Rate distortion coevolutionary dynamics and the flow nature of cognitive epigenetic systems

James F. Glazebrook, PhD
Department of Mathematics and Computer Science
Eastern Illinois University

and

Rodrick Wallace, PhD
Division of Epidemiology
The New York State Psychiatric Institute

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Abstract

We outline a model for a cognitive epigenetic system based on elements of the Shannon theory of information and the statistical physics of the generalized Onsager relations. Particular attention is paid to the concept of the rate distortion function and from another direction as motivated by the thermodynamics of computing, the fundamental homology with the free energy density of a physical system. A unifying aspect of the dynamic framework involves the concept of a groupoid and of a groupoid atlas. From a stochastic differential equation we postulate a multidimensional Itô process for an epigenetic system from which a stochastic flow may permeate through components of this atlas.

Key words Rate distortion function, epigenetic system, free energy density, groupoid, Onsager relations, Itô process.

1 Introduction

Living systems as far-from-equilibrium open systems, are essentially cognitive, and conversely, in order to grasp the essence of cognition, one may attempt to understand the ontology of the former processes. Often this is viewed in the framework of a cell-like, structured, self-organizing system engaged in a two-way interaction between its functional mechanisms and that of its neighboring environment. There are several theories that revolve around this principle. Readers may be familiar with a central relationship as explained by autopoiesis, a seemingly enduring theory as developed by Maturana and Varela (1980a, 1980b) over several decades. Though a much earlier and somewhat different hypothesis of Bertalanffy (1972) had proposed that living systems are maintained in non-equilibrium states by a flow-like patterns when drawing matter and energy from their environment, and adjust accordingly in a “flowing balance” (Bertalanffy, 1972; Capra, 1996). Further scientific rigor aimed at understanding this hypothesis was achieved by Prigogine (1980) who formulated similar ideas cast within a theory of dissipative structures. Whereas living systems are continuously maintained in far-from equilibrium, dissipative structures do likewise while being capable of evolving. They are destabilized in the increase of information and energy, though as they remain self-organized and self-perpetuating, the complexity of their structure increases; often this is simply for the sake of survival within the environment. Elucidating a possible synthesis of these separate approaches, commencing from the Bertalanffy hypothesis, is the main topic of Capra (1996). But cognition is also a function of a prevailing culture: by means of social and historical patterns of behavior traditions, etc., human cognition feeds back into that culture through the course of social interaction, adaptive technologies, policies, memetic trends, etc., and inevitably alters it in time (see e.g. Clark, 1997; Hollan et al., 2000; Hutchins, 1994; Richerson and Boyd, 2004; Wallace and Fullilove, 2008).

There are several common factors at stake here, and in seeking to understand these, the ‘immunology–language’ viewpoint of Atlan and Cohen (1998) (see also Cohen, 2000) views human organizations at all levels as perceiving patterns of threat or opportunity, comparing those patterns with some internal, learned or inherited, picture of the world, and then choosing one or a small number of responses from a vastly larger repertory of that which is possible to them. Putting it another way,

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*Department of Mathematics and Computer Science, Eastern Illinois University, 600 Lincoln Avenue, Charleston IL 61920–3099 USA, email jfglazebrook@eiu.edu

†549 W. 123 St., Suite 16F, New York, NY, 10027, USA. Telephone (212) 865-4766, email rdwall@ix.netcom.com. Affiliation is for identification only.
consider a basic observation concerning the immune system: the latter incorporates its own system of options and responds
cognitively at the level of information processing such that the meaning of an antigen is defined by the response of the immune
system, somewhat reminiscent of earlier work of Jerne (1974) who had postulated such ‘meaning’. The broader cognitive
model that results from this can said to be a ‘reactive’ system determined by contextual factors in Cohen and Harel (2007)
as depicted in Figure 1.

Starting from this basic perspective, Wallace (2005) has formulated a model of cognition that takes ‘meaning’ in the
sense of Dretske’s theory of semantic communication (Dretske, 1981, 1988) claiming that the immune perception/response
information networks of any cognitive system must be constrained by Shannon’s fundamental limit theorems of information
theory (see e.g. Ash, 1990; Berger, 1971; Cover and Thomas, 1991). These networks comprising of cognitive modules interact
within the framework of a kind of broadcasting system relaying within ‘a theater of consciousness’ – the basic operative
hypothesis of the Global (Neuronal) Workspace theory as developed in (Baars, 1988; Baars and Franklin, 2003) that forms
an integral part of the model in (Wallace, 2005; Wallace and Fullilove, 2008).

Here we take a further step forward that incorporates a number of related factors mainly following the (generalized)
Onsager relations of non-equilibrium thermodynamics combined with the principles of rate distortion theory which is one of
the mainstays of Shannon’s pioneering work. One particular observation is that distortion in communication between
interacting cognitive modules, is patently stochastic, in particular, it is manifestly a process of Brownian motion. This
observation is developed in several steps relative to a corresponding stochastic differential equation that eventually unfolds
to a kind of master equation. By considering critical solutions for the distortion, we argue on a case-by-case basis that
the ensuing Brownian motion (viz Weiner process) can be locally one of bounded variation. Accordingly, we are motivated
to introduce the necessary conditions that permit integration of these equations in order to obtain a multidimensional Itô
process; further we discuss those formal conditions that describe an associated stochastic solution flow. Thus in a way we
have recovered Bertalanffy’s perception of the “flow” of living systems, the kind of which information-tied epigenetic systems
are certain representations.

Whereas such model conditions are mathematically stated for a flow in a smooth manifold geometry, we do not claim
such ‘smoothness’ in general. Indeed, such epigenetic ‘flow systems’ in living organisms and brain-environment interactions
may be at best continuous over time and are most likely to be ‘singular’ in some sense. The latter requires harder topological
techniques and possible complications that detract from a more natural interpretation. Eventually one seeks a workable
common ground between various mathematical abstractions and the actual empirical properties of the systems themselves.

2 The basis of the epigenetic model

2.1 Gene-environment interaction

Another slant emerging from this cognitive paradigm goes towards epigenetic information sources as providing a tunable
catalyst directed to gene expression by which the embedding of information sources can direct developmental pathways
within the ontology of enveloped structures via the process of information (Wallace and Wallace, 2008). This leads to further
developments of several cognitive paradigms for gene expression, in part furthering the scientific reason underlying that of
Jablonka and Lamb (2005) who consider epigenetic inheritance systems in which information may be transmitted through
generations, not just simply through the base sequence of DNA, but also transmitted via cultural and behavioral means in
higher animals, and by epigenetic means in cell lineages. This is initiated by memory systems that enable somatic cells of
differing phenotype, but of identical genotype, to transmit their phenotypes to their descendants, even in the absence of the
original stimuli that had engaged these phenotypes.

Conditional upon an individual’s phenotype, environmental factors may trigger alterations in behavior and health, eventu-
ally impinging upon the nervous system with the likely consequence of mental disarray. In this respect Moffitt et al. (2006)
(cf Caspi and Moffitt, 2006) make several observations: (1) a heritable versus environmental influence on phenotype vari-
ation across a given environment, (2) altered gene expressions via epigenetic programming geared in response to subsequent
health-behavioral reactions towards the environment, (3) how an individual’s phenotype determines a risk-level towards the
environment, and (4) behavioral effects due to interdependence between specified variations in the DNA sequence specific to
a measured environment. Indispensable to understanding and extending these findings is the parallel question of how gene-
cultural interaction, two distinct but interacting hereditary systems, compares psychopathology across oriental and western
cultures in basic perceptual processes, often an important piece missing from the larger picture (Nisbett, 2003; Richerson
and Boyd, 2004; Wallace, 2005; Wallace, 2009). As such distinct interacting systems of information influencing action and
behavior, both kinds (genes and culture) are claimed in Durham (1991) to create a real and unambiguous symmetry: between
genes and phenotypes on the one hand, and culture and phenotypes on the other, whereby genes and culture may be reason-
ably viewed as two parallel lines of hereditary influence on phenotypes. From the perspective of (Wallace, 2005; Wallace and
Wallace, 2008; Wallace and Wallace, 2009) both can be realized as generalized languages in the sense they have their
own intrinsic recognizable grammar and syntax (see §). Likewise, on reflecting upon the fundamental mechanisms of serial
endosymbiotic theory (see e.g. Margulis, 2004), Witzany (2006) argues a case for extending the latter via biosemiotic cell-cell type interactions as 'signed' language-communication processes subject to a range of syntactic, pragmatic and semantic rules as applied to protein coding DNA, RNA editing, DNA splicing, transcription and other essential functions.

2.2 Autopoiesis and the structuralist approach

A more general perspective is to consider those central relations characterizing the system and its various structural components, and how a state is altered under perturbations by the environment. An autopoietic system is organizationally closed and structurally determined. The system’s autopoiesis is preserved within the living state, adaptable only to structural fluctuations for as long as the living entity survives within and is structurally coupled to its environment; otherwise there is termination. The autopoiesis of the nervous system (though itself is not strictly autopoietic) functionally self-(creates) replicates in order to engage in cognition. Though structurally dependent, the nervous system affords an innate plasticity and with appropriate alterations, it is conducive to learning and can adapt itself towards broader interactions and human self-consciousness (Maturana and Varela, 1980a, 1980b). Such broader interpretations of this theory are addressed in Mingers (1991), such as the family nexus which via its idiosyncratic behavioral and linguistic interactions (in relationship to hereditary factors, socioeconomic status, environment, culture, etc.), creates a peculiar structured reality that for the best part is only accessible to the members themselves, and through which a prospective psychotherapist must explore in order to fathom out those recurrent patterns of conversation and behavior influencing the cognitive malfunctioning of any concerned (cf Laing and Esterson, 1964).

