A new class of interpolatory $L$-splines with adjoint end conditions

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Abstract. A thin plate spline surface for interpolation of smooth transfinite data prescribed along concentric circles was recently proposed by Bejancu, using Kounchev’s polyspline method. The construction of the new ‘Beppo Levi polyspline’ surface reduces, via separation of variables, to that of a countable family of univariate $L$-splines, indexed by the frequency integer $k$. This paper establishes the existence, uniqueness and variational properties of the ‘Beppo Levi $L$-spline’ schemes corresponding to non-zero frequencies $k$. In this case, the resulting $L$-spline end conditions are formulated in terms of adjoint differential operators, unlike the usual ‘natural’ $L$-spline end conditions, which employ identical operators at both ends. Our $L$-spline error analysis leads to an $L^2$-error bound for transfinite surface interpolation with Beppo Levi polysplines.

Keywords: interpolation, $L$-spline, Beppo Levi polyspline, approximation order

1 Introduction

The thin plate spline (TPS) surface for scattered data interpolation was defined by Duchon [10] as the unique minimizer of the squared seminorm

$$\|F\|^2_{BL} := \int_{R^2} \left( |F_{xx}|^2 + 2|F_{xy}|^2 + |F_{yy}|^2 \right) dx dy,$$

subject to $F$ taking prescribed values at a finite number of scattered locations. The minimization takes place in the Beppo Levi space of continuous functions $F$ with generalized second-order partial derivatives in $L^2(R^2)$.

Recently, Bejancu [5] proposed a new type of TPS surface, passing through several continuous curves prescribed along concentric circles. The new surface minimizes, for $F \in C^2(R^2 \setminus \{0\})$, the polar coordinate version of (1):

$$\|f\|^2_{BL} := \int_{0}^{\infty} \int_{-\pi}^{\pi} \left( |f_{rr}|^2 + 2 \left| \frac{f_{r\theta}}{r} \right|^2 + \left| \frac{f_{\theta\theta}}{r^2} + \frac{f_{rr}}{r} \right|^2 \right) r d\theta dr,$$

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where $f(r, \theta) := F(r \cos \theta, r \sin \theta)$ denotes the polar form of $F$. Similar surfaces for transfinite interpolation have also been studied in [34] in the case of continuous periodic data prescribed along parallel lines or hyperplanes (see also the survey [10]).

The ‘transfinite TPS’ surfaces belong to the class of multivariate polysplines introduced by Kounchev [13]. In the context of data prescribed on concentric circles $r = r_j$, $j \in \{1, \ldots, n\}$, with $0 < r_1 < \ldots < r_n$, let us denote $\rho := \{r_1, \ldots, r_n\}$ and $\Omega := \{(r, \theta) : r_1 \leq r \leq r_n, -\pi \leq \theta \leq \pi\}$. A function $S : \Omega \to \mathbb{R}$ is termed a biharmonic polyspline on annuli determined by $\rho$ if two conditions hold: first, $S$ and its polar form $s$ are piecewise biharmonic, i.e.,

$$\left(\partial_{xx} + \partial_{yy}\right)^2 S(x, y) = \left(\partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}\right)^2 s(r, \theta) = 0,$$

on each annulus $r_j < r < r_{j+1}$, $-\pi \leq \theta \leq \pi$, for $1 \leq j \leq n - 1$; and second, $S \in C^2(\Omega)$, i.e. neighbouring pieces join up $C^2$-continuously across the interface circles. For sufficiently smooth periodic data functions $u, v, \mu_j : [-\pi, \pi] \to \mathbb{R}, 1 \leq j \leq n$, Kounchev proved that such a polyspline surface is uniquely determined by transfinite interpolation conditions

$$s(r_j, \theta) = \mu_j(\theta), \quad \forall \theta \in [-\pi, \pi], \forall j \in \{1, \ldots, n\},$$

(3)

together with boundary conditions $\partial_s(r_1, \theta) = u(\theta)$ and $\partial_s(r_n, \theta) = v(\theta)$, $\forall \theta \in [-\pi, \pi]$. He also extended this result to polysplines of higher orders and more general interface configurations in arbitrary dimension.

In [5], Bejancu proposed a global polyspline $S : \mathbb{R}^2 \to \mathbb{R}$ for which boundary conditions on the extreme circles $r = r_1$ and $r = r_n$ are replaced by the requirement that the polar Beppo Levi energy [2] is finite for $f := s$. This Beppo Levi polyspline has two additional biharmonic pieces over the extreme annuli $0 < r < r_1$ and $r > r_n$, such that $S \in C^2(\mathbb{R}^2\setminus\{0\})$. The new surface is automatically continuous at 0, but its partial derivatives can have a singularity at 0.

For sufficiently smooth data, it turns out that there exists a one-parameter family of such Beppo Levi polysplines on annuli determined by $\rho$, each satisfying the transfinite interpolation conditions [3]. Two surfaces $S^A$ and $S^B$ of this family are uniquely determined in [3 Theorem 1] by the following additional conditions: $S^A$ takes an arbitrarily prescribed value at 0, while $S^B$ is biharmonic at 0 (hence, non-singular). Both $S^A$ and $S^B$ are then characterized as genuine TPS surfaces, i.e. minimizers of [2], subject to their respective interpolation conditions.

Following the method of separation of variables used by Kounchev [13], the construction of the Beppo Levi polysplines $S^A$, $S^B$ is obtained in [5 section 4] via the absolutely convergent Fourier representation in polar form

$$s(r, \theta) = \sum_{k \in \mathbb{Z}} \hat{s}_k(r) e^{ik\theta}, \quad (r, \theta) \in [0, \infty) \times [-\pi, \pi].$$

(4)

For each frequency $k$, the amplitude coefficient $\hat{s}_k$ of this representation is a univariate $L_k$-spline for an ordinary differential operator operator $L_k$, as described
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in the next section. Moreover, the form of $\hat{s}_k$ on the extreme intervals $(0, r_1)$ and $(r_n, \infty)$ is determined by the condition that the corresponding Plancherel component of the Beppo Levi energy \([2]\) of $s$ is finite.

