used. This map can be extended uniquely to $M$. The weight $\omega$ is said to be semifinite if $\{ A \in M^+ : \omega(A) < \infty \}$ generates $M$; faithful if $\omega(A) \neq 0$ for nonzero $A \in M^+$; normal if $\omega(\sup A_i) = \sup \omega(A_i)$ for bounded increasing nets $\{ A_i \}$ in $M^+$; a trace if $\omega(A^* A) = \omega(A A^*)$ for $A \in M$. See, e.g., [24, Sect.10.14] and [26, Sect.VII.1] for details.

The canonical weight $\Omega_l$ on $L(G)$ is the normal semifinite faithful (n.s.f.) weight associated to the left convolution Hilbert algebra $C_c(G)$ [26, Sect.VII.3]:
$$\Omega_l(A) := \begin{cases} \| f \|^2, & \text{if } A = L_f^* L_f, \text{ f left bounded}, \\ +\infty, & \text{otherwise}. \end{cases}$$

$\Omega_l$ is a trace iff $G$ is unimodular.

Let $\Delta$ be the modular operator on $L^2(G)$, multiplication by the modular function $\delta_G$, and $J$ the modular conjugation.
$$J : L^2(G) \to L^2(G), \quad (Jf)(s) := \delta_{G}^{-1/2}(s)\overline{f(s^{-1})}.$$ The modular automorphism group $\{ \sigma_t^{\Omega_l} \}_{t \in \mathbb{R}}$ on $L(G)$ associated with the canonical n.s.f. weight $\Omega_l$ (see [20]) is of the form
$$\sigma_t^{\Omega_l}(A) := \Delta^{it} A \Delta^{-it}, \quad A \in L(G).$$

The centralizer of $\Omega_l$ is the fixed-point set of the modular group $\{ \sigma_t^{\Omega_l} \}_{t \in \mathbb{R}}$ and is denoted by $L(G)^{\Omega_l}$, i.e., $A \in L(G)$ belongs to $L(G)^{\Omega_l}$ iff $\sigma_t^{\Omega_l}(A) = A$, $t \in \mathbb{R}$. $L(G)^{\Omega_l}$ is a von Neumann subalgebra of $L(G)$. See [26, VIII.3].

The canonical n.s.f. weight $\Omega_r$ on $\mathcal{R}(G)$ is given by $\Omega_r(A) := \Omega_l(JAJ)$, for $A \in \mathcal{R}(G)$, and the corresponding modular automorphism group $\{ \sigma_t^{\Omega_r} \}_{t \in \mathbb{R}}$ on $\mathcal{R}(G)$ by
$$\sigma_t^{\Omega_r}(A) := \Delta^{-it} A \Delta^{it}, \quad A \in \mathcal{R}(G).$$

Connes [7] introduced the notion of spatial derivative of a n.s. weight defined on a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ with respect to a n.s.f weight defined on the commutant $M' \subset \mathcal{B}(\mathcal{H})$. In particular, for a n.s. weight $\omega$ on $L(G)$, the spatial derivative $d\omega/d\Omega_r$ is the unique (not necessarily bounded) positive selfadjoint operator $T$ such that
$$\omega(L_f^* L_f) = \begin{cases} \| T^{1/2} f \|^2, & \text{if } f \text{ left bounded, } f \in D(T^{1/2}), \\ +\infty, & \text{otherwise}. \end{cases}$$

For the definition and properties of the cocycle derivative $(D \omega : D\Omega_l)$ of a n.s weight $\omega$ relative to a n.s.f weight $\Omega$ defined on a von Neumann algebra $M$ see, e.g., [26, VIII.3] [23, Sect.3].

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1 The support $s(\omega)$ of a normal weight defined on a von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is the (orthogonal) projection $P \in M$ such that $I - P$ is the lower upper bound of the family of projections $P' \in M$ satisfying $\omega(P') = 0$, where $I$ denotes the identity operator on $\mathcal{H}$; see [24, 5.15].
2.3 Noncommutative $L^p$-Fourier transforms

We introduce the theory of noncommutative $L^p$-spaces associated with a weight on the basis of spatial derivatives, as proposed by Connes [7] and Hilsum [16]. These spatial $L^p$-spaces are the basis for the Fourier analysis on l.c. groups given by Terp [28]. For a survey of the theory of noncommutative $L^p$-spaces see, e.g., [21] and references therein.

In what follows, if $T$ is a (not necessarily bounded) positive selfadjoint operator and $P$ the projection onto $\ker(T)^\perp$, by definition, $T^\dagger$, $t \in \mathbb{R}$, is the partial isometry coinciding with the unitary operator $(TP)^{it}$ on $\ker(T)^\perp$ and 0 on $\ker(T)$.

Let $\alpha \in \mathbb{R}$ and let $T$ be a closed densely defined operator on $L^2(G)$ with polar decomposition $T = V[T]$. $T$ is called $\alpha$-homogeneous if it satisfies the following equivalent conditions [28 Lem.2.7]:

(i) $\rho(s)T = \delta_0^{-\alpha}(s)T\rho(s)$, for all $s \in G$.

(ii) $V \in \mathcal{L}(G)$ and $\sigma_{\alpha t}^0(A)|T|^\dagger = |T|^\dagger A$, for all $A \in \mathcal{R}(G)$ and $t \in \mathbb{R}$.

A positive selfadjoint operator $T$ on $L^2(G)$ is $(-1)$-homogeneous iff $T = d\omega/d\Omega_r$ for a (necessarily unique) n.s. weight $\omega$ on $\mathcal{L}(G)$ [7 Th.13]. In such case, the integral of $T$ with respect to $\Omega_r$ is defined by

$$\int T d\Omega_r := \omega(I) \in [0, \infty],$$

where $I$ denotes the identity operator on $L^2(G)$.

For each $p \in [1, \infty)$, the noncommutative $L^p$-space $L^p(\Omega_r)$ is the set of $(-1/p)$-homogeneous operators $T$ on $L^2(G)$ satisfying $\|T\|_p^p := \int |T|^p d\Omega_r < \infty$. And $L^\infty(\Omega_r)$ is identified with $\mathcal{L}(G)$ with the usual operator norm $\|\cdot\|_\infty := \|\cdot\|_{\mathcal{B}(L^2(G))}$.

In the sequel, for a preclosed operator $T$, we denote by $[T]$ the closure of $T$. With the obvious scalar multiplication, sum $(S, T) \mapsto [S + T]$ and norm $\|\cdot\|_p$, $L^p(\Omega_r)$ is a Banach space. The operation $T \mapsto T^*$ is an isometry of $L^p(\Omega_r)$. By linearity, the integral $\int T d\Omega_r$ defined on $L^1(\Omega_r)^+$ extends to a linear form on $L^p(\Omega_r)$.

For $p \in [1, 2]$ and $1/p + 1/q = 1$, the $L^p$-Fourier transform of $f \in L^p(G)$ is the operator $\mathcal{F}_p(f)$ on $L^2(G)$ given by

$$\mathcal{F}_p(f)g := f \ast \Delta^{1/q}g, \quad g \in D(\mathcal{F}_p(f)).$$

$\mathcal{F}_1(f)g = f \ast g$ with domain $D(\mathcal{F}_1(f)) = L^2(G)$. $\mathcal{F}_2$ is a unitary transformation from $L^2(G)$ onto $L^2(\Omega_r)$ [28 Th.3.2].

Let $p \in [1, 2]$ and $1/p + 1/q = 1$. For each $T \in L^p(\Omega_r)$, $\mathcal{F}_p(T)$ denotes the unique function in $L^q(\Omega)$ such that

$$\int h(t)\mathcal{F}_p(T)(t) dt = \int [\mathcal{F}_p(h)T] d\psi_0, \quad h \in C_c(\Omega_r).$$

For $p \in (1, 2]$, $\mathcal{F}_p$ is simply the transpose of $\mathcal{F}_p$. For $p = 1$, the mapping $\mathcal{F}_1$ takes an element $T \in L^1(\Omega_r)$ into the unique function $\varphi \in A(G)$ that defines the same element of $\mathcal{L}(G)^*$ as $T$ does; in particular,

$$\mathcal{F}_1\left(\frac{d\omega}{d\Omega_r}\right) = \varphi, \quad \varphi \in \mathcal{L}(G)^* \simeq A(G)^+. $$

The mapping $\mathcal{F}_1$ is an isometry from $L^1(\Omega_r)$ onto $A(G)$ [28 Th.5.2]. For $p = 2$, the contranformation $\mathcal{F}_2$ is not exactly the inverse of $\mathcal{F}_2$; they are related by the formula $\mathcal{F}_2(T) = \mathcal{F}_2^{-1}(T^*)$, $T \in L^2(\Omega_r)$. It follows that $\mathcal{F}_2 : L^2(\Omega_r) \rightarrow L^2(G)$ is unitary.

In the sequel we shall say that a bounded operator $A$ on a Hilbert space $\mathcal{H}$ is nonsingular on a closed subspace $\mathcal{H}_0$ of $\mathcal{H}$ if the support and range of $A$ are included in $\mathcal{H}_0$ and the restriction $A|_{\mathcal{H}_0}$ admits a (bounded or not) inverse in $\mathcal{H}_0$. In such case, the inverse $(A|_{\mathcal{H}_0})^{-1}$ shall be denoted simply by $A^{-1}$. 


