Radon transform in finite Hilbert space

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Abstract – A novel analysis of finite-dimensional Hilbert space is outlined. The approach bypasses the general, inherent, difficulties present in handling angular variables in finite-dimensional problems: the finite-dimensional, $d$, odd prime, Hilbert space operators are underpinned with a finite geometry which provides intuitive perspectives to the physical operators. The analysis emphasizes a central role for projectors of mutual unbiased bases (MUB) states, extending thereby their use in finite-dimensional quantum-mechanics studies. Interrelations among the Hilbert space operators revealed via their (finite) dual affine plane geometry (DAPG) underpinning are displayed and utilized in formulating the finite-dimensional ubiquitous Radon transformation and its inverse illustrating phase-space-like physics encoded in lines and points of the geometry. The finite geometry required for our study is outlined.

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Introduction. – The mathematics of Radon transform was introduced for studies of astronomical data [1–3]. Today it finds an extensive application in CAT (computer-assisted tomography) scan [3–5], state reconstruction [6–9] as well as within the fundamental phase space formulation of quantum mechanics [7,10]. The methodology is geometrically based: A phase space image of an arbitrary Hilbert space operator $A$ is given by [11,12],

$$A \rightarrow W_A(q,p) = \int dq dp e^{i(pq-y)} A(q+y).$$

(1)

The Wigner function [13], $W_\rho(q,p)$, being the image of the density operator, $\rho$, although not non-negative, does enjoy many of the attributes of a phase space distribution. In particular its marginal,

$$\tilde{\rho}(x',\theta) = \int dq dp \delta(Cq+Sp-x')W_\rho(q,p),$$

(2)

$$C = \cos \theta, \quad S = \sin \theta,$$

is its Radon transform [7,14]:

$$R[W_\rho](x,\theta) = \tilde{\rho}(x,\theta); \quad R^{-1}[\tilde{\rho}](q,p) = W_\rho(q,p).$$

(3)

The explicit expression for $W_\rho(q,p)$ in terms of principal value integral of $\tilde{\rho}(x,\theta)$ is given in [3,6–9]. We offer in the last section an interpretation for this singular relation gained via the finite-dimensional approach.

The tomographic luster of the Radon transform suggests viewing it via mutual unbiased bases (MUB). Informationally complete set of MUB, i.e. such that allows complete accounting of an arbitrary state, are $\{|x',\theta\}; -\infty \leq x' \leq \infty; 0 \leq \theta \leq \pi$. These are the eigenstates of the dynamical variables $\hat{X}_\theta = \hat{X} + \hat{S}, [3,15]$. Now

$$\text{tr} \hat{\rho}P(x',\theta) = \int dq dp \delta(Cq+Sp-x')W_\rho(q,p) = \tilde{\rho}(x',\theta),$$

(4)

illustrating a quasi-distributional attribute of $W_\rho(q,p)$, as the phase space mappings of $P(x',\theta) \equiv |x',\theta\rangle\langle x',\theta|$ is $\delta(x' - Cq - Sp)$. Noteworthy is the pivotal role of the delta-function: it identifies the Radon transformation, defines the marginal distribution and constitutes the phase space mapping of an MUB projector.

The present intuitive approach involves the replacement of phase space with finite geometry’s points and lines. The role of the Dirac delta-function is played by what we signify with $\Lambda_{\alpha,j}$, where $\alpha$ and $j$ refer to finite geometry point and line, respectively. In this way we eschew in the finite-dimensional intrinsically sticky issue of angular variables. On the physical side the salient feature is the mapping of finite-dimensional Hilbert space operators onto $c$ number functions of the phase-space-like geometrical points and lines and, in particular, the definition of quasi-distribution and Radon transformation over this phase-space-like points and lines. These ideas are clarified below.

Finite geometry and Hilbert space operators. – We now briefly review the essential features of the finite geometry required for our study [16–23]. A finite plane
affine and projective. We shall confine ourselves to affine plane geometry (APG) which is defined as follows. An APG is a non-empty set whose elements are called points. These are grouped in subsets called lines subject to:

1) Given any two distinct points there is exactly one line containing both.
2) Given a line $L$ and a point $S$ not in $L$ ($S \not\equiv L$), there exists exactly one line $L'$ containing $S$ such that $L \cap L' = \varnothing$. This is the parallel postulate.
3) There are 3 points that are not collinear.

