DISTINGUISHING NUMBER OF UNIVERSAL HOMOGENEOUS URYSOHN METRIC SPACES

ANTHONY BONATO, CLAUDE LAFLAMME, MICHEAL PAWLIUK, AND NORBERT SAUER

Abstract. The distinguishing number of a structure is the smallest size of a partition of its elements so that only the trivial automorphism of the structure preserves the partition. We show that for any countable subset of the positive real numbers, the corresponding countable Urysohn metric space, when it exists, has distinguishing number two or is infinite.

While it is known that a sufficiently large finite primitive permutation group has distinguishing number two, unless it is not the full symmetric group or alternating group, the infinite case is open these countable Urysohn metric spaces provide further confirmation toward the conjecture that all primitive homogeneous countably infinite structures have distinguishing number two or else is infinite.

1. Introduction

The asymmetric colouring number of a graph was introduced by Babai long ago in [2], and it resurfaced more recently as the distinguishing number by Albertson and Collins in [1]. The distinguishing number can generally be defined for any structure $S$ as the smallest positive integer $d$ such that a partition of its elements in $d$ pieces exists so that only the trivial automorphism of $S$ preserves the partition, and is denoted by either $D(S)$ (or $ACN(S)$). The distinguishing number is a measure of the symmetry level found within a structure, leading to interesting structural information that comes from the investigation of what is needed to break that symmetry.

Of particular interest to us are countable homogeneous structures; these structures have a transitive permutation group, in fact by definition transitive on all finite structures of the same types, and thus, highly symmetric. While the rationals are easily seen to have infinite distinguishing number, Imrich, Klavzar and Trofimov showed in [11] that the distinguishing number of the Rado graph (or infinite random graph) is two, which is the smallest it can be since the Rado graph is not rigid. The distinguishing number of
various other finite and countable homogeneous structures was determined in [3, 4, 13], including for all simple and directed homogeneous graphs and posets. In the infinite case of homogeneous simple and directed graphs, it was shown in [13] that the distinguishing number is either two or infinite, with only obvious exceptions having imprimitive automorphism groups. The following was conjectured.

**Conjecture ([13]).** The distinguishing number of all primitive homogeneous countably infinite structures is two or infinite.

The conjecture is very much in the spirit of the finite situation in general. Indeed, Cameron, Neumann and Saxl proved in [6] that a sufficiently large, finite primitive permutation group has distinguishing number two, unless it is not the full symmetric group or alternating group. The 43 exceptions were determined by Seress in [18]. One of these exceptions is the dihedral group $D_{10}$, which is primitive and the automorphism group of the homogeneous graph $C_5$, its distinguishing number is 3 and hence, we have the necessity for the conjecture to address only infinite structures. A tool developed in [13] appears in the right direction to confirm the conjecture, namely, that of a fixing type for the action of a group $G$ on a set $A$. If the action does have such a fixing type, then the distinguishing number of $G$ acting on $A$ is two. It may be possible that more general result for all primitive groups exists (homogeneous or not) but we have no insight in that direction.

We reference also the closely related work in permutation groups by Imrich et al. in [12], where in particular they conjecture that any closed group of $S_\infty$ having infinite motion and where all orbits of its point stabilizers are finite has distinguishing number two; this is the so-called *Infinite Motion Conjecture for Permutation Groups*. Recall that, up to (topological group) isomorphism, the closed subgroups of the infinite symmetric $S_\infty$ are exactly the automorphism groups of countable structures. Of particular interest is that such extremely amenable groups are exactly those automorphism groups of Fraïssé ordered classes with the Ramsey property. See [10] for background on Fraïssé limits.

In this paper, we consider the case of countable homogeneous metric spaces. In particular, we consider the case of the countable homogeneous Urysohn spaces $U_S$ for a given countable spectrum $S \subseteq \mathbb{R}^+$, which are the Fraïssé limits of all finite metric spaces whose spectrum a subset of $S$. Note that not every subset $S$ can be the spectrum of such a Urysohn metric space, and a necessary and sufficient condition is known as the “4-value” condition, which is precisely when metric triangles amalgamate; when this is the case we call $S$ a *universal spectrum*. Depending on $S$, the automorphism group of $U_S$ may or may not be primitive.

These countable metric spaces are very much related to the well known (uncountable) Urysohn space, the complete separable metric space which is
both homogeneous and universal; it is the completion of the countable homogeneous Urysohn spaces using the rationals as spectrum. See for example [5][14].

The main result of the paper is as follows, with proof given at the end of Section 3 below. We use $\omega$ to denote the set of non-negative integers.

**Theorem** (Main Theorem). Let $S \subseteq \mathbb{R}^+$ be a countable universal spectrum and $U_S$ the countable homogeneous structure with spectrum $S$. Then $D(U_S) = 2$ or $\omega$, and the following items hold.

1. If $S$ has a positive limit, then $D(U_S) = 2$.
2. If $S$ has no positive limits but has 0 as a limit, then $D(U_S) = 2$ if and only if $S$ contains arbitrarily large elements of arbitrarily small distance.
3. If $S$ does not have a limit, then $D(U_S) = 2$ if and only if $S$ contains two elements of distance smaller than the minimum positive element of $S$.

2. General Notions and Preliminaries

A relational structure $R$ is **rigid** if its group of automorphisms $\text{Aut}(R)$ consists only the identity. The idea behind the distinguishing number is to find the smallest number of predicates $\langle P_i : i < d \rangle$ such that the expanded structure $(R; P_i : i < d)$ becomes rigid.

For a metric space $M = (M, d_M)$, let $\text{spec}(M)$, the **spectrum** of $M$, be the set of the distances between points of $M$. A metric space $M$ is **universal** if it embeds every finite metric space $N$ with spectrum $\text{spec}(N) \subseteq \text{spec}(M)$.

For $S$ a set of reals and $r \in \mathbb{R}$, let $S_{>r} = \{ s \in S : s > r \}$, and similarly for $S_{\geq r}$. If $S \subseteq \mathbb{R}_{\geq 0}$ is countable, then let $\mathfrak{A}(S)$ denote the set of finite metric spaces whose spectrum is a subset of $S$. Note that $\mathfrak{A}(S)$ is an age (meaning that it is closed under isomorphism and substructures, and up to isomorphism has only countably many members), and we will need conditions on $S$ for which the age $\mathfrak{A}(S)$ has the amalgamation property.

**Definition 2.1.** We call a pair of metric spaces $A$ and $B$ an **amalgamation instance** if $d_A(x, y) = d_B(x, y)$ for all $x, y \in A \cap B$. If so, then we define:

$$\Pi(A, B) = \{ C = (A \cup B; d_C) : C \mid A = A \text{ and } C \mid B = B \}.$$

For $S \subseteq \mathbb{R}_{\geq 0}$ and $\{A, B\} \subseteq \mathfrak{A}(S)$ let:

$$\Pi_S(A, B) = \{ C \in \Pi(A, B) : \text{spec}(C) \subseteq S \}.$$

Finally, we say that the set $S \subseteq \mathbb{R}_{\geq 0}$ has the amalgamation property if $\Pi_S(A, B) \neq \emptyset$ for all amalgamation instances $\{A, B\} \subseteq \mathfrak{A}(S)$.

We now define the “4-values” condition, which is the description that triangles amalgamate.

