Physical properties of black di-ring

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Abstract. We study the stationary asymptotically flat five-dimensional vacuum solution describing black di-ring: two concentric black rings. We give an outline of solution generating methods to construct the black di-ring and introduce the rod structures of the black di-ring and the seed solutions. We investigate the phase structure of black di-ring to show the infinite non-uniqueness of the solution. We analyze the distributions of black di-rings whose Komar angular momentum of inner ring vanishes and whose angular velocity of inner ring vanishes in the phase diagram. We also investigate temperatures of horizons for these two special families to consider the existence of the thermal equilibrium black di-rings.

1. Introduction

One of the most important recent findings of the higher-dimensional general relativity is a single-rotational black ring solution by Emparan and Reall [1]. (See also [2].) This solution is a vacuum, axially symmetric and asymptotically flat solution of the five-dimensional general relativity. The topology of the event horizon is $S^1 \times S^2$. The black ring rotates along the direction of $S^1$. The balanced black ring which has no conical singularity has a minimum of angular momentum for a fixed mass parameter. When the angular momentum is near this minimum, there are two different black rings with the same angular momentum. They are called fat and thin black rings according to their shapes. In addition we have a single-rotational spherical black hole [3] with the same asymptotic parameters. As a result, there can be three different configurations for the fixed mass and angular momentum parameter. This fact is often called the discrete non-uniqueness of the five dimensional vacuum solutions.

In this several years, we found important five-dimensional vacuum solutions by using solitonic solution generating techniques. The solution of black ring with $S^2$ rotation was constructed by the Bäcklund transformation [4]. The same solution was found by another method [5]. It was shown that the solution of $S^1$ rotating black ring can be generated by the solitonic technique [6, 7]. The black ring solution with two angular momentum components was also constructed by the inverse scattering method [8]. In five dimensions, in addition to the solutions with single horizon, there exist solutions with disconnected event horizons. Black Saturn which is a spherical black hole surrounded by a black ring was constructed by the inverse scattering method [9]. It was shown that the black rings can be superposed concentrically [10, 11] and orthogonally [12, 13].

The existence of multi-black hole configurations implies continuous non-uniqueness of five-dimensional black holes. The phase diagram of the black Saturn was investigated in [9, 14]. The plot of random sets of points in the phase diagram showed that the black Saturn covers...
the wide region of the phase diagram. The phases of black Saturn were investigated based on
the thin and long ring approximation in which the black Saturn can be modelled as a simple
superposition of an MP black hole and a black ring [9]. It was argued that the configurations
that approach maximal entropy for fixed mass and angular momentum are black Saturns with
a nearly static black hole and a very thin black ring.

The black di-ring also indicates infinite non-uniqueness [10]. It was confirmed that there are
infinite number of black di-rings for same mass and same angular momentum. Distributions of
black di-rings in the phase diagram have not been fully investigated. When we approximate the
black di-ring as a simple superposition of two concentric black rings, we can roughly estimate
the region covered by the black di-ring in the phase diagram. The maximum of the area would
be smaller than the one of black Saturn for the same mass and angular momentum. Because of
the strong non-linearity, however, we need precise analyses for the distributions of black di-ring
in the phase diagram for the decisive conclusion.

In this article we analyze the phase structure of black di-ring by numerical plots in the phase
diagram. In addition, we investigate the phases of two particular family of the black di-ring.
One is the solution whose Komar angular momentum of inner ring is zero. The other is the
solution whose angular velocity of inner ring is zero.

2. Black di-ring solution
The black di-ring solution was generated by using the Bäcklund transformation [10] and
reconstructed by the inverse scattering method [15]. The solution has two disconnected $S^1 \times S^2$
event horizons with single angular momentum component. In this section we briefly review
the rod structure analysis and shortly see the solution generating methods which were used to
generate the black di-ring.

2.1. Rod structure analysis
The rod structure is a powerful tool to describe an axially symmetric vacuum solutions. In
the recent investigations for the uniqueness theorem of higher dimensional black holes, the rod
structure plays an important role. Here we give a brief explanation of the rod structure analysis
elaborated by Harmark [16]. See [16] for complete explanations.

