Random version of Dvoretzky’s theorem in $\ell^n_p$

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Abstract
We study the dependence on $\varepsilon$ in the critical dimension $k(n, p, \varepsilon)$ for which one can find random sections of the $\ell^n_p$-ball which are $(1 + \varepsilon)$-spherical. We give lower (and upper) estimates for $k(n, p, \varepsilon)$ for all eligible values $p$ and $\varepsilon$ as $n \to \infty$, which agree with the sharp estimates for the extreme values $p = 1$ and $p = \infty$. Toward this end, we provide tight bounds for the Gaussian concentration of the $\ell_p$-norm.

1 Introduction
The fundamental theorem of Dvoretzky from [8] in geometric language states that every centrally symmetric convex body on $\mathbb{R}^n$ has a central section of large dimension which is almost spherical. The optimal form of the theorem, which was proved by Milman in [20], reads as follows. For any $\varepsilon \in (0, 1)$ there exists $\eta = \eta(\varepsilon) > 0$ with the following property: for every $n$-dimensional symmetric convex body $A$ there exist a linear image $A_1$ of $A$ and $k$-dimensional subspace $F$ with $k \geq \eta(\varepsilon) \log n$ such that
\[
(1 - \varepsilon)B_F \subseteq A_1 \cap F \subseteq (1 + \varepsilon)B_F,
\]
where $B_F$ denotes the Euclidean ball in $F$. The example of the cube $A = B^n_\infty$ shows that this result is best possible with respect to $n$ (see [29] for the details). The

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approach of [20] is probabilistic in nature and shows that most of the $k$-dimensional sections are $(1 + \varepsilon)$-spherical (or Euclidean). Here “most” means with overwhelming probability in terms of the Haar probability measure $\nu_{n,k}$ on the Grassmann manifold $G_{n,k}$. More precisely, given a centrally symmetric convex body $A$ on $\mathbb{R}^n$ and $\varepsilon \in (0,1)$ the random $k$-dimensional subspace $F$ satisfies:

$$\frac{1 - \varepsilon}{M} B_2 \subseteq A \cap F \subseteq \frac{1 + \varepsilon}{M} B_2$$

with probability greater than $1 - e^{-k}$ as long as $k \leq c(\varepsilon) k(A)$. Here $c(\varepsilon)$ stands for the function of $\varepsilon$ in the probabilistic formulation and $k(A)$ is usually referred to the “critical dimension” of the body $A$. The latter can be computed in terms of the global parameters $M = M(A) = \int_{S_n} ||\theta|| A \, d\vartheta(\theta)$ and $b = b(A) = \max_{0 \leq r < 1} ||\theta|| A$; that is $k(A) \simeq n(M/b)^2$. Recall that $1/b$ is the radius of the maximal centered inscribed ball in $A$. Next, one may select a good position of the body $A$ for which the $k(A)$ is large enough with respect to $n$ (see [21] for further details).

It has been proved in [22] that this formulation is optimal with respect to the dimension $k(A)$ in the following sense: the maximal dimension $m$ for which the random $m$-dimensional sections are 4-Euclidean with probability greater than $2^{-m}$ is less than $C k(A)$ for some absolute constant $C > 0$, i.e. $m \leq k(A)$\footnote{For any two quantities $\Gamma, \Delta$ depending on $n, p$, etc. we write $\Gamma \leq \Delta$ if there exists numerical constant $C > 0$ - independent of everything - such that $\Gamma \leq C \Delta$. We write $\Gamma \geq \Delta$ if $\Delta \leq \Gamma$ and $\Gamma \simeq \Delta$ if $\Gamma \leq \Delta$ and $\Delta \leq \Gamma$. Accordingly we write $\Gamma \simeq p \Delta$ if the constants involved are depending only on $p$.}. (Here and everywhere else $C, c, C_1, C_2, \ldots$ stand for positive absolute constants whose values may change from line to line).

The proof in [20] provides the lower bound $c(\varepsilon) \geq c \varepsilon^2 / \log \frac{n}{\varepsilon}$ and this is improved to $c(\varepsilon) \geq c \varepsilon^2$ by Gordon in [10] and an alternative approach is given by Schechtman in [27]. This dependence is known to be optimal. The recent works of Schechtman in [28] and Tikhomirov in [34] established that the dependence on $\varepsilon$ in the randomized Dvoretzky for $B_n^p$ is of the exact order $\varepsilon / \log \frac{1}{\varepsilon}$.

As far as the dependence on $\varepsilon$ in the existential version of Dvoretzky’s theorem is concerned, Schechtman proved in [29] that one can always $(1 + \varepsilon)$-embed $\ell_p^n$ in any $n$-dimensional normed space $E$ with $k(E, \varepsilon) \geq c \varepsilon \log n / (\log \frac{1}{\varepsilon})^2$. Tikhomirov in [35] proved that for 1-symmetric spaces $E$ we may have $k(E, \varepsilon) \geq c \log n / (\log \frac{1}{\varepsilon})^2$ complementing the previously known result due to Bourgain and Lindenstrauss from [2]. Recall that a normed space $(\mathbb{R}^n, || \cdot ||)$ is said to be 1-symmetric if the norm satisfies $\| \sum_i \varepsilon_i a_i, n(i) \| = \| \sum_i \varepsilon_i a_i \|$ for all scalars $(a_i)$, for all choices of signs $\varepsilon_i = \pm 1$ and for any permutation $\pi$, where $(e_i)$ is the standard basis in $\mathbb{R}^n$. Tikhomirov’s result was subsequently extended by Fresen in [9] for permutation invariant spaces with uniformly bounded basis constant. In this note we will not deal with the existential form of Dvoretzky’s theorem. Related results for $\ell_p$ spaces are presented in [15]. For more detailed information on the subject, explicit statements and historical remarks the reader is referred to the recent monograph [1].

Our goal here is to study the random version for the spaces $\ell_p^n$ and to give bounds on the dimension $k(n, p, \varepsilon) \equiv k(\ell_p^n, \varepsilon)$ for which the $k$-dimensional random section of
Theorem 1.1. For all sufficiently large $n$ and for any $1 \leq p \leq \infty$ one has:

$$P \left( \|X\|_p - \mathbb{E}\|X\|_p \geq \varepsilon \mathbb{E}\|X\|_p \right) \leq C_1 \exp(-c_0 \beta(n, p, \varepsilon)), \quad 0 < \varepsilon < 1,$$

where $X$ is standard $n$-dimensional Gaussian vector and $C_1, c_1 > 0$ are absolute constants. The function $\beta(n, p, \varepsilon)$ is defined as follows:

$$\beta(n, p, \varepsilon) = \begin{cases} \varepsilon^2 n, & 1 \leq p \leq 2, \\ \max\left\{ \min\left\{ p^{2-p} \varepsilon^2 n, (\varepsilon n)^{2/p}, \varepsilon n^2 \right\}, 2 < p \leq c_0 \log n \right. \\ \left. p > c_0 \log n \right\},$$

where $0 < c_0 < 1$ is suitable absolute constant. Furthermore, for $p \leq c_0 \log n$ we have:

$$P \left( \|X\|_p - \mathbb{E}\|X\|_p \geq \varepsilon \mathbb{E}\|X\|_p \right) \leq \frac{C_1}{1 + p^{2-\ell} \varepsilon^2 n},$$

for all $\varepsilon > 0$.

The bound we retrieve in the case of fixed $p$ is not new. The corresponding estimates have been studied by Naor [23] in an even more general probabilistic context. Also, for $p = \infty$ we recover the same bound proved by Schechtman in [28]. Therefore, the above concentration result interpolates between the sharp concentration estimates for fixed $1 \leq p < \infty$ and $p = \infty$ and is derived in a unified way. However, our methods are different from the techniques used in [23] and [28] and utilize Gaussian functional inequalities. Actually, following the same ideas as in [27] we will prove a distributional inequality for Gaussian random matrices similar to the concentration inequality described above. Using this inequality and a chaining argument we prove the second main result which is the critical dimension $k(n, p, \varepsilon)$ in the randomized Dvoretzky for the $B^n_p$ balls.

Theorem 1.2. For all large enough $n$, for any $1 \leq p \leq \infty$ and for any $0 < \varepsilon < 1$ the random $k$-dimensional section of $B^n_p$ with dimension $k \leq k(n, p, \varepsilon)$ is $(1 + \varepsilon)$-Euclidean with probability greater than $1 - C \exp(-ck(n, p, \varepsilon))$, where $k(n, p, \varepsilon)$ is defined as:

i. If $1 \leq p < 2$, then

$$k(n, p, \varepsilon) \geq \varepsilon^2 n.$$

ii. If $2 < p < c_0 \log n$, then

$$k(n, p, \varepsilon) \geq \begin{cases} (Cp)^{p/2} \varepsilon^2 n, & 0 < \varepsilon \leq (Cp)^{p/2} n^{-\frac{p-2}{2p}} \\ p^{1/2} \varepsilon^2 n^{2/p}, & (Cp)^{p/2} n^{-\frac{p-2}{2p}} < \varepsilon \leq 1/p \\ \varepsilon n^{2/p} / \log^2 \varepsilon, & 1/p < \varepsilon < 1 \end{cases}$$

Furthermore for $p < c_0 \log n$ we have:

$$k(n, p, \varepsilon) \geq \log n / \log \frac{1}{\varepsilon}.$$
iii. If \( p \geq c_0 \log n \), then

\[
k(n, p, \varepsilon) \gtrsim \varepsilon \log n / \log \frac{1}{\varepsilon}.
\]

where \( C, c, c_0 > 0 \) are absolute constants.

As one observes the dependence on \( \varepsilon \) in \( 1 \leq p \leq 2 \) is \( \varepsilon^2 \) as predicted by V. Milman’s proof (and its improvement by [11] and [27]). However, for \( p > 2 \) the dependence on \( \varepsilon \) is much better than \( \varepsilon^2 \) for all values of \( p \). This permits us to find sections of \( B^p_n \) of polynomial dimension which are closer to the Euclidean ball than previously obtained. Observe that Theorem 1.2 retrieves the right dependence on \( c(\varepsilon) \) at \( p = 1 \) (actually when \( p \) is fixed) and at \( p = \infty \).

The rest of the paper is organized as follows: In Section 2 we fix the notation, we give the required background material and we include some basic probabilistic inequalities. Gaussian functional inequalities as logarithmic Sobolev inequality, Talagrand’s \( L_1 - L_2 \) inequality and Pisier’s Gaussian inequality are also included.

Before the proof of Theorem 1.1 we prefer to deal with an easier problem first; the problem of determining the right order of the Gaussian variance of the \( \ell_p \) norm. We study this question in Section 3. This is a warm-up for the concentration result we will investigate in Section 4. The main techniques that we will use, as well as the main problems we have to resolve, will be apparent already in Section 3. This estimate will be used to obtain the dependence \( \log n / \log \frac{1}{\varepsilon} \) for \( p \leq c_0 \log n \), but still proportional to \( \log n \) in Theorem 1.2.

In Section 4 we present the proof of Theorem 1.1. Moreover, efforts have been made to provide lower estimates for the probability described in Theorem 1.1 (see also the Appendix by Tikhomirov).

In Section 5 we prove Theorem 1.2 and we show that in several cases the result is best possible up to constants.

We conclude in Section 6 with further remarks and open questions.

## 2 Notation and background material

We work in \( \mathbb{R}^n \) equipped with the standard inner product \( \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \), for \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in \( \mathbb{R}^n \). The \( \ell_p \)-norm in \( \mathbb{R}^n \) (\( 1 \leq p < \infty \)) is defined as:

\[
\|x\|_{\ell_p} \equiv \|x\|_p := \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \quad x = (x_1, \ldots, x_n)
\]

and for \( p = \infty \) as:

\[
\|x\|_{\ell_\infty} \equiv \|x\|_{\infty} := \max_{1 \leq i \leq n} |x_i|, \quad x = (x_1, \ldots, x_n).
\]

The Euclidean sphere is defined as: \( S^{n-1} = \{ x \in \mathbb{R}^n : \|x\|_2 = 1 \} \). The normed space \( (\mathbb{R}^n, \| \cdot \|_p) \) is denoted by \( \ell_p^n \), for \( 1 \leq p \leq \infty \) and its unit ball by \( B^n_p \), i.e. \( B^n_p = \{ x \in \mathbb{R}^n : \|x\|_p \leq 1 \} \).
\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q \text{,}

\text{for all } x \in \mathbb{R}^n. \text{ We write } \| \cdot \| \text{ for an arbitrary norm on } \mathbb{R}^n \text{ and } \| \cdot \|_A \text{ if the norm is induced by the centrally symmetric convex body } A \text{ on } \mathbb{R}^n. \text{ For any subspace } F \text{ of } \mathbb{R}^n \text{ we write: } S_F := S^{n-1} \cap F \text{ and } B_F := B_2^n \cap F.\]

The random variables in some probability space \((\Omega, \mathcal{A}, P)\) are denoted by \(\xi, \eta, \ldots\) while the random vectors by \(X = (X_1, \ldots, X_n)\) or simply \(X, Y, Z, \ldots\). The random vectors under consideration are going to be Gaussian unless it is stated otherwise. If \(\mu\) is a probability measure we write \(E\) and \(\text{Var}\) for the expectation and the variance respectively with respect to \(\mu\). If the measure is prescribed the subscript is omitted.

We shall make frequent use of the Paley-Zygmund inequality (for a proof see [3]):

**Lemma 2.1.** Let \(\xi\) be a non-negative random variable defined on some probability space \((\Omega, \mathcal{A}, P)\) with \(\xi \in L^2(\Omega, \mathcal{A}, P)\). Then,

\[P(\xi \geq tE\xi) \geq (1 - t)^2 \frac{(E\xi)^2}{E\xi^2},\]

for all \(0 < t < 1\).

Also the multivariate version of Chebyshev’s association inequality due to Harris will be useful:

**Proposition 2.2.** Let \(Z = (\zeta_1, \ldots, \zeta_k)\) where \(\zeta_1, \ldots, \zeta_k\) are i.i.d. random variables taking values almost surely in \(A \subseteq \mathbb{R}\). If \(F, G : A^k \subseteq \mathbb{R}^k \rightarrow \mathbb{R}\) are coordinatewise non-decreasing\(^2\) functions, then we have:

\[E[F(Z)G(Z)] \geq E[F(Z)]E[G(Z)].\]

Harris’ inequality can be derived from consecutive applications of Chebyshev’s association inequality and conditioning. For a detailed proof we refer the reader to [3]. For some measure space \((\Omega, \mathcal{E}, \mu)\) we write

\[\|f\|_{L^p(\mu)} := \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p}, \quad 1 \leq p < \infty,\]

for any measurable function \(f : \Omega \rightarrow \mathbb{R}\). If \(\mu\) is Borel probability measure on \(\mathbb{R}^n\) and \(K\) is a centrally symmetric convex body on \(\mathbb{R}^n\) we also use the notation

\[L(\mu, K) := \left( \int_{\mathbb{R}^n} \|x\|_K^r \, d\mu(x) \right)^{1/r}, \quad -n < r \neq 0,\]

\(^2\text{A real valued function } H \text{ defined on } U \subseteq \mathbb{R}^k \text{ is said to be coordinatewise non-decreasing if it is non-decreasing in each variable while keeping all the other variables fixed at any value.}
and for $r = 0$

$$I_0(\mu, K) := \exp\left(\int_{\mathbb{R}^n} \log \|x\|_K \, d\mu(x)\right).$$

If $\sigma$ is the (unique) probability measure on $S^{n-1}$ which is invariant under orthogonal transformations and $A$ is centrally symmetric convex body on $\mathbb{R}^n$, then we write:

$$(2.2) \quad M_q(A) := \left(\int_{S^{n-1}} \|\theta\|_A^q \, d\sigma(\theta)\right)^{1/q}, \quad q \neq 0.$$

For $q = 1$ we simply write $M(A) = M_1(A)$.

For the random version of Dvoretzky’s theorem recall V. Milman’s formulation from [20] (see also [21] or [1]) and see [11] and [27] for the dependence on $\varepsilon$:

**Theorem 2.3.** Let $A$ be a centrally symmetric convex body on $\mathbb{R}^n$. Define the critical dimension $k(A)$ of $A$ as follows:

$$(2.3) \quad k(A) = \frac{\mathbb{E}[\|Z\|_A^2]}{b^2(A)} \approx n \left(\frac{M(A)}{b(A)}\right)^2,$$

where $b(A)$ is the Lipschitz constant of the map $x \mapsto \|x\|_A$, i.e. $b = \max_{\theta \in S^{n-1}} \|\theta\|_A$ and $Z$ is a standard Gaussian $n$-dimensional random vector. Then, the random $k$-dimensional subspace $F$ of $(\mathbb{R}^n, \|\cdot\|_A)$ satisfies:

$$\frac{1}{(1 + \varepsilon)M}B_F \subseteq A \cap F \subseteq \frac{1}{(1 - \varepsilon)M}B_F$$

with probability greater than $1 - e^{-ck}$ provided that $k \leq k(A, \varepsilon)$, where $k(A, \varepsilon) \approx \varepsilon^2 k(A)$ and $M \equiv M(A)$.

Here the probability is considered with respect to the Haar probability measure $\nu_{n,k}$ on the Grassmann manifold $G_{n,k}$, which is invariant under the orthogonal group action.

With some abuse of terminology for a subspace $F$ of a normed space $(\mathbb{R}^n, \|\cdot\|)$ (or equivalently for a section $A \cap F$ of a centrally symmetric convex body $A$ on $\mathbb{R}^n$) we say that is $(1 + \varepsilon)$-spherical (or Euclidean) if:

$$\max_{\theta \in S_F} \|\theta\| / \min_{\theta \in S_F} \|\theta\| < 1 + \varepsilon \quad \text{or} \quad \max_{z \in S_F} \|z\|_A / \min_{z \in S_F} \|z\|_A < 1 + \varepsilon.$$

Thus, the previous theorem states that the random $k$-dimensional subspace of $(\mathbb{R}^n, \|\cdot\|_A)$ is $\frac{1}{(1 + \varepsilon)M}$-spherical with probability greater than $1 - e^{-ck}$ as long as $k \leq \varepsilon^2 k(A)$. In the next paragraph we provide asymptotic estimates for $k_{p,n} := k(p, n) \equiv k(B_p^n)$ in terms of $n$ and $p$. 


