Design of arbitrarily shaped acoustic cloaks through partial differential equation-constrained optimization satisfying sonic-metamaterial design requirements

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We develop an optimization framework for the design of acoustic cloaks, with the aim of overcoming the limitations of usual transformation-based cloaks in terms of microstructure complexity and shape arbitrariness of the obstacle. This is achieved by recasting the acoustic cloaking design as a nonlinear optimal control problem constrained by a linear elliptic partial differential equation. In this setting, isotropic material properties’ distributions realizing the cloak take the form of control functions and a system of the first-order optimality conditions is derived accordingly. Such isotropic media can then be obtained in practice with simple hexagonal lattices of inclusions in water. For this reason, the optimization problem is directly formulated to consider suitable partitions of the control domain. Two types of inclusions are analysed, and long-wavelength homogenization is used to define the feasible set of material properties that is employed as a constraint in the optimization problem. In this manner, we link the stage of material properties optimization with that of microstructure design, aiming at finding the optimal implementable solution. The resulting cloak is numerically tested via coupled structural/acoustic simulations. As a further test...
benchmark, a concave target and the silhouette of a ship are considered other than the usual axisymmetric cloak.

1. Introduction

Inspired by the development of transformation theory, the quest for the implementation of invisibility devices has spread during the last decade over diverse research fields [1], in which governing partial differential equations have been shown to be invariant under coordinate transformations. Started in electromagnetism [2,3], this theory has indeed unlocked the possibility to achieve perfect concealment from detection in acoustics [4–6], elastodynamics [7,8], surface water waves [9], heat conduction [10] and even matter waves [11]. The beauty and power of this analytical method stand in the fact that the obtained cloak is theoretically exact for all frequencies and incoming directions of the probing incident radiation.

In acoustics [6,12], it has also been shown that the solution of the problem is not unique in terms of material parameters distributions: inertial cloaks [13–19] are made with anisotropic inertial properties, pure pentamode cloaks [20,21] are obtained with solids exhibiting singular anisotropic elasticity tensors, while the most general acoustic cloak can comprise both mass and elasticity anisotropy. On the flip side, however, these material distributions are hard to achieve in practice, and one is often forced to resort to complex microstructures designed by homogenization-based optimization techniques [22–24] to obtain the required anisotropic material behaviour. More than that, analytical solutions are available for simple geometries only, such as the axisymmetric case [25], and the literature dealing with arbitrarily shaped cloaks based on exact solutions of transformation theory is limited and almost entirely restricted to the inertial cloak case [26–28]. For the pentamode case, Hu and co-workers have shown that a numerical method based on the elasticity equation can be used to find quasi-symmetric deformations to design arbitrarily shaped cloaks [29], in the same way as the solution of the Laplace equation can be used in electromagnetism to find quasi-isotropic transformations [30,31]. Analytical solutions that lead to approximate pentamode cloaks can be found in the case of elliptical targets using transformation in suitable curvilinear coordinates [32,33].

Several attempts have been made to overcome such restrictions and allow for simplified design, for instance using quasi-conformal cloaks [34], in which the transformation is specifically constructed in such a way that anisotropy is avoided in the obtained material distributions. However, the geometries that allow for application of this technique are limited and the cloak should in principle comprise the overall space; thus a truncation is required that makes the solution not exact.

Scattering cancellation, instead, is an alternative technique that relies on surrounding the target with a distribution of small obstacles, in such a way that the resulting multiple scattering solution has no influence on the incident field. Such distribution of scatterers can be obtained by setting a priori their number and shape and optimizing for their location either with evolutionary algorithms [35] or with gradient based optimization [36]. Increased degrees of freedom can instead be considered in the optimization if not only the location, but the shape also is not fully determined a priori: in this case one can use parametric optimization of Bezier shapes [37], or even topology optimization [38] which has recently allowed the consideration of acoustic–elastic interactions [39] in the optimal design of acoustic cloaks. The simplicity of construction unlocked by these techniques has also allowed for the design and validation of three-dimensional cloaks of axisymmetric obstacles [40], whose practical demonstration is still lacking when considering classic transformation theory-based cloaks. The downside of these methods is that they are inherently narrowband and work for a limited set of incident angles. The broadbandedness and the number of working directions can be increased by augmenting the set of cases considered in the cost function, accepting a trade-off between performance and number of working frequencies/directions.
In the search for simplified configurations, one can progressively rely more on optimization and less on model-based intuition, for example, by exploiting neural networks to compute the physical properties of a set of layers of isotropic homogeneous fluids [41] or on structural topology optimization [42,43].

In this paper, we follow another route and reformulate the design phase such that the properties of the cloak are obtained as the solution of a partial differential equation (PDE)-constrained optimization problem, that is, an optimal control problem (OCP). The control functions are infinite-dimensional, space-varying fields of material properties that nullify the scattered wave. The state equation is represented by the inhomogeneous Helmholtz equation [44] describing the scattered wave in the domain. A similar PDE-constrained optimization framework is considered in [45] with the additional complexity of adding uncertainty in the problem formulation and in [46] to solve active thermal cloaking problems. Instead of considering the wave propagation velocity as control function as in [45], we consider as separate control functions both the density and bulk modulus fields, and introduce constraints in the optimization for such controls, thus taking into account the fact that in practical implementations these two parameters can hardly be chosen independently. This in turns allows us to derive an elegant and concise expression for the reduced gradient of the cost functional with respect to these two control variables. More than that, our formulation is intended to facilitate the link between the design of the macrostructure, i.e. the material property distribution, and that of the microstructure that implements via long-wavelength homogenization the required density and bulk modulus, thus unlocking the marriage between the two stages of the design of such two-scale optimization problem. Indeed, the standard approach to implement inhomogeneous material property distributions in acoustic cloaking is to discretize them and fill each resulting sub-domain with an appropriately optimized microstructure [20,21,33]. This approach leads to sub-optimal solutions depending on the chosen discretization: provided that the sub-domains are sufficiently small compared to the wavelength considered, the wave ‘feels’ a gradient of refraction index that might be different from the required one. In this work, we instead make use of appropriate control basis functions that allow us to obtain optimal solutions taking into account the size and shape of the cloak sub-domains at the level of the optimization problem. Finally, considering inhomogeneous but isotropic material distributions considerably reduces the complexity of the required microstructure, which can be simply obtained considering hexagonal lattices of solid inclusions in the hosting water medium. The paper is organized as follows. In the next two sections the optimization problem is introduced, and the optimality conditions are derived. The OCP is then discretized with the finite-element method (FEM) in order to allow for numerical solutions and the solution of the usual axisymmetric cloak is shown. An analysis of the performance of the method with respect to design parameters as the thickness of the cloak and the number of frequencies/directions considered in the definition of the cost objective is also provided. In the fourth section, an in-depth analysis of the reachable set of homogenized material properties is conducted on simple hexagonal lattices of solid inclusions in water, in order to build a set of constraints for the OCP that allows for practical implementations. In §5, such constraints are introduced in the formulation of the problem, and constrained solutions are compared to those obtained previously with the unconstrained problem. Before drawing conclusions, §6 deals with the numerical validation of the cloak implemented with the microstructures analysed in §4. As further test cases, concave and boat-shaped obstacles are considered to show the effectiveness of the proposed method in dealing with arbitrarily shaped targets.

2. Problem statement

We consider a two-dimensional acoustic scattering problem in an inhomogeneous medium consisting of water as background fluid and of a cloaking region modelled as an inhomogeneous yet isotropic equivalent fluid. The computational domain \( \Omega \subset \mathbb{R}^2 \) is divided in two sub-domains: \( \Omega_c \) is the domain occupied by the cloak, \( \Omega_a \) corresponds to the surrounding ambient, and it is occupied by the fluid. The domain’s boundary is \( \Gamma = \partial \Omega = \Gamma_i \cup \Gamma_e \), where \( \Gamma_i \) is the obstacle’s
shape and $\Gamma_e$ the external boundary. The interface between cloak and fluid domains is denoted as $\Gamma_c$, whereas the external boundary $\Gamma_e$ is needed for computational purposes and its role will be detailed in the following. The domain $\Omega_a$ is filled with water with standard physical properties ($\rho_0 = 998 \text{ kg m}^{-3}, \kappa_0 = 2.2 \text{ MPa}$). We denote as $\rho_0$ and $\kappa_0$ its properties in the background domain $\Omega_a$. On the other hand, the physical properties in the domain $\Omega_c$ are assumed as control functions and denoted as $\rho$ and $\kappa$. These are considered as functions of the space variable $x \in \Omega_c$. This layout is shown in Figure 1.

