THE COHOMOLOGY OF \((S(n, k))\) RELEVANT TO MORAVA STABILIZER ALGEBRA

LIMAN CHEN, XIANGJUN WANG AND XUEZHI ZHAO

Abstract. In this paper we redefine a increasing filtration on the the Hopf algebra \(S(n, k)\), From which we get a spectral sequence called May spectral sequence. As an application we computed \(H^\ast S(n, n)\) at prime 2, \(H^\ast S(3, 2)\) at prime 3 and \(H^\ast S(4, 2)\) at prime \(p \geq 5\).

1. Introduction

In stable homotopy theory, the “chromatic” point of view plays an important role (cf. \([3, 11, 14]\)). Fix a prime \(p\). Let \(E(n)_\ast\), \(n \geq 0\) be the Johnson-Wilson homology theories and let \(L_n\) be localization functor with respect to \(E(n)_\ast\). Then there are natural transformations \(L_nX \longrightarrow L_{n-1}X\), and the chromatic tower

\[
\cdots \longrightarrow L_nX \longrightarrow L_{n-1}X \longrightarrow \cdots \longrightarrow L_2X \longrightarrow L_1X \longrightarrow L_0X.
\]

By the Hopkins-Ravenel chromatic convergence theorem, the homotopy inverse limit of this tower is the \(p\)-localization of \(X\).

\[X \longrightarrow \text{Holim}L_nX.\]

Thus the homotopy groups \(\pi_\ast(L_nX)\) is the part of homotopy groups \(\pi_\ast(X)\) one could see from \(E(n)_\ast\).

To determine the homotopy groups \(\pi_\ast(L_nX)\), one has the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum \(BP\), whose \(E_2\)-term is

\[E_2^{s,t} = \text{Ext}^{s,t}_{BP,BP}(BP_\ast,BP_\ast(L_nX)).\]

(cf. \([1, 10, 11, 14]\))

To determine the Adams-Novikov \(E_2\)-term \(\text{Ext}^{s,t}_{BP,BP}(BP_\ast,BP_\ast(L_nX))\) one has the Bockstein spectral sequence. This is an argument based on the cohomology of the Morava stabilizer algebra \(S(n)\) at each prime \(p\) (cf. \([11, 16, 17, 19]\)). Here the Hopf algebra \(S(n)\) is defined as

\[S(n) = \mathbb{Z}/p \otimes_{K(n)_\ast} K(n)_\ast, K(n)_\ast \otimes_{K(n)_\ast} \mathbb{Z}/p,\]

where \(K(n)_\ast = \mathbb{Z}/p[v_n, v_n^{-1}]\),

\[K(n)_\ast, K(n) = K(n)_\ast \otimes_{BP} BP, BP \otimes_{BP} K(n)_\ast = K(n)_\ast[t_1, t_2, \ldots]/(v_n t_n^p - v_n^p t_n),\]

\(K(n)_\ast\) acts on \(\mathbb{Z}/p\) by sending \(v_n\) to 1. Thus

\[S(n) = \mathbb{Z}/p[t_1, t_2, \ldots, t_s]/(t_n^p - t_n).\]
We write \( S(n,k) = S(n)/(t_j : j < k) = \mathbb{Z}/p[t_k, t_{k+1}, \ldots, t_s, \ldots]/(t^n - t_s) \). The Hopf algebra structure of \( S(n) \) determines that of \( S(n,k) \), while \( S(1) = S(n) \). Let \( V(n-1) \) and \( T(k-1) \) denote the Smith-Toda spectra and the Ravenel spectra respectively characterized by

\[
BP_*V(n-1) = BP_*/I_n = BP_*(p, v_1, \ldots, v_{n-1}) \quad \text{and} \quad BP_*T(k-1) = BP_*[t_1, t_2, \ldots, t_{k-1}].
\]

If \( L_nV(n-1) \wedge T(k-1) \) exist, (although \( V(n-1) \) does not exist (cf. \([10]\)), but \( V(n-1) \wedge T(k-1) \) might exist), then by the change of rings theorem, the \( E_2 \)-term of the Adams-Novikov spectral sequence converging to \( \pi_*(L_n V(n-1) \wedge T(k-1)) \) is

\[
\Ext^{*,*}_{BP_*, BP_*}(BP_*, BP_*(L_n V(n-1) \wedge T(k-1))) \\
\cong \Ext^{*,*}_{S(n,k)}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes K(n)_*[v_{n+1}, \ldots, v_{n+k-1}].
\]

In this paper, we will use \( H^{*,*}(S(n,k)) \) to denote the \( \Ext \) groups \( \Ext^{*,*}_{S(n,k)}(\mathbb{Z}/p, \mathbb{Z}/p) \).

In \([5, 13]\), Ravenel and Henn determined \( H^{*,*}(S(1)), H^{*,*}(S(2)) \) at all primes, and \( H^{*,*}(S(3)) \) at the odd primes \( p \geq 5 \). \( H^{*,*}(S(n,k)) \) is known from \([11]\) for \( k \geq n \) at odd primes and \( k > n \) at the prime 2. In \([15]\) Shimomura and Tokashiki computed \( H^{*,*}(S(n), n-1) \) at odd primes \( p > 3 \). In this paper we will be concentrated on the case \( k \leq n \).

Consider the cohomology of the Hopf algebra \( S(n,k) \) at all primes. In section 2 of this paper, we follow Ravenel’s ideal (cf. \([11]\) 3.2.5 Theorem), redefined the May filtration in \( S(n,k) \) and its cobar complex \( C^{*,*}(S(n,k)) \). This filtration induces a spectral sequence so called May spectral sequence \( E^{*,*,M}_{r}(n,k), d_r \) that converges to \( H^{*,*}(S(n,k)) \). Then in section 3 we prove that the \( E_2 \)-term of the May spectral sequence is isomorphic to the cohomology of

\[
\{E|h_{i,j}|k \leq i \leq s_0, j \in \mathbb{Z}/n| \otimes P|b_{i,j}|k \leq i \leq s_0 - n, j \in \mathbb{Z}/n|, d_1\}
\]

where \( s_0 = \max \left\{ \left\lfloor \frac{2pm + p - 2}{2(p - 1)} \right\rfloor, n + k - 1 \right\} \) and \( \left\lfloor \frac{2pm + p - 2}{2(p - 1)} \right\rfloor \) is the integer part of \( \frac{2pm + p - 2}{2(p - 1)} \). In particular, if

\[
n + k - 1 \geq \left\lfloor \frac{2pm + p - 2}{2(p - 1)} \right\rfloor,
\]

the May’s \( E_2 \)-term becomes the cohomology of

\[
\{E|h_{i,j}|k \leq i \leq n + k - 1, j \in \mathbb{Z}/n|, d_1\}.
\]

The homological dimension of each element is given by

\[
s(h_{i,j}) = 1, \quad s(b_{i,j}) = 2.
\]

For the May differentials, one has \( d_r : E^{*,*,M}_{r}(S(n,k)) \rightarrow E^{*,*,M-1}_{r}(S(n,k)) \) and if \( x \in E^{*,*,S}_{r}(S(n,k)) \) then

\[
d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).
\]

The first May differential \( d_1 \) is given by

\[
d_1(h_{i,j}) = - \sum_{k \leq m \leq i-k} h_{m,j} h_{i-m,j+k}, \quad \text{and} \quad d_1(b_{i,j}) = 0.
\]

We analyze the higher May differentials and give a collapse theorem in section 4. As an consequence we compute the cohomology of \( S(n,n) \) at the prime 2, \( S(3,2) \) at the prime 3 and \( S(4,2) \) at the prime \( p \geq 5 \) in section 5.
2. The May spectral sequence

