Symplectic embedding and Hamilton-Jacobi analysis of Proca model

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Abstract

Following the symplectic approach we show how to embed the Abelian Proca model into a first-class system by extending the configuration space to include an additional pair of scalar fields, and compare it with the improved Dirac scheme. We obtain in this way the desired Wess-Zumino and gauge fixing terms of BRST invariant Lagrangian. Furthermore, the integrability properties of the second-class system described by the Abelian Proca model are investigated using the Hamilton-Jacobi formalism, where we construct the closed Lie algebra by introducing operators associated with the generalized Poisson brackets.

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I. INTRODUCTION

The standard Dirac quantization method (DQM) [1] has been widely used in order to quantize Hamiltonian systems involving first- and second-class constraints. However, the resulting Dirac brackets may be field-dependent and nonlocal, and thus pose serious ordering problems for the quantization of the theory. On the other hand, the Becci-Rouet-Stora-Tyutin (BRST) [2,3] quantization of constrained systems along the lines originally established by Batalin, Fradkin, and Vilkovisky [4,5], and then reformulated in a more tractable and elegant version by Batalin, Fradkin, and Tytin [6], does not suffer from these difficulties, as it relies on a simple Poisson bracket structure. As a result, the embedding of second-class systems into first-class ones (gauge theories) has received much attention in the past years and the DQM improved in this way, has been applied to a number of models [6–13] in order to obtain the corresponding Wess-Zumino (WZ) actions [14,15]. In fact, the earlier work on this subject is based on the traditional Dirac’s pioneering work [1], which has been criticized for introducing “superfluous” primary constraints, and has been avoided in more recent treatments, based on the symplectic structure of phase space. That this approach is of particular advantage in the case of first-order Lagrangians such as Chern-Simons theories has been emphasized by Faddeev and Jackiw [16]. This symplectic scheme has been applied to a number of models [17,18] and has recently been used to implement the improved DQM embedding program in the context of the symplectic formalism [19,20].

Based on the Carathéodory equivalent Lagrangians method [21], an alternative Hamilton-Jacobi (HJ) scheme for constrained systems was also proposed [22] and exploited to quantize singular systems [23–25]. One of the most interesting applications of the HJ scheme is system with second-class constraints [1,26], since the set of differential equations derived from the corresponding HJ equation is not integrable [26], being incomplete. They become complete with the addition of suitable “integrability conditions,” which turn out to be Dirac “consistency conditions” requiring time independence of the constraints [25].

In this paper, we wish to illustrate the above quantization schemes in the case of the
Abelian Proca model. The material is organized as follows. In section 2, we briefly recapitulate the Proca model in the framework of the standard and the improved DQMs. In section 3, after briefly reviewing the gauge non-invariant symplectic formalism for this model [16], we then show how the improved DQM program for embedding this second-class system into a first-class one is realized in the framework of the symplectic formalism, and obtain in this way the corresponding Wess-Zumino and gauge fixing terms of the BRST invariant Lagrangian. In section 4, we finally apply the HJ quantization scheme to the Proca model and comment on the integrability conditions and the closed Lie algebra obtained by introducing operators associated with the generalized Poisson brackets. Our conclusion is given in section 5.

II. DIRAC QUANTIZATION METHOD

Standard Dirac quantization method

In this section, we briefly recapitulate the massive Proca model described by the Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu, \quad (2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $g_{\mu\nu} = \text{diag}(+,-,-,-)$. The canonical momenta conjugate to the fields $A^\mu$ are given by $\pi_0 = 0$ and $\pi_i = F_{i0}$ with the Poisson algebra $\{ A^\mu(x), \pi_\nu(y) \} = \delta_\mu^\nu \delta(x-y)$. The canonical Hamiltonian then reads

$$\mathcal{H}_0 = \frac{1}{2} \pi_i^2 + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} m^2 (A^0)^2 + \frac{1}{2} m^2 (A^i)^2 - A^0 (\partial^i \pi_i + m^2 A^0). \quad (2.2)$$

Since we have one primary constraint

$$\Omega_1 = \pi_0 \approx 0, \quad (2.3)$$

the total Hamiltonian is given by

$$\mathcal{H}_T = \mathcal{H}_0 + u \Omega_1 \quad (2.4)$$
with $u$ a Lagrange multiplier. The requirement $\dot{\Omega}_1 \approx 0$ leads to the secondary, Gauss’ law constraint

$$\Omega_2 = \partial^i \pi_i + m^2 A^0 \approx 0. \quad (2.5)$$

Note that the time evolution of this constraint with $\mathcal{H}_T$ generates no further constraint, but only fixes the multiplier $u$ to be $u = -\partial_i A^i$, so that $\mathcal{H}_T$ no longer involves arbitrary parameters:

$$\mathcal{H}_T = \mathcal{H}_0 - \Omega_1 \partial_i A^i. \quad (2.6)$$

As a result, the full set of constraints of this model is $\Omega_i \ (i, j = 1, 2)$. They satisfy the second-class constraint algebra

$$\Delta_{ij}(x, y) = \{\Omega_i(x), \Omega_j(y)\} = -m^2 \epsilon_{ij} \delta(x - y) \quad (2.7)$$

with $\epsilon_{12} = -\epsilon_{21} = 1$. The consistent quantization of the Proca model is then obtained in terms of the Dirac brackets \cite{28}

$$\{A^0(x), A^j(y)\}_D = \frac{1}{m^2} \partial^j \delta(x - y), \quad \{A^0(x), A^0(y)\}_D = 0,$$

$$\{A^i(x), A^k(y)\}_D = 0, \quad \{\pi_\mu(x), \pi_\nu(y)\}_D = 0,$$

$$\{A^i(x), \pi_j(y)\}_D = \delta^i_j \delta(x - y), \quad \{A^0(x), \pi_\nu(y)\}_D = 0,$$

$$\{A^i(x), \pi_0(y)\}_D = 0. \quad (2.8)$$

For later comparison we also list the equations of motion following from the time evolution of the fields $A^\mu$ and $\pi_\mu$ with $\mathcal{H}_T$:

$$\dot{A}_0 = -\partial_i A^i, \quad \dot{A}^i = \pi_i + \partial^i A^0,$$

$$\dot{\pi}_0 = \partial^i \pi_i + m^2 A^0, \quad \dot{\pi}_i = \partial_j F^{ij} - m^2 A^i. \quad (2.9)$$

which, together with the constraints $\Omega_i$, reproduce the well-known equations

$$(\partial_\nu \partial^\nu + m^2) A^\mu = 0. \quad (2.10)$$

**Improved Dirac Quantization Method**
For late comparison we now briefly review the improved DQM \[7–10\], which implements the conversion of the second-class constraints of a system \[6\] to the first-class constraints, for the case of the Abelian Proca model \[27,28\]. To this end we extend phase space by introducing a pair of auxiliary fields \((\theta, \pi_\theta)\) satisfying the canonical Poisson brackets

\[
\{ \theta(x), \pi_\theta(y) \} = \delta(x - y).
\] (2.11)

