Hall equilibrium of thin Keplerian discs embedded in mixed poloidal and toroidal magnetic fields

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ABSTRACT

Axisymmetric steady-state weakly ionized Hall–magnetohydrodynamic (MHD) Keplerian thin discs are investigated by using asymptotic expansions in the small disc aspect ratio $\epsilon$. The model incorporates the azimuthal and poloidal components of the magnetic fields in the leading order in $\epsilon$. The disc structure is described by an appropriate Grad–Shafranov equation for the poloidal flux function $\psi$ that involves two arbitrary functions of $\psi$ for the toroidal and poloidal currents. The flux function is symmetric about the mid-plane and satisfies certain boundary conditions at the near-horizontal disc edges. The boundary conditions model the combined effect of the primordial as well as the dipole-like magnetic fields. An analytical solution for the Hall equilibrium is achieved by further expanding the relevant equations in an additional small parameter $\delta$ that is inversely proportional to the Hall parameter. It is thus found that the Hall equilibrium discs fall into two types: Keplerian discs with (i) small ($R_d \sim \delta^0$) and (ii) large ($R_d \gtrsim \delta^{-k}$, $k > 0$) radius of the disc. The numerical examples that are presented demonstrate the richness and great variety of magnetic and density configurations that may be achieved under the Hall–MHD equilibrium.

Key words: plasmas – planetary systems: protoplanetary discs.

1 INTRODUCTION

In the present study, the Hall equilibrium of thin Keplerian discs embedded in 3D axially symmetric magnetic fields is investigated. Protoplanetary discs have been the subject of numerous investigations in search of possible instabilities that may play a significant role in such yet not totally understood phenomena as the outwards transfer of angular momentum and planet formation. However, in most astrophysics-related hydromagnetic stability studies that include accretion, jet and wind effects, the models for the underlying equilibrium states are based on the local approximation or a ‘cylindrical disc’ that is uniform in the axial direction. A more realistic model of thin discs is adopted in Regev (1983), Kluzniak & Kita (2000) and Umurhan et al. (2006) for equilibrium non-magnetized rotating discs [see also Ogilvie 1997 for Keplerian discs of ideal magnetohydrodynamic (MHD) plasmas], and is based on an asymptotic approach in the small aspect ratio of the disc. Traditionally, a big variety of geometrical configurations of axially symmetric steady-state equilibria are described through a few arbitrary functions of the poloidal magnetic flux. The latter satisfies the Grad–Shafranov (GS) type equations within both one-fluid and multifluid models of the plasmas (e.g. McClements & Thyagaraja 2001; Thyagaraja & McClements 2006; see also the earlier study by Lovelace et al. 1986, where in particular thin rotating discs have been discussed, and references therein). Such solutions provide a base for their stability studies.

Although the magnetic field configuration in real discs is largely unknown and, as observations and numerical simulations indicate, toroidal magnetic component may be of the same order of magnitude, or sometimes dominate the poloidal field due to the differential rotation of the disc (see e.g. Terquem & Papaloizou 1996; Papaloizou & Terquem 1997; Hawley & Krolik 2002; Proga 2003). As shown in the present study, there are three kinds of possible equilibrium in thin rotating discs, which cannot be reduced from one to the other: (i) pure poloidal, (ii) pure toroidal and (iii) mixed toroidal and poloidal magnetic field equilibria. The first case has been considered within the ideal MHD model by Lovelace et al. (1986) and Ogilvie (1997). The second case, i.e. the pure toroidal magnetic field, has been investigated within the Hall MHD model by Shtemler, Mond & Liverts (2007). The present study is aimed to describe numerous axially symmetric equilibria in 3D...
axially symmetric magnetic fields that contain both toroidal and poloidal components, i.e. case (iii) within the Hall MHD model in thin disc approximation.

While most of the recent research has focused on the MHD description of Magneto-rotational instability (MRI), the importance of the Hall electric field to such astrophysical objects as protoplanetary discs has been pointed out by Wardle (1999). Since then, such works as Balbus & Terquem (2001), Salmeron & Wardle (2003, 2005), Desch (2004), Urpin & Rüdiger (2005), Rüdiger & Kitchatinov (2005) and Pandey & Wardle (2008) have shed light on the role of the Hall effect in the modification of the MRIs. However, in addition to modifying the MRIs, the Hall electromotive force gives rise to a new family of instabilities in a density-stratified environment. That new family of instabilities is characterized by the Hall parameter $\pi_H$ of the order of unity ($\pi_H$ is the ratio of the Hall drift velocity $V_{H0}$ to the characteristic velocity, where $V_{H0} = l_i^2 \Omega_i / H_i$. $l_i$ and $\Omega_i$ are the inertial length and Larmor frequency of the ions, respectively, and $H_i$ is the density inhomogeneity length (see Huba 1991). As shown by Liverts & Mond (2004) and Kolberg, Liverts & Mond (2005), the Hall electric field combined with radial density stratification gives rise to two modes of wave propagation. The first one is the stable fast magnetic penetration mode, while the second is a slow quasi-electrostatic mode that may become unstable for a short enough density inhomogeneity length $H_i$. The latter is termed as the Hall instability. Density stratifications also play a crucial role in the recently discussed starto-rotational instability. That instability is excited by the combined effect of vertical and radial density stratification.

The aim of the present work is, therefore, to describe the various Hall equilibrium configurations of magnetic field and density in thin magnetized Keplerian discs as a first step towards a further study of their various density stratification-dependent instabilities. The following simplifying assumptions are adopted throughout the present work: the plasma is treated in a 3D axially symmetric magnetic field within the Hall MHD model. The problem inside the disc is treated within the thin-disc approximation, while the effects of the outer region are modelled by the boundary condition for the poloidal magnetic flux. Influence of the accretion, jet and wind effects is neglected in the thin-disc approximation (Ogilvie 1997). A near-Keplerian region of the rotating disc is considered that starts at some radial distance away from a central body. The influence of the latter on the Keplerian portion of the disc is modelled by a dipole-like contribution (singular at the disc axis) to the boundary condition for the poloidal magnetic flux at the near-horizontal disc edges. In addition, axial electrical currents that are localized in the central non-Keplerian regions of the disc are taken into account through the resulting toroidal magnetic field components within the rotating disc. Influence of the interstellar (primordial) magnetic field is modelled by specifying the contribution of the latter on the near-horizontal disc edges.

This paper is organized as follows. The dimensional governing equations, the normalization procedure and the resulting non-dimensional system are presented in the next section. Section 3 contains an asymptotic analysis of the Hall equilibrium in the small aspect ratio as well as the derivation of the appropriate GS equation for the flux function and the corresponding boundary conditions at the disc edge. In Section 4, an analytic model is developed for the Hall equilibrium with the aid of asymptotic expansions in a newly defined small parameter inversely proportional to the Hall parameter. Numerical examples for different regimes of the equilibrium are presented in Section 5. Summary and discussion are in Section 6.

2 THE PHYSICAL MODEL FOR THIN KEPLERIAN DISCS

A rotating thin disc is considered that is under the influence of a general 3D axially symmetric magnetic field, as well as the gravitational field due to a massive central body. Electron inertia and pressure, displacement current, viscosity and radiation effects are neglected in the electrons momentum equation. Consequently, the latter is reduced to the generalized Ohm’s law that takes into account the Hall effect. In addition, the plasma is assumed to be quasi-neutral.

