Efficiency statistics of a quantum Otto cycle

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The stochastic efficiency [G. Verley et al., Nat. Commun. 5, 4721 (2014)] was introduced to evaluate the performance of energy-conversion machines in micro-scale. However, such an efficiency generally diverges when no heat is absorbed while work is produced in a thermodynamic cycle. As a result, any statistical moments of the efficiency do not exist. In this study, we come up with a different version of the definition for the stochastic efficiency which is always finite. Its mean value is equal to the conventional efficiency, and higher moments characterize the fluctuations of the cycle. In addition, the fluctuation theorems are re-expressed via the efficiency. For working substance satisfying the equipartition theorem, we clarify that the thermodynamic uncertainty relation for efficiency is valid in an Otto engine. To demonstrate our general discussions, the efficiency statistics of a quantum harmonic-oscillator Otto engine is systematically investigated. The probability that the stochastic efficiency surpasses the Carnot efficiency is explicitly obtained. This work may shed new insight for optimizing micro-machines with fluctuations.

I. INTRODUCTION

For a heat engine operating between a hot and cold reservoir, the conventional efficiency is defined by the ratio of the output work and the heat absorbed from the hot reservoir, which characterizes the performance of the engine. As the size of the engine decreases, the thermal fluctuations [1, 2] and quantum fluctuations [3, 4] become more significant. From the point view of stochastic thermodynamics, the work, heat, and entropy of microscopic systems are all stochastic quantities. Hence, at a microscopic level, it is natural to expect that the efficiency, introduced to evaluate the ability of energy conversion for various thermal machines, is also a stochastic quantity.

Recently, the stochastic efficiency, defined as the ratio of the stochastic output work and the stochastic heat absorbed from the hot reservoir in a cycle, has been widely studied for classical heat engines [5–9]. This stochastic efficiency is also applied to a quantum Otto cycle in Refs. [10, 11]. However, such a definition of the stochastic efficiency seems weird for the following three reasons: (1) The mean value of the stochastic efficiency is not equal to the conventional efficiency in general. In contrary, the mean values of stochastic work, heat, and entropy are equal to their counterparts in the conventional thermodynamics; (2) The efficiency approaches infinity with a non-zero probability. Such a result is due to the possibility that no heat is absorbed from the hot reservoir while work is produced in one realization of the cycle [11]; (3) Due to the divergent efficiency distribution, any moments of the efficiency are ill-defined [9]. Thus, one fails to evaluate the performance of the heat engine by the moments of this version of stochastic efficiency.

To avoid such weirdness, meanwhile to evaluate the fluctuations in a practical heat engine, we come up with a different version of the stochastic efficiency. Then, the fluctuation theorems [1–4, 12, 13] is re-expressed via the stochastic efficiency. Moreover, the thermodynamic uncertainty relation (TUR) [14, 15] for efficiency is investigated. For working substance satisfying the equipartition theorem, we obtain the TUR for a quantum Otto cycle in the quasistatic limit (a general proof) and in a finite-time Otto cycle (numerical simulations). As a specific example, we apply our version of the stochastic efficiency to study a quantum harmonic-oscillator Otto cycle. We find that both the probability that the stochastic efficiency surpasses the Carnot efficiency and the probability that the stochastic efficiency is negative increase as the temperatures of the reservoirs decrease.

This paper is arranged as follows. In Sec. II, we introduce the quantum Otto cycle and the joint distribution of input work and absorbed heat from the hot reservoir. In Sec. III, a different version of stochastic efficiency is given. The fluctuation theorems and the TUR are also re-expressed via the stochastic efficiency. In Sec. IV, we demonstrate our general discussions in a quantum Otto cycle with the harmonic oscillator being the working substance. And we systematically investigate the statistics of the stochastic efficiency. Section V is the summary and discussion.
II. THE JOINT DISTRIBUTION OF WORK AND HEAT IN A QUANTUM OTTO CYCLE

As illustrated in Fig. 1, we consider a quantum Otto cycle which involves four strokes: two adiabatic processes and two isochoric processes [16, 17]. In the adiabatic compression (expansion) process, the Hamiltonian of the working substance is changed from $H(\lambda_0)$ to $H(\lambda_1)$ (from $H(\lambda_1)$ to $H(\lambda_0)$) during time $\tau_c$ ($\tau_h$) through a time-dependent parameter $\lambda$. In the two isochoric processes, during time $t_h$ ($t_c$), the working substance contacts a hot (cold) reservoir at the inverse temperature $\beta_h$ ($\beta_c$) with fixed $\lambda$. For simplicity, we assume that a complete thermalization is achieved in the two isochoric processes. Namely, the working substance is thermal equilibrium with the corresponding reservoir at the end of each isochoric process.

