Splitting automorphisms of prime power orders of free Burnside groups

V. S. Atabekyan
March 27, 2014

Abstract

We prove that if the order of a splitting automorphism of free Burnside group $B(m, n)$ of odd period $n \geq 1003$ is a prime power, then the automorphism is inner. Thus, we give an affirmative answer to the question on the coincidence of splitting and inner automorphisms of free Burnside groups $B(m, n)$ for automorphisms of orders $p^k$ ($p$ is a prime number). This question was posed in the Kourovka Notebook in 1990 (see 11th ed., Question 11.36. b).

Keywords: Splitting automorphism, inner automorphism, normal automorphism, free Burnside group

MSC: 20B27, 20F50, 20E36, 20F28

1 Introduction

An automorphism $\varphi$ of a group $G$ is called a splitting automorphism of period $n$, if $\varphi^n = 1$ and $g \varphi g^2 \cdots g^{n-1} = 1$ for all $g \in G$. Various authors studied groups with splitting automorphism. The known theorem of O.Kegel, proved in 1961, states that any finite group possessing a nontrivial splitting automorphism of prime order is nilpotent (see [8]). E.Khukhro proved that any solvable group possessing a nontrivial splitting automorphism of prime order is also nilpotent (see [9]). E.Jabara in [10] established that a finite group with the splitting automorphism of order 4 is solvable.

It is not difficult to check that if $\varphi$ is a splitting automorphism of period $n$ of $G$, then $g^{\varphi^{n-1}} \cdots g^{\varphi^2} g \varphi g = 1$ for any $g \in G$. Let $\varphi$ be a splitting automorphism of period $n$ of some group $G$. Since $(\varphi g)^n = \varphi^n g^{\varphi^{n-1}} \cdots g^{\varphi^2} g \varphi g$, this precisely means that the relation $(\varphi g)^n = 1$ holds.

The author was supported in part by State Committee Science MES RA grant in frame of project 13-1A246
holds in the holomorph \( Hol(G) \) of \( G \) for each \( g \in G \). In particular, the identity automorphism of a given group \( G \) is a splitting automorphism of period \( n \) if and only if the identity \( x^n = 1 \) holds in \( G \).

It is easy to check that if in a given group \( G \) the identity \( x^n = 1 \) is valid, then each inner automorphism of \( G \) is an inner splitting automorphism of period \( n \). However, the converse statement is false. Indeed, let \( F_2 \) be an absolutely free group with free generators \( a \) and \( b \) and let \( C_n \) be a cyclic group of order \( n \) with generator element \( c \). Let us consider the group \( \Gamma = F_2/F_2^n \times C_n \) of period \( n \), where \( F_2^n \) stands for the subgroup generated by all possible \( n \)th powers of the elements of \( F_2 \). It can readily be seen that for any \( n > 1 \), the automorphism \( \alpha : \Gamma \to \Gamma \) given on the generators by the formulae \( \alpha(a) = ac \), \( \alpha(b) = bc \), and \( \alpha(c) = c \), is a splitting automorphism of period \( n \). However, this is an outer automorphism because it is clear that no relation of the form \( u^{-1}au = ac \) can hold in \( \Gamma \).

S.V.Ivanov in [11] posed the following problem: Let \( n \) be large enough odd number and \( m > 1 \). Is it true that each automorphism \( \varphi \) of \( B(m,n) \), that satisfies the relations \( \varphi^n = 1 \) and \( g \varphi g^2 \cdots \varphi^{n-1} = 1 \) for any \( g \) in \( B(m,n) \), is inner (see [11], Question 11.36. b))? In fact, Ivanov’s problem concerns splitting automorphisms of period \( n \) of the groups \( B(m,n) \).

By definition, the free Burnside group \( B(m,n) \) of period \( n \) and rank \( m \) has the following presentation
\[
B(m,n) = \langle a_1, a_2, \ldots, a_m \mid X^n = 1 \rangle,
\]
where \( X \) ranges over the set of all words in the alphabet \( \{a_1^{\pm 1}, a_2^{\pm 1}, \ldots, a_m^{\pm 1}\} \). The group \( B(m,n) \) is a quotient group of the free group \( F_m \) of rank \( m \) by normal subgroup \( F_m^n \), generated by all \( n \)th powers of the elements of \( F_m \). Every periodic group of period \( n \) with \( m \) generators is a quotient group of \( B(m,n) \).

