Joint Correlation Detection and Alignment of Gaussian Databases

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Abstract

In this work, we propose an efficient two-stage algorithm solving a joint problem of correlation detection and permutation recovery between two Gaussian databases. Correlation detection is an hypothesis testing problem; under the null hypothesis, the databases are independent, and under the alternate hypothesis, they are correlated, under an unknown row permutation. We develop relatively tight bounds on the type-I and type-II error probabilities, and show that the analyzed detector performs better than a recently proposed detector, at least for some specific parameter choices. Since the proposed detector relies on a statistic, which is a sum of dependent indicator random variables, then in order to bound the type-I probability of error, we develop a novel graph-theoretic technique for bounding the $k$-th order moments of such statistics. When the databases are accepted as correlated, the algorithm also outputs an estimation for the underlying row permutation. By comparing to known converse results for this problem, we prove that the alignment error probability converges to zero under the asymptotically lowest possible correlation coefficient.

Index Terms: Correlation detection, Gaussian model, permutation recovery.

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1 Introduction

Two fundamental problems in the statistical analysis of Gaussian databases are correlation detection (or testing) and alignment (also termed recovery in the literature). Correlation detection of two databases is basically an hypothesis testing problem; under the null hypothesis, the databases are independent, and under the alternate hypothesis, there exists a permutation, for which the databases are correlated. In this task, the main objective is to attain the best trade-off between the type-I and type-II error probabilities. In the problem of database alignment, we make an assumption that the two databases are correlated, and want to estimate the underlying permutation. The objective is to minimize the probability of alignment error.

While alignment of databases with $n$ sequences, each containing $d$ Gaussian entries has been recently studied in [1] and correlation detection of such Gaussian databases has been lately explored in [2], it seems very natural to tackle the two individual problems together as a joint problem of detection and alignment. In this manuscript, we propose a simple two-stage algorithm that solves jointly the two problems of detection and alignment of correlated Gaussian databases. In his first stage, the algorithm first calculates a statistic that is based on local decisions for all $n^2$ pairs of sequences, and then takes a final decision in favor of one of the hypotheses based on this statistic. If the algorithm has accepted the alternate hypothesis (i.e., that the databases are probably correlated), then in his second stage, it outputs an estimation on the underlying permutation.

The main problem in [2] is identifying the lowest possible correlation coefficient between each consecutive entries of the databases, as a function of $n$ and $d$, such that the sum of the type-I and type-II error probabilities can be driven to zero as $n$ and $d$ grow to infinity. While the detector that was proposed in [2] can be used also for non-asymptotic correlation values, we show in Section 6 that its performance is inferior to the performance of the new proposed detector, which relies on many local decisions and a final global decision. Moreover, those local decisions that are made in the detection stage of the proposed algorithm, are in fact the basis for estimating the underlying permutation (in case the alternate hypothesis is accepted). As opposed to the proposed correlation detector, the detector in [2], which is merely based on summing up all $n^2$ inner products between the pairs of sequences, obviously cannot be used as a basis for permutation recovery. Still, regarding the problem of correlation detection alone
(when there is no need for a second stage of permutation recovery), we also suggest a third detector, which is based on a unification of the detector in [2] and the one proposed here. We conjecture that this detector is able to close the information-theoretic gap in [2], and leave its full analysis as an open problem.

Information-theoretic limits in Gaussian database alignment have been studied in [1]. It has been proved using maximum likelihood estimation that the probability of alignment error can be made as low as desired as long as \( \rho^2 = 1 - o(n^{-\frac{4}{7}}) \) (with \( \rho \) being the correlation coefficient between each pair of entries). A matching converse bound has also been supplied in [1]. Quite surprisingly, the proposed algorithm, although being suboptimal, attains a vanishing error probability for the same asymptotic lower bound on \( \rho^2 \). The main advantage in this respect is the fact that the proposed algorithm has a much lower computational complexity than the maximum likelihood estimator. The proposed permutation recovery algorithm has another advantage. While ordinary permutation recovery algorithms normally outputs the best matching between all the sequences in the two databases, the proposed algorithm “connects” only pairs of sequences from the two databases which seems to be sufficiently correlated. This way, the proposed algorithm avoids from connecting pairs of sequences with too low empirical correlation, which could have been mistakenly aligned by other algorithms.

The database alignment problem was originally introduced by Cullina et al. [3]. The discrete case was studied in [3], which derived achievability and converse bounds in terms of mutual information. Exact recovery of the underlying permutation for correlated Gaussian databases was studied in [1], and follow-up work extended the results to partial recovery [1]. A typicality-based framework for permutation estimation was investigated in [5]. Random feature deletions and repetitions were researched respectively in [6] and [7]. The Gaussian database alignment recovery problem is equivalent to a certain idealized tracking problem studied in [8, 9]. The recovery problem between two correlated random graphs has also been investigated in the past few years. A starting point on this problem proposed the correlated Erdős–Rényi graph model with dependent Bernoulli edge pairs [10]. A subsequent work studied the recovery problem under the Gaussian setting [11]. More recent papers have investigated the corresponding detection problem for correlation between graphs [12, 13]. It has been lately proved [14, 15] that detecting whether Gaussian graphs are correlated is as difficult as recovering the node labeling.

The remaining part of the paper is organized as follows. In Section 2 we establish notation
conventions. In Section 3, we formalize the model, formulate the problem, and state some results from previous work. In Section 4, we state the new proposed algorithm. In Section 5, we provide and discuss the main results of this work, and in Section 6, we compare our results to a previous work. Finally, in Section 7, we shortly discuss another detector as an option for future work. All the results in this work are proved in the Appendixes.

2 Notation Conventions

Throughout the paper, random variables will be denoted by capital letters and specific values they may take will be denoted by the corresponding lower case letters. Random vectors and their realizations will be denoted, respectively, by capital letters and the corresponding lower case letters, both in the bold face font. For example, the random vector \( \mathbf{X} = (X_1, X_2, \ldots, X_d) \), \((d – \text{ positive integer})\) may take a specific vector value \( \mathbf{x} = (x_1, x_2, \ldots, x_d) \) in \( \mathbb{R}^d \). When used in the linear-algebraic context, these vectors should be thought of as column vectors, and so, when they appear with superscript \( T \), they will be transformed into row vectors by transposition. Thus, \( \mathbf{x}^T \mathbf{y} \) is understood as the inner product of \( \mathbf{x} \) and \( \mathbf{y} \). The notation \( \| \mathbf{x} \| \) will stand for the Euclidean norm of vector \( \mathbf{x} \). As customary in probability theory, we write \( \mathbf{X} = (X_1, \ldots, X_d) \sim \mathcal{N}(\mathbf{0}_d, \mathbf{I}_d) \) (\( \mathbf{0}_d \) being a vector of \( d \) zeros and \( \mathbf{I}_d \) being the \( d \times d \) identity matrix) to denote that the probability density function of \( \mathbf{X} \) is

\[
P_X(\mathbf{x}) = (2\pi)^{-d/2} \cdot \exp\left\{ -\frac{1}{2} \| \mathbf{x} \|^2 \right\}.
\]

For \( \mathbf{X} = (X_1, \ldots, X_d) \) and \( \mathbf{Y} = (Y_1, \ldots, Y_d) \), we write

\[
(\mathbf{X}, \mathbf{Y}) \sim \mathcal{N}^{\otimes d} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)
\]

to denote the fact that \( \mathbf{X} \) and \( \mathbf{Y} \) are jointly Gaussian with IID pairs, where each pair \((X_i, Y_i), i \in \{1, \ldots, d\}\), is a Gaussian vector with a zero mean and the specified covariance matrix.

Logarithms are taken to the natural base. The probability of an event \( \mathcal{E} \) will be denoted by \( \mathbb{P}\{\mathcal{E}\} \) and the indicator function by \( \mathbb{1}\{\mathcal{E}\} \). The expectation operator will be denoted by \( \mathbb{E}[\cdot] \).
3 Settings, Problem Formulation, and Prior Art

3.1 Settings and Problem Formulation

We consider the following binary hypothesis testing problem. Under the null hypothesis $H_0$, the Gaussian databases $X^n$ and $Y^n$ are generated independently with $X_1, \ldots, X_n, Y_1, \ldots, Y_n \overset{iid}{\sim} \mathcal{N}(0_d, I_d)$. Let us denote by $P_0$ the probability distribution that governs $(X^n, Y^n)$ under $H_0$. Under the alternate hypothesis $H_1$, the databases $X^n$ and $Y^n$ are correlated with some unknown permutation $\sigma \in S_n$ and some known correlation coefficient $\rho > 0$. Let us denote by $P_{1|\sigma}$ the probability distribution that governs $(X^n, Y^n)$ under $H_1$, for some permutation $\sigma \in S_n$. Summarizing:

$$H_0 : (X_1, Y_1), \ldots, (X_n, Y_n) \overset{iid}{\sim} \mathcal{N}^d \left( \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right),$$

$$H_1 : (X_1, Y_{\sigma(1)}), \ldots, (X_n, Y_{\sigma(n)}) \overset{iid}{\sim} \mathcal{N}^d \left( \begin{bmatrix} 0 & 1 \\ \rho & 1 \end{bmatrix} \right),$$

for some permutation $\sigma \in S_n$.

We would like to consider a joint problem of correlation detection and low-complexity permutation recovery. Let us denote the set

$$L_n = \left\{ \{ (k_1, \ell_1), (k_2, \ell_2), \ldots, (k_m, \ell_m) \} \left| m \in \{1, 2, \ldots, n\}, \forall i \neq j : k_i \neq k_j, \ell_i \neq \ell_j \right. \right\}. \quad (4)$$

Given the databases $X^n$ and $Y^n$ (and not the permutation $\sigma$), a joint detector-recover $\phi : \mathbb{R}^{d \times n} \times \mathbb{R}^{d \times n} \to \{0, L_n\}$ decides whether the null hypothesis or the alternate hypothesis occurred, and if it accepts the latter, it outputs a member $\psi(X^n, Y^n)$ of $L_n$, i.e., a list of all probably matched pairs. Note that $\psi(X^n, Y^n)$ must not be a permutation.

For a given $n$ and $d$, the type-I probability of error of a test $\phi$ is

$$P_{EA}(\phi) \overset{\Delta}{=} P_0 \left\{ \phi(X^n, Y^n) \neq 0 \right\}, \quad (5)$$

and the type-II probability of error is

$$P_{MD}(\phi) \overset{\Delta}{=} \max_{\sigma \in S_n} P_{1|\sigma} \left\{ \phi(X^n, Y^n) = 0 \right\}. \quad (6)$$

Assuming that $X^n$ and $Y^n$ are indeed correlated, the probability of alignment error is defined by

$$P_{e|H_1}(\psi) \overset{\Delta}{=} \max_{\sigma \in S_n} P_{1|\sigma} \{ \psi(X^n, Y^n) \neq \{(1, \sigma(1)), (2, \sigma(2)), \ldots, (n, \sigma(n))\} \}. \quad (7)$$
For the specific algorithm that will be defined in Section 4, which solves the above defined joint detection-recovery problem, our main objective is to find sufficient conditions on the parameters $n$, $d$, and $\rho$, under which these error probabilities can be driven to zero when $n, d \to \infty$.

### 3.2 Prior Art

In this subsection, we present the achievability and converse results for the correlated Gaussian database detection problem that were recently reported in [2], as well as the achievability and converse results for the correlated Gaussian database alignment problem from [1].

Concerning the detection problem, the sum-of-inner-products statistic is defined by

$$T \triangleq \sum_{i=1}^{n} \sum_{j=1}^{n} X_i^T Y_j,$$

and the test itself is defined by comparing $T$ to a threshold as follows:

$$\phi_{T,t}(X^n, Y^n) = \begin{cases} 0 & T < t \\ 1 & T \geq t \end{cases}.$$  

(9)

Define the risk of the optimal test w.r.t. the threshold $t$ by

$$R(\phi_T) = \inf_t \{P_{FA}(\phi_{T,t}) + P_{MD}(\phi_{T,t})\}.$$  

(10)

Then, the following result is proved in [2, Section 4].