Let \( p \) be a prime, \( BP_s = \mathbb{Z}_p[v_1, v_2, \cdots] \) and \( BP_sBP = BP_t[t_1, t_2, \cdots] \). For the Hazewinkel’s generators described inductively by \( v_s = pm_s - \sum_{i=1}^{k-1} v_i p^i m_i \) (cf. [4] and [10]) 1.2), the diagonal map \( \Delta : BP_sBP \to BP_sBP \otimes BP_sBP \) is given by
\[
\sum_{i+j=s} m_i(\Delta t_j) = \sum_{i+j+k=s} m_i t_j^i \otimes t_k^{i+j}.
\]
One can easily prove that
\[
\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1
\]
\[
\Delta(t_2) = \sum_{i+j=2} t_i \otimes t_j^i - v_1 b_{1,0},
\]
where \( p \cdot b_{1,0} = \Delta(t_1^p) - t_1^p \otimes 1 - 1 \otimes t_1^p \). Inductively define
\[
p \cdot b_{s,k-1} = \Delta(t_s^p) - \sum_{i+j=s} t_i^p \otimes t_j^{i+p} + \sum_{0<i<s} v_i^{p^i} b_{s-i,k+i-1},
\]
one has
\[
\Delta(t_{s+1}) = \sum_{i+j=s+1} t_i \otimes t_j^i - \sum_{0<i<s+1} v_i b_{s+1-i,j-1}.
\]
Thus for the \( n \)-th Morava K-theory \( K(n) = \mathbb{Z}/p[v_n, v_n^{-1}] \), the Hopf algebra
\[
K(n)_* K(n) = K(n)_* \otimes_{BP_sBP} BP_sBP \otimes_{BP_sBP} K(n)_*
\]
is isomorphic to
\[
K(n)_* K(n) = \mathbb{Z}/p[t_1, t_2, \cdots, t_s, \cdots]/(v_n t_s^{p^n} - v_n^{p^s} t_s).
\]
And \( S(n) = \mathbb{Z}/p \otimes K(n)_* K(n) \otimes K(n)_* \), \( \mathbb{Z}/p \) is isomorphic to
\[
S(n) = \mathbb{Z}/p[t_1, t_2, \cdots, t_n, t_{n+1}, \cdots]/(t_s^{p^n} - t_s).
\]
The inner degree of \( t_s \) in \( S(n) \) is
\[
|t_s| \equiv 2(p-1)(1 + p + \cdots + p^{s-1}) \quad \text{mod } 2(p-1)(1 + p + \cdots + p^{n-1})
\]
because \( v_n \) is sent to 1. The structure map \( \Delta : S(n) \to S(n) \otimes S(n) \) acts on \( t_s \) as follows
\[
\Delta(t_s) = \sum_{0 \leq i \leq s} t_i \otimes t^{p^i}_{s-i} \quad \text{for } s \leq n
\]
\[
\Delta(t_s) = \sum_{0 \leq i \leq s} t_i \otimes t^{p^i}_{s-i} - b_{s-n,n-1} \quad \text{for } s > n
\]
where \( b_{i,j} = \sum_{0 < m < p} (p \cdot t^{m p^i}_{i} \otimes t^{p-m p^j}_{j} \) at odd primes and \( b_{i,j} = t_{i}^{2j} \otimes t_{i}^{j} \) at the prime 2. For the integer \( k \geq 1 \), let \( S(n,k) = S(n)/(t_s | s < k) \). We have
\[
S(n,k) = \mathbb{Z}/p[t_k, t_{k+1}, \cdots, t_{n+k}, t_{n+k+1}, \cdots]/(t_s^{p^n} - t_s),
\]
the structure map $\Delta : S(n,k) \to S(n,k) \otimes S(n,k)$ acts on $t_s$ as

$$\Delta(t_s) = 1 \otimes t_s + \sum_{k \leq i \leq s-k} t_i \otimes t^{p^k}_{s-k} + t_s \otimes 1 \quad \text{for } s \leq n + k - 1,$$

(2.2) \hspace{1cm} \Delta(t_s) = 1 \otimes t_s + \sum_{k \leq i \leq s-k} t_i \otimes t^{p^k}_{s-k} + t_s \otimes 1 - b_{s-n,n-1} \quad \text{for } s \geq n + k.

In the resulting May spectral sequence, we want to have the 0-th May differential is

$$d_0(t^{p^n}_s) = 0$$

and the first May differential is given by

$$d_1(t_s) = t_k \otimes t^{p^k}_{s-k} + t_{k+1} \otimes t^{p^{k+1}}_{s-k-1} + \cdots + t_{s-k} \otimes t^{p^{s-k}}_1$$

for $s \leq n + k - 1$, and for $s \geq n + k$

$$d_1(t_s) = \begin{cases} t_k \otimes t^{p^k}_{s-k} + \cdots + t_{s-k} \otimes t^{p^{s-k}}_1 & \text{if the May filtration } M(t_k \otimes t^{p^k}_{s-k}) > M(b_{s-n,n-1}), \\ -b_{s-n,n-1} & \text{if the May filtration } M(t_k \otimes t^{p^k}_{s-k}) \leq M(b_{s-n,n-1}). \end{cases}$$

So we define the May filtration on $S(n,k)$ as:

**Definition 2.3** In the Hopf algebra $S(n,k)$, we define May filtration $M$ as follows:

1. For $k \leq s \leq n + k - 1$, set the May filtration of $t^p_s$ as $M(t^p_s) = 2s - 1$.
2. For $n + k \leq s$, inductively set the May filtration of $t^p_s$ as

$$M(t^p_s) = \max\{2s - 1, \ pM(t^p_{s-n}) + 1\},$$

3. For the monomial $t^{j_1}_s \cdot t^{j_2}_{s_2} \cdots t^{j_m}_{s_m}$, where $s_i \neq s_j$ define its May filtration as

$$M(t^{j_1}_s \cdot t^{j_2}_{s_2} \cdots t^{j_m}_{s_m}) = \sum_{1 \leq i \leq m} M(t^{j_i}_s).$$

**Lemma 2.4** Let $s_0 = \max \left\{ \left\lfloor \frac{2pn + p - 2}{2(p-1)} \right\rfloor, n + k - 1 \right\}$ where $\left\lfloor \frac{2pn + p - 2}{2(p-1)} \right\rfloor$ is the integer part of $\frac{2pn + p - 2}{2(p-1)}$. Then the May filtration of $t^p_s$ satisfies

1. $M(t^p_s) > M(t^p_{s-1}) + 1$ and
2. For $s \leq s_0$, the May filtration $M(t^p_s) = 2s - 1$.
3. For $s > s_0$, $pM(t^p_{s-n}) + 1 \geq 2s - 1$ and the May filtration $M(t^p_s) = pM(t^p_{s-n}) + 1$.

**Proof.**

1. If $s_0 = \max \left\{ \left\lfloor \frac{2pn + p - 2}{2(p-1)} \right\rfloor, n + k - 1 \right\} = n + k - 1$. From its definition, we see that for $s \leq n + k - 1 = s_0$, the May filtration of $t^p_s$ is $2s - 1$ and $M(t^p_s) > M(t^p_{s-1}) + 1$.

