Debiased Inference on Identified Linear Functionals of Underidentified Nuisances via Penalized Minimax Estimation

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Abstract

We study generic inference on identified linear functionals of nonunique nuisances defined as solutions to underidentified conditional moment restrictions. This problem appears in a variety of applications, including nonparametric instrumental variable models, proximal causal inference under unmeasured confounding, and missing-not-at-random data with shadow variables. Although the linear functionals of interest, such as average treatment effect, are identifiable under suitable conditions, nonuniqueness of nuisances pose serious challenges to statistical inference, since in this setting common nuisance estimators can be unstable and lack fixed limits. In this paper, we propose penalized minimax estimators for the nuisance functions and show they enable valid inference in this challenging setting. The proposed nuisance estimators can accommodate flexible function classes, and importantly, they can converge to fixed limits determined by the penalization, regardless of whether the nuisances are unique or not. We use the penalized nuisance estimators to form a debiased estimator for the linear functional of interest and prove its asymptotic normality under generic high-level conditions, which provide for asymptotically valid confidence intervals.

1 Introduction

Many causal or structural parameters of interest can be expressed as linear functionals of unknown functions that satisfy certain conditional moment restrictions. For example, in a nonparametric instrumental variable (NPIV) model, parameters such as average policy effects, weighted average derivatives, and average partial effects are all linear functionals of the NPIV regression that is characterized by a conditional moment equation given by the exclusion restriction [Newey and Powell, 2003]. Similarly, in proximal causal inference under unmeasured confounding [Tchetgen Tchetgen et al., 2020], the average treatment effect and various policy effects can be expressed as linear functionals of two bridge functions defined as solutions to some conditional moment restrictions [Cui et al., 2022, Kallus et al., 2021]. Similarly, in missing-not-at-random data problems with shadow variables, some parameters for the missing subpopulation can be also written as linear functionals of an unknown density ratio function that solves a certain conditional moment restriction [Li et al., 2022, Miao et al., 2015].

In this paper, we tackle the commonplace problem when these conditional moment restrictions are underidentified, giving rise to multiplicity of valid nuisances, all of which identify the same unique functional. This multiplicity can easily occur in many applications. For example, the identification of NPIV regressions requires the so-called “completeness condition.” This condition
can be easily violated if the instrumental variables are not very strong, as is common in practice [e.g., Andrews and Stock, 2005, Andrews et al., 2019]. Even with additional restrictions on the function space, Santos [2012] shows that NPIV regressions are unidentifiable in a variety of models. Moreover, Canay et al. [2013] argues that completeness conditions are generally not testable, so nuisance multiplicity may not be diagnosable. Similar phenomena are also common in proximal causal inference and missing data problems with shadow variables (see examples 2 and 3).

Fortunately, the linear functionals of interest can still be identifiable, even if nuisances are not. In particular, these functionals often capture some identifiable aspects of the unidentifiable nuisance functions. For example, in proximal causal inference, merely the existence – but not uniqueness – of bridge functions is sufficient for identification of the average treatment effect [Miao and Tchetgen, 2018, Cui et al., 2022, Kallus et al., 2021]. But even if the functional is identifiable, the multiplicity of valid nuisance functions for identifying it raises significant challenges for statistical inference. In this setting, common nuisance estimators may be unstable and may not necessarily converge to any fixed limit when the sample size grows to infinity. Estimators for the functional based on these nuisance estimates would then generally have intractable asymptotic distributions, and indeed exhibit non-normality in practice. It is therefore very difficult to use these estimators to construct confidence intervals or hypothesis tests.

To tackle this challenge in generality, we study continuous linear functionals of nuisance functions characterized by linear conditional moment restrictions. We allow the conditional moment restrictions to have nonunique solutions and impose conditions that ensure identification only of the linear functional, regardless of the identifiability of the nuisances. We develop estimation and inference methods robust to possible nonuniqueness of the nuisance functions. This provides a general and unified solution to a wide range of applications where concerns regarding nonunique nuisances arise naturally.

Our contributions are summarized as follows:

- We formulate a general statistical inference problem for an identified linear functional of an underidentified nuisance function. This general formulation includes a wide variety of problems in IV models, proximal causal inference, and missing-not-at-random data (Examples 1 to 3). We establish the sufficient and necessary condition for the identification of the target linear functional (Lemma 1) and provide three different identification formulae (Lemma 2) with one satisfying Neyman orthogonality (Lemma 3).

- We propose penalized minimax estimators for the unknown nuisance function and an additional nuisance function related to the Reisz representer of the linear functional (Assumption 1). These estimators are generic and admit highly flexible function classes like reproducing kernel Hilbert space (RKHS) and neural networks. Moreover, these nuisance estimators do not require deriving the closed-form of the Reisz representer, just like the automatic debiased machine learning methods in recent literature (see a review in Section 2). By virtue of the penalization, we prove that these minimax estimators converge to fixed limits in terms of $L_2$-norm, even though the nuisances are underidentified (Theorems 1 and 2).

- We also derive the convergence rates of the penalized nuisance estimators in terms of a weaker projected norm. We further propose novel ill-posedness measures (Definition 2) to convert the projected norm error rates into $L_2$-norm error rates. Compared to existing ones, our proposed ill-posedness measures additionally incorporate penalization, which we show can better capture the asymptotic behaviors of our penalized estimators and are better suited to the underidentified setting (Lemmas 7 and 8).
• We construct debiased estimators for the functional of interest by plugging our penalized nuisance estimates into our Neyman-orthogonal identification formula. We prove that the resulting functional estimator has an asymptotic normal distribution under some high-level conditions (Theorem 3). We also propose a consistent estimator for the asymptotic variance and use it to construct asymptotically calibrated confidence intervals (Theorem 4).

The rest of this paper is organized as follows. We first review the related literature in Section 2 and set up our problem in Section 3. In particular, we characterize identifiability of linear functionals of underidentified nuisances in Section 3.1, discuss desirable properties of a doubly robust identification formula in Section 3.2, and illustrate the statistical challenges raised by underidentified nuisances in Section 3.3. Then we propose our penalized minimax nuisance estimators in Section 4. We further construct debiased functional estimators, establish their asymptotic normality, and discuss variance estimation and confidence intervals in Section 5. Finally, we conclude this paper and discuss future directions in Section 6.

Notation. For a generic random vector \( W \in \mathcal{W} \), we use \( \mathcal{L}_2(W) \) to denote the space of square integrable functions of \( W \) with respect to the probability distribution of \( W \). For any \( f(W), g(W) \in \mathcal{L}_2(W) \), we denote the \( L_2 \)-norm by \( \|f\|_2 = \sqrt{\mathbb{E}[f^2(W)]} \) and inner product by \( \langle f, g \rangle = \mathbb{E}[f(W)g(W)] \). We denote the empirical \( L_2 \)-norm with respect to data \( W_1, \ldots, W_n \) by \( \|f\|_n = \sqrt{\frac{\sum_{i=1}^n f^2(W_i)}{n}} \). We let \( \mathbb{P}(f(W)) = \int f(w) d\mathbb{P}(w) \) be the expectation with respect to \( W \) alone. We differentiate this from \( \mathbb{E}[f(W; W_1, \ldots, W_n)] \), which we use to denote full expectation with respect to both \( W \) and data \( W_1, \ldots, W_n \). Thus if \( \tilde{h} \) depends on the data \( W_1, \ldots, W_n \), then \( \mathbb{P}(f(W; \tilde{h})) \) remains a function of \( \tilde{h} \) (and thus the data) but \( \mathbb{E}[f(W; \tilde{h})] \) is a nonrandom scalar. We use \( \mathbb{P}_n \) to denote the empirical expectation with respect to \( W \) given data \( W_1, \ldots, W_n \): \( \mathbb{P}_n(f(W)) = \frac{1}{n} \sum_{i=1}^n f(W_i) \). We further define the empirical process \( \mathbb{G}_n \) by \( \mathbb{G}_n(f) = \sqrt{n} (\mathbb{P}_n - \mathbb{P})(f) \). For any linear operator \( L : \mathcal{A} \rightarrow \mathcal{B} \) where \( \mathcal{A} \) and \( \mathcal{B} \) are Hilbert spaces, we denote its range space as \( \mathcal{R}(L) = \{ L(a) : a \in \mathcal{A} \} \subseteq \mathcal{B} \) and its null space as \( \mathcal{N}(L) = \{ a : L(a) = 0 \} \subseteq \mathcal{A} \). Unless otherwise stated, the default norm for the \( \mathcal{L}_2(W) \) space is \( \| \cdot \|_2 \). Thus the convergence of functions in \( \mathcal{L}_2(W) \), the compactness of subsets of \( \mathcal{L}_2(W) \), and the continuity of functionals defined on \( \mathcal{L}_2(W) \) are all understood in terms of the norm \( \| \cdot \|_2 \).

For any set \( D \), we denote its closure as \( \text{cl}(D) \). We say that \( D \) is star-shaped if for any element \( d \in D \) and any constant \( \alpha \in [0, 1] \), the element \( \alpha d \) also belongs to \( D \). The star hull of \( D \) is defined as \( \{ \alpha d : d \in D, \alpha \in [0, 1] \} \). For a function class \( \mathcal{G} \subseteq \mathcal{L}_2(W) \), we say it is \( b \)-uniformly bounded if \( |g(W)| \leq b \) almost surely for any \( g \in \mathcal{G} \). The localized Rademacher complexity of a function class \( \mathcal{G} \) is defined as \( \mathcal{R}_n(\delta; \mathcal{G}) = \mathbb{E} \left[ \sup_{g \in \mathcal{G}, \|g\|_2 \leq \delta} \left| \frac{1}{n} \sum_{i=1}^n \epsilon_i g(W_i) \right| \right] \), where \( W_1, \ldots, W_n, \epsilon_1, \ldots, \epsilon_n \) are independent with \( W_i \sim W \) and \( \epsilon_i \) taking values in \( \{-1, 1\} \) equiprobably. The critical radius \( \eta_n \) of the function class \( \mathcal{G} \) is defined as any solution to the inequality \( \mathcal{R}_n(\delta; \mathcal{G}) \leq \delta^2 \). We use \( |\mathcal{G}| \) to denote the cardinality of \( \mathcal{G} \) up to equality almost everywhere.

2 Related Literature

Our paper is related to the literature on the estimation of and inference on point-identified (finitely and/or infinitely dimensional) parameters defined by conditional moment restrictions [e.g., Newey, 1990, Chamberlain, 1987, Newey and Powell, 2003, Ai and Chen, 2003, Blundell et al., 2007, Chen and Pouzo, 2012, 2009, Chen and Reiss, 2011, Hall and Horowitz, 2005, Darolles et al., 2011]. Some other literature study partial identification sets and inference thereon when unconditional or conditional moment restrictions underidentify the parameters [e.g., Andrews and Shi, 2013, 2014,
Our paper studies the estimation and inference of identifiable functionals of unknown nuisance functions satisfying certain conditional moment restrictions. We do not restrict the nuisance functions to be parametric and focus on inference after using flexible nonparametric estimation of nuisances. Existing literature usually study this problem in the point-identified setting [e.g., Ai and Chen, 2007, 2012, Chen and Pouzo, 2015, Chen and Christensen, 2018, Brown and Newey, 1998, Newey and Stoker, 1993, Chen et al., 2021]. Recently, a few works further consider functionals of partially identified nuisance functions. In particular, Severini and Tripathi [2006], Escanciano and Li [2013], Freyberger and Horowitz [2015] study the point identification and partial identification of continuous linear functionals of partially identified NPIV regressions. Severini and Tripathi [2012] derive the semiparametric efficiency bound for point identified linear functionals, without requiring the identification of nuisances. Some works also study the estimation of identifiable linear functionals of unidentifiable nuisances and derive their asymptotic distributions, either in the IV setting [Santos, 2011, Babii and Florens, 2017, Chen, 2021, Escanciano and Li, 2021] or in the shadow variable setting [Li et al., 2022]. These works are reviewed in more detail in Section 3.1 and discussion below Lemma 4. Our paper builds on these literature and develops them in two aspects. First, our paper proposes a unified solution to a wider variety of problems, including IV, shadow variables, and proximal causal inference [Tchetgen Tchetgen et al., 2020]. Second, these existing literature are restricted to series or kernel estimation for conditional moment models, while our paper adopts the minimax estimation framework to accommodate generic function classes and thus more flexible machine learning methods like RKHS methods and neural networks.

Our proposed estimation method is based on the minimax estimation framework formalized in Dikkala et al. [2020]. Such minimax methods have been employed in average treatment effect estimation under unconfoundedness [Hirshberg and Wager, 2021, Kallus, 2020, Chernozhukov et al., 2020] and policy evaluation [Kallus, 2018, Feng et al., 2019, Yang et al., 2020, Uehara et al., 2021], but in these problem setting the nuisances are inherently unique regression functions. The minimax framework and its variants have also been successfully applied to causal inference under unmeasured confounding, including IV estimation [Lewis and Syrgkanis, 2018, Zhang et al., 2020, Liao et al., 2020, Bennett et al., 2019, Bennett and Kallus, 2020, Muandet et al., 2020] and proximal causal inference [Kallus et al., 2021, Ghassami et al., 2021, Mastouri et al., 2021], but this literature typically assumes that the unknown nuisance functions are uniquely identified whenever considering estimation and inference.

In contrast, our paper tackles the challenge of inference when the unknown functions are not unique solutions. To this end, we employ penalization to target certain unique nuisance function among all possible ones. Penalization is a common technique for solving ill-posed inverse problems [Carrasco et al., 2007, Engl et al., 1996], which has been in particular applied to series or kernel estimation for underidentified conditional moment models [Chen and Pouzo, 2012, Santos, 2011, Babii and Florens, 2017, Chen, 2021, Escanciano and Li, 2021, Li et al., 2022, Florens et al., 2011]. Our paper shows the effectiveness of penalization in the general minimax estimation framework and investigates the impact of penalization on the estimation of linear functionals. We also propose novel ill-posedness measures to better characterize the convergence of penalized estimators in terms of strong norms, which can be of independent interest.

Our paper is also related to the debiased machine learning literature [see Chernozhukov et al., 2018a, and the references therein]. This literature typically studies the estimation and inference of smooth functionals of certain regression functions that are inherently unique, in contrast to solutions of general moment restrictions. To alleviate the inherent bias of machine learning regression estimators, this literature leverages Neyman orthogonal estimating equations for functionals
of interest, which requires estimating some Reisz representers first. The Reisz representers can be estimated by fitting regressions according to their analytic forms (like propensity scores in average treatment effect estimation) [e.g., Farrell, 2015, Farrell et al., 2021, Chernozhukov et al., 2017, 2018a, Semenova and Chernozhukov, 2021]. Alternatively, some recent literature propose to estimate the Reisz representers by exploiting the representer property directly. These methods do not need to derive the analytic forms of Reisz representers on a case-by-case basis, and are therefore termed automatic debiased machine learning [Chernozhukov et al., 2020, 2018b, 2022b, 2019, 2021, 2022a]. Our proposed method also avoids deriving Reisz representers explicitly and is therefore automatic in the same sense. In contrast, however, existing automatic debiased machine learning methods focus on exogenous/unconfounded settings where nuisances are naturally well-posed and are unique.

3 Problem Setup

Consider independent and identically distributed data \( W_1, \ldots, W_n \sim W \). Let \( S = S(W) \) and \( T = T(W) \) be two \( W \)-measurable variables (usually potentially-overlapping subsets of the components of the vector \( W \)). Our parameter of interest is a functional of some unknown nuisance function \( h^* \in \mathcal{L}_2(S) \):

\[
\theta^* = \mathbb{E} [m(W; h^*)],
\]

where \( m \) is a given function such that \( h \mapsto \mathbb{E} [m(W; h)] \) is a continuous linear functional.

