Consistent axial–like gauge fixing on hypertori

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Abstract

We analyze the Gribov problem for SU($N$) and U($N$) Yang-Mills fields on $d$-dimensional tori, $d = 2, 3, \ldots$. We give an improved version of the axial gauge condition and find an infinite, discrete group $\mathcal{G}' = \mathbb{Z}^{dr} \rtimes \mathbb{Z}_2$ for $\mathcal{G} = SU(N)$ and $r = N - 1$ for $\mathcal{G} = U(N)$, containing all gauge transformations compatible with that condition. This residual gauge group $\mathcal{G}'$ provides (generically) all Gribov copies and allows to explicitly determine the space of gauge orbits which is an orbifold. Our results apply to Yang-Mills gauge theories either in the Lagrangian approach on $d$-dimensional space-time $T^d$, or in the Hamiltonian approach on $(d+1)$-dimensional space-time $T^d \times \mathbb{R}$. Using the latter, we argue that our results imply a non-trivial structure of all physical states in any Yang-Mills theory, especially if also matter fields are present.

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1. The separation of gauge and physical degrees of freedom is important for understanding gauge theories because the physical configuration space of a gauge theory is the space of gauge orbits. A complete gauge fixing should provide a (convenient) choice of representatives for these orbits. In non-Abelian Yang-Mills theories this is highly non-trivial. Although in the formal perturbative expansion the gauge can be fixed by imposing a gauge condition such as the Coulomb- or the Landau gauge, such a procedure is not really a gauge fixing because of the existence of Gribov copies, i.e. gauge equivalent field configurations all obeying the gauge condition [1]. Obviously this is insufficient in the non-perturbative regime.

Although Gribov ambiguities are absent on \( \mathbb{R}^d \) in the axial gauge [2], they cannot be avoided on \( d \)-spheres \( S^d \) [3, 4], and the same is to be expected for any other compact manifold \( M^d \). As any deeper investigation of quantum Yang-Mills theory on \( \mathbb{R}^d \) will (in an intermediate step) require the introduction of an infrared cutoff which can be achieved by replacing \( \mathbb{R}^d \) by some compact manifold \( M^d \), the Gribov problem seems closely related to non-perturbative infrared properties of these models. Indeed, Gribov has argued that these ambiguities might give a clue to understanding confinement [1]. This argument was much substantiated and extended by Zwanziger, who analyzed the fundamental domain for the Landau gauge using a Morse functional [5] and showed that it contains only fields of bounded size. This has important consequences for the infrared properties of the theory defined such that the fields are constrained to lie in that region [3]. An explicit characterization of the fundamental domain for covariant gauges is difficult because, for topological reasons, there must be some boundary identifications [7].

In a recent investigation of gauge theories on a cylinder in the Hamiltonian framework a complete gauge fixing procedure for \( U(N) \) and \( SU(N) \) Yang-Mills theories on a circle \( S^1 \) was found [8, 9]. Although the Gribov problem prevents a complete gauge fixing by a linear gauge condition, it was shown that a modified version of the Coulomb gauge provides a maximal gauge reduction so that all gauge transformations compatible with that gauge are (generically) in a discrete group \( \mathcal{G}' \) independent of the gauge orbit. This residual gauge group \( \mathcal{G}' \) allows an explicit construction of the space of gauge orbits and plays an essential role for the non-perturbative structure of Yang-Mills theories coupled to matter on a cylinder [8].

In this paper we analyze the Gribov problem for \( U(N) \) and \( SU(N) \) Yang-Mills gauge theories on \( d \)-dimensional tori \( T^d \), \( d \geq 2 \). We give a complete gauge fixing procedure generalizing the one of [8, 9] to higher dimensions. Our interest in the tori is partly motivated by [10], in which the ultraviolet limit of Yang–Mills theory is constructed nonperturbatively on a 4-dimensional compact space using the axial gauge. On the torus the ‘axial gauge’ \( A_1 = 0 \) is not really a gauge condition because it fixes not only gauge but also physical degrees of freedom, namely the Wilson–Polyakov loops in \( x^1 \)-direction. Our gauge fixing procedure uses a consistent analogue of the axial gauge (consistent in the sense that it really is a gauge condition). We show by construction that for any Yang-Mills configuration \( A \), there is a gauge transformation \( U \) such that \( A'^U = U^{-1}(-i d + A)U \)

\footnote{Note that \( S^1 = T^1 \), and that the Coulomb- and the axial gauge in (1+1)-dimensions are essentially the same}
obeys this gauge condition. Moreover, \( U \) is unique up to an element in a residual gauge group \( G' \), and for generic field configurations (as explained in detail below), this group is discrete. It gives all Gribov copies for a given generic gauge configuration obeying our gauge condition. Moreover, it allows us to write down the space of gauge orbits explicitly.

Our results apply to Yang-Mills gauge theories either in the Lagrangian formulation on \( d \)-dimensional space-time \( T^d \), or in the Hamiltonian approach in \((d+1)\)-dimensional space-time \( T^d \times \mathbb{R} \). In the latter case, we give an alternative derivation of our result by explicitly solving the Gauss’ law. We furthermore argue that the existence of a non-trivial residual gauge group \( G' \) implies a non-trivial structure of all physical states.