From $n + k - 1 \geq \left\lfloor \frac{2pn + p - 2}{2(p-1)} \right\rfloor$, one sees that

$$s_0 + 1 = n + k > \frac{2pn + p - 2}{2(p-1)} \quad \text{and} \quad p(2k - 1) + 1 > 2(n + k) - 1.$$
Thus from \(M(t^p_{n+k}) = 2k - 1\), one knows that the May filtration of \(t^p_{n+k}\) is \(pM(t^p_k) + 1\) and
\[
M(t^p_{n+k}) = p(2k - 1) + 1 > 2(n + k) - 1 = M(t^p_{n+k-1}) + 1.
\]
Inductively suppose that \(M(t^p_s) > M(t^p_{s-1}) + 1\) and for \(s_0 < s \leq m\),
\[
pM(t^p_{s-n}) + 1 > 2s - 1,
\]
so the May filtration \(M(t^p_s) = pM(t^p_{s-n}) + 1\). Then from \(M(t^p_{m+1-n}) > M(t^p_{m-n}) + 1\) one get
\[
pM(t^p_{m+1-n}) + 1 > p\left(M(t^p_{m-n}) + 1\right) + 1 = pM(t^p_{m-n}) + p + 1
\]
\[
> 2m - 1 + p \geq 2(m + 1) - 1.
\]
The May filtration of \(t^p_{m+1}\) is \(pM(t^p_{m+1-n}) + 1\).

If \(s_0 = \left[\frac{2pm + p - 2}{2(p-1)}\right] > n + k - 1\), then for \(k \leq s \leq s_0\), \(s \leq \frac{2pm + p - 2}{2(p-1)}\). This implies
\[
p(2(s-n) - 1) + 1 \leq 2s - 1.
\]
From \(\frac{2pm + p - 2}{2(p-1)} \leq 2n\) we see that \(s - n \leq n \leq n + k - 1\). Thus the May filtration
\[
M(t^p_s) = 2(s - n) - 1 \quad \text{and} \quad M(t^p_s) + 1 < 2s - 1.
\]
This implies that the May filtration of \(t^p_s\) is \(2s - 1\) and \(M(t^p_s) > M(t^p_{s-1}) + 1\).

Notice that \(s_0 + 1 = \left[\frac{2pm + p - 2}{2(p-1)}\right] + 1 > \frac{2pm + p - 2}{2(p-1)}\), this implies
\[
p(2(s_0 + 1 - n) - 1) + 1 > 2(s_0 + 1 - 1).
\]
The May filtration of \(t^p_{s_0+1-n}\) is \(2(s_0 + 1 - n) - 1\), so the May filtration
\[
M(t^p_{s_0+1}) = pM(t^p_{s_0+1-n}) + 1.
\]
Similarly, by induction we get the Lemma. \(\square\)

**Example:** The May filtration in \(S(4, 2)\) is given by:

\[
\begin{array}{cccccccccc}
  t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 & t_9 & t_{10} & \ldots \\
  p = 2 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 19 & 23 & \ldots \\
  p = 3 & 3 & 5 & 7 & 9 & 11 & 16 & 22 & 28 & 34 & \ldots \\
  p \geq 5 & 3 & 5 & 7 & 9 & 3p+1 & 5p+1 & 7p+1 & 9p+1 & p(3p+1)+1 & \ldots \\
\end{array}
\]

Let \(F^{*,M}(n, k)\) be the sub-module of \(S(n, k)\) generated by the elements with May filtration \(\leq M\). Set \(E^{*,M}(n, k) = F^{*,M}(n, k)/F^{*,M-1}(n, k)\). One can see from Lemma 2.4 that
\[
(2.5) \quad E^{*,*}(n, k) \cong \bigotimes_{k \leq s} T[t^p_s | j \in Z/n]
\]
is a bigraded Hopf algebra, where \(T[\ ]\) denote the truncated polynomial algebra of height \(p\) on the indicated generators. The structure map
\[
\Delta : E^{*,*}(n, k) \rightarrow E^{*,*}(n, k) \otimes E^{*,*}(n, k)
\]
acts the the generators \(t^p_s\) as \(\Delta(t^p_s) = 1 \otimes t^p_s + t^p_s \otimes 1\).
Let $C^{s,t}(n,k) = \bigotimes \mathcal{S}(n,k)$ denote the cobar construction of $S(n,k)$ where $\mathcal{S}(n,k) = Ker \epsilon$ denote the augmentation ideal of $S(n,k)$. The differential

$$d : C^{s,t}(n,k) \rightarrow C^{s+1,t}(n,k)$$

is given on the generators as

$$d(\alpha_1 \otimes \cdots \otimes \alpha_s) = \sum_{1 \leq i \leq s} (-1)^i \alpha_1 \otimes \cdots \otimes (\Delta(\alpha_i) - \alpha_i \otimes 1 - 1 \otimes \alpha_i) \otimes \cdots \otimes \alpha_s.$$

(2.6)

In general, the generator $\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_s$ of $C^{s,t}(n,k)$ is denoted by $[\alpha_1|\alpha_2|\cdots|\alpha_s]$. For the generator $[\alpha_1|\alpha_2|\cdots|\alpha_s]$, define its May filtration as

$$M([\alpha_1|\alpha_2|\cdots|\alpha_s]) = M(\alpha_1) + M(\alpha_2) + \cdots + M(\alpha_s).$$

Let $FC^{s,t,M}(n,k)$ denote the sub-complex of $C^{s,t}(n,k)$ generated by the elements with May filtration $\leq M$. Then we get a short exact sequence

$$0 \rightarrow FC^{s,t,M-1}(n,k) \rightarrow FC^{s,t,M}(n,k) \rightarrow E^{s,t,M}_0(n,k) \rightarrow 0$$

of cochain complexes. The cochain complex

$$E^{s,t,M}_0(n,k) = FC^{s,t,M}(n,k)/FC^{s,t,M-1}(n,k)$$

is isomorphic to the cobar complex of $E^{*,*}(n,k)$ given in (2.5). Let $E^{s,t,M}_1(n,k)$ be the homology of $(E^{s,t,M}_0(n,k), d_0)$. Then (2.7) gives rise to a spectral sequence (so called the May spectral sequence)

$$\{E^{s,t,M}_r(n,k), d_r\}$$

that converges to

$$H^{s,t}(C^{s,t}(n,k), d) = Ext^{s,t}_{S(n,k)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

**Theorem 2.8** For $k \leq n$ the Hopf algebra $S(n,k)$ can be given an increasing filtration as in Definition (2.3). The associated bigraded Hopf algebra $E^{s,t,M}(n,k)$ is primitively generated with the algebra structure of (2.5). In the associated spectral sequence, the $E_1$-term $E^{s,t,M}_1(n,k)$ is isomorphic to

$$E[h_{i,j} | k \leq i, j \in \mathbb{Z}/n] \otimes P[h_{i,j} | k \leq i, j \in \mathbb{Z}/n].$$

The homological dimension of each element is given by $s(h_{i,j}) = 1, s(b_{i,j}) = 2$ and the degree is given by

$$h_{i,j} \in E^{1,2(p'-1)p',*}_{1}(n,k),$$

$$b_{i,j} \in E^{2,2(p'-1)p'+1,*}_{1}(n,k)$$

where $h_{i,j}$ corresponds to $\ell^p_i$ and $b_{i,j}$ corresponds to $\sum (\begin{pmatrix} p \end{pmatrix}/m_i^{mp'_i} \otimes \ell_i^{(p-m)p'})$. One has

$$d_r : E^{s,t,M}_r(n,k) \rightarrow E^{s+1,t,M-r}_{r}(n,k)$$

and if $x \in E^{s,t,*}_r(n,k)$ then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

