Super Rogers-Szegö polynomials associated with $BC_N$ type of Polychronakos spin chains

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Abstract

As is well known, multivariate Rogers-Szegö polynomials are closely connected with the partition functions of the $A_{N-1}$ type of Polychronakos spin chains having long-range interactions. Applying the ‘freezing trick’, here we derive the partition functions for a class of $BC_N$ type of Polychronakos spin chains containing supersymmetric analogues of polarized spin reversal operators and subsequently use those partition functions to obtain novel multivariate super Rogers-Szegö (SRS) polynomials depending on four types of variables. We construct the generating functions for such SRS polynomials and show that these polynomials can be written as some bilinear combinations of the $A_{N-1}$ type of SRS polynomials. We also use the above mentioned generating functions to derive a set of recursion relations for the partition functions of the $BC_N$ type of Polychronakos spin chains involving different numbers of lattice sites and internal degrees of freedom.

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1. Introduction

Exact solutions of one-dimensional quantum integrable spin chains with long-range interactions and their supersymmetric analogues [1–22] have recently attracted a lot of interest due to their connection with surprisingly large number of topics in physics as well as mathematics. Indeed, this type of quantum spin chains have found important applications in subjects like condensed matter systems obeying generalized exclusion statistics [5, 22, 23], quantum electric transport phenomena [24, 25], ‘infinite matrix product states’ in conformal field theory [26–31], planar $\mathcal{N} = 4$ super Yang–Mills theory [32–34], random matrix theory [35], Yangian quantum groups [4, 5, 10, 17, 36–39] and Rogers-Szegő (RS) polynomials [10, 40–43]. Haldane and Shastry have pioneered the study of quantum integrable spin models with long-range interaction, by deriving the exact spectrum of a spin-$\frac{1}{2}$ chain with equally spaced lattice sites on a circle [1, 2]. All spins of this su(2) symmetric Haldane-Shastry (HS) chain interact with each other through a pairwise exchange interaction whose strength is inversely proportional to the square of their chord distances. A remarkable feature of this su(2) HS chain and its su($m$) generalization is that they exhibit Yangian quantum group symmetry even for finite number of lattice sites. As a result, the energy eigenvalues of these spin chains can be expressed in an elegant way by using certain sequences of the binary digits ‘0’ and ‘1’, which are called as ‘motifs’ in the literature [3, 4]. Furthermore, the complete spectra of these HS spin chains, including the degeneracy factors of all energy levels, can be reproduced from the energy functions of some one-dimensional classical vertex models [17].

A close relation between the Hamiltonian of the su($m$) HS spin chain and that of the su($m$) spin Sutherland model can be established by using the ‘freezing trick’ [6, 7]. Each particle of the later model, with $m$ number of ‘spin’ degrees of freedom, moves on a circle for any finite value of the corresponding coupling constant. However, in the strong coupling limit, the coordinates of such particles ‘freeze’ at the minimum value of the scalar part of the potential, which coincides with the equally spaced lattice points of the HS spin chain. Moreover, the spin degrees of freedom of these particles essentially decouples from their coordinate degrees of freedom. As a result, the su($m$) spin Sutherland model naturally yields the Hamiltonian of the su($m$) HS model which governs the dynamics of the decoupled spin degrees of freedom. Such a method of deriving the Hamiltonian of a quantum spin chain from that of a spin dynamical model has also been applied [7] to the case of su($m$) spin Calogero model with confining harmonic potential. This led to another quantum integrable spin chain with long-range interaction, which is known in the literature as the su($m$) Polychronakos or Polychronakos–Frahm (PF) spin chain. The sites of this PF spin chain, which are inhomogeneously spaced on a line, coincide with the zeros of the Hermite polynomial [8]. Indeed, the Hamiltonian
of the ferromagnetic su(m) PF spin chain is given by

\[ H_{PF}^{(m)} = \sum_{1 \leq i < j \leq N} \frac{1 - P_{ij}^{(m)}}{(\rho_i - \rho_j)^2}, \tag{1.1} \]

where \( P_{ij}^{(m)} \) denotes the exchange operator which interchanges the spins of the \( i \)-th and \( j \)-th lattice sites, and \( \rho_i \) is the \( i \)-th zero point of the Hermite polynomial of degree \( N \). Similar to the case of HS spin model, the PF chain (1.1) exhibits \( Y(gl_m) \) Yangian quantum group symmetry for any value of \( N \) [10]. Consequently, the energy eigenvalues of this spin chain can be expressed through the motifs and the corresponding spectrum can be reproduced from the energy functions of a one-dimensional classical vertex model.

Due to the decoupling of the spin and coordinate degrees of freedom in the case of su(m) spin Calogero model for large values of its coupling constant, the canonical partition function of the su(m) PF spin chain (1.1) can be derived by using the freezing trick. More precisely, this partition function can be obtained by dividing the canonical partition function of the su(m) spin Calogero model through that of the spinless Calogero model. Thus, for the purpose of deriving the partition function of the su(m) PF spin chain by using the freezing trick, it is necessary to calculate at first the canonical partition function of the su(m) spin Calogero model. This partition function has been computed in the literature by using two different approaches — a direct one and an indirect one. Polychronakos has originally computed this partition function in an indirect way by expanding the corresponding grand canonical partition function (which can be obtained easily from the grand canonical partition function of the spinless Calogero model) as a power series of the fugacity parameter [9]. Finally, by applying the freezing trick, the partition function of the ferromagnetic su(m) PF spin chain (1.1) has been derived in the form

\[ Z_{A,N}^{(m)}(q) = \sum_{\sum_{i=1}^{m} a_i = N} \frac{N!}{(q)_{a_1} (q)_{a_2} \cdots (q)_{a_m}} \left[ a_1, a_2, \cdots, a_m \right]_q, \tag{1.2} \]

where \( q \equiv e^{-1/(k_B T)} \), summation is taken over all \( a_i \) (which are non-negative integers) satisfying the condition \( \sum_{i=1}^{m} a_i = N \), and the \( q \)-multinomial coefficients are defined as

\[ \left[ \begin{array}{c} N \\ a_1, a_2, \cdots, a_m \end{array} \right]_q = \frac{(q)_N}{(q)_{a_1} (q)_{a_2} \cdots (q)_{a_m}}, \]

with \((q)_n \equiv (1-q)(1-q^2) \cdots (1-q^n)\). Since each \( q \)-multinomial coefficient is a polynomial of \( q \), the partition function \( Z_{A,N}^{(m)}(q) \) in (1.2) can also be expressed as a polynomial of \( q \). A supersymmetric generalization of the PF spin chain (1.1), containing both bosonic and fermionic spin degrees of freedom, has also been studied and the corresponding partition function has been computed with the help of the freezing trick via the indirect approach as described above [11]. However, it is also possible to directly compute the

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canonical partition function of the $su(m)$ spin Calogero model from the knowledge of its spectrum. Proceeding in this way and subsequently applying the freezing trick, Barba et al. have derived \[15\] the canonical partition function of the $su(m)$ PF spin chain (1.1) in a form which apparently looks quite different from $Z^{(m)}_{A,N}(q)$ in (1.2).

In this context it may be noted that, the classical Rogers-Szeg"{o} (RS) polynomial in a single variable (say, $x$) is defined as $H_N(x,q) = \sum_{k=0}^{N} \left[ \frac{N!}{k!(N-k)!} \right] x^k$ [44]. This RS polynomial has been studied in connection with the well known Rogers-Ramanujan identities in number theory. Moreover, this RS polynomial can be viewed as a $q$-deformed version of the Hermite polynomial, which provides a basis for the coordinate representation of the $q$-oscillator algebra \[45, 46\]. Different types of homogeneous and inhomogeneous multivariate generalizations of the classical RS polynomial have also been studied in the literature. In particular, Hikami has observed that the homogeneous multivariate RS polynomials (depending on only one type of variables) of the form

$$H^{(m)}_{A,N}(x_1, x_2, \cdots, x_m; q) = \sum_{\sum_{i=1}^{m} a_i = N \atop a_i \geq 0} \left[ \frac{N!}{a_1!a_2!\cdots a_m!} \right] q^{a_1} x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m},$$  

(1.3)

reproduce the partition function (1.2) of the PF spin chain in the limit $x_1 = x_2 = \cdots = x_m = 1$ \[10, 40, 41\]. Consequently, a representation of the $Y(gl_m)$ Yangian invariant motifs associated with the PF spin chain (1.1) can be constructed by using a recursion relation satisfied by the RS polynomials (1.3). It may also be noted that, super RS (SRS) polynomials containing two different types of variables have been proposed in Ref. \[43\] for the purpose of analyzing the spectra and partition functions of the supersymmetric PF spin chains on the basis of their $Y(gl_{(m|n)})$ super Yangian symmetry. From the above discussion it is clear that RS and SRS polynomials play an important role in the study of PF spin chains and their supersymmetric generalizations. However, since the partition function of the $su(m)$ PF spin chain obtained by using the freezing trick via the direct approach appears to be quite different from (1.2), such a form of the partition function cannot be connected with the RS polynomials (1.3) in a straightforward way.

As is well known, root systems associated with Lie algebras are widely used in the classification of quantum integrable systems with long-range interaction. In particular, the above discussed $su(m)$ PF spin chains with $N$ number of lattice sites and their supersymmetric generalization are related to the $A_{N-1}$ root system. However, it is possible to construct exactly solvable variants of the PF spin chain (1.1) associated with the $BC_N$ and $D_N$ root systems \[19, 21, 47\]. One remarkable feature of the Hamiltonians of the PF spin chains associated with the latter root systems is that they contain reflection operators like $S_i$ ($i = 1, \ldots, N$), which satisfy the relation $S_i^2 = 1$ and yield a representation of some elements appearing in the $BC_N$ or $D_N$ type of Weyl algebra. In the special case when $S_i$ is taken as the spin reversal operator $P$, which changes the sign of the spin component on the $i$-th lattice site, the partition functions of PF spin
chains associated with the $BC_N$ and $D_N$ root systems have been computed by using the freezing trick via the direct approach [21, 47]. Furthermore, by taking reflection operators as supersymmetric analogue of spin reversal operators (SASRO), partition functions of PF spin chains associated with the $BC_N$ root system have been computed by using the freezing trick via the direct as well as indirect approaches [48].

However, it has been found recently that reflection operators can be chosen in more general way than the above mentioned spin reversal operators and their supersymmetric analogues. For example, these reflection operators may be taken as arbitrarily polarized spin reversal operators (PSRO) like $P_i^{(m_1, m_2)}$, which acts as the identity on the first $m_1$ elements of the spin basis and as minus the identity on the rest of the $m_2$ elements of the spin basis [49]. As a result, depending on the action of $P_i^{(m_1, m_2)}$, the basis vectors of the $(m_1 + m_2)$-dimensional spin space on each lattice site can be classified into two groups — $m_1$ elements with positive parity and $m_2$ elements with negative parity. For the special cases like $m_1 = m_2$ or $m_1 = m_2 + 1$, $P_i^{(m_1, m_2)}$ reduces to the spin reversal operator $P_i$ (up to a sign factor) through a similarity transformation. Choosing the reflection operators as such PSRO, new exactly solvable spin Calogero models of $BC_N$ and $D_N$ type have been constructed and the partition functions of the related PF chains have also been computed by using the freezing trick via the direct approach [49, 50]. Furthermore, exactly solvable spin Calogero models of $BC_N$ type have been constructed by taking reflection operators as supersymmetric analogues of PSRO (SAPSRO) [51]. The strong coupling limit of such spin Calogero models yields a large class of $BC_N$ type of PF spin chains with SAPSRO, which can reproduce all of the previously studied $BC_N$ type of PF spin chains at certain limits.

In spite of the above mentioned developments on different variants of the $BC_N$ type of PF spin chains, it is not clear till now whether the spectra of these spin chains can be described by some motif like objects related to the symmetry of these spin chains. Furthermore, one may interesting ask whether there exist some one-dimensional classical vertex models whose energy functions would generate the complete spectra of these $BC_N$ type of PF spin chains. However it is known that, in the cases of $A_{N−1}$ type of PF spin chains and their supersymmetric generalizations, multivariate RS and SRS polynomials play a key role in solving such problems. Hence, as a first step towards solving these problems for the case of $BC_N$ type of PF spin chains, in this article our aim is to construct the corresponding multivariate SRS polynomials and explore some of their properties. Since all of the previously studied PF spin chains of $BC_N$ type can be obtained by taking certain limits of the PF spin chains with SAPSRO, it is expected that canonical partition functions of the later spin chains would help us in finding out the general form of the $BC_N$ type of multivariate SRS polynomials. In this context it may be noted that, the canonical partition functions of the $BC_N$ type of PF chains with SAPSRO have been computed earlier by using the freezing trick via the direct approach [51]. However it may be recalled that, for the case of $A_{N−1}$ type of PF spin
chains, partition functions obtained by using the above mentioned procedure cannot be connected with the multivariate RS polynomials in a straightforward way. Therefore, in this article we shall derive a new expression for the canonical partition functions of the $BC_N$ type of PF chains with SAPSRO by using the freezing trick via the indirect approach, and subsequently use those partition functions to construct the corresponding multivariate SRS polynomials.

