Long time and Painleve-type asymptotics for the Sasa-Satsuma equation in solitonic space time regions

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Abstract

The Sasa-Satsuma equation with $3 \times 3$ Lax representation is one of the integrable extensions of the nonlinear Schrödinger equation. In this paper, we consider the Cauchy problem of the Sasa-Satsuma equation with generic decaying initial data. Based on the Riemann-Hilbert problem characterization for the Cauchy problem and the $\partial$-nonlinear steepest descent method, we find qualitatively different long time asymptotic forms for the Sasa-Satsuma equation in three solitonic space-time regions:

1. For the region $x < 0, |x/t| = O(1)$, the long time asymptotic is given by

$$q(x,t) = u_{sol}(x,t|\sigma_d(I)) + t^{-1/2}h + O(t^{-3/4}).$$

in which the leading term is $N(I)$ solitons, the second term the second $t^{-1/2}$ order term is soliton-radiation interactions and the third term is a residual error from a $\partial$ equation.

2. For the region $x > 0, |x/t| = O(1)$, the long time asymptotic is given by

$$u(x,t) = u_{sol}(x,t|\sigma_d(I)) + O(t^{-1}).$$

in which the leading term is $N(I)$ solitons, the second term is a residual error from a $\partial$ equation.

3. For the region $|x/t^{1/3}| = O(1)$, the Painleve asymptotic is found by

$$u(x,t) = \frac{1}{t^{1/3}}u_P\left(\frac{x}{t^{1/3}}\right) + O\left(t^{2/(3p)-1/2}\right), \quad 4 < p < \infty,$$

in which the leading term is a solution to a modified Painleve II equation, the second term is a residual error from a $\partial$ equation.

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Keywords: Sasa-Satsuma equation; Riemann-Hilbert problem; $\bar{\partial}$ steepest descent analysis; soliton resolution; Painleve equation.

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1 Introduction

In this paper, we focus on the long time asymptotics of the Cauchy problem for the Sasa-Satsuma equation [1]

\[ u_t + u_{xxx} + 6|u|^2u_x + 3u(|u|^2)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \quad (1.1) \]
\[ u(x,0) = u_0(x) \in \mathcal{S}(\mathbb{R}), \quad (1.2) \]

where initial data \( u_0(x,0) \) belongs to the Schwarz space \( \mathcal{S}(\mathbb{R}) \). The Sasa-Satsuma equation is one of the integrable extensions of the nonlinear Schrödinger (NLS) equation, and it plays an important role in a number of physical science areas due to its rich mathematical structure and physical background such as deep water waves [2] and dispersive nonlinear media [3]. In the past few years, Sasa-Satsuma equation has attracted much attention and widely been studied. Early in 1991, Sasa and Satsuma present the Sasa-Satsuma equation as a new-type high-order NLS equation. It was shown that the Sasa-Satsuma equation is solvable by the means of inverse scattering transformation (IST), and the soliton solution can propagate steadily [1]. In 2003, Gilson et al studied the bilinearization and multisoliton solutions of the Sasa-Satsuma equation [4]. In addition, the squared eigenfunctions of Sasa-Satsuma equation has been derived via the Riemann-Hilbert (RH) method [5]. In 2007, Sergyeyev and Demskoi studied the recursion operator and nonlocal symmetries of Sasa-Satsuma and the complex sine-Gordon II equation [6]. Zhai and Geng constructed finite gap solutions of the Sasa-Satsuma equation via the algebro-geometric method [11]. The initial-boundary value problem for the Sasa-Satsuma equation on the half line was investigated by using the Fokas method [10]. Other important properties such as infinite conservation laws, the Painlevé property and rogue wave etc may refer to [7–9].

The IST plays an important role in the study of integrable systems. It can exactly solve the integrable systems under reflectionless potentials condition. The study on long-time asymptotic behaviors is a formidable challenge in integrable system. The nonlinear steepest descent method for RH problem developed by Deift and Zhou made an indelible contribution to asymptotic analysis of integrable systems [12]. In 1997, Deift and Zhou studied the long-time behaviors of the Cauchy problem of small dispersion KdV by generalizing the steepest descent method [13]. In 2002, Vartanian considered the long-time asymptotic behaviors of defocusing NLS equation with finite-density initial data [14]. In addition, de Monvel has studied the long-time behaviors of WKI-type short-pulse equation with sufficiently decaying initial conditions.
A powerful $\partial$-steepest descent method was first introduced by McLaughlin and Miller to analyze the asymptotics of orthogonal polynomials [16, 17]. In recent years, this method has been successfully used to obtain the long-time asymptotics and the soliton resolution conjecture for some integrable systems [18, 20–23]. But most work was concentrated on integrable systems with $2 \times 2$ Lax pair such as Camassa-Holm, short-pulse etc by using the nonlinear steepest descent analysis method [23–27].

In recent years, de Monvel et al studied long-time asymptotic analysis for the Degasperis-Procesi and Novikov equations with $3 \times 3$ Lax pairs [28, 29]. For Schwartz initial data and without consideration of solitons, Liu, Geng and Xue applied Deift-Zhou nonlinear steepest descent method to study long time asymptotics for the Sasa-Satsuma equation with $3 \times 3$ Lax pair [30]. Huang and Lenells further found modified Painleve asymptotic for the Sasa-Satsuma equation without soliton region [31].

In this article, we consider the effect of solitons on the long time asymptotics of the Sasa-Satsuma equation. We extend the $\partial$-steepest descent method to the Sasa-Satsuma equation and find qualitatively different long time asymptotic forms in three solitonic space-time regions I, II and III, see Figure 1. Here we list our main results obtained in this paper.

- **In the region I:** $u(x,t) = u_{sol}(x,t|\sigma_d(I)) + t^{-1/2}h + O(t^{-3/4})$.
- **In the region II:** $u(x,t) = u_{sol}(x,t|\sigma_d(I)) + O(t^{-1})$.
- **In the region III:** $u(x,t) = \frac{1}{t^{1/3}}u_p \left( \frac{x}{t^{1/3}} \right) + O \left( t^{-2/(3p) - 1/2} \right)$, $4 < p < \infty$.

![Figure 1: Three space-time regions. Region I: Oscillatory region, $x < 0$, $|x/t| = O(1)$; Region II: Soliton region, $x > 0$, $|x/t| = O(1)$; Region III: Painleve region, $|x/t|^{1/3} = O(1)$.](image)

The organization of this work is as follows: In Section 2, we quickly recall direct scattering theory on the Sasa-Satsuma equation, such as analyticity, symmetries and asymptotics for the eigenfunctions of Lax pair. Further the the Cauchy problem of the Sasa-Satsuma equation (1.1) is characterized with a RH problem. For detailed, my refer to [30, 31]. In Section 3, we consider
the space-time Region I: $x < 0, |x/t| = O(1)$, in which there two phase points on the real axis. We introduce a series of effective transformation to transfer the RH problem into some solvable model problems, which allow us to set up the long-time asymptotic behaviors of the Sasa-Satsumo equation in Region I. In Section 4, we consider the space-time Region II: $x > 0, |x/t| = O(1)$, in which there are two phase points on the imaginary axis. In this case, the long-time asymptotic result is similar with that in Region I. The difference is that the contribution from the jump contours can be decayed exponentially as $t \to +\infty$ in the Region II. Finally, in Section 5, we self-similar region or Painlevé Region III: $|x/t^{1/3}| = O(1)$. In this case, the contribution from the solitons can be decayed exponentially and we only need to consider the influence caused by the jump condition and pure $\overline{\sigma}$-problem. Furthermore, the RH problem corresponding to the jump condition can be transformed into a solvable modified Painlevé-II model. Based on these results, we can easily obtain the long time asymptotic behaviors of the solutions in self-similar region. As byproducts of large-time asymptotic analysis, we verify that the soliton solutions of Sasa-Satsumo equation are asymptotically stable.

2 Direct scattering transformation

2.1 Spectral analysis on Lax pair

As we konw, Sasa-Satsumo equation is equivalent to the compatible condition of the following lax pair:

\[ \Phi_x = (-ik\sigma + U)\Phi, \quad \Phi_t = (-4ik^3\sigma + \overline{U})\Phi, \]  

(2.1)

where $\Phi(k, x, t)$ is a matrix-valued function of $k, x, t$ and $k \in \mathbb{C}$ is the spectral parameter,

\[ \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 \\ -q^+ \\ 0 \end{pmatrix}, \quad q = (u, u^*)^T, \]

(2.2)

\[ \overline{U} = 4k^2U + 2ik\sigma(U_x - U^2) + 2U^3 - U_{xx} + [U_x, U], \]

where "$^\ast$" and "$^\dagger$" denote the complex conjugation and Hermite transformation, respectively. In what follows, we introduce unknown function

\[ \Psi(k, x, t) = \Phi(k, x, t)e^{i(kx + 4k^3t)}\sigma, \]

which satisfies the following equation

\[ \Psi_x = -ik[\sigma, \Psi] + U\Psi, \quad [\sigma, \Psi] = \sigma\Psi - \Psi\sigma. \]  

(2.3)
The above equation has a pair of Jost solutions $\Psi_+(k, x, t)$ and $\Psi_-(k, x, t)$ which can be written as follows:

$$\Psi_+(k, x, t) = I - \int_x^{+\infty} e^{i(k-y)x} U(y, t) \Psi_+(k, y, t) dy,$$

$$\Psi_-(k, x, t) = I + \int_x^{-\infty} e^{i(k-y)x} U(y, t) \Psi_-(k, y, t) dy. \quad (2.4)$$

If we rewrite $\Psi_\pm$ as block form $\Psi_{\pm L}(k, x, t), \Psi_{\pm R}(k, x, t)$, where $\Psi_{\pm L}(k, x, t)$ and $\Psi_{\pm R}(k, x, t)$ represent the first two column and third column of $\Psi_\pm$, respectively. By analyzing the expression of $\Psi_\pm$, we obtain that $\Psi_{-L}, \Psi_{+R}$ are analytic in $C_+$, and $\Psi_{+L}, \Psi_{-R}$ are analytic in $C_-$. Moreover,

$$(\Psi_{-L}(k, x, t), \Psi_{+R}(k, x, t)) = I + O \left( \frac{1}{k} \right), \quad k \in C_+ \to \infty, \quad (2.6)$$

$$(\Psi_{+L}(k, x, t), \Psi_{-R}(k, x, t)) = I + O \left( \frac{1}{k} \right), \quad k \in C_- \to \infty. \quad (2.7)$$

According to the fact that the trace of $U$ equals zero and Abel lemma, we have that $\det \Psi_\pm(k, x, t)$ are independent of variable $x$ and $\det \Psi_\pm = 1$. Furthermore, since $\Psi_\pm e^{-i(kx+4k^3t)}$ satisfy the same first order linear differential equation, they are linearly dependent, i.e. there exists an $x,t$-independent transition matrix $S(k)$, such that

$$\Psi_-(k) = \Psi_+(k)e^{-i(kx+4k^3t)}S(k), \quad \det S = 1. \quad (2.8)$$

According to the symmetries

$$U^\dagger = -U, \quad \zeta U \zeta = U^*, \quad \zeta = \zeta^{-1} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.9)$$

we obtain that $\Psi_\pm$ and $S(k)$ satisfy

$$\Psi_\pm(k^*; x, t) = \Psi_\mp^{-1}(k; x, t), \quad \Psi_\pm(k; x, t) = \zeta \Psi_\mp(-k^*, x, t) \zeta, \quad (2.10)$$

$$s^\dagger(k^*) = s^{-1}(k), \quad s(k) = \zeta s^*(-k^*) \zeta. \quad (2.11)$$

From the above symmetries of the scattering matrix, we can rewrite $s(k)$ as following form

$$s(k) = \begin{pmatrix} a(k) & -\text{adj}[a^+(k^*)]b^+(k^*) \\ b(k) & \text{det}[a^+(k^*)] \end{pmatrix}, \quad (2.12)$$

where $a(k)$ denotes a $2 \times 2$ matrix, $\text{adj}[a^+(k^*)]$ represents the adjoint matrix of matrix $a^+(k^*)$, and $b(k)$ is a 2 dimensional row vector. Moreover, the continuous spectrum $a(k)$ and $b(k)$
satisfy $a(k) = \sigma_1 a^*(-k^*) \sigma_1$ and $b^*(-k^*) \sigma_1 = b(k)$. Next, evaluating equation (2.8) at $t = 0$, we find

$$s(k) = \left( \begin{array}{c} a(k) \\ b(k) \end{array} \right) = \lim_{x \to \infty} e^{ikx} \Psi_-(k; x, 0).$$

(2.13)

Furthermore, we note that

$$a(k) = I + \int_{-\infty}^{+\infty} q(x, 0) \Psi_-(k, x, 0) dx,$$

(2.14)

$$b(k) = -\int_{-\infty}^{+\infty} e^{-2ikx} q^+(x, 0) \Psi_+(k, x, 0) dx,$$

(2.15)

from the analyticity of $\Psi(k, x, t)$, we have $a(k)$ is analytic in $C_+$.

