Sketch-Based Estimation of Subpopulation-Weight

Edith Cohen
AT&T Labs–Research
180 Park Avenue
Florham Park, NJ 07932, USA
edith@research.att.com

Haim Kaplan
School of Computer Science
Tel Aviv University
Tel Aviv, Israel
haimk@cs.tau.ac.il

ABSTRACT

Summaries of massive data sets support approximate query processing over the original data. A basic aggregate over a set of records is the weight of subpopulations specified as a predicate over records’ attributes. Bottom-k sketches are a powerful summarization format of weighted items that includes priority sampling (PS), (PIT) and the classic weighted sampling without replacement (WS). They can be computed efficiently for many representations of the data including distributed databases and data streams.

We derive novel unbiased estimators and efficient confidence bounds for subpopulation weight. Our estimators and bounds are tailored by distinguishing between applications (such as data streams) where this is not the case. Our rank conditioning (RC) estimator, is applicable when the total weight is not provided. This estimator generalizes the known estimator for PIT sketches [18] and its derivation is simpler. When the total weight is available we suggest another estimator, the subset conditioning (SC) estimator which is tighter.

Our rigorous derivations, based on clever applications of the Horvitz-Thompson estimator (that is not directly applicable to bottom-k sketches), are complemented by efficient computational methods. Performance evaluation using a range of Pareto weight distributions demonstrate considerable benefits of the WS SC estimator over larger subpopulations (over all other estimators); of the WS RC estimator (over existing estimators for this basic sampling method); and of our confidence bounds (over all previous approaches).

Overall, we significantly advance the state-of-the-art estimation of subpopulation weight queries.

1. INTRODUCTION

Sketches or statistical summaries of massive data sets are an extremely useful tool. Sketches are obtained by applying a probabilistic summarization algorithm to the data set.

The algorithm returns a sketch that has smaller size than the original data set but supports approximate query processing on the original data set.

Consider a set of records \( I \) with associates weights \( w(i) \) for \( i \in I \). A basic aggregate over such sets is subpopulation weight. A subpopulation weight query specifies a subpopulation \( J \subseteq I \) as a predicate on attributes values of records in \( I \). The result of the query is \( w(J) \), the sum of the weights of records in \( J \). This aggregate can be used to estimate other aggregates over subpopulations such as selectivity \( (w(J)/w(I)) \), and variance and higher moments of \( \{w(i) | i \in J \} \).

As an example consider a set of all IP flows going through a router or a network during some time period. Flow records containing this information are collected at IP routers by tools such as Cisco’s NetFlow [25] (now emerging as an IETF standard). Each flow record contains the number of packets and bytes in each flow. Possible subpopulation queries in this example are numerous. Some examples are “the bandwidth used for an application such as p2p or Web traffic” or “the bandwidth destined to a specified Autonomous System.” The ability to answer such queries is critical for network management and monitoring, and for anomaly detection.

Another example is census database that includes a record for each household with associated weight equal to the household income. Example queries are to find total income by region or by the gender of the head of the household.

To support subpopulation selection with arbitrary predicates, the summary must retain content of some individual records. Two common summarization methods are \( k\)-mins and bottom-k sketches. Bottom-k sketches are obtained by assigning a rank value, \( r(i) \), for each \( i \in I \) that is independently drawn from a distribution that depends on \( w(i) \geq 0 \). The bottom-k sketch contains the \( k \) records with smallest rank values [7]. The distribution of the sketches is determined by the family of distributions that is used to draw the rank values: By appropriately selecting this family, we can obtain sketches that are distributed as if we draw records without replacement with probability proportional to their weights (WS), which is a classic sampling method with a special structure that allows sketches to be computed more efficiently than other bottom-k sketches. A different selection corresponds to the recently proposed priority sketches (PS) [18], which have estimators that minimize the sum of per-record variances [30]. \( k\)-mins sketches are obtained by assigning independent random ranks to records (again, the distribution used for each record depends on the weight…
of the record). The record of smallest rank is selected, and this is repeated \( k \) times, using \( k \) independent rank assignments. \( k \)-mins sketches include weighted sampling with replacement (WSR). Bottom-\( k \)-sketches are more informative than respective \( k \)-mins sketches (WSR bottom-\( k \) sketches can mimic WSR \( k \)-mins sketches [13]) and in most cases can be derived much more efficiently.

Before delving into the focus of this paper, which is estimators and confidence bounds for subpopulation weight, we overview classes of applications where these sketches are produced, and which benefit from our results.

Bottom-\( k \) and \( k \)-mins sketches are used as summaries of a single weighted set or as summaries of multiple subsets that are defined over the same ground set. In the latter case, the sketches of different subsets are “coordinated” in the sense that each record obtains a consistent rank value across all the subsets it is included in. These coordinated sketches support subpopulation selection based on subsets’ memberships (such as set union and intersection).

We distinguish between explicit or implicit representations of the data [14]. Explicit representations list the occurrence of each record in each subset. They include a representation of a single weighted set (for example, a distributed data set or a data stream [15] [1]) or when there are multiple subsets that are represented as item-subset pairs (for example, item-basket associations in a market basket data, links in web pages, features in documents [5] [3] [24] [29] [2]). Bottom-\( k \) sketches can be computed much more efficiently than \( k \)-mins sketches when the data is represented explicitly [3] [24] [12] [14]).

Implicit representations are those where the multiple subsets are specified compactly and implicitly (for example, as neighborhoods in a metric space [7] [16] [15] [23] [22] [13]). In these applications, the summarization algorithm is applied to the compact representation. Beyond computation issues, the distinction between data representations is also important for estimation: In applications with explicit representation, the summarization algorithm can provide the total weight of the records without a significant processing or communication overhead. In applications with implicitly represented data, and for sketches computed for subset relations, the total weight is not readily available.

An important variant uses hash values of the identifiers of the records instead of random ranks. For \( k \)-mins sketches, families of min-wise independent hash functions or \( \epsilon \)-min-wise functions have the desirable properties [5] [6] [17]. Hashing had also been used with bottom-\( k \) sketches [1] [21] [2]. This variant has the property that all copies of the same record obtain the same rank value across subsets without the need for coordination between copies or additional bookkeeping. Therefore hashing allows to perform aggregations over distinct occurrences (see [19]).

For records associated with points in some metric space such as a graph, the Euclidean plane, a network, or the time axis (data streams) [7] [11] [22], sketches are produced for neighborhoods of locations of interest. For example, all records that lie within some distance from a location or happened within some elapsed time from the current time. For such metric applications, we do not want to explicitly store a separate sketch for each possible distance. This is addressed by all-distances sketches. The all-distances sketch of a location is a succinct representation of the sketches of neighborhoods of all distances from the location. All-distances \( k \)-mins sketch were introduced in [7] [11]. All-distances bottom-\( k \) sketches were proposed and analyzed in [13]. All-distances sketches also support spatially or temporally decaying aggregation [22] [11]. One application of decaying aggregates is kernel density estimators [24] and typicality estimation [21] – The estimated density is a linear combination of the subpopulation weight over neighborhoods.

**Overview.** Section 2 contains some background and definitions. In Section 3 we apply the Maximum Likelihood principle to derive WSR ML estimators. These estimators are applicable to WS sketches as our derivation exploits special properties of the exponential distribution used to produce these sketches. While biased, WS ML estimators can be computed efficiently and perform well in practice.

Section 4 introduces a variant of the Horvitz-Thompson (HT) estimator [20]. The HT estimators assign a positive adjusted weight to each record that is included in the sketch. Records not included in the sketch have zero adjusted weight. The adjusted weight has the property that for each record, the expectation of its adjusted weight over sketches is equal to its actual weight. The adjusted weight is therefore an unbiased estimator of the weight of the record. From linearity of expectation, the sum of the adjusted weights of records in the sketch that are members of a subpopulation constitutes an unbiased estimate of the weight of the subpopulation.

The HT estimator assigns to each included record an adjusted weight equal to its actual weight divided by the probability that it is included in a sketch. This estimator minimizes the per-record variance of the adjusted weight for the particular distribution over sketches. The HT estimator, however, cannot be computed for bottom-\( k \) sketches, since the probability that a record is included in a sketch cannot be determined from the information available in the sketch alone [29] [28]. Our variant, which we refer to as HT on a partitioned sample space (HTP), overcomes this hurdle by applying the HT estimator on a set of partitions of the sample space such that this probability can be computed in each subspace.

We apply HTP to derive Rank Conditioning estimators (RC) for general bottom-\( k \) sketches (that is, sketches produced with arbitrary families of rank distributions). Our derivation generalizes and simplifies one for PRI sketches (PRI RC estimator) [15] and reveals general principles. It provides tighter and simpler estimators for WS sketches than previously known. We show that the covariance between adjusted weights of different records is zero and therefore the variance of the subpopulation weight estimator is equal to the sum of the variances of the records.

In Section 5 we again apply HTP and derive subset conditioning estimators for WS sketches (WS SC). These estimators use the total weight \( w(f) \) in the computation of the adjusted weights. The WS SC estimator is superior to the WS RC estimator, with lower variance on any subpopulation: The variance for each record is at most that of the WS RC estimator, covariances of different records are negative, and the sum of all covariances is zero. These properties give the WS SC estimator a distinct advantage as the relative variance decreases for larger subpopulations. The SC derivation exploits special properties of WS sketches – there is no known PRI estimator with negative covariances. Moreover, the WS SC estimator is strictly better than any WSR estimator: it
has a lower sum of per-record variances than the HT wsr estimator (that minimizes the sum of per-record variances for wsr but covariances do not cancel out) and is also better than the wsr “ratio” estimator based on the sum of multiplicities in the sample of records that are members of the subpopulation (which does have negative covariances that cancel out but a much higher sum of per-record variances on skewed distributions).

The ws SC estimator is expressed as a definite integral. We provide an efficient approximation method that is based on a Markov chain that converges to this estimator. After any fixed number of steps of the Markov chain we get an unbiased estimate that is at least as good as ws RC. We implemented and compared the performance of a k-mins estimator (wsr), ws ML, pri RC, ws RC, and the approximate ws SC estimators on Pareto weight distributions with a range of skew parameters (see Section 7). When the total weight is unknown or is not used, the performances of ws ML, ws RC, and pri RC are almost indistinguishable. They outperform ws r and the performance gain grows with the skew of the data. Therefore, our estimator for ws sketches nearly match the best estimators on an optimal sketch distribution.

When the total weight is provided, the ws SC estimator has a significant advantage (smaller variance) on larger subpopulations and emerges as the best estimator. The simulations also show that the approximate ws SC estimator is very effective even with a small number of steps.

Confidence intervals are critical for many applications. In Section 5 we derive confidence intervals (tailored to applications where the total weight is or is not provided) and develop methods to efficiently compute these bounds. In Section 7 we compare our confidence bounds with previous approaches (a bound for pri sketches [31] and known WSR estimators) using a range of Pareto distributions with different skew parameters. Our bounds for ws sketches are significantly tighter than the pri bounds, even when the total weight is not used. This may seem surprising since combined with our results, the pri RC estimator has nearly optimal variance [20] among all RC estimators. The explanation is that the confidence intervals do not reflect this near optimality. Our ws confidence bounds derivation, based on some special properties of ws sketches, exploits the information available in the sketch. We point on the sources of slack in the pri confidence bounds of [31] that explain its inferior behavior. We propose approaches to address some non-inherent sources of slack. Our ws bounds that use the total weight are tighter, in particular for large subpopulations, than those that do not use the total weight.

A short summary of some of the results in this paper appeared in [10].

2. PRELIMINARIES

Let \((I, w)\) be a weighted set. A rank assignment maps each item \(i\) to a random rank \(r(i)\). The ranks of items are drawn independently using a family of distributions \(F_w\) \((w \geq 0)\), where the rank of an item with weight \(w(i)\) is drawn from \(F_w(i)\). For a subset \(J\) of items and a rank assignment \(r()\) we define \(B_r(r(), J)\) to be the item in \(J\) with smallest rank according to \(r()\) and \(r_1(J) = r(B_r(r(), J))\) to be the \(i\)th smallest rank value of an item in \(J\).

Definition 2.1. \(k\)-mins sketches are produced from \(k\) independent rank assignments, \(r^{(1)}(), \ldots, r^{(k)}().\) The sketch of a subset \(J\) is the \(k\)-vector \((r^{(1)}(J), r^{(2)}(J), \ldots, r^{(k)}(J))\).

For some applications, we use a sketch that includes with each entry an identifier or some other attributes such as the weight of the items \(B_1(r^{(j)}(), J)\) \((j = 1, \ldots, k)\).

Definition 2.2. Bottom-\(k\) sketches are produced from a single rank assignment \(r()\). The bottom-\(k\) sketch \(s(r(), J)\) of the subset \(J\) is a list of entries \((r_i(J), w(B_i(r(), J)))\) for \(i = 1, \ldots, k.\) (If \(|J| < k\) then the list contains only \(|J|\) items.) The list is ordered by rank, from smallest to largest. In addition to the weight, the sketch may include an identifier and attribute values of items \(B_i(r(), J)\) \((i = 1, \ldots, k)\). We also include with the sketch the \((k+1)st\) smallest rank value \(r_{k+1}(J)\) (without additional attributes of the item with this rank value).