3 Admissible vectors. General case.

Let \( G \) be a lc group with fixed left Haar measure \( ds \) and let \( \pi \) be a (continuous) unitary representation of \( G \) on a Hilbert space \( \mathcal{H}_\pi \) with inner product \( (\cdot,\cdot)_\pi \) and norm \( \|\cdot\|_\pi \). Given an element \( \eta \in \mathcal{H}_\pi \), the orbit \( \{\pi(s)\eta\}_{s \in G} \) is called a \textbf{covariant frame} with \textbf{(frame) bounds} \( 0 < \alpha \leq \beta < \infty \) when

(F1) \( G \ni s \mapsto (\xi,\pi(s)\eta)_\pi \in \mathbb{C} \) is a measurable function for all \( \xi \in \mathcal{H}_\pi \);

(F2) \( \alpha \|\xi\|_\pi^2 \leq \int_G |(\xi,\pi(s)\eta)_\pi|^2 \, ds \leq \beta \|\xi\|_\pi^2 \) for all \( \xi \in \mathcal{H}_\pi \).

In such case there exists \( \psi \in \mathcal{H}_\pi \) giving rise to a \textit{dual covariant frame} \( \{\pi(s)\psi\}_{s \in G} \) with the same \textbf{frame bounds} and every \( \xi \in \mathcal{H}_\pi \) has the representations

\[
\xi = \int_G (\xi,\pi(s)\eta)\pi(s)^{-}\psi \, ds = \int_G (\xi,\pi(s)^{-}\psi)\pi(s)\eta \, ds. \tag{7}
\]

These representations must be interpreted in weak sense. Relations of this type are known as \textbf{covariant frame expansions} in harmonic analysis [3, Section 5.8] or \textbf{covariant coherent state expansions} in mathematical physics [1, Chapters 7 and 8]. The pair of vectors \( \{\xi,\psi\} \) is then called an \textbf{admissible pair} for \( (\pi,\mathcal{H}_\pi) \) with bounds \( \alpha, \beta \). If \( \eta = \psi \), it is said that \( \eta \) is an \textbf{admissible vector}.

The inequalities in (F2) have their images in terms of positive elements of the Fourier algebra \( A(G) \), weights on the left group von Neumann algebra \( \mathcal{L}(G) \), the corresponding \textbf{spatial derivatives} and \textbf{holomorphic extensions} of the \textbf{cocycle derivatives}:

**Theorem 1** Let \( G \) be a lc group, \( (\pi,\mathcal{H}_\pi) \) a unitary representation of \( G \), \( \eta \in \mathcal{H}_\pi \) and

\[
\varphi_\eta(s) := (\pi(s)\eta|\eta)_\pi, \quad s \in G.
\]

The following statements are equivalent:

(i) \( \{\pi(s)\eta\}_{s \in G} \) is a covariant frame for \( (\pi,\mathcal{H}_\pi) \) with bounds \( \alpha, \beta \).

(ii) \( \varphi_\eta \in A(G)^+ \) and

\[
\alpha(P_0 f|f) \leq (f^\beta * f, \varphi_\eta) \leq \beta(P_0 f|f), \quad f \in C_c(G), \tag{8}
\]

with \( P_0 \) an (orthogonal) projection in the centre \( Z(G) = \mathcal{L}(G) \cap R(G) \).

(iii) The normal finite functional \( \omega_\eta \in \mathcal{L}(G)^+_\eta \), corresponding to \( \varphi_\eta \in A(G)^+ \), has support \( s(\omega_\eta) \in Z(G) \) and

\[
\alpha \Omega_t(A) \leq \omega_\eta(A) \leq \beta \Omega_t(A), \quad A \in (\mathcal{L}(G)s(\omega_\eta))^+. \tag{9}
\]

In particular, \( \omega_\eta \) is faithful on \( \mathcal{L}(G)s(\omega_\eta) \). Moreover, \( s(\omega_\eta) = P_0 \).

(iv) The spatial derivative \( d\omega_\eta/d\Omega_t \) satisfies

\[
\alpha \Delta s(\omega_\eta) \leq \frac{d\omega_\eta}{d\Omega_t} \leq \beta \Delta s(\omega_\eta). \tag{10}
\]

(v) \( \omega_\eta \in \mathcal{L}(G)^+_\eta \) is faithful on \( \mathcal{L}(G)s(\omega_\eta) \) and the \textbf{cocycle derivatives} \( (D\omega_\eta : D\Omega_t), t \in \mathbb{R} \), can be extended to \( \mathcal{L}(G)s(\omega_\eta) \)-valued \( \sigma \)-weakly continuous bounded functions on the horizontal strip \( \mathbb{D}_{1/2} := \{ z \in \mathbb{C} : -1/2 \leq \text{Im}(z) \leq 0 \} \) which are holomorphic in the interior of the strip and such that

\[
\|(D\omega_\eta : D\Omega_t)_{-i/2}\| \leq \sqrt{\beta} \quad \text{and} \quad \|(D\Omega_t : D\omega_\eta)_{-i/2}\| \leq 1/\sqrt{\alpha}.
\]
If this is the case, \((D \omega_\eta : D \Omega_t)_{-i/2}\) is nonsingular on \(s(\omega_\eta)L^2(G)\) with bounded inverse, one has \((D \Omega_t : D \omega_\eta)_{-i/2} = (D \omega_\eta : D \Omega_t)_{-i/2}\) and

\[
\omega_\eta(A) = \Omega_t((D \omega_\eta : D \Omega_t)^*_{-i/2}A(D \omega_\eta : D \Omega_t)_{-i/2}), \quad A \in \mathcal{L}(G),
\]

\[
\Omega_t(A) = \omega_\eta((D \omega_\eta : D \Omega_t)^{-1}_{-i/2}A(D \omega_\eta : D \Omega_t)^{-1}_{-i/2}), \quad A \in \mathcal{L}(G)s(\omega_\eta).
\]

**Proof:** (i)\(\iff\)(ii)\(\iff\)(iii): If \(\{\pi(s)\eta\}_{s \in G}\) is a covariant frame with frame bounds \(\alpha, \beta\), for a fixed \(\xi \in \mathcal{H}_\pi\), Cauchy-Schwarz inequality implies that the map \(\mathcal{H}_\pi \ni \zeta \mapsto \int_G(\xi|\pi(s)\eta)_\pi(\pi(s)|\xi)_\pi ds \in \mathbb{C}\) is conjugated linear and bounded. By Riesz representation theorem, there exists a unique element \(S_\eta \xi \in \mathcal{H}_\pi\) such that

\[
(S_\eta\xi|\xi)_\pi = \int_G(\xi|\pi(s)\eta)_\pi(\pi(s)|\xi)_\pi ds, \quad \xi \in \mathcal{H}_\pi.
\]

The mapping \(S_\eta : \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi\) so defined is linear and, by (F2), satisfies

\[
\alpha \||\xi||^2 \leq (S_\eta\xi|\xi)_\pi \leq \beta ||\xi||^2, \quad \xi \in \mathcal{H}_\pi,
\]

i.e., \(\alpha I_{\mathcal{H}_\pi} \leq S_\eta \leq \beta I_{\mathcal{H}_\pi}\), where \(I_{\mathcal{H}_\pi}\) denotes the identity operator on \(\mathcal{H}_\pi\). Thus, \(S_\eta\) is a bounded positive operator with bounded inverse. Put \(\psi := S^{-1}_\eta\eta\). Then the equalities \(\xi = S^{-1}_\eta S_\eta \xi = S_\eta S^{-1}_\eta \xi\) coincide with the representation formulas given in (17). In other words, \(\{\pi(s)\eta\}_{s \in G}\) is a covariant frame with frame bounds \(\alpha, \beta\) iff the operator

\[
V_\eta : \mathcal{H}_\pi \rightarrow L^2(G), \quad (V_\eta \xi)(s) := (\xi|\pi(s)\eta)_\pi,
\]

is a bounded map and \(V^*_\eta V_\eta = S_\eta\) satisfies (14). Furthermore, (13) implies that \(S_\eta\) commutes with \(\pi(s)\) for all \(s \in G\). Hence \(V_{S^{-1}_\eta} = V_\eta S^{-1}_\eta\) is also bounded and the representation formulas (17) are equivalent to

\[
V^*_\eta V_\eta = S^{-1}_\eta V^*_\eta V_\eta = I_{\mathcal{H}_\pi}, \quad V^*_\eta V_{S^{-1}_\eta} = V^*_\eta V_\eta S^{-1}_\eta = I_{\mathcal{H}_\pi}.
\]

From (17) it is clear that \(\eta\) and \(\psi\) are cyclic vectors for \((\pi, \mathcal{H}_\pi)\). A similar calculation shows that

\[
V^*_\eta V_{S^{-1/2}_\eta} = I_{\mathcal{H}_\pi},
\]

that is, \(S^{-1/2}_\eta \eta\) is an admissible vector for \((\pi, \mathcal{H}_\pi)\).