In the $E_1$-term of this spectral sequence, we have the following relations:

$$h_{i,j} \cdot h_{i,j} = -h_{i,j} \cdot h_{i,j}, \quad h_{i,j} \cdot b_{i,j} = b_{i,j} \cdot h_{i,j}, \quad b_{i,j} \cdot b_{i,j} = b_{i,j} \cdot b_{i,j}.$$
Proof. It is a routine calculation in homology algebra that for the truncated polynomial algebra $\Gamma = T[x]$ with $|x| \equiv 0 \mod 2$ and $x$ primitive,

$$Ext^{1}(Z/p, Z/p) = E[h] \otimes P[b]$$

where $h \in Ext^{1}$ is represented in the cobar complex by $x$ and $b \in Ext^{2}$ is represented by $\sum \binom{n}{m}/p(x^{m} \otimes x^{p-m})$ ($b = h^{2}$ represented by $x \otimes x$ at the prime 2). Notice that the $E_{0}$-term of the spectral sequence is isomorphic to the cobar complex of $E^{*,M}(n,k)$. Theorem 4.3.22, in the cobar complex of $BP/I_{n}$ Theorem 3.3 and Lemma 3.4 and 3.8).

Thus (3.1) and (3.2) are given by

$$H^{*,M}(E_{0}^{*,M},(n,k),d_{0}) = Ext^{*,M}_{T[x^{p}]}(Z/p,Z/p) = \bigotimes_{k \leq s} Ext^{*,*}_{T[x^{p}]}(Z/p,Z/p)$$

Thus the May’s $E_{1}$-term

$$E_{1}^{*,M}S(n,k) = E[h_{i,j}|i \leq s, j \leq Z/n] \otimes P[b_{i,j}|k \leq i,j \leq Z/n].$$

Notice that $d_{0}(t_{i}^{p^{j}} \cdot t_{1}^{p^{j}}) = -t_{i}^{p^{j}} \otimes t_{1}^{p^{j}} - t_{1}^{p^{j}} \otimes t_{i}^{p^{j}}$, we get $h_{i,j}h_{i_{1},j_{1}} = h_{i_{1},j_{1}}h_{i,j}$. In a similar way, one can prove that $h_{i,j} \cdot b_{i_{1},j_{1}} = b_{i_{1},j_{1}} \cdot h_{i,j}$ and $b_{i,j} \cdot b_{i_{1},j_{1}} = b_{i_{1},j_{1}} \cdot b_{i,j}$ (cf. [6] Lemma 3.4 and 3.8).

3. The first May differentials

Now suppose $k \leq n$, then $s_{0} \leq 2n$. From (2.2) and Lemma 2.4 one has

$$d_{1}(h_{i,j}) = - \sum_{k \leq r \leq i} h_{r,j}h_{i-r,j+r} \quad \text{for } i \leq s_{0}$$

$$d_{1}(h_{i,j}) = b_{i-n,j+n-1} \quad \text{for } s_{0} < i.$$

Thus for $i > s_{0} - n$, $b_{i,j}$ is the boundary of the first May differentials. Recall from [10] Theorem 4.3.22, in the cobar complex of $BP/I_{n}$ one has

$$d_{1}(b_{i,j}) = \sum_{0 < r < i} \left(b_{r,j} \otimes t_{i-r}^{p^{j+1}} - t_{r}^{p^{j+1}} \otimes b_{i-r,j+r}ight)$$

Thus for $i \leq s_{0} - n$, the first May differential $d_{1}(b_{i,j}) = 0$. This implies:

**Theorem 3.3** Let $k \leq n$ and $s_{0}$ be given in Lemma 2.4. The $E_{2}$-term of the May spectral sequence is isomorphic to the cohomology of

$$E_{1}^{*,*}S(n,k) = E[h_{i,j}|k \leq i \leq s_{0}, j \leq Z/n] \otimes P[b_{i,j}|k \leq i \leq s_{0} - n, j \leq Z/n].$$

The first May differential are given by

$$d_{1}(h_{i,j}) = - \sum_{k \leq r \leq i} h_{r,j}h_{i-r,j+r} \quad \text{for } i \leq s_{0}$$

$$d_{1}(b_{i,j}) = 0 \quad \text{for } k \leq i \leq s_{0} - n.$$

At the prime $p = 2$, $s_{0} = 2n$. The reduced May $E_{1}$-term becomes

$$E_{1}^{*,*}S(n,k) = E[h_{i,j}|n < i \leq 2n, j \leq Z/n] \otimes P[h_{i,j}|k \leq i \leq n, j \leq Z/n]$$

and the first May differential of $h_{2n,j}$ is given by

$$d_{1}(h_{2n,j}) = - \sum_{k \leq i \leq 2n-k} h_{i,j}h_{2n-i,j+i} + h_{n,j+n-1}^{2}$$
Proof. We define a filtration in the May’s $E_1$-term
\[ E_1^{*,*}(n, k) = E[h_{i,j} | k \leq i, j \in \mathbb{Z}/n] \otimes P[b_{i,j} | k \leq i, j \in \mathbb{Z}/n] \]
as follows: for each $s \geq k$, define
\[
F^s(n, k) = \begin{cases} 
E[h_{i,j} | k \leq i \leq s] & \text{for } k \leq s \leq n + k - 1 \\
E[h_{i,j} | k \leq i \leq s] \otimes P[b_{i,j} | k \leq i \leq s - n] & \text{for } n + k - 1 < s.
\end{cases}
\]
From (3.1), we see that for each $s \geq k$, $F^s(n, k)$ is a sub-complex of $E_1^{*,*}(n, k)$ that satisfies
\[ F^{s_0}(n, k) = \tilde{E}_1^{*,*}(n, k) = E[h_{i,j} | k \leq i \leq s_0, j \in \mathbb{Z}/n] \otimes P[b_{i,j} | k \leq i \leq s_0 - n, j \in \mathbb{Z}/n] \]
and
\[ F^k(n, k) \hookrightarrow F^{k+1}(n, k) \hookrightarrow \cdots \hookrightarrow F^s(n, k) \hookrightarrow F^{s+1}(n, k) \hookrightarrow \cdots \hookrightarrow E_1^{*,*}(n, k). \]
Indeed, for $s > n + k - 1$,
\[ F^s(n, k) = F^{s-1}(n, k) \bigotimes (E[h_{s,j} | j \in \mathbb{Z}/n] \otimes P[b_{s-n,j} | j \in \mathbb{Z}/n]). \]
For $s > s_0$ one has $d_1(h_{s,j}) = b_{s-n,j+n-1}$. Thus
\[ E[h_{s,j} | j \in \mathbb{Z}/n] \otimes P[b_{s-n,j} | j \in \mathbb{Z}/n] \]
is a sub-complex of $F^s(n, k)$ whose cohomology is $\mathbb{Z}/p$ concentrated at dimensional 0. This implies
\[ H^* F^{s_0}(n, k) \cong H^* F^{s+1}(n, k) \cong \cdots \cong H^* F^s(n, k) \cong \cdots \cong H^* E_1^{*,*}(n, k) \]
At prime $p = 2$, $s_0 = \left\lfloor \frac{2 \times 2n}{2} \right\rfloor = 2n > n + k - 1$. The first May differentials are deduced from (2.2). \hfill \Box

As a corollary one can easily see that if $\frac{2pn + p - 2}{2(p - 1)} \leq n + k - 1$, then the reduced May’s $E_1$-term becomes
\[ \tilde{E}_1^{*,*}(n, k) = E[h_{i,j} | k \leq i \leq n + k - 1, j \in \mathbb{Z}/n]. \]

**Theorem 3.4** If $\frac{2pn + p - 2}{2(p - 1)} \leq n + k - 1$, then the cohomology of $S(n, k)$ is of dimensional $n^2$.