In fact, bottom-\(k\) sketches must include the items’ weights but do not need to store all rank values and it suffices to store \(r_{k+1}\). Using the weights of the items with \(k\) smallest ranks and \(r_{k+1}\), we can redraw rank values to items in \(S\) using the density function \(f_w(x) / F_w(r_{k+1})\) for \(0 \leq x \leq r_{k+1}\) and 0 elsewhere, for item with weight \(w\) [14].

Lemma 2.3. This process of re-assigning ranks is equivalent to drawing a random rank assignment \(r()\) and taking \(s(r()(), J)\) from the probability subspace where \({B_1(r^{(1)}(), J), \ldots, B_k(r^{(k)}(), J)} = \{B_1(r(J), J), \ldots, B_k(r(J), J)\}\) (the same subset of items with \(k\) smallest ranks, not necessarily in the same order) and \(r_{k+1}(J) = r_{k+1}(J)\).

Bottom-\(k\) and \(k\)-mins sketches have the following useful property: The sketch of a union of two sets can be generated from the sketches of the two sets. Let \(J, H\) be two subsets. For any rank assignment \(r(), r(J\cup H) = \min\{r(J), r(H)\}\). Therefore, for \(k\)-mins sketches we have \(r_1(J\cup H), \ldots, r_k(J\cup H) = (\min\{r_1(J), r_1(H)\}, \ldots, \min\{r_k(J), r_k(H)\})\). This property also holds for bottom-\(k\) sketches. The \(k\) smallest ranks in the union \(J\cup H\) are contained in the union of the sets of the \(k\)-smallest ranks in each of \(J\) and \(H\). That is, \(B_k(r(J\cup H), J\cup H) \subseteq B_k(r(J), J) \cup B_k(r(H), H)\). Therefore, the bottom-\(k\) sketch of \(J\cup H\) can be computed by taking the pairs with \(k\) smallest ranks in the combined sketches of \(J\) and \(H\). To support subset relation queries and subset unions, the sketches must preserve all rank values.

 ws sketches. The choice of which family of random rank functions to use matters only when items are weighted. Otherwise, we can map (bijectively) the ranks of one rank function to ranks of another rank function in a way that preserves the bottom-\(k\) sketch.1 Rank functions \(F_w\) with some convenient properties are exponential distributions with parameter \(w\) [4]. The density function of this distribution is \(F_w(x) = \text{e}^{-wx}\), and its cumulative distribution function is \(F_w(x) = 1 - \text{e}^{-wx}\). The minimum rank \(r(J) = \min_{i \in J} r(i)\) of an item in a subset \(J \subset I\) is exponentially distributed with \(X\) as we shall see in Section 5.2 if \(w(J)\) is provided and we use ws sketches, we can redraw all rank values, effectively obtaining a rank assignment from the probability subspace where the subset of items with \(k\) smallest ranks is the same.

We map \(r\) such that \(F_1(r) = \alpha\) to \(r'\) such that \(F_2(r') = \alpha\), where \(F_1\) is the CDF of the first rank function and \(F_2\) is the CDF of the other (assuming the CDFs are continuous).
parameter \( w(J) = \sum_{i \in J} w(i) \) (the minimum of independent exponentially distributed random variables is exponentially distributed with parameter equal to the sum of the parameters of these distributions). Cohen \[7\] used this property to obtain unbiased low-variance estimators for both the weight and the inverse weight of the subset.\(^3\)

With exponential ranks the item with the minimum rank \( r(J) \) is a weighted random sample from \( J \): The probability that an item \( i \in J \) is the item of minimum rank is \( w(i)/w(J) \). Therefore, a \( k \)-mins sketch of a subset \( J \) corresponds to a weighted random sample of size \( k \), drawn with replacement from \( J \). We call \( k \)-mins sketch using exponential ranks a wsr sketch. On the other hand, a bottom-\( k \)-sketch of a subset \( J \) with exponential ranks corresponds to a weighted \( k \)-sample drawn without replacement from \( J \) \[13\]. We call such a sketch a ws sketch.

The following property of exponentially-distributed ranks is a consequence of the memoryless nature of the exponential distribution.

**Lemma 2.4.** \[14\] Consider a probability subspace of rank assignments over \( J \) where the \( k \) items of smallest ranks are \( i_1, \ldots, i_k \) in increasing rank order. The rank differences \( r_1(J) - r_2(J), \ldots, r_{k+1}(J) - r_k(J) \) are independent random variables, where \( r_j(J) - r_{j-1}(J) (j = 1, \ldots, k+1) \) is exponentially distributed with parameter \( w(J) - \sum_{i=1}^{j-1} w(i) \). (We formally define \( r_0(J) \equiv 0 \).

WS sketches can be computed more efficiently than other bottom-\( k \) sketches in some important settings. One such example is unaggregated data (each item appears in multiple “pieces”) \[9\] \[8\] that is distributed or resides in external memory. Computing a bottom-\( k \) sketch generally requires pre-aggregating the data, so that we have a list of all items and their weight, which is a costly operation. A key property of exponential ranks is that we can obtain a rank value for an item by computing independently a rank value for each piece, based on the weight of the piece. The rank value of the item is the minimum rank value of its pieces.

The WS sketch contains the items of the \( k \) pieces of distinct items with smallest ranks and can be computed in two communication rounds over distributed data or in two linear passes: The first pass identifies the \( k \) items with smallest rank values. The second pass is used to add up the weights of the pieces of each of these \( k \) items.

Another example is when items are partitioned such that we have the weight of each part. In this case, a WS sketch can be computed while processing only a fraction of the items. A key property is that the minimum rank value over a set of items depends only on the sum of the weights of the items. Using this property, we can quickly determine which parts contribute to the sketch and eliminate chunks of items that belong to other parts.

The same property is also useful when sketches are computed online over a stream. Bottom-\( k \)-sketches are produced using a priority queue that maintains the \( k+1 \) items with smallest ranks. We draw a rank for each item and update the queue if this rank is smaller than the largest rank in the queue. With WS sketches, we can simply draw directly from a distribution the accumulated weight of items that can be “skipped” before we obtain an item with a smaller rank value than the largest rank in the queue. The stream algorithm simply adds up the weight of items until it reaches one that is incorporated in the sketch.

**PRI sketches.** With priority ranks \[13\] \[11\] the rank value of an item with weight \( w \) is selected uniformly at random from \([0, 1/w]\). This is equivalent to choosing a rank value \( r/w \), where \( r \in U[0,1] \), the uniform distribution on the interval \([0,1]\). It is well known that if \( r \in U[0,1] \) then \(-\ln(r)/w \) is an exponential random variable with parameter \( w \). Therefore, in contrast with priority ranks, exponential ranks correspond to using rank values \(-\ln(r)/w \) where \( r \in U[0,1] \).

PRI sketches are of interest because one can derive from them an estimator that (nearly) minimizes the sum of per-item variances \( \sum_{i \in J} \text{var}(\hat{w}(i)) \). More precisely, Szegedy showed that the sum of per-item variances using PRI sketches of size \( k \) is no larger than the smallest sum of variances attainable by an estimator that uses sketches with average size \( k-1 \).

Some of our results apply to arbitrary rank functions. Some basic properties that hold for both PRI and WS ranks are monotonicity — if \( w_1 \geq w_2 \) then for all \( x \geq 0 \), \( F_{w_1}(x) \leq F_{w_2}(x) \) (items with larger weight are more likely to have smaller ranks) and invariance to scaling — scaling of all the weights does not change the distribution of subsets selected to the sketch.

**Review of weight estimators for wsr sketches.** For a subset \( J \), the rank values in the \( k \)-mins sketch \( r_1(J), \ldots, r_k(J) \) are independent samples from an exponential distribution with parameter \( w(J) \). The quantity \( \sum_{k=1}^{k-1} r_k(J) \) is an unbiased estimator of \( w(J) \). The standard deviation of this estimator is equal to \( w(J)/\sqrt{k-2} \) and the average (absolute value of the) relative error is approximately \( \sqrt{2/(\pi(k-2))} \) \[11\]. The quantity \( \sum_{k=1}^{k} r_k(J) \) is the maximum likelihood estimator of \( w(J) \). This estimator is \( k/(k-1) \) times the unbiased estimator. Hence, it is obviously biased, and the bias is equal to \( w(J)/(k-1) \). Since the standard deviation is about \((1/\sqrt{k})w(J)\), the bias is not significant when \( k \gg 1 \). The quantity \( \sum_{k=1}^{k} r_k(J) \) is an unbiased estimator of the inverse weight \( 1/w(J) \). The standard deviation of this estimate is \( 1/(\sqrt{k}w(J)) \).

Subpopulation weight estimators for wsr sketches when the total weight is known are the HT estimator, where the adjusted weight is the ratio of the weight of the item and the probability \( 1 - (1 - w(i)/w(J))^k \) that it is sampled. This estimator minimizes the sum of per-item variances but covariances do not cancel out. Another estimator is the sum of multiplicities of items in the sketch that are members of the subpopulation, multiplied by total weight, and divided by \( k \). This estimator has covariances that cancel out, but higher per-item variances. With WS sketches it is not possible to obtain an estimator with minimum sum of per-item variances and covariances that cancel out.

3. **MAXIMUM LIKELIHOOD ESTIMATORS FOR WS SKETCHES**

\(^3\)Estimators for the inverse-weight are useful for obtaining unbiased estimates for quantities where the weight appears in the denominator such as the weight ratio of two different subsets.

\(^4\)Szegedy’s proof applies only to estimators based on adjusted weight assignments. It also does not apply to estimators on the weight of subpopulations.
Estimating the total weight. Consider a set $I$ and its bottom-$k$ sketch $s$. Let $i_1, i_2, \ldots, i_k$ be the items in $s$ ordered by increasing ranks (we use the notation $r(i_{k+1})$ for the $(k+1)$st smallest rank). If $|I| \leq k$ (and we can determine this) then $w(I) = \sum w(i_j)$.

Consider the rank differences, $r(i_1), r(i_2) - r(i_1), \ldots, r(i_{k+1}) - r(i_k)$. From Lemma 2.3, they are independent exponentially distributed random variables. The joint probability density function of this set of differences is therefore the product of the above densities. The maximum likelihood estimator of the product of the above densities. The maximum likelihood estimator for $w(I)$ is the value that maximizes this function. To find the maximum, take the natural logarithm (for simplification) of the expression and look at the value which makes the derivative zero. We obtain that the maximum likelihood estimator $\hat{w}(I)$ is the solution of the equation

$$\sum_{i=0}^{k} \frac{1}{\hat{w}(I) - s_i} = r(i_{k+1}).$$

The left hand side is a monotone decreasing function, and the equation can be solved by a binary search on the range $[s_k + 1/r(i_{k+1}), s_k + (k+1)/r(i_{k+1})]$. We can obtain a tighter estimator (smaller variance) by redrawing the rank values of the items $i_1, i_2, \ldots, i_k$ (see Lemma 2.3) and taking the expectation of the solution of Eq. 1 (or average over multiple draws).

Estimating a subpopulation weight. We derive maximum likelihood subpopulation weight estimators that use and do not use the total weight $w(I)$. Let $J \subset I$ be a subpopulation. Let $j_1, j_2, \ldots, j_a$ be the items in $s$ that are in $I \setminus J$.

Let $r'_1, \ldots, r'_a$ be their respective rank values and let $s'_0 = \sum_{h \leq i} w(j_h)$ ($i = 1, \ldots, a$). Define $s'_0 \equiv 0$. Let $i_1, i_2, \ldots, i_c$ be the items in $J \cap s$. Let $r_1, \ldots, r_c$ be their respective rank values and let $s_0 = \sum_{i \leq c} w(i_k)$ ($i = 1, \ldots, c$). Define $s_0 \equiv 0$.

WS ML subpopulation weight estimator that does not use $w(I)$: Consider rank assignments such that rank values in $I \setminus J$ are fixed and the order of ranks of the items in $J$ is fixed. The probability density of the observed ranks of the first $k$ items in $J$ is that of seeing the same rank differences (probability density is $w(J) - s_i \exp(-w(J) - s_i)(r_{i+1} - r_{i})$ for the $i$th difference) and of the rank difference between the $c+1$ and $c$ smallest ranks in $J$ being at least $\tau - r_c$ (where $\tau$ is the $(k+1)$st smallest rank in the sketch), which is $\exp(-w(J) - s_i)(\tau - r_c))$. Rank differences are independent, and therefore, the probability density as a function of $w(J)$ is the product of the above densities. The maximum likelihood estimator for $w(J)$ is the value that maximizes this probability. If $c = 0$, the expression $\exp(-w(J)\tau)$ is maximized for $w(J) = 0$. Otherwise, by taking the natural logarithm and deriving we find that the value of $w(J)$ that maximizes the probability density is the solution of $\sum_{h=0}^{c-1} \frac{1}{w(J) - s'_h} = \tau$. As with the estimator of the total weight, we can obtain a tighter estimator by redrawing the rank values.

WS ML subpopulation weight estimator that uses $w(I)$: We compute the probability density, as a function of $w(J)$, of the event that we obtain the sketch $s$ with these ranks given that the prefix of sampled items from $I \setminus J$ is $j_1, \ldots, j_a$ and the prefix of sampled items from $J$ is $i_1, \ldots, i_c$.