2.1 Gaussian random variables

If $g$ is a standard Gaussian random variable we set $\sigma_p^p := \mathbb{E}|g|^p$ for every $p > 0$. The next asymptotic estimate follows easily by Stirling’s formula:

$$\sigma_p^p = \mathbb{E}|g|^p = \frac{2^{p/2}}{\sqrt{\pi}} \left( \frac{p + 1}{2} \right) \sim \sqrt{2} \left( \frac{p}{e} \right)^{p/2}, \quad p \to \infty. \quad (2.4)$$

The $n$-dimensional standard Gaussian measure with density $(2\pi)^{-n/2} e^{-\|x\|^2/2}$ is denoted by $\gamma_n$. In the next Proposition, the asymptotic estimate (2.5) is a special case of a more general result from [30].

**Proposition 2.4.** Let $1 \leq p \leq \infty$ and let $Z$ be distributed according to $\gamma_n$. Then, we have:

$$\mathbb{E}\|Z\|_p = \int_{\mathbb{R}^n} \|x\|_p d\gamma_n(x) \approx \begin{cases} \frac{n^{1/p} \sqrt{p}}{\sqrt{\log n}}, & p < \log n \\ \frac{pn^{2/p}}{\log n}, & 2 \leq p \leq \log n \\ \frac{\log n}{p}, & p \geq \log n \end{cases}. \quad (2.5)$$

Therefore, for the critical dimension of $B^p_n$, we have:

$$k_{p,n} = k(B^p_n) = \begin{cases} n & 1 \leq p \leq 2 \\ pn^{2/p} & 2 \leq p \leq \log n \\ \log n & p \geq \log n \end{cases}. \quad (2.6)$$

We shall need Gordon’s lemma for Mill’s ratio from [10]:

**Lemma 2.5.** For any $a > 0$ we have:

$$\frac{a}{1 + a^2} \leq e^{a^2/2} \int_a^\infty e^{-t^2/2} dt \leq \frac{1}{a}. \quad (2.6)$$

Equivalently, we have:

$$1 \leq \frac{\Phi(a)}{a(1 - \Phi(a))} \leq 1 + \frac{1}{a^2}, \quad (2.7)$$

for $a > 0$, where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ and $\phi = \Phi'$.

The following technical lemma will be useful:

**Lemma 2.6.** Let $2 \leq p < \infty$ and let $g_1, g_2$ be i.i.d. standard normal variables. The following properties hold:

i. The function $t \mapsto P\left(\|g_1|^p - |g_2|^p > t\right)$ is log-convex in $(0, \infty)$.

ii. For any $r \geq 1$ we have:

$$\left(\mathbb{E}|g_1|^p - |g_2|^p\right)^{1/r} \approx r^{p/2} \sigma_p^p. \quad (2.8)$$
Proof. (i) Set \( H_p(t) := P \left( \| g_1 \|^p - \| g_2 \|^p > t \right) \). Then, we may check that:

\[
H_p(t) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{R}} H_p(x, t) \, d\gamma(x),
\]

where

\[
H_p(x, t) := \int_{\|x\|^p + t}^\infty e^{-\gamma/2} \, dy \quad (x, t) \in \mathbb{R} \times (0, \infty).
\]

We have the following:

Claim 1. For fixed \( x \in \mathbb{R} \), the map \( t \mapsto H_p(x, t) \) is log-convex on \( (0, \infty) \).

To this end it suffices to check that \( H_p(x, t) \geq (H_p(x, t))^2 / H_p(x, t) \) for all \( t > 0 \), equivalently:

\[
\int_{\|x\|^p + t}^\infty e^{-\gamma/2} \, dy \geq \exp \left( \frac{1}{2} (\|x\|^p + t)^{2/p} \right) \frac{(\|x\|^p + t)^{1/p}}{p - 1 + (\|x\|^p + t)^{2/p}}.
\]

The latter follows by \( (2.9) \) (for \( a = (\|x\|^p + t)^{1/p} \)) in Lemma \( \text{[2.4]} \).

The first assertion now follows by Hölder’s inequality.

(ii) The upper estimate is a consequence of the triangle inequality and the fact that \( \sigma_p \approx r^{p/2} \sigma_p \) (see estimate \( (2.4) \)). For the lower bound we have to elaborate more.

Using polar coordinates we may write:

\[
\mathbb{E}[\| g_1 \|^p - \| g_2 \|^p] = \frac{2^{\frac{p}{2}+2}}{\pi} \Gamma \left( \frac{p}{2} + 1 \right) \int_0^{\pi/4} \left( \cos^p \theta - \sin^p \theta \right)^r \, d\theta.
\]

We have the following:

Claim 2. For \( r \geq 1 \) we have:

\[
\int_0^{\pi/4} \left( \cos^p \theta - \sin^p \theta \right)^r \, d\theta \geq (2/3)^r / \sqrt{pr}.
\]

Indeed, we may write:

\[
\int_0^{\pi/4} \left( \cos^p \theta - \sin^p \theta \right)^r \, d\theta \geq \int_0^{\pi/6} \left( \cos^p \theta - \sin^p \theta \right)^r \, d\theta \geq \left( 1 - 3^{-p/2} \right)^r \int_0^{\pi/6} \left( \cos \theta \right)^p \, d\theta,
\]

where we have used the fact that \( \sin \theta \leq 3^{-1/2} \cos \theta \) for any \( \theta \in [0, \pi/6] \). Next, we have:

\[
\int_0^{\pi/6} \left( \cos \theta \right)^p \, d\theta = \frac{1}{2} B \left( \frac{pr + 1}{2}, \frac{1}{2} \right) - \int_0^{\pi/6} \left( \cos \theta \right)^p \, d\theta \geq \frac{1}{2} B \left( \frac{pr + 1}{2}, \frac{1}{2} \right) - \frac{2 \cos^{p+1}(\pi/6)}{pr + 1}.
\]

A standard approximation for the Beta function provides:

\[
B \left( \frac{pr + 1}{2}, \frac{1}{2} \right) \approx (pr)^{-1/2},
\]

and thus, the Claim 2 follows.

Finally, Stirling’s approximation formula yields \( 2^{p/2} \Gamma \left( \frac{p}{2} + 1 \right) \approx (pr)^{1/2} (pr/e)^{p/2} \) and the result follows. \( \Box \)
2.2 Functional inequalities on Gauss’ space

First we refer to the logarithmic Sobolev inequality. In general, if $\mu$ is a Borel measure on $\mathbb{R}^n$ it is said that $\mu$ satisfies a log-Sobolev inequality with constant $\rho$ if for any smooth function $f$ we have:

$$\text{Ent}_\mu(f^2) := \mathbb{E}_\mu(f^2 \log f^2) - \mathbb{E}_\mu f^2 \log(\mathbb{E}_\mu f^2) \leq \frac{2}{\rho} \int \|\nabla f\|_2^2 \, d\mu.$$  

It is well known (see [17]) that the standard $n$-dimensional Gaussian measure $\gamma_n$ satisfies the log-Sobolev inequality with $\rho = 1$. The next lemma, based on the classical Herbst’s argument, is a useful estimate which holds for any measure satisfying a log-Sobolev inequality:

**Lemma 2.7.** Let $\mu$ be a measure satisfying the log-Sobolev inequality with constant $\rho > 0$. Then, for any Lipschitz map $f$ and for any $2 \leq p < q$ we have:

$$\|f\|_{L^p(\mu)}^2 - \|f\|_{L^2(\mu)}^2 \leq \frac{\|f\|_{L^p(\mu)}^2}{\rho} (q - p). \tag{2.9}$$

In particular, we have:

$$\frac{\|f\|_{L^2(\mu)}}{\|f\|_{L^p(\mu)}} \leq \sqrt{1 + \frac{q - 2}{pk(f)}}, \tag{2.10}$$

for $q \geq 2$ where $k(f) := \|f\|_{L^2(\mu)}/\|f\|_{L^p(\mu)}^2$. Furthermore,

$$\frac{\|f\|_{L^p(\mu)}}{\|f\|_{L^q(\mu)}} \leq \exp\left(\frac{1}{p} - \frac{1}{2} \frac{1}{pk(f)}\right), \tag{2.11}$$

for $0 < p \leq 2$.

**Proof.** The proof of the first estimate is essentially contained in [31]. The second one is direct application of the first for $p = 2$. For the last assertion, note that by Lyapunov’s convexity theorem (see [12]) the map $p \mapsto \log \|f\|_p^2$ is convex. Moreover, we have: $p\phi'(p) - \phi(p) = \frac{\text{Ent}_\mu(f^2)}{\int |f|^p \, d\mu}$. Hence, for any $0 < p < 2$, the convexity of $\phi$ and the log-Sobolev inequality yield:

$$\frac{2\phi(2) - \phi(p)}{2 - p} \leq 2\phi'(2) = \frac{\text{Ent}_\mu(f^2)}{\|f\|_2^2} + \phi(2) \leq \frac{2}{2\rho k} + \phi(2),$$

where $k \equiv k(f)$. The result follows. \hfill \Box

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3Recall that for a Lipschitz map $f : (X, \rho) \to \mathbb{R}$ on some metric space $(X, \rho)$ the Lipschitz constant of $f$ is defined by $\|f\|_{Lip} = \sup_{x \neq y \in X} \frac{|f(x) - f(y)|}{\rho(x, y)}$. 

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Note 2.8. When \( f \) is a Lipschitz map with \( k(f) \geq 1 \), the above two estimates imply

\[
\frac{\|f\|_{L^q(A)}}{\|f\|_{L^1(A)}} \leq \sqrt{1 + c_1 \frac{q - 1}{k(f)}}, \quad q \geq 1.
\]

In the case \( A \) is a centrally symmetric convex body on \( \mathbb{R}^n \), integration in polar coordinates yields:

\[
I_r(y_n, A) = c_{n,r} M_r(A),
\]

where \( c_{n,r} := \sqrt{2} \frac{\Gamma\left(\frac{2-n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \) and \( M_r(A) := \int_{S^{n-1}} |\theta| r^m \, d\theta \). Applying this for \( A = B_2^n \) we readily see that \( c_{n,r} = I_r(y_n, B_2^n) \). Therefore, for \(-n < s < r\) we obtain:

\[
\max \left\{ \frac{M_r(A)}{M_s(A)} I_r(y_n, B_2^n) \right\} \leq \frac{M_r(A)}{M_s(A)} I_s(y_n, B_2^n) = I_r(y_n, A)/I_s(y_n, A).
\]

It follows that:

\[
M_q(A)/M_1(A) \leq \sqrt{1 + c_1 \frac{q - 1}{k(A)}}, \quad q \geq 1.
\]

This estimate improves considerably upon the estimate presented in \[18\] Statement 3.1 or \[17\] Proposition 1.10, (1.19) in the range \( 1 \leq q \leq k(A) \). For a purely probabilistic approach of this fact we refer the reader to \[24\].

It is immediate that

\[
\|f\|_{L^r(A)} \leq \begin{cases} \|f\|_{L^1(A)}, & 1 \leq r \leq k(f) \\ \sqrt{\frac{r}{k(f)}} \|f\|_{L^1(A)}, & r \geq k(f) \end{cases}
\]

for any Lipschitz function \( f \) in \( (\mathbb{R}^n, \gamma_n) \). In \[18\] it is proved that for norms this estimate can be reversed:

**Lemma 2.9.** Let \( \| \cdot \|_A \) be a norm on \( \mathbb{R}^n \). Then, we have:

\[
I_r(y_n, A) \asymp \begin{cases} I_r(y_n, A), & r \leq k(A) \\ \sqrt{\frac{r}{k(A)}} I_{1}(y_n, A), & r \geq k(A) \end{cases}
\]

This result implies the next well known fact:

**Proposition 2.10.** Let \( \| \cdot \| \equiv \| \cdot \|_A \) be a norm on \( \mathbb{R}^n \). Then, we have:

\[
c \exp(-C r^2 k) \leq P \left( \|X\| > (1 + t) \|\mathbb{E}[X]\| \right) \leq C \exp(-c r^2 k),
\]

for \( t \geq 1 \). Moreover, one has:

\[
\left( \mathbb{E}[\|X\|] - \mathbb{E}[\|X\|] \right)^{1/r} \asymp \sqrt{\frac{r}{k}} \mathbb{E}[\|X\|],
\]

for all \( r \geq k \), where \( k \equiv k(A) \) and \( X \) is a standard Gaussian \( n \)-dimensional random vector.
Sketch of proof of Proposition 2.10. Set \( I_r \equiv I_r(\gamma_n, A) \). There exists \( c_1 \in (0, 1) \) such that \( I_s \geq c_1 \sqrt{s/k} I_l \) for all \( s > k \) by Lemma 2.9. Thus, for \( t \geq 1 \), if we choose \( r > k \) by \( c_1 \sqrt{r/k} = 4t \), we may write:

\[
P\left( \|X\| > \frac{1}{2} I_r \right) \leq P\left( \|X\| > \frac{c_1}{2} \sqrt{r/k} I_l \right) \leq P(\|X\| \geq (1 + r) I_l).
\]

On the other hand, the Paley-Zygmund inequality (Lemma 2.1) yields:

\[
P\left( \|X\| > \frac{1}{2} I_r \right) \geq \left(1 - 2^{-r}\right)^2 \left( \frac{I_r}{I_2} \right)^{2r} \geq c_2 e^{-C_2 r} \geq c_2 \exp(-C_2 r^2 k),
\]

where we have also used the fact that \( I_r \approx I_2 \), which follows by Lemma 2.9. For the second assertion, we apply integration by parts and we use the first estimate. \( \square \)

The above estimate shows that the large deviation estimate for norms with respect to \( \gamma_n \) is completely settled. Therefore for the concentration inequalities we are interested in, we may restrict ourselves to the range \( 0 < \varepsilon < 1 \).

Other important functional inequalities related to the concentration of measure phenomenon are the Poincaré inequalities. Using a standard variational argument (see [17]) one can show that any measure which satisfies a log-Sobolev inequality with constant \( \rho \) also satisfies a Poincaré inequality with constant \( \rho \), i.e.

\[
\rho \text{Var}_{\mu}(f) \leq \int_{\mathbb{R}^n} \|\nabla f\|_2^2 \ d\mu,
\]

(2.16)

for any smooth function \( f \).

A refinement of the Poincaré inequality was proved by Talagrand in [32] for the discrete cube \( \{-1, 1\}^n \) (see also [3] for a recent exposition) and its continuous version, in the Gaussian context, was presented in [5] (see also [4]):

**Theorem 2.11** (Talagrand’s \( L_1 - L_2 \) bound). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth function. If \( A_i := \|\partial_i f\|_{L_2(\gamma_n)} \) and \( a_i := \|\partial_i f\|_{L_1(\gamma_n)} \), then one has:

\[
\text{Var}_{\gamma_n}(f) \leq C \sum_{i=1}^n \frac{A_i^2}{1 + \log(A_i/a_i)},
\]

where \( \partial_i f \) stands for the partial derivative \( \partial f / \partial x_i \).

This inequality will be used in order to prove concentration for the \( \ell_p \) norm when \( p \) is sufficiently large.

Pisier discovered in [26] another Gaussian inequality which contains the \((r, r)\)-Poincaré inequalities and the Gaussian concentration inequality as a special case (see Remarks 2.13).

**Theorem 2.12.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) be a convex function and let \( f : \mathbb{R}^n \to \mathbb{R} \) be \( C^1 \)-smooth. Then, if \( X, Y \) are independent copies of a Gaussian random vector, we have:

\[
\mathbb{E} \phi \left( f(X) - f(Y) \right) \leq \mathbb{E} \phi \left( \frac{\sqrt{2}}{2} \langle \nabla f(X), Y \rangle \right).
\]
Remarks 2.13. 1. \((r, r)-\text{Poincaré inequalities.}\) For \(\phi(t) = |t|^r,\) \(r \geq 1\) we get:

\[
\|f - \mathbb{E} f\|_{L^r(\gamma_n)} \asymp \left( \mathbb{E} \|f(X) - f(Y)^r\right)^{1/r} \leq \frac{\pi}{2} \sigma_r \left( \mathbb{E} \|\nabla f(X)\|_2^r \right)^{1/r}.
\]

In particular for \(r = 2\) we have \(\text{Var}(f(X)) \leq \frac{\pi}{8} \mathbb{E} \|\nabla f(X)\|_2^2,\) which is the Gaussian Poincaré inequality with non-optimal constant.

2. \(\text{Gaussian concentration.}\) The choice \(\phi(t) = \exp(\lambda t), \lambda > 0\) and a standard optimization argument on \(\lambda\) (see [26] for the details) yield:

\[
P(|f(X) - \mathbb{E} f(X)| > t) \leq 2 \exp(-t^2/(2\pi^2 \mathbb{E} \|f\|_{\text{Lip}}^2)),
\]

for all \(t > 0.\) Alternatively, we may conclude a similar estimate by equations (2.17) and Markov’s inequality.

2.3 Negative moments of norms

The next result is due to Klartag and Vershynin from [14] (see also [16] for a similar estimate as (2.20) with \(k(A)\) instead of \(d(A)):\)

**Proposition 2.14.** Let \(A\) be a centrally symmetric convex body on \(\mathbb{R}^n.\) We define:

\[
d(A) := \min \left\{ n, -\log \gamma_n \left( \frac{m}{2} A \right) \right\},
\]

where \(m\) is the median of \(x \mapsto \|x\|_{\text{A}}\) with respect to \(\gamma_n.\) Then, one has:

\[
\gamma_n (\{ x : \|x\|_{\text{A}} \leq c \varepsilon \mathbb{E} \|X\|_{\text{A}} \}) \leq (C \varepsilon)^{d(A)},
\]

for all \(0 < \varepsilon < \varepsilon_0\) where \(\varepsilon_0 > 0\) is an absolute constant. Moreover, for all \(0 < k < d(A)\) we have: \(I_{-k}(\gamma_n, A) \geq c I_1(\gamma_n, A).\) Note that \(d(A) > c_1 k(A).\)

Note that this result implies that the negative moments exhibit stable behavior up to the point \(d(A).\) However, one can show that up to the critical dimension the moments of any norm with respect to the Gaussian (or the uniform on the sphere) measure are almost constant, thus complementing the estimates (2.12) and (2.15). In order to quantify the latter we need the next consequence of Proposition 2.14.