Following the improved DQM, we obtain for the Abelian conversion of the second-class constraints, Eqs. (2.3) and (2.5), to the first-class constraints

\[
\tilde{\Omega}_1 = \pi_0 + m^2 \theta, \quad \tilde{\Omega}_2 = \partial^i \pi_i + m^2 A^0 + \pi_\theta
\] (2.12)

satisfying the rank-zero algebra

\[
\{ \tilde{\Omega}_i, \tilde{\Omega}_j \} = 0.
\] (2.13)

Similarly, we obtain for the first-class physical fields in the extended phase space

\[
\tilde{A}^\mu = (A^0 + \frac{1}{m^2} \pi_\theta, A^i + \partial^i \theta), \quad \tilde{\pi}_\mu = (\pi_0 + m^2 \theta, \pi_i).
\] (2.14)

Since an arbitrary functional of the first-class physical fields is also first-class \[6\], we can directly obtain the desired first-class Hamiltonian \(\tilde{H}\) corresponding to the Hamiltonian \(H_T\) in Eq. (2.6) via the substitution \(A^\mu \rightarrow \tilde{A}^\mu, \pi_\mu \rightarrow \tilde{\pi}_\mu:\)

\[
\tilde{H} = H_T + \frac{1}{2} m^2 (\partial_i \theta)^2 + \frac{1}{2m^2} \pi_\theta^2 + \partial_i^2 \theta \tilde{\Omega}_1 - \frac{1}{m^2} \pi_\theta \tilde{\Omega}_2.
\] (2.15)

On the other hand, one easily recognizes that the Poisson brackets between the first-class fields in the extended phase space are formally identical with the Dirac brackets of the corresponding second-class fields \[29\]. Note that the symplectic formalism \[16,17\] also gives the same result. (See next section for more details.)

Next, one can consider the partition function of the model in order to present the Lagrangian corresponding to \(\tilde{H}\) in the canonical Hamiltonian formalism as follows

\[
Z = \int \mathcal{D}A^\mu \mathcal{D}\pi_\mu \mathcal{D}\theta \mathcal{D}\pi_\theta \prod_{i,j=1}^2 \delta(\tilde{\Omega}_i) \delta(\Gamma_j) \det \{ \tilde{\Omega}_i, \Gamma_j \} e^{iS},
\] (2.16)
\[
S = \int d^4 x \left( \pi_\mu \dot{A}^\mu + \pi_\theta \dot{\theta} - \tilde{H} \right),
\]
(2.17)

with the Hamiltonian \( \tilde{H} \) in Eq. (2.13). Then, after performing tedious integrations over all momenta, one obtains the Lagrangian
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta).
\]
(2.18)

Up to a total divergence term this is just the manifestly gauge invariant Stückelberg Lagrangian, with \( \theta \) the Stückelberg scalar, which is invariant under the gauge transformations as \( \delta A_\mu = \partial_\mu \Lambda \) and \( \delta \theta = -\Lambda \).

### III. SYMPLECTIC EMBEDDING FORMALISM

**Gauge noninvariant symplectic scheme**

In order to set the stage for the symplectic embedding of the Proca model into a gauge theory, we briefly review [16,28] the gauge noninvariant symplectic formalism for this model. Following ref. [16], we rewrite the second-order Lagrangian (2.1) as the first-order Lagrangian
\[
\mathcal{L}_0 = \pi_0 \dot{A}^0 + \pi_i \dot{A}^i - \mathcal{H}_0,
\]
(3.1)

where the Lagrangian \( \mathcal{L}_0 \) is to be regarded as a function of the configuration–space variables \( A^i \) and \( \pi_i \), and \( \mathcal{H}_0 \) is the canonical Hamiltonian density in Eq. (2.2). Since \( \mathcal{H}_0 \) depends on \( \pi_i \) and \( A^i \), but not on their time derivatives, it can be regarded as the (level zero) symplectic potential \( \mathcal{H}^{(0)} \).

In order to find the symplectic brackets we introduce the sets of the symplectic variables \( \xi^{(0)\alpha} = (\vec{A}, \vec{\pi}, A^0) \) and their conjugate momenta \( a^{(0)}_\alpha = (\vec{\theta}, \vec{\pi}, 0) \), which are directly read off from the canonical sector of the first-order Lagrangian (3.1), written in the form
\[
\mathcal{L} = a^{(0)}_\alpha \xi^{(0)\alpha} - \mathcal{H}^{(0)}.
\]
(3.2)
The dynamics of the model is then governed by the symplectic two-form matrix:

\[ f^{(0)}_{\alpha\beta}(x, y) = \frac{\partial a^{(0)}_{\beta}(y)}{\partial \xi^{(0)\alpha}(x)} - \frac{\partial a^{(0)}_{\alpha}(x)}{\partial \xi^{(0)\beta}(y)}, \quad (3.3) \]

via the equations of motion

\[ \int d^3 y f^{(0)}_{\alpha\beta}(x, y) \dot{\xi}^\beta(y) = \frac{\delta}{\delta \xi^{(0)\alpha}(x)} \int d^3 y \mathcal{H}_0(y). \quad (3.4) \]

In the Proca model the symplectic two-form matrix is given by

\[ f^{(0)}_{\alpha\beta}(x, y) = \begin{pmatrix} O & -I & \vec{b} \\ I & O & \vec{b} \\ \vec{b}^T & \vec{b}^T & 0 \end{pmatrix} \delta(x - y), \quad (3.5) \]

where \( O, \vec{b} \) and \( \vec{b}^T \) stand for a \( 3 \times 3 \) null (identity) matrix, a column vector and its transpose, respectively, showing the matrix \( f^{(0)}_{\alpha\beta}(x, y) \) is singular. Here the symplectic two-form matrix has a zero mode \( \nu^{(0)T}_{\alpha, y}(1, x) = (\vec{0}, \vec{0}, 1) \delta(x - y) \), which generates the constraint \( \Omega_2 \) in the context of the symplectic formalism [16] as follows

\[ \int d^3 y \delta(x - y) \frac{\delta}{\delta A^0(y)} \int d^3 z H_0(z) = -\Omega_2(x) = 0, \quad (3.6) \]

where \( \Omega_2 \) is given by Eq. (2.5). Following the symplectic algorithm, we add the constraint \( \Omega_2 \) to the canonical sector of the Lagrangian (3.1), by enlarging the symplectic phase space with the addition of a Lagrange multiplier \( \rho \). The once iterated first-label Lagrangian is then given as

\[ \mathcal{L}^{(1)} = \pi_0 \dot{A}^0 + \pi_i \dot{A}^i + \Omega_2 \dot{\rho} - \mathcal{H}^{(1)}, \quad (3.7) \]

where the first-iterated Hamiltonian \( \mathcal{H}^{(1)} = \mathcal{H}^{(0)}|_{\Omega_2=0} \) is given by