2. In the following, the structure group \( G \) of the Yang-Mills field is \( U(N) \) or \( SU(N) \) in the fundamental representation and \( g \) is the Lie algebra of \( G \). For the \( d \)-dimensional torus \( T^d \), \( A \) is the set of all one–forms \((\text{Yang-Mills field configurations}) \) \( A = \sum_{i=1}^{d} A_i dx^i \) with \( A_i \) continuous mappings \( T^d \to g \), and \( G = \text{Map}(T^d; G) \) is the gauge group containing all continuously differentiable mappings \( T^d \to G \). Parametrizing points \( \vec{x} \in T^d \) by \( \vec{x} = (x^1, \ldots, x^d) \), \( 0 \leq x^1, \ldots, x^d \leq 2\pi \), we shall write such functions on \( T^d \) as periodic functions on \([0,2\pi]^d\) with the appropriate continuity and differentiability conditions at the boundary of \([0,2\pi]^d\).

We also introduce an algebraic basis in the Lie algebra \( g \) of \( G = SU(N) \) as follows \footnote{Our arguments do not change if one imposes additional, mutually compatible, regularity conditions on the \( A_i \)'s and \( U \)'s.}: Let \( e_{ij} \) be the \( N \times N \) matrix with the elements \((e_{ij})_{kl} = \delta_{ik}\delta_{jl} \). We define \( H_i = e_{i,i} - e_{i+1,i+1} \) for \( i = 1, \ldots, N-1 \), spanning the Cartan subalgebra \( h \) of \( g \). Moreover, \( E^+_1 = e_{1,2}, E^+_2 = e_{2,3}, \ldots, E^+_N = e_{N-1,N}, E^-_N = e_{1,3}, \ldots, E^+_{\frac{1}{2}N(N-1)} = e_{1,N} \), and \( E^-_j = (E^+_j)^* \) for \( j = 1, 2, \ldots, \frac{1}{2}N(N-1) \). These matrices obey

\[
[H_i, H_j] = 0 \\
[H_i, E^+_j] = \pm a_{ij} E^+_j \quad \forall i, j
\]  \hspace{1cm} (1)

with \( a_{ij} \) the elements of \((N-1) \times \frac{1}{2}N(N-1)\) matrix given by \( a_{ij} = \delta_{k(j)} - \delta_{l(j)} - \delta_{i+1,k(j)} + \delta_{i+1,l(j)} \) where \( k(j), l(j) \) are determined from \( E^+_j = e_{k(j),l(j)} \). Moreover, \([E^+_j, E^-_k] \) is in \( h \) if and only if \( j = k \), and

\[
[E^+_k, E^-_k] = \sum_{j=1}^{N-1} c_{kj} H_j
\]  \hspace{1cm} (2)

where \( c_{kj} \) is a \( \frac{1}{2}N(N-1) \times (N-1)\)-matrix with \( c_{kj} = \delta_{kj} \) for \( k, j = 1, 2, \ldots N-1 \).

For \( G = U(N) \) there is also \( H_0 = 1 \). Every \( X \in g \) has the decomposition

\[
X = \sum_j X^{0,j} H_j + \sum_j \left( X^{+,j} E^-_j + X^{-,j} E^+_j \right),
\]  \hspace{1cm} (3)

and \( X^{0,j} = (X^{0,j})^* \) and \( X^{+,j} = (X^{+,j})^* \) are uniquely determined by \( X \).

3. For simplicity, we start with \( d = 2 \). The generalization to \( d > 2 \) will be easy.
As mentioned, the axial gauge $A_1 = 0$ is not really a gauge condition as it fixes the eigenvalues of the Wilson–Polyakov loops ($\mathcal{P}$ the usual path ordering symbol)

$$\mathcal{P} \exp \left( -i \int_0^{2\pi} dy^1 A_1(y^1, x^2) \right),$$

which are gauge–invariant quantities, and in general different from one. The simplest similar condition that does not fix them is to require that $A_1(x^1, x^2)$ is independent of $x^1$ and lies in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, i.e.

$$A_1^{0,j}(x^1, x^2) = Y_j^j(x^2)$$
$$A_1^{+j}(x^1, x^2) = 0 \quad \forall j.$$  \hspace{1cm} (5)

We prove that this is indeed a consistent gauge fixing condition by showing that for an arbitrary $A \in \mathcal{A}$, there is an $U \in \mathcal{G}$ such that $A^U$ has the form (3). This amounts to solving

$$(A^U)_1(\vec{x}) = U^{-1}(\vec{x})[-i\partial_1 + A_1(\vec{x})]U(\vec{x}) = \sum_j Y_1^j(x^2)H_j \equiv Y_1(x^2),$$  \hspace{1cm} (6)

or equivalently $[-i\partial_1 + A_1]U = UY_1$, which can be solved by the ansatz $U = u_0C$ where $u_0$ satisfies the homogeneous eq. $[-i\partial_1 + A_1]u_0 = 0$ and $C$ obeys $-i\partial_1 C = CY_1$. A solution of the former for $\vec{x} \in [0, 2\pi]^2$ is provided by the parallel transporter

$$u_0(x^1, x^2) = \mathcal{P} \exp \left( -i \int_0^{x^1} dy^1 A_1(y^1, x^2) \right),$$

whereas the general solution of the latter is $C = V \exp (ix^1Y_1)$ with $V \in \mathcal{G}$ independent of $x^1$. Thus (3) is solved by

$$U(x^1, x^2) = u_0(x^1, x^2)V(x^2) \exp \left( ix^1Y_1(x^2) \right)$$  \hspace{1cm} (8)

