RENNORMALIZED SOLUTIONS TO A REACTION-DIFFUSION SYSTEM APPLIED TO IMAGE DENOISING

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Abstract. This paper concerns the Neumann problem of a reaction-diffusion system with a variable exponent Laplacian term and could be applied to image denoising. It is shown that the problem admits a unique renormalized solution for each integrable initial datum.

1. Introduction. In this paper, we study the existence and uniqueness of the renormalized solution to the following problem

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div} \left( g(x) |\nabla u|^{p(x)-2} \nabla u \right) - 2\lambda w, \quad (x, t) \in Q_T, \\
\frac{\partial w}{\partial t} &= \Delta w - (f(x) - u), \quad (x, t) \in Q_T, \\
\frac{\partial u}{\partial \vec{n}}(x, t) &= \frac{\partial w}{\partial \vec{n}}(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= f(x), \quad w(x, 0) = 0, \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with \( \partial \Omega \) being Lipschitz continuous, \( T > 0 \), \( Q_T = \Omega \times (0, T) \) and \( \vec{n} \) is the unit outward normal vector to \( \partial \Omega \), \( \lambda > 0 \), \( f \in L^1(\Omega) \), while

\[
g(x) = \frac{1}{1 + k_1 |(\nabla G_{\sigma_1} * f)(x)|^2}, \quad p(x) = 1 + \frac{1}{1 + k_2 |(\nabla G_{\sigma_2} * f)(x)|^2}, \quad x \in \Omega
\]

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with $k_1, k_2, \sigma_1, \sigma_2 > 0$ and $G_\sigma$ being the Gaussian kernel, i.e.,

$$G_\sigma(x) = \frac{1}{(4\pi \sigma)^{N/2}} \exp \left( -\frac{|x|^2}{4\sigma} \right), \quad x \in \mathbb{R}^N \quad (\sigma > 0).$$

It is clear that $g, p \in C^\infty(\bar{\Omega})$ and

$$g_1 \leq g(x) \leq g_2, \quad 1 < p(x) < 2, \quad x \in \bar{\Omega}$$

with $g_1, g_2 > 0$. The system (1)–(4) is used to restore the noisy image, where $u$ represents the restoration image describing a real scene (the unknown), $f$ is the observed noisy image and $w$ represents the oscillatory of $u$.

Image restoration is a fundamental problem in both image processing and computer vision with numerous applications. In the last twenty years, nonlinear partial differential equations have become a major method for image restoration. Y. Chen, S. Levine and M. Rao [11] firstly proposed a partial differential equation with variable exponent for image restoration and established the well-posedness of weak solutions. Later, many authors proposed similar equations to refine the models ([1, 17, 18]). Particularly, in [17] we consider the following model for image restoration, which combined the $H^{-1}$ norm for oscillatory functions with the variable exponent regularization

$$\min \tilde{E}_{H^{-1}}(u) = \int_\Omega \frac{g(x)}{p(x)} |\nabla u|^p(x) \, dx + \lambda \|u - f(x)\|_{H^{-1}(\Omega)}^2,$$

where $f \in L^2(\Omega)$. Minimizing $\tilde{E}_{H^{-1}}$ and using gradient descent algorithm, we could formally deduce the Euler-Lagrange equations (1) and (2). In [17] we proved the well-posedness of weak solutions and showed the effectiveness of the model in image restoration by numerical experimental. While in this present paper, we treat the general case $f \in L^1(\Omega)$, i.e., the noisy data is allowed to be an integrable function.

In the mathematical study, for the case $f \in L^2(\Omega)$, one can establish well-posedness of weak solutions to the problem (1)–(4) in a suitable Sobolev space, which satisfy the equations in the distribution sense. However, there may not exist such a weak solution for an integrable datum. Therefore, a different solution should be defined for the problem (1)–(4) with $f \in L^1(\Omega)$. In this paper, we select its renormalized solutions, which were firstly introduced by Di Perna and Lions [13] for the study of Boltzmann equation and then are used to study parabolic equations ([2], [8], [9], [14], [24], [26]) and even parabolic equations with variable exponent Laplacian term ([6, 29]). In practice, in order to avoid nonsensical results such as negative gray levels, ones need to cut off gray levels to reduce the loss of dynamic range in practical computation ([15]). So, renormalized solutions are also more suitable than weak solutions. Compared with previous results, the contribution of our work is to study a reaction-diffusion system with variable exponent Laplacian term and Neumann boundary conditions. So the methods used in [6] and [29] are not suited entirely for our problem. This is worthy mentioning that the similar problem for the case of $p(x) \equiv p$ and Neumann boundary conditions is well studied in [2, 3, 4, 5]. Finally, we consider the regularity with the initial data in $L^r(\Omega)$ for $1 \leq r < 2$, which has been discussed in [25] and [27] for $p$-Laplacian equation and in [6] for $p(x)$-Laplacian equation.

The paper is arranged as follows. In Section 2, we first recall some mathematical preliminaries about variable exponent spaces, and then state the definition of the
renormalized solutions and the main results. Section 3 is devoted to the proof of the main results.

2. Mathematical preliminaries and main results. In this section, we first recall some results on generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ which could be found in [12] and then state the definition of renormalized solutions and the main results on the well-posedness. We always assume $p(x)$ is a continuous function on $\Omega$ satisfying $p^- = \min_{x \in \Omega} p(x) > 1$ and $p^+ = \max_{x \in \Omega} p(x) < +\infty$.

2.1. Generalized Lebesgue-Sobolev spaces. First, denote the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \{ u \text{ is a measurable function in } \Omega : \varrho_{p(x)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \},$$

which is equipped with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf \left\{ \lambda > 0, \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \, dx \leq 1 \right\}.$$

The space $L^{p(x)}(\Omega)$ is also called a generalized Lebesgue space.

Next, define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p(x)}(\Omega), i = 1, \cdots, N \right\},$$

which is a Banach space equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p(x)}}.$$

The function spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ have the following properties:

- Both $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

- For $f \in L^{p(x)}(\Omega)$,

$$\min(\|f\|_{L^{p(-)}(\Omega)}, \|f\|_{L^{p(+)}(\Omega)}) \leq \varrho_{p(x)}(f) \leq \max(\|f\|_{L^{p(-)}(\Omega)}, \|f\|_{L^{p(+)}(\Omega)})�.$$

- Hölder’s inequality. For $f \in L^{p(x)}(\Omega)$ and $g \in L^{q(x)}(\Omega)$ with $q(x) = p(x)/(p(x) - 1)$ in $\Omega$,

$$\int_{\Omega} |fg|dx \leq \left( \frac{1}{p^-} + \frac{1}{q} \right) \|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^{q(x)}(\Omega)} \leq 2 \|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^{q(x)}(\Omega)}.$$

- Particularly, for each $q \in (1, p^-)$,

$$\|f\|_{L^{q}(\Omega)} \leq 2\|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^{q(x)}(\Omega)} \|f\|_{L^{p(x)}(\Omega)}.$$

- Let $q \in C(\bar{\Omega})$ satisfy

$$1 < q(x) \leq \max_{\Omega} q(x) < \min_{\Omega} p_*(x), \quad x \in \Omega$$

with

$$p_*(x) = \begin{cases} \frac{p(x)N}{N - p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) > N. \end{cases}$$

Then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact.
2.2. The definition of renormalized solutions. Recall the following space introduced in [3, 23]. Denote $T_K$ the truncation function at height $K \geq 0$:

$$T_K(r) = \min(K, \max(r, -K)) = \begin{cases} r, & \text{if } |r| \leq K, \\ K \frac{r}{|r|}, & \text{if } |r| > K, \end{cases} \quad r \in \mathbb{R}$$

and define the space

$$\mathcal{T}^{1,p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable, } T_K(u) \in L^{p(\cdot)}(\Omega) \text{ for all } K > 0 \right\}.$$

Similar to the result of Lemma 2.1 in [7] and Proposition 4.1 in [23], we have the following lemma.