4. **The higher May differentials in the MSS for $S(n, k)$**

From (3.2) we see that the first non-trivial May differential of $b_{i,j}$ appears at
\[
d_r(b_{i,j}) = \begin{cases} 
0 & \text{if } i < 2k. \\
\xi_{i-k,j} h_{k,j} + \xi_{i-k,j+1} h_{k,j+1} & \text{if } i \geq 2k.
\end{cases}
\]
In [6] (2.10) and (2.11), a collapse theorem is given for the higher May differentials in the exterior part $E[h_{i,j} | i > 0, j \geq 0]$ of the MSS for the steenrod algebra $A$ at odd primes. In this section, we will give a similar collapse theorem for the higher May differentials of $E_1^{*,*}(n, k)$.

Let $p$ be an odd prime. We define a Hopf algebra $T(n, k)$ as
\[
T(n, k) = P[\xi_i | k \leq i \leq n + k - 1].
\]
The inner degree of $\xi_i$ is defined to be $|\xi_i| = 2(p-1)(1+p+\cdots+p^{i-1})$ and the structure map $\Delta : T(n,k) \to T(n,k) \otimes T(n,k)$ acts on $\xi_i$ by

$$\Delta(\xi_i) = \xi_i \otimes 1 + \sum_{k \leq r \leq i-k} \xi_r \otimes \xi_{i-r}^p + 1 \otimes \xi_i.$$ 

There is a Hopf algebra reduction homomorphism $\Phi : T(n,k) \to S(n,k)$ which send $\xi_i$ to $t_i$. The image of $\Phi$ is $P[t_i | k \leq i \leq n+k-1]/(t_i^{p^k} - t_i)$ and $\text{Ker} \Phi$ is the idea generated by $(\xi_i^p - \xi_i)$. Further more the homomorphism $\Phi$ also induces homomorphism in cobar complexes and cohomologies

$$\Phi : \text{Ext}^{*,*}_{T(n,k)}(Z/p, Z/p) \to \text{Ext}^{*,*}_{S(n,k)}(Z/p, Z/p).$$

Similar to that of definition 2.3, we set May filtration on $T(n,k)$ as

$$M(\xi_i^{p^k}) = 2i - 1$$

and let $F^{*,*}T(n,k)$ be the sub-module of $T(n,k)$ generated by the elements with May filtration $\leq M$. Then $E^{*,*}T(n,k) = F^{*,*}T(n,k)/F^{*,*}T(n,k)$ becomes a bigraded Hopf algebra with the structure of

$$E^{*,*}T(n,k) = \bigotimes T[\xi_i^{p^k} | k \leq i \leq n+k-1, \ j \geq 0]$$

and $\Delta(\xi_i^{p^k}) = \xi_i^{p^k} \otimes 1 + 1 \otimes \xi_i^{p^k}$.

Consider the cobar construction $C^{*,*}T(n,k)$ of $T(n,k)$. Similarly for the generator $[\beta_1|\beta_2|\cdots|\beta_s]$ of $C^{*,*}T(n,k)$ define its May filtration as

$$M([\beta_1|\beta_2|\cdots|\beta_s]) = M(\beta_1) + M(\beta_2) + \cdots + M(\beta_s)$$

and let $FC^{*,*}T(n,k)$ denote the sun-complex generated by elements with May filtration $\leq M$. We get a spectral sequence $E_r^{*,*}T(n,k)$, $d_r$ with $E_0$-term

$$E_0^{*,*}T(n,k) = FC^{*,*}T(n,k)/FC^{*,*}T(n,k)$$

which is isomorphic to the cobar complex of $E^{*,*}T(n,k)$. The $E_1$-term of this spectral sequence is isomorphic to

$$E_1^{*,*}T(n,k) = E[h_{i,j} | k \leq i \leq n+k-1, \ j \geq 0] \otimes P[b_{i,j} | k \leq i \leq n+k-1, \ j \geq 0].$$

Noticed that the reduction map $\Phi : T(n,k) \to S(n,k)$ is May filtration preserving, it induces a homomorphism of May spectral sequences

$$\Phi : E^{*,*}_r T(n,k) \to E^{*,*}_r S(n,k).$$

**Theorem 4.4** The reduction map $\Phi : T(n,k) \to S(n,k)$ induces a homomorphism between May spectral sequences $\Phi : E^{*,*}_1 T(n,k) \to E^{*,*}_1 S(n,k)$ which sends $h_{i,j}$ and $b_{i,j}$ to $h_{i,j}$ and $b_{i,j}$ respectively. It sends infinite cocycles of $E^{*,*}_r T(n,k)$ to that of $E^{*,*}_r S(n,k)$.

Similar to [6] (2.10) and (2.11) we give a collapse theorem in the MSS for $T(n,k)$. To the generators $h_{i,j}$, $b_{i,j} \in E^{*,*}_1 T(n,k)$ define their index as

$$SI(h_{i,j}) = SI(b_{i,j}) = i.$$ 

given a monomial $g = x_1 x_2 \cdots x_m \in E^{*,*}_1 T(n,k)$ where each $x_i$ is of the generators $h_{i,j}$ or $b_{i,j}$, define its sum of index as

$$SI(g) = SI(x_1) + SI(x_2) + \cdots + SI(x_m).$$

For example the sum of index of $h'_{4,0} h'_{3,0} b'_{2,1}$ is 9.
We use \( s(x) \) to denote the homological dimension of \( x \). Noticed that the May filtration of \( h'_{i,j} \), \( b'_{i,j} \) satisfies

\[
M(h'_{i,j}) = 2i - 1 = 2SI(h'_{i,j}) - 1 = 2SI(h'_{i,j}) - s(h'_{i,j})
\]
\[
M(b'_{i,j}) = p(2i - 1) > 2SI(b'_{i,j}) - 2 = 2SI(b'_{i,j}) - s(h'_{i,j})
\]
we see that for the monomial \( g = x_1x_2 \cdots x_m \in E^*_T(n,k) \) of homological dimension \( s \), its May filtration satisfies

\[
M(g) = M(x_1) + M(x_2) + \cdots + M(x_m)
\]
\[
\geq 2SI(x_1) - s(x_1) + 2SI(x_2) - s(x_2) + \cdots + 2SI(x_m) - s(x_m)
\]
\[
= 2SI(g) - s
\]
and the equality holds if and only if \( g \) is a monomial in \( E[h'_{i,j}] | k \leq i \leq n + k - 1, j \geq 0 \).

