Physical Essence of Propagable Fractional-Strength Optical Vortices in Free Space

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Fractional-order vector vortex beams (VVBs) have recently been demonstrated as new carriers of fractional-strength optical vortices. However, the inherent mechanism of the new vortex beams, formed by the combination of both unstable states, to propagate stably in free space requires clarification. In this study, this scientific problem is solved by revealing the physical essence of propagable fractional-strength optical vortices in free space. Therefore, three new viewpoints regarding these unique vortex beams are proposed: Abbe’s diffraction limit, the phase evolution of the vortex beam, and the binary time vector property of phase. First, due to Abbe’s diffraction limit, the inherent polarization modes are intertwined, thereby maintaining the entire unique vortex beam in free space. In the second case, the phase evolution of the vortex beam is the physical cause for the polarization rotation of fractional-order VVBs. Third, the phase is not merely a scalar attribute of the light beam and demonstrates a binary time vector property. This study provides entirely different physical viewpoints on the phase of the vortex beam and Abbe’s diffraction limit, which may deepen the knowledge on the behavior of light beams in classical optics.

1. Introduction

A vortex beam is a light beam carrying an optical vortex \( \exp(i\varphi l) \),[1] where \( \varphi \) and \( l \) denote the vortex phase and topological charge, respectively. Generally, conventional vortex beams, e.g., the Bessel–Gaussian beam and Laguerre–Gaussian beams, are special solutions of the wave function that always possess an integer topological charge \( l \).[1–2] Therefore, optical vortices carried by these light beams can induce an orbital angular momentum (AM) equivalent to \( \hbar \) per photon and maintain stability in free space during propagation.[3] It is important that an optical vortex can propagate in free space, mainly because the information adhered to the propagable vortex beams can be transferred without distortion in free space.[4] For more than 30 years, significant success has been achieved in the study of optical vortices, including spin and orbital AM conversion,[4,5] rotational Doppler effects,[6,7] advanced lasers,[8] optical imaging,[9] and orbital AM optical communications.[1,10,11] However, most studies on orbital AM were conducted within the framework of vortex beams with integer-orbital AM.

One may wonder whether there is an alternative type of propagable optical vortex in optics that can not only possess a noninteger orbital AM but also stably propagate in free space. Recently, we solved this problem by demonstrating a propagable vortex beam with a topological charge of \( l + 0.5 \).[12,13] It should be noted that this type of vortex beam is a solution to the wave function. That is, optical vortices with fractional topological charges naturally exist in free space. Generally, conventional vortex beams with integer-orbital AM possess a homogeneous polarization state, e.g., linear, circular, or elliptic

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polarization.\[1,2\] Unlike conventional vortices, fractional-strength optical vortices have different carriers, namely, vector vortex beams (VVBs) with an order of \(m + 0.5\).\[13\] Although a mathematical derivation for the generation of propagable fractional-strength optical vortices was provided in our previous study, the physical essence of these unique vector beams is unclear.\[12\] It is unclear as to why the combination of both unstable states, namely, the \(m + 0.5\) order VVB and vortex phase with topological charge of \(l + 0.5\), can form a stable vortex beam with fractional-order and fractional topological charges. Furthermore, it is unclear as to how the polarization and vortex phases of the light beam remain stable in free space; however, the discontinuity of the light beam caused by the phase and polarization jumps is missing. In addition to the abovementioned aspects, it is unclear as to why the modulation symmetry of the light beam is broken. However, the inherent polarization modes within the fractional-order VVB are intertwined.\[14\]

In this study, we revealed the physical essence of a propagable vortex beam with a topological charge of \(l + 0.5\) by reunderstanding Abbe’s diffraction limit, investigating the phase evolution of the vortex beam, and proposing a binary time vector property of phase. First, although the modulation symmetry of the fractional-order VVB is naturally broken down by the optical vortices with fractional topological charges, the inherent polarization modes are intertwined due to Abbe’s diffraction limit, thereby leading to a stable state of fractional-order VVBs in free space. Second, we demonstrated the phase evolution of the vortex beam, which is the physical cause of the polarization rotation of fractional-order VVBs. Third, the phase is not merely a scalar attribute of the light beam and demonstrates a binary time vector property. Therefore, the entire electric field of the fractional-order VVBs is continuous, whereas the topological charge of the vortex phase and order of polarization are fractions. This paper provides entirely different physical viewpoints on a propagable vortex beam with a topological charge of \(l + 0.5\), which clarifies the behavior of light beams in classical optics.

2. Results

As two stable states in classical optics, a vortex beam with an integer topological charge and an integer-order VVB represent two extremes of a light beam. In the former, a vortex beam with an integer topological charge is generally homogeneously polarized and can be considered as a scalar light beam.\[1,2\] Therefore, these light beams can stably carry an optical vortex with an integer topological charge in free space. For the latter, an integer-order VVB can be obtained by combining left and right circularly polarized beams with inverse optical vortices.\[13\] In particular, the inherent modulation symmetry of a light beam can be broken down by imposing an additional optical vortex.\[14\] Therefore, an integer-order VVB can only maintain stability without an optical vortex in free space. Unlike both light beams, a propagable vortex beam with a topological charge of \(l + 0.5\) can be considered as the middle state. The stability of these unique vortex beams is dependent on the interaction between the vortex phase and the polarization of the fractional-order VVB.\[12\] In the absence of both, the fractional-order VVB and vortex beam with a noninteger topological charge cannot propagate individually in free space. However, it is unclear as to why does the middle state of a light beam exists naturally in free space.

