Test of the Kugo-Ojima Confinement Criterion 
in the Lattice Landau Gauge
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We present the first results of numerical test of the Kugo-Ojima confinement criterion in the lattice Landau gauge. The Kugo-Ojima criterion of colour confinement in the BRS formulation of the continuum gauge theory is given by \( u^a_\beta(0) = -\delta^a_\beta \), where

\[
\int dxe^{i(x-y)}(0)[T_{\mu}e^a(x)g(A_\mu \times e^b)(y)]0 = (g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})u^a_\nu(p^2). (\ast)
\]

We measured the lattice version of \( u^a_\beta(0) \) in use of \( 1/(-D(A)) \) where \( D(A) \) is a lattice covariant derivative in the new definition of the gauge field as \( U = e^A \). We obtained that \( u^a_\beta(0) \) is consistent with \( -c\delta^a_\beta \), \( c = 0.7 \) in \( SU(3) \) quenched simulation data of \( \beta = 5.5 \), on \( 8^4 \) and \( 12^4 \). We report the \( \beta \) dependence and finite-size effect of \( c \).

1. INTRODUCTION

The colour confinement problem in the continuum gauge theory was extensively analysed in use of the BRS formulation by Kugo and Ojima\textsuperscript{[1]}. The QCD lagrangian is invariant under the BRS transformation and the physical space is specified as the one that satisfies the condition \( \mathcal{V}_{phys} = \{ phys \} \)

\[ Q_B|phys = 0. \]

where

\[ Q_B = \int d^3x \left[ B^a D_0 e^a - \partial_0 B^a \cdot e^a + \frac{i}{2} \partial_\mu \tilde{e}^a \cdot (e \times e)^a \right] \]

and \( (F \times G)^a = f_{abc} F^b G^c \).

Under the assumption that BRS singlets have positive metric, it is proved that \( \mathcal{V}_{phys} \) has positive semidefinite in such a way that BRS quartet particles appear only in zero norm.

One finds from the BRS transformation that for each colour \( a \), a set of massless asymptotic fields \( \chi^a, \beta^a, \gamma^a, \tilde{\gamma} \) form a BRS quartet.

The Noether current corresponding to the conservation of the colour symmetry is \( gJ^a_\mu = \partial^\nu F^a_{\mu\nu} + \{ Q_B, D_\mu \tilde{e} \} \), where its ambiguity by divergence of antisymmetric tensor should be understood, and this ambiguity is utilised so that massless contribution may be eliminated for the charge, \( Q^a \), to be well defined.

Denoting \( g(A_\mu \times \tilde{e})^a \to u^a_\mu \partial_\mu \tilde{e}^b \), and then \( D_\mu \tilde{e}^a \to (1 + u)^a_\mu \partial_\mu \tilde{e}^b \), one obtains the eq.\textsuperscript{(*)} provided \( A_\mu \) has a vanishing expectation value. The current \( \{ Q_B, D_\mu \tilde{e} \} \) contains the massless component, \( (1 + u)^a_\mu \partial_\mu \tilde{e}^b(x) \). We can modify the Noether current for colour charge \( Q^a \) such that

\[ gJ^a_\mu = gJ^a_\mu - \partial^\nu F^a_{\mu\nu} = \{ Q_B, D_\mu \tilde{e} \}. \]

In the case of \( 1 + u = 0 \), massless component in \( gJ^a_\mu_0 \) is vanishing and the colour charge

\[ Q^a = \int d^3x \{ Q_B; g^{-1} D_0 \tilde{e}^a(x) \} \]

becomes well defined.

The physical state condition \( Q_B|phys = 0 \) together with the equation \textsuperscript{[1]} implies that all BRS singlet one particle states \( | f \rangle \in \mathcal{V}_{phys} \) are colour singlet states. This statement implies that all coloured particles in \( \mathcal{V}_{phys} \) belong to BRS quartet
and have zero norm. This is the colour confinement.

2. LATTICE CALCULATION OF $u_0^a$

The Faddeev-Popov operator is

$$\mathcal{M}[U] = -(\partial \cdot D(A)) = -(D(A) \cdot \partial),$$

(2)

where the new definition of the gauge field is adopted as $U = e^A$, and the lattice covariant derivative $D_\mu(A) = \partial_\mu + \text{Ad}(A_\mu)$ is given in [3].

The inverse, $\mathcal{M}^{-1}[U] = (M_0 - M_1[U])^{-1}$, is calculated perturbatively by using the Green function of the Poisson equation $M_0^{-1} = (-\partial^2)^{-1}$ and

$$\mathcal{M}^{-1} = M_0^{-1} + \sum_{k=0}^{N_{\text{max}}} (M_0^{-1} M_1)^k M_0^{-1}. \quad (3)$$

In use of colour source $|\lambda^a x\rangle$ normalised as $Tr\langle \lambda^a x|\lambda^b x_0\rangle = \delta^{ab}\delta_{x,x_0}$, the ghost propagator is given by

$$G^{ab}(x,y) = \langle Tr\langle \lambda^a x\rangle\mathcal{M}[U]^{-1}\langle \lambda^b y\rangle \rangle$$

(4)

where the outmost $\langle \rangle$ specifies average over samples $U$.

