EIGENVALUES OF
AN ELLIPTIC SYSTEM

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Abstract

We describe the spectrum of a non-self-adjoint elliptic system on a finite interval. Under certain conditions we find that the eigenvalues form a discrete set and converge asymptotically at infinity to one of several straight lines. The eigenfunctions need not generate a basis of the relevant Hilbert space, and the larger eigenvalues are extremely sensitive to small perturbations of the operator. We show that the leading term in the spectral asymptotics is closely related to a certain convex polygon, and that the spectrum does not determine the operator up to similarity. Two elliptic systems which only differ in their boundary conditions may have entirely different spectral asymptotics. While our study makes no claim to generality, the results obtained will have to be incorporated into any future general theory.

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1 Introduction

Consider a first order elliptic differential system acting in \( L^2 := L^2((\alpha, \beta), C^n) \) according to the formula
\[
(Lf)(x) := A(x)f'(x)
\]
where \( A \) is an \( n \times n \) complex matrix depending on \( x \). (We show in Section 2 that our methods also apply to certain second and higher order elliptic systems.) If \( A(x) \) is a piecewise continuous function of \( x \) then the solution to the equation \( Lf = zf \) can be written in the form
\[
f(x) = U(z, x)f(\alpha)
\]
where \( U(z, x) \) is an invertible \( n \times n \) matrix which depends continuously on \( x \) and analytically on \( z \). The operator has domain contained in \( W^{1,2} := W^{1,2}((\alpha, \beta), C^n) \).
Since $W^{1,2} \subseteq C[\alpha, \beta]$, we can impose general boundary conditions of the type

$$Sf(\alpha) + Tf(\beta) = 0.$$  \hspace{1cm} (1)

where $S, T$ are any $n \times n$ matrices.

Problems of this type arise in several contexts. One relates to non-equilibrium thermodynamics, \([20]\) and another concerns the study of generalized determinants for elliptic operators on manifolds with boundary, \([13, 17, 18]\). Of course problems involving higher order differential operators can also be reformulated in terms of first order systems; we do not, however, assume Hamiltonian structure, which would force $n$ to be even and also imply strong constraints on the solutions of the equations. The present paper provides a class of nearly exactly soluble examples which illustrate some of the phenomena such studies have to face.

The spectral behaviour of $L$ has obvious relevance to the time-dependent system

$$\frac{\partial f}{\partial t} = A(x) \frac{\partial f}{\partial x}.$$  \hspace{1cm} (2)

If each matrix $A(x)$ has only real eigenvalues $a_r(x)$ then (2) is called a hyperbolic system and $a_r(x)$ are the characteristic speeds at $x$. This case has special features by comparison with the case in which the eigenvalues are in general position in the complex plane, and is only partially analyzed in this paper. See Section 7 on semigroup properties.

**Example 1** Consider the ‘Dirichlet’ boundary conditions $f(\alpha) \in U, f(\beta) \in V$, where $U, V$ are linear subspaces in $C^n$ with

$$\text{dim}(U) + \text{dim}(V) = n.$$

This falls within the above scheme if we choose $S, T$ as follows: $S$ should have kernel $U$ and range $W$ while $T$ should have kernel $V$ and range $X$ where $W \cap X = \{0\}$ and $W + X = C^n$. In particular if $U$ is the linear span of the single vector $u \neq 0$ and $V$ is the annihilator of the vector $v \neq 0$ then the boundary value problem reduces to

$$F(z) := \langle U(z, \beta)u, v \rangle = 0.$$

Returning to the general context, the embedding of $W^{1,2}$ into $L^2$ is compact, so the operator $L$ has compact resolvent if its spectrum is not equal to $C$, and its spectrum must consist of discrete eigenvalues of finite multiplicity. If $A(x) = aI$ for all $x \in [\alpha, \beta]$ then the only possible solution of $Lf = zf$ is $f(x) = ce^{zx/a}$ for some $c \in C^n$; in the context of Example \([1]\) we deduce that $\text{Spec}(L) = C$ if $U \cap V \neq \{0\}$, but in all other cases the spectrum is empty. The situation changes entirely if we assume that $A(x)$ is constant but its eigenvalues $a_r$ are all different and all non-zero.

It follows immediately from the above discussion that $z$ is an eigenvalue of $L$ if and only if

$$F(z) := \det (S + TU(z, \beta)) = 0.$$
where $F$ is an entire function of $z$. There are well-developed numerical procedures for evaluating $F(z)$ for any $z$ and for finding the points at which $F$ vanishes. However, the asymptotic behaviour of the eigenvalues of $L$ and the study of such quantities as the regularized determinant are difficult subjects. There is one case in which they can be analyzed fairly directly, namely when $A(x)$ is piecewise constant, an assumption which we make for the remainder of the paper.

In the following sections we present a detailed analysis of the function $F(z)$. We prove in the ‘generic case’ that there are several series of eigenvalues diverging to infinity along straight lines which are determined by a certain convex polygon. The asymptotic form of the spectral counting function depends upon the length of the boundary of this polygon. There are exceptional cases in which almost periodic structure arises, and which we do not analyze fully. The implications of our results for the spectral analysis are explained with a series of simple examples.

A much more complete analysis of the second order case with $n = 2$ has been performed by Boulton, [4, 5], and his results have informed our approach to the case $n \geq 2$. Our results describe how the more general analysis goes in the generic case, and what further phenomena have to be considered for a complete analysis.

2 The function $F(z)$

We assume henceforth that

$$\alpha = \alpha_0 < \alpha_1 < ... < \alpha_m = \beta$$

and that $A(x) = A_s$ if $\alpha_{s-1} < x \leq \alpha_s$. We also assume that each $A_s$ is invertible and diagonalizable. Then

$$U(z, \beta) = U_m U_{m-1} ... U_2 U_1$$

where

$$U_s = e^{zA_s^{-1}(\alpha_s - \alpha_{s-1})} = V_s e^{zD_s} V_s^{-1}$$

and each matrix $D_s$ is diagonal. We then have

$$F(z) = \det \left( S + TV_m e^{zD_m} V^{-1}_m V_{m-1} e^{zD_{m-1}} V_{m-1}^{-1} ... V_1 e^{zD_1} V_1^{-1} \right)$$

$$= \sum_{r=1}^{R} \delta_r e^{zr}.$$  \hspace{1cm} (3)

It might be thought that the case in which $A(x)$ is continuous can be obtained from the above by an approximation procedure. In Example 9 we show that taking such a limit would not be a straightforward matter. We discuss the spectral asymptotics of such piecewise constant operators in Section 4, and make a conjecture about the case in which $A(x)$ depends continuously on $x$. 