It can be shown [16,18,19] that for $d = p^m$ (a power of prime) APG can be constructed (our study here is for $d = p$) and the following properties are, necessarily, built in:

a) The number of points is $d^2$; $S_{\alpha}$, $\alpha = 1, 2, \ldots, d^2$ and the number of lines is $d(d+1)$; $L_{j}$, $j = 1, 2, \ldots, d(d+1)$.

b) A pair of lines may have at most one point in common.

c) Each line is made of $d$ points and each point is common to $d+1$ lines: $L_j = \bigcup_{\alpha} S_\alpha$, $S_\alpha = \bigcap_{j=1}^{d+1} L_j$.

d) If a line $L_j$ is parallel to the distinct lines $L_k$ and $L_l$, then $L_k \parallel L_l$. The $d^2$ points are grouped in sets of $d$ parallel lines. There are $d+1$ such groupings.

e) Each line in a set of parallel lines intersect each line of any other set.

The existence of APG implies [16–19] the existence of its dual geometry DAPG wherein the points and lines are interchanged. Since we shall study extensively this, DAPG, we list the corresponding properties for it. We shall refer to these by DAPG($y$):

a) The number of lines is $d^2$; $L_{\alpha}$, $\alpha = 1, 2, \ldots, d^2$. The number of points is $d(d+1)$, $S_\alpha$, $\alpha = 1, 2, \ldots, d(d+1)$.

b) A pair of points on a line determine a line uniquely. Two (distinct) lines share one and only one point.

c) Each point is common to $d$ lines. Each line contain $d+1$ points.

d) The $d(d+1)$ points may be grouped in sets, $R_\alpha$, of $d$ points each, no two points of a set shares a line. Such a set is designated by $\alpha' \in \{\alpha \cup M_\alpha\}$, $\alpha' = 1, 2, \ldots, d$ ($M_\alpha$ contain all the points not connected to $\alpha$—they are not connected among themselves) i.e. such a set contains $d$ disjoint (among themselves) points. There are $d+1$ such sets:

$$\bigcup_{\alpha=1}^{d(d+1)} R_\alpha = \bigcup_{\alpha=1}^{d} R_\alpha; \quad R_\alpha = \bigcup_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'}; \quad R_\alpha \cap R_{\alpha'} = \varnothing, \alpha \neq \alpha'. \quad (5)$$

DAPG($c$) allows the identification, which we adopt,

$$S_\alpha = \frac{1}{d} \sum_{j=1}^{d} L_j \Rightarrow \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d} \sum_{j=1}^{d} L_j. \quad (6)$$

We list now some direct DAPG implied interrelation subject to eq. (6): DAPG($d$) implies

$$\sum_{\alpha=1}^{d+1} \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{d+1}{d} \sum_{j=1}^{d} L_j \Rightarrow \sum_{\alpha=1}^{d} \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha'} = \frac{1}{d+1} \sum_{\alpha=1}^{d} \sum_{\alpha' \in \alpha \cup M_\alpha} S_{\alpha}. \quad (7)$$

Finite-dimensional mutual unbiased bases, MUB, brief review. – In a finite, $d$-dimensional, Hilbert space two complete, orthonormal vectorial bases, $B_1, B_2$, are said to be MUB if and only if ($B_1 \neq B_2$) [21,24–35]:

$$\forall |u\rangle, |v\rangle \in B_1, B_2, \text{ respectively, } \langle u|v\rangle = 1/\sqrt{d}. \quad (8)$$

The physical meaning of this is that the knowledge of a system is in a particular state in one basis implies complete ignorance of its state in the other basis.

Ivanovic [36] proved that there are at most $d+1$ MUB, pairwise, in a $d$-dimensional Hilbert space and gave an explicit formulae for the $d+1$ bases in the case of $d = p$ (prime number). Wootters and Fields [25] constructed such $d+1$ bases for $d = p^m$ with $m$ an integer. A variety of methods for construction of the $d+1$ bases for $d = p^m$ is now available [27–29]. Our present study is confined to $d = p \neq 2$.