**Proposition 1 (Detection Achievability).** Let $t = \sqrt{\frac{dn}{2}}$ with $\gamma \in (0, 4\rho^2)$. The risk of the sum-of-inner-products test $\phi_T$ for the binary hypothesis testing problem (3) is upper-bounded by

$$R(\phi_T) \leq \min_{\gamma \in (0, 4\rho^2)} \left\{ \exp \left( -\frac{d}{2} G_{FA}(\gamma) \right) + \exp \left( -\frac{d}{2} G_{MD}(\gamma) \right) \right\}$$

(11)

$$\leq 2 \exp \left( -\frac{d\rho^2}{60} \right),$$

(12)

where,

$$G_{FA}(\gamma) \triangleq \sqrt{1 + \gamma} - 1 - \ln \left( \frac{1 + \sqrt{1 + \gamma}}{2} \right),$$

(13)

$$G_{MD}(\gamma) \triangleq \frac{1}{1 - \rho^2} \left( \sqrt{(1 - \rho^2)^2 + \gamma} - \sqrt{\rho^2 \gamma} \right) - 1 - \ln \left( \frac{1 - \rho^2 + \sqrt{(1 - \rho^2)^2 + \gamma}}{2} \right).$$

(14)

It follows from the simple upper bound in (12) that, if $\rho^2 = \omega(1/d)$, then $R(\phi_T) \to 0$ as $d \to \infty$. The following converse result, which relies on a truncated second-moment method, is proved in [2, Section 5].
Proposition 2 (Detection Converse). For \( n \geq e^2 \), if
\[
\rho^2 = o \left( \frac{1}{d/\sqrt{n}} \right) \quad \text{and} \quad d = \Omega(\ln n),
\]
then the minimax risk for the binary hypothesis testing problem goes to 1 as \( d, n \to \infty \).

For the sake of comparison, we now revisit the key results from \( \Pi \) for the alignment problem.

Proposition 3 (Alignment Achievability). The probability of error of the ML decoder \( \hat{\sigma}_{ML} = \arg \max_{\sigma \in S_n} P_{\text{err}}(X^n, Y^n) \) is upper-bounded by
\[
P_{\text{err}}(\hat{\sigma}_{ML}) \leq n(1 - \rho^2)^{\frac{d}{4}} \frac{1 - \left( n(1 - \rho^2)^{\frac{d}{4}} \right)^n}{1 - n(1 - \rho^2)^{\frac{d}{4}}}. \tag{16}
\]

Therefore, if \( \rho^2 = 1 - o(n^{-\frac{d}{4}}) \), then the ML decoder returns the true permutation with high probability.

Proposition 4 (Alignment Converse). The minimax probability of error is lower-bounded by
\[
P_{\text{err}}^* \geq 1 - \left( n(1 - \rho^2)^{\frac{d}{4}(1+\varepsilon(d))} \right)^{-2} - 4 \left( n(1 - \rho^2)^{\frac{d}{4}(1+\varepsilon(d))} \right)^{-1}, \tag{17}
\]
where \( \varepsilon(d) \to 0 \) as \( d \to \infty \).

Consequently, if \( \rho^2 = 1 - \omega(n^{-\frac{d}{4}}) \), then the probability of error of any decoder is close to 1.

4 The Proposed Algorithm

Denote the normalized vectors
\[
\tilde{X}_i = \frac{X_i}{\|X_i\|}, \quad i \in \{1, 2, \ldots, n\},
\]
and
\[
\tilde{Y}_j = \frac{Y_j}{\|Y_j\|}, \quad j \in \{1, 2, \ldots, n\},
\]
and consider the normalized databases \( \tilde{X}^n \) and \( \tilde{Y}^n \). For \( \rho \in (0, 1] \), define the statistic
\[
N(\rho) \triangleq \sum_{i=1}^n \sum_{j=1}^n 1 \left\{ \tilde{X}_i^T \tilde{Y}_j \geq \frac{\rho}{2} \right\}. \tag{20}
\]
such that in the detection phase, the proposed algorithm compares \( N(\rho) \) to a threshold:

\[
\phi^N(\tilde{X}_n, \tilde{Y}_n) = \begin{cases} 
0 & N(\rho) < \beta n P \\
1 & N(\rho) \geq \beta n P 
\end{cases},
\]

(21)

where \( \beta \in (0, 1) \) and \( P \) is the probability of \( \tilde{X}_i^T \tilde{Y}_j \geq \frac{\rho}{2} \) when \( X \) and \( Y \) are a matched pair.

If \( \phi^N(\tilde{X}_n, \tilde{Y}_n) = 1 \), then the algorithm has to provide a list of probably matching pairs. First, it makes a list \( L^* \) of all index pairs \((i,j)\) for which \( \tilde{X}_i^T \tilde{Y}_j \geq \frac{\rho}{2} \). Since this list may contain any number of index pairs between \( \beta n P \) and \( n^2 \), while the final output must be a member of the set \( L_n \) (defined in (4)), the algorithm expurgates the list \( L^* \) by removing the minimal number of index pairs until a member of \( L_n \) is resulted. We denote the final output by \( \psi_e \).

Our proposed algorithm may be presented in a simple graphical way. In order to perform both detection and alignment, one has to draw a table with \( n \) raws and \( n \) columns. In this table, raw \( i \) is for \( X_i \) and column \( j \) is for \( Y_j \). The cell \((i,j)\) is filled with a dot only if \( \tilde{X}_i^T \tilde{Y}_j \geq \frac{\rho}{2} \).

The dots are then counted and one declare that the two databases are correlated if the total number of dots exceeds a threshold of \( \beta n \) (since \( P \approx 1 \)). Typical tables for \( n = 10 \) under \( H_0 \) and under \( H_1 \) with the identity permutation are given in Figure 1.

![Figure 1: Examples of typical tables under \( H_0 \) (left) and under \( H_1 \) with the identity permutation.](image)

When the algorithm declares that the databases are correlated, it still has to expurgate the original table before outputting the decoded permutation (more precisely, a member of \( L_n \)). Basically, each raw and each column must contain at most one dot. The algorithm expurgates the minimal number of dots such that each raw and each column will not contain more than a
single dot. Figure 2 below presents an example for an original table and an expurgated table, which presents the decoded permutation.

Figure 2: An example of an original table (left) and an expurgated table. The expurgated dots are in blue. Note that any other attempt to expurgate the original table will erase more than three dots. Also note that the expurgated table must not represent a permutation, rather it must represent a member of $L_n$.

5 Main Results

In order to present our first result, we need some definitions. Assume, without loss of generality, that under $H_1$, $\sigma$ is the identity permutation, i.e., $\sigma(i) = i$, for every $i \in \{1, 2, \ldots, n\}$. Let us denote $\Psi_n = \{(m, m') : m, m' \in \{1, 2, \ldots, n\}\}$. For any $i \in \{1, \ldots, n\}$ and $j \neq i$, denote the probabilities

$$P \triangleq P_{1|\sigma} \left\{ \mathbf{X}_i^T \mathbf{Y}_i \geq \frac{\rho}{2} \right\}, \quad (22)$$

and

$$Q \triangleq P_{1|\sigma} \left\{ \mathbf{X}_i^T \mathbf{Y}_j \geq \frac{\rho}{2} \right\}, \quad (23)$$

and note that $P \geq Q$. In addition, it follows that for any $(i, j) \in \Psi_n$,

$$P_0 \left\{ \mathbf{X}_i^T \mathbf{Y}_j \geq \frac{\rho}{2} \right\} = Q. \quad (24)$$

In the following result, which is proved in Appendix A, we present upper bounds on the type-I and type-II error probabilities for a general set of parameters. Let us denote the Stirling
numbers of the second kind:
\[ S(k, \ell) = \frac{1}{\ell!} \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} (\ell - i)^k, \]  
(25)
and define the numbers
\[ B(k) = \sum_{d=1}^{k} S(k, d) \cdot \frac{k^{2d}}{d!}. \]  
(26)

**Theorem 1** (type-I and II probability bounds). Let \( n, d \in \mathbb{N} \) and \( \rho \in (0, 1) \) be given. Let \( P = P(d, \rho) \) and \( Q = Q(d, \rho) \) as defined above. Then, for any \( \beta \in (0, 1) \),
\[ P_{FA}(\phi_N) = \inf_{k \in \mathbb{N}} \frac{k(k+1)B(k) \left[ (n^2Q)^k \cdot 1\{n^2Q \geq 1\} + n^2Q \cdot 1\{n^2Q < 1\} \right]}{(\beta n P)^k}, \]  
(27)
and,
\[ P_{MD}(\phi_N) \leq \exp \left\{ -\min \left( (1 - \beta)^2 \frac{nP}{16nQ + 2}, (1 - \beta) \frac{n}{12} \right) \right\}. \]  
(28)

**Discussion**

The threshold \( \beta \in (0, 1) \) trades-off between the two error probabilities. When \( \beta \) is relatively low, then relatively few “dots” are required to accept \( H_1 \), hence the type-I error probability is high and the type-II error probability is low. When \( \beta \) grows towards 1, an increasing number of dots are needed to accept \( H_1 \); the type-I error probability gradually decreases while the type-II error probability increases.

While the two error probabilities of the sum-of-inner-products tester \([8]-[9]\) were upper-bounded using standard large-deviations techniques, i.e., the Chernoff’s bound, this cannot be similarly done with the new proposed tester, since calculating the moment generating function (MGF) of the double summation in \([20]\) seems to be hopeless. Thus, alternative tools from large deviations theory have to be invoked. Starting with the type-II probability of error, which is given explicitly by
\[ P_{MD}(\phi_N) = \mathbb{P}_{1|\sigma} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} 1\left\{ X_i^T Y_j \geq \frac{\rho}{2} \right\} \leq \beta n P \right\}. \]  
(29)
The main difficulty in analyzing this lower tail probability is the statistical dependencies between the indicator random variables. Nevertheless, an existing large deviations result by Janson \([16, \text{Theorem 10}]\) provides the appropriate tool to handle the situation in hand. Other recently
treated settings where tools from [16] proved useful can be found in [17, Appendixes B and I] and [18, Appendix K].

The behavior of the bound in (28) depends on $n$, $d$, and $\rho$. As long as $d$ is sufficiently large, such that $16nQ(d, \rho) \ll 2$, then the bound decays exponentially with $n$, when the rate function is given by

$$\min \left\{ \frac{(1 - \beta)^2}{2}, \frac{1 - \beta}{12} \right\},$$

(30)

since $P(d, \rho) \approx 1$ for large $d$ (and not too low $\rho$). On the other hand, when $16nQ(d, \rho) \gg 2$, then the bound also converges rapidly to zero, but now with an exponent function that depends on $\min \left\{ \frac{P(d, \rho)Q(d, \rho)}{n}, n \right\}$ (note that $Q(d, \rho) \to 0$ as $d \to \infty$).

Regarding the type-I probability of error, the situation is somewhat more complicated, since no general large deviation bounds exist to assess upper tail probabilities of the form

$$P_{\phi_A}(\phi_N) = \mathbb{P}_0 \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} 1 \left\{ \hat{X}_i^T \hat{Y}_j \geq \frac{\rho}{2} \right\} \geq \beta n P \right\}.$$  

(31)

Hence, the best we could do is to upper-bound this probability using a generalization of Chebyshev’s inequality:

$$P_{\phi_A}(\phi_N) \leq \frac{\mathbb{E}_0 \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} 1 \left\{ \hat{X}_i^T \hat{Y}_j \geq \frac{\rho}{2} \right\} \right]^k}{(\beta n P)^k},$$

(32)

where $k \in \mathbb{N}$ may be optimized to yield the tightest bound. In order to derive (or at least, to upper-bound) the $k$-th moment in (32), new techniques, that relies on graph-counting considerations, are developed in Appendix B to prove the general Lemma 1 that appears in Appendix A. The relatively interesting fact (at least to the writer of these lines) is the appearing dichotomy between two regions in the parameter space $\{(n, d, \rho)\}$. On the one hand, if $n^2Q(d, \rho)$, which is the expected number of dots under $H_0$, is smaller than 1, then the $k$-th moment in (32) is proportional to $n^2Q(d, \rho)$, for any $k \in \mathbb{N}$. In this case, the type-I error probability is relatively small, since the typical number of “correlated” pairs of vectors is zero. On the other hand, if $(n, d, \rho)$ are such that $n^2Q(d, \rho) \geq 1$, then the $k$-th moment in (32) is proportional to $[n^2Q(d, \rho)]^k$. In this case, any increment in $n^2Q(d, \rho)$ (due to increasing $n$ or decreasing $\rho$), causes a relatively sharp increment in the type-I error probability. Such phenomena, where moments of enumerators (i.e., sums of IID or weakly dependent indicator random variables)
undergo phase transitions have already been encountered multiple times, e.g. in information theory \cite[Appendix B, Lemma 3]{17}, \cite[p. 168]{19}, \cite[Appendix A, Lemma 1.1]{20}.

Moving further, we provide sufficient conditions under which the various error probabilities converge to zero as the parameters $n$ and $d$ grow to infinity. The following result is proved in Appendix C.

