Temporal Behavior of the Conditional and Gibbs’ Entropies

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We study the temporal approach to equilibrium of the Gibbs’ and conditional entropies for both invertible deterministic dynamics as well as non-invertible stochastic systems in the presence of white noise. The conditional entropy will either remain constant or monotonically increase to its maximum of zero. However, the Gibbs’ entropy may have a variety of patterns of approach to its final value ranging from a monotone increase or decrease to an oscillatory approach. We have illustrated all of these behaviors using examples in which both entropy dynamics can be determined analytically.

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I. INTRODUCTION

A variety of various measures of dynamic behavior carry the name of entropy. Two have proved to be especially intriguing in the examination of the temporal evolution of dynamical systems when considered from an ensemble point of view.

One of these is known as the conditional entropy. Convergence properties of the conditional entropy have been extensively studied because ‘entropy methods’ have been known for some time to be useful for problems involving questions related to convergence of solutions in partial differential equations \[ E \in R^2 \]. Their utility can be traced, in some instances, to the fact that the conditional entropy may serve as a Liapunov functional \[ \ell \].

Another type of entropy is the Gibbs’ entropy, which is strongly related to an extension of the equilibrium entropy that was introduced by Gibbs \[ 8 \] to a time dependent situation. This has been considered by a number of authors recently Ruelle \[ 2, 10 \], Nicolis and Daems \[ 11 \], Daems and Nicolis \[ 12 \] and Bag et al. \[ 13, 14 \], Bag \[ 15, 17 \].

Here we compare and contrast the temporal evolution of the conditional and Gibbs’ entropies in a variety of dynamical settings. Our primary considerations are stochastic non-invertible systems with additive white noise, but we also do discuss the two entropy behaviors in systems with invertible dynamics.

The organization of the paper is as follows. Section II gives some basic background, definition of steady state Gibbs’ entropy, and extension of this to time dependent situations, and shows how the conditional entropy can be considered a generalization of the time dependent Gibbs’ entropy. Section III looks at the behavior of the Gibbs’ entropy and the conditional entropy in systems with invertible dynamics. The general results of this section are illustrated with three specific examples. This is extended to non-invertible systems in Section IV where we cite a number of results from \[ R \] on the behavior of the conditional entropy and contrast these with the behavior of the Gibbs’ entropy. These considerations are illustrated with two detailed examples drawn from dynamical systems perturbed by noise. The paper concludes with a summary and discussion in Section V where we show how our results illuminate the connection between a previously postulated dynamic analog of the non-equilibrium thermodynamic entropy and the entropy increase of the second law of thermodynamics.

II. GIBBS’ AND CONDITIONAL ENTROPIES

Let \( X \) be a phase space and \( \mu \) a reference measure on \( X \). Denote the corresponding set of densities by \( D(X) \), or \( D \) when there will be no ambiguity, so \( f \in D \) means \( f \geq 0 \) and \( \int_X f(x) \, dx = 1 \) (for integrals with respect to the reference measure we use the notation \( \int f(x) \, dx \) rather than \( \int f(x) \, d\mu(x) \)). Let \( \{ P^t \}_{t \geq 0} \) be a semigroup of Markov operators on \( L^1(X) \), i.e. \( P^t f_0 \geq 0 \) for an initial density \( f_0 \geq 0 \), \( \int P^t f_0(x) \, dx = \int f_0(x) \, dx \), and \( P^{t+s} f_0 = P^t(P^s f_0) \). If the group property holds for \( t, s \in \mathbb{R} \), then we say that \( P^t \) is invertible. If it holds only for \( t, s \in \mathbb{R}^+ \) we say that \( P^t \) is non-invertible. If there is a density \( f_\ast \) such that \( P^t f_\ast = f_\ast \) for all \( t > 0 \), \( f_\ast \) is called a stationary density of \( P^t \).

In his seminal work Gibbs \[ 8 \], assuming the existence of a system steady state density \( f_\ast \) on the phase space \( X \), introduced the concept of the index of probability given by \( \log f_\ast(x) \) where “log” denotes the natural logarithm. He then identified the entropy in a steady state situation with the average of the index of probability

\[ H_G(f_\ast) = - \int_X f_\ast(x) \log f_\ast(x) \, dx, \quad (2.1) \]
and we call this the equilibrium or steady state Gibbs’ entropy. If entropy is to be an extensive quantity (in accord with experimental evidence) then Definition 2.1 is unique up to a multiplicative constant. It is for this reason that we extend the definition of the steady state Gibbs’ entropy to time dependent (non-equilibrium) situations and say that the time dependent Gibbs’ entropy of a density \( f(t,x) \) is defined by

\[
H_G(f) = - \int_X f(t,x) \log f(t,x) \, dx.
\]

We define the conditional entropy as

\[
H_c(f|f_s) = - \int_X f(t,x) \log \frac{f(t,x)}{f_s(x)} \, dx.
\]

This section considers the behavior of the Gibbs’ entropy and the conditional entropy in situations where the dynamics are invertible in the sense that they can be run forward or backward in time without ambiguity. To make this clearer, consider a phase space \( X \) and a dynamics \( S_t : X \to X \). For every initial point \( x_0 \), the sequence of successive points \( S_t(x_0) \), considered as a function of time \( t \), is called a trajectory. In the phase space \( X \), if the trajectory \( S_t(x_0) \) is nonintersecting, or intersecting but periodic, then at any given time \( t \) such that \( x_t = S_t(x_0) \) we could change the sign of time by replacing \( t \) by \(-t\), and run the trajectory backward using \( x_t \) as a new initial point in \( X \). Thus in this case we have a dynamics that may be reversed in time completely unambiguously. Dynamics with this character are known variously as time reversal invariant or reversible.

We formalize this by introducing the concept of a dynamical system \( \{S_t\}_{t \in \mathbb{R}} \), which is simply any group of transformations \( S_t : X \to X \) having the two properties: 1. \( S_0(x) = x \); and 2. \( S_t(S_{t'}(x)) = S_{t+t'}(x) \) for \( t, t' \in \mathbb{R} \) or \( \mathbb{Z} \). Since, from the definition, for any \( t \in \mathbb{R} \), we have \( S_t(S_{-t}(x)) = x = S_{-t}(S_t(x)) \), dynamical systems are invertible in the sense discussed above since they may be run either forward or backward in time.

Our first result from [18] shows that the conditional entropy of any invertible system is uniquely determined by the system preparation and does not change with time. This is formalized in

**Theorem 1** [18, Theorem 3] If \( P^t \) is an invertible Markov operator and has a stationary density \( f_s \), then the conditional entropy is constant and equal to the value determined by \( f_s \) and the choice of the initial density \( f_0 \) for all time \( t \). That is,

\[
H_c(P^tf_0|f_s) = H_c(f_0|f_s)
\]

for all \( t \).

More specifically, when considering a deterministic dynamics \( S^t : X \to X \), the corresponding Markov operator is also known as the Frobenius Perron operator [21] and is given by

\[
P^t f_0(x) = f_0(S^{-t}(x))|J^{-t}(x)|,
\]

where \( J^{-t}(x) \) denotes the Jacobian of \( S^{-t}(x) \). Further,

\[
H_c(P^tf_0|f_s) = - \int_X P^t f_0(x) \log \left[ \frac{P^t f_0(x)}{f_s(x)} \right] \, dx
\]

\[
= - \int_X \frac{f_0(S^{-t}(x))}{|J^{-t}(x)|} \log \left[ \frac{f_0(S^{-t}(x))}{f_s(S^{-t}(x))} \right] \, dx
\]

\[
= - \int_X f_0(y) \log \left[ \frac{f_0(y)}{f_s(y)} \right] \, dy
\]

\[
= H_c(f_0|f_s)
\]

as expected from Theorem 1. This behavior is, however, quite different from what is seen in the Gibbs’ entropy since

\[
H_G(P^tf_0) = - \int_X P^t f_0(x) \log [P^t f_0(x)] \, dx
\]

\[
= - \int_X \frac{f_0(S^{-t}(x))}{|J^{-t}(x)|} \log \left[ \frac{f_0(S^{-t}(x))}{|J^{-t}(x)|} \right] \, dx
\]

\[
= - \int_X f_0(y) \log \left[ \frac{f_0(y)}{|J^{-t}(y)|} \right] \, dy
\]

\[
= H_G(f_0) + \int_X f_0(y) \log |J^{-t}(y)| \, dy.
\]

Thus, in spite of the fact that the conditional entropy is constant for an invertible system, the Gibbs’ entropy may continually change, and satisfies

\[
H_G(P^tf_0) - H_G(f_0) = \int_X f_0(y) \log |J^t(y)| \, dy.
\]

Note in particular that for the Gibbs’ entropy to be an increasing function of time, we must have an expanding dynamics in the sense that

\[
\int_X f_0(y) \left\{ \log |J^t(y)| - \log |J^{t'}(y)| \right\} \, dy > 0 \quad \text{for} \quad t > t'.
\]
When the Jacobian is constant,
\[ H_G(P^t f_0) - H_G(f_0) = \log |J^t|, \tag{3.4} \]
illustrating that the Gibbs’ entropy \( H_G(P^t f_0) \) may either deviate from or approach the initial entropy \( H_G(f_0) \) depending on the value of \(|J^t|\). If the Lebesgue measure is preserved so \(|J^t| = 1\), then \( H_G \) is constant.

