Abstract. We investigate a self-interacting random walk, whose dynamically evolving environment is a random tree built by the walker itself, as it walks around. At time \(n = 1, 2, \ldots\), right before stepping, the walker adds a random number (possibly zero) \(Z_n\) of leaves to its current position. We assume that the \(Z_n\)'s are independent, but, importantly, we do not assume that they are identically distributed.

We obtain non-trivial conditions on their distributions under which the random walk is recurrent. This result is in contrast with some previous work in which, under the assumption that \(Z_n \sim \text{Ber}(p)\) (thus i.i.d.), the random walk was shown to be ballistic for every \(p \in (0, 1]\). We also obtain results on the transience of the walk, and the possibility that it “gets stuck.”

From the perspective of the environment, we provide structural information about the sequence of random trees generated by the model when \(Z_n \sim \text{Ber}(p_n)\), with \(p_n = \Theta(n^{-\gamma})\) and \(\gamma \in (2/3, 1]\). We prove that the empirical degree distribution of this random tree sequence converges almost surely to a power-law distribution of exponent 3, thus revealing a connection to the well known preferential attachment model.

1. Introduction

The study of random walks on graphs that change over time has received increasingly more attention in the past decades, and has been the source of many new results in theoretical probability. Random walks on dynamic graphs encompasses several models: random walks in dynamic random environment \([4,18,25]\), reinforced random walks \([3,14,16,24,29]\) and excited random walk \([7,8,21,22,30]\).

In all of these models, the underlying graph structure, more precisely the set of edges, is fixed and the graph dynamics reduces to a change in time of the transition probabilities of the walker. In random walk in dynamic random environment, the change in time is driven by a random process independent of the walker dynamics, whereas in reinforced and excited random walks (a.k.a., *self-interacting* random walks), it is coupled with the random walk trajectories.

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Recently, models of random walks which build their graph while walking have been introduced, mutually coupling the random walk and the graph dynamics [17, 19, 20]. While they can be thought of as self-interacting random walks, in a different way than in the predecessor models, they do not a-priori constrain the graph structure; the set of edges (as well as the vertex-set) changes with time and strongly depends on the random walk trajectories (and vice versa). In this paper, we study a model which belongs to this latter class; it is a particular case of the Tree Builder Random Walk (TBRW), introduced in [20].

Notation: As usual, for two sequences of positive numbers, \( a_n = O(b_n) \) will mean that \( a_n / b_n \) remains bounded from above; \( a_n = o(b_n) \) will mean that \( a_n / b_n \to 0 \) and \( a_n = \Theta(b_n) \) will mean that \( a_n/b_n \) remains bounded between two positive constants. By \( X_n = O(Y_n) \) for two sequences of random variables we mean that \( \exists K > 0 \) for which \( \limsup_n X_n / Y_n \leq K \) a.s.; we use \( o, \Theta \) for random variables in a similar manner.

For typographical reasons we will often write \( 1\{A\} \) instead of \( \mathbb{1}_A \). The Bernoulli distribution with parameter \( p \) will be denoted by \( \text{Ber}(p) \), and \( d_{TV} \) will denote the total variation distance between probability measures. Finally, for us \( \mathbb{N} \) will include zero, that is, \( \mathbb{N} := \{0, 1, 2, \ldots\} \).

1.1. The model. The model is parsimonious and depends on a sequence of probability laws \( \mathcal{L} := L_1, L_2, \ldots, \) each \( L_n \) supported on nonnegative integers, and a pair \( (T_0, x_0) \), where \( T_0 \) is a locally finite rooted tree with a self-loop \(^1\) attached at the root and \( x_0 \) is a vertex of \( T_0 \). The model is a stochastic process \( \{(T_n, X_n)\}_{n \geq 0} \) on trees with a marked vertex (the current position of the walker), defined inductively. Given \( (T_n, X_n) \) we obtain \( (T_{n+1}, X_{n+1}) \) according to the rule below:

(1) Generate \( T_{n+1} \): create a nonnegative random number of new vertices, independently of the history of the process up to time \( n \), according to \( L_n \) and connect them to \( X_n \);
(2) Obtain \( X_{n+1} \): given \( T_{n+1} \), choose uniformly (and independently from everything else) a neighbor of \( X_n \) in \( T_{n+1} \); this vertex will be \( X_{n+1} \).

Note that at every time \( n \), first the tree \( T_n \) may be modified (by the possible addition of new leaves) and then the random walk takes a step on the possibly modified tree \( T_{n+1} \).

We refer to this model as \( \mathcal{L} \)-TBRW to emphasize the dependence on the sequence \( \mathcal{L} := \{L_n\}_{n \geq 1} \), which accounts for different probabilities of adding new vertices to the tree along the evolution. We denote by \( \mathbb{P}_{x_0, T_0; \mathcal{L}} \) the law of \( \{(T_n, X_n)\}_{n \geq 0} \) when \( (T_0, X_0) = (T_0, x_0) \) and by \( \mathbb{E}_{x_0, T_0; \mathcal{L}} \) the corresponding expectation.

It will be helpful to introduce also a sequence of independent nonnegative integer valued random variables \( Z := \{Z_n\}_{n \geq 1} \), such that \( Z_n \sim L_n \). That is, \( Z_n \) is the number of leaves added at time \( n \).

We reserve the letters \( P \) and \( E \) for \( P := L_1 \times L_2 \times \ldots \) and for the corresponding expectation.

\(^1\)The role of the self-loop is to avoid periodicity.
Finally, $\mathcal{L}^{(m)}$ will denote the shifted sequence of laws $\mathcal{L}^{(m)} = \{L_{m+n}\}_{n \geq 1}$. When $\tau$ is an $\mathbb{N}$-valued stopping time with respect to the filtration generated by $Z_1, Z_2, \ldots$, we will also use the notation $\mathcal{L}^{(\tau)}$ for the randomly shifted sequence $\mathcal{L}^{(\tau)} = \{L_{\tau+n}\}_{n \geq 1}$.

The behavior of the process $\mathcal{L}$-TBRW may be studied from different perspectives. For example, one may look at $\mathcal{L}$-TBRW as a non-markovian, self-interacting random walk $\{X_n\}_{n \geq 0}$ whose environment is dynamically built by the walker trajectories. From this first perspective, understanding the dichotomy of transience/recurrence and questions such as ballisticity and localization are natural.

Another interesting point of view consists of looking at $\mathcal{L}$-TBRW as a random graph model. From this second perspective questions concerning the structure and degree distribution of the random sequence of trees $\{T_n\}_{n \geq 0}$ stand out.

There is also a third perspective, which, in a way, is between the above two. The model $\mathcal{L}$-TBRW may be seen as a Markov chain $\{(T_n, X_n)\}_{n \geq 0}$, in the Polish space of locally finite rooted trees (see [11] for an introduction to this space), i.e., each pair $(T_n, X_n)$ may be interpreted as a tree $T_n$ rooted at $X_n$. From this perspective, the existence of stationary measures and the long-time behavior of the random rooted tree $(T_n, X_n)$, are typical questions. For this approach, we refer the reader to [17] where the authors prove that, when $L_n = \text{Ber}(p)$, $\forall n \geq 1$ and $p \in (0, 1]$ the sequence $\{(T_n, X_n)\}_{n \geq 0}$ converges, in a suitable sense, to a random infinite rooted tree.

The model $\mathcal{L}$-TBRW belongs to a general class of models of random walks that build their trees, called Tree Builder Random Walks (TBRW) [20]. However, in [20], the authors assume a sort of uniform ellipticity condition, namely, that $\inf_n P(Z_n \geq 1) = \kappa > 0$, i.e., that the probability of adding at least one new leaf is bounded away from zero. Under this assumption, they prove that the corresponding random walk $\{X_n\}_{n \geq 0}$ is ballistic. The $\mathcal{L}$-TBRW , with $L_n = \text{Ber}(p)$, $\forall n \geq 1$, $p \in (0, 1]$, meets the uniform ellipticity condition, and in fact, the ballisticity of the walker had already been proven in [17].

Uniform ellipticity is a key assumption because, in essence, it induces a regeneration structure, similar in spirit to the one introduced in [1] for random walks in random environments on $\mathbb{Z}^d$, which allows the walker to forget fixed proportions of the space and regenerate the environment by starting a completely “new” tree. A natural question from the random walk’s perspective is:

(Q1) How does the random walk in $\mathcal{L}$-TBRW behave in the absence of the uniform ellipticity condition?

When looking at $\mathcal{L}$-TBRW as a random graph model, it is important to mention that random walks which build their graphs first appeared in the Network Science literature (see, [2, 13, 27, 28] and references therein) as an attempt to generate scale-free random graphs (graphs whose degree distribution is close to a power-law), while relaxing the assumption of global knowledge present in the preferential attachment model of Barabási-Albert [5] (global, here, refers to the fact the degree of every existing vertex must be known in order to decide the attachment probability). A natural question in this regard is:

(Q2) Can $\mathcal{L}$-TBRW generate scale-free random graphs?
Note that, the ballisticity of the random walk in $\mathcal{L}$-TBRW, when $L_n = \text{Ber}(p), p \in (0, 1]$, is strongly intertwined with the structure of the random trees $\{T_n\}_{n \geq 0}$. In particular, since the walker is moving away from its initial position fast, the trees generated are path-like (or vice versa); at time $n$ the tree has a height of order $n$, and the degree distributions of $\{T_n\}_{n \geq 0}$ have exponential tails. This suggests that the scale-free nature of the sequence $\{T_n\}_{n \geq 0}$ for $\mathcal{L}$-TBRW may only emerge when the uniform ellipticity condition fails.

Here we drop uniform ellipticity, by considering a sequence of laws $\{L_n\}_{n \geq 1}$ such that $\lim_{n \to \infty} L_n(\{0\}) \to 1$, and in such a case, we begin addressing the two questions above: we show the recurrence/transience of the random walk $\{X_n\}_{n \geq 0}$ under certain assumptions on $\mathcal{L}$, and also the power-law degree distribution for the tree sequence $\{T_n\}_{n \geq 0}$, when $L_n := \text{Ber}(n^{-\gamma})$ with $\gamma \in (2/3, 1]$.

**Note:** In this paper by “recurrence” we mean the property that the walker, starting at the root $x_0$, visits any vertex (even those eventually added) infinitely often, with probability one, while by “transience” we mean that the graph distance between the walker and the root tends to infinity almost surely.

**Remark 1.1** (Strong Markov Property). *We will often use the Strong Markov Property for the Markov chain $\{(T_n, X_n)\}_{n \geq 0}$. Although it is quite obvious, it is worth noting that since time is discrete, this property automatically holds even though the state space is somewhat unusual (the space of marked/rooted trees).*

### 1.2. Main results

We say that the walker $X$ is *recurrent* if for all $m \in \mathbb{N}$ and $v \in T$

$$\inf_{(T, x)} \mathbb{P}_{T, x; \mathcal{L}(m)} (X \text{ visits } v \text{ i.o.}) = 1,$$

that is, if starting at the vertex $x$ of the tree $T$, the walker visits the vertex $v$ infinitely many times almost surely, when the tree growth is governed by the shifted sequence of laws $\mathcal{L}(m) = \{L_{m+n}\}_{n \geq 1}$. On the other hand, we say that the walker is *transient* if for all $m \in \mathbb{N}$,

$$\inf_{(T, x)} \mathbb{P}_{T, x; \mathcal{L}(m)} \left( \lim_{n \to \infty} d(x, X_n) = \infty \right) = 1,$$

where $d$ denotes the graph distance.

Our first results regard the behavior of the random walker in $\mathcal{L}$-TBRW.

**Theorem 1.2** (Recurrent Regime). *Let $m_n$ denote the first moment of $L_n$, and assume the following about $\mathcal{L}$:

1. $m_n < \infty, n \geq 1$;
2. $q_n := L_n(\{0\}) \nearrow 1$, as $n \to \infty$;
3. $(1 - q_n) \cdot M_n^2 \to 0$, as $n \to \infty$, where $M_n := \sum_1^n m_k$;

Then, the random walk in $\mathcal{L}$-TBRW is recurrent.

**Corollary 1.3.** *Let $L_n := \text{Ber}(p_n)$, with $p_n := 1 - q_n := n^{-\gamma}$. Then the random walk in $\mathcal{L}$-TBRW is recurrent for $\gamma > 2/3$.*

Theorem 1.2 cannot be applied to the case $L_n = \text{Ber}(p_n)$ with $p_n = \Theta(n^{-\gamma})$, for $\gamma \leq 2/3$, since assumption A3 is not met. However, in the specific situation $L_n = \text{Ber}(p_n)$...
with \( p_n = \Theta(n^{-\gamma}) \) we manage to extend recurrence of the walker for all \( \gamma > 1/2 \), as stated in the following theorem.