### 2.2 A Riemann-Hilbert problem

Suppose that $a(k)$ has $2N$ simple zeros $k_1, \ldots, k_{2N}$ in $C_+$. Since $a(k) = \sigma_1 a^*(-k^*) \sigma_1$, we note that the simple zeros satisfy $k_{N+j} = -k_j^*, j = 1, \ldots, N$. We define the following sectionally meromorphic matrix

$$M(k; x, t) = \begin{cases} 
\left( \begin{array}{c} \Psi_-(k)a(k)^{-1}(k), \Psi_+(k) \end{array} \right), & k \in C_+, \\
\left( \begin{array}{c} \Psi_+(k), \Psi_-(k) \end{array} \right), & k \in C_-,
\end{cases}$$

(2.16)

which has $2N$ simple poles $\mathcal{K} = \{k_j, j = 1, \ldots, 2N\}$ in $C^+$ and $2N$ simple poles $\overline{\mathcal{K}} = \{k_j^*, j = 1, \ldots, 2N\}$ in $C^-$. Besides, matrix-valued function $M(k; x, t)$ satisfies the following Riemann-Hilbert problem:

**RHP1.** Find a matrix-valued function $M(k) = M(k; x, t)$ which solves:

(a) Analyticity: $M(k)$ is meromorphic in $C \setminus \mathbb{R}$ and has simple poles at $k_j \in \mathcal{K}$ and $k_j^* \in \overline{\mathcal{K}}$;

(b) Jump condition: $M(k)$ has continuous boundary values $M_{\pm}(k)$ on $\mathbb{R}$ and

$$M_+(k) = M_-(k) J(k), \quad k \in \mathbb{R},$$

(2.17)

where

$$J(k) = \left( \begin{array}{cc} I + \gamma^+(k^*) \gamma(k) & \gamma^+(k^*) e^{-2it\theta} \\
\gamma^+(k) & 1 \end{array} \right).$$

(2.18)

(c) Asymptotic behaviors:

$$M(k; x, t) = I + O \left( \frac{1}{k} \right), \quad k \to \infty;$$

(2.19)
(d) Residue conditions: \( M_2(k;x,t) \) has simple poles at each point in \( \mathcal{X} \cup \mathcal{X}^c \) with:

\[
\text{Res}_{k = k_j} M(k) = \lim_{k \to k_j} M(k) \begin{pmatrix} 0 & 0 \\ c_j e^{2i\theta(k)} & 0 \end{pmatrix}, \\
\text{Res}_{k = k_j^\ast} M(k) = \lim_{k \to k_j^\ast} M(k) \begin{pmatrix} 0 & -c_j^\ast e^{-2i\theta(k)} \\ 0 & 0 \end{pmatrix},
\]

where

\[
c_j = \frac{b(k_j) \text{adj}[a(k_j)]}{\det[a(k_j)]}, \quad c_j^\ast = \frac{\text{adj}[a^\dagger(k_j)]b^\dagger(k_j^\ast)}{\det[a^\dagger(k_j)]}, \quad \gamma(k) = \gamma^\ast(-k^\ast)\sigma_2,
\]

\[
\theta = \frac{kx}{t} + 4k^3, \quad \gamma(k) = b(k)a^{-1}(k), \quad j = 1, \ldots, 2N
\]

Substituting the asymptotic form of \( M_2(k;x,t) \) into equation \((2.3)\), we find

\[
q(x,t) = (u(x,t), u^\ast(x,t))^T = 2i \lim_{k \to \infty} (kM(k;x,t))_{12},
\]

where \( u(x,t) \) is the solution of Sasa-Satsuma equation \((1.1)\).

3 Long time asymptotics in region I: \( x < 0, |x/t| = O(1) \)

3.1 Deformation of the RH problem

3.1.1 Conjugation

In this subsection, we first consider the oscillatory term

\[
e^{2i\theta(k)} = e^{2i(4k^3 + \frac{x}{12t})}, \quad \theta(k) = 4(k^3 - 3k_0^2k),
\]

one will observe that the long time asymptotic of RHP 1 is influenced by the growth and decay of the oscillatory term. In this section, our aim is to introduce a transform of \( M(k;x,t) \) \( \to M^{(1)}(k;x,t) \) so that \( M^{(1)}(k;x,t) \) is well behaved as \(|t| \to \infty\) along the characteristic line.

The \( \theta(k) \) admits phase points \( \pm k_0 = \pm \sqrt{-\frac{x}{12t}} \). We mainly focus on the physically interesting region \( 0 < k_0 \leq C \), where \( C \) is a constant. From \((3.1)\), we have

\[
\text{Re}i\theta(k) = 4 \left( \text{Im}^2 k - 3\text{Re}^2 k + 3k_0^2 \right) \text{Im} k,
\]

and the signature table of \( \text{Re}i\theta(k) \) is shown in Figure 3.
In order to analyze the long time asymptotic behavior of RHP, we first divide the all poles into two parts:

\[
\Delta^- = \left\{ k \mid 3Re^2k - Im^2k < 3k_0^2 \right\}, \quad \Delta^+ = \left\{ k \mid 3Re^2k - Im^2k > 3k_0^2 \right\}.
\]

Due to technical problems, we assume that there are no poles corresponding to the region \(\Delta^-\) in subsequent studies of all space-time region I, II, III.

Based on the signature table of \(Im\theta(k)\) and the above partition, we can factorize the jump matrix \(J(k, x, t)\) as two different forms:

\[
J(k, x, t) = \begin{cases} 
\begin{pmatrix} I & e^{-2it\theta}(k^*) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{2it\theta}(k) & 0 \\ 0 & 1 \end{pmatrix}, & k \in (-\infty, -k_0) \cup (k_0, +\infty), \\
\begin{pmatrix} I & \gamma(k) \\ 1+\gamma(k) & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1+\gamma(k)^\dagger(k^*) & 0 \end{pmatrix} \begin{pmatrix} I & e^{-2it\theta}(k^*) \\ 0 & 1 \end{pmatrix}, & k \in (-k_0, k_0),
\end{cases}
\]

We introduce \(2 \times 2\) matrix function \(\delta(k)\) which satisfy the matrix RHP

\[
\begin{align*}
\delta_+(k) &= \delta_-(k)(I + \gamma(k)^\dagger(k^*)\gamma(k)), & k \in (-k_0, k_0), \\
\delta(k) &= I + O\left(\frac{1}{k}\right), & k \to \infty.
\end{align*}
\]

Take the determinant on both sides of the above equation, therefore,

\[
\begin{align*}
\det \delta_+(k) &= \det \delta_-(k)(I + |\gamma(k)|^2), & k \in (-k_0, k_0), \\
\det \delta(k) &= 1 + O\left(\frac{1}{k}\right), & k \to \infty.
\end{align*}
\]
Combining with the fact that $I + \gamma^+(k^+)\gamma(k)$ is positive definite, the vanishing lemma makes sure of the existence and uniqueness of $\delta(k)$. Moreover, through the Plemelj formula, we get

$$\det \delta(k) = \left( \frac{k - k_0}{k + k_0} \right)^{i\nu} e^{\chi(k)},$$

where

$$\nu = -\frac{1}{2\pi} \log \left( 1 + |\gamma(k_0)|^2 \right),$$

$$\chi(k) = \frac{1}{2\pi i} \int_{k_0}^{k} \log \left( \frac{1 + |\gamma(\xi)|^2}{1 + |\gamma(k_0)|^2} \right) \frac{d\xi}{\xi - k}.$$ (3.7)

By symmetry and uniqueness, we obtain that

$$\delta(k) = \sigma_2 \delta^*(-k^*) \sigma_2 = (\delta^+(k^+))^{-1}.$$ (3.10)

Moreover, by direct calculation and the maximum principle, we have

$$|\delta(k)| \lesssim 1, \quad |\det \delta(k)| \lesssim 1, \quad k \in \mathbb{C},$$ (3.11)

where $|A| = (\text{tr}A^+A)^{1/2}$ denotes the Frobenius norm for any matrix $A$. Next, we summarize the properties of $\delta(k)$ and $\det \delta(k)$ as follows:

**Proposition 1.** The matrix function $\delta(k)$ and scalar function $\det \delta(k)$ satisfy the following properties

(a) $\delta(k)$ and $\det \delta(k)$ are analytic in $\mathbb{C} \setminus [-k_0, k_0]$.

(b) For $k \in \mathbb{C} \setminus [-k_0, k_0]$, $\delta(k)\delta^+(k^+) = I$, $\det \delta(k) \det \delta^+(k^+) = 1$;

(c) For $k \in (-k_0, k_0)$,

$$\delta_+(k) = \delta_-(k)(1 + \gamma^+(k)\gamma(k)), \quad \det \delta_+(k) = \det \delta_-(k)(1 + |\gamma(k)|^2);$$ (3.12)

(d) As $|k| \to \infty$ with $|\arg(k)| \leq c < \pi$,

$$\det \delta(k) = 1 + \frac{i}{k} \left[ -2\nu k_0 + \frac{1}{2\pi} \int_{-k_0}^{k_0} \log \left( \frac{1 + |\gamma(\xi)|^2}{1 + |\gamma(k_0)|^2} \right) d\xi \right] + O(k^{-2});$$ (3.13)

(d) Along the ray $k = \pm k_0 + e^{i\phi} R_+$ where $|\phi| \leq c < \pi$, as $k \to \pm k_0$

$$|\det \delta(k) - \left( \frac{k - k_0}{k + k_0} \right)^{i\nu} e^{\chi(\pm k_0)}| \lesssim |k \mp k_0|^{1/2}. $$ (3.14)
Proof. The proof of above properties is similar to the proof of Proposition 3.1 provided by Borghese et al. \(\square\)

Now we define
\[
T(k) = \begin{pmatrix} T_1^{-1}(k) & 0 \\ 0 & T_2(k) \end{pmatrix} = \begin{pmatrix} \delta^{-1}(k) & 0 \\ 0 & \det \delta(k) \end{pmatrix},
\]
and introduce the matrix transformation
\[
M^{(1)}(k; x, t) = M(k; x, t)T(k),
\]
then \(M^{(1)}(k; x, t)\) solves the following RHP:

RHP2. Find a matrix-valued function \(M^{(1)}(K) = M^{(1)}(K; x, t)\) such that

(a) \(M^{(1)}(k)\) is analytic in \(\mathbb{C} \setminus (\mathbb{R} \cup \mathcal{S} \cup \overline{\mathcal{S}})\);

(b) \(M^{(1)}(k)\) has the following jump condition \(M^{(1)}_+(k) = M^{(1)}_-(k)V^{(1)}(k), \ k \in \mathbb{R}\), where
\[
V^{(1)}(k) = \begin{cases} 
\begin{pmatrix} I & T_1 T_2 \gamma^+(k)e^{-2i\theta(k)} \\ 0 & 1 \end{pmatrix} & k \in \mathbb{R} \setminus [-k_0, k_0], \\
\begin{pmatrix} I & 0 \\ \gamma(k) T_1^{-1} T_2^{-1} \gamma(k) & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ T_1 T_2 \gamma^+(k)e^{-2i\theta(k)} \\ \gamma(k) T_1^{-1} T_2^{-1} \gamma(k) & 1 \end{pmatrix} & k \in (-k_0, k_0),
\end{cases}
\]
(c) \(M^{(1)}(k) = I + O(k^{-1})\), as \(k \to \infty\);

(d) \(M^{(1)}(k)\) satisfies the following residue conditions at double poles \(k_j \in \mathcal{S}\) and \(k^*_j \in \overline{\mathcal{S}}:\)
\[
\text{Res}_{k=k_j} M^{(1)}(k) = \lim_{k \to k_j} M^{(1)}(k) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad j \in \Delta^+,
\]
\[
\text{Res}_{k=k^*_j} M^{(1)}(k) = \lim_{k \to k^*_j} M^{(1)}(k) = \begin{pmatrix} 0 & -T_1 T_2 \gamma^+(k)e^{-2i\theta(k)} \\ 0 & 0 \end{pmatrix}, \quad j \in \Delta^+.
\]

3.1.2 A mixed \(\overline{\mathbb{S}}\)-RH problem

Next, in order to make continuous extension to the jump matrix \(V^{(1)}(k)\), we introduce new contours defined as follows:
\[
\Sigma_{1}^{\pm} = \pm k_0 + e^{\pm \frac{2\pi i}{3} \mathbb{N}} \mathbb{R}_+ \quad \Sigma_{3}^{\pm} = \pm k_0 + e^{\pm \frac{2\pi i}{3} h}, \quad h \in (0, \sqrt{2}k_0),
\]
\[
\Sigma_{2}^{\pm} = \pm k_0 + e^{\pm \frac{2\pi i}{3} \mathbb{N} \mathbb{R}_+} \quad \Sigma_{4}^{\pm} = \pm k_0 + e^{\pm \frac{2\pi i}{3} h}, \quad h \in (0, \sqrt{2}k_0),
\]
Therefore, the complex plane $\mathbb{C}$ is divided into eight open domains. Naturally, we apply $\partial$-stepest descent method to extend the scattering data into eight regions so that the matrix function has no jumps on $\mathbb{R}$. Denote these regions according to symmetry relation by $\Omega_j^\pm, j = 1, 2, 3, 4$ and $\Omega_5, \Omega_6$, which are shown in Figure 3.

![Figure 3: Deformation from $\mathbb{R}$ to the new contour $\Sigma$.](image)

Let
\[ \rho = \frac{1}{2} \min_{\lambda \neq \mu \in \mathcal{K} \cup \overline{\mathcal{K}}} |\lambda - \mu|, \quad (3.21) \]
for any $k_j \in \mathcal{K}$, since the discrete spectrum conjugate occur in pairs and not on $\mathbb{R}$, we get that $dist(\mathcal{K}, \mathbb{R}) \geq \rho$. In what follows, we introduce the characteristic function near the discrete spectrum
\[ \chi_{\mathcal{K}}(k) = \begin{cases} 1, & \text{dist}(k, \mathcal{K} \cup \overline{\mathcal{K}}) < \frac{\rho}{3}, \\ 0, & \text{dist}(k, \mathcal{K} \cup \overline{\mathcal{K}}) > \frac{2\rho}{3}, \end{cases} \quad (3.22) \]
In order to deform the contour $R(k)$ to the contour $\Sigma^{(2)}$, we make the following matrix transformation:
\[ M^{(2)}(k) = M^{(1)}(k) R^{(2)}(k), \quad (3.23) \]
where $R^{(2)}(k)$ is chosen to satisfy three specific properties:
- Compared to $M^{(1)}(k)$, $M^{(2)}(k)$ has no more jumps on $\mathbb{R}$;
The norm of $R^{(2)}(k)$ can be well controlled;

The transformation keep the residue conditions unchanged.