When the system is forced by time harmonic waves, the steady-state acoustic pressure $P(x,t)$ can be separated as $\Re\{p(x)e^{i\omega t}\}$, where $\Re\{\cdot\}$ denotes the real component of its argument. The complex amplitude $p(x)$ satisfies the Helmholtz equation for inhomogeneous media [44]:

$$\nabla \cdot (a(x)\nabla p(x)) = -b(x)\omega^2 p(x),$$

(2.1)

where $p(x) \in \mathbb{C}$ is the pressure field phasor and $\omega$ the circular frequency of the forcing wave. The coefficients $a$ and $b$ are defined as $a := \rho^{-1}$ and $b := \kappa^{-1}$, where $\rho$ is the local mass density and $\kappa$ the local bulk modulus. The definition of $a$ and $b$ will turn out to be useful in manipulating equation (2.1) and setting up the resulting OCP.

Notice that (2.1) governs the dynamics of an ideal, inviscid, still fluid at homogeneous constant temperature. Although such an acoustic wave equation is not able to capture typical phenomena of ocean acoustics (absorption, multipath propagation due to temperature gradients, scattering by air bubbles, etc.), it is commonly adopted in the literature related to underwater cloaking [16,18,19,21] because of its invariance under curvilinear coordinate transformations [12], and because it is sufficiently accurate to allow for the design of experimental demonstrators. We thus limit ourselves to the same case with the aim to introduce an alternative approach still comparable to the ones commonly adopted. In passing, we also note that the acoustic wave equation describes the propagation of SH elastic waves in solids and the propagation of TE and TM modes in electromagnetism [47]. It follows that upon suitable reinterpretation of the controls $a$ and $b$ the introduced optimization approach can be equivalently used to design cloaks that work with respect to such types of incident radiation.

The total pressure field can be decomposed into an incident and a scattered field, that is:

$$p(x) = p_s(x) + p_i(x),$$

(2.2)
where \( p_i \) is the solution of the Helmholtz equation obtained considering a homogeneous fluid without obstacles. That is, \( p_i \) satisfies equation (2.1) with homogeneous properties

\[
a_0 \Delta p_i(x) = -b_0 \omega^2 p_i(x).
\]

A plane wave solution to equation (2.3) is \( p_i = e^{-i k_0 a \cdot x} \), where \( k_0 = \omega / c_0 \) is the homogeneous wave number, \( c_0 = \sqrt{\mu_0 / \rho_0} \) is the undisturbed sound velocity and \( a \in \mathbb{R}^2 \) is the unit vector associated with the direction of the incident wave. Equation (2.1) can be rewritten in terms of the scattered pressure \( p_s \):

\[
- \nabla \cdot \left( a(x) \nabla p_s(x) \right) - \omega^2 b(x) p_s(x) = \omega^2 \left( b(x) - b_0 \right) p_i(x) + \nabla \cdot \left[ \left( a(x) - a_0 \right) \nabla p_i(x) \right],
\]

which is obtained by plugging equation (2.3) into equation (2.1) and rearranging the terms. We remark that the incident wave \( p_i \) is a datum of the problem. On the obstacle surface, we consider the general boundary condition

\[
\nabla p \cdot n = - \frac{j \omega \rho}{z} p_r,
\]

where \( z \) is the acoustic impedance at the interface. Here (2.5) can be written in terms of scattered and incident wave as

\[
\nabla p_s \cdot n + \frac{j \omega \rho}{z} p_s = - \nabla p_i \cdot n - \frac{j \omega \rho}{z} p_r,
\]

and corresponds to an inhomogeneous boundary condition of Robin type. Sound soft Dirichlet or sound hard Neumann boundary conditions are obtained as limiting cases when \( z \) tends to zero or infinity, respectively.

In order to approximate computationally an unbounded domain we need to guarantee that the scattered wave is outgoing by satisfying the Sommerfeld radiation condition [48]:

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial p_s(x)}{\partial r} + j k_0 p_s(x) \right) = 0,
\]

where \( r = \|x\| \). For the sake of simplicity, we substitute equation (2.6) with the first-order Bayliss and Turkel approximation for two-dimensional domains, that is [49]:

\[
\nabla p_s(x) \cdot n(x) + \left( j k_0 + \frac{1}{2R} \right) p_s(x) = 0 \quad \text{on } \Gamma_e,
\]

where \( R \) is the radius of \( \Gamma_e \). Note that equation (2.7) is a homogeneous Robin boundary condition. This approximation guarantees a perfect absorption when the wave direction is normal to the surface \( \Gamma_e \), and a reflection coefficient that increases up to 0.01 when the angle between the propagation direction and the normal is 30° and to 0.1 when it is 60° [50]. In the following, we will make the assumption that the acoustic centre of the target–cloak pair is close to the centre of the domain, such that the wave direction on \( \Gamma_e \) is nearly orthogonal to the external surface, so that the resulting artificial reflections are contained. This allows one to retain the aforementioned approximation for a computationally efficient implementation when compared to more sophisticated domain truncations [51,52]. In the following, we will omit the explicit dependence on the space variable when it is clear from the context.

\[3. \ The \ optimal \ control \ problem\]

In this section, the acoustic cloaking problem is formulated as an OCP where the state dynamics consists of the scattered field \( p_s(x) \) that solves the linear elliptic PDE (2.4). Space modulated density and bulk modulus in the cloaking region take the role of control functions. Hence, the overall OCP is nonlinear due to the way the control affects the state. The cloaking objective is achieved if the intensity of the scattered wave vanishes, which is equivalent to minimizing the quadratic objective \( \|p_s(x)\|^2 = \bar{p}_s(x)p_s(x) \) in the ambient domain \( \Omega_o \), where \( \bar{p}_s(x) \) represents the complex conjugate of \( p_s(x) \). This objective can be encoded in a quadratic cost functional which aims at finding the optimal trade-off minimizing the scattered wave with control functions which
deviate as little as possible from the background properties of water. Then, the OCP can be written as follows:

$$\min_{v,u,p_s} J(v,u,p_s) = \frac{\lambda_v}{2} \int_{\Omega_c} v^2 \, d\Omega + \frac{\lambda_u}{2} \int_{\Omega_c} u^2 \, d\Omega + \frac{1}{2} \int_{\Omega_v} \tilde{p}_s p_s \, d\Omega \quad (3.1)$$

subject to

$$\begin{align*}
- \nabla \cdot (a \nabla p_s) - b \omega^2 p_s &= f \quad \text{in } \Omega, \\
a \nabla p_s \cdot n + aj_{\text{op}} p_s &= g \quad \text{on } \Gamma_u, \\
a \nabla p_s \cdot n + \alpha p_s &= 0 \quad \text{on } \Gamma_e, 
\end{align*} \quad (3.2)$$

where

$$f = \omega^2 (b - b_0) p_i + \nabla \cdot [(a - a_0) \nabla p_i],$$

$$g = -a \nabla p_i \cdot n - aj_{\text{op}} p_i$$

and

$$\alpha = a \left( jk_0 + \frac{1}{2R} \right)$$

and the functional relationships between the control functions $u$ and $v$ and the perturbed material properties are

$$a = a_0 e^{-v} \quad \text{and} \quad b = b_0 e^{-u}.$$  

In this way the positivity of the density $\rho$ and bulk modulus $\kappa$ is ensured for any choice of the control functions $u$ and $v$. The exponential change of variables to ensure positivity of the control variables is standard and was used in [45] when controlling the wave propagation velocity.

We now derive a set of first-order optimality conditions applying the Lagrangian method [53]. Using this idea, we obtain an explicit expression for the gradient of the cost functional in the continuous setting. We briefly recall the main steps of the Lagrange method in appendix A. First of all, we define suitable functional spaces for state and control functions. We select the logarithm of background properties we can select as control space

$$\phi \text{ derivative with respect to an arbitrary state variation } \psi \in H^1(\Omega).$$

In other words, for each $\psi \in \Omega_c$ we associate a real-valued control pair $(u(x), v(x))$ whose elements are bounded.