**Lemma 2.15.** Let \(A\) be a centrally symmetric convex body on \(\mathbb{R}^n\) which satisfies the small ball probability estimate:

\[
\gamma_n (\varepsilon I_1 A) < (K \varepsilon)^{ad},
\]

for all \(0 < \varepsilon < \varepsilon_0 (K, a > 0).\) Then, for all \(r, s > 0\) with \(r + s < ad/3\) we have:

\[
I_{r-s}(\gamma_n, A) \leq \left( \frac{CK}{I_1} \right)^s I_r(\gamma_n, A),
\]

where \(I_1 = I_1(\gamma_n, A)\) and \(C > 0\) is an absolute constant.
Proof. We set \( I_q = I_q(\gamma_n, A) \). For any \( 0 < \varepsilon < \varepsilon_0 \) we may write:

\[
I_{r-s}^{r-}\gamma = \int \frac{1}{|x|^n} \, d\gamma(x) \leq \frac{1}{(\varepsilon I_A)^{r-s}} \int \frac{1}{|x|^n} \, d\gamma(x) + \int \frac{1}{J_{2s} I_A |x|^s} \, d\gamma(x)
\]

\[
\leq \frac{1}{(\varepsilon I_A)^{r-s}} + (Ke)^{ad/2} I_{r-s}^{r-s} I_{2(r-s)},
\]

by the Cauchy-Schwarz inequality. Note that the small ball probability assumption implies that: \( I_{r-s} \geq c_0 I_1 \) for all \( 0 < s < 2d/3 \). Thus, if \( r + s < ad/3 \) we get \( I_{2(r-s)} > c_1 I_{r-s} \) and previous estimate yields:

\[
I_{r-s}^{r-s} < \frac{1}{(\varepsilon I_A)^{r-s}} (Ke)^{ad/2} c_1^{-r} I_{r-s}^{r-s}.
\]

Choosing \( \varepsilon \) small enough so that \( (Ke)^{ad/2} < c_1^{-r} / 2, \) say \( 0 < \varepsilon \leq c_1/(2K) \), we conclude the result. \( \square \)

**Theorem 2.16.** Let \( A \) be a centrally symmetric convex body on \( \mathbb{R}^n \). Then, one has:

\[
\frac{I_q(\gamma_n, A)}{I_q(\gamma_n, A)} \leq 1 + \frac{Cr}{k(A)},
\]

for all \( 0 < r < ck(A) \), where \( C, c > 0 \) are absolute constants.

Proof. We present the argument in two steps:

1. **Step 1.** (positive moments). We use the log-Sobolev inequality to estimate the growth of the moments. The basic observation is that:

\[
\frac{d}{dr} \left( \log \|f\|_{L_p(\nu)} \right) = \frac{\text{Ent}_r(\|f\|)}{r^2 \|f\|_{L_p(\nu)}^2},
\]

for any Lipschitz function \( f \). Apply this for the function \( f = \| \cdot \|_A \) to get:

\[
(\log I_1)^r \leq \frac{1}{2rP} \|X\|_{L_p}^2 \| \nabla \|X\|_A \| \leq \frac{b^2}{2P} I_{-2}^{r-2},
\]

for all \( r > 0 \), where \( b = b(A) \) the Lipschitz constant of \( \| \cdot \|_A \). It is easy to see that \( (\log I_1)^r \leq \frac{1}{2rP} \) for \( r \geq 2 \), while for \( 0 < r < 2 \) we may write:

\[
(\log I_1)^r \leq \frac{b^2}{2P} I_{-2}^{r-2} \leq \frac{C b^2}{P^2} \leq \frac{C_1}{k(A)},
\]

where we have used Proposition 2.14. Using 2.21 we may write:

\[
\log(I_1/I_0) = \int_0^r (\log I_1)^r dt \leq \int_0^\infty \frac{C_1}{k} dt = \frac{C r}{k},
\]

for all \( r > 0 \).
Step 2. (negative moments). As before, using the log-Sobolev inequality, for all $0 < r < c_1 d(A)$ we may write:

\[(\log I_r)' \geq -\frac{b^2}{2I_r} I_{r,-2} \geq -\frac{C_2 b^2}{I_1^2} \geq -\frac{C_2}{k(A)},\]

where we have used Lemma 2.15. The same reasoning applied to (2.22) shows that

\[\log (I_r/I_0) \geq -C_2 k(A),\]

for all $0 < r < c_1 d(A)$. Combining the two steps and restricting to $0 < r < c_2 k(A)$ we conclude the result. \(\square\)

3 The Gaussian variance of the $\ell_p$ norm

A standard method for bounding the variance is the concentration inequality (2.18), e.g. see [18] or [17, Proposition 1.9]. An integration by parts argument implies that if $f : \mathbb{R}^n \to \mathbb{R}$ is a $L$-Lipschitz function, then $\text{Var}(f) \leq L^2$. In particular, if $f(x) = \|x\|_p$ this estimate yields:

\[\text{Var}(\|X\|_p) \leq b^2 (B^n_p) = \max\{n^{2/p-1}, 1\}, \quad 1 \leq p \leq \infty.\]

For $1 \leq p \leq 2$ this estimate turns out to be the correct one. But, for $2 < p \leq \infty$ this method gives bounds which are far from the actual ones. The purpose of this Section is to compute the correct order of magnitude for the Gaussian variance of the $\ell_p$ norm. Our first approach lies in determining the limit distribution of the sequence of variables $(\|g\|_{\ell^n_p})_{n=1}^{\infty}$. Here $\|g\|_{\ell^n_p}$ stands for the $\ell_p$ norm of the $n$-dimensional “truncation” of the sequence $(g_j)_{j=1}^{\infty}$ of i.i.d. standard Gaussian random variables, i.e. $\|g\|_{\ell^n_p} := (\sum_{j=1}^{n} |g_j|^p)^{1/p}$.

3.1 The variance of the $\ell_p$ norm for $1 \leq p < \infty$

In this case we use the next Proposition known in Statistics as the “Delta Method” (for a proof see [18]):

**Proposition 3.1.** Let $\theta, \sigma \in \mathbb{R}$ and let $(Y_n)$ be a sequence of random variables that satisfies $n^{1/2}(Y_n - \theta) \rightarrow N(0, \sigma^2)$ in distribution. For the differentiable function $h$ assume that $h'(\theta) \neq 0$. Then,

\[n^{1/2}(h(Y_n) - h(\theta)) \rightarrow N(0, \sigma^2(h'(\theta))^2)\]

in distribution.

Now we may prove the next asymptotic estimate:

**Theorem 3.2.** Let $1 \leq p < \infty$. Let $(\xi_j)_{j=1}^{\infty}$ be sequence of i.i.d random variables with $m^{3p}_{\|\cdot\|_{\ell_p}} := \mathbb{E}|\xi|^3_{\ell^p} < \infty$. Then, there exist positive constants $c_p, C_p$ depending only on $p$ and the distribution of $(\xi_j)$ such that:

\[c_p n^{2/p-1} \leq \text{Var}(\|\xi\|_{\ell_p}) \leq C_p n^{2/p-1},\]

for all $n$, where $\|\xi\|_{\ell_p}^p = \sum_{j=1}^{n} |\xi_j|^p$. 



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Proof. Let $Y_n := \frac{1}{n} \sum_{j=1}^{n} |\xi_j|^p$. Then by the Central Limit Theorem we know that:

$$\sqrt{n}(Y_n - m_p) \longrightarrow N(0, \nu_p^2)$$

in distribution, where $\nu_p^2 := \text{Var}(\xi_1|^p)$. Consider the function $h(t) = t^{1/p}$, $t > 0$ and apply Proposition 3.1 to get:

$$\xi_n := \sqrt{n}(n^{-1/p}||\xi||_p - m_p) \longrightarrow N\left(0, \frac{\nu_p^2}{p^2}m_p^{2(1-p)}\right),$$

in distribution. Using the fact that $m_{3p} < \infty$ we may conclude the uniform integrability of $(\xi_n^2)^{\infty}_{n=1}$.

Claim. For all $n \geq 1$ we have:

$$\mathbb{E}|\xi_n|^3 \leq m_{3p}'/m_p^{3(p-1)}.$$

Proof of Claim. We may write:

$$\mathbb{E}|\xi_n|^3 = n^{\frac{3}{2}-\frac{3}{p}} \mathbb{E}||\xi||_p - n^{1/p}m_p||^3 \leq \frac{n^{\frac{3}{2}-\frac{3}{p}}}{(n^{1/p}m_p)^{3(p-1)}} \mathbb{E}||\xi||_p^n - nm_p^3 \leq \frac{n^{3/2}}{m^{3(p-1)}} \mathbb{E}||\xi||_p^n - ||\xi'||_p^3,$$

where $\xi'$ is an independent copy of $\xi$ and we have also used the numerical inequality $a^{p-1}|z-a| \leq |z^p - a^p|$ for $z \geq 0, a > 0, p \geq 1$ and Jensen’s inequality. Finally, a standard symmetrization argument yields:

$$\mathbb{E} \left| \sum_{j=1}^{n} (|\xi_j|^p - |\xi'_j|^p) \right|^3 \leq \mathbb{E} \left| \sum_{j=1}^{n} (|\xi_j|^p - |\xi'_j|^p)^2 \right|^{3/2} \leq n^{3/2} \mathbb{E}||\xi||_p^n - ||\xi'||_p^3 \leq n^{3/2}m_{3p}'^3,$$

where we have also used Jensen’s inequality, again. This proves the claim.

Hence, we may conclude:

$$(3.1) \quad n^{1-\frac{3}{2}} \text{Var}(||\xi||_p) = \text{Var}\left(n^{1-\frac{3}{2}}||\xi||_p \right) = \text{Var}\left[ \sqrt{n}(n^{-1/p}||\xi||_p - m_p) \right] \longrightarrow \frac{\nu_p^2}{p^2}m_p^{2(1-p)},$$

as $n \to \infty$ and the result follows. □

Remark 3.3. The reader should notice that, for fixed $p \geq 1$, the dependence we obtain on the dimension is the same regardless the randomness we choose for the underlying variables $(\xi_i)$. In addition the argument is essentially based on the stochastic independence. Moreover, in the case that $(\xi_i)$ are standard normals, the above limit value is estimated as:

$$(3.2) \quad \frac{\nu_p^2}{p^2}m_p^{2(1-p)} \sim \frac{1}{e\sqrt{2}} \frac{2^p}{p}, \quad p \to \infty.$$

This suggests that the constants $c_p, C_p$ should depend exponentially on $p$. 

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3.2 The variance of the $\ell_\infty$ norm

Of course the variance in that case can be computed by employing the tail estimates for the $\ell_\infty$-norm proved in [23]. However, we prefer here to give a proof of a more “probabilistic flavor”. Actually, the argument we present below works for all i.i.d. random variables with exponential tails, but we shall focus on Gaussians. Let $(g_i)_{i=1}^\infty$ be independent, standard Gaussian random variables and let $Y_n := \max_{i \leq n} |g_i|, \ n \geq 2$. We set $a_n := -\Phi^{-1}(\frac{1}{2n}) > 0$. Note that $a_n \to \infty$ and Gordon’s inequality (2.7) shows that $a_n \sim \sqrt{2\log n}$ as $n \to \infty$. We define $W_n := a_n(Y_n - a_n)$ and we have the next well known fact (see [7, §9.3]):

**Proposition 3.4.** Let $\eta$ be a Gumbel random variable, that is the cumulative distribution function of $\eta$ is given as:

$$F_\eta(t) := \exp(-e^{-t}), \ t \in \mathbb{R}.$$  

If $(W_n)$ is the sequence defined above, then for every $t \in \mathbb{R}$ we have:

$$\mathbb{P}(W_n \leq t) \to \exp(-e^{-t}),$$

that is $W_n$ converges to the Gumbel variable in distribution.

For the random variable $\eta$ it is known that $\mathbb{E}(\eta) = \gamma$ (the Euler-Mascheroni constant) and $\text{Var}(\eta) = \pi^2/6$. Therefore, we obtain:

$$a_n^2 \text{Var}(Y_n) = \text{Var}(W_n) \to \text{Var}(\eta),$$

as $n \to \infty$. This proves the following:

**Theorem 3.5.** If $Z$ is an $n$-dimensional standard Gaussian random vector, we have:

$$\text{Var} \|Z\|_{\ell_\infty} = \text{Var}_{\gamma_n} \|x\|_{\ell_\infty} \simeq (\log n)^{-1}.$$  

It should be noticed that the dependence on dimension we get for fixed $1 \leq p < \infty$ is polynomial in $n$ while for $p = \infty$ is logarithmic in $n$. As we have already explained this “skew” behavior relies on the fact that as $p$ grows, the constants in the equivalence should be expected to be exponential in $p$ (see (3.1) and (3.2)). In the rest of the paragraph we try to study and quantify this phenomenon. Our aim is to give as sharp bounds as possible and describe the behavior of $p$ along $n$, too.

3.3 Tightening the bounds

The purpose of this subsection is to provide continuous bounds in terms of $p$ for the variance of the $\ell_p$ norm when dimension $n \to \infty$ and $p$ varies from 1 to $\infty$ (along with $n$). One can easily see that:

$$c_1p \leq n^{1-2/p} \text{Var}\|X\|_p \leq c_2p \text{Var}|g|_p \approx p(2p/e)^p,$$

by just comparing with the variance of the $\ell_2$ norm and the $p$-th power of the $\ell_p$ norm. Below, we show that one can always have better estimates. In order to prove these estimates we will use the following:
Lemma 3.6. Let $4 \leq p \leq \infty$. Then one has:

$$I_r(y_n, B_p^n)/I_r(y_n, B_p^n) \leq \exp \left( \frac{C}{r} \log n \right), \quad 0 < r < c_1 \sqrt{k_{p,n} \log n},$$

where $k_{p,n} = k(B_p^n)$.

We postpone the proof of this Lemma to Section 4 (Theorem 4.9).

3.3.1 Upper bound (via Talagrand’s inequality)

For $p > 1$ we have: $\partial_i\|x\|_p = \frac{|x_i|^{p-1}}{\|x\|_p} \text{sgn}(x_i)$ a.s. Thus, one has:

$$A^2 := \left\| \partial_i \cdot \|p \right\|^2 \leq \sigma^{2(p-2)} r^{-2(p-1)}(y_n, B_p^n), \quad a := \left\| \partial_i \cdot \|p \right\| \leq \sigma^{-1} r^{-1} r^{-1}(y_n, B_p^n).$$

Set $I_r(y_n, B_p^n) \equiv I_r$. Thus, direct application of Theorem 2.1 yields:

$$\text{Var}(\|X\|_p) \leq Cn \frac{\sigma^{2(p-2)} r^{-2(p-1)} \theta^{2(p-1)}(2p-1)}{1 + \log \frac{\sigma^{2(p-2)} \theta^{2(p-1)}}{\sigma^{2(p-2)} \theta^{2(p-1)}}} \leq Cn \frac{\sigma^{2(p-2)} \theta^{2(p-1)}(2p-1)}{p},$$

where we have used the fact that $(\sigma^{2p-2} / \sigma^{p-1})^{p-1} \approx 2^{p}$, which follows by (2.4). As long as $2p < c_1 \sqrt{k_{p,n} \log n}$, which is satisfied when $p \leq c_0 \log n$ for some sufficiently small absolute constant $c_0 > 0$ in view of Proposition 2.4, we may apply Lemma 3.6 to get:

$$r^{2(p-1)} \geq e^{-\frac{c_p^2}{n \log n}} \theta^{p-1} \geq c_1 \theta^{2(p-1)}(n-1)^{2-2/p}.$$ 

Plug this estimate in (3.3) we derive the upper bound:

$$\text{Var}(\|X\|_p) \leq C_2 \frac{\sigma^{2(p-2)} \theta^{2(p-1)}(2p-1)}{\sigma^{2(p-2)} \theta^{2(p-1)}} \leq \frac{2^p}{p} n^{2/p-1}.$$ 

Note that this is exactly of the same order as the one we obtained at the limit value using the Delta Method.

