A VARIATIONAL APPROACH TO STRONGLY DAMPED WAVE EQUATIONS

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Abstract. We discuss a Hilbert space method that allows to prove analytical well-posedness of a class of linear strongly damped wave equations. The main technical tool is a perturbation lemma for sesquilinear forms, which seems to be new. In most common linear cases we can furthermore apply a recent result due to Crouzeix–Haase, thus extending several known results and obtaining optimal analyticity angle.

1. Introduction

Of concern of this note are complete second order abstract Cauchy problems of the form

\[
\begin{aligned}
\ddot{u}(t) + Au(t) + B\dot{u}(t) &= 0, \quad t \geq 0, \\
u(0) &= u_{10}, \\
\dot{u}(0) &= u_{20},
\end{aligned}
\]

where the elastic operator \(A\) is in the literature usually assumed to be a self-adjoint, strictly positive definite operator on a Hilbert space \(H\). It is known that such elastic systems exhibit good properties whenever \(B\) is a multiplication operator: e.g., they are forward as well as backward solvable, they admit energy decay estimates if \(B\) is dissipative, or else blow-up estimates if \(B\) is accretive, see e.g. [21, 22, 25] and references therein.

It is interesting to note that, in particular, the standard model of an electrical transmission line by means of the telegraph equation fits this framework, the case of \(B\) negative multiplication operator corresponding to \textit{viscous} damping.

In [8], Chen–Russell proposed a family of different, strongly (or \textit{structural}) damping effects: theoretical arguments and empirical studies motivated them to consider damping operators that are unbounded on \(H\), cf. references in [8]. For the sake of simplicity, they mostly investigated the special cases of \(B = A\) and \(B = 2\rho A^{1/2}\). However, they also pointed out that the crucial property is the so-called \textit{frequency response} estimate

\[
||\lambda R(i\lambda, A)|| \leq M, \quad \lambda \in \mathbb{R},
\]

satisfied by the resolvent operator of \(A\), where

\[
A := \begin{pmatrix} 0 & -I \\ A & B \end{pmatrix}
\]

is the reduction matrix associated with (1.1). Thus, following Chen–Russell the issue becomes to find conditions on \(A, B\) ensuring that \(A\) (or rather its closure) generates an analytic semigroup in the candidate phase space \(H := D(A^{\frac{1}{2}}) \times H\).

Ever since, several authors including Dautray–Lions, Chen–Triggiani, Xiao–Liang, and Chill–Srivastava have further investigated these kind of parabolic systems, significantly extending the results of Chen–Russell. Chen–Triggiani still imposed the assumption that the damping effect is at most as strong as the

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elastic one, i.e., that

$$B = \rho A^\alpha, \quad \text{for } \alpha \in [0, 1] \text{ and } \rho \in (0, \infty),$$

and then showed, by methods based on spectral analysis, that the semigroup generated by the closure (of a suitable part) of $-A$ is analytic if and only if $\alpha \in [\frac{1}{2}, 1]$, cf. [8, Thm. 1.1]. Successively, Xiao–Liang have proved similar results in the slightly more general case where $B = f(A)$ for a suitable class of functions $f$, cf. [32, Thm. 6.4.2]. Similar, less sharp results have also been obtained in [15, § 6.3] by a technique based on the theory of operator matrices. We observe that strongly damped wave equations are also of interest in the framework of control theory, see e.g. [14, § XVIII.6], and references therein. Energy decay estimates have also been extensively investigated, see e.g. [20, 5].

More recently, Chill–Srivastava have discussed $L^p$-maximal regularity properties for the solution to

$$\begin{cases}
\ddot{u}(t) + Au(t) + B\dot{u}(t) = f(t), & t \in [0, T], \\
u(0) = 0, & \dot{u}(0) = 0.
\end{cases}$$

While they are not directly interested in the analyticity of the semigroup generated by $-A$, their results in some sense extend those of [32]: if (1.3) holds and $A$ is a sectorial operator on an $L^q$-space, $q \in (1, \infty)$, and under further technical assumptions, it turns out that (1.4) has maximal $L^p$-regularity if $\alpha \in (\frac{1}{2}, 1]$, cf. [11, Thm. 4.1]. Observe that, in particular, if (1.4) has $L^p$-maximal regularity, then a differentiable semigroup governs (1.1) on a certain phase space, cf. [11, Cor. 2.5].

The case of $D(B) \subset D(A)$ has been treated less frequently, see e.g. [31, 20]; moreover, most authors have not discussed analyticity properties. In [27, Thm. 6.2], we have showed that if $B$ generates a cosine operator function with phase space $V \times H$, and if $A$ is bounded from $V$ to $H$, then (1.1) is governed by an analytic semigroup of angle $\frac{\pi}{2}$. In many relevant cases this amounts to saying that $A = \rho B^\alpha$, for $\alpha \leq \frac{1}{2}$ and $\rho \in \mathbb{C}$.