We posit that \( h^* \) solves the following linear conditional moment restriction:

\[
\mathbb{E} [g_1(W)h(S) \mid T] = \mathbb{E} [g_2(W) \mid T].
\]

for some given functions \( g_1 \in \mathcal{L}_2(W) \) and \( g_2 \in \mathcal{L}_2(W) \). Note that \( h(S) \) in Equation (2) is a function of variables that may not appear in the conditioning variable \( T \). In this sense, solving Equation (2) is an ill-posed inverse problem [Carrasco et al., 2007]. And, the solution may not be unique, meaning Equation (2) does not identify \( h^* \).

In this paper, we are interested in the estimation of and inference on the parameter \( \theta^* \) when there may be many solutions \( h_0 \) to Equation (2) but all of them lead to the same \( \theta^* = \mathbb{E} [m(W; h_0)] \), so that \( \theta^* \) is identifiable from the system of equations given by Equations (1) and (2), while \( h^* \) need not be. Defining the linear operator \( P : \mathcal{L}_2(S) \mapsto \mathcal{L}_2(T) \) by \( Ph = \mathbb{E} [g_1(W)h(S) \mid T] \), the set of all solutions to Equation (2) is given by

\[
\mathcal{H}_0 = \{ h \in \mathcal{L}_2(S) : [Ph](T) = \mathbb{E} [g_2(W) \mid T] \} = h^* + \mathcal{N}(P).
\]

Henceforth we will use the convention that \( h^* \) is some specific, albeit possibly unidentifiable, function, whereas \( h_0 \in \mathcal{H}_0 \) is any one solution to Equation (2). Note \( \mathcal{H}_0 \) is identifiable by construction.

By Equation (3), the conditional moment restriction in Equation (2) uniquely identifies the nuisance function \( h^* \) only when the linear operator \( P \) is injective so that \( \mathcal{N}(P) = \{0\} \). Nevertheless, the injectivity of the linear operator \( P \) often requires very strong conditions that can fail in practice (see Examples 1 to 3), and are known to be untestable [Canay et al., 2013]. Therefore, it is desirable to avoid having to assume that the nuisance function \( h^* \) is uniquely identified.

**Example 1** (Functionals of NPIV Regression). Consider a causal inference problem with an observed outcome \( Y \), confounded treatments \( A \), and instrumental variables \( Z \). We are interested in the NPIV regression model [Newey and Powell, 2003]:

\[
Y = h^*(A) + \epsilon, \quad \text{where } \mathbb{E} [\epsilon \mid Z] = 0.
\]
The NPIV regression solves the following conditional moment restriction:

$$E[h(A) \mid Z] = E[Y \mid Z],$$

which is an example of Equation (2) with $g_1(W) = 1$, $g_2(W) = Y$, $S = A$, and $T = Z$.

We are interested in linear functionals of the NPIV regression $h^*$. One example is

$$\theta^* = E[m(W; h^*)] = E[Ah^*(A)].$$

This parameter is of interest because it allows us to identify the coefficients of the best linear approximation to the IV regression function:

$$\beta^* = \arg\min_{\beta} E \left[ (h^*(A) - \beta^\top A)^2 \right] = \left( E \left[ AA^\top \right] \right)^{-1} \theta^*.$$

Alternatively, we can also consider other linear functionals of the NPIV regression, such as the weighted average derivatives described in Ai and Chen [2007].

It is known that the IV regression $h^*$ is identifiable if and only if a completeness condition on the distribution of $A \mid Z$ is satisfied (Proposition 2.1 in Newey and Powell, 2003). However, as Severini and Tripathi [2006] showed, the completeness condition can be easily violated, especially when the instrumental variables are not very strong. Moreover, the completeness condition is impossible to test in general nonparametric models [Canay et al., 2013], so the failure of identifying $h^*$ may not be detectable. Fortunately, even when the NPIV regression $h^*$ is unidentifiable, the functional of interest can be still identifiable as we discuss in Section 3.1.

**Example 2** (Proximal Causal Inference). Consider a causal inference problem with potential outcomes $Y(a)$ that would be realized if the treatment assignment were equal to $a \in \{0,1\}$. We are interested in the average treatment effect, $\theta^* = E[Y(1) - Y(0)]$. The actual treatment assignment is denoted as $A$, the corresponding observed outcome is $Y = Y(A)$, and some additional covariates $X$ are also observed, but these do not account for all confounders and there exist unmeasured confounders $U$. We consider the proximal causal inference framework [e.g., Tchetgen Tchetgen et al., 2020, Miao et al., 2018] that requires two different sets of proxy variables $Z, V$ strongly dependent with the unobserved confounders. The so-called negative control treatment $Z$ cannot directly affect the outcome $Y$, and the so-called negative control outcome $V$ cannot be affected by either the treatment $A$ or the negative control treatment $Z$ (see Assumptions 4 to 7 in Cui et al., 2022 for formal statements).

If $h^*$ is a so-called bridge function satisfying

$$E[Y - h^*(V,X,A) \mid U,X,A] = 0,$$

then our target parameter is identified as a linear functional of it:

$$\theta^* = E[m(W; h^*)] = E[h^*(V,X,1) - h^*(V,X,0)].$$

Equation (7) involves unobserved variables, but it implies that $h^*$ also solves the following (observable) conditional moment restriction

$$E[Y - h(V,X,A) \mid Z,X,A] = 0.$$

Equation (9) is an example of Equation (2) with $g_1(W) = 1$, $g_2(W) = Y$, $S = (V,X,A)$, and $T = (Z,X,A)$. Similarly, we can consider various average policy effects in the proximal causal inference
framework, even with continuous treatments, as they can also be written as linear functionals of bridge functions [Kallus et al., 2021, Qi et al., 2021].

Kallus et al. [2021] points out that the solution to Equation (9) is very likely to be nonunique and give various concrete examples. This particularly occurs in data-rich settings where there are more proxy variables than the unobserved confounders. Since the unobserved confounders are unknown in practice, it is generally impossible to know a priori whether bridge functions are unique or not. Fortunately, it is well known that even with nonunique bridge functions, any of them can still lead to the same average treatment effect or average policy effect under suitable conditions [Cui et al., 2022, Miao and Tchetgen, 2018, Kallus et al., 2021]. In Section 3.1, we will derive identification conditions for the target linear functionals.

**Example 3** (Missing-Not-at-Random Data with Shadow Variables). Consider a partially missing outcome $Y$ and an indicator $A \in \{0,1\}$ denoting whether $Y$ is observed, so that we only observe $V = AY$. We are interested in the average missing outcome, $\theta^* = \mathbb{E}[(1-A)Y]$ (which also gives the outcome mean via $\mathbb{E}[Y] = \theta^* + \mathbb{E}[V]$). However, the outcome is missing not at random, namely, while we observed some covariates $X$, we generally have $Y \not\perp A \mid X$. Nonetheless, if $h^*$ were the $Y$-conditional missingness propensity ratio $h^*(X,Y) = \mathbb{P}(A=0 \mid X,Y)/\mathbb{P}(A=1 \mid X,Y)$, then $\theta^*$ is a linear functional of it, using only the observables $W = (X,A,V)$:

$$\theta^* = \mathbb{E}[m(W;h^*)] = \mathbb{E}[Vh^*(X,V)].$$

(10)

Since the definition of $h^*$ involves conditioning on unobservables, we cannot generally learn it. We therefore consider additional so-called shadow variables $Z$ satisfying $Z \perp A \mid X,Y$ and $Z \not\perp Y \mid X$ [e.g., Li et al., 2022, d’Haultfoeuille, 2010, Miao et al., 2015, Miao and Tchetgen Tchetgen, 2016]. These conditions are particularly relevant when the missingness is directly driven by the outcome $Y$ and the shadow variables $Z$ are strong proxies for the outcome $Y$. Under these conditions, $h^*$ will necessarily satisfy the following conditional moment equation [Li et al., 2022]:

$$\mathbb{E}[Ah(X,V) \mid X,Z] = \mathbb{E}[1-A \mid X,Z].$$

(11)

This is an example of Equation (2) with $g_1(W) = A$, $g_2(W) = 1-A$, $S = (X,V)$, and $T = (X,Z)$.

Again, the conditional moment restriction in Equation (11) admits multiple solutions (i.e., does not identify $h^*$) unless a strong completeness condition on the distribution of $Y \mid X,Z$ holds. In Section 3.1, we will discuss conditions that ensure the identification of $\theta^*$ even when $h^*$ is not identified.

### 3.1 Identification of Linear Functionals of Underidentified Nuisances

Fortunately, even when the nuisance function $h^*$ is not uniquely identified, the target parameter $\theta^*$ can still be identifiable. We next establish sufficient and necessary conditions for the identification of the target parameter $\theta^*$ by generalizing the analyses in Severini and Tripathi [2006, 2012].

Note that, since $\mathbb{E}[m(W;\cdot)]$ is continuous linear, the Reisz representer theorem [Section 5.3, Luenberger, 1997] gives the existence of a unique function $\alpha \in \mathcal{L}_2(S)$ (known as the Riez representer) such that for any $h \in \mathcal{L}_2(S)$:

$$\mathbb{E}[m(W;h)] = \mathbb{E}[\alpha(S)h(S)].$$

(12)

Based on this, we can state sufficient and necessary conditions for the identifiability of $\theta^*$.

**Lemma 1.** The parameter $\theta^*$ is identifiable if and only if $\alpha \in \mathcal{N}(P)^\perp = \text{cl}(\mathcal{R}(P^*))$. 

7
In the above, \( P^* : \mathcal{L}_2(T) \mapsto \mathcal{L}_2(S) \) denotes the adjoint of \( P \), which can be shown to be given by \( P^* q = \mathbb{E} [ g_1(W) q(T) \mid S ] \).

According to Lemma 1, even if the nuisance function \( h^* \) is underidentified, the parameter \( \theta^* \) may be still identifiable, provided that the corresponding Reisz representer is orthogonal to the null space of \( P \), or equivalently is in the closure of the range of \( P^* \). In this case, the parameter \( \theta^* \) captures only certain identifiable parts of the nuisance \( h^* \). Of course, if the nuisance function \( h^* \) is identified to begin with, then \( \mathcal{N}(P) = \{ 0 \} \) and \( \mathcal{N}(P)^\perp = \mathcal{L}_2(S) \), so that any parameter \( \theta^* \) of the form in Equation (1) is trivially identifiable. In fact, the converse is also true: \( h^* \) is identifiable if and only if every continuous linear function of \( h^* \) is identifiable (see Lemma 9 in Appendix A.1). But for a given parameter of interest, Lemma 1 shows that the identifiability of \( h^* \) is not necessary for the identifiability of \( \theta^* \).

Although the condition in Lemma 1 guarantees that the target parameter \( \theta^* \) is identifiable, to also enable inference, we will need a slightly stronger form where we rule out \( \alpha \) being on the boundary of \( \mathcal{R}(P^*) \), analogously to Severini and Tripathi [2012].

**Assumption 1.** The Reisz representer \( \alpha \) in Equation (12) satisfies that \( \alpha \in \mathcal{R}(P^*) \). Namely, there exists a solution \( q^* \in \mathcal{L}_2(T) \) to the following conditional moment equation:

\[
\mathbb{E} [ g_1(W) q(T) \mid S ] = \alpha(S). \tag{13}
\]

Again, the conditional moment restriction in Equation (13) may have multiple solutions, and we only require at least one to exist. The set of solutions can be written as follows:

\[
\mathcal{Q}_0 = \{ q \in \mathcal{L}_2(T) : [P^* q](S) = \alpha(S) \} = q^* + \mathcal{N}(P^*). \tag{14}
\]

Under Assumption 1, the parameter \( \theta^* \) can be identified by any solution \( h_0 \) to Equation (2), or any solution \( q_0 \) to Equation (13), or both \( h_0 \) and \( q_0 \), corresponding to three different identification formulae shown in Lemma 2 below. This lemma shows that we do not need to focus on the particular nuisance function \( h^* \) that defines the target parameter \( \theta^* \); instead, we only need to solve the conditional moment restrictions in Equation (2) or in Equation (13), and any one valid solution thereof can recover the target parameter.

**Lemma 2.** Under Assumption 1, for any \( h_0 \in \mathcal{H}_0 \) and \( q_0 \in \mathcal{Q}_0 \),

\[
\theta^* = \mathbb{E} [ m(W; h_0) ] = \mathbb{E} [ q_0(T) g_2(W) ] = \mathbb{E} [ \psi(W; h_0, q_0) ],
\]

where \( \psi(W; h, q) := m(W; h) + q(T) (g_2(W) - g_1(W) h(S)) \).

We refer to the last identification formula, given by \( \psi \), as the doubly robust identification formula, as it enjoys certain robustness properties described in the next section.

We next revisit our examples and discuss the identification of \( \theta^* \) even when \( h^* \) is not identified.

**Example 1.** Cont’d (Functionals of NPIV Regression). Consider the parameter \( \theta^* = \mathbb{E} [ Ah^*(A) ] \) given in Equation (5). Assumption 1 posits the existence of a function \( q^* \in \mathcal{L}_2(Z) \) that solves

\[
\mathbb{E} [ q(Z) \mid A ] = A. \tag{15}
\]

Under this condition, even when the NPIV regression \( h^* \) is unidentifiable, the parameter \( \theta^* \) can be identified by any \( h_0 \) solving Equation (4) and any \( q_0 \) solving Equation (15):

\[
\theta^* = \mathbb{E} [ Ah_0(A) ] = \mathbb{E} [ q_0(Z) Y ] = \mathbb{E} [ Ah_0(A) + q_0(Z) (Y - h_0(A)) ].
\]
Escanciano and Li [2021] study the estimation of and inference on the closely related parameter $\beta^*$ in Equation (6), allowing the NPIV regression $h^*$ to be unidentifiable. They also assume the existence of solutions to Equation (15), which is a special example of our Assumption 1. To estimate $\beta^*$, they propose a penalized series estimator for the nuisance function $q^*$. They show that their proposed nuisance estimator converges to one particular solution to Equation (15), and the resulting estimator for $\beta^*$ has good asymptotic properties. In our paper, we will accommodate general flexible hypothesis classes. This permits going beyond sieve estimation and its involved technical assumptions (Assumptions A.2–A.6 in Escanciano and Li, 2021) and allows us to rely instead on high-level conditions for approximation by general hypothesis classes. To enable this, we instead incorporate penalization into the general minimax estimation framework with general hypothesis classes [Dikkala et al., 2020], we estimate not only $q^*$ but also $h^*$, and we employ the doubly robust identification formula in Lemma 2 to cancel out estimation errors in these nuisances so that we do not need strong assumptions to characterize their behavior.

**Example 2, Cont’d (Proximal Causal Inference).** For the average treatment effect $\theta^*$ identified via Equation (8), Assumption 1 corresponds to the existence of another bridge function $q^*(Z, X, A)\text{ }$ solving

$$
\mathbb{E}[q(Z, X, A) | W, X, A] = \frac{A - \mathbb{P}(A = 1 | V, X)}{\mathbb{P}(A = 1 | V, X) (1 - \mathbb{P}(A = 1 | V, X))}.
$$

(16)

Lemma 2 implies that $\theta^*$ can be identified by any $h_0$ solving Equation (9) and any $q_0$ solving Equation (16):

$$
\theta^* = \mathbb{E}[h_0(W, X, 1) - h_0(W, X, 0)] = \mathbb{E}[q_0(Z, X, A)Y] = \mathbb{E}[h_0(W, X, 1) - h_0(W, X, 0) + q_0(Z, X, A)(Y - h_0(W, X, A))].
$$

Although any solution to Equations (9) and (16) identifies $\theta^*$, multiplicity of solutions raises significant challenges to statistical inference. Indeed, even if uniqueness is not assumed for identification, the existing proximal causal inference literature largely assumes uniqueness for statistical inference [e.g., Cui et al., 2022, Kallus et al., 2021, Ghassami et al., 2021, Mastouri et al., 2021, Singh, 2020, Miao and Tchetgen, 2018]. One exception is Imbens et al. [2021], which handles the nonunique nuisances by a penalized generalized method of moment estimator, but their approach only applies in their specific panel-data setting where the nuisance is linearly parameterized. In this paper, we will add penalization to the minimax estimation methods in Kallus et al. [2021] for proximal causal inference and develop new estimators and inferential procedures that are robust to the nonuniqueness of general nonparametric nuisance functions.