The present paper studies the class of such Beppo Levi $L_k$-splines corresponding to non-zero frequencies $k$ (see section 2). In this case, the restrictions satisfied by $\hat{s}_k$ on the extreme intervals exhibit a twisted symmetry, expressed in terms of adjoint differential operators. Different features appear in the radial case $k = 0$, treated in the companion paper \([7]\), which is connected to Rabut’s work on radially symmetric thin plate splines \([16]\).

In section 3, we prove the existence, uniqueness and variational characterization of interpolation schemes with Beppo Levi $L_k$-splines, as required by the construction of \([5]\). A part of these results, corresponding to $|k| \geq 2$, has first been obtained in the MSc thesis \([2]\). Further, in section 4, we apply an error analysis of the Beppo Levi $L_k$-spline schemes to establish the $L^2$-approximation order $O(h^2)$ for transfinite surface interpolation with Beppo Levi polysplines on annuli, where $h$ is the maximum distance between successive interface circles. The extension of this work to higher order Beppo Levi polysplines on annuli and their $L$-spline Fourier coefficients will be addressed in a separate paper.

2 Preliminaries

2.1 Energy spaces

For each $r \geq 0$, define the Fourier coefficients of $f(r, \theta)$ with respect to $\theta$ by

$$\hat{f}_k(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ik\theta} f(r, \theta) d\theta, \quad k \in \mathbb{Z}. \quad (5)$$

The following observation shows the effect on $\hat{f}_k$ of the condition that the polar Beppo Levi integral \([2]\) is finite. Namely, if $f$ is the polar form of $F \in C^2(\mathbb{R}^2 \setminus \{0\})$, then $\hat{f}_k \in C^2(0, \infty)$ and Plancherel’s formula implies the identity \([5\; (3.2)]\):

$$\|f\|_{BL}^2 = 2 \pi \sum_{k \in \mathbb{Z}} \|\hat{f}_k\|_k^2, \quad (6)$$

where, for each $k \in \mathbb{Z}$, we denote

$$\|\psi\|_k^2 := \int_0^\infty \left\{ \left|\frac{d^2\psi}{dr^2}\right|^2 + 2k^2 \left|\frac{\psi}{r^2} - \frac{1}{r} \frac{d\psi}{dr}\right|^2 + 2k^2 \left|\frac{\psi}{r^2} - \frac{1}{r^2} \frac{d\psi}{dr}\right|^2 \right\} r \, dr. \quad (7)$$

Let $AC_{loc}(0, \infty)$ be the vector space of functions $\psi : (0, \infty) \to \mathbb{C}$ that are absolutely continuous on any interval $[a, b], 0 < a < b < \infty$. For each non-zero integer $k$, our analysis in section 3 takes place naturally in the space of functions for which the energy \([7]\) is finite. Specifically, for $k = \pm 1$, the corresponding energy space is the vector space $A_1$ of functions $\psi \in C^1(0, \infty)$ with $\psi' \in AC_{loc}(0, \infty)$, such that $r^{1/2}\psi''$ and $r^{-1/2}\psi' - r^{-3/2}\psi$ belong to $L^2(0, \infty)$. For $|k| \geq 2$, the
space associated to \( \mathfrak{m} \) is the vector space \( \Lambda \) of functions \( \psi \in C^1(0, \infty) \) with \( \psi' \in AC_{\text{loc}}(0, \infty) \), such that \( r^{1/2} \psi'' \), \( r^{-1/2} \psi' \), and \( r^{-3/2} \psi \) all belong to \( L^2(0, \infty) \). Note that \( \| \cdot \|_k \) is a norm for \( |k| \geq 2 \) and a semi-norm for \( k = \pm 1 \).

The results of section 3 employ the following properties of functions from the spaces \( \Lambda_1 \) and \( \Lambda_2 \).

Lemma 1. (i) If \( \psi \in \Lambda_2 \), there exist non-negative constants \( C_\psi \) and \( \tilde{C}_\psi \), such that

\[
|\psi(r)| \leq C_\psi \left( r^{3/2} + r |1 - r|^{1/2} \right), \quad \forall r > 0.
\]

(ii) If \( \psi \in \Lambda_1 \), there exist non-negative constants \( C_\psi \) and \( \tilde{C}_\psi \), such that

\[
|\psi(r)| \leq C_\psi r \left( 1 + |\ln r|^{1/2} \right), \quad \forall r > 0.
\]

Proof. (i) For each \( r > 0 \), we use the Leibniz-Newton formula

\[
r^{-3/2} \psi(r) - \psi(1) = \int_1^r \left[ r^{-3/2} \psi'(t) \right] \, dt
= \int_1^r \left[ t^{-3/2} \psi'(t) - \frac{3}{2} t^{-5/2} \psi(t) \right] \, dt.
\]

Via Cauchy-Schwarz, the last integral is bounded above in modulus by

\[
\left| \int_1^r t^{-1} \left[ t^{-1/2} \psi'(t) \right] \, dt \right| + \frac{3}{2} \left| \int_1^r t^{-1/2} \psi'(t) \, dt \right|
\leq \int_1^r t^{-2} \, dt \left\{ \left\| t^{-1/2} \psi'(t) \right\|_{L^2(0, \infty)} \right\}^{1/2} + \frac{3}{2} \left\| t^{-3/2} \psi(t) \right\|_{L^2(0, \infty)}^{1/2}
\leq \left[ 1 - r^{-1} \right] \left\{ \left\| r^{-1/2} \psi' \right\|_{L^2(0, \infty)} + \left( 3/2 \right) \left\| r^{-3/2} \psi \right\|_{L^2(0, \infty)} \right\},
\]

which implies the first of inequalities (8). For the second inequality, we similarly start with