Taking an even more specific example, if the dynamics corresponding to the invertible Markov operator are described by the system of ordinary differential equations
\[
\frac{dx}{dt} = F_i(x) \quad i = 1, \ldots, d \tag{3.5}
\]
operating in a region \( X \subset \mathbb{R}^d \) with initial conditions \( x_i(0) = x_{i,0} \), then \[21\] the evolution of \( f(t, x) \equiv P^t f_0(x) \) is governed by the generalized Liouville equation
\[
\frac{\partial f}{\partial t} = -\sum_i \frac{\partial (f F_i)}{\partial x_i}. \tag{3.6}
\]
If the stationary density \( f_s \) exists, it is given by the solution of
\[
\sum_i \frac{\partial (f_s F_i)}{\partial x_i} = 0. \tag{3.7}
\]
Note that the constant function \( f_s \equiv 1 \), meaning that the flow defined by Eq. \[3.5\] preserves the Lebesgue measure, is a stationary solution of Eq. \[3.6\] if and only if
\[
\sum_i \frac{\partial F_i}{\partial x_i} = 0, \tag{3.8}
\]
but if \( X \) has an infinite Lebesgue measure then \( f_s \) is not integrable, so there will be no stationary density.

The rate of change of the Gibbs’ entropy is given by
\[
\frac{dH_G}{dt} = \int_X f \sum_i \frac{\partial F_i}{\partial x_i} dx, \tag{3.9}
\]
and so it is only in Lebesgue measure preserving dynamics (like Hamiltonian systems), for which Eq. \[3.1\] holds, that \( H_G \) will be constant. This was first noted by Gibbs \[3\] pp. 143-4 and much later pointed out in \[28\] and proved in general in \[20\], as emphasized in \[11\]. If
\[
\sum_i \frac{\partial F_i}{\partial x_i} > 0, \tag{3.10}
\]
then the Gibbs’ entropy will increase.

**Example 1** To illustrate these points, consider the continuous time dynamical system on \( \mathbb{R}^2 \)
\[
\frac{dx}{dt} = Fx \tag{3.11}
\]
where \( F = (F_{ij}) \) is a \( 2 \times 2 \) matrix. It can be solved exactly and its solution is given by
\[
x(t) = e^{tF} x(0), \quad \text{where} \quad e^{tF} = \sum_{n=0}^{\infty} \frac{t^n}{n!} F^n.
\]
The evolution of the density \( f \) under the action of this flow is determined by the solution of the Liouville equation
\[
\frac{\partial f}{\partial t} = -\frac{\partial (F_1(x)f)}{\partial x_1} - \frac{\partial (F_2(x)f)}{\partial x_2}, \tag{3.12}
\]
where \( F_j(x) = F_{j1} x_1 + F_{j2} x_2, \ j = 1, 2 \). If the initial density is given by \( f_0(x) \), then the general solution of Eq. \[3.13\] is given by
\[
f(t, x) = |\det e^{-tF}| f_0(e^{-tF} x).
\]
Since
\[
|\det e^{-tF}| = e^{-t \text{Tr} F},
\]
where \( \text{Tr} F = F_{11} + F_{22} \) is the trace of the matrix \( F \), we have
\[
P^t f_0(x) = e^{-t \text{Tr} F} f_0(e^{-tF} x).
\]
Consequently, we obtain
\[
H_G(P^t f_0) = H_G(f_0) + t \text{Tr} F.
\]
If \( \lambda_1, \lambda_2 \) are the eigenvalues of \( F \) then \( \text{Tr} F = \lambda_1 + \lambda_2 \). Thus when \( \text{Tr} F < 0 \) then the system has a one dimensional attractor and \( H_G(P^t f_0) \rightarrow -\infty \) as \( t \rightarrow \infty \), while if \( \text{Tr} F > 0 \) then the dynamics are sweeping \[21\]. However, in this example there will be no stationary density and the conditional entropy is thus not defined.

**Example 2** Consider the second order system
\[
m \frac{d^2 y}{dt^2} + \gamma \frac{dy}{dt} + \omega^2 y = 0 \tag{3.13}
\]
with constant coefficients \( m, \gamma \) and \( \omega \). Introduce the velocity \( v = \dot{y} \) as a new variable. Then Eq. \[3.13\] is equivalent to the system
\[
\frac{dy}{dt} = v \tag{3.14a}
\]
\[
m \frac{dv}{dt} = -\gamma v - \omega^2 y, \tag{3.14b}
\]
thus by writing
\[
x = \begin{pmatrix} y \\ v \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma/m \end{pmatrix}
\]
we recover Eq. \[3.11\] Since \( \text{Tr} F = -\gamma/m \), we obtain
\[
P^t f_0(x) = e^{\gamma t/m} f_0(e^{-tF} x)
\]
and

\[ H_G(P^tf_0) = H_G(f_0) - \frac{\gamma t}{m}. \]

As in Example 1 when there is damping so \( \text{Tr } F = -\gamma/m < 0 \), then \( H_G(P^tf_0) \to -\infty \) as \( t \to \infty \). Further, as in the previous example there is no stationary density \( f_\ast \) and so \( H_c(f_\ast) \) is not defined.

**Example 3** Let \( X \) be the unit circle in \( \mathbb{R}^2 \). If

\[ F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

then Eq. (3.11) has the general solution

\[ x(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x(0) \]

and \( x(t) \in X \) for \( x(0) \in X \). If \( \mu \) is the Lebesgue measure on \( X \) then the corresponding Perron-Frobenius operator is given by

\[ P^t f_0(x) = f_0(x(-t)) \quad \text{for} \quad f_0 \in L^1(X) \]

and \( f_\ast(x) = \frac{1}{2\pi} 1_X(x) \) is the stationary density of \( P^t \). We then have \( H_c(P^t f_0|f_\ast) = H_G(P^t f_0) - \log 2\pi \)

\[ = H_G(f_0) - \log 2\pi = H_c(f_0|f_\ast) \]

and both entropies are constant and fixed by the initial system preparation \((f_0)\).

**IV. ENTROPY BEHAVIOR AND NON-INVERTIBLE DYNAMICS**

**A. Asymptotic stability and conditional entropy**

A semigroup of Markov operators \( P^t \) on \( L^1(X) \) is said to be asymptotically stable if there is a stationary density \( f_\ast \) of \( P^t \) such that for all initial densities \( f_0 \)

\[ \lim_{t \to \infty} P^t f_0 = f_\ast \]

(here the limit denotes convergence in \( L^1(X) \)). Systems with dynamics that are asymptotically stable must, by necessity, be non-invertible \[21\] Remark 4.3.1].

**Theorem 2** \[30\] Let \( P^t \) be a semigroup of Markov operators on \( L^1(X) \) and \( f_\ast \) be a stationary density. Then for every density \( f_0 \) the conditional entropy \( H_c(P^t f_0|f_\ast) \) is a nondecreasing function of \( t \).

For a given density \( f_0 \) the conditional entropy \( H_c(P^t f_0|f_\ast) \) is bounded above by zero. Thus we know that it has a limit as \( t \to \infty \). Our next result connects the temporal convergence properties of \( H_c \) with those of \( P^t \).

**Theorem 3** \[18, Theorem 1\] Let \( P^t \) be a semigroup of Markov operators on \( L^1(X) \) and \( f_\ast \) be a stationary density. Then

\[ \lim_{t \to \infty} H_c(P^t f_0|f_\ast) = 0 \]

for all \( f_0 \) with \( H_c(f_0|f_\ast) > -\infty \) if and only if \( P^t \) is asymptotically stable.

A consequence of the convergence of the conditional entropy to zero is that

\[ \lim_{t \to \infty} \int h(x) P^t f_0(x) \, dx = \int h(x) f_\ast(x) \, dx \]

for any measurable function \( h \) for which the integral

\[ \int e^{r h(x)} f_\ast(x) \, dx \]

is finite for all \( r \) in some neighborhood of zero \[31\] Lemma 3.1. Since the conditional and Gibbs’ entropies are related by

\[ H_G(P^t f_0) = H_c(P^t f_0|f_\ast) - \int P^t f_0(x) \log f_\ast(x) \, dx, \]

Theorem 3 implies

**Theorem 4** Let \( P^t \) be an asymptotically stable semigroup of Markov operators on \( L^1(X) \) with a stationary density \( f_\ast \) such that \( \int f_\ast^{1+r}(x) \, dx < \infty \) for all \( r \) in some neighborhood of zero. Then

\[ \lim_{t \to \infty} H_G(P^t f_0) = H_G(f_\ast) \]

for all \( f_0 \) with \( H_c(f_0|f_\ast) > -\infty \).