We take the natural logarithms of the joint probability density and derive with respect to $w(J)$. If $c = 0$, the derivative is positive and the probability density is maximized for $w(J) = 0$. If $a = 0$, the derivative is negative and the probability density is maximized for $w(J) = w(I)$. Otherwise, if $a > 0$ and $c > 0$, the probability density is maximized for $\hat{w}(J)$ that is the solution of

$$\sum_{h=0}^{c-1} \frac{1}{\hat{w}(J) - s'_h} - \sum_{h=0}^{\infty} \frac{1}{w(I) - w(J) - s'_h} = 0.$$ 

The equation is easy to solve numerically, because the left hand side is a monotone decreasing function of $w(J)$.

4. ADJUSTED WEIGHTS

Definition 4.1. Adjusted-weight summarization (AWS) of a weighted set $(I, w)$ is a probability distribution $\Omega$ over weighted sets $b$ of the form $b = (J, a)$ where $J \subset I$ and $a$ is a weight function on $J$, such that for all $i \in I$, $E(a(i)) = w(i)$.

(To compute this expectation we extend the weight function $a$ to $\Omega$ by assigning $a$ to all $i \in I \setminus J$.) For $i \in J$ we call $a(i)$ the adjusted weight of $i$ in $b$.

An AWS algorithm is a probabilistic algorithm that inputs a weighted set $(I, w)$ and returns a weighted set according to some AWS of $(I, w)$. An AWS algorithm for $(I, w)$ provides unbiased estimators for the weight of $I$ and for the weight of subsets of $I$: By linearity of expectation, for any $H \subset I$, the sum $\sum_{i \in H} a(i)$ is an unbiased estimator of $w(H)$.

Let $\Omega$ be a distribution over sketches $s$, where each sketch consists of a subset of $I$ and some additional information such as the rank values of the items included in the subset. Suppose that given the sampled sketch $s$ we can compute $Pr[i \in s|s \in \Omega]$ for all $i \in s$ (since $I$ is a finite set, these probabilities are strictly positive for all $i \in s$). Then we can make $\Omega$ into an AWS using the Horvitz-Thompson (HT) estimator which provides for each $i \in s$ the adjusted weight

$$a(i) = \frac{w(i)}{Pr[i \in s|s \in \Omega]}.$$ 

It is well known and easy to see that these adjusted weights are unbiased and have minimal variance for each item for the particular distribution $\Omega$ over subsets.

HT on a partitioned sample space (HTP) is a method to derive adjusted weights when we cannot determine $Pr[i \in s|s \in \Omega]$ from the sketch $s$ alone. For example if $\Omega$ is a distribution of bottom-$k$ sketches, then the probability $Pr[i \in s|s \in \Omega]$ generally depends on all the weights $w(i)$ for $i \in I$ and therefore, it cannot be determined from the information contained in $s$ alone.

For each item $i$ we partition $\Omega$ into subsets $P_0, P_2, P_4, \ldots$. This partition satisfies the following two requirements

A useful property of adjusted weights is that they provide unbiased aggregations over any other numeric attribute: For weights $h(i)$, $\sum_{i \in H} h(i)a(i)/w(i)$ is an unbiased estimator of $h(J)$.
1. Given a sketch \( s \), we can determine the set \( P_j^i \) containing \( s \).

2. For every set \( P_j^i \) we can compute the conditional probability \( p_j^i \) = \( \Pr \{ i \in s | s \in P_j^i \} \).

For each \( i \in s \), we identify the set \( P_j^i \) and use the adjusted weight \( a(i) = w(i)/p_j^i \) (which is the HT adjusted weight in \( P_j^i \)). Items \( i \notin s \) get an adjusted weight of 0. The expected adjusted weight of each item \( i \) within each subspace of the partition is \( w(i) \) and therefore its expected adjusted weight over \( \Omega \) is \( w(i) \).

**Rank Conditioning (RC) adjusted weights** for bottom-

k sketches are an HTP estimator. The probability space \( \Omega \) includes all rank assignments. The sketch includes the \( k \) items with smallest rank values and the \( (k+1) \)st smallest rank \( r_{k+1} \). The partition \( P_1^i, \ldots, P_j^i \) which we use is based on rank conditioning. For each possible rank value \( r \) we have a set \( P_j^i \) containing all rank assignments in which the \( k \)th rank assigned to an item other than \( i \) is \( r \). (If \( i \in s \) then this is the \( (k+1) \)st smallest rank.

The probability that \( i \) is included in a bottom-k sketch given that the rank assignment is from \( P_j^i \) is the probability that its rank value is smaller than \( r \). For \( WS \) sketches, this probability is equal to \( 1 - \exp(-w(i)r) \). Assume \( s \) contains \( i_1, \ldots, i_k \) and that the \( (k+1) \)st smallest rank \( r_{k+1} \) is known.

Then for item \( i_j \), the rank assignment belongs to \( P_j^i \) and therefore the adjusted weight of \( i_j \) is \( \frac{w(i_j)}{1 - \exp(-w(i_j)r_{k+1})} \).

The WS RC estimator on the total weight is \( \sum_{j=1}^{k} \frac{w(i_j)}{1 - \exp(-w(i_j)r_{k+1})} \).

The PTH RC adjusted weight for an item \( i_j \) (obtained by a tailored derivation in \( \Pi \)), is \( \max \{ w(i_j), 1/r_{k+1} \} \).

**Variance of RC adjusted weights**

**Lemma 4.2.** Consider RC adjusted weights and two items \( i, j \). Then, \( \text{COV}(a(i), a(j)) = 0 \) (The covariance of the adjusted weight of \( i \) and the adjusted weight of \( j \) is zero.)

**Proof.** It suffices to show that \( E(a(i)a(j)) = w(i)w(j) \). Consider a partition of the sample space of all rank assignments according to the \( (k-1) \)th smallest rank of an item in \( I \setminus \{ i, j \} \). Consider a subset in the partition and let \( r_{k-1} \) denote the value of the \( (k-1) \)th smallest rank of an item in \( I \setminus \{ i, j \} \). We show that in this subset \( E(a(i)a(j)) = w(i)w(j) \). The product \( a(i)a(j) \) is positive in this subset only when \( r(i) < r_{k-1} \) and \( r(j) < r_{k-1} \), which (since rank assignments are independent) happens with probability \( \Pr \{ r(i) < r_{k-1} \} \Pr \{ r(j) < r_{k-1} \} \).

In this case the \( k \)th smallest rank in \( I \setminus \{ i, j \} \) is \( r_{k-1} \) and therefore, \( a(i) = \frac{w(i)}{\Pr \{ r(i) < r_{k-1} \}} \) and \( a(j) = \frac{w(j)}{\Pr \{ r(j) < r_{k-1} \}} \).

It follows that

\[
E(a(i)a(j)) = \frac{w(i)}{\Pr \{ r(i) < r_{k-1} \}} \frac{w(j)}{\Pr \{ r(j) < r_{k-1} \}} = w(i)w(j).
\]

This proof also extends to show that for any subset \( J \subset I \), \( E(\prod_{i \in J} a(i)) = \prod_{i \in J} w(i) \).

**Corollary 4.3.** For a subset \( J \subset I \),

\[
\text{VAR}(a(J)) = \sum_{j \in J} \text{VAR}(a(j)).
\]

Therefore, with RC adjusted weights, the variance of the weight estimate of a subpopulation is equal to the sum of the per-item variances, just like when items are selected independently. This Corollary, combined with Szegedy’s result \( \| \), shows that when we have a choice of a family of rank functions, PRI weights are the best rank functions to use when using RC adjusted weights.

**Selecting a partition.** The variance of the adjusted weight \( a(i) \) obtained using HTP depends on the particular partition in the following way.

**Lemma 4.4.** Consider two partitions of the sample space, such that one partition is a refinement of the other, and the AWSSs obtained by applying HTP using these partitions. For each \( i \in I \), the variance of \( a(i) \) using the coarser partition is at most that of the finer partition.

**Proof.** We use the following simple property of the variance. Consider two random variables \( A_1 \) and \( A_2 \) over a probability space \( \Omega \). Suppose that there is a partition \( \{ B_j \} \) of \( \Omega \) such that for every \( B_j \) and for every \( s \in B_j \), \( A_2(s) = E(A_1(s)|s \in B_j) \). Then \( \text{VAR}(A_2) \leq \text{VAR}(A_1) \).

Let \( P_j^i \) be the sets in the fine partition, and let \( C_j^i \) be the sets in the coarse partition such that \( C_j^i = \bigcup P_j^i \). Let \( \tilde{P}_j^i \) be the subset containing all \( s \in P_j^i \) such that \( i \in s \). Similarly, let \( \tilde{C}_j^i \) be the subset containing all \( s \in C_j^i \) such that \( i \in s \).

Let \( a(i, s) \) be the adjusted weight of \( i \) in a sketch \( s \) according to the partition \( P_j^i \), and let \( \overline{a}(i, s) \) be the adjusted weight of \( i \) in a sketch \( s \) according to the partition \( C_j^i \). We will show that for \( s \in \tilde{C}_j^i \) such that \( i \in s \), \( \overline{a}(i, s) = E_{s' \in \tilde{P}_j^i} (a(i, s')) \).

From this and the property of the variance stated above the lemma follows. We remove the superscript \( i \) from the sets \( P_j^i \), \( C_j^i \), \( \tilde{P}_j^i \), and \( \tilde{C}_j^i \) in the rest of the proof.

Let \( p_j = \Pr(s \in \tilde{P}_j | s \in P_j) \) and \( \overline{p}_j = \Pr(s \in \tilde{C}_j | s \in C_j) \).

Now,

\[
E_{s' \in \tilde{C}_j} (a(i, s')) = \sum_{s'} \Pr(s \in \tilde{C}_j | s \in \tilde{P}_j) \frac{w(i)}{\Pr(s \in \tilde{P}_j)} = \sum_{s'} \Pr(s \in P_j) \frac{w(i)}{\Pr(s \in P_j)} = \frac{w(i)}{\Pr(s \in \tilde{C}_j)} \overline{p}_j = \overline{a}(i, s) = E_{s' \in \tilde{P}_j} (a(i, s')).
\]

It follows from Lemma 4.5 that when applying HTP, it is desirable to use the coarsest partition for which we can compute the probability \( p_j^i \) from the information in the sketch. In particular a partition that includes a single component minimizes the variance of \( a(i) \) (This is the HT estimator). The RC partition yields the same adjusted weights as conditioning on the rank values of all items in \( I \setminus i \), so it is in a sense also the finest partition we can work with. It turns
out that when the total weight \( w(I) \) is available we can use a coarser partition.

5. USING THE TOTAL WEIGHT

When the total weight is available we can use HTp estimators defined using a coarser partition of the sample space than the one used by the RC estimator. The Prefix conditioning estimator computes the adjusted weights of item \( i \) by partitioning the sample space according to the sequence (prefix) of \( k-1 \) items with smallest ranks drawn from \( I \setminus \{i\} \).

The subset conditioning estimator (SC) uses an even coarser partition defined by the unordered set of the first \( k-1 \) items that are different from \( i \). By Lemma 4.3, subset conditioning is the best in terms of per-item variances. Another advantage of these estimators is that they do not need \( r_{k+1} \) and thereby require one less sample.

5.1 Prefix conditioning estimator.

For an item \( i \in s \) we partition the sample space according to the sequence (prefix) of \( k-1 \) items with smallest ranks drawn from \( I \setminus \{i\} \). That is if \( i \notin s \), then \( s \) belongs to the partition associated with the \( k-1 \) items in \( s \) of smallest ranks. If \( i \in s \), then \( s \) belongs to the partition associated with the sequence of \( k-1 \) items in \( s \setminus \{i\} \).

We assign adjusted weights as follows. Consider a sketch \( s \) and \( i \in s \). Let \( P \) be the set of sketches with the same prefix of \( k-1 \) items from \( I \setminus \{i\} \) as in \( s \). We compute the probability \( \Pr\{i \in s \mid s \in P\} \), that is, the probability that \( i \) is in a sketch from \( P \). We compute the probability of \( i \) occurring in each of the positions \( j \in 1, \ldots, k \) and the probability that it does not occur at all. We use the notation \( f_{s \setminus \{i\}}(j_1, \ldots, j_k) \) for the event that the first \( k \) items drawn by weighted sampling without replacement from a subset \( J \) are \( j_1, \ldots, j_k \).

We denote by \( i_\ell \) (1 \( \leq \ell \leq k-1 \)) the \( \ell \)th item in \( s \setminus \{i\} \). For each \( j = 1, \ldots, k \), the probability \( e_j \) that \( i \) appears in the \( j \)th position in a sketch from \( P \) is

\[
p(i \rightarrow j \mid s \in P) = \Pr\{f_{s \setminus \{i\}}(i_1, i_2, i_{j-1}, i, i_j, i_{k-1}) = \}
\]

\[
\frac{w(i)}{w(I)} - \frac{w(i_1)}{w(I)} - \frac{w(i_{j-1})}{w(I)} - \sum_{m=1}^{j-2} \frac{w(i_m)}{w(I)} - \frac{w(i_{k-1})}{w(I)} - \frac{w(i_k)}{w(I)} - \ldots - \frac{w(i_{k-1})}{w(I)} - \frac{w(i_k)}{w(I)} - \frac{w(i_k)}{w(I)} .
\]

The probability that the sketch is from \( P \) but \( i \) does not appear in it (technically, appears in a position \( k+1 \) or beyond) is

\[
p(i \notin s \mid s \in P) = \Pr\{ \bigcup_{\ell \in \{j+1\}} f_{s \setminus \{i\}}(i_1, i_2, \ldots, i_{j-1}, \ell) = \}
\]

\[
\frac{w(i_1)}{w(I)} - \frac{w(i_2)}{w(I)} - \frac{w(i_{j-1})}{w(I)} - \frac{w(i_{k-1})}{w(I)} - \sum_{m=1}^{k-j} \frac{w(i_m)}{w(I)} .
\]

Therefore,

\[
\Pr\{i \in s \mid s \in P\} = \sum_{j=1}^{k} p(i \rightarrow j \mid s \in P) + p(i \notin s \mid s \in P) .
\]

The computation of the prefix conditioning adjusted weights is quadratic in \( k \) for each item \( i \). RC adjusted weights, on the other hand, can be computed in constant number of operations per item.

