Analytic Fluid Approximation for Warm Dark Matter.

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We present the full evolution of the velocity for a massive particle, along with the equation of state when we are able to analytically compute the energy density and pressure evolution. Therefore, it is easy to compute the perturbation equations for any massive decoupled particle, i.e. warm dark matter (WDM) or neutrinos, treated as fluid. Using this approach we analytically calculate the moment when the WDM stop being relativistic, $a_{nr}$, which is just 15% difference with respect to the exact Boltzmann solution. Using the fluid approximation the matter power spectrum is computed fast and with great accuracy, the cut-off in structure formation due to the free-streaming, $\lambda_{fs}$, of the particle, characteristic for a WDM particle, is replicated in both matter power spectrum and halo mass function, in which we found up to 30% correction on the Jeans mass. We also show that the matter power can be computed using a fitting formula that involves only the cut-off scale, $k_{fs}$. This formulation can be integrated in comprehensive numerical modeling reasonable increasing the performance in the calculations.

I. INTRODUCTION

The most successful model to describe the Universe is the ΛCDM model, which supported by observational evidence such as the Cosmic Microwave Background (CMB) anisotropies [1], galaxy redshift surveys [2], type Ia Supernovae [3] arrive to the conclusion that the content of the Universe is composed of 65% dark energy driving the accelerated expansion of the Universe, 31% dark matter (DM) whose clustering feature influence the large scale structure formation, and the rest 4% is baryonic matter. Therefore is important to have useful tools to study the main components of the Universe in order to get a glance on its nature.

Most DM particles that have been proposed have non-negligible velocities in the early Universe where it is assume that DM particles are in thermal contact with the primordial bath, decouple when still relativistic, and then become non-relativistic when the Universe is still radiation-dominated, when the primordial clumps begins to cluster to form large scale structures. Therefore, the moment when DM become non-relativistic is important and directly proportional to its mass. Depending on how large is the mass the DM is known to be cold, CDM, if $m_{cdm} \sim O(MeV)$. This kind of DM particles stop being relativistic and start clustering object at a very early time. DM with mass around $m_{wdm} \sim O(KeV)$ are known to be warm, WDM, whose main attribute is that its dispersion velocity wipes out some density concentrations of matter and, therefore, induce a cut-off scale into the mass halo function [4].

Having a cut-off scale is an appealing dark matter feature because it conciliate observations with theoretical predictions, for instance, the number of satellite galaxies in our Galaxy is smaller than the expected from CDM simulations, the so-called missing satellite problem [5–7]. Solutions to this problem had also been pursued through baryon physics - star formation and halo evolution in the galaxy may be suppressed due to some baryonic process and the discussion is still in progress [8–10].

Perhaps the two most important quantities to study the cosmological impact of DM are the, the amount of energy density today $\Omega_{dm0}$, and the moment when these particles become non-relativistic (given by the scale factor $a_{nr}$). There may be a third parameter, the velocity dispersion of the DM particles at $a_{nr}$, its value may reflect the nature of the DM, for instance, an abrupt transition to the non-relativistic DM epoch that may suggest of DM subject to a phase transition [11].

Rough approximation are usually made to account for the evolution when being relativistic ($a < a_{nr}$) and when DM became non-relativistic ($a > a_{nr}$). But here we present a simple analytic approach for a massive particle which is valid for all times characterised by having a non-negligible thermodynamic velocity dispersion [12–24]. With this approach we are able to compute the fluid approximation for the perturbation equations for any massive particles, i.e. WDM, given the analytic solution for the energy density evolution we were able to reproduce the most appealing feature of WDM, the cut-off in the matter power spectrum [12–15]. For example, a $m_{wdm} = 3KeV$ is analytically computed to have a $a_{nr} = 3.14 \times 10^{-7}$ while the Boltzmann exact solution gives $a_{nr} = 2.73 \times 10^{-7}$, just a 15% difference. In general, the percentage difference between the numerical value obtained from Boltzmann equations and the analytic one of $a_{nr}$ is on average 18% in a mass range 1-10 keV.

