The controllability and observability analysis of the one-dimensional heat flow

Sutrima, Muslich, and S Wibowo
Mathematics Department of Mathematics and Natural Sciences Faculty of Sebelas Maret University, Indonesia

E-mail: sutrima@mipa.uns.ac.id, zutrima@yahoo.co.id

ABSTRACT. In this paper controllability and observability analysis are performed for the heat flow on the state-space of one dimension. The process is described by a semilinear parabolic partial differential equation. We report that the associated linear infinite-dimensional operator is a Riesz-spectral operator and generates a $C_0$-semigroup of bounded linear operators. Then we construct a sufficient and necessary condition for the approximate controllability and observability of the system. In particular, a finite number of dominant modes of the system are approximately observable when the input is measured at the heat output by an appropriate sensor.

1. Introduction

The $C_0$-semigroup of bounded linear operators is a powerful tool for study of the distributed parameter systems, see [3] or [12]. A typical framework is the following class of systems:

$$\dot{z}(t) = Az(t) + Bu(t), \quad t \geq 0$$

$$y(t) = Cz(t)$$ (1) (2)

where $z,u$ and $y$ are the system state, input and output, respectively. The operator $A$ is a densely defined differential linear operator on an infinite-dimesional Hilbert space $Z = L_2(a,b), a,b \in \mathbb{R}$, which generates a $C_0$-semigroup, and $B,C$ are bounded linear operators on $Z$. Furthermore, if $A$ is a Riesz-spectral operator, then system (1)-(2) to be a Riesz-spectral system. In this system, the operator has several interesting properties, regarding in particular controllability and observability.

The study of controllability and observability may carried out through finite-dimensional models by the use of discretization methods. However, an infinite dimensional system theory allows us to take explicitly into account the distributed nature of heat flow model. Many results regarding the controllability and observability of distributed parameter systems are available in the scientific literature (see, e.g., [2], [4], [7], [8], [11], [13], [17], and [18]).

In this paper controllability and observability analysis are performed for a one dimensional heat flow model. The dynamics of the process are described by a semilinear partial differential equation (PDE). Firstly, we govern the Riesz-spectral system (1)-(2) in manner from the PDE with suitable operators on the Hilbert space $Z$. The controllability and observability analysis are performed on the Riesz-spectral. More precisely, we mean that the controllability and observability in this study by
approximately controllable and observable. Finally, we discuss about the modal concept of the system, regarding observability.

2. Perturbed heat flow model

First let us recall how the perturbed heat equation is governed (see, e.g., [6], [14]). Consider a metal bar of length one that can be heated along its length. Suppose we use a heating element around point $x_0$ as control temperature and measure its temperature around the point $x_1$. Suppose there are no heating and cooling at either ends, then the perturbed heat equation given by:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t) - \alpha \frac{\partial z}{\partial x}(x,t) + b(x)u(t),$$  \hspace{1cm} (3)

$$\frac{\partial z}{\partial x}(0,t) = \frac{\partial z}{\partial x}(1,t) = 0$$

$$y(t) = \int_0^1 c(x)z(x,t) \, dx$$ \hspace{1cm} (4)

where $z(x,t)$ represents the temperature at position $x$ at time $t$, $u(t)$ is the input of heat along the bar, $\alpha$ is a constant of perturbation (assumed positive), $y(t)$ is an output at time $t$, and $b$ and $c$ are the “shaping functions” around the control point $x_0$ and the sensing point $x_1$, respectively, and both elements in $L_2(0,1)$.

Many processes involving a diffusion phenomenon can be modeled by linear systems on the state-space, (see e.g., [1], [5], [9] and [10]). They investigate the dynamical analysis of the system use $C_0$-semigroup theory. In particular, the system (3)-(4) is a diffusion equation, by referring to the previous research we will analyze the dynamical of the system, regarding the controllability and observability.

