Integrated Fluctuation Theorems for Generic Quantum State and Quantum Channel

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We derive the integrated fluctuation theorems for a quantum state of an arbitrary rank with a generic quantum channel described by a generic completely positive and trace-preserving map. We consider two different measurement schemes: two-time measurement scheme, which is a standard approach to the quantum fluctuation theorems, and the one-time measurement scheme, in which the entropy gain distribution is constructed by utilizing the quantum cross entropy. Deriving two lower bounds on the entropy gain from the derived integrated fluctuation theorems, we prove that the lower bound derived from the one-time measurement scheme is tighter, from which we demonstrate that the quantum cross entropy with respect to the output state gives an upper bound on the loss of the maximum amount of classical information transmittable through a quantum channel. We also discuss their properties in quantum autoencoder and the thermodynamics of the system undergoing a heat-emitting process which is described by a unital map.

I. Introduction

Quantum thermodynamics is a rapidly growing field to explore the fundamental laws of the thermodynamics in the nanoscale from the perspectives of the quantum information science [1–10]. In the nanoscale, the statistical fluctuations become more significant, which has been principally taken into account in the integrated fluctuation theorem [11–14].

The discovery of the integrated fluctuation theorem is the most important milestone in thermodynamics to date [15]. With its compact form, the integrated fluctuation theorem can be regarded as a first principle in the thermodynamics, from which many fundamental principles describing the thermodynamic phenomena can be derived, such as arrow of time [16] and the response theory [17, 18]. Also, the integrated fluctuation theorems, both in quantum and classical scenarios, can characterize the protocols applied on the system to achieve some tasks. For example, Sagawa and Ueda [19], and Fujitani and Suzuki [20], related the integrated fluctuation theorem with an efficacy of the feedback control for the manipulation of the total entropy gain via measurements. The relation between the integrated fluctuation theorems and the adiabaticity of the process was revealed by considering the state distinguishability [21–24]. Furthermore, in the context of quantum computing and communications, Gardas and Deffner [25] demonstrated that the integrated fluctuation theorem can be used to determine the dynamics of the quantum systems and the susceptibility to the thermal noise. Also, Kafri and Deffner [26] demonstrated the relation between the integrated fluctuation theorem and the Holevo information [27–29], which gives an upper limit on the amount of classical information that can be transmitted through the quantum channels.

A standard approach to the integrated fluctuation theorems in the quantum regime is called two-time measurement (TTM) scheme [30–37], in which the distribution of the measurement outcomes is constructed by the projection measurements on the system before and after the process. While this scheme has a good correspondence to the standard classical approach to the stochastic thermodynamics [38], in the quantum regime, it is considered to be inconsistent because it does not taken into account the quantum coherence [39] and the informational contribution of back-action of projection the measurements [21].

In order to improve the TTM scheme, focusing on the closed quantum systems, Deffner, Paz and Zurek in Ref. [21] proposed the one-time measurement (OTM) scheme for the quantum work, in which the work distribution is constructed by the average energy conditioned on the initial projection measurement outcome. The OTM scheme is thermodynamically consistent because it takes into account the back-action from the projection measurements and the quantum coherence [21]. In this scheme, we measure the initial state with the observable with the same eigenbases of the initial state, so that the wavefunction collapses into one of the eigenstate of the initial state. Then, the distribution of the measurement outcomes is constructed by evaluating the expectation value of the final and initial observable, which is conditioned on the initial measurement outcome. Since the state after the evolution from the eigenstate contains the quantum coherence, the second projection measurement

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will destroy this coherence. Therefore, the scheme avoiding the second projection measurement provides more precise description of the process.

Therefore, the integrated fluctuation theorems are essentially significant not only from the fundamental perspective, but also from the practical perspective. The better understanding of the integrated fluctuation theorems requires the most general treatment, which requires the formalism to be general enough for any quantum states and quantum channels. In detail, the generality of the formalism requires two requirements. First, we need to consider a state of an arbitrary rank. Second, the quantum channel in our formalism has to be a generic completely positive and trace-preserving (CPTP) map. However, to the best of our knowledge, there has not been such an integrated fluctuation theorem taking into account both of these elements.

In this paper, taking into account both of these two elements, we construct the entropy gain distributions in both the TTM and OTM scheme, and derive the corresponding integrated fluctuation theorems. From them, we derive the lower bounds on the entropy gain, which particularly becomes significant when we need to consider the information flow of the system into its bath [40, 41]. Finally, we prove that the OTM yields a tighter bound, which can be characterized by the quantum cross entropy of the output state, and relate it to the entropic disturbance, which is the loss of the Holevo information through a quantum channel. This also demonstrates that the quantum cross entropy plays a role in quantifying the least entropy gain and the upper bound on the loss of the maximum amount of the classical information transmittable in the quantum communication [42]. While there was less attention on the quantum cross entropy alone, the role of the quantum cross entropy has been rigorously studied in the contexts of quantum machine learning [43, 44]. In our paper, instead of focusing on some particular protocols, we provide an operational meaning of the quantum cross entropy in the quantum dynamics from the integrated fluctuation theorems.

This paper is organized as the following. In Sec. II, we first construct the entropy gain distributions in TTM scheme and OTM scheme, and derive the corresponding integrated fluctuation theorems in Theorem 1 for the TTM and Theorem 2 for the OTM scheme. Second, deriving the lower bounds on the entropy gain from the integrated fluctuation theorems, we prove that the OTM scheme yields a tighter bound in Theorem 3, and explain the operational meaning of the quantum cross entropy in terms of the classical information transmission in a generic quantum channel in Theorem 4. Third, in Sec. III, we illustrate two examples of the application of our results, which includes the quantum autoencoder (QAE) protocol [45, 46], which is a quantum data compression assisted by the near-term quantum algorithms [47–52], and the second law of thermodynamics, followed by the conclusion in Sec. IV.

II. Main Results

Let us consider a finite-dimensional Hilbert space $\mathcal{H}$ with its dimension $d \equiv \text{dim}(\mathcal{H})$. We consider a general setup, in which an input state $\rho_{\text{in}}$ has rank $r \equiv \text{rank}(\rho_{\text{in}}) \leq d$,

$$\rho_{\text{in}} = \sum_{i=1}^{r} p_i |p_i\rangle \langle p_i|,$$

where $\{p_i, |p_i\rangle\}_{i=1}^{r}$ is the eigensystem of $\rho_{\text{in}}$. Let $\Phi : \mathcal{H} \rightarrow \mathcal{H}$ be a CPTP map. Through this quantum channel $\Phi$, the output state $\rho_{\text{out}}$ becomes

$$\rho_{\text{out}} \equiv \Phi(\rho_{\text{in}}) \equiv \sum_{j=1}^{r'} q_j |q_j\rangle \langle q_j|,$$

where $r' \equiv \text{rank}(\rho_{\text{out}}) \leq d$ and $\{q_j, |q_j\rangle\}_{j=1}^{r'}$ is the eigensystem of $\rho_{\text{out}}$.