**Theorem 1.4** (Recurrent Regime for \( L_n = \text{Ber}(n^{-\gamma}) \)). Consider a \( \mathcal{L} \)-TBRW where \( L_n = \text{Ber}(n^{-\gamma}) \) and \( \gamma > 1/2 \). Then, the walk is recurrent.

In Section 6 we will give conditions under which the walk is transient (see, Theorem 6.2 and the corollary afterwards.) Specifically, we will see that when there are infinitely many growth times (i.e. \( p_n \) is not summable), and there are sufficiently many edges grown at those times, the walk is never recurrent, and under a mild condition on \( p_n \) (Condition 3 of Theorem 6.2) it is transient.

In our next result, we focus on the tree structure rather than the walker, and show that for \( L_n = \text{Ber}(p_n) \) with \( p_n = \Theta(n^{-\gamma}) \), in the regime \( \gamma \in (2/3, 1] \), \( \mathcal{L} \)-TBRW generates trees whose degree distributions converge to the very same limiting distribution as in the celebrated Barabási-Albert model [5, 10]. Note that by Corollary 1.3 in this regime recurrence holds.

**Theorem 1.5** (Power-law degree distribution). Let \( \{T_n\}_{n \geq 0} \) denote the sequence of random trees in \( \mathcal{L} \)-TBRW for \( L_n := \text{Ber}(p_n) \) with \( p_n = \Theta(n^{-\gamma}) \). Then, for \( \gamma \in (2/3, 1] \), any initial condition \((T_0, x_0)\) and \( \forall d \in \mathbb{N} \setminus \{0\} \), it holds that

\[
\lim_{n \to \infty} \frac{1}{|V_n|} \sum_{v \in V_n} 1\{\deg_{T_n}(v) = d\} = \frac{4}{d(d+1)(d+2)}, \quad \mathbb{P}_{T_0,x_0;\mathcal{L}} \text{-a.s.}
\]

**1.3. Open questions.** We list below a couple of questions about the \( \mathcal{L} \)-TBRW which are yet to be answered.

1) For \( L_n = \text{Ber}(p_n) \) with \( p_n = \Theta(n^{-\gamma}) \) we expect that for \( \gamma > 0 \) sufficiently small the random walk is transient. We conjecture that there is a phase transition for recurrence/transience according to whether \( \gamma > 1/2 \) or \( \gamma < 1/2 \); is this really the case? What happens at \( \gamma = 1/2 \)? Here by “transience” we mean that the distance of the walker from the root tends to infinity almost surely. Of course it is far from being obvious a-priori that this is exactly the negation of recurrence.

2) Concerning the tree structure, for \( L_n = \text{Ber}(p_n) \) with \( p_n = \Theta(n^{-\gamma}) \) and \( \gamma > 0 \) sufficiently small we expect a tree sequence with exponential tail degree distribution. Is \( \gamma = 1/2 \) also a phase transition point for power-law/exponential tail degree distribution?

**2. Recurrence of the walker in TBRW (proof of Theorem 1.2)**

In this section we prove Theorem 1.2 which states that the under certain assumptions on \( \{Z_n\}_{n \geq 1} \) the random walk is recurrent. Without the loss of generality, throughout this section we may and will assume that our initial tree \( T_0 \) consists of just a single vertex, denoted by \( o \). As will become clear along the proof, the initial tree plays no important role in the long-time behavior of the walker.
Proof of Theorem 1\textsuperscript{2}. We begin noticing that by (A3), we can choose a sequence \( g_n \) such that
\[
g_n \rightarrow \infty \text{ and } (1 - q_n) \cdot g_n \cdot M_n^2 \rightarrow 0, \text{ as } n \rightarrow \infty.
\]
Also consider the following random variable
\[
W_n = \sum_{k=n}^{n + g_n M_n^2} \mathbf{1}\{Z_k \geq 1\}.
\]
In words, \( W_n \) counts how many times the walker has added at least one leaf during the time interval \([n, n + g_n M_n^2]\). By (A2), it follows that
\[
\mathbb{E}_{T_0,x;L}[W_n] = \sum_{k=n}^{n + g_n M_n^2} 1 - q_k \leq (1 - q_n) \cdot g_n \cdot M_n^2 \rightarrow 0, \tag{2.1}
\]
as \( n \) goes to infinity. Now, let \( A_n \) denote the following event
\[
A_n = \{\text{no grow in the interval } [n, n + g_n M_n^2]\},
\]
that is, the walker did not add any new leaves to the tree in the time interval \([n, n + g_n M_n^2]\). By the bound in (2.1) and Markov Inequality we have that
\[
\mathbb{P}_{T_0,x;L}(A_n^c) = \mathbb{P}_{T_0,x;L}(W_n \geq 1) \leq \mathbb{E}_{T_0,x;L}[W_n] = o(1). \tag{2.2}
\]
Finally, let \( B_n \) denote the following event
\[
B_n := \{X \text{ visits the root some time in } [n, n + g_n M_n^2]\}.
\]
Using the fact (see [12]) that the cover time of a tree of size \( k \) is at most \( 2k^2 \) gives us
\[
\mathbb{P}_{T_0,x;L}(B_n^c, A_n | V_n \leq g_n^{1/4}M_n) \leq \frac{\sqrt{g_n M_n^2}}{g_n M_n^2} = o(1). \tag{2.3}
\]
By the simple Markov property we have that on the event \( A_n \), \( \{X_k\}_{n+g_n M_n^2} \) is distributed as a SSRW on \( T_n \), which is a tree with at most \( g_n^{1/4} M_n \) vertices. And \( B_n^c \) means that this SSRW did not cover the entire graph in \( g_n M_n^2 \) steps. The bound then follows by Markov Inequality. Finally,
\[
\mathbb{P}_{T_0,x;L}(B_n^c) \leq \mathbb{P}_{T_0,x;L}(B_n^c, A_n, V_n \leq g_n^{1/4}M_n) + \mathbb{P}_{T_0,x;L}(A_n^c) + \mathbb{P}_{T_0,x;L}(V_n \geq g_n^{1/4}M_n) = o(1),
\]
since the first term at the RHS is \( o(1) \) by (2.3), the second one is \( o(1) \) by (2.2), whereas the third one is \( o(1) \) by Markov Inequality. The above inequality yields
\[
\mathbb{P}_{T_0,x;L}(X \text{ visits the root some time in } [n, n + g_n M_n^2]) = P(B_n) = 1 - o(1).
\]
Finally, by Fatou’s Lemma
\[
\mathbb{P}_{T_0,x;L}(\limsup_{n \rightarrow \infty} B_n) \geq \limsup_{n \rightarrow \infty} \mathbb{P}_{T_0,x;L}(B_n) = 1,
\]
which proves that the root is visited infinitely many times for any initial condition \((T_0,x)\). To extend the result to a vertex which is eventually added to the tree, we apply Strong
3. Recurrence of the walker in TBRW with $L_n = \text{Ber}(p_n)$ with $p_n = \Theta(n^{-\gamma})$

(proof of Theorem 1.4)

Let us begin recalling that the recurrence for $\gamma > 2/3$ follows by Corollary 1.3, thus we need to show it for $\gamma \in (1/2, 2/3]$. Before we go to the proof of Theorem 1.4, let us say some words about the general idea. The main idea behind the proof of Theorem 1.2 is that the walker $X$ most of the time mixes on $T_n$ before adding new leaves. This allows us to rely on general bounds for cover time. However, mixing before adding new leaves is a strong condition which is not satisfied for the sequence of laws $L_n = \text{Ber}(n^{-\gamma})$ with $\gamma \leq 2/3$. So, the strategy behind the proof of Theorem 1.4 is to try to mix in a smaller tree. For this we must find a sequence of time intervals $[t_n, t_n + s_n]$ with the following characteristics:

1. $s_n$ is small enough so that the trees $T_{t_n}$ and $T_{t_n + s_n}$ are comparable in size;
2. $X$ spends a large enough amount of time on $T_{t_n}$ in the time interval $[t_n, t_n + s_n]$;
3. $s_n$ is large enough so that the time spent on $T_{t_n}$ is enough for $X$ to cover $T_{t_n}$, although it may not mix over $T_{t_n + s_n}$.

Given $t, s \in \mathbb{N}$, we will say that a vertex $v$ added between times $t$ and $t + s$ is red and define $N_{t,t+s}$ as the number of visits to red vertices between times $t$ and $t + s$, i.e.,

$$N_{t,t+s} := \sum_{j=t}^{t+s} 1\{X_j \notin V(T_t)\}.$$  

The main ingredient in the proof of Theorem 1.4 is the lemma below which provides the right order of the time window to observe $X$. Specifically, it states that, for some small $\delta$, the sequence of time intervals $[n, n + n^{2(1-\gamma)+\delta}]$ satisfies the properties 1) – 3) above with $t_n = n$ and $s_n = n^{2(1-\gamma)+\delta}$.

Lemma 3.1. Consider a TBRW where $L_n = \text{Ber}(n^{-\gamma})$ and $\gamma \in (1/2, 1]$. Then, for any initial condition $(T, x)$, any $m \in \mathbb{N}$ (time shift) and $0 < \delta < 2\gamma - 1$, it holds that

$$\mathbb{E}_{T,x;L^{(m)}}[N_{n,n+n^{2(1-\gamma)+\delta}}] = o(n^{2(1-\gamma)+\delta}).$$

The proof of the above lemma is given in Subsection 3.1

Proof of Theorem 1.4. For a fixed $n$, let us denote by $S_{n,n+n^{2(1-\gamma)+\delta}}$ the number of transitions of the random walk $X$ on $T_n$, between time $n$ and $n + n^{2(1-\gamma)+\delta}$, i.e.,

$$S_{n,n+n^{2(1-\gamma)+\delta}} := \sum_{j=n+1}^{n+n^{2(1-\gamma)+\delta}} 1\{X_{j-1} \in V(T_n)\}\{X_j \in V(T_n)\}.$$
Observe that if $S_{n,n+n^{2(1-\gamma)+\delta}} \leq n^{2(1-\gamma)+\delta}/2$, then $N_{n,n+n^{2(1-\gamma)+\delta}} \geq n^{2(1-\gamma)+\delta}/4$. Thus, by Markov’s inequality and Lemma 3.1, for $\delta < 2\gamma - 1$, we have that

$$\mathbb{P}_{T,x,\mathcal{L}} \left( S_{n,n+n^{2(1-\gamma)+\delta}} \leq n^{2(1-\gamma)+\delta}/2 \right) = o(1).$$

Let $\tilde{X}$ denote the walker $X$ seen only when it makes transitions over $T_n$. Specifically, let $\phi_0 \equiv n$ and, for $i \geq 1$, define recursively $\phi_i := \inf\{\ell > \phi_{i-1} : X_\ell \in V(T_n)\}$. Now, for $m \geq 0$, consider the process $Y_m := X_{\phi_m}$, which corresponds to the walker $X$ seen only when visits vertices in $T_n$. Note that $Y$ is a lazy random walk on $T_n$ and may not be symmetric since the probability of taking a self-loop depends on the red substructure dangling from the corresponding vertex of $T_n$. Setting $\sigma_0 \equiv 0$ and recursively for $j \geq 1$, $\sigma_j := \inf\{m > \sigma_{j-1} : Y_m \neq Y_{\sigma_{j-1}}\}$, we can define $\tilde{X}_k := Y_{\sigma_k}$. Note that the process $\{\tilde{X}_k\}_{k \geq 0}$ is a SSRW on $T_n$.