5.2 Subset conditioning estimator.

The SC estimator has the following two additional important properties. In contrast with RC, the adjusted weights of different items have negative covariances, and the covariances cancel out: the sum of the adjusted weights equals the total weight of the set. This implies that the variance of the estimator of a large subset is smaller than the sum of the variances of the individual items in the subset, and in particular, the variance of the estimator for the entire set is zero. We now define this estimator precisely.

For a set \( s = \{i_1, i_2, \ldots, i_k\} \) and \( \ell \geq 0 \), we define

\[
f(s, \ell) = \int_0^\infty \ell \exp(-\ell x) \prod_{j \in s} (1 - \exp(-w(i_j) x)) dx .
\]

This is the probability that a random rank assignment with exponential ranks for items in \( s \) and an additional set of items \( X \) such that \( w(X) = \ell \), assigns the \(|s|\) smallest ranks to the items in \( s \) and the \((|s|+1)\)st smallest rank to an item from \( X \). For exponential ranks, this probability depends only on \( w(X) \) (the total weight of items in \( X \)), and does not depend on how the weight of \( X \) is divided between items. This is a critical property that allows us to compute adjusted weights with subset conditioning.

Recall that for an item \( i \), we use the subspace with all rank assignments in which among the items in \( I \setminus \{i\} \), the items in \( s \setminus \{i\} \) have the \((k-1)\)st smallest ranks. The probability, conditioned on this subspace, that item \( i \) is contained in the sketch is \( \frac{f_{s \setminus \{i\}}(\ell x)}{f_{s \setminus \{i\}}(\ell x, w(I \setminus s))} \), and so the adjusted weight assigned to \( i \) is

\[
a(i) = w(i) \frac{f(s \setminus \{i\}, w(I \setminus s))}{f(s, w(I \setminus s))} .
\]

The following lemma shows that SC estimate the entire set with zero variance.

**Lemma 5.1.** Let \( s \) be a ws sketch of \( I \) and let \( a(i) \) be SC adjusted weights. Then, \( \sum_{i \in s} a(i) = w(I) \).

**Proof.** Observe that for any sketch \( s \), \( i \in s \), and \( \ell \geq 0 \)

\[
f(s, \ell) = f(s \setminus \{i\}, \ell) - f(s \setminus \{i\}, \ell + w(i)) \frac{\ell}{\ell + w(i)} .
\]

This relation follows by manipulating Eq. (2), or by the following argument: Let \( X = I \setminus s \) and \( w(X) = \ell \). The probability that the items with smallest ranks in \( s \cup X \) are the items in \( s \) is equal to the probability that the \(|s| - 1 \) items of smallest ranks in \( s \setminus \{i\} \cup X \) are \( s \setminus \{i\} \) minus the probability that the \(|s| - 1 \) items of smallest ranks in \( s \cup X \) are \( s \setminus \{i\} \) and the \(|s|\)st smallest rank is from \( X \setminus \{i\} \). This latter probability is equal to

\[
f(s \setminus \{i\}, w(X \cup \{i\})) \frac{\ell}{\ell + w(i)} .
\]

Using Equation (3) we obtain that

\[
\sum_{i \in s} a(i) = \sum_{i \in s} \frac{\ell}{\ell + w(i)} .
\]
To verify the last equality, observe that
\[
\frac{w(i)}{w(i) + w(I \setminus s)} \cdot f(s \setminus \{i\}, w(I \setminus \{s \setminus \{i\}\}))
\]
is the probability that the first \(|s| - 1\) items drawn from \(I\) are \(s', i\) and the \(|s|\)th item is \(i\). These are disjoint events and their union is the event that the first \(|s|\) items drawn from \(I\) are \(s\). The probability of this union is \(f(s, w(I \setminus s))\).

**Lemma 5.2.** Consider SC adjusted weights of two items \(i \neq j\). Then, \(\text{cov}(a(i), a(j)) < 0\).

**Proof.** Consider a partition of rank assignments according to the items in \(I \setminus \{i, j\}\) that have the \(k - 2\) smallest ranks. Consider a part in this partition and denote this set of \(k - 2\) items by \(c\). We compute the expectation of \(a(i)a(j)\) conditioned on this part. Let \(\ell = w(I) - w(c) - w(i) - w(j)\). The probability of this part is \(f(c, \ell)\). The probability that \(a(i)a(j) > 0\) in \(c\) is equal to \(f(c \cup \{i, j\}, \ell)\). Therefore, the conditional probability is \(\frac{f(c \cup \{i, j\}, \ell)}{f(c, \ell)}\). In this case, the adjusted weight assigned to \(i\) is set according to items \(c \cup \{j\}\) having the \((k - 1)\) smallest ranks in \(I \setminus i\). Therefore, this weight is
\[
a(i) = w(i) \cdot \frac{f(c \cup \{j\}, \ell)}{f(c \cup \{i, j\}, \ell)}.
\]
Symmetrically for \(j\),
\[
a(j) = w(j) \cdot \frac{f(c \cup \{i\}, \ell)}{f(c \cup \{i, j\}, \ell)}.
\]
We therefore obtain that \(E(a(i)a(j))\) conditioned on this part is
\[
w(i)w(j) \cdot \frac{f(c \cup \{j\}, \ell)f(c \cup \{i\}, \ell)}{f(c \cup \{i, j\}, \ell)f(c, \ell)}.
\]
It suffices to show that
\[
\frac{f(c \cup \{j\}, \ell)f(c \cup \{i\}, \ell)}{f(c \cup \{i, j\}, \ell)f(c, \ell)} \leq 1.
\]
To show that, we apply Eq. 3 and substitute in the numerator
\[
f(c \cup \{j\}, \ell) = f(c, \ell) - f(c, \ell + w(j)) \frac{\ell}{\ell + w(j)}
\]
and in the denominator
\[
f(c \cup \{i, j\}, \ell) = f(c \cup \{i\}, \ell) - f(c \cup \{i\}, \ell + w(j)) \frac{\ell}{\ell + w(j)}
\]
The numerator being at most the denominator therefore follows from the immediate inequality
\[
f(c, \ell)f(c \cup \{i\}, \ell + w(j)) \leq f(c, \ell + w(j))f(c \cup \{i\}, \ell).
\]

**Lemma 5.3.** Consider weighted sketches of a weighted set \((I, w)\) and subpopulation \(J \subseteq I\). The SC estimator for the weight of \(J\) has smaller variance than the RC estimator for the weight of \(J\).

**Proof.** By Lemma 1.2, the variance of the RC estimator for \(J\) is \(\sum_{j \in J} \text{var}_\text{RC}(a(j))\). So using Lemma 4.2, we obtain that \(\sum_{j \in J} \text{var}_\text{SC}(a(j))\) is no larger than the variance of the RC estimator for \(J\). Finally since
\[
\text{var}_\text{SC}\left(\sum_{j \in J} a(j)\right) = \sum_{j \in J} \text{var}_\text{SC}(a(j)) + \sum_{i \neq j, j \in J} \text{cov}_\text{SC}(a(i), a(j)),
\]
and Lemma 5.2 that implies that the second term is negative the lemma follows.

### 5.3 Computing SC adjusted weights.

The adjusted weights can be computed by numerical integration. We propose (and implement) an alternative method based on a Markov chain that is faster and easier to implement. The method converges to the subset conditioning adjusted weights as the number of steps grows. It can be used with a fixed number of steps and provides unbiased adjusted weights.

As an intermediate step we define a new estimator as follows. We partition the rank assignments into subspaces, each consisting of all rank assignments with the same ordered set of \(k\) items of smallest ranks. Let \(P\) be a subspace in the partition. For each rank assignment in \(P\) and item \(i\) the adjusted weight of \(i\) is the expectation of the RC adjusted weight of \(i\) over all rank assignments in \(P\).

These adjusted weights are unbiased because the underlying RC adjusted weights are unbiased. By the convexity of the variance, they have smaller per-item variance than RC.

It is also easy to see that the variance of this estimator is higher than the variance of the prefix conditioning estimator: Rank assignments with the same prefix of items from \(I \setminus i\), but where the item \(i\) appears in different positions in the \(k\)-prefix, can have different adjusted weights with this assignment, whereas they have the same adjusted weight with prefix conditioning.

The distribution of \(r_{k+1}\) in each subspace \(P\) is the sum of \(k\) independent exponential random variables with parameters \(w(I), w(I) - w(i_1), \ldots, w(I) - \sum_{k=1}^{i_k} w(i_k)\) where \(i_1, \ldots, i_k\) are the items of \(k\) smallest ranks in rank assignments of \(P\) (see Lemma 2.3). So the adjusted weight of \(i_j (j = 1, \ldots, k)\) is \(a(i_j) = E(w(i_j) / (1 - \exp(-w(i_j)r_{k+1}))\) where the expectation is over this distribution of \(r_{k+1}\).

Instead of computing the expectation, we average the RC adjusted weights \(w(i_j) / (1 - \exp(-w(i_j)r_{k+1}))\) over multiple draws of \(r_{k+1}\). This average is clearly an unbiased estimator of \(w(i_j)\) and its variance decreases with the number of draws. Each repetition can be implemented in \(O(k)\) time (drawing and summing \(k\) random variables).

We define a Markov chain over permutations of the \(k\) items \(\{i_1, \ldots, i_k\}\). Starting with a permutation \(\pi\) we continue to a permutation \(\pi'\) by applying the following process. We draw \(r_{k+1}\) as described above from the distribution of \(r_{k+1}\) in the subspace corresponding to \(\pi\). We then redraw rank values for the items \(\{i_1, \ldots, i_k\}\) as described in Section 5 following Definition 2.2. The permutation \(\pi'\) is obtained by reordering \(\{i_1, \ldots, i_k\}\) according to the new rank values. This Markov chain has the following property.

**Lemma 5.4.** Let \(P\) be a (unordered) set of \(k\) items. Let \(p_\pi\) be the conditional probability that in a random rank assignment whose prefix consists of items of \(P\), the order of these items in the prefix is as in \(\pi\). Then \(p_\pi\) is the stationary distribution of the Markov chain described above.

**Proof.** Suppose we draw a permutation \(\pi\) of the items in \(P\) with probability \(p_\pi\) and then draw \(r_{k+1}\) as described above. Then this is equivalent to drawing a random rank assignment whose prefix consists of items in \(P\) and taking \(r_{k+1}\) of this assignment.

Similarly assume we draw \(r_{k+1}\) as we just described, draw ranks for items in \(P\), and order \(P\) by these ranks. Then this

\footnote{Note that this is not an instance of HTD, we simply average another estimator in each part.}
is equivalent to drawing a permutation $\pi$ with probability $p_r$.  

Our implementation is controlled by two parameters $\text{INPERM}$ and $\text{PERMNUM}$. $\text{INPERM}$ is the number of times the rank value $r_{k+1}$ is redrawn for a permutation $\pi$ (at each step of the Markov chain). $\text{PERMNUM}$ is the number of steps of the Markov chain (number of permutations in the sequence).

We start with the permutation $(i_1, \ldots, i_k)$ obtained in the ws sketch. We apply this Markov chain to obtain a sequence of $\text{PERMNUM}$ permutations of $(i_1, \ldots, i_k)$. For each permutation $\pi_j$, $1 \leq j \leq \text{PERMNUM}$, we draw $r_{k+1}$ from $P_{\pi_j} \text{INPERM}$ times as described above. For each such draw we compute the RC adjusted weights for all items. The final adjusted weight is the average of the RC adjusted weights assigned to the item in the $\text{PERMNUM} \cdot \text{INPERM}$ applications of the RC method.

We redraw a permutation in this Markov chain in $O(k \log k)$ time ($O(k)$ time to re-sort $k$ rank values and $O(k \log k)$ to sort). Redrawing $r_{k+1}$ given a permutation takes $O(k)$ time. Therefore, the total running time is $O(\text{PERMNUM} \cdot k \log k + \text{INPERM} \cdot k)$.

The expectation of the RC adjusted weights over the stationary distribution is the subset conditioning adjusted weight. An important property of this process is that if we apply it for a fixed number of steps, and average over a fixed number of draws of $r_{k+1}$ within each step, we still obtain unbiased estimators. Our experimental section shows that these estimators perform very well.