We want $U$ to be continuously differentiable on the torus, hence $V$ must be in $\text{Map}(T^1; \mathcal{G})$. The only other non-trivial condition implying this is continuity at the endpoints in $x^1$-direction, i.e. $U(0, x^2) = U(2\pi, x^2) \forall x^2$. This leads to the following condition

$$V(x^2)^{-1} \mathcal{P} \exp \left( -i \int_0^{2\pi} dy^1 A_1(y^1, x^2) \right)V(x^2) = \exp \left( -i2\pi \sum_j Y_1^j(x^2)H_j \right).$$  \hspace{1cm} (9)

Thus $Y_1^j$ are determined by the eigenvalues of the Wilson loops (3), and $V(x^2) \in \text{Map}(T^1; \mathcal{G})$ has to be such that it diagonalizes these loops. Obviously such a $V$ exists and (8) has always a solution. (6) and (8) then imply that $\partial_1 U$ exists and is continuous if $U$ is continuous.

Moreover, the conditions on $A$ and the explicit expressions (6) and (8) for $u_0$ and $U$ show that $U$ will be continuously differentiable on the torus if $V$ and $Y_1$ are continuously differentiable in $x^2$.

\[^3\partial_1 \equiv \partial/\partial x^1\]

\[^4\text{note that this is in general not a gauge transformation!}\]
This holds if all the diagonal entries of the diagonal matrix on the right side of (3) are different. The gauge condition (3) does not fix the gauge completely. That is to say, (3) has several solutions for any given \(A_1(x)\). We now show how to find all solutions of (3). We first note that the \(Y_1^j(x^2)\) are not unique but can be shifted by arbitrary integers,

\[
Y_1^j(x^2) \rightarrow Y_1^j(x^2) + n_1^j, \quad n_1^j \in \mathbb{Z}.
\] (10)

This can be achieved by the special gauge transformations \(\exp \left( i x^1 \sum_j n_1^j H_j \right) \). These transformations provide a representation of the Abelian group \(\mathbb{Z}^r\) with \(r = N\) for \(G = U(N)\) and \(r = N - 1\) for \(G = SU(N)\). Moreover, if \(V(x^2)\) diagonalizes the Wilson loops (4) then so does \(\tilde{V}(x^2)\) if

\[
\tilde{V}(x^2) = V(x^2) P(x^2)
\] (11)

where \(P(x^2) \in \text{Map}(T^1; \mathcal{P}_G)\) and \(\mathcal{P}_G = \{ p \in G \mid \forall h \in \mathbb{H} : p^{-1} h p \in \mathbb{H} \}\) is the Peter-Weyl group of \(G\) (see appendix). We further note that \(P(x^2) \exp \left( -i 2\pi \sum_j Y_1^j(x^2) H_j \right) P^{-1}(x^2) = \exp \left( -i 2\pi \sum_j Y_1^j(x^2) H_j \right)\).

In general, (11) is also sufficient, that is, given a particular solution of (3) as \(V(x^2)\) and \(Y_1^j(x^2)\), any other solution of (3) has the form \(U(x^1, x^2) = U_1(x^1, x^2) P(x^2)\), where

\[
U_1(x^1, x^2) = u_0(x^1, x^2) V(x^2) \exp \left( i x^1 Y_1^1(x^2) \right) \exp \left( i x^1 \sum_j n_1^j H_j \right),
\] (12)

and where \(P(x^2) \in \text{Map}(T^1; \mathcal{P}_G)\) and \(n_1^j \in \mathbb{Z} \forall j\) are arbitrary. If a degeneracy occurs, e.g. if \(Y_1^i(x^2) = Y_1^j(x^2) - Y_1^k(x^2)\) (modulo \(\mathbb{Z}\)), then \(\exp \left( -i 2\pi \sum_j Y_1^j(x^2) H_j \right)\) has two degenerate eigenvalues and therefore commutes with an SU(2)-subgroup of SU(\(N\)). More generally, as \([E_j^\pm, Y_1^i(x^1)] = \mp \sum_i Y_1^i(x^1) a_{ij}\), the condition

\[
\sum_i Y_1^i(x^1) a_{ij} \in \mathbb{Z}
\] (13)

is necessary and sufficient for \(\exp \left( -i 2\pi \sum_j Y_1^j(x^2) H_j \right)\) to commute with the SU(2)-subgroup generated by \(E_j^\pm\), and if \(q\) of these conditions hold there is a \(q\)-fold degeneracy and an additional (obvious) SU(\(q + 1\)). In the latter case, one has eq. (11) with \(P(x^2) \in \text{Map}(T^1; \mathcal{P}')\) where \(\mathcal{P}'\) is the group generated by \(\mathcal{P}_G\) and this SU(\(q + 1\)).

