Evolution of Curvature in Riemannian Geometry

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Abstract

In this paper we shall follow two different routes that embody the existence of a point of exclusivity - the opposite of a singularity. First, we show that this follows from Raychaudhuri’s equation. Then, we substantiate that the evolution of the Riemann-Christoffel tensor can be expressed entirely by an arbitrary timelike vector field and that the curvature tensor returns to its initial value with respect to change in a particular index. It has been shown that geodesics can diverge just as they can converge, resulting in a point of exclusivity and point of singularity.

1 Introduction

General theory of relativity, the geometric theory of gravitation is based on the plinth of Riemannian geometry that studies differentiable manifolds. There is a plethora of literature on these two intricately related topics. As is known, there are a multitude of implications resulting from Riemannian geometry when applied to the general relativity - the Einstein field equations, solutions to such equations in the form of Schwarzschild metric, Friedmann-Lemaître-Robertson-Walker metric and others, singularities and black holes.

However, in essence, the current paper is not innately related to the general theory of relativity. It is essentially devoted to the geometric aspect of space-time, somewhat in contradistinction to a recent paper that was based on a Hamiltonian formulation. The inception begins with the feasibility of existence of a point of exclusivity (the opposite of a singularity), that is corroborated considering the Raychaudhuri equation. And then, we consider parallel transported vectors and thereby manipulating of their connection with the geometry of space-time we are led to an evolution equation for the Riemann-Christoffel tensor. As a consequence, several interesting implications are drawn out. Most importantly, it has been argued that with necessary conditions fulfilled one can indeed discern the existence of a point of exclusivity.
2 The positive and negative values of the expansion scalar

In his seminal paper of 1955 [13], Raychaudhuri had obtained an equation that led to the focusing theorem which in turn substantiated the existence of singularities and black holes, eventually. We would find in the present section that the result derived by Raychaudhuri plays a significant role in some novel aspects. Let us commence with the expansion scalar \( \theta \) of the Raychaudhuri equation. We know that

\[
\theta = \frac{\partial}{\partial t} (\ln G)
\]

where, \( G = \sqrt{-g} \). Also, we know that

\[
A^i_i = \frac{1}{\sqrt{-g}} \left[ A^i \frac{\partial}{\partial x^i} \sqrt{-g} + \sqrt{-g} \frac{\partial}{\partial x^i} A^i \right]
\]

So, considering the coordinate \( t \) \((i = 0)\) we may write

\[
A^0_0 = [A^0 \theta + A^0_0]
\]

Now, since the Kronecker delta function is independent of both the normal and covariant derivatives the last equation can be equivalently expressed as

\[
A_{k,0} = A_k \theta + A_{k,0}
\]

Again, we also know that

\[
A_{k,i} = A_{k,i} - \Gamma^l_{ki} A_l
\]

Therefore, using equation (1) we obtain

\[
A_k \theta + \Gamma^l_{k0} A_l = 0 \tag{2}
\]

or,

\[
A_k \theta + \Gamma^l_{k0} \delta^k_l A_k = 0
\]

Since, the vector fields \( A_k \)'s are arbitrary, we have the following equation for the expansion scalar

\[
\theta + \Gamma^l_{k0} \delta^k_l = 0 \tag{3}
\]
One can make two immediate observations from equation (\ref{three}). If $k \neq l$, we have

$$\theta = 0$$ \hspace{1cm} (4)

And, if $k = l$, we have

$$\theta + \Gamma_{l0}^l = 0$$ \hspace{1cm} (5)