**Theorem 2** (Sufficient conditions for convergence). Let $d, n \in \mathbb{N}$ and $\rho \in (0, 1)$.

1. Regarding the binary hypothesis testing problem \cite[3]{3}, for every $\rho(n, d)$ such that

$$\forall d, n \in \mathbb{N} : \quad n^2 \sqrt{d} \cdot \left(1 - \left(\frac{\rho}{2}\right)^2\right)^{d/2} < 1,$$  

then,

$$P_{FA}(\phi_N) \xrightarrow{n \to \infty} 0$$  

and,

$$P_{MD}(\phi_N) \xrightarrow{n \to \infty} 0.$$  

2. Regarding the permutation recovery problem, as long as $\rho(n, d)$ converges to zero sufficiently slowly, such that

$$n^2 \sqrt{d} \cdot \left(1 - \left(\frac{\rho}{2}\right)^2\right)^{d/n} \xrightarrow{d,n \to \infty} 0,$$  

and assuming that $n$ and $d$ are of the same order, then it holds that

$$P_{e|H_1}(\psi_e) \xrightarrow{d,n \to \infty} 0.$$  

A few remarks are now in order.

**Remark 1.** It is important to note that the proposed algorithm does not necessarily outputs a permutation, since it is primarily based on local detections of correlated pairs. In fact, the algorithm outputs a member of the set $L_n$ (defined in \cite[4]{4}), which can be regarded as a partial-permutation. Although this may be thought of as a major drawback, since one would like to have a perfect alignment of the two databases, we explain why, nonetheless, this is an advantage. Assume that the two databases are indeed correlated and that the true permutation is the
identity permutation. Furthermore, assume that the two pairs \((X_1, Y_1)\) and \((X_2, Y_2)\) have been drawn with very low correlation, i.e., that \(\tilde{X}_i^T \tilde{Y}_i \approx 0\), \(i = 1, 2\), and that the rest of the correct pairs have a typical correlation, i.e., \(\tilde{X}_i^T \tilde{Y}_i \approx \rho\), \(i = 3, \ldots, n\). In such a case, it is likely that an algorithm that aims to recover the entire permutation, will output one out of two possible permutations - the true permutation or an incorrect permutation where \((X_1, Y_2)\) and \((X_2, Y_1)\) are declared as matching pairs (see Figures 3a and 3b below). On the other hand, the proposed algorithm will declare only on the \(n - 2\) pairs with relatively high correlation, while avoiding from “connecting” \(X_1, X_2\) to \(Y_1, Y_2\) in any way (see Figure 3c). Indeed, this case is regarded as an error, because the output is not a perfect permutation, but this error is still different from the case in Figure 3b, where two pairs are confused. It is important to remember that if the condition in (36) holds, then these error events have negligible probabilities.

\[
n^2 \sqrt[d]{d} \left(1 - \left(\frac{\rho}{2}\right)^2\right)^{\frac{d}{2}}
\]  

(38)

Figure 3: Outputs of different alignment algorithms when two pairs of vectors have been drawn with relatively low correlation, while the rest with a typical correlation. An algorithm that outputs a perfect permutation will output one of the above permutations. In this case, the proposed algorithm will output a member of \(L_n\) which is not a perfect permutation.

**Remark 2.** The expression of
turns out to play a major roll in both problems; in the hypothesis testing problem and the permutation recovery problem. This expression stands for the average number of independent pairs of feature vectors that seems to be correlated. In order to achieve low type-I and type-II error probabilities, we only require this expression to be bounded by one, such that the typical number of incorrect pairs is zero, and the probability that the number of incorrect pairs will cross the threshold $\beta n$ converges to zero relatively fast. For the alignment error probability to converge to zero, it is not enough that this expression is bounded - it must converge to zero as $d, n \to \infty$. This phenomenon that permutation recovery is generally harder than correlation detection between databases has already been pointed out in [2].

**Remark 3.** In order to compare our result with those reported in [1], we first have to indicate that the factor of $\sqrt[10]{d}$ in (38) seems to be artificial. Indeed, by making different choices of the Hölder conjugates $p, q$ in the proof of Lemma 3 in Appendix C, this factor can be replaced by a factor of $A \sqrt[10]{d}$, where $A \in \mathbb{R}$ and $k \gg 10$. Another option is to use the Cauchy-Schwarz inequality (i.e., choosing $p = q = 2$), and then the factor of $\sqrt[10]{d}$ is replaced by $\sqrt{\log \frac{2}{\rho}}$. This way, the conditions in (38) and (39) will be similar, but now w.r.t. the expression

$$n^2 \sqrt{\frac{\log \frac{2}{\rho}}{d}} \left(1 - \left(\frac{\rho}{2}\right)^2\right)^{\frac{d}{2}}. \quad (39)$$

Isolating $\rho$ as a function of $n, d$ in (38) is immediate, but not possible analytically in (39), hence (38) may be more attractive in real-life scenarios, where $\rho$ has to be written as a function of $d$, $n$, and the probability of alignment error.

Since each one of $\sqrt[10]{d}$ and $\sqrt{\log \frac{2}{\rho}}$ are negligible relative to the exponentially decaying factor $[1 - (\rho/2)^2]^d/2$, we shall assume that the dominant behavior in both conditions in Theorem 2 is related to the expression

$$n^2 \cdot \left(1 - \left(\frac{\rho}{2}\right)^2\right)^{\frac{d}{2}}. \quad (40)$$

Hence, we conclude that as long as $(\rho/2)^2 = 1 - o(n^{-\frac{d}{2}})$, our new proposed decoder returns the true permutation with high probability, and most importantly, thanks to the converse result in Proposition 4, this (suboptimal) decoder attains a vanishing error probability for the lowest possible decay rate of $\rho$ as a function of $d$ and $n$. This is obviously very good news, since the computational complexity of our proposed decoder grows like $n^2$, while the ML decoder, which checks all possible permutations, has a computational complexity of the order of $n!$. 

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6 Comparison with a Previous Work

In this section, we show that for some different parameter choices, the new proposed tester in (20)-(21) achieves lower type-I and type-II error probabilities than the sum-of-inner-products tester in (8)-(9), that was proposed in [2]. The fact that the new tester has lower error probabilities is not trivial, since the analysis in [2] is based on Chernoff’s bound, which is known to be relatively tight in many different cases, while the analysis in the current manuscript follows other methods, that are able to accommodate the statistical dependencies between the indicator random variables in (20).

Concerning the results of [2], recall from Proposition 1 that for $\gamma \in (0, 4\rho^2)$,

$$- \log P_{FA}(\phi_T) \leq \frac{d}{2}G_{FA}(\gamma), \quad (41)$$
$$- \log P_{MD}(\phi_T) \leq \frac{d}{2}G_{MD}(\gamma), \quad (42)$$

with

$$G_{FA}(\gamma) = \sqrt{1 + \gamma} - 1 - \ln \left( \frac{1 + \sqrt{1 + \gamma}}{2} \right), \quad (43)$$
$$G_{MD}(\gamma) = \frac{1}{1 - \rho^2} \left( \sqrt{(1 - \rho^2)^2 + \gamma} - \sqrt{\rho^2\gamma} \right) - 1 - \ln \left( \frac{1 - \rho^2 + \sqrt{(1 - \rho^2)^2 + \gamma}}{2} \right). \quad (44)$$

Regarding the results in this manuscript, Theorem 1 provides that for the type-I error probability,

$$P_{FA}(\phi_N) \leq \inf_{k \in \mathbb{N}} \frac{k(k + 1)B(k) \left\{ (n^2Q(d, \rho))^k \cdot 1\{n^2Q(d, \rho) \geq 1\} + n^2Q(d, \rho) \cdot 1\{n^2Q(d, \rho) < 1\} \right\}}{(\beta nP(d, \rho))^k}, \quad (45)$$

where $B(k)$ is defined in (26). For the type-II error probability, Theorem 1 yields

$$- \log P_{MD}(\phi_N) \leq \min \left\{ (1 - \beta)^2 \frac{n^2Q(d, \rho)}{16nQ(d, \rho) + 2}, (1 - \beta) \frac{n}{12} \right\}. \quad (46)$$

The quantity $Q(d, \rho)$ is upper-bounded using (C.3) in Lemma 3 and $P(d, \rho)$ is lower-bounded using (C.7) in Lemma 4. Since the numbers $B(k)$ grow quite rapidly with $k$, when calculating the bounds in (45), we switched the infimum over $\mathbb{N}$ with a minimization over $\{1, 2, \ldots, 40\}$.

In Figure 4 below, we present plots of the trade-off between $- \log P_{FA}(\phi_T)$ and $- \log P_{MD}(\phi_T)$ (in solid line) and the trade-off between $- \log P_{FA}(\phi_N)$ and $- \log P_{MD}(\phi_N)$ (in dashed line), for different choices of the parameters $n$, $d$, and $\rho$. 15
In Figures 4a and 4b we fixed $n = 2000$ and $d = 5000$ and examined the behavior of the two testers for two values of $\rho$. For relatively high $\rho = 0.6$, the sum-of-inner-products tester (9) dominates the new proposed tester (21), but as $\rho$ decreases, the effective signal-to-noise ratio of the sum-of-inner-products tester also decreases, the quantity $Q(d, \rho)$ increases, and $P(d, \rho)$ decreases. As a consequence, the performance of both testers degrades, but now, no tester dominates the other; at low $P_{FA}$ values, the proposed tester has higher $P_{MD}$ values, but at high $P_{FA}$ values, the proposed tester is better, i.e., it attains lower $P_{MD}$ values. Specifically at $\rho = 0.15$, when balancing the two error probabilities, we find that the risks are given by $R(\phi_T) \approx e^{-14}$ and $R(\phi_N) \approx e^{-20}$, i.e., the new proposed tester performs better.

In Figures 4c and 4d we fixed $n = 1000$ and $\rho = 0.8$ and examined the behavior of the two
testers for two values of $d$. While the error probabilities of the tester in (9) depends merely on $d$ and $\rho$, the error probabilities of the new proposed tester (21) depends on all three parameters $n$, $d$, and $\rho$. The type-I probability of error depends strongly on all three, as can be seen in (45), but the type-II probability of error depends strongly on $n$ and weakly on $d$ and $\rho$. Because of that, when increasing from $d = 200$ to $d = 1000$, both error probabilities of the sum-of-inner-products tester can be made smaller, while for the new tester, only the range of the type-I error probability increases. On the one hand, for the relatively low $d = 200$, the new proposed tester (21) almost dominates the tester in (9). On the other hand, for the relatively high $d = 1000$, none of the testers dominates the other, but still, $R(\phi_T) \approx e^{-72}$ and $R(\phi_N) \approx e^{-77}$, i.e., the new proposed tester has a lower risk hence it performs better.

7 Future Work

Concerning correlation detection alone, it follows by the comparison in Section 6 that our detector, which is based on the statistic in (20), may outperform the detector proposed in [2], which is based on the sum-of-inner-products statistic in [8]. Yet, a question on the lowest possible $\rho^2$ that allows a vanishing risk still remains open. Regarding this issue, we conjecture that a unification of the two statistics may result in a new detector that will outperform the one in [2]. Specifically, we would recommend on using the following truncated statistic:

$$N_{trun}(\rho) \triangleq \sum_{i=1}^{n} \sum_{j=1}^{n} X_i^T Y_j 1 \left\{ X_i^T Y_j \geq \frac{d \rho^2}{2} \right\}, \quad (47)$$

and performing a test by comparing $N_{trun}(\rho)$ to a threshold:

$$\phi_{\theta}(X^n, Y^n) = \begin{cases} 0 & N_{trun}(\rho) < \theta \\ 1 & N_{trun}(\rho) \geq \theta \end{cases}, \quad (48)$$

where $\theta > 0$. To see why the test defined in (8)-(9) may be improved by the test in (48), first note that the double summation in (8) can also be written by

$$T = \sum_{i=1}^{n} X_i^T Y_{\sigma(i)} + \sum_{i=1}^{n} \sum_{j \neq \sigma(i)} X_i^T Y_j \quad (49)$$

for some permutation $\sigma \in S_n$. If the two databases are indeed correlated for some $\sigma \in S_n$, then the expectation of the first term on the right-hand-side of (49) is given by $n d \rho$, but the second term, which is a sum over $n(n-1)$ relatively small terms, acts as an effective noise
that increases the probability of missed detection. Alternatively, if the two databases are independent, then the sum \( \sum_{i=1}^{n} \mathbf{X}_i^T \mathbf{Y}_{\sigma(i)} \) is expected to be relatively small for every \( \sigma \in S_n \), so the second term on the right-hand-side of (49) increases the probability of false alarm. Then, by eliminating this effective noise using the truncation operation in (47), we expect to improve the achievability result in Proposition 1. While the proof of Proposition 1 in [2, Section 4] relies on the opportunity to derive the MGF of \( T \), calculating the MGF of \( N_{\text{true}}(\rho) \) seems to be much more challenging. Nevertheless, the type-II error probability of the test in (48) can be bounded using Chernoff inequality.