Based on the above analysis, we define $R^{(2)}$ as follows:

$$R^{(2)}(k) = \begin{cases} 
\begin{pmatrix} I & 0 \\ R^\pm_j e^{\pm 2i\theta(k)} & 1 \end{pmatrix}, & j = 1, 4, \\
\begin{pmatrix} I & R^\pm_j e^{-2i\theta(k)} \\ 0 & 1 \end{pmatrix}, & j = 2, 3, \\
\begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, & \text{otherwise}, 
\end{cases}$$

(3.24)

where vector function $R_j(k)$ are defined in following proposition:

**Proposition 2.** There exists a function $R^\pm_j : \overline{\Omega}_j \to C, j = 1, 2, 3, 4$ such that

$$R^\pm_1(z) = \begin{cases} 
-\gamma(k) T_1^{-1}(k) T_2^{-1}(k), & k \in I_\pm, \\
-\gamma(\pm k_0) T_1^{-1}(k) e^{-\mathcal{X}(\pm k_0)} \left( \frac{k - k_0}{k + k_0} \right)^{-i\nu} (1 - \mathcal{X}(\mathcal{X}(k))), & k \in \Sigma^\pm_1, 
\end{cases}$$

(3.25)

$$R^\pm_2(z) = \begin{cases} 
T_1(k) T_2(k) \gamma^+(k), & k \in I_\pm, \\
T_1(k) e^{\mathcal{X}(\pm k_0)} \left( \frac{k - k_0}{k + k_0} \right)^{i\nu} \gamma^+(\pm k_0) (1 - \mathcal{X}(\mathcal{X}(k))), & k \in \Sigma^\pm_2, 
\end{cases}$$

(3.26)

$$R^\pm_3(z) = \begin{cases} 
-\gamma(k) (k_0) T_1^{-1}(k) T_2^{-1}(k), & k \in I, \\
T_1(k) e^{\mathcal{X}(\pm k_0)} \left( \frac{k - k_0}{k + k_0} \right)^{i\nu} \gamma^+(\pm k_0) (1 - \mathcal{X}(\mathcal{X}(k))), & k \in \Sigma^\pm_3, 
\end{cases}$$

(3.27)

$$R^\pm_4(z) = \begin{cases} 
\gamma(k) (k_0) T_1^{-1}(k) T_2^{-1}(k), & k \in I, \\
\gamma^+(\pm k_0) T_1^{-1}(k) e^{-\mathcal{X}(\pm k_0)} \left( \frac{k - k_0}{k + k_0} \right)^{-i\nu} (1 - \mathcal{X}(\mathcal{X}(k))), & k \in \Sigma^\pm_4, 
\end{cases}$$

(3.28)

and $R_j$ admit estimates

$$|R^j_1(k)| \lesssim 1 + |\text{Re}(k)|^{-1/2}, j = 1, 2, 3, 4,$$

(3.30)

$$|\partial R^j_1(k)| \lesssim |\partial \mathcal{X}(\mathcal{X}(k))| + |(p^\pm_j)'(\text{Re}(k))| + |k + k_0|^{-1/2}, j = 1, 2, 3, 4,$$

(3.31)

$$|\partial R^j_1(k)| \lesssim |\partial \mathcal{X}(\mathcal{X}(k))| + |(p^\pm_j)'(\text{Re}(k))| + |k - k_0|^{-1/2}, j = 1, 2, 3, 4,$$

(3.32)

$$\partial R_j(k) = 0, \text{ if } k \in \Omega_5 \cup \Omega_6 \text{ or } \text{dist}(k, \mathcal{X} \cup \overline{\mathcal{X}}) < \rho/3.$$  

(3.33)
where
\[
p_1^+(k) = \gamma(k), \quad p_2^+(k) = \frac{\gamma^+(k^*)}{1 + \gamma(k)\gamma^+(k^*)},
\]
\[
p_3^+(k) = \gamma^+(k^*), \quad p_4^+(k) = \frac{\gamma(k)}{1 + \gamma(k)\gamma^+(k^*)},
\]
\[I = (-k_0, k_0), \quad I_- = (-\infty, -k_0), \quad I_+ = (k_0, +\infty),\]

The above proposition can be shown in a similar way as the reference \[20\].

We make the transformation (3.23) and find that \(M^2\) satisfies a special \(\mathfrak{d}\)-RH problem.

\(\mathfrak{d}\)-RHP1. Find a matrix-valued function \(M^2(k) = M^2(k; x, t)\) which satisfies

(a) \(M^2(k)\) is continuous in \(C \setminus (\Sigma^2 \cup \mathcal{K} \cup \overline{\mathcal{K}})\).

(b) \(M^2(k)\) has the following jump condition
\[
V^{(2)}(k) = \begin{cases}
(I \quad 0), & k \in \Sigma^2, \\
(-R_1^+e^{2i\theta} \quad 1), & k \in \Sigma^2_1, \\
(I \quad R_2^+e^{-2i\theta} \quad 0), & k \in \Sigma^2_2,
\end{cases}
\]
\[k \in \Sigma^2, \quad \theta \in \mathbb{R}, \quad \theta \neq 0, \pi.
\]

As \(k \in (-ik_0, ik_0)\), the jump matrix \(V^{(2)}\) has the following form
\[
V^{(2)}(k) = \begin{cases}
(I \quad (R_3^+ - R_3^-)e^{2i\theta} \quad 1), & k \in (ik_0 \tan(\frac{\pi}{12}), ik_0), \\
(I \quad (R_4^+ - R_4^-)e^{2i\theta} \quad 0), & k \in (-ik_0 \tan(\frac{\pi}{12}), k_0 \tan(\frac{\pi}{12})),
\end{cases}
\]
\[k \in (-ik_0, ik_0), \quad \theta \in \mathbb{R}, \quad \theta \neq 0, \pi.
\]

(c) \(M^2(k) \to I, \quad k \to \infty;\)

(d) For any \(k \in C \setminus (\Sigma^2 \cup \mathcal{K} \cup \overline{\mathcal{K}})\), we have
\[
\mathfrak{d}M^{(2)}(k) = M^{(1)}(k)\mathfrak{d}R^{(2)}(k),
\]
\[\text{where}
\]
\[
\mathfrak{d}R^{(2)}(k) = \begin{cases}
0, & j = 1, 3, \\
0 \quad 0, & j = 2, 4, \\
0 \quad 0, & \text{otherwise,}
\end{cases}
\]
(e) \( M^{(2)}(k; x, t) \) has simple poles at \( k_j \) and \( k_j^* \) with

\[
\text{Res}_{k = k_j} M^{(2)}(k) = \lim_{k \to k_j} M^{(2)}(k) = \left( \begin{array}{cc} 0 & 0 \\ T^{-1}_2 e^{2it\theta} & 0 \end{array} \right), \quad j \in \Delta^+,
\]

\[
\text{Res}_{k = k_j^*} M^{(2)}(k) = \lim_{k \to k_j^*} M^{(2)}(k) = \left( \begin{array}{cc} 0 & -T_1 T_2 c_j^T e^{-2it\theta(k)} \\ 0 & 0 \end{array} \right), \quad j \in \Delta^+.
\]

### 3.2 Analysis on a pure RH problem

In this section, our aim is to decompose \( \text{RHP1} \) into a pure RHP with \( \partial \text{R}^{(2)} = 0 \) and a pure \( \partial \)-problem with \( \partial \text{R}^{(2)} \neq 0 \). We express the decomposition as follows:

\[
M^{(2)}(k; x, t) = \begin{cases} 
\text{RHP}, & \partial \text{R}^{(2)} = 0 \rightarrow M^{(2)}_{\text{RHP}}, \\
M^{(3)} = M^{(2)} M^{(2)}_{-1}, & \partial \text{R}^{(2)} \neq 0 \rightarrow M^{(3)} \end{cases}
\]

Here \( M^{(2)}_{\text{RHP}}(k) \) corresponds to the pure RHP part which has the same poles and residue condition with \( M^{(2)}(k) \), and \( M^{(3)}(k) \) corresponds to the pure \( \partial \) part without jumps and poles. Now we first consider the pure RHP part \( M^{(2)}_{\text{RHP}}(k) \), which solves the following RHP:

**RHP3.** Find a matrix-valued function \( M^{(2)}_{\text{RHP}}(k) \) which satisfies

(a) \( M^{(2)}_{\text{RHP}}(k) \) is continuous in \( C \setminus \left( \Sigma^{(2)} \cup \mathcal{A} \cup \overline{\mathcal{A}} \right) \).

(b) \( M^{(2)}_{\text{RHP}}(k) \) has the following jump condition \( M^{(2)}_{+\text{RHP}}(k) = M^{(2)}_{-\text{RHP}}(k) V^{(2)}(k), \quad k \in \Sigma^{(2)} \).

(c) \( M^{(2)}_{\text{RHP}}(k) \to I, \quad k \to \infty \).

(d) \( \partial \text{R}^{(2)} = 0, \quad k \in C \setminus \left( \Sigma^{(2)} \cup \mathcal{A} \cup \overline{\mathcal{A}} \right) \).

Next, we managed to separate the jumps and poles into two parts. Define open neighborhood of the stationary point \( \pm k_0 \)

\[
\mathcal{U}_{\pm k_0} = \{ k : |k \pm k_0| < \rho / 2 \},
\]

Moreover, we decomposition \( M^{(2)}_{\text{RHP}}(k) \) into two parts:

\[
M^{(2)}_{\text{RHP}}(k) = \begin{cases} 
\text{Er}(k) M^{(\text{out})}(k), & k \in C \setminus \mathcal{U}_{\pm k_0}, \\
\text{Er}(k) M^{(\text{LC})}(k) = \text{Er}(k) M^{(\text{out})}(k) M^{(\text{SA})}(k), & k \in \mathcal{U}_{\pm k_0},
\end{cases}
\]
where $M^{(\text{out})}$ corresponds to the pure soliton solutions outside the neighborhood $U_{\pm k_0}$, which is defined in $\mathbb{C}$ and has only discrete spectrum with no jumps. $M^{(SA)}$ is the model RHP which considered by Liu in [30], which is defined in $U_{\pm k_0}$ without discrete spectrum.

RHP4. Find a matrix-valued function $M^{(\text{out})}(k)|\sigma_d^{\text{out}}|$ which satisfies

(a) $M^{(\text{out})}(k)|\sigma_d^{\text{out}}|$ is continuous in $\mathbb{C} \setminus \left( \Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^c \right)$.

(b) $M^{(\text{out})}(k)|\sigma_d^{\text{out}}| \to I, \; k \to \infty$;

(c) $M^{(\text{out})}(k)|\sigma_d^{\text{out}}|$ has simple poles at $k_j$ and $k_j^*$ with

\[
\text{Res } M^{(\text{out})}(k) = \lim_{k \to k_j} M^{(\text{out})}(k)|\sigma_d^{\text{out}}| \left( \begin{array}{cc} 0 & 0 \\ c_j T_1^{-1} T_2^{-1} e^{2i\theta(k)} & 0 \end{array} \right), \quad j \in \Delta^+,
\]

\[
\text{Res } M^{(\text{out})}(k) = \lim_{k \to k_j^*} M^{(\text{out})}(k)|\sigma_d^{\text{out}}| \left( \begin{array}{cc} 0 & -e^{-2i\theta(k)} \\ 0 & 0 \end{array} \right), \quad j \in \Delta^+.
\]

Proposition 3. The jump matrix $V^{(2)}(k)$ in the above RHP satisfies the following estimate

\[
||V^{(2)}(k) - I||_{L^\infty(\Sigma^{(2)})} = \begin{cases} 
O \left( e^{-\sqrt{2} |k + k_0|^3 - 24i k_0 k + k_0|^2} \right), & |k + k_0| > \rho/2, \; k \in \Sigma_{\frac{i}{4}}, \Sigma_{\frac{i}{2}}, \\
O \left( e^{-16i k_0^2 |k + k_0|} \right), & |k + k_0| > \rho/2, \; k \in \Sigma_{\frac{i}{3}}, \Sigma_{\frac{i}{4}}, \\
0, & k \in [-i k_0 \tan(\frac{\pi}{12}), i k_0 \tan(\frac{\pi}{12})], \\
O \left( e^{-14i \text{Im} k^3} \right), & k \in [\pm i k_0, \pm i k_0 \tan(\frac{\pi}{12})].
\end{cases}
\]

Proof. Note that the phase function $\theta(k)$ can be written as

\[
i\theta(k) = 4it \left((k \mp k_0)^3 \pm 3k_0(k \mp k_0)^2 \pm 2k_0^3 \right),
\]

(3.45)

Combine the above form and the boundedness of non-exponential term in $R_j(k)$, we obtain the final estimate of $||V^{(2)}(k) - I||_{L^\infty(\Sigma^{(2)})}$.

Thus, the jump matrix outside the neighborhood $U_{\pm k_0}$ will decay exponentially to the identity matrix at $t \to +\infty$, we can ignore the jump condition of $M^{(2)}_{RHP}$ on $\Sigma_2$. Here, the main distribution to the RHP are from the soliton solutions corresponding to the scattering data

\[
\sigma_d^{\text{out}} = \{(k^*_j, \bar{c}_j), \; k_j \in \mathcal{K}^{2N}, \; j=1 \\ldots k \}, \quad \bar{c}_j(k) = c_j \delta^{-1}(k_j)(\det \delta(k_j))^{-1}.
\]

Moreover, outside $U_{\pm k_0}$, the error between $M^{(2)}_{RHP}(k)$ and $M^{(\text{out})}(k)$ can be expressed by the error matrix function $E(k)$.  

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3.2.1 Soliton solutions

In this subsection, we build a outer model RH problem and show that its solution can be approximated with a finite sum of solitons. We first recall the RHP corresponding to the matrix function $M^{(\text{out})}(k; \sigma_d^{\text{out}})$:

**RHP5.** Find a matrix-valued function $M^{(\text{out})}(k; \sigma_d^{\text{out}})$ which satisfies

(a) $M^{(\text{out})}(k; \sigma_d^{\text{out}})$ is continuous in $\mathbb{C} \setminus \left( \Sigma^{(2)} \cup \mathscr{K} \cup \mathscr{K}^* \right)$.

(b) $M^{(\text{out})}(k; \sigma_d^{\text{out}}) \to I$, $k \to \infty$.

