Basic results of fractional Orlicz-Sobolev space and applications to non-local problems

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Abstract

In this paper, we study the interplay between Orlicz-Sobolev spaces $L^M$ and $W^{1,M}$ and fractional Sobolev spaces $W^{s,p}$. More precisely, we give some qualitative properties of the new fractional Orlicz-Sobolev space $W^{s,M}$, where $s \in (0,1)$ and $M$ is an $N$–function. We also study a related non-local operator, which is a fractional version of the nonhomogeneous $M$–Laplace operator. As an application, we prove existence of weak solution for a non-local problem involving the new fractional $M$–Laplace operator.

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1 Introduction

Recently, great attention has been focused on the study of fractional and non-local operators of elliptic type, both for pure mathematical research and in view of concrete real-world applications. In most of these applications a fundamental tool to treat these type of problems is the so-called fractional order Sobolev spaces that for $0 < s < 1 \leq p < \infty$, are defined as

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{u(x) - u(y)}{|x-y|^N p+sp} \in L^p(\Omega \times \Omega) \right\},$$

where $\Omega \subset \mathbb{R}^N$ is an open set. The literature on non-local operators and on their applications is very interesting and, up to now, quite large. After the seminal papers by Caffarelli et al. [12, 13, 14], a large amount of papers were written on problems involving the fractional diffusion operator $(-\Delta)^s$ ($0 < s < 1$). We can quote [4, 15, 28, 30, 35, 36] and the references therein. We also refer to the recent monographs [15, 29] for a thorough variational approach of non-local problems.

On the other hand, for some nonhomogeneous materials (such as electrorheological fluids, sometimes referred to as “smart fluids”), the standard approach based on Lebesgue and Sobolev spaces $L^p$ and $W^{1,p}$, is not adequate. This leads to the study of variable exponent Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where $p$ is a real-valued function. Variable exponent Lebesgue spaces appeared in the literature in 1931 in the paper by Orlicz [31]. We refer the reader to [17, 18, 33, 34, 39] for more details on Sobolev space with variable exponent.

A natural question is to see what results can be recovered when the standard $p(x)$–Laplace operator is replaced by the fractional $p(x)$–Laplacian. It is worth mentioning that there are some
papers concerning related equations involving the fractional $p(x)$-Laplace operator. In fact, results for the fractional Sobolev spaces with variable exponent and fractional $p(x)$-Laplace equations are few, for example, we refer to [31, 6, 21, 38].

In the theory of PDEs, when trying to relax some conditions on the operators, such as growth conditions, the problem can not be formulated with classical Lebesgue and Sobolev spaces with variable exponents. Hence, the adequate functional spaces is the so-called Orlicz spaces. More precisely, if in the definition of the ordinary Sobolev space $W^{1,p}(\Omega)$, the role played by the Lebesgue space $L^p(\Omega)$ is assumed instead by a more general Orlicz space $L^M(\Omega)$, the resulting structure is called an Orlicz-Sobolev space and denoted $W^{1,M}(\Omega)$, where $M$ is an $N$-function admitting an integral representation $M(t) = \int_0^{|t|} m(s) \, ds$ and $m$ assumed some conditions (see section 2).

Classical Sobolev and Orlicz-Sobolev spaces play a significant role in many fields of mathematics, such as partial differential equations. For more details on the theory of Orlicz and Orlicz-Sobolev, we can cite [1, 10, 20, 22, 23, 32] and the references therein.

It is therefore a natural question to see what results can be “recovered” when the $M$-Laplace operator is replaced by the fractional $M$-Laplacian. As far as we know, the only results about the fractional Orlicz-Sobolev spaces and the fractional $M$-Laplacian are obtained in [2, 8]. In particular, the authors generalize the $M$-Laplace operator to the fractional case. They also introduce a suitable functional space to study an equation in which a fractional $M$-Laplace operator is present.

A bridge between fractional order theories and Orlicz-Sobolev settings is provided in [8], where the authors define the fractional order Orlicz-Sobolev space associated to an $N$-function $M$ and a fractional parameter $0 < s < 1$ as

$$W^{s,M}(\Omega) = \left\{ u \in L^M(\Omega) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M\left( \frac{u(x) - u(y)}{|x-y|^s} \right) \frac{dxdy}{|x-y|^N} < \infty \right\}.$$ 

The previous definition creates problems in the calculus and in the embedding results, for example, the Borel measure defined as $d\mu = \frac{dxdy}{|x-y|^N}$ is not finish in the neighbourhood of the origin, that’s why, in [2], the authors introduced another definition of the fractional Orlicz-Sobolev space, i.e,

$$W^{s,M}(\Omega) = \left\{ u \in L^M(\Omega) : \exists \lambda > 0 / \int_{\Omega} \int_{\Omega} M\left( \frac{\lambda(u(x) - u(y))}{|x-y|^sM^{-1}(|x-y|^N)} \right) dxdy < \infty \right\}.$$ 

They also define the fractional $M$-Laplacian operator as,

$$(-\Delta)_m^su(x) = 2PV \int_{\mathbb{R}^N} m\left( \frac{u(x) - u(y)}{|x-y|^sM^{-1}(|x-y|^N)} \right) \frac{u(x) - u(y)}{|u(x) - u(y)|} \frac{dy}{|x-y|^sM^{-1}(|x-y|^N)}, \quad (1.1)$$

this operator is a direct generalization of the fractional $p$-Laplacian. In [2], the main result is to show the continuous and compact embedding for these spaces. As an application, the authors has studied the existence and uniqueness of a solution for a non-local problem involving the fractional $M$-Laplacian.