The arrangement of this paper is as follows. In Sec. 2 we shall briefly review some results of Ref. [51] which are relevant for our purpose, like the construction of SAPSRO by using the $BC_N$ type of Weyl algebra and the method of generating PF spin chains with SAPSRO from the related spin Calogero models by applying the freezing trick. Furthermore, we shall describe the Hilbert space associated with the $BC_N$ type of spin Calogero models with SAPSRO and the procedure of deriving the spectra of these models by choosing a partially ordered set of basis vectors where the corresponding Hamiltonians can be expressed in a triangular form. In Sec. 3, we shall compute the grand canonical partition functions of the $BC_N$ type of ferromagnetic as well as anti-ferromagnetic spin Calogero models with SAPSRO, and expand those grand canonical partition functions as some power series of the fugacity parameter to obtain the corresponding canonical partition functions. Applying the freezing trick, subsequently we shall derive novel expressions for the canonical partition functions of the $BC_N$ type of ferromagnetic and anti-ferromagnetic PF chains with SAPSRO. Inspired by the form of such partition functions, in Sec. 4 we shall define $BC_N$ type of homogeneous multivariate SRS polynomials and also find out the corresponding generating functions. Using these generating functions, we shall show that the $BC_N$ type of SRS polynomials can be expressed as some bilinear combinations of the $A_{N-1}$ type of SRS polynomials. Furthermore, we shall derive a set of recursion relations for the partition functions of the $BC_N$ type of PF spin chains involving different numbers of lattice sites and internal degrees of freedom. Sec. 5 is the concluding section.

2. $BC_N$ type of spin models with SAPSRO

It is well known that the $BC_N$ type of Weyl algebra is generated by the elements like $\mathcal{W}_{ij}$ and $\mathcal{W}_i$, which satisfy the relations

$$\mathcal{W}_{ij}^2 = 1, \quad \mathcal{W}_{ij} \mathcal{W}_{jk} = \mathcal{W}_{ik} \mathcal{W}_{lj} = \mathcal{W}_{jk} \mathcal{W}_{ik}, \quad \mathcal{W}_{ij} \mathcal{W}_{kl} = \mathcal{W}_{kl} \mathcal{W}_{ij},$$

$$\mathcal{W}_i^2 = 1, \quad \mathcal{W}_i \mathcal{W}_j = \mathcal{W}_j \mathcal{W}_i, \quad \mathcal{W}_{ij} \mathcal{W}_k = \mathcal{W}_k \mathcal{W}_{ij}, \quad \mathcal{W}_{ij} \mathcal{W}_j = \mathcal{W}_i \mathcal{W}_{ij}, \quad (2.1)$$

where $i, j, k, l \in \{1, 2, \cdots, N\}$ are all different indices. Representations of this Weyl algebra play an important role in constructing $BC_N$ type of quantum integrable spin models with long-range interaction. For the purpose of describing a class of representations of the $BC_N$ type of Weyl algebra (2.1) on a superspace, let us consider a set of operators like $C_{j\alpha}^l (C_{j\alpha})$ which creates (annihilates) a particle of species $\alpha$ on the $j$-th
lattice site. These creation (annihilation) operators are assumed to be bosonic when 
\( \alpha \in [1, 2, \ldots, m] \) and fermionic when \( \alpha \in [m + 1, m + 2, \ldots, m + n] \). Hence, the parity of 
these operators are defined as

\[
\pi(C_{j\alpha}) = \pi(C_{j\alpha}^\dagger) = 0 \text{ for } \alpha \in [1, 2, \ldots, m], \\
\pi(C_{j\alpha}) = \pi(C_{j\alpha}^\dagger) = 1 \text{ for } \alpha \in [m + 1, m + 2, \ldots, m + n],
\]

and they satisfy commutation (anti-commutation) relations given by

\[
[C_{j\alpha}, C_{k\beta}]_\pm = 0, \ [C_{j\alpha}^\dagger, C_{k\beta}]_\pm = 0, \ [C_{j\alpha}, C_{k\beta}^\dagger]_\pm = \delta_{jk}\delta_{\alpha\beta}, \tag{2.2}
\]

where \( [C, D]_\pm \equiv CD - (-1)^{\pi(C)\pi(D)}DC \). Let us now consider a finite dimensional 
subspace of the related Fock space, where each lattice site is occupied by only one 
particle, i.e., \( \sum_{\alpha=1}^{m+n} C_{j\alpha}^\dagger C_{j\alpha} = 1 \) for all \( j \in \{1, 2, \cdots, N\} \). The supersymmetric exchange 
operator \( \hat{P}_{ij}^{(m|n)} \) is defined on such subspace of the Fock space as [5]

\[
\hat{P}_{ij}^{(m|n)} \equiv \sum_{\alpha,\beta=1}^{m+n} C_{i\alpha}^{\dagger} C_{j\beta}^{\dagger} C_{i\beta} C_{j\alpha}. \tag{2.3}
\]

The supersymmetric exchange operator (2.3) can equivalently be expressed as an 
operator on the total internal space of \( N \) number of spins, which is defined in the 
following way [37, 51]. Let us denote such total internal space as \( \Sigma^{(m_1,m_2|m_1,n_2)} \), where 
m_1, m_2, n_1, n_2 are some arbitrary non-negative integers satisfying the relations \( m_1 + 
m_2 = m \) and \( n_1 + n_2 = n \). The space \( \Sigma^{(m_1,m_2|m_1,n_2)} \) is spanned by orthonormal state 
vectors of the form \( |s_1, \ldots, s_i, \ldots, s_N\rangle \), where \( s_i \equiv (s_i^1, s_i^2, s_i^3) \) has three components 
which take discrete values like \( s_i^1 = \pi(s_i) \in \{0, 1\} \), \( s_i^2 = f(s_i) \in \{0, 1\} \), and

\[
s_i^3 \in \begin{cases}
{1, 2, \ldots, m_1}, & \text{if } \pi(s_i) = 0 \text{ and } f(s_i) = 0, \\
{1, 2, \ldots, m_2}, & \text{if } \pi(s_i) = 0 \text{ and } f(s_i) = 1, \\
{1, 2, \ldots, n_1}, & \text{if } \pi(s_i) = 1 \text{ and } f(s_i) = 0, \\
{1, 2, \ldots, n_2}, & \text{if } \pi(s_i) = 1 \text{ and } f(s_i) = 1.
\end{cases} \tag{2.4}
\]

Hence, each local spin vector \( s_i \) may be chosen in \((m + n)\) number of different ways and 
\( \Sigma^{(m_1,m_2|m_1,n_2)} \) can be expressed in a direct product form given by

\[
\Sigma^{(m_1,m_2|m_1,n_2)} \equiv \mathcal{C}_{m+n} \otimes \mathcal{C}_{m+n} \otimes \cdots \otimes \mathcal{C}_{m+n} \otimes \cdots \otimes \mathcal{C}_{m+n}, \tag{2.5}
\]

where \( \mathcal{C}_{m+n} \) denotes an \((m + n)\)-dimensional complex vector space. It is evident that 
this \( \Sigma^{(m_1,m_2|m_1,n_2)} \) is isomorphic to the subspace of the Fock space, on which \( \hat{P}_{ij}^{(m|n)} \) in 
(2.3) is defined. A supersymmetric spin exchange operator \( P_{ij}^{(m|n)} \) is defined on the space 
\( \Sigma^{(m_1,m_2|m_1,n_2)} \) as

\[
P_{ij}^{(m|n)} |s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N\rangle = (-1)^{\alpha_{ij}(s)} |s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N\rangle, \tag{2.6}
\]
where \( \alpha_{ij}(s) = \pi(s_i)\pi(s_j) + (\pi(s_i) + \pi(s_j)) h_{ij}(s) \) and \( h_{ij}(s) = \sum_{k=i+1}^{j-1} \pi(s_k) \). From Eq. (2.6) it follows that, the exchange of two spins with \( \pi(s_i) = \pi(s_j) = 0 \) or \( \pi(s_i) = \pi(s_j) = 1 \) produces a phase factor of 1 or \(-1\) respectively. So we may call \( s_i \) as a 'bosonic' spin if \( s_i^1 \equiv \pi(s_i) = 0 \) and a 'fermionic' spin if \( s_i^1 \equiv \pi(s_i) = 1 \). However, it should be noted the exchange one bosonic spin with one fermionic spin (or, vice versa) produces a nontrivial phase factor of \(-1^{h_{ij}(s)}\), where \( h_{ij}(s) \) represents the number of fermionic spins within the \( i \)-th and \( j \)-th lattice sites. Using the commutation (anti-commutation) relations in (2.2), it can be shown that \( \hat{P}^{(m|n)}_{ij} \) in (2.3) is completely equivalent to \( P^{(m|n)}_{ij} \) in (2.6) [37]. The action of SAPSRO (denoted by \( P^{(m_1,m_2|n_1,n_2)}_i \)) is defined on the space \( \Sigma^{(m_1,m_2|n_1,n_2)} \) as [51]

\[
P^{(m_1,m_2|n_1,n_2)}_i |s_1, \cdots, s_i, \cdots, s_N\rangle = (-1)^{f(s_i)} |s_1, \cdots, s_i, \cdots, s_N\rangle, \quad (2.7)
\]

Hence, the second component of the spin \( s_i \) determines its parity under the action of SAPSRO. It is easy to verify that \( P^{(m|n)}_{ij} \) in (2.6) and \( P^{(m_1,m_2|n_1,n_2)}_i \) in (2.7) respectively yield representations of the elements \( \hat{W}_{ij} \) and \( W_i \) appearing in the \( BC_N \) type of Weyl algebra (2.1). These representations of the Weyl algebra can be used to construct a large class of exactly solvable \( BC_N \) type of ferromagnetic PF spin chains with Hamiltonians given by

\[
H^{(m_1,m_2|n_1,n_2)} = \sum_{i \neq j} \left[ \frac{1 - P^{(m|n)}_{ij}}{(\xi_i - \xi_j)^2} + \frac{1 - \tilde{P}^{(m_1,m_2|n_1,n_2)}_{ij}}{\xi_i^2} \right] + \beta \sum_{i=1}^{N} \frac{1 - P^{(m_1,m_2|n_1,n_2)}_i}{\xi_i^2}, \quad (2.8)
\]

where \( \beta > 0 \) is a real parameter, \( \xi_i = \sqrt{2y_i} \), and \( y_i \) being the \( i \)-th zero point of the generalized Laguerre polynomial \( L_N^{\beta-1} \). It may be noted that, the Hamiltonian (2.8) can reproduce all of the previously studied \( BC_N \) type of PF spin chains for some specific values of the discrete parameters \( m_1, m_2, n_1 \) and \( n_2 \). For example, in the presence of only bosonic or fermionic spins, i.e., when either \( n_1 = n_2 = 0 \) or \( m_1 = m_2 = 0 \), \( H^{(m_1,m_2|n_1,n_2)} \) reduces to the non-supersymmetric PF spin chain associated with PSRO [49]. In another special case, where the discrete parameters in (2.8) satisfy the relations

\[
m_1 = \frac{1}{2} (m + \epsilon \tilde{m}), \quad m_2 = \frac{1}{2} (m - \epsilon \tilde{m}), \quad n_1 = \frac{1}{2} (n + \epsilon' \tilde{n}), \quad n_2 = \frac{1}{2} (n - \epsilon' \tilde{n}), \quad (2.9)
\]

with \( \epsilon, \epsilon' = \pm 1 \), \( \tilde{m} \equiv m \mod 2 \) and \( \tilde{n} \equiv n \mod 2 \), the exactly solvable Hamiltonian (which depends on the parameters \( m, n, \epsilon, \epsilon' \)) of the \( BC_N \) type of PF spin chains with SASRO [48] can be obtained from \( H^{(m_1,m_2|n_1,n_2)} \) through a unitary transformation [51].

Applying the freezing trick, the Hamiltonians (2.8) of the \( BC_N \) type of PF spin chains with SAPSRO can be derived from those of \( BC_N \) type of spin Calogero models containing both coordinate and spin degrees of freedom. The Hamiltonians of such spin
Calogero models are given by

\[
H^{(m_1,m_2|n_1,n_2)} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{a^2}{4} r^2 + a \sum_{i \neq j} \left[ \frac{a - P_{ij}^{(m_1n_1)}}{(x_{ij}^-)^2} + \frac{a - \tilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(x_{ij}^+)^2} \right] + \beta a \sum_{i=1}^{N} \frac{\beta a - P_i^{(m_1,m_2|n_1,n_2)}}{x_i^2},
\]

where \( a > \frac{1}{2} \) is a real coupling constant, \( x_{ij}^- \equiv x_i - x_j \), \( x_{ij}^+ \equiv x_i + x_j \), and \( r^2 \equiv \sum_{i=1}^{N} x_i^2 \).

