Comment about quasi-isotropic solution of Einstein equations near cosmological singularity

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Abstract

We generalize for the case of arbitrary hydrodynamical matter the quasi-isotropic solution of Einstein equations near cosmological singularity, found by Lifshitz and Khalatnikov in 1960 for the case of the radiation-dominated Universe. It is shown that this solution always exists, but dependence of terms in the quasi-isotropic expansion acquires a more complicated form.

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In the paper \cite{1} by Lifshitz and Khalatnikov, the quasi-isotropic solution of the Einstein equations near a cosmological singularity was found provided the Universe was filled by radiation with the equation of state $p = \frac{\epsilon}{3}$. The metric of this solution was written down in the synchronous system of reference

$$ds^2 = dt^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta,$$

where spatial metric $\gamma_{\alpha\beta}$ near the singularity has the form

$$\gamma_{\alpha\beta} = ta_{\alpha\beta} + t^2b_{\alpha\beta} + \cdots$$
where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are functions of spatial coordinates. The functions $a_{\alpha\beta}$ are chosen arbitrary, and then the functions $b_{\alpha\beta}$ and also the energy and velocity distributions for matter can be expressed through these functions (for details see [1] and also [2, 3]).

In correspondence with the standard cosmological model of the hot Universe, it was supposed that the natural equation of state for the matter near the cosmological singularity is that of radiation: $p = \varepsilon/3$. However, nowadays the situation has changed in connection with the development of inflationary cosmological models, which as an important ingredient contain inflaton scalar field or/and other exotic types of matter [4]. One can add also that the appearance of brane and M theory cosmological models [5] and the discovery of the cosmic acceleration [6] suggests that the matter playing essential role on different stages of cosmological evolution can obey very different equations of state [4]. Thus, generalization of the old quasi-isotropic solution of the Einstein equations near the cosmological singularity can be useful in this new context.

In this note we make such a generalization for the equation of state:

$$p = k\varepsilon, \quad (3)$$

where $p$ denotes pressure and $\varepsilon$ denotes energy density [1]. The Friedmann isotropic solution near the singularity for such a matter behaves as

$$a \sim a_0 t^m, \quad (4)$$

where

$$m = \frac{4}{3(1 + k)}. \quad (5)$$

We look for an expression for a spatial metric in the following form:

$$\gamma_{\alpha\beta} = t^m a_{\alpha\beta} + t^n b_{\alpha\beta}, \quad (6)$$

where the power index $m$ is given by Eq. (5). We leave the power index $n$ free for some time, requiring only that

$$n > m. \quad (7)$$

1Actually, the authors have known this generalization for a long time but have never published it in detail in regular journals.
The inverse metric reads

$$\gamma^{\alpha\beta} = \frac{a^{\alpha\beta}}{t^m} - \frac{b^{\alpha\beta}}{t^{2m-n}},$$

(8)

where $a^{\alpha\beta}$ is defined by the relation

$$a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\gamma$$

(9)

while the indices of all the other matrices are lowered and raised by $a_{\alpha\beta}$ and $a^{\alpha\beta}$, for example,

$$b^\beta_\alpha = a^{\alpha\gamma} b_{\gamma\beta}.$$  

(10)

Let us write down also expressions for the extrinsic curvature, its contractions and its derivatives:

$$\kappa_{\alpha\beta} \equiv \frac{\partial \gamma_{\alpha\beta}}{\partial t} = mt^{m-1} a_{\alpha\beta} + nt^{n-1} b_{\alpha\beta},$$

(11)

$$\kappa_\alpha^\beta = \frac{m \delta_\alpha^\beta}{t} + \frac{(n-m) b^\beta_\alpha}{t^{m-n+1}},$$

(12)

$$\kappa^\alpha_\alpha = \frac{3m}{t} + \frac{(n-m) b}{t^{m-n+1}},$$

(13)

$$\frac{\partial \kappa^\beta_\alpha}{\partial t} = -m \delta^\beta_\alpha - \frac{(m-n+1)(n-m) b^\beta_\alpha}{t^{m-n+2}},$$

(14)

$$\frac{\partial \kappa^\alpha_\alpha}{\partial t} = -3m \frac{b}{t^2} - \frac{(m-n+1)(n-m) b}{t^{m-n+2}},$$

(15)

$$\kappa^\beta_\alpha \kappa^\alpha_\beta = \frac{3m^2}{t^2} + \frac{2m(n-m) b}{t^{m-n+2}}.$$  

