TWO-SIDED DIRICHLET HEAT ESTIMATES OF SYMMETRIC STABLE PROCESSES ON HORN-SHAPED REGIONS

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Abstract. In this paper, we consider symmetric α-stable processes on (unbounded) horn-shaped regions which are non-uniformly $C^{1,1}$ near infinity. By using probabilistic approaches extensively, we establish two-sided Dirichlet heat estimates of such processes for all time. The estimates are very sensitive with respect to the reference function corresponding to each horn-shaped region. Our results also cover the case that the associated Dirichlet semigroup is not intrinsically ultracontractive. A striking observation from our estimates is that, even when the associated Dirichlet semigroup is intrinsically ultracontractive, the so-called Varopoulos-type estimates do not hold for symmetric stable processes on horn-shaped regions.

Keywords: Dirichlet heat kernel; fractional Laplacian; horn-shaped region; Lévy system

MSC 2020: 60G51; 60G52; 60J25; 60J76.

1. BACKGROUND AND MAIN RESULTS

Dirichlet heat kernel is the fundamental solution of the heat equation with zero exterior conditions, which plays an important role in the study of Cauchy or Poisson problems with Dirichlet conditions. While the research on estimates and properties for the Dirichlet heat kernel of the Laplacian has a long history and fruitful results (see [30] and the references therein), the corresponding work for the fractional Laplacian or more general non-local operators was powerfully attracted and extendedly developed in recent few years.

Let $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ be the fractional Laplacian on $\mathbb{R}^d$ with $\alpha \in (0, 2)$, which is the infinitesimal generator of the (rotationally) symmetric α-stable process $X := \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$. The fractional Laplacian $\Delta^{\alpha/2}$ is a non-local operator and can be written in the form

$$\Delta^{\alpha/2} f(x) = \lim_{\varepsilon \to 0} \int_{\{|y-x|>\varepsilon\}} (f(y) - f(x)) \frac{c_{d,\alpha}}{|y-x|^{d+\alpha}} \, dy, \quad f \in C_c^\infty(\mathbb{R}^d),$$

where $c_{d,\alpha}$ is a positive constant depending only on $d$ and $\alpha$, and $C_c^\infty(\mathbb{R}^d)$ is the space of smooth functions with compact support in $\mathbb{R}^d$. Throughout this paper, we denote by $p(t, x, y)$ the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ (or equivalently the transition density function of the symmetric α-stable process $X$) on $\mathbb{R}^d$. It is well known (e.g. see [30]) that

$$p(t, x, y) \simeq t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \quad \text{for all } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Here and below, we denote $a \wedge b := \min\{a, b\}$ and $f \simeq g$ if the quotient $f/g$ remains bounded between two positive constants.

For every open subset $D \subset \mathbb{R}^d$, we denote by $X_D$ the subprocess of $X$ killed upon leaving $D$. The infinitesimal generator of $X_D$ is the Dirichlet fractional Laplacian $\Delta^{\alpha/2}_D$ (the fractional Laplacian with zero exterior condition). It is known (see [23]) that $X_D$ has the transition density $p_D(t, x, y)$ with respect to the Lebesgue measure (which is called the Dirichlet heat kernel) that is jointly continuous on $(0, \infty) \times D \times D$. The first breakthrough on two-sided estimates of the
transition density for the Dirichlet fractional Laplacian (which we will call Dirichlet heat kernel estimates later) was done by the second named author jointly with Zhen-Qing Chen and Renming Song in [11].

To state the main results in [11] explicitly, we first recall the definition of uniform $C^{1,1}$ open set. An open set $D$ in $\mathbb{R}^d$ with $d \geq 2$ is said to be $C^{1,1}$ at $z \in \partial D$, if there are a localization radius $R > 0$ and a constant $\Lambda > 0$ (both of them may depend on $z \in D$) such that there exist a $C^{1,1}$-function $\psi := \psi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0, \ldots, 0) = 0$, $\nabla \psi(0, \ldots, 0) = (0, \ldots, 0)$, $\|\nabla \psi\|_{\infty} \leq \Lambda$ and $|\nabla \psi(x) - \nabla \psi(y)| \leq \Lambda |x - y|$ for all $x, y \in \mathbb{R}^{d-1}$, and an orthonormal coordinate system $CS_z$ with its origin at $z$ such that

$$B(z, R) \cap D = \{ y := (y_1, \tilde{y}) \in CS_z : |y_1 - R, y_1 > \psi(\tilde{y}) \}.$$  

The pair $(R, \Lambda)$ is called the $C^{1,1}$ characteristics of $D$ at $z$. An open set $D$ in $\mathbb{R}^d$ with $d \geq 2$ is said to be a (uniform) $C^{1,1}$ open set, if there exist $R, \Lambda > 0$ such that $D$ is $C^{1,1}$ at every $z \in \partial D$ with the same $C^{1,1}$ characteristics $(R, \Lambda)$ of $D$. The pair $(R, \Lambda)$ is called the characteristics of the $C^{1,1}$ open set $D$. It is known that any $C^{1,1}$ open set $D$ with the characteristics $(R, \Lambda)$ satisfies the (uniform) interior ball condition; that is, there exists $r < R$ such that for every $x \in D$ with $\delta_D(x) \leq r$, it holds that $B(\xi^*_r, r) \subset D$, where $\delta_D(x)$ is the Euclidean distance between $x$ and $D^c$, and $\xi^*_r := x + r(x - x_0)/|x - x_0|$ with $x_0 \in \partial D$ such that $|x - x_0| = \delta_D(x)$.

Let $D$ be a $C^{1,1}$ open subset of $\mathbb{R}^d$. It was shown in [11, Theorem 1.1] that

(i) For every $T > 0$, on $(0, T) \times D \times D$,

$$p_D(t, x, y) \approx p(t, x, y) \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right). \tag{1.2}$$

(ii) Suppose in addition that $D$ is bounded. Then, for every $T > 0$, on $(T, \infty) \times D \times D$,

$$p_D(t, x, y) \approx \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} e^{-\lambda_D t}, \tag{1.3}$$

where $\lambda_D > 0$ is the smallest eigenvalue of the Dirichlet fractional Laplacian $(-\Delta)^\alpha/2 D$. (i) says that, until any finite time, the Dirichlet heat kernel $p_D(t, x, y)$ is comparable with the global heat kernel $p(t, x, y)$ multiplied by some weighted functions $\delta_D(x)^{\alpha/2}/\sqrt{t} \wedge 1$ and $\delta_D(y)^{\alpha/2}/\sqrt{t} \wedge 1$, which are determined by the dependency between time and position of the points $x, y \in D$. The uniform $C^{1,1}$-property of the open set $D$ plays a key role in the proof of (i). On the other hand, the estimate of $p_D(t, x, y)$ for large time given in (ii) is based on the result (i) and the so-called intrinsic ultracontractivity of $p_D(t, x, y)$, i.e., $p_D(t, x, y) \leq c \phi_1(x) \phi_1(y) e^{-\lambda_D t}$, where $\phi_1$ is the ground state (i.e., the positive eigenfunction corresponding to the first eigenvalue $\lambda_D$) and satisfies $\phi_1(x) \approx \delta_D(x)^{\alpha/2}$. The notion of intrinsic ultracontractivity was first introduced by Davies and Simon in [25].

The idea and the approach in [11] later were extensively adopted to study Dirichlet heat kernel estimates for censored stable-like processes in [12], for relativistic stable processes in [13], for $\Delta^\alpha/2 + \Delta^\beta/2$ in [14], for $\Delta + \Delta^\alpha/2$ in [15], for subordinate Brownian motions with Gaussian components in [20], for unimodal Lévy processes in [17], for a large class of symmetric pure jump Markov processes dominated by isotropic unimodal Lévy processes with weak scaling conditions in [29] [32], and so on.

As mentioned above, the uniform $C^{1,1}$-property of $D$ is crucial for the estimate (1.2). When $D$ has lower regularity, (1.2) may not be available but Dirichlet heat kernel estimates can be established in terms of the survival probability $P^x(\tau_D > t)$ instead of $\delta_D(x)^{\alpha/2}/\sqrt{t} \wedge 1$, where $\tau_D$ is the first exit time from $D$ of the process $X$, i.e., $\tau_D = \inf\{ t > 0 : X_t \notin D \}$. That is, in these cases one would expect that for any $T > 0$, on $(0, T] \times D \times D$,

$$p_D(t, x, y) \approx p(t, x, y) P^x(\tau_D > t) P^y(\tau_D > t), \quad x, y \in D, \ 0 < t \leq T. \tag{1.4}$$

(1.4) are called the Varopoulos-type estimates in the literature, and they can be traced back to the paper [36] by Varopoulos, where (1.4) are proved to be satisfied for Dirichlet heat kernels of a divergence and nondivergence form elliptic operator (even with time-dependent coefficients)
on bounded Lipschitz domains. Nowadays, (1.4) have been obtained for a quite large class of discontinuous processes. See [6, Theorem 1] for Dirichlet heat kernel estimates of symmetric α-stable process when $D$ is $κ$-fat (including domain above the graph of a Lipschitz function), and see [19, Theorem 1.3 and Corollary 1.4] and [25, Theorems 2.22 and 2.23] for the corresponding results for rotationally symmetric Lévy processes and more general jump processes with critical killings, respectively. On the other hand, as indicated above, the estimate (1.3) for large time is a direct consequence of the intrinsic ultracontractivity of the associated Dirichlet semigroup, which is satisfied when $C^{1,1}$ open set $D$ is bounded. Indeed, the intrinsic ultracontractivity holds for symmetric α-stable process on any bounded open set $D$; see [33, 9].

When $D$ is unbounded, (1.3) would fail. For example, it was proved in [24, Theorem 1.2] that when $D$ is a half-space-like $C^{1,1}$ open set of $\mathbb{R}^d$, (1.2) holds for all $(t, x, y) \in (0, \infty) \times D \times D$. See [24] for more details and [16, 17, 18, 21, 31] for related developments on other (general) symmetric jump processes.

**Notation** We will use the symbol “:=” to denote a definition, which is read as “is defined to be”. In this paper, for $a, b \in \mathbb{R}$ we denote $a \land b := \min\{a, b\}$ and $a \lor b := \max\{a, b\}$. We also use the convention $0^{-1} = +\infty$. We write $h(s) \simeq f(s)$, if there exist constants $c_1, c_2 > 0$ such that $c_1 f(s) \leq h(s) \leq c_2 f(s)$ for the specified range of the argument $s$. Similarly, we write $h(s) \asymp f(s) g(s)$, if there exist constants $c_1, c_2, c_3, c_4 > 0$ such that $f(c_1 s) g(c_2 s) \leq h(s) \leq f(c_3 s) g(c_4 s)$ for the specified range of $s$. Upper case letters with subscripts denote constants that will be fixed throughout the paper. Let $C_{i,j}, \, C_{i,j}, \, c_{i,j}, \, i, j = 0, 1, 2, \ldots$ with subscripts denote constants from Lemma $i,j$ or Proposition $i,j$ or the equation $(i,j)$, which are also fixed throughout the paper. Lower case letters $c$’s without subscripts denote strictly positive constants whose values are unimportant and which may change even within a line, while values of lower case letters with subscripts $c_i, i = 0, 1, 2, \ldots$ are fixed in each proof, and the labeling of these constants starts anew in each proof. $c_i = c_i(a, b, c, \ldots)$, $i = 0, 1, 2, \ldots$, denote constants depending on $a, b, c, \ldots$. The dependence on the dimension $d \geq 2$ and the index $\alpha \in (0, 2)$ may not be mentioned explicitly.

Without any mention, the constants $C, C, C, C_{i,j}, C_{i,j}, c, c, c_i, c_{i,j}$ are independent of $x, y \in D$ and $t > 0$. For $x \in D$ we use $z_x$ to denote a point $z_x$ in $\partial D$ such that $|x - z_x| = \delta_D(x)$. For a Borel subset $V$ in $\mathbb{R}^d$, $|V|$ denotes the Lebesgue measure of $V$. We use the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = 0$.

### 1.1. Setting and main result.

The aim of this paper is to study two-sided Dirichlet heat kernel estimates of symmetric α-stable processes on horn-shaped regions (see below for the definition). We emphasize that horn-shaped regions are non-uniformly $C^{1,1}$ near infinity and usually unbounded, so the corresponding Dirichlet heat kernel estimates go beyond the scope of all the papers quoted above.

In fact, due to the non-uniform $C^{1,1}$-property of horn-shaped regions, new ideas and much more efforts are required to achieve the sharp Dirichlet heat kernel estimates. Furthermore, on the one hand, our two-sided Dirichlet heat kernel estimates are for full time. On the other hand, our results cover the case that the associated Dirichlet semigroup is not intrinsically ultracontractive. To the best of our knowledge, this is the first result on explicit estimates for Dirichlet heat kernel on non-uniformly $C^{1,1}$ and unbounded domains. Even we did not find the corresponding results for Brownian motions in the literature.

Throughout our paper, we always let $f : \mathbb{R} \to (0, \infty)$ be a continuous function satisfying the following conditions:

\begin{align}
& f(-t) \equiv f(0) \text{ for } t > 0 \text{ and } f \in C^{1,1}((0, \infty)); \\
& f \text{ is non-increasing on } (0, \infty) \text{ with } \lim_{r \to \infty} f(r) = 0; \\
& \text{for any } c \geq 1, f(cs) \simeq f(s) \text{ on } \mathbb{R}.
\end{align}

Note that the above properties imply that $f(s - 2) \leq cf(s)$ for all $s$. The function $f$ is served as the reference function for the horn-shaped region, which will be defined explicitly below.
Let \( d \geq 2 \), and write \( x = (x_1, \hat{x}) \in \mathbb{R}^d \), where \( \hat{x} = (x_2, x_3, \ldots, x_d) \). For any \( a > 0 \), denote \( D_f^a := \{ x \in \mathbb{R}^d : x_1 > a, |\hat{x}| < f(x_1) \} \).

**Definition 1.1.** For any \( d \geq 2 \), let \( D \) be an open set of \( \mathbb{R}^d \).

(1) We say that \( D \) is a horn-shaped region with the reference function \( f \), if there exists \( M \geq 2f(0) \) such that

(i) \( D \cap \{ x \in \mathbb{R}^d : x_1 < M \} \) is bounded;

(ii) \( \{ x \in D : x_1 > M \} = D_f^M \);

(iii) there exist \( c_1 \in (0, 1) \) and \( \Lambda > 0 \) such that for all \( x \in D_f^M \), \( D \) is \( C^{1,1} \) at \( z_x \in \partial D_f^M \) with the characteristics \((c_1 f(x_1), \Lambda)\).

(2) We say that \( D \) is a horn-shaped \( C^{1,1} \) region with the reference function \( f \), if \( D \) is a horn-shaped region with the reference function \( f \) and there exist \( c_1 \in (0, 1] \) and \( \Lambda > 0 \) such that for all \( x \in D \), \( D \) is \( C^{1,1} \) at \( z_x \in \partial D \) with the characteristics \((c_1 f(x_1), \Lambda)\).

See Figure 1 for a horn-shaped \( C^{1,1} \) region \( D \) when \( d = 2 \).

**Remark 1.2.** It is easy to see that, for every horn-shaped region \( D \) with the reference function \( f \), there exist horn-shaped \( C^{1,1} \) regions \( U_1 \) and \( U_2 \) with the same reference function \( f \) such that \( U_1 \subset D \subset U_2 \), and \( \delta_{U_1}(x) = \delta_{U_2}(x) = \delta_D(x) \) for \( x \in D_f^M \) with some constant \( M > 0 \).

For both mathematical and physical backgrounds on the study of analytic properties related to horn-shaped regions, readers are referred to [1, 2, 3, 8, 26, 28, 35]. We note that the properties (1.5)–(1.7) of the reference function \( f \) essentially are also imposed in [1, 2, 3, 26, 28, 35], when explicit two-sided estimates for Dirichlet eigenfunctions for horn-shaped regions are concerned.

In the following, we fix a \( C^{1,1} \) horn-shaped region \( D \) with the reference function \( f \), and set

\[
\Psi(t, x) := \frac{\delta_D(x)^{\alpha/2} (f(x_1)^{\alpha/2} \land t^{1/2})}{t \land 1} \land 1, \quad x \in D, t > 0. \tag{1.8}
\]

The function \( \Psi(t, x) \) will be used to describe the behavior of Dirichlet heat kernels near the boundary of \( D \). Note that, by the definition of \( D \), there exists a constant \( c_0 > 0 \) such that \( \delta_D(x) \leq c_0 f(x_1) \) for all \( x \in D \). Thus, there exist \( c_1, c_2 > 0 \) such that for all \( x \in D \) and \( t > 0 \),

\[
\Psi(t, x) \simeq \begin{cases} 
1 & \text{if } \delta_D(x) \geq c_1 t^{1/\alpha}, \\
\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} & \text{if } c_1 t^{1/\alpha} \leq \delta_D(x) \leq c_2 f(x_1), \\
\frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \land 1} & \text{if } c_2 f(x_1) \leq c_1 t^{1/\alpha} \geq \delta_D(x). 
\end{cases}
\]

![Figure 1. A horn-shaped \( C^{1,1} \) region in \( \mathbb{R}^2 \).](image)
We also set
\[ \phi(x) := \frac{\delta_D(x)^{\alpha/2}f(x_1)^{\alpha/2}}{(1 + |x|)^{d+\alpha}}, \quad x \in D, \] (1.9)
which is comparable to the ground state of Dirichlet fractional Laplacian \((-\Delta)^{\alpha/2}|D for the horn-shaped region \(D); see [33] Theorem 1 and Proposition 1 | or [33] Theorem 6.1 | for more details.

For any fixed constant \(c > 0\), let \(t_0(x) := t_0(c, x) \in (0, \infty)\), which is defined for all \(x \in D\), such that
\[ e^{-ct_0(x)f(x_1)^{-\alpha}} = t_0(x)(1 + |x|)^{-(d+\alpha-1)}, \quad x \in D. \] (1.10)

Since the function \(t \mapsto e^{-ct_0(x)f(x_1)^{-\alpha}}\) is continuous and strictly decreasing on \((0, \infty)\) with values on \((0, 1)\) and the function \(t \mapsto t(1 + |x|)^{-(d+\alpha-1)}\) is continuous and strictly increasing on \((0, \infty)\) with values on \((0, \infty)\), \(t_0(x)\) exists and is unique for all \(x \in D\). The functions \(e^{-ct_0(x)f(x_1)^{-\alpha}}\) and \(t(1 + |x|)^{-(d+\alpha-1)}\) come from estimates of the survival probability \(P^x(\tau_D > t)\); see Lemma 2.8 below. One can see that there is a constant \(c_1 > 0\) such that for all \(x \in D\), \(f(x_1)^{\alpha} \leq c_1 t_0(x)\). Usually it is not easy to obtain the explicit value of \(t_0(x)\); however, we possibly can get explicit estimates of \(t_0(x)\) for all \(x \in D\) under some mild assumption on the reference function \(f\). For example, if \(f(r) \geq c(1+r)^{-p}\) for some constants \(c\) and \(p > 0\), then \(t_0(x) \asymp f(x_1)^{\alpha} \log(2 + |x|)\) for all \(x \in D\).

The main result of this paper is as follows.

**Theorem 1.3.** Suppose that \(d \geq 2\) and \(D\) is a \(C^{1,1}\) horn-shaped region of \(\mathbb{R}^d\) associated with the reference function \(f\) satisfying (1.5), (1.6) and (1.7). Let \(p_D(t, x, y)\) be the transition density of killed symmetric \(\alpha\)-stable process \(X^D\) with \(\alpha \in (0, 2)\). Then, there exist constants \(c_{1,3,0}, c_{1,3,1} > 0\) such that the following two statements hold with \(t_0(\cdot) := t_0(c_{1,3,0}, \cdot)\).

1. For any \(x, y \in D\) and any \(0 < t \leq c_{1,3,1}(f(x_1) \wedge f(y_1))^{\alpha} \leq 1\),
   \[ p_D(t, x, y) \asymp p(t, x, y)\Psi(t, x)\Psi(t, y). \] (1.11)

2. Suppose in addition that \(f(s) \geq c(1+s)^{-p}\) on \((0, \infty)\) for some \(c, p > 0\), and the function \(s \mapsto f(s)^{\alpha} \log(2+s)\) is comparable to some monotone function \(g\) on \((0, \infty)\) (i.e. \(g(s) \asymp f(s)^{\alpha} \log(2+s)\)).
   (i) If \(g\) is non-increasing on \((0, \infty)\) so that \(\lim_{s \to \infty} g(s) = 0\), then there exist positive constants \(c_{1,3,i} (2 \leq i \leq 10)\) such that for any \(x, y \in D\) and any \(c_{1,3,1}(f(x_1) \vee f(y_1))^{\alpha} \leq t \leq c_{1,3,2}(t_0(x) \wedge t_0(y)) \leq c_{1,3,2}\|t_0(\cdot)\|_{\infty} < \infty\),
   \[ p_D(t, x, y) \asymp p(t, x, y)\Psi(t, x)\Psi(t, y) \exp \{-t(f(x_1) \vee f(y_1))^{-\alpha}\} \] (1.12)
   and for any \(x, y \in D\) and any \(t \geq c_{1,3,2}(t_0(x) \vee t_0(y))\),
   \[ c_{1,3,3}\phi(x)\phi(y) \max \left\{ \int_{0}^{c_{1,3,4}s_1(c_{1,3,5})} f(s)^{d-1}e^{-c_{1,3,6}sf(s)^{-\alpha}} ds, e^{-c_{1,3,6}t} \right\} \]
   \[ \leq p_D(t, x, y) \]
   \[ \leq c_{1,3,7}\phi(x)\phi(y) \max \left\{ \int_{0}^{c_{1,3,8}s_1(c_{1,3,9})} f(s)^{d-1}e^{-c_{1,3,10}sf(s)^{-\alpha}} ds, e^{-c_{1,3,10}t} \right\}, \] (1.13)
   where \(s_1(t) = g^{-1}(t) \vee 2\) and \(g^{-1}(t) = \inf\{s \geq 0 : g(s) \leq t\}\) for \(t > 0\).

(ii) If \(g\) is non-decreasing on \((0, \infty)\) so that \(\lim_{s \to \infty} g(s) = 0\), then there exists a constant \(c_{1,3,2} > 0\) such that for any \(x, y \in D\) and any \(c_{1,3,1}(f(x_1) \vee f(y_1))^{\alpha} \leq t \leq c_{1,3,2}(t_0(x) \wedge t_0(y))\),
   \[ p_D(t, x, y) \asymp p(t, x, y)\Psi(t, x)\Psi(t, y) \exp \{-t(f(x_1) \vee f(y_1))^{-\alpha}\}; \] (1.14)
   and for any \(x, y \in D\) and any \(t \geq c_{1,3,2}(t_0(x) \wedge t_0(y))\),
   \[ p_D(t, x, y) \asymp \phi(x)\phi(y)e^{-t}. \] (1.15)
Remark 1.4. Let us give some remarks on Theorem 1.3. 

(i) It is clear from Remark 1.2 and the proof of Theorem 1.3 that, for horn-shaped region $D$ (not necessarily $C^{1,1}$ near the origin), the conclusions of Theorem 1.3 still hold true for all $x, y \in D$ with $|x| \vee |y|$ large enough.