Aim of this paper is to discuss (1.1) under assumptions on $B$ that complement, or perhaps interpolate, those of the above mentioned papers. In fact, we will assume $B$ to be at least as unbounded as $A$. The quoted results suggest that $\alpha = \frac{1}{2}$ is a critical exponent, whenever (1.3) holds. In fact, we will show that the exponent $\alpha = 1$ is critical, too. More precisely if $\alpha = 1$, then the leading term in (1.1) is not $A$ anymore, but $B$. In fact, we show that (1.1) is governed by an analytic semigroup under quite weak boundedness assumptions on $A$, whenever $B$ is associated with a closed, $H$-elliptic form. In particular, we show that no closedness or spectral conditions on $A$ are necessary. Our method is based on the introduction of a suitable weak formulation of (1.1), and then on the application of the theory of sesquilinear forms on complex Hilbert spaces. We refer to [30, 1] for comprehensive treatments of this mature theory that goes back to Kato and Lions, and to [14] for a similar, slightly less general approach to damped wave equations due to Dautray–Lions.

In Section 2 we introduce our general framework and show a first well-posedness result for (1.1). To this aim we prove a perturbation lemma for sesquilinear forms that may be of independent interest. We also obtain a first estimate on the angle of analyticity. In Section 3 we impose slightly stronger conditions and, by means of a recent result due to Crouzeix–Haase, we find sufficient condition in order that the semigroup is analytic of angle $\frac{\pi}{2}$: this includes the relevant case of self-adjoint damping operator $B$. Some applications to semilinear problems are also considered. Finally, in Section 4 we briefly discuss how our theory can be adapted in order to discuss

It is fair to add that a variational approach to linear damped wave equations has also been pursued in [14, § XVIII.5.1], see also [13, § XVIII.6]. Indeed, Dautray–Lions’ methods are quite similar to those presented in Section 2 below, and they also consider the neutral equation $Cu(t) + Au(t) + B\dot{u}(t) = 0, t \geq 0$, even in the nonautonomous case, where $D(B) \subset D(A)$. Though, the assumptions in [14, § XVIII.5.1] are restricted to the case of $A, B$ differential operators whose principal part is self-adjoint and (in the case of
any interpolation space between $V$ well-known perturbation result for operators due to Desch–Schappacher. In the following we denote by
\begin{equation}
\| (2.1) \parallel
\end{equation}
and likewise for the operator associated with $b$.

Lemma 2.1. Let $a : V \times V \rightarrow \mathbb{C}$, $b : V \times V \rightarrow \mathbb{C}$ be sesquilinear forms.\footnote{One can often think of sesquilinear forms in terms of physical quantities. In fact, if $a, b$ are symmetric, $2a(u, u), 2b(u, u)$ merely represent the energy functionals associated with the elastic and damping operators $A, B$ that appear in \cite{11}, respectively.}

More precisely, we recall that the operator associated with $a$ is by definition given by
\begin{align*}
D(A) &= \{ f \in V : \exists h \in H \text{ s.t. } a(f, g) = (h \mid g)_H \forall g \in V \}, \\
Af &= h,
\end{align*}
and likewise for the operator associated with $b$.

The following perturbation lemma seems to be of independent interest. It is the form equivalent of a well-known perturbation result for operators due to Desch–Schappacher. In the following we denote by $H_\alpha$ any interpolation space between $V$ and $H$ that verifies the interpolation inequality
\begin{equation}
\| f \|_{H_\alpha} \leq M_\alpha \| f \|_V \| f \|_{H_\alpha}^{1-\alpha}, \quad f \in V.
\end{equation}

Lemma 2.1. Let $a : V \times V \rightarrow \mathbb{C}$ be a sesquilinear mapping. Let $\alpha \in [0, 1)$ such that $a_1 : V \times H_\alpha \rightarrow \mathbb{C}$ and $a_2 : H_\alpha \times V \rightarrow \mathbb{C}$ be continuous sesquilinear mappings. Then $a$ is $H$-elliptic if and only if $a + a_1 + a_2 : V \times V \rightarrow \mathbb{C}$ is $H$-elliptic.

Proof. Let $a$ be $H$-elliptic and let
\begin{equation}
|a_1(f, g)| \leq M \| f \|_V \| g \|_{H_\alpha} \quad \text{and} \quad |a_2(g, f)| \leq M \| g \|_V \| f \|_{H_\alpha},
\end{equation}
for some constant $M > 0$ and for all $f \in V, g \in H_\alpha$, so that by (2.1) we can estimate both $|a_1(f, g)|$ and $|a_2(g, f)|$ by $M M_\alpha \| f \|_V \| f \|_{H_\alpha}^{1-\alpha}$.

