Inverse Frame Domination in Graphs

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Abstract:
The generalization of the concept of inverse frame domination on graphs is the fundamental point of this work. It has been shown to present modern properties on inverse frame domination and some corresponding theorems related to delete, add edges or remove vertices. The relationship between the initial graph and a graph obtaining from the edges of the contraction was also discussed.

Key words: Domination, Frame domination, Inverse frame domination.

Mathematical subject classification: 05C69

1. Introduction:

We’re dealing with undirected and simple graphs throughout the whole study. For a graph \( G = (V, E) \), let \( V(G) \) and \( E(G) \), respectively, denote to its vertex and edge set. The complement of \( G \), denoted by \( \overline{G} \), is the graph in which two vertices are adjacent if and only if they are not adjacent in \( G \). \( G[D] \) is the induced subgraph of \( G \) with \( D \) vertices. \( G − v \) and \( G − e \) are the graphs that obtained by removing the vertex \( v \) or edge \( e \) from \( G \) respectively. \( G + e \) is also the graph obtained by \( G \) by adding the edge \( e \) where \( e \in \overline{G} \). Let \( e = uv \in E(G) \), the operation \( G / e \) referred to contraction \( e \) obtained by removing the edge \( e \) from the graph and merging the vertices \( u, v \) to a new vertex. All other edges incident to \( u \) or \( v \) become a new vertex incident. For more information one can see (1), (2), (3), (4).

Domination is one of the most important concept in graph theory that has attracted many researchers recently because it has the potential to solve many real-life issues involving communication networks design and analysis, as well as defense surveillance. Furthermore, there are different fields of graph theory as a topological graph (5) and labeled graph (6), (7), and the others. In a simple graph \( G(V, E) \), a subset \( D \subseteq V \) is called a dominating set if every vertex in \( V − D \) is adjacent to a vertex in \( D \). The minimum cardinality of a dominating set is called a domination number, and is denoted by \( \gamma(G) \) (8). If \( V − D \) contains a dominating set \( D^{-1} \) of \( G \), then \( D^{-1} \) is called an inverse dominating set. The minimum cardinality of an inverse dominating set is called an inverse domination number \( \gamma^{-1}(G) \) (9). For more details
about inverse domination see (10) and (11). The existing literature contains many variant of domination models. Hedetniemi and Laskar in 1990 have been provide a comprehensive bibliography of articles on the concept of dominance (12). Ore in 1962 introduced the concept of domination number \( \gamma(G) \) (13). In 2017, a frame domination has been introduced by A. Omran and Y. Rajihy (14), while a total frame domination has been introduced by Y. Rajihy in 2018 (15). Finally, a new domination named Hn-Domination in Graphs has been proposed by A. Omran and H. Oda in 2019 (16). In this work, we define the inverse frame domination in graphs. We also address the effect of adding, removing and contract an edge. In addition, we evaluate changing and unchanging of inverse frame domination number in some special of graphs.

2. Inverse Frame Domination in Graphs

We start with the definition of inverse frame dominating set and inverse frame domination number of graphs

**Definition 2.1 (14)**: Let \( G = (V, E) \) be graph and let \( F \subseteq V(G) \) is a minimum frame dominating set of \( G \). The set \( \hat{F} \) is called an inverse frame dominating set with respect to \( F \) if \( V - F \) contains a frame dominating set of \( G \).

**Definition 2.2 (14)**: The minimum cardinality of an inverse frame dominating set of \( G \) is called the inverse frame domination number, and is denoted by \( \gamma_f^{-1}(G) \).

**Example 2.3**: Let \( G = (V, E) \) be graph shown in figure in 2.1 below

\[ F = \{v_3, v_6\} \] is a frame dominating set of \( G \) and \( \gamma_f(G) = 2 \).

\[ V - F = \{v_1, v_2, v_4, v_5\} \] contain a frame dominating set.

\[ \hat{F} = \{v_2, v_4\} \] is minimum inverse frame dominating set and \( \gamma_f^{-1}(G) = 2 \).