**B. Effects of noise in continuous time systems**

In this section, we consider the behavior of the entropies \( H_G(P^t f_0) \) and \( H_c(P^t f_0|f_\ast) \) when the dynamics are described by the stochastically perturbed system

\[ \frac{dx_i}{dt} = F_i(x) + \sum_{j=1}^d \sigma_{ij}(x) \xi_j, \quad i = 1, \ldots, d \]  

(4.1)

with the initial conditions \( x_i(0) = x_{i,0} \). \( \sigma_{ij}(x) \) is the amplitude of the stochastic perturbation and \( \xi_j = \frac{dw_j}{dt} \) is
a white noise term that is the derivative of a Wiener process. It is assumed that the Itô, rather that the Stratonovich, calculus, is used. (For the differences see \[37, 21\] and \[38\]. If the $\sigma_{ij}$ are independent of $x$ then the Itô and the Stratonovich approaches yield identical results.)

The Fokker-Planck equation governing the evolution of the density function $f(t, x)$ is given by

\[
\frac{\partial f}{\partial t} = -\sum_{i=1}^{d} \frac{\partial [F_i(x)f]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x)f]}{\partial x_i \partial x_j} \tag{4.2}
\]

where

\[
a_{ij}(x) = \sum_{k=1}^{d} \sigma_{ik}(x)\sigma_{jk}(x).
\]

If $k(t, x, x_0)$ is the fundamental solution of the Fokker-Planck equation, i.e. for every $x_0$ the function $(t, x) \mapsto k(t, x, x_0)$ is a solution of the Fokker-Planck equation with the initial condition $\delta(x-x_0)$, then the general solution $f(t, x)$ of the Fokker-Planck equation (4.2) with the initial condition $f(x, 0) = f_0(x)$ is given by

\[
f(t, x) = \int k(t, x, x_0)f_0(x_0) \, dx_0, \tag{4.3}
\]

and defines a Markov semigroup by $P^t f_0(x) = f(t, x)$. If a stationary (steady state) density $f_*(x)$ exists, it is the stationary solution of Eq. (4.2)

\[
-\sum_{i=1}^{d} \frac{\partial [F_i(x)f]}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 [a_{ij}(x)f]}{\partial x_i \partial x_j} = 0. \tag{4.4}
\]

Differentiating Eq. (4.2) with respect to time, and using Eq. (4.2) with integration by parts along with the fact that since $f_*$ is a stationary density it satisfies (4.4), we obtain

\[
\frac{dH_c}{dt} = \frac{1}{2} \int \left( \frac{f^2}{f_*} \right) \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial}{\partial x_i} \left( \frac{f}{f_*} \right) \frac{\partial}{\partial x_j} \left( \frac{f}{f_*} \right) \, dx. \tag{4.5}
\]

Since the matrix $(a_{ij}(x))$ is nonnegative definite, one concludes that $\frac{dH_c}{dt} \geq 0$. Using the identity

\[
\frac{\partial}{\partial x_i} \left( \log \frac{f}{f_*} \right) = \frac{f_*}{f} \frac{\partial}{\partial x_i} \left( \frac{f}{f_*} \right),
\]

we can rewrite Eq. (4.5) in the equivalent form

\[
\frac{dH_c}{dt} = \frac{1}{2} \int f \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial}{\partial x_i} \left( \log \frac{f}{f_*} \right) \frac{\partial}{\partial x_j} \left( \log \frac{f}{f_*} \right) \, dx. \tag{4.6}
\]

The right hand side of Eq. (4.6) appears in various expressions describing entropy balance equations in Daems and Nicolis \[12\] and Bag \[15\].

A similar calculation for the Gibbs’ entropy yields, however, something additional. Namely, we have

\[
\frac{dH_G}{dt} = \int f \left( \sum_i \frac{\partial F_i(x)}{\partial x_i} - \frac{1}{2} \sum_{i,j} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} \right) \, dx + \frac{1}{2} \int \frac{1}{f} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \, dx. \tag{4.7}
\]

If $a_{ij}$ are independent of $x$ then we obtain

\[
\frac{dH_G}{dt} = \int f \sum_i \frac{\partial F_i(x)}{\partial x_i} \, dx + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \int \frac{1}{f} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \, dx. \tag{4.8}
\]

As pointed out in \[12\], the first term is of indeterminate sign, while the second is positive definite so the temporal behavior of the Gibbs’ entropy in this non-invertible system is unclear. It has become customary \[12, 13, 14, 15, 17, 18\] to refer to the first term in Eq. (4.7) as the ‘entropy flux’ and the second term as the ‘entropy production’.

There are a number of results giving conditions such that the general solution $f(t, x)$ of the Fokker Planck equation is asymptotically stable and thus the conditional entropy evolves monotonically to zero, e.g. Theorem 11.9.1 in \[21\]. In particular, assume that the stationary density is of the form

\[
f_*(x) = e^{-B(x)}.
\]

From Theorem \[8\] it follows that

\[
\lim_{t \to \infty} H_c(f_0|f_*) = 0
\]

and from Theorem \[4\] that

\[
\lim_{t \to \infty} H_G(f) = H_G(f_*)
\]

for all $f_0$ with $H_c(f_0|f_*) > -\infty$ provided that $\int e^{-(1+r)B(x)} \, dx < \infty$ for $r$ in some neighborhood of zero.
From the definition of the conditional entropy we may write
\[ H_c(f|f_*) = H_G(f) + \int_X f(t, x) \log f_*(x) \, dx \quad (4.9) \]
so the derivative of the Gibbs' entropy is
\[ \frac{dH_G}{dt} = \frac{dH_c}{dt} - \int Lf \log f_*, dx, \quad (4.10) \]
where the operator \( L \) is given by
\[ Lf = -\sum_{i=1}^d \frac{\partial(F_i(x)f)}{\partial x_i} + \sum_{i,j=1}^d \frac{\partial^2(a_{ij}(x)f)}{\partial x_i \partial x_j}. \quad (4.11) \]
Since \( \log f_*(x) = -B(x) \), we may write
\[ \frac{dH_G}{dt} = \frac{dH_c}{dt} + \int f L^- B \, dx, \quad (4.12) \]
where the operator \( L^- \) is the formal adjoint of the operator \( L \) in Eq. 4.11. If \( e^{-B(x)+L^-B(x)} \, dx < \infty \) for \( r \) in some neighborhood of zero, then
\[ \lim_{t \to \infty} \int f L^- B \, dx = \int f_* L^- B \, dx = \int Lf_* B \, dx = 0, \]
which implies
\[ \lim_{t \to \infty} \frac{dH_G}{dt} = \lim_{t \to \infty} \frac{dH_c}{dt}. \]

1. The one dimensional case

In a one dimensional system (\( d = 1 \)) the stochastic differential Eq. 4.1 becomes
\[ \frac{dx}{dt} = F(x) + \sigma(x) \xi, \quad (4.13) \]
where \( \xi \) is a (Gaussian distributed) perturbation with zero mean and unit variance, and \( \sigma(x) \) is the amplitude of the perturbation. The corresponding Fokker-Planck equation 4.2 is
\[ \frac{\partial f}{\partial t} = -\frac{\partial[F(x)f]}{\partial x} + \frac{1}{2} \frac{\partial^2[\sigma^2(x)f]}{\partial x^2}. \quad (4.14) \]
If stationary solutions \( f_*(x) \) of 4.14 exist, they are defined by \( P^t f_* = f_* \) for all \( t \) and given as the generally unique (up to a multiplicative constant) solution of
\[ -\frac{\partial[F(x)f_*]}{\partial x} + \frac{1}{2} \frac{\partial^2[\sigma^2(x)f_*]}{\partial x^2} = 0. \quad (4.15) \]
The integrable solution is given by
\[ f_*(x) = \frac{K}{\sigma^2(x)} \exp \left[ \int^x 2F(z) \, dz \right], \quad (4.16) \]
where \( K > 0 \) is a normalizing constant and the semigroup \( P^t \) is asymptotically stable.

It is known [13, Section IV] that under relatively mild conditions there exists a constant \( \lambda > 0 \) such that
\[ H_c(P^t f_0|f_*) \geq e^{-2\lambda t} H_c(f_0|f_*). \]
Specific examples of \( \sigma(x) \) and \( F(x) \) for which one can determine the solution \( f(t, x) \) of Eq. 4.14 are few. One is that for an Ornstein-Uhlenbeck process which we consider in our next example.