On the other hand, \(\varphi_\eta \in \mathcal{P}(G)\) by its own definition [10 Prop.3.15]. Let \((\pi', \mathcal{H}_{\pi'})\) be the cyclic representation of \(G\), with cyclic vector \(\eta'\), induced by \(\varphi_\eta\) [10 Sect.3.3]. Then, for any \(f \in L^1(G)\),

\[
\langle f, \varphi_\eta \rangle = \int_G f(s)\varphi_\eta(ds) = (\pi'(f)\eta'|\eta')_{\pi'} = \int_G f(s)(\pi'(s)\eta'|\eta')_{\pi'} ds.
\]

Hence, being \(\varphi_\eta\) continuous,

\[
\varphi_\eta(s) = (\pi(s)\eta|\eta)_\pi = (\pi'(s)\eta'|\eta')_{\pi'}, \quad s \in G.
\]

By (15), the cyclic representations \((\pi', \mathcal{H}_{\pi'}, \eta')\) and \((\pi, \mathcal{H}_\pi, \eta)\) of \(G\) are equivalent and the unitary operator \(U : \mathcal{H}_{\pi'} \rightarrow \mathcal{H}_\pi\) such that \(U(\pi'(s)\eta') = \pi(s)\eta\) resolves the equivalence [10 Prop.3.23]. Thus, \(\{\pi'(s)\eta'\}_{s \in G}\) and \(\{\pi(s)\eta\}_{s \in G}\) are simultaneously covariant frames with frame bounds \(\alpha, \beta\) or not.

Now, let us assume that

\[
\langle f^2 \ast f, \varphi_\eta \rangle \leq \beta \langle f|f \rangle, \quad f \in C_c(G).
\]

Then, for \(f \in C_c(G)\),

\[
||\pi'(f)\eta'||^2_{\pi'} = (\pi'(f)\eta'|\pi'(f)\eta')_{\pi'} = (\pi'(f)^*\pi'(f)\eta'|\eta')_{\pi'} = \langle f^2 \ast f, \varphi_\eta \rangle \leq \beta ||f||^2.
\]
This implies that there is a bounded operator $T : L^2(G) \to \mathcal{H}_{\pi'}$ such that $Tf = \pi'(f)\eta'$ for $f \in C_c(G)$. $||T|| \leq \sqrt{\beta}$ and $T$ has dense range since $\eta'$ is cyclic. Moreover, by the definition of $\pi'$, for $f, g \in C_c(G)$,

$$TLfg = T(f * g) = \pi'(f * g)\eta' = \pi'(f)\pi'(g)\eta' = \pi'(f)Tg,$$

that is, $T$ intertwines $\pi'$ and the left regular representation:

$$TL_f = \pi'(f)T, \quad f \in C_c(G).$$

Then one has, for $f \in C_c(G)$,

$$T^*TL_f = T^*\pi'(f)T = (\pi'(f)^*T)^*T = (TL_f^*)^*T = LfT^*T.$$ 

Thus, $T^*T$ belongs to the commutant $\mathcal{R}(G)$ of $\mathcal{L}(G)$.

Let $T = V[T]$ be the polar decomposition of $T$, where $|T| = (T^*T)^{1/2}$. Then $VV^* = I_{\mathcal{H}_{\pi'}}$ and $V^*V = P_0$, where $P_0$ is the projection from $L^2(G)$ onto the support of $|T|$, $(\text{Ker}|T|)^\perp =: \mathcal{H}_0$. For $f \in C_c(G)$ and $g \in L^2(G)$,

$$VL_f|T|g = V|T|L_fg = TL_fg = \pi'(f)Tg = \pi'(f)V|T|g.$$ 

Being $|T|L^2(G)$ dense in $\mathcal{H}_0$, $V$ puts into equivalence $\{\pi', \mathcal{H}_{\pi'}\}$ and the subrepresentation $\{\lambda|_{\mathcal{H}_0}, \mathcal{H}_0\}$ of the left regular representation. Let $\eta_0 := V^*\eta' \in L^2(G)$. Then $\eta_0$ is a cyclic vector for $\{\lambda|_{\mathcal{H}_0}, \mathcal{H}_0\}$ and, for $s \in G$,

$$\varphi_\eta(s) = (\pi(s)\eta|\eta_0) = (\pi'(s)\eta'|\eta_0') = (\lambda(s)\eta|\eta_0).$$

Note that $(\lambda(s)\eta|\eta_0) = \int_G \eta_0(s^{-1}t)\eta_0(t) \, dt = (\eta_0|\eta_0^\perp)(s)$. This proves that $\varphi_\eta \in A(G)^+$. Moreover, for $f \in C_c(G)$,

$$f * \eta_0 = L_f\eta_0 = L_fV^*\eta' = V^*\pi'(f)\eta' = V^*Tf = |T|f.$$ 

Hence, $\eta_0$ is right bounded and $R_{\eta_0} = |T|$. In particular, since $||T|| \leq \sqrt{\beta}$,

$$0 \leq R_{\eta_0} \leq \sqrt{\beta}P_0. \quad (21)$$

Moreover, being $R_{\eta_0}$ positive, $\eta_0 = \eta_0^\perp \quad [20 \text{ Prop.2.5}].$

Since, for $f \in C_c(G)$,

$$\langle f, \varphi_\eta \rangle = \int_G f(s)\varphi_\eta(s) \, ds = \int_G f(s)(\lambda(s)\eta|\eta_0) \, ds = (L_f\eta_0|\eta_0),$$

the positive normal functional $\omega_\eta \in \mathcal{L}(G)^+_+$ is of the form

$$\omega_\eta(A) = (A\eta_0|\eta_0), \quad A \in \mathcal{L}(G). \quad (22)$$

Then, for $f \in L^2(G)$, $f$ left bounded,

$$\omega_\eta(L_f^*L_f) = ||L_f\eta_0||^2 = ||R_{\eta_0}f||^2, \quad (23)$$

and \text{(21)} implies that $\omega_\eta(A) \leq \beta \Omega_l(A)$ for $A \in (P_0\mathcal{L}(G)P_0)^+$ and $\omega_\eta$ is null on $((I - P_0)\mathcal{L}(G)(I - P_0))^+$.

In the opposite direction, assume now that $\varphi_\eta \in A(G)^+$ and

$$\omega_\eta(A) \leq \beta \Omega_l(A), \quad A \in \mathcal{L}(G)^+_+. \quad (24)$$

Recall that, since $\omega_\eta$ is normal and $\omega_\eta(\lambda(\cdot)) = \varphi_\eta(\cdot)$, for $f \in C_c(G)$,

$$\omega_\eta(L_f^*L_f) = \omega_\eta\left(\int_G f^t(s)\lambda(s) \, ds \int_G f(t)\lambda(t) \, dt\right)$$

$$= \int_G \int_G f^t(s)f(t) \omega_\eta(\lambda(st)) \, ds \, dt$$

$$= \int_G \int_G \int_G f^t(s)f(s^{-1}t)\varphi_\eta(t) \, dtds = \langle f^2 * f, \varphi_\eta \rangle. \quad (25)$$
Thus, \(24\) implies \(19\) and then, repeating the above arguments, for \(f \in C_c(G)\),
\[
\omega_\eta(L_f^* L_f) = (L_f \eta_0 | L_f \eta_0) = (R_{\eta_0} f | R_{\eta_0} f) = \langle f^* \ast f, \varphi_\eta \rangle, \tag{26}
\]
so that, by \(24\), condition \(24\) really implies
\[
\langle f^* \ast f, \varphi_\eta \rangle \leq \beta (P_0 f | f), \quad f \in C_c(G).
\]

Similar arguments can be found in Combes \([5\) Lemma 2.3] and Haagerup \([14\) Prop.2.4].

Reasoning as before with \(15\), due to \(20\), the cyclic representations \((\lambda_{|H_0}, H_0, \eta_0)\), \((\pi', H_\pi, \eta')\) and \((\pi, H_\pi, \eta)\) of \(G\) are unitarily equivalent and the orbits \(\{\lambda_{|H_0}(s) \eta_0\}_{s \in G}\), \(\{\pi'(s) \eta'\}_{s \in G}\) and \(\{\pi(s) \eta\}_{s \in G}\) are simultaneously covariant frames with frame bounds \(\alpha, \beta\) or not. Thus, it only remains to deal with the first inequalities in \(27\), \(8\) and \(9\).

For \((\lambda_{|H_0}, H_0, \eta_0)\) the corresponding operator \(V_{\eta_0}\) given in \(15\) is of the form
\[
(V_{\eta_0} f)(s) = \langle f | \lambda(s) \eta_0 \rangle = \int_G f(t) \eta_0(t^{-1}) dt = \int_G f(t) \eta_0(t^{-1}) dt = (f * \eta_0)(s) = (f * \eta_0)(s),
\]
because \(\eta_0 = \eta_0^2\). That is, \(V_{\eta_0} = R_{\eta_0}|_{H_0} = |T|_{H_0}\) and \(S_{\eta_0} = |T|^2_{H_0}\). Recall also that \(H_0\) is just the support of \(|T|\). Using \(20\) and reasoning as above, the first inequality in \(14\),
\[
\alpha ||f||^2 \leq (S_{\eta_0} f, f) = (|T|^2 f, f) = (R_{\eta_0} f, R_{\eta_0} f), \quad f \in H_0,
\]
is equivalent to \(\sqrt{\alpha} P_0 \leq R_{\eta_0}\) and also to the first inequalities in \(8\) and \(9\) on \((P_0 L(G) P_0)^+\), being \(\omega_\eta\) null on \((|I - P_0| L(G) (|I - P_0|))^+\). By the first inequality in \(29\), \(\omega_\eta\) is faithful on \((P_0 L(G) P_0)^+\) since \(1_2\) is. Thus, the support \(s(\omega_\eta)\) coincides with \(P_0\).