**Definition 2.2.** A set $S \subseteq \mathbb{R}_{\geq 0}$ satisfies the 4-values condition if $\Pi_S(A, B) \neq \emptyset$ for any two amalgamation instances of the form $A = (\{x, y, z\}; d_A)$ and $B = (\{x, y, w\}; d_B)$ in $\mathfrak{A}(S)$.
There are several equivalent definitions of the 4-values condition. The first version has been given in [8] together with Theorem 2.4 below. In [16] there is an equivalent version of the 4-values condition given and Definition 2.2 is stated as Lemma 3.3 of [16]. Definition 2.2 here is the one most suitable for our purposes.

We provide the condition on $S$ leading to non-trivial Urysohn metric spaces in the following definition.

**Definition 2.3.** A countable set $S \subseteq \mathbb{R}_{\geq 0}$ is a universal spectrum if the following items hold.

1. The element 0 is in $S$ and $S$ contains at least one positive number; and
2. The set $S$ satisfies the 4-values condition.

It follows from Theorem 3.8 of [16] that if a set $S \subseteq \mathbb{R}_{\geq 0}$ satisfies the 4-values condition, then $\mathfrak{A}(S)$ is an age with amalgamation. Hence, the following theorem follows from the general Fraïssé theory.

**Theorem 2.4.** Let $S \subseteq \mathbb{R}_{\geq 0}$ be a universal spectrum. Then $\mathfrak{A}(S)$ is an age with amalgamation and there exists a countable homogeneous, universal, metric space $U_S$ whose spectrum is $S$.

Even though by definition an amalgamation instance can be amalgamated, we will in many cases want to do so controlling the new distances. We therefore have the following.

**Lemma 2.5.** Let $S \subseteq \mathbb{R}_{\geq 0}$ be a universal spectrum, and $A$ and $B$ an amalgamation instance in $\mathfrak{A}(S)$. If there exists a number $s \in S$ so that

$$s \leq d_A(a, x) + d_B(x, b)$$

for all $a \in A \setminus B$, $x \in A \cap B$, and $b \in B \setminus A$, then there exists a metric space $C \in \Pi_S(A, B)$ so that $d_C(a, b) \geq s$ for all $a \in A \setminus B$ and $b \in B \setminus A$.

**Proof.** Because $S \subseteq \mathbb{R}_{\geq 0}$ satisfies the 4-values condition, there is a metric space $D \in \Pi_S(A, B)$. Let $C$ be the binary relational structure obtained from $D$ by replacing the new distances as follows:

$$d_C(a, b) = \max\{d_D(a, b), s\}$$

for every $a \in A \setminus B$ and every $b \in B \setminus A$.

We claim that $C$ is indeed a metric space in $\mathfrak{A}(S)$, and hence, all triangles of $C$ not in $A$ and not in $B$ must be verified to be metric.

One type of such triangles is of the form $\{a, x, b\}$ with $a \in A \setminus B$ and $b \in B \setminus A$ and $x \in A \cap B$. As a triangle of $D$ it is metric, and together with the assumption on $s$ we derive that

$$d_C(a, b) = \max\{d_D(a, b), s\} \leq d_A(a, x) + d_B(x, b).$$

The other type of triangles we need to verify is of the form either $\{a, a', b\}$ with $\{a, a'\} \subseteq A \setminus B$ and $b \in B \setminus A$, or the other way around of the form
\{a, b, b'\} \text{ with } a \in A \setminus B \text{ and } \{b, b'\} \subseteq B \setminus A; \text{ it suffices to consider the first case.}

If both \(d_D(a, b) \leq s\) and \(d_D(a', b') \leq s\), then \(d_C(a, b) = d_C(a', b) = s\), and \(d_C(a, a') = d_D(a, a') \leq d_D(a, b) + d_D(a', b) \leq d_C(a, b) + d_D(a', b')\) shows that \(\{a, a', b\}\) is metric. If both \(d_D(a, b) \geq s\) and also \(d_D(a', b') \geq s\), then the side lengths have not changed from \(D\) to \(C\) and hence, again metric.

For the remaining case, say \(d_D(a, b) < s\) but \(d_D(a', b') \geq s\). Then \(d_C(a, b) = s \leq d_D(a', b) = d_C(a', b) \leq d_C(a, a') + d_C(a', b')\). The verification of the other two sides is immediate. \(\Box\)

The following general notions will useful to analyze various universal spectrum.

**Definition 2.6.** Let \(S\) be a universal spectrum. An element \(s \in S\) is called:

1. an initial number of \(S\) if \([s, 2s) \cap S = \emptyset\),
2. a jump number of \(S\) if \((s, 2s] \cap S = \emptyset\),
3. an insular number of \(S\) if it both an initial and a jump number of \(S\).

A subset \(B \subseteq S\) is a block of \(S\) if:

1. \(B\) is an interval of \(S\),
2. \(\min B\) is a positive initial number of \(S\),
3. either \(B\) is unbounded, or \(\max B\) is a jump number of \(S\), and
4. \(\max B\) (if it exists) is the only jump number (of either \(B\) of \(S\)).

Hence, if \(s \in S\) is insular, then \(B = \{s\}\) is a block consisting of only one element. Note also for future reference that if \(s > 0\) is a jump number of \(S\), then the relation \(\preceq\) given by \(x \preceq y\) if \(d(x, y) \leq s\) is an equivalence relation on \(U_S\), this will soon play an important role. But for now let us introduce further terminology.

2.1. Universal spectra without positive limits. In this subsection we develop tools to handle the case where the spectrum does not have a positive limit; meaning that no positive element \(r \in \mathbb{R}\) is a limit of \(S\), whether \(r \in S\) or not.

To handle this case, we first recall that a (universal spectrum) set \(S\) is *inversely well ordered* if every non-empty bounded above subset of \(S\) has a maximum. We now recall the \(\oplus\) operation from \[17\].

**Definition 2.7.** \[17\] Let \(S\) be an inversely well ordered spectrum, and for \(r, t \in \mathbb{R}\) define

\[ r \oplus t = \max\{s \in S : s \leq r + t\}. \]

Note that the \(\oplus\) operation on \(S\) is commutative and monotone. Further it was shown \[17\] (Theorem 5) that the \(\oplus\) operation is associative on any finite \(S\). Via the same argument we have the following result.

**Lemma 2.8.** If \(S\) is an inversely well ordered (spectrum) set, then the \(\oplus\) operation on \(S\) is associative.
Observe that if \( \{r, s, t\} \subseteq S \), then \( r \oplus t \geq s \) if and only if \( r + t \geq s \); and hence, we have the following.

**Lemma 2.9.** A triangle is metric if and only if the \( \oplus \) sum of any two of the three side lengths is larger than or equal to the third side length.

We use this operation to construct a specific and controlled amalgamation.

**Lemma 2.10.** Let \( S \) be an inversely ordered universal spectrum, and \( A \) and \( B \) in \( \mathfrak{A}(S) \) an amalgamation instance. Then there exists a unique metric space \( C \in \Pi_S(A, B) \), which we denote by \( \Pi_S^\oplus(A, B) \), such that:

\[
d_C(a, b) = \min\{d_A(a, x) \oplus d_B(x, b) : x \in A \cap B\}
\]

for all \( a \in A \setminus B \) and \( b \in B \setminus A \). In this case, let \( \mu(a, b) \in A \cap B \) be such that

\[
d_C(a, b) = d_A(a, \mu(a, b)) \oplus d_B(\mu(a, b), b)
\]

and such that \( d_A(a, \mu(a, b)) + d_B(\mu(a, b), b) \) is as small as possible.