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1 We label the zero modes as follows: \( \nu^{(l)}_{\alpha, y}(\sigma, x) \), \( (\sigma = 1, ..., N) \), where “\( l \)” refers to the “level”, \( \alpha, y \) stand for the component, while \( \sigma, x \) label the \( N \)-fold infinity of zero modes in \( \mathcal{R}^3 \). For simplicity we refer to the zero modes only according to their discrete labelling \( \sigma \).
The situation at this stage is exactly the same as before except for the replacement, $\mathcal{L}_0 \to \mathcal{L}^{(1)}$ and $\mathcal{H}_0 \to \mathcal{H}^{(1)}$. In other words, we now have for the symplectic variables and their conjugate momenta

$$\xi^{(1)\alpha} = (\vec{A}, \vec{\pi}, A^0, \rho), \quad \phi^{(1)}_\alpha = (\vec{\pi}, \vec{\rho}, 0, \Omega_2).$$

The first iterated symplectic nonsingular two-form matrix is now given by

$$f^{(1)}_{\alpha\beta}(x, y) = \begin{pmatrix}
O & -I & \vec{0} & \vec{0} \\
I & O & \vec{0} & -\vec{\nabla}_x \\
\vec{\nabla}_y & \vec{\nabla}_y & 0 & m^2 \\
\vec{\nabla}_x & -m^2 & 0 & 0
\end{pmatrix} \delta(x - y).$$

Its inverse matrix is easily obtained

$$(f^{(1)}_{\alpha\beta})^{-1}(x, y) = \begin{pmatrix}
O & I & \frac{1}{m^2} \vec{\nabla}_x & \vec{0} \\
-I & O & \vec{0} & \vec{0} \\
-\frac{1}{m^2} \vec{\nabla}_y & \vec{\nabla}_y & 0 & -\frac{1}{m^2} \\
\vec{\nabla}_x & \vec{\nabla}_x & \frac{1}{m^2} & 0
\end{pmatrix} \delta(x - y).$$

Now, this inverse symplectic two-form matrix gives the symplectic brackets of the Proca model

$$\{\xi^{(1)\alpha}(x), \xi^{(1)\beta}(y)\}_{\text{symp}} = (f^{(1)}_{\alpha\beta})^{-1}(x, y),$$

which are recognized to be identical with the Dirac brackets in Eq. (2.8).

**Gauge invariant symplectic embedding**

Embedding a second-class system into a first-class one is, on Lagrangian level, equivalent to finding the Wess-Zumino(WZ) action for the Lagrangian in question. This is what we propose to do next in the context of the symplectic formalism, taking the Proca model as an illustration. The starting point is provided by the Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + \mathcal{L}_{WZ}. \]  

(3.13)

The symplectic procedure is greatly simplified, if we make the following “educated guess” for the WZ Lagrangian, respecting Lorentz symmetry,

\[ \mathcal{L}_{WZ} = \frac{1}{2} c_1 \partial_\mu \theta \partial^\mu \theta + c_2 A_\mu \partial_\mu \theta + c_3 f, \]  

(3.14)

with \( f \) an arbitrary polynomial of \( \theta \). As an Ansatz we shall take \( c_i \ (i = 1, 2, 3) \) to be constants to be fixed by the symplectic embedding procedure. After partial integration of the second term in Eq. (3.14) in order to coincide with the constraint \( \tilde{\Omega}_1 \), in terms of the canonical momenta conjugate to \( A^0, A^i \) and \( \theta \)

\[ \pi_0 = -c_2 \dot{\theta}, \quad \pi_i = F_{i0}, \quad \pi_\theta = c_1 \dot{\theta}, \]  

(3.15)

the canonical Hamiltonian reads

\[ \mathcal{H}^{(0)} = \frac{1}{2} \pi_i^2 + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} m^2 (A^0)^2 + \frac{1}{2} m^2 (A^i)^2 - A^0 (\partial^i \pi_i + m^2 A^0) + \frac{1}{2} c_1 \pi_\theta^2 + \frac{1}{2} c_1 (\partial_\theta)^2 - c_2 A^i \partial_i \theta - c_3 f. \]  

(3.16)

Here note that the equation for the canonical momentum \( \pi_0 \) in Eq. (3.15) yields the constraint \( \tilde{\Omega}_1 \), which will be shown to be equivalent to the corresponding first-class constraint in Eq. (2.12).

Following the canonical procedure for obtaining the equivalent symplectic first-order Lagrangian with the WZ term, we have

\[ \mathcal{L}^{(0)} = \pi_0 \dot{A}^0 + \pi_i \dot{A}^i + \pi_\theta \dot{\theta} - \mathcal{H}^{(0)}, \]  

(3.17)

where the initial set of symplectic variables \( \xi^{(0)\alpha} \) and their conjugate momenta \( a^{(0)}_\alpha \) are now given by

\[ \xi^{(0)\alpha} = (\vec{A}, \vec{\pi}, \theta, \pi_\theta, A^0), \quad a^{(0)}_\alpha = (\vec{\pi}, \vec{0}, \pi_\theta, 0, -c_2 \theta). \]  

(3.18)

From Eq. (3.18) we read off the symplectic singular two-form matrix to be
having a non-trivial zero mode given by

$$\nu^{(0)T}_{\alpha,y}(1, x) = (\vec{0}, \vec{0}, 0, -c_2, 1) \delta(x - y).$$  \hspace{1cm} (3.20)$$

Applying this zero mode from the left to the equation of motion, we are led to a constraint

$$\tilde{\Omega}_2$$

\[ \begin{align*}
\int d^3 y \nu^{(0)T}_{\alpha,y}(1, x) \frac{\delta}{\delta \xi^{(0)\alpha}(y)} \int d^3 z \mathcal{H}^{(0)}(z) &= -\tilde{\Omega}_2(x) = 0, \hspace{1cm} (3.21)
\end{align*} \]

where \( \tilde{\Omega}_2 \) is given by

$$\tilde{\Omega}_2 = \partial^i \pi_i + m^2 A^0 + \frac{c_2}{c_1} \pi_\theta,$$  \hspace{1cm} (3.22)$$

which will be shown to be equal to the corresponding constraint in Eq. (2.12) with the fixed values of \( c_1 \) and \( c_2 \).