To complete the proof of (i)\(\Rightarrow\)(ii)\(\Rightarrow\)(iii) we must see that \(P_0\) belongs to the centre \(Z(G) = L(G) \cap R(G)\). Since \((\lambda_{|H_0}, H_0)\) is a subrepresentation of the left regular representation of \(G\), \(H_0\) is left invariant, so that \(P_0 \in R(G)\). On the other hand, according to \(23\) Th.7.4, the supports of \(\omega_\eta\) and \(d\omega_\eta/d\Omega_\pi\) coincide. Since \(d\omega_\eta/d\Omega_\pi\) is \((-1\)-homogeneous), \(\ker(d\omega_\eta/d\Omega_\pi)\) is invariant under all \(p(s), s \in G\); thus, \(P_0 \in L(G)\).

(iii)\(\Rightarrow\)(iv): Let us recall that \(d\Omega_\pi/d\Omega_\pi = \Delta\) on \(L^2(G)\) and that the centralizer \(L(G)^\Omega\) can also be defined as the set of elements in \(L(G)\) that commute with \(\Delta\) \([19\) page 61]. In particular, the centre \(Z(G)\) of \(L(G)\) is contained in \(L(G)^\Omega\) \([20\) V1.1.23]. Since \(P_0 \in \Delta(G)\), \(P_0\) commutes with \(\Delta\).

This implies that \(d\Omega_\pi/d\Omega_\pi = \Delta P_0\) on \(L(G)\). On the other hand, \(\omega_1 \leq \omega_2\) on a von Neumann algebra \(M \subseteq B(H)\) and a n.s.f. weight \(\Omega'\) on \(M'\), \(\omega_1 \leq \omega_2\) if \(d\omega_1/d\Omega' \leq d\omega_2/d\Omega'\).

(iii)\(\Rightarrow\)(v): By \(9\), \(\omega_\eta \leq \beta \Omega_1\) as n.f.s weights on \(L(G) P_0\). According to \(20\) VIII.3.17 (a result due to Connes \([6\]), this is equivalent to the fact that the cocycle derivative \((D\omega_\eta : D\Omega_\pi)_t\) can be extended to an \((L(G)P_0\)-valued \(\sigma\)-weakly continuous bounded function on the horizontal strip \(\mathcal{D}_{1/2}\) which is holomorphic in the interior of the strip and such that \(||(D\omega_\eta : D\Omega_\pi)_t|| \leq \sqrt{\alpha}\).

Furthermore, in such case, \((11\) is satisfied. Now, by the uniqueness of the cocycle derivative and the chain rule for cocycle derivatives of n.f.s weights \(20\) VIII.3.7,\]
\[
(D\Omega_\pi : D\omega_\eta)_t = (D\omega_\eta : D\Omega_\pi)_t^{-1}, \quad t \in \mathbb{R},
\]
and, using \(20\) VIII.3.17 again, the inequality \(\alpha \Omega_1 \leq \omega_\eta\) as n.f.s weights on \(L(G) P_0\) is equivalent to the extension of \((D\Omega_\pi : D\omega_\eta)_t\) on \(\mathcal{D}_{1/2}\) such that the inequality \(||\Delta^{1/2} R_{\eta_0}^{-1}|| \leq 1/\sqrt{\alpha}\) and \(12\) hold.

The proof of Theorem \(1\) uses a right selfdual bounded element \(\eta_0\) of \(L^2(G)\). Furthermore, \(\eta_0\) is also left bounded and all the ingredients in statements (i)-(v) of Theorem \(1\) can be written in terms of \(\eta_0\) and its \(L^2\)-Fourier transform \(F_2(\eta_0)\):
Theorem 2 Under the equivalent conditions (i)-(v) of Theorem \ref{thm:equivalent}, the representation \((\pi, \mathcal{H}_\pi)\) is unitarily equivalent to a cyclic subrepresentation \((\lambda|_{\mathcal{H}_0}, \mathcal{H}_0)\) of the left regular representation of \(G\) with a cyclic vector \(\eta_0\) such that:

(a) \(P_0\) is the projection of \(L^2(G)\) onto \(\mathcal{H}_0\).

(b) \(\eta_0\) is right bounded, \(\eta_0 = \eta_0^*\) and
\[
\sqrt{\alpha} P_0 \leq R_{\eta_0} \leq \sqrt{\beta} P_0,
\]
(27)

Thus, \(R_{\eta_0}\) is positive and nonsingular on \(\mathcal{H}_0\) with bounded inverse.

(c) \(\{\lambda|_{\mathcal{H}_0}(s)\eta_0\}_{s \in G}\) is a covariant frame for \((\lambda|_{\mathcal{H}_0}, \mathcal{H}_0)\) with bounds \(\alpha, \beta\).

(d) \(\varphi_s(s) = (\lambda(s)\eta_0\eta_0^*),\) for \(s \in G\).

(e) \(\omega_\eta(A) = (A\eta_0|\eta_0)\), for \(A \in \mathcal{L}(G)\).

(f) \(\eta_0\) is left bounded, \(L_{\eta_0}\) is nonsingular on \(\mathcal{H}_0\) with bounded inverse and
\[
(D\omega_\eta : D\Omega_1)_{-i/2} = L_{\eta_0}, \quad (D\Omega_1 : D\omega_\eta)_{-i/2} = L_{\eta_0}^{-1}.
\]

Therefore,
\[
\omega_\eta(A) = \Omega_1(L_{\eta_0}^*AL_{\eta_0}), \quad A \in \mathcal{L}(G),
\]
(28)
\[
\Omega_1(A) = \omega_\eta((L_{\eta_0}^{-1})^*AL_{\eta_0}^{-1}), \quad A \in \mathcal{L}(G)P_0.
\]
(29)

(g) \(\varphi_s(s) = \Omega_1(L_{\eta_0}^*\lambda(s)L_{\eta_0})\), for \(s \in G\).

(h) The spatial derivative \(d\omega_\eta/d\Omega_1\) is given by
\[
\frac{d\omega_\eta}{d\Omega_1} = F_2(\eta_0)F_2(\eta_0)^* = F_2(\eta_0)F_2(J\eta_0) = L_{\eta_0}\Delta L_{\eta_0}^*.
\]

(i) The cocycle derivative \((D\omega_\eta : D\Omega_1)_t \in \mathcal{L}(G)P_0\) is
\[
(D\omega_\eta : D\Omega_1)_t = (F_2(\eta_0)F_2(\eta_0)^*)^t\Delta^{-it} = (L_{\eta_0}\Delta L_{\eta_0}^*)^t\Delta^{-it}, \quad t \in \mathbb{R}.
\]

(j) \(\{\eta_0, R_{\eta_0}^{-2}\eta_0\}\) is an admissible pair for \((\lambda|_{\mathcal{H}_0}, \mathcal{H}_0)\) with bounds \(\alpha, \beta\).

Proof: (a)-(e) are implicit in the proof (i)\(\Leftrightarrow\)(ii)\(\Leftrightarrow\)(iii) of Theorem \ref{thm:equivalent}

(f): According to \[26\] VIII.3.18.(ii), for \(f \in C_c(G)\), the element in \(L^2(G)\) corresponding to \((D\omega_\eta : D\Omega_1)_{-i/2}\) (we are using the GNS representation of \(\mathcal{L}(G)\) associated with \(\Omega_1\)) is given by
\[
J\delta_{-i/2}((D\omega_\eta : D\Omega_1)_{-i/2}\eta_0)Jf = J\Delta^{1/2}(D\omega_\eta : D\Omega_1)_{-i/2}\Delta^{-1/2}Jf.
\]

On the other hand, \[28\] says that \(\omega_\eta(L_{\eta_0}^*L_{\eta_0}) = (R_{\eta_0}f|R_{\eta_0}f)\), for \(f \in C_c(G)\). These facts together with \[11\] imply that \(J\Delta^{1/2}(D\omega_\eta : D\Omega_1)_{-i/2}\Delta^{-1/2}J = R_{\eta_0}\) and, then,
\[
(D\omega_\eta : D\Omega_1)_{-i/2} = \Delta^{-1/2}JR_{\eta_0}J\Delta^{1/2} = L_{\eta_0}^{-1}L_{\eta_0}^*.
\]

Now, \[28\] and \[29\] are just \[11\] and \[12\].

(g): The result follows from \[28\] and \(\varphi_s(s) = \omega_\eta(\lambda(s))\) for \(s \in G\).
(h): According to (20), for $s \in G$,
\[
\varphi_\eta(s) = (\lambda(s)\eta_0|\eta_0) = \int_G \eta_0(s^{-1}t)\eta_0(t)\,dt = (\eta_0^* \ast \eta_0')(s).
\]

Since $f \ast g = \mathcal{F}_1([\mathcal{F}_2(\overline{g})\mathcal{F}_2(\overline{f})])$ for all $f, g \in L^2(G)$ [28 Cor.5.7], we have
\[
\varphi_\eta = \mathcal{F}_1(\mathcal{F}_2(\eta_0)\mathcal{F}_2(\eta_0)^*),
\]
that is, $d\omega_\eta/d\Omega_\tau = \mathcal{F}_2(\eta_0)\mathcal{F}_2(\eta_0)^*$. Now, the second equality follows from the fact that $\mathcal{F}_2(f) = \mathcal{F}_2(f)^*$ for all $f \in L^2(G)$ [28 Prop.3.3]. Also, by (f), $L_{\eta_0}$ is nonsingular on $\mathcal{H}_0$ with bounded inverse and the n.s.f. weights $\omega_\eta$ and $\Omega_\tau$ on $\mathcal{L}(G)P_0$ satisfy $\omega_\eta(\cdot) = \Omega_\tau(L_{\eta_0}^* \cdot L_{\eta_0})$. Under these conditions, according to [23 Prop.7.13],
\[
\frac{d\omega_\eta}{d\Omega_\tau} = \frac{d\Omega_\tau(L_{\eta_0} \cdot L_{\eta_0})}{d\Omega_\tau} = L_{\eta_0} \frac{d\Omega_\tau}{d\Omega_\tau} L_{\eta_0}^* = L_{\eta_0} \Delta L_{\eta_0}^*.
\]
(i): According to [7 Th.9.(2)] (see also [23 Th.7.4]),
\[
(d\omega_\eta/d\Omega_\tau)^it = (D\omega_\eta : D\Omega_\tau)_t(d\Omega_\tau/d\Omega_\tau)^it, \quad t \in \mathbb{R}.
\]
Then, (i) follows from (h) and $d\Omega_\tau/d\Omega_\tau = \Delta$ [28 page 551].

