Correction to “An Efficient Game Form for Unicast Service Provisioning”

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Abstract—A correction to the specification of the mechanism proposed in [1] is given.

Index Terms—Budget balance, game form/mechanism, individual rationality, Nash implementation, Unicast service provisioning.

Due to an error, the mechanism presented in [1] has a tax function which is not differentiable with respect to the allocations. We need a tax function which is differentiable with respect to the allocations so that we can have Nash implementation. We correct this error as follows.

We consider the problem formulated in [1]. We use the same notation as in [1].

Specification of the game form/mechanism:
Message space: The message space is the same as that of the mechanism presented in [1]. A message of user \( i \in \mathcal{N} \) (\( \mathcal{N} \) denotes the set of users) is of the form
\[
m_i = (x_i, p_i^1, p_i^2, \ldots, p_i^{l(|R_i|)}),
\]
where \( x_i \) denotes the (non-negative) bandwidth user \( i \) requests at all the links of his route, and \( p_i^{l_{jk}} \geq 0 \) denotes the price user \( i \) is willing to pay per unit of bandwidth at link \( l_{jk} \) of his route \( R_i \).

Outcome function: For any \( m \in \mathcal{M} \), the outcome function is defined as follows:
\[
f(m) = (x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_n)
\]
where \( t_i \) is the tax paid by user \( i \) for using link \( l \). The form of \( t_i \) is the same as the tax function defined in [1] excluding the term that is of the form described by relation (23) in [1]. For example, if \( |\mathcal{G}^l| > 3 \), \( (\mathcal{G}^l) \) denotes the set of users using link \( l \) the tax function in Eq. (13) of [1] now becomes,
\[
t_i^l = P^l_{-i}x_i + (p_i^l - P^l_{-i} - c_l)^2
- 2P^l_{-i}(p_i^l - P^l_{-i}) \left( \frac{\mathcal{E}^l_{-i} + x_i}{\gamma} \right) + \Phi_i^l,
\]
where
\[
\zeta_*^l = \max \{0, \frac{\sum_{i \in \mathcal{G}^l} x_i - c_l}{\gamma}\}, \quad (2)
\]
c\( ^l \) is the capacity of link \( l \), \( \Phi_i^l \) is defined by Eq. (14) in [1].

\[
P^l_{-i} = \frac{\sum_{j \in \mathcal{G}^l \setminus i} p_j^*}{|\mathcal{G}^l| - 1}, \quad \mathcal{E}^l_{-i} = \sum_{j \in \mathcal{G}^l \setminus i} x_j - c_l, \quad (3)
\]

\( \mathcal{P}^l_{-i} \) and \( \mathcal{E}^l_{-i} \) are the same as in [1] and \( \gamma, \tilde{\gamma} \), are positive constants.

This completes the specification of the mechanism.

Based on the above specification, the proof of Lemma 2 in [1] is updated as follows.

**Proof of Lemma 2 in [1]:** Let \( m^* = (m_1^*, m_2^*, \ldots, m_n^*) \) be a NE of the game induced by the mechanism. Since user \( i \) does not control \( \Phi_i^l \), it implies \( \frac{\partial \mathcal{E}_i}{\partial p_i^l} = 0 \), (as in Eq. (34) of [1]). By following the same steps as in equations (35-38) of [1], we obtain for any \( l \in L \):
\[
\frac{\partial t_i^l}{\partial p_i^l} \bigg|_{m=m^*} = 2 \left[ (p_i^l - P^l_{-i} - \zeta_+^l) - P^l_{-i} \left( \frac{\mathcal{E}^l_{-i} + x_i^*}{\gamma} \right) \right] = 0. \tag{4}
\]

Summing (4) over all \( i \in \mathcal{G}^l \), we get
\[
\sum_{i \in \mathcal{G}^l} \frac{\partial t_i^l}{\partial p_i^l} \bigg|_{m=m^*} = \sum_{i \in \mathcal{G}^l} \left[ (p_i^l - P^l_{-i} - \zeta_+^l) - P^l_{-i} \left( \frac{\mathcal{E}^l_{-i} + x_i^*}{\gamma} \right) \right] = -|\mathcal{G}^l| \zeta_+^l - \sum_{i \in \mathcal{G}^l} P^l_{-i} \left( \frac{\mathcal{E}^l_{-i} + x_i^*}{\gamma} \right) = 0. \tag{5}
\]

Suppose \( \sum_{i \in \mathcal{G}^l} x_i^* > c_l \). Then we must have, \( \zeta_+^l > 0 \) and \( \sum_{i \in \mathcal{G}^l} P^l_{-i} \left( \frac{\mathcal{E}^l_{-i} + x_i^*}{\gamma} \right) \geq 0 \). But this contradicts Eq. (5). Therefore, we must have
\[
\sum_{i \in \mathcal{G}^l} x_i^* \leq c_l. \tag{6}
\]

This implies,
\[
\zeta_*^l = 0. \tag{7}
\]

Combining (7) along with (5) we obtain
\[
\sum_{i \in \mathcal{G}^l} P^l_{-i} \left( \frac{\mathcal{E}^l_{-i} + x_i^*}{\gamma} \right) = 0. \tag{8}
\]

Moreover, combining (6) and (8) we obtain
\[
P^l_{-i} \left( \frac{\mathcal{E}^l_{-i} + x_i^*}{\gamma} \right) = 0. \tag{9}
\]

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for every $i \in \mathcal{G}^l$. Using (7) and (9) in (4) we obtain

$$p^*_{li} = P^*_{li}. \tag{10}$$

Since (10) is true for all $i \in \mathcal{G}^l$, it implies,

$$p^*_{li} = p^*_{lj} = P^*_{li} =: p^*_l, \tag{11}$$

and along with (9) it implies

$$p^*_lE^*_{li} = 0 \tag{12}$$

where $E^*_{li} = \sum_{i \in \mathcal{G}^l} x^*_i - c^l$ ($E^*_{li}$ is the same as in (1)).

Furthermore, since

$$\frac{\partial \Phi^l}{\partial x_i} = 0 \tag{13}$$

(Eq. (34) in [1]), it follows from (1) that

$$\frac{\partial h^l}{\partial x_i} \big|_{m=m^*} = p^*_l. \tag{14}$$

because of (7), (11), (12), and (13).

Remark 1. The proof of Theorem 5 follows when $x^*_i > 0$. Note that, when $x^*_i = 0$, since user $i$ does not have incentive to increase its demand, it follows that

$$\frac{\partial U_i(x_i)}{\partial x_i} - \sum_{i \in \mathcal{R}_i} p^*_l \big|_{m=m^*} \leq 0. \tag{15}$$

Now, set $\lambda^*_{li} = p^*_{li}$. Then (12) and (15) are consistent with the KKT conditions (68-70) of [1].

I. PROPERTIES OF THE MECHANISM

Existence of Nash equilibria (NE): The proof of existence of NE of the game induced by the mechanism is the same as in [1] (see Theorem 6, page 398, and its proof in [1]; also see the proof of Theorem 7).

Feasibility of allocations at NE: Because of the specification of the mechanism and Eq. (7), the allocations corresponding to all NE are in the feasible set.

Budget Balance at any feasible allocation: Budget balance at any feasible allocation follows by Lemma 3 of [1].

Individual Rationality: Individual rationality follows by Theorem 4 of [1].

Nash implementation: Nash implementation follows by Theorem 5 of [1].

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REFERENCES

[1] A. Kakhbod, D. Teneketzis, An efficient game form for unicast service provisioning. IEEE Transactions on Automatic Control, Vol 57, No. 2, February 2012, pp. 392-404.