There are related theories with varying degrees of overlap to the overall concepts of Maturana and Varela (1980a, 1980b). One example is a biogenetic-structuralist theory (Laughlin and d’Aquili, 1974) where connections within an evolutionary context are made between neural organization/brain–functional activity and the environment. The result is “acquired models of reality”, again, a cognitive outcome of the response to and engagement with the environment, proposing a concept of ‘neurognosis’, a kind of ‘holographic’ model based upon biogenetically induced rudimentary information embedded in various associated regions of the brain, as taken from birth towards the later stages of maturity. The neurognostic model depends upon ontogenetic feedback from an external reality according to which acquired components are continually engaged with sensory input. Such modes are claimed to be represented in neural processes by probability expectations which we shall see, lie at the heart of information theory, and which encode the behavior of these process in relationship to the environment. Thus it may be reasonable to suppose that enriched environments of some sort reciprocate to a greater advantage than others, and the structures governing the alignment of the self to the environment to an extent evolved in accordance to some degree of mastery over the latter. Take for instance, the hypothalamic–pituitary–adrenal (HPA) axis (as governing the neurophysiology of the “flight or fight” mechanism) is cognitive in the sense of Atlan and Cohen (1998). If there is an arousal of the individual’s close environment, then mind, memory and emotional cognition engage, evaluate and select appropriate responses. The HPA accelerates this process and possible malfunctioning may induce hyper–reactivity as observed in cases of post–traumatic stress disorder. Depression, as another example, may be partly viewed as the evolution of a structure conducive to a negative
alignment of the self with an external reality. If we think of a child as gradually mastering its environment via a symbiotic relation with its mother, then take away the mother, the inability to further manage the environment activates a negative kind of (neurognostic) structure such as those that have been researched in the context of various evolutionary theories of depression that are founded upon the occurrence of attachment-defeat-loss, diminished opportunities, down-regulation of foraging capability, social/professional rank, etc. incurred with varying risk factors within a culturally influenced environment (see e.g. Gilbert, 2006; Moffitt et al., 2006; Wallace, 2009). Another example is to consider the body’s blood pressure control system consisting as a network of cognitive systems which compare a set of incoming signals with an internal reference structure in order to select a suitable level of blood pressure from possible levels; hence as claimed in (Wallace and Fullilove, 2008; Wallace and Wallace, 2009) an elaborate tumor control strategy must be at least as cognitive as the immune system itself.

2.3 Trail systems and Roman roads

Thus granted that most (if not all) given classes of cognitive modules interact within their cultural environment, we may proceed to consider what happens when the environment is the communication medium itself. For instance, in typical AI laboratory models where multi-agent, inter-sensing systems function in local coordinated tasks (often with no explicit communication between agents), the eventual net effect may induce a ‘shared memory’ (Cao et al., 1997; Krieger et al., 2000) where, for instance, it was claimed that more energetic and efficient foraging tasks were typical of multi-agent (robotic) systems compared to individual agents, and the former tended to produce behavioral patterns similar to those of ‘aul–like’ decentralized control systems. On the other hand, biological regulatory networks besides being susceptible to alterations in the environment and/or intracellular conditions, may operate stochastically in varying degrees as seen along the pathways of neuronal signalling transduction (Manninen et al., 2006) that provides an analogy motivating part of this paper (see §4). A similar scenario is that of neuronal ‘Trail Systems’ (TS) in Glade et al. (2009): single wire and logical gates in a self–organized bioprocessor along which self–propelled particles communicate via traces etched out in the environment, thus creating the TS. The claim of Glade et al. (2009) is that such systems, although not precisely defined, are capable of programming and function on the basis of a Turing machine. It suggests an evolutionary factor by which various living beings, by activating their respective nervous systems, have trained themselves to use models like the TS towards evolution within and possible mastery over their environment, by means of simulating ‘trail’ signals. More from an information–theory viewpoint, Wallace (2009b) envisages similar ideas to trail systems as reminiscent of ‘Roman roads’: decision making and tasking within small local communities, eventually creating ‘roads’, manifestly inter–connected cognitive modules having different time constraints, but eventually creating extensions of ‘local consciousness’. A rate distortion argument applies here to account for the mutual crosstalk between different modules using the homology of the rate distortion function with free energy as will be described later. Next we proceed with some specifics.

3 Rate distortion and source entropy

3.1 The rate distortion function

As is well–known, distortion arises when there is a fast relay of information through some channel which exceeds the latter’s capacity. One of the principles of the Shannon theory is that in order to reproduce a message transmitted from a source to a receiver, it is necessary to know what sort of information should be transmitted and how. For the purpose of engineering a communication system, one needs to figure out a suitable encoding/decoding system once the nature of the channel is specified. Following Berger (1971), we briefly recall some of the basic principles involved.

Source encoder: We may consider some output $x(t)$ emanating from the source as projected to a finite set of preselected images, namely, the space of possible source outputs is partitioned into a set of equivalence classes and the source encoder informs the channel encoder of that class containing the particular source output observed. Once the channel encoder is informed that the source output belongs to say, the $m$-th equivalence class, it transforms the corresponding waveform $\tilde{x}_m(t)$ across the channel.

Source decoder: Within the system is a cascade of a channel encoder and a source decoder. The channel decoder receives a waveform $\tilde{y}(t)$ of a corresponding function $y(t)$ over some time interval and decides upon the nature of the message as transmitted. Then it sends its approximation $\tilde{y}_m(t)$ to register the system’s estimate of $x(t)$ over that time interval. Initially, we may think of $x(t)$ and $y(t)$ as ‘waveforms’, but in our case, we consider these as consisting of a language with its own intrinsic grammar/syntax, as well as ‘meaning’ – to be made more specific in §5.3 Analogous considerations apply to the channel signals $\tilde{x}(t)$ and $\tilde{y}(t)$ (see Figure 2).

One of Shannon’s notable results was that a communication system can be designed such that it achieves a level of fidelity $D$ once the rate distortion $R(D) \leq C$, where $C$ denotes the channel capacity. Putting it another way, if the receiver can
tolerate an average amount of distortion $D$, the rate distortion $R(D)$ is the effective rate at which the source can relay information with that level of tolerance.

The rate at which a source produces information subject to insisting upon perfect reproduction, is the source entropy $H$. Given a distortion measure such that perfect reproduction is assigned zero distortion, then we have $R(0) = H$. As $D$ increases, $R(D)$ becomes a monotonically decreasing (convex) function which eventually is zero, typically at a maximum value for $D$ (see Berger, 1971, Chapter 1). This is a very basic observation, and typically in rate distortion theory one seeks a reduction of $H$ by either slowing down the emission of coding, or encoding the relevant languages at a lower rate. In view of Shannon’s theorem, as long as $C > H$, we will obtain appropriate fidelity in transmission. Inherent difficulties are clear, since the source rate may be corrupted due to low memory and coding congestion, hence the need for a communicating system to evolve so as recover the source data at the channel output satisfying the Shannon estimate.

### 3.2 Average mutual information

Having mentioned the rate distortion function $R(D)$ we now follow Berger (1971) to give its specific definition in terms of average mutual information (an alternative, and equivalent definition of $R(D)$ and a statement of the Rate Distortion Theorem will be given in Appendix I). Firstly, for $k, j$ running over a suitable alphabet, let us write a given conditional probability assignment as $Q(k|j)$ such that in the usual way we have an associated joint distribution $P(j, k) = P(j)Q(k|j)$. We express the average distortion as

$$d(Q) = \sum_{j,k} P(j)Q(k|j) \ d(j,k), \quad (3.1)$$

where $d(\ , \)$ denotes the distortion measure. A conditional probability assignment $Q(k|j)$ is said to be $D$–admissible if and only if $d(Q) \leq D$. The set of all $D$–admissible conditional probability assignments we denote by

$$Q_D = \{Q(k|j) : d(Q) \leq D\}. \quad (3.2)$$

Along with an average distortion $d(Q)$, we also have an average mutual information

$$I(Q) = \sum_{j,k} P(j)Q(k|j) \log \left( \frac{Q(k|j)}{Q(k)} \right). \quad (3.3)$$

Then for fixed $D$, the rate distortion function is defined as

$$R(D) = \min_{Q \in Q_D} I(Q). \quad (3.4)$$

Observe that if a parameter $s$ represents the slope of the function $R(D)$ at a point $(D_s, R_s)$ generated parametrically, we have $R'(D) = s$ (this is not exactly trivial: see Berger, 1971, Theorem 2.5.1).

### 3.3 Meaningful paths

More formally, a pattern of sensory input is mixed in an unspecified but systematic algorithmic manner with a pattern of internal ongoing activity to create a path of combined signals $x = (a_0, a_1, \ldots, a_n, \ldots)$. Each $a_k$ thus represents some functional composition of internal and external signals. Wallace (2005) provides some neural network examples.
This path is fed into a highly nonlinear, but otherwise similarly unspecified, decision oscillator, $h$, which generates an output $h(x)$ that is an element of one of two disjoint sets $B_0$ and $B_1$ of possible system responses. Let

$$B_0 \equiv b_0, \ldots, b_k,$$

$$B_1 \equiv b_{k+1}, \ldots, b_m.$$  \hspace{1cm} (3.5)

Assume a graded response, supposing that if

$$h(x) \in B_0,$$  \hspace{1cm} (3.6)

the pattern is not recognized, and if

$$h(x) \in B_1,$$  \hspace{1cm} (3.7)

the pattern is recognized, and some action $b_j, k+1 \leq j \leq m$ takes place. Such oscillators may be influenced by ‘forcing’ when a signal is subjected to some impulse such that its frequency, and hence the response, adjusts accordingly with respect to that applied impulse. More familiar oscillating physical models react to this by exhibiting ‘beats’ and ‘resonance’ for instance.