(i) and (k): At the end of the proof proof (i)\(\Leftrightarrow\)(ii)\(\Leftrightarrow\)(iii) of Theorem 1 we got $V_{\eta_0} = R_{\eta_0}\mid_{\mathcal{H}_0}$.

Then, the arguments preceding (16) and (17) show that $\{\eta_0, R_{\eta_0}^{-2}\eta_0\}$ is an admissible pair for $(\lambda|_{\mathcal{H}_0}, \mathcal{H}_0)$ with bounds $\alpha, \beta$ and $R_{\eta_0}^{-1}\eta_0$ is an admissible vector for $(\lambda|_{\mathcal{H}_0}, \mathcal{H}_0)$. Let us note that, since $L_{\eta_0}$ and $R_{\eta_0}$ commute, $L_{\eta_0}^{-1}\eta_0 = R_{\eta_0}^{-1}\eta_0 \Leftrightarrow L_{\eta_0}\eta_0 = R_{\eta_0}\eta_0 \Leftrightarrow \eta_0^* \ast \eta_0 = \eta_0 \ast \eta_0$.

**Remark 3** It is possible to dualize the above discussion entirely using the modular conjugation $J$ in such a way that the roles of the left and right regular representations and the corresponding von Neumann algebras are interchanged. See [13] for details.

### 4 $\sigma^\Omega$-invariance

Note that the result in Theorem 1(v), concerning cocycle derivatives, is a Radon-Nikodym theorem for the weights $\omega_\eta$ and $\Omega_\tau$; see Equations (11) and (12). In this section we study particular cases of Theorems 1 and 2 closely related to the Pedersen-Takesaki work [19] on noncommutative Radon-Nikodym theorems. They turn out when the weight $\omega_\eta$ is invariant under the action of the automorphism group $\sigma^\Omega$. This is the case when the reduced von Neumann algebra $\mathcal{L}(G)P_0$ is semifinite and $\{(D\omega_\eta : D\Omega_\tau)_t\}_{t \in \mathbb{R}}$ is an one-parameter group of unitary elements of $\mathcal{L}(G)P_0$.

**Theorem 4** Assume that the equivalent statements (i)-(v) of Theorem 4 are satisfied. Put

\[
H := (D\omega_\eta : D\Omega_\tau)_{-i/2}(D\omega_\eta : D\Omega_\tau)_{-i/2} = L_{\eta_0} L_{\eta_0}^*.
\]

Then the following additional conditions are equivalent:

(PT.1) $(D\omega_\eta : D\Omega_\tau)_{-i/2} = L_{\eta_0}$ commutes with $\Delta$, that is, it belongs to the centralizer $\mathcal{L}(G)^{\Omega_\tau}$.

(PT.2) $\omega_\eta$ is invariant with respect to $\{\sigma_\tau^\Omega\}_{t \in \mathbb{R}}$ on $\mathcal{L}(G)$, that is, $\omega_\eta(A) = \omega_\eta(\sigma_\tau^\Omega(A))$, for $A \in \mathcal{L}(G)$ and $t \in \mathbb{R}$.

(PT.3) $\Omega_\tau$ is invariant with respect to $\{\sigma_\tau^{\omega_\eta}\}_{t \in \mathbb{R}}$ on $\mathcal{L}(G)P_0$, that is, $\Omega_\tau(A) = \Omega_\tau(\sigma_\tau^{\omega_\eta}(A))$, for $A \in \mathcal{L}(G)P_0$ and $t \in \mathbb{R}$.
(PT.4) \((D\omega_\eta : D\Omega_t)_t \in \mathcal{L}(G)^{\Omega_t}, \text{ for } t \in \mathbb{R}\).

(PT.5) \((D\omega_\eta : D\Omega_t)_t \in (\mathcal{L}(G)P_0)^{\omega_\eta}, \text{ for } t \in \mathbb{R}\).

(PT.6) \((D\Omega_t : D\omega_\eta)_t \in \mathcal{L}(G)^{\Omega_t}, \text{ for } t \in \mathbb{R}\).

(PT.7) \((D\Omega_t : D\omega_\eta)_t \in (\mathcal{L}(G)P_0)^{\omega_\eta}, \text{ for } t \in \mathbb{R}\).

(PT.8) \((D\omega_\eta : D\Omega_t)_t \in \{\mathcal{L}(G)P_0\}^\omega_\eta, \text{ for } t \in \mathbb{R}\).

(PT.9) \((D\Omega_t : D\omega_\eta)_t \in \{\mathcal{L}(G)P_0\}^{\omega_\eta}, \text{ for } t \in \mathbb{R}\).

(PT.10) \omega_\eta(A) = \Omega_t(H^{1/2}AH^{1/2}), \text{ for } A \in \mathcal{L}(G).

(PT.11) \Omega_t(A) = \omega_\eta(H^{-1/2}AH^{-1/2}), \text{ for } A \in \mathcal{L}(G)P_0.

If this is the case, \(\alpha P_0 \leq H \leq \beta P_0\).

**Proof:** Let us begin by recalling some results of the Pedersen-Takesaki work [19]. Let \(\mathcal{M}\) be a von Neumann algebra and \(\Omega\) a n.s.f weight on \(\mathcal{M}\). For each \(H \in \mathcal{M}_\Omega^\omega\),

\[
\Omega^H(A) := \Omega(H^{1/2}AH^{1/2}), \quad A \in \mathcal{M}_+,
\]

is a s.n. weight on \(\mathcal{M}\). If \(H\) is a positive selfadjoint operator affiliated with \(\mathcal{M}_\Omega\) and \(H_\varepsilon := H(1 + \varepsilon H)^{-1}\) for \(\varepsilon > 0\),

\[
\Omega^H(A) := \lim_{\varepsilon \downarrow 0} \Omega(H_\varepsilon^{1/2}AH_\varepsilon^{1/2}), \quad A \in \mathcal{M}_+,
\]

also defines a n.s. weight on \(\mathcal{M}\); this weight is faithful iff \(H\) is nonsingular. If \(\omega\) is a n.s. weight on \(\mathcal{M}\), the following conditions are equivalent:

(i) \(\omega = \omega \circ \sigma^\omega_t\), for \(t \in \mathbb{R}\).

(ii) \((D\omega : D\Omega)_t \in \mathcal{M}_\omega, \text{ for } t \in \mathbb{R}\).

(iii) \((D\omega : D\Omega)_t \in \mathcal{M}_\Omega, \text{ for } t \in \mathbb{R}\).

(iv) \{(D\omega : D\Omega)_t\}_t \in \{\mathcal{L}(G)\}_{t \in \mathbb{R}}\) is a strong(operator)-continuous group of unitary elements of \(\mathcal{L}(\omega).\mathcal{M}_\omega(\omega)\),

, where \(\mathcal{L}(\omega)\) denotes the support of \(\omega\).

(v) \(\omega = \Omega^H\) for some nonsingular positive selfadjoint operator \(H\) affiliated with \(\mathcal{M}_\Omega\).

If moreover \(\omega\) is faithful, then also the following statement is equivalent to those above:

(vi) \(\Omega = \Omega \circ \sigma^\omega_t\), for \(t \in \mathbb{R}\).

See also [20] Lem.VIII.2.7], [26] Lem.VIII.2.8], [26] Cor.VIII.3.6] and [23] Th.4.10].

In our context, under the equivalent equivalent conditions (i)-(v) of Theorem 11 by Theorem 2(i), when \(L_{m_0}\) commutes with \(\Delta\) one has

\[
(D\omega_\eta : D\Omega_t)_t = (L_{m_0} \Delta L_{m_0}^*)^{it} \Delta^{-it} = (L_{m_0} L_{m_0}^*)^{it} = H^{it}, \quad t \in \mathbb{R},
\]

and, using 23 Cor.3.4] on \(\mathcal{L}(G)P_0\),

\[
(D\Omega_t : D\omega_\eta)_t = (D\omega_\eta : D\Omega_t)_t^{-1} = H^{-it}, \quad t \in \mathbb{R}.
\]
In this case it is clear that the extensions of these cocycle derivatives on the horizontal strip \( \overline{D}_{1/2} \)
given in Theorem 1(v) lead to

\[
(D\omega_\eta : D\Omega_t)_{-1/2} = H^{1/2} \quad \text{and} \quad (D\Omega_t : D\omega_\eta)_{-1/2} = H^{-1/2},
\]

so that

\[
\|H^{1/2}\| \leq \sqrt{\beta} \quad \text{and} \quad \|H^{-1/2}\| \leq 1/\sqrt{\alpha}
\]
or, in other words, \( \alpha P_0 \leq H \leq \beta P_0 \). Now, (PT.10) and (PT.11) follows, respectively, from [11]
and [12].

