Axelrod’s Model with Surface Tension

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In this work we propose a subtle change in Axelrod’s model for the dissemination of culture. The mechanism consists of excluding non-interacting neighbours from the set of neighbours out of which an agent is drawn for potential cultural interactions. Although the alteration proposed does not alter topologically the configuration space, it yields significant qualitative changes, specifically the emergence of surface tension, driving the system in some cases to metastable states. The transient behaviour is considerably richer, and cultural regions have become stable leading to the formation of different spatio-temporal structures. A new metastable “glassy” phase emerges between the globalised phase and the polarised, multicultural phase.

I. INTRODUCTION

The interest in phenomena involving information flow on networks has been increasing over the last decades. A variety of mathematical techniques have been combined to model brain activity, biochemical networks, excitable media, ecosystems and the like [1]. Naturally amongst them are the social systems, including human societies and culture. In an attempt to escape from reductionism, and inspired by the successes of statistical mechanics, many different approaches are being proposed to understand the collective behaviour that emerges from the interaction of many subparts that constitute the complex system under consideration. There is already a considerable amount of literature on social modelling, from applause synchronisation to disease spreading, opinion formation and urban spatial segregation. For a general review, refer to [2]. In this context, Axelrod proposed in 1997 [3] a discrete vector model for cultural dissemination that has been attracting scientists for its very simple description but rich behaviour.

Axelrod’s model fits in a broad class of models based on agents, where each agent has an internal state and a set of rules determine the mechanisms of interaction with other agents. Although very simple in Axelrod’s case, this framework is very powerful and its possibilities of representing multilevel structures and capturing the essence of quite different systems are vast. In a certain way, Axelrod’s model extends some of the basic models proposed in the previous decades, such as the voter model, to a discrete vector representation of the internal state of each site (or node) of the network. Within this description, agents can be compared by overlapping their vectors. Axelrod’s basic premise comes from the fact that more similar individuals tend to exchange culturally more often, usually referred to as homophily. Though one agent gets more similar to its neighbour after the interaction (local convergence), the global state of the cultural network may in some cases tend to a polarised configuration, where groups of different cultures form.

Probably one of the most important characteristics of Axelrod’s model is its out-of-equilibrium phase transition, capturing two different regimes: one leading to a globalised state and the other to a polarised one, depending on the model’s parameters. Conversely, not much attention has been given to its transient behaviour and how the spatial structures develop over time. In fact, although homophily has been the source of inspiration for Axelrod’s mechanics, during time evolution the information flows through space - cultures hop through agents quite freely - and cultural borders in general tend to break, as if people from the same culture did not tend to stick together. To account for a stronger homophily we should think of a surface tension effect, for which we propose a mechanism, that is not directly related to a memory or majority rule, but on the contrary, emerges naturally from a slight and justifiable change in the updating rules of the model.

In this work we will outline an analysis of the effect of introducing a sort of agent optimisation, in the sense that agents only use their time to interact with whom the interaction is possible. In the next section, we will shortly describe Axelrod’s original model and some important quantities to characterise the macroscopic state of the system. In the following section we introduce our slight variation in the original dynamics and the motivation for doing it. The subsequent section contains a description of the difference attained both in transient and in stationary behaviour as well as some results from numeric simulations. Finally in the last section we will discuss the resulting differences and present some of the new issues that have arisen from the introduction of surface tension.
II. AXELROD’S MODEL

In the original paper [3], Axelrod defined culture - in a simple fashion - as the set of people’s characteristics subject to other individuals’ influence. In this model, every agent $i$ is represented by a discrete vector $\sigma_i = (\sigma_i^1, \sigma_i^2, \ldots, \sigma_i^q)$ of $F$ components (or cultural features), each of which can express one of $q$ possible cultural traits ($\sigma_i^j \in \{0, 1, \ldots, q - 1\}$). Each feature denotes one cultural aspect, such as musical preferences, sports, and so forth, and the cultural traits can be thought of as different expressions of these features. The fact that the number of possible traits per feature is the same for all features is just a consideration of the model to make it simpler.