However, the matrices with degenerate eigenvalues form a set of measure zero in the set of all matrices, and it is conceivable that this will extend to the functional measure of field configurations that are degenerate in the above sense, since there is no symmetry that enforces such a degeneracy. We therefore expect that the additional gauge transformations associated to these non-generic field configurations obeying at least one of the conditions (11) do not play a dynamical role (two other reasons will be given in the sequel).

We now show that it is possible (and in fact natural) to further restrict the gauge freedom by imposing the additional gauge condition that

\[
\int_0^{2\pi} \frac{dy^1}{2\pi} A_2^{0,j}(y^1, x^2) = Y_2^j
\] (14)
independent of \( x^2 \), i.e. that the up-to-now arbitrary gauge transformation \( P(x^2) \) can be chosen such that \( A^U \) obeys this condition (note that this does not restrict the \( A^\pm_2 \)). Since \( U(\vec{x}) = U_1(\vec{x})P(x^2) \) this is equivalent to demonstrating that the following eq. has a solution \( P(x^2) \in \text{Map}(T^1; P) \):

\[
[-i\partial_2 + B_2(x^2)]P(x^2) = P(x^2) \sum_j Y_2^j H_j
\]

(15)

where

\[
B_2(x^2) \equiv \int_0^{2\pi} \frac{dy^1}{2\pi} \sum_j (A^{U_1})_{2,j}(y^1, x^2)H_j.
\]

(16)

The general solution of this eq. is obtained as above (and path–ordering is not necessary) as \( P(x^2) = U_2(x^2)p \) with

\[
U_2(x^2) = \exp \left( -i \int_0^{x^2} dy^2 B_2(y^2) \right) \exp \left( ix^2 \sum_j Y_2^j H_j \right) \exp \left( ix^2 \sum_j n_2^j H_j \right)
\]

(17)

where the \( Y_2^j \) are determined by the continuity condition \( P(0) = P(2\pi) \), i.e.

\[
\exp \left( -i \int_0^{2\pi} dy^2 B_2(y^2) \right) = \exp \left( -i2\pi \sum_j Y_2^j H_j \right)
\]

(18)

and \( n_2^j \in \mathbb{Z} \) \( \forall j \) and \( p \in P \) are arbitrary.

The Peter-Weyl group \( P \) still contains continuous 1-parameter subgroups (see the appendix). This suggests that it should be possible to reduce the gauge freedom further, and we do this as follows. In the appendix we show that for an arbitrary \( X \in g \), there is a \( p \in P \) such that \( p^{-1}XP \) obeys the conditions

\[
(p^{-1}XP)^{+,j} = (p^{-1}XP)^{-,j} \quad \forall j = 1, 2, \ldots N - 1.
\]

(19)

Moreover, for \( G = \text{SU}(N) \) this \( p \) is generically unique up to an element in the subgroup of \( \text{SU}(N) \) generated by the elements \( z_i, i = 1, 2, \ldots N \) and \( \sigma \) with matrix elements

\[
(z_{i})_{kl} = e^{i\pi/N} \delta_{kl} \begin{cases} -1 & \text{for } k = i \\ 1 & \text{otherwise} \end{cases} \\
(\sigma)_{kl} = \delta_{k,N+1-l}
\]

(20)

obeying the relations \( z_1z_2 \cdots z_N = 1, \sigma^2 = 1, z_i^2 = 1, z_i^2 = z_i^2, \) and \( \sigma z_i\sigma = z_{N+1-i} \) \( \forall i, j \). This group obviously is a subgroup of \( \mathbb{Z}_{2N}^{N-1} \supseteq \mathbb{Z}_2 \) and we denote it as \( \mathbb{Z}_{\text{SU}(N)} \). For \( G = U(N) \), \( p \) can in addition be multiplied by an arbitrary phase, hence the corresponding group is \( \mathbb{Z}_{\text{U}(N)} = U(1) \times \mathbb{Z}_{\text{SU}(N)} \). In the non-generic case where some of the elements \( X^{+,j} \) are zero, there is obviously additional freedom in the choice of \( p \) corresponding to continuous subgroups of \( P \).

Since \( U(\vec{x}) = U_1(\vec{x}, x^2)U_2(x^2)p \), we can write

\[
\int_0^{2\pi} \frac{dy^1}{2\pi} \int_0^{2\pi} \frac{dy^2}{2\pi} (A^{U_1}_2(y^1, y^2) = p^{-1}XP \text{ with } X = \int_0^{2\pi} \frac{dy^1}{2\pi} \int_0^{2\pi} \frac{dy^2}{2\pi} (A^{U_1U_2}_2(y^1, y^2). \]

It follows that we can impose the additional gauge condition

\[
\int_0^{2\pi} \frac{dy^1}{2\pi} \int_0^{2\pi} \frac{dy^2}{2\pi} \left( A^{+,j}_2(y^1, y^2) - A^{-,j}_2(y^1, y^2) \right) = 0 \quad \text{for } j = 1, 2, \ldots N - 1,
\]

(21)
and in the generic case this condition fixes the $p \in P_G$ in eq. (17) up to an element in $\mathbb{Z}_G$.