The coefficient of the \( a^2 \) order term in the r.h.s. of (2.10) may be written as

\[
U(x) = \sum_{i \neq j} \left[ \frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2} \right] + \beta^2 \sum_{i=1}^{N} \frac{1}{x_i^2} + \frac{r^2}{4}.
\]

Since this \( a^2 \) order term in \( H^{(m_1,m_2|n_1,n_2)} \) dominates in the strong coupling limit \( a \rightarrow \infty \), the particles of this spin Calogero model concentrate at the minimum \( \xi \) of the potential \( U(x) \). Consequently, the coordinate and spin degrees of freedom of these particles decouple from each other. Furthermore one can show that, within the configuration space of the Hamiltonian (2.10), the coordinates \( \xi_i \) of the unique minimum \( \xi \) of the potential \( U(x) \) are given by \( \xi_i = \sqrt{2y_i} \), where \( y_i \)’s denote the zeros of the generalized Laguerre polynomial \( L_N^{\beta-1} \) [52]. Consequently, \( H^{(m_1,m_2|n_1,n_2)} \) in (2.10) can be written in \( a \rightarrow \infty \) limit as

\[
H^{(m_1,m_2|n_1,n_2)} \approx H_{sc} + a \mathcal{H}^{(m_1,m_2|n_1,n_2)},
\]

where \( H_{sc} \) is the scalar (spinless) Calogero model of \( BC_N \) type given by

\[
H_{sc} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{a^2}{4} r^2 + a(a - 1) \sum_{i \neq j} \left[ \frac{1}{(x_{ij}^-)^2} + \frac{1}{(x_{ij}^+)^2} \right] + \sum_{i=1}^{N} \frac{\beta a(\beta a - 1)}{x_i^2}. \tag{2.13}
\]

Thus the Hamiltonians (2.8) of the \( BC_N \) type of PF spin chains with SAPSRO emerge naturally from the strong coupling limit of the corresponding spin Calogero models (2.10). Due to Eq. (2.12), the eigenvalues of \( H^{(m_1,m_2|n_1,n_2)} \) satisfy the relation

\[
E_{ij}^{(m_1,m_2|n_1,n_2)} \approx E_{i}^{sc} + a \mathcal{E}_{j}^{(m_1,m_2|n_1,n_2)}, \tag{2.14}
\]

where \( E_{i}^{sc} \) and \( \mathcal{E}_{j}^{(m_1,m_2|n_1,n_2)} \) are two eigenvalues of \( H_{sc} \) and \( \mathcal{H}^{(m_1,m_2|n_1,n_2)} \) respectively. With the help of Eq. (2.14), one can derive an exact formula for the canonical partition function \( Z_{B,N}^{(m_1,m_2|n_1,n_2)}(T) \) of the \( BC_N \) type of PF spin chain (2.8) at a given temperature \( T \) as

\[
Z_{B,N}^{(m_1,m_2|n_1,n_2)}(T) = \lim_{a \rightarrow \infty} \frac{Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)}{Z_{B,N}(aT)}, \tag{2.15}
\]

where \( Z_{B,N}(aT) \) and \( Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT) \) represent canonical partition functions (at the temperature \( aT \)) of the \( BC_N \) type of spinless Calogero model (2.13) and spin Calogero model (2.10) respectively.
The $BC_N$ type of spinless Calogero model (2.13) is a well known exactly solvable system with ground state wave function of the form

$$\mu(x) = e^{-\frac{4}{3}r^2} \prod_i |x_i|^{\beta a} \prod_{i<j} |x_i^2 - x_j^2|^{\alpha},$$

(2.16)

and ground state energy given by

$$E_0 = Na\left(\beta a + a(N-1) + \frac{1}{2}\right).$$

(2.17)

An exact expression for the canonical partition function of the $BC_N$ type of spinless Calogero model (2.13) has been derived earlier as [21]

$$Z_{B,N}(aT) = \frac{1}{N} \prod_{j=1}^N (1 - q^{2j}),$$

(2.18)

where $q \equiv e^{-1/(k_BT)}$ and the contribution from the ground state energy has been ignored without any loss of generality.

The exact spectrum of the $BC_N$ type of spin Calogero Hamiltonian (2.10) can be computed by expressing it in a triangular form while acting on a partially ordered set of basis vectors of the corresponding Hilbert space [51]. As found in the later reference, the Hilbert space associated with this spin Calogero Hamiltonian is the closure of the linear subspace spanned by the wave functions of the form

$$\psi_{r}^s \equiv \psi_{r_1,...,r_i,...,r_j,...,r_N}^s = \Lambda^{m_1,m_2|n_1,n_2}_{r_1,...,r_i,...,r_j,...,r_N} (\phi_r(x)|s\rangle),$$

(2.19)

where $r_i$'s are arbitrary non-negative integers, $\phi_r(x) \equiv \mu(x) \prod_{i=1}^N x_i^{r_i}, |s\rangle \equiv |s_1,\ldots,s_N\rangle$ represents an arbitrary basis element of the spin space $\Sigma^{m_1,m_2|n_1,n_2}$, and $\Lambda^{m_1,m_2|n_1,n_2}_{r_1,...,r_i,...,r_j,...,r_N}$ is a completely symmetric projector related to the $BC_N$ type of Weyl algebra. It can be shown that $\psi_{r}^s$'s in (2.19) satisfy the symmetry conditions

$$\psi_{r_1,...,r_i,...,r_j,...,r_N}^{s_1,...,s_i,...,s_j,...,s_N} = (-1)^{\alpha_{ij}(s)} \psi_{r_1,...,r_i,...,r_j,...,r_N}^{s_1,...,s_j,...,s_i,...,s_N},$$

(2.20)

and

$$\psi_{r_1,...,r_i,...,r_N}^{s_1,...,s_i,...,s_N} = (-1)^{r_i+f(s_i)} \psi_{r_1,...,r_i,...,r_N}^{s_1,...,s_i,...,s_N},$$

(2.21)

where $\alpha_{ij}(s)$ is defined after (2.6) and $1 \leq i < j \leq N$. Due to these symmetry conditions, $\psi_{r}^s$'s corresponding to all possible values of $r$ and $s$ do not form a set of linearly independent basis vectors for the Hilbert space associated with the spin Calogero Hamiltonian $H^{m_1,m_2|n_1,n_2}$. However, $\psi_{r}^s$'s in (2.19) would lead to a complete set of basis vectors if the following three conditions are imposed on the possible values of $r$ and $s$.  

10
i) The lower index $r$ in $\psi^s_r$ is chosen in an ordered form which separately arranges its even and odd components into two non-increasing sequences:

$$r \equiv (r_e, r_o) = (2l_1, \ldots, 2l_1, \ldots, 2l_s, \ldots, 2l_s, 2p_1 + 1, \ldots, 2p_1 + 1, \ldots, 2p_t + 1), \quad (2.22)$$

where $0 \leq s, t \leq N$, $l_1 > l_2 > \ldots > l_s \geq 0$ and $p_1 > p_2 > \ldots > p_t \geq 0$. Since any given $r$ can be brought in the ordered form (2.22) through an appropriate permutation of its components, one can choose this ordered form as a consequence of the symmetry condition (2.20).

ii) The second component of $s_i$ corresponding to each $r_i$ is given by

$$s^2_i \equiv f(s_i) = \begin{cases} 0, & \text{for } r_i \in r_e, \\ 1, & \text{for } r_i \in r_o. \end{cases} \quad (2.23)$$

This is a direct consequence of the symmetry condition (2.21).

iii) If $r_i = r_j$ for $i < j$, then from Eq. (2.23) it follows that the second components of the corresponding spins $s_i$ and $s_j$ must have the same value. In that case, one can further apply Eq. (2.20) to obtain an ordering among $s_i$ and $s_j$, by using the rule $\pi(s_i) \leq \pi(s_j)$, and subsequently, for the case $\pi(s_i) = \pi(s_j)$, by using the rule $s^3_i \geq s^3_j + \pi(s_j)$.

All $\psi^s_r$'s in (2.19), satisfying the above mentioned three conditions, represent a set of (non-orthonormal) basis vectors of the Hilbert space associated with the spin Calogero Hamiltonian in (2.10). If a partial ordering is defined among these basis vectors like $\psi^s_r > \psi^s_{r'}$, for $|r| > |r'|$, where $|r| \equiv \sum_{i=1}^{N} r_i$, it can be shown that $H^{(m_1, m_2|n_1, n_2)}$ in (2.10) acts as an upper triangular matrix on such partially ordered basis vectors:

$$H^{(m_1, m_2|n_1, n_2)} \psi^s_r = E^s_r \psi^s_r + \sum_{|r'| < |r|} C_{r'r} \psi^s_{r'}, \quad (2.24)$$

where $C_{r'r}$'s are real constants, $s'$ is a suitable permutation of $s$ and

$$E^s_r = a|r| + E_0. \quad (2.25)$$

Consequently, all eigenvalues in the spectrum of $H^{(m_1, m_2|n_1, n_2)}$ are given by $E^s_r$ in (2.25), where the quantum numbers $r$ and $s$ satisfy the conditions i)-iii). Since the r.h.s. of (2.25) does not depend on the quantum number $s$, $E^s_r$ has an 'intrinsic degeneracy' which is obtained by counting the number of all possible choice of spin degrees of freedom corresponding to a given $r$. By using the energy levels (2.25) and corresponding intrinsic degeneracy factors, it is possible to directly compute the canonical partition function $Z^{(m_1, m_2|n_1, n_2)}_{B, N}(aT)$ of the spin Calogero model (2.10). Furthermore, by inserting such
expression of $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$ and $Z_{B,N}(aT)$ given in (2.18) to the relation (2.15), one can evaluate the canonical partition function $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(T)$ of the spin chains (2.8) [51].

However the partition functions of the $BC_N$ type of PF spin chains with SAPSRO, obtained in the above mentioned way, have a rather complicated form which can not be expressed through the $q$-multinomial coefficients in a straightforward way. Hence, for the purpose of constructing $BC_N$ type of multivariate SRS polynomials, in the next section we shall derive a new expression for the canonical partition functions of the $BC_N$ type of PF spin chains (2.8) through the indirect approach. More precisely, we shall first compute the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SAPSRO (2.10) and expand such grand canonical partition functions as a power series of the fugacity parameter to obtain the corresponding the canonical partition functions. Substitution of those canonical partition functions to the relation (2.15) would lead to the desired expressions for the canonical partition functions of the spin chains (2.8).

3. Partition functions of $BC_N$ type of spin models with SAPSRO

A remarkable feature of the grand canonical partition functions associated with the $A_{N-1}$ type of spin Calogero models (with harmonic confining potentials) and their supersymmetric generalizations is that such partition functions can be expressed as some simple products of the corresponding ‘basic modes’. For example, the grand canonical partition function $Z_A^{(m|0)}$ of the $m$-flavor bosonic spin Calogero model can be written through the corresponding basic mode, i.e., the grand canonical partition function of the one-flavor (spinless) bosonic Calogero model as [9]

$$Z_A^{(m|0)} = \left(Z_A^{(1|0)}\right)^m.$$  

(3.1)

It is well known that, up to a constant shift of all energy levels, the spectrum of one-flavor bosonic Calogero model of $A_{N-1}$ type coincides with that of the $N$ number of free bosonic oscillators. Dropping the zero-point energy of these oscillators and using an identity given by

$$\sum_{k_1 \geq k_2 \geq \ldots \geq k_N \geq 0} \frac{\sum_{i=1}^{N} k_i}{q} = \frac{1}{(q)_N},$$  

(3.2)

the canonical partition function $Z_A^{(1|0)}(q)$ of such one-flavor bosonic Calogero model with $N$ number of particles can be obtained as

$$Z_A^{(1|0)}(q) = \frac{1}{(q)_N}.$$  

(3.3)
As a result, the corresponding grand canonical partition function may be expressed as

$$Z_A^{(1|0)} = \sum_{N=0}^{\infty} y^N \cdot Z_{A,N}^{(1|0)}(q) = \sum_{N=0}^{\infty} \frac{y^N}{(q)_N},$$  

(3.4)

where \( y \equiv q^{-\mu} \) (with \( \mu \) being the chemical potential) denotes the fugacity parameter. Inserting this expression of \( Z_A^{(1|0)} \) into Eq. (3.1), one can derive the grand canonical partition function of the \( m \)-flavor bosonic spin Calogero model [9].