(ii) When $0 < t \leq c_1(f(x_1)^\alpha \wedge f(y_1)^\alpha)$, $p_D(t, x, y)$ satisfies (1.14), which is of the same form as (1.2): that is, $p_D(t, x, y)$ is comparable with the global heat kernel $p(t, x, y)$ multiplied by weighted functions $\Psi(t, x)$ and $\Psi(t, y)$, which are comparable to $\frac{\delta_{2p(x)^{\alpha/2}}}{\sqrt{t}}$ and $\frac{\delta_{2p(y)^{\alpha/2}}}{\sqrt{t}}$ respectively. This assertion is reasonable since $C^{1,1}$ horn-shaped region $D$ enjoys the “semi-uniform” interior ball condition in the sense that for any $x \in D$ and $r \in (0, c_1f(x_1))$ (with possibly small $c_1$), $B(x, r) \subset D$ with $x \in D$.

(iii) For $t \geq c_1(f(x_1)^\alpha \wedge f(y_1)^\alpha)$, estimates for $p_D(t, x, y)$ heavily rely on the asymptotic property of the reference function $f$. According to [33, Theorem 5], under assumptions of case (i) in (2) the associated Dirichlet semigroup $(P^D_t)_{t\geq0}$ is intrinsically ultracontractive. Note that $s_1(t) = 2$ for large $t$ under assumptions of case (i) in (2). Hence, similar to (1.3), the estimate indicated in (1.13) for $t \geq 1$ essentially is a direct consequence of the intrinsic ultracontractivity of $(P^D_t)_{t\geq0}$. However, when $c_1(f(x_1)^\alpha \wedge f(y_1)^\alpha) \leq t \leq 1$, estimates for $p_D(t, x, y)$ are much more delicate.

(iv) It will be shown in Lemma 2.8 that the following upper bound on survival probability holds true: for any $x \in D$ and $t > 0$,

$$
P^x(\tau_D > t) \leq c_1\Psi(t, x) \min\{e^{-c_2f(x_1)^{-\alpha}} + t(1 + |x|)^{-d+\alpha-1}, e^{-c_3t}\}.
$$

In particular, when $t = T_0 := c_1.32|t_0|\|y\| < \infty$ and $10T_0 < |x| < 2|y|$, (1.16) implies that

$$
p(T_0, x, y)P^x(\tau_D > T_0)P^y(\tau_D > T_0) \leq c_3(T_0)\frac{\phi(x)\phi(y)}{|y|^{d+\alpha-1}}.
$$

On the other hand, (1.13) implies that $p_D(T_0, x, y) \propto \phi(x)\phi(y)$. Therefore, the so-called Varopoulos-type estimates (1.4) do not hold true under assumptions of case (i) in (2), which is different from [11, Theorem 1.1] and [22, Theorem 1.1].

(v) Under assumptions of case (ii) in (2), the associated Dirichlet semigroup $(P^D_t)_{t\geq0}$ is not intrinsically ultracontractive, see also [33, Theorem 5]. Though (1.14) is of the same form as that for (1.12), the ranges of time variable are different; that is, $c_2(t_0(x) \wedge t_0(y)) \geq 1$ in (1.14), while $c_2(t_0(x) \vee t_0(y)) \leq 1$ in (1.12). Also by this reason, the estimates (1.13) and (1.15) are different too, even both of them enjoy the same form (by neglecting constants in the exponential term) when $t \to \infty$.

The proof of Theorem 1.3 is completely different from those in [11] and [23], where two-sided Dirichlet heat kernel estimates for fractional Laplacians in uniformly $C^{1,1}$ open sets and half-space-like open sets were established respectively. For example, because of the non-uniformity on $C^{1,1}$ characteristics, the boundary Harnack principle can not be applied to $C^{1,1}$ horn-shaped regions, and so the approach of [11, Theorem 1.1 (i)] does not work in the present setting. In order to obtain Dirichlet heat kernel estimates of horn-shaped regions, we need to take into accounts carefully the interaction between jumping kernel of symmetric $\alpha$-stable processes and the characterization (heavily depending on the reference function $f$) of the horn-shaped region. Roughly speaking, the proof of Theorem 1.3 is split into three cases according to different ranges of time and space.

(1) When $0 < t \leq c_1(f(x_1)^\alpha \wedge f(y_1)^\alpha)$, we make use of the Chapman-Kolmogorov equation and a general formula for upper bounds of Dirichlet heat kernels (see [7, Lemma 1.10], [29, Lemma 5.1] and [20, Lemma 3.1]). Note that, in this case the estimates for exit probability (see Lemma 2.6) are different from those implied by (1.2) when $c_1(f(x_1)^\alpha \wedge f(y_1)^\alpha) \leq t \leq c_1(f(x_1)^\alpha \wedge f(y_1)^\alpha)$.

(2) When $c_1(f(x_1)^\alpha \wedge f(y_1)^\alpha) \leq t \leq c_2(t_0(x) \vee t_0(y))$ or $c_2(f(x_1)^\alpha \wedge f(y_1)^\alpha) \leq t \leq c_2(t_0(x) \wedge t_0(y))$, we will adopt the chain argument to derive lower bounds and apply the split technique combined with the survival probability (2.10) to obtain upper bounds. In particular, in arguments for both cases above, instead of the boundary Harnack principle, we make use of the Lévy system. (3) When $t \geq c_1(t_0(x) \vee t_0(y))$ or $t \geq c_2(t_0(x) \wedge t_0(y))$, the dominant behaviour (with the largest probability) of the killed process taking time $t$ from $x$ to $y$ is that, the process jumps form $x$ to
the origin, and then jumps to $y$ after spending more than $t/2$ at a neighborhood of origin or at another neighborhood inside $D$ with the largest survival probability. This gives us the intuitive meanings of (1.13) and (1.15). In this case, lower bounds are derived by using assertions in cases (1) and (2); however, the proofs of upper bounds are much more involved. In particular, we will use the iteration arguments based on the survival probability.

1.2. Relation with intrinsic ultracontractivity. Recall that in the present setting the Dirichlet semigroup $(P^D_t)_{t \geq 0}$ is intrinsically ultracontractive, if for every $t > 0$ there is a constant $C_{D,t} > 0$ such that

$$p_D(t, x, y) \leq C_{D,t} \phi(x) \phi(y), \quad x, y \in D,$$

where $\phi$ is defined by (1.9) that is comparable with the ground state of $(P^D_t)_{t \geq 0}$.

The intrinsic ultracontractivity of Markov semigroups (including Dirichlet semigroups and Feynman-Kac semigroups) has been intensively established for various Lévy type processes. For more details, see [9, 10] and the references therein. The intrinsic ultracontractivity and two-sided estimates of ground state for symmetric $\alpha$-stable processes and more general symmetric jump processes on unbounded open sets were investigated in [34] and [8], respectively. We note that the two-sided Dirichlet heat kernel estimates are much more complex than estimates of ground state. Informally, to obtain Dirichlet heat kernel estimates we need to consider the relationship between time and space carefully; for ground state estimates we only just take time $t = 1$ and make use of estimates for $p(1, x, y)$; see [3] Sections 5 and 6.

In the following, we deduce explicit estimates for the intrinsic ultracontractivity under assumptions in (i) of (2) in Theorem 1.3 by directly applying two-sided Dirichlet heat kernel estimates. Recall that $g(s) \simeq f(s)^{\alpha \log(2 + s)}$.

**Proposition 1.5.** Under assumptions in (i) of (2) in Theorem 1.3, (1.17) holds with

$$C_{D,t} = c_{1.5.2} \left\{ \begin{array}{ll}
t^{-2d/\alpha} (1 + g^{-1}(c_{1.5.3}t))^{2d+2}, & 0 < t \leq c_{1.5.1}(t_0(x) \vee t_0(y)); \\
\max \left\{ \int_0^{c_{1.5.4} s^t (c_{1.5.5} e)} f(s)^{-d} e^{-c_{1.5.7} t} ds, e^{-c_{1.5.7} t} \right\}, & t > c_{1.5.1}(t_0(x) \vee t_0(y)).
\end{array} \right.$$  

**Proof.** According to (1.11) and (1.12), there are constants $c_0, c_1 > 0$ such that $p_D(t, x, y) \leq c_1 p(t, x, y) \Phi(t, x) \Phi(t, y)$ for any $x, y \in D$ and $0 < t \leq c_0(t_0(x) \vee t_0(y)) \leq 1$.

In the following, without loss of generality, we may assume that $x, y \in D$ with $x_1 \geq y_1$. According to the non-increasing property of the function $g$ and $\lim_{s \to \infty} g(s) = 0$ as well as $t_0(y) \simeq g(|y|)$, $t \leq c_2 t_0(y)$ for some $c_2 > 0$ implies that $|y| \leq c_3 g^{-1}(c_4 t)$. In particular,

$$|y|^{d+\alpha} \leq (1 + c_3 g^{-1}(c_4 t))^{d+\alpha}. \tag{1.18}$$

Thus, if $|y|/2 < |x| < 2|y|$, then, for $0 < t \leq c_0(t_0(x) \vee t_0(y)) \leq 1$,

$$p_D(t, x, y) \leq c_0 t^{-d/\alpha} \delta_D(x)^{\alpha/2} (f(x_1)^{\alpha/2} \wedge t^{1/2}) \delta_D(y)^{\alpha/2} (f(y_1)^{\alpha/2} \wedge t^{1/2})$$

$$\leq c_0 t^{-2d/\alpha} (1 + g^{-1}(c_7 t))^{2d+2\alpha} \delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2} \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2} \left(1 + |x|^{d+\alpha}\right)^{1/2} \left(1 + |y|^{d+\alpha}\right)^{1/2}$$

$$\leq c_0 t^{-2d/\alpha} (1 + g^{-1}(c_7 t))^{2d+2\alpha} \phi(x) \phi(y),$$

where in the second inequality we used the fact that $|y|/2 \leq |x| < 2|y|$ and (1.18); if $|x| \geq 2|y|$, then, for $0 < t \leq c_0(t_0(x) \vee t_0(y))$, we can argue as follows

$$p_D(t, x, y) \leq c_0 t^{-1} (1 + g^{-1}(c_10 t))^{d+\alpha} \delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2} \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2} \left(1 + |x|^{d+\alpha}\right)^{1/2} \left(1 + |y|^{d+\alpha}\right)^{1/2}$$

$$\leq c_0 t^{-1} (1 + g^{-1}(c_10 t))^{d+\alpha} \phi(x) \phi(y),$$
where the first inequality follows from the fact that $|x| \geq 2|y|$, and the second inequality is due to (1.18). Similarly, we can prove that if $|x| \leq |y|/2$, then, for $0 < t \leq c_6(t_0(x) \lor t_0(y))$, 

$$p_D(t, x, y) \leq c_1 t^{-1}(1 + g^{-1}(c_12t))^{d+\alpha}(x)\phi(y).$$

Combining all the estimates above with (1.13), we can obtain that (1.17) holds for all $x, y \in D$ and $t > 0$ with the desired estimates for $C_{D,t}$.

We would like to mention that the arguments above (in particular, (1.18)) fail, under assumptions in (ii) of (2) in Theorem 1.3, i.e., when the function $g(s)$ is non-decreasing on $(0, \infty)$.

1.3. A toy example. In this part, we present the following example to illustrate how powerful Theorem 1.3 is.

**Example 1.6.** Let $f(s) = \log^{-\theta}(2 + s)$ with $\theta > 0$ for all $s \in [0, \infty)$. For any $x, y \in D$, set $t_1(x, y) = \log^{-\theta_1}(e + (|x| \land |y|))$ and $t_2(x, y) = \log^{-\theta_1}(e + (|x| \land |y|))$. Then, we have the following two statements:

(i) Assume that $\theta > 1/\alpha$. Then, there exist positive constants $c_{1.6.1}$, $c_{1.6.2}$ and $c_{1.6.3}$ such that for all $x, y \in D$,

$$p_D(t, x, y) \asymp \begin{cases} p(t, x, y) \left( \frac{\delta_D(x)^{\alpha/2} \left( \log^{-\theta_a/2}(e + |x|) \land t^{1/2} \right)}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2} \left( \log^{-\theta_a/2}(e + |y|) \land t^{1/2} \right)}{t} \right) \land 1 & \text{for all } 0 < t \leq c_{1.6.1} t_1(x, y); \\
 p(t, x, y) \left( \frac{\delta_D(x)^{\alpha/2} \log^{-\theta_a/2}(e + |x|) \delta_D(y)^{\alpha/2} \log^{-\theta_a/2}(e + |y|)}{(1 + |x|)^{d+\alpha} (1 + |y|)^{d+\alpha}} \right) \exp \left( 1 \right) & \text{for all } c_{1.6.1} t_1(x, y) < t \leq c_{1.6.2} t_2(x, y); \\
 \left( \frac{\delta_D(x)^{\alpha/2} \log^{-\theta_a/2}(e + |x|) \delta_D(y)^{\alpha/2} \log^{-\theta_a/2}(e + |y|)}{(1 + |x|)^{d+\alpha} (1 + |y|)^{d+\alpha}} \right) \exp(-t) & \text{for all } t > c_{1.6.3}. 
\end{cases}$$

(ii) Assume that $\theta \leq 1/\alpha$. Then, there exist positive constants $c_{1.6.4}$ and $c_{1.6.5}$ such that for all $x, y \in D$,

$$p_D(t, x, y) \asymp \begin{cases} \left( \frac{\delta_D(x)^{\alpha/2} \left( \log^{-\theta_a/2}(e + |x|) \land t^{1/2} \right)}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2} \left( \log^{-\theta_a/2}(e + |y|) \land t^{1/2} \right)}{t} \right) \land 1 & \text{for all } 0 < t \leq c_{1.6.4} t_1(x, y); \\
 p(t, x, y) \left( \frac{\delta_D(x)^{\alpha/2} \log^{-\theta_a/2}(e + |x|) \delta_D(y)^{\alpha/2} \log^{-\theta_a/2}(e + |y|)}{(1 + |x|)^{d+\alpha} (1 + |y|)^{d+\alpha}} \right) \exp \left( - t \log(2 + s) \right) & \text{for all } c_{1.6.4} t_1(x, y) < t \leq c_{1.6.5} t_2(x, y); \\
 \left( \frac{\delta_D(x)^{\alpha/2} \log^{-\theta_a/2}(e + |x|) \delta_D(y)^{\alpha/2} \log^{-\theta_a/2}(e + |y|)}{(1 + |x|)^{d+\alpha} (1 + |y|)^{d+\alpha}} \right) \exp(-t) & \text{for all } t > c_{1.6.5} t_2(x, y). 
\end{cases}$$

**Proof.** This directly follows from Theorem 1.3. Here we give some details on the case that $\theta > 1/\alpha$ and $c_{1.6.2} t_2(x, y) \leq t \leq c_{1.6.3}$. For any $t > 0$, define

$$s_1(t) = \inf \{ s > 0 : f(s)^{\alpha} \log(2 + s) \leq t \} \lor 2. $$

Then, for $0 < t \leq 1$, 

$$s_1(t) \asymp \exp \left( \left( t^{-1/(\theta a - 1)} \right) \right).$$
Hence, for any $c_i > 0$ (1 ≤ $i$ ≤ 3) and $t \in (0, 1]$,
\[
\int_0^{c_1 s_1(t)} f(s)^{d-1} e^{-c_3 t s^{\alpha}} ds \propto \exp\left( t^{-1/(\theta \alpha - 1)} \right).
\]
Indeed, it is clear that for all $t \in (0, 1]$,
\[
\int_0^{c_1 s_1(t)} f(s)^{d-1} e^{-c_3 t s^{\alpha}} ds \leq (\log 2)^{-(d-1)} \int_0^{c_1 s_1(t)} ds \leq c_4 \exp(c_5 t^{-1/(\theta \alpha - 1)}).
\]
On the other hand, noting that $c_1 s_1(t) \geq c_6 \exp(c_7 t^{-1/(\theta \alpha - 1)})$ for all $t \in (0, 1]$ with some $c_6, c_7 > 0$ that satisfies $2c_3 c_7^{\theta \alpha - 1} \leq 1$, and also that the function $s \mapsto f(s)^{d-1} e^{-c_3 t s^{\alpha}}$ is decreasing on $(0, \infty)$, we have
\[
\int_0^{c_1 s_1(c_2 t)} f(s)^{d-1} e^{-c_3 t s^{\alpha}} ds \\
\geq \int_0^{c_6 \exp(c_7 t^{-1/(\theta \alpha - 1)})} f(s)^{d-1} e^{-c_3 t s^{\alpha}} ds \\
\geq \log^{-\theta(d-1)} \left( 2 + c_6 \exp(c_7 t^{-1/(\theta \alpha - 1)}) \right) \cdot \exp(-c_3 t \log^{\alpha \theta} (2 + c_6 \exp(c_7 t^{-1/(\theta \alpha - 1)})) \cdot c_6 \exp(c_7 t^{-1/(\theta \alpha - 1)}) \\
\geq c_8 t^{-\theta(d-1)/(\theta \alpha - 1)} \exp(-c_3 t - c_6 c_7^{\theta \alpha - 1} t^{-1/(\theta \alpha - 1)}) \cdot \exp(c_7 t^{-1/(\theta \alpha - 1)}) \\
\geq c_{10} t^{-\theta(d-1)/(\theta \alpha - 1)} \exp\left( \frac{c_7}{2} t^{-1/(\theta \alpha - 1)} \right) \geq \exp(c_{11} t^{-1/(\theta \alpha - 1)})
\]
for all $t \in (0, 1]$.

With these at hand, we can get the required assertions in Example 1.6.

Note that, for this example, the associated Dirichlet semigroup $(P^D_t)_{t \geq 0}$ is intrinsically ultra-contractive, if and only if $\theta > 1/\alpha$; see [34, Example 2] or [8, Theorem 1.1(1)]. On the other hand, it is easy to see that \( \limsup_{|x|, |y| \to \infty} t_2(x, y) = 0 \), if and only if $\theta > 1/\alpha$. This explains why there is a threshold at $\theta = 1/\alpha$ for two-sided estimates of $p_D(t, x, y)$.

The rest of this paper is arranged as follows. The next section serves as preparations for main proofs. Results in Sections 2 will be frequently used in the proof of Theorem 1.3. In particular, upper bound estimates of survival probabilities for full time are presented here. Sections 3, 4 and 5 are devoted to the proof of Theorem 1.3, according to different ranges of time. Proof of Theorem 1.3 and further remarks are briefly given in Section 6.

2. Preparations

2.1. Preliminary estimates. In this part, we collect some (mostly known) results which will be frequently used in proofs of our paper. Throughout this paper, let $X := \{X_t, t \geq 0; P^x, x \in \mathbb{R}^d\}$ be a (rotationally) symmetric $\alpha$-stable process in $\mathbb{R}^d$ with $d \geq 2$, whose transition density is denoted by $p(t, x, y)$. For any open subset $U$, let $X^U$ be the subprocess of $X$ killed upon leaving $U$, whose transition density is denoted by $p_U(t, x, y)$. Let $\tau_U := \inf\{t \geq 0 : X_t \notin U\}$ be the first exit time from $U$ for the process $X$. It is well known (cf. see [11, Lemma 3.2]) that, for any $\kappa_1, \kappa_2 > 0$, there exists a constant $c_1 := c_1(\kappa_1, \kappa_2) > 0$ such that for all $x \in \mathbb{R}^d$ and $t > 0$,
\[
P^x(\tau_{B(x, \kappa_1 t^{1/\alpha})} > \kappa_2 t) \geq c_1.
\] (2.1)

Recall that the Lévy system of $X$ describes the behaviors of jumps for the process $X$. In particular, given a non-negative function $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ with $f(s, x, x) = 0$ for all $s > 0$ and $x \in \mathbb{R}^d$, it holds for any stopping time $\tau$ that
\[
\mathbb{E}^x \left[ \sum_{s \leq \tau} f(s, X_{s-}, X_s) \right] = \mathbb{E}^x \int_0^\tau \int_{\mathbb{R}^d} f(s, X_s, y) \frac{c_{d, \alpha}}{|X_s - y|^{d+\alpha}} dy ds.
\] (2.2)

We refer the reader to [22, Lemma 4.7] for more details about the property of Lévy system. On the other hand, according to [34, Lemma 2], we have
Lemma 2.1. There exists a constant $c_{2.1.1} > 0$ such that for any open set $U \subset \mathbb{R}^d$, $x \in U$ and $t > 0$, 
\begin{equation}
\mathbb{P}^x(\tau_U > t) \le \exp(-c_{2.1.1}\eta_U t),
\end{equation}
where $\eta_U := \inf_{z \in U} \int_{U} |x - z|^{-d-\alpha} \, dz$.

Throughout the remainder of this paper, let $f : \mathbb{R} \to (0, \infty)$ satisfy (1.5), (1.6) and (1.7). For fixed constants $c_* \in (0, 1/5)$ and $\Lambda > 0$, let $D$ be a horn-shaped $C^{1,1}$ region with the reference function $f$ so that for all $x \in D$, $D$ is $C^{1,1}$ at $x \in \partial D$ with the characteristics $(5c_* f(x_1, \Lambda))$.

To save notations in the proofs, without loss of generality, we may assume that the following conditions are satisfied:

(i) $f(0) \le 2^{-2}$, and for all $x \in D$, $\delta_D(x) \le 2^{-1}$;
(ii) $D \cap \{x \in \mathbb{R}^d : x_1 < 2\} \subset B(0,2)$, and $\{x \in D : x_1 > 2\} = D^2_c$;
(iii)(non-uniform) Interior ball condition: for every $x \in D$ and $0 < r \le 5c_* f(x_1)$, $B(\xi^*_{x,r}, r) \subset D$, where $\xi^*_{x,r} := z_x + r(x - z_x)/|x - z_x|$.

We remark here that, clearly the arguments below work for general $C^{1,1}$ horn-shape regions without the additional assumptions (i)–(iii).

Lemma 2.2. There exists a constant $c_{2.2.1} > 0$ such that for all $x \in D$, $0 < t \le c_{2.2.1} f(x_1)^\alpha$ and $\lambda_1 > 0$ ($i = 1, 2, 3$), there is a constant $c_{2.2.2} := c_{2.2.1}(c_{2.1.1}, \lambda_1, \lambda_2, \lambda_3)$ so that when $\delta_D(x) \ge \lambda_1 t^{1/\alpha}$, $p_D(t, x, y) \ge c_{2.2.2}t^{-d/\alpha}$ holds for all $y \in D$ with $\delta_D(y) \ge \lambda_2 t^{1/\alpha}$ and $|x - y| \le \lambda_3 t^{1/\alpha}$.

Proof. Since for any $x \in D$, $D$ is $C^{1,1}$ at $x \in \partial D$ with the characteristics $(5c_* f(x_1, \Lambda))$, the desired assertion can be proven by the arguments for the proof of [11] Proposition 3.3 or [19] Proposition 3.6].

The next lemma is partially motivated by [32] Lemma 5.4] and [29] Lemma 7.4].

Lemma 2.3. For every $\lambda \in (0, 1]$, there exists a constant $c_{2.3.1} := c_{2.3.1}(\lambda) > 0$ such that for all $t > 0$ and $x \in D$ with $0 < t^{1/\alpha} \le c_* f(x_1)$, there is $\xi^*_{x} \in D$ so that $B(\xi^*_{x}, 4\lambda t^{1/\alpha}) \subset D$ and 
\begin{equation}
\int_{B(\xi^*_{x}, 2\lambda t^{1/\alpha})} p_D(t, x, z) \, dz \ge c_{2.3.1} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right),
\end{equation}
where 
\begin{align*}
\xi^*_{x} := \left\{ \begin{array}{ll}
\xi^*_{x,4\lambda t^{1/\alpha}} = z_x + 4\lambda t^{1/\alpha}(x - z_x)/|x - z_x| & \text{when } \delta_D(x) \le 4\lambda t^{1/\alpha}, \\
\xi^*_{x,4\lambda t^{1/\alpha}} = z_x + 4\lambda t^{1/\alpha}(x - z_x)/|x - z_x| & \text{when } \delta_D(x) > 4\lambda t^{1/\alpha}.
\end{array} \right.
\end{align*}

Proof. Fix $\lambda \in (0, 1]$. If $\delta_D(x) > 4\lambda t^{1/\alpha}$, then $B(x, 4\lambda t^{1/\alpha}) \subset D$, and so 
\begin{equation}
\int_{B(x, 2\lambda t^{1/\alpha})} p_D(t, x, z) \, dz \ge \int_{B(x, 2\lambda t^{1/\alpha})} p_{B(x, 4\lambda t^{1/\alpha})}(t, x, z) \, dz = \mathbb{P}^x(\tau_{B(x, 4\lambda t^{1/\alpha})} > t) \ge c_4,
\end{equation}
where the last inequality follows from (2.1) with $\kappa_1 = 2\lambda$ and $\kappa_2 = 1$. Thus (2.4) holds for this case.