The stochastic work and heat in a quantum Otto cycle are defined under the two-point measurement scheme [10]. At time $t = 0, \tau_c, \tau_c + t_h, \tau_c + t_h + \tau_h$ ($t = 0$ is the initial time of the adiabatic compression process), we apply the projective measurements of energy on the working substance according to the corresponding instantaneous Hamiltonian. Then, the stochastic work $w_c$ ($w_e$) in the adiabatic compression (expansion) process, and the stochastic absorbed heat $q$ in the hot isochoric process are defined as

$$w_c = E^1_m - E^0_n,$$
$$q = E^1_k - E^1_m,$$
$$w_e = E^1_l - E^0_c,$$

where $E^0_n, E^1_m, E^1_k, E^0_c$ are the measured energy of the four projective measurements corresponding to the times $t = 0, \tau_c, \tau_c + t_h, \tau_c + t_h + \tau_h$ respectively ($n, m, k, l$ denote the corresponding quantum numbers). Thus, the joint probability distribution $P(w,q)$ of the total stochastic input work $w = w_c + w_e$, and the stochastic absorbed heat from the hot reservoir $q$ is given by

$$P(w,q) = \sum_{n,m,k,l} \delta(w - E^1_m + E^0_n - E^1_l + E^0_c)\delta(q - E^1_k + E^1_m) \times |\langle m | U_c | n \rangle|^2 |\langle l | U_h | k \rangle|^2 \frac{e^{-\beta_c E^0_c - \beta_h E^1_k} Z^0(\beta_c) Z^1(\beta_h)}{Z^0(\beta_e) Z^1(\beta_h)},$$

where $U_c, U_h$ are the unitary evolution operators corresponding to the compression and expansion processes, $|j\rangle_{0(1)}$ is the eigenstate of the Hamiltonian $H(\lambda_0)$ ($H(\lambda_1)$) and $Z^0(\beta_c) = \text{Tr}[e^{-\beta_c H(\lambda_0)}], Z^1(\beta_h) = \text{Tr}[e^{-\beta_h H(\lambda_1)}]$ are the partition functions corresponding to the equilibrium states at $t = 0$ and $t = \tau_c + t_h$ respectively.

III. STOCHASTIC EFFICIENCY

A. Definition

We define the stochastic efficiency of a (classical) quantum heat engine as

$$\eta = -\frac{w}{\langle q \rangle},$$

where $\langle \cdot \rangle$ denotes the mean value over numerous measurements, i.e.,

$$\langle q \rangle = \int dw dq P(w,q).$$

For heat engines, the denominator is always non-zero ($\langle q \rangle > 0$), so the stochastic efficiency $\eta$ in Eq. (3) is finite. Moreover, it follows from Eq. (4) that $\langle \eta \rangle = -\langle w \rangle / \langle q \rangle$, which is just the conventional efficiency.

From the joint distribution $P(w,q)$, the distribution of the stochastic efficiency $P(\eta)$ is obtained by

$$P(\eta) = \int dw dq P(w,q) \delta(\eta + w / \langle q \rangle).$$

The fluctuation of the stochastic efficiency is determined by the output work, which characterizes the reliability of the heat engine.

For practical calculation of the joint distribution of work and heat, we show the characteristic function of $P(w,q)$ in the following

$$\chi(u,v) \equiv \langle e^{iuw + iuv} \rangle = \chi_c(u,v) \chi_h(u,v),$$

where

$$\chi_c(u,v) = \frac{\text{Tr}[U_c e^{(iu-iv)H(\lambda_1)}] U_c e^{(-iu-iv)H(\lambda_0)}]}{Z^0(\beta_c)},$$

$$\chi_h(u,v) = \frac{\text{Tr}[U_c e^{iuH(\lambda_0)}] U_c e^{(-iu+iv-\beta_h)H(\lambda_1)}}{Z^1(\beta_h)}.$$
The cumulant moments of work and heat are obtained from $\chi(u, v)$, such as the average input work

$$\langle w \rangle = -i \frac{\partial \ln \chi(u, v)}{\partial u} \bigg|_{u=v=0},$$

(9)

the average heat absorbed form the hot reservoir

$$\langle q \rangle = -i \frac{\partial \ln \chi(u, v)}{\partial v} \bigg|_{u=v=0},$$

(10)

and the variance of input work

$$\Delta w^2 = -\frac{\partial^2 \ln \chi(u, v)}{\partial u^2} \bigg|_{u=v=0},$$

(11)

where $\Delta(\cdot) = \sqrt{\langle \cdot^2 \rangle - \langle \cdot \rangle^2}$ denotes the standard deviation.