We now give explicitly the MUB states in conjunction with the algebraically complete operators [32,37] set:

$$\hat{Z}, \hat{X}. \text{ Thus we label the } d \text{ distinct states spanning the Hilbert space, termed the computational basis, by } |n\rangle, n = 0, 1, \ldots, d-1; |n + d\rangle = |n\rangle$$

$$\hat{Z}|n\rangle = \omega^n|n\rangle; \quad \hat{X}|n\rangle = |n + 1\rangle, \quad \omega = e^{i2\pi/d}. \quad (9)$$

The $d$ states in each of the $d+1$ MUB bases [27,32] are the states of the computational basis (CB) and

$$|m; b\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \omega^{k (n-1) - km}|n\rangle; \quad b, m = 0, 1, \ldots, d-1. \quad (10)$$

Here $b$ labels the other $d$ bases and $m$ labels the states within a basis. We have [27]

$$\hat{X} \hat{Z}^b|m; b\rangle = \omega^m|m; b\rangle. \quad (11)$$

For later reference we shall refer to the computational basis (CB) by $b = -1$. Thus the above gives $d+1$ bases,
$b = -1, 0, 1, \ldots, d - 1$. The total number of states $d(d + 1)$
are grouped in $d + 1$ sets each of $d$ states. We have of
course,

$$
\langle m; b m'; b \rangle = \delta_{m,m'}; \quad \langle m; b m'; b' \rangle = \frac{1}{\sqrt{d}} \quad b \neq b'.
$$

(12)

we remark at this junction that the eigenvalues of the CB
might be considered finite-dimensional modulated position
values (“qu”) and the eigenvalues of shifting operator,
$X$, can be considered modulated momentum (“pu”). This
completes our discussion of MUB.

**DAPG underpinning of $d$-dimensional Hilbert space.** – Now the underpinning of Hilbert space
operators with DAPG will be undertaken. We consider $d = p$, a
prime, $\neq 2$. For $d = p$ we may construct $d + 1$ MUB
[21,24,27,36]. Points will be associated with MUB state
projectors. To this end we recall that we designate the MUB
states by $| m, b \rangle$, with $b = 0, 1, 2, \ldots d - 1$ labels the
eigenfunction of, respectively, $XZ^b$. $m$ labels the state
within a basis. We designate the computational basis, CB,
by $b = -1$. (Note that the first column label of $-1$ is for
convenience and does not designate a negative value of a
number.) The projection operator defined by

$\hat{A}_m | m, b \rangle | b, m \rangle; \quad \alpha = \{ m, b \}; \quad b = -1, 0, 1, 2, \ldots, d - 1; \quad m = 0, 1, 2, \ldots, d - 1.$

(13)

The point label, $\alpha = \{ m, b \}$ is now associated with the
projection operator, $\hat{A}_m$. We now consider a realization,
possible for $d = p$, a prime, of a $d$-dimensional DAPG, as
points marked on a rectangular whose horizontal width
($x$-axis) is made of $d + 1$ columns of points. Each column
is labelled by $b$, and its vertical height ($y$-axis) is made of
$d$ points each marked with $m$. The total number of points
is $d(d + 1)$ — there are $d$ points in each of the $d + 1$
columns. We associate the $d$ points $m = 0, 1, 2, \ldots, d - 1$ in
each set, labelled by $b$, ($\alpha \sim (m, b)$) to the disjoint
points of DAPG($d$), $R_n$ viz. for fixed $b \alpha \in \alpha \cup M_{\alpha}$. Form a
column. The columns are arranged according to their basis
label, $b$. The first, i.e. the leftmost, being $b = -1$, $\alpha_{-1} =
(m, -1); m = 0, 1, \ldots, d - 1$. Next we place the columns
with increasing values of $b$, the basis label, as we move to
the right. Thus the rightmost column is for $b = d - 1$. All
the columns are now made of $d + 1$ points, each of different
$b$ (necessarily, DAPG($d$)). A point $S_{\alpha}$ underpins a Hilbert
space state projector, $\hat{A}_m$, i.e. $\hat{A}_m^2 = \hat{A}_m$, and $tr \hat{A}_m = 1$. We
designate the line operator underpinned with $b_{j \prime}$ by
$\hat{P}_{j \prime}$. Thus the above relations now hold with $S_{\alpha} \leftrightarrow \hat{A}_m$;
$L_j \leftrightarrow \hat{P}_{j \prime}$.