Let us look at the case of Dirichlet boundary conditions in more detail. Assuming that $A$ is independent of $x$, invertible and diagonal with eigenvalues $a_r$, the solution of the eigenvalue equation is

$$f_r(x) = c_r e^{z(x-\alpha)/a_r}$$

where $c \in \mathbb{C}^n$ must be chosen so that $f$ satisfies the boundary conditions. If $u_1, \ldots, u_{n-p}$ is a basis of the annihilator of $U$ in the dual space and $v_1, \ldots, v_p$ is a basis of the annihilator of $V$ then the conditions are that $(c, u_i) = 0$ for all $1 \leq i \leq n - p$ and $(c, w_j) = 0$ for all $1 \leq j \leq p$, where $w_j \in \mathbb{C}^n$ is defined by

$$w_{j,r} = v_{j,r} e^{z(\beta-\alpha)/a_r}.$$  

The existence of a non-zero eigenvector then reduces to

$$\det\{v_1, \ldots, v_{n-p}, w_1, \ldots, w_p\} = 0.$$  

This may be rewritten in the form $F(z) = 0$ where $F$ is defined by $[\square]$. In this case each of the coefficients $\gamma_r$ is a sum of $p$ distinct terms of the form $(\beta-\alpha)/a_r$,

In the simplest case $\dim(U) = 1$ and there are $n$ distinct $\gamma_r$. In general the number of $\gamma_r$ is given by a combinatorial expression. However, some of the $\gamma_r$ may coincide, and some of the $\delta_r$ may vanish.

We next consider a similar problem for second order elliptic systems. Let $H$ act in $L^2 := L^2((\alpha, \beta), \mathbb{C}^n)$ according to the formula

$$(Hf)_r(x) := a_r^2 \frac{d^2 f_r}{dx^2}$$

for $1 \leq r \leq n$, where $a_r \neq 0$ for all $r$. The operator has domain contained in $W^{2,2} := W^{2,2}((\alpha, \beta), \mathbb{C}^n)$. Since $W^{2,2} \subseteq C^{(1)}[\alpha, \beta]$, the boundary conditions may involve $f(\alpha), f'(\alpha), f(\beta), f'(\beta)$. We do not consider the most general choice of boundary conditions, which leads to a somewhat complicated function $F(z)$, but follow Boulton in assuming a combination of Dirichlet or Neumann boundary conditions in the following sense, $[\square, \square]$. We suppose that $f(\alpha) \in U_1$, $f'(\alpha) \in U_2$, $f(\beta) \in V_1$ and $f'(\beta) \in V_2$ where $U_1, U_2, V_1, V_2$ are linear subspaces of $\mathbb{C}^n$. We impose the minimal further condition

$$\dim(U_1) + \dim(U_2) + \dim(V_1) + \dim(V_2) = 2n.$$  

The eigenvalue problem may be written in the form $Hf = z^2 f$, and its solution is of the form

$$f_r(x) = c_r e^{(x-\alpha)z/a_r} + d_r e^{-(x-\alpha)z/a_r}$$

where $(c, d) \in \mathbb{C}^{2n}$, provided $z \neq 0$. The case $z = 0$ is dealt with separately. The boundary conditions lead to $2n$ linear constraints on the vector $(c, d)$ which have a non-zero solution if $F(z) = 0$, where $F(z)$ is a function of the form $[\square]$.

As a particular case we mention the possibility $H := L^2$ where $L$ is the first order operator already discussed. This corresponds to the choice of boundary conditions $f(\alpha) \in U$, $f'(\alpha) \in A^{-1}U$, $f(\beta) \in V$, $f'(\beta) \in A^{-1}V$, where $A_{i,j} = \delta_{i,j} a_i$. The eigenvalues of $H$ are the squares of the eigenvalues of $L$, and it follows from our analysis below that they are asymptotic to certain parabolae at infinity.
3 Zeros of $F(z)$

Our task is to study the asymptotic distribution of the zeros of entire functions of the form

$$F(z) = \sum_{r=1}^{R} \delta_r e^{z \gamma_r} = \sum_{r=1}^{R} F_r(z)$$

as $|z| \to \infty$. We assume that all $\gamma_r$ are different, that all $\delta_r$ are non-zero and that $R \geq 2$. We will see that the asymptotic structure of the set of zeros depends heavily on the convex hull $K$ of the set of $\gamma_r$. We will only consider the generic case, defined as follows. After relabelling, the convex hull has vertices $\gamma_1, \ldots, \gamma_Q$ written in anticlockwise order, while the remainder of the $\gamma_r$ lie in the interior of $K$; we assume that no $\gamma_r$ lies on a edge of $K$ unless it is already a vertex of $K$. If $1 \leq r \leq Q$ we define

$$r_+ := \begin{cases} r + 1 & \text{if } 1 \leq r \leq Q - 1 \\ 1 & \text{if } r = Q, \end{cases}$$

$$r_- := \begin{cases} r - 1 & \text{if } 2 \leq r \leq Q \\ Q & \text{if } r = 1. \end{cases}$$

We will prove that, in addition to a finite number of zeros near to the origin, $F(z)$ has $Q$ series of zeros, each of which converges asymptotically at infinity towards one of the sets on which

$$|F_r(z)| = |F_{r_-}(z)|.$$ 

where $1 \leq r \leq Q$. Rewriting this in the form

$$|\delta_r e^{z \gamma_r}| = |\delta_{r_-} e^{z \gamma_{r_-}}|$$

we see that it is a straight line

$$z \cdot (\gamma_r - \gamma_{r_-}) = k_r$$

perpendicular to the edge $(\gamma_{r_-}, \gamma_r)$ of $K$.

If one or more of the $\gamma_r$ with $Q < r \leq R$ lie within an edge of $K$ then the spectral behaviour of $L$ at infinity is much more complicated. Boulton has shown that the behaviour depends upon whether those $\gamma_r$ which lie within an edge of $K$ divide that edge into parts whose lengths have rational or irrational ratios. In the former case the eigenvalues still converge at infinity to one of several straight lines, but there are several lines perpendicular to the relevant edge instead of just one. In the irrational case the eigenvalues have almost periodic structure at infinity within one of several strips, with one strip perpendicular to each edge. Boulton’s analysis is only presented in a typical second order case, but the ideas clearly extend as stated, \[4, 5\].

Returning to the generic case, we study the asymptotic location of the zeros of $F(z)$ by dividing the exterior of $K$ into $4Q$ subregions. Given $\varepsilon > 0$ there are 3
semi-infinite strips $S_r^\varepsilon$, $S_r^+$, $S_r^-$ associated with each edge $(\gamma_r^-, \gamma_r)$ of $K$, and also a wedge $W_r$ associated with each vertex $\gamma_r$. Let $e_r$ be the outward pointing unit normal to the edge $(\gamma_r^-, \gamma_r)$.

Given $1 \leq r \leq Q$ we define

$$S_r^\varepsilon := \{z \in \mathbb{C} : z \cdot e_r \geq \gamma_r \cdot e_r \text{ and } k_r - \varepsilon \leq z \cdot (\gamma_r - \gamma_r^-) \leq k_r + \varepsilon\}$$

$$S_r^+ := \{z \in \mathbb{C} : z \cdot e_r \geq \gamma_r \cdot e_r \text{ and } k_r + \varepsilon \leq z \cdot (\gamma_r - \gamma_r^-) \leq k_r + 1\}$$

$$S_r^- := \{z \in \mathbb{C} : z \cdot e_r \geq \gamma_r \cdot e_r \text{ and } k_r - 1 \leq z \cdot (\gamma_r - \gamma_r^-) \leq k_r - \varepsilon\}$$

$$W_r := \{z \notin K : k_r + 1 \leq z \cdot (\gamma_r - \gamma_r^-) \text{ and } k_{r+} - 1 \geq z \cdot (\gamma_{r+} - \gamma_r)\}.$$ 

**Theorem 2** Given $0 < \varepsilon < 1/9$ there exists $T_\varepsilon$ such that any eigenvalue $z$ of $L$ satisfying $|z| > T_\varepsilon$ must lie in $S_r^\varepsilon$ for some $r$ such that $1 \leq r \leq Q$.