\[
r^{-1/2} \psi'(r) - \psi'(1) = \int_1^r \left[ t^{-1/2} \psi'(t) \right] \, dt
= \int_1^r \left[ t^{-1/2} \psi''(t) - \frac{1}{2} t^{-3/2} \psi'(t) \right] \, dt,
\]

which holds for each \( r > 0 \), since \( \psi' \in AC_{\text{loc}}(0, \infty) \). Hence, we obtain the following upper bound on the modulus of last integral:

\[
\left| \int_1^r t^{-1} \left[ t^{1/2} \psi''(t) \right] \, dt \right| + \frac{1}{2} \left| \int_1^r t^{-1/2} \psi'(t) \, dt \right|
\]
of (7) should satisfy the Euler-Lagrange equation

\[ t' \left( t^{1/2} \psi''(t) \right) - \frac{1}{2} \int_1^r t^{-1/2} \psi'(t)^2 \, dt \]

\[ \leq |1 - r^{-1}|^{1/2} \left\{ \left\| r^{1/2} \psi'' \right\|_{L^2(0, \infty)} + (1/2) \left\| r^{-1/2} \psi' \right\|_{L^2(0, \infty)} \right\}. \]

(ii) For the first inequality, we employ the Leibniz-Newton formula

\[ r^{-1} \psi(r) - \psi(1) = \int_1^r [t^{-1} \psi(t)]' \, dt, \]

together with the estimate

\[ \left| \int_1^r t^{-1/2} \left( t^{1/2} \left[ t^{-1} \psi(t) \right]' \right) \, dt \right| \]
\[ \leq \left| \int_1^r t^{-1} \, dt \right|^{1/2} \left| \int_1^r t^{1/2} \left[ t^{-1} \psi(t) \right]' \, dt \right|^{1/2} \]
\[ \leq |\ln r|^{1/2} \left\| r^{1/2} (r^{-1} \psi)' \right\|_{L^2(0, \infty)}, \]

the last norm being finite due to \( r^{1/2} (r^{-1} \psi)' = r^{-1/2} \psi' - r^{-3/2} \psi. \) Since \( \psi' \in AC_{\text{loc}}(0, \infty), \) the second inequality is obtained via

\[ \psi'(r) - \psi'(1) = \int_1^r \psi''(t) \, dt, \]

followed by a similar estimate, this time in terms of \( \left\| r^{1/2} \psi'' \right\|_{L^2(0, \infty)}. \)

\[ \square \]

### 2.2 Beppo Levi \( L_k \)-splines

As observed in [6], due to the Plancherel-type formula [6], to obtain the variational characterization of the Beppo Levi polyspline \( s \) as minimizer of the polar thin plate energy (2), it is sufficient to show that, for each \( k \in \mathbb{Z}, \) the amplitude coefficient \( s_k \) minimizes the corresponding energy component (7). Letting \( Q(r, g, g', g'') \) denote the integrand of (7), classical calculus of variations considerations imply that, except at the interpolation locations \( r_1, \ldots, r_n, \) a minimizer of (7) should satisfy the Euler-Lagrange equation

\[ \partial_g Q - \frac{d}{dr} \partial_{g'} Q + \frac{d^2}{dr^2} \partial_{g''} Q = 0. \]

The resulting left-hand side Euler-Lagrange differential operator is given, up to a constant factor, by

\[ L_k := \frac{d^4}{dr^4} + 2 \frac{d^3}{dr^3} - \frac{2k^2 + 1}{r} \frac{d^2}{dr^2} + \frac{2k^2 + 1}{r^2} \frac{d}{dr} + \frac{k^4 - 4k^2}{r^3} \]

\[ = r \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{k^2}{r^3} \right)^2. \]
Therefore, \( \hat{s}_k \) should necessarily be annihilated by \( L_k \) on each subinterval \((0, 1), (r_1, r_2), \ldots, (r_n, \infty) \).

The null-space \( \text{Ker} L_k \) is computed in [13] via the substitution \( r = e^t, \frac{d}{dr} = r \frac{d}{dt} \), which transforms \( L_k \) into a differential operator with constant coefficients in variable \( t \). Standard factorization then implies

\[
L_k = \frac{1}{r^3} \left( r \frac{d}{dr} - |k| \right) \left( r \frac{d}{dr} - |k| - 2 \right) \left( r \frac{d}{dr} + |k| \right) \left( r \frac{d}{dr} + |k| - 2 \right),
\]

hence

\[
\text{Ker} L_k = \begin{cases} \text{span} \{ r^2, r^2 \ln r, 1, \ln r \}, & \text{if } k = 0, \\ \text{span} \{ r^3, r^2 \ln r, r^{-1} \}, & \text{if } |k| = 1, \\ \text{span} \{ r^{|k|+2}, r^{|k|}, r^{-|k|+2}, r^{-|k|} \}, & \text{if } |k| \geq 2. \end{cases}
\]

Moreover, the condition that the polar Beppo Levi energy component (7) is finite further restricts the form of \( \hat{s}_k \) on the extreme intervals \((0, r_1) \) and \((r_n, \infty) \).

Specifically, for \( k \neq 0 \), evaluating (7) for each of the four generating functions of \( \text{Ker} L_k \), we obtain the necessary conditions

\[
\hat{s}_k (r) \in \begin{cases} \text{span} \{ r^{|k|+2}, r^{|k|} \}, & \text{for } r \in (0, r_1), \\ \text{span} \{ r^{-|k|+2}, r^{-|k|} \}, & \text{for } r \in (r_n, \infty). 
\end{cases}
\]

Note that \( \text{span} \{ r^{|k|+2}, r^{|k|} \} = \text{Ker} G_k \) and \( \text{span} \{ r^{-|k|+2}, r^{-|k|} \} = \text{Ker} R_k \), where

\[
G_k := \frac{1}{r} \left[ \frac{d^2}{dr^2} - \frac{2 |k|}{r} \left( \frac{d}{dr} + \frac{|k|}{r^2} \right) \right]
= \frac{1}{r^3} \left( r \frac{d}{dr} - |k| \right) \left( r \frac{d}{dr} - |k| - 2 \right),
\]

\[
R_k := \frac{1}{r} \left[ \frac{d^2}{dr^2} + \frac{2 |k|}{r} - 1 \left( \frac{d}{dr} + \frac{|k|}{r^2} \right) \right]
= \frac{1}{r^3} \left( r \frac{d}{dr} + |k| \right) \left( r \frac{d}{dr} + |k| - 2 \right).
\]