Let $v \in V(T_n)$ and denote by $A_n$ the following event:

$$A_n := \{X \text{ visits } v \text{ in the interval } [n, n + n^{2(1-\gamma)+\delta}]\}.$$ 

Recall that $V_n := |V(T_n)|$ is a sum of $n$ independent Bernoulli random variables for which it holds that

$$\mathbb{E}_{T,x,\mathcal{L}}[V_n] = \Theta(n^{1-\gamma}).$$

Thus, by the above identity and Chernoff bounds there exist positive constants $C_1$ and $C_2$ depending on $T$ and $\gamma$ only such that

$$\mathbb{P}_{T,x,\mathcal{L}} \left( V_n \geq C_1 n^{1-\gamma} \right) \leq e^{-C_2 n^{1-\gamma}}.$$ 

On the event $\{V_n \leq C_1 n^{1-\gamma}\}$, $\tilde{X}$ is a SSRW on a tree with at most $C_1 n^{1-\gamma}$ vertices. By Theorem 2 in [12], which states that the expected cover time of a SRRW in a tree with $m$ vertices is at most $2m^2$, Markov’s inequality yields that

$$\mathbb{P}_{T,x,\mathcal{L}} \left( A_n^c, S_{n,n+n^{2(1-\gamma)+\delta}} \geq n^{2(1-\gamma)+\delta}/2, V_n \leq C_1 n^{1-\gamma} \right) = o(1).$$

Indeed, on the event $\{A_n^c, S_{n,n+n^{2(1-\gamma)+\delta}} \geq n^{2(1-\gamma)+\delta}/2, V_n \leq C_1 n^{1-\gamma}\}$, $\tilde{X}$ takes at least $n^{2(1-\gamma)+\delta}/2$ steps in a tree with at most $C_1 n^{1-\gamma}$ vertices and did not cover it (specifically, it did not visit the vertex $v$). So, on this event, the time to cover $T_n$ is greater than $n^{2(1-\gamma)+\delta}/2$, whereas its expected value is $O(n^{2(1-\gamma)})$. From the above we deduce that

$$\mathbb{P}_{T,x,\mathcal{L}}(A_n^c) = o(1).$$

Finally, by reverse Fatou, it follows that

$$\mathbb{P}_{T,x,\mathcal{L}} \left( \limsup_n A_n \right) \geq \limsup_n \mathbb{P}_{T,x,\mathcal{L}}(A_n) = 1,$$

which proves that the vertex $v$ of $T_n$ is visited infinitely often with probability one. Finally, observe that our argument does not depend on $(T,x)$, nor on the location of $v$ in the tree. Our proof only depends on the asymptotic behavior of $V_n$ and $N_{n,n+n^{2(1-\gamma)+\delta}}$, which do not change if we consider $C^{(m)}$ instead of $\mathcal{L}$. So, the above actually proves that the walker visits any vertex i.o., even those eventually added by the process. \qed
3.1. Proof of Lemma 3.1. Lemma 3.1 states that in the time window \([n, n + n^{2(1-γ)+δ}]\) the walker spends \(o(n^{2(1-γ)+δ})\) steps on the red vertices – the ones added in the same time window. The natural direction to prove such result would be to prove that the expected number of visits to each new vertex is small enough and then sum over the random number of vertices we could add from \(n\) to \(n + n^{2(1-γ)+δ}\). However, the expected number of visits to a vertex is sensitive to its degree, which, in our model, can increase over time. So, our proof consists in the following two steps:

i) bounding from above the expected number of visits to a red vertex in the time window \([n, n + n^{2(1-γ)+δ}]\) by a factor times its expected degree at time \(n + n^{2(1-γ)+δ}\) (see, Equation (3.3));

ii) controlling the evolution of the degree by showing that there exists \(d_0 = d(γ)\) such that is extremely unlikely that a red vertex reaches degree greater than \(d_0\) (see, Proposition 3.2).

Before proving Lemma 3.1 we introduce some instrumental notions and an auxiliary result which is a quantitative version of step ii).

For \(k \in \mathbb{N}\) we let \(v_k\) be the vertex possibly added at time \(k\), depending on the value of \(Z_k\) (if \(Z_k = 0\) the vertex \(v_k\) is not added to the tree). For \(t \geq k\), let \(D_{k,t}\) denote the degree of vertex \(v_k\) at time \(t\), i.e.,

\[
D_{k,t} := \begin{cases} \deg_t(v_k), & \text{if } Z_k = 1; \\ 0, & \text{otherwise}. \end{cases}
\] (3.1)

Note that if a vertex \(v_k\) is not added (i.e., \(Z_k = 0\)) then \(D_{k,t} = 0\) for all \(t\). As mentioned above (step ii)), the proof of Lemma 3.1 relies on controlling the evolution of the degree and this is formally stated in the proposition below (whose quite-technical proof is deferred to Subsection 3.1.1).

Proposition 3.2. Let \(γ \in (1/2, 1]\) and \(δ < 2γ - 1\). Fix natural numbers \(m\) (time shift), \(d\) (degree), \(k\) (vertex index) with \(k \geq n\) and \(n\) sufficiently large. Then, there exists a positive constant \(C_1\) depending on \(γ\) and \(d\), and there exists \(ε > 0\) (depending on \(γ\) and \(δ\), but not on \(d\)), such that

\[
P_{T,x;\mathcal{L}^{(m)}}(D_{k,n+n^{2(1-γ)+δ}} \geq d) \leq \frac{C_1}{k^{γn^ε(d-1)}}.
\]

Given \(k \in \mathbb{N}\) (vertex index), \(t, s \in \mathbb{N}\) (times) with \(k \geq t\) and \(d \geq 1\) (degree), we denote by \(N^{(d)}_{t,t+s}(v_k)\) the number of visits to \(v_k\) when it has degree \(d\) in the time interval \([t, t+s]\). Formally,

\[
N^{(d)}_{t,t+s}(v_k) := \begin{cases} \sum_{j=t}^{t+s} 1\{X_j = v_k, D_{k,j} = d\}, & \text{if } Z_k = 1; \\ 0, & \text{otherwise}. \end{cases}
\] (3.2)

For convenience, when \(d = 1\), we drop the superscript \((d)\) on the above definition.
Proof of Lemma 3.1. Recall that \( N_{n, n + n^{2(1-\gamma)+\delta}} \) denotes the number of visits to red vertices from \( n \) to \( n + n^{2(1-\gamma)+\delta} \). The latter can be written as

\[
N_{n, n + n^{2(1-\gamma)+\delta}} = \sum_{k=n}^{n+n^{2(1-\gamma)+\delta}} N_{n, n + n^{2(1-\gamma)+\delta}}(v_k).
\]

Moreover, to avoid clutter, let us denote by \( D_{n,k}^{(\delta)} := D_{k,n+n^{2(1-\gamma)+\delta}} \), i.e., the degree of vertex \( v_k \) (with \( k \geq n \)) at time \( n + n^{2(1-\gamma)+\delta} \). Recall that, by the definition in (3.3), \( D_{n,k}^{(\delta)} = Z_k D_{n,k} \).

Moreover, since \( p_n \) is decreasing, it follows that

\[
Z_k D_{n,k}^{(\delta)} \succ Z_k \left( 1 + \text{Bin} \left( N_{n, n + n^{2(1-\gamma)+\delta}} (v_k), \frac{1}{(n + n^{2(1-\gamma)+\delta} + m)\gamma} \right) \right),
\]

under \( P_{T,x;\mathcal{L}(m)} \). Here \( \succ \) denotes stochastic domination and \( \text{Bin} \) the binomial distribution. The above stochastic domination implies that

\[
\mathbb{E}_{T,x;\mathcal{L}(m)} \left[ N_{n, n + n^{2(1-\gamma)+\delta}} (v_k) \right] \leq (n + n^{2(1-\gamma)+\delta} + m)\gamma \mathbb{E}_{T,x;\mathcal{L}(m)} \left[ Z_k \left( D_{n,k}^{(\delta)} - 1 \right) \right] \leq n^{\gamma} \left( 1 + o(1) \right) \mathbb{E}_{T,x;\mathcal{L}(m)} \left[ Z_k \left( D_{n,k}^{(\delta)} - 1 \right) \right],
\]

(3.3)

where, the last inequality holds since \( \gamma > 1/2 \) and \( \delta < 2\gamma - 1 \). In order to control the RHS of (3.3), note that

\[
\mathbb{E}_{T,x;\mathcal{L}(m)} \left[ Z_k \left( D_{n,k}^{(\delta)} - 1 \right) \right] \leq d_0 \mathbb{P}_{T,x;\mathcal{L}(m)} \left( D_{n,k}^{(\delta)} \geq 2 \right),
\]

\[
\mathbb{E}_{T,x;\mathcal{L}(m)} \left[ Z_k \left( D_{n,k}^{(\delta)} - 1 \right) 1 \{ D_{n,k}^{(\delta)} \geq 2 \} \right] \leq d_0 \mathbb{P}_{T,x;\mathcal{L}(m)} \left( D_{n,k}^{(\delta)} \geq 2 \right),
\]

where, the bottom inequality follows from noticing that \( D_{n,k}^{(\delta)} \) is at most \( n^{2(1-\gamma)+\delta} \) (since a vertex added after time \( n \) can, by time \( n + n^{2(1-\gamma)+\delta} \), have at most \( n^{2(1-\gamma)+\delta} \) neighbors). Moreover, by Proposition 3.2, we have that for every \( d \geq 1 \) there exists \( \varepsilon \in (0, \gamma) \) (which does not depend on \( d \)) and a positive constant \( C \) (depending on \( \gamma \) and \( d \)) such that

\[
\mathbb{P}_{T,x;\mathcal{L}(m)} \left( D_{n,k}^{(\delta)} \geq d \right) \leq \frac{C}{k^{-\varepsilon}(d-1)},
\]

which implies

\[
\mathbb{E}_{T,x;\mathcal{L}(m)} \left[ Z_k \left( D_{n,k}^{(\delta)} - 1 \right) \right] \leq d_0 \frac{C_1}{k\gamma n^{\varepsilon}} + n^{2(1-\gamma)+\delta} \frac{C_2}{k\gamma n^{\varepsilon}(d_0-1)}.
\]

Thus,

\[
\mathbb{E}_{T,x;\mathcal{L}(m)} \left[ N_{n, n + n^{2(1-\gamma)+\delta}} \right] \leq n^{\gamma} \left( 1 + o(1) \right) \left( C_1 n^{-\varepsilon} d_0 + C_2 \frac{n^{2(1-\gamma)+\delta}}{n^{\varepsilon}(d_0-1)} \right)^{n+n^{2(1-\gamma)+\delta}} \sum_{k=n}^{n+n^{2(1-\gamma)+\delta}} k^{-\gamma}
\]

\[
\leq (1 + o(1)) \left( C_1 n^{-\varepsilon} d_0 + C_2 \frac{n^{2(1-\gamma)+\delta}}{n^{\varepsilon}(d_0-1)} \right) n^{2(1-\gamma)+\delta},
\]
where, in the last inequality we use the trivial bound \( \sum_{k=n}^{n^2} \frac{1}{k^2} \leq n^{2(1-\gamma)+\delta-\gamma} \).

Since \( \varepsilon > 0 \) we clearly have that \( \frac{n^{2(1-\gamma)+\delta}}{\varepsilon (d_0 - 1)} = o(n^{2(1-\gamma)+\delta}) \), while choosing \( d_0 \) large enough such that \( \varepsilon (d_0 - 1) > 2(1-\gamma) + \delta \), then one has

\[
\frac{n^{2(1-\gamma)+\delta + 2(1-\gamma) + \delta}}{\varepsilon (d_0 - 1)} = o(n^{2(1-\gamma)+\delta}).
\]

\[ \square \]

3.1.1. Proof of Proposition 3.2. The proof of Proposition 3.2 relies on three auxiliary results, namely Lemma 3.3 and Lemmas 3.4 and 3.5, which will be presented below.

The first lemma is a general result for SSRW on fixed trees that will be useful to our proposes.

**Lemma 3.3.** Let \( T \) be a finite tree and \( v \) a vertex of \( T \). Let \( d := \deg_T(v) \) be the degree of \( v \) and denote by \( N_t(v) \) the number of visits to \( v \) in \( t \) steps of a SSRW on \( T \). Then, for every \( \varepsilon > 0 \) there exist \( t_0 \) such that for all \( t \geq t_0 \) the following bound holds

\[
E_v[N_t(v)] \leq \frac{(d + \varepsilon) \cdot t}{2(|T| - 1)},
\]

where \( |T| \) denotes the number of vertices in \( T \).

**Proof.** Define recursively the following sequence of return times: \( H_0 \equiv 0 \) and for \( j \geq 1 \), \( H_j := \inf \{ n > H_{j-1} : Y_n = v \} \), i.e., \( H_j \) is the \( j \)-th time \( Y \) returns to \( v \).

For \( v \) and \( w \) neighboring vertices in \( T \) we let \( T_v(w) \) be the subtree of \( T \) containing \( w \) and obtained by removing the edge between \( w \) and \( v \). By a first-step analysis, we have that

\[
E_v[H_1] = \sum_{w \sim v} \frac{1 + E_w[H_1]}{d} = \sum_{w \sim v} \frac{1 + (2|T_v(w)| - 1)}{d} = \frac{2(|T| - 1)}{d},
\]

where, in the second equality we use that \( E_w[H_1] = 2|T_v(w)| - 1 \) (see, e.g., Corollary 10 in [23]).

By the Renewal Theorem we have that

\[
\lim_{t \to +\infty} \frac{N_t(v)}{t} = \frac{1}{E_v[H_1]}, \text{ a.s.}, \quad \text{and} \quad \lim_{t \to +\infty} \frac{E_v[N_t(v)]}{t} = \frac{1}{E_v[H_1]} = \frac{d}{2(|T| - 1)}.
\]

\[ \square \]

Since in TBRW the domain of the walker changes as it walks, we will need to keep track of the random times when new vertices are added and the times when they have their degrees increased. For this reason we will need some further definitions.