**Proof** If $z \in S_r^+$ then

$$|F_r^-(z)/F_r(z)| \leq e^{-\varepsilon} < 1 - \varepsilon/2.$$ 

If $s \neq r, r-\pm$ then $(\gamma_r - \gamma_s) \cdot e_r > 0$ and for large enough $|z|$ we have

$$|F_s(z)/F_r(z)| \leq \varepsilon/2R.$$ 

Combining these bounds we see that for such $z$

$$|F(z)| \geq |F_r(z)|\varepsilon(1/2 - (R - 2)/2R) > 0.$$ 

A similar argument applies to $S_r^-$. If $s \neq r, r_{\pm}$ then either

$$\gamma_r - \gamma_s = p(\gamma_r - \gamma_r^-) + q e_r$$

where $p > 0$ and $q > 0$, or

$$\gamma_r - \gamma_s = p(\gamma_r - \gamma_{r+}) + q e_{r+}$$

where $p > 0$ and $q > 0$. It follows that

$$\lim_{|z| \to \infty, z \in W_r} \{(\gamma_r - \gamma_s) \cdot z\} = +\infty.$$ 

Now let $z \in W_r$. We have

$$|F_{r\pm}(z)/F_r(z)| \leq e^{-1}$$

and if $|z|$ is large enough we also have

$$|F_s(z)/F_r(z)| \leq \varepsilon/2R.$$ 

provided $s \neq r, r_{\pm}$. Combining these inequalities we obtain

$$|F(z)| \geq |F_r(z)|(1 - 2e^{-1} - \varepsilon(R - 3)/2R) > 0.$$
Note 3 If one of the $\gamma_s$ for $Q < s \leq R$ is very close to the boundary of $K$ then $T_\varepsilon$ will be correspondingly large in the above proof. This is inevitable because the possible asymptotic directions of the spectrum change as $\gamma_s$ moves through an edge of $K$, whereupon $K$ changes so that $\gamma_s$ becomes another vertex of $K$.

Note 4 The semi-infinite lines (3) divide the complex plane into $Q$ sectors $S_r$. If $|z| \to \infty$ within $S_r$ then $F(z) \sim \delta_r e^{\overline{\gamma}r}$, so

$$\log(F(z)) \sim \log(\delta_r) + z\overline{\gamma}r.$$  

This formula may be used to compute the regularized determinant in the sense of [13, 17, 18].

4 Spectral Asymptotics

We have proved that the zeros of $F(z)$ converge asymptotically towards one of $Q$ straight lines as $|z| \to \infty$. To obtain more detailed information we introduce the functions

$$G_r(z) := F_r(z) + F_{r-}(z).$$

Theorem 5 The zeros of $G_r(z)$ lie on the line (3) with constant distance $2\pi/|\gamma_r - \gamma_{r-}|$ between any two consecutive zeros.

Proof If we put $\gamma_r - \gamma_{r-} = \rho e^{i\theta}$ where $\rho > 0$ and $\theta \in \mathbb{R}$ and $c_r = \log(\delta_{r-}/\delta_r)$, then the zeros of $G_r(z)$ are given by

$$z = (c_r + (2n + 1)\pi i)e^{i\theta}/\rho$$

where $n \in \mathbb{Z}$. The statements of the lemma all follow from this.

Theorem 6 The zeros of $F(z)$ which lie in $S_r^\varepsilon$ converge as $|z| \to \infty$ to the zeros of $G_r(z)$, in the sense that the modulus of the differences of corresponding zeros converges to zero.

Proof We have already proved that the zeros of $F(z)$ converge to one of the lines (3). The statement is a straightforward application of Rouche’s theorem, since we have already noted that the remaining terms in the series are asymptotically negligible as $|z| \to \infty$ within $S_r^\varepsilon$.

Corollary 7 If $N(E)$ is the number of zeros of $F(z)$ such that $|z| \leq E$ then

$$N(E) = b(K)E/2\pi + O(1)$$

as $E \to \infty$, where $b(K)$ is the length of the boundary of $K$.
Proof The number of zeros of $F(z)$ satisfying $|z| \leq E$ associated with each line is $|\gamma_r - \gamma_{r-1}|E/2\pi + O(1)$ by the above theorem, since the zeros of $F(z)$ only converge to the zeros of $G_r(z)$ in one direction. The corollary follows by summing over $r$.

Example 8 If $L_i$ are two elliptic systems acting in $L^2((\alpha, \beta), C^n)$ respectively for $i = 1, 2$ then we may consider their direct sum $L = L_1 \oplus L_2$. By applying Corollary 7 to each of the components we obtain

$$N(E) = (b(K_1) + b(K_2))E/2\pi + O(1)$$

in an obvious notation. On the other hand we have $F(z) = F_1(z)F_2(z)$ so the exponents $\gamma_r$ in $F(z)$ are the sums of the exponents in $F_1(z)$ and in $F_2(z)$. This implies that $K = K_1 + K_2$. These different approaches to the asymptotics are reconciled by the classical but non-trivial fact that

$$b(K_1 + K_2) = b(K_1) + b(K_2)$$

for any two plane convex sets. This example may be used to construct counterexamples to various conjectures.

Example 9 Let us consider the entire function

$$F_n(z) = n^{-1} \sum_{r=1}^{n} \exp\left(e^{2\pi ir/n}z\right).$$

According to Corollary 7

$$N_n(E) = b_n E/2\pi + O(1)$$

as $E \to \infty$, where $b_n \to 2\pi$ as $n \to \infty$. On the other hand

$$\lim_{n \to \infty} F_n(z) = F(z) := \frac{1}{2\pi} \int_{0}^{2\pi} \exp\left(e^{i\theta}z\right) \, d\theta.$$ 

Since $F(z)$ is rotationally invariant and entire with $F(0) = 1$ it must be identically equal to 1, and so it has no zeros at all. This establishes that even for this class of entire functions, the asymptotic distribution of the zeros does not vary continuously with the function.

When combined with Corollary 7, the following theorem proves that the eigenvalues of $L$ move off to infinity as the eigenvalues of $A$ coalesce. The singular behaviour of the spectrum in the limit $\delta \to 0$ is in accordance with the behaviour for $\delta = 0$ described in the Introduction.

Theorem 10 Let $A = \alpha I + \delta B$, where $\alpha \neq 0$, the eigenvalues of $B$ are all distinct, and $\delta \in C$ is sufficiently small. Let $L$ be defined in the usual manner with $\dim(U) = p$. Then there exist distinct constants $\sigma_r$ such that

$$\gamma_r = p(\beta - \alpha)/\alpha + \sigma_r\delta + O(\delta^2)$$

and there exists $\kappa > 0$ such that

$$b(K) = \kappa\delta + O(\delta^2).$$
The eigenvalues of $A$ are $a_r = \alpha + \delta b_r$ for $1 \leq r \leq n$ where $b_r$ are the distinct eigenvalues of $B$. The formula for the $\gamma_r$ in Section 2 yields (8) immediately. This implies (7) with
\[
\kappa = \sum_{r=1}^{Q} |\sigma_r - \sigma_{r-1}|/2\pi.
\]

In the remainder of this section we show how to apply Corollary 7 to certain elliptic systems with variable coefficients. We assume that
\[
Lf(x) = A(x)f'(x)
\]
in $L^2((\alpha, \beta), C^n)$, where $A(x)$ is invertible and diagonalizable for each $x \in (\alpha, \beta)$.