**Theorem 3.** For the binary hypothesis testing problem (3), the type-II error probability of the test \( \phi_{\theta} \), with \( \theta = \beta \rho dn, \beta \in [0, \frac{1}{2}] \), is upper-bounded by

\[
P_{\text{MD}}(\theta) \leq \left( \frac{1}{2} + \delta(\rho) \right)^n \exp \{ n \cdot H_2(\beta) \},
\]

where \( H_2(\cdot) \) is the binary entropy function and \( \delta(\rho) \to 0 \) as \( \rho \to 0 \).

At the moment, the type-I error probability of the test in (48) is still an open problem. We hope that the test (48) will be able to close the existing gap between Propositions 1 and 2.

**Appendix A - Proof of Theorem 1**

**Analysis of False Alarms**

For \( k \in \mathbb{N} \), consider the following

\[
P_{\text{FA}}(\phi_N) = \mathbb{P}_0 \{ N(\rho) \geq \beta n P \} = \mathbb{P}_0 \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} 1 \left\{ \tilde{\mathbf{X}}_i^T \tilde{\mathbf{Y}}_j \geq \frac{\rho}{2} \right\} \geq \beta n P \right\} \leq (\beta n P)^{-k} \cdot \mathbb{E}_0 \left[ \left( \sum_{i=1}^{n} \sum_{j=1}^{n} 1 \left\{ \tilde{\mathbf{X}}_i^T \tilde{\mathbf{Y}}_j \geq \frac{\rho}{2} \right\} \right)^k \right] .
\]
where (A.4) is due to Markov’s inequality. Since $k \in \mathbb{N}$ is arbitrary, it holds that

$$P_{\mathcal{F}_A}(\phi_N) \leq \inf_{k \in \mathbb{N}} (\beta n P)^{-k} \cdot \mathbb{E}_0 \left[ \sum_{i=1}^n \sum_{j=1}^n 1 \left\{ \tilde{X}_i^T \tilde{Y}_j \geq \rho \right\} \right]^k,$$

so the main task is to evaluate the $k$-th moment of $N(\rho)$.

In order to upper-bound the $k$-th moment of $N(\rho)$, we will make use of the following result, which is proved in Appendix B.

**Lemma 1.** Let $n, d \in \mathbb{N}$. Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be two sets of IID random variables, taking values in $\mathcal{X}^d$ and $\mathcal{Y}^d$, respectively. Let $J : \mathcal{X} \times \mathcal{Y} \to \{0, 1\}$ and assume that

$$\mathbb{P}(J(x, Y) = 1) \leq Q,$$  \hspace{1cm} (A.6)

$$\mathbb{P}(J(X, y) = 1) \leq Q,$$  \hspace{1cm} (A.7)

$$\mathbb{P}(J(X, Y) = 1) \leq Q,$$  \hspace{1cm} (A.8)

where $Q \in [0, 1]$. Let $k \in \mathbb{N}$. Then,

$$\mathbb{E} \left[ \left( \sum_{m=1}^n \sum_{k=1}^n J(X_m, Y_k) \right)^k \right] \leq k(k + 1) B(k) \cdot \left\{ \begin{array}{ll} (n^2 Q)^k & \text{if } n^2 Q \geq 1, \\ n^2 Q & \text{if } n^2 Q < 1. \end{array} \right.$$  \hspace{1cm} (A.9)

Continuing from (A.5) using the result of Lemma 1 yields

$$P_{\mathcal{F}_A}(\phi_N) \leq \inf_{k \in \mathbb{N}} k(k + 1) B(k) \left[ (n^2 Q)^k \cdot 1 \{ n^2 Q \geq 1 \} + n^2 Q \cdot 1 \{ n^2 Q < 1 \} \right]$$  \hspace{1cm} (A.10)

which is exactly (27) in Theorem 1.

**Analysis of Miss-Detection**

Under $\mathcal{H}_1$, the expectation of $N(\rho)$ is given by

$$\Delta_{n, d} = \mathbb{E}_1[N(\rho)] = \mathbb{E}_1 \left[ \sum_{i=1}^n \sum_{j=1}^n 1 \left\{ \tilde{X}_i^T \tilde{Y}_j \geq \rho \right\} \right]$$  \hspace{1cm} (A.11)

$$= \sum_{i=1}^n \sum_{j=1}^n \mathbb{P}_{1|\sigma} \left\{ \tilde{X}_i^T \tilde{Y}_j \geq \rho \right\}$$  \hspace{1cm} (A.12)

$$= n P + n(n - 1) Q.$$  \hspace{1cm} (A.13)
Consider the following

\[
P_{MD}(\phi_N) = \mathbb{P}_{1|\sigma} \{ N(\rho) \leq \beta nP \} \tag{A.14}
\]

\[
\leq \mathbb{P}_{1|\sigma} \{ N(\rho) \leq \beta [nP + n(n - 1)Q(\alpha)] \} \tag{A.15}
\]

\[
= \mathbb{P}_{1|\sigma} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{1} \left\{ \bar{X}_{i}^T \bar{Y}_{j} \geq \frac{\rho}{2} \right\} \leq \beta \Delta_{n,d} \right\}. \tag{A.16}
\]

Let us denote in short \( T(i,j) = \mathbb{1} \left\{ \bar{X}_{i}^T \bar{Y}_{j} \geq \frac{\rho}{2} \right\} \).

In order to upper-bound (A.16), we borrow one result from [16] concerning large deviations behavior of sums of dependent indicator random variables.

Let \( \{U_k\}_{k \in \mathcal{K}} \), where \( \mathcal{K} \) is a set of multidimensional indexes, be a family of Bernoulli random variables. Let \( G \) be a dependency graph for \( \{U_k\}_{k \in \mathcal{K}} \), i.e., a graph with vertex set \( \mathcal{K} \) such that if \( \mathcal{A} \) and \( \mathcal{B} \) are two disjoint subsets of \( \mathcal{K} \), and \( G \) contains no edge between \( \mathcal{A} \) and \( \mathcal{B} \), then the families \( \{U_k\}_{k \in \mathcal{A}} \) and \( \{U_k\}_{k \in \mathcal{B}} \) are independent. Let \( S = \sum_{k \in \mathcal{K}} U_k \) and \( \Delta = \mathbb{E}[S] \). Moreover, we write \( i \sim j \) if \((i,j)\) is an edge in the dependency graph \( G \). Let

\[
\Omega = \max_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}, j \sim i} \mathbb{E}[U_j], \tag{A.17}
\]

and

\[
\Theta = \frac{1}{2} \sum_{i \in \mathcal{K}} \sum_{j \in \mathcal{K}, j \sim i} \mathbb{E}[U_i U_j]. \tag{A.18}
\]

**Theorem 4** ([16], Theorem 10). *With notations as above, then for any \( 0 \leq \beta \leq 1 \),

\[
\mathbb{P}\{S \leq \beta \Delta\} \leq \exp \left\{ -\min \left( (1 - \beta)^2 \frac{\Delta^2}{8\Theta + 2\Delta}, (1 - \beta) \frac{\Delta}{6\Omega} \right) \right\}. \tag{A.19}
\]

In our case, we have \( \Delta = \Delta_{n,d} \), and it only remains to assess the quantities \( \Theta \) and \( \Omega \). One can easily check that the indicator random variables \( T(i,j) \) and \( T(k,\ell) \) are independent as long as \( i \neq k \) and \( j \neq \ell \). Thus, we define our dependency graph in a way that each vertex \((i,j)\) is connected to exactly \( 2n - 2 \) vertices of the form \((i,\ell), \ell \neq j\) or \((k,j), k \neq i\). If the vertices \((i,j)\) and \((k,\ell)\) are connected, we denote it by \((i,j) \sim (k,\ell)\).
Concerning $\Theta$ and $\Omega$, we get that

$$\Theta = \frac{1}{2} \sum_{(i,j) \in \Psi_n} \sum_{(k,\ell) \sim (i,j)} \mathbb{E}[I(i,j)I(k,\ell)]$$  \hspace{1cm} (A.20)

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{(k,\ell) \in \Psi_n} \mathbb{E}[I(i,i)I(k,\ell)] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{(k,\ell) \sim (i,j)} \mathbb{E}[I(i,j)I(k,\ell)] \hspace{1cm} (A.21)$$

$$= \frac{1}{2} PQ(2n-2)n + \frac{1}{2} [2Q^2(n-2) + 2PQ]n(n-1) \hspace{1cm} (A.22)$$

$$= PQ(n-1)n + [Q^2(n-2) + PQ]n(n-1) \hspace{1cm} (A.23)$$

$$= 2PQn(n-1) + Q^2n(n-1)(n-2), \hspace{1cm} (A.24)$$

and

$$\Omega = \max_{(i,j) \in \Psi_n} \sum_{(k,\ell) \in \Psi_n} \sum_{(k,\ell) \sim (i,j)} \mathbb{E}[I(k,\ell)]$$ \hspace{1cm} (A.25)

$$= \max \left\{ \sum_{(k,\ell) \in \Psi_n} \mathbb{E}[I(k,\ell)], \sum_{(k,\ell) \sim (i,i)} \mathbb{E}[I(k,\ell)], \sum_{(k,\ell) \sim (i,j), i \neq j} \mathbb{E}[I(k,\ell)] \right\} \hspace{1cm} (A.26)$$

$$= \max \{(2n-2)Q, 2P + (2n-4)Q \} \hspace{1cm} (A.27)$$

$$= 2P + (2n-4)Q. \hspace{1cm} (A.28)$$

Then,

$$\frac{\Delta_{n,d}}{6\Omega} = \frac{nP + n(n-1)Q}{6[2P + (2n-4)Q]} \hspace{1cm} (A.29)$$

$$= \frac{n[P + (n-1)Q]}{12[P + (n-2)Q]} \hspace{1cm} (A.30)$$

$$\geq \frac{n[P + (n-2)Q]}{12[P + (n-2)Q]} \hspace{1cm} (A.31)$$

$$= \frac{n}{12}, \hspace{1cm} (A.32)$$

and,

$$\frac{\Delta_{n,d}^2}{8\Theta + 2\Delta_{n,d}} = \frac{[nP + n(n-1)Q]^2}{8[2PQn(n-1) + Q^2n(n-1)(n-2)] + 2[nP + n(n-1)Q]} \hspace{1cm} (A.33)$$

$$\geq \frac{[nP + n(n-1)Q]^2}{8[2PQn^2 + 2Q^2n^2(n-1)] + 2[nP + n(n-1)Q]} \hspace{1cm} (A.34)$$

$$= \frac{[nP + n(n-1)Q]^2}{16Qn[P + Qn(n-1)] + 2[nP + n(n-1)Q]} \hspace{1cm} (A.35)$$
\[ P_{md}(\phi_N) \leq P \mid_{\sigma} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} 1 \{ \tilde{X}_i^T \tilde{Y}_j \geq \frac{\mu}{2} \} \leq \beta \Delta_{n,d} \right\} \quad (A.38) \]
\[ \leq \exp \left\{ -\min \left( (1 - \beta)^2 \frac{\Delta_{n,d}^2}{8\Theta + 2\Delta_{n,d}}, (1 - \beta) \frac{\Delta_{n,d}}{6\Omega} \right) \right\} \quad (A.39) \]
\[ \leq \exp \left\{ -\min \left( (1 - \beta)^2 \frac{nP}{16Qn + 2}, (1 - \beta) \frac{n}{12} \right) \right\}, \quad (A.40) \]

which completes the proof of Theorem 1.