The eigenvalues of $\rho_{\text{in}}$ and $\rho_{\text{out}}$ satisfy the following conditions $0 < p_i \leq 1$ ($1 \leq i \leq r$), $p_i = 0$ ($r + 1 \leq i \leq d$) and $0 < q_j \leq 1$ ($1 \leq j \leq r'$), $q_j = 0$ ($r' + 1 \leq j \leq d$). Also, we have

$$\sum_{i=1}^{r} p_i = \sum_{j=1}^{r'} q_j = 1$$

and

$$\sum_{i=1}^{d} |p_i\rangle \langle p_i| = \sum_{j=1}^{d} |q_j\rangle \langle q_j| = 1,$$

where $1$ denotes the $d \times d$ identity matrix acting on $\mathcal{H}$.

We define the entropy gain in this process as

$$\Delta S \equiv S(\rho_{\text{out}}) - S(\rho_{\text{in}}),$$

where $S(\rho) \equiv -\text{Tr} [\rho \ln \rho]$ denotes the von-Neumann entropy of a state $\rho$.

We define $\Pi_{\text{in}}$ and $\Pi_{\text{out}}$ as the projectors onto $\text{supp}(\rho_{\text{in}})$ and $\text{supp}(\rho_{\text{out}})$ the support of the states $\rho_{\text{in}}$ and $\rho_{\text{out}}$, respectively,

$$\Pi_{\text{in}} \equiv \sum_{i=1}^{r} |p_i\rangle \langle p_i|,$n

$$\Pi_{\text{out}} \equiv \sum_{j=1}^{r'} |q_j\rangle \langle q_j|.$$

We also define the projectors onto the null spaces as

$$\Pi_{\text{in}} \equiv 1 - \Pi_{\text{in}} = \sum_{i=r+1}^{d} |p_i\rangle \langle p_i|,$n

$$\Pi_{\text{out}} \equiv 1 - \Pi_{\text{out}} = \sum_{j=r'+1}^{d} |q_j\rangle \langle q_j|.$$
A. Two-time measurement scheme

For the TTM scheme, we propose the following entropy gain distribution

\[
P(\sigma) \equiv \sum_{i,j=1}^{d} p_i(q_j \mid \Phi(p_i | p_i)) q_j \delta(\sigma - (b_j - a_i)), \quad (8)
\]

where \(a_i\) and \(b_j\) are defined as

\[
a_i = \begin{cases} 
- \ln p_i & (1 \leq i \leq r) \\
0 & \text{(otherwise)}
\end{cases}
\]

\[
b_j = \begin{cases} 
- \ln q_j & (1 \leq j \leq r') \\
0 & \text{(otherwise)}
\end{cases}
\]

(9)

We can find that \(P(\sigma) > 0\) and \(\int d\sigma P(\sigma) = 1\). Here, note that \(\sigma\) can be regarded as a classical random variable, which is the information gain \(b_j - a_i\) along a single trajectory \(i \to j\). With this distribution, from Eqs. (1), (2), and (4), and the linearity of the CPTP map, the von-Neumann entropy gain \(\langle \sigma \rangle_P \equiv \int d\sigma P(\sigma) \sigma\) is given by

\[
\langle \sigma \rangle_P = \sum_{i,j=1}^{d} p_i(q_j \mid \Phi(p_i | p_i)) q_j (b_j - a_i)
\]

\[
= - \sum_{i=1}^{r} \sum_{j=1}^{r'} p_i(q_j \mid \Phi(p_i | p_i)) q_j \ln q_j \\
+ \sum_{i=1}^{r} \sum_{j=1}^{r'} p_i(q_j \mid \Phi(p_i | p_i)) q_j \ln p_i
\]

\[
= - \sum_{j=1}^{r'} q_j \ln q_j + \sum_{i=1}^{r} p_i \ln p_i
\]

\[
=S(\rho_{out}) - S(\rho_{in})
\]

\[
=\Delta S,
\]

which is the exact entropy gain. This means that \(\sigma\) can be identified to be a random variable as a source of the entropy gain \(\Delta S\). Then, we can obtain our first main result:

**Theorem 1** (Integrated fluctuation theorem in the two-time measurement scheme). The integrated fluctuation theorem in the two-time measurement scheme is

\[
\langle e^{-\sigma} \rangle_P = \text{Tr} [\Phi(\Pi_{in})\rho_{out}] + r - \text{Tr} [\Pi_{out}\Phi(\Pi_{in})], \quad (11)
\]

which results in

\[
\Delta S \geq - \ln (\text{Tr} [\Phi(\Pi_{in})\rho_{out}) + r - \text{Tr} [\Pi_{out}\Phi(\Pi_{in})]). \quad (12)
\]

**Proof.** From Eqs. (1), (2), (8), and (9), we have

\[
\langle e^{-\sigma} \rangle_P = \int d\sigma P(\sigma) e^{-\sigma}
\]

\[
= \sum_{i,j=1}^{d} p_i(q_j \mid \Phi(p_i | p_i)) q_j e^{-b_j} e^{a_i},
\]

\[
= \sum_{i=1}^{r} \left( \sum_{j=1}^{d} q_j \mid \Phi(p_i | p_i)) q_j e^{-b_j} \right)
\]

\[
= \sum_{i=1}^{r} \sum_{j=1}^{r'} q_j \mid \Phi(p_i | p_i)) q_j \\
+ \sum_{i=1}^{r} \sum_{j=r'+1}^{d} q_j \mid \Phi(p_i | p_i)) q_j
\]

\[
= \text{Tr} [\Phi(\Pi_{in})\rho_{out}] + \text{Tr} [\Pi_{out}\Phi(\Pi_{in})],
\]

Because

\[
\text{Tr} [\Pi_{out}\Phi(\Pi_{in})] = \text{Tr} [\Phi(\Pi_{in})] - \text{Tr} [\Pi_{out}\Phi(\Pi_{in})],
\]

and

\[
\text{Tr} [\Phi(\Pi_{in})] = \sum_{i=1}^{r} \text{Tr} [\Phi(\rho_{in})] = r, \quad (15)
\]

we can obtain Eq. (11). From Jensen’s inequality \(\langle e^{-\sigma} \rangle_P \geq e^{-\langle \sigma \rangle_P}\), and Eq. (10), we can obtain Eq. (12). \(\square\)