The Lagrangian functional $\mathcal{L} : \mathcal{V} \times \mathcal{U} \times \mathcal{W}^* \to \mathbb{R}$ can be formed as

$$\mathcal{L} := J + \Re \left\{ \int_{\Omega} (\nabla \cdot (a \nabla p_s) + b \omega^2 p_s + f) \lambda \, d\Omega \right\}, \quad (3.3)$$

where the adjoint function $\lambda : \Omega \to \mathbb{C}$ belongs to $H^1(\Omega)$, that is, we can identify $\mathcal{W}^* = H^1(\Omega)$. Note that the Lagrangian is defined as a real-valued functional and an equivalent formulation can be recovered by using the imaginary part.

A system of the first-order necessary conditions for optimality is obtained by taking the Gâteaux derivatives of the Lagrangian with respect to state, control and adjoint variables independently (e.g. [53]). The adjoint dynamics is obtained by setting to zero the Lagrangian derivative with respect to an arbitrary state variation $\varphi \in H^1(\Omega)$. Applying the divergence theorem and substituting the boundary conditions, the Lagrangian can be rewritten as

$$\mathcal{L} = J + \Re \left\{ \int_{\Omega} -a \nabla p_s \cdot \nabla \lambda + b \omega^2 p_s \lambda + f \lambda \, d\Omega + \int_{\Gamma_u} \left( g - aj_{\text{op}} p_s \right) \lambda \, d\Gamma - \int_{\Gamma_e} \alpha p_s \lambda \, d\Gamma \right\}. \quad (3.4)$$

$\mathcal{L}$ is a functional which maps complex-valued functions to real numbers; therefore to compute its Gâteaux derivatives we make use of basic results from complex analysis, that is, we apply Wirtinger’s calculus rules [55]. In particular recall that $d\Re(cz)/dz = c/2$ and $d(\bar{z}z)/dz = \bar{z}$ for $cz \in \mathbb{C}$.  

\[ \text{Proc. R. Soc. A 478 (20210750) royalsocietypublishing.org/journal/rspa} \]
Hence, the Gâteaux derivative of $\mathcal{L}$ with respect to $p_s$ is

$$
\mathcal{L}_{p_s}^{\prime}[\varphi] = \frac{1}{2} \int_{\Omega_s} \tilde{p}_s \varphi \, d\Omega + \frac{1}{2} \int_{\Omega} -a \nabla \tilde{\lambda} \cdot \nabla \varphi + b \omega^2 \tilde{\phi} \, d\Omega
- \int_{\Gamma_c} a \tilde{\phi} \, d\Gamma - \frac{1}{2} \int_{\Gamma_c} \tilde{\lambda} \, d\Gamma = 0 \quad \forall \varphi \in H^1(\Omega)
$$

which, by applying the divergence theorem in the reverse direction, can be seen to be the weak formulation of the adjoint dynamics:

$$
\nabla \cdot (a \nabla \lambda) - b \omega^2 \lambda = p_s \chi_{\Omega_c} \quad \text{in} \ \Omega,
\nabla \lambda \cdot \mathbf{n} - j \frac{\omega_0}{z} \lambda = 0 \quad \text{on} \ \Gamma_d,
\nabla \lambda \cdot \mathbf{n} + \left( \frac{1}{2R} - jk_0 \right) \lambda = 0 \quad \text{on} \ \Gamma_e.
$$

(3.5)

and

$$
\chi_{\Omega_c}(x) \text{ being the indicator function of the domain } \Omega_c.
$$

We now turn to the optimality conditions involving the control functions $u$ and $v$. The Lagrangian (3.3) can be rewritten substituting the explicit form of $f$ and considering satisfied the boundary conditions of state and adjoint PDEs as

$$
\mathcal{L} = f + \Re \left\{ \int_{\Omega} b \omega^2 (p_s + p_i) \tilde{\lambda} \, d\Omega - \int_{\Omega} a \nabla (p_s + p_i) \cdot \nabla \tilde{\lambda} \, d\Omega + \int_{\Gamma_c} a \nabla (p_s + p_i) \cdot \mathbf{n} \tilde{\lambda} \, d\Gamma \right\},
$$

(3.6)

so that we can easily take control variations $\psi \in L^\infty(\Omega_c)$. Physically, the control variations cannot modify the background properties outside of the cloak. Hence, $a = a_0$ on $\Gamma_d$, $\psi^0_a[\psi] = b^0_a[\psi] = 0$ on $\Omega_d$ and

$$
\psi^0_u[\psi] = (a_0 e^{-\psi}) \psi = -a_0 e^{-\psi} \psi = -\psi,
$$

and similarly $b^0_u[\psi] = -b^0 v$, so that the control necessary conditions (i.e. the reduced gradient) in variational form results in

$$
\begin{align*}
\mathcal{L}_u^\prime[\psi] &= \lambda_u \int_{\Omega_c} v \varphi \, d\Omega + \Re \left\{ \int_{\Omega_c} a \nabla (p_s + p_i) \cdot \nabla \tilde{\lambda} \varphi \, d\Omega \right\} = 0 \quad \forall \varphi \in L^\infty(\Omega_c) \\
\mathcal{L}_v^\prime[\psi] &= \lambda_v \int_{\Omega_c} u \varphi \, d\Omega - \Re \left\{ \int_{\Omega_c} b \omega^2 (p_s + p_i) \tilde{\lambda} \psi \, d\Omega \right\} = 0 \quad \forall \varphi \in L^\infty(\Omega_c).
\end{align*}
$$

(3.7)

The strong form of the reduced gradient can be identified as

$$
\begin{align*}
\nabla J_v = \lambda_u v + \Re \left\{ a \nabla (p_s + p_i) \cdot \nabla \tilde{\lambda} \right\} \\
\nabla J_u = \lambda_v u - \Re \left\{ b \omega^2 (p_s + p_i) \tilde{\lambda} \right\}.
\end{align*}
$$

(3.8)

Equation (3.8) together with the adjoint equation (3.5) and the state equation (2.4) constitute a system of the first-order necessary conditions for optimality. Note that we did not make any assumption on the structure of the control basis functions other than belonging to the space $L^\infty(\Omega_c)$. However, the actual controlled material properties will be realized with piece-wise constant functions at the microstructure level. In order to preserve the optimal properties at the microstructure, we express the control functions as linear combinations of indicator functions describing the cell domain. In particular, let us define a subdivision of the control domain $\Omega_c$ in $N_c$ disjoint sets whose elements $\Omega_{c,j}$ satisfy

$$
\bigcup_{j=1}^{N_c} \Omega_{c,j} \subseteq \Omega_c \quad \text{and} \quad \Omega_{c,j} \cap \Omega_{c,i} = \emptyset \quad \text{for} \ i \neq j,
$$

and define the functions $\psi_j(x) = \chi_{\Omega_{c,j}}(x)$ as the indicator functions of such sets. Then it is natural to express to control variables $u$ and $v$ as

$$
u = \sum_{j=1}^{N_c} \psi_j(x) v_j = \psi(x)^\top v
$$

(3.9)
where the shape functions $\psi_j$ are defined according to the cell shape and distribution in the domain $\Omega_c$ and the constant coefficients $u_i$ and $v_i$ of the linear combination are the control variables of the optimization problem. The control discretization layout is shown in figure 2. This formulation allows one to preserve the cloak’s optimal properties at the microstructure level, as was mentioned in the Introduction.

The optimization problem is still set in the infinite-dimensional space for the state and adjoint variables. It is also clear that $u = \sum_{j=1}^{N_c} \psi_j(x) u_j \in L^\infty(\Omega_c)$ and the functional setting of the OCP is still consistent. Regarding the optimality conditions, it is easy to see that state and adjoint dynamics are unchanged. Slightly more care is needed to recover the form of the reduced gradients $\nabla J_v$ and $\nabla J_u$ for $j = 1, \ldots, N_c$. We substitute equation (3.9) in the Lagrangian formulation (3.6) as

$$
L = \frac{\lambda_v}{2} v^T \left( \int_{\Omega_c} \psi \psi^T \, d\Omega \right) v + \frac{\lambda_u}{2} u^T \left( \int_{\Omega_c} \psi \psi^T \, d\Omega \right) u + \frac{1}{2} \int_{\Omega_e} p_s p_s \, d\Omega
+ \Re \left\{ \int_{\Omega_c} \omega^2 b(u)(p_s + p_l) \, d\Omega - \int_{\Omega} a(v) \nabla (p_s + p_l) \cdot \nabla \tilde{\lambda} \, d\Omega + \int_{\Gamma} a(v) \nabla (p_s + p_l) \cdot \hat{n} \, d\Gamma \right\},
$$