In our next theorem we assume boundary conditions of the form $f(\alpha) = u$ and $\langle f(\beta), v \rangle = 0$, where $u, v \in C^n$ are both non-zero.

The meaning of the word ‘generically’ below will be explained during the proof.

**Theorem 11** Let $b(x)$ be the length of the boundary of $K(x)$, where $K(x)$ is the convex hull of the eigenvalues of $A(x)^{-1}$. If also $A(\cdot)$ is piecewise constant then generically one has
\[
N(E) = E^2/2\pi \int_{\alpha}^{\beta} b(x) \, dx + O(1) \quad (8)
\]
as $E \to \infty$.

**Proof** Following the notation of Section 2, equation (3) becomes
\[
F(z) = \langle V_m e^{zD_m} V^{-1}_m.V_{m-1} e^{zD_{m-1}} V^{-1}_{m-1}...V_1 e^{zD_1} V^{-1}_1 u, v \rangle
\]
\[
= \sum_{r=1}^{R} \delta_r e^{z \gamma_r}. \quad (9)
\]

Denoting the eigenvalues of $A_s$ by $\{a_{s,t}\}_{t=1}^{n}$, $D_s$ is a diagonal matrix with entries $\{a_{s,t}(\alpha_s - \alpha_{s-1})\}_{t=1}^{n}$. Each $\gamma_r$ in (9) is of the form
\[
\gamma_r = \sum_{s=1}^{m} a_{s,t(s)}^{-1} (\alpha_s - \alpha_{s-1}) \quad (10)
\]
where $t(\cdot)$ is a function from $\{1, ..., m\}$ to $\{1, ..., n\}$. The parameter $r$ is a relabelling of the set of all such functions, so $R \leq n^m$, with equality unless two sums of the form (10) happen to be equal. If $K$ is the convex hull of the $\gamma_r$ then our generic assumption is that $\delta_r \neq 0$ for all vertices $\gamma_r$ of $K$, and that no $\gamma_r$ lie within any edge of $K$.

By Corollary 7 we have to prove that
\[
b(K) = \int_{\alpha}^{\beta} b(x) \, dx. \quad (11)
\]
The right hand side equals
\[
\sum_{s=1}^{m} b(K_s)(\alpha_s - \alpha_{s-1})
\]
where \( A(x) = A_s \) and \( K(x) = K_s \) if \( \alpha_{s-1} < x \leq \alpha_s \) as in Section 2. The identity (11) follows by combining the facts that

\[
\begin{align*}
    b(tK_1) &= tb(K_1) \quad (12) \\
    b(K_1 + K_2) &= b(K_1) + b(K_2) \quad (13)
\end{align*}
\]

for any convex sets \( K_1, K_2 \) and any \( t > 0 \), with the identity

\[
K = \sum_{s=1}^{m} (\alpha_s - \alpha_{s-1})K_s.
\]

**Conjecture 12** We conjecture that Theorem 11 remains generically valid for piecewise continuous coefficients \( A(x), x \in [\alpha, \beta] \).

**Note 13** The leading coefficient in (8) depends only on the symbol of \( L \). However, the truth of the theorem depends upon the genericity assumption for the following reason. Let \( L, \tilde{L} \) be two operators with the same symbol but different boundary conditions. Since the coefficients \( \delta_r \) in the expansion (1) may vanish for different values of \( r \) for the two operators, it may happen that \( K \neq \tilde{K} \) in an obvious notation. The leading coefficient in the spectral counting function (8) will then usually be different.

This phenomenon is illustrated by the following theorem. We make the same assumptions as in Theorem 11, except that we now allow general boundary conditions of the form (1). The following theorem may be regarded as a version of Weyl’s formula for the operator associated with the matrix-valued symbol

\[
p_{r,s}(x, \xi) = A_{r,s}(x)\xi
\]

where \( x \in [\alpha, \beta] \) and \( \xi \in \mathbb{R} \). Once again the meaning of the word ‘generically’ will be explained during the proof. We warn the reader that the asymptotic form obtained here is different from that of Theorem 11; this is possible because the boundary conditions of that theorem are all non-generic in the sense used in the present theorem.

**Theorem 14** Under the above assumptions one generically has

\[
N(E) = \frac{E}{2\pi} \int_{\alpha}^{\beta} b(x) \, dx + O(1)
\]

where

\[
b(x) = 2 \sum_{r=1}^{n} |a_r(x)|^{-1}
\]

and \( \{a_r(x)\}_{r=1}^{n} \) are the eigenvalues of \( A(x) \) for each \( x \in (\alpha, \beta) \).
Proof. A more detailed understanding of (3) may be obtained by considering
\[ \tilde{F}(z_1, ..., z_m) = \det \left( S + TV_m e^{z_m D_m} V_m^{-1} V_{m-1} e^{z_{m-1} D_{m-1}} V_{m-1}^{-1} ... V_1 e^{z_1 D_1} V_1^{-1} \right) \]
\[ = \sum_{r=1}^{R} \delta_r \exp \left( \sum_{s=1}^{m} z_s \gamma_{s,r} \right). \quad \text{(14)} \]

The coefficient $\gamma_{t,r}$ can be determined by putting
\[ z_s = \begin{cases} w & \text{if } s = t \\ 0 & \text{otherwise} \end{cases} \]
to obtain
\[ \sum_{r=1}^{R} \delta_r \exp \left( w \gamma_{t,r} \right) = \det \left( S + TV_t e^{w D_t} V_t^{-1} \right) \]
\[ = \det \left( V_t^{-1} \right) \det \left( SV_t + TV_t e^{w D_t} \right). \]

By expanding the final determinant we deduce that each $\gamma_{t,r}$ is the sum of some subset of the numbers $a_{t,j}^{-1}(\alpha_t - \alpha_{t-1})$ where $1 \leq j \leq n$ and $\{a_{t,j}\}_{j=1}^{n}$ are the eigenvalues of $A_t$. Since $F(z) = \tilde{F}(z, ..., z)$ we also have
\[ \gamma_r = \sum_{s=1}^{m} \gamma_{s,r}. \]

We say that we are in the generic case if all possible $\delta_r$ are non-zero, and also no $\gamma_r$ lies within an edge of the convex hull $K$ of $\{\gamma_r\}_{r=1}^{R}$.

We now apply Corollary 7 exactly as in the proof of Theorem 11 but with different choices for $K_s$. Generically each $K_s$ is the convex hull of all numbers of the form
\[ v_{s,J} = \sum_{j \in J} a_{s,j}^{-1} \]
where $J$ ranges over all subsets of $\{1, ..., n\}$. Hence
\[ K_s = I_{s,1} + ... + I_{s,n} \]
where $I_{s,j}$ is the interval with end-points $0, a_{s,j}^{-1}$. It follows by (13) that
\[ b(K_s) = \sum_{j=1}^{n} b(I_{s,j}) = 2 \sum_{j=1}^{n} |a_{s,j}|^{-1}. \]

This yields the statement of the theorem.