We denote the D-dimensional axially symmetric stationary metric as

$$ds^2 = G_{ij} dx^i dy^j + e^\nu (d\rho^2 + dz^2)$$

where $G_{ij}$ and $\nu$ are functions only of $\rho$ and $z$ and $i, j = 0, 1, \ldots, D - 3$. The $D - 2$ by $D - 2$
matrix field $G$ satisfies the following constraint

$$\rho = \sqrt{|\det G|}.$$  (2)

The equations for the matrix field $G$ can be derived from the Einstein equation $R_{ij} = 0$ as

$$G^{-1} \nabla G = (G^{-1} \nabla G)^2,$$  (3)

where the differential operator $\nabla$ is the gradient in three-dimensional unphysical flat space with
metric

$$d\rho^2 + \rho^2 d\omega^2 + dz^2.$$  (4)

Because of the constraint $\rho = \sqrt{|\det G|}$, at least one eigenvalue of $G(\rho, z)$ goes to zero for
$\rho \to 0$. However it was shown that if more than one eigenvalue goes to zero as $\rho \to 0$, we have
a curvature singularity there. Therefore we consider solutions which have only one eigenvalue
goes to zero for $\rho \to 0$, except at isolated values of $z$. Denoting these isolated values of $z$ as
a_1, a_2, \ldots, a_N$, we can divide the $z$-axis into the $N + 1$ intervals $[-\infty, a_1], [a_1, a_2], \ldots, [a_N, \infty]$, which is called as rods. These rods correspond to the source added to the equation (3) at $\rho = 0$ to prevent the break down of the equation there.

The eigenvector for the zero eigenvalue of $G(0, z)$

$$v = v^i \frac{\partial}{\partial x^i},$$

which satisfies

$$G_{ij}(0, z)v^i = 0,$$

determines the direction of the rod. If the value of $\frac{G_{ij}v^iv^j}{\rho^2}$ is negative (positive) for $\rho \to 0$ the rod is called timelike (spacelike). Each rod corresponds to the region of the translational or rotational invariance of its direction. The timelike rod corresponds to a horizon. The spacelike rod corresponds to a compact direction.

The rod structure of black di-ring is composed of two finite timelike rods, two finite spacelike rods and two semi-infinite spacelike rods. Both timelike rods have non-trivial direction vectors. For the single spin black di-ring, there are six parameters, of which four are the lengths of the finite rods and two are the non-trivial components of direction vectors of timelike rods. Figure 1 shows the rod structure of the single spin black di-ring.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{rod_structure}
\caption{Rod structure of black di-ring with single angular momentum component.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{seed_solution}
\caption{Rod structure of seed solution to generate a black di-ring by Bäcklund transformation.}
\end{figure}

2.2. Solution generating methods

To solve the five-dimensional Einstein equations for the axially symmetric vacuum spacetime with single spin, we use the fact that a part of the equations can be reduced to the Ernst equation in the four dimensions. A solution for the Ernst equation can be obtained from the Bäcklund transformation for an seed solution. To generate the black di-ring solution by the Bäcklund transformation [10], we start from the static seed solution whose rod structure is described in Fig.2. By the Bäcklund transformation, the finite spacelike rod with direction vector $(0, 1, 0)$ is transformed to the timelike rod with non-trivial direction vector $(1, \Omega_1, 0)$. In addition the trivial direction vector of finite timelike rod of seed solution is transformed to the non-trivial one. See [10] for the details of the methods and the exact expression of the solution.

The black di-ring solution was regenerated by the inverse scattering method [15]. In the analysis the seed solution denoted in Fig. 3 was used. To construct the black di-ring from this seed, two (anti-)solitons with trivial vectors are removed at first. Next two (anti-)solitons with non-trivial vectors are added at the same points. Another type of the seed solution can become a seed of the black di-ring. For example, we can construct the black di-ring from the seed solution whose rod structure is given by Fig.4. The expressions of these two black di-ring solutions are

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different each other. However it can be shown that these two solutions exactly coincide with each other. The expression by Evslin and Krishnan [15] is simpler than the others. We can easily make a correlation between the parameters of the solution constructed by the Bäcklund transformation and the one constructed by inverse scattering method from the seed of Fig 4.

\[
\begin{array}{ccc}
(0,0) & (1,0) & (1,0) \\
(0,1) & (1,0) & (1,0) \\
(0,0) & (0,1) & (0,1) \\
(0,1) & (0,1) & (0,1)
\end{array}
\]

**Figure 3.** Rod structure of the seed metric to generate the black di-ring by the inverse scattering method in [15]. Two finite spacelike rods with direction vectors \((0,1,0)\) have negative line density.