2.1 The basic equations

Under the assumptions mentioned above, the physical model of the rotating magnetized disc is given by

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}) = 0, \tag{1}$$

$$m_i n \frac{D \mathbf{V}}{Dt} = -\nabla P + \frac{1}{c} \mathbf{j} \times \mathbf{B} - m_i n \nabla \Phi, \tag{2}$$

$$\frac{D}{Dt} (P n^{-\gamma}) = 0, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + c \nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\mathbf{E} = -\frac{1}{c} \mathbf{V} \times \mathbf{B} + \frac{1}{c} \frac{\mathbf{j} \times \mathbf{B}}{n_e}, \quad \mathbf{j} = \frac{e}{4\pi} \nabla \times \mathbf{B}. \tag{5}$$

Here, the standard cylindrical coordinates \{r, \theta, z\} are adopted throughout the paper with the associated unit basic vectors \{\mathbf{i}_r, \mathbf{i}_\theta, \mathbf{i}_z\}; \mathbf{V}$ is the plasma velocity; $t$ is time; $D/Dt = \partial/\partial t + (\mathbf{V} \cdot \nabla)$ is the material derivative; $\Phi(R) = -GM/R$ is the gravitational potential of the
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central object: \( R^2 = r^2 + z^2 \); \( G \) is the gravitational constant; \( M_\odot \) is the total mass of the central object. The electric field \( E \) is described by the generalized Ohm's law, which is derived from the momentum equation for the electrons fluid by neglecting the electrons inertia and pressure; \( B \) is the magnetic field; \( j \) is the current density; \( \gamma \) is the polytropic coefficient (\( \gamma = 5/3 \) in the adiabatic case). \( P = P_e + P_i + P_n \) is the total plasma pressure; \( c \) is speed of light; \( P_i \) and \( m_i \) are the species pressures and masses (\( l = e, i, n \)), subscripts \( e \) and \( i \) denote the electrons, ions and neutrals, respectively; \( e \) is the electron charge; \( m_i = Zm_p \), \( m_p \) is the proton mass (\( Z = 1 \) for simplicity). Since the plasma is assumed to be quasi-neutral and partially ionized with small ionization degree and strongly coupled ions and neutrals
\[
n_e \approx n_i \approx \alpha n_n, n \approx n_n, \alpha = n_e/(n_e + n_n) \ll 1, V_i \sim V_n. \tag{6}
\]

In real protoplanetary discs, the electron density is determined by ionization versus recombination, and the fractional ionization may vary significantly in space. In such cases, rate equations that describe the ionization degree should be employed. However, in the present study the ionization fraction is assumed to be constant. This simplification is made to avoid the widely uncertain physics of ionization and recombination. Nevertheless, such approximation allows one to roughly estimate the real characteristics of the equilibrium system (see a discussion in section 5 in Shemler et al. 2007).

2.2 Scaling procedure

The physical variables are now transformed into non-dimensional variables:
\[
f_{nd} = f/f_s, \tag{7}
\]
where \( f \) and \( f_{nd} \) stand for any of the physical dimensional and non-dimensional variables, while the characteristic scales \( f_s \) are defined as follows:
\[
V_s = \Omega_s r_s, \quad t_s = \frac{1}{\Omega_s^2}, \quad \Phi_s = V_s^2, \quad m_s = m_i, \quad n_s = n_n, \quad P_s = K(m_s, n_s)^\gamma, \quad j_s = \frac{c B_s}{4\pi r_s}, \quad E_s = \frac{V_s B_s}{c}. \tag{8}
\]
Here, \( \Omega_s = (GM_\odot/r_s^3)^{1/2} \) is the Keplerian angular velocity of the fluid at the characteristic radius \( r_s \) that belongs to the Keplerian portion of the disc; \( K \) is the dimensionless constant in the steady-state polytropic law \( P = K n^\gamma \). The characteristic values of the electric current and field, \( j_s \) and \( E_s \), have been chosen consistently with Maxwell equations. The characteristic dimensional magnetic field \( B_s \) is specified below. Note that a preferred direction is tacitly defined here, namely, the positive direction of the \( z \)-axis is chosen according to positive Keplerian rotation.

The resulting dimensionless system (omitting the subscript 'nd' from the dimensionless variables) is given by:
\[
\frac{\partial n}{\partial t} + \nabla \cdot (nV) = 0, \tag{9}
\]
\[
\frac{n DV}{Dt} = -\frac{1}{M_s} \nabla P + \frac{1}{\beta M_s^2} \frac{j \times B}{n} - n \nabla \Phi, \quad \Phi(r, z) = -\frac{1}{(r^2 + z^2)^{1/2}}, \tag{10}
\]
\[
\frac{D(P n^{-\gamma})}{Dt} = 0, \tag{11}
\]
\[
\frac{\partial B}{\partial t} + \nabla \times E = 0, \quad \nabla \cdot B = 0, \tag{12}
\]
\[
E = -V \times B + \Pi_H \frac{j \times B}{n}, \quad j = \nabla \times B. \tag{13}
\]
Here, \( M_s \) and \( \beta \) are the Mach number and plasma beta, and \( \Pi_H \) is the Hall coefficient in the generalized Ohm's law (13):
\[
M_s = \frac{V_s}{c_s}, \quad \beta = 4\pi e m_i B_s^2 / \Pi_e \Omega_s, \quad \Pi_e = \frac{\Omega_s}{\Omega_e} \left( \frac{l_i}{r_s} \right)^2 \equiv \frac{B_e c}{4\pi e n_e \Omega_e r_s^2}, \tag{14}
\]
\[
c_s = \sqrt{P_s/(m_s n_s)} \] is the characteristic sound velocity, \( l_i = c/\omega_{pi} \) and \( \Omega_e = eB_e/(m_e c) \) are the inertial length and the Larmor frequency of ions, respectively, \( \omega_{pi} = \sqrt{4\pi e^2 \alpha n_e / m_e} \) is the plasma frequency of the ions. The increasing importance of the Hall term for weakly ionized discs is apparent as \( \Pi_H \) is inversely proportional to the ionization degree \( \alpha \).

A common property of thin Keplerian discs is their highly compressible motion with large Mach numbers \( M_s \). Furthermore, the characteristic effective semithickness \( H_n = H(r_n) \) of the disc \([H = H(r) \text{ is the local disc height}] \) is such that the disc aspect ratio \( \varepsilon \) equals the inverse Mach number:
\[
1/M_s = \varepsilon = H_n/r_n < 1. \tag{15}
\]
The smallness of $\epsilon$ means that dimensional axial coordinate $|z| \leq H$ is also small, i.e. $z/r_s \sim \epsilon$, and the following rescaled values of the order of unity in $\epsilon$ may be introduced in order to further apply the asymptotic expansions in $\epsilon$ (Shtemler et al. 2007):

$$\xi = \frac{z}{\epsilon}, \quad h = \frac{H}{\epsilon}, \quad \pi_H = \frac{\Pi_H}{\epsilon},$$

(16)

where, as was mentioned in the Introduction, the rescaled Hall parameter $\pi_H$ is determined in terms of the characteristic disc thickness $H_s = \epsilon r_s$ as the density inhomogeneity length.

### 2.3 Physical conditions in protoplanetary discs

Before turning to the detailed description of the Hall equilibrium of rotating Keplerian discs, it is instructive to estimate some of the parameters that have been introduced in the previous section. Typical protoplanetary discs extend up to the order of 100 au and consist of molecular gas with characteristic ion masses between 30$m_p$ to 40$m_p$. The temperature within a radius of about 0.1 au may exceed 10$^4$ K were as further away from the central star the temperature decreases and may reach values of about 10 K in the outer regions of the disc. As a result, while in the inner regions of the disc thermal collisions provide the dominant ionization mechanism, in regions beyond about 0.1 au the only sources of ionization are non-thermal, like cosmic rays and the decay of radioactive elements (Umebayashi & Nakano 1988; Gammie 1996; Igea & Glassgold 1999). Taking that into account, the frequently used model is employed, in which the radial temperature profile is given by

$$T(r) = 280 \left(\frac{r}{1 \text{ au}}\right)^{-1/2} \text{ K},$$

(17)

while the column mass density is given by the minimum mass model as

$$\Sigma(r) = 1700 \left(\frac{r}{1 \text{ au}}\right)^{-3/2} \text{ g cm}^{-2}.$$  

(18)

The thickness of the disc as well as the neutral number density radial profiles may be calculated from the two functions above. The ionization fraction is given by

$$\alpha = \frac{\xi}{n \beta_s},$$

(19)

where $\xi$, the ionization rate due to the non-thermal processes, is given by

$$\xi = 10^{-17} \exp(-\Sigma/192) + 7 \times 10^{-23} \text{ s}^{-1},$$

(20)

and $\beta_s$ is the dissociative recombination rate given by

$$\beta_s = 1.1 \times 10^{-7} \left(\frac{T}{300 \text{K}}\right)^{-1} \text{ cm}^3 \text{ s}^{-1}.$$  

(21)

A detailed and critical study of the model as well as some other suggestions may be found in Hayashi, Nakazawa & Nakagawa (1985), Gammie (1996), Fromang et al. (2002) and Sano & Stone (2002). Here however, equations (17)–(21) are employed in order to obtain a rough estimate for the physical conditions in the disc. To do that, attention is focused on two representative radii, $r_1 = 1$ au and $r_2 = 20$ au. A characteristic magnetic field of 0.1$G$ is assumed while the mass of the central star has been assumed to be one solar mass. The results are summarized in Table 1. The increase in the rescaled Hall parameter is due to the decrease in the neutrals number density that compensates for the increase in the ionization fraction and the angular momentum per unit mass. In particular, it is evident that the entire disc resides within the Hall regime.