Again, on account of the relation

$$\Gamma_{l0}^l = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} \sqrt{-g} = \frac{\partial}{\partial x^0} (\ln \sqrt{-g})$$

we derive from equation (5)

$$\theta + \theta = 0$$ \hspace{1cm} (6)

which is, in essence, similar to equations (\ref{three}/\ref{seven}). Either the expansion scalar is zero, or

$$|\theta| = \pm \theta$$ \hspace{1cm} (7)

which seems erroneous from a mathematical perspective. But, suppose $\theta$ can actually have the same value with opposite signs; then the last two equations imply some physical meaning. This shall be further corroborated by the following methodology. Equation (\ref{two}) can also be written as (with, $A_k^k = \Gamma_{k0}^l A_l$)

$$A_k \theta + A_0^k = 0$$

Differentiating with respect to $x^0$ and using the above equation again we have

$$A_k \dot{\theta} - \frac{A_{k0} A_k^k A_0^k}{A^2} + A_{00}^k = 0$$ \hspace{1cm} (8)

where, $A^2 = A^k A_k$. Again, Raychaudhuri’s equation is as follows:

$$\dot{\theta} = \frac{\theta^2}{3} - 2\sigma^2 + 2\omega^2 - R_s$$ \hspace{1cm} (9)

where, the symbols have their usual meanings. Thus, from equations (\ref{eight}) and (\ref{nine}) we have

$$\frac{A^2 \theta^2}{3} + A^2 (2\omega^2 - \sigma^2 + R_s) + A_{k0} A^k \theta + A^k A_0^k = 0$$

Writing $\epsilon = 2\omega^2 - 2\sigma^2 - R_s$, and solving the quadratic equation we finally derive

$$\theta = \frac{3}{2} \left[ - \frac{A_{k0} A^k}{A^2} \pm \sqrt{\left( \frac{A_{k0} A^k (A_{k0} A^k)}{A^4} - \frac{4}{3} \left( \frac{A_{k0} A^k}{A^2} + \epsilon \right) \right)} \right]$$ \hspace{1cm} (10)
Here, we can derive some interesting conclusions. When the discriminant of the above solution is zero, we would have another quadratic equation of the form:

\[ p^2 - \frac{4}{3}p - \frac{4}{3} \epsilon = 0 \]

where, \( p = \frac{A_{k,0}A^k}{A^2} \). The solution is of the form

\[ p = \frac{2}{3} \pm \frac{2}{3} \sqrt{1 - 3\epsilon} \]

which implies that \( \epsilon \geq \frac{1}{3} \). On the other hand, if we have

\[ \frac{(A_{k,0}A^k)(A_{k,0}A^k)}{A^4} = \frac{4}{3} \left\{ \frac{A_{k,0}A^k}{A^2} + \epsilon \right\} \]

then, in such a scenario

\[ \theta = \frac{3 A_{k,0}A^k}{2 A^2} \]

This may be considered the initial value of \( \theta \) during the birth of the universe when all geodesics were focused at a particular point and then after the big bang as the universe began to evolve the value of \( \theta \) evolved - accumulating the discriminant term. Again, if the Raychaudhri scalar is such that

\[ R_s > \frac{A_{k,0}A^k}{A^2} + 2\omega^2 - 2\sigma^2 \]

then

\[ \sqrt{\frac{(A_{k,0}A^k)(A_{k,0}A^k)}{A^4} - \frac{4}{3} \left\{ \frac{A_{k,0}A^k}{A^2} + \epsilon \right\}} > \frac{A_{k,0}A^k}{A^2} \]

So, \( \theta \) can have positive values as well. Therefore, writing

\[ \epsilon = -\frac{A_{k,0}A^k}{A^2} \pm \sqrt{\frac{(A_{k,0}A^k)(A_{k,0}A^k)}{A^4} - \frac{4}{3} \left\{ \frac{A_{k,0}A^k}{A^2} + \epsilon \right\}} \]

we would have

\[ \theta = \pm \frac{3}{2} \epsilon \quad (11) \]

This is what insinuates from equation (7) too. Hence, essentially, \( \dot{\theta} \) can diverge to both positive and negative infinity. The physical significance of this conclusion is novel and
significant - geodesics can diverge just as they can converge. As a consequence, there will exist a point of exclusivity akin to a point of singularity. This point of exclusivity can be looked upon as the origin of all matter and energy and the formation of spacetime as we know it. We understand immediately how important the Raychaudhuri scalar and expansion scalar are.