**Appendix B - Proof of Lemma 1**

For any \( k \in \mathbb{N} \), let \( S(k,d) \) be the number of ways to partition a set of \( k \) labeled objects into \( d \in \{1,2,\ldots,k\} \) nonempty unlabeled subsets, which is given by the Stirling numbers of the second kind [21]:

\[ S(k,d) = \frac{1}{d!} \sum_{i=0}^{d} (-1)^i \binom{d}{i} (d-i)^k. \quad (B.1) \]

Obviously, \( S(k,1) = S(k,k) = 1 \). Let us denote \( \Psi_n = \{(m,m') : m,m' \in \{1,2,\ldots,n\}\} \). Now,

\[ \mathbb{E} \left[ N^k \right] = \mathbb{E} \left[ \left( \sum_{(m,m') \in \Psi_n} J(X_m,Y_{m'}) \right)^k \right] \quad (B.2) \]
\[ = \sum_{(m_1,m'_1) \in \Psi_n} \cdots \sum_{(m_k,m'_k) \in \Psi_n} \mathbb{E} \left[ J(X_{m_1},Y_{m'_1}) \cdots J(X_{m_k},Y_{m'_k}) \right] \quad (B.3) \]
\[ = \sum_{d=1}^{k} \sum_{\{ (m_i,m'_i) \in \Psi_n, 1 \leq i \leq d, \text{ divided into } d \text{ subsets of identical pairs} \}} \mathbb{E} \left[ J(X_{m_1},Y_{m'_1}) \cdots J(X_{m_k},Y_{m'_k}) \right] \quad (B.4) \]
\[ = \sum_{d=1}^{k} S(k,d) \sum_{\{ (m_i,m'_i) \in \Psi_n, 1 \leq i \leq d, (m_i,m'_i) \neq (m_j,m'_j) \forall i \neq j \}} \mathbb{E} \left[ J(X_{m_1},Y_{m'_1}) \cdots J(X_{m_d},Y_{m'_d}) \right], \quad (B.5) \]

where in the inner summation in (B.4), we sum over all possible \( k \) pairs of codewords’ indices, which are divided in any possible way into exactly \( d \) subsets, all pairs in each subset are
identically\footnote{Two pairs of indices \((m_1, m'_1)\) and \((m_2, m'_2)\) are said to be identical if and only if \(m_1 = m_2\) and \(m'_1 = m'_2\), otherwise, they said to be distinct.}. In \((B.5)\), we use the Stirling numbers of the second kind, and sum over exactly \(d\) distinct pairs of codewords’ indices, where all the identical pairs of indices in \((B.4)\) have been merged together, using the trivial fact that multiplying any number of identical indicator random variables is equal to any one of them.

Let us handle the inner sum of \((B.5)\). The idea is as follows. Instead of summing over the set \(\{ (m_i, m'_i) \in \Psi_n, \ 1 \leq i \leq d, \ (m_i, m'_i) \neq (m_l, m'_l) \ \forall i \neq l \}\) of \(d\) distinct pairs of indices of codewords, we represent each possible configuration of indices in this set as a \emph{graph} \(G\), and sum over all different \emph{graphs} with exactly \(d\) distinct edges. In our graph representation, each codeword index \(m_i \in \{1, 2, \ldots, n\}\) and \(m'_i \in \{1, 2, \ldots, n\}\) is denoted by a vertex and each pair of indices \((m_i, m'_i), m_i \neq m'_i\), is connected by an edge. Hence, the number of edges is fixed, but the numbers of vertices and subgraphs (i.e., disconnected parts of the graph) are variable.

Now, given \(d \in \{1, 2, \ldots, k\}\), we sum over the set \(\mathcal{V}(d) = \{(v_x, v_y)\}\) of pairs of integers, which consists of all possible pairs of vertices needed in order to support a graph with \(d\) different edges. Of course, if all of the \(d\) edges are disconnected, then we must have \((v_x, v_y) = (d, d)\) vertices.

Next, given \(d \in \{1, 2, \ldots, k\}\) and \((v_x, v_y) \in \mathcal{V}(d)\), we sum over the range of possible number of subgraphs. Let \(S_{\min}(d, v_x, v_y)\) \((S_{\max}(d, v_x, v_y))\) be the minimal (maximal) number of subgraphs that a graph with \(d\) edges and \((v_x, v_y)\) vertices can has. For a given quadruplet \((d, v_x, v_y, s)\), where \(s \in [S_{\min}(d, v_x, v_y), S_{\max}(d, v_x, v_y)]\) is the number of subgraphs within \(G\), note that one can create many different graphs (see Figure 5), and we have to take all of them into account. Hence, let \(T(d, v_x, v_y)\) \((T(d, v_x, v_y, s))\) be the number of distinct ways to connect a graph with \(d\) edges and \((v_x, v_y)\) vertices (and \(s\) subgraphs). Finally, for any \(1 \leq i \leq T(d, v_x, v_y, s)\), let \(G_i(d, v_x, v_y, s)\) be the set of different graphs with \(d\) edges, \((v_x, v_y)\) vertices, and \(s\) subgraphs, that can be defined on the set \(\Psi_n\) of pairs of codewords, as we explained before.

We now prove that the cardinality of the set \(G_i(d, v_x, v_y, s)\) is upper–bounded by an expression that depends only on the numbers of vertices \((v_x, v_y)\). First, let \(N(v_x)\) be the number of options to choose \(v_x\) different vectors from a set of cardinality \(n\). Then,

\[
N(v_x) = n \cdot (n - 1) \cdot (n - 2) \cdots (n - v_x + 1) \leq n^{v_x}. \quad (B.6)
\]

\footnote{Of course, indices that belong to distinct pairs may be joined together, e.g., if \((m_1, m'_1)\) and \((m_2, m'_2)\) are distinct, then it may be that \(m_1 = m_2\) or \(m'_1 = m'_2\), but not both.}
Thus,

$$|G_i(d, v_x, v_y, s)| = N(v_x) \cdot N(v_y) \quad (B.7)$$

$$\leq n^{v_x} \cdot n^{v_y}. \quad (B.8)$$

Let $\Theta(G)$ be an indicator random variable that equals one if and only if all of the pairs of codewords that are linked by the edges of $G$ have $J(X, Y) = 1$. Using the above definitions, the inner sum of (B.5) can now be written as:

$$\sum_{\{v_x, v_y\} \in V(d)} \sum_{i=1}^{S_{\text{max}}(d,v_x,v_y)} \sum_{s=1}^{S_{\text{min}}(d,v_x,v_y)} \sum_{G \in G_i(d,v_x,v_y)} \mathbb{E}[\Theta(G)], \quad (B.10)$$

i.e., for any $d \in \{1, 2, \ldots, k\}$, we first sum over the numbers of vertices, then over the number of subgraphs, later on, for a fixed quadruplet $(d, v_x, v_y, s)$, over all possible $T(d, v_x, v_y, s)$ topologies, and finally, over all specific graphs $G \in G_i(d, v_x, v_y, s)$ with a given topology. One should note, that all limits of the three outer sums in (B.10) depend only on $k$, while $|G_i(d, v_x, v_y, s)|$ is the only one that depends also on $n$. It turns out that the expectation of $\Theta(G)$ can be easily evaluated if all subgraphs of $G$ are trees (also known as a forest in the terminology of graph theory). If at least one subgraph of $G$ contains loops, we apply a process of graph pruning, in which we cut out the minimal amount of edges while keeping all vertices intact, until we get a forest (for example, see Figure 5).

3In fact, this procedure is equivalent to upper–bounding some of the indicator functions in (B.9) by one.
Denote by \( \mathcal{P}(G) \) the pruned graph of \( G \). Notice that the expectation of \( \Theta(G) \) is upper-bounded\(^4\) by the expectation of \( \Theta(\mathcal{P}(G)) \), which can be evaluated in a simple iterative process of graph reduction. In the first step, we take the expectation with respect to all vectors that are labels of leaf vertices in \( \mathcal{P}(G) \), while we condition on the realizations of all other vectors, those corresponding to inner vertices in \( \mathcal{P}(G) \); afterwards, we erase leaf vector vertices and corresponding edges. Successive steps are identical to the first one on the remaining (unerasd) graph, continuing until all vectors that are attributed to \( G \) have been considered (for example, see Figure 7).

In each step of the graph reduction, the expectation with respect to each of the leaf vectors is given by \( \mathbb{P}\{J(x, Y_{\text{leaf}}) = 1\} \leq Q \), since we condition on the realizations of vectors that are attributed to the inner vertices. For any graph \( G \in \mathbb{G}_i(d, v_x, v_y, s) \), we conclude that the expectation of \( \Theta(\mathcal{P}(G)) \) is upper-bounded by \( Q^{\mathcal{E}(G)} \), where \( \mathcal{E}(G) \) is the number of edges in \( \mathcal{P}(G) \). Of course, if \( G \) is already a forest, then \( \mathcal{E}(G) = d \). For any \( G \in \mathbb{G}_i(d, v_x, v_y, s) \) which is not a forest, we find \( \mathcal{E}(G) \) as follows. Assume that \( v_x(1), v_x(2), \ldots, v_x(s) \) and \( v_y(1), v_y(2), \ldots, v_y(s) \) are the numbers of vertices in each of the \( s \) subgraphs of \( G \). Then, in the process of graph pruning, each of the \( j \in \{1, 2, \ldots, s\} \) subgraphs of \( G \) will transform into a tree with exactly \( v_x(j) + v_y(j) - 1 \) edges. Hence,

\[
\mathcal{E}(G) = \sum_{j=1}^{s} (v_x(j) + v_y(j) - 1) = v_x + v_y - s. \quad (B.11)
\]

The innermost sum of (B.10) can be treated as follows:

\[
\sum_{G \in \mathbb{G}_i(d, v_x, v_y, s)} \mathbb{E}[\Theta(G)] \leq \sum_{G \in \mathbb{G}_i(d, v_x, v_y, s)} \mathbb{E}[\Theta(\mathcal{P}(G))] \quad (B.12)
\]

\(^4\)It follows from the fact that \( \Theta(G) \leq \Theta(\mathcal{P}(G)) \) with probability one.
Figure 7: Example for the process of graph reduction.

\[
\sum_{G \in \mathcal{G}_i(d,v_x,v_y,s)} Q^{E(G)} \geq \sum_{G \in \mathcal{G}_i(d,v_x,v_y,s)} Q^{v_x,v_y,s} \subseteq \sum_{s=s_{\min}(d,v_x,v_y)} \sum_{i=1} n^v \cdot n^y \cdot Q^{v_x,v_y,s},
\]

where (B.14) follows from (B.11) and (B.16) is due to (B.8). Next, we substitute (B.16) back into (B.10) and get that

\[
S_{\text{max}}(d,v_x,v_y) \frac{T(d,v_x,v_y)}{T(d,v_x,v_y,s)} \sum_{s=s_{\min}(d,v_x,v_y)} \sum_{i=1} n^v \cdot n^y \cdot Q^{v_x,v_y,s},
\]

where (B.14) follows from (B.11) and (B.16) is due to (B.8). Next, we substitute (B.16) back into (B.10) and get that

\[
S_{\text{max}}(d,v_x,v_y) \frac{T(d,v_x,v_y)}{T(d,v_x,v_y,s)} \sum_{s=s_{\min}(d,v_x,v_y)} \sum_{i=1} n^v \cdot n^y \cdot Q^{v_x,v_y,s},
\]
\[ S_{\text{max}}(d,v_x,v_y) \]
\[ = \sum_{s=S_{\text{min}}(d,v_x,v_y)}^{S_{\text{max}}(d,v_x,v_y)} T(d,v_x,v_y, s) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x+v_y-s} \]  
\[ \leq \left( \sum_{s=S_{\text{min}}(d,v_x,v_y)}^{S_{\text{max}}(d,v_x,v_y)} T(d,v_x,v_y, s) \right) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x+v_y-S_{\text{max}}(d,v_x,v_y)} \]  
\[ = T(d,v_x,v_y) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x+v_y-S_{\text{max}}(d,v_x,v_y)} , \]  
\[ \text{(B.19)} \]

where (B.18) is true since \( Q \in [0,1] \), such that replacing \( s \) by its maximal value \( S_{\text{max}}(d,v_x,v_y) \) provides an upper bound. The passage (B.19) follows from the definitions of \( T(d,v_x,v_y) \) and \( T(d,v_x,v_y,s) \). Before we substitute it back into (B.10) and then into (B.5), we summarize some minor results that will be needed in the sequel. Let us define \( d^* = \max\{v_x,v_y\} \).

**Lemma 2.** We have the following.

1. For fixed \((v_x,v_y)\), \( S_{\text{max}}(d,v_x,v_y) \) is a non-increasing sequence of \( d \).

2. For any \((v_x,v_y)\), we have that \( S_{\text{max}}(d^*,v_x,v_y) = \min\{v_x,v_y\} \).

**Proof:**

1. For fixed \((v_x,v_y)\), if we add to the graph an edge, we have only two options. On the one hand, we may connect vertices that belong to the same subgraph, such that the number of subgraphs remains the same. On the other hand, we can connect vertices that belong to different subgraphs, and then, the number of subgraphs decreases.

2. Without loss of generality, assume that \( v_x \leq v_y \). Then \( d^* = v_y \), hence it follows by definition that each column contains exactly one edge, such that the set of edges in each row provides a subgraph. Thus, the number of subgraphs equals exactly \( v_x \).