Given an integer \( t = 2(p-1)(c_0 + c_1p + \cdots + c_mp^m) \) with \( 0 \leq c_i < p \), we define its sum of degree as

\[
Sd(t) = c_0 + c_1 + \cdots + c_m
\]
and for an element \( g \in E^*_T(n,k) \), express its inner degree \(|g|\) as \(|g| = 2(p-1)(c_0 + c_1p + \cdots + c_mp^m)\), where \( 0 \leq c_i < p \) and define its sum of degree to be

\[
Sd(g) = Sd(|g|) = c_0 + c_1 + \cdots + c_m.
\]

Then from

\[
|h'_{i,j}| = 2(p-1)(p^j + p^{j+1} + \cdots p^{j+i-1})
\]
\[
|b'_{i,j}| = 2(p-1)(p^{j+1} + p^{j+2} + \cdots + p^{j+i})
\]
we see that \( SI(h'_{i,j}) = Sd(h'_{i,j}), SI(b'_{i,j}) = Sd(b'_{i,j}). \) But for the reason of the \( p \)-adic numbers one has

\[
SI(x_1x_2 \cdots x_s) \geq Sd(x_1x_2 \cdots x_m).
\]

**Theorem 4.9** In the May spectral sequence for \( T(n,k) \),

1. If the inner degree \( t = 2(p-1)(c_0 + c_1p + \cdots + c_mp^m) \) and the May filtration 
   \[ M < 2Sd(t) - s = 2(c_0 + c_1 + \cdots + c_m) - s, \]
   then the May’s \( E_1 \)-term \( E_1^{s,t,M}T(n,k) = 0. \)
2. If a cocycle \( g \in E[h'_{i,j}] | k \leq i \leq n + k - 1, j \geq 0 \) in the exterior part of May’s \( E_1 \)-term satisfies \( SI(g) = Sd(g) \), then it is an infinite cocycle in the MSS for \( T(n,k) \) and \( \Phi(g) \) is an infinite cocycle in the MSS for \( S(n,k) \).

**Proof.** (1) follows from (4.6) and (4.8).

Suppose \( g \in E[h'_{i,j}] | k \leq i \leq n + k - 1, j \geq 0 \) is a cocycle in the exterior part of May’s \( E_1 \)-term \( E_1^{s,t,M}T(n,k) \) that satisfies \( SI(g) = Sd(g) \). Then its May filtration \( M = 2SI(g) - s = 2Sd(t) - s \). Consider the higher May differentials

\[
d_r : E_r^{s,t,M}T(n,k) \to E_r^{s+1,t,M-r}T(n,k),
\]
we see that \( M - r < 2Sd(t) - (s + 1) \) for \( r > 1 \). Thus the target \( E_r^{s+1,t,M-r}T(n,k) \) and then \( E_r^{s+1,t,M-r}T(n,k) \) is zero. \( \square \)
Example Let $p \geq 5$. The $E_2$-term of the May spectral sequence for $H^{*,*}S(4,2)$ is isomorphic to the homology of

$$E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j} | j \in \mathbb{Z}/4]$$

with first May differentials

$$d_1(h_{2,j}) = 0, \quad d_1(h_{3,j}) = 0, \quad d_1(h_{4,j}) = h_{2,j}h_{2,j+2}, \quad d_1(h_{5,j}) = h_{2,j}h_{3,j+2} + h_{3,j}h_{2,j+3}.$$ 

So $h_{5,0}h_{4,0}h_{3,0}h_{2,0}$ is a cohomology class in May’s $E_2$-term.

To prove that $h_{5,0}h_{4,0}h_{3,0}h_{2,0}$ is a 4-dimensional cocycle in the MSS $E^{4,*}_1S(4,2)$, consider the MSS for $T(4,2)$. $h_{5,0}h'_{4,0}h'_{3,0}h'_{2,0}$ is a 4-dimensional cocycle in the exterior part of May’s $E_1$-term $E^{4,*,M}_1T(4,2)$.

$$\deg(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0}) = 2(p-1)(4 + 4p^2 + 2p^3 + p^4),$$

$$SI(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0}) = 14 = Sd(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0}).$$

Thus it is an infinite cocycle in the MSS for $T(4,2)$ and $h_{5,0}h_{4,0}h_{3,0}h_{2,0} = \Phi(h'_{5,0}h'_{4,0}h'_{3,0}h'_{2,0})$ is an infinite cocycle in the MSS for $S(4,2)$.

5. The Cohomology of $S(n, n)$ at $p = 2$ and of $S(3, 2)$ at $p = 3$

As an application of Theorem 3.3, we will compute $H^{*,*}S(n, n)$ at $p = 2$, $H^{*,*}S(3, 2)$ at prime $p = 3$ and $H^{*,*}S(4, 2)$ at prime $p \geq 5$ in this section.

5.1. The cohomology of $S(n, n)$ at prime two. Consider the cohomology of $S(n, n)$ at $p = 2$. The reduced Mays $E_1$-term becomes

$$\tilde{E}_1^{*,*}S(n, n) = P[h_{n,j} | j \in \mathbb{Z}/n] \otimes E[h_{s,j} | n < s < 2n, j \in \mathbb{Z}/n]$$

(cf. Theorem 3.3). Noticed that the only non-trivial first May differential is

$$d_1(h_{2n,j}) = h_{2n,j-1} + h_{2n,j}.$$ 

We see that the $E_2$-term is the tensor product of $E[h_{s,j}] | n < s < 2n]$ and the cohomology of

$$\{ P[h_{n,j} | j \in \mathbb{Z}/n] \otimes E[h_{2n,j} | j \in \mathbb{Z}/n], \quad d_1 \}.$$ 

Lemma 5.2 The May’s $E_2$-term $E_2^{*,*,*}S(n, n)$ at $p = 2$ is isomorphic to the tensor product of $E[h_{s,j}] | n < s < 2n, j \in \mathbb{Z}/n]$ and $E[h_{n,j}, \rho_{2n} | j \in \mathbb{Z}/n] \otimes P[h_{n,j-1}]$, where $\rho_{2n} = \sum_{0 \leq j < n} h_{2n,j}$ and $h_{2n,j} = h_{n,n-1} - h_{2n,j}$.

Proof. We define $b_{n,j} = h_{n,j} + h_{n,j+1}$ for $0 \leq j \leq n - 2$ and define $b_{n,n-1} = h_{n}^{2}$ for $j = n$. It is easy to see that $P[h_{n,j} | j \in \mathbb{Z}/n]$ could be divided as the tensor product of $P[b_{n,j} | 0 \leq j < n]$ and $E[h_{n,j} | j \in \mathbb{Z}/n]$ as $\mathbb{Z}/2$-modules.

$$P[h_{n,j} | 0 \leq j \leq n - 1] = P[b_{n,j} | 0 \leq j \leq n - 1] \otimes E[h_{n,j} | 0 \leq j \leq n - 1].$$

From (5.1) we see that

$$d_1(h_{2n,j}) = \begin{cases} b_{n,j-1} & \text{if } 1 \leq j < n \\ \sum_{0 \leq i < n-2} b_{n,i} & \text{if } j = n. \end{cases}$$

The cohomology of $\{ P[h_{n,j} | j \in \mathbb{Z}/n] \otimes E[h_{2n,j} | j \in \mathbb{Z}/n], \quad d_1 \}$ is isomorphic to the tensor product of $E[h_{n,j} | j \in \mathbb{Z}/n]$ and the cohomology of

$$P[b_{n,j} | 0 \leq j < n] \otimes E[h_{2n,j} | j \in \mathbb{Z}/n].$$
The generator of \( P[b_{n,j}|j \in \mathbb{Z}/n] \) are denoted as
\[
b^{s_1}_{n,i_1} b^{s_2}_{n,i_2} \cdots b^{s_m}_{n,i_m}
\]
such that \( s_i > 0 \), \( 0 \leq i_1 < i_2 < \cdots < i_m < n \) and the generators of \( E[h_{2n,j}|j \in \mathbb{Z}/n] \) are denoted as
\[
h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1}
\]
such that \( n \geq j_k > \cdots > j_2 > j_1 > 0 \).