### B. Fluctuation theorems

Fluctuation theorems indicate the equality relation in a general nonequilibrium process. According to Ref. [18], the fluctuation theorems are reexpressed via the efficiency for the Hamiltonian of the working substance involving time-reversal symmetry:

$$\langle e^{-\delta s_q} \rangle = 1,$$

(12)

$$P(-\eta \langle q \rangle, q) = P_B(\eta \langle q \rangle, -q) e^{\delta s_{q}(\eta,q)},$$

(13)

where $\delta s_{q}(\eta,q) = \beta_c(\eta_C q - \eta \langle q \rangle)$ is the total stochastic entropy production expressed in terms of $\eta$ and $q$, and $\eta_C = 1 - \beta_h/\beta_c$ is the Carnot efficiency. The subscript $R$ denotes the reverse process of the cycle (the clockwise direction in Fig. 1). Then, using the Jensen’s inequality $e^{x} \leq \langle e^x \rangle$, we have $\langle \eta \rangle \leq \eta_C$ for heat engines ($\langle q \rangle > 0$), which is the second law of thermodynamics. It is worth mentioning that this inequality is not sharp. In fact, for quantum systems without energy-level crossing when changing the parameter $\lambda$, we obtain a sharper inequality $\langle \eta \rangle \leq \eta_O$ as a result of the minimum work principle [19], where $\eta_O$ is the Otto efficiency, i.e., the efficiency of an Otto cycle in the quasistatic limit (see Appendix A).

### C. Thermodynamic uncertainty relation

Since the fluctuation theorems always imply the genralized TUR [20], it follows from Eq. (13) that

$$\frac{\Delta \eta^2}{\langle \eta \rangle^2} \geq f(\langle \delta s \rangle),$$

(14)

where $f(x) = \text{csch}^2[xg(x/2)]$, and $g(x)$ is the inverse function of $x \tanh x$. Equation (14) expresses a trade-off between the relative fluctuation of the efficiency and the dissipation quantified through the entropy production in a cycle. When $\langle \delta s \rangle \to 0$, $f(\langle \delta s \rangle) = 2/\langle \delta s \rangle$, which reproduces the TUR [14, 15] for the efficiency.

For the spectra of the working substance with scale property, i.e., $E_n^3 = E_0^3/e$ [10] ($e$ is $n$-independent), the general expression of the joint characteristic function (Eq. (6)) is obtained in the quasistatic limit (see Appendix B). From Eqs. (9,11), we obtain

$$\frac{\Delta \eta^2}{\langle \eta \rangle^2} = \frac{\Delta w^2}{\langle w \rangle^2} = \frac{1}{\langle \delta s \rangle} \left( \frac{1 - \eta_O}{1 - \eta_C} + \frac{1 - \eta_C}{1 - \eta_O} \right) \geq \frac{2}{\langle \delta s \rangle},$$

(16)

with the equal sign saturated at $\eta_O = \eta_C$. The inequality (16) is consistent with the TUR in steady states [14, 15] or in a specific Otto cycle [22].

Moreover, due to the third law of thermodynamics, $\langle \delta s \rangle \to 0$ in the low-temperature limit. Using the property of the function $f(x)$ in Eq. (14), the TUR for the efficiency is also reproduced in the low-temperature limit. Consequently, we expect that the TUR for efficiency is valid for an arbitrary temperature under these conditions. For a finite-time cycle, we numerically study the TUR in a specific model below.

### IV. QUANTUM HARMONIC-O SCILLATOR HEAT ENGINE

In this section, we illustrate our general discussions above with a specific example: a quantum harmonic-oscillator being the working substance of the Otto cycle. The frequency is changed from $\omega_0$ to $\omega_1$ ($\omega_1 > \omega_0$) in the adiabatic compression process. Then, according to Refs. [21, 23], $\chi_c(u,v)$ and $\chi_h(u,v)$ of the joint characteristic function in Eq. (6) are explicitly obtained as (see Appendix C for detailed derivation)
\[
\chi_c (u, v) = 2 \sinh \left( \frac{\beta_c \omega_0}{2} \right) \left\{ 2 \cos [(u-v)\omega_1] \cos [(u-i\beta_c)\omega_0] + 2Q_c \sin [(u-v)\omega_1] \sin [(u-i\beta_c)\omega_0] - 2 \right\}^{-\frac{1}{2}}, \tag{17}
\]

and

\[
\chi_h (u, v) = 2 \sinh \left( \frac{\beta_h \omega_1}{2} \right) \left\{ 2 \cos (u\omega_0) \cos [(u-v-i\beta_h)\omega_1] + 2Q_h \sin (u\omega_0) \sin [(u-v-i\beta_h)\omega_1] - 2 \right\}^{-\frac{1}{2}}, \tag{18}
\]

where \( Q_{c(h)} \geq 1 \) is the corresponding non-adiabatic factor \([24, 25]\). The equal sign is hold when the quantum adiabatic condition is satisfied. In the following, we study the efficiency statistics of the Otto cycle in different circumstances.

