Discreteness of dyonic dilaton black holes

E. A. Davydov

Bogoliubov Laboratory of Theoretical Physics, JINR,
6 Joliot-Curie St, Dubna, Moscow region, 141980, Russian Federation,
and
Peoples’ Friendship University of Russia (RUDN University),
6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation

October 15, 2018

Abstract

We show that there are two classes of solutions that describe static spherically symmetric dyonic dilaton black holes with two nonsingular horizons. The first class includes only the already known solutions that exist for a few special values of the dilaton coupling constant. Solutions belonging to the second class have essentially different properties. They exist for continuously varying values of the dilaton coupling constant, but arise only for discrete values of the dilaton field at the horizon. For each given value of the dilaton coupling constant, there may exist several such solutions differing by the number of zeros of the shifted dilaton function in the subhorizon region and separating the domains of singular solutions.

1 Introduction

Now, after decades of purely theoretical research, a new era begins in the study of black holes. Thanks to such projects as the Event Horizon Telescope, LIGO, VIRGO it will be possible to obtain observational data and to test various theoretical models. Therefore, it is important to close some gaps in the theories of black holes that have not been resolved in due time. One of these not fully investigated questions is the dyonic black hole model in the presence of a dilaton field. Static spherically symmetric black holes with electric and / or magnetic charges are well studied, and solutions for them are represented by the Reissner-Nordström metric. However, in the presence of the dilaton field, the situation becomes much more complicated.

Dilatonic scalar field arises in a variety of theories including dimensional reductions, low-energy limit of string theory, various models of supergravity. A dilaton field naturally couples to gauge fields and thus can influence various physical phenomena. Such a dilaton effect is clearly manifested in models of black holes with gauge fields. Now purely electrically or magnetically charged black holes do not contain the RN solution as a special case. The analytical solutions for single-charged dilatonic black hole with any value of the dilaton coupling constant are known [1, 2], and in non-extremal case they are all singular on the inner horizon (except for the case of the zero dilaton coupling constant, which is trivial). If the black hole carries both magnetic and electric charges it admits the RN solution with a constant dilaton. Non-extremal dyonic solutions also turn out to be singular in the general case on one of the horizons, however, with the exception of some special cases. The study of the conditions for the existence of regular solutions (which do not have singularities on the horizons) is a rather important problem. It was not brought to an end so far, and we hope that this study will shed light on this issue.

*davydov@theor.jinr.ru
It should be mentioned that by now only two exact solutions for the static spherically symmetric dyon-dilaton black hole are known [1, 3, 4, 5]. They arise for special values of the dilaton coupling constant: $a = 1$ and $a = \sqrt{3}$. These solutions are regular. In the study of numerical solutions, it was found in [6] that for values of the dilaton coupling constant close (or coincident) to a sequence of triangular numbers,

$$a^2 = n(n + 1)/2, \quad n = 0, 1, 2, \ldots,$$

the corresponding solutions are also regular on both horizons. Taking into account that the first two terms of this sequence coincide with the values of $a$ corresponding to known exact solutions, it was suggested that regular black hole solutions in the dyon-dilaton system are possible only for these discrete values of the dyon coupling constant. This conjecture we shall call for brevity the triangular hypothesis.

The first two values of the dilaton coupling from this triangular sequence have a clear physical meaning: they arise in string theory and with the Kaluza-Klein dimensional reduction. Attempts were made to find physical theories in which the values of the dilaton constant, corresponding to other members of this sequence, may naturally appear. However, these attempts were unsuccessful [7, 8]. Therefore, the triangular hypothesis was in a suspended state: without an explicit physical justification, but with some purely mathematical indications in its favor. For example, in the study of extreme dyon black holes, it was found [9] that the solution for the dilaton is analytic on the horizon only for triangular values of the dilaton constant.

In this paper we verify the validity of the triangular hypothesis and find out the conditions for the existence of regular solutions for arbitrary values of the dilaton coupling constant in the model of a static spherically-symmetric dyonic dilaton black hole. In Section II we give a description of the model and find the reasons for the appearance of a triangular sequence (1). Since all known analytical solutions in this model are associated with solutions to Toda lattice equations, in Section III we try to find all configurations that can have a connection with Toda lattices and construct the corresponding solutions. Section IV is devoted to the investigation of numerical solutions for a much more extensive domain in the parameter space than was done in [6]. In the Conclusion we summarize and discuss the results obtained.

2 The Model

Dyon system with dilaton field can be described by the following action in $D \geq 4$ dimensions [6]:

$$S = \int \left[ R - \frac{D - 2}{4(D - 3)} (\nabla \varphi)^2 - \frac{e^{a\varphi}}{2} \left( F^2 + \frac{2}{(D - 2)!} G^2 \right) \right] \sqrt{-g} d^D x. \quad (2)$$

The gauge part of the action (2) contains the standard Maxwell two-form $F_{\mu\nu}$ and the additional Abelian $(D - 2)$-form $G_{\mu_1 \ldots \mu_{D-2}}$. Gauge fields interact with dilaton $\varphi$ via the exponential potential characterized by the coupling constant $a$. In comparison with the notations often encountered in the literature, we multiplied the dilaton by a constant factor $\frac{D - 2}{4\sqrt{D - 3}}$, depending on the number of spacetime dimensions, $D$. This led to the appearance of an unusual factor in the dilaton kinetic term, but many of the further formulas will have a simpler form.

A static spherically symmetric metric can be written as:

$$ds^2 = -h(r) dt^2 + h(r) s^2(r) dr^2 + \rho^{2/(D-2)}(r) d\Omega^2_{D-2}, \quad (3)$$

where $d\Omega^2_{D-2} = g_{MN}(y) dy^M dy^N$ is a metric on $(D - 2)$-dimensional sphere. The variable $s(r)$ represents the remaining gauge degree of freedom which will be fixed later. The curvature scalar density for such metric reads as:

$$R\sqrt{-g} = \frac{h' \rho'}{hs} + (D - 3)(D - 2)hs \rho^{2/(D-2)} + (D - 3)\rho^2 \frac{1}{(D - 2)\rho s} + \text{tot. div.} \quad (4)$$
The equations of motion for gauge field are quite simple:
\begin{align*}
\nabla_\mu (e^{a\varphi} F^{\mu\nu}) &= 0, \\
\nabla_\mu (e^{a\varphi} G^{\mu_1..\mu_{D-2}}) &= 0.
\end{align*}

The appropriate field ansatz preserving spherical symmetry can be written in the following form:
\begin{align*}
F_{tr} &= \frac{Q}{\rho} h se^{-a\varphi} \varepsilon_{tr}, \\
G_{M_1..M_{D-2}} &= P \varepsilon_{M_1..M_{D-2}},
\end{align*}

where $Q$ and $P$ are some constants, which can be called electric and magnetic charges; $\varepsilon_{tr}$ and $\varepsilon_{M_1..M_{D-2}}$ are antisymmetric tensors. We consider only the dyonic configuration, for which both charges are non-vanishing. Mention that in $D = 4$ case, one can assume that $F_2$ and $G_2$ are identical and describe the same electromagnetic field.