**Proof.** Let \( C \) be defined as above, we must show that every triangle of \( C \) is metric. Let \( D \in \Pi(A, B) \). Note first that \( d_D(a, b) \leq d_C(a, b) \) for all \( a \in A \setminus B \) and \( b \in B \setminus A \); this is because \( d_D(a, b) \leq d_A(a, x) + d_B(x, b) \) for any \( x \in A \cap B \), and therefore, \( d_D(a, b) \leq d_A(a, x) \oplus d_B(x, b) \) by definition of \( \oplus \).

The first kind of triangles to consider are of the form \( \{a, x, b\} \) for \( a \in A \setminus B \), \( x \in A \cap B \) and \( b \in B \setminus A \). Since \( D \) is metric and \( d_D(a, b) \leq d_C(a, b) \), all we need to verify is the inequality \( d_C(a, b) \leq d_A(a, x) + d_B(x, b) \). Let \( x' = \mu(a, b) \). By definition, we have that \( d_C(a, b) = d_A(a, x') \oplus d_B(b, x') \leq d_A(a, x') + d_B(b, x') \leq d_A(a, x) + d_B(b, x) \).

Now, by symmetry, the only other case is a triangle \( \{a, a', b\} \) for \( \{a, a'\} \subseteq A \setminus B \) and \( b \in B \setminus A \). Since \( d_D(a, b) \leq d_C(a, b) \) and \( d_D(a', b) \leq d_C(a', b) \), we then have that

\[
d_C(a, a') = d_D(a, a') \\
\leq d_D(a, b) + d_D(b, a') \\
\leq d_D(a, b) + d_C(b, a').
\]

For the other sides, it remains to show without loss of generality that

\[
d_C(a', b) \leq d_C(a', a) \oplus d_C(a, b),
\]

or equivalently that \( d_C(a', b) \leq d_C(a', a) \oplus d_C(a, b) \). For this let \( z = \mu(a, b) \). But \( d_A(a', z) \leq d_A(a', a) + d_A(a, z) \), equivalently \( d_C(a', z) \leq d_C(a', a) \oplus d_C(a, z) \). Hence,

\[
d_C(a', b) \leq d_C(a', z) \oplus d_C(z, b) \\
\leq (d_C(a', a) \oplus d_C(a, z)) \oplus d_C(z, b) \\
= d_C(a', a) \oplus (d_C(a, z) \oplus d_C(z, b)) \\
= d_C(a', a) \oplus d_C(a, b).
\]

This completes the proof.

We are now ready to further analyze the structure of the universal spectrum without positive limits, but first some useful terminology.

**Definition 2.11.** Let \( S \) a universal spectrum without positive limits.
For $s \in S$, let $s^-$ be the largest number in $S$ smaller than $s$ if $s > 0$, and $0^- = 0$.

If $s \neq \max S$, then let $s^+$ be the smallest number in $S$ larger than $s$, and let $s^+ = s$ if $s = \max S$.

Two numbers $s < t \in S$ are said to be consecutive if $s^+ = t$ (or if $t^- = s$).

The cover of $\{r, t\}$, is the number (in $S$):

$$\min\{s \in S : |r - t| \leq s\}$$

The gap at $s \in S$, denoted by $\text{gap}(s)$, is the number (in $\mathbb{R}$):

$$\min\{|s - t| : t \in S \setminus \{s\}\}$$

If $T \subseteq S_{>0}$, then $\text{gap}(T) = \min\{\text{gap}(t) : t \in T\}$.

Note that if $\{s, t\} \in S$ with $s \neq t$, then $\text{gap}(s) \leq |s - t|$. Hence, we immediately have the following general fact.

**Fact 2.12.** Let $S$ be a universal spectrum, and $M$ be a metric space with $\text{spec}(M) \subseteq S$. Then for any three distinct points $\{x, y, z\} \subseteq M$, if $d(x, y) < \text{gap}(d(x, z))$, then $d(x, z) = d(y, z)$.

**Lemma 2.13.** If $S$ is a universal spectrum without positive limits, then the cover $c$ of two consecutive numbers $r < t \in S$ with $r + r \geq t$ is an initial number of $S$.

**Proof.** Note that $c \leq r$ because $r + r \geq t$ and $c$ is the smallest number in $S$ with $r + c \geq t$. Assume that $c$ is not initial. Then there exists a number $p < c$ with $p + p \geq c$; this implies that the two triangles, $T_0$ with side lengths $p, p, c$, and $T_1$ of side lengths $\{r, t, c\}$, are metric. These two triangles form an amalgamation instance via the common side $c$. Hence, because $S$ satisfies the 4-values condition there exists a number $s \in S$ and a metric space $M \in \Pi(T_0, T_1)$ with amalgamation distance $s$.

This is not possible. Indeed first note that $r + p < t$ since $p < c$ and again $c$ is the smallest number in $S$ with $r + c \geq t$. But now if $s \leq r$, then $s + p \leq r + p < t$ and the triangle $\{t, p, s\}$ is not metric. If on the other hand $s > r$, then $s \geq t$ since $r < t$ are consecutive; but now $r + p < t \leq s$ and the triangle $\{r, p, s\}$ is not metric. \qed

This gives the following.

**Lemma 2.14.** Let $S$ be a universal spectrum without positive limits. If $0$ is a limit of $S$, then $0$ is also a limit of the set of initial numbers of $S$, and also a limit of the set of jump numbers of $S$.

**Proof.** Let $r < t < \ell \in S$ be two consecutive numbers, and let $c$ be their cover. If $r + r \geq t$, then $c \leq r$ and it follows from Lemma 2.13 that $c$ is initial. But, if $r + r < t$, then $t$ itself is initial by definition. Thus, $S$ contains arbitrarily small initial numbers.

Moreover, is $s$ is an initial number, then $s^-$ is a jump number. We therefore have that $S$ contains arbitrarily small jump numbers as well. \qed
2.2. Fraïssé theory for universal homogeneous spaces. In this subsection we continue with $S$ a universal spectrum without positive limits, but we will focus on $U_S$ the homogeneous structure with spectrum $S$ and construct some well chosen automorphisms.

Recall that if $s > 0$ is a jump number of $S$, then the relation $\sim^s$ given by $x \sim^s y$ if $d(x,y) \leq s$ is an equivalence relation on $U_S$. If $E$ is an $\sim^s$ equivalence class and $M$ the metric space induced by $E$, then $\text{spec}(M) = \{ r \in S : r \leq s \} = S_{\leq s}$. On the other hand, it is evident that $\text{spec}(M)$ satisfies the 4-values condition, and hence, it is a universal spectrum. It follows that $M$ is isomorphic to the universal homogeneous metric space $U_{S_{\leq s}}$.

We now define the notion of dense subset of $U_S$, and show that similar to the rationals, if $U_S$ is partitioned into finitely many dense sets, then there is a non-trivial automorphism preserving the partition.

Definition 2.15. Let $S$ be a universal spectrum. A subset $A$ of $U_S$ is dense if $A \cap E \neq \emptyset$ for every jump number $s \in S_{>0}$ and for every equivalence class $E$ of the relation $\sim^s$.

We will in fact build a non-trivial automorphism which is an involution.