**Example 4** In considering the Ornstein-Uhlenbeck process, developed in thinking about perturbations to the velocity of a Brownian particle, we denote the dependent variable by \( v \) so \( \sigma(v) \equiv \sigma \) a constant, and \( F(v) = -\gamma v \) with \( \gamma \geq 0 \). Now Eq. 4.14 becomes
\[ \frac{dv}{dt} = -\gamma v + \sigma \xi \quad (4.17) \]
with the Fokker-Planck equation
\[ \frac{\partial f}{\partial t} = \frac{\partial[\gamma v f]}{\partial v} + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial v^2}. \quad (4.18) \]
The unique stationary solution is
\[ f_*(v) = \frac{e^{-\gamma v^2/\sigma^2}}{\int_{-\infty}^{+\infty} e^{-\gamma v^2/\sigma^2} \, dv} = \sqrt{\frac{\gamma}{\pi \sigma^2}} e^{-\gamma v^2/\sigma^2}. \quad (4.19) \]
If the initial density \( f_0 \) is a Gaussian of the form
\[ f_0(v) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{(v-\bar{v})^2}{2\sigma^2} \right\}, \quad (4.20) \]
where \( \sigma > 0 \) and \( \bar{v} \in \mathbb{R} \), then
\[ P^t f_0(v) = \frac{1}{\sigma_t \sqrt{2\pi}} \exp \left\{ -\frac{(v-\bar{v}(t))^2}{2\sigma_t^2} \right\}, \quad (4.21) \]
wherein
\[ \sigma_t^2 = \sigma^2 + (\bar{v}^2 - \sigma^2) e^{-2\gamma t} \quad (4.22) \]
with \( \sigma_t^2 = \sigma^2/2\gamma \) and
\[ \bar{v}(t) = \bar{v} e^{-\gamma t}. \quad (4.23) \]
The Gibbs' entropy is
\[ H_G(P^t f_0) = \log \sigma_t \sqrt{2\pi} + \frac{1}{2} \quad (4.24) \]
Also
\[ \int_{-\infty}^{+\infty} P^t f_0(x) \log f_*(x) \, dx = -\log \sigma_* \sqrt{2\pi} - \frac{1}{2} \int \sigma^2 \, dz, \quad (4.25) \]
so
\[ H_c(P^t f_0|f_*) = \frac{1}{2} \log \left[ \frac{\sigma_t^2}{\sigma_*^2} \right] + \frac{1}{2} \left[ 1 - \frac{\sigma_t^2}{\sigma_*^2} \right] \quad (4.26) \]
and
\[ H_c(P^t f_0|f_*) = -\frac{1}{2} e^{-2\gamma t} \left[ \frac{\sigma_t^2}{\sigma_*^2} - 1 \right]. \]
We may examine how the two different types of entropy behave. First, we may show that $H_c(P^tf_0|f_*) \geq 0$ with
\[
\left( \frac{dH_c(P^tf_0|f_*)}{dt} \right)_{t=0} = \frac{\gamma (R - 1)^2}{R} > 0
\]
and $H_c(P^tf_0|f_*)$ increasing in a monotone fashion, with
\[
\lim_{t \to \infty} \frac{dH_c(P^tf_0|f_*)}{dt} = 0,
\]
wherein $R \equiv \bar{\sigma}^2/\sigma^2$.

This is not the case with the Gibbs’ entropy, for
\[
\frac{dH_G(P^tf_0)}{dt} \begin{cases} 
> 0 & \text{for } \sigma^2 < \bar{\sigma}^2 \\
= 0 & \text{for } \sigma^2 = \bar{\sigma}^2 \\
< 0 & \text{for } \sigma^2 > \bar{\sigma}^2,
\end{cases}
\]
implicating that the evolution of the Gibbs’ entropy in time is a function of the statistical properties ($\bar{\sigma}^2$) of the initial ensemble. All of these conclusions concerning the dynamics of $H_G(P^tf_0)$ are implicit in the work of Bag [18] but not explicitly stated.

Similar effects can be observed for the Rayleigh process considered in [18, Section IV].

2. Multidimensional Ornstein-Uhlenbeck process

Consider the multidimensional Ornstein-Uhlenbeck process
\[
\frac{dx}{dt} = Fx + \Sigma \xi,
\]
where $F$ is a $d \times d$ matrix, $\Sigma$ is a $d \times d$ matrix an $\xi$ is $d$ dimensional vector. The formal solution to Eq. (4.28) is given by
\[
x(t) = e^{tF}x(0) + \int_0^t e^{(t-s)F} \Sigma dw(t),
\]
where $e^{tF} = \sum_{n=0}^{\infty} \frac{t^n}{n!} F^n$ is the fundamental solution to $\dot{X}(t) = FX(t)$ with $X(0) = I$, and $w(t)$ is the standard $d$-dimensional Wiener process. From the properties of stochastic integrals it follows that
\[
\eta(t) = \int_0^t e^{(t-s)F} \Sigma dw(t)
\]
has mean 0 and covariance
\[
R(t) = E\eta(t)\eta(t)^T = \int_0^t e^{sF} \Sigma \Sigma^T e^{sF^T} ds,
\]
where $F^T$ is the transpose of the matrix $F$. The matrix $R(t)$ is nonnegative definite but not necessarily positive definite. We follow the presentation of [40] and [41]. For each $t > 0$ the matrix $R(t)$ has constant rank equal to the dimension of the space $[F, \Sigma] := \{ F^j \Sigma \epsilon_j : l, j = 1, \ldots, d, \epsilon_j = (\delta_{j1}, \ldots, \delta_{jD})^T \}$.

If $l = \text{rank } R(t)$ then $d - l$ coordinates of the process $\eta(t)$ are equal to 0 and the remaining $l$ coordinates constitute an $l$-dimensional Gaussian process. Thus if $l < d$ there is no stationary density. If rank $R(t) = d$ then the transition probability function of $x(t)$ is given by the Gaussian density
\[
k(t, x, x_0) = \frac{1}{\sqrt{(2\pi)^d \det R_\ast}} \exp \left\{ -\frac{1}{2} (x - \bar{e}F x_0)^T R^{-1}(t)(x - \bar{e}F x_0) \right\},
\]
where $R_\ast$ is a positive definite matrix given by
\[
R_\ast = \int_0^\infty e^{sF} \Sigma \Sigma^T e^{sF^T} ds,
\]
and is a unique symmetric matrix satisfying
\[
FR_\ast + R_\ast F^T = -\Sigma \Sigma^T.
\]

We conclude that if $[F, \Sigma]$ contains $d$ linearly independent vectors and all eigenvalues of $F$ have negative real parts, then the corresponding semigroup of Markov operators is asymptotically stable. From Theorem [3] it follows that
\[
\lim_{t \to \infty} H_c(P^tf_0|f_*) = 0
\]
and from Theorem [4] that
\[
\lim_{t \to \infty} H_G(P^tf_0) = H_G(f_*)
\]
for all $f_0$ with $H_c(P^tf_0|f_*) > -\infty$.

Now let $f_0$ be a Gaussian density of the form
\[
f_0(x) = \frac{1}{\sqrt{(2\pi)^d \det Q_0}} \exp \left\{ \frac{1}{2} x^T Q_0^{-1} x \right\},
\]
where $Q_0$ is a positive definite symmetric matrix. From Eq. (4.28) it follows that $x(t)$ is Gaussian with zeroth mean vector and the following covariance matrix
\[
Q(t) = e^{tF} Q_0 e^{tF^T} + R(t).
\]
Hence the density of $x(t)$ is given by
\[
P^tf_0(x) = \frac{1}{\sqrt{(2\pi)^d \det Q(t)}} \exp \left\{ -\frac{1}{2} x^T Q(t)^{-1} x \right\}.
\]
Since $\int P^t f_0(x) x^T Q(t)^{-1} x \, dx = d$, the Gibbs’ entropy of $P^t f_0$ is

$$H_G(P^t f_0) = \frac{1}{2} \log(2\pi)^d \det Q(t) + \frac{d}{2}. \quad (4.37)$$

By Eq. $4.39$ and the formula

$$\int P^t f_0(x) x^T R_*^{-1} x \, dx = \text{Tr} (R_*^{-1} Q(t))$$

we obtain the conditional entropy

$$H_c(P^t f_0 | f_*) = H_G(P^t f_0) - \frac{1}{2} \log(2\pi)^d \det R_* - \frac{1}{2} \text{Tr} (R_*^{-1} Q(t)) \quad (4.38)$$

for all $t \geq 0$ and every $f_0$ of the form given by Eq. $4.34$. Formula $4.37$ remains valid when we start with a Gaussian density $f_0$ with non-zero mean vector but then in the formula for the conditional entropy one additional term appears, see [18, Section IV].

**Example 5 Noisy harmonic oscillator.**

Consider the second order system

$$m \frac{d^2 y}{d t^2} + \gamma \frac{dy}{dt} + \omega^2 y = \sigma \xi \quad (4.39)$$

with constant positive coefficients $m$, $\gamma$ and $\sigma$. Introduce the velocity $v = \frac{dy}{dt}$ as a new variable. Then Eq. $4.39$ is equivalent to the system

$$\frac{dy}{dt} = v \quad (4.40a)$$
$$m \frac{dv}{dt} = -\gamma v - \omega^2 y + \sigma \xi, \quad (4.40b)$$

and the corresponding Fokker-Planck equation is

$$\frac{\partial f}{\partial t} = -\frac{\partial [vf]}{\partial y} + \frac{1}{m} \frac{\partial [(\gamma v + \omega^2 y)v]}{\partial v} + \frac{\sigma^2}{2m} \frac{\partial^2 f}{\partial v^2}. \quad (4.41)$$

We can assume in what follows that $m = 1$, as introducing the constants $\tilde{\gamma} = \gamma/m$, $\tilde{\omega}^2 = \omega^2/m$ and $\tilde{\sigma}^2 = \sigma^2/m^2$ leads to

$$\frac{\partial f}{\partial t} = -\frac{\partial [vf]}{\partial y} + \frac{\partial [(\tilde{\gamma} v + \tilde{\omega}^2 y)v]}{\partial v} + \frac{\tilde{\sigma}^2}{2} \frac{\partial^2 f}{\partial v^2}. \quad (4.42)$$