Next, following the symplectic algorithm outlined before, we obtain the first-iterated Lagrangian by enlarging the canonical sector with the constraint \( \tilde{\Omega}_2 \) and its associated Lagrangian multiplier \( \rho \) as follows

$$L^{(1)} = \pi_0 \dot{A}^0 + \pi_i \dot{A}^i + \pi_\theta \dot{\theta} + \tilde{\Omega}_2 \dot{\rho} - \mathcal{H}^{(1)}$$  \hspace{1cm} (3.23)$$

where \( \mathcal{H}^{(1)} = \mathcal{H}^{(0)}|_{\tilde{\Omega}_2=0} \) is the first-iterated Hamiltonian. We have now for the first-level symplectic variables \( \xi^{(1)\alpha} \) and their conjugate momenta \( a^{(1)}_\alpha \)

$$\xi^{(1)\alpha} = (\vec{A}, \vec{\pi}, \theta, \pi_\theta, A^0, \rho), \hspace{1cm} a^{(1)}_\alpha = (\vec{\pi}, \vec{0}, \pi_\theta, 0, -c_2 \theta, \tilde{\Omega}_2).$$  \hspace{1cm} (3.24)$$

and the first-iterated symplectic two-form matrix now reads

\[
\begin{pmatrix}
O & -I & \vec{0} & \vec{0} & \vec{0} \\
I & O & \vec{0} & \vec{0} & \vec{0} \\
\vec{0}^T & \vec{0}^T & 0 & -1 & -c_2 \\
\vec{0}^T & \vec{0}^T & 1 & 0 & 0 \\
\vec{0}^T & \vec{0}^T & c_2 & 0 & 0
\end{pmatrix}
\]

having a non-trivial zero mode given by

$$\nu^{(0)T}_{\alpha,y}(1, x) = (\vec{0}, \vec{0}, 0, -c_2, 1) \delta(x - y).$$  \hspace{1cm} (3.20)$$

Applying this zero mode from the left to the equation of motion, we are led to a constraint \( \tilde{\Omega}_2 \)

\[ \begin{align*}
\int d^3 y \nu^{(0)T}_{\alpha,y}(1, x) \frac{\delta}{\delta \xi^{(0)\alpha}(y)} \int d^3 z \mathcal{H}^{(0)}(z) &= -\tilde{\Omega}_2(x) = 0, \hspace{1cm} (3.21)
\end{align*} \]

where \( \tilde{\Omega}_2 \) is given by

$$\tilde{\Omega}_2 = \partial^i \pi_i + m^2 A^0 + \frac{c_2}{c_1} \pi_\theta,$$  \hspace{1cm} (3.22)$$

which will be shown to be equal to the corresponding constraint in Eq. (2.12) with the fixed values of \( c_1 \) and \( c_2 \).

Next, following the symplectic algorithm outlined before, we obtain the first-iterated Lagrangian by enlarging the canonical sector with the constraint \( \tilde{\Omega}_2 \) and its associated Lagrangian multiplier \( \rho \) as follows

$$L^{(1)} = \pi_0 \dot{A}^0 + \pi_i \dot{A}^i + \pi_\theta \dot{\theta} + \tilde{\Omega}_2 \dot{\rho} - \mathcal{H}^{(1)}$$  \hspace{1cm} (3.23)$$

where \( \mathcal{H}^{(1)} = \mathcal{H}^{(0)}|_{\tilde{\Omega}_2=0} \) is the first-iterated Hamiltonian. We have now for the first-level symplectic variables \( \xi^{(1)\alpha} \) and their conjugate momenta \( a^{(1)}_\alpha \)

$$\xi^{(1)\alpha} = (\vec{A}, \vec{\pi}, \theta, \pi_\theta, A^0, \rho), \hspace{1cm} a^{(1)}_\alpha = (\vec{\pi}, \vec{0}, \pi_\theta, 0, -c_2 \theta, \tilde{\Omega}_2).$$  \hspace{1cm} (3.24)$$

and the first-iterated symplectic two-form matrix now reads
In order to realize a gauge symmetry, this matrix must have at least one zero-mode which
does not imply a new constraint. To start out with, we introduce two zero-modes for the
first-iterated symplectic two-form matrix

$$\nu_{\alpha, y}^{(1)} (1, x) = (\vec{0}, \vec{0}, -c_2, 1, 0) \delta (x - y),$$

$$\nu_{\alpha, y}^{(1)} (2, x) = (\vec{0}, -\frac{c_2}{c_1}, 0, 0, 1) \delta (x - y).$$

We require that these zero-modes should not generate any new constraint upon applying it
from the left to the equation of motion

$$\int d^3y \, \nu_{\alpha, y}^{(1)} (1, x) \frac{\delta}{\delta \xi^{(1)\alpha} (y)} \int d^3z \, \mathcal{H}^{(1)} (z) = (m^2 - \frac{c_2^2}{c_1} ) A^0,$$

$$\int d^3y \, \nu_{\alpha, y}^{(1)} (2, x) \frac{\delta}{\delta \xi^{(1)\alpha} (y)} \int d^3z \, \mathcal{H}^{(1)} (z) = (m^2 - \frac{c_2^2}{c_1} ) \partial_i A^i + \frac{c_2 c_3}{c_1} \frac{df}{d\theta}. \quad (3.27)$$

Hence no new constraint is generated provided we choose for the free adjustable coefficients:

$$c_1 = c_2 = m^2, \quad c_3 = 0. \quad (3.28)$$

As a result, we arrived at the final result in the form of the St"uckelberg Lagrangian (2.18),
which manifestly displays the gauge invariance under $\delta A^\mu = \partial^\mu \Lambda$ and $\delta \theta = -\Lambda$. Note that
with the above fixed $c_i$, $\tilde{\Omega}_2$ in Eq. (3.22) is isomorphic to $\tilde{\Omega}_2$ given in Eq. (2.12).

Now, in order to discuss gauge transformation, we consider the skew symmetric matrix

$$f_{\alpha\beta}^{(1)} (x, y) = \left( \begin{array}{cccc}
O & -I & \vec{0} & \vec{0} \\
I & O & \vec{0} & \vec{0} \\
\vec{0}^T & \vec{0}^T & 0 & -1 -c_2 & 0 \\
\vec{0}^T & \vec{0}^T & 1 & 0 & \frac{c_2}{c_1} \\
\vec{0}^T & \vec{0}^T & c_2 & 0 & m^2 \\
\vec{0}^T & \vec{0}^T & 0 & -\frac{c_2}{c_1} & -m^2 & 0
\end{array} \right) \delta (x - y). \quad (3.25)$$

$$f_{\alpha\beta}^{(1)} (x, y) = \left( \begin{array}{cccc}
O & -I & \vec{0} & \vec{0} \\
I & O & \vec{0} & \vec{0} \\
\vec{0}^T & \vec{0}^T & 0 & -1 -c_2 & 0 \\
\vec{0}^T & \vec{0}^T & 1 & 0 & \frac{c_2}{c_1} \\
\vec{0}^T & \vec{0}^T & c_2 & 0 & m^2 \\
\vec{0}^T & \vec{0}^T & 0 & -\frac{c_2}{c_1} & -m^2 & 0
\end{array} \right) \delta (x - y). \quad (3.25)$$