### 3 HALL EQUILIBRIUM IN THIN DISCS IN 3D AXISYMMETRICAL MAGNETIC FIELDS

#### 3.1 General relations

In magnetized discs, the solution of the steady-state equilibrium problem ($\partial / \partial t \equiv 0$) may be obtained within the Keplerian portion of a disc by asymptotic expansions in small $\epsilon$ with the aid of equations (9)–(16) (similar to Regev 1983; Ogilvie 1997; Kluzniak & Kita 2000; Umurhan et al. 2006; Shtemler et al. 2007). This yields for the gravitational potential in the Keplerian portion of the disc:

$$\Phi(r, \zeta) = -\frac{1}{r} + \frac{\epsilon^2 \zeta^2}{2r^3} + O\left(\frac{\epsilon^3}{r^4}\right), \quad \epsilon \ll r, \quad \zeta \sim \epsilon^0.$$  

(22)

Table 1. Estimation of the physical conditions at two radii within the disc.

| $r_s$ au | $\alpha$ | $l_\text{H} \text{ Km}$ | $\pi_H$ | $\omega_{\text{pl}} \text{ s}^{-1}$ | $\Omega_\star \text{ s}^{-1}$ |
|---------|---------|-------------------------|--------|-------------------------------|-----------------------------|
| 1       | $1.12 \times 10^{-14}$ | $1.24 \times 10^3$          | 1.03   | 241.6                         | $2.00 \times 10^{-7}$     |
| 20      | $4.14 \times 10^{-13}$ | $1.25 \times 10^4$          | 11.2   | 23.82                         | $2.23 \times 10^{-9}$    |
To leading order in $\epsilon$ within the thin disc approximation, the toroidal velocity $V_\theta$ is described by the Keplerian law, which follows from the leading order radial component of the momentum equation:

$$V = V_\theta i_\theta, \quad V_\theta = \Omega(r) r, \quad \Omega(r) = r^{-3/2}, \quad V_\theta \sim \epsilon^0, \quad V_\theta \sim \epsilon^2.$$

The asymptotic orders in $\epsilon$ in equilibrium relations (23) are similar to those in Ogilvie (1997). They may be inferred by observing that due to equation (9) $\nabla \cdot (nV) = 0$ the axisymmetric fluid momentum per unit volume is expressed in terms of a scalar stream function $\chi(r, \zeta)$:

$$nV = nV_\theta i_\theta + \frac{1}{r} \hat{\nabla} \times i_\theta, \quad \hat{\nabla} = i_r \frac{\partial}{\partial r} + i_\theta \frac{1}{r} \frac{\partial}{\partial \zeta}.$$  

(24)

Assuming now that the toroidal velocity component is dominant and noting that $\partial / \partial z \sim \epsilon^{-1}$ imply that $\chi \sim \epsilon^2$, which immediately results in the ordering presented in equation (23) (see also discussion just after equation 41).

By a similar way, the divergent free axisymmetric magnetic field $\mathbf{B}$, $(\nabla \cdot \mathbf{B} = 0)$ is written in terms of a poloidal flux function $\Psi(r, \zeta)$ in the following way:

$$\mathbf{B} = B_\theta i_\theta + \frac{1}{r} \hat{\nabla} \Psi \times i_\theta, \quad \Psi(r, \zeta) = \Psi_0(r) + \epsilon \psi(r, \zeta), \quad B_\zeta = -\frac{1}{\epsilon} \frac{\partial \psi}{\partial \zeta}, \quad B_r = \frac{1}{r} \frac{d\Psi_0}{dr} + \epsilon \frac{1}{r} \frac{\partial \psi}{\partial r}.$$  

(25)

Here, the poloidal flux function $\Psi$ is presented as sum of $\Psi_0(r)$ and $\epsilon \psi(r, \zeta)$, which are scaled in $\epsilon$ by such a way to produce $B_\zeta \sim B_r \sim \epsilon^0$ to leading order in the thin disc approximation.

Proceeding further, it is noted that equation (12) means that the electric field is derived from a potential $\phi(r, \zeta)$

$$E = \epsilon \hat{\nabla} \phi.$$  

(26)

Taking now the dot product of both sides of equation (13) with the magnetic field reveals that $\mathbf{B} \cdot \hat{\nabla} \phi = 0$, and therefore $\phi$ is a function of the magnetic flux, i.e.

$$\phi = \phi(\Psi),$$  

(27)

which immediately, due to axisymmetry, means that the toroidal component of the electric field vanishes, i.e. $E_\theta = 0$. In order to impose that condition, Ampere’s law is first employed in order to write the electric current density in the following way:

$$j = j_\theta i_\theta + \frac{1}{r} \hat{\nabla} \psi(r, \zeta) \times i_\theta, \quad j_\theta = -\frac{1}{\epsilon} \frac{\partial B_\zeta}{\partial \zeta}, \quad j_\zeta = -\frac{1}{\epsilon} \frac{d}{dr} \left( \frac{d\Psi_0}{dr} \right) - \frac{\epsilon}{r} \frac{\partial \psi}{\partial \zeta} - \epsilon \frac{\partial}{\partial r} \left( \frac{\epsilon}{r} \frac{\partial \psi}{\partial r} \right), \quad j_r = -\frac{\partial B_\zeta}{\partial r}.$$  

(28)

Returning now to equation (13), the toroidal component of the electric field is given by:

$$E_\theta = \mathbf{B} \cdot \hat{\nabla} (r B_\theta) = 0.$$  

(29)

The leading order in $\epsilon$ of the last equation yields

$$\frac{\partial B_\zeta}{\partial \zeta} \frac{d}{dr} \left( \frac{\partial \psi}{\partial \zeta} \right) = 0,$$

(30)

which is satisfied if either $\partial B_\zeta / \partial \zeta = 0$ or $d\Psi_0/dr = 0$. The former case was considered by Ogilvie (1997) within the classical MHD model for the equilibrium of a differentially rotating thin disc containing a pure poloidal magnetic field ($B_\theta \equiv 0$). The second case, for the pure toroidal magnetic field ($\Psi \equiv 0$), has been investigated within the Hall MHD model by Shemler et al. (2007). Thus, in the leading order in $\epsilon$, there are three possible equilibrium states: (i) pure toroidal, (ii) pure poloidal and (iii) mixed magnetic field equilibrium, which cannot be reduced from one to the other. In the present work, the case $\Psi_0 \equiv 0$ is investigated within the Hall MHD model allowing for both toroidal as well as poloidal components of the magnetic field in the leading order. This results in

$$\mathbf{B} = B_\theta i_\theta + \frac{1}{r} \hat{\nabla} \Psi \times i_\theta, \quad \Psi(r, \zeta) = \epsilon \psi(r, \zeta), \quad B_r = -\frac{1}{\epsilon} \frac{\partial \psi}{\partial \zeta}, \quad B_\zeta \equiv B_r \equiv \frac{1}{\epsilon} \frac{\partial \psi}{\partial \zeta}, \quad \phi = \phi(\Psi)$$  