**Observation 2.4**: \( \gamma_f^{-1}(C_n) = \gamma_f^{-1}(K_{m,n}) = \gamma_f^{-1}(K_n) = \gamma_f^{-1}(W_n) = 1 \)

The bounded differ from one graph to another.

**Proposition 2.5**: Let \( G = (V, E) \) be a graph of order \( n \):

1) If \( G \) has a vertex not contains in a cycle, then \( G \) has no inverse frame dominating set.
2) If \( G \) has an inverse frame domination number, then \( \gamma_f^{-1}(G) \geq \gamma_f(G) \)
3) If \( G \) is a Hamilton graph, then \( G \) has an inverse frame dominating set and \( \gamma_f^{-1}(G) = 1 \).
4) If $G$ contains $m$ bridges and has an inverse frame domination number, then $\gamma_f^{-1}(G) = m + 1$.

5) If $G$ is a disconnected graph with $H$ components and has an inverse frame domination number, then $\gamma_f^{-1}(G) \geq H$.

**Proof 1**) It's clear that each vertex of a degree less or equal to one does not contain any cycle, we can't dominate those vertices by inverse frame dominating set. 2), 3), 4), and 5) it is obvious.

**Theorem 2.6**: The inverse frame domination number of some special graphs is

1) $\gamma_f^{-1}(\overline{C_n}) = 1, n \geq 5$

2) $\gamma_f^{-1}(\overline{K_{n,m}}) = 2, n, m \geq 3$

3) $\gamma_f^{-1}(\overline{P_n}) = 1, n \geq 5$

4) $\gamma_f^{-1}(\overline{N_n}) = 1$

**Proof:**

1) If $G = C_n; n \geq 5$, then $\overline{G} = \overline{C_n}$ is a cycle or cycle with common edge (edges). Therefore, from observation 2.4 $\gamma_f^{-1}(\overline{C_n}) = 1$ (see figure 2.2. a).

2) If $G = K_{n,m}, n, m \geq 3$, then $\overline{G} = \overline{K_{n,m}}$ is a disconnected graph with two components $k_1$ and $k_2$ such that every component represents a complete graph, on other word $k_1 = K_n$ and $k_2 = K_m$. By observation 2.4 $\gamma_f^{-1}(K_N) = \gamma_f^{-1}(K_m) = 1$. Hence $\gamma_f^{-1}(\overline{K_{n,m}}) = \gamma_f^{-1}(k_1) + \gamma_f^{-1}(k_2) = \gamma_f^{-1}(K_n) + \gamma_f^{-1}(K_m) = 2$ (see figure 2.2. b).

3) If $G = P_n; n \geq 5$, then $\overline{P_n}$ is a cycle with common edges. Hence $\gamma_f^{-1}(\overline{P_n}) = 1$ (see figure 2.2. c).

4) If $G = N_n; n \geq 3$, then $\overline{N_n} = K_n$. From observation 2.4 $\gamma_f^{-1}(\overline{K_n}) = 1$.
Remark 2.7: The complement graph of $[C_n; n = 3,4, P_n; n \leq 4, K_{2,2}, K_{2,m}, K_n, N_n; n \leq 2, W_n, S_n]$ has no inverse frame dominating set.

Theorem 2.8: If a graph $G = (V, E)$ has an inverse frame domination number $\gamma_f^{-1}(G)$ and $\forall v \in V(G)$, then $G - v$ either has no inverse frame domination number or $\gamma_f^{-1}(G - v) \geq \gamma_f^{-1}(G)$.

Proof: Let a graph $G = (V, E)$ has an inverse frame domination number and $v \in V(G)$, we have two different cases as follows.

Case 1. If $v$ belong to a cycle of order $n$, then $G - v$ has a vertex does not belong to any cycle in $G - v$, according to proposition 2.5(1), $G - v$ doesn't have an inverse frame dominating set (see figure 2.3.a.1).