With the ansatz (7), the terms with gauge fields in the action (2) can be replaced by an effective potential. As a result, the effective one-dimensional action for the dyon-dilaton model (2) can be written as
\begin{align*}
S_{\text{eff}} &= \int \left( \frac{h'\rho'}{sh} + \frac{(D-3)\rho^2}{(D-2)\rho s} - \frac{(D-2)\rho \varphi^2}{4(D-3)s} - hs U(\rho, \varphi) \right) dr,
\end{align*}

where
\begin{align*}
U(\rho, \varphi) &= \frac{1}{\rho} \left( P^2 e^{a\varphi} + Q^2 e^{-a\varphi} \right) - (D-3)(D-2)\rho \frac{D-4}{D-2}.
\end{align*}

Varying it with respect to metric and dilaton variables, we obtain the remaining equations of motion. The metric function $s$ is just a Lagrange multiplier, and variation with respect to it provides the Hamiltonian constraint:
\begin{align*}
H &= \frac{h'\rho'}{sh} + \frac{(D-3)\rho^2}{(D-2)\rho s} - \frac{(D-2)\rho \varphi^2}{4(D-3)s} + hs U(\rho, \varphi) = 0.
\end{align*}

After that, one can choose the gauge as desired. The conventional choice is Schwarzschild gauge $s = h^{-1}$, in which the equations of motion for $\rho$, $h$ and $\varphi$ read as:
\begin{align*}
\frac{h''}{h} + \frac{2(D-3)}{D-2} \left( \frac{h\rho'}{\rho} \right)' &= -\frac{(D-3)h\rho^2}{(D-2)\rho^2} - \frac{(D-2)h\varphi^2}{4(D-3)s} - U_\rho, \\
\rho'' &= \frac{(D-3)\rho^2}{(D-2)\rho} - \frac{(D-2)\rho \varphi^2}{4(D-3)}, \\
(h\rho \varphi')' &= \frac{2(D-3)}{D-2} U_\varphi.
\end{align*}

As usual, $U_\rho$ and $U_\varphi$ are the corresponding partial derivatives of the potential $U$.

The key property of the dyonic configuration is that the derivative of the potential, $U_\varphi = \frac{a}{\rho} \left( P^2 e^{a\varphi} - Q^2 e^{-a\varphi} \right)$, vanishes when
\begin{align*}
\varphi &= a^{-1} \ln |Q/P| \equiv \bar{\varphi}.
\end{align*}

Therefore the constant dilaton $\varphi = \bar{\varphi}$ is a solution to the dilaton equation of motion Eq. (13), which is impossible for a single-charged configuration. As a result, the dyonic dilaton black hole admits Reissner-Nordström solution as a special case.

Indeed, the equation for $\rho$ reads as:
\begin{align*}
\rho'' &= \frac{(D-3)\rho^2}{(D-2)\rho}.
\end{align*}
It can be easily integrated, and the solution with Minkowski asymptotic is $\rho = r^{D-2}$. Then from the constraint (10) we find the remaining metric component:

$$h = 1 - \frac{2M}{r^{D-3}} + \frac{2|QP|}{(D-3)(D-2)r^{2(D-3)}},$$

where the new integration constant $M$ appears.

The event horizon in Schwarzschild gauge emerges when $h$ is vanishing. The equation (17) for $h = 0$ has two solutions, $r = r_\pm$:

$$r_\pm = \left(\frac{M}{A_1 \mp A_2}\right)^{1/(D-3)}, \quad \text{where} \quad A_1 = \frac{(D-3)(D-2)M^2}{2|QP|}, \quad A_2 = \sqrt{A_1^2 - A_1}.$$

With $A_2^2 < 0$ there would be a naked singularity, and with $A_2^2 = 0$ the above solution describes an extremal Reissner-Nordström black hole, when the two horizons coincide. However, our study is focused on the non-extremal case, when there are two distinct horizons, so in what follows we assume that $A_2 > 0$.

### 2.1 Dilaton ‘quantization’ on RN background

We see that static dilaton field provides Reissner-Nordström metric. But is the solution stable if dilaton acquires some dynamics? In order to check this, let us consider small deviations of the dilaton field from constant solution: $\varphi = \bar{\varphi} + \delta \varphi$, where $\delta \varphi \ll \bar{\varphi}$. For simplicity, we will not carry out a full stability analysis of the entire system, just trace the behavior of solutions for $\delta \varphi$ on a fixed RN background.

Then the dilaton equation (13) with the given RN metric and linearized with respect to the variable $\delta \varphi$ will look like:

$$\left(\left[(Mr^{3-D} - A_1)^2 - A_2^2\right] r^{D-2} \delta \varphi'\right)' = 2(D-3)^2 M^2 a^2 r^{2-D} \delta \varphi,$$

where we used the formula $A_1 h = (Mr^{3-D} - A_1)^2 - A_2^2$ following from the Eqs. (17–18). After the coordinate change, $x \equiv Mr^{3-D} - A_1$, the equation (19) takes a notably simple form:

$$\frac{d}{dx} \left[ (x^2 - A_2^2) \frac{d}{dx} \delta \varphi \right] = 2a^2 \delta \varphi.$$

In non-extremal case $A_2^2 > 0$, this is a Legendre differential equation. Its non-singular solutions in the region $x^2 \leq A_2^2$ between the two horizons are the Legendre polynomials $P_n$:

$$\delta \varphi = \delta \varphi_{\text{hor}} P_n(x A_2^{-1} x), \quad n = 0, 1, 2, \ldots.$$