Now, we have that

\[ E[N^k] \leq \sum_{d=1}^{k} S(k,d) \sum_{(v_x,v_y)\in\mathcal{V}(d)} T(d,v_x,v_y) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x+v_y-S_{\text{max}}(d,v_x,v_y)} \]  
\[ = \sum_{d=1}^{k} \sum_{(v_x,v_y)\in\mathcal{V}(d)} S(k,d) \cdot T(d,v_x,v_y) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x+v_y-S_{\text{max}}(d,v_x,v_y)} \]  
\[ = \sum_{v_x=1}^{k} \sum_{v_y=1}^{k} \sum_{d=\max\{v_x,v_y\}}^{\min(k,v_x,v_y)} S(k,d) \cdot T(d,v_x,v_y) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x+v_y-S_{\text{max}}(d,v_x,v_y)} \]  
\[ \text{(B.22)} \]
Moving forward, we use the trivial bound thus, according to the first point in Lemma 2, we attain an upper bound by substituting \( d = d^* \). Finally, where (B.25) follows from the second point in Lemma 2 and (B.28) follows from the definition in (B.23).

where in (B.22) we changed the order of summation, and (B.23) is true since \( Q \in [0, 1] \), and thus, according to the first point in Lemma 2, we attain an upper bound by substituting \( d = d^* \). Moving forward, we use the trivial bound

\[
\mathcal{T}(d, v_x, v_y) \leq \left( \frac{v_x v_y}{d!} \right) \leq \frac{(v_x v_y)^d}{d!},
\]

and arrive at

\[
\mathbb{E} \left[ N^k \right] \leq \sum_{v_x=1}^{k} \sum_{v_y=1}^{k} \left( \sum_{d=\max\{v_x, v_y\}}^{\min\{k, v_x, v_y\}} S(k, d) \cdot \frac{(v_x v_y)^d}{d!} \right) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x + v_y - \min\{v_x, v_y\}} \quad \text{(B.25)}
\]

\[
\leq \sum_{v_x=1}^{k} \sum_{v_y=1}^{k} \left( \frac{k^2 d}{d!} \right) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x} \cdot Q^{v_y} \quad \text{(B.26)}
\]

\[
= \sum_{v_x=1}^{k} \sum_{v_y=1}^{k} B(k) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{\max\{v_x, v_y\}} \quad \text{(B.28)}
\]

where (B.25) follows from the second point in Lemma 2 and (B.28) follows from the definition in (B.26). Finally,

\[
\mathbb{E} \left[ N^k \right] \leq \sum_{v_x=1}^{k} \sum_{v_y=1}^{k} B(k) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{\max\{v_x, v_y\}}
\]

\[
+ \sum_{v_y=1}^{k} \sum_{v_x=1}^{k} B(k) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{\max\{v_x, v_y\}}
\]

\[
= \sum_{v_x=1}^{k} \sum_{v_y=1}^{k} B(k) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_x}
\]

\[
+ \sum_{v_y=1}^{k} \sum_{v_x=1}^{k} B(k) \cdot n^{v_x} \cdot n^{v_y} \cdot Q^{v_y}
\]

\[
\leq \sum_{v_x=1}^{k} v_x \cdot B(k) \cdot (n^2 Q)^{v_x} + \sum_{v_y=1}^{k} v_y \cdot B(k) \cdot (n^2 Q)^{v_y}
\]

\[
= 2B(k) \sum_{\ell=1}^{k} \ell \cdot (n^2 Q)^{\ell}.
\]
Now, if $n^2Q \geq 1$, then
\[
\mathbb{E}[N^k] \leq \left( 2B(k) \sum_{\ell=1}^{k} \right) \cdot (n^2Q)^k \leq k(k + 1)B(k) \cdot (n^2Q)^k,
\] (B.33)
and otherwise, if $n^2Q < 1$, then
\[
\mathbb{E}[N^k] \leq k(k + 1)B(k) \cdot n^2Q.
\] (B.35)

Thus, Lemma 1 is proved.

**Appendix C - Proof of Theorem 2**

**Preliminaries**

Before we analyze the various error probabilities, we first present bounds on $P$ and $Q$. Regarding $Q$, which is the probability for a non-matched pair to cross a threshold, we would expect it to converge to zero for an appropriate choice of parameters. The following generic lemma is proved in Appendix D.

**Lemma 3.** Let $d \in \mathbb{N}$ and $\rho \in (0, 1)$. Let
\[
(X, Y) \sim N^{\otimes d}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),
\] (C.1)
and
\[
\tilde{X} = \frac{X}{\|X\|}, \quad \tilde{Y} = \frac{Y}{\|Y\|}.
\] (C.2)

Then,
\[
\mathbb{P}\left\{ \tilde{X}^T \tilde{Y} \geq \frac{\rho}{2} \right\} \leq \frac{e^d}{\sqrt{d}} \cdot \left( 1 - \left( \frac{\rho}{2} \right)^2 \right)^{\frac{d}{2}}.
\] (C.3)

Regarding $P$, which is the probability for a matched pair to cross a threshold, we would expect it to converge to one for an appropriate choice of parameters. The following lemma is proved in Appendix E.

**Lemma 4.** Let $d \in \mathbb{N}$ and $\rho \in (0, 1)$. Let
\[
(X, Y) \sim N^{\otimes d}\left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right),
\] (C.4)
and
\[
\tilde{X} = \frac{X}{\|X\|}, \quad \tilde{Y} = \frac{Y}{\|Y\|}.
\] (C.5)

Then,
\[
P\{\tilde{X}^T \tilde{Y} \geq \frac{\rho}{2}\} \geq \frac{1}{8}. \] (C.6)

Furthermore,
\[
P\{\tilde{X}^T \tilde{Y} \geq \frac{\rho}{2}\} \geq 1 - \frac{\sqrt{d}}{2} \cdot \left(1 - \left(\frac{\rho a_d}{2}\right)^2\right)^{\frac{d}{2}} - 2 \exp \left\{\frac{-\sqrt{d}}{2} \cdot \left(\frac{1}{2} - \frac{1}{3\sqrt{d}}\right)\right\}, \] (C.7)
where,
\[
a_d = 2 \sqrt{\frac{\sqrt{d} - 1}{\sqrt{d} + 1}} - 1. \quad \text{(C.8)}
\]

Detection Error Probabilities

We shall assume that \(\rho\), as a function of \(n\) and \(d\), decreases sufficiently slowly such that \(n^2Q(d, \rho) < 1\) holds for all \(n\) and \(d\) (thanks to Lemma 3 this holds whenever (33) is true). This assumption implies that \(nQ(d, \rho) < 1\) holds for all \(n\) and \(d\) as well. As to the type-I error probability,
\[
P_{\text{FA}}(\phi_N) \leq \inf_{k \in \mathbb{N}} \frac{k(k+1)B(k)}{(\beta n P)^k} \left(\frac{n^2Q}{k} \cdot 1\{n^2Q \geq 1\} + n^2Q \cdot 1\{n^2Q < 1\}\right) \quad \text{(C.9)}
\]
\[
\leq \inf_{k \in \mathbb{N}} \frac{k(k+1)B(k)}{(\beta n P)^k} \quad \text{(C.10)}
\]
\[
\leq \inf_{k \in \mathbb{N}} \frac{k(k+1)B(k)}{(\beta n)^k} \xrightarrow{n \to \infty} 0, \quad \text{(C.11)}
\]
where (C.10) holds due to the assumption that \(n^2Q < 1\) for all \(n\) and \(d\) and (C.11) is due to (C.6) in Lemma 4. The type-II error probability can be upper-bounded as follows:
\[
P_{\text{MD}}(\phi_N) \leq \exp \left\{ -\min \left( (1 - \beta)^2 \frac{nP}{16nQ + 2}, (1 - \beta)^2 \frac{n}{12} \right) \right\} \quad \text{(C.12)}
\]
\[
\leq \exp \left\{ -\min \left( (1 - \beta)^2 \frac{n}{144}, (1 - \beta)^2 \frac{n}{12} \right) \right\} \xrightarrow{n \to \infty} 0, \quad \text{(C.13)}
\]
where (C.13) is due to (C.6) in Lemma 4 and the fact that \(nQ(d, \rho) < 1\) holds for all \(n\) and \(d\).
Alignment Error Probability

Define the error events:

\[ E_1 = \bigcup_{i=1}^{n} \{ \hat{X}_i^T \hat{Y}_i \leq \rho \frac{2}{\sqrt{2}} \}, \]  
(C.14)

and

\[ E_2 = \bigcup_{i=1}^{n} \bigcup_{j \neq i} \{ \hat{X}_i^T \hat{Y}_j \geq \rho \frac{2}{\sqrt{2}} \}. \]  
(C.15)

Then, the probability of alignment error is upper-bounded by

\[ P_{e|H_1}(\psi_e) \leq P_{1|\sigma} \{ E_1 \cup E_2 \}, \]  
(C.16)

and can be further upper-bounded using the union bound as

\[ P_{e|H_1}(\psi_e) \leq P_{1|\sigma} \left( \bigcup_{i=1}^{n} \{ \hat{X}_i^T \hat{Y}_i \leq \rho \frac{2}{\sqrt{2}} \} \right) + P_{1|\sigma} \left( \bigcup_{i=1}^{n} \bigcup_{j \neq i} \{ \hat{X}_i^T \hat{Y}_j \geq \rho \frac{2}{\sqrt{2}} \} \right) \]  
(C.17)

\[ \leq \sum_{i=1}^{n} P_{1|\sigma} \{ \hat{X}_i^T \hat{Y}_i \leq \rho \frac{2}{\sqrt{2}} \} + \sum_{i=1}^{n} \sum_{j \neq i} P_{1|\sigma} \{ \hat{X}_i^T \hat{Y}_j \geq \rho \frac{2}{\sqrt{2}} \} \]  
(C.18)

\[ = \sum_{i=1}^{n} \left[ 1 - P_{1|\sigma} \{ \hat{X}_i^T \hat{Y}_i \geq \rho \frac{2}{\sqrt{2}} \} \right] + \sum_{i=1}^{n} \sum_{j \neq i} P_{1|\sigma} \{ \hat{X}_i^T \hat{Y}_j \geq \rho \frac{2}{\sqrt{2}} \} \]  
(C.19)

\[ \leq n(1 - P) + n^2 Q. \]  
(C.20)

Upper-bounding (C.20) using Lemma 3 and (C.7) in Lemma 4 yields that

\[ P_{e|H_1}(\psi_e) \leq n^{10} \sqrt{d} \cdot \left( 1 - \left( \frac{\rho a d}{2} \right)^2 \right)^\frac{d}{2} + 2 \exp \left\{ -\frac{\sqrt{d}}{2} \left( \frac{1}{2} - \frac{1}{3 \sqrt{d}} \right) \right\} + n^{10} \sqrt{d} \cdot n^2 \cdot \left( 1 - \left( \frac{\rho}{2} \right)^2 \right)^\frac{d}{2} \]  
(C.21)

\[ = n^{10} \sqrt{d} \cdot \left( 1 - \left( \frac{\rho a d}{2} \right)^2 \right)^\frac{d}{2} + 2n \cdot \exp \left\{ -\frac{\sqrt{d}}{2} \left( \frac{1}{2} - \frac{1}{3 \sqrt{d}} \right) \right\} + n^2 n^{10} \sqrt{d} \cdot \left( 1 - \left( \frac{\rho}{2} \right)^2 \right)^\frac{d}{2}. \]  
(C.22)

Finally, if the condition (36) holds, then the first and the third terms in (C.22) converge to zero as \( n, d \to \infty \). If, in addition, \( n \) and \( d \) grow at the same rate, then the second term in (C.22) converges to zero as \( n, d \to \infty \) and (37) follows. The proof of Theorem 2 is now complete.
Appendix D - Proof of Lemma 3

Thanks to symmetry, $P\left\{ \hat{X}^T\hat{Y} \geq \theta \right\}$ is equal to the probability that a uniformly distributed vector on a $d$-dimensional hypersphere of unit norm will fall in a $d$-dimensional hyper-spherical cap with half-angle

$$\varphi = \arccos(\theta).$$ (D.1)

This probability is given by \[22\]

$$\frac{1}{2} B\left(\sin^2(\varphi); \frac{d-1}{2}, \frac{1}{2}\right),$$ (D.2)

where the complete beta function

$$B(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,$$ (D.3)

and the incomplete beta function

$$B(x; a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt.$$ (D.4)