For \( d \in \mathbb{N} \), define inductively the following sequence of stopping times

\[
\eta_{k,1} := \begin{cases} 
  k, & \text{if } Z_k = 1; \\
  +\infty, & \text{otherwise},
\end{cases}
\]
and for \( d \geq 2 \),
\[
\eta_{k,d} := \begin{cases} 
\inf \{ t > \eta_{k,d-1} : D_{k,t} = d \}, & \text{if } \eta_{k,d-1} < \infty; \\
+\infty, & \text{otherwise}.
\end{cases}
\]

So, \( \eta_{k,d} \) is the first time \( v_k \) reaches degree \( d \) (if \( v_k \) is not added then \( \eta_{k,d} = +\infty \), for all \( d \geq 1 \)).

Recall that, given \( k \in \mathbb{N} \) (vertex index), \( t, s \in \mathbb{N} \) (times) with \( k \geq t \) (we are interested in vertices possibly added after time \( t \)) and \( d \geq 1 \) (degree), \( N_{t,t+s}^{(d)}(v_k) \) denotes the number of visits to \( v_k \) when it has degree \( d \) in the time interval \([t, t+s]\). Regarding \( N_{t,t+s}^{(d)}(v_k) \) we have the following result:

**Lemma 3.4.** Fix natural numbers \( m \) (time shift), \( k \) (vertex index), \( d \) (degree), \( t, s \) (times) with \( k \geq t \) and \( s \) sufficiently large. Then there exist positive constants \( C_1 \) and \( C_2 \) depending on \( \gamma \) only such that
\[
\mathbb{E}_{T,x;L(m)} \left[ N_{t,t+s}^{(d)}(v_k) \right] \leq C_1 \frac{d \cdot s}{t^{1-\gamma}} \mathbb{P}_{T,x;L(m)}(\eta_{k,d} < t + s) + se^{-C_2 t^{1-\gamma}}.
\]

**Proof.** By definition of \( N_{t,t+s}^{(d)}(v_k) \) (see, (3.2)) we have that \( N_{t,t+s}^{(d)}(v_k) = N_{t,t+s}^{(d)}(v_k) 1\{\eta_{k,d} < t + s\} \). Moreover,
\[
1\{\eta_{k,d} < t + s\} N_{t,t+s}^{(d)}(v_k) \leq 1\{\eta_{k,d} < t + s\} N_{0,s}^{(d)}(v_k) \circ \theta_{\eta_{k,d}} \cdot \mathbb{P}_{T,x;L(m)}\text{-a.s.}
\]

It is important to point out that \( v_k \) in \( N_{0,s}^{(d)}(v_k) \) is not the \( k \)-th vertex added by the shifted process, but the vertex \( v_k \) which belongs to \( T_{\eta_{k,d}} \) on the event \( \{\eta_{k,d} < t + s\} \). Thus, by the strong Markov property it follows that
\[
\mathbb{E}_{T,x;L(m)} \left[ N_{t,t+s}^{(d)}(v_k) \right] \leq \mathbb{E}_{T,x;L(m)} \left[ \mathbb{E}_{T_{\eta_{k,d}};X_{\eta_{k,d}};L(m+\eta_{k,d})} \left[ N_{0,s}^{(d)}(v_k) \right] 1\{\eta_{k,d} < t + s\} \right].
\]

Our next step is to handle
\[
\mathbb{E}_{T_{\eta_{k,d}};X_{\eta_{k,d}};L(m+\eta_{k,d})} \left[ N_{0,s}^{(d)}(v_k) \right]. \tag{3.5}
\]

Let \( \tilde{X} \) denote the random walk \( X \) only when it makes transitions over \( T_{\eta_{k,d}} \). Let \( W \) denote a SSRW on \( T_{\eta_{k,d}} \), starting from \( v_k \) and let \( N_s^W(v_k) \) be the number of visits to \( v_k \) by the walker \( W \) in \( s \) steps. We can couple \( \tilde{X} \) (staring from \( v_k \)) with \( W \) in such a way that all possible subsequent visits of \( X \) to \( v_k \) counted by \( N_{0,s}^{(d)}(v_k) \) are counted by \( N_s^W(v_k) \) as well. Since \( v_k \) may reach degree \( d + 1 \) before \( X \) takes \( s \) steps, or \( \tilde{X} \) may take less than \( s \) steps over \( T_{\eta_{k,d}} \), we just ‘complete’ the remaining steps of \( W \) by letting it walking on \( T_{\eta_{k,d}} \) independently of \( X \) whenever \( v_k \) reaches degree \( d + 1 \) and/or after \( \tilde{X} \) takes \( s \) steps.

Thus, by Lemma 3.3 on the event \( \{\eta_{k,d} < t + s\} \) we can bound the random variable in (3.5) as
\[
\mathbb{E}_{T_{\eta_{k,d}};X_{\eta_{k,d}};L(m+\eta_{k,d})} \left[ N_{0,s}^{(d)}(v_k) \right] \leq \mathbb{E}_{v_k} \left[ N_s^W(v_k) \right] + 1 \leq \frac{2d \cdot s}{2(|T_{\eta_{k,d}}| - 1)} \leq \frac{d \cdot s}{|T_{k,d}| - 1},
\]
Lemma 3.5. Fix natural numbers $k \geq t$ assures that $T_{\eta_k,d}$ contains $T_t$. Replacing the above on $|T_t|$ leads us to

$$
\mathbb{E}_{T,x;L^{(m)}} \left[ N_{t,t+s}(v_k) \right] \leq \mathbb{E}_{T,x;L^{(m)}} \left[ \left( \frac{d \cdot s}{|T_t| - 1} \right) \mathbbm{1}_{\{\eta_k,d < t + s\}} \right].
$$

(3.6)

Given the initial condition $(T, x)$, since the tree growth is governed by the shifted sequence of laws $L^{(m)} = \{L_{m+n}\}_{n \geq 1}$, it follows that $|T_t|$ is

$$
|T_t| = |T| + \sum_{r=1}^{t} Z_{r+m} \implies \mathbb{E}_{T,x;L^{(m)}} [|T_t|] = |T| + \Theta \left( (t + m)^{1-\gamma} \right),
$$

where the constants involved in the $\Theta$ notation depend only on $\gamma$. By Chernoff bounds, there exist positive constants $C_2$ and $C_3$ such that

$$
\mathbb{P}_{T,x;L^{(m)}} \left( |T_t| \leq C_2(|T| + (t + m)^{1-\gamma}) \right) \leq e^{-C_3 t^{1-\gamma}}.
$$

Combining the above inequality with (3.6) yields

$$
\mathbb{E}_{T,x;L^{(m)}} \left[ N_{t,t+s}(v_k) \right] \leq \left( \frac{d \cdot s}{C_2(|T| + (t + m)^{1-\gamma}) - 1} \right) \mathbb{P}_{T,x;L^{(m)}} \left( \eta_k,d < t + s \right) + se^{-C_3 t^{1-\gamma}},
$$

which is enough to prove the lemma. $\square$

The next result is a recursion for the tail probability of $D_{k,t+s}$.

Lemma 3.5. Fix natural numbers $m$ (time shift), $k$ (vertex index), $d$ (degree), $t, s$ (times) with $k \geq t$ and $s$ sufficiently large, and fix $\varepsilon < \gamma$. Then, there exist positive constants $C_1$ and $C_2$ depending on $\gamma$ such that

$$
\mathbb{P}_{T,x;L^{(m)}} (D_{k,t+s} \geq d + 1) \leq I + II,
$$

where,

$$
I := \left\{ \left( \frac{(t + s + m)^\gamma}{(t + m)^\gamma} \right) \left[ 1 - \left( \frac{1}{(t + s + m)^\gamma} \right)^{t^{1-\gamma} - \varepsilon} \right] + C_1 \frac{d \cdot s}{(t^{1-\gamma} - 1)t^{\gamma-\varepsilon}} \right\} \mathbb{P}_{T,x;L^{(m)}} (D_{k,t+s} \geq d);
$$

and

$$
II := \frac{se^{-C_2 t^{1-\gamma}}}{t^{\gamma-\varepsilon}}.
$$

Proof. Let us begin noticing the following identity of events:

$$
\{\eta_k,d \leq t + s\} = \{D_{k,t+s} \geq d\}.
$$

Then, we have that

$$
\mathbb{P}_{T,x;L^{(m)}} (D_{k,t+s} \geq d + 1) \leq \mathbb{P}_{T,x;L^{(m)}} (D_{k,t+s} \geq d + 1, N_{t,t+s}(v_k) \leq t^{\gamma-\varepsilon}) + \mathbb{P}_{T,x;L^{(m)}} (N_{t,t+s}(v_k) \geq t^{\gamma-\varepsilon}).
$$

(3.7)
By Markov’s inequality and Lemma 3.4 we have
\[ \Pr_{T,x;L,(m)}(N_{t,t+s}^{(d)}(v_k) \geq t^{\gamma-\varepsilon}) \leq \frac{C_1d \cdot s}{(t^{1-\gamma} - 1)t^{\gamma-\varepsilon}} \Pr_{T,x;L,(m)}(\eta_{k,d} < t + s) + \frac{se^{-C_2t^{1-\gamma}}}{t^{\gamma-\varepsilon}}. \quad (3.8) \]
For the first term of the RHS of (3.7), fix \( j \leq t^{\gamma-\varepsilon} \), then we have the following identity
\[ \{ D_{k,t+s} \geq d + 1, N_{t,t+s}^{(d)}(v_k) = j \} = \{ Z_{H'_j+1} = 1, Z_{H'_{j-1}+1} = 0, \ldots, Z_{H'_1+1} = 0, \eta_{k,d} < t + s \}, \]
where \( H'_i \) denotes the time of the \( i \)-th visit to \( v_k \) after time \( \eta_{k,d} \), i.e., after reaching degree \( d \). In words, the LHS of the above identity denotes the event in which \( v_k \) has degree at least \( d + 1 \) before time \( t + s \) and has been visited \( j \) times while it had degree \( d \). This means that at each of these \( j \) visits to \( v_k \), in the next step it has failed \( j - 1 \) times to increase its degree to \( d + 1 \) and only succeed after the \( j \)-th visit, that is, at the time \( H'_j + 1 \). The failures and success are described formally by the \( Z_{H'_1+1} \)'s. Using the independent nature of \( Z \)'s and the fact that \( p_n \) is decreasing in \( n \), we have that
\[ \Pr_{T,x;L,(m)}(D_{k,t+s} \geq d + 1, N_{t,t+s}^{(d)}(v_k) = j) \leq \frac{1}{(t + m)^\gamma} \left( 1 - \frac{1}{(t + s + m)^\gamma} \right)^{j-1} \Pr_{T,x;L,(m)}(D_{k,t+s} \geq d). \]
Summing over \( j \) from 1 to \( t^{\gamma-\varepsilon} \) leads to
\[ \Pr_{T,x;L,(m)}(D_{k,t+s} \geq d + 1, N_{t,t+s}^{(d)}(v_k) \leq t^{\gamma-\varepsilon}) \leq \frac{(t + s + m)^\gamma}{(t + m)^\gamma} \left[ 1 - \left( 1 - \frac{1}{(t + s + m)^\gamma} \right)^{t^{\gamma-\varepsilon}} \right] \Pr_{T,x;L,(m)}(D_{k,t+s} \geq d), \]
which combined with (3.8) proves the result. \( \square \)

We are finally ready to prove Proposition 3.2.