The main Theorem 2.1 of the present paper gives a positive answer to the above mentioned question for all splitting automorphisms of prime power order of free Burnside groups \( B(m,n) \) for odd periods \( n \geq 1003 \). In the paper [6], we established some properties of splitting automorphisms of the groups \( B(m,n) \). Those properties have proved useful in our proof of the main result 2.1. Theorem 2.1 strengthens the result of [6], where the Ivanov’s problem was solved only for automorphisms of prime orders.

### 1.1 On simple periodic groups

S.I.Adian and I.G.Lysenok in [2] proved, that for any \( m > 1 \) and odd \( n \geq 1003 \) there exists a maximal normal subgroup \( N \) of free Burnside group \( B(m,n) \) such that the quotient group \( B(m,n)/N \) is an infinite group, every proper subgroup of which is contained in some cyclic subgroup of order \( n \). The groups constructed in [2] are infinite simple groups in which the
identity relation \(x^n = 1\) holds. This groups are called 'Tarski monsters' since Tarski has formulated a question on the existence of such groups. The first examples of Tarski monsters were constructed by A.Yu.Olshanskii [12] for prime periods \(n > 10^{75}\). It is now known that for every odd \(n \geq 1003\) there are continuum many non-isomorphic Tarski-monsters of period \(n\) (see [3], [13, Theorem 28.7], [4]). Each of these Tarski monsters is a result of a factorization of the group \(B(m, n)\) by some maximal normal subgroup of, which leads to an infinite group containing only cyclic proper subgroups.

We denote by \(M_n\) the set of all normal subgroups \(N \trianglelefteq B(m, n)\) for which the quotient group \(B(m, n)/N\) is a Tarski monster.

The following two statements were proved in [5] (see also [7]) and [6] respectively. They play the key role in the proof of main Theorem 2.1.

**Lemma 1.1.** (see [5, Corrolary 2]) Let \(n \geq 1003\) be an odd number and let \(\varphi\) be an automorphism of \(B(m, n)\) such that for any normal subgroup \(N \in M_n\) the equality \(N^{\varphi} = N\) holds. Then \(\varphi\) is an inner automorphism.

**Lemma 1.2.** (see [6, Lemma 4]) If \(\varphi\) is an arbitrary nontrivial splitting automorphism of period \(n\) of the group \(B(m, n)\), where \(n \geq 1003\) is odd, then the stabilizer of any normal subgroup \(N \in M_n\) under the action of the cyclic group \(\langle \varphi \rangle\) is nontrivial.

2 The main result

**Theorem 2.1.** Let \(\phi\) be a splitting automorphism of period \(n\) of the group \(B(m, n)\), where \(n \geq 1003\) is an odd number. If the order of automorphism \(\phi\) is a power of some prime number, then \(\phi\) is an inner automorphism.

**Proof.** According to Lemma 1.1, to prove the Theorem 2 it suffices to show that the equality \(N^{\phi} = N\) holds for any normal subgroup \(N \in M_n\).

Suppose that there is a normal subgroup \(A \in M_n\) which is not \(\phi\)-invariant, that is \(A^{\phi} \neq A\). On the other hand, by Lemma 1.2 the centralizer of each such subgroup \(A\) is not trivial. Let \(p^r\) be the order of automorphism \(\phi\), where \(p\) is some prime number. The number \(p^r\) divides \(n\) by definition.

Since the subgroups of the cyclic group \(\langle \phi \rangle\) of order \(p^r\) are linearly ordered by inclusion, one can choose some subgroup \(N\) with the minimal centralizer among all non-\(\phi\)-invariant subgroups \(A \in M_n\). Being a subgroup of the group \(\langle \phi \rangle\) of order \(p^r\), this minimal nontrivial centralizer is generated by some automorphism of the form \(\phi^{pk}\), where \(1 < k < r\).

By virtue of the minimality, the subgroup \(\langle \phi^{pk} \rangle\) is contained in the centralizer of any subgroup \(A \in M_n\). Hence, the automorphism \(\phi^{pk}\) centralizes all subgroups \(A \in M_n\). According to
Lemma 1.1 we obtain that automorphism $\phi^k$ is inner.

We shall use the following lemma, that was proved in the paper [6].

**LEMMA 2.1.** (see [6, Lemma 3]) Let $\varphi : G \to G$ be an arbitrary automorphism and $H$ be a normal subgroup of the group $G$ such that the quotient group $G/H$ is a non-abelian and simple. In that case, if the subgroups $H, H^{\varphi}, ..., H^{\varphi_{k-1}}$ are pairwise distinct and $H^{\varphi_k} = H$, then the quotient group $G/ \bigcap_{i=1}^{k} H^{\varphi_i}$ is decomposed into the direct product of normal subgroups $H_j/ \bigcap_{i=1}^{k} H^{\varphi_i}$, $j = 1, 2, ..., k$, wherein each quotient group $H_j/ \bigcap_{i=1}^{k} H^{\varphi_i}$ is isomorphic to $G/H$ and $H_j = \bigcap_{i \neq j} H^{\varphi_i}$.