We had expected that the above theorem would be valid for periodic boundary conditions, but the following example shows that this need not be the case.
Example 15 Let $\alpha = 0$, $\beta = 2$, $n = 2$, $A(x) = A_1$ for $0 \leq x < 1$ and $A(x) = A_2$ for $1 \leq x \leq 2$. Suppose also that

$$A_i = V_i \begin{bmatrix} a_{1,i} & 0 \\ 0 & a_{2,i} \end{bmatrix} V_i^{-1}$$

for $i = 1, 2$, where all $a_{j,i}$ are non-zero. Finally put

$$D_i = \begin{bmatrix} u_i & 0 \\ 0 & v_i \end{bmatrix}$$

for $i = 1, 2$, where $u_i = a_{1,i}^{-1}$ and $v_i = a_{2,i}^{-1}$.

Periodic boundary conditions correspond to the choice $S = I$ and $T = -I$ and lead to the formula

$$F(z) = \det \left( I - V_2 e^{D_2 z} V_2^{-1} V_1 e^{D_1 z} V_1^{-1} \right)$$

$$= c \det \left( X - e^{D_2 z} X e^{D_1 z} \right)$$

where $c = \det(V_2) \det(V_1^{-1}) \neq 0$ and $X = V_2^{-1} V_1$ is invertible. Hence

$$F(z) = c \det \begin{bmatrix} X_{1,1} (1 - e^{(u_2 + u_1) z}) & X_{1,2} (1 - e^{(u_2 + v_1) z}) \\ X_{2,1} (1 - e^{(v_2 + u_1) z}) & X_{2,2} (1 - e^{(v_2 + v_1) z}) \end{bmatrix}.$$ (15)

If all $X_{i,j}$ are non-zero we deduce that the possible values of $\gamma_r$ are $0, u_1 + u_2, u_1 + v_2, v_1 + u_2, v_1 + v_2$ and $u_1 + u_2 + v_1 + v_2$. Depending on the positions of these points $\mathcal{K}$ may have several shapes, and hence $b(\mathcal{K})$ may have several values. One possibility is

$$b(\mathcal{K}) = 2|u_2 + u_1| + 2|v_2 + v_1|$$

which is quite different from the formula obtained in Theorem 14.

In the very special case $A_2 = -A_1$ every solution of $Lf = zf$ is periodic and $\text{Spec}(L) = \mathbb{C}$. But in this case $u_2 + u_1 = v_2 + v_1 = 0$ and $X = I$, so (15) yields $F(z) = 0$ for all $z$.

5 Basis Problems

The fact that one may be able to determine the eigenvalues and eigenfunctions of an operator does not imply that the eigenfunctions form a basis in the relevant Banach space, [2, 9]. If this fails then the positions of the eigenfunctions may be very unstable with respect to small perturbations of the operator, and the significance of the spectrum becomes moot. Boulton [4, 5] has investigated a closely related problem for second order elliptic systems in the language of pseudospectral theory, [3, 7, 8, 15, 16, 21], which amounts to estimating the resolvent norms of the operators concerned.
In this section we show that such problems do indeed occur in the context of this paper. For some operators of the type we consider the eigenfunctions form a basis, but for many others, and we believe most, they do not. We do not attempt a complete analysis, but just discuss the simplest example, of a first order system with two components. The operator $H = L^2$ provides a second order system with the same eigenvectors as $L$ and hence the same basis problems.

**Example 16** Let $n = 2$ and $0 \neq t \in \mathbb{R}$ and let $L$ be defined by

$$(L_f)_1(x) := tf_1'(x)$$

$$(L_f)_2(x) := -ti f_2'(x)$$

for all $f \in W^{1,2}((0,\pi),\mathbb{C}^2)$, subject to the boundary conditions $f_1(0) = f_2(0)$ and $f_1(\pi) = f_2(\pi)$. A direct calculation shows that $\text{Spec}(L) = t\mathbb{Z}$ and that the eigenfunctions form a complete orthonormal set. Thus $L$ is self-adjoint.

The behaviour of the following slightly more general example is quite different.

**Example 17** Let $n = 2$, $s > 0$, $t > 0$ and $u := s + it$. Let $L$ be defined by

$$(L_f)_1(x) := uf_1(x)$$

$$(L_f)_2(x) := \pi f_2'(x)$$

for all $f \in W^{1,2}((0,\pi),\mathbb{C}^2)$, subject to the boundary conditions $f_1(0) = f_2(0)$ and $f_1(\pi) = f_2(\pi)$. A direct calculation shows that $\lambda$ is an eigenvalue if

$$e^{\pi\lambda(1/u-1/\pi)} = 1$$

or equivalently if

$$\lambda_n = (s^2 + t^2)n/t$$

for some $n \in \mathbb{Z}$. The corresponding eigenfunction is

$$f_{n,1}(x) = e^{x\lambda_n/u}$$

$$f_{n,2}(x) = e^{x\lambda_n/\pi}$$

**Lemma 18** Let $S$ be a compact subset of $\mathbb{C}$ with zero Lebesgue measure and suppose that $C \setminus S$ has two components, one containing $0$ and the other unbounded. Then the linear span of the functions $\{z^n\}_{n \in \mathbb{Z}}$ is uniformly dense in $C(S)$.

**Proof** By Hartogs-Rosenthal lemma, [10, p 47], $C(S) = R(S)$ where the latter is the uniform closure in $C(S)$ of the space of rational functions which do not have a pole on $S$. Each such rational function may be written as a linear combination of functions $z^m$, $(z-\sigma)^{-n}$ where $m,n \geq 0$ and $\sigma \not\in S$. If $\sigma$ is in the unbounded component of $C \setminus S$ then $(z-\sigma)^{-n}$ may be uniformly approximated by polynomials in $z$ by Runge’s theorem, [10, p 28]. If $\sigma$ is in the bounded component then $(z-\sigma)^{-n}$ may be uniformly approximated by polynomials in $z^{-1}$ by using inversion and Runge’s theorem. Putting these facts together completes the proof.
**Theorem 19** The set of eigenfunctions \( \{ f_n \}_{n \in \mathbb{Z}} \) is complete in the sense that its linear span is dense in \( L^2((0, \pi), \mathbb{C}^2) \).

**Proof** Define \( w : [0, \pi] \to \mathbb{C} \) by

\[
w(x) = f_{1,1}(x) = e^{x(\tau-i)}
\]

where \( \tau = s/t \). Let \( S = T \cup \bar{T} \) where \( T = w([0, \pi]) \). Then \( S \) is a closed curve in \( \mathbb{C} \) surrounding the origin and satisfies the conditions of Lemma 18.

It is sufficient to prove that if \( \phi \in C([0, \pi], \mathbb{C}^2) \) satisfies \( \phi(0) = \phi(\pi) = 0 \) then \( \phi \) may be uniformly approximated by finite linear combinations of \( \{ f_n \}_{n \in \mathbb{Z}} \). Given such a \( \phi \) we define \( \psi \in C(S) \) by

\[
\psi(w(x)) = \begin{cases} 
\phi_1(x) & \text{if } x \in [0, \pi] \\
\phi_2(x) & \text{if } x \in [0, \pi] 
\end{cases}
\]

Given \( \varepsilon > 0 \) there exists an approximation

\[
\| \psi(z) - \sum_{r=-N}^{N} \delta_r z^r \|_\infty < \varepsilon
\]

by Lemma 18. Putting \( z = w(x) \) where \( x \in [0, \pi] \) we obtain

\[
\| \phi_1 - \sum_{r=-N}^{N} \delta_r f_{r,1} \|_\infty < \varepsilon.
\]

Putting \( z = w(x) \) where \( x \in [0, \pi] \) we obtain

\[
\| \phi_2 - \sum_{r=-N}^{N} \delta_r f_{r,2} \|_\infty < \varepsilon.
\]

Combining these we obtain the required estimate

\[
\| \phi - \sum_{r=-N}^{N} \delta_r f_r \|_\infty < 2\varepsilon.
\]

The fact that the eigenvalues of \( L \) are real does not imply that it is similar to a self-adjoint operator.