Now, we turn to the case that $\delta_D(x) \le 4\lambda t^{1/\alpha}$. Since $5\lambda t^{1/\alpha} \le 5c_* f(x_1)$, according to the (non-uniform) interior ball condition of $D$, $B(\xi^*_{x}, 4\lambda t^{1/\alpha}) \subset D$ and $B(\xi^*_{x}, 5\lambda t^{1/\alpha}) \subset D$ with $\xi^*_{x} := \xi^*_{x,5\lambda t^{1/\alpha}} = z_x + 5\lambda t^{1/\alpha}(x - z_x)/|x - z_x|$. In particular, $B(\xi^*_{x}, 2\lambda t^{1/\alpha}) \subset B(\xi^*_{x}, 5\lambda t^{1/\alpha}) \subset D$. Since $\delta_D(x) \le 4\lambda t^{1/\alpha}$, we have $x \in B(\xi^*_{x}, 5\lambda t^{1/\alpha})$ with $\delta_B(\xi^*_{x}, 5\lambda t^{1/\alpha})(x) = \delta_D(x)$, and, for any $z \in B(\xi^*_{x}, 2\lambda t^{1/\alpha})$, $|x - z| \le \left| x - \xi^*_{x} \right| + \left| \xi^*_{x} - z \right| \le 2\lambda t^{1/\alpha}$ and $\delta_B(\xi^*_{x}, 5\lambda t^{1/\alpha})(z) \ge 2\lambda t^{1/\alpha}$. Thus, according to (1.2), 
\begin{align*}
&\int_{B(\xi^*_{x}, 2\lambda t^{1/\alpha})} p_D(t, x, z) \, dz \ge \int_{B(\xi^*_{x}, 2\lambda t^{1/\alpha})} p_{B(\xi^*_{x}, 5\lambda t^{1/\alpha})}(t, x, z) \, dz \\
&\ge c_2 \frac{\delta_B(\xi^*_{x}, 5\lambda t^{1/\alpha})(x)^{\alpha/2}}{\sqrt{t}} t^{-d/\alpha} \int_{B(\xi^*_{x}, 2\lambda t^{1/\alpha})} \frac{\delta_B(\xi^*_{x}, 5\lambda t^{1/\alpha})(z)^{\alpha/2}}{\sqrt{t}} \, dz \ge c_3 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.
\end{align*}
Lemma 2.4. There exist constants $c_{2.4.1} \in (0, 1)$ and $c_{2.4.2} > 0$ such that for all $t > 0$ and $x \in D$ with $0 < t \leq c_{2.4.1} f(x_1)^\alpha$, 
\[ P^x(\tau_D > t) \leq c_{2.4.2} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \land 1 \right) . \] (2.5) 

Proof. It suffices to prove (2.5) for the case $\delta_D(x) \leq c_1 t^{1/\alpha}$ with arbitrary fixed $c_1 > 0$.

On the one hand, note that for any $x \in D$, $D$ is $C^{1,1}$ at $z_x \in \partial D$ with the characteristics $(5c_4 f(x_1), \Lambda)$. We can follow the proof of [32] (2.11) in Theorem 2.6 to find constants $c_2, c_3 \in (0, 1)$ such that for every $x \in D$ and $0 < t \leq c_2 f(x_1)^\alpha$ with $\delta_D(x) \leq c_3 t^{1/\alpha}$,
\[ E^x[\tau_V] \leq c_4 t^{1/2} \delta_D(x)^{\alpha/2} , \] (2.6)
where $V_t := B(z_x, 2c_3 t^{1/\alpha}) \cap D$.

On the other hand, according to [21, Lemma 2.4], it holds that for every $t > 0$ and $x \in D$ with $\delta_D(x) \leq c_3 t^{1/\alpha}$,
\[ P^x(X_{\tau_V} \in D) \leq P^x(X_{\tau_V} \in B(z_x, 2c_3 t^{1/\alpha})^c) \leq c_5 t^{-1} E^x[\tau_V] . \] (2.7)
Combining both estimates above together yields that, for any $x \in D$ and $0 < t \leq c_2 f(x_1)^\alpha$ with $\delta_D(x) \leq c_3 t^{1/\alpha}$, (by noting that $f \leq 2^{-2}$),
\[ P^x(\tau_D > t) = P^x(\tau_V \geq t) + P^x(\tau_D > t > \tau_V) \leq P^x(\tau_V \geq t) + P^x(X_{\tau_V} \in D) \leq c_6 t^{-1} E^x[\tau_V] \leq c_7 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}, \]
proving the desired assertion. \[ \square \]

Lemma 2.5. For all $\lambda \in (0, 1]$, there exist constants $c_{2.5.1} := c_{2.5.1}(\lambda)$ and $c_{2.5.2} := c_{2.5.2}(\lambda) \in (0, 1)$ such that for any $t > 0$ and $x \in D$ with $0 < t \leq c_{2.5.1} f(x_1)^\alpha$ and $\delta_D(x) \leq \lambda t^{1/\alpha}$,
\[ P^x(\tau_{B(z_x, 10 \lambda t^{1/\alpha}) \cap D} > t) \geq c_{2.5.2} \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} . \] (2.8)
Proof. This follows from the proof of [32] Lemma 5.2], thanks to the fact that for any $x \in D$, $D$ is $C^{1,1}$ at $z_x \in \partial D$ with the characteristics $(5c_4 f(x_1), \Lambda)$. \[ \square \]

Lemma 2.6. There exist constants $c_{2.6.1} \in (0, 1)$ and $c_{2.6.2} > 0$ such that for all $t > 0$ and $x \in D$ with $\delta_D(x) \leq c_{2.6.1} t^{1/\alpha}$,
\[ E^x[\tau_{B(z_x, c_{2.6.1} t^{1/\alpha} \Lambda \cap D})] \leq c_{2.6.2} \delta_D(x)^{\alpha/2} (f(x_1)^{\alpha/2} \land t^{1/2}) . \] (2.9) 

Proof. Let $c_2, c_3 \in (0, 1]$ be the constants in (2.6), and set $c_{2.6.1} = c_3$. When $0 < t \leq c_2 f(x_1)^\alpha$, (2.9) follows from (2.6). If $t > c_2 f(x_1)^\alpha$, then, according to [8, Lemma 6.2] (by choosing $c_{2.6.1}$ small if necessary),
\[ E^x[\tau_{B(z_x, c_{2.6.1} t^{1/\alpha} \Lambda \cap D})] \leq E^x[\tau_{B(z_x, c_{2.6.1} \Lambda \cap D})] \leq c_4 \delta_D(x)^{\alpha/2} f(x_1 - 2)^\alpha . \]
Combining both estimates above with the fact that $f(x_1 - 2) \leq c_5 f(x_1)$ for $x \in D$ immediately yields (2.9). \[ \square \]

Recall that $\Psi(t, x)$ is defined in (1.8).

Lemma 2.7. There exists a constant $c_{2.7.1} > 0$ such that for every $x, y \in D$ and $t > 0$ with $t^{1/\alpha} \leq 2|x - y|$, 
\[ p_D(t, x, y) \leq c_{2.7.1} \frac{t}{|x - y|^{d+\alpha}} \Psi(t, x) . \]
Proof. (i) Case 1: \( \delta_D(x) \geq 2^{-t^{1/\alpha}} \). For any \( x, y \in D \) and \( t > 0 \),
\[
p_D(t, x, y) \leq p(t, x, y) \leq \frac{c_1 t}{|x-y|^{|d+\alpha|}} \leq \frac{c_2 t}{|x-y|^{|d+\alpha|}} \left( \frac{\delta_D(x)^{\alpha/2} \left( f(x_1)^{\alpha/2} \land t^{1/2} \right)}{t} \land 1 \right),
\]
where in the last inequality we used the fact that \( 2^{-t^{1/\alpha}} \leq \delta_D(x) \leq c_3 f(x_1) \) for all \( x \in D \).

(ii) Case 2: \( \delta_D(x) < 2^{-t^{1/\alpha}} \). Without loss of generality, we may assume that the constant \( c_{2.6.1} \) in Lemma 2.6 is smaller than \( 2^{-4} \). For fixed \( x, y \in D \) such that \( t^{1/\alpha} \leq 2|x-y| \), let \( V_1 = B(z_x, c_4 c_{2.6.1} (t^{1/\alpha} \land 1)) \cap D \) with \( c_4 \in (0, 1) \) small enough, \( V_3 = \{ z \in D : |z-x| \geq |x-y|/2 \} \) and \( V_2 = D \setminus (V_1 \cup V_3) \). Since \( |z-x| \geq |x-y|/2 \geq t^{1/\alpha}/4 \) for all \( z \in V_3 \) and \( c_{2.6.1} \leq 2^{-4} \), we have \( dist(V_1, V_3) > 0 \). Then, by \[29\] Lemma 5.1] (see \[7\] Lemma 1.10 and \[20\] Lemma 3.1] for the proof) we find that
\[
p_D(t, x, y) \leq P^x(\tau_{V_1} \in V_2) \sup_{0 \leq s \leq t, z \in V_2} p_D(s, z, y) + c_5 (t \land E^x[\tau_{V_1}]) \sup_{v \in V_1, z \in V_3} \frac{1}{|v-z|^{|d+\alpha|}} \leq \frac{c_6}{|x-y|^{|d+\alpha|}} \left( \frac{E^x[\tau_{V_1}]}{t} \land 1 \right) \sup_{0 \leq s \leq t, z \in V_2} p_D(s, z, y) + c_5 (t \land E^x[\tau_{V_1}]) \sup_{v \in V_1, z \in V_3} \frac{1}{|v-z|^{|d+\alpha|}} \leq \frac{c_7 t}{|x-y|^{|d+\alpha|}} \left( \frac{E^x[\tau_{V_1}]}{t} \land 1 \right) \leq \frac{c_8 t}{|x-y|^{|d+\alpha|}} \left( \frac{\delta_D(x)^{\alpha/2} \left( f(x_1)^{\alpha/2} \land t^{1/2} \right)}{t} \land 1 \right),
\]
where the second inequality is due to \([2.7]\), in the third inequality we used the facts that
\[
p_D(s, z, y) \leq p(s, z, y) \leq \frac{c_9 s}{|z-y|^{|d+\alpha|}} \leq \frac{c_{10} t}{|x-y|^{|d+\alpha|}}, \quad z \in V_2, 0 < s \leq t
\]
(thanks to \( |z-x| \leq |x-y|/2 \) for all \( z \in V_2 \)) and
\[
\sup_{v \in V_1, z \in V_3} \frac{1}{|v-z|^{|d+\alpha|}} \leq \sup_{v \in V_1, z \in V_3} \frac{1}{|z-x| - |v-x|)^{|d+\alpha|}} \leq \frac{1}{(|x-y|/2 - 2^{-t^{1/\alpha}})^{|d+\alpha|}} \leq \frac{c_{11}}{|x-y|^{|d+\alpha|}},
\]
(thanks to the fact that \( |x-y| \geq t^{1/\alpha}/2 \)), and the fourth inequality follows from \([2.9]\). The proof is complete. \( \square \)

2.2. Estimate of the survival probability. In this part, we will present the following estimate for the survival probability, which extends Lemma 2.4 for all \( t > 0 \).

Lemma 2.8. There are positive constants \( c_{2.8.1} \) and \( c_{2.8.2} \) such that for any \( t > 0 \) and \( x \in D \),
\[
P^x(\tau_D > t) \leq c_{2.8.1} \Psi(t, x) \min \left\{ e^{-c_{2.8.2} f(x)} \land t + t(1 + |x|)^{-(d+\alpha-1)}, e^{-c_{2.8.2} t} \right\}. \tag{2.10}
\]
Proof. (i) We will first show that for all \( t > 0 \) and \( x \in D \),
\[
P^x(\tau_D > t) \leq c_1 \min \left\{ e^{-c_2 f(x_1)} \land t + t(1 + |x|)^{-(d+\alpha-1)}, e^{-c_2 t} \right\}. \tag{2.11}
\]
By \[3\] (2.10) in Proposition 2.8] and the fact that \( \delta_D(x) \leq c_3 f(x_1) \) for all \( x \in D \), we know that for any \( U \subset D \) and \( z \in U \),
\[
\int_U \frac{1}{|z-y|^{|d+\alpha|}} dy \geq \int_D \frac{1}{|z-y|^{|d+\alpha|}} dy \geq c_4 \delta_D(z)^{\alpha} \geq c_5 f(z_1)^{\alpha}.
\]
In particular, by \([2.3]\), for all \( t > 0 \) and \( x \in D \), \( P^x(\tau_D > t) \leq e^{-c_{4.2} t} \). Thus, in order to verify \([2.11]\), we only need to prove that for all \( t > 0 \) and \( x \in D \) with \( |x| \) large enough,
\[
P^x(\tau_D > t) \leq c_1 \left( e^{-c_2 f(x_1)} \land t(1 + |x|)^{-(d+\alpha-1)} \right).
\]
For any \( x \in D \) with \( |x| \) large enough, let \( U = B(x, |x|/2) \cap D \). Then, for \( t > 0 \),
\[
P^x(\tau_D > t) = P^x(\tau_U > t) + P^x(\tau_D > t \geq \tau_U) \leq P^x(\tau_U > t) + P^x(X_{t_U} < t, X_t \in B(x, |x|/3) \cap D) + P^x(X_t \in B(x, |x|/3)^c \cap D)
\]
\[
= I_1 + I_2 + I_3.
\]
First, by \((2.3)\) and \((2.12)\),

\[
I_1 \leq \mathbb{P}^x(\tau_U > t) \leq \exp \left( -c_7 t \inf_{z \in D, |z| > |x|/2} f(z_1)^{-\alpha} \right) \leq \exp \left( -c_8 f(x_1)^{-\alpha} t \right),
\]

where the last inequality above is due to \((1.7)\).

Second, due to the strong Markov property and \((1.7)\),

\[
I_2 \leq \mathbb{E}^x \left[ \mathbb{P}^{X_U}(X_{t - \tau_U} \in B(x, |x|/3) \cap D) : \tau_U \leq t, X_{\tau_U} \in D \right]
\leq \sup_{0 < s < t, \tau \in \mathbb{U} \cap D} \mathbb{P}^x(X_s \in B(x, |x|/3) \cap D) \leq \sup_{0 < s < t, \tau \in \mathbb{U} \cap D} \int_{B(x, |x|/3) \cap D} p(s, z, y) \, dy
\leq \frac{c_9 t}{|x|^{d+\alpha}} |B(x, |x|/3) \cap D|
\leq \frac{c_{10} f(x_1)^{d-1} t}{(1 + |x|)^{d+\alpha-1}} \leq \frac{c_{11} t}{(1 + |x|)^{d+\alpha-1}},
\]

where in the fourth inequality we used the fact that \(|z - y| \geq |x|/6\) for any \(x, y, z \in \mathbb{R}^d\) with \(|z - x| \geq |x|/2\) and \(|y - x| \leq |x|/3\) (and so \(p(s, z, y) \leq c_{12} s^{-d-\alpha}\) for all \(s > 0\)).

Third, it holds that

\[
I_3 \leq \int_{B(x, |x|/3) \cap D} p(t, x, z) \, dz \leq \int_{B(x, |x|/3) \cap D} \frac{c_{13} t}{|x - z|^{d+\alpha}} \, dz
\leq c_{14} \int_{|z|/3}^{\infty} \frac{1 + f(s)^{d-1}}{s^{d+\alpha}} \, ds \leq \frac{c_{15} t}{(1 + |x|)^{d+\alpha-1}}.
\]

Combining all the estimates above, we prove \((2.13)\), and so \((2.11)\) holds true.

(ii) In the following, we set

\[
L(x, t) = \min \left\{ e^{-c_2 t f(x_1)^{-\alpha}} + t(1 + |x|)^{-(d+\alpha-1)}, e^{-c_2 t} \right\}.
\]

We first consider the case \(\delta_D(x) \leq c_{2.6.1} t^{1/\alpha}\) (where \(c_{2.6.1} > 0\) is the constant in Lemma 2.6). Letting \(V_1 = B(x, c_{2.6.1} (t^{1/\alpha} \wedge 4^{-1}) \cap D)\), we have

\[
\mathbb{P}^x(\tau_D > t) \leq \mathbb{P}^x(\tau_{V_1} > t/2, \tau_D > t) + \mathbb{P}^x(0 < \tau_{V_1} \leq t/2, X_{\tau_{V_1}} \in D, \tau_D > t) =: J_1 + J_2.
\]

By the strong Markov property, \((2.9)\) and \((2.11)\), we get

\[
J_1 = \mathbb{E}^x \left[ \mathbb{P}^{X_{t/2}}(\tau_D > t/2) \right] \leq \mathbb{P}^x(\tau_{V_1} > t/2) \sup_{z \in V_1} \mathbb{P}^z(\tau_D > t/2)
\leq c_{16} \left( \frac{\mathbb{E}^x[\tau_{V_1}]}{t} \wedge 1 \right) \sup_{z \in V_1} \mathbb{P}^z(\tau_D > t/2) \leq c_{17} \Psi(t, x) \sup_{z \in V_1} L(z, t/2).
\]

Let \(V_3 = \{ z \in D : |z - x| \geq 1 + |x|/2 \} \) and \(V_2 = D \setminus (V_1 \cup V_3) \). If \(z \in B(z_2, 2^{-1})\), then \(|z - x| \leq |z - z_2| + \delta_D(x) \leq 2^{-1} + 2^{-1} = 1\), which implies that dist \((V_1, V_3) > 0\). (Here we note that \(\delta_D(x) \leq 1/2\) for all \(x \in D\) by our assumption). Using \(V_1, V_2\) and \(V_3\), we bound \(J_2\) as

\[
J_2 = \mathbb{E}^x \left[ \mathbb{P}^{X_{t/2}}(\tau_D > t/2) : 0 < \tau_{V_1} \leq t/2, X_{\tau_{V_1}} \in V_2 \right] + \mathbb{E}^x \left[ \mathbb{P}^{X_{t/2}}(\tau_D > t/2) : 0 < \tau_{V_1} \leq t/2, X_{\tau_{V_1}} \in V_3 \right] =: J_{2.1} + J_{2.2}.
\]

We find that

\[
J_{2.1} \leq \mathbb{P}^x(X_{t/2} \in V_2) \sup_{z \in V_2} \mathbb{P}^z(\tau_D > t/2) \leq c_{18} \left( \frac{\mathbb{E}^x[\tau_{V_1}]}{t} \wedge 1 \right) \sup_{z \in V_2} L(z, t/2)
\leq c_{19} \Psi(t, x) \sup_{z \in V_2} L(z, t/2),
\]

where the second inequality above follows from \((2.7)\) and \((2.11)\), and the last inequality is due to \((2.9)\).

For \(J_{2.2}\), we use the Lévy system \((2.2), (2.11)\) and \((2.9)\) again and obtain that

\[
J_{2.2} \leq c_{20} e^{-c_2 t} \mathbb{E}^x \left[ \int_0^{\tau_{V_1} \wedge (t/2)} \int_{V_3} \frac{1}{X_{t/2} - z}^{d+\alpha} \, dz \, ds \right]
\]
\[ \leq c_{21} e^{-c_{22} t} (E^t [\tau_{V_1}] \wedge t) \int_{V_3} \frac{dz}{|x - z|^{d+\alpha}} \leq c_{22} e^{-c_{22} t} \left( \delta_D(x)^{\alpha/2} (f(x_1)^{\alpha/2} \wedge t^{1/2}) \wedge t \right) \int_{1+|x|/2}^{\infty} \frac{ds}{s^{d+\alpha}} \]

\[ \leq c_{23} \left( \frac{\delta_D(x)^{\alpha/2}}{t} \left( f(x_1)^{\alpha/2} \wedge t^{1/2} \right) \wedge 1 \right) \frac{te^{-c_{24} t}}{(1 + |x|)^{d+\alpha-1}} \leq c_{24} \Psi(t, x) \min \left\{ \frac{t}{(1 + |x|)^{d+\alpha-1}}, e^{-c_{24} t/2} \right\}, \]

where in the second inequality we used the fact that for any \( y \in V_1 \) and \( z \in V_3, \)

\[ |y - z| \geq |z - y| \geq |z - x| - |x - z| - |y - z| \geq |x - z| - 1/2 - 1/4 \geq |x - z|/4. \]

Note that \( |z| \leq 3|x|/2 + 1 \) for every \( z \in V_1 \cup V_2. \) Then, by the fact that \( f(s - 2) \leq c_{25} f(s) \) for all \( s > 0 \) and (1.17),

\[ \sup_{z \in V_1 \cup V_2} L(z, t/2) \leq c_{26} \min \left\{ e^{-c_{27} tf(x_1)^{1-\alpha} + t(1 + |x|)^{-(d+\alpha-1)}}, e^{-c_{27} t} \right\}. \]

Therefore, the desired assertion (2.10) for the case \( \delta_D(x) \leq c_{26,1} t^{1/\alpha} \) follows from all the estimates above.

Next, we turn to the case that \( \delta_D(x) \geq c_{26,1} t^{1/\alpha} \) (which is possible only when \( t^{1/\alpha} \leq c_{27} f(x_1) \), thanks to the fact that \( \delta_D(x) \leq c_{27} f(x_1) \) for all \( x \in D \)). Then, according to (2.11), we have

\[ \mathbb{P}^x (\tau_D > t) \leq c_1 \min \left\{ e^{-c_3 f(x_1)^{1-\alpha} + t(1 + |x|)^{-(d+\alpha-1)}}, e^{-c_3 t} \right\} \]

\[ \leq c_{29} \Psi(t, x) \min \left\{ e^{-c_3 f(x_1)^{1-\alpha} + t(1 + |x|)^{-(d+\alpha-1)}}, e^{-c_3 t} \right\}, \]

where in the second inequality we used the fact that \( t^{1/\alpha} \leq c_{27} f(x_1) \). Thus, we establish (2.10) for all \( x \in D. \) The proof is complete. \( \square \)

From the next section to Section 5 we will prove Theorem 1.3 which is exactly split into three cases according to different ranges of time \( t. \) By the symmetry of \( p_{D}(t, x, y) \) with respect to \( (x, y), \) without loss of generality, we will assume that \( x_1 \geq y_1 \) throughout Sections 3–5.

3. Case 1: \( t \leq C_0 (y_1)^{\alpha} \) for some small constant \( C_0 > 0 \)

In this section, we will consider the case that \( 0 < t \leq C_0 (y_1)^{\alpha}, \) where \( C_0 \in (0,1) \) is a small positive constant to be fixed later.