(31)

where the magnetic flux function $\Psi$ is scaled with $\epsilon$ in such a way that the resulting radial component of the magnetic field is of the order of the toroidal one ($\sim \epsilon^0$) in the thin disc approximation, while the axial magnetic field $B_\zeta$ is of the order of $\sim \epsilon$. Consequently, equations (29) and (31) yield

$$r B_\theta = I(\psi),$$  

(32)

where according to Ampere’s law $I(\psi)$ is the current flowing through a circular area in the plane $\zeta = \text{const}$. Furthermore, now Ampere’s law may be written as follows:

$$E \equiv \hat{\nabla} \phi = -\mathbf{V} \times \mathbf{B} + \epsilon \pi n \frac{\mathbf{j} \times \mathbf{B}}{n}.$$  

(33)

Finally, note that the case of the pure toroidal magnetic field $\Psi \equiv 0$ [i.e. $\mathbf{B} = (0, B_\theta, 0)$] requires a separate consideration. In that case, the relation $E_\theta = 0$ is satisfied identically, while Faraday’s law reduces, with the help of equation (33), to

$$\hat{\nabla} \times \left[ \frac{r B_\theta}{N} \hat{\nabla} (r B_\theta) \right] = 0,$$

(34)
where \( N = r^2 n \) is the inertial moment density through the Keplerian disc. This leads to the familiar result up to the scaling factor \( \pi_H \) (Shtemler et al. 2007):

\[
r B_0 = I(N), \quad \pi_H \phi = - \int_0^N \frac{I(N) I(N)}{N} \, dN.
\] (35)

Here and everywhere below, the upper dot denotes a derivative with respect to the argument.

Everything is ready now to derive the relevant GS equation in the general case \( \psi \neq 0 \). The latter is indeed readily obtained by projecting equation (13) on to the \( \hat{\nabla} \psi \) direction and taking into account the above relations. To leading order in \( \epsilon \), the result is the following non-linear differential equation for \( \psi \), which depends parametrically on the radial coordinate \( r \):

\[
\frac{\partial^2 \psi}{\partial r^2} = -I(\psi) I(\psi) + \frac{1}{\pi_H} N(r, \gamma) [\Omega(r) + \phi(\psi)].
\] (36)

This is the GS equation (similar to Lovelace et al. 1986; McClements & Thyagaraja 2001, and references therein) that is modified for the special case of a thin disc in the Hall MHD equilibrium. It is interesting to note that in the limit of negligible Hall effect (\( \pi_H \to 0 \)), equation (36) is reduced to the algebraic limit of the familiar isorotation law of ideal MHD that expresses the fluid’s angular momentum constancy on magnetic surfaces.

In general, a coupled problem should be considered inside and outside the Keplerian disc such that the edge of the disc \( \zeta = h(r) \) is determined self-consistently. However, to avoid the solution of the outer problem, the following assumption that is supported by the observation data for some protoplanetary discs (Calvet et al. 2002) is adopted:

\[
h(r) = \text{const} = 1.
\] (37)

Furthermore, the differential equation (36) must be complemented by an appropriate boundary conditions at the disc edge \( \zeta = h(r) \), as well as by symmetry condition at the mid-plane (assuming \( \hat{B}_r \), as well as \( B_0 \) being even functions of \( \zeta \)):

\[
n = 0, \quad \psi = \Gamma(r) \quad \text{at} \quad \zeta = h(r), \quad \frac{\partial \psi}{\partial \zeta} = 0\quad \text{at} \quad \zeta = 0,
\] (38)

where \( \Gamma(r) \) is a specified function. Since there is no direct observational information on how magnetic fields are distributed on disc edges, the following specific equilibrium configurations are conjectured, which are idealized representations of Keplerian discs. The value of \( \hat{B}_r \) at the horizontal disc edges is assumed to be a sum of a constant primordial (Ogilvie 1997) and a dipole-like (Lepeltier & Aly 1996; see also Lovelace et al. 2002; Matt et al. 2002) magnetic fields. The latter qualitatively reflects the effect of the central body on the Keplerian portion of the disc. The following two configurations are examined:

\[
\Gamma(r) = 0.5 Br^2 - Mr^{-1}, \quad (B = \pm 1),
\] (39)

\[
\Gamma(r) = -Mr^{-1}, \quad (B = 0, \quad M = \pm 1),
\] (40)

where the constants \( B \) and \( M \) are the dimensionless primordial magnetic field strength and the moment of the dipole-like magnetic field, respectively. In the case (39), the characteristic magnetic field \( B_r \) is chosen to be equal to the primordial magnetic field, while in the case (40) of zero primordial magnetic field, \( B_r \) is chosen to make the dimensionless magnetic moment equal to unity. The signs of \( B \) and \( M \) depend on their directions with respect to positive direction determined by Keplerian rotation. Both \( B \) and \( M \) correspond to the magnetic flux function scaled by \( \epsilon \), so the corresponding unscaled values are small constant of the order of \( \epsilon \). Since the primordial magnetic field is likely much smaller than that of the dipole-like magnetic field, the mixed case (39) with \( B \sim M \) is rather a qualitative demonstration of the method ability to describe a wide variety of structures, while the case (40) assumes implicitly that the primordial magnetic field creates a much smaller input into the boundary conditions than the magnetic dipole. Although, in general, the magnetic axis of the dipole is not aligned with the disc axis, some of the important problems may be investigated analytically in an axisymmetric approach, which is valuable for subsequent full 3D analysis.

Note that Ohm’s law generalized for the Hall equilibrium allows us to explicitly integrate the vertical momentum balance equation in thin disc approximation. However, this procedure is singular in the limit of small Hall parameter. Employing equations (33) and (23) in the axial component of the momentum equation (10) provides an explicit relation for the number density to the leading order in \( \epsilon \):

\[
n(r, \zeta) = v \left[ \frac{2}{\beta \pi_H} \left( \phi(\psi) - \phi(\Gamma) \right) + \frac{2\Omega}{\beta \pi_H} \left( \psi - \Gamma \right) + \frac{1 - \zeta^2}{r^3} \right]^{\frac{\gamma}{\gamma - 1}}, \quad N = r^2 n, \quad v = \left( \frac{v - 1}{2\gamma} \right)^{\frac{\gamma}{\gamma - 1}}.
\] (41)

Thus to the leading order, the momentum equation in the axial direction yields the density distribution (41), the momentum equation in the radial direction provides the Keplerian toroidal velocity \( \sim \epsilon^0 \), while the momentum equilibrium in the azimuthal direction along with the mass balance relation determines the radial and axial velocities to be small of the orders of \( \epsilon \) and \( \epsilon^2 \), respectively. Expression (41), singular for vanishing \( \pi_H \) and everywhere below the Hall parameter \( \pi_H \), is assumed to be of the order of or much larger than unity.

Inspecting the resulting problem (36) and (41) reveals that it contains the Hall parameter \( \pi_H \) in the following two combinations: as the ratio with inertial moment density \( N/\pi_H \) in equation (36), and as a factor with the plasma beta \( \beta \pi_H \) in equation (41). Since both plasma beta and Hall parameter are independent physical parameters, and since they are generally unknown in astrophysical systems, a scaled Hall plasma beta parameter is naturally introduced:

\[
\rho_{\text{HI}} = \beta \pi_H.
\] (42)
Note also that for large Hall plasma beta $\beta_H$, the arbitrary function $\phi(\psi)$ is dropped from equation (41) and the number density is approaching its pure hydrodynamical limit.