**E.g.** for $d = 3$ the underpinning’s schematics is

$$
\begin{pmatrix}
 m & b & -1 & 0 & 1 & 2 \\
 0 & \hat{A}_{0(-1)} & \hat{A}_{0(0)} & \hat{A}_{0(1)} & \hat{A}_{0(2)} \\
 1 & \hat{A}_{1(-1)} & \hat{A}_{1(0)} & \hat{A}_{1(1)} & \hat{A}_{1(2)} \\
 2 & \hat{A}_{2(-1)} & \hat{A}_{2(0)} & \hat{A}_{2(1)} & \hat{A}_{2(2)} \\
\end{pmatrix}
$$

Now DAPG($c$) (and eq. (13), (7)) implies that $A_{\alpha} =
0, 1, 2, \ldots, d - 1; \alpha \in \alpha' \cup M_{\alpha'}$, forms an orthonormal basis
for the $d$-dimensional Hilbert space:

$$
\sum_{m} \langle m; b | b, m \rangle = \sum_{\alpha' \in \alpha \cup M_{\alpha}} \hat{A}_{\alpha'} = I; \quad \sum_{\alpha} \hat{A}_{\alpha} = (d + 1)I.
$$

(14)

Equation (6) implies

$$
A_{\alpha} = \frac{1}{d} \sum_{j \in \alpha} P_j.
$$

(15)

Evaluating

$$
\sum_{\alpha \in \alpha} A_{\alpha} = \frac{1}{d} \sum_{\alpha \in \alpha} \sum_{j' \in \alpha} P_{j'} = \frac{1}{d} \left[ \sum_{j' \neq j} (d - 1)(d + 1) P_{j'} + (d + 1) P_{j} \right] =
$$

$$
I + P_{j} \Rightarrow P_{j} = \sum_{\alpha \in \alpha} A_{\alpha} - I.
$$

(16)

Equation (12) implies

$$
tr \hat{A}_{\alpha} \hat{A}_{\alpha'} = \begin{cases}
1; & \alpha = \alpha'; \\
0; & \alpha \neq \alpha'; \alpha \in \alpha' \cup M_{\alpha'}, \\
\frac{1}{d}; & \alpha \neq \alpha'; \alpha \notin \alpha' \cup M_{\alpha'}.
\end{cases}
$$

(17)

Hence, using eqs. (6), (12),

$$
tr \hat{A}_{\alpha} P_{j} = \begin{cases}
\sum_{\alpha \notin \alpha'} tr \hat{A}_{\alpha} \hat{A}_{\alpha'} = 1; & \alpha \in \alpha; \\
\sum_{\alpha \notin \alpha'} tr \hat{A}_{\alpha} \hat{A}_{\alpha'} - A_{\alpha} = 0; & \alpha \notin \alpha.
\end{cases}
$$

(18)

Trivially

$$
tr P_{j} = \sum_{\alpha \in \alpha} tr A_{\alpha} - 1 = 1,
$$

(19)

$$
tr P_{j} = \sum_{\alpha \in \alpha} tr A_{\alpha} - 1 = \Rightarrow tr P_{j} P_{j'} = \sum_{\alpha \in \alpha} tr P_{j} A_{\alpha} - 1 =
$$

$$
\begin{cases}
d + 1 & j = j' \\
0 & j \neq j'
\end{cases}
$$

(20)

i.e.

$$
tr P_{j} P_{j'} = d\delta_{j, j'}.
$$

(21)

An alternative view of the Lambda function is gained via

$$
tr \hat{A}_{\alpha} P_{j} = \frac{1}{d} \sum_{\alpha} P_{j} =
\begin{cases}
\frac{d}{d} (tr P_{j}^2 + tr P_{j} \sum_{\alpha \notin \alpha} P_{j} P_{j}) = 1; & j \in \alpha; \\
\sum_{\alpha \notin \alpha} P_{j} P_{j} = 0; & j \notin \alpha.
\end{cases}
$$

(22)

Note that the case of $j \notin \alpha$ implies $j \in M_{\alpha}$. These are
summarized by

$$
\hat{A}_{\alpha, j} \equiv tr \hat{A}_{\alpha} P_{j} = \begin{cases}
1; & \alpha \in \alpha; \\
0; & \alpha \notin \alpha.
\end{cases}
$$

(23)
Geometric underpinning of MUB quantum operators: The line operator. – We now consider a particular realization of DAPG of dimensionality \( d = p, \neq 2 \) which is the basis of our present study. Thus \( \alpha = m(b) \) designates a point by its row, \( m \), and its column, \( b \); when \( b \) is allowed to vary it designates the point's row position in every column. We now assert that the \( d+1 \) points, \( m_j(b), b = 0, 1, 2, \ldots, d-1 \), and \( m_j(-1) \), form the line \( j \) which contains the two (specific) points \( m(-1) \) and \( m(0) \).