However, it should be borne in the mind that the notion and methodology innovated here is in stark contrast with the theory of Big Rip [7], particularly for the fact that no speculative, mysterious energy has been talked about - here it is essentially the geometry trying to explain the evolution of the universe.

3 The evolution equation

Now, we shall substantiate the novel result of the preceding section taking a different route. At first, let us consider a covariant vector $B_i$ whose transport between two different points of a Riemannian manifold is independent of the path and that is not covariantly constant; then it is known that the derivative of this vector field would be given as [18]

$$\frac{\partial B_i}{\partial x^k} = \Gamma^l_{ik} B_l$$

(12)

We assume that $B_i$ is thrice differentiable. Thus we have the following from relation (12)

$$\frac{\partial^2 B_i}{\partial x^n \partial x^k} = \frac{\partial \Gamma^l_{ik}}{\partial x^n} B_l + \Gamma^l_{ik} \frac{\partial B_l}{\partial x^n}$$

(13)

and

$$\frac{\partial^3 B_i}{\partial x^m \partial x^n \partial x^k} = B_{i,mnk} = \frac{\partial^2 \Gamma^l_{ik}}{\partial x^m \partial x^n} B_l + \Gamma^l_{ik} \frac{\partial^2 B_l}{\partial x^m \partial x^n} + \frac{\partial \Gamma^l_{ik}}{\partial x^m} \frac{\partial B_l}{\partial x^n} + \frac{\partial \Gamma^l_{ik}}{\partial x^n} \frac{\partial B_l}{\partial x^m}$$

(14)

Now, computing $B_{i,mnk}$, subtracting the resultant equation from (14) and then changing the indices as $i \rightarrow j$, we have the equation

$$B_{i,mnk} - B_{i,mkn} = \Gamma^j_{ik,mn} B_j - \Gamma^j_{in,mk} B_j + \Gamma^j_{ik} B_{j,mn} - \Gamma^j_{in} B_{j,mk} + \theta$$

(15)

where, $\theta = \Gamma^j_{ik,n} B_{j,m} + \Gamma^j_{ik,m} B_{j,n} - \Gamma^j_{in,k} B_{j,m} - \Gamma^j_{in,m} B_{j,k}$ and the 'comma' implies normal derivative. Now, we know the following relations regarding the Riemann-Christoffel (RC) tensor

$$B_{i,km} - B_{i,mk} = -R^m_{k\,\cdot\,m} B_m$$

and

$$B_{i,km} - B_{i,km} = -R^m_{k\,\cdot\,m} B_m$$
where, the ‘semi-colon’ implies covariant derivative. Therefore, we can write the following from equation (15):

\[ \partial_m [B_{i,nk} - B_{i,kn}] = \partial_m [2R^j_{i \, kn} B_j + \eta B_i] = \Gamma^j_{ik,mn} B_j - \Gamma^j_{in,mk} B_j \Gamma^i_{ik,mn} - \Gamma^j_{in} B_{j,mk} + \theta \]

where, \( \eta B_i = B_{i,kn} - B_{i,nk} \) (\( \eta \) being a covariant derivative operator). This yields the premature evolution equation for the RC tensor or the curvature tensor as

\[ 2\partial_m [R^j_{i \, kn}] B_j + 2R^j_{i \, kn} B_{j,m} + [\Gamma^j_{in,mk} - \Gamma^j_{ik,mn}] B_j + \Gamma^j_{in} B_{j,mk} - \Gamma^j_{in} B_{j,mn} + \partial_m (\eta B_i) - \theta = 0 \] (16)

It is worth noting that the vector fields \( B_i \) can be related to all the necessary features of the manifold and do not depend explicitly on the Christoffel symbols, in this regard. Now, with the differential equation

\[ \frac{\partial B_i}{\partial x^m} = \Gamma^m_{in} B_m \]

we shall have equation (13) as

\[ B_{i,kn} = \Gamma^m_{in,k} B_m + \Gamma^m_{in} \Gamma^p_{mk} B_p \]