3.3.2 Lower bound (via Talagrand’s inequality)

Here we will use the next numerical result:

Lemma 3.7. Let $a, b > 0$ and $0 < \theta \leq 1$. Then, we have:

$$\theta |a - b| \left( \frac{2}{a + b} \right)^{1-\theta} \leq |a^\theta - b^\theta| \leq \theta |a - b| \frac{a^{\theta - 1} + b^{\theta - 1}}{2}.$$
Proof. We may assume without loss of generality that \(0 < a < b\) and \(0 < \theta < 1\). If we set \(f(t) = t^{p-1}\), \(t > 0\), note that \(f\) is convex in \([a, b]\), hence the estimate follows by the Hermite-Hadamard inequality (see [12]). \(\square\)

Applying the lower bound of Lemma 3.7 for \(a = ||X||_p^p\), \(b = ||Y||_p^p\) and \(\theta = 1/p\), where \(X, Y\) are independent and \(X, Y \sim N(0, I_n)\), we obtain:

\[
2 \operatorname{Var}[|X|_p] = E(|X|_p - |Y|_p)^2 \geq \frac{2^{2/q}}{p^2} \frac{E(|X|_p - |Y|_p)^2}{E(|X|_p + |Y|_p)^{2/q}} \geq \frac{1}{p^2} \sum_{i=1}^n \frac{|X|_p^p - |Y|_p^p}{S^{2/q}},
\]

where \(q\) is the conjugate exponent of \(p\), i.e. \(1/p + 1/q = 1\) and

\[
S := ||X||_p^p + ||Y||_p^p = ||Z||_p^p, \quad Z = (Z_1, \ldots, Z_{2n}) \sim N(0, I_{2n}).
\]

Now we observe that the variables \(\eta_j := \frac{|X|_p^p - |Y|_p^p}{S^{2/q}}\) have the same distribution and satisfy \(E(\eta_1^2) = 0\) for \(i \neq j\). Therefore, we have:

\[
E \left| \sum_{i=1}^n \frac{|X|_p^p - |Y|_p^p}{S^{2/q}} \right|^2 = E \left| \sum_{i=1}^n \eta_i \right|^2 = \sum_{i=1}^n E\eta_i^2 = nE\eta_1^2.
\]

Hence, estimate (3.4) becomes:

\[
\operatorname{Var}[|X|_p] \geq \frac{n}{2p^2} \left( \frac{E(|X|_p^p - |Y|_p^p)^2}{S^{2/q}} \right) = \frac{n}{p^2} \left( \frac{E|X|_p^p|Y|_p^p}{S^{2/q}} - \frac{E|X|_p^p|Y|_p^p}{S^{2/q}} \right).
\]

Let \(T := \sum_{i=1}^n |X|_p^p + \sum_{i=1}^n |Y|_p^p\). Note that \(T \leq S\), thus we obtain:

\[
\operatorname{Var}[|X|_p] \geq \frac{n}{p^2} \left( \frac{E|Z|_p^{2p}}{S^{2/q}} - \sigma_p^{2p}E(T^{-2/q}) \right).
\]

An application of Lemma 3.6 yields

\[
E(T^{-2/q}) \leq \frac{1}{\sigma_p^{2p} (n-1)^{2-2/p}},
\]

as long as \(p \leq c_0 \log n\). For the term \(E|Z|_p^{2p}/S^{2/q}\), we may write:

\[
E|Z|_p^{2p}/S^{2/q} = (2n)^{-1} E|Z|_p^{2p}/S^{2/q} = (2n)^{-1} E|Z|_p^{2p}/|Z|_p^{2p}\geq \frac{1}{2n E|Z|_p^{2p}/|Z|_p^{2p-1}},
\]

where we have used that the variables \(|Z|_p^{2p}/S^{2/q}\) are equidistributed and the Cauchy-Schwarz inequality. Now by using Lemma 3.8 again we obtain:

\[
E|Z|_p^{2p} \leq e^{\frac{\alpha^2}{2}} (E|Z|_p^{2p})^2 \leq C_1 (E|Z|_p^{2p})^2.
\]

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and similarly we have: \( \mathbb{E}[\|Z\|_{p}^{2(p-1)}] \leq C_{2}(\mathbb{E}[\|Z\|_{p}]^{2(p-1)/p}) \), as long as \( p \leq c_{0} \log n \). Therefore, we get:

\[
(3.7) \quad \frac{\mathbb{E}[\|Z\|_{p}^{2p}]}{S^{2/q}} \leq \frac{c_{3}}{n} \frac{\mathbb{E}[\|Z\|_{p}^{2p}]}{(\mathbb{E}[\|Z\|_{p}]^{2(p-1)/p})^{2}} \leq \frac{\sigma_{2p}^{2p}}{n^{2-2/p}(\sigma_{p}^{2p})^{2}}.
\]

Inserting (3.6) and (3.7) in (3.5) we get:

\[
\text{Var}[\|X\|_{p}] \geq \frac{c_{4}n}{p^{2}} \left[ c_{5} \frac{\sigma_{2p}^{2p}}{n^{2-2/p}(\sigma_{p}^{2p})^{2}} - c_{6} \frac{\sigma_{p}^{2p}}{n^{2-2/p}} \right] = \frac{c_{4}c_{5}\sigma_{p}^{2p}}{p^{2}n^{-2/p}} \left[ \frac{\sigma_{2p}^{2p}}{\sigma_{p}^{2p}} - c_{6} \right] .
\]

Taking into account that \((\sigma_{2p}/\sigma_{p})^{2p} \approx 2^p\) we may conclude:

\[
(3.8) \quad \text{Var}(\|X\|_{p}) \geq \frac{c_{7}2^p}{p} n^{2/p-1},
\]

provided that \( p \) is greater than some large absolute constant.

Finally, for much larger values of \( p \), namely for \( p \geq c_{0} \log n \), we employ Theorem 2.11 again. This is an extension of the known argument for \( \ell_{\infty} \), which can be found in [4]. As before, if \( a_{i} := \|\partial_{i} f\|_{L_{1}(\gamma)} \) we may write:

\[
a_{i} = \int_{\mathbb{R}^{d}} \frac{|x|^{p-1}}{\|x\|_{p}} d\gamma_{\alpha}(x) = \frac{1}{n} \int_{\mathbb{R}^{d}} \left( \frac{\|x\|_{p-1}}{\|x\|_{p}} \right)^{p-1} d\gamma_{\alpha}(x) \leq \frac{n^{1/p}}{n} = n^{-1/q},
\]

where in the last step we have used estimate (2.4) and \( q \) is the conjugate exponent of \( p \). Moreover, we have:

\[
A_{i}^{2} := \|\partial_{i} f\|_{L_{2}(\gamma)}^{2} = \int_{\mathbb{R}^{d}} \frac{|x|^{2p-2}}{\|x\|_{p}^{2p-2}} d\gamma_{\alpha}(x) = \frac{1}{n} \int_{\mathbb{R}^{d}} \left( \frac{\|x\|_{p-2}}{\|x\|_{p}} \right)^{2p-2} d\gamma_{\alpha}(x) \leq 1/n,
\]

by the estimates (2.4) again. These bounds and Theorem 2.11 yield:

\[
(3.9) \quad \text{Var}(\|X\|_{p}) \leq C \sum_{i=1}^{n} \frac{A_{i}^{2}}{1 + \frac{q}{\log n} + \log A_{i}} \leq \frac{C}{\log n}, \quad p \geq c_{0} \log n
\]

where we have used the monotonicity of \( t \mapsto \frac{t^{2}}{1 + \frac{q}{\log n} + \log t} \) and that \( q \ll 2 \).

Finally, let us note that the variance of the \( \ell_{p} \) norm stabilizes around \( \frac{1}{\log n} \) for \( p > (\log n)^{2} \). This is a special case of the next reverse concentration estimate:

**Proposition 3.8.** Let \( p > (\log n)^{2} \) and let \( X \) be an \( n \)-dimensional standard Gaussian random vector. Then we have:

\[
P \left( \|X\|_{p} - \mathbb{E}[\|X\|_{p}] > c \mathbb{E}[\|X\|_{p}] \right) \geq ce^{-C_{p} \log n},
\]

for all \( 0 < c < 1 \), where \( C, c > 0 \) are absolute constant. In particular, we have:

\[
\text{Var}[\|X\|_{p}] \approx \frac{1}{\log n}.
\]
Proof. Consider \( \frac{2}{\log n} < \varepsilon < 1 \) and write:

\[
P(||X||_p > (1 + \varepsilon)||X||_\infty) \geq P(||X||_\infty > (1 + \varepsilon)n^{1/p}||X||_\infty) \geq P(||X||_\infty > (1 + 2\varepsilon)||X||_\infty) > ce^{-C\varepsilon \log n},
\]

where we have used (2.1) and at the last step the concentration from [28]. Hence,

\[
P \left( ||X||_p - E||X||_p \right) \geq \varepsilon e^{-C\varepsilon \log n},
\]

for all \( 0 < \varepsilon < 1 \). For the second assertion we may write:

\[
\operatorname{Var} ||X||_p = 2(E||X||_p)^2 \int_0^\infty tP \left( ||X||_p - E||X||_p > tE||X||_p \right) dt \geq 2c'(E||X||_p)^2 \int_0^1 te^{-Ct \log n} dt \geq \frac{(E||X||_p)^2}{(\log n)^2}.
\]

The result follows by Proposition 2.4. \( \square \)

The results of this paragraph can be summarized in the next:

Theorem 3.9. There exist absolute constants \( c_0, c_1, C_1 > 0 \) with the following property: For all \( n \) large enough and for any \( 1 \leq p \leq c_0 \log n \) we have:

\[
(3.10) \quad \frac{2^p}{p} c_1 \frac{2^p}{p} \leq n^{1-\frac{1}{p}} \operatorname{Var} ||X||_p \leq C_1 \frac{2^p}{p}.
\]

If \( p > c_0 \log n \) then we have:

\[
(3.11) \quad \operatorname{Var} ||X||_p \leq \frac{C_1}{\log n},
\]

whereas for \( p \geq (\log n)^2 \) we also have:

\[
(3.12) \quad \operatorname{Var} ||X||_p \geq \frac{c_1}{\log n},
\]

where \( X \sim N(0, I_n) \).

Note 3.10. While this paper was under review, Tikhomirov [36] improved Proposition 3.8 by extending the range to \( p \geq c_0 \log n \) (his proof gives \( C_0 = 12 \)). In particular, \( \operatorname{Var} ||X||_p \geq (\log n)^{-1} \) for \( p \geq c_0 \log n \). We present his argument in the Appendix. This only leaves a relatively small interval \( (c_0 \log n, C_0 \log n) \), for which the behavior of the variance is not exactly determined. In other words we are not aware for which constant \( c_1 > 0 \) the phase of transition from polynomial to logarithmic behavior occurs. Our bounds strongly suggest that the value of this constant seems plausible to be \( c_1 = 1/\log 2 \).
We close this section with some discussion on the methods used for bounding the variance. If we are interested in giving sufficient upper bounds, we may use the Poincaré inequality \[ \text{(2.10)} \] which estimates the variance by the $L_2$ average of the Euclidean norm of the gradient of $f$. In principle the latter average is smaller than the Lipschitz constant: $\|\nabla f\|_2 \leq ||\nabla f||_{L_\infty(\gamma_n)} = L$. The reader may check that for $2 < p < \infty$ and $f = \| \cdot \|_p$, we have:

$$\int_{\mathbb{R}^n} \| \nabla f(x) \|_2^p \, d\gamma_n(x) = \int_{\mathbb{R}^n} \| x \|_{2p-2}^p \, d\gamma_n(x) \approx_p \frac{1}{n^{1-\gamma}} \ll 1 = b^2(B^n_2) \equiv \text{Lip}(f)^2.$$ 

In case $p = \infty$ we have $\| \nabla \|_{\infty} \|_2 \equiv 1$ a.e., hence:

$$\int_{\mathbb{R}^n} \| \nabla \|_{\infty} \|_2^2 \, d\gamma_n(x) = 1 = b(B^n_\infty).$$

Thus, the Poincaré inequality also fails to give the sharp upper bound for the variance in this case. The recovery of the correct estimate is promised by the different order of magnitude for the $L_1-L_2$ norms of the partial derivatives of $x \mapsto \|x\|_\infty$ and Talagrand’s inequality (see \[ 3 \] for the details). The phenomenon that $\text{Var}(\|X\|_{\infty}) \approx 1/\log n$ while $\mathbb{E}(\|\nabla \|_{\infty})^2 \approx 1$ is referred to super-concentration following \[ 41 \]. For recent results on the related subject see \[ 33 \].

### 4 Gaussian concentration for $\ell_p$ norms

In this Section we study the Gaussian concentration for the $\ell_p$-norms for $1 \leq p \leq \infty$. First we show how we may employ the log-Sobolev inequality in order to get concentration results.

#### 4.1 An argument via the log-Sobolev inequality

Note that for the $\ell_p$ norm with $1 \leq p \leq 2$ the estimate \[ \text{(2.12)} \] implies:

$$\frac{\mathbb{L}_r(\gamma_n, B^n_p)}{\mathbb{L}_r(\gamma_n, B^n_p)} \leq \sqrt{1 + \frac{C_1 r}{k(B^n_p)}} \leq \exp \left( \frac{C_2 r}{n} \right),$$

for all $r \geq 1$. Therefore, for any $0 < \varepsilon < 1$ we apply Markov’s inequality to get:

$$P(\|X\|_p > (1 + \varepsilon)I_1) \leq P(\|X\|_p > e^{\varepsilon/2}I_1) \leq e^{-\varepsilon^2/2}I_1 \leq \exp(-\varepsilon^2/2 + C_2^2/n).$$

Choosing $r = en/(4C_2)$ (as long as $\varepsilon > 4C_2/n$) we obtain:

$$P(\|X\|_p > (1 + \varepsilon)I_1) \leq \exp \left( -\frac{1}{16C_2^2} \varepsilon^2 n \right).$$

Taking into account Theorem \[ 2.16 \] and arguing similarly we find:

$$P(\|X\|_p < (1 - \varepsilon)I_1) \leq \exp(-c_2\varepsilon^2 n).$$
Proposition 4.1. Let $2 < p < c \log n$. Then, for every $r > 0$ we have:

$$\frac{d}{dr} (\log I_r) \leq \frac{C_p}{n} \left( 1 + \frac{r}{k(B^n_{2p-2})} \right)^{p-1} \leq \begin{cases} \frac{C_p}{n}, & 0 < r \leq k(B^n_{2p-2}) \\ \frac{C_p}{r} \left( \frac{C_1 r}{k(B^n_p)} \right)^p, & k(B^n_{2p-2}) \leq r < k(B^n_p)/C_1, \end{cases}$$

while for $0 < r < cd(B^n_p)$ we have:

$$-\frac{d}{dr} (\log I_{-r}) \leq \frac{C_p}{n},$$

where $c, C, C_1 > 0$ are absolute constants and $I_s \equiv I_s(\gamma_n, B^n_p)$.

**Proof.** First we prove the growth condition on the positive moments. Our starting point is the next estimate:

$$\frac{d}{dr} (\log I_r) = \frac{1}{r^2 I_r} \text{Ent}_{I_r}(\|x\|^{2p}) \leq \frac{2}{r^2 I_r} E \|\nabla(\|X\|^{p/2})\|^2 \leq \frac{1}{2I_r} E \|X\|^{2p-2} \|X\|^{-2p},$$

where we have used the log-Sobolev inequality. We distinguish two cases:

**Case 1:** $0 < r \leq 2p$. We may write:

$$\frac{d}{dr} (\log I_r) \leq \frac{n}{2I_r} \frac{\|X\|^{2p-2}}{\|X\|_{\frac{2p}{p-r}}} \leq \frac{n(c p)^{p-1}}{I_r(B^n_p)^{\frac{p}{2p-r} - (B^n_p)^{\frac{2p-r}{2p}}}} \leq \frac{n(c p)^{p}}{I_r^2(B^n_p)^{\frac{p}{2p}}},$$

by Proposition 2.14 and Hölder’s inequality. By Proposition 2.14 for $0 < s < c_1 k_{p,n}$ we have: $L_s \geq c_2 I_1$. Since, $p < c_1 k_{p,n}$ for $p \leq \log n$ we get: $(\log I_r)' \leq C_p/n$.

**Case 2:** $r > 2p$. We may write:

$$\frac{d}{dr} (\log I_r) \leq \frac{1}{2I_r} E \|X\|^{2p-2} \|X\|^{-2p} \leq \frac{I_r^{2p-2}(\gamma_n, B^n_{2p-2})}{2I_r^2}.$$
Proposition 4.2. Let \( \varepsilon \) follow. Then, for all \( 0 < \varepsilon < 1 \) we get:
\[
\frac{d}{dr} (\log L_r) \leq \frac{\int_{-\varepsilon}^{\varepsilon} r^{-2p} \, dr}{2I_p} \left( 1 + \frac{r}{k_{2p-2,n}} \right)^{p-1} = \frac{\sigma_{2p-2,p}^2}{2n} \left( 1 + \frac{r}{k_{2p-2,n}} \right)^{p-1} \leq \frac{C_3}{n} \left( 1 + \frac{r}{k_{2p-2,n}} \right)^{p-1},
\]
for some absolute constant \( C_3 > 0 \).

Now we turn to providing bounds for the negative moments. Here the argument is simpler. Using the log-Sobolev inequality again and Proposition 4.1 we have:
\[
\frac{d}{dr} (\log L_r) \geq -\frac{1}{2I_r} \mathbb{E} ||X||^{2p-2}_r ||X||_r^{-2p} \geq -\frac{1}{2I_r} \mathbb{E} ||X||^{2p-2}_r \mathbb{E} ||X||_r^{-2p} \geq -C_2 \frac{\sigma_{2p-2}^2 n}{1^p} \geq -C_3 / n,
\]
for \( r \leq C_4 d(B^p_r) \), where in the last step we have used Lemma 2.15. The result easily follows.

We are ready to prove the next concentration inequality. Note that the dependence we get on \( \varepsilon \) is better than the one we get if we employ (2.18).

**Proposition 4.2.** Let \( 4 \leq p < c_0 \log n \). Then, one has:

\[
P \left( ||X||_p - \mathbb{E} ||X||_p > \varepsilon \mathbb{E} ||X||_p \right) \leq C_1 \exp \left( -c_1 \varepsilon^{1+\frac{1}{2}} k(B^p_r) \right),
\]
for all \( 0 < \varepsilon < 1 \). Moreover, we have:

\[
P \left( ||X||_p > (1 - \varepsilon) \mathbb{E} ||X||_p \right) \leq C_2 \exp \left( -c_2 \varepsilon k(B^p_r) \right),
\]
for \( 0 < \varepsilon < 1 \).

**Proof.** Let \( 4 \leq p \leq c \log n \), where \( c > 0 \) is the constant from Proposition 4.1. Then, for each \( 0 < \varepsilon < 1 \) using Markov’s inequality we may write:

\[
P(||X||_p > (1 + \varepsilon)I_0) \leq e^{-\varepsilon r/2} \exp (r \log (I_r/I_0)) = \exp \left[ -r \left( \frac{\varepsilon}{2} - \log (I_r/I_0) \right) \right],
\]
for all \( r > 0 \). Using Proposition 4.1 we obtain:

\[
\log (I_r/I_0) \leq \frac{C_p}{n} \int_0^r \left( 1 + \frac{s}{k_{2p-2,n}} \right)^{p-1} ds < \frac{C_p k_{2p-2,n}}{pn} \left( 1 + \frac{r}{k_{2p-2,n}} \right)^p \leq \frac{(2C)^p k_{2p-2,n}}{pn} \left( \frac{r}{k_{2p-2,n}} \right)^p,
\]
for \( r > k_{2p-2,n} \). Therefore, (4.1) becomes:

\[
P(||X||_p > (1 + \varepsilon)I_0) \leq \exp \left( -\frac{\varepsilon r}{2} + \frac{(2C)^p}{pn k_{2p-2,n}^{p+1}} r^{p+1} \right),
\]

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for \( r > k_{2p-2,n} \). Minimizing the right-hand side with respect to \( r \), we find that \( r_{\min} = r_0 \) satisfies:

\[
(4.2) \quad \frac{(2C)^p}{p n k_{2p-2,n}^{p-1}} (p+1) r_0^p - \frac{\varepsilon}{2} = 0 \implies r_0 \approx \varepsilon^{1/p} k_{p,n},
\]

and in order for this value to be admissible we ought to have \( r_0 > k_{2p-2,n} \). Hence, the value \( r_0 \) is admissible if \( \varepsilon \) satisfies:

\[
r_0 > k_{2p-2,n} \iff (2C)^{-1} \frac{\varepsilon n}{2} \frac{p}{p+1} > k_{2p-2,n} \iff \varepsilon > (2C)^{p} \frac{2(p+1)}{pn} k_{2p-2,n}.
\]

Note that Proposition 2.4 implies that:

\[
(4.3) \quad k_{q,n} \leq c_2 q n^{2/q}, \quad \forall 2 \leq q \leq \log n.
\]

Since \( p \geq 4 \) it suffices to have \( \varepsilon > (2C)^p 8c_2 p n^{-\frac{2}{p+1}} \) or equivalently to have \( \varepsilon > (16 c_2 c) p n^{-\frac{p+1}{p+2}} \).