The dynamic of the model is driven by the empirical observation that similar individuals are more likely to interact than dissimilar ones (homophily). The agents are situated in the nodes of a network. In this work, we have studied the case of a square lattice with periodic boundary conditions, but other topologies have been used [4, 5]. This is a discrete stochastic and asynchronous model. In every time step one agent $i$ is randomly selected along with one of its neighbours $j$. The set of neighbours of a node $i$ (its neighbourhood) is denoted as $\nu_i$. An interaction occurs with probability $\omega_{ij}$ equal to their similarity - the similarity between two vectors $\sigma_i$ and $\sigma_j$ is calculated as their overlap normalised by the number of features $\omega_{ij} = \frac{1}{F} \sum_{j=1}^{F} \delta_{\sigma_i^j, \sigma_j^j}$, where $\delta_{\sigma_i^j, \sigma_j^j}$ is the Kronecker delta - and it consists of randomly selecting one of the features ($f$) whose trait is different between the agents and making the agent $j$ adopt $i$’s trait for that feature after the interaction, that is, $\sigma_j^f = \sigma_i^f$. It is worth noting that, in Axelrod’s model, it does not matter whether $i$ transfers one trait to $j$ or the other way around, it is symmetric with respect to that.

Thus, two agents $i$ and $j$ are able to interact if their similarity lies in the range $0 < \omega_{ij} < 1$, and we will call this link active. If $\omega_{ij} = 0$, their similarity is null and so is their probability of interacting, according to the definition. If $\omega_{ij} = 1$, there is no more cultural difference between them, and no more possible information exchange. In both cases, the link between these two agents is said to be inactive. One site (agent) is said to be active if it has at least one neighbour with whom the link is active.

Under these rules the system always orbits to an absorbing state (without any more possible interactions) which can - depending on the two parameters $F$ and $q$ - be a globalised or a polarised absorbing state. The globalised (monocultural, ordered) state is characterised by the majority of agents sharing the same culture (a giant component of one culture), whereas the polarised (multicultural, disordered) state is characterised by many local cultural clusters but no global dominant culture. On most topologies, a first order (except for the special case $F = 2$) out-of-equilibrium phase transition separates these regimes: there is a critical line in the parameter space $q_c(F)$ separating these two different phases. Several order parameters have been proposed to characterise the transition, but the most widely used is the average fraction of the dominant culture’s size, $\langle \frac{S_{\text{max}}}{N} \rangle$. As the lattice size increases, the transition gets sharper across the critical value (figure 1), typical of a first-order transition. It is worth mentioning that the (out-of-equilibrium) transition occurs going from one system to another: $q$ is part of the definition of the system, and not a collective emergent property such as the temperature. The transition happens throughout the absorbing states of these different systems as the parameters vary.

III. THE MODEL WITH SURFACE TENSION

The introduction of surface tension in models on networks is sometimes achieved by the use of a majority rule [6] or a local memory [7]. In the former case, the configuration space topology is altered - some transitions otherwise possible are no longer possible with the majority rule. In this work we propose a subtle modification of the model’s dynamics that leads to the emergence of surface tension without changing the topology of the configuration space. All the possible transitions are preserved, only the probabilities associated with these transitions change.

The main idea behind this modification is a sort of optimisation from the agent’s point of view. Suppose - in the Axelrod’s case - an agent $i$ was selected and it has 4 neighbours, one of which ($j$) has a state such that $\omega_{ij} = 0$ or $\omega_{ij} = 1$. Even though there is no possible interaction, there is a finite probability ($\frac{1}{4}$) that this neighbour is selected and the iteration will have to be discarded. People usually do not try to interact culturally with whom there is no possible interaction, and this is the only alteration
introduced in the original model. We can now formalise the rules for the model.

The agents are placed on a network with a specific topology (in our case, a 2D regular lattice with periodic boundary conditions). The concept of active sites (or agents) and active links according to the definition in the previous section is still used here. The initial conditions are randomly selected from a uniform distribution and the dynamics consist of iterating these two steps:

**Step 1:** Randomly choose one site $i$. Then choose one of its neighbours $j$ such that $0 < \omega_{ij} < 1$, in case there is (otherwise go back to step 1).

**Step 2:** With a probability equal to their similarity, there is an interaction. The interaction consists of randomly selecting an $f$ such that $\sigma^f_i \neq \sigma^f_j$ and making site $j$ adopt $\sigma^f_j$.