3.1. Near diagonal estimates, i.e., \( |x - y| \leq t^{1/\alpha}. \)

Lemma 3.1. (Lower bound) There exist constants \( c_{3,1,1}, c_{3,1,2} \in (0,1) \) such that for all \( t > 0 \) and \( x, y \in D \) with \( 0 < t \leq c_{3,1,1} f(y_1)^{\alpha} \) and \( |x - y| \leq t^{1/\alpha}, \)

\[ p_D(t, x, y) \geq c_{3,1,2} t^{-d/\alpha} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right). \]

Proof. (i) Case 1: \( \delta_D(y) > 3t^{1/\alpha}. \) Since \( B(x, 2t^{1/\alpha}) \subset D \) in this case, by (1.2)

\[ p_D(t, x, y) \geq p_{B(x, 2t^{1/\alpha})}(t, x, y) \geq c_{1} t^{-d/\alpha}. \]

(ii) Case 2: \( \delta_D(y) \leq 3t^{1/\alpha}. \) It is obvious that \( \delta_D(x) \leq 4t^{1/\alpha}. \) Recall that we have assumed that \( f \leq 1/4. \) Then, \( |x_1 - y_1| \leq |x - y| \leq t^{1/\alpha} \leq c_{3,1,1} f(y_1) \leq 1/4, \) and so \( f(x_1) \asymp f(y_1). \) In particular, \( t \leq c_{3,1,1} f(x_1)^{\alpha}. \) Hence, by choosing \( c_{3,1} \in (0,1) \) small if necessary, we get from Lemma 2.3 that \( B(x, 2t^{1/\alpha}) \subset D, B(\xi_x^t, 2(t/3)^{1/\alpha}) \subset D, \) and

\[ \int_{B(\xi_x^t, (t/3)^{1/\alpha})} p_D(t/3, x, z) \, dz \geq c_3 \delta_D(x)^{\alpha/2} \sqrt{t}, \quad \int_{B(\xi_y^t, (t/3)^{1/\alpha})} p_D(t/3, y, z) \, dz \geq c_3 \delta_D(y)^{\alpha/2} \sqrt{t}, \quad (3.1) \]

where \( \xi_x^t := z_x + 2(t/3)^{1/\alpha}(x - z_x)/|x - z_x| \) and \( \xi_y^t := z_y + 2(t/3)^{1/\alpha}(y - z_y)/|y - z_y|. \)

On the other hand, for every \( z_1 \in B(\xi_x^t, (t/3)^{1/\alpha}) \) and \( z_2 \in B(\xi_y^t, (t/3)^{1/\alpha}), \) we have \( \delta_D(z_1) \geq (t/3)^{1/\alpha}, \delta_D(z_2) \geq (t/3)^{1/\alpha} \) and

\[ |z_1 - z_2| \leq |z_1 - \xi_x^t| + |\xi_x^t - x| + |x - y| + |\xi_y^t - y| + |z_2 - \xi_y^t| \leq c_4 t^{1/\alpha}. \]
Thus, by Lemma 2.2,
\[ p_D(t/3, z_1, z_2) \geq c_5 t^{-d/\alpha}, \quad (z_1, z_2) \in B((\xi^t, (t/3)^{1/\alpha}) \times B((\xi^t, (t/3)^{1/\alpha}). \]
Combining this with (3.1) in turn gives us
\[
p_D(t, x, y) \geq \int_{B(\xi^t_{x,y}, (t/3)^{1/\alpha})} p_D(t/3, x, z_1) p_D(t/3, z_1, z_2) p_D(t/3, z_2, y) dz_1 dz_2
\]
\[
\geq c_5 t^{-d/\alpha} \left( \int_{B(\xi^t_{x,y}, (t/3)^{1/\alpha})} p_D(t/3, x, z) dz \right) \left( \int_{B(\xi^t_{x,y}, (t/3)^{1/\alpha})} p_D(t/3, y, z) dz \right)
\]
\[
\geq c_5 t^{-d/\alpha} \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}.
\]
Using both estimates in (i) and (ii), we obtain the desired assertion. \( \square \)

Lemma 3.2. (Upper bound) There exist constants \( c_{3.2.1} \in (0,1) \) and \( c_{3.2.2} > 0 \) such that for all \( t > 0 \) and \( x, y \in D \) with \( 0 < t \leq c_{3.2.1} f(y)^{\alpha} \) and \( |x - y| \leq t^{1/\alpha} \),
\[ p_D(t, x, y) \leq c_{3.2.2} t^{-d/\alpha} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right). \]
Proof. As explained in the beginning of part (ii) of the proof for Lemma 3.1 above, we can apply Lemma 2.4 and obtain that for all \( t > 0 \) and \( x, y \in D \) with \( 0 < t \leq c_{3.2.1} f(y)^{\alpha} \) and \( |x - y| \leq t^{1/\alpha} \) (by choosing \( c_{3.2.1} \) small enough if necessary), it holds that \( t \leq c_1 c_{3.2.1} f(y)^{\alpha} \), and
\[ P^x(\tau_D > t/3) \leq c_2 \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right), \quad P^y(\tau_D > t/3) \leq c_2 \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right). \]
Hence,
\[
p_D(t, x, y) = \int_D \int_D p_D(t/3, x, z_1) p_D(t/3, z_1, z_2) p_D(t/3, z_2, y) dz_1 dz_2
\]
\[
\leq c_3 t^{-d/\alpha} \left( \int_D p_D(t/3, x, z_1) dz_1 \right) \left( \int_D p_D(t/3, z_2, y) dz_2 \right)
\]
\[
= c_4 t^{-d/\alpha} P^x(\tau_D > t/3) P^y(\tau_D > t/3) \leq c_4 t^{-d/\alpha} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right).
\]
The proof is complete. \( \square \)

3.2. Off-diagonal estimates, i.e., \( |x - y| \geq t^{1/\alpha} \).

Lemma 3.3. (Lower bound when \( 0 < t \leq C_0 f(x_1)^{\alpha} \).) There exist constants \( c_{3.3.1}, c_{3.3.2} \in (0,1) \) such that for all \( t > 0 \) and \( x, y \in D \) with \( 0 < t \leq c_{3.3.1} f(x_1)^{\alpha} \) and \( |x - y| \geq t^{1/\alpha} \),
\[ p_D(t, x, y) \geq c_{3.3.2} \frac{t}{|x - y|^{d+\alpha}} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right). \]
Proof. (i) Case 1: \( \delta_D(x) \leq 40^{-1} t^{1/\alpha} \) and \( \delta_D(y) \geq 2^{-2} t^{1/\alpha} \). Note that \( B(y, 2^{-4} t^{1/\alpha}) \subset D \) and \( 0 < t \leq c_{3.3.1} f(x_1)^{\alpha} \leq c_{3.3.1} f(y)^{\alpha} \). Then, by choosing \( c_{3.3.1} > 0 \) small enough, we can obtain from Lemma 2.5 that
\[ P^x(\tau_{B(z^2, 4^{-1} t^{1/\alpha}) \cap D} > t) \geq c_1 \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}. \]
In the following, we set \( V_1 = B(z^2, 2^{-4} t^{1/\alpha}) \cap D \), \( V_2 = B(y, 2^{-3} t^{1/\alpha}) \) and \( V_2' = B(y, 2^{-4} t^{1/\alpha}) \). Since \( |x - y| \geq t^{1/\alpha} \) and \( \delta_D(x) \leq 40^{-1} t^{1/\alpha} \), we have \( V_1 \cap V_2 = 0 \), and for \( (v, z) \in V_1 \times V_2 \),
\[ |v - z| \leq |v - z_x| + |z_x - x| + |x - y| + |y - z| \leq |x - y| + 3t^{1/\alpha}/4 \leq 2|x - y|. \]
On the other hand, since \( \delta_D(y) \geq 2^{-2} t^{1/\alpha} \) and \( t \leq c_{3.3.1} f(y)^{\alpha} \), by choosing \( c_{3.3.1} \leq c_{2.2.1} \), it follows from Lemma 2.2 that
\[ p_D(t/2, z, y) \geq c_3 t^{-d/\alpha}, \quad z \in V_2, \]
(3.4)
where we used the fact that $\delta_D(z) \geq 2^{-t^{1/\alpha}} \geq |z - y|/2$ for all $z \in V_2$.

Therefore, by the strong Markov property and (3.3),

$$
p_D(t, x, y) = \mathbb{E}^x \left[ p_D(t/2, X_{t/2}^x, y) \right] = \mathbb{E}^x \left[ p_D(t/2, X_{t/2}, y) : t/2 < \tau_D \right]
\geq \mathbb{E}^x \left[ p_D(t/2, X_{t/2}, y) : 0 \leq \tau_{V_1} \leq t/4, X_{\tau_{V_1}} \in V_2', X_s \in V_2 \text{ for all } s \in [\tau_{V_1}, \tau_{V_1} + t/2] \right] \geq c_3 t^{-d/\alpha} \mathbb{E}^x \left[ 0 \leq \tau_{V_1} \leq t/4, X_{\tau_{V_1}} \in V_2', X_s \in V_2 \text{ for all } s \in [\tau_{V_1}, \tau_{V_1} + t/2] \right]
\geq c_3 t^{-d/\alpha} \mathbb{E}^x \left[ \mathbb{P}^{\tau_{V_1}} \left( \tau_{B(x, s; \cdot, \cdot)} > t/2 \right) : 0 \leq \tau_{V_1} \leq t/4, X_{\tau_{V_1}} \in V_2' \right]
\geq c_3 t^{-d/\alpha} \mathbb{E}^x \left[ \mathbb{P}^{\tau_{V_1}} \left( \tau_{B(x, s; \cdot, \cdot)} > t/2 \right) \right] \inf_{z \in V_2'} \mathbb{P}^z \left( \tau_{B(x, s; \cdot, \cdot)} > t/2 \right)
\geq c_3 t^{-d/\alpha} \int_0^{t/4} \int_{V_2'} p_v(s, x, z) \int_{V_2} \frac{1}{|z - u|^{d + \alpha}} du dz ds
\geq c_5 \frac{t^{-d/\alpha} |V_2'|}{|x - y|^{d + \alpha}} \int_0^{t/4} \mathbb{P}^x(\tau_{V_1} > s) ds \geq \frac{c_6 t}{|x - y|^{d + \alpha}} \mathbb{P}^x(\tau_{V_1} > t/4) \geq \frac{c_7 t}{|x - y|^{d + \alpha}} \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}},
$$

where in the fifth inequality we used the Lévy system (2.2) and (2.1) (since $V_1 \cap V_2 = \emptyset$), the sixth inequality is due to (3.3), and the last inequality follows from (3.2).

(ii) Case 2: $\delta_D(x) \geq 40^{-1} t^{1/\alpha}$ and $\delta_D(y) \geq 2^{-2} t^{1/\alpha}$. Following the argument of part (i) with $V_1$ replaced by $V_1 = B(x, 40^{-1} t^{1/\alpha})$ and noting that $\mathbb{P}^x(\tau_{V_1} > t/4) \geq c_8$, we can prove that

$$
p_D(t, x, y) \geq \frac{c_9 t}{|x - y|^{d + \alpha}}.
$$

(iii) Case 3: $\delta_D(y) \leq 2^{-2} t^{1/\alpha}$. According to Lemma 2.3 (by choosing $c_{3.3.1}$ small if necessary), for every $0 < t \leq c_{3.3.1} f(y)^{\alpha/4}$, there is $\xi^t : \approx y + 2^{-(t/2)^{1/\alpha}} (y - z_g) / |y - z_g| \in D$ such that $B(\xi^t, 2^{-1}(t/2)^{1/\alpha}) \subset D$, $|\xi^t - y| \leq |\xi^t - z_g| + |z_g - y| \leq \frac{3y^{1/\alpha}}{4}$, and

$$
\int_{B(\xi^t, 2^{-1}(t/2)^{1/\alpha})} p_D(t/2, z, y) dz \geq c_{10} \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}.
$$

On the other hand, for all $z \in B(\xi^t, 2^{-2}(t/2)^{1/\alpha})$, we have $t \leq c_{11} c_{3.3.1} f(z_1)^{\alpha}, \delta_D(z) \geq \delta_D(\xi^t) - 2^{-2}(t/2)^{1/\alpha} \geq 2^{-2}(t/2)^{1/\alpha}$,

$$
|x - z| \geq |x - y| - |y - \xi^t| - |\xi^t - z| \geq \frac{1}{4} (1 - 2^{-\alpha}) t^{1/\alpha} = \frac{1}{4} (2^\alpha - 1)(t/2)^{1/\alpha}
$$

and

$$
|x - z| \leq |x - y| + |y - \xi^t| + |\xi^t - z| \leq |x - y| + t^{1/\alpha} \leq 2|x - y|.
$$

Hence, by conclusions in parts (i) and (ii) (after adjusting constants), we can obtain that for any $z \in B(\xi^t, 2^{-2}(t/2)^{1/\alpha})$,

$$
p_D(t/2, x, z) \geq \frac{c_{12} t}{|x - z|^{d + \alpha}} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \geq \frac{c_{13} t}{|x - y|^{d + \alpha}} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right).
$$

Therefore, putting both estimates together, we arrive at

$$
p_D(t, x, y) = \int_D p_D(t/2, x, z) p_D(t/2, z, y) dz \geq \int_{B(\xi^t, 2^{-2}(t/2)^{1/\alpha})} p_D(t/2, x, z) p_D(t/2, z, y) dz
\geq \frac{c_{13} t}{|x - y|^{d + \alpha}} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \int_{B(\xi^t, 2^{-2}(t/2)^{1/\alpha})} p_D(t/2, z, y) dz
\geq \frac{c_{14} t}{|x - y|^{d + \alpha}} \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}.
$$

By all the conclusions above, we can obtain the desired assertion. □
Lemma 3.4. (Lower bound when $C_0 f(x_1)^\alpha \leq t \leq C_0 f(y_1)^\alpha$.) There exists $c_{3.4.0} \in (0,1)$ such that, for all $c_{3.4.1}, c_{3.4.2} \in (0,c_{3.4.0})$ and for all $x, y \in D$ and $t > 0$ satisfying $c_{3.4.1} f(x_1)^\alpha \leq t \leq c_{3.4.2} f(y_1)^\alpha$ and $|x - y| \geq t^{1/\alpha}$, there is $c_{3.4.3} \in (0,1)$ so that

$$p_D(t, x, y) \geq \frac{c_{3.4.3} t}{|x - y|^{d+\alpha}} \left( \frac{\delta_D(x)_{\alpha/2} f(x_1)^{\alpha/2}}{t} \wedge 1 \right) \wedge 1.$$ 

Proof. We may assume that $c_{3.4.1}, c_{3.4.2} \in (0,c_{3.4.0})$, where $c_{3.4.0}$ is a small positive constant less than $c_{2.5.1} = c_{2.5.1}(1/10)$ which will be chosen later.

(i) Case 1: $\delta_D(x) \leq 10^{-1}(c_{3.4.1}/4)^{1/\alpha} f(x_1)$ and $\delta_D(y) \geq 2^{-2} t^{1/\alpha}$ (and that $c_{3.4.1} f(x_1)^\alpha \leq t \leq c_{3.4.2} f(y_1)^\alpha$ and $|x - y| \geq t^{1/\alpha}$). Set $V_1 = B(x, c_{3.4.1}/4)^{1/\alpha} f(x_1) \cap D$, $V_2 = B(y, 2^{-3} t^{1/\alpha}) \subset D$ and $V'_2 = B(y, 2^{-4} t^{1/\alpha})$. Since $|x - y| \geq t^{1/\alpha}$, it is easy to verify that $V_1 \cap V_2 = \emptyset$. Then, by the Markov property,

$$p_D(t, x, y) = E^x \left[ p_D(t/2, X_{t/2}^D, y) \right] = E^x \left[ p_D(t/2, X_{t/2}, y) : \tau_D > t/2 \right] \geq E^x \left[ p_D(t/2, X_{t/2}, y) : 0 \leq \tau_1 \leq 2^{-2} c_{3.4.1} f(x_1)^\alpha, X_{\tau_1} \in V'_2, X_s \in V_2 \text{ for all } s \in [\tau_1, \tau_1 + t/2] \right].$$

According to arguments in part (i) of the proof of Lemma 3.3, (3.3) and (3.4) still hold by choosing $c_{3.4.0}$ less than $c_{2.2.1}$. Therefore, by the strong Markov property again,

$$p_D(t, x, y) \geq c_1 t^{-d/\alpha} P^x \left( 0 \leq \tau_1 \leq c_{3.4.1} f(x_1)^\alpha/4, X_{\tau_1} \in V'_2, X_s \in V_2 \text{ for all } s \in [\tau_1, \tau_1 + t/2] \right) \geq c_1 t^{-d/\alpha} E^x \left[ P^{X_{\tau_1}} \left( \tau_{B(x_{\tau_1}, 2^{-4} t^{1/\alpha})} > t \right) : 0 \leq \tau_1 \leq c_{3.4.1} f(x_1)^\alpha/4, X_{\tau_1} \in V'_2 \right] \geq c_1 t^{-d/\alpha} \int_0^{2^{-2} c_{3.4.1} f(x_1)^\alpha} p_{\tau_1}(s, x, z) \int_{V'_2} \frac{1}{|z - u|^{d+\alpha}} \, du \, dz \, ds \geq c_2 t^{-d/\alpha} \int_0^{2^{-2} c_{3.4.1} f(x_1)^\alpha} P^x(\tau_{V_1} > s) \, ds \geq c_3 f(x_1)^\alpha \int_0^{2^{-2} c_{3.4.1} f(x_1)^\alpha} \delta_D(x)_{\alpha/2} f(x_1)^{\alpha/2} \geq c_6 \left( \frac{\delta_D(x)_{\alpha/2} f(x_1)^{\alpha/2}}{t} \wedge 1 \right) \wedge 1 \frac{t}{|x - y|^{d+\alpha}},$$

where in the first inequality we used (3.4), in the fourth inequality we used the Lévy system (2.2) and (2.1), the fifth step follows from (3.3), and the seventh inequality is due to (2.8) with $t = 2^{-2} c_{3.4.1} f(x_1)^\alpha$ and $\lambda = 1/10$.

(ii) Case 2: $\delta_D(x) \geq 10^{-1}(c_{3.4.1}/4)^{1/\alpha} f(x_1)$ and $\delta_D(y) \geq 2^{-2} t^{1/\alpha}$ (and that $c_{3.4.1} f(x_1)^\alpha \leq t \leq c_{3.4.2} f(y_1)^\alpha$ and $|x - y| \geq t^{1/\alpha}$). Following the arguments as in part (i) with $V_1$ replaced by $V_1 = B(x, 10^{-1}(c_{3.4.1}/4)^{1/\alpha} f(x_1))$, and using the fact that $P^x(\tau_{V_1} > 2^{-2} c_{3.4.1} f(x_1)^\alpha) \geq c_7$, we can prove

$$p_D(t, x, y) \geq \frac{c_8 f(x_1)^\alpha}{|x - y|^{d+\alpha}} \geq \frac{c_9 t}{|x - y|^{d+\alpha}} \frac{\delta_D(x)_{\alpha/2} f(x_1)^{\alpha/2}}{t} \wedge 1,$$

where in the last inequality above we used the fact that $\delta_D(x) \leq c_{10} f(x_1)$ for all $x \in D$.

(iii) Case 3: $\delta_D(y) \leq 2^{-2} t^{1/\alpha}$ (and that $c_{3.4.1} f(x_1)^\alpha \leq t \leq c_{3.4.2} f(y_1)^\alpha$ and $|x - y| \geq t^{1/\alpha}$). Following arguments in part (iii) in the proof of Lemma 3.3, we can verify that there exists $\xi'_y \in D$ such that $B(\xi'_y, 2^{-1}(t/2)^{1/\alpha}) \subset D$, (3.5), (3.6) and (3.7) are satisfied, and $(c_{3.4.1}/2) f(x_1)^\alpha \leq t/2 \leq c_{11} c_{3.4.2} f(z_1)^\alpha$ holds for all $z \in B(\xi'_y, 2^{-2}(t/2)^{1/\alpha})$. Therefore, according to conclusions in parts (i) and (ii) (by adjusting the constant $c_{3.4.0}$ smaller properly if necessary), we can obtain for all $z \in B(\xi'_y, 2^{-2}(t/2)^{1/\alpha})$,

$$p_D(t/2, x, z) \geq \frac{c_{12} t}{|x - z|^{d+\alpha}} \left( \frac{\delta_D(x)_{\alpha/2} f(x_1)^{\alpha/2}}{t} \wedge 1 \right) \geq \frac{c_{13} t}{|x - y|^{d+\alpha}} \left( \frac{\delta_D(x)_{\alpha/2} f(x_1)^{\alpha/2}}{t} \wedge 1 \right).$$
where we used \((3.7)\) in the last inequality. Combining this with \((3.5)\), we have

\[
p_D(t, x, y) = \int_D p_D(t/2, x, z)p_D(t/2, z, y) \, dz \geq \int_{B(\|t\|^{1/2})} p_D(t/2, x, z)p_D(t/2, z, y) \, dz
\]

\[
\geq c_{13} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right)
\]

\[
\geq c_{14} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right).
\]

Therefore, the desired assertion follows from all the conclusions above.

\[\square\]

**Lemma 3.5. (Upper bound)** There exist \(c_{3.5.1} \in (0, 1]\) and \(c_{3.5.2} > 0\) such that for all \(t > 0\) and \(x, y \in D\) with \(0 < t \leq c_{3.5.1} f(y_1)^{\alpha}\) and \(|x - y| \geq t^{1/\alpha}\),

\[
p_D(t, x, y) \leq c_{3.5.2} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) t^{d+\alpha}.
\]

**Proof.** Since \(|x - y| \geq t^{1/\alpha}\), for every \(u \in D\) such that \(|u - x| \leq |y - x|/2\), we have \(|y - u| \geq |x - y| - |u - x| \geq |x - y|/2 \geq t^{1/\alpha}/2\). Thus, by Lemma 2.7, we have that for every \(s \leq t\) and \(y, u \in D\) such that \(|u - x| \leq |y - x|/2\),

\[
p_D(s, y, u) \leq c_1 \left( \frac{s}{|u - y|^{d+\alpha}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{s^{1/2}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) t^{d+\alpha}.
\]

We first consider the case that \(\delta_D(x) \leq 2^{-1} c_{2.6.1} t^{1/\alpha}\). Let \(V_1 = B(z, 2^{-1} c_{2.6.1} t^{1/\alpha}) \cap D\), \(V_3 = \{z \in D : |z - x| \geq |x - y|/2\}\) and \(V_3 = D\setminus(V_1 \cup V_3)\). It is easy to check that \(\text{dist}(V_1, V_3) > 0\). Then, applying Lemma 5.1 and (2.7), we can get

\[
p_D(t, x, y) \leq c_2 \left( \frac{\text{E}^x[\tau_{V_1}]^{1/2}}{t} \right) \sup_{0 \leq s \leq t, z \in V_2} p_D(s, z, y)
\]

\[
+ c_3 \left( \int_0^t \text{P}^x(\tau_{V_1} > s) \text{P}^y(\tau_D > t - s) \, ds \right) \sup_{u \in V_1, z \in V_3} \frac{1}{|u - z|^{d+\alpha}} =: I_1 + I_2.
\]

On the one hand, note that \(|z - y| \geq |x - y| - |z - x| \geq |x - y|/2\) for all \(z \in V_2\). Combining (3.8) with (2.9) yields that

\[
I_1 \leq c_4 \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) t^{d+\alpha}.
\]

On the other hand, we write

\[
\int_0^t \text{P}^x(\tau_{V_1} > s) \text{P}^y(\tau_D > t - s) \, ds
\]

\[
\leq \int_0^{t/2} \text{P}^x(\tau_{V_1} > s) \text{P}^y(\tau_D > t/2) + \int_{t/2}^t \text{P}^y(\tau_D > t - s) \, ds \text{P}^x(\tau_{V_1} > t/2) =: I_{2.1} + I_{2.2}.
\]

Note that \(t \leq c_{3.5.1} f(y_1)^{\alpha} \leq c_{3.5.1}\), and let \(c_{3.5.1}\) small enough if necessary. By (2.5) and (2.9),

\[
I_{2.1} \leq c_6 \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right)
\]

\[
\leq c_7 \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right).
\]
Similarly, also by \((2.5)\) and \((2.9)\),
\[
I_{2,2} \leq c_9 \left( \mathbb{E}^x[\tau_{V_1}] / t \right) \int_0^{t/2} \mathbb{P}(\tau_D > s) \, ds \leq c_{10} \left( \mathbb{E}^x[\tau_{V_1}] / t \right) \int_0^{t/2} \left( \delta_D(y)^{\alpha/2} / \sqrt{s} \right) \, ds
\]
\[
\leq c_{11} \left( \delta_D(x)^{\alpha/2}(f(x_1)^{\alpha/2} \wedge t^{1/2}) / t \right) \left( (t^{1/2}\delta_D(y)^{\alpha/2}) \wedge t \right).
\]
Note that for all \(u \in V_1\) and \(z \in V_3\),
\[
|u - z| \geq |x - y| - |x - u| - |z - y| \geq |x - y| - t^{1/\alpha}/16 - |x - y|^2/2 \geq c_{12}|x - y|.
\]
Combining with all the estimates above, we have
\[
I_2 \leq c_{13} \left( \delta_D(x)^{\alpha/2}(f(x_1)^{\alpha/2} \wedge t^{1/2}) / t \right) \left( \delta_D(y)^{\alpha/2} / \sqrt{t} \right) \wedge 1 \leq \frac{t}{|x - y|^{d + \alpha}}
\]
and so
\[
p_D(t, x, y) \leq c_{14} \left( \delta_D(y)^{\alpha/2} / \sqrt{t} \right) \wedge 1 \leq \frac{t}{|x - y|^{d + \alpha}}.
\]
When \(\delta_D(x) \geq 2^{-4}c_{2.6.1}t^{1/\alpha}\), we can follow the arguments for the case \(\delta_D(x) \leq 2^{-4}c_{2.6.1}t^{1/\alpha}\) above with \(V_1\) replaced by \(V_1 = B(x, 2^{-4}c_{2.6.1}t^{1/\alpha})\) (by noticing that \(\mathbb{E}[\tau_{V_1}] \leq c_{15}t\) to prove that
\[
p_D(t, x, y) \leq c_{16} \left( \delta_D(y)^{\alpha/2} / \sqrt{t} \right) \wedge 1 \leq \frac{t}{|x - y|^{d + \alpha}}.
\]
where in the second inequality we used the fact that \(t^{1/\alpha} \leq c_{18} f(x_1)\), thanks to the property that \(\delta_D(x) \leq c_{19} f(x_1)\) for all \(x \in D\). The proof is complete.