**Lemma 2.1.** Let $u \in \mathcal{T}^{1,p(\cdot)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_K(u) = v \chi_{\{|u| < K\}}$, for all $K > 0$. The function $v$ is denoted by $\nabla u$. Moreover if $u \in W^{1,1}_{\text{loc}}(\Omega)$, then $v \in (L^{p(\cdot)}(\Omega))^N$ and $v = \nabla u$ in the usual weak sense.

Note that this definition of derivative is not a definition in the sense of distributions.

Now, we give the definition of renormalized solutions of the problem (1)—(4).

**Definition 2.2.** A pair of functions $(u, w) \in C([0, T], L^1(\Omega))^2$ is called a renormalized solution to the problem (1)—(4) if

$$T_K(u) \in L^p(0, T; W^{1,p(x)}(\Omega)) \quad \text{and} \quad \nabla T_K(u) \in \left( L^{p(x)}(Q_T) \right)^N,$$

$$\lim_{m \to \infty} \int_{\{m \leq |u| \leq m+1\}} |\nabla w|^2 dx dt = 0,$$

and for each $S \in C^\infty(\mathbb{R})$ with $S' \in C_0(\mathbb{R})$, $\varphi \in C^1(Q_T)$ and $\phi \in C^1(Q_T)$, the following integral equalities

$$\int_\Omega S(u)(x, t)\varphi(x, t)dx - \int_\Omega S(0)\varphi(x, 0)dx - \int_0^t \int_\Omega S(u)\frac{\partial \varphi}{\partial \tau} dxdt$$
$$+ \int_0^t \int_\Omega g(x)S'(u)|\nabla u|^{p(x)-2}\nabla u\nabla \varphi + g(x)S''(u)|\nabla u|^{p(x)}\varphi dxdt$$
$$= -2\lambda \int_0^t \int_\Omega wS'(u)\varphi dxdt$$

and

$$\int_\Omega S(w)(x, t)\phi(x, t)dx - \int_\Omega S(0)\phi(x, 0)dx - \int_0^t \int_\Omega S(w)\frac{\partial \phi}{\partial \tau} dxdt$$
$$+ \int_0^t \int_\Omega S'(w)|\nabla w|\nabla \phi + S''(w)|\nabla w|^2\phi dxdt = - \int_0^t \int_\Omega (f - u)S'(w)\phi dxdt,$$

hold a.e. for $t \in (0, T)$.

The main results concerning the renormalized solution to the problem (1)—(4) is as follows.
Theorem 2.3. For $f \in L^1(\Omega)$, there exists a unique renormalized solution to the problem (1)–(4). Furthermore, if $f \in L^r(\Omega)$ with $1 \leq r < 2$ and $p^- > 2 - 1/(N + 1)$, then $|u|^p(x) \in L^1(Q_T)$ and $\nabla u_q(x) \in L^1(Q_T)$ for $q_0, q_1 \in C(\Omega)$ satisfying

$$1 \leq q_0(x) < p(x) \left(1 + \frac{r}{N}\right) - 2 + r, \quad 1 \leq q_1(x) < p(x) - \frac{N(2 - r)}{N + r}, \quad x \in \Omega.$$

3. The proof of the main results. Denote $C$ to be generic constants depending only on $T$, $\lambda$, $|\Omega|$, $g_1$, $g_2$ and $\|f\|_{L^1(\Omega)}$ in this section. They may take different values at different positions. We first prove the existence of renormalized solution to the problem (1)–(4).

For each positive integer $n$, consider the approximation problem:

$$\frac{\partial u_n}{\partial t} = \text{div} \left(g(x)\nabla u_n \abs{\nabla u_n}^{p(x) - 2}\nabla u_n\right) - 2\lambda w_n, \quad (x, t) \in Q_T, \quad (10)$$

$$\frac{\partial w_n}{\partial t} = \Delta w_n - (f_n(x) - u_n), \quad (x, t) \in Q_T, \quad (11)$$

$$\frac{\partial u_n}{\partial n} = \frac{\partial w_n}{\partial n} = 0, \quad (x, t) \in \partial \Omega \times (0, T), \quad (12)$$

$$u_n(x, 0) = f_n(x), \quad w_n(x, 0) = 0, \quad x \in \Omega, \quad (13)$$

where $f_n \in C_0^\infty(\Omega)$ satisfies

$$f_n \rightarrow f \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}.$$

According to the result in [17], there exists a unique weak solution $(u_n, w_n)$ for the problem (10)–(13). Furthermore, based on the methods developed in [28] (See Lemma 2.2.3) or [22] (See Ch 2.1.6) and following the same lines as the proof for the case if $p(x) \equiv p$ in [19] Lemma 3, we get $u_n \in L^\infty(0, T; W^{1, p(x)}(\Omega)) \cap W^{1, 2}(0, T; L^2(\Omega))$ and $w_n \in L^\infty(0, T; W^{1, 2}(\Omega)) \cap W^{1, 2}(0, T; L^2(\Omega))$ such that for each $\phi \in L^\infty(0, T; W^{1, p(x)}(\Omega)) \cap L^2(Q_T)$ and $\varphi \in L^\infty(0, T; W^{1, 2}(\Omega)) \cap W^{1, 2}(0, T; L^2(\Omega))$,

$$\int_Q \left(\frac{\partial u_n}{\partial t} \phi + g(x)\nabla u_n \abs{\nabla u_n}^{p(x) - 2}\nabla u_n \cdot \nabla \phi\right) dx dt = -2\lambda \int_Q w_n \phi dx dt, \quad (14)$$

$$\int_Q \left(\frac{\partial w_n}{\partial t} \varphi + \nabla w_n \cdot \nabla \varphi\right) dx dt = -\int_Q (f_n - u_n) \varphi dx dt, \quad (15)$$

and (13) hold in the trace sense.

First, we give some energy estimates for $u_n$ and $w_n$.

Lemma 3.1. For each $K > 0$,

$$\int_\Omega |u_n| + |w_n| dx \leq C, \quad a.e. \ t \in (0, T), \quad (16)$$

$$\int_Q |\nabla T_K(u_n)|^{p(x)} + |\nabla T_K(w_n)|^2 dx dt \leq C. \quad (17)$$

Proof. For $t \in (0, T)$, choosing $\phi = T_K(u_n)\chi_{[0, t]}$ in (14) leads to

$$\int_Q T_K(u_n)(t) dx + \int_0^t \int_\Omega g(x)|\nabla u_n|^{p(x) - 2}\nabla u_n \nabla T_K(u_n) dx ds$$

$$= -2\lambda \int_0^t \int_\Omega w_n T_K(u_n) dx ds,$$
where
\[
T_K(r) = \int_0^r T_K(s)ds = \begin{cases} 
\frac{r^2}{2}, & \text{if } |r| \leq K, \\
K|r| - K^2, & \text{if } |r| > K,
\end{cases} \quad r \in \mathbb{R}.
\]
So
\[
\int_\Omega T_K(u_n(t))dx + \int_0^t \int_\Omega |\nabla T_K(u_n)|^{p(x)}dxds \leq C \int_0^t \int_\Omega |w_n|dxds. \tag{18}
\]
Similarly, choosing \( \varphi = T_K(w_n)\chi_{[0,t]} \) in (15) yields
\[
\int_\Omega T_K(w_n(t))dx + \int_0^t \int_\Omega |\nabla T_K(w_n)|^2dxds \leq C \left( \int_0^t \int_\Omega |u_n|dxds + 1 \right). \tag{19}
\]
Adding (18) and (19), one gets
\[
\int_\Omega T_K(u_n(t)) + T_K(w_n(t))dx \leq C \left( \int_0^t \int_\Omega |w_n|dxds + 1 \right)\tag{20}.
\]
Hence
\[
\int_\Omega |u_n| + |w_n|dx = \int_\{u_n > K\} \cup \{u_n \leq K\} |u_n|dx + \int_\{u_n > K\} \cup \{u_n \leq K\} |w_n|dx \\
\leq \frac{1}{K} \int_\Omega T_K(u_n(t)) + T_K(w_n(t))dx + 3K|\Omega| \\
\leq C \left( \int_0^t \int_\Omega |w_n| + |u_n|dxds + 1 + 3K^2|\Omega| \right).
\]
Then (16) follows from the Gronwall inequality and (17) follows from (18) and (19).