B. One-time measurement scheme

Here, we propose the following entropy gain distribution in the OTM scheme

\[
P(\sigma) \equiv \sum_{i=1}^{r} p_i(\delta(\sigma - C(\Phi(\rho_i | \rho_i), \rho_{out}) - \ln p_i), \quad (16)
\]

where \(C(\rho_1, \rho_2) \equiv -\text{Tr} [\rho_1 \ln \rho_2]\) denotes the quantum cross entropy of \(\rho_1\) with respect to \(\rho_2\). Note that \(C(\rho_1, \rho_2) < \infty\) (supp(\(\rho_1\) \subseteq supp(\(\rho_2\))) and \(C(\rho_1, \rho_2) = \infty\) (otherwise). Also, for a state \(\rho\), by definition, we have \(C(\rho, \rho) = S(\rho)\). In Eq. (16), from Eqs. (1) and (2), we have supp(\(\Phi(\rho_i | \rho_i)\)) \subseteq supp(\(\rho_{out}\)), so that \(C(\Phi(\rho_i | \rho_i), \rho_{out}) < \infty\). Again, we also have \(\hat{P}(\sigma) > 0\) and \(\int d\sigma P(\sigma) = 1\). Now, the random variable \(\sigma\) has a different meaning than that in the TTM scheme. Here, let us define the transition probability

\[
P(j|i) \equiv \langle q_j | \Phi(\rho_i | \rho_i)\rangle q_j. \quad (17)
\]

Then, in OTM scheme, \(\sigma\) randomly takes \(\sum_j b_j P(j|i) - a_i = \sum_j (b_j - a_i) P(j|i)\), which is the conditional expectation of the information gain conditioned on the initial measurement outcome. With this distribution, similar to
the TTM scheme, the entropy gain \( \langle \sigma \rangle_{\bar{P}} \equiv \int d\sigma \bar{P}(\sigma) \sigma \) is given by

\[
\langle \sigma \rangle_{\bar{P}} = \sum_{i=1}^{r} p_i C(\Phi(|p_i|, p_{i\text{out}})) + \sum_{i=1}^{r} p_i \ln p_i \\
= - \sum_{i=1}^{r} \text{Tr} [\Phi(|p_i|, p_{i\text{out}}) \ln \rho_{i\text{out}}] + \sum_{i=1}^{r} p_i \ln p_i \\
= - \text{Tr} [\rho_{\text{out}} \ln \rho_{\text{out}}] + \text{Tr} [\rho_{\text{in}} \ln \rho_{\text{in}}] \\
= S(\rho_{\text{out}}) - S(\rho_{\text{in}}) \equiv \Delta S ,
\]

which is again the exact entropy gain. Here, \( \sigma \) can be also identified to be a random variable as a source of the entropy gain \( \Delta S \). Therefore, \( \bar{P}(\sigma) \) is a good definition. Then, we can obtain our second main result:

**Theorem 2** (Integrated fluctuation theorem in the one-time measurement scheme). The general integrated fluctuation theorem in the one-time measurement scheme is

\[
\langle e^{-\sigma} \rangle_{\bar{P}} = \sum_{i=1}^{r} e^{-C(\Phi(|p_i|, p_{i\text{out}}), \rho_{\text{out}})} ,
\]

which results in

\[
\Delta S \geq - \ln \left(\sum_{i=1}^{r} e^{-C(\Phi(|p_i|, p_{i\text{out}}), \rho_{\text{out}})}\right) .
\]

**Proof.** From Eqs. (1), (2), and (16), we have

\[
\langle e^{-\sigma} \rangle_{\bar{P}} = \int d\sigma \bar{P}(\sigma) e^{-\sigma} \\
= \sum_{i=1}^{r} p_i e^{-C(\Phi(|p_i|, p_{i\text{out}}), \rho_{\text{out}})} e^{-\ln p_i} \\
= \sum_{i=1}^{r} e^{-C(\Phi(|p_i|, p_{i\text{out}}), \rho_{\text{out}})} ,
\]

which proves Eq. (19). From Jensen’s inequality \( \langle e^{-\sigma} \rangle_{\bar{P}} \geq e^{-\langle \sigma \rangle_{\bar{P}}} \) and Eq. (18), we obtain Eq. (20). \( \square \)

**C. Comparison of the lower bounds**

From Theorems 1 and 2, we obtain two different bounds. Then, it is natural to ask which bound is tighter. Let us define

\[
L_{\text{ttm}} \equiv - \ln \left(\text{Tr} [\Phi(\Pi_\text{in}) \rho_{\text{out}}] + r - \text{Tr} [\Pi_{\text{out}} \Phi(\Pi_\text{in})]\right) ,
\]

\[
L_{\text{otm}} \equiv - \ln \left(\sum_{i=1}^{r} e^{-C(\Phi(|p_i|, p_{i\text{out}}), \rho_{\text{out}})}\right) .
\]

Then, we can obtain our third main result:

**Theorem 3** (Comparison of the lower bounds). The one-time measurement scheme yields tighter lower bound on entropy gain than the two-time measurement scheme

\[
\Delta S \geq L_{\text{otm}} \geq L_{\text{ttm}} .
\]

**Proof.** Because the quantum cross entropy can be lower bounded by using the state overlap [43],

\[
C(\Phi(|p_i|, \rho_{\text{out}})) \geq - \ln \text{Tr} [\Phi(|p_i|, p_{\text{out}}) \rho_{\text{out}}] ,
\]

and due to the linearity of the CPTP map, we have

\[
\sum_{i=1}^{r} e^{-C(\Phi(|p_i|, p_{i\text{out}}), \rho_{\text{out}})} \leq \text{Tr} [\Phi(\Pi_\text{in}) \rho_{\text{out}}] .
\]

Also, recall that

\[
r - \text{Tr} [\Pi_{\text{out}} \Phi(\Pi_\text{in})] = \text{Tr} [\Pi_{\text{out}} \Phi(\Pi_\text{in})] \geq 0 .
\]

Therefore, we can write

\[
\sum_{i=1}^{r} e^{-C(\Phi(|p_i|, p_{i\text{out}}), \rho_{\text{out}})} \leq \text{Tr} [\Phi(\Pi_\text{in}) \rho_{\text{out}}] + r - \text{Tr} [\Pi_{\text{out}} \Phi(\Pi_\text{in})] .
\]

From the definitions in Eq. (22), we can obtain

\[
e^{-L_{\text{otm}}} \leq e^{-L_{\text{ttm}}} ,
\]

which results in Eq. (23). \( \square \)

Therefore, the OTM scheme outperforms the TTM scheme in describing the model in the most general scenario. Given that \( \langle \sigma \rangle_{\bar{P}} \) and \( \langle \sigma \rangle_{\bar{P}} \) are the expectation of the information gain in the TTM scheme and the conditional information gain in the OTM scheme, respectively, we can say that the conditional information gain approach enables a more precise description to the lower bound on the entropy gain \( \Delta S \). From this, we can conclude that the quantum cross entropy in a generic quantum communication setup plays a role in quantifying the least entropy gain of the system through a quantum channel, which is an operational meaning of the quantum cross entropy in the context of the quantum communication.