(16)

We need also an explicit expression for the determinant of the spatial metric:

$$\gamma \equiv \det \gamma_{\alpha\beta} = t^{3m} (1 + t^{n-m} b) \det a,$$

(17)

$$\dot{\gamma} \equiv \frac{\partial \gamma}{\partial t} = (3mt^{3m-1} + b(2m + n)t^{2m+n-1}) \det a,$$

(18)

$$\frac{\dot{\gamma}}{\gamma} = \frac{3m}{t} \left( 1 - \frac{b(n-m)t^{n-m}}{3m} \right).$$

(19)

Now, using well-known expressions for the components of the Ricci tensor

$$R^\alpha_0 = -\frac{1}{2} \frac{\partial \kappa^\alpha_\alpha}{\partial t} - \frac{1}{4} \kappa^\beta_\alpha \kappa^\alpha_\beta,$$

(20)

\[\text{[Footnote]}\]
\[ R^0_\alpha = \frac{1}{2}(\kappa^\beta_{\alpha;\beta} - \kappa^\beta_{\beta;\alpha}), \]  
\[ R^\beta_\alpha = -P^\beta_\alpha - \frac{1}{2} \frac{\partial \kappa^\beta_\alpha}{\partial t} - \frac{\dot{\gamma}^\beta}{4\gamma} \kappa^\beta_\alpha, \]  
where \( P^\beta_\alpha \) is a three-dimensional part of the Ricci tensor, and substituting into Eqs. \( (20)-(22) \) the expressions \( (11)-(19) \), one get

\[ R^0_0 = 3 \frac{m(2 - m)}{2t^2} + \frac{(n - 1)(n - m)b}{2t^{m-n+2}}, \]

\[ R^0_\alpha = \frac{n - m}{2t^{m-n+1}}(v^\beta_{\alpha;\beta} - b_\alpha), \]

\[ R^\beta_\alpha = -\frac{\ddot{P}^\beta_\alpha}{t^m} + \frac{m(2 - 3m)\delta^\beta_\alpha}{4t^2} + \frac{(n - m)(2 - 2n - m)b^\beta_\alpha}{4t^{m-n+2}} - \frac{m(n - m)b\delta^\beta_\alpha}{4t^{m-n+2}}. \]

Notice, that in Eq. \( (23) \) \( \ddot{P}^\beta_\alpha \) denotes a three-dimensional Ricci tensor constructed by using the metrics \( a_{\alpha\beta} \). The terms in the curvature tensor \( P^\beta_\alpha \), which are proportional to \( \beta_{\alpha\beta} \) have the time dependence \( \sim \frac{1}{t^{2m-n}} \) and are less divergent than the first term in the right-hand side of Eq. \( (25) \) provided the condition \( (4) \) is satisfied.

Now, let us write down the expressions for the components of the energy-momentum tensor of the perfect fluid

\[ T_{ik} = (\varepsilon + p)u_i u_k - pg_{ik}, \]

satisfying the equation of state \( (3) \). Up to higher-order corrections, they have the following form:

\[ T^0_0 = \varepsilon \]

\[ T^0_\alpha = \varepsilon(k + 1)u_\alpha, \]

\[ T^\beta_\alpha = -k\varepsilon\delta^\beta_\alpha, \]

\[ T = T^i_i = \varepsilon(1 - 3k). \]

Using the Einstein equations

\[ R^j_i = 8\pi G(T^j_i - \frac{1}{2}\delta^j_i T), \]
one has from 00-component of these equations:

$$8\pi G\varepsilon = \frac{1}{3k+1}\left(\frac{3m(2-m)}{2t^2} - \frac{(n-1)(n-m)b}{t^{m-n+2}}\right), \quad (32)$$

and from 0α-component of these equations one has

$$u_\alpha = \frac{(n-m)(3k+1)(b^2_{\alpha\beta} - b_\alpha)t^{n+1-m}}{3m(2-m)(k+1)}. \quad (33)$$

Now, writing down the spatial components of the Einstein equations, using the expressions (29)-(30) and the expression (32) for the energy density $\varepsilon$, one get:

$$-\frac{\tilde{\mathcal{P}}^\beta_{\alpha}}{t^m} + \frac{m(2-3m)\delta^\beta_\alpha}{4t^2} + \frac{(n-m)(2-2n-m)b^\beta_\alpha}{4t^{m-n+2}} - \frac{m(n-m)b\delta^\beta_\alpha}{4t^{m-n+2}} = \frac{(k-1)\delta^\beta_\alpha}{3k+1}\left(\frac{3m(2-m)}{2t^2} - \frac{(n-1)(n-m)b}{t^{m-n+2}}\right). \quad (34)$$