2.1. Intertwinement of Inherent Polarization Modes

Mathematically, a propagable vortex beam with a natural noninteger topological charge can be expressed as follows\[12\]

\[
E = \exp[i(l + 0.5)\varphi] \begin{bmatrix} \cos[(m + 0.5)\varphi + \beta] \\ \sin[(m + 0.5)\varphi + \beta] \end{bmatrix}
\]  

(1)

where \(\varphi\) is the azimuthal angle; and \(l, m\) are two integers that relate to the topological charge and order of entire vortex beam, respectively; \(\beta\) determines the polarization direction of a \(m + 0.5\)-order VVB. Without loss of generality, \(\beta = 0\) in this study. Generally, the above vortex beam can be simplified as follows

\[
E = \exp[i(m + l + 1)\varphi]|R\rangle + \exp[-i(m - l)\varphi]|L\rangle
\]  

(2)

Here, \(|L\rangle=[1\ i^]\) and \(|R\rangle=[1\ -i^]\) denote the left and right circularly polarized modes, respectively; and \(t\) denotes the matrix transposition operator.

As demonstrated in our previous study, Equation (2) represents stable propagable vortex beams with fractional topological charges.\[12\] Therefore, the inherent left- and right-circular polarization modes in Equation (2) are intertwined, even when the light beams propagate over an infinite distance in free space. However, the topological charge of \(|L\rangle\) differed from that of \(|R\rangle\) (refer to Equation (2)). Different topological charges lead to different positions for \(|R\rangle\) and \(|L\rangle\), and the maximum distance between the two polarized modes can be obtained at infinite distances in free space. In particular, the first scientific problem is how both circular polarization modes in Equation (2) are intertwined, such that the entire vortex beam remains stable in free space.

To verify the intertwinement of the inherent polarization modes in Equation (2), we investigated the influence of the vortex phase on \(|L\rangle\) and \(|R\rangle\) when the entire vortex beam propagated over an infinite distance in free space. According to information optics, the image of a light beam at an infinite distance can be obtained in the focal region of an objective lens (OL) obeying the sine condition shown in Figure 1.\[15\] It should be noted that \(\Omega\) is

![Figure 1. Schematic of vortex beam with different topological charges in a focusing system, where \(\Omega\) is the focal sphere of the objective lens (OL), with its center at \(O\) and radius \(f\); A and B are two arbitrary points in \(\Omega\); \(O_1\) is the point on the vortex beam in the focal plane; \(P\) is the vortex phase of incident light beam; and \(\theta\) is the convergent angle of OL.](https://www.advancedsciencenews.com)
a focal sphere with its center at O and radius \( f \) (the focal length of the OL), where \( O_1 \) is an arbitrary point in the focal region of the lens. As shown in Figure 1, when the topological charge of the vortex phase (\( P \)) is zero, constructive interference occurs at the focal point O due to the equivalent optical paths of the light beams between the points in \( \Omega \) and O. Therefore, a bright focal spot can be obtained at Point O. In contrast, the light intensity of the focus is transformed into a donut when the topological charge is an integer \( l \). It should be noted that the donut in the focal plane indicates the image of a vortex beam over an infinite distance in free space.

Due to the phase symmetry of the vortex beam, the maximum light intensity on the donut is located at Point \( O_1 \). In contrast, the light intensity of the focal point \( O \) due to the equivalent optical paths of the light beams is expressed as follows:

\[
\begin{align*}
L_1 &= \sqrt{(\sin \theta \cos \varphi - \rho \cos \varphi_s)^2 + (\sin \theta \sin \varphi - \rho \sin \varphi_s)^2 + (\cos \theta)^2} \\
L_2 &= \sqrt{(-\sin \theta \cos \varphi - \rho \cos \varphi_s)^2 + (-\sin \theta \sin \varphi - \rho \sin \varphi_s)^2 + (\cos \theta)^2}
\end{align*}
\]

The OPD \( L_2 - L_1 \) can be calculated as follows:

\[
\Delta s = \frac{4 \rho \sin \theta \cos(\varphi - \varphi_s)}{\sqrt{1 + \eta_{\rho}^2 + 2 \eta_{\rho} \sin \theta \cos(\varphi - \varphi_s)} + \sqrt{1 + \eta_{\rho}^2 - 2 \eta_{\rho} \sin \theta \cos(\varphi - \varphi_s)}}
\]

where \( \theta \) is the convergent angle and \( \eta_{\rho} = \rho / f \). Given that \( O_1 \) is a point on the vortex beam near the focus, \( \rho \ll f \). In particular, \( \eta_{\rho} \approx 0 \). Finally, Equation (5) can be simplified as follows:

\[
\Delta s = 2 \rho \sin \theta \cos(\varphi - \varphi_s)
\]

As shown in Figure 1, A and O1 are the two corresponding field points in the wavefront and focal planes of the OL, respectively. Therefore, \( \varphi = \varphi_s \) and Equation (6) can be simplified as follows:

\[
\Delta s = 2 \rho \sin \theta
\]

According to the principle of interferometry, the above OPD for the vortex beams are modulated by the vortex phase \( \exp(i l \varphi) \), which can be expressed as follows:

\[
\Delta s = \Delta \varphi / 2 = M \lambda
\]

where \( M \) is the topological charge of the vortex beam, and \( \lambda \) indicates the order of constructive and deconstructive interference on the vortex beam. Furthermore, Equation (8) can be simplified as follows:

\[
\Delta s = |M + l/2| \lambda
\]

It should be noted that the high-order constructive interference of the vortex beam always occurs at the outer ring. For this reason, for \( l > 0 \) and \( M > 0 \) and for \( l < 0 \) and \( M < 0 \).