The line is given by (we forgo the subscript \( j \); it is implicit),

\[
m(b) = \frac{b}{2}(c-1) + m(0), \mod[d] \ b \neq -1, \\
m(-1) = c/2.
\] (24)

The rationale for this particular form will be clarified below. Thus a line \( j \) is parameterized fully by \( j = (m(-1), m(0)) \). (Note: since our line labelling is based on \( b \) values \(-1 \) and \( 0 \) a more economic label for \( m \) is \( \alpha = m(c) \) without \( b \).) We shall use either when no confusion should arise.) We now prove that the \( d+1 \) points \( m(-1), m(0) \), can have \( d \) values the number of lines is \( d^2 \); the number of points in a line is evidently \( d+1 \): a point for each \( b \).

1) Since each of the parameters, \( m(-1) \) and \( m(0) \), can have \( d \) values the number of lines is \( d^2 \); the number of points in a line is evidently \( d+1 \) a point for each \( b \).

DAPG(a).

2) The linearity of the equation precludes having two points with a common value of \( b \) on the same line, DAPG(\( d \)). Now consider two points on a given line, \( m(b_1), m(b_2) ; b_1 \neq b_2 \). We have from eq. (24), \( b \neq -1, b_1 \neq b_2 \)

\[
m(b_1) = \frac{b_1}{2}(c-1) + m(0), \mod[d] \\
m(b_2) = \frac{b_2}{2}(c-1) + m(0), \mod[d].
\] (25)

These two equations determine uniquely \( (d = p) \) \( m(-1) \) and \( m(0) \). DAPG(b).

For fixed point, \( m(b), \ c \to m(0) \) i.e. the number of free parameters is \( d \) (the number of points on a fixed column). Thus each point is common to \( d \) lines. That the line contain \( d+1 \) is obvious. DAPG(c).

3) As is argued in 2 above no line contain two points in the same column \( (i.e. \ with \ equal \ b) \). Thus the \( d \) points, \( \alpha \), in a column form a set \( R_\alpha = \bigcup_{\alpha', \alpha' \in M, \alpha \neq \alpha'} S_{\alpha'} \), with trivially \( R_\alpha \cap R_{\alpha'} = \emptyset, \alpha \neq \alpha' \), and \( \bigcup_{\alpha = 1}^{d+1} S_\alpha = \bigcup_{\alpha = 1}^{d+1} R_\alpha \).

4) Consider two arbitrary points \( \not \in \) the same set, \( R_\alpha \) defined above: \( m(b_1), m(b_2) ; b_1 \neq b_2 \). The argument of 2 above states that, for \( d = p \), there is a unique solution for the two parameters that specify the line containing these points. DAPG(e).

We illustrate the above for \( d = 3 \), where we explicitly specify the points contained in the line \( j = (m(-1) \) \( (1, -1), m(0) = (2, 0) \)

\[
\begin{pmatrix}
m \backslash b & -1 & 0 & 1 & 2 \\
0 & 1 & \cdot & \cdot & (0, 2) \\
1 & (1, -1) & \cdot & (1, 1) & \cdot \\
2 & \cdot & (2, 0) & \cdot & \cdot 
\end{pmatrix}
\]

For example the point \( m(1) \) is gotten from

\[
m(1) = \frac{1}{2}(2 - 1) + 2 = 1 \mod[3] \to m(1) = (1, 1)
\]

Similar calculation gives the other point: \( m(2) = (0, 2) \), i.e. the line \( j = (1, 2) \) contains the points \((1, -1), (2, 0), (1, 1) \) and \((0, 2) \).