Using the relation (5) it is easy to check that the terms proportional to $\frac{1}{t^2}$ in the left- and right-hand sides of Eq. (34) cancel each other. On the other hand, the only way to cancel the term $\frac{\tilde{\mathcal{P}}^\beta_{\alpha}}{t^m}$ is to require that the terms proportional to $\frac{1}{t^{m-n+2}}$ behave as the term $\frac{1}{t^m}$, i.e.

$$n = 2. \quad (35)$$

In this case, the condition of cancellation of terms proportional to $\frac{1}{t^m}$ gives the following expression for the tensor $b^{\beta}_{\alpha}$:

$$b^{\beta}_{\alpha} = \frac{4\tilde{\mathcal{P}}^\beta_{\alpha}}{m^2-4} + \frac{\tilde{\mathcal{P}}\delta^\beta_\alpha(-3m^2 + 12m - 4)}{3m(m-3)(m^2-4)}. \quad (36)$$

Using the relation (5) one can rewrite the expression (36) in the following form:

$$b^{\beta}_{\alpha} = -\frac{9(k+1)^2}{(3k+5)(3k+1)}\left(\frac{\tilde{\mathcal{P}}^\beta_{\alpha}}{9k+5} + \frac{(3k^2 - 6k - 5)\delta^\beta_\alpha\mathcal{P}}{9k+5}\right). \quad (37)$$

It is easy to see that the Eq. (37) expressing the second-order correction for the spatial metric (2) is well defined for all the types of hydrodynamical matter with $0 \leq k \leq 1$, including the stiff matter, i.e. a fluid with the equation of state $p = \varepsilon$. 


Now, using the relation
\[ \tilde{P}\beta^{\alpha} = \frac{1}{2} \tilde{P}_{\beta}, \]  
(38)

and the formulae (37) and (33) we arrive to the following expression for the three-dimensional velocity \( u_\alpha \):
\[ u_\alpha = -\frac{27k(k+1)^3 \tilde{P}_{\alpha}}{8(3k+5)(9k+5)} t^{3-\frac{4}{3(k+1)}}. \]  
(39)

Note that the velocity flow is potential. Actually, it can be shown that this important property remains in all higher orders of perturbative expansion for the quasi-isotropic solution. Similarly, the expression for the energy density of matter (32) can be rewritten as
\[ 8\pi G\varepsilon = \frac{4}{3(k+1)^2 t^2} + \frac{9(k+1) \tilde{P}}{2(9k+5)t^{\frac{4}{3(k+1)}}}. \]  
(40)

It is straightforward to check that further terms of perturbative expansion (2) for the metric \( \gamma_{\alpha\beta} \) have the following form:
\[ \gamma_{\alpha\beta} = t^m a_{\alpha\beta} + t^2 b_{\alpha\beta} + t^{2+(2-m)} c_{\alpha\beta} + \cdots. \]  
(41)

Thus, we have seen that this expansion has a curious feature. The order of its first term \( t^m \) is defined by the equation of state of the matter (3), the second order term always has the behavior \( \sim t^2 \), while logarithmic distance between orders is equal to \( 2 - m \).

It is easy to understand that the quasi-isotropic expansion does work if and only if the first term in the right-hand side of Eq. (3) is smaller than the second one. Remembering that \( n = 2 \) for any value of \( m \) and using the equation (5), we get the following restriction on the parameter \( k \) from the equation of state (3):
\[ k > -\frac{1}{3}. \]  
(42)

Thus, for the values \( k \leq -1/3 \) the quasi-isotropic expansion at small times may not be constructed. It is interesting to notice, however, that for \( k = const < -\frac{1}{3} \), a quasi-isotropic-like solution arises as a late-time \( (t \rightarrow \infty) \) attractor for generic inhomogeneous evolution of space-time \( \mathbb{R}^4 \) (this regime is the power-law inflation [4] actually). Of course, perturbative expansion is
made in inverse powers of $t$ in that case. Also, we would like to note that
different aspects of relation between the quasi-isotropic expansion and other
approximation schemes were considered in detail in the papers [10].

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