From Equation (7) and (9), the radius \( \rho_M \) of the maximum light intensities of the vortex beam in the focal plane of OL can be obtained, namely, OO1 in Figure 1, which can be expressed as follows:

\[
\rho_M = \frac{|M + l/2| \lambda}{2 \sin \theta}
\]

For convenience, \( \rho_{M1} \) and \( \rho_{M0.51} \) in Equation (10) denote the radii of the bright rings (constructive interference) and dark rings (deconstructive interference) of the vortex beam with topological charge \( l \), respectively. Here, \( M = 0, \pm 1, \pm 2 \ldots \).

Equation (10) implies that a vortex beam with topological charge \( l \) is composed of a series of bright and dark rings and demonstrates that vortex beams with topological charges of \( \pm l \) possess an identical radius \( \rho_{M, \pm l} \). Therefore, the left and right circular polarization modes with inverse optical vortices are always superimposed, thereby generating a stable integer-order VVB in free space\(^{[16]} \). The order is determined by the topological charges of both circularly polarized beams. For a vortex beam with fractional topological charge, as expressed in Equation (2), the inherent left and right circular polarization modes possess different topological charges, thus leading to different trajectories of the bright rings in the OL focal plane. Without loss of generality, the brightest inner ring is considered below as an example, to explain how both circular polarization modes are intertwined to form a stable vortex beam, as expressed by Equation (2).

Figure 2 presents the brightest inner ring with radius \( \rho \) of the vortex beam with \( M = 0 \), as expressed by Equation (10). According to Equation (10), \( \rho_{M1} = |M + l/2| \lambda / (2 \sin \theta) \) denotes the radius of the bright rings, and \( \rho_{M0.51} = |M + 0.5 + l/2| \lambda / (2 \sin \theta) \) represents the radius of the dark rings. Finally, the thickness of each bright ring of the vortex beam can be obtained as follows.
D = 2(\rho_{M_1} - \rho_{M_2}) = \frac{\lambda}{2\sin \theta} \quad (11)

At \( l = 0 \), a bright spot was observed in the focal plane. In particular, \( \rho_{M_0} = 0 \) and \( \rho_{M_0} = \frac{\lambda}{4\sin \theta} \). Finally, Abbe’s diffraction limit in free space can be derived by calculating the size of the focus in Figure 2a using Equation (11), namely \( D = 2\rho_{M_0} = \frac{\lambda}{2\sin \theta} \).

Under the condition of \( l \neq 0 \), the vortex beam was transformed into a donut in the focal plane. As can be seen from Equation (11), all the bright rings of the vortex beam exhibited different thickesses, which is consistent with Abbe’s diffraction limit in free space, as shown in Figure 2b. According to Equation (10), the distance between the brightest rings generated with topological charge differences \( l_1 \) and \( l_2 \) can be expressed as follows

\[
F = (\rho_{M_1} - \rho_{M_2}) = \frac{|l_1| - |l_2|)}{4 \sin \theta} \quad (12)
\]

Optical vortices with topological charges \( l_1 \) and \( l_2 \) were carried by left and right circularly polarized beams, respectively. When \( l_1 = -l_2 = m \), both the left and right circular polarization modes were located on the same ring, thereby intertwining to form an integer \( m \)-order VVB in free space. When \( |l_1| - |l_2| = \pm 1 \) in Equation (2), the distance between the left and right circular polarization modes was \( F = \frac{\lambda}{4\sin \theta} \). In this case, the left and right circular polarization modes with a topological charge difference of \( l \) exhibited different trajectories in the focal plane. However, due to Abbe’s diffraction limit, the two polarization modes could not be distinguished, as shown in Figure 2b. Therefore, the modulation symmetry of the light beam in Equation (2) was broken; however, the inherent left- and right-polarization modes with a topological charge difference of \( l \) were still intertwined together. Therefore, Equation (1) represents only two stable vortex beams in free space. The first is a vortex beam that possesses an optical vortex with a topological charge of \( \pm 0.5 \), in addition to a polarization state of an \( m + 0.5 \)-order VVB (see Equation (13)); and the second is a vortex beam that has a constant polarization state, namely, a \( \pm 0.5 \)-order VVB, but an optical vortex with arbitrary topological charge of \( l + 0.5 \) (see Equation (14)).

\[
E_{m+0.5} = \exp(\pm i0.5\varphi) \left[ \frac{\cos((m+0.5)\varphi + \beta)}{\sin((m+0.5)\varphi + \beta)} \right] \quad (13)
\]

\[
E_{l+0.5} = \exp((l+0.5)\varphi) \left[ \frac{\cos(\pm 0.5\varphi + \beta)}{\sin(0.5\varphi + \beta)} \right] \quad (14)
\]

For many years, we focused on overcoming Abbe’s diffraction limit to achieve high resolutions in optical imaging and optical lithography. However, in terms of light-beam generation, Abbe’s diffraction limit is the physical essence of the intertwining of the inherent left and right circular polarization modes in free space. Without Abbe’s diffraction limit, the intertwining of inherent polarization modes cannot occur, and the unique vortex beams in Equation (13) and (14) cannot be generated in free space.

2.2. Phase of Inherent Polarization Modes

As a middle state between conventional vortex beams with integer topological charges and integer-order VVBs, vortex beams with fractional topological charges exhibit unique propagation behaviors in free space. Their polarization states rotate continuously during propagation in free space; however, their order is maintained. The second scientific problem is the mechanism of polarization rotation within these unique vortex beams.