Below we point out several operators that can be incorporated to (1.1) by using the following functions which satisfy the hypotheses that will be considered in this work,

- $m(t) = pt^{p-1}, \ t > 0, \text{ with } p > 1.$
• \( m(t) = pt^{p-1} + qt^{q-1}, \ t > 0, 1 < p < q, \)
• \( m(t) = 2\gamma(1 + t^2)^{\gamma-1}t, \ t > 0 \) and \( \gamma > 1. \)
• \( m(t) = \gamma \frac{(\sqrt{1 + t^2} - 1)^{\gamma-1}}{\sqrt{1 + t^2}}, \ t > 0 \) and \( \gamma \geq 1. \)
• \( m(t) = \frac{pt^{p-1}(1+t)\ln(1+t) + tp^p}{1+t}, \ t > 0. \)

The \( N \)–functions \( M \) associated to the above functions arise in several areas, for example quantum-physics, nonlinear elasticity problems, minimal surfaces theory and plasticity problems. Regarding the described applications we quote the references \([7, 16, 19, 21]\).

The main purpose of this paper is to present some further basic results both on the function spaces \( W^{s,M} (\Omega) \) and the fractional \( M \)–Laplace operator. Then, we study the existence of solutions to the non-local problem

\[
\begin{cases}
(\Delta)^{s}_mu = \lambda g(x,u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

(1.2)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( 0 < s < 1 \), \( \lambda > 0 \) is a parameter and the nonlinear term \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function that satisfy

(A) \(|g(x,t)| \leq C_0|t|^{q-1}, \ \forall x \in \Omega, \ t \in \mathbb{R}. \)

(B) \( C_1|t|^q \leq G(x,t) := \int_0^t g(x,s)ds \leq C_2|t|^q, \ \forall x \in \Omega, \ t \in \mathbb{R}, \)

where \( C_0, C_1 \) and \( C_2 \) are positive constants and \( 1 < q < p^* = \frac{Np}{N-p} \).

(Q) \( C|t|^p \leq M(t), \ \forall t \geq 0, \) where \( 1 < p < N \) and \( C \) is a positive constant.

Regarding the hypotheses (A) and (B) we point out that the following functions \( g \) and \( G \) satisfy such hypotheses:

1. \( g(x,t) = q|t|^{q-2}t \) and \( G(x,t) = |t|^q, \) where \( 2 < q < p^* \) for all \( x \in \Omega. \)

2. \( g(x,t) = q|t|^{q-2}t + (q-2)[\log(1 + t^2)]|t|^{q-4}t + \frac{1}{1+t^2}|t|^{q-2} \) and \( G(x,t) = \log(1 + t^2)|t|^{q-2}, \) where \( 4 < q < p^* \) for all \( x \in \Omega. \)

The main difficulty to consider problem (1.2) by the Variational approach is to prove that the corresponding energy functional, i.e,

\[
I_\lambda(u) = F(u) - \lambda \int_{\Omega} G(x,u)dx,
\]

defined in a suitable space where

\[
F(u) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M\left( \frac{u(x) - u(y)}{|x - y|^s M^{-1}(|x - y|^N)} \right) dx dy,
\]

belong to \( C^1 \). In fact the problem is to prove that the term \( F \) belongs to \( C^1 \). The \( C^1 \) regularity of \( F \) is obtained in Lemma \([32]\). With such regularity we prove the multiplicity result below.
**Theorem 1.1.** Suppose that \((A), (B), (Q), (S)\) and (2.4) are satisfied. Furthermore, we assume that \(q < \min(p^*, m_0)\). Then there exists \(\lambda^* > 0\) such that for any \(\lambda \in ]0, \lambda^*[\) problem (1.2) has at least two distinct, non-trivial weak solutions.

This paper is organized as follows. In Section 2, we give some definitions and fundamental properties of the spaces \(L^G(\Omega)\) and \(W^{1,G}(\Omega)\). In Section 3, we prove some basic properties of the fractional Sobolev-Orlicz space and its associated operator. Finally, in Section 4, using a direct variational method, we give an application of our abstract results.

2 Preliminaries

In this preliminary section, for the reader’s convenience, we make a brief overview on the classical Orlicz-Sobolev spaces, as well as we introduce the Fractional order Orlicz-Sobolev Spaces, studied in [2], and the associated fractional \(M\)-laplacian operator.