In the non-generic case where at least one of the following conditions holds,

\[
\int_{0}^{2\pi} \frac{dy^1}{2\pi} \int_{0}^{2\pi} \frac{dy^2}{2\pi} A_{2}^{\pm,j}(y^1, y^2) = 0 \text{ for } j = 1, 2, \ldots \text{ or } N - 1,
\]

there are additional gauge transformations compatible with (21). By a similar argument as above we expect that these additional gauge transformations can be ignored.

The group $\mathbb{Z}_G$ obviously contains the center $\mathbb{C}_G$ of $G$ ($\mathbb{Z}_N$ for $G = SU(N)$ and $U(1)$ for $G = U(N)$) which acts trivially on all Yang-Mills field configurations $A \in \mathcal{A}$. For the Yang-Mills sector the residual gauge freedom associated with the center is therefore irrelevant and the relevant group of gauge transformations compatible with (21) is $\mathbb{Z}_G/\mathbb{C}_G$ identical with $\mathbb{Z}_2^{N-1} \times \mathbb{Z}_2$ for $G = U(N)$ and $G = SU(N)$. If matter fields are present, invariance under gauge transformations in the center $\mathbb{C}_G$ can be imposed in the matter sector.

To summarize, we have shown that (13), (14) and (21) provide a complete gauge reduction on the torus $T^2$. For every $A \in \mathcal{A}$, there is a $U \in G$ such that $A^U$ obeys all these conditions, and for generic $A$ and gauge group $SU(N)$ this $U$ is unique up to the discrete residual group $G' = \mathbb{Z}^{2r} \times (\mathbb{Z}_2^{N-1} \times \mathbb{Z}_2)$. In the generic case, i.e. for those field configurations $A$ which do not satisfy any of the conditions (13) and (22), $G'$ provides all Gribov copies of our gauge condition. For non-generic field configurations obeying at least one of these conditions one has additional Gribov copies, and it is straightforward to write down a complete list of these. However, as argued already above, we do not consider these additional Gribov copies as important. This is also suggested by the fact that the latter are coordinate dependent in the sense that they change if we replace the $x^1$- by the $x^2$-directions, i.e. diagonalize the Wilson loops in 2-direction instead of those in 1-direction, or impose the condition (21) on other $j'$s, hence they do not have a gauge invariant meaning. This suggests that the additional Gribov copies associated with these field configurations can be regarded as arising from ‘coordinate singularities’, in contrast to those corresponding to the residual gauge group $G'$ which do have a gauge invariant significance.

4. For $d > 2$, the complete gauge fixing condition is given by

\[
\begin{align*}
A_1^{0,j}(x^1, x^2, \ldots, x^d) &= Y_1^{j}(x^2, \ldots, x^d) \\
A_1^{\pm,j}(x^1, x^2, \ldots, x^d) &= 0 \quad \forall j
\end{align*}
\]

\[
\begin{align*}
\int_{0}^{2\pi} \frac{dy^1}{2\pi} A_2^{0,j}(y^1, x^2, \ldots, x^d) &= Y_2^{j}(x^3, \ldots, x^d) \\
\int_{0}^{2\pi} \frac{dy^1}{2\pi} \int_{0}^{2\pi} \frac{dy^2}{2\pi} A_3^{0,j}(y^1, y^2, x^3, \ldots, x^d) &= Y_3^{j}(x^4, \ldots, x^d) \\
& \quad \vdots \\
\int_{0}^{2\pi} \frac{dy^1}{2\pi} \int_{0}^{2\pi} \frac{dy^2}{2\pi} \cdots \int_{0}^{2\pi} \frac{dy^{d-1}}{2\pi} A_d^{0,j}(y^1, y^2, \ldots, y^{d-1}, x^d) &= Y_d^{j} \quad \forall j
\end{align*}
\]
\[
\int_0^{2\pi} \frac{dy_1}{2\pi} \int_0^{2\pi} \frac{dy_2}{2\pi} \cdots \int_0^{2\pi} \frac{dy_d}{2\pi} \left( A_{d}^{+ j}(y_1^i, \ldots, y_d^i) - A_{d}^{- j}(y_1^i, \ldots, y_d^i) \right) = 0
\]
for \( j = 1, 2, \ldots, N - 1 \). \tag{25}

It is straightforward to extend our argument for \( d = 2 \) above and construct for an arbitrary \( A \in \mathcal{A} \) the \( U \in \mathcal{G} \) such that \( A^U \) obeys these conditions, and to show that this \( U \) is unique up to an element in the residual gauge group \( \mathcal{G}' = \mathbb{Z}^d \times_1 (\mathbb{Z}_2^{N-1} \times \mathbb{Z}_2) \).

It is easy to write down explicitly the action of all elements in \( \mathcal{G}' \) on Yang-Mills configurations \( A \) obeying (23)–(25). As the resulting list of equations is quite lengthy and not very illuminating, we refrain from doing this here.

The result for \( d = 1 \) is somewhat different \cite{9} but easily recoverd as follows: in this case there is only one gauge condition \( A_1(x^1) = \sum_j Y_j^1 H_j \), and the space of all configurations obeying this condition is obviously \( \mathbb{R}^r \). Moreover, the gauge transformation \( U \) transforming a general \( A \in \mathcal{A} \) to one obeying this condition is generically unique up to an element in \( \mathbb{Z}^r \times_1 \mathcal{P}_G \), but as \( H_G \subset \mathcal{P}_G \) acts trivially on all field configuration obeying the gauge condition, the residual gauge group is \( \mathcal{G}' = \mathbb{Z}^r \times_1 \mathcal{P}_G / H_G = \mathbb{Z}^r \times_1 S_N \) (see appendix).