2.2.1. Phase Evolution of Different Vortex Beams

According to Equation (2), the order of the vortex beams in Equation (1) is determined by the topological charges of the left and right circular-polarization modes. Given that both the left and right circular-polarization modes are unique solutions to the wave function, their vortex phases can be stably maintained in free space. Therefore, the order of the light beams in Equation (1) remains at \( m + 0.5 \) due to the unchanged topological charge. In contrast, unlike the order of the vortex beams in Equation (1), the polarization rotation of the light beam is not dependent on the value of the topological charge and instead on the phase difference between the inherent left and right circular polarization modes. Therefore, the key to revealing the polarization rotation of these unique vortex beams in Equation (13) and (14) is to determine the phase evolution of the different vortex beams, as expressed by Equation (2).

Theoretically, the propagable vortex beams in Equation (13) and (14) can be considered as the superposition of two conventional optical vortices with circular polarization and topological charges with a difference of 1. As shown in Figure 1, Points O and O₁ are the geometric focus of OL and points at the bright ring of the vortex beam, which can be expressed as \((0,0,0)\) and \((\rho, \cos \varphi, \rho \sin \varphi, z)\), respectively. Here, \( z = 0 \) denotes the focal plane of the OL. When the topological charge \( l \) is zero, all the light beams at the focal sphere are focused on the geometric focus O (see Figure 1). As there is no vortex phase in the wavefront of the OL, the phase of the light beam is zero in the

![Figure 2](https://www.advancedsciencenews.com)

Figure 2. Schematic of vortex beam with different radii \( \rho \): a) the light beam with a topological charge \( l = 0 \) and b) two adjacent vortex beams with \( l = 2 \) and \( l = -3 \). The combination of both vortex beams forms a 2.5-order VVB with a topological charge of \(-0.5\), as expressed by Equation (2).
It should be noted that $\Delta$ follows difference, as indicated by Equation (18), can be expressed as wavenumber and $n$ charges can be obtained by propagating in free space. For clarity, Equation (17) can be simplified as follows

$$L_3 = \sqrt{(f \sin \theta \cos \varphi - \rho_1 \cos \varphi_3)^2 + (f \sin \theta \sin \varphi - \rho_1 \sin \varphi_3)^2 + (f \cos \theta)^2}$$

Moreover, OPD $(L_0-L_3)$ can therefore be obtained as follows

$$\Delta \delta = L_0 - L_3 = \frac{-\rho_1^2 / f + 2 \rho_1 \sin \theta \cos(\varphi - \varphi_3)}{1 + \sqrt{1 + \rho_1^2 / f^2 - 2 \sin \theta \cos(\varphi - \varphi_3) \rho_1 / f}}$$

(17)

Given that O and O$_1$ are points on the vortex beam near the focus, $\rho_1 \ll f$. In particular, $(\rho_1^2)/f \approx 0$, $\rho_1/f \approx 0$. Finally, Equation (17) can be simplified as follows

$$\Delta \delta = \rho_1 \sin \theta \cos(\varphi - \varphi_3)$$

(18)

According to the relationship between the phase and OPD, the phase difference between vortex beams with different topological charges can be obtained by $\Delta \varphi = k \Delta \delta$, where $k = 2\pi n / \lambda$ is the wavenumber and $n = 1$ is the refractive index in the focusing space, as shown in Figure 1. Using Equation (10), the phase difference, as indicated by Equation (18), can be expressed as follows

$$\Delta \varphi = \pi |M + 1/2|$$

It should be noted that $\varphi = \varphi_3$ as shown in Figure 1.

In the case of $l \geq 0$, $M \geq 0$. In particular

$$\Delta \varphi_{l=0} = \pi (M + 1/2)$$

It should be noted that $M = 0, 1, 2, 3...$. In the case of $l < 0$, $M < 0$. In particular

$$\Delta \varphi_{l=0} = - \pi (M + 1/2)$$

(20)

(21)

It should be noted that $M = 0, -1, -2, -3...$. Please refer to Equation (9).

Without loss of generality, we considered $M = 0$ as an example to investigate the phase difference between different vortex beams. Here, $M = 0$ indicates the brightest ring at the center, as shown in Figure 1. Figure 3 presents the phase of the vortex beam $\Delta \varphi$ with different topological charges at infinite distances. As shown in Figure 3, light beams carrying an optical vortex $\exp(i\varphi)$ attain an additional phase $\Delta \varphi$ at an infinite distance. Therefore, the transmittances of the phases are symmetrical about the light beam with $l = 0$ in Figure 3, which can be expressed as follows

$$T = \exp(i\Delta \varphi) = i^l$$

(22)

Equation (22) denotes the additional phase change at an infinite distance and implies the phase evolution of vortex beam during propagation in free space. For clarity, Figure 4 presents the phase evolution of vortex beam when vortex beam with topological charge $l$ propagates from the wavefront of OL to an infinite distance in free space, namely, the focal plane. In the case of $l \geq 0$, $\Delta \varphi = 0.5\pi l$. Therefore, the optical vortex changes from $\exp(i\varphi)$ at the wavefront of OL to $\exp(i(l\varphi + 0.5\pi l))$ at an infinite distance in free space, and the entire vortex phase of light beam rotates clockwise during propagation in free space. However, in the case of $l < 0$, $\Delta \varphi = +0.5\pi l$, and the optical vortex is transformed into $\exp[i(l\varphi - 0.5\pi l)]$ at an infinite distance in free space. In particular, unlike the optical vortex with positive topological charge, the entire vortex phase with negative topological charge rotates counterclockwise from the wavefront of OL to an infinite distance in free space. In addition, because the optical vortex at an infinite distance can be simplified to
exp[±i(φ ± 0.5π)], the entire vortex phase can only rotate a quarter of a circle at most, as indicated by the purple arrow in Figure 4.