The grand canonical partition function of the \( \text{su}(m|n) \) supersymmetric spin Calogero model of \( A_{N-1} \) type can also be written as the product of two types of basic modes as [11]

$$Z_A^{(m|n)} = \left( Z_A^{(1|0)} \right)^m \left( Z_A^{(0|1)} \right)^n,$$  

(3.5)

where \( Z_A^{(0|1)} \) represents the grand canonical partition function of the one-flavor (spinless) fermionic Calogero model. By using the identity

$$\sum_{k_1>k_2>\cdots>k_N\geq 0} q^{k_i} = q^{N(N-1)/2} \cdot \frac{1}{(q)_N},$$  

(3.6)

the canonical partition function of such Calogero model with \( N \) number of particles can be derived as

$$Z_{A,N}^{(0|1)}(q) = q^{N(N-1)/2} \cdot \frac{1}{(q)_N},$$  

(3.7)

and the corresponding grand canonical partition function may be obtained as

$$Z_A^{(0|1)} = \sum_{N=0}^{\infty} y^N Z_{A,N}^{(0|1)}(q) = \sum_{N=0}^{\infty} y^N q^{N(N-1)/2} \cdot \frac{1}{(q)_N}.$$  

(3.8)

Inserting \( Z_A^{(1|0)} \) in (3.4) and \( Z_A^{(0|1)} \) in (3.8) into Eq. (3.5), one can derive the grand canonical partition function of the \( \text{su}(m|n) \) supersymmetric spin Calogero model of \( A_{N-1} \) type [11].

It may be noted that, grand canonical partition functions of the \( BC_N \) type of spin Calogero models with SASRO have been computed earlier in Ref. [48]. Those spin Calogero models with SASRO may be considered as some special cases of the \( BC_N \) type of spin Calogero models with SAPSRO (2.10), since the former models can be obtained from the latter ones by imposing the condition (2.9) and also using a unitary transformation. However it has been found in the later reference that, instead of only \( BC_N \) type of basic modes, both \( BC_N \) and \( A_{N-1} \) types of basic modes appear in the expressions of grand canonical partition functions of the spin Calogero models with SASRO. Such a mixture of two different types of basic modes in the expression of the grand canonical partition function is clearly not suitable for our present purpose of constructing \( BC_N \) type of multivariate RS polynomials. In the following, our aim is to
derive the grand canonical partition functions of the \( BC_N \) type of spin Calogero models with SAPSRO (2.10) as simple products of only \( BC_N \) types of basic modes.

In the previous section it has been mentioned that \( \psi^s_r \)'s in (2.19), with indices \( r \) and \( s \) satisfying the rules i)-iii), represent a set of (non-orthonormal) basis vectors of the Hilbert space associated with the spin Calogero Hamiltonian with SAPSRO (2.10).

While these rules for ordering \( r \) and \( s \) are very convenient for computing the canonical partition function of the Hamiltonian (2.10), they are not suitable for computing the corresponding grand canonical partition function and they do not uniquely follow from the symmetry conditions (2.20) and (2.21). Indeed, for the purpose of computing the grand canonical partition function of the Hamiltonian (2.10) from its spectrum, it is necessary to order at first the upper index \( s \) of \( \psi^s_r \) in an appropriate way and subsequently find out the rules which the lower index \( r \) should obey. Hence, instead of using the rules i)-iii), we order the indices \( s \) and \( r \) of the state vectors (2.19) by using the following equivalent set of rules to obtain essentially the same set of complete basis vectors:

1) Let us define the difference between two local spin vectors \( s \equiv (s^1, s^2, s^3) \) and \( \bar{s} \equiv (\bar{s}^1, \bar{s}^2, \bar{s}^3) \) as \( s - \bar{s} = (s^1 - \bar{s}^1, s^2 - \bar{s}^2, s^3 - \bar{s}^3) \), and assume that \( s < \bar{s} \) if the first nonvanishing component of \( s - \bar{s} \) is negative. Using the symmetry condition (2.20), we order the index \( \hat{s} \equiv (s^1_1, s^1_2, \cdots, s^1_N) \) such that \( s^1_i \leq s^1_j \) for \( i < j \).

2) If \( s^1_i = s^1_j \) for \( i < j \), then by using (2.20) we order the components \( r_i \) and \( r_j \) within \( r \equiv (r_1, r_2, \cdots, r_N) \) such that \( r_i \geq r_j + \pi(s^1_i) \).

3) Due to the condition (2.21), \( r_i \) is taken as an even non-negative integer if \( s^2_i \equiv f(s^1_i) = 0 \), and \( r_i \) is taken as an odd positive integer if \( s^2_i \equiv f(s^1_i) = 1 \).

As before, a partial ordering may be defined among these relabeled basis vectors as: \( \psi^s_r > \psi^s_r' \), for \( |r| > |r'| \). It is evident that, in analogy with (2.24), the spin Calogero Hamiltonian (2.10) would act as an upper triangular matrix on such partially ordered basis vectors. As a result, all eigenvalues in the spectrum of \( H^{(m_1, m_2 | n_1, n_2)} \) can equivalently be given by \( E^s_r \) in (2.25), where the indices \( r \) and \( s \) are ordered by using the new set of rules 1)-3).

Next, we assume that the local spin \( s \equiv (s^1, s^2, s^3) \) occurs \( \gamma^{s^1, s^2, s^3} \) times in the configuration \( s \equiv (s_1, s_2, \cdots, s_N) \). It is evident that \( N \) can be written as

\[
N = \sum_{s_1, s^2, s^3} \gamma^{s_1, s^2, s^3}.
\]

Using the condition 1), we explicitly order the configuration \( s \) as

\[
s = (S_1, S_2, S_3, S_4),
\]

(3.10)
where

\[ S_1 = (001), \ldots, (001), \ldots, (00m_1), \ldots, (00m_1), \]
\[ S_2 = (011), \ldots, (011), \ldots, (01m_2), \ldots, (01m_2), \]
\[ S_3 = (101), \ldots, (101), \ldots, (10n_1), \ldots, (10n_1), \]
\[ S_4 = (111), \ldots, (111), \ldots, (11n_2), \ldots, (11n_2). \]  \hspace{1cm} (3.11)

Due to Eqs. (3.12) and (3.13), we obtain a restriction on \( r_i^{s^1, s^2, s^3} \) as

\[ r_i^{s^1, s^2, s^3} \geq r_{i+1}^{s^1, s^2, s^3} + s^1, \]  \hspace{1cm} (3.12)

where \( i \in \{1, 2, \ldots, \gamma^{s^1, s^2, s^3} - 1\} \). Furthermore, by using the condition 3), we can express any \( r_i^{s^1, s^2, s^3} \) in the form

\[ r_i^{s^1, s^2, s^3} = 2k_i^{s^1, s^2, s^3} + \delta_{s^2,1}, \]  \hspace{1cm} (3.13)

where \( k_i^{s^1, s^2, s^3} \) is a non-negative integer. Hence, the lower index \( r \) corresponding to the upper index \( s \) in (3.10) and (3.11) may be written as

\[ r = (R_1, R_2, R_3, R_4), \]  \hspace{1cm} (3.14)

where

\[ R_1 = 2k_1^{001}, \ldots, 2k_1^{001}, \ldots, 2k_1^{00m_1}, \ldots, 2k_1^{00m_1}, \]
\[ R_2 = 2k_1^{011} + 1, \ldots, 2k_1^{011} + 1, \ldots, 2k_1^{01m_2} + 1, \ldots, 2k_1^{01m_2} + 1, \]
\[ R_3 = 2k_1^{101}, \ldots, 2k_1^{101}, \ldots, 2k_1^{10n_1}, \ldots, 2k_1^{10n_1}, \]
\[ R_4 = 2k_1^{111} + 1, \ldots, 2k_1^{111} + 1, \ldots, 2k_1^{11n_2} + 1, \ldots, 2k_1^{11n_2} + 1. \]  \hspace{1cm} (3.15)

Due to Eqs. (3.12) and (3.13), we obtain a restriction on \( k_i^{s^1, s^2, s^3} \) appearing in Eq. (3.15) as

\[ k_i^{s^1, s^2, s^3} \geq k_{i+1}^{s^1, s^2, s^3} + s^1, \]  \hspace{1cm} (3.16)

where \( i \in \{1, 2, \ldots, \gamma^{s^1, s^2, s^3} - 1\} \).
Let us now try to evaluate the grand canonical partition function of the $BC_N$ type of spin CS model with SAPSRO (2.10). To this end, we note that $|r|$ can be written as

$$|r| = \sum_{s_1^1,s_2^2,s_3^3} \sum_{i=1}^{\gamma_{s_1^1,s_2^2,s_3^3}} r_i^{s_1^1,s_2^2,s_3^3}. \quad (3.17)$$

By using the above form of $|r|$, the expression of $N$ given in (3.9) and the energy eigenvalue relation (2.25), we define the grand canonical partition function associated with the Hamiltonian (2.10) as

$$Z_B^{(m_1,m_2|n_1,n_2)}(aT, \mu) = \sum_{\{\gamma_i^{1,2,3}\} \geq 0} \sum_{\{r_j^{1,2,3}\} \geq 0} q^{\sum_{i=1}^{\gamma_{s_1^1,s_2^2,s_3^3}} \left( 2k_i^{s_1^1,s_2^2,s_3^3} + (\delta_{s_2,1} - \mu) \gamma_{s_1^1,s_2^2,s_3^3} \right)} \quad (3.18)$$

where the symbol $\sum_{\{\gamma_i^{1,2,3}\} \geq 0}$ implies multiple sums over all $\gamma_{i}^{1,2,3}$ ranging from 0 to $\infty$, the symbol $\sum_{\{r_j^{1,2,3}\} \geq 0}$ implies restricted multiple sums over all $r_j^{1,2,3}$ (ranging from 0 to $\infty$) which satisfy the conditions (3.12) and (3.13), and the contribution from the ground state energy has been ignored as before. Using (3.13), one can rewrite $|r|$ in (3.17) as

$$|r| = \sum_{s_1^1,s_2^2,s_3^3} \left\{ \sum_{i=1}^{\gamma_{s_1^1,s_2^2,s_3^3}} 2k_i^{s_1^1,s_2^2,s_3^3} + \delta_{s_2,1} \gamma_{s_1^1,s_2^2,s_3^3} \right\}.$$

Inserting the above expression of $|r|$ along with $N$ given in (3.9) into Eq. (3.18), we obtain

$$Z_B^{(m_1,m_2|n_1,n_2)} = \sum_{\{\gamma_i^{1,2,3}\} \geq 0} \sum_{\{r_j^{1,2,3}\} \geq 0} q^{\sum_{i=1}^{\gamma_{s_1^1,s_2^2,s_3^3}} 2k_i^{s_1^1,s_2^2,s_3^3} + (\delta_{s_2,1} - \mu) \gamma_{s_1^1,s_2^2,s_3^3}}, \quad (3.19)$$

where $Z_B^{(m_1,m_2|n_1,n_2)} = Z_B^{(m_1,m_2|n_1,n_2)}(aT, \mu)$, and the symbol $\sum_{\{k_j^{1,2,3}\} \geq 0}$ implies restricted multiple sums over all $k_j^{1,2,3}$ (ranging from 0 to $\infty$) which satisfy the condition (3.16). It is possible to express $Z_B^{(m_1,m_2|n_1,n_2)}$ in (3.19) in a factorized form like

$$Z_B^{(m_1,m_2|n_1,n_2)} = \prod_{s_1^1,s_2^2,s_3^3} \sum_{\gamma_{s_1^1,s_2^2,s_3^3} = 0}^{\infty} q^{(\delta_{s_2,1} - \mu) \gamma_{s_1^1,s_2^2,s_3^3}} \left\{ \sum_{k_1^{1,2,3},\ldots,k_3^{1,2,3} \geq 0} q^{\sum_{i=1}^{\gamma_{s_1^1,s_2^2,s_3^3}} 2k_i^{s_1^1,s_2^2,s_3^3}} \right\}, \quad (3.20)$$
where \( \sum_{k_1^{s_1,s_2,s_3}, \ldots, k_N^{s_1,s_2,s_3}} \) implies restricted multiple sums over the variables \( k_1^{s_1,s_2,s_3}, k_2^{s_1,s_2,s_3}, \ldots, k_N^{s_1,s_2,s_3} \) (ranging from 0 to \( \infty \)) which satisfy the condition (3.16). Let us now rewrite Eq. (3.20) in the form

\[
\mathcal{Z}^{(m_1,m_2|n_1,n_2)}_B = \prod_{s_1,s_2} \left( \prod_{s_3} \mathcal{Z}^{s_1,s_2,s_3}_B \right),
\]

(3.21)

where \( \mathcal{Z}^{s_1,s_2,s_3}_B \) is given by

\[
\mathcal{Z}^{s_1,s_2,s_3}_B = \sum_{\gamma^{s_1,s_2,s_3} = 0}^{\infty} q^{(\delta_{s_2,1}-\mu)} \gamma^{s_1,s_2,s_3} \left\{ \sum_{k_1^{s_1,s_2,s_3}, \ldots, k_1^{s_1,s_2,s_3} \geq 0} q \sum_{n=1}^{\infty} 2k_1^{s_1,s_2,s_3} \right\}. \quad (3.22)
\]