**Example 3, Cont’d (Missing-Not-at-Random Data with Shadow Variables).** For the parameter $\theta^*$ in Equation (10), Assumption 1 requires the existence of a function $q^*(X, Z)$ that solves

$$
\mathbb{E}[Aq(X, Z) - V | X, V] = 0.
$$

(17)

Under this assumption, the parameter $\theta^*$ can be identified by any solutions $h_0, q_0$ to the conditional moment restrictions in Equations (11) and (17):

$$
\theta^* = \mathbb{E}[Vh_0(X, V)] = \mathbb{E}[(1 - A)q_0(X, Z)] = \mathbb{E}[Vh_0(X, V)] + \mathbb{E}[q_0(X, Z)((1 - A) - Ah_0(X, V))].
$$

Li et al. [2022] assume a condition equivalent to Equation (17) and develop estimation and inferential methods robust to nonunique nuisances. Their method extends that in Santos [2011]: they first use series estimators proposed by Chernozhukov et al. [2007] to estimate the set of solutions
to Equation (17), and then pick a unique element therein that maximizes a certain criterion. In contrast, the methods we will propose can accommodate flexible hypothesis classes, rely on high-level conditions about these classes, and avoid the challenging task of estimating solution sets to conditional moment restrictions.

### 3.2 Product Bias and Neyman Orthogonality for Doubly Robust Identification

Among the three identification formulae in Lemma 2, we are particularly interested in the doubly robust one. This is because the doubly robust formula has a special product bias property established below, which provides certain robustnesses when used for estimation and inference.

**Lemma 3.** For any \( h \in L^2(S) \), \( q \in L^2(T) \) and any \( h_0 \in H_0, q_0 \in Q_0 \),

\[
E \left[ \psi(W; h, q) \right] - \theta^* = E \left[ g_1(W) (q(T) - q_0(T)) (h(S) - h_0(S)) \right].
\]  

(18)

Consequently,

\[
\frac{\partial}{\partial t} E \left[ \psi(W; h_0 + th, q_0) \right] \bigg|_{t=0} = \frac{\partial}{\partial t} E \left[ \psi(W; h_0, q_0 + tq) \right] \bigg|_{t=0} = 0.
\]  

(19)

In Section 5, we will construct estimators for the parameter \( \theta^* \) by plugging nuisance estimates for \( h^*, q^* \) into a sample analogue of the doubly robust identification formula in Lemma 2. The results in Lemma 3 will be crucial for establishing the asymptotic property of the resulting estimator. In particular, Equation (18) ensures that the bias due to estimating the two nuisances is bounded by the product of the estimation errors of the two nuisance estimators (see Lemma 6). This property directly implies the Neyman orthogonality property shown in Equation (19), which shows that the doubly robust identification formula is insensitive to perturbations to the nuisances. Neyman orthogonality plays a pivotal role in the recent debiased machine learning literature that has enabled the use of nuisances estimated by flexible machine learning methods \(\text{e.g., Chernozhukov et al., 2016, 2018a, 2021, 2022b}\). The Neyman orthogonality in Equation (19) is particularly notable in that it holds around any valid solution pair to eqs. (2) and (13), even when there are infintely many of these.

### 3.3 Challenges with Nonunique Nuisances

In Lemma 3, we show that our doubly robust identification formula satisfies the Neyman orthogonality property. Following the recent literature on debiased machine learning cited above, it can therefore be hoped we can simply plug in any flexible nuisance estimators and use the debiased machine learning inference algorithm.

However, because of nonunique nuisances, statistical inference on the target parameter based on asymptotic normality is still very challenging. To illustrate the challenge, consider some generic nuisance estimators \( \hat{q}, \hat{h} \) and the corresponding doubly robust estimator for the target parameter:

\[
\tilde{\theta} = \frac{1}{n} \sum_{i=1}^{n} \psi(W_i; \hat{h}, \hat{q}).
\]

When the nuisances are nonunique, common nuisance estimators \( \hat{h}, \hat{q} \) typically do not converge to any fixed asymptotic limits in terms of strong norms, that is, that \( ||h - h_0||_2 \) and \( ||\hat{q} - q_0||_2 \) converge for some \( h_0 \in H_0, q_0 \in Q_0 \). Instead, we can at most establish that the \( \hat{h}, \hat{q} \) converge in terms of some weak norms. For example, if we apply the sieve methods in Chen [2007] or the minimax
methods in Dikkala et al. [2020] to nuisance estimation, then we can only establish convergence of the violation of the conditional moment restrictions in Equations (2) and (13), which is invariant to nonunique solutions. Namely, for any $h_0 \in \mathcal{H}_0, q_0 \in \mathcal{Q}_0$, we get convergence of the projected norm of errors, which are equal to the conditional-moment-restriction violation:

$$
\|P[h - h_0]\|_2 = \|E[g_1(W)h(S) - g_2(W) | T]\|_2, \quad \|P^*[\hat{q} - q_0]\|_2 = \|E[g_1(W)\hat{q}(T) - \alpha(S)]\|_2.
$$

Even when the projected-norm errors converge to 0, estimators $\hat{h}, \hat{q}$ do not necessarily converge to any fixed solutions to Equations (2) and (13). Since the sets of solutions to Equations (2) and (13) can be very large, these nuisance estimators can be very unstable even for large sample size.

That $\hat{h}, \hat{q}$ converge in terms of weak norm as above is generally not enough for deriving the asymptotic distribution of the estimator $\hat{\theta}$. Note that the estimation error of estimator $\hat{\theta}$ can be decomposed as follows: for any $h_0 \in \mathcal{H}_0, q_0 \in \mathcal{Q}_0$,

$$
\sqrt{n}(\hat{\theta} - \theta^*) = \mathcal{G}_n (\psi(W; h_0, q_0) - \theta^*) + \sqrt{n}\mathbb{P} \left( \psi(W; \hat{h}, \hat{q}) - \psi(W; h_0, q_0) \right) + \mathcal{G}_n \left( \psi(W; \hat{h}, \hat{q}) - \psi(W; h_0, q_0) \right).
$$

The first two terms on the right hand side of the last display are easily handled. By Central Limit Theorem, the first term has an asymptotic normal distribution. The second term quantifies the bias due to the estimation errors of $\hat{h}$ and $\hat{q}$. Thanks to the product bias and Neyman orthogonality property established in Lemma 3, we can show that this second term is negligible under suitable conditions (see Lemma 6 and Theorem 3). Therefore, to show the asymptotic normality of the estimator $\hat{\theta}$, we only need to establish that the third term is negligible (called the stochastic equicontinuity term). This typically requires that the nuisance estimators $\hat{h}, \hat{q}$ converge to fixed $h_0 \in \mathcal{H}_0, q_0 \in \mathcal{Q}_0$, in terms of strong norms like the $L_2$ norm [e.g., Lemma 19.24 in Van der Vaart, 2000]. This remains the case even if we employ cross fitting as described in Definition 1 below [e.g., see discussions below Assumption 3.2 in Chernozhukov et al., 2018a]. Without this, the stochastic equicontinuity term is generally not negligible, and the resulting asymptotic estimator can easily have an intractable asymptotic distribution (see section 3.1 in Chen, 2021 for a concrete example in the IV setting). This is why statistical inference on the target parameter $\theta^*$ is very challenging when nuisances are nonunique.

4 Penalized Minimax Estimation of Nuisances

We now turn to the estimation of $h_0, q_0$ using penalized minimax optimization. In Section 3.3, we illustrate the main problem with nonunique nuisances: common nuisance estimators do not converge to fixed limits in strong $L_2$ norm, which leads to an intractable stochastic equicontinuity term. This suggests we construct alternative nuisance estimators that do converge to fixed solution to Equations (2) and (13) in terms of strong norms. In this section, we realize this by incorporating penalization into the minimax estimation framework [Dikkala et al., 2020], which leads to highly flexible machine learning nuisance estimators converging to fixed limits in the $L_2$ norm.

First, in Section 4.1, we formulate the fixed limits we hope to converge to. Then, in Sections 4.2 and 4.3 we develop our penalized estimators for solutions to the conditional moment restrictions in Equations (2) and (13), respectively, and characterize their strong convergence properties even under solution multiplicity. Throughout this section, we will refer to any function $h_0 \in \mathcal{H}_0$ as a primary nuisance, and any function $q_0 \in \mathcal{Q}_0$ as a representer nuisance.

4.1 Targeting Unique Nusances via Penalization

According to Equations (3) and (14), any functions in the linear varieties $\mathcal{H}_0$ and $\mathcal{Q}_0$ are valid solutions to Equations (2) and (13). But in estimation, we need to restrict to solutions in certain
hypothesis classes $\mathcal{H}$ and $\mathcal{Q}$, such as functions with certain degrees of smoothness, balls of an RKHS, or neural networks of a given architecture. We consider using penalty functions $R_1 : \mathcal{L}_2(S) \to [0, \infty)$ and $R_2 : \mathcal{L}_2(T) \to [0, \infty)$ to target particular solutions in $\mathcal{H} \cap \mathcal{H}_0$ and $\mathcal{Q} \cap \mathcal{Q}_0$ respectively.

**Assumption 2.** The function classes $\mathcal{H} \cap \mathcal{H}_0$ and $\mathcal{Q} \cap \mathcal{Q}_0$ are non-empty, and the penalty functions $R_1$ and $R_2$ have unique minimizers $h_R \in \mathcal{L}_2(S)$ and $q_R \in \mathcal{L}_2(T)$ over $\mathcal{H} \cap \mathcal{H}_0$ and $\mathcal{Q} \cap \mathcal{Q}_0$, respectively:

$$h_R = \arg \min_{h \in \mathcal{H} \cap \mathcal{H}_0} R_1(h), \quad q_R = \arg \min_{q \in \mathcal{Q} \cap \mathcal{Q}_0} R_2(q).$$

For example, we can consider the $L_2$ norm penalties $R_1(h) = \|h\|^2_2$ and $R_2(q) = \|q\|^2_2$. If $\mathcal{H}$, $\mathcal{Q}$ are closed convex sets, then the unique existence of minimizers $h_R, q_R$ is guaranteed by the projection theorem [Theorem 1, Section 3.12 in Luenberger, 1997]. This means that even if there are many functions in $\mathcal{H} \cap \mathcal{H}_0$ and $\mathcal{Q} \cap \mathcal{Q}_0$, the $L_2$ norm penalties can pin down the single minimum-norm solution therein. In practice, these penalties may be unknown and need to be estimated from data, e.g., via $R_{1,n}(h) = \|h\|^2_n$ and $R_{2,n}(q) = \|q\|^2_n$. More generally, if $\mathcal{H}$, $\mathcal{Q}$ are convex, closed, and bounded, then Assumption 2 holds for any strictly convex and lower semicontinuous penalty function on $(\mathcal{H}, \| \cdot \|_2)$ and $(\mathcal{Q}, \| \cdot \|_2)$ (Theorem A.1 in Chen and Pouzo [2012]).

In Assumption 2, the target primary and representer nuisances $h_R$ and $q_R$ are defined as solutions to two constrained minimization problems. Below we prove that they can be alternatively formulated as solutions to penalized minimization problems.

**Lemma 4.** Suppose Assumption 2 holds.

1. If the penalty functions $R_1(\cdot)$ is continuous and the function class $\mathcal{H}$ is compact, then

$$h_R = \lim_{\mu_1 \downarrow 0} h_{\mu_1}, \quad \text{where} \quad h_{\mu_1} = \arg \min_{h \in \mathcal{H}} \mathbb{E} \left[ (\mathbb{E} [g_2(W) - g_1(W)h(S) | T])^2 \right] + \mu_1 R_1(h). \quad (20)$$

2. If the penalty functions $R_2(\cdot)$ is continuous and the function class $\mathcal{Q}$ is compact, then

$$q_R = \lim_{\mu_2 \downarrow 0} q_{\mu_2}, \quad \text{where} \quad q_{\mu_2} = \arg \min_{q \in \mathcal{Q}} \mathbb{E} \left[ (\mathbb{E} [\alpha(S) - g_1(W)q(T) | S])^2 \right] + \mu_2 R_2(q). \quad (21)$$

As we review in Examples 1 and 3, a few existing works (mostly focusing on NPIV) already propose to use penalization in series estimation to target unique nuisances. They can be viewed as finite-sample approximations for the constrained minimization formulation in Assumption 2 or the penalized minimization formulation in Lemma 4. Take the estimation of $h_R$ as an example. One approach is to first estimate the partial identification set $\mathcal{H}_0$ via series estimators proposed in Chernozhukov et al. [2007], and then minimize the penalty function over the estimated set and a series hypothesis class [Santos, 2011, Li et al., 2022]. The other approach is to first estimate the minimum distance criterion (i.e., the first term of the objective function) in Equation (20) using series, and then minimize the estimated criterion plus the penalty function over a series hypothesis class [Appendix A in Chen and Pouzo, 2012, Escanciano and Li, 2021, Chen, 2021].

In this paper, we will move beyond series estimation methods, and show how penalization can be combined with the flexible minimax estimation framework [Dikkala et al., 2020] to deal with nonunique solutions to conditional moment equations. In particular, we will build on the penalized minimization formulations in Lemma 4, so we avoid the tasks of estimating the partial identification sets $\mathcal{H}_0, \mathcal{Q}_0$. 

12
4.2 Estimating the Primary Nuisance Target

In this section, we estimate the primary nuisance target \( h_R \) based on the penalized minimization formulation in Equation (20). This requires first estimating the minimum distance criterion. According to Dikkala et al. [2020], Kallus et al. [2021], this criterion has an equivalent variational formulation:

\[
\frac{1}{4\lambda_1} \mathbb{E} \left[ (\mathbb{E} [g_2(W) - g_1(W)h(S) \mid T])^2 \right] = \max_{q \in L_2(T)} \mathbb{E} [q(T) (g_2(W) - g_1(W)h(S))] - \lambda_1 \|q\|_2^2, \quad \forall \lambda_1 > 0, h \in L_2(S). \tag{22}
\]

In this formulation, an auxiliary function \( q \) is used to detect the extent of a given nuisance \( h \) violating the conditional moment equation in Equation (2). Following Kallus et al. [2021], we call the auxiliary function \( q \) a critic function, and call the additional term \(-\lambda_1 \|q\|_2^2\) a stabilizer.

Motivated by the formulation in Equation (22), we propose the following estimator:

\[
\hat{h} \in \arg \min_{h \in \mathcal{H}_n} \sup_{q \in \mathcal{Q}'_n} \frac{1}{n} \sum_{i=1}^{n} q(T_i) (g_2(W_i) - g_1(W_i)h(S_i)) - \lambda_1 \|q\|_n^2 + \mu_{1,n} R_{1,n}(h). \tag{23}
\]

Here \( \mathcal{H}_n \) is the function class for the nuisance estimator, \( \mathcal{Q}'_n \) is the class of critic functions. We allow the function classes to vary with the sample size \( n \), and require that they approximate \( \mathcal{H}, \mathcal{Q}' \) in the limit (see condition 2 in Theorem 1). Moreover, the term \(-\lambda_1 \|q\|_n^2\) is an estimated stabilizer, and \( \mu_{1,n} R_{1,n}(h) \) is an estimated penalty term with a penalization parameter \( \mu_{1,n} > 0 \). Recall that the penalty and the stabilizer play very different roles. The penalty term is used to target the unique primary nuisance \( h_R \) among all solutions to the conditional moment equation in Equation (2). According to Equation (20), the penalization parameter \( \mu_{1,n} \) must vanish as \( n \to \infty \). In contrast, the stabilizer is used to ensure that the variational formulation in Equation (22) can recover the minimum distance criterion in Equation (20). Its coefficient \( \lambda_1 \) has to be strictly positive so we set it as a constant rather than a vanishing sequence.