In order to realize a gauge symmetry, this matrix must have at least one zero-mode which
does not imply a new constraint. To start out with, we introduce two zero-modes for the
first-iterated symplectic two-form matrix

$$\nu_{\alpha, y}^{(1)} (1, x) = (\vec{0}, \vec{0}, 0, -c_2, 1, 0) \delta (x - y),$$

$$\nu_{\alpha, y}^{(1)} (2, x) = (\vec{0}, -\frac{c_2}{c_1}, 0, 0, 1) \delta (x - y).$$

(3.26)
where the submatrix \( f_{\hat{\alpha}, \hat{\beta}} \) refers to the \( \xi_{\hat{\alpha}} = (\vec{A}, \vec{p}, \theta, \pi) \) sector, and \( M_{\hat{\alpha} \sigma} \) is a \( 2 \times 8 \) matrix defined as \( M_{\hat{\alpha} \sigma} \delta(x - y) = \frac{\partial \Omega_{\sigma}(y)}{\partial \xi_{\hat{\alpha}}(x)} \). Following ref. [17], the zero-modes \( \nu^{(1)}_{\alpha, y}(\sigma, x) \) of \( f^{(1)}(x, y) \) are of the general form

\[
\nu^{(1)}_{\alpha, y}(1, x) = \begin{pmatrix}
-f_{\hat{\alpha}, \hat{\beta}}^{-1} M_{\hat{\beta} 1} & 1 & \delta(x - y), \\
0 & 0 & \end{pmatrix}
\]

\[
\nu^{(1)}_{\alpha, y}(2, x) = \begin{pmatrix}
0 & 1 & \delta(x - y).
\end{pmatrix}
\]

From Eq. (3.25) we have

\[
f_{\hat{\alpha}, \hat{\beta}}^{-1} = \begin{pmatrix}
O & I & \vec{0} & \vec{0} \\
-I & O & \vec{0} & \vec{0} \\
\vec{0} & \vec{0} & 0 & 1 \\
\vec{0} & \vec{0} & -1 & 0
\end{pmatrix}
\]

so that the zero-modes are (we have now \( c_2/c_1 = 1 \))

\[
\nu^{(1) T}_{\alpha, y}(1, x) = (\vec{0}, \vec{0}, 0, -1, 1, 0) \delta(x - y),
\]

\[
\nu^{(1) T}_{\alpha, y}(2, x) = (\vec{\nabla}_x, \vec{0}, -1, 0, 0, 1) \delta(x - y)
\]

in agreement with Eq. (3.26).

As has been shown in ref. [17], the “trivial” zero modes generate gauge transformation on \( \xi_{1}^{(1)} \) in the sense

\[
\delta \xi_{\hat{\alpha}}(x) = \Sigma_{\sigma} \int d^{3}y \nu^{(1) T}_{\alpha, y}(\sigma, x) \epsilon_{\sigma}(y).
\]

we thus conclude from Eq. (3.32),

\[\text{2The Dirac algorithm as applied to the symplectic Lagrangian shows that the gauge transformation on } \xi_{\hat{\alpha}} \text{ are generated by the first-class constraints } \Omega_{\sigma}(x) \text{ (} \sigma = 1, 2 \text{) with respect to the symplectic Poisson bracket: } \delta \xi_{\hat{\alpha}}(x) = \{ \xi_{\hat{\alpha}}(x), G \}_{\text{symp}} = \int d^{3}y \frac{\delta \xi_{\hat{\alpha}}(x)}{\delta \xi_{\hat{\beta}}(y)} f^{-1}_{\hat{\alpha}, \hat{\beta}} \frac{\delta G}{\delta \xi_{\hat{\gamma}}(y)} \text{ where } G = \Sigma_{\sigma} \int d^{3}y \epsilon_{\sigma}(y) \Omega_{\sigma}(y). \text{ Hence, we have } \delta \xi_{\hat{\alpha}}(x) = \Sigma_{\sigma} \int d^{3}y f^{-1}_{\hat{\alpha}, \hat{\beta}} \frac{\delta \Omega_{\sigma}(y)}{\delta \xi_{\hat{\gamma}}(x)} \epsilon_{\sigma}(y) \text{ or the } \hat{\alpha} \text{-components of the zero-modes (3.32) are seen to generate the gauge transformation on } \xi_{\hat{\alpha}}.\]
\[ \delta A^0 = \epsilon_1, \quad \delta A^i = \partial^i \epsilon_2, \quad \delta \theta = -\epsilon_2, \]
\[ \delta \pi_i = 0, \quad \delta \pi_\theta = -m^2 \epsilon_1, \quad \delta \rho = \epsilon_2. \quad (3.34) \]

As one readily checks, these only represent a symmetry transformation of the symplectic first level Lagrangian \( \mathcal{L}^{(1)} \) if \( \epsilon_1 = \dot{\epsilon}_2 \). In that case it is also a symmetry of the St"{u}ckelberg Lagrangian (2.18). As was shown in ref. [30], this condition also follows from the requirement that the gauge transformation commutes with the time-derivative operation in Hamilton’s equations of motion.

**BRST invariant symplectic embedding**

Following the Batalin-Fradkin-Vilkovisky formalism [4,5] in the extended phase space, we introduce a minimal set of the ghost and antighost fields together with auxiliary fields as follows
\[ (\mathcal{C}, \bar{\mathcal{P}}), \quad (\mathcal{P}, \bar{\mathcal{C}}), \quad (N, B). \quad (3.35) \]

Similar to the previous WZ action case, we now construct the BRST invariant gauge-fixed Lagrangian in the symplectic scheme by including the above auxiliary fields with ghost terms in the Lagrangian,
\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{WZ} + \mathcal{L}_{GF}, \]
\[ \mathcal{L}_0 + \mathcal{L}_{WZ} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta), \quad (3.36) \]
where we take an Ansatz for the gauge-fixing (GF) Lagrangian respecting Lorentz symmetry
\[ \mathcal{L}_{GF} = d_1 A^\mu \partial_\mu B + d_2 \partial_\mu \bar{\mathcal{C}} \partial^\mu \mathcal{C} + d_3 B^2 + d_4 g. \quad (3.37) \]

with the \( \theta \)-dependent function \( g \). Here note that we have used the canonical momentum field \( B \) instead of the multiplier field \( N \) in this Ansatz to construct the desired well-known ghost Lagrangian in the Proca model. Through the Legendre transformation we can obtain the canonical Hamiltonian of the form
\[ \mathcal{H}^{(0)} = \frac{1}{2} \pi_i^2 + \frac{1}{4} F_{ij} F^{ij} + \frac{1}{2} m^2 (A^0)^2 + \frac{1}{2} m^2 (A^i)^2 - A^0 (\partial^i \pi_i + m^2 A^0) + \frac{1}{2 m^2} \pi_\theta^2 + \frac{1}{2 m^2} (\partial^i \theta)^2 - m^2 A^i \partial_i \theta - (d_1 - 1) A^i \partial_i B - \frac{1}{d_2} \bar{P} \mathcal{P} + d_2 \partial_i \bar{C} \partial_i \mathcal{C} - d_3 B^2 - d_4 g \]  