**Lemma 3.2.** For each positive integer \( m \),
\[
\int_{\{m \leq |u_n(x,t)| \leq m+1\}} g(x)|\nabla u_n|^{p(x)}dxdt \leq -2\lambda \int_{QT} w_n\theta_m(u_n)dxdt, \tag{20}
\]
\[
\int_{\{m \leq |w_n(x,t)| \leq m+1\}} |\nabla w_n|^2dxdt \leq -\int_{QT} (f_n - u_n)\theta_m(w_n)dxdt, \tag{21}
\]
where
\[
\theta_m(r) = T_{m+1}(r) - T_m(r), \quad r \in \mathbb{R}
\]
and
\[
\overline{\theta}_m(r) = \int_0^r \theta_m(s)ds \geq 0, \quad r \in \mathbb{R}.
\]

**Proof.** Choosing \( \phi = \theta_m(u_n) \) in (14) gives
\[
\int_\Omega \overline{\theta}_m(u_n(T))dx + \int_{QT} \theta'_m(u_n)g(x)|\nabla u_n|^{p(x)}dxdt = -2\lambda \int_{QT} w_n\theta_m(u_n)dxdt.
\]
Since \( \theta'_m(r) = 1 \) when \( m \leq |r| \leq m + 1 \), and \( \theta'_m(r) = 0 \) otherwise, one gets (20). Similarly, (21) follows by choosing \( \psi = \theta_m(w_n) \) in (15).

Next, we study the convergence of \( \{u_n\} \) and \( \{w_n\} \) in \( C([0,T]; L^1(\Omega)) \).

**Lemma 3.3.** Both \( \{u_n\} \) and \( \{w_n\} \) are Cauchy sequences in \( C([0,T]; L^1(\Omega)) \).
Proof. Let $n$ and $m$ be two positive integers. It follows from (14) and (15) that
\[
\int_0^t \int_\Omega \frac{\partial (u_n - u_m)}{\partial s} \phi dx ds + \int_0^t \int_\Omega g(x)|\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \phi dx ds = -2\lambda \int_0^t \int_\Omega (w_n - w_m) \phi dx ds
\]
and
\[
\int_0^t \int_\Omega \frac{\partial (w_n - w_m)}{\partial s} \phi dx ds + \int_0^t \int_\Omega (\nabla w_n - \nabla w_m) \nabla \phi dx ds = \int_0^t \int_\Omega (u_n - u_m) \phi dx ds - \int_0^t \int_\Omega (f_n - f_m) \phi dx ds,
\]
where $\phi \in L^\infty(0, T; W^{1,p(x)}(\Omega)) \cap L^2(Q_T)$ and $\varphi \in W^{1,2}(0, T; L^2(\Omega))$. For each $\delta > 0$, taking $\phi = 1/\delta \mathcal{T}_\delta(u_n - u_m)$ in (22) and $\varphi = 1/\delta \mathcal{T}_\delta(w_n - w_m)$ in (23), one gets
\[
\frac{1}{\delta} \int_\Omega \mathcal{T}_\delta(u_n - u_m) + \frac{1}{\delta} \mathcal{T}_\delta(w_n - w_m) dx
\leq C \int_0^t \int_\Omega |u_n - u_m| + |w_n - w_m| dx ds + (T + 1) \int_\Omega |f_n - f_m| dx.
\]
Hence
\[
\int_{\{|u_n - u_m| \geq \delta\}} |u_n - u_m| dx + \int_{\{|w_n - w_m| \geq \delta\}} |w_n - w_m| dx - \delta |\Omega|
\leq C \int_0^t \left[ \int_{\{|u_n - u_m| \geq \delta\}} |u_n - u_m| dx + \int_{\{|w_n - w_m| \geq \delta\}} |w_n - w_m| dx \right] ds
+ 2CT\delta |\Omega| + (T + 1) \int_\Omega |f_n - f_m| dx.
\]
The Gronwall inequality shows
\[
\int_0^t \left[ \int_{\{|u_n - u_m| \geq \delta\}} |u_n - u_m| dx + \int_{\{|w_n - w_m| \geq \delta\}} |w_n - w_m| dx \right] ds
\leq e^{CT}(2CT + 1)T\delta |\Omega| + e^{CT}(T + 1)T \int_\Omega |f_n - f_m| dx.
\]
Letting $n, m \to \infty$ and then $\delta \to 0^+$ in (24), one obtains
\[
\lim_{\delta \to 0^+} \lim_{n, m \to \infty} \left\{ \int_{\{|u_n - u_m| \geq \delta\}} |u_n - u_m| dx + \int_{\{|w_n - w_m| \geq \delta\}} |w_n - w_m| dx \right\} = 0,
\]
which yields
\[
\lim_{n, m \to \infty} \left\{ \int_\Omega |u_n - u_m| dx + \int_\Omega |w_n - w_m| dx \right\} = 0.
\]
By lemma 3.1 and lemma 3.3, we conclude that there exist two subsequences of $\{u_n\}$ and $\{w_n\}$, still denoted by themselves for convenience, such that $u_n$ converges
to a function \( u \) in \( C([0, T]; L^1(\Omega)) \), \( w_n \) converges to a function \( w \) in \( C([0, T]; L^1(\Omega)) \) and

\[
\nabla T_K(u_n) \rightharpoonup \nabla T_K(u) \text{ weakly in } L^{p(x)}(Q_T)^N, \\
|\nabla T_K(u_n)|^{p(x)-2}\nabla T_K(u_n) \nabla \chi_K \text{ weakly in } L^{q(x)}(Q_T)^N, \\
\nabla T_K(w_n) \rightharpoonup \nabla T_K(w) \text{ weakly in } L^2(Q_T)^N,
\]

as \( n \to \infty \) for each \( K > 0 \), where \( q(x) = p(x)/(p(x) - 1) \) in \( \Omega \).

To study the strong convergence of \( \nabla T_K(u_n) \) and \( \nabla T_K(w_n) \), we first list some results about the time regularization of \( T_K(u) \) (for fixed \( K > 0 \)), which have been exploited in [20], [9], [6], [29], etc. For any fixed \( K > 0 \) and \( \mu > 0 \), as the method of [6], we take a sequence of functions defined in \( \Omega \) such that \( v^n_0 \to T_K(f) \) a.e. in \( \Omega \) as \( \mu \to \infty \) with

\[
v^n_0 \in L^\infty(\Omega) \cap W^{1,p(x)}(\Omega), \quad \|v^n_0\|_{L^\infty(\Omega)} \leq K \text{ for all } \mu > 0,
\]

and

\[
\frac{1}{\mu}\|v^n_0\|_{W^{1,p(x)}} \to 0, \quad \text{as } \mu \to \infty.
\]

Let

\[
T_K(u)_\mu = \mu \int_{-\infty}^t e^{\mu(s-t)} T_K(u(x, s)) ds + e^{\mu t} v^n_0
\]

by extending \( T_K(u) \) by 0 for \( s < 0 \). Then \( T_K(u)_\mu \) is a unique solution of the monotone problem:

\[
\left\{ \\
\frac{\partial T_K(u)_\mu}{\partial t} + \mu(T_K(u)_\mu - T_K(u)) = 0, \\
(T_K(u)_\mu)|_{t=0} = v^n_0, \quad \text{in } \Omega.
\right.
\]

with \( T_K(u)_\mu \in L^\infty(Q_T) \) and \( \nabla (T_K(u)_\mu) \in (L^{p(x)}(Q_T))^N \). It is also clear that

\[
\|(T_K(u)_\mu)\|_{L^\infty(Q_T)} \leq K \text{ for each } \mu > 0 \text{ and } K > 0,
\]

(28)

\[
(T_K(u)_\mu) \to T_K(u) \quad \text{a.e. in } Q_T \text{ and weakly-* in } L^\infty(Q_T) \text{ as } \mu \to \infty,
\]

(29)

\[
(\nabla T_K(u)_\mu) \to \nabla T_K(u) \text{ in } L^{p(x)}(Q_T) \text{ as } \mu \to \infty.
\]

(30)

The proof of the following lemma is firstly proved in [9] with constant exponents and Dirichlet boundary cases. To make this paper self-contained, we include a proof in the variable exponents and Neumann boundary cases with the similar method.