(c) $M^{(\text{out})}(k; \sigma_d^{\text{out}})$ has simple poles at $k_j$ and $k_j^*$ with

\[
\operatorname{Res} M^{(\text{out})}(k; \sigma_d^{\text{out}}) = \lim_{k \to k_j} M^{(\text{out})}(k; \sigma_d^{\text{out}}) \begin{pmatrix} 0 & e^{-2i\theta(k)} \\ 0 & 0 \end{pmatrix}, \quad j \in \Delta^+, \quad (3.47)
\]
\[
\operatorname{Res} M^{(\text{out})}(k; \sigma_d^{\text{out}}) = \lim_{k \to k_j^*} M^{(\text{out})}(k; \sigma_d^{\text{out}}) \begin{pmatrix} 0 & -e^{2i\theta(k)} \\ 0 & 0 \end{pmatrix}, \quad j \in \Delta^+. \quad (3.48)
\]

In order to show the existence and uniqueness of solution corresponding to the above RHP, we need to study the existence and uniqueness of RHP1 in the reflectionless case. In this special case, $M(k; x, t)$ has no contour, the RHP1 reduces to the following RH problem.

**RHP6.** Given scattering data $\sigma_d = \{(k_j, c_j)\}_{k=1}^{2N}$ and $\mathscr{K} = \{k_j\}_{j=1}^{2N}$. Find a matrix-valued function $M(k; x, t|\sigma_d)$ with following condition:

(a) Analyticity: $M(k; x, t|\sigma_d)$ is analytical in $\mathbb{C} \setminus \left( \Sigma^{(2)} \cup \mathscr{K} \cup \mathscr{K}^* \right)$;

(b) Asymptotic behaviors: $M(k; x, t|\sigma_d) = I + O(k^{-1})$, $k \to \infty$;

(c) Residue conditions: $M(k; x, t|\sigma_d)$ has simple poles at $k_j$ and $k_j^*$ with

\[
\operatorname{Res} M(k; x, t|\sigma_d) = \lim_{k \to k_j} M(k; x, t|\sigma_d) N_j, \quad N_j = \begin{pmatrix} 0 & 0 \\ \gamma_j & 0 \end{pmatrix}, \quad \gamma_j = c_j e^{2i\theta(k_j)}, \quad (3.49)
\]
\[
\operatorname{Res} M(k; x, t|\sigma_d) = \lim_{k \to k_j^*} M(k; x, t|\sigma_d) \bar{N}_j, \quad \bar{N}_j = \begin{pmatrix} 0 & 0 \\ \bar{\gamma}_j & 0 \end{pmatrix}, \quad \bar{\gamma}_j = -c_j^* e^{-2i\theta(k)}. \quad (3.50)
\]

**Proposition 4.** Given scattering data $\sigma_d = (k_j, c_j)_{k=1}^{2N}$ and $\mathscr{K} = \{k_j\}_{j=1}^{2N}$. The RH problem have unique solutions

\[
q_{\sigma_d}(x, t|\sigma_d) = (u(x, t), u^*(x, t))^T = 2i \lim_{k \to \infty} (kM(k|\sigma_d))_{12}. \quad (3.51)
\]
Proof. The uniqueness of the solution can be guaranteed by Liouville’s theorem. As for the RHP

\[
M_+(k; x, t | \sigma_d) = M_-(k; x, t | \sigma_d)V(k), \tag{3.52}
\]

By Plemelj formula, we have

\[
M(k | \sigma_d) = I + \sum_{j=1}^{2N} \frac{\text{Res}_{k=k_j} M(k | \sigma_d)}{k-k_j} + \sum_{j=1}^{2N} \frac{\text{Res}_{k=k^*_j} M(k | \sigma_d)}{k-k^*_j}. \tag{3.53}
\]

Since \(N_j(k; x, t)\) and \(\bar{N}_j(k; x, t)\) are nilpotent matrices, we can rewrite the residue condition as the following form:

\[
\text{Res}_{k=k_j} M(k | \sigma_d) = a(k_j)N_j = \begin{pmatrix} a_{11}(k_j) & a_{12}(k_j) & a_{13}(k_j) \\ a_{21}(k_j) & a_{22}(k_j) & a_{23}(k_j) \\ a_{31}(k_j) & a_{32}(k_j) & a_{33}(k_j) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_{j,1} & \gamma_{j,2} & 0 \end{pmatrix} \delta = \begin{pmatrix} a_{j,11} & a_{j,12} & 0 \\ a_{j,21} & a_{j,22} & 0 \\ a_{j,31} & a_{j,32} & 0 \end{pmatrix}.
\]

Similarly, we get

\[
\text{Res}_{k=k^*_j} M(k | \sigma_d) = b(k_j)\bar{N}_j = \begin{pmatrix} b_{11}(k_j) & b_{12}(k_j) & b_{13}(k_j) \\ b_{21}(k_j) & b_{22}(k_j) & b_{23}(k_j) \\ b_{31}(k_j) & b_{32}(k_j) & b_{33}(k_j) \end{pmatrix} \begin{pmatrix} 0 & 0 & \tilde{\gamma}_{j,1} \\ 0 & 0 & \tilde{\gamma}_{j,2} \\ 0 & 0 & 0 \end{pmatrix} \delta = \begin{pmatrix} 0 & 0 & \bar{\beta}_{j,1} \\ 0 & 0 & \bar{\beta}_{j,2} \\ 0 & 0 & \bar{\beta}_{j,3} \end{pmatrix}.
\]

Substitute the above residue condition into Eq. (3.53), the RHP has the following formal solution

\[
M(k | \sigma_d) = I + \sum_{j=1}^{2N} \frac{1}{k-k_j} \begin{pmatrix} a_{j,11} & a_{j,12} & 0 \\ a_{j,21} & a_{j,22} & 0 \\ a_{j,31} & a_{j,32} & 0 \end{pmatrix} + \sum_{j=1}^{2N} \frac{1}{k-k^*_j} \begin{pmatrix} 0 & 0 & \bar{\beta}_{j,1} \\ 0 & 0 & \bar{\beta}_{j,2} \\ 0 & 0 & \bar{\beta}_{j,3} \end{pmatrix}. \tag{3.54}
\]

As for the above formal solution, calculate the residue condition at \(k_j\) and \(k^*_j\), we have

\[
\begin{pmatrix} a_{j,11} & a_{j,12} & 0 \\ a_{j,21} & a_{j,22} & 0 \\ a_{j,31} & a_{j,32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma_{j,1} & \gamma_{j,2} & 0 \end{pmatrix} + \sum_{l=1}^{2N} \frac{1}{k_l-k_j} \begin{pmatrix} \bar{\beta}_{l,1} & \bar{\beta}_{l,1} & \gamma_{l,1} \\ \bar{\beta}_{l,2} & \bar{\beta}_{l,2} & \gamma_{l,2} \\ \bar{\beta}_{l,3} & \bar{\beta}_{l,3} & \gamma_{l,2} \end{pmatrix}, \quad j = 1, 2, \ldots, 2N,
\]

\[
\begin{pmatrix} 0 & 0 & \bar{\beta}_{j,1} \\ 0 & 0 & \bar{\beta}_{j,2} \\ 0 & 0 & \bar{\beta}_{j,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \tilde{\gamma}_{j,1} \\ 0 & 0 & \tilde{\gamma}_{j,2} \\ 0 & 0 & 0 \end{pmatrix} + \sum_{l=1}^{2N} \frac{1}{k^*_l-k^*_j} \begin{pmatrix} 0 & 0 & \alpha_{l,11} \tilde{\gamma}_{l,1} + \alpha_{l,12} \tilde{\gamma}_{l,2} \\ 0 & 0 & \alpha_{l,21} \tilde{\gamma}_{l,1} + \alpha_{l,22} \tilde{\gamma}_{l,2} \\ 0 & 0 & \alpha_{l,31} \tilde{\gamma}_{l,1} + \alpha_{l,32} \tilde{\gamma}_{l,2} \end{pmatrix}, \quad j = 1, 2, \ldots, 2N,
\]

\[
\tag{3.55}
\]

by solving the linear system (3.55), we can obtain the important parameters \(a_{j,11}, a_{j,12}, a_{j,21}, a_{j,22}, a_{j,31}, a_{j,32}, \bar{\beta}_{j,1}, \bar{\beta}_{j,2}\) and \(\bar{\beta}_{j,3}\), thus, we complete the proof of the uniqueness of RHP5. □

Furthermore, we make the following transformation

\[
M^{(out)}(k | \sigma_d^{(out)}) = M(k | \sigma_d) \begin{pmatrix} \delta^{-1}(k) & 0 \\ 0 & \det \delta(k) \end{pmatrix}. \tag{3.56}
\]
Then $M^{\text{out}}(k | \sigma_d^{\text{out}})$ satisfies the following RH problem

**RHP7.** Find a matrix-valued function $M^{\text{out}}(k | \sigma_d^{\text{out}})$ with following condition:

(a) $M^{\text{out}}(k | \sigma_d^{\text{out}})$ is continuous in $\mathbb{C} \setminus \left( \Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}' \right)$.

(b) $M^{\text{out}}(k | \sigma_d^{\text{out}}) \to I$, $k \to \infty$;

(c) $M^{\text{out}}(k | \sigma_d^{\text{out}})$ has simple poles at $k_j$ and $k_j^*$ with

\[
\text{Res}_{k=k_j} M^{\text{out}}(k | \sigma_d^{\text{out}}) = \lim_{k \to k_j} M^{\text{out}}(k | \sigma_d^{\text{out}}) \begin{pmatrix} 0 & 0 \\ -T^1_1 T^1_2 e^{-2it\theta(k)} & 0 \end{pmatrix}, \quad j \in \Delta^+, \quad (3.57)
\]

\[
\text{Res}_{k=k_j^*} M^{\text{out}}(k | \sigma_d^{\text{out}}) = \lim_{k \to k_j^*} M^{\text{out}}(k | \sigma_d^{\text{out}}) \begin{pmatrix} 0 & -T^1_1 T^1_2 e^{-2it\theta(k)} \\ 0 & 0 \end{pmatrix}, \quad j \in \Delta^+. \quad (3.58)
\]

**Proposition 5.** RHP7 has uniqueness solution, which also satisfy

\[
q_{\text{sol}}(x, t | \sigma_d^{\text{out}}) = q_{\text{sol}}(x, t | \sigma_d). \quad (3.59)
\]

**Proof.** Since matrix-valued function $M^{\text{out}}(k | \sigma_d^{\text{out}})$ can be obtained from $M(k | \sigma_d)$ by an explicit transformation

\[
M^{\text{out}}(k | \sigma_d^{\text{out}}) = M(k | \sigma_d) \begin{pmatrix} \delta^{-1}(k) & 0 \\ 0 & \det \delta(k) \end{pmatrix}, \quad (3.60)
\]

the uniqueness of $M^{\text{out}}(k | \sigma_d^{\text{out}})$ are follows from that of $M(k | \sigma_d)$. Moreover, according to the asymptotic behavior of $\Delta(k)$, we have

\[
q_{\text{sol}}(x, t | \sigma_d^{\text{out}}) = 2i \lim_{k \to \infty} (k M^{\text{out}}(k | \sigma_d^{\text{out}}))_{12} = 2i \lim_{k \to \infty} (k M(k | \sigma_d) \Delta(k))_{12}
\]

\[
= 2i \lim_{k \to \infty} (k M(k | \sigma_d))_{12} = q_{\text{sol}}(x, t | \sigma_d). \quad (3.61)
\]

We now consider the long-time behavior of soliton solutions. Firstly, we define a space-time cone

\[
\mathcal{C}(v_1, v_2) = \left\{ (x, t) \in \mathbb{R}^2 | x = vt, v \in [v_1, v_2] \right\}, \quad (3.62)
\]
where \( v_2 \leq v_1 < 0 \). Denote

\[
\mathcal{I} = \left[ -\frac{v_1}{4}, -\frac{v_2}{4} \right), \quad \mathcal{K}(\mathcal{I}) = \left\{ k_j \in \mathcal{K} | -\frac{v_1}{4} \leq 3\Re^2 k_j - \Im^2 k_j \leq -\frac{v_2}{4} \right\},
\]

\[
N(\mathcal{I}) = |\mathcal{K}(\mathcal{I})|, \quad \mathcal{K}^-(\mathcal{I}) = \left\{ k_j \in \mathcal{K} | 3\Re^2 k_j - \Im^2 k_j < -\frac{v_2}{4} \right\},
\]

\[
\mathcal{K}^+(\mathcal{I}) = \left\{ k_j \in \mathcal{K} | 3\Re^2 k_j - \Im^2 k_j > -\frac{v_2}{4} \right\},
\]

\[
c_j(\mathcal{I}) = c_j \delta^{-1}(k_j) \exp \left[ -\frac{1}{2\pi i} \int_{-k_j}^{k_j} \frac{\log(1 + |\gamma(\zeta)|^2)}{\zeta - k_j} d\zeta \right],
\]

\[
(3.63)
\]

\[
M(\mathcal{I}; x, t|\sigma_d) = \left( I + O \left( e^{-\mu t} \right) \right) M^{\Delta_\mathcal{I}}(\mathcal{I}; x, t|\sigma_d),
\]

\[
(3.64)
\]

\[
_M = \min_{k_j \in \mathcal{K} / \mathcal{K}(\mathcal{I})} \left\{ \Im k_j \cdot \text{dist} \left( 3\Re^2 k_j - \Im^2 k_j, \mathcal{I} \right) \right\}.
\]

**Proof.** Denote

\[
\Delta^- = \left\{ j \in \Delta_{\mathcal{K}} | 3\Re^2 k_j - \Im^2 k_j < -\frac{v_1}{4} \right\}, \quad \Delta^+ = \left\{ j \in \Delta_{\mathcal{K}} | 3\Re^2 k_j - \Im^2 k_j > -\frac{v_2}{4} \right\},
\]

\[
(3.65)
\]
As for \( t > 0, (x, t) \in \mathcal{C}(x_1, x_2, y_1, y_2) \), we obtain
\[
v_1 \leq v = \frac{x}{t} \leq v_2,
\]
thus
\[
-\frac{v_1}{12} \leq (\pm k_0)^2 \leq -\frac{v_2}{12}, \quad \text{i.e.} \quad -\frac{v_1}{4} \leq 3\text{Re}^2(\pm k_0) - \text{Im}^2(\pm k_0) \leq -\frac{v_2}{4},
\]
which means that \( \pm k_0 \) lie in the strip domain \( \{k| -\frac{v_1}{4} \leq 3\text{Re}^2k - \text{Im}^2k \leq -\frac{v_2}{4}\} \).