The system (3)-(4) can be formulated as a linear systems $\Sigma(A, B, C)$ (1)-(2) with

$$Ax := \frac{d^2 z}{dx^2} - \alpha \frac{dz}{dx},$$ \hspace{1cm} (5)

on domain

$$\mathcal{D}(A) = \left\{ z \in L_2(0,1) : \frac{dz}{dx} \text{ is absolutely continuous, } \frac{d^2 z}{dx^2} \in L_2(0,1), \frac{dz}{dx}(0) = \frac{dz}{dx}(1) = 0 \right\},$$ \hspace{1cm} (6)

and $B$ and $C$ are the control and observation operator on $L_2(0,1)$, respectively, defined as

$$Bu(t) := b(x)u(t), \quad Cz(t) := \int_0^1 c(x)z(x,t) \, dx.$$ \hspace{1cm} (7)

In order to investigate the approximate controllability and approximate observability, we need $A$ to be a Riesz-spectral operator as a generator of $C_0$-semigroup. This operator also facilitates investigations dominant modes. This existence will be proved by using some tools from functional analysis, which are recalled from the standard text books.

3. Main Results

In the following analysis, we consider the approximate controllability and approximate observability in Definition 4.1.3 and 4.1.12 of [3].

**Definition 3.1** For the state linear system $\Sigma(A, B, C)$, we define the following concepts:

1. The controllability map of $\Sigma(A, B, C)$ on $[0, \tau]$ (for some finite $\tau > 0$) is the bounded linear map $B^\tau: L_2([0, \tau]; U) \to Z$ defined by

$$B^\tau u = \int_0^\tau T(\tau - s)Bu(s) \, ds.$$
(2) $\sum(A,B,-)$ is approximately controllable on $[0,\tau]$ (for some finite $\tau > 0$) if given an arbitrary $\varepsilon > 0$ it is possible to steer from the origin to within a distance $\varepsilon$ from all points in the state space at time $\tau$, i.e., if

$$\operatorname{ran} B^\tau = Z.$$

(3) The observability map of $\sum(A,-,C)$ on $[0,\tau]$ (for some finite $\tau > 0$) is the bounded linear map $\mathcal{C}^\tau : Z \to L_2([0,\tau]; Y)$ defined by

$$\mathcal{C}^\tau z = CT(z).$$

(4) $\sum(A,-,C)$ is approximately observable on $[0,\tau]$ (for some finite $\tau > 0$) if knowledge of the output in $L_2([0,\tau]; Y)$ determines the initial state uniquely, i.e., if

$$\ker \mathcal{C}^\tau = \{0\}.$$

Our analysis consists of a use of $C_0$-semigroup theory in the Hilbert space $L_2(0,1)$ which endowed with the weighted inner product

$$z_1, z_2 \alpha = \int_0^1 e^{-\alpha x} z_1(x) \overline{z_2(x)} \, dx.$$

**Theorem 3.2** If $A_0 = -A$, then $A_0$ is self-adjoint and nonnegative on $L_2(0,1)$ with the weighted inner product

$$\langle z_1, z_2 \rangle_\alpha = \int_0^1 e^{-\alpha x} z_1(x) \overline{z_2(x)} \, dx.$$

**Proof.** Let $z, w \in \mathcal{D}(A_0)$. By using integration by part twice and a fact that $\mathcal{D}(A_0) = \mathcal{D}(A)$, we have

$$\langle A_0 z, w \rangle_\alpha = \int_0^1 \left( -\frac{d^2 z}{dx^2} + \alpha \frac{dz}{dx} \right) \overline{w(x)} e^{-\alpha x} \, dx$$

$$= \left[ e^{-\alpha x} \frac{dz}{dx} \right]_0^1 + \int_0^1 \frac{dz}{dx} \frac{d\bar{w}}{dx} e^{-\alpha x} \, dx$$

$$= \int_0^1 \frac{dz}{dx} \frac{d\bar{w}}{dx} e^{-\alpha x} \, dx.$$
Therefore, \( \langle Az, z \rangle_a \leq 0 \) for all \( z \in \mathcal{D}(A) \). Thus, \( A \) generates a \( C_0 \)-semigroup \( T(t) \) on \( L_2(0,1) \) with the weighted inner product, by Proposition 3.3.5 and Theorem 3.8.4 of [15].

On the following we need to find the eigenvalues of \( A \). Given any \( \lambda \in \mathbb{C} \) such that \( Az = \lambda z \). By the definition of \( A \) we have

\[
\frac{d^2z}{dx^2} - a \frac{dz}{dx} = \lambda z(x), \quad z \in \mathcal{D}(A)
\]

By solving and evaluating at the boundary value we get eigenvalues of \( A \) are

\[
\lambda_n = -\left(\frac{\alpha^2}{4} + n^2\pi^2\right), \quad n \in \mathbb{N}
\]

and the corresponding eigenfunctions form the orthonormal basis \( \{\phi_n\} \) on the \( L_2(0,1) \) with the weighted inner product, where \( \phi_n(x) = \sqrt{2}e^{\alpha x}[2n\pi \cos(n\pi x) - \alpha \sin(n\pi x)], n \in \mathbb{N} \).