Even though the above expression of \( \mathcal{Z}^{s_1,s_2,s_3}_B \) implicitly depends on \( s_1 \) through the condition (3.16) and explicitly depends on \( s_2 \), it does not depend at all on the value of \( s_3 \). Consequently, by replacing each value of \( s_3 \) with 1, we can express Eq. (3.21) in a factorized form like

\[
\mathcal{Z}^{(m_1,m_2|n_1,n_2)}_B = \left\{ \mathcal{Z}^{0,0,1}_B \right\}^{m_1} \left\{ \mathcal{Z}^{0,1,1}_B \right\}^{m_2} \left\{ \mathcal{Z}^{1,0,1}_B \right\}^{n_1} \left\{ \mathcal{Z}^{1,1,1}_B \right\}^{n_2}. \quad (3.23)
\]

For small values of the discrete parameters satisfying the condition \( m_1 + m_2 + n_1 + n_2 = 1 \) (like \( m_1 = 1, m_2 = n_1 = n_2 = 0 \)), Eq. (3.23) leads to the relations

\[
\mathcal{Z}^{(1,0,0,0)}_B = \mathcal{Z}^{0,0,1}_B, \quad \mathcal{Z}^{(0,1,0,0)}_B = \mathcal{Z}^{0,1,1}_B, \quad \mathcal{Z}^{(0,0,1,0)}_B = \mathcal{Z}^{1,0,1}_B, \quad \mathcal{Z}^{(0,0,0,1)}_B = \mathcal{Z}^{1,1,1}_B. \quad (3.24a,b,c,d)
\]

Hence, we can write the grand canonical partition function (3.23) of the \( BC_N \) type of spin CS model (2.10) completely in terms of the corresponding basic modes as

\[
\mathcal{Z}^{(m_1,m_2|n_1,n_2)}_B = \left\{ \mathcal{Z}^{(1,0,0,0)}_B \right\}^{m_1} \left\{ \mathcal{Z}^{(0,1,0,0)}_B \right\}^{m_2} \left\{ \mathcal{Z}^{(0,0,1,0)}_B \right\}^{n_1} \left\{ \mathcal{Z}^{(0,0,0,1)}_B \right\}^{n_2}. \quad (3.25)
\]

Let us now try to evaluate the four \( BC_N \) type of basic modes appearing in the above relation. Using Eqs. (3.24a) and (3.22), along with the condition (3.16) (for the case \( s_1 = 0, s_2 = 0, s_3 = 1 \)), we obtain

\[
\mathcal{Z}^{(1,0,0,0)}_B = \sum_{N=0}^{\infty} y^N \sum_{k_1^{001} \geq k_2^{001} \geq \cdots \geq k_N^{001} \geq 0} \sum_{q=1}^{N} 2k_1^{001}. \quad (3.26)
\]

Using the identity (3.2), the above equation can be written as

\[
\mathcal{Z}^{(1,0,0,0)}_B = \sum_{N=0}^{\infty} y^N \frac{1}{(q^2)_N}. \quad (3.26)
\]

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Next, by using Eqs. (3.24b) and (3.22), along with the condition (3.16) (for the case \( s^1 = 0, s^2 = 1, s^3 = 1 \)), we obtain

\[
Z_B^{(0,1|0,0)} = \sum_{N=0}^{\infty} y^N q^N \sum_{k_1^{011} \geq k_2^{011} \geq \ldots \geq k_N^{011} \geq 0} \sum_{i=1}^{N} 2k_i^{011}.
\]

Again, using the identity (3.2), the above equation can be written as

\[
Z_B^{(0,1|0,0)} = \sum_{N=0}^{\infty} y^N \frac{q^{N(N-1)}}{(q^2)_N}. \tag{3.27}
\]

Next, by using Eqs. (3.24c) and (3.22), along with the condition (3.16) (for the case \( s^1 = 1, s^2 = 0, s^3 = 1 \)), we obtain

\[
Z_B^{(0,0|1,0)} = \sum_{N=0}^{\infty} y^N \sum_{k_1^{101} > k_2^{101} > \ldots > k_N^{101} \geq 0} \sum_{i=1}^{N} 2k_i^{101},
\]

which, due to the identity (3.6), leads to

\[
Z_B^{(0,0|1,0)} = \sum_{N=0}^{\infty} y^N \frac{q^{N(N-1)}}{(q^2)_N}. \tag{3.28}
\]

Finally, by using Eqs. (3.24d) and (3.22), along with the condition (3.16) (for the case \( s^1 = 1, s^2 = 1, s^3 = 1 \)), we obtain

\[
Z_B^{(0,0|0,1)} = \sum_{N=0}^{\infty} y^N q^N \sum_{k_1^{111} > k_2^{111} > \ldots > k_N^{111} \geq 0} \sum_{i=1}^{N} 2k_i^{111},
\]

which, due to the identity (3.6), leads to

\[
Z_B^{(0,0|0,1)} = \sum_{N=0}^{\infty} y^N \frac{q^{N^2}}{(q^2)_N}. \tag{3.29}
\]

The grand canonical partition function \( Z_B^{(m_1,m_2|n_1,n_2)} \) can be formally expanded as a power series of the fugacity parameter \( y \) as

\[
Z_B^{(m_1,m_2|n_1,n_2)} = \sum_{N=0}^{\infty} y^N Z_N^{(m_1,m_2|n_1,n_2)}(aT).
\]

Inserting the expressions of the basic modes given in Eqs. (3.26), (3.27), (3.28) and (3.29) into the r.h.s. of Eq. (3.25), and comparing this r.h.s. of the latter equation with
that of Eq. (3.30), we obtain a new expression for the canonical partition function of the $BC_N$ type of spin CS model (2.10) as

$$Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT) = \sum_{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N} \frac{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}}{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}} \cdot (3.31)$$

Inserting the above expression of $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)$, along with $Z_{B,N}(aT)$ given in (2.18), into the relation (2.15), we also get a new expression for the canonical partition function of the $BC_N$ type of ferromagnetic PF chain (2.8) as

$$Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \sum_{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N} \frac{(q^2)^N \prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}}{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}} \cdot (3.32)$$

where, for the sake of convenience, the variable $q$ is used (instead of $T$) as the argument of $Z_{B,N}^{(m_1,m_2|n_1,n_2)}$. It may be noted that, $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ in (3.32) can be rewritten by using the $q^2$-multinomial coefficients as

$$Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q) = \sum_{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N} \frac{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}}{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}} \cdot (3.33)$$

where the notations $\{a\}_{m_1} = a_1, \ldots, a_{m_1}$, $\{b\}_{m_2} = b_1, \ldots, b_{m_2}$, $\{c\}_{n_1} = c_1, \ldots, c_{n_1}$ and $\{d\}_{n_2} = d_1, \ldots, d_{n_2}$ are used.

We like to make a comment here on the rather surprising appearance of both $BC_N$ and $A_{N-1}$ types of basic modes in the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SASRO, as found in Ref. [48]. Choosing $m$ as any odd number, $n$ as any even number and taking two discrete parameters as $\epsilon = \epsilon' = 1$, the grand canonical partition function $Z_{11}^{(m|n)}$ of the $BC_N$ type of spin Calogero model with SASRO has been computed in the latter reference as

$$Z_{11}^{(m|n)} = Z_B^{(1,0|0,0)} \left\{ Z_A^{(1|0)} \right\}_{m-1}^{\frac{m-1}{2}} \left\{ Z_A^{(0|1)} \right\}_{n-1}^{\frac{n-1}{2}} \cdot (3.34)$$

where we have used the notations of the present paper for all basic modes appearing in the r.h.s. of the above equation. Since spin Calogero models with SASRO can be reproduced from the more general class of spin Calogero models with SAPSRO (2.10).
by imposing the condition (2.9), it should be possible to obtain the grand canonical partition functions of the former models from those of the later models by imposing the same condition. If \( m \) is an odd number, \( n \) is an even number and \( \epsilon = \epsilon' = 1 \), the condition (2.9) yields \( m_1 = (m+1)/2 \), \( m_2 = (m-1)/2 \) and \( n_1 = n_2 = n/2 \). Substituting these values of \( m_1 \), \( m_2 \), \( n_1 \) and \( n_2 \) in Eq. (3.25), and also replacing the corresponding \( Z_B^{(m_1,m_2|m_1,n_2)} \) with the notation \( Z_B^{(m|n)} \), we find that

\[
Z_{11}^{(m|n)} = Z_B^{(1,0|0,0)} \left\{ Z_B^{(1,0|0,0)} Z_B^{(0,1|0,0)} \right\}^{\frac{m-1}{2}} \left\{ Z_B^{(0,0|1,0)} Z_B^{(0,0|0,1)} \right\}^{\frac{n}{2}}. \quad (3.35)
\]

Equating the r.h.s. of Eq. (3.34) with that of Eq. (3.35), we obtain novel relations like

\[
Z_A^{(1|0)} = Z_B^{(1,0|0,0)} Z_B^{(0,1|0,0)}, \quad Z_A^{(0|1)} = Z_B^{(0,0|1,0)} Z_B^{(0,0|0,1)}, \quad \text{(3.36a,b)}
\]

which connects the \( A_{N-1} \) and \( BC_N \) types of basic modes associated with the grand canonical partition functions of the corresponding one-flavor Calogero models. Inserting the expressions of \( Z_A^{(1|0)} \) in (3.4), \( Z_B^{(1,0|0,0)} \) in (3.26), and \( Z_B^{(0,0|0,1)} \) in (3.27) into Eq. (3.36a), and comparing the coefficients of \( q^N \) from both sides of the later equation, we obtain a \( q \)-identity of the form

\[
\frac{1}{(q)_N} = \sum_{r=0}^{N} \frac{q^{N-r}}{(q^2)_r \cdot (q^2)_{N-r}}. \quad (3.37)
\]

Similarly, inserting the expressions of \( Z_A^{(0|1)} \) in (3.8), \( Z_B^{(0,0|1,0)} \) in (3.28), and \( Z_B^{(0,0|0,1)} \) in (3.29) into Eq. (3.36b), and comparing the coefficients of \( q^N \) from both sides of the later equation, we obtain another \( q \)-identity of the form

\[
\frac{1}{(q)_N} = \sum_{r=0}^{N} \frac{q^{4(N-2r)(N-2r+1)}}{(q^2)_r \cdot (q^2)_{N-r}}. \quad (3.38)
\]

One can easily verify that, for all possible choice of the parameters \( m, n, \epsilon \) and \( \epsilon' \), the grand canonical partition functions of the spin Calogero models with SASRO [48] can be reproduced in a similar way from Eq. (3.25) by using the condition (2.9) and the relations (3.36a,b). Hence, the appearance of both \( BC_N \) and \( A_{N-1} \) types of basic modes in the grand canonical partition functions of the \( BC_N \) type of spin Calogero models with SASRO can be explained by employing the relations (3.36a,b).