Remark 1. It can be verified that \( r^{-3} \) is the only factor of the form \( r^{|k|} \) which, when inserted in front of the last two brackets in the right-hand side of the above formulae, turns \( G_k \) and \( R_k \) into mutually adjoint operators. Indeed, the formal adjoint of \( G_k \) is

\[
G_k^* = \frac{d^2}{dr^2} \left( \frac{1}{r} \right) + \left( |k| + 1 \right) \frac{d}{dr} \left( \frac{1}{r^2} \right) + \frac{|k| (|k| + 2)}{r^3} = R_k
\]

and a similar computation shows \( R_k^* = G_k \).

Recall the notation \( \rho := \{r_1, \ldots, r_n\} \) used in the Introduction.

Definition 1. Let \( k \neq 0 \). A function \( \eta : [0, \infty) \to \mathbb{C} \) is called a Beppo Levi \( L_k \)-spline on \( \rho \) if the following conditions hold:
Karlin and Ziegler \([12]\), via the representation

be described as an extended complete Chebyshev (ECT) system in the sense of

This differs from (13) for

Due to conditions (ii), \(S_k (\rho)\) is a subspace of \(A_1\) if \(|k| = 1\), and of \(A_2\) if \(|k| \geq 2\).

For \(k = 0\), the related notion of a Beppo Levi \(L_0\)-spline is treated in [7]. In this case, the correct left/right operators \(G_0\) and \(R_0\) on the extreme intervals are not obtained by just letting \(k = 0\) in (12). Also, \(G_0\) and \(R_0\) are not anymore mutually adjoint.

The proof of the next result follows from the definition of biharmonic Beppo Levi polysplines on annuli \([5]\).

**Proposition 1.** A univariate function \(\eta : [0, \infty) \to \mathbb{C}\) is a Beppo Levi \(L_k\)-spline on \(\rho\), i.e. \(\eta \in S_k (\rho)\), if and only if the polar surface \(s (r, \theta) := \eta (r) e^{-ik\theta}\) is a biharmonic Beppo Levi polyspline on annuli determined by \(\rho\).

We now review some relevant literature. It was pointed out by Kounchev \([13,\ \text{p. 91}]\) that, on any interval of positive real numbers, the null space of \(L_k\) can be described as an extended complete Chebyshev (ECT) system in the sense of Karlin and Ziegler \([12]\), via the representation

\[
 r^{2k} |L_k| = D_4 D_3 D_2 D_1 = \frac{d}{dr} r^{2k+1} \frac{d}{dr} \frac{1}{r^{2k+1}} \frac{d}{dr} r^{2k+1} \frac{d}{dr} \frac{1}{r^{2k+1}},
\]

where \(D_1 = \frac{d}{dr} \left( \frac{1}{r} \right)\), \(D_2 = D_4 = \frac{d}{dr} \left( r^{2k+1} \right)\), \(D_3 = \frac{d}{dr} \left( \frac{1}{r^{2k+1}} \right)\). We also observe, following Schumaker \([18,\ \text{p. 398}]\), that \(L_k\) possesses the factorization

\[
 L_k = M_k^* M_k, \tag{13}
\]

where \(M_k^*\) denotes the formal adjoint of

\[
 M_k := \frac{1}{\sqrt{r^{2k+1}}} \frac{d}{dr} \frac{1}{r^{2k+1}} \frac{d}{dr} r^{2k+1} \frac{1}{r^{2k+1}} = \frac{1}{r^{3k/2}} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{k^2}{r^2} \right).
\]

Due to \([13]\), a function that satisfies conditions (i) and (iii) of Definition \([1]\) can be characterized as a ‘generalized spline’ or ‘\(M_k\)-spline’ on \([r_1, r_n]\) in the sense of Ahlberg, Nilson, and Walsh \([1]\), Schultz and Varga \([17]\). However, our labeling such a function as a ‘\(L_k\)-spline’ agrees with the terminology of Lucas \([15]\) and Jerome and Pierce \([11]\), which is more adequate, in view of the fact that \(L_k\) may possess other factorizations of the type \([13]\). Indeed, for \(k \neq 0\), our adjoint boundary operators \(G_k\) and \(R_k\) actually generate, via \([10]\), the factorization

\[
 L_k = G_k r^3 R_k = \tilde{L}_k \tilde{R}_k, \tag{14}
\]

where \(\tilde{L}_k := r^{3/2} R_k\).

This differs from \([13]\) for \(|k| \geq 2\), while it coincides with \([13]\) for \(|k| = 1\).
On the other hand, the ‘natural’ end conditions of $L$-spline literature (see [18]) are always formulated in terms of a single differential operator at both ends of the interpolation domain. It is thus remarkable that adjoint boundary operators as in condition (ii) of our definition have also occurred in [4], in the context of exponential $L$-splines generated as Fourier coefficients of Beppo Levi polyspline surfaces on parallel strips. Such exponential $L$-splines coincide in fact with Matérn kernels on the full real line (for Matérn kernels on a compact interval, see [9]). As shown in [3], adjoint $L$-spline end conditions are intimately connected to Wiener-Hopf factorizations for semi-cardinal interpolation.

3 Interpolation with Beppo Levi $L_k$-splines

3.1 A fundamental identity

We employ the notations introduced in the previous section.