The principal objects of formal interest are paths $x$ which, through information flow, trigger pattern recognition-and-response. That is, given a fixed initial state $a_0$, we examine all possible subsequent paths $x$ beginning with $a_0$ and leading to the event $h(x) \in B_1$. Thus $h(a_0, \ldots, a_j) \in B_0$ for all $0 < j < m$, but $h(a_0, \ldots, a_m) \in B_1$.

For each positive integer $n$, let $N(n)$ be the number of high probability grammatical/syntactical paths of length $n$ which begin with some particular $a_0$ and further leading to the condition $h(x) \in B_1$. These are paths of combined signals as above, that are structured to some language. For short, we call such paths ‘meaningful’, assuming, not unreasonably, that $N(n)$ will be considerably less than the number of all possible paths of length $n$ leading from $a_0$ to the condition $h(x) \in B_1$.

One critical assumption which permits an inference on the necessary conditions constrained by the asymptotic limit theorems of information theory, is that the finite limit

$$H \equiv \lim_{n \rightarrow \infty} \frac{\log[N(n)]}{n},$$  \hspace{1cm} (3.8)

(the ‘uncertainty’) both exists and is independent of the path $x$. The rate distortion principle applies as follows (Wallace, 2005): the restriction to meaningful sequences of symbols increases the rate at which information can be transmitted with arbitrary small error, and that the grammar/syntax of the path can be associated with a dual information source. Here we may assume a typical information source $X$ to be ‘adiabatic’, ‘piece-wise stationary’ and ‘ergodic’ (APSE), and that a system engaging in a cognitive process is describable as such. We list here the explanations:

1. ‘Adiabatic’ means that the changes are slow enough to allow the necessary limit theorems to function.

2. ‘Stationary’ means that between pieces the probabilities hardly change, and ‘piecewise’ means that these properties hold between phase transitions which are described using renormalization methods (see Wallace, 2005).

3. ‘Ergodic’ means that in the long term, correlated sequences of symbols are generated at an average rate equal to their (joint) probabilities.

More specifically, the essence of ‘adiabatic’ is that, when the information source is parametrized according to some appropriate scheme, within continuous ‘pieces’ of that parametrization, alterations in parameter values occur slowly enough so that the information source $X$ remains as close to stationary and ergodic as needed to put to work the fundamental limit theorems of information theory. In view of (3.8), the Shannon uncertainty of $X$ can be stated more specifically by (see e.g. Cover and Thomas, 1991):

$$H[X] = \lim_{n \rightarrow \infty} \frac{\log[N(n)]}{n}.$$  \hspace{1cm} (3.9)

### 3.4 The fundamental homology

We recall how the information source uncertainty was defined as in equation (3.8). This is quite analogous to the free energy density of a physical system, equation (5.2), and the relevance to a cognitive process can be explained by the following steps. For instance, Feynman (1996) provides a series of physical examples (based in part on the research of C. H. Bennett into the thermodynamics of computing (Bennett, 1982) where this homology is, in fact, an identity, at least for very simple systems. Bennett (1982) argues, in terms of idealized irreducibly elementary computing machines, that the information contained in a message can be viewed as the work saved by not needing to recompute what has been transmitted, or as Feynman (1996) puts it: the information contained in a message is proportional to the amount of free energy density needed to erase it. The
essential argument is that computing, in any form, takes work. Thus the more complicated a cognitive process, measured by its information source uncertainty, the greater its energy consumption, and our ability to provide energy to the brain is limited: typically, a unit of brain tissue consumes an order of magnitude more energy than a unit of any other tissue.

The less information available to us concerning an event, the higher is its entropy, and information retrieved is not without a cost in expenditure of energy, where ‘cost’ may be interpreted as the necessary number of bits needed to encode a message. The thermodynamic minimum of energy in terms of bits of information is $k_B T \log_2 e$ erg/bit ($= k_B T$ erg/nat). So efficiency in an information system is essentially when there is the minimum amount of energy expended in retrieving information. Specifically, if $F$ is taken to denote the free energy, then taking $\Lambda$ is to denote the minimum number of nats/sec, the efficiency of the system is given by $\eta = k_B TF^{-1} \Lambda$ (see e.g. Berger, 1971).

In a similar spirit to Bennett’s work, Li and Vitányi (1992) consider the thermodynamic costs of computation and how certain thermodynamic considerations can give a recursively invariant notion of ‘cognitive distance’ using a kind of billiard dynamics approach. In this case a minimal cognitive distance between two objects will correspond to the minimal amount of work expended for a given cognitive transformation of objects, either by some computational procedure or by some neurocognitive function of the brain. A higher descriptive level leading to more complex and protracted algorithms, then leads to greater Kolmogorov complexity (Li and Vitányi, 1993). As a particular application of the Bennett/Feynman ideas in the Global Workspace setting, Wallace (2007) argues that the cognitive disorder of ‘inattentive blindness’ emerges as a thermodynamic limit on processing capacity in a topologically-fixed global workspace, i.e. one which has been strongly configured about a particular task. Institutional and machine generalizations seem clear.

### 4 Dynamic groupoids and their atlases

#### 4.1 Concept of a groupoid

Many cognitive processes exhibit the patterns of dynamical systems (see e.g. Glazebrook and Wallace, 2009a). In such systems one aims to unify the internal and external symmetries, and to be able to reduce vast myriad-like network configurations into manageable schemes involving the corresponding equivalence classes analogous to those already mentioned in source encoding/decoding, etc. in §3.1 (see also §6.2 below). A precise way of doing this lies within the categorical concept known as a groupoid (see Brown, 2006; Connes, 1994; Weinstein, 1996). In essence a groupoid $G$ consists of both a set of objects $X$ and a set of morphisms, or ‘arrows’, each of which project to an object in $X$, and all such morphisms admit an inverse.

**Remark 4.1.** The most familiar example of a groupoid, as known to students of algebra, is that of a ‘group’ where there is a single object (‘the identity’). Hence groupoids can be viewed as extensions of the ‘group’ concept to sets of multiple identities thus providing a wide scope of applications to the dynamics of neurocognitive and socio–bioinformatic systems (see e.g. Baianu et al., 2006; Glazebrook and Wallace, 2009a, 2009b; Golubitsky and Stewart, 2006; Stewart et al., 2003; Wallace 2005; Wallace and Fullilove, 2008).

A groupoid can be depicted by

$$\alpha, \beta : G \xrightarrow{\alpha} X$$

where the groupoid morphisms $(\alpha, \beta)$ onto objects, are called the range and source maps, respectively. Informally, the groupoid represents a feature of built in reciprocity between its algebraic structures, internalizing and externalizing the prevailing symmetries. The morphisms $\alpha, \beta$ satisfy certain algebraic relations of associativity, existence of two-sided identities, etc. (details can be seen in e.g. Brown, 2006; Connes, 1994; Weinstein, 1996). A groupoid can here be understood in relationship to a linkage by a meaningful path of an information source dual to a cognitive process for which the underlying principle is that: states $a_j, a_k$ in a set $A$ are related by the groupoid morphism if and only if there exists a high probability grammatical path connecting them to the same base point, and the tuning across the various possible ways in which that can happen – the different cognitive languages – parametrizes the set of equivalence relations and creates the groupoid.

**Example 4.1.** Since we have already mentioned equivalence classes in the context of source encoding/decoding, it seems appropriate to see how an equivalence relation $\mathcal{R}$ defined on (a set) $X$ takes shape as a groupoid. Here we have the two projections $\alpha, \beta : \mathcal{R} \rightarrow X$, and a product $(x, y)(y, z) = (x, z)$ whenever $(x, y), (y, z) \in \mathcal{R}$ together with an identity, namely $(x, x)$, for each $x \in X$. Moreover, the essential equivalence relations (classes) derived from a systems space (network) arise from the orbit equivalence relation of some groupoid $G$ acting on that space (see e.g. Weinstein, 1996). In the context of connected (sub)networks/graphs which can reduced to equivalence classes, natural groupoid structures come about in accordance with equivalence classes of relations $\mathcal{R}(xy)$, as above, that is simply interpreted as having an edge linking node $x$ to node $y$. Conversely, a groupoid (of equivalence relations) admits an underlying graph structure via its implicit scheme of objects and morphisms between objects (for details, see e.g. Brown, 2006). Thus we have the two-way associations whereby...
‘objects’ can be identified with ‘nodes’, and ‘morphisms’ identified with ‘edges’ in groupoids (of equivalence relations) and networks, respectively:

- Network $\xrightarrow{\text{equivalence relation}}$ Groupoid
- Network $\xleftarrow{\text{underlying graph}}$ Groupoid

### 4.2 Groupoid atlases

An important observation about multi-tasking in institutional and distributed cognitive systems concerns how the various submodules interact. When a given subnetwork is represented in its groupoid form, then such interactions naturally can be realized in terms of groupoid actions, a topic that warrants further attention. Then we would like to see the overall cognitive system in terms of such interacting groupoids and designed by an ‘atlas’ of the latter, and one, as shown in Glazebrook and Wallace (2009a), containing the representation of several possible emergent ‘giant components’ induced by the outcome of local group(oid) actions within the Workspace (see Figure 3). A workable concept seems to be that of a groupoid atlas (Bak et al., 2006) which provides a schematic representation for coupling interactions between multi-agent systems and uses a pasting together of the local dynamic groupoid actions with the net effect of a ‘global’ groupoid.