The results of Section $4V.B2$ in the two dimensional setting apply with $x = (y, v)^T$,

$$F = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}, \quad \text{and} \quad \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}. \quad (4.43)$$

Since

$$[F, \Sigma] = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \sigma \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} \right\},$$

the transition density function is given by Eq. $4.31$. The eigenvalues of $F$ are equal to

$$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4\omega^2}}{2}, \quad (4.41a)$$
$$\lambda_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4\omega^2}}{2}, \quad (4.41b)$$

and are either negative real numbers when $\gamma^2 \geq 4\omega^2$ or complex numbers with negative real parts when $\gamma^2 < 4\omega^2$. Thus the stationary density is given by Eq. $4.32$. As is easily seen $R_*$, being a solution to Eq. $4.33$, is given by

$$R_* = \frac{\sigma^2}{2\gamma \omega^2} \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix}. \quad (4.44)$$

The inverse of the matrix $R_*$ is

$$R_*^{-1} = \frac{2\gamma}{\sigma^2} \left( \begin{array}{cc} \omega^2 & 0 \\ 0 & 1 \end{array} \right) \quad (4.45)$$

and the unique stationary density becomes

$$f_*(y, v) = \frac{\gamma \omega}{\pi \sigma^2} e^{-\frac{\gamma \omega}{\sigma^2} \left| \tilde{\omega}^2 y^2 + v^2 \right|}.$$

If the initial density $f_0$ is the Gaussian

$$f_0(y, v) = \frac{1}{2\pi \tilde{\sigma}_1 \tilde{\sigma}_2} \exp \left\{ -\frac{y^2}{2\tilde{\sigma}_1^2} - \frac{v^2}{2\tilde{\sigma}_2^2} \right\},$$

where $\tilde{\sigma}_1 > 0, \tilde{\sigma}_2 > 0$, then $P^t f_0$ is as in Eq. $4.30$ with

$$Q(t) = e^{t F} Q_0 e^{t F^T} + R(t),$$

where

$$Q_0 = \frac{\tilde{\sigma}_1^2}{\sigma^2} \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\sigma}_2^2 \end{pmatrix}. \quad (4.46)$$

The formula for the covariance matrix $R(t)$ is given by Chandrasekhar [72, pp. 27-30]. The Gibbs’ entropy is

$$H_G(P^t f_0) = 1 + \log(2\pi) + \frac{1}{2} \log \det Q(t) \quad (4.47)$$

and the conditional entropy is

$$H_c(P^t f_0 | f_*) = 1 + \frac{1}{2} \log \det Q(t) - \frac{1}{2} \log \det R_* - \frac{1}{2} \text{Tr} (R_*^{-1} Q(t)). \quad (4.48)$$

We are going to show that the Gibbs entropy need not be a monotonic function of time, so we need to calculate $\det Q(t)$ to have the analytic formula for the Gibbs
entropy. The calculations depend on the nature of eigenvalues \( \lambda_1 \) and \( \lambda_2 \) in Eq. (4.44), so we must distinguish between three cases: (i) \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 \neq \lambda_2 \), (ii) \( \lambda_1, \lambda_2 \in \mathbb{R} \) with \( \lambda_1 = \lambda_2 \), and (iii) \( \lambda_1, \lambda_2 \) are complex.

In what follows we use the following notation

\[
\begin{align*}
\sigma_* &= \frac{\sigma^2}{2\gamma \omega^2}, \\
\alpha_1 &= \bar{\sigma}_1^2 - \sigma_*, \\
\alpha_2 &= \bar{\sigma}_2^2 - \omega^2 \sigma_*,
\end{align*}
\]

(4.46a)

(4.46b)

(4.46c)

Observe that \( \alpha_1 \alpha_2 = \det(Q_0 - R_s) \) and \( \sigma_*^2 \omega^2 = \det R_s \).

(i) Let us first consider the overdamped case

\[
\gamma^2 > 4\omega^2,
\]

so the eigenvalues in Eq. (4.41) are real and \( \lambda_1 \neq \lambda_2 \). Define, for \( t \geq 0 \),

\[
\begin{align*}
c_1(t) &= \frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}, \\
c_2(t) &= \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}.
\end{align*}
\]

(4.47a)

(4.47b)

Then

\[
e^{tF} = \begin{pmatrix} c_1(t) & c_2(t) \\ \bar{c}_1(t) & \bar{c}_2(t) \end{pmatrix}
\]

and the covariance matrix \( R(t) \) is given by

\[
R(t) = R_s - \frac{\sigma^2}{2\gamma \omega^2} \left( c_1^2 + \omega^2 c_2^2, -\gamma \omega^2 c_2^2, (c_1')^2 + \omega^2 (c_2')^2 \right),
\]

where we suppressed the dependence of \( c_1 \) and \( c_2 \) on \( t \). Accordingly, for \( Q_0 \) as in Eq. (4.42) we have

\[
e^{tF} Q_0 e^{tF^T} = \begin{pmatrix} c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 & c_1' c_1^2 \sigma_1^2 + c_2' c_2^2 \sigma_2^2 \\ c_1' c_1^2 \sigma_1^2 + c_2' c_2^2 \sigma_2^2 & (c_1')^2 \sigma_1^2 + (c_2')^2 \sigma_2^2 \end{pmatrix}.
\]

(4.47)

From Eq. (4.47) it follows that

\[
c_1 c_1' + \omega^2 c_2 c_2' = -\gamma \omega^2 c_2^2.
\]

Combining the three preceding equations and introducing the values of \( \sigma_* \), \( \alpha_1 \), and \( \alpha_2 \) from Eq. (4.40) we obtain for the matrix \( Q(t) \) the formula

\[
Q(t) = \begin{pmatrix} c_1^2 \alpha_1 + c_2^2 \alpha_2 + \sigma_* & c_1' c_1^2 \alpha_1 + c_2' c_2^2 \alpha_2 \\ c_1' c_1^2 \alpha_1 + c_2' c_2^2 \alpha_2 & (c_1')^2 \alpha_1 + (c_2')^2 \alpha_2 + \omega^2 \sigma_* \end{pmatrix}.
\]

Hence

\[
\det Q(t) = \omega^2 \sigma_*^2 + \alpha_1 \alpha_2 (c_1' c_1^2 - c_1' c_2^2)^2 + \sigma_* \left((\omega^2 c_1^2 + (c_1')^2) \alpha_1 + (\omega^2 c_2^2 + (c_2')^2) \alpha_2\right).
\]

Making use of Eq. (4.47) together with the relations \( \lambda_1 \lambda_2 = \omega^2 \) and \( \lambda_1 + \lambda_2 = -\gamma \), we arrive at

\[
\det Q(t) = \omega^2 \sigma_*^2 + \alpha_1 \alpha_2 e^{-2\gamma t} - \frac{\sigma_*}{(\lambda_1 - \lambda_2)^2} (\gamma \lambda_1 (\lambda_2^2 \alpha_1 + \alpha_2) e^{2\lambda_1 t} + 4 \omega^2 (\omega^2 \alpha_1 + \alpha_2) e^{-\gamma t} + \gamma \lambda_2 (\lambda_1^2 \alpha_1 + \alpha_2) e^{2\lambda_2 t}).
\]

Consequently, after some algebra we obtain

\[
\frac{dH(P^t f_0)}{dt} = -\frac{\gamma}{\det Q(t)} \left( \alpha_1 \alpha_2 e^{-2\gamma t} + \gamma (\lambda_2^2 \alpha_1 + \alpha_2) e^{2\lambda_2 t} + \gamma \lambda_1 (\lambda_2^2 \alpha_1 + \alpha_2) e^{2\lambda_1 t} \right)
\]

and

\[
\left( \frac{dH_G(P^t f_0)}{dt} \right)_{t=0} = -\frac{\gamma \alpha_2}{\sigma^2} \left\{ \begin{array}{ll} < 0 & \text{for } \alpha_2 > 0, \\
 0 & \text{for } \alpha_2 = 0, \\
 > 0 & \text{for } \alpha_2 < 0. \end{array} \right.
\]

(4.48)

(4.49)

Since \( \gamma > 0 \) and \( \det Q(t) > 0 \), the sign of the derivative of \( H_G(P^t f_0) \) is completely determined by the remaining parts and depends on the sign of \( \alpha_1 \) and \( \alpha_2 \) and their mutual relations. In the case of \( \alpha_1 \alpha_2 = 0 \) we conclude from Eqs. (4.48) and (4.49) that

\[
\frac{dH_G(P^t f_0)}{dt} < 0 \quad \text{for } \alpha_1^2 > \sigma_*^2, \sigma_2^2 > \omega^2 \sigma_*.
\]

(4.50)

for all \( t \geq 0 \). Now assume that \( \alpha_1 \alpha_2 \neq 0 \). It also follows directly from Eq. (4.48) that

\[
\frac{dH_G(P^t f_0)}{dt} < 0 \quad \text{for } \alpha_1^2 > \sigma_*^2, \sigma_2^2 > \omega^2 \sigma_*.
\]

(4.51)

This behavior is illustrated in Fig. 7.