The subset conditioning estimator has powerful properties. Unfortunately, it seems specific to ws sketches. Use of subset conditioning requires that given a weighted set $(H, w)$ of $k - 1$ weighted items, an item $i$ with weight $w(i)$, and a weight $\ell > 0$, we can compute the probability that the bottom-$k$ sketch of a set $I$ that includes $H \cup \{i\}$ and has total weight $\ell + w(H) + w(i)$ contains the items $H \cup \{i\}$. This probability is determined from the distribution of the smallest rank of items with total weight $\ell$. In general, however, this probability depends on the weight distribution of the items in $I \setminus \{H \cup \{i\}\}$. The exponential distribution has the property that the distribution of the smallest rank depends only on $\ell$ and not on the weight distribution.

### 6. CONFIDENCE BOUNDS

Let $r$ be a rank assignment of a weighted set $Z = (H, w)$. Recall that for $H' \subseteq H$, $r(H')$ is the minimum rank of an item in $H'$. In this section it will be useful to denote by $\mathcal{P}(H')$ the maximum rank of an item in $H'$. We define $r(\emptyset) = +\infty$ and $\mathcal{P}(\emptyset) = 0$. For a distribution $D$ over a totally ordered set (by $<$) and $0 < \alpha < 1$, we denote by $Q_\alpha(D)$ the $\alpha$-quantile of $D$. That is, $\Pr_{y \in D} \{ y < Q_\alpha(D) \} \leq \alpha$ and $\Pr_{y \in D} \{ y \geq Q_\alpha(D) \} \geq 1 - \alpha$.

#### 6.1 Total weight

For two weighted sets $Z_1 = (H_1, w_1)$ and $Z_2 = (H_2, w_2)$, let $\Omega(Z_1, Z_2)$ be the weighted subspace that contains all rank assignments $r$ over $Z_1 \cup Z_2$ such that $\mathcal{P}(H_1) < r(H_2)$.

Let $(I, w)$ be a weighted set, let $r$ be a rank assignment for $(I, w)$, $s$ be the bottom-$k$ sketch that corresponds to $r$ (we also use $s$ as the set of $k$ items with smallest ranks). Let $\overline{w}(s, w, r_{k+1}, \delta)$ be the set containing all weighted sets $Z' = (H, w')$ such that $\Pr\{ r'(H) \geq r_{k+1} \mid r' \in \Omega(s, w), Z' \} \geq \delta$. Define $\overline{\mathcal{P}}(s, w, r_{k+1}, \delta)$ as follows. If $\overline{w}(s, w, r_{k+1}, \delta) = \emptyset$, then $\overline{\mathcal{P}}(s, w, r_{k+1}, \delta) = 0$. Otherwise, let $\overline{\mathcal{P}}(s, w, r_{k+1}, \delta) = \sup\{ w'(H) \mid (H, w') \in \overline{w}(s, w, r_{k+1}, \delta) \}$. (This supremum is well defined for “reasonable” families of rank functions, otherwise, we allow it to be $+\infty$)

Let $\overline{w}(s, w, r_{k+1}, \delta)$ be the set of all weighted sets $Z' = (H, w')$ such that $\Pr\{ r'(H) \leq r_{k+1} \mid r' \in \Omega(s, w), Z' \} \geq \delta$. Let $w(s, w, r_{k+1}, \delta)$ be as follows: We have $\overline{w}(s, w, r_{k+1}, \delta) \neq \emptyset$ for “reasonable” families of rank functions, but if it is empty, we define $w(s, w, r_{k+1}, \delta) = +\infty$. Otherwise, let $w(s, w, r_{k+1}, \delta) = \inf\{ w'(H) \mid (H, w') \in \overline{w}(s, w, r_{k+1}, \delta) \}$. (This infimum is well defined since weighted sets have non-negative weights.)

#### Lemma 6.1. Let $r$ be a rank assignment for the weighted set $(I, w)$, and let $s$ be the bottom-$k$ sketch that corresponds to $r$. Then $w(s, w) + \overline{\mathcal{P}}(s, w, r_{k+1}, \delta)$ is a $(1 - \delta)$-confidence upper bound on $w(I)$, and $w(s, w) + w(s, w, r_{k+1}, \delta)$ is a $(1 - \delta)$-confidence lower bound on $w(I)$.

**Proof.** We prove (1). The proof of (2) is analogous.

We show that in each subspace $\Omega(s, w)$, $(I \setminus s, w)$ of rank assignments our bound is correct with probability $1 - \delta$. Since these subspaces, specified by $s \subset I$ of size $|s| = k$, form a partition of the rank assignments over $(I, w)$, the lemma follows.

Let $D_{k+1}$ be the distribution of the $(k + 1)$st smallest rank over rank assignments in $\Omega(s, w), (I \setminus s, w)$ (the smallest rank in $I \setminus s$). Assume that $r$ is a rank assignment in $\Omega(s, w), (I \setminus s, w)$. We show that if $r_{k+1} \leq Q_{1-\delta}(D_{k+1})$ then our upper bound is correct. Since by the definition of a quantile $r_{k+1} \leq Q_{1-\delta}(D_{k+1})$ with probability $\geq 1 - \delta$ in $\Omega(s, w), (I \setminus s, w)$, it follows that our bound is correct with probability $\geq (1 - \delta)$ in $\Omega(s, w), (I \setminus s, w)$.

If $r_{k+1} \leq Q_{1-\delta}(D_{k+1})$ then

$$
\Pr\{ r'(I \setminus s) \geq r_{k+1} \mid r' \in \Omega(s, w), (I \setminus s, w) \} \geq \Pr\{ r'(I \setminus s) \geq Q_{1-\delta}(D_{k+1}) \mid r' \in \Omega(s, w), (I \setminus s, w) \} \geq \delta.
$$

So we obtain that $(I \setminus s, w) \in \overline{w}(s, w, r_{k+1}, \delta)$ and therefore $w(I \setminus s) \leq \overline{w}(s, w, r_{k+1}, \delta)$. □

This lemma also holds for a variant, where we consider rank assignments $r$ (and corresponding subspaces) where the items in $s$ appear in the same order as in $r'$.

#### 6.2 Subpopulation weight

We derive confidence bounds for the weight of a subpopulation $J \subset I$. The arguments are more delicate, as the number of items from $J$ that we see in the sketch can vary between 0 and $k$ and we do not know if the $(k+1)$th smallest rank belongs to an item in $J$ or in $I \setminus J$. We will work with weighted lists instead of weighted sets.

A weighted list $(H, w, \pi)$ consists of a weighted set $(H, w)$ and a linear order (permutation) $\pi$ on the elements of $H$. We will find it convenient to sometimes specify the permutation $\pi$ as the order induced by a rank assignment $r$ on $H$.

The concatenation $(H^{(1)}, w^{(1)}, \pi^{(1)}) \oplus (H^{(2)}, w^{(2)}, \pi^{(2)})$ of two weighted lists, is a weighted list with items $H^{(1)} \cup H^{(2)}$, corresponding weights as defined by $w^{(1)} \oplus w^{(2)}$ and order such that each $H^{(i)}$ is ordered according to $\pi^{(i)}$ and the elements of $H^{(1)}$ precede those of $H^{(2)}$. Let $\Omega(H, w, \pi)$ be the probability subspace of rank assignments over $(H, w)$ such that the rank order is according to $\pi$.

Let $r$ be a rank assignment, $s$ be the corresponding sketch, and $\ell$ be the weighted list $\ell = (J \cap s, w, r)$. Let $\overline{w}(\ell, r_{k+1}, \delta)$
be the set of all weighted lists $h = (H, w', \pi)$ such that

$$\Pr\{r'(H) \geq r_{k+1}\} \subseteq r' \in \Omega(\ell \oplus h) \geq \delta .$$

Let $\mathfrak{w}(\ell, r_{k+1}, \delta) = \sup\{w'(H) | (H, w', \pi) \in \mathfrak{w}(\ell, r_{k+1}, \delta)\}$.

Let $W(\ell, r_{k+1}, \delta) = 0$, then $\mathfrak{w}(\ell, r_{k+1}, \delta) = 0$. If unbounded, then $\mathfrak{w}(\ell, r_{k+1}, \delta) = +\infty$.) Let $W(\ell, r_k, \delta)$ be the set of all weighted lists $h = (H, w', \pi)$ such that

$$\Pr\{r'(H) \bigcap s \leq r_k \} \subseteq r' \in \Omega(\ell \oplus h) \geq \delta .$$

Let $w(\ell, r_k, \delta) = \inf\{w'(H) | (H, w', \pi) \in W(\ell, r_k, \delta)\}$. If $W(\ell, r_k, \delta) = 0$, then $w(\ell, r_k, \delta) = +\infty$. We prove the following.

**Lemma 6.2.** Let $r$ be a rank assignment, $s$ be the corresponding sketch, and $\ell$ be the weighted list $\ell = (J \cap s, w, r)$. Then $w(J \cap s) + \mathfrak{w}(\ell, r_{k+1}, \delta)$ is a $(1 - \delta)$-confidence upper bound on $w(J)$ and $w(J \cap s) + w(\ell, r_k, \delta)$ is a $(1 - \delta)$-confidence lower bound on $w(J)$.

**Proof.** The bounds are conditioned on the subspace of rank assignments over $(I, w)$ where the ranks of items in $I \cap J$ are fixed and the order of the ranks of the items in $J$ is fixed. These subspace are a partition of the sample space of rank assignments over $(I, w)$. We show that the confidence bounds hold within each subspace.

Consider such a subspace $\Phi \equiv \Phi(I, \pi : J, a : (I \cup J))$, where $\pi : J$ is a permutation over $J$, representing the order of the ranks of the items in $J$ and $a : (I \cup J)$ are the rank values of the elements in $I \cup J$.

Let $D_k$ be the distribution of $r_k$ for $r \in \Phi$ and let $D_k$ be the distribution of $r_k$ for $r \in \Phi$. Over rank assignments in $\Phi$ we have $\Pr\{r_k \leq Q_{1-\delta}(D_k)\} \geq 1 - \delta$ and $\Pr\{r_k \geq Q_{\delta}(D_k)\} \leq 1 - \delta$.

We show that

- The upper bound is correct for rank assignments $r \in \Phi$ such that $r_k \leq Q_{1-\delta}(D_k)$. Therefore, it is correct with probability at least $(1 - \delta)$.

- The lower bound is correct for rank assignments $r \in \Phi$ such that $r_k \geq Q_{\delta}(D_k)$. Therefore, it is correct with probability at least $(1 - \delta)$.

Consider a rank assignment $r \in \Phi$. Let $s$ be the items in the sketch. Let $\ell = (J \cap s, w, r)$ and $\ell'' = (J \cup s, w, r)$ be the weighted lists of the items in $J \cap s$ or $J \cup s$, respectively, as ordered by $r$. There is bijection between rank assignments in $\Omega(\ell \oplus \ell'')$ and rank assignments in $\Phi$ by augmenting the rank assignment in $\Omega(\ell \oplus \ell'')$ with the ranks $a(j)$ for items $j \in I \cup J$. For a rank assignment $r \in \Phi$, let $\ell'' \in \Omega(\ell \oplus \ell'')$ be its restriction to $J$.

A rank assignment $r' \in \Phi$ has $r'_{k+1} \geq r_{k+1}$ if and only if $\mathfrak{w}(\ell(J \cap s)) \geq r_{k+1}$. So if $r' \in \Phi$ such that $r_{k+1} \leq Q_{1-\delta}(D_k)$ then

$$\Pr_{r' \in \Omega(\ell \oplus \ell'')}\{r'(J \setminus s) \geq r_{k+1}\} = \Pr_{r' \in \Phi}\{r'_{k+1} \geq r_{k+1}\} \geq \Pr_{r' \in \Phi}\{r'_{k+1} \geq Q_{1-\delta}(D_k)\} \geq \delta .$$

Therefore, $\ell'' \in \mathfrak{w}(\ell, r_{k+1}, \delta)$, and hence $w(J \setminus s) \leq \mathfrak{w}(\ell, r_{k+1}, \delta)$ and the upper bounds hold.

**A rank assignment $r' \in \Phi$ has $r' \equiv r_k$ if and only if the maximum rank that $r'$ gives to an item in $J \cap s$ is $r_k$. So if $r \in \Phi$ such that $r_k \geq Q_{\delta}(D_k)$ then $\Pr_{r' \in \Omega(\ell \oplus \ell'')}\{r'(J \cap s) \leq r_k\} = \Pr_{r' \in \Phi}\{r' \leq r_k\} \geq \Pr_{r' \in \Phi}\{r' \leq Q_{\delta}(D_k)\} \geq \delta$.

Therefore, $\ell'' \in \mathfrak{w}(\ell, r_k, \delta)$, and hence $w(J \setminus s) \leq \mathfrak{w}(\ell, r_k, \delta)$ and the lower bound holds. $\square$

### 6.3 Subpopulation weight using $w(I)$

We derive tight confidence intervals that use the total weight $w(I)$. For weighted lists $H^{(i)} = (H^{(i)}, w^{(i)}, \pi^{(i)})$ ($i = 1, 2$), the probability space $\Omega(h^{(1)}, h^{(2)})$ contains all rank assignments $r$ over the weighted set $(H^{(1)}, w^{(1)}) \cup (H^{(2)}, w^{(2)})$ such that for each $i = 1, 2$, the order of $H^{(i)}$ induced by the rank values $r : H^{(i)} = \pi^{(i)}$. We define the functions $c_{h^{(1)}, h^{(2)}}(r) = d_h^{(1)}(r) = \pi^{(2)}$ for $r \in \Omega(h^{(1)}, h^{(2)})$ as follows: $c_{h^{(1)}, h^{(2)}}(r)$ is the number of items amongst those with $k$ smallest ranks that are in $H^{(1)}$ (equivalently, it is $i$ such that $r_i(H^{(1)}) < r_{k-1}(H^{(2)})$ and $r_k(H^{(1)}) > r_{k-1}(H^{(2)})$).

$$d_{h^{(1)}, h^{(2)}}(r) = r_{k-1}(H^{(1)}) - r_{k-1}(H^{(2)}) \geq \delta$$

is the difference between the largest rank values of items in $H^{(2)}$ and $H^{(1)}$ that are amongst the $k$ least ranked items.