**Proof of Proposition 3.2** Setting \( t = n \) and \( s = n^{2(1-\gamma)+\delta} \), with \( \delta < 2\gamma - 1 \) and \( \varepsilon < \gamma \) in Lemma 3.5 and using the shorthand \( D_{n,k}^{(\delta)} := D_{k,n+n^{2(1-\gamma)+\delta}} \), we obtain that the probability \( \Pr_{T,x;L,(m)}(D_{n,k}^{(\delta)} \geq d + 1) \) is bounded from above by
\[ \left( \frac{n + n^{2(1-\gamma)+\delta} + m}{n + m} \right)^\gamma \left( 1 - \left( 1 - \frac{1}{(n + n^{2(1-\gamma)+\delta} + m)^\gamma} \right)^{n^{\gamma-\varepsilon}} \right) \Pr_{T,x;L,(m)}(D_{n,k}^{(\delta)} \geq d) \]
\[ + \frac{C_1dn^{2(1-\gamma)+\delta}}{(n^{1-\gamma} - 1)n^{\gamma-\varepsilon}} \Pr_{T,x;L,(m)}(D_{n,k}^{(\delta)} \geq d) + \frac{n^{2(1-\gamma)+\delta}}{n^{\gamma-\varepsilon}} e^{-C_2n^{1-\gamma}}. \]
Since \( \gamma > 1/2 \) and \( n \) is sufficiently large we obtain:
\[ \Pr_{T,x;L,(m)}(D_{n,k}^{(\delta)} \geq d + 1) \leq \left( \frac{C'_1}{n^\varepsilon} + \frac{C_1dn^{2(1-\gamma)+\delta}}{(n^{1-\gamma} - 1)n^{\gamma-\varepsilon}} \right) \Pr_{T,x;L,(m)}(D_{n,k}^{(\delta)} \geq d) + \frac{n^{2(1-\gamma)+\delta}}{n^{\gamma-\varepsilon}} e^{-C_2n^{1-\gamma}}, \]
By applying the following bound to $P_{T;L} (D_{n,k}^{(\delta)} \geq d)$ and continuing the recursion backwards all the way to $d = 1$ and recalling that
\[
P_{T;L} (D_{n,k}^{(\delta)} \geq d) = 1 \quad (k + m)^{\gamma},
\]
we obtain that $P_{T;L} (D_{n,k}^{(\delta)} \geq d + 1)$ is bounded from above by
\[
\frac{1}{k^{\gamma}} \prod_{j=2}^{d} \left( \frac{C_1 n^{2(1-\gamma)+\delta}}{(n^{1-\gamma} - 1)n^{\gamma-\delta}} + d^{\gamma} n^{\gamma} \right)^{d-1} \leq 1 \quad (k^{\gamma} n^{\gamma})^{d-1},
\]
for some $\beta > 0$. Now, by choosing $\varepsilon$ so that
\[
1 - \varepsilon - 2(1 - \gamma) - \delta > \varepsilon \iff \varepsilon < \frac{1 - 2(1 - \gamma) - \delta}{2},
\]
there exists $C_2$ depending on $\gamma$ and $d$ such that
\[
P_{T;L} (D_{n,k}^{(\delta)} \geq d) \leq \frac{C_2}{k^{\gamma} n^{\gamma(d-1)}},
\]
which proves the result.

4. Power-law degree distribution in TBRW trees (proof of Theorem 1.5)

In this section, we prove that the degree distribution of the random tree sequence $\{T_n\}_{n \geq 0}$ in the $L$-TBRW with $L_n = \text{Ber}(p_n)$, $p_n = \Theta(n^{-\gamma})$ and $\gamma \in (2/3, 1]$ converges to a power-law distribution with exponent 3, the very same limiting distribution as in the Barabási-Albert model of preferential attachment (PA) [5,10].

It will be useful to look at the random tree sequence $\{T_n\}_{n \geq 0}$ only at the random times when the TBRW adds new vertices to the tree. For this reason, we begin by recursively defining the following sequence of stopping times:
\[
\tau_k := \inf \{ n > \tau_{k-1} : Z_n = 1 \}, \tag{4.1}
\]
where $\tau_0 \equiv 0$. In words, $\tau_k$ is the time when the $k$-th (new) vertex is added.

It will be useful to understand the asymptotic behavior of the stopping times $\{\tau_k\}_{k \geq 0}$. The following lemma, whose proof will be postponed to the end of this section, essentially tells us that under the regime $\gamma > 2/3$ we need to wait large amounts of time to see the walker adding another leaf.

**Lemma 4.1** (Growth times are rare). Consider a $L$-TBRW, where $L_n = \text{Ber}(p_n)$ with $p_n = \Theta(n^{-\gamma})$ and $\gamma > 2/3$. Then, there exists a sufficiently small $\delta > 0$, depending only on $\gamma$, such that
\[
\lim_{k \to \infty} k^{\gamma + \delta} \tau_k^{-\gamma} = 0, \quad P_{T_0;L} \text{-a.s.} \tag{4.2}
\]
For the rest of this proof fix $\delta > 0$ such that (4.2) holds.
Definition 4.2 (Good and bad time intervals). With Lemma 4.1 in mind, we say that $\Delta \tau_k := \tau_k - \tau_{k-1}$ is good if
\[ \Delta \tau_k \geq k^{2+d} + 1, \tag{4.3} \]
and we say that it is bad otherwise.

To avoid clutter, we will write $\tilde{T}_k := T_{\tau_k}$, and $\tilde{V}_k$ for the set of vertices of $\tilde{T}_k$ and $\tilde{F}_k$ for $\mathcal{F}_{\tau_k}$. Notice that for any $n \in (\tau_{k-1}, \tau_k)$, $T_n$ and $\tilde{T}_{k-1}$ has the same degree distribution. For this reason we focus our attention on the process $\{\tilde{T}_k\}_k$.

Let us begin looking at the evolution of the random graph model $\{\tilde{T}_k\}_k$ from a slightly different, although equivalent, perspective. At any “time” step $k$, a new vertex $v_k$ and two half-edges, $h_{k,1}$ and $h_{k,2}$, are added to $\tilde{T}_{k-1}$. The two half-edges account for the edge which connects the vertex $v_k$ to the tree $\tilde{T}_{k-1}$. While the half-edge $h_{k,1}$ is always incident to the new vertex $v_k$, the half-edge $h_{k,2}$ will be attached to the vertex in $\tilde{V}_{k-1}$ where the random walk resides at time $\tau_k - 1$, thus determining to which of the existing vertices the new vertex $v_k$ will be connected to in $\tilde{T}_{k}$.

Leveraging on the above-mentioned equivalent perspective of the evolution of $\{\tilde{T}_k\}_k$, we are going to introduce a color-assignment process for the half-edges which will be instrumental for proving the result. Specifically, at any time $k$, we color the two half-edges $h_{k,1}$ and $h_{k,2}$ in either blue or red according to the following rule: Let $B_{k-1}$ and $R_{k-1}$ denote the total number of half-edges blue and red in $\tilde{T}_{k-1}$, respectively (w.l.o.g, we assume $B_1 = 2, R_1 = 0$, which means that $T_0$ is a single edge whose half-edges are both blue). Then, for $k \geq 2$,

- if $\Delta \tau_k$ is bad, we color both $h_{k,1}$ and $h_{k,2}$ red;
- if $\Delta \tau_k$ is good, we color $h_{k,1}$ blue, while we flip a biased coin to decide the color of $h_{k,2}$. The probability that $h_{k,2}$ is blue is $B_{k-1}/2(k-1)$, and red otherwise.

Note that $B_{k-1}/2(k-1)$ is the ratio of blue half-edges in $\tilde{T}_{k-1}$. Note also that the color assignment depends on the evolution of $\{\tilde{T}_k\}_k$ through the conditions $\Delta \tau_k$ being bad or good, which in turns depends on $\gamma$ (see, Equation (4.3)), but it does not depend on the position of the random walk at time $\tau_k - 1$.

As we shall soon show, in order to prove Theorem 1.5, it suffices to show that the empirical blue degree distribution of $\tilde{T}_k$ converges to $4/d(d+1)(d+2)$. The first ingredient is to show that the total number of red half-edges is small compared to the total number of blue half-edges. This is addressed in the next lemma, whose proof is deferred to the end of this section.

Lemma 4.3 (Not many red edges). There exist $\varepsilon, \delta' > 0$ depending on $\gamma$ only, such that for $k$ sufficiently large
\[ \mathbb{P}_{\tilde{T}_k, x : \mathcal{L}} (R_k > k^{1-\varepsilon}) \leq e^{-k\delta'}. \]

Next, we introduce some notation which will be instrumental in proving Theorem 1.5. Let $b_k(v)$ (resp. $r_k(v)$) be the blue (resp. red) degree of a vertex $v$ in $\tilde{V}_k$, that is, $b_k(v)$ (resp. $r_k(v)$) counts the number of blue (resp. red) half-edges incident to $v$ in $\tilde{T}_k$. Also, for a fixed
Let \( d \in \mathbb{N} \setminus \{0\} \), let \( B_k(d) \) denote the number of vertices in \( \tilde{T}_k \) whose blue degree is exactly \( d \), i.e.,
\[
B_k(d) := \sum_{v \in \tilde{V}_k} \mathbf{1}\{b_k(v) = d\}.
\]
Moreover, we say that a vertex \( v \in \tilde{V}_k \) is blue if all half-edges incident to it are blue.

Our main argument will rely on stochastic approximation techniques \cite{6,26} to deal with the error arising from the fact that the blue degree of a vertex does not evolve according to a "pure" linear preferential attachment scheme, i.e., the probability of a vertex increasing its blue degree by one is not simply proportional to its blue degree, but there is an (additive) error which must be controlled. To keep the paper self-contained, we state below the specific stochastic approximation framework we shall use, which is taken \textit{ad litteram} from \cite{15}.

\textbf{Lemma 4.4} (Stochastic Approximation; Lemma 3.1 in \cite{15}). Let \( \{Q_n\}_{n \geq 0} \) be a non-negative stochastic process satisfying the following recursion:
\[
Q_n - Q_{n-1} = \frac{1}{n} (\Psi_{n-1} - Q_{n-1} \Phi_{n-1}) + M_n - M_{n-1},
\]
where, \( \{\Psi_n\}_{n \geq 0}, \{\Phi_n\}_{n \geq 0} \) are almost-surely convergent processes with deterministic limits \( \psi > 0 \) and \( \phi > 0 \), respectively, and \( \{M_n\}_{n \geq 0} \) is an almost-surely convergent process. Then, \( \lim_{n \to \infty} Q_n = \frac{\psi}{\phi} \), almost surely.

Together with the stochastic approximation framework, we will need a lemma which formalizes the discussion made in the previous paragraph. Roughly speaking, it says that whenever \( \Delta \tau_k \) is good, the position of \( X_{\tau_{k-1}} \) is selected according to a mixture of a preferential attachment scheme and some random error.

\textbf{Lemma 4.5} (Almost PA at good times). There exists \( \delta > 0 \) such that for every \( k \in \mathbb{N} \) and \( t > k^{2+\delta} \),
\[
\mathbb{P}_{T_{k-1},X_{\tau_{k-1}};\mathcal{L}(\tau_{k-1})}(X_{\tau_{k-1}} = v \mid \tau_1 = t) = \frac{\deg_{\tau_{k-1}}(v)}{2(k-1)} + \text{error}(v,k), \text{ for } v \in \tilde{T}_{k-1},
\]
where the term \( \text{error}(v,k) \) represents a \( \mathcal{F}_{\tau_{k-1}} \)-measurable random variable bounded by \( e^{-k^\delta} \), \( \mathbb{P}_{T_0,x;\mathcal{L}} \)-almost surely.

Again, we will defer the proof of the above lemma to the end of this section. Now we will show how Theorem \ref{thm:main} follows from the aforementioned lemmas.

\textbf{Proof of Theorem \ref{thm:main}}. We begin by noticing that, if \( N_k(d) \) denotes the number of vertices of degree \( d \) in \( \tilde{T}_k \) (ignoring the colors), then for all \( d \in \mathbb{N} \setminus \{0\} \),
\[
0 \leq B_k(d) - \sum_{v \in \tilde{V}_k} \mathbf{1}\{b_k(v) = d, v \text{ is blue}\} \leq R_k.
\]

\textbf{4.4}
Note also that
\[
0 \leq \sum_{v \in \mathcal{V}_k} \mathbb{1}\{\deg_{\tilde{T}_k}(v) = d\} - \sum_{v \in \mathcal{V}_k} \mathbb{1}\{b_k(v) = d, \text{\textit{is blue}}\} \leq R_k. \tag{4.5}
\]
By virtue of Lemma 4.3, \(\lim_k R_k/k = 0, \quad \mathbb{P}_{T_0,x;\mathcal{L}}\)-a.s. Putting this together with (4.4) and (4.5), we see that in order to prove Theorem 1.5, it is enough to show the following claim.

**Claim 4.6** (Limit distribution for blue degrees). For any \(d \in \mathbb{N} \setminus \{0\}\),
\[
\lim_{k \to \infty} \frac{B_k(d)}{k} = \frac{4}{d(d + 1)(d + 2)}, \quad \mathbb{P}_{T_0,x;\mathcal{L}}\)-a.s.
\]

**Proof of the claim:** We want to show that the evolution of the blue degrees behaves much like a preferential attachment scheme. This is not a-priori clear because mixing on a long “good” time interval only yields\(^2\) that the full degrees (including the red degrees) of the vertices behave like that scheme. However, we are going to show that the value of
\[
\mathbb{P}_{T_0,x;\mathcal{L}}\left(\Delta b_k(v) = 1 \mid \Delta \tau_k \text{ is good, } \tilde{\mathcal{F}}_{k-1}\right),
\]
is “close” to \(\frac{b_{k-1}(v)}{2(k-1)}\), where here and in the sequel, \(v \in \tilde{\mathcal{V}}_{k-1}\).

This means that \(v\) receives the new blue half edge (if there is one) with a probability that is roughly proportional to its existing blue degree.