Notice that, if \(t \leq C_0 f(y_1)\) and \(|x - y| \leq t^{1/\alpha}\), then \(t \leq c_1 C_0 f(x_1)^{\alpha}\) for some constant \(c_1 \geq 1\). Therefore, putting all the previous lemmas in this section together yields the following statement.

**Proposition 3.6.** There is a constant \(C_0 \in (0, 1]\) such that for all \(t > 0\) and \(x, y \in D\) such that for all \(0 < t \leq C_0 f(y_1)^{\alpha}\),
\[
p_D(t, x, y) \simeq p(t, x, y) \left( \delta_D(x)^{\alpha/2} (f(x_1)^{\alpha/2} \wedge t^{1/2}) / t \right) \left( \delta_D(y)^{\alpha/2} / \sqrt{t} \right) \wedge 1 \simeq p(t, x, y) \Psi(t, x) \Psi(t, y),
\]
where \(\Psi(t, x)\) is defined by \((1.8)\).

The next two sections are devoted to estimates of \(p_D(t, x, y)\) for the case that \(t \geq C_0 f(y_1)\), where \(C_0\) is the fixed constant given in Proposition 3.6. For this, we define for any \(y \in D\),
\[
t_0(y) := \inf \left\{ t > 0 : e^{-C_* f(y_1)^{\alpha} t} \leq t(1 + |y|)^{-(d + \alpha - 1)} \right\},
\]
where \(C_* = c_{2.9.2} > 0\) is given in \((2.10)\).

**Remark 3.7.** As mentioned in the remark below \((1.10)\), \(t_0(y) \in (0, \infty)\) is unique and satisfies that
\[
e^{-C_* f(y_1)^{\alpha} t_0(y)} = t_0(y)(1 + |y|)^{-(d + \alpha - 1)}.
\]
In particular, we can check that there is a constant \(C_{3.10} > 0\) such that for all \(y \in D\),
\[
f(y_1)^{\alpha} \leq C_{3.10} t_0(y).
\]
Usually it is not easy to obtain the explicit value of \(t_0(y)\); however, we are able to get explicit estimates of \(t_0(y)\) under some mild assumption on \(f\). For example, if \(f(r) \geq c(1 + r)^{-p}\) for some constants \(c\) and \(p > 0\), then \(t_0(y) \simeq f(y_1)^{\alpha} \log(2 + |y|)\) for all \(y \in D\).
4. Case II: $C_0f(y_1)^a \leq t \leq C_1t_0(y)$ for any given constant $C_1 > 0$.

Throughout this section, we always let $C_0$ be the constant in Proposition [3.6] and $t_0(y)$ be defined by (3.9) for any $y \in D$.

**Lemma 4.1. (Lower bound)** There exist constants $c_{4.1.1} \in (0, 1)$ and $c_{4.1.2} > 0$ such that for all $x, y \in D$ and $C_0f(y_1)^a \leq t$,

$$p_D(t, x, y) \geq c_{4.1.1} \left( t^{-d/a} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \left( \frac{\delta_D(x)\alpha/2 f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( \frac{\delta_D(y)\alpha/2 f(y_1)^{\alpha}}{t \wedge 1} \right) e^{-c_{4.1.2}t f(y_1)^{-\alpha}}.$$  

**Proof.** (i) Case 1: $|x - y| \leq t^{1/a} \wedge 1$. According to Lemma 2.3, we can find $\xi_x, \xi_y \in D$ and a constant $\lambda := \lambda(C_0) \in (0, 1)$ small enough such that $V_x := B(\xi_x, \lambda f(x_1)) \subset B(\xi_x, 4\lambda f(x_1)) \subset D$, $V_y := B(\xi_y, \lambda f(y_1)) \subset B(\xi_y, 4\lambda f(y_1)) \subset D$, and

$$\int_{V_x} p_D(2^{-2}C_0f(x_1)^a, x, z) \, dz \geq c_1 \delta_D(x)^{\alpha/2} f(x_1)^{-\alpha/2},$$

$$\int_{V_y} p_D(2^{-2}C_0f(y_1)^a, y, z) \, dz \geq c_1 \delta_D(y)^{\alpha/2} f(y_1)^{-\alpha/2}. \quad (4.1)$$

Here, we used the fact that $\delta_D(x) \leq c_2 f(x_1)$ for all $x \in D$ with some constant $c_2 > 0$.

On the other hand, for any $z \in V_x$, $w \in V_y$ and $t \geq 2^{-1}C_0f(y_1)^a$, taking $n := n(t, y) = [2t/(C_0f(y_1)^a)] + 1$ and $\bar{c} := \bar{c}(t, y) = t f(y_1)^{-\alpha} n^{-1}$, we have

$$p_D(t, z, w) = \int_D \cdots \int_D p_D(t/n, z_1) \cdots p_D(t/n, z_{n-1}, w) \, dz_1 \cdots dz_{n-1}$$

$$\geq \int_{V_x} \cdots \int_{V_y} p_D(\bar{c} f(y_1)^a, z, z_1) \cdots p_D(\bar{c} f(y_1)^a, z_{n-1}, w) \, dz_1 \cdots dz_{n-1},$$

where in the inequality above we used the facts that $V_y \subset D$.

The assumption $|x - y| \leq t^{1/a} \wedge 1$ implies that $f(x_1) \simeq f(y_1)$. Using this and $C_0f(y_1)^a \leq t \wedge 1$, we have that, for all $z \in V_x$ and $u \in V_y$, $|z - u| \leq c_3(t^{1/a} \wedge 1)$, $\delta_u(u) \geq \lambda f(y_1)$ and $\delta_D(z) \geq \lambda f(x_1) \geq c_4 \lambda f(y_1)$. Hence, according to Lemma 2.2, we obtain that for $z \in V_x$ and $u \in V_y$,

$$p_D(\bar{c} f(y_1)^a, z, u) \geq c_5 \left( \frac{f(y_1)^a}{|z - u|^{d+\alpha}} \wedge (f(y_1)^a)^{-d/a} \right) \geq c_6 f^{-d(a)/\alpha} f(y_1)^a,$$

where the last inequality is due to the fact that $|z - u| \leq c_3(t^{1/a} \wedge 1)$ and $t \geq C_0f(y_1)^a$. We mention that, since $\bar{c} \in [C_0/4, C_0/2]$ (i.e., $\bar{c}$ may depend on $y$ and $t$ but it is uniformly bounded between $C_0/4$ and $C_0/2$), $c_5 > 0$ here is independent of $y$ and $t$ due to the argument in Proposition 3.3. Similarly, we have $p_D(\bar{c} f(y_1)^a, w, u) \geq c_7 f(y_1)^{-d}$ for $w, u \in V_y$. Hence, putting all the estimates above together yields that for all $z \in V_x$, $w \in V_y$ and $t \geq 2^{-1}C_0f(y_1)^a$,

$$p_D(t, z, w) \geq \left( c_7 f(y_1)^{-d} |B(\xi_y, \lambda f(y_1))| \right)^{n-1} c_6 (f(y_1)^{-d/\alpha} f(y_1)^a) \geq c_8 f^{-d(a)/\alpha} f(y_1)^a \geq c_9 f^{-d(a)/\alpha} f(y_1)^a \geq c_{10} e^{-c_{11}t f(y_1)^{-\alpha}} \quad (4.2)$$

where the last inequality follows from the facts that $n = [2t/(C_0f(y_1)^a)] + 1$ and $(t f(y_1)^{-\alpha})^{-(d+\alpha)/\alpha} \geq c_{10} e^{-c_{11}t f(y_1)^{-\alpha}}$ for each $t \geq 2^{-1}C_0f(y_1)^a$. Therefore, for all $t \geq C_0f(y_1)^a$,

$$p_D(t, x, y) = \int_D \int_D p_D(2^{-2}C_0f(x_1)^a, x, z) p_D(t - 2^{-2}C_0f(x_1)^a - 2^{-2}C_0f(y_1)^a, z, w)$$

$$\times p_D(2^{-2}C_0f(y_1)^a, w, y) \, dz \, dw$$

$$\geq \left[ \int_{V_x} p_D(2^{-2}C_0f(x_1)^a, x, z) \, dz \right] \left[ \int_{V_y} p_D(2^{-2}C_0f(y_1)^a, w, y) \, dw \right]$$

$$\times \inf_{z \in V_x, w \in V_y} p_D(t - 2^{-2}C_0f(x_1)^a - 2^{-2}C_0f(y_1)^a, z, w)$$

$$\geq c_{12} (\delta_D(x)^{\alpha/2} f(x_1)^{-\alpha/2})(\delta_D(y)^{\alpha/2} f(y_1)^{-\alpha/2})$$

$$\times \exp \left( -c_9 (t - 2^{-2}C_0f(y_1)^a - 2^{-2}C_0f(x_1)^a) f(y_1)^{-d} \right) f(y_1)^{-d}$$
which is a direct consequence of the argument for (4.2) (by choosing $t$
which is deduced from the scaling property of symmetric
where the second inequality follows from [4.1], [4.2] and the fact that
(t - 2^{-2}C_0f(x_1)^{\alpha} - 2^{-2}C_0f(y_1)^{\alpha}) \geq t/2 \quad \text{for } t \geq C_0f(y_1)^{\alpha},
(thanks to $t \geq C_0f(y_1)^{\alpha} \geq C_0f(x_1)^{\alpha}$, and the last inequality is due to
$\left(\frac{t}{t+1}\right)^2 (tf(y_1)^{-\alpha})^{2+d/\alpha} \geq c_{15}e^{-C_1tf(y_1)^{-\alpha}}$ for $t \geq C_0f(y_1)^{\alpha}$).

(ii) Case 2: $|x - y| \geq t^{1/\alpha} \wedge 1$. Let $V_x = B(\xi_x, \lambda f(x_1))$ and $V_y = B(\xi_y, \lambda f(y_1))$ be those defined
in part (i). By Lemma [2.3], we have $|x - \xi_x| \leq c_1\lambda f(x_1)$ and $|y - \xi_y| \leq c_1\lambda f(y_1)$. Choosing $\lambda > 0$
small enough if necessary, we find that for every $z \in B(\xi_x, 2\lambda f(x_1))$ and $w \in B(\xi_y, 2\lambda f(y_1))$,
$|z - w| \geq |x - y| - |x - \xi_x| - |z - \xi_x| - |y - \xi_y| \geq |x - y| - c_{18}\lambda f(y_1) \geq c_{19}|x - y|$
and, similarly,
$|z - w| \leq |x - y| + c_{18}\lambda f(y_1) \leq c_{20}|x - y|$, \hspace{1cm} (4.3)
where we have used the fact that $|x - y| \geq (t^{1/\alpha} \wedge 1) \geq C_0^{1/\alpha}f(y_1)$ (because $C_0, f \in (0, 1]$). In particular,
$B(z, \lambda f(x_1)) \cap B(w, \lambda f(y_1)) = \emptyset$ for every $z \in V_x$ and $w \in V_y$. Therefore, for any $z \in V_x, w \in V_y$ and $t > C_0f(y_1)^{\alpha}/2$,
$p_d(t, z, w) = E[p_d(t/2, X(t/2), w)]$ 
$\geq E\left[p_d(t/2, X(t/2), w) : 0 < \tau_{B(z, \lambda f(x_1))} < 2^{-2}C_0f(x_1)^{\alpha}, X_{\tau_{B(z, \lambda f(x_1))}} \in B(w, \lambda f(y_1)/2),
X_s \in B(w, \lambda f(y_1)) \text{ for all } s \in [\tau_{B(z, \lambda f(x_1))}, \tau_{B(z, \lambda f(x_1))} + t]\right]$ 
$\geq c_{21}e^{-c_2f(y_1)^{-\alpha}}f(y_1)^{-d} \inf_{u \in B(w, \lambda f(y_1)/2)} P^u(\tau_{B(u, \lambda f(y_1)/2)} > t)$ 
$\times \left(\int_0^{2^{-2}C_0f(x_1)^{\alpha}} \int_{B(\lambda f(x_1))}(s, z, u) \int_{B(w, \lambda f(y_1)/2)} \frac{1}{u - v|^{d+\alpha}} du dv ds\right)$ 
$\geq c_{23}e^{-c_2f(y_1)^{-\alpha}}f(y_1)^{-d}e^{-c_4tf(y_1)^{-\alpha}}$ 
$\times f(x_1)^{\alpha} P^x(\tau_{B(z, \lambda f(x_1)/2)} > 2^{-2}C_0f(x_1)^{\alpha}) |B(w, \lambda f(y_1))/| \frac{1}{|x - y|^{d+\alpha}}$ 
$\geq c_{25}e^{-c_2f(y_1)^{-\alpha}} \frac{1}{|x - y|^{d+\alpha}} f(x_1)^{\alpha}$.

Here the first inequality is due to Lévy system [2.2], the second inequality follows from
$\inf_{u, w \in B(\xi_y, 2\lambda f(y_1))} p_d(t/2, u, w) \geq c_{21}f(y_1)^{-d}e^{-c_2tf(y_1)^{-\alpha}},$
which is a direct consequence of the argument for (4.2) (by choosing $\lambda$ small enough if necessary),
in the third inequality we have used (4.3) and the estimate as follows
$P^u(\tau_{B(u, r)} > t) = P^u(\tau_{B(u, 1)} > t/r^{\alpha}) = \int_{B(u, 1)} p_d(t/r^{\alpha}, u, z) dz \geq c_{26}e^{-c_2t^{\alpha}r^{\alpha}},$ \hspace{1cm} (4.4)
which is deduced from the scaling property of symmetric $\alpha$-stable processes and [1.3], and the fourth inequality is due to (2.1). Therefore, combining this with (4.1), we arrive at that for all $t \geq C_0f(y_1)^{\alpha}$,
$p_d(t, x, y) \geq \int_{V_x} \int_{V_y} p_d \left(2^{-2}C_0f(x_1)^{\alpha}, x, z\right) p_d \left(t - 2^{-2}C_0f(x_1)^{\alpha} - 2^{-2}C_0f(y_1)^{\alpha}, z, w\right)$
\[
\times p_D(2^{-2}C_0 f(x_1)^\alpha, w, y) \, dz \, dw \\
\geq \left( \int_{V_y} p_D(2^{-2}C_0 f(x_1)^\alpha, x, z) \, dz \right) \left( \int_{V_y} p_D(2^{-2}C_0 f(x_1)^\alpha, w, y) \, dw \right) \\
\times \inf_{z \in V_y, w \in V_y} p_D(t - 2^{-2}C_0 f(x_1)^\alpha - 2^{-2}C_0 f(y_1)^\alpha, z, w) \\
\geq c_{28} \left( \delta_D(x)^{\alpha/2} f(x_1)^{-\alpha/2} \right) \left( \delta_D(y)^{\alpha/2} f(y_1)^{-\alpha/2} \right) \left( 1 \right) \left( e^{-c_{29}t f(y_1)^{-\alpha}} \right) \left( \frac{1}{|x - y|^{d + \alpha}} \right) \left( f(x_1)^{\alpha} \right) \\
= c_{28} \left( \delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2} \right) \left( \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2} \right) \left( \frac{t}{t \wedge 1} \right) \left( \frac{t}{t \wedge 1} \right) \left( e^{-c_{29}t f(y_1)^{-\alpha}} \right) \left( \frac{t}{|x - y|^{d + \alpha}} \right),
\]

where the last inequality follows from the inequality \( (t \wedge 1)^2 (tf(y_1)^{-\alpha}) \geq c_{31} e^{-c_{29}t f(y_1)^{-\alpha}} \) for \( t \geq C_0 f(y_1)^\alpha \). We complete the proof. \( \Box \)

**Lemma 4.2. (Upper bound)** For any \( c_0 > 0 \), there exists a constant \( c_{4.2.1} := c_{4.2.1}(c_0) > 0 \) such that for all \( x, y \in D \) and \( C_0 f(y_1)^\alpha \leq t \leq c_{0}\),

\[
p_D(t, x, y) \leq c_{4.2.1} \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d + \alpha}} \right) \left( \delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2} \right) \left( \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2} \right) \left( \frac{t}{t \wedge 1} \right) \left( \frac{t}{t \wedge 1} \right) e^{-c_{4.2.1}(C_0 f(y_1)^{-\alpha})},
\]

where \( C_* = c_{2.9.2} \) is given in (2.10).

**Proof.** Without loss of generality we may assume that \( c_0 \geq 2 \).

(i) Case 1: \( |x - y| \leq t^{1/\alpha} \). Using (2.10) and considering the cases \( C_0 f(y_1)^\alpha < t/2 \leq c_0(y) \) and \( t_0(y) < t/2 \leq c_0 t_0(y)/2 \), we know that for any \( y \in D \) and \( t > 0 \) with \( C_0 f(y_1)^\alpha < t \leq c_0 y_0(y) \),

\[
P^y(\tau_D > t/2) \leq c_1 \left( \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2} \right) \left( \frac{t}{t \wedge 1} \right) e^{-2^{-1}(C_* f(y_1)^{-\alpha})t},
\]

where we used the fact that for every \( t_0(y) < t/2 \leq c_0 y_0(y)/2, \)

\[
\frac{t}{(1 + |y|)^{d + \alpha - 1}} \leq \frac{c_0 y_0(y)}{(1 + |y|)^{d + \alpha - 1}} = c_0 e^{-C_* t_0(y) f(y_1)^{-\alpha}} \leq c_0 e^{-C_* t_0(y) f(y_1)^{-\alpha}}.
\]

Let \( c_2 := c_2(c_0) = C_* / c_0 \). Then, for any \( y, z \in D \) and \( t > 0 \) with \( C_0 f(y_1)^\alpha < t \leq c_0 y_0(y), \)

\[
p_D(2t/3, z, y) = \int_D p_D(t/6, z, u) p_D(t/2, u, y) \, du \\
\leq c_3 t^{-d/\alpha} P^y(\tau_D > t/2) \\
\leq c_4 t^{-d/\alpha} \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) e^{-c_2 f(y_1)^{-\alpha} t/2}.
\]

Hence, for any \( x, y \in D \) and \( t > 0 \) with \( C_0 f(y_1)^\alpha < t \leq c_0 t_0(y), \)

\[
p_D(t, x, y) = \int_D p_D(t/3, x, z) p_D(2t/3, z, y) \, dz \\
\leq c_4 t^{-d/\alpha} \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) e^{-c_2 f(y_1)^{-\alpha} t/2} \int_D p(t/3, x, z) \, dz \\
\leq c_5 t^{-d/\alpha} \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) e^{-c_2 f(y_1)^{-\alpha} t/2},
\]

where in the last inequality we have used (2.10) and the fact that \( C_0 f(x_1)^\alpha \leq C_0 f(y_1)^\alpha \leq t. \)

(ii) Case 2: \( |x - y| > t^{1/\alpha} \). Let \( V_1 = \{ z \in D : |z - y| > |x - y| / 2 \} \) and \( V_2 = \{ z \in D : |z - y| \leq |x - y| / 2 \} \). Then, it holds that for any \( x, y \in D \) and \( t > 0, \)

\[
p_D(t, x, y) = \int_{V_1} p_D(t/2, x, z) p_D(t/2, z, y) \, dy + \int_{V_2} p_D(t/2, x, z) p_D(t/2, z, y) \, dy =: I_1 + I_2.
\]
On the one hand, for any $z \in V_1$, $|z - y| \geq |x - y|/2 \geq t^{1/\alpha}/2$. Then, by Lemma 2.7 for any $z \in V_1$, $y \in D$ and $t > 0$ with $|x - y| \geq t^{1/\alpha}$,

$$p_D(t/2, z, y) \leq c_7 \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|z - y|^{d + \alpha}} \leq c_8 \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|x - y|^{d + \alpha}}.$$  

According to (2.10), for all $x, y \in D$ with $x_1 \geq y_1$ and $t > 0$ with $C_0 f(y_1)^\alpha \leq t \leq c_{0}(y)$,

$$P^x(\tau_D > t/2) \leq c_9 \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( e^{-C_0 f(x_1)^{1-\alpha} t/2} + t(1 + |x|)^{(d + \alpha - 1)} \right) \leq c_{10} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( e^{-C_0 f(y_1)^{1-\alpha} t/2} + t(1 + |y|)^{(d + \alpha - 1)} \right) \leq c_{11} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) e^{-c_{2}(y_1)^{1-\alpha} t},$$

where in the second inequality we used the fact that $1 + |x| \geq c_{12}(1 + |y|)$ for all $x, y \in D$ with $x_1 \geq y_1$, and the last inequality follows from (4.6). Hence, for all $x, y \in D$ and $t > 0$ with $C_0 f(y_1)^\alpha \leq t \leq c_{0}(y)$,

$$I_1 \leq c_8 \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|x - y|^{d + \alpha}} P^x(\tau_D > t/2) \leq c_{13} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|x - y|^{d + \alpha}} e^{-c_{2}(y_1)^{1-\alpha} t/2}.$$ 

On the other hand, for every $z \in V_2$, $|z - x| \geq |x - y|/2 \geq t^{1/\alpha}/2$. So, according to Lemma 2.7 we obtain that for every $z \in V_2$,

$$p_D(t/2, x, z) \leq c_{14} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|x - y|^{d + \alpha}}.$$  

This along with (4.5) yields that for all $x, y \in D$ and $t > 0$ with $C_0 f(y_1)^\alpha \leq t \leq c_{0}(y)$,

$$I_2 \leq c_{14} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|x - y|^{d + \alpha}} P^y(\tau_D > t/2) \leq c_{15} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|x - y|^{d + \alpha}} e^{-c_{2}f(y_1)^{1-\alpha}/2}.$$ 

Therefore, according to all the estimates above, for any $x, y \in D$ and $t > 0$ with $C_0 f(y_1)^\alpha \leq t \leq c_{0}(y)$,

$$p_D(t, x, y) \leq c_{16} \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) \frac{t}{|x - y|^{d + \alpha}} e^{-c_{2}f(y_1)^{1-\alpha}/2}.$$ 

Now, the required assertion follows from both conclusions above. \qed

We summarize both lemmas above as follows.