A. Average efficiency and efficiency distribution of the heat engine

The average output work and average absorbed heat per cycle can be obtained using Eqs. (9),(10),(17),(18) as

\[
-\langle w \rangle = -\frac{1}{2} \left[ (\omega_1 Q_c - \omega_0) \vartheta_c - (\omega_1 - \omega_0 Q_h) \vartheta_h \right], \tag{19}
\]

and

\[
\langle q \rangle = \frac{\omega_1}{2} \left( \vartheta_h - Q_c \vartheta_c \right), \tag{20}
\]

where

\[
\vartheta_h \equiv \coth \frac{\beta_h \omega_1}{2}, \vartheta_c \equiv \coth \frac{\beta_c \omega_0}{2}. \tag{21}
\]

Substituting Eqs. (19) and (20) into Eq. (3), the average efficiency is obtained as

\[
\langle \eta \rangle = \frac{(\omega_1 - \omega_0 Q_h) \vartheta_h - (\omega_1 Q_c - \omega_0) \vartheta_c}{\omega_1 (\vartheta_h - Q_c \vartheta_c)}. \tag{22}
\]

In the quasistatic limit, the non-adiabatic factors \( Q_{c,h} = 1 \), and the average efficiency is the Otto efficiency \( \eta_O = 1 - \epsilon (\epsilon = \omega_0/\omega_1) \). Then, Eq. (22) is further expressed with \( \eta_O \) as

\[
\langle \eta \rangle = \eta_O - \frac{\epsilon \sum_{\alpha=h,c} \vartheta_\alpha (Q_\alpha - 1)}{\vartheta_h - Q_c \vartheta_c}. \tag{23}
\]

This result means that the non-adiabatic effect decreases the average efficiency of the engine, which is demonstrated in Fig. 2.

On the other hand, the variance of the efficiency is

\[
\Delta \eta^2 = \Delta w^2 / \langle q \rangle^2, \tag{24}
\]

where the variance of work is obtained from Eq. (11) as

\[
\Delta w^2 = \frac{\omega_1^2}{4} (1 - \epsilon)^2 \left( \vartheta_h^2 + \vartheta_c^2 - 2 \right) + \frac{\omega_1^2}{2} \left[ \vartheta_h^2 (Q_h^2 - 1) \epsilon^2 + \vartheta_c^2 (Q_c^2 - 1) - \epsilon \sum_{\alpha=h,c} (Q_\alpha - 1) (\vartheta_\alpha^2 - 1) \right]. \tag{25}
\]

In the quasistatic limit, the variance of efficiency accordingly becomes

\[
\Delta \eta^2_{ads} = \frac{(1 - \epsilon)^2 \left( \vartheta_h^2 + \vartheta_c^2 - 2 \right)}{(\vartheta_h - \vartheta_c)^2}. \tag{25}
\]

It is worth mentioning that the efficiency fluctuation does not vanish for a quantum harmonic-oscillator Otto cycle in the quasistatic limit, while the fluctuation of the previous version of the stochastic efficiency vanishes in this case \([10]\).
FIG. 2: Average efficiency as the function of \(Q_c\) and \(Q_h\). In this figure, we use \(\eta_0 = 0.5\), \(T_h = 1\) \(\eta_C = 0.8\), and \(\omega_1 = 1\).

Then, the non-adiabatic factor (adiabatic case) and Fig. 5 (non-adiabatic case) with the black dots. As comparisons, the Otto efficiency and Carnot efficiency are respectively represented with the blue dash-dotted line and the red dotted line. And we show the probability that the efficiency of a stochastic Otto cycle surpasses the Carnot efficiency in the figure. In addition, one can infer that lower temperature leads to greater probability of the heat engine surpassing the Carnot efficiency. Meanwhile, the lower temperature increases the probability of the engine to be useless, namely, the engine outputs negative work.

**B. Finite-time performance of the heat engine**

To further explore the finite-time performance of the cycle, we first analyze the explicit time dependence of the non-adiabatic factors for a specific protocol. For an adiabatic process with frequency changed from \(\omega_1\) to \(\omega_f\) during time \(t \in [0, \tau]\), the time dependence of the frequency of the harmonic oscillator is [22, 26, 27]:

\[
\omega(t) = \frac{\omega_1}{(\omega_1/\omega_f - 1)t/\tau + 1}.
\]  

(26)

Then, the non-adiabatic factor \(Q(\tau)\) is obtained as (See Appendix D for detailed derivation)

\[
Q(\tau) = 1 + \frac{1 - \cos \left[ \sqrt{\frac{a^2 \tau^2}{\omega_f^2 - 1}} \ln(\omega_f/\omega_1) \right]}{a^2 \tau^2 - 1},
\]  

(27)

where

\[
a \equiv \frac{2\omega_f/\omega_1}{\omega_f - \omega_i}.
\]  

(28)

As shown in Fig. 6, the non-adiabatic factor \(Q(\tau)\) (blue solid line) oscillates with the driving time \(\tau\), reflecting the quantum coherence effect in the non-adiabatic transition. The orange dashed line represents \(Q(\tau) = 1\), which is achieved for the quantum adiabatic driving or with some special values of \(\tau\) [26, 27].