Note that the only difference compared to Axelrod’s original model is the constraint that the randomly selected neighbour has $0 < \omega_{ij} < 1$. This choice makes the model asymmetric with respect to the trait transfer direction. If we invert and make site $i$ adopt $\sigma^f_j$ in step 2, the opposite effect arises: the total interface length tends to increase and a physically implausible cultural mixing occurs. Suppose now a site $i$ is selected and only one of the four links connecting it to its neighbours is inactive. In this case, there are only three possible neighbours to be randomly drawn, with probability $\frac{1}{3}$ for each. After the choice, there is still a probability that no interaction occurs, but all the selected pairs may potentially exchange cultural information. This single modification is responsible for the emergence of surface tension (see figure 2).

A colour scheme was devised for the simulations in order to observe the spatiotemporal patterns. Different colours were assigned to different cultural states. Being the RGB colour space three-dimensional, it is impossible to assign similar colour to similar cultures based on distances. Thus, the colour to culture mapping does not indicate that similar colours represent similar cultures.

**IV. NUMERICAL RESULTS**

While seemingly subtle, there are some radical changes in the overall properties of the model. In Axelrod’s model, the absence of surface tension leads to instability of cultural regions, and bubbles tend to break (figure 3). In our model, cultures have a tendency to organise in space, forming well-defined cultural regions throughout the trajectory in time. This can be interpreted as cultural cohesion, or a tendency of people from the same culture to stick together: the principle of homophily in a stronger sense. As a consequence, instead of local cultural interfaces there are spatially extended interfaces between cultures, macroscopic membranes, which usually do not break. These membranes have very peculiar interacting mechanics, and a more careful investigation is yet to be made. The isolation of these membranes is possible exporting these rules to the voter model, according to [9]. Another remarkable consequence of the formation of domains is that the cultural exchange happens in the borders between regions, and the borders tend to concentrate cultural diversity (see figure 4).

![FIG. 2. Detail of a square border between two cultures, one represented in white and the other in orange. The agents lying on the straight part of the interface have the same number of active neighbours on both sides (green), thus being the probability of trait transfer the same in both directions. In the corner, however, that symmetry no longer holds (blue). There are two agents which can be drawn to transfer their traits to the corner agent, making it twice more probable for the corner to be invaded, producing two new corners and continuously “rounding” the sharp corners. This illustrates the curvature-driven surface tension of the model, not present in Axelrod’s original model.](image1)

![FIG. 3. Same initial conditions (square), the first row corresponding to the case with surface tension and the second to the Axelrod’s original model.](image2)
branes) are flexible enough, allowing bigger ones to coalesce progressively, becoming even bigger at the expense of annihilating the small ones (figure 4). For large enough values of $q$, there are some nuclei with possible culture exchange immerse in a vast disordered ocean of cultural disagreement. In this region, the transient is very similar to Axelrod’s, despite the formation of membranes within the islands of cultural exchange. The absorbing state reached is also similar to Axelrod’s in this case.

Useful to describe the macroscopic state are the densities of links, divided in three categories: (inactive) links between nodes with no feature in common, (active) links between nodes with some features - but not all - in common, and (inactive) links between nodes with the same cultural state (all features in common). The densities of these links will be called $\rho_0$, $\rho_a$ and $\rho_F$ respectively, and are given by the following expressions on a regular lattice with periodic boundary conditions and $N$ nodes.

$$\rho_0 = \frac{1}{4N} \sum_{i=1}^{N} \sum_{j \in \varepsilon_i} \delta_{\omega_{ij},0}$$

$$\rho_F = \frac{1}{4N} \sum_{i=1}^{N} \sum_{j \in \varepsilon_i} \delta_{\omega_{ij},1}$$

$$\rho_a = 1 - \rho_0 - \rho_F$$

One can see how these quantities develop during the transient until it reaches a metastable state in the figure 5.

A function we used to detect metastable configurations is the number of surviving traits, proposed in [10], which is a Lyapunov function for both models. It is calculated by counting the traits that are still circulating on the network, and has maximum value $qF$ and minimum value $F$. Observe that if the numeric value of two different cultural features is the same, it is counted twice. The fact that they have the same symbol does not mean anything since they stand for different cultural features, thus being incomparable.

$$ST = qF - \sum_{f=1}^{F} \sum_{i=0}^{q-1} [\delta_0 \sum_{\sigma_i} \delta_{\omega,\sigma_i}]$$

This function, besides being monotonically decreasing in time, may provide insights into the mechanics of the models. At each interaction, one agent adopts one of the cultural traits of its influencing neighbour and discards the previous value for that feature. In this way, there are eventual trait extinctions during the system’s trajectory in time. Once there is an extinction of a certain trait, it will never get back to the network and many configurations previously accessible become inaccessible. Kuperman has also studied two different Lyapunov functions for his model with majority rule [6], and the use of these functions to understand stability [11] or detect metastable configurations are of great importance.