By Young’s inequality one has for all $\alpha \in [0, 1)$ and all $x, y > 0$
\begin{equation}
xy \leq \frac{1 + \alpha}{2} x^{\frac{1}{1+\alpha}} + \frac{1 - \alpha}{2} y^{\frac{1}{1-\alpha}}.
\end{equation}
Thus, for all $\epsilon > 0$ letting $x = (\sqrt{\epsilon} \| f \|_V)^{1+\alpha}$ and $y = (\frac{1}{\sqrt{\epsilon}} \| f \|_V)^{1-\alpha}$ one obtains
\begin{equation}
\| f \|_V^{1+\alpha} \| f \|_{H}^{1-\alpha} \leq \frac{1 + \alpha}{2} \| f \|_V^{\frac{1}{1+\alpha}} + \frac{1 - \alpha}{2 \epsilon} \| f \|_H^{2}, \quad f \in V.
\end{equation}
Accordingly, for all $\epsilon > 0$ there exists $M(\epsilon) > 0$ such that
\begin{equation}
-\epsilon \| f \|_V^2 + M(\epsilon) \| f \|_H^2 \leq a_1(f, f) + a_2(f, f), \quad f \in V.
\end{equation}
By assumption $a$ is $H$-elliptic, i.e., $\text{Re}(a(f, f)) \geq \alpha \| f \|_V^2 - \omega \| f \|_H^2$ for some $\alpha > 0$ and $\omega \in \mathbb{R}$. Thus, that for $\epsilon = \alpha/2$
\begin{equation}
\text{Re}(a + a_1 + a_2)(f, f) = \text{Re}(a(f, f) + a_1(f, f) + a_2(f, f)) \geq \alpha \| f \|_V^2 - \omega \| f \|_H^2 - \epsilon \| f \|_V^2 - M(\epsilon) \| f \|_H^2 \geq \frac{\alpha}{2} \| f \|_V^2 - (\omega + M(\epsilon)) \| f \|_H^2,
\end{equation}
for all $f \in V$. This completes the proof. \qed
With the aim of discussing the abstract damped wave equation \[ \text{[12]} \] we introduce \( V := V \times V \) as well as the candidate energy space \( H := V \times H \). Observe that \( V \) is continuously and densely imbedded into \( H \) and that both \( V \) and \( H \) have a canonical Hilbert space structure. Define

\[
a(u, v) := - (u_2 \mid v_1)_V + a(u_1, v_2) + b(u_2, v_2),
\]

where we have considered

\[
u = (u_1, u_2)^\top, \quad v = (v_1, v_2)^\top \in V,
\]
i.e., \( a \) is a sesquilinear form with domain \( V \). Observe that \( a \) is in general not symmetric.

**Lemma 2.2.** The following assertions hold.

1) The form \( a \) is continuous with respect to \( V \) if and only if \( a, b \) are continuous with respect to \( V \).
2) The form \( a \) is \( H \)-elliptic if and only if \( b \) is \( H \)-elliptic.
3) Let \( \text{Re} u, v = \text{Re}(u \mid v)_V \) for all \( u, v \in V \). If \( b \) is accretive, then \( a \) is accretive.
4) If \( a \) is accretive, then \( b \) is accretive.

Observe that, as a direct consequence of the sesquilinearity of \( a \), \( \text{Re} u, v = \text{Re}(u \mid v)_V \) for all \( u, v \in V \) if and only if \( \text{Re}(u, v) \geq \text{Re}(u \mid v)_V \) for all \( u, v \in V \).

**Proof.**

1) Let \( a \) be continuous. Then for some constant \( M_a > 0 \) and all \( u, v \in V \) one has

\[
|b(u, v)| = |a(u, v)| \leq M_a \|u\|_V \|v\|_V = M_a \|u\|_V \|v\|_V,
\]
where we have set \( u := (0, u)^\top \) and \( v := (0, v)^\top \). Similarly, setting \( u := (u, 0)^\top \) and \( v := (v, 0)^\top \) we obtain that

\[
|a(u, v)| = |a(u, v)| \leq M_a \|u\|_V \|v\|_V = M_a \|u\|_V \|v\|_V.
\]

Let now \( a, b \) be continuous, i.e., assume that for some \( M_a, M_b \geq 0 \) there holds

\[
|a(u, v)| \leq M_a \|u\|_V \|v\|_V, \quad u, v \in V,
\]
as well as

\[
|b(u, v)| \leq M_b \|u\|_V \|v\|_V, \quad u \in V.
\]

A tedious computation then shows that

\[
|a(u, v)|^2 \leq \|u_2\|_V^2 \|v_1\|_V^2 + M_a^2 \|u_1\|_V^2 \|v_2\|_V^2 + M_b^2 \|u_2\|_V^2 \|v_2\|_V^2 + 2M_a M_b \|u_1\|_V \|v_2\|_V \|v_1\|_V \|v_2\|_V + 2M_b \|u_2\|_V \|v_1\|_V \|v_1\|_V \|v_2\|_V + 2M_a M_b \|u_1\|_V \|v_2\|_V \|v_2\|_V
\]

\[
\leq M_a^2 (\|u_1\|_V^2 + \|u_2\|_V^2)(\|v_1\|_V^2 + \|v_2\|_V^2),
\]
i.e., \( |a(u, v)| \leq M_a \|u\|_V \|v\|_V \), where

\[
M_a^2 := \frac{M_a^2}{2} + M_a M_b + \max \{M_a^2, 1, M_b^2\}.
\]