**Figure 4.** Rod structure of the seed metric to generate the black di-ring by the inverse scattering method. The position of the rod with negative line density is different from the seed of Evslin and Krishnan [15].

3. **Phase structure of black di-ring**

We can calculate physical variables of black di-ring from the exact expressions of the solution. To investigate the phase structure of the black di-ring, we define the following dimensionless variables,

\[
j^2 = \frac{27\pi}{32G} J^2, \quad a_h = \frac{3}{16} \sqrt{\frac{3}{\pi}} \frac{A_h}{(GM)^{3/2}}, \quad \tau_i = \sqrt{\frac{32\pi}{3}} T_i(GM)^{1/2}.
\]

(7)

Here \(M\) and \(J\) are the ADM mass and angular momentum of the black di-ring, respectively. \(A_h\) is a total area of the event horizons and \(T_i\) is a temperature of \(i\)'s horizon.

To analyze the phase structure of black di-ring, we solve the two balance conditions to decide the appropriate parameters for the regular solution, e.g., six lengths of finite rods in Fig. 3 and plot the corresponding point in the phase diagram. Figure 5 shows a part of the plots for about 25000 sets of parameters. The black di-ring can become larger entropy than the black ring with the same mass and angular momentum. There is a black di-ring with zero angular momentum as similar as the black Saturn.

In Fig. 6, we plot the relation between the reduced angular momentum \(j\) and the reduced area \(a_h\) of the black di-ring whose inner ring has zero Komar angular momentum. In Fig. 7, we plot the relation between the reduced temperatures of outer ring \(\tau_1\) and inner ring \(\tau_2\) for the same black di-rings. We can see that there is a thermal equilibrium black di-ring whose temperature of both horizons are equivalent with each other. It can be shown that there is no solution whose angular velocities of event horizons are equivalence with each other in this plot.

In Fig. 8, we plot the relation between the reduced angular momentum \(j\) and the reduced area \(a_h\) of the black di-ring whose inner ring has zero angular velocity. In Fig. 9, we plot the relation between the reduced temperatures of outer ring \(\tau_1\) and inner ring \(\tau_2\) for the same black di-rings. We can see that there is a thermal equilibrium black di-ring whose temperature of both horizons are equivalent with each other. It can be shown that there is no solution whose angular velocities of event horizons are equivalent with each other in this plot.
Figure 5. The distribution of black di-rings in the phase diagram. Black bold lines shows the behaviors of Myers-Perry black hole and black ring.

Figure 6. Distribution of black di-ring for zero inner angular momentum. Black bold lines shows the behaviors of Myers-Perry black hole and black ring.

Figure 7. Relation of reduced temperatures of horizons for zero inner angular momentum.

4. Concluding remarks
We analyzed the physical properties of black di-ring. We numerically solved the balance conditions to decide the parameters set for the regular solution and plotted the corresponding point in the phase diagram. We reconfirmed the infinite non-uniqueness of the black di-ring. We confirmed that the black di-ring can satisfy the balance conditions even if the total angular momentum is zero. We also investigated the phase structures of black di-rings for two special cases. One is the black di-ring with zero inner angular momentum and the other is zero inner angular velocity. We showed that the temperatures of inner and outer ring can become equivalent for the both cases. The angular velocities of the horizons can not become equivalent with each other in these cases.

We have recently confirmed the existence of the thermodynamically equilibrium black di-ring
whose temperatures and angular velocities of horizons are equivalent with each other. This configuration can not be expected from a simple superposition of black rings because fixing temperature and angular velocity uniquely determines a black ring. There seems little chance to impose equality of all temperature and all angular velocities on solutions of multi black rings without non-linearity of the solution. The existence of black di-ring in thermodynamical equilibrium may suggest the strong coupling of the rings in the configuration.

Acknowledgments
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