where

$$a(v) = a_0 e^{-\psi^Tv} \quad \text{and} \quad b(u) = b_0 e^{-\psi^Tv}.$$  

Furthermore, since $\forall x \in \Omega_c$ there is at most one index $k$ such that $\psi_k(x) \neq 0$ we have

$$e^{-\psi^Tv} = e^{-\sum_{j=1}^{N_c} v_j \psi_j} = \sum_{j=1}^{N_c} e^{-v_j \psi_j}$$

for every vector $v \in \mathbb{R}^{N_c}$. Note also that the gradient of $a$ and $b$ can be written as

$$\nabla_v a = -\psi a \quad \text{and} \quad \nabla_u b = -\psi b$$

so that the reduced gradients can be expressed by taking the finite-dimensional gradient of the Lagrangian with respect to $v$ and $u$, that is:

$$\nabla_v J = \lambda_v \left( \int_{\Omega_c} \psi \psi^T \, d\Omega \right) v + \Re \left\{ \int_{\Omega_c} \psi a \nabla (p_s + p_l) \cdot \nabla \tilde{\lambda} \, d\Omega \right\}$$

and

$$\nabla_u J = \lambda_u \left( \int_{\Omega_c} \psi \psi^T \, d\Omega \right) u + \Re \left\{ \int_{\Omega_c} \psi b \omega^2 (p_s + p_l) \, d\Omega \right\}. $$

Figure 2. The obstacle is surrounded by hexagonal domains $\Omega_{c,j}$. Notice that $\bigcup_{j=1}^{N_c} \Omega_{c,j} \subset \Omega_c$ is the control domain. (Online version in colour.)
Note that $\int_{\Omega} \psi \psi^T \, d\Omega$ is a diagonal matrix whose entries are the areas of the respective cells. We can now turn to the full discretization of the problem.

(a) Discretization of the optimal control problem

For the numerical solution of the OCP we employ the FEM. We select piecewise quadratic, globally continuous ansatz functions $\varphi_i (\mathbb{P}_2$ finite elements) for the space approximation of state and adjoint in $\Omega$ while the control basis functions do not need any spatial approximation since their functional form is expressed by equation (3.9). The FEM approximation of the state equation reads

$$A(u, v) p = f(u, v),$$

where

$$A_{ij} = \int_{\Omega} a(v) \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega - \int_{\Omega} \omega^2 b(u) \varphi_i \varphi_j \, d\Omega - \int_{\Gamma_c} \alpha \varphi_i \varphi_j \, d\Gamma - \int_{\Gamma_i} \frac{\alpha p}{z} \varphi_i \varphi_j \, d\Gamma.$$

Since $a = a_0$ in $\Omega \setminus \Omega_c$, it is useful to rewrite

$$a = a_0 + a_0(e^{-\psi^T v} - 1) \quad \text{and} \quad b = b_0 + b_0(e^{-\psi^T u} - 1)$$

so that using equation (3.10) the components of $A$ can be separated as

$$A_{ij} = \int_{\Omega} a_0 \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega + \sum_{k=1}^{N_c} (e^{-u_k} - 1) \int_{D_{c,k}} a_0 \nabla \varphi_i \cdot \nabla \varphi_j \, d\Omega - \int_{\Omega} b_0 \omega^2 \varphi_i \varphi_j \, d\Omega$$

$$- \sum_{k=1}^{N_c} (e^{-u_k} - 1) \int_{D_{c,k}} b_0 \omega^2 \varphi_i \varphi_j \, d\Omega - \int_{\Gamma_c} a_0 (j k_0 + \frac{1}{2 R}) \varphi_i \varphi_j \, d\Gamma - \int_{\Gamma_i} \frac{\alpha p}{z} \varphi_i \varphi_j \, d\Gamma$$

$$= (A_0)_{ij} + \sum_{k=1}^{N_c} (e^{-u_k} - 1)(A_k)_{ij} + (B_0)_{ij} + \sum_{k=1}^{N_c} (e^{-u_k} - 1)(B_k)_{ij} + C_0 + C_i,$$  \hspace{1cm} (3.12)

where the matrices $A_k, B_k, C_0$ and $C_i$ can be precomputed and only their sum must be performed when varying the control vectors $u$ and $v$. Besides the presence of the exponential function that enforces the positive definiteness of the material properties, equation (3.12) highlights the bilinear structure of the control problem. Finally, it is easy to notice that the matrix $A$ is symmetric being the sum of symmetric matrices. The components of the right-hand side $f$ can be written as

$$f_i = \int_{\Omega} (a - a_0) \nabla p_i \cdot \nabla \varphi_i \, d\Omega + \int_{\Omega} (b - b_0) \omega^2 \varphi_i \varphi_i \, d\Omega - \int_{\Gamma_c} a_0 \left( \nabla p_i \cdot n + j \frac{\alpha p}{z} p_i \right) \varphi_i \, d\Gamma$$

$$= \sum_{k=1}^{N_c} (e^{-u_k} - 1) \int_{D_{c,k}} a_0 \nabla p_i \cdot \nabla \varphi_i \, d\Omega + \sum_{k=1}^{N_c} (e^{-u_k} - 1) \int_{D_{c,k}} b_0 \omega^2 \varphi_i \varphi_i \, d\Omega$$

$$- \int_{\Gamma_c} a_0 \left( \nabla p_i \cdot n + j \frac{\alpha p}{z} p_i \right) \varphi_i \, d\Omega$$

$$= \sum_{k=1}^{N_c} (e^{-u_k} - 1)(l_k)_i + \sum_{k=1}^{N_c} (e^{-u_k} - 1)(d_k)_i - q_i$$

which again shows the same bilinear structure in the way the control functions enter the right-hand side. The adjoint discretization follows the same steps for the left-hand side while the right-hand side corresponds to the FEM discretization of the state projected in $L^2(\Omega)$. That is, we have

$$A^\dagger(u, v) \lambda = M_{\Omega_c} p,$$

where $M_{\Omega_c}$ is the restriction of the usual mass matrix to the observation domain $\Omega_c$, that is, the domain in which we want to minimize the scattered field, and $(\cdot)^\dagger$ is the Hermitian operator. Note that since $A^\dagger = A$ the discretized version of the state operator remains self-adjoint. Finally, the
FEM discretization of state \( p \) and adjoint \( \lambda \) can be plugged in equation (3.11) to obtain the fully discrete version of the reduced gradient, that at component level of \( v_k \) can be written as

\[
\nabla J_{v_k} = \lambda_v |D_{c,k}| v_k + e^{-v_k} \int_{D_{c,k}} a_0 \nabla (p_s + p_j) \cdot \nabla \lambda \, d\Omega \\
= \lambda_v |D_{c,k}| v_k + e^{-v_k} \lambda^t (A_k p + l_k),
\]

(3.13)

where \( |D_{c,k}| \) is the measure of the set associated with the \( k \)th cell. For the \( u_k \) control vectors, we have

\[
\nabla J_{u_k} = \lambda_u |D_{c,k}| u_k + e^{-u_k} \int_{D_{c,k}} b_0 \omega^2 (p_s + p_j) \lambda \, d\Omega \\
= \lambda_u |D_{c,k}| u_k + e^{-u_k} \lambda^t (B_k p + d_k).
\]

Once the fully discretized version of the optimality conditions is obtained, we set up algorithm 1 using an iterative steepest descent method to solve the OCP with microstructure specified by the functions \( \psi \). The algorithm is forced to stop if either the number of iterations reaches the maximum \( \text{maxIter} = 6000 \) or \( \| \nabla J(u') \| < \text{tol} \), where \( \text{tol} \) is set to \( 1 \times 10^{-3} \).