**Lemma 3.4.** Let \( K \geq 0 \) be fixed. If \( S \in C^1(\mathbb{R}) \) with \( S' \in C_0(\mathbb{R}) \) is an increasing function satisfying \( S(r) = r \text{ for } |r| \leq K \), then

\[
\lim_{\mu \to 0} \liminf_{n \to \infty} \int_0^T \int_{\Omega} \frac{\partial S(u_n)}{\partial t} (T_K(u_n) - T_K(u)_\mu) dx dt \geq 0.
\]

(31)

**Proof.** Since \( S \) is increasing and \( S(r) = r \text{ for } |r| \leq K \),

\[
T_K(S(u_n)) = T_K(u_n), \quad \text{and } T_K(S(u)) = T_K(u) \quad \text{a.e. in } Q_T.
\]

(32)

As a consequence

\[
(T_K(S(u))_\mu = (T_K(u)_\mu, \quad \text{a.e. in } Q_T,
\]

(33)

for any \( \mu > 0 \). It follows that under the notation

\[
v_n = S(u_n), \quad \text{and } v = S(u),
\]

(34)
we have
\[ I = \int_0^T \int_0^t \int_\Omega \frac{\partial S(u_n)}{\partial \tau} (T_K(u_n) - T_K(u)) \, dx \, d\tau \, dt \]
\[ = \int_0^T \int_0^t \int_\Omega \frac{\partial v_n}{\partial \tau} (v_n - T_K(v)) \, dx \, d\tau \, dt \]
\[ = \int_0^T \int_0^t \int_\Omega \left( \frac{\partial (v_n - (T_K(v)_\mu))}{\partial \tau} (v_n - T_K(v)) \right) \, dx \, d\tau \, dt \]
\[ - \int_0^T \int_0^t \int_\Omega \frac{\partial v_n}{\partial \tau} (v_n - T_K(v)) \, dx \, d\tau \, dt \]
\[ + \int_0^T \int_0^t \int_\Omega \frac{\partial (T_K(v)_\mu)}{\partial \tau} (v_n - T_K(v)) \, dx \, d\tau \, dt. \]

Since
\[ \int_0^T \int_0^t \int_\Omega \frac{\partial T_K(v)_\mu}{\partial \tau} (v_n - T_K(v)) \, dx \, d\tau \, dt = 0, \]
we get
\[ I = \frac{1}{2} \int_0^T \int_\Omega |v_n - T_K(v)_\mu|^2 \, dx \, dt - \frac{T}{2} \int_\Omega |v_n - (T_K(v))_\mu|^2 (t = 0) \, dx \]
\[ - \frac{1}{2} \int_0^T \int_\Omega |v_n - T_K(v)|^2 \, dx \, dt + \frac{T}{2} \int_\Omega |v_n - T_K(v)|^2 (t = 0) \, dx \]
\[ + \int_0^T \int_0^t \int_\Omega \frac{\partial (T_K(v)_\mu)}{\partial \tau} (v_n - T_K(v)) \, dx \, d\tau \, dt. \]

Since $S$ is bounded, we have
\[ v_n \to v \quad \text{strongly in } L^2(Q) \quad \text{and in } L^\infty \text{ weak-* as } n \to \infty. \]

And since
\[ v_n(t = 0) = S(u_n)(t = 0) = S(u_0) \quad \text{a.e. in } Q_T, \]
the strong convergence of $u_n$ to $u_0$ in $L^1(\Omega)$ implies that
\[ v_n(t = 0) \to S(u_0) \quad \text{in } L^2(\Omega) \text{ as } n \to \infty. \]

Then
\[ \lim_{n \to \infty} I = \frac{1}{2} \int_0^T \int_\Omega |v - T_K(v)_\mu|^2 \, dx \, dt - \frac{T}{2} \int_\Omega |S(u_0) - (T_K(v))_\mu|^2 (t = 0) \, dx \]
\[ - \frac{1}{2} \int_0^T \int_\Omega |v - T_K(v)|^2 \, dx \, dt + \frac{T}{2} \int_\Omega |S(u_0) - T_K(S(u_0))|^2 (t = 0) \, dx \]
\[ + \int_0^T \int_0^t \int_\Omega \frac{\partial (T_K(v)_\mu)}{\partial \tau} (v - T_K(v)) \, dx \, d\tau \, dt. \]  

(35)

Since $T_K(S(u_0)) = T_K(u_0)$ a.e. in $\Omega$, we have
\[ T_K(v)_\mu \to T_K(v) \quad \text{in } L^2(Q_T) \]
and
\[ T_K(v)_\mu(t = 0) \to T_K(S(u_0)) \quad \text{in } L^2(Q_T). \]
as $\mu \to \infty$. So by (35), it follows that
\[ \liminf_{\mu \to \infty} \lim_{n \to \infty} I = \liminf_{\mu \to \infty} \int_0^T \int_0^t \int_\Omega \frac{\partial (T_K(v)_\mu)}{\partial \tau} (v - T_K(v)_\mu) \, dx \, d\tau \, dt. \]
By (32)–(34), we have
\[ \frac{\partial T_K(v)_\mu}{\partial t} = \mu(T_K(v) - T_K(v)_\mu) \]
and then
\[ \lim_{m \to \infty} \liminf_{n \to \infty} I = \liminf_{m \to \infty} \int_0^T \int_\Omega (T_K(v) - T_K(v)_\mu)(v - T_K(v)_\mu) \, dx \, d\tau \, dt \geq 0. \]

\[ \Box \]

**Lemma 3.5.** \( \nabla T_K(u_n) \) converges to \( \nabla T_K(u) \) in \( (L^p(x)(Q_T))^N \), and \( \nabla T_K(w_n) \) converges to \( \nabla T_K(w) \) in \( (L^2(Q_T))^N \).

**Proof.** For positive integer \( m \), choose \( S_m \in C^\infty(\mathbb{R}) \) such that \( S_m(r) = r \) for \( |r| \leq m \), supp\( S_m' \subset [-m+1, (m+1)] \) and \( \|S_m''\|_{L^\infty(\mathbb{R})} \leq 1 \). Denote \( W^m_{\mu} = T_K(u_n) - (T_K(u))_\mu \). Choosing \( \phi = S_m'(u_n)W^m_{\mu} \) in (14) leads to
\[ \int_0^T \int_\Omega \frac{\partial S_m(u_n)}{\partial t} W^m_{\mu} \, dx \, d\tau \]
\[ + \int_0^T \int_\Omega g(x)|\nabla u_n|^{p(x)-2}\nabla u_n S_m'(u_n) \nabla W^m_{\mu} \, dx \, d\tau \]
\[ + \int_0^T \int_\Omega g(x)|\nabla u_n|^{p(x)} S_m''(u_n) W^m_{\mu} \, dx \, d\tau \]
\[ + 2\lambda \int_0^T \int_\Omega w_n S_m'(u_n) W^m_{\mu} \, dx \, d\tau = 0. \]  
(36)

Owing to supp\( S_m'' \subset [-m+1, -m] \cap [m, m+1] \),
\[ \left| \int_0^T \int_\Omega g(x)|\nabla u_n|^{p(x)} S_m''(u_n) W^m_{\mu} \, dx \, d\tau \right| \]
\[ \leq T\|S_m''\|_{L^\infty(\mathbb{R})}\|W^m_{\mu}\|_{L^\infty(Q_T)} \int_{\{m\leq|u_n(x,t)|\leq m+1\}} g(x)|\nabla u_n|^{p(x)} \, dx. \]