2) To begin with, consider the form \( a_0 : V \times V \to \mathbb{C} \) defined by

\[
a_0(u, v) := b(u_2, v_2).
\]

A direct computation shows that \( a_0 \) is \( H \)-elliptic if and only if \( b \) is \( H \)-elliptic. Similarly, define the continuous sesquilinear mappings \( a_1 : H \times V \to \mathbb{C} \) and \( a_2 : V \times H \to \mathbb{C} \) by

\[
a_1(u, v) := - (u_2 \mid v_1)_V \quad \text{and} \quad a_2(u, v) := a(u_1, v_2).
\]

By Lemma 2.3 we conclude that \( a = a_0 + a_1 + a_2 \) is \( H \)-elliptic if and only if \( a_0 \) is \( H \)-elliptic if and only if \( b \) is \( H \)-elliptic.

3) If \( b \) is accretive and \( \text{Re} u, v = \text{Re}(u \mid v)_V \) for all \( u, v \in V \), then

\[
\text{Re}(u, u) = \text{Re}(b(u_2, u_2)) \geq 0, \quad u = (u_1, u_2)^\top \in V,
\]
i.e., \( \mathbf{a} \) is accretive.

4) Conversely, if \( \mathbf{a} \) is accretive, we obtain that for all \( u \in V \)

\[
\text{Re}(b(u, u)) = \text{Re} \langle u, u \rangle \geq 0,
\]

where we have set \( u := (0, u) \).

By \cite{30} Prop. 1.51 and Thm. 1.52] we can now state the following.

**Theorem 2.3.** Let \( a, b \) be continuous. Let further \( b \) be \( H \)-elliptic. Then the operator associated with \( \mathbf{a} \) is closed. It generates a \( C_0 \)-semigroup \( (e^{-\mathbf{a}t})_{t \geq 0} \) on \( H \) which is analytic of angle \( \frac{\pi}{2} - \arctan M \), where \( M_a \) is defined as in (2.3). The semigroup \( (e^{-\mathbf{a}t})_{t \geq 0} \) is contractive if \( b \) is accretive and \( \text{Re} \langle u, v \rangle = \text{Re} \langle v, u \rangle \) for all \( u, v \in V \).

We emphasize that in the above theorem we are assuming \( a \) neither to be \( H \)-elliptic, nor to be (quasi)accretive. In other words, the operator \( A \) associated with \( a \) need not be closed or (quasi)dissipative. Thus, in the limiting case of \( A \) bounded from \( D(B) \) to \( H \), where \( B \) is the operator associated with \( b \), Theorem 2.3 extends the well-posedness results of \cite{9, 32, 11}. In this sense, we say that the leading term in (1.1) is not the elastic, but rather the damping one.

**Remark 2.4.** 1) Let \( V \neq \{0\} \). The form \( \mathbf{a} \) is self-adjoint if and only if \( b \) is self-adjoint and \( \langle \cdot, \cdot \rangle = -\langle \cdot | \cdot \rangle_V \). Let in fact \( u := (u, 0) \) and \( v := (0, v) \), with \( u, v \in V \), \( v \neq 0 \). Then, one has

\[
\mathbf{a}(u, v) = a(u, v) \quad \text{and} \quad \mathbf{a}(v, u) = -\langle v, u \rangle_V.
\]

On the other hand, if \( u := (0, u) \) and \( v := (0, v) \), with \( u, v \in V \), \( v \neq 0 \neq u \), then

\[
\mathbf{a}(u, v) = b(u, v) \quad \text{and} \quad \mathbf{a}(v, u) = b(v, u).
\]

To prove the converse implication, it suffices to observe that if \( b \) is self-adjoint and \( \langle \cdot, \cdot \rangle = -\langle \cdot | \cdot \rangle_V \), then

\[
\mathbf{a}(u, v) = b(u_2, v_2) = \overline{b(v_2, u_2)} = \mathbf{a}(v, u).
\]

2) The form \( \mathbf{a} \) is not coercive, unless \( V = \{0\} \). Let in fact \( u := (u, 0) \), with \( 0 \neq u \in V \). Then one has

\[
\text{Re}(\mathbf{a}(u, u)) = 0 < \|u\|^2_V.
\]

This shows that there exists no \( \epsilon > 0 \) such that the estimate \( \|e^{-\mathbf{a}t}\| \leq e^{-\epsilon t} \) holds for all \( t \geq 0 \). This should be compared with the exponential stability result in \cite[Thm. 1.1]{31}.

3) In the relevant case of \( \dim V = \infty \) the imbedding of \( V \) in \( H \) is not compact. Thus if Theorem 2.3 applies, then \( (e^{-\mathbf{a}t})_{t \geq 0} \) is not compact.