One commences from a family of dynamically interacting groupoids $(G_\alpha) = \{G_1, G_2, \ldots\}$ where each groupoid has the same set of objects; this family is called a single domain or multiple groupoid. A groupoid atlas is then defined as a set with a covering by patches each of which comprise a single domain with global action, representing the local processing which is then globalized across the atlas. This is a desirable effect and one particularly suited to logically inscribing processors or sensors (the ‘agents’) within the cognitive modules of the Workspace. As a descriptive mechanism, this atlas has the advantage of admitting a weaker structure compared with that of a conventional manifold since no condition of compatibility between arbitrary overlaps of the patches is necessary. This is a key property relevant to the structure of cognitive modules that can be geared to equivalence class representations where flexibility in the structure is a natural characteristic. In this way, the atlas provides a convenient description of a web of complexity representing the dynamic reciprocity of tightly-knitted functional systems as was applied to small world networks (Glazebrook and Wallace, 2009a). Time and space does not permit including a mathematical outline of the construction; for the technical details we refer the reader to Bak et al. (2006) and del Hoyo and Minian (2009). However, we intend to apply this concept with some essential details in §5.3 below.

### 5 Information and the Onsager relations

#### 5.1 A fundamental homology and the Onsager relations

The information supported interactive cognitive modules we have in mind are assumed to possess their own internal metabolisms and mechanisms of self-organization as reflective of vital biochemical processes. Just as for the latter, the evolution of these ‘cognitive cells’ is characterized by reactive states of thermal nonequilibrium, in accord with the laws of thermodynamics, and the capability to assimilate information which we study in respect of source uncertainty. This we can achieve by applying the Onsager relations of nonequilibrium thermodynamics (Kurzynski, 2006; Landau and Lifshitz, 2007). The reasoning starts by observing how a fundamental homology between the information source uncertainty dual to a cognitive process and the free energy density of a physical system can arise mainly from the formal similarity between their definitions in the asymptotic limit.

#### 5.2 The Groupoid Free Energy Density

Recall that for a thermodynamic state of a given system at fixed temperature $T$ with energy $E$ and entropy $S$, the free energy density $F$ is defined to be

$$F = E - TS.$$  \hspace{1cm} (5.1)

In the Hamiltonian formulism one takes the volume $V$ and the partition function $Z(K)$ derived from the system’s Hamiltonian at inverse temperature $K$ (Kurzynski, 2006; Landau and Lifshitz, 2007). The free energy density is then defined to be

$$F[K] = \lim_{V \to \infty} \frac{1}{K} \log[Z(K,V)]$$

$$= \lim_{V \to \infty} \frac{\log[\tilde{Z}(K,V)]}{V}, \text{ where } \tilde{Z} = Z^{-\frac{1}{K}}. \hspace{1cm} (5.2)$$

Consider now an information source $H_{G_\alpha}$ over a corresponding groupoid $G_\alpha$. The probability of $H_{G_\alpha}$ is given by:

$$P(H_{G_\alpha}) = \frac{\exp[-H_{G_\alpha}K]}{\sum_\beta \exp[-H_{G_\beta}K]}, \hspace{1cm} (5.3)$$
Figure 3: Displayed are groupoids $G_1, G_2, G_3$ derived from the equivalence relations of their respective networks of multi-agent type cognitive modules. They are viewed as the components of a groupoid atlas. The groupoid actions are indicated by dotted arrows, in this case, between $G_1$ and $G_2$, and may represent the formation of network linkages as the system is shifted via crosstalk, and for instance, how a ‘giant component’ emerges (see e.g. Glazebrook and Wallace, 2009a). This is a descriptive mechanism that could be seen, for instance, as enveloping the underlying networks of the ‘reactive’ system in Figure 1, and is applicable to the content of §5.3.
where the normalizing sum is over all possible subgroupoids of the largest available symmetry groupoid. Now let

$$Z_G = \sum_\alpha \exp[-H_{G_\alpha}].$$ \hspace{1cm} (5.4)

The groupoid free energy density (GFE) of the system $F_G$ at inverse normalized equivalent temperature $K$ is defined as

$$F_G[K] = -\frac{1}{K} \log[Z_G(K)].$$ \hspace{1cm} (5.5)

With each such groupoid $G_\alpha$ of the (large) cognitive groupoid, we can associate a dual information source $H_{G_\alpha}$. We recall the rate distortion function between the message sent by the cognitive process and the observed impact, while noting that both $H_{G_\alpha}$ and $R(D)$ may be considered as free energy density measures. In a sense, $R(D)$ constitutes a sort of ‘thermal bath’ for the process of cognition. Then the probability of the dual information source can be expressed by

$$P(H_{G_\alpha}) = \frac{\exp[-H_{G_\alpha}/\kappa R(D)\tau]}{\sum_\beta \exp[-H_{G_\beta}/\kappa R(D)\tau]},$$ \hspace{1cm} (5.6)

where $\kappa$ denotes a suitable dimensionless constant characteristic of the system in the context of a fixed machine response time $\tau$. The sum is over all possible subgroupoids of the largest available symmetry groupoid. Accordingly, the term $R(D)\kappa$ represents a ‘rate distortion energy’, in this case, a kind of ‘temperature analog’. In the context of a fixed $\tau$, a decline in $R(D)$ (on increase in average distortion), acts to ‘lower the machine temperature’, driving it to more simple and less rich behaviors.

### 5.3 A groupoid atlas of information sources

The groupoids $G_\alpha$ can indeed be taken to comprise a groupoid atlas $A$ on an appropriate set $X_A$. In Bak et al. (2006) this is motivated by considering a group $G$ in the standard algebraic sense, and a family

$$\{(G_A)_\alpha \simeq (X_A)_\alpha : \alpha \in \Psi_A\}$$

(5.7)

of group actions ‘$\simeq$’ on subsets $(X_A)_\alpha \subseteq X_A$, where the local groups $(G_A)_\alpha$ and the corresponding subsets $(X_A)_\alpha$ are indexed by an indexing set $\Psi_A$ called the coordinate system of $A$ which is seen to satisfy certain conditions (Bak et al., 2006). Now the family of local groups can be replaced by a family of local groupoids $(G_A)$ defined with respective object sets $(X_A)_\alpha$, and with a coordinate systems $\Psi_A$ that is equipped with a reflexive relation denoted by $\leq$. This data is to satisfy the following conditions (Bak et al., 2006):

1. If $\alpha \leq \beta$ in $\Psi_A$, then $(X_A)_\alpha \cap (X_A)_\beta$ is a union of components of $(G_A)$, that is, if $x \in (X_A)_\alpha \cap (X_A)_\beta$ and $g \in (G_A)_\alpha$ acts as $g : x \rightarrow y$, then $y \in (X_A)_\alpha \cap (X_A)_\beta$.
2. If $\alpha \leq \beta$ in $\Psi_A$, then there is a groupoid morphism defined between the restrictions of the local groupoids to intersections

$$\phi^\beta : (G_A)_\alpha|_{(X_A)_\alpha \cap (X_A)_\beta} \rightarrow (G_A)_\beta|_{(X_A)_\alpha \cap (X_A)_\beta},$$

and which is the identity morphism on objects.

Thus each of the $G_\alpha$ with its associated dual information source $H_{G_\alpha}$ constitutes a component of an atlas which incorporates the dynamics of an (inter)active system through information sources, by means of the intrinsic (groupoid) actions. For simplicity, let us refer to this atlas as $A$. Suppose we have another such atlas $B$ representing a separate system which can be related to $A$ via a suitable transformation. To make matters precise we consider then a morphism $f : A \rightarrow B$ prescribed by a triple $(X_f, \Phi_f, G_f)$ satisfying (del Hoyo and Minian, 2009):

1. $X_f : X_A \rightarrow X_B$ is a set-theoretic function.
2. $\Phi_f : \Phi_A \rightarrow \Phi_B$ is a function that preserves the relation $\leq$.
3. $G_f : G_A \rightarrow G_B$ is a (generalized) natural transformation of groupoid diagrams over the function $\Phi_f$ which restricts to $X_f$ on objects.
These conditions can summarized in the following straightforward way. For each $\alpha$, a function $G_{f(\alpha)}: G_{\alpha} \rightarrow G_{\Phi f(\alpha)}$ is given, such that for objects $\text{Obj}(G_{f(\alpha)}) = X_f|X_{\alpha}$, and if $\alpha \leq \beta$, the diagram

$$
\begin{array}{cc}
G_{\alpha} & \longrightarrow & G_{\Phi f(\alpha)} \\
\downarrow & & \downarrow \\
G_{\beta} & \longrightarrow & G_{\Phi f(\beta)}
\end{array}
$$

(5.9)

is commutative.