FIG. 1: Entropy behavior for the overdamped noisy harmonic oscillator. The left hand panels show plots of \( H_G(P^t f_0) \) as a function of time as given by Eq. (4.44) and the right hand panels show \( H_s(P^t f_0 | f_s) \) (as in Eq. 4.35) plus \( H_G(f_s) \), i.e. \( H_s(P^t f_0 | f_s) + H_G(f_s) \) as a function of time. The parameters used were \( m = 1, \gamma = 3, \omega = 2 \), and \( \sigma_* = 1 \). Upper panels correspond to the range of parameters as in Eq. (4.51) with specific values \( \sigma_1 = 2, \sigma_2 = 2 \), while the lower panels correspond to parameters as in Eq. (4.50) with \( \sigma_1 = 0.5, \sigma_2 = 1 \).

To study the remaining cases we rewrite Eq. (4.48) in the form

\[
\frac{dH_G(P^t f_0)}{dt} = \frac{\gamma}{\det Q(t)} e^{-2\gamma t} h_1(t),
\]

(4.52)
where
\[ h_1(t) = -\alpha_1 \alpha_2 - \sigma_* \left( \lambda_1 \beta_1 e^{-2\lambda_2 t} + \lambda_2 \beta_2 e^{-2\lambda_1 t} \right) - 2\sigma_* \frac{\omega^2}{\gamma} (\beta_1 + \beta_2) e^{-\gamma t} \]
(4.53)
and
\[ \beta_1 = \frac{\lambda_1 (\lambda_2^2 \alpha_1 + \alpha_2)}{(\lambda_1 - \lambda_2)^2}, \]
(4.54a)
\[ \beta_2 = \frac{\lambda_2 (\lambda_1^2 \alpha_1 + \alpha_2)}{(\lambda_1 - \lambda_2)^2}. \]
(4.54b)
Since \( \lambda_1 \lambda_2 = \omega^2 \), we obtain
\[ h_1(t) = 2\omega^2 \sigma_* (\beta_1 e^{-2\lambda_2 t} - (\beta_1 + \beta_2) e^{-\gamma t} + \beta_2 e^{-2\lambda_1 t}), \]
which leads to
\[ h_1(t) = 2\omega^2 \sigma_* e^{-2\lambda_2 t} \left(e^{(\lambda_1 - \lambda_2)t} - 1 \right) (\beta_1 e^{(\lambda_1 - \lambda_2)t} - \beta_2). \]
For \( t > 0 \) such that \( \beta_1 e^{(\lambda_1 - \lambda_2)t} = \beta_2 \) we have
\[ h_1(t_*) = -\alpha_1 \left( \alpha_2 + \omega^2 \sigma_* \left( \frac{\beta_2}{\beta_1} \right)^{\gamma/(\lambda_1 - \lambda_2)} \right). \]
(4.55)
Returning to formulae (4.54) we note that
\[ \frac{\beta_2}{\beta_1} = 1 + \frac{(\lambda_1 - \lambda_2)(\omega^2 \alpha_1 - \alpha_2)}{\lambda_1 (\lambda_2^2 \alpha_1 + \alpha_2)}. \]

We can now continue to study the of behavior of \( H_G(P^t f_0) \). First, we consider the case of \( \alpha_1 < 0 \) and \( \alpha_2 < 0 \). If \( \omega^2 \alpha_1 \geq \alpha_2 \) then \( h_1(t) \geq h_1(0) \) and \( h_1(0) > 0 \) by Eq. (4.44). Now if \( \omega^2 \alpha_1 < \alpha_2 \) then \( h_1(t) \geq h_1(t_*) \) and from Eq. (4.59) it follows that \( h_1(t_*) > 0 \). Consequently, we obtain
\[ \frac{dH_G(P^t f_0)}{dt} > 0 \quad \text{for} \quad \bar{\sigma}_1^2 < \sigma_*, \bar{\sigma}_2^2 < \omega^2 \sigma_. \]
(4.56)

Consider now the case of \( \alpha_1 > 0 \) and \( \alpha_2 < 0 \). Then we know that \( h_1(0) > 0 \). Now if \( \lambda_2^2 \alpha_1 + \alpha_2 \geq 0 \) then \( \beta_1 \leq 0 \) and \( \beta_1 \leq \beta_2 \). Thus \( h_1 \) is decreasing and diverges to \( -\infty \) as \( t \to \infty \). Consequently, if \( \lambda_2^2 \alpha_1 \geq -\alpha_2 > 0 \), or equivalently
\[ \lambda_2^2 \bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \gamma \lambda_2 \sigma_* \geq 0 \quad \text{and} \quad \bar{\sigma}_2^2 < \omega^2 \sigma_, \]
(4.57)
then there is \( t_0 > 0 \) such that
\[ \frac{dH_G(P^t f_0)}{dt} \begin{cases} > 0 & \text{for} \quad t < t_0, \\ < 0 & \text{for} \quad t > t_0. \end{cases} \]
(4.58)

If \( \lambda_2^2 \alpha_1 + \alpha_2 < 0 \) then \( \beta_1 > 0 \) and \( \beta_2/\beta_1 > 1 \). From Eq. (4.57) it follows that \( h_1(t_*) < 0 \). Thus \( h_1 \) starting from a positive value at 0 decreases to a negative value at \( t_* \) and then increases and diverges to \( \infty \). Hence we conclude that, if \( 0 < \lambda_2^2 \alpha_1 < -\alpha_2 \), or equivalently
\[ \lambda_2^2 \bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \gamma \lambda_2 \sigma_* < 0 \quad \text{and} \quad \bar{\sigma}_1^2 > \sigma_*, \]
(4.59)
then there are \( t_1, t_2 > 0 \) such that
\[ \frac{dH_G(P^t f_0)}{dt} \begin{cases} > 0 & \text{for} \quad 0 < t < t_1, \\ < 0 & \text{for} \quad t_1 < t < t_2, \\ > 0 & \text{for} \quad t > t_2. \end{cases} \]
(4.60)

These behaviors are illustrated in Fig. 2.
A symmetric behavior is observed when \( \alpha_1 < 0 \) and \( \alpha_2 > 0 \), and graphically shown in Fig. 3. We then have
\[ h_1(0) < 0 \] and a similar analysis leads to the following conclusions. If \( \lambda_2^2 \alpha_1 \leq -\alpha_2 < 0 \), or equivalently
\[ \lambda_2^2 \bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \gamma \lambda_2 \sigma_* \leq 0 \quad \text{and} \quad \bar{\sigma}_2^2 > \omega^2 \sigma_*, \]
(4.61)
then there is \( t_0 > 0 \) such that

\[
\frac{dH_G(P^t f_0)}{dt} \begin{cases} < 0 & \text{for } t < t_0, \\ > 0 & \text{for } t > t_0 \end{cases} \tag{4.62}
\]

and if \( 0 > \lambda_2^2 \alpha_1 > -\alpha_2 \), or equivalently

\[
\lambda_2^2 \bar{\sigma}_1^2 + \bar{\sigma}_2^2 + \gamma \alpha_2 \sigma_* > 0 \quad \text{and} \quad \bar{\sigma}_1^2 < \sigma_*, \tag{4.63}
\]

then there are \( t_1, t_2 > 0 \) such that

\[
\frac{dH_G(P^t f_0)}{dt} \begin{cases} < 0 & \text{for } 0 < t < t_1, \\ > 0 & \text{for } t_1 < t < t_2, \\ < 0 & \text{for } t > t_2. \end{cases} \tag{4.64}
\]

(ii) Let us now consider the critical damping situation when

\[
\gamma^2 = 4 \omega^2.
\]

so that \( \lambda_1 = \lambda_2 \), and set

\[
\lambda = -\frac{\gamma}{2}.
\]

In this case we have

\[
F = \begin{pmatrix} 0 & 1 \\ -\lambda & 2\lambda \end{pmatrix} \quad \text{and} \quad e^{tF} = e^{\lambda t} \begin{pmatrix} 1 - \lambda t & t \\ -\lambda^2 t & 1 + \lambda t \end{pmatrix},
\]

so that the corresponding covariance matrix \( R(t) \) is given by

\[
R(t) = R_* + \frac{\sigma_2^2 \lambda t}{4 \lambda^3} \begin{pmatrix} (1 - \lambda t)^2 + \lambda^2 t^2 & 2 \lambda^3 t^2 \\ 2 \lambda^3 t^2 & (\lambda + 2 \lambda t)^2 + \lambda^4 t^2 \end{pmatrix}.
\]

We also have

\[
e^{tF}Q_0 e^{tF^T} = e^{2\lambda t} \begin{pmatrix} \bar{\sigma}_1^2 (1 - \lambda t)^2 + \bar{\sigma}_2^2 t^2 - \sigma_2^2 \lambda^2 t (1 - \lambda t) + \sigma_*^2 (1 + \lambda t) \\ -\bar{\sigma}_1^2 \lambda^2 t (1 - \lambda t) + \sigma_*^2 (1 + \lambda t) \end{pmatrix}.
\]