2.1 Orlicz and Orlicz-Sobolev spaces

We start by recalling some basic facts about Orlicz spaces.

Let \(\Omega\) be an open subset of \(\mathbb{R}^N\). Let \(M : \mathbb{R} \to \mathbb{R}\) be an \(N\)-function, i.e,

1. \(M\) is even, continuous, convex, with \(M(t) > t\) for \(t > 0\),

2. \(\frac{M(t)}{t} \to 0\) as \(t \to 0\) and \(\frac{M(t)}{t} \to +\infty\) as \(t \to +\infty\).

Equivalently, \(M\) admits the representation:

\[
M(t) = \int_0^{|t|} m(s)ds,
\]

where \(m : \mathbb{R}_+ \to \mathbb{R}_+\) is non-decreasing, right continuous, with \(m(0) = 0, m(t) > 0 \forall t > 0\) and \(m(t) \to \infty\) as \(t \to \infty\). The conjugate \(N\)-function of \(M\) is defined by

\[
\overline{M}(t) = \int_0^{|t|} \overline{m}(s)ds,
\]

where \(\overline{m} : \mathbb{R}_+ \to \mathbb{R}_+\) is given by \(\overline{m}(t) = \sup\{s : m(s) \leq t\}\). Evidently we have

\[
st \leq M(s) + \overline{M}(t),
\]

which is known as the Young inequality. Equality holds in (2.3) if and only if either \(t = m(s)\) or \(s = \overline{m}(t)\).

If \(A\) and \(B\) are two \(N\)-functions, we say that \(A\) is stronger than \(B\) if

\[
B(x) \leq A(ax), \ x \geq x_0 \geq 0,
\]

for each \(a > 0\) and \(x_0\) (depending on \(a\)), \(B \ll A\) in symbols. This is the case if and only if for every positive constant \(k\)

\[
\lim_{t \to +\infty} \frac{B(kt)}{A(t)} = 0.
\]

Throughout this paper we assume that
Due to assumption (2.4), we may define the numbers

\[ m_0 = \inf_{t > 0} \frac{tm(t)}{M(t)} \quad \text{and} \quad m^0 = \sup_{t > 0} \frac{tm(t)}{M(t)}. \]

The above relation implies that \( M \) and \( M^\ast \) satisfy the \( \triangle_2 \)-condition, i.e.

\[ M(2t) \leq KM(t) \quad \forall \ t \geq 0. \] (2.5)

Furthermore, in this paper we shall assume that the function \( M \) satisfies the following condition:

the function \( t \mapsto M(\sqrt{t}), \ t \in [0, \infty[ \) is convex, \( (S) \)

and

\( m \) is an odd, increasing homeomorphism from \( \mathbb{R} \) onto \( \mathbb{R} \), \( (2.6) \)
in this case, \( M^\ast \) became

\[ M^\ast(t) = \int_0^{[t]} m^{-1}(s)ds. \]

The Orlicz class \( K^M(\Omega) \) (resp. the Orlicz space \( L^M(\Omega) \)) is defined as the set of (equivalence classes of) real-valued measurable functions \( u \) on \( \Omega \) such that

\[ \rho(u; M) = \int_\Omega M(u(x))dx < \infty \quad (\text{resp.} \quad \int_\Omega M(\lambda u(x))dx < \infty \text{ for some } \lambda > 0). \]

\( L^M(\Omega) \) is a Banach space under the Luxemburg norm

\[ \|u\|_{(M)} = \inf \left\{ \lambda > 0 : \int_\Omega M\left(\frac{u}{\lambda}\right) \leq 1 \right\}, \] (2.7)

and \( K^M(\Omega) \) is a convex subset of \( L^M(\Omega) \).

**Proposition 2.1.** Let \( (u_n)_{n \in \mathbb{N}} \) be a sequence in \( L^M \) and \( u \in L^M \). If \( M \) satisfies the \( \triangle_2 \) condition and \( \rho(u_n; M) \to \rho(u; M) \), then \( u_n \to u \) in \( L^M \).

Next, we introduce the Orlicz-Sobolev spaces. We denote by \( W^{1,M}(\Omega) \) the Orlicz-Sobolev space defined by

\[ W^{1,M}(\Omega) := \left\{ u \in L^M(\Omega) : \frac{\partial u}{\partial x_i} \in L^M(\Omega), \ i = 1, \ldots, N \right\}. \]

This is a Banach space with respect to the norm

\[ \|u\|_{1,M} = \|u\|_{(M)} + \|\nabla u\|_{(M)}. \]
2.2 Fractional Orlicz-Sobolev spaces

**Definition 2.2.** Let $M$ be an $N$-function. For a given domain $\Omega$ in $\mathbb{R}^N$ and $0 < s < 1$, we define the fractional Orlicz-Sobolev space $W^{s,M}(\Omega)$ as follows,

$$W^{s,M}(\Omega) = \left\{ u \in L^M(\Omega) : \exists \lambda > 0 / \int_{\Omega} \int_{\Omega} M\left( \frac{\lambda(u(x) - u(y))}{|x-y|^s M^{-1}(|x-y|^N)} \right) dxdy < \infty \right\}. \quad (2.8)$$