We use the notation $c_{e_1, d_1} \leq c_{e_2, d_2}$ for the lexicographic order over pairs.

Let $r$ be a rank assignment, and let $s$ be the sketch corresponding to $r$. Let $\Delta = \pi(I \setminus J \cap s) - \pi(J \cap s)$, and let $\ell = (J \cap s, w, r : J \cap s)$ and $\ell_2 = (I \cup s, w, r : (J \cup s) \cap s)$. Then $\mathfrak{w}(\ell, \ell_2, \Delta, \delta)$ be the set of all pairs $(h_1, h_2)$ of weighted lists $h_1 = (H_1, w_1, \pi_1)$ and $h_2 = (H_2, w_2, \pi_2)$ such that $w_1(H_1) + w_2(H_2) = w(I) - w(s)$ and

$$\Pr\{c_{e_1, d_1}(h_1, h_2) \leq c_{e_2, d_2}(h_1, h_2) \} \geq (|J \cap s|, \Delta) \geq \delta$$

over the probability space of all $\ell \in \Omega(\ell_1 \oplus h_1, \ell_2 \oplus h_2)$. If $\mathfrak{w}(\ell, \ell_2, \Delta, \delta) = 0$, then $\mathfrak{w}(\ell, \ell_2, \Delta, \delta) = 0$. Otherwise, $\mathfrak{w}(\ell, \ell_2, \Delta, \delta) = \sup\{w(H_1) | (h_1, h_2) \in \mathfrak{w}(\ell, \ell_2, \Delta, \delta)\}$. Then $W(\ell, \ell_2, \Delta, \delta)$ be the set of all pairs $(h_1, h_2)$ of weighted lists $h_1 = (H_1, w_1, \pi_1)$ and $h_2 = (H_2, w_2, \pi_2)$ such that $w_1(H_1) + w_2(H_2) = w(I) - w(s)$ and

$$\Pr\{c_{e_1, d_1}(h_1, h_2) \leq c_{e_2, d_2}(h_1, h_2) \} \leq \sup\{w(H_1) | (h_1, h_2) \in \mathfrak{w}(\ell, \ell_2, \Delta, \delta)\}$$

over the probability space of all $\ell \in \Omega(\ell_1 \oplus h_1, \ell_2 \oplus h_2)$. If $\mathfrak{w}(\ell, \ell_2, \Delta, \delta) = 0$, then $\mathfrak{w}(\ell, \ell_2, \Delta, \delta) = w(I) - w(s)$.

**Proof.** The lower bound on $w(I)$ is equal to $w(I)$ minus a $(1 - \delta)$-confidence upper bound, $w(I \setminus J \cap s) + w(I \setminus J, \ell, \Delta, \delta)$ on $w(I \setminus J)$. Therefore it suffices to prove the upper bound.

We show that the bound holds with probability at least $(1 - \delta)$ in the subspace of rank assignments over $(I, w)$ where the rank order of the items in $J$ and the rank order of the items in $I \setminus J$ are fixed. These subspaces are a partition of
the space of rank assignments. Consider such a subspace $\Phi = \Omega(\ell_1, \ell_2)$. Let $\ell'_1 = (J, w, \pi_1)$ and $\ell'_2 = (I \setminus J, w, \pi_2)$ be the weighted lists that corresponds to the rank order of the items in $J$ and in $I \setminus J$, respectively, for $r \in \Phi$.

Let $D$ be the distribution over the pairs $(\ell'_1, \ell'_2(r), d_{\ell'_1, \ell'_2}(r))$ for $r \in \Phi$. We define the quantile $Q_{1-\delta}(D)$ with respect to the lexicographic order over the pairs.

We show that the upper bound is correct for all $r \in \Phi$ such that $(\ell'_1, \ell'_2(r), d_{\ell'_1, \ell'_2}(r)) \preceq Q_{1-\delta}(D)$. Therefore, it holds with probability at least $1 - \delta$.

Let $r \in \Phi$ such that $(\ell'_1, \ell'_2(r), d_{\ell'_1, \ell'_2}(r)) \preceq Q_{1-\delta}(D)$. Let $s$ be the corresponding sketch, $\ell_1 = (J \cap s, w, r), \ell_2 = ((I \setminus J) \cap s, w, r)$. By definition, $\ell'_1, \ell'_2(r) = (J \cap s)$, $\Delta = d_{\ell'_1, \ell'_2}(r) = \mathbb{R}(I \setminus J) \cap s, \ell'_1 = \ell_1 + \ell'_1^c$, and $\ell'_2 = \ell_2 + \ell'_2^c$. It follows that

$$\Pr((\ell'_1, \ell'_2(r), d_{\ell'_1, \ell'_2}(r)) \preceq (\ell_1 \cap s), \Delta) \geq \delta$$

Therefore, $(\ell'_1, \ell'_2(r)) \preceq \mathbb{R}(\ell_1, \ell_2, \Delta, \delta)$, and hence,

$$w(J \setminus s) \leq \mathbb{R}(\ell_1, \ell_2, \Delta, \delta).$$

We formulate the conditions in the statement of Lemma 6.3 in terms of predicates on the rank assignment. Inequality 6.1 is equivalent to $\Pr(U_{h_1, h_2}(r) \cap \Omega(1, h_1 \cup h_2 \cup h_2) \geq \delta$, where $U_{h_1, h_2}(r)$ is the predicate (that depends on $\ell_1, \ell_2, \Delta$): $U_{h_1, h_2}(r) = (r(H_1) > \mathbb{R}((J \cap s), \Delta) \land \left(\begin{array}{l}(r(H_1) < \mathbb{R}((s \cap (I \setminus J))) \\
                        (r(H_1) > \mathbb{R}((s \cap (I \setminus J))) \land \mathbb{R}((I \setminus J) \cap s) - \mathbb{R}(J \cap s) > \Delta)\end{array}\right))$. (6)

The first line guarantees that we have at least $|J \cap s|$ items of $J$ among the $k$ items of smallest ranks. If the second line holds then we have strictly more than $|J \cap s|$ items of $J$ among the $k$ items of smallest ranks. If the third line holds then we have exactly $|J \cap s|$ items of $J$ among the $k$ items of smallest ranks and $(\mathbb{R}(I \setminus J) \cap s, \mathbb{R}(J \cap s) > \Delta)$.

Similarly, the condition in Inequality 6.2 is equivalent to $\Pr(L_{h_1, h_2}(r) \cap \Omega(1, h_1 \cup h_2 \cup h_2) \geq \delta$, where $L_{h_1, h_2}(r)$ is the predicate: $L_{h_1, h_2}(r) = (r(H_2) > \mathbb{R}((s \cap (I \setminus J))) \land \left(\begin{array}{l}(r(H_2) < \mathbb{R}((s \cap (I \setminus J))) \\
                        (r(H_2) > \mathbb{R}((s \cap (I \setminus J))) \land \mathbb{R}((I \setminus J) \cap s) - \mathbb{R}(J \cap s) < \Delta)\end{array}\right))$. (7)

Either the $k$ items with smallest ranks include strictly less than $|J \cap s|$ items from $J$ or they include exactly $|J \cap s|$ items from $J$ and $\mathbb{R}(I \setminus J) \cap s, \mathbb{R}(J \cap s) < \Delta$.

6.4 Confidence bounds for wsr sketches

In our simulations, we apply the normal approximation to obtain confidence bounds on total weight using wsr sketches: The average of the $k$ minimum ranks $T = \sum_{i=1}^{k} r_i/k$ is an average of $k$ independent exponential random variables with (the same) parameter $w(I)$ (This is a Gamma distribution). The expectation of the sum is $k/w(I)$ and the variance is $k/w^2(I)$. The confidence bounds are the $\delta$ and $1-\delta$ quantiles of $T$. Let $\alpha$ be the Z-value that corresponds to confidence level $1-\delta$ in the standard normal distribution. By applying the normal approximation, the approximate upper bound is the solution of $k/w(I) + \alpha \sqrt{k/w^2(I)} = kT$, and the approximate lower bound is the solution of $k/w(I) - \alpha \sqrt{k/w^2(I)} = kT$. Therefore, the approximate bounds are $(1 \pm \alpha/\sqrt{k})/T$.

6.5 Confidence bounds for ws sketches

The confidence bounds make “worst case” assumptions on the weight distribution of “unseen” items. ws sketches have the nice property that the distribution of the $i$th largest rank in a weighted set, conditioned on either the set or the list of the $i - 1$ items of smallest rank values, depends only on the total weight of the set (and not on the particular partition of the “unseen” weight into items). Therefore, the confidence bounds are tight in the respective probability subspaces: for any distribution and any subset, the probability that the bound is violated is exactly $\delta$.

Bounds on the total weight $(w(I))$. We apply Lemma 6.1. For a weighted set $(s, w, |s| = k, \ell \geq 0)$, consider a weighted set $U$ of weight $w(s) + \ell$ containing $(s, w)$. Let $y$ be the $(k + 1)$th smallest rank value, over rank assignments over $U$ such that the $k$ items with smallest rank values are the elements of $s$. The probability density function of $y$ is (see Section V.B and Eq. 4)

$$D(\ell, y) = \frac{\exp(-\ell y) \prod_{j=1}^{\ell} \left(1 - \exp(-w(i_j)y)\right)}{\int_0^{\infty} \exp(-\ell x) \prod_{j=1}^{\ell} \left(1 - \exp(-w(i_j)x)\right) dx}$$

Let $\ell_{k+1}$ be the observed $k + 1$ smallest rank. The $(1 - \delta)$-confidence upper bound is $w(s) + \ell$ plus the value of $\ell$ that solves the equation $\int_{\ell_{k+1}}^{\infty} D(\ell, y) dy = 1 - \delta$. The function $D_{\ell_{k+1}}(0, y)$ is an increasing function of $\ell$ (the probability of the $(k+1)$st smallest rank being at most $\ell_{k+1}$ is increasing with $\ell$). If $\int_{0}^{\infty} D(0, y) dy > 1 - \delta$, then there is no solution and the upper bound is $w(s)$.

The lower bound is $w(s) + \ell$ plus the value of $\ell$ that solves the equation $\int_{\ell_{k+1}}^{\infty} D(\ell, y) dy = \delta$. If there is no solution ($\int_{0}^{\infty} D(0, y) dy > \delta$), then the lower bound is $w(s)$.

Conditioning on the order of items. We consider bounds that use the stronger conditioning, where we fix the rank order of the items. For $0 \leq s_0 \leq \ldots \leq s_h < t$, we use the notation $v(t, s_0, \ldots, s_h)$ for the random variable that is the sum of $h + 1$ independent exponential random variables with parameters $t - s_j$ ($j = 0, \ldots, h$). From linearity of expectation,

$$E(v(t, s_0, \ldots, s_h)) = \sum_{j=0}^{h} 1/(t - s_j).$$

From independence, the variance is the sum of variances of the exponential random variables and is

$$\text{VAR}(v(t, s_0, \ldots, s_h)) = \sum_{j=0}^{h} 1/(t - s_j)^2.$$
by solving an equation of the form:

$$\Pr\{v(x, s_0, \ldots, s_k) \leq \tau\} = \delta$$

for $x > s_h$ (where $0 \leq s_0 < \cdots < s_h$, $\tau > 0$, and $0 < \delta < 1$ are provided).

Since for $x > y > s_h$ and any $\tau$, $\Pr\{v(x, s_0, \ldots, s_h) \leq \tau\} \geq \Pr\{v(x, s_0, \ldots, s_k) \leq \tau\}$, it is easy to approximately solve equations like this numerically. Observe that the probability $\Pr\{v(x, s_0, \ldots, s_h) \leq \tau\}$ is minimized as $x$ approaches $s_h$ from above. If the limit is at least $\delta$, then the equation has no solution.

**The weight $w(I)$**. Let $i_1, i_2, \ldots, i_k$ be the items in the current sketch, ordered by increasing rank values, and let $s_j = \sum_{i=1}^{j} w(i)$. The distribution of $(k + 1)$ smallest rank (for any fixed possible order of the remaining items) is the random variable $v(w(I), s_0, \ldots, s_k)$. Using an ordered variant of Lemma 6.4, we obtain that the $(1-\delta)$-confidence lower bound is the solution of the equation

$$\Pr\{v(x, s_0, \ldots, s_k) \leq r_{k+1}\} = \delta$$

and is $s_k$ if there is no solution $x > s_k$. The $(1-\delta)$-confidence upper bound is the solution of the equation

$$\Pr\{v(x, s_0, \ldots, s_k) \leq r_{k+1}\} = 1 - \delta$$

(and is $s_k$ if there is no solution $x > s_k$.)