Here $\delta \varphi_{\text{hor}}$ is the integration constant: $\delta \varphi$ is equal to $\pm \delta \varphi_{\text{hor}}$ on horizons located at $|x| = A_2$. From the Eq. (20) we find that the non-negative integer parameter $n$ is related to the dilaton coupling constant as following:

$$a^2 = n(n + 1)/2, \quad n = 0, 1, 2, \ldots.$$

Thus, here we encounter triangular numbers in the expression for the dilaton coupling constant. Earlier this sequence arose when considering series expansions and numerical solutions. Now, in addition to previous approaches, we find a physical reason for the appearance of triangular numbers: on a Reissner-Nordström background the non-diverging solutions for $\delta \varphi$ emerge only for these discrete values of dilaton coupling constant.
Unlike the two horizons case, on the extremal RN background with $A_2 = 0$, the non-singular solution to the Eq. (20) exists for any value of dilaton coupling constant:

$$\delta \varphi = C x^\nu, \quad a^2 = \nu(\nu + 1)/2, \quad C = \text{const}. \quad (23)$$

Obviously, the solution (23) is analytic only in case of triangular values of $a$. In [9] the same result was derived by considering the series expansions at the extremal horizon of the solution to the full system (11–13). Now we see that the non-analyticity of dilaton solution for $\nu \neq n$ may be derived from the dilaton behavior on RN background, both in the extremal and non-extremal cases.

3 The equivalent dynamical system

The results obtained in the linear approximation can change when the nonlinear dynamics is switched on. So now we consider the full dyon-dilaton system described by the effective action (8). However, instead of solving a cumbersome system of equations (11–13), we transform the action (8) to a more convenient form. It is easy to notice that its kinetic part simplifies in the gauge $s = \rho$, in which horizons $h = 0$ are located at $r \to \pm \infty$. Then the kinetic part is quadratic in $h'/h$ and $\rho'/\rho$. It is convenient to choose exponential variables for the metric:

$$\sigma e^\chi = \frac{2(D-3)}{D-2} |QP| h, \quad \sigma e^\gamma = 2(D-3)^2 h \rho^{2(D-3)/(D-2)}, \quad \sigma \equiv \text{sign}(h). \quad (24)$$

In addition, we replace the dilaton function $\varphi$ by its deviation from constant solution:

$$\psi = \varphi - \bar{\varphi}. \quad (25)$$

With the new variables, the effective action reads as:

$$\frac{2(D-3)}{D-2} S_{\text{eff}} = \int L_1(\gamma, \gamma') \, dr - \int L_2(\chi, \chi', \psi, \psi') \, dr. \quad (26)$$

In this form it represents the sum of two independent actions. And the latter describe just a one- and two-dimensional motion of ‘particles’ in the exponential potentials:

$$L_1 = \frac{\gamma'^2}{2} + \sigma e^\gamma, \quad (27)$$

$$L_2 = \frac{\chi'^2}{2} + \frac{\psi'^2}{2} + \sigma \left( e^{\chi + a\psi} + e^{\chi - a\psi} \right), \quad (28)$$

where the coordinate $r$ plays the role of ‘time’. It has long been observed that spherically symmetric dyon-dilaton black hole can be reformulated as a ‘Toda molecule’ [1, 4]. Mention that black hole’s exterior region corresponds to $h > 0$, and in subhorizon region one has $h < 0$. Therefore the lagrangians (27–28) are actually different in the two regions.

Obviously, the energy is conserved in both systems $L_1$ and $L_2$. The conservation laws are:

$$E_1 = \frac{\gamma'^2}{2} - \sigma e^\gamma = \text{const}, \quad (29)$$

$$E_2 = \frac{\chi'^2}{2} + \frac{\psi'^2}{2} - \sigma \left( e^{\chi + a\psi} + e^{\chi - a\psi} \right) = \text{const}. \quad (30)$$

However, the constants $E_1$ and $E_2$ are not independent: they should obey the Hamiltonian constraint (10) for the full system:

$$H = E_1 - E_2 = 0, \quad \Rightarrow \quad E_1 = E_2 \equiv E. \quad (31)$$
The one-dimensional system given by the Eq. (27) can be easily integrated:

\[
\begin{align*}
    h < 0, \text{ subhorizon:} & \quad e^\gamma = E \cosh^{-2} \left( \sqrt{E/2} (r - r_0) \right), \\
    h > 0, \text{ exterior:} & \quad e^\gamma = E \sinh^{-2} \left( \sqrt{E/2} (r - r_0) \right).
\end{align*}
\]

(32) (33)

In subhorizon region \( h < 0 \), the integration constant \( r_0 \) is not important. The solution vanishes at \( r \to \pm \infty \), which corresponds to the horizons \( h = 0 \). In the exterior region \( h > 0 \), the outer horizon is located at \( r \to +\infty \) or \( r \to -\infty \), where the solution approaches zero. The Minkowski asymptotic corresponds to the vicinity of the boundary \( r \to r_0 \). Indeed, with \( h \to 1 \), \( r \to r_0 \) the line interval for solution (33) can be written as:

\[
ds^2 = -dt^2 + \tilde{r}^2 d\Omega^2_{D-2}, \quad \text{where} \quad \tilde{r} \equiv [(D-3)|r - r_0|]^{-1/(D-3)},
\]

(34)

which is indeed a Minkowski space.

The two-dimensional system (28) seems to be non-integrable, in general. Only two exact solutions with \( a = 1 \) and \( a = \sqrt{3} \) are known so far. In the first case the solution can be easily found in terms of the variables \( \chi \pm \psi \). It coincides up to integration constants with the formulas for \( \gamma \) given by the Eqs. (32–33). The second known integrable case, \( a = \sqrt{3} \), is not so simple. The solution was found by reducing the equations of motion to a system of equations for the integrable Toda lattice [4, 5]. In our research, we want to put the question differently: instead of establishing a correspondence between the equations of motion of the dyon-dilaton system and the integrable Toda lattice for a given dilaton coupling constant, we will determine at what values of the coupling constant this correspondence exists.