**Proposition 4.3.** Let $\Psi(t, x)$ be defined by (1.3), and $C_* = c_{2.9.2}$ be given in (2.10). Then the following hold.

(i) There exist constants $c_{4.3.1}, c_{4.3.2}, c_{4.3.3} > 0$ such that for all $x, y \in D$ with $C_0 f(y_1)^\alpha \leq t$,

$$p_D(t, x, y) \geq c_{4.3.1} p(t, x, y) \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) e^{-c_{4.3.2}f(y_1)^{1-\alpha}} \geq c_{4.3.3} p(t, x, y) \Psi(t, x) \Psi(t, y) e^{-c_{4.3.2}f(y_1)^{1-\alpha}}.$$ 

(ii) For any $c_0 \geq 1$, there exist constants $c_{4.3.4} := c_{4.3.4}(c_0), c_{4.3.5} := c_{4.3.5}(c_0) > 0$ such that for all $x, y \in D$ with $C_0 f(y_1)^\alpha \leq t \leq c_{0}(y) = c_{0}(x, y)$,

$$p_D(t, x, y) \leq c_{4.3.4} p(t, x, y) \left( \frac{\delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2}}{t \wedge 1} \right) \left( \frac{\delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}}{t \wedge 1} \right) e^{-c_{4.3.5}(c_0) f(y_1)^{1-\alpha}}.$$
\[ \leq c_{4.3.5}p(t, x, y)\Psi(t, x)\Psi(t, y)e^{-(2\alpha\sqrt{4})^{-1}C_tf(y)}^{-\alpha}. \]

Remark 4.4. In Proposition 4.3, we do not require \( t_0(y) \) to be bounded. Actually, we will treat all cases including \( \lim_{y \to 0} t_0(y) > 0 \) (which in particular includes the case that \( \lim_{y \to 0} t_0(y) = \infty \)) in the next section. When \( \lim_{y \in D, |y| \to 0} t_0(y) > 0 \), Proposition 4.3 has shown the explicit heat kernel estimates for any finite time.

5. Case III: \( t \geq C_1 t_0(y) \) for some \( C_1 > 0 \)

In this section, we will make additional assumptions on the reference function \( f \) as in Theorem 4.3

(i) There exist constants \( c, p > 0 \) such that \( f(s) \geq c(1 + s)^{-p} \) for all \( s > 0 \);

(ii) There is a monotone function \( g \) on \( (0, \infty) \) such that \( g(s) \geq f(s)\alpha \log(2 + s) \).

As mentioned in Remark 3.7 under (i), for any \( y \in D \), \( t_0(y) \geq f(y_1)\alpha \log(2 + |y|) \), where \( t_0(y) = t_0(C_*, y) \) is defined by (3.9) and \( C_* = c_{2.9.2} \) is the constant in (2.10). According to the different monotone property of \( g \), we will split this section into two parts.

5.1. Case III-1: \( g \) is non-increasing on \( (0, \infty) \) such that \( \lim_{s \to \infty} g(s) = 0 \). In this part, we are concerned with the case that \( g \) is non-increasing on \( (0, \infty) \) and \( \lim_{s \to \infty} g(s) = 0 \). Since \( t_0(y) \geq f(y_1)\alpha \log(2 + |y|) \), we have \( \lim_{y \in D, |y| \to 0} t_0(y) = 0 \).

For any \( t > 0 \), define
\[
\begin{align*}
  s_0(t) &= \inf\{ s > 0 : f(s) \leq t \} \vee 2, \\
  s_1(t) &= g^{-1}(t) \vee 2,
\end{align*}
\]
where \( g^{-1}(t) = \inf\{ s > 0 : g(s) \leq t \} \) and we use the convention that \( \inf \emptyset = \infty \). It is clear that there exists a constant \( C_{5.2} \in (0, 1) \) such that
\[
C_{5.2}s_0(t) \leq s_1(t) \quad \text{for all } t > 0.
\]

Recall that \( C_0 \) is the constant in Proposition 3.6, \( C_{3.10} \) is the constant given in (3.10) and \( \phi \) is the function defined in (1.9).

Lemma 5.1. (Lower bound when \( C_1 t_0(y) \leq t \leq C \) for any \( C_1 \geq C_0 C_{3.10} \) and any \( C > 0 \))

Suppose that \( g \) is non-increasing on \( (0, \infty) \) such that \( \lim_{s \to \infty} g(s) = 0 \). Then, for every \( C_1 \geq C_0 C_{3.10} \) and \( C_2 > 0 \), there exist positive constants \( c_{5.1.1}, c_{5.1.2}, c_{5.1.3}, c_{5.1.4} \) (depending on \( C_1 \) and \( C_2 \)) such that for every \( y \in D \) and \( C_1 t_0(y) \leq t \leq C_2 \),
\[
\begin{align*}
  p_D(t, x, y) &\geq c_{5.1.1} \phi(x) \phi(y) \int_0^{c_{5.1.2}s_1(c_{5.1.3}t)} f(s)^{d-1}e^{-c_{5.1.4}f(s)\alpha} ds.
\end{align*}
\]

Proof. Fix \( C_1 \geq C_0 C_{3.10} \) and \( C_2 > 0 \). Since \( s_1(t_0(y)) \leq |y| \vee 2 \), there exist \( c_3, c_4 \) such that \( c_3 s_1(c_4 t) \leq (|y| \vee 2)/2 \) for all \( t \geq t_1(y) \). Recall that we take \( C_1 \geq C_0 C_{3.10} \), and assume that \( D \) is a \( C_{1.1} \)-horn-shaped region satisfying \( \{ x \in D : x_1 > 2 \} = D_2 \) and \( C_0 f(y_1) \alpha \leq c_1 t_0(y) \) for all \( y \in D \). Recall also that we assume that \( x_1 \geq y_1 \). Then, one can choose \( M \geq 2 \) large enough so that \( |x| \geq 2|y|/3 \) for every \( |y| > M \).

We first consider the case that \( |y| \leq M \). Note that there exists \( c_0 > 0 \) such that \( t_0(y) > c_0 \) for all \( y \in D \) with \( |y| \leq M \). Since \( s_1(c_4 t) \leq (2c_3)^{-1}(|y| \vee 2) \leq (2c_2)^{-1}M \) and \( (c_1 c_0) \vee (C_0 f(y_1)) \leq c_1 t_0(y) \), by Proposition 4.3 (i) we have that for every \( y \in D \) with \( |y| \leq M \) and any \( C_1 t_0(y) \leq t \leq C_2 \),
\[
\begin{align*}
  &p_D(t, x, y) \geq c_{5.1.1} \delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2} \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2} \left( 1 + \frac{1}{|x-y|^\alpha} \right) e^{-c_0 f(M)\alpha} \\
  &\quad \geq c_{5.1.1} \phi(x) \phi(y) \int_0^{s_1(c_4 t)} f(s)^{d-1}e^{-c_0 f(s)\alpha} ds.
\end{align*}
\]
where we used the fact that $|x - y| \leq c_0 (1 + |x|)$ and $s_1(c_4 t) \leq (2c_3)^{-1} (|y| \vee 2) \leq (2c_3)^{-1} M$ for every $|y| \leq M$ yielding

$$\int_0^{s_1(c_4 t)} f(s) d^{-1} e^{-c_{44} s} ds \leq \int_0^{(2c_3)^{-1} M} f(s) d^{-1} e^{-c_{44} f(s)} ds \leq c_{10}.$$

For the remainder of the proof, we assume that $|y| > M$. Recall that $c_3 s_1(c_4 t) \leq |y|/2$ for all $t \geq c_1 t_0(y)$. According to Propositions 3.6 and 4.3 for all $x, y, z \in D$ and $c_2 \geq t \geq c_1 t_0(y)$ with $z_1 \leq c_3 s_1(c_4 t) \leq |y|/2$ (which implies that $t \leq c_1 t_0(z)$), we have that

$$p_D(t, x, z) \geq c_{12} \Psi(t, x) \Psi(t, z) \left( \frac{t}{|x - z|^{d+\alpha}} \wedge v^{d/\alpha} \right) e^{-c_{44} t f(z)} \geq c_{14} \Psi(t, x) \Psi(t, z) \left( \frac{t}{1 + |x|^{d+\alpha}} \right) e^{-c_{44} t f(z)}$$

and

$$p_D(t, y, z) \geq c_{15} \Psi(t, y) \Psi(t, z) \left( \frac{t}{1 + |y|^{d+\alpha}} \right) e^{-c_{44} t f(z)},$$

where we used the fact that $z_1 \leq |y|/2 \leq |x|/4$.

Now, we let $\tilde{D} := \{ z : (z_1, \tilde{z}) \in D : |\tilde{z}| \leq 2c_1 f(z_1) \} \subseteq D$ for some constant $c_{17} > 0$ (small enough) such that $\tilde{D}(z) \geq c_{17} f(z_1)$ for all $z \in \tilde{D}$. In particular, for any $z \in \tilde{D}$ with $z_1 \leq c_3 s_1(c_4 t)$ and $c_1 t_0(y) \leq t \leq c_2$, $\Psi(t, z) \geq c_{18} f(z_1)^{21/\alpha} / t \geq c_{19} e^{-c_{44} f(z_1)}$. Then, combining both estimates above together yields that for all $x, y \in D, c_1 t_0(y) \leq t \leq c_2$ and $z \in \tilde{D}$ with $z_1 \leq c_3 s_1(c_4 t)$,

$$p_D(t, x, z) \geq c_{20} \frac{\Psi(t, x) t}{(1 + |x|)^{d+\alpha}} e^{-c_{44} t f(z)} \quad \text{and} \quad p_D(t, y, z) \geq c_{20} \frac{\Psi(t, y) t}{(1 + |y|)^{d+\alpha}} e^{-c_{44} t f(z)}.$$

Hence, for all $c_1 t_0(y) \leq t \leq c_2$,

$$p_D(2t, x, y) \geq \int \{ z \in \tilde{D} : z_1 \leq c_3 s_1(c_4 t) \} p_D(t, x, z) p_D(t, y, z) d z \geq c_{22} \int_{\{ z \in \tilde{D} : z_1 \leq c_3 s_1(c_4 t) \}} \frac{\Psi(t, x) t}{(1 + |x|)^{d+\alpha}} \frac{\Psi(t, y) t}{(1 + |y|)^{d+\alpha}} e^{-c_{44} t f(z)} d z \geq c_{22} \phi(x) \phi(y) \int_{\{ z \in \tilde{D} : z_1 \leq c_3 s_1(c_4 t) \}} e^{-c_{44} t f(z)} d z,$$

where the last inequality follows from the definition of $\Psi(t, x)$ and the fact that $t \geq C_0 f(y_1)^{21/\alpha} \geq C_0 f(x_1)^{21/\alpha}$.

Furthermore, note that for all $c_1 t_0(y) \leq t \leq c_2$, it holds that $0 < c_{24} := C_{5.2} c_3 s_0(c_2 c_4) \leq C_{5.2} c_3 s_0(c_4 t) \leq c_3 s_1(c_4 t)$. Thus, by the fact $\{ x \in D : x_1 > 2 \} = D^2_2$, we have

$$\int_{\{ z \in \tilde{D} : z_1 \leq c_3 s_1(c_4 t) \}} e^{-c_{44} t f(z)} d z \geq c_{24} \left( \int \left( \int_{c_3 s_1(c_4 t)} f(s) d^{-1} e^{-c_{44} t f(s)} d s \right) 1_{c_3 s_1(c_4 t) > 2} + \left( \int_{\{ z \in \tilde{D} : z_1 \leq c_2 \}} e^{-c_{44} t f(z)} d z \right) \right) \geq c_{25} \int_{c_3 s_1(c_4 t)} f(s) d^{-1} e^{-c_{44} t f(s)} d s,$$

where the last inequality follows from the property that for every $c_1 t_0(y) \leq t \leq c_2$,

$$\int_{\{ z \in \tilde{D} : z_1 \leq c_2 \}} e^{-c_{44} t f(z)} d z \geq c_{26} \geq c_{27} \int_0^{1/2} f(s) d^{-1} e^{-c_{44} t f(s)} d s.$$

By now we have obtained the desired assertion. \qed

Since $t_0(y) \simeq f(y_1)^{21/\alpha} \log(2 + |y|)$ for all $y \in D$ and the function $g(s) \simeq f(s)^{21/\alpha} \log(2 + s)$ is non-increasing on $(0, \infty)$, for any $y, z \in D$ with $|z| \geq |y|/8$, $t_0(y) \geq c_0 t_0(z)$ holds for some constant.
$c_0 > 0$ independent of $y$ and $z$. In particular, according to (2.10), we know that for any $z \in D$ such that $|z| \geq |y|/8$ and any $c_1 t_0(y) \leq t \leq c_2$ (with any fixed $c_1$ and $c_2$),

$$
\mathbb{P}^z(\tau_D > t) \leq c_3 \Psi(t, z) \min\left\{e^{-c_2 z f(z_1)^{-\alpha} t} + \frac{t}{(1 + |z|)^{d+\alpha-1}}, e^{-c_2 z f(z_1)^{-\alpha} t}\right\}
$$

$$
\leq c_3 \Psi(t, z) \left(e^{-c_2 z f(z_1)^{-\alpha} t} + \frac{t}{(1 + |z|)^{d+\alpha-1}}\right) \leq c_4 \Psi(t, z) \left(\frac{t}{(1 + |z|)^{d+\alpha-1}}\right)^q, \quad (5.4)
$$

where $q = c_0 c_1 \wedge 1 \leq 1$ and in the last inequality we used the facts that $c_0 c_1 t_0(z) \leq c_1 t_0(y) \leq t \leq c_2 (1 + |z|)$ and

$$
e^{-c_2 z f(z_1)^{-\alpha} t} \leq e^{-c_0 c_1 c_2 z f(z_1)^{-\alpha} t_0(z)} = \left(\frac{t_0(z)}{(1 + |z|)^{d+\alpha-1}}\right)^{c_0 c_1} \leq c_5 \left(\frac{t}{(1 + |z|)^{d+\alpha-1}}\right)^{c_0 c_1}.
$$

To consider upper bounds of $p^D(t, x, y)$ we will frequently use (5.4).

**Lemma 5.2. (Upper bound when $C_1 t_0(y) \leq t \leq C$ for some $C_1$ and for any $C > C_1$.)**

Suppose that $g$ is non-increasing on $(0, \infty)$ such that $\lim_{s \to \infty} g(s) = 0$. Then there exists $c_1 > C_0 C_{3.10}$ such that for every $c_2 > c_1$, we can find positive constants $c_{5.2.1}$, $c_{5.2.2}$, $c_{5.2.3}$ and $c_{5.2.4}$ (depending on $c_1$ and $c_2$) so that for every $y \in D$ with $c_1 t_0(y) \leq t \leq c_2$,

$$
p_D(t, x, y) \leq c_{5.2.1} \phi(x) \phi(y) \int_0^{c_{5.2.2} s_1(c_{5.2.3})} f(s)^{d-1} e^{-c_{5.2.4} f(s)^{-\alpha}} ds.
$$

**Proof.** Recall that for any $z, y \in D$ with $|z| \geq |y|/8$, $t_0(y) \geq c_0 t_0(z)$ holds for some constant $c_0 > 0$ independent of $z, y$. As explained in the proof of Lemma 5.1, $C_0 f(y_1)^{\alpha} \leq c_1 t_0(y)$ for every $y \in D$ and $c_1 > C_0 C_{3.10}$, and we can choose $M$ large enough such that $|x| \geq 2 |y|/3$ for every $y \in D$ with $|y| > M$.

Note that there exists $c_3 > 0$ such that $t_0(y) > c_3$ for $y \in D$ with $|y| \leq M$. Thus, for every $y \in D$ with $|y| \leq M$ and $c_1 t_0(y) \leq t \leq c_2$, it holds that

$$
1 + \frac{1}{|x - y|^{d+\alpha}} \leq \frac{c_4}{(1 + |x|)^{d+\alpha}} \leq \frac{c_4 (1 + M)^{d+\alpha}}{(1 + |x|)^{d+\alpha} (1 + |y|)^{d+\alpha}} \quad (5.5)
$$

and $(c_1 c_3 \wedge C_0 f(y_1)^{\alpha}) \leq t \leq c_2 (1 \wedge c_3^{-1} t_0(y))$. Thus, by applying Proposition 4.3 (ii) and (5.5), we get that

$$
p_D(t, x, y) \leq c_5 \delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2} \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2} \left(1 + \frac{1}{|x - y|^{d+\alpha}}\right) e^{-c_6 f(M)^{-\alpha}}
$$

$$
\leq c_7 \phi(x) \phi(y) \int_0^{c_8 s_1(c_9 t)} f(s)^{d-1} e^{-c_{10} f(s)^{-\alpha}} ds. \quad (5.6)
$$

Here in the second inequality we have used the facts that for $y \in D$ with $|y| \leq M$ and $c_1 c_3 \leq c_1 t_0(y) \leq t \leq c_2$, (by noting that $s_1(t) \geq 2$ for all $t > 0$),

$$
\int_0^{c_8 s_1(c_9 t)} f(s)^{d-1} e^{-c_{10} f(s)^{-\alpha}} ds \geq \int_0^{c_8 s_1(c_9 t)} f(s)^{d-1} e^{-c_{10} c_{11} c_3 f(s)^{-\alpha}} ds \geq c_{11}. \quad (5.7)
$$

Next, we suppose that $|y| > M$. It follows from the assumption $f(s) \geq c(1 + s)^{-p}$ that, for any $t \geq c_1 t_0(y) \geq C_0 f(y_1)^{\alpha}$ and $v, u \in D$,

$$
p_D(t, v, u) \leq c_{12} t^{-d/\alpha} \leq c_{13} f(y_1)^{-d} \leq c_{14} (1 + |y|)^{dp}. \quad (5.8)
$$

Fix large $N$ such that $(N - 1) q_0 - dp \geq d + \alpha$ and $(N - 1) q \geq 1$, where $q_0 := q(d + \alpha - 1)$ and $q > 0$ is the constant in (5.4). Suppose that $z, u \in D$ satisfies $|z| \geq |y|/2$ and $c_1 t_0(y) \leq t \leq c_2$.}
Choose $M$ larger if necessary such that $3|z|/4 \geq 3|y|/8 \geq 3M/8 \geq (3Nc_2)^{1/\alpha} \geq (3Nt)^{1/\alpha}$. Then,

$$p_D(2t, z, u) \leq \left\{ \begin{array}{ll}
p_D(t, z, v)p_D(t, v, u) dv, & |u| \geq |z|/4 \\
c_{15} \Psi(t, z) t |z - u|^{-(d+\alpha)}, & |u| < |z|/4 \\
\end{array} \right.$$  

(5.9)

$$\leq \left\{ \begin{array}{ll}
c_{14}(1 + |y|)^{dp} \int_D p_D(t, z, v) dv, & |u| \geq |z|/4 \\
c_{15} \Psi(t, z) t |z - u|^{-(d+\alpha)}, & |u| < |z|/4 \\
\end{array} \right.$$  

where the first inequality follows from Lemma 2.7 because $|z - u| \geq 3|z|/4 \geq (Nt)^{1/\alpha}$ for every $|u| < |z|/4$, the second inequality is due to (5.8), in the third inequality we have used (5.4), and the last inequality is due to $|z| \geq |y|/2$ and $t \leq c_2$.

Furthermore, we can obtain that for any $z, u \in D$ with $|z| \geq |y|/2$ and any $c_1 t_0(y) \leq t \leq c_2$,

$$p_D(3t, z, u) \leq \left\{ \begin{array}{ll}
p_D(2t, z, v)p_D(t, v, u) dv, & |u| \geq |z|/4 \\
c_{18} \Psi(t, z) t |z - u|^{-(d+\alpha)}, & |u| < |z|/4 \\
\end{array} \right.$$  

(5.10)

$$\leq \left\{ \begin{array}{ll}
c_{19} \Psi(t, z) \left( \frac{t^{q}}{(1 + |z|)^{2q - dp}} \vee \frac{t}{(1 + |z|)^{d+\alpha}} \right) \int_D p_D(t, v, u) dv, & |u| \geq |z|/4 \\
c_{18} \Psi(t, z) t |z - u|^{-(d+\alpha)}, & |u| < |z|/4 \\
\end{array} \right.$$  

where the first inequality is due to Lemma 2.7, the second inequality follows from (5.9), and we have used (5.4) again and the fact that $|u| \geq |z|/4 \geq |y|/8$ in the third inequality.