(3.38)

where the canonical momenta conjugate to \( A^0, A^i, \theta, \mathcal{C} \) and \( \bar{\mathcal{C}} \) are given as

\[ \pi_0 = -m^2 \theta, \quad \pi_i = F_{i0}, \quad \pi_\theta = m^2 \dot{\theta}, \quad \bar{P} = d_2 \dot{\bar{C}}, \quad \mathcal{P} = -d_2 \bar{\mathcal{C}}, \]

(3.39)

and satisfy the super-Poisson algebra

\[ \{ \mathcal{C}(x), \bar{P}(y) \} = \{ \mathcal{P}(x), \bar{\mathcal{C}}(y) \} = \{ N(x), B(y) \} = \delta(x - y). \]

Here note that since in the GF Lagrangian (3.37) we have already introduced the momenta field \( B \), we do not have any specific relation between \( B \) and \( N \) in Eq. (3.37), and thus we still have the covariant term proportional to \( A^\mu \partial_\mu B \) in Eq. (3.38), where we have also used the identification

\[ \bar{N} = -\partial_\mu A^\mu, \]

(3.40)

which plays a crucial role in construction of the BRST symmetry and is also related to the integrability condition in Eqs. (4.8) and (4.9) in the next section.

Following the canonical procedure for obtaining the symplectic first-order Lagrangian, we have

\[ \mathcal{L}^{(0)} = \pi_0 \dot{A}^0 + \pi_i \dot{A}^i + \pi_\theta \dot{\theta} + B \bar{N} + \bar{\mathcal{P}} \bar{\mathcal{C}} + \bar{\mathcal{C}} \bar{\mathcal{P}} - \mathcal{H}^{(0)}, \]

(3.41)

where the initial set of symplectic variables \( \xi^{(0)\alpha} \) and their conjugate momenta \( \alpha^{(0)} \) are now given by

\[ \xi^{(0)\alpha} = \{ A, B \} = \frac{\delta A}{\delta q}_r \frac{\delta B}{\delta p}_l - (-1)^{n_A n_B} \frac{\delta B}{\delta q}_r \frac{\delta A}{\delta p}_l \]

where \( n_A \) denotes the ghost number in \( A \) and the subscript \( r \) and \( l \) imply right and left derivatives, respectively.

Note that, differently from the identification \( N = -A^0 \) in the literature [28], here we have used a highly nontrivial relation.
\[
\xi^{(0)\alpha} = (A, \pi, \theta, \pi_{\theta}, A^0, N, B, C, \bar{\mathcal{P}}, \bar{\mathcal{C}}, \mathcal{P}),
\]
\[
\alpha^{(0)}_\alpha = (\bar{\pi}, \bar{\theta}, \pi_{\theta}, 0, -m^2 \theta, B, 0, \bar{\mathcal{P}}, 0, \mathcal{P}, 0).
\] (3.42)

From Eq. (3.42) we read off the symplectic singular two-form matrix to be
\[
f^{(0)}_{\alpha\beta}(x, y) = \begin{pmatrix} f_{\hat{\alpha}\hat{\beta}} & O \\ O & f^{GF}_{\mu\nu} \end{pmatrix} \delta(x - y),
\] (3.43)

where \(f_{\hat{\alpha}\hat{\beta}}\) is a \(9 \times 9\) submatrix, which can be read off from Eq. (3.19), and \(f^{GF}_{\mu\nu}\) is a \(6 \times 6\) submatrix defined as
\[
f^{GF}_{\mu\nu} = \begin{pmatrix} -\epsilon & O & O \\ O & -\epsilon & O \\ O & O & -\epsilon \end{pmatrix},
\] (3.44)

with \(\epsilon\) being the Levi-Civita tensor with \(\epsilon_{12} = 1\) and \(O\) being the \(2 \times 2\) null matrix. As in Eq. (3.21), using a non-trivial zero mode
\[
\nu^{(0)T}_{\alpha, y}(1, x) = (0, \bar{\pi}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \delta(x - y).
\] (3.45)

we obtain a constraint \(\tilde{\Omega}_2\)
\[
\tilde{\Omega}_2 = \partial^i \pi_i + m^2 A^0 + \pi_{\theta} + (d_1 - 1) \dot{B},
\] (3.46)

which will be shown to be equal to the corresponding constraint in Eq. (2.12) with the fixed values of \(d_1\).

Next, as in the gauge invariant symplectic embedding case, we obtain the first-iterated Lagrangian by enlarging the canonical sector with the constraint \(\tilde{\Omega}_2\) and its associated Lagrangian multiplier \(\rho\)
\[
\mathcal{L}^{(1)} = \pi_0 \dot{A}^0 + \pi_i \dot{A}^i + \pi_{\theta} \dot{\theta} + B \dot{N} + \bar{\mathcal{P}} \dot{\mathcal{C}} + \bar{\mathcal{C}} \dot{\mathcal{P}} + \tilde{\Omega}_2 \dot{\rho} - \mathcal{H}^{(1)}
\] (3.47)

where \(\mathcal{H}^{(1)} = \mathcal{H}^{(0)}|_{\tilde{\Omega}_2=0}\) is the first-iterated Hamiltonian. We now obtain the first-level symplectic variables \(\xi^{(1)\alpha}\) and their conjugate momenta \(a^{(1)}_\alpha\).
\[ \zeta^{(0)\alpha} = (\vec{A}, \vec{\pi}, \theta, \pi_\theta, A^0, N, B, C, \bar{C}, \bar{C}, \rho), \]
\[ a^{(0)}_\alpha = (\vec{\pi}, \vec{0}, \pi_\theta, 0, -m^2\theta, B, 0, \bar{P}, 0, \bar{P}, 0, \bar{\Omega}_2). \quad (3.48) \]

and the first-iterated symplectic two-form matrix
\[
f_{\alpha\beta}^{(1)}(x, y) = \begin{pmatrix}
    f_{\tilde{\alpha}\tilde{\beta}} & O & \vec{m} \\
    O & f_{\mu\nu}^{GF} & \vec{m}_{GF} \\
    -\vec{m}^T & -\vec{m}_{GF}^T & 0
\end{pmatrix} \delta(x - y),
\]
with \( \vec{m}^T = (\vec{\theta}, -\vec{\nabla}_x, 0, 1, m^2) \) and \( \vec{m}_{GF}^T = (0, -(d_2 - 1)\partial_t, 0, 0, 0, 0, 0). \)