When \( \Phi \) is particularly a unital map, it is well known that \( \Delta S \geq 0 \) [53]. For this case, with our formalism, we can obtain the following corollary:

**Corollary 1** (Lower bound from OTM scheme under a unital map). When \( \Phi \) is a unital map, \( L_{\text{otm}} \) is lower bounded as

\[
L_{\text{otm}} \geq - \ln (1 - \text{Tr} [\Phi(\Pi_\text{in}) \rho_{\text{out}}]) \geq 0 .
\]

**Proof.** Because \( \Phi \) is a unital map, we have

\[
\Phi(\mathbb{1}) = \Phi(\Pi_\text{in}) + \Phi(\Pi_\text{in}) = \mathbb{1} .
\]
Therefore,\[
\sum_{i=1}^{r} \text{Tr} \left[ \Phi(|p_i\rangle\langle p_i|) \rho_{\text{out}} \right] = \text{Tr} \left[ \Phi(\Pi_{\text{in}}) \rho_{\text{out}} \right] = 1 - \text{Tr} \left[ \Phi(\Pi_{\text{in}}) \rho_{\text{out}} \right] \leq 1.
\]
From Eqs. (22) and (24), we can finally obtain Eq. (29).

This result becomes important when we deal with the second law of thermodynamics in Sec. III B.

D. Entropic disturbance

The quantum cross entropy has another operational meaning in terms of the classical information transmitted through the quantum channel $\Phi$. The Holevo information sets the upper limit on the amount of classical information transmittable through a quantum channel [27–29]. The loss of Holevo information through the quantum channel $\Phi$ is called entropic disturbance [40, 54], which is defined as follows.

Let $E \equiv \{|p_i, \rho_i\}_{i=1}^{r}$ denotes an ensemble of input state $\rho_{\text{in}} \equiv \sum_{i=1}^{r} p_i \rho_i$ and $\Phi(E) \equiv \{|p_i, \Phi(\rho_i)\}_{i=1}^{r}$ denotes the ensemble of output state $\rho_{\text{out}} \equiv \Phi(\rho_{\text{in}}) = \sum_{i=1}^{r} p_i \Phi(\rho_i)$. Entropic disturbance is defined as $\Delta \chi \equiv \chi(\rho_{\text{in}}) - \chi(\Phi(E))$, where $\chi(E) \equiv S(\rho_{\text{in}}) - \sum_{i=1}^{r} p_i S(\rho_i)$ and $\chi(\Phi(E)) \equiv S(\rho_{\text{out}}) - \sum_{i=1}^{r} p_i S(\Phi(\rho_i))$ are the Holevo information of $\rho_{\text{in}}$ and $\rho_{\text{out}}$, respectively. In our case, we have $\rho_{\text{in}} = |p_i\rangle\langle p_i|$ so that $S(\rho_{\text{in}}) = S(|p_i\rangle\langle p_i|) = 0$.

In Ref. [40], a lower bound on $\Delta \chi$ was derived. Here, we provide an upper bound on the entropic disturbance, which is summarized in our fourth main result:

**Theorem 4 (Upper bound on entropic disturbance).** The entropic disturbance $\Delta \chi$ is upper bounded by

$$\Delta \chi \leq -L_{\text{otm}} + \sum_{i=1}^{r} p_i S(\Phi(|p_i\rangle\langle p_i|)).$$

**Proof.** Because $S(|p_i\rangle\langle p_i|) = 0$, from $\Delta S \equiv S(\rho_{\text{out}}) - S(\rho_{\text{in}})$, the entropic disturbance becomes $\Delta \chi = -\langle \sigma \rangle + \sum_{i=1}^{r} p_i S(\Phi(|p_i\rangle\langle p_i|)).$ From Eq. (23), can finally obtain Eq. (33).

Therefore, the upper bound of the entropic disturbance, which means the maximum amount of the loss of the Holevo information via the quantum channel $\Phi$, can be characterized by the quantum cross entropy [55].

III. Examples

In this section, we illustrate two examples of the applications of our results: quantum autoencoder and quantum thermodynamics.

A. Quantum Autoencoder

As our first example, we demonstrate the application of the integrated fluctuation theorems to the characterization of the quantum autoencoder (QAE) proposed by Romero, Olson and Aspuru-Guzik in Ref. [45]. The QAE is a quantum analogue of the (classical) variational autoencoder [56]. In the QAE, the encoding and decoding operations are described by a parameterized quantum circuit. The original quantum data is compressed to the latent system by tracing over the other subsystem. Then, one prepares the fresh qubits, and decompressed the quantum data through the decoding operation acting on the fresh-qubit system and the latent system. The goal of the protocol is to recover the quantum data in the output, implying that a low-dimensional feature quantum state is well extracted through the encoding process; thus, we can use the resulting decoding process as a generative model to produce a quantum state outside the training quantum dataset by fluctuating the feature state. The cost function dependent on these tunable parameters, which measures the distance between the output and input state, is constructed by the quantum computer, and the set of the parameters is optimized through training the cost function with the classical computers. Recently, as a practical near-term quantum algorithm, the QAE has been widely explored both theoretically and experimentally [57–68]. Here, we focus on the integrated fluctuation theorems with the quantum autoencoder channel, and explore the relation between the total entropy gain and the information contents of the compressed states as well as the fresh-qubit state.

We consider a composite Hilbert space $H = H_A \otimes H_B$, where $H_A$ ($H_B$) denotes the Hilbert space of the reduced quantum system $A$ ($B$). For the followings, let us regard $H_A$ as the latent Hilbert space, into which we compress our quantum data. Also, let us write $d_j$ as the dimension of the reduced Hilbert space $H_j$, $d_j \equiv \dim(H_j)$ ($j = A, B$), so that the dimension of the total system is given by $d = d_A d_B$.