It may be noted that, following a procedure similar to the case of \( BC_N \) type of ferromagnetic PF chain (2.8), one can also calculate the partition function of the \( BC_N \) type of anti-ferromagnetic PF chain with Hamiltonian given by

\[
\tilde{\mathcal{H}}^{(m_1,m_2|n_1,n_2)} = \sum_{i \neq j} \left[ \frac{1 + P_{ij}^{(m|n)}}{(\xi_i - \xi_j)^2} + \frac{1 + \tilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(\xi_i + \xi_j)^2} \right] + \beta \sum_{i=1}^{N} \frac{1 + P_{ij}^{(m_1,m_2|n_1,n_2)}}{\xi_i^2}. \quad (3.39)
\]
By using the freezing trick, the above Hamiltonian can be obtained from the \(BC_N\) type of spin Calogero Hamiltonian like

\[
\tilde{H}^{(m_1,m_2|n_1,n_2)} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{a^2}{4} r_i^2 + a \sum_{i \neq j} \left[ \frac{a + P_{ij}^{(m_n)}}{(x_{ij})^2} + \frac{a + \tilde{P}_{ij}^{(m_1,m_2|n_1,n_2)}}{(x_{ij})^2} \right] + \beta a \sum_{i=1}^{N} \frac{\alpha a + \tilde{P}_{i}^{(m_1,m_2|n_1,n_2)}}{x_i^2}. \quad (3.40)
\]

Hence, we can derive an analogue of Eq. (2.15) as

\[
\tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(T) = \lim_{a \to \infty} \frac{Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)}{Z_{B,N}(aT)}, \quad (3.41)
\]

where \(\tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(T)\) and \(Z_{B,N}^{(m_1,m_2|n_1,n_2)}(aT)\) represent canonical partition functions of the \(BC_N\) type of anti-ferromagnetic PF spin chain (3.39) and spin Calogero model (3.40) respectively. The Hilbert space of \(\tilde{H}^{(m_1,m_2|n_1,n_2)}\) in (3.40) can be obtained as the closure of the linear subspace spanned by the wave functions which are quite similar to their ferromagnetic counterparts (2.19). More precisely, the state vectors associated with the Hilbert space of \(\tilde{H}^{(m_1,m_2|n_1,n_2)}\) are given by

\[
\tilde{\psi}_{r}^{s} \equiv \tilde{\psi}_{r_1,\ldots,r_1,\ldots,r_{j},\ldots,r_{N}}^{s_1,\ldots,s_{i},\ldots,s_{j},\ldots,s_{N}} = \tilde{\Lambda}^{(m_1,m_2|n_1,n_2)}(\phi_{r}(x)|s)), \quad (3.42)
\]

where the completely symmetric projector \(\Lambda^{(m_1,m_2|n_1,n_2)}\) in (2.19) is replaced by the completely antisymmetric projector \(\tilde{\Lambda}^{(m_1,m_2|n_1,n_2)}\) related to the \(BC_N\) type of Weyl algebra. Due to this change of the projector, the symmetry conditions (2.20) and (2.21) of the wave functions are now modified as

\[
\tilde{\psi}_{r_1,\ldots,r_1,\ldots,r_{j},\ldots,r_{N}}^{s_1,\ldots,s_{i},\ldots,s_{j},\ldots,s_{N}} = (-1)^{\alpha_{ij}(s)+1} \tilde{\psi}_{r_1,\ldots,r_{j},\ldots,r_{i},\ldots,r_{N}}^{s_1,\ldots,s_{j},\ldots,s_{i},\ldots,s_{N}}, \quad (3.43)
\]

and

\[
\tilde{\psi}_{r_1,\ldots,r_{j},\ldots,r_{i},\ldots,r_{N}}^{s_1,\ldots,s_{i},\ldots,s_{j},\ldots,s_{N}} = (-1)^{r_i+f(s_i)+1} \tilde{\psi}_{r_1,\ldots,r_{i},\ldots,r_{j},\ldots,r_{N}}^{s_1,\ldots,s_{i},\ldots,s_{j},\ldots,s_{N}}, \quad (3.44)
\]

where \(1 \leq i < j \leq N\). Due to these modified symmetry conditions, it is possible to obtain a set of (non-orthonormal) basis vectors for the Hilbert space of \(\tilde{H}^{(m_1,m_2|n_1,n_2)}\) by ordering the indices \(s\) and \(r\) of the state vectors \(\tilde{\psi}_{r}^{s}\) in a suitable way. More precisely, the rule 1) described in this section for ordering \(s\) in the ferromagnetic case remains unchanged in the present case, while the rules 2) and 3) for ordering \(r\) in the ferromagnetic case are modified in the following way:

2') If \(s_i = s_j\) for \(i < j\), then by using (3.43) the components \(r_i\) and \(r_j\) within \(r \equiv (r_1, r_2, \ldots, r_N)\) are ordered such that \(r_i \geq r_j + 1 - \pi(s_j)\).

3') Due to the condition (3.44), \(r_i\) is taken as an odd positive integer if \(s_i^2 \equiv f(s_i) = 0\) and \(r_i\) is taken as an even nonnegative integer if \(s_i^2 \equiv f(s_i) = 1\).
If one defines a partial ordering among the above mentioned basis vectors as: \( \tilde{\psi}_r^s > \tilde{\psi}_r^{s'} \), for \( |r| > |r'| \), then \( \tilde{H}^{(m_1,m_2|n_1,n_2)} \) in (3.40) would act as an upper triangular matrix on such partially ordered basis vectors. Consequently, all eigenvalues in the spectrum of \( \tilde{H}^{(m_1,m_2|n_1,n_2)} \) are related to their ferromagnetic counterparts as

\[
\tilde{Z}_B^{(m_1,m_2|n_1,n_2)} = \left\{ \tilde{Z}_B^{(1,0,0,0)} \right\}^{m_1} \left\{ \tilde{Z}_B^{(0,1,0,0)} \right\}^{m_2} \left\{ \tilde{Z}_B^{(0,0,1,0)} \right\}^{n_1} \left\{ \tilde{Z}_B^{(0,0,0,1)} \right\}^{n_2},
\]

where the anti-ferromagnetic basic modes appearing in the r.h.s. of the above equation are related to their ferromagnetic counterparts as

\[
\tilde{Z}_B^{(1,0,0,0)} = Z_B^{(0,0,1,0)}, \quad \tilde{Z}_B^{(0,0,0,1)} = Z_B^{(1,0,0,0)}.
\]

Expanding the grand canonical partition function (3.45) as a power series of the fugacity parameter, we derive the corresponding canonical partition function as

\[
\tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)} (aT) = \sum_{\substack{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N \atop a_i \geq 0, b_j \geq 0, c_k \geq 0, d_l \geq 0}} \frac{\prod_{i=1}^{m_1} (q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}}{\prod_{i=1}^{m_1} (q^{a_i}) \prod_{j=1}^{m_2} (q^{b_j}) \prod_{k=1}^{n_1} (q^{c_k}) \prod_{l=1}^{n_2} (q^{d_l})}. \tag{3.47}
\]

Substituting this expression of \( \tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)} (aT) \), along with \( Z_{B,N}(aT) \) given in (2.18), to the relation (3.41), we finally obtain the canonical partition function of the \( BC_N \) type of anti-ferromagnetic PF chain (3.39) as

\[
\tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)} (q) = \sum_{\substack{\sum_{i=1}^{m_1} a_i + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N \atop a_i \geq 0, b_j \geq 0, c_k \geq 0, d_l \geq 0}} \frac{(q^2)_{a_i} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} (q^2)_{c_k} \prod_{l=1}^{n_2} (q^2)_{d_l}}{\prod_{i=1}^{m_1} (q^{a_i}) \prod_{j=1}^{m_2} (q^{b_j}) \prod_{k=1}^{n_1} (q^{c_k}) \prod_{l=1}^{n_2} (q^{d_l})}. \tag{3.48}
\]

In the next section, we shall use the expressions (3.32) and (3.48) for the partition functions of the \( BC_N \) type of PF spin chains for the purpose of constructing corresponding multivariate SRS polynomials.

4. SRS polynomials associated with \( BC_N \) type of PF spin chains

We have seen earlier that the homogeneous multivariate RS polynomial (1.3) is closely related to the partition function (1.2) of the \( A_{N-1} \) type of su(\( m \)) PF spin chain.
Before starting the discussion on the $BC_N$ type of homogeneous multivariate SRS polynomial, let us briefly review the connection between the partition function of the $A_{N-1}$ type of su$(m|n)$ supersymmetric PF spin chain and the corresponding SRS polynomial [43]. The partition functions of the $A_{N-1}$ type of su$(m|n)$ supersymmetric PF spins have been computed by using the freezing trick as [11]

\[ Z_{A,N}^{(m|n)}(q) = \sum_{\substack{m \atop a_i \geq 0, b_j \geq 0}} \frac{\left(\sum_{i=1}^{m} b_i (b_i - 1) \right)}{2^{m}} \prod_{i=1}^{m} (q)_{a_i} \prod_{j=1}^{n} (q)_{b_j}. \] (4.1)

It may be noted that, in the absence of the fermionic spin degrees of freedom (i.e., for the case $n = 0$), $Z_{A,N}^{(m|n)}(q)$ reduces to $Z_{A,N}^{(m)}(q)$ in (1.2). Motivated by the form of the partition functions (4.1), the $A_{N-1}$ type of SRS polynomials have been defined as [43]

\[ H_{A,N}^{(m|n)}(x, y; q) = \sum_{\substack{m \atop a_i \geq 0, b_j \geq 0}} \frac{\left(\sum_{i=1}^{m} b_i (b_i - 1) \right)}{2^{m}} \prod_{i=1}^{m} (x)_{a_i} \prod_{j=1}^{n} (y)_{b_j}. \] (4.2)

(along with $H_{A,0}^{(m|n)}(x, y; q) = 1$), where $x \equiv x_1, x_2, \ldots, x_m$ and $y \equiv y_1, y_2, \ldots, y_n$ represent two different types of variables. It is evident that the partition functions (4.1) can be obtained from the SRS polynomials (4.2) as $Z_{A,N}^{(m|n)}(q) = H_{A,N}^{(m|n)}(x = 1, y = 1; q)$. Moreover, for the special case $n = 0$, the SRS polynomial (4.2) reduces to its bosonic counterpart (1.3). By using the relation

\[ (q^{-1})_l = (-1)^l q^{\frac{l(l+1)}{2}} (q)_l, \] (4.3)

the SRS polynomials (4.2) may be rewritten as

\[ H_{A,N}^{(m|n)}(x, y; q) = \sum_{\substack{m \atop a_i \geq 0, b_j \geq 0}} (q)_N \cdot \prod_{i=1}^{m} (x)_{a_i} \prod_{j=1}^{n} (-q^{-1} y_j)_{b_j}. \] (4.4)

The above form of the SRS polynomials can be obtained from a power series expansion of the generating function given by [43]

\[ G_{A}^{(m|n)}(x, y; q, t) = \frac{1}{\prod_{i=1}^{m} (tx_i; q)_\infty \prod_{j=1}^{n} (-tq^{-1} y_j; q^{-1})_\infty}, \] (4.5)

where $(t; q)_0 \equiv 1$ and $(t; q)_l \equiv (1 - t)(1 - qt) \cdots (1 - q^{l-1} t)$ for $l > 0$. Indeed, by using the identity [44]

\[ \frac{1}{(t; q)_\infty} = \sum_{N=0}^{\infty} \frac{t^N}{(q)_N}, \] (4.6)
it is easy to check that the generating function in (4.5) can be expanded as a power series of the parameter \( t \) as

\[
G_A^{(m_1)}(x, y; q, t) = \sum_{N=0}^{\infty} \frac{G_{A,N}^{(m_1)}(x, y; q)}{(q)_N} t^N.
\]  

(4.7)

Inspired by the form of partition functions (3.32) of the \( BC_N \) type of ferromagnetic PF chains with SAPSRO, we define \( BC_N \) type of homogeneous multivariate SRS polynomials of the first kind as

\[
\mathcal{H}_{B,N}^{(m_1, m_2|n_1, n_2)}(x, \bar{x}, y, \bar{y}; q) = 1, \quad \text{where } x \equiv x_1, x_2, \ldots, x_{m_1}, \bar{x} \equiv \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{m_2},
\]

\[
y \equiv y_1, y_2, \ldots, y_{n_1}, \bar{y} \equiv \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n_2}
\]

and set \( \mathcal{H}_{B,0}^{(m_1, m_2|n_1, n_2)}(x, \bar{x}, y, \bar{y}; q) = 1 \), where \( x \equiv x_1, x_2, \ldots, x_{m_1}, \bar{x} \equiv \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{m_2}, \)

\[
y \equiv y_1, y_2, \ldots, y_{n_1}, \bar{y} \equiv \bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{n_2}
\]

represent four different types of variables. The partition functions in (3.32) can be obtained from these \( BC_N \) type of SRS polynomials as

\[
Z_{B,N}^{(m_1, m_2|n_1, n_2)}(q) = \mathcal{H}_{B,N}^{(m_1, m_2|n_1, n_2)}(x = 1, \bar{x} = 1, y = 1, \bar{y} = 1; q).
\]  

(4.9)

In the absence of the fermionic spin degrees of freedom, i.e., for the case \( n_1 = n_2 = 0 \), the \( BC_N \) type of SRS polynomials (4.8) reduce to that type of RS polynomials of the form

\[
\mathcal{H}_{B,N}^{(m_1, m_2)}(x, \bar{x}; q) = \sum_{\sum_{j=1}^{m_1} a_j + \sum_{j=1}^{m_2} b_j = 0} (q^2)_N \cdot \mathcal{Q}^{(m_1, m_2)}(x_i^{a_i} \prod_{j=1}^{m_2} (\bar{x}_j)^{b_j}).
\]  

(4.10)

Interestingly, in another special case like \( m_1 = m, m_2 = 0, n_1 = n, n_2 = 0 \), the \( BC_N \) type of SRS polynomials (4.8) can be connected with the \( A_{N-1} \) type of SRS polynomials (4.2) as

\[
\mathcal{H}_{B,N}^{(m, 0|n, 0)}(x, y; q) = \mathcal{H}_{A,N}^{(m|n)}(x, y; q^2).
\]  

(4.11)