Due to \( \|S_m''\|_{L^\infty(\mathbb{R})} \leq 1 \), (28)–(30) and (20), one gets
\[ \lim_{m \to \infty} \limsup_{\mu \to +\infty} \limsup_{n \to \infty} \left| \int_0^T \int_\Omega g(x)|\nabla u_n|^{p(x)} S_m''(u_n) W^m_{\mu} \, dx \, d\tau \right| \]
\[ \leq C \lim_{m \to \infty} \limsup_{n \to \infty} \int_{\{m\leq|u_n(x,t)|\leq m+1\}} g(x)|\nabla u_n|^{p(x)} \, dx \]
\[ \leq C \lim_{m \to \infty} \limsup_{n \to \infty} \left\{ -2\lambda \int_{Q_T} w_n \theta_m(u_n) \, dx \right\} \]
\[ = 0. \]  
(37)

The dominated convergence theorem gives
\[ \lim_{\mu \to 0} \lim_{n \to \infty} 2\lambda \int_0^T \int_\Omega w_n S_m'(u_n) W^m_{\mu} \, dx \, d\tau = 0. \]  
(38)

Combining (31) and (36)–(38) shows
\[ \lim_{m \to \infty} \limsup_{\mu \to \infty} \limsup_{n \to \infty} \int_0^T \int_\Omega g(x)|\nabla u_n|^{p(x)-2}\nabla u_n S_m'(u_n) \nabla W^m_{\mu} \, dx \, d\tau \leq 0. \]
Due to $S_m(r) = 1$ for $|r| \leq m$, one gets
\[ \lim_{n \to \infty} \sup_{t} \int_0^T \int_{\Omega} g(x) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla T_K(u_n) \, dx \, dt \leq \lim_{m \to \infty} \lim_{n \to \infty} \sup_{t} \int_0^T \int_{\Omega} g(x) |\nabla u_n|^{p(x)-2} \nabla u_n S'_m(u_n) \nabla (T_K(u))_\mu \, dx \, dt. \] (39)

Noting
\[ g(x) |\nabla u_n|^{p(x)-2} \nabla u_n S'_m(u_n) = g(x) |\nabla T_{m+1}(u_n)|^{p(x)-2} \nabla T_{m+1}(u_n) S'_m(u_n) \]
a.e. in $Q_T$, one obtains
\[ g(x) |\nabla u_n|^{p(x)-2} \nabla u_n S'_m(u_n) \to g(x) \chi_{m+1} S'_m(u) \]
weakly in $L^{q(x)}(Q_T)^N$ as $n \to \infty$. Owing to $S'_m(r) = 1$ for $|r| \leq m$, one gets that for any $m \geq K$,
\[ \lim_{\mu \to \infty} \lim_{n \to \infty} \int_0^T \int_{\Omega} g(x) |\nabla u_n|^{p(x)-2} \nabla u_n S'_m(u_n) \nabla (T_K(u))_\mu \, dx \, dt = \int_0^T \int_{\Omega} g(x) \chi_{m+1} S'_m(u) \nabla T_K(u) \, dx \, dt. \] (40)

Note that for $m \geq K$,
\[ |\nabla T_{m+1}(u_n)|^{p(x)-2} \nabla T_{m+1}(u_n) = |\nabla T_K(u_n)|^{p(x)-2} \nabla T_K(u_n), \]
a.e. $(x, t) \in Q_T \cap \{|u_n| < K\}$.

Thus for $m \geq K$,
\[ \chi_{m+1} \nabla T_K(u) = \chi_K \nabla T_K(u) \] a.e. $(x, t) \in Q_T$.

(41)

Combining (39)–(41) leads to
\[ \lim_{n \to \infty} \sup_{t} \int_0^T \int_{\Omega} g(x) |\nabla u_n|^{p(x)-2} \nabla u_n \nabla T_K(u_n) \, dx \, dt \leq \int_0^T \int_{\Omega} g(x) \chi_K \nabla T_K(u) \, dx \, dt. \]

Then
\[ \lim_{n \to \infty} \sup_{t} \int_0^T \int_{\Omega} g(x) \left( |\nabla T_K(u_n)|^{p(x)-2} \nabla T_K(u_n) - |\nabla T_K(u)|^{p(x)-2} \nabla T_K(u) \right) \cdot \left( \nabla T_K(u_n) - \nabla T_K(u) \right) \, dx \, dt = 0. \]

Similarly to Lemma 5 in [10], one could get that $\nabla T_K(u_n)$ converges to $\nabla T_K(u)$ a.e. in $(Q_T)$ and then in $(L^{p(x)}(Q_T)^N)$ for any $T' < T$. By extending the functions to a large interval (see [9]), it is then sufficient to prove that $\nabla T_K(u_n)$ converges to $\nabla T_K(u)$ in $(L^{p(x)}(Q_T)^N)$. 

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Turn to the convergence of $\nabla T_K(w_n)$. Denote $V_\mu^n = T_K(w_n) - (T_K(w))_\mu$ and choose $\psi = S'_m(w_n)V_\mu^n$ in (15). One can get that
\[
\int_0^T \int_0^T \int_0^T \frac{\partial S_m(w_n)}{\partial \tau} V_\mu^n d\sigma d\tau dt + \int_0^T \int_0^T \nabla w_n S'_m(w_n) \nabla V_\mu^n d\sigma d\tau dt \\
+ \int_0^T \int_0^T \int_0^T \nabla w_n^2 S''_m(w_n) V_\mu^n d\sigma d\tau dt \\
= - \int_0^T \int_0^T \int_0^T (f_n - u_n) S'_m(w_n) V_\mu^n d\sigma d\tau dt.
\]
By a similar discussion, one has
\[
\liminf_{\mu \to \infty} \limsup_{n \to \infty} \int_0^T \int_0^T \frac{\partial S_m(w_n)}{\partial \tau} V_\mu^n d\sigma d\tau dt \geq 0,
\]
\[
\limsup_{\mu \to \infty} \limsup_{n \to \infty} \int_0^T \int_0^T \nabla w_n^2 S''_m(w_n) V_\mu^n d\sigma d\tau dt = 0
\]
and
\[
\limsup_{\mu \to \infty} \limsup_{n \to \infty} \int_0^T \int_0^T (f_n - u_n) S'_m(w_n) V_\mu^n d\sigma d\tau dt = 0.
\]
Hence
\[
\limsup_{\mu \to \infty} \limsup_{n \to \infty} \int_0^T \int_0^T \nabla w_n S'_m(w_n) \nabla V_\mu^n d\sigma d\tau dt \leq 0.
\]
Similarly, one gets that $\nabla T_K(w_n)$ converges to $\nabla T_K(w)$ in $(L^2(Q_T))^N$. \qed

**Proof of the existence.** Let us verify that $(u, w)$ satisfies the definition of the weak solution. Owing to Lemma 3.3, (25)–(27), (5) and (6) hold. For fixed positive integer $m$,
\[
\iint_{\{m \leq |u_n| \leq m+1\}} g(x)|\nabla u_n|^{p(x)} dxdt \\
= \iint_{Q_T} g(x)|\nabla u_n|^{p(x)} - 2 \nabla u_n (\nabla T_{m+1}(u_n) - \nabla T_m(u_n)) dxdt \\
= \iint_{Q_T} g(x)|\nabla T_{m+1}(u_n)|^{p(x)} dxdt - \iint_{Q_T} g(x)|\nabla T_m(u_n)|^{p(x)} dxdt.
\]
Letting $n \to \infty$ and then $m \to \infty$, it follows from Lemma 3.2 that
\[
\lim_{m \to \infty} \iint_{\{m \leq |u| \leq m+1\}} g(x)|\nabla u|^{p(x)} dxdt = 0.
\]
Similarly, one can prove
\[
\lim_{m \to \infty} \iint_{\{m \leq |w| \leq m+1\}} |\nabla w|^2 dxdt = 0.
\]
For $S \in C^\infty(\mathbb{R})$ with $S' \in C_0(\mathbb{R})$, multiplying (10) by $S'(u_n)$ and (11) by $S'(w_n)$, one gets
\[
\frac{\partial S(u_n)}{\partial t} - \text{div}(g(x)S'(u_n)|\nabla u_n|^{p(x)} - 2 \nabla u_n) \\
+ g(x)S''(u_n)|\nabla u_n|^{p(x)} = -2\lambda w_n S'(u_n)
\]
(42)
Using (44)–(51), one gets (8) and (9) by letting $
abla w_n = (f_n - u_n)S'(w_n)$. (43)