Note that

$$\sin^2(\varphi) = 1 - \cos^2(\varphi) = 1 - \theta^2,$$ (D.5)

and then, it follows from Hölder’s inequality that

$$B\left(1 - \theta^2; \frac{d-1}{2}, \frac{1}{2}\right) \leq \left[ \int_0^{1-\theta^2} t^{(d-3)/2}dt \right]^{1/p} \left[ \int_0^{1-\theta^2} \left( \frac{1}{\sqrt{1-t}} \right)^q dt \right]^{1/q},$$ (D.7)

where $\frac{1}{p} + \frac{1}{q} = 1$ and $p, q > 1$. For the second integral in (D.7), for every $q \neq 2$,

$$\left[ \int_0^{1-\theta^2} \left( \frac{1}{\sqrt{1-t}} \right)^q dt \right]^{1/q} = \left( \frac{1 - \theta^2 - q}{2 - q} \right)^{1/q},$$ (D.8)

and substituting $q^* = \frac{15}{8}, 2 - q^* = \frac{1}{8},$ yields

$$\left[ \int_0^{1-\theta^2} \left( \frac{1}{\sqrt{1-t}} \right)^{q^*} dt \right]^{1/q^*} = [16 \left( 1 - \frac{\theta}{\sqrt{1}} \right)]^{\frac{8}{15}}$$ (D.9)

$$\leq 16^{\frac{8}{15}}$$ (D.10)

$$< 5.$$ (D.11)
For the first integral in (D.7) with $p^* = \frac{15}{7},$

$$\left[ \int_0^{1-\theta^2} p^*^{(d-3)/2}dt \right]^{1/p^*} = \left[ \frac{(1 - \theta^2)p^*(d-3)/2 + 1}{p^*(d-3)/2 + 1} \right]^{1/p^*} \leq \left( \frac{1 - \theta^2)^{d/2}}{(d - 2)^7/15}, \right.$$ (D.12)

and then,

$$B\left(1 - \theta^2, \frac{d - 1}{2}, \frac{1}{2}\right) \leq \frac{5(1 - \theta^2)^{d/2}}{(d - 2)^7/15}. \quad \text{(D.15)}$$

In order to bound the beta function, we use the following identity

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}, \quad \text{(D.16)}$$

where $\Gamma(\cdot)$ is the gamma function:

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx. \quad \text{(D.17)}$$

In general,

$$B\left(\frac{d - 1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \quad \text{(D.18)}$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}. \quad \text{(D.19)}$$

In order to bound the ratio of gamma functions, let us invoke Gautschi’s inequality:

**Lemma 5.** Let $x$ be a positive real number, and let $s \in (0, 1)$. Then,

$$x^{1-s} \leq \frac{\Gamma(x + 1)}{\Gamma(x + s)} \leq (x + 1)^{1-s}. \quad \text{(D.20)}$$

We get the following

$$B\left(\frac{d - 1}{2}, \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)} \quad \text{(D.21)}$$

$$= \frac{\sqrt{\pi}}{\Gamma\left(\frac{d-1+1}{2}\right)\Gamma\left(\frac{d-1+1}{2}\right)} \quad \text{(D.22)}$$

$$\geq \frac{\sqrt{\pi}}{(\frac{d}{2} - 1 + 1)^{1-\frac{d}{2}}} \quad \text{(D.23)}$$

$$= \sqrt{\frac{2\pi}{d^3}}. \quad \text{(D.24)}$$
where (D.23) is due to the upper bound in Lemma 5.

Now, combining (D.15) and (D.24) yields the upper bound
\[
\mathbb{P}\left\{ \tilde{X}^T \tilde{Y} \geq \theta \right\} \leq \frac{1}{2} \sqrt{\frac{d}{2\pi} \left( \frac{5(1 - \theta^2)^{d/2}}{(d-2)^{7/15}} \right)} \leq \frac{\sqrt{d}}{(d-2)^{7/15}} \cdot (1 - \theta^2)^{d/2} \leq \frac{\sqrt{d}}{d^{1/10}} \cdot (1 - \theta^2)^{d/2}, \tag{D.25}
\]
where (D.27) holds for any \( d \geq 8 \). Finally, substituting \( \theta = \frac{\rho}{2} \) yields
\[
\mathbb{P}\left\{ \tilde{X}^T \tilde{Y} \geq \frac{\rho}{2} \right\} \leq \sqrt{\frac{10}{d^{4/10}} \cdot \left( 1 - \left( \frac{\rho}{2} \right)^2 \right)^{d/2}}, \tag{D.29}
\]
which completes the proof of Lemma 3.

Appendix E - Proof of Lemma 4

Observe that under the assumption of
\[
(X, Y) \sim \mathcal{N}^\otimes d \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right), \tag{E.1}
\]
it holds that \( Y = \rho X + \sqrt{1 - \rho^2} Z \) with \( Z \sim \mathcal{N}(0_d, I_d) \) and independent of \( X \). Consider the following
\[
\mathbb{P}\left\{ \tilde{X}^T \tilde{Y} \geq \frac{\rho}{2} \right\}
= \mathbb{P}\left\{ \frac{X^T (\rho X + \sqrt{1 - \rho^2} Z)}{\|X\| \|\rho X + \sqrt{1 - \rho^2} Z\|} \geq \frac{\rho}{2} \right\} \tag{E.2}
= \mathbb{P}\left\{ \frac{\rho \|X\|^2 + \sqrt{1 - \rho^2} X^T Z}{\|X\| \sqrt{\rho^2 \|X\|^2 + 2\rho \sqrt{1 - \rho^2} X^T Z + (1 - \rho^2) \|Z\|^2}} \geq \frac{\rho}{2} \right\} \tag{E.3}
= \mathbb{P}\left\{ \frac{\rho^2 \|X\|^4 + 2\rho \sqrt{1 - \rho^2} \|X\|^2 \|X^T Z\| + (1 - \rho^2) (X^T Z)^2}{\rho^2 \|X\|^4 + 2\rho \sqrt{1 - \rho^2} \|X\|^2 \|X^T Z\| + (1 - \rho^2) \|X\|^2 \|Z\|^2} \geq \rho^2 \right\} \tag{E.4}
= \mathbb{P}\left\{ \frac{\rho^2 \|X\|^4 + 2\rho \sqrt{1 - \rho^2} \|X\|^3 \|Z\| \cdot \cos(\Phi_{XZ}) + (1 - \rho^2) \|X\|^2 \|Z\|^2 \cdot \cos^2(\Phi_{XZ})}{\rho^2 \|X\|^4 + 2\rho \sqrt{1 - \rho^2} \|X\|^3 \|Z\| \cdot \cos(\Phi_{XZ}) + (1 - \rho^2) \|X\|^2 \|Z\|^2} \geq \rho^2 \right\} \tag{E.5}
= \mathbb{P}\left\{ \frac{\rho^2 \|X\|^4 + 2\rho \sqrt{1 - \rho^2} \|X\|^3 \|Z\| \cdot \cos(\Phi_{XZ}) + (1 - \rho^2) \|X\|^2 \|Z\|^2 \cdot \cos^2(\Phi_{XZ})}{\rho^2 \|X\|^4 + 2\rho \sqrt{1 - \rho^2} \|X\|^3 \|Z\| \cdot \cos(\Phi_{XZ}) + (1 - \rho^2) \|X\|^2 \|Z\|^2} \geq \rho^2 \right\} \tag{E.6}
\]
\[
\begin{align*}
\mathbb{P}\left\{ \rho^2 \|X\|^2 + 2\rho \sqrt{1 - \rho^2} \|X\| \|Z\| \cdot \cos(\Phi_{xz}) + (1 - \rho^2) \|Z\|^2 \cdot \cos^2(\Phi_{xz}) \geq \theta^2 \right\} \\
\geq \mathbb{P}\left\{ \frac{\rho^2 \|X\|^2 + 2\rho \sqrt{1 - \rho^2} \|X\| \|Z\| \cdot \cos(\Phi_{xz})}{\rho^2 \|X\|^2 + 2\rho \sqrt{1 - \rho^2} \|X\| \|Z\| \cdot \cos(\Phi_{xz}) + (1 - \rho^2) \|Z\|^2} \geq \theta^2 \right\} \\
= \mathbb{P}\left\{ \cos(\Phi_{xz}) \geq \frac{(1 - \rho^2) \theta^2 \|Z\|^2 - \rho^2 (1 - \theta^2) \|X\|^2}{2 \rho \sqrt{1 - \rho^2} \|X\| \|Z\|} \right\} \quad \text{(E.9)}
\end{align*}
\]

Substituting \( \theta = \frac{\theta}{2} \) yields
\[
\begin{align*}
\mathbb{P}\left\{ \tilde{X}^T \tilde{Y} \geq \frac{\theta}{2} \right\} &\geq \mathbb{P}\left\{ \cos(\Phi_{xz}) \geq \frac{\sqrt{1 - \rho^2} \|Z\|}{2 \rho (1 - \theta^2) \|X\|} - \frac{\rho \|X\|}{2 \sqrt{1 - \rho^2} \|Z\|} \right\} \quad \text{(E.11)}
\end{align*}
\]

Since \( \Phi_{xz} \) is independent of the pair \( \|X\| \) and \( \|Z\| \), we may first calculate the probability in \( \text{E.14} \) w.r.t. \( \|X\| \) and \( \|Z\| \) and only then w.r.t. \( \Phi_{xz} \). In order to prove \( \text{E.6} \), let us denote the median of \( \|X\| \) and \( \|Z\| \) by \( m \). Averaging w.r.t. \( \|X\| \) and \( \|Z\| \) yields
\[
\begin{align*}
\mathbb{P}\left\{ \tilde{X}^T \tilde{Y} \geq \frac{\theta}{2} \right\} &= \int_0^\infty dx \int_0^\infty dz f_{X_d}(x) f_{Z_d}(z) \mathbb{P}\left\{ \cos(\Phi_{xz}) \geq \frac{\rho z}{8 x} - \frac{\rho x}{2 z} \right\} \quad \text{(E.15)}
\end{align*}
\]

In order to prove \( \text{E.17} \), we continue from \( \text{E.11} \) and arrive at
\[
\begin{align*}
\mathbb{P}\left\{ \tilde{X}^T \tilde{Y} \geq \theta \right\} &= \mathbb{P}\left\{ \frac{\rho^2 \|X\|^2 + 2\rho \sqrt{1 - \rho^2} \|X\| \|Z\| \cdot \cos(\Phi_{xz}) + (1 - \rho^2) \|Z\|^2 \cdot \cos^2(\Phi_{xz})}{\rho^2 \|X\|^2 + 2\rho \sqrt{1 - \rho^2} \|X\| \|Z\| \cdot \cos(\Phi_{xz}) + (1 - \rho^2) \|Z\|^2} \geq \theta^2 \right\} \quad \text{(E.21)}
\end{align*}
\]

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\[ \mathbb{P} \left\{ \rho^2(1 - \theta^2) \|X\|^2 - (1 - \rho^2)\theta^2 \|Z\|^2 + 2\rho \sqrt{1 - \rho^2} (1 - \theta^2) \|X\| \cdot \cos(\Phi_{xz}) + (1 - \rho^2) \|Z\|^2 \cdot \cos^2(\Phi_{xz}) \geq 0 \right\} \]
\[ = \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{-\rho(1 - \theta^2)\|X\| + \theta \sqrt{-\rho^2(1 - \theta^2)\|X\|^2 + (1 - \rho^2)\|Z\|^2}}{\sqrt{1 - \rho^2} \|Z\|} \right\} \]
\[ = \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{-\rho(1 - \theta^2)\|X\| + \theta \sqrt{1 - \rho^2} \|Z\| \sqrt{1 - \frac{\rho^2(1 - \theta^2)\|X\|^2}{(1 - \rho^2)\|Z\|^2}}}{\sqrt{1 - \rho^2} \|Z\|} \right\} \]
\[ = \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \theta \sqrt{1 - \frac{\rho^2(1 - \theta^2)\|X\|^2}{(1 - \rho^2)\|Z\|^2}} - \frac{\rho(1 - \theta^2)\|X\|}{\sqrt{1 - \rho^2} \|Z\|} \right\} \]
\[ \geq \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \theta - \frac{\rho(1 - \theta^2)\|X\|}{\sqrt{1 - \rho^2} \|Z\|} \right\} \]
\[ \geq \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \theta - \frac{\rho(1 - \theta^2)\|X\|}{(1 - \rho^2)^2 \|Z\|^2} \right\} . \]