Case 2. If $v$ belong to more cycles, we discuss two subcase as following:

Subcase 1. If $G$ contains a cycle of order $n \geq 4$ and all vertices that dominated by the vertex $v$ belong to either one cycle or two (more) cycles and there is a vertex common with these cycles, then $\gamma_f^{-1}(G - v) = \gamma_f^{-1}(G)$ (see figure 2.3.a.2)

Subcase 2. If $G$ connected graph and all vertices which dominated by the vertex $v$ constitute a partite from two or more cycles, then $G - v$ became disconnected graph $\gamma_f^{-1}(G - v) > \gamma_f^{-1}(G)$ (see figure 2.3.b)

From all cases above, the result is obtained.
Theorem 2.9: Let $G = (V, E)$ be a graph has an inverse frame domination number $\gamma_f^{-1}(G)$ and $e = uv \in \bar{G}$, then $\gamma_f^{-1}(G + e) \leq \gamma_f^{-1}(G)$.

Proof: There are three cases as follows.

Case 1. If $u$ and $v$ belong to a cycle of order $n \geq 4$, then $G + e$ contain a cycle with common edge. This adding does not effect on inverse frame domination number $\gamma_f^{-1}(G + e) = \gamma_f^{-1}(G)$ (see figure 2.4. a).

Case 2. If $u$ and $v$ belong to different cycles have only one common vertex, then $G + e$ contain a cycle join all vertex belong to $G$. Hence $\gamma_f^{-1}(G + e) < \gamma_f^{-1}(G)$ (see figure 2.4.b).

Case 3. If $u$ and $v$ belong to disjoint cycles, then $e$ becomes a bridge in $G + e$, then this adding edge does not effect on the inverse frame domination number. Hence $\gamma_f^{-1}(G + e) = \gamma_f^{-1}(G)$

From all cases above, the result is obtained.

Figure 2.3: Effect of deleting a vertex from a graph

\begin{itemize}
  \item (1) $G - v_5 = P_4$
  \item (2) $G - v_4 = C_3$
\end{itemize}
Theorem 2.10: Let $G = (V, E)$ be a graph has an inverse frame domination number $\gamma_{f}^{-1}(G)$ and $e \in E(G)$, then $G - e$ either has no inverse frame dominating set or $\gamma_{f}^{-1}(G - e) \geq \gamma_{f}^{-1}(G)$.

Proof: Let $G = (V, E)$ has an inverse frame dominating set and $uv = e \in E(G)$, we will study the following different cases:

Case 1. If $u$ and $v$ belong to the same cycle of order $n$, then $G - e$ represent a path, according to proposition 2.5(1), $G - e$ has no inverse frame dominating set (see figure 2.5. a).

Case 2. If $u$ and $v$ belong to more cycles of order $n \geq 4$, then $G - e$ either a cycle and this deleting not effect on inverse frame domination number hence $\gamma_{f}^{-1}(G - e) = \gamma_{f}^{-1}(G)$ or a cycle with common edge, according to proposition 2.5(1), $G - e$ has no inverse frame dominating set.

Case 3. If $e = uv$ is a bridge between two different cycles, then $G - e$ became disconnected graph $\gamma_{f}^{-1}(G - e) = \gamma_{f}^{-1}(G)$ (see figure 2.5.b).

Case 4. If $u$ and $v$ join two different cycles has only common vertex $w; u \neq w \neq v$, then $G - e$ became connected graph by common vertex. Hence $\gamma_{f}^{-1}(G - e) > \gamma_{f}^{-1}(G)$(see figure 2.5.c).

Figure 2.4: Effect of adding an edge to a graph
**Theorem 2.11:** Let \( G(V, E) \) has an inverse frame domination number and \( e \in E(G) \). Then \( G/e \) either has no inverse frame dominating set or \( \gamma_f^{-1}(G/e) \geq \gamma_f^{-1}(G) \).

**Proof:** We have two cases:

**Case 1.** There are two subcases such that \( G/e \) has no inverse frame dominating set as follows.

- **Subcase 1.** If \( e \) belong to cycle of order 3, then \( G/e \equiv P_2 \), then according to proposition 2.5(1), \( G/e \) has no inverse frame dominating set.