By using the relation (4.3), we express the \( BC_N \) type of SRS polynomials (4.8) in a more compact form as

\[
\mathcal{H}_{B,N}^{(m_1, m_2|n_1, n_2)}(x, \bar{x}, y, \bar{y}; q)
\]

\[
= \sum_{\sum_{j=1}^{m_1} a_j + \sum_{j=1}^{m_2} b_j + \sum_{k=1}^{n_1} c_k + \sum_{l=1}^{n_2} d_l = N} (q^2)_N \cdot \mathcal{Q}^{(m_1, m_2)}(x_i^{a_i} \prod_{j=1}^{m_2} (\bar{x}_j)^{b_j} \prod_{k=1}^{n_1} (-q^{-2} y_k)^{c_k} \prod_{l=1}^{n_2} (-q^{-1} \bar{y}_l)^{d_l}).
\]  

(4.12)
Let us now define a generating function of the form
\[
G^{(m_1,m_2|n_1,n_2)}_B(x, \bar{x}, y, \bar{y}; q, t) = G^{(m_1)}_1(x; q, t) \cdot G^{(m_2)}_2(\bar{x}; q, t) \cdot G^{(n_1)}_3(y; q, t) \cdot G^{(n_2)}_4(\bar{y}; q, t),
\]  
(4.13)

where
\[
G^{(m_1)}_1(x; q, t) = \frac{1}{\prod_{i=1}^{m_1} (tx_i; q^2)_\infty}, \tag{4.14a}\\
G^{(m_2)}_2(\bar{x}; q, t) = \frac{1}{\prod_{j=1}^{m_2} (tq \bar{x}_j; q^2)_\infty}, \tag{4.14b}\\
G^{(n_1)}_3(y; q, t) = \frac{1}{\prod_{k=1}^{n_1} (-tq^{-2}y_k; q^{-2})_\infty}, \tag{4.14c}\\
G^{(n_2)}_4(\bar{y}; q, t) = \frac{1}{\prod_{l=1}^{n_2} (-tq^{-1}\bar{y}_l; q^{-2})_\infty}. \tag{4.14d}
\]

Expanding all terms appearing in the r.h.s. of Eq. (4.13) by using the identity (4.6) and subsequently using the expression of the \( BC_N \) type of SRS polynomials given in (4.12), we obtain
\[
G^{(m_1,m_2|n_1,n_2)}_B(x, \bar{x}, y, \bar{y}; q, t) = \sum_{N=0}^{\infty} \frac{\mathbb{H}^{(m_1,m_2|n_1,n_2)}_{B,N}(x, \bar{x}, y, \bar{y}; q)}{(q^2)_N} t^N. \tag{4.15}
\]

Thus \( G^{(m_1,m_2|n_1,n_2)}_B(x, \bar{x}, y, \bar{y}; q, t) \) in (4.13) represents the generating function of the \( BC_N \) type of SRS polynomials.

We have already seen in Eq. (4.11) that, in a particular case, the \( BC_N \) type of SRS polynomial can be expressed through the \( A_{N-1} \) type of SRS polynomial. For the purpose of exploring such connection between the \( BC_N \) and \( A_{N-1} \) types of SRS polynomials in a general case, we use Eqs. (4.14a), (4.14c) and (4.5) to find that
\[
G^{(m_1)}_1(x; q, t) \cdot G^{(n_1)}_3(y; q, t) = G^{(m_1|n_1)}_A(x, y; q^2, t). \tag{4.16}
\]

Hence, by using the power series expansion (4.7), we obtain
\[
G^{(m_1)}_1(x; q, t) \cdot G^{(n_1)}_3(y; q, t) = \sum_{N_1=0}^{\infty} \frac{\mathbb{H}^{(m_1|n_1)}_{A,N_1}(x, y; q^2)}{(q^2)_{N_1}} t^{N_1}. \tag{4.17}
\]

Next, by using Eqs. (4.14b), (4.14d) and (4.5), we find that
\[
G^{(m_2)}_2(\bar{x}; q, t) \cdot G^{(n_2)}_4(\bar{y}; q, t) = G^{(m_2|n_2)}_A(\bar{x}, \bar{y}; q^2, t), \tag{4.18}
\]
where $\tilde{x} \equiv q \cdot \bar{x}$ and $\tilde{y} \equiv q \cdot \bar{y}$. Hence, by using (4.7), we obtain

$$G^{(m_2)}_2(x; q, t) \cdot G^{(n_2)}_4(y; q, t) = \sum_{N_2=0}^{\infty} \frac{\mathbb{H}^{(m_2|n_2)}_{A,N_2}(x; \tilde{y}; q^2)}{(q^2)^{N_2}} t^{N_2}. \quad (4.19)$$

Since $\mathbb{H}^{(m_2|n_2)}_{A,N_2}(x; \tilde{y}; q^2)$ is a homogeneous polynomial of the variables $\tilde{x}$, $\tilde{y}$ of order $N_2$, the above equation can be rewritten as

$$G^{(m_2)}_2(x; q, t) \cdot G^{(n_2)}_4(y; q, t) = \sum_{N_2=0}^{\infty} \frac{q^{N_2}}{(q^2)^{N_2}} \mathbb{H}^{(m_2|n_2)}_{A,N_2}(\tilde{x}, \tilde{y}; q^2) t^{N_2}. \quad (4.20)$$

Inserting the series expansions (4.17) and (4.20) in Eq. (4.13), it is easy to find that

$$G^{(m_1,m_2|n_1,n_2)}_B(x, \tilde{x}, y, \tilde{y}; q, t) = \sum_{N=0}^{\infty} t^N \sum_{N_1=0}^{N} \frac{q^{N-N_1}}{(q^2)^{N_1} (q^2)^{N-N_1}} \mathbb{H}^{(m_1|n_1)}_{A,N_1}(x, y; q^2) \cdot \mathbb{H}^{(m_2|n_2)}_{A,N-N_1}(\tilde{x}, \tilde{y}; q^2). \quad (4.21)$$

Comparing the coefficients of $t^N$ in the r.h.s. of (4.15) and (4.21), we finally obtain a relation between the $BC_N$ and $A_{N-1}$ types of SRS polynomials as

$$\mathbb{H}^{(m_1,m_2|n_1,n_2)}_{B,N}(x, \tilde{x}, y, \tilde{y}; q, t) = \sum_{N_1=0}^{N} q^{N-N_1} \binom{N}{N_1} \mathbb{H}^{(m_1|n_1)}_{A,N_1}(x, y; q^2) \cdot \mathbb{H}^{(m_2|n_2)}_{A,N-N_1}(\tilde{x}, \tilde{y}; q^2), \quad (4.22)$$

where the notation $\binom{N}{N_1/q^2} \equiv \left[\frac{N}{N_1,N-N_1}\right] q^2$ has been used.

It may be noted that, even though the $BC_N$ type of partition function given in (3.32) is rather complicated in form, it can be easily computed for arbitrary values of $N$ and for some small values of the discrete parameters $m_1$, $m_2$, $n_1$ and $n_2$. In particular, by using (3.32), it is easy to find that

$$Z_{B,N}^{(1,0|0,0)}(q) = 1, \quad Z_{B,N}^{(0,1|0,0)}(q) = q^N, \quad Z_{B,N}^{(0,0|1,0)}(q) = q^{N(N-1)}, \quad Z_{B,N}^{(0,0|0,1)}(q) = q^{N^2}. \quad (4.23)$$

In this context it is interesting to ask whether there exists some recursion relations such that, by taking the partition functions given in (4.23) as the initial conditions, it is possible to compute $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ for arbitrarily values of the discrete parameters $m_1$, $m_2$, $n_1$, $n_2$ and $N$. To answer this question, we define a generating function as

$$G^{(m_1,m_2|n_1,n_2)}_B(q, t) \equiv G^{(m_1,m_2|n_1,n_2)}_B(x = 1, \tilde{x} = 1, y = 1, \tilde{y} = 1; q, t). \quad (4.24)$$

Inserting $x = 1$, $\tilde{x} = 1$, $y = 1$, $\tilde{y} = 1$ in (4.15) and also using (4.9), one can expand $G^{(m_1,m_2|n_1,n_2)}_B(q, t)$ in a power series of $t$ as

$$G^{(m_1,m_2|n_1,n_2)}_B(q, t) = \sum_{N=0}^{\infty} \frac{Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q)}{(q^2)_N} t^N, \quad (4.25)$$
where it is assumed that $Z_{B,0}^{(m_1,m_2|n_1,n_2)}(q) = 1$. Therefore, $G_B^{(m_1,m_2|n_1,n_2)}(q,t)$ may be considered as the generating function for the partition function $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$. Next, by using Eqs. (4.13), (4.14) and (4.24), we find that

$$G_B^{(m_1,m_2|n_1,n_2)}(q,t) = \frac{1}{\{(t; q^2)_\infty\}^{m_1} \cdot \{(tq; q^2)_\infty\}^{m_2} \cdot \{(-tq^2; q^{-2})_\infty\}^{n_1} \cdot \{(-tq^{-1}; q^{-2})_\infty\}^{n_2}}.$$  

Consequently, this generating function satisfies a factorization relation given by

$$G_B^{(m_1+m'_1,m_2+m'_2|n_1+n'_1,n_2+n'_2)}(q,t) = G_B^{(m_1,m_2|n_1,n_2)}(q,t) \cdot G_B^{(m'_1,m'_2|n'_1,n'_2)}(q,t). \quad (4.26)$$

Expanding both sides of the above equation by using (4.25) and comparing the coefficients of $t^N$, we find that

$$Z_{B,N}^{(m_1+m'_1,m_2+m'_2|n_1+n'_1,n_2+n'_2)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} q^{N_1} Z_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q) \cdot Z_{B,N_1}^{(m'_1,m'_2|n'_1,n'_2)}. \quad (4.27)$$

 Appropriately choosing the values of discrete variables $m'_1$, $m'_2$, $n'_1$, $n'_2$ in Eq. (4.27) and also using Eq. (4.23), we derive a set of recursion relations like

$$Z_{B,N}^{(m_1+1,m_2|n_1,n_2)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} q^{N_1} Z_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q),$$

$$Z_{B,N}^{(m_1,m_2+1|n_1,n_2)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} q^{N_1} Z_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q),$$

$$Z_{B,N}^{(m_1,m_2|n_1+1,n_2)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} q^{N_1(N_1-1)} Z_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q),$$

$$Z_{B,N}^{(m_1,m_2|n_1,n_2+1)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} q^{N_1^2} Z_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q). \quad (4.28)$$

By using this set of recursion relations and also using the initial conditions in (4.23), it is possible to compute $Z_{B,N}^{(m_1,m_2|n_1,n_2)}(q)$ for arbitrarily values of the discrete parameters $m_1$, $m_2$, $n_1$, $n_2$ and $N$. Furthermore, it is easy to check that, all initial conditions appearing in (4.23) can also be derived from the recursion relations (4.28) by using only one initial condition given by

$$Z_{B,N}^{(0,0|0,0)}(q) = \delta_{N,0}. \quad (4.29)$$

Next, we define $BC_N$ type of homogeneous multivariate SRS polynomials of the
second kind as

\[
\mathcal{H}_{B,N}^{(m_1,m_2|m_1,n_2)}(x, \bar{x}, y, \bar{y}; q) = \sum_{i=1}^{m_1} (q^2)_N \cdot q^{i-1} \prod_{j=1}^{m_2} (q^2)_{a_j} \prod_{j=1}^{m_2} (q^2)_{b_j} \prod_{k=1}^{n_1} \frac{y^c_k}{(q^2)_k} \prod_{l=1}^{n_2} \frac{(\bar{y}^d_l)}{(q^2)_l},
\]

(4.30)

which is related to the partition function (3.48) associated with the BC$_N$ type of antiferromagnetic PF spin chain as

\[
\mathcal{Z}_{B,N}^{(m_1,m_2|m_1,n_2)}(q) = \mathcal{H}_{B,N}^{(m_1,m_2|m_1,n_2)}(x = 1, \bar{x} = 1, y = 1, \bar{y} = 1; q).
\]

(4.31)

Comparing (4.30) with (4.8), we find that BC$_N$ type of SRS polynomials of the first kind and the second kind are related as

\[
\mathcal{H}_{B,N}^{(m_1,m_2|m_1,n_2)}(x, \bar{x}, y, \bar{y}; q) = \mathcal{H}_{B,N}^{(m_2,n_1|m_2,m_1)}(y, y, x; q).
\]

(4.32)

It may be noted that, by using Eqs. (4.32) and (4.22), one can easily derive a relation between BC$_N$ type of SRS polynomials of the second kind and A$_{N-1}$ type of SRS polynomials. Let us now try to find out the generating function for \(\mathcal{H}_{B,N}^{(m_1,m_2|m_1,n_2)}(x, \bar{x}, y, \bar{y}; q)\), which would satisfy the relation

\[
\mathcal{G}_B^{(m_1,m_2|m_1,n_2)}(x, \bar{x}, y, \bar{y}; q, t) = \sum_{N=0}^{\infty} \frac{\mathcal{H}_{B,N}^{(m_1,m_2|m_1,n_2)}(x, \bar{x}, y, \bar{y}; q)}{(q^2)_N} t^N.
\]