The arbitrary functions $\phi(\psi)$ and $I(\psi)$ reflect the uncertainty of the steady-state equilibrium solution, which, as assumed here, describes the equilibrium solution at long times. In general, the corresponding initial value problem must be solved, which shifts that uncertainty to the arbitrariness of the initial data. The magnetic field and density structure are governed by the choice of the functions $\phi(\psi)$ and $I(\psi)$, here taken as following trial functions in order to qualitatively describe the equilibrium states:

$$I(\psi) = I^{(0)} + I^{(1)} \psi, \quad \phi(\psi) = \phi^{(0)} + \phi^{(1)} \psi.$$  \hfill (43)

The constant $I^{(0)}$ is the total current through the disc that is localized along the axis (Shtemler et al. 2007), while $\phi^{(0)} = 0$ without loss of generality due to the physical meaning of the electric field potential (see also equations 36 and 41). Such kinds of simplest trial function are commonly used for ideal MHD model (see e.g. Wu & Chen 1989; Kaiser & Lortz 1995; Bogoyavlenskij 2000 and references therein for astrophysical applications). Sometimes, the above relations are considered as first terms of generic expressions in which the arbitrary functions can be expanded in power series in $\psi$ (Bogoyavlenskij 2000). Clearly, approximations (43) are biased by their partial form. However, in the present study it is conjectured that the main features of the equilibrium are rather general since relations (43) may be considered as a best-fitting linear approximations rather than first terms in appropriate Taylor sets.

To summarize, in the present model Hall MHD axisymmetric equilibria are determined up to the Hall parameter $\pi_H$, the Hall plasma beta parameter $\beta_H$, and the free constants $B$ and $M$ that characterize the primordial magnetic field and an effective magnetic dipole moment induced by the central body. In such equilibria, the electric potential and the toroidal magnetic field are arbitrary functions of the magnetic flux, while the density is an explicit function of the coordinates and the magnetic flux function, which satisfies a GS-type equation modified for the special case of the Hall equilibrium in thin-disc approximation.

### 4 ANALYSIS OF THE EQUILIBRIUM IN TERMS OF THE SCALED INVERSE HALL PARAMETER

Inspecting expression (41) for the inertial moment density $N$ and the GS equation (36) for the magnetic flux $\psi$ that contains $N/\pi_H$ reveals the existence of the following small parameter $\delta$:

$$\delta = \frac{\nu}{\pi_H} \lesssim \frac{1}{2}.$$  \hfill (44)

In the adiabatic case with $\gamma = 5/3$ adopted everywhere below, this parameter is indeed sufficiently small within the range from large values $\pi_H$ up to $\pi_H = 1$ when it equals $\delta = \nu \approx 0.1$ (see definition of $\nu$ in equation 36). Note also that validity for using $\delta = \nu/\pi_H$ as the small parameter is violated in the classical MHD limit ($\pi_H \to 0$), which requires a separate consideration.

Since the parameters of the equilibrium are widely arbitrary, two different families of equilibrium solutions are found, which correspond to either (i) small ($\sim \delta$) or (ii) finite ($\sim \delta^0$) deviations of the magnetic flux $\psi(r, \zeta)$ from its boundary value $\Gamma - \delta^0$. Note that the scaling in $\delta$ is applied to the flux function that is already scaled in $\nu$ (see equations 25), so that the dimensionless unscaled value $|\psi - \Gamma|$ is either (i) $\sim \delta \epsilon$ or (ii) $\sim \delta^0$. Consequently, as will be seen below, the Hall equilibrium discs fall into two types which are characterized by quite different orders in $\delta$: Keplerian discs with (i) small ($R_{K} \sim \delta^0$) and (ii) large ($R_{K} \gtrsim \delta^k, k > 0$) radii of the disc.

Thereafter, the two families of the equilibrium solutions are named, respectively, Keplerian discs with small and large radii. In both families, the angular velocity satisfies the Keplerian law in the leading order in $\epsilon$ independently of $\delta$. Thus, using the small parameter $\delta$ not only enables and simplifies further calculations, but also allows us to clearly distinguish two quite different Hall equilibrium families of solutions that are considered below separately.

#### 4.1 First family of solutions: small-radius discs

Assuming that $|\psi - \Gamma| \sim \delta^0$ the solution is expanded in power series in $\delta$:

$$\psi(r, \zeta) = \psi_0 + \delta \psi_1 + \cdots, \quad N(r, \zeta)/\nu = N_0 + \delta N_1 + \cdots, \quad n(r, \zeta)/\nu = n_0 + \delta n_1 + \cdots, \quad I(\psi) = I(\psi_0) + \cdots, \quad \phi(\psi) = \phi(\psi_0) + \cdots.$$  \hfill (45)

Substituting equations (45) into (36)--(41) yields to leading order in $\delta$ (omitting the subscript 0 for brevity)

$$\frac{\partial^2 \psi}{\partial \zeta^2} = -I(\psi) \dot{\psi}(\psi) \quad \psi = \Gamma(r) \text{ at } \zeta = 1, \quad \frac{\partial \psi}{\partial \zeta} = 0 \text{ at } \zeta = 0.$$  \hfill (46)

Thus, in the lowest order in $\delta$ the electric field potential is dropped out from the equilibrium problem. Transforming then to a new independent variable $Y = \partial \psi/\partial \zeta$ and integrating the transformed differential equation in (46) under the appropriate conditions at $\zeta = 0$ yields

$$\frac{\partial \psi}{\partial \zeta} = \sqrt{I^2(\psi(r, 0)) - I^2(\psi(r, \zeta))}.$$  \hfill (47)
Integrating equation (47) once more under the conditions at \( \zeta = 1 \) in (46) yields to leading order in \( \delta \) the following implicit dependence of \( \psi \) and \( N = r^2 n \) on \( r \) and \( \zeta \):

\[
1 - \zeta = - \frac{d\psi}{\sqrt{I^2(\Gamma) - I^2(\psi)}} \tag{48}
\]

and

\[
N(r, \zeta) = r^2 n(r, \zeta) = r^2 \left[ \frac{2}{\beta_\parallel} (\phi(\psi) - \phi(\Gamma)) + \frac{2\Omega(\psi) - \Omega(\Gamma)}{\beta_\parallel} + \frac{1 - \zeta^2}{r^3} \right]^{3/2} \tag{49}
\]

Thus, both poloidal and toroidal magnetic fields are completely characterized by arbitrary functions \( I(\psi), \phi(\psi) \) and the edge magnetic flux function \( \Gamma(r) \) (with the characteristic amplitudes \( B \) and \( M \)). The function \( N(r, \zeta) \) depends also on \( \phi(\psi), \Gamma(r) \) and on the Hall plasma beta \( \beta_\parallel \).

Substituting (43) into equations (48) and (49) yields for \( \gamma = 5/3 \):

\[
\psi = \Gamma(r) + \left( \frac{I(0)}{I(0) + \Gamma(r)} \right) \left( \frac{\cos(I(1)\cos(I(1)))}{\cos(I(1))} - 1 \right),
\]

\[
n = \left[ \frac{2}{\beta_\parallel} \left( \frac{I(0)}{I(0) + \Gamma(r)} \right) (\phi(\psi) + \Omega(r)) \left( \frac{\cos(I(1))}{\cos(I(1))} - 1 \right) + \frac{1 - \zeta^2}{r^3} \right]^{3/2},
\]

\[
r B_r = \frac{(I(0) + I(1)\Gamma(r))}{\cos(I(1))}, \quad r B_\theta = \frac{(I(0) + I(1)\Gamma(r))}{\cos(I(1))} \quad B_z = \frac{1}{r} \frac{d\Gamma}{dr} \frac{\cos(I(1))}{\cos(I(1))},
\]

In particular, in the mid-plane \( \zeta = 0 \) equations (50)–(52) results in

\[
n = \left[ \frac{2}{\beta_\parallel} \left( \frac{I(0)}{I(0) + \Gamma(r)} \right) (\phi(\psi) + \Omega(r)) \right]^{3/2}, \quad B_r = 0, \quad r B_\theta = \frac{(I(0) + I(1)\Gamma(r))}{\cos(I(1))}, \quad B_z = \frac{1}{r} \frac{d\Gamma}{dr} \frac{1}{\cos(I(1))},
\]

while on the disc’s edges \( \zeta = \pm 1 \) the magnetic field and number density are given by

\[
n = 0, \quad r B_r = \pm \left( \frac{I(0)}{I(0) + \Gamma(r)} \right) \tan(I(1)), \quad r B_\theta = \frac{(I(0) + I(1)\Gamma(r))}{\cos(I(1))}, \quad B_z = \frac{1}{r} \frac{d\Gamma}{dr}
\]

The parameters \( I(0), I(1), \phi(1), \beta_\parallel \) and \( B \) and/or \( M \) are needed for a complete description of the Hall equilibrium state. Furthermore, the solutions (50)–(52) are invariant with respect to simultaneous change of signs of \( I(0) \) and \( I(1) \), so further analysis is restricted to positive values of \( I(0) \).