For the generators of \( E[h_{2n,j}|0 < j \leq n] \otimes P[b_{n,j}|0 \leq j < n] \) described as above, one has

1. For \( j_1 > i_1 + 1 \),
\[
h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1} b^{s_1}_{n,i_1} b^{s_2}_{n,i_2} \cdots b^{s_m}_{n,i_m}
\]
is the leading term of the first May differential
\[
d_1(h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1} h_{2n,i_1-1} b^{s_1}_{n,i_1} b^{s_2}_{n,i_2} \cdots b^{s_m}_{n,i_m}).
\]
While for \( i_1 < n - 1 \),
\[
b^{s_1}_{n,i_1} b^{s_2}_{n,i_2} \cdots b^{s_m}_{n,i_m}
\]
is the leading term of the first May differential
\[
d_1(h_{2n,i_1+1} b^{s_1}_{n,i_1} b^{s_2}_{n,i_2} \cdots b^{s_m}_{n,i_m}).
\]

2. For \( j_1 \leq i_1 + 1 \) and \( j_1 < n \), the leading term of the first May differential
\[
d_1(h_{2n,j_k} \cdots h_{2n,j_2} h_{2n,j_1} b^{s_1}_{n,i_1} b^{s_2}_{n,i_2} \cdots b^{s_m}_{n,i_m})
\]
is
\[
h_{2n,j_k} \cdots h_{2n,j_2} b_{2n,j_1-1} b^{s_1}_{n,i_1} b^{s_2}_{n,i_2} \cdots b^{s_m}_{n,i_m}.
\]

3. For \( j_1 = n = i_1 + 1 \),
\[
d_1(h_{2n,n} b^{s_1}_{n,n-1} \cdot x) = \sum_{i=0}^{n-2} b_{n,i} b^{s_1}_{n,n-1} \cdot x = d_1(\sum_{i=0}^{n-2} h_{2n,i+1} b^{s_1}_{n,n-1} \cdot x)
\]

Thus the cohomology of \( E[h_{2n,j}|j \in \mathbb{Z}/n] \otimes P[b_{n,j}|j \in \mathbb{Z}/n] \) is isomorphic to \( E[p_{2n}|j \in \mathbb{Z}/n] \otimes P[b_{n,n-1}] \), where \( \rho_{2n} = \sum_{0 \leq j < n} h_{2n,j} \). The Lemma follows.

**Theorem 5.3** The Mayer E_\( \infty \)-term \( E^{\ast,\ast}_{\infty} S(n,n) \) is isomorphic to its \( E_2 \)-term. Thus the cohomology of \( S(n,n) \) at prime 2 isomorphic to the tensor product of \( E[h_{s,j}|n < s < 2n] \) and \( E[h_{n,j}, \rho_{2n}, j \in \mathbb{Z}/p] \otimes P[h_{n,0}] \)

**Proof.** It is easy to see from (2.2) that for \( n \leq s < 2n \), \( h_{s,j} \) is an infinite cocycle. From \( d(t_{2n} + t_{2n}^2 + \cdots + t_{2n}^{2n-1}) = 0 \) we get the infinite cocycle \( \rho_{2n} \). The Theorem follows.

### 5.2. The cohomology of \( S(3,2) \) at prime 3.

Now consider the cohomology of \( S(3,2) \) at prime \( p = 3 \). From Lemma 2.4 we see that the \( s_0 = 4 \). Thus from Theorem 3.3 we see that the reduced May’s \( E_1 \)-term is

\[
E^{\ast,\ast}_{1} S(3,2) = E[h_{2,j}, h_{3,j}, h_{4,j}|j \in \mathbb{Z}/3]
\]

and the first May differentials are given by
\[
d_1(h_{2,j}) = 0 \quad d_1(h_{3,j}) = 0 \quad \text{and} \quad d_1(h_{4,j}) = -h_{2,j} h_{2,j+2}.
\]
The May’s $E_2$-term is isomorphic to

$$E_2^{*,*,*}S(3,2) = H^{*,*,*}(E[h_{2,j}, h_{4,j}|j \in Z/3], d_1) \otimes E[h_{3,j}|j \in Z/3]$$

**Lemma 5.5** The May $E_2$-term for $H^*S(3,2)$ at $p = 3$ has Poincare series $(x^6 + 3x^5 + 6x^4 + 9x^3 + 6x^2 + 3x + 1)(x + 1)^3$. It is the tensor product of $E[h_{3,j}|j \in Z/3]$ and the $Z/3$ module $C$ generated by the following element.

| Dimension | 0 | 1 | 2 | 3 | 4 | 5 | 6 | A |
|-----------|---|---|---|---|---|---|---|---|
| Generators | $h_{2,j}$ | $g_j$ | $l_j$ | $k_j$ | $l_j'$ | $k_jk_{j+1}$ | $g_jg_{j+1}$ | $g_jl_{j+1}$ | \(A\) |

where \(j \in Z/3\), \(g_j = h_{4,j}h_{2,j}\), \(k_j = h_{4,j}h_{2,j+2}\), \(l_j = h_{4,j}h_{4,j+1}h_{2,j}\) and

\[
\begin{align*}
l_j' &= h_{4,j}h_{4,j+1}h_{2,j+1} + h_{4,j+1}h_{4,j+2}h_{2,j} \\
A &= h_{4,0}h_{4,1}h_{4,2}h_{2,0}h_{2,1}h_{2,2} = -g_0g_1g_2.
\end{align*}
\]

**Proof.** From (5.4), it is easy to see that $d_1(h_{4,j}h_{2,j}) = 0$, $d_1(h_{4,j}h_{2,j+2}) = 0$ and from $d_1(h_{4,j+1}) = h_{2,j+1}h_{2,j+3} = h_{2,j+1}h_{2,j}$, we see that $d_1(h_{4,j}h_{4,j+1}h_{2,j}) = 0$. These gives the cohomology classes $g_j$, $k_j$ and $l_j$. From

\[
\begin{align*}
d_1(h_{4,j}h_{4,j+1}h_{2,j+1}) &= -h_{2,j}h_{2,j+2}h_{4,j+1}h_{2,j+1} = -h_{4,j+1}h_{2,j}h_{2,j+1}h_{2,j+2} \\
d_1(h_{4,j+1}h_{4,j+2}h_{2,j}) &= h_{4,j+1}h_{2,j+2}h_{2,j+4}h_{2,j} = h_{4,j+1}h_{2,j}h_{2,j+1}h_{2,j+2}
\end{align*}
\]

we get $l_j'$. A routine computation shows that $H^*(E[h_{2,j}, h_{4,j}|j \in Z/3]) = C$. \(\square\)

**Theorem 5.6** The May $E_2$-term for $H^{*,*}S(3,2)$ at $p = 3$ is the $E_\infty$-term, thus $H^{*,*}S(3,2)$ is the tensor product of $E[h_{3,j}|j \in Z/3]$ and $C$.