In the following, we adopt the protocol of Eq. (26) for the finite-time adiabatic processes in the Otto cycle, then we use of the explicit form of \(Q(\tau)\) given in Eq. (27) to study the power at maximum efficiency (PME) and efficiency at maximum power (EMP) of the cycle. In this sense, the non-adiabatic factors become \(Q_{c(h)} = Q(\tau_{c(h)})\), and then the average power \(\langle P(\tau_c, \tau_h)\rangle\) is given in Eq. (27) and the efficiency \(\langle \eta(\tau_c, \tau_h)\rangle\) of the Otto engine are respectively

\[
\langle P(\tau_c, \tau_h)\rangle = \frac{\omega_1 \{ Q(\tau_c) - \epsilon \} \vartheta_c - [1 - \epsilon Q(\tau_h)] \vartheta_h}{\tau_h + \tau_c}.
\]  

(29)

and

\[
\langle \eta(\tau_c, \tau_h)\rangle = \eta_0 - \frac{\epsilon \sum_{a=h,c} \vartheta_a [Q(\tau_a) - 1] \vartheta_{a}}{\vartheta_h - Q(\tau_c) \vartheta_c},
\]  

(30)

where the total duration of the two isochoric processes, i.e., \(t_c + t_h\), is assumed to be much smaller than \(\tau_c + \tau_h\), and is thus ignored.
Since $Q_{c(h)} = 1$ can be achieved within finite time, the average efficiency of some cycles approach the Otto efficiency $\eta_O$ with non-vanishing power. These cycles happen to have the special operation time sets $(\tau^*_c, \tau^*_h)$ corresponding to $Q_h(\tau^*_h) = Q_c(\tau^*_c) = 1$. With the help of Eq. (27), one finds the special operation time follows as

$$\sqrt{a^2 \tau^2 - \ln \epsilon} = 2n\pi, n = 1, 2, 3,...$$

namely,

$$\tau^*_{h,c} = \frac{\eta_O}{\omega_0} \sqrt{\frac{1}{4} + \left( \frac{n_h c \pi}{\ln \epsilon} \right)^2}.$$  

Therefore, the PME is

(31)

(32)

FIG. 4: Efficiency distribution in adiabatic case with $Q_h = Q_c = 1$. The probability distribution of efficiency is plotted with the black dots. The Otto efficiency and Carnot efficiency are respectively represented with the blue dash-dotted line and the red dotted line. The (gray) area in the left side denotes the negative work output regime; the (light red) area in the right side of the red dotted line represents the regime of $\eta > \eta_C$. The parameters are chosen as: (a) $T_h = 10$, $T_c = 2$; (b) $T_h = 1$, $T_c = 0.2$. In this figure, we choose $\omega_0 = 0.5$, $\omega_1 = 1$, and Carnot efficiency is fixed at 0.8.

FIG. 5: Efficiency distribution in non-adiabatic case with $Q_h = Q_c = 1.2$. The probability distribution of efficiency is plotted with the black dots. The Otto efficiency and Carnot efficiency are respectively represented with the blue dash-dotted line and the red dotted line. The (gray) area in the left side denotes the negative work output regime; the (light red) area in the right side of the red dotted line represents the regime of $\eta > \eta_C$. The parameters are chosen as: (a) $T_h = 10$, $T_c = 2$; (b) $T_h = 1$, $T_c = 0.2$. In this figure, we choose $\omega_0 = 0.5$, $\omega_1 = 1$, and Carnot efficiency is fixed at 0.8.

FIG. 6: Time dependence of the non-adiabatic factor. In this figure, the blue solid curve represents $Q(\tau)$ in Eq. (D7), the orange dashed line is $Q(\tau) = 1$. The initial and final frequencies of the harmonic oscillator in the adiabatic process are chosen as $\omega_0 = 0.5$ and $\omega_1 = 1$. 