Due to Lemma 3.3, as $n \to \infty$, we have $(S(u_n), S(w_n))$ converges to $(S(u), S(w))$ in $C([0,T]; \Omega)$ and weakly−* in $L^\infty(Q_T)$ and

\[ \int_\Omega S(u_n)(x,t)\varphi(x,t)dx - \int_\Omega S(u)(x,t)\varphi(x,t)dx \to \int_\Omega S(u_n)(x,t)\varphi(x,t)dx - \int_\Omega S(u)(x,t)\varphi(x,t)dx \]

for any $\varphi \in C^1(Q_T)$, $\phi \in C^1(Q_T)$, a.e. for $t \in (0,T)$. Assume that $S' \subset [-K, K]$. Note

\[ S'(u_n)|\nabla u_n|^{\frac{p(x)}{2}}\nabla u_n = S'(u_n)|\nabla T_K(u_n)|^{\frac{p(x)}{2}}\nabla T_K(u_n) \] a.e. in $Q_T$

and

\[ S'(w_n)\nabla w_n = S'(w_n)\nabla T_K(w_n) \] a.e. in $Q_T$.

Owing to Lemma 3.3 and Lemma 3.5, one gets

\[ S'(u_n)|\nabla u_n|^{\frac{p(x)}{2}}\nabla u_n \to S'(u)|\nabla u|^{\frac{p(x)}{2}}\nabla u \] weakly in $L^{\frac{q(x)}{2}}(Q_T)$ (46)

and

\[ S'(w_n)\nabla w_n \to S'(w)\nabla w \] weakly in $L^2(Q_T)$. (47)

It follows from Lemma 3.5 that

\[ S''(u_n)|\nabla u_n|^{\frac{p(x)}{2}} \to S''(u)|\nabla u|^{\frac{p(x)}{2}} \] weakly in $L^1(Q_T)$ (48)

and

\[ S''(w_n)|\nabla w_n|^2 \to S''(w)|\nabla w|^2 \] weakly in $L^1(Q_T)$. (49)

Lemma 3.3 shows that

\[ -2\lambda w_nS'(u_n) \to -2\lambda wS'(u) \] weakly in $L^1(Q_T)$ (50)

and

\[ -(f_n - u_n)S'(w_n) \to -(f - u)S'(w) \] weakly in $L^1(Q_T)$. (51)

Using (44)–(51), one gets (8) and (9) by letting $n \to \infty$ in (42) and (43).

Now, we shall give the proof of the uniqueness result of Theorem 2.3. We first give the following integration by parts lemma, which is slight modification of Lemma 4.1 in [5].
Lemma 3.6. Let \((u, w)\) be the renormalized solution of the problem (1)–(4). Then for any continuously differentiable nondecreasing functions \(G(r)\) and \(H(r)\), we have
\[
\begin{align*}
\int_{\Omega} \tilde{G}(S(u))(x, t)dx &- \int_{\Omega} \tilde{G}(S(u(t)))dx \\
&+ \int_{0}^{t} \int_{\Omega} g(x)S'(u)|\nabla u|^p|z(x, t)|dxdt \\
&+ \int_{0}^{t} \int_{\Omega} g(x)S''(u)|\nabla u|^p(z(x, t))dxdt = -2\lambda \int_{0}^{t} \int_{\Omega} wS'(u)G(S(u))dxdt
\end{align*}
\]
and
\[
\begin{align*}
\int_{\Omega} \tilde{H}(S(w)(x, t))dx &- \int_{\Omega} \tilde{G}(S(0))dx + \int_{0}^{t} \int_{\Omega} S'(w)|\nabla w|^2H(S(w))dxdt \\
&+ \int_{0}^{t} \int_{\Omega} S''(w)|\nabla w|^2H(S(w))dxdt = -\int_{0}^{t} \int_{\Omega} (f - u)S'(w)H(S(w))dxdt,
\end{align*}
\]
where \(\tilde{G}(s) = \int_{0}^{s} G(r)dr\) and \(\tilde{H}(s) = \int_{0}^{s} H(r)dr\).

Based on the method in [9], for \(s > 0\) and \(\sigma > 0\), define \(S^\sigma_s \in W^{2,\infty}(\mathbb{R})\) by
\[
S^\sigma_s (r) = \begin{cases} r, & \text{if } |r| < s, \\ (|r| - \frac{(|r| - s)^2}{2\sigma}) \frac{r}{|r|}, & \text{if } s \leq |r| \leq s + \sigma, \quad r \in \mathbb{R}. \end{cases}
\]
(52)

It is obvious that
\[
(S^\sigma_s)'(r) = \begin{cases} 1, & \text{for } |r| < s, \\ \frac{s + \sigma - |r|}{\sigma}, & \text{for } s \leq |r| \leq s + \sigma, \quad r \in \mathbb{R}. \end{cases}
\]
(53)

Then the following technical lemma for renormalized solutions can be proved.

Lemma 3.7. Let \((u, w)\) be a renormalized solution to the problem (1)–(4). Then for \(0 < \sigma < 1\),
\[
\lim_{s \to +\infty} \frac{1}{\sigma} \int_{\{s - \sigma < |u| < s + \sigma\}} |\nabla u|^p dxdt = 0,
\]
(54)
\[
\lim_{s \to +\infty} \frac{1}{\sigma} \int_{\{s - \sigma < |w| < s + \sigma\}} |\nabla w|^2 dxdt = 0.
\]
(55)

Proof. For \(s > 1\) and \(0 < \sigma < 1\), set
\[
R^\sigma_s (r) = \frac{1}{\sigma} (T_{s+\sigma}(r) - T_{s-\sigma}(r)), \quad r \in \mathbb{R}.
\]
Choosing \(S = S^1_{s+1}\) in (8) and \(G(r) = R^\sigma_s (t)dt\), noticing that since \(0 < \sigma < 1\), \(R^\sigma_s (u) = R^\sigma_s (S^1_{s+1}(u))\), one gets
\[
\int_{\Omega} \tilde{R}^\sigma_s (S^1_{s+1}(u))(T)dx - \int_{\Omega} \tilde{R}^\sigma_s (S^1_{s+1}(f))dx
\]
Due to supp$(G) = \int_{0}^{\infty} R_{s}^{\sigma}(r) dt \geq 0$. It is obvious that

\[
\int_{\Omega} \tilde{R}_{s}^{\sigma}(S_{s+1}^{1}(u))(T) dx - \int_{\Omega} \tilde{R}_{s}^{\sigma}(S_{s+1}^{1}(f)) dx \geq -3 \int_{\Omega} |f| \chi_{\{|f|>s-1\}} dx.
\]  
Due to supp$(R_{s}^{\sigma}) \subset [-s - \sigma, -s + \sigma] \cup [s - \sigma, s + \sigma]$,

\[
\frac{2g_{1}}{\sigma} \int \{s-\sigma \leq |u| \leq s+\sigma\} |\nabla u|^{p(x)} dx dt
\leq \int_{\Omega} g(x)(S_{s+1}^{1})^{\prime}(u) |\nabla u|^{p(x)-2} \nabla u \nabla R_{s}^{\sigma} dx dt.
\]

Owing to supp$(S_{s+1}^{1})^{\prime\prime} \subset [-s - 1, -s + 1] \cup [s - 1, s + 1]$,

\[
\int_{\Omega} g(x)(S_{s+1}^{1})^{\prime\prime}(u) |\nabla u|^{p(x)} R_{s}^{\sigma} dx dt \leq 2g_{2} \int_{\{s \leq |u| \leq s+1\}} |\nabla u|^{p(x)} dx dt.
\]