This space is equipped with the norm,

$$\|u\|(s,M) = \|u\|(M) + [u]_{(s,M)}, \quad (2.9)$$

where $[.]_{(s,M)}$ is the Gagliardo semi-norm, defined by

$$[u]_{(s,M)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} M\left( \frac{u(x) - u(y)}{\lambda|x-y|^s M^{-1}(|x-y|^N)} \right) dxdy \leq 1 \right\}. \quad (2.10)$$

Let $W^{s,M}_0(\Omega)$ denote the closure of $C_\infty^0(\Omega)$ in the norm $\|u\|(s,M)$ defined in (2.9).

**Theorem 2.3.** [(Generalized Poincaré inequality)] Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and let $s \in ]0,1[$. Let $M$ be an $N$-function. Then there exists a positive constant $\mu$ such that,

$$\|u\|(M) \leq \mu [u]_{(s,M)}, \quad \forall u \in W^{s,M}_0(\Omega).$$

Therefore, if $\Omega$ is bounded and $M$ be an $N$-function, then $[u]_{(s,M)}$ is a norm of $W^{s,M}_0(\Omega)$ equivalent to $\|u\|(s,M)$.

Let $M$ be a given $N$-function, satisfying the following conditions:

$$\int_0^1 \frac{M^{-1}(\tau)}{\tau^{\frac{N-s}{N}}} d\tau < \infty \quad (2.11)$$

and

$$\int_1^{+\infty} \frac{M^{-1}(\tau)}{\tau^{\frac{N-s}{N}}} d\tau = \infty. \quad (2.12)$$

If (2.12) is satisfied, we define the inverse Sobolev conjugate $N$-function of $M$ as follows,

$$M_*^{-1}(t) = \int_0^t \frac{M^{-1}(-\tau)}{\tau^{\frac{N-s}{N}}} d\tau. \quad (2.13)$$

**Theorem 2.4.** Let $M$ be an $N$-function and $s \in ]0,1[$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity and bounded boundary. If (2.11) and (2.12) hold, then

$$W^{s,M}(\Omega) \hookrightarrow L^M(\Omega). \quad (2.14)$$

Moreover,

$$W^{s,M}(\Omega) \hookrightarrow L^B(\Omega) \quad (2.15)$$

is compact for all $B \ll M_*$. 

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The fractional $M$-Laplacian operator is defined as
\[
(-\Delta)^s_M u(x) = 2P.V \int_{\mathbb{R}^N} M \left( \frac{u(x) - u(y)}{|x - y|^s M^{-1}(|x - y|^N)} \right) \frac{dy}{|u(x) - u(y)| |x - y|^s M^{-1}(|x - y|^N)},
\]
where $P.V$ is the principal value.

This operator is well defined between $W^{s,M}(\mathbb{R}^N)$ and its dual space $W^{-s,\overline{M}}(\mathbb{R}^N)$. In fact, in [2], lemma 3.5] the following representation formula is provided
\[
\langle (-\Delta)^s_M u, v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M \left( \frac{u(x) - u(y)}{|x - y|^s M^{-1}(|x - y|^N)} \right) \frac{v(x) - v(y)}{|u(x) - u(y)| |x - y|^s M^{-1}(|x - y|^N)} dx \, dy,
\]
for all $v \in W^{s,M}(\mathbb{R}^N)$.

3 Basic results of $W^{s,M}(\mathbb{R}^N)$ and fractional $M$–Laplacian operator

In this section, we point out certain useful auxiliary results.

Let $E$ denote the generalized Sobolev space $W^{0,M}_0(\mathbb{R}^N)$. We define the functional $F : E \to \mathbb{R}$ by
\[
F(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M \left( \frac{u(x) - u(y)}{|x - y|^s M^{-1}(|x - y|^N)} \right) dx \, dy.
\]

**Lemma 3.1.** The following properties hold true:

(i) $[u]^{m_0}_{(s,M)} \leq F(u) \leq [u]^{m_0}_{(s,M)} \quad \forall \ u \in E, \ [u]_{(s,M)} < 1$;

(ii) $[u]^{m_0}_{(s,M)} \leq F(u) \leq [u]^{m_0}_{(s,M)} \quad \forall \ u \in E, \ [u]_{(s,M)} > 1$;

(iii) $F \left( \frac{u}{|u|_{(s,M)}} \right) \leq 1, \quad \forall \ u \in E$.