First consider the case \( k_{p,n}^{\varepsilon n} < \varepsilon < 1 \). In this case the above restriction is satisfied as long as \( p \leq c_3 \log n \) for some sufficiently small absolute constant \( c_3 > 0 \). Indeed one needs to check that: \( k_{p,n}^{\varepsilon n} > (16 c_2 c) p n^{-\frac{p+1}{p+2}} \) and by taking into account (4.3) again it suffices to have \( \frac{\varepsilon n}{2(p+1)} > (16 c_2 c)^p \) or it is enough to have:

\[
\frac{\varepsilon n^{p-1}}{2(p+1)} > (16 e^2 c_2 c)^p = \varepsilon^{p/c_4}.
\]

Thus, if \( c_0 := \min\{c_3, c_4/4\} > 0 \) we have all the requirements so that we may conclude:

\[
P(\|X\|_p > (1 + \varepsilon) I_0) \leq \exp\left( -\frac{\varepsilon r_0}{2} + \frac{(2C)^p}{p n k_{2p-2,n}^{p-1}} r_0^{p+1} \right) \leq \exp\left( -\frac{\varepsilon r_0}{2} + \frac{\varepsilon r_0}{2(p+1)} \right)
= \exp\left( -\frac{p}{2(p+1)} \varepsilon r_0 \right)
\leq \exp\left( -c e^{1+\frac{1}{c_4}} k_{p,n} \right),
\]

for all \( 4 \leq p \leq c_0 \log n \) and for all \( k_{p,n}^{\varepsilon n} < \varepsilon < 1 \). By adjusting the constants we get:

\[
P(\|X\|_p > (1 + \varepsilon) I_0) \leq C \exp\left( -c e^{1+\frac{1}{c_4}} k_{p,n} \right),
\]

for the whole range \( 0 < \varepsilon < 1 \) and for \( 4 \leq p \leq c_0 \log n \).

Now we turn to bounding the probability \( P(\|X\| \leq (1 - \varepsilon) I_0) \). Proposition 4.1 shows that \( (\log l_{c, r})' \geq -c^r / n \) for \( 0 < r \leq c_l k_p \). Hence, we get:

\[
P(\|X\|_p \leq (1 - \varepsilon) I_0) \leq P(\|X\|_p \leq e^{-\varepsilon} I_0) \leq e^{-c' \left( \frac{I_0}{l_{c', r}} \right)^{\varepsilon}} \leq \exp(-c r + r^2 C^p / n),
\]

\[
\text{for all } 4 \leq p \leq c_0 \log n \text{ and for all } k_{p,n}^{\varepsilon n} > \varepsilon < 1.
\]
for all $0 < r < c_1 k_{p,n}$, where we have used the bound:

$$\log(I_0/I_r) = - \int_0^r (\log I_s)' \, ds \leq \frac{C_p}{n} r,$$

for $0 < r < c_1 k_{p,n}$. Finally, choosing $r = k_{p,n}$ we see that $C_p^2 k_{p,n}^2/n < (2eC_p)^{n^{4/p-1}} \leq C'$ as long as $4 \leq p \leq c_1' \log n$, hence we conclude:

$$P(\|X\|_p \leq (1 - \epsilon) I_0) \leq C' \exp(-c' \epsilon k_{p,n}),$$

for $0 < \epsilon < 1$. □

Although this concentration result improves upon the one we get by just using (2.18), it is still suboptimal. It turns out that although the $L_2$ average of the Euclidean norm of the gradient is the proper quantity to be estimated for the concentration result, yet it should not be used in order to bound the growth of the high moments of the norm, in this range of $p$.

### 4.2 Estimating centered moments

In this paragraph we study centered moments of the $\ell_p$ norm, i.e. $(\mathbb{E} \left|\left|X\right|\right|_p - \mathbb{E} \left|\left|Y\right|\right|_p)^{1/r}$.

For this end we distinguish three cases: (a) $1 \leq p \leq 2$, (b) $2 < p < c_0 \log n$ and (c) $c_0 \log n \leq p \leq \infty$, where $c_0 > 0$ is sufficiently small absolute constant. While in the first two cases we estimate directly the centered moments in terms of $n, p, r$, in the last we have to argue differently and study the almost constant behavior of the noncentered moments. This is because when $p$ grows along with $n$ the estimates collapse. To overcome this obstacle we use Talagrand’s $L_1 - L_2$ bound.

#### 4.2.1 The case $1 \leq p \leq 2$

In this subsection we sketch the proof of the next theorem:

**Theorem 4.3.** Let $1 \leq p \leq 2$. Then, one has:

$$c_1 \exp(-c_1^2 n) \leq P \left( \left| \left| X \right| \left| \right|_p - \mathbb{E} \left| \left| X \right| \right|_p \right| > \epsilon \mathbb{E} \left| \left| X \right| \right|_p \right) \leq C_2 \exp(-c_2 \epsilon^2 n),$$

for $0 < \epsilon < 1$, where $C_1, c_1, C_2, c_2 > 0$ are absolute constants.

**Proof (Sketch).** The rightmost inequality follows by the Gaussian concentration inequality (2.18), Proposition 2.4 and the fact that $\text{Lip}(\|\cdot\|_p) = b(B^n_p) = n^{1/p-1/2}$ for $1 \leq p \leq 2$. Now we focus on the left-hand side inequality. We have the next:

**Proposition 4.4.** Let $1 \leq p \leq 2$. Then, we have:

$$\left( \mathbb{E} \left|\left|X\right|\right|_p - \mathbb{E} \left|\left|X\right|\right|_p \right)^{1/r} \geq \sqrt{r/n} \mathbb{E} \left|\left|X\right|\right|_p,$$

for all $r \geq 1$. 25
Proof. Indeed; the estimate

\begin{equation}
\left( \frac{1}{n} \sum_{i=1}^{n} \| X_i \|_p^r \right)^{1/r} \leq C_3 \sqrt{\frac{r}{n}} \mathbb{E} \| X \|_p, \quad r \geq 1
\end{equation}

is well known and follows by integration by parts combined with the right-hand side estimate in (4.4). For the estimate

\begin{equation}
\left( \mathbb{E} \| X \|_p - \mathbb{E} \| Y \|_p \right)^{1/r} \geq C_3 \sqrt{\frac{r}{n}} \mathbb{E} \| X \|_p
\end{equation}

we may apply the triangle inequality, Lemma 3.7 and finally the Cauchy-Schwarz inequality to write:

\[ 2 \left( \mathbb{E} \| X \|_p - \mathbb{E} \| Y \|_p \right)^{1/r} \geq \left( \mathbb{E} \| X \|_p - \| Y \|_p \right)^{1/r} \geq \frac{1}{2p} \left( \mathbb{E} \| X \|_p^p - \| Y \|_p^p \right)^{1/r} \frac{1}{\mathbb{E} \| Y \|_p^{p-1}} \frac{1}{\mathbb{E} \| Y \|_p^{p-1}} \right)^{1/r}.
\]

Note that (4.5) already implies \( \left( \mathbb{E} \| X \|_p \right)^{1/r} \leq 2C_3 \mathbb{E} \| X \|_p \approx n^{1/p} \) for all \( 1 \leq s \leq n \). Moreover, we have:

\[ \left( \mathbb{E} \| X \|_p^s - \| Y \|_p^s \right)^{1/s} \geq \mathbb{E} \| X \|_p^s - \| X \|_p^s \cdot \left( \mathbb{E} \sum_{i=1}^{n} |X_i| \right)^{1/s} \approx \sqrt{sn}, \]

where we have used the facts that the joint distribution of \( (X_i - |X_i|)^2 \) is the same as \( (|X_i| - |Y_i|)^2 \), Jensen’s inequality and at the last step, that \( \left( \mathbb{E} \sum_{i=1}^{n} |X_i| \right)^{1/s} \approx \sqrt{sn} \) for \( 1 \leq s \leq n \) (see e.g. [19]). Putting them all together we see:

\[ \left( \mathbb{E} \| X \|_p - \mathbb{E} \| X \|_p \right)^{1/r} \geq C_4 \frac{\sqrt{sn}}{n^{1/p}} \geq \sqrt{\frac{r}{n}} \mathbb{E} \| Y \|_p, \]

which completes the proof. \( \square \)

Now we turn to proving the lower bound in the probabilistic estimate (4.4): For every \( n^{-1/2} < \varepsilon < 2C_3 \) consider \( r \in [1, n] \) so that \( \varepsilon = 2C_3 \sqrt{r/n} \) to write:

\[ P \left( \| X \|_p - \mathbb{E} \| X \|_p \geq \varepsilon \mathbb{E} \| X \|_p \right) \geq P \left( \| X \|_p - \mathbb{E} \| X \|_p \geq \frac{1}{2} \left( \mathbb{E} \| X \|_p^p - \mathbb{E} \| X \|_p^p \right)^{1/r} \right) \]

\[ = P \left( \zeta \geq 2^{-r} \mathbb{E} \zeta \right) \geq (1 - 2^{-r})^2 \frac{\mathbb{E} \zeta^2}{\mathbb{E} \zeta^2}, \]

by Lemma 2.1 where \( \zeta := \| X \|_p - \mathbb{E} \| X \|_p \right)^{1/r} \). Employing the estimates (4.5) and (4.6) we conclude:

\[ P \left( \| X \|_p - \mathbb{E} \| X \|_p \geq \varepsilon \mathbb{E} \| X \|_p \right) \geq C_5 e^{-C_5 r}, \]

as required. \( \square \)
4.2.2 The case $2 < p \leq c_0 \log n$

It is clear from the argument of the previous paragraph that in order to obtain sharp concentration inequalities it is enough to get sharp estimates for the centered moments: $(\mathbb{E} \|X\|^p - \|Y\|^p)^{1/r}$. In view of Lemma 3.7 it is also obvious that estimates for the centered moments $(\mathbb{E} \|X\|^p - \|Y\|^p)^{1/r}$ will provide estimates for the moments $(\mathbb{E} \|X\|^p - \|Y\|^p)^{1/r}$. Note that in order to estimate the centered moments from above one may also employ Theorem 2.12 in the form of an $(r, r)$-Poincaré inequality (2.17). We use this method in the next Section in order to derive the optimal dependence on $\varepsilon$ in the critical dimension of randomized Dvoretzky. Here we shall prove the next result (see [23] for a different approach):

**Proposition 4.5.** Let $2 < p < \infty$. Then, we have:

$$
(4.7) \quad (\mathbb{E} \|X\|^p - \|Y\|^p)^{1/r} = \sigma_p^p \max \left\{2^{p/2}(rn)^{1/2}, r^{p/2}n^{1/r}\right\},
$$

for all $r \geq 2$.

**Proof.** Note that if $X = (X_1, \ldots, X_n)$ is a Gaussian random vector and $Y$ an independent copy of it, the variables $\xi_i := |X_i|^p - |Y_i|^p$ are i.i.d. and the functions $t \mapsto P(|\xi_i| > t)$ are log-convex on $(0, \infty)$ by Lemma 2.6. Then we may apply the main result from [13] to get:

$$
\left(\mathbb{E} \|X\|^p - \|Y\|^p\right)^{1/r} \equiv \left\|\sum_{i=1}^n \xi_i\right\|_r \approx \left(\sum_{i=1}^n \|\xi_i\|_r\right)^{1/r} + \sqrt{\sum_{i=1}^n \|\xi_i\|_r^{1/2}} \\
\approx n^{1/r}\|\xi_1\|_r + \sqrt{rn}\|\xi_1\|_2 \\
\approx n^{1/r}r^{p/2}\sigma_p^p + \sqrt{rn}2^{p/2}\sigma_p^p,
$$

where we have used Lemma 2.6 again. The proof is complete. \hfill \Box

Now we are ready to prove the following:

**Theorem 4.6.** Let $n > C$ and let $2 < p \leq c_0 \log n$. Then, we have:

$$
P \left(\|X\|^p - \mathbb{E}\|X\|^p > \varepsilon \mathbb{E}\|X\|^p\right) \leq C \exp \left(-\varepsilon \min \left\{\frac{\varepsilon^2 p^2 n}{2p}, (\varepsilon n)^{2/p}\right\}\right),$$

for all $0 < \varepsilon < 1/p$, where $C, c, c_0 > 0$ are absolute constants.

**Proof.** Define $a(n, p, r) := \max(2^{p/2}(rn)^{1/2}, r^{p/2}n^{1/r})$, $r \geq 2$. Note that for fixed $n, p$ the map $r \mapsto a(n, p, r)$ is strictly increasing with inverse $A(n, p, s) \approx \min\left(\frac{s}{2n}, s^{2/p}\right)$. Then, Proposition 4.5 shows that:

$$
(4.8) \quad c_1 \sigma_p^p a(n, p, r) \leq \left(\mathbb{E} \|X\|^p - \mathbb{E}\|X\|^p\right)^{1/r} \leq C_1 \sigma_p^p a(n, p, r),
$$

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for all \( r \geq 2 \). Applying Markov's inequality we get:

\[
P\left( \left| \|X\|_p^p - \mathbb{E} \|X\|_p^p \right| > r \mathbb{E} \|X\|_p^p \right) \leq \left( \frac{C_1 \alpha(n,p,r)}{t n} \right)^r = \exp \left( -A(n,p,etn/C_1) \right)
\]

\[
\leq \exp \left( -c_2 \min \left( \frac{r^2 n}{2^p}, (tn)^{2/p} \right) \right),
\]

provided that \( etn/C_1 > \alpha(n,p,2) = 2^{p/2} n^{1/2} \). It follows that

\[
P\left( \left| \|X\|_p^p - \mathbb{E} \|X\|_p^p \right| > r \mathbb{E} \|X\|_p^p \right) \leq \varepsilon^2 \exp \left( -c_2 \min \left( \frac{r^2 n}{2^p}, (tn)^{2/p} \right) \right),
\]

for all \( r > 0 \). Now fix \( 0 < \varepsilon < 1/p \). Then, we may write:

\[
P\left( \left| \|X\|_p^p - \mathbb{E} \|X\|_p^p \right| > \mathbb{E} \|X\|_p^p \right) \leq \exp \left( -c_3 \min \left( \frac{\varepsilon^2 p^2 n}{2^p}, (en)^{2/p} \right) \right),
\]

by the previous estimate. Arguing similarly we show the upper estimate. Thus,

\[
P\left( \left| \|X\|_p^p - \mathbb{E} \|X\|_p^p \right| > \mathbb{E} \|X\|_p^p \right) \leq \varepsilon^2 \exp \left( -c_4 \min \left( \frac{\varepsilon^2 p^2 n}{2^p}, (en)^{2/p} \right) \right),
\]

for all \( 0 < \varepsilon < 1/p \). The result follows.

\[\square\]

**Remark 4.7.** For fixed \( 2 < p < \infty \) the estimate can be reversed. The argument is similar to that in Theorem 4.3. Let \( \frac{2^{p/2}}{2^{1/p} \varepsilon} < t < 1 \) and choose \( r \geq 2 \) with \( \alpha(n,p,r) = 2c_1 nt \). Then, in view of the lower estimate in (1.8) we get:

\[
P\left( \left| \|X\|_p^p - \mathbb{E} \|X\|_p^p \right| > r \mathbb{E} \|X\|_p^p \right) \geq P\left( \|X\|_p^p < \frac{1}{2} \left( \mathbb{E} \|X\|_p^p - \mathbb{E} \|X\|_p^p \right)^{1/r} \right)
\]

\[
\geq (1 - 2^{-r}) \left( \frac{\mathbb{E} \|X\|_p^p - \mathbb{E} \|X\|_p^p}{\mathbb{E} \|X\|_p^p - \mathbb{E} \|X\|_p^p} \right)^{2/r}
\]

\[
\geq \frac{1}{4} e^{-c_3 r} \geq \frac{1}{4} \exp \left( -c_5 pA(n,p,2c_1 nt) \right).
\]

It follows, as before, that:

\[
P\left( \left| \|X\|_p^p - \mathbb{E} \|X\|_p^p \right| > \mathbb{E} \|X\|_p^p \right) \geq c_7 \exp \left( -c_7 pA(n,p,2c_1 npe) \right),
\]

for \( \frac{2^{p/2}}{2^{1/p} \varepsilon} < \varepsilon < 1/p \). Next recall that Proposition 4.4 implies \( I_p/I_1 \leq 1 + \frac{2^{p/2}}{4^{1/p} \varepsilon^p} \) for \( p \leq c_0 \log n \), where \( I_r \equiv I_r(y_n, B_r^p) \). Thus, we may replace \( I_p \) by \( I_1 \) in the above concentration estimate. This yields the following double estimate:

**Proposition 4.8.** For all sufficiently large \( n \) and for \( 2 < p < c_0 \log n \) one has:

\[
(4.9) \quad c \exp \left( -C \theta(n,p,\varepsilon) \right) \leq P\left( \left| \|X\|_p^p - \mathbb{E} \|X\|_p^p \right| > \mathbb{E} \|X\|_p^p \right) \leq C \exp \left( -c \theta(n,p,\varepsilon) \right),
\]

for all \( 0 < \varepsilon < 1/p \), where \( \theta(n,p,\varepsilon) := \min \left\{ \frac{p^2 e^\gamma n}{2^p}, (en)^{2/p} \right\} \) and \( C, c, c_0 > 0 \) are absolute constants.