Following Ref. [45], we consider the following scenario (see Fig. 1). In this setup, we apply a parameterized unitary $U$ to the input state $\rho_{\text{in}}$ and perform the partial trace over $H_B$ to compress the quantum data into the latent Hilbert space $H_A$. Then, we use the fresh qubits prepared in the state $\rho_B$ to decompress the data by ap-
plying the unitary $U^\dagger$ to generate the output state $\rho_{\text{out}}$. In this case, the CPTP map $\Phi$ becomes

$$\rho_{\text{out}} = \Phi(\rho_{\text{in}}) \equiv U^\dagger \left( \text{Tr}_B \left[ U \rho_{\text{in}} U^\dagger \right] \otimes \rho_B \right) U .$$  \hfill (34)

In the QAE, there are two protocols. One is setting the tunable parameters to construct $U$ and $U^\dagger$ for the compression and decompression, respectively. Here, note that we classify these two operations into one protocol because the tunable parameters are shared between $U$ and $U^\dagger$. The other one is the state preparation of $\rho_\text{out}$ in the fresh-qubit system $\mathcal{H}_B$. Therefore, in order to characterize the protocol, our integrated fluctuation theorems need to include the information contents characterizing these two protocols. Particularly, we want to explore how the information contents of the compressed states are taken into account. In order to explicitly discuss the compressed state, let us first define the compressed states

$$\rho_A \equiv \text{Tr}_B \left[ U \rho_{\text{in}} U^\dagger \right]$$ \hfill (35)

$$\rho_A^{(i)} \equiv \text{Tr}_B \left[ U |p_i\rangle\langle p_i| U^\dagger \right] .$$ \hfill (36)

Therefore, we can write $\rho_A = \sum_{i=1}^d p_i \rho_A^{(i)}$, so that we have $\text{supp}(\rho_A^{(i)}) \subseteq \text{supp}(\rho_A)$. Now, we are ready to discuss the integrated fluctuation theorems in the QAE.

First, let us consider the TTM scheme. The integrated fluctuation theorem for the QAE in the TTM scheme is given by (see Appendix A for the proof),

$$\langle e^{-\sigma} \rangle_P = e^{-S_2(\rho_B)} \sum_{i=1}^{r} \text{Tr} \left[ \rho_A^{(i)} \rho_A \right] - \text{Tr} \left[ \Pi_{\text{out}} \Phi(\Pi_{\text{in}}) \right] + r ,$$ \hfill (37)

where $S_2(\rho) \equiv - \ln \text{Tr} [\rho^2]$ denotes the Rényi-2 entropy of the state $\rho$ [69, 70].

Let us interpret the meaning of each term appearing in Eq. (37), which characterizes the QAE protocols. First, $S_2(\rho_B)$ is the Rényi-2 entropy of the prepared fresh-qubit state $\rho_B$, which is associated with the state preparation. Second, $\text{Tr} \left[ \rho_A^{(i)} \rho_A \right]$ is the state overlap of the compressed states in the latent Hilbert space, which is associated with the compressing protocol. Third, $\text{Tr} \left[ \Pi_{\text{out}} \Phi(\Pi_{\text{in}}) \right]$ is associated with the whole process of compressing and decompressing because this corresponds to the sum of the transition probabilities $P(j|i) \equiv \langle q_j | \Phi(|p_i\rangle\langle p_i|) | q_j \rangle$. The final term is $r$, which is the rank of the input state.

As a matter of fact, the rank $r$ has an important meaning in terms of the optimal compression rate, which has been rigorously studied by Ma et al. in Ref. [68]. In their result, when $\rho_B$ is a pure state and $r \geq d_A$, and when the eigenvalues of the input state are in the descending order, the maximally achievable compression rate over all the possible unitaries $U$ is given by $J_{\text{max}}(U) = \sum_{i=1}^{r} p_i d_A$, which is lower bounded by $J_{\text{max}}(U) \geq d_A/r$. The equality holds when $p_i = 1/r$. This bound can be interpreted as follows. Given a set of all the input state $\{ \rho_{\text{in}}^{(1)}, \rho_{\text{in}}^{(2)}, \cdots \}$ with a fixed rank $r = \text{rank}(\rho_{\text{in}}^{(k)}) \ (\forall k)$, we have a set of maximally achievable compression rates $\{ J_{\text{max}}^{(1)}(U), J_{\text{max}}^{(2)}(U), \cdots \}$, where $J_{\text{max}}^{(k)}(U)$ denotes the maximally achievable compression rates for $\rho_{\text{in}}^{(k)}$. Defining the lower bound as $\xi \equiv d_A/r$, then we can write $\xi = \min \{ J_{\text{max}}^{(1)}(U), J_{\text{max}}^{(2)}(U), \cdots \}$ [68]. Therefore, the rank of the input state in the integrated fluctuation theorem of the TTM scheme in Eq. (37) gives the maximally achievable compression rate in the worst scenario for the QAE when the fresh-qubit is in the pure state.

While the integrated fluctuation theorem of the TTM scheme can characterize the QAE protocol. It also has an issue when $\rho_{\text{in}}$ and $\rho_{\text{out}}$ have full rank, i.e. $r = r' = d$. In this case, Eq. (37) becomes (see Appendix B for the proof)

$$\langle e^{-\sigma} \rangle_P = d_B \text{Tr} \left[ \rho_B^2 \right] ,$$ \hfill (38)

which does not include the information about the compressed states in the latent Hilbert space.

In order to circumvent this issue, we can employ the OTM scheme. The integrated fluctuation theorem of the OTM scheme for the QAE is (see Appendix C for the proof)

$$\langle e^{-\sigma} \rangle_P = e^{-S(\rho_B)} \sum_{i=1}^{r} e^{-C(\rho_A^{(i)} \rho_A)} .$$ \hfill (39)

An important observation is that $\langle e^{-\sigma} \rangle_P$ includes two terms which characterize the protocols of the QAE. One is the von-Neumann entropy $S(\rho_B)$, which is the informational contribution from the state preparation protocol in the fresh-qubit system $\mathcal{H}_B$. The other one is associated with the quantum cross entropy $C(\rho_A^{(i)} \rho_A)$ with respect to the latent Hilbert space $\mathcal{H}_A$. This quantity can be regarded as a term characterizing the compression protocol of the QAE. Recall that from Theorem 3, the OTM scheme gives tighter bound on the entropy gain. Therefore, the OTM scheme can be regarded as providing a more precise characterization of the QAE protocol.

Equations (37) and (39) become useful when we want to estimate the information of the reduced state and the input state. Suppose that $\rho_{\text{in}}$ is an unknown state. Particularly, we also assume that its rank $r$ is initially unknown. In this case, $\langle e^{-\sigma} \rangle_P$ and $\langle e^{-\sigma} \rangle_P$ can be computed by taking the average of a large number of samples of $\sigma$, which leads us to extract the information-theoretic properties of $\rho_A^{(i)}$ the feature state at the middle of the channel and $\rho_{\text{in}}$ the input state.