**Theorem 1.** (i) Let $k \in \mathbb{Z}$, $|k| \geq 2$, and an arbitrary Beppo Levi $L_k$-spline $\eta \in \mathcal{S}_k(\rho)$. Also, assume that $\psi \in \Lambda_2$ vanishes on the knot-set $\rho$:

$$\psi(r_j) = 0, \quad \forall j \in \{1, \ldots, n\}. \quad (15)$$

Then the following orthogonality relation holds:

$$\int_0^\infty r^3 |R_k \eta(r)| \left[ R_k \overline{\psi}(r) \right] dr = 0. \quad (16)$$

(ii) The same conclusion holds if $k = \pm 1$ and $\psi \in \Lambda_1$ satisfies (15).

**Proof.** For convenience, let us denote the left-hand side of (16) by $I_k := I_k(\eta, \psi)$. Note that, for any $k \neq 0$, $\eta \in \mathcal{S}_k(\rho)$ implies $R_k \eta(r) = 0$, $\forall r > r_n$, hence we can work with integral $I_k$ on the integration domain $(0, r_n]$. Since

$$r^{3/2} R_k \psi = r^{1/2} \psi'' + (2 |k| - 1) r^{-1/2} \psi' + |k| ((|k| - 2) r^{-3/2} \psi,$$

the hypotheses imply, via Cauchy-Schwarz inequality, that $I_k$ is an absolutely convergent integral. Using the factorization of the operator $R_k$ and making the notation

$$\eta_1(r) := (r \frac{d}{dr} + |k| - 2) \eta(r),$$

$$\psi_1(r) := (r \frac{d}{dr} + |k| - 2) \psi(r),$$

we have

$$I_k = \int_0^{r_n} r^{-3} \left[ \left( r \frac{d}{dr} + |k| \right) \eta_1(r) \right] \left[ \left( r \frac{d}{dr} + |k| \right) \overline{\psi_1}(r) \right] dr$$

$$= \sum_{j=1}^{n} \int_{r_{j-1}}^{r_j} r^{-2} \left[ \left( r \frac{d}{dr} + |k| \right) \eta_1(r) \right] \frac{d}{dr} \overline{\psi_1}(r) dr$$

$$+ \sum_{j=1}^{n} \int_{r_{j-1}}^{r_j} r^{-3} \left[ \left( r \frac{d}{dr} + |k| \right) \eta_1(r) \right] |k| \overline{\psi_1}(r) dr,$$
where all integrals remain absolutely convergent and \( r_0 := 0 \). Next, we apply integration by parts in each term of the first sum, which is permitted due to the fact that \( \psi \in AC_{\loc}(0, \infty) \). Since
\[
\frac{d}{dr} \left( r^{-2} \left( \frac{d}{dr} + |k| \right) \eta_1 (r) \right) = r^{-3} \left( \frac{d}{dr} - 2 \right) \left( \frac{d}{dr} + |k| \right) \eta_1 (r),
\]
we obtain
\[
I_k = \sum_{j=1}^{n} \left[ \psi_j (r) r^{-2} \left( \frac{d}{dr} + |k| \right) \eta_1 (r) \right]^{r_j}_{r_{j-1}}
- \sum_{j=1}^{n} \int_{r_{j-1}}^{r_j} \frac{r^{-3}}{n} \psi_j (r) \left( \frac{d}{dr} - |k| - 2 \right) \left( \frac{d}{dr} + |k| \right) \eta_1 (r) \, dr.
\]
Since \( \psi \) has continuity \( C^1 \) and \( \eta \) has continuity \( C^2 \), the first sum of the last display is telescopic, hence we only have to evaluate the boundary terms corresponding to \( r := r_0 \) and \( r := r_0 = 0 \). Note that the boundary term at \( r_0 \) is zero, since the condition \( R_k \eta (r) = 0, \forall r > r_0 \), of a Beppo Levi \( L_k \)-spline implies, by continuity, the relation \( \left( \frac{d}{dr} + |k| \right) \eta_1 (r) \bigg|_{r=r_0} = 0 \).

For the boundary term at 0, consider first the case \( |k| \geq 2 \). Then the left end condition \( G_k \eta (r) = 0 \), i.e. \( \eta \in \text{span} \{r^{|k|+2}, r^{|k|}\} \), for \( r \in (0, r_0) \), implies
\[
r^{-2} \left( \frac{d}{dr} + |k| \right) \eta_1 (r) = O \left( r^{|k|-2} \right), \quad \text{as} \ r \to 0.
\]
Since, by Lemma \( \text{[1]} \) \( \psi_1 (r) = O (r) \), as \( r \to 0 \), we deduce that the boundary term at 0 vanishes if \( |k| \geq 2 \). If \( |k| = 1 \), the left end condition implies \( \eta \in \text{span} \{r^3, r^4\} \), for \( r \in (0, r_0) \), hence
\[
r^{-2} \left( \frac{d}{dr} + 1 \right) \eta_1 (r) = cr, \quad \forall r \in (0, r_0),
\]
for some constant \( c \). Since, by Lemma \( \text{[1]} \) in this case \( \psi_1 (r) = O \left( r |\ln r|^{1/2} \right) \), as \( r \to 0 \), it follows that the boundary term at 0 also vanishes if \( |k| = 1 \).

On the other hand, for each \( j \in \{1, \ldots, n\} \), since \( \eta \in \text{Ker} L_k \) on the interval \( (r_{j-1}, r_j) \), there exists a constant \( c_j \) such that
\[
\left( \frac{d}{dr} - |k| - 2 \right) \left( \frac{d}{dr} + |k| \right) \eta_1 (r) = c_j r^{|k|}, \quad \forall r \in (r_{j-1}, r_j).
\]
Hence
\[
I_k = \sum_{j=1}^{n} c_j \int_{r_{j-1}}^{r_j} r^{|k|-3} \left( \frac{d}{dr} + |k| - 2 \right) \psi_j (r) \, dr
= \sum_{j=1}^{n} c_j \int_{r_{j-1}}^{r_j} \frac{d}{dr} \left[ r^{|k|-2} \psi_j (r) \right] \, dr = \sum_{j=1}^{n} c_j \left[ r^{|k|-2} \psi_j (r) \right]_{r_{j-1}}^{r_j}.
\]
For \(|k| \geq 2\), since Lemma 1 implies \(r^{1-2|k|}\psi(r) = O(r)\), as \(r \to 0\), and, by hypothesis, \(\psi(r_j) = 0, \forall j \in \{1, \ldots, n\}\), we deduce \(I_k = 0\), as stated. For \(|k| = 1\), we reach the same conclusion without the need to investigate \(r^{-1}\psi(r)\) as \(r \to 0\), since in this case \(c_1 = 0\).