Lemma 2.16. Let $S$ be a universal spectrum without positive limits but with 0 as a limit, and let $\{A_i : i \in n\}$ form a finite partition of $U_S$ into dense sets. Then there exists, for every number $s \in S_{>0}$, an automorphism $f$ of $U_S$ such that:

1. $f$ preserves the partition, that is $f[A_i] = A_i$ for every $i \in n$, and
2. $f(f(x)) = x$ and $d(x, f(x)) = s$ for all $x \in U_S$.

Proof. The proof is an inductive construction on the countable domain of $U_S$, and is a consequence of the following claim handling the inductive step.

Claim. Let $A$ and $B$ be two disjoint and finite subspaces of $U_S$ for which there exists an automorphism $g$ of the subspace induced by $A \cup B$ so that:

1. $g$ preserves the partition restricted to $A \cup B$, and
2. $g(x) \in B$, $g(g(x)) = x$, and $d(x, g(x)) = s$ for all $x \in A$.

Let $u \in U_S \setminus (A \cup B)$. Then there exists a point $v \in U_S$ and an automorphism $g'$ of the subspace induced by $A \cup B \cup \{u, v\}$ so that:

1. $g'$ extends $g$, that is $g'(x) = g(x)$ for all $x \in A \cup B$,
2. $d(u, v) = s$,
3. $u$ and $v$ are in the same member of the partition, and
4. $g'(u) = v$ and $g'(v) = u$.

To prove the claim, we first show that there exists a metric space $M$ with $M = A \cup B \cup \{u, v\}$ so that the following hold.

1. $M$ restricted to $A \cup B \cup \{u\}$ is equal to $U_S$ restricted to $A \cup B \cup \{u\}$.
2. $d_M(u, v) = s$. 


(3) \(d_M(v, x) = d(u, g(x))\) (and so \(d_M(v, g(x)) = d(u, x)\)) for all \(x \in A \cup B\).

To verify that \(M\) will indeed be a metric space under these conditions, it suffices that every triangle of \(M\) is metric. Let \(\{x, y, z\}\) be a triangle of \(M\). If \(v \not\in \{x, y, z\}\), then the triangle \(\{x, y, z\}\) is metric because every triangle of \(U_S\) is metric. Now let \(\{x, y, v\}\) be a triangle of \(M\) with \(u \not\in \{x, y\}\); but the triangle \(\{g(x), g(y), u\}\) is metric and has the same side lengths as the triangle \(\{x, y, v\}\), hence, the latter is metric. Let \(\{x, u, v\}\) be a triangle of \(M\). The sides have lengths \(d(x, u), d_M(x, v) = d(g(x), u)\), and \(s\); but the triangle \(\{x, g(x), u\}\) is metric and has the same side lengths.

Now the bijection \(\tilde{g}\) of \(A \cup B \cup \{u, v\}\) extending \(g\) and interchanging \(u\) and \(v\) is an automorphism of \(M\) because \(d_M(u, x) = d(u, x) = d_M(v, g(x))\) for all \(x \in A \cup B\).

Because \(U_S\) is homogeneous there exists an embedding \(h\) of \(M\) into \(U_S\) with \(h(x) = x\) for all \(x \in A \cup B \cup \{u\}\). By Lemma 2.14 let \(0 < r \in S\) be a jump number with \(0 < r < \text{gap}(\{d(h(v), x) : x \in A \cup B \cup \{u\}\})\), and let \(E\) be the \(\tilde{e}\) equivalence class containing the point \(h(v)\). If \(i \in n\) is such that \(h(v) \in A_i\), then choose \(w \in A_i \cap E\); this is possible since \(A_i\) is assumed to be dense. It follows from the choice of \(r\) and from Fact 2.12 that \(d(w, x) = d(h(v), x) = d_M(v, x)\) for all \(x \in A \cup B \cup \{v\}\).

The required automorphism \(g'\) is simply the map corresponding to \(\tilde{g}\) interchanging \(u\) and \(w\). 

\[\square\]

3. Distinguishing Number of Homogeneous Urysohn Metric Spaces

We now consider the general case of a universal spectrum \(S\), and let \(U_S\) be the homogeneous metric space with spectrum \(S\). Our aim is to show that in all cases the distinguishing number \(U_S\) is either two or infinite. When it is two, we will show this is so by decomposing \(U_S\) into a rigid subspace particularly constructed so that all automorphisms fixing this subspace also fix its complement.

In all but one cases, the rigid subspace is made from a rigid forest. First we show how to use the graph structure of a metric space.

**Definition 3.1.** Consider a metric space \(M\) with distances in \(S \subseteq \mathbb{R}\). For \(s \in S\), the \(s\)-distance graph of \(M\) is the (simple) graph on the elements of \(M\) (as vertices), and two vertices are adjacent if and only if their distance is \(s\).

This following observation will play a crucial role in building rigid subspaces.

**Observation.** If the \(s\)-distance graph of a metric space is rigid, then the metric space is rigid.
3.1. Basic Construction. Here is the first such construction.

Lemma 3.2. Let \( S \) be a universal spectrum. Then the distinguishing number of \( U_S \) is two if there exists a positive number \( s \in S \) for which the following hold.

1. The element \( s \) is not a jump number; that is, there exists a number \( r \in S \) with \( s < r \leq s + s \).
2. For every positive \( t \in S \), there exist numbers \( \{h_t, k_t\} \subseteq S \) so that:
   a. \( s < h_t < k_t \).
   b. \( h_t + k_t \geq t \geq k_t - h_t \).

Proof. We say that a set \( P \) of pairs of points in \( U_S \setminus M \) is stabilized by the subspace \( M \) of \( U_S \) if for all \( (x, y) \in P \):

1. There exists a point \( z \in M \) with \( d(x, z) \neq d(y, z) \).
2. The \( s \)-distance graph of \( M \) is a rigid forest.

The proof of the lemma is an inductive construction on the countable domain of \( U_S \), and is a consequence of the following claim handling the inductive step.

Claim. Let \( P \) be a set of pairs of points in \( U_S \) which is stabilized by the finite subspace \( M \) of \( U_S \), and let \( \{x, y\} \) be two points in \( U_S \setminus M \). Then there exists a finite subspace \( N \) of \( U_S \) containing \( M \) and stabilizing \( P' = P \cup \{(x, y)\} \).

To prove the claim, if there exists a point \( z \in M \) with \( d(x, z) \neq d(y, z) \), then we can let \( N = M \). Hence, we assume that \( d(x, z) = d(y, z) \) for all \( z \in M \).

Let \( r \in S \) with \( s < r \leq s + s \). Let \( C \) be a metric space with spectrum \( \{s, r\} \) whose \( s \)-distance graph is a rigid tree \( G \) which is not isomorphic to one of the trees of the \( s \)-distance graph of the space \( M \). Let \( e \) be an endpoint of the tree \( G \). Let \( T \) be the metric space with \( T = \{x, y, e\} \) so that \( d_T(x, y) = t \) and \( d_T(x, e) = h_t \) and \( d_T(y, e) = k_t \) as per the hypothesis. Then \( T \) is indeed a metric space because \( h_t + k_t \geq t \geq k_t - h_t \) and \( h_t < k_t \). The pair of metric spaces \( (T, C) \) forms an amalgamation instance. Now because \( s < r \) and \( s < h_t \), then \( s < r' = \min\{r, h_t\} \). It follows from Lemma 2.5 that there exists a metric space \( C' \in \Pi_S(T, C) \) with \( d_{C'}(v, x) \geq r' \) and with \( d_{C'}(v, y) \geq r' \) for all \( v \in C' \) and so that \( d_{C'}(v, x) \leq d_{C'}(v, y) \) for every \( v \in C \).