3.1 Toda-related solutions

For a more general consideration, we consider the case when the dilaton may interact differently with the gauge forms \( F_{[2]} \) and \( G_{[D-2]} \), as was discussed, for example, in [1]. The corresponding lagrangian can be written as

\[
L_2 = \frac{\chi^2}{2} + \frac{\psi^2}{2} + \sigma \left( e^{\chi + a_2 \psi} + e^{\chi - a_1 \psi} \right),
\]

(35)

where there are two dilaton coupling constants \( a_1 \) and \( a_2 \). They must be of the same sign, otherwise the system will not have a constant dilaton solution, which is a distinctive feature of the dyon configuration that we would like to preserve. The lagrangian is invariant under the transformation \( a_1 \leftrightarrow a_2, \psi \to -\psi \). Therefore without loss of generality we can assume that \( a_1 \geq a_2 > 0 \).

In order to make the formulas simpler, we introduce a vector \( X = (\chi, \psi) \) and write the Hamiltonian for the system (35):

\[
H_2 = \frac{1}{2} \sum_{i=1}^{2} P_i^2 - \sigma \sum_{i=1}^{2} \exp \left( A_{ik} X^k \right).
\]

(36)

Here

\[
P_i = \frac{dX^i}{dr}, \quad A_{ik} = \begin{pmatrix} 1 & a_2 \\ 1 & -a_1 \end{pmatrix},
\]

(37)

and the expression \( A_{ik} X^k \) implies the sum for \( k = 1, 2 \). Exactly the same Hamiltonians in form were considered in the study of generalized Toda lattices. Not all such systems are integrable. However, a sufficient condition for integrability is well known. In order to use it, we can calculate the following matrix:

\[ \mathcal{C}_{ik} = 2 \frac{\{ A_{ik} X^l, \{ A_{im} X^m, H_2 \} \}}{\{ A_{il} X^l, \{ A_{km} X^m, H_2 \} \}}, \]

(38)
where \{\cdot, \cdot\} is a Poisson bracket. As was found in the study of Toda lattices, if \(C_{ik}\) is a Cartan matrix of some Lie algebra, then the system (36) can be explicitly integrated [10].

For the Hamiltonian (36) we find that
\[
C_{ik} = \begin{pmatrix}
2 & 2(1-a_1a_2) \\
2(1-a_1a_2) & 1+a_1^2/2
\end{pmatrix}.
\]

(39)

It is not difficult to calculate that \(C_{ik}\) can be a Cartan matrix only in four cases. The corresponding algebras and values of dilaton couplings are listed below:

| A_1 \oplus A_1 | A_2 | C_2 | G_2 |
|-----------------|-----|-----|-----|
| (2 0) \(a_1a_2 = 1\) | (2 -1) \(a_1 = a_2 = \sqrt{3}\) | (2 -1) \(a_1 = 3, a_2 = 2\) | (2 -1) \(a_1 = 3\sqrt{3}, a_2 = 5/\sqrt{3}\) |

(40)

Exactly the same list of Lie algebras and dilaton coupling constants (up to rescaling) was obtained in a series of papers by Ivashchuk and co-authors [11, 12, 13]. Note that, with the exception of the case \(A_1 \oplus A_1\), there are only individual values corresponding to integrable solutions. And when \(a_1 = a_2\) there are only two values: 1 and \(\sqrt{3}\). Of course, we can not say that there are no other integrable cases. However, there are no signs indicating the presence of a triangular sequence \(a^2 = n(n + 1)/2\), or any other sequence.

In the works [11, 12, 13], the authors considered solutions only in the exterior region. And we also need to obtain solutions in the subhorizon area, in order to verify the regularity of both horizons. Moreover, in these studies the authors did not construct explicit solutions for \(C_2\) and \(G_2\) cases. Therefore, now we partially reproduce, and partially supplement their results, using the same approach that was developed by Ivashchuk and co-authors in [14, 15].

The equations of motion for the lagrangian (35) are:
\[
\frac{d^2X^i}{dt^2} = \sigma \sum_{k=1}^{2} (A^T)^{ik} \exp(A_{kl}X^l), \quad i = 1, 2.
\]

(41)

First we need to find a linear transformation of variables \(X = KY\) with a constant matrix \(K_{ik}\), that reduces this system of equations to the form
\[
\frac{d^2Y^i}{dt^2} = -\sum_{k=1}^{2} B^{ik} \exp(C_{kl}Y^l), \quad i = 1, 2,
\]

(42)

where \(B^{ik}\) is some constant diagonal matrix. This system coincides with open Toda lattice equations corresponding to the Lie algebra with the Cartan matrix \(C_{ik}\).

It is easy to find that
\[
K = A^{-1}C = \begin{pmatrix}
\frac{2}{1+a_1^2} & \frac{2}{1+a_1^2} \\
\frac{2}{1+a_2^2} & \frac{2}{1+a_2^2}
\end{pmatrix}, \quad B = -\sigma C^{-1}AA^T = -\frac{\sigma}{2} \text{diag} \left(1 + a_2^2, 1 + a_1^2\right).
\]

(43)

An important parameters of the solution are components of the twice dual Weyl vector in the basis of simple roots of the corresponding Lie Algebra:
\[
n_i = 2 \sum_{k=1}^{2} (C^{-1})^{ik}.
\]

(44)
For the Cartan matrix 39 we have

\[
\begin{align*}
    n = & \left( \begin{array}{cc}
    \frac{a_1(1+a_2^2)}{a_1+a_2}, & \frac{a_2(1+a_2^2)}{a_1+a_2}
    \end{array} \right),
\end{align*}
\]

\[
\begin{array}{|c|c|c|c|c|}
    \hline
    & A_1 & A_1 & A_2 & C_2 \hline
    n_1 & 1 & 2 & 3 & 6 \\
    n_2 & 1 & 2 & 4 & 10 \\
    \hline
\end{array}
\]

The key part of the solution are the so-called fluxbrane polynomials:

\[
H_i(z) = c_i + \sum_{k=1}^{n_i} p_{ik} z^k(r), \quad i = 1, 2, \quad \text{where } z \equiv -\frac{2\sigma}{E}e^{-\sqrt{2E}(r-r_0)}. \quad (46)
\]

The four constants \(c_i > 0\), \(E > 0\) and \(r_0\) are the free parameters of solution which can be found from initial or boundary conditions. Mention that \(E\) and \(r_0\) are common parameter for systems \(L_1\) and \(L_2\). The ansatz for the system 42 reads as

\[
Y^i = -n_i \sqrt{E/2}(r - r_0) - \ln H_i(z), \quad i = 1, 2. \quad (47)
\]

Substituting it into the equations, we obtain the values of the parameters \(p_{ik}\) for which the ansatz satisfies equations. The resulting fluxbrane polynomials are given in the Appendix by the Eqs 58–63.