To use Lemma 2.1, suppose that $G = B(m, n)$, $\varphi = \phi$ and let $H = N$ be the normal subgroup of the group $B(m, n)$ with the minimal normalizer $\langle \phi^k \rangle$. The quotient group $B(m, n)/N$ is a Tarski monster, since $N \in M_n$. In particular, $B(m, n)/N$ is a non-abelian and simple group. By Lemma 2.1, the quotient group $B(m, n)/K$ is decomposed into the direct product of subgroups $N_0/K, N_1/K, ..., N_{p^k-1}/K$, where $K = \bigcap_{i=0}^{p^k-1} N^{\phi^i}$.

As it was mentioned above, the automorphism $\phi^k$ is inner, that is for some element $u \in B(m, n)$ we have the equality $\phi^k = i_u$, where $i_u$ is the inner automorphism generated by $u$. Since the automorphism $\phi^k$ has order $p^{r-k}$, the relation $(i_u)^{p^{r-k}}(x) = x$ holds for any $x \in B(m, n)$. In the other words the element $u^{p^{r-k}}$ belongs to the center of $B(m, n)$. From Adian’s theorem on the triviality of the center of the group $B(m, n)$ (see [1, Theorem 3.4]) it follows the equality $u^{p^{r-k}} = 1$. Moreover, the element $u$ has order $p^{r-k}$ because the automorphism $\phi^k$ has order $p^{r-k}$.

For the element $uK$ of the group $B(m, n)/K$ there exist uniquely defined elements $u_0K, u_1K, ..., u_{p^k-1}K$ from the subgroups $N_0/K, N_1/K, ..., N_{p^k-1}/K$ respectively such that

$$uK = u_0K \cdot u_1K \cdot ... \cdot u_{p^k-1}K.$$ (1.1)

Note that the relation $u^{p^{r-k}} = 1$ implies immediately the relations $u_i^{p^{r-k}}K = K$ for all $i = 0, 1, ..., p^k - 1$. In particular, we have $u_0^{p^{r-k}}K = K$.

According to Lemma 2.1 the groups $N_0/K$ and $B(m, n)/N$ are isomorphic. By definition of the set $M_n$ any element of the quotient group $B(m, n)/N$ is contained in some cyclic subgroup of order $n$ (see [2, Proposition 5.2]). Therefore, there exists an element $a \in B(m, n)$ such that $aK \in N_0/K$ and the element $au_0K$ has the order $n$.

Besides, the elements $u_1K, ..., u_{p^k-1}K$ commute with both of the elements $aK$ and $u_0K$ in $B(m, n)/K$ since $u_1K, ..., u_{p^k-1}K$ belong to the direct factors $N_1/K, ..., N_{p^k-1}/K$ respectively. Hence, the equality (1.1) implies the relations

$$u^sau^{-s}K = u_0^sau_0^{-s}K$$ (1.2)

for all integers $s$. Since $\phi$ is a splitting automorphism, we have the equality $aa^{\phi}a^{\phi^2} \cdot ... \cdot a^{\phi^{n-1}} = 1$. 

4
Taking into account $\phi$-invariance of the subgroup $K$, we obtain that the equality

$$aK \cdot a^\phi K \cdot a^{\phi^2} K \cdots a^{\phi^{n-1}} K = K$$  \hspace{1cm} (1.3)

holds in the quotient group $B(m, n)/K$.

Recall that according to choice of subgroup $N$ we have $a^{\phi^k} K \in N_0/K$. This allows us to rewrite the equality (1.3) in the form

$$bK \cdot b^\phi K \cdot b^{\phi^2} K \cdots b^{\phi^{k-1}} K = K,$$  \hspace{1cm} (1.4)

where

$$b = aa^{\phi^k} a^{\phi^{2k}} \cdots a^{\phi^{(n/p^k-1)p^k}}.$$  \hspace{1cm} (1.5)

Moreover, it is easy to see that the elements $bK$, $b^\phi K$, $b^{\phi^2} K$, $\ldots$, $b^{\phi^{k-1}} K$ belong to the direct components $N_0/K$, $N_1/K$, $\ldots$, $N_{p^k-1}/K$ respectively. Hence, the equality (1.4) immediately implies that all the factors on the left-hand side of the equality (1.4) are trivial. In particular, we have the equality $bK = K$.