Inspecting equation (51), it is readily seen that a class of equilibria may be found which is characterized by a finite radius that appears as a cut-off value at which the plasma density vanishes. The finite radius of the discs, \( R_d \), may be inferred from the condition of zero number density that according to equation (51) is given by

\[
\frac{2}{\beta_\parallel} \left( \frac{I(0)}{I(1)} + \Gamma(R_d) \right) \left( \phi(\psi) + \Omega(R_d) \right) \left( \frac{\cos(I(1)) - 1}{\cos(I(1))} + \frac{1 - \zeta^2}{r^3} \right)^{3/2} = 0.
\]

Note that the frequently used limit of high plasma beta reduces equation (55) to the classical hydrodynamic limit and yields \( R_d = \infty \). Generally, the parameters \( I(0), I(1), \phi(1), \beta_\parallel \) and \( B \) and/or \( M \) may be correlated by using equation (55) with a given, e.g. observed, value of the disc radius. Neglecting the term \( (1 - \zeta^2)/R_d^2 \) in equation (55) (that is a plausible assumption if the dimensionless disc radius is sufficiently large, although corresponding values of \( R_d \) will be larger for larger values of the Hall plasma beta \( \beta_\parallel \)), the value \( R_d \) may be explicitly estimated as the root of the following algebraic equation that is independent on the Hall plasma beta:

\[
\left( \frac{I(0)}{I(1)} + \Gamma(R_d) \right) \left( \phi(\psi) + \Omega(R_d) \right) \approx 0.
\]

Equation (56), satisfied by either its first or second co-factors, is equal to zero

\[
R_{dl} + 2 \frac{I(0)}{B I(1)} R_{dl} - 2 \frac{M}{B} = 0,
\]

\[
R_{dl} = \left( - \frac{1}{\phi(1)} \right)^{2/3}, \quad (\phi(1) < 0),
\]

where the first root \( R_{dl} \) results also in zero values of both \( B_{dl} \) and \( B_{d0} \) at the lateral disc edge. As follows from equation (55), the first root is governed by the effect of both toroidal and poloidal magnetic fields, while the second root is determined by the combined effect of rotation and poloidal magnetic field.

In two limiting cases of predominantly either permodial (\( M = 0 \)) or dipole (\( B = 0 \)) origin, the cubic equation (57) has the following simple explicit solutions, respectively,

\[
R_{dl}^{(B)} \approx \left( - \frac{2 I(0)}{B I(1)} \right)^{1/2}, \quad \left( \frac{I(0)}{B I(1)} < 0 \right).
\]
Hall equilibrium of thin Keplerian discs

\[ R_{d1}^{(M)} = \frac{M I^{(1)}}{f^{(0)}}, \quad \left( \frac{M I^{(1)}}{f^{(0)}} > 0 \right), \]

where \( R_{d1}^{(B)} \to \infty \) as \( f^{(1)} \to 0 \), and \( R_{d1}^{(H)} \to \infty \) as \( f^{(0)} \to 0 \).

Another interesting special solution of equation (57) valid for mixed magnetic flux at the horizontal flux and sufficiently low total currents \( f^{(0)} \) is

\[ R_{d2} \approx \sqrt{\frac{2M}{B}}, \quad \left( \frac{2M}{B} > 0, \quad \left| \frac{f^{(0)}}{f^{(1)}} \right| < \sqrt{\frac{|M|B|}{2}} \right), \]

which means that \( \Gamma(R_d) \to 0 \). This corresponds to zero net poloidal flux, configurations that have been recently employed in order to study the viability of internal dynamo action (Brandenburg et al. 1995; Fromang & Papaloizou 2007). It should finally be noted that the finite radius solutions obtained above are the direct result of the requirement of finite thickness of discs that are vertically held by the balance between pressure and the Lorentz force, combined with the particular functional dependencies of the total current and electric potential on the magnetic flux function in equation (43). Alternatively, requiring a vertically diffused disc with exponentially decreasing density (as is done in Coppi & Keyes 2003; Livers & Mond 2009) will result also in exponentially decreasing radial density profiles.

### 4.2 Second family of solutions: large-radius discs

For small deviations of the magnetic flux function from its edge value, the above asymptotic expansions (in \( \delta \)) fail. Thus, assuming that the system parameters \( f^{(0)} = \delta f^{(0)}, \Gamma = \delta \Gamma, (B = \delta B, M = \delta M) \) are scaled in \( \delta \), and omitting the bars with no confusion, it may be written:

\[ \psi(r, \zeta) - \delta \Gamma(r) = \delta \psi(r, \zeta) + \cdots, \quad N(r, \zeta)/v = N_0 + \delta N_1 + \cdots, \quad n/v = n_0 + \delta n_1 + \cdots, \quad I(\psi) = \delta I^{(0)} + I^{(1)} \delta \Gamma + \cdots. \]

Substituting (62) into (36)–(41) yields in the leading order in \( \delta \) for density and magnetic flux

\[ n_0 = \left( \frac{1 - \zeta^2}{r^3} \right)^{3/2}, \quad N_0 = r^2 n_0, \]

\[ \delta^2 \psi_1 = -I(\Gamma) \hat{I}(\Gamma) - N_0(r, \zeta) \Omega(\rho) + \hat{\phi}(\Gamma)|, \psi_1 = 0 \quad \text{for} \quad \zeta = 1, \quad \frac{\partial \psi_1}{\partial \zeta} = 0 \quad \text{for} \quad \zeta = 0. \]

Using equations (43) and integrating relations (64) yields

\[ \psi_1 = I^{(1)} \left[ f^{(0)} + I^{(1)} \Gamma \right] \left[ 1 - \frac{\zeta^2}{2} + \frac{\Omega(\rho) + \phi(\Gamma)}{8 r^5/2} \left( \frac{1}{2} - \frac{\zeta^2}{2} \right)^{1/2} - \frac{3}{5} \left( \zeta \sin \theta - \pi/2 + (1 - \zeta^2) \right)^{1/2} \right]. \]

In that approximation, the parameters \( f^{(0)}, f^{(1)}, \phi^{(1)} \) and \( B \) and/or \( M \) are needed for the complete description of the Hall equilibrium state (the parameters \( \phi^{(0)} \) and \( \beta_H \) are dropped from that problem). Solution (65) is invariant with respect to the simultaneous change of \( f^{(0)}, f^{(1)} \) by \(-f^{(0)}, -f^{(1)}\) so that further analysis may be restricted by positive values of \( f^{(0)} \).

It is finally remarked that in the case under consideration the density distribution (63) is the same as in the pure hydrodynamic case, in particular, it vanishes at \( r \to \infty \) and the disc is unbounded in such approximation. It may be shown that infinite radial size disc in this approximation follows from non-uniformity at large radius of the asymptotic expansions (62) in \( \delta \). Thus, according to equation (65) \( \psi(r, \zeta) - \delta \Gamma(r) = \delta \psi_1(r, \zeta) \sim (f^{(0)} + I^{(1)} \Gamma) \delta \) at \( r \to \infty \). Substituting that estimate into equation (41) for the density yields that the terms in equation (41) \( (f^{(0)} + I^{(1)} \Gamma) \delta \) neglected in the approximation (63) are of the same order as the last term in equation (41) \( \sim 1/r^3 \) at sufficiently large radius \( r \gg 1 \). This yields that asymptotic expansions (62) fail for \( r \sim \delta^{-2} (k > 0) \) with \( k \) depending on the form of \( \Gamma(r) \), and the proportionality coefficient depends on the Hall plasma beta parameter. For instance, if \( B = 0 \) and \( \Gamma(r) = r/M \), then \( r \sim \delta^{-2/3} \gg 1 \). Consequently, the disc radius \( R_d \) may be estimated as \( R_d \gg \delta^{-2}, (k > 0) \). A more exact determination of the disc radius (finite or infinite) requires an additional effort.