In this work we make use of natural units, $c = 1$. We present the work as follows: in Sec[1] we present the the-
Theoretical warm dark matter framework, compute the perturbations of the WDM model and the mass halo function and finally compute the free streaming scale. We present our conclusions in Sec. [IV]

II. DM FRAMEWORK

Relativistic particles with peculiar velocity, \( v \), and mass \( m \) have a momentum \( p = \gamma mv \), and energy \( E^2 = p^2 + m^2 \), where \( \gamma \equiv 1/\sqrt{1 - v^2} \). Solving for \( v \) we obtain,

\[
v = \frac{p^2}{\sqrt{m^2 + p^2}}
\]

The particle is relativistic when the velocity is \( v \sim 1 \) or equivalently when \( p \gg m \). The particle is non-relativistic when \( v \ll 1 \). It is common to establish that a particle becomes non relativistic when \( p = m \) [12], when this happens, from Eq. (1), the velocity is simply \( v_{nr} = 1/\sqrt{2} \) with \( \gamma_{nr} = \sqrt{2} \), thus \( \gamma_{nr} v_{nr} = 1 \). Quantities with subindex \( nr \) are evaluated at \( a_{nr} \).

In an expanding FRW Universe, the momentum of a relativistic particle redshift as \( p(a) = p_{nr}(a_{nr}/a) = m(a_{nr}/a) \). Therefore, the velocity at all moments in an expanding Universe evolves as

\[
v(a) = \frac{(a_{nr}/a)}{\sqrt{1 + (a_{nr}/a)^2}}.
\]

Eq.(2) describes the exact velocity evolution of a decoupled massive particle. The transition between relativistic to non-relativistic is smooth and continuous, see [? ] for a generalize transition. This evolution is general and valid for any massive decoupled particles (WDM, CDM or massive HDM, neutrinos). If \( a \gg a_{nr} \) we expand into series the denominator of Eq. (2), we have \( (1 + (a_{nr}/a)^2)^{-1/2} \sim 1 - a_{nr}^2/2a^2 + \ldots \), it is clear that Eq.(2) reduce to the non-relativistic limit where \( v_{nr}(a) \sim a_{nr}/a \).

The pressure of any generic particle is given by \( P = \langle |\vec{p}|^2 \rangle/n/3 \langle E \rangle \) and the energy density is given by \( \rho = \langle E \rangle n \), with \( n \) being particle number density, \( \langle |\vec{p}|^2 \rangle \) is the average quadratic momentum and \( \langle E \rangle \) the average energy of the particles. Therefore the equation of state (EoS) is given by

\[
\omega = \frac{\langle |\vec{p}|^2 \rangle}{3\langle E \rangle^2} = \frac{v(a)^2}{3}
\]

The EoS of DM have been investigates using the CMB and large scale structure (LSS) [23], and gravitational lensing data. [26 27] confirming that DM should be cold when structure began to cluster. We integrate the continuity equation, \( \rho = -3H(\rho + P) \), using Eq. (3) to obtain the analytic evolution of the background \( \rho_{bdm}(a) \). For all \( a \) we have,

\[
\rho_{bdm}(a) = \rho_{bdm}(a_o)^{-1} \left( \frac{v_o}{a_o} \right)^4 \left( \frac{v(a)}{v(a_o)} \right)
\]

FIG. 1: Plot of the equation of state for a non-cold dark matter. The continuous black line is obtained from solving Boltzmann equations using CLASS, the red dashed line is the analytic expression for the EoS of WDM, Eq. (2) along with Eq. (3).