(4.33)

Using Eqs. (4.15), (4.32) and (4.33), it is easy to find that

\[
\mathcal{G}_B^{(m_1,m_2|m_1,n_2)}(x, \bar{x}, y, \bar{y}; q, t) = \mathcal{G}_B^{(m_2,n_1|m_2,m_1)}(y, y, x; q, t).
\]

(4.34)

By using the above relation along with (4.13), we get an expression for this generating function as

\[
\mathcal{G}_B^{(m_1,m_2|m_1,n_2)}(x, \bar{x}, y, \bar{y}; q, t) = \mathcal{G}_1^{(n_2)}(y; q, t) \cdot \mathcal{G}_2^{(m_1)}(y; q, t) \cdot \mathcal{G}_3^{(m_2)}(\bar{x}; q, t) \cdot \mathcal{G}_4^{(m_1)}(x; q, t),
\]

(4.35)

where the factors appearing in the r.h.s. can be obtained from Eq. (4.14). Using the \(x = \bar{x} = y = \bar{y} = 1\) limit of this generating function and following a procedure similar to the ferromagnetic case, it can be shown that \(\mathcal{Z}_{B,N}^{(m_1,m_2|m_1,n_2)}(q)\) satisfies a relation exactly of the form (4.27). For some small values of the discrete parameters \(m_1, m_2, n_1, n_2\), and for arbitrary values of \(N\), Eq. (3.48) yields

\[
\mathcal{Z}_{B,N}^{(0,0|0,0)}(q) = q^{N^2}, \mathcal{Z}_{B,N}^{(0,1|0,0)}(q) = q^{N(N-1)}, \mathcal{Z}_{B,N}^{(0,0|1,0)}(q) = q^N, \mathcal{Z}_{B,N}^{(0,0|0,1)}(q) = 1.
\]

(4.36)
Using these partition functions and an equation of the form (4.27) corresponding to the anti-ferromagnetic case, we derive a set of recursion relations like

\[
\tilde{Z}_{B,N}^{(m_1+1,m_2|n_1,n_2)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} \cdot q^{N_1^2} \cdot \tilde{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q),
\]

\[
\tilde{Z}_{B,N}^{(m_1,m_2+1|n_1,n_2)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} \cdot q^{N_1(N_1-1)} \cdot \tilde{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q),
\]

\[
\tilde{Z}_{B,N}^{(m_1,m_2|n_1+1,n_2)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} \cdot q^{N_1} \cdot \tilde{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q),
\]

\[
\tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2+1)}(q) = \sum_{N_1=0}^{N} \binom{N}{N_1} \cdot \tilde{Z}_{B,N-N_1}^{(m_1,m_2|n_1,n_2)}(q).\]

(4.37)

By using this set of recursion relations and the initial conditions given in (4.36) (or, alternatively, a single initial condition of the form (4.29)), in principle it is possible to compute \(\tilde{Z}_{B,N}^{(m_1,m_2|n_1,n_2)}(q)\) for arbitrarily values of the discrete parameters \(m_1, m_2, n_1, n_2\) and \(N\). Indeed by using the symbolic software package Mathematica we have seen that, in comparison to the direct use of the expressions (3.32) and (3.48), it is much more efficient to use the set of corresponding recursion relations (4.28) and (4.37) to obtain explicit forms of the ferromagnetic and anti-ferromagnetic partition functions as some polynomials of the variable \(q\). Hence, the set of recursion relations (4.28) and (4.37) might be useful in studying various spectral properties like level density distribution and nearest neighbour spacing distribution for the \(BC_N\) type of ferromagnetic and anti-ferromagnetic PF chains.

5. Conclusions

Here we derive the canonical partition functions of the \(BC_N\) type of PF spin chains with SAPSRO by employing the freezing trick via the indirect approach, and subsequently construct the related \(BC_N\) type of homogeneous multivariate SRS polynomials. More precisely, we compute the grand canonical partition functions of the \(BC_N\) type of ferromagnetic as well as anti-ferromagnetic spin Calogero models with SAPSRO, and expand those grand canonical partition functions as some power series of the fugacity parameter to obtain the corresponding canonical partition functions. Applying the freezing trick, subsequently we derive novel expressions for the canonical partition functions of the related \(BC_N\) type of PF spin chains. Inspired by the form of such partition functions, we define \(BC_N\) type of homogeneous multivariate SRS polynomials and also find out the corresponding generating functions. Using these generating functions, we show
that the $BC_N$ type of SRS polynomials can be expressed as some bilinear combinations of the $A_{N-1}$ type of SRS polynomials.

It is worth noting that, the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SAPSRO are expressed as some simple products of only $BC_N$ types of basic modes in Eqs. (3.25) and (3.45). Such expressions of the grand canonical partition functions play an important role in our construction of the $BC_N$ type of SRS polynomials. Even though the grand canonical partition functions of the $BC_N$ type of spin Calogero models with SASRO have been computed earlier [48], it was found that both $BC_N$ and $A_{N-1}$ types of basic modes appear in the expressions of such grand canonical partition functions. Comparing our results with this earlier work, we find novel relations like (3.36a,b), which connect the basic modes of the $A_{N-1}$ and $BC_N$ types of grand canonical partition functions and also lead to interesting $q$-identities of the form (3.37) and (3.38).

In this paper, we also derive a set of recursion relations (4.28) and (4.37) for the partition functions of the $BC_N$ type of PF spin chains involving different numbers of lattice sites and internal degrees of freedom. In this context it may be noted that, another type of recursion relations, involving different numbers of lattice sites and fixed values of the internal degrees of freedom, have been computed earlier for the partition functions of the $A_{N-1}$ type of PF (supersymmetric PF) spin chains and the corresponding RS (SRS) polynomials [10, 40, 42, 43]. The later type of recursion relations play a key role in expressing the spectra of the $A_{N-1}$ type of PF spin chains and their supersymmetric generalizations through the motifs and in constructing the related one-dimensional vertex models. It can be shown that the $BC_N$ type of SRS polynomials considered in the present paper also satisfy the later type of recursion relations, involving different values of $N$ and fixed values of the internal parameters $m_1$, $m_2$, $n_1$ and $n_2$. As a result, the spectra of $BC_N$ type of PF spin chains with SAPSRO can be described by some motif like objects similar to the case of $A_{N-1}$ type of spin chains. Furthermore, it is possible to construct some one-dimensional classical vertex models whose energy functions would generate the complete spectra of these $BC_N$ type of PF spin chains. We plan to describe such exciting developments on the $BC_N$ type of PF spin chains and related SRS polynomials in a forthcoming publication.

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References

[1] F. D. M. Haldane, Phys. Rev. Lett. 60 (1988) 635–638.
[2] B. S. Shastry, Phys. Rev. Lett. 60 (1988) 639–642.
[3] F. D. M. Haldane, Z. N. C. Ha, J. C. Talstra, D. Bernard, V. Pasquier, Phys. Rev. Lett. 69 (1992) 2021–2025.
[4] D. Bernard, M. Gaudin, F. D. M. Haldane, V. Pasquier, J. Phys. A: Math. Gen. 26 (1993) 5219–5236.
[5] F. D. M. Haldane, in: A. Okiji, N. Kawakami (Eds.), Correlation Effects in Low-dimensional Electron Systems, volume 118 of Springer Series in Solid-state Sciences, pp. 3–20.
[6] B. Sutherland, B. S. Shastry, Phys. Rev. Lett. 71 (1993) 5–8.
[7] A. P. Polychronakos, Phys. Rev. Lett. 70 (1993) 2329–2331.
[8] H. Frahm, J. Phys. A: Math. Gen. 26 (1993) L473–L479.
[9] A. P. Polychronakos, Nucl. Phys. B 419 (1994) 553–566.
[10] K. Hikami, Nucl. Phys. B 441 (1995) 530–548.
[11] B. Basu-Mallick, H. Ujino, M. Wadati, J. Phys. Soc. Jpn. 68 (1999) 3219–3226.
[12] F. Finkel, A. González-López, Phys. Rev. B 72 (2005) 174411(6).
[13] B. Basu-Mallick, N. Bondyopadhaya, Nucl. Phys. B 757 (2006) 280–302.
[14] B. Basu-Mallick, N. Bondyopadhaya, D. Sen, Nucl. Phys. B 795 (2008) 596–622.
[15] J. C. Barba, F. Finkel, A. González-López, M. A. Rodríguez, Europhys. Lett. 83 (2008) 27005(6).
[16] A. Enciso, F. Finkel, A. González-López, Phys. Rev. E 82 (2010) 051117.
[17] B. Basu-Mallick, N. Bondyopadhaya, K. Hikami, SIGMA 6 (2010) 091–13.
[18] D. Bernard, V. Pasquier, D. Serban, Europhys. Lett. 30 (1995) 301–306.
[19] T. Yamamoto, O. Tsuchiya, J. Phys. A: Math. Gen. 29 (1996) 3977–3984.
[20] A. Enciso, F. Finkel, A. González-López, M. A. Rodríguez, Nucl. Phys. B 707 (2005) 553–576.
[21] J. C. Barba, F. Finkel, A. González-López, M. A. Rodríguez, Phys. Rev. B 77 (2008) 214422(10).

[22] Y. Kuramoto, Y. Kato, Dynamics of one-dimensional quantum systems: inverse-square interaction models, *Cambridge Studies in Advanced Mathematics* 29, Cambridge University Press, New York, 2009.

[23] A. P. Polychronakos, J. Phys. A: Math. Gen. 39 (2006) 12793–12845.

[24] C. W. J. Beenakker, B. Rajaei, Phys. Rev. B 49 (1994) 7499–7510.

[25] M. Caselle, Phys. Rev. Lett. 74 (1995) 2776–2779.

[26] J. I. Cirac, G. Sierra, Phys. Rev. B 81 (2010) 104431(4).

[27] A. Nielsen, J. I. Cirac, G. Sierra, J. Stat. Mech. (2011) P11014.

[28] H.-H. Tu, A. E. B. Nielsen, G. Sierra, Nucl. Phys. B 886 (2014) 328–363.

[29] R. Bondesan, T. Quella, Nucl. Phys. B 886 (2014) 483–523.

[30] H.-H. Tu, G. Sierra, Phys. Rev. B 92 (2015) 041119(R).

[31] B. Basu-Mallick, F. Finkel, A. González-López, Phys. Rev. B 93 (2016) 155154.

[32] N. Beisert, C. Kristjansen, M. Staudacher, Nucl. Phys. B 664 (2003) 131–184.

[33] T. Bargheer, N. Beisert, F. Loebbert, J. Phys. A: Math. Theor. 42 (2009) 285205(58).

[34] N. Beisert et al., Lett. Math. Phys. 99 (2012) 3–32.

[35] N. Taniguchi, B. S. Shastry, B. L. Altshuler, Phys. Rev. Lett. 75 (1995) 3724–3727.

[36] B. Basu-Mallick, A. Kundu, Nucl. Phys. B 509 (1998) 705–728.

[37] B. Basu-Mallick, Nucl. Phys. B 540 (1999) 679–704.

[38] B. Basu-Mallick, N. Bondyopadhaya, K. Hikami, D. Sen, Nucl. Phys. B 782 (2007) 276–295.

[39] N. Beisert, D. Erkal, J. Stat. Mech. 0803 (2008) P03001.

[40] K. Hikami, J. Phys. Soc. Jpn. 64 (1995) 1047–1050.

[41] K. Hikami, J. Phys. A 30 (1997) 2447–2456.

[42] A. Kirillov, A. Kuniba, T. Nakanishi, Commun. Math. Phys. 185 (1997) 441–465.
[43] K. Hikami, B. Basu-Mallick, Nucl. Phys. B 566 (2000) 511–528.

[44] G. E. Andrews, The Theory of Partitions, Addison–Wesley, Reading, Mass., 1976.

[45] A. J. Macfarlane, J. Phys. A 22 (1989) 4581–4588.

[46] S. Odake, R. Sasaki, Phys. Lett. B 663 (2008) 141–145.

[47] B. Basu-Mallick, F. Finkel, A. González-Lópe, Nucl. Phys. B 812 (2009) 402–423.

[48] J. C. Barba, F. Finkel, A. González-López, M. A. Rodríguez, Nucl. Phys. B 806 (2009) 684–714.

[49] B. Basu-Mallick, N. Bondyopadhaya, P. Banerjee, Nucl. Phys. B 883 (2014) 501–528.

[50] B. Basu-Mallick, C. Datta, F. Finkel, A. González-López, Nucl. Phys. B 898 (2015) 53–77.

[51] P. Banerjee, B. Basu-Mallick, N. Bondyopadhaya, C. Datta, Nucl. Phys. B 904 (2016) 297–326.

[52] E. Corrigan, R. Sasaki, J. Phys. A: Math. Gen. 35 (2002) 7017–7061.