Nevertheless, the above non-uniform solution may be employed with no restriction for smaller values of radial coordinate (e.g. \( r \sim \delta^3 \)) in the local analysis of the disc stability (see e.g. Shtemler et al. 2007).

### 5 NUMERICAL EXAMPLES

The density contour lines and the poloidal magnetic-field lines are calculated for linear approximations of the trial functions \( I(\psi) = f^{(0)} + f^{(1)} \psi \) and \( \phi^H(\psi) = \phi^{(1)} \psi \) as well as for a model magnetic flux at the horizontal edge of the disc \( \Gamma(r) = Br^2 + M/r \). The parameters \( f^{(0)}, f^{(1)}, \phi^{(1)} \) and/or \( M \) are chosen in such a way to cover all possible combinations of their signs taking into account the symmetry properties. All calculations for equilibrium solutions considered below may be restricted to positive values of \( f^{(0)} > 0 \) with no loss of generality.

Furthermore, the parameters are selected by assuming that for physically accepted solution density should either equal to zero at a finite radius of the disc (first family of equilibria) or vanish asymptotically with growing value of radius (second family of equilibria). In all calculations, a typical value of the Hall plasma beta \( \beta_H = 5 \) has been fixed. In several figures, \( \theta \) denotes a small value \( \theta = 0.005 \).
5.1 Numerical examples for the first family of equilibrium solutions

For the first family of equilibrium solutions, the disc radius $R_d$ is exhibited as a cut-off value at which density drops to zero. The numerical solutions are restricted to values of the parameters of the trial and edge boundary functions, which yield the dimensionless disc radius $R_d$ of the order of unity. Note that the values of the disc radius in Figs 3(b), 4(b) and (d) correspond to zero net poloidal flux, and they are in the fair agreement with equation (61). The physically acceptable equilibrium solutions in Fig. 1 ($I^{(0)} > 0$) are found only either for negative primordial magnetic field $B < 0$, $I^{(1)} > 0$ and $\phi^{(1)} > 0$, or otherwise for $B > 0$, $I^{(1)} < 0$ and $\phi^{(1)} < 0$. For the dipole-like magnetic field along with qualitatively similar magnetic-field configurations, new types appear in Fig. 2 with quite different topology from that in Fig. 1. The equilibrium solutions in Fig. 2 ($I^{(0)} > 0$) are found only either for positive moment of magnetic dipole, $M > 0$ for $I^{(1)} > 0$ and $\phi^{(1)} < 0$ or otherwise for $M < 0$ at $I^{(1)} < 0$ and $\phi^{(1)} > 0$. For the case of mixed magnetic flux at the horizontal edges along with qualitatively similar magnetic-field configurations, new types appear now in Figs 3 and 4 with quite different topology from that in Figs 1 and 2 for primordial and dipole-like at the horizontal edges.

It is easy to distinguish two kinds of density contour lines topology:

(i) monotonically decreasing density starting from the disc centre to the disc radius $r = R_d$ (Figs 1–4 except of Fig. 3d),
(ii) a density core in the disc centre that is separated from an outer density ring by an internal plasma-free cavity inside the disc (Fig. 3d).

The appearance of a density ring in Fig. 3(d) is associated with three roots of the cubic equation (57), which determine two lateral boundaries.
Figure 3. Magnetic field lines and density contours for mixed magnetic flux at the horizontal edges: (a–b) $M = 4, B = 1, f^{(0)} = 2, f^{(1)} = 1, \phi^{(1)} = -0.5, R_d \approx 2$; (c–d) $M = 4, B = 1, f^{(0)} = 2, f^{(1)} = -1, \phi^{(1)} = -0.5, R_d \approx 2$; (e–f) $M = -4, B = 1, f^{(0)} = 2, f^{(1)} = 1, \phi^{(1)} = -0.5, R_d \approx 2$; (g–h) $M = -4, B = 1, f^{(0)} = 2, f^{(1)} = 1, \phi^{(1)} = -0.5, R_d \approx 1.9$.

Figure 4. Magnetic field lines and density contours for mixed magnetic flux at the horizontal edges: (a–b) $B = -1, M = -4, f^{(0)} = 2, f^{(1)} = 1, \phi^{(1)} = -0.5, R_d \approx 2.2$; (c–d) $B = -1, M = 4, f^{(0)} = 2, f^{(1)} = -1, \phi^{(1)} = -0.5, R_d \approx 0.5$.

of the external ring and of the internal cavity inside which the density is identically zero [the inner and external radii in Fig. 3d of the disc ring are $R_d^{(in)}$ and $R_d$, and the ratio of the outer-to-inner radius is of the order of $R_d/R_d^{(in)} \sim 3$].

In Figs 1–4, there are two kinds (i) and (ii) of magnetic field structures both of which are characterized by condition (60), while the other two kinds (iii) and (iv) of the magnetic field structures are obtained by employing equation (59).

(i) Magnetic structures for zero dipole magnetic flux at the horizontal edge. The magnetic structures are formed by a magnetic island within the disc that is centred on an O-point located at the origin (Figs 1a and c). When a magnetic line intersects the horizontal disc edges, it is changed by the near-vertical magnetic lines that are oriented from bottom to top disc edges.

(ii) Magnetic structures for dipole-like and mixed edge magnetic fluxes. In that case, the magnetic structures are formed by near-vertical magnetic lines oriented from bottom to top disc edges (Figs 2a, c, 3a, c, 4a and c).

(iii) Magnetic structures for zero dipole and mixed edge magnetic fluxes. In that case, magnetic structures are generated which have an X-point located either at the origin of the coordinates (Figs 1e and g) or even inside the Keplerian portion of the disc (Figs 3e and g).

(iv) Magnetic structures for pure dipole-like edge magnetic flux. In that case, magnetic structures are generated which have two multivalued points located in two symmetrical points with respect to the mid-plane on the disc axis ($r = 0$, not included in the Keplerian portion of the disc $R_d \gg \epsilon$), where all magnetic-field lines $\psi = const$ converge (Fig. 2e).

Finally, note that the radii of the small discs are independent of the Hall parameter $\pi_\text{Hi}$; moreover, they are slightly dependent on the Hall plasma beta $\beta_\text{H} = \beta_\text{Hi}$ for a wide range of the parameters of the trial $I(\psi)$, $\phi(\psi)$ and edge $\Gamma(r)$ functions for which $R_d$ is well approximated by the solution of the approximate equation (56). Thus, the values of $R_d$ are found to be in a fair agreement with those given by equation (56) for most of Figs 1–4 [except for Figs 1f and 4d where $R_d$ is so small that the exact equation (55) must be solved].

It is interesting that the approximate conditions (57) are not satisfied for such structures as X-points as well as other topologically unstable configurations depicted in Figs 1(h), 2(f), 3(f) and (h).

5.2 Numerical examples for the second family of equilibrium solutions

For the second family of equilibrium solutions, the density is given by a pure hydrodynamic Keplerian distribution (see Fig. 5), and it is the same for all admissible magnetic field configurations in Figs 6–8 (with parameters $M, B, f^{(0)}$ scaled by $\delta$, and restricted to $\phi^{(1)} > 0$). To
Figure 5. Density contours for the second family of solutions.

Figure 6. Perturbed magnetic field lines for pure primordial magnetic flux at the horizontal edges, $M = 0$, $\phi^{(1)} = 1$: (a) $B = -1$, $I^{(0)} = 1$, $I^{(1)} = -1$; (b) $B = -1$, $I^{(0)} = 1$, $I^{(1)} = 1$; (c) $B = 1$, $I^{(0)} = 1$, $I^{(1)} = 1$; (d) $B = 1$, $I^{(0)} = 1$, $I^{(1)} = -1$.