\[\begin{align*}
\end{align*}\]

### 3.2 Existence, uniqueness, and optimality

**Theorem 2.** Let \(\nu_1, \ldots, \nu_n\) be arbitrary real values, where \(n \geq 2\). For each \(k \neq 0\), there exists a unique Beppo Levi \(L_k\)-spline \(\sigma \in S_k(\rho)\), such that

\[\sigma(j) = \nu_j, \quad j \in \{1, \ldots, n\}.
\]

**Proof.** It is sufficient to prove the existence of a unique function \(\bar{\sigma} \in C^2 [r_1, r_n]\) such that \(\bar{\sigma} \in \text{Ker} L_k\) on each subinterval \((r_{j-1}, r_j)\) with \(j \in \{2, \ldots, n\}\). \(\bar{\sigma}\) satisfies the interpolation conditions (17) in place of \(\sigma\), and the following endpoint conditions hold:

\[\begin{align*}
\left[\left(r_{j-1}^{-|k|} - |k|\right) \left(r_j^{-|k|} - |k| - 2\right) \bar{\sigma}(r)\right]_{r \to r_j^+} &= 0, \\
\left[\left(r_{j-1}^{-|k|} + |k|\right) \left(r_j^{-|k|} + |k| - 2\right) \bar{\sigma}(r)\right]_{r \to r_j^-} &= 0.
\end{align*}\]

Indeed, such a function \(\bar{\sigma}\) can be uniquely extended to the required Beppo Levi \(L_k\)-spline \(\sigma \in S_k(\rho)\) by defining

\[\sigma(r) := \begin{cases}
\bar{\sigma}(r), & \text{if } 0 < r < r_1, \\
0, & \text{if } r_1 \leq r \leq r_n, \\
c_1 r + c_2 r^{-|k|} + c_3 r^{-|k|+2} + c_4 r^{-|k|+2}, & \text{if } r_n < r.<r<.\]

To verify this, note that \(c_1, c_2\) (respectively, \(c_3\) and \(c_4\)) are uniquely determined by the conditions that \(\sigma\) and \(\sigma'\) are continuous at \(r_1\) (respectively, at \(r_n\)). The continuity of \(\sigma''\) at \(r_1\) and \(r_n\) then follows automatically from (18) and from the properties \(G_k \sigma (r) = 0, \forall r \in (0, r_1)\), and \(R_k \sigma (r) = 0, \forall r > r_n\).

Now, a function \(\bar{\sigma}\) with the properties stated in the previous paragraph is determined by four coefficients on each of the \(n\) subintervals \((r_{j-1}, r_j)\), \(j \in \{2, \ldots, n\}\). These coefficients are coupled by three \(C^2\)-continuity conditions at each interior knot \(r_2, \ldots, r_{n-1}\), the endpoint conditions (18), and the \(n\) interpolation conditions (17), which amount to a \(4(n-1) \times 4(n-1)\) system of linear equations.

To show that this system has a unique solution, we assume zero interpolation data: \(\nu_j = 0, j \in \{1, \ldots, n\}\). Then the system becomes homogeneous, since the endpoint conditions and the continuity conditions at the interior knots were already homogeneous linear equations. Let \(\bar{\sigma}\) be determined by an arbitrary solution of this homogeneous system and let \(\sigma \in S_k(\rho)\) be the unique extension of \(\bar{\sigma}\) to a Beppo Levi \(L_k\)-spline. Taking \(\eta = \psi := \sigma\) in (16), we obtain \(R_k \sigma (r) = 0, i.e., \(\sigma \in \text{span} \{r^{-|k|+2}, r^{-|k|}\}\), for \(r \in (0, \infty)\). Since \(\sigma (r_j) = 0, j \in \{1, \ldots, n\}\), and \(n \geq 2\), we deduce \(\sigma \equiv 0\). Therefore the above homogeneous system admits only the trivial solution, which concludes the proof. \(\square\)
Remark 2. Theorem 2 also extends to the case \( n = 1 \). Indeed, for each integer \( k \neq 0 \), it is straightforward to verify that there exists a unique function \( \varphi_k \) with the properties: \( G_k \varphi_k (r) = 0 \) for \( 0 < r < 1 \), \( R_k \varphi_k (r) = 0 \) for \( r > 1 \), \( \varphi_k \) is \( C^2 \)-continuous at \( r = 1 \), and \( \varphi_k (1) = 1 \). Its expression

\[
\varphi_k (r) = \frac{1}{2} \begin{cases} 
\rho \left[ (1 + |k|) + (1 - |k|) r^2 \right], & 0 \leq r \leq 1, \\
\rho^{-1} \left[ (1 - |k|) + (1 + |k|) r^2 \right], & 1 < r.
\end{cases}
\]

was given in [5] (3.10) for \( |k| \geq 2 \) and is also seen to hold for \( |k| = 1 \). Hence, if \( \rho = \{ r_1 \} \), then \( \sigma := \nu_k \varphi_k (\cdot / r_1) \) is the unique Beppo Levi \( L_k \)-spline in \( S_k (\rho) \), such that \( \sigma (r_1) = \nu_1 \). As shown by the next result, if \( |k| \geq 2 \) and \( n \geq 2 \), the dilates of \( \varphi_k \) also provide a basis for a linear representation of the interpolant of Theorem 2.