Differentiating this again and rearranging we get

\[ \Gamma^m_{in,jk} B_m = B_{i,jkn} - \Gamma^m_{in,k} \Gamma^r_{mj} B_r - \Gamma^m_{in} \Gamma^p_{mk,j} B_p - \Gamma^m_{in,j} \Gamma^p_{mk} B_p - \Gamma^m_{in} \Gamma^p_{mk,j} B_p + \Gamma^m_{in} \Gamma^p_{mk,j} B_p \]

Now, interchanging the indices \( m \) and \( j \) (\( m \leftrightarrow j \)) we have

\[ \Gamma^j_{in,mk} B_j = B_{i,mkn} - \Gamma^j_{in,k} \Gamma^r_{jm} B_r - \Gamma^j_{in} \Gamma^p_{jm,k} B_p - \Gamma^j_{in,m} \Gamma^p_{jm} B_p - \Gamma^j_{in} \Gamma^p_{jm,k} B_p + \Gamma^j_{in} \Gamma^p_{jm,k} B_p \] (17)

Similarly, we would have

\[ \Gamma^j_{ik,mn} B_j = B_{i,mnk} - \Gamma^j_{ik,n} \Gamma^r_{jm} B_r - \Gamma^j_{ik,m} \Gamma^p_{jm,n} B_p - \Gamma^j_{ik,n} \Gamma^p_{jm} B_p - \Gamma^j_{ik,m} \Gamma^p_{jm,n} B_p + \Gamma^j_{ik,n} \Gamma^p_{jm} B_p \] (18)

Again, since \( \Gamma^j_{in,k} B_j = B_{i,kn} - \Gamma^j_{in} \Gamma^p_{jk} B_p \), the equations (17) and (18) can be rewritten respectively as

\[ \Gamma^j_{in,mk} B_j = B_{i,mkn} - \Gamma^j_{in} \Gamma^p_{jm,k} \Gamma^q_{pm} B_q - \delta^j_v \Gamma^r_{jm} \{B_{i,kn} - \Gamma^j_{in} \Gamma^p_{jm} B_p\} \\
\quad - \Gamma^j_{in} \{B_{i,mk} - \Gamma^p_{jm} \Gamma^q_{pm} B_q\} - \delta^j_p \Gamma^p_{jk} \{B_{i,mn} - \Gamma^j_{in} \Gamma^p_{jm} B_p\} \] (19)

and

\[ \Gamma^j_{ik,mn} B_j = B_{i,mnk} - \Gamma^j_{ik} \Gamma^p_{jn} \Gamma^q_{pm} B_q - \delta^j_v \Gamma^r_{jm} \{B_{i,nk} - \Gamma^j_{ik} \Gamma^p_{jn} B_p\} \\
\quad - \Gamma^j_{ik} \{B_{j,mn} - \Gamma^p_{jn} \Gamma^q_{pm} B_q\} - \delta^j_p \Gamma^p_{jn} \{B_{j,mk} - \Gamma^j_{ik} \Gamma^p_{jn} B_p\} \] (20)

6
Now, subtracting equation (20) from equation (19) and rearranging, we have

\[ B_j[\Gamma_{in,mk}^j - \Gamma_{ik,mn}^j] = (B_{i,mnk} - B_{i,mnk}) + \Gamma_{jm}(B_{i,nk} - B_{i,kn}) + \Gamma_{ik}B_{j,mn} - \Gamma_{in}B_{j,mk} \]

\[ + \delta_p^j(\Gamma_{jn}^pB_{i,mk} - \Gamma_{jk}^pB_{i,mn}) + 2\delta_p^jB_{j,m}(\Gamma_{jn}^p - \Gamma_{ik}^p) \]  

(21)

Let us consider the second term with parenthesis on the right hand side of the last equation. Multiplying by \( B^2B_j = B^iB_jB_j = B^jB_j^2 \) we shall have

\[ B^2B_j\Gamma_{jm}(B_{i,nk} - B_{i,kn}) = B^jB_jB_j\Gamma_{jm}(B_{i,nk} - B_{i,kn}) = B^2B_{j,m}(B_{i,nk} - B_{i,kn}) \]