Without the penalization, the remaining part of the objective function in Equation (23) is exactly the minimax objective function proposed in Dikkala et al. [2020] when applied to the conditional moment equation in Equation (2). Thanks to the penalization, our estimator can converge to the fixed target \( h_R \) in \( L_2 \) norm even when Equation (2) admits many different solutions. This is shown in the following theorem, which builds on the theoretical analyses in Chen and Pouzo [2012], Dikkala et al. [2020].

**Theorem 1.** Suppose Assumption 2, the conditions in Lemma 4 statement 1, and the following assumptions hold:

1. The functions \( g_1, g_2 \) in Equation (2) are bounded, and for simplicity, we assume \( |g_1(W)| \leq 1 \) and \( |g_2(W)| \leq 1 \) almost surely.

2. The classes \( \mathcal{H}_n \) and \( \mathcal{Q}_n \) can approximate \( \mathcal{H} \) and \( \mathcal{Q}' \) up to a vanishing error \( \delta_{1,n} \to 0 \), namely, for any \( h \in \mathcal{H} \) and \( q \in \mathcal{Q}' \), there exist \( \Pi_n h \in \mathcal{H}_n \) and \( \Pi_n q \in \mathcal{Q}'_n \) such that

\[
\|\Pi_n h - h\|_2 \leq \delta_{1,n}, \quad \|\Pi_n q - q\|_2 \leq \delta_{1,n}.
\]

3. The function classes \( \mathcal{H}, \mathcal{Q}', \mathcal{H}_n, \mathcal{Q}'_n \) are all closed and \( b \)-uniformly bounded. Moreover, \( \mathcal{Q}'_n \) and \( \mathcal{Q}' \) are star-shaped, and the class \( \mathcal{Q}' \) satisfies that \( P[h - h_R] \in \mathcal{Q}' \) for any \( h \in \mathcal{H} \).
There exists a vanishing positive sequence $\eta_{1,n} \to 0$ that upper bounds the critical radii of the class $Q'_n$ and the star hull of the class \{\begin{align*} W &\mapsto g_1(W)q(T)(h(S) - h_R(S)) : q \in Q'_n, h \in \mathcal{H}_n \end{align*}\}.  

The estimated penalty function $R_{1,n}(\cdot)$ is continuous and it is uniformly consistent, namely, 
\[ \delta_{R_{1,n}} := \sup_{h \in \mathcal{H}} |R_{1,n}(h) - R_{1}(h)| = o_p(1). \]

5. $\mu_{1,n} \to 0$, $\max \{\eta_{1,n}^2, \delta_{1,n}^2\} = o(\mu_{1,n})$, and $1/3 < \lambda_1 < \infty$.

Then $\hat{h}$ in Equation (23) is a consistent estimator for $h_R$ in $L_2$ norm, i.e.,
\[ \|\hat{h} - h_R\|_2 = o_p(1). \]  
Moreover, there exist universal positive constants $c_1$ to $c_5$ such that the following holds with probability at least $1 - c_1 \exp(-c_2 n \eta_{1,n}^2) - c_4 \exp(-c_5 n \eta_{1,n}^2/h^2)$:
\[ \|P(\hat{h} - h_R)\|_2^2 \leq 12 \lambda_1 \gamma_{1,n} - 12 \lambda_1 \mu_{1,n} \left( R_{1}(h_R) - R_{1}(\hat{h}) \right) = o_p(\mu_{1,n}), \]  
\[ R_{1}(\hat{h}) - R_{1}(h_R) \leq 12 \lambda_1 \gamma_{1,n}/\mu_{1,n} = o_p(1), \]
where $\gamma_{1,n} = (2c_3 + \lambda_1 + 11c_3^2/6\lambda_1) \eta_{1,n}^2 + 3\delta_{1,n}^2/\lambda_1 + \mu_{1,n} \left( R_{1}(\Pi_n h_R) - R_{1}(h_R) \right) + 2\mu_{1,n} \delta_{R_{1,n}} = o_p(\mu_{1,n})$.

In Theorem 1, condition 2 is trivial if we set $\mathcal{H}_n = \mathcal{H}$, $Q_n = Q$, and is otherwise standard in sieve literature \[e.g., Chen, 2007\] but can even be satisfied by neural networks as an approximation of Sobolev balls \[Yarotsky, 2017\]. Conditions 3 and 4 are directly adapted from assumptions in Dikkala et al. \[2020\], but they remove a symmetry condition on the critic function classes assumed in Dikkala et al. \[2020\]. In particular, condition 3 requires that $P(\hat{h} - h_R) \in Q'$ for any $h \in \mathcal{H}$. This is a well-specification condition on the critic function class $Q'$. It is needed to ensure that critic functions restricted to the class $Q'$ (which is in turn approximated by $Q'_n$) are enough to recover the minimum distance criterion in Equation (20) for any hypothesis $h \in \mathcal{H}$ \[see Equation (22)\]. Notably, our proof technique is different and more direct than that in Dikkala et al. \[2020\], and in particular this allows us to remove a symmetry condition.

Related to the penalization are conditions 5 and 6 in Theorem 1. In condition 5, we assume that the estimated penalty function is continuous and uniformly consistent. For the empirical $L_2$ norm penalty $R_{1,n}(h) = \|h\|_2^2$, the continuity condition obviously holds, and the uniform consistency can be easily established by uniform laws for the empirical norm \[see Theorem 14.1 in Wainwright \[2019\] or Lemma 10 in Appendix A.1\]. Finally, condition 6 states that $\max \{\eta_{1,n}^2, \delta_{1,n}^2\} = o(\mu_{1,n})$.

Here $\eta_{1,n}^2$ quantifies the complexity of the nuisance and critic function classes and $\delta_{1,n}^2$ quantifies the function approximation error. These roughly characterize the estimation variance and (squared) bias in absence of the penalization. Condition 6 thus requires that they are asymptotically dominated by the penalization; otherwise the penalization is too weak to pick a unique solution to Equation (2). In contrast, Chen and Pouzo \[2012, 2015\] recommend a fast vanishing penalty \[e.g., $\mu_{1,n} = o(n^{-1})]\) when the conditional moment equation has a unique solution. This weak penalty improves finite-sample estimation performance, but does not impact the asymptotic properties of the resulting estimator. Therefore, this recommendation is not effective in tackling conditional moment restrictions with nonunique solutions.

Theorem 1 Equation (24) shows the consistency of our proposed estimator $\hat{h}$ for the target $h_R$ in the strong $L_2$ norm. This is a direct consequence of the penalization. Without penalization, we can only establish consistency in terms of the weak projected norm for nonunique nuisances \[e.g., Dikkala et al., 2020\]. The proof of this consistency result builds on the analysis in Theorem A.1
in Chen and Pouzo [2012] and the minimax estimation theory in Dikkala et al. [2020]. Moreover, we also derive the projected norm convergence in Equation (25). In Equation (26) we further upper bound $R_1(h)$ in terms of $R_1(h_R)$ and a vanishing estimation error. This will be useful in characterizing the ill-posedness of the conditional moment restriction in Equation (2) when applying our penalized minimax estimator. See Equation (32) for details.

One may wonder how the penalization impacts the projected norm convergence rate. Note that if the conditional moment equation happens to have a unique solution within $H$ (i.e., $H_0 \cap H = \{h_R\}$), then we can show that the unpunished minimax estimator (i.e., with $\mu_{1,n} = 0$) would have a squared projected norm convergence rate $O(\eta^2_{1,n} + \delta^2_{1,n})$, while our penalized estimator would have rate $O(\eta^2_{1,n} + \delta^2_{1,n} + \mu_{1,n}(R_1(\Pi_nh_R) - R_1(h_R)) + \mu_{1,n}\delta_{R_1,n})$. These two rates are not directly comparable without further conditions on the penalty, even under the assumption that $\max\{\eta^2_{1,n}, \delta^2_{1,n}\} = o(\mu_{1,n})$. But even if the penalization may lead to slightly slower convergence rate for a unique solution, it enables robustness to possible nonunique solutions. This is very important since it is generally unknown whether the solution is unique, and nonuniqueness can be quire common as discussed in Examples 1 to 3.

### 4.3 Estimating the Representer Nuisance Target

Now we estimate the representer nuisance target $q_R$ based on the penalized minimization formulation in Equation (21). Again, we first give an equivalent variational formulation for the minimum distance criterion in Equation (21):

$$
\frac{1}{4\lambda^2} \mathbb{E} \left[ (\mathbb{E} [\alpha(S) - g_1(W)q(T) \mid S])^2 \right]
= \sup_{h \in L_2(S)} \mathbb{E} [h(S)(\alpha(S) - g_1(W)q(T))] - \lambda_2 \|h\|^2_2
= \sup_{h \in L_2(S)} \mathbb{E} [m(W; h)] - \mathbb{E} [g_1(W)q(T)h(S))] - \lambda_2 \|h\|^2_2, \quad \forall \lambda_2 > 0, q \in L_2(T). \tag{27}
$$

In Equation (27), the final formulation involves only the functional $\mathbb{E} [m(W; h)]$ but not the Reisz representer $\alpha(S)$. This obviates the need to derive the form of the Reisz representer $\alpha(S)$, so it is particularly convenient for estimation. This is in line with the spirit of the recent automatic debiased machine learning methods for functionals of regression functions [Chernozhukov et al., 2020, 2022b, 2021]. These methods propose ways to learn Reisz representers solely based on the functionals of interest, without needing to derive the form of the Reisz representers on a case-by-case basis. Thus these methods are considered generic and “automatic”. Our formulation in Equation (27) has the same advantages, but we will use it to solve a conditional moment equation that involves the Reisz representer, rather than estimate the Reisz representer itself.

Motivated by the formulation in Equation (27), we propose the following estimator:

$$
\hat{q} \in \arg \min_{q \in Q_n, h \in H_n} \sup_{h \in L_2(S)} \frac{1}{n} \sum_{i=1}^n m(W_i; h) - g_1(W_i)q(T_i)h(S_i) - \lambda_2 \|h\|^2_n + \mu_{2,n}R_{2,n}(q). \tag{28}
$$

Here $Q_n$ is the function class for the nuisance estimator, $H_n$ is the class of critic functions. Again, we allow the function classes to vary with the sample size $n$, only requiring that they approximate $Q, H'$ in the limit. We also have an estimated stabilizer $-\lambda_2 \|h\|^2_n$ and an estimated penalty $\mu_{2,n}R_{2,n}(q)$.

In the following theorem, we show the convergence of the penalized minimax estimator $\hat{q}$ in terms of both the strong $L_2$ norm and the weak projected norm.
Theorem 2. Suppose Assumption 2, the conditions in Lemma 4 statement 2, and the following assumptions hold:

1. The function \( g_1 \) in Equation (2) is bounded, and for simplicity, we assume \( |g_1(W)| \leq 1 \) almost surely. Moreover, \( m(W; h) \) is almost surely Lipschitz in \( h \) with a Lipschitz constant \( L_m \).

2. The classes \( Q_n \) and \( \mathcal{H}'_n \) can approximate \( Q \) and \( \mathcal{H}' \) up to a vanishing error \( \delta_{2,n} \to 0 \), namely, for any \( q \in Q \) and \( h \in \mathcal{H}' \), there exist \( \Pi_n q \in Q_n \) and \( \Pi_n h \in \mathcal{H}'_n \) such that

\[
\| \Pi_n q - q \|_2 \leq \delta_{2,n}, \quad \| \Pi_n h - h \|_2 \leq \delta_{2,n}.
\]

3. The function classes \( Q, \mathcal{H}', Q_n, \mathcal{H}'_n \) are all closed and \( b \)-uniformly bounded. Moreover, \( \mathcal{H}'_n \) and \( \mathcal{H}' \) are star-shaped, and the class \( \mathcal{H}' \) satisfies that \( P^*[q - q_R] \in \mathcal{H}' \) for any \( q \in Q \).

4. There exists a vanishing positive sequence \( \eta_{2,n} \to 0 \) that upper bounds the critical radii of the class \( \mathcal{H}'_n \) and the star hull of the class \( \{W \mapsto g_1(W)h(S)(q_R(T) - q(T)) : q \in Q_n, h \in \mathcal{H}'_n\} \).

5. The estimated penalty function \( R_{2,n}(\cdot) \) is continuous and it is uniformly consistent, namely, \( \delta_{R_{2,n}} := \sup_{h \in \mathcal{H}} \left[ R_{2,n}(h) - R_2(h) \right] = o_p(1) \).

6. As \( n \to \infty, \mu_{2,n} \to 0 \), max \( \{ \eta_{2,n}^2, \delta_{2,n}^2 \} = o(\mu_{2,n}) \), and \( 1/3 < \lambda_2 < \infty \).

Then \( \hat{q} \) in Equation (28) is a consistent estimator for \( q_R \) in \( L_2 \) norm, i.e.,

\[
\| \hat{q} - q_R \|_2 = o_p(1).
\]

Moreover, there exist universal positive constants \( \tilde{c}_1 \) to \( \tilde{c}_5 \) such that the following holds with probability at least \( 1 - \tilde{c}_1 \exp \left( -\tilde{c}_2 n \eta_{2,n}^2 \right) - \tilde{c}_4 \exp \left( -\tilde{c}_5 n \eta_{2,n}^2 / b^2 \right) \):

\[
\| P^*[\hat{q} - q_R] \|_2^2 \leq 12 \lambda_2 \gamma_{2,n} - 12 \lambda_2 \mu_{2,n} (R_2(q_R) - R_2(\hat{q})) = o_p(\mu_{2,n}),
\]

\[
R_2(\hat{q}) - R_2(q_R) \leq 12 \lambda_2 \gamma_{2,n} - o_p(1),
\]

where \( \gamma_{2,n} = (2 \tilde{c}_3 + \lambda_2 + 11 \tilde{c}_3^2 / 6 \lambda_2) \eta_{2,n}^2 + 3 \delta_{2,n}^2 / \lambda_2 + \mu_{2,n} (R_2(\Pi_n q_R) - R_2(q_R)) + 2 \mu_{2,n} \delta_{R_{2,n}} = o_p(\mu_{2,n}). \)

The conditions and conclusions in Theorem 2 are analogous to those in Theorem 1. The only new one is the requirement that \( m(W; h) \) is Lipschitz in \( h \) in condition 1. This condition trivially holds for Example 2. It also holds for Example 1 if the endogenous treatment variable \( A \) is bounded. And, it holds for Example 3 if the missing outcome \( Y \) is bounded.

5 Debiased Inference on the Parameter of Interest

In Section 4, we propose penalized minimax nuisance estimators and prove their convergence in Theorems 1 and 2. In this section, we show how to use these penalized estimators to construct a debiased machine learning estimator for the target parameter \( \theta^* \) in Equation (1).

Below, we propose a estimator based on cross-fitting the penalized estimators in Equations (23) and (28). This cross-fitting technique is a popular approach to overcome overfitting bias of complex machine learning nuisance estimators [e.g., Chernozhukov et al., 2019, Zheng and Laan, 2011].

Definition 1 (Debiased Machine Learning Estimator). Fix an integer \( K \geq 2 \).
1. Randomly split the data into $K$ (approximately) even folds, whose index sets are denoted by $I_1, \ldots, I_K$, respectively.

2. For $k = 1, \ldots, K$, use all data except those in $I_k$ to construct nuisance estimators $\hat{h}_k$ and $\hat{q}_k$ as described in Equations (23) and (28), respectively.