**Theorem 3.** Assume that\( |k| \geq 2 \), \( n \geq 2 \), and let \( \sigma \) be the Beppo Levi \( L_k \)-spline satisfying the interpolation conditions \( 17 \) of Theorem 2 for given values \( \nu_1, \ldots, \nu_n \) at the knot-set \( \rho \). Then there exist unique coefficients \( a_1, \ldots, a_n \), such that

\[
\sigma (r) = \sum_{j=1}^{n} a_j \varphi_k \left( \frac{r}{r_j} \right), \quad \forall r \geq 0.
\]

This result has been established in [5, Lemma 3] for the special case in which \( \sigma \) satisfies Lagrange interpolation conditions. The proof given there also applies to our general interpolation conditions \( 17 \). Note that representation \( 20 \) does not hold for \( |k| = 1 \), but a similar representation for \( k = 0 \) appears in [7] Theorem 4.

The next result shows that our Beppo Levi \( L_k \)-spline interpolants minimize the functional \( 7 \), subject to the interpolation conditions.

**Theorem 4.** Given \( k \neq 0 \) and arbitrary real values \( \nu_1, \nu_2, \ldots, \nu_n \), let \( \sigma \) denote the unique Beppo Levi \( L_k \)-spline obtained in Theorem 2. Then \( ||\sigma||_k < ||g||_k \) whenever \( g \) satisfies the same interpolation conditions \( 17 \) as \( \sigma \) and \( g \neq \sigma \), where \( g \in A_1 \) if \( |k| = 1 \), while \( g \in A_2 \) if \( |k| \geq 2 \).

**Proof.** Letting \( \eta := \sigma, \psi := g - \sigma \), the hypotheses imply that \( \psi \) satisfies \( 15 \), hence \( 13 \) holds by Theorem 1. Since \( \psi' \in AC_{loc} (0, \infty) \) and \( \psi \in A_1 \) if \( |k| = 1 \), while \( \psi \in A_2 \) if \( |k| \geq 2 \), we can use the proof of [5, Formulas (5.3)] for \( k \neq 0 \) to show that

\[
\int_0^\infty r^3 [R_k \eta (r)] [R_k \psi (r)] \, dr = \left\{ \eta, \psi \right\}_k,
\]

where

\[
\left\{ \eta, \psi \right\}_k := \int_0^\infty \left\{ \eta' \psi' + 2k^2 \left[ \frac{\eta}{r^2} - \frac{\eta'}{r} \right] \left[ \frac{\psi}{r^2} - \frac{\psi'}{r} \right] \\
+ \left[ \frac{k^2 \eta}{r^2} - \frac{\eta'}{r} \right] \left[ \frac{k^2 \psi}{r^2} - \frac{\psi'}{r} \right] \right\} r \, dr.
\]
Therefore (10) implies the orthogonality property
\[ \langle \sigma, g - \sigma \rangle_k = 0, \]
from which
\[ \|g\|_k^2 = \|\sigma\|_k^2 + \|g - \sigma\|_k^2, \quad (21) \]
and \( \|g\|_k \geq \|\sigma\|_k \), with equality only if \( \|g - \sigma\|_k = 0 \). The last relation implies \( g \equiv \sigma \) if \( |k| \geq 2 \), since \( \|\cdot\|_k \) is a norm in this case. If \( |k| = 1 \), the semi-norm \( \|g - \sigma\|_k \) vanishes if and only if \( g(r) - \sigma(r) = ar, \forall r \in (0, \infty) \), for some constant \( a \). Since \( g - \sigma \) takes zero values at the knots \( r_1, \ldots, r_n \), we deduce again \( g \equiv \sigma \), which completes the proof. \( \Box \)

4 Approximation orders

For each \( k \neq 0 \), the following result establishes \( L^\infty \) and \( L^2 \)-error bounds for interpolation with Beppo Levi \( L_k \)-splines to data functions from \( A_1 \) or \( A_2 \).

**Theorem 5.** Let \( \rho := \{r_1, \ldots, r_n\} \) be a set of nodes with \( 0 < r_1 < \ldots < r_n \), \( n \geq 2 \), and \( h := \max_{1 \leq j \leq n-1} (r_{j+1} - r_j) \). For an integer \( k \neq 0 \), let \( g : (0, \infty) \to \mathbb{R} \) be a data function such that \( g \in A_1 \) if \( |k| = 1 \), while \( g \in A_2 \) if \( |k| \geq 2 \). Let \( \sigma \in S_k(\rho) \) be the Beppo Levi \( L_k \)-spline of Theorem 4 corresponding to data values \( \nu_j := g(r_j), 1 \leq j \leq n \). Then, for \( m \in \{0,1\} \), we have the error bounds:

\[ \left\| \frac{d^m}{dr^m}(g - \sigma) \right\|_{L^\infty[r_1,r_n]} \leq \frac{1}{2^{1-m} \sqrt{r_1}} h^{3/2-m} \|g\|_k, \quad (22) \]

\[ \left\| \frac{d^m}{dr^m}(g - \sigma) \right\|_{L^2[r_1,r_n]} \leq \frac{1}{2^{1-m} \sqrt{r_1}} h^{2-m} \|g\|_k. \quad (23) \]

**Proof.** Similar error bounds for \( k = 0 \) were obtained in [7] Theorems 5 & 6, along the lines of the classical error analysis for generalized splines [1]. The same arguments are also seen to apply to the present case \( k \neq 0 \), by replacing the semi-norm \( \|\cdot\|_0 \) of [7] with \( \|\cdot\|_k \) and using the inequality \( \int_0^\infty r |g''(r)|^2 \, dr \leq \|g\|_k^2 \), valid for any data function \( g \) as in the hypothesis. \( \Box \)

**Remark 3.** As in [7], the bounds (22) and (23) also imply an \( L^p \)-error bound for \( p \in (2, \infty) \). Moreover, a similar analysis to that of [7] Theorem 7 shows that the exponents of \( h \) in the above error bounds cannot be increased for the classes \( A_1 \) and \( A_2 \) of data functions.