The ghost propagators of $\beta = 5$ and $5.5$ are almost the same and they are infrared divergent which can be parameterised as $\frac{1}{p^2 + 2}$. We observed that the ghost propagators of $\beta = 6$ is similar to that of $\beta = 5.5$ and its finite-size effect is small [3].

In the similar way, one can calculate the Kugo-Ojima parameter at $p = 0$ as,

$$(g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2})u_0^a(p^2)|_{p=0} = \langle Tr\langle \lambda^a p\rangle D_\mu(A)\mathcal{M}[U]^{-1}\langle \text{Ad}(A_\nu)\rangle|\lambda^b p\rangle \rangle |_{p=0} \quad (5)$$

We observed that off-diagonal element of $u_0^a$ is consistent to zero, but there are statistical fluctuations. The projection operator $g_{\mu\nu} - \frac{p_{\mu} p_{\nu}}{p^2}$ in equation (*) is treated such that it has an expectation value $\frac{1}{4}$ in the limit of $p_{\mu} \to 0$.

Making the accuracy of the covariant Laplacian equation solver higher, we observe the tendency that the expectation value of $|u_0^a|$ increases.

At $\beta = 8$, direct measurement of $u_0^a$ gives a large fluctuation, but suitable $Z_3$ twisting treatment for each sample so that the Polyakov scatter plot should be concentrated around $\arg z = 0$, suppresses the fluctuation and makes the quality of the data better. We consider that this treatment is indispensable in the simulation where $Z_3$ symmetry persists and the $Z_3$ factor affects the observed quantity. The similar behaviour is observed in $\beta = 6, 8^4$ lattice. The minimum Landau gauge fixing via smeared gauge fixing performed at $\beta = 6, 8^4$ lattice does not change the expectation value obtained after the $Z3$ twisting but reduces the standard deviation.

The absolute value of $u_0^a$ is plotted as the function of the spatial extent of the lattice $aL$ where $a$ is calculated by assuming $\Lambda_{\overline{MS}} = 100 MeV$. We find for $aL < 2fm$, there exists large finite-size effect DWe expect that by making $L$ large and $a$ small, such that $aL > 2fm$, the absolute value of $u_0^a$ becomes closer to 1.

Non-symmetric lattice $8^3 \times 16$ yields non-symmetric data in $\mu$ of (*). This fact shows necessity of tuning lattice constants according to the

![Figure 1](image-url)
Table 1

Kugo-Ojima parameter $u_a^\nu$.

| $\beta$        | diag | off-diag | diag1 | diag2 | diag3 | diag4 |
|---------------|------|----------|-------|-------|-------|-------|
| $5.5, 8^4 \times 16$ |      |          |       |       |       |       |
| $5.5, 12^4$    |      |          |       |       |       |       |
| $5.5, 8^4$     |      |          |       |       |       |       |
| $6.0, 12^4$    |      |          |       |       |       |       |
| $6.0, 8^4, \text{with } Z_3$ |      |          |       |       |       |       |
| $6.0, 8^4, \text{with } Z_3, \text{min}$ |      |          |       |       |       |       |
| $6.0, 8^4, \text{no } Z_3$ |      |          |       |       |       |       |
| $8.0, 8^4, \text{with } Z_3$ |      |          |       |       |       |       |
| $8.0, 8^4, \text{no } Z_3$ |      |          |       |       |       |       |

Figure 2. The finite-size effect of the Kugo-Ojima parameter $|u_a^\nu|$ as the function of the spatial extent of the lattice $aL(fm)$.

3. SUMMARY AND DISCUSSION

Proof of Kugo-Ojima colour confinement is accomplished successfully only in case of $u_a^\nu = -\delta_a^\nu$, and this condition is suggested to be a necessary condition as well. We did the first numerical tests of this criterion by the nonperturbative dynamics of lattice Landau gauge. We observed that the value at $\beta = 5.5$ is around $-0.7$. Its absolute value decreases as $\beta$ increases.

We observed the gluon propagator is infrared finite and the ghost propagator is infrared divergent, suggested to be more singular than $\frac{1}{p^2}$, but less singular than $\frac{1}{p^4}$. These results qualitatively agree with the Gribov-Zwanziger’s conjecture, and are consistent with the results of Dyson-Schwinger equation. It is nice to observe that the infrared finiteness of the gluon propagator is in accordance with the Kugo-Ojima colour confinement. As stated in their inverse Higgs mechanism theorem, if we have no massless vector poles in all channels of the gauge field, $A_\mu^\nu$, and if the colour symmetry is not broken at all, it follows that $1 + u = 0$.

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