It follows from (56)–(59) that

\[
\frac{2g_{1}}{\sigma} \int \{s-\sigma \leq |u| \leq s+\sigma\} |\nabla u|^{p(x)} dx dt
\leq 2g_{2} \int \{s \leq |u| \leq s+1\} |\nabla u|^{p(x)} dx dt + 2 \int \{|u| \geq s-1\} |w| dx dt + 3 \int_{\Omega} |f| \chi_{\{|f|>s-1\}} dx,
\]

which, together with (7) and $w \in L^{1}(Q_{T})$, leads to (54). Similarly, one can get (55).

\[\text{Proof of the uniqueness.} \]

Let $(u_{1}, w_{1})$ and $(u_{2}, w_{2})$ be two renormalized solutions of the problem (1)–(4). For $\delta > 0$ and $s > 1$, we take $S = S_{s}^{\sigma}$ in (8). Similar to the proof of Lemma 3.6, we have

\[
\frac{1}{\delta} \int_{Q_{T}} \tilde{R}_{\delta}(S_{s}^{\sigma}(u_{1}) - S_{s}^{\sigma}(u_{2})) = A_{s, \delta}^{\sigma} + B_{s, \delta}^{\sigma} + C_{s, \delta}^{\sigma},
\]

where

\[
A_{s, \delta}^{\sigma} = -\frac{1}{\delta} \int_{0}^{T} \int_{0}^{t} g(x)(S_{s}^{\sigma})^{\prime}(u_{1}) |\nabla u_{1}|^{p(x)-2} \nabla u_{1} - (S_{s}^{\sigma})^{\prime}(u_{2}) |\nabla u_{2}|^{p(x)-2} \nabla u_{2} \]
\[
\cdot \nabla T_{\delta}(S_{s}^{\sigma}(u_{1}) - S_{s}^{\sigma}(u_{2})) dx dt,
\]

\[
B_{s, \delta}^{\sigma} = -\frac{1}{\delta} \int_{0}^{T} \int_{0}^{t} g(x)(S_{s}^{\sigma})^{\prime\prime}(u_{1}) |\nabla u_{1}|^{p(x)} - (S_{s}^{\sigma})^{\prime\prime}(u_{2}) |\nabla u_{2}|^{p(x)} \]
\[
\cdot T_{\delta}(S_{s}^{\sigma}(u_{1}) - S_{s}^{\sigma}(u_{2})) dx dt,
\]

\[
C_{s, \delta}^{\sigma} = -\frac{2\lambda}{\delta} \int_{0}^{T} \int_{0}^{t} [w_{1}(S_{s}^{\sigma})^{\prime}(u_{1}) - w_{2}(S_{s}^{\sigma})^{\prime}(u_{2})] T_{\delta}(S_{s}^{\sigma}(u_{1}) - S_{s}^{\sigma}(u_{2})) dx dt,
\]

since $S_{s}^{\sigma}(u_{i})(t=0) = S_{s}^{\sigma}(u_{i})(t=0) = S_{s}^{\sigma}(f)$.

Due to (52) and (53), it follows by letting $\sigma \to 0^{+}$ that

\[
S_{s}^{\sigma}(u_{i}) \to T_{\delta}(u_{i}) \quad \text{a.e. in } Q_{T} \text{ and in } L^{p(x)}(0, T; W^{1,p(x)}(\Omega))
\]
and
\[(S^\sigma)^{i}(u_1) \to \chi_{\{|u_1| \leq s\}} \text{ a.e. in } Q_T \text{ and in } L^q(Q_T) \text{ for any } q < +\infty, \] for \(i = 1, 2.\) Then
\[
\lim_{\sigma \to 0^+} A^\sigma_{s,\delta} = -\frac{1}{\delta} \int_0^T \int_0^t \int_\Omega g(x) \left| \nabla T_{s}(u_1) \right|^2 \nabla T_{s}(u_1) - \left| \nabla T_{s}(u_2) \right|^2 \nabla T_{s}(u_2) \left| \nabla T_{\delta}(T_{s}(u_1) - T_{s}(u_2)) \right| dx d\tau dt \\
\leq 0.
\] (63)\)

Owing to \(T_{\delta} \geq 0\) and \(\text{supp}(S^\sigma_{s})'' \subset [-s + \sigma, -\infty) \cup [s, s + \sigma],\)
\[
|B^\sigma_{s,\delta}| \leq \frac{T}{\sigma} \left[ \int \int_{\{s - \sigma \leq |u_1| \leq s + \sigma\}} \left| \nabla u_1 \right|^2 dx dt + \int \int_{\{s - \sigma \leq |u_2| \leq s + \sigma\}} \left| \nabla u_2 \right|^2 dx dt \right].
\] (64)\)

It follows from (62) that
\[
\lim_{\sigma \to 0^+} |C_{s,\delta}| \leq 2\lambda \int_0^T \int_0^t \int_\Omega |w_1 \chi_{\{|u_1| \leq s\}} - w_2 \chi_{\{|u_2| \leq s\}}| dx d\tau dt.
\] (65)\)

Letting \(\sigma \to 0^+\) and then \(\delta \to 0^+\) to take superior limit in (60), one gets from (54), (60), (61) and (63)–(65) that
\[
\int \int_{Q_T} |T_{s}(u_1) - T_{s}(u_2)| dx dt \leq 2\lambda \int_0^T \int_0^t \int_\Omega |w_1 \chi_{\{|u_1| \leq s\}} - w_2 \chi_{\{|u_2| \leq s\}}| dx d\tau dt.
\]

Letting \(s \to +\infty\) to take inferior limit in (60) leads to
\[
\int \int_{Q_T} |u_1 - u_2| dx dt \leq 2\lambda \int_0^T \int_0^t \int_\Omega |w_1 - w_2| dx d\tau dt.
\] (66)\)

For \(\delta > 0\) and \(s > 1,\) let \(S = S^\sigma_{s}\) in (9). Similar to the previous proof, we have
\[
\frac{1}{\delta} \int \int_{Q_T} T_{\delta}(S^\sigma_{s}(w_1) - S^\sigma_{s}(w_2)) = D^\sigma_{s,\delta} + E^\sigma_{s,\delta} + F^\sigma_{s,\delta} + G^\sigma_{s,\delta},
\] (67)\)

where
\[
D^\sigma_{s,\delta} = -\frac{1}{\delta} \int_0^T \int_0^t \int_\Omega \left( (S^\sigma_{s})' (w_1) \nabla w_1 - (S^\sigma_{s})' (w_2) \nabla w_2 \right) \cdot \nabla T_{\delta}(S^\sigma_{s}(w_1) - S^\sigma_{s}(w_2)) dx d\tau dt,
\]
\[
E^\sigma_{s,\delta} = -\frac{1}{\delta} \int_0^T \int_0^t \int_\Omega \left( (S^\sigma_{s})'' (w_1) \nabla w_1 |^2 - (S^\sigma_{s})'' (w_2) |\nabla w_2|^2 \right) \cdot T_{\delta}(S^\sigma_{s}(w_1) - S^\sigma_{s}(w_2)) dx d\tau dt,
\]
\[
F^\sigma_{s,\delta} = \frac{1}{\delta} \int_0^T \int_0^t \int_\Omega \left( u_1 (S^\sigma_{s})' (w_1) - u_2 (S^\sigma_{s})' (w_2) \right) T_{\delta}(S^\sigma_{s}(w_1) - S^\sigma_{s}(w_2)) dx d\tau dt,
\]
\[
G^\sigma_{s,\delta} = \frac{1}{\delta} \int_0^T \int_0^t \int_\Omega f(S^\sigma_{s})' (w_1) - (S^\sigma_{s})' (w_2) \right) T_{\delta}(S^\sigma_{s}(w_1) - S^\sigma_{s}(w_2)) dx d\tau dt.
\]