**Theorem 3.4** The operator \( A \) is a Riesz-spectral operator on \( L_2(0,1) \) with the weighted inner product.

**Proof.** From the results before we conclude that the eigenvalues are simple. Consequently, the set \( \{\lambda_n, n \in \mathbb{N}\} \) is totally disconnected. Because \( \{\phi_n, n \in \mathbb{N}\} \) forms orthonormal basis, then this basis is a Riesz basis. The operator \( A \) is closed, by self-adjointness. Thus, \( A \) is Riesz-spectral operator on \( L_2(0,1) \) with the weighted inner product. Furthermore, by Theorem 2.3.5 of [3] \( A \) has representation

\[
Az = \sum_{n=1}^{\infty} \left(-\frac{\alpha^2}{4} + n^2\pi^2\right) \langle z, \phi_n \rangle_a \phi_n
\]

The last theorem shows us that \( \Sigma(A,B,C) \) where \( A \) is the linear operator defined by (5) on its domain \( \mathcal{D}(A) \) given by (6), and \( B,C \) are the operator given by (7) is a Riesz-spectral system. Therefore, we have a lot of convenience to investigate the controllability and observability of the system.

**Theorem 3.5** Consider the state linear system \( \Sigma(A,B,C) \), where \( A \) is the linear operator defined by (5) on its domain \( \mathcal{D}(A) \) given by (6), and \( B,C \) are the operator on \( L_2(0,1) \) with the weighted inner product given by (7). Then \( \Sigma(A,B,\cdot) \) is approximately controllable if and only if

\[
\langle b, \phi_n \rangle_{a} \neq 0, \quad \text{for all } n \in \mathbb{N}.
\]

and \( \Sigma(A,\cdot,C) \) is approximately observable if and only if

\[
\langle c, \phi_n \rangle_{a} \neq 0, \quad \text{for all } n \in \mathbb{N}.
\]

**Proof.** The operator \( A \) is a Riesz-spectral operator on \( L_2(0,1) \) with the weighted inner product. We know that \( A \) has eigenfunctions \( \phi_n(x) = \sqrt{2}e^{\alpha x}[2n\pi \cos(n\pi x) - \alpha \sin(n\pi x)], n \in \mathbb{N} \). So Theorem 4.2.3 of [3] shows that \( \Sigma(A,B,\cdot) \) is approximately controllable if and only if

\[
\langle b, \phi_n \rangle_{a} \neq 0, \quad \text{for all } n \in \mathbb{N}
\]

and \( \Sigma(A,\cdot,C) \) is approximately observable if and only if

\[
\langle c, \phi_n \rangle_{a} \neq 0, \quad \text{for all } n \in \mathbb{N}.
\]

Suppose we get \( b \) and \( c \) as the “shaping functions” around the control point \( x_0 \) and the sensing point \( x_1 \), respectively

\[
b(x) = \frac{1}{2\varepsilon} 1_{(x_0-\varepsilon,x_0+\varepsilon)}(x),
\]

\[
c(x) = \frac{1}{2\nu} 1_{(x_1-\nu,x_1+\nu)}(x),
\]

where \( 1_{(a,b)}(x) = \begin{cases} 1 & \text{for } \alpha < x < \beta \\ 0 & \text{elsewhere} \end{cases} \). Then (8) and (9) are equivalent to
\[ \tan(n\pi x_0) \tan(n\pi \epsilon) \neq \frac{2n\pi}{\alpha}, \quad \text{for all } n \in \mathbb{N} \]

and

\[ \tan(n\pi x_1) \tan(n\pi \nu) \neq \frac{2n\pi}{\alpha}, \quad \text{for all } n \in \mathbb{N}, \]

respectively.

We see that if we try to control at points for which \( \tan(n\pi x_0) = \frac{2n\pi}{\alpha} \) for some \( n \in \mathbb{N} \), we lose approximate controllability. Notice that these points are the zeros of the eigenfunctions of \( A \) and sometimes called as *nodes* of the systems.