Note that, since \( u_t = (D\omega_\eta : D\Omega_t)_{t} \in \mathcal{L}(G)^{\Omega_t} \) for \( t \in \mathbb{R} \), one has \( u_{s+t} = u_su_t \), for \( s, t \in \mathbb{R} \), that is, \( \{(D\omega_\eta : D\Omega_t)_{t}\} \in \mathbb{R} \) is a group. \( \square \)

**Remark 5** The modular automorphism group \( \{\omega^u_\eta\}_{t \in \mathbb{R}} \) on \( \mathcal{L}(G)P_0 \) is given by [7] Th.9.(1):

\[
\omega^u_\eta(A) = \left( \frac{d\omega_\eta}{dt} \right)^iu \left( \frac{d\omega_\eta}{dt} \right)^{-it}, \quad A \in \mathcal{L}(G)P_0, \ t \in \mathbb{R}.
\]

It is well known that every von Neumann algebra \( \mathcal{M} \) is uniquely decomposable into the direct sum of those of type I, II_1, II_\infty and III. If there is not summand of type III, then \( \mathcal{M} \) is said to be **semifinite**. See, e.g., [25] Sect.V.1 for details.

Pedersen-Takesaki work includes the following characterization of semifinite von Neumann algebras [10] Th.7.4: There exists a n.s.f. weight \( \omega \) on \( \mathcal{M} \) such that \( \{\sigma^\omega_t\}_{t \in \mathbb{R}} \) is **inner** in the sense that there exists a strong-continuous one parameter unitary group \( \{u_t\}_{t \in \mathbb{R}} \) in \( \mathcal{M} \) such that \( \sigma^\omega_t(\cdot) = u_t \cdot u_t^* \), \( t \in \mathbb{R} \). If this is the case, then with a fixed n.s.f. trace \( \tau \) on \( \mathcal{M} \), every s.n. weight on \( \mathcal{M} \) is written uniquely in the form \( \tau^H \), with \( H \) a positive selfadjoint operator affiliated with \( \mathcal{M} \), where \( \tau^H \) is given by [22]. See also [26] Th.VIII.3.14.

The above comments and Theorem 4(4) imply the following result.

**Corollary 6** Assume that the equivalent statements (i)-(v) of Theorem 7 and the additional equivalent conditions (PT.1)-(PT.11) of Theorem 1 are satisfied. Then the von Neumann algebra \( \mathcal{L}(G)P_0 \) is semifinite.

Now, let \( \mathcal{M} \) be a von Neumann algebra equipped with a one parameter automorphism group \( \{\sigma_t\}_{t \in \mathbb{R}} \). Let \( \omega \) be an s.n. weight on \( \mathcal{M} \) and put \( \mathcal{N}_\omega := \{A \in \mathcal{M} : \omega(A^*A) < \infty \} \). It is said that \( \omega \) satisfies the (Kubo-Martin-Schwinger) **KMS condition** for \( \{\sigma_t\}_{t \in \mathbb{R}} \) if the following two conditions hold:

(i) \( \omega = \omega \circ \sigma_t \), for \( t \in \mathbb{R} \).

(ii) For every pair \( A, B \in \mathcal{N}_\omega \cap \mathcal{N}_\omega^* \), there exists a function \( f = f_{A,B} \) defined, continuous and bounded on the closed horizontal strip \( \overline{D}_{-1} := \{z \in \mathbb{C} : 0 \leq \text{Im}(z) \leq 1 \} \), holomorphic on the open strip \( D_{-1} \) and such that

\[
f(t) = \omega(\sigma_t(A)B), \quad f(t + i) = \omega(B\sigma_t(A)), \quad t \in \mathbb{R}.
\]

**Corollary 7** Assume that the equivalent statements (i)-(v) of Theorem 7 are satisfied. Then the following additional conditions are equivalent:

(KMS.1) \( \omega_\eta \) satisfies the KMS condition with respect to \( \{\sigma^\Omega_t\}_{t \in \mathbb{R}} \). (Here, \( \mathcal{N}_{\omega_\eta} = \mathcal{L}(G) \), since \( \omega_\eta \in \mathcal{L}(G)^{\Omega_1} \).

(KMS.2) \( \sigma^\omega_\eta = \sigma^\Omega_t|\mathcal{L}(G)P_0 \) for all \( t \in \mathbb{R} \).

\[2\text{Really, these cocycle derivatives admit holomorphic extensions on } \mathbb{C}, \text{ since } H \in \mathcal{L}(G)^{\Omega_1} \text{ is an entire analytical element. See [23] Sect.2.15}.\]
(KMS.3) $H := L_{\eta_0} L_{\eta_0}^*$ belongs to the centre $\mathcal{Z}(G)$.

**Proof:** Again Pedersen-Takesaki work [19] implies the following result (see also [23, Cor.4.11]): For a pair $\omega, \Omega$ of n.s.f. weights on a von Neumann algebra $\mathcal{M}$ with centre $\mathcal{Z}$, such that $\omega$ is finite, the following conditions are equivalent:

(i) $\omega$ satisfies the KMS condition with respect to $\{\sigma_t^{\Omega}\}_{t \in \mathbb{R}}$.

(ii) $s(\omega) \in \mathcal{Z}$ and $\sigma_t^\omega = \sigma_t^{\Omega} |_{\mathcal{M} s(\omega)}$ for all $t \in \mathbb{R}$.

(iii) $\omega(\cdot) = \Omega(H^{1/2} \cdot H^{1/2})$ for some nonsingular positive selfadjoint operator $H \in \mathcal{Z}$.

The result is a straightforward consequence of Theorem 4 and these equivalences. 

\[ \Box \]

5 Unimodular case

Assume now that $G$ is a lc unimodular group, i.e., the modular function is $\delta_G = 1$. Then:

(U1) The modular operator is $\Delta = I$.

(U2) $J f = f^\sharp = f^\flat$, for all $f \in L^2(G)$. The sets of left bounded and right bounded elements of $L^2(G)$ coincide, since $L_f = J R_f J$ and $R_g = J L_g J$ on them. In the sequel, left and right bounded elements shall be called simply bounded.

(U3) The canonical weight $\Omega_l$ is a trace on $\mathcal{L}(G)$ and the corresponding modular automorphism group $\{\sigma_t^{\Omega_l}\}_{t \in \mathbb{R}}$ is trivial, that is, $\sigma_t^{\Omega_l}(A) = A$, for $A \in \mathcal{L}(G)$ and $t \in \mathbb{R}$.

(U4) The von Neumann algebra $\mathcal{L}(G)$ is semifinite.

(U5) The centralizer $\mathcal{L}(G)^{\Omega_l}$ coincides with $\mathcal{L}(G)$.

(U6) The $\alpha$-homogeneous operators for any $\alpha \in \mathbb{R}$ are simply the operators affiliated with $\mathcal{L}(G)$. The spaces $L^p(\Omega_r)$ reduce to the noncommutative $L^p(\mathcal{L}(G), \Omega_l)$ spaces associated with a trace on a von Neumann algebra. This theory was laid out in the early 50’s by Segal [22] and Dixmier [8]. See also [18, 27, 21] and [26, Sect.IX.2].

(U7) For $p \in [1, 2]$, the $L^p$-Fourier transform of $f \in L^p(G)$ is the left-convolution by $f$ operator on $L^2(G)$:

$F_p(f)g := f \ast g, \quad g \in D(F_p(f))$.

(U8) Fixed the n.s.f. trace $\Omega_l$ on $\mathcal{L}(G)$, every s.n. weight on $\mathcal{L}(G)$ is written uniquely in the form $\Omega_l^H$, with $H$ a positive selfadjoint operator affiliated with $\mathcal{L}(G)$, where $\Omega_l^H$ is given by (32). See [26, Th.VIII.3.14].

(U9) For all positive selfadjoint operators $H$ affiliated with $\mathcal{L}(G)$,

\[ \int H \, d\Omega_l = \Omega_l^H(I), \quad \frac{d\Omega_l^H}{d\Omega_l} = H, \]

\[ (D\Omega_l^H : D\Omega_l)_t = \left(\frac{d\Omega_l^H}{d\Omega_l}\right)^{it} = H^{it}, \quad t \in \mathbb{R}. \]

See [28, Rem.3.2] and [7, Th.9.(2)].