Clearly, there is a breakdown of index notation, pertinent to the index 'j'. We shall elaborate this scenario now and we shall find the above equation to be useful subsequently. We make an ansatz that in this special scenario, the first term in the parenthesis in equation (21), namely \((B_{i,mnk} - B_{i,mnk})\), becomes explicitly independent of the index j present in the left hand side. The rationale can be attributed to when some structure preserving endomorphism that preserves the geometry of the manifold breaks down and thereby the RC tensor accrues a new upper index and the index notation breaks down.

The feasibility of the rationale introduced above will be evident later while elucidating equation (30). So, essentially, the RC tensor arising from the breakdown will be independent of the index j. This causes the breakdown of the 'index notation' mentioned earlier. Therefore, considering this ansatz we may write

\[ B_{i,mnk} - B_{i,mnk} = -\partial_m[2R_{i,\kappa\lambda}^\rho B_\rho + \eta B_i] = -2\partial_m[R_{i,\kappa\lambda}^\rho B_\rho - 2R_{i,\kappa\lambda}^\rho B_{\rho,m} - \partial_m(\eta B_i) \]

where we have considered a new index - \( \rho \), such that \( j \neq \rho \). Thus, using (21) and multiplying both sides of equation (16) by \( B^2 = B^iB_j = \delta^i_jB_jB_j \) we derive

\[ 2B^jB^2_j[\partial_m\{R_{i,\kappa\lambda}^\rho\}B_j - \partial_m\{R_{i,\kappa\lambda}^\rho\}B_\rho] + 2B^2B_j\{\{R_{i,\kappa\lambda}^\rho\}B_j,m - \{R_{i,\kappa\lambda}^\rho\}B_{\rho,m}\} \]

\[ + B^2B_{j,m}(B_{i,nk} - B_{i,kn}) + \delta^j_pB^j\theta + \Gamma_{jn}^\rho - \Gamma_{ik}^\rho \]

\[ + 2\delta^j_pB^jB^2_jB_{j,m}(\Gamma_{jn}^\rho - \Gamma_{ik}^\rho) - B^jB^2_j\theta = 0 \]  

(22)

Now, we shall use the relation \( \Gamma_{jn}^\rho_{in,k}B_j = B_{i,\kappa\lambda}^\rho - \Gamma_{jn}^\rho_{ik\lambda}B_\rho \), and compute \( \theta \) as follows:

\[ B^2_j\theta = B_jB_{j,m}(B_{i,nk} - \Gamma_{jn}^\rho_{ik\lambda}B_\rho) + B_jB_{j,n}(B_{i,km} - \Gamma_{jn}^\rho_{ik\lambda}B_\rho) \]

\[ - B_jB_{j,m}(B_{i,nk} - \Gamma_{jn}^\rho_{ik\lambda}B_\rho) - B_jB_{j,k}(B_{i,nm} - \Gamma_{jn}^\rho_{ik\lambda}B_\rho) \]  

(23)

from which we obtain

\[ B^2_j\theta = B_{j,m}(B_{i,nk} - B_{i,k}B_{j,n}) + B_{j,n}(B_{i,km} - B_{i,m}B_{j,k}) \]

\[ - B_{j,m}(B_{j,k}B_{i,n} - B_{i,n}B_{j,k}) - B_{j,k}(B_{j,n}B_{i,m} - B_{i,m}B_{j,n}) \]  

(24)

Rearranging the terms we have

\[ B^2_j\theta = B_{j,m}(B_{i,nk} - B_{i,k}B_{j,n}) + B_{j,n}(B_{i,km} - B_{i,k}B_{j,n}) + B_{j,m}(B_{i,n}B_{j,k} - B_{i,k}B_{j,n}) \]
Also, we have

\[ B^j B_j \delta_p^j B_j (\Gamma^p_{jk} B_{i,mk} - \Gamma^p_{jk} B_{i,mn}) = B^2 (B^j B_{i,mk} - B_{j,k} B_{i,mn}) \]

and

\[ B^j \delta_p^j B^j B_{j,m} (\Gamma^p_{jn} \Gamma^p_{jk} - \Gamma^p_{ik} \Gamma^p_{jn}) = B^j B_{j,m} (B_{i,n} B_{j,k} - B_{i,k} B_{j,n}) \]