The geometrical line, \( L_j, j = (1, 2) \) given above upon being transcribed to its operator formula is via eq. (16),

\[
P_j = A_{(1,-1)} + A_{(2,0)} + A_{(1,1)} + A_{(0,2)} - \hat{I}. \quad (26)
\]

Evaluating the point operators, \( \hat{A}_\alpha \),

\[
A_{(1,-1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{(2,0)} = \frac{1}{3} \begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix}, \\
A_{(1,1)} = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega \\ \omega^2 & 1 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}, \quad A_{(0,2)} = \frac{1}{3} \begin{pmatrix} 1 & 1 & \omega' \\ \omega^2 & 1 & 1 \\ \omega' & \omega^2 & 1 \end{pmatrix}
\] (27)

and evaluating the sum, eq. (26), gives

\[
P_j = A_{(m(-1)=1}, m(0)=2) = \begin{pmatrix} 0 & 0 & \omega \\ 0 & 1 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix}.
\] (28)

This operator obeys \( P_{j=1,2} = \hat{I}. \) That this is quite general, viz. \( P_j^2 = \hat{I}, \forall j \) is shown in [15]. In appendix A we show that \( \hat{P}_j^2 = \hat{I}, \forall j \) implies the operator relation,

\[
\sum_{\alpha \neq \alpha' \in \mathcal{E}} \hat{A}_\alpha \hat{A}_{\alpha'} = \sum_{\alpha \in \mathcal{E}} \hat{A}_\alpha.
\]

It is, perhaps, of interest that, if we associate the CB states with the position variable, \( q \), of the continuous problem and its Fourier transform state, viz. \( b = 0 \) (cf. eq.(12)), with the momentum, \( p \), we have that the line of the finite-dimension problem is parameterized with “initial” values of “\( q \)” and “\( p \)”, i.e. \( m(-1) \) and \( m(0) \).

Mapping onto phase space. – We now define a mapping of Hilbert space operators, e.g. an arbitrary operator, \( B \), onto the phase-space-like lines of DAPG. The mapping is defined by [15]

\[
B \Rightarrow V_B(j) \equiv \text{tr} \ B P_j. \quad (29)
\]

Here \( P_j \) is a line operator within DAPG. (Alternatively we could have cast the mappings within APG as is clear
from the discussion in the previous section.) The density operator may be expressed in terms of \( V_{\alpha}(j) \):

\[
\rho = \frac{1}{d} \sum_{j} (\text{tr} \rho P_j) P_j. \tag{30}
\]

\( V_{\alpha}(j) \) is quasi-distribution [15] in the phase-space-like lines and points of DAPG. Like its continuum analogue, \( W_{\alpha}(q,p) \), it is a real function (since both \( \hat{\rho} \) and \( P_j \) are Hermitians). It is not non-negative as can be seen by noting that for \( \hat{\rho} = \hat{A}_\alpha \) and \( \hat{\rho}^{\prime} = \hat{A}_{\alpha^{\prime}}^{\prime} \)

\[
\text{tr} \rho \hat{\rho}^{\prime} = 0 = \frac{1}{d} \sum_{j} V_{\alpha}(j) V_{\alpha^{\prime}}(j); \quad |V_{\alpha}(j)| \neq 0, |V_{\alpha^{\prime}}(j)| \neq 0. \tag{31}
\]

These quasi-distributions attribute in several texts [6–9]. Quite generally the expectation value of an arbitrary operator \( B \) is given by

\[
\text{tr} \rho \hat{B} = \frac{1}{d} \sum_{j} V_{\alpha}(j) V_{\alpha}(j). \tag{32}
\]

The quasi-distribution may be reconstructed from the expectation values of the point operator \( \hat{A}_\alpha \), i.e. MUB state-projector’s expectation value (obtained, e.g. by measurements)

\[
\text{tr} \rho \hat{A}_\alpha = \frac{1}{d} \sum_{j} V_{\alpha}(j) V_{\alpha}(j) = \frac{1}{d} \sum_{j} V_{\alpha^{\prime}}(j) \Lambda_{\alpha,j}. \tag{33}
\]

Thence

\[
\frac{1}{d} \sum_{\alpha \in \alpha^{\prime} \cup M_{\alpha}} \sum_{j} V_{\alpha}(j) \Lambda_{\alpha,j} = V_{\alpha}(j). \tag{34}
\]

These equations are the finite-dimensional Radon transform and its inverse.