when \( a_{nr} \ll a_o \) and \( a_{nr} \ll a \) we have

\[
\frac{v_o}{v(a)} \sim \frac{a}{a_o},
\]

then, the fluid behave as matter. Moreover, when \( a_{nr} \ll a_o \) and \( a \ll a_{nr} \) we have

\[
\frac{v_o}{v(a)} \sim \frac{a_{nr}}{a},
\]

since the last quantity is a constant the fluid behaves as radiation. As seen in Eq. (3) a massive particle (WDM or CDM) becomes non-relativistic at \( a_{nr} \) with \( v(a_{nr}) = 1/\sqrt{2} \) and has only one free parameter, the scale factor \( a_{nr} \). Therefore the physics of any massive particle with smooth continuous velocity transition can be described by \( a_{nr} \) or the mass of the particle, i.e. \( m_{wdm} \). We plot \( \omega_{wdm} \) in Fig. 1.

More complex approaches have been studied [28 31] where they take generalized properties of DM such as the sound speed and viscosity and put constrains with observational datasets.

A. \( m_{wdm} - a_{nr} \) relation

Several constraints has been placed around the mass of the WDM based on different methods then it would be useful to have a relation between the mass \( (m_{wdm}) \) and the momento when DM become non relativistic \( (a_{nr}) \). Among current constrains on the mass of WDM are the ones based on the abundance of redshift \( z = 6 \) galaxies in the Hubble Frontier Fields, \( m_{wdm} > 2.4 \) keV [32]. Based on galaxy luminosity function at \( z \sim 6 - 8 \), \( m_{wdm} > 1.5 \) keV [33]. Using lensing surveys such as CLASH, \( m_{wdm} > 0.9 \) keV [34]. The highest lower limit is given by the high redshift Ly-\( \alpha \) forest data which put lower bounds of \( m_{wdm} > 3.3 \) keV [35].
The evolution of energy density of WDM particle for all the evolution of the Universe is given by Eq. (4), the WDM energy density evolve as matter with \( \rho(a) \propto a^{-3} \) when \( a \gg a_{nr} \). The non-relativistic limit of the EoS is \( w \approx 3T/M \) at \( a_{nr} \), and use Eq. (3) to approximate \( w = 3T/M = v^2/3 \) to obtain the relationship \( T = Mv^2/9 \) at \( a_{nr} \). The relativistic energy density is given by \( \rho(T) = (\pi^2g_x/30)T^4 \) valid \( a \leq a_{nr} \) which becomes \( \rho(T_{nr}) \approx (\pi^2g_x/30)(Mv^2/9)^4 \).

We equate both equations for the energy density evaluated at \( a_{nr} \), \( \rho(a_{nr}) = \rho(T_{nr}) \), and obtain \( \rho_{\text{dm}}(a_0/a_{nr})^3\sqrt{2} = (\pi^2g_x/30)(Mv(a_{nr})^2/9)^4 \). We know that \( v(a_{nr}) = 1/\sqrt{2} \) for WDM, and we assume \( g_x = 7/4 \) for a neutrino type fermion, the scale factor where WDM becomes non-relativistic is then

\[
\frac{a_{nr}}{a_0} = 3.14 \times 10^{-7} \left( \frac{\rho_{\text{dm}}}{0.120} \right)^{1/3} \left( \frac{3\, \text{keV}}{M} \right)^{4/3} \left( \frac{7/4}{g} \right)^{1/3}.
\]  

With the numerical code CLASS we obtained the EoS for a 3 keV WDM particle and look at the moment where it becomes non-relativistic, to find that the numerical value \( a_{nr}^{\text{num}} = 2.73 \times 10^{-7} \), just a 13% different with respect to the expected value.

We can relate the time when two different WDM become non-relativistic from Eq. (7) and we find

\[
\frac{a_{nr}}{a_{nr}^{\text{num}}} = \left( \frac{M'}{M} \right)^{4/3}.
\]  

In Table I we show some WDM cases (\( m_{\text{wdm}} = 1.3, 10 \) keV) in which we compare the numerical results for \( a_{nr} \) obtained from this analytical calculations and the ones obtained from solving the Boltzmann equations using the numerical code CLASS for a non-cold DM.