In terms of the classical information transmission, from Eq. (33), the entropic disturbance in the QAE is given by (see Appendix D for the proof)

$$\Delta \chi \equiv \sum_{i=1}^{r} p_i S(\rho_A^{(i)}) + \ln \left( \sum_{i=1}^{r} e^{-C(\rho_A^{(i)} \rho_A)} \right) .$$ \hfill (40)

Therefore, the information contents of the compressed states contributes to setting an upper bound on the loss
of the maximum amount of classical information transmittable through the QAE. Also, note that for the QAE, \( \Delta \chi \) can be explicitly written as

\[
\Delta \chi = S(\rho_{in}) - S(\rho_A) - \sum_{i=1}^{r} p_i S(\rho_A^{(i)}),
\]

which implies that the loss of the maximum amount of classical information in the QAE protocol is independent of the choice of \( \rho_B \).

**B. Second law of thermodynamics**

For the second example, we explore the role of \( L_{otm} \) the lower bound from OTM scheme in the quantum thermodynamics. Let us consider a system \( \mathcal{H}_s \) initially prepared in \( \rho_{in} \) coupled to a heat bath \( \mathcal{H}_b \), whose initial state is prepared in the Gibbs state

\[
\rho_{eq}^0 \equiv \frac{e^{-\beta H_b}}{Z},
\]

where \( Z \equiv \text{Tr}[e^{-\beta H_b}] \) is the canonical partition function with inverse temperature \( \beta \) and \( H_b \) the time-independent bare Hamiltonian of the bath. Then, when \( \Phi \) is a thermal operation \([4-6]\) from \( t = 0 \) to \( t = \tau \),

\[
\rho_{out} = \Phi(\rho_{in}) = \text{Tr}_b [U_\tau (\rho_{in} \otimes \rho_{eq}) U_\tau^\dagger],
\]

where \( \langle Q_\tau \rangle \) is defined as the heat dissipation from the bath to the system;

\[
\langle Q_\tau \rangle = \text{Tr}[\rho_{eq}^0 H_b] - \text{Tr}[U_\tau (\rho_{in} \otimes \rho_{eq}^0) U_\tau^\dagger H_b].
\]

In this case, we can obtain the following corollary:

**Corollary 2.** For any systems undergoing the heat-emitting process which is described by a unital map, \( L_{otm} \) is always tighter than the Clausius bound \([4]\)

\[
\Delta S \geq L_{otm} \geq \beta \langle Q_\tau \rangle.
\]

**Proof.** When the system \( \mathcal{H}_s \) undergoes the heat-emitting process, the heat bath absorbs the energy, so that we have \( \langle Q_\tau \rangle \leq 0 \). From Corollary 1, we have \( L_{otm} \geq 0 \). Therefore, we obtain Eq. (46).

The spin-boson model \([71]\) with a boson heat bath is an example of the system undergoing the heat-emitting process which is described by a unital map (see Fig. 2). In this case, \( \mathcal{H}_s \) and \( \mathcal{H}_a \) denote the two-level atomic system and the boson heat bath, respectively. This model can be described by the following time-independent Hamiltonian (we set \( \hbar = 1 \))

\[
H = \frac{\omega_0}{2} \sigma_z + H_b + \sigma_z \otimes \sum_k (g_k a_k + g_k^* a_k^\dagger),
\]

where

\[
H_b = \sum_k \omega_k a_k^\dagger a_k
\]

described by the following interaction strength \( g_k \). The Hamiltonian of the spin-boson model is described in Eq. (47).

**FIG. 2. Spin-boson model:** A two-level atom is interacting with multiple boson modes. Each boson mode is decoupled from each other, and has different angular frequency \( \omega_k \). The atomic system is coupled to each mode with different interaction strength \( g_k \). The Hamiltonian of the spin-boson model is described in Eq. (47).
Then, we can write the heat-emitting process.

\[ \langle \mathcal{H}_0 \rangle = \frac{1}{t} \int_0^t H(t_1) dt_1 = \sigma_z \otimes \sum_k \left( G_k(t) a_k - G^*_k(t) a_k^\dagger \right), \]  

(51)

with

\[ G_k(t) \equiv g_k \frac{\sin(\omega_k t/2)}{\omega_k t/2} e^{-i\omega_k t/2}. \]  

(52)

The term \( \mathcal{H}_1 \) is given by (See Appendix F for the proof)

\[ \mathcal{H}_1 = -i \int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)], \]  

(53)

which is just a real number. Then, the propagator becomes

\[ U_t = \exp \left[ -it \sum_k \left( \sigma_z \otimes (G_k(t) a_k + G^*_k(t) a_k^\dagger) \right) \right] e^{-i\mathcal{H}_1}. \]  

(54)

Let us define \( u_t \) as

\[ u_t \equiv \exp \left[ -it \sum_k \left( G_k(t) a_k + G^*_k(t) a_k^\dagger \right) \right]. \]  

(55)

Then, we can write

\[ U_t = \begin{pmatrix} u_t e^{-i\mathcal{H}_1} & 0 \\ 0 & a_k^\dagger e^{-i\mathcal{H}_1} \end{pmatrix}. \]  

(56)

With this propagator, from Baker-Hausdorff-Campbell’s formula, we have

\[ U_t^\dagger a_k U_t = a_k + itG_k(t) \sigma_z \]  

(57)

\[ U_t^\dagger a_k^\dagger U_t = a_k^\dagger - itG^*_k(t) \sigma_z. \]  

(58)

Therefore, we can write (See Appendix G for the proof)

\[ U_t^\dagger H_b U_t = H_b + it \sum_k \omega_k \sigma_z \otimes (G_k(t) a_k^\dagger - G^*_k(t) a_k) \]  

(59)

\[ + \sum_k \omega_k |G_k(t)|^2 t^2. \]

Therefore, from Eqs. (59) and (52), we can explicitly write \( \langle Q_+ \rangle \) as

\[ \langle Q_+ \rangle = -\sum_k \omega_k |g_k|^2 \left( \sin \frac{\omega_k \tau}{2} \right)^2 \leq 0, \]  

(60)

which implies the atomic system undergoes the heat-emitting process.

From Eq. (56), the quantum channel \( \Phi \) is a unital map (see Appendix H for the proof):

\[ \Phi(\mathbb{1}) \equiv \text{Tr}_{b} [U_r (\mathbb{1} \otimes \rho_b^\text{eq}) U_r^\dagger] = \mathbb{1}. \]  

(61)

Thus, from Corollary 1, we must have \( L_{\text{tot}} \geq 0 \). Therefore, from Eq. (60), for the spin-boson model, we can obtain Eq. (46). These results demonstrate that the quantum cross entropy appearing in the integrated fluctuation theorems from the OTM scheme characterizes the heat operation more precisely than the Clausius bound when the system undergoes the heat-emitting process which is described by a unital map.