Next, for \( k_j \in \mathcal{H} \setminus \mathcal{H}(I) \), by analyzing the residue coefficients, we have the following estimate for \( N \Delta^\pm_{\mathcal{I}} \) and \( \tilde{N} \Delta^\pm_{\mathcal{I}} \)
\[
N \Delta^\pm_{\mathcal{I}} = \begin{cases} O(1), & k_j \in \mathcal{H}(I), \\ O(e^{-8\mu t}), & k_j \in \mathcal{H} \setminus \mathcal{H}(I), \end{cases} \quad \tilde{N} \Delta^\pm_{\mathcal{I}} = \begin{cases} O(1), & k_\ast_j \in \mathcal{H}(I), \\ O(e^{-8\mu t}), & k_\ast_j \in \mathcal{H} \setminus \mathcal{H}(I). \end{cases}
\] (3.68)

For every \( k_j \in \mathcal{H} \setminus \mathcal{H}(I) \), we suppose that \( D_j \) is a small disk centered on \( k_j \), which has a sufficiently small radius so that the disks don’t intersect each other. Define matrix-valued function
\[
\Gamma(k) = \begin{cases} I - \frac{1}{k - k_j} N \Delta^\pm_{\mathcal{I}}, & z \in D_j, \\ I - \frac{1}{k - k_\ast_j} \tilde{N} \Delta^\pm_{\mathcal{I}}, & z \in \overline{D}_j, \\ I, & \text{otherwise}. \end{cases}
\] (3.69)

Then we introduce a new transformation to convert the poles \( k_j \in \mathcal{H} \setminus \mathcal{H}(I) \) to jumps which will decay to identity matrix exponentially,
\[
\tilde{M}(k|\sigma_d) = M(k|\sigma_d) \Gamma(k)
\] (3.70)

Direct calculation shows that \( \tilde{M} \Delta^\pm(k|\sigma_d) \) satisfies the following jump condition
\[
\tilde{M}_+(k|\sigma_d) = \tilde{M}_-(k|\sigma_d) \tilde{V}(k) = \tilde{M}_-(k|\sigma_d) \Gamma(k), \quad k \in \partial D_j \cup \partial \overline{D}_j,
\] (3.71)

where jump matrices \( \tilde{V}(k) \) satisfies
\[
||\tilde{V}(k) - I||_{L^\infty(\Sigma)} = O(e^{-8\mu t}).
\] (3.72)

Note that \( \tilde{M}(k|\sigma_d) \) has the same poles and residue conditions with \( M \Delta^\pm(k|\sigma_d(I)) \), thus we can get
\[
\mathcal{E}(k) = \tilde{M}(k|\sigma_d) \left[M \Delta^\pm(k|\sigma_d(I))\right]^{-1}
\] (3.73)
has no poles and satisfies jump condition

\[ \mathcal{E}_+(k) = \mathcal{E}_-(k) V_{\mathcal{E}}(k), \]  
(3.74)

where \( V_{\mathcal{E}}(k) = M^{A^L}(k|\sigma_d(I))\tilde{V}(k) \left[ M^{A^L}(k|\sigma_d(I)) \right]^{-1} \sim \tilde{V}(k) \) satisfies

\[ \|V_{\mathcal{E}}(k) - 1\|_{L^\infty(\Sigma)} = \mathcal{O}(e^{-8\mu t}), \quad t \to +\infty. \]  
(3.75)

Based on the properties of small norm RHP, we know that \( \mathcal{E}(k) \) exists and

\[ \mathcal{E}(k) = I + \mathcal{O}\left(e^{-8\mu t}\right), \quad t \to +\infty. \]  
(3.76)

Finally, according to Eqs. (3.70) and (3.74), we obtain the following conclusion

\[ M(k; x, t|\sigma_d) = \left(I + \mathcal{O}\left(e^{-8\mu t}\right)\right) M^{A^L}(k; x, t|\sigma_d(I)), \]  
(3.77)

\[ \quad \square \]

**Corollary 1.** Suppose that \( q_{\text{sol}} \) is the soliton solutions corresponding to the scattering data \( \sigma_d = \{(k_j, c_j)\} \) of Sasa-Satsuma equation, as \( t \to +\infty \),

\[ q_{\text{sol}}(x, t|\sigma_d^{(\text{out})}) = q_{\text{sol}}(x, t|\sigma_d(I)) + \mathcal{O}\left(e^{-8\mu t}\right), \]  
(3.78)

where \( q_{\text{sol}}(x, t|\sigma_d(I)) \) is the soliton solution corresponding to the scattering data \( \sigma_d(I) \) of Sasa-Satsuma equation.

**Local solvable RH problem**

**Proposition 7.**

\[ ||V_{(2)}^{(2)}(k) - I||_{L^\infty(\Sigma^{(2)})} = \begin{cases} \mathcal{O}\left(e^{-6tk_0^2}\right), & k \in \Sigma_j^\pm \setminus \mathcal{U}_{\pm k_0}, \quad j = 1, 2, \\ \mathcal{O}\left(e^{-8tk_0^2}\right), & k \in \Sigma_j^\pm \setminus \mathcal{U}_{\pm k_0}, \quad j = 3, 4, \\ \mathcal{O}\left(t^{-1/2}k_0^{-1}|k \mp k_0|^{-1}\right), & k \in \Sigma^{(2)} \cap \mathcal{U}_{\pm k_0}, \\ 0, & k \in \left[-ik_0 \tan\left(\frac{\pi}{12}\right), ik_0 \tan\left(\frac{\pi}{12}\right)\right], \\ \mathcal{O}\left(e^{-14tk_0^2\tan^2\left(\frac{\pi}{12}\right)}\right), & k \in \left[\pm ik_0, \pm ik_0 \tan\left(\frac{\pi}{12}\right)\right]. \end{cases} \]  
(3.79)

**Remark 1.** In the open neighborhood \( \mathcal{U}_{\pm k_0} \), \( M_{\text{RHP}}^{(2)} \) has no poles, and for \( k \in \Sigma^{(2)} \cap \mathcal{U}_{\pm k_0} \), the jump matrix \( V_{(2)}^{(2)} \) is uniformly bounded point by point but does not uniformly decay to the identity matrix. Nevertheless, for \( k \in \Sigma^{(2)} \setminus \mathcal{U}_{\pm k_0} \), the jump matrix \( V_{(2)}^{(2)} \) uniformly decays to the identity matrix.

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In the absence of solitons, the original RHP3 can be reduced to a solvable model for the Sasa-Satsuma equation [30].

**RHP8.** Find a matrix-valued function \( M^{(SA)}(k; x, t) \) which satisfies

(a) Analyticity: \( M^{(SA)}(k; x, t) \) is continuous in \( \mathbb{C} \setminus \Sigma^{(2)} \);

(b) Symmetry: \( M^{(SA)}(k) = \nu M^{(SA)*}(-k*) \nu \);

(c) Asymptotic behaviors: \( M^{(SA)}(k) \to I, \quad k \to \infty \);

(d) Jump condition: \( M^{(SA)}(k; x, t) \) has the following jump condition

\[
M_+^{(SA)}(k) = M_-^{(SA)}(k)V^{(SA)}(k), \quad k \in \Sigma^{(2)},
\]

where jump matrix \( V^{(SA)}(k) = V^{(2)}(k) \) is given by Eq. (3.35).

The proposition 6 indicates that the jump matrix \( V^{(SA)} \) uniformly decays to the identity matrix outside \( U_{\pm k_0} \), thus the main contribution to RHP6 comes from the local solvable RHP which is defined in the neighborhood \( U_{\pm k_0} \). Here, we mainly adopt the final results of solving the model RHP, see [30] for more details.

Based on the Beals-Coifman theory and Deift-Zhou steepest descent method, we can obtain
the following formula corresponding to $M^{(SA)}$:

$$M^{(SA)}(k) = I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{\kappa(\xi) \omega\left(\frac{\xi}{\zeta} \right)}{\xi - z} d\xi$$

$$= I + \frac{1}{2\pi i} \int_{\Sigma_{A}^{(2)}} \frac{\kappa_{A}(\xi) \omega\left(\frac{\xi}{\zeta} \right)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{\Sigma_{B}^{(2)}} \frac{\kappa_{B}(\xi) \omega\left(\frac{\xi}{\zeta} \right)}{\xi - z} d\xi + \mathcal{O}\left(\frac{\log t}{t}\right) \quad (3.81)$$

Note that the symmetry $M_{1}^{B_{0}} = -\zeta (M_{1}^{A_{0}})^{*} \zeta$, we have

$$M^{(SA)}(k) = I - \frac{1}{\sqrt{48tk_{0}(k + k_{0})}} M_{1}^{A_{0}} + \frac{1}{\sqrt{48tk_{0}(k - k_{0})}} \zeta (M_{1}^{A_{0}})^{*} \zeta + \mathcal{O}\left(\frac{\log t}{t}\right), \quad (3.82)$$

where

$$M_{1}^{A_{0}} = \begin{pmatrix} 0 & i(\delta_{A})^{-2} \beta_{12} \\ -i(\delta_{A})^{2} \beta_{21} & 0 \end{pmatrix}, \quad \delta_{A} = e^{\tau(-k_{0}) - 8i\tau(192\tau)^{4}},$$

$$\beta_{12} = \frac{\sqrt{2\pi}}{12e^{\pi/4}e^{-\frac{3i}{2}}} e^{-i\tau} \gamma(k_{0}), \quad \beta_{21} = -\beta_{12} = \frac{\sqrt{2\pi}}{12e^{\pi/4}e^{-\frac{3i}{2}}} \gamma(-k_{0}). \quad (3.83)$$

At the same time, we obtain that $||M^{(SA)}||_{\infty} \lesssim 1$. Clearly, RHP3 and RHP5 have the same jump contour and jump matrices, thus, we can define a local solvable model RHP inside $U_{\pm k_{0}}$

$$M^{(LC)}(k) = M^{(out)}(k)M^{(SA)}(k), \quad k \in U_{\pm k_{0}}. \quad (3.84)$$

Moreover, $M^{(LC)}(k)$ is a bounded function in $U_{\pm k_{0}}$ and has the same jump condition as $M_{RHP}^{(2)}(k)$.

### 3.2.2 A small norm RH problem

In this subsection, we mainly consider the small norm RHP corresponding to the error function $Er(k)$. Firstly, according to the definition of $M_{RHP}^{(2)}(k)$ and $M^{(LC)}(k)$, we can obtain that $Er(k)$ satisfies the following RHP:

**RHP9.** Find a matrix-valued function $Er(k)$ which satisfies

(a) Analyticity: $Er(k)$ is continuous in $C \setminus \Sigma^{(Er)}$, where $\Sigma^{(Er)} = \partial U_{\pm k_{0}} \cup \left(\Sigma^{(Er)} \setminus U_{\pm k_{0}}\right)$;

(b) Asymptotic behaviors: $Er(k) \rightarrow I, \quad k \rightarrow \infty$;

(c) Jump condition: $Er(k)$ has the following jump condition

$$Er_{+}(k) = Er_{-}(k) V^{(Er)}(k), \quad k \in \Sigma^{(Er)}, \quad (3.85)$$

$$\int_{\Sigma^{(Er)}} \frac{1}{2\pi i} \int_{\Sigma_{A}^{(2)}} \frac{\kappa_{A}(\xi) \omega\left(\frac{\xi}{\zeta} \right)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{\Sigma_{B}^{(2)}} \frac{\kappa_{B}(\xi) \omega\left(\frac{\xi}{\zeta} \right)}{\xi - z} d\xi + \mathcal{O}\left(\frac{\log t}{t}\right) \quad (3.81)$$
where matrix $V^{(Er)}(k)$ is defined by

$$ V^{(Er)}(k) = \begin{cases} M^{(out)}(k) V^{(2)}(k) M^{(out)}(k)^{-1}, & k \in \Sigma^{(2)} \setminus \mathcal{U}_{\pm k_0}, \\ M^{(out)}(k) M^{(SA)}(k) M^{(out)}(k)^{-1}, & k \in \partial \mathcal{U}_{\pm k_0}. \end{cases} $$

(3.86)

**Proposition 8.** The jump matrix $V^{(Er)}(k)$ has the following uniform estimate

$$ |V^{(Er)}(k) - I| = \begin{cases} \mathcal{O} \left( e^{-6i k_0 \nu^2} \right), & k \in \Sigma_j \setminus \mathcal{U}_{\pm k_0}, \quad j = 1, 2, \\ \mathcal{O} \left( e^{-8i k_0 \nu^2} \right), & k \in \Sigma_j \setminus \mathcal{U}_{\pm k_0}, \quad j = 3, 4, \\ 0, & k \in [-i k_0 \tan \left( \frac{\pi}{12} \right), i k_0 \tan \left( \frac{\pi}{12} \right)], \\ \mathcal{O} \left( e^{-14i k_0 \nu \tan^3 \left( \frac{\pi}{12} \right)} \right), & k \in [\pm i k_0, \pm i k_0 \tan \left( \frac{\pi}{12} \right)], \\ \mathcal{O}(t^{-1/2}), & k \in \partial \mathcal{U}_{\pm k_0}. \end{cases} $$

(3.87)

**Proof.** Using the definition of $Er(k)$, the estimate of $V^{(2)}$ and the expression of $M^{(SA)}(k)$, we can easily complete the proof of the above proposition.