### III. LARGE SCALE STRUCTURE IN BDM SCENARIO

In order to compute the cut-off scale, first we compute the Boltzmann equations for WDM, using the fluid approximation in CLASS.

#### A. Perturbations

We follow \[36\] to compute the fluid limit to the perturbed equations in k-space in the synchronous gauge for the WDM

\[
\frac{\Delta \rho}{\rho} = \left( \frac{1}{\omega} - 3 \frac{\dot{H}}{H} \right) \frac{\Delta \omega}{\omega},
\]

where \( \delta \) is the contrast, \( \theta \) is the velocity parameter and \( \sigma \) is the anisotropic stress perturbations. The dot represent the derivative respect to the conformal time, \( \tau \equiv \int dt/a(t), H \) is the Hubble parameter.

In Eq. (11) we have taken the anisotropic stress approximation for massive neutrinos \[28\] and ignore the \( \eta \) term that slightly the computation of the matter power spectrum \[37\]. We have also used the relation \( \dot{\omega} = -2H\omega(1 - 3\omega) \).

Throughout this paper, we adopt Planck 2018 cosmological parameters \[1\] unless is specify otherwise. For the several simulations we adopt a flat Universe with \( \Omega_m h^2 = 0.12 \), and \( \Omega_b h^2 = 0.02237 \) as the CDM matter and baryonic omega parameter. \( h = 0.6736 \) is the Hubble constant in units of 100 km/s/Mpc, \( n_s = 0.965 \) is the tilt of the primordial power spectrum. \( z_{\text{reio}} = 7.67 \) is the redshift of reionization and \( \ln(10^{10}A_s) = 3.044 \), where \( A_s \) is the amplitude of primordial fluctuations.

In Fig. 2 we show the dimensionless matter power spectrum obtained with CLASS code \[38\] taking into account WDM fluid approximation, Eqs. (9)-(11). We show the
matter power spectrum for different values of \( m_{\text{wDM}} = \{1, 2, 3, 4, 10\} \) keV. The bigger the mass the colder the DM is, therefore, for bigger masses the difference with CDM decrease. In the Fig. 2 solid lines is the spectrum obtained by solving the Boltzmann equation while dashed lines are the ones obtained from fluid approximation equations. The percentage difference between Λ-CDM and Λ-WDM power spectrum is show in the bottom panel of Fig. 2.

The effect of the free-streaming (computed below in subsection III.C) is to suppress structure formation below a threshold scale, therefore the matter power spectrum show a cut-off at small scales depending the value of \( m_{\text{wDM}} \), equivalently \( a_{nr} \). The smaller the scale of the transition, the mass is bigger, the power is damped at smaller scales. Transitions of the order of \( a_{nr} \gtrsim 10^{-8} \) WDM is indistinguishable from CDM at observable scales, \( k \sim \mathcal{O}(10) \), this corresponds to \( m_{\text{wDM}} \gtrsim 100 \) keV.

The CMB power spectrum can also be computed, but the difference with respect the fiducial ΛCDM model is barely perceptible one can notice an increased the height of the acoustic peaks of less than 1% because difference respect CDM, this increment is the free-streaming also increase the acoustic oscillations.

### B. Halo Mass Function

The change in the matter power spectrum is known to strongly affect large scale structure, we compute the the abundance of structure using the Press-Schechter approach [39]. With the linear matter power spectrum (see Sec III] as an input we compute the halo mass function as

\[
\frac{dn}{d\log M} = M \frac{dm}{dM} = \frac{1}{2M} \frac{\overline{\rho}}{M} \sigma^2 \frac{d\log \sigma^2}{d\log M} \quad (12)
\]

where \( n \) is the number density of haloes, \( M \) the halo mass and the the peak-height of perturbations is given by

\[
\nu = \frac{\delta_c^2(z)}{\sigma^2(M)},
\]

where \( \delta_c = 1.686 \) is the overdensity required for spherical collapse model in a ΛCDM cosmology. The average density is \( \overline{\rho} = \Omega_m \rho_c \), where \( \rho_c \) is the critical density of the Universe. Here \( \Omega_m = \Omega_c + \Omega_k \). The variance of the linear density field on mass-scale, \( \sigma^2(M) \), can be computed from the following integrals

\[
\sigma^2(M) = \int_0^\infty dk \frac{k^2 P_{\text{lin}}(k)}{2\pi^2} |W(kR)|^2. \quad (14)
\]