Theorem 3.5 is a powerful tool for the controllability and observability analysis of any perturbed heat dynamics with its boundary conditions. The theorem also justifies the concept of modal controllable and modal observable. The system \( \Sigma(A, B, -) \) that satisfies condition (8) is called modal controllable, as it is equivalent to being able to control each eigenmode. Analogous, the system \( \Sigma(A, -, C) \) satisfying condition (9) is called modal observable.

In the rest works we focus on modal observable. The framework of the works follows [4]. In Riesz-spectral system, modal observability is equivalent to approximate observability. By this reason [16] provide the following definition.

**Definition 3.6** Let \( A \) be a Riesz spectral operator. The \( n \)th mode of \( A \) is \((\lambda_n, \phi_n)\), where \( \lambda_n \) is the \( n \)th eigenvalue of \( A \), and \( \phi_n \) its corresponding eigenfunction. Moreover, let \( C: L_2(0,1) \to \mathbb{R} \) be a bounded linear observation operator. Then the \( n \)th mode of \( A \) is said to be \( C \)-observable whenever the condition (9) holds.

Now we locate the measurement \( y(t) \) at the heat output. Suppose observation occurs at around end point, so we can choose \( c \) in (10) to be

\[ c(x) = \frac{1}{v} e^{-\alpha x} \phi_n(x). \]

Therefore, we have a bounded linear observation operator \( C \) in (7) at the point measurement as:

\[ y(t) = (C_v z)(t) = \frac{1}{v} \int_{1-v}^{1} z(x, t) \, dx \quad (11) \]

where \( v > 0 \) is very small.

According to Definition 3.6 and Theorem 3.5, the \( n \)th mode \((\lambda_n, \phi_n)\) of the system \( \Sigma(A, -, C_v) \) is observable if and only if the following condition holds:

\[ \frac{1}{v} \int_{1-v}^{1} e^{-\alpha x} \phi_n(x) \, dx \neq 0 \quad \text{for all } n \in \mathbb{N}. \]

Since every \( \phi_n \) is continuous, and \( e^{-\alpha x} \) is positive on \([1 - v, 1]\), we have the following results.

**Lemma 3.7** Consider the system \( \Sigma(A, -, C_v) \) where \( A \) is defined by (5) on its domain (6), and \( C_v \) is given by (11). Then its \( n \)th mode \((\lambda_n, \phi_n)\) is \( C_v \)-observable if for all \( x \in [1 - v, 1] \) we have

\[ \phi_n(x) \neq 0. \]

The result states that the observability analysis can be investigated via the nodes of the system or zeros of the eigenfunctions.

**Lemma 3.8** Let \( \{\phi_n\} \) be the Riesz basis of eigenfunctions of \( A \), which is given by (5) and defined on \( \mathcal{D}(A) \) (6). Then

1. for every \( n \in \mathbb{N} \), \( \phi_n \) has finite number of zeros in \([0,1]\).
2. for all \( n \in \mathbb{N} \), there exists \( v > 0 \) such that

\[ \phi_n(x) \neq 0, \quad \text{for all } x \in [1 - v, 1]. \]
**Proof.** (1) Since \(-\phi_n\) is a solution of the Sturm-Liouville problem, then by the statement of theorems in the Section 5.3.2 of [6], \(-\phi_n\) has exactly \(n - 1\) zeros in \((0,1)\). Hence, \(\phi_n\) has finite number of zeros in \([0,1]\).

(2) It is clear that \(\phi_n(1) \neq 0\) for all \(n \in \mathbb{N}\). Since \(\phi_n\) is continuous, we can find \(\nu > 0\) such that for \(x \in [1 - \nu, 1]\) holds
\[
0 < \phi_n(x) < 2\phi_n(1), \text{ for } n \text{ even}
\]
and
\[
2\phi_n(1) < \phi_n(x) < 0, \text{ for } n \text{ odd}.
\]
These conclude that for any \(n \in \mathbb{N}\) holds \(\phi_n(x) \neq 0\), for all \(x \in [1 - \nu, 1]\).

The result states that \(\nu\) decreases as \(n\) gets larger. In other words, the largest zero of \(\phi_n\) is closer to 1 as \(n\) gets larger.

The dominant mode has an important role in the diffusion system. The dominant mode is the mode which explains the largest part of a system's response. The mode can also be defined as the mode at which the spectral response contains the maximal power. The following theorem provide the dominant modes in observability term.