**Subpopulation weight (with unknown $w(I)$)**. Let $J$ be a subpopulation. For a rank assignment, let $s$ be the corresponding sketch and let $s_h$ ($1 \leq h \leq |J \cap s|$) be the sum of the weights of the $h$ items of smallest rank values from $J$ (we define $s_0 \equiv 0$). Specializing Lemma 6.2 to $w$s sketches we obtain that the $(1-\delta)$-confidence upper bound on $w(J)$ is the solution of the equation

$$\Pr\{v(x, s_0, \ldots, s_{|J \cap s|}) \leq r_{k+1}\} = 1 - \delta$$

(and is $s_{|J \cap s|}$ if there is no solution $x > s_{|J \cap s|}$.) The $(1-\delta)$-confidence lower bound is $0$ if $|J \cap b| = 0$. Otherwise, let $x > s_{|J \cap b|-1}$ be the solution of

$$\Pr\{v(x, s_0, \ldots, s_{|J \cap b|-1}) \leq r_k\} = \delta.$$ 

The lower bound is max($s_{|J \cap s|}, x$) if there is a solution and is $s_{|J \cap b|}$ otherwise.

To solve these equations, we either used the normal approximation to the respective sum of exponentials distribution or used the quantile method which we developed.

**Normal approximation.** We apply the normal approximation to the quantities of a sum of exponentials distribution. For $\delta \ll 0.5$, let $\alpha$ be the $Z$-value that corresponds to confidence level $1 - \delta$. The approximate $\delta$-quantile of $v(x, s_0, \ldots, s_h)$ is $E(v(x, s_0, \ldots, s_h)) - \alpha \sqrt{\text{VAR}(v(x, s_0, \ldots, s_h))}$ and the approximate $(1-\delta)$-quantile is $E(v(x, s_0, \ldots, s_h)) + \alpha \sqrt{\text{VAR}(v(x, s_0, \ldots, s_h))}$.

To approximately solve $\Pr\{v(x, s_0, \ldots, s_h) \leq \tau\} = \delta$ (such that $\tau$ is the $\delta$-quantile of $v(x, s_0, \ldots, s_h)$), we solve the equation

$$E(v(x, s_0, \ldots, s_h)) - \alpha \sqrt{\text{VAR}(v(x, s_0, \ldots, s_h))} = \tau.$$ 

To approximately solving $\Pr\{v(x, s_0, \ldots, s_h) \leq \tau\} = 1 - \delta$, we solve

$$E(v(x, s_0, \ldots, s_h)) + \alpha \sqrt{\text{VAR}(v(x, s_0, \ldots, s_h))} = \tau.$$ 

We solve these equations (to the desired approximation level) by searching over values of $x > s_h$ using standard numerical methods. The function $E(v(x)) + \alpha \sqrt{\text{VAR}(v(x))}$ is monotonic decreasing in the range $x > s_h$. The function $E(v(x)) - \alpha \sqrt{\text{VAR}(v(x))}$ is decreasing or bitonic (first increasing then decreasing) depending on the value of $\alpha$.

**The quantile method.** Let $D(x)$ be a parametric family of probability spaces such that there is a total order $\preceq$ over the union of the domains of $\{D(x)\}$. Let $\tau$ be a value in the union of the domains of $\{D(x)\}$ such that the probability $\Pr\{y \preceq \tau \mid y \in D(x)\}$ is increasing with $x$. So the solution ($x$) to the equation $\Pr\{y \preceq \tau \mid y \in D(x)\} = \delta$ is unique. (We refer to this property as monotonicity of $\{D(x)\}$ with respect to $\tau$.)

We assume the following two “black box” ingredients. The first ingredient is drawing independent monotone parametric samples $s(x) \in D(x)$. That is, for any $x$, $s(x)$ is a sample from $D(x)$ and if $x \geq y$ then $s(x) \preceq s(y)$. Two different parametric samples are independent: That is for every $x$, $s^1(x)$ and $s^2(x)$ are independent draws from $D(x)$. The second ingredient is a solver of equations of the form $s(x) = \tau$ for a parametric sample $s(x)$.

We define a distribution $\overline{D}(\tau)$ such that a sample from $\overline{D}(\tau)$ is obtained by drawing a parametric sample $s(x)$ and returning the solution of $s(x) = \tau$. The two black box ingredients allow us to draw samples from $\overline{D}(\tau)$. Our interest in $\overline{D}(\tau)$ is due to the following property:

**Lemma 6.4.** For any $\delta$, the solution of $Q_\delta(D(x)) = \tau$ is the $\delta$-quantile of $\overline{D}(\tau)$.

The quantile method for approximately solving equations of the form $\Pr\{y \preceq \tau \mid y \in D(x)\} = \delta$ draws multiple samples from $\overline{D}(\tau)$ and returns the $\delta$-quantile of the set of samples.

We apply the quantile method to approximately solve Equations of the form

Eq. 8 (as an alternative to the normal approximation).

The family of distributions that we consider is $D(x) = v(x, s_0, \ldots, s_h)$. This family has the monotonicity property with respect to any $\tau > 0$. A parametric sample $s(x)$ from $v(x, s_0, \ldots, s_h)$ is obtained by drawing $h + 1$ independent random variables $v_0, \ldots, v_h$ from $U[0, 1]$. The parametric sample is $s(x) = \sum_{j=0}^{h} v_j/(x - s_j)$ and is a monotone decreasing function of $x$. A sample from $\overline{D}(\tau)$ is then the solution of the equation $\sum_{j=0}^{h} v_j/(x - s_j) = \tau$. Since $s(x)$ is monotone, the solution can be found using standard search.

**Subpopulation weight using $w(I)$**. We specialize the conditions in Lemma 6.3 to $w$s sketches. Consider the distribution of $(c_{t_1 \oplus h_1, t_2 \oplus h_2}(r), d_{t_1 \oplus h_1, t_2 \oplus h_2}(r))$ for $r \in \Omega(t_1 \oplus h_1, t_2 \oplus h_2)$. We shall refer to items of $h_1$ as items of $J$ and to items of $h_2$ as items of $I \setminus J$. This distribution in general depends on the decomposition of the weighted lists $h_1$ and $h_2$ into items. However from Equation 7 we learn that

$$\Pr\{c_{t_1 \oplus h_1, t_2 \oplus h_2}(r), d_{t_1 \oplus h_1, t_2 \oplus h_2}(r) \preceq (|J \cap s|, \Delta)\}$$

where $\Delta = \tau(|J \setminus s| - |J \cap s|)$, depends only on $x = w(H_1)$, and $w(H_2) = w(I) - x$. Indeed, let $\tau = (|J \cap s|, \Delta)$, $\Pr\{c_{t_1 \oplus h_1, t_2 \oplus h_2}(r), d_{t_1 \oplus h_1, t_2 \oplus h_2}(r) \leq \tau\}$ is the probability of the predicate $L_{h_1, h_2}$ stated in Eq. (7). This predicate
depends on the rank values of $|J \cap s|$ and $|J \cap s| + 1$ smallest ranks in $J$ and of the $(|I \setminus J| \cap s)$ and $(|I \setminus J| \cap s) + 1$ smallest ranks in $I \setminus J$. For WS sketches, the distribution of these ranks is determined by the weighted lists $\ell_1, \ell_2$ and $x$.

So we pick a weight list $h_1$ with a single item of weight $x$, and a weighted list $h_2$ with a single item of weight $w(I) - x$, and let $D(x)$ be the distribution of

$$
(c_{\ell_1 \oplus h_1, t_2 \oplus h_2}(r), d_{\ell_1 \oplus h_1, t_2 \oplus h_2}(r))
$$

for $r \in \Omega(\ell_1 \oplus h_1, t_2 \oplus h_2)$. To emphasize the dependency of $r$ on $x$ we shall denote by $D^{(x)}$ a rank assignment drawn from $\Omega(\ell_1 \oplus h_1, t_2 \oplus h_2)$ where $w(H_1) = x$.

Since the largest rank in $J \cap s$ and the smallest rank of an item in $H_1$ decrease with $x$, and the largest rank in $(\setminus J) \cap s$ and the smallest rank in $H_2$ increase with $x$ (decrease with $w(I) - x$) it follows that the family $D^{(x)}$ has the monotonic property with respect to $\tau = (\setminus J \cap s, \Delta)$.

Obviously, $w(J \setminus s) \in [0, w(I) - w(s)]$. Therefore, we can truncate the bounds to be in this range. So the upper bound on $w(J \setminus s)$ is the minimum of $w(I) - w(s)$ and $x$ such that $Q_{1-\delta}(D^{(x)}) = (|J \cap s|, \Delta)$. If there is no solution then the upper bound is $0$. The lower bound on $w(J \setminus s)$ is the value of $x$ such that $Q_\delta(D^{(x)}) = (|J \cap s|, \Delta)$. If there is no solution, then the lower bound is $0$. The respective (upper or lower) bounds on $w(J \setminus s)$ are plus the bound on $w(J \cap s)$.

We apply the quantile method to solve the equations

$$Q_{1-\delta}(D^{(x)}) = (|J \cap s|, \Delta),$$

and

$$Q_\delta(D^{(x)}) = (|J \cap s|, \Delta).$$

The second black box ingredient that we need for the quantile method is drawing a monotone parametric sample $s(x)$ from $D^{(x)}$. Let $s_i (i \in (0, 1, \ldots, |J \cap s|))$ be the sum of the weights of the first $i$ items from $J$ in $\ell_1$. Let $s_i' (i \in (0, 1, \ldots, k - |J \cap s|))$ be the respective sums for $I \setminus J$. We draw a rank assignment $r^{(x)} \in \Omega(\ell_1 \oplus h_1, t_2 \oplus h_2)$ as follows. We draw $k + 2$ independent random variables $v_0, \ldots, v_{|J \cap s|}, v'_0, \ldots, v'_{k - |J \cap s|}$ from $U[0, 1]$. We let the $j$th rank difference between items from $J$ be $-\ln(v_j)/(x - s_j)$, and the $j$th rank difference between items from $(I \setminus J)$ be $-\ln(v'_j)/(x - s'_j)$. These rank differences determine $\tau(J \cap s)$ and $r(H_1)$ (sums of $|J \cap s|$ and $|J \cap s| + 1$ first rank differences from $J$, respectively), and $\tau((I \setminus J) \cap s)$ and $r(H_2)$ (sums of $(|I \setminus J| \cap s)$ and $(|I \setminus J| \cap s) + 1$ first rank differences from $I \setminus J$, respectively). Then $s(x)$ is the pair $(c(r^{(x)}), d(r^{(x)}))$.

The second black box ingredient is solving the equation $s(x) = \tau$. Let $i = |J \cap s|$ and let $i' = k - i = |(I \setminus J) \cap s|$ as before. The solver has three phases: We first compute the range $(L, U)$ of values of $x$ such that the first coordinate of the pair $s(x)$ is equal to $|J \cap s|$. That is, the rank assignment $r$ has exactly $|J \cap s|$ items from $J$ among the first $k$ items. Let $d(r^{(x)}) = r(J \setminus J) - r(J)$ denote the second coordinate in the pair $s(x)$. In the second phase we search for a value $x \in (L, U)$ (if there is one) such that $d(r^{(x)}) = \Delta$ (the second coordinate of $s(x)$ is equal to $\Delta$). The function $d(r^{(x)})$ is monotone increasing in this range, which simplifies numeric solution. The third phase is truncating the solution to be in $[0, w(I) - w(s)]$. Details are provided in Figure 11.

The precise statements here is that the probability that $r(J \cap s)$ is smaller than some threshold $t$, increases with $x$.

Computing the range $(L, U)$.

- If $i' = 0$, let $U = w(I) - w(s)$. Otherwise $(i' > 0)$, $U$ is the solution of $\sum_{k=0}^{i'} \frac{-\ln(v'_k)}{w(I) - w(s)} - \sum_{k=0}^{i'-1} \frac{-\ln(v'_k)}{w(I) - w(s)} = 0$. (There is always a solution \( U \in (s_i, w(I) - s'_i) \)).

- If $i = 0$, let $L = 0$. Otherwise $(i > 0)$, $L$ is the solution of $\sum_{k=0}^{i-1} \frac{-\ln(v'_k)}{w(I) - w(s)} - \sum_{k=0}^{i'-1} \frac{-\ln(v'_k)}{w(I) - w(s)} = 0$. (There is always a solution $L \in (s_{i-1}, w(I) - s'_i)$).

Search for $x \in (L, U)$ such that $d(x) = \Delta$.

- If $i = 0$ (we must have $\Delta > 0$) we set $M$ to be the solution of $\sum_{k=0}^{i'-1} \frac{-\ln(v'_k)}{w(I) - w(s)} = \Delta$ in the range $(L, U)$. If there is no solution, we set $M = -L$.

- If $i' = 0$ (we must have $\Delta < 0$), we set $M$ to be the solution of $\sum_{k=0}^{i} \frac{-\ln(v'_k)}{w(I) - w(s)} = -\Delta$ in the range $(L, U)$. If there is no solution, we set $M = -U$.

- Otherwise, if $i > 0$ and $i' > 0$, we set $M$ to be the solution of $\sum_{k=0}^{i-1} \frac{-\ln(v'_k)}{w(I) - w(s)} - \sum_{k=0}^{i'-1} \frac{-\ln(v'_k)}{w(I) - w(s)} = \Delta$. There must be a solution in the range $(L, U)$.

Truncating the solution.

- We can have $L \in (s_{i-1}, s_i)$ and hence possibly $M < s_i$. In this case we set $M = s_i$. Similarly, we can have $U \in (w(I) - s'_i, w(I) - s_{i-1})$ and hence possibly $M > w(I) - s'_i$. In this case we set $M = w(I) - s'_i$.

- We return $M$.

Figure 1: Solver for $s(x) = \tau$ for subpopulation weight with known $w(I)$.

6.6 Confidence bounds for priority sketches

We review the confidence bounds for PRI sketches obtained by Thorup and Ziegler. We denote $p_{r}(i) = PR\{r(i) < \tau\}$. The number of items in $J \cap s$ with $p_{r}(i) < 1$ is used to bound $\sum_{i \in J \cap s, p_{r}(i) < 1} w(i)$ (the expectation of the sum of independent Poisson trials). These bounds are then used to obtain bounds on the weight $\sum_{i \in J, p_{r}(i) < 1} w(i)$, exploiting the correspondence (specific for PRI sketches) between $\sum_{i \in J, p_{r}(i) < 1} w(i)$ and $\sum_{i \in J \cap s, p_{r}(i) < 1} w(i)$: For PRI sketches, $p_{r}(i) = \min(1, w(i)/\tau)$. If $w(i)/\tau \geq 1$ then $p_{r}(i) = 1$ (item is included in the sketch) and if $w(i)/\tau < 1$ then $p_{r}(i) = w(i)/\tau$. Therefore, $p_{r}(i) < 1$ if and only if $w(i) > (w(i)/\tau^r)$. Since

$$\sum_{i \in J, p_{r}(i) < 1} w(i) = \tau^{-1} \sum_{i \in J \cap s, p_{r}(i) < 1} p_r(i).$$

For $n' \geq 0$, define $\overline{\tau}_3(n')$ (respectively, $\underline{\tau}_3(n')$) to be the infimum (respectively, supremum) over all $\mu$, such that for all sets of independent Poisson samples with sum of expectations $\mu$, the sum is less than $\delta$ likely to be at most $n'$ (respectively, at least $n'$). If $n' = \{i \in J \cap s| w(i) > 1\}$, then $\overline{\tau}_3(n')$ and $\underline{\tau}_3(n')$ are $(1 - \delta)$-confidence bounds on $\sum_{i \in J \cap s, w(i) > 1} p_{r}(i)$. Since

$$w(J) = \sum_{i \in J \cap s, w(i) > 1} w(i) + \tau^{-1} \sum_{i \in J \cap s, w(i) > 1} p_{r}(i),$$

we obtain $(1 - \delta)$-confidence upper and lower bounds on $w(J)$ by substituting $\overline{\tau}_3(J)$ and $\underline{\tau}_3(J)$ for $\sum_{i \in J \cap s, w(i) > 1} p_{r}(i)$ in this formula, respectively.

Chernoff bounds provide an upper bound on $\overline{\tau}_3(n')$ of
\[ D^{(\text{PRI}, \text{low})}(\ell, y) = \frac{\ell \prod_{j \in s} \min(1, w(i_j)y)}{\int_{x=0}^{\ell} \ell \prod_{j \in s} \min(1, w(i_j)x) \, dx} = \prod_{j \in s} \min(1, w(i_j)y) \int_{x=0}^{\ell} \exp(-xy) \prod_{j \in s} \min(1, w(i_j)x) \, dx \]

The lower bound on \( w(\ell \setminus s) \) is the value of \( \ell \) that satisfies the equation \( \int_{x=0}^{\ell} D^{(\text{PRI}, \text{low})}(\ell, y) \, dy = 1 - \delta \neq 0 \), for a fixed \( \ell \) (which is the tentative bound on the weight of \( I \setminus s \)), consider the maximum probability that the minimum rank of an item in a set \( Z (= I \setminus s) \) with total weight \( \ell \) and maximum weight \( 1/y \), is at most \( y \). This probability is maximized if we make the items of \( Z \) as large as possible: It is 1 if \( \ell \geq 1/y \) (we put in \( Z \) at least one item of weight \( 1/y \)), and it is \( y^\ell \) if \( \ell < 1/y \) (\( Z \) consists of one item of weight \( \ell \)).

The respective probability density of the minimum rank \( y \) as a function of \( \ell \) is 0 for \( y > 1/\ell \) and \( y^\ell \) otherwise. Applying a similar derivation to that of Eq. [8], we obtain that the probability density of the event that the items in \( s \) have smaller ranks than items in \( I \setminus s \) and the smallest rank among items in \( I \setminus s \) is equal to \( 1 - \delta \).

### 7. SIMULATIONS

#### Total weight.
We compare estimators and confidence bounds on the total weight \( w(I) \) using three distributions of 1000 items each with weights independently drawn from Pareto distributions with parameters \( \alpha \in \{1, 1.2, 2\} \), and also on a uniform distribution.

#### Estimators.
We evaluate the maximum likelihood \( w(I) \) estimator (WS ML), the rank conditioning \( w(I) \) estimator (WS RC), the rank conditioning PRI estimator (PRI RC) and 1, and the WSR estimator \( \mathbb{I} \) (Section 2).

Figure 2 (left) shows the absolute value of the relative error, averaged over 100 runs, as a function of \( k \). We can see that all three bottom-k based estimators outperform the WSR estimator, demonstrating the advantage of the added information when sampling “without replacement” over sampling “with replacement” (see also [13]). The advantage of these estimators grows with the skew. The quality of the estimate is similar among the bottom-k estimators (WS ML, WS RC, and PRI RC). The maximum likelihood estimator (WS ML), which is biased, has worse performance for very small values of \( k \) where the bias is more significant. PRI RC has a slight advantage especially if the distribution is more skewed. This is because, in this setting, with unknown \( w(I) \), PRI RC is a nearly optimal adjusted-weight based estimator.
Confidence bounds. We compare the Chernoff based PRI confidence bounds from [31] and the WS and WSR confidence bounds we derived. We apply the normal approximation with the stricter (but easier to compute) conditioning on the order for the WS confidence bounds and the normal approximation for the WSR confidence bounds (see Sections 6.4 and 6.5). The 95%-confidence upper and lower bounds and the 90% confidence interval (the width, which is the difference between the upper and lower bounds), averaged over 1000 runs, are shown in Figure 2 (middle and right). We can see that the WS confidence bounds are tighter, and often significantly so, than the PRI confidence bounds. In fact PRI confidence bounds were worse than the WSR-based bounds on less-skewed distributions (including the uniform distribution on 1000 items). This surprising behavior is explained by the large “slack” between the bounds in [31] and the actual variance of the (nearly-optimal) PRI RC estimator.

The WS bounds in Eq. (5) (that do not use conditioning on the order) should be tighter than the bounds that use this conditioning. The PRI bounds in Eq. (11) and Eq. (10) (that address some of the “slack” factors) may be tighter. We have not implemented these alternative bounds and leave this comparisons for future work.

The normal approximation provided fairly accurate confidence bounds for the total weight. The WS and WSR bounds were evidently more efficient, with real error rate that closely corresponded to the desired confidence level. For the 90% confidence interval, across the three distributions with $\alpha = 1, 1.2, 2$, and value of $k$, the highest error rate was 12%. The true weight was within the WS confidence bounds on average in 90.5%, 90.2%, 90% of the time for the different values of $\alpha$. The corresponding in-bounds rates for WSR were 90.6%, 90.3%, and 90.0%, and for PRI 99.2%, 99.1%, and 98.9%. (The high in-bounds rate for the PRI bounds reflects the slack in these bounds).

Subpopulation weight. Estimators. We implemented an approximate version of WS SC using the Markov chain and averaging method. We showed that this approximation provides unbiased estimators that are better than the plain WS RC estimator (better per-item variances and negative covariances for different items), but attains zero sum of covariances only at the limit. We quantified this improvement of WS SC over WS RC and its dependence on the size of the subpopulation. We evaluated the quality of approximate WS SC as a function of the parameters INPERM, and PERNUM (see Section 6.5), and we compared WS SC to the PRI RC estimator.

To evaluate how the quality of the estimator depends on the size of the subpopulation we introduce a group size parameter $g$. We order the items by their weights and partition them sequentially into $\lceil I \rceil / g$ groups each consisting of $g$ items. For each group size, we compute the sum, over subsets in this partition, of the square error of the estimator (averaged over multiple runs). This sum corresponds to the sum of the variances of the estimator over the subsets of the partition. For $g = 1$, this sum corresponds to the sum of the variances of the items.

The RC estimators have zero covariances, and therefore, the sum of square errors should remain constant when sweeping $g$. The WS SC estimator has negative covariances and therefore we expect the sum to decrease as a function of $g$. For $g = n$, we obtain the variance of the sum of the adjusted weights, which should be 0 for the WS SC estimator (but not for the approximate versions).

We used two distributions generated by drawing $n = 20000$ items from a Pareto distribution with parameter $\alpha \in \{1.2, 2\}$. The sum of square errors, as a function of $g$, is constant for the RC estimators, but decreases with the WS SC estimator. For $g = 1$, the PRI RC estimator (that obtains the minimum sum of per-item variances by a sketch of size $k + 1$) performs slightly better than the WS RC estimator when the data is more skewed (smaller $\alpha$). The WS SC estimator, however, performs very closely and better for small values of $k$ (it uses one fewer sample). For $g > 1$, the WS SC estimator outperforms both RC estimators and has significantly smaller variance for larger subpopulations. Figure 3 shows the results for $k \in \{4, 40, 500\}$. For each value of $k$, we show the sum of square errors over subsets in the partition, averaged over 1000 repetitions, as a function of the partition parameter $g$. Figure 4 shows the sum of square errors (again, averaged over 1000 repetitions) as a function of $k$ for partitions with $g \in \{1, 5000\}$.

We conclude that in applications when $w(I)$ is provided, the WS SC estimator emerges as a considerably better choice than the RC estimators. It also shows that the metric of the sum of per-item variances, that PRI RC is nearly optimal [30] with respect to it, is not a sufficient notion of optimality.

Figure 5 compares different choices of the parameters INPERM, and PERNUM for the approximate (Markov chain based) WS SC estimator. We denote each such choice as a pair (INPERM, PERNUM). We compare estimators with parameters $(400, 1)$, $(20, 20)$, $(1, 400)$, and $(5, 2)$. We conclude the following: (i) A lot of the benefit of WS SC on moderate-size subsets is obtained for small values: $(5, 2)$ performs nearly as well as the variants that use more steps and iterations. (ii) There is a considerable benefit of redrawing within a permutation: $(400, 1)$ that iterates within a single permutation performs well. (iii) Larger subsets, however, benefit from larger PERNUM: $(1, 400)$ performs better than $(20, 20)$ which in turn is better than $(400, 1)$.

Confidence bounds. We evaluate confidence bounds on
for many data representations including data streams; and tighter confidence bounds across summarization formats. Our derivations are complemented with the design of interesting and efficient computation methods, including a Markov chain based method to approximate the \\

The confidence bounds, intervals, and square errors, were normalized using the weight of the corresponding subpopulation. For each distribution and values of k and g, the normalized bounds were then averaged across 500 repetitions and across all subpopulations of size g. Across these distributions, the \( w + w(I) \) confidence bounds are tighter (more so for larger g) than \( w = w(I) \) and both are significantly tighter than the PRI confidence bounds. Representative results are shown in Figure 6.

8. CONCLUSION

We consider the fundamental problem of processing approximate subpopulation weight queries over summaries of a set of weighted records. Summarization methods supporting such queries include the k-mins format, which includes weighted sampling with replacement (WSR or PPSWR Probability Proportional to Size With Replacement) and the bottom-K format which includes weighted sampling without replacement (WS, also known as PPSWOR - PPS Without Replacement) and priority sampling (PRI) which is related to IPPS (Inclusion Probability Proportion to Size). Surprisingly perhaps, the vast literature on survey sampling and PPS and IPPS estimators (e.g. [20, 25]) is mostly not applicable to our common database setting: subpopulation-weight estimation, skewed (Zipf-like) weight distributions, and summaries that can be computed efficiently over massive datasets (such as data streams or distributed data). Existing unbiased estimators are the HT and ratio estimators for PPSWR, the PRI estimator, and a WS estimator based on mimicking WSR sketches.

We derive novel and significantly tighter estimators and confidence bounds on subpopulation weight: better estimators for the classic WS sampling method; better estimators than all known estimators/summarizations (including PRI)
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Figure 2: Left: Absolute value of the relative error of the estimator of $w(I)$ averaged over 1000 repetitions. Middle: 95% confidence upper and lower bounds for estimating $w(I)$. Right: width of 90% confidence interval for estimating $w(I)$. We show results for $\alpha = 1$ (top row), $\alpha = 1.2$ (second row), $\alpha = 2$ (third row), and uniform weights (bottom row).
Figure 3: Sum of variances over a partition as a function of group size for fixed values of $k$. We used 20000 items drawn from Pareto distributions with $\alpha = 1.2$ (top) and $\alpha = 2$ (bottom). To compute the variance in a group we averaged over 1000 repetitions. We used the approximation of $\text{ws SC}$ with $\text{inperm} = 20$, $\text{permnum} = 20$. 

$k = 500$  

$k = 40$  

$k = 4$
Figure 6: Subpopulation 95% confidence bounds (top), 90% confidence intervals (middle), and (normalized) squared error of the 95% confidence bounds (bottom) for $g = 200$. 