4) An advantage of dealing with sesquilinear forms instead of operators is the flexibility of this theory. Let us briefly discuss the case of time-dependent damped wave equations. Consider families \( (a_t)_{t \in [0, T]} \) and \( (b_t)_{t \in [0, T]} \) of sesquilinear forms with joint (time-independent) dense domain \( V \). Assume them to be equicontinuous. Let furthermore the mappings \( t \mapsto a_t(u, v) \) and \( t \mapsto b_t(u, v) \) be measurable for all \( u, v \in V \). If finally \( (b_t)_{t \geq 0} \) is equi-\( H \)-elliptic, i.e.,

\[
\text{Re}(b_t(u, u)) + \omega \|u\|^2_H \geq \alpha \|u\|^2_V, \quad u \in V, \ t \geq 0,
\]

for some \( \omega \in \mathbb{R}, \alpha > 0 \), then it is easy to see that the family of sesquilinear forms \( (a_t)_{t \geq 0} \) defined by

\[
a_t(u, v) := -(u_2 | v_1)_V + a_t(u_1, v_2) + b_t(u_2, v_2),
\]

fits the framework of Lions’ theory of time-dependent forms, cf. \cite{24}, and we conclude that the nonautonomous abstract Cauchy problem associated with \( (a_t)_{t \geq 0} \) is well-posed in a suitably weak sense.
In order to interpret Theorem [23] as a well-posedness result for (1.1), we still have to determine the operator \((A, D(A))\) associated with \(a\), which by definition is

\[
D(A) := \{ u \in V : \exists z \in H \text{ s.t. } a(u, v) = (z | v)_V \text{ for all } v \in V \},
\]

\[
Au := z.
\]

In fact, the expression \("Au + Bu\) in (1.1) is in general purely formal, as the solution \(u\) to (1.1) need not satisfy \(u \in C(R_+, D(A)) \cap C^1(R_+, D(B))\). However, in our framework a direct computation shows that the following holds.

**Proposition 2.5.** The operator \(A\) on \(H\) associated with the form \(a\) is given by

\[
D(A) = \{ u \in V : \exists w \in H \text{ s.t. } b(u_1, v_1) + b(u_2, v_2) = (w | v)_H \text{ for all } v \in V \},
\]

\[
Au = (u_2, u_1)\top.
\]

In the remainder of this section we assume \(V, H\) to be function spaces over a measure space \((X, \mu)\). The following is a direct consequence of the above proposition and should be compared with the results of [10].

**Corollary 2.6.** Let \(\rho \in H\) such that \(\rho u \in V\) and \(a(u, v) = b(\rho u, v)\) for all \(u, v \in V\). Then

\[
D(A) = \{ u \in V : \exists w \in H \text{ s.t. } b(\rho u_1 + u_2, v) = (w | v)_H \text{ for all } v \in V \}
\]

\[
Au = (u_2, B(\rho u_1 + u_2))\top.
\]

where \(B\) denotes the operator associated with \(b\).

While throughout the paper we consider complex Hilbert spaces, it is of interest for applications to ensure that solutions to (1.1) are in fact real whenever the initial data are real. In the following we denote the closed convex subsets \(V_R\) and \(H_R\) defined by the real-valued functions belonging to \(V\) and \(H\), respectively.

**Proposition 2.7.** Let \(a, b\) be continuous and \(b\) be \(H\)-elliptic. Assume further that \(Reu \in V\) and moreover \(a(Reu, Imu), (Reu | Imu)_V\) \(\in R\) for all \(u \in V\). Then \((e^{-t a})_{t \geq 0}\) is real (i.e., it leaves invariant \(V_R \times H_R\)) if and only if the semigroup associated with \(b\) is real (i.e., it leaves invariant \(H_R\)).

**Proof.** Without loss of generality we can assume both \(b\) and \(a\) to be accretive, since reality of a semigroup is invariant under rescaling. Let the semigroup associated with \(b\) be real. Then by [30, Prop. 2.5] one has \(Reu \in V\) for all \(u \in V\) and \(b(Reu, Imu) \in R\). Thus, for an arbitrary \(u = (u_1, u_2)\top \in V\), one has \(Reu = (Reu_1, Reu_2)\top \in V\) and moreover

\[
a(Reu, Imu) = -(Reu_2 | Imu_1)_V + a(Reu_1, Imu_2) + b(Reu_2, Imu_2) \in R.
\]

Since the projection \(P\) of \(H\) onto \(H_R\) is given by

\[
Pu = (Reu_1, Reu_2), \quad u = (u_1, u_2)\top \in H,
\]

the claim follows by [30, Thm. 2.2]. Conversely, let \((e^{-t a})_{t \geq 0}\) be real and let \(u \in V\). Set \(u := (0, u)\top \in V\). Then, \(Reu = (0, Reu) \in V\) and \(b(Reu, Imu) = a(Reu, Imu) \in R\). \(\Box\)