**Theorem 3.9** For all \(N \in \mathbb{N}\), there exists \(\nu > 0\) (in equation (11)) such that the modes \((\lambda_n, \phi_n)\) of \(A\) are \(C_\nu\)-observable for any \(n \leq N\).

**Proof.** In fact that \(\phi_n\) has exactly \(n - 1\) zeros in \([0,1]\). This theorem follows from Lemma 3.8.

We see from this theorem that the lower modes are dominant in the system. In generally, it holds in some parabolic PDE. We note here that Theorem 3.9 does not recommend that any mode above \(N\) is unobservable.

### 4. Conclusions

In this paper we have discussed the controllability and observability of a perturbed heat flow of the state-space of one dimensional. The problem is modeled by PDE, the analysis based on \(C_0\)-semigroup theory. It is reported that the associated linear infinite-dimensional operator \(A\) is a Riesz-spectral operator and generates a \(C_0\)-semigroup of bounded linear operators. Then it is shown that the sufficient and necessary condition for the approximately controllability and observability of the system. In particular, a finite number of dominant modes of the system are approximately observable when the input is measured at the heat output by an appropriate sensor.

### Acknowledgment

The authors would like to thank the Institute for Research and Community Services of Sebelas Maret University for funding to this research in the academic year of 2015.

### References

[1] Achhab M E, Laabissi M, Wilkin J and Dochain D 2000 Analysis of nonlinear dynamical model of an axial dispersion nonisothermal reactor Proc 14th Int. Symp. Math. Theory of Networks and Systems (MTNS 2000), CD-ROM, paper B82, 7p

[2] Cindea N and Tucsnak M 2010 Internal Exact Observability of a perturbed Euler-Bernoulli equation Annals of the Acad. of Romanian Sci. Series on Math. and its Appl. 2 205 – 221

[3] Curtain R F and Zwart H J 1995 *Introduction to Infinite-Dimensional Linear Systems Theory* (New York: Springer)
[4] Delattre C, Dochain D and Wilkin J 2004 Observability analysis of nonlinear tubular bioreactor: a case study J. Pro. Cont. 13 661 – 669
[5] Delattre C, Dochain D and Wilkin J 2003 Sturm-Liouville systems are Riesz-spectral systems Int. J. Appl. Math. Comput. Sci. 13 481 – 484
[6] Haberman R 1987 Elementary Applied partial Differential Equations: with Fourier Series and Boundary Value Problems (New Jersey: Prentice-Hall)
[7] Jacob B and Zwart H 2001 Exact observability of diagonal systems with a one-dimensional output operator Int. J. Appl. Math. Comput. Sci. 11 1277 – 1283
[8] Jacob B and Schnaubelt R 2007 Observability of polynomially stable systems Syst. Contr. Lett. 56 277 – 284
[9] Laabissi M, Wilkin J, Dochain D and Achhab M E 2005 Dynamical analysis of a tubular biochemical reactor infinite-dimensional nonlinear model Proc. Of the 44th IEEE Conference on Decision and Control, and the European Control Conference (Sevilla, Spain, 12-15 December 2014) pp 5965–5970
[10] Laabissi M, Achhab M E, Wilkin J and Dochain D 2001 Trajectory analysis of nonisothermal tubular Reactor Nonlinear Models Syst. Contr. Lett. 42 169 – 184
[11] Miller M 2005 Controllability cost of conservative systems: resolvent condition and transmutation Journal of Functional Analysis 218 425 – 444
[12] Pazy A 1983 Semigroups of Linear Operators and Applications to Partial Differential Equations (New York: Springer)
[13] Rabah R and Skylar G 2014 Observability and controllability for linear neutral type systems 21st International Symposium on Mathematical Theory of Networks and Systems (Groningen, The Netherlands, 7-11 July 2014) pp 223–229
[14] Renardy M and Rogers R C 2004 An Introduction to Partial Differential Equations (New York: Springer)
[15] Tucsnak M and Weiss G 2009 Observation and Control for Operator Semigroups (Berlin: Birkhauser)
[16] Winkin J, Dochain D and Ligaras Ph 2000 Dynamical analysis of distributed parameter tubular reactors Automatica 36 349 – 361
[17] Zhao X and Weiss G 2011 Controllability and observability of a well-posed system coupled with a finite-dimensional system IEEE Transaction on Automatic Control 56 1 12p
[18] Zuazua E 2005 Controllability and Observability of Partial Differential Equations: Some results and open problems Preprint nd.edu/~mtns/papers/12197