Note that now \( \sigma_* = -\frac{\sigma_2^2}{4 \lambda^3} \) and \( \omega^2 = \lambda^2 \). Thus

\[
Q(t) = \begin{pmatrix} e^{2\lambda t} (\alpha_1 (1 - \lambda t)^2 + \alpha_2 t^2) + \sigma_* & e^{2\lambda t} (1 - \lambda t) \alpha_1 + t (1 + \lambda t) \alpha_2 \\ e^{2\lambda t} (-\lambda^2 \bar{\sigma}_1^2,(1 - \lambda t) \alpha_1 + t (1 + \lambda t) \alpha_2) & e^{2\lambda t} (\alpha_1 \lambda^4 t^2 + \alpha_2 (1 + \lambda t)^2) + \lambda^2 \sigma_* \end{pmatrix},
\]

where \( \alpha_1 \) and \( \alpha_2 \) are given by Eq. \( 4.49 \). Hence

\[
det Q(t) = \lambda^2 \sigma_2^2 + \alpha_1 \alpha_2 e^{4\lambda t} + \sigma_* e^{2\lambda t} (\alpha_1 \lambda^2 ((1 - \lambda t)^2 + \lambda^2 t^2)) + \alpha_2 ((1 + \lambda t)^2 + \lambda^2 t^2)
\]

and after some algebra we obtain

\[
\frac{dH_G(P^t f_0)}{dt} = \frac{2\lambda}{\det Q(t)} e^{4\lambda t} (\alpha_1 \alpha_2 + \sigma_* \alpha_1 \lambda^4 t^2 e^{-2\lambda t}}
\]

\[
+ \sigma_* \alpha_2 (1 + \lambda t)^2 e^{-2\lambda t}). \tag{4.65}
\]

Since \( \lambda = -\gamma/2 \), Eq. \( 4.47 \) remains valid. Now the analysis and conclusions are similar to the overdamped case. First, observe that from Eq. \( 4.65 \) follow Eq. \( 4.51 \) in the case of \( \alpha_1 \alpha_2 = 0 \) and Eq. \( 4.52 \) in the case of positive \( \alpha_1 \) and \( \alpha_2 \), so assume that \( \alpha_1 \alpha_2 \neq 0 \). Let us rewrite Eq. \( 4.65 \) in the form

\[
\frac{dH_G(P^t f_0)}{dt} = \frac{\gamma}{\det Q(t)} e^{-2\gamma t} p_2(t),
\]

where now

\[
h_2(t) = -\alpha_1 \alpha_2 - \sigma_* e^{-2\lambda t} (\alpha_1 \lambda^4 t^2 + \alpha_2 (1 + \lambda t)^2).
\]

Then

\[
h_2'(t) = 2 \lambda^2 \sigma_* e^{-2\lambda t} (\lambda^2 \alpha_1 + \alpha_2) t - (\lambda \alpha_1 - \alpha_2).
\]

Note that for

\[
t_* = \frac{\lambda \alpha_1 - \alpha_2}{\lambda \alpha_1 + \alpha_2}
\]

we have

\[
h_2(t_*) = -\alpha_1 (\alpha_2 + \lambda^2 \sigma_* e^{-2\lambda t_*}).
\]

A similar analysis as in the overdamped case leads to the same conclusions so that Eq. \( 4.50 \) remains valid in the case of negative \( \alpha_1 \) and \( \alpha_2 \) and also Eqs. \( 4.53 \) hold in the same ranges of parameters in the case of \( \alpha_1 \alpha_2 < 0 \).

(iii) Finally, let us consider the underdamped case

\[
\gamma^2 < 4 \omega^2,
\]
so that $\lambda_1, \lambda_2$ are complex, and set

$$\lambda = -\frac{\gamma}{2} \quad \text{and} \quad \beta = \sqrt{\omega^2 - \lambda^2}.$$

Then $\lambda_1 = \lambda + i\beta$ and $\lambda_2 = \lambda - i\beta$. The fundamental matrix in this case is equal to

$$e^{t\mathbf{F}} = \begin{pmatrix} \beta \cos(\beta t) - \lambda \sin(\beta t) & \sin(\beta t) \\ -\omega^2 \cos(\beta t) & \beta \cos(\beta t) + \lambda \sin(\beta t) \end{pmatrix}.$$

Let us rewrite the matrix $e^{t\mathbf{F}}$ as

$$e^{t\mathbf{F}} = \frac{e^{t\mathbf{F}_1}}{\beta} \begin{pmatrix} c_3(t) & \sin(\beta t) \\ -\omega^2 \sin(\beta t) & c_4(t) \end{pmatrix},$$

where

$$c_3(t) = \beta \cos(\beta t) - \lambda \sin(\beta t), \quad (4.66a)$$
$$c_4(t) = \beta \cos(\beta t) + \lambda \sin(\beta t). \quad (4.66b)$$

Observe that $\sigma_*$ as defined in Eq. 4.46a is equal to $-\sigma^2/4\lambda\omega^2$. The covariance matrix $R(t)$ is equal to

$$R_+ = \frac{\sigma_* e^{2t\lambda}}{\beta^2} \begin{pmatrix} c_3(t) + \omega^2 \sin^2(\beta t) & 2\lambda \omega^2 \sin^2(\beta t) \\ 2\lambda \omega^2 \sin^2(\beta t) & \omega^4 \sin^2(\beta t) + \omega^2 c_4(t) \end{pmatrix}.$$ Further

Making use of expressions 4.40 and 4.66 the sum of the matrices in the two preceding equations gives

$$Q(t) = \begin{pmatrix} \sigma_0^2 + \frac{e^{2t\lambda}}{\beta^2} (\alpha_1 c_3(t) + \omega^2 \sin(\beta t)) \\ \omega^2 \lambda \alpha_1 \alpha_2 + \frac{e^{2t\lambda}}{\beta^2} (\alpha_1 c_3(t) + \omega^2 \sin(\beta t)) \end{pmatrix} \begin{pmatrix} \sin(\beta t) (\alpha_2 c_4(t) - \omega^2 \alpha_1 c_3(t)) \\ -\omega^2 \alpha_1 \sin(\beta t) \end{pmatrix},$$

which after some algebra leads to

$$\det Q(t) = \omega^2 \sigma_*^2 + e^{4t\lambda} \alpha_1 \alpha_2 + \frac{\sigma_*^2 e^{2t\lambda}}{\beta^2} (\omega^2 \alpha_1 + \alpha_2)$$
$$- \lambda^2 (\omega^2 \alpha_1 + \alpha_2) \cos(2\beta t)$$
$$- \lambda \beta (\omega^2 \alpha_1 - \alpha_2) \sin(2\beta t)). \quad (4.67)$$

We have

$$\frac{dH_G(P_f f_0)}{dt} = \frac{2\lambda}{\det Q(t)} e^{2t\lambda} (\alpha_1 \alpha_2 e^{2t\lambda})$$
$$+ \frac{\sigma_*^2}{\beta^2} (\omega^4 \alpha_1 \sin^2(\beta t) + \omega^2 c_4(t))). \quad (4.68)$$

Since $\lambda = -\frac{\gamma}{2}$, Eq. 4.54 holds. Again, Eq. 4.65 implies Eq. 4.56 in the case of $\alpha_1 \alpha_2 = 0$. In the case of positive $\alpha_1$ and $\alpha_2$ the Gibbs entropy is decreasing. This corresponds to

$$\sigma_1^2 > \sigma_* \quad \text{and} \quad \sigma_2^2 > \omega^2 \sigma_*, \quad (4.69)$$

and is illustrated in Fig. 4.

Let us rewrite Eq. 4.68 in the form

$$\frac{dH_G(P_f f_0)}{dt} = \frac{\gamma}{\det Q(t)} e^{4t\lambda} h_3(t),$$

where

$$h_3(t) = -\alpha_1 \alpha_2 - \frac{\sigma_*}{\beta^2} e^{-2t\lambda} (\omega^4 \alpha_1 \sin^2(\beta t) + \omega^2 c_4(t)).$$

We have

$$h_3'(t) = \frac{2\omega^2 \sigma_*}{\beta^2} e^{-2t\lambda} \sin(\beta t) (\lambda (\omega^2 \alpha_1 + \alpha_2) \sin(\beta t) + \beta (\alpha_2 - \omega^2 \alpha_1) \cos(\beta t)).$$

The function $h_3$ has extreme values at all $t$ for which either $\sin(\beta t) = 0$ or

$$\beta \cos(\beta t) = \lambda \sin(\beta t) \left(\frac{\omega^2 \alpha_1 + \alpha_2}{\omega^2 \alpha_1 - \alpha_2}\right). \quad (4.70)$$

Making use of the relations $\omega^2 = \lambda^2 + \beta^2$ and $\lambda = -\frac{\gamma}{2}$ it is seen that for every nonnegative integer $k$ we have

$$h_3(k\pi/\beta) = -\alpha_2 (\alpha_1 + \sigma_* e^{k\pi/\beta}) \quad (4.71)$$

and

$$h_3(t_* + k\pi/\beta) = -\alpha_1 \left(\alpha_2 + \sigma_* e^{(t_* + k\pi/\beta)}\right), \quad (4.72)$$

and
where $t_*$ is the smallest positive solution of Eq. 4.70. Thus, if $\alpha_2 < 0$ and $\alpha_2 > 1$ then $h_3(k\pi/\beta) < 0$ and $h_3(t_* + k\pi/\beta) > 0$ for all $k$. Consequently, if
\[
\tilde{\sigma}_1^2 < \sigma_* \quad \text{and} \quad \tilde{\sigma}_2^2 > \omega^2\sigma_* \quad (4.73)
\]
then there are two infinite sequences of points $t_k$ and $\tilde{t}_k$ such that
\[
\frac{dH_G(P^t f_0)}{dt} \begin{cases} < 0 \text{ for } t_k < t < \tilde{t}_k, \\ > 0 \text{ for } \tilde{t}_k < t < t_{k+1}. \end{cases} \quad (4.74)
\]