In order to achieve this, by Lemma 4.5 and recalling that \(\Delta \tau_k\) is good if \(\Delta \tau_k \geq k^{2+\delta} + 1\), we have that
\[
\mathbb{P}_{T_0,x;\mathcal{L}}\left(X_{\tau_{k-1}} = v \mid \Delta \tau_k \text{ is good, } \mathcal{F}_{\tau_{k-1}}\right) = \frac{\deg_{\tau_{k-1}}(v)}{2(k-1)} + \text{error}(v,k), \quad \mathbb{P}_{T_0,x;\mathcal{L}}\)-a.s. \tag{4.6}
\]
Where \(\text{error}(v,k)\) represents a \(\mathcal{F}_{\tau_{k-1}}\)-measurable random variable bounded by \(e^{-k^\delta}\), \(\mathbb{P}_{T_0,x;\mathcal{L}}\)-almost surely. From the definition of the color-assignment process, we have that
\[
\mathbb{P}_{T_0,x;\mathcal{L}}\left(\Delta b_k(v) = 1 \mid X_{\tau_{k-1}} = v, \Delta \tau_k \text{ is good, } \tilde{\mathcal{F}}_{k-1}\right) = \frac{B_{k-1}}{2(k-1)},
\]
which, combined with (4.6), and using that \(\deg_{\tau_{k-1}}(v) = b_{k-1}(v) + r_{k-1}(v)\) and \(B_k + R_k = 2k\), leads to
\[
\mathbb{P}_{T_0,x;\mathcal{L}}\left(\Delta b_k(v) = 1 \mid \Delta \tau_k \text{ is good, } \mathcal{F}_{\tau_{k-1}}\right) = \frac{b_{k-1}(v)}{2(k-1)} + \frac{B_{k-1}r_{k-1}(v) - b_{k-1}(v)R_{k-1}}{(2(k-1))^2} \pm \frac{B_{k-1}}{2(k-1)} \cdot \text{error}(v,k), \tag{4.7}
\]
where \(F_{k-1}(v)\) is a \(\tilde{\mathcal{F}}_{k-1}\)-measurable random variable satisfying that
\[
F_{k-1}(v) = \mathcal{O}\left(\frac{B_{k-1}r_{k-1}(v) - b_{k-1}(v)R_{k-1}}{(k-1)^2}\right). \tag{4.8}
\]
\(^2\)Since the stationary distribution on a fixed graph is proportional with the degrees.
Here $O$ refers to a limit as $k \to \infty$. Note that although the collection of $v$’s to which this applies, keeps changing, the term $\text{error}(v, k)$ is uniformly bounded in $v$. Denote by $B_k(d)$ the subset of vertices in $V_k$ whose blue degree is exactly $d$; note that $|B_k(d)| = B_k(d)$. Clearly,

$$
\Delta B_k(d) = \begin{cases} 
1 \{\Delta \tau_k \text{ is good}\} + \sum_{v \in B_k(0)} 1 \{\Delta b_k(v) = 1\} - \sum_{v \in B_k(1)} 1 \{\Delta b_k(v) = 1\}, & d = 1, \\
\sum_{v \in B_k(d-1)} 1 \{\Delta b_k(v) = 1\} - \sum_{v \in B_k(d)} 1 \{\Delta b_k(v) = 1\}, & d > 1.
\end{cases}
$$

To avoid clutter, henceforth we set

$$W_{k-1} := \mathbb{P}_{T_0,x,L} (\Delta \tau_k \text{ is good} \mid \mathcal{F}_{\tau_{k-1}}).$$

Notice that for $v \in B_{k-1}(0)$ Equation (4.7) gives us that

$$\mathbb{P}_{T_0,x,L} (\Delta b_k(v) = 1 \mid \Delta \tau_k \text{ is good}, \mathcal{F}_{\tau_{k-1}}) = F_{k-1}(v),$$

which together with the fact vertices can increase their blue degree only at good times, $B_k \leq 2k$ and (4.8) implies that

$$\mathbb{E}_{T_0,x,L} \left[ \sum_{v \in B_{k-1}(0)} 1 \{\Delta b_k(v) = 1\} \mid \tilde{F}_{k-1} \right] = W_{k-1} \sum_{v \in B_{k-1}(0)} F_{k-1}(v) = W_{k-1} \mathcal{O} \left( \frac{R_{k-1}}{k-1} \right).$$

Then, using the above identity and (4.7) we obtain

$$\mathbb{E}_{T_0,x,L} [\Delta B_k(d) \mid \tilde{F}_{k-1}] = W_{k-1} \left( \frac{(d-1)B_{k-1}(d-1) - d B_{k-1}(d)}{2(k-1)} + W_{k-1} (Y_{k-1}(d-1) - Y_{k-1}(d)) \right),$$

where, $Y_{k-1}(0) \equiv 1$ and for every $d \geq 1$,

$$Y_{k-1}(d) := \begin{cases} 
\sum_{v \in B_{k-1}(d)} F_{k-1}(v) - \sum_{v \in B_{k-1}(0)} F_{k-1}(v), & d = 1, \\
\sum_{v \in B_{k-1}(d)} F_{k-1}(v), & d > 1.
\end{cases}$$

Notice that for all $d \geq 1$, by (4.8) we have that

$$Y_{k-1}(d) = \mathcal{O} \left( \frac{R_{k-1}}{k-1} \right).$$

Now, in order to simplify our notation, set $Q_k(d) := B_k(d)/k$. Then, writing

$$Q_k(d) - Q_{k-1}(d) = Q_k(d) - \mathbb{E}_{T_0,x,L} [Q_k(d) \mid \tilde{F}_{k-1}] + \mathbb{E}_{T_0,x,L} [Q_k(d) \mid \tilde{F}_{k-1}] - Q_{k-1}(d),$$

and applying (4.9), we obtain the following recursion for $Q_k(d)$:

$$Q_k(d) - Q_{k-1}(d) = \frac{1}{k} (\Psi_{k-1}(d-1) - Q_{k-1}(d) \Phi_{k-1}(d)) + \Delta M_{k-1}(d),$$

where, for every $d \geq 1$, we have that
\begin{itemize}
  \item \( \Psi_{k-1}(d-1) = \left( \frac{d}{2} Q_{k-1}(d-1) + Y_{k-1}(d-1) - Y_{k-1}(d) \right) W_{k-1} \);
  \item \( \Phi_{k-1}(d) = 1 + \frac{d}{2} W_{k-1} = 1 + \frac{d}{2} \mathbb{P}_{T_0,x;\mathcal{L}} \left( \Delta \tau_k \text{ is good} \mid \mathcal{F}_{k-1} \right) \);
  \item \( \Delta M_{k-1}(d) = Q_k(d) - \mathbb{E}_{T_0,x;\mathcal{L}} \left[ Q_k(d) \mid \mathcal{F}_{k-1} \right] \).
\end{itemize}

In light of Lemma 4.4, our goal is to guarantee that both of the processes \( \{ \Psi_n(d-1) \}_{n \geq 0} \) and \( \{ \Phi_n(d) \}_{n \geq 0} \) have positive and deterministic limits and \( \{ M_n(d) \}_{n \geq 0} \) is convergent, for all fixed \( d \).

The convergence of \( \{ \Phi_n(d) \}_{n \geq 0} \) to \( 1+d/2 \) (i.e. that \( W_{k-1} \to 1 \)) comes as a consequence of Lemma 4.1. Indeed, observe that

\[
W_{k-1} = \mathbb{P}_{T_0,x;\mathcal{L}} \left( \Delta \tau_k \geq k^{2+\delta} + 1 \mid \mathcal{F}_{\tau_{k-1}} \right) = \prod_{s=\tau_{k-1}+1}^{\tau_{k-1}+k^{2+\delta}+2} (1-p_s) \to 1 ,
\]

\( \mathbb{P}_{T_0,x;\mathcal{L}} \)-a.s. as \( k \) goes to infinity, since \( p_s = \Theta(s^{-\gamma}) \) and by Lemma 4.1 \( k^{2+\delta}/\tau_k^\gamma \) goes to zero almost surely.

Convergence of \( \{ M_n(d) \}_{n \geq 0} \): For a fixed \( d \), the terms \( \Delta M_k(d) \) define a martingale difference sequence. Setting \( M_0(d) = 0 \), let the process \( \{ M_n(d) \}_{n \geq 0} \) be the corresponding martingale. Diving both sides of (4.9) yields to

\[
\mathbb{E}_{T_0,x;\mathcal{L}} \left[ Q_k(d) \mid \mathcal{F}_{k-1} \right] = \frac{B_{k-1}(d)}{k} + \mathcal{O} \left( \frac{B_{k-1}(d) + B_{k-1}(d-1) + R_{k-1}}{k^2} \right) ,
\]

which implies that

\[
|\Delta M_{k-1}(d)| = \frac{\Delta B_k(d)}{k} + \mathcal{O} \left( \frac{B_{k-1}(d) + B_{k-1}(d-1) + R_{k-1}}{k^2} \right) .
\]

On the other hand, since \( \Delta B_k(d) \leq 2 \) and \( B_k(d) \) and \( R_k \) are less than \( 2k \) for all \( d \), it follows that

\[
(\Delta M_k(d))^2 = \mathcal{O} \left( \frac{1}{k^2} \right) ,
\]

which is sufficient to guarantee that \( \{ M_n(d) \}_{n \geq 0} \) is bounded in \( L^2 \) and thus converges \( \mathbb{P}_{T_0,x;\mathcal{L}} \)-almost surely.

Convergence of \( \{ \Psi_k(d-1) \}_{k \geq 0} \): This part requires induction on \( d \), since \( \Psi_k(d-1) \) is defined via \( Q_k(d-1) \). Thus, our first step is to prove it for \( d = 1 \).

Observe that as a consequence of Lemma 4.3 and (4.10), we have that \( \{ Y_n(d) \} \) converges to zero for \( d \geq 1 \). Moreover, when \( d = 1 \), Lemma 4.1 implies that

\[
\lim_{k \to \infty} \Psi_{k-1}(0) = \lim_{k \to \infty} (1 - Y_{k-1}(1)) \mathbb{P}_{T_0,x;\mathcal{L}} \left( \Delta \tau_k \text{ is good} \mid \mathcal{F}_{\tau_{k-1}} \right) = 1 .
\]

Combining the above with our previous results and Lemma 4.4 gives us that \( \{ Q_n(1) \}_{n \geq 0} \) converges to \( \frac{1}{1+1/2} = 2/3 \). Now let \( d \geq 2 \). Using induction we show that \( \lim_{k \to \infty} Q_k(d) \) exists
a.s., and calculate the limit recursively. So let us assume that \( \lim_{k \to \infty} Q_k(d-1) =: Q(d-1) \) exists almost surely. It follows that
\[
\lim_{k \to \infty} \Psi_k(d-1) = (d-1)Q(d-1)/2, \text{ a.s.}
\]
Using Lemma 4.4 again, we conclude that
\[
Q(d) := \lim_{k \to \infty} Q_k(d) = \frac{d-1}{d+2} \cdot Q(d-1), \quad \mathbb{P}_{T_0,x;L}-\text{a.s.}
\]
which implies that for \( d \geq 1 \),
\[
\lim_{k \to \infty} Q_k(d) = \frac{4}{d(d+1)(d+2)}, \quad \mathbb{P}_{T_0,x;L}-\text{a.s.}
\]
and proves Claim 4.6.

Returning now to the proof of Theorem 1.5, recall that, as observed at the beginning of our argument, the statement follows from Claim 4.6, and we are done. \( \square \)

Finally, having shown how Theorem 1.5 follows from our lemmas, we still owe the proofs of Lemmas 4.1, 4.3 and 4.5, and this is what we are going to do below.