Now let \( M' \) be the subspace of \( U_S \) induced by the set \( M \cup \{x, y\} \) of points. The metric spaces \( M' \) and \( C' \) form an amalgamation instance. It follows from Lemma 2.5 again that there exists a metric space \( M'' \in \Pi_S(M', C') \) so that \( d_{M''}(v, z) \geq r' > s \) for every \( v \in C \) and every \( z \in M \).

Because \( U_S \) is homogeneous there exists an embedding \( f \) of \( M'' \) into \( U_S \) with \( f(z) = z \) for all \( z \in M' \). Then the image \( N \) of \( M'' \) under the embedding \( f \) is as required. \( \square \)

This yields the following case.
Lemma 3.3. Let $S$ be a universal spectrum. If $S$ contains a positive number $s$ which is not a jump number, and has a limit $r$ (not necessarily in $S$) with $s < r$, then the distinguishing number of $U_S$ is two.

Proof. By Lemma 3.2 if suffices to show that for every positive $t \in S$ there exist numbers $\{h_t, k_t\} \subseteq S$ so that $s < h_t < k_t,$ and $h_t + k_t \geq t \geq k_t - h_t.$

But if $t \leq r$, then because $r$ is a limit point of $S$ one can find the required numbers $s < h_t < k_t \in S$ (close enough to $r$). If $t > r$, then choose $h_t \in S$ close enough to $r$ so that $s < h_t < t$, and let $k_t = t$; again this is possible since $r$ is a limit point of $S$. □

The case of $S$ having a positive limit in $S$ can be handle in a similar manner, but the more general case of the positive limit not necessarily in $S$ is more delicate. In that case the rigid forest will be replaced by a “crab nest”.

3.2. Crab Nest. In the more general situation of $S$ having a limit point not in $S$ and all points below are jump numbers, we may not be able to retain the connected components in the intended rigid s-graph, and therefore, we need a new structure.

First call a finite graph $S$ a spider if it is a tree which contains exactly one vertex, the centre of $S$, of degree larger than two and then all of the other vertices have degree two or are endpoints having degree one.

Two cliques $C$ and $C'$ of a graph $G$ are called adjacent if they are vertex disjoint, the order of one of them (say $C'$) is less than or equal to the order of the other ($C$), and there exists an injection $f$ of $V(C')$ to $V(C)$ so that a vertex $x \in V(C')$ is adjacent to a vertex $y \in V(C)$ if and only if $f(x) = y$.

A finite graph $G$ is a crab if there exists a rigid spider $S$ and an integer $n \geq 5$, the heft of $G$ denoted by heft($G$), so that for the set $C$ of maximal cliques of the graph $G$:

1. Every vertex of $G$ is a vertex of exactly one of the cliques in $C$.
2. Exactly one of the cliques in $C$, the centre clique of $G$, has order $n + 1$ and all other cliques in $C$ have order $n$.
3. If $C$ is the centre clique, then there exists for every vertex $x \in V(C)$ exactly one clique $C' \in C$ which is adjacent to $C$ and for which no vertex in $V(C')$ is adjacent to $x$.
4. There exists a bijection $f$ of $C$ to $V(S)$ which maps the centre clique of $G$ to the centre of $S$, and such that $C$ and $C'$ are adjacent if and only if $f(C)$ is adjacent to $f(C')$.

Note that the degree of the centre of the spider $S$ is equal to $n + 1$. A clique $C \in C$ is an end clique of the crab $G$ if $f(C)$ is an endpoint of the spider $S$.

Lemma 3.4. Every crab $G$ is rigid.

Proof. Let $G$ be a crab with associated spider $S$, heft $n$ and $C$ the set of maximal cliques. Let $g$ be an automorphism of $G$. Then $g$ induces a permutation $C$, and fixes the centre clique $C$ since it is the only one of size
$n + 1$; hence, $g$ induces an automorphism of its associated spider and since the latter is assumed to be rigid then $C$ are actually fixed.

Further we claim that $g$ is the identity map. First let $x \in V(C)$, and assume that $x \neq f(x)$. There exists a (unique) clique $C'$ which is adjacent to the clique $C$ and which does not contain a vertex which is adjacent to $f(x)$. But $V(C')$ contains a vertex $x'$ which is adjacent to $x$, implying that $f(x') \notin V(C')$, a contradiction. Hence, $f(x) = x$ for all $x \in V(C)$, implying inductively on the distance from the centre that $f(x) = x$ for all $x \in V(G)$. \hfill \Box

The rigid subspaces we are looking for will be build of crabs into what we call crab nests.

**Definition 3.5.** A graph $G$ is a crab nest if it has a subgraph $H$ on the same vertices as $G$, the crab graph of $G$, together with an enumeration of its connected components $\{H_i, i \in I\}$ for $I = \omega$ or $I = n \in \omega$, and if there exists a set $R \subseteq V(G)$, the distinguished endpoint set of $G$, so that for every $i \in I$:

1. $H_i$ is a crab, and an induced subgraph of $G$.
2. $\text{heft}(H_i) + 2 < \text{heft}(H_{i+1})$.
3. $R$ contains exactly one vertex $r_i \in V(H_i)$, and this vertex is a vertex of an end clique of $H_i$.
4. If $(x, y) \in E(G) \setminus E(H)$ then the following hold.
   a. If $x \in V(H_i)$ and $y \in V(H_j)$ for some $j < i$, then $x = r_i$.
   b. If $x = r_i$, $(r_i, z) \in E(G) \setminus E(H)$ and $z \in H_k$ with $k < i$, then $y = z$.

**Lemma 3.6.** Every crab nest is rigid.

**Proof.** Let $G$ be a crab nest with crab graph $H$, the corresponding enumeration $\{H_i, i \in I\}$ of the connected components of $H$, corresponding endpoint set $R$. Let $f$ be an automorphism of $G$ and we show that $f$ must be the identity.

Assume that there exists a vertex $x \in V(H_i)$ with $i > 0$ such that $f(x) \in V(H_0)$. Let $C$ be the maximal clique (thus, of size at least three) of $H_i$ with $x \in V(C)$. Let

$$j = \max\{k \in I : \text{ for some } y \in V(C), f(y) \in V(H_j)\}.$$ 

Assume that $j > 0$. Note that if $y \in V(C)$ is such that $f(y) \in V(H_j)$, then $(f(y), f(x)) \in E(G) \setminus E(H)$, and thus, by item 4a there can be only one such element $y$ and we must have $f(y) = r_j$. We then have that $f(z) \in H_k$ for some $k < j$ for all other $z \in V(C)$, now contradicting to Item 4b since we would have both $(f(y), f(x))$ and $(f(z), f(x))$ in $E(G) \setminus E(H)$.

It follows that $j = 0$. That is, the automorphism $f$ maps every element of $V(C)$ into $V(H_0)$. But every triangle of $H_0$ is in one of the cliques of $H_0$. Implying that $f$ maps the clique $C$ into a clique of $H_0$. But this is not possible because $\text{heft}(H_i) > \text{heft}(H_0) + 2$. It follows that $f$ does not map
any vertex of $V(G) \setminus V(H_0)$ into $V(H_0)$. Implied that if $x \in V(H_0)$, then $f(x) \in V(H_0)$.