3. INTERPOLATION SPACES AND NONLINEAR PROBLEMS

In Theorem [23] we have shown that if \(a, b\) are continuous and \(b\) is \(H\)-elliptic, the form \(a\) is associated with an analytic semigroup on \(H\). We can sharpen this result under the additional assumption that for some constant \(M_b > 0\)

\[
(3.1) \quad \|Imb(u, u)\| \leq M_b\|u\|_H\|u\|_V, \quad u \in V.
\]
Theorem 3.1. If (H.1) holds, then the operator $A$ associated with $a$ generates a cosine operator function on $H$. Moreover, the form domain $V$ is isometric to the fractional power domain $D(\lambda + A)^\gamma$, for $\lambda > 0$ large enough.

Proof. We first show that $|\text{Im}(a, u)| \leq M_0 |u|_H$ for some constant $M_0$ and all $u \in V$. To let this aim $u = (u_1, u_2)^T \in V$. Since $|a(u, v)| \leq M_0 |u|_V |v|_V$ for some $M_0 > 0$ and all $u, v \in V$, there holds

$$|\text{Im}(a, u)|^2 \leq (1 + M_0^2) |u_1|^2_V |u_2|^2_V + M_0^2 |u_2|^2_H |u_2|^2_V$$

$$+ 2M_0 |u_1|^2_V |u_2|^2_V + 2M_0 (1 + M_0) |u_1|_V |u_2|_H |u_2|^2_V$$

$$\leq (1 + M_a + M_b) (|u_1|^2_V + |u_2|^2_H) |u_2|^2_V$$

$$\leq (1 + M_a + M_b) |u|^2_H |u|^2_V.$$ 

This shows in particular that the numerical range of $a$ is contained in a parabola (see [18 p. 204]) and thus, applying a result due to Crouzeix [13], we promptly obtain that $A$ generates a cosine operator function on $H$.

Moreover, by Haase’s converse of Crouzeix’s theorem (see [1 § 5.6.6]) there exists an equivalent scalar product $\langle \cdot, \cdot \rangle_H$ on $H$ and $\lambda > 0$ such that the numerical range of $a(\lambda) := a + \lambda(\cdot, \cdot)_H$ lies in a parabola.

Now it follows by a result due to McIntosh (see again [1 § 5.6.6]) that $A$ has the square root property. This concludes the proof. □

The following result should be compared with [14 Thm. XVIII.5.1].

Corollary 3.2. Let $B = B_0 + B_1$, where $B_0$ is a self-adjoint and strictly positive definite operator. Assume $A$ to be bounded from $D(B_0^\frac{1}{2})$ to $D(B_0^{-\frac{1}{2}})$ and $B_1$ to be bounded from $D(B_0^\frac{1}{2})$ to $H$. Then problem (1.1) is governed by an analytic semigroup of angle $\frac{\pi}{2}$ on $D(B_0^\frac{1}{2}) \times H$.

In particular, (1.1) admits a unique mild solution for all initial data $u_{10} \in D(B_0^\frac{1}{2})$ and $u_{20} \in H$. If $A = \rho B$ for $\rho \in \mathbb{C}$, then (1.1) admits a unique classical solution for all $u_{10}, u_{20} \in D(B_0^\frac{1}{2})$ such that $\rho u_{10} + u_{20} \in D(B)$.

Proof. Let $b_0 : D(B_0) \times D(B_0) \to \mathbb{C}$ the coercive, symmetric sesquilinear form associated with $B_0$. In particular, $B_0$ has the square root property (cf. [1 § 5.5.1]) and therefore the form norm of $b_0$ is isomorphic to $D(B_0^\frac{1}{2})$. Since now for the sesquilinear form $b$ associated with $B$ holds

$$|\text{Im}(u, u)| = |\text{Im}(B_0 u | u) H + \text{Im}(B_1 u | u)| \leq \|B_1 u\|_H \|u\|_H \leq M |u|_{D(B_0)} |u|_H$$

for some constant $M > 0$, one sees that (3.1) is satisfied. After defining by $a$ the sesquilinear form associated with $A$, Theorem 3.1 can be applied. Since every cosine operator function generator also generates an analytic semigroup of angle $\frac{\pi}{2}$ (see [2 Thm. 3.14.17]), the claim holds. □

Example 3.3. For an open bounded domain $\Omega \subset \mathbb{R}^n$ with $C^2$-boundary $\partial \Omega$ consider the complete second order problem

$$\begin{aligned}
\ddot{u}(t, x) &= \nabla \cdot \left( \alpha(x) \nabla u(t, x) + \beta(x) \nabla \dot{u}(t, x) \right), & t \geq 0, \ x \in \Omega, \\
\ddot{u}(t, z) &= \ddot{u}(t, z) = 0, & t \geq 0, \ z \in \partial \Omega, \\
u(0, x) &= u_{10}(x), & x \in \Omega, \\
\dot{u}(0, x) &= u_{20}(x), & x \in \Omega,
\end{aligned}$$

where $\alpha, \beta \in C^1(\Omega)$ such that $0 < \beta(x)$ for all $x \in \Omega$. 