**Algorithm 1. Steepest Descent for Optimal Cloak.**

1: 
2: \( \psi, N_c \leftarrow \text{Define microstructure shape and domain} \)
3: FEM Model \( \leftarrow \text{Assemble constant FEM matrices} \)
4: \( u^0, v^0 \leftarrow \text{Assign control initial guesses} \)
5: 
6: for \( t = 1 : \text{maxIter} \) do
7: \( p^t \leftarrow \text{Solve state equation: } A(u^t, v^t) p = f(u^t, v^t) \)
8: \( \lambda^t \leftarrow \text{Solve adjoint equation: } \tilde{A}(u^t, v^t) \lambda = M_{\Omega, p^t} \)
9: 
10: \( \nabla J(u^t)_k \leftarrow \lambda_u |D_{c,k}| u^t_k + e^{-u_k} \lambda^t (B_k p^t + d_k) \)
11: \( \nabla J(v^t)_k \leftarrow \lambda_v |D_{c,k}| v^t_k + e^{-v_k} \lambda^t (A_k p^t + l_k) \)
12: 
13: if \( \left\| \left( \nabla J(u^t), \nabla J(v^t) \right) \right\| < \text{tol} \) then
14: return
15: end if
16: 
17: \( \tau \leftarrow \text{ArmijoBacktracking}(f, \nabla J(u^t), \nabla J(v^t), u^t, v^t) \)
18: 
19: \( u^{t+1} \leftarrow u^t - \tau \nabla J(u^t) \)
20: \( v^{t+1} \leftarrow v^t - \tau \nabla J(v^t) \)
21: 
22: end for

The OCP is solved for a circular target surrounded by the set of \( N_c = 390 \) hexagonal unit cells as shown in figure 3 when probed by acoustic illumination from left to right at an angular frequency \( \omega \) corresponding to \( \lambda/\tau = 0.69 \), with \( \lambda = 2\pi c_0 / \omega \) and \( \tau \) being the radius of the target. The inner boundary \( \Gamma_i \) is set to be sound soft \( (z \to 0) \). Indeed, while no material can show an acoustic impedance sufficiently greater than that of water to approximate an infinitely rigid constraint, a sound soft boundary can be physically realized employing a thin cylinder filled with air, as
Figure 3. Optimal distribution of material properties computed solving the OCP when the shape of the target $\Gamma$ is a circle and the angular frequency $\omega$ is selected such that $\lambda/r = 0.69$, $r$ being the radius of the target. The incidence direction is $\mathbf{a} = [1, 0]$. 390 hexagonal unit cells of edge $l = 8.7\% \lambda$ are employed. The external radius of the cloak is $1.57r$. Algorithm 1 converges at the cost $\tilde{J}/\tilde{J}_0 = 1.5 \times 10^{-4}$ after 5446 iterations lasting 7.3 h. (Online version in colour.)

Figure 4. (a) Total pressure field, obstacle case. The pressure is normalized with respect to the amplitude of the incident wave. (b) Scattered pressure field, obstacle case. (c) Decibel reduction in acoustic intensity computed for the scattered field at 1 m from the centre of the domain, with respect to the incident intensity. (d) Total pressure field, cloaked case, when the material properties in the cloak are the optimal ones shown in figure 3. (e) Scattered pressure field, cloak case. (f) Decibel reduction in acoustic intensity with respect to the incident one, cloak case. (Online version in colour.)

The size of the unit cells is chosen such that it satisfies the condition of scale separation between their characteristic length and the wavelength, in such a way that long-wavelength homogenization can then be employed to realize the cloak. From a numerical point of view, this entails the solution of a Dirichlet problem for both state and adjoint variables. Figure 3 shows the results in terms of non-dimensional material properties $\hat{\rho} = \rho/\rho_0$ and $\hat{\kappa} = \kappa/\kappa_0$. Figure 4 compares the total (figure 4a versus figure 4d) and scattered pressure fields (figure 4b versus figure 4e) between the cloaked and uncloaked case. The mean scattered intensity $I_{\text{mean}}$ at 1 m from the centre of $\Omega$ is also computed.
for all the azimuthal angles \( \theta \) and in figure 4c,f is shown for comparison in terms of Decibel reduction with respect to the incident intensity \( I_{\text{inc}} \):

\[
\Delta(\theta) = 10 \log_{10} \left( \frac{I_{\text{mean}(\theta)}}{I_{\text{inc}}} \right) \quad [\text{dB}]
\]

An average reduction of 40 dB on the scattered intensity is obtained using the hexagonal microstructure discretization. Note that, as shown in figure 3, one needs to implement equivalent controlled properties both higher and smaller than those of the background fluid.

The influence of the thickness of the cloak (that together with the upper bound on the characteristic length of the cells sets the overall number of control variables) is depicted in figure 5a. The part of the final cost related to the scattered field distribution,

\[
\tilde{J} = \frac{1}{2} \int_{\Omega} \bar{p} \dot{p}_s \, d\Omega,
\]

normalized with respect to that obtained in the uncloaked scenario is computed when the obstacle is surrounded by cloaks of increasing thickness. The associated number of cells is also reported on the horizontal axis. For each configuration, the number of iterations required to reach convergence is also depicted, showing that increasing the thickness (and thus the overall number of design variables) not only allows for better performance, but actually improves convergence.

In figure 5b, a blue line shows instead what happens when taking the 390 unit cells cloak of figure 3 and computing \( \tilde{J}/J_0 \) in a frequency band around the frequency \( f_0 \) targeted in the optimization (highlighted with a dot in the graph). It is shown that the scattered field increases rapidly as one moves away from the design frequency. A band wide 7.1% of \( f_0 \) can be identified, in which \( \tilde{J}/J_0 \) lies below 0.1. A similar plot can be produced probing the cloaked obstacle with waves incident from directions distinct from the design direction, as shown in figure 5c. An aperture angle of 6.9° can be computed for \( \tilde{J}/J_0 < 0.1 \).

Multiple frequencies and directions can also be taken into account in the optimization. The definition of the OCP can be modified as follows. Let us consider a number \( N_f \) of incident pressure fields \( p_{i,h} \), \( h \in \{1, \ldots, N_f\} \). The governing equations are linear with respect to the pressure, thus the
superposition principle holds and we can modify the objective functional by weighting the sum of the scattered fields for each probing frequency. Indeed, we can select as

\[
J_{N_f} = \frac{\lambda_v}{2} \int_{\Omega_c} v^2 \, d\Omega + \frac{\lambda_u}{2} \int_{\Omega_c} u^2 \, d\Omega + \frac{1}{2} \sum_{h=1}^{N_f} \bar{p}_{s,h} \bar{p}_{s,h} \, d\Omega,
\]

where each scattered pressure \( p_{s,h} \) satisfies the state dynamics (3.2) with frequency \( \omega_h \) and forcing terms determined by \( p_{i,h} \). Note that the PDE constraints are now \( N_f \). With similar arguments as for the previous derivation, we can form a Lagrangian which comprises the sum of the PDE constraints. From the latter, we can compute \( N_f \) adjoint equations of the form (3.5) where the right-hand side depends on \( p_{s,h} \) only. Note that the control functions are the same for each state and adjoint equation. In this way, the reduced gradients can be computed as

\[
\nabla J_v = \lambda_v \left( \int_{\Omega_c} \psi \psi^\top \, d\Omega \right) v + \sum_{h=1}^{N_f} \Im \left\{ \int_{\Omega_c} \psi a \nabla (p_{s,h} + p_{i,h}) \cdot \nabla \bar{\lambda}_h \, d\Omega \right\}
\]

and

\[
\nabla J_u = \lambda_u \left( \int_{\Omega_c} \psi \psi^\top \, d\Omega \right) u + \sum_{h=1}^{N_f} \Im \left\{ \int_{\Omega_c} \psi b \omega^2 \bar{p}_{s,h} (p_{s,h} + p_{i,h}) \bar{\lambda}_h \, d\Omega \right\}.
\]

Figure 5b reports the behaviour of the 390 unit cell cloak when designed to cloak more than one frequency. Three cases are analysed: the red and yellow lines refer to the optimization targeting two distinct frequencies, while the purple one refers to the optimization targeting three frequencies at the same time. On each curve, the dots represent the location of the design frequencies. It can be clearly appreciated a trade-off between the number of frequencies considered in the optimization and the performance in terms of scattered field amplitude at the desired frequencies, i.e. for example in each scenario the computed \( \tilde{J}/\tilde{J}_0 \) in correspondence of the dots is higher than that obtained in \( f_0 \) in the single frequency optimization. Nonetheless, targeting two closely spaced frequencies (red curve) is shown to improve the broadbandness of the cloak, enlarging the percentage amplitude of the frequency band where \( \tilde{J}/\tilde{J}_0 < 0.1 \) to 27.9%. If the spacing between the two frequencies is too large, two valleys instead of one single are obtained in the frequency dependence of the acoustic performance, as shown with the yellow curve. The addition of a third target frequency in between the two (purple curve) can restore the presence of a single minimum. Similar considerations can be applied when a multi-directional objective is considered, as shown in figure 5c. Again, dots on the polar plot represent the directions included in the definition of the cost.