This in turn implies, because the crab $H_0$ is rigid, that $f(x) = x$ for all $x \in V(H_0)$. Then via induction on the index set $I$ it follows that $f(x) = x$ for all $x \in V(G)$. \hfill \Box

We are now ready to handle the case of $S$ having a positive limit not necessarily in $S$.

**Lemma 3.7.** Let $S$ be a universal spectrum. If $S$ has a positive limit (not necessarily in $S$), then the distinguishing number of $U_S$ is two.

**Proof.** If $S$ has a limit $r$ (not necessarily in $S$) and one can find a non-jump number $s < r$, then Lemma 3.3 applies. This is the case if $S$ has two positive limits $r' < r$, in which case one can find such an $s$ close to $r'$. Similarly if $r$ is a limit of the elements of $S$ below $r$. Hence, we may assume that $S$ has only one positive limit $r$, every number in $S$ less than $r$ is insular, and the elements of $S$ above $r$ form a (possibly two way) sequence converging to $r$. In particular this means that every non-empty and bounded above subset of $S$ has a maximum, and thus, the operation $\oplus$ is defined for $S$.

We are now ready to undertake the construction of the rigid subspace using a rigid $s$-graph, where $s$ will be chosen close enough to $r$ and such that $r < s < r + r$. For the sake of this proof, we say that a set $P$ of pairs of points in $U_S \setminus M$ is stabilized by the subspace $M$ if the following properties hold.

1. The $s$-distance graph of $M$ is a crab nest $G$ with crab graph $H$ and enumeration $H_0,H_1,H_2,\ldots$ of the connected components of $H$ and distinguished endpoint set $R$ of $G$.
2. If $u \neq v$ are two points of $M$, then $d(u,v) \geq r$.
3. For every pair $(x,y) \in P$, there exists a crab $H_i$ and point $r_i \in R \cap H_i$ such that $d(x,r_i) \neq d(y,r_i)$.

The proof of the Lemma is an inductive construction on the countable domain of $U_S$, and is a consequence of the following claim handling the inductive step.

**Claim.** Let $P$ be a set of pairs of points in $U_S$ which is stabilized by the finite subspace $M$ of $U_S$. Let $\{x,y\}$ be two points in $U_S \setminus M$ such that $d(x,y) = t > 0$, and $d(x,z) = d(y,z)$ for all $z \in M$. Then there exists a finite subspace $N$ of $U_S$ containing $M$ and stabilizing $P' = P \cup \{(x,y)\}$.

To prove the claim, let $G$ be the $s$-distance graph of $M$ with crab graph $H$ and enumeration $\{H_i : i \in \mathbb{N}\}$ of the connected components of $H$ and distinguished endpoint set $R$.

If $t < r$, then let $r < h_t < k_t$ be two numbers in $S$ (close to $r$) with $k_t - h_t \leq t$. Otherwise, consider $s' < s'' \in S$ such that $r < s < s' < s'' < r + r$ (this is why we picked $s$ close enough to $r$). If $r \leq t < r + r$, let $h_t = s'$ and $k_t = s''$; if $r + r \leq t$, then let $h_t = s'$ and $k_t = t$. 

Let $C$ be a metric space with spectrum $\{r, s\}$ whose $s$-distance graph is a crab $H_n$, for which $\text{heft}(H_{n-1}) + 2 < \text{heft}(H_n)$. Let $r_n$ be a vertex of an end clique of the crab $H_n$ and let $R' = R \cup \{r_n\}$. Let $T$ be the metric space with $T = \{x, y, r_n\}$ so that $d_T(x, y) = t$ and $d_T(x, r_n) = h_t$ and $d_T(y, r_n) = k_t$. Note that $T$ is in all cases indeed a metric space. The pair of metric spaces $(T, C)$ forms an amalgamation instance. It follows from Lemma 2.5 and from $s' \leq r + r \leq r + h_t < r + k_t$ that there exists a metric space $C' \in \mathcal{T}(T, C)$ with

(i.) $d_{C'}(v, x) \geq s'$ and $d_{C'}(v, y) \geq s'$ for all $r_n \neq v \in C$

and so that $d_{C'}(v, x) \leq d_{C'}(v, y)$ for every $v \in C$.

Now let $M'$ be the subspace of $U_S$ induced by the set $M \cup \{x, y\}$. The metric spaces $M'$ and $C'$ form an amalgamation instance, and let $M'' = \Pi_S(M', C')$ provided by Lemma 2.10 Then for $v \in C$ and $z \in M$:

(iii.) $d_{M''}(v, z) \geq d_{C'}(v, x) \oplus d(x, z) \geq d_{C'}(v, x)$.

Hence, for $v \in C$ and $z \in M$, if $t \geq r$ or if $v \neq r_n$, then

(iv.) $d_{M''}(v, z) \geq s' > s$,

and if $t < r$, then

(v.) $d_{M''}(r_n, z) = h_t \oplus d(x, z)$.

We claim that the $s$-distance graph of $M'' \setminus \{x, y\}$ is a crab nest $G''$ with distinguishing endpoint set $R'$. The crab graph $H''$ of $G''$ is the crab graph $H$ of $G$ together with the additional connected component $H_n$. Item (4) of Definition 3.5 remains to be verified. Let $(z, v) \in E(G'') \setminus E(H'')$. Then $z$ and $v$ are two points of $M''$ with $d_{M''}(z, v) = s$. If $\{z, v\} \subseteq M$, then Item (4) of Definition 3.5 will be satisfied. If $\{z, v\} \subseteq C$, then $\{z, v\} \in E(H_n)$ because the spectrum of $C$ is $\{s, r\}$ and the $s$-distance graph of $C$ is the crab $H_n$. Hence, we may assume that $v \in C$ and $z \in M$. If $t \geq r$ or if $v \neq r_n$ it follows from (iv) above that $d_{M''}(v, z) \geq s' > s$.

Thus, $t < r$ and $v = r_n$. If $r \leq d(x, z) = d(y, z)$, then $d_{M''}(r_n, z) = h_t \oplus d(x, z) \geq r \oplus r \geq s' > s$. If $p = d(x, z) < r$, then it is possible that $d_{M''}(r_n, z) = h_t \oplus d(x, z) = h_t \oplus p = s$. But now assume that $q = d(x, w) < r$ for a point $w \in M$. Then $p$ and $q$ are insular numbers of $S$ and both smaller than $r$. If $z \neq w$, then $d(z, w) \geq r$ by assumption, which is a contradiction because $p \oplus q = \max\{p, q\} < r$ since they are insular points. Hence, $z = w$ verifying Item (4c) of Definition 3.5 and $G''$ is a crab nest.

The space $M''$ is not a subspace of $U_S$ but otherwise meets the conditions for stabilizing the set $P \cup \{(x, y)\}$. Because $U_S$ is homogeneous, there exists an embedding $f$ of $M''$ into $U_S$ with $f(a) = a$ for all $a \in M'$. Then the image $N$ of $M''$ under the embedding $f$ is as required to prove the claim. $\square$

3.3. The Remaining Cases. There are a few remaining cases to handle, made possible from previous results and techniques.
Lemma 3.8. Let $S$ be a universal spectrum. If there exist numbers $a < b \in S$ with $b - a \leq \inf S > 0$, then the distinguishing number of $S$ is two.