Due to these facts, Theorems 1 and 2 together with Theorem 4 lead to the following result for unimodular lc groups:
Theorem 8 Let $G$ be an unimodular lc group, $(\pi, \mathcal{H}_\pi)$ a unitary representation of $G$, $\eta \in \mathcal{H}_\pi$ and

$$\varphi_\eta(s) := (\pi(s)\eta|\eta)_\pi, \quad s \in G.$$ 

The following statements are equivalent:

1. \{$(\pi(s)\eta)_s \in G$ is a covariant frame for $(\pi, \mathcal{H}_\pi)$ with bounds $\alpha, \beta$. $(u.i)$
2. $\varphi_\eta \in A(G)^+$ and
   $$\alpha(P_0 f|f) \leq \langle f^2, \varphi_\eta \rangle \leq \beta(P_0 f|f), \quad f \in C_c(G),$$ \quad (33)
   with $P_0$ a projection in $Z(G)$.
3. The normal finite functional $\omega_\eta \in \mathcal{L}(G)^+_*$, corresponding to $\varphi_\eta \in A(G)^+$, has support $s(\omega_\eta) = P_0$ and satisfies
   $$\alpha \Omega_t(A) \leq \omega_\eta(A) \leq \beta \Omega_t(A), \quad A \in (\mathcal{L}(G)P_0)^+. \quad (34)$$
4. $(D\omega_\eta : D\Omega_t)_t = \mathcal{F}_2(\eta_0) = L_{\eta_0} \in \mathcal{L}(G)^+, \quad d\omega_\eta/d\Omega_t = L_{\eta_0}^2$, and
   $$\alpha P_0 \leq L_{\eta_0}^2 \leq \beta P_0. \quad (35)$$
5. The cocycle derivatives are of the form
   $$(D\omega_\eta : D\Omega_t)_t = L_{\eta_0}^{2it}, \quad (D\Omega_t : D\omega_\eta)_t = L_{\eta_0}^{-2it}, \quad t \in \mathbb{R}, \quad (36)$$
   where $L_{\eta_0}$ satisfies \textbullet, and one has $\omega_\eta = \Omega_t^{L_{\eta_0}}$ on $\mathcal{L}(G)$ and $\Omega_t(A) = \omega_\eta^{L_{\eta_0}}$ on $\mathcal{L}(G)P_0$, that is,
   $$\omega_\eta(A) = \Omega_t(L_{\eta_0}AL_{\eta_0}), \quad A \in \mathcal{L}(G), \quad (37)$$
   $$\Omega_t(A) = \omega_\eta(L_{\eta_0}^{-1}AL_{\eta_0}^{-1}), \quad A \in \mathcal{L}(G)P_0. \quad (38)$$

Proof: Theorem \textbullet is of applicability thanks to (U5). $L_{\eta_0}$ is positive in this case, since $\eta_0 = \eta_0^* = J\eta_0$ and $L_{\eta_0}^{*} = L_{\eta_0}^{-1} = L_{\eta_0}$.

Corollary 9 Assume that the equivalent statements (u.i)-(u.v) of Theorem 8 are satisfied. Then the following additional conditions are equivalent:

(KMSu.1) For every pair $A, B \in \mathcal{L}(G)$ there exists a function $f = f_{A,B}$ defined, continuous and bounded on the closed horizontal strip $\overline{D}_{-1} := \{ z \in \mathbb{C} : 0 \leq \text{Im}(z) \leq 1 \}$, holomorphic on the open strip $D_{-1}$ and such that

$$f(t) = \omega_\eta(AB), \quad f(t+i) = \omega_\eta(BA), \quad t \in \mathbb{R}.$$

(KMSu.2) $\omega_\eta$ is a trace.

(KMSu.3) $L_{\eta_0} \in Z(G)$.

Proof: Note that (KMSu.1) is just the KMS condition for $\omega_\eta$ with respect to the trivial automorphism group $\{ \sigma^D_t \}_{t \in \mathbb{R}}$. With respect to (KMSu.2), $\omega_\eta$ is a trace iff the corresponding automorphism group $\{ \sigma_t^\pi \}_{t \in \mathbb{R}}$ is trivial.
6 Commutative case

Let $G$ be a commutative lc group. Then:

(C1) $\lambda = \rho$, $L(G) = R(G)$ and, obviously, $G$ is unimodular.

(C2) The irreducible unitary representations of $G$ are all one-dimensional $[10]$ Cor.3.6]. They are continuous homomorphisms $\gamma$ from $G$ into the multiplicative group $\mathbb{T}$ of complex numbers with modulus 1, the (unitary) characters of $G$. With the topology of compact convergence on $G$, the set $\hat{G}$ of all characters of $G$ is a commutative lc group called the dual group of $G$.

(C3) $\hat{G}$ can be identified with the spectrum of $L^1(G)$ $[10]$ Th.4.2]. The Gelfand transform on $L^1(G)$ then becomes the map from $L^1(G)$ to $C(\hat{G})$ defined by

$$\mathcal{F} f(\gamma) := \hat{f}(\gamma) := \int_{\hat{G}} \overline{\gamma(s)} f(s) \, ds.$$  

The map $\mathcal{F}$ is the Fourier transform on $G$ and $\mathcal{F} (f \ast g) = \hat{f} \cdot \hat{g}$, for $f, g \in L^1(G)$.

(C4) Let $M(\hat{G})$ denote the set of bounded Radon complex-valued measures on $\hat{G}$. If $\mu \in M(\hat{G})$ we define the bounded continuous function $\varphi_\mu$ on $G$ by

$$\varphi_\mu(s) := \int_{\hat{G}} \gamma(s) \, d\mu(\gamma).$$  

Bochner theorem $[10]$ Th.4.18] asserts that if $\varphi \in \mathcal{P}(G)$, there is a unique positive $\mu \in M(\hat{G})$ such that $\varphi = \varphi_\mu$ as in (39).

(C5) The Fourier-Stieltjes algebra $B(G)$ is given by $B(G) := \{ \varphi_\mu : \mu \in M(\hat{G}) \}$. The correspondence $\mu \mapsto \varphi_\mu$ is a bijection from $M(\hat{G})$ to $B(G)$. We shall denote its inverse by $\varphi \mapsto \mu_\varphi$. By Bochner’s theorem, $B(G)$ is the linear span of $\mathcal{P}(G)$. The Fourier algebra is $A(G) = B(G) \cap L^1(G)$.

(C6) One of the Fourier inversion theorems in this context reads as follows $[10]$ Th.4.21]: If $f \in A(G)$ then $\hat{f} \in L^1(\hat{G})$, and if Haar measure $d\gamma$ on $\hat{G}$ is suitably normalized relative to the given Haar measure $ds$ on $G$, we have $d\mu_\gamma (\gamma) = \hat{f}(\gamma) \, d\gamma$; that is,

$$f(s) = \int_{\hat{G}} \gamma(s) \hat{f}(\gamma) \, d\gamma.$$  

From now on, it will always be tacitly assumed that the Haar measures of $G$ and $\hat{G}$ are so adjusted that the inversion theorem holds.

(C7) In this case, Plancherel theorem $[10]$ Th.4.25] says that the Fourier transform $\mathcal{F}$ on $L^1(G) \cap L^2(G)$ extends uniquely to a unitary isomorphism from $L^2(G)$ onto $L^2(\hat{G})$.

Let us come back to the context of Theorem 8. Since $\mathcal{H}_0$ is invariant under convolution and $\mathcal{F}$ transform convolution into usual product, there exists a measurable closed subset $S_0$ of $\hat{G}$ such that

$$\mathcal{H}_0 := \mathcal{F} \mathcal{H}_0 = \{ \hat{g} \in L^2(\hat{G}) : \text{supp}(\hat{g}) \subseteq S_0 \}.$$  

Let $\chi_{S_0}$ denote the characteristic function of $S_0$.

The following result shows that, when $G$ is commutative, covariant frames are characterized by means of the classical Radon-Nikodym theorem on the dual group $\hat{G}$.

**Theorem 10** If $G$ is a commutative lc group, the equivalent statements (a.i)-(a.v) of Theorem 8 are equivalent to the following ones:

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(c.vii) \( \hat{\eta}_0 \geq 0; \eta_0 * \eta_0 \in A(G)^+ \) or, equivalently, \( \hat{\eta}_0^2 \in L^1(\hat{G})^+ \); and

\[ \alpha \chi_{S_0} \leq \hat{\eta}_0^2 \leq \beta \chi_{S_0}. \]  

(c.viii) One has

\[ \alpha \int_G \hat{f} \, d\gamma \leq \int_G \hat{f} \, d\mu_{\eta_0 * \eta_0} \leq \beta \int_G \hat{f} \, d\gamma, \quad \hat{f} \in \mathcal{H}_0^+. \]  

(c.ix) The positive measures

\[ d\mu_{\eta_0 * \eta_0} = \hat{\eta}_0^2 \, d\gamma, \quad d\gamma|_{S_0} = \hat{\eta}_0^{-2} \, d\mu_{\eta_0 * \eta_0} \]  

are absolutely continuous one with respect to the other on \( S_0 \) and, moreover, the Radon-Nikodym derivative \( \hat{\eta}_0^2 \) satisfies (40).

**Proof:** \( \hat{\eta}_0 \geq 0 \) is equivalent to \( L_{\eta_0} \in \mathcal{L}(G)^+ \) in (u.v). \( \varphi_\eta \in A(G)^+ \) in (u.iv) means that \( \eta_0 * \eta_0 \in A(G)^+ \). The equivalence \( \eta_0 * \eta_0 \in A(G)^+ \Leftrightarrow \hat{\eta}_0^2 \in L^1(\hat{G})^+ \) follows from the Fourier inversion theorem in (C6). Equation (40) is just (55) moved to the dual group by means of the Fourier transform given in (C3). The equivalence of Equations (41) and (34) is again a consequence of the Fourier inversion theorem in (C6). Finally, (57) and (58) move to (12). \( \square \)

7 Some remarks

We finish by including some brief remarks about connections of this work with the *tracial condition* in [11], *dual integrable representations and brackets* in [2] and the *formal degree* in [9].