**Theorem 20** Let \( P_n \) be the spectral projection associated with the eigenvalue \( \lambda_n \) of \( L \). Then \( \| P_n \| \) diverges at an exponential rate as \( n \to \infty \). The eigenfunctions \( f_n \) therefore cannot constitute a basis of \( L^2((0, \pi), \mathbb{C}^2) \).

**Proof** We have

\[
P_n \phi = \frac{\langle \phi, g_n \rangle f_n}{\langle f_n, g_n \rangle}
\]
where $g_n$ is the appropriate eigenfunction of $L^*$, and hence

$$
\|P_n\| = \frac{\|g_n\| \|f_n\|}{|\langle f_n, g_n \rangle|}.
$$

A direct calculation shows that the eigenfunction $g_n$ is given by

$$
g_{n,1}(x) = ue^{-x\lambda_n/\pi},
g_{n,2}(x) = -ue^{-x\lambda_n/\pi}.
$$

Evaluating the relevant integrals we find that

$$
\langle f_n, g_n \rangle = -2\pi it,
$$

$$
\|f_n\|^2 \sim \frac{t}{sn} e^{2\pi sn/t},
$$

$$
\|g_n\|^2 \sim \frac{t}{sn} (s^2 + t^2)
$$

as $n \to +\infty$, with a similar formula as $n \to -\infty$. This implies the statement about $\|P_n\|$. The final statement is a consequence of the fact that if the eigenfunctions form a basis then $\|P_n\|$ must be a bounded sequence, [2, 9, 11].

**Note 21** If $\phi \in L^2$ and $\phi_n = P_n \phi$ then it is possible that

$$
\phi \sim \sum_{n \in \mathbb{Z}} \phi_n
$$

in the sense of some Abel-type summation scheme, [14, 4, 19]. In the particular case in which $\phi(x) = (1, 0)$ for all $x \in (0, \pi)$ one finds that $\|\phi_n\| \to \infty$ exponentially fast as $n \to \pm \infty$. It is not clear that the convergence of (16) using a summation scheme would have much numerical significance, because of the high instability of the spectrum under small perturbations of the operator.

### 6 Numerical Range Problems

The properties which we have discussed so far are similarity invariants. In other words if $L$ has the property and $T$ is a bounded invertible operator then $TLT^{-1}$ also has the property. In this section we discuss properties which depend upon the particular norm chosen out of a similarity class. These are important because the norm is often given by physical considerations, and even if it is not, one might not know whether some better equivalent norm exists or how to find it.

Let $Hf = -Af''$ in $L^2((0, \beta), \mathbb{C}^n)$ where $A$ is a bounded invertible $n \times n$ matrix. We impose any linear boundary condition at $x = \beta$ and a boundary condition at $x = 0$ of the form $f(0) \in U$, $f'(0) \in V$, where $U, V$ are linear subspaces of $\mathbb{C}^n$.

The following theorem, which extends results of Boulton, [4, 5], establishes that Kato’s theory of sectorial forms, one of the main tools in non-self-adjoint semigroup
theory, cannot be applied to such operators even if the eigenvalues all lie in a half
plane \( \{ z \in \mathbb{C} : \text{Re}(z) \geq k \} \) for some \( k \), as one might expect if the eigenvalues of
\( A \) all have positive real parts.

**Theorem 22** If there exist \( c \in U \) and \( d \in V \) such that \( \langle Ad, c \rangle \neq 0 \) then the
numerical range of \( H \) equals the entire complex plane.

**Proof** Let \( \phi \in C^\infty(\mathbb{R}) \) satisfy \( \phi(x) = 1 \) if \( x \leq 1/3 \) and \( \phi(x) = 0 \) if \( x \geq 2/3 \). Let
also \( \psi \in \mathbb{R} \) and \( n \in \mathbb{Z}_+ \), and define

\[
f_n(x) = \left( c + e^{i\psi} d \left[ (x + 1/n)^{2/3} - (1/n)^{2/3} \right] \right) \phi(x)
\]

so that \( f \in \text{Dom}(H) \). An easy calculation shows that

\[
\lim_{n \to \infty} \|f_n\|^2 = \int_0^\beta \left| \left( c + e^{i\psi} dx^{2/3} \right) \phi(x) \right|^2 dx
\]

which we denote by \( k_1 \neq 0 \). An integration by parts establishes that

\[
\langle Hf_n, f_n \rangle = Q(f_n) + B_n
\]

where

\[
Q(f_n) = \int_0^\beta \langle Af'_n, f'_n \rangle \, dx
\]

\[
\to \int_0^\beta \left| \left( c + e^{i\psi} dx^{2/3} \right) \phi'(x) + \frac{2}{3} e^{i\psi} dx^{-1/3} \phi(x) \right|^2 dx
\]

\[
= k_2
\]

as \( n \to \infty \), and

\[
B_n = \langle Af'_n(0), f_n(0) \rangle = \frac{2}{3} e^{i\psi} n^{1/3} \langle Ad, c \rangle.
\]

We deduce that

\[
\frac{\langle Hf_n, f_n \rangle}{\|f_n\|^2} \sim \frac{k_2 + \frac{2}{3} e^{i\psi} n^{1/3} \langle Ad, c \rangle}{k_1}
\]

as \( n \to \infty \). By varying \( \psi \in \mathbb{R} \) and using the fact that the numerical range is a
convex set, [6, Theorem 6.1], we deduce that it must equal \( \mathbb{C} \).

**Example 23** We consider the operator \( H \) defined on \( L^2((0, \pi), \mathbb{C}^2) \) by

\[
(Hf)_1 = -e^{2i\alpha} f_1''
\]

\[
(Hf)_2 = -e^{-2i\alpha} f_2''
\]

subject to the four boundary conditions \( f_1(0) = 0, f_2'(0) = 0 \) and

\[
\cos(\theta) f_1(\pi) + \sin(\theta) f_2(\pi) = 0
\]

\[
-\sin(\theta) f_1'(\pi) + \cos(\theta) f_2'(\pi) = 0.
\]

To avoid redundancy we assume that \( -\pi/2 < \theta \leq \pi/2 \).
In the particular case $\theta = 0$ the two components of $H$ are independent and its eigenvalues are $n^2 e^{2i\alpha}$ where $n = 1, 2, \ldots$ and also $m^2 e^{-2i\alpha}$ where $m = 0, 1, 2, \ldots$ Moreover $H$ is normal and its eigenfunctions form a complete orthonormal set in $L^2$.