Here we will use the sharp-k window function \( W(x) = \Theta(1 - kR) \), with \( \Theta \) being a Heaviside step function, and \( R = (3cM/4\pi\overline{\rho})^{1/3} \), where the value of \( c = 2.5 \) is proved to be best for cases similar as the WDM [40]. Finally for the first crossing distribution \( F(\nu) \) we adopt [41], that has the form

\[
F(\nu) = A \left( 1 + \frac{1}{\nu'} \right) \sqrt{\nu \over 2\pi} e^{-\nu'/2} \quad (15)
\]

with \( \nu' = 0.707\nu, \ p = 0.3, \) and \( A = 0.322 \) determined from the integral constraint \( \int f(\nu) d\nu = 1 \).

### C. Free-streaming scale

The thermal velocities of the dark matter particles have a direct influence on structure formation. While DM particles are still relativistic, primordial density fluctuations are suppressed on scales of order the Hubble horizon at that time. This is call the free-streaming scale and depends on the moment when a massive particle becomes non-relativistic \( a_{nr} \).

The smoothing comoving free streaming scale \( \lambda_{fs} \) is defined by

\[
\lambda_{fs} = \int_0^t \frac{v(t)dt}{a(t)} \quad (16)
\]

\[
= \frac{2t_{nr}}{a_{nr}^2} \int_0^{a_{eq}} v(a) da + \frac{3a_{eq}^2 t_{nr}}{2a_{nr}^2} \int_0^a \frac{v(a)}{a^{1/2}} da. \quad (17)
\]

In the last equation, the first integral assume a radiation dominated Universe with \( t \propto a^3 \), and the second one a matter dominated Universe with \( t \propto a^{2/3} \). The free-streaming scale is defined by the mode \( k_{fs} \) and a mass \( M_{fs} \) contained in sphere of radius \( \lambda_{fs}/2 \) given by

\[
k_{fs} = \frac{2\pi}{\lambda_{fs}}, \quad M_{fs} = \frac{4\pi}{3} \left( \frac{\lambda_{fs}}{2} \right)^3 \rho_{m0}. \quad (18)
\]

For mass-scales less than Jeans mass, \( M \lesssim M_{fs} \), free-streaming erases all peaks in the initial density field therefore the number of structures below this mass scale.
should be significantly reduce in numbers. We show this behavior in Fig. 3, where we compare CDM and WDM mass functions.

Let us now determine the comoving free streaming scale, \( \lambda_{fs} \), using the fiducial approximations. It is standard to assume that in the relativistic regime \( v_{w,\text{dm}} = 1 \) for \( a < a_{nr} \). While in the non-relativistic regime \( (a > a_{nr}) \) it is assume \( v_{w,\text{dm}} = a_{nr} / a \). With these choices of \( v \) one gets the usual free streaming scale

\[
\lambda_{fs}(a_{eq}) = \frac{2t_{nr}}{a_{nr}} \left[ 1 + \log \left( \frac{a_{eq}}{a_{nr}} \right) \right] \tag{19}
\]

However the velocity equation presented here, Eq. (2), has some serious advantages for WDM over the assumptions taken to compute Eq. (19). First, we do not need to distinguish between the relativistic and non-relativistic regime, Eq. (2) is a general equation valid for all \( a \). Second, Eq. (2) integrates the concept of stop being relativistic, this happens when \( v_{w,\text{dm}} = 1/\sqrt{2} \), which in turn came from establishing that \( p = m \) hold in that moment. Third, Eq. (2) is simple enough that we can integrate Eq. (17) to obtain,

\[
\lambda_{fs}(a) = \frac{2t_{nr}}{a_{nr}} \log \left[ \frac{a_{eq}}{a_{nr}} + \sqrt{1 + \left( \frac{a_{eq}}{a_{nr}} \right)^2} \right] \tag{20}\]