Consider now the case of $\alpha_2 < 0$. From Eq. 4.71 it follows that $h_3(k\pi/\beta) > 0$. When $\alpha_1 < 0$ the values of $h_3$ at $t_* + k\pi/\beta$ are positive. Therefore $h_3(t) > 0$ for all $t > 0$. Consequently, if
\[
\tilde{\sigma}_1^2 < \sigma_* \quad \text{and} \quad \tilde{\sigma}_2^2 > \omega^2\sigma_* \quad (4.75)
\]
then the Gibbs’ entropy increases. Finally, when $\alpha_1 > 0$ then $\alpha_2 < \omega^2\alpha_1$, the function $h_3$ decreases from a positive value at $k\pi/\beta$ to a negative value at $t_* + k\pi/\beta$ and then increases back to a positive value. Consequently, if
\[
\tilde{\sigma}_1^2 > \sigma_* \quad \text{and} \quad \tilde{\sigma}_2^2 < \omega^2\sigma_* \quad (4.76)
\]
then there are two infinite sequences of points $t_k$ and $\tilde{t}_k$ such that
\[
\frac{dH_G(P^t f_0)}{dt} \begin{cases} > 0 \text{ for } t_k < t < \tilde{t}_k, \\ < 0 \text{ for } \tilde{t}_k < t < t_{k+1}. \end{cases} \quad (4.77)
\]
These behaviors are illustrated in Fig. 5.

V. SUMMARY AND DISCUSSION

From the most general properties of the conditional entropy, it may remain constant or increase (Theorem 2). In invertible systems (e.g. measure preserving systems of differential equations or invertible maps) the conditional entropy is fixed at the value with which the system is prepared (Theorem 2, see also [28, 29, 43, 44]). This property is illustrated by Example 2. The addition of noise can reverse this invertibility property and induce the dynamic property of asymptotic stability. Asymptotic stability is necessary and sufficient for the monotonic evolution of the conditional entropy to a maximum value of zero, c.f. Theorem 3. This has been amply and fully illustrated in Examples 1 and 2 for the Ornstein-Uhlenbeck process and a noisy harmonic oscillator respectively.

The situation is much different for the Gibbs’ entropy, however, and it is often difficult to make general statements about what the temporal behavior will be. The rate of change of the Gibbs’ entropy in invertible systems depends on the Jacobian (c.f. Eq. 3.9), and it is only for Lebesque measure preserving flows that the Gibbs’ entropy is constant. In considering the Gibbs’ entropy in invertible systems, Example 1 treats a general two dimensional system. When the steady state is stable ($\lambda_1 + \lambda_2 < 0$) then the Gibbs’ entropy diverges to $-\infty$. Alternately, when the steady state is unstable ($\lambda_1 + \lambda_2 > 0$) then the Gibbs’ entropy diverges to $\infty$. Example 3 considers the specific case of a damped harmonic oscillator in which the Gibbs entropy diverges to $-\infty$. Example 4 treats the measure preserving rotation on the circle and shows that the Gibbs’ entropy is constant and fixed at the value corresponding to the way in which the system was prepared—as is the conditional entropy.

The situation with the temporal behavior of the Gibbs’ entropy becomes even more curious when an invertible system is subjected to noise and thus rendered non-invertible. A number of authors have considered aspects
of this recently, notably Ruelle [3, 10], Nicolis and Daems [11], Daems and Nicolis [12], Bag et al. [13, 14], Bag [15, 16, 17], and Garbaczewski [18]. As we have shown in Example [1] in contrast to the conditional entropy that increases monotonically to approach zero, the Gibbs’ entropy monotonically approaches the equilibrium value of $H_G(f_s)$ by either increasing or decreasing and the direction of movement is totally determined by the variance $\sigma^2$ of the initial ensemble. The temporal behavior of the Gibbs’ entropy can, however, have even more complicated patterns as illustrated by Example [6]. There, we have shown that when the harmonic oscillator is either over damped or critically damped the approach of $H_G(P^t f_0)$ to $H_G(f_s)$ may be either monotonic increasing or increasing (Fig. 1), or approach the equilibrium value with an undershoot or overshoot (Figs. 2 and 3). When the harmonic oscillator is under damped then the approach of the Gibbs’ entropy to $H_G(f_s)$ may even be oscillatory as shown in Figs. 4 and 5. All of these patterns of temporal behavior are, as we have shown, totally dependent on the relation of the variance of the initial ensemble to the variance of the equilibrium state. Remember that in all of these cases (over, critically, and under damped) the conditional entropy smoothly approaches zero so $H_c(P^t f_0|f_s) + H_G(f_s)$, as shown in the right hand panels of Figs. 4 through 5 monotonically increases to approach $H_G(f_s)$.

The concept of entropy originally arose in the context of the second law of thermodynamics. Following Landau and Lifshitz [19], we may formulate the second law of thermodynamics as follows. Let $S_{TD}(t)$ be defined as the time dependent thermodynamic entropy. Then for an isolated system

$$S_{TD}(t_2) \geq S_{TD}(t_1) \quad \text{for all} \quad t_2 > t_1,$$

and there is a unique steady state

$$S_{TD}^* = \lim_{t \to +\infty} S_{TD}(t)$$

for all initial system preparations. The entropy difference satisfies

$$\Delta S(t) \equiv S_{TD}(t) - S_{TD}^* \leq 0$$

and

$$\lim_{t \to +\infty} \Delta S(t) = 0. \quad (5.4)$$

In other words, the system entropy evolves to a unique maximum for all system preparations.

In attempts to give a dynamical interpretation of the second law, it is assumed that a thermodynamic system has states distributed in the phase space $X$. The distribution of these states is characterized by a (time dependent) density $f(t, x)$. A thermodynamic equilibrium is assumed be characterized by a stationary (time independent) density $f_s(x)$.

The Gibbs’ equilibrium entropy definition Eq. 2.1 has repeatedly proven to yield correct results when applied to a variety of equilibrium situations. This is why it is the gold standard for equilibrium computations in statistical mechanics and thermodynamics. Thus it makes total sense to identify the equilibrium Gibbs’ entropy $H_G(f_s)$ with the equilibrium thermodynamic entropy $S_{TD}^*$

$$S_{TD}^* = H_G(f_s).$$

Do the results in Sections III and IV on the dynamic behavior of the conditional and Gibbs’ entropies that we have determined analytically, and illustrated with examples, offer any insight into dynamic analogs of $S_{TD}(t)$ and $\Delta S(t)$?

The question of how a time dependent non-equilibrium entropy should be defined has interested investigators for some time, and specifically the question of whether the Gibbs’ entropy $H_G(f)$ can be taken to coincide with the time dependent entropy $S_{TD}(t)$ of the second law of thermodynamics has occupied many researchers. Various aspects of this question have been considered in [3, 10, 11, 12, 13, 14, 15, 16, 17, 18].

The non-equilibrium Gibbs’ entropy $H_G(f)$ is manifestly not a good candidate for $S_{TD}(t)$ because its dynamical behavior is at odds with what is demanded by the Second Law of Thermodynamics. As we have demonstrated and summarized, concrete examples can be constructed in which the direction of the temporal change in $H_G(f)$ depends on the initial preparation of the system and others can be constructed in which $H_G(f)$ oscillates in time. Thus there is good reason to search for a different analog of $S_{TD}(t)$.

A number of authors, among them de Groot and Mazur [19, pp. 122-129, Eq. 247], van Kampen [20, pp. 111-114 and 185], and Penrose [21, p. 213] have suggested that $S_{TD}(t)$ should be associated dynamically with

$$H_{NE}(f) \equiv H_c(f|f_s) e^{H_G(f_s)} = H_c(f|f_s) + H_G(f_s) \quad (5.5)$$

as an extension of Gibbs [8] pp. 44-45 and 168] discussion of entropy. This also goes under the name of the “Gibbs’ entropy postulate” [22, 23, 24, 25, 26, 27, 28].

Here, we have shown that $H_{NE}(f) = H_c(f|f_s) + H_G(f_s)$ has the temporal behavior required for the entropy $S_{TD}(t)$ of the second law of thermodynamics. This is a consequence of the temporal behavior of $H_c(f|f_s)$. Namely $H_{NE}(f)$ is either constant for invertible dynamics, or monotone increasing to the equilibrium value of $H_G(f_s)$ for non-invertible asymptotically stable dynamics induced by noise perturbations. Once this identification is granted, then it follows that $\Delta S(t)$ should be identified with $H_c(f|f_s)$:

$$\Delta S(t) \equiv H_c(f|f_s) = - \int_X f(t, x) \log \frac{f(t, x)}{f_s(x)} dx, \quad (5.6)$$

as has been previously suggested [13, 58].
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