All coefficients of the fluxbrane polynomials 58–63 are positive when \(c_1, c_2 > 0\). They depend on the variable \(z\) which is non-negative in subhorizon region. As a result, the functions \(\ln H_i(z)\) are well-defined everywhere on the interval between the two horizons. In the exterior region \(z \leq 0\), and polynomials \(H_i(z)\) can take negative values. However, \(H_i(0) > 0\) which corresponds to the horizon \(r \to +\infty\). Therefore there exist such interval \(r_s < r < +\infty\), where \(H_i(z) > 0\). Recall that in the chosen gauge \(s = \rho\) the exterior region (from outer horizon to Minkowski space) corresponds to the interval \(r_0 \leq r < +\infty\), as it follows from the Eq. 43. With a suitable choice of parameters \(r_0\) and \(c_i\) we can always ensure that \(r_s < r_0\), and the solutions for \(Y^i\) are well-defined in the exterior region also.

It is interesting that by construction the asymptotics of \(Y^i\) are similar on both horizons:

\[
Y^i \big|_{r \to \pm \infty} = -n_i \sqrt{E/2}|r|. \quad (48)
\]

Now let us turn to the original functions, \(\chi\) and \(\psi\). Calculating \(X = KY\) we find:

\[
\begin{align*}
    \chi &= \frac{1}{a_1 + a_2} \sum_{i=1}^{2} a_i \ln \left( -\frac{\sigma E}{2} z H_i H_i^{-\frac{1}{n_i}}(z) \right), \\
    \psi &= \frac{2a_1a_2}{a_1 + a_2} \ln \left( H_1^{-\frac{1}{n_1}}(z) H_2^{-\frac{1}{n_2}}(z) \right).
\end{align*}
\]

As it should be, the function \(\chi\) goes to \(-\infty\) on both horizons:

\[
\chi \big|_{r \to \pm \infty} = -\sqrt{2E}|r|. \quad (51)
\]

The dilaton function on both horizons approaches constants:

\[
\begin{align*}
    r \to -\infty : \quad \psi &\to \psi_{\text{in.hor.}} = \frac{2a_1a_2}{a_1 + a_2} \ln \left( \frac{1}{p_{1n_1}} \frac{1}{p_{2n_2}} \right), \\
    r \to +\infty : \quad \psi &\to \psi_{\text{out.hor.}} = \frac{2a_1a_2}{a_1 + a_2} \ln \left( \frac{1}{c_1^{n_1}} \frac{1}{c_2^{n_2}} \right).
\end{align*}
\]

Thus, analytical solutions turn out to be regular on both horizons and outside the black hole.
It is worth mentioning that for \( a_1 = a_2 \) the lagrangian \( L_2 \) becomes invariant with respect to the reflection \( \varphi \rightarrow -\varphi \). As a result, the dilaton function \( \psi \) assumes the same modulus values on the horizons. For the expressions for the fluxbrane polynomials \( \tilde{c}_1, \tilde{c}_2 \) given in the Appendix we have in \( A_1 \otimes A_1 \) case: \( \psi_{\text{out.hor.}} = -\psi_{\text{in.hor.}} = \ln(c_2/c_1) \); in \( A_2 \) case: \( \psi_{\text{in.hor.}} = \psi_{\text{out.hor.}} = \ln(c_2/c_1) \).

The Minkowski asymptotic is achieved at \( \sigma = 1, r = r_0 \), which implies \( z_0 = -2/E \). Then the condition \( h(r_0) = 1 \) imposes an additional constraint

\[
H_1^{-2} h_2^{-2} (z_0) = \frac{2(D-3)}{D-2} |QP| \tag{54}
\]
on the parameters \( c_i \) and \( E \). Next, the asymptotic for the metric \( h = 1 - 2M \tilde{r}^{3-D} + \ldots \), where \( \tilde{r} \) is given by the Eq. (31), provides the relation between the mass parameter \( M \) and \( c_i, E \):

\[
M = \frac{1}{(D-3)(a_1 + a_2)} \sqrt{\frac{E}{2}} \sum_{i=1}^{2} a_i \left( 1 + \frac{4}{n_i E} \left. \frac{d \ln H_i(z)}{dz} \right|_{z=z_0} \right). \tag{55}
\]

Expanding dilaton function \( \varphi \) as \( \varphi = \varphi_\infty + \Sigma \tilde{r}^{3-D} + \ldots \), we can find that

\[
\varphi_\infty = \varphi + \psi(z_0), \quad \Sigma = \frac{2}{D-3} \sqrt{\frac{2}{E}} \left. \frac{d \psi}{dz} \right|_{z=z_0}. \tag{56}
\]
Thus, there are four equations (54) - (56) that allow us to express three parameters \( c_i, E \) in terms of four parameters \( \varphi_\infty, M, |QP|, \Sigma \). This means that those four parameters are not independent, as expected.

It is also possible to calculate the Hawking temperature of the black holes. We give the computations for the case \( D = 4 \), to avoid cumbersome factors that depend on the dimension of the spacetime:

\[
T_H = \frac{1}{2\pi} \left( \sqrt{g^{rr}} \frac{\partial}{\partial r} \sqrt{-g_{tt}} \right)_{r\rightarrow +\infty} = \frac{1}{2\pi} \left. (e^{\chi} e^\gamma) \right|_{r\rightarrow +\infty} = \frac{1}{4\pi \sqrt{2E}} \frac{1}{c_1} \frac{1}{c_2} \frac{1}{c_2^{-1} + c_2^{-1}} \tag{57}
\]
In case of single dilaton constant, \( a_1 = a_2 = a \), one has \( T = (4\pi \sqrt{2E})^{-1} \exp(a^{-1} \psi_{\text{out.hor.}}) \).