**Note.** Let us mention that the extra \( p \) on the exponent in the lower estimate can be removed if we restrict the range to \( p^{-1} 2^{p/2} n^{1/2} \leq \varepsilon \leq p^{-1} 2^{p/2} n^{-\frac{p-2}{2p}} \).
4.2.3 The case $c_0 \log n < p \leq \infty$

In this Subsection we deal with the large values of $p$ in terms of the dimension, namely for $p \geq \log n$. We have the following:

**Theorem 4.9.** Let $4 < p \leq \infty$. Then, for any $0 < r < s \leq c_1 \sqrt{k_{p,n}} \log n$ we have:

\[
\frac{I_s(y_n, B_p^n)}{I_r(y_n, B_p^n)} \leq \exp \left( \frac{c_2(2s - r)}{k_{p,n} \log n} \right), \quad \frac{I_{s-1}(y_n, B_p^n)}{I_{r-1}(y_n, B_p^n)} \geq \exp \left( - \frac{c_2(2s - r)}{k_{p,n} \log n} \right),
\]

where $c_1, c_2 > 0$ are absolute constants.

**Proof.** Set $I_s = I_s(y_n, B_p^n)$. If $a = a_i := ||\partial_i f||_{L_2(y_n)}$ we get:

\[
(4.10) \quad a_i = \frac{|r|}{n} \int_{\mathbb{R}^n} \|x\|_{p}^{2n-2} \|x\|_{p}^{-n} d\gamma_n(x) \leq \frac{|r|}{n^{1/2} r^{r-1}},
\]

where we have used (2.1). Similarly, for $A = A_i := ||\partial_i f||_{L_2(y_n)}$ we have that:

\[
(4.11) \quad \frac{|r|}{n^{1/2} r^{r-2}} \leq A_i = \frac{|r|}{n^{1/2}} \left( \int_{\mathbb{R}^n} \|x\|_{p}^{2n-2p} \|x\|_{p}^{2n-2} d\gamma_n(x) \right)^{1/2} \leq \frac{|r|}{n^{1/2} r^{r-2}}.
\]

We apply Theorem 2.11 for $f(x) := \|x\|_{p}^r$, $r \neq 0$ to obtain:

\[
\text{Var}_{y_n}(f) \leq C_1 n \frac{A^2}{1 + \log(A/a)}.
\]

The function $t \mapsto \frac{\log(t)}{t}$, $t > a$ is increasing, thus (4.10) and (4.11) imply that:

\[
(4.12) \quad I_{2r} - I_r = \text{Var}_{y_n}(f) \leq C_1 r^2 \frac{I_{2r-2}}{1 + \log \left( n^{1/q-1/2} \left( \frac{I_{2r-2}}{I_{2r}} \right)^{r-1} \right)} \leq C_2 r^2 \frac{I_{2r-2}}{\log n},
\]

for all $r \neq 0$, since $1 \leq q < 4/3$ and $\log(I_{2r-2}/I_{2r-1}) \geq 0$.

**Claim.** For $r > -k_{p,n}$, $r \neq 0$ we have:

\[
I_{2r-2} \leq C_3 I_{2r}/k_{p,n}.
\]

We distinguish three cases:

- For $0 < r < 1$ we have: $I_{2r-2} \leq \frac{I_{2r}}{I_{2r-2}} \leq c_1 I_{2r} = c_1 I_{2r}$.
- For $r \geq 1$ we may write: $I_{2r-2} = \frac{I_{2r}}{I_{2r-2}} \leq c_3 I_{2r} = c_3 I_{2r}$, since $I_1 \approx I_0$.
- Finally, for $-k_{p,n} < r < 0$ we have: $I_{2r-2} \leq \frac{I_{2r}}{I_{2r-2}} = c_4 I_{2r}$, by Lemma 2.15.
Thus, (4.12) yields:

\[(4.13)\]

\[I_{2r} \leq C \frac{I_{2r}^2}{k_{p,n} \log n},\]

for \(r > -k_p, r \neq 0\). We only prove the stability for the positive moments (the negative moments are treated similarly): As long as \(0 < r < \sqrt{k_{p,n} \log n} / C\) we may write

\[I_{2r} \leq \left(1 + \frac{Cr^2}{k_{p,n} \log n}\right) I_{2r}.\]

Iterating the last one we find:

\[I_{2m} \leq \exp\left(C \sum_{j=0}^{m-1} \frac{2^j r}{k_{p,n} \log n}\right) \leq \exp\left(C \left(2^m r - r\right) / k_{p,n} \log n\right),\]

for \(m = 1, 2, \ldots\) as long as \(2^m r \leq \sqrt{k_{p,n} \log n} / C\). The result follows. \(\Box\)

The next corollary is immediate:

**Corollary 4.10.** Let \(c_0 \log n < p \leq \infty\). Then, one has:

\[P\left(\|X\|_p - \mathbb{E}\|X\|_p > \varepsilon \mathbb{E}\|X\|_p\right) \leq \frac{C}{\exp} \left(-c \varepsilon \log n\right),\]

for all \(\varepsilon \in (0, 1)\), where \(C, c, c_0 > 0\) are absolute constants.

**Proof.** Let \(K := k_{p,n} \log n\). Using Markov’s inequality and Theorem 4.9 we may write:

\[P\left(\|X\|_p > (1 + \varepsilon)I_1\right) \leq P\left(\|X\|_p \geq e^{c_{1/2}I_1}\right) \leq \exp\left(-c_{1/2} I_1^r / I_1\right) \leq \exp\left(-c_{1/2} r^2 / 2 + c_{2r^2} / K\right),\]

for all \(0 < r < c_1 \sqrt{K}\). The choice \(r \approx \sqrt{K}\) yields the one-sided estimate:

\[P\left(\|X\|_p > (1 + \varepsilon)I_1\right) \leq C_1 \exp\left(-c_{1/2} \sqrt{K}\right).\]

Working similarly with the probability \(P(\|X\|_p < (1 - \varepsilon)I_1)\) and taking into account the fact that \(k(B^p_{2r}) = \log n\) for \(p \geq \log n\), we conclude the asserted estimate. \(\Box\)

Summarizing the results of this paragraph (by taking into account Theorem 4.3, Theorem 4.6 and Proposition 4.2 and the variance estimates from Section 3) we may have a concentration inequality which interpolates between the concentration estimates for fixed \(p \geq 1\) and \(p = \infty\):

**Theorem 4.11.** For all large enough \(n\) and for any \(1 \leq p \leq \infty\) one has:

\[P\left(\|X\|_p - \mathbb{E}\|X\|_p > \varepsilon \mathbb{E}\|X\|_p\right) \leq C_1 \exp\left(-c_1 \beta(n, p, \varepsilon)\right),\]

where \(\beta(n, p, \varepsilon)\) is some function of \(n, p, \varepsilon\) that depends on the variance estimates from Section 3. 

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for every $0 < \varepsilon < 1$, where $\beta(n, p, \varepsilon)$ is defined as follows:

$$\beta(n, p, \varepsilon) = \begin{cases} 
\varepsilon^2 n, & 1 \leq p \leq 2 \\
\max \left\{ \min \left\{ p^{2-p} \varepsilon^2 n, (\varepsilon n)^{2/p} \right\}, \varepsilon p n^{2/p} \right\}, & 2 < p \leq c_0 \log n \\
\varepsilon \log n, & p > c_0 \log n
\end{cases}$$

where $c_0 \in (0, 1)$ and $C_1, c_1 > 0$ are suitable absolute constants. Furthermore, for $p \leq c_0 \log n$ we have the estimate:

$$P \left( \left| |X|_p - \mathbb{E}|X|_p \right| > \varepsilon \mathbb{E}|X|_p \right) \leq \exp \left( -\log \left( 1 + c_1 \varepsilon^2 n \right) \right),$$

for every $\varepsilon \in (0, 1)$.

\section{The critical dimension in random Dvoretzky for $\ell_p^n$}

In this paragraph we study the critical dimension $k(n, p, \varepsilon)$ (and in particular the dependence on $\varepsilon$) in the random version of Dvoretzky’s theorem for $\ell_p^n$ spaces. Our method is inspired by Schechtman’s approach in [27]. The key point is a distributional inequality for rectangular matrices with independent standard Gaussian entries. In [27] it is proved that, if $G = (g_{ij})_{i,j=1}^{n,k}$ is a Gaussian matrix then the process $(\|Gx\|_{\ell_p^n})_{x \in \ell_2^{n-1}}$ is sub-Gaussian with constant $b = \max_{x \in \ell_2^{n-1}} \|x\|$. The proof of [27, Lemma] is based on an orthogonal splitting, combined with a conditioning argument and inequality [2,18].

Here we use similar ideas to prove a functional inequality which generalizes [27, Lemma]. Once again, the advantage of this new inequality is that it involves $\|\nabla f\|_2$ instead of the Lipschitz constant of $f$. Our result reads as follows:

\textbf{Theorem 5.1.} Let $a, b \in S^{k-1}$ and $G = (g_{ij})_{i,j=1}^{n,k}$ be random matrix with standard i.i.d. Gaussian entries. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is $C^1$-smooth, then we have:

$$\left( \mathbb{E}|f(Ga) - f(Gb)|^r \right)^{1/r} \leq \pi r \|a - b\|_2 \left( \mathbb{E}\|\nabla f(W)\|_2^2 \right)^{1/r},$$

for all $r \geq 1$, where $W \sim N(0, I_n)$.

\textbf{Proof.} Fix $a, b \in S^{k-1}$ and assume without loss of generality that $a \neq \pm b$. Define $p := \frac{\|a\|_2}{\|b\|_2}$ and note that since $\|a\|_2 = \|b\|_2$ the vector $u := a - p$ is perpendicular to $p$. If we set $X := G(u)$ and $Z := G(p)$ then $X, Z$ are independent Gaussian random vectors in $\mathbb{R}^n$ with $X \sim N(0, \|u\|_2^2 I_n)$, $Z \sim N(0, \|p\|_2^2 I_n)$ and $G(a) = Z + X$ while $G(b) = Z - X$. Thus, we may write:

$$\mathbb{E}|f(Ga) - f(Gb)|^r = \mathbb{E}_X \mathbb{E}_Z \left| f(Z + X) - f(Z - X) \right|^r.$$

For $x, z \in \mathbb{R}^n$ we define $F(x, z) := f(x + z) - f(z - x)$. Note that for fixed $z$ we have $\mathbb{E}_X F(X, z) = 0$ since, $X$ is symmetric random vector. Applying Theorem [2.12] for
\( \phi(t) = |t|^{r}, \ r \geq 1 \) and \( x \mapsto F(x, z) \) instead of \( f \) we derive:

\[
\mathbb{E}|F(X, z)| = \mathbb{E}|f(z + X) - f(z - X)| \leq \left( \frac{r}{2} \right)^{r} \mathbb{E}_{X, Y} |(\nabla f(z + X), Y) + (\nabla f(z - X), Y)|^{r}
\]

\[
\leq \pi^{r} \mathbb{E}_{X, Y} |(\nabla f(z + X), Y)|^{r}
\]

\[
= \pi^{r} |a - b|^{r} \sigma_{r}^{r} \mathbb{E}_{X} \|\nabla f(z + X)\|_{2}^{r}.
\]

Moreover, note that \( W := X + Z \sim N(0, I_{n}) \), thus we get:

\[
\mathbb{E} |f(Ga) - f(Gb)|^{r} = \mathbb{E}|F(X, Z)| \leq \pi^{r} |a - b|^{r} \sigma_{r}^{r} \mathbb{E}_{X, Z} \|\nabla f(Z + X)\|_{2}^{r},
\]

as required. \( \square \)

**Remarks 5.2.**

1. If we assume that \( f \) is \( L \)-Lipschitz and applying Markov's inequality we may conclude the more general form of [27, Lemma]:

\[
\text{Prob}\left( \left| f(G(a)) - f(G(b)) \right| > t \right) \leq 2 \exp\left( -\frac{2}{r^{2} L^{2}}\|a - b\|^{2} \right), \quad t > 0.
\]

2. The same proof provides the following variant of Theorem 5.1 for the \( r \)-norm.

**Theorem 5.3.** Let \( \phi : \mathbb{R} \to \mathbb{R} \) be convex function and let \( f : \mathbb{R}^{n} \to \mathbb{R} \) be \( C^{1} \)-smooth. If \( G = (g_{i})_{i=1}^{n,k} \) is Gaussian matrix and \( a, b \in S^{k-1} \), then we have:

\[
\mathbb{E}\phi(f(Ga) - f(Gb)) \leq \mathbb{E}\phi\left( \frac{\pi}{2} |a - b| \|\nabla f(X)\| \right),
\]

where \( X, Y \) are independent copies of a Gaussian \( n \)-dimensional random vector.

The proof is left as an exercise to the interested reader (see also [25]).

3. For \( a, b \in S^{k-1} \) with \( \langle a, b \rangle = 0 \) the above statements are reduced to the inequalities we discussed in Section 2.

The next result is an application of Theorem 5.1 for the \( \ell_{p} \) norm.

**Theorem 5.4.** Let \( n \) be large enough and let \( 2 < p < c_{0} \log n \). Let \( a, b \in S^{k-1} \) and let \( G = (g_{i})_{i=1}^{n,k} \) be standard Gaussian random variables. Then,

\[
\left( \mathbb{E} \|G a\|_{p} - \|G b\|_{p} \right)^{1/r} \leq \|a - b\|_{2} \psi(n, p, r) \mathbb{E} \|Z\|_{p} ,
\]

for \( r \geq 2 \), where \( \psi(n, p, r) \) is defined as:

\[
\psi(n, p, r) := \sqrt{r} \min\left\{ \frac{1}{\sigma_{n}^{p} n^{1/p}}, \frac{\sigma_{p-2}^{p-1}}{m^{1/2} \sigma_{p}^{p}} \left( 1 + \frac{p \sigma_{p}^{p}}{\sigma_{2p-2}^{p-1} n^{r}} \right)^{\frac{1}{r}} \right\}
\]

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Moreover, for any $\varepsilon > 0$ one has:

$$P \left( \|G a\|_p - \|G b\|_p > \varepsilon \|Z\|_p \right) \leq C \exp \left( -c \tau \left( n, p, \frac{\varepsilon}{\|a - b\|_2} \right) \right),$$

where

$$\tau(n, p, t) := \max \left\{ t^2 p n^{2/p}, \min \left\{ \frac{t^2 n}{C_p}, (tn)^{2/p} \right\} \right\}, \quad t > 0$$

and $C, c > 0$ are absolute constants.

**Proof.** In view of Theorem 5.1 we need an upper estimate for the quantity:

$$\left( \mathbb{E} \left\| \nabla |X|_p \right\|_2^r \right)^{1/r} \leq \frac{P_{\tau(p-1)}(y_n, B_{2p-2}^n)}{P_{\tau(p-1)}(y_n, B_p^n)},$$

where in the last step we have used Proposition 2.2. A standard application of Lemma 2.7 (we use (2.9)) yields:

$$\frac{P_{\tau(p-1)}(y_n, B_{2p-2}^n)}{P_{\tau(p-1)}(y_n, B_{2p-2}^n)} \leq \left( 1 + \frac{(p - 1)(r - 2)}{\sigma_{2p-2}^2 n^{r-1}} \right).$$

Moreover, from Proposition 2.14 we see that:

$$\frac{P_{\tau(p-1)}(y_n, B_p^n)}{P_{\tau(p-1)}(y_n, B_p^n)} \geq 1,$$

for $r \leq c_k(B_p^n)$. Plugging estimates 5.2 and 5.3 in 5.1 we find:

$$\left( \mathbb{E} \left\| \nabla |X|_p \right\|_2^r \right)^{1/r} \leq \frac{(\sigma_{2p-2}/\sigma_p)^{p-1}}{n^{2/p-1}} \left( 1 + \frac{p(r - 2)}{\sigma_{2p-2}^2 n^{r-1}} \right),$$

for $2 \leq r \leq c_k(B_p^n)$. Taking into account that $\|\nabla |X|_p\|_2 \leq 1$ a.s. we conclude the first assertion. For the distributional inequality we argue as in the proof of Theorem 4.6 i.e. we use Markov’s inequality and the previous estimate.

**The chaining method: Dudley-Fernique decomposition.** For each $j = 1, 2, \ldots$ consider $\delta_j$-nets $N_j$ on $S^{k-1}$ with cardinality $|N_j| \leq (3/\delta_j)^k$ (see [21, Lemma 2.6]). Note that for any $\theta \in S^{k-1}$ and for all $j$ there exist $u_j \in N_j$ with $\|\theta - u_j\|_2 \leq \delta_j$ and by the triangle inequality it follows that $\|u_j - u_{j-1}\|_2 \leq \delta_j + \delta_{j-1}$. Moreover, if we assume that $\delta_j \to 0$ as $j \to \infty$ and $(t_j)$ is sequence of numbers with $t_j \geq 0$ and $\sum_j t_j \leq 1$ then, for any $\varepsilon > 0$ we have the following:
Fact. Set $E := \mathbb{E}[|X|]$. If we define the following sets:

$$A := \left\{ \omega \mid \exists \theta \in S^{k-1} : \|G_\omega(\theta)\| - E > \varepsilon E \right\},$$

$$A_1 := \left\{ \omega \mid \exists u_1 \in N_1 : \|G_\omega(u_1)\| - E > t_1 \varepsilon E \right\},$$

and for $j \geq 2$

$$A_j := \left\{ \omega \mid \exists u_j \in N_j, u_{j-1} \in N_{j-1} : \|G_\omega(u_j)\| - \|G_\omega(u_{j-1})\| > t_j \varepsilon E \right\},$$

then one has: $A \subseteq \bigcup_{j=1}^\infty A_j$ (see also \cite{27}).