### 5.4 Biochemical data compression

Further motivation is provided by considering phases as chemically and thermodynamically homogeneous when formally compared with average mutual information such as applied to living systems that are capable of assimilating and using information reaped either from their environment or from information that is intrinsic to their particular system via genetic characteristics. Here we reflect in part upon the theme of §3.4 in which we discussed certain physical means by which information runs at the cost of expending free energy. On course with the fundamental homology, it befits us to consider some physical equations that are homologous to the variational calculus of the rate distortion function, and follow to some extent Berger (1971, §6.4). The idea is that when a living information system is confronted with an environment, it reacts towards it at an atomic-molecular level. Given then some thermodynamic system, let $n_{jk}$ be the number of atomic weights of substance $j$ that end up in phase $k$, and let $n_j$ be the number of atomic weights that were introduced originally, so that $n_j = \sum_k n_{jk}$. The multiphase (chemical) equilibrium problem involves determining the $n_{jk}$. To proceed, let us express the free energy as $F = F_1 + F_2$, where

$$
F_1 = \sum_{j,k} n_{jk} c_{jk},
$$

$$
F_2 = \sum_{j,k} n_{jk} \log \left[ \frac{n_{jk}}{\sum_j n_{jk}} \right],
$$

(5.10)

where the $c_{jk}$ are the free energy constants, and where the term contained in the logarithm is the chemical potential of the reactant $j$ in phase $k$. Let now $n = n_0 + \cdots + n_{M-1}$ (for some $M$), $P_j = n_j/n$, $Q_{kj} = n_{jk}/n_j$ and $Q_k = \sum_j n_{jk}/n$, then

$$
F = F_1 + F_2 = n \sum_{j,k} P_j Q_{kj} [c_{ij} + \log \left( \frac{Q_{kj}}{Q_k} \right)] - \sum_j P_j \log P_j.
$$

(5.11)

Since in principle the $n$ and $P_j$ can be determined, the minimization of $F$ reduces to minimizing the double summation on the right side of (5.11). Letting $d_{jk} = -c_{jk}/s$, then ‘nature’ selects the $n_{jk}$ so as to minimize the quantity

$$
\mathcal{V} = \sum_{j,k} P_j Q_{kj} \left[ \log \left( \frac{Q_{kj}}{Q_k} \right) - s d_{jk} \right].
$$

(5.12)

The crucial observation here is that (5.12) is formally the same as

$$
\mathcal{V} = I(Q) - s d(Q),
$$

(5.13)

in other words, the function to be minimized is the difference between the average mutual information and $s$ times the average distortion. This is minimized by an appropriate choice of $Q = Q(k|j)$ such as to determine a point on the $R(D)$ curve where, as in (3.2) we have $R'(D) = s$. The multiphase chemical equilibrium is a local process; the overall system itself may in general remain in a state far-from-equilibrium.

According to Berger (1971), ‘nature’ then automatically performs the analogous minimization in multiphase chemical equilibrium, and when an information system encounters an environment, the reaction on the molecular level follows these principles. Thus the free energy of the combined system and the environment is minimized, although this is at the cost of the system’s free energy capacity where there will inevitably be some heat dissipation. Eventually the system can fine-tune its coding mechanism and re-configures itself for the task of gaining more specialized knowledge about its environment, and thanks to newly acquired information, it may engage the latter to its advantage along with detecting other drifting cells of information. How such optimizing procedures can be realized in an evolutionary context, is of course one of the main tasks of applying rate distortion arguments (cf Wallace and Wallace, 1998, 1999, 2008, 2009). For instance, a scenario studied in Tlusty (2007, 2008) concerns how the genetic code maps the 64 nucleotide triplets (codons) to 20 amino acids. In terms of this mapping, the code is viewed as a noisy information channel and the claim is that evolutionary characteristics determine the emergence of the code via appropriate selection of amino acids which minimise the risk of errors, and subsequently the code emerges at a ‘supercritical’ phase transition once the mapping ceases to be random.
6 The Onsager relations in the context of information

6.1 The basic equations

Understanding the time dynamics of cognitive systems away from phase transition critical points thus requires a phenomenology similar to the Onsager relations. If the dual source uncertainty of a cognitive process is parametrized by some vector of quantities $K \equiv (K_1, \ldots, K_m)$, then, in analogy with nonequilibrium thermodynamics, the gradients in the $K_j$ of the disorder, defined as

$$S = H(K) - \sum_{j=1}^{m} K_j \frac{\partial H}{\partial K_j}, \quad (6.1)$$

become of central interest. Equation (6.1) is similar to the definition of entropy in terms of the free energy density of a physical system, as suggested by the homology between free energy density and information source uncertainty described above. Pursuing the homology further, the generalized Onsager relations defining temporal dynamics become

$$\frac{dK_j}{dt} = \sum_i L_{ji} \frac{\partial S}{\partial K_i}, \quad (6.2)$$

where the (kinetic coefficients) $L_{ji}$ are, in first order, constants interpreted as reflecting the nature of the underlying cognitive phenomena. The partial derivatives $\partial S/\partial K$ are analogous to thermodynamic forces in a chemical system, and may be subject to override by external physiological driving mechanisms as shown in (Wallace, 2005; Wallace and Fullilove, 2008) along with further extensions of these dynamical procedures.

Remark 6.1. Equation (6.2) is ‘general’ in the sense that we do not necessarily assume the symmetry condition $L_{ji} = L_{ij}$ which in this latter case expresses Onsager’s 4-th law of thermodynamics (see e.g. (3.7) in Kurzynski, 2006). The matrix $L = [L_{ij}]$ is to be viewed empirically, in the same spirit as the slope and intercept of a regression model, and may have a structure far different when compared to the more basic, more familiar chemical or physical processes. Generally, information sources are notoriously one–way in time, as exemplified by the patent linguistic scarcity of palindromic structures that do actually make some sense.

6.2 Equivalence classes of information sources

Equations (6.1) and (6.2) can be derived in a simple parameter-free covariant manner which relies on the underlying topology of the information source space implicit to a process. Different cognitive phenomena have, according to our development, dual information sources, and we are interested in the local properties of the system near a particular reference state. We impose a topology on the system, so that, near a particular ‘language’ $A$, dual to an underlying cognitive process, there is (in some sense) an open set $U$ of closely similar languages $\hat{A}$, such that $A$ and $\hat{A}$ are subsets of $U$. Note that it may be necessary to coarse-grain the system’s responses to define these information sources. The problem is to proceed in such a way as to preserve the underlying essential topology, while eliminating ‘high frequency noise’. The formal tools for this can be found in e.g. Burago et al. (2001).

Since the information sources dual to the cognitive processes are similar, for all pairs of languages $A, \hat{A}$ in $U$, it is possible to make use of the following:

(1) Create an embedding alphabet which includes all symbols allowed to both of them.

(2) Define an information-theoretic distortion measure in that extended, joint alphabet between any meaningful (i.e. high probability grammatical/syntactical) paths in $A$ and $\hat{A}$, which we write as $d(Ax, \hat{A}x)$ (Ash, 1990; Cover and Thomas, 1991). Note that these languages do not interact, in this approximation.

(3) Define a metric on $U$, for example,

$$\mathcal{M}(A, \hat{A}) = \lim \frac{\int_{A,\hat{A}} d(Ax, \hat{A}x)}{\int_{A,A} d(Ax, Ax)} - 1, \quad (6.3)$$

using an appropriate integration limit argument over the high probability paths. The usual metric properties apply as in Burago et al. (2001).

Note that these conditions can be used to define equivalence classes of languages, where previously we defined equivalence classes of states which could be linked by meaningful paths to some base point. This led to the characterization of different information sources from which a formal topological manifold, which is an equivalence class of information sources, can
be constructed. As a working hypothesis we may assume this to be a standard differentiable manifold in which the set of such equivalence classes generates the dynamical groupoid (cf Glazebrook and Wallace, 2009a, 2009b), and then study those mechanisms, internal or external, which can break that groupoid symmetry. In particular, the imposition of a metric structure on this groupoid, and on its base set, would permit a nontrivial interaction between orbit equivalence relations and isotropy groups, leading to interesting algebraic structures.

Since $H$ and $M$ are both scalars, a ‘covariant’ derivative can be defined directly as

$$dH/dM = \lim_{\hat{A} \to A} \frac{H(A) - H(\hat{A})}{M(A, \hat{A})},$$

where $H(A)$ is the source uncertainty of language $A$. Suppose the system is set in some reference configuration $A_0$. To obtain the unperturbed dynamics of that state, impose a Legendre transform using this derivative, defining another scalar

$$S \equiv H - M dH/dM.$$  

(6.5)

The simplest possible Onsager relation – here seen as an empirical, fitted, equation like a regression model – in this case becomes

$$dM/dt = L dS/dM,$$  

(6.6)

where $t$ is the time and $dS/dM$ represents an analog to the thermodynamic force in a chemical system (cf §5.3). Relevant here are patterns of oscillatory-like behavior where a weak signal is amplified by the presence of noise as result of some synchronized hopping around local extrema. The standard terminology for this phenomenon is stochastic resonance (Gammaitoni et al., 1998) and we will proceed to give some idea of how this can be related to certain types of cognitive processes. Since we are working with stochastic differential equations, the first step is to modify the equation of thermodynamic force accordingly. To this extent equation (6.6) is rewritten as

$$dM/dt = LdS/dM + \sigma W(t),$$  

(6.7)

where $\sigma$ is a constant and $W(t)$ represents a white noise term. Again, the quantity $S$ is seen as a function of the parameter $M$. This leads directly to a family of classic stochastic differential equations expressed as differential 1-forms

$$dM_i = L(t, dS/dM) \, dt + \sigma(t, dS/dM) \, dB_i,$$  

(6.8)

where $L$ and $\sigma$ are appropriately regular functions of $t$ and $M$, and $dB_i$ represents the noise structure.