3. Construct the final debiased machine learning estimator:

$$
\hat{\theta} = \frac{1}{K} \sum_{k=1}^{K} \frac{1}{|I_k|} \sum_{i \in I_k} \psi(W_i; \hat{h}_k, \hat{q}_k), \quad \psi(W; h, q) = m(W; h) + q(T)(g_2(W) - g_1(W)h(S)).
$$

In the remainder of the section we will build toward a theorem to characterize the asymptotic normality of this estimator and then estimate the corresponding asymptotic variance, enabling reliable inference.

Controlling the Stochastic Equicontinuity Term

In Section 3.3, we show that with nonunique nuisances, a certain stochastic equicontinuity term is intractable because nuisance estimators often have no fixed limits. Fortunately, our penalized minimax estimators $\hat{h}_k, \hat{q}_k$ can converge to fixed targets $h_R, q_R$ in $L_2$ norm, even when the nuisances are nonunique. We next show that that when we enjoy such strong-norm convergence then the stochastic equicontinuity term for our estimator $\hat{\theta}$ is negligible.

**Lemma 5.** Suppose that for $k = 1, \ldots, K$, $\|\hat{h}_k - h_R\|_2 \to 0$ and $\|\hat{q}_k - q_R\|_2 \to 0$ in probability as $n \to \infty$. Then

$$
\sqrt{n} \left( \hat{\theta} - \theta^* \right) = \mathcal{G}_n (\psi(W; h_R, q_R) - \theta^*) + \frac{\sqrt{n}}{K} \sum_{k=1}^{K} \Delta_k + o_p(1),
$$

where $\Delta_k = \mathbb{P} \left( \psi(W; \hat{h}_k, \hat{q}_k) - \psi(W; h_R, q_R) \right)$.

Lemma 5 establishes that our estimator behaves like a sample average of $\psi(W; h_R, q_R)$ (i.e., asymptotic linearity), up to some bias terms $\Delta_k$ for $k = 1, \ldots, K$.

Controlling the Bias Terms

To obtain asymptotic linearity, it only remains to show that the bias terms are negligible, i.e., $|\Delta_k| = o_p \left( n^{-1/2} \right)$ for $k = 1, \ldots, K$. Toward that end, we apply Equation (18) to each bias term $\Delta_k$, and obtain the following upper bound.

**Lemma 6.** For $k = 1, \ldots, K$, the bias term $\Delta_k$ in Lemma 5 satisfies that

$$
|\Delta_k| \leq \min \left\{ \|P[h_k - h_R]\|_2 \|\hat{q}_k - q_R\|_2, \|P^*[\hat{q}_k - q_R]\|_2 \|h_k - h_R\|_2 \right\}.
$$

Lemma 6 shows that we need to bound the estimation errors of nuisance estimators $\hat{h}_k, \hat{q}_k$ in terms of both the $L_2$ norm and the projected norm. In Theorems 1 and 2, we already upper bound the projected norm errors and show consistency in terms of the $L_2$ norm error. Thus, if the rate on either $\|P[h_k - h_R]\|_2$ or $\|P^*[\hat{q}_k - q_R]\|_2$ is $O_p(1/\sqrt{n})$, then we are done, as we will obtain $|\Delta_k| = o_p(1)O_p(1/\sqrt{n}) = o_p(1/\sqrt{n})$ and therefore asymptotic linearity. Obtaining such a rate from Theorems 1 and 2, however, requires that $\min \{\delta_{1,n} + \eta_{1,n}, \delta_{2,n} + \eta_{2,n}\} = O(1/\sqrt{n})$, which is generally only possible when either $Q$ or $\mathcal{H}$ are relatively “small” hypothesis classes such as parametric, VC-subgraph, etc.
Ill-Posedness Measures for Better $L_2$ Rates

To go beyond such classes we need to also obtain a rate on the $L_2$ norm error bound, so that we can obtain a $o_p(1/\sqrt{n})$ product of projected and $L_2$ norm errors. However, without further conditions, the $L_2$ norm error may converge to 0 at an arbitrarily slow rate, and no explicit rate is available. To address this problem, a standard approach is to translate the project norm error bound into a rate on the ill-posedness measure via a measure that quantifies the extent of ill-posedness of the inverse problem defined by the conditional moment equation of interest [e.g., Chen and Pouzo, 2012, Dikkala et al., 2020, Blundell et al., 2007]. In the following assumption, we introduce ill-posedness measures for the conditional moment equations in Equations (2) and (13).

**Definition 2** (Ill-posedness Measures with Penalty). Define the following ill-posedness measures for the conditional moment equations in Equations (2) and (13), relative to the hypothesis classes $H_n, Q_n$ and penalty functions $R_1, R_2$:

$$\omega_{H,n}(\gamma, \varepsilon) = \sup_{h \in H_n, 0 < \|P[h-h_R]\|_2 \leq \gamma} \|h - h_R\|_2, \quad \omega_{Q,n}(\gamma, \varepsilon) = \sup_{q \in Q_n, 0 < \|P[q-q_R]\|_2 \leq \gamma} \|q - q_R\|_2$$  \hspace{1cm} (32)

Importantly, the ill-posedness measures in Definition 2 are defined relative to both the hypothesis classes $H_n, Q_n$ and the penalty functions $R_1, R_2$. It is well-known that either restricting the hypothesis classes $H_n, Q_n$ or imposing penalization can regularize the inverse problems and reduce the extent of ill-posedness [Chen and Pouzo, 2012, Carrasco et al., 2007]. Standard ill-posedness measures in the existing literature for conditional moment restrictions with unique solutions usually only take into account the hypothesis class restrictions [e.g., Chen and Pouzo, 2012, Dikkala et al., 2020, Blundell et al., 2007]. In particular, the ill-posed measure in Chen and Pouzo [2012] can be viewed as a special example of our proposed ill-posedness measures with $\varepsilon = +\infty$ and a linear sieve hypothesis class. We next show that assuming well-behavedness in terms of ill-posedness without penalization information necessarily implies the conditional moment restrictions have unique solutions, and therefore these measures are implausible in our settings.

**Lemma 7.** Let $\{H_n\}$ be a sequence of convex sets satisfying $H_n \subseteq H_{n+1}$. Suppose $|H_0 \cap H_n| > 1$ and $H_n \cap \mathcal{N}(P) \neq \emptyset$. Then for any positive sequence $\gamma_n \to 0$, $\omega_{H,n}(\gamma_n, +\infty) \neq 0$.

Let $\{Q_n\}$ be a sequence of convex sets satisfying $Q_n \subseteq Q_{n+1}$. Suppose $|Q_0 \cap Q_n| > 1$ and $Q_n \cap \mathcal{N}(P^*) \neq \emptyset$. Then for any positive sequence $\gamma_n \to 0$, $\omega_{Q,n}(\gamma_n, +\infty) \neq 0$.

According to Lemma 7, if we do not incorporate the penalization information in the ill-posedness measures (i.e., $\varepsilon = +\infty$ in Equation (32)), then consistency in terms of the projected norm (i.e., $\gamma_n \to 0$) cannot be translated into consistency in terms of the $L_2$ norm (i.e., $\omega_{H,n}(\gamma_n, +\infty) \neq 0$ and $\omega_{Q,n}(\gamma_n, +\infty) \neq 0$). Conversely, if we assume that the ill-posedness measures without penalization are well behaved (i.e., $\omega_{H,n}(\gamma_n, +\infty) \to 0$ and $\omega_{Q,n}(\gamma_n, +\infty) \to 0$), then as a consequence $h_R = h^*$ and $q_R = q^*$ must be unique solutions to the moment restrictions in eqs. (2) and (13). The same is true if we assume a finite ratio ill-posedness measure: assuming $\sup_{h \in H_n} \|h - h_R\|_2/\|P[h - h_R]\|_2 < \infty$ would also imply that eq. (2) uniquely identified $h^*$. Therefore, the existing ill-posedness measures either fail to to characterize the key consistency property of our penalized estimator or they fail to capture the possibility of underidentified nuisances.

Below we show this can be fixed by further incorporating the penalization information, as we do in definition 2. Therefore, we expect that our ill-posedness measures can lead to more meaningful statistical guarantees in the setting with nonunique nuisances.

**Lemma 8.** Suppose Assumption 2 and the conditions in Lemma 4 hold and that $H_n, Q_n$ are compact. Then, for any positive sequence $\gamma_n \to 0$ and $\varepsilon_n \to 0$, we have $\omega_{H,n}(\gamma_n, \varepsilon_n) \to 0$ and $\omega_{Q,n}(\gamma_n, \varepsilon_n) \to 0$. 

The Asymptotic Characterization of the Debiased Estimator

Given the ill-posedness measures in Definition 2, we can now translate the projected norm error bounds in Theorems 1 and 2 into $L_2$ norm error bounds. Under conditions in Theorems 1 and 2, Equations (26) and (31) show that our nuisance estimator $\hat{h}_k, \hat{q}_k$ satisfy $R_1(h_R) - R_1(h_R) \leq 12\lambda\gamma_{1,n}/\mu_{1,n}$ and $R_2(q_R) - R_2(q_R) \leq 12\lambda\gamma_{2,n}/\mu_{2,n}$ with high probability. Accordingly, we have the following $L_2$ norm error bounds with high probability:

$$\||h_k - h_R\||_2 \leq \omega_{H,n} (\|P[h_k - h_R]\|_2, 12\lambda\gamma_{1,n}/\mu_{1,n}), \||q_k - q_R\|_2 \leq \omega_{Q,n} (\|P[q_k - q_R]\|_2, 12\lambda\gamma_{2,n}/\mu_{2,n}).$$

Moreover, Equations (25) and (30) in Theorems 1 and 2 can guarantee that $\|P[h_k - h_R]\| = o_p(\sqrt{\mu_{1,n}})$ and $\|P[q_k - q_R]\| = o_p(\sqrt{\mu_{2,n}})$. By combining these results with Lemma 6, we can provide conditions that guarantee the bias terms $\{\Delta_k\}_{k=1}^K$ are asymptotically negligible, and finally establish the asymptotic normality of our proposed estimator.

**Theorem 3.** Suppose the conditions of Theorems 1 and 2 hold. Further assume that

$$\min \{\omega_{H,n}(\sqrt{\mu_{1,n}}, 12\lambda\gamma_{1,n}/\mu_{1,n})\sqrt{\mu_{2,n}}, \omega_{Q,n}(\sqrt{\mu_{2,n}}, 12\lambda\gamma_{2,n}/\mu_{2,n})\sqrt{\mu_{1,n}}\} = O(1/\sqrt{n}),$$

$V := \mathbb{E}\left[(\psi(W; h_R, q_R) - \theta^*)^2\right] < \infty.$

Then, the estimator $\hat{\theta}$ in Definition 1 satisfies that, as $n \to \infty$

$$\sqrt{n}(\hat{\theta} - \theta^*) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\psi(W_i; h_R, q_R) - \theta^*\right] + o_p(1) \sim \mathcal{N}(0, V).$$

Equation (33) permits a tradeoff in the rates for $\hat{h}$ and $\hat{q}$. As discussed below Lemma 6, this condition is automatically satisfied when the projected norm error rates are $O_p(1/\sqrt{n})$ (as a consequence of Lemma 8), as would be obtained, for example, when one hypothesis class is parametric. For more complex hypothesis classes, Equation (33) depends on both the projected norm error rates (as determined by the the classes’ complexity and approximation error) and the ill-posedness of the conditional moment equations in Equations (2) and (13). Notably, there is a tradeoff for the penalization parameters: smaller penalization parameters (provided that they satisfy condition 6 in each of Theorems 1 and 2) lead to faster projected-norm convergence of $\hat{h}_k, \hat{q}_k$, but they also correspond to more severe ill-posedness. When the penalization parameters are properly chosen so that conditions 6 in Theorems 1 and 2 and Equation (33) hold, our proposed estimator $\hat{\theta}$ is asymptotically normal.

Variance Estimation and Calibrated Confidence Intervals

To construct confidence intervals based on the asymptotic normality in eq. (35), we also need to estimate the variance, $V$. We can easily construct a plug-in estimator for it:

$$\hat{V} = \frac{1}{K} \sum_{k=1}^K \frac{1}{|I_k|} \sum_{i \in I_k} \left[\psi(W_i; \hat{h}_k, \hat{q}_k) - \hat{\theta}\right]^2,$$

Below we show that this variance estimator is consistent and we can use it to construct asymptotically calibrated confidence intervals.

**Theorem 4.** Suppose the conditions in Theorem 3 hold. Then as $n \to \infty$,
1. The variance estimator \( \hat{V} \) in Equation (36) converges to the asymptotic variance \( V \) in Equation (34) in probability.

2. The probability of the confidence interval \( \text{CI} = [\hat{\theta} - \Phi^{-1}(1 - \alpha/2) \hat{V}, \hat{\theta} + \Phi^{-1}(1 - \alpha/2) \hat{V}] \) covering the true parameter value \( \theta^* \) converges to \( 1 - \alpha \), where \( \Phi^{-1} \) is the quantile function of the standard normal distribution.

6 Conclusions and Future Directions

In this paper, we study estimation of and inference on uniquely identified linear functionals of underidentified nuisance functions. This challenge arises in a variety of applications in causal inference and missing data, including instrumental variable analysis, proximal causal inference under unmeasured confounding, and shadow variables for missing-not-at-random data. We propose penalized minimax estimators for the unknown nuisances, and use them to construct debiased machine learning estimators for the functionals of interest. Our penalized nuisance estimators can converge to fixed limits in terms of the \( L_2 \) norm error even when the nuisances are underidentified. Importantly, these nuisance estimators can accommodate a wide variety of flexible machine learning methods like RKHS methods or neural networks. We further provide high level conditions under which our functional estimator has an asymptotic normal distribution. We finally propose a consistent variance estimator, and use it to construct an asymptotically valid confidence interval.

There are several interesting future directions of research. Our paper currently focuses on linear functionals of nuisance functions defined by linear conditional moment equations. It would be interesting to explore more general nonlinear functionals and/or nonlinear conditional moment equations without requiring point identification of the nuisances. For example of the former, we may consider inference on consumer surplus and deadweight loss functionals of a demand function estimated using an IV for an endogenous price [e.g., Chen and Christensen, 2018]. For example of the latter, we may consider functionals of nonparametric IV quantile regressions [e.g., Example 3.3 in Ai and Chen, 2009]. One important challenge in this direction is to establish the identifiability of the functionals of interest [Chen et al., 2014]. Moreover, our paper studies point-identified functionals (though the nuisances are not identified). We may relax the identification restriction on the functionals, and instead study partial identification bounds of the functionals [Escanciano and Li, 2013]. In particular, debiased inference on partial identification bounds is an area of growing interest [Dorn et al., 2021, Kallus et al., 2022, Kallus, 2022, Yadlowsky et al., 2018].

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A Proofs

A.1 Supporting Lemmas

Lemma 9. The following three conditions are equivalent: (i) the function $h^*$ is identifiable, (ii) $\mathcal{N}(P) = \{0\}$, (iii) $\mathbb{E}[m(W; h^*)]$ is identifiable for every function $m$ such that $\mathbb{E}[m(W; \cdot)]$ is continuous linear.

Lemma 10 (Theorem 14.1 in Wainwright [2019]). Given a star-shaped and $b$-uniformly bounded function class $\mathcal{G}$, let $\eta_n$ be any positive solution of the inequality $R_n(\mathcal{G}^0) \leq \eta^2 / b$. Then there exist universal positive constants $c_1, c_2$, such that for any $t \geq \eta_n$, we have

$$||g||_n^2 - ||g||_2^2 \leq \frac{1}{2} ||g||_2^2 + \frac{1}{2} t^2, \ \forall g \in \mathcal{G},$$

with probability at least $1 - c_1 \exp\left(-c_2 \frac{t^2}{n}\right)$.