**Proof of Lemma 4.1.** Let us begin by noticing that the claim is true for \( \gamma > 1 \), since in this regime one has \( \tau_k = +\infty \), for all sufficiently large \( k \), \( \mathbb{P}_{T_0,x;L}-\text{a.s.} \). In the sequel, we therefore address the case \( \gamma \in (2/3, 1] \). Let \( V_m = |V_m| \) and note that \( V_m := Z_1 + \cdots + Z_m \), which implies that, as \( m \to \infty \),
\[
\mathbb{E}_{T_0,o;L}(V_m) = \Theta(m^{1-\gamma}).
\]
Since \( \tau_k \) is the first time of having \( k + 1 \) vertices, if \( T_0 = \{o\} \), one has the identity
\[
\{ \tau_k \leq k^{1/(1-\gamma)-\varepsilon} \} = \{ V_{k^{1/(1-\gamma)-\varepsilon}} \geq k \},
\]
for any \( \varepsilon < 1/(1-\gamma) \). Recalling the independence of the \( Z_n \)'s, using that \( \log(1+x) < x, \ x > 0 \), and bounding the sum by an integral, it follows that for every \( \lambda > 0 \),
\[
\log \mathbb{E}_{T_0,x;L}(e^{\lambda V_m}) \leq \begin{cases} (e^\lambda - 1)m^{1-\gamma}/(1-\gamma), & \gamma < 1, \\ (e^\lambda - 1)(\log m + 1), & \gamma = 1. \end{cases}
\]
Thus, by the exponential Markov inequality there is a positive constant \( C \) depending on \( \gamma \) only, such that
\[
\mathbb{P}_{T_0,x;L}(V_{k^{1/(1-\gamma)-\varepsilon}} \geq k) \leq e^{-k} e^{Ck^{1-(1-\gamma)\varepsilon}}.
\]
This and the Borel-Cantelli lemma yields that \( \tau_k < k^{1/(1-\gamma)-\varepsilon} \) occurs only finitely many times, almost surely. We conclude the proof by choosing a small enough \( \varepsilon > 0 \) and noting that \( \gamma > 2/3 \) if and only if \( \gamma/(1-\gamma) > 2 \). \( \square \)

Now we prove Lemma 4.5 first and prove Lemma 4.3 at the end of this section.
Proof of Lemma 4.5. Recall that \( \{G_n\}_{n \geq 1} \) denotes the natural filtration induced by the variables \( \{Z_n\}_{n \geq 1} \). Observe that conditioned on \( G_{k-1} \), \( \Delta \tau_k \) dominates a random variable \( Y_k \) following geometric distribution of parameter \( c/\tau_k^\gamma \), where \( c \) is a positive constant depending on \( (p_n)_n \) only. Thus, with for \( k \) sufficiently large, there exists a positive constant \( C \) such that, almost surely,

\[
\mathbb{P}_{T_0,x;\mathcal{L}} \left( \Delta \tau_k \text{ is bad } \mid \bar{\mathcal{F}}_{k-1} \right) \leq \mathbb{P}_{T_0,x;\mathcal{L}} \left( Y_k < k^{2+\delta} + 1 \mid \bar{\mathcal{F}}_{k-1} \right) \leq C \frac{k^{2+\delta}}{\tau_k^\gamma}.
\]

(4.13)

Note that Lemma 4.1 allows us to choose \( \delta \) small enough so the RHS of the above inequality goes to zero almost surely when \( k \) goes to infinity.

Now, let \( \Pi_k \) be the (random) stationary distribution of a simple random walk on the rooted tree \( T_{\tau_k} \) with a self-loop attached to the root \( o \) (we introduce a self-loop at the root just to avoid periodicity). More specifically, \( \Pi_k \) is the random distribution over \( V_{\tau_k} \) which assigns to each \( v \in V_{\tau_k} \) the weight

\[
\Pi_k(v) := \frac{\text{deg}_{\tau_k}(v)}{2k}.
\]

Let \( P_t^{x,T} \) denote the distribution of a random walk over \( T \) started at \( x \) after \( t \) steps. It is known (see [12]) that the mixing time of a random walk on a tree \( T \) with \( k \) vertices satisfies the upper bound

\[
t_{\text{mix}} \leq 2k^2,
\]

uniformly in the initial state \( x \in T \). Recall that \( d_{\text{TV}} \) denotes the total variation distance. It is well known that for any \( \varepsilon > 0 \) the upper bound

\[
t_{\text{mix}}(\varepsilon) \leq \log \left( \frac{1}{\varepsilon} \right) t_{\text{mix}},
\]

holds, and in particular, for \( \varepsilon := \exp\{-k^\delta\} \) by (4.14), we obtain that

\[
d_{\text{TV}} \left( \Pi_{k-1}, P_{X_{\tau_{k-1}},T_{\tau_{k-1}}}^t \right) \leq e^{-k^\delta}, \quad \mathbb{P}_{T_0,x;\mathcal{L}} \text{-a.s.}
\]

(4.15)

for all \( t > k^{2+\delta} \). Next, the Strong Markov property applied to \( \mathcal{L} \)-TBRW at time \( \tau_{k-1} \) yields

\[
\mathbb{P}_{T_0,x;\mathcal{L}} \left( X_{\tau_{k-1}} = v \mid \mathcal{F}_{\tau_{k-1}} \right) = \mathbb{P}_{X_{\tau_{k-1}},T_{\tau_{k-1}};\mathcal{L}(\tau_{k-1})} \left( X_{\tau_{k-1}} = v \right).
\]

(4.16)

Observe that under \( \mathbb{P}_{X_{\tau_{k-1}},T_{\tau_{k-1}};\mathcal{L}(\tau_{k-1})} \) conditioned on \( \tau_1 = t \), \( X_{\tau_1} \) has distribution \( P_{X_{\tau_{k-1}},T_{\tau_{k-1}};\mathcal{L}(\tau_{k-1})}^t \). The reader may check this fact recalling that the event \( \tau_1 = t \) depends only on the random variables \( \{Z_n\}_{n \geq 1} \) together with the independence of \( Z_n \) from \( \mathcal{F}_{n-1} \), for all \( n \).

Using (4.16) and (4.15) for \( t > k^{2+\delta} \), we have that

\[
\mathbb{P}_{X_{\tau_{k-1}},T_{\tau_{k-1}};\mathcal{L}(\tau_{k-1})} \left( X_{\tau_1} = v \mid \tau_1 = t \right) = \frac{\text{deg}_{\tau_{k-1}}(v)}{2(k-1)} \pm \text{error}(v,k),
\]

where \( \text{error}(v,k) \) represents an \( \mathcal{F}_{\tau_{k-1}} \)-measurable random variable for which

\[
\mathbb{P}_{T_0,x;\mathcal{L}} \left( \text{error}(v,k) \leq e^{-k^\delta} \right) = 1.
\]
This concludes the proof. □

Proof of Lemma 4.3. We know that $\Delta R_k = 2$ when $\Delta \tau_k$ is bad, while it equals 1 when $\Delta \tau_k$ is good and the half-edge $h_{k,2}$ is colored red. Therefore,

$$
\mathbb{E}_{T_0, x; \mathcal{L}} [\Delta R_k \mid \mathcal{F}_{\tau_{k-1}}] = 2\mathbb{P}_{T_0, x; \mathcal{L}} (\Delta \tau_k \text{ is bad} \mid \mathcal{F}_{\tau_{k-1}}) + \frac{R_{k-1}}{2(k-1)} \mathbb{P}_{T_0, x; \mathcal{L}} (\Delta \tau_k \text{ is good} \mid \mathcal{F}_{\tau_{k-1}})
$$

(4.17)

$$
= \frac{R_{k-1}}{2(k-1)} + \text{error}(k),
$$

where, error$(k)$ is an $\mathcal{F}_{\tau_{k-1}}$-measurable variable such that

$$
|\text{error}(k)| \leq 2\mathbb{P}_{T_0, x; \mathcal{L}} (\Delta \tau_k \text{ is bad} \mid \mathcal{F}_{\tau_{k-1}}), \mathbb{P}_{T_0, x; \mathcal{L}} \text{-a.s.} \tag{4.18}
$$

Now, define the function $\phi : \{2, 3, \ldots\} \to (1, \infty)$ by

$$
\phi(k) := \prod_{j=1}^{k-1} \left(1 + \frac{1}{2j}\right) = 2^{1-k} \frac{(2k-1)!!}{(k-1)!} = \Theta(\sqrt{k}),
$$

where the last step uses Stirling’s formula and that $(2k-1)!! = \frac{(2k)!}{2^k k!}$. In fact, even $\phi(k)/\sqrt{k} \to \text{const}$, which can be computed. Let $R'_k := \frac{R_k}{\phi(k)}$. Using (4.17), a simple computation shows that

$$
\mathbb{E}_{T_0, x; \mathcal{L}} [\Delta R'_k \mid \mathcal{F}_{\tau_{k-1}}] = \frac{\text{error}(k)}{\phi(k)}.
$$

Next, exploiting Doob’s decomposition, we may write $R'_k$ as

$$
R'_k = M_k + \sum_{m=1}^{k} \frac{\text{error}(m)}{\phi(m)},
$$

(4.19)

where the process $M$ is a mean zero martingale whose increments satisfy that

$$
|M_{j+1} - M_j| = \left| \frac{R_{j+1} - (1 + \frac{1}{2j}) R_j - \text{error}(j+1)}{\phi(j+1)} \right| \leq \frac{5}{\phi(j+1)}, \mathbb{P}_{T_0, x; \mathcal{L}} \text{-a.s.}
$$

Thus, there exists a positive constant $C_1$ such that

$$
\sum_{j=1}^{k} |M_{j+1} - M_j|^2 \leq C_1 \log k. \tag{4.20}
$$

Combining (4.20) with the Azuma–Hoeffding inequality [9] yields a positive constant $C_2$ depending on $\gamma$ and $C_1$ such that

$$
\mathbb{P}_{T_0, x; \mathcal{L}} (M_k > C_2 k^{\delta'/2} \log^{1/2} k) \leq e^{-k^{\delta'}}.
$$
To control the predictable term in Doob’s decomposition (i.e. the sum in (4.19)), we break the sum into two terms as \( \sum_{m=1}^{k^{\delta'}} \frac{\text{error}(m)}{\phi(m)} \leq 2 \sum_{m=1}^{k^{\delta'}} \frac{1}{\phi(m)} \leq C_3 k^{\delta'/2} \), for some universal constant \( C_3 \). For the rest of the terms, recall from the proof of Lemma 4.1 (see (4.12)) that for any \( \epsilon' < 1/(1 - \gamma) \) and for large enough \( j \),

\[
P_{T_0, x; L} \left( \tau_j \leq j^{1/(1 - \gamma) - \epsilon'} \right) \leq e^{-C_4 j}. \tag{4.21}
\]

Thus, recalling (4.18) and (4.13), on the event \( \bigcap_{j=k^{\delta'}} \{ \tau_j \geq j^{1/(1 - \gamma) - \epsilon'} \} \) we have that

\[
\sum_{j=k^{\delta'}}^{k} \frac{\text{error}(j)}{\phi(j)} \leq \sum_{j=k^{\delta'}}^{k} C_5 j^{2 + \delta - (\gamma/(1 - \gamma) - \gamma' \epsilon') - 1/2} \leq C_6 k^{1/2 - \epsilon},
\]

where \( \epsilon > 0 \) depends on \( \gamma \), which is fixed and on \( \delta \) and \( \epsilon' \), which can be chosen properly. On the other hand, by the union bound and (4.21) we obtain that

\[
P_{T_0, x; L} \left( \bigcup_{j=k^{\delta'}}^{k} \{ \tau_j \leq j^{1/(1 - \gamma) - \epsilon'} \} \right) \leq \exp\{-C_7 k^{\delta'}\}.
\]

Regarding the \( L^{(\delta)} \)-TBRW process we have the following result:

5. Example: recurrence without power-law degree distribution

In order to illustrate that for the TBRW random walk recurrence is not equivalent to observing a scale-free tree sequence, we provide an example of a sequence \( \{Z_n\}_{n \geq 1} \) under which the walker is recurrent but the corresponding tree sequence has a limiting degree distribution which is very different from the power-law. In fact, we prove that in this setup, the proportion of leaves converges to one almost surely.

To carry out this program, for a fixed \( \delta > 0 \), consider the sequence of real numbers \( (p_n^{(\delta)})_{n \geq 1} \), with \( p_n^{(\delta)} := n^{-\delta} \), and the laws

\[
L_n^{(\delta)} := \begin{cases} 
\log n, & \text{with probability } p_n^{(\delta)}, \\
0, & \text{with probability } 1 - p_n^{(\delta)}.
\end{cases} \tag{5.1}
\]

Regarding the \( L^{(\delta)} \)-TBRW process we have the following result:
Proposition 5.1 (Recurrence with leaves only). For the $L^{(δ)}$-TBRW process satisfying (5.1), the random walk is recurrent for all $δ \in (2/3, 1]$. Furthermore, if $N_n(1)$ denotes the number of leaves and $|V_n|$ the total number of vertices in $T_n$, then

$$\lim_{n \to \infty} \frac{N_n(1)}{|V_n|} = 1, \quad \mathbb{P}_{T_0,x;L^{(δ)}}\text{-a.s.}$$

Intuition: By (5.1), when the walker adds new leaves at time $n$, it adds an amount of $\log n$. Heuristically, Proposition 5.1 says that this quantity is not large enough to "trap" the walker in some neighborhood of a vertex, so the walk is still recurrent. Nonetheless, it is large enough to change the power-law degree distribution dramatically, producing trees whose empirical degree distribution converges to the degenerate distribution over $\mathbb{N}$ which puts all the mass at 1.

Proof of Proposition 5.1. We check recurrence first.