5. Our result above allows us to determine the space of all gauge orbits explicitly: Let \( \mathcal{A}_0 \) be the set of all \( A \in \mathcal{A} \) obeying the gauge condition (23). Then obviously the orbit space \( \mathcal{A}/\mathcal{G}' \) is (generically) identical to \( \mathcal{A}_0 / \mathcal{G}' = (\mathcal{A}_0 / \mathbb{Z}^d) / (\mathbb{Z}_2^{N-1} \times \mathbb{Z}_2) \). \( \mathcal{A}_0 \) is a manifold, but as there are field configurations in \( \mathcal{A}_0 \) that are fixed points of \( \sigma \) (20), \( \mathcal{A}_0 / \mathcal{G}' \) is only an orbifold, and thus has singularities which cannot be removed by a mere choice of coordinates. Thus some of the elements in the residual gauge group, or in other words some of the Gribov copies, produce more than just coordinate singularities. This agrees with the results of Babelon and Viallet \cite{11}.

6. In the following, we discuss Yang-Mills theory on space-time \( T^d \times \mathbb{R} \) in the Hamiltonian framework. In this case, the consistency of the gauge fixing conditions can also be seen from the Gauss’ law \( G = \sum_{i=1}^d (-i \partial_i E_i + [A_i, E_i]) + \rho = 0 \) (with \( \rho \) the temporal component of some matter current as usual).

To avoid clumsy notation, the following discussion is given for \( d = 2 \); the extension to \( d > 2 \) is again trivial. Introducing the notation

\[
X(n_1, x^2) = \int_0^{2\pi} \frac{dy_1}{2\pi} e^{-in_1 y_1^1} X(y_1^1, x^2)
\]

\[
X(n_1, n_2) = \int_0^{2\pi} \frac{dy_2}{2\pi} e^{-in_2 y_2^2} X(n_1, y_2^2)
\]

and using the basis \( \{ H_j, E^{\pm j} \} \) in \( g \) introduced above, the gauge condition (3) allows to write the Gauss’ law as

\[
i \left( n_1 \pm \sum_i Y_i^1(x^2) a_{ij} \right) E_1^{\pm j}(n_1, x^2) + \partial_2 E_2^{\pm j}(n_1, x^2) + \rho^{\pm j}(n_1, x^2) = 0
\]

\[
\text{for } j = 1, 2, \ldots, N - 1 \tag{26}
\]
with

\[ \rho = \rho + [A_2, E_2]. \] (27)

These equations can be solved (in the generic case),

\[
E^{0, j}_1(n_1, x^2) = -\frac{\tilde{\rho}^{0, j}(n_1, x^2) - i\partial_2 E^{0, j}_2(n_1, x^2)}{\imath n_1} \quad \forall n_1 \neq 0
\]

\[
E^{\pm, j}_1(n_1, x^2) = -\frac{\tilde{\rho}^{\pm, j}(n_1, x^2) - i\partial_2 E^{\pm, j}_2(n_1, x^2)}{\imath (n_1 \pm \sum_i Y_i(x^2)a_{ij})} \quad \forall n_1 \in \mathbb{Z},
\] (28)

and determine those and only those components of \( E_1 \) conjugate to components of \( A_1 \) set to zero by the gauge condition (5). This shows that (5) is a consistent gauge condition.

By now we have taken into account all components of the Gauss’ law except \( G^{0, j}(n_1 = 0, x^2) = 0 \). Using the gauge condition (14), we can write the latter as

\[
in_2 E^{0, j}_2(n_1 = 0, n_2) + \tilde{\rho}^{0, j}(n_1 = 0, n_2) = 0,
\] (29)

which can be solved

\[
E^{0, j}_2(n_1 = 0, n_2) = -\frac{\tilde{\rho}^{0, j}(n_1 = 0, n_2)}{\imath n_2} \quad \forall n_2 \neq 0,
\] (30)

thus showing that also (14) is consistent. We are finally left with the Gauss’ law component \( G^{0, j}(n_1 = 0, n_2 = 0) \). Using (4), this can be written as

\[
\frac{1}{2} N(N-1) \sum_{k=1}^{\frac{1}{2}} c_{kj} \left[ E^{+, k}_2 A^{-k}_2 - E^{-k}_2 A^{+, k}_2 \right] (n_1 = 0, n_2 = 0) + \rho^{0, j}(n_1 = 0, n_2 = 0)
\]

\[
\equiv \left( E^{+, j}_2(0, 0) A^{-j}_2(0, 0) - E^{-j}_2(0, 0) A^{+, j}_2(0, 0) \right) + \{\cdots\} = 0
\] (31)

(we used \( c_{kj} = \delta_{kj} \) for \( k, j = 1, 2, \ldots, N - 1 \) where \( \{\cdots\} \) does not depend on \( E(A)^{\pm, j}_2(0, 0) \equiv E(A)_2^{\pm, j}(n_1 = 0, n_2 = 0) \). With the gauge condition (21), i.e. \( A^{+, j}_2(0, 0) = A^{-j}_2(0, 0) \), this is (in the generic case) easily solved for the component of \( E_2 \) conjugate to \( A^{+, j}_2(0, 0) - A^{-j}_2(0, 0) \),

\[
E^{+, j}_2(0, 0) - E^{-j}_2(0, 0) = -\frac{\{\cdots\}}{A^{+, j}_2(0, 0)}
\] (32)

showing that also (21) is consistent.