2.2.2. Phase of Vortex Beam with Different M

Based on the principle of interference, a vortex beam with topological charge \( l \) possesses the inner brightest ring, as shown in Figure 5, in addition to other bright rings located at the periphery of the light beam. In general, it is difficult to observe the outer rings due to the weak diffraction effect of the vortex beam, as shown in Figure 5a. However, when the modulation by a high-pass pupil filter or topological charge is sufficiently large, the diffraction effect of the light beam becomes large, and the outer rings cannot be neglected accordingly, as shown in Figures 5b,c. To determine the phase evolution of the entire vortex beam, we investigated the phases for different values of \( M \).

As discussed above, the phase of the geometric focus \( O \) is zero because there is no vortex phase at the wavefront of the OL. Therefore, the phases of different \( M \) values in a vortex beam can be obtained by comparison with that of the zero topological charge. As shown in Figure 5, \( O_1 \) and \( O_2 \) are two points at two arbitrary bright rings of a vortex beam with topological charge \( l \), which can be expressed as \((\rho_{M}, \cos \varphi_{M}, \sin \varphi M, z)\). Here, \( M \) denotes the order of the bright ring and \( z = 0 \) indicates the OL focal plane.

Similar to the derivation in Figure 1, the phases with different \( M \) values can be obtained by simply calculating the OPD between \( A \) and \( O_1 \), \( A \) and \( O_2 \), or \( B \) and \( O_1 \) or \( B \) and \( O_2 \). Specifically, according to Equation (18) and (19), the phases with different \( M \) values can be rewritten as follows

\[
\phi = \pi [M + l/2]
\]

where \( \phi \) is the additional phase of the vortex beam with topological charge \( l \) in the OL focal plane. It should be noted that \( M = 0, -1, -2, -3 \ldots \) for \( l < 0 \), whereas \( M = 0, 1, 2, 3 \ldots \) for \( l > 0 \), as shown in Figure 5b,c.

For a particular \( l \), the phase difference between the \( M \) order bright ring and the brightest inner ring can be expressed as follows

\[
\Delta \phi_l = \pi (|M| - M_0)
\]

where \( M_0 = 0 \) represents the brightest inner ring in Figure 5b,c. As demonstrated in Section 2.2.1, different \( l \) values denote different initial phase changes, with \( M = 0 \). This phase is referred to as \( \phi_0 \), which can be observed in Figure 3. Considering \( l = -1 \) as example, \( \phi_0 = 0.5\pi \), as shown in Figure 3. Figure 6 presents the phase \( \Delta \phi_{l-1} \) of a vortex beam with different \( M \) values at an infinite distance. Although the initial phase is different for different vortex beams, the phase differences between adjacent rings \(|\Delta \phi_l|\) are always equal to \( \pi \). As shown in Figure 6, the phase changes caused by the propagation of the vortex beam do not influence the stability of the entire optical vortex, thereby leading to an unchanged topological charge of the entire light beam for different \( M \) values.

2.2.3. Polarization Rotation of Vortex Beam in Equation (1)

Sections 2.2.1 and 2.2.2 revealed that the phase of the vortex beam changes continuously during propagation in free space. Therefore, the mathematical form of the vortex beam in the wavefront of the OL is slightly different from that of the vortex beam in the focal plane of the OL, i.e., at an infinite distance in free space. Before explaining the polarization rotation of the vortex beam in Equation (1), we investigated the polarization state of its counterpart in integer order for comparison.

In terms of the integer-order VVB, its polarization state in the wavefront of the OL can be considered as a combination of the left and right circular polarization modes with topological charge \( \pm m \), which can be expressed as follows

\[
E_{WF} = \exp(i m \varphi)|R\rangle + \exp(-i m \varphi)|L\rangle
\]

where \(|L\rangle = |1, i\rangle^t \) and \(|R\rangle = |1, i\rangle^t \) denote the left- and right-circularly polarized modes, respectively, and \( t \) denotes the matrix transposition operator.

When an \( m \)-order VVB propagates to an infinite distance, an additional phase change occurs at the inherent polarization modes \(|L\rangle\) and \(|R\rangle\). According to Equation (22), the polarization state at an infinite distance is transformed to the following

\[
E_t = i^m \exp(i m \varphi)|R\rangle + i^{-m} \exp(-i m \varphi)|L\rangle
\]
By comparison with the light beam in Equation (25), the phase-change \( i^m = l - m \) is independent of the polarization state of the light beam in Equation (26). Therefore, polarization rotation cannot be realized, and an \( m \)-order VVB always maintains its stability during propagation in free space.

A propagable vortex beam with a fractional topological charge has two stable states in free space, as expressed by Equation (13) and (14), respectively. Both propagable light beams can be divided into a combination of left and right circularly polarized modes. Specifically, the VVB in the order of \( m + 0.5 \) can be simplified to

\[
E_{m+0.5,+} = i^{m+1} \exp[i(m+1)\varphi] \left[ \frac{1}{-i} \right] + i^{-m} \exp(-im\varphi) \left[ \frac{1}{i} \right]
\]  

(27)

\[
E_{m+0.5,-} = i^m \exp(im\varphi) \left[ \frac{1}{-i} \right] + \exp[-i(m+1)\varphi] \left[ \frac{1}{i} \right]
\]  

(28)

where \( E_{m+0.5,+} \) and \( E_{m+0.5,-} \) indicate \( m + 0.5 \)-order VVBs with topological charges of \(+0.5\) and \(-0.5\), respectively. After propagating to an infinite distance, the \( m + 0.5 \)-order VVBs in Equation (27) and (28) are transformed into the following [see Equation (22)]

\[
E_{m+0.5,+} = i^{m+1} \exp[i(m+1)\varphi] \left[ \frac{1}{-i} \right] + i^{-m} \exp(-im\varphi) \left[ \frac{1}{i} \right]
\]  