IV. Conclusion

By constructing the entropy gain distributions for a quantum state of arbitrary rank and a quantum channel described by a generic completely positive and trace-preserving map in two-time measurement and one-time measurement scheme, we derived the corresponding integrated fluctuation theorems for both measurement schemes (Theorem 1 and 2). We further proved that the one-time measurement scheme yields a tighter lower bound (Theorem 3), which is characterized by the quantum cross entropy with respect to the output state. We also demonstrate the relation between the entropic disturbance and the quantum cross entropy (Theorem 4). These results indicate that the quantum cross entropy plays a role in quantifying the least entropy gain and the upper bound on the loss of Holevo information, which is the maximum amount of classical information transmittable through a quantum channel. These are the operational meanings of the quantum cross entropy in the quantum communication.

In order to explore the application of our results in near-term quantum algorithms and quantum thermodynamics, we focused on the quantum autoencoder and second law of thermodynamics with a system undergoing the heat-emitting process which is described by a unital map. We have demonstrated the characterization of the quantum autoencoder protocols with the integrated fluctuation theorems, and the role of the one-time measurement scheme as well as the quantum cross entropy in the heat operation.

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Appendix

A. Proof of Eq. (37)

From Eq. (34), we have

\[\sum_{i=1}^{r} \sum_{j=1}^{r'} q_j\langle q_j|\Phi(|p_i\rangle\langle p_i|)|q_j\rangle = \text{Tr} [\Phi(\Pi_{\text{in}})\rho_{\text{out}}] \]

\[= \text{Tr} [U^\dagger (U\Pi_{\text{in}}U^\dagger) \otimes \rho_B] UU^\dagger (\rho_A \otimes \rho_B)U \]

\[= \text{Tr} [\rho_B^2] \sum_{i=1}^{r} \text{Tr} [(U|p_i\rangle\langle p_i|U^\dagger)\rho_A] \]

\[= \text{Tr} [\rho_B^2] \sum_{i=1}^{r} \rho_A^{(i)} \rho_A \] .  

(A1)

From the definition of the Rényi-2 entropy,

\[S_2(\rho_B) = -\ln \text{Tr} [\rho_B^2] \] ,

we have

\[\sum_{i=1}^{r} \sum_{j=1}^{r'} q_j\langle q_j|\Phi(|p_i\rangle\langle p_i|)|q_j\rangle = e^{-S_2(\rho_B)} \sum_{i=1}^{r} \text{Tr} [\rho_A^{(i)} \rho_A] \] .  

(A3)

From Eq. (11) in Theorem 1, we obtain

\[\langle e^{-\sigma}\rangle_P = e^{-S_2(\rho_B)} \sum_{i=1}^{r} \text{Tr} [\rho_A^{(i)} \rho_A] - \text{Tr} [\Pi_{\text{out}}\Phi(\Pi_{\text{in}})] + r . \]  

(A4)

B. Proof of Eq. (38)

When \(r = d\), we have

\[\sum_{i=1}^{r} \text{Tr} [\rho_A^{(i)} \rho_A] = \sum_{i=1}^{d} \text{Tr} [U|p_i\rangle\langle p_i|U^\dagger] \rho_A] \]

\[= \text{Tr} [U|\mathbb{1}\rangle\langle \mathbb{1}|U^\dagger] \rho_A] \]

\[= d_B \] .  

(B1)

Also, when \(r = r' = d\), we have

\[\Pi_{\text{out}} = \Pi_{\text{in}} = \mathbb{1} . \]  

(B2)

Therefore,

\[d - \text{Tr} [\Pi_{\text{out}}\Phi(\Pi_{\text{in}})] = d - \text{Tr} [\Phi(\mathbb{1})] \]

\[= d - \text{Tr} [U^\dagger (U\Pi_{\text{out}}U^\dagger) \otimes \rho_B] U \]

\[= d - d_B \text{Tr} [\mathbb{1} \otimes \rho_B] \]

\[= d - d_A d_B \]

\[= d - d \]

\[= 0 . \]  

(B3)
From Eq. (A4), we can finally obtain
\[
\langle e^{-\sigma} \rangle_P = d_B e^{-S_2(\rho_B)}.
\] (B4)

C. Proof of Eq. (39)

From Eq. (34), we have
\[
C(\Phi(|p_i\rangle\langle p_i|), \rho_{\text{out}}) = C(\left( U^\dagger(\text{Tr}_B[U|p_i\rangle\langle p_i|U^\dagger] \otimes \rho_B)U, U^\dagger(\rho_A \otimes \rho_B)U \right)
= C(\rho_A^{(i)} \otimes \rho_B, \rho_A \otimes \rho_B)
= C(\rho_A^{(i)}, \rho_A) + C(\rho_B, \rho_B)
= S(\rho_B) + C(\rho_A^{(i)}, \rho_A).
\] (C1)

Note that \(C(\rho_A^{(i)}, \rho_A) < \infty\) due to
\[
\rho_A = \sum_{i=1}^{r} p_i \rho_A^{(i)},
\] (C2)
meaning that
\[
\text{supp}(\rho_A^{(i)}) \subseteq \text{supp}(\rho_A).
\] (C3)

Therefore, from Eq. (19) in Theorem 2, we can obtain
\[
\langle e^{-\sigma} \rangle_{\tilde{P}} = e^{-S(\rho_B)} \sum_{i=1}^{r} e^{-C(\rho_A^{(i)}, \rho_A)},
\] (C4)

D. Proof of Eq. (40)

For the QAE, we have
\[
L_{\text{otm}} = S(\rho_B) - \ln \left( \sum_{i=1}^{r} e^{-C(\rho_A^{(i)}, \rho_A)} \right).
\] (D1)

Also, due to
\[
S(\Phi(|p_i\rangle\langle p_i|)) = S(\rho_A^{(i)}) + S(\rho_B),
\] (D2)
we have
\[
\sum_{i=1}^{r} p_i S(\Phi(|p_i\rangle\langle p_i|)) = \sum_{i=1}^{r} p_i S(\rho_A^{(i)}) + S(\rho_B).
\] (D3)

Therefore, the upper bound on \(\Delta \chi\) in the QAE is given by
\[
\Delta \chi \leq \sum_{i=1}^{r} p_i S(\rho_A^{(i)}) + \ln \left( \sum_{i=1}^{r} e^{-C(\rho_A^{(i)}, \rho_A)} \right).
\] (D4)
E. Proof of Eq. (51)

In this section, we provide details of deriving Eq. (51). $\mathcal{P}_0$ is defined as

$$\mathcal{P}_0 = \frac{1}{t} \int_0^t H(t_1) dt_1.$$  \hfill (E1)

Because in the interaction picture, we have

$$H(t) = \sigma_z \sum_k (g_k a_k e^{-i\omega_k t} + g_k^* a_k^\dagger e^{i\omega_k t}),$$  \hfill (E2)