**Proof.** (i) By lemma 1 in [27], we now that
\[
\|u\|^{m_0}_{(M)} \leq \int_{\mathbb{R}^N} M(u) \, dx \leq \|u\|^{m_0}_{(L^M(\mathbb{R}^N))}, \quad \forall u \in L^M(\mathbb{R}^N), \|u\|_{(M)} < 1.
\]

It follows that
\[
\|h_u\|^{m_0}_{L^M(\mathbb{R}^{2N})} \leq F(u) \leq \|h_u\|^{m_0}_{L^M(\mathbb{R}^{2N})}, \quad \forall u \in L^M(\mathbb{R}^N), \|h_u\|_{L^M(\mathbb{R}^{2N})} < 1,
\]
where $h_u = h_u(x,y) = \frac{u(x) - u(y)}{|x - y|^s M^{-1}(|x - y|^N)}$. Having in mind that, $\|h_u\|_{L^M(\mathbb{R}^{2N})} = [u]_{(s,M)}$, we obtain
\[
[u]^{m_0}_{(s,M)} \leq F(u) \leq [u]^{m_0}_{(s,M)} \quad \forall \ u \in E, \ [u]_{(s,M)} < 1.
\]

(ii) A similar reasoning allow us to claim that
\[
[u]^{m_0}_{(s,M)} \leq F(u) \leq [u]^{m_0}_{(s,M)} \quad \forall \ u \in E, \ [u]_{(s,M)} > 1.
\]

(iii) Let $(\lambda_k)$ be a sequence such that $\lambda_k \to [u]_{(s,M)}$ as $k \to \infty$. Then, the definition of the norm, yields
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M \left( \frac{u(x) - u(y)}{\lambda_k|x - y|^s M^{-1}(|x - y|^N)} \right) dx \, dy \leq 1.
\]
Passing by limit in the above inequality and using Fatou’s Lemma, we can deduce that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M\left( \frac{1}{|u|_{(s,M)}} \frac{|u(x) - u(y)|}{|x-y|^s M^{-1}(|x-y|^N)} \right) dx dy \leq 1.
\]
This ends the proof.

**Lemma 3.2.** The functional $F$ is of class $C^1(E, \mathbb{R})$ and
\[
\langle F'(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m\left( \frac{u(x) - u(y)}{|x-y|^s M^{-1}(|x-y|^N)} \right) \frac{|u(x) - u(y)|}{|x-y|^s M^{-1}(|x-y|^N)} \frac{v(x) - v(y)}{|u(x) - u(y)|} dx dy = \langle (-\Delta)^s_m u, v \rangle.
\]

**Proof.** We denote $h_u = h_u(x,y) = \frac{u(x) - u(y)}{|x-y|^s M^{-1}(|x-y|^N)}$. We observe that for $u, v \in E$ and $t > 0$
\[
\frac{1}{t}(F(u + tv) - F(u)) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{t} \int_{|h_u|}^{h_u + |tv|} m(s) ds dx dy.
\]
Now, since $m$ is increasing, and for $0 < t < 1$ small enough, we get
\[
\left| \frac{1}{t} \int_{|h_u|}^{h_u + |tv|} m(s) ds \right| \leq m(|h_u| + |h_v|)|h_v|.
\]
Thus, the claim $m(h_u) \in L^\infty(\mathbb{R}^{2N})$. Indeed, by using (2.4) and the fact that $m^{-1}$ is increasing, we obtain that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M(m(|h_u| + |h_v|)) dx dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m(|h_u| + |h_v|) m^{-1}(s) ds dx dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m\left(m(|h_u| + |h_v|) m(|h_u| + |h_v|) dx dy \right)
\]
\[
(3.19)
\]
On the other hand, we have $h_v \in L^M(\mathbb{R}^{2N})$. Hence, due to the Hölder’s inequality for Orlicz spaces, $m(|h_u| + |h_v|)|h_v| \in L^1(\mathbb{R}^{2N})$. Thus, by the dominated convergence theorem, we infer that
\[
\langle F'(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m(h_u) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_v(x,y) dx dy
\]
\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m(h_v(x,y)) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_v(x,y) dx dy
\]
\[
= \langle (-\Delta)^s_m u, v \rangle.
\]
We note that this Gâteaux derivative is linear with respect to $v$. It remains to show that $F'$ is continuous. Indeed, if we let $(u_n) \subset E$ be a sequence such that $u_n \rightarrow u$ in $E$ and consider
\[
|\langle F'(u_n) - F'(u), v \rangle| = \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m(h_{u_n}) \frac{u_n(x) - u_n(y)}{|u_n(x) - u_n(y)|} - m(h_u) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_v dx dy \right|
\]
\[
\leq \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} m(h_{u_n}) \left( \frac{u_n(x) - u_n(y)}{|u_n(x) - u_n(y)|} - \frac{u(x) - u(y)}{|u(x) - u(y)|} \right) h_v dx dy \right|
\]
\[
+ \left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (m(h_{u_n}) - m(h_u)) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_v dx dy \right|
\]
\[
:= I_{1,n} + I_{2,n}.
\]
We denote by \( X_n = \frac{u_n(x) - u_n(y)}{|u_n(x) - u_n(y)|} - \frac{u(x) - u(y)}{|u(x) - u(y)|} \). We have
\[
|M(|X_n h_v|)| \leq |M(2|h_v|)| \in L^1(\mathbb{R}^{2N}). \tag{3.20}
\]
Now, since \( u_n \to u \) a.e in \( \mathbb{R}^N \), we can deduce that
\[
M(|X_n h_v|) \to 0 \text{ a.e in } \mathbb{R}^N. \tag{3.21}
\]
Combining (3.20) and (3.21) and applying dominated convergence theorem, we infer that
\[
\|X_n h_v\|_{(M)} \to 0, \text{ as } n \to +\infty. \tag{3.22}
\]
Taking \( v = 0 \) in (3.19) and using lemma 3.1, we can deduce that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \overline{M}(m(|h_{u_n}|)) \, dx \, dy \leq m_0 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} M(|h_{u_n}|) \, dx \, dy
\]
\[
= m_0 F(u_n) \leq \|u_n\|_{(s,M)}^{n_0} + \|u_n\|_{(s,M)}^{m_0} + 1 
\]
\[
\leq c,
\]
for some positive constant \( c \). Here we used the boundedness of \((u_n)\) in \( E \).
Combining (3.22), (3.23) and Hölder’s inequality, we get
\[
I_{1,n} \leq \|m(h_{u_n})\|_{(M)} \|X_n h_v\|_{(M)} \to 0, \text{ as } n \to +\infty.
\]
We deal now with \( I_{2,n} \). We observe that
\[
I_{2,n} \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |m(h_{u_n}) - m(h_u)| |h_v| \, dx \, dy,
\]
then, by Hölder inequality, we obtain
\[
\sup_{\|v\|_{(s,M)} \leq 1} I_{2,n} \leq \sup_{\|v\|_{(s,M)} \leq 1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |m(h_{u_n}) - m(h_u)| |h_v| \, dx \, dy
\]
\[
\leq \|m(h_{u_n}) - m(h_u)\|_{L^\infty(\mathbb{R}^{2N})}.
\]
Now, combining the fact that \( \overline{M} \) satisfies the \( \Delta_2 \) condition, proposition 2.1 (3.23) and applying dominated convergence theorem we get
\[
\|m(h_{u_n}) - m(h_u)\|_{L^\infty(\mathbb{R}^{2N})} \to 0,
\]
and therefore \( \|F'(u_n) - F'(u)\|_{E'} \to 0 \) as required. The proof of Lemma 3.2 is complete. \( \square \)