(29)

\[
E_{m+0.5,-} = i^m \exp(im\varphi) \left[ \frac{1}{-i} \right] + \exp[-i(m+1)\varphi] \left[ \frac{1}{i} \right]
\]  

(30)

Supposing \( m > 0 \), the two equations above can finally be simplified to

\[
E_{m+0.5,+} = i^m \exp(i0.25\pi) \exp(i0.5\varphi) \frac{\cos[(m+0.5)\varphi + 0.25\pi]}{\sin[(m+0.5)\varphi + 0.25\pi]}
\]  

(31)

\[
E_{m+0.5,-} = i^m \exp(i0.25\pi) \exp(-i0.5\varphi) \frac{\cos[(m+0.5)\varphi - 0.25\pi]}{\sin[(m+0.5)\varphi - 0.25\pi]}
\]  

(32)

A vortex beam with a topological charge of \( l + 0.5 \) is an additional stable state of the light beam in Equation (14). Similarly, it can be expressed as follows

\[
E_{l+0.5,+} = \exp[i(l+1)\varphi] \left[ \frac{1}{-i} \right] + \exp(il\varphi) \left[ \frac{1}{i} \right]
\]  

(33)

\[
E_{l+0.5,-} = \exp(il\varphi) \left[ \frac{1}{-i} \right] + \exp[(l+1)\varphi] \left[ \frac{1}{i} \right]
\]  

(34)

In particular, when both vortex beams propagate to an infinite distance, their polarization states are transformed to the following

\[
E_{l+0.5,+} = i^{l+1} \exp[i(l+1)\varphi] \left[ \frac{1}{-i} \right] + i^l \exp(il\varphi) \left[ \frac{1}{i} \right]
\]  

(35)

\[
E_{l+0.5,-} = i^l \exp(il\varphi) \left[ \frac{1}{-i} \right] + i^{l+1} \exp[(l+1)\varphi] \left[ \frac{1}{i} \right]
\]  

(36)

Supposing \( l > 0 \), the two equations above can finally be simplified to

\[
E_{l+0.5,+} = i^l \exp[i(l+0.5)\varphi] \exp(i0.25\pi) \frac{\cos(0.5\varphi + 0.25\pi)}{\sin(0.5\varphi + 0.25\pi)}
\]  

(37)

\[
E_{l+0.5,-} = i^l \exp[i(l+0.5)\varphi] \exp(-i0.25\pi) \frac{\cos(-0.5\varphi - 0.25\pi)}{\sin(-0.5\varphi - 0.25\pi)}
\]  

(38)

According to Equation (31), (32), (37), and (38), a schematic of the polarization rotation is presented in Figure 7. Due to the phase change caused by the propagation of the vortex beam, the polarization states of \( m + 0.5 \)-order VVBs are rotating continuously from \( \beta = 0 \) in the wavefront of the OL to \( \beta = \pm 0.25\pi \) at the focal plane, i.e., at an infinite distance. The direction of the polarization rotation is determined by the sign of the VVB order \( m + 0.5 \) and \( \beta = \pm 0.25\pi \). Specifically, \( m+\beta > 0 \) implies a clockwise rotation of the polarization, whereas \( m+\beta < 0 \) denotes a counterclockwise rotation of the polarization, as shown in Figure 7. In addition to the polarization rotation of the vortex beam, these unique light beams possess an additional phase change similar to that of conventional vortex beams, which changes continuously from 0 in the wavefront of the OL to \( i^m \exp(i0.25\pi) \) and \( i^l \exp(i0.25\pi) \) in Equation (31), (32), (37), and (38) in the focal plane. However, the order and topological charge of the light beam was maintained, thus rendering these unique vortex beams propagable in free space.

### 2.3. Electric Field Continuity of Fractional-Strength Optical Vortices

Generally, an \( m + 0.5 \)-order VVB and vortex phase with a topological charge of \( l + 0.5 \) cannot remain stable individually in free space. However, when both the unstable states overlap, two propagable vortex beams can be obtained using Equation (13) and (14), which possess a stable polarization of \( m + 0.5 \)-order VVB and a stable vortex phase with a topological charge of \( l + 0.5 \). Why can the interaction between the two unstable states maintain the entire propagable vortex beams in Equation (13), (14) stably in free space? Why is the discontinuity of light beam caused by
the phase and polarization jump missing? As detailed in this section, we solved the third scientific problem by determining the relationship between phase and polarization.

2.3.1. Binary Time Vector Property of Phase

In theory, a light beam propagating along the +z axis can be expressed in a generalized form as follows

\[ E = A \cos(\phi + \alpha t)E_0 \]  

(39)

where \( A \) is the amplitude of the light beam; \( \phi = kwz \) is the phase of the entire light beam; \( k = 2\pi/\lambda \) is the wavenumber; \( n = 1 \) is the refractive index in free space; \( \omega = 2\pi/T \), where \( T \) is the period of the electric field of the light beam; and \( E_0 \) denotes the polarization state of the light beam. For example, a conventional linearly polarized beam can be obtained using \( \phi = l\varphi \) and \( E_0 = [1 \ 0]^T \). Here, \( E_0 = [1 \ 0]^T \) indicates an x-linearly polarized beam.