Since $(N - 1)q_0 - dp \geq d + \alpha$, $(N - 1)q \geq 1$ and $|z - u| \geq (Nt)^{1/\alpha}$ for every $|u| < |z|/4$ and $|z| \geq |y|/2$, we can iterate the argument above $N$ times to obtain that for all $z, u \in D$ with $|z| \geq |y|/2$ and all $c_1 t_0(y) \leq t \leq c_2$,

$$p_D(Nt, z, u) \leq c_{22} t \Psi(t, z)(1 + |z|)^{-d-\alpha}.$$  

Combining this with (2.11), we further obtain that for any $u, v \in D$ with $|z| \geq |y|/2$ and $c_1 t_0(y) \leq t \leq c_2$,

$$p_D((N + 1)t, u, z) = \int_D p_D(t, u, v)p_D(Nt, v, z) dv$$  

$$\leq c_{22} t \Psi(t, z)(1 + |z|)^{-d-\alpha} \int_D p_D(t, u, v) dv$$  

(5.10)

$$\leq c_{23} t \Psi(t, z)(1 + |z|)^{-d-\alpha} \left( e^{-c_{24} t f(u_1)^{-\alpha}} + t(1 + |u|)^{-(d+\alpha-1)} \right).$$  

Since $|x| \geq 2|y|/3$, by (5.10) we arrive at that for every $c_1 t_0(y) \leq t \leq c_2$,

$$p_D(2(N + 1)t, x, y) = \int_D p_D((N + 1)t, x, u)p_D((N + 1)t, y, u) du$$  

$$\leq c_{25} t^2 \Psi(t, x)\Psi(t, y)(1 + |x|)^{-d-\alpha}(1 + |y|)^{-d-\alpha}L(t),$$  

(5.11)
where

\[ L(t) := \int_D K(t, z)^2 \, dz \quad \text{and} \quad K(t, z) := e^{-c_2t f(z_1)^{-\alpha}} + t(1 + |z|)^{-(d+\alpha-1)}. \]

Moreover, thanks to the non-increasing property of \( g(s) \simeq f(s)^\alpha \log(2 + s) \), it is not difficult to verify that for all \( c_1 t_0(y) \leq t \leq c_2 \)

\[ K(t, z) \leq c_{26} \begin{cases} 
1 & \text{if } 0 < z_1 \leq c_{27} s_0(c_{28} t); \\
\frac{t(1 + |z|)^{-(d+\alpha-1)}}{e^{-c_2t f(z_1)^{-\alpha}}} & \text{if } c_{27} s_0(c_{28} t) < z_1 \leq c_{29} s_1(c_{30} t); \\
\end{cases} \]

Write

\[ L(t) = \int_{\{z \in D : z_1 \leq c_{27} s_0(c_{28} t)\}} K(t, z)^2 \, dz + \int_{\{z \in D : c_{27} s_0(c_{28} t) < z_1 \leq c_{29} s_1(c_{30} t)\}} K(t, z)^2 \, dz \\
+ \int_{\{z \in D : z_1 > c_{29} s_1(c_{30} t)\}} K(t, z)^2 \, dz =: L_1 + L_2 + L_3. \]

Therefore, using the facts that \( s_0(t) \geq 2 \) and \( \{x \in D : x_1 > 2\} = D_f^1 \),

\[ L_1 \leq c_{31} \int_{c_{27}}^{c_{29} s_0(c_{28} t)} f(s)^{d-1} \, ds + 1 \leq c_{32} \left( \int_0^{c_{27} s_0(c_{28} t)} f(s)^{d-1} e^{-t f(s)^{-\alpha}} \, ds + 1 \right), \]

\[ L_2 \leq c_{33} \int_{c_{29} s_1(c_{30} t)}^{c_{27} s_0(c_{28} t)} f(s)^{d-1} e^{-2c_2 t f(s)^{-\alpha}} \, ds, \]

\[ L_3 \leq c_{34} \int_{\{z \in D : z_1 > c_{29} s_1(c_{30} t)\}} (1 + |z|)^{-2(d+\alpha-1)} \, dz \leq c_{35}, \]

where the inequality for \( L_1 \) follows from the argument of \([5.7]\) and the fact that \( e^{-t f(s)^{-\alpha}} \geq c_{36} \) for every \( 0 \leq s \leq c_{27} s_0(c_{28} t) \). Hence, according to the proof of \([5.7]\) again,

\[ L(t) \leq c_{36} \int_0^{c_{29} s_1(c_{30} t)} f(s)^{d-1} e^{-c_3 t f(s)^{-\alpha}} \, ds. \]

This, along with \([5.11]\) (by replacing \( 2(N + 1)t \) with \( t \)), the definition of \( \Psi(t, x) \) and \([1.7]\), yields that for all \( 2c_1(N + 1)t_0(y) \leq t \leq c_2 \),

\[ p_D(t, x, y) \leq c_{38} \Psi(2^{-1}(N + 1)^{-1} t, x) \Psi(2^{-1}(N + 1)^{-1} t, y) \frac{t}{(1 + |x|)^{d+\alpha} (1 + |y|)^{d+\alpha}} \times \int_0^{c_{29} s_1(c_{30} 2^{-1}(N + 1)^{-1} t)} f(s)^{d-1} e^{-c_3 t 2^{-1}(N + 1)^{-1} f(s)^{-\alpha}} \, ds \\
\leq c_{39} \phi(x) \phi(y) \int_0^{c_{32} s_1(c_{40} t)} f(s)^{d-1} e^{-c_4 t f(s)^{-\alpha}} \, ds, \]

proving the desired assertion. \( \square \)

By Lemmas \([5.1]\) and \([5.2]\) we further have the following statement.

**Proposition 5.3.** Suppose that \( g \) is non-increasing on \((0, \infty)\) such that \( \lim_{s \to \infty} g(s) = 0 \). Then there are constants \( c_{5,3,i} > 0 \) (\( i = 1, 2, \cdots, 8 \)) such that for all \( x, y \in D \) and \( t \geq c_{5,3,1} t_0(y) \),

\[ c_{5,3,2} \phi(x) \phi(y) \max \left\{ \int_0^{c_{5,3,3} s_1(c_{5,3,4} t)} f(s)^{d-1} e^{-c_5 t f(s)^{-\alpha}} \, ds, e^{-c_{5,3,6} t} \right\} \]

\[ \leq p_D(t, x, y) \leq c_{5,3,7} \phi(x) \phi(y) \max \left\{ \int_0^{c_{5,3,8} s_1(c_{5,3,9} t)} f(s)^{d-1} e^{-c_5 t f(s)^{-\alpha}} \, ds, e^{-c_{5,3,11} t} \right\}. \]
Proof. Since the function $g(s) \simeq f(s)^\alpha \log(1 + s)$ is non-increasing with $\lim_{s \to \infty} g(s) = 0$, $s_1(t) = 2$ for $t > 0$ large enough. Thus, by Lemmas 5.1 and 5.2 we only need to verify the required assertion for all $t \geq c_0$ with any given $c_0 > 0$.

According to [33, Theorem 5], the associated Dirichlet semigroup $(P^D_t)_{t \geq 0}$ is intrinsically ultracontractive when $\lim_{y \in D, |y| \to \infty} t_0(y) = 0$. Hence, it follows from [27, Theorem 4.2.5] that for all $t \geq c_0$ and $x, y \in D$, $p_D(t, x, y) \simeq e^{-\lambda t} \phi_1(x) \phi_1(y)$ where $\phi_1(x)$ is the ground state (i.e., the first strictly positive eigenfunction corresponding to the smallest eigenvalue $\lambda_1$ of the Dirichlet fractional Laplacian $(-\Delta)^\alpha|_D$) of the semigroup $(P^D_t)_{t \geq 0}$. On the other hand, by [33, Theorem 6.1] and its proof, for all $x \in D$, $\phi_1(x) \simeq \phi(x) = \delta_D(x)^{\alpha/2} f(x)^{\alpha/2}(1 + |x|)^{-d-\alpha}$. Putting both estimates together, we can obtain the desired assertion.

5.2. Case III-2: $g$ is non-decreasing on $(0, \infty)$. In this part, we are concerned with the case that $g$ is non-decreasing on $(0, \infty)$. In particular, $\lim_{s \to \infty} g(s) > 0$. Because of $t_0(y) \simeq f(y)^\alpha \log(2 + |y|)$, we have $\lim_{y \in D, |y| \to \infty} t_0(y) > 0$.

Lemma 5.4. (Lower bound) Suppose that $g$ is non-decreasing on $(0, \infty)$. Then there exist constants $c_{5.4.1} > 0$ large enough and $c_{5.4.2}, c_{5.4.3} > 0$ such that for every $y \in D$ with $t \geq c_{5.4.1} t_0(y) \geq 1$,

$$p_D(t, x, y) \geq c_{5.4.2} e^{-c_{5.4.3} t} \phi(x) \phi(y).$$

(5.12)

Proof. We choose $c_1 > C_0 C_{3.10}$ and $M > 20$ large enough so that $c_1 t_0(y) \geq 2 \vee C_0 f(y)^\alpha$ and, that if $|y| > M$ then $|x| \geq 2|y|/3$. Note that, for $|y| \leq M$ and $t \geq 2$,

$$t^{-d/\alpha} \land \frac{t}{|x - y|^{d+\alpha}} \geq c_2 \frac{t^{-d/\alpha}}{|x| + 1} \geq c_2 t^{-d/\alpha}(M + \delta)^{d+\alpha}.$$ 

By Proposition 4.3(i), for every $y \in D$ with $|y| \leq M$ and $t \geq c_1 t_0(y)$,

$$p_D(t, x, y) \geq c_3 t^{-d/\alpha} \delta_D(x)^{\alpha/2} f(x)^{\alpha/2} \delta_D(y)^{\alpha/2} f(y)^{\alpha/2} \frac{e^{-ct_0 f(M)^{-\alpha}}}{(|x| + 1)^{d+\alpha} (|y| + 1)^{d+\alpha}}. $$

Thus, (5.12) holds if $|y| \leq M$.

Next, we assume that $|y| > M$. Fix a ball $B(x_0, 4\lambda_1) \subset D$ with $x_0 \in D$ such that $|x_0| \leq 6$ and $\lambda_1 > 0$. As shown in the beginning of the proof for Lemma 4.1, there are $\xi_x, \xi_y \in D$ and $\lambda_2 > 0$ such that $B(\xi_x, 4\lambda_2 f(x_1)) \subset D$, $B(\xi_y, 4\lambda_2 f(y_1)) \subset D$, and (4.1) holds true with $V_x := B(\xi_x, 2\lambda_2 f(x_1))$ and $V_y := B(\xi_y, 2\lambda_2 f(y_1))$.

On the other hand, we find that for all $z \in V_x$, $w \in B(x_0, \lambda_1)$ and $t \geq c_1 t_0(y) \geq 1$,

$$p_D(t, z, w) = E^z \left[p_D(t/2, X^{D}_{t/2}, w)\right]$$

$$\geq E^z \left[p_D(t/2, X_{t/2}, w) : 0 < \tau_{B(z, \lambda_2 f(x_1))} < 2^{-2} C_0 f(x_1)^\alpha, X_{\tau_{B(z, \lambda_2 f(x_1))}} \in B(w, \lambda_1), \right.$$ 

$$X_s \in B(w, 2\lambda_1) \text{ for all } s \in [\tau_{B(z, \lambda_2 f(x_1))}, \tau_{B(z, \lambda_2 f(x_1))} + t] \right]$$

$$\geq c_6 e^{-ct_0} \left( \int_0^{2^{-2} C_0 f(x_1)^\alpha} \int_{B(z, \lambda_2 f(x_1))} p_{B(z, \lambda_2 f(x_1))}(s, z, u) \int_{B(w, \lambda_1)} \frac{1}{|u - v|^{d+\alpha}} dv du ds \right)$$

$$\times \inf_{u \in B(w, \lambda_1)} \mathbb{P}^u (\tau_{B(w, \lambda_1)} > t)$$

$$\geq c_6 e^{-ct_0} f(x_1)^\alpha \mathbb{P}^\tau_{B(z, \lambda_2 f(x_1))} > C_0 f(x_1)^\alpha/4 \frac{1}{(1 + |x|)^{d+\alpha}} \geq c_{10} e^{-c_0 t} f(x_1)^\alpha \frac{1}{(1 + |x|)^{d+\alpha}},$$

where the second inequality follows from Lévy system (2.2) and the fact that

$$\inf_{w, v \in B(z_0, 3\lambda_1)} p_D(t/2, w, v) \geq \inf_{w, v \in B(z_0, 3\lambda_1)} p_{B(z_0, 4\lambda_1)}(t/2, w, v) \geq c_6 e^{-ct_0}, \quad t \geq 1$$

thanks to (1.3), the third inequality is due to (4.4) (also by (1.3)) and the fact that $|u - v| \leq c_{11}(1 + |x|)$ for all $u \in B(z, \lambda_2 f(x_1))$ with $z \in V_x$ and $v \in B(w, \lambda_1)$, and in the last inequality we have used (2.1).
Combining the estimate above with (4.1) yields that for all \( w \in B(x_0, \lambda_1) \) and \( t \geq 2c_1t_0(y) \geq 1, \)

\[
p_D(t, x, w) \geq \int_{V_\epsilon} p_D(2^{-2}C_0f(x)^{\alpha}, x, z)p_D(t - 2^{-2}C_0f(x)^{\alpha}, z, w) \, dz
\geq c_12\delta_D(x)^{\alpha/2}f(x)^{-\alpha/2}e^{-c_1st}f(x)^{\alpha} \frac{f(x)^{\alpha}}{(1 + |x|)^{d+\alpha}} = c_12\phi(x)e^{-c_1st}.
\]

Similarly, for every \( w \in B(x_0, \lambda_1) \) and \( t \geq 2c_1t_0(y) \geq 1, \)

\[
p_D(t, y, w) \geq c_14\phi(y)e^{-c_1st}. \quad \text{Hence, for all } \ t \geq 2c_1t_0(y) \geq 1,
\]

\[
p_D(t, x, y) \geq \int_{B(x_0, \lambda_1)} p_D(t/2, x, w)p_D(t/2, w, y) \, dw \geq c_16e^{-c_1st}\phi(x)\phi(y).
\]

Now we have proved the desired assertion. \( \square \)

**Lemma 5.5. (Upper bound)** Suppose that \( g \) is non-decreasing on \((0, \infty)\). Then, there exist constants \( c_{5.5.1}, c_{5.5.2}, c_{5.5.3} > 0 \) such that for all \( t > 0 \) and \( y \in D \) with \( t \geq c_{5.5.1}t_0(y)(\geq 1), \)

\[
p_D(t, x, y) \leq c_{5.5.2}e^{-c_{5.5.3}t}\phi(x)\phi(y). \tag{5.13}
\]

**Proof.** Since \( t_0(z) \simeq f(z)^{\alpha} \log(2 + |z|) \) and the function \( s \mapsto g(s) \simeq f(s)^{\alpha} \log(2 + s) \) is non-decreasing, we have \( t_0(z) \geq c_0 > 0 \) for all \( z \in D \). Choose \( M_0 > 20 \) and \( c_1 > C_0C_{3.10} \) large enough so that \( |x| \geq 2|y|/3 \) for every \( |y| > M_0, \) \( f(z) \simeq f(y_1) \) for every \( |z| \geq 2|y| > 2M_0, \) and \( c_1t_0(y) \geq 2 \lor C_0f(y_1)^{\alpha}. \)

Since \( t_0(y) \geq c_0 > 0, \) for every \( y \in D \) with \( |y| \leq M_0 \) and \( t \geq c_1t_0(y), \)

\[
t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \simeq \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}} \leq \frac{c_2t}{(1 + |x|)^{d+\alpha}},
\]

so, by applying Proposition 4.3(ii), we get that for every \( y \in D \) with \( |y| \leq M_0 \) and \( t \geq c_1t_0(y), \)

\[
p_D(t, x, y) \leq c_3\delta_D(x)^{\alpha/2}f(x)^{\alpha/2}\delta_D(y)^{\alpha/2}f(y)^{\alpha/2}e^{-c_4t(M_0)^{-\alpha}} \leq c_5\phi(x)\phi(y)e^{-c_4t(M_0)^{-\alpha}/2}.
\]

Thus, we only need to consider the case that \( |y| > M_0 \) and \( t \geq c_1t_0(y). \) For this, we will split the proof into two parts.

(1) For any \( t > 0, \) define

\[
s_2(t) := \sup\{ s > 0 : C_{5.14.1} \log(2 + s) \leq t \} \lor 2M_0,
\]

\[
s_3(t) := \sup\{ s > 0 : C_{5.14.2}g(s) \leq t \} \lor s_2(t), \tag{5.14}
\]

where we use the convention that \( \sup\emptyset = 0. \) It is easy to see that \( s_3(t) \geq s_2(t) > 0, \) and the constants \( C_{5.14.1}, C_{5.14.2} \) (both of which are large) are to be determined later.

Again by the assumption that the function \( s \mapsto g(s) \simeq f(s)^{\alpha} \log(2 + s) \) is non-decreasing on \((0, \infty), \) \( t_0(z) \simeq f(z)^{\alpha} \log(2 + |z|) \) and the definition of \( s_3(t), \) (by choosing \( c_1, C_{5.14.1}, C_{5.14.2} \) large enough if necessary,) we can find a positive constant \( c_6 \) such that for every \( t \geq c_1t_0(y), \)

\[
t \leq c_6t_0(z) \text{ when } |z| \geq s_3(t)/2, \quad \text{and } t \geq 33t_0(z) \text{ when } |z| \leq 8s_3(t). \tag{5.15}
\]
This along with Lemma 2.8 yields that for all \( z \in D \) and \( t \geq c_1 t_0(y) \),

\[
P^z(\tau_D > t) \leq c_7 \Psi(t, z) \min \left\{ e^{-c_{9.2} f(z_1) - \alpha t} + \frac{t}{(1 + |z|)^{d+\alpha-1}}, e^{-c_{9.2} t} \right\}
\]

\[
\leq c_7 \Psi(t, z) \times \left\{ e^{-c_{9.2} f(z_1) - \alpha t} + \frac{c_6 t_0(z)}{(1 + |z|)^{d+\alpha-1}}, |z| \geq s_3(t) \right\}
\]

\[
\leq c_8 \Psi(t, z) \times \left\{ e^{-c_{10} f(z_1) - \alpha t} + \frac{t_0(z)}{(1 + |z|)^{d+\alpha-1}}, |z| \geq s_3(t) \right\}
\]

\[
\leq c_9 \Psi(t, z) \times \left\{ e^{-c_{10} f(z_1) - \alpha t} + \frac{t_0(z)/2}{(1 + |z|)^{d+\alpha-1}}, \right\}
\]

\[
\leq c_9 \Psi(t, z) \times \left\{ e^{-c_{10} f(z_1) - \alpha t} + \frac{t_0(z)/2}{(1 + |z|)^{d+\alpha-1}}, s_2(t)/2 \leq |z| \leq 8 s_3(t) \right\}
\]

\[
\leq c_9 \Psi(t, z) \times \left\{ e^{-c_{10} f(z_1) - \alpha t} + \frac{t_0(z)/2}{(1 + |z|)^{d+\alpha-1}}, s_2(t)/2 \leq |z| \leq 8 s_3(t) \right\}
\]

\[
\leq c_9 \Psi(t, z) \times \left\{ e^{-c_{10} f(z_1) - \alpha t} + \frac{t_0(z)/2}{(1 + |z|)^{d+\alpha-1}}, |z| \leq 2 s_2(t) \right\}
\]

where in the last inequality we have used (5.15). Thus, for all \( t \geq c_1 t_0(y) \), \( u, z \in D \) with \( |z| \geq s_3(t) \) and \( N \geq 2 \),

\[
p_D(N t, z, u) = \int_D p_D(t, z, v)p_D((N - 1)t, v, u) \, dv \leq c_{11} ((N - 1)t)^{-d/\alpha} \int_D p_D(t, z, v) \, dv
\]

\[
\leq c_{12} P^z(\tau_D > t) \leq c_{13} \Psi(t, z) e^{-c_{10} f(z_1) - \alpha t}.
\]

Below, we will further refine the estimate above. For every \( |u| \geq 2 |z|, |z| \geq s_3(t) \geq 2 \) and \( N \geq 2 \), we have

\[
p_D((N + 1)t, z, u) = \left( \int_{\{v \in D: |v - z| \leq |u|/4\}} + \int_{\{v \in D: |v - z| > |u|/4\}} \right) p_D(t, z, v)p_D(Nt, v, u) \, dv =: I_1 + I_2.
\]

On the one hand, for \( |z| \geq s_3(t) \) and \( |u| \geq 2 |z| \),

\[
I_1 \leq c_{14} \int_{\{v \in D: |v - z| \leq |u|/4\}} p_D(t, z, v) \frac{N t}{|v - u|^{d+\alpha}} \, dv
\]

\[
\leq c_{15} t \int_D p_D(t, z, v) \, dv \leq c_{16} \Psi(t, z) \frac{t}{(1 + |u|)^{d+\alpha}} e^{-c_{10} f(z_1) - \alpha t},
\]

where the second inequality follows from the fact that \( |v - u| \geq |u| - |v - z| - |z| \geq |u|/4 \) for all \( v \in D \) with \( |v - z| \leq |u|/4 \), and in the last inequality we have used (5.16). On the other hand, for \( |z| \geq s_3(t) \) and \( |u| \geq 2 |z| \),

\[
I_2 \leq c_{17} \Psi(t, z) \int_{\{v \in D: |v - z| > |u|/4\}} \frac{t}{|z - v|^{d+\alpha}} p_D(Nt, v, u) \, dv \leq c_{18} \Psi(t, z) \frac{t}{(1 + |u|)^{d+\alpha}} e^{-c_{10} f(u_1) - \alpha t},
\]

where the first inequality follows from Lemma 2.7 (since \( |z - v| \geq |u|/4 \geq |z|/2 \geq s_3(t)/2 \geq s_2(t)/2 \geq t^{1/\alpha} \) for all \( t \geq c_1 t_0(y) \) by taking \( c_1 \) large enough if necessary), and in the second inequality we used (5.16), \( P^z(\tau_D > N t) \leq P^z(\tau_D > t) \) and the fact \( |u| \geq 2 |z| \geq 2 s_2(t) \). Combining with both estimates above, we arrive at that for all \( z, u \in D \) with \( |z| \geq s_3(t) \) and \( |u| \geq 2 |z| \),

\[
p_D((N + 1)t, z, u) \leq c_{19} \Psi(t, z) e^{-c_{10} f(z_1) - \alpha t} \frac{t}{(1 + |u|)^{d+\alpha}},
\]

where we used the fact that \( f(u_1) \leq f(z_1) \) for \( |u| \geq 2 |z| > 2 M_0 \) due to the choice of \( M_0 \).

Meanwhile, for all \( z, u \in D \) with \( |z| \geq s_3(t) \) and \( |u| \leq |z|/2 \), we have

\[
p_D((N + 1)t, z, u) = \left( \int_{\{v \in D: |v - z| \leq |z|/4\}} + \int_{\{v \in D: |v - z| > |z|/4\}} \right) p_D(Nt, z, v)p_D(t, v, u) \, dv =: J_1 + J_2.
\]
Then, replacing (5.16) by $P^z(\tau_D > t) \leq c_{20}\Psi(t, z)e^{-c_{2.9}t^2}$ (due to (2.10)) and following the arguments above for $I_1$ and $I_2$, we can obtain immediately that for all $z, u \in D$ with $|z| \geq s_3(t)$ and $|u| \leq |z|/2$, and for all $N \geq 2$,

$$p_D((N + 1)t, z, u) \leq c_{21}\Psi(t, z)e^{-c_{2.9}t^2} \frac{t}{(1 + |z|)^{d+\alpha}}.$$ 

Therefore, putting all the cases together, we finally get that for any $N \geq 2$ and $z, u \in D$ with $|z| \geq s_3(t)$,

$$p_D((N + 1)t, z, u) \leq c_{22}\Psi(t, z)L_1(z, u, t),$$

where

$$L_1(z, u, t) = \frac{te^{-c_{2.9}t^2}}{(1 + |z|)^{d+\alpha}}I_{\{|u| \leq |z|/2\}} + e^{-c_{10}(f(z))^{-\alpha}t}I_{\{|z|/2 \leq |u| \leq |z|\}} + \frac{te^{-c_{10}f(z_1)^{-\alpha}t}}{(1 + |u|)^{d+\alpha}}I_{\{|u| > |z|\}}.$$ 

Note that, for the case $|z|/2 \leq |u| \leq |z|$ above, we used (5.17) directly.