Now we apply the above chaining method for the $\ell_p$ norm with $p > 2$ and we employ the distributional inequality of Theorem 5.4 to prove our second main result:

**Theorem 5.5** (Random Dvoretzky for $\ell_p^n$). For all large $n$, for any $1 \leq p \leq \infty$ and for every $0 < \varepsilon < 1$ there exists $k(n, p, \varepsilon)$ with the following property: the random $k$-dimensional subspace of $\ell_p^n$ with $k \leq k(n, p, \varepsilon)$ is $(1 + \varepsilon)$-Euclidean with probability greater than $1 - C \exp(-c(n, p, \varepsilon))$, where $k(n, p, \varepsilon)$ is estimated as follows:

(i) For $1 \leq p < 2$ we have:

$$k(n, p, \varepsilon) \geq \varepsilon^2 n,$$

(ii) For $2 < p < c_0 \log n$ we have:

$$k(n, p, \varepsilon) \geq \begin{cases} (Cp)^{-p} \varepsilon^2 n, & 0 < \varepsilon \leq (Cp)^{p/2} n^{-\frac{2}{p-1}} \\ \frac{1}{p} \log n^{2/p}, & (Cp)^{p/2} n^{-\frac{2}{p-1}} < \varepsilon \leq 1/p \\ C \exp(\frac{2}{p} / \log \frac{1}{\varepsilon}), & 1/p < \varepsilon < 1. \end{cases}$$

Moreover, for $p < c_0 \log n$ we have:

$$k(n, p, \varepsilon) \geq \log n / \log \frac{1}{\varepsilon},$$

(iii) For $c_0 \log n < p \leq \infty$ we have:

$$k(n, p, \varepsilon) \geq \varepsilon \log n / \log \frac{1}{\varepsilon},$$

where $C, c, c_0 > 0$ are absolute constants.

**Sketch of proof.** For $1 \leq p < 2$ the assertion follows from Theorem \cite{28} and the fact that $k(B_p^n) = n$. Let $2 < p < c_0 \log n$ and fix $0 < \varepsilon < 1/p$. Choose $\delta_j = \varepsilon^{-j}$, $t_j = s_p^{-1} j_p^{p/2} e^{-j}$, with $s_p := \sum_{j=1}^\infty j_p^{p/2} e^{-j}$. Then, according to the previous chaining method we may write:

$$P(A) \leq C |N_1| \exp(-c_1 \tau(p, \varepsilon t_1)) + C \sum_{j=2}^\infty |N_{j-1}| \cdot |N_j| \exp(-c_1 \tau(p, \varepsilon s_p^{-1} t_j e^j / 4))$$

$$\leq C \sum_{j=1}^\infty (3e^j)^{2j} \exp(-c_2 \tau(p, s_p^{-1} e j_p^{p/2})),$$
that the ratio between the $\ell_1$ norm of a vector and its $\ell_2$ norm, $\tau(n, p, t)$, was defined in Theorem 5.4, hence:

$$\tau(n, p, t) \approx \min \left\{ \frac{n^2}{Ct}, \frac{(en)^{2/p}}{p} \right\}, \quad t > 0.$$  

Note that

$$\tau(n, p, s_p^{-1}p^{p/2}) \geq j \min \left\{ \frac{e^2}{(Cp)^p}, \frac{(en)^{2/p}}{p} \right\} = jk(n, p, \epsilon),$$

where we have used the fact that $s_p \leq \sqrt{p} \frac{p^{p/2}}{en}$. Therefore, we have:

$$P(A) \leq C \sum_{j=1}^\infty \exp \left( c_3 jk - c_4 jk(n, p, \epsilon) \right) \leq \sum_{j=1}^\infty \exp \left( - \frac{c_4}{2} jk(n, p, \epsilon) \right) \leq C' \exp \left( - \frac{c_4}{2} k(n, p, \epsilon) \right).$$

as long as $k \leq \frac{c_2}{c_4} k(n, p, \epsilon)$. 

In the case that $p < c_0 \log n$ and $p \gg 1$ for the range $1/p < \epsilon < 1$ we have for any fixed $\theta \in S^{k-1}$ the concentration inequality

$$P \left( \|G\theta\|_p - \|X\|_p > \epsilon \|X\|_p \right) \leq C \exp(-ck(B^n_p)),$$

by Proposition 4.2. Thus, the classical net argument yields the estimate: $k(n, p, \epsilon) \geq \epsilon k(B^n_p)/\log \frac{1}{\epsilon}$. We omit the details.

Moreover, for $2 < p < c_0 \log n$ but $p \approx \log n$, the main result of Section 2 shows that $\text{Var}\|X\|_p \leq n^{-c_1}$ for some absolute constant $c_1 > 0$. Therefore, Chebyshev's probabilistic inequality and the net argument as before implies $k(n, p, \epsilon) \geq \log n/\log \frac{1}{\epsilon}$.

Finally, for $p \gg \log n$ we employ Corollary 4.10 combined with the net argument again to get $k(n, p, \epsilon) \geq \epsilon \log n/\log \frac{1}{\epsilon}$.

Below we show that the dependence on $\epsilon$ we get for the randomized Dvoretzky in $\ell^n_p$, for fixed $2 < p < \infty$, is essentially optimal. We have the following:

**Theorem 5.6 (Optimality in the Random Dvoretzky for $\ell^n_p$).** Let $2 < p < c_0 \log n$. Assuming that with probability larger than $1 - e^{-\beta k}$, a $k$-dimensional subspace satisfies that the ratio between the $\ell^n_p$ norm and a multiple of the $\ell^n_2$ norm is $(1 + \epsilon)$ equivalent for all vectors in the subspace, with $\frac{2\epsilon^2}{p} n^{-\frac{1}{1+p}} < \epsilon < 1/p$, then $k \leq \beta^{-1} e^{2/p} k(B^n_p)$.

For the proof we will need the next lemma from [28]:

**Lemma 5.7.** Let $1 \leq k \leq n-1$ and let $\mathcal{A} \subset G_{n,k}$ be a $\nu_{n,k}$-measurable set. Then, for $U_{\mathcal{A}} := \bigcup \{F \mid F \in \mathcal{A} \}$ we have:

$$\nu_{n,k}(\mathcal{A}) \leq [\gamma_{n}(U_{\mathcal{A}})]^k.$$
Proof of Theorem 5.6: Let $0 < \varepsilon < 1/3$ and define the collection of all $k$-dimensional subspaces of a space $(\mathbb{R}^n, ||\cdot||)$ for which the restricted norm there has distortion (with respect to the Euclidean norm) at most $1 + \varepsilon$:

$$\mathcal{A}_\varepsilon := \{ F \in G_{n,k} | \exists \lambda_F : \lambda_F \leq ||\theta|| \leq (1 + \varepsilon)\lambda_F, \forall \theta \in S_F \}.$$ 

Note that for $F \in \mathcal{A}_\varepsilon$ we have: $(1 + \varepsilon)^{-1}M_F \leq \lambda_F \leq M_F$. Thus instead of working with $\lambda_F$ we may define $\mathcal{A}_\varepsilon$ using $M_F := M(F \cap B)$ (here $B = \{x : ||x|| \leq 1\}$) namely, if

$$\mathcal{T}_\varepsilon := \{ F \in G_{n,k} | (1 + \varepsilon)^{-1}M_F \leq ||\theta|| \leq (1 + \varepsilon)M_F \forall \theta \in S_F \},$$

then we get $\mathcal{A}_\varepsilon \subset \mathcal{T}_\varepsilon$. Define further:

$$\mathcal{B}_\varepsilon := \left\{ F \in \mathcal{T}_\varepsilon | (1 - 2\varepsilon)\frac{\|g\|}{\|g\|_2} \leq M_F \leq (1 + 2\varepsilon)\frac{\|g\|}{\|g\|_2} \right\}$$

and note that $\mathcal{T}_\varepsilon, \mathcal{B}_\varepsilon$ are measurable. Hence, an application of Lemma 5.7 yields:

$$\nu_{n,k}(\mathcal{T}_\varepsilon) = \nu_{n,k}(\mathcal{T}_\varepsilon \setminus \mathcal{B}_\varepsilon) + \nu_{n,k}(\mathcal{B}_\varepsilon)$$

$$\leq \gamma_n \left( \left\{ x : ||x|| \geq \frac{1 + 2\varepsilon}{1 + \varepsilon} \frac{\|g\|}{\|g\|_2} ||x||_2 \text{ or } ||x|| \leq (1 + \varepsilon)(1 - 2\varepsilon)\frac{\|g\|}{\|g\|_2} \right\} \right)^k + \gamma_n \left( \left\{ x : \frac{1 - 2\varepsilon}{1 + \varepsilon} ||x||_2 \frac{\|g\|}{\|g\|_2} \leq ||x|| \leq (1 + \varepsilon)(1 + 2\varepsilon)\frac{\|g\|}{\|g\|_2} \right\} \right)^k.$$ 

Apply this argument for the $\ell_p$ norm with $2 < p < c_0 \log n$ and consider the next claim which follows easily by Theorem 4.3 and Proposition 4.8.

Claim. For every $2^{p/2}p^{-1}n^{-\frac{p^2}{2p+1}} < t \leq \varepsilon < 1/p$ we have:

$$ce^{-Cp(\varepsilon n)^{2/p}} \leq p \left( \|g\|_p \leq (1 - t)\frac{\|g\|_p}{\|g\|_2} \|g\|_2 \text{ or } \|g\|_p \geq (1 + t)\frac{\|g\|_p}{\|g\|_2} \|g\|_2 \right) \leq Ce^{-c(\varepsilon n)^{2/p}}.$$ 

Now assume that $2^{p/2}p^{-1}n^{-\frac{p^2}{2p+1}} < t < 1/p$, so by the previous claim we get:

$$\nu_{n,k}(\mathcal{T}_\varepsilon) \leq C^k e^{-ck(\varepsilon n)^{2/p}} + (1 - ce^{-Cp(\varepsilon n)^{2/p}})^k \leq e^{-ck(\varepsilon n)^{2/p}} + 1 - ce^{-Cp(\varepsilon n)^{2/p}}.$$ 

Now employing the assumption that $\nu_{n,k}(\mathcal{T}_\varepsilon) \geq 1 - e^{-\beta k}$ for some absolute constant $\beta > 0$ and that $\beta \ll (\varepsilon n)^{2/p}$, we conclude:

$$1 - ce^{-Cp(\varepsilon n)^{2/p}} \geq 1 - e^{-\beta k} - e^{-ck(\varepsilon n)^{2/p}} \geq 1 - 2e^{-c\beta k},$$

which implies $k \leq \frac{\log n}{2\beta}p(\varepsilon n)^{2/p},$ as required. \[\square\]
6  Further remarks and questions

1.  Instability of the variance. It is worth mentioning that the variance is not an isomorphic invariant. One can observe that:

There exists absolute constant $0 < c_0 < 1$ with the following property: for every $n \geq 2$ there exist 1-symmetric convex bodies $K$ and $L$ on $\mathbb{R}^n$ such that:

$$\text{Var}||Z||_K \approx \frac{1}{n^\delta \log n}, \quad \text{Var}||Z||_L \approx \frac{1}{\log n} \quad \text{and} \quad e^{-1/c_0}L \subseteq K \subseteq L,$$

where $\delta = 1 - c_0 \log 2$ and $Z \sim N(0, I_n)$.

Indeed; for $p_0 := c_0 \log n$, where $0 < c_0 < 1$ as in Theorem 3.9, we consider the bodies $K := B_n^{p_0}$ and $L := B_\infty$. We can easily see that these bodies enjoy the aforementioned properties.

2.  Non-centered moments. We know that for any centrally symmetric convex body $T$ on $\mathbb{R}^n$ one has:

$$c_1 r \leq \frac{I_r(\gamma_n, T)}{I_1(\gamma_n, T)} - 1 \leq c_2 r \frac{k(T)}{k(T)}$$

for all $r \geq 2$, where $c_1, c_2 > 0$ are absolute constants. This follows from the lower estimate in (2.14) and Lemma 2.7. In particular, for $1 \leq r \leq k(T)$ we obtain:

$$c_1' r \leq \frac{I_r(\gamma_n, T)}{I_1(\gamma_n, T)} - 1 \leq c_2' r \frac{k(T)}{k(T)}$$

and when $k(T) \approx n$ we readily see that this estimate is sharp up to constants, in particular for the $\ell_p$ norms with $1 \leq p \leq 2$. Furthermore, one can show that the same behavior holds true for $2 < p < c_0 \log n$ even though the critical dimension $k(B_n^p)$ in that case is much smaller than $n$. For $2 < p < c_0 \log n$ we have:

$$\frac{I_r(\gamma_n, B_n^p)}{I_1(\gamma_n, B_n^p)} \leq 1 + \frac{C^p}{n} r,$$

for all $1 \leq r \leq k(B_n^p)/C$. In fact for the negative moments this is already clear if we take into account Theorem 2.16, Proposition 4.1 and Theorem 4.9. More precisely we have: For $1 \leq p < c_0 \log n$ and for any $1 \leq r \leq c k(B_n^p)$ we get:

$$\max \left\{ \frac{I_r(\gamma_n, B_n^p)}{I_{-r}(\gamma_n, B_n^p)}, \frac{I_r(\gamma_n, B_n^p)}{I_1(\gamma_n, B_n^p)} \right\} \leq 1 + \frac{C^p}{n} r$$

and for $p \geq c_0 \log n$ and $1 \leq r \leq c k(B_n^p)$ we have:

$$\max \left\{ \frac{I_r(\gamma_n, B_n^p)}{I_{-r}(\gamma_n, B_n^p)}, \frac{I_r(\gamma_n, B_n^p)}{I_1(\gamma_n, B_n^p)} \right\} \leq 1 + \frac{C}{(\log n)^{2r}}.$$

We should note here the next threshold phenomenon when $2 < p \leq \infty$: 

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• $2 < p \leq c_0 \log n$: It is $I_1/I_1 - 1 \leq_p r/n = O_p(k^{2/p-1})$ for $1 \leq r \leq c_1 k(B_p^n)$ while for $r \geq c_2 k(B_p^n)$ we have $I_1/I_1 - 1 \approx 1$.

• $p > c_0 \log n$: It is $I_1/I_1 - 1 \leq r/\log n)^2 = O((\log n)^{-1})$ for $1 \leq r \leq c_1 k(B_p^n)$, while for $r \geq c_2 k(B_p^n)$ we have $I_1/I_1 - 1 \approx 1$.

for absolute constants $0 < c_1 < c_2$. The detailed study of this phenomenon will be presented elsewhere. Let us also note that although the behavior of the quantities $I_1/I_1 - 1, I_r/I_1 - 1$ is completely determined for the $\ell_p$ norms – it is of the order $r/n$ for $1 \leq r \leq c k(B_p^n)$ – combining this information with Markov’s inequality we still do not derive the optimal concentration inequality in the whole range $2 < p < \infty$.

3. Gaussian concentration and randomized Dvoretzky. One can show that the Gaussian concentration for norms $\| \cdot \|_A$ with $k(A) \approx n$ is essentially optimal:

**Lemma 6.1.** Let $\alpha \in (0,1)$ and let $A$ be centrally symmetric convex body on $\mathbb{R}^n$ with $k = k(A) \geq an$. Then,

$$P \left( \|Z\|_A - \mathbb{E}\|Z\|_A \geq \epsilon \mathbb{E}\|Z\|_A \right) \geq c e^{-C\alpha^2 k/a^2},$$

for all $n^{-1/2} < \epsilon < 1$.

**Proof.** Set $I_\alpha = \mathbb{E}|Z|_A^\alpha$. Taking into account (6.1) we may write:

$$1 + \frac{c_1 r}{n} \leq \frac{I_r}{I_1} \leq \sqrt{1 + \frac{C_1 r}{k}},$$

for all $r \geq 2$. Let $n^{-1/2} < \epsilon < 1$. If we set $r_0 := \frac{2\epsilon k}{c_1}$, then by previous estimates and the Paley-Zygmund inequality we have:

$$P \left( \|Z\| > (1 + \epsilon)I_1 \right) \geq P \left( \|Z\| > \frac{1 + \epsilon}{1 + \frac{C_1 r_0}{n}} I_0 \right) = P \left( \|Z\| > \delta I_0 \right) \geq (1 - \delta^2)^2 \frac{I_{2r_0}}{I_{2r_0}} \geq c_2 e^{-C\delta^2 \frac{k}{a^2}},$$

where $\delta := \frac{1 - \epsilon}{1 + \epsilon}$. The result easily follows. \(\square\)

Although the Gaussian concentration for spaces $E = (\mathbb{R}^n, \| \cdot \|)$ with $k(E) \approx n$ is sharp, the argument provided in Section 5 fails to give the optimal dependence on $\epsilon$ in randomized Dvoretzky. The reason is that in Gauss’ space, norms with concentration estimate less than $e^{-\epsilon^2 n}$ cannot be distinguished from the Euclidean norm. Therefore it is more appropriate to work on the sphere when we study almost spherical sections in normed spaces.

4. Refined Gaussian concentration and “new dimensions”. The reader should notice that the refined form of the Gaussian concentration for $2 < p < \infty$ (Theorem 4.11) and moreover Theorem 5.5 provide random, almost Euclidean subspaces of relatively large dimensions in which the norm has very small distortion. Previously, that phenomenon could not be observed if one was using the classical concentration inequality in terms of the Lipschitz constant. In order to illustrate this let us consider
an example, say the $\ell_p$ norm with $p = 5$. The classical setting yields the existence of random $k$-dimensional sections of $B_2^k$ which are $(1 + \varepsilon)$-isomorphic to a multiple of $B_2^k$ as long as $k \leq \varepsilon^2 n^{2/5}$. The latter is relatively large when $\varepsilon \gg n^{-1/5}$. Now, we may consider distortions smaller than $n^{-1/5}$, in fact as small as $n^{-1/2}$, since $\tau(n, 5, \varepsilon) \approx \min\{\varepsilon^2 n, (\varepsilon n)^{2/5}\}$. For instance (for $\varepsilon \approx n^{-2/5}$) the random $k$-dimensional section of $B_2^k$ with $k \approx n^{1/5}$, is $(1 + n^{-2/5})$-isomorphic to a multiple of $B_2^k$ with probability greater than $1 - e^{-c_\varepsilon n}$.

5. The existence of $\log(1/\varepsilon)$ as $p \to \infty$. Note that Theorem 4.9 and furthermore Corollary 4.10 suggest that the concentration of the $\ell_p$ norm with $p \geq \log n$ is similar with the one we get for the $\ell_\infty$ norm. This means that the classical net argument yields random subspaces which are $(1 + \varepsilon)$-spherical as long as $k \leq \varepsilon \log n / \log \frac{1}{\varepsilon}$. We do not know if this $\log(1/\varepsilon)$ term is needed, for this range of $p$. As an easy corollary of the main result of 3.4, we have:

**Proposition 6.2.** Let $p > (\log n)^2$ and $\varepsilon \in (0, 1/3)$. If the random $k$-dimensional subspace of $\ell_p^n$ is $(1 + \varepsilon)$-spherical with probability greater than $3/4$, then $k \leq C\varepsilon \log n / \log \frac{1}{\varepsilon}$, where $C > 0$ is an absolute constant.