As shown in Figure 8, Equation (39) represents the oscillation of the electric field over time \( t \) for a light beam. The adjustment of the phase \( \phi \) did not influence the waveform of oscillation and instead the starting point of electric field at \( t \) (see Points A, B, C, D in Figure 8). In particular, the starting point of electric field at \( t_2 \) (Point C) can be obtained by controlling the phase \( \phi = kwz \) with \( z = 0.25t \) at \( t_0 \) (Point F). Hence, the phase of light beam actually represents the starting point of electric field oscillation of the light beam at a particular \( t \). From this viewpoint, it is not appropriate to consider the phase by ignoring the entire electric field in Equation (39). In particular, \( \exp(i\phi)E_0 \) should be comprehensively considered for a light beam instead of merely the phase \( \exp(i\phi) \) alone.

In terms of a vortex beam with a topological charge of \( l \), the entire light beam can be considered as a combination of many different light beams modulated by the vortex phase \( \phi = l\varphi \), as indicated by the yellow arrows A, B, C, and D in Figure 9a. Considering \( l = 1 \) as an example, due to the spatial variation of the phase along the radius \( R \) in Figure 9b, different starting points of the electric field oscillation can be simultaneously obtained in the transverse plane of vortex beam in Figure 9c,d. It should be noted that the period in Figure 9c,d is determined by the topological charge \( l \). For clarity, the spatial domain of the electric field oscillation in Figure 9d can be converted to the time domain. Here, the x-coordinate \( t \) in Figure 9e,f corresponds to a different starting point of the electric field oscillation, as shown in Figure 8. According to Equation (39), the phase represents the electric field oscillation of the light beam at a particular \( t \). For an entire oscillation period \( T \), the electric field of the light beam can be divided into two areas along the radius \( R \) in Figure 9f, where the directions of the electric fields are upward indicated by the blue arrow in Area A and downward indicated by the yellow arrow in Area B, respectively. Both inverse electric fields demonstrate that the oscillation of the electric field caused by the change in the vortex phase leads to the reversal of the polarization. It should be noted that the vector property induced by the phase change possesses a binary direction, as shown in Figure 9f. This property of a light beam is referred to as the binary time vector property of phase.

Figure 9. Schematic of the binary time vector property of phase. Subfigures (a,b,g) are the vortex phases \( \phi = l\varphi \). Subfigure (c) shows the electric field along the radius \( R \) of vortex phase, which is transformed into the electric fields along the phase \( \varphi \) and time \( t \) in (d,e), respectively. The electric field of the light beam in (e) is divided into Area A and B indicated in (f), where the directions of the electric fields are upward and downward denoted by the blue and yellow arrow, respectively. Here, Arrows A, B, C, D denote different light beams in the transverse plane of vortex phase in (a). The pink arrow \( R \) (b) and green arrow \( t \) (g) represent the azimuth direction of vortex beam in spatial and time domain, respectively.
2.3.2. Coexistence of the Phase and Polarization Discontinuity

According to Equation (39), \( \mathbf{E}_0 \) denotes the polarization state of the light beam. For example, \( \mathbf{E}_0 = [1 \ 0]^T \) denotes the linear polarization state of a light beam, whereas \( \mathbf{E}_0 = [\cos m \phi \quad \sin m \phi]^T \) indicates the polarization state of an \( m \)-order VVB. As shown in Figure 10, the phase represents the electric field oscillation of the light beam indicated by the red arrow, whereas its oscillation trajectory is determined by the unit vector \( \mathbf{E}_0 \), i.e., the direction of the electric field indicated by the blue arrow. In principle, the electric field oscillation is time variant and belongs to the time domain. Therefore, the electric field of the light beam changed periodically with time \( t \), as shown in Figure 9f. In contrast, the polarization state is spatially variant and belongs to the spatial domain.

For a stable polarized beam, the electric field oscillation in the transverse plane is synchronous, regardless of the spatial distribution of the polarization state. When modulating the phase of an \( m \)-order VVB, the polarization of the original \( m \)-order VVB cannot be maintained because the phase influences the entire electric field in Equation (39). However, in terms of propagable vortex beams in Equation (13) and (14), the stability of the vortex beam is dependent on the interaction between the vortex phase with a topological charge of \( l + 0.5 \) and the polarization with an \( m + 0.5 \)-order VVB. As shown in Figure 11a,b, there were polarization discontinuities and phase jumps along the \( x \)-axis. For the polarization of \( m + 0.5 \)-order VVB, as shown in Figure 11a, the electric field oscillation along the \( x \)-axis demonstrated two inverse oscillation processes. The first is indicated by the red curve and the second by the blue curve, as shown in Figure 11d. Given that both processes are synchronous [blue and yellow arrows in Figure 11d], a polarization discontinuity is always obtained along the \( x \)-axis, and the light intensity along the \( x \)-axis is zero, as shown in Figure 11g. For the vortex phase with a topological charge of \( l + 0.5 \), as shown in Figure 11b, due to the binary time vector property of phase, the oscillation process along the \( x \)-axis is formed by the combination of two inverse electric fields. For the electric field oscillation indicated by the green arrow in Figure 11b, the starting point is located in Area A in Figure 11e. In particular, during \( t = 0 \), the direction of the electric field is upward, as indicated by the green arrow. Similarly, the starting point of the inverse counterpart, as indicated by the red arrow, was equal to that of the green arrow at time \( t = 0 \), with \( \phi = \pi \) in Equation (39). Therefore, the direction of the electric field is downward. Due to the inverse electric field caused by the binary time vector property of phase, a dark line along the \( x \)-axis was created in the transverse plane of the light beam, as shown in Figure 11h.