$0 10 20 30 40
0 0.1 0.2 0.3 0.4 0.5 0.6
$
FIG. 7: Efficiency at maximum power of the Otto engine as the function of $\eta_O$. In this figure, $\eta_O/\eta_C = 0.8$ is fixed. The blue solid curve represents the EMP of the Otto engine. The black dash-dotted curve is the upper bound for EMP, $\eta_+ = 2\eta_O/(3 - \eta_O)$, of the Otto cycle obtained in Ref. [31] without considering the oscillation of work. The red dotted line denotes the Otto efficiency.

$$P(\tau_c^*, \tau_h^*) = \frac{\omega_0 \omega_1 (\theta_h - \theta_c)}{2 \sum_{\alpha=h,c} \sqrt{1/4 + (n_{\alpha} \pi / \ln \epsilon)^2}}$$  \hspace{1cm} (33)

It should be noted that in the usual finite-time thermodynamic cycles, the PME generally approaches zero [28–31]. Here, thanks to the special protocol we have chosen to realize the quantum adiabatic process in finite time, the current quantum Otto cycle outputs non-zero or even relatively large power (comparable to the maximum power) when the Otto efficiency is reached. Obviously, the maximum $P(\tau_c^*, \tau_h^*)$ is

$$P_{\text{max}}(\tau_c^*, \tau_h^*) = \frac{\omega_0 \omega_1 (\theta_h - \theta_c)}{2 \sqrt{1 + (2\pi / \ln \epsilon)^2}}$$  \hspace{1cm} (34)

which is achieved at $n_c = n_h = 1$. Besides, the second largest and third largest power are reached at $(n_c = 1, n_h = 2)$ and $(n_c = 2, n_h = 2)$, respectively. For $(n_{\alpha} \pi / \ln \epsilon)^2 \gg 1/4$, Eq. (33) can be approximated as

$$P(\tau_c^*, \tau_h^*) \approx \frac{\omega_0 \omega_1 \ln \epsilon (\theta_h - \theta_c)}{2\pi (n_c + n_h)},$$  \hspace{1cm} (35)

which shows that $P(\tau_c^*, \tau_h^*)$ is a monotonically decreasing quasi-continuous function of $n_c$ and $n_h$.

In addition, the EMP of this Otto engine as the function of $\eta_O$ is illustrated in Fig. 7. As shown in this figure, the EMP of our cycle (blue solid curve) is found to surpass the upper bound, $\eta_+ = 2\eta_O/(3 - \eta_O)$ (black dashed curve), of the Otto cycle’s EMP without considering the oscillation of the output work [31]. This indicates that the oscillation of the output work (due to quantum coherence) are conducive to improving the EMP.

C. Thermodynamic uncertainty relation (TUR) for efficiency

Because the spectra of a quantum harmonic oscillator have scale property and the system follows the equipartition theorem in the high-temperature limit, we conclude that in the quasistatic limit, the TUR (Eq. (16)) is valid according to the discussions in Sec. III C.

For the non-adiabatic driving cycle, the results are shown in Fig. 8. The TUR (Eq. (16)) is still valid since $\Delta \eta^2 / (\eta \tau^2) \left(\frac{2}{(3\pi S)}\right)$ increases monotonically with $Q_h$ and $Q_c$. Here, without loss of generality, we take $Q_h$ as the independent variable in the figure. On the contrary, the TUR may be violated due to the incomplete thermalization in the isochoric processes [22]. One can conclude from Fig. 8(a) that higher temperature makes $\Delta \eta^2 / (\eta \tau^2) \left(\frac{2}{(3\pi S)}\right)$ lower. Moreover, as shown in Fig. 8(b),
when $Q_h \rightarrow 1$, $\frac{\Delta \eta^2}{\langle \eta \rangle^2} \left(\frac{2}{5\sqrt{7}}\right)$ is closer to 1 in the case with $\eta_0 = 0.7$. This is consistent with the discussions in Sec. III C that the condition for $\frac{\Delta \eta^2}{\langle \eta \rangle^2} \left(\frac{2}{5\sqrt{7}}\right) \rightarrow 1$ is $\eta_O \rightarrow \eta_C$.

V. SUMMARY AND DISCUSSION

In this paper, we come up with a new definition of the stochastic efficiency for heat engine in micro scale. The moments of the efficiency always exist, and its mean value is equal to the conventional efficiency. Moreover, the fluctuation theorems are reexpressed via the efficiency. For spectra of the working substance with scale property, the statistics of the efficiency is fully determined by the partition functions of the working substance in the quasistatic limit. Importantly, we reveal the connection between the TUR and the equipartition theorem.

For a quantum Otto cycle with a harmonic oscillator being the working substance, we obtain the exact expression of the joint characteristic function of work and heat. We find that the Otto efficiency can be reached with a finite output power (the power at maximum efficiency) with some special duration and the EMP surpasses the upper bound obtained in Ref. [31].

The theoretical predictions of current study can be tested on some state-of-art experiments, such as the Brownian particle system [8] and trapped ion system [33]. As a direct extention, similarly to the stochastic Brownian particle system [8] and trapped ion system [33].

This work is supported by the National Natural Science Foundation of China (NSFC) (Grants No. 11534002, No. 11875049, No. U1730449, No. U1530401, and No. U1930403), and the National Basic Research Program of China (Grants No. 2016YFA0301201). Y. H. Ma is supported by the China Postdoctoral Science Foundation (Grant No. BX2021030).