Lemma 11 (Lemma 11 in Foster and Syrgkanis [2019]). Let $\mathcal{F} : \mathcal{X} \to \mathbb{R}^d$ be a 1-uniformly bounded function class, whose $t$th coordinate projection is denoted as $\mathcal{F}_t$. Let $\ell : \mathbb{R}^d \times \mathcal{Z} \to \mathbb{R}$ be a loss function Lipschitz in its first argument with a Lipschitz constant $L$. We receive an i.i.d. sample set $S = \{Z_1, \ldots, Z_n\}$. Let $L_f$ denote the random variable $\ell(f(X), Z)$ and let

$$\mathbb{P}_nL_f = \mathbb{E}[\ell(f(X), Z)], \quad \text{and} \quad \mathbb{P}_nL_f = \frac{1}{n} \sum_{i=1}^{n} \ell(f(X_i), Z_i) .$$

There exists universal positive constants $c_1, c_2, c_3$ such that for any $\delta_n^2 \geq \frac{4d \log(41 \log(2c_1 n))}{c_1 n}$ that solves the inequalities $\mathbb{R}(\mathcal{F}_t - f^*_t, \delta) \leq \delta^2$ for any $t \in \{1, \ldots, d\}$, we have

$$|\mathbb{P}_n(L_f - L_f^*) - \mathbb{P}(L_f - L_f^*)| \leq 18Ld\delta_n \left\{ \sum_{t=1}^{d} ||f_t - f^*_t||_2 + \delta_n \right\}, \ \forall f \in \mathcal{F},$$

with probability at least $1 - c_2 \exp\left(-c_3 n \delta^2\right)$.

A.2 Proofs for Section 3

Proof for Lemma 2. First, note that

$$\theta^* = \mathbb{E}[m(W; h^*)] = \mathbb{E}[\alpha(S)h^*(S)] = \mathbb{E}[\alpha(S)h_0(S)] + \mathbb{E}[\alpha(S)(h^*(S) - h_0(S))].$$

According to the definition of $H_0$, we have that $h^* - h_0 \in \mathcal{N}(P)$. Because we assumed that $\alpha \in \mathcal{R}(P^*) \subseteq \mathcal{N}(P)^\perp$, we must have $\mathbb{E}[\alpha(S)(h^*(S) - h_0(S))] = 0$. Therefore,

$$\theta^* = \mathbb{E}[\alpha(S)h_0(S)] = \mathbb{E}[m(W; h_0)].$$

Second, note that

$$\mathbb{E}[q_0(T)g_2(W)] = \mathbb{E}[q_0(T)\mathbb{E}[g_2(W) | T]] = \mathbb{E}[q_0(T)\mathbb{E}[g_1(W)h_0(S) | T]] = \mathbb{E}[\mathbb{E}[g_1(W)q_0(T) | S]h_0(S)] = \mathbb{E}[\alpha(S)h_0(S)] = \mathbb{E}[m(W; h_0)] = \theta^*. $$
Third, note that
\[
\mathbb{E}[m(W; h_0) + q_0(T)(g_2(W) - g_1(W)h_0(S))] \\
= \mathbb{E}[m(W; h_0)] + \mathbb{E}[q_0(T)\mathbb{E}[g_2(W) - g_1(W)h_0(S) | T]] \\
= \mathbb{E}[m(W; h_0)] = \theta^*.
\]

Proof for Lemma 3. We first prove Equation (18). Note that for any \( h \in \mathcal{L}_2(S) \), \( q \in \mathcal{L}_2(T) \) and \( h_0 \in \mathcal{H}_0, q_0 \in \mathcal{Q}_0 \),
\[
\mathbb{E} [\psi(W; h, q)] - \theta^* = \mathbb{E} [\psi(W; h, q)] - \mathbb{E} [\psi(W; h_0, q_0)] \\
= \mathbb{E} [m(W; h - h_0)] + \mathbb{E} [q(T)(g_2(W) - g_1(W)h(S))] \\
= \mathbb{E} [\alpha(S)(h(S) - h_0(S))] + \mathbb{E} [q(T)(\mathbb{E}[g_2(W) | T] - g_1(W)h(S))] \\
= \mathbb{E} [g_1(W)q(T)(h(S) - h_0(S))] + \mathbb{E} [g_1(W)q(T)(h_0(S) - h(S))] \\
= \mathbb{E} [g_1(W)q(T) - q_0(T)](h(S) - h_0(S)).
\]

Now we prove Equation (19). Note that
\[
\frac{\partial}{\partial t} \mathbb{E} [\psi(W; h_0 + th, q_0)] \big|_{t=0} = \frac{\partial}{\partial t} \{ \mathbb{E} [\psi(W; h_0 + th, q_0)] - \theta^* \} \big|_{t=0} = 0,
\]
where we use the fact that \( \mathbb{E} [\psi(W; h_0 + th, q_0)] - \theta^* = 0 \) for any \( t \in \mathbb{R} \) and \( h \in \mathcal{L}_2(S) \) according to Equation (18). We can similarly show that
\[
\frac{\partial}{\partial t} \mathbb{E} [\psi(W; h_0, q_0 + tq)] \big|_{t=0} = 0.
\]

A.3 Proofs for Section 4

Proof for Lemma 4. We prove this conclusion by contradiction. Suppose \( \lim_{\mu_1 \downarrow 0} h(\mu_1) \neq h_R \), then there exists \( \epsilon > 0 \) and \( \{h_{\mu_1,k}\}_{k=1}^{\infty} \) such that \( \mu_{1,k} \downarrow 0 \) but \( \|h_{\mu_1,k} - h_R\|_2 \geq \epsilon \). Since \( \mathcal{H} \) is compact, without loss of generality, we can assume that \( \{h_{\mu_1,k}\}_{k=1}^{\infty} \) converges to a limit \( h^\dagger \) such that \( \|h^\dagger - h_R\|_2 \geq \epsilon \). Moreover, by the definition of \( h_{\mu_1,k} \),
\[
\frac{1}{\mu_{1,k}} \mathbb{E} \left[ \left( \mathbb{E} \left[ Y - h_{\mu_1,k}(S) | T \right] \right)^2 \right] + R_1(h_{\mu_1,k}) - R_1(h_R) \leq 0, \forall k.
\]
We now show that inequality is impossible. We consider two possible cases.

• First, if \( \mathbb{E} \left[ \left( \mathbb{E} \left[ Y - h^\dagger(S) | T \right] \right)^2 \right] \neq 0 \), then \( R_1(h_{\mu_1,k}) - R_1(h_R) \to R_1(h^\dagger) - R_1(h_R) \), and \( \mathbb{E} \left[ \left( \mathbb{E} \left[ Y - h_{\mu_1,k}(S) | T \right] \right)^2 \right] / \mu_{1,k} \to +\infty \). So the inequality Equation (37) above must be violated for large enough \( k \).

• Second, if \( \mathbb{E} \left[ \left( \mathbb{E} \left[ Y - h^\dagger(S) | T \right] \right)^2 \right] = 0 \), then \( h^\dagger \in \mathcal{H}_0 \cap \mathcal{H} \), and
\[
\liminf_{k \to \infty} \frac{1}{\mu_{1,k}} \mathbb{E} \left[ \left( \mathbb{E} \left[ Y - h_{\mu_1,k}(S) | T \right] \right)^2 \right] + R_1(h_{\mu_1,k}) - R_1(h_R) \geq R_1(h^\dagger) - R_1(h_R) > 0.
\]

The final strict inequality holds because \( h^\dagger \neq h_R \) and \( R_1 \) has a unique minimizer over \( \mathcal{H}_0 \cap \mathcal{H} \). This means that Equation (37) must be also violated for large enough \( k \).
We use \( \Psi_n(h, q) = \frac{1}{n} \sum_{i=1}^{n} q(T_i) (g_2(W_i) - g_1(W_i)h(S_i)), \)
\( \Psi(h, q) = \mathbb{E} [\Psi_n(h, q)] = \mathbb{E} [g_1(W)q(T) (h_R(S) - h(S))]. \)

Note that
\[
|\Psi_n(h, q) - \Psi(h, q)| \leq \left| \frac{1}{n} \sum_{i=1}^{n} q(T_i) (g_2(W_i) - g_1(W_i)h_R(S_i)) - g_1(W_\lambda)q(T) (h_R(S) - h(S)) \right|.
\]

Given that \( \eta_{1,n} \) upper bounds the critical radius of the star hull of
\( \{ W \mapsto g_1(W) (h(S) - h_R(S)) q(T) : q \in \mathcal{Q}'_n, h \in \mathcal{H}_n \}, \)
Lemma 11 implies that with probability at least \( 1 - c_1 \exp (-c_2\eta_{1,n}^2), \forall q \in \mathcal{Q}'_n, h \in \mathcal{H}_n, \)
\[
\left| \frac{1}{n} \sum_{i=1}^{n} q(T_i) (g_2(W_i) - g_1(W_i)h_R(S_i)) - g_1(W_\lambda)q(T) (h_R(S) - h(S)) \right| \leq 18\eta_{2,n} \| g_1(W) (h(S) - h_R(S)) q(T) \|_2 + 18\eta_{1,n}^2.
\]

Moreover, since \( \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} q(T_i) (g_2(W_i) - g_1(W_i)h_R(S_i)) \right] = 0, q(T) (g_2(W) - g_1(W)h_R(S)) \) is Lipschitz in \( q \) with a lipschitz constant \( \sqrt{b^2 + 1} \), and \( \eta_{1,n} \) upper bounds the critical radius of \( \mathcal{Q}'_n \), Lemma 11 implies that with probability at least \( 1 - c_1 \exp (-c_2\eta_{1,n}^2), \forall q \in \mathcal{Q}'_n, \)
\[
\left| \frac{1}{n} \sum_{i=1}^{n} q(T_i) (g_2(W_i) - g_1(W_i)h_R(S_i)) - g_1(W_\lambda)q(T) (h_R(S) - h(S)) \right| \leq 18\sqrt{b^2 + 1} \eta_{1,n} \| q \|_2 + 18\eta_{1,n}^2.
\]

Therefore, with probability at least \( 1 - 2c_1 \exp (-c_2\eta_{1,n}^2), \forall h \in \mathcal{H}_n, q \in \mathcal{Q}'_n, \)
\[
|\Psi_n(h, q) - \Psi(h, q)| \leq c_3 \eta_{2,n} \| h \|_2 + c_3 \eta_{2,n}^2, \quad c_3 = 36b + 18\sqrt{b^2 + 1}.
\] (38)

Moreover, according to Lemma 10, we have that with probability at least \( 1 - c_4 \exp (-c_5\eta_{1,n}^2/b^2), \)
\[
\| q \|_n^2 - \| q \|_2^2 \leq \frac{1}{2} \| q \|_2^2 + \frac{1}{2} \eta_{1,n}^2, \quad \forall q \in \mathcal{Q}'_n.
\] (39)

We use \( \mathcal{E} \) to denote the event that both Equations (38) and (39) hold. So \( \mathbb{P}(\mathcal{E}^C) \leq c_1 \exp (-c_2\eta_{1,n}^2) + c_4 \exp (-c_5\eta_{1,n}^2/b^2) \to 0 \) as \( n \to \infty. \)

Let \( \mathcal{H}_n(\epsilon) = \{ h \in \mathcal{H}_n : \| h - h_R \|_2 \geq \epsilon \} \) and \( \mathcal{H}(\epsilon) = \{ h \in \mathcal{H} : \| h - h_R \|_2 \geq \epsilon \}. \) By the definition of \( \mathcal{H} \) and the fact that \( \mathcal{H}_n(\epsilon) \subseteq \mathcal{H}(\epsilon), \) we have that
\[
\mathbb{P} \left( \| h - h_R \|_2 \geq \epsilon \right) \leq \mathbb{P} \left( \inf_{h \in \mathcal{H}_n(\epsilon)} \sup_{q \in \mathcal{Q}'_n} \Psi_n(h, q) - \lambda \| q \|_n^2 + \mu_{1,n} R_{1,n}(h) \leq \sup_{q \in \mathcal{Q}'_n} \Psi_n(\Pi_n h_R, q) - \lambda \| q \|_n^2 + \mu_{1,n} R_{1,n}(\Pi_n h_R) \right)
\leq \mathbb{P} \left( \inf_{h \in \mathcal{H}(\epsilon)} \sup_{q \in \mathcal{Q}'_n} \Psi_n(h, q) - \lambda \| q \|_n^2 + \mu_{1,n} R_{1,n}(h) \leq \sup_{q \in \mathcal{Q}'_n} \Psi_n(\Pi_n h_R, q) - \lambda \| q \|_n^2 + \mu_{1,n} R_{1,n}(\Pi_n h_R) \right).
\]
We will show that when $\mathcal{E}$ holds, for any $\epsilon > 0$, the probability of the event in the right hand side of the inequality above converges to 0. This, together with the vanishing probability of $\mathbb{P}(\mathcal{E}^C)$, in turn implies that for any $\epsilon > 0$, $\mathbb{P}(\|\hat{h} - R\|_2 \geq \epsilon) \to 0$ as $n \to \infty$.

Since $\mathcal{Q}_n'$ is a compact function class, by Birge’s maximum theorem, $\sup_{q \in \mathcal{Q}_n} \Psi_n(h, q) - \lambda \|q\|^2_n + \mu_1, n R_{1,n}(h)$ is continuous in $h$. Moreover, $\mathcal{H}(\epsilon)$ is also compact, so there exists $h_{n, \epsilon} \in \mathcal{H}(\epsilon)$ that attains the infimum in $\inf_{h \in \mathcal{H}(\epsilon)} \sup_{q \in \mathcal{Q}_n} \Psi_n(h, q) - \lambda \|q\|^2_n + \mu_1, n R_{1,n}(h)$ for any $n$. Again, because $\mathcal{H}(\epsilon)$ is compact, there exist a subsequence $\{h_{n, \epsilon}, \ldots, h_{n_k, \epsilon}, \ldots\} \subseteq \{h_1, \ldots, h_n, \ldots\}$ and $h_\epsilon \in \mathcal{H}(\epsilon)$ such that $\|h_{n_k, \epsilon} - h_\epsilon\|_2 \to 0$ as $k \to \infty$.

Consider

$$\inf_{h \in \mathcal{H}(\epsilon)} \sup_{q \in \mathcal{Q}_n} \Psi_n(h, q) - \lambda \|q\|^2_n + \mu_1, n R_{1,n}(h) = \sup_{q \in \mathcal{Q}_n} \Psi_n(h_{n, \epsilon}, q) - \lambda \|q\|^2_n + \mu_1, n R_{1,n}(h_{n, \epsilon})$$

$$\leq \sup_{q \in \mathcal{Q}_n} \Psi_n(\Pi_n h_R, q) - \lambda \|q\|^2_n + \mu_1, n R_{1,n}(\Pi_n h_R).$$

The estimation error of the penalty is given as follows:

$$(R_{1,n} (\Pi_n h_R) - R_1 (\Pi_n h_R)) - (R_{1,n} (h_{n, \epsilon}) - R_1 (h_{n, \epsilon})) \leq 2\delta_{1,n} := 2 \sup_{h \in \mathcal{H}} |R_{1,n}(h) - R_1(h)| = o_p(1).$$

Under the event $\mathcal{E}$, we have

$$\sup_{q \in \mathcal{Q}_n} \Psi(h_{n, \epsilon}, q) - 1.5\lambda \|q\|^2_2 - c_3 \eta_{1,n} \|q\|_2 + \mu_1, n (R_1(h_{n, \epsilon}) - R_1(h_R))$$

$$\leq \sup_{q \in \mathcal{Q}_n} \Psi(\Pi_n h_R, q) - 0.5\lambda \|q\|^2_2 + c_3 \eta_{1,n} \|q\|_2 + (2\lambda + \lambda) \eta_{1,n}^2 + \mu_1, n (R_1(\Pi_n h_R) - R_1(h_R)) + 2\mu_1, n \delta_{R_1,n}$$

$$\leq \sup_{q \in \mathcal{Q}_n} \Psi(\Pi_n h_R - h_R, q) - 0.5\lambda \|q\|^2_2 + c_3 \eta_{1,n} \|q\|_2 + (2\lambda + \lambda) \eta_{1,n}^2 + \mu_1, n (R_1(\Pi_n h_R) - R_1(h_R)) + 2\mu_1, n \delta_{R_1,n}$$

$$= (c_3 \eta_{1,n} + \delta_{1,n})^2 / (2\lambda) + (2\lambda + \lambda) \eta_{1,n}^2 + \mu_1, n (R_1(\Pi_n h_R) - R_1(h_R)) + 2\mu_1, n \delta_{R_1,n}$$

where the second inequality follows from $\Psi(h, q) = 0$, the third inequality follows from $\Psi(\Pi_n h_R - h_R, q) \leq \|\Pi_n h_R - h_R\|_2 \|q\|_2 \leq \delta_{1,n} \|q\|_2$, and the fourth inequality follows from the fact that for any $a > 0, b > 0$, $a \|q\|_2 - b \|q\|_2^2 \leq a^2 / 4b$.