Therefore, the revised form of equation \((22)\) is given as

\[
2B^2 B_j [\partial_m \{ R^j_{i,kn} \} B_j - \partial_m \{ R^p_{i,kn} \} B_p] + 2B^2 B_j [\{ R^j_{i, kn} \} B_{j,m} - \{ R^p_{i, kn} \} B_{p,m}]
+ B^2 B_j (B_{j,n} B_{i,mk} - B_{j,k} B_{i,mn}) + 2B^2 B_j (B_{i,n} B_{j,k} - B_{i,k} B_{j,n})
- B^2 B_j (B_{j,n} B_{i,km} - B_{j,k} B_{i,mn}) - B^2 B_{j,m} (B_{i,n} B_{j,k} - B_{i,k} B_{j,n}) = 0
\]

Again, as we have seen before

\[ (B_{i,n} B_{i,n} - B_{i,k} B_{i,k}) = 0 \]

where, \( \eta B_j = B_{j:k} - B_{j,n} \). And, we know the relation for the contravariant vector as

\[ B_{i,ik}^n - B_{i,ki}^n = R_{i,kl}^n B_l \]

Now since, the metric tensor is invariant with respect to the covariant derivative, lowering the index of the vector we obtain

\[ B_{l;ik} - B_{l;ki} = R_{i,kl}^n B_n \]

Thus

\[ (B_{i,n} B_{i,n} - B_{i,k} B_{i,k}) = 2R_{i, kn}^j B_j + R_{i, ij}^q B_q \]

Using this relation we finally derive the curvature evolution equation as follows:

\[
2B^2 B_j [\partial_m \{ R^j_{i,kn} \} B_j - \partial_m \{ R^p_{i,kn} \} B_p] + 2B^2 B_j [\{ R^j_{i, kn} \} B_{j,m} - \{ R^p_{i, kn} \} B_{p,m}]
+ B^2 B_j B_{j,n} (2R_{i,km}^j B_j + R_{k,ij}^q B_q) + B^2 B_j B_{j,k} (2R_{i,mn}^j B_j + R_{m,ij}^q B_q)
+ B^2 B_{j,m} (B_{i,n} B_{j,k} - B_{i,k} B_{j,n}) = 0
\]

The mathematical significance is immediately apparent. The physical significance will be manifest in the subsequent parts of the paper. Let us consider the special case where \( k = n \).
In such a scenario, the preceding equation reduces to
\[ 2B^2 B_j \partial_m \{ R^i_j \} B_j - \partial_m \{ R^p_i \} B_p \] + \[ 2B^2 B_j \{ R^i_j \} B_{j,m} - \{ R^p_i \} B_{p,m} \] + \[ B^2 B_j B_{j,n} B_q (R^q_{n,ij} + R^q_{m,ij}) = 0 \] \quad (27)

Again, \( R^p_i = \delta^p_j R^i_j \) and \( B_p = \delta^j_p B_j \). Also, taking into consideration another special case: \( n = q = m \), we have
\[ 2B^2 B^2 \partial_m R^i_j [1 - \delta^p_j \delta^j_p] + 2B^2 B_j B_{j,m} R^i_j [1 - \delta^p_j \delta^j_p] + B^2 B_j B_{j,m} B_{j,n} R_{ij} = 0 \] \quad (28)
which is another form of the evolution equation (26), with \( m = n = q = k \). Now, if the ansatz breaks down and \( j = p \) then we have from (28)
\[ \delta_{j,m} \delta_{ij} B_{j,m} B^2 R_{ij} B^i B^j = 0 \]

where, \( R_s = R_{ij} B^i B^j \) is the Raychaudhuri scalar. So
\[ B_{j,m} B^2 R_s = \partial_m (B^2 R_s) - B^2 \partial_m R_s = 0 \]
\[ \Rightarrow B^2 = \text{const.} \]