Such cognitive-epigenetic systems which are driven by stochastic and noise driven diffusion processes, may be suitably conditioned to admitting further noise perturbations that lead to a degree of stochastic resonance capable of amplifying a relatively weak signal or actually reducing the level of randomness in the system. Such resonance may function as a catalyst towards the system’s self-organization and complexity, in the same way as open systems far-from-equilibrium require internal amplification in order to reach a macroscopic dynamical structure (Gammaitoni et al., 1998; Prigogine, 1980; West et al., 2005).

6.3 Rate distortion dynamics

Recall that the rate distortion function $R(D)$ defines the minimum channel capacity necessary for the system to have an average distortion $\leq D$, thus imposing a limit on the information source uncertainty and suggesting how distortion measures can drive information system dynamics. In other words, $R(D)$ affords a homological relation to free energy density, very much along the lines of the above relation between free energy density and information source uncertainty. Accordingly, it is proposed that the dynamics of cognitive modules interacting in characteristic real-time $\tau$ will be constrained by the system as described in terms of $R(D)$, but now we generalize matters as in Wallace and Wallace (2008) by producing a vector–valued function $R(Q)$ where in the vector $Q = (Q_1, \ldots, Q_k)$ the first component is defined to be the average distortion, and then (cf (6.1))

$$S_R = R(Q) - \sum_{i=1}^m Q_i \frac{\partial R}{\partial Q_i},$$  

(6.9)

which leads to the deterministic and stochastic systems of equations analogous to the Onsager relations of nonequilibrium thermodynamics

$$dQ_j/dt = \sum_i L_{ji} \frac{\partial S_R}{\partial Q_i},$$  

(6.10)
together with
\[ dQ_i^j = L_i^j(Q_1, \ldots, Q_k, t) \, dt + \sum_i \sigma_{ij}^i(Q_1, \ldots, Q_k, t) \, dB_i^j, \tag{6.11} \]
where the \( dB_i^j \) represents often highly structured stochastic noise whose properties may be described in terms of Brownian motion and quadratic variation (see e.g. Kunita, 1990; Protter, 1995).

At this stage we introduce several examples for which part of the purpose will be to motivate introduction of the Itô principle which we will do so below.

**Example 6.1.** Firstly, for a simple Gaussian channel with noise having zero mean and variance \( \sigma^2 \), we have
\[ S_R = R(D) - DdR(D)/dD = \frac{1}{2} \log(\sigma^2 / D) + \frac{1}{2}. \tag{6.12} \]
The simplest possible Onsager relation becomes
\[ dD/dt = -\mu dS_R/dD = \frac{\mu}{2D}, \tag{6.13} \]
in which the term \(-dS_R/dD\) represents the force of an ‘entropic wind’ which is a kind of internal dissipation inevitably driving the real–time system of interacting cognitive information sources toward greater distortion. Equation (6.13) has a solution \( D = \sqrt{\mu t} \), showing in this case that the average distortion increases monotonically with time. Following Wallace (2009a, §7.2), this example shows that such a system will inevitably succumb to a relentless entropic force, requiring a constant free energy expenditure for maintenance of some fixed average distortion within the system’s communication between them. The distortion in this case will, without free energy input, have a time dependence \( D = f(t) \), with \( f(t) \) monotonically increasing in \( t \), eventually leading to the punctuated failure of the system. Further, in the Einstein diffusion equation, a straightforward argument of Wallace (2009a, §7.2) shows that the standard deviation of the particle position increases in proportion to \( \mu t \).

Thus, whereas we do not expect the high correlations of an information source to exhibit typical Brownian motion, it does seem to be the case that the distortion in communication between the interacting cognitive modules within the appropriate context of the Onsager relations, does display Brownian motion which may be of bounded variation in certain cases.

### 6.4 Rate distortion coevolutionary dynamics

Here we consider different cognitive developmental subprocesses of gene expression characterized by information sources \( H_m \) interacting through chemical or other signals, and assume that different processes become each other’s principal environments which is a suitable hypothesis within a broad coevolutionary context. Let
\[ H_m = H_m(K_1, \ldots, K_s, \ldots, H_j, \ldots), \tag{6.14} \]
where the \( K_s \) represent other relevant parameters, and \( j \neq m \). We regard the dynamics of this system as driven by a recursive network of stochastic differential equations. Letting the \( K_j \) and \( H_m \) all be represented as parameters \( Q_j \) (with the caveat that \( H_m \) does not depend on itself), we follow the generalized Onsager formulation of Wallace and Wallace (2009), in terms of the equation
\[ S^m = H_m - \sum_i Q_i \partial H_m / \partial Q_i, \tag{6.15} \]
to obtain a recursive system of phenomenological Onsager relations, in terms of a system of stochastic differential equations
\[ dQ_i^j = \sum_i \left[ L_{ij}(t, \ldots, \partial S^m / \partial Q^i, \ldots) \, dt + \sigma_{ij}(t, \ldots, \partial S^m / \partial Q^i, \ldots) \, dB_i^j \right], \tag{6.16} \]
in which, for ease of notation, both the terms \( H_j \) and the external \( K_j \)’s are expressed by the same symbol \( Q_j \). As \( m \) ranges over the \( H_m \) we could allow different kinds of ‘noise’ \( dB_i^j \), having particular forms of quadratic variation which may represent a projection of environmental factors within the scope of a rate distortion manifold (Glazebrook and Wallace, 2009b).

The next step in extending (6.16) is to bring in rate distortion functions for mutual crosstalk between a set of interacting cognitive modules by using the homology of \( R(D) \) itself. To this extent, consider different cognitive processes indexed \( 1, \ldots, s \), and take the mutual rate distortion functions \( R_{ij} \) characterizing communication (and distortion) between them. At the same time the essential parameters remain the characteristic time constants of each process, \( \tau_j \), for \( 1 \leq j \leq s \), together with an overall embedding free energy density \( F \). Taking the \( Q^m \) to run over all the relevant parameters and mutual rate distortion functions (along with the distortion measures \( D_{ij} \)), then (6.13) now takes shape as
\[ S_R^m = R_{ij} - \sum_k Q^k \partial R_{ij} / \partial Q_k, \tag{6.17} \]
and accordingly (6.16) becomes
\[ \sum_{\beta=(ij)} \{ L_\beta(t, \ldots, \partial S_R^\beta / \partial Q^\alpha, \ldots) dt + \sigma_\beta(t, \ldots, \partial S_R^\beta / \partial Q^\alpha, \ldots) dB_t^\beta \}. \] (6.18)

This last equation generalizes the treatment in terms of crosstalk, its distortion, the inherent time constants of the different cognitive modules, and the overall available free energy density.

Example 6.2. For a Gaussian channel and fixed embedded communication free energy density \( F \) representing the richness of incoming information from the interacting cognitive modules, we extend (6.13) to
\[ \frac{dD}{dt} = \frac{\mu}{2D} - \alpha F, \quad \alpha > 0, \] (6.19)
that represents the communication distortion between the modules. The equilibrium solution is \( D_{\text{equil}} = \frac{\mu}{2g(F)} \). The difference between (6.13) and (6.19) is that whereas in the former case, the distortion grows directly as the square root of the elapsed time, equation (6.19) reveals there is a finite, equilibrium, average distortion that is inversely proportional to the available environmental or informational free energy that the interacting systems can implement in order to navigate their actions.

The above situation can be generalized to \( D_{\text{equil}} = \frac{1}{g(F)} \), where \( g(F) \) is monotonically increasing in \( F \). On introducing a characteristic response time variable \( \tau \), so that
\[ \frac{dD}{dt} = \frac{\mu}{2D} - g(F)h(\tau), \] (6.20)
where \( h(\tau) \) is also monotonically increasing, leads to
\[ D_{\text{equil}} = \frac{\mu}{2g(F)h(\tau)}. \] (6.21)
This example reveals that given a fixed rate of available information free energy, the increasing allowable response time decreases average distortion in the interaction.

Example 6.3. Suppose now that feedback is allowed so that the system actively seeks information in proportion to the distortion between intent and impact, then the Onsager relation for a Gaussian channel becomes
\[ \frac{dD}{dt} = \frac{\mu}{2D} - g(F)h(\tau)D, \] (6.22)
and
\[ D_{\text{equil}} = \left( \frac{\mu}{2g(F)h(\tau)} \right)^\frac{1}{2}, \] (6.23)
which is significantly smaller than (6.21), and is effectively the classic result for Brownian motion in a harmonic central field (e.g. equation (54) of Wang and Uhlenbeck, 1945).

6.5 Multidimensional Itô process

Together with the multiphase equilibrium problem of §5.4 we have so far pursued a theme of how the stochastic simulation of biochemical systems closely parallels that of evolutionary–genetic systems. An initial observation here, and one that will motivate further ideas, is that a stochastic differential equation of the type (6.18) should in principle model the dynamics of large, intricate networks that are constrained by the costs of actual computational time, which seems relevant in certain senses to many epigenetic processes. The general setting for such processes often involves that of an Itô stochastic DE (viz Itô process) and this is how we intend to view (6.18) as a kind of ‘master equation’.