Moreover, we have $P[h_{n, \epsilon} - h_R] \in \mathcal{Q}_n$. Accordingly, there exist $\Pi_n P[h_{n, \epsilon} - h_R] \in \mathcal{Q}_n$. By the star-shape of $\mathcal{Q}_n$, we have $1/3\lambda \Pi_n P[h_{n, \epsilon} - h_R] \in \mathcal{Q}_n$. Therefore,

$$\sup_{q \in \mathcal{Q}_n} \Psi(h_{n, \epsilon}, q) - 1.5\lambda \|q\|^2_2 - c_3 \eta_{1,n} \|q\|_2$$

$$\geq \Psi(h_{n, \epsilon}, 1/3\lambda \Pi_n P[h_{n, \epsilon} - h_R]) - 1.5\lambda \|1/3\lambda \Pi_n P[h_{n, \epsilon} - h_R]\|^2_2 - c_3 \eta_{1,n} \|1/3\lambda \Pi_n P[h_{n, \epsilon} - h_R]\|_2.$$

Denote $A = P[h_{n, \epsilon} - h_R]$ and $B = \Pi_n P[h_{n, \epsilon} - h_R] - P[h_{n, \epsilon} - h_R]$. Then $A + B = \Pi_n P[h_{n, \epsilon} - h_R]$ and $\|B\|_2 \leq \delta_{1,n}$. Note that

$$\Psi(h_{n, \epsilon}, 1/3\lambda \Pi_n P[h_{n, \epsilon} - h_R]) = \Psi(h_{n, \epsilon}, A) + \Psi(h_{n, \epsilon}, B) = \frac{1}{3\lambda} \|A\|^2_2 + \frac{1}{3\lambda} E[AB]$$

$$\geq \frac{1}{3\lambda} \|A\|^2_2 - \frac{1}{3\lambda} \|A\|_2 \|B\|_2.$$
It follows that
\[ \sup_{q \in \mathcal{Q}_n} \Psi(h_{n, \epsilon}, q) - 1.5 \lambda \|q\|_2^2 - c_3 \eta_{1,n} \|q\|_2 \]
\[ \geq \frac{1}{3\lambda} \|A\|^2 - \frac{1}{3\lambda} \|A\|_2 \|B\|_2 - \frac{1}{6\lambda} \|A + B\|^2 - \frac{c_3 \eta_{1,n}}{3\lambda} \|A + B\|_2 \]
\[ \geq \frac{1}{6\lambda} \|A\|^2 - \frac{1}{3\lambda} (c_3 \eta_{1,n} + 2 \|B\|_2) \|A\|_2 - \frac{1}{6\lambda} \|B\|^2 - \frac{c_3 \eta_{1,n}}{3\lambda} \|B\|. \]

Here
\[ \frac{1}{3\lambda} (c_3 \eta_{1,n} + 2 \|B\|_2) \|A\|_2 \leq \frac{1}{6\lambda} \left[ 2 (c_3 \eta_{1,n} + 2 \|B\|_2)^2 + \|A\|_2^2 / 2 \right] \leq \frac{1}{12\lambda} \|A\|^2 + \frac{2}{3\lambda} (c_3^2 \eta_{1,n}^2 + 4 \|B\|_2^2) \]
and
\[ \frac{c_3 \eta_{1,n}}{3\lambda} \|B\| \leq \frac{1}{6\lambda} (c_3^2 \eta_{1,n}^2 + \|B\|_2^2). \]

Therefore,
\[ \sup_{q \in \mathcal{Q}_n} \Psi(h_{n, \epsilon}, q) - 1.5 \lambda \|q\|_2^2 - c_3 \eta_{1,n} \|q\|_2 \geq \frac{1}{12\lambda} \|A\|^2 - \frac{3}{\lambda} \|B\|^2 - \frac{5}{6\lambda} c_3 \eta_{1,n}^2 \]
\[ \geq \frac{1}{12\lambda} \|P[h_{n, \epsilon} - h_R]\|^2 - \frac{3}{\lambda} \eta_{1,n}^2 - \frac{5}{6\lambda} c_3 \eta_{1,n}^2 \]

Therefore, we have that
\[ \mathbb{P} \left( \left\{ \|\hat{h} - h_R\|_2 > \epsilon \right\} \cap \mathcal{E} \right) \]
\[ \leq \mathbb{P} \left( \frac{1}{12\lambda} \|P[h_{n, \epsilon} - h_R]\|^2 + \mu_{1,n} (R_1(h_{n, \epsilon}) - R_1(h_R)) \right) \]
\[ \leq \left( 2c_3 + \lambda + 11c_3^2 / 6\lambda \right) \eta_{1,n}^2 + 3 \delta_{1,n}^2 / \lambda + \mu_{1,n} (R_1(\Pi_n h_R) - R_1(h_R)) + 2 \mu_{1,n} \delta_{R_1,n} \right) \]
\[ \leq \mathbb{P} \left( \frac{1}{12\lambda} \|P[h_{n, \epsilon} - h_R]\|^2 + \mu_{1,n} (R_1(h_{n, \epsilon}) - R_1(h_R)) \leq \gamma_n \right), \]
where
\[ \gamma_n = (2c_3 + \lambda + 11c_3^2 / 6\lambda) \eta_{1,n}^2 + 3 \delta_{1,n}^2 / \lambda + \mu_{1,n} (R_1(\Pi_n h_R) - R_1(h_R)) + 2 \mu_{1,n} \delta_{R_1,n}. \]

According to our assumptions, we have \( \eta_{1,n}^2 = \mathcal{O}(\mu_{1,n}) \) and \( \delta_{1,n}^2 = \mathcal{O}(\mu_{1,n}). \) Moreover, \( \|h_R - \Pi_n h_R\|_2 = \mathcal{O}(1) \) so by the continuity of \( R_1, \) we have \( \mu_{1,n} (R_1(\Pi_n h_R) - R_1(h_R)) = \mathcal{O}(\mu_{1,n}). \) Also, \( \delta_{R_1,n} = \mathcal{O}(1) \) according to our assumption, which implies that \( 2 \mu_{1,n} \delta_{R_1,n} = \mathcal{O}(\mu_{1,n}). \) Therefore, \( \gamma_n = \mathcal{O}(\mu_{1,n}). \)

Now consider \( k \to \infty, \mu_{1,n,k} \to 0, \|h_{n,k, \epsilon} - h_{\epsilon}\|_2 \to 0, \) and
\[ \lim_{k \to \infty} \frac{1}{12\lambda} \|P[h_{n,k, \epsilon} - h_R]\|^2 + \mu_{1,n,k} (R_1(h_{n,k, \epsilon}) - R_1(h_R)) \]

There are two cases: either \( \|P[h_{\epsilon} - h_R]\|_2 = 0 \) or \( \|P[h_{\epsilon} - h_R]\|_2 > 0. \) We now analyze each case separately.
When \( \|P[h_\epsilon - h]\|_2 = 0 \), we have \( h_\epsilon \in \mathcal{H}_0 \); but since \( \|h_\epsilon - h\|_2 \geq \epsilon \), we have \( R_1(h_\epsilon) - R_1(h) > 0 \). Therefore,

\[
\lim_{{k \to \infty}} \frac{1}{12\lambda} \frac{\|P[h_{n_k, \epsilon} - h]\|_2^2 + \mu_{1,n_k} (R_1(h_{n_k, \epsilon}) - R_1(h))}{\mu_{1,n_k}} \geq R_1(h_\epsilon) - R_1(h) > 0.
\]

When \( \|P[h_\epsilon - h]\|_2 > 0 \), we have

\[
\lim_{{k \to \infty}} \frac{1}{12\lambda} \frac{\|P[h_{n_k, \epsilon} - h]\|_2^2 + \mu_{1,n_k} (R_1(h_{n_k, \epsilon}) - R_1(h))}{\mu_{1,n_k}} = R_1(h_\epsilon) - R_1(h) + \lim_{{k \to \infty}} \frac{1}{12\lambda \mu_{1,n_k}} \|P[h_{n_k, \epsilon} - h]\|_2^2 = +\infty.
\]

In either case, because \( \gamma_n / \mu_{1,n} \) converges to 0 in probability, we always have

\[
\lim_{{n \to \infty}} \Pr \left( \frac{1}{6\lambda} \|P[h_n, \epsilon - h]\|_2^2 + \mu_{1,n} (R_1(h_n, \epsilon) - R_1(h)) \leq \gamma_n \right) = 0.
\]

This shows that for any \( \epsilon > 0 \), \( \Pr \left( \|\hat{h} - h_R\|_2 \geq \epsilon \right) \to 0 \) as \( n \to \infty \). Namely, \( \|\hat{h} - h_R\|_2 = o_p(1) \).

Moreover, following our analysis above, we can analogously show that when \( E \) is true (which holds with high probability),

\[
\frac{1}{12\lambda} \|P[\hat{h} - h]\|_2^2 + \mu_{1,n} (R_1(h_R) - R_1(\hat{h})) \leq \gamma_n
\]

This implies our asserted conclusions. \( \square \)

**Proof for Theorem 2.** Denote

\[
\tilde{\Psi}_n(h, q) = \frac{1}{n} \sum_{i=1}^{n} m(W_i; h) - g_1(W_i)q(T_i)h(S_i),
\]

\[
\tilde{\Psi}(h, q) = \mathbb{E} \left[ \tilde{\Psi}_n(h, q) \right] = \mathbb{E} \left[ m(W; h) - g_1(W)q(T)h(S) \right].
\]

Note that

\[
\left| \tilde{\Psi}_n(h, q) - \tilde{\Psi}(h, q) \right| \leq \left| \frac{1}{n} \sum_{i=1}^{n} m(W_i; h) - g_1(W_i)q_R(T_i)h(S_i) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} g_1(W_i)h(S_i) (q_R(T_i) - q(T_i)) - \tilde{\Psi}(h, q) \right|.
\]

Given that \( \eta_{1,n} \) upper bounds the critical radius of the star hull of

\[
\{ W \mapsto g_1(W)h(S) (q_R(T) - q(T)) : q \in \mathcal{Q}_n, h \in \mathcal{H}_n' \},
\]

Lemma 11 implies that with probability at least \( 1 - \tilde{c}_1 \exp \left( -\tilde{c}_2 \eta_{2,n}^2 \right) \), \( q \in \mathcal{Q}_n, h \in \mathcal{H}_n' \),

\[
\left| \frac{1}{n} \sum_{i=1}^{n} (q(T_i) - q_R(T_i)) h(S_i) - \tilde{\Psi}(h, q) \right| \leq 18\eta_{2,n} \parallel (q(T) - q_R(T))h(S) \parallel_2 + 18\eta_{2,n}^2 \leq 36b\eta_{2,n} \parallel h(S) \parallel_2 + 18\eta_{2,n}^2.
\]

32
Moreover, since $\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} m(W_i; h) - g_1(W)q_R(T_i)h(S_i) \right] = 0$, $m(W; h) - g_1(W)q_R(T)h(S)$ is Lipschitz in $h$ with a lipschitz constant $\sqrt{b^2 + L_m^2}$, and $\eta_{2,n}$ upper bounds the critical radius of $\mathcal{H}'_n$, Lemma 11 implies that with probability at least $1 - \tilde{c}_1 \exp \left( -\tilde{c}_2 m n_{2,n}^2 \right)$, $\forall h \in \mathcal{H}'_n$,

$$\left| \frac{1}{n} \sum_{i=1}^{n} q_R(T_i)h(S_i) - m(W_i; h) \right| \leq 18 \sqrt{b^2 + L_m^2} \eta_{2,n} \|h\|_2 + 18 n_{2,n}^2.$$  

Therefore, with probability at least $1 - 2\tilde{c}_1 \exp \left( -\tilde{c}_2 m n_{2,n}^2 \right)$, $\forall h \in \mathcal{H}'_n, q \in \mathcal{Q}_n$,

$$\left| \tilde{\Psi}_n(h, q) - \tilde{\Psi}(h, q) \right| \leq \tilde{c}_3 \eta_{2,n} \|h\|_2 + \tilde{c}_3 \eta_{2,n}^2, \quad \tilde{c}_3 = 36b + 18 \sqrt{b^2 + L_m^2}. \quad (40)$$

Moreover, according to Lemma 10, we have that with probability at least $1 - \tilde{c}_4 \exp \left( -\tilde{c}_5 n_{2,n}^2 \right)$,

$$\left| \|h\|_2^2 - \|\hat{h}\|_2^2 \right| \leq \frac{1}{2} \|h\|_2^2 + \frac{1}{2} \|h\|_2^2, \quad \forall h \in \mathcal{H}'_n. \quad (41)$$

We use $\mathcal{E}$ to denote the event that both Equations (40) and (41) hold. So $\mathbb{P} \left( \mathcal{E}^C \right) \leq 2\tilde{c}_1 \exp \left( -\tilde{c}_2 m n_{2,n}^2 \right) + \tilde{c}_4 \exp \left( -\tilde{c}_5 n_{2,n}^2 / b^2 \right) \rightarrow 0$ as $n \rightarrow \infty$.

Let $\mathcal{Q}_n(\epsilon) = \{q \in \mathcal{Q}_n : \|q - q_R\|_2 \geq \epsilon \}$ and $\mathcal{Q}(\epsilon) = \{q \in \mathcal{Q} : \|q - q_R\|_2 \geq \epsilon \}$. By the definition of $q$ and the fact that $\mathcal{Q}_n(\epsilon) \subseteq \mathcal{Q}(\epsilon)$, we have that

$$\mathbb{P} \left( \|\hat{q} - q_R\|_2 \geq \epsilon \right) \leq \mathbb{P} \left( \inf_{q \in \mathcal{Q}_n(\epsilon)} \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, q) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(q) \leq \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, \Pi_n h_R, q) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(\Pi_n q_R) \right)$$

$$\leq \mathbb{P} \left( \inf_{q \in \mathcal{Q}(\epsilon)} \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, q) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(q) \leq \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, \Pi_n h_R, q) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(\Pi_n q_R) \right).$$

We will show that when $\mathcal{E}$ holds, for any $\epsilon > 0$, the probability of the event in the right hand side of the inequality above converges to 0. This, together with the vanishing probability of $\mathbb{P} \left( \mathcal{E}^C \right)$, in turn implies that for any $\epsilon > 0$, $\mathbb{P} (\|\hat{q} - q_R\|_2 \geq \epsilon) \rightarrow 0$ as $n \rightarrow \infty$.