Let us now take the limit \( a_{nr} / a_{eq} \ll 1 \) to get

\[
\lambda_{fs}(a_{eq}) \approx \frac{2t_{nr}}{a_{nr}} \log \left[ \frac{2a_{eq}}{a_{nr}} \right] = \frac{2t_{nr}}{a_{nr}} \left( \log[2] + \log \left[ \frac{a_{eq}}{a_{nr}} \right] \right) \tag{21}\]

with \( \log 2 \approx 0.69 \). Eq. (20) or its limit Eq. (21) capture the full evolution of the velocity \( v(a) \) of a massive particle.

The analytic solution of the comoving free streaming after the moment of equivalence is given by

\[
\lambda_{fs} = \frac{3t_{nr}}{a_{nr}} \left[ 2F_1 \left( \frac{1}{4}, \frac{1}{2}; \frac{5}{4}; -\frac{a_{eq}^2}{a_{eq}^2} \right) - \left( \frac{a_{eq}}{a} \right)^{1/2} \left( \frac{a_{eq}^2}{a} \right)^{1/2} \right] \tag{22}\]

\[
\lambda_{fs} \approx \frac{3t_{nr}}{a_{nr}} \left( 1 - \left( \frac{a_{eq}}{a} \right)^{1/2} - \frac{a_{eq}^2}{2a_{eq}^2} \right) \tag{23}\]

The last term in Eq. (23) is the highest relativistic correction but it accounts for less than 1% for the value of \( \lambda_{fs} \).

However, the correction in Eq. (20) has an impact on the free streaming of 5% less compared with previous results (Eq. (19)), this seems to be low, however, we may consider its implication when computing the mass contained within a sphere of radius \( R = \lambda_{fs} / 2 \), or \( M_{fs} \). The correction in \( M_{fs} \) could be around 30% increment in some cases. For instance, for a 10 keV mass WDM using Eq. (21) the free-streaming is \( \lambda_{fs} = 0.79 \) Mpc/h.

Table I: In this table we show the moment when a WDM stop being relativistic, \( a_{nr} \), the free streaming scale, \( \lambda_{fs} \) [Mpc/h] from Eqs. (21) and (23), the correspondent mode \( k_{fs} \) [h/Mpc] and Jeans mass, \( M_{fs} [M_{\odot}/h] \) for different masses \( m_{w,\text{dm}} \), \( 1.3, 10 \) keV. For each case we show the \( a_{nr} \) obtained from solving the Boltzmann equations using CLASS, and the one obtained from the analytic expression Eq. (7).

| Mass (keV) | \( a_{nr} \) | \( \lambda_{fs} \) | \( k_{fs} \) | \( M_{fs} \) |
|----------|---------|---------|---------|---------|
| 1 keV - Boltzmann | \( 8.17 \times 10^{-7} \) | 3.42 | 1.84 | \( 6.96 \times 10^{-11} \) |
| 1 keV - analytic | \( 1.36 \times 10^{-6} \) | 2.19 | 2.86 | \( 1.84 \times 10^{-11} \) |
| 3 keV - Boltzmann | \( 2.73 \times 10^{-7} \) | 0.94 | 6.66 | \( 1.46 \times 10^{-10} \) |
| 3 keV - analytic | \( 3.14 \times 10^{-7} \) | 0.83 | 7.55 | \( 1.01 \times 10^{-10} \) |
| 10 keV - Boltzmann | \( 8.20 \times 10^{-8} \) | 0.22 | 28.15 | \( 1.94 \times 10^{6} \) |
| 10 keV - analytic | \( 6.30 \times 10^{-8} \) | 0.28 | 22.19 | \( 3.96 \times 10^{6} \) |