If $\theta \neq 0$ then we may apply Theorem 22 with $c = (\sin(\theta), -\cos(\theta))$ and $d = (\cos(\theta), \sin(\theta))$. We obtain

$$\langle Ad, c \rangle = i \sin(2\theta) \sin(2\alpha)$$

which is generically non-zero. We deduce that the numerical range of $H$ equals $\mathbb{C}$. This fact cannot, however, be discovered simply by finding the eigenvalues. If $\theta \neq 0$ then 0 is not an eigenvalue of $H$. The eigenfunction associated to the eigenvalue $\lambda = z^2$ must be of the form

$$f_1(x) = c_1 \sinh(\frac{xz e^{-i\alpha}}{2})$$
$$f_2(x) = c_2 \cosh(\frac{xz e^{i\alpha}}{2})$$

if it is to satisfy the first two boundary conditions. The existence of a non-zero eigenfunction satisfying the other two boundary conditions then forces $F(z) = 0$ where

$$F(z) := e^{i\alpha} \cos(\theta)^2 \sinh(\pi z e^{-i\alpha}) \sinh(\pi z e^{i\alpha})$$
$$+ e^{-i\alpha} \sin(\theta)^2 \cosh(\pi z e^{-i\alpha}) \cosh(\pi z e^{i\alpha}).$$

This is of the canonical form

$$F(z) = \sum_{r=1}^{4} \delta_r e^{-i\gamma_r}$$

where $\gamma_1 = 2\pi \cos(\alpha)$, $\gamma_2 = 2\pi i \sin(\alpha)$, $\gamma_3 = -2\pi \cos(\alpha)$ and $\gamma_4 = -2\pi i \sin(\alpha)$. The points $\gamma_r$ are the vertices of a rhombus, and the directions of the four outward pointing normals are $\pm ie^{\pm i\alpha}$. For $|\alpha| < \pi/4$ this implies that all except finitely many of the eigenvalues $\lambda = z^2$ lie in the half plane $\mathbb{C}^- = \{z : \text{Re}(z) < 0\}$. We conjecture that all the eigenvalues of $H$ satisfy $\text{Re}(\lambda) < 0$.

7 Semigroup Properties

In spite of the above, there are cases in which one can prove that the operator $H$ is the generator of a contraction semigroup on $L^2((\alpha, \beta), \mathbb{C}^\mu)$ provided this space is given a new equivalent norm. The following two theorems describe the abstract situation.

**Theorem 24** Let $Z$ be a closed operator on the Hilbert space $\mathcal{H}$ and let $A$ be a bounded invertible operator. If $0 \notin \text{Spec}(Z)$ and $AZ$ is accretive then $-AZ$ is the generator of a contraction semigroup and hence $\text{Spec}(AZ) \subseteq \mathbb{C}^+$, the set of complex numbers with non-negative real parts.
Proof The operator $AZ$ is closed and invertible, so there exists $\varepsilon > 0$ such that $z \in \text{Spec}(AZ)$ for all $|z| < \varepsilon$. The theorem now follows by applying [6, Theorem 2.25].

**Theorem 25** Let $Z$ be a closed operator on the Hilbert space $\mathcal{H}$ and let $A$ be a bounded invertible operator. If $Z + \lambda I$ is invertible for some $\lambda > 0$ and $AZ$, $A$ are both accretive, then $-AZ$ is the generator of a contraction semigroup and hence $\text{Spec}(AZ) \subseteq \mathbb{C}^+.$

**Proof** By the hypotheses the operator $A(Z + \lambda I)$ is invertible and accretive. Hence $-A(Z + \lambda I)$ generates a contraction semigroup by Theorem 24. Since $A$ is a bounded perturbation, $-AZ$ generates a strongly continuous semigroup. This must be a contraction semigroup by [6, Theorem 2.27].

We now return to a more concrete context. Let $Z = -d^2/dx^2$ act in $L^2((\alpha, \beta), \mathbb{C}^n)$ subject to the boundary conditions $f(\alpha) \in U$, $f(\beta) \in U$, $f'(\alpha) \in V$, $f'(\beta) \in V$ where $\dim(U) + \dim(V) = n$.

**Theorem 26** If $U \cap V \neq \{0\}$ then $\text{Spec}(Z) = \mathbb{C}$. If $U \cap V = \{0\}$ then $Z$ is similar to a self-adjoint operator with non-negative discrete spectrum.

**Proof** Let $0 \neq c \in (U \cup V)^\perp$. Then for any $z \in \mathbb{C}$ the function

$$g(x) = ce^{zx}$$

is orthogonal to the range of $Z - z^2 I$, so $z^2 \in \text{Spec}(Z)$.

If, on the other hand, $U \cap V = \{0\}$ then we may write $Z$ as the (non-orthogonal) direct sum of its restrictions to $L^2((\alpha, \beta), U)$ and $L^2((\alpha, \beta), V)$. In the first we are imposing Neumann boundary conditions, and in the second Dirichlet boundary conditions, so the spectrum is as stated. Another way to express the same idea is to give $\mathbb{C}^n$ an inner product which makes $U$ and $V$ orthogonal, so that $Z$ becomes self-adjoint with respect to this inner product. The change from one inner product to an equivalent one amounts to the same as a similarity transformation.

**Note 27** In the above theorem $0 \in \text{Spec}(Z)$ unless $U = \{0\}$ and $V = \mathbb{C}^n$, the full Dirichlet case. This is why we need both of Theorems 24 and 25.

**Note 28** If the angle between $U$ and $V$ is very small then the condition number of the similarity transformation is large and it will be difficult to distinguish between the two alternative conclusions of the above theorem. In other words $Z$ will have bad pseudospectral properties.

The following provides a partial converse to Theorem 22. We have not been able to find a similar result if the boundary conditions are different at the two ends of the interval.
Theorem 29 Let $Hf = -Af''$ in $L^2((\alpha, \beta), \mathbb{C}^n)$ where $A$ is a bounded invertible $n \times n$ matrix. We impose boundary conditions $f(\alpha) \in U$, $f(\beta) \in U$, $f'(\alpha) \in V$, $f'(\beta) \in V$ where $U \cap V = \{0\}$ and $U + V = \mathbb{C}^n$. Then if $A$ is accretive and $AV \perp U$, the operator $-H$ generates a contraction semigroup in $L^2((\alpha, \beta), \mathbb{C}^n)$.

Proof. We can apply Theorem 25 with $Zf = -f''$ subject to the above boundary conditions provided we show that $H$ is accretive. If $f \in \text{Dom}(H)$ then integration by parts yields

$$\langle Hf, f \rangle = \int_\alpha^\beta \langle Af', f' \rangle \, dx.$$ 

This has non-negative real part since $A$ is accretive.

The condition $AV \perp U$ depends upon the particular inner product used in $\mathbb{C}^n$ and is not satisfied generically. On the other hand the condition $AV \cap U = \{0\}$ does hold generically. The second condition implies the first for a suitable choice of the inner product on $\mathbb{C}^n$, but one then needs to check whether $A$ is accretive for this new inner product. Assuming this does happen, if the condition number relating the two inner products is $\kappa$ then one only gets

$$\|e^{-Ht}\| \leq \kappa$$

with respect to the standard inner product for all $t \geq 0$. This may be of value if $\kappa$ is fairly small but it is of little computational use if $\kappa$ is sufficiently large.