Summarizing, we can see that the configurations with one coupling constant, \( a = 1 \) and \( a = \sqrt{3} \), are the only ones that can be easily associated with the generalized Toda lattices related to Lie algebras. Therefore, there is no reason to extend the properties of these solutions to cases of other dilaton coupling values. In the next section, we show that, in fact, solutions with different values of the parameter \( a \) have essentially different properties.

4 Numerical solutions

The full dyon-dilaton system is most likely non-integrable for arbitrary values of dilaton coupling constant. However, it can be divided into two independent subsystems (27) and (28). The first one is explicitly integrated, so we need to solve numerically only the second one. This section is devoted to the construction of numerical solutions for the dynamical system given by the lagrangian \( L_2 \). Our goal is to build the regular solutions, i.e. for which the values of the metric and dilaton functions and their derivatives are finite on horizons. Therefore we will only consider a subhorizon region with \( h \leq 0 \). This means that we choose \( \sigma = -1 \) in the expression (28) for \( L_2 \).

In previous section we considered the connection between the system \( L_2 \) and the Toda molecule, but now it is more convenient to treat \( L_2 \) as describing a particle moving in the \((\chi, \psi)\)-plane in the potential \( e^{\chi-a\psi} + e^{\chi+a\psi} \). This allows to predict in advance the basic properties of solutions.
The free asymptotic motion at \( r \to \pm \infty \) happens when potential vanishes. The potential goes to zero if \( \chi \to -\infty \) and \( a|\psi| < |\chi| \). So, asymptotically the particle moves with constant velocity \( \chi', \psi' = \text{const} \), and the direction of the velocity vector can be any, provided that \( a|\psi'| < |\chi'| \). The regular horizon in the dyon-dilaton system implies \( \chi \to -\infty \), \( \psi \equiv \psi_{\text{hor.}} = \text{const} \). It corresponds to an asymptotic motion with a velocity vector parallel to the \( \chi \)-axis. We can choose the initial condition in such a way as to satisfy the requirement of regularity of one of the horizons: \( \psi'|_{r\to-\infty} = 0 \) and \( \psi|_{r\to-\infty} = \psi_{\text{hor.}} \). However, in the general case, the asymptotics near the second horizon can be different: the velocity vector can be unparallel to the \( \chi \)-axis, as \( r \to +\infty \). Only potentials possessing a hidden internal symmetry can guarantee a transition from one nonsingular asymptotic to another. We examined such symmetries in the previous section and made sure that they are rather rare.

The potential \( e^{\chi-a\psi} + e^{\chi+a\psi} \) has form of a symmetric valley. Thus the particle trajectory from \( r \to -\infty \) to \( r \to +\infty \) oscillates in \( \psi \)-direction and crosses the \( \chi \)-axis for several times. The number of intersections of the trajectory with the \( \chi \)-axis divides the set of solutions into families, each of which is characterized by its number of intersections, \( n \). The solutions which separate families with \( n \) and \( (n+1) \) intersections are separatrixes. They separate the solutions with different asymptotics at \( r \to +\infty \), because when \( n \) is odd, the signs of the asymptotic values \( \psi|_{r\to+\infty} \), \( \psi'|_{r\to+\infty} \) are opposite to the sign of the initial value, \( \psi_{\text{hor.}} \); when \( n \) is even, the signs are the same. The separatrix beginning at \( (\infty, \psi_{\text{hor.}}) \) will end at \( (-\infty, (-1)^n\psi_{\text{hor.}}) \) due to symmetry of the lagrangian \( L_2 \) with respect to reflection \( \psi \to -\psi \) and time inversion \( r \to -r \). Therefore the separatrix solutions have finite \( \psi \)-coordinate everywhere and correspond to the dyon-dilaton black holes with two regular horizons. Our goal is to construct such solutions numerically and find what values of the system parameters they correspond to.

The parameter \( E \) can be set equal to unity by the rescaling \( \chi \to \chi + \ln E \), \( r \to r/\sqrt{E} \). Thus, we have two parameters that determine the behavior of the solution: \( \psi_{\text{hor.}} \) and \( a \). For the convenience of testing the triangular hypothesis, we will use the parameter \( \nu \) instead of \( a \). They are related as \( \nu(\nu+1)/2 = a^2 \). According to the hypothesis, separatrix solutions must exist only for integer values of the parameter \( \nu \).

On the Fig. [1] we have depicted the sets of points on \((\nu, \psi_{\text{hor.}})\)-plane to which the numerically constructed separatrix solutions correspond. They form a set of lines, each of which corresponds to a family of solutions with \( n = 1, 2, 3, \ldots \) intersections of the \( \chi \)-axis. However, these families can be combined into two classes.

- The first class contains the two vertical lines \( \nu = 1 \) and \( \nu = 2 \). The corresponding solutions are the known integrable ones (with \( a = 1, \sqrt{3} \)). They correspond to discrete values of the dilaton coupling constant, while the parameter \( \psi_{\text{hor.}} \) can be arbitrary.

- The lines belonging to the second class are the curves that, for small values of the parameter \( \psi_{\text{hor.}} \), are close to vertical lines \( \nu = 3, 4, 5, \ldots \), but with an increase in the parameter, they are noticeably deviated from them. Thus, regular solutions for \( \nu > 2 \) can be found in the intervals \( \nu_{\text{min}} \leq \nu < n \), where \( n \) is a number of intersections with the \( \chi \)-axis for a given family of solutions. For \( \psi_{\text{hor.}} \to 0 \) the curves approach \( \nu = n \) straights, which corresponds to the solutions obtained in the linear approximation for small values of the dilaton. However, for large \( \psi_{\text{hor.}} \), the non-linearity deforms the curves, which then tend to straight \( \nu = \nu_{\text{min}}^n < n \). For relatively small values of dilaton coupling constant (\( \nu \leq 6 \)), the intervals \( (\nu_{\text{min}}^n, n) \) are very narrow. Apparently, therefore, it was conjectured that they reduce to discrete values \( \nu = n \). For large \( \nu \), the intervals become quite wide.