4. Unit cell design

The required material parameters distribution obtained through the solution of the OCP introduced in the previous section has to be practically realized with opportunely designed microstructures that show the appropriate equivalent density and bulk modulus when homogenized. It is well known [56] that hexagonal lattices of solid inclusions in water behave in the long-wavelength limit as isotropic acoustic fluids, whose properties can be tailored upon control on the material and shape of the inclusion itself. For this very reason, the cloak subdomains have been chosen to be shaped as hexagons: in this way they can naturally be filled by hexagonal lattices. The basic configuration considered in the two-dimensional setting consists thus of a circular inclusion placed in each lattice point and made by a material with high contrast with respect to the hosting medium, e.g. a metal. This allows one to obtain a wide range of material properties with densities and bulk moduli that are generally higher than that of water. Preliminary results shown in the previous section (figure 3) underline the need to go also for \( \rho \) and \( \kappa \) smaller than those of water: it is thus implied that some kind of porosity has to be contemplated
in the solid inclusion. Indeed, since resonance phenomena are not exploited in this application, the density can be simply evaluated with the rule of mixtures:

$$\rho_{\text{hom}} = \sum_i \chi_i \rho_i,$$

where $\rho_i$ is the density of the constituents and $\chi_i$ is the cell volume filling fraction of each constituent. This in turn implies that a third light phase has to be included in the mix other than the fluid and the solid. The simplest configuration considered consists thus of a hollow cylinder filled by air ($\rho = 1.23$ kg m$^{-3}$, $\kappa = 0.14$ MPa; figure 6a).

The equivalent bulk modulus is instead computed via inspection of the dispersion relation of each considered lattice, computed via Bloch analysis on the unit cell [56]. A typical dispersion relation is shown in figure 6b: in the long wavelength limit, the linearity of the branch justifies the evaluation of $\kappa_{\text{hom}}$ as

$$\kappa_{\text{hom}} = c_{\text{ph}}^2 \rho_{\text{hom}},$$

where $c_{\text{ph}}$ is the phase speed computed as the slope of the very branch emanating from the origin. In order to compute the set of obtainable $\rho_{\text{hom}}, \kappa_{\text{hom}}$, the geometry is parametrized with the two characteristic adimensional parameters $\hat{r}_{\text{out}} = r_{\text{out}}/L$ and $\hat{r}_{\text{in}} = r_{\text{in}}/L$ (figure 6a), whose variation is considered to be bounded in the following way:

$$\begin{cases} 
\hat{r}_{\text{in}} \geq \delta \hat{r}_1 \\
\hat{r}_{\text{out}} \leq \frac{\sqrt{3}}{2} - \delta \hat{r}_1 \\
\hat{r}_{\text{out}} \geq \hat{r}_{\text{in}} + \delta \hat{r}_2,
\end{cases}$$

where $\delta \hat{r}_{1,2}$ are the non-dimensional minimum feature sizes, that is, the thinnest gap and wall allowed. These constraints define a closed feasible region $C^\circ$ in the plane $\hat{r}_{\text{in}} \times \hat{r}_{\text{out}}$, that is shown in figure 7a. By computing the homogenized properties of the associated lattices, the contour $\partial C^\circ$ going across the extremal points ABC is mapped to a curve $\partial S^\circ$ joining A’B’C’ in the $\hat{\rho} \times \hat{\kappa}$ space, with $\hat{\rho} = \rho_{\text{hom}}/\rho_0$ and $\hat{\kappa} = \kappa_{\text{hom}}/\kappa_0$. This defines the set of the obtainable material properties $S^\circ$. In figure 7b, such curve is computed for a configuration where the solid phase is chosen to be

Figure 6. (a) Schematic of a unit cell comprising a circular inclusion made by a solid phase filled with air. (b) Dispersion relation computed along the boundary of the irreducible Brillouin zone. The reduced frequency is computed as $\hat{f} = fL/c_0$, while the reduced wavenumber stands for the adimensional $kL$, which spans between the high symmetry points $\Gamma, K$ and M. (Online version in colour.)
aluminium ($\rho = 2700$ (kg m$^{-3}$), Young’s modulus $YM = 70$ (GPa), Poisson’s ratio $\nu = 0.3$) and the minimum features are selected as $\delta r_2 = 5\%$ and $\delta r_1 = 4 \delta r_2$. It can be seen how the inclusion of the light phase allows for obtaining $\hat{\rho} < 1$; notice however how it is hard to reach the region where $\hat{\kappa}$ is less than 1. To enlarge the feasible region, another configuration is thus considered: the inclusion is now shaped as an $N$-pointed star, $N$ being a multiple of 3; other than maintaining the invariance of the lattice upon rotation of $\pi/3$, i.e. the symmetry required for isotropy, the oblique walls allow one to reduce the tangential stiffness of the inclusion. When considering hydrostatic loads, this in turn increases the compressibility with respect to the case of the hollow cylinder. An $N$-pointed star is completely characterized by the lengths of the internal and external tips $\hat{P} = P/L$ and $\hat{p} = p/L$, by the fillet radii and by the thickness of the wall (figure 8a). The latter two parameters are considered fixed and are chosen to be 2.5$\%$ and 5$\%$, respectively. The bounds on the remaining two geometrical features are

$$\begin{align*}
\hat{p} &\geq \hat{p}_{\min} \\
\hat{p} &\leq \hat{P} \\
\hat{p} &\leq \hat{P}_{\max}
\end{align*}$$

and
These also define a feasible $\hat{\rho} \times \hat{\kappa}$ region $\mathcal{C}^*$ (figure 9a) whose boundary $\partial \mathcal{C}^*$ can be mapped to a path $\partial \mathcal{S}^*$ in the $\hat{\rho} \times \hat{\kappa}$ space. In figure 9b, it is shown how adopting this type of unit cell the feasible set of material properties is enlarged also in the region that is not reachable with the hollow cylinder.

Note that, the higher the number $N$, the more similar is the $N$-pointed star to a hollow cylinder when $\hat{\rho} \to \hat{\rho}$. For this reason, the $D'E'$ curve for a 12-pointed star almost overlaps with the $A'B'$ curve of the circular inclusion. This allows one to obtain a connected feasible set $\mathcal{S} := \mathcal{S}^\oplus \cup \mathcal{S}^\star$ in the $\hat{\rho} \times \hat{\kappa}$ space, as shown in figure 10, that will be considered in the following reachable region for the equivalent material properties.

5. Constrained optimal control problem

In this section, we reformulate the fully discrete PDE-constrained optimization problem in order to satisfy the constraints imposed by the realization of the actual microstructure. That is, we solve a reduced constrained optimization problem where the constrained control region generates equivalent material properties that lie in the reachable region $\mathcal{S}$ of the $\hat{\rho} \times \hat{\kappa}$ space. Furthermore, we include a regularization term in the control weightings to impose a smoother transition of material properties between neighbouring cells. First of all, the optimal material properties obtained in §3 are plotted in figure 10 as black markers in the $\hat{\rho} \times \hat{\kappa}$ plane. It can be noticed how part of them falls outside of the set of material properties that can be practically implemented by means of the microstructures described in the previous section.

In order to constrain the control variables to lie on the feasible set described by the region $\mathcal{S}$, we equip the steepest descent algorithm 1 with an additional projection step thus employing a standard projected gradient (PG) method [57]. For each component-wise control pair $(v_k, u_k)$, the corresponding point $P_k = (\hat{\rho}_k, \hat{\kappa}_k) = (e^{v_k}, e^{u_k})$ must lie in the region of the $\hat{\rho} \times \hat{\kappa}$ plane defined by $\mathcal{S}$.

The feasible region in the control space is defined as $\mathcal{S}' = \{(v, u) \in \mathbb{R}^2 : (e^v, e^u) \in \mathcal{S} \subseteq \mathbb{R}^2\}$ and we denote the projection onto $\mathcal{S}'$ as $\Pi_{\mathcal{S}'}$. The pairwise vector projection $\Pi_{\mathcal{S}'}$ is defined as

$$\left(\Pi_{\mathcal{S}'}(v, u)\right)_k = \Pi_{\mathcal{S}'}(v_k, u_k).$$

The PG method consists of replacing the gradient update in algorithm 1 with

$$(v, u)^{t+1} = \Pi_{\mathcal{S}'}(v^t - \tau \nabla J(v^t), u^t - \tau \nabla J(u^t)),$$
where the step-size $\tau$ satisfies the Armijo backtracking line-search along the projected directions [57].