Next, from Beals-Coifman theorem, we can construct the solution of RHP7. At first, we consider the trivial factorization of the jump matrix $V^{(Er)}$

$$ V^{(Er)} = (b_-)^{-1} b_+, \quad b_- = I, \quad b_+ = V^{(Er)}, $$

(3.88)
and

$$(\omega_{Er})_- = I - b_- = 0, \quad (\omega_{Er})_+ = b_+ - I = V^{(Er)} - I,$$

$$\omega_{Er} = (\omega_{Er})_+ + (\omega_{Er})_- = V^{(Er)} - I. \tag{3.89}$$

$$C_{\omega Er} f = C_-(f(\omega_{Er})_+) + C_+(f(\omega_{Er})_-) = C_-(f(V^{(Er)} - I)),$$

where $C_-$ denotes the Cauchy projection operator,

$$C_- f(k) = \lim_{\zeta \to k \in \Sigma(E)} \frac{1}{2\pi i} \int_{\Sigma(E)} \frac{f(\xi)}{\xi - k} d\xi, \tag{3.90}$$

and $||C_-||_{L^2}$ is a finite value. Therefore, the solution of RHP7 can be expressed by

$$Er(k) = I + \frac{1}{2\pi i} \int_{\Sigma(E)} \frac{\kappa_{Er}(\xi)(V^{(Er)}(\xi) - I)}{\xi - k} d\xi, \tag{3.91}$$

where $\kappa_{Er} \in L^2(\Sigma(E))$ satisfies $(I - C_{\omega Er})\kappa_{Er} = I$. Since

$$||C_{\omega Er}||_{L^2(\Sigma(E))} \lesssim ||C||_{L^2(\Sigma(E))} ||V^{(Er)} - I||_{L^2(\Sigma(E))} \lesssim O(t^{-1/2}), \tag{3.92}$$

matrix function $\kappa_{\omega Er}$ exists and is unique, and the solution $Er(k)$ of RHP exists and is unique.

**Proposition 9.** For $V^{(Er)}$ and $\kappa_{Er}$, we have the following important estimate

$$||\kappa_{Er} - I||_{L^2(\Sigma(E))} = O(t^{-1/2}),$$

$$||V^{(Er)} - I||_{L^p} = O(t^{-1/2}), \quad p \in [1, +\infty), \quad k \geq 0. \tag{3.93}$$

In order to recover the solution $q(x, t)$, it is necessary to consider the long time asymptotic of $Er(k)$ as $k \to \infty$.

**Proposition 10.** As $k \to \infty$, $Er(k)$ has the following asymptotic expansion

$$Er(k) = I + k^{-1} Er_1(k) + O(k^{-2}), \tag{3.94}$$

where

$$Er_1(k) = -\frac{1}{2\pi i} \int_{\Sigma(E)} \kappa_{Er}(\xi)(V^{Er} - I) d\xi. \tag{3.95}$$

Moreover, $Er_1(k)$ has the following explicit expression

$$Er_1(x, t) = \frac{1}{\sqrt{48k_0}} M^{(out)}(k_0) M_1^{(out)}(k_0) M^{(out)-1}(k_0) + \frac{1}{\sqrt{48k_0}} M^{(out)}(-k_0) M_1^{(out)}(-k_0) M^{(out)-1}(-k_0) + O(t^{-1} \log t). \tag{3.96}$$
Proof. From Eq. (3.95)

\[
Er_1(x, t) = -\frac{1}{2\pi i} \int_{\Sigma^{(\varepsilon)}} K_{Er}(\xi) (V^{Er} - I) d\xi
\]

\[
-\frac{1}{2\pi i} \oint_{\partial U_{\pm k_0}} (V^{Er} - I) d\xi - \frac{1}{2\pi i} \int_{\Sigma^{(\varepsilon)}} (V^{Er} - I) d\xi
\]

\[
-\frac{1}{2\pi i} \int_{\Sigma^{(\varepsilon)}} (K_{Er}(\xi) - I) (V^{Er} - I) d\xi
\]

(3.97)

Using Eq. (3.87), one obtain

\[
\left| -\frac{1}{2\pi i} \int_{\Sigma^{(\varepsilon)}} (V^{Er} - I) d\xi \right| \lesssim O(t^{-p/2}), \quad (p > 1).
\]

(3.98)

Moreover, according to proposition 8, we have

\[
\left| -\frac{1}{2\pi i} \int_{\Sigma^{(\varepsilon)}} (K_{Er}(\xi) - I) (V^{Er} - I) d\xi \right| \lesssim \|K_{Er} - I\|_{L^2(\Sigma^{(\varepsilon)})} \|V^{Er} - I\|_{L^2(\Sigma^{(\varepsilon)})} = O(t^{-1}).
\]

(3.99)

Combining the above analysis, we obtain

\[
Er_1(k) = -\frac{1}{2\pi i} \oint_{\partial U_{\pm k_0}} (V^{Er} - I) d\xi + O(t^{-1}).
\]

(3.100)

Substituting the expression of \(M^{(SA)}(k; x, t)\) and use the residue theorem, the final result is as follows:

\[
Er_1(k) = -\frac{1}{2\pi i} \oint_{\partial U_{\pm k_0}} M^{(out)}(\xi)(M^{(SA)} - I)M^{(out)}(\xi)^{-1} d\xi + O(t^{-1})
\]

\[
= \frac{1}{\sqrt{48t_k}} M^{(out)}(k_0)M_1^{(out)}(k_0)M^{(out)}(k_0)^{-1} + \frac{1}{\sqrt{48t_k}} M^{(out)}(-k_0)M_1^{(out)}(-k_0)M^{(out)}(-k_0)^{-1} + O(t^{-1} \log t).
\]

(3.101)

3.3 Analysis on a pure \(\bar{\partial}\)-problem

In this subsection, we mainly consider the pure \(\bar{\partial}\)-problem which is obtained by removing the RHP part with \(\bar{\partial} R^{(2)} = 0\).

Define

\[
M^{(3)}(k) = M^{(2)}(k)M^{(2)}_{RHP}(k)^{-1},
\]

(3.102)
we have that $M^{(3)}$ is continuous and has no jumps in the complex plane. Therefore, we obtain a pure $\overline{\sigma}$-problem.

$\overline{\sigma}$-problem. Find a matrix-valued function $M^{(3)}(k)$ which satisfies

(a) $M^{(3)}(k)$ is continuous in $\mathbb{C} \setminus \Sigma^{(2)}$;
(b) $M^{(3)}(k) \sim I, \quad k \to \infty$;
(c) $\overline{\partial} M^{(3)}(k) = M^{(3)}(k) W^{(3)}(k), \quad k \in \mathbb{C}$, where

$$W^{(3)} = M^{(2)}_{RHP}(k) \overline{\partial} R^{(2)}(k) M^{(2)}_{RHP}(k)^{-1}.$$

The solution of pure $\overline{\sigma}$-problem is given by the following integral equation

$$M^{(3)}(k) = I - \frac{1}{\pi} \int_{\mathbb{C}} \frac{M^{(3)}(\xi) W^{(3)}(\xi)}{\xi - k} dA(\xi), \quad (3.103)$$

where $dA(\xi)$ is the Lebesgue measure. Further, we write the equation (3.103) in operator form

$$(I - S) M^{(3)}(k) = I, \quad (3.104)$$

where $S$ is the Cauchy operator

$$S[f](k) = - \frac{1}{\pi} \int_{\mathbb{C}} \frac{M^{(3)}(\xi) W^{(3)}(\xi)}{\xi - k} dA(\xi). \quad (3.105)$$

Proposition 11. For large time $t$,

$$||S||_{L^{\infty} \to L^{\infty}} \lesssim (k_0 t)^{-1/4}, \quad (3.106)$$

which implies that the operator $(I - S)^{-1}$ is invertible and the solution of pure $\overline{\sigma}$-problem exists and is unique.

As $k \to \infty$, the asymptotic expansion of $M^{(3)}(k)$ is given by

$$M^{(3)} = I + M^{(3)}_1 \frac{1}{k} - \frac{1}{\pi} \int_{\mathbb{C}} \frac{\xi M^{(3)}(\xi) W^{(3)}(\xi)}{k(\xi - k)} dA(\xi), \quad (3.107)$$

where

$$M^{(3)}_1 = \frac{1}{\pi} \int_{\mathbb{C}} M^{(3)}(\xi) W^{(3)}(\xi) dA(\xi). \quad (3.108)$$

To reconstruct the solution $u(x, t)$ of Sasa-Satsuma equation, it is necessary to consider the long time asymptotic behavior of $M^{(3)}_1$. Thus, we give the following proposition
Proposition 12. For large time $t$, there exists the estimate for $M_1^{(3)}$

$$|M_1^{(3)}| \lesssim (k_0 t)^{-3/4}. \tag{3.109}$$

Remark 2. The Propositions 11 and 12 can be shown in a similar way in reference [20].

3.4 Long time asymptotic behaviors in Region I

Theorem 1. Let $\sigma_d = \{ (k_j, c_j), k_j \in \mathcal{K} \}_{j=1}^{2N}$ denote the scattering data generated by initial value $u_0(x) \in \mathcal{S}(\mathbb{R})$. For fixed $v_2 \leq v_1 \in \mathbb{R}$, define $I = [-v_1/4, -v_2/4]$ and a space-time cone $C(v_1, v_2)$ for variables $x$ and $t$. Let $u_{sol}(x, t, \sigma_d(I))$ be the N(I) solution corresponding to the modified scattering data $\sigma_d(I) = \{ (k_j, c_j(I)), k_j \in \mathcal{K}(I) \}$. Then as $t \to +\infty$ with $(x, t) \in C(v_1, v_2)$, we have

$$u(x, t) = u_{sol}(x, t|\sigma_d(I)) + t^{-1/2}h + \mathcal{O}(t^{-3/4}), \tag{3.110}$$

where

$$h = \frac{1}{\sqrt{48k_0}} \left(M_{out}^{(out)}(k_0)M_{out}^{A_0}(k_0)M^{out-1}(k_0) + M^{out}(-k_0)M_{out}^{B_0}(-k_0)M^{out-1}(-k_0) \right). \tag{3.111}$$

Proof. Recalling a series of transformations (3.16), (3.23), (3.40) and (3.42), we obtain

$$M = M^{(3)}ErM^{(out)}R(2)^{-1}T^{-1}. \tag{3.112}$$

Particularly taking $k \to \infty$ along the imaginary axis, then we have

$$M = \left( I + \frac{M_1^{(3)}}{k} + \ldots \right) \left( I + \frac{Er_1}{k} + \ldots \right) \left( I + \frac{M_1^{(out)}}{k} + \ldots \right) \left( I + \frac{T_1}{k} + \ldots \right),$$

which leads to

$$M_1 = M_1^{(out)} + Er_1 + M_1^{(3)} + T_1. \tag{3.113}$$

According to the reconstruct formula (2.23) and the Proposition 11, one can get

$$q(x, t) = 2i \left( M_1^{(out)} \right)_{12} + 2i (Er_1)_{12} + \mathcal{O}(t^{-3/4}). \tag{3.114}$$

Note that

$$2i \left( M_1^{(out)} \right)_{12} = q_{sol}(x, t|\sigma^{(out)}_d), \tag{3.115}$$

which combine with Proposition 9 gives

$$(Er_1)_{12} = t^{-1/2}h + \mathcal{O}(t^{-1}\log t), \tag{3.116}$$
where
\[ h = \frac{1}{\sqrt{48k_0}} \left( M^{(out)}(k_0) M_1^{A_0}(k_0) M^{(out)-1}(k_0) + M^{(out)}(-k_0) M_1^{B_0}(-k_0) M^{(out)-1}(-k_0) \right)_{12}. \]

Substitute (3.113) and (3.114) into (3.112), we can get
\[ q(x, t) = q_{sol}(x, t|\sigma_d^{(out)}) + t^{-1/2}h + O(t^{-3/4}). \] (3.115)

Again by using (3.78), we obtain the final asymptotic expression with \((x, t) \in C(v_1, v_2)\)
\[ q(x, t) = q_{sol}(x, t|\sigma_d(I)) + t^{-1/2}h + O(t^{-3/4}). \] (3.116)

**Remark 3.** The large time asymptotic formula (3.116) indicates that the main contribution to the soliton resolution of the Cauchy initial value problem of the Sasa-Satsuma equation in Region I comes from three parts: 1) Leading term \(q_{sol}(x, t|\sigma_d(I))\) corresponds to \(N(I)\)-soliton whose parameters are controlled by a sum of localized soliton-soliton interactions as one moves through the once; 2) \(t^{-1/2}h\) comes from soliton-radiation interactions on continuous spectrum; 3) The final term \(O(t^{-3/4})\) is derived from the error estimate of the pure \(\partial\)-problem. The final results have also helped prove an important conclusion that soliton solutions of Sasa-Satsuma equation are asymptotically stable.