What is entirely new is a metastable “glassy” phase for intermediate values of $q$. The transient regime is characterised by the non-homogeneous nucleation and growth of bubbles (figure 4), and the nucleation usually occurs at neighbour sites with orthogonal cultures (in the case of non-periodic boundary conditions, the edges and corners become nucleation regions). Therefore, the average density of nucleation regions also varies with $q$. After growing, these bubbles eventually touch each other, forming well defined domains whose borders are almost completely rigid; yet, there might still be some occasional
changes. In case of dense nucleation, the mean domain size reached is smaller and the domains become more susceptible to trait invasion from neighbour ones, provoking some extra domain concatenation and leading to ultimate irregular domain shapes.

These states are metastable in the sense that they are not absorbing states (there are still cultural exchanges taking place), but are quasi-stationary, remaining confined in certain attractor-like regions of the configuration space. In [11] a completely different notion of metastability is used, not to be mistaken with the one used here. It is important to note that what defines these regions as attractors are the transition probabilities, and not the topology: if we use these metastable configurations as the initial conditions for the Axelrod’s case, these domains break, and the consensus state is reached. Under Axelrod’s dynamics, there are only fixed point attractors, but in our case there are other kinds, characterising a different attractor structure, albeit the common topology. The absorbing states or metastable states characterising the three different phases are represented in figure 6.

![Image of absorbing states or metastable states](image)

**FIG. 6.** Absorbing states or metastable states for choices of parameters lying in the three different phases. $F = 25$ and $L = 500$ for all of them. First row: $q = 2$, $q = 10$ and $q = 20$. Second row: $q = 50$, $q = 100$ and $q = 125$.

The next step would be to characterise both transitions and find the critical lines in the parameter space. This however is not such a simple task. Using the typical order parameter $\langle S_{\text{max}} \rangle$ apparently gives us a good initial idea about the critical values but when the lattice size increases, the middle plateau (corresponding to the metastable phase) progressively lowers, tending eventually to zero. We have also tried to use the cluster entropy proposed in [12], which works pretty well for Axelrod’s original model but not for ours with surface tension. One of the difficulties of dealing with this out-of-equilibrium transition involving metastable states is deciding when to interrupt the simulation to measure the order parameter of that (truncated) configuration, or even to understand the difference between long transients or metastable states.

### V. DISCUSSION

Axelrod’s model is a simplification of a very complex problem, and it was not proposed in an attempt to describe or explain the intricate process of cultural exchange and formation. That said, the model is interesting in itself, and that probably is the great motivation for studying it. According to our view, one of its main limitations is related to the absence of surface tension during its time evolution. It is possible to observe that cultural traits flow through the agents in a way that the cultural borders tend to break, forming very different spatiotemporal patterns from what we expect due to the homophilic nature of social interactions - affine agents staying together in space. In Axelrod’s model, cultural bubbles are however unstable and usually dissolve or break, depending on the surroundings. Besides that, both models still lack a mechanism for creation of novelty, since the traits only get extinct in the course of time.

We proposed a modification in the rules that results in surface tension without changing topologically the configuration space, clearly implying a modification of the transient behaviour, making it much richer: cultural borders do not break anymore, cultural regions become connected in space, spatial patterns emerge such as bubble nucleation and surface-tension-driven polygonal cells. The new model still preserves the globalised and polarised phase of Axelrod’s model, but a new “glassy” metastable phase emerges between them, with cultural regions resembling political maps whose (active) borders concentrate all the cultural exchange and diversity. Unlike Axelrod’s model, this phase has the noteworthy characteristic of not reaching a frozen state. This novel outcome accounts for social interactions in a more realistic way, with spontaneous formation of cultural niches and, more importantly, the boundaries that separate and define them.

An important issue to be pointed out is the dependence of system’s evolution on the uniform random distribution as the initial condition. The polarised phase actually stems from this random initial condition in situations where $q$ is large enough such that it is quite improbable that any two neighbours have something in common. Much of our understanding of the model comes from this widespread choice, but some carefully chosen initial conditions lead to significantly different situations that must be understood. We have used some of these particular configurations as initial conditions to unravel the mechanics of the membranes, and they can be quite useful.

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