One can insert (28), (30) and (32) into the Hamiltonian for Yang-Mills theory coupled to matter. Similarly as discussed in 1 for \( d = 1 \), one thereby obtains the Hamiltonian in the restricted Hilbert space \( \mathcal{H}' \) obtained by restricting the Yang-Mills configurations to \( \mathcal{A}_0 \), i.e. imposing (3), (14) and (21). On \( \mathcal{H}' \) we still are left with a representation of the residual gauge group \( \mathcal{G}' \). As the latter is a discrete group and has a convenient semi-direct product structure, one can explicitly construct all states in \( \mathcal{H}' \) which are invariant under \( \mathcal{G}' \), and it is this set of states which span the physical Hilbert space \( \mathcal{H}_{\text{phys}} \) of the model (4).

It is interesting to note that in the present approach the non-generic field configurations obeying at least one of the conditions (13) and (24) appear as singularities in the eqs. (28), (32). As shown
in detail in Ref. [8] for $d = 1$ (see eq. (18)), they lead to singularities in the matter ($\rho-\rho$) interactions on the physical Hilbert space $\mathcal{H}_{\text{phys}}$ of the model. As these singular interactions are repulsive, we believe that they are not ‘dangerous’ but an intrinsic feature of non-Abelian Yang-Mills theories. 

This is supported by the fact that the effective Hamiltonian for Yang-Mills theory on a cylinder coupled to an external matter current $\rho(x) = \rho\delta(x^1)$ reduces to a completely integrable Sutherland Hamiltonian $^6$ 

$$H = -\frac{1}{8\pi} \sum_{j,k} b_{jk} \frac{\partial^2}{\partial Y^j_1 \partial Y^k_1} + \frac{\pi}{2} \sum_j \left(\frac{\rho^{+j} \rho^{-j}}{\sin^2(\pi \sum_i a_{ij})} + \frac{\rho^{-j} \rho^{+j}}{\sin^2(\pi \sum_i Y^1_i a_{ij})}\right)$$  

(33)

(we set the coupling constant to 1) where $b_{kl} = \text{tr}(H_j H_k)$ [8] (insert into eq. (18) of Ref. [8] and use the formula $\sum_{n \in \mathbb{Z}} 1/(n - y)^2 = \pi^2 / \sin^2(\pi y)$; to see for $G = U(N)$ that this is indeed a Sutherland Hamiltonian $^6$ use the basis $(H_i)_{kl} = \delta_{ik} \delta_{ij}$. For $G = SU(N)$ one can also use this basis leading to the same result with the additional constraint $\sum_{j=1}^N Y^j_1 = 0$ which just fixes the ‘center-of-mass motion’ of the $Y^1_i$’s). This Hamiltonian is positive, and it is self-adjoint in spite of the singularities $^6$.

7. We have given a complete gauge fixing procedure for $U(N)$ and $SU(N)$ gauge theories on tori $T^d$. It is similar to the axial gauge but does not fix physical degrees of freedom. By complete we mean that the residual gauge group is generically discrete. It would be interesting to find a similar procedure for other manifolds $M^d$, e.g. $M^d = S^d$. In Ref. [4] it is proven that on spheres there is no gauge condition with a residual gauge group which is discrete and the same for all gauge field configurations obeying the condition. We note that our result would not contradict a similar theorem on tori due to the existence of the non-generic field configurations with larger residual gauge groups containing continuous subgroups. One hopes that in the limit in which all linear dimensions of a manifold become infinite, one obtains the same ‘thermodynamic limit’ for any reasonable manifold. We note that the proper treatment of the axial gauge is even non-trivial on an infinite space $\mathbb{R}^d$ [14]. In our opinion, even in that case the axial gauge cannot be fixed completely, the reason being basically the same as for the case we have discussed.

The gauge transformation $U$ that maps every $A$ to its representative obeying our gauge condition is a product of transformations $U_1 \cdot \ldots \cdot U_d$ applied consecutively. The map $U_1$ diagonalizes the Wilson–loops in $x^1$–direction and is similar in spirit to a rewriting of the theory in terms of loop variables $^6$, but we do not try to give a complete reformulation of the theory in terms of the latter. This partial rewriting seems advantageous because the Jacobian of the complete change of variables has not been calculated, whereas in our case it can be calculated as the product of $d$ Jacobians.

For pure Yang-Mills theory not only periodic functions $[2, \pi]^d \to G$ correspond to gauge transformations, but in fact also all those which are periodic only up to an element in the center $\mathbb{Z}_G$.

\footnotesize

$^5$we thank E.Seiler for discussions about this

$^6$E.L. is grateful to A. Gorsky for a discussion on this point
However, these latter transformations must not be fixed as they correspond to physical symmetries of the system which may be broken, and our construction applies also to this case.