$\mathcal{P}_0$ can be explicitly given by

$$\mathcal{P}_0 = \frac{1}{t} \int_0^t \sigma_z \otimes \sum_k \left( g_k a_k e^{-i\omega_k t_1} + g_k^* a_k^\dagger e^{i\omega_k t_1} \right) dt_1$$

$$= \frac{1}{t} \int_0^t \sigma_z \otimes \sum_k \left( g_k a_k \left( \frac{e^{-i\omega_k t} - 1}{-i\omega_k} \right) + g_k^* a_k^\dagger \left( \frac{e^{i\omega_k t} - 1}{+i\omega_k} \right) \right) dt_1$$

$$= \frac{1}{t} \sigma_z \otimes \sum_k \frac{i}{\omega_k} \left( g_k a_k e^{-i\omega_k t/2} \left( e^{-i\omega_k t /2} - e^{+i\omega_k t/2} \right) - g_k^* a_k^\dagger e^{+i\omega_k t/2} \left( e^{i\omega_k t/2} - e^{-i\omega_k t/2} \right) \right)$$

$$= \frac{1}{t} \sigma_z \otimes \sum_k \frac{2}{\omega_k} \left( g_k a_k e^{-i\omega_k t/2} \sin \left( \frac{\omega_k t}{2} \right) - g_k^* a_k^\dagger e^{+i\omega_k t/2} \sin \left( \frac{\omega_k t}{2} \right) \right)$$

$$= \sigma_z \otimes \sum_k \left( g_k a_k e^{-i\omega_k t/2} \frac{\sin(\omega_k t/2)}{\omega_k t/2} - g_k^* a_k^\dagger e^{+i\omega_k t/2} \frac{\sin(\omega_k t/2)}{\omega_k t/2} \right).$$

Defining

$$G_k(t) = g_k \frac{\sin(\omega_k t/2)}{\omega_k t/2} e^{-i\omega_k t/2},$$  \hfill (E4)

we can finally obtain

$$\mathcal{P}_0 = \sigma_z \otimes \sum_k \left( G_k(t) a_k - G^*(t) a_k^\dagger \right).$$  \hfill (E5)

F. Proof of Eq. (53)

In this section, we provide the details of deriving Eq. (53). $\mathcal{P}_1$ is defined as

$$\mathcal{P}_1 = -\frac{i}{2t} \int_0^t dt_1 \int_0^{t_1} dt_2 [H(t_1), H(t_2)].$$  \hfill (F1)

Because

$$[H(t_1), H(t_2)] = \sum_{k,k'} \left[ g_k a_k e^{-i\omega_k t_1} + g_k^* a_k^\dagger e^{i\omega_k t_1}, g_k' a_{k'} e^{-i\omega_{k'} t_2} + g_{k'}^* a_{k'}^\dagger e^{i\omega_{k'} t_2} \right]$$

$$= \sum_k |g_k|^2 \left( e^{-i\omega_k (t_1 - t_2)} - e^{i\omega_k (t_1 - t_2)} \right)$$

$$= -2i \sum_k |g_k|^2 \sin(\omega_k (t_1 - t_2)).$$  \hfill (F2)
we have
\[ \int_0^{t_1} \sin(\omega_k(t_1 - t_2)) \, dt_2 = \frac{1}{\omega_k} \left( 1 - \cos(\omega_k t_1) \right), \] (F3)
and
\[ \int_0^{t} dt_1 \int_0^{t_1} dt_2 \sin(\omega_k(t_1 - t_2)) = \int_0^{t} \frac{1}{\omega_k} \left( 1 - \cos(\omega_k t) \right) \, dt_1 = \frac{1}{\omega_k} \left( t - \frac{1}{\omega_k} \sin(\omega_k t) \right). \] (F4)
Therefore, we can finally obtain
\[ \bar{\Pi}_1 = -\frac{i}{2t} (\omega - 2i) \cdot \sum_k |g_k|^2 \left( t - \frac{1}{\omega_k} \sin(\omega_k t) \right) = -\sum_k |g_k|^2 \left( 1 - \frac{\sin(\omega_k t)}{\omega_k t} \right). \] (F5)

G. Proof of Eq. (59)

In this section, we provide the details of deriving Eq. (59). From Eqs. (57) and (58), we can write
\[ U_t^\dagger H_b U_t = \sum_k \omega_k \left( U_t^\dagger a_k^\dagger U_t \right) \left( U_t^\dagger a_k U_t \right) \]
\[ = \sum_k \omega_k \left( a_k^\dagger - i t G_k^*(t) \sigma_z \right) \left( a_k + i t G_k(t) \sigma_z \right) \]
\[ = H_b + i t \sum_k \omega_k \sigma_z \otimes (G_k(t) a_k^\dagger - G_k^*(t) a_k) + \sum_k \omega_k |G_k(t)|^2 t^2. \] (G1)

H. Proof of Eq. (61)

In this section, we provide the details of deriving Eq. (61). From Eq. (56), we have
\[ U_\tau (\mathbb{1} \otimes \rho_b^{eq}) U_\tau^\dagger = \begin{pmatrix} u_\tau e^{-i\bar{\Pi} \tau} & 0 & 0 \\ 0 & u_\tau^\dagger e^{-i\bar{\Pi} \tau} & 0 \\ 0 & 0 & u_\tau e^{i\bar{\Pi} \tau} \end{pmatrix} \begin{pmatrix} \rho_b^{eq} & 0 & 0 \\ 0 & \rho_b^{eq} & 0 \\ 0 & 0 & \rho_b^{eq} \end{pmatrix} \begin{pmatrix} u_\tau e^{i\bar{\Pi} \tau} & 0 & 0 \\ 0 & u_\tau^\dagger e^{-i\bar{\Pi} \tau} & 0 \\ 0 & 0 & u_\tau^\dagger \rho_b^{eq} u_\tau \end{pmatrix}. \] (H1)
Let the computational bases be \(|0\rangle \equiv (1 ~ 0)^T\) and \(|1\rangle \equiv (0 ~ 1)^T\). Then, we have
\[ U_\tau (\mathbb{1} \otimes \rho_b^{eq}) U_\tau^\dagger = |0\rangle\langle 0| \otimes (u_\tau \rho_b^{eq} u_\tau^\dagger) + |1\rangle\langle 1| \otimes (u_\tau^\dagger \rho_b^{eq} u_\tau). \] (H2)
Because \(\text{Tr} \left[ u_\tau \rho_b^{eq} u_\tau^\dagger \right] = \text{Tr} \left[ u_\tau^\dagger \rho_b^{eq} u_\tau \right] = 1\), we can finally obtain
\[ \Phi(\mathbb{1}) \equiv \text{Tr}_b \left[ U_\tau (\mathbb{1} \otimes \rho_b^{eq}) U_\tau^\dagger \right] = \mathbb{1}. \] (H3)

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