Since $\phi^k = i_u$, one can rewrite the equality (1.5) in the form

$$b = a \cdot uau^{-1} \cdot u^2 au^{-2} \cdot u^{n/p^k-1} au^{-(n/p^k-1)}.$$  

Accordingly, we obtain equality

$$bK = aK \cdot uau^{-1} K \cdot u^2 au^{-2} K \cdots u^{n/p^k-1} au^{-(n/p^k-1)} K$$

in the quotient group. Then using the relations (1.2), we get the equality

$$bK = a \cdot uau^{-1} \cdot u_0^{2} au_{0}^{-2} \cdots u_{0}^{n/p^k-1} au_{0}^{-(n/p^k-1)} K.$$

Next, we use the following identity

$$a \cdot uau_{0}^{-1} \cdot u_0^{2} au_{0}^{-2} \cdots u_{0}^{n/p^k-1} au_{0}^{-(n/p^k-1)} = (au_0)^{n/p^k} \cdot u_{0}^{-n/p^k}. $$

The number $n/p^k$ is divided by $p^{r-k}$ since $n$ is divided by $p^r$. By virtue of the equality $u_0^{p^{r-k}} K = K$, we finally obtain $bK = (au_0)^{n/p^k} K$. Hence, we obtain $(au_0)^{n/p^k} K = K$ because $bK = K$. This contradicts to condition of choice of the the element $a$, according to which the element $au_0 K$ has order $n$ in the group $B(m, n)/K$. Theorem is proved.

**Corollary 2.1.** If $p$ is an odd prime number and $n = p^k \geq 1003$, then any splitting automorphism of period $n$ of the group $B(m, n)$ is an inner automorphism.
References

[1] S. I. Adian, *The Burnside problem and identities in groups*, Nauka, Moscow 1975; English transl., Ergeb. Math. Grenzgeb., vol. 95, Springer-Verlag, Berlin-New York 1979.

[2] S. I. Adyan; I. G. Lysenok, Groups, all of whose proper subgroups are finite cyclic. *Izv. Akad. Nauk SSSR Ser. Mat.* 55 (1991), no. 5, 933–990 (in russian); translation in Math. USSR-Izv. 39 (1992), no. 2, 905–957.

[3] V. S. Atabekyan, Simple and free periodic groups. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* (1987), no. 6, 76–78 (in russian); translation in Mosc. Univ. Math. Bull. 42(6) (1987) 80–82.

[4] V. S. Atabekyan, On periodic groups of odd period \( n \geq 1003 \). *Mat. Zametki*, 82:4 (2007), 495500 (in russian); translation in Math. Notes, 82:4 (2007) 443447.

[5] V. S. Atabekyan, Normal automorphisms of free Burnside groups. *Izv. RAN. Ser. Mat.* 75:2 (2011) 3–18 (in russian); translation in Izv. Math. 75:2 (2011) 223–237.

[6] V. S. Atabekyan, Splitting automorphisms of free Burnside groups. *Mat. Sb.*, 204(2) (2013) 31–38 (in russian); translation in Sbornik: Mathematics, (2013), 204:2, 182–189.

[7] E. A. Cherepanov, Normal automorphisms of free Burnside groups of large odd exponents, *Internat. J. Algebra Comput* 16(5) (2006) 839–847.

[8] O. H. Kegel, Die Nilpotenz der \( H_p \)-Gruppen, *Math. Z.*, v. 75 (1961), 373–376.

[9] E. I. Khukhro, Nilpotency of solvable groups admitting a splitting automorphism of prime order, *Algebra i Logika* 19:1 (1980), 118-129; English transl. in Algebra and Logic 19:1 (1980), 77-84.

[10] E. Jabara, Groups admitting a 4-splitting automomorphism, *Rend. Circ. Mat. Palermo*, II. Ser., 45:1 (1996), 84–92.

[11] V. D. Mazurov, Yu. I. Merzlyakov and V. A. Chirkin (eds.), *The Kourovka notebook. Unsolved problems in group theory*, 11th ed., Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences, Novosibirsk 1990. (Russian)

[12] A.Yu. Ol’shanskiñ, Groups of bounded period with subgroups of prime order, *Algebra i Logika* 21:5 (1982), 553–618; English transl. in Algebra and Logic 21:5 (1982), 369–418.

[13] A. Yu. Olshanskii, *The Geometry of Defining Relations in Groups*. Kluwer Academic Publishers, 1991.