Appendix A: Proof of $\langle \eta \rangle \leq \eta_O$

As a result of the minimum work principle [19], when the energy levels do not cross during the driving, the average work under finite-time driving $\langle w_c(e) \rangle$ is not less than it under quantum adiabatic driving $\langle w_c(e) \rangle_{adi}$.

Namely, $\delta w_{c(e)} \equiv \langle w_c(e) \rangle - \langle w_c(e) \rangle_{adi} \geq 0$. Thus, the average efficiency of a quantum Otto cycle with the complete thermalization satisfies

$$\langle \eta \rangle = -\frac{\langle w \rangle}{\langle q \rangle} = -\frac{\langle w_c \rangle_{adi} + \langle w_c \rangle_{adi} + \delta w_c + \delta w_c}{\langle q \rangle_{adi} - \delta w_c} \leq -\frac{\langle w_c \rangle_{adi} + \langle w_c \rangle_{adi}}{\langle q \rangle_{adi}} = \eta_O,$$

for $\langle q \rangle_{adi} > -\langle w_c \rangle_{adi} - \langle w_c \rangle_{adi} > 0$, and $\delta w_c \geq 0, \delta w_c \geq 0$, where $\langle q \rangle_{adi}$ denotes the heat absorbed from the hot reservoir in the quasistatic limit.

Appendix B: The joint characteristic function for a quantum Otto cycle in the quasistatic limit

According to Eq. (6), the expression of the joint characteristic function $\chi(u, v)$ is obtained by a transformation of the characteristic function of work $\chi_c(u) = \chi_c(u, 0)$ and $\chi_h(u) = \chi_h(u, 0)$, i.e.,

$$\chi_c(u, v) = \chi_c(u)|_{u-u-v, \beta_c-\beta_c+iv} \chi_h(u, v) = \chi_h(u)|_{\beta_c-\beta_c-iv}.$$  \hspace{1cm} (B1)

with scale property ($E_n^0 = E_0^0/e$), in the quasistatic limit, the expressions of the characteristic function $\chi_c(u)$ and $\chi_h(u)$ read

$$\chi_c(u) = \sum_n e^{-\beta_c E_n^0} Z^0(\beta_c) e^{iu(E_n^1-E_n^0)}$$

$$= \sum_n e^{-\beta_c-\beta_c-iu(e^{-1}-1)} Z^0(\beta_c)$$

$$= Z^0[\beta_c - iu(e^{-1} - 1)] Z^0(\beta_c),$$

and

$$\chi_h(u) = \sum_n e^{-\beta_h E_n^1} Z^1(\beta_h) e^{iu(E_n^0-E_n^1)}$$

$$= \sum_n e^{-\beta_h-\beta_h-iu(e^{-1}-1)} Z^1(\beta_h)$$

$$= Z^1[\beta_h - iu(e^{-1} - 1)] Z^1(\beta_h).$$

Then, it follows from Eq. (B1) that the joint characteristic function $\chi(u, v)$ reads

$$\chi(u, v) = \frac{Z^0[\beta_c + iv - i(u-v)(e^{-1}-1)] Z^1[\beta_c - iv - iu(e^{-1})]}{Z^0(\beta_c) Z^1(\beta_h)}.$$  \hspace{1cm} (B4)
Appendix C: The joint characteristic function for a
quantum harmonic oscillator heat engine

For a harmonic oscillator with time-dependent frequency in an adiabatic process during time $[0, \tau]$, the Hamiltonian is

$$H(t) = \frac{\dot{p}^2}{2m} + \frac{1}{2}m\omega(t)^2x^2. \quad (C1)$$

where

$$Q_c = \frac{\omega_1}{2\omega_0} \left[ y_1(\tau_c)^2 + y_2(\tau_c)^2 + \frac{y_1(\tau_c)^2 + y_2(\tau_c)^2}{\omega_1^2} \right], \quad (C4)$$

the overhead dot denotes the time derivative, $y_1$ and $y_2$ are the two general solutions of the classical harmonic oscillator, i.e.,

$$\ddot{y}(t) + \omega(t)^2y(t) = 0, \quad (C5)$$

and

$$\chi_c(u, v) = 2\sinh\left(\frac{\beta\omega_0}{2}\right) \{2\cos(u\omega_1)\cos[(u - i\beta_c)\omega_0] + 2Q_c \sin(u\omega_1) \sin[(u - i\beta_c)\omega_0] - 2\}^{-\frac{1}{2}}, \quad (C2)$$