Recurrence of the walk. By Theorem 1.2, we need to show that $\{Z_n^{(δ)}\}_n$ satisfies all the assumptions (A1)-(A3) of that theorem. Assumptions (A1) and (A2) trivially follow from the definitions of $p_n^{(δ)}$ and $Z_n^{(δ)}$. For (A3), recalling that $δ \in (2/3, 1]$, use that $E(Z_n^{(δ)}) = \log n/n^δ$, yielding

$$M_n = \sum_{k=1}^{n} \frac{\log k}{k^δ} \leq \log n \sum_{k=1}^{n} \frac{1}{k^δ} \leq \log n \int_0^n x^{-δ} dx = (1 - δ)^{-1} \log n \cdot n^{1-δ},$$

hence

$$(1 - q_n) \cdot M_n^2 \leq n^{-δ} \cdot (1 - δ)^{-2} n^{2-2δ}(\log n)^2 = (1 - δ)^{-2} n^{2-3δ}(\log n)^2 \to 0,$$

since $δ > 2/3$. This proves recurrence.

Next we check the degree distribution.

Proportion of leaves. Define the stopping times

$$\tau_0 \equiv 0; \quad \tau_k := \inf\{n > \tau_{k-1} : Z_n^{(δ)} = \log n\}, \quad k \geq 1.$$  \hspace{1cm} (5.2)

Since $δ \leq 1$, it follows that all the $\tau_k$ are finite $P$-a.s. Introduce the shorthands

$$\tilde{N}_k := N_{\tau_k}(1); \quad \tilde{V}_k := |V_{\tau_k}|.$$

By (5.2), we have that $\tilde{V}_k = |V_0| + \sum_{j=1}^{k} \log(\tau_j)$ where $|V_0|$ is size of the initial tree $T_0$. Moreover, observe that whenever a certain amount of new leaves is added, even in the “worst case scenario” when new vertices are added to a leaf, only one leaf has to be subtracted from the total number of leaves. This implies that

$$\tilde{N}_k \geq \sum_{j=1}^{k} \log(\tau_j) - k = \tilde{V}_k - |V_0| - k, \quad \mathbb{P}_{T_0,x;L^{(δ)}}\text{-a.s.}$$
Using that \( \tau_k \geq k \) almost surely, we have that
\[
\tilde{V}_k \geq \sum_{j=1}^{k} \log j \geq \int_1^k \log x \, dx = k(\log k - 1) + 1,
\]
yielding that
\[
\frac{\tilde{N}_k}{\tilde{V}_k} \geq 1 - \frac{(|V_0| + k)}{\tilde{V}_k} \geq 1 - \frac{(|V_0| + k)}{k(\log k - 1) + 1}
\]
\[
\implies \liminf_{k \to \infty} \frac{\tilde{N}_k}{\tilde{V}_k} \geq 1, \quad \mathbb{P}_{T_0, x; \mathcal{L}(\delta)} - \text{a.s.}
\]

To conclude the proof use that \( \tilde{N}_k \leq \tilde{V}_k \) along with the obvious fact that the sequence \( \{N_n(1)/|V_n|\}_n \) only changes its values at the stopping times \( \tau_k \)'s. \( \square \)

6. Transience when many edges are grown

The purpose of this section is to demonstrate that when there are infinitely many growth times (i.e. \( p_n \) is not summable), and there are sufficiently many edges grown at those times, the walk is never recurrent, and under a mild condition on \( p_n \) (ruling out that the walk “gets stuck”) it is transient.

Let \( Z_n = w_n \) with probability \( p_n \) and \( Z_n = 0 \) otherwise. Here \( \{w_n\}_{n \geq 1} \) is a non-decreasing numerical sequence such that \( w_n \in \mathbb{N} \setminus \{0\} \).

We will denote \( \Xi_n := \sum_{k=1}^{n} p_k \), and for a sequence \( \{a_n\}_{n \geq 1} \), we will write \( \Delta a_i := a_i - a_{i-1} \). The graph distance between the root \( o \) and vertex \( v \) will be denoted by \( d(o, v) \). Finally, denote \( \tau_n := \inf\{n > \tau_{n-1} : Z_n > 0\} \), with \( \tau_0 \equiv 0 \), and \( r_{i,j} := \mathbb{P}_{T_0, o; \mathcal{L}(\tau_i > j)}, i, j \in \mathbb{N} \); note that \( r_{i,j} \) is a function of the sequence \( \{p_n\}_{n \geq 1} \) only.

**Remark 6.1** (Poisson-Binomial). The \( r_{i,j} \) can be computed for a fixed sequence of \( p_n \)'s using the identities of the Poisson-Binomial distribution, since the number of growths up to time \( m \) follows a Poisson-Binomial distribution with parameters \( m, p_1, ..., p_m \) and mean \( \Xi_m \), for \( m \geq 1 \). If this number is \( K^{m, p_1, ..., p_m} \) then
\[
r_{i,j} = P(K^{j, p_1, ..., p_j} \leq i - 1).
\]

Since we only need an upper bound in the next result, we can simply use Chebyshev. Indeed, if \( Y_i := \mathbb{1}\{Z_i > 0\}, \) i.e., the indicator of the \( i \)-th growth then \( r_{i,j} = P(Y_1 + ... + Y_j \leq i - 1) \), and by the Chebyshev inequality,
\[
r_{i,j} \leq \frac{\sum_{k=1}^{j} p_k(1 - p_k)}{(\Xi_j - i + 1)^2} \leq \frac{\Xi_j}{(\Xi_j - i + 1)^2}, \tag{6.1}
\]
provided \( \Xi_j > i - 1 \). \( \diamond \)

**Theorem 6.2** (Transience). Assume that \( \sum_{n \geq 1} p_n = \infty \) to rule out obvious recurrence, and that there exists a numerical sequence \( 0 < a_1 < a_2, ... \) such that
(1) \[
\sum_{i=1}^{\infty} r_{i,a_i} < \infty;
\]

(2) \[
\sum_{i=1}^{\infty} \frac{\Delta a_i}{w_{i-1} + 1} < \infty.
\]

Then the $\mathcal{L}$-TBRW is not recurrent.

If furthermore we assume that

(3) \[
\sum_{n \geq 1} \min\{p_n, p_{n+1}\} = \infty. \quad (6.2)
\]

then the $\mathcal{L}$-TBRW is transient: $P_{\tau_0,o,\mathcal{L}}\left(\lim_{n \to \infty} d(o, X_n) = \infty\right) = 1$.

Corollary 6.3. For every non-summable sequence $\{p_n\}_{n \geq 1}$ there exists some sequence $\{w_n\}_{n \geq 1}$, making the TBRW non-recurrent, and under Condition 3 of Theorem 6.2 even transient.

Example 6.4 (Transience). Let $p_i := \frac{1}{i+1}$, hence $P_n = O(\log n)$. Then for any $\{w_i\}$ growing so fast that with some $\epsilon > 0$

\[
\sum_{i=1}^{\infty} \frac{2^{i+\epsilon}}{w_{i-1} + 1} < \infty,
\]

holds, the $\mathcal{L}$-TBRW is transient. To verify this, pick $a_i := 2^{i+\epsilon}$, $\epsilon > 0$. Then by (6.1),

\[
r_{i,a_i} \leq \frac{P_{a_i}}{(P_{a_i} - i + 1)^2},
\]

which is $O(i^{-1-\epsilon})$, thus summable.

The next example shows that the condition (6.2) is indeed essential for transience.

Example 6.5 (The walk is not recurrent and the walker is “stuck”). Consider the sequence of $p_n$’s

\[
1/2, 1/2, 1/3, 1/4, 1/4, 1/8, 1/5, 1/16, ...
\]

which is obtained by “combing” the harmonic series $1/2, 1/3, 1/4, ...$ and the convergent series $1/2, 1/4, 1/8, ...$ together; condition (6.2) is not satisfied in this case. A close look at the proof of Theorem 6.2 reveals that for this choice of $p_n$ and sufficiently large $w_n$ (so that the first two conditions of Theorem 6.2 are met) with positive probability $d(o, X_n)$ can be stuck between two positive integers forever.

Proof of Theorem 6.2. We prove the result in three steps.

STEP 1: We now assume that (6.2) holds. Recall that $\tau_i$ is the $i$-th time of growth, $i \geq 1$, and for $k \geq 1$, let

\[
v_k := \text{the “father vertex” launching } w_k \text{ new edges at } \tau_k;
\]
\( A_k := \{ \exists i \geq 1 : d(o, X_n) > d(o, v_k) \forall n \geq \tau_k + i \} . \)

Note the following. Writing simply \( d(v) \) instead of \( d(o, v) \), on the event \( \limsup_k A_k \), the walker \( X \) eventually leaves all closed (!) balls of radius \( d(v_k) \) around \( o \). In particular then, \( d(v_{k+1}) > d(v_k) \). Hence, in fact \( X \) eventually leaves arbitrarily large closed balls, and this implies transience. So, if we show that

\[
P_{T_0, o, L} \left( \limsup_{k \to \infty} A_k \right) = 1 ,
\]

then transience follows.

We start by showing that for any \( \epsilon > 0 \) there is a deterministic integer \( k_0 = k_0(\epsilon) \equiv 1 \) such that if \( k \geq k_0 \) then

\[
\pi_k := P(A_k) \geq 1 - \epsilon .
\]

To achieve this, first note that

\[
P_{T_0, o, L}(\cap_{i=k}^{\infty} \{ \tau_i \leq a_i \}) = 1 - P_{T_0, o, L}(\cup_{i=k}^{\infty} \{ \tau_i > a_i \}) \geq 1 - \sum_{i=k}^{\infty} r_{i,a_i} .
\]

Heuristic explanation for estimating \( \pi_k \) from below: First note that clearly, \( \tau_k \geq k \), thus \( \bar{w}_{\tau_k} \geq w_k \), given our monotonicity assumption on the \( w_i \)'s. Consider the following strategy. For each \( k \geq 0 \), on the time interval \( [\tau_k, \tau_k + 1 - 1] \) the walker chooses to increase her distance from \( o \) every time she is at a vertex of degree \( 1 + w_k \) or larger; each time this happens with probability at least \( w_k / (w_k + 1) \).

We now claim that this strategy guarantees that \( A_k \) occurs almost surely, that is, that the walker, while implementing this strategy, eventually leaves the ball of radius \( d(v_k) \), whatever \( k \geq 1 \) is.

Indeed, for any \( k \geq 1 \), the event \( v_k = v_{k+1} = v_{k+2} = \ldots \) has probability zero. To see this recall (6.2) and note that at least one time out of any two consecutive times, the walker is at a vertex different from \( v_k \). This means that the walker eventually stays at a distance from the origin which is not just at least \( d(v_k) \) but also at least \( d(v_{k+i}) > d(v_k) \), with some \( i \geq 1 \).

Since the above strategy guarantees that \( A_k \) occurs a.s., by (6.5) one has that

\[
\pi_k \geq \left( 1 - \sum_{i=k}^{\infty} r_{i,a_i} \right) \cdot \prod_{i=k}^{\infty} \left( \frac{w_{i-1}}{w_{i-1} + 1} \right)^{\Delta a_i} , \quad k \geq 1 .
\]

Using that for some \( c > e \),

\[
\left( \frac{w_{i-1}}{w_{i-1} + 1} \right)^{\Delta a_i} > c^{- \frac{\Delta a_i}{w_{i-1} + 1}} , \quad i \geq 2 ,
\]

it follows that

\[
\prod_{i=k}^{\infty} \left( \frac{w_{i-1}}{w_{i-1} + 1} \right)^{\Delta a_i} \geq \prod_{i=k}^{\infty} c^{- \frac{\Delta a_i}{w_{i-1} + 1}} = c^{- \sum_{i=k}^{\infty} \frac{\Delta a_i}{w_{i-1} + 1}} .
\]
Exploiting (6.6) and (6.7) the bound \( \pi_k > 1 - \epsilon \) follows if, with some appropriately small \( \delta(\epsilon) > 0 \),
\[
\sum_{i=k}^{\infty} r_{i,a} < \delta(\epsilon) \quad \text{and} \quad \sum_{i=k}^{\infty} \frac{\Delta a_i}{w_{i-1} + 1} < \delta(\epsilon),
\]
which, in turn, are guaranteed for large enough \( k \)'s by our summability assumptions.

**STEP 2:** So far, we have shown that, assuming (6.2), for some (deterministic) integer \( k_0(\epsilon) \geq 1 \),
\[
P_{T_0,o,L}(A_k) \geq 1 - \epsilon, \quad \forall k \geq k_0(\epsilon).
\]
By the reverse Fatou inequality then,
\[
P_{T_0,o,L} \left( \limsup_{k \to \infty} A_k \right) = P_{T_0,o,L} \left( \limsup_{i \to \infty} A_{k_0+i} \right) \geq 1 - \epsilon.
\]
Since \( \epsilon > 0 \) was arbitrary, (6.3) follows, and hence we are done with showing transience under conditions 1-3.

**STEP 3:** To complete the proof of the theorem, a careful look at the first two steps reveals that if we drop (6.2), then although the argument for the walker going out to infinity breaks down, it still follows that she will be bounded away from the root a.s., hence the process is not recurrent. \( \square \)

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