**4 Long time asymptotics in region II:** \(x > 0, |x/t| = O(1)\)

In the region \(x > 0, |x/t| = O(1)\), we have the stationary points lie on the imaginary axis, i.e.
\[ \pm k_0 = \pm \sqrt{-\frac{x}{12t}} = \pm i \sqrt{\frac{x}{12t}}, \] (4.1)
which have a fixed distance from the real axis. The signature table of the phase function is as shown in the Figure 7.
Next, in order to make continuous extension to the jump matrix $V^{(1)}$, we introduce new contours defined as follows:

$$\Sigma^{(1)}_1 = \left( ih + e^{\frac{\pi}{4}} R_+ \right) \cup \left( ih + e^{\frac{3\pi}{4}} R_+ \right),$$

$$\Sigma^{(1)}_2 = R,$$

$$\Sigma^{(1)}_3 = \left( -ih + e^{-\frac{\pi}{4}} R_+ \right) \cup \left( -ih + e^{-\frac{3\pi}{4}} R_+ \right),$$

where $h > 0$ such that $12k_0^2 + 12h^2 = -c < 0$. Therefore, the complex plane $\mathbb{C}$ is divided into four open domains. Naturally, we apply $\partial$ steepest descent method to extend the scattering data into eight regions so that the matrix function has no jumps on $\mathbb{R}$. From the top to the bottom, these open regions are denoted as $\Omega_1, \Omega_2, \Omega_3$ and $\Omega_4$, where $\Omega_2$ and $\Omega_3$ can be divided into three parts $\Omega_{k,1}, \Omega_{k,2}$ and $\Omega_{k,3}$, which are shown in Fig. 8.
Proposition 13. There exists functions $R_j : \Omega_{j} \rightarrow \mathbb{C}$ satisfying the following boundary conditions

\[
R_1(k) = \begin{cases} 
-\gamma(k), & k \in \mathbb{R}, \\
-\gamma(0)(1 - \mathcal{K}(k)), & k \in \Sigma^{(1)} \setminus \Omega_{1,4},
\end{cases}
\quad (4.3)
\]

\[
R_2(k) = \begin{cases} 
\gamma^\dagger(k^*), & k \in \mathbb{R}, \\
\gamma^\dagger(0)(1 - \mathcal{K}(k)), & k \in \Omega_{3},
\end{cases}
\quad (4.4)
\]

and $R_1, R_2$ satisfy the following estimate

\[
|\bar{\partial}R_j(k)| \lesssim |\gamma'(Re k)| + |k|^{-1/2} + \bar{\partial}\mathcal{K}(k),
\quad (4.5)
\]

and

\[
\bar{\partial}R_j(k) = 0, \quad \text{if } k \in \Omega_{1,4} \text{ or } \text{dist}(k, \mathcal{K} \cup \overline{\mathcal{K}}) < \rho/3. \quad (4.6)
\]

Based on the above analysis, we define $R^{(1)}$ as follows:

\[
R^{(1)}(k) = \begin{cases} 
\begin{pmatrix} 1 \\ Re^e^{2i\theta(k)} & 0 \\ 0 & 1 \end{pmatrix}, & k \in \Omega_{2}, \\
\begin{pmatrix} 1 \\ Re^{-2i\theta(k)} & 0 \\ 0 & 1 \end{pmatrix}, & k \in \Omega_{3}, \\
\begin{pmatrix} 1 \\ 0 & 0 \end{pmatrix}, & k \in \Omega_{1,4},
\end{cases}
\quad (4.7)
\]
In order to deform the contour $\mathbb{R}$ to the contour $\Sigma^{(2)} = \Sigma^{(1)}_1 \cup \Sigma^{(1)}_3$, we make the following matrix transformation:

$$M^{(1)}(k) = M(k)R^{(1)}(k), \quad (4.8)$$

then we can transform the RH problem on $\mathbb{R}$ into that on $\Sigma^{(2)}$.

**$\mathfrak{b}$-RHP2.** Find a matrix-valued function $M^{(1)}(k) = M^{(1)}(k; x, t)$ which satisfies

(a) $M^{(1)}(k)$ is continuous in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{K} \cup \overline{\mathcal{K}})$.

(b) $M^{(1)}(k)$ has the following jump condition $M^{(1)}_+(k) = M^{(1)}_-(k)V^{(1)}(k)$, $k \in \Sigma^{(2)}$, where

$$V^{(1)} = \begin{cases} 
(I & 0) , 
& k \in \Sigma^{(1)}_1, \\
(-R_1e^{2it\theta} & 1) , 
& k \in \Sigma^{(1)}_1, \\
(I & 0) , 
& k \in \Sigma^{(1)}_3, \\
0 & 1 
\end{cases} \quad (4.9)$$

(c) $M^{(1)} \to I, \quad k \to \infty$;

(d) For any $k \in \mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{K} \cup \overline{\mathcal{K}})$, we have

$$\mathfrak{b}M^{(1)}(k) = M(k)\mathfrak{b}R^{(1)}(k), \quad (4.10)$$

where

$$\mathfrak{b}R^{(1)}(k) = \begin{cases} 
0 & 0 
& k \in \Omega_2, \\
\overline{\mathfrak{b}R}_1e^{2it\theta} & 0 
& k \in \Omega_3, \\
0 & \overline{\mathfrak{b}R}_2e^{-2it\theta} 
& k \in \Omega_4, \\
0 & 0 
\end{cases} \quad (4.11)$$

(e) $M^{(1)}(k; x, t)$ has simple poles at $k_j$ and $k_j^+$ with

$$\text{Res} M^{(1)}(k) = \lim_{k \to k_j} M^{(1)}(k) \begin{pmatrix} 0 & 0 
\end{pmatrix}, \quad j \in \Delta^+, \quad (4.12)$$

$$\text{Res} M^{(1)}(k) = \lim_{k \to k_j^+} M^{(1)}(k) \begin{pmatrix} 0 & -c_j^*e^{-2it\theta(k)} 
0 & 0 
\end{pmatrix}, \quad j \in \Delta^+.$$
4.2 Solution of the mixed $\overline{\partial}$-RH problem

Similar to Region I, we will decompose $\overline{\partial}$-RHP2 into a pure RHP with $\overline{\partial}R^{(1)} = 0$ and a pure $\overline{\partial}$-problem with $\overline{\partial}R^{(1)} \neq 0$. We express the decomposition as follows:

$$M^{(1)}(k; x, t) = \begin{cases} 
\overline{\partial}R^{(1)} = 0 & \to M^{(1)}_{RHP} \\
\overline{\partial}R^{(1)} \neq 0 & \to M^{(2)} = M^{(1)}M^{(1)-1}_{RHP},
\end{cases} \tag{4.13}$$

here $M^{(1)}_{RHP}$ corresponds to the pure RHP part which has the same poles and residue condition with $M^{(1)}(k)$, and $M^{(2)}$ corresponds to the pure $\overline{\partial}$ part without jumps and poles.

Firstly, we first consider the solution of pure RH problem for $M^{(1)}_{RHP}$.

RHP10. Find a matrix-valued function $M^{(1)}_{RHP}(k) = M^{(1)}_{RHP}(k; x, t)$ which satisfies

(a) $M^{(1)}_{RHP}$ is analytical in $\mathbb{C} \setminus (\Sigma^{(2)} \cup \mathcal{K} \cup \mathcal{K}^c)$;

(b) $M^{(1)}_{RHP} \to I$, $k \to \infty$;

(c) $M^{(1)}_{RHP}$ has the following jump condition

$$M^{(1)}_{RHP+}(k) = M^{(1)}_{RHP-}(k)V^{(2)}(k), \quad k \in \Sigma^{(2)},$$

where

$$V^{(2)} = \begin{cases} 
(I \ -R_1 e^{2it\theta} & 0) \\
-R_2 e^{-2it\theta} & 1 \\
0 & 1
\end{cases}, \quad k \in \Sigma^{(1)},$$

$$V^{(2)} = \begin{cases} 
(I \ R_2 e^{2it\theta} & 0) \\
0 & 1
\end{cases}, \quad k \in \Sigma^{(2)},$$

(d) $M^{(1)}_{RHP}(k)$ has simple poles at $k_j$ and $k_j^*$ with

$$\text{Res}_k M^{(1)}_{RHP} = \lim_{k \to k_j} M^{(1)}_{RHP} \begin{pmatrix} 0 & 0 \\
c_j e^{2it\theta} & 0
\end{pmatrix}, \quad j \in \Delta^+,$n

$$\text{Res}_k M^{(1)}_{RHP} = \lim_{k \to k_j^*} M^{(1)}_{RHP} \begin{pmatrix} 0 & -c_j^* e^{-2it\theta(k)} \\
0 & 0
\end{pmatrix}, \quad j \in \Delta^+.$$

(4.14)

(4.15)

Note that as $k \in \Sigma^{(1)}$, we have

$$\text{Re}(2it\theta(k)) = 2t \left( 4(-3u^2(v+h)+3)^{3}+12k_0^2(v+h) \right) \leq 2t \left( -8u^2v+12h^2+12k_0^2(v+h) \right) \leq -2cht, \tag{4.16}$$

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where \( k = u + i(v + h) \), \( u = v \geq 0 \) and \( c, h \) are fixed positive real value. Moreover, as \( k \in \Sigma_3^{(1)} \), we have the same result. Therefore, as for \( M_{RHP}^{(1)} \), the jump matrices across \( \Sigma_1^{(1)} \) and \( \Sigma_3^{(1)} \) enjoy the property of exponential decay as \( t \to +\infty \). Furthermore, with a small exponential error, we can approximate the RH problem \( M_{RHP}^{(1)} \) as the RH problem \( \tilde{M}_{RHP}^{(1)} \) which only has simple poles without jump conditions.

**RHP11.** Find a matrix-valued function \( \tilde{M}_{RHP}^{(1)} \) which satisfies

(a) \( \tilde{M}_{RHP}^{(1)} \) is analytical in \( \mathbb{C} \setminus (\mathcal{K} \cup \overline{\mathcal{K}}) \);

(b) \( \tilde{M}_{RHP}^{(1)} \to I, \ k \to \infty \);

(c) \( \tilde{M}_{RHP}^{(1)}(k) \) has simple poles at \( k_j \) and \( k_j^* \) with

\[
\text{Res}_{k=k_j} \tilde{M}_{RHP}^{(1)} = \lim_{k \to k_j} \tilde{M}_{RHP}^{(1)} \left( \begin{array}{cc} 0 & 0 \\ c_j e^{2i\theta(k)} & 0 \end{array} \right), \quad j \in \Delta^+,
\]

\[
\text{Res}_{k=k_j^*} \tilde{M}_{RHP}^{(1)} = \lim_{k \to k_j^*} \tilde{M}_{RHP}^{(1)} \left( \begin{array}{cc} 0 & -c_j^* e^{-2i\theta(k)} \\ 0 & 0 \end{array} \right), \quad j \in \Delta^+.
\]  

(4.17)

For the RH problem 10, we will adopt the method appearing in the analysis of asymptotic property of soliton solutions in Region I. Similarly, we define a space-time cone

\[
\mathcal{C}(v_1, v_2) = \left\{ (x, t) \in \mathbb{R}^2 | x = vt, v \in [v_2, v_1] \right\},
\]

(4.18)

where \( 0 < v_2 \leq v_1 \). Denote

\[
\mathcal{I} = \left[ -\frac{v_1}{4}, -\frac{v_2}{4} \right], \quad \mathcal{K}(\mathcal{I}) = \left\{ k_j \in \mathcal{K} | -\frac{v_1}{4} \leq 3\text{Re}^2 k_j - \text{Im}^2 k_j \leq -\frac{v_2}{4} \right\},
\]

\[
N(\mathcal{I}) = |\mathcal{K}(\mathcal{I})|, \quad \mathcal{K}^- (\mathcal{I}) = \left\{ k_j \in \mathcal{K} | 3\text{Re}^2 k_j - \text{Im}^2 k_j < -\frac{v_1}{4} \right\},
\]

\[
\mathcal{K}^+ (\mathcal{I}) = \left\{ k_j \in \mathcal{K} | 3\text{Re}^2 k_j - \text{Im}^2 k_j > -\frac{v_2}{4} \right\},
\]

(4.19)

\[
c_j(\mathcal{I}) = c_j \delta^{-1}(k_j) \exp \left[ -\frac{1}{2i} \int_{-k_0}^{k_0} \frac{\log(1 + |\gamma(\zeta)|^2)}{\zeta - k_j} d\zeta \right].
\]
Figure 9: The poles distribution diagram. The points (•)(•)(•) represent breather solitons and the points (●) represent single solitons. Moreover, points (●) denote the poles outside the band $I$ and points (●)(●)(●) denotes the poles within the band $I$. Besides, points (●) represent the poles which lie on the critical line $\Re \theta(k) = 0$.

**Proposition 14.** Given scattering data $\sigma_d = \{(k_j, c_j)\}_{j=1}^{2N}$ and $\sigma_d(I) = \{(k_j, c_j(I))|k_j \in \mathcal{H}(I)\}$.

At $t \to +\infty$ with $(x,t) \in C(v_1, v_2)$, we have

$$ M^{\Delta_+}(k; x, t|\sigma_d) = \left( I + O \left( e^{-8\mu t} \right) \right) M^{\Delta_+}(k; x, t|\sigma_d(I)), $$

(4.20)

where $\mu(I) = \min_{k_j \in \mathcal{H}(I)} \{ \Im k_j \cdot \text{dist} \left( 3\Re^2 k_j - \Im^2 k_j, I \right) \}$.

**Corollary 2.** Suppose that $q_{sol}$ is the soliton solutions corresponding to the scattering data $\sigma_d = \{(k_j, c_j)\}$ of Sasa-Satsuma equation, as $t \to +\infty$,

$$ q_{sol}(x, t|\sigma_d) = q_{sol}(x, t|\sigma_d(I)) + O \left( e^{-8\mu t} \right), $$

(4.21)

where $q_{sol}(x, t|\sigma_d(I))$ is the soliton solution corresponding to the scattering data $\sigma_d(I)$ of Sasa-Satsuma equation.

Next, we consider the pure $\overline{\partial}$-problem for $M^{(2)}(k)$. Naturally, $M^{(2)}$ fulfills the following condition

**$\overline{\partial}$-problem 2.** Find a matrix-valued function $M^{(2)}(k)$ which satisfies

(a) $M^{(2)}(k)$ is continuous in $C \setminus \{\Sigma^{(2)} \cup \mathcal{H} \cup \mathcal{\overline{H}}\}$;
(b) \( M^{(2)}(k) \sim I, \; k \to \infty; \)
(c) \( \nabla M^{(2)}(k) = M^{(2)}(k)W^{(2)}(k), \; k \in \mathbb{C}, \) where \( W^{(2)} = M_{RHP}^{(1)}(k)\nabla R^{(1)}(k)M_{RHP}^{(1)}(k)^{-1}. \)

Similar to the analysis of \( \nabla \)-problem, according to the theory of the Cauchy operator, we can prove the existence of the solution of \( M^{(2)}(k) \). Furthermore, based on the estimate of \( \nabla R_j \), we obtain the key estimate for the coefficient to the negative first power in the expansion of \( M^{(2)}(k) \) as \( k \to +\infty \).

**Proposition 15.** For large time \( t \),
\[
||S||_{L^\infty \to L^\infty} \lesssim t^{-1/2}, \tag{4.22}
\]
thus, the operator \( (I - S)^{-1} \) is invertible. Furthermore, the solution of pure \( \nabla \)-problem exists and is unique.