In order to realize the BRST symmetry, we introduce two zero-modes
\[
\nu_{\alpha,y}^{(1)T}(1, x) = (\vec{\theta}, 0, 0, -m^2, 1, 0, 0, 0, 0, 0, 0, 0, 0)\delta(x - y),
\]
\[
\nu_{\alpha,y}^{(1)T}(2, x) = (\vec{\nabla}_x, \vec{\theta}, -1, 0, 0, -\partial_t, 0, 0, 0, 0, 0, 0, 1)\delta(x - y). \quad (3.50)
\]

We require that these zero-modes should not generate any new constraint upon applying it from the left to the equation of motion
\[
\int d^3y \nu_{\alpha,y}^{(1)T}(2, x) \frac{\delta}{\delta \zeta^{(1)\alpha}(y)} \int d^3z \mathcal{H}^{(1)}(z) = (d_1 - 1)\partial_t \bar{\partial} B + d_4 \frac{\partial g}{\partial \theta}, \quad (3.51)
\]
and the equation corresponding to the zero-mode \( \nu_{\alpha,y}^{(1)T}(1, x) \) yields trivial identity. In order to guarantee no new constraint, we choose for the free adjustable coefficients:
\[
d_1 = 1, \quad d_3 = -\frac{1}{2} \alpha, \quad d_4 = 0, \quad (3.52)
\]
where we have used the conventional form for \( d_3 \) to be consistent with the BRST gauge fixing term. As a result, we have arrived at the Lorentz invariant Lagrangian of the form
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 (A_\mu + \partial_\mu \theta)^2 + A^\mu \partial_\mu B + d_2 \partial_\mu \bar{C} \partial^\mu C - \frac{1}{2} \alpha B^2. \quad (3.53)
\]

Exploiting the transformation rules \( (3.33) \), with the generalized zero-modes \( \nu_{\alpha,y}^{(1)T}(\sigma, x) \) in Eq. \( (3.50) \), the replacement of \( \epsilon_2 = -\lambda \bar{C} \) and an additional BRST transformation rule for the \( \bar{C} \), we obtain the BRST transformation rules having nilpotent property \( \delta_B^2 = 0 \) as follows,
\[
\delta_B A^\mu = -\lambda \partial^\mu \bar{C}, \quad \delta_B \theta = \lambda \bar{C}, \quad \delta_B \bar{C} = -\lambda B, \quad \delta_B C = \delta_B B = 0, \quad (3.54)
\]
under which we obtain
\[ \delta_B \mathcal{L} = -(d_2 + 1) \lambda \partial_\mu \mathcal{C} \partial^\mu B. \] (3.55)

With the fixed value of \(d_2:\)
\[ d_2 = -1 \] (3.56)
we have finally obtained the desired BRST invariant Lagrangian
\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 (A_\mu + \partial_\mu \theta)(A^\mu + \partial^\mu \theta) + A_\mu \partial_\mu B - \partial_\mu \bar{\mathcal{C}} \partial^\mu \mathcal{C} - \frac{1}{2} \alpha B^2. \] (3.57)

Note that in this symplectic formalism, one can avoid algebra of complicated structure
having fermionic gauge fixing function and minimal Hamiltonian needed in the standard
BRST formalism \[28\].

### IV. HAMILTON-JACOBI ANALYSIS

In this section we apply the HJ method \[22,23,25\] to the Proca model where the general-
ized HJ PDEs are given as
\[ \mathcal{H}'_0 = p_0 + \mathcal{H}_0 = 0, \quad \mathcal{H}'_1 = \pi_0 + \mathcal{H}_1 = 0, \] (4.1)
where \(\mathcal{H}_0\) is a canonical Hamiltonian density (2.2) and \(\mathcal{H}_1\) is actually zero for the Proca case. Note that \(\mathcal{H}'_1\) is the primary constraint in the Dirac terminology \[1\]. From Eq. (4.1) one obtains after some calculation,
\[ dq_\mu = \frac{\partial \mathcal{H}_0'}{\partial p_\mu} dt, \quad dp_\mu = -\frac{\partial \mathcal{H}_0'}{\partial q_\mu} dt, \] (4.2)
where \(q_\mu = (t, A^0, A^i)\), \(p_\mu = (p_0, \pi_0, \pi_i)\) and \(t_\alpha = (t, A^0)\).

Exploiting the Hamilton equations (4.2), we obtain the set of equations of motion
\[ dA_0 = dA_0, \quad dA^i = (\pi_i + \partial^i A^0) dt, \]
\[ d\pi_0 = (\partial_\mu \pi_i + m^2 A^0) dt, \quad d\pi_i = (\partial_j F^{ij} - m^2 A^i) dt. \] (4.3)
Note that, since the equation for $A^0$ is trivial, one cannot obtain any information about the variable $A^0$ at this level, and the set of equations is not integrable.

In order to remedy these unfavorable problems, we have to investigate the integrability conditions:

$$\dot{\mathcal{H}}'_\alpha = \{\mathcal{H}'_\alpha, \mathcal{H}'_0\} + \{\mathcal{H}'_\alpha, \mathcal{H}'_\beta\} \dot{q}_\beta = 0,$$  \hspace{1cm} (4.4)

where, unlike in the usual case, the Poisson bracket is defined as

$$\{A, B\} = \frac{\partial A}{\partial q_\mu} \frac{\partial B}{\partial p_\mu} - \frac{\partial B}{\partial q_\mu} \frac{\partial A}{\partial p_\mu},$$  \hspace{1cm} (4.5)

with the extended index $\mu$ corresponding to $q_\mu = (t, A^0, A^i)$. Eq. (4.4) then imply that

$$\dot{\mathcal{H}}'_0 = -\mathcal{H}'_2, \quad \dot{\mathcal{H}}'_1 = \mathcal{H}'_2,$$  \hspace{1cm} (4.6)

where $\mathcal{H}'_2$ is a secondary constraint in Dirac terminology given as

$$\mathcal{H}'_2 = \partial^i \pi_i + m^2 A^0 = 0.$$  \hspace{1cm} (4.7)