Since $\mathcal{H}'_n$ is a compact function class, by Birge’s maximum theorem, $\sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, q) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(q)$ is continuous in $q$. Moreover, $\mathcal{Q}(\epsilon)$ is also compact, so there exists $q_{n_k,\epsilon} \in \mathcal{Q}$ that attains the infimum in $\inf_{q \in \mathcal{Q}(\epsilon)} \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, q) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(q)$ for any $n$. Again, because $\mathcal{Q}(\epsilon)$ is compact, there exist $\{q_{n_1,\epsilon}, \ldots, q_{n_k,\epsilon}, \ldots \} \subseteq \{q_1, \ldots, q_n, \ldots \}$ and $q_\epsilon \in \mathcal{Q}(\epsilon)$ such that $\|q_{n_k,\epsilon} - q_\epsilon\|_2^2 \rightarrow 0$ as $k \rightarrow \infty$.

Consider

$$\inf_{q \in \mathcal{Q}(\epsilon)} \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, q) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(q) = \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, q_{n,\epsilon}) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(\Pi_n q_{n,\epsilon})$$

$$\leq \sup_{h \in \mathcal{H}'_n} \tilde{\Psi}_n(h, \Pi_n q_R) - \lambda \|h\|_2^2 + \mu_2,n R_{2,n}(\Pi_n q_R).$$

The estimation error of the regularizer the following:

$$(R_{2,n}(\Pi_n q_R) - R_2(\Pi q_R)) - (R_{2,n}(q_{n,\epsilon}) - R_2(q_{n,\epsilon})) \leq 2\delta_{R_{2,n}} := 2 \sup_{q \in \mathcal{Q}} |R_{2,n}(q) - R_2(q)| = o_p(1).$$
By following the proof of Theorem 1, we have that under the event \( \tilde{\mathcal{E}} \),
\[
\mathbb{P}\left( \{\|\hat{q} - q_R\|_2 > \epsilon\} \cap \tilde{\mathcal{E}} \right)
\leq \mathbb{P}\left( \frac{1}{12\lambda} \|P^*[q_{n,\epsilon} - q_R]\|_2^2 + \mu_{2,n} (R_2(q_{n,\epsilon}) - R_2(q_R)) \leq \gamma_{2,n} \right),
\]
where
\[
\gamma_{2,n} = (2\bar{c}_3 + \lambda + 11\bar{c}_3^2 / 6\lambda) n_{2,n}^2 + 3\delta_{2,n}^2 / \lambda + \mu_{2,n} (R_2(\Pi_n q_R) - R_2(q_R)) + 2\mu_{2,n} \delta_{R_2,n}.
\]
Again, by following the analyses in the proof of Theorem 1, we have \( \tilde{\gamma}_n = o_p(\mu_{2,n}) \) and
\[
\liminf_{k \to \infty} \frac{1}{12\lambda} \|P^*[q_{n,k} - q_R]\|_2^2 + \mu_{2,n} (R_2(q_{n,k}) - R_2(q_R)) > 0.
\]
Therefore,
\[
\mathbb{P}\left( \frac{1}{12\lambda} \|P^*[q_{n,\epsilon} - q_R]\|_2^2 + \mu_{2,n} (R_2(q_{n,\epsilon}) - R_2(q_R)) \leq \tilde{\gamma}_n \right) \to 0.
\]
This in turn implies that for any \( \epsilon > 0 \), \( \mathbb{P}(\|\hat{q} - q_R\|_2 \geq \epsilon) \to 0 \) as \( n \to \infty \).

Similarly, we can show that when \( \tilde{\mathcal{E}} \) is true (which holds with high probability),
\[
\frac{1}{12\lambda} \|P^*[\hat{q} - q_R]\|_2^2 + \mu_{2,n} (R_2(\hat{q}) - R_2(q_R)) \leq \gamma_{2,n}.
\]
This immediately implies our asserted conclusions. \( \square \)

A.4 Proofs for Section 5

Proof for Lemma 5. For any function \( f(W) \), denote \( \mathbb{P}f = \int f(w)p(w)\,dw \), and \( \mathbb{P}_{n,k}f = \sum_{i \in I_k} f(W_i) / |I_k| \).

Also denote
\[
\hat{\theta}_k = \frac{1}{|I_k|} \sum_{i \in I_k} \psi(W_i; \hat{h}_k, \hat{q}_k).
\]

We obviously have
\[
\hat{\theta}_k - \theta^* = \mathbb{P}_{n,k} [\psi(W; h_R, q_R) - \theta^*] + (\mathbb{P}_{n,k} - \mathbb{P}) \left[ \psi(W; \hat{h}_k, \hat{q}_k) - \psi(W; h_R, q_R) \right] + \mathbb{P} \left[ \psi(W; \hat{h}_k, \hat{q}_k) - \psi(W; h_R, q_R) \right].
\]

By simple algebra, we can show that
\[
\mathbb{P} \left[ \left( \psi(W; \hat{h}_k, \hat{q}_k) - \psi(W; h_R, q_R) \right)^2 \right] \leq \left( \frac{L^2_{\text{m}} + b^2}{n} \right) \|h_k - h_R\|_2^2 + (b + 1)^2 \|\hat{q} - q\|_2^2 = o_p(1).
\]

Thus by Markov inequality, we have
\[
\left| (\mathbb{P}_{n,k} - \mathbb{P}) \left[ \psi(W; \hat{h}_k, \hat{q}_k) - \psi(W; h_R, q_R) \right] \right| = o_p \left( \frac{1}{\sqrt{|I_k|}} \right) = o_p \left( \frac{1}{\sqrt{n}} \right).
\]
Therefore,
\[
\sqrt{n} \left( \hat{\theta} - \theta^* \right) = \frac{1}{K} \sum_{k=1}^{K} \sqrt{n} \left( \hat{\theta}_k - \theta^* \right)
\]
\[
= \mathbb{G}_n \left( \psi(W; h_R, q_R) - \theta^* \right) + \frac{\sqrt{n}}{K} \sum_{k=1}^{K} \mathbb{P} \left( \psi(W; \hat{h}_k, \hat{q}_k) - \psi(W; h_0, q_0) \right) + o_p(1).
\]

**Proof for Lemma 6.** According to Equation (18) in Lemma 3, we have
\[
|\Delta_k| = \left| \mathbb{P} \left[ g_1(W) (\hat{q}_k(T) - q_R(T)) \left( \hat{h}_k(S) - h_R(S) \right) \right] \right|.
\]
Thus we have
\[
|\Delta_k| = \left| \mathbb{P} \left[ P[\hat{h}_k - h_R](T) (\hat{q}_k(T) - q_R(T)) \right] \right| \leq \|P[\hat{h}_k - h_R]\|_2 \|\hat{q}_k - q_R\|_2,
\]
and
\[
|\Delta_k| = \left| \mathbb{P} \left[ P^*[\hat{h}_k - h_R](T) \left( \hat{h}_k(S) - h_R(S) \right) \right] \right| \leq \|P^*[\hat{q}_k - q_R]\|_2 \|\hat{h}_k - h_R\|_2.
\]
Therefore, we have
\[
|\Delta_k| \leq \min \left\{ \|P[\hat{h}_k - h_R]\|_2 \|\hat{q}_k - q_R\|_2, \|P^*[\hat{q}_k - q_R]\|_2 \|\hat{h}_k - h_R\|_2 \right\}.
\]

**Proof for Lemma 7.** We prove the conclusion for \( \omega_{H,n}(\gamma_n, +\infty) \). Fix \( h_0 \in \mathcal{H}_0 \cap \mathcal{H}_n \) with \( \|h_0 - h_R\|_2 \neq 0 \) and \( h^\perp \in \mathcal{H}_n \cap \mathcal{N}(P)^\perp \). By the assumed monotonicity, we have \( h_0 \in \mathcal{H}_0 \cap \mathcal{H}_{n'} \) and \( h^\perp \in \mathcal{H}_{n'} \cap \mathcal{N}(P)^\perp \) for \( n' \geq n \). By the convexity of \( \mathcal{H}_n \), we have
\[
h_n := (1 - c_n) h_0 + c_n h^\perp \in \mathcal{H}_n, \quad c_n = \frac{\gamma_n}{\|h^\perp - h_R\|_2}.
\]
Obviously, we have \( \|h_n - h_0\|_2 \to 0 \) as \( n \to \infty \). Moreover, we have
\[
\|P[h_n - h_R]\|_2 = \|c_n P[h^\perp - h_R]\|_2 \in (0, \gamma_n).
\]
Moreover, we have
\[
\|h_n - h_R\|_2 \to \|h_0 - h_R\|_2.
\]
It follows that
\[
\omega_{H,n}(\gamma_n, +\infty) \geq \|h_n - h_R\|_2 \to \|h_0 - h_R\|_2 > 0.
\]
This implies the asserted conclusion. The conclusion for \( \omega_{Q,n}(\gamma_n, +\infty) \) can be proved analogously.
Proof for Lemma 8. We prove this conclusion by contradiction. Suppose that \( \omega_{\mathcal{H},n}(\gamma_n, \varepsilon_n) \neq 0 \). Then there exists \( h_n \in \mathcal{H}_n \) such that \( \|P[h_n - h_R]\|_2 \leq \gamma_n \) and \( R_1(h_n) \leq R_1(h_R) + \varepsilon_n \) but \( \|h_n - h_R\|_2 \) does not converge to 0. Since \( \mathcal{H} \) is compact, without loss of generality, we can assume that \( \{h_{n_k}\}_{k=1}^\infty \) converge to a limit \( h^\dagger \) such that \( \|h^\dagger - h_R\|_2 \geq \varepsilon \). Moreover, our assumptions imply that \( \|P[h^\dagger - h_R]\|_2 = 0 \) (i.e., \( h^\dagger \in \mathcal{H}_0 \cap \mathcal{H} \)) and \( R_1(h^\dagger) \leq R_1(h_R) \). But this contradicts Assumption 2 that requires \( h_R \) to be the unique minimizer of \( R_1 \) over \( \mathcal{H}_0 \cap \mathcal{H} \). So we must have \( \omega_{\mathcal{H},n}(\gamma_n, \varepsilon_n) \to 0 \). Similarly we can prove \( \omega_{\mathcal{Q},n}(\gamma_n, \varepsilon_n) \to 0 \).

Proof for Theorem 3. Under conditions in Theorems 1 and 2, the following hold with high probability at least \( 1 - K \left( \bar{c}_1 \exp(-\bar{c}_2 n \eta_{1,n}^2/b^2) - \bar{c}_4 \exp(-\bar{c}_5 n \eta_{1,n}^2/b^2) \right) - K (c_1 \exp(-c_2 n \eta_{1,n}^2) - c_4 \exp(-c_5 n \eta_{1,n}^2/b^2)) \) for \( k = 1, \ldots, K \):

\[
\|P[\hat{h}_k - h_R]\|_2 = o_p\left(\sqrt{n \mu_{1,n}}\right), \quad \|P^*[\hat{q}_k - q_R]\|_2 = o_p\left(\sqrt{n \mu_{2,n}}\right),
\]

and

\[
R_1(\hat{h}_k) - R_1(h_R) \leq 12 \lambda_1 \gamma_{1,n}/\mu_{1,n}, \quad R_2(\hat{q}_k) - R_2(q_R) \leq 12 \lambda_2 \gamma_{2,n}/\mu_{2,n}.
\]

It then follows from Lemma 6 that

\[
|\Delta_k| = o_p\left(\min\left\{\omega_{\mathcal{H},n}(\sqrt{n \mu_{1,n}}, 12 \lambda_1 \gamma_{1,n}/\mu_{1,n}), \omega_{\mathcal{Q},n}(\sqrt{n \mu_{2,n}}, 12 \lambda_2 \gamma_{2,n}/\mu_{2,n})\right\}\right).
\]

Thus under the condition in Equation (33), we have

\[
|\Delta_k| = o_p\left(n^{-1/2}\right).
\]

Then the final conclusion follows from Lemma 5 and the Central Limit Theorem.

Proof for Theorem 4. We only need to prove that \( \hat{V} \to V \) in probability. Once this is true, the Slutsky’s theorem implies that

\[
\frac{\sqrt{n} \left( \hat{\theta} - \theta^* \right)}{\hat{V}} \to \mathcal{N}(0,1).
\]

It follows that as \( n \to \infty \),

\[
P(\theta^* \in \text{CI}) \to 1 - \alpha.
\]

Now we show the consistency of \( \hat{V} \). First, we have

\[
\hat{V} - V = \frac{1}{K} \sum_{k=1}^K \frac{1}{|I_k|} \sum_{i \in I_k} \left[ (\psi(W_i; \hat{h}_k, \hat{q}_k) - \hat{\theta})^2 - (\psi(W_i; h_R, q_R) - \theta^*)^2 \right] + \frac{1}{n} \sum_{i=1}^n (\psi(W_i; h_R, q_R) - \theta^*)^2 - \mathbb{E} \left[ (\psi(W; h_R, q_R) - \theta^*)^2 \right].
\]
Note that
\[
\frac{1}{K} \sum_{k=1}^{K} \frac{1}{|I_k|} \sum_{i \in I_k} \left[ \left( \psi(W_i; \hat{h}_k, \hat{q}_k) - \hat{\theta} \right)^2 - (\psi(W_i; h_R, q_R) - \theta^*)^2 \right] \\
\leq \frac{1}{K} \sum_{k=1}^{K} \frac{1}{|I_k|} \sum_{i \in I_k} \left[ \left( \psi(W_i; \hat{h}_k, \hat{q}_k) - \psi(W_i; h_R, q_R) \right) - (\hat{\theta} - \theta^*) \right] \\
\times \left[ \left( \psi(W_i; \hat{h}_k, \hat{q}_k) - \psi(W_i; h_R, q_R) \right) - (\hat{\theta} - \theta^*) + (2\psi(W_i; h_R, q_R) + 2\theta^*) \right] \\
\leq \frac{1}{K} \sum_{k=1}^{K} \left( \frac{1}{|I_k|} \sum_{i \in I_k} \mathcal{R}_i^2 \right)^{1/2} \left[ \left( \frac{1}{|I_k|} \sum_{i \in I_k} \mathcal{R}_i^2 \right)^{1/2} + \left( \frac{1}{|I_k|} \sum_{i \in I_k} (\psi(W_i; h_R, q_R) + \theta^*)^2 \right)^{1/2} \right],
\]
where
\[
\frac{1}{|I_k|} \sum_{i \in I_k} \mathcal{R}_i^2 \leq \frac{1}{|I_k|} \sum_{i \in I_k} (\psi(W_i; \hat{h}_k, \hat{q}_k) - \psi(W_i; h_R, q_R))^2 + (\hat{\theta} - \theta^*^2).
\]

From Theorem 3, we already know that \(|\hat{\theta} - \theta^*| = O_p(n^{-1/2})\). Moreover,
\[
\mathbb{E} \left[ \left( \psi(W_i; \hat{h}_k, \hat{q}_k) - \psi(W_i; h_R, q_R) \right)^2 \mid (W_i)_{i \in I_k} \right] \\
\leq ||m(W; \hat{h}_k) - m(W; h_R)||^2_2 + ||\hat{q}_k - q_R||^2_2 ||g_2 - g_1 h||^2_2 + ||\hat{h}_k - h_R||_2 \cdot \|q_1\|.
\]

According to Theorems 1 and 2, the last display is \(o_p(1)\). Then by Markov inequality, we have
\[
\frac{1}{|I_k|} \sum_{i \in I_k} \mathcal{R}_i^2 = o_p\left(n^{-1/2}\right).
\]

Since \(\mathbb{E} \left[ (\psi(W; h_R, q_R))^2 \right] < +\infty\), we have
\[
\mathbb{E} \left[ (\psi(W; h_R, q_R) \pm \theta^*)^2 \right] < \infty.
\]

It follows that
\[
\left( \frac{1}{|I_k|} \sum_{i \in I_k} (\psi(W_i; h_R, q_R) + \theta^*)^2 \right) = O_p\left(1\right),
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} (\psi(W_i; h_R, q_R) - \theta^*)^2 - \mathbb{E} \left[ (\psi(W; h_R, q_R) - \theta^*)^2 \right] = O_p\left(n^{-1/2}\right).
\]

Therefore, we have
\[
\hat{V} - V = o_p\left(1\right).
\]