**Lemma 3.3.** The functional \( F \) is weakly lower semi-continuous.

**Proof.** By Corollary III.8 in [7], it is enough to show that \( F \) is inferior semi-continuous. For this purpose, we fix \( u \in E \) and \( \epsilon > 0 \). Since \( F \) is convex, we deduce that for any \( v \in E \) the following inequality holds
\[
F(v) \geq F(u) + (F'(u), v - u).
\]
Using Hölder inequality we have

\[
F(v) \geq F(u) - \langle F'(u), u - v \rangle \\
= F(u) - \int_\Omega \int_\Omega m(h_u) \frac{u(x) - u(y)}{|u(x) - u(y)|} h_{u-v} \, dx \, dy \\
\geq F(u) - \int_\Omega \int_\Omega m(h_u) |h_{u-v}| \, dx \, dy \\
\geq F(u) - \|m(h_u)\|_{(M)} \|h_{u-v}\|_{(M)} \\
\geq F(u) - \|m(h_u)\|_{(M)} \|u - v\|_{(s,M)} \\
= F(u) - C \|u - v\|_{(s,M)} \geq F(u) - \epsilon
\]

for all \( v \in E \) with \( \|u - v\|_{(s,M)} < \delta = \frac{\epsilon}{C} \), where \( C \) is positive constant. We conclude that \( F \) is weakly lower semi-continuous. \( \square \)

**Lemma 3.4.** Assume that the sequence \((u_n)\) converges weakly to \( u \) in \( E \) and

\[
\limsup_{n \to +\infty} (F'(u_n), u_n - u) \leq 0.
\]

Then \((u_n)\) converges strongly to \( u \) in \( E \).

**Proof.** Since \((u_n)\) converges weakly to \( u \) in \( E \) implies that \( \{u_n\}_{(s,M)} \) is a bounded sequence of real numbers. That fact and relations (i) and (ii) from lemma 3.1 imply that the sequence \((F(u_n))\) is bounded. Then, up to a subsequence, we deduce that \( F(u_n) \to c \). Furthermore, the weak lower semi-continuity of \( F \) implies

\[
F(u) \leq \liminf_{n \to \infty} F(u_n) = c.
\]

On the other hand, since \( F \) is convex, we have

\[
F(u) \geq F(u) + \langle F'(u_n), u - u_n \rangle.
\]

Therefore, combingings (3.25) and (3.26) and the hypothesis (3.24), we conclude that \( F(u) = c \).