Proof. Let $p = \inf S_{> 0} \geq b - a$, where $a < b \in S$. If $S$ has a positive limit, then the distinguishing number of $U_S$ is two according to Lemma 3.7. Thus, we assume that $S$ does not have a positive limit. In particular, $p \in S$.

Moreover, the set $S$ must contain an initial non-jump number. For otherwise, since $a < b \leq p + a \leq a + a = 2a$ and thus, $a$ is a non-jump number and must be a non-initial numbers. We then have that there is $a_1 \in S \cap [a/2, a)$, a non-jump number which for the same reason must be a non-initial number. Continuing in this manner yields a positive limit in $S$, a contradiction.

Now, let $s$ be the smallest positive initial non-jump number of $S$, and let $r$ be the smallest number in $S$ larger than $s$. We cannot have $a < s$ since being a non-jump number and small that $s$ would make $a$ again a non-initial number; and similar to above would yield $S$ with a positive limit point. If $s < a$, then with $h_t = a$ and $k_t = b$ if $t \leq r$, and with $h_t = r$ and $k_t = t$ if $t > r$, the conditions of Lemma 3.2 are satisfied and hence, $U_S$ has then distinguishing number two.

Hence, we may assume that $s = a$ and $r$ can be taken for $b$, implying that $r - s \leq p$, and of course $q \geq p$ for all $q \in S$. We then proceed analogously to the proof of Lemma 3.2.

For the sake of this proof, we say that a set of pairs $P$ in $U_S \setminus M$ is stabilized by the subspace $M$ of $U_S$ if for all $(x, y) \in P$:

1. There exists a point $z \in M$ with $d(x, z) \neq d(y, z)$.
2. The $s$-distance graph of $M$ is a rigid forest.

The proof of the Lemma is an inductive construction on the countable domain of $U_S$, and is a consequence of the following claim handling the inductive step.

Claim. Let $P$ be a set of pairs of points in $U_S \setminus M$ which is stabilized by the finite subspace $M$ of $U_S$. Let $\{x, y\}$ be two points in $U_S \setminus M$ with $t = d(x, y) > 0$. Then there exists a finite subspace $N$ of $U_S$ containing $M$ and stabilizing $P' = P \cup \{(x, y)\}$.

To prove the claim, if there exists a point $z \in M$ with $d(x, z) \neq d(y, z)$ let $N = M$. Hence, we may assume that $d(x, z) = d(y, z)$ for all $z \in M$.

Let $C$ be a metric space with spectrum $\{s, r\}$ whose $s$-distance graph is a rigid tree $S$ which is not isomorphic to one of the trees of the $s$-distance graph of the space $M$. Let $e$ be an endpoint of the tree $S$. If $t \leq r$ let $h_t = s$ and $k_t = r$; and if $t > r$ let $h_t = r$ and $k_t = t$. Let $T$ be the metric space with $T = \{x, y, e\}$ so that $d_T(x, y) = t$, $d_T(x, e) = h_t$ and $d_T(y, e) = k_t$. Note that $T$ is indeed a metric space in all cases. The pair of metric spaces $(T, C)$ forms an amalgamation instance, and it follows from Lemma 3.5 and from $r \leq s + h_t < s + k_t$ that there exists a metric space $C' \in \Pi_S(T, C)$ with $d_C'(v, x) \geq r$ and with $d_C'(v, y) \geq r$ for all $e \neq v \in C$ and so that

Distinguishing number of Urysohn metric spaces

15

Proof. Let $p = \inf S_{> 0} \geq b - a$, where $a < b \in S$. If $S$ has a positive limit, then the distinguishing number of $U_S$ is two according to Lemma 3.7. Thus, we assume that $S$ does not have a positive limit. In particular, $p \in S$.

Moreover, the set $S$ must contain an initial non-jump number. For otherwise, since $a < b \leq p + a \leq a + a = 2a$ and thus, $a$ is a non-jump number and must be a non-initial numbers. We then have that there is $a_1 \in S \cap [a/2, a)$, a non-jump number which for the same reason must be a non-initial number. Continuing in this manner yields a positive limit in $S$, a contradiction.

Now, let $s$ be the smallest positive initial non-jump number of $S$, and let $r$ be the smallest number in $S$ larger than $s$. We cannot have $a < s$ since being a non-jump number and small that $s$ would make $a$ again a non-initial number; and similar to above would yield $S$ with a positive limit point. If $s < a$, then with $h_t = a$ and $k_t = b$ if $t \leq r$, and with $h_t = r$ and $k_t = t$ if $t > r$, the conditions of Lemma 3.2 are satisfied and hence, $U_S$ has then distinguishing number two.

Hence, we may assume that $s = a$ and $r$ can be taken for $b$, implying that $r - s \leq p$, and of course $q \geq p$ for all $q \in S$. We then proceed analogously to the proof of Lemma 3.2.

For the sake of this proof, we say that a set of pairs $P$ in $U_S \setminus M$ is stabilized by the subspace $M$ of $U_S$ if for all $(x, y) \in P$:

1. There exists a point $z \in M$ with $d(x, z) \neq d(y, z)$.
2. The $s$-distance graph of $M$ is a rigid forest.

The proof of the Lemma is an inductive construction on the countable domain of $U_S$, and is a consequence of the following claim handling the inductive step.

Claim. Let $P$ be a set of pairs of points in $U_S \setminus M$ which is stabilized by the finite subspace $M$ of $U_S$. Let $\{x, y\}$ be two points in $U_S \setminus M$ with $t = d(x, y) > 0$. Then there exists a finite subspace $N$ of $U_S$ containing $M$ and stabilizing $P' = P \cup \{(x, y)\}$.

To prove the claim, if there exists a point $z \in M$ with $d(x, z) \neq d(y, z)$ let $N = M$. Hence, we may assume that $d(x, z) = d(y, z)$ for all $z \in M$.

Let $C$ be a metric space with spectrum $\{s, r\}$ whose $s$-distance graph is a rigid tree $S$ which is not isomorphic to one of the trees of the $s$-distance graph of the space $M$. Let $e$ be an endpoint of the tree $S$. If $t \leq r$ let $h_t = s$ and $k_t = r$; and if $t > r$ let $h_t = r$ and $k_t = t$. Let $T$ be the metric space with $T = \{x, y, e\}$ so that $d_T(x, y) = t$, $d_T(x, e) = h_t$ and $d_T(y, e) = k_t$. Note that $T$ is indeed a metric space in all cases. The pair of metric spaces $(T, C)$ forms an amalgamation instance, and it follows from Lemma 3.5 and from $r \leq s + h_t < s + k_t$ that there exists a metric space $C' \in \Pi_S(T, C)$ with $d_C'(v, x) \geq r$ and with $d_C'(v, y) \geq r$ for all $e \neq v \in C$ and so that

Distinguishing number of Urysohn metric spaces

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jump number and 
p only if
function of
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The metric spaces
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Lemma 3.10.
A, BONATO, C. LAFLAMME, M. PAWLIUK, AND N. SAUER
assume that
p
as a limit. If
S
the smallest positive number of
U
is not a jump number, since if
p < q
Lemma 3.9.
Let
S
be a universal spectrum which does not have a limit, then the distinguishing number of
U is two or infinite. If
p = \min(S_{>0}),
then the distinguishing number of
U is two if and only if there exist numbers
a < b ∈ S with b − a ≤ p.