Let $B = -\nabla \cdot (\beta \nabla)$ and $A = -\nabla \cdot (\alpha \nabla)$ on $H := L^2(\Omega)$, both with domain $H^2(\Omega) \cap H_0^1(\Omega)$. Accordingly introduce the forms
\[
b(f, g) := \int_{\Omega} \beta(x) \nabla f(x) \nabla g(x) \quad \text{and} \quad a(f, g) := \int_{\Omega} \alpha(x) \nabla f(x) \nabla g(x).\]

Then $D(B^\sharp) = H_0^1(\Omega)$ and by Corollary 7.2 and Corollary 2.10 one concludes that the operator
\[
D(A) = \{(u_1, u_2)^T \in (H_0^1(\Omega))^2 : \alpha \nabla u_1 + \beta \nabla u_2 \in H_0^1(\Omega)\},
\]
\[
Au = (u_2, \nabla (\alpha \nabla u_1 + \beta \nabla u_2))^T.
\]
generates on $H_0^1(\Omega) \times L^2(\Omega)$ an analytic semigroup of angle $\frac{\pi}{2}$. This semigroup is contractive if $\alpha \equiv 1$ (and more generally also whenever $\alpha > 0$, up to considering weighted phase space). It yields the solutions to the above problem, which are real valued whenever $u_{10} \in H_0^1(\Omega)$ and $u_{20} \in L^2(\Omega)$ are real valued.

The analytical well-posedness of the above problem has been shown in [9] only in the case of $\alpha$ strictly positive, whereas we allow for $\alpha$ to be a complex-valued function.

We can now exploit the technique developed in [20, Chapt. 7] for semilinear parabolic problems, which heavily relies on interpolation theory. In order to avoid technicalities, we consider in the remainder of this section the special case of $A = \rho B$ for some $\rho \in \mathbb{C}$. This case is relevant in many concrete contexts, e.g., whenever investigating semilinear strongly damped equations like the Klein–Gordon one, see e.g. [19, 4, 16, 9]. As an example of a possible application, we formulate the following, which is a direct consequence of [20, Thm. 7.1.3 and 7.1.10]. More refined results, also yielding global well-posedness, can be obtained by applying further tools from [20, § 7.2].

**Corollary 3.4.** Let $B$ satisfy the assumptions of Corollary 3.2. Assume $G : [0, T] \times D(B^\sharp) \times D(B^\sharp) \to H$ to be a continuous mapping that is locally Hölder continuous with respect to the first variable and locally Lipschitz continuous with respect to the second and third ones. Then for small initial data $u_{10}, u_{20} \in D(B^\sharp)$
\[
\begin{cases}
\ddot{u}(t) + B(pu + \dot{u})(t) = G(t, u_1(t), \dot{u}_2(t)), & t \in [0, T], \\
u(0) = u_{10}, & \dot{u}(0) = u_{20},
\end{cases}
\]
has a unique classical solution, locally in time.

Theorem 3.1 also allows to apply the theory developed in [12] for quasilinear parabolic problems, where determining interpolation spaces is a crucial step, too. A prototypical result is the following, which can be compared with [11, Thm. 5.1].

**Corollary 3.5.** Let $D$ be a subspace of $H$ with $D \to V$. Let the mapping
\[
B : V \times V \to \mathcal{L} \{(u, v)^T \in V \times V : pu + v \in D\}, H
\]
be well-defined and locally Lipschitz continuous. Let $u_{10}, u_{20} \in V$ and assume the operator $B(u_{10}, u_{20})$ to satisfy the assumptions of Corollary 3.2 with $D(B(u_{10}, u_{20})^\sharp) = V$. Then for all $f \in L^2(R_+, H) \text{ and all } g \in \text{Lip}(R_+ \times V, H) \text{ there exists } \tau > 0 \text{ such that the problem}
\[
\begin{cases}
\ddot{u}(t) + B(u(t), \dot{u}(t))(pu(t) + \dot{u}(t)) = f(t) + g(t, u(t)), & t \in (0, \tau), \\
u(0) = u_{10}, & \dot{u}(0) = u_{20},
\end{cases}
\]
has a solution $u \in H^2((0, \tau), H) \cap H^1((0, \tau), V)$ with $pu + \dot{u} \in L^2((0, \tau), D)$.

4. Dynamic boundary conditions

We introduce a new Hilbert space, which we denote $H$ because in the applications we have in mind this is often a space of boundary values of functions in $H$. We also consider a bounded linear operator $L : V \to \partial H$ with dense range (in $\partial H$) and dense kernel (in $V$) and the two Hilbert spaces
\[
\mathcal{V} := \{(f_1, f_2, f_3)^T \in V \times V \times \partial H : Lf_2 = f_3\} \quad \text{and} \quad \mathcal{H} := V \times H \times \partial H.
\]
It follows from \[28\], Lemma 5.6] that \( \mathcal{V} \) is dense in \( \mathcal{H} \).