For the readers sake, we remark why such a level of mathematical formality is necessary by recalling a basic difficulty: the sample paths \( B_t^\beta \) of Brownian motion are not in general functions of bounded variation, so that \( dB_t^\beta \) is not defined as for that of the usual Riemann-Stieltjes integral. One may start by supposing that for almost all samples, \( Q_t \) in (6.18) is independent of future Brownian motion \( B_u^\beta - B_t^\beta, \ u \geq t \); otherwise said \( Q_t^\beta \) is an adapted stochastic process. Putting it another way, the information available at a given time includes the history of the process at that time. More generally, the stochastic process may be taken to be predictable in order to define the Itô integral of the equation (see e.g. §2.3 of Kunita, 1990 and Appendix § here).

On the other hand, there is part justification for making assumptions of (local) boundedness of variation in the Brownian motion incurred via distortion, and in particular, the adapted condition, points to those examples and observations already quoted that concern the dispensation of available free energy. Firstly, in view of Example 6.1 we may expect a constant
free energy expenditure for maintenance of some fixed average distortion in communication between the interacting cognitive modules. Secondly, given a fixed rate of available information free energy, the increasing allowable response time decreases average distortion in the interaction (Example 6.22), and thirdly, the possibility that finite, equilibrium, average distortion is actually inversely proportional to the available environmental or informational free energy that the interacting systems can utilize (Example 6.23). However, to make matters precise, it is appropriate to consider the formalities of some filtered probability space \((\Omega, \mathcal{F}, P)\), where \(\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}\) (see Appendix 8) and postulate a multidimensional stochastic process given by the Itô integral of the master equation (6.18):

\[
Q_t^\alpha = Q_0^\alpha + \sum_{\beta = \{ij\}} \int_0^t L_\beta(s, \ldots, \partial S^\beta_R/\partial Q^\alpha, \ldots) \, ds + \int_0^t \sigma_\beta(s, \ldots, \partial S^\beta_R/\partial Q^\alpha, \ldots) \, dB^\beta_s. \tag{6.24}
\]

In order to state the conditions for which this process is well-defined, we first express (6.24) in the simplified form

\[
Q_t^\alpha = Q_0^\alpha + A^\alpha(t) + \int_0^t \sigma(s) \, dB^\alpha_s. \tag{6.25}
\]

Then following Harrison (1985, Chapter 4):

1. \(Q_0^\alpha\) is measurable with respect to \(F_0\).
2. \(\sigma_\beta\) is an adapted stochastic process, and \(\sigma(\int_0^t \sigma_\beta(s) \, ds < \infty) = 1\), for all \(t \geq 0\).
3. The integral of ‘drift’, \(A^\alpha(t) = \int_0^t L_\beta \, ds\), is a continuous and adapted variation-finite (VF) process.

Granted that our observations about the local boundedness of the (rate distortion) Brownian motion as essentially fulfilling these conditions, we now have an explicit stochastic model for the role of cross-talk, its distortion, the inherent time constraints of the different cognitive modules, as well as the overall available free energy density, where the \(Q\) parameter structure represents the full-scale fragmentation of the system in the presence of some Weiner noise.

**Remark 6.2.** A further possible generalization of (6.18) is to introduce into that expression a matrix valued function \(V_\beta : \mathbb{R}_+^n \rightarrow \mathbb{R}^{n \times n}\) to describe the intrinsic reaction rates (cf Manninen et al., 2006):

\[
\frac{dQ_t^\alpha}{\partial Q} = \sum_{\beta = \{ij\}} [L_\beta(t, \ldots, \partial S^\beta_R/\partial Q^\alpha, \ldots) \, dt + V_\beta(t, x, \ldots, \partial S^\beta_R/\partial Q^\alpha, \ldots) \, dB^\beta_t].
\]

### 6.6 The stochastic flow

Towards the possibility of a stochastic flow generated by (6.18) (such as a Brownian flow of diffeomorphisms), we opt to simplify (6.18) accordingly. Following Kunita (1990) we write (6.18) in a simplified form as

\[
\frac{dQ_t}{\partial Q} = \sum_{\alpha} dQ_t^\alpha = f_0(Q_t, \partial S_R/\partial Q, t) \, dt + \sum_{k=1}^m f_k(Q_t, \partial S_R/\partial Q, t) \, dB^k_t. \tag{6.26}
\]

Typically, we would seek a solution starting from some \(x\) at some time \(s\); let us call these solutions \(Q_{s,t}\). Then a stochastic flow can be represented as the solution of a stochastic differential equation of the type

\[
Q_{s,t}(x) = x + \int_s^t F(Q_{s,r}(x), \partial S_R/\partial Q, dr), \tag{6.27}
\]

where in this case, Brownian motion \(F(x, \partial S_R/\partial Q, t)\) valued in vector fields, is given by

\[
F(x, \partial S_R/\partial Q, t) = \int_0^t f_0(x, \partial S_R/\partial Q, r) \, dr + \sum_{k=1}^m \int_0^t f_k(x, \partial S_R/\partial Q, r) \, dB^k_r. \tag{6.28}
\]

The formal conditions for (6.27) to produce a Brownian flow of diffeomorphisms are (Kunita, 1990):

1. \(Q_{s,t}(x)\) is continuous in \(s, t, x\).
2. The map \(Q_{s,t} : \mathbb{R}^d \rightarrow \mathbb{R}^d\) (for suitable \(d\)) is a diffeomorphism for any \(s < t\).
3. \(Q_{s,u} = Q_{t,u}(Q_{s,t})\), for any \(s < t < u\).
Note that conditions (1)-(3) are ‘almost everywhere (surely)’ conditions. Also, such a flow generates a holonomy or geometric phase (transition) which can be explained by the process of tracking internal states in relationship to a spatiotemporal orientation. In more precise differential-geometric terms, holonomy results from the parallel transport of vectors around a closed path, thus leading to a representation of the space of the latter into a group of global symmetries. The procedure for constructing a holonomy groupoid associated to this flow concerns some mathematical technicalities, but is nevertheless standard (see e.g. Connes, 1994; Moerdijk and Mrčun, 2003). Whereas this case is fairly well tempered, we would in general expect ‘singularities’ in the flow. The holonomy groupoid can be still be constructed, but this involves deeper mathematics outside of the scope of this paper (see e.g. Debord (2001) for details).

7 Discussion and conclusions

We have given here a descriptive account of cognitive modules as components of epigenetic-evolutionary systems (as far-from-equilibrium open systems) embedded within the context of environment and culture. Using dynamic groupoids of network equivalence classes we have put into atlas form the various constituent (inter)reactive systems based on rate distortion principles of the Shannon theorems and the groupoid free energy density. The rate distortion function \( R(D) \) determines a channel capacity that is measurable in an analogous way to a free energy that regulates many of the (inter)reactive /reciprocating processes that have been described. Rate distortion arguments suggest that if an external information source is pathogenic, then sufficient exposure to it within a developmental stage will likely result in a image inscribed on mind and body in a punctuated fashion, subsequently causing a developmental dysfunction. In this analogy, the reduction of the \( R(D) \) amounts to ‘lower temperature’ which in turn directs the system to behavioral patterns which are less enriched and are less complex. Further, accurate and efficient communicating systems require a greater channel capacity, and keeping in mind the analogy with free energy density, a higher rate of metabolism is necessary and further costs are incurred (cf [34]). Failure to provide such resources equates to a decline in processing, possibly to a point of disintegration. On the other hand, increased communication between the system’s cognitive modules, depending on the availability of free energy, will usually be followed up by a phase transition (in essence, this is what the holonomy groupoid encodes as shown in Glazebrook and Wallace (2009a)) inducing further complexity into the systems behavior. In principle one might envisage an associated holonomy groupoid atlas for interacting cognitive-dynamical systems based on synchronous (geometric) phase transitions of the various constituents. This would amount to a broad-scale descriptive artifact for the purpose of understanding the cumulative transitional mechanisms of living processes that may eventually uncover some even deeper conceptual issues.

Stochastic processes are perhaps more in keeping with what the world expects compared to a strictly deterministic approach, though the former are likely to entail higher computational costs and some neuroscientific work in this area is aimed at reducing such costs (cf Manninen et al., 2006). Integration of the Onsager stochastic differential equation towards a multidimensional Ito process, leads naturally to a stochastic flow which in our formulation diffuses across the atlas through mainly noisy channels (cf [34]). Such evidence is provided by Tlusty’s rate-distortion analysis of the genetic coding map (1988), Baars and Franklin (2003), Cohen and Harel (2007) and Maturana and Varela (1980a, 1980b).

8 Appendix: Probability space and Brownian motion

We state some basic details as to be found in Harrison (1985), Kunita (1990) and Protter (1985). Let \( \Omega \) be a set. A collection \( \mathcal{F} \) of subsets of \( \Omega \) is called a \( \sigma \)-field if it contains an empty set and it is closed under the operations of countable unions and complements. It is customary to call \( (\Omega, \mathcal{F}) \) a measurable space in which members of \( \Omega \) are called samples and those of \( \mathcal{F} \) are called events. Let \( P \) be a \( \sigma \)-additive measure on \((\Omega, \mathcal{F})\). It is called a probability if \( P(\Omega) = 1 \). The triple \((\Omega, \mathcal{F}, P)\) is then called a probability space. Let \( \mathcal{F} = \{ \mathcal{F}_t : t \geq 0 \} \) be a family of \( \sigma \)-algebras on \( \Omega \) such that a) \( \mathcal{F}_t \subseteq \mathcal{F} \), for all \( t \geq 0 \), and b) \( \mathcal{F}_s \subseteq \mathcal{F}_t \), if \( s \leq t \). Then \( \mathcal{F} \) is said to be a filtration (an increasing sequence of sub-\( \sigma \)-algebras) of \((\Omega, \mathcal{F})\). The filtration \( \mathcal{F} \) characterizes how information arises (how uncertainty is resolved) and \( \mathcal{F}_t \) may be interpreted as the set of all events whose occurrence or nonoccurrence will be determined at time \( t \). In a filtered probability space it is usually understood that \( \mathcal{F} = \mathcal{F}_\infty \).