Taking into account that \( \frac{u_n + u}{2} \) converges weakly to \( u \) in \( E \) and using again the weak lower semi-continuity of \( F \) we find

\[
c = F(u) \leq \liminf_{n \to \infty} F\left(\frac{u_n + u}{2}\right).
\]

We assume by contradiction that \((u_n)\) does not converge to \( u \) in \( E \). Then by (i) in lemma 3.1 it follows that there exist \( \epsilon > 0 \) and a subsequence \((u_{nm})\) of \((u_n)\) such that

\[
F\left(\frac{u_{nm} - u}{2}\right) \geq \epsilon \ \forall \ m \in \mathbb{N}.
\]

On the other hand, relations (2.5) and (S) enable us to apply [25, theorem 2.1] in order to obtain

\[
\frac{1}{2} F(u) + \frac{1}{2} F(u_{nm}) - F\left(\frac{u_{nm} + u}{2}\right) \geq F\left(\frac{u_{nm} - u}{2}\right) \geq \epsilon, \ \forall m \in \mathbb{N}.
\]

Letting \( m \to \infty \) in the above inequality we obtain

\[
c - \epsilon \geq \limsup_{m \to \infty} F\left(\frac{u_{nm} + u}{2}\right).
\]

and that is a contradiction with (3.27). It follows that \((u_n)\) converges strongly to \( u \) in \( E \) and lemma 3.4 is proved. \( \square \
4 Application to non-local fractional problems

The main task of this Section is to prove Theorem 1.1.

We shall work in the closed linear subspace

\[ \tilde{W}^s,M_0(\Omega) = \{ u \in W^{s,M}(\mathbb{R}^N) : u = 0 \text{ a.e in } \mathbb{R}^N \setminus \Omega \} \]
equivalently renormed by setting \( \| \cdot \| := [\cdot]_{s,M} \), which is a reflexive separable Banach space.

Remark 4.1. We note that by condition \((Q)\) we can deduce that \( \tilde{W}^s,M_0(\Omega) \) is continuously embedded in \( L^M(\Omega) \). Consequently, \( \tilde{W}^s,M_0(\Omega) \) is compactly embedded in \( L^r(\Omega) \) for any \( 1 \leq r < p^* \).

This makes the following definition well-defined.

Definition 4.2. We say that \( u \) is a weak solution to (1.2) if \( u \in \tilde{W}^s,M_0(\Omega) \) and

\[ \langle F'(u), v \rangle - \lambda \int_\Omega g(x,u)v dx = 0, \]

for every \( v \in \tilde{W}^s,M_0(\Omega) \).

For each \( \lambda > 0 \) we define the energy functional \( I_\lambda : \tilde{W}^s,M_0(\Omega) \rightarrow \mathbb{R} \) associated to (1.2) given by

\[ I_\lambda(u) = F(u) - \lambda \int_\Omega G(x,u) dx. \]

We first establish some basic properties of \( I_\lambda \).

Proposition 4.3. For each \( \lambda > 0 \) the functional \( I_\lambda > 0 \) is well-defined on \( \tilde{W}^s,M_0(\Omega) \) and \( I_\lambda \in C^1(\tilde{W}^s,M_0(\Omega),\mathbb{R}) \) with the derivative given by

\[ \langle I_\lambda'(u), v \rangle = \langle F'(u), v \rangle - \lambda \int_\Omega g(x,u)v dx, \]

for all \( u, v \in \tilde{W}^s,M_0(\Omega) \).

Proof. The proof follows from Lemma 3.2 and condition \((A)\). \( \square \)

Proposition 4.4. The functional \( I_\lambda \) is coercive.

Proof. Let \( u \in \tilde{W}^s,M_0(\Omega) \) with \( \| u \| > 1 \). By combining \((ii)\) in Lemma 3.1 and hypothesis \((B)\), we get

\[ I_\lambda(u) = F(u) - \lambda \int_\Omega G(x,u) dx \]

\[ \geq \| u \|^{m_0} - \lambda C_2 \| u \|^{q}_{L^q(\Omega)}. \]

Since \( q < m_0 \) the above inequality implies that \( I_\lambda(u) \rightarrow -\infty \) as \( \| u \| \rightarrow \infty \), that is, \( I_\lambda \) is coercive. \( \square \)

Proposition 4.5. The functional \( I_\lambda \) is weakly lower semi-continuous.
Proof. Let $u_n \subset \tilde{W}^{s,M}_0(\Omega)$ be a sequence which converges weakly to $u$ in $\tilde{W}^{s,M}_0(\Omega)$. By Lemma 2.11 we deduce that

$$F(u) \leq \liminf_{n \to +\infty} F(u_n).$$

(4.31)

On the other hand, Remark 4.1 and conditions (A) and (B) imply

$$\lim_{n \to +\infty} \int_{\Omega} G(x, u_n)dx = \int_{\Omega} G(x, u)dx. $$

(4.32)

Thus, from (4.31) and (4.32), we find

$$I_\lambda(u) \leq \liminf_{n \to +\infty} I_\lambda(u_n).$$

Therefore, $I_\lambda$ is weakly lower semi-continuous and Proposition 4.5 is verified.