Proof. If there exist number
a < b ∈ S with b − a ≤ p, then the distinguishing number of
U is two again according to Lemma 3.8. This is the case if
p is not a jump number, since if
p < q ≤ p + p, then
q − p ≤. Thus, we may assume that
p is a jump number, and hence, an insular number since being the smallest positive number of
S it is an initial number.

Thus, we assume that
p is insular and
p < b − a for all
a < b ∈ S, and we show that the distinguishing number of
U is not finite. Let
γ be a colouring function of
U with
n ∈ \omega colours, that is \gamma[U_S] ≤ n. Note that since
p is a jump number and
p = \min(S_{>0}),
the relation ∼ on
U given by
x ∼ y if and only if
d(x, y) = p
is an equivalence relation. Let
E be a ∼ equivalence class of
U. Then by our assumption
d(x, z) = d(y, z) for all points
z of
U \setminus E and all points
x, y ∈ E.
The set
E is infinite because every finite metric space
M with
\text{spec}(M) = \{p\}
is an element of the age \mathfrak{A}(S). Hence, there are two points
x ≠ y in
E with
γ(x) = γ(y).
The function
f with
f(z) = z
for all points
z of
U \setminus \{x, y\}, and
f(x) = y
and
f(y) = x,
is then a colour preserving automorphism of
U.

The following is another instance where we can show that the distinguishing number is infinite.

Lemma 3.10. Let
S be a universal spectrum with no positive limits but with
0 as a limit. If
\text{gap}(S_{≥s}) > 0 for every positive number
s ∈ S, then there exists for every
n ∈ \omega and every partition
P = \{A_i : i ∈ \}\\ of
U a non-trivial automorphism
f of
U preserving
P.

Proof. There exists a subset
I ⊆ n and a jump number
s of
S and an ≈ equivalence class
E so that for all
i ∈ I the set
A_i is dense for the homogeneous metric space
E, and so that
A_i ∩ E = ∅ for all
i \not∈ I.

Let
r ∈ S with
0 < r < \text{gap}(S_{≥s}). According to Lemma 2.16 there exists an automorphism
f of
E with
d(x, f(x)) = s
for all
x ∈ E which preserves
the partition of $E$ induced by the partition $P$ of $U_S$. It follows from Fact 2.12 that $d(y, x) = d(y, f'(x))$ for all points $y$ in $U_S \setminus E$ and all points $x \in E$. It follows that the function $f : U_S \to U_S$ with $f(x) = f'(x)$ if $x \in E$ and with $f(y) = y$ if $y \not\in E$ is a non-trivial automorphism of $U_S$ preserving $P$. □

We can then characterize the case of a universal spectrum with no positive limits but with 0 as a limit.

**Lemma 3.11.** Let $S$ be a universal spectrum with no positive limits but with 0 as a limit, then the distinguishing number of $U_S$ is two or infinite. The distinguishing number of $U_S$ is infinite if and only if $\operatorname{gap}(S_{\geq s}) > 0$ for every positive number $s \in S$.

**Proof.** On account of Lemma 3.10 it remains to prove that if there exists a positive number $s \in S$ for which $\operatorname{gap}(S_{\geq s}) = 0$, then the distinguishing number of $U_S$ is equal to two.

But if every number in $\operatorname{gap}(S_{\geq s})$ is insular, then $\operatorname{gap}(S_{\geq s}) \geq s > 0$. Let $r$ be the smallest non-insular initial number larger than or equal to $s$. Then $\operatorname{gap}(S_{\geq r}) = 0$ and in turn then $\operatorname{gap}(S_{\geq r+}) = 0$. By Lemma 3.2, the distinguishing number of $U_S$ is two. □

### 3.4. Proof of the Main Theorem

We now obtain the characterization for the distinguishing number of universal homogeneous Urysohn metric spaces, which we restate for convenience.

**Theorem.** Let $S$ be a countable universal spectrum and $U_S$ the countable homogeneous structure with spectrum $S$. Then $D(U_S) = 2$ or $\omega$, and the following items hold.

1. If $S$ has a positive limit (not necessarily in $S$), then $D(U_S) = 2$.
2. If $S$ has no positive limits but has 0 as a limit, then $D(U_S) = 2$ if and only if $S$ contains arbitrarily large elements of arbitrarily small distance.
3. If $S$ does not have a limit, then $D(U_S) = 2$ if and only if $S$ contains two elements of distance smaller than the minimum positive element of $S$.

**Proof.** Let $S$ be a universal spectrum. If $S$ has a positive limit, then $D(U_S) = 2$ by Lemma 3.7.

If $S$ has no positive limits but has 0 as a limit, then by Lemma 3.11 $D(U_S) = 2$ if and only if $\operatorname{gap}(S_{\geq s}) = 0$ for every positive number $s \in S$. However, this implies that $S$ contains arbitrarily large elements of arbitrarily small distance.

Finally, if $S$ does not have a limit, then by Lemma 3.9 $D(U_S) = 2$ if and only if there exist numbers $a < b \in S$ with $b - a \leq \min(S_{>0})$. □
4. Conclusion

We have shown that the distinguishing number of every universal homogeneous Urysohn metric spaces is either two or infinite, and moreover characterized when each case occurs by structural properties of the corresponding universal spectrum. It is interesting that this is the case even though the permutation group of these Urysohn metric spaces is often imprimitive; this is for example the case when the spectrum contains a jump number. In these cases, it is due to the homogeneity and universality that the distinguishing number passes directly from two to infinity.

However, one cannot expect the distinguishing number of every metric space to always be either two or infinite, even for homogeneous metric spaces. This is the case of the pentagon $C_5$ equipped with the graph distance, making it into an homogeneous metric space with primitive automorphism group $D_{10}$ and distinguishing number 3. One can also produce an infinite homogeneous metric space of distinguishing number 3, but with imprimitive automorphism group. Indeed consider the Rado (homogeneous) graph, first turn it into a metric space $R$ with spectrum $\{0,3,5\}$ by assigning distance 5 to every edge, distance 3 to non-edges, and then consider the wreath product $R[M]$ for $M$ the metric space consisting of two points of distance 1. This creates an homogeneous metric space $R[M]$ with spectrum $\{0,1,3,5\}$. Since the Rado graph has distinguishing number 2, one easily sees that one must use 3 colours to obtain two different set of two different colours to assign to elements of $M$. Finally, the automorphism group of $R[M]$ is imprimitive since points of distance 1 form a non-trivial equivalence relation.

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A. Bonato: Department of Mathematics, Ryerson University, 350 Victoria St., Toronto, ON, Canada, M5B 2K3  
*E-mail address: abonato@ryerson.ca*

C. Laflamme: University of Calgary, Department of Mathematics and Statistics, Calgary, Alberta, Canada T2N 1N4  
*E-mail address: laflamme@ucalgary.ca*

M. Pawliuk: University of Calgary, Department of Mathematics and Statistics, Calgary, Alberta, Canada T2N 1N4  
*E-mail address: mpawliuk@ucalgary.ca*

N. Sauer: University of Calgary, Department of Mathematics and Statistics, Calgary, Alberta, Canada T2N 1N4  
*E-mail address: nsauer@ucalgary.ca*