Assuming that this constant term doesn’t change the structural form and properties of \( R_s \), we can write without loss of any generality
\[ B^2 R_s \sim R_s \] \quad (29)
which can be looked upon as an endomorphism of the set of Raychaudhuri scalar, in this particular cosmology (with respect to the index \( j \)), given as
\[ B^2 : R_s \mapsto R_s \] \quad (30)

which implies
\[ B^2 \circ R_s = R_s \]

So, when the ansatz breaks down it corresponds to the consideration of an endomorphism. On the other hand, we assumed earlier that when an endomorphism breaks down the ansatz will hold. This substantiates the plausibility of why we had associated the breakdown of an endomorphism with the ansatz we introduced. Essentially, these two can be considered to be correlated and complementary.

Another interpretation with regard to the endomorphism is that \( B_j \) is a timelike unit vector field with the respect to the index \( j \), which insinuates that the Raychaudhuri scalar precludes all vector fields the self scalar products of which are not constant.
Now, let us get back to equation \((26)\). Using the Kronecker delta it can be rewritten as

\[
2B^2 B^2_j \partial_m \{ R^i_{i,kn} \}[1 - \delta^i_j \delta^j_p] + 2B^2 B_j B_{j,m} \{ R^i_{i,kn} \}[1 - \delta^i_j \delta^j_p]
+ B^2 B_j B_{j,n}(2R^i_{i,kn} B_j + R^i_{kij} B_q) + B^2 B_j B_{j,k}(2R^i_{m,n} B_j + R^i_{m,ij} B_q)
+ B^2 B_{j,m}(B_{i,n} B_{j,k} - B_{i,k} B_{j,n}) = 0
\]

(31)

Again, we know that the Brouwer’s fixed point theorem \([19]\) states: For any continuous function \(f\) mapping a compact convex set to itself there is a fixed point. Therefore, considering a continuous mapping \(f\) in our Riemannian manifold and a compact, geodesically convex vector field \((B)\) comprised of timelike vectors \(B_i\), we shall have

\[
f : B \mapsto B
\]

and a fixed point under this automorphism. Incidentally, choosing the index \(j\) we can infer that there is a fixed point with respect to this index in the vector field (or the geodesic field), through which the family of vectors \(B_j\) are parallel transported. Consequently, the vectors \(B_j\) will be constant irrespective of the coordinates and the geometric structure of the manifold. Thus, equation \((31)\) becomes

\[
2B^2 B^2_j \partial_m \{ R^i_{i,kn} \}[1 - \delta^i_j \delta^j_p] = 0
\]

\[
R^i_{i,kn} = \text{const.}
\]

(32)

Hence, the curvature tensor returns to its initial state with respect to the index \(j\). This corresponds to the statement of Poincare’s recurrence theorem \([20]\), if one considers the whole manifold to be a system.

Now, let us consider equation \((28)\). Writing, \(1 - \delta^i_j \delta^j_p = \delta\), we have

\[
2B^2 B^2_j \delta^{ijk} \partial_m R_{ik} + 2B^2 B_j B_{j,m} \delta^{ijk} R_{ik} + B^2 B_j B_{m} B_{j,m} \delta^k_j R_{ik} = 0
\]

or,

\[
2B^2 B^2_j \delta^{ijk} \partial_m R_{ik} + 2B^j B_{j,m} \delta_{ji} R_{ik} B^i B^k + B^2 B_{j,m} \delta^k_j \delta_{mk} R_{ik} B^i B^k = 0
\]

Since, we have considered the case where \(j \neq p\), \(\delta = 1\). So, using the expression for the Raychaudhuri scalar \((R_s)\) and after rearranging we have finally

\[
B_{j,m} R_s = -\rho B^2 B^2_j R_{ik,m}
\]