From Proposition 4.4 and 4.5 and Theorem 1.2 in [37] we deduce that there exists $u_1 \in \tilde{W}^{s,M}_0(\Omega)$ a global minimizer of $I_\lambda$. The following result implies that $u_1 \neq 0$.

**Proposition 4.6.** For every $\lambda > 0$ we have $\inf_{\tilde{W}^{s,M}_0(\Omega)} I_\lambda < 0$.

**Proof.** Fix $v \in \tilde{W}^{s,M}_0(\Omega)$, $v \neq 0$ and $v \geq 0$ in $\Omega$. Using relation (i) in Lemma 3.1 and condition (B) we obtain

$$I_\lambda(tv) = F(tv) - \lambda \int_{\Omega} G(x, tv)dx$$

$$\leq t^{m_0} \|v\|^{m_0} - \lambda C_2 t^q \|v\|^q,$$

for $t$ small enough. Taking into account $q < m_0$, we infer that $I_\lambda(tv) < 0$. The proof of Proposition 4.6 is complete.

Since Proposition 4.6 holds it follows that $u_1 \in \tilde{W}^{s,M}_0(\Omega)$ is a non-trivial weak solution of problem (1.2).

**Lemma 4.7.** Assume the hypotheses of Theorem 1.1 are fulfilled. Then there exists $\lambda^* > 0$ such that for any $\lambda \in ]0, \lambda_*[$ there exist $\rho, \alpha > 0$ such that $I_\lambda(u) \geq \alpha > 0$ for any $u \in \tilde{W}^{s,M}_0(\Omega)$ with $\|u\| = \rho$.

**Proof.** In light of Remark 4.1 there exists a positive constant $c_1$ such that

$$\|u\|_{L_\theta(\Omega)} \leq c_1 \|u\|, \forall u \in \tilde{W}^{s,M}_0(\Omega).$$

We fix $\rho \in ]0, \|u_1\|[$.

**Case 1:** $\|u_1\| < 1$. Invoking (i) in Lemma 3.1 and (A), we deduce that

$$I_\lambda(u) \geq \|u\|^{m_0} - \lambda C_2 c_1^q \|u\|^q$$

$$= \rho^q (\rho^{m_0-q} - \lambda C_2 c_1^q),$$

for any $u \in \tilde{W}^{s,M}_0(\Omega)$ with $\|u\| = \rho$. Put $\lambda^* = \frac{\rho^{m_0-q}}{3C_2 c_1^q}$. Then, for any $\lambda \in ]0, \lambda_*[$, we obtain

$$I_\lambda(u) \geq \alpha > 0, \forall u \in \tilde{W}^{s,M}_0(\Omega) \text{ and } \|u\| = \rho,$$

where $\alpha = \frac{\rho^{m_0-q}}{3}$.  

**Case 2:** $1 < \|u_1\|$. It sufficient to replace $m_0$ by $m_0$ in the previous case. This ends the proof. 

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Proof of Theorem 1.1 completed. Using Lemma 4.7 and the Mountain Pass Theorem (see Theorem 2.1 in [10]) we deduce that there exists a sequence \((u_n) \subset \tilde{W}^{s,M}_0(\Omega)\) such that

\[ I_\lambda(u_n) \to c > 0 \quad \text{and} \quad I'_\lambda(u_n) \to 0, \quad (4.33) \]

where

\[ c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) \]

and

\[ \Gamma = \{ \gamma \in C([0,1], X), \gamma(0) = 0, \gamma(1) = u_1 \} . \]

By relation (4.33) and proposition 4.4 we obtain that \((u_n)\) is bounded and thus passing eventually to a subsequence, still denoted by \((u_n)\), we may assume that there exists \(u_2 \in \tilde{W}^{s,M}_0(\Omega)\) such that \(u_n\) converges weakly to \(u_2\). Hence

\[
\langle I'_\lambda(u_n) - I'_\lambda(u_2), u_n - u_2 \rangle = \langle F'(u_n) - F'(u_2), u_n - u_2 \rangle \\
- \lambda \int_\Omega [g(x,u_n) - g(x,u_2)](u_n - u_2)dx \to 0, \quad n \to +\infty,
\]

where \(F\) is defined in relation (3.18). Therefore, by combining Remark 4.1 and Lemma 3.4, we can deduce that \(u_n\) converges strongly to \(u_2\) in \(\tilde{W}^{s,M}_0(\Omega)\). It follows, in view of relation (4.33), that

\[ I_\lambda(u_2) = c > 0 \quad \text{and} \quad I'_\lambda(u_2) = 0. \]

We conclude that \(u_2\) is a critical point of \(I_\lambda\) and so it is a non trivial second solution of (1.2). Since \(I_\lambda(u_1) < 0\), we can conclude that \(u_2 \neq u_1\). The proof of Theorem 1.1 is now complete. \( \square \)

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