Similarly, letting \(\sigma \to 0^+\) and then \(\delta \to 0^+\) to take superior limit, and then letting \(s \to +\infty\) to take inferior limit in (67), one gets
\[
\int \int_{Q_T} |w_1 - w_2| dx dt \leq \int_0^T \int_0^t \int_\Omega |u_1 - u_2| dx d\tau dt.
\] (68)
Combining (66) and (68) leads to
\[ \int \int_{Q_T} |u_1 - u_2| + |w_1 - w_2| dx dt \leq (2\lambda + 1) \int_0^T \int_\Omega |u_1 - u_2| + |w_1 - w_2| dx dt. \]
Then, the uniqueness result follows from the Gronwall inequality. \( \square \)

Turn to the regularity of the renormalized solution to the problem (1)–(4) under the assumption that \( f \in L^r(\Omega), 1 \leq r < 2 \) and \( p^- > 2 - 1/(N + 1) \). Since we prove the existence of the renormalized solutions by an approximation technique, we can assume additionally that the solution is smooth enough. So by the standard theory of parabolic equations [21], one has

**Lemma 3.8.** If \( u \in L^r(Q_T), f \in L^r(\Omega) \) and \( w \) is the solution to (2), then
\[ \|w\|_{L^r(Q_T)} \leq C(\|u\|_{L^r(Q_T)} + \|f\|_{L^r(\Omega)}). \]

Furthermore, we prove

**Lemma 3.9.** If \( f \in L^r(\Omega) \) and \( u \) is the solution to (1), then
\[ \|u\|_{L^\infty(0,T;L^r(\Omega))} + \int \int_{Q_T} \frac{\|\nabla u\|^p(x)}{|u|^{2-r} + 1} dx dt \leq C. \]

**Proof.** Taking \( \varphi = (|u| + 1)^{r-1}\text{sign}(u) \) in (1) yields
\[ \frac{d}{dt} \int_\Omega (|u| + 1)^r dx + \int_\Omega (r - 1)|\nabla u|^p dx \leq 2\lambda \int_\Omega |w||(u| + 1)^{r-1} dx. \]
Integrating over \((0,t)\) and Using the Hölder inequality, one gets
\[ \int_\Omega (|u| + 1)^r dx + \int_\int_{Q_T} \frac{|\nabla u|^p}{(|u| + 1)^{2-r}} dx dt \leq C \left( \int_\Omega |w|^r dx \right)^{\frac{r}{r-(r-1)}} \left( \int_\int_{Q_T} (|u| + 1)^r dx dt \right)^{\frac{1}{r}} + C \int_\Omega (|f| + 1)^r dx \]
\[ \leq C \int_\int_{Q_T} (|u| + 1)^r dx dt + \int_\Omega |f|^r dx. \]
Then, the lemma is proved by the Gronwall inequality. \( \square \)

**Proof of the regularity.** Take \( \alpha, \beta \in \bar{C}(\Omega) \) such that
\[ 1 \leq \alpha(x) < q_0(x), \quad 1 \leq \beta(x) < q_1(x), \quad x \in \Omega. \]
Since \( \Omega \) is bounded, \( \varepsilon = \min_{x \in \Omega} \{q_0(x) - \alpha(x), q_1(x) - \beta(x)\} > 0 \), and there exists \( \delta > 0 \) such that for each \( x, y \in \Omega \), if \( |y - x| \leq \delta \), then
\[ |p(y) - p(x)| + |\alpha(y) - \alpha(x)| + |\beta(y) - \beta(x)| < \frac{\varepsilon}{3}. \]
Note that \( \bar{\Omega} \) can be covered by a finite number of balls \( \{B_i\}_{i=1,2,...,I} \) with \( B_i = \{y| |y - x_i| < \delta \} \)
\[ \text{meas}(B_i \cap \Omega) > d_i, \quad i = 1, 2, \ldots, I, \]
where \( d_i \) is a small positive number. Set
\[ B_i = B_i \cap \Omega, \quad Q_i = B_i \times (0,T), \quad i = 1, 2, \ldots, I. \]
Define $p_i^-$ and $p_i^+$ to be the minimum and the maximum of $p$ on $\overline{B}_i$, respectively. Moreover, $\alpha_i^+, \alpha_i^-, \beta_i^-$ and $\beta_i^+$ are defined similarly. Then

$$\alpha_i^+ = \max_{y \in \mathcal{B}_i} \alpha(y) < \alpha(x_i) + \frac{\epsilon}{3}$$

$$< q_0(x_i) - \frac{\epsilon}{3} < \min_{y \in \mathcal{B}_i} q_0(y) = p_i^- \left(1 + \frac{r}{N}\right) - 2 + r.$$ 

Similarly,

$$\beta_i^+ = \min_{y \in \mathcal{B}_i} q_i(y) = p_i^- - \frac{N(2 - r)}{N + r}.$$ 

Let $p_i^* = Np_i^-/(N - p_i^-)$ and $\sigma_i = p_i^*(1 + r/N)$. For sufficiently large $K > 0$, it follows from the interpolation in Lebesgue spaces, the Poincaré inequality, Lemma 3.9 and Lemma 3.1 that

$$\int_0^T \int_{\mathcal{B}_i} |T_K(u)|^{\sigma_i} dx dt$$

$$\leq \left(\sup_{t \in (0, T)} \int_{\mathcal{B}_i} |T_K(u)|^T dx\right)^{rp_i^-/N} \int_0^T \left(\int_{\mathcal{B}_i} |T_K(u)|^{p_i^-} dx\right)^{rp_i^-/p_i^+} dt$$

$$\leq C \int_0^T \left(\int_{\mathcal{B}_i} |T_K(u) - T_K(u)|^{p_i^-} + |T_K(u)|^{p_i^-} dx\right)^{p_i^-/p_i^+} dt$$

$$\leq C \int_0^T \int_{\mathcal{B}_i} |\nabla T_K(u)|^{p_i^-} dx dt + C$$

$$\leq CK^{2-r} \int_{Q_i \cap \{|u| \leq K\}} \frac{|D_u|^{p_i^-}}{(1 + |u|)^{2-r}} dx dt + C$$

$$\leq CK^{2-r} \int_{Q_i \cap \{|u| \leq K\}} \frac{|D_u|^{p_i^-} + 1}{(1 + |u|)^{2-r}} dx dt + C$$

$$\leq CK^{2-r},$$

where

$$\overline{T_K(u)} = \frac{1}{|\mathcal{B}_i|} \int_{\mathcal{B}_i} T_K(u) dx.$$ 

The Tchebycheff inequality gives

$$\text{meas}\{|u| > K, x \in Q_i\} \leq CK^{2-r-\sigma_i}.$$ 

So $u \in M^{\sigma_i-2+r}(\overline{B}_i)$, which is the Marcinkiewicz space of exponent $\sigma_i - 2 + r$. Since $\alpha_i^+ < \sigma_i - 2 + r$, one has $u \in L^{\alpha_i^+}(Q_i)$ and thus

$$\int_{Q_x} |u|^{\alpha_i(x)} dx dt \leq \sum_{i=1}^l \int_{Q_i} (|u|^{\alpha_i^+} + 1) dx dt < C.$$ 

Turn to the regularity of $|\nabla u|$. For any $l > 1$ and $K > 1$, it is obvious that

$$\{|\nabla u| > K\} \cap Q_i \subset \{|u| > l\} \cup \{|\nabla T_i(u)| > K\} \cup A \cap Q_i,$$

where $\text{meas}(A) = 0$ due to $\nabla u = \nabla T_i(u)$ a.e. on $\{|u| \leq l\}$. The Tchebycheff inequality gives

$$\text{meas}\{|\nabla u| > K\} \cap Q_i \leq Cl^{2-r-\sigma_i} + CK^{-p_i^-}l^{2-r}.$$
Choosing $l = K^{p_i}/\sigma_i$, leads to meas\{ $|\nabla u| \geq K$ $\cap Q_i$ $\leq CK^{p_i-n(2-r)/(n+r)}$ \} and thus

$$\iint_{Q_T} |\nabla u|^{\beta(x)} dxdt \leq \sum_{i=1}^{l} \iint_{Q_i} (|\nabla u|^{\beta_i} + 1) dxdt < C.$$ 

\[ \square \]

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