(33)

where, \(\rho = 2\delta^{jk}(2\delta_{ji} B_j^{-1} + \delta^k_j \delta_{mk})^{-1} = 2\delta^{jk}(2\delta_{ji} B_j^{-1} + \delta_{mi})^{-1}\). Thus, we have obtained the result that under certain conditions the Raychaudhuri scalar depends on the derivative of the Ricci tensor \((R_{ik})\), the associated timelike vector field \((B_j)\) and its derivatives.
Since, it is known that $R_s$ is the trace of the tidal tensor epitomizing the relative acceleration due to gravity of two objects separated by an infinitesimal distance and that $R_{ik}$ measures the change in geometry as an object moves along geodesics in the space, we can conclude: For some particular timelike vector field in a Riemannian or pseudo-Riemannian manifold, the relative acceleration due to gravity decreases with the increase in curvature and vice versa.

4 Sectional curvature

In this section, we analyze and discuss the results of the preceding section. Firstly, let us consider the parameter $\rho$ of equation (33). Now, for $j = k \neq i$ and $m = i$ we have

$$B_{j,i}R_s = -2B^2 B_j^2 R_{ij,i}$$

(34)

where, $R_s = R_{ij} B^i B^j$. On the other hand, for $i = j = k, i \neq m$ we have with $R_s = R_{jj} B^j B^j$. Hence, we shall obtain

$$B_{j,m}R_s = -B^2 B_j^3 R_{jj,m}$$

(35)

Now, equation (34) can also be written as

$$B_{j,i}R_s = -B^2 B_j^2 [R_{ij,i} + R_{ij,i}]$$

(36)

Now, if we take into consideration the automorphism introduced in the previous section, then

$$B_{j,i} = 0$$

Consequently, we have

$$R_{ij,i} + R_{ij,i} = 0$$

(37)

The general implication of this equation is that the Ricci curvature, $R_{ij}$, is constant with respect to the index $i$. However, there can be another implication that the first term gives a positive value and the second gives a negative value - a notion that might seem erroneous, but can be used as an alternative explanation. To be precise

$$|R_{ij,i}| = \pm R_{ij,i}$$

(38)

which essentially insinuates that

$$|R_{ij}| = \pm R_{ij}$$

(39)

i.e. the Ricci curvature can have both positive and negative values. Again, since $R_{ij}$ is obtained by contracting the RC tensor which in turn is related to the sectional curvature of
a Riemannian manifold (with respect to the given manifold and two linearly independent tangent vectors at the same point), we can conclude that the sectional curvature also will have both positive and negative values. This bespeaks for both geodesic convergence and divergence on account of Rauch’s comparison theorem [21][22] that states: for positive sectional curvature, geodesics tend to converge and for negative sectional curvature, geodesics tend to diverge.

Essentially, under special circumstances, geodesics can diverge. Geodesic convergence leads to a singularity; similarly geodesic divergence would lead to a point of exclusivity as shown previously by resorting to Raychaudhuri’s equation.

It is interesting to point out that during the inflationary era, curvature played a significant role pertinent to the dynamics, as researchers have found out [23].

5 Discussions

In the present article, we have established that an equation epitomizing the evolution of curvature, resorting to an ansatz originating from the breakdown of an endomorphism correlated to the Raychaudhuri scalar. It is also shown that the ansatz and the endomorphism are interrelated in the sense that one precludes the other.

From the aforementioned considerations we have also found that the Riemann-Christoffel curvature tensor tends to follow Poincare’s recurrence theorem and thereby the curvature returns to its initial value after certain period of time.

Another interesting consequence is that the existence and feasibility of negative curvature which entails geodesic divergence. This is also validated by using the expansion scalar which is show to have both positive and negative values. This negative value and that of the Riemann-Christoffel tensor indicates that there might exist a point of exclusivity which is the opposite of the point of singularity - a result that has been derived from the Raychaudhuri equation too.

Ostensibly, the notion of such a point is in a sense a speculative extrapolation and demands ample amount of study and research. But, the prospect is something worth investigating, at the very least.

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