Figure 7. Perturbed magnetic field lines for pure dipole magnetic flux at the horizontal edges, $B = 0$, $\phi^{(1)} = 1$: (a) $M = 1$, $I^{(0)} = 1$, $I^{(1)} = 1$; (b) $M = -1$, $I^{(0)} = 1$, $I^{(1)} = -1$; (c) $M = -1$, $I^{(0)} = 1$, $I^{(1)} = 1$; (d) $M = 1$, $I^{(0)} = 1$, $I^{(1)} = -1$.

Figure 8. Perturbed magnetic field lines for mixed magnetic flux at the horizontal edges, $\phi^{(1)} = 1$: (a) $B = -1$, $M = 1$, $I^{(0)} = 1$, $I^{(1)} = -1$; (b) $B = -1$, $M = 1$, $I^{(0)} = 1$, $I^{(1)} = 1$; (c) $B = -1$, $M = -1$, $I^{(0)} = 1$, $I^{(1)} = 1$; (d) $B = -1$, $M = -1$, $I^{(0)} = 1$, $I^{(1)} = -1$; (e) $B = 1$, $M = 1$, $I^{(0)} = 1$, $I^{(1)} = -1$; (f) $B = 1$, $M = 1$, $I^{(0)} = 1$, $I^{(1)} = 1$; (g) $B = 1$, $M = -1$, $I^{(0)} = 1$, $I^{(1)} = -1$; (h) $B = 1$, $M = -1$, $I^{(0)} = 1$, $I^{(1)} = 1$.

Illustrate the qualitative behaviour of the magnetic field, the field lines of the scaled magnetic flux $\psi_1(r, \zeta) = [\psi(r, \zeta) - \delta \Gamma(r)]/\delta$ are depicted in Figs 6–8 for different kinds of the edge magnetic flux: with zero dipole-moment, pure dipole-like magnetic field and mixed magnetic fields. The second family of equilibria generates structures that are quite similar to those in the first family, e.g. X-points in Figs 6(c), (d) and 8(f–h), and several configurations with a single O-point structures in Figs 6(a), (b), 7(b), 8(c) and (d). In addition, new kinds of the structures arise and may be seen in Figs 7(a), (c), (d) and Figs 8(a), (b), (e). Thus, Fig. 7(c) demonstrates a magnetic island with radius that increases infinitely with decreasing magnetic flux value. Then, double O-point structures appear with magnetic islands additional to those centred on
the O-point at the coordinate origin. Figs 7(a), (d) and 8(e) demonstrate the presence of two magnetic islands (‘plasmoids’), first with central O-points located at the origin of the coordinate, and second with both finite radius of magnetic islands and coordinate of O-point inside the disc in Figs 7(a) and 8(e), or with both the radius of magnetic island and coordinate of O-point which tend to infinity in Figs 7(d). Figs 8(a) and (b) with a single O-points located at the origin separated from the right by a separatrix \( \psi_1 = 0 \) demonstrate the magnetic lines with strong variations of their gradients from the right of the separatrix. Fig. 8(e) demonstrates the presence of O-points, closed magnetic lines confined both from the left and right by the separatrix \( \psi_1 = 0 \), while X-points are in Figs 8(f–h). Finally, note that some of the equilibrium magnetic lines obtained above are quite similar to the instantaneous magnetic lines obtained experimentally in a tokomak (e.g. fig. 6 in Iisuka, Minamitani & Tanaka 1986).

6 D SUMMARY AND DISCUSSION

3D axially symmetric steady-state equilibria of weakly ionized polytropic Hall-MHD plasmas are investigated for the Keplerian portion of thin discs. Asymptotic expansions in small aspect ratio \( \epsilon \) provide an efficient way to construct an equilibrium model for differentially rotating discs. There are three principal Hall equilibrium states that cannot be reduced from one to the other, classified according to the following orientations of the magnetic field lines: (i) pure toroidal, (ii) pure poloidal and (iii) mixed toroidal and poloidal equilibria. The three classes of equilibria differ from each other by the different ordering with respect to \( \epsilon \) of the various components of the magnetic field. In the present study, the most interesting case is investigated that involves both toroidal and poloidal components of the magnetic field that are of the same zero order in \( \epsilon \). The disc structure is described by the GS equation for the poloidal flux that involves two arbitrary functions \( I(\psi) \) and \( \phi(\psi) \) for the toroidal electric current and electric potential, respectively. The arbitrariness of functions \( I(\psi) \) and \( \phi(\psi) \) in steady-state problems reflects the uncertainty of the initial data in the general initial value problem. The flux function is symmetric about the mid-plane and satisfies certain boundary conditions at the horizontal edges of the disc (radial variations of the disc height are neglected). The boundary conditions for the magnetic flux \( \psi = \Gamma(r) \) at the horizontal disc edges express the joint effect of a primordial magnetic field, and a dipole-like magnetic field that reflects the influence of the central body on the Keplerian disc. Solutions for different configurations of the magnetic field and density are obtained explicitly for trial linear approximations of the current \( I(\psi) = I^{(0)} + I^{(1)} \psi \) flowing through a circular area in a plane and of the electric potential \( \phi(\psi) = \phi^{(1)} \psi \) [where \( I^{(0)} \) is the total current concentrated along the disc axis], as well as several forms of the boundary magnetic flux \( \Gamma(r) \).

A particular noteworthy new feature of the present model is the finite radius of the rotating discs, which is inherent for the Hall equilibrium. By using a small parameter \( \delta \) proportional to the inverse Hall parameter with a small coefficient of proportionality, it is established that the Hall equilibria discs fall into two types which are characterized by quite different orders in \( \delta \): Keplerian discs with (i) small \( (R_0 \sim \delta^k) \) and (ii) large \( (R_0 \gtrsim \delta^k, k > 0) \) radius of the disc. For discs of the first family, a finite radius of the disc appears as a cut-off value at which the plasma density vanishes. Discs of the second family have large (finite or infinite) radii due to non-uniformity of the asymptotic expansions in \( \delta \).

The method developed here allows us to investigate analytically the equilibrium states with a large number of possible combinations of boundary conditions for the magnetic flux. Thus, it is demonstrated that all possible mutual orientations of the rotating axis, the primordial magnetic field and the magnetic moment of the central body, as well as the relative strength of the two latter give rise to a great richness of possible topologies of the magnetic field lines. Such configurations include geometries with O-points, X-points, material gaps and rings, detached plasmoids, magnetic islands and zero net induced magnetic flux. Note that magnetic islands centred on O-points are typical for steady-state equilibria in Keplerian disc. For instance, they have been simulated in thin disc approximation within MHD model for the pure poloidal magnetic field (Lovelace et al. 1986). On the other hand, equilibria that contain X-points are more subtle and generally have a strong tendency to undergo a significant change in their topology due to possible reconnection processes (Priest & Forbes 2000). The latter, however, is commonly speculated to be responsible for widely observed astrophysical phenomena like jets and winds. Describing such processes requires an extensive stability analysis. Since the primordial poloidal flux on the disc is likely too small to be responsible for the observed total magnetic flux in disc systems (Colgate & Li 2000), the possible flux enhancing due to the instability of the axisymmetric Hall equilibria becomes an important issue. Additionally, it is natural then to incorporate multipoles of higher odd orders (Lepeltier & Aly 1999) instead of the primordial poloidal flux, in addition to the dipole adopted here, into the boundary flux sources at the origin that describe the central object effect on the Keplerian disc. The stability study of the developed equilibrium disc may provide further selection of admissible magnetic configurations. For instance, development of X-point configurations may lead to an instability of such equilibria, since X-point equilibrium solutions are rather structurally unstable and a small perturbation of the equilibrium likely leads to an essential change in its form, such as magnetic lines reconnection and transition to the neighbourhood stable (smooth) configuration. The present Hall equilibrium solution for thin Keplerian discs may be incorporated to the leading order in small aspect ratio into a more sophisticated problem accounting for accretion and jet inflow (similar to Ogilvie 1997).

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