TABLE I: In this table we show the moment when a WDM stop being relativistic, \( a_{nr} \), the free streaming scale, \( \lambda_{fs} \) [Mpc/h] from Eqs. (21) and (23), the correspondent mode \( k_{fs} \) [h/Mpc] and Jeans mass, \( M_{fs} [M_{\odot}/h] \) for different masses \( m_{w,\text{dm}} \), \( 1.3, 10 \) keV. For each case we show the \( a_{nr} \) obtained from solving the Boltzmann equations using CLASS, and the one obtained from the analytic expression Eq. (7).

A good parametrization for the MPS for WDM can be found in [12]. Inspired in this parametrization we propose to fit the transfer function as

\[
T_X(k) = \left[ \frac{P_{w,\text{dm}}}{P_{\text{lin}}} \right]^{1/2} = \left[ 1 + \alpha \left( \frac{k}{k_{fs}} \right)^{\beta} \right] \tag{24}\]

so all three parameters \( \alpha, \beta \) and \( \gamma \) remains dimensionless.

Using the exact value for \( a_{nr} \) to compute \( k_{fs} \), computing the \( \lambda_{fs} \) using Eqs. (21) and (23), we get that the values that best fit the transfer functions are

\[
\alpha = 0.29 \pm 0.01 \quad \beta = 2.31 \pm 0.04 \quad \gamma = -3.05 \pm 0.13 \tag{25}
\]

in the mass range (1 - 10) keV for WDM. Notice that the standard error for all three parameters are small, specially for \( \alpha \) and \( \beta \). The value of \( \beta \) is actually close to previous works [42] that obtained \( \beta = 2.24 \), however we find a significantly difference previous values of \( \gamma = -4.46 \). We also want to highlight that \( \alpha \) is almost a constant value for different masses of WDM, while previously was a function that depends on the mass of the WDM. Therefore the dependance of the transfer function on the mass of the WDM particle is in the free-streaming scale, \( k_{fs} \), and not in the parameter \( \alpha, \beta \) or \( \gamma \).

IV. CONCLUSION

We have presented a generalization for the velocity dispersion of particles, Eq. (2), which is valid for any massive particle, such as, WDM and massive neutrinos, generically also known as non-cold DM. This velocity is function of the scale factor and it also depends on the moment the particle become non-relativistic, \( a_{nr} \). The general expression for the velocity has some serious advantages that
must be taken into account and can be enumerated: (1) Eq. (2) embrace the concept that a particle stopped being relativistic when \( p = m \), therefore it has been proven that one-to-one relation between \( a_n \) and the mass of the particle, i.e. \( m_{\text{wdm}} \), can be found. (2) It is a simple and describe a smooth transition between relativistic to non-relativistic regimes for the particle. (3) The continuity equation can be solve and therefore is straightforward to compute the perturbation equations treated as fluid, which is terms of computational effort could save a significant amount of time. (4) A slight difference in the CMB power spectrum can be found, less than 1%, but more importantly, the cut-off in the linear matter power spectrum can be found, only 13% above the value obtained from solving the Boltzmann equations encoded in CLASS. In general, the percentage difference between the numerical value computed from Boltzmann equations and the analytic one is on average 18% in the mass range 1-10 keV. Calculation on the free-streaming scale, \( \lambda_{fs} \), show that Eq. (2) implies a 5% correction on its value, which is reflected in a 30% correction in the Jeans mass, \( M_{fs} \), this order of correction is been found it the interesting range of WDM masses.

This framework where we include the dispersion velocity of the dark matter particle may be incorporated in a broad number of observational cosmological probes, theoretical analyzes and N-body simulations, including forecasts for large scale structure measures, i.e. weak lensing [13], future galaxy clustering measures of the power spectrum [14]. From future observation from large to small-scale clustering of dark and baryonic matter may be able to put more feasible constrains on \( a_n \) and therefore on the WDM mass, \( m_{\text{wdm}} \).

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