Example 30 One can prove that $L$ generates a one-parameter group in the follow situation. Let $Lf = Af'$ subject to the quasi-periodic boundary conditions $f(\beta) = Sf(\alpha)$, where $A$ is a diagonal matrix with real, non-zero eigenvalues and $S$ is invertible. It appears that $L$ generates a periodic flow on $[\alpha, \beta]$, but with different flow rates in the different components and mixing at the end points. One can also regard this as a flow in a generally non-trivial vector bundle over the circle. We restrict attention to the simple case $A = I$, but note that Boulton’s ideas allow one to treat the case in which the eigenvalues of $A$ are real and rationally related, 4, 5.

Under the above assumptions we have

$$F(z) = \det \left\{ S - e^{z(\beta-\alpha)}I \right\}.$$ 

Putting $\sigma = e^{z(\beta-\alpha)}$ and finding the eigenvalues of $S$ we obtain $n$ generally distinct solutions $\sigma = \sigma_r$. We conclude that the eigenvalues of $L$ lie on one of $n$ distinct lines parallel to the $y$-axis.

Our next theorem implies that it is only possible to prove that $e^{Lt}$ is a strongly continuous one-parameter group using numerical range ideas if $L$ is already skew-adjoint. In spite of the failure of this method, the subsequent theorem shows that $L$ does generate a one-parameter group whenever the eigenvalues of $A$ are all real.
**Theorem 31** The numerical range of \( L \) is only contained in a strip of the form

\[
\{ z \in \mathbb{C} : c_1 \leq \text{Re} (z) \leq c_2 \}
\]

if \( L \) is already skew adjoint, in other words if \( S \) is unitary.

**Proof** For general functions in \( \text{Dom}(L) \) we have

\[
\text{Re} \left\langle Lf, f \right\rangle = \| f(\beta) \|^2 - \| f(\alpha) \|^2 = \left\langle (S^* S - I)f(\alpha), f(\alpha) \right\rangle.
\]

If \( S \) is not unitary then this may be arbitrarily large (positive or negative or both) for functions in \( \text{Dom}(L) \) of unit norm; the proof is similar to that of Theorem 22. This prevents the numerical range from lying in a strip of the stated kind.

**Theorem 32** For all operators \( L \) defined as in Example 30, \( L \) is the generator of a strongly continuous one-parameter group acting on \( L^2(\alpha, \beta, \mathbb{C}^n) \).

**Proof** We define the bounded invertible operator \( W : L^2 \to L^2 \) by

\[
Wf(x) = S(x)f(x)
\]

where \( S(x) \) is any smooth function with values in the invertible \( n \times n \) matrices satisfying \( S(\alpha) = I \) and \( S(\beta) = S \). Now let \( D \) be the space of functions in \( W^{1,2} \) such that \( f(\alpha) = f(\beta) \) and define \( M : D \to L^2 \) by

\[
Mf = W^{-1}LWf.
\]

A direct calculation shows that

\[
Mf(x) = f'(x) + Y(x)f(x)
\]

where \( Y \) is a bounded smooth matrix-valued function on \( [\alpha, \beta] \). Writing this is the more abstract form \( M = M_0 + Y \) we see that \( M \) is a bounded perturbation of \( M_0 \), which generates a one-parameter group of isometries on \( L^2 \). Hence \( M \), and then \( L \), generate one-parameter groups on \( L^2 \).

8 Inverse Spectral Theory

In [4, 5], Boulton showed that a second order elliptic system \( H \) may have real spectrum without being similar to a self-adjoint operator. In this section we investigate the extent to which one can reconstruct a first order elliptic system from its spectrum. We start with a partial positive result and then give an example which shows that in general one cannot go any further.

We continue in our earlier framework including the generic assumption of Section 3. The first condition of the following theorem is probably unnecessary. We conjecture
that in general the multiplicity of any eigenvalue of \( L \) equals the order of the corresponding zero of \( F(z) \). Note that we have already proved that all zeros of \( F(z) \) of large enough modulus are simple, so the condition only concerns a finite number of smaller eigenvalues.

**Theorem 33** Assume that the zeros of \( F(z) \) are all simple. Then the spectrum of \( L \) determines the function \( F(z) \) up to an exponential factor, and hence the set of constants \( \gamma_r, 1 \leq r \leq R \), up to a common additive factor.

**Proof** Although we do not use this in the proof, we note that by Theorems 5 and 6 the asymptotics of the spectrum of \( L \) determine the length and direction of each edge of \( K \). Since \( K \) is a convex polygon it can only be reconstructed in one way up to translations in \( \mathbb{C} \).

The statement of the theorem is a consequence of the fact that \( F(z) \) is an entire function of order 1, and can therefore be written in the form

\[
F(z) = z^m e^{hz} \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \right\}
\]

for some \( h \in \mathbb{C} \), where \( z_n \) are the zeros of \( F(z) \). See [12, p. 199]. The fact that \( F(z) \) determines all of the \( \gamma_r, 1 \leq r \leq R \), up to the additive factor \( h \) is elementary.

**Note 34** One may break the product in (17) into sporadic terms together with terms associated with \( z_n \) which are close to one of the straight lines already described. The products associated with each line are asymptotically similar to certain formulae involving gamma functions, namely

\[
\frac{e^{(\gamma+\delta)z} \Gamma(a+1)}{\Gamma(a-z+1)} = \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{z}{a+n}\right) e^{z/(a+n)} \right\}
\]

where \( \gamma \) is Euler’s constant and

\[
\delta = \sum_{n=1}^{\infty} \frac{a}{(a+n)n}.
\]

This follows immediately from Whittaker and Watson, [22, p 236].

In certain cases one can go further than Theorem 33. If \( \dim(W) = 1 \) then we showed in Section 2 that \( \gamma_r = (\beta - \alpha)/a_r \) for \( 1 \leq r \leq R \) where \( a_r \) are the eigenvalues of \( A \). If in addition we know the centre of \( K \) and the value of \( \beta - \alpha \), then we can evaluate all of the eigenvalues of \( A \). The following example shows that in general one cannot go any further.

**Example 35** Given \( k > 0 \) let \( u = s + it \) lie on the circle \( \mathcal{C} = \{ u : s^2 + t^2 = kt \} \) and consider again Example 17. We exclude the case \( s = t = 0 \) for obvious reasons; the case \( s = 0, t = \sqrt{k} \) is covered by Example 14. The two vertices of \( K \) are \( \gamma_{\pm} = \pi(v \pm ik) \) where \( v = s/(s^2 + t^2) = s/kt \). We thus see that \( K \) is the same, up
to a translation, for all of these operators as \( u \) varies on \( C \). By Examples 16 and 17 the spectrum of \( L \) is \( k\mathbb{Z} \) for all choices of \( u \) on \( C \).

In order to see that \( L \) cannot be reconstructed from its spectrum up to a similarity transformation, we need to prove the following theorem.

**Theorem 36** No two of the operators of Example 17 are similar as \( u \) varies in \( C \).

**Proof** We use the notation of Theorem 20. If two such operators \( L \) and \( \tilde{L} \) determined by \( u, \tilde{u} \in C \) were similar then it would follow that \( \|P_n\|/\|P_n\| \) would be bounded above and below by positive constants uniformly with respect to \( n \).

Now the proof of Theorem 20 implies that

\[
\|P_n\| \sim \frac{e^{\pi knv}|u|}{2\pi |t|knv}
\]

as \( n \to \infty \). We deduce that if \( L \) and \( \tilde{L} \) are similar then \( v = \tilde{v} \). This implies that \( u = \tilde{u} \).

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