This $\mathcal{H}'_2$ then yields an additional integrability condition as

$$\dot{\mathcal{H}}'_2 = m^2 \mathcal{H}'_3,$$  \hspace{1cm} (4.8)

where

$$\mathcal{H}'_3 = \partial_i A^i + A_0 = 0,$$  \hspace{1cm} (4.9)

which provides the missing information for the Hamilton equations for $A_0$. As a result, one can easily get the desired equations of motion (2.10). Moreover, in Eqs. (1.8) and (1.9), time evolution of $\mathcal{H}'_2$ can be rewritten in the nontrivial covariant form: $\dot{\mathcal{H}}'_2 = m^2 \partial_\mu A^\mu$ and such somehow unusual structure has been already seen in Eq. (3.40) in the symplectic

5Note that even though $\mathcal{H}'_{\tilde{\alpha}}$ carry the extended index $\tilde{\alpha}$ ($\tilde{\alpha} = 0, 1, 2, 3$) with the additional constraints, the coordinates $t_{\tilde{\alpha}}$ carry only the index $\tilde{\alpha}$ since one cannot generate coordinates themselves.
embedding. Note that $\mathcal{H}_3'$ yields the value of $\dot{A}_0$ which is exactly the same as the above fixed value of $u$, and also the Poisson brackets in the HJ scheme are the same as those in the DQM since $\Omega_i$ do not depend on time explicitly. Moreover, if the generalized HJ PDEs in Eqs. (4.4) and (1.7) are rewritten in terms of $\Omega_1(=\mathcal{H}_1')$ and $\Omega_2(=\mathcal{H}_2')$ and $u(=\dot{A}_0)$, one can easily reproduce the integrability conditions in Eqs. (4.6) and (4.8), thus showing that the integrability conditions in HJ scheme are equivalent to the consistency conditions in DQM.

Now we consider the closeness of the Lie algebra involved in the HJ scheme by introducing operators $X_\alpha$ ($\alpha = 0, 1$) corresponding to $\mathcal{H}_\alpha'$, formally defined by

$$ X_\alpha f = \frac{\partial f}{\partial t} + \left(\pi_i + \partial_i A^0\right) \frac{\delta f}{\delta A_i} + \left(\partial_j F^{ij} - m^2 A^i\right) \frac{\delta f}{\delta \pi_i}. $$

(4.10)

Using Eq. (4.10) we then obtain

$$ X_0 f = \frac{\partial f}{\partial t} + \left(\pi_i + \partial_i A^0\right) \frac{\delta f}{\delta A_i} + \left(\partial_j F^{ij} - m^2 A^i\right) \frac{\delta f}{\delta \pi_i}, $$

$$ X_1 f = \frac{\delta f}{\delta A_0}, $$

(4.11)

to yield the commutator relation among the operators $X_\alpha$

$$ [X_0, X_1] f = - \left(\frac{\delta f}{\delta A_i}\right) \delta(x - y). $$

(4.12)

Since the above commutator relation is not closed, we need to extend the set $\{X_\alpha\}$ to a set of operators $\{X_\alpha\}$ ($\bar{\alpha} = 0, 1, 2, 3$) by introducing new operators. In fact, after some algebra, we can construct two new operators $X_2$ and $X_3$:

$$ X_2 f = \partial_i \frac{\delta f}{\delta A_i}, $$

$$ X_3 f = \partial_i \frac{\delta f}{\delta \pi_i}, $$

(4.13)

to yield a closed Lie algebra

$$ [X_0, X_1] f = -X_2 f \delta(x - y), \quad [X_0, X_2] f = -m^2 X_3 f \delta(x - y), $$

$$ [X_0, X_3] f = [X_1, X_2] f = [X_1, X_3] f = [X_2, X_3] f = 0, $$

(4.14)
which automatically guarantee the integrability conditions discussed above.

Finally, we discuss the integrability conditions in terms of action. In fact, Eq. (4.2) yields

\[
dS = dt \alpha \int d^3 x \left( -\mathcal{H}_\alpha + \pi_i \frac{\delta \mathcal{H}'_\alpha}{\delta \pi_i} \right),
\]

(4.15)

from which we have obtained the action of the form

\[
S = \int d^3 x \left( -\mathcal{H}_0 dt + \pi_0 dA^0 + \pi_i dA^i \right).
\]

(4.16)

Since we can now have full equations of motion for \( A^i \) and \( A^0 \) from Eqs. (4.3) and (4.7), respectively, \( dA^\mu \) can be integrable to yield the desired expression \( dA^\mu = \dot{A}^\mu dt \). We can thus arrive at the desired standard action

\[
S = \int d^4 x \mathcal{L}_0
\]

(4.17)

where \( \mathcal{L}_0 \) is exactly the same as the first-order Lagrangian (3.1) in the symplectic formalism. Note that it was through the introduction of the secondary constraint obtained by the integrability condition (4.4) that one could construct the action (4.17) even in the second-class system.

V. CONCLUSION

It has been the primary objective of this paper to demonstrate in form of a simple model how the improved Dirac quantization method (DQM) program of embedding second-class Hamiltonian systems into first-class ones in the context of Dirac’s quantization procedure can be realized in the framework of the symplectic approach to constrained systems. Rather than proceeding iteratively as in the improved DQM approach, we have greatly simplified the calculation by making use of manifest Lorentz invariance in our Ansatz for the Wess-Zumino (WZ) term. Just as in the case of the improved DQM procedure, this Ansatz clearly shows, that the embedding procedure requires the introduction of an even number of additional fields, which, following the Faddeev-Jackiw prescription [16] can be chosen.
to be canonically conjugate pairs. Indeed, the number of second-class constraints is always even, and we know from the improved DQM embedding procedure that phase space must be augmented by one degree of freedom for each secondary constraint. This fact has not been recognized in a recent paper on this subject [20] where in our notation \( \pi_\theta \) has been taken to be a function of \( A^i, \pi_i \) and \( \theta \). Correspondingly they did not obtain the desired Wess-Zumino Lagrangian. On the other hand, our procedure treating \( \pi_\theta \) as an independent field, led in a natural way to the Stückelberg Lagrangian, in agreement with earlier calculations using the improved DQM [19]. Similar to the WZ term case, we introduced the additional ghost and antighost fields together with the auxiliary fields in the symplectic scheme to construct the BRST invariant gauge-fixed Lagrangian and its nilpotent BRST transformation rules. In this approach we could avoid complicated calculations of the minimal Hamiltonian associated with the fermionic gauge fixing function.

Finally, we have also demonstrated explicitly for the model in question that the “integrability conditions” of the Hamilton-Jacobi (HJ) scheme are just the “consistency conditions” of the standard Dirac quantization procedure. Next, we have constructed operators in terms of the generalized Poisson brackets, to obtain the closed Lie algebra associated with the commutator relations among these operators, and to guarantee the integrability conditions using the closed Lie algebra. Moreover, using the integrability conditions in the HJ formalism, we have reconstructed the desired standard action, equivalent to the first-order Lagrangian in the symplectic scheme.

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