To summarize we give a comparative list of the central formulae: the expressions for the continuous phase space vs. the finite plane geometry ones:

a) Phase space mapping,

\[
\rho \Rightarrow W_{\rho}(q,p), \quad \int \frac{dq dp}{2\pi} W_{\rho}(q,p) = 1,
\]

\[
\rho \Rightarrow V_{\rho}(j), \quad \frac{1}{d} \sum_{j} V_{\rho}(j) = 1.
\]

b) MUB state map (\( \alpha = (m,b) \))

\[
|x,\theta\rangle \langle \theta,x | \Rightarrow W_{|x,\theta\rangle} = \delta(x - Cq - Sp),
\]

\[
A_\alpha \Rightarrow \text{tr} A_\alpha P_j = \Lambda_{\alpha,j}.
\]

c) Radon transform (\( \alpha = (m,b) \))

\[
\mathcal{R}[W_{\rho}](x,\theta) = \int \frac{dq dp}{2\pi} W_{\rho}(q,p) \delta(x - Cq - Sp), \tag{35}
\]

Thus, mapping of finite dimensional Hilbert space operators onto phase-space-like lines and points of finite geometry was used to define a finite-dimensional phase-space physics and applied to give the Radon transform and its inversion.

**Summary and concluding remarks.** – Finite geometry stipulates interrelations among lines and points. The stipulations for the (finite) dual affine plane geometry (DAPG) was shown to conveniently accommodate association of geometric lines and points with projectors of states of mutual unbiased bases (MUB). The latter act in a (finite-dimensional, d) Hilbert space. This underpinning of Hilbert space operators with DAPG reveals some novel inter operators relations. Noteworthy among these are Hilbert space operators, \( P_j, j = 1,2,\ldots,d^2 \), which are underpinned with DAPG lines, \( L_j \); they abide by \( P_j^2 = \hat{I}, \forall j \), and are mutually orthogonal, \( \text{tr} P_j P_j^{\prime} = d \delta_{j,j^{\prime}} \). These allow their utilization for general mapping of Hilbert space operators onto the phase-space-like lines and points of DAPG in close analogy with the mappings within the continuum of Hilbert space operators onto phase space via the well-known Wigner function [11,38]. The physics of phase space involves Weyl-Wigner (W-W) mapping of Hilbert space operators to functions in phase space. The W-W image for \( \hat{\rho}(x,\theta) \) is the Radon transform of the Wigner function, \( W_{\rho}(q,p) \). This comes about because the W-W map of the mutual unbiased state (MUB) involved, \( |x,\theta\rangle \), is \( \delta(x - Cq - Sp) \) which specifies the phase space point, \( q,p \), that lies on the line stipulated within the \( \delta \)-function. Thence the state reconstruction, gotten via the inversion of the Radon transform requires the properly normalized sum (integral) of all the lines going through that phase space point. This involves the handling of singularity in the continuum problem. Finite-dimensional mapping onto the phase-space-like lines of the dual affine plane geometry (DAPG) involves, likewise, mapping of an MUB state, \( |m,b\rangle \). This map now relates DAPG points \( \alpha \equiv (m,b) \) and a line, \( j \), via \( \Lambda_{\alpha,j} \) (eq. (23)) which replaces the continuum \( \delta \)-function. Mapping and inversion in the finite-dimensional analysis, in close analogy to the Wigner function of the continuum, is conveniently given in terms of a quasi-distribution, \( V_{\rho}(j) \) —the image of the state \( \rho \).

The approach allows an easy transcription from dual affine plane geometry (DAPG) to affine plane geometry (APG) and vice versa.

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Appendix: “fluctuation distillation” formula. —

Given, eq. (16), \( P_j = \sum_{\alpha \in j} \hat{A}_\alpha - \hat{I} \) and, [15], \( P_j^2 = \hat{I} \), implies

\[
\left( \sum_{\alpha \in j} \hat{A}_\alpha - \hat{I} \right) \left( \sum_{\alpha' \in j} \hat{A}_{\alpha'} - \hat{I} \right) = \hat{I}.
\]

Thus,

\[
\sum_{\alpha, \alpha' \in j} \hat{A}_\alpha \hat{A}_{\alpha'} = 2 \sum_{\alpha \in j} \hat{A}_\alpha.
\]

Recalling that, eq. (13), \( \hat{A}_\alpha^2 = \hat{A}_\alpha \) allows

\[
\sum_{\alpha \neq \alpha' \in j} \hat{A}_\alpha \hat{A}_{\alpha'} = \sum_{\alpha \in j} \hat{A}_\alpha.
\]

QED

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