Regarding the strong variation of material properties obtained in §3, we add a regularizing weighting and force neighbouring cells to have similar properties. The computed homogenized properties, indeed, refer to infinite repetition of equal unit cells, while in the most simple implementable configuration each hexagonal sub-domain is filled by a single unit cell which is thus surrounded by different ones. Limiting the difference between adjacent cells is thus beneficial for the equivalence of the behaviour of the graded index metamaterial to the expected one.

The map from the geometrical parametrization to the equivalent properties being regular and one-to-one, we can limit the geometrical dissimilarity between neighbouring cells by constraining the equivalent properties on the $\hat{\rho} \times \hat{k}$ plane or equivalently on the control space $u$ and $v$.

Thus, a penalty factor that weights the difference in the control intensity can be introduced in the cost functional as

$$\frac{1}{2} \sum_{j=1}^{N_c} \sum_{i \in \Lambda_j} (u_i - u_j)^2 = u^T H u,$$

where $\Lambda_j$ is the set of cells adjacent to the $j$th cell and $|\Lambda_j|$ its cardinality; the matrix $H$ is defined as

$$H_{ij} = \begin{cases} |\Lambda_j| & \text{if } i = j \\ -1 & \text{if } j \in \Lambda_i \\ 0 & \text{otherwise,} \end{cases}$$

and we have used the identity $\sum_{j=1}^{N_c} \sum_{i \in \Lambda_j} u_i^2 = \sum_{j=1}^{N_c} |\Lambda_j| u_j^2$. Note that the matrix $H$ corresponds to the Laplacian associated with the graph induced by the topology of the cells where an edge is present if the cells are neighbours. The graph is fully connected and hence its eigenvalues are nonnegative (e.g. [58]). The eigenvalue zero appears with multiplicity one and corresponds to the eigenvector space spanned by a vector of ones. Intuitively, this corresponds to the same control for all the cells.

Figure 10. (a) The overall reachable set of material properties given as the union of those obtained separately with the hollow cylinder inclusion and the 12-pointed star inclusion. (b) A magnification around the material properties of the background fluid shows how the $A'B'$ and $D'E'$ curves almost overlap creating a connected set. Black markers in the graph are used to underline the location of the optimal material properties computed with the unconstrained OCP, that are shown in figure 3. (Online version in colour.)
As a result, the fully discrete cost function can be written as

$$J(v, u, p, s) = \lambda_v \frac{1}{2} v^\top (H + D)v + \lambda_u \frac{1}{2} u^\top (H + D)u + \frac{1}{2} p^\top M_{\Omega_k} p,$$

where $D$ is the diagonal matrix whose entries are the areas of the associated cell. Owing to the structure of $H$, it is clear that $H + D$ is positive definite. The fully discretized reduced gradients become

$$\nabla J_v = \lambda_v ((Hv)_k + |D_{c,k}|v_k) + e^{-\nu} \lambda^\top (A_k p + I_k)$$

and

$$\nabla J_u = \lambda_u ((Hu)_k + |D_{c,k}|u_k) + e^{-\nu} \lambda^\top (B_k p + d_k).$$

The solution of the constrained optimization problem obtained by the PG method is shown in figure 11. In particular, in figure 11a–c are depicted the total field, the scattered field and the polar dependence of the decibel gain in scattered intensity computed with respect of the incident intensity, as previously done in the unconstrained scenario.

The performances in terms of scattering reduction are comparable to those obtained without the constraints. Moreover, figure 11d,e shows the obtained solution of the constrained optimization in terms of material properties distribution, i.e. the normalized bulk modulus and density, respectively. Finally, figure 11f shows the location of each unit cell as black markers in...
Note that the obtained material properties lie inside the reachable set $S$ or on its boundary $\partial S$ whenever the feasibility constraint is active.

### 6. Design of the microstructured cloak and validation

Once the optimal required material properties are found, the inverse engineering problem of finding the microstructure geometry that exhibits those $\hat{\rho}$ and $\hat{\kappa}$ pairs has to be solved. This being a much more difficult problem than the direct one, it is usually tackled adopting optimization algorithms, either parametric of evolutionary, that employ as cost function the distance between the required desired material properties and those obtained by homogenization on the considered lattice [21,33]. In the case at hand, the simplicity of the geometry of the considered unit cells, which is univocally determined in both configurations by a pair of parameters, allows for a direct mapping of the whole $C^0$ and $C^*$ spaces into the $\hat{\rho} \times \hat{\kappa}$ one. Once this map is computed, it can subsequently be used to solve the inverse engineering; in particular, the homogenized material properties are computed for the grid of points shown on the $\mathbf{r}_{\text{in}} \times \mathbf{r}_{\text{out}}$ and $\hat{\mathbf{p}} \times \hat{\mathbf{P}}$ spaces in figure 12 and the resulting discrete map is used for a first guess of the cell geometrical parameters when required $(\hat{\rho}, \hat{\kappa})$ values are specified. An optimization routine allows one then to refine the properties of each cell with few iterations.

Following the aforementioned design procedure, the entire cloak geometry obtained from the solution of the constrained OCP is defined and the resulting microstructure is depicted in figure 13, where colours are used to distinguish between domains filled by air, aluminium or water. A fully coupled structural/acoustic frequency domain finite-element simulation of the designed cloak is carried out by means of the commercial software COMSOL Multiphysics®, in order to test its performances when considering the actual implemented structure.

A first-order approximation of the Sommerfeld absorbing condition is assigned on the boundary $\Gamma_c$ to approximate an unbounded domain. The results are shown in figure 14. By looking at the scattered intensity plot of figure 14c, we can state that the outgoing energy is two orders of magnitude lower with respect to the uncloaked case; then the obstacle is undetectable. The discrepancies with respect to the simulation performed with the homogenized properties can be attributed to the fact that one single unit cell has been considered to fill each cloak sub-domain, while an infinite microstructure should ideally be placed there instead.

To validate the performance of the method against arbitrarily shaped obstacles and cloaks, a concave target is now considered. The total and scattered fields obtained for a probing incident wave are shown in figure 15, along with the associated scattered acoustic intensity. When a concave cloak whose outer surface consists of a scaled version of the shape of the target is considered, a mean reduction in the scattered field intensity of $25 \text{ dB}$ is obtained, as shown in figure 16d–f. As a further test case, a constrained OCP is set to find the optimal material properties’ distribution to cloak the silhouette of a ship, i.e. an obstacle with pointed edges. The probing acoustic field consists of the superposition of an incident plane wave with wavelength $\lambda_1 = 20.4\%$ of the ship characteristic length $L$ and direction $a = [1, 0]$ (horizontal incidence) and a plane wave with wavelength $\lambda_2 = 18.9\% L$ and direction $a = [0, 1]$ (vertical incidence). The size of each hexagonal sub-domain is $8.7\% \lambda_1 = 9.3\% \lambda_2$. Figure 17 shows the uncloaked case scenario in terms of total fields, scattered fields and scattered intensity for both horizontal and vertical incidence.

The multi-frequency problem is solved with the PG method and the results are shown in figure 18a,b in terms of material properties distributions while in figure 18c it is shown that they all lie in the feasible set $S$. The corresponding acoustic fields and scattered intensity are shown in figure 19 for comparison with the uncloaked scenario. A $15 \text{ dB}$ reduction of scattered intensity is obtained in both the backward and forward scattering directions with respect to the uncloaked case. Figure 20 shows the convergence of the three main case studies considered; in particular, since the ship silhouette scenario takes into account two distinct incident pressure waves, it is the most complicated and entails a slower convergence.
Figure 12. Direct mapping between the space of geometrical parameters and the space of homogenized material properties, that is inverted to solve for the design of the microstructure once the constrained OCP is solved. Colours help to trace visually each different point from the space of homogenized material properties back to that of its geometrical features. (Online version in colour.)

Figure 13. (a) Schematic of the geometry of the cloak made by the hexagonal lattice of inclusions. Aluminium part is depicted in grey, air in yellow and water in light blue. (b) Three-dimensional render of the extruded geometry useful for experimental validation. (Online version in colour.)

Figure 14. Fully coupled structural acoustic finite-element simulation of the microstructured cloak. (a) Total pressure field. (b) Associated scattered pressure field. (c) Decibel reduction in scattered acoustic intensity. The cost on the scattered pressure field in \( \Omega_g \) is \( \tilde{J}/\tilde{J}_0 = 3.7 \times 10^{-2} \). (Online version in colour.)
Figure 15. (a) Total pressure field for a wave incident from left on a concave bean-shaped target. (b) Associated scattered pressure field. (c) Decibel reduction in scattered acoustic intensity with respect to the incident intensity. (Online version in colour.)