Substituting \( \theta = \frac{\rho}{2} \) yields
\[ \mathbb{P} \left\{ X^T Y \geq \frac{\rho}{2} \right\} \geq \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{\rho}{2} - \frac{\rho(1 - \frac{\rho^2}{4})\|X\|}{(1 - \rho^2)^2 \|Z\|^2} \right\} \]
\[ \geq \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{\rho}{2} - \rho \sqrt{\frac{\|X\|^2}{\|Z\|^2}} \right\} . \]
\[ \text{Note that the expectations of } \|X\|^2 \text{ and } \|Z\|^2 \text{ are given by } d \text{ and let } \varepsilon > 0. \text{ Averaging w.r.t. } \|X\| \text{ and } \|Z\| \text{ yields} \]
\[ \mathbb{P} \left\{ X^T Y \geq \frac{\rho}{2} \right\} \]
\[ = \int_0^\infty dx \int_0^\infty dz f_{x_\rho}(x)f_{x_\rho}(z) \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{\rho}{2} - \rho \sqrt{\frac{x}{z}} \right\} \]
\[ \geq \int_{(1 - \varepsilon)d}^\infty \int_0^{(1 + \varepsilon)d} dx dz f_{x_\rho}(x)f_{x_\rho}(z) \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{\rho}{2} - \rho \sqrt{\frac{x}{z}} \right\} \]
\[ \geq \int_{(1 - \varepsilon)d}^\infty \int_0^{(1 + \varepsilon)d} dx dz f_{x_\rho}(x)f_{x_\rho}(z) \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{\rho}{2} - \rho \sqrt{\frac{(1 - \varepsilon)d}{(1 + \varepsilon)d}} \right\} \]
\[ = \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{\rho}{2} - \rho \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} \right\} \times \mathbb{P} \left\{ \|X\|^2 \geq (1 - \varepsilon)d \right\} \cdot \mathbb{P} \left\{ \|Z\|^2 \leq (1 + \varepsilon)d \right\} \]
\[ = \mathbb{P} \left\{ \cos(\Phi_{xz}) \geq \frac{\rho}{2} - \rho \sqrt{\frac{1 - \varepsilon}{1 + \varepsilon}} \right\} \times (1 - \mathbb{P} \left\{ \|X\|^2 \leq (1 - \varepsilon)d \} \right) \cdot (1 - \mathbb{P} \left\{ \|Z\|^2 \geq (1 + \varepsilon)d \right\} \). \]
\[ \text{(E.33)} \]

Now, the next task is to prove that each of the three factors in (E.33) converges to one as
Concerning the last two factors in (E.33), we have the following result, which is proved in Appendix F.

**Lemma 6.** Let $d \in \mathbb{N}$ and $\varepsilon > 0$. Let $X$ be a $\chi^2$-distributed random variable with $d$ degrees of freedom. Then, it holds that

\[
\mathbb{P}\{X \geq (1 + \varepsilon)d\} \leq \exp\left\{-\frac{d\varepsilon^2}{2} \left(\frac{1}{2} - \frac{\varepsilon}{3}\right)\right\}, \quad (E.34)
\]

\[
\mathbb{P}\{X \leq (1 - \varepsilon)d\} \leq \exp\left\{-\frac{d\varepsilon^2}{4}\right\}. \quad (E.35)
\]

Let us choose the sequence $\varepsilon(d) = \frac{1}{\sqrt[4]{d}}$, such that the right-hand sides of (E.34) and (E.35) converge to zero as $d \to \infty$, and denote the respective exponent functions by $E_1(d)$ and $E_2(d)$.

For this choice of $\varepsilon(d)$, we have that

\[
\frac{\rho}{2} - \rho \sqrt{\frac{1 - \varepsilon(d)}{1 + \varepsilon(d)}} = \frac{\rho}{2} - \rho \sqrt{\frac{1 - 1/\sqrt[4]{d}}{1 + 1/\sqrt[4]{d}}}
\]

\[
= \frac{\rho}{2} - \rho \sqrt{\frac{\sqrt[4]{d} - 1}{\sqrt[4]{d} + 1}}
\]

\[
= \frac{\rho}{2} \cdot \left(2 \sqrt{\frac{\sqrt[4]{d} - 1}{\sqrt[4]{d} + 1}} - 1\right)
\]

\[
\triangleq -\frac{\rho a_d}{2}, \quad (E.39)
\]

where $a_d \to 1$ as $d \to \infty$. Hence,

\[
\mathbb{P}\left\{\mathbf{X}^T \tilde{Y} \geq \frac{\rho}{2}\right\} \geq \mathbb{P}\left\{\cos(\Phi_{XZ}) \geq -\frac{\rho a_d}{2}\right\} \cdot \left(1 - e^{-E_1(d)}\right) \cdot \left(1 - e^{-E_2(d)}\right). \quad (E.40)
\]

Regarding the first factor in (E.40),

\[
\mathbb{P}\left\{\cos(\Phi_{XZ}) \geq -\frac{\rho a_d}{2}\right\} = 1 - \mathbb{P}\left\{\cos(\Phi_{XZ}) \leq -\frac{\rho a_d}{2}\right\}
\]

\[
= 1 - \mathbb{P}\left\{\cos(\Phi_{XZ}) \geq \frac{\rho a_d}{2}\right\}
\]

\[
\geq 1 - \frac{\sqrt[4]{d}}{\sqrt[4]{d}} \cdot \left(1 - \left(\frac{\rho a_d}{2}\right)^2\right)^\frac{4}{7}, \quad (E.43)
\]

where (E.42) follows by symmetry and (E.43) is due to Lemma 3.

In order to continue, let us recall the following elementary result:

**Lemma 7** ([23]). If $A_1, A_2, \ldots, A_n$ are numbers in $[0, 1]$ whose sum is denoted by $S_n$, then the Weierstrass product inequality states that

\[
(1 - A_1)(1 - A_2) \cdot \ldots \cdot (1 - A_n) \geq 1 - S_n. \quad (E.44)
\]
Now, lower-bounding (E.40) with (E.43) and then applying (E.44) to the resulted product, we arrive at
\[
\mathbb{P}\left\{\tilde{X}^T\tilde{Y} \geq \rho/2\right\} \geq \left[1 - \sqrt[n]{d} \cdot \left(1 - \left(\frac{\rho d}{2}\right)^2\right)^{\frac{d}{2}}\right] \cdot \left(1 - e^{-E_1(d)}\right) \cdot \left(1 - e^{-E_2(d)}\right) \quad \text{(E.45)}
\]
\[
\geq 1 - \sqrt[n]{d} \cdot \left(1 - \left(\frac{\rho d}{2}\right)^2\right)^{\frac{d}{2}} - e^{-E_1(d)} - e^{-E_2(d)} \quad \text{(E.46)}
\]
\[
\geq 1 - \sqrt[n]{d} \cdot \left(1 - \left(\frac{\rho d}{2}\right)^2\right)^{\frac{d}{2}} - 2 \exp\left\{-\frac{\sqrt[n]{d}}{2} \left(\frac{1}{2} - \frac{1}{3\sqrt[n]{d}}\right)\right\}, \quad \text{(E.47)}
\]
which completes the proof of Lemma 4.

Appendix F - Proof of Lemma 6

Starting with (E.34),
\[
\mathbb{P}\left\{\|Z\|^2 \geq (1 + \varepsilon)d\right\} = \mathbb{P}\left\{\sum_{i=1}^{d} Z_i^2 \geq (1 + \varepsilon)d\right\}, \quad \text{(F.1)}
\]
where \(Z_i \sim \mathcal{N}(0, 1)\) are IID. Consider the following for \(s \geq 0,\)
\[
\mathbb{P}\left\{\sum_{i=1}^{d} Z_i^2 \geq (1 + \varepsilon)d\right\} = \mathbb{P}\left\{\prod_{i=1}^{d} \exp\{sZ_i^2\} \geq \exp\{s(1 + \varepsilon)d\}\right\} \quad \text{(F.2)}
\]
\[
\leq \mathbb{E}\left[\prod_{i=1}^{d} \exp\{sZ_i^2\}\right] \quad \text{(F.3)}
\]
\[
= \prod_{i=1}^{d} \mathbb{E}\left[\exp\{sZ_i^2\}\right] \quad \text{(F.4)}
\]
where (F.3) follows from Markov's inequality and (F.4) from independence. Since (F.4) holds for every \(s \geq 0,\) it follows that
\[
\mathbb{P}\left\{\sum_{i=1}^{d} Z_i^2 \geq (1 + \varepsilon)d\right\} \leq \inf_{s>0} \prod_{i=1}^{d} \mathbb{E}\left[\exp\{sZ_i^2\}\right] \quad \text{(F.5)}
\]
For a random variable \(X \sim \mathcal{N}(0, \sigma^2),\) it holds that
\[
\mathbb{E}\left[\exp\{\alpha X^2\}\right] = \left\{\begin{array}{ll}
\sqrt{\frac{1}{1-2\alpha\sigma^2}} & \alpha < \frac{1}{2\sigma^2} \\
\infty & \alpha \geq \frac{1}{2\sigma^2}
\end{array}\right. \quad \text{(F.6)}
\]
and then
\[
\mathbb{P}\left\{\sum_{i=1}^{d} Z_i^2 \geq (1 + \varepsilon)d\right\} \leq \inf_{s \in (0, 1/2)} \frac{(1 - 2s)^{-d/2}}{\exp\{s(1 + \varepsilon)d\}} \quad \text{(F.7)}
\]
\[
= \inf_{s \in (0, 1/2)} \exp\left\{-\frac{d}{2} \log(1 - 2s) - s(1 + \varepsilon)d\right\}. \quad \text{(F.8)}
\]
It is rather easy to check that the minimizer in (F.8) is $s^* = \frac{\varepsilon}{2(1+\varepsilon)}$, for which

$$
\mathbb{P} \left\{ \sum_{i=1}^{d} Z_i^2 \geq (1+\varepsilon)d \right\} \leq \exp \left\{ -\frac{d}{2} \varepsilon - \log(1+\varepsilon) \right\} \quad (F.9)
$$

$$
\leq \exp \left\{ -\frac{d\varepsilon^2}{2} \left( \frac{1}{2} - \frac{\varepsilon}{3} \right) \right\}, \quad (F.10)
$$

where (F.10) is due to the fact that $\log(1+t) \leq t - \frac{t^2}{2} + \frac{t^3}{3}$ for all $t \in \mathbb{R}$.

Continuing to (E.35),

$$
\mathbb{P} \left\{ \left\| X \right\|^2 \leq (1-\varepsilon)d \right\} = \mathbb{P} \left\{ \sum_{i=1}^{d} X_i^2 \leq (1-\varepsilon)d \right\}, \quad (F.11)
$$

where $X_i \sim \mathcal{N}(0,1)$ are IID. Consider the following for $u \geq 0$,

$$
\mathbb{P} \left\{ \sum_{i=1}^{d} X_i^2 \leq (1-\varepsilon)d \right\} = \mathbb{P} \left\{ \prod_{i=1}^{d} \exp \{ -uX_i^2 \} \geq \exp \{ -u(1-\varepsilon)d \} \right\} \quad (F.12)
$$

$$
\leq \mathbb{E} \left[ \prod_{i=1}^{d} \exp \{ -uX_i^2 \} \right] \quad \text{exp} \{ -u(1-\varepsilon)d \} \quad (F.13)
$$

$$
= \prod_{i=1}^{d} \mathbb{E} \left[ \exp \{ -uX_i^2 \} \right], \quad (F.14)
$$

where (F.13) follows from Markov’s inequality and (F.14) from independence. Since (F.14) holds for every $u \geq 0$, it follows that

$$
\mathbb{P} \left\{ \sum_{i=1}^{d} X_i^2 \leq (1-\varepsilon)d \right\} \leq \inf_{u>0} \frac{\prod_{i=1}^{d} \mathbb{E} \left[ \exp \{ -uX_i^2 \} \right]}{\exp \{ -u(1-\varepsilon)d \}} \quad (F.15)
$$

$$
\leq \inf_{u>0} \frac{(1+2u)^{-d/2}}{\exp \{ -u(1-\varepsilon)d \}} \quad (F.16)
$$

$$
= \inf_{u>0} \exp \left\{ -\frac{d}{2} \log(1+2u) + u(1-\varepsilon)d \right\}. \quad (F.17)
$$

The minimizer in (F.17) is $u^* = \frac{\varepsilon}{2(1-\varepsilon)}$, for which

$$
\mathbb{P} \left\{ \sum_{i=1}^{d} X_i^2 \leq (1-\varepsilon)d \right\} \leq \exp \left\{ \frac{d}{2} \left[ \log(1-\varepsilon) + \varepsilon \right] \right\} \quad (F.18)
$$

$$
\leq \exp \left\{ -\frac{d\varepsilon^2}{4} \right\}, \quad (F.19)
$$

where (F.19) is due to the fact that $\log(1-v) \leq -v - \frac{v^2}{2}$ for all $v > 0$.  

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