A stochastic process \( W = \{ W_t \}, \ t \in \mathbb{T} \), is said to be adapted (relative to \((\Omega, \mathcal{F}, P)\)) if \( W_t \) is measurable with respect to \( \mathcal{F}_t \), for all \( t \geq 0 \). Loosely speaking, this means that the information available at time \( t \) includes the history of \( W \) up to that point. The predictable \( \sigma \)-field is the least \( \sigma \)-field in the product space \([0, T] \times \Omega \) for which all continuous \( \mathcal{F}_t \)-adapted processes are measurable. A predictable process is then defined as a process that is measurable with respect to the predictable \( \sigma \)-field.
An example is a continuous $\mathcal{F}_t$-adapted process. Recall that a process of random variables $W = \{W_t\}, t \in \mathbb{T}$ is Brownian motion (viz a Weiner process) if and only if

1. $W_0 = 0$ with probability 1.
2. For $0 \leq s < t < \infty$ the increment $W_t - W_s$ is normally distributed $N(0, |t - s|)$.
3. For $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, the set of increments
   $$\{W_{t_0}, W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}, \text{for } 1 \leq j \leq k\},$$
   is a set of independent random variables (that is, the increments are independent of the past).

A process $Q$ is called a $(\mu, \sigma)$-Brownian motion if it has the form $Q_t = Q_0 + \mu t + \sigma W_t$ where $W$ is a Weiner process and $Q_0$ is independent of $W$. Then we have $Q_{t+s} - Q_t \sim N(\mu s, \sigma^2 s)$.

9 Appendix: Basic results of information theory

9.1 The Shannon uncertainties

Invoking the spirit of the Shannon-McMillan Theorem, it is possible to define an APSE information source $X$ associated with stochastic variates $X_j$, having joint and conditional probabilities $P(a_0, \ldots, a_n)$ and $P(a_n|a_0, \ldots, a_{n-1})$ such that appropriate joint and conditional Shannon uncertainties satisfy the classic relations

$$H[X] = \lim_{n \to \infty} \frac{\log[N(n)]}{n} = \lim_{n \to \infty} \frac{H(X_0, \ldots, X_{n-1})}{n} = \lim_{n \to \infty} \frac{H(X_0, \ldots, X_n)}{n}.$$  \hspace{1cm} (9.1)

This information source is defined as dual to the underlying ergodic cognitive process (Wallace, 2005).

Recall that the Shannon uncertainties $H(\ldots)$ are cross-sectional law-of-large-numbers sums of the form $-\sum_k P_k \log[P_k]$, where the $P_k$ constitute a probability distribution (for the basic details, see Ash, 1990; Berger, 1971; Cover and Thomas, 1991; Khinchin, 1957). Messages from an information source, seen as symbols $x_j$ from some alphabet, each having probabilities $P_j$ associated with a random variable $X$, are ‘encoded’ into the language of a ‘transmission channel’, a random variable $Y$ with symbols $y_k$, having probabilities $P_k$, possibly with error. Someone receiving the symbol $y_k$ then retranslates it (without error) into some $x_j$, which may or may not be the same as the $x_j$ that was sent. More formally, the message sent along the channel is characterized by a random variable $X$ having the distribution

$$P(X = x_j) = P_j, j = 1, \ldots, M.$$  \hspace{1cm} (9.2)

The channel through which the message is sent is characterized by a second random variable $Y$ having the distribution

$$P(Y = y_k) = P_k, k = 1, \ldots, L.$$  \hspace{1cm} (9.3)

Let the joint probability distribution of $X$ and $Y$ be defined as

$$P(X = x_j, Y = y_k) = P(x_j, y_k) = P_{jk},$$  \hspace{1cm} (9.4)

and the conditional probability of $Y$ given $X$ as

$$P(Y = y_k | X = x_j) = P(y_k | x_j).$$  \hspace{1cm} (9.5)

Then the Shannon uncertainty of $X$ and $Y$ independently and the joint uncertainty of $X$ and $Y$ together are defined respectively as

$$H(X) = - \sum_{j=1}^M P_j \log(P_j),$$

$$H(Y) = - \sum_{k=1}^L P_k \log(P_k),$$

$$H(X, Y) = - \sum_{j=1}^M \sum_{k=1}^L P_{jk} \log(P_{jk}).$$  \hspace{1cm} (9.6)
The conditional uncertainty of $Y$ given $X$ is defined as

$$H(Y|X) = - \sum_{j=1}^{M} \sum_{k=1}^{L} P_{jk} \log[P(y_k|x_j)].$$  \hspace{1cm} (9.7)

For any two stochastic variates $X$ and $Y$, we have the inequality $H(Y) \geq H(Y|X)$, as the knowledge of $X$ generally gives some knowledge of $Y$. Equality occurs only in the case of stochastic independence. Since $P(x_j, y_k) = P(x_j)P(y_k|x_j)$, it is deduced

$$H(X|Y) = H(X, Y) - H(Y).$$  \hspace{1cm} (9.8)

The information transmitted by translating the variable $X$ into the channel transmission variable $Y$ – possibly with error – and then retranslating without error the transmitted $Y$ back into $X$, is defined as

$$I(X|Y) = H(X) - H(X|Y)$$

$$= H(X) + H(Y) - H(X, Y),$$  \hspace{1cm} (9.9)

where we refer to Berger (1971), Cover and Thomas (1991) and Khinchin (1957) for details. The essential point is that if there is no uncertainty in $X$ given the channel $Y$, then there is no loss of information through transmission. In general this will not be true, and herein lies the essence of the theory.

### 9.2 The Rate Distortion Theorem

Following Wallace (2005), suppose we have an (ergodic) information source $Y$ with output from a particular alphabet generating sequences of the form

$$y^n = y_1, \ldots, y_n$$  \hspace{1cm} (9.10)

‘digitalized’ in some sense, and induce a chain of ‘digitalized’ values

$$b^n = b_1, \ldots, b_n$$  \hspace{1cm} (9.11)

where the $b$-alphabet is considered more restricted than the $y$-alphabet. In this way, $b^n$ is deterministically retranslated into a reproduction of the signal $y^n$. That is, each $b^n$ is mapped onto a unique $n$-length $y$-sequence in the alphabet of $Y$:

$$b^n \rightarrow \hat{y}^n = \hat{y}_1, \ldots, \hat{y}_n.$$  \hspace{1cm} (9.12)

We remark that many $y^n$ sequences may be mapped onto the same retranslation sequence $\hat{y}^n$, the set of which is denoted $\hat{Y}$; this may be interpreted as a loss of information.

A distortion measure $d: Y \times \hat{Y} \rightarrow \mathbb{R}^+$, between paths $y^n$ and $\hat{y}^n$ is defined as

$$d(y^n, \hat{y}^n) = \frac{1}{n} \sum_{i=1}^{n} d(y_i, \hat{y}_i),$$  \hspace{1cm} (9.13)

for some suitable distance function $d$ (such as the Hamming distance). Suppose that with each path $y^n \in Y$ and each $b^n$-path retranslation $\hat{y}^n \in \hat{Y}$ into the $y$-language, we consider the associated individual, joint, and conditional probability distributions

$$p(y^n), \ p(\hat{y}^n), \ p(y^n|\hat{y}^n).$$  \hspace{1cm} (9.14)

The average distortion is then defined to be

$$D = \sum_{y^n} p(y^n) \ d(y^n, \hat{y}^n).$$  \hspace{1cm} (9.15)

For the corresponding strings $Y$ (incoming), $\hat{Y}$ (outgoing), applying the Shannon uncertainty rule of (9.9) gives

$$I(Y, \hat{Y}) \equiv H(Y) - H(Y|\hat{Y})$$

$$= H(Y) + H(\hat{Y}) - H(Y, \hat{Y}).$$  \hspace{1cm} (9.16)

The information rate distortion function $R(D)$ for a source sequence $Y$, retranslated sequence $\hat{Y}$ with distortion measure $d: Y \times \hat{Y} \rightarrow \mathbb{R}^+$, is defined as follows. Let $Y = \sum_{(y, \hat{y})} p(y) \ p(y|\hat{y}) \ d(y, \hat{y})$. Then

$$R(D) = \sum_{p(y, \hat{y}) : Y \leq D} I(Y, \hat{Y}).$$  \hspace{1cm} (9.17)
To explain this notation, the minimization is over all conditional distributions $p(y|\hat{y})$, for which the joint distribution $p(y, \hat{y}) = p(y) p(y|\hat{y})$ satisfies average distortion less than or equal to $D$. The Rate Distortion Theorem (see e.g. Berger, 1971; Cover and Thomas, 1991) states that $R(D)$ is the minimum necessary rate of information transmission (effectively the channel capacity) so that the average distortion does not exceed the distortion $D$.

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