The polarization of the \( m + 0.5 \) order VVB in Figure 11a and the vortex phase with a topological charge of \( l + 0.5 \) in Figure 11b can induce a polarization discontinuity and phase jump along the \( x \)-axis, respectively. However, when overlapping, the upward starting points of the electric fields in Figure 11d (blue arrow) and in Figure 11e (green arrow) were maintained. In contrast, their downward counterparts (yellow arrow in Figure 11d and red arrow in Figure 11e) interacted. Therefore, the combined starting point was reversed, and an identical electric field oscillation of the entire stable vortex beam could be obtained, as shown in Figure 11f. Therefore, the dark line along the \( x \)-axis caused by both discontinuities disappeared in Figure 11i, and

![Figure 10](image1.png)

**Figure 10.** Schematic of polarization and phase function of light beam.

![Figure 11](image2.png)

**Figure 11.** Coexistence of phase jump and polarization discontinuity. Subfigures (a) and (b) denote the polarization of the \( m + 0.5 \) order VVB and the vortex phase with a topological charge of \( l + 0.5 \), respectively. Due to the inverse electric fields along the \( x \)-axis in (d) and (e), the polarization discontinuity and phase jump are obtained in (g) and (h), respectively. When overlapping, the entire electric field along the \( x \)-axis changes continuously, see (c,f,i). The blue, yellow, green, and red arrows in (a,b,d,e) indicate the direction of electric fields.
the entire electric field changed continuously in the transverse plane of the propagable vortex beam in Equation (1). Moreover, the polarization and phase were in the spatial and temporal domains, respectively. Thus, the electric field of the propagating vortex beam in Equation (1) was continuous, whereas the phase and polarization discontinuities in Figure 11a,b could coexist within these unique light beams, as shown in Figure 11c.

3. Discussion

This study reveals the physical essence of propagable vortex beams with a topological charge of \( l + 0.5 \), in addition to generalized rules regarding the propagative behavior of optical vortices. A discussion is presented below on several optical phenomena in classical optics based on the results above. Considering photonic skyrmions in a confined electromagnetic field as example, photonic skyrmions are formed by the transverse and longitudinal components of the electric field in the focal region of OL when focusing a radially polarized beam with different topological charges \( l \). Without loss of generality, we ignored the SPP excitation process. Given that the transverse and longitudinal components are both from a radially polarized beam, their topological charges possess a difference of \( l \). As shown in Figure 2, due to Abbe’s diffraction limit, both polarization components cannot be distinguished, thereby intertwining together in the focal region of OL. Besides, according to Equation (19), the phase difference between both components is always \( 0.5\pi \). Therefore, transverse spin can be obtained by the natural combination of the transverse and longitudinal components. It should be noted that the phase differences between adjacent rings for both electrics field component are equal to \( \pi \), as shown in Figure 6. In particular, there are four particular states of transverse spins, namely, \([1, i], [-1, i], [-1, -i], [1, -i]\), and \([1, -i]\). Finally, the sequential change of these four transverse spins forms the photonic skyrmions in the focal region of OL.

In addition to the photonic skyrmions, the phase evolutions of the vortex beam in Figure 3 and 6 are consistent with those of the electric fields in the focal region. Considering the \( m \)-order VVB as an example, based on Debye vectorial diffraction theory, the electric fields in the focal region of the OL can be simplified into a series of Bessel functions \( J_m(\rho)^{(14,19)} \), with reference to Equation (40)–(42), where \( J_m(\cdot) \) and \( J_{m-1}(\cdot) \) denote the Bessel functions of the transverse and longitudinal components of the electric field, respectively.

\[
E_x = \int_0^{2\pi} \cos(m\phi) \exp(i\rho \cos(\phi - \gamma)) d\phi = 2\pi i^m J_m(\rho) \cos(m\gamma)
\]

(40)

\[
E_y = \int_0^{2\pi} \sin(m\phi) \exp(i\rho \cos(\phi - \gamma)) d\phi = 2\pi i^m J_m(\rho) \sin(m\gamma)
\]

(41)

\[
E_z = \int_0^{2\pi} \cos((m - 1)\phi) \exp(i\rho \cos(\phi - \gamma)) d\phi
\]

\[
= 2\pi i^{m-1} J_{m-1}(\rho) \cos((m - 1)\gamma)
\]

(42)

where \( E_x \) and \( E_y \) are the two transverse components in the \( x \)- and \( y \)-directions of the electric fields, respectively, and \( E_z \) denotes the longitudinal component. According to Equation (40)–(42), the transverse and longitudinal components of the electric field possess the phases \( i^m \) and \( i^{m-1} \), which correspond to those in Equation (22). In particular, the phases in the above mathematical equations can be explained using the derivation of Equation (22) from physics.

In conclusion, we revealed the physical essence of a propagable vortex beam with a topological charge of \( l + 0.5 \) by reunderstanding Abbe’s diffraction limit, investigating the phase evolution of the vortex beam, and proposing the binary time vector property of phase. In the first case, due to Abbe’s diffraction limit, the fractional-order VVBs can remain stable, although the modulation symmetries are naturally broken down by optical vortices with fractional topological charges. In the second case, we demonstrated the phase evolution of the vortex beam, which is the physical cause of the polarization rotation of fractional-order VVBs. Third, the phase is not merely a scalar attribute of the light beam and demonstrates a binary time vector property. Therefore, the electric field of the entire fractional-order VVBs is continuous, whereas the topological charge of the vortex phase and order of polarization are fractional. This study provides entirely different physical viewpoints on the vortex phase of the light beam and Abbe’s diffraction limit, which may deepen our knowledge on the behavior of light beams in classical optics.

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Conflict of Interest

The authors declare no conflict of interest.

Author Contributions

X.W. conceived of the research and carried out the numerical simulations. Y.M., Y.C., and X.W. analyzed all of the data. X.W. wrote the paper with advice regarding development. All of the authors participated in the analysis and discussion of the results.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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