**Proof.** It is easy to see that $h_{2,j}$ and $h_{3,j}$ are infinite cycles. To prove that all the higher May differentials are trivial, consider the May filtration of each generator in the $E_2$-term and the differentials

$$d_r : E_r^{*,*,M}S(3,2) \to E_r^{*,1,M-r}S(3,2).$$

One has

\[
\begin{align*}
g_j &\in E_2^{2,*,10}S(3,2) \\
l_j &\in E_2^{3,*,17}S(3,2) \\
h_{2,j} &\in E_2^{1,*,3}S(3,2) \\
k_j &\in E_2^{2,*,10}S(3,2) \\
l_j' &\in E_2^{3,*,17}S(3,2) \\
h_{3,j} &\in E_2^{1,0,5}S(3,2).
\end{align*}
\]

The May filtration of $g_j$ and $k_j$ are 10. Beside, it is easy to check that each generator in the 3rd dimension $E_2^{3,*,*}S(3,2)$ listed as below

$$l_j, \quad l_j', \quad k_jh_{2,j}, \quad g_jh_{3,i}, \quad k_jh_{3,i}, \quad h_{3,i}h_{3,j}h_{2,k}, \quad h_{3,0}h_{3,1}h_{3,2}$$

has May filtration $\geq 10$. Thus $g_j$ and $k_j$ are infinite cycles. Similarly one can prove that $l_j$ and $l_j'$ are infinite cycles. This complete the proof. \(\square\)
5.3. The cohomology of $S(4,2)$ at the primes $p > 3$. In this case, $s_0 = 5$ and the reduced May’s $E_1$-term is

$$
E_1^{t,*} \cong S(4,2) = E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j}| j \in \mathbb{Z}/4].
$$

To compute the $E_2$-term, we set a filtration on the exterior algebra $E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j}| j \in \mathbb{Z}/4]$ as follows:

$$
F^k = \bigoplus_{0 \leq r \leq k} Z/p\{h_{5,j}, \ldots h_{5,j_r}\} \otimes E[h_{2,j}, h_{3,j}, h_{4,j}| j \in \mathbb{Z}/4]
$$

where $h_{5,j_1} \cdots h_{5,j_r}$’s are the generators of the $r$-dimensional module of the exterior algebra $E[h_{5,j}| j \in \mathbb{Z}/4]$. This filtration gives raise to a spectral sequence with

$$
E_0^k = F^k/F^{k-1} = Z/p\{h_{5,j_1} \cdots h_{5,j_r}\} \otimes E[h_{2,j}, h_{3,j}, h_{4,j}| j \in \mathbb{Z}/4].
$$

The $E_1$-term of this spectral sequence is

$$
E_1^k = Z/p\{h_{5,j_1} \cdots h_{5,j_r}\} \otimes H^*E[h_{2,j}, h_{3,j}, h_{4,j}| j \in \mathbb{Z}/4],
$$

and the differentials are given by

$$
d_r : E_r^k \rightarrow E_r^{k-r}.
$$

By a routine computation, we get

**Theorem 5.7** The cohomology of $E[h_{2,j}, h_{3,j}, h_{4,j}| j \in \mathbb{Z}/4]$ is the tensor product of $E[h_{3,j}, \rho_0, \rho_1]$ and $\mathbb{N}$, where

$$
\rho_0 = h_{4,0} + h_{4,2}, \quad \rho_1 = h_{4,1} + h_{4,3},
$$

and $\mathbb{N}$ is the direct sum of the modules generated by the following cohomology classes:

$$
\begin{align*}
1; & \quad h_{2,j}; & e_j &= h_{2,j}h_{2,j+1}, & g_j &= h_{4,j}h_{2,j}; \\
\rho_0 = h_{2,j}g_{j+1}, & \quad h_{2,j}g_{j+1}; & h_{2,j}g_{j+3}; & \quad g_jg_{j+1}, & e_jg_{j+2}, & h_{2,j}g_{j+1}g_{j+2}; & e_0g_2g_3
\end{align*}
$$

with $j \in \mathbb{Z}/4$. Beside, we also have the following relations:

$$
h_{2,i}h_{2,i+2} = 0, \quad h_{2,i}g_{i+2} = h_{2,i+2}g_i, \quad h_{2,i}g_{i+2}g_{i+3} = h_{2,i+2}g_i + 3g_i.
$$

With the add of a personal computer, we compute that

**Theorem 5.8** The cohomology of the exterior algebra $E[h_{2,j}, h_{3,j}, h_{4,j}, h_{5,j}]$ has Poincaré series

$$(1 + t)^4(1 + 6t + 18t^2 + 59t^3 + 92t^4 + 176t^5 + 161t^6 + 176t^7 + 92t^8 + 59t^9 + 18t^{10} + 6t^{11} + t^{12}).$$

The ranks at each cohomological dimension are listed as

$$
\begin{align*}
0, & \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad \cdots \quad 16, \\
1, & \quad 10, \quad 48, \quad 171, \quad 461, \quad 976, \quad 1671, \quad 2303, \quad 2558, \quad 2303, \quad \cdots \quad 1
\end{align*}
$$

From the collapse Theorem 4.9, we claim that the MSS for the cohomology of $S(4,2)$ collapse at $E_2$-term.
ON $H^*(S(n,k))$

References

[1] J. F. Adams, _Stable Homotopy and Generalised Homology_. University of Chicago Press, Chicago 1974.
[2] H. Cartan and S. Eilenberg, _Homological Algebra_, Princeton University Press 1956.
[3] P. Goerss, H. W. Henn, M. Mahowald and C. Rezk, A resolution of the $K(2)$–local sphere at the prime 3. _Ann. Math._ **162** (2005) 777-822.
[4] M. Hazewinkel, A universal formal group law and complex cobordism, _Bull. A. M. S._ **81** (1975), 930-933.
[5] H. W. Henn, Centralization of abelian $p$-subgroups and mod-$p$ cohomology of profinite groups, _Duke Math. J._ **91** (1998), 561-585.
[6] S. Nave, Lee, The Smith-Toda complex $V((p+1)/2)$ does not exist, _Ann. Math._ **171** (2010) 491-509.
[7] X. Liu and X. Wang, A four-filtered May spectral sequence and its applications, _Acta. Math. Sin. (English Ser.)_ **24** (2008) 1507-1524.
[8] J. P. May, The cohomology of restricted Lie algebras and of Hopf algebras. _J. Algebra_ **3** (1966) 123-146.
[9] J. P. May, The cohomology of restricted Lie algebras and of Hopf algebras; application to the Steenrod algebra. _Princeton Univ., 1964 Thesis._
[10] H. Miller, D. C. Ravenel and S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence _Ann. of Math._ **106** (1977), 469-516.
[11] D. C. Ravenel, _Nilpotence and Periodicity in Stable Homotopy Theory_, Ann. of Math. Studies **128**, Princeton Univ. Press, Princeton, NJ, 1992.
[12] D. C. Ravenel, The cohomology of the Morava stabilizer algebras _Math. Z._ **152** (1977), 287-297.
[13] D. C. Ravenel, Localization with respect to certain periodic homology theories. _Amer. J. Math._ **106** (1984) 351-414.
[14] Shimomura, K., and Tokashiki, S., The cohomology of $S(n,n-1)$ relevant to the Morava Stabilizer algebra at odd prime, _Kochi Journal of Mathematics_ **7** (2012), 109-118.
[15] X. Wang, The homotopy groups $\pi_*(L_2S^0)$ at the prime 3, _Topology_ **34** (2002), 1183-1198.
[16] K. Shimomura and X. Wang, The Adams-Novikov $E_2$-term for $\pi_*(L_2S^0)$ at the prime 2, _Math. Z._ **241** (2002) 271-311.
[17] Tangora, M. C. On the cohomology of the Steenrod algebra, _Math. Z._ **116** (1970) 18–64.
[18] X. Wang, $\pi_*(L_2T(1)/(v_1))$ and its applications in computing $\pi_*(L_2T(1))$ at the prime two, _Forum Math._ **19** (2007), 127-147.

School of Mathematical Science, Nankai University, Tianjin 300071, P. R. China
E-mail address: chenlimanstar1@163.com

School of Mathematical Science and LPMC, Nankai University, Tianjin 300071, P. R. China
E-mail address: xjwang@nankai.edu.cn

Department of Mathematics & Institute of mathematics and interdisciplinary science, Capital Normal University, Beijing 100048, P. R. China
E-mail address: zhaoxve@mail.cnu.edu.cn