Figure 16. (a) Total pressure field for a wave incident from left on a concave bean-shaped target. (b) Associated scattered pressure field. (c) Decibel reduction in scattered acoustic intensity with respect to the incident intensity. (d) Normalized bulk modulus distribution inside the cloak. (e) Normalized density distribution. (f) Each unit cell in the cloak represented as a $\hat{\rho} \times \hat{\kappa}$ pair falling inside the set $S$. 539 hexagonal unit cells of edge $l = 8.7\% \lambda$ are employed and the algorithm converges at the cost $\tilde{J}/\tilde{J}_0 = 9.6 \times 10^{-3}$ after 598 iterations lasting 0.7 h. (Online version in colour.)

7. Conclusion
In this paper, we have introduced a general acoustic cloaking design strategy that simultaneously aims at reducing the complexity of the required microstructures and enlarging the set of geometries that can be cloaked with respect to traditional transformation-based methods. This is achieved by synergic use of PDE-constrained optimization, to find the isotropic material distribution that minimizes scattering, and parametric structural optimization, to design simple
Figure 17. (a) Total pressure field for a wave incident from left with wavelength $\lambda_1 = 20.4\%$ of the ship characteristic length. (b) Associated scattered pressure field. (c) Decibel reduction in scattered acoustic intensity with respect to the incident intensity. (d) Total pressure field for a wave incident from bottom with wavelength $\lambda_2 = 18.9\%$ of the ship characteristic length. (e) Associated scattered pressure field. (f) Decibel reduction in scattered acoustic intensity with respect to the incident intensity. (Online version in colour.)

Figure 18. (a) Normalized bulk modulus distribution inside the cloak. (b) Normalized density distribution. (c) Each unit cell in the cloak represented as a $\hat{\rho} \times \hat{\kappa}$ pair falling inside the set $S$. 545 hexagonal unit cells of edge $l = 8.7\% \lambda_1 = 9.4\% \lambda_2$ are employed and the algorithm converges at the costs $(\tilde{J}/\tilde{J}_0)_1 = 8.7 \times 10^{-2}$ and $(\tilde{J}/\tilde{J}_0)_2 = 3.9 \times 10^{-2}$ for the pressure fields with wavelengths of $\lambda_1$ and $\lambda_2$, respectively, after 2450 iterations lasting 8.4 h. (Online version in colour.)

hexagonal lattices of inclusions that match the required densities and bulk moduli. More than that, such two-scale optimization problem is formulated in such a way that the two stages, i.e. the computation of the macroscale material properties distribution and the microscale design, are not disconnected steps but intimately linked together, in order to retain the optimality of the solution found. This is done at the OCP level by considering as control space a suitable
Figure 19. Acoustic fields obtained with the cloak made by the properties depicted in figure 18. (a) Total pressure field for horizontal incidence. (b) Associated scattered pressure field. (c) Decibel reduction in scattered acoustic intensity. (d) Total pressure field for vertical incidence. (e) Associated scattered pressure field. (f) Decibel reduction in scattered acoustic intensity. (Online version in colour.)

Figure 20. The cost reduction over algorithm 1 iterations shows a good convergence in the three main case studies. In particular, the axisymmetric obstacle shows the best result probably due to its simple set-up. (Online version in colour.)

linear combination of indicator functions which corresponds to the topology of the hexagonal lattices, and constraining the controls to take values inside a feasible region that is pre-computed analysing all the possible considered unit cell geometries. The method is tested against the usual axisymmetric cloaking scenario, producing a two-order-of-magnitude mean reduction of
intensity over the whole azimuthal scattering directions. The effect of considering multiple frequencies and directions is discussed; then, more complicated scenarios are considered: an obstacle with a concave shape and one with pointed edges. The solution is found to reduce the backscattered and forward scattered wave with performances comparable to those obtained in the simple axisymmetric scenario. With the simplicity of the considered unit cells’ geometry, this paper paves the way for experimental validation of the acoustic cloaking principle with arbitrary obstacle shapes.

Data accessibility. The code to generate the figures for the optimization problem is available in repository.

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Appendix A. Lagrange method

In this appendix, we briefly recall the main steps of the Lagrange method to recover a first-order system of necessary optimality conditions for control problems constrained by general PDEs. The rigorous formulation based on functional analysis tools is not treated here but we point the reader to standard textbooks on the subject (e.g. [53]). Without loss of generality, a PDE-constrained optimization problem can be written as

$$\min_{y,u} \ J(y,u)$$

s.t. \[ G(y,u) = 0, \]

where \( y \in \mathcal{Y}, u \in \mathcal{U} \) represent the state and control variables together with their suitably chosen functional spaces, \( J(y,u) \) is real-valued cost functional \( J: \mathcal{Y} \times \mathcal{U} \mapsto \mathbb{R} \) and \( G: \mathcal{Y} \times \mathcal{U} \mapsto \mathcal{W} \) is an abstract PDE constraint which in our case is the inhomogeneous Helmholtz equation with suitable boundary conditions. Our aim is to find a set of first-order necessary optimality conditions which locally extremizes the problem. In this setting, the Lagrangian method is the infinite-dimensional analogue of the one used in standard, finite-dimensional optimization when deriving the KKT system of optimality conditions (e.g. [57]). We define the Lagrangian as the real-valued functional

$$\mathcal{L} = J(y,u) + \langle G(y,u), p \rangle_{\mathcal{W}^*,\mathcal{W}},$$

where \( p \in \mathcal{W}^* \) is the adjoint variable and \( \langle \cdot, \cdot \rangle_{\mathcal{W},\mathcal{W}^*} \) is the duality pairing in \( \mathcal{W} \) which, in most applications, consist of a scalar product in a suitable space. Note that the Lagrangian is linear in \( p \) and that \( J \) and \( \mathcal{L} \) have the same extrema when the constraint \( G(y,u) = 0 \) is satisfied. Loosely speaking, the adjoint variable quantifies the cost variations associated with the state constraint. Therefore, by taking Gâteaux derivatives of the Lagrangian, the infinite-dimensional analogues of standard derivatives in the finite-dimensional settings, with respect to state, control and adjoint variables, it is possible to derive a system of optimality conditions which does not explicitly compute the sensitivity \( y'(u) \) which is in general difficult to derive and not convenient from a numerical standpoint. The triple \((\hat{y}, \hat{u}, \hat{p})\) which locally extremizes \( \mathcal{L} \) satisfies

$$\mathcal{L}'(\hat{y}, \hat{u}, \hat{p})[\xi] = \frac{d}{d \varepsilon} \mathcal{L}(\hat{y}, \hat{u}, \hat{p} + \varepsilon \xi)\bigg|_{\varepsilon = 0} = 0 \quad \forall \xi \in \mathcal{W}^*, \quad \forall y \in \mathcal{Y}, \quad \forall u \in \mathcal{U}, \quad \forall p \in \mathcal{W}^* \hspace{1cm} (A1)$$

and

$$\mathcal{L}'(\hat{y}, \hat{u}, \hat{p})[\psi] = \frac{d}{d \varepsilon} \mathcal{L}(\hat{y}, \hat{u} + \varepsilon \psi, \hat{p})\bigg|_{\varepsilon = 0} = 0 \quad \forall \psi \in \mathcal{Y}, \quad \forall u \in \mathcal{U}, \quad \forall p \in \mathcal{W}^*$$

where \( \mathcal{L}' \) indicates the Gâteaux derivative with respect to the variable written as subscript in the direction of the respective variation. Roughly speaking, it consists of an infinite-dimensional version of a directional derivative where the direction is an arbitrary yet small function. The
triple $(\xi, \varphi, \psi)$ consists of arbitrary state, adjoint and control variations respectively which are modulated by $\varepsilon \in \mathbb{R}$. Equations (A 1) constitute the variational or weak formulation of a set of first-order necessary conditions for optimality which in abstract form can be expressed as

$$
\begin{align*}
G(\hat{y}, \hat{u}) &= 0, \\
G'(\hat{y}, \hat{u})^*p &= -J'(\hat{y}, \hat{u})
\end{align*}
$$

and

$$
J'(\hat{y}, \hat{u}) + G'(\hat{y}, \hat{u})^*\hat{p} = 0
$$

and represent the state, adjoint and Euler equations, respectively.

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