Similarly, replacing (5.16) by $P^z(\tau_D > t) \leq c_{23}\Psi(t, z)e^{-c_{2.9}t^2}$ and following the argument for (5.19), we can obtain that for every $N \geq 2$ and $u, z \in D$ with $|z| \leq 4s_2(t)$,

$$p_D((N + 1)t, z, u) \leq c_{24}\Psi(t, z)L_3(z, u, t),$$

where

$$L_3(z, u, t) = \frac{te^{-c_{2.9}t^2}}{(1 + |z|)^{d+\alpha}}I_{\{|u| \leq |z|/2\}} + e^{-c_{2.9}t^2}I_{\{|z|/2 \leq |u| \leq |z|\}} + e^{-c_{2.9}t^2}t \frac{t}{(1 + |u|)^{d+\alpha}}I_{\{|u| > |z|\}}.$$ 

In particular, by choosing $C_{3.14.1}$ large enough so that $e^{-c_{2.9}t^2} \leq c_{25}t(1 + |z|)^{-\alpha}$ for every $|z| \leq 4s_2(t)$, it holds that

$$L_3(z, u, t) \leq c_{25}e^{-c_{2.9}t^2}t \frac{t}{(1 + |z|)^{d+\alpha}}.$$ 

Next, let $u, z \in D$ with $8s_3(t) \geq |z| \geq s_2(t)/2$. Then, by (5.16),

$$P^z(\tau_D > t) \leq c_9\Psi(t, z) \frac{t}{(1 + |z|)^{d+\alpha-1}}, \quad s_2(t)/2 \leq |z| \leq 8s_3(t).$$

Hence, for every $t \geq c_{14.0}(y)$ and $u, z \in D$ with $8s_3(t) \geq |z| \geq s_2(t)/2$,

$$p_D(2t, z, u) \leq \begin{cases} \int_D p_D(t, z, v)p_D(t, v, u) dv \leq c_{27}P^z(\tau_D > t) & \text{if } |u| \geq |z|/4 \\ c_{28}\Psi(t, z) |z - u|^{-(d+\alpha)} \leq c_{29}\Psi(t, z)|z - u|^{-\alpha} & \text{if } |u| < |z|/4 \end{cases} \leq \begin{cases} c_{28}\Psi(t, z) \frac{t}{(1 + |z|)^{d+\alpha-1}}, & |u| \geq |z|/4 \\ c_{27, 5}\Psi(t, z) (1 + |z|)^{-\alpha} & |u| < |z|/4 \end{cases} \leq \begin{cases} c_{28}\Psi(t, z) \frac{t}{(1 + |z|)^{d+\alpha-1}}, & \text{if } |u| \geq |z|/4 \\ c_{27, 5}\Psi(t, z) (1 + |z|)^{-\alpha} & \text{if } |u| < |z|/4 \end{cases}$$

where the first inequality is due to Lemma 2.7 (since $|z - u| \geq 3|z|/4 \geq 3s_2(t)/8 \geq t^{1/\alpha}$ by choosing $c_1$ large enough if necessary), in the second inequality we have used that $p_D(t, u, v) \leq p(t, u, v) \leq c_{29}t^{-\alpha} \leq c_{29}$ for every $t \geq c_{14.0}(y)$. 

Now, applying (5.22) and following the same iteration arguments for (5.10), we can find an integer $M \geq 3$ such that for all $u, z \in D$ with $8s_3(t) \geq |z| \geq s_2(t)/2$,

$$p_D((M - 1)t, z, u) \leq c_{30}\Psi(t, z) \frac{t}{(1 + |z|)^{d+\alpha}}.$$ 

Then, according to (5.23) and (5.16), for every $t \geq c_{14.0}(y)$ and every $u, z \in D$ with $s_2(t)/2 \leq |z| \leq 4s_3(t)$ and $s_2(t)/2 \leq |u| \leq 8s_3(t)$,

$$p_D(Mt, z, u) = \int_D p_D((M - 1)t, z, v)p_D(t, v, u) dv \leq c_{30}\Psi(t, z) \frac{t}{(1 + |z|)^{d+\alpha}} \int_D p_D(t, v, u) dv \leq c_{31}\Psi(t, z) \frac{t}{(1 + |u|)^{d+\alpha}}.$$
Meanwhile, following the same arguments for (5.18), (in particular, applying
\[ \mathbb{P}^x(\tau_D > t) \leq c_9 \frac{\Psi(t, z)t}{(1 + |z|)^{d+\alpha-1}} \]
in the estimate of \( I_1 \) for every \( s_2(t)/2 \leq |z| \leq 4s_3(t) \), and
\[ \mathbb{P}^u(\tau_D > t) \leq c_8 \left( e^{-c_{2,9.2}f(u_1)^{-\alpha}t} + \frac{t_0(u)}{(1 + |u|)^{d+\alpha-1}} \right) \]
in the estimate of \( J_2 \) for every \(|u| \geq 8s_3(t)\), which are due to the last and the third inequalities in (5.16) respectively, we can obtain that for every \( u, z \in D \) with \( s_2(t)/2 \leq |z| \leq 4s_3(t) \) and \(|u| \geq 8s_3(t) \geq 2|z|\),
\[
p_D(Mt, z, u) \leq c_{32} \frac{\Psi(t, z)t}{(1 + |u|)^{d+\alpha}} \left( \frac{t}{(1 + |z|)^{d+\alpha-1}} + \frac{t_0(u)}{(1 + |u|)^{d+\alpha-1}} + e^{-c_{2,9.2}f(u_1)^{-\alpha}t} \right)
\leq c_{33} \frac{\Psi(t, z)t}{(1 + |u|)^{d+\alpha}} \left( \frac{t}{(1 + |z|)^{d+\alpha-1}} + \frac{\log(2 + |u|)}{(1 + |u|)^{d+\alpha-1}} + e^{-c_{2,9.2}f(z_1)^{-\alpha}t_0(z)} \right)
= c_{34} \frac{\Psi(t, z)t}{(1 + |u|)^{d+\alpha}} \left( \frac{t + t_0(z)}{(1 + |z|)^{d+\alpha-1}} + \frac{\log(2 + |u|)}{(1 + |u|)^{d+\alpha-1}} \right) \leq c_{35} \frac{\Psi(t, z)t}{(1 + |u|)^{d+\alpha}} \left( \frac{t + \log(2 + |u|)}{(1 + |u|)^{d+\alpha-1}} \right).
\]
Here in the second inequality we have used the facts that \( t_0(u) \leq c_{35} \log(2 + |u|) \), \( f(u_1) \leq f(z_1) \) (which is due to \(|u| \geq 2|z| \geq 2s_2(t) \geq 2M_0 \) and \( t \geq t_0(z) \) (which is due to (5.15)), and the last inequality follows from \(|u| \geq 2|z|\).

Following the same arguments above for (5.24), and using (5.23) as well as (5.16), we can obtain that for every \( u, z \in D \) with \( s_2(t)/2 \leq |z| \leq 4s_3(t) \) and \(|u| \leq s_2(t)\),
\[
p_D(Mt, z, u) \leq c_{36} \frac{\Psi(t, z)t}{(1 + |z|)^{d+\alpha}} e^{-c_{2,9.2}t}.
\]

Combining all above estimates together, we know that there exists \( M \geq 3 \) such that for all \( u, z \in D \) with \( s_2(t)/2 \leq |z| \leq 4s_3(t) \) and \( t \geq c_1 t_0(y) \),
\[
p_D(Mt, z, u) \leq c_{37} \frac{\Psi(t, z)}{(1 + |z|)^{d+\alpha}} L_2(z, u, t),
\]
where
\[
L_2(z, u, t) = e^{-c_{2,9.2}t} \frac{t}{(1 + |z|)^{d+\alpha}} 1_{\{|u| \leq s_2(t)\}} + \frac{t + \log(2 + |u|)}{(1 + |z|)^{d+\alpha-1}} 1_{\{|u| \geq s_2(t)\}}.
\]

(2) According to (5.15), (by taking \( c_1 \) large enough if necessary), we have \(|y| \leq s_3(t)\) when \( t \geq c_1 t_0(y) \). Now, we will prove the desired upper bounds of \( p_D(t, x, y) \) for \(|y| \geq M_0 \) and \( t \geq c_1 t_0(y) \).

We first note that, since \( t \geq c_1 t_0(y) \geq 2 \vee C_0 f(y_1)^\alpha \geq 2 \vee C_0 f(x_1)^\alpha \), we have
\[
\frac{\Psi(t, x)}{(1 + |x|)^{d+\alpha}} \simeq \phi(x) \quad \text{and} \quad \frac{\Psi(t, y)}{(1 + |y|)^{d+\alpha}} \simeq \phi(y).
\]

We consider the following five cases separately.

(i) Case 1: \( s_2(t) \leq |y| \leq s_3(t) \) and \( s_2(t)/2 \leq |x| \leq 4s_3(t) \) (since \(|x| \geq 2|y|/3 \) for all \(|y| > M_0\).

In this case, letting \( C_{5.14.1} > 4/c_{2,9.2} \), we get from (5.25) that
\[
p_D(2Mt, x, y) = \int_D p_D(Mt, x, u)p_D(Mt, u, y) \, du \leq c_{38} \frac{\Psi(t, x)}{(1 + |x|)^{d+\alpha}} \frac{\Psi(t, y)}{(1 + |y|)^{d+\alpha}} \int_D L_2(x, u, t)L_2(y, u, t) \, du
\]
\[
= c_{38} \frac{\Psi(t, x)}{(1 + |x|)^{d+\alpha}} \frac{t}{(1 + |y|)^{d+\alpha}} \int_D \left( \int_{\{|u| \leq s_2(t)\}} \frac{t + \log(2 + |u|)^2}{(1 + |u|)^{2(d+\alpha-1)}} \, du + \int_{\{|u| \geq s_2(t)\}} \frac{(t + \log(2 + |u|)^2}{(1 + |u|)^{2(d+\alpha-1)}} (1 + |u|)^{2(d+\alpha-1)} \, du \right) \, du
\]
\[
\leq c_{39} \frac{\phi(x)}{\phi(y)} (t)^2 \left( e^{-2c_{2,9.2}t} s_2(t) + (t + \log s_2(t))^2 s_2(t)^{-2d-2\alpha+3} \right) \leq c_{40} \phi(x) \phi(y) e^{-c_{41}t},
\]
where in the last inequality we used the facts that \( s_2(t) \leq e^{t/C_{5.14.1}} \) for large \( t \) and \( C_{5.14.1} > 4/c_{2,9.2} \).
(ii) Case 2: \( s_2(t) \leq |y| \leq s_3(t) \) and \( |x| \geq 4s_3(t) \). In this case, we write

\[
p_D(2Mt, x, y) = \left( \int_{\{u \in D: |u| < |x|/2\}} + \int_{\{u \in D: |x|/2 \leq |u| \leq 2|x|\}} + \int_{\{u \in D: |u| > 2|x|\}} \right) \times p_D(Mt, x, u)p_D(Mt, u, y) \, du =: H_1 + H_2 + H_3.
\]

By (5.19) and (5.25), for all \( t \geq c_1t_0(y) \),

\[
H_1 \leq c_{42}\Psi(t, x)\Psi(t, y) \int_{\{u \in D: |u| < |x|/2\}} L_1(x, u, t)L_2(y, u, t) \, du
\]

\[
\leq c_{43}\Psi(t, x)\Psi(t, y)e^{-c_{2.9.2t}t} \int_{\{u \in D: |u| \leq s_2(t)\}} t (1 + |x|)^{d+\alpha} \, du \int_{\{u \in D: |u| > s_2(t)\}} L_2(y, u, t) \, du
\]

\[
\leq c_{44}\Psi(t, x)t \int_{\{u \in D: |u|/2 \leq |u| \leq 2|x|\}} e^{-c_{2.9.2t}t} \int_{\{u \in D: |u| > s_2(t)\}} e^{-c_{2.9.2t}t} L_2(y, u, t) \, du
\]

\[
\leq c_{45}\phi(x)\phi(y)t^2 e^{-c_{2.9.2t}t} \int_{\{u \in D: |u| > s_2(t)\}} e^{-c_{2.9.2t}t} \left( t + \log(2 + s) \right) \, du
\]

where in the last inequality we have used the fact that \( s_2(t) \leq e^{t/C_{5.14.1}} \) for large \( t \) and we have chosen \( C_{5.14.1} > 4/c_{2.9.2} \). On the other hand,

\[
H_2 \leq c_{48}\Psi(t, y) \int_{\{u \in D: |x|/2 \leq |u| \leq 2|x|\}} p_D(Mt, x, u) \frac{t}{|u - y|^{d+\alpha}} \, du \leq c_{49}\Psi(t, y) \int_{\{u \in D: |x|/2 \leq |u| \leq 2|x|\}} t (1 + |x|)^{d+\alpha} \, du
\]

\[
\leq c_{50}\Psi(t, x)\Psi(t, y) \int_{\{u \in D: |x|/2 \leq |u| \leq 2|x|\}} e^{-c_{51}f(x_1)^{-\alpha}t} \leq c_{52}\phi(x)\phi(y)e^{-c_{51}t},
\]

where the first inequality follows from Lemma 2.7 and the fact that \( |y| \leq s_3(t) \leq |x|/4 \) and so \( |u - y| \geq |u| - |y| \geq |x|/4 > s_3(t) \geq |x|/4 \) for all \( u \in D \) with \( |x|/2 \leq |u| \leq 2|x| \) (by taking \( c_1 \) large enough if necessary), in the second inequality we have used again the fact \( |u - y| \geq |x|/4 \), in the third inequality we have applied (5.16), and in the last inequality we have used the facts that \( f(x_1) \leq f(y_1) \), and for every \( y \in D \) and \( t \geq c_1t_0(y) \) with large enough \( c_1 > 0 \),

\[
e^{-c_{51}f(x_1)^{-\alpha}t/2} \leq e^{-c_{51}f(y_1)^{-\alpha}t} \leq \left( \frac{t_0(y)}{(1 + |y|)^{d+\alpha}} \right)^{c_{51}/(2c_{2.9.2})} \leq \left( \frac{t_0(y)}{(1 + |y|)^{d+\alpha}} \right)^{(c_{51}^{-1})c_{51}/(2c_{2.9.2})}.
\]

Furthermore, applying (5.19) and (5.25), we can easily verify

\[
H_3 \leq c_{54}\Psi(t, x)\Psi(t, y) \int_{\{u \in D: |u| \geq 2|x|\}} L_2(y, u, t)L_1(x, u, t) \, du
\]

\[
\leq c_{55}\Psi(t, x)\Psi(t, y) \int_{\{u \in D: |u| \geq 2|x|\}} \frac{t}{(1 + |x|)^{d+\alpha}} \int_{\{u \in D: |u|/2 \leq |u| \leq 2|x|\}} t e^{-c_{56}f(x_1)^{-\alpha}t} \, du
\]

\[
\leq c_{57}\phi(x)\phi(y)e^{-c_{56}t} \int_{\{u \in D: |u| \geq 2|x|\}} \frac{t}{(1 + |x|)^{d+\alpha}} \int_{\{u \in D: |u|/2 \leq |u| \leq 2|x|\}} \frac{t}{(1 + |u|)^{d+\alpha}} \, du
\]

\[
\leq c_{58}\phi(x)\phi(y)e^{-c_{56}t} \int_{\{u \in D: |u| \geq 2|x|\}} \frac{t}{(1 + |x|)^{d+\alpha}} \int_{\{u \in D: |u|/2 \leq |u| \leq 2|x|\}} \frac{t}{(1 + |u|)^{d+\alpha}} \, du
\]

where in the last inequality we used the facts that \( t^2 e^{-c_{56}f(x_1)^{-\alpha}t} \leq c_{56}e^{-c_{56}t} \). Therefore, according to all estimates for \( H_1, H_2 \) and \( H_3 \), we can obtain the desired conclusion in this case.

(iii) Case 3: \( |y| \leq s_2(t) \) and \( s_2(t)/2 \leq |x| \leq 4s_3(t) \). According to (5.20), (5.21) and (5.25), we have

\[
p_D(2Mt, x, y) = \int_D p_D(Mt, x, u)p_D(Mt, u, y) \, du
\]

\[
\leq c_{61}\Psi(t, y)e^{-c_{2.9.2t}/2} \Psi(t, x) \frac{t}{(1 + |y|)^{d+\alpha}} \int_D L_2(x, u, t) \, du \leq c_{62}\phi(x)\phi(y)e^{-c_{56}t},
\]

where in the last inequality we used the fact that \( \int_D L_2(x, u, t) \, du \leq c_{61}\Psi(t, x)(1 + |x|)^{-d-\alpha}e^{-c_{56}t} \) (that has been verified in the proof of cases (i) and (ii) above).
(iv) Case 4: $|y| \leq s_2(t)$ and $|x| \geq 4s_3(t)$. Define $H_1, H_2$ and $H_3$ as those in case (ii). According to (5.19), we arrive at

$$H_1 \leq c_{66} \Psi(t, x) \int_{\{u \in D: |u| \leq |x|/2\}} L_1(x, u, t) p_D(Mt, u, y) \, du \leq c_{67} \frac{\Psi(t, x)}{(1 + |x|)^{d+\alpha}} t e^{-c_2.9.2^{t}} P^y(\tau_D > t)$$

$$\leq c_{68} \phi(x) \Psi(t, y) t e^{-c_2.9.2^{t}} \leq c_{69} \phi(x) \phi(y) t e^{-c_2.9.2^{t}} \leq c_{70} \phi(x) \phi(y) t e^{-c_2.9.2^{t}},$$

where the third inequality is due to (2.10), and in the fourth inequality we have used the fact that given $\frac{d+\alpha}{c_{5.14.1}} \leq c_{2.9.2}$ (by choosing $C_{5.14.1}$ large enough if necessary), it holds

$$\frac{1}{(1 + |y|)^{d+\alpha}} \geq c_7 s_2(t)^{-d-\alpha} \geq c_7 e^{-c_{5.14.1}^{\alpha}} \geq c_7 e^{-c_2.9.2^{t}} \quad \text{for every } |y| \leq s_2(t).$$

(5.27)

Following the arguments in case (ii) and using (5.20), (5.27) instead of (5.25), we also can obtain the desired estimates for $H_2$ and $H_3$.

(v) Case 5: $|y| \leq s_2(t)$ and $|x| \leq s_2(t)$. According to (5.21) and (2.10), we arrive at

$$p_D(2Mt, x, y) = \int_D p_D(Mt, u, y) p_D(Mt, u, y) \, du$$

$$\leq c_{72} \Psi(t, x) t e^{-c_2.9.2^{t}} (1 + |x|)^{d+\alpha} P^y(\tau_D > t) \leq c_{73} \phi(x) \Psi(t, y) t e^{-c_2.9.2^{t}} \leq c_{74} \phi(x) \phi(y) t e^{-c_7 t},$$

where the last step follows from (5.27).

Therefore, by all the conclusions above and the definition of $\Psi(t, x)$, we complete the proof. \(\square\)

Putting Lemmas 5.4 and 5.5 together, we obtain

**Proposition 5.6.** Suppose that $g$ is non-decreasing on $(0, \infty)$. Then, there exists a constant $c_{5.6.1} > 0$ large enough such that for all $x, y \in D$ and $t \geq c_{5.6.1} t_0(y) \geq 1$,

$$c_{5.6.2} \phi(x) \phi(y) t e^{-c_{5.6.3} t} \leq p_D(t, x, y) \leq c_{5.6.4} \phi(x) \phi(y) t e^{-c_{5.6.5} t},$$

where $c_{5.6.i} (i = 2, \ldots, 5)$ are independent of $t, x$ and $y$.

6. **Further Remarks for Theorem 1.3**

Theorem 1.3 immediately follows from Proposition 3.6, Proposition 4.3, Proposition 5.3 and Proposition 5.6 in the previous three sections.

Below, we present one more example to further illustrate Theorem 1.3.

**Example 6.1.** Let $f(s) = (1 + s)^{-\theta}$ with $\theta > 0$ for all $s \in [0, \infty)$. For any $x, y \in D$, set $t_1(x, y) = (1 + (|x| \wedge |y|))^{-\theta_1}$ and $t_2(x, y) = (1 + (|x| \wedge |y|))^{-\theta_1} \log(2 + (|x| \wedge |y|))$. Then there exist positive constants $c_{6.1.1}$, $c_{6.1.2}$ and $c_{6.1.3}$ such that for all $x, y \in D$,

$$p_D(t, x, y) \approx \begin{cases} p(t, x, y) \left( \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) \left( \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \wedge 1 \right) & \text{for all } 0 < t \leq c_{6.1.1} t_1(x, y); \\
\frac{\delta_D(x)^{\alpha/2}}{t} (1 + |x|)^{-\theta_1} \frac{\delta_D(y)^{\alpha/2}}{t} (1 + |y|)^{-\theta_1} & \exp(-t (1 + (|x| \wedge |y|))^{\theta_1}) \text{ for all } c_{6.1.1} t_1(x, y) < t \leq c_{6.1.2} t_2(x, y); \\
\frac{\delta_D(x)^{\alpha/2}}{(1 + |x|)^{d+\alpha}} (1 + |y|)^{-\theta_1} & \frac{\delta_D(y)^{\alpha/2}}{(1 + |y|)^{d+\alpha}} F(t), \quad \text{for all } c_{6.1.2} t_2(x, y) < t \leq c_{6.1.3}; \\
\frac{\delta_D(x)^{\alpha/2}}{(1 + |x|)^{d+\alpha}} (1 + |y|)^{-\theta_1} & \exp(-t), \quad \text{for all } t > 1,
\end{cases}$$

where $F(t) = (1 \vee t^{1+\eta(1-\frac{d}{\alpha})}) I_{\{\theta \neq \frac{d}{\alpha}\}} + \log(1 + t^{-1}) I_{\{\theta = \frac{d}{\alpha}\}}$. 


Proof. For the reference $f$ given in the example, the associated Dirichlet semigroup $(P_t^D)_{t \geq 0}$ is intrinsically ultracontractive. We note that for $s_0(t)$ and $s_1(t)$ defined in the proof of Example 1.6, $s_0(t) \simeq t^{-1/(\theta_0)}$ and $s_1(t) \simeq t^{-1/(\theta_0)} \log^{1/(\theta_0)}(1 + t^{-1})$. Hence,

\[ \int_0^{c_1 s_0(t)} f(s)^{d-1} \, ds \simeq F(t) \quad \text{and} \quad \int_0^{c_2 s_1(t)} f(s)^{d-1} e^{-c_3 f(s)^{-\alpha}} \, ds \leq c_4 F(t). \]

Then, the assertion follows from Theorem 1.3. \qed

Remark 6.2. In the proof of Theorem 1.3(2)(i), the condition $f(s) \geq c(1+s)^{-p}$ is only required to derive upper bounds of $p_D(t, x, y)$ when $c_2(t_0(x) \vee t_0(y)) \leq t \leq c_3$ involved in the estimate (1.13). Indeed, by carefully tracking the proofs in Section 5.1, without the conditions $f(s) \geq c(1+s)^{-p}$ and $\lim_{s \to \infty} f(s)^{\alpha} \log(2 + s) = 0$, one can still obtain two sided bounds for $p_D(t, x, y)$ in this special time-space region with the assumption $\lim_{y \in D, |y| \to \infty} t_0(y) = 0$. For example, if $f(s) = \exp(-c_0(1+s)^{\kappa})$ for some $c_0 > 0$ and $\kappa > 0$, then for any $x, y \in D$ and $c_2(t_0(x) \vee t_0(y)) \leq t \leq c_3$,

\[ p_D(t, x, y) \asymp \phi(x) \phi(y) \int_0^{s_1(t)} f(s)^{d-1} e^{-tf(s)^{-\alpha}} \, ds \asymp \delta_D(x)^{\alpha/2} f(x_1)^{\alpha/2} \delta_D(y)^{\alpha/2} f(y_1)^{\alpha/2}. \]

In particular, the term $(1 + |x|)^{-d-\alpha}(1 + |y|)^{-d-\alpha}$ arising from $\phi(x) \phi(y)$ in (1.13) and respecting the spatial decay disappears in this case, since it is absorbed into the boundary decay term $f(x_1)^{\alpha/2} f(y_1)^{\alpha/2}$.

Acknowledgements. We thank the anonymous referee for his/her helpful comments. The research of Xin Chen is supported by the National Natural Science Foundation of China (Nos. 11501361 and 11871338). The research of Panki Kim is supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (No. 2016R1E1A1A01941893). The research of Jian Wang is supported by the National Natural Science Foundation of China (Nos. 11831014 and 12071076), the Program for Probability and Statistics: Theory and Application (No. IRTL1704) and the Program for Innovative Research Team in Science and Technology in Fujian Province University (IRSTTFJ).

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