The main result of this section applies (23) and the corresponding error bound of [7] for \( k = 0 \) to obtain a \( L^2 \)-convergence order for transfinite surface interpolation with biharmonic Beppo Levi polysplines on annuli. To state this result, let \( W^2 \) be the Wiener-type algebra of continuous periodic functions \( \mu : [-\pi, \pi] \to \mathbb{R} \) with Fourier coefficients \( \hat{\mu}_k, k \in \mathbb{Z} \), such that \( \sum_{k \in \mathbb{Z}} |\hat{\mu}_k| (1 + |k|)^2 < \infty \). Note that \( W^2 \subset C^2[-\pi, \pi] \) and, as observed in [5] Remark 1, any periodic cubic spline belongs to \( W^2 \).
Theorem 6. Given $F \in C^2([\mathbb{R}^2 - \{0\}] \cap C([\mathbb{R}^2])$ of polar form $f$ such that $\Box$ is finite, assume that $f(r_j, \cdot) \in W^2$ along each domain circle $r = r_j$, $j \in \{1, \ldots, n\}$. Let $S$ be either one of the Beppo Levi polysplines $S^A$ or $S^B$ determined in [1] Theorem 1, satisfying the transfinite interpolation conditions [3] for $\mu_j := f(r_j, \cdot)$, $j \in \{1, \ldots, n\}$, where also $S^A(0) = F(0)$ and $S^B$ is biharmonic at 0. Then, for $m \in \{0, 1\}$, we have the $L^2$-error bound

$$
\left(\int_{r_1}^{r_n} \int_{-\pi}^{\pi} \left| \frac{\partial^m}{\partial r^m} (f_\mu - s) (r, \theta) \right|^2 r \, d\theta \, dr \right)^{1/2} \leq \frac{r_n}{r_1} h^{2-m} \| f \|_{BL}. \tag{24}
$$

Proof. For each $r \geq 0$, let $\hat{f}_k(r)$, $k \in \mathbb{Z}$, be the Fourier coefficients of $f(r, \theta)$ with respect to $\theta$. The smoothness assumptions on $F$ imply that $\hat{f}_k \in C^2([0, \infty))$. $\hat{f}_k$ is continuous at $r = 0$, $\forall k \in \mathbb{Z}$, and identity [3] holds. Since $\frac{\partial^m}{\partial r^m} (f - s) (r, \cdot) \in C[-\pi, \pi] \subset L^2[-\pi, \pi]$, the following Plancherel formula is also valid for $m \in \{0, 1\}$ and $r \in [r_1, r_n]$:

$$
\frac{1}{2\pi} \int_{-\pi}^\pi \left| \frac{\partial^m}{\partial r^m} (f - s) (r, \theta) \right|^2 d\theta = \sum_{k \in \mathbb{Z}} \left| \frac{d^m}{dr^m} \left( \hat{f}_k - \hat{s}_k \right) (r) \right|^2.
$$

Moreover, since $\sqrt{2} \frac{\partial^m}{\partial r^m} (f - s) \in C([r_1, r_n] \times [-\pi, \pi]) \subset L^2([r_1, r_n] \times [-\pi, \pi])$, we may multiply the above relation by $r$ and integrate both sides to obtain, via Fubini’s theorem,

$$
\frac{1}{2\pi} \int_{r_1}^{r_n} \int_{-\pi}^\pi \left| \frac{\partial^m}{\partial r^m} (f - s) (r, \theta) \right|^2 r \, d\theta \, dr = \sum_{k \in \mathbb{Z}} \int_{r_1}^{r_n} \left| \frac{d^m}{dr^m} \left( \hat{f}_k - \hat{s}_k \right) (r) \right|^2 r \, dr. \tag{25}
$$

Note that, for each $j \in \{1, \ldots, n\}$, the transfinite interpolation condition $s(r_j, \theta) = f(r_j, \theta)$, $\forall \theta \in [-\pi, \pi]$, is equivalent to $\hat{s}_k(r_j) = \hat{f}_k(r_j)$, $\forall k \in \mathbb{Z}$. Hence, for $k \neq 0$, the error bound [24] implies, for $m \in \{0, 1\}$,

$$
\left\| \frac{d^m}{dr^m} \left( \hat{f}_k - \hat{s}_k \right) \right\|_{L^2([r_1, r_n])} \leq \frac{1}{2^{1-m} \sqrt{r_1}} h^{2-m} \left\| \hat{f}_k \right\|_k. \tag{26}
$$

In addition, it follows from [7] Theorem 6 that this error bound also holds for $k = 0$, since $\hat{s}_A^0(0) = \hat{f}_0(0) = F(0)$ and $\hat{s}_B^0 \in \text{span} \{r^2, 1\}$ for $r \in (0, r_1)$. Therefore [24], [25], and [6] imply

$$
\frac{1}{2\pi} \int_{r_1}^{r_n} \int_{-\pi}^\pi \left| \frac{\partial^m}{\partial r^m} (f - s^{A,B} \cdot) (r, \theta) \right|^2 r \, d\theta \, dr \leq \sum_{k \in \mathbb{Z}} \int_{r_1}^{r_n} \left| \frac{d^m}{dr^m} \left( \hat{f}_k - \hat{s}_k \right) (r) \right|^2 r \, dr \leq Ch^{2(2-m)} \sum_{k \in \mathbb{Z}} \left\| \frac{d^m}{dr^m} \right\|_k^2 = C \frac{2^m h^{2(2-m)}}{2\pi} \| f \|_{BL}^2,
$$

where $C = 2^{2(m-1)} h_n/r_1$, which establishes [24]. \qed
A new class of interpolatory $L$-splines

A similar approximation order for transfinite interpolation via biharmonic Beppo Levi polysplines on parallel strips has recently been proved in [6]. Related Plancherel representations of the error have been employed before by Kounchev and Render [14] for cardinal polysplines on annuli and by Sharon and Dyn [19] for interpolatory subdivision schemes.

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