$\chi_h(u, v) = 2\sinh\left(\frac{\beta\omega_0}{2}\right) \{2\cos(u\omega_0)\cos[(u - i\beta_h)\omega_1] + 2Q_h \sin(u\omega_0) \sin[(u - i\beta_h)\omega_1] - 2\}^{-\frac{1}{2}}, \quad (C3)$$

with the initial value $\{y_1(0), y_2(0), \dot{y}_1(0), \dot{y}_2(0)\} = \{1, 0, 0, \omega_0\}$. Similarly, the expression of $Q_h$ is given by the replacement: $\omega_0 \leftrightarrow \omega_1, \ t \in [0, \tau_c] \rightarrow t \in [\tau_c + t_h, \tau_c + t_h + \tau_h].$

Then, the characteristic functions of work $\chi_c(u), \chi_h(u)$ (see Appendix B) reads $[21, 23]$

Then, the expression of $\chi(u, v)$ is obtained by $\chi(u, v) = \chi_c(u, v)\chi_h(u, v)$, where (Eq. (B1))

$$\chi_c(u, v) = \chi_c(u)|_{u \rightarrow u - v, \beta_c \rightarrow \beta_c + iv} = 2\sinh\left(\frac{\beta\omega_0}{2}\right) \{2\cos[(u - v)\omega_1]\cos[(u - i\beta_c)\omega_0] + 2Q_c \sin[(u - v)\omega_1] \sin[(u - i\beta_c)\omega_0] - 2\}^{-\frac{1}{2}},$$

$$\chi_h(u, v) = \chi_h(u)|_{\beta_h \rightarrow \beta_h - iv} = 2\sinh\left(\frac{\beta\omega_1}{2}\right) \{2\cos(u\omega_0)\cos[(u - v - i\beta_h)\omega_1] + 2Q_h \sin(u\omega_0) \sin[(u - v - i\beta_h)\omega_1] - 2\}^{-\frac{1}{2}}.$$
\[ m\omega^2(t)x^2/2, \text{ and } D(t) = \omega(t)(xp + px)/2 \] are respectively the Hamiltonian, the Lagrangian, and the generator of the scale transformation. The time-dependent matrix reads

\[
\mathcal{M}(t) = \begin{pmatrix}
\dot{\omega}/\omega^2 & -\dot{\omega}/\omega^2 & 0 \\
-\dot{\omega}/\omega^2 & \dot{\omega}/\omega^2 & -2 \\
0 & 2 & \dot{\omega}/\omega^2
\end{pmatrix}.
\] (D3)

The general solution of Eq. (D1) follows as

\[
\overleftarrow{\phi}(\tau) = \mathcal{T}_+ \exp \left[ \int_0^\tau \mathcal{M}(t)dt \right] \overleftarrow{\phi}(0),
\] (D4)

where \( \mathcal{T}_+ \) denotes the time-ordered operation. For the specific protocol in Eq. (26), the matrix \( \mathcal{M}(t) \) is independent of \( t \). For the thermal equilibrium initial state, \( \overleftarrow{\phi}(0) = \langle H(0) \rangle 0 \rangle \), we find

\[
\overleftarrow{\phi}(\tau) = \left( \begin{array}{c}
-\frac{\omega^2}{\tau} \left[ \left( \frac{\omega_i}{\omega_f} \right)^{-\sqrt{2-a^2\tau^2}} + \left( \frac{\omega_f}{\omega_i} \right)^{\sqrt{2-a^2\tau^2}} \right] \\
\frac{\omega^2}{\tau} \left[ 1 - \left( \frac{\omega_i}{\omega_f} \right)^{\sqrt{2-a^2\tau^2}} \right]
\end{array} \right) \langle H(0) \rangle,
\] (D5)

where \( a = 2\omega_f/(\omega_f - \omega_i) \). Thus, the internal energy of the system at \( t = \tau \) is

\[
\langle H(\tau) \rangle = \frac{a^2\tau^2 - \cos(a^2\tau^2 - 1)\ln(\omega_f/\omega_i)}{a^2\tau^2 - 1} \langle H(0) \rangle.
\] (D6)

Consequently, the non-adiabatic factor is obtained by [24]

\[
Q(\tau) = \frac{\langle H(\tau) \rangle}{\langle H \rangle_{adi}} = 1 + \frac{1 - \cos(a^2\tau^2 - 1)\ln(\omega_f/\omega_i))}{a^2\tau^2 - 1},
\] (D7)

where \( \langle H \rangle_{adi} = \langle H(a\tau \to \infty) \rangle = \langle H(0) \rangle \) \( \omega_f/\omega_i \) is the internal energy of the system at the end of the process under quantum adiabatic driving. In the short-time limit \( a\tau \to 0 \), and long-time limit \( a\tau \to \infty \), it is easy to check that

\[
\lim_{a\tau \to 0} Q(\tau) = 1 + \frac{\ln(\omega_f/\omega_i)^2}{2}, \quad \lim_{a\tau \to \infty} Q(\tau) = 1.
\] (D8)
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