**Remark 11** For unimodular lc groups \( G \) and subrepresentations \((\lambda_{|H_0}, H_0)\) of the left regular representation of \( G \), Führ [11 Th.2.2] gives a characterization of admissible vectors \( \eta \) as tracial ones on the reduced right von Neumann algebra \( P_0 \mathcal{R}(G) P_0 \), i.e., such that

\[ \Omega_r(B) = (B\eta|\eta), \quad B \in P_0 \mathcal{R}(G) P_0. \]  

In this work, for arbitrary lc groups \( G \), according to the definition of \( \Omega_t \), (12) and Theorem 2(e) and (k), for left bounded \( f \),

\[(P_0 f|f) = \Omega_t(L^{-1}_f L_f P_0) = \omega_\eta((L^{-1}_{\eta_0})^* L_f L_f P_0 L^{-1}_{\eta_0}) \]  

\[ = (L^{-1}_{\eta_0})^* L_f L_f P_0 L^{-1}_{\eta_0} \eta_0|\eta_0) = (P_0 L^{-1}_f L_f L^{-1}_{\eta_0} \eta_0 L^{-1}_{\eta_0} \eta_0).\]

Since the set of left bounded vectors is dense in \( L^2(G) \), this means that the weight \( \Omega_t \) on the reduced left von Neumann algebra \( \mathcal{L}(G) P_0 \) is recovered by means of the admissible vector \( L^{-1}_{\eta_0} \eta_0 = R^{-1}_{\eta_0} \eta_0 \). In other words, \( R^{-1}_{\eta_0} \eta_0 = P_0 \), as required in [13 Th.5]. These results generalize the tracial condition (43); see Remark 4.

**Remark 12** Barbieri, Hernández and Parcet [2] analyze the properties of principal shift invariant spaces in a Hilbert space given by the action of an arbitrary countable discrete group. To be precise, let \( G \) be a countable discrete group and \( \{ \delta_s \}_{s \in G} \) the standard unit vector basis of \( l^2(G) \). Consider the usual normalized trace \( \tau \) on \( \mathcal{L}(G) \):

\[ \tau(A) := (A \delta_e|\delta_e), \quad A \in \mathcal{L}(G), \]  

where \( e \) denotes the identity element of \( G \). A unitary representation \((\pi, \mathcal{H}_\pi)\) of \( G \) is called **dual integrable** whenever there exists a map \([, , ] : \mathcal{H} \times \mathcal{H} \to L^1(\mathcal{L}(G), \tau) \) such that

\[ (\phi|\pi(s)\psi)_\pi = \tau([\phi, \psi]|\lambda(s)^*), \quad \phi, \psi \in \mathcal{H}_\pi, s \in G. \]
Given any $\eta \in \mathcal{H}_{\pi}$, let us consider the principal invariant subspace generated by $\eta$:

$$\langle \eta \rangle := \text{span}\{\pi(s)\eta : s \in G\}.$$  

Let $P_T$ denote the orthogonal projection in $l^2(G)$ onto $(\ker T)^\perp$ for any densely defined operator $T$ on $l^2(G)$. One of the main results is [2 Th.A]: Given any $\eta \in \mathcal{H}_{\pi}$, the system $\{\pi(s)\eta\}_{s \in G}$ is

i) An orthonormal basis for $\langle \eta \rangle$ iff $[\eta, \eta] = I$.

ii) A Riesz basis for $\langle \eta \rangle$ with frame bounds $0 < \alpha \leq \beta < \infty$ iff

$$\alpha I \leq [\eta, \eta] \leq \beta I.$$  

iii) A frame for $\langle \eta \rangle$ with frame bounds $0 < \alpha \leq \beta < \infty$ iff

$$\alpha P_{[\eta, \eta]} \leq [\eta, \eta] \leq \beta P_{[\eta, \eta]}.$$  

A central tool in the proof of this result is the Hilbert space $L^2(\mathcal{L}(G), [\eta, \eta])$ defined as follows:

The functional

$$||A||_{2,\eta} := \left(\tau(|A|^2[\eta, \eta])\right)^{1/2} = ||A[\eta, \eta]^{1/2}||_2, \quad A \in \mathcal{L}(G),$$

is a seminorm on $\mathcal{L}(G)$. Let $N_{\eta}$ be the null space associated with the seminorm $|| \cdot ||_{2,\eta}$. The space $L^2(\mathcal{L}(G), [\eta, \eta])$ is the closure of $\mathcal{L}(G)/N_{\eta}$ in the $|| \cdot ||_{2,\eta}$ norm. To each $0 \neq \eta \in \mathcal{H}_{\pi}$ there corresponds an isometric isomorphism $S_\eta : \langle \eta \rangle \to L^2(\mathcal{L}(G), [\eta, \eta])$ satisfying $S_\eta[\pi(s)\eta] = \lambda(s)$, for $s \in G$; see [2 Prop.3.4].

Here, clearly, we can identify $L^2(\mathcal{L}(G), [\eta, \eta])$ with the GNS representation associated with the weight $\omega_{\eta}$ (see, e.g., [23 Sect.1.2]) and the bracket $[\eta, \eta]$ itself with the spatial derivative $d\omega_{\eta}/d\Omega_{\eta}$.

In [2] the authors apply these results in the concrete framework of unitary representations given by measurable actions of $G$ on $\sigma$-finite measure spaces $(X, \mu)$. In this scenario, a noncommutative Zak transform, defined as a measurable field of operators over $X$, and a tiling property play the central roles. The generalization of these ideas to arbitrary groups deserves further study.

**Remark 13** Given a representation $(\pi, \mathcal{H}_{\pi})$ of a lc group $G$, the coefficients of $\pi$ are the functions $c_{\phi, \psi}, \phi, \psi \in \mathcal{H}_{\pi}$, where

$$c_{\phi, \psi}(s) := (\phi|\pi(s)\psi)_{\pi}, \quad s \in G.$$  

Duflo and Moore [9 Th.2] prove that an irreducible representation $\pi$ of $G$ is equivalent to a subrepresentation of the left regular representation $\lambda$ iff it has a nonzero square integrable coefficient. Representations satisfying these conditions are usually called square integrable. To determine which coefficients are square integrable and certain relations of orthogonality, they extend to nonunimodular groups the notion of formal degree, which is a positive number in the unimodular case.

For it, recall that a character $\gamma$ of $G$ is a continuous homomorphism of $G$ in $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$. Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be representations of $G$. A densely defined closed operator $T$ from $\mathcal{H}$ to $\mathcal{H}'$ is called semi-invariant with weight $\gamma$ if

$$\pi'(s)T = \gamma(s)T\pi(s), \quad s \in G.$$  

Given an arbitrary unit vector $\eta$ of $\mathcal{H}_{\pi}$, let us consider

$$T_{\eta} : D(T_{\eta}) \subseteq \mathcal{H}_{\pi} \to L^2(G) : \psi \mapsto \overline{c_{\eta\psi}} = (\pi(s)\psi|\eta),$$

where the domain $D(T_{\eta})$ is the set of $\psi \in \mathcal{H}_{\pi}$ such that $c_{\eta\psi} \in L^2(G)$. $T_{\eta}$ is a closed operator semi-invariant relative to $\pi$ and $\rho$ with weight $\delta_{\mu}^{1/2}$, that is,

$$\rho(s)T_{\phi} = \delta_{\mu}^{1/2}(s)T_{\phi}\pi(s), \quad s \in G.$$  

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The formal degree of the representation \((\pi, \mathcal{H}_\pi)\) is the operator \(K\) defined by

\[
K := (T_\eta^* T_\eta)^{-1}
\]

and the result reads as follows \([9, \text{Th.3}]\): If \((\pi, \mathcal{H}_\pi)\) is a square integrable representation of \(G\), then there exists a unique operator \(K\) in \(\mathcal{H}_\pi\), selfadjoint positive, semi-invariant with weight \(\delta_G^{-1}\), and satisfying the following conditions:

(a) If \(\phi, \psi \in \mathcal{H}_\pi\), \(f \neq 0\), then \(c_{\phi\psi}\) is square integrable iff \(\psi \in \text{dom}\ K^{-1/2}\).

(b) If \(\phi, \phi' \in \mathcal{H}_\pi\) and \(\psi, \psi' \in \text{dom}\ K^{-1/2}\),

\[
(c_{\phi\psi}|c_{\phi'\psi'}) = (\phi|\phi')_{\pi} (K^{-1/2}\psi|K^{-1/2}\psi')_{\pi}.
\]

The uniqueness of \(K\) in the case of square integrable representations is due to an extension of Schur lemma \([9, \text{Sect.2}]\): If \(\eta' \in \mathcal{H}_\pi\), \(\eta' \neq 0\), then \(T_\eta = c_{\eta'} U_{\eta'} K^{-1/2}\), where \(c_{\eta'} \in (0, \infty)\) and \(U_{\eta'}\) is an isometry that intertwines \(\pi\) and \(\rho\).

Here, for an arbitrary representation \((\pi, \mathcal{H}_\pi)\) of \(G\) and \(\eta \in \mathcal{H}_\pi\), we consider the operator \(T_{\eta}T_{\eta}^*\) on \(L^2(G)\), which is selfadjoint positive and semi-invariant with respect to \(\rho\) with weight \(\delta_G\). That is, using the terminology of noncommutative \(L^p\)-spaces, \(T_{\eta}T_{\eta}^*\) is \((-1)\)-homogeneous and, thus, there exists a unique n.s. weight \(\omega_\eta\) on \(L(G)\) such that

\[
T_{\eta}T_{\eta}^* = \frac{d\omega_\eta}{d\Omega_G}.
\]

Theorem \([1]\) says that \(\{\pi(s)\eta\}_{s \in G}\) is a covariant frame for \((\pi, \mathcal{H}_\pi)\) with bounds \(\alpha, \beta\) iff \(\omega_\eta\) satisfies any of the equivalent statements (iii)-(v) of the theorem. In this context, orthogonality relations have a clear description dealing with standard forms; see \([13, \text{Sect.4}]\) for details.

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