**Proposition 16.** Suppose that \( M^{(2)}(k) \) has asymptotic expansion as follows:
\[
M^{(2)} = I + \frac{M_1^{(2)}}{k} - \frac{1}{\pi} \int_{\mathbb{C}} \frac{\nabla M^{(2)}(\xi)W^{(2)}(\xi)}{k(\xi - k)} dA(\xi), \; k \to \infty, \tag{4.23}
\]
where
\[
M_1^{(2)} = \frac{1}{\pi} \int_{\mathbb{C}} M^{(2)}(\xi)W^{(2)}(\xi) dA(\xi). \tag{4.24}
\]
Moreover, for the large time \( t \), we have the following estimate
\[
|M_1^{(2)}| \lesssim O(t^{-1}). \tag{4.25}
\]

### 4.3 Long time asymptotic behaviors in region II

Base on the above analysis of mixed \( \nabla \)-RH problem, we can obtain the final result corresponding to long time asymptotic behaviors of soliton solution region of Sasa-Satsuma equation as \( x > 0 \) and \( |x/t| = O(1) \).

**Theorem 2.** Assume that initial data \( u_0(x) \in S(\mathbb{R}) \) and the corresponding scattering data is \( \sigma_d = \{(k_j, c_j), k_j \in \mathcal{X}\}_{j=1}^{2N}. \) For fixed \( v_2 \leq v_1 \in \mathbb{R}^+ \), we set \( \mathcal{I} = [-v_1/4, -v_2/4] \) and a space-time cone \( S(v_1, v_2) \) for time variable \( t \) and space variable \( x \). Let \( q_{sol}(x, t, \sigma_d(I)) \) be the \( N(I) \) solution of Sasa-Satsuma equation with the scattering data \( \sigma_d(I) = \{(k_j, c_j(I)), k_j \in \mathcal{X}(I)\}. \) As \( t \to +\infty \) with \( (x, t) \in S(v_1, v_2) \), we have
\[
u(x, t) = u_{sol}(x, t|\sigma_d(I)) + O(t^{-1}). \tag{4.26}
\]
Remark 4. The large time asymptotic formula (4.26) indicates that the main contribution to the soliton resolution of the Cauchy initial value problem of the Sasa-Satsuma equation in Region II comes from two parts: 1) Leading term $u_{sol}(x,t|\sigma_d(I))$ corresponds to $N(I)$-soliton whose parameters are controlled by a sum of localized soliton-soliton interactions as one moves through the once; 2) The remaining term $O(t^{-1})$ is derived from the error estimate of the pure $\bar{\delta}$-problem. The final results prove that the soliton solution of Sasa-Satsuma equation in Region II is asymptotically stable.

5 Painleve asymptotics in Region III: $|x/t^{1/3}| = O(1)$

In this region, we first consider the case for $x < 0$ and $|x/t^{-1/3}| = O(1)$, as for $x > 0$, the long time asymptotic result will follows from a similar analysis. As for $x < 0$, we obtain the stationary points

$$\pm k_0 = \pm \sqrt{\frac{x}{12t}} = \pm \sqrt{-\frac{x}{12t^{1/3}}} t^{-1/3} \to 0, \quad as \quad t \to +\infty,$$

therefore, we get the signature table of the phase function $\text{Re}(i\theta)$ as follows:

```
|   | Imk   | Re $i\theta(k)$ |
|---|-------|-----------------|
|   |       | $> 0$           |
| Re $\theta(k)$ | $< 0$ | $< 0$           |
|   |       | $> 0$           |
```

Figure 10: The signature table of $\text{Re}(i\theta(k))$.

Here, we first note that there can be no poles on the sign demarcation line in the current situation, we can transfer the residual condition at all poles into the jump condition on a sufficiently small circle near the poles. Moreover, these jump conditions on the circles have a uniform upper bound so that they can decay into identity matrices exponentially as $t \to +\infty$. 
Based on this, we can reduce RH problem of $M(k; x, t)$ to the following form:

**RHP12.** Find a matrix-valued function $M^{(1)}(k; x, t)$ which solves:

(a) Analyticity : $M^{(1)}(k; x, t)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$;

(c) Asymptotic behaviors:

\[ M^{(1)}(k; x, t) = I + O\left(\frac{1}{k}\right), \quad k \to \infty; \quad (5.2) \]

(b) Jump condition: $M^{(1)}(k; x, t)$ has continuous boundary values $M^{(1)}_{\pm}$ on $\mathbb{R}$ and

\[ M^{(1)}_{+}(k) = M^{(1)}_{-}(k) V^{(1)}(k), \quad k \in \Sigma^{(1)}, \quad (5.3) \]

where $V^{(1)}(k) = J(k)$ and $\Sigma^{(1)} = \mathbb{R}$.

Next, we will focus on the solutions of the RH problem corresponding to $M^{(1)}$. Firstly, we carry out a scaling transform:

\[ k \to \zeta t^{-1/3}, \quad (5.4) \]

Then the jump condition of $M^{(1)}$ becomes as

\[
\begin{pmatrix}
1 & \gamma^+(\zeta t^{-1/3}) e^{-2it\theta} \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\gamma(\zeta t^{-1/3}) e^{2it\theta} & 1 \\
\end{pmatrix}, \quad k \in \mathbb{R}
\]

\[ (5.5) \]

where

\[ 2it\theta(\zeta t^{-1/3}) = 2i(4\zeta^3 + x_0 t^{-1/3}) = 8i(\zeta^3 - 3\tau^2/3\zeta). \quad (5.6) \]

Next, we make continuous extensions off the real axis onto $\Sigma^{(2)}$ to obtain a mixed $\overline{\partial}$-RH problem. And then, we will again decompose the mixed problem into a pure $\overline{\partial}$-part and a pure RH-part.
For the pure $\overline{\partial}$-part, we focus on the case in region $\Omega_1$. In $\Omega_1$, we denote
\[
\zeta = u + k_0 t^{1/3} + iv, \quad u \geq \sqrt{3}v, \quad (5.7)
\]
and
\[
\text{Re} \left( 2i t \theta (\zeta t^{-1/3}) \right) = 8 \left( -3(u + k_0 t^{1/3})^2 v + v^3 + 3v^{2/3} \right) \\
\leq 8 \left( -3u^2 v - 6uvk_0 t^{1/3} + v^3 \right) \leq -21u^2 v. \quad (5.8)
\]

Similar to the previous analysis, we can obtain that there exists $R_1 : \overline{\Omega}_1 \rightarrow \mathbb{C}$ which enjoys the boundary condition
\[
R_1 = \left\{ \begin{array}{ll}
\gamma(t^{-1/3}), & \zeta \in (k_0 t^{1/3}, \infty), \\
\gamma(k_0), & \zeta \in \Sigma^{(2)}_1,
\end{array} \right. \quad (5.9)
\]
and the interpolation is given by
\[
\gamma(k_0) + \left( \gamma(\text{Re} \zeta^{-1/3}) - \gamma(k_0) \right) \cos 3\phi, \quad 0 \leq \phi \leq \frac{\pi}{6}. \quad (5.10)
\]
Thus, we get the $\overline{\partial}$-derivative in $\Omega_1$ in the $\zeta$ variable:
\[
|\overline{\partial} R_1| \lesssim |t^{-1/3} \gamma'(u t^{-1/3})| + \frac{||\gamma'||_L^2}{t^{1/3} |t^{-1/3} - k_0|^{1/2}}. \quad (5.11)
\]
We proceed as in the previous section and study the integral equation related to the $\tilde{d}$-problem. Setting $z = \alpha + i\beta$ and $\zeta = u + k_0 t^{1/3} + iv$, the region $\Omega_1$ corresponds to $u \geq \sqrt{3} v \geq 0$. We can apply the fundamental theorem of calculus to prove that

$$\int_{\Omega_1} \frac{1}{|z - \zeta|} |W(\zeta)| d\zeta \lesssim t^{2/(3p) - 1/6}, \quad 4 < p < \infty,$$

and we also show that

$$\int_{\Omega_1} |W(\zeta)| d\zeta \lesssim t^{2/(3p) - 1/6}, \quad 4 < p < \infty.$$  

So far, we complete the discussion of the pure $\tilde{d}$-problem. As for the RH-part, we first give the pure RH problem.

**RHP13.** Find a matrix-valued function $M^{(1)}_{RHP}(\zeta) = M^{(1)}_{RHP}(\zeta, x, t)$ which satisfies

(a) $M^{(1)}_{RHP}$ is analytic in $C \setminus \Sigma^{(2)}$;

(b) $M^{(1)}_{RHP} \to I$, $\zeta \to \infty$;

(c) $M^{(1)}_{RHP}$ has the following jump condition $M^{(1)}_{RHP+}(\zeta) = M^{(1)}_{RHP-}(\zeta) V^{(2)}(\zeta)$, $\zeta \in \Sigma^{(2)}$, where

$$V^{(2)} = \begin{cases} 
I & \zeta \in \Sigma^{(2)}_1, \\
\left( \begin{array}{cc} I & 0 \\
-\gamma(-k_0)e^{2i\theta(\xi^{-1/3})} & 1 \\
0 & 1 \\
1 & \gamma(-k_0)e^{-2i\theta(\xi^{-1/3})} \\
0 & 1 \\
1 + \gamma^+(\xi^{-1/3})\gamma(\xi^{-1/3}) & \gamma(\xi^{-1/3})e^{-2i\theta(\xi^{-1/3})} \\
\gamma^+(\xi^{-1/3})e^{2i\theta(\xi^{-1/3})} & 1 \\
\zeta \in (-k_0 t^{-1/3}, k_0 t^{-1/3}).
\end{array} \right) & \zeta \in \Sigma^{(2)}_2,
\end{cases}$$

where

$$\gamma = \gamma^+ \gamma^{-1}.$$

Direct calculation shows that

$$|\gamma(\xi^{-1/3})e^{2i\theta} - \gamma(0)e^{2i\theta}| \lesssim t^{-1/6}, \quad |\gamma(\pm k_0)e^{2i\theta} - \gamma(0)e^{2i\theta}| \lesssim t^{-1/6}.$$

Thus, the above RH problem can be reduced to a problem on the following contour with jump matrices:
RHP14. Find a matrix-valued function \( \tilde{M}_{RHP}^{(1)}(\zeta) \) which satisfies

(a) \( \tilde{M}_{RHP}^{(1)}(\zeta) \) is analytic in \( \mathbb{C} \setminus \Sigma^{(2)} \);

(b) \( \tilde{M}_{RHP}^{(1)}(\zeta) \to I, \quad \zeta \to \infty \);

(c) \( \tilde{M}_{RHP}^{(1)}(\zeta) \) has the following jump condition

\[
\tilde{M}_{RHP+}^{(1)}(\zeta) = \tilde{M}_{RHP-}^{(1)}(\zeta) \tilde{V}^{(2)}(\zeta), \quad \zeta \in \Sigma^{(2)},
\]

where

\[
\tilde{V}^{(2)}(\zeta) = \begin{cases}
\begin{pmatrix} I & 0 \\ \gamma(0)e^{2i\theta(\zeta^{-1/3})} & 1 \end{pmatrix}, & \zeta \in \Sigma^{(2)}_1, \\
\begin{pmatrix} I & 0 \\ \gamma(0)e^{2i\theta(\zeta^{-1/3})} & 1 \end{pmatrix}, & \zeta \in \Sigma^{(2)}_2, \\
\begin{pmatrix} I & \gamma(0)e^{-2i\theta(\zeta^{-1/3})} \\ 0 & 1 \end{pmatrix}, & \zeta \in \Sigma^{(2)}_3, \\
\begin{pmatrix} 1 + \gamma(0) & \gamma(0)e^{-2i\theta(\zeta^{-1/3})} \\ \gamma(0)e^{2i\theta(\zeta^{-1/3})} & 1 \end{pmatrix}, & \zeta \in (-k_0t^{-1/3}, k_0t^{-1/3}).
\end{cases}
\] (5.16)

In order to solve the above RH problem, we will introduce the model problem for Sector \( \mathcal{P}_\geq \) which appears in the Appendix B [31]. Matched with the model problem as \( p(t, \zeta) = s = \gamma(0) \), the solution of the RHP has the following asymptotic expansion:

\[
\tilde{M}_{RHP}^{(1)}(y, t, \zeta) = I + \sum_{j=1}^{N} \sum_{l=0}^{\infty} \frac{\tilde{M}_{10}^{(1)}(y)}{\zeta^{l/3}} \mathcal{O}\left( \frac{t^{-(N+1)/3} + t^{-1/3}}{\zeta^{N+1}} \right), \quad \zeta \to \infty. \quad (5.17)
\]

where \( y = x/t^{1/3}, \zeta = kt^{1/3} \) and \( x < 0, t \to +\infty \). More importantly, the (13)-entry of the leading coefficient \( \tilde{M}_{10}^{(1)} \) is given by

\[
(\tilde{M}_{10}^{(1)}(y))_{13} = u_P(y; s), \quad (5.18)
\]

where \( u_P(y; s) \) is the smooth solution of the modified Painlevé II equation

\[
3u''_P(y) - yu_P(y) + 24u_P(y)^2u_P(y) = 0. \quad (5.19)
\]
Finally, considering the solutions of the $\bar{\sigma}$-problem and the RH problem, we obtain

**Theorem 3.** In the Region III, as $t \to \infty$, the solution for Sasa-Satsuma equation has Painleve asymptotic

$$u(x,t) = \frac{1}{t^{1/3}} u_P \left( \frac{x}{t^{1/3}} \right) + \mathcal{O} \left( t^{2/(3p) - 1/2} \right), \quad 4 < p < \infty.$$  

**Remark 5.** The large time asymptotic formula (5.20) indicates that the main contribution to the soliton resolution of the Cauchy initial value problem of the Sasa-Satsuma equation in Region III comes from three parts: 1) Leading term $\frac{1}{t^{1/3}} u_P \left( \frac{x}{t^{1/3}} \right)$ corresponds to the contribution from the jump contours which are matched with the modified Painlevé II model; 2) The second term $\mathcal{O} \left( t^{2/(3p) - 1/2} \right)$ is obtained from the error estimate of the pure $\bar{\sigma}$-problem. The final results have also helped prove an important conclusion that soliton solutions of Sasa-Satsuma equation in Region III are also asymptotically stable.
Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant No. 11671095, 51879045).

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