**APPENDIX**

In this appendix we (a) explicitly construct the Peter-Weyl group

$$P_G = \{ p \in G \mid \forall h \in \mathfrak{h} : p^{-1}hp \in \mathfrak{h} \} \quad (A1)$$

and (b) prove its properties needed for the final gauge reduction in the eqs. following (21).

(a) As $U(N) = U(1) \times SU(N)$ implies $P_{U(N)} = U(1) \times P_{SU(N)}$, it suffices to consider $G = SU(N)$. Then $\mathfrak{h}$ consists of all traceless diagonal real matrices $h = \text{diag}\{h_1, \ldots, h_N\}$ (and the $h_i$ are the eigenvalues of $h$). Let $p \in P$ and take any $h \in \mathfrak{h}$ such that all $h_i$ are different. Since $h = p\hat{h}$ and $\hat{h}$ have the same eigenvalues, there is a permutation $\pi$ of $(1, 2, \ldots, N)$ such that $\hat{h} = \hat{h}p\hat{\pi}^{-1}$ with $\hat{\pi} \in SU(N)$ representing $\pi$, $(\hat{\pi})_{ij} = e^{i\pi/N\text{deg}(\pi)}\delta_{i,\pi(j)}$ and $\text{deg}(\pi) = 0$ for $\pi$ even and 1 otherwise. So $H = \hat{\pi}^{-1}p$ commutes with $h$ and thus is diagonal. Since $p$ and $\hat{\pi}$ are in $SU(N)$, $\det H = 1$ and so $H \in \mathfrak{h}$, and $p$ has the decomposition $p = \hat{\pi}H$. As obviously $\hat{\pi}^{-1}H\hat{\pi} \in \mathfrak{h} \forall H \in \mathfrak{h}$, $\pi \in S_N$, and since the representation $\pi \mapsto \hat{\pi}$ is faithful,

$$PSU(N) = H_{SU(N)} \times S_N \quad (A2)$$

(b) By definition of $E_i^\pm$, (A9) is equivalent to

$$(p^{-1}Xp)_{i,i+1} = (p^{-1}Xp)_{i+1,i} \text{ for } i = 1, 2, \ldots N - 1 \quad (A3)$$

for $X \in g$ and $p \in P$. $X$ is hermitean and $p$ unitary, so (A3) is a reality condition on either side. Since by (a) $P$ is a semidirect product, every $p \in P$ has a unique decomposition $p = H\hat{\sigma}$ with $H \in \mathfrak{h}$ and $\sigma \in S_N$, and we can solve (A3) for $H$ and $\sigma$ separately. $X \in g$ is arbitrary, so we can assume $X_{i,i+1} \neq 0$ for all $i$, thus $X_{k,k+1} = R_k e^{i\xi_k}$ with $R_k > 0$ and $\xi_k \in \mathbb{R}$. Inserting $H_{ij} = \delta_{ij}e^{i\alpha_i}$ into (A3) gives

$$\exp(2i(\xi_i - \alpha_i + \alpha_{i+1})) = 1 \text{ for } i = 1, 2, \ldots N - 1 \quad (A4)$$

which amounts to the recursion $\alpha_{k+1} = \alpha_k - \xi_k + m_k\pi$ with $m_k \in \{0, 1\}$. Denoting

$$\alpha_i^{(0)} = \frac{1}{N} \sum_{i=1}^{N-1} (N - i)\xi_i, \quad \alpha_i^{(0)} = \alpha_1 - \sum_{l=1}^{i-1} \xi_l \quad \text{for } i \in \{2, 3, \ldots N\} \quad (A5)$$

all $p \in H$ obeying (A3) are given by $p = H_0z\zeta$ with

$$(H_0)_{ij} = \delta_{ij} \exp(\lambda i\alpha_i^{(0)}) \quad (A6)$$

and

$$z_{ij} = \delta_{ij} e^{i\pi\ell_i} \text{ with } \ell_1 = 0, \ell_i \in \{0, 1\} \text{ for } i \geq 2 \text{ and } \lambda = \exp\left(-i\pi/N\sum_{k=2}^{N} \ell_k\right) \quad (A7)$$
and ζ ∈ Z_N for G = SU(N), ζ ∈ U(1) for G = U(N). σ ∈ S_N is determined by \((\hat{\sigma}^{-1} Z \hat{\sigma})_{i,i+1} = (\hat{\sigma}^{-1} Z \hat{\sigma})_{i+1,i}\) for Z_{i,i+1} = Z_{i+1,i}. Taking Z_{i,j+1} \neq Z_{j+1,i} for any \((i,j)\) with \(i \neq j\), gives the condition \(\sigma(i + 1) = \sigma(i) \pm 1\), the only two solutions of which are \(\sigma(i) = i\) and \(\sigma(i) = N + 1 - i\). We conclude that all \(p \in P\) obeying (A3) are given by \(p = H_{0}z\zeta\) where \(z\) is in the subgroup of \(P\) generated by \(\hat{\sigma}\) and \(\zeta\) is in the center of the group. We denote this latter group as \(\mathbb{Z}_{SU(N)}\).

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