In recent years it has become increasingly clear that the right functional setting in order to discuss equations with dynamic boundary conditions is obtained by enlarging the state space of the corresponding equation with non-dynamic boundary conditions, see e.g. \[17\].

We define the sesquilinear form \( \mathfrak{a} \) with dense domain \( \mathcal{V} \) by

\[
\mathfrak{a}(u, v) := -(u_2 | v_1)_H + a(u_1, v_2) + b(u_2, v_2),
\]

i.e., \( \mathfrak{a} \) acts formally on the first two coordinates of the vectors in its domain just like \( \mathfrak{a} \). To begin with, we deduce a result analogous to Lemma \[2.2\].

**Lemma 4.1.** The following assertions hold.

1) The form \( \mathfrak{a} \) is continuous with respect to \( \mathcal{V} \) if and only if \( a, b \) are continuous with respect to \( V \).

2) The form \( \mathfrak{a} \) is \( \mathbf{H} \)-elliptic if the sesquilinear form

\[
(4.1) \quad \{(f_2, f_3)^T \in \mathbb{V} \times \partial \mathbb{H} : Lf_2 = f_3\} \ni ((f_2, Lf_2), (g_2, Lg_2)) \mapsto b(f_2, g_2) \in \mathbb{C}
\]

is \( \mathbb{H} \times \partial \mathbb{H} \)-elliptic.

3) Let \( \text{Re} (u, v) = \text{Re}(u | v)_V \) for all \( u, v \in \mathcal{V} \). If the form introduced in \( (4.1) \) is accretive, then \( \mathfrak{a} \) is accretive.

We can finally prove a generation result in this setting, too.

**Theorem 4.2.** Let \( a, b \) be continuous. Let further the form introduced in \( (4.1) \) be \( \mathbb{H} \times \partial \mathbb{H} \)-elliptic. Then the operator associated with \( \mathfrak{a} \) is closed. It generates an analytic \( C_0 \)-semigroup \( (e^{-t\mathfrak{a}})_{t \geq 0} \) on \( \mathcal{H} \). This is contractive if the form in \( (4.1) \) is accretive and \( \text{Re} (u, v) = \text{Re}(v | u)_V \) for all \( u, v \in \mathcal{V} \). If additionally \( (3.1) \) holds, then the operator associated with \( \mathfrak{a} \) generates a cosine operator function on \( \mathcal{H} \), too.

Identifying the operator associated with \( \mathfrak{a} \) is generally difficult. However, this can be accomplished in many concrete cases by a suitable application of the Gauß–Green formulae.

**Example 4.3.** Consider again the forms \( a, b \) (this time with domain \( H^1(\Omega) \)) and the operators \( A, B \) introduced in Example \[3.3\] for the sake of simplicity with \( \alpha \in \mathbb{C} \) and \( \beta \equiv 1 \). Moreover, let \( L \) be the trace operator. By Theorem \[4.2\] one concludes that the operator \( A \) defined by

\[
D(A) := \left\{ f := (\alpha f_1, f_2, Lf_2)^T \in (H^1(\Omega))^2 \times L^2(\partial \Omega) : \Delta(\alpha f_1 + f_2) \in L^2(\Omega), \partial_n(\alpha f_1 + f_2) \in L^2(\partial \Omega) \right\},
\]

\[
Af := (f_2, \Delta(\alpha f_1 + f_2), \partial_n(\alpha f_1 + f_2))^T.
\]

(\( \partial_n \) denotes the normal derivative) generates on \( H^1(\Omega) \times L^2(\Omega) \times L^2(\partial \Omega) \) a cosine operator function and hence a (contractive) analytic semigroup of angle \( \pi \). A direct computation shows that this semigroup solves the initial-boundary value problem

\[
\begin{align*}
\dot{u}(t, x) &= \Delta(\alpha u(t, x) + \dot{u}(t, x)), & t \geq 0, & x \in \Omega, \\
\dot{u}(t, z) &= \partial_n(\alpha u(t, z) + \dot{u}(t, z)), & t \geq 0, & z \in \partial \Omega, \\
u(0, x) &= u_{10}(x), & x \in \Omega, \\
\dot{u}(0, x) &= u_{20}(x), & x \in \Omega, \\
\dot{u}(0, z) &= u_{30}(z), & z \in \partial \Omega.
\end{align*}
\]

Admittedly, this well-posedness result could also have been obtained combining the results of \[3\] and Corollary \[1.2\].

**Remark 4.4.** For the sake of simplicity, we have avoided to consider an additional term describing further dynamic processes on the boundary space \( \mathbb{H} \). However, such a generalization can easily be achieved by standard perturbation theory.
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