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### A An anti-concentration estimate by Tikhomirov

After the second named author presented the results of the paper, to the Functional Analysis Seminar at the Math Department in University of Alberta, Tikhomirov was motivated to study the lower bound of the variance for $p \geq \log n$. He proved that the upper bound $(\log n)^{-1}$ is tight. Moreover, he proved that the concentration we obtain in Corollary 4.10 is sharp. We are indebted to him for kindly allowing us to present his argument here:

**Theorem A.1 (Tikhomirov, 2016).** Let $p \geq C_0 \log n$. Then, one has:

\[(A.1) \quad P\left(\|Z\|_p - \mathbb{E}\|Z\|_p > \varepsilon \mathbb{E}\|Z\|_p\right) \geq c e^{-C \varepsilon \log n},\]

for all $\varepsilon \in (0, 1)$, where $c, C, C_0 > 0$ are absolute constants and $Z \sim N(0, I_n)$.

In fact something more is true. In order to formulate it we need a little bit of terminology. Let $(\Omega, \Sigma, P)$ be the probability space. In what follows, we let $X = (x_1, x_2, \ldots, x_n) = (|g_1|, |g_2|, \ldots, |g_n|)$, where $g_1, g_2, \ldots, g_n$ are i.i.d. standard Gaussian variables. Further, for a random variable $\eta$ let $Q(\eta, t)$ be its Lévy concentration function defined by

\[Q(\eta, t) := \sup_{\lambda \in \mathbb{R}} P(|\eta - \lambda| \leq t), \quad t > 0.\]

By $z^*$ we denote the non-increasing rearrangement of a vector $z$. 

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Proposition A.2. There is a universal constant $C > 0$ such that for any $n \geq C$, $p \geq 12 \log n$ and any $\varepsilon \in (0,1)$ we have

$$Q\left(\|X\|_p, \varepsilon \sqrt{\log n}\right) \leq 1 - 0.07n^{-120\varepsilon}.$$ 

By $o(1)$ we mean any quantity which is a function of $n$ and becomes arbitrarily small for large enough $n$. The dimension $n$ is always assumed to be large. Further, for any $s \in (0,1)$ let $\xi_s$ be the quantile of order $s$ with respect to the distribution of $|g_1|$, i.e. the number satisfying the equation

$$\sqrt{\frac{2}{\pi}} \int_0^{\xi_s} \exp(-t^2/2) dt = P(|g_1| \leq \xi_s) = s.$$ 

By $y = (y_1, y_2, \ldots, y_n)$ we denote a non-random vector of quantiles, where

$$y_i := \xi_{1-(i-0.5)/n}, \ i = 1, 2, \ldots, n.$$ 

It follows from Lemma 2.5 that

$$(A.2) \quad \left(a^{-1} - a^{-3}\right) \exp(-a^2/2) < \int_a^\infty \exp(-t^2/2) dt < a^{-1} \exp(-a^2/2), \ a > 0.$$ 

Using this estimate, it is elementary to check that

$$y_1 = (1 + o(1)) \sqrt{2 \log n}.$$ 

Lemma A.3. We have $y_1^2 - y_i^2 \geq \log i$ for all $1 \leq i \leq 0.317n$.

Proof. By the definition of the quantiles, we have

$$\int_{y_a}^\infty \exp(-t^2/2) dt = \sqrt{\frac{2}{\pi}} \frac{2m-1}{2n}, \ m = 1, 2, \ldots, n.$$ 

Together with (A.2), it gives:

$$\left(y_1^{-1} - y_2^{-3}\right) \exp(-y_1^2/2) < \frac{y_1^{-1}}{2i-1} \exp(-y_i^2/2),$$

whence

$$\frac{y_1}{y_i} \exp\left(y_1^2/2 - y_i^2/2\right) > (1 - o(1))(2i-1).$$

It can be checked that, under the assumption that $y_i \geq 1$ (which holds true since $i \leq 0.317n$), we have

$$\frac{y_1}{y_i} \exp\left(y_1^2/2 - y_i^2/2\right) \leq \exp\left(y_1^2 - y_i^2\right).$$

Plugging the estimate into the previous formula and using the rough bound $(1 - o(1))(2i-1) \geq i$ for $i \geq 2$, we obtain the result. 

The next lemma is checked by a direct computation:
Lemma A.4. Denote $y_0 := \xi_{1-1/(4\kappa)}$. Then

$$P(x_i^* \in [y_i, y_0]) \geq 0.17.$$  

Lemma A.5. For every $i \geq \varepsilon^2$ we have

$$P(x_i^* \geq y_{[i/\varepsilon^2])}) \leq \exp(-i).$$

Proof. Recall that by Chernoff’s inequality we have for any $i$ and any $s \in (1 - i/n, 1)$:

$$\sum_{j=0}^{n-i} \left( \begin{array}{c} n \\ j \end{array} \right) (1 - s)^{n-j}s^j \leq \exp \left( (n - i) \log \frac{sn}{n-i} + i \log \frac{(1-s)n}{i} \right) \leq \exp(i) \left( \frac{(1-s)n}{i} \right)^i.$$  

Hence, denoting $s := 1 - \frac{\lceil \varepsilon^2 \rceil i - 1/2}{n}$, we get

$$P(x_i^* \geq y_{[i/\varepsilon^2])}) = P(x_i^* \geq \xi_i) = \sum_{j=0}^{n-i} \left( \begin{array}{c} n \\ j \end{array} \right) P(\|g\| \geq s_j) - P(\|g\| \leq s_j) = \sum_{j=0}^{n-i} \left( \begin{array}{c} n \\ j \end{array} \right) (1-s)^{n-j}s^j$$  

$$\leq \exp(i) \left( \frac{(1-s)n}{i} \right)^i$$  

$$\leq \exp(-i).$$

\[\square\]

Let us fix any $\varepsilon \in (0, 1]$ and denote

$$Q_1 := \{(z_1, \ldots, z_n) \in \mathbb{R}_+^n : z_i^* \in [y_i, y_0], z_i^* \leq y_{[i/\varepsilon^2])}, \forall i \geq 2\};$$

$$Q_2 := \{(z_1, \ldots, z_n) \in \mathbb{R}_+^n : z_i^* \in [y_i + 60\varepsilon \sqrt{\log n}, y_0 + 60\varepsilon \sqrt{\log n}], z_i^* \leq y_{[i/\varepsilon^2])}, \forall i \geq 2\}.$$

Further, let $E_k := \{\omega \in \Omega : X(\omega) \in Q_k \} (k = 1, 2)$. It is easy to see from Lemmas A.4 and A.5 that $P(E_0) \geq 0.15$ (note that for $i < \varepsilon^2$ the condition $x_i^* \leq y_{[i/\varepsilon^2])} = y_0$ is fulfilled automatically provided that $x_i^* \leq y_0$).

Lemma A.6. Let $p \geq 12 \log n$ and $z \in Q_1$. Then,

$$\|z\|^p \leq 3e^2(z_i^*)^p.$$  

Proof. We have

$$\sum_{i=2}^{n} (z_i^*)^p \leq (e^2 - 1)(z_i^*)^p + \sum_{k=2}^{[n/e^2]+1} \sum_{i=[k^2(k-1)+1]}^{[n/e^2]} (z_i^*)^p \leq (e^2 - 1)(z_i^*)^p + (e^2 + 1) \sum_{k=2}^{[n/e^2]+1} y_k^{p/2}.$$  

Applying Lemma A.3 we get

$$\sum_{k=2}^{[n/e^2]} y_k^{p/2} \leq \sum_{k=2}^{[n/e^2]} (y_k^2 - \log(k-1))^{p/2} \leq y_1^p \sum_{k=2}^{\infty} (k-1)^{-p/(4\log n)} < 1.2y_1^p.$$  

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Finally, we get

\[ \|z\|^p_p \leq e^{2(z_1^*)^p} + (e^{2} + 1)y_1^p \leq 3e^{2(z_1^*)^p}. \]

Next, consider an operator \( T : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) which acts by adding \( 60\varepsilon \sqrt{\log n} \) to the largest coordinate of a vector, i.e. for \( z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n_+ \) with \( k = \min \{ s : s = \text{argmax}\{z_1, z_2, \ldots, z_n\} \} \) we have \( Tz = (z_1, z_2, \ldots, z_k + 60\varepsilon \sqrt{\log n}, \ldots, z_n) \). Obviously, \( T \) maps the set \( Q_1 \) into \( Q_2 \).

**Lemma A.7.** Let \( z \in Q_1 \) and \( p \geq 12\log n \). Then,

\[ \|Tz\|^p_p - \|z\|^p_p > 2\varepsilon \sqrt{\log n}. \]

**Proof.** By Lemma [A.6](#) we have

\[ \sum_{i=2}^{n} (z_i^*)^p \leq (3e^2 - 1)(z_1^*)^p. \]

On the other hand, \( (\{Tz\}^*)^p = \left( 1 + 60\varepsilon \sqrt{\log n/z_1^*} \right)^p (z_1^*)^p \). Thus,

\[ \frac{\|Tz\|^p_p}{\|z\|^p_p} \geq 1 + \frac{\left( 1 + 60\varepsilon \sqrt{\log n/z_1^*} \right)^p - 1}{3e^2} \geq \left( 1 + 60\varepsilon \sqrt{\log n/z_1^*} \right)^{p/(3e^2)}, \]

whence

\[ \frac{\|Tz\|^p_p}{\|z\|^p_p} \geq \left( 1 + 60\varepsilon \sqrt{\log n/z_1^*} \right)^{1/(3e^2)} \geq 1 + \frac{2\varepsilon \sqrt{\log n}}{z_1^*} > 1 + \frac{2\varepsilon \sqrt{\log n}}{\|z\|^p_p}. \]

**Proof of Proposition [A.2](#)** Denote by \( \rho(z) (z \in \mathbb{R}^n_+) \) the probability density function of the vector \( X \). Then from the definition of \( T \) and \( Q_1 \) we have for any \( z \in Q_1 \):

\[ \rho(z) \geq \rho(Tz) \geq \frac{\exp\left( -y_0 + 60\varepsilon \sqrt{\log n}/2 \right)}{\exp(-y_0^2/2)} \rho(z) \geq n^{-120\varepsilon} \rho(z). \]

Fix any \( \lambda \in \mathbb{R} \). Then from Lemma [A.3](#) it follows that for any \( z \in Q_1 \) we have

\[ \max \left\{ \|z\|^p_p - \lambda, \|Tz\|^p_p - \lambda \right\} > \varepsilon \sqrt{\log n}. \]

Denote \( W_\lambda := \left\{ z \in \mathbb{R}^n_+: \|z\|^p_p - \lambda > \varepsilon \sqrt{\log n} \right\} \) and \( \bar{W}_\lambda := \left\{ z \in \mathbb{R}^n_+: \|Tz\|^p_p - \lambda > \varepsilon \sqrt{\log n} \right\} \). Note that \( T(\bar{W}_\lambda) \subset W_\lambda \) and \( T \) is volume preserving transformation, therefore

\[ \int_{\bar{W}_\lambda} \rho(z) \, dz \geq \int_{T(\bar{W}_\lambda)} \rho(z) \, dz = \int_{\bar{W}_\lambda} \rho(Tz) \, dz. \]

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Moreover, from Lemma A.7 it follows that $Q_1 \subset W_\lambda \cup \tilde{W}_\lambda$, hence we may write:

\[
P \left( \|X\|_p - A \geq \epsilon \sqrt{\log n} \right) = \int_{W_\lambda} \rho(z) \, dz \geq \frac{1}{2} \left[ \int_{W_\lambda} \rho(z) \, dz + \int_{\tilde{W}_\lambda} \rho(Tz) \, dz \right]
\geq \frac{1}{2} \int_{W_\lambda \cup \tilde{W}_\lambda} \min \{ \rho(z), \rho(Tz) \} \, dz
\geq \frac{1}{2} \int_{Q_1} \rho(z) \, dz
\geq \frac{1}{2} n^{-120\epsilon} \int_{Q_1} \rho(z) \, dz
\geq \frac{1}{2} n^{-120\epsilon} \mathbb{P}(E_1) \geq 0.07 n^{-120\epsilon}.
\]

\[\square\]

References

[1] S. Artstein-Avidan, A. Giannopoulos and V. D. Milman, *Asymptotic Geometric Analysis, Part I*, AMS-Mathematical Surveys and Monographs 202 (2015).

[2] J. Bourgain and J. Lindenstrauss, *Almost euclidean sections in spaces with a symmetric basis*, Israel Seminar (GAFA) 1987-88, Lecture Notes in Mathematics 1376, (1989), 278–288.

[3] S. Boucheron, G. Lugosi and P. Massart, *Concentration Inequalities: A non-asymptotic theory of independence*, Oxford University Press (2013).

[4] S. Chatterjee, *Superconcentration and related topics*, Springer Monographs in Mathematics (2013).

[5] D. Cordero-Erausquin and M. Ledoux, *Hypercontractive Measures, Talagrand’s inequality, and Influences*, Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics 2050, (2012), 169–189.

[6] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press (1946).

[7] H. A. David, *Order Statistics*, 2nd edition, Wiley, New York, (1981).

[8] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proc. Int. Symp. on linear spaces, Jerusalem (1961), 123–160.

[9] D. Fresen, *Explicit Euclidean embeddings in permutation invariant normed spaces*, Adv. Math. 266, 1–16 (2014).

[10] R. D. Gordon, *Values of Mills’ ratio of area to bounding ordinate and of the normal probability integral for large values of the argument*, Ann. Math. Statistics 12, no. 3, (1941), 364–366.

[11] Y. Gordon, *Some inequalities for Gaussian processes and applications*, Israel J. Math. 50 (1985), 265–289.

[12] G. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press 1988.
[13] P. Hitczenko, S. J. Montgomery-Smith and K. Oleszkiewicz. Moment inequalities for linear combinations of certain i.i.d. symmetric random variables, Studia Math. 123 (1997), 15–42.

[14] B. Klartag and R. Vershynin. Small ball probability and Dvoretzky theorem, Israel J. Math., Vol. 157, no. 1 (2007), 193–207.

[15] H. König. Isometric imbeddings of Euclidean spaces into finite-dimensional $\ell_p$-spaces, Panoramas of Mathematics (Colloquia 93-94), 79–87, Banach Center Publ. 34, Polish Acad. Sci., Warszawa, 1995.

[16] R. Latała and K. Oleszkiewicz. Small ball probability estimates in terms of width, Studia Math. 169 (2005), 305–314.

[17] M. Ledoux. The concentration of measure phenomenon, Mathematical Surveys and Monographs 89, American Mathematical Society, Providence, RI, (2001).

[18] A. Litvak, V. D. Milman and G. Schechtman. Averages of norms and quasi-norms, Math. Ann. 312 (1999), 95–124.

[19] S. J. Montgomery-Smith. The distribution of Rademacher sums, Proc. Amer. Math. Soc. 109 (1990), 517–522.

[20] V. D. Milman. New proof of the theorem of A. Dvoretzky on sections of convex bodies, (Russian), Funkcional. Anal. i Prilozen. 5 (1971) 28–37.

[21] V. D. Milman and G. Schechtman. Asymptotic theory of finite dimensional normed spaces, Lecture Notes in Math. 1200 (1986), Springer, Berlin.

[22] V. D. Milman and G. Schechtman. Global versus Local asymptotic theories of finite-dimensional normed spaces, Duke Math. Journal 90 (1997), 73–93.

[23] A. Naor. The Surface Measure and Cone Measure on the Sphere of $\ell^m_p$, Transactions of the American Mathematical Society 359 (2007), 1045–1079.

[24] G. Paouris, P. Pivovarov and P. Valettas. On a quantitative reversal of Alexandrov's inequality, (2017), preprint; available at https://arxiv.org/abs/1702.05762.

[25] G. Paouris and P. Valettas. On Dvoretzky's theorem for subspaces of $L_p$, (2015), preprint; available at https://arxiv.org/abs/1510.07289.

[26] G. Pisier. Probabilistic Methods in the Geometry of Banach Spaces, CIME, Varenna, 1985, Springer, Lecture Notes in Mathematics 1206 (1986), 167–241.

[27] G. Schechtman. A remark concerning the dependence on $\epsilon$ in Dvoretzky's theorem, Geometric Aspects of Functional Analysis (1987-88), 274–277, Lecture Notes in Mathematics 1376, Springer, Berlin (1989).

[28] G. Schechtman. The random version of Dvoretzky's theorem in $\ell^m_p$, GAFA Seminar 2004-2005, 265–270, Lecture Notes in Math., 1910, Springer-Verlag (2007).

[29] G. Schechtman. Two observations regarding embedding subsets of Euclidean spaces in normed spaces, Advances in Mathematics 200 (2006) 125–135.

[30] G. Schechtman and J. Zinn. On the volume of the intersection of two $L_p^n$ balls, Proc. Amer. Math. Soc. 110, No.1, (1990), 217–224.
[31] P. Stavrakakis and P. Valettas, On the geometry of log-concave probability measures with bounded log-Sobolev constant, Proceedings of the Asymptotic Geometric Analysis Programme, Fields Institute Communications 68 (2013), 359–380.

[32] M. Talagrand, On Russo’s approximate zero-one law, Ann. Prob. 22, (1994), 1576–1587.

[33] K. Tanguy, Some superconcentration inequalities for extrema of stationary Gaussian processes, in Statistics and Probability Letters, (2015).

[34] K. E. Tikhomirov, The Randomized Dvoretzky’s theorem in $\ell^m_n$ and the $\chi$-distribution, Geometric Aspects of Functional Analysis, Israel Seminar (GAFA) 2011–2013 (eds. B. Klartag and E. Milman), Lecture Notes in Mathematics 2116, (2013) Springer.

[35] K. E. Tikhomirov, Almost Euclidean sections in symmetric spaces and concentration of order statistics, J. Funct. Anal. 265 (9), 2074–2088, (2013).

[36] K. E. Tikhomirov, Private Communication, Edmonton, AB, April 2016.