The qualitative behavior of the solutions depends not only on the parameter \( \nu \), but also on \( \psi_{\text{hor.}} \). For given \( \nu \) there is only a discrete set of values \( \{\psi_{\text{hor.}}^1, \ldots, \psi_{\text{hor.}}^k\} \), \( k = k(\nu) = 0, 1, 2, \ldots \), which corresponds to regular solutions. This set can be empty for \( \nu \lesssim 12.07 \), because there
Figure 1: The curves $\psi_{\text{hor}}(\nu)$ corresponding to non-extremal regular black hole solutions. The regions between the curves are domains of solutions singular on one of the horizons.

are gaps between the intervals $(\nu_{\text{min}}^n, \nu = n)$, where there are no regular solutions. But for $\nu \gtrsim 12.07$ the intervals completely cover the $\nu$-axis and even start overlapping each other for $\nu \gtrsim 12.96$, so there can be one or two distinct regular solutions for given $\nu$. In the case of overlap, the solutions differ in the number of zeros of the dilaton function, which can be equal to $[\nu + 1]$ or $[\nu + 2]$, where $\lceil \cdot \rceil$ denotes the integer part of a number. For $\nu \gtrsim 30$ the overlap can be double (resulting in three distinct solutions with $[\nu + 1]$, $[\nu + 2]$ and $[\nu + 3]$ zeroes of the dilaton functions), e.t.c. Probably, for $\nu \to \infty$ the overlap can be infinite, however, it is difficult to verify by numerical calculations.

We can formulate the result of our numerical study as follows: the spectrum of dilaton coupling constant corresponding to regular solutions is discrete for $n \leq 2$, continuous with gaps for $2 < \nu \lesssim 12$, and continuous for $\nu \gtrsim 12.07$. This disproves the triangular hypothesis that the spectrum should only be discrete. However, now numerical results allow us to put forward the hypothesis that the spectrum of a parameter $\psi_{\text{hor}}$ must be discrete, with the exception of two special cases with $a = 1, \sqrt{3}$, for which the spectrum is continuous.

It is important to mention that the authors of the triangular hypothesis themselves wrote in their work [6] that “the precise numerical value of $a$ begins to show some dependence on the initial data for large values of $\psi_{\text{hor}}$.” (they actually used another notations for dilaton coupling constant and dilaton value at horizon). They found that “even for large $a$ the largest values of $\psi_{\text{hor}}$ consistent with asymptotically flat solutions yield numerical values of $a$ which agree with the [triangular] series to 0.1%. We have numerically checked terms up to $a = \sqrt{21}$ in the series”. Then they concluded that “it would appear to be merely the result of numerical errors”. However, in present work, we checked the sequence up to $a = \sqrt{465}$ and for much larger values of $\psi_{\text{hor}}$. Look, for example, on the Fig. 1 where we present the graphics for $a \leq \sqrt{105}$. It is easy to see that for the range $a \leq \sqrt{21}$ ($\nu \leq 6$) considered in [6], deviations from the triangular sequence are really small. However, then they become more and more significant, and they can not be attributed in any way to the errors of numerical integration.

By the way, our choice of the gauge $s = \rho$ is especially convenient for numerical integration, since the most problematic areas for numerical integration are the areas near the horizons. And in this gauge they are stretched to infinite intervals, which greatly simplifies the verification of the convergence of the numerical scheme. For example, for each numerical solution we checked the
feasibility of the energy constraint \((30)\). If the solutions contained errors, they would not satisfy it. However, for all solutions the constraint was satisfied with sufficient accuracy. We used the standard MAPLE tools for the numerical integration of differential equations with an absolute and relative errors of not more than \(10^{-13}\).

Let us make some more remarks. Consider the transition to extremal black hole solution, i.e. when the distance between the horizons goes to zero. For every solution the number of intersection with \(\chi\)-axis (zeroes of the dilaton function) does not depend on the distance between the horizons. As a result, the dilaton function will oscillate rapidly, with a ‘period’ of oscillations tending to zero. Its contribution into energy density, \(\psi'^2/2\), will diverge if the amplitude of the oscillations associated with the parameter \(\psi_{\text{hor}}\) does not tend to zero. However, for a given value of the dilaton coupling constant, the parameter \(\psi_{\text{hor}}\) can be set to zero, provided that the regularity of solutions is maintained, for only two cases: \(a = 1, \sqrt{3}\). Otherwise, the spectrum of \(\psi_{\text{hor}}\) is discrete, and the transition to an extreme black hole is singular.

5 Conclusion

We studied static spherically-symmetric non-extremal black hole solutions in the dyon-dilaton system, and found that the condition for the existence of regular solutions (non-singular on both horizons and with Minkowski asymptotic) leads to discreteness in the parameter space of the model. However, the entire set of regular solutions is divided into two essentially different classes.

The first class includes solutions arising in configurations with a few special values of the dilaton coupling constant(s), which allow one to establish a connection with solutions for Toda lattices. Even in the presence of two dilaton coupling constants there are only four families of solutions in this class that correspond to Lie algebras \(A_1 \oplus A_1\), \(A_2\), \(C_2\) and \(G_2\). From this point of view, there is absolutely no confirmation of the triangular hypothesis.

For the existence of regular solutions belonging to the second class, it is not required that the dilaton coupling constant takes on any special values. At small values of \(a\), there are intervals in which there are no regular solutions, but these ‘voids’ disappear after \(a \gtrsim 8.9\). In this class there is, apparently, a countable number of families of regular solutions. However, in contrast to solutions of the first class, solutions from the second class are separatrices that separate singular solutions with different numbers of zeros of the dilaton function. For a given value of the dilaton coupling constant, there can be several regular solutions differing by the number of zeros. Without changing the dilaton coupling constant, it is impossible to go from one regular solution to another with a continuous change in the parameters of the system.

This property resembles a kind of quantization for dyonic dilaton black holes. Indeed, the dilaton coupling constant is an external parameter in the model, in the real world it is most probably fixed and its value is determined by some fundamental theory. If its value is large enough, for example, \(a = 9.52\), then there may exist regular dyonic black holes with 12 and 13 zeroes of dilaton function in the subhorizon region. A nontrivial question arises: how can a transition from one state to another occur if they are separated by singular configurations? Of course, the non-stationary processes go far beyond our study of static solutions. Nonetheless, the process of transition between regular solutions, separated by singular states, can be accompanied by intriguing physical phenomena. Taking into account that the dyon-dilaton models are used to describe condensed matter systems \([16, 17]\), configurations with regular solutions separated by singular states, similar to those studied in this paper, can find application in this field.