- **Subcase 2.** If there is an edge join two vertices in a cycle say \( u \) and \( v \) and \( d(u, v) = 2 \), then \( G/e \) has at least one vertex doesn’t contain a cycle in \( G/e \), according to proposition 2.5(1), \( G/e \) has no inverse frame dominating set (as an example, see figure 2.6.a).

**Case 2.** If \( e \) belongs to one or more cycles, we have two subcases as following:

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**Figure 2.5:** Effect of deleting an edge from a graph
Subcase 1. If \( G \) contains a cycle of order \( n; n \geq 4 \) and \( e \) belong to this cycle, then the contraction edge does not effect on inverse frame domination number. Hence, \( \gamma_{f}^{-1}(G/e) = \gamma_{f}^{-1}(G) \).

Subcase 2. If \( G \) contains a cycle of order \( n \) and there is an edge \( e = uv \) such that \( d(u, v) \geq 3 \), then this cycle convert to two cycles common by one vertex. Thus, \( \gamma_{f}^{-1}(G/e) > \gamma_{f}^{-1}(G) \) (see figure 2.6.b)

3. Conclusion:

In this work the main concept of inverse frame domination has been generalized to the different types of graphs. Also we have added some advanced properties through different proved theorems related to inverse frame domination number. In addition, we determined the upper and lower bound on inverse frame domination number for different types of graph like, cycle, null, complete, and wheel graph. Finally, we concluded that the domination number of some graphs is affects when added, removed or contraction an edge as well as when removed a vertex.

References

1. Harary F. Graph theory. USA: Addison-Wesley Reading MA USA.; 1969.
2. Gross JL, Yellen J, Zhang P. Handbook of graph theory. Second Edition. Journal of Development Economics. 2013.
3. West DB. Introduction To Graph Theory ,Solution Manual. Prentice Hall. 2005.
4. Thulasiraman K, Arumugam S, Brandstädt A, Nishizeki T. Handbook of graph theory,
combinatorial optimization, and algorithms. Handbook of Graph Theory, Combinatorial Optimization, and Algorithms. 2016.

5. A. A. Jabor., A. A. Omran, Domination in Discrete Topology Graph, AIP, Third International Conference of Science(ICMS2019), Vol.2183(2019): 030006-1–030006-3; https://doi.org/10.1063/1.5136110.

6. M.N. Al-Harere, A. A. Omran, Binary operation graphs, AIP conference proceeding vol.2086, Maltepe University, Istanbul, Turkey,030008, 31 July - 6 August (2018). https://doi.org/10.1063/1.5095093.

7. M.N. Al-Harere, A. A. Omran, On binary operation graphs, Boletim da Sociedade Paranaense de Matematica, Vol 38 No7, 59-67, 2020.

8. Haynes TW. Fundamentals of Domination in Graphs. Fundamentals of Domination in Graphs. 2013.

9. Kulli VR, Sigarkanti SC. Inverse domination in graphs (incorrect proof!). Nat Acad Sci Lett [Internet]. 1991;14(12):473–5. Available from: http://scholar.google.com/scholar?hl=en&btnG=Search&q=intitle:Inverse+domination+in+graphs #0

10. Cynthia VJA, Kavitha A. INVERSE DOMINATION NUMBER OF GENERALIZED PETERSEN. 2017;109(10):69–77.

11. Kulli VR. Inverse and Disjoint Secure Total Domination in Graphs. 2016;12(1):23–9.

12. Hedetniemi ST, Laskar RC. Bibliography on Domination in Graphs and Some Basic Definitions of Domination Parameters. Ann Discret Math. 1991;48(C):257–77.

13. Ore O. Theory of graphs. American Mathematical Soc.; 1965.

14. A. A. Omran and Y. Rajihy, Some properties of frame domination in graphs, Journal of Engineering and Applied Sciences, 12, 8882-8885, (2017).

15. Rajihy Y. Some properties of total frame domination in graphs. J Eng Appl Sci. 2018;13(Specialissue1).

16. A. A. Omran and Haneen H. Oda, Hn-Domination in Graphs, Baghdad Science Journal 16.1, 242-247, (2019).