The discreteness in the parameter space corresponding to regular solutions is also observed with various modifications of the model, such as adding a cosmological constant \([6]\) or higher-curvature corrections \([18]\). Indeed, as we learned from the verification of the triangular hypothesis, the discreteness arises when considering the dilaton on the Reissner-Nordström background, and the
smooth inclusion of extra terms will inevitably coexist with the discreteness, at least to some extent.

A possible topic for further research is also the study of rotating dyonic dilaton black holes [19, 20]. They are also characterized by the presence of a special value of the dilaton coupling constant, which is $a = \sqrt{3}$ [21]. One can expect that for rotating solutions the discreteness in the parameter space also manifests itself in a nontrivial fashion. It would be especially interesting to explore the system of dyonic diholes [22, 23, 24] in presence of a dilaton field.

This work was financially supported by the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.0008) and by the Russian Foundation for Fundamental Research under grant 17-02-01299.

Appendix: Fluxbrane polynomials

Here we give fluxbrane polynomials that are part of the solutions (47) of the open Toda lattice equations (42) corresponding to the Lie algebras arising in the models of dyon-dilaton black holes.

Case $A_1 \oplus A_1$:

$$H_1 = c_1 + \frac{(1 + a_2^2)}{8c_1}z, \quad H_2 = c_2 + \frac{(1 + a_1^2)}{8c_2}z, \quad \text{where} \quad a_1a_2 = 1.$$  \hspace{1cm} (58)

Case $A_2$:

$$H_1 = c_1 + \frac{c_2}{2c_1}z + \frac{1}{16c_2}z^2, \quad H_2 = c_2 + \frac{c_1}{2c_2}z + \frac{1}{16c_1}z^2.$$  \hspace{1cm} (59)

Case $C_2$:

$$H_1 = c_1 + \frac{5c_2}{8c_1}z + \frac{5^2c_1}{2c_2}z^2 + \frac{5^3}{210^2c_1}z^3,$$
$$H_2 = c_2 + \frac{5c_1^2}{4c_2}z + \frac{5^2}{26}z^2 + \frac{5^3c_2}{28^2c_1^2}z^3 + \frac{5^4}{2143^2c_2}z^4.$$  \hspace{1cm} (60)

Case $G_2$:

$$H_1 = c_1 + \frac{7c_2}{6c_1}z + \frac{7^2c_2}{48c_2}z^2 + \frac{7^3}{24^2}z^3 + \frac{7^4c_2}{28^3c_1}z^4 + \frac{7^5c_1}{20^23^5c_2}z^5 + \frac{7^6}{212^23^7c_1}z^6,$$
$$H_2 = c_2 + \frac{7c_1^2}{2c_2}z + \frac{7^2c_1}{16}z^2 + \frac{7^3c_2}{23^3c_1}z^3 + \frac{7^4(c_1^5 + c_2^3)}{28^33^8c_1c_2}z^4 + \frac{7^6}{23^412^35^2c_2}z^5 + \frac{7^6(3^3c_1^5 + 5^2c_2^3)}{212^33^7c_1^2c_2^2}z^6 + \frac{7^7c_1}{2113^75^2c_2}z^7 + \frac{7^8}{2163^85^2c_1}z^8 + \frac{7^9c_2}{217312^5c_1^2}z^9 + \frac{7^{10}}{220^312^54c_2}z^{10}.$$  \hspace{1cm} (63)

References

[1] G. W. Gibbons and K. i. Maeda, Nucl. Phys. B 298 (1988) 741.
[2] D. Garfinkle, G. T. Horowitz and A. Strominger, Phys. Rev. D 43 (1991) 3140. Erratum: [Phys. Rev. D 45 (1992) 3888].
[3] P. Dobiasch and D. Maison, Gen. Rel. Grav. 14 (1982) 231.
[4] G. W. Gibbons, Nucl. Phys. B 207 (1982) 337.
[5] S. C. Lee, *Phys. Lett.* **149B** (1984) 98.

[6] S. J. Poletti, J. Twamley and D. L. Wiltshire, *Class. Quant. Grav.* **12** (1995) 1753, [Erratum: *Class. Quant. Grav.* **12** (1995) 2355]

[7] G. W. Gibbons, D. Kastor, L. A. J. London, P. K. Townsend and J. H. Traschen, *Nucl. Phys. B* **416** (1994) 850.

[8] M. Nozawa, *Class. Quant. Grav.* **28** (2011) 175013.

[9] D. Gal’tsov, M. Khramtsov and D. Orlov, *Phys. Lett. B* **743** (2015) 87.

[10] B. Constant, *Adv. Math.* **34** (1979) 195.

[11] V. D. Ivashchuk, *J. Geom. Phys.* **86** (2014) 101.

[12] M. E. Abishev, K. A. Boshkayev, V. D. Dzhunushaliev and V. D. Ivashchuk, *Class. Quant. Grav.* **32** (2015) no.16, 165010.

[13] M. E. Abishev, K. A. Boshkayev and V. D. Ivashchuk, *Eur. Phys. J. C* **77** (2017) no.3, 180.

[14] V. D. Ivashchuk, *Class. Quant. Grav.* **19** (2002) 3033.

[15] A. A. Golubtsova and V. D. Ivashchuk, [arXiv:0804.0757 [nlin.SI]](https://arxiv.org/abs/0804.0757).

[16] N. Kundu, P. Narayan, N. Sircar and S. P. Trivedi, *JHEP* **1303** (2013) 155.

[17] A. Amoretti, M. Baggioli, N. Magnoli and D. Musso, *JHEP* **1606** (2016) 113.

[18] C. M. Chen, D. V. Gal’tsov and D. G. Orlov, *Phys. Rev. D* **78** (2008) 104013.

[19] G. Clément, *Phys. Lett. A* **118** (1986) 11.

[20] D. Rasheed, *Nucl. Phys. B* **454** (1995) 379.

[21] D. V. Galtsov, A. A. Garcia and O. V. Kechkin, *Class. Quant. Grav.* **12** (1995) 2887.

[22] I. Cabrera-Munguia, C. Lmmerzahl and A. Macas, *Phys. Lett. B* **743** (2015) 357.

[23] H. García-Compeán and V. S. Manko, *Phys. Lett. B* **748** (2015) 366.

[24] G